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October 21, 2020

Abstract

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Contents

Chapter 1

Introduction

In March 2018, the insurance and pension industry association Forsikring & Pension published new guidelines for the pension prognoses presented to policy holders. The purpose was to secure more consistent methods of calculation between companies and to present the financial risk associated with the prognoses.

The biometric risks are, however, somewhat neglected, as the prognoses still make the (rather optimistic) assumption that the policy holder remains active and premium paying up until retirement – the possibility of spells of disability or unemployment is ignored. Also, the prognoses give the policy holder no information on how she can influence them – e.g. by postponing retirement or by saving at a higher rate.

The thesis attempts to remedy these two shortcomings. The idea is to take the financial market as given (deterministic or simulated scenario) and handle the realization risks with classic actuarial techniques. This will result in differential equations describing the prognoses, which can then be computed numerically. The differential equations are then themselves differentiated with respect to the premium level and time of retirement, thereby arriving at new sets of differential equations for the derivatives, which again can be solved numerically. With these derivatives in hands, we can give the policy holder valuable information on how she can influence her expected benefits by retiring later or paying more premiums.

Literature review? In spite of these initiatives, the subject of prognoses has received relatively little attention in the actuarial literature. Norberg...

The main theoretical workhorse is the notion of retrospective reserves first introduced in Norberg and expanded in Lollike. When doing valuation, it is prospective, when doing ALM, it is retrospective, and also when doing prognoses, as they both project into the future.

LOLLIKE, KALASHNIKOV OG REITZEL. extends lollike with lump sum and cash dividends

Structure

Her en gennemgang af kapitlerne og deres sigter.

Section PROGNOSES will make rigorous exactly what is meant by *prognosis* and is in

Each section begins with a short summary of the key points contained therein.

Remarks are nonessential and can be skipped.

Acknowledgements

Chapter 2

General Setup

This section describes the key concepts used throughout the thesis. Throughout the text, we follow the convention that

- The contract is signed at time 0.
- The prognosis is calculated at time $t_0 \geq 0$ (with the information available at that time).
- The benefit that is being prognosticated is paid out at time $T \geq t_0$.
- The insured retires at time R.
- The contract terminates at time n. There are no payments beyond this time point.

2.1 The Classic Markov Chain Life Insurance Setup

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. The state of the policy holder is modelled in the classical fashion, i.e. as a continuous time Markov process $Z = \{Z(t)\}_{t\geq 0}$ on a finite state space $\mathcal{J} = \{0, 1, ..., J\}$, deterministically starting in state 0, i.e. $Z(0) = 0$.[1]

Define the transition probabilities

$$p_{ij}(t,s) = \mathbb{P}\left(Z(s) = j \,\middle|\, Z(t) = i\right), \quad i,j \in \mathcal{J},\ t < s,$$

and assume that they are continuously differentiable in t and s, i.e. the function $(t,s) \mapsto p_{ij}(t,s)$ is $C^{1,1}$ for $t < s$. It is then possible to define the transition intensities

$$\mu_{ij}(t) = \lim_{h \searrow 0} \frac{p_{ij}\left(t, t+h\right)}{h}, \quad i,j \in \mathcal{J},\ i \neq j.$$

Finally, let $\mathbf{N} = \{N_k\}_{k\in\mathcal{J}}$ be the vector counting process associated with Z. It has entries $N_k = \{N_k(t)\}_{t\geq 0}$ given by

$$N_k(t) = \#\{s \in (0,t] \,:\, Z(s-) \neq k, Z(s) = k\},$$

[1] State 0 is typically the active state, but in theory it could be anything.

counting the number of transitions into state k up until and including time t. It is known from e.g. [Pedersen, 2017] that

$$M_k(t) = N_k(t) - \int_0^t 1_{(Z(s-)\neq k)} \mu_{Z(s-)k}(s) ds$$

is an F^Z-martingale. Recall that $F^Z = \{F^Z(t)\}_{t\geq 0}$ is the filtration naturally generated by the process Z, i.e. $F^Z(t) = \sigma(Z(s) : s \leq t)$.

2.2 The Financial Market

The financial market enters into the insurance setting via the return on investment of the payments made by the policy holder to the company. Denote by $S = \{S(t)\}_{t\geq 0}$ the state process of the financial market. The rate of return is then the FV-process $r = \{r(t)\}_{t\geq 0}$. This rate of return is *not* a market rate, but rather the rate ascribed to the individual policy holder by the company. It may comprise returns on bonds, stocks and other investments depending on the portfolio profile which in turn may depend on the contract type and risk appetite of the policy holder. Also, it may depend on the age and the *current* state of the policy. Formally,

$$r(t) = r_{Z(t)}(t),$$

for S-adapted FV-processes $r_j(t)$, $j \in \mathcal{J}$ with continuous sample paths.

We make no assumptions about the market process S, but instead take it as exogenously given in economic scenarios. These scenarios also include the transition intensities μ governing the state process Z. Interesting scenarios include:

- Best estimate scenarios, e.g. the Common Return Expectations (Samfundsforudsætninger) published by the Council for Return Expectations (Rådet for Afkastforventninger) – see [**?**] for the return expectations calibrated in 2020.
- Stress scenarios, e.g. market crashes, drawn out recessions or adverse mortality developments.
- Simulated scenarios. Having decided on some stochastic model for (S, μ), one can simulate a large number of scenarios, calculate the prognoses in each of them and then apply a suitable average or quantile.

Since all computations takes place within a given scenario – meaning that S and μ is known – we will suppress this in the notation.

Remark 1.
Remark omkring at diskretisering gør dt okay. kontra G(s)/G(t) ○

Remark 2.
Remark omkring P og Q ○

Valg om ingen diffusions-modellering.

2.3 Payment Streams

Throughout the thesis we will need a number of different payment processes. We make the small departure from the classical construction that the time of retirement R is stressed as the only time point where payment rates are discontinuous and lump sums are paid out. We follow the convention that positive payments are benefits, while negative payments are premiums.

A payment process consists of three types of payments:

- Sojourn payment rates $b_j(t)$ paid continuously during sojourns in state j. The payment rates can be decomposed as

$$b_j(t) = 1_{(t<R)} b_j^{\Uparrow}(t) + 1_{(R\leq t)} b_j^{\Downarrow}(t),$$

  where the $b_j^{\Uparrow}, b_j^{\Downarrow} : [0,n] \to \mathbb{R}$ are FV-continuous functions. $b_j^{\Uparrow}$ denotes the rates paid continuously before the time of retirement R (up-arrow for "opsparingsfase" in Danish), while $b_j^{\Downarrow}$ are the rates paid after retirement ("nedsparingsfase").

- Transition payments $b_{jk}(t)$ paid upon transition from j to k. The payment functions can be decomposed as

$$b_{jk}(t) = 1_{(t<R)} b_{jk}^{\Uparrow}(t) + 1_{(R\leq t)} b_{jk}^{\Downarrow}(t),$$

  where the $b_{jk}^{\Uparrow}, b_{jk}^{\Downarrow} : [0,n] \to \mathbb{R}$ are continuous FV-functions.

- A lump sum retirement benefit paid out deterministically at time R. The size of this benefit may depend on the time of retirement and so is a function of t, i.e. $\triangle B_j : [0,n] \to \mathbb{R}$. It is assumed to be differentiable in t.

All accounted for, the payment processes take the form

$$dB(t) = b_{Z(t)}(t)dt + \triangle B_{Z(t)}(t) d\varepsilon_R(t) + \sum_{k\neq Z(t-)} b_{Z(t-)k}(t) dN_k(t). \;\; B(0) = 0, \tag{2.3.1}$$

Remark 3.
For the sake of readability, only the case with a single discontinuity in the payment processes and a single deterministic lump sum benefit $\triangle b_j$ (both at time of retirement R) will be treated in this thesis. However, the results can be extended by more discontinuities and more deterministic lump sums. ○

Remark 4.
$b_j^{\Uparrow}$ and $b_j^{\Downarrow}$ do not always have different signs. For instance, if k is the disability state, $b_k^{\Uparrow}$ corresponds to a disability benefit, while $b_k^{\Downarrow}$ corresponds to a pension benefit – both positive rates. ○

Remark 5.
It would of course be possible to capture the distinction between pre- and post-retirement payments in the accumulated payment processes (i.e. work with a $B^{\Uparrow}$ and a $B^{\Downarrow}$) instead of in the payment functions b_j and b_{jk}. ○

Having introduced the state process Z, the financial and biometric bases (r, μ) and (r^*, μ^*) and the contractual payments B, we are now ready to review a number of classical results, which will prove useful in the later chapters.

2.4 Valuation of Payment Streams

The contract is settled on the technical basis (sometimes called the first order basis) which consists of prudently determined interest rate and transition intensities (r^*, μ^*). As time passes, and if all goes well, the discrepancy between the technical basis and the realized rates creates a surplus that partially belongs to the policy holders. In order to specify the reallocation via dividends from the surplus to the policy holders – and equally important to determine guaranteed benefits – we need some theory on how to valuate payment streams. The results are well-known from the literature, see e.g. [Norberg, 1991], but are included here for the sake of completeness.

Definition 6.
Given a payment stream B, the prospective technical reserve at time $t \geq 0$ is

$$V^*_{Z(t)}(t) = \mathbb{E}^* \left[\int_t^n e^{-\int_t^s r^*(u)du} dB(s) \middle| Z(t) \right],$$

where $\mathbb{E}^*$ is integration with respect to the measure $\mathbb{P}^*$, under which Z is Markov with intensities μ^*.

Note that the time t technical reserve only depend on the state process Z through $Z(t)$. It is know from equation (16.2) in [Pedersen, 2017] that the reserve has dynamics

$$\begin{aligned} dV^*_{Z(t)}(t) =& r^*(t)V^*_{Z(t)}dt - b_{Z(t-)}(t)dt - \sum_{\substack{k\in\mathcal{J} \\ k\neq Z(t-)}} R^*_{Z(t-)k}(t)\mu^*_{Z(t-)k}(t)dt \\ &+ \sum_{\substack{k\in\mathcal{J} \\ k\neq Z(t-)}} \left(V^*_k(t) - V^*_{Z(t-)}(t)\right) dN_k(t), \;\; V^*_{Z(t)}(R) = V^*_{Z(t)}(R-) - \triangle B_{Z(t-)}(R) \end{aligned}$$

where R^*_{jk} is the sum at risk for the transition $j \mapsto k$ given by

$$R^*_{jk}(t) = \left(b_{jk}(t) + V^*_k(t) - V^*_j(t)\right).$$

As noted in [Bruhn and Lollike, 2020], the dynamics can be rewritten as

$$\begin{aligned} dV^*_{Z(t)}(t) =& r^*(t)V^*_{Z(t)} - b_{Z(t-)}(t)dt - \sum_{\substack{k\in\mathcal{J} \\ k\neq Z(t-)}} b_{Z(t)k}(t)dN_k(t) \\ &- \sum_{\substack{k\in\mathcal{J} \\ k\neq Z(t-)}} \rho_{Z(t-)k}(t)dt + \sum_{\substack{k\in\mathcal{J} \\ k\neq Z(t-)}} R^*_{Z(t-)k}(t)\left(dN_k(t) - \mu_{Z(t-)k}(t)dt\right), \end{aligned} \tag{2.4.1}$$

where ρ_{jk} is the surplus risk contribution for the transition $j \mapsto k$ given by

$$\rho_{jk}(t) = R^*_{jk}(t)\left(\mu^*_{jk}(t) - \mu_{jk}(t)\right).$$

This rate is the contribution from the policy holder to the surplus.
Proposition 7.
Måske Cash-Flow varianten af den prospektive reserve

Proof. See XXX. □

The preceding chapter . In the next chapter, the concept of prognoses will be discussed in depth – , after which formal definitions will be proposed

Chapter 3

Prognoses

A prognosis should be a useful summary of the level of benefits that the policy holder can expect to receive in the future. In other words, a prognosis is a functional of the payment stream $B : [t_0, n] \times \Omega \to \mathbb{R}$. A good prognosis should be useful and understandable by the policy holder. But it should also be feasibly computable - it must be possible to churn out an actual number in the end. The challenge is finding a functional that is both these things.

In the literature, many different functionals have been proposed. [Norberg, 2001] suggests expected total benefits paid out – discounted or undiscounted. He also suggests simply expected benefits rates at certain times. [Jensen, 2016] extends this last idea to an entire (expected) benefit stream.

The thesis takes the view that it is benefit rates (e.g. monthly benefits) and single lump sum benefits (e.g. pension endowment or term insurance payout) that is of greatest interest to the policy holder – not accumulated benefits. Questions such as "What benefit rate can I expect in my first month of retirement?" or "What can I expect to be paid out to my family if I die in 10 years?" are more likely to interest the policy holder than "How many benefits can I expect to receive across my entire lifetime?" (although this is also an interesting question, see Section XXX).

The thesis also takes the view that the benefits prognosticated should be real as opposed to nominal. The purpose of pension and life insurance is essentially to ensure that the policy holder has sufficient purchasing power in all stages and events of her life. For this reason, prognoses of real benefits in time t_0-currency that she can tie to her everyday consumption are more useful than the nominal equivalents. Therefore, all payments are assumed to be real and the rate of future return r_{t_0} is assumed to be a real rate as well (nominal less inflation measured by some price index). Of course, this rate might not be easy to model, but that discussion is outside the scope of this thesis.

When dealing with the realization risk of Z in the context of valuation, the expectation has a privileged position as the summarizing functional of choice. This is because the law of large numbers ensures that the average liability converges towards the expectation as the portfolio increases in size.

In the context of prognoses – however – it is not entirely obvious why the expectation should be *the* most interesting functional to the policy holder. After all, few people would probably be satisfied with the justification that "this quantity is your average pension rate across an infinite number of lifetimes". Still, the expectation is the functional pursued in this thesis. The choice is made on practical grounds: Most actuarial techniques for treating the state risk of Z are based on moments so in order to build on these techniques, the approach will be moment-based. This is not entirely a bad thing, as moments in most cases give us much information on a distribution. In section XXX, just before the curtains close, an approach for obtaining higher-order moment for the prognoses is sketched.

By the above considerations, we are led to formulate the following candidates for prognoses:

- Sojourn benefit rate prognoses

$$\mathbb{E}\left[b_{Z(t-)}(t)\middle|\cdot\right]$$

- Retirement lump sum prognoses

$$\mathbb{E}\left[\triangle B_{Z(R-)}(R)\middle|\cdot\right]$$

- State k transition benefit prognoses

$$\mathbb{E}\left[b_{Z(t-)k}(t)\middle|\cdot\right]$$

In the next section, it will be discussed which events it is meaningful to insert in place of the dot.
Remark 8.
Quantiles are an alternative to moments. They are useful to the policy holder, as they answer the natural question: "What is a lower bound for my pension benefits that I can put into my budget with peace of mind". Unfortunately, a quantile approach to the state risk Z is outside the scope of this thesis. However, the scenario approach to the financial risk allows for expectation-quantile hybrids, where the state risk of Z is handled inside each scenario ○

3.1 Conditioning for prognoses

We now turn our attention to the question of what to choose as the conditioning event for prognoses. At the one extreme is no conditioning at all. This will result in prognoses that are arguably too low. Take the sojourn prognosis for some time point $t > R$ as an example. There are many $\omega \in \Omega$ where the policy holder dies before time t, which would pull the prognosis towards 0, since there are no pension payments in the death state. When assessing the liabilities of the company these ω's should of course be included. But when giving a policy holder a realistic picture of what to expect in pension benefits, they should not[1]. For this reason, it seems reasonable that prognoses should at least condition on not having made the transition to the death state before time t.

At the other extreme we may condition on the entire state path up until the time of interest, i.e. $\{Z(s)\}_{0\leq s<t}$. Many current prognoses do something similar by conditioning on the policy holder staying active. However, this conditioning eradicates the risk of the state process Z completely. Thus, it may lead to overly optimistic prognoses, as Z is assumed to remain in the premium paying active state until retirement, which may have an effect on the level of benefits.

Instead of simply staying active, one could condition on some other specific path that includes periods of invalidity or unemployment. The path could be chosen by the police holder in a "what if"-scenario. As Z is then conditionally deterministic, there is no realization risk at all, and calculating these prognoses would be no more difficult than solving a deterministic forward differential equation. These types of prognoses are not the main topic of this thesis, but are treated briefly in Remark XX.

The conditioning proposed in this thesis is an attempt to bridge the two extremes of no conditioning and conditioning on an entire state path. The idea is to condition on the event that the policy holder does not make some specified transitions until the payout time t (e.g. die, surrender or convert to free policy), i.e. remains in some subset of states $\hat{\mathcal{J}} \subset \mathcal{J}$ until time t. Formally put, we will condition on events on the form

$$\left(Z(s) \in \hat{\mathcal{J}} : 0 \leq s < t\right) \tag{3.1.1}$$

[1] An exception is if a benefit rate is paid out to the bereaved for some time after her death

With this conditioning, we retain the risk of the state process, but – as we shall argue – only the events relevant for prognoses.

We restrict ourselves to subsets $\hat{\mathcal{J}} \subset \mathcal{J}$ satisfying

- Z is in $\hat{\mathcal{J}}$ at the time of prognostication, i.e. $Z(t_0) \in \hat{\mathcal{J}}$.
- Z cannot return to which $\hat{\mathcal{J}}$ once left. Formally $\mu_{kj} = 0$ for all $k \notin \hat{\mathcal{J}}$ and $j \in \hat{\mathcal{J}}$. With this assumption, we have

$$\Big(Z(s) \in \hat{\mathcal{J}} : 0 \leq s < t\Big) = \Big(Z(t-) \in \hat{\mathcal{J}}\Big), \tag{3.1.2}$$

because if Z is in $\hat{\mathcal{J}}$ just before time t, then it has been there for all $s < t$. Since we only treat subsets $\hat{\mathcal{J}}$ with this restriction, we will prefer to write the shorter formulation on the right hand side of (3.1.2) throughout the thesis. The restriction is made to make the mathematics more tractable, but from a practical standpoint it is not overly restrictive, as the examples XX, XX and XX will demonstrate.

