

Optimization, Auto-Differentiation, and Tomlab

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Auto-Differentiation(AD)

Derivatives and Numerical Methods

There are two general types of algorithms for optimizers/solvers/etc.:

1 Derivative-free:

- e.g. Simplex and Nelder-Mead. This is Matlab's `fminsearch`
- Also, “costly global function” optimization
- Avoid at all costs (though sometimes don't have a choice)

2 Derivatives

- Pretty much every other algorithm, especially for large number of variables/constraints
- Including global optimization techniques (which use derivatives locally)

Key derivatives to calculate are:

- Gradient of objective
- Hessian of objective (nonlinear least squares and some algorithms only use gradient)
- Jacobian of constraints

Calculating Derivatives

How to calculate derivatives for the objective and constraints?

1 Calculate by hand

- Sometimes, though not always, the most accurate and fastest option
- But algebra is error prone for non-trivial setups
- (note: many optimizers have a way to check your analytical derivatives)

2 Finite-differences:

- $\partial_{x_i} f(x_1, \dots, x_N) \approx \frac{f(x_1, \dots, x_i + \Delta, \dots, x_N) - f(x_1, \dots, x_i, \dots, x_N)}{\Delta}$
- Evaluates function at least N extra times to get a gradient
- # evaluations for Jacobians with M constraints even worse
- Large Δ is numerically stable but inaccurate, small Δ is unstable
- **Avoid like the plague!** (and is what matlab does out of the box)

3 Auto-differentiation

- Not a form of finite-differences or numeric differentiation
- Essentially analytical. Repeated use of the chain-rule
- Does not work for every function, but only evaluates $f(\cdot)$ once if it works—i.e. $O(1)$ not $O(N \times M)$ for $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$

Auto-differentiation (adapted from Wikipedia)

- Remember the chain rule: $\frac{dy}{dx} = \frac{dy}{dw} \frac{dw}{dx}$
- Consider functions composed of calculations with fundamental operations (with known analytical derivatives)
- For example, consider function: $f(x_1, x_2) = x_1 x_2 + \sin(x_1)$

Operations to compute value	Operations to compute $\frac{df(x_1, x_2)}{dx_1}$
$w_1 = x_1$	$\frac{dw_1}{dx_1} = 1$ (seed)
$w_2 = x_2$	$\frac{dw_2}{dx_1} = 0$ (seed)
$w_3 = w_1 \cdot w_2$	$\frac{dw_3}{dx_1} = w_2 \cdot \frac{dw_1}{dx_1} + w_1 \cdot \frac{dw_2}{dx_1}$
$w_4 = \sin w_1$	$\frac{dw_4}{dx_1} = \cos w_1 \cdot \frac{dw_1}{dx_1}$
$w_5 = w_3 + w_4$	$\frac{dw_5}{dx_1} = \frac{dw_3}{dx_1} + \frac{dw_4}{dx_1}$

- Generalizes to multiple variables. AD takes source code and generates the derivatives at the same time (i.e. doesn't increase with # variables)

Implementations of AD

- A field unto itself. Do not implement directly
- Implementation is language dependent. Two approaches:
 - *Source code transformation*: utility (outside of the language itself) reads in the code for your function, and generates a function which calculates value and derivative. Rerun if you change your code
 - *Operator Overloading*: Takes your existing functions, and passes variables that act like numbers, but are actually recording and tracing the chain rule steps/etc. Can be magical, or infuriating
- Implementation depends on the language:
 - *Fortran*: usually needs SCT. Many choices: e.g.
<http://tapenade.inria.fr:8080/tapenade/index.jsp>
 - *Python*: <https://github.com/LowinData/pyautodiff> and <https://pythonhosted.org/algopy/>
 - *C++*: overloading <http://www.fadbad.com/fadbad.html>,...
 - *R*: <https://cran.r-project.org/web/packages/madness/index.html>
 - *Matlab*: open source SCT (e.g. AdiMat) not very good. Use Tomlab/MAD instead, coupled with the Tomlab optimizer.

