Visualising the Function

Lets take a function $g(x, y)$, which gives us the relationship between x, y and g. If we input the value of x and y into the function, we get $g(x, y)$ the output value of the function. We can visualise this function in a three dimensional space such that x and y determine the coordinates in the horizontal plane and g is the height of the function.

Think of the Royal Albert Hall. Imagine that the entrance is the origin $(0, 0)$. The x axis the lies along the invisible line connecting the entrance and the stage. The y axis is perpendicular to the x axis and passes through the entrance. Now superimpose a chess like grid on the floor and walk anywhere in the hall and calculate your (x, y) in meters. To find g, all you have to do is look up at the roof. The height of the roof gives you the value of g. So, any coordinate (x, y), you can find the corresponding value of $g(x, y)$ by just looking up to the roof. The roof of the Royal Albert Hall in this case is described by the function $g(x, y)$

Full and Partial Differential

When we differentiate $g(x, y)$ partially with respect to x, i.e. $\left(\frac{\partial g}{\partial x}\right)$, we obtain the rate at which g changes when x changes by a infinitesimally small amount given that y remains constant at a given value. The partial of $g(x, y)$ with respect to x can be denoted as follows.

$$\frac{\partial g}{\partial x} = g_x$$

Put another way, the partial gives us the marginal impact of x on g. In visual terms, think of a three dimensional representation of $g(x, y)$ described above. We can think of taking a vertical cross-section of the three dimensional function such that there is a constant value of y along this cross-section. The cross-section will give us a line which describes the g as a function of y. The partial in just the slope of the line. In the Royal Albert Hall example, it is just the slope of the roof along the direction x. Of course the slope of the line changes are both x and y changes.

When we differentiate fully, we are looking to find the full impact changes in all variable have on the g. That is, full differentiation gives you dg, the change in g due to dx and dy which are the changes in x and y respectively.

$$dg = g_x dx + g_y dy$$

The partial gives us the change in the height of the function due to infinitesimally changes along x and y dimensions.

$g(x, y) = k$ can be represented by a horizontal cross-section of the function at the height k. If we imagine cutting such a cross-section, we will find that it would also give us a line, which shows how x and y vary with each in that particular cross-section. So, by constraining ourselves to a horizontal plane, we can obtain a relationship between two entirely independent variables x and y. The constraint or the cross-section force a relationship between x and y. It is important to realise that every new horizontal cross-section will gives us an new relationship between x and y.

Turns out that we can actually obtain the rate at which x and y vary with each other in this forced relationship by using the partials. Intuitively, this is because each partial tells you the rate at which the height of the function changes at x and y change. If we would like to remain in the same horizontal cross-section, the height cannot change. Consequently, the height changes due to changes in x and y would have to cancel each other out. We find that the partials of the function at a particular coordinate fully determines the rate at which y changes due to change in x or vice versa. Fully differentiating $g(x, y) = k$, we get

$$\begin{aligned} dg = g_x dx + g_y dy &= 0 \\ \Rightarrow \quad \frac{dy}{dx} &= -\frac{g_x}{g_y} \end{aligned}$$

Useful to note that this tells us that $\frac{dy}{dx}$ is such that changes in the height of the function cancels out $g_y dy = -g_x dx$.

Constraint Optimisation and Lagrange Multipliers

The next step is to think of how we can maximise a function $f(x, y)$ subject to the constraints described by another function $g(x, y) = 0$. This entails finding the optimal point in the correct horizontal cross-section where $f(x, y)$ is maximised, while still satisfying the constraint of the function $g(x, y) = 0$.

The slope of a function in the cross-section is the negative ratio of its respective partials. E.g., Marginal rate of substitution between two goods is the negative ratio of the marginal utilities of the two goods.

Take a function $g(x, y)$ and set it equal to 0 to ensure that you are analysing it in its cross-section. The function $g(x, y) = 0$ describes a implicit relationship between x and y. Lets for convinience sake, represent that as an explicit relation $y = h(x)$.

$$g = g(x, y) = 0$$

$$
\begin{aligned}
dg &= g_x dx + g_y dy = 0 \\
\Rightarrow\ h'(x) &= \frac{dy}{dx} = -\frac{g_x}{g_y}
\end{aligned}
$$

Our objective in this exercise is to maximise $f(x, y)$ subject to the constraint $g(x, y) = 0$. In pursuing this objective, it is useful to transform $f(x, y)$ as a function of only x. y is eliminated from the function by using the explicit relation $y = h(x)$ obtained from the constraint $g(x, y) = 0$.

$$
F(x) = f\big(x, h(x)\big)
$$

$F(x)$ is a function which accounts for the fact that the constraint is going to bind at the optima. Maximising just entails find the first order condition of $F(x)$.

$$
\begin{aligned}
F'(x) &= f_x + f_y h'(x) \\
&= f_x - \left[\frac{f_y}{g_y}\right] g_x \\
&= f_x - \lambda g_x
\end{aligned}
$$

Similarly, we can write $f(x, y)$ as $\tilde{F}(y)$ using the explicit function $x = k(y)$ obtained from the constraint $g(x, y) = 0$ and find the first order conditions.

$$
\begin{aligned}
\tilde{F}'(y) &= f_y + f_x k'(y) \\
&= f_y - \left[\frac{f_x}{g_x}\right] g_y \\
&= f_y - \lambda g_y
\end{aligned}
$$

Turns out that at the point of optima $\left[\frac{f_x}{f_y}\right] = \left[\frac{g_x}{g_y}\right]$. That is the $\frac{dy}{dx}$ of the objective function and constraint in the cross-section is equal at the point of optima. Thus, we can write the above fist order condition as

$$
\begin{aligned}
F'(x) &= f_x - \lambda g_x \\
\tilde{F}'(y) &= f_y - \lambda g_y
\end{aligned}
$$

where the lagrange multiplier $\lambda = \left[\frac{f_y}{g_y}\right] = \left[\frac{f_x}{g_x}\right]$ is the ratio of the f and g's partials with respect to both x and y.

