The Essentials of Classical Mechanics

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1. Introduction

Classical mechanics is a theory that describes the motion of objects: how planets move around the sun, how a skier moves down the slope, or how an electron moves around the nucleus of an atom. Classical mechanics refers to the theories developed by Newton and his three laws of motion as well as the Hamiltonian and Lagrangian formulations of mechanics, which are completely equivalent to Newton's but provide simpler solutions to many complicated problems.

The faults of classical mechanics lies in its inability to describe the motion of objects moving at speeds close to the speed of light nor the movement of microscopic particles inside atoms and molecules. The description of these two regimes require the development of new theories: relativistic mechanics to describe high-speed motions and quantum mechanics to describe microscopic particles.

This text borrows heavily from John R. Taylor's book Classical Mechanics. The structure of this text and many sections are essentially copied over from Classical Mechanics. The role of this text is to present only the fundamental concepts of classical mechanics, keeping ancillary information to the minimum.

2. Fundamental Assumptions

We first begin with Newton's theory of classical mechanics, which is formulated in terms of four underlying concepts: space, time, mass, and force, which culminate in his three laws of motion. Mathematical tools will be employed to define the concepts and assumptions of Newton's theory.

2.1. Space

All objects reside in a three-dimensional space. A point P within such a space can be labeled by a position vector $\mathbf{r}$ which specifies the distance and direction of point P relative to the chosen origin O. Mathematically, we can write the position vector as

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \tag{1}$$

2.2. Time

The classical view is that time is a single universal parameter t on which all observers agree. This means that all observers equipped with accurate and synchronized clocks will agree as to the time at which any given event occurred.

2.3. Reference Frames

Our definitions of space and time imply that a reference frame will be chosen to describe any motion. A choice of a spatial origin and axes must be made to label positions and a choice of a temporal origin must be made to measure times. For example, the choice of a reference frame for time could simply be the choice for the origin of time: whether that be at $t = 0$ or $t = t_0$. The choice of a reference frame for position could be the location of the origin or the orientation of the three spatial axes.

In certain special frames called inertial frames, Newton's basic laws hold true in their standard, simple form. If a second frame is accelerating or rotating relative to an inertial frame, then this second frame is noninertial, and Newton's basic laws do not hold in their standard form in this second frame.

2.4. Mass

The mass of an object characterizes the object's inertia - its resistance to being accelerated. Objects with large mass are harder to accelerate than objects with smaller mass.

In order to quantify our conception of mass we have to first define a unit of mass and then give a prescription for measuring the mass of any object in terms of the chosen unit. The internationally agreed unit of mass is the kilogram and is defined arbitrarily to be the mass of a chunk of platinum-iridium stored at the International Bureau of Weights and Measures outside Paris.

To measure the mass of any other object, we need a means of comparing masses. One way is to compare the weight (gravitational force on the object) of objects. Two objects have the same mass if and only if they have the same weight, which could be deduced by using a balance scale.

Now that we have a way of comparing the equality of masses, we can measure an unknown mass by putting it on one end of the balance scale and loading known masses (e.g. multiple 1-kg mass objects) on the other end of the balance scale until the scale is balanced to any desired precision.

2.5. Force

A force is an interaction that influences the motion of an object. One way to define the magnitude of a force is to adopt the newton, which is the force that accelerates a standard kilogram mass with an acceleration of 1 m/s^2. A given force of 2 N would accelerate a standard kilogram with an acceleration of 2 m/s^2.

Lastly, we must define the direction of a force. The direction of a force is the direction of acceleration caused by the force on the object.

2.6. Newton's Three Laws of Motion

Newton's three laws of motion apply to a point mass, or particle. A point mass is an object with mass but no size and no internal degrees of freedom. We will later build up the mechanics of extended bodies from the mechanics of particles by considering the extended body as a collection of many separate particles.

Note that in many problems the object of interest can be realistically approximated as a point mass. Atomic and subatomic particles can often be considered point masses. Even a planet orbiting around the sun can be approximated in such a way.

First Law: In the absence of forces, a particle moves with constant velocity $\mathbf{v}$.

