ECON 3350/7350: Applied Econometrics for Macroeconomics and Finance

Tutorial 2: Univariate Time Series - I

This tutorial aims to get you familiar with the fundamental features of univariate time series models.

1. Derive the expected value, variance, covariance, autocorrelation function (ACF), and partial autocorrelation function (PACF) for the time series y_t having the following data generating processes (DGP):
   (a) AR(1): $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.
   (b) MA(1): $y_t = \beta_0 + \beta_1 \epsilon_{t-1} + \epsilon_t$.
   (c) ARMA$(1, 1)$: $y_t = a_0 + a_1 y_{t-1} + \beta_1 \epsilon_{t-1} + \epsilon_t$, $0 \leq |a_1| < 1$.

Solution:

(a)
- The Expected Value

$$y_t = a_0 + a_1 y_{t-1} + \epsilon_t; \quad 0 \leq |a_1| < 1$$

$$\begin{aligned} E\{y_t\} &= \mu = a_0 + a_1 E\{y_{t-1}\} + E\{\epsilon_t\} \\ \mu &= \frac{a_0}{1 - a_1}; \text{ since } E\{y_{t-1}\} = \mu \end{aligned}$$

- The Variance

$$\begin{aligned} V\{y_t\} &= \gamma_0 = a_1^2 V\{y_{t-1}\} + V\{\epsilon_t\} + +2cov\{a_1 y_{t-1}, \epsilon_t\} \\ \gamma_0 &= \frac{\sigma^2}{1 - a_1^2}; \text{ since } V\{y_{t-1}\} = \gamma_0,\ cov(y_{t-1}, \epsilon_t) = 0 \end{aligned}$$

- Covariance:
  - Set $a_0 = 0$ without loss of generality

$$\begin{aligned} cov\{y_t, y_{t-k}\} &= \gamma_k = E\{y_t y_{t-k}\} \\ &= E\{(a_1 y_{t-1} + \epsilon_t) y_{t-k}\} \end{aligned}$$

- γ_1 $(k = 1)$

$$\begin{aligned}\gamma_1 &= E\{(a_1 y_{t-1} + \epsilon_t) y_{t-1}\} \\ &= a_1 \frac{\sigma^2}{1 - a_1^2} = a_1 \gamma_0\end{aligned}$$

- γ_2 $(k = 2)$

$$\begin{aligned}\gamma_2 &= E\{(a_1 y_{t-1} + \epsilon_t) y_{t-2}\} \\ &= a_1^2 \frac{\sigma^2}{1 - a_1^2} = a_1^2 \gamma_0\end{aligned}$$

- γ_k $(k > 2)$

$$\begin{aligned}\gamma_k &= E\{(a_1 y_{t-1} + \epsilon_t) y_{t-k}\} \\ &= a_1^k \frac{\sigma^2}{1 - a_1^2} = a_1^k \gamma_0\end{aligned}$$

- Autocorrelation:
  - ρ_1

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = a_1$$

  - ρ_2

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = a_1^2$$

  - ρ_k, $k > 2$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = a_1^k$$

- Partial Autocorrelation
  - ϕ_{11}

$$\phi_{11} = \rho_1 = a_1$$

  - ϕ_{22}

$$\begin{aligned}\phi_{22} &= (\rho_2 - \rho_1^2)/(1 - \rho_1^2) \\ &= (a_1^2 - a_1^2)/(1 - a_1^2) \\ &= 0\end{aligned}$$

  - ϕ_{33}

$$\begin{aligned}\phi_{33} &= \frac{\rho_3 - \sum_{j=1}^{3-1} \phi_{2,j} \rho_{3-j}}{1 - \sum_{j=1}^{3-1} \phi_{3-1,j} \rho_j} \\ &= \frac{a_1^3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1 - \sum_{j=1}^{3-1} \phi_{3-1,j} \rho_j} \\ &= \frac{a_1^3 - a_1 a_1^2 + 0}{1 - \sum_{j=1}^{3-1} \phi_{3-1,j} \rho_j} \\ &= 0\end{aligned}$$

    since

$$\begin{aligned}\phi_{21} &= \phi_{1,1} - \phi_{22}\phi_{1,1} \\ &= \phi_{1,1}\end{aligned}$$