FIGURE

Having formalized the class of conditioning events, we are now ready to give a formal definition of sojourn prognoses and retirement lump sum prognoses:

Definition 9.
A sojourn benefit rate prognosis is a quantity on the form

$$\mathbb{E}\left[b_{Z(t-)}(t) \middle| Z(t-) \in \hat{\mathcal{J}} \right],$$

for some subset $\bar{\mathcal{J}} \subset \mathcal{J}$.

Definition 10.
A retirement lump sum prognosis is a quantity on the form

$$\mathbb{E}\left[\triangle B_{Z(R-)}(R) \middle| Z(R-) \in \bar{\mathcal{J}} \right],$$

for some subset $\bar{\mathcal{J}} \subset \mathcal{J}$.

For the sojourn prognoses and the retirement lump sum prognoses, the conditioning in (3.1.1) will suffice. For the state k transition prognosis ($k \notin \hat{\mathcal{J}}$), however, we introduce the event

$$\Big(\{Z(s)\}_{0 \leq s < t} \subset \hat{\mathcal{J}}, Z(t) = k\Big) \tag{3.1.3}$$

for some state $k \notin \hat{\mathcal{J}}$. With assumption XX, the event (3.1.3) becomes

$$\Big(Z(t-) \in \hat{\mathcal{J}}, Z(t) = k\Big). \tag{3.1.4}$$

The difference between the event (3.1.2) and the event in (3.1.4) is subtle but important. The event (3.1.2) is equivalent to stating that Z leaves $\hat{\mathcal{J}}$ at time t or later. The event (3.1.4) states that Z leaves $\hat{\mathcal{J}}$ *exactly* at time t through the state k. Both events are interesting in their own right:

- For a prognosis based on the event (3.1.2), it could be interesting to consider the mean time until exit from $\hat{\mathcal{J}}$, given that exit has not happened at time t and given that the eventual exit happens through state k. Further pursuit of this idea is however outside the scope of the thesis.
- Instead, we let (3.1.4) be the conditioning event of the state k transition prognoses. This is a way of formally answering the policy holder's question "If I die at time t exactly, how much can I expect to be paid out to the bereaved".

With this is mind, we can define the transition prognoses:

Definition 11.
A state k transition benefit prognosis is a quantity on the form

$$\mathbb{E}\left[b_{Z(t-)k}(t) \middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = k \right],$$

for some subset $\hat{\mathcal{J}} \subset \mathcal{J}$ and $k \notin \hat{\mathcal{J}}$.

Having formally defined the three different prognoses, we close this chapter by exemplifying their use in three classical models:

- The 2-state alive-dead model.
- The 3-state alive-disabled-dead model.
- The multi-state model with policy holder behavior.

First up is the alive-dead model:

Example 12.
In the simple alive-dead model, as we shall see, the proposed conditioning approach actually coincides with the classical conditioning (i.e. staying active).

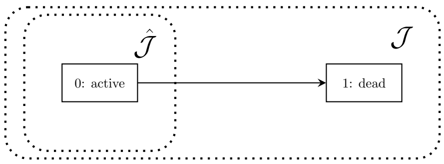


Figure 3.1: Alive-dead model

For the time t sojourn benefit rate prognosis, we condition on the policy holder not having died before time t (because the events where she has already died by then doesn't interest her with respect to her prognoses). So we let $\hat{\mathcal{J}} = \{0\}$ and the prognosis is

$$\mathbb{E}\left[b_{Z(t-)}(t) \middle| Z(t-) \in \hat{\mathcal{J}} \right] = \mathbb{E}\left[b_0(t) \middle| Z(t-) = 0 \right],$$

which is the same conditioning as the classical prognoses. Likewise for the retirement lump sum prognosis:

$$\mathbb{E}\left[\triangle b_{Z(R-)}(R) \middle| Z(R-) \in \hat{\mathcal{J}} \right] = \mathbb{E}\left[\triangle b_0(R) \middle| Z(R-) = 0 \right],$$

which is also the classical conditioning. With regard to the only transition benefit prognosis, the state 1 transition, we have

$$
\begin{aligned}
\mathbb{E}\left[b_{Z(t-)1}(t) \middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = 1 \right] =& \mathbb{E}\left[b_{01}(t) \middle| Z(t-) = 0, Z(t) = 1 \right] \\
=& \mathbb{E}\left[b_{01}(t) \middle| \triangle N_1(t) = 1 \right],
\end{aligned}
$$

This prognosis gives the policy holder an estimate of what she will leave to her bereaved in the event of her death *exactly* at time t. △

Example 12 demonstrates that in the alive-death model, the conditioning approach proposed doesn't really bring anything new to the table, as it coincides with the classical conditioning of staying active until payout time t. This is because "not dying" is the same event as "staying active". This is no longer the case in the active-disabled-death model, as the next example shows.

Example 13.

Consider the active-disabled-dead model:

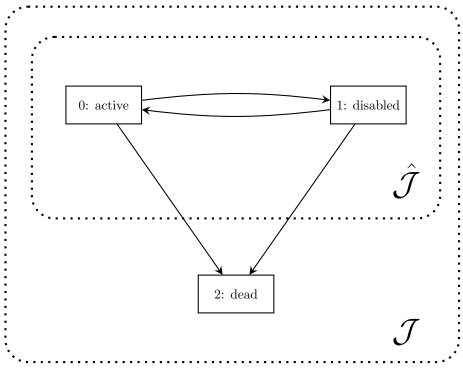


Figure 3.2: Active-disabled-dead model

In this model, most classical prognoses – e.g. [Jensen, 2016] – condition on the policy holder staying active all the way up to the time of payout t, e.g. the event $\big(Z(s)0 : 0 \leq s < t\big)$. Instead we only assume that the policy holder does not die – she might become disabled. So we let $\hat{\mathcal{J}} = \{0, 1\}$, such that the time t sojourn benefit prognosis becomes

$$
\mathbb{E}\left[b_{Z(t-)}(t) \middle| Z(t-) \in \hat{\mathcal{J}} \right].
$$

Again, and for the same reason as given above, this conditioning excludes death before time t. However, it retains two important elements of risk that the classical conditioning does not:

- The time t sojourn benefit rate may not be the same in states 0 and 1, e.g. $b_0 \neq b_1$.
- The benefit may also depend on the past state path $\{Z(s)\}_{0 \leq s < t}$, which is fixed in the classical conditioning, but only restricted to $\hat{\mathcal{J}}$ in the proposed conditioning. This means that Z may leave the premium paying state 0 for periods before t, which for some types of contracts may reduce the benefit at time t.

The same arguments go for the retirement lump sum benefit prognosis, which is simply

$$
\mathbb{E}\left[\triangle B_{Z(R-)}(R) \middle| Z(R-) \in \hat{\mathcal{J}} \right].
$$

With respect to the transition prognosis for the dead state 2, the policy holder is interested in knowing what she can expect to have paid out to her bereaved in case of her death at time t. The prognosis is

$$\mathbb{E}\left[b_{Z(t-)2}(t) \middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = 2 \right], \tag{3.1.5}$$

which is the expected death sum paid out given death at exactly time t. The prognosis takes into account that the transition can be made from state 0 or from state 1 (which matters if $b_{02} \neq b_{12}$), and that the state process can have moved freely between 0 and 1 up until t, which for some contracts affect the benefit paid out. △

The advantages of the proposed conditioning showed themselves already in the active-disabled-death model just presented. However, as we move to a more complex model with policy holder options, new interesting choices for $\hat{\mathcal{J}}$ show up.

Example 14.
Consider a multi-state model with policy holder options:

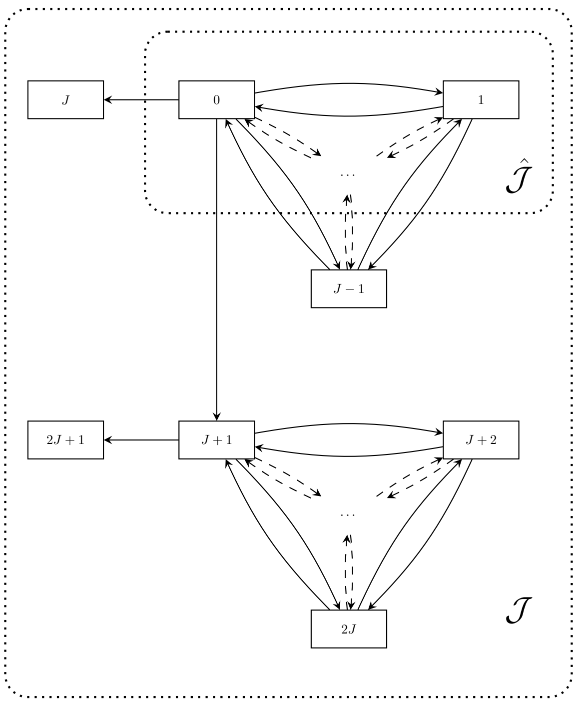


Figure 3.3: Multi-state model including policyholder options

In this model, we let $\hat{\mathcal{J}} = \{0, 1, ..., J-2\}$. In words, we condition on the policy holder not having died, surrendered or converted to free policy until time t. The difference to the classical prognoses is – again – that the policy holder is no longer assumed to be active from the time of prognostication to the time of payment. Indeed, she may lose her ability to work – or she may simply lose her job for a period (if the state space includes a "not employed" state). Both are relevant events to take into consideration when presenting her a prognosis.

The reason for excluding the surrender and free policy states from $\hat{\mathcal{J}}$ is not the same as the reason for excluding death. The policy holder is in control of the surrender and free policy transitions. So to her the transitions are not risks at all and shouldn't be treated as such when communicating prognoses. In contrast – from the point of view of a company doing valuation or solvency calculations – the transitions are risks and should be treated as such.

The relevant sojourn benefit prognosis and retirement lump sum prognosis are, respectively,

$$\mathbb{E}\left[b_{Z(t-)}(t) \middle| Z(t-) \in \hat{\mathcal{J}} \right]$$
$$\mathbb{E}\left[\triangle B_{Z(R-)}(R) \middle| Z(R-) \in \hat{\mathcal{J}} \right],$$

i.e. the expected rate (respectively lump sum) paid out at time t (respectively time R), given that the policy holder has not died, surrendered or invoked the free policy option until time t.

An interesting prognosis for the surrender payment is

$$\mathbb{E}\left[b_{0J}(t) \middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = J \right].$$

for $t \in [0, n]$ the time of surrender. This is exactly the expected sum paid out upon surrender at time t, given that the policy holder has not died, surrendered or invoked the free policy option up until time t.

Likewise, a prognosis for the free policy factor ρ is

$$\mathbb{E}\left[\rho(t) \middle| (Z(s) \in \hat{\mathcal{J}}, s < t) \cup (Z(t) = J + 1) \right].$$

for $t \in [t_0, n]$ the time of free policy conversion. We stress again that these prognoses are only really interesting, if the surrender payment $b_{0J}(t)$ and the free policy factor $\rho(t)$ depend on the past state path $\{Z(s)\}_{0 \leq s \leq t}$, which is not always the case. △

Remark 15.
Remark on Markov and conditioning until time n. ○

Remark 16.
Remark on transitions inside $\hat{\mathcal{J}}$. Trick med at $\sigma(Z(t)) = F^Z(t)$ for nogle kæder ○

In the preceding chapter, we have motivated and presented definitions for three different prognoses, one for each type of payment typical for life insurance contracts:

- Sojourn benefit rate prognoses (def. XX).
- Retirement lump sum benefit prognoses (def. XX).
- Transition benefit prognoses (def. XX).

In the two following chapters, the focus is on the computation of these prognoses. Chapter 4 deals with the prognostication of guaranteed payments. In this case, closed form expressions

In Chapter 5, we deal with the case of reserve-dependent benefits, which encompasses many popular contract types, including Unit Link and Participating Life Insurance. In this case,

Chapter 4

Guaranteed benefits

In this section, we will calculate the three types of prognoses for a life insurance contract with guaranteed benefits only. In this setting, all payments are settled at the signing of the contract (time 0) in accordance with the equivalence principle. Thus the payments at time t do not depend on the the past state path of the policy holder $\{Z(s)\}_{0<s<t}$, but only on the state just before time t, i.e. $Z(t-)$. As we shall see, this feature gives the prognoses a particularly simple form compared to the case with reserve-dependent benefits.

Assume that the payment process is on the form (2.3.1), i.e.

$$dB(t) = b_{Z(t-)}(t)dt + \triangle B_{Z(t-)}(t)d\varepsilon_R(t) + \sum_{k\neq Z(t-)} b_{Z(t-)k}(t)dN_k(t), \;\; B(0) = \pi_0,$$

The prognoses are then readily calculated:
Proposition 17.
The time t sojourn benefit rate prognosis in the case with guaranteed benefits is given by

$$\mathbb{E}\left[b_{Z(t-)}(t) \middle| Z(t-) \in \hat{\mathcal{J}} \right] = \frac{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)b_j(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)}$$

for some subset $\hat{\mathcal{J}} \subset \mathcal{J}$.

Proof.

$$\begin{aligned}
\mathbb{E}\left[b_{Z(t-)}(t) \middle| Z(t-) \in \hat{\mathcal{J}} \right] &= \sum_{j\in\mathcal{J}} \mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}} \middle| Z(t-) \in \hat{\mathcal{J}} \right] b_j(t) \\
&= \frac{\sum_{j\in\hat{\mathcal{J}}} \mathbb{E}\left[1_{\{Z(t-)=j\}} \right] b_j(t)}{\mathbb{P}\left(Z(t-) \in \hat{\mathcal{J}} \right)} \\
&= \frac{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)b_j(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)}
\end{aligned}$$

where we used the continuity of $p_{0j}(0,t)$. □

We see that the prognosis is simply the weighted average over $j \in \hat{\mathcal{J}}$ of the sojourn payment functions $b_j(t)$, with weights given by the probability of being in state j at time t (recall that $\hat{\mathcal{J}}$ is to be thought of as the full state space less the surrender, free policy and death state). The reason for this simplicity is that the only risk the policy holder is exposed to, is her state at time t. Since all the payments are fixed in the contract, it makes no difference e.g. how long she has been in a premium paying state.

The retirement lump sum prognosis takes a similar form:
Proposition 18.
The retirement lump sum prognosis in the case without bonus is given by

$$\mathbb{E}\left[\triangle b_{Z(R-)}(R) | Z(R-) \in \hat{\mathcal{J}}\right] = \frac{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(0,R) \triangle b_j(R)}{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(0,R)},$$

for some subset $\bar{\mathcal{J}} \subset \mathcal{J}$.

Proof.

$$\begin{aligned}\mathbb{E}\left[\triangle b_{Z(R-)}(R) \middle| Z(R-) \in \hat{\mathcal{J}}\right] &= \sum_{j \in \mathcal{J}} \mathbb{E}\left[\mathbb{1}_{\{Z(R-)=j\}} \middle| Z(R-) \in \hat{\mathcal{J}}\right] \triangle b_j(R) \\ &= \frac{\sum_{j \in \hat{\mathcal{J}}} \mathbb{E}\left[1_{\{Z(R-)=j\}}\right] \triangle b_j(R)}{\mathbb{P}\left(Z(R-) \in \hat{\mathcal{J}}\right)} \\ &= \frac{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(0,R) \triangle b_j(R)}{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(0,R)}\end{aligned}$$

where we again used the continuity of $p_{0j}(0,t)$. □

Again, the prognosis is simply a weighted average of the lump sum payouts of the states j, with weights given by the probability of being in state j at time of retirement R.
Remark 19.
We can arrive at these two first prognoses – sojourn and retirement lump sum – via a slightly different approach. Define a new Markov chain $\hat{Z}$ on the state space $\hat{\mathcal{J}}$ with transition probabilities

$$\hat{p}_{ij}(s,t) = \frac{p_{ij}(s,t)}{\sum_{j \in \hat{\mathcal{J}}} p_{ij}(s,t)}, \quad i,j \in \hat{\mathcal{J}},\ s < t,$$

We then obtain the sojourn prognosis

$$\begin{aligned}\mathbb{E}\left[b_{\hat{Z}(t)}(t)\right] &= \sum_{j \in \hat{\mathcal{J}}} \hat{p}_{0j}(0,t) b_j(t) \\ &= \frac{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(t_0,t) b_j(t)}{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(t_0,t)},\end{aligned}$$

which is identical to Proposition 17. So the sojourn and retirement lump sum prognoses can be obtained by reducing the state space and normalizing the transition probabilities accordingly. However, as we shall see, this approach does not work for the transition prognoses, because the intensities out of $\hat{\mathcal{J}}$ matter. So reducing the chain from Z to $\hat{Z}$ throws away needed information. ○

As we have seen, the transition benefit prognosis takes a slightly different form, as the conditioning event differs from the sojourn and the lump sum case – recall the discussion in Section 3.1. To calculate the transition prognoses, we need a lemma:

Lemma 20.

Given an F^Z-adapted process W, we have for $j \in \hat{\mathcal{J}}$ and $k \notin \hat{\mathcal{J}}$

$$\mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}W(t-)\middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = k\right] = \frac{\sum_{j\in\hat{\mathcal{J}}} \mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}W(t-)\right]\mu_{jk}(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(t_0,t)\mu_{jk}(t)}$$

Proof. See Appendix XXX. □

Proposition 21.

The state k transition benefit prognosis in the case without bonus is given by

$$\mathbb{E}\left[b_{Z(t-)k}(t)\middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = k\right] = \frac{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)\mu_{jk}(t)b_{jk}(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)\mu_{jk}(t)},$$

for some subset $\bar{\mathcal{J}} \subset \mathcal{J}$ and $k \notin \bar{\mathcal{J}}$.

