Optimising with Inequality Constraints

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A Simple Example

Solving the following problem using the Lagrangian methods allows us to develop an intuition for the Lagrange multiplier.

$$\max_x[-x^2] \qquad \text{s.t.} \qquad x \geqslant k$$

$-x^2$ is the objective function and $x \geqslant k$ is the constraint.

We need to maximise the objective function $-x^2$ while confining ourselves to the constraint area $x \geqslant k$.

The Lagrangian tells us whether the constraint matters or not. $-x^2$ is maximised at $x = 0$. This solution is not in the constraint area if $k > 0$ or $k < 0$. Thus, if the constraint matters, then the solution will be in the constraint area and not at $x = 0$. Conversely, if the constraint does not matter, then we can simply ignore it and the solution will at $x = 0$.

If the objective function is increasing at the constraint boundary, then the maximum will be within the constrained area and the constraint would not matter, i.e., the constraint will not bind. Conversely, if the objective function is decreasing at the constraint boundary, then the maximum will be at the constraint boundary and the constraint will bind, i.e., the constraint will become an equality. In the above example, a binding constraint would be $x = k$.

Let's now set up the Lagrange.

$$\mathscr{L} = -x^2 - \lambda[x - k]$$

When setting up the Lagrangian, it is usually good to follow the convention of the constraint being greater or equal to zero, i.e., $x - k \geqslant 0$.

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x} &= -2x - \lambda = 0 && \Rightarrow \quad \lambda = -2x \\ \frac{\partial \mathscr{L}}{\partial \lambda} &= x - k = 0 && \Rightarrow \quad x = k \end{aligned}$$

The lagrangian multiplier λ is simply the slope of the objective function. $\lambda = -2x$ suggests that the objective function is decreasing in x if $x > 0$. Conversely, the objective function is increasing in x if $x < 0$ and the objective function is stationary if $x = 0$.

If we evaluate λ at the constraint boundary, we get the slope of the objective function at the constraint boundary.

$$\lambda\big|_{x=k} = -2k$$

This implies that the objective function is decreasing at the constraint boundary $x = k$ if $k > 0$. Thus, if $k > 0$, the constraint binds and the solution will be at the constraint boundary $x = k$.

Conversely, the objective function is increasing if $k < 0$. In this case, the constraint will not bind and we can ignore the constraint in finding the solution. The convention is as follows:

- If the value of λ you obtain is positive, then set $\lambda^* = 0$.
- If the value of λ you obtain is negative, then set $\lambda^* = \lambda$.

The final lagrangian is as follows:

$$\mathscr{L} = \begin{cases} -x^2, & \text{if } \lambda \geqslant 0 \\ -x^2 - \lambda^*[x - k], & \text{if } \lambda < 0 \end{cases}$$

Thus, if $\lambda \geqslant 0$, i.e., the constraint is not binding, the Lagrangian is simply the objective function $-x^2$, which is maximised at $x = 0$.

If $\lambda < 0$, the Langrangian is $-x^2 - (-2k)[x-k]$ where we have replaced λ^* with $-2k$. First order condition on $-x^2 - (-2k)[x-k]$ gives us the solution $x = k$.

Exercises

Exercise 1.

$$\max_x [-x] \qquad \text{s.t.} \qquad x \geqslant k$$

Exercise 2.

$$\max_x [\ln x] \qquad \text{s.t.} \qquad x \geqslant 2$$