(b) • The Expected Value

$$\begin{aligned}E\{y_t\} &= \beta_0 + \beta_1 E\{\epsilon_{t-1}\} + E\{\epsilon_t\} \\ &= \mu\end{aligned}$$

• The Variance

$$\begin{aligned}V\{y_t\} = \gamma_0 &= V\{\beta_0\} + \beta_1^2 V\{\epsilon_{t-1}\} + V\{\epsilon_t\} + 2cov\{\epsilon_t, \epsilon_{t-1}\} \\ \gamma_0 &= \sigma^2(1+\beta_1^2)\end{aligned}$$

• Covariance:

  - Set $\mu = 0$ without loss of generality

$$\begin{aligned}cov\{y_t, y_{t-k}\} = \gamma_k &= E\{y_t y_{t-k}\} \\ &= E\{(\beta_1\epsilon_{t-1} + \epsilon_t)y_{t-k}\}\end{aligned}$$

$cov(y_t, y_{t-k}) > 0$ for $k = 1$, $cov(y_t, y_{t-k}) = 0$ for $k > 1$

  - γ_1 ($k = 1$)

$$\begin{aligned}\gamma_1 &= E\{(\beta_1\epsilon_{t-1} + \epsilon_t)y_{t-1}\} \\ &= E\{\beta_1\epsilon_{t-1}(\beta_1\epsilon_{t-2} + \epsilon_{t-1}) + \epsilon_t y_{t-1}\} \\ &= \beta_1\sigma^2 \\ &= \frac{\beta_1}{1+\beta_1^2} \times \gamma_0; \text{ since } \sigma^2 = \gamma_0/(1+\beta_1^2)\end{aligned}$$

  - γ_2 ($k = 2$)

$$\begin{aligned}\gamma_2 &= E\{(\beta_1\epsilon_{t-1} + \epsilon_t)y_{t-2}\} \\ &= 0; \text{ since } y_{t-2} \text{ is not a function of } \epsilon_t \text{ or } \epsilon_{t-1}\end{aligned}$$

  - γ_k ($k > 2$)

$$\begin{aligned}\gamma_k &= E\{(\beta_1\epsilon_{t-1} + \epsilon_t)y_{t-k}\} \\ &= 0\end{aligned}$$

• Autocorrelation:

  - ρ_1

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\beta_1}{1+\beta_1^2}$$

  - ρ_k, $k > 1$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = 0$$

• Partial Autocorrelation

- ϕ_{11}

$$\phi_{11} = \rho_1$$

- ϕ_{22}

$$\begin{aligned}\phi_{22} &= (\rho_2 - \rho_1^2)/(1-\rho_1^2)\\ &= (0-\rho_1^2)/(1-\rho_1^2)\\ &= -\rho_1^2/(1-\rho_1^2)\end{aligned}$$

- ϕ_{33}

$$\begin{aligned}\phi_{33} &= \frac{\rho_3 - \sum_{j=1}^{3-1}\phi_{2,j}\rho_{3-j}}{1-\sum_{j=1}^{3-1}\phi_{2,j}\rho_j}\\ &= \frac{\rho_3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1-\sum_{j=1}^{3-1}\phi_{2,j}\rho_j}\\ &= \frac{\rho_1^3/(1-\rho_1^2)}{1-\sum_{j=1}^{3-1}\phi_{2,j}\rho_j};\text{ since } \rho_2=\rho_3=0\end{aligned}$$

(c)

- The Expected Value

$$\begin{aligned}E\{y_t\} &= a_0 + a_1E\{y_{t-1}\} + \beta_1E\{\epsilon_{t-1}\} + E\{\epsilon_t\}\\ \mu &= \frac{a_0}{1-a_1};\text{ since } E\{y_t\} = E\{y_{t-1}\} = \mu\end{aligned}$$

