Notes on Envelope Theorem

Kumar Aniket

Envelope Theorem

Creating an Envelope for a Function.

- A ladder of of length L is is leaned against a vertical wall at any angle. The wall and ground is taken as $y-$ and $x-$ axes. The top of the ladder is distance b from the ground and foot of the ladder is distance a from the wall. The straight line equation that describes the ladder can be written as:

$$y = b - \left[\frac{b}{a}\right] x$$

$$\Rightarrow \qquad \frac{x}{a} + \frac{y}{b} = 1 \tag{1}$$

such that $x, y \geqslant 0$. The relationship between a and b can be described as:

$$a^2 + b^2 = L^2 \tag{2}$$

Using (1) and (2), we get

$$f(x,y,a) = \left[\frac{x}{a} + \frac{y}{(L^2 - a^2)^{1/2}} - 1\right] = 0 \tag{3}$$

$$\frac{\partial f}{\partial a} = -\frac{x}{a^2} + \frac{ay}{(L^2 - a^2)^{3/2}} = 0 \tag{4}$$

From (4) we get

$$\begin{aligned} a &= \frac{Lx^{1/3}}{(x^{2/3} + y^{2/3})^{1/2}} \\ \left(L^2 - a^2\right)^{1/2} &= \frac{Ly^{2/3}}{(x^{2/3} + y^{2/3})^{1/2}} \end{aligned} \tag{5}$$

By substituting (5) in (3) we get the following

$$x^{2/3} + y^{2/3} = L^{2/3}$$

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If you change the angle of the ladder through the all possible angles, the ladder would always be below this equation and thus this curve envelopes all possible position of the ladder. Incidentally, this curve is call an astroid.

Illustrating the Envelope Theorem with an Example

Maximising a function. The function $y = f(x, a)$ has to be maximised with respect to x where a is just a parameter.

$$\begin{aligned} y &= f(x, a) \\ &= -\frac{1}{2}x^2 + (\alpha + \beta a)x \end{aligned}$$

- First and second order conditions

$$\frac{\partial y}{\partial x} = -x + (\alpha + \beta a) = 0 \tag{6}$$
$$\frac{\partial^2 L}{\partial x^2} = -1 < 0$$

The first order condition (6) gives us the relationship between x and a that maximising the function $y = f(x, a)$.

$$x^*(a) = \alpha + \beta a \tag{7}$$

- Think of $x^*(a)$ as the optimal path along which the function $y = f(x, a)$ is maximised. Substituting $x^*(a)$ into $y = f(x, a)$ gives us:

$$\begin{aligned} y^*(a) = f(x^*(a), a) &= -\frac{1}{2}\Big(\alpha + \beta a\Big)^2 + \Big(\alpha + \beta a\Big)^2 \\ &= \frac{1}{2}(\alpha + \beta a)^2 \end{aligned} \tag{8}$$

Thus, $y^*(a) = f(x^*(a), a)$ gives us the relationship between the maximised value of y and parameter a. We can see exactly how the maximised value of y changes when we change the parameter a by differentiating the function $y^*(a) = f(x^*(a), a)$. Differentiating (8) with respect to a we get the following.

$$\frac{dy^*(a)}{da} = (\alpha + \beta a) \cdot \beta \tag{9}$$

Envelope Theorem Route. Now the *Envelope theorem* tells us that we did not have to go through this long winded process to obtain the relationship between the maximised value of y and a. We could have got $\frac{dy^*(a)}{da}$ in (9) by simply differentiating the original function with respect to a and evaluating it along the optimal path (7).

- Lets start with the original function again and differentiate it partially with respect to a.

$$
\begin{aligned}
y &= f(x,a) \\
&= -\frac{1}{2}x^2 + (\alpha + \beta a)x \\
\frac{\partial y}{\partial a} &= x \cdot \beta
\end{aligned} \tag{10}
$$

If we evaluate (10) along the optimal path (7), we get

$$
\begin{aligned}
\left.\frac{\partial y}{\partial a}\right|_{x^*(a)=\alpha+\beta a} &= x^*(a) \cdot \beta \\
&= (\alpha + \beta a) \cdot \beta
\end{aligned} \tag{11}
$$

(11) and (11) both essentially get you the same answer but obtaining (11) is much simpler. Thus, the *Envelope theorem* just makes it easier for us to evaluate how the maximised value of a function as it changes with the parameter.

Stating the Envelope Theorem Formally. To see how this relationship hold formally just fully differentiate the function $y(x, a)$ with respect to a assuming that $x = x(a)$, i.e., x is a function of a.

$$
\begin{aligned}
y &= f(x,a) \\
&= f\Big(x(a), a\Big)
\end{aligned}
$$

- Fully differentiate this function with respect to a.

$$
\frac{dy}{da} = \frac{\partial f}{\partial x}\frac{dx}{da} + \frac{\partial f}{\partial a} \tag{12}
$$

We will evaluate (12) along the optimal path $x^*(a) = \alpha + \beta a$ from (7).

$$
\left.\frac{dy}{da}\right|_{x=x*(a)} = \left.\frac{\partial f}{\partial x}\right|_{x=x*(a)} \left.\frac{dx}{da}\right|_{x=x*(a)} + \left.\frac{\partial f}{\partial a}\right|_{x=x*(a)}
$$

We should note that along $x^*(a)$ by definition $\frac{\partial f}{\partial x} = 0$ (see equation 6). That is because $\frac{\partial f}{\partial x} = 0$ was obtained by setting $\frac{\partial f}{\partial x} = 0$. By doing this we get the *Envelope Theorem* formally.

$$
\left.\frac{dy}{da}\right|_{x=x*(a)} = \left.\frac{\partial f}{\partial a}\right|_{x=x*(a)}
$$

In essence, the*Envelope Theorem* tells us that the rate at which the maximised value of a function changes with a parameter is given by the the partial of the function evaluated along the optimal path. To see this in the context of the example above.

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- $\frac{\partial f}{\partial x}$ is given by the following:

$$\frac{\partial f}{\partial x} = -x + (\alpha + \beta a)$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x*(a)} = -x^*(a) + (\alpha + \beta a) = 0$$

- $\frac{\partial f}{\partial a}$ is given by the following:

$$\left.\frac{\partial f}{\partial a}\right|_{x=x*(a)} = x^*(a) \cdot \beta = (\alpha + \beta a) \cdot \beta$$