Second Law: For any particle of mass m, the net force $\mathbf{F}$ on the particle is always equal to the mass m times the particle's acceleration:

$$\boxed{\mathbf{F} = m\mathbf{a}.} \tag{2}$$

Here the force **F** denotes the vector sum of all the forces on the particle and **a** is the particle's acceleration. The second law can also be restated in terms of the particle momentum, which is defined as

$$\mathbf{p} = m\mathbf{v}. \tag{3}$$

Since

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} = m\mathbf{a} \tag{4}$$

then the Second Law can rephrased to say that

$$\boxed{\mathbf{F} = \dot{\mathbf{p}}.} \tag{5}$$

Note that the first two laws do not hold in all references frames. Supposed we take the view of reference frame $\mathcal{S}$ where a frictionless puck is placed on a smooth horizontal surface and is subject to zero force. If the puck moves with a constant velocity within our frame then we call it an inertial frame. Now consider a reference frame $\mathcal{S}'$ that is accelerating relative to frame $\mathcal{S}$. Then from the view of $\mathcal{S}'$ the puck will be accelerating despite having no force placed on it. Since the First and Second Law do not hold within frame $\mathcal{S}'$ we say that the frame is noninertial.

Thus, we see that we may use the First Law to identify whether the reference frame is inertial or not. Having identified an inertial frame we may then claim that as an experimental fact that the second law holds in these same inertial frames.

Third Law: If object 1 exerts a force $\mathbf{F}_{21}$ on object 2, then object 2 always exerts a reaction force $\mathbf{F}_{12}$ on object 1 given by

$$\boxed{\mathbf{F}_{12} = -\mathbf{F}_{21}.} \tag{6}$$

While the first two laws are concerned with the response of a single object to applied forces, the third law is concerned with the fact that every force on an object inevitably involves a second object - the object that exerts the force.

3. Momentum

The momentum, **p**, of a particle is defined as

$$\mathbf{p} = m\mathbf{v}. \tag{7}$$

Note that here by momentum we will refer specifically to mechanical momentum. In physical scenarios such as the magnetic field force on charges we will find that the conservation of mechanical momentum doesn't necessarily hold. We can still preserve our notion of momentum conservation, however, by including other sources of momentum such as the momentum from electromagnetic fields. In the scenario referenced above, the mechanical momentum can be converted into the electromagnetic momentum of the fields.

3.1. Conservation of Momentum

The conservation of momentum for systems of particles can be derived from Newton's Third Law. Consider a system of N particles whose index is labeled by either α or β, either of which can take on any of the values $1, 2, ..., N$. The net force on particle α can be divided into two sources. Firstly, each of the other $N-1$ particles can exert a force which will be denoted as $\mathbf{F}_{\alpha\beta}$. And secondly, there may be a net external force which will be denoted as $\mathbf{F}_\alpha^{\text{ext}}$. Thus, the net force on particle α is

$$\mathbf{F}_\alpha = \sum_{\beta\neq\alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} \tag{8}$$

where we note by referencing the Second Law that

$$\dot{\mathbf{p}}_\alpha = \mathbf{F}_\alpha. \tag{9}$$

Now consider the total momentum of our N-particle system

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha \sum_{\beta\neq\alpha} \mathbf{F}_{\alpha\beta} + \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} \tag{10}$$

where we note

$$\sum_\alpha \sum_{\beta\neq\alpha} \mathbf{F}_{\alpha\beta} = 0 \tag{11}$$

so that

$$\boxed{\dot{\mathbf{P}} = \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} = \mathbf{F}^{\text{ext}}.} \tag{12}$$

This result says that the internal forces have no effect on the evolution of the total momentum $\mathbf{P}$. The rate of change of the total momentum of the system $\dot{\mathbf{P}}$ is determined by the net external force on the system. This is the principle of conservation of momentum. It is equivalent to Newton's Third Law (Newton's Third Law implies the conservation of momentum and vice versa).

Conservation of Momentum: If the net external force $\mathbf{F}^{\text{ext}}$ on an N-particle system is zero, the system's total momentum $\mathbf{P} = \sum m_\alpha \mathbf{v}_\alpha$ is constant.

3.2. Center of Mass

We will now introduce a concept called the center of mass. The center of mass of a system of particles is defined as

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha \tag{13}$$

where M denotes the total mass of all the particles ($M = \sum m_\alpha$). The center of mass is a position vector that is the weighted average, using mass as the weight, of all the positions of the particles in the system.