- The Variance

$$\begin{aligned}V\{y_t\} = \gamma_0 &= V\{a_0\} + a_1^2V\{y_{t-1}\} + \beta_1^2V\{\epsilon_{t-1}\} + V\{\epsilon_t\}\\ &\quad + 2cov\{a_1y_{t-1},\beta_1\epsilon_{t-1}\} + 2cov\{a_1y_{t-1},\epsilon_t\} + 2cov\{\beta_1\epsilon_{t-1},\epsilon_t\}\\ \gamma_0 &= \frac{1+\beta_1^2+2a_1\beta_1}{1-a_1^2}\sigma^2,\ \text{ since } cov(a_1y_{t-1},\beta_1\epsilon_{t-1}) = a_1\beta_1E(\epsilon_{t-1}^2)\end{aligned}$$

  - To show $cov(a_1y_{t-1},\beta_1\epsilon_{t-1}) = a_1\beta_1E(\epsilon_{t-1}^2)$ you can proceed as follows

$$\begin{aligned}cov(a_1y_{t-1},\beta_1\epsilon_{t-1}) &= E[(a_1y_{t-1})(\beta_1\epsilon_{t-1})]\\ &= E\{[a_1(a_1y_{t-2}+\beta_1\epsilon_{t-2}+\epsilon_{t-1})](\beta_1\epsilon_{t-1})\}\\ &= E\{a_1\epsilon_{t-1}\beta_1\epsilon_{t-1}\}\end{aligned}$$

  is the only non-zero expected value.

- Covariance:
  - Set $\mu = 0$ without loss of generality

$$\begin{aligned}cov\{y_t,y_{t-k}\} = \gamma_k &= E\{y_ty_{t-k}\}\\ &= E\{(a_1y_{t-1}+\beta_1\epsilon_{t-1}+\epsilon_t)y_{t-k}\}\end{aligned}$$

- γ_1 $(k = 1)$

$$\gamma_1 = \frac{(1 + a_1\beta_1)(a_1 + \beta_1)}{1 - a_1^2}\sigma^2$$

- γ_k $(k > 2)$

$$\gamma_2 = a_1\gamma_1$$

- Autocorrelation:
  - ρ_1

$$\rho_1 = \frac{(1 + a_1\beta_1)(a_1 + \beta_1)}{1 + \beta_1^2 + 2a_1\beta_1}$$

  - ρ_k, $k \geq 2$

$$\rho_k = a_1\rho_{k-1}$$

  - Autoregressive pattern dominates from $k > 1$
- Partial Autocorrelation
  - ϕ_{11}

$$\phi_{11} = \rho_1$$

  - ϕ_{22}

$$\begin{aligned}\phi_{22} &= (\rho_2 - \rho_1^2)/(1 - \rho_1^2) \\ &= (a_1\rho_1 - \rho_1^2)/(1 - \rho_1^2)\end{aligned}$$

  - ϕ_{33}

$$\begin{aligned}\phi_{33} &= \frac{\rho_3 - \sum_{j=1}^{3-1}\phi_{2,j}\rho_{3-j}}{1 - \sum_{j=1}^{3-1}\phi_{2,j}\rho_j} \\ &= \frac{a_1^2\rho_1 - \phi_{21}a_1\rho_1 - \phi_{22}\rho_1}{1 - \sum_{j=1}^{3-1}\phi_{2,j}\rho_j}\end{aligned}$$

where

$$\begin{aligned}\phi_{21} &= \phi_{11} - \phi_{22}\phi_{11} \\ &= \rho_1[1 - (a_1\rho_1\text{-}\rho_1^2)/(1\text{-}\rho_1^2)]\end{aligned}$$

  - Moving Average pattern dominates after $k > 1$

2. Compute the true ACF values for the following DGPs:

   (1) DGP1: $y_t = 0.75y_{t-1} + \epsilon_t$

   (2) DGP2: $y_t = -0.75y_{t-1} + \epsilon_t$

   (3) DGP3: $y_t = 0.95y_{t-1} + \epsilon_t$

   (4) DGP4: $y_t = 0.5y_{t-1} + 0.25y_{t-2} + \epsilon_t$

   (5) DGP5: $y_t = 0.25y_{t-1} - 0.5y_{t-2} + \epsilon_t$

(6) DGP6: $y_t = 0.75\epsilon_{t-1} + \epsilon_t$

(7) DGP7: $y_t = 0.75\epsilon_{t-1} - 0.5\epsilon_{t-2} + \epsilon_t$

(8) DGP8: $y_t = 0.75y_{t-1} + 0.5\epsilon_{t-1} + \epsilon_t$

Solution:

(1) $\rho_0 = 1$, $\rho_1 = 0.75$, ..., $\rho_k = 0.75^k$. The ACF will decay geometrically.

