Dynamic Optimisation

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Pemberton and Rau, Chapter 27, page 563.

Difference equation describing the evolution of y_t over time.

$$y_{t+1} - y_t = g_t(x_t, y_t) \tag{State Equation}$$

y_t state variables

x_t control variable

λ_t co-state variable

- Agent's t period reward is $f_t(x_t, y_t)$.
- **Assume** that the agents is maximising the sum of single period (discounted) rewards.
- *Endpoint conditions* that y_0 and y_{T+1} are given.
- Agent chooses $x_0, x_1, \dots, x_T$ and also affect the path $y_0, y_1, \dots, y_T$, in order to solve the following problem:

$$\max \sum_{t=0}^{T} f_t(x_t, y_t)$$

$$\text{s.t.} \qquad y_{t+1} - y_t = g_t(x_t, y_t) \qquad \forall \qquad t = 0, 1, \dots, T$$

0.1. **Hamiltonian.**

$$\begin{aligned}&\mathscr{L}(x_0, x_1, \dots, x_T, y_0, y_1, \dots, y_T, \lambda_0, \lambda_1, \dots, \lambda_T) \\ &\qquad = \sum_{t=0}^{T} f_t(x_t, y_t) - \lambda_t(y_{t+1} - y_t - g_t(x_t, y_t))\end{aligned}$$

Hamiltonian:

$$H(x_t, y_t, \lambda_t) = f_t(x_t, y_t) + \lambda_t g_t(x_t, y_t)$$

$$\begin{aligned}
\mathscr{L} &= \sum_{t=0}^{T} H_t(x_t, y_t, \lambda_t) + \sum_{t=0}^{T} \lambda_t y_t - \sum_{t=0}^{T} \lambda_t y_{t+1} \\
&= \sum_{t=1}^{T} \Big(H_t(x_t, y_t, \lambda_t) + [\lambda_t - \lambda_{t+1}] y_t \Big) + \Big(H_0(x_0, y_0, \lambda_0) + \lambda_0 y_0 - \lambda_T y_{T+1} \Big)
\end{aligned}$$

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First Order Conditions. Hamiltonian is the period period reward $f_t(x_t, y_t)$ adjusted for the effect of the constraint $\lambda_t g_t(x_t, y_t)$.

$$H(x_t, y_t, \lambda_t) = f_t(x_t, y_t) + \lambda_t g_t(x_t, y_t)$$

First order conditions:

- x_t is the control variables. The control equation describes how at each t, the control variables is chosen by the agent to optimise their period reward H_t.

$$\frac{\partial H_t}{\partial x_t} = 0 \tag{Control Equation}$$

- λ_t is the co-state variables and the equation that describes the way the co-state variable evolves is given by

$$\frac{\partial H_t}{\partial y_t} = \lambda_{t-1} - \lambda_t \tag{Co-state equation}$$

- State Equation rewritten as:

$$\frac{\partial H_t}{\partial \lambda_t} = y_{t+1} - y_t \tag{State Equation}$$

Consumption Problem.

- $$\max \sum_{t=0}^{T} \left[\frac{u(c_t)}{(1+\rho)^t} \right]$$

$$\text{s.t.} \qquad a_{t+1} - a_t = r_t a_t + w_t - c_t \tag{State Equation}$$

- Hamiltonian:

$$H_t(c_t, a_t, \lambda_t) = \left[\frac{u(c_t)}{(1-\rho)^t} \right] + \lambda_t (r_t a_t + w_t - c_t)$$

- First Order Conditions
  - Control Equation:

$$\frac{\partial H_t}{\partial c_t} = \frac{u'(c_t)}{(1+\rho)^t} - \lambda_t = 0$$

The discounted marginal utility at each period t is equal to the value of the co-state variable, which measures how much the state equation binds or effects the objective function.

$$u'(c_t) = (1+\rho)^t \lambda_t$$

– Co-state Equation:

$$\frac{\partial H_t}{\partial a_t} = \lambda_t r_t = \lambda_{t-1} - \lambda_t$$

The co-state Equation describes how the co-state equation evolves.

$$\frac{\lambda_t}{\lambda_{t-1}} = \frac{1}{1+r_t}$$

– State Equation:

$$\frac{\partial H_t}{\partial \lambda_t} = a_{t+1} - a_t = r_t a_t + w_t - c_t$$

The state equation can just be recovered by differentiating the Hamiltonian with respect to the co-state variable.