An interesting feature of the center of mass is that we can write the total mentum $\mathbf{P}$ of any N-particle system in terms of the system's center of mass:

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}. \tag{14}$$

This means that the total momentum of the N particles is exactly the same as that of a single particle of mass M and velocity equal to that of the center of mass. Moreover, if we differentiate both sides of the above equation, we get

$$\boxed{\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}.} \tag{15}$$

This means that the center of mass $\mathbf{R}$ moves exactly as if it were a single particle of mass M subjected to the net external force on the system. This result is why we often treat extended bodies, such as baseballs and planets, as if they were point particles. The sum of the forces on a baseball (all of its constituent particles), for example, can be understood to act on a particle of mass M whose position is the center of mass of the baseball. And consequently, the motion of the baseball as a result of the external forces on its constellation of particles can be understood in relation to the motion of its center of mass as a result of the external forces on the center of mass particle.

4. Angular Momentum

So far, we have been concerned with the conservation of momentum $\mathbf{p}$ which is more specifically defined as "linear" momentum. The angular momentum, $\boldsymbol{\ell}$, of a particle is also a conserved quantity and is defined as

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} \tag{16}$$

where $\mathbf{r}$ is the particle's position vector relative to the chosen origin O and $\mathbf{p}$ is the particle's momentum. Since $\mathbf{r}$ depends on the choice of origin so does $\boldsymbol{\ell}$. This means that we should, if we want to be detailed, refer to $\boldsymbol{\ell}$ as the angular momentum relative to O.

Differentiating $\boldsymbol{\ell}$ relative to time gives us

$$\dot{\boldsymbol{\ell}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \boldsymbol{\Gamma} \tag{17}$$

where we have defined

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}. \tag{18}$$

$\boldsymbol{\Gamma}$ here is called the net torque on the particle relative to O. What (17) tells us is that the rate of change of a particle's angular momentum about the origin O is equal to the net applied torque about O.

Thus now we see that angular momentum about O is conserved (is constant) so long as the net torque $\boldsymbol{\Gamma}$ about O is zero. In many one particle systems, we can define an origin O such that the net torque is zero. This is useful to exploit when solving problems. For example, consider a single planet orbiting the sun. The only force on the planet is the gravitational pull of the sun which is a central force. If we define O to be the location of the sun, then the gravitational force is parallel to the position vector $\mathbf{r}$ measured from the sun so that $\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F} = 0$.

4.1. Conservation of Angular Momentum

We will now generalize our results for a system of N particles labeled $\alpha = 1, 2, ..., N$, each with angular momentum $\boldsymbol{\ell}_\alpha$. The total angular momentum, $\mathbf{L}$, is

$$\mathbf{L} = \sum_{\alpha=1}^{N} \boldsymbol{\ell}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha. \tag{19}$$

Differentiating with respect to time, we get

$$\dot{\mathbf{L}} = \sum_{\alpha=1}^{N} \dot{\boldsymbol{\ell}}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_\alpha. \tag{20}$$

We see that the rate of change of L is just the net torque on the whole system. However, it is important to separate the effects of internal and external forces. The net force on particle α is

$$\mathbf{F}_\alpha = \sum_{\beta\neq\alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} \tag{21}$$

where $\mathbf{F}_{\alpha\beta}$ is the force exerted on particle α by particle β and $\mathbf{F}_{\text{ext}}$ is the net force exerted on particle α by sources outside of the N particle system. Plugging this back into (20) we get

$$\dot{\mathbf{L}} = \sum_{\alpha} \sum_{\beta\neq\alpha} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \sum_{\alpha} \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}}. \tag{22}$$

By noting Newton's Third Law ($\mathbf{F}_{\alpha\beta} = -\mathbf{F}_{\beta\alpha}$ and assuming that all of the forces $\mathbf{F}_{\alpha\beta}$ are central, then we are left with

$$\dot{\mathbf{L}} = \sum_{\alpha} \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} = \sum_{\alpha} \boldsymbol{\Gamma}^{\text{ext}} = \boldsymbol{\Gamma}^{\text{ext}}. \tag{23}$$

This result is the principle of conservation of angular momentum.

Conservation of Angular Momentum: If the net external torque on an N-particle system is zero, the system's total angular momentum $\mathbf{L} = \sum m_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha$ is constant.

Note that this result depends on our two assumptions that all interval forces $\mathbf{F}_{\alpha\beta}$ are central and satisfy the third law.

