1. A real symmetric matrix $A = A^T$ is *positive definite* if any of the following equivalent conditions hold:

   - The *quadratic form* $x^T Ax > 0$ for all nonzero vectors x.
   - All *determinants* formed from submatrices of any order centered on the diagonal of A are positive.
   - All *eigenvalues* of A are positive.
   - There is a lower triangular matrix L such that $A = LL^T$, called Cholesky decomposition of A.

   As you can see, the best way to check the positive definiteness is with Cholesky decomposition.

   (a) Let $n = 3$ and write the formulas for computing the entries ℓ_{ij} of L for a given 3×3 symmetric positive definite matrix A.

   (b) Use the observations in (a) to derive formulas to compute the Cholesky decomposition for an $n \times n$ symmetric positive definite (spd) matrix A.

   (c) Program your formulas to compute the Cholesky decomposition of an $n \times n$ spd matrix A. Check the correctness of your program by comparing with MATLAB's built-in function `chol` for the matrices $A = (a_{ij})$ with $a_{ij} = \frac{1}{i+j-1}$ with $n = 3, 4, 5$.

2. Read section 2.9, and present your error analysis for the two "computed" solutions $\widehat{x}_1 = \begin{bmatrix} 1.01 \\ 1.01 \end{bmatrix}$ and $\widehat{x}_2 = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$ of the linear system of equations

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} x = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}.$$

3. Assume you have a base-2 computer that stores floating-point numbers using a 6 bit normalized mantissa and a 4 bit exponent, and a sign bit for each.

   (a) For this machine, what is machine precision?

   (b) What is the smallest positive normalized number that can be represented in this machine?

4. Consider the following program

```
x = 1;
delta = 1 / 2^(53);
for j = 1 : 2^(20),
    x = x + delta;
end
```

   By mathematical reasoning, what is the expected final value of `x`? Use your knowledge of floating-point arithemtic to predict what it actually is. Verify by running the program and explain the result.

5. Using mathematical reasoning, we know that for any positive number x, $2x$ is a number greater than x. Is this true of floating-point numbers? Run the following program and explain your result

```
x = 1;
twox = 2*x;
k = 0;
while (twox > x)
```

```
    x = twox;
    twox = 2*x;
    k = k + 1;
end
```

6. The polynomial $p_1(x) = (x-1)^6$ has the value zero at $x = 1$ and is positive elsewhere. The expanded form of the polynomial

$$p_2(x) = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1,$$

is mathematically equivalent. Plot $p_1(x)$ and $p_2(x)$ for 101 equally spaced points in the interval $[0.995, 1.005]$. Explain the plots. (you should evaluate the polynomial $p_2(x)$ by Horner's rule).

7. (a) Write a MATLAB function that computes the two roots of a quadratic polynomial $q(x) = x^2 + bx + c$ with good precision.

   (b) Compare your computed results with MATLAB's built-in function `roots([a b c])` for the following set of data:

   1) $b = -56, c = 1$
   2) $b = -10^8, c = 1.$