Formal Geometry and Bordism Operations

Lecture notes

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Class information

Course ID: MATH 278 (159627).

Meeting times: Spring 2016, MWF 12pm–1pm.

Goals: The primary goal of this class is to teach students to view results in algebraic topology through the lens of (formal) algebraic geometry.

Grading: This class won't have any official assignments. I'll give references as readings for those who would like a deeper understanding, though I'll do my best to ensure that no extra reading is required to follow the arc of the class.

I do want to assemble course notes from this class, but it's unlikely that I will have time to type *all* of them up. Instead, I would like to "crowdsource" this somewhat: I'll type up skeletal notes for each lecture, and then we as a class will try to flesh them out as the semester progresses. As incentive to help, those who contribute to the document will have their name included in the acknowledgements, and those who contribute *substantially* will have their name added as a coauthor. Everyone could use more CV items. (Publication may take a while. I suspect the course won't run perfectly smoothly the first time, so this may takes a second semester pass to become fully workable. But, since topics courses only come around once in a while, this will necessarily mean a delay.)

The source for this document can be found at

`https://github.com/ecpeterson/FormalGeomNotes`.

If you're taking the class or otherwise want to contribute, you can write me at

`ecp@math.harvard.edu`

to request write access.

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Hood Chatham and Geoffrey Lee, Catherine Ray, Kevin Wray—Berkeley students who I talked to a lot

Various unpublished sources: COCTALOS, the Crystals notes, Charles's 512 notes, Charles's *Notes on the Hopkins–Miller theorem*, Jacob's lecture notes, Haynes's notes on cobordism, Neil's *Functorial Philosophy*, ...

Contributors: Eva Belmont, Hood Chatham (spectral sequence package), Jun Hou Fung, Jeremy Hahn (Yuli Rudyak's alternative proof, lots of discussion about Div_n^+, the AHR appendix, Tate construction), Mauro Porta, Krishanu Sankar, Danny Shi, Allen Yuan (the transfer, AHR appendix)

Regular participants (whoever I could think of on March 24, again missing people): Eva Belmont, Lukas Brantner, Christian Carrick, Hood Chatham, Jun Hou Fung, Meng Guo, Jeremy Hahn, Erick Knight, Jake McNamara, Akhil Mathew, Krishanu Sankar, Jay Shah, Danny Shi, Allen Yuan

Course registrants (collected March 24, possibly missing some MIT people): Colin Aitken, Adam Al-Natsheh, Eva Belmont, Jason Bland, Dorin Boger, Lukas Brantner, Christian Carrick, Hood Chatham, David Corwin, Jun Hou Fung, Meng Guo, Jeremy Hahn, Changho Han, Chi-Yun Hsu, Erick Knight, Benjamin Landon, Gabriel Long, Yusheng Luo, Jake Marcinek, Jake McNamara, Max Menzies, Morgan Opie, Alexander Perry, Mauro Porta, Kishanu Sankar, Jay Shah, Ananth Shankar, Danny Shi, Koji Shimizu, Geoffrey Smith, Hunter Spinik, Philip Tynan, Yi Xie, David Yang, Zijian Yao, Lynnelle Ye, Chenglong Yu, Allen Yuan, Adrian Zahariuc, Yifei Zhao, Rong Zhou, Yihang Zhu.

People who contributed to the GitHub repository (to be collected later, make sure there's no overlap with the above list):

Other readers: Jon Beardsley, Ben Gadoua, Denis Nardin, Sune Precht Reeh, Andrew Senger ("stalkwise" remarks), Kevin Wray

Contents

I would like as few section titles as possible to involve people's names.

A bunch of broken displaymode tombstones can have their positions fixed by using http://tex.stackexchange.com/a/66221/2671.

Compile an index by replacing all the `textit` commands in definition environments with some more fancy macro that tags it for inclusion. There's information on how indices are compiled here: https://en.wikibooks.org/wiki/LaTeX/Indexing.

Remember to use $f \colon A \to B$ everywhere.

Should sections have subsections? Does that help organize the TOC?

Hood wrote a macro called sumfgl (see also sumF and sumG) that will make a bunch of formal group law expressions typeset better. Propagate that change through. Consider also the adjunct macro in the preamble. (Jay says that left-adjoints should be on top in general.)

Make sure you use either id or 1 everywhere to denote the identity morphism.

You're not very consistent about using $\widehat{\mathbb{G}}$ or Γ to denote an arbitrary formal group. It seems like you use one or the other based on your preference of whether it has finite height or not.

Eliminate contractions.

Eliminate filler words like "things".

Make sure that "Case Study" and "Lecture" are OK names by the publisher's standards. If they aren't, do a careful search-and-replace for them.

Double check that you're careful about choosing consistent names for your objects: $S = \operatorname{Spec} R$ is the base scheme, that sort of thing.

Be consistent about $\mathcal{O}_X$ vs $\mathcal{O}(X)$, and similarly with $\mathcal{I}_D$ versus $\mathcal{I}(D)$.

Be consistent about "S" versus "S^0" for the standard sphere spectrum.

The AGT style guide says that "—" should not have spaces on either side.

We should use the convention that $\varphi \in \mathcal{M}_{\mathbf{fgl}}(T)$ corresponds to $x +_\varphi y \in T[\![x, y]\!]$ throughout.

Jon suggested that we include backreferences from the bibliography, using the package `backref`. More information on getting this to work well is at this tex.se url: http://tex.stackexchange.com/questions/54541/precise-back-reference-target-with-hyperref-and-backref .

Use Hood's HFp macro where appropriate. (Maybe modify it also to do HF2 appropriately.)

Encourage spectral sequence figures to occur on facing pages.

Clark's use of MinionPro font is really nice (cf. `https://dl.dropboxusercontent.com/u/1741495/teaching/2017spring.18.917/gamma-blurb.pdf`). Generally, we should make an effort to pick a nice font. http://web.willbenton.com/writing/2008/better-latex has more info about converting FF fonts for use in LaTeX, including Minion.

Use booktabs (http://www.ctan.org/tex-archive/macros/latex/contrib/booktabs/) to typeset any tables appearing in this book. (I don't think there are very many.)

In several places you abuse the `align*` environment to put references into a right-hand column. Is there a less hack-y way to do this?

Chapter 0

Introduction

The goal of this book is to communicate a certain *weltanschauung* uncovered in pieces by many different people working in bordism theory, and the goal just for this introduction is to tell a story about one theorem where it is especially apparent.

To begin, we will define a homology theory called "bordism homology". Recall that the singular homology of a space X comes about by probing X with simplices: beginning with the collection of continuous maps $\sigma\colon \Delta^n \to X$, we take the free $\mathbb{Z}$–module on each of these sets and construct a chain complex

$$\cdots \xrightarrow{\partial} \mathbb{Z}\{\Delta^n \to X\} \xrightarrow{\partial} \mathbb{Z}\{\Delta^{n-1} \to X\} \xrightarrow{\partial} \cdots.$$

Bordism homology is constructed analogously, but using manifolds Z as the probes instead of simplices:[1]

$$\begin{aligned} \cdots &\xrightarrow{\partial} \{Z^n \to X \mid Z^n \text{ a compact } n\text{–manifold}\} \\ &\xrightarrow{\partial} \{Z^{n-1} \to X \mid Z^{n-1} \text{ a compact } (n-1)\text{–manifold}\} \\ &\xrightarrow{\partial} \cdots. \end{aligned}$$

Lemma 0.0.1 ([Koc78, Section 4]). *This forms a chain complex of monoids under disjoint union of manifolds, and its homology is written $MO_*(X)$. These are naturally abelian groups, and moreover they satisfy the axioms of a generalized homology theory.* □

In fact, we can define a bordism theory MG for any suitable family of structure groups $G(n) \to O(n)$. The coefficient ring of MG, or its value $MG_*(*)$ on a point, gives the ring of G–bordism classes, and generally $MG_*(Y)$ gives a kind of "bordism in families over the space Y". There are comparison morphisms for the most ordinary kinds of bordism, given by replacing a chain of manifolds with an equivalent simplicial chain:

$$MO \to H\mathbb{Z}/2, \qquad\qquad MSO \to H\mathbb{Z}.$$

[1] One doesn't need to take the free abelian group on anything, since the disjoint union of two manifolds is already a (disconnected) manifold, whereas the disjoint union of two simplices is not a simplex.

In both cases, we can evaluate on a point to get ring maps, called "genera":

$$MO_*(*) \to \mathbb{Z}/2, \qquad MSO_*(*) \to \mathbb{Z},$$

neither of which is very interesting, since they're both zero in positive degrees.

However, having maps of homology theories (rather than just maps of coefficient rings) is considerably more data then just the genus. For instance, we can use it to extract a theory of integration as follows. Consider the following special case of oriented bordism, where we evaluate MSO_* on an infinite loopspace:

$$\begin{aligned} MSO_n K(\mathbb{Z}, n) &= \{\text{oriented } n\text{–manifolds mapping to } K(\mathbb{Z}, n)\} \,/ \sim \\ &= \left\{ \begin{array}{c} \text{oriented } n\text{–manifolds } Z \\ \text{with a specified class } \omega \in H^n(Z; \mathbb{Z}) \end{array} \right\} \Big/ \sim . \end{aligned}$$

Associated to such a representative (Z, ω), the yoga of stable homotopy theory then allows us to build a composite

$$\begin{aligned} \mathbb{S} &\xrightarrow{(Z,\omega)} MSO \wedge (\mathbb{S}^{-n} \wedge \Sigma^\infty_+ K(\mathbb{Z}, n)) \\ &\xrightarrow{\text{colim}} MSO \wedge H\mathbb{Z} \\ &\xrightarrow{\varphi \wedge 1} H\mathbb{Z} \wedge H\mathbb{Z} \\ &\xrightarrow{\mu} H\mathbb{Z}, \end{aligned}$$

where φ is the orientation map. Altogether, this composite gives us an element of $\pi_0 H\mathbb{Z}$, i.e., an integer.

I used to think that we got a generalized Stokes's theorem too, but now I'm not sure. Stokes's theorem is the statement that the chain and cochain differentials are adjoint: $\langle d\omega, \sigma \rangle = \langle \omega, \partial\sigma \rangle$, where the pairing is the integration pairing. It would be neat to interpret this in generality, but it might be a stretch.

Lemma 0.0.2. *The integer obtained by the above process is $\int_Z \omega$.* $\square$

Cite me: Where is this proven?.

This definition of $\int_Z \omega$ via stable homotopy theory is pretty nice, in the sense that many theorems accompany it for free. It is also very general: given a ring map off of any bordism spectrum, a similar sequence of steps will furnish us with an integral tailored to that situation.

Now take $G = e$ to be the trivial structure group, which is the bordism theory of framed manifolds, i.e., those with trivialized tangent bundle. In this case, the Pontryagin–Thom construction gives an equivalence $\mathbb{S} \xrightarrow{\simeq} Me$. It is thus possible (and some people have indeed taken up this viewpoint) that stable homotopy theory can be done solely through the lens of framed bordism. We will prefer to view this the other way: the sphere spectrum $\mathbb{S}$ often appears to us as a natural object, and we will occasionally replace it by Me, the framed bordism spectrum. For example, given a ring spectrum E with unit map $\mathbb{S} \to E$, we can reconsider this as a ring map

$$Me \xrightarrow{\simeq} \mathbb{S} \to E.$$

Following along the lines of the previous paragraph, we learn that any ring spectrum E is automatically equipped with a theory of integration for framed manifolds.

Sometimes, as in the examples above, this unit map factors:

$$\mathbb{S} \simeq Me \to MO \to H\mathbb{Z}/2.$$

This is a witness to the overdeterminacy of $H\mathbb{Z}/2$'s integral for framed bordism: if the framed manifold is pushed all the way down to an unoriented manifold, there is still enough residual data to define the integral.[2] Given any ring spectrum E, we can ask the analogous question: If we filter O by a system of structure groups, through what stage does the unit map $Me \to E$ factor? For instance, the map

$$\mathbb{S} = Me \to MSO \to H\mathbb{Z}$$

considered above does *not* factor further through MO—an orientation is *required* to define the integral of an integer–valued cohomology class. Recognizing $SO \to O$ as the 0^{th} Postnikov–Whitehead truncation of O, we are inspired to use the rest of the Postnikov filtration as our filtration of structure groups. Here is a diagram of this filtration and some interesting minimally-factored integration theories related to it:

$$\begin{tikzcd}
Me \arrow[r] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr] & \cdots \arrow[r] & MString \arrow[r] & MSpin \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\
 & & & kO & H\mathbb{Z} & H\mathbb{Z}/2.
\end{tikzcd}$$

This is the situation homotopy theorists found themselves in some decades ago, when Ochanine and Witten proved the following mysterious theorem using analytical and physical methods:

Theorem 0.0.3 (Ochanine [Och87], Witten [Wit87, Wit88])**.** *There is a map of rings*

$$\sigma : MSpin_* \to \mathbb{C}((q)).$$

Moreover, if Z is a Spin manifold such that twice its first Pontryagin class vanishes—that is, if Z lifts to a String–manifold—then $\sigma(Z)$ lands in the subring $MF \subseteq \mathbb{Z}[\![q]\!]$ of modular forms with integral coefficients. □

However, neither party gave indication that their result should be valid "in families" (in our sense), and no theory of integration was formally produced (in our sense). From the perspective of the homotopy theorist, it wasn't even totally clear what such a claim would mean: to give a topological enrichment of these theorems would mean finding a ring spectrum E such that $E_*(*)$ had something to do with modular forms.

[2] It's literally more information than this: even unframeable unoriented manifolds acquire a compatible integral.

Around the same time, Landweber, Ravenel, and Stong began studying "elliptic cohomology" for independent reasons; sometime much earlier, Morava had constructed an object "K^{Tate}" associated to the Tate elliptic curve; and a decade later Ando, Hopkins, and Strickland put all these things together in the following theorem:

Cite me: Landweber–Ravenel–Stong, Morava's *Forms of K–theory*, and Ando–Hopkins–Strickland.

