Scientific Computing Language
Homework 2

Problem 1 (10 pts)**.** Let $Ax = b$ Write $\tilde{x} = x + \delta x$ be the computed estimate of x where x is perturbed by δx. Although we can not compute δx, we can estimate the residual, defined as

$$r = b - A^{\tilde{}\,x.}$$

Show that

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.$$

Does small residual mean the error is also small?

Problem 2 (10 pts)**.** Show that the multiplication of two $n \times n$ matrices takes $\sim 2n^3$ flops.

Problem 3 (15 pts)**.** Suppose that $A \in \mathbb{R}^{n \times n}$ and that $\|A\| < 1$ in some induced matrix norm.

1. Show that $I - A$ is nonsingular. (Show that $(I - A)x = 0$ for nonzero x implies that $\|A\| \geq 1$, using the definition of an induced matrix norm.
2. Show that $\lim_{m \to \infty} A^m = 0$. (We say $\lim_{B \to \infty} B_k = B^*$ if $\lim_{B \to \infty} \|B_k - B^*\| = 0$.)
3. Use the above result to show $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Problem 4 (15 pts)**.** Here are population figures for three countries over the same 30 years period.

Year	United States	China	Germany
1980	227.225	984.736	78.298
1990	249.623	1148.364	79.380
2000	282.172	1263.638	82.184
2010	308.282	1330.141	81.644

1. Use cubic polynomial interpolation to estimate the population of China in 1992.
2. Use cubic polynomial interpolation to estimate the population of China in 1984.
3. Use cubic polynomial interpolation to make a plot of the German population from 1985 to 2000. Your plot should show a smooth curve and be well annotated.

Problem 5 (10 pts)**.** Consider the matrix $A \in \mathbb{R}^{n \times n}$, with entries $a_{ij} = 1$ if $i = j$ or $j = n$, $a_{ij} = -1$ if $i > j$, zero otherwise. Show that A admits an LU decomposition, with $|L_{ij}| \leq 1$ and $u_{nn} = 2^{n-1}$.

Problem 6 (15 pts)**.** In Matlab, define

$$A = \begin{bmatrix} 1 & & & & 10^{12} \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 0 \end{bmatrix}, \hat{x} = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \\ 1 \\ 4/3 \end{bmatrix}, b = A\hat{x}.$$

1. The `format` command can adjust output appearance in Matlab; enter `format long` in Matlab. Using `lufact` function and triangular substitutions, solve the linear system $Ax = b$ and let Matlab print out the result. About how many (to the nearest integer) accurate digits are in the result? (The answer is much less than the default 16 of double precision)

2. Repeat part 1) with 10^{20} as the corner element. (The result is even less accurate).

Problem 7 (15 pts)**.** In this problem you will explore the backslash and how it handles banded matrices. To do this you will generate tridiagonal matrices using the following code:

```
A = triu( tril(rand(n),1), -1);
A(:1:n+1:end) = a;
```

The result is $n \times n$, with each entry on the sub- and super-diagonals chosen randomly from $(0, 1)$ and each diagonal entry equaling a.

1. Write a script that solves 200 linear systems whose matrices are generated as above, with $n = 1000$ and $a = 2$. Record the total time used by the solution process `A\b` only, using the built-in `cputime`.

2. Repeat the experiment of 1), but add the command `A=sparse(A);` right after the two lines above. how does timing change?

3. Based on these observations, state a hypothesis on how backslash solves tridiagonal linear systems given in standard dense form and in sparse form.

Problem 8 (10 pts)**.** Based on the `lufact` function, write a function

```
function [L, U] = luband(A, upper, lower)
```

that accepts upper and lower bandwidth values and computes LU factors in a way that avoids doing arithmetic using the locations that are known to stay zero.