Proof. The result follows immediately from Lemma 20:

$$\begin{aligned}\mathbb{E}\left[b_{Z(t-)k}(t)\middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = k\right] &= \sum_{j\in\hat{\mathcal{J}}} \mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}\middle| Z(t-) \in \hat{\mathcal{J}}, Z(t) = k\right] b_{jk}(t)\\ &= \frac{\sum_{j\in\hat{\mathcal{J}}} \mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}\right]\mu_{jk}(t)b_{jk}(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)\mu_{jk}(t)}\\ &= \frac{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)\mu_{jk}(t)b_{jk}(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t)\mu_{jk}(t)},\end{aligned}$$

where we also used the continuity of $p_{0j}(0,t)$ □

Again, the prognosis take the form of a weighted average of payment functions $b_{jk}(t)$. This time, however, the weights are the probabilities – given a transition into state k at time t – that the transition happened from state j. These weights $p_{0j}(0,t)\mu_{jk}(t)$ resemble probability density functions – i.e. the (generalized) survival function $p_{0j}(0,t)$ times the intensity $\mu_{jk}(t)$. This is rather intuitive, since what we're after is the probability of making the transition $j \mapsto k$ in a small time interval around t.

Remark 22.

From Propositions XX, XX and XX, it is clear that the problem of calculating prognoses in the case without bonus is only as complex as calculating the transition probabilities $p_{0j}(0,t)$. Outside of very simple models, where these probabilities can be found analytically, this requires solving the Kolmogorov forward differential equations in a one-dimensional time grid. As we shall see the case with bonus requires solving a forward differential equation containing the transition probabilities.... ○

We will now give an example blah blah.

Example 23.

Consider a policy modelled as in Example 13, i.e. the classical alive-disabled-dead model with reactivation:

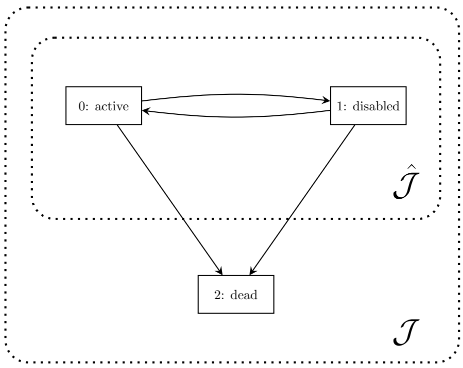


Figure 4.1: Active-disabled-dead model

The contract is specified as follows:

- Before retirement, the policy holder pays a constant premium rate π if active, and receives a constant disability benefit rate b_1 if disabled.
- In case of death before retirement, a transition lump sum is paid out to the bereaved: b_{01} if she dies as active, b_{12} if she dies as disabled.
- At retirement, she is paid a lump sum pension benefit $\triangle b_0$ if and only if she is active at the point of retirement R. As retired, she is paid a constant pension benefit $b_0 = b_1$ as long as she remains alive.

Mathematically put, the payment stream takes the form

$$\begin{aligned} dB(t) =& \mathbb{1}_{\{t<R\}} \left(-\mathbb{1}_{\{Z(t-)=0)\}} \pi + \mathbb{1}_{\{Z(t-)=1\}} b_1 \right) dt \\ &+ \mathbb{1}_{\{t<R\}} \left(\mathbb{1}_{\{Z(t-)=0\}} b_{02} + \mathbb{1}_{\{Z(t-)=1\}} b_{12} \right) dN_2(t) \\ &+ \mathbb{1}_{\{Z(t-)=0\}} \triangle b_0 d\varepsilon_R(t) + \mathbb{1}_{\{t\geq R\}} \mathbb{1}_{\{Z(t-)\neq 2\}} b_0 dt \end{aligned}$$

The payments are determined such that the equivalence principle is satisfied under the technical basis, i.e.

$$\mathbb{E}^* \left[\int_0^n e^{-\int_0^s r^*(u)du} dB(s) \right] = 0$$

We can now calculate prognoses using Propositions 17, 18 and 21. Due to the simplicity of the model and the payments, some prognoses are trivial. For instance, the sojourn pension benefit prognosis (so let $t \geq R$) is calculated with Proposition 17 as

$$\begin{aligned} \mathbb{E} \left[b_{Z(t-)}(t) \middle| Z(t-) \in \{0,1\} \right] =& \frac{\sum_{j=0}^1 p_{0j}(0,t) b_j(t)}{\sum_{j=0}^1 p_{0j}(0,t)} \\ =& b_0 \end{aligned}$$

because the pension benefit b is the same for the active and disabled states.

A nontrivial prognosis, however, is the retirement lump sum prognosis. By Proposition 18,

$$\begin{aligned}\mathbb{E}\left[\triangle b_{Z(R-)}(R)\Big|\, Z(R-)\in\{0,1\}\right] =& \frac{\sum_{j=0}^{1} p_{0j}(0,R)\triangle b_j(R)}{\sum_{j=0}^{1} p_{0j}(0,R)}\\ =& \frac{p_{00}(0,R)\triangle b_0 + p_{01}(0,R)\cdot 0}{p_{00}(0,R)+p_{01}(0,R)}\\ =& \frac{p_{00}(0,R)}{p_{00}(0,R)+p_{01}(0,R)}\triangle b_0\\ =& \mathbb{P}\left(Z(R)=0\big|\, Z(R)\in\{0,1\}\right)\cdot\triangle b_0,\end{aligned}$$

which is just the probability of staying active until retirement (given survival until retirement) times the lump sum $\triangle b_0$ rewarded for doing so. Recalling Remark 19, $\mathbb{P}\left(Z(R)=0\big|\, Z(R)\in\{0,1\}\right)$ can be interpreted as the probability – in a model with only two states: 0 and 1 – of being in state 0 at time t. It is worth noting that this probability does not depend on the intensities out of $\hat{\mathcal{J}}$, μ_{02} and μ_{12}.

If we assume that no reactivation can occur, i.e. $\mu_{10}=0$, the prognosis simplify to

$$\begin{aligned}\mathbb{E}\left[\triangle b_{Z(R)}(R)\Big|\, Z(R)\in\{0,1\}\right] =& \frac{p_{00}(0,R)}{p_{00}(0,R)+p_{01}(0,R)}\triangle b_0\\ =& e^{-\int_0^t \mu_{01}(u)du}\triangle b_0,\end{aligned}$$

where the transition probabilities were found with Kolmogorovs forward differential equation[1]. Again recalling Remark 19, this can be interpreted as the probability – in a model with only two states: 0 and 1 – of not making the transition from state 0 to state 1 before time R, times the lump sum reward for not doing so.

Finally, using Proposition 21, we can calculate the state 2 transition prognosis, i.e. the expected death sum paid out given death at time $t<R$:

$$\begin{aligned}\mathbb{E}\left[b_{Z(t-)2}(t)\Big|\, Z(t-)\in\{0,1\}, Z(t)=2\right] =& \frac{\sum_{j=0}^{1} p_{0j}(0,t)\mu_{j2}(t)b_{j2}(t)}{\sum_{j=0}^{1} p_{0j}(0,t)\mu_{j2}(t)}\\ =& \frac{p_{00}(0,t)\mu_{02}(t)b_{02}+p_{01}(0,t)\mu_{12}(t)b_{12}}{p_{00}(0,t)\mu_{02}(t)+p_{01}(0,t)\mu_{12}(t)},\end{aligned}$$

which is again a weighted average of the transition payments with weights given by the 'densities' $p_{0j}(0,t)\mu_{j2}(t)$. In contrast to the retirement lump sum prognosis, the transition prognosis depends on the intensities out of $\hat{\mathcal{J}}$ – i.e. μ_{02} and μ_{12} – and so does not simplify nicely with the assumption of no reactivation.

However, if we also assume that $\mu_{02}=\mu_{12}$ the prognosis simplifies to

$$\begin{aligned}\mathbb{E}\left[b_{Z(t-)2}(t)\Big|\, Z(t-)\in\{0,1\}, Z(t)=2\right] =& \frac{p_{00}(0,t)b_{02}+p_{01}(0,t)b_{12}}{p_{00}(0,t)+p_{01}(0,t)}\\ =& e^{-\int_0^t \mu_{01}(u)du}b_{02}+\left(1-e^{-\int_0^t \mu_{01}(u)du}\right)b_{12}\end{aligned}$$

[1] See Appendix .3 for proof.

which, again recalling Remark 19, can be interpreted as a weighted average in a two state model – the first weight being the probability of there being no transition from state 0 to state 1 before t, the second being the probability of a transition taking place.

TABLE FOR COMPARISON △

In the preceding chapter, we saw how to calculate .

However, many modern insurance contracts include benefits that are not guaranteed, but instead depend on the performance of a number of individual or collective reserves and accounts.

Chapter 5

Prognoses with reserve-dependent benefits

The defining feature of the case without bonus was that all payments were completely determined at the time of signing of the contract and so did not depend on the state path of Z. In the real world, however, payment streams often takes the form

$$\begin{aligned} dB(t, \mathbf{W}(t-)) =& \mathbf{W}(t-) d\mathbf{B}(t) \\ =& \sum_{i=1}^{n} W^i(t-) dB^i(t), \end{aligned} \tag{5.0.1}$$

where the B^i are profile payment streams on the form (2.3.1), scaled linearly by the entries of an $\mathcal{F}$-adapted n-dimensional process $\mathbf{W}$. It is the object of this chapter to develop a theory for computing prognoses for payment streams on the form (5.0.1).

- We start out by formulating the theory in general terms and then show that the guaranteed benefits from the last chapter can be treated as a special case.
- Then, we show that a broad class of With Profit contracts – as presented in Chapter VI in [Asmussen and Steffensen, 2020] – also fits into the general theory, with (X, Y) (X being the technical reserve of guaranteed payments and Y being the surplus) playing the role of $\mathbf{W}$.
- Finally, the differences between the classical prognoses (without event risk) and the proposed are discussed in the context of a numerical example.

Given a payment stream on the form (5.0.1), the three prognoses take the forms

$$\hat{b}(t) = \mathbb{E}\left[\mathbf{W}(t-)\mathbf{b}_{Z(t-)} \,\middle|\, \{Z(s)\}_{0 \le s < t} \in \bar{\mathcal{J}}\right] \tag{5.0.2}$$

$$\triangle \hat{B}(R) = \mathbb{E}\left[\mathbf{W}(R-)\triangle \mathbf{B}_{Z(R-)} \,\middle|\, \{Z(s)\}_{0 \le s < R} \in \bar{\mathcal{J}}\right] \tag{5.0.3}$$

$$\hat{b}_{\cdot k}(t) = \mathbb{E}\left[\mathbf{W}(t-)\mathbf{b}_{Z(t-)k} \,\middle|\, \{Z(s)\}_{0 \le s < t} \in \bar{\mathcal{J}}, Z(t) = k\right] \tag{5.0.4}$$

Rewriting, we obtain for the sojourn prognosis

$$\begin{aligned}
\hat{b}(t) =&\mathbb{E}\left[\sum_{i=1}^{n} W_i(t-)b^i_{Z(t)}\middle| \{Z(s)\}_{0\leq s<t} \in \bar{\mathcal{J}}\right] \\
=&\mathbb{E}\left[\sum_{i=1}^{n} W_i(t-)b^i_{Z(t)}\middle| Z(t-) \in \bar{\mathcal{J}}\right] \\
=&\frac{\sum_{i=1}^{n}\sum_{j\in\hat{\mathcal{J}}}\mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}W_i(t-)\right]b^i_j(t)}{\sum_{j\in\hat{\mathcal{J}}}p_{Z(t_0)j}(t_0,t)},
\end{aligned}$$

and for the retirement lump sum prognosis

$$\begin{aligned}
\triangle\hat{B}(t) =&\mathbb{E}\left[\sum_{i=1}^{n} W_i(t-)\triangle B^i_{Z(t-)}\middle| \{Z(s)\}_{0\leq s<t} \in \bar{\mathcal{J}}\right] \\
=&\mathbb{E}\left[\sum_{i=1}^{n} W_i(t-)\triangle B^i_{Z(t-)}\middle| Z(t-) \in \bar{\mathcal{J}}\right] \\
=&\frac{\sum_{i=1}^{n}\sum_{j\in\hat{\mathcal{J}}}\mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}W_i(t-)\right]\triangle B^i_j(t)}{\sum_{j\in\hat{\mathcal{J}}}p_{Z(t_0)j}(t_0,t)},
\end{aligned}$$

and – using Lemma 20 – for the state k transition prognosis

$$\begin{aligned}
\hat{b}_{\cdot k}(t) =&\mathbb{E}\left[\sum_{i=1}^{n} W^i(t-)b^i_{Z(t-)k}\middle| \{Z(s)\}_{0\leq s<t} \in \bar{\mathcal{J}}, Z(t)=k\right] \\
=&\mathbb{E}\left[\sum_{i=1}^{n} W^i(t-)b^i_{Z(t-)k}\middle| Z(t-) \in \bar{\mathcal{J}}, Z(t)=k\right] \\
=&\sum_{i=1}^{n}\sum_{j\in\hat{\mathcal{J}}}\mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}W^i(t-)\middle| Z(t-)\in\hat{\mathcal{J}}, Z(t)=k\right]b^i_{jk}(t) \\
=&\frac{\sum_{i=1}^{n}\sum_{j\in\hat{\mathcal{J}}}\mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}W^i(t-)\right]\mu_{jk}(t)b^i_{jk}(t)}{\sum_{j\in\hat{\mathcal{J}}}p_{Z(t_0)j}(t_0,t)\mu_{jk}(t)}
\end{aligned}$$

It is clear from the above that if we can calculate the vectors

$$\tilde{\mathbf{W}}_j(t-) = \mathbb{E}\left[\mathbb{1}_{\{Z(t-)=j\}}\mathbf{W}(t-)\right],\ \ j\in\hat{\mathcal{J}}$$

we can use the entries to calculate the prognoses $\hat{b}_{Z(t)}(t)$, $\triangle\hat{B}_{Z(R)}(R)$ and $\hat{b}_{Z(t-)k}(t)$. The $\tilde{\mathbf{W}}_j$'s are sometimes referred to as *state-wise retrospective reserves.* In [Bruhn and Lollike, 2020], the authors derive a system of differential equations characterizing the $\tilde{\mathbf{W}}_j$'s and use them in an asset liability management context to project expected liabilities and other balance sheet items into the future. In this thesis however, their use is in determining expected future payments for the purpose of prognoses. Also, the results from [Bruhn and Lollike, 2020] are here extended to allow for a lump sum payment at retirement.

To illustrate the method for obtaining the differential equations, assume that $p_{0i}(0,t) > 0$ for all $i \in \mathcal{J}$ and $t > 0$ and that W is a 1-dimensional process with dynamics

$$\begin{aligned} dW(t) =& W(t-)f_{Z(t-)}(t)dt \\ &+ W(t-)h_{Z(t-)}(t)d\varepsilon_R(t). \end{aligned}$$

In integral form, W is given by

$$\begin{aligned} W(t) =& w_0 + \int_{[0,t]} W(s-)f_{Z(s-)}(s)ds \\ &+ \int_{[0,t]} W(s-)h_{Z(s-)}(s)d\varepsilon_R(s). \end{aligned}$$

The state wise reserves are then ($j \in \mathcal{J}$)

$$\begin{aligned} \tilde{W}_j(t) =& \mathbb{E}\left[W(t)\mathbb{1}_{\{Z(t)=j\}}\right] \\ =& \mathbb{E}\left[w_0\mathbb{1}_{\{Z(t)=j\}}\right] + \mathbb{E}\left[\int_{[0,t]} \mathbb{1}_{\{Z(t)=j\}} W(s-)f_{Z(s)}(s)ds\right] \\ &+ \mathbb{E}\left[\int_{[0,t]} \mathbb{1}_{\{Z(t)=j\}} W(s-)h_{Z(s-)}(s)d\varepsilon_R(s)\right] \\ =& p_{0j}(0,t)w_0 + \int_{[0,t]} \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} W(s-)f_{Z(s-)}(s)\,\middle|\, Z(s-)\right]\right] ds \\ &+ \mathbb{E}\left[\mathbb{E}\left[\int_{[0,t]} \mathbb{1}_{\{Z(t)=j\}} W(s-)h_{Z(s-)}(s)d\varepsilon_R(s)\,\middle|\, Z(s-)\right]\right] \\ =& p_{0j}(0,t)w_0 + \int_{[0,t]} \sum_{i\in\mathcal{J}} p_{0i}(0,s)f_i(s)\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} W(s-)\,\middle|\, Z(s-)=i\right] ds \\ &+ \sum_{i\in\mathcal{J}} p_{0i}(0,R-)\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z(t)=j\}} W(s-)h_i(s)d\varepsilon_R(s)\,\middle|\, Z(R-)=i\right]. \end{aligned} \tag{5.0.5}$$

Since $W(s-) \in \mathcal{F}_{(0,s)}$ and $\mathbb{1}_{\{Z(t)=j\}} \in \mathcal{F}_{(s,\infty)}$, it follows by the Markovianity of Z that $U(s-) \perp\!\!\!\perp \mathbb{1}_{\{Z(t)=j\}} | Z(s)$. And so

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} W(s-)\,\middle|\, Z(s-)=i\right] =& \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}}\,\middle|\, Z(s-)=i\right]\mathbb{E}\left[W(s-)\,\middle|\, Z(s-)=i\right] \\ =& p_{ij}(s,t)\frac{\tilde{W}_i(s-)}{p_{0i}(0,s)}, \end{aligned}$$

By the same token,

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z(t)=j\}} W(s-)h_i(s)d\varepsilon_R(s) \middle| Z(R-)=i\right] =& \mathbb{1}_{\{t\geq R\}}\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} W(R-)h_i(R) \middle| Z(R-)=i\right] \\
=& \mathbb{1}_{\{t\geq R\}} h_i(R)\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} \middle| Z(R-)=i\right] \mathbb{E}\left[W(R-)\middle| Z(R-)=i\right] \\
=& \mathbb{1}_{\{t\geq R\}} h_i(R) p_{ij}(R,t)\frac{\tilde{W}_i(R-)}{p_{0i}(0,R)}.
\end{aligned}$$

Plugging these two results into (5.0.5) and cancelling the p_{0i} gives us

$$\begin{aligned}
\tilde{W}_j(t) =& p_{0j}(0,t)w_0 + \int_{[0,t]} \sum_{i\in\mathcal{J}} f_i(s)p_{ij}(s,t)\tilde{W}_i(s)ds \\
&+ \mathbb{1}_{\{t\geq R\}} \sum_{i\in\mathcal{J}} h_i(R)p_{ij}(R,t)\tilde{W}_i(R-).
\end{aligned} \tag{5.0.6}$$