Theorem 0.0.4 (Ando–Hopkins–Strickland)**.** *If E is an "elliptic cohomology theory", then there is a canonical map of homotopy ring spectra $MString \to E$ called the σ–orientation (for E). Additionally, there is an elliptic spectrum K^{Tate} whose σ–orientation gives Witten's genus $MString_* \to K^{\mathrm{Tate}}_*$.* □

We now come to the motivation for this class: the homotopical σ–orientation was actually first constructed using formal geometry. The original proof of Ando–Hopkins–Strickland begins with a reduction to maps of the form

$$MU[6,\infty) \to E.$$

They then work to show that in especially good cases they can complete the missing arrow in the diagram

$$\begin{tikzcd} MU[6,\infty) \arrow[r] \arrow[dr] & MString \arrow[d, dotted] \\ & E. \end{tikzcd}$$

Leaving aside the extension problem for the moment, their main theorem is the following description of the cohomology ring $E^*MU[6,\infty)$:

Theorem 0.0.5 (Ando–Hopkins–Strickland [AHS01], cf. Singer [Sin68] and Stong [Sto63])**.** *For E an even–periodic cohomology theory,*

$$\operatorname{Spec} E_*MU[6,\infty) \cong C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0)),$$

where "$C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0))$" is a certain scheme. When E is taken to be elliptic, so that there is a specified isomorphism $\widehat{\mathbb{G}}_E \cong C_0^\wedge$ for C an elliptic curve, the theory of elliptic curves furnishes the scheme with a canonical point. Hence, there is a preferred class $MU[6,\infty) \to E$, natural in the choice of elliptic E. □

Our real goal is to understand theorems like this last one, where algebraic geometry asserts some real control over something squarely in the domain of homotopy theory.

The structure of the class will be to work through a sequence of case studies where this perspective shines through most brightly. We'll start by working through Thom's calculation of the homotopy of MO, which holds the simultaneous attractive features of being approachable while revealing essentially all of the structural complexity of the general situation, so that we know what to expect later on. Having seen that through to the end, we'll then venture on to other examples: the complex bordism ring, structure theorems for finite spectra, unstable cooperations, and, finally, the σ–orientation and its extensions. The overriding theme of the class will be that algebraic geometry is a good organizing principle that gives us one avenue of insight into how homotopy theory functions. In particular, it allows us to organize "operations" of various sorts between spectra derived from bordism theories.

We should also mention that we will specifically *not* discuss the following aspects of this story:

- Analytic techniques will be completely omitted. Much of modern research stemming from the above problem is an attempt to extend index theory across Witten's genus, or to find a "geometric cochains" model of certain elliptic cohomology theories. These often mean heavy analytic work, and we will strictly confine ourselves to the domain of homotopy theory.

- As sort of a sub-point (and despite the motivation provided in this Introduction), we will also mostly avoid manifold geometry. Again, much of the contemporary research about *tmf* is an attempt to find a geometric model, so that geometric techniques can be imported—including equivariance and the geometry of quantum field theories, to name two.

- In a different direction, our focus will not linger on actually computing bordism rings MX_*, nor will we consider geometric constructions on manifolds and their behavior after imaging into the bordism ring. This is also the source of active research: the structure of the symplectic bordism ring remains, to large extent, mysterious, and what we do understand of it comes through a mix of formal geometry and raw manifold geometry. This could be a topic that fits logically into this document, were it not for time limitations and the author's inexpertise.

- The geometry of E_∞ rings will also be avoided, at least to the extent possible. Such objects become inescapable by the conclusion of our story, but there are better resources from which to learn about E_∞ rings, and the pre–E_∞ story is not told so often these days. So, we will focus on the unstructured part and leave E_∞ rings to other authors.

- There will be plenty of places where we will avoid stating things in maximum generality or with maximum thoroughness. The story we are interested in telling draws from a blend of many others from different subfields of mathematics, many of which have their own topics courses. Sometimes this will mean avoiding stating the most beautiful theorem in a subfield in favor of a theorem we will find more useful. Other times this will mean abbreviating someone else's general definition to one more specialized to the task at hand. In any case, we will give references to other sources where you can find these things cast in starring roles.

Finally, we must mention that there are several good companions to these notes. Essentially none of the material here is original—it's almost all cribbed either from published or unpublished sources—but the source documents are quite scattered and individually dense. We will make a point to cite useful references as we go. One document stands out above all others, though: Neil Strickland's *Functorial Philosophy for Formal Phenomena* [Str]. These lecture notes can basically be viewed as an attempt to make it through this paper in the span of a semester.

0.1 Conventions

Throughout this book, we use the following conventions:

- $\mathsf{C}(X, Y)$ will denote the mapping object of arrows $X \to Y$ in a category C. If C is an ∞–category, this will often be interpreted as a mapping *space*. If C has a self-enrichment, we will often write $\underline{\mathsf{C}}(X, Y)$ (or, e.g., $\underline{\mathsf{Aut}}(X)$) to distinguish the internal mapping object from $\mathsf{C}(X, Y)$ the classical mapping set. As an exception to this uniform notation, we will sometimes abbreviate $\underline{\mathsf{Spaces}}(X, Y)$ to $F(X, Y)$, and similarly we will sometimes abbreviate $\underline{\mathsf{Spectra}}(X, Y)$ to $F(X, Y)$.

- Following Lurie, for an object $X \in \mathsf{C}$ we will write $\mathsf{C}_{/X}$ for the slice category of objects *over* X and $\mathsf{C}_{X/}$ for the slice category of objects *under* X.

- For a ring spectrum E, we will write $E_* = \pi_* E$ for its coefficient ring, $E^* = \pi_{-*} E$ for its coefficient ring with the opposite grading, and $E_0 = E^0 = \pi_0 E$ for the 0^{th} degree component of its coefficient ring. In particular, this allows us to make sense of expressions like "$E^*[\![x]\!]$", which we interpret as

$$E^*[\![x]\!] = (E^*)[\![x]\!] = (\pi_{-*}E)[\![x]\!].$$

- For a space or spectrum, we will write $X[n, \infty) \to X$ for the upward n^{th} Postnikov truncation over X and $X \to X(-\infty, n)$ for the downward n^{th} Postnikov truncation under X. There is thus a natural fiber sequence

$$X[n, \infty) \to X \to X(-\infty, n).$$

  This notation extends naturally to objects like $X(a, b)$ or $X[a, b]$, where $-\infty \leq a \leq b \leq \infty$ denote the (closed or open) endpoints of any interval.

This notation is visible in Greenlees–May, but probably doesn't originate there.

Case Study 1

Unoriented bordism

This Case Study culminates in the calculation of $MO_*(*)$, the bordism ring of unoriented manifolds, but we mainly take this as an opportunity to introduce several key concepts that will serve us throughout the book. First and foremost, we will require a definition of bordism spectrum that we can manipulate computationally, using just the tools of abstract homotopy theory. Once that is established, we immediately begin to bring algebraic geometry into the mix: the main idea is that the cohomology ring of a space is better viewed as a scheme (with plenty of extra structure), and the homology groups of a spectrum are better viewed as representation for a certain elaborate algebraic group. This data actually finds familiar expression in homotopy theory: we show that a form of group cohomology for this representation forms the input to the classical Adams spectral sequence. Finally, we calculate this representation structure for $H\mathbb{F}_{2*}MO$, find that it is suitably free, and thereby gain control of the Adams spectral sequence computing $MO_*(*)$.

Thread Crefs to the relevant theorems below through this introduction.

1.1 Thom spectra and the Thom isomorphism

Our goal is a sequence of theorems about the unoriented bordism spectrum MO. We will begin by recalling a definition of the spectrum MO using just abstract homotopy theory, because it involves ideas that will be useful to us throughout the semester and because we cannot compute effectively with the chain-level definition given in the Introduction.

Definition 1.1.1. For a spherical bundle $S^{n-1} \to \xi \to X$, its Thom space is given by the cofiber

$$\xi \to X \xrightarrow{\text{cofiber}} T_n(\xi).$$

"Proof" of definition. There is a more classical construction of the Thom space: take the associated disk bundle by gluing an n–disk fiberwise, and add a point at infinity by collapsing ξ:

$$T_n(\xi) = (\xi \sqcup'_{S^{n-1}} D^n)^+.$$

To compare this with the cofiber definition, recall that the thickening of ξ to an n–disk bundle is the same thing as taking the mapping cylinder on $\xi \to X$. Since the inclusion into the mapping cylinder is now a cofibration, the quotient by this subspace agrees with both the cofiber of the map and the introduction of a point at infinity. $\square$

Before proceeding, here are two important examples:

Example 1.1.2. If $\xi = S^{n-1} \times X$ is the trivial bundle, then $T_n(\xi) = S^n \wedge (X_+)$. This is supposed to indicate what Thom spaces are "doing": if you feed in the trivial bundle then you get the suspension out, so if you feed in a twisted bundle you should think of it as a *twisted suspension*.

Example 1.1.3. Let ξ be the tautological S^0–bundle over $\mathbb{RP}^\infty = BO(1)$. Because ξ has contractible total space, $EO(1)$, the cofiber degenerates and it follows that $T_1(\xi) = \mathbb{RP}^\infty$. More generally, arguing by cells shows that the Thom space for the tautological bundle over $\mathbb{RP}^n$ is $\mathbb{RP}^{n+1}$.

Cite me: Give a reference for this general construction of classifying spaces for fibrations..

Now we catalog a bunch of useful properties of the Thom space functor. Firstly, recall that a spherical bundle over X is the same data as a map $X \to BGL_1S^{n-1}$, where GL_1S^{n-1} is the subspace of $F(S^{n-1}, S^{n-1})$ expressed by the pullback of spaces

$$\begin{tikzcd}
GL_1S^{n-1} \arrow[r] \arrow[d] & \mathsf{Spaces}(S^{n-1}, S^{n-1}) \arrow[d] & \\
h\mathsf{Spaces}(S^{n-1}, S^{n-1})^\times \arrow[r] & h\mathsf{Spaces}(S^{n-1}, S^{n-1}) \arrow[r, equal] & \pi_0\mathsf{Spaces}(S^{n-1}, S^{n-1}).
\end{tikzcd}$$

Lemma 1.1.4. *The construction T_n can be viewed as a functor from the slice category over BGL_1S^{n-1} to* Spaces. *Maps of slices*

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "f^*\xi"'] & & X \arrow[dl, "\xi"] \\
& BGL_1S^{n-1} &
\end{tikzcd}$$

induce maps $T_n(f^*\xi) \to T_n(\xi)$, and T_n is suitably homotopy-invariant. $\square$

Next, the spherical subbundle of a vector bundle gives a common source of spherical bundles. The action of $O(n)$ on $\mathbb{R}^n$ preserves the unit sphere, and hence gives a map $O(n) \to GL_1S^{n-1}$. These are maps of topological groups, and the block–inclusion maps $i^n \colon O(n) \to O(n+1)$ commute with the suspension map $GL_1S^{n-1} \to GL_1S^n$. In fact, much more can be said:

Lemma 1.1.5. *The block–sum maps $O(n) \times O(m) \to O(n+m)$ are compatible with the join maps $GL_1S^{n-1} \times GL_1S^{m-1} \to GL_1S^{n+m-1}$.* $\square$

Again taking a cue from K–theory, we take the colimit as n grows large, using the maps

$$\begin{tikzcd} BGL_1 S^{n-1} \arrow[r, equal] \arrow[d] & BGL_1 S^{n-1} \times * \arrow[r, "\mathrm{id} \times \mathrm{triv}"] \arrow[d] & BGL_1 S^{n-1} \times BGL_1 S^0 \arrow[r, "*"] \arrow[d] & BGL_1 S^n, \arrow[d] \\ BO(n) \arrow[r, equal] & BO(n) \times * \arrow[r, "\mathrm{id} \times \mathrm{triv}"] & BO(n) \times BO(1) \arrow[r, "\oplus"] & BO(n+1). \end{tikzcd}$$

Corollary 1.1.6. *The operations of block–sum and topological join imbue the colimiting spaces BO and $BGL_1\mathbb{S}$ with the structure of H–groups. Moreover, the colimiting map*

Should you justify "group" rather than "space"?

$$J_{\mathbb{R}} \colon BO \to BGL_1\mathbb{S},$$

called the stable J–homomorphism, is a morphism of H–groups. □

Finally, we can ask about the compatibility of Thom constructions with all of this. In order to properly phrase the question, we need a version of the construction which operates on stable spherical bundles, i.e., whose source is the slice category over $BGL_1\mathbb{S}$. By calculating

Does this calculation need justification?

$$T_{n+1}(\xi * \mathrm{triv}) \simeq \Sigma T_n(\xi),$$

we are inspired to make the following definition:

Definition 1.1.7. For ξ an S^{n-1}–bundle, we define the *Thom spectrum* of ξ to be

$$T(\xi) := \Sigma^{-n}\Sigma^{\infty} T_n(\xi).$$

By filtering the base space by compact subspaces, this begets a functor

$$T \colon \mathsf{Spaces}_{/BGL_1\mathbb{S}} \to \mathsf{Spectra}.$$

Lemma 1.1.8. *T is monoidal: it carries external fiberwise joins to smash products of Thom spectra. Correspondingly, $T \circ J_{\mathbb{R}}$ carries external direct sums of stable vector bundles to smash products of Thom spectra.* □

Definition 1.1.9. The spectrum MO arises as the universal example of all these constructions, strung together:

$$MO := T(J_{\mathbb{R}}) = \operatorname*{colim}_n T(J^n_{\mathbb{R}}) = \operatorname*{colim}_n \Sigma^{-n} T_n J^n_{\mathbb{R}}.$$

Cite me: There should be a reference here (to Pontryagin, presumably) saying that we recover MO as defined on the first day..

The spectrum MO has several remarkable properties. The most basic such property is that it is a ring spectrum, and this follows immediately from $J_{\mathbb{R}}$ being a homomorphism of H–spaces. Much more excitingly, we can also deduce the presence of Thom isomorphisms just from the properties stated thus far. That $J_{\mathbb{R}}$ is a homomorphism means that the following square commutes:

$$\begin{tikzcd}
BO \times BO \arrow[r, "\sigma", "\simeq"'] \arrow[drr, bend right] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}} \times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
& BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}$$

σ almost shows up in giving a categorical definition of a G–torsor. I wish I understood this, but I always get tangled up.

