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Section TA & Time: Mulkin, 11 Am

Problem 1.1: Evaluate the integrals of Ex 1.1 from the lecture slides.

$$\frac{1}{3!}\int_{55}^{60}\left(\frac{1}{15}\right)^4 t^3 e^{-\frac{t}{15}}\,dt = \left(-\frac{t^3}{15^3}e^{-\frac{t}{15}} - \frac{3t^2}{15^2}e^{-\frac{t}{15}} - \frac{6t}{15}e^{-\frac{t}{15}} - 6e^{-\frac{t}{15}}\right)\Big|_{55}^{60}\cdot\frac{1}{6}$$

$$\begin{array}{lll} & t^3 & \frac{1}{15^4}e^{-\frac{t}{15}} \\ - & 3t^2 & -\frac{1}{15^3}e^{-\frac{t}{15}} \\ & 6t & \frac{1}{15^2}e^{-\frac{t}{15}} \\ - & 6 & -\frac{1}{15}e^{-\frac{t}{15}} \\ & 0 & e^{-\frac{t}{15}} \end{array}$$

$= -64(.018316) - 48(.018316) - 24(.018316) - 6(.018316)$

$+ 49.2963(0.02556153) + 40.\overline{33}(0.02556153) + 22(0.02556153) + 6(0.02556153) =$

$= (3.006 - 2.60087) = 0.399178\left(\frac{1}{6}\right) \approx 0.068$

Problem 1.2: Starting at 9 am, patients arrive at a doctor's office according to a Poisson process. On average, 3 patients arrive every hour.

a) find the prob. that at least two patients arrive by 9:30 am

$P(N_{0.5} \geq 2) = 1 - P(N_{0.5} < 2) = 1 - P(N_{0.5} = 1) - P(N_{0.5} = 0)$ $\quad N_{0.5} \sim Pois(\lambda = 3\cdot(0.5)) \stackrel{d}{=} Pois(\lambda = 1.5)$

$P(N_{0.5} \geq 2) = 1 - e^{-1.5}(1 + 1.5) = 0.4422$

b) find the prob that 10 patients arrive by noon and 8 come before 11 am

$P(N_3 = 10, N_2 = 8) = P(N_2 = 8, N_3 - N_2 = 2)$. Note $N_3 - N_2 \perp\!\!\!\perp N_2$ thus $P = P(N_2 = 8)P(N_3 - N_2 = 2)$

$N_2 \sim Pois(\lambda = 6)$, $N_3 - N_2 \sim Pois(\lambda = 3) \Rightarrow p = \frac{e^{-6}\cdot 6^8}{8!}\cdot\frac{e^{-3}3^2}{2!} = 0.0231$

c) if 6 patients arrive by 10 am, find the prob. that only one arrives by 9:15

$P(N_{0.25} = 1 \mid N_1 = 6) = P(N_{0.25} = 1, N_1 = 6)/P(N_1 = 6) = P(N_{0.25} = 1, N_1 - N_{0.25} = 5)/P(N_1 = 6)$

$N_{0.25} \sim Pois(\lambda = 0.75)$, $N_1 - N_{0.25} \sim Pois(\lambda = 2.25)$ and $N_{0.25} \perp\!\!\!\perp N_3 - N_{0.25}$

$\Rightarrow p = e^{-.75}.75 \cdot e^{-2.25}\cdot 2.25^5\left(\frac{1}{120}\right)/(e^{-3}\cdot 3^6/720) = 4.5\cdot 2.25^5/3^6 = 0.356$

Problem 1.3. Let $(N_t)_{t\geq 0}$ be a $Pois(\lambda = 2)$. Denote $\{X_i\}$ the interarrival times for N and $\{S_i\}$ the arrival times. Find

a) $E[X_3 X_4]$

Interarrival times are iid $\sim exp(\lambda)$, thus $Cov(X_3, X_4) = 0 \Rightarrow E[X_3]E[X_4] = E[X_3X_4], = E[X_1]^2 = \frac{1}{\lambda^2} = \boxed{\frac{1}{4}}$

b) $E[S_3 S_4]$

$E[S_3S_4] = E[S_3(S_3 + S_4 - S_3)] = E[S_3(S_3 + X_4)] = E[S_3^2] + E[S_3X_4]$ $\quad$ Note $S_3 \perp\!\!\!\perp X_4$ thus $E[S_3X_4] = E[S_3]E[X_4]$