Now, $\tilde{W}_j(t)$ is obviously not differentiable at R, partly due to the lump sum occurring via h_i, partly because the integral is not differentiable at R, as this is potentially a point of discontinuity for f_i. On $(0,R)\cup(R,n)$, however, we have by the Kolmogorov forward equations and Leibniz' integral rule,

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{W}_j(t) =& \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \big(p_{0k}(0,t)\mu_{kj}(t) - p_{0j}(0,t)\mu_{jk}(t)\big)\, w_0 \\
&+ \sum_{i\in\mathcal{J}} f_i(t)p_{ij}(t,t)\tilde{W}_i(t) \\
&+ \int_{[0,t]} \sum_{i\in\mathcal{J}} f_i(s)\frac{\partial}{\partial t}p_{ij}(s,t)\tilde{W}_i(s)ds \\
&+ \mathbb{1}_{\{t\geq R\}} \sum_{i\in\mathcal{J}} h_i(R)\frac{\partial}{\partial t}p_{ij}(R,t)\tilde{W}_i(R-) \\
=& \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \big(p_{0k}(0,t)\mu_{kj}(t) - p_{0j}(0,t)\mu_{jk}(t)\big)\, w_0 \\
&+ f_j(t)\tilde{W}_j(t) \\
&+ \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \int_{[0,t]} \sum_{i\in\mathcal{J}} f_i(s)p_{ik}(s,t)\mu_{kj}(t)\tilde{W}_i(s)ds - \int_{[0,t]} \sum_{i\in\mathcal{J}} f_i(s)p_{ij}(s,t)\mu_{jk}(t)\tilde{W}_i(s)ds \\
&+ \mathbb{1}_{\{t\geq R\}} \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \left(\sum_{i\in\mathcal{J}} h_i(R)p_{ik}(R,t)\mu_{kj}(t)\tilde{W}_i(R) - \sum_{i\in\mathcal{J}} h_i(R)p_{ij}(R,t)\mu_{jk}(t)\tilde{W}_i(R) \right)
\end{aligned}$$

We can now recognize $\tilde{W}_k(t)$ and $\tilde{W}_j(t)$ from equation (5.0.6) to arrive at

$$\frac{\partial}{\partial t}\tilde{W}_j(t) = f_j(t)\tilde{W}_j(t) + \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \mu_{kj}(t)\tilde{W}_k(t) - \mu_{jk}(t)\tilde{W}_j(t),$$

which holds for $t \in (0, R) \cup (R, n)$ and $j \in \mathcal{J}$. The initial value is

$$\begin{aligned}\tilde{W}_j(0) =& p_{0j}(0,0)w_0 \\ =& \mathbb{1}_{\{j=0\}} w_0\end{aligned}$$

and the gluing condition can be derived from equation (5.0.6) as

$$\begin{aligned}\tilde{W}_j(R) - \tilde{W}_j(R-) =& \sum_{i \in \mathcal{J}} h_i(R) p_{ij}(R, R-) \tilde{W}_i(R-) \\ =& h_j(R) \tilde{W}_j(R-)\end{aligned}$$

We have now arrived at a system of differential equations for a one-dimensional reserve W with relative simple dynamics. For a multidimensional reserve with more general dynamics – including event risk via N_k – the system of differential equations is stated in the following proposition.

Proposition 24.

Let W be an $\mathcal{F}^Z$-adapted n-dimensional process with initial value $W(0) = w_0$ and dynamics

$$\begin{aligned}dW(t) =& f_{Z(t-)}(t, W(t-))dt \\ &+ \sum_{\substack{k \in \mathcal{J} \\ k \neq Z(t-)}} g_{Z(t-)k}(t, W(t-))dN_k(t) \\ &+ h_{Z(t-)}(t, W(t-))d\varepsilon_R(t),\end{aligned} \tag{5.0.7}$$

where f, g and h are n-dimensional deterministic functions of finite variation that are $\mathcal{C}^0$ on $(0, R) \cup (R, n)$ and on the form

$$\begin{aligned}f_j(t, w) =& f_j^0(t) + f_j^1(t) \cdot w \\ g_{jk}(t, w) =& g_{jk}^0(t) + g_{jk}^1(t) \cdot w \\ h_j(t, w) =& h_j^0(t) + h_j^1(t) \cdot w,\end{aligned}$$

for $n \times n$-matrices f_j^1, g_{jk}^1 and h_j^1 and vectors f_j^0, g_{jk}^0 and h_j^0 of length n.

Then

$$\tilde{W}_j(t) = \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} W(t)\right]$$

is characterized by the system of differential equations

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{W}_j(t) =& f_j^1(t)\tilde{W}_j(t) + p_{0j}(0,t) f_j^0(t) \\ &+ \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \mu_{kj}(t) \left(g_{kj}^1(t) \tilde{W}_k(t) + p_{0k}(0,t) g_{kj}^0(t) \right) \\ &+ \sum_{\substack{k \in \mathcal{J} \\ k \neq j}} \left(\mu_{kj}(t) \tilde{W}_k(t) - \mu_{jk}(t) \tilde{W}_j(t) \right),\end{aligned}$$

for $t \in (0, R) \cup (R, n)$ and $j \in \mathcal{J}$. The initial values are

$$\tilde{W}_j(0) = \mathbb{1}_{\{j=0\}} w_0$$

and the gluing conditions are

$$\tilde{W}_j(R) = \tilde{W}_j(R-) + h_j^1(R)\tilde{W}_j(R-) + p_{0j}(0, R)h_j^0(R)$$

Proof. The proof closely mimics the one in [Bruhn and Lollike, 2020], but extends it to allow for the deterministic jumps in W at time R via the h-function. The proof can be found in Appendix .4. $\square$

In words, the process W has dynamics affine in itself, with changes that depend on the state process Z and happen

- continuously during state sojourns,
- upon transitions between states,
- deterministically at the time of retirement R.

As is the case with the payment processes, the time of retirement R is a point of discontinuity for the functions f, g and h.

As we shall see, the formulation of a payment stream on the form (5.0.1) and an account on the form of Definition 24 is general enough to encompass many classical contracts, such as With Profit (with special cases Participating Life Insurance and Unit Link).

The reader is encouraged to consult equations (**??**) and (**??**) on page XXX for the With Profit case (where the account plays the role of the amount of purchased bonus benefits, often denoted Q in the literature) and equation (5.0.7) on page XXX for the Unit Link case (where it is simply the asset account of the policy holder).

Remark 25.

We can replicate the prognosis of the last chapter – i.e. prognoses for guaranteed benefits – in the present framework. Recall that the prospective technical reserve $V^*_{Z(t)}(t)$ for some payment stream B has dynamics

$$\begin{aligned} dV^*_{Z(t)}(t) =& r^*(t)V^*_{Z(t)}dt - b_{Z(t-)}(t)dt - \sum_{\substack{k \in \mathcal{J} \\ k \neq Z(t-)}} R^*_{Z(t-)k}(t)\mu^*_{Z(t-)k}(t)dt \\ &+ \sum_{\substack{k \in \mathcal{J} \\ k \neq Z(t-)}} \left(V^*_k(t) - V^*_{Z(t-)}(t)\right) dN_k(t) - \triangle B_{Z(t-)}(R)d\varepsilon_R(t). \end{aligned}$$

which is clearly on the form (5.0.7). Assuming that B satisfies the equivalence principle under the technical basis (r^*, μ^*), the initial value is $V^*_{Z(0)}(0) = 0$. Then, by Proposition 24,

$$\tilde{V}^*_j(t) = \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} V^*_{Z(t)}(t)\right] = p_{0j}(0, t)V^*_j(t)$$

is characterized by the system of differential equations STJERNER PÅ P'er

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{V}_j^*(t) =& XXX \left(r^*(t) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}^*(t) \right) \tilde{V}_j^*(t) - p_{0j}(0,t) \left(b_j(t) + \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{jk}^*(t) \left(b_{jk}(t) + V_k^*(t) \right) \right) \\
&+ \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{kj}(t) \left(\tilde{V}_k^*(t) + p_{0k}(0,t) V_k^*(t) \right) \\
&+ \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \left(\mu_{kj}(t)\tilde{V}_k^*(t) - \mu_{jk}(t)\tilde{V}_j^*(t) \right) XXX \\
=& r^*(t)\tilde{V}_j^*(t) - p_{0j}(0,t)b_j(t) - \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} p_{0k}(0,t)\mu_{kj} b_{kj}(t) \\
&+ \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \left(\mu_{kj}(t)\tilde{V}_k^*(t) - \mu_{jk}(t)\tilde{V}_j^*(t) \right),
\end{aligned}
$$

with gluing condition

$$
\tilde{V}_j^*(R) = \tilde{V}_j(R-) - p_{0j}(0,R)\triangle B_j(R)
$$

and initial value

$$
\begin{aligned}
\tilde{V}_j^*(0) =& \mathbb{1}_{\{j=0\}} V_{Z(0)}^*(t) \\
=& 0
\end{aligned}
$$

In [Bruhn and Lollike, 2020], the authors show that XXX

To obtain the prognoses of the last chapter, define the payment stream

$$
d\bar{B}(t) = \frac{1}{V_{Z(t)}^*(t)} dB(t),
$$

we can prognosticate the payment stream

$$
V_{Z(t)}^*(t) d\bar{B}(t),
$$

which is clearly on the form (5.0.1). By (XXX), the sojourn benefit prognosis is then

$$
\begin{aligned}
\hat{\bar{b}}(t) =& \frac{\sum_{j\in\hat{\mathcal{J}}} \tilde{V}_j^*(t)\bar{b}_j(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{Z(t_0)j}(t_0,t)} \\
=& \frac{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t) V_j^*(t) \frac{b_j(t)}{V_j^*(t)}}{\sum_{j\in\hat{\mathcal{J}}} p_{Z(t_0)j}(t_0,t)} \\
=& \frac{\sum_{j\in\hat{\mathcal{J}}} p_{0j}(0,t) b_j(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{Z(t_0)j}(t_0,t)},
\end{aligned}
$$

which we recognize as the sojourn prognosis for the guaranteed payments from the last chapter, cf. (XXX). So in effect, the case with reserve-dependent benefits is a generalization of the case with guaranteed benefits. ○

Remark 26.
Dynamikken må gerne afhænge af hele den fortidige Z-sti ○

Remark 27.
We can replicate the prognoses without bonus in the present with-bonus framework. Simply let $w_0 = 1$ and $f = g = h = 0$. Then by Proposition 24, $\tilde{W}_j(t) = \mathbb{E}\left[W(t)1_{\{Z(t)=j\}}\right]$ is characterized by the system of differential equations

$$\frac{\partial}{\partial t}\tilde{W}_j(t) = \sum_{\substack{k\in\mathcal{J}\\k\neq j}} \left(\mu_{kj}(t)\tilde{W}_k(t) - \mu_{jk}(t)\tilde{W}_j(t)\right), \tilde{W}_j(0) = \mathbb{1}_{\{j=0\}},$$

for $j \in \mathcal{J}$, which we recognize as Kolmogorovs forward equations for the transition probabilities. The unique solution is of course

$$\tilde{W}_j(t) = p_{Z(t_0)j}(t_0, t)$$

and so by Proposition XXX the prognoses are

$$xxx$$

which are exactly the prognoses in the case without bonus, see XX.

This example shows that we were actually already employing Proposition 24 in Example XXX when calculating prognoses in the case without bonus, because the transition probabilities $p_{Z(t_0)j}$ are just a special case of $\tilde{W}_j(t)$. ○

Remark on defining new payment functions.

Having formulated the necessary theory for a general process W, we are now ready to treat a broad class of contract types known as With Profit (as presented in [Asmussen and Steffensen, 2020]).

5.1 With Profit

In this section, we shall see how the With Profit class of contracts – as described in e.g. [Asmussen and Steffensen, 2020] – fits into general theory of the preceding section. Specifically, we will let $W = (X, Y)$, X being the technical value of future guaranteed payments and Y being the surplus, and let payments depend linearly on these two processes in the sense of 5.0.1. We shall see how the specification of the dividend process D allows us to model many classical With Profit pension and life insurance contracts, including participating life insurance and unit-link.

The first ingredient in a With Profit contract is the guaranteed payment stream B°. The payment stream comprises both benefits and premiums, balanced such that the equivalence principle is satisfied on the technical basis, i.e.

$$0 = V_0^{\circ *}(0) = \mathbb{E}^*\left[\int_0^n e^{-\int_0^s r^*(u)du} dB^\circ(s)\right],$$

where the technical interest rate r^* is deterministic and $\mathbb{E}^*$ is integration with respect to $\mathbb{P}^*$, which is a probability measure under which Z has intensities μ^*.

With premiums determined under this prudent technical basis, a systematic surplus will emerge if everything goes well. Part of this surplus belongs to the insured and is paid back in the form of dividends. Let $D = \{D(t)\}_{0\leq t\leq n}$ be the process of accumulated dividends. Assume that it is $\mathcal{F}$-adapted and that it can be decomposed as

$$D = D_1 + D_2,$$

where D_1 is paid out directly to the policy holder and D_2 is used to instantaneously purchase units of a profile payment stream $B^\dagger$, also on the form 2.3.1, but consisting only of benefits, i.e. $B^{\dagger-} = 0$. Assume that D_2 is absolutely continuous, i.e.

$$D_2(t) = \delta_2(t)dt$$

for some process $\delta_2 = \{\delta_2(t, Z(t))\}_{0\leq t\leq n}$. Denote by $Q(t)$ the amount of units of $B^\dagger$ purchased at time t. Since the units are priced on the technical basis, we have the budget equation

$$\delta_2(t)dt = V^{\dagger *}_{Z(t)}(t)dQ(t),$$

stating that the funds available in a small interval around t, i.e. $\delta_2(t)dt$, is used to purchase $dQ(t)$ units of $B^\dagger$ with unit price $V^{\dagger *}_{Z(t)}(t)$. Assume that $V^{\dagger *}_{Z(t)}(t) > 0$ whenever $\delta_2(t) > 0$ such that we never try to purchase a free payment stream. It follows that

$$dQ(t) = \frac{\delta_2(t)}{V^{\dagger *}_{Z(t)}(t)}dt, \quad Q(0) = 0. \tag{5.1.1}$$

Often, δ_2 is assumed to be non-negative such that Q is non-decreasing, meaning that the additional benefits, once granted, are guaranteed. But this is not a requirement.

The total payment stream experienced by the policy holder is then

$$dB(t) = dB^\circ(t) + Q(t-)dB^\dagger(t) + dD_1(t). \tag{5.1.2}$$

We now introduce the technical value of future guaranteed payments

$$X(t) = V^{\circ *}_{Z(t)}(t) + Q(t)V^{\dagger *}_{Z(t)}(t),$$

and note that

$$\begin{aligned} X(0) =& V_0^{\circ *}(0) + Q(0)V_0^{\dagger *}(0) \\ =& 0 + 0 \cdot V_0^{\dagger *}(0) \\ =& 0. \end{aligned}$$

By integration by parts for FV-functions (Theorem 4.1 in [Pedersen, 2017]) and equation (2.4.1), the dynamics of X are

$$\begin{aligned}
dX(t) =& dV^{\circ *}_{Z(t)}(t) + Q(t-)dV^{\dagger *}_{Z(t)}(t) + V^{\dagger *}_{Z(t)}(t)dQ(t) \\
=& r^*(t)X(t)dt + \delta_2(t)dt \\
& - \left(b^{\circ}_{Z(t-)}(t) + Q(t-)b^{\dagger}_{Z(t-)}(t)\right) dt - \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(t-)}} \left(b^{\circ}_{Z(t-)k}(t) + Q(t-)b^{\dagger}_{Z(t-)k}(t)\right) dN_k(t) \\
& - \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(t-)}} \left(\rho^{\circ}_{Z(t-)k}(t) + Q(t-)\rho^{\dagger}_{Z(t-)k}(t)\right) dt \\
& + \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(t-)}} \left(R^{\circ *}_{Z(t-)k}(t) + Q(t-)R^{\dagger *}_{Z(t-)k}(t)\right) \left(dN_k(t) - \mu_{Z(t-)k}(t)dt\right),
\end{aligned} \tag{5.1.3}$$

We can now define the surplus

$$Y(t) = -\int_0^t e^{\int_s^t r(u)du} dB(s) - X(t),$$

which is the value of past payments accumulated with financial returns less the technical value of future guaranteed payments. Again using integration by parts and Leibniz' rule, we obtain

$$\begin{aligned}
dY(t) =& r(t)\left(-\int_0^t e^{\int_s^t r(u)du} dB(s)\right) - dB(t) - dX(t) \\
=& r(t)\left(Y(t) + X(t)\right) dt - dB(t) - dX(t) \\
=& r(t)Y(t)dt + dC(t) - \delta_2(t)dt - dD_1(t) \\
& - \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(t-)}} \left(R^{\circ *}_{Z(t-)k}(t) + Q(t-)R^{\dagger *}_{Z(t-)k}(t)\right) \left(dN_k(t) - \mu_{Z(t-)k}(t)dt\right),
\end{aligned} \tag{5.1.4}$$

where the surplus contribution process C is given by

$$dC(t) = \left(r(t) - r^*(t)\right) X(t)dt + \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(t-)}} \left(\rho^{\circ}_{Z(t-)k}(t) + Q(t-)\rho^{\dagger}_{Z(t-)k}(t)\right) dt,$$

representing the systematic contributions from X to Y. We discard the martingale term in the last line of (5.1.4). It is left for the company to cover, as the policy holder is not supposed to participate in the performance of diversifiable risk; only the systematic. Finally, note that

$$\begin{aligned}
Y(0) =& -0 - X(0) \\
=& 0.
\end{aligned}$$

Recall that the total payments experienced by the policy holder is given by

$$
\begin{aligned}
dB(t) =& dB^\circ(t) + Q(t-)dB^\dagger(t) + dD_1(t) \\
=& dB^\circ(t) + \frac{X(t-) - V^{\circ *}_{Z(t-)}(t-)}{V^{\dagger *}_{Z(t-)}(t-)} dB^\dagger(t) + dD_1(t)
\end{aligned}
$$

This is the payment stream that we want to prognosticate. If we make the assumption that the cash dividends D_1 are linear in X and Y, i.e.

$$
dD_1(t) = dB^{D,0}(t) + X(t-) \cdot dB^{D,1}(t) + Y(t-) \cdot dB^{D,2}(t), \tag{5.1.5}
$$

we get the total payment stream

$$
\begin{aligned}
dB(t) =& dB^\circ(t) + \frac{X(t-) - V^{\circ *}_{Z(t-)}(t-)}{V^{\dagger *}_{Z(t-)}(t-)} dB^\dagger(t-) + dB^{D,0}(t) + X(t-) \cdot dB^{D,1}(t) + Y(t-) \cdot dB^{D,2}(t) \\
=& \left(dB^\circ(t) - \frac{V^{\circ *}_{Z(t-)}(t-)}{V^{\dagger *}_{Z(t-)}(t-)} dB^\dagger(t) + dB^{D,0}(t) \right) && (5.1.6) \\
& + X(t-) \left(\frac{1}{V^{\dagger *}_{Z(t-)}(t-)} dB^\dagger(t) + dB^{D,1}(t) \right) && (5.1.7) \\
& + Y(t-) \cdot dB^{D,2}(t). && (5.1.8)
\end{aligned}
$$

The line (5.1.6) can be handled as guaranteed payments, whereas the lines (5.1.7) and (5.1.8) are on the form (5.0.1) for $W = (X, Y)$. So if we can calculate

$$
\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \right], \quad j \in \mathcal{J},
$$

we can calculate prognoses with Propositions XX and XX.