We have extended this square very slightly by a certain shearing map σ defined by $\sigma(x, y) = (xy^{-1}, y)$. It is evident that σ is a homotopy equivalence, since just as we can de-scale the first coordinate by y we can re-scale by it. We can calculate directly the behavior of the long composite:

$$J_{\mathbb{R}} \circ \mu \circ \sigma(x, y) = J_{\mathbb{R}} \circ \mu(xy^{-1}, y) = J_{\mathbb{R}}(xy^{-1}y) = J_{\mathbb{R}}(x).$$

It follows that the second coordinate plays no role, and that the bundle classified by the long composite can be written as $J_{\mathbb{R}} \times 0$.[1] We are now in a position to see the Thom isomorphism:

Lemma 1.1.10 (Thom isomorphism, universal example)**.** *As MO–modules,*

$$MO \wedge MO \simeq MO \wedge \Sigma^\infty_+ BO.$$

Proof. Stringing together the naturality properties of the Thom functor outlined above, we can thus make the following calculation:

$$\begin{aligned}
T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}})) &\simeq T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}}) \circ \sigma) && \text{(homotopy invariance)} \\
&\simeq T(J_{\mathbb{R}} \times 0) && \text{(constructed lift)} \\
&\simeq T(J_{\mathbb{R}}) \wedge T(0) && \text{(monoidality)} \\
&\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(Example 1.1.2)} \\
T(J_{\mathbb{R}}) \wedge T(J_{\mathbb{R}}) &\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(monoidality)} \\
MO \wedge MO &\simeq MO \wedge \Sigma^\infty_+ BO. && \text{(definition of } MO\text{)}
\end{aligned}$$

The equivalence is one of MO–modules because the MO–module structure of both sides comes from smashing with MO on the left. □

From here, the general version of Thom's theorem follows quickly:

Definition 1.1.11. A map $\varphi\colon MO \to E$ of homotopy ring spectra is called an *orientation* of E (by MO).[2]

[1] This factorization does *not* commute with the rest of the diagram, just with the little lifting triangle it forms.

[2] Later, we will refer to analogous ring spectrum maps $MU \to E$ off of the complex bordism spectrum as *complex-orientations* of E. However, calling ring maps $MO \to E$ "unoriented-orientations" is rightfully considered distasteful.

Theorem 1.1.12 (Thom isomorphism)**.** *Let $\xi\colon X \to BO$ classify a vector bundle and let $\varphi\colon MO \to E$ be a map of ring spectra. Then there is an equivalence of E–modules*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X.$$

Modifications to above proof. To accommodate X rather than BO as the base, we redefine $\sigma\colon BO \times X \to BO \times X$ by

$$\sigma(x, y) = \sigma(x\xi(y)^{-1}, y).$$

Follow the same proof as before with the diagram

$$\begin{tikzcd}
BO \times X \arrow[r, "\sigma", "\simeq"'] \arrow[drrr, bend right] & BO \times X \arrow[r, "\mathrm{id}\times\xi", "\simeq"'] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}}\times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
 & & BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}$$

This gives an equivalence $\theta_{MO}\colon MO \wedge T(\xi) \to MO \wedge \Sigma^\infty_+ X$. To introduce E, note that there is a diagram

$$\begin{tikzcd}
E \wedge T(\xi) \arrow[d, "\eta_{MO}\wedge\mathrm{id}\wedge\mathrm{id}=f"] & E \wedge \Sigma^\infty_+ X \arrow[d, "\eta_{MO}\wedge\mathrm{id}\wedge\mathrm{id}"] \\
MO \wedge E \wedge T(\xi) \arrow[r, "\theta_{MO}\wedge E"] \arrow[d, "(\mu\circ(\varphi\wedge\mathrm{id}))\wedge\mathrm{id}=g"] & MO \wedge E \wedge \Sigma^\infty_+ X \arrow[d, "(\mu\circ(\varphi\wedge\mathrm{id}))\wedge\mathrm{id}=h"] \\
E \wedge T(\xi) \arrow[r, "\theta_E"] & E \wedge \Sigma^\infty_+ X
\end{tikzcd}$$

The bottom arrow θ_E exists by applying the action map to both sides and pushing the map $\theta_{MO} \wedge E$ down. Since θ_{MO} is an equivalence, it has an inverse α_{MO}. Therefore, the middle map has inverse $\alpha_{MO} \wedge E$, and we can similarly push this down to a map α_E, which we now want to show is the inverse to θ_E. From here it is a simple diagram chase: we have renamed three of the maps in the diagram to f, g, and h for brevity. Noting that $g \circ f$ is the identity map because of the unit axiom, we conclude

$$\begin{aligned}
g \circ f &\simeq g \circ (\alpha_{MO} \wedge E) \circ (\theta_{MO} \wedge E) \circ f \\
&\simeq \alpha_E \circ h \circ (\theta_{MO} \wedge E) \circ f && \text{(action map)} \\
&\simeq \alpha_E \circ \theta_E \circ g \circ f && \text{(action map)} \\
&\simeq \alpha_E \circ \theta_E.
\end{aligned}$$

It follows that α_E gives an inverse to θ_E. □

Remark 1.1.13. One of the tentpoles of the theory of Thom spectra is that Theorem 1.1.12 has a kind of converse: if a ring spectrum E has suitably natural and multiplicative Thom isomorphisms for Thom spectra formed from real vector bundles, then one can define an essentially unique ring map $MO \to E$ realizing these isomorphisms via the machinery of Theorem 1.1.12.

Have you really proved this? It should be easy, but I don't think "this just works" alone suffices.

Remark 1.1.14. There is also a cohomological version of the Thom isomorphism. Suppose that E is a ring spectrum under MO and let ξ be the spherical bundle of a vector bundle on a space X. The spectrum $F(\Sigma^\infty_+ X, E)$ is a ring spectrum under E (hence under MO), so there is a Thom isomorphism as well as an evaluation map

$$F(\Sigma^\infty_+ X, E) \wedge T(\xi) \xrightarrow{\simeq} F(\Sigma^\infty_+ X, E) \wedge \Sigma^\infty_+ X \xrightarrow{\text{eval}} E.$$

Passing through the exponential adjunction, the map

$$F(\Sigma^\infty_+ X, E) \xrightarrow{\simeq} F(T(\xi), E)$$

gives the cohomological Thom isomorphism

$$E^* X \cong E^* T(\xi).$$

Example 1.1.15. We will close out this section by using this to actually make a calculation. Recall from Example 1.1.3 that $T(\mathcal{L} \downarrow \mathbb{RP}^n) = \mathbb{RP}^{n+1}$. Because MO is a connective spectrum, the truncation map

$$MO \to MO(-\infty, 0] = H\pi_0 MO = H\mathbb{F}_2$$

Cite me: Find a reference for this, which I actually am not sure how to prove. Allen said Corollary 6.6 of https://arxiv.org/pdf/0806.3983v1.pdf is relevant.

is a map of ring spectra. Hence, we can apply the Thom isomorphism theorem to the mod–2 homology of Thom complexes coming from real vector bundles:

$$\begin{aligned} \pi_*(H\mathbb{F}_2 \wedge T(\mathcal{L} - 1)) &\cong \pi_*(H\mathbb{F}_2 \wedge T(0)) && \text{(Thom isomorphism)} \\ \pi_*(H\mathbb{F}_2 \wedge \Sigma^{-1}\Sigma^\infty \mathbb{RP}^{n+1}) &\cong \pi_*(H\mathbb{F}_2 \wedge \Sigma^\infty_+ \mathbb{RP}^n) && \text{(Example 1.1.3)} \\ \widetilde{H\mathbb{F}}_{2*+1} \mathbb{RP}^{n+1} &\cong H\mathbb{F}_{2*} \mathbb{RP}^n. && \text{(generalized homology)} \end{aligned}$$

This powers an induction that shows $H\mathbb{F}_{2*}\mathbb{RP}^\infty$ has a single class in every degree. The cohomological version of the Thom isomorphism in Remark 1.1.14, together with the $H\mathbb{F}_2^*\mathbb{RP}^n$–module structure of $H\mathbb{F}_2^* T(\mathcal{L} - 1)$, also gives the ring structure:

$$H\mathbb{F}_2^* \mathbb{RP}^n = \mathbb{F}_2[x]/x^{n+1}.$$

1.2 Cohomology rings and affine schemes

An abbreviated summary of this book is that we are going to put "Spec" in front of rings appearing in algebraic topology and see what happens. Before doing any algebraic topology, let me remind you what this means on the level of algebra. The core idea is to replace a ring R by the functor it corepresents, $\operatorname{Spec} R$. For any "test $\mathbb{F}_2$–algebra" T, we set

$$(\operatorname{Spec} R)(T) := \mathsf{Algebras}_{\mathbb{F}_2/}(R, T) \cong \mathsf{Schemes}_{/\mathbb{F}_2}(\operatorname{Spec} T, \operatorname{Spec} R).$$

More generally, we have the following definition:

Definition 1.2.1. An *affine* $\mathbb{F}_2$*–scheme* is a functor $X : \mathsf{Algebras}_{\mathbb{F}_2/} \to \mathsf{Sets}$ which is (non-canonically) isomorphic to $\operatorname{Spec} R$ for some $\mathbb{F}_2$–algebra R. Given such an isomorphism, we will refer to $\operatorname{Spec} R \to X$ as a *parameter* for X and its inverse $X \to \operatorname{Spec} R$ as a *coordinate* for X.

Lemma 1.2.2. *There is an equivalence of categories*

$$\operatorname{Spec} : \mathsf{Algebras}^{\mathrm{op}}_{\mathbb{F}_2/} \to \mathsf{AffineSchemes}_{/\mathbb{F}_2}. \quad \square$$

The centerpiece of thinking about rings in this way, for us and for now, is to translate between a presentation of R as a quotient of a free algebra and a presentation of $(\operatorname{Spec} R)(T)$ as selecting tuples of elements in T subject to certain conditions. Consider the following example:

Example 1.2.3. Set $R_1 = \mathbb{F}_2[x]$. Then

$$(\operatorname{Spec} R_1)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x], T)$$

is determined by where x is sent—i.e., this Hom–set is naturally isomorphic to T itself. Consider also what happens when we impose a relation by passing to $R_2 = \mathbb{F}_2[x]/(x^{n+1})$. The value

$$(\operatorname{Spec} R_2)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x]/(x^{n+1}), T)$$

of the associated affine scheme is again determined by where x is sent, but now x can only be sent to elements which are nilpotent of order $n+1$. These schemes are both important enough that we give them special names:

$$\mathbb{A}^1 := \operatorname{Spec} \mathbb{F}_2[x], \qquad\qquad \mathbb{A}^{1,(n)} := \operatorname{Spec} \mathbb{F}_2[x]/(x^{n+1}).$$

The symbol "$\mathbb{A}^1$" is pronounced "the affine line"—reasonable, since the value $\mathbb{A}^1(T)$ is, indeed, a single T's worth of points. Note that the quotient map $R_1 \to R_2$ induces an inclusion $\mathbb{A}^{1,(n)} \to \mathbb{A}^1$ and that $\mathbb{A}^{1,(0)}$ is a constant functor:

$$\mathbb{A}^{1,(0)}(T) = \{f : \mathbb{F}_2[x] \to T \mid f(x) = 0\}.$$

Accordingly, we pronounce "$\mathbb{A}^{1,(0)}$" as "the origin on the affine line" and "$\mathbb{A}^{1,(n)}$" as "the $(n+1)^{\mathrm{st}}$ order (nilpotent) neighborhood of the origin in the affine line".

We can also use this language to re-express another common object arising in algebraic topology: the Hopf algebra, which appears when taking the mod–2 cohomology of an H–group. In addition to the usual ring structure on cohomology groups, the H–group multiplication, unit, and inversion maps induce an additional diagonal map Δ, an augmentation map ε, and an antipode χ respectively. Running through the axioms, one quickly checks the following:

Lemma 1.2.4. *For a Hopf $\mathbb{F}_2$–algebra R, the functor* Spec R *is naturally valued in groups. Such functors are called group schemes. Conversely, a choice of group structure on* Spec R *endows R with the structure of a Hopf algebra.*

Proof sketch. This is a matter of recognizing the product in $\mathsf{Algebras}^{\mathrm{op}}_{\mathbb{F}_2/}$ as the tensor product, then using the Yoneda lemma to transfer structure around. □

Example 1.2.5. The functor $\mathbb{A}^1$ introduced above is naturally valued in groups: since $\mathbb{A}^1(T) \cong T$, we can use the addition on T to make it into an abelian group. When considering $\mathbb{A}^1$ with this group scheme structure, we notate it as $\mathbb{G}_a$. Applying the Yoneda lemma, one deduces the following formulas for the Hopf algebra structure maps:

$$\begin{aligned} \mathbb{G}_a \times \mathbb{G}_a &\xrightarrow{\mu} \mathbb{G}_a & x_1 + x_2 &\leftarrow\!\shortmid x, \\ \mathbb{G}_a &\xrightarrow{\chi} \mathbb{G}_a & -x &\leftarrow\!\shortmid x, \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \mathbb{G}_a & 0 &\leftarrow\!\shortmid x. \end{aligned}$$

As an example of how to reason this out, consider the following diagram:

$$\begin{tikzcd} \mathbb{G}_a \times \mathbb{G}_a \arrow[rr, "\mu"] & & \mathbb{G}_a \\ \operatorname{Spec} \mathbb{F}_2[x_1] \times \operatorname{Spec} \mathbb{F}_2[x_2] \arrow[u, "x_1 \simeq \; x_2 \simeq"'] & & \\ \operatorname{Spec} \mathbb{F}_2[x_1, x_2] \arrow[u, equal] \arrow[rru, bend right, "x_1+x_2"] \arrow[rr, "\Delta^*"] & & \operatorname{Spec} \mathbb{F}_2[x]. \arrow[uu, "x"', "\simeq"] \end{tikzcd}$$

It follows that the bottom map of affine schemes is induced by the algebra map

$$\mathbb{F}_2[x] \xrightarrow{\Delta} \mathbb{F}_2[x_1, x_2], \qquad\qquad x \mapsto x_1 + x_2.$$

Remark 1.2.6. In fact, $\mathbb{A}^1$ is naturally valued in *rings*. It models the inverse functor to Spec in the equivalence of categories above, i.e., the elements of a ring R always form a complete collection of $\mathbb{A}^1$–valued functions on some affine scheme Spec R.

Example 1.2.7. We define the *multiplicative group scheme* by

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{F}_2[x,y]/(xy-1).$$

Its value $\mathbb{G}_m(T)$ on a test algebra T is the set of pairs (x,y) such that y is a multiplicative inverse to x, and hence $\mathbb{G}_m$ is valued in groups. Applying the Yoneda lemma, we deduce the following formulas for the Hopf algebra structure maps:

$$\begin{aligned}
\mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m & \qquad & x_1 \otimes x_2 \leftarrow\!\shortmid x \\
& & y_1 \otimes y_2 \leftarrow\!\shortmid y, \\
\mathbb{G}_m \xrightarrow{\chi} \mathbb{G}_m & & (y,x) \leftarrow\!\shortmid (x,y), \\
\operatorname{Spec} R \xrightarrow{\eta} \mathbb{G}_m & & 1 \leftarrow\!\shortmid x, y.
\end{aligned}$$

Remark 1.2.8. As presented above, the multiplicative group comes with a natural inclusion $\mathbb{G}_m \to \mathbb{A}^2$. Specifically, the subset $\mathbb{G}_m \subseteq \mathbb{A}^2$ consists of pairs (x,y) in the graph of the hyperbola $y = 1/x$. However, the element x also gives an $\mathbb{A}^1$–valued function $x\colon \mathbb{G}_m \to \mathbb{A}^1$, and because multiplicative inverses in a ring are unique, we see that this map too is an inclusion. These two inclusions have rather different properties relative to their ambient spaces, and we will think harder about these essential differences later on.