Sparsity

Sparse Matrices and Methods

- Many algorithms are specialized for matrices (or Jacobians or Hessians) with many 0s—e.g. Gaussian elimination
- Only store non-zero values, but $0 \neq 0.0$ for optimizers
- Not (usually) for storage, but rather specialized algorithms
- For Jacobians and Hessians, can solve enormous (e.g. hundreds of thousands or millions) of variable systems
 - But the more non-zeros, the more likely dense methods are preferable.
- For example, $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ with $f(x) = \sqrt{x}$ point-wise
 - Jacobian has N non-zeros, while dense has N^2
 - Optimizers/solvers can use this to step in the right direction
 - Auto-differentiation will figure out the sparsity pattern of derivatives—i.e., which values are always 0 for all inputs

Sparse Matrices in Matlab

%First, can convert dense matrix, and it drops the 0's.

```
X = [1.0 0 0;  
      2.0 1.0 0];
```

```
S = sparse(X)
```

```
%S =
```

```
%(1,1)      1
```

```
%(2,1)      2
```

```
%(2,2)      1
```

%Or can take lists of indices and values,

```
x_indices = [1; 2; 2];
```

```
y_indices = [1; 1; 2];
```

```
values = [1; 2; 1];
```

```
S2 = sparse(x_indices, y_indices, values)
```

%Or can preallocate and just reference in loops/etc.

```
S3 = sparse(0,3);
```

```
S3(1,1) = 1;
```

```
S3(2,1) = 2;
```

```
S3(2,2) = 1;
```

Tomlab

What is in Tomlab?

- Sadly, the Operations Research community keeps the best implementations closed-source
- Collection of sparse/dense linear/nonlinear local/global constrained/unconstrained continuous/mixed-integer optimizers
- Nonlinear methods have built in auto-differentiation
- Repackages and resells state-of-the-art commercial products, and adds a few of its own which are high quality
- Several methods to solve the same type of problem, because you never know which one will work best. Easy to swap

What Types of Problems?

See http://tomopt.com/docs/TOMLAB_QUICKGUIDE.pdf.

- Programming = Optimizer in OR
- Linear Programming (LP) and Mixed-Integer LP (MILP)
- Constrained Nonlinear Programming (NLP)
- Unconstrained Global Optimization (glb)
- Linear Least Squares (LLS)
- Nonlinear Least Squares (NLLS)
- Solving systems of equations generally uses NLLS
- ... and many others (semi-definite, quadratic, etc.)

Most have sparse vs. dense algorithms, and constrained vs. unconstrained

- Read docs to find best fit for your particular problem
- Always use appropriate constraints (none, box-bounded, linear, etc.)
- For borderline sparse problems, sometimes dense works better

Purchased Packages

See <http://tomopt.com/tomlab/products/>

- Stanford Systems Optimization Laboratory (SOL): SNOPT, NPSOL, NLSSOL, LSSOL...
- Knitro. Good for big problems, and complementarity conditions
- MAD (auto-differentiation)
- LGO and CGO (global optimizers, costly and otherwise)
- For a given problem type, tomlab will list available algorithms

Setting up Tomlab

- Use the UBC Matlab license
 - If you aren't sure what you have, open matlab, type 'license' and check the nubmer is '924490'. Otherwise, follow department/ubc instructions
- Download Tomlab from the site or
 - <http://jesseperla.com/tomlab/tomlab-win64-setup.exe>
 - or <http://jesseperla.com/tomlab/tomlab-osx64-setup.dmg>
- Run to install it
- Put <http://jesseperla.com/tomlab/tomlab.lic> into main tomlab directory (with the startup.m file)
- You now have two choices:
 - Whenever you want to use it, run the startup.m script in tomlab
 - Or, put the tomlab directory in your matlab path (with "Set Path" menu) and it runs on its own when starting matlab
- Finally, Tomlab replaces the fmincon, fsolve etc. in matlab. Avoid by delete/rename in the /optim folder in installation

