Lagrangian

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Lagrangian

Pemberton Rau. § 18.3.

Problem:

$$f(x, y) \qquad \text{s.t.} \qquad g(x, y) = 0$$
$$x \geqslant 0$$
$$y \geqslant 0$$

- If implicit function $g(x, y) = 0$ can be transformed into explicit function $y = h(x)$:

$$dg = g_x dx + g_y dy = 0$$
$$\frac{dy}{dx} = -\left[\frac{g_x}{g_y}\right] = h'(x)$$

$\left[-\frac{g_x}{g_y}\right]$ is the slope of the constraint function $g(x, y) = 0$.

$$F(x) = f(x, y)$$
$$= f(x, h(x))$$
$$F'(x) = f_x + f_y \cdot h'(x)$$

First order condition $F'(x) = 0$ gives us

$$F'(x) = f_x + f_y \cdot h'(x) = 0$$
$$\frac{f_x}{f_y} = -h'(x) = \frac{g_x}{g_y}$$
$$\left.\frac{dy}{dx}\right|_{f(x,y)=\bar{f}} = \left.\frac{dy}{dx}\right|_{g(x,y)=0}$$

Change is $F(\,\cdot\,)$ is due to two channels.

f_x: x has a *direct* change through the first argument of $F(\,\cdot\,)$.

f_y: Change is x also *indirectly* affect $F(\,\cdot\,)$ through y. Change in x leads to change in y, which leads to change in $F(\,\cdot\,)$. This indirect change is constrained when it is forced to go through the constraint $g(x, y) = 0$.

$$
\begin{aligned}
F'(x) &= f_x + f_y \cdot \left[-\frac{g_x}{g_y}\right] \\
&= f_x - \left[\frac{f_y}{g_y}\right] g_x \\
&= f_x - \lambda g_x \qquad \text{where } \lambda = \left[\frac{f_y}{g_y}\right] \quad \text{or } f_y - \lambda g_y = 0
\end{aligned}
$$

- So, first order condition $F'(x) = 0$ gives us the two equations following.

$$
\begin{aligned}
f_x - \lambda g_x &= 0 \\
f_y - \lambda g_y &= 0
\end{aligned}
$$

This implies that if $F'(x) = 0$

$$
\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y}
$$

With a little abuse of notation we can write the following:

$$
\begin{aligned}
\lambda &= \frac{\left[\frac{\partial f}{\partial x}\right]}{\left[\frac{\partial g}{\partial x}\right]} = \left.\frac{\partial f}{\partial g}\right|_{\Delta x} \\
&= \frac{\left[\frac{\partial f}{\partial y}\right]}{\left[\frac{\partial g}{\partial y}\right]} = \left.\frac{\partial f}{\partial g}\right|_{\Delta y}
\end{aligned}
$$

λ is the rate of change of the objective function f with respect to the constraint function g along the x and the y dimensions.

  - So, in utility maximisation subject to budget constraint, it is $\left.\frac{\partial U}{\partial I}\right|_i$ rate at which Utility changes when budget changes due to increase in price of one particular good i. At optimum, $\left.\frac{\partial U}{\partial I}\right|_i = \left.\frac{\partial U}{\partial I}\right|_i$ for all goods. [???]

Necessary condition for constrained maxima.

$$
\begin{aligned}
\frac{\partial L}{\partial x} &\leqslant 0 \\
&= 0 \quad \text{if} \quad x > 0 \\
\frac{\partial L}{\partial y} &\leqslant 0 \\
&= 0 \quad \text{if} \quad y > 0
\end{aligned}
$$

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Martin Osborne's Tutorial.

- Problem

$$\max_{x,y} f(x,y) \qquad \text{s.t.} \qquad g(x,y) = c$$

$$\mathscr{L} = f(x,y) - \lambda \left[g(x,y) - c\right]$$

- First Order condition:

$$\frac{f_1(x,y)}{g_1(x,y)} = \frac{f_2(x,y)}{g_2(x,y)} = \lambda$$

- Let the solution be $x*(c), y^*(c)$ with Lagrange multiplier $\lambda(c^*)$.

The solution exists if

  - x^*, y^* and λ^* are differentiable.
  - either $g_1\big(x^*(c), y^*(c)\big) \neq 0$ or $g_2\big(x^*(c), y^*(c)\big) \neq 0$.