Example 1.2.9 (cf. Example 4.4.12). This example showcases the complications that algebraic geometry introduces to this situation, and is meant as discouragement from thinking of the theory of affine group schemes as a strong analogue of the theory of linear complex Lie groups. We set $\alpha_2 = \operatorname{Spec} \mathbb{F}_2[x]/(x^2)$, with group scheme structure given by

$$\begin{aligned}
\alpha_2 \times \alpha_2 \xrightarrow{\mu} \alpha_2 & \qquad & x_1 + x_2 \leftarrow\!\shortmid x, \\
\alpha_2 \xrightarrow{\chi} \alpha_2 & & -x \leftarrow\!\shortmid x, \\
\operatorname{Spec} \mathbb{F}_2 \xrightarrow{\eta} \alpha_2 & & 0 \leftarrow\!\shortmid x.
\end{aligned}$$

This group scheme has several interesting properties, which we will merely state for now, reserving their proofs for Example 4.4.12.

1. α_2 has the same underlying structure ring as $\mu_2 := \mathbb{G}_m[2]$, the 2–torsion points of $\mathbb{G}_m$, but is not isomorphic to it. (For instance, $\underline{\mathsf{GroupSchemes}}(\mu_2, \mu_2)$ gives the constant group scheme $\underline{\mathbb{Z}/2}$, but $\underline{\mathsf{GroupSchemes}}(\alpha_2, \mu_2) = \alpha_2$.)

2. There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.

3. If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence

$$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$

However, this short exact sequence does not split (even after base change).

4. The subgroups of $\alpha_2 \times \alpha_2$ of order 2 are parameterized by the scheme $\mathbb{P}^1$, i.e., for R an $\mathbb{F}_2$-algebra the subgroup schemes of $\alpha_2 \times \alpha_2$ of order two *which are defined over* R are parameterized by the set $\mathbb{P}^1(R)$.

Jeremy had some motivation for this, that quite generally one wants to consider ind-systems of compact objects. Why does one want this? Is it better motivation than just dropping into it?

We now turn to a different class of examples, which will wind up being the key players in our upcoming topological story. To begin, consider the colimit of the sets $\operatorname{colim}_{n\to\infty} \mathbb{A}^{1,(n)}(T)$, which is of use in algebra: it is the collection of nilpotent elements in T. These kinds of conditions which are "unbounded in n" appear frequently enough that we are moved to give these functors a name too:

How are finite schemes characterized in the functor of points perspective? They should commute with sequential colimits or something.

Erick points out that the morphisms in this system should be infintesimal thickenings. Is there a functor-of-points way to recognize such things, without reaching all the way up to talking about ideals? (He also thinks that we should allow finite type in addition to finite, but I don't think I want that. I've been wrong before, though.)

Definition 1.2.10. An *affine formal scheme* is an ind-system of finite affine schemes.[3] The morphisms between two formal schemes are computed by

$$\mathsf{FormalSchemes}(\{X_\alpha\}, \{Y_\beta\}) = \lim_\alpha \operatorname*{colim}_\beta \mathsf{Schemes}(X_\alpha, Y_\beta).$$

Given affine charts $X_\alpha = \operatorname{Spec} R_\alpha$, we will glibly suppress the system from the notation and write

$$\operatorname{Spf} R := \{\operatorname{Spec} R_\alpha\}.$$

Example 1.2.11. The individual schemes $\mathbb{A}^{1,(n)}$ do not support group structures. After all, the sum of two elements which are nilpotent of order $n+1$ can only be guaranteed to be nilpotent of order $2n+1$. It follows that the entire ind-system $\{\mathbb{A}^{1,(n)}\} =: \widehat{\mathbb{A}}^1$ supports a group structure, even though none of its constituent pieces do. We call such an object a *formal group scheme*, and this particular formal group scheme we denote by $\widehat{\mathbb{G}}_a$.

Example 1.2.12. Similarly, one can define the scheme $\mathbb{G}_m[n]$ of elements of unipotent order n:

$$\mathbb{G}_m[n] = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy-1, x^n-1)} \subseteq \mathbb{G}_m.$$

Additionally, the only $\mathbb{G}_m[n]$ which are infinitesimal thickenings of $\mathbb{G}_m[1]$ are those with $n = 2^j$.

These *are* all group schemes, and they nest together in a complicated way: there is an inclusion of $\mathbb{G}_m[n]$ into $\mathbb{G}_m[nm]$. There is also a second filtration along the lines of the one considered in Example 1.2.11:

$$\mathbb{G}_m^{(n)} = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy-1, (x-1)^n)}.$$

... of infinitesimal thickenings

These schemes form a sequential system, but they are only occasionally group schemes. Specifically, $\mathbb{G}_m^{(2^j)}$ is a group scheme, in which case $\mathbb{G}_m^{(2^j)} \cong \mathbb{G}_m[2^j]$. We define $\widehat{\mathbb{G}}_m$ using this common subsystem:

$$\widehat{\mathbb{G}}_m := \{\mathbb{G}_m^{(2^j)}\}_{j=0}^\infty.$$

[3] This has the effect of formally adjoining colimits of filtered diagrams to the category of finite affine schemes.

Let us now consider the example that we closed with last time, where we calculated $H\mathbb{F}_2^*(\mathbb{R}P^n) = \mathbb{F}_2[x]/(x^{n+1})$. Putting "Spec" in front of this, we could reinterpret this calculation as

$$\operatorname{Spec} H\mathbb{F}_2^*(\mathbb{R}P^n) \cong \mathbb{A}^{1,(n)}.$$

This is such a useful thing to do that we will give it a notation all of its own:

Definition 1.2.13. Let X be a finite cell complex, so that $H\mathbb{F}_2^*(X)$ is a ring which is finite–dimensional as an $\mathbb{F}_2$–vector space. We will write

$$X_{H\mathbb{F}_2} = \operatorname{Spec} H\mathbb{F}_2^* X$$

for the corresponding finite affine scheme.

Example 1.2.14. Putting together the discussions from this time and last time, in the new notation we have calculated

$$\mathbb{R}P^n_{H\mathbb{F}_2} \cong \mathbb{A}^{1,(n)}.$$

So far, this example just restates things we knew in a mildly different language. Our driving goal for the next section is to incorporate as much information as we have about these cohomology rings $H\mathbb{F}_2^*(\mathbb{R}P^n)$ into this description, which will result in us giving a more "precise" name for this object. Along the way, we will discover why X had to be a *finite* complex and how to think about more general X. For now, though, we will content ourselves with investigating the Hopf algebra structure on $H\mathbb{F}_2^*\mathbb{R}P^\infty$, the cohomology of an infinite complex.

Example 1.2.15. Recall that $\mathbb{R}P^\infty$ is an H–space in two equivalent ways:

1. There is an identification $\mathbb{R}P^\infty \simeq K(\mathbb{F}_2, 1)$, and the H–space structure is induced by the sum on cohomology.

2. There is an identification $\mathbb{R}P^\infty \simeq BO(1)$, and the H–space structure is induced by the tensor product of real line bundles.

I thought we came up with an instructive third example of where to find the H–space structure.

In either case, this induces a Hopf algebra diagonal

$$H\mathbb{F}_2^*\mathbb{R}P^\infty \otimes H\mathbb{F}_2^*\mathbb{R}P^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{R}P^\infty$$

which we would like to analyze. This map is determined by where it sends the class x, and because it must respect gradings it must be of the form $\Delta x = ax_1 + bx_2$ for some constants $a, b \in \mathbb{F}_2$. Furthermore, because it belongs to a Hopf algebra structure, it must satisfy the unitality axiom

$$\begin{tikzcd} H\mathbb{F}_2^*\mathbb{R}P^\infty & H\mathbb{F}_2^*\mathbb{R}P^\infty \otimes H\mathbb{F}_2^*\mathbb{R}P^\infty \arrow[l, "\begin{pmatrix}\varepsilon\otimes\mathrm{id}\\ \mathrm{id}\otimes\varepsilon\end{pmatrix}"'] & H\mathbb{F}_2^*\mathbb{R}P^\infty. \arrow[l, "\Delta"'] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}$$

and hence it takes the form

$$\Delta(x) = x_1 + x_2.$$

Noticing that this is exactly the diagonal map in Example 1.2.5, we tentatively identify "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" with the additive group. This is extremely suggestive but does not take into account the fact that $\mathbb{RP}^\infty$ is an infinite complex, so we have not yet allowed ourselves to write "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$". In light of the rest of the material discussed in this section, we have left open a very particular point: it is not clear if we should use the name "$\mathbb{G}_a$" or "$\widehat{\mathbb{G}}_a$". We will straighten this out in the subsequent Lecture.

1.3 The Steenrod algebra

We left off in the previous Lecture with an ominous finiteness condition in Definition 1.2.13, and we produced a pair of reasonable guesses as to what "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" could mean in Example 1.2.15. We will decide which of the two guesses is reasonable by rigidifying the target category so as to incorporate the following extra structures:

1. Cohomology rings are *graded*, and maps of spaces respect this grading.
2. Cohomology rings receive an action of the Steenrod algebra, and maps of spaces respect this action.
3. Both of these are made somewhat more complicated when taking the cohomology of an infinite complex.
4. (Cohomology rings for more elaborate cohomology theories are only skew-commutative, but "Spec" requires a commutative input.)

In this Lecture, we will fix all these deficiencies of $X_{H\mathbb{F}_2}$ except for #4, which does not matter with mod–2 coefficients but which will be something of a bugbear throughout the rest of the book.

We will begin by considering the grading on $H\mathbb{F}_2^* X$, where X is a finite complex. In algebraic geometry, the following standard construction is used to track gradings:

Definition 1.3.1 ([Str99b, Definition 2.95]). A $\mathbb{Z}$–*grading* on a ring R is a system of additive subgroups R_k of R satisfying $R = \bigoplus_k R_k$, $1 \in R_0$, and $R_j R_k \subseteq R_{j+k}$. Additionally, a map $f\colon R \to S$ of graded rings is said to *respect the grading* if $f(R_k) \subseteq S_k$.[4]

Lemma 1.3.2 ([Str99b, Proposition 2.96]). *A graded ring R is equivalent data to an affine scheme* $\operatorname{Spec} R$ *with an action by $\mathbb{G}_m$. Additionally, a map $R \to S$ is homogeneous exactly when the induced map* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is $\mathbb{G}_m$–equivariant.*

[4]The terminology "$\mathbb{Z}$–filtering" might be more appropriate, but this is the language commonly used.

Proof. A $\mathbb{G}_m$–action on $\operatorname{Spec} R$ is equivalent data to a coaction map

$$\alpha^* : R \to R \otimes \mathbb{F}_2[x^{\pm}].$$

Define R_k to be those points in r satisfying $\alpha^*(r) = r \otimes x^k$. It is clear that we have $1 \in R_0$ and that $R_j R_k \subseteq R_{j+k}$. To see that $R = \bigoplus_k R_k$, note that every tensor can be written as a sum of pure tensors. Conversely, given a graded ring R, define the coaction map on R_k by

$$(r_k \in R_k) \mapsto x^k r_k$$

and extend linearly. $\square$

This notion from algebraic geometry is somewhat different from what we are used to in algebraic topology, essentially because the algebraic topologist's "cohomology ring" is not *really* a ring at all—one is only allowed to consider sums of homogeneous degree elements. This restriction stems directly from the provenance of cohomology rings: recall that

$$H\mathbb{F}_2^n X := \pi_{-n} F(\Sigma_+^\infty X, H\mathbb{F}_2).$$

One can only form sums internal to a *particular* homotopy group, using the cogroup structure on $\mathbb{S}^{-n}$. On the other hand, the most basic ring of algebraic geometry is the polynomial ring, and hence their notion is adapted to handle, for instance, the potential degree drop when taking the difference of two (nonhomogeneous) polynomials of the same degree.

We can modify our perspective very slightly to arrive at the algebraic geometers', by replacing $H\mathbb{F}_2$ with the periodified spectrum

$$H\mathbb{F}_2 P = \bigvee_{j=-\infty}^{\infty} \Sigma^j H\mathbb{F}_2.$$

This spectrum becomes a ring in the homotopy category by using the factorwise-defined multiplication maps

$$\Sigma^j H\mathbb{F}_2 \wedge \Sigma^k H\mathbb{F}_2 \simeq \Sigma^{j+k}(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \xrightarrow{\Sigma^{j+k}\mu} \Sigma^{j+k} H\mathbb{F}_2.$$

This spectrum has the property that $H\mathbb{F}_2 P^0(X)$ is isomorphic to $\bigoplus_n H\mathbb{F}_2^n(X)$ as ungraded rings, but now we can make topological sense of the sum of two classes which used to live in different $H\mathbb{F}_2$–degrees. At this point we can manually craft the desired coaction map α^* from Lemma 1.3.2, but we will shortly find that algebraic topology gifts us with it on its own.

Our route to finding this internally occurring α^* is by turning to the next supplementary structure: the action of the Steenrod algebra. Naively approached, this does not fit into the framework we have been sketching so far: the Steenrod algebra arises as the homotopy endomorphisms of $H\mathbb{F}_2$ and so is a *noncommutative* algebra. In turn, the action map

$$\begin{array}{ccc} \mathcal{A}^* \otimes H\mathbb{F}_2^* X & \longrightarrow & H\mathbb{F}_2^* X \\ \| & & \| \\ [H\mathbb{F}_2, H\mathbb{F}_2]_* \otimes [X, H\mathbb{F}_2]_* & \xrightarrow{\circ} & [X, H\mathbb{F}_2] \end{array}$$

will be difficult to squeeze into any kind of algebro-geometric framework. Milnor was the first person to see a way around this, with two crucial observations. First, the Steenrod algebra is a Hopf algebra[5], using the map

$$[H\mathbb{F}_2, H\mathbb{F}_2]_* \xrightarrow{\mu^*} [H\mathbb{F}_2 \wedge H\mathbb{F}_2, H\mathbb{F}_2]_* \cong [H\mathbb{F}_2, H\mathbb{F}_2]_* \otimes [H\mathbb{F}_2, H\mathbb{F}_2]_*$$

as the diagonal. This Hopf algebra structure is actually cocommutative—this is a rephrasing of the symmetry of the Cartan formula:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y).$$

It follows that the linear-algebraic dual of the Steenrod algebra $\mathcal{A}_*$ is a commutative ring, and hence $\operatorname{Spec} \mathcal{A}_*$ would make a reasonable algebro-geometric object.

