ECS130 Scientific Computation **Handout C** January 20, 2017

1. Norms are an indispensable tool to provide vectors and matrices with measures of size, length and distance.

2. A ***vector norm*** on $\mathbf{C}^n$ is a mapping that maps each $x \in \mathbf{C}^n$ to a real number $\|x\|$, satisfying

   (a) $\|x\| > 0$ for $x \neq 0$, and $\|0\| = 0$ (positive definite property)

   (b) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathcal{C}$ (absolute homogeneity)

   (c) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

3. Commonly used vector norms:

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i|, \quad \text{“Manhattan” or “taxi cab” norm} \\ \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^H x}, \quad \text{Euclidean length} \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i|. \end{aligned}$$

4. The geometry of the closed unit “ball”: $\{x \in \mathbf{C}^2 : \|x\|_p \leq 1\}$ for $p = 1, 2, \infty$.

5. Norm equivalence: Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be any two vector norms. There are constants $c_1, c_2 > 0$ such that
$$c_1 \|\cdot\|_\alpha \leq \|\cdot\|_\beta \leq c_2 \|\cdot\|_\alpha$$
   For examples, it can be easily shown that
$$\begin{aligned} \|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty \\ \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \\ \|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty \end{aligned}$$

6. Cauchy-Schwarz inequality:
$$|x^H y| \leq \|x\|_2 \|y\|_2$$
   with equality if and only if x and y are linearly dependent.

7. A ***matrix norm*** on $\mathbf{C}^{m \times n}$ is a mapping that maps each $A \in \mathbf{C}^{m \times n}$ to a real number $\|A\|$, satisfying

   (a) $\|A\| > 0$ for $A \neq 0$, and $\|0\| = 0$ (positive definite property)

   (b) $\|\alpha A\| = |\alpha| \, \|A\|$ for $\alpha \in \mathcal{C}$ (absolute homogeneity)

   (c) $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)

8. Example: for $A = (a_{ij}) \in \mathbf{C}^{m\times n}$, the Frobenius norm $\|A\|_{\mathrm{F}}$ is defined by

$$\|A\|_{\mathrm{F}} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \sqrt{\operatorname{tr}(A^H A)}.$$

9. The *induced matrix norm* $\|\cdot\|$:

   A vector norm $\|\cdot\|$ induces a matrix norm, denoted by the same notation:

$$\|A\| \stackrel{\text{def}}{=} \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

   (Exercise. verify that $\|A\|$ is indeed a norm on $\mathcal{C}^{m\times n}$

10. Useful property: $\|Ax\| \leq \|A\|\,\|x\|$. Therefore, $\|A\|$ is the maximal factor by which A can "strech" a vector.

11. The vector p-norms induce the matrix p-norms, in particular, for $p = 1, 2, \infty$, we have

$$\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1\leq j\leq n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\} = \text{ max absolute column sum,}$$

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \text{the largest singular value of } A,$$

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} = \text{ max absolute row sum.}$$

12. An application: sensitivity analysis of linear system of equations $Ax = b$.