But Proposition 24 tells us just how to do that *if* the process (X, Y) has affine dynamics. Inspecting equations (5.1.3) and (5.1.4), this is certainly the case if

$$
\delta_2(t) = \delta_2^0(t) + X(t) \cdot \delta_2^1(t) + Y(t) \cdot \delta_2^2(t)
$$

holds *and* (5.1.5) holds. With these assumptions, the process (X, Y) has affine dynamics and so we can employ Proposition 24 and conclude that

$$
\begin{pmatrix} \tilde{X}_j(t) \\ \tilde{Y}_j(t) \end{pmatrix} = \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \right]
$$

is described by the system of differential equations

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix}\tilde{X}_j(t)\\ \tilde{Y}_j(t)\end{pmatrix} =& f_j^1(t)\begin{pmatrix}\tilde{X}_j(t)\\ \tilde{Y}_j(t)\end{pmatrix} + p_{0j}(0,t)f_j^0(t)\\
&+\sum_{\substack{k\in\mathcal{J}\\ k\neq j}}\mu_{kj}(t)\left(g_{kj}^1(t)\begin{pmatrix}\tilde{X}_k(t)\\ \tilde{Y}_k(t)\end{pmatrix} + p_{0k}(0,t)g_{kj}^0(t)\right)\\
&+\sum_{\substack{k\in\mathcal{J}\\ k\neq j}}\left(\mu_{kj}(t)\begin{pmatrix}\tilde{X}_k(t)\\ \tilde{Y}_k(t)\end{pmatrix} - \mu_{jk}(t)\begin{pmatrix}\tilde{X}_j(t)\\ \tilde{Y}_j(t)\end{pmatrix}\right),
\end{aligned}
$$

for $t \in (0,R)\cup(R,n)$ and functions

$$f, g, h$$

The initial value is

$$\begin{pmatrix}\tilde{X}_j(0)\\ \tilde{Y}_j(0)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

and the gluing condition is

$$\begin{pmatrix}\tilde{X}_j(R)\\ \tilde{Y}_j(R)\end{pmatrix} = h_j^1(R)\begin{pmatrix}\tilde{X}_j(R-)\\ \tilde{Y}_j(R-)\end{pmatrix} + p_{0j}(0,R)h_j^0(R)$$

for functions

$$
\begin{aligned}
h_j^1(t) &= \\
h_j^0(R) &= a
\end{aligned}
$$

By specifying the dividend process D, we can arrive at many typical contract types. In the following example, we treat the case of a unit link insurance contract. Crucially, we shall see how the inclusion of insurance event risk – the main difference between the prognosis approach proposed in this thesis and the classical ditto – in certain cases leads to more conservative benefit prognoses.

5.2 Numerical examples

The state of the policy holder is modeled as in Example XX, i.e. on the classical alive-disabled-dead model with reactivation:

FIGURE

Assume that the guaranteed payments B° consists of a premium $b_0^{\circ}(t)$ paid in as active, a disability sojourn benefit $b_1^{\circ}(t)$ paid out as disable until retirement and a death sum $b_{02}^{\circ}(t) = b_{12}^{\circ}(t)$. Formally,

$$
\begin{aligned}
dB^\circ(t) =& \mathbb{1}_{\{t<R\}} \left(-\mathbb{1}_{\{Z(t)=0\}} b_0^\circ(t) + \mathbb{1}_{\{Z(t)=1\}} b_1^\circ(t) \right) dt \\
& \mathbb{1}_{\{t<R\}} b_{02}^\circ(t) dN_2(t)
\end{aligned}
$$

Together, the payments satisfy the equivalence principle under the technical basis, i.e. $V_0^{\circ *}(0) = 0$.

In addition to the disability and death covers, the policy holder saves for her pension via the cash-dividend payment stream D_1 in the following manner

$$
\begin{aligned}
dD_1(t) =& - \mathbb{1}_{\{t<R\}} \mathbb{1}_{\{Z(t)=0\}} \left(\pi(t) - b_0^\circ(t)\right) \\
& + \mathbb{1}_{\{t\geq R\}} \left(\mathbb{1}_{\{Z(t)=0\}} + \mathbb{1}_{\{Z(t)=1\}} \right) \frac{Y(t)}{a(t)}
\end{aligned}
$$

In words, the policy holder pays the total premiums $\pi(t)$. After paying for the disability and death covers, $\pi(t) - b_0^\circ(t)$ remains, which is paid directly into the surplus Y. As retired, a pension benefit is then paid out as a fraction of the surplus Y. Since the surplus is forfeited upon death, we let Y accrue not only with the rate of return r, but also with the mortality rate μ_{02} respectively μ_{12}. This means that the surpluses of the deceased are inherited by the survivors.

Note that in this setup no additional units of any payment stream $B^\dagger$ are purchased, so $D_2 = Q = 0$.

Finally – to simplify expressions a bit – assume that $\mu^* = \mu$, such that $\rho^\circ = \rho^\dagger = 0$. This means that

$$
dC(t) = \left(r(t) - r^*(t)\right) V_{Z(t)}^{\circ *}(t) dt
$$

which just states that the only contribution to the surplus from the guaranteed payments are through the realized financial performance, which hopefully exceed the technical.

The surplus then has dynamics

$$
\begin{aligned}
dY(t) =& \left(r(t) + \mu_{Z(t)2}(t) \right) Y(t) dt \\
& + \mathbb{1}_{\{t<R\}} \mathbb{1}_{\{Z(t)=0\}} \left(\pi(t) - b_0^\circ(t)\right) dt \\
& - \mathbb{1}_{\{t\geq R\}} \left(\mathbb{1}_{\{Z(t)=0\}} + \mathbb{1}_{\{Z(t)=1\}} \right) \frac{Y(t)}{a(t)} dt \\
& + \left(r(t) - r^*(t)\right) V_{Z(t)}^{\circ *}(t) dt \\
& - Y(t-) dN_2(t).
\end{aligned}
\tag{5.2.1}
$$

The quantity a is the S-adapted annuity

$$
a(t) = \int_t^n e^{-\int_t^s \bar{r}_t(u) + \bar{\mu}_t(u) du} ds
$$

where $\bar{r}_t(u)$ is the $S(t)$-measurable payout interest rate curve and $\bar{\mu}_t(u)$ is the $S(t)$-measurable payout mortality rate curve. These curves are updated regularly (e.g. yearly) and are examples of future management actions relevant in the context of prognoses. As we shall see, the choice of $\bar{r}_t(u)$ and $\bar{\mu}_t(u)$ has great impact on how the surplus is paid out over the course of retirement:

- If $\bar{r}_t(u) < r(u)$ and $\bar{\mu}_t(u) < \mu_{02}(u), \mu_{12}(u)$ for $u \geq t$, the pension benefit rate $Y(t)/a(t)$ will tend to *increase* over time.
- If $\bar{r}_t(u) > r(u)$ and $\bar{\mu}_t(u) > \mu_{02}(u), \mu_{12}(u)$ for $u \geq t$, the pension benefit rate $Y(t)/a(t)$ will tend to *decrease* over time.
- If $\bar{r}_t(u) \approx r(u)$ and $\bar{\mu}_t(u) \approx \mu_{02}(u), \mu_{12}(u)$ for $u \geq t$, the pension benefit rate $Y(t)/a(t)$ will be approximately constant (see XXX for proof'ish).

Since $a(t) \underset{t \to n}{\longrightarrow} 0$, it follows that $a^{-1}(t) \underset{t \to n}{\longrightarrow} \infty$, which ensures that the surplus is eventually paid out.

Say we want to prognosticate the pension rate at some time $t \geq R$, i.e.

$$
\begin{aligned}
\hat{b}(t) =& \mathbb{E}\left[\frac{Y(t-)}{a(t)} \middle| Z(t-) \in \bar{\mathcal{J}} \right] \\
=& \frac{\sum_{j \in \hat{\mathcal{J}}} \tilde{Y}_j(t)}{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(0,t)} \frac{1}{a(t)}.
\end{aligned}
$$

By XXX, $\tilde{Y}_j(t) = \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} Y(t) \right]$ is described by the system of differential equations

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{Y}_0(t) &= \left(r(t) + \mu_{02}(t) - \mathbb{1}_{\{t \geq R\}} \frac{1}{a(t)} \right) \tilde{Y}_0(t) \\
&\quad + p_{00}(0,t) \left(\mathbb{1}_{\{t<R\}} \left(\pi(t) - b_0^*(t) \right) + \left(r(t) - r^*(t) \right) V_0^{\circ *}(t) \right) \\
&\quad + \mu_{10}(t)\tilde{Y}_1(t) - \mu_{01}(t)\tilde{Y}_0(t) - \mu_{02}(t)\tilde{Y}_0(t), \\
\frac{\partial}{\partial t}\tilde{Y}_1(t) &= \left(r(t) + \mu_{12}(t) - \mathbb{1}_{\{t \geq R\}} \frac{1}{a(t)} \right) \tilde{Y}_1(t) \\
&\quad + p_{01}(0,t) \left(r(t) - r^*(t) \right) V_1^{\circ *}(t) \\
&\quad + \mu_{01}(t)\tilde{Y}_0(t) - \mu_{10}(t)\tilde{Y}_1(t) - \mu_{12}(t)\tilde{Y}_1(t).
\end{aligned}
\tag{5.2.2}
$$

Obviously, $\tilde{Y}_0(0) = \tilde{Y}_1(0) = 0$ and $\tilde{Y}_2(t) = 0$.

We now solve this scheme numerically for four different scenarios. For each, scenario, we also compute the "classical" prognosis, i.e. the prognosis obtained by assuming that the insured remains active and premium paying until retirement. This prognosis is simply

$$
\hat{b}_{Classical}(t) = \frac{\bar{Y}(t)}{a(t)}
$$

where $\bar{Y}(t)$ satisfies the differential equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\bar{Y}(t) &= \left(r(t) + \mu_{02}(t) - \mathbb{1}_{\{t \geq R\}} \frac{1}{a(t)} \right) \bar{Y}(t) \\
&\quad + \mathbb{1}_{\{t<R\}} \pi(t) + \left(r(t) - r^*(t) \right) V_0^{\circ *}(t)
\end{aligned}
$$

with $\bar{Y}(0) = 0$. The scenarios are:

- **Baseline**: A best estimate scenario with premiums set at $\pi = 80$ t.DKK annually until retirement at $R = 65$, HUSK DØD og invalidydelser!!!! a real return rate of 3% per annum and transition intensities given as in (Section 5 of [Buchardt and Møller, 2015]):

$$\begin{aligned}
\mu_{01}^{BE}(t) &= \left(0.0004 + 10^{4.54+0.06t-10}\right) \mathbb{1}_{\{t<R\}} \\
\mu_{10}^{BE}(t) &= \left(2.0058 \cdot e^{-0.117t}\right) \mathbb{1}_{\{t<R\}} \\
\mu_{02}^{BE}(t) &= 0.0005 + 10^{5.88+0.038t-10} \\
\mu_{12}^{BE}(t) &= \mu_{02}^{BE}(t) \left(1 + \mathbb{1}_{\{t<R\}}\right)
\end{aligned}$$

Hvad med tekniske???

- **Disability Stress**: The baseline scenario, *but* with a higher disability intensity.

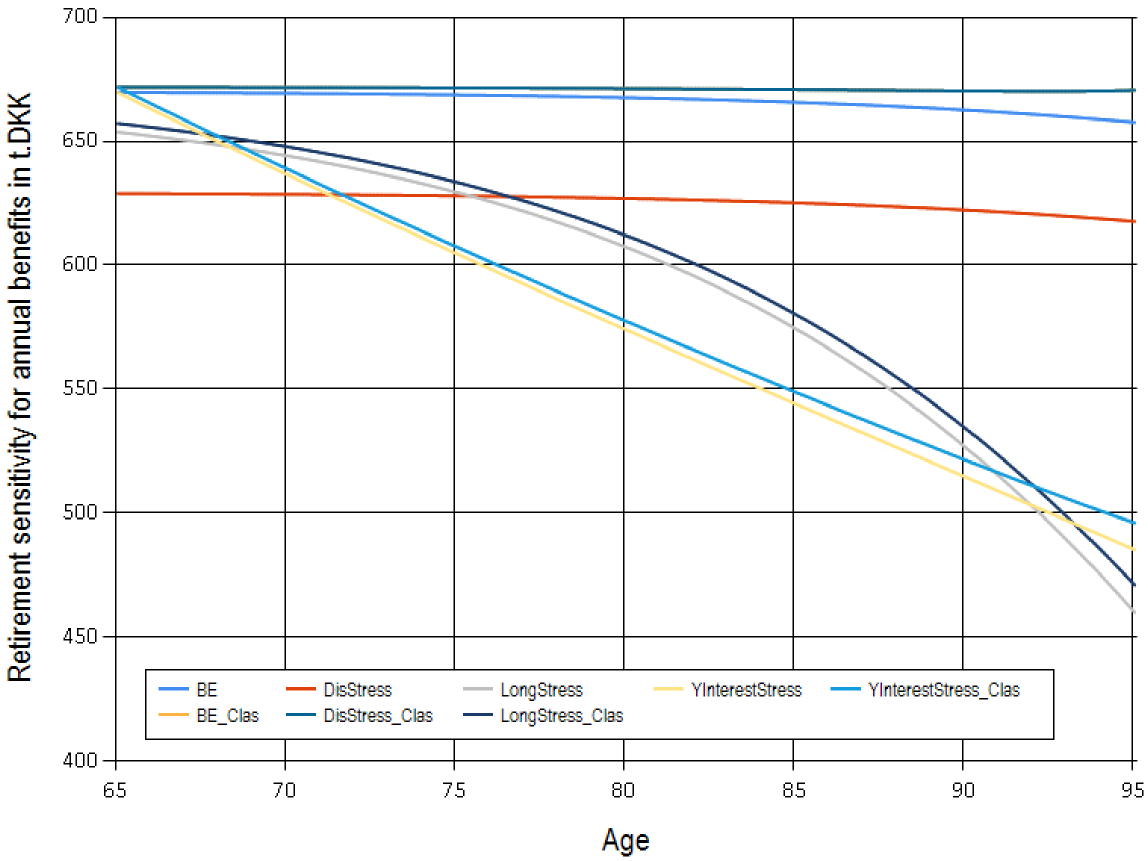
- **Longevity Stress**: The baseline scenario, *but* with a lower mortality.

- **Interest Stress**: The baseline scenario, *but* with a lower rate of return r in the retirement phase.

Formally, the scenarios are:

Scenario	μ_{01}	μ_{10}	μ_{02}	μ_{12}	r	$\bar{r}$	$\bar{\mu}$
Baseline	$\mu_{01}^{BE}(t)$	$\mu_{10}^{BE}(t)$	$\mu_{02}^{BE}(t)$	$\mu_{12}^{BE}(t)$	0.03	0.03	$\mu_{01}^{BE}(t)$
Disability Stress	$1.5 \cdot \mu_{01}^{BE}(t)$	$\mu_{10}^{BE}(t)$	$\mu_{02}^{BE}(t)$	$\mu_{12}^{BE}(t)$	0.03	0.03	$\mu_{01}^{BE}(t)$
Longevity Stress	$\mu_{01}^{BE}(t)$	$\mu_{10}^{BE}(t)$	$0.8 \cdot \mu_{02}^{BE}(t)$	$0.8 \cdot \mu_{12}^{BE}(t)$	0.03	0.03	$\mu_{01}^{BE}(t)$
Interest Stress	$\mu_{01}^{BE}(t)$	$\mu_{10}^{BE}(t)$	$\mu_{02}^{BE}(t)$	$\mu_{12}^{BE}(t)$	$\mathbb{1}_{\{t<R\}}0.03 + \mathbb{1}_{\{t\geq R\}}0.02$	0.03	$\mu_{01}^{BE}(t)$

Figure 5.1: Annual benefit prognoses (classical and proposed) for the four scenarios.



A few things are worth noting in figure XXX:

- For each scenario, the classical prognosis is higher than the proposed prognosis $\hat{b}(t)$. This is what was expected and one of the principal reason for building the theory in the first place. The difference is, however, quite small – and the inter-scenario difference is much larger.
- The difference between the BE- and the Disability Stress-scenario shows the promise of the proposed prognoses. When the disability rate is high, the insured is more likely to exit the premium paying state, which translates to a lower surplus at retirement. As noted earlier, an "unemployed"-state can be added to the state space, with an intensity that is probably much higher than the disability intensity.
- The Interest Stress scenario shows that if the payout interest rate $\bar{r}$ is set high than the realized rate, benefits decrease over time because the surplus is paid out "too quickly" in the early phase of retirement.
- The Longevity Stress scenario tells the same story, but with $\bar{\mu}$. Also, the start rate (at age 65) is lower in this scenario, because fewer people have died before retirement (their death would benefit the surpluses of the survivors). BEMÆRK KONKAVITET.
- Finally, note that BE_Clas and DisStress_Clas are exactly equal (and thus only one is visible on the graph) because only the technical disability rate μ^*_{01} is involved in the classical prognoses and this rate is not stressed.