4.2. Moment of Inertia

The calculation of angular momenta for a system does not always require us to use the basic definition

$$\mathbf{L} = \sum_{\alpha=1}^{N} \boldsymbol{\ell}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha. \tag{24}$$

The angular momenta for many systems rotating about a fixed axis can be expressed in terms of the moment of inertia of the system and the angular velocity of rotation.

Add section detailing moment of inertia stuff. Ch. 10 has more angular momentum stuff, like center of mass simplifications

5. Energy of Systems with One Particle

Central to the study of the motion of objects due to different forces is the concept of energy. Energy is a type of quantitative tally that we can keep track of for a system. It can take many forms: kinetic, several kinds of potential, thermal, and more. Different types of energy can also transform from one type to another. Understanding how energy transforms will help us understand the implications of such transformations on the system. Central to the energy formalism is the conservation of energy.

5.1. Kinetic Energy and Work

The first type of energy that we will formulate is the kinetic energy. The kinetic energy of a single particle of mass m traveling with speed v is defined to be

$$T = \frac{1}{2}mv^2. \tag{25}$$

Let's differentiate the kinetic energy relative to time to get

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}m(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) = m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v} \tag{26}$$

and then multiply both sides by $\mathrm{d}t$ to get

$$\mathrm{d}T = \mathbf{F}\cdot\mathrm{d}\mathbf{r}\,. \tag{27}$$

We define the expression on the right ($\mathbf{F}\cdot\mathrm{d}\mathbf{r}$) as the work done by the force $\mathbf{F}$ in the displacement $\mathrm{d}\mathbf{r}$. This result is the Work-Kinetic Energy theorem which states that the change in the particle's kinetic energy between two points is equal to the work done by the net force as it moves between the two points. Since our results so far only apply to infinitesimal distance $\mathrm{d}\mathbf{r}$, we can generalize the result to larger displacements by summing up the infinitesimal contributions on both sides of the equation (essentially, integrating both sides) to get

$$\boxed{\Delta T = T_A - T_B = \int_A^B \mathbf{F}\cdot\mathrm{d}\mathbf{r} = W(A\to B)} \tag{28}$$

where T_i is the kinetic energy at point i and the integral is a line integral from point A to point B. The right side being a line integral means that the change in kinetic energy depends on the path that the object takes.

Now it becomes clear why we defined the kinetic energy in the way that we did: it allows us to define changes in kinetic energy as the the work done by external forces.

5.2. Potential Energy and Conservative Forces

Now we will develop the concept of potential energy. For forces that are conservative, we can define a corresponding potential energy. A force F acting on a particle is conservative if

1. $\mathbf{F}$ depends only on the particle's position $\mathbf{r}$; that is to say, $\mathbf{F} = \mathbf{F}(\mathbf{r})$.
2. For any points A and B, the work $W(A\to B)$ done by $\mathbf{F}$ is the same for all paths between A and B.

The first condition ensures that the work done by the force is not dependent on the state of the particle as it travels from point A to point B. For example, that the speed at which the particle travels from point A to point B does not affect the work done by the force. Examples of conservative forces include

- The gravitational force from a stationary object, such as the force of the sun on a planet, where we have $\mathbf{F}(\mathbf{r}) = \frac{GmM}{r^2}\hat{\mathbf{r}}$.

- The electrostatic force on a particle with charge q by a static electric field, where we have $\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r})$.

Examples of nonconservative forces include

- Air resistance (depends on velocity)
- Friction (depends on direction of motion)
- Magnetic force (depends on velocity)
- Time-varying forces

For a given conservative force, we can define the potential energy $U(\mathbf{r})$ in the following way. First, choose a reference point $\mathbf{r}_o$ at which U is defined to be zero. Then, we can define $U(\mathbf{r})$, the potential energy at an arbitrary point $\mathbf{r}$, to be

$$\boxed{U(\mathbf{r}) = -W(\mathbf{r}_o \to \mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'\,.} \tag{29}$$

Using our definition of potential energy, we see that the work by the force when an object moves from an arbitrary point A to point B is given by

$$W(\mathbf{r}_A \to \mathbf{r}_B) = -[U(\mathbf{r}_B) - U(\mathbf{r}_A)] = -\Delta U. \tag{30}$$

Now note that the Work-Kinetic Energy theorem states that

$$\Delta T = W(\mathbf{r}_A \to \mathbf{r}_B). \tag{31}$$

Comparing the two above equations, we see that

$$\Delta T = -\Delta U \tag{32}$$

which implies that

$$\Delta(T + U) = 0 \tag{33}$$

for conservative forces. If we define the mechanical energy of a system to be

$$E = T + U, \tag{34}$$

then it becomes clear that the **mechanical energy of a system is conserved if the force on the particle is conservative**. In our case, the system is that of a single particle and whatever is causing the force.