Second, we want to identify the role of $\mathcal{A}_*$ in acting on $H\mathbb{F}_2^* X$. By assuming that X is a finite complex, we can write it as the Spanier–Whitehead dual $X = DY$ of some other finite complex Y. Starting with the action map on $H\mathbb{F}_2^* Y$:

$$\mathcal{A}^* \otimes H\mathbb{F}_2^* Y \to H\mathbb{F}_2^* Y$$

we take the $\mathbb{F}_2$–linear dual to get a coaction map

$$\mathcal{A}_* \otimes H\mathbb{F}_{2*} Y \leftarrow H\mathbb{F}_{2*} Y,$$

then use $X = DY$ to return to cohomology

$$\mathcal{A}_* \otimes H\mathbb{F}_2^* X \xleftarrow{\lambda^*} H\mathbb{F}_2^* X.$$

Finally, we re-interpret this as an action map

$$\operatorname{Spec} \mathcal{A}_* \times X_{H\mathbb{F}_2} \xrightarrow{\alpha} X_{H\mathbb{F}_2}.$$

Having produced the action map α, we are now moved to study α as well as the structure group $\operatorname{Spec} \mathcal{A}_*$ itself. Milnor works out the Hopf algebra structure of $\mathcal{A}_*$ by defining elements $\xi_j \in \mathcal{A}_*$ dual to $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} \in \mathcal{A}^*$. Taking $X = \mathbb{R}\mathrm{P}^n$ and $x \in H\mathbb{F}_2^1(\mathbb{R}\mathrm{P}^n)$ the generator, then since $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} x = x^{2^j}$ he deduces the formula

$$\lambda^*(x) = \sum_{j=0}^{\lfloor \log_2 n \rfloor} x^{2^j} \otimes \xi_j \quad (\text{in } H\mathbb{F}_2^* \mathbb{R}\mathrm{P}^n).$$

[5]The construction of both the Hopf algebra diagonal here and the coaction map below is somewhat ad hoc. We will give a more robust presentation in Lecture 3.1.

Noticing that we can take the limit $n \to \infty$ to get a well-defined infinite sum, he then makes the following calculation, stable in n:

$$\begin{aligned}
(\lambda^* \otimes \mathrm{id}) \circ \lambda^*(x) &= (\mathrm{id} \otimes \Delta) \circ \lambda^*(x) && \text{(coassociativity)} \\
(\lambda^* \otimes \mathrm{id}) \left(\sum_{j=0}^{\infty} x^{2^j} \otimes \xi_j \right) &= \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^i} \otimes \xi_i \right)^{2^j} \otimes \xi_j &= && \text{(ring homomorphism)} \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= && \text{(characteristic 2).}
\end{aligned}$$

Then, turning to the right-hand side:

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= (\mathrm{id} \otimes \Delta) \left(\sum_{m=0}^{\infty} x^{2^m} \otimes \xi_m \right) \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= \sum_{m=0}^{\infty} x^{2^m} \otimes \Delta(\xi_m),
\end{aligned}$$

from which it follows that

$$\Delta \xi_m = \sum_{i+j=m} \xi_i^{2^j} \otimes \xi_j.$$

Finally, Milnor shows that this is the complete story:

Theorem 1.3.3 (Milnor [Mil58, Theorem 2], [MT68, Chapter 6])**.** $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_j, \ldots]$.

Flippant proof. There is at least a map $\mathbb{F}_2[\xi_1, \xi_2, \ldots] \to \mathcal{A}_*$ given by the definition of the elements ξ_j above. This map is injective, since these elements are distinguished by how they coact on $H\mathbb{F}_2^* \mathbb{R}\mathrm{P}^\infty$. Then, since these rings are of graded finite type, Milnor can conclude his argument by counting how many elements he has produced, comparing against how many Adem and Cartan found (which we will do ourselves in **??**), and noting that he has exactly enough. □

We are now in a position to uncover the desired map α^* from earlier. In order to retell Milnor's story with $H\mathbb{F}_2 P$ in place of $H\mathbb{F}_2$, note that there is a topological construction involving $H\mathbb{F}_2$ from which $\mathcal{A}_*$ emerges:

$$\mathcal{A}_* := \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2).$$

Performing substitution on this formula gives the periodified dual Steenrod algebra:

$$\mathcal{A}P_0 := \pi_0(H\mathbb{F}_2 P \wedge H\mathbb{F}_2 P) = H\mathbb{F}_2 P_0(H\mathbb{F}_2 P) = \mathcal{A}_*[\xi_0^{\pm}].$$

Lemma 1.3.4 ([Goe08, Formula 3.4, Remark 3.14]). *Projecting to the quotient Hopf algebra $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^{\pm}]$ gives exactly the coaction map α^*.*

This could be clearer.

Calculation. Starting with an auxiliary cohomology class $x \in H\mathbb{F}_2^n(X)$, we produce a homogenized cohomology class $x \cdot u^n \in H\mathbb{F}_2P^0(X)$. Under the coaction map, this is sent to

$$\begin{array}{ccccc} H\mathbb{F}_2P^0(X) & \xrightarrow{\alpha^*} & H\mathbb{F}_2P^0(X) \otimes \mathcal{A}P_0 & \longrightarrow & H\mathbb{F}_2P^0(X) \otimes \mathbb{F}_2[\xi_0^{\pm}] \\ x \cdot u^n & \longmapsto & & & x \cdot u^n \otimes \xi_0^n. \end{array}$$

Applying Lemma 1.3.2 to this coaction thus selects the original degree n classes. $\square$

Early on in this discussion, trading the language "graded map" for "$\mathbb{G}_m$–equivariant map" did not seem to have much of an effect on our mathematics. The thrust of this Lemma is that "Steenrod–equivariant map" already includes "$\mathbb{G}_m$–equivariant map", which is a visible gain in brevity. To study the rest of the content of Steenrod equivariance algebro–geometrically, we need only identify what the series $\lambda^*(x)$ embodies. Note that this necessarily involves some creativity, and the only justification we can supply will be moral, borne out over time, as our narrative encompasses more and more phenomena. With that caveat in mind, here is one such description. Recall the map induced by the H–space multiplication

$$H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^\infty \otimes H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^\infty \leftarrow H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^\infty.$$

Taking a colimit over finite complexes, we produce an coaction of $\mathcal{A}_*$, and since the map above comes from a map of spaces, it is equivariant for the coaction. Since the action on the left is diagonal, we deduce the formula

$$\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2).$$

Lemma 1.3.5. *The series $\lambda^*(x) = \sum_{j=0}^\infty x^{2^j} \otimes \xi_j$ is the universal example of a series satisfying $\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2)$. The set $(\operatorname{Spec} \mathcal{A}P_0)(T)$ is identified with the set of power series f with coefficients in the $\mathbb{F}_2$–algebra T satisfying*

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Proof. Given a point $f \in (\operatorname{Spec} \mathcal{A}P_0)(T)$, we extract such a series by setting

$$\lambda_f^*(x) = \sum_{j=0}^\infty f(\xi_j) x^{2^j} \in T[\![x]\!].$$

Conversely, any series $\lambda(x)$ satisfying this homomorphism property must have nonzero terms appearing only in integer powers of 2, and hence we can construct a point f by declaring that f sends ξ_j to the $(2^j)^{\text{th}}$ coefficient of λ. $\square$

We close our discussion by codifying what Milnor did when he stabilized against n. Each $\mathbb{RP}^n_{H\mathbb{F}_2}$ is a finite affine scheme, and to make sense of the object $\mathbb{RP}^\infty_{H\mathbb{F}_2}$ Milnor's technique was to consider the ind-system $\{\mathbb{RP}^n_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite affine schemes. We will record this as our technique to handle general infinite complexes:

Definition 1.3.6 (cf. Definition 2.1.14)**.** When X is an infinite complex, filter it by its subskeleta $X^{(n)}$ and define $X_{H\mathbb{F}_2}$ to be the ind-system $\{X^{(n)}_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite schemes.

This choice to follow Milnor resolves our uncertainty about the topological example from last time:

Example 1.3.7 (cf. Examples 1.2.11 and 1.2.15). Write $\widehat{\mathbb{G}}_a$ for the ind-system $\mathbb{A}^{1,(n)}$ with the group scheme structure given in Example 1.2.15. That this group scheme structure filters in this way is a simultaneous reflection of two facts:

1. Algebraic: The set $\widehat{\mathbb{G}}_a(T)$ consists of all nilpotent elements in T. The sum of two nilpotent elements of orders n and m is guaranteed to itself be nilpotent with order at most $n+m$.

2. Topological: There is a factorization of the multiplication map on $\mathbb{RP}^\infty$ as $\mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{n+m}$ purely for dimensional reasons.

As group schemes, we have thus calculated

$$\mathbb{RP}^\infty_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a.$$

Example 1.3.8. Given the appearance of a homomorphism condition in Lemma 1.3.5, we would like to connect $\operatorname{Spec} \mathcal{A}P_0$ with $\widehat{\mathbb{G}}_a$ more directly. Toward this, we define a "hom functor" for two formal schemes:

$$\underline{\mathsf{FormalSchemes}}(X,Y)(T) = \left\{ (u,f) \,\middle|\, \begin{array}{c} u : \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2, \\ f : u^*X \to u^*Y \end{array} \right\}.$$

Restricting attention to homomorphisms, we see that a proper name for $\operatorname{Spec} \mathcal{A}P_0$ is

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

To check this, consider a point $g \in (\operatorname{Spec} \mathcal{A}P_0)(T)$ for an $\mathbb{F}_2$–algebra T. The $\mathbb{F}_2$–algebra structure of T (which is uniquely determined by a property of T) gives rise to a map $u\colon \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2$. The rest of the data of g gives rise to a power series in $T[\![x]\!]$ as in the proof of Lemma 1.3.5, which can be re-interpreted as an automorphism $g\colon u^*\widehat{\mathbb{G}}_a \to u^*\widehat{\mathbb{G}}_a$ of formal group schemes.[6]

[6]This description, too, is sensitive to the difference between $\widehat{\mathbb{G}}_a$ and $\mathbb{G}_a$. The scheme $\underline{\operatorname{End}}\, \mathbb{G}_a$ is populated by *polynomials* satisfying a homomorphism condition, and essentially none of them have inverses.

Remark 1.3.9. The projection $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^{\pm}]$ is split as Hopf algebras, and hence there is a decomposition

$$\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a \cong \mathbb{G}_m \times \underline{\mathrm{Aut}}_1\,\widehat{\mathbb{G}}_a,$$

where $\underline{\mathrm{Aut}}_1\,\widehat{\mathbb{G}}_a$ consists of those automorphisms with leading coefficient ξ_0 exactly equal to 1. This can be read to mean that the "interesting" part of the Steenrod algebra, $\underline{\mathrm{Aut}}_1\,\widehat{\mathbb{G}}_a$, consists of stable operations, in the sense that their action is independent of the degree–tracking mechanism.

Example 1.3.10. Remembering the slogan

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$$

also makes it easy to recall the structure formulas for the dual Steenrod algebra. For instance, consider the antipode map, which has the effect on $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$ of sending a power series to its compositional inverse. That is:

$$\sum_{j=0}^{\infty} \chi(\xi_j) \left(\sum_{k=0}^{\infty} \xi_k x^{2^k} \right)^{2^j} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \chi(\xi_j) \xi_k^{2^j} x^{2^{j+k}} = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} \chi(\xi_j) \xi_k^{2^j} \right) x^{2^n} = 1,$$

from which we can extract formulas like

$$\chi(\xi_0) = \xi_0^{-1}, \qquad \chi(\xi_1) = \xi_0^{-3}\xi_1, \qquad \chi(\xi_2) = \xi_0^{-7}\xi_1^3 + \xi_0^{-5}\xi_2, \qquad \dots.$$

In summary, the formula $\mathbb{RP}^{\infty}_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a$ is meant to point out that this language of formal schemes has an extremely good compression ratio—you can fit a lot of information into a very tiny space. This formula simultaneously encodes the cohomology ring of $\mathbb{RP}^{\infty}$ as the formal scheme, its diagonal as the group scheme structure, and the coaction of the dual Steenrod algebra by the identification with $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$. As a separate wonder, it is also remarkable that there is a single cohomological calculation—that of $\mathbb{RP}^{\infty}_{H\mathbb{F}_2}$—which exerts such enormous control over mod–2 cohomology itself (e.g., the entire structure of the dual Steenrod algebra). This will turn out to be a surprisingly common occurrence as we progress.

1.4 Hopf algebra cohomology

In this section, we will focus on an important classical tool: the Adams spectral sequence. We are going to study this in greater earnest later on, so we will avoid giving a satisfying construction in this Lecture. But, even without a construction, it is instructive to see how such a thing comes about from a moral perspective.

Cite me: I first saw this presentation from Matt Ando. He must have learned it from someone. I'd like to know who to attribute this to.

Remark 1.4.1. Throughout this Lecture, we will work with graded homology groups, rather than with periodified cohomology as was the case in Lecture 1.3. This choice will remain mysterious for now, but we can at least reassure ourselves that it carries the same data as we were studying previously. Referring to our discussion of the construction of the coaction map, we see that without taking Spanier–Whitehead duals we already have an analogous coaction map on homology:

$$H\mathbb{F}_{2*}X \to H\mathbb{F}_{2*}X \otimes \mathcal{A}_*.$$

Additionally, building on the discussion in Remark 1.3.9, the splitting of the Hopf algebra shows that we are free to work gradedly or work with the periodified version of mod–2 homology, while still retaining the rest of the framework.