- Let

$$f^* = f^*(x^*(c), y^*(c))$$

differentiating the above

$$\begin{aligned} f^{*\prime}(c) &= f'_x(x^*(c), y^*(c)) \cdot x^{*\prime}(c) + f'_y(x^*(c), y^*(c)) \cdot y^{*\prime}(c) \\ &= \lambda^*(c)\Big[g'_x(x^*(c), y^*(c))x^{*\prime}(c) + g'_y(x^*(c), y^*(c))y^{*\prime}(c)\Big] \\ &= \lambda^*(c) \end{aligned}$$

using the first order condition

- That is because

$$g\big(x^*(c), y^*(c)\big) = c \quad \forall\ c$$

$$g'_x(x^*(c), y^*(c))x^{*\prime}(c) + g'_y(x^*(c), y^*(c))y^{*\prime}(c) = 1 \quad \forall\ c$$

- that is,

$\lambda^*(c)$, the value of the Lagrange multiplier at the solution of the problem is equal to $f^{*\prime}(c)$, the rate of change in the maximal value of the objective function as the constraint $g(x^*(c), y^*(c)) = c$ is relaxed at the margin.

- Examples

  Utility maximisation problem: optimal value of Lagrange multiplier measures the *marginal utility of income*, the rate of increase in maximised utility as income increases.

  Problem 1.

$$\max_x f(x) = x^2 \qquad \text{s.t.} \qquad x = c$$

  The solution to Problem 1 is $x^* = c$.

  Maximised value of the function is. $f(x^*) = c^2$.

  Derivative of maximised value value of the function. $\frac{df(x^*)}{dc} = 2c$

$$\begin{aligned} \mathscr{L}(x) &= x^2 - \lambda(x - c) \\ \frac{d\mathscr{L}(x)}{dx} &= 2x - \lambda = 0 \qquad\qquad\qquad \lambda = 2x \end{aligned}$$

  Constraint: $x = c$

  (x, λ) that satisfies the first order condition is $x = c$ and $\lambda = 2c$. λ is equal to the derivative of the maximised value of the function $f(x^*)$ with respect to c.

- Alternative problem:

$$\begin{aligned} &\max_x f(x) = x^2 \qquad \text{s.t.} \qquad x \geqslant c \\ &\mathscr{L}(x) = x^2 - \lambda(x - c) && \text{(Using +)} \\ &(x^*, \lambda^*) = (c, 2c) \end{aligned}$$

  where * are the values at the constraint boundary. $\lambda > 0$ implies that $\frac{df(x^*)}{dc} > 0$, that is objective function increases when constraint increases at the margin. Given that the objective function has to be maximised, the constraint is not binding. (Increasing the value

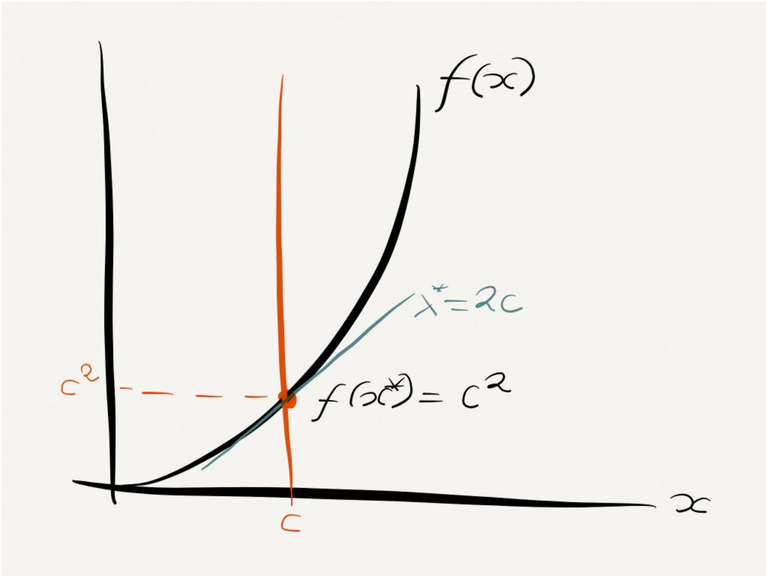

FIGURE 1. Maximise $f(x) = x^2$ subject to $x = c$. λ is the rate at which the maximised value of objective function changes with respect to the constraint c.

of the constraint function would only increasing the objective function.)

$$\max_x f(x) = x^2 \qquad \text{s.t.} \qquad x \leqslant c$$

$$\mathscr{L}(x) = x^2 - \lambda(c - x) \qquad \text{(Using +)}$$

$$(x^*, \lambda^*) = (c, -2c)$$

$\lambda < 0$ implies that $\frac{df(x^*)}{dc} < 0$, that is objective function decreases when constraint increases at the margin. Given that the objective function has to be maximised, the constraint is binding. (Increasing the value of the constraint function would only decrease the objective function. Of course, if the function was being minimised, the constraint would bind.)