Duality

We look at a dual problem in this section. A dual problem emerges when you solve a problem and compares the solution to the mirror image of the problem, one where the objective function and the constraint gets interchanged. Turns out that the solutions to the dual problem do not neccessarily have similar characteristics.

To explore the duality, in the first instance we would maximise the utility subject to the an income constraint and the follow that up with minimising the expenditure of buying a consumption bundle given the constraint that a certain utility level has to be reached. Then we look at the respective demand functions obtained and compares the characteristics of these respective demand functions.

Maximising Utility

In this section we explore how we can obtain the demand curves for a Cobb-Douglas utility function by maximising utility subject to the budget constraint.

$$\begin{aligned} \text{Maximise} \quad & U = x_1^{\alpha} x_2^{1-\alpha} \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 = m \end{aligned}$$

Setting up the langrange function allows us to obtain the first order condition.

$$L = x_1^{\alpha} x_2^{1-\alpha} + \lambda \left[m - p_1 x_1 - p_2 x_2 \right]$$

$$\frac{\partial L}{\partial x_1} = \alpha \left(\frac{x_2}{x_1} \right)^{1-\alpha} - \lambda p_1 = 0 \tag{1}$$

$$\frac{\partial L}{\partial x_2} = (1-\alpha) \left(\frac{x_1}{x_2} \right)^{\alpha} - \lambda p_2 = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x = 0 \tag{3}$$

From (1) and (2) we get the following.

$$\frac{p_2 x_2}{p_1 x_1} = \frac{1-\alpha}{\alpha} \tag{4}$$

Substituting (4) into (3) we obtain the demand curves for x_1 and x_2.

$$x_1 = \frac{\alpha m}{p_1} \tag{5}$$
$$x_2 = \frac{(1-\alpha)m}{p_2} \tag{6}$$

Minimising Expenditure

In this section we explore how we can obtain the demand curves for a Cobb-Douglas utility function by minimising the expenditure on the goods subject to a given level of utility.

$$\begin{aligned} \text{Maximise} \quad & e = p_1x_1 + p_2x_2 \\ \text{subject to} \quad & \bar{u} = x_1^\alpha x_2^{1-\alpha} \end{aligned}$$

Setting up the langrange function allows us to obtain the first order condition.

$$L = p_1x_1 - p_2 + \mu\left[\bar{u} - x_1^\alpha x_2^{1-\alpha}\right]$$
$$\frac{\partial L}{\partial x_1} = p_1 - \mu\alpha\left(\frac{x_2}{x_1}\right)^{1-\alpha} = 0 \tag{7}$$
$$\frac{\partial L}{\partial x_2} = p_2 - \mu(1-\alpha)\left(\frac{x_1}{x_2}\right)^{\alpha} - \lambda p_2 = 0 \tag{8}$$
$$\frac{\partial L}{\partial \lambda} = \bar{u} - x_1^\alpha x_2^{1-\alpha} = 0 \tag{9}$$

From (7) and (8) we get the following.

$$\frac{p_2x_2}{p_1x_1} = \frac{1-\alpha}{\alpha} \tag{10}$$

Substituting (10) into (9) we obtain the demand curves for x_1 and x_2.

$$x_1 = \left[\frac{p_2}{p_1}\frac{\alpha}{(1-\alpha)}\right]^{(1-\alpha)}\bar{u} \tag{11}$$
$$x_2 = \left[\frac{p_1}{p_2}\frac{(1-\alpha)}{\alpha}\right]^{\alpha}\bar{u} \tag{12}$$

The striking thing to note is that the demand curves in (5) and (6) depend on their own prices only whereas the demand curves in (11) and (12) depend on the relative prices. (5) and (6) give us the *uncompensated* demand function whereas (11) and (12) give us the *compensated* demand function.

Change prices lead to changes in consumer's income in real terms. If the price of a particular good falls, the consumer becomes richer in real terms. Thus, the consumer reacts to two different impulses.

Compensated Demand Function The first impulse is the change in the relative prices of good which gives us the compensated demand function. The compensated demand function gives us the change in the good's demand due to the relative prices.[1]

Uncompensated Demand Function Added to the change in relative prices is also change in the consumer's real income. The fall in price of a good makes the consumer feel that she has become richer over all in real terms. So, add the income effect to the compensated demand function and you get the uncompensated demand function, which tells you the overall change in demand due to change in the price of a good. Thus the uncompensated demand function takes into account the change in relative prices as well as the change in consumer's income in real terms.

Try to visualise the dual problem in the three dimensions and explore the reason why the dual problem gives us such a different answer.

[1] These demand functions also give us the substitution effect.