With this caveat out of the way, begin by considering the following three self-maps of the stable sphere:

$$\mathbb{S}^0 \xrightarrow{0} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{1} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{2} \mathbb{S}^0.$$

If we apply mod–2 homology to each line, the induced maps are

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2, \qquad \mathbb{F}_2 \xrightarrow{\mathrm{id}} \mathbb{F}_2, \qquad \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2.$$

We see that mod–2 homology can immediately distinguish between the null map and the identity map just by its behavior on morphisms, but it cannot distinguish between the null map and the multiplication-by-2 map. To try to distinguish between these two, we use the only other tool available to us: homology theories send cofiber sequences to long exact sequences, and moreover the data of a map f and the data of the inclusion map $\mathbb{S}^0 \to C(f)$ into its cone are equivalent in the stable category. So, we trade our maps 0 and 2 for the following cofiber sequences:

$$\mathbb{S}^0 \longrightarrow C(0) \longrightarrow \mathbb{S}^1, \qquad \mathbb{S}^0 \longrightarrow C(2) \longrightarrow \mathbb{S}^1.$$

The homology groups of these spectra $C(0)$ and $C(2)$ are more complicated than just that of $\mathbb{S}^0$, and we will draw them according to the following conventions: each "$\bullet$" in the row labelled "$[j]$" indicates an $\mathbb{F}_2$–summand in the j^{th} $H\mathbb{F}_2$–homology of the spectrum. Applying homology to these cofiber sequences and drawing the results, these again appear to be identical:

$$\begin{array}{lccccccc}
[1] & & & \bullet & \longrightarrow & \bullet & \qquad & & & \bullet & \longrightarrow & \bullet \\
[0] & \bullet & \longrightarrow & \bullet & & & \qquad & \bullet & \longrightarrow & \bullet & & \\
& & & & & & & & & & & \\
& H\mathbb{F}_{2*}\mathbb{S}^0 & \to & H\mathbb{F}_{2*}C(0) & \to & H\mathbb{F}_{2*}\mathbb{S}^1, & \qquad & H\mathbb{F}_{2*}\mathbb{S}^0 & \to & H\mathbb{F}_{2*}C(2) & \to & H\mathbb{F}_{2*}\mathbb{S}^1,
\end{array}$$

However, if we enrich our picture with the data we discussed in Lecture 1.3, we can finally see the difference. Recall the topological equivalences

It would be nice if the dots aligned directly beneath the spaces in the cofiber sequences above. This can't be accomplished by a "column sep" attribute, since this doesn't control the width of a column but rather its literal separation from its neighbor. This means finding some kind of "intertext" analogue for tikzcd.

$$C(0) \simeq \mathbb{S}^0 \vee \mathbb{S}^1, \quad C(2) \simeq \Sigma^{-1}\Sigma^\infty \mathbb{R}P^2.$$

In the two cases, the coaction map λ_* is given by

$$\begin{aligned} \lambda_* : H\mathbb{F}_{2*}C(0) &\to H\mathbb{F}_{2*}C(0) \otimes \mathcal{A}_* \\ \lambda^* : e_0 &\mapsto e_0 \otimes 1 \\ \lambda^* : e_1 &\mapsto e_1 \otimes 1, \end{aligned} \qquad \begin{aligned} \lambda_* : H\mathbb{F}_{2*}C(2) &\to H\mathbb{F}_{2*}C(2) \otimes \mathcal{A}_* \\ \lambda^* : e_0 &\mapsto e_0 \otimes 1 + e_1 \otimes \xi_1 \\ \lambda^* : e_1 &\mapsto e_1 \otimes 1. \end{aligned}$$

We use a vertical line to indicate the nontrivial coaction involving ξ_1:

$$\begin{tikzcd} {[1]} & & \bullet \arrow[r] & \bullet & & \bullet \arrow[r] \arrow[d, dash, "\xi_1"] & \bullet \\ {[0]} & \bullet \arrow[r] & \bullet & & \bullet \arrow[r] & \bullet & \end{tikzcd}$$

$$H\mathbb{F}_{2*}\mathbb{S}^0 \to H\mathbb{F}_{2*}C(0) \to H\mathbb{F}_{2*}\mathbb{S}^1, \qquad H\mathbb{F}_{2*}\mathbb{S}^0 \to H\mathbb{F}_{2*}C(2) \to H\mathbb{F}_{2*}\mathbb{S}^1,$$

We can now see what trading maps for cofiber sequences has bought us: mod–2 homology can distinguish the defining sequences for $C(0)$ and $C(2)$ by considering their induced extensions of comodules over $\mathcal{A}_*$. The Adams spectral sequence bundles this thought process into a single machine:

Can this be phrased so as to indicate how this works for longer extensions? I've never tried to think about even what happens for $C(4)$.

Theorem 1.4.2 ([Rav86, Definition 2.1.8, Lemma 2.1.16], [MT68, Chapter 18])**.** *There is a convergent spectral sequence of signature*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow (\pi_*\mathbb{S}^0)^\wedge_2. \quad \square$$

In effect, this asserts that the above process is *exhaustive*: every element of $(\pi_*\mathbb{S}^0)^\wedge_2$ can be detected and distinguished by some representative class of extensions of comodules for the dual Steenrod algebra. Mildly more generally, if X is a bounded-below spectrum, then there is even a spectral sequence of signature

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X) \Rightarrow \pi_*X^\wedge_2.$$

We could now work through the construction of the Adams spectral sequence, but it will fit more nicely into a story later on in Lecture 3.1. Before moving on to other pursuits, however, we will record the following utility Lemma. It is believable based on the above discussion, and we will need to use it before we get around to examining the guts of the construction.

Lemma 1.4.3 (cf. Remark 3.1.17)**.** *The 0–line of the Adams spectral sequence consists of exactly those elements visible to the Hurewicz homomorphism.* □

For the rest of the section, we will focus on the algebraic input "$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$", which will require us to grapple with the homological algebra of comodules for a Hopf algebra. To start that discussion, it's both reassuring and instructive to see that homological algebra can, in fact, even be done with comodules. In the usual development of homological algebra for *modules*, the key observations are the existence of projective and injective modules, and there is something at work similar here.

Remark 1.4.4 ([Rav86, Appendix A1]). Much of the results below do not rely on working with a Hopf algebra over the field $k = \mathbb{F}_2$. In fact, k can usually be taken to be a ring rather than a field. More generally, the theory goes through in the context of comodules over flat Hopf algebroids, cf. also Lemma 3.1.16.

Lemma 1.4.5 ([Rav86, Definition A1.2.1])**.** *Let A be a Hopf k–algebra, let M be an A–comodule, and let N be a k–module. There is a cofree adjunction:*

$$\mathsf{Comodules}_A(M, N \otimes_k A) \cong \mathsf{Modules}_k(M, N),$$

where $N \otimes_k A$ is given the structure of an A–comodule by the coaction map

$$N \otimes_k A \xrightarrow{\mathrm{id} \otimes \Delta} N \otimes_k (A \otimes_k A) = (N \otimes_k A) \otimes_k A.$$

Proof. Given a map $f \colon M \to N$ of k–modules, we can build the composite

$$M \xrightarrow{\psi_M} M \otimes_k A \xrightarrow{f \otimes \mathrm{id}_A} N \otimes_k A.$$

Alternatively, given a map $g \colon M \to N \otimes_k A$ of A–comodules, we build the composite

$$M \xrightarrow{g} N \otimes_k A \xrightarrow{\mathrm{id}_N \otimes \varepsilon} N \otimes_k k = N. \qquad \square$$

Corollary 1.4.6 ([Rav86, Lemma A1.2.2])**.** *The category $\mathsf{Comodules}_A$ has enough injectives. Namely, if M is an A–comodule and $M \to I$ is an inclusion of k–modules into an injective k–module I, then $M \to I \otimes_k A$ is an injective A–comodule under M.* □

Remark 1.4.7. In our case, M itself is always k–injective, so there's already an injective map $\psi_M : M \to M \otimes A$: the coaction map. The assertion that this map is coassociative is identical to saying that it is a map of comodules.

Satisfied that "Ext" at least makes sense, we're free to pursue more conceptual ends. Recall from algebraic geometry that a module M over a ring R is equivalent data to quasicoherent sheaf $\widetilde{M}$ over $\operatorname{Spec} R$. We now give a definition of "quasicoherent sheaf" that fits with our functorial perspective:

Definition 1.4.8 ([Hov02, Definition 1.1], [Str99b, Definition 2.42]). A presheaf (of modules) over a scheme X is an assignment $\mathcal{F}\colon\ X(T) \to \mathsf{Modules}_T$, satisfying a kind of functoriality in T: for each map $f\colon\ T \to T'$, there is a compatible choice of natural transformation

$$\begin{tikzcd} X(T) \arrow[r, "\mathcal{F}(T)"] \arrow[d, "X(f)"'] \arrow[dr, phantom, "\Swarrow \tau(f)"] & \mathsf{Modules}_T \arrow[d, "-\otimes_T T'"] \\ X(T') \arrow[r, "\mathcal{F}(T')"] & \mathsf{Modules}_{T'}. \end{tikzcd}$$

(We think of the image of a particular point $t\colon\ \operatorname{Spec} T \to X$ in $\mathsf{Modules}_T$ as the module of "sections over t".) Such a presheaf is said to be a *quasicoherent sheaf* when these natural transformations are all natural isomorphisms.

Lemma 1.4.9 ([Str99b, Proposition 2.47]). *An R–module M gives rise to a quasicoherent sheaf $\widetilde{M}$ on* $\operatorname{Spec} R$ *by the rule*

$$(\operatorname{Spec} T \to \operatorname{Spec} R) \mapsto M \otimes_R T.$$

Conversely, every quasicoherent sheaf over an affine scheme arises in this way. □

The tensoring operation appearing in the definition of a presheaf appears more generally as an operation on the category of sheaves.

Jay was frustrated with which adjoint I put on top (and perhaps which went on which side). Apparently there's some convention, which I should look up and obey.

Definition 1.4.10. A map $f\colon\ \operatorname{Spec} S \to \operatorname{Spec} R$ induces maps $f^* \dashv f_*$ of categories of quasicoherent sheaves. At the level of modules, these are given by

$$\begin{tikzcd} \mathsf{QCoh}_{\operatorname{Spec} R} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \mathsf{QCoh}_{\operatorname{Spec} S} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\ \mathsf{Modules}_R \arrow[r, shift left, "M \mapsto M \otimes_R S"] & \mathsf{Modules}_S. \arrow[l, shift left, "N \mapsfrom N"] \end{tikzcd}$$

One of the main uses of these operations is to define the cohomology of a sheaf. Let $\pi\colon\ X \to \operatorname{Spec} k$ be a scheme over $\operatorname{Spec} k$, k a field, and let $\mathcal{F}$ be a sheaf over X. The adjunction above induces a derived adjunction

$$\operatorname{Ext}_X(\pi^* k, \mathcal{F}) \cong \operatorname{Ext}_{\operatorname{Spec} k}(k, R\pi_* \mathcal{F}),$$

which is used to translate the *definition* of sheaf cohomology to that of the cohomology of the derived pushforward $R\pi_* \mathcal{F}$, itself interpretable as a mere complex of k–modules. This pattern is very general: the sense of "cohomology" relevant to a situation is often accessed by taking the derived pushforward to a suitably terminal object.[7] To invent a notion of cohomology for comodules over a Hopf algebra, we are thus moved to produce push and pull functors for a map of Hopf algebras, and this is best motivated by another example.

[7] This perspective often falls under the heading of "six–functor formalism".

Example 1.4.11. A common source of Hopf algebras is through group–rings: given a group G, we can define the Hopf k–algebra $k[G]$ consisting of formal k–linear combinations of elements of G. This Hopf algebra is commutative exactly when G is abelian, and $k[G]$–modules are naturally equivalent to k–linear G–representations. Dually, the ring k^G of k–valued functions on G is always commutative, using pointwise multiplication of functions, and it is *cocommutative* exactly when G is abelian. If G is finite, then k^G and $k[G]$ are k–linear dual Hopf algebras, and hence finite–dimensional k^G–comodules are naturally equivalent to finite–dimensional k–linear G–representations.[8]

A map of groups $f\colon G \to H$ induces a map $k^f\colon k^H \to k^G$ of Hopf algebras, and it is reasonable to expect that the induced push and pull maps of comodules mimic those of G– and H–representations. Namely, given an H–representation M, we can produce a corresponding G–representation by precomposition with f. However, given a G–representation N, two things may have to be corrected to extract an H–representation:

1. If f is not surjective, we must decide what to do with the extra elements in H.

2. If f is not injective—say, $f(g_1) = f(g_2)$—then we must force the behavior of the extracted H–representation to agree on $f(g_1)$ and $f(g_2)$, even if g_1 and g_2 act differently on N. In the extreme case of $f\colon G \to 1$, we expect to recover the fixed points of N, since this pushforward computes $H^0_{\mathrm{gp}}(G; N)$.

These concerns, together with the definition of a tensor product as a coequalizer, motivate the following:

Definition 1.4.12. Given A–comodules M and N, their cotensor product is the k-module defined by the equalizer

$$M \square_A N \to M \otimes_k N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} M \otimes_k A \otimes_k N.$$

Lemma 1.4.13. *Given a map $f\colon A \to B$ of Hopf k–algebras, the induced adjunction $f^* \dashv f_*$ is given at the level of comodules by*

$$\begin{tikzcd} \mathsf{QCoh}_{\operatorname{Spec} k /\!/ \operatorname{Spec} A} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \mathsf{QCoh}_{\operatorname{Spec} k /\!/ \operatorname{Spec} B} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\ \mathsf{Comodules}_A \arrow[r, shift left, "M \mapsto M"] & \mathsf{Comodules}_B. \arrow[l, shift left, "N \square_B A \leftarrow\!\shortmid N"] \end{tikzcd} \qquad \square$$

[8]There is a variation on this equivalence that uses fewer dualities and which is instructive to expand. The Hopf algebra $k^G = \prod_{g \in G} k$ is the ring of functions on the constant group scheme G, and its k–points $(\operatorname{Spec} k^G)(k)$ biject with points in G. Namely, given $g \in G$ we can form a projection map $g\colon k^G \to k$ and hence a composite $M \to M \otimes_k k^G \xrightarrow{\mathrm{id} \otimes g} M \otimes_k k \cong M$. Collectively, this determines a map $G \times M \to M$ witnessing M as a G–representation. In the other direction, if G is finite then we can construct a map $M \to M \otimes_k \prod_{g \in G} k$ sending $m \in M$ to $g \cdot m$ in the g^{th} labeled component of the target.

Remark 1.4.14. In Lecture 3.1 (and Definition 3.1.14 specifically), we will explain the notation "Spec k// Spec A" used above. For now, suffice it to say that there again exists a functor-of-points notion of "quasicoherent sheaf" associated to a Hopf k–algebra A, and such sheaves are equivalent to A–comodules.

As an example application, cotensoring gives rise to a concise description of what it means to be a comodule map:

Lemma 1.4.15 ([Rav86, Lemma A1.1.6b]). *Let M and N be A–comodules with M projective as a k–module. Then there is an equivalence*

$$\mathsf{Comodules}_A(M, N) = \mathsf{Modules}_k(M, k)\Box_A N. \quad \Box$$

From this, we can deduce a connection between the push–pull flavor of comodule cohomology described above and the input to the Adams spectral sequence.