$E[X_4] = \frac{1}{2}$, $E[S_3] = E[\sum_{i=1}^{3} X_i] = 3\left(\frac{1}{2}\right) = \frac{3}{2} \Rightarrow E[S_3X_4] = \frac{3}{4}$

$E[S_3^2]$ if $S_3 = X_1 + X_2 + X_3 \sim Gamma(3, \beta = \frac{1}{2}) = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$

Thus, $E[S_3^2] + E[S_3X_4] = \frac{9}{4} + \frac{3}{4} = \boxed{3}$

Problem 1.4 Ben, Max, and Yolanda are in the front of three separate lines in the cafeteria. They follow independent Poisson processes with parameters $\lambda = 1, 2, 3$, respectively.

a) Find the prob. that Yolanda is served first

Denote the processes $\{S_{B_i}\}$, $\{S_{M_i}\}$, and $\{S_{Y_i}\}$ the waiting times for the ith occurrence of their respective processes. We want $P(\min\{S_{B_1}, S_{M_1}, S_{Y_1}\} = S_{Y_1}) = P$. Note that this is equal to $P(\min\{X_{B_1}, X_{M_1}, X_{Y_1}\} = X_{Y_1})$

Recall from lecture 2, $P(\min\{X_{B_1}, X_{M_1}, X_{Y_1}\} = X_{Y_1}) = \frac{\lambda_Y}{\lambda_Y + \lambda_M + \lambda_B}$, thus, $P = \frac{3}{1+2+3} = \boxed{\frac{1}{2}}$

b) Find the prob. that Ben is served before Yolanda

$P(\min(\{X_{B_1}, X_{Y_1}\}) = X_{B_1}) = \frac{\lambda_B}{\lambda_B + \lambda_Y} = \boxed{\frac{1}{4} = 0.25}$

c) Find the expected waiting time for the first person to be served

$E[\min(\{X_{B_1}, X_{M_1}, X_{Y_1}\})] = E[X_{(1)}]$, $X_{(1)} \sim \exp(\lambda_B + \lambda_M + \lambda_Y)$

$\sim \exp(\lambda = 6)$

if $X_{(1)} \sim \exp(\lambda = 6)$, then $E[X_{(1)}] = \boxed{1/6}$

Problem 1.5 For a Pois. process with param. $\lambda > 0$, show that for $0 \le s < t$, the correlation between N_s, N_t is

$$\mathrm{Corr}(N_s, N_t) = \sqrt{\frac{s}{t}}$$

$\mathrm{Corr}(N_s, N_t) = \frac{\mathrm{Cov}(N_s, N_t)}{SD(N_s)SD(N_t)} = \frac{\mathrm{Cov}(N_s, N_t)}{\sqrt{\lambda^2 s \cdot t}} = \frac{\mathrm{Cov}(N_s, N_t)}{\lambda\sqrt{s}\sqrt{t}}$

$\mathrm{Cov}(N_s, N_t) = E[N_s N_t] - E[N_s]E[N_t] = E[N_s N_t] - \lambda^2 \cdot s \cdot t$

$E[N_s N_t] = E[N_s(N_t - N_s + N_s)] = E[N_s(N_t - N_s) + N_s^2] = E[N_s(N_t - N_s)] + E[N_s^2]$

$s < t \Rightarrow N_t - N_s \perp N_s \Rightarrow \mathrm{Cov}(N_t - N_s, N_s) = 0 \Rightarrow E[(N_t - N_s)N_s] = E[N_t - N_s]E[N_s]$

$N_t - N_s \sim \mathrm{Pois}((t-s)\lambda)$ and $N_s \sim \mathrm{Pois}(s\lambda) \Rightarrow E[N_t - N_s]E[N_s] = \lambda^2(s)(t-s)$

$E[N_s^2] = \mathrm{Var}(N_s) + (E[N_s])^2 = \lambda s(1 + \lambda s)$

Thus, $E[N_s N_t] = \lambda^2 s(t-s) + \lambda s(1+\lambda s) = \lambda^2 st - \lambda^2 s^2 + \lambda s + \lambda^2 s^2 = \lambda^2 st + \lambda s$

and $\mathrm{Cov}(N_s, N_t) = \lambda^2 st + \lambda s - \lambda^2 st = \lambda s$

Finally $\mathrm{Corr}(N_s, N_t) = \frac{\lambda s}{\lambda\sqrt{s}\sqrt{t}} = \frac{\sqrt{s}}{\sqrt{t}} = \sqrt{\frac{s}{t}}$