Lagrangian and Kuhn Tucker Conditions

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1. Langragian

$$\max f(x,y) \text{ s. t. } g(x,y) = 0$$

Constraint: Transform $g(x,y) = 0$ into $y = h(x)$ so that the maximisation problem becomes

$$\begin{aligned} g(x,y) &= 0 \\ h'(x) &= \frac{dy}{dx} = -\frac{g_x}{g_y} \qquad \text{(total differentiation)} \end{aligned}$$

$$\begin{aligned} \max F(x) &= f(x, h((x)) \\ F'(x) &= f_x + f_y h'(x) = 0 \\ &= f_x - f_y \frac{g_x}{g_y} = 0 \\ &= f_x - \lambda g_x = 0 \end{aligned}$$

where $\lambda = \frac{f_y}{g_y}$. λ is the rate at which f and g change with respect of y.
For real number λ, this leads to the following two equations.

$$\begin{aligned} f_x - \lambda g_x &= 0 \\ f_y - \lambda g_y &= 0 \end{aligned}$$

Another interpretation is that at the optima, the slope of f in $x - y$ plane and the slope of g in the $x - y$ plane is identical. This implies that

$$\left|\frac{dy}{dx}\right|_f = -\frac{\left[\frac{\partial f}{\partial y}\right]}{\left[\frac{\partial f}{\partial x}\right]} = -\frac{\left[\frac{\partial g}{\partial y}\right]}{\left[\frac{\partial g}{\partial x}\right]} = \left|\frac{dy}{dx}\right|_g$$

Langrangian: Maximise $f(x,y)$ subject to the constraint $g(x,y) = 0$, where λ is the langrangian multiplier.

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

The solution is given by the first order conditions.

$$\frac{\partial L}{\partial x} = f_x - \lambda g_x = 0$$
$$\frac{\partial L}{\partial y} = f_y - \lambda g_y = 0$$
$$\frac{\partial L}{\partial \lambda} = g(x, y) = 0$$

The condition on the second order conditions is a little more complicated.

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