(2) $\rho_0 = 1$, $\rho_1 = -0.75$, ..., $\rho_k = (-1)^k 0.75^k$. The ACF will decay in a dampened oscillatory path.

(3) $\rho_0 = 1$, $\rho_1 = 0.95$, ..., $\rho_k = 0.95^k$. The ACF will decay geometrically but at a much slower rate than DGP1.

(4) For AR(2) model $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \epsilon_t$, $\rho_0 = 1$, $\rho_1 = a_1/(1 - a_2)$, ..., $\rho_k = a_1\rho_{k-1} + a_2\rho_{k-2}$. Thus, $\rho_0 = 1$, $\rho_1 = 2/3$, $\rho_2 = 7/12$, ..., $\rho_k = a_1\rho_{k-1} + a_2\rho_{k-2}$ for $k \geq 2$.

(5) $\rho_0 = 1$, $\rho_1 = 1/6$, $\rho_2 = -11/24$, ..., $\rho_k = a_1\rho_{k-1} + a_2\rho_{k-2}$ for $k \geq 2$.

(6) For MA(q) model $y_t = \beta_0 + \beta_1\epsilon_{t-1} + \cdots + \beta_q\epsilon_{t-q} + \epsilon_t$, , the ACF cuts off at $k = q$, i.e., $\rho_k = 0$ for all $k > q$. Thus, $\rho_0 = 1$, $\rho_1 = \beta_1/(1 + \beta_1^2) = 12/25$, $\rho_k = 0$, for $k \geq 2$.

(7) $\rho_0 = 1$ and $\rho_k = 0$ for $k \geq 3$. $\rho_1 = \beta_1(1 + \beta_2)/(1 + \beta_1^2 + \beta_2^2) = 6/29$ and $\rho_2 = \beta_2/(1 + \beta_1^2 + \beta_2^2) = -8/29$.

(8) For ARMA(1,1) model $y_t = a_0 + a_1 y_{t-1} + \beta_1\epsilon_{t-1} + \epsilon_t$, $\rho_0 = 1$, $\rho_1 = (1 + a_1\beta_1)(a_1 + \beta_1)/(1 + \beta_1^2 + 2a_1\beta_1)$, $\rho_k = a_1\rho_{k-1}$ for all $k \geq 2$. Thus, $\rho_0 = 1, \rho_1 = 0.859$, $\rho_2 = 0.645$, ...

3. The data file `arma.csv` contains (simulated) data for each of the DGPs in Question 2. Import the data to Stata and use the variable t to declare time series. Compute, plot, and describe the behavior of the ACF and PACF of each DGP. Discuss the effects of parameter signs. Hint: Use the *ac* and *pac* commands, respectively.

Solution: See the do-file `tutorial2.do`.

(1)
  (a) ACF: Decay geometrically as parameter is positive.
  (b) PACF: One non-zero peak.

(2)
  (a) ACF: Decay in a dampened oscillatory path as parameter is negative.
  (b) PACF: One non-zero peak.

(3)
  (a) ACF: Decay geometrically but slower than DGP1.

(b) PACF: One non-zero peak.

(4) (a) ACF: Decay geometrically as parameter is positive.

(b) PACF: Two non-zero peaks.

(5) (a) ACF: Decay in a oscillatory path as one parameter is negative (and large in absolute value).

(b) PACF: Two non-zero peaks.

(6) (a) ACF: One non-zero peak.

(b) PACF: Decay in a oscillatory path.

(7) (a) ACF: Two non-zero peak.

(b) PACF: Decay in a oscillatory path.

(8) (a) ACF: Decay geometrically from $k = 2$ onwards as AR(1) process dominates.

(b) PACF: Decay in a oscillatary path from $k = 2$ as MA(1) dominates.