Scientific Computing Language
Homework 4

Problem 1 (Fixed point iteration, 10 pts)**.** In each case, show that the given $g(x)$ has a fixed point at the given r and show that the fixed point iteration can converge to it (Apply error analysis and check the derivative $g'(x)$). a)$g(x) = \frac{1}{2}\left(x + \frac{9}{x}\right)$, $r = 3$. b) $g(x) = x + 1 - \tan\left(\frac{x}{4}\right)$, $r = \pi$. Moreover, apply fixed point iteration in Matlab and use a log-linear graph of the error to verify linear convergence.

Problem 2 (Newton's method, 10 pts)**.** Consider the equation $f(x) = x^{-2} - \sin x = 0$ on the interval $x \in [0.1, 4\pi]$. Use a plot to approximately locate the roots of f. To which roots do the following initial guesses converge when using Newton's method. Is the root obtained the one that is closest to that guess? a). $x_0 = 1.5$. b) $x_0 = 2$. c) $x_0 = 4$. d) $x_0 = 5$. e) $x_0 = 2\pi$.

Problem 3 (Newton system, 15 pts)**.** In this problem you are to fit a function of the form: $P(t) = a_1 + a_2 e^{a_3 t}$to a subset of U.S. census data for the twentieth century:

Year	1910	1930	1950	1970	1990
Population (millions)	90.0	122.8	150.7	205.0	248.7

1. Determine the unknown parameter a_1, a_2, a_3 in P by requiring that P exactly reproduce the data in the years 1910, 1950, 1990. This creates three nonlinear equations for a_1,a_2, a_3 that may be solved using Newton system.

2. To obtain convergence, rescale the data using the time variable $t = (year - 1900)/100$ and divide the population numbers above by 100. Using your model $P(t)$, predict the result of the 2000 census, and compare it to the true figure of 284.1 millions.

Problem 4 (Cubic splines, 15 pts)**.** For each of the function, inteval and value of n, define $n + 1$ evely spaced nodes. Then plot the cubic spline interpolant at those nodes together with the original function over the given interval.

1. $\cos(\pi^2 x^2)$, $x \in [0, 1]$, $n = 7$.

2. $\ln(x)$, $x \in [1, 3]$, $n = 5$.

Next, for each of the function, apply the not-a-knot spline function for equispaced nodes with $n = 10, 20, 40, 80, 160$. In each case compute the interpolant at 1600 equally spaced points in the interval and use it to estimate the error

$$E(n) = \|f - S\|_\infty = \max_x |f(x) - S(x)|.$$

Make a log-log plot of E as a function of n and compare it graphically to fourth-order convergence.

Problem 5 (Simpson, 20pts)**.** We can derive Simpson formula $S_{2n}(f)$ without appealing to extrapolation.

1. Find a quadratic function $p(x)$ to interpolate the three points $(-h, \alpha)$, $(0, \beta)$ and (h, γ).

2. Compute $\int_{-h}^{h} p(s)\, ds$ in terms of α, β, h, γ computed in part 1).

3. Assume equally spaced nodes in the form $x_i = a + ih$ for $h = (b-a)/n$ and $i = 0, 1, ..., n$. Suppose f is approximated by $p(x)$ over the subinterval $[x_{i-1}, x_{i+1}]$. Apply the result from part 2) to find

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\, dx \approx \frac{h}{3}\big[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\big].$$

4. Now also assume that $n = 2m$ for an integer m. Derive Simpson's formula

$$\int_a^b f(x)\, dx \approx \frac{h}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 4f(x_{n-1}) + f(x_n)\big].$$

Show that the above formula is equivalent to $S_{2m}(f)$ in (10.4) in lecture notes.

Problem 6 (Integration, 15 pts)**.** For each of the integral, use trapezoid formula to estimate the integral for $n = 10 \cdot 2^k$ nodes for $k = 1, 2, ..., 10$. Make a log-log plot of the errors and confirm or refute second-order accuracy. a) $\int_0^1 x \log(1+x)\, dx = \frac{1}{4}$. b) $\int_0^{\pi/2} e^x \cos x, dx = \frac{e^{\pi/2}-1}{2}$. c) $\int_0^1 \sqrt{x} \log(x)\, dx = -\frac{4}{9}$.

Problem 7 (Polynomial interpolation, 15 pts)**.** The Chebyshev points can be used when the interval of interpolation is $[a, b]$ rather than $[-1, 1]$ by means of the change of variable

1. Find a linear function $\psi : [-1, 1] \to [a, b]$ such that $\psi(-1) = a$, $\psi(1) = b$ and ψ is strictly increasing on $[-1, 1]$.

2. Let $\{x_i\}_{0 \le i \le n}$ be standard Chebyshev points (second kind). Then a polynomial in x can be used to interpolate the function values $f(\psi(x_i))$. Denote $z = \psi(x)$. This in turn implies an interpolation $\widetilde{p}(z) = p(\psi^{-1}(z))$. Show that $\widetilde{p}$ is a polynomial in z.

3. Implement the idea of part 2) to plot a polynomial interpolant of $f(x) = \cosh(\sin x)$ over $[0, 2\pi]$ using $n + 1$ Chebyshev nodes with $n = 40$.