Corollary 1.4.16. *Let $N = N' \otimes_k A$ be a cofree comodule. Then $N\Box_A k = N'$.*

Proof. Picking $M = k$, we have

$$\begin{aligned} \mathsf{Modules}_k(k, N') &= \mathsf{Comodules}_A(k, N) \\ &= \mathsf{Modules}_k(k, k)\Box_A N \\ &= k\Box_A N. \end{aligned} \qquad \Box$$

Corollary 1.4.17. *There is an isomorphism*

$$\mathsf{Comodules}_A(k, N) = \mathsf{Modules}_k(k, k)\Box_A N = k\Box_A N$$

and hence

$$\operatorname{Ext}_A(k, N) \cong \operatorname{Cotor}_A(k, N) \, (= H^* R\pi_* N) \, .$$

Proof. Resolve N using the cofree modules described above, then apply either functor $\mathsf{Comodules}_A(k, -)$ or $k\Box_A -$. In both cases, you get the same complex. $\Box$

Example 1.4.18. In Lecture 1.3, we identified $\mathcal{A}_*$ with the ring of functions on the group scheme $\underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a)$ of strict automorphisms of $\widehat{\mathbb{G}}_a$, which is defined by the kernel sequence

$$0 \to \underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

The punchline is that this is analogous to Example 1.4.11 above: $\operatorname{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$ is thought of as "the derived fixed points" of "$G = \underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a)$" on the "$G$–module" $H\mathbb{F}_{2*}X$.

We now give several examples to get a sense of how the Adams spectral sequence behaves.

Example 1.4.19. Consider the degenerate case $X = H\mathbb{F}_2$. Then $H\mathbb{F}_{2*}(H\mathbb{F}_2) = \mathcal{A}_*$ is a cofree comodule, and hence Cotor is concentrated on the 0–line:

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}(H\mathbb{F}_2)) = \mathbb{F}_2.$$

The Adams spectral sequence collapses to show the wholly unsurprising equality $\pi_* H\mathbb{F}_2 = \mathbb{F}_2$, and indeed this is the element in the image of the Hurewicz map $\pi_* H\mathbb{F}_2 \to H\mathbb{F}_{2*}H\mathbb{F}_2$.

Example 1.4.20. Next, we consider the more computationally serious case of $X = kO$, the connective real K–theory spectrum. The main input we need is the structure of $H\mathbb{F}_{2*}kO$ as an $\mathcal{A}_*$–comodule, so that we can compute

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}kO) \Rightarrow \pi_* kO^\wedge_2.$$

There is a slick trick for doing this: by working in the category of kO–modules rather than in all spectra, we can construct a relative Adams spectral sequence

$$\mathrm{Cotor}^{*,*}_{\pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2}(\mathbb{F}_2, \pi_* H\mathbb{F}_2 \wedge_{kO} (kO \wedge H\mathbb{F}_2)) \Rightarrow \pi_*(kO \wedge H\mathbb{F}_2).$$

The second argument is easy to identify:

$$\pi_* H\mathbb{F}_2 \wedge_{kO} (kO \wedge H\mathbb{F}_2) = \pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 = \mathcal{A}_*.$$

The Hopf algebra requires further input. Consider the following trio of cofiber sequences[9]:

$$\Sigma kO \xrightarrow{\cdot\eta} kO \to kU, \qquad \Sigma^2 kU \xrightarrow{\cdot\beta} kU \to H\mathbb{Z}, \qquad H\mathbb{Z} \xrightarrow{\cdot 2} H\mathbb{Z} \to H\mathbb{F}_2.$$

These combine to give a resolution of $H\mathbb{F}_2$ via an iterated cofiber of free kO–modules, with Poincaré series

$$((1+t^2) + t^3(1+t^2)) + t((1+t^2) + t^3(1+t^2)) = 1 + t + t^2 + 2t^3 + t^4 + t^5 + t^6.$$

Repeatedly using the identity $kO \wedge_{kO} H\mathbb{F}_2 \simeq H\mathbb{F}_2$ gives a small presentation of the Hopf algebra $\pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2$: it is a commutative Hopf algebra over $\mathbb{F}_2$ with the above Poincaré series. The Borel–Milnor–Moore [MM65, Theorem 7.11] classification of commutative Hopf algebras over $\mathbb{F}_p$ shows that the algebra structure is either

$$\frac{\mathbb{F}_2[a,b,c]}{(a^2 = 0, b^2 = 0, c^2 = 0)} \quad \text{or} \quad \frac{\mathbb{F}_2[a,b,c]}{(a^2 = b, b^2 = 0, c^2 = 0)}$$

for $|a| = 1$, $|b| = 2$, and $|c| = 3$. By knowing that the natural map $\mathcal{A} \to \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2$ winds up inducing an isomorphism $\pi_{*\le 2}\mathbb{S} \to \pi_{*\le 2} kO$, we conclude that we are in the latter case, which gives a presentation of the Hopf algebra as a whole:

[9] This first sequence, known as the Wood cofiber sequence, is a consequence of a very simple form of Bott periodicity [Har80, Section 5]: there is a fiber sequence of infinite-loopspaces $O/U \to BO \to BU$, and $\underline{kO}_1 = O/U$.

$$\begin{tikzcd}
\pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 \arrow[r] \arrow[d, equal] & \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2 \arrow[d, equal] \\
\mathcal{A}_* \arrow[r] & \frac{\mathbb{F}_2[\xi_1, \xi_2, \xi_3, \xi_4, \ldots]}{(\xi_1^4, \xi_2^2), (\xi_n \mid n \geq 3)}.
\end{tikzcd}$$

This Hopf algebra is commonly denoted $\mathcal{A}(1)_*$, and its corresponding subgroup scheme $\operatorname{Spec} \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2 \subseteq \underline{\mathrm{Aut}}_1 \widehat{\mathbb{G}}_a$ admits easy memorization: it is the subscheme of automorphisms of the form $x + \xi_1 x^2 + \xi_2 x^4$, with exactly the additional relations imposed on ξ_1 and ξ_2 so that this set is stable under composition and inversion.[10,11] Its cohomology is periodic with period 8, and it is pictured through a range in Figure 1.1.

This doesn't connect back with the first Adams SS.

Example 1.4.21. At the other extreme, we can pick the extremely nondegenerate case $X = \mathbb{S}$, where $\underline{\mathrm{Aut}}_1 \widehat{\mathbb{G}}_a$ acts maximally nonfreely on $\operatorname{Spec} \mathbb{F}_2$. The resulting spectral sequence is pictured through a range in Figure 1.2.

Jon asked: spectral sequences coming from π_* of a Tot tower increase Tot degree. ANSS differentials decrease degree: they run against the multiplicative structure in pictures. What's going on with this? I think this is a duality effect: working with the Steenrod algebra versus its dual.

Another thing that might belong here is the deformation picture: $H\mathbb{F}_{2*}H\mathbb{Z}$ is also computable using the same methods, and the change-of-rings theorem then says that the Adams spectral sequence computing $H\mathbb{Z}_* X$ from $H\mathbb{F}_{2*} X$ is controlled by Ext over the exterior algebra generated by that one Bockstein. (This might also belong much later, say, when we talk about the Morava K–theory cooperations or if we ever describe $H\mathbb{F}_{2*} BP$.)

1.5 The unoriented bordism ring

Our goal in this section is to use our results so far to make a calculation of $\pi_* MO$, the unoriented bordism ring. Our approach is the same as in the examples at the end of the previous section: we will want to use the Adams spectral sequence of signature

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)\tilde{}) \Rightarrow \pi_* MO,$$

which requires understanding $H\mathbb{F}_2 P_0(MO)$ as a comodule for the dual Steenrod algebra.

Our first step toward this is the following calculation:

Lemma 1.5.1 ([Swi02, Theorem 16.17])**.** *The natural map*

$$\operatorname{Sym} \widetilde{H\mathbb{F}_2 P_0}(BO(1)) \to H\mathbb{F}_2 P_0(BO).$$

induces a map

$$\operatorname{Sym} \widetilde{H\mathbb{F}_2 P_0}(BO(1)) = \frac{\operatorname{Sym} H\mathbb{F}_2 P_0(BO(1))}{\beta_0 = 0} \xrightarrow{\simeq} H\mathbb{F}_2 P_0(BO)$$

which is an isomorphism of Hopf algebras and of comodules for the dual Steenrod algebra.

Proof. This follows from a combination of standard facts about Stiefel–Whitney classes.

Cite me: You could cite these standard facts..

[10] A similar analysis shows that $H\mathbb{F}_{2*} H\mathbb{Z}$ corepresents the subscheme of automorphisms of the form $x + \xi_1 x^2$ which are stable under composition and inversion.

[11] There is also an accidental isomorphism of this Hopf algebra with $\mathbb{F}_2^{D_4}$, where D_4 is the dihedral group with 8 elements.

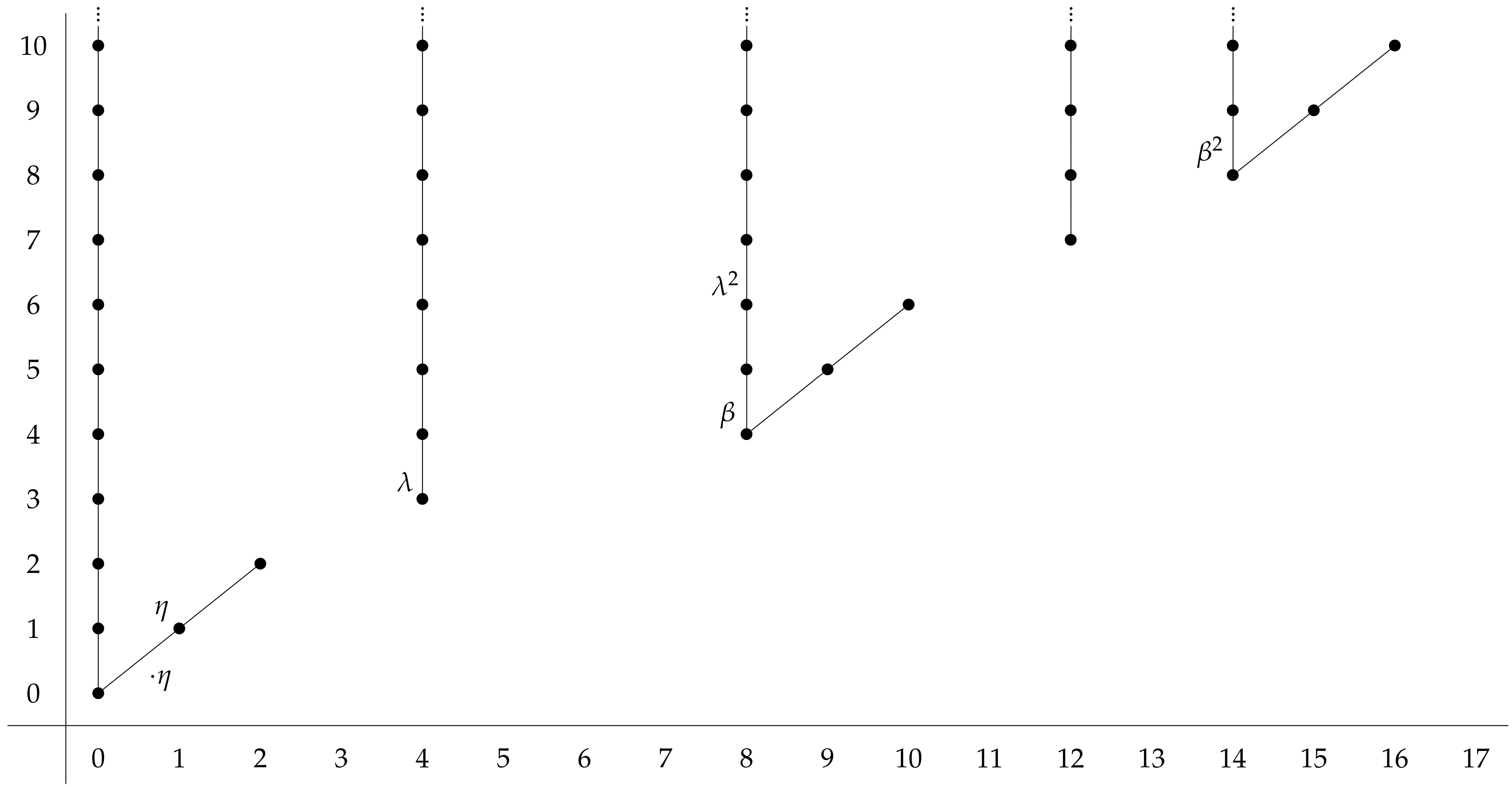

Figure 1.1: The $H\mathbb{F}_2$–Adams spectral sequence for kO, which collapses at the second page. North and north-east lines denote multiplication by 2 and by η.

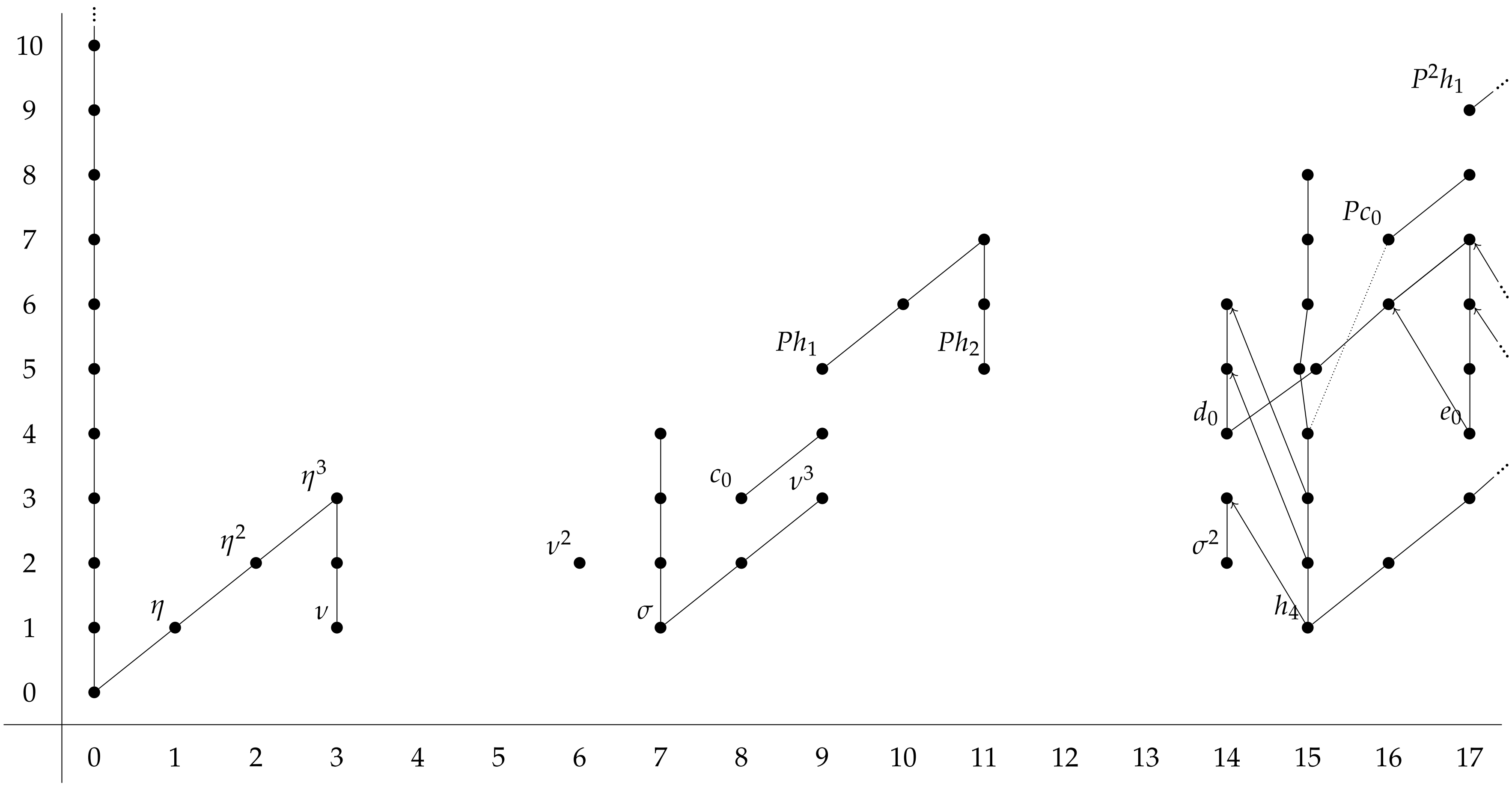

Figure 1.2: A small piece of the $H\mathbb{F}_2$–Adams spectral sequence for the sphere, beginning at the second page [Rav78, pg. 412]. North and north-east lines denote multiplication by 2 and by η, north-west lines denote d_2– and d_3–differentials.