5.2.1. Nonconservative Forces

For forces on our particle that are nonconservative, we cannot define corresponding potential energies nor can we define a conserved mechanical energy. Nonetheless, we can still extend our definition of kinetic energy and work to include the contribution of nonconservative forces. By noting the Work-Kinetic Energy theorem and separating the work from conservative versus nonconservative forces, we get

$$\Delta T = W = W_{\text{cons}} + W_{\text{nc}}. \tag{35}$$

By noting that $W_{\text{cons}} = -\Delta U$, we can rewrite our equation to get

$$\Delta(T + U) = W_{\text{nc}} \tag{36}$$

where by noting that the mechanical energy is $E = T + U$ we finally arrive at

$$\boxed{\Delta E = W_{\text{nc}}.} \tag{37}$$

We see that mechanical energy only changes to the extent of how much work nonconservative forces do on our particle.

5.2.2. Force as the Gradient of Potential Energy

Defining the potential energy $U(\mathbf{r})$ corresponding to a force $\mathbf{F}(\mathbf{r})$ in terms of an integral of $\mathbf{F}(\mathbf{r})$ as in (29) suggests that we can write $\mathbf{F}(\mathbf{r})$ as some kind of derivative of $U(\mathbf{r})$. To see how we can do that, we first note the definition of the work done by $\mathbf{F}(\mathbf{r})$ in a small displacement from $\mathbf{r}$ to $\mathbf{r} + \mathrm{d}\mathbf{r}$.

$$W(\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = F_x \,\mathrm{d}x + F_y \,\mathrm{d}y + F_z \,\mathrm{d}z \tag{38}$$

Next, we note that the work $W(\mathbf{r} \to \mathbf{r}+\mathrm{d}\mathbf{r})$ is also the same as the negative of the change in potential energy in the displacement (see (30)).

$$W(\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}) = -\,\mathrm{d}U = -\frac{\partial U}{\partial x}\,\mathrm{d}x - \frac{\partial U}{\partial y}\,\mathrm{d}y - \frac{\partial U}{\partial z}\,\mathrm{d}z \tag{39}$$

Now we can equate (38) and (39) to see that

$$F_x = -\frac{\partial U}{\partial x}, \qquad F_x = -\frac{\partial U}{\partial x}, \qquad F_x = -\frac{\partial U}{\partial x} \tag{40}$$

which can be written as

$$\boxed{\mathbf{F} = -\nabla U} \tag{41}$$

where the vector differential operator ∇ is defined as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{42}$$

Thus, we have shown that for any conservative force we can define a scalar function called the potential energy from which we can derive the force from.

5.3. Time-Dependent Potential Energy

In many cases, a force $\mathbf{F}(\mathbf{r}, t)$ may be time-dependent which means that it does not satisfy the first condition of a conservative force. However, the second condition may still hold for such a force $\mathbf{F}(\mathbf{r}, t)$ since we can define a potential energy

$$U(\mathbf{r}, t) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}' \tag{43}$$

which has the property of

$$\mathbf{F}(\mathbf{r}, t) = -\nabla U(\mathbf{r}, t). \tag{44}$$

Note that now our potential energy has a t-dependence. The difference here, for a time-dependent force, is that the mechanical energy $E = T + U$ is no longer conserved. Remember that the Work-Kinetic Energy theorem states that

$$\mathrm{d}T = \mathbf{F} \cdot \mathrm{d}\mathbf{r}\,. \tag{45}$$

Meanwhile since the potential energy now has a t-dependence, we note that

$$\mathrm{d}U = \frac{\partial U}{\partial x}\,\mathrm{d}y + \frac{\partial U}{\partial y}\,\mathrm{d}z + \frac{\partial U}{\partial z}\,\mathrm{d}x + \frac{\partial U}{\partial t}\,\mathrm{d}t \tag{46}$$

which, when noting (44), we can rewrite as

$$\mathrm{d}U = -\mathbf{F} \cdot \mathrm{d}\mathbf{r} + \frac{\partial U}{\partial t}\,\mathrm{d}t\,. \tag{47}$$

