Eigen Values

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Eigenvalues and Eigen Vectors

Matrix: Take a determinant $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

Eigenvalues: *Find the Eigenvalues first.*

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{1}$$

$$\Rightarrow [\lambda \mathbf{I} - \mathbf{A}]\, \mathbf{x} = 0 \tag{2}$$

$|\lambda \mathbf{I} - \mathbf{A}| = 0$ gives a quadratic equation with roots $\lambda = \{2, -1\}$. These roots are called Eigenvalues.

This also implies that $[2\mathbf{I} - \mathbf{A}]$ and $[-1\mathbf{I} - \mathbf{A}]$ and are thus *singular* matrices.

Eigenvectors: *Find the Eigenvectors associated with Eigenvalues.*

It follows that $[2\mathbf{I} - \mathbf{A}]\, \mathbf{x} = 0$ and $[-1\mathbf{I} - \mathbf{A}]\, \mathbf{x} = 0$ are simultaneous equations which we can solve and obtain the Eigenvectors.

$\mathbf{Ax} = 2\mathbf{x}$ if and only if $x_1 = x_2$ with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Similarly, $\mathbf{Ax} = -1\mathbf{x}$ if and only if $x_1 = 4x_2$.

The Eigenvectors corresponding to the Eigenvalues 2 and -1 are $\mathbf{x}^T = [1, 1]$ and $[1, 4]$.

Putting Eigenvalues and Eigenvectors together: *Using Eigenvectors to diagonalise a matrix and find the associated diagonalised matrix composed of Eigenvalue diagonal elements.*

We can show that $\mathbf{AS} = \mathbf{SD}$ where $\mathbf{S}$ is matrix constructed from Eigenvectors and $\mathbf{D}$ is a diagonal matrix constructed from Eigenvalues.

$$\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

This can also be written as:

$$\mathbf{A} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{4}$$

where λ_1 and λ_2 are the Eigenvalues of $\mathbf{A}$.

Punchline: Put another way $\mathbf{x^{-1}Ax} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix with Eigenvalues as its diagonal elements. So, Eigenvectors are used to transform a matrix to a diagonal matrix.

Theorem: A real symmetric matrix $\mathbf{A}$ (a Hessian is always symmetric) is:

i. *positive definite* if and only if, all its eigenvalues $\lambda_i > 0$;
   positive semi-definite if and only if, all its eigenvalues are $\lambda_i \geqslant 0$;
ii. *negative definite* if and only if, all its eigenvalues $\lambda_i < 0$;
   negative semi-definite if and only if, all its eigenvalues are $\lambda_i \leqslant 0$;

1. Taylor's Expansion

Taylor's Expansion: getting $f(x)$ from a.

$$f(x) = f(a) + f'(a)\frac{(x-a)^1}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots + \\ \ldots + f^n(a)\frac{(x-a)^n}{n!}.$$

$$f(x) = f(0) + f'(0)\frac{x^1}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots + \\ \ldots + f^n(0)\frac{x^n}{n!}.$$