Figure 5.2: The development of the classical and proposed reserves for the four scenarios.

- bemærkninger
- bemærkninger

Remark om at lade premium afhænge af Y

Remark om at man nok skal have en unemployed state for at forskelle er store nok

5.3 Premium waiver

In the last section, we saw how periods of disability can negatively impact the pension benefits paid out. It is in the interest of most policy holders to get rid of this risk. One way of achieving this is with a premium waiver ("indbetalingssikring"/"præmiefritagelse"). The idea is that the company assumes the premium payments in case of disability, ensuring that savings increase at the same pace as in the active state.

Continuing the contract from the last section, we formalize the idea of a premium waiver as follows: The policy holder is still paying the total premium $\pi(t)$ as active. From this is deducted the payment $\bar{b}_0^\circ(t)$, which now not only covers the death and disability benefits, but also the premium waiver. The premium waiver is an additional liability to the company, i.e. a payment stream $\pi(t) - \bar{b}_0^\circ(t)$ paid in the disabled state. The payment $\bar{b}_0^\circ(t)$ can be determined by the equivalence principle as a solution to (MANGLER rente)

$$\begin{aligned} 0 =& V_0^*(0) \\ =& \int_0^R p_{00}(0,s)\left(-b_0^\circ(s) + \mu_{02}(s)b_{02}^\circ\right) \\ &+ p_{01}^*(0,s)\left(\left(\pi(s) - b_0^\circ(s)\right) + b_1^\circ(t) + \mu_{12}^*(s)b_{12}^\circ\right) ds \end{aligned}$$

With the simplifying assumption that all payment functions are constant, we can isolate b_0°

$$\bar{b}_0^\circ = \frac{\int_0^R p_{00}^*(0,s)\mu_{02}^*(s)b_{02}^\circ + p_{01}^*(0,s)\left(\pi + b_1^\circ + \mu_{12}^*(s)b_{12}^\circ\right) ds}{\int_0^R p_{00}^*(0,s) + p_{01}^*(0,s) ds}.$$

The rate paid into the surplus Y – as active as well as disabled – is then $\pi(t) - \bar{b}_0^\circ(t)$. Consulting equation (5.2.1), we see that Y then has dynamics

$$\begin{aligned} dY(t) =& \left(r(t) + \mu_{Z(t)2}(t)\right) Y(t)dt \\ &+ \mathbb{1}_{\{t<R\}}\left(\mathbb{1}_{\{Z(t)=0\}} + \mathbb{1}_{\{Z(t)=1\}}\right)\left(\pi(t) - \bar{b}_0^\circ(t)\right) dt \\ &- \mathbb{1}_{\{t\geq R\}}\left(\mathbb{1}_{\{Z(t)=0\}} + \mathbb{1}_{\{Z(t)=1\}}\right)\frac{Y(t)}{a(t)}dt \\ &+ \left(r(t) - r^*(t)\right) V_{Z(t)}^{\circ *}(t)dt \\ &- Y(t-)dN_2(t). \end{aligned}$$

With the added assumptions that

- $\mu_{02}(t) = \mu_{12}(t)$, meaning that the accrual from the surpluses of deceased happens at the same rate as active and as disabled.

- $r(t) = r^*(t)$, meaning that is no contribution from the guaranteed payments to the surplus.

Then there is no distinction between the policy holder being in the active or the disabled state. The premium waiver effectively erases the risk from the transitions $0 \to 1$ (disability) and $1 \to 0$ (reactivation) from the point of view of the policy holder. No matter her state path inside $\hat{\mathcal{J}} = \{0, 1\}$, her surplus will develop at the same rate.

The computation of the prognosis can also be simplified by noting that

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\tilde{Y}_0(t) + \tilde{Y}_1(t)\right) = &\left(r(t) + \mu_{02}(t) - \mathbb{1}_{\{t \geq R\}} \frac{1}{a(t)}\right)\left(\tilde{Y}_0(t) + \tilde{Y}_1(t)\right) \\
&+ \left(p_{00}(t) + p_{01}(t)\right) \mathbb{1}_{\{t < R\}} \left(\pi(t) - \bar{b}_0^{\circ}(t)\right) \\
&- \mu_{02}(t)\left(\tilde{Y}_0(t) + \tilde{Y}_1(t)\right) \\
= &\left(r(t) - \mathbb{1}_{\{t \geq R\}} \frac{1}{a(t)}\right)\left(\tilde{Y}_0(t) + \tilde{Y}_1(t)\right) \\
&+ \left(p_{00}(t) + p_{01}(t)\right) \mathbb{1}_{\{t < R\}} \left(\pi(t) - \bar{b}_0^{\circ}(t)\right),
\end{aligned}
$$

which is obtained by adapting (5.2.2) to the current setup. Solving this, the prognosis can be readily calculated as

$$
\begin{aligned}
\hat{b}(t) = &\mathbb{E}\left[\frac{Y(t-)}{a(t)} \,\middle|\, Z(t-) \in \bar{\mathcal{J}}\right] \\
= &\frac{\sum_{j \in \hat{\mathcal{J}}} \tilde{Y}_j(t)}{\sum_{j \in \hat{\mathcal{J}}} p_{0j}(0, t)} \frac{1}{a(t)} \\
= &\frac{\tilde{Y}_0(t) + \tilde{Y}_1(t)}{p_{00}(t) + p_{01}(t)} \frac{1}{a(t)}.
\end{aligned}
$$

The simplification is that we no longer have to solve a system of differential equations – only the single one for $\tilde{Y}_0(t) + \tilde{Y}_1(t)$.

Remark 28.
It can be shown that $p_{00}(t) + p_{01}(t) = e^{-\int_0^t \mu_{02}(s)ds}$ ○

Chapter 6

Accumulated prognoses

So far we have looked at prognoses for rates and single benefits. However, if interested in the accumulated benefits. Let t be the time of death and assume that $0 < t_0 < R < t < n$, then the net benefits received given survival until time t is

$$
\begin{aligned}
a =& \mathbb{E}\left[\int_{t_0}^{t} dB(s, W(s-)) \middle| \{Z(s)\}_{t_0 \le s < t} \in \hat{\mathcal{J}}, Z(t) = k\right] \\
=& \mathbb{E}\left[\int_{t_0}^{t} W(s-) b_{Z(s-)}(s) ds \middle| \{Z(s)\}_{t_0 \le s < t} \in \hat{\mathcal{J}}, Z(t) = k\right] \\
&+ \mathbb{E}\left[\int_{t_0}^{t} W(s-) \sum_{h \neq Z(s-)} b_{Z(s-)h}(s) dN_h(s) \middle| \{Z(s)\}_{t_0 \le s < t} \in \hat{\mathcal{J}}, Z(t) = k\right] \\
&+ \mathbb{E}\left[W(R-) \triangle B_{Z(R)}(R) \middle| \{Z(s)\}_{t_0 \le s < t} \in \hat{\mathcal{J}}, Z(t) = k\right]
\end{aligned}
$$

Treating them one by one we have

$$
\begin{aligned}
&= \mathbb{E}\left[\int_{t_0}^{t} W(s-) b_{Z(s-)}(s) ds \middle| \{Z(u)\}_{t_0 \le u < t} \in \hat{\mathcal{J}}, Z(t) = k\right] \\
&= \int_{t_0}^{t} \mathbb{E}\left[W(s-) b_{Z(s-)}(s) \middle| \{Z(u)\}_{t_0 \le u < t} \in \hat{\mathcal{J}}, Z(t) = k\right] ds \\
&= \int_{t_0}^{t} \mathbb{E}\left[W(s-) b_{Z(s-)}(s) \middle| \{Z(u)\}_{t_0 \le u < s} \in \hat{\mathcal{J}}\right] ds \\
&= \int_{t_0}^{t} \mathbb{E}\left[W(s-) b_{Z(s-)}(s) \middle| Z(s-) \in \hat{\mathcal{J}}\right] ds
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\int_{t_0}^{t} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]\\
=&\mathbb{E}\left[\int_{[t_0,t)} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]\\
&+\mathbb{E}\left[\int_{\{t\}} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]\\
=&\mathbb{E}\left[\int_{[t_0,t)} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}\right]\\
&+\mathbb{E}\left[W(t-)b_{Z(t-)k}(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]
\end{aligned}
$$

where

$$
\begin{aligned}
a=&\mathbb{E}\left[\int_{\{t\}} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]\\
=&\mathbb{E}\left[W(t-)b_{Z(t-)k}(t)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]
\end{aligned}
$$

and

$$
\begin{aligned}
a=&\mathbb{E}\left[\int_{[t_0,t)} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}, Z(t)=k\right]\\
=&\mathbb{E}\left[\int_{[t_0,t)} W(s-)\sum_{h\neq Z(s-)} b_{Z(s-)h}(s)dN_h(s)\middle|\{Z(u)\}_{t_0\le u<t}\in\hat{\mathcal{J}}\right]
\end{aligned}
$$

Chapter 7

Sensitivities

Having defined the prognoses, we will now formalize the concept of prognosis sensitivities. The point is to give the policy holder an idea of how much she can expect her benefits to increase if she postpones her time of retirement or if she increases premiums paid. And – conversely – how much she can expect her benefits to *decrease* if she retires at an *earlier* point in time or if she *decreases* premiums paid.

In recent years, the question of retirement age has been vividly discussed in the Danish public. The main issue is whether workers of certain professions – typically hard physical labor – can (or rather, should) remain in the workforce until the rather high pension ages laid down in the 2006 political settlement Velfærdsforliget. For these groups, sensitivities are of great interest[1], as they reveal the trade-offs between benefit level, premium level and time of retirement.

Although we settle for two kinds of sensitivities in this thesis – time of retirement and premium level – the techniques presented can be used to obtain sensitivities for many other contract elements.

The point of retirement is already formalized as R. We will now formalize what we mean by premium level. Given a payment stream B, we assume that the premium (i.e. negative) part of the payment stream takes the form

$$dB^-(t,\alpha) = \alpha \cdot dB^-(t),$$

where $\alpha > 0$ is the premium level. The sensitivities are now simply defined as the derivatives of the prognoses in R and α, respectively. For the case with reserve-dependent benefits, these are

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathbb{E}\left[W(t-)b_{Z(t-)}\middle|\{Z(s)\}_{0\le s<t}\in\bar{\mathcal{J}}\right],\\
\frac{\partial}{\partial\theta}\mathbb{E}\left[W(R-)\triangle B_{Z(R-)}\middle|\{Z(s)\}_{0\le s<R}\in\bar{\mathcal{J}}\right],\\
\frac{\partial}{\partial\theta}\mathbb{E}\left[W(t-)b_{Z(t-)k}\middle|\{Z(s)\}_{0\le s<t}\in\bar{\mathcal{J}},Z(t)=k\right].
\end{aligned}$$

for $\theta = R,\alpha$. Since we have already seen that the case with guaranteed benefits is a special case of the case with reserve-dependent benefits, we only treat the latter.

[1]Of course, the public discussion has focused on state pensions which are pay-as-you-go systems.

7.1 Reserve-dependent benefits

For the case with reserve-dependent benefits, the quantities quickly become quite involved. For this reason, we start out in an extremely simple setting and incrementally move to more sophisticated modelling.

Consider a model without any insurance risk (let $\mathcal{J} = \{0\}$ so there is only *one* state-wise retrospective reserve; $W(t, R)$, which is the value of the reserve at time t, given retirement at time R). The contract simply consists of the reserve W accruing with interest and with premium paid into it until retirement. Starting at time R, pension is paid out continuously as a fraction $a^{-1}(t)$ of the reserve. The reserve then has the continuous dynamics

$$W(dt, R) = r(t)W(t-, R)dt + 1_{t<R}\pi(t)dt - 1_{t\geq R}\frac{W(t-, R)}{a(t)}dt \tag{7.1.1}$$

and initial value $W(0, R) = 0$. The quantity a is the annuity

$$a(t) = \int_t^n e^{-\int_t^s \bar{r}_t(u)du} ds$$

where $\bar{r}_t$ is some continuous payout interest rate curve decided by the company at time t. Since $a(t) \underset{t\to n}{\longrightarrow} 0$, it follows that $a^{-1}(t) \underset{t\to n}{\longrightarrow} \infty$, which ensures that the reserve is eventually paid out.

Since there is only one benefit in this setting – and since there is only one state – the only prognosis is of course the pension benefit

$$\hat{b}(t) = \frac{W(t-, R)}{a(t)}, \ t \geq R$$

In the following, we will derive sensitivities for this prognosis.

7.1.1 Retirement sensitivity

With regard to the retirement sensitivity, i.e. infinitesimally postponing retirement, there are two interesting, but sensitivities to consider:

- The sensitivity of the pension benefit *start* rate, i.e.

$$\frac{\partial}{\partial R}\hat{b}(R) = \frac{\partial}{\partial R}\frac{W(R-, R)}{a(R)}. \tag{7.1.2}$$

- The effect on the pension benefit rate at some fixed time point t *after* retirement, i.e.

$$\frac{\partial}{\partial R}\hat{b}(t) = \frac{\partial}{\partial R}\frac{W(t-, R)}{a(t)}, \ t > R. \tag{7.1.3}$$

The distinction is that in the first case, both the time of retirement and the time of payment are being moved infinitesimally. In the second case, only the time of retirement is moved whereas the time of payment is fixed.

For the first case (7.1.2), we can solve the differential equation (7.1.1) to obtain

$$W(R-,R) = \int_0^R e^{\int_s^R r(u)du} \pi(s)ds \tag{7.1.4}$$

Differentiating, we obtain

$$\begin{aligned}\frac{\partial}{\partial R} W(R-,R) =& e^{\int_R^R r(u)du} \pi(R) + \int_0^R \frac{\partial}{\partial R} e^{\int_s^R r(u)du} \pi(s)ds \\ =& \pi(R) + r(R)W(R-,R)\end{aligned}$$

which has the intuitive interpretation that postponing retirement has the effect of increasing the account *at* retirement by the premium rate $\pi(R)$ and the rate of interest accrual $r(R)W(R-,R)$. By the quotient rule – and assuming the necessary differentiability – the sensitivity of the benefit rate at retirement is then

$$\begin{aligned}\frac{\partial}{\partial R}\hat{b}(R) =& \frac{\partial}{\partial R}\frac{W(R-,R)}{a(R)} \\ =& \frac{\big(\pi(R) + r(R)W(R-,R)\big)\, a(R) - W(R-,R)\Big(\big(\bar{r}_R(R) - \gamma(R)\big)\, a(R) - 1\Big)}{\big(a(R)\big)^2} \\ =& \frac{\pi(R)a(R) + W(R-,R) + \Big(r(R) - \big(\bar{r}_R(R) - \gamma(R)\big)\Big)\, W(R-,R)a(R)}{\big(a(R)\big)^2},\end{aligned} \tag{7.1.5}$$

where

$$\gamma(R) = \int_R^n e^{-\int_R^s \bar{r}_R(u)du} \int_R^s \frac{\partial}{\partial R}\bar{r}_R(u)duds$$

It is hard to give an intuitive interpretation of (7.1.5). However, if we further make the two further assumptions that

- $\bar{r}_R(R) \approx r(R)$, i.e. that the payout interest rate curve initial value at retirement is close to the actual return rate at retirement, (which doesn't seem unreasonable).
- $\frac{\partial}{\partial R}\bar{r}_R(u) \approx 0$ for $u > R$, i.e. that the information gained by postponing R does not greatly change the payout interest curve $\bar{r}_R(u)$. This assumption is certainly eligible for criticism and is an interesting research question in its own right. It is however outside the scope of this thesis. Having made the assumption, we have $\gamma(R) \approx 0$.

We then obtain

$$\frac{\partial}{\partial R}\hat{b}(R) \approx \frac{\pi(R) + W(R-,R)a^{-1}(R)}{a(R)},$$

which is interpreted as follows: The numerator is the increment additional premium paid in $\pi(R)$ plus the increment benefit $W(R-,R)a^{-1}(R)$ *not* paid out, i.e. saved (hence the positive sign). So the sensitivity is just the ratio between the infinitesimal additional savings and the annuity $a(R)$.

For the second case (7.1.3), we can solve the differential equation (7.1.1) for $t > R$ as

$$W(t-,R) = W(R-,R)e^{\int_R^t r(u)-a^{-1}(u)du}, \tag{7.1.6}$$

which just says that the account at time $t > R$ is the account at retirement, positively compounded with interest and negatively compounded with benefits paid out. Differentiating, we obtain

$$\begin{aligned}\frac{\partial}{\partial R}W(t-,R) &= \left(\pi(R) + r(R)W(R-,R)\right)e^{\int_R^t r(u)-a^{-1}(u)du} \\ &\quad + W(R-,R)\left(-r(u) + a^{-1}(u)\right)e^{\int_R^t r(R)-a^{-1}(R)du} \\ &= e^{\int_R^t r(u)-a^{-1}(u)du}\left(\pi(R) + W(R-,R)a^{-1}(R)\right)\end{aligned}$$

which is interpreted as follows: By postponing retirement, an additional increment of $\pi(R)$ is paid in. Also, an increment *less* of the benefit rate $W(R-,R)a^{-1}(R)$ is paid out, i.e. saved (hence the positive sign). These additional savings are then compounded (positively with r and negatively with a^{-1}) until time $t > R$. Note that, contrary to the first case, there is no interest term $r(R)W(R-,R)$. This is because we – in this case – are keeping t fixed.

