Differential Equation

Dr. Kumar Aniket
University of Cambridge

1. Differential Equation

1.1. Linear First Order Differential Equation. The simplest linear first order differential equation.

$$\dot{x}(t) = ax(t)$$
$$\frac{\dot{x}(t)}{x(t)} = a$$

Solving it

$$\int \frac{\dot{x}(t)}{x(t)} dt = \int a dt$$
$$x(t) = x(0)e^{at} \qquad \text{where} \quad x(0) = e^{(c_1 - c_0)}$$

Using the fact that

$$\ln x(t) + c_0 = at + c_1$$
$$\frac{d \ln x(t)}{dt} = \frac{\dot{x}(t)}{x(t)}$$

Differential Equations.

i. Solving Separable differential equations.

$$\frac{dy}{dt} = F(t)G(t) \tag{1}$$

$$\int \frac{1}{G(t)} \frac{dy}{dt} = \int F(t) \tag{2}$$

ii. Solving differential equation when the rate of change of y is constant.

$$\frac{dy}{dt} = -ay \tag{3}$$

$$\int \frac{dy}{y} = -a \int dt \tag{4}$$

$$\ln y = -at + C \tag{5}$$

$$y = Ae^{-at} \qquad \text{(where } A = \pm e^C\text{)}$$

Constant rate of change is obtained from an underlying exponential process. This equation is also called the *complementary solution*.

iii. Solving differential equation when the rate of change of y is constant.

$$\frac{dy}{dt} = b - ay \tag{6}$$

Lets think of a particular function $y^P = c$, where c is a constant. The above equation can be written down as:

$$\frac{dy^P}{dt} = b - ay^P \quad \left(\Rightarrow \quad y^P = \frac{b}{a} \text{ as } \frac{dy^P}{dt} = 0\right) \tag{7}$$

Take a difference of the two equations with $z = y - y^P$.

$$\frac{dy}{dt} - \frac{dy^P}{dt} = -a(y - y^P) \tag{8}$$

$$\frac{dz}{dt} = -az \tag{9}$$

The solution is the sum of particular solution $y^P = \frac{b}{a}$ and the dynamic convergence ($a > 0$) or divergence ($a < 0$) process Ae^{-at}.

$$z = Ae^{-at} \tag{10}$$

$$y = \frac{b}{a} + Ae^{-at} \tag{11}$$

iv. Solving

$$\frac{dy}{dt} = -ay + g(t) \tag{12}$$

For a particular function y^P and $z = y - y^P$ it can be written as:

$$\frac{dy^P}{dt} = -ay^P + g(t) \tag{13}$$

$$\frac{dy}{dt} - \frac{dy^P}{dt} = -a(y - y^P) \tag{14}$$

$$\frac{dz}{dt} = -az \tag{15}$$

and then the general solution is

$$y = y^P + z \tag{16}$$

Strategy should be to choose y^P which is similar to $g(t)$ so that we can solve for the parameters of y^P.

v. For example, solving:

$$\frac{dy}{dt} = -5y + 4e^{3t} \tag{17}$$

Take the particular solution $y^P = Be^{3t}$ and using it in the equation gives us $B = \frac{1}{2}$, which means that our particular solution is $y^P = \frac{1}{2}e^{3t}$. The complementary solution is the general solution to:

$$\frac{dz}{dt} = -5z \tag{18}$$

which gives us $z = Ae^{-5t}$. The general solution is therefore

$$y = y^P + z = \frac{1}{2}e^{3t} + Ae^{-5t} \tag{19}$$

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