Linear Least Squares (i.e. Regression)

$$\begin{aligned} \min_x \quad & \frac{1}{2} \|Cx - d\|_2 \\ \text{s.t.} \quad & x_L \leq x \leq x_U \\ & b_L \leq Ax \leq b_U \end{aligned}$$

- Little benefit over stata until problems get large or sparse
- Though if there are linear constraints, A , may be helpful
- Major benefit for large, sparse C
- See “Section 11. LLS Problem” in
http://tomopt.com/docs/TOMLAB_QUICKGUIDE.pdf

Example: Two-way fixed Effect

- Use student, i , and instructor, j fixed effects with observables

$$\text{grade}_{ij} = \text{observables}_{ij} + \text{student}_i + \text{instructor}_j + \epsilon_{ij}$$

- 300K observations for 36K students and 945 instructors
 - $\approx 37\text{K}$ variables, **but only 2 non-zero** (plus observables)
- Took about a day to solve in Stata
- See `teacher_student_fixed_effect.m` example for generating sparse matrix. Given `id1` student id, and `id1` instructor id. Key code:

```
%Preallocate a sparse matrix
```

```
%Total number of observables with indicators for the two types.
```

```
X = sparse(N_observations, N_observables + N_students + N_teachers);
```

```
%Filling in indicators for the matches
```

```
for i=1:N_observations
```

```
X(i, N_observables + id1(i)) = 1; %set student indicator
```

```
X(i, N_observables + N_students + id2(i)) = 1; %sets instructor indicator
```

```
end
```


Solving LLS in Tomlab

- Given X and y such that $\min_{\beta} \|X\beta - y\|_2$
- Ensure X loaded sparse, with each row having 2 indicators

*%linear least squares, can pass in sparse matrices or use dense
help llsAssign %Can see options, if you wish. Or use manual*

% tomlab convention, call XXXAssign for problem type XXX

```
Prob = llsAssign(X, y, [], [], 'LLS Example');
```

%Can change settings. See tomlab documentation

```
Prob.optParam.MaxIter = 5000; %optional, increase iterations
```

```
Prob.PriLevOpt = 1; %optional, gives more information if higher
```

%Tomlab convention: tomRun, passing in the algorithm type and problem

%Takes about 10-20 seconds to run, instead of a day. Not even tweaked

```
Result = tomRun('Tlsqr', Prob); %intended for sparse unconstrained LLS
```

```
beta = Result.x_k;
```

%Tried alternative methods. Easy to swap

```
%Result = tomRun('snopt', Prob); %Tlsqr works much better here
```

Constrained Optimization

Nonlinear Programming (NLP)

Given $f : \mathbb{R}^N \rightarrow \mathbb{R}$ and $c : \mathbb{R}^N \rightarrow \mathbb{R}^M$

$$\min_x \{f(x)\}$$

$$\text{s.t. } x_L \leq x \leq x_U$$

$$b_L \leq Ax \leq b_U$$

$$c_L \leq c(x) \leq c_U$$

- See “Section 7. NLP Problem” in http://tomopt.com/docs/TOMLAB_QUICKGUIDE.pdf
- Can solve large problems in general with dense solvers (e.g. tens, hundreds, thousands of variables depending on structure)
- Can solve enormous problems if Hessian of $f(x)$, Jacobian of $c(x)$ sparse (e.g. hundreds of thousands or millions of variables)
- If constraints are bounds, use x_L, x_U . If linear, use A . Equality set $b_L(m) = b_U(m)$ where appropriate. No bounds, leave unconstrained

Example: Portfolio Choice under Rational Inattention

- Entropy constraint, κ and signal variances, σ_n , weights α_n
- Choose precision x_n

$$\begin{aligned} \min_{\{x_n\}} \quad & \sum_{n=1}^N \alpha_n^2 x_n^2 \\ \text{s.t.} \quad & \frac{1}{2} \sum_{n=1}^N \left(\log \sigma_n^2 - \log x_n^2 \right) \leq \kappa \\ & 0 < x_n < \infty \end{aligned}$$