The sensitivity of the pension benefit rate paid out at time t is then

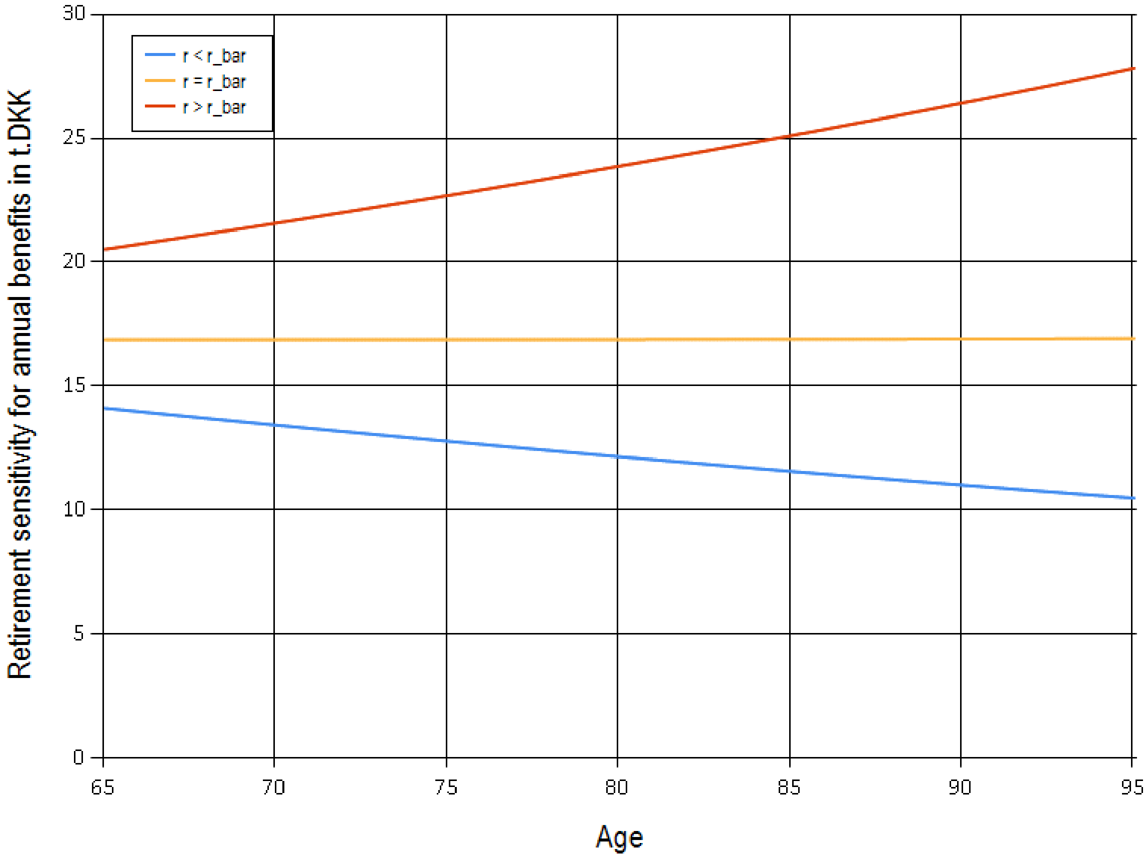
$$\begin{aligned}\frac{\partial}{\partial R}\hat{b}(t) &= \frac{\partial}{\partial R}\frac{W(t-,R)}{a(t)} \\ &= \frac{e^{\int_R^t r(u)-a^{-1}(u)du}\left(\pi(R) + W(R-,R)a^{-1}(R)\right)}{a(t)}\end{aligned}$$

Note that this is consistent with the
Example 29.
Premiums are set to 80 t.DKK (real) per annum starting at age 25. We consider three scenarios with varying real return rates; Low (1%), Mean (3%) and High (5%). The table below then shows the value of the reserve at retirement at age 65, i.e. $W(R-,R)|_{R=65}$ along with the start benefit $\hat{b}(R)|_{R=65}$. Tallene er små idet der ikke er arv fra døde, mu er 0.

Scenario	r	$\bar{r}$	$W(R-,R)$ in t.DKK	$\hat{b}(R)$ in t.DKK
Low	0.02	0.03	4,905	226
Mean	0.03	0.03	6,188	285
High	0.04	0.03	7,902	364

Note that the payout interest rate $\bar{r}$ is the same in all three scenarios. This means that the scenarios differ not only on the realized return rates r, but also on whether the payout interest rate $\bar{r}$ is fixed above or below the realized return rate r. The result is the following figure:

Figure 7.1: Retirement sensitivity for the simple savings contract.



There are at least two features worth noting on Figure 29:

- As can be seen from the starting points of the three curves, the effect on the annual start benefit rate $\hat{b}(R)$ of postponing retirement one year is around 15-20 t.DKK (or equivalently 1.3-1.7 t.DKK effect on the monthly rate) – depending on the rate of return in the savings phase.
- This postponement effect then increases/decreases/remains constant for later benefit rates $\hat{b}(t)$, $t > R$ depending on the scenario. For instance – when $r < \bar{r}$ – the retirement sensitivity decreasess in time because the benefit rate does (as we saw in Example XXX). Intuitively, when most of the reserve is paid out in the early phase of retirement, increasing the reserve by postponing retirement matters most for these early benefit rates.

△

Remark 30.
As we saw in Chapter 5, If $\bar{r} = r$,

$$\frac{\partial}{\partial R}\frac{W(t-,R)}{a(t)} = \frac{\pi(R) + W(R-,R)a^{-1}(R)}{a(R)}$$

○

Comment
Example 31.
Assume that

$$
\begin{aligned}
x =& 25 \\
R =& 68 \\
r = \bar{r} =& 0.05 \\
\pi =& 80,000
\end{aligned}
$$

By numerical computations, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial R} W(R-, R) =& XXX \\
\pi =& 50,000
\end{aligned}
$$

$\triangle$

7.1.2 Premium sensitivity

Intuitively – for the simple contract considered here – if the policy holder, say, doubles her premiums, the prognosis should double as well. This makes sense because the only flow into or from the reserve until retirement is the premium stream. This is exactly what the following calculations show.

Assume that the premium rate takes the form

$$\pi(t, \alpha) = \alpha \cdot \pi(t)$$

where α is the premium level and $\pi(t)$ is some baseline premium plan. Denote by $W(t, R, \alpha)$ the reserve at time t given retirement at time R and a premium level of α.

Consulting equations (7.1.4) and (7.1.6), we see that

$$W(t, R, \alpha) = \alpha W(t, R, 1)$$

and so

$$\frac{\partial}{\partial \alpha} W(t-, R, \alpha) = W(t-, R, 1) \tag{7.1.7}$$

which just states that increasing the premium intensity increases the reserve at a rate constant in α, which is equal to the reserve with the unscaled baseline premium plan. The sensitivity is then (for $t \geq R$)

$$\frac{\partial}{\partial \alpha} \frac{W(t-, R, \alpha)}{a(t)} = \frac{W(t-, R, 1)}{a(t)}, \tag{7.1.8}$$

which, again, is constant in α.

7.1.3 A contour map of benefits

Having calculated the sensitivities (i.e. derivatives) of the prognosis benefit $\hat{b}(t, R, \alpha)$ in R and α, we have access to the gradient

$$\nabla \hat{b}(t, R, \alpha) = \begin{bmatrix} \frac{\partial}{\partial \alpha} \hat{b}(t, R, \alpha) \\ \frac{\partial}{\partial R} \hat{b}(t, R, \alpha) \end{bmatrix}$$

for each $t \geq R$. With this, we can obtain a contour map of benefits, illustrating the trade-off between premium level α and time of retirement R. Samme tal som sidste eksempel.

Figure 7.2: A contour map of start benefits rates (i.e. the rate at retirement) as a function of premium level α and time of retirement R.

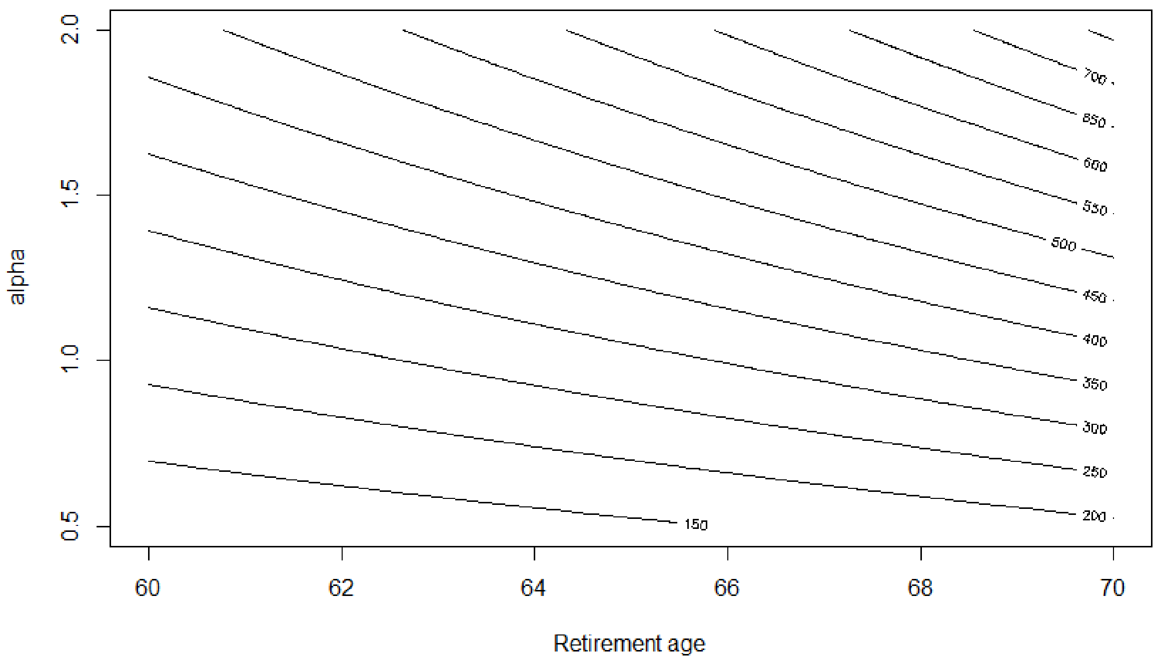


Figure XXX shows the different avenues available (given that the pension scheme legislation is flexible enough) for obtaining a certain benefit level. This information is useful to the policy holder planning his retirement.

7.1.4 Exchange ratios

Closely related to the contour map of the previous subsection, we consider the ratio between the sensitivities and denote it the *exhange ratio*. The exchange ratio is of interest to the policy holder as it informs her how much – infinitesimally – she will have to increase her premiums in order to retire earlier *while keeping the same benefit prognosis level.* This is a key piece of information for a policy holder deciding on her retirement timing.

Definition 32.
Given a benefit prognosis $\hat{b}(t, R, \alpha)$*, the exchange ratio is defined as*

$$e(t, R, \alpha) = \frac{\frac{\partial}{\partial \alpha}\hat{b}(t, R, \alpha)}{\frac{\partial}{\partial R}\hat{b}(t, R, \alpha)}$$

In the simple contract pursued in this section, the exchange ratio for the prognosis atretirement is simply

$$\begin{aligned} e(R, R, \alpha) =& \frac{\frac{\partial}{\partial \alpha}\frac{W(R-,R,\alpha)}{a(t)}}{\frac{\partial}{\partial R}\frac{W(R-,R,\alpha)}{a(t)}} \\ =& \frac{\frac{W(t-,R,1)}{a(R)}}{\frac{\pi(R)+W(R-,R,\alpha)a^{-1}(R)}{a(R)}} \\ =& \frac{W(t-, R, 1)}{\pi(R) + W(R-, R, \alpha)a^{-1}(R)} \end{aligned}$$

Remark 33.
Midle før eller efter man tager brøken ○

In the preceding section, intuitive results on sensitivities and exchange ratios were obtained for a simple savings contract. However, the setting had been without insurance event risk. The natural next step is to adapt the results of the preceding section to a setting with insurance risk.

7.2 Re-introducing insurance risk

When adding insurance risk, most quantities – reserves, sensitivities, exchange ratios – no longer have closed form solutions. Consequently, we have to settle for deriving systems of differential equations describing the sensitivities and solving them numerically.

Recall that the sojourn benefit rate prognosis is given by

$$\begin{aligned} \hat{b}(t) =& \mathbb{E}\left[\sum_{i=1}^{n} W_i(t-)b^i_{Z(t)} \middle| \{Z(s)\}_{0\leq s<t} \in \bar{\mathcal{J}}\right] \\ =& \frac{\sum_{j\in\hat{\mathcal{J}}} \tilde{W}_j(t-)b_j(t)}{\sum_{j\in\hat{\mathcal{J}}} p_{Z(t_0)j}(t_0, t)}. \end{aligned}$$

It it clear then that in order to calculate the sensitivities $\frac{\partial}{\partial\theta}\hat{b}(t, \theta)$ for $\theta = R, \alpha$, we need the sensitivities of the state-wise reserves, i.e.

$$\frac{\partial}{\partial\theta}\tilde{W}_j(t, \theta) = \frac{\partial}{\partial\theta}\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}}W(t)\right]$$

Since we already have a system of differential equations describing $\tilde{W}_j(t, \theta)$, we can obtain an analogue system describing $\frac{\partial}{\partial\theta}\tilde{W}_j(t, \theta)$ by simply differentiating the differential equations, initial value and gluing condition of Proposition 24 with respect to θ. The result is the following corollary:

Corollary 34.
For a process W with initial value w_0 and dynamics

$$\begin{aligned} dW(t,\theta) =& f_{Z(t-)}(t, W(t-))dt \\ &+ \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(t-)}} g_{Z(t-)k}(t, W(t-))dN_k(t) \\ &+ h_{Z(t-)}(t, W(t-))d\varepsilon_R(t), \end{aligned}$$

the θ-sensitivity of the reserve

$$\frac{\partial}{\partial\theta}\tilde{W}_j(t) = \frac{\partial}{\partial\theta}\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}}W(t)\right]$$

is characterized by the system of differential equations NOT DONE

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial\alpha}\tilde{W}_j(t) =& f_j^1(t)\tilde{W}_j(t) + p_{0j}(0,t)f_j^0(t) \\ &+ \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \mu_{kj}(t)\left(g_{kj}^1(t)\tilde{W}_k(t) + p_{0k}(0,t)g_{kj}^0(t)\right) \\ &+ \sum_{\substack{k\in\mathcal{J}\\ k\neq j}} \left(\mu_{kj}(t)\tilde{W}_k(t) - \mu_{jk}(t)\tilde{W}_j(t)\right), \end{aligned}$$

for $t \in (0,R)\cup(R,n)$ and $j \in \mathcal{J}$. The initial values are

$$\tilde{W}_j(0) = \mathbb{1}_{\{j=0\}}w_0$$

and the gluing conditions are

$$\tilde{W}_j(R) = \tilde{W}_j(R-) + h_j^1(R)\tilde{W}_j(R-) + p_{0j}(0,R)h_j^0(R)$$

Example 35.
As a sanity check of the corollary, we can reproduce the results of the last section. So assume that $\mathcal{J} = \{0\}$ and that

$$W(dt,R,\alpha) = r(t)W(t-,R,\alpha) + 1_{t<R}\alpha\cdot\pi(t) - 1_{t\geq R}W(t-,R,\alpha)b(t)$$

Then by Corollary 34,

$$\begin{aligned} \frac{\partial}{\partial\alpha}\tilde{W}_0(t,R,\alpha) =& \frac{\partial}{\partial\alpha}\mathbb{E}\left[\mathbb{1}_{\{Z(0)=j\}}W(t,R,\alpha)\right] \\ =& \frac{\partial}{\partial\alpha}W(t,R,\alpha), \end{aligned}$$

is characterized by the differential equation

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \alpha}W(t,R,\alpha) = \begin{cases} r(t)\frac{\partial}{\partial \alpha}W(t,R,\alpha) + \pi(t) & \text{if } t < R \\ \big(r(t) - b(t)\big)\frac{\partial}{\partial \alpha}W(t,R,\alpha) & \text{if } t > R \end{cases},$$

with gluing condition $\frac{\partial}{\partial \alpha}W(R,R,\alpha) = \frac{\partial}{\partial \alpha}W(R-,R,\alpha)$.

Solving this differential equation, we obtain

$$\begin{aligned}\frac{\partial}{\partial \alpha}W(t,R,\alpha) = & \begin{cases} \int_0^t e^{\int_s^t r(u)du}\pi(s)ds & \text{if } t \le R \\ \int_0^R e^{\int_s^R r(u)du}\pi(s)ds \cdot e^{\int_R^t r(u)-b(u)du} & \text{if } t > R \end{cases} \\ = & W(t,R,1),\end{aligned}$$

where we used equation (**??**). We see that this is exactly the sensitivity obtained in the last subsection (see equation (7.1.7)). This check gives us confidence blah blah

It may come as a surprise that $W(t,R,\alpha)$ does not depend on α at all. Note that this is *only* the case because there are no benefits before retirement.... △

As noted earlier, one can replace α with another contract element to obtain other interesting sensitivities.

The derivative in R, however, poses a problem. Writing

$$W(t) = W(t,R)$$

to stress that the process depends on the fixed time of retirement R, we saw in chapter XXX that the function

$$(t,R) \mapsto \tilde{W}(t,R) = \mathbb{E}\left[\mathbb{1}_{\{Z(t)\}}W(t,R)\right]$$

is not differentiable on the line $t = R$, which means that the function

$$(t,R) \mapsto \left.\frac{\partial}{\partial y}\tilde{W}(x,y)\right|_{(x,y)=(t,R)}$$

is not well defined on the line $t = R$. However, the limits $\frac{\partial}{\partial R}\tilde{W}(t,R)$.

Example 36.

Consider a simple savings contract modelled on an alive-dead model. blah blah. By Proposition XX, we have △

Remark 37.

It is of course possible to decompose the premiums in the same manner as the benefits, i.e. into a scaled part (e.g. the premium rate) and a fixed part (e.g. the initial lump sum premium). ○

Remark 38.