- Problem 2

  Production: $f(x, y) = x^a y^b$.

  Price of output, x and y are p, w_x and w_y respectively.

  Constraint imposed by law of the land: $x = y$.

  Problem:

$$\max_{x,y} \quad f(x, y) = px^a y^b - w_x x - w_y y \qquad \text{s.t.} \qquad y - x = 0$$

$$\mathscr{L} = \Big[px^a y^b - w_x x - w_y y\Big] - \lambda[y - x]$$

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First order conditions

$$\lambda = w_x - apx^{a-1}y^b$$

$$\lambda = bpx^a y^{b-1} - w_y$$

$$x = y$$

$$\Rightarrow \qquad x = y = \left[\frac{w_x + w_y}{b(a+b)}\right]^{\frac{1}{a+b-1}}$$

$$\lambda = \frac{bw_x - aw_y}{a+b}$$

No value of (x, y) for which $g_1'(x, y) = g_2'(x, y) = 0$, so first order condition should give us the solution.

Since λ is rate at which the maximised objective function changes with the constraint, $\lambda > 0$ implies that the objective function's maximised value will increase if constraint is relaxed at the margin.

$\lambda > 0$ implies that firm's profits $\left[px^a y^b - w_x x - w_y y\right]$ increases when the constraint is relaxed to $y - x = \varepsilon$ or $y = x + \varepsilon$ for $\varepsilon > 0$.

$$\lambda > 0 \qquad \text{if} \qquad bw_x > aw_y$$

The firm would like to increase y over x at the margin such that $y = x + \varepsilon$. The value this create for the firm is given by $\varepsilon\lambda = \varepsilon\left[\frac{bw_x - aw_y}{a+b}\right]$. Thus, $\varepsilon\left[\frac{bw_x - aw_y}{a+b}\right]$ is the maximum bribe the firm would pay the inspector if she is ready to overlook the increase in y over x.

Conversely, if $\lambda < 0$, the firm would bribe the inspector to allow the firm to use reduce y compared to x such that $y - x = -\varepsilon$.

0.1. **Lagrangian Examples.** Getting the intuition about when the constraint binds in the Lagrangian. To sum up, it binds when you obtain a $\lambda > 0$.

- Maximising $y = -x^2$ subject to $x \leqslant 2$. This would imply that the constraint does not bind and the maxima is simply the unconstrained maxima.

$$\max_x -x^2 \qquad \text{s.t.} \qquad x \leqslant 2$$

Setting up the Lagrangian:

$$\mathscr{L} = -x^2 + \lambda\,[2 - x]$$

Constraint — The constraint is always written as greater or equal to and there is positive sign in front of the λ.

$$\frac{\partial \mathscr{L}}{\partial x} = -2x - \lambda = 0 \qquad\qquad \lambda = -2x = -4$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = 2 - x = 0 \qquad\qquad x = 2$$

$\lambda \leqslant 0$ implies that the constraint is not binding and we replace it with $\lambda = 0$. The general slackness condition reads as follows.

$$\lambda(2 - x) = 0$$

Either λ or $(2 - x)$ has to be equal to zero. Since $\lambda = 0$, we can have $x \leqslant 2$. The problem to be solved remains to be as follows:

$$\begin{aligned}\mathscr{L} &= -x^2 + 0 \cdot [2 - x] \\ &= -x^2\end{aligned}$$

- Maximising $y = -x^2$ subject to $x \geqslant 2$. This would imply that the constraint **binds** and the maxima is the **constrained maxima**.

$$\max_x -x^2 \qquad \text{s.t.} \qquad x \geqslant 2$$

Setting up the Lagrangian:

$$\mathscr{L} = -x^2 + \lambda [x - 2]$$

Constraint — The constraint is always written as greater or equal to and there is positive sign in front of the λ.

$$\frac{\partial \mathscr{L}}{\partial x} = -2x + \lambda = 0 \qquad \lambda = 2x = 4$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = 2 - x = 0 \qquad x = 2$$

$\lambda > 0$ implies that the constraint is binding. The general slackness condition reads as follows.

$$\lambda(x - 2) = 0$$

Either λ or $(x - 2)$ has to be equal to zero. Since $\lambda > 0$, we can have $x = 2$. The problem to be solved remains to be as follows:

$$\mathscr{L} = -x^2 + \lambda \cdot [x - 2]$$
$$= -x^2 + 4 \cdot [x - 2]$$

  - We can either replace the value of λ in $\mathscr{L}$

$$\frac{\partial \mathscr{L}}{\partial x} = -2x + 4 = 0 \qquad x = 2$$

  - Or we can just solve the binding constraint.

$$x - 2 = 0 \qquad x = 2$$