- Generate some $\{\alpha_n, \sigma_n\}$ and solve
- How large for N ? Using `fminsearch` naively, got to $N \approx 30$. Sample code I give you solves with $N = 100,000$ in ≈ 20 seconds
- Since the hessian is sparse, could potentially use specialized methods (but didn't even bother to play with that, just using AD)

Solving in Tomlab (with Anonymous Functions)

```
%...define kappa, alpha vector of length N, sigma vector of length n...
objective = @(x) sum((x.^2).*(alpha.^2)); %Objective
constraint = @(x) (1/2)*sum( log(sigma.^2) - log(x.^2) ); %Constraint

%Bounds on x, and initial guess
x_L = 1E-10 * ones(N,1); %Bounding a little above 0 since it takes logs.
x_U = inf(N,1); %Upper bounds for x.
x_0 = 1.1*ones(N,1); %Some initial conditions

%Generate problem. Use `help conAssign' to see arguments, or use manual
madinitglobals; %Need to run for auto-differentiation to work.
Prob = conAssign(objective, [], [], [], x_L, x_U, 'Portfolio example',...
    x_0, [], [], [], [], [], constraint, [], [], [], kappa, kappa);
Prob.ADObj = 1; % Gradient with AD. ADObj = -1 for Hessian
Prob.ADCons = 1; % Jacobian with AD. ADCons = -1 for constraint Lagrangian

%Run type
Result = tomRun('knitro', Prob); %Choose algorithm. See tomlab for options
x_optimal = Result.x_k;

%Result = tomRun('npsol', Prob, 1) %Could try another algorithm
```

Using External Functions and Fixed Parameters

*%Alternatively, the objective/constraint can be in separate files
%Assumption is that vectors alpha/sigma/kappa attached to problem*

%File: portfolio_objective.m

```
function f = portfolio_objective(x, Prob)
    f = sum((x.^2).*(Prob.alpha .^2));
end
```

%File: portfolio_constraint.m

```
function c = portfolio_constraint(x, Prob)
    c = (1/2)*sum( log(Prob.sigma.^2) - log(x.^2) ) - Prob.kappa;
end
```

Calling with External Functions

%Changed to have $c(x) = 0$ for the constraint since in $c(x)$

```
Prob = conAssign(@portfolio_objective, [], [], [], x_L, x_U, 'Example', ...  
    x_0, [], [], [], [], [], @portfolio_constraint, [], [], [], 0, 0);
```

%Put any constants into the Prob, available in the function

%Can throw anything you want only Prob (e.g. vectors, cells, etc.)

```
Prob.alpha = alpha;
```

```
Prob.sigma = sigma;
```

```
Prob.kappa = kappa;
```

%Setup AD

```
Prob.ADObj = 1; % Gradient with AD. ADObj = -1 for Hessian
```

```
Prob.ADCons = 1; % Jacobian with AD. ADCons = -1 for constraint Lagrangian
```

%Run the optimizer

```
Result = tomRun('knitro', Prob); %The last
```

Systems of Equations and NLLS

Nonlinear Least Squares (NLLS)

Given residual $r : \mathbb{R}^N \rightarrow \mathbb{R}^M$ and $c : \mathbb{R}^N \rightarrow \mathbb{R}^P$

$$\min_x \left\{ \frac{1}{2} r(x)^T r(x) \right\}$$

$$\text{s.t. } x_L \leq x \leq x_U$$

$$b_L \leq Ax \leq b_U$$

$$c_L \leq c(x) \leq c_U$$

Also for solving system of equations (potentially with inequalities):

$$r(x) = \mathbf{0}$$

- See “Section 13. NLLS Problem” in http://tomopt.com/docs/TOMLAB_QUICKGUIDE.pdf
- Can solve very large problems, with/without constraints