How much R decrease if premiums increase ○

Remark 39.

remark om at man kunne bruge gennemsnitsløn istedet for at midle ud over Z ○

Example 40.
Continuation of Example **??** (Unit Link). Premium paid in is now $\alpha \cdot \pi(t)dt$

$\tilde{W}_j^\alpha(t)$ is characterized by

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{W}_0^\alpha(t) =& \tilde{W}_0^\alpha(t)\left(R(t) - \mathbb{1}_{\{t\geq R\}}b^\dagger(t) + \mu_{02}(t)\right) \\
&+ p_{00}(0,t)\mathbb{1}_{\{t<R\}}\pi(t) \\
&+ \mu_{10}(t)\tilde{W}_1^\alpha(t) - \mu_{01}(t)\tilde{W}_0^\alpha(t) - \mu_{02}(t)\tilde{W}_0^\alpha(t),\ \tilde{W}_0^\alpha(0) = 0, \\
\frac{\partial}{\partial t}\tilde{W}_1^\alpha(t) =& \tilde{W}_1^\alpha(t)\left(R(t) - \mathbb{1}_{\{t\geq R\}}b^\dagger(t) + \mu_{12}(t)\right) \\
&+ \mu_{01}(t)\tilde{W}_0^\alpha(t) - \mu_{10}(t)\tilde{W}_1^\alpha(t) - \mu_{12}(t)\tilde{W}_1^\alpha(t),\ \tilde{W}_1^\alpha(0) = 0,
\end{aligned}$$

which is obtained by differentiating the system of differential equations for $\tilde{U}_j(t)$ with respect to α. We obtain numerically the sensitivities

$$\begin{aligned}
\frac{\partial}{\partial \alpha}\mathbb{E}\left[W(t-)b_{Z(t)}(t)|Z(t)\in\hat{\mathcal{J}}\right] =& \frac{\sum_{j\in\hat{\mathcal{J}}}\frac{\partial}{\partial\alpha}\mathbb{E}\left[W(t-)1_{\{Z(t)=j\}}\right]b_j^\dagger(t)}{\sum_{j\in\hat{\mathcal{J}}}p_{0j}(0,t)} \\
=& \frac{\sum_{j\in\hat{\mathcal{J}}}\tilde{W}_j^\alpha(t)b_j^\dagger(t)}{\sum_{j\in\hat{\mathcal{J}}}p_{0j}(0,t)},
\end{aligned}$$

Figure 7.3: 5 farver, 1 for hvert scenarie. Viser pensionsraters (årlige) afledte i α, altså $\frac{\partial}{\partial\alpha}\mathbb{E}\left[W(t-)b_{Z(t)}(t)|Z(t)\in\{0,1\}\right]$.Med andre ord stiger prognosen med ca 8.000 kr. årligt, hvis man øger sin indbetaling med 1%. Årsagen til at prognosen ikke bare stiger lineært med α er naturligvis, at risikopræmierne til invalidedækningerne ikke afhænger af α. Så når man betaler en krone marginalt går den direkte til livrenten. Så hvis hæver man sine præmier med 1%, så hæver man prognosen med mere end 1%.

△

7.2.1 Exchange ratios

In recent years, there has

Having calculated the prognosis sensitivities in time of retirement R and premium level α, an interesting derived quantity is the exchange ratio

$$\frac{d\alpha(E)}{dR(E)}$$

that is the rate by which the policy holder must increase her premium level in order to retire one year earlier with the same benefit rate.

Chapter 8

Suggestions for further research

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.1 Proof of Lemma 20

Let $h > 0$. Then

$$\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\middle| Z(t-h)\in\hat{\mathcal{J}}, Z(t)=k\right] = \frac{\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}\mathbb{1}_{\{Z(t)=k\}}W(t-h)\right]}{\mathbb{P}\left(Z(t-h)\in\hat{\mathcal{J}}, Z(t)=k\right)} \tag{.1.1}$$

For the numerator in (.1.1), note that

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}\mathbb{1}_{\{Z(t)=k\}}W(t-h)\right] =& \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}\mathbb{1}_{\{Z(t)=k\}}W(t-h)\middle| Z(t-h)\right]\right] \\
=& \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\middle| Z(t-h)\right]\mathbb{E}\left[\mathbb{1}_{\{Z(t)=k\}}\middle| Z(t-h)\right]\right] \\
=& \sum_{i\in\mathcal{J}} p_{Z(t_0)i}(t_0, t-h)\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\middle| Z(t-h)=i\right]p_{ik}(t-h,t) \\
=& \mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\right]p_{jk}(t-h,t)
\end{aligned}$$

and for the denominator in (.1.1), note that

$$\mathbb{P}\left(Z(t-h)\in\hat{\mathcal{J}}, Z(t)=k\right) = \sum_{i\in\mathcal{J}} p_{Z(t_0)i}(t_0,t-h)p_{ik}(t-h,t).$$

Plugging these two results into (.1.1), we obtain

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\middle| Z(t-h)\in\hat{\mathcal{J}}, Z(t)=k\right] =& \frac{\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\right]p_{jk}(t-h,t)}{\sum_{i\in\mathcal{J}} p_{Z(t_0)i}(t_0,t-h)p_{ik}(t-h,t)} \\
=& \frac{\mathbb{E}\left[\mathbb{1}_{\{Z(t-h)=j\}}W(t-h)\right]\frac{p_{jk}(t-h,t)}{h}}{\sum_{i\in\mathcal{J}} p_{Z(t_0)i}(t_0,t-h)\frac{p_{ik}(t-h,t)}{h}}
\end{aligned}$$

Now letting h tend to zero, we obtain the desired result by

.2 Bevis for blah blah

Since $x \mapsto p_{jk}(x,t)$ is continuous on $[0,t]$ and differentiable on $(0,t)$, there exists by the mean value theorem $\xi \in (t-h,t)$ such that

$$\begin{aligned}\frac{-p_{jk}(t-h,t)}{h} =& \frac{p_{jk}(t,t)-p_{jk}(t-h,t)}{t-(t-h)}\\ =& \frac{\partial}{\partial x}p_{jk}(x,t)|_{x=\xi}\\ =& \mu_{j\bullet}(\xi)p_{jk}(\xi,t)-\sum_{\substack{i\in\mathcal{J}\\ i\neq j}}\mu_{ji}(\xi)p_{ik}(\xi,t),\end{aligned}$$

where we also used Kolmogorov's backward differential equation. Now when $h \searrow 0$, we have $\xi \nearrow t$, and so

$$\lim_{h\searrow 0}\frac{p_{jk}(t-h,t)}{h} = \mu_{jk}(t),$$

since $\mu_{jk}(t)$ is continuous (as $p_{jk}(x,t)$ is $C^{1,1}$).

.3 Proof for blah blahdsa

MANGLER

.4 Proof of Proposition 24

Assume that $p_{Z(t_0)j}(t_0,t) > 0$ for all $t > t_0$ and all $j \in \mathcal{J}$. We then have

$$
\begin{aligned}
\tilde{W}_j(t) =& \mathbb{E}\left[W(t)\mathbb{1}_{\left\{Z(t)=j\right\}}\right] \\
=& \mathbb{E}\left[w_0\mathbb{1}_{\left\{Z(t)=j\right\}}\right] + \mathbb{E}\left[\int_0^t \mathbb{1}_{\left\{Z(t)=j\right\}} dW(s)\right] \\
=& p_{Z(t_0)j}(t_0,t)\cdot w \\
&+ \mathbb{E}\left[\int_0^t \mathbb{1}_{\left\{Z(t)=j\right\}} f_{Z(s-)}(s,W(s-))ds\right] \\
&+ \mathbb{E}\left[\int_0^t \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(s-)}} \mathbb{1}_{\left\{Z(t)=j\right\}} g_{Z(s-)k}(s,W(s-))dN_k(s)\right] \\
&+ \mathbb{E}\left[\int_0^t \mathbb{1}_{\left\{Z(t)=j\right\}} h_{Z(s-)}(s,W(s-))d\varepsilon_R(s)\right] \\
=& p_{Z(t_0)j}(t_0,t)\cdot w \\
&+ \int_0^t \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\left\{Z(t)=j\right\}} f_{Z(s-)}(s,W(s-))\middle| Z(s-)\right]\right] ds \\
&+ \mathbb{E}\left[\mathbb{E}\left[\int_0^t \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(s-)}} \mathbb{1}_{\left\{Z(t)=j\right\}} g_{Z(s-)k}(s,W(s-))dN_k(s)\middle| Z(t)\right]\right] \\
&+ \mathbb{E}\left[\mathbb{E}\left[\int_0^t \mathbb{1}_{\left\{Z(t)=j\right\}} h_{Z(s-)}(s,W(s-))d\varepsilon_R(s)\middle| Z(R-)\right]\right] \\
=& p_{Z(t_0)j}(t_0,t)\cdot w \\
&+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s-)\mathbb{E}\left[\mathbb{1}_{\left\{Z(t)=j\right\}} f_i(s,W(s-))\middle| Z(s-)=i\right] ds \\
&+ \mathbb{E}\left[\int_0^t \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(s-)}} \mathbb{1}_{\left\{Z(t)=j\right\}} g_{Z(s-)k}(s,W(s-))\mathbb{1}_{\left\{Z(s-)=i\right\}}\mu_{jk}(s)\frac{p_{kj}(s,t)}{p_{ij}(s,t)}ds\right] \\
&+ \sum_{i\in\mathcal{J}} p_{0i}(0,R-)\mathbb{E}\left[\int_0^t \mathbb{1}_{\left\{Z(t)=j\right\}} h_i(s,W(s-))d\varepsilon_R(s)\middle| Z(R-)=i\right] \\
=& p_{Z(t_0)j}(t_0,t)\cdot w \\
&+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s-)\mathbb{E}\left[\mathbb{1}_{\left\{Z(t)=j\right\}} f_i(s,W(s-))\middle| Z(s-)=i\right] ds \\
&+ \int_0^t \mathbb{E}\left[\sum_{\substack{k\in\mathcal{J}\\ k\neq Z(s-)}} \mathbb{1}_{\left\{Z(t)=j\right\}} g_{Z(s-)k}(s,W(s-))\mathbb{1}_{\left\{Z(s-)=i\right\}}\right]\mu_{jk}(s)\frac{p_{kj}(s,t)}{p_{ij}(s,t)}ds XXXNOGETHER \\
&+ \mathbb{1}_{\{t\geq R\}}\sum_{i\in\mathcal{J}} p_{0i}(0,R-)\mathbb{E}\left[\mathbb{1}_{\left\{Z(t)=j\right\}} h_i(R,W(R-))\middle| Z(R-)=i\right]
\end{aligned}
$$

Now since $W(s-) \in \mathcal{F}_{(0,s-)}$ and $\mathbb{1}_{\{Z(t)=j\}} \in \mathcal{F}_{(s-,\infty)}$, it follows by the Markovianity of Z that $W(s-) \perp\!\!\!\perp \mathbb{1}_{\{Z(t)=j\}} | Z(s-)$ and so

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} W(s-) \middle| Z(s-)=i\right] =& \mathbb{E}\left[\mathbb{1}_{\{Z(t)=j\}} \middle| Z(s-)=i\right] \mathbb{E}\left[W(s-) \middle| Z(s-)=i\right] \\ =& p_{ij}(s-,t)\frac{\tilde{W}_i(s-)}{p_{0i}(0,s-)}\end{aligned}$$

Using this and the affine structure of f, g and h we obtain

$$\begin{aligned}\tilde{W}_j(t) =& p_{Z(t_0)j}(t_0,t)\cdot w && (.4.1)\\ &+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s)p_{ij}(s,t)\left(f_i^0(s) + \frac{\tilde{W}_i(s-)}{p_{0i}(0,s)} f_i^1(s)\right) ds && (.4.2)\\ &+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s) \sum_{\substack{k\in\mathcal{J}\\k\neq i}} \mu_{ik}(s)p_{kj}(s,t)\left(g_{ik}^0(s) + \frac{\tilde{W}_i(s-)}{p_{0i}(0,s)} g_{ik}^1(s)\right) ds && (.4.3)\\ &+ \mathbb{1}_{\{t\geq R\}} \sum_{i\in\mathcal{J}} p_{0i}(0,R)p_{ij}(R,t)\left(h_i^0(R) + \frac{\tilde{W}_i(R-)}{p_{0i}(0,R)} h_i^1(R)\right) && (.4.4)\\ =& p_{Z(t_0)j}(t_0,t)\cdot w && (.4.5)\\ &+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s)p_{ij}(s,t)f_i^0(s)ds && (.4.6)\\ &+ \int_0^t \sum_{i\in\mathcal{J}} p_{ij}(s,t)\tilde{W}_i(s-)f_i^1(s)ds && (.4.7)\\ &+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s) \sum_{\substack{k\in\mathcal{J}\\k\neq i}} \mu_{ik}(s)p_{kj}(s,t)g_{ik}^0(s)ds && (.4.8)\\ &+ \int_0^t \sum_{i\in\mathcal{J}} \sum_{\substack{k\in\mathcal{J}\\k\neq i}} \mu_{ik}(s)p_{kj}(s,t)\tilde{W}_i(s-)g_{ik}^1(s)ds && (.4.9)\\ &+ \mathbb{1}_{\{t\geq R\}} \sum_{i\in\mathcal{J}} p_{0i}(0,R)p_{ij}(R,t)h_i^0(R) && (.4.10)\\ &+ \mathbb{1}_{\{t\geq R\}} \sum_{i\in\mathcal{J}} p_{ij}(R,t)\tilde{W}_i(R-)h_i^1(R) && (.4.11)\end{aligned}$$

$\tilde{W}_j(t)$ is not differentiable at R for two reasons. First, the lump sum added via the h function potentially renders $\tilde{W}_j(t)$ discontinuous at this point (and hence not differentiable). Second, even if $h=0$, the point R is potentially a point of discontinuity for f and g, which renders the integrals above non-differentiable. On $(0,R)\cup(R,n)$, however, we have by Leibniz' integral rule

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{W}_j(t) =& \frac{\partial}{\partial t}p_{Z(t_0)j}(t_0,t)w_0 \\
&+ p_{Z(t_0j)}(t_0,t)f_j^0(t) \\
&+ \tilde{W}_i(t-)f_j^1(t) \\
&+ \sum_{\substack{i\in\mathcal{J}\\ i\neq j}} p_{Z(t_0)i}(t_0,t)\mu_{ij}(t)g_{ij}^0(t) \\
&+ \sum_{\substack{i\in\mathcal{J}\\ i\neq j}} \tilde{W}_i(t)\mu_{ij}(t)g_{ij}^1(t) \\
&+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s)\frac{\partial}{\partial t}p_{ij}(s,t)f_i^0(s)ds \\
&+ \int_0^t \sum_{i\in\mathcal{J}} \frac{\partial}{\partial t}p_{ij}(s,t)\tilde{W}_i(s-)f_i^1(s)ds \\
&+ \int_0^t \sum_{i\in\mathcal{J}} p_{0i}(0,s)\sum_{\substack{k\in\mathcal{J}\\ k\neq i}} \mu_{ik}(s)\frac{\partial}{\partial t}p_{kj}(s,t)g_{ik}^0(s)ds \\
&+ \int_0^t \sum_{i\in\mathcal{J}} \sum_{\substack{k\in\mathcal{J}\\ k\neq i}} \mu_{ik}(s)\frac{\partial}{\partial t}p_{kj}(s,t)\tilde{W}_i(s-)g_{ik}^1(s)ds \\
&+ \mathbb{1}_{\{t\geq R\}}\sum_{i\in\mathcal{J}} p_{0i}(0,R)\frac{\partial}{\partial t}p_{ij}(R,t)h_i^0(R) \\
&+ \mathbb{1}_{\{t\geq R\}}\sum_{i\in\mathcal{J}} \frac{\partial}{\partial t}p_{ij}(R,t)\tilde{W}_i(R-)h_i^1(R)
\end{aligned}
$$

We can now use the Kolmogorov forward equations

$$
\frac{\partial}{\partial t}p_{Z(t_0)j}(t_0,t) = \sum_{\substack{i\in\mathcal{J}\\ i\neq j}} \Big(p_{Z(t_0)i}(t_0,t)\mu_{ij}(t) - \mu_{ji}(t)p_{Z(t_0)j}(t_0,t)\Big)
$$

to recognize $\tilde{W}_j(s)$ and $\tilde{W}_i(s-)$ from equations (.4.5) through (.4.11) to arrive at

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{W}_j(t) =& p_{Z(t_0j)}(t_0,t)f_j^0(t) + \tilde{W}_i(t-)f_j^1(t) \\
&+ \sum_{\substack{i\in\mathcal{J}\\ i\neq j}} \mu_i j(t)\Big(p_{Z(t_0)i}(t_0,t)g_{ij}^0(t) + \tilde{W}_i(t)g_{ij}^1(t)\Big) \\
&+ \sum_{\substack{i\in\mathcal{J}\\ i\neq j}} \Big(\tilde{W}_i(t)\mu_{ij}(t) - \mu_{ji}(t)\tilde{W}_j(t)\Big)
\end{aligned}
$$

The initial value is just

$$
\begin{aligned}
\tilde{W}_j(t_0) =& \mathbb{E}\left[\mathbb{1}_{\{Z(t_0=j)\}}W(0)\right] \\
=& \mathbb{1}_{\{j=Z(t_0)\}}\cdot w
\end{aligned}
$$

and by inspecting equations (.4.5) through (.4.11), the gluing condition at time R follows from the continuity of the transition probabilities

$$\tilde{W}_j(R) - \tilde{W}_j(R-) = p_{Z(t_0)j}(t_0, R)h_j^0(R) + \tilde{W}_j(R-)h_j^1(R)$$

For the second line of XXX, we note that because $\left(g_{Z(s-)k}(s) + U(s-)g^U_{Z(s-)k}(s)\right)$ is predictable, we may replace $dN_k(t)$ with its predictable compensator to obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z(t)=j\}} \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(s-)}} \left(g_{Z(s-)k}(s) + U(s-)g^U_{Z(s-)k}(s)\right) dN_k(s)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z(t)=j\}} \sum_{\substack{k\in\mathcal{J}\\ k\neq Z(s-)}} \left(g_{Z(s-)k}(s) + U(s-)g^U_{Z(s-)k}(s)\right) dN_k(s)\,\middle|\, Z(t)\right]\right]
\end{aligned}$$