The first term on the right is just dT by the Work-Kinetic Energy theorem so that we now just left with

$$\boxed{\mathrm{d}(T + U) = \frac{\partial U}{\partial t}\,\mathrm{d}t\,.} \tag{48}$$

What we have shown is that mechanical energy is only conserved when $\frac{\partial U}{\partial t} = 0$ or, in other words, when U is independent of time. If U has a time-dependence, then the mechanical energy is not conserved. Of course, while mechanical energy may not be conserved, the *total* energy *is* conserved. The loss in mechanical energy is balanced by the gain in some other form of energy.

6. Energy of Systems with Multiple Particles

In the previous section, we have only focused on the energy of systems with a single particle. Now we will extend our analysis to systems of several particles that can interact with each other.

6.1. System of Two Particles

Let's start by analyzing a system with two particles to get a feel for the dynamics of multiple particle systems, from which we will later generalize. Consider an isolated system of two particles interacting via forces $\mathbf{F}_{12}$ (force on particle 1 by particle 2) and $\mathbf{F}_{21}$ (force on particle 2 by particle 1) with no other external forces. In general, the force $\mathbf{F}_{12}$ could depend on the positions of both the particles so that it can be written as

$$\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2). \tag{49}$$

By Newton's Third Law, we also must have

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{50}$$

An isolated system must be translationally invariant. That is to say, if we bodily translate the system to a new position, without changing the relative positions of the particles, then the interparticle forces should remain the same. This means that we can more specifically say that the force depends on $\mathbf{r}_1 - \mathbf{r}_2$ so that

$$\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2). \tag{51}$$

This result simplifies our analysis because we can learn everything about the force $\mathbf{F}_{12}$ by fixing $\mathbf{r}_2$ at a convenient point. By fixing $\mathbf{r}_2$ at the origin, (51) simply reduces to $\mathbf{F}_{12}(\mathbf{r}_1)$. Now, if the force on particle 1 is a conservative force, then we can define a potential energy $U(\mathbf{r}_1)$ such that

$$\mathbf{F}_{12} = -\nabla_1 U(\mathbf{r}_1) \tag{52}$$

where

$$\nabla_1 = \hat{\mathbf{x}}\frac{\partial}{\partial x_1} + \hat{\mathbf{y}}\frac{\partial}{\partial y_1} + \hat{\mathbf{z}}\frac{\partial}{\partial z_1} \tag{53}$$

This gives the force on particle 1 in the case that particle 2 is at the origin. To find the force on particle 1 for when particle 2 is in an arbitrary location, we simply have to replace $\mathbf{r}_1$ with $\mathbf{r}_1 - \mathbf{r}_2$ to get

$$\mathbf{F}_{12} = -\nabla_1 U(\mathbf{r}_1 - \mathbf{r}_2). \tag{54}$$

Usually we would also have to change the operator to act on $\mathbf{r}_1 - \mathbf{r}_2$ however since we treat $\mathbf{r}_2$ as constant in this case we can leave the operator as ∇_1 since it is unchanged by the addition of a constant $(-\mathbf{r}_2)$ to $\mathbf{r}_1$.

By Newton's Third Law ($\mathbf{F}_{21} = -\mathbf{F}_{12}$) we have

$$\nabla_1 U(\mathbf{r}_1 - \mathbf{r}_2) = -\nabla_2 U(\mathbf{1}_1 - \mathbf{r}_2) \tag{55}$$

where

$$\mathbf{F}_{21} = -\nabla_2 U(\mathbf{r}_1 - \mathbf{r}_2). \tag{56}$$

Equations (54) and (56) tell us that there exists a single potential energy function U from which we can derive the force on either particle.