Solving Functional Equations

- One important use of this is solving functional equations of the form

$$\Phi(f) = \mathbf{0}$$

- Where $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ and Φ is an operator $\Phi : \mathbb{C}(\mathbb{R}^N) \rightarrow \mathbb{R}^P$
- Could include differential equations, difference equations, etc.
- To solve, can approximate f with a basis. Collocation methods, etc.
 - e.g. $f(x) \approx \sum_{q=1}^Q d_q P_q(x)$
 - Where $P_q(x)$ is a polynomial/spline/finite-element basis
 - d_q are the unknown coefficients to solve for
- Then, problem is to find the coefficients

$$\min_d \left\{ \frac{1}{2} \Phi(d)^T \Phi(d) \right\}$$

- Since for fixed x_n nodes, can usually evaluate P_q through linear algebra can use AD on $\Phi(d)$
- Note: if $\Phi(d)$ is a linear operator (e.g. linear PDEs) can use sparse LLS. How huge numerical PDEs with millions of points are solved.

AD with Tomlab

Using AD Directly in Tomlab

- Keep in mind that optimizers/solvers in Tomlab do this automatically
- But if having trouble with optimizer calls, can test function separately

%Example function. Also works fine with separate files/function defs

```
f = @(x) 3*x + exp(x);
```

%Evaluating function

```
x_val = 2.1;
```

```
f(x_val)
```

%Evaluating with derivative at x_val

```
x = fmad(x_val, 1); %Seed, since  $dx/dx = 1$ 
```

```
f_val = f(x)
```

%Extract (both calculated at same time)

```
getvalue(f_val)
```

```
getderivs(f_val)
```

Black Magic? Is it Always so Easy?

- Auto-differentiation works seamlessly for functions composed of an arbitrarily complicated graph of simple functions
 - Just need analytical derivatives for the lowest-level functions
 - Functions of vector and matrices are no problem at all. In fact, the field was designed for large numbers of variables/constraints and sparsity
- Can you call other functions (with operator overloading)?
 - Depends on how they were written. Often no problem at all
 - If the functions assume arguments are numbers, there can be problems
 - Sometimes can fix the underlying code to make more generic (example)
- Verboten: Iterations and fixed-points *within a function*
 - e.g. it can't differentiate an optimization step within a function
 - However, many algorithms can be re-written without nesting (e.g. nested fixed-point vs. MPEC for discrete-choice estimation)
 - Possible that simulation could be embedded (e.g. mixed-logit)...

Keep Functions Generic

- Remember, MAD replaces arguments with things that look like variables. Keep everything generic, don't overwrite with other types
- Some internal matlab functions do this sort of thing
- Sometimes can copy/paste others sourcecode and tweak

```
madinitglobals; %Need to run for auto-differentiation to work
```

```
x_val = [2.1;3.0];
```

```
x = fmad(x_val, eye(2,2)); %Seeds with derivatives
```

```
%Extract (both calculated at same time)
```

```
f_val = f_func(x)
```

```
function y = f_func(x)
```

```
    y = x.^2; %This leaves x, y generic
```

```
    %x = 1; %Don't do this!!!!
```

```
    %y = zeros(1,1) %Don't do this!!!!
```

```
    %y(1) = x.^2; %indexing is fine (as long as you do not preallocate)
```

```
    %One trick is to allocate as something like: y = x, then index
```

```
end
```

Missing Function (with Analytical Derivative)

- See MAD manual, http://tomopt.com/docs/TOMLAB_MAD.pdf
- Section: "Adding Functions to the fmad Class"
- Example, normcdf isn't there, could add something like (untested):

%Add to the appropriate location in tomlab

*%This should work for vectors/matrices since using .**

function y=normcdf(x)

 y = x; *%Needs to copy*

 y.value= normcdf(x.value); *%Evaluate given double*

 y.deriv = normpdf(x.value) .* x.deriv; *%Note chain rule*

end