Now consider the total work done on the two particles:

$$W_{\text{tot}} = \mathrm{d}\mathbf{r}_1 \cdot \mathbf{F}_{12} + \mathrm{d}\mathbf{r}_2 \cdot \mathbf{F}_{21}. \tag{57}$$

Using Newton's Third Law and (54) we can rewrite this as

$$W_{\text{tot}} = \mathrm{d}(\mathbf{r}_1 - \mathbf{r}_2) \cdot [-\nabla_1 U(\mathbf{r}_1 - \mathbf{r}_2)]. \tag{58}$$

Renaming $\mathbf{r}_1 - \mathbf{r}_2$ as $\mathbf{r}$ (the vector pointing from particle 2 to particle 1) we end up with

$$W_{\text{tot}} = -\,\mathrm{d}\mathbf{r} \cdot \nabla U(\mathbf{r}) = -\,\mathrm{d}U\,. \tag{59}$$

What this result tells us is that the potential energy defined here accounts for the work done on both particles, the total work.

Regarding the conservation of energy, remember that for kinetic energy we have

$$\mathrm{d}T = \mathrm{d}T_1 + \mathrm{d}T_2 = (\text{work on particle 1}) + (\text{work on particle 2}) = W_{\text{tot}}. \tag{60}$$

Noting (59), we have

$$\mathrm{d}T = -\,\mathrm{d}U \tag{61}$$

which can be rewritten to become

$$\mathrm{d}(T + U) = 0. \tag{62}$$

Thus, the total mechanical energy,

$$E = T + U = T_1 + T_2 + U \tag{63}$$

of our two-particle system is conserved. **To summarize, the important concept to note here is that for an isolated system with no external forces, the interaction forces between the two particles in the system must be translationally invariant and obey Newton's Third Law. If the interaction forces are conservative, then translation invariance means that the interaction forces are only a function of the relative positions between the two particles. In this case, we can define a potential energy function that accounts for the work done on both particles (and has the usual feature that we can take the gradient of the potential energy to find the force on any specific particle) showing that we can formulate a mechanical energy that is conserved for the system.**

6.1.1. Elastic Collision

An elastic collision is a collision between two particles (or bodies that can be treated as particles) that interact via a conservative force that goes to zero as their separation $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$. Since

$$\mathbf{F}_i = -\nabla_i U, \tag{64}$$

we see that the potential U approaches a constant, which we may take as zero, as $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$. An example of this type of interaction could be in the collision of two billiard balls which are manufactured so that they behave like springs, which is a conservative force, when forced together. Because the forces are conservative, the total energy is conserved so that $T + U =$ constant.

Now imagine that the particles are far apart where $U = 0$. When the two particles collide (get close to each other), kinetic energy is transferred into potential energy as the particles slow down. If the two particles re-emerge after the collision so that they are now far apart again, then $U = 0$ again. Thus, if the subscript "in" labels when the particles are far apart initially and "fin" labels when the particles are also far apart after the collision, then we see that

$$\boxed{T_{\mathrm{in}} = T_{\mathrm{fin}}.} \tag{65}$$

Elastic collisions prove useful in the analysis of the collisions between bodies such as billiard balls since the kinetic energy is conserved before and after the collision. Of course, not all collisions are elastic. For example, the billiard balls could shatter if they collide with each other. In this case, kinetic energy is not conserved as some of it is transformed into thermal energy.

6.2. System of N Particles

We will now extend the definition of mechanical energy to be defined for a system of N particles using the methods that we have employed for the system of two particles. Our N particles will be labeled by $\alpha = 1, \ldots, N$. The total kinetic energy is just the sum of the N separate kinetic energies:

$$T = \sum_{\alpha} T_{\alpha} = \sum_{\alpha} \frac{1}{2} m_{\alpha} v_{\alpha}^2. \tag{66}$$

Assuming all the forces are conservative, there are two types of forces acting on the particles:

1. Interaction forces between particles. For a pair of particles, $\alpha\beta$, we introduce a potential energy $U_{\alpha\beta}^{\text{int}}$ that describes the interaction between the two particles.
2. External forces that act on a particle. For a particle α, we introduce a potential energy U_{α}^{ext} to describe the net external force on that particle.

The total potential energy is then

$$U = U^{\text{int}} + U^{\text{ext}} = \sum_{\alpha} \sum_{\beta > \alpha} U_{\alpha\beta} + \sum_{\alpha} U_{\alpha}^{\text{ext}} \tag{67}$$

where the condition $\beta > \alpha$ in the double sum is to make sure we don't couble count the interaction potential energy (for example, we include U_{12} but not U_{21}).