First, these classes generate the cohomology ring $H\mathbb{F}_2^* BO(n)$:

$$H\mathbb{F}_2^* BO(n) \cong \mathbb{F}_2[\![w_1, \ldots, w_n]\!].$$

Second, the total Stiefel–Whitney classis exponential, in the sense of ... which you haven't defined.

$$w(V \oplus W) = w(V) \cdot w(W).$$

From this, it follows that the natural map

$$H\mathbb{F}_2^* BO(n) \xrightarrow{\bigoplus_{j=1}^n \mathcal{L}_j} H\mathbb{F}_2^* BO(1)^{\times n} \cong (H\mathbb{F}_2^* BO(1))^{\otimes n}$$

is the inclusion of the symmetric polynomials, by calculating the total Stiefel–Whitney class

$$w\left(\bigoplus_{j=1}^n \mathcal{L}_j\right) = \prod_{j=1}^n (1 + w_1(\mathcal{L}_j)) = \sum_{j=0}^n \sigma_j(w_1(\mathcal{L}_1), \ldots, w_1(\mathcal{L}_n)) t^j.$$

Dually, the homological map

$$(H\mathbb{F}_{2*} BO(1))^{\otimes n} \to H\mathbb{F}_{2*} BO(n)$$

is surjective, modeling the quotient from the tensor product to the symmetric tensor product. Stabilizing as $n \to \infty$, we recover the statement of the Lemma. □

With this in hand, we now turn to the homotopy ring $H\mathbb{F}_2 P_0 MO$. There are two equivalences that we might consider employing. We have the Thom isomorphism:

$$\begin{array}{ccc} H\mathbb{F}_2 P_0(BO(1)) & = & H\mathbb{F}_2 P_0(MO(1)) \\ \beta_j, j \geq 0 & \longmapsto & \beta'_j, j \geq 0, \end{array}$$

and we also have the equivalence induced by the topological map in Example 1.1.3:

$$\begin{array}{ccc} \widetilde{H\mathbb{F}_2} P_0(BO(1)) & = & H\mathbb{F}_2 P_0(\Sigma MO(1)) \\ \beta_j, j \geq 1 & \longmapsto & \beta'_{j-1}, j \geq 1. \end{array}$$

We will use them both in turn.

Corollary 1.5.2 ([Ada95, Section I.3], [Hop, Proposition 6.2])**.** *There is an isomorphism*

$$H\mathbb{F}_2 P_0(MO) \cong \frac{\operatorname{Sym} H\mathbb{F}_2 P_0 MO(1)}{b'_0 = 1}.$$

Proof. The block sum maps

$$BO(n) \times BO(m) \to BO(n+m)$$

Thomify to give compatible maps

$$MO(n) \wedge MO(m) \to MO(n+m).$$

Taking the colimit, this gives a ring structure on MO compatible with that on $\Sigma^\infty_+ BO$ and compatible with the Thom isomorphism. $\square$

We now seek to understand the utility of the scheme $\operatorname{Spec} H\mathbb{F}_2P_0(MO)$, as well as its action of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$. The first of these tasks comes from untangling some of the topological dualities we've been using thus far.

Lemma 1.5.3. *The following square commutes:*

$$\begin{tikzcd}
\mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[r, equal] & \mathsf{Spectra}(MO, H\mathbb{F}_2P) \\
\mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[u, hook] \arrow[r, equal] & \mathsf{RingSpectra}(MO, H\mathbb{F}_2P). \arrow[u, hook]
\end{tikzcd}$$

Proof. The top isomorphism asserts only that $\mathbb{F}_2$–cohomology and $\mathbb{F}_2$–homology are linearly dual to one another. The second follows immediately from investigating the effect of the ring homomorphism diagrams in the bottom-right corner in terms of the subset they select in the top-left. $\square$

Corollary 1.5.4. *There is a bijection between homotopy classes of ring maps $MO \to H\mathbb{F}_2P$ and homotopy classes of factorizations*

$$\begin{tikzcd}
\mathbb{S}^0 \arrow[r] \arrow[dr] & MO(1) \arrow[d, dotted] \\
& H\mathbb{F}_2P.
\end{tikzcd}$$

Proof. We extend the square in the Lemma 1.5.3 using the following diagram:

$$\begin{tikzcd}
\mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO(1)), \mathbb{F}_2) & \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[l] \\
\{f\colon H\mathbb{F}_2P_0(MO(1)) \to \mathbb{F}_2 \mid f(\beta_0') = 1\} \arrow[u, hook] \arrow[r, equal] & \mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2), \arrow[u, hook]
\end{tikzcd}$$

where the equality at bottom follows from the universal property of $H\mathbb{F}_2P_0(MO)$ in $\mathbb{F}_2$–algebras expressed in Corollary 1.5.2. Noting that β_0' is induced by the topological map $\mathbb{S}^0 \to MO(1)$, the condition $f(\beta_0') = 1$ is exactly the condition expressed in the statement of the Corollary. $\square$

Cite me: I wish I could find this in the literature.

Corollary 1.5.5. *There is an* $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$*–equivariant isomorphism of schemes*

$$\operatorname{Spec} H\mathbb{F}_2P_0(MO) \cong \mathrm{Coord}_1(\mathbb{RP}^\infty_{H\mathbb{F}_2P}),$$

where the latter is the subscheme of functions $\mathbb{RP}^\infty_{H\mathbb{F}_2P} \to \widehat{\mathbb{A}}^1$ *which are coordinates (i.e., which are* isomorphisms *of formal schemes—or, equivalently, which restrict to the canonical identification of tangent spaces* $\mathbb{RP}^1_{H\mathbb{F}_2P} = \widehat{\mathbb{A}}^{1,(1)}$*).*

Proof. The conclusion of the previous Corollary is that the $\mathbb{F}_2$–points of $\operatorname{Spec} H\mathbb{F}_2P_0(MO)$ biject with classes $H\mathbb{F}_2P^0MO(1) \cong \widetilde{H\mathbb{F}_2P}^0\mathbb{RP}^\infty$ satisfying the condition that they give an isomorphism $\mathbb{RP}^\infty_{H\mathbb{F}_2P}$. Because $H\mathbb{F}_2P_0(MO)$ is a polynomial algebra, this holds in general: for $u\colon \mathbb{F}_2 \to T$ an $\mathbb{F}_2$–algebra, the T–points of $\operatorname{Spec} H\mathbb{F}_2P_0(MO)$ will biject with coordinates on $u^*\mathbb{RP}^\infty_{H\mathbb{F}_2P}$. The isomorphism of schemes follows, though we have not yet discussed equivarience.

To compute the action of $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$, we turn to the map in Example 1.1.3:

$$\Sigma^\infty BO(1) \xrightarrow{c,\simeq} \Sigma MO(1).$$

Writing $\beta(t) = \sum_{j=0}^\infty \beta_j t^j$ and $\xi(t) = \sum_{k=0}^\infty \xi_k t^{2^k}$, the dual Steenrod coaction on $H\mathbb{F}_2P_0BO(1)$ is encoded by the formula

$$\sum_{j=0}^\infty \psi(\beta_j)t^j = \psi(\beta(t)) = \beta(\xi(t)) = \sum_{j=0}^\infty \beta_j \left(\sum_{k=0}^\infty \xi_k t^{2^k}\right)^j.$$

Because $c_*(\beta_j) = \beta'_{j-1}$, this translates to the formula $\psi(\beta'(t)) = \beta'(\xi(t))$, where

$$\beta'(t) = \sum_{j=0}^\infty \beta'_j t^{j+1}.$$

Passing from $H\mathbb{F}_2P_0(MO(1))$ to $H\mathbb{F}_2P_0(MO) \cong \operatorname{Sym} H\mathbb{F}_2P_0(MO(1))/(\beta'_0 = 1)$, this is precisely the formula for precomposing a coordinate with a strict automorphism—i.e., a point in $\mathrm{Aut}_1(\widehat{\mathbb{G}}_a)$ acts on a point in $\mathrm{Coord}(\mathbb{RP}^\infty_{H\mathbb{F}_2P})$ in the way claimed. $\square$

We are now ready to analyze the group cohomology of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ with coefficients in the comodule $H\mathbb{F}_2P_0(MO)$. This is the last piece of input we need to assess the Adams spectral sequence computing π_*MO.

Theorem 1.5.6 ([Str06, Theorem 12.2], [Mit83, Proposition 2.1])**.** *The action of* $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ *on* $\mathrm{Coord}_1(\widehat{\mathbb{G}}_a)$ *is free:*

$$\mathrm{Coord}_1(\widehat{\mathbb{G}}_a) \cong \operatorname{Spec} \mathbb{F}_2[b_j \mid j \neq 2^k - 1] \times \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a).$$

Proof. Recall, again, that $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ is defined by the (split) kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Consider a point $f \in \mathrm{Coord}_1(\widehat{\mathbb{G}}_a)(R)$, which in terms of the standard coordinate can be expressed as

$$f(x) = \sum_{j=1}^{\infty} b_{j-1} x^j,$$

where $b_0 = 1$. Decompose this series as $f(x) = f_2(x) + f_{\mathrm{rest}}(x)$, with

$$f_2(x) = \sum_{k=0}^{\infty} b_{2^k-1} x^{2^k}, \qquad\qquad f_{\mathrm{rest}}(x) = \sum_{j \neq 2^k} b_{j-1} x^j.$$

Because we assumed $b_0 = 1$ and f_2 is concentrated in power–of–2 degrees, it follows that f_2 gives a point $f_2 \in \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$. We can use it to de-scale and get a new coordinate $g(x) = f_2^{-1}(f(x))$, which has an analogous decomposition into series $g_2(x)$ and $g_{\mathrm{rest}}(x)$. Finally, note that $g_2(x) = x$ and that f_2 is the unique point in $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$ that has this property. □

Corollary 1.5.7 ([Str06, Remark 12.3])**.** *$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1, j \geq 1]$ with $|b_j| = j$.*

Proof. Set $M = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]$, and write $\mathcal{A}P_0'$ for the ring of functions on $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$. It follows from Corollary 1.4.16 applied to Theorem 1.5.6 that the $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$–cohomology of $H\mathbb{F}_2 P_0(MO)$ has amplitude 0:

$$\begin{aligned}\mathrm{Cotor}^{*,*}_{\mathcal{A}P_0'}(\mathbb{F}_2, H\mathbb{F}_2 P_0(MO)) &= \mathrm{Cotor}^{*,*}_{\mathcal{A}P_0'}(\mathbb{F}_2, \mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\ &= \mathbb{F}_2 \square_{\mathcal{A}_*} (\mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\ &= \mathbb{F}_2 \otimes_{\mathbb{F}_2} M = M.\end{aligned}$$

Since the Adams spectral sequence

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)) \Rightarrow \pi_* MO$$

is concentrated on the 0–line, it collapses. Using the residual $\mathbb{G}_m$–action to infer the grading, we deduce

$$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]. \qquad □$$

This is pretty remarkable: some statement about manifold geometry came down to understanding how we could reparametrize a certain formal group, itself a (fairly simple) purely algebraic problem. The connection between these two problems seems fairly miraculous: we needed a small object, $\mathbb{RP}^\infty$, which controlled the whole story; we needed to be able to compute everything about it; and we needed various other "generation" or "freeness" results to work out in our favor. It is not obvious that we will get this lucky twice, should we try to reapply these ideas to other cases. Nevertheless, trying to push our luck as far as possible is the main thrust of the rest of the book. We could close this section with this accomplishment, but there are two easy consequences of this calculation that are worth recording before we leave.

Lemma 1.5.8. *MO splits as a wedge of shifts of $H\mathbb{F}_2$.*

Proof. Referring to Lemma 1.4.3, we find that the Hurewicz map induces a π_*–injection $MO \to H\mathbb{F}_2 \wedge MO$. Pick an $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha$ for $\pi_* MO$ and extend it to a $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha \cup \{w_\beta\}_\beta$ for $\pi_* H\mathbb{F}_2 \wedge MO$. Altogether, this larger basis can be represented as a single map

$$\bigvee_\alpha \Sigma^{|v_\alpha|}\mathbb{S} \vee \bigvee_\beta \Sigma^{|w_\beta|}\mathbb{S} \xrightarrow{\bigvee_\alpha v_\alpha \vee \bigvee_\beta w_\beta} H\mathbb{F}_2 \wedge MO.$$

Smashing through with $H\mathbb{F}_2$ gives an equivalence

$$\bigvee_\alpha \Sigma^{|v_\alpha|} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{|w_\beta|} H\mathbb{F}_2 \xrightarrow{\simeq} H\mathbb{F}_2 \wedge MO.$$

The composite map

$$MO \to H\mathbb{F}_2 \wedge MO \xleftarrow{\simeq} \bigvee_\alpha \Sigma^{|v_\alpha|} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{|w_\beta|} H\mathbb{F}_2 \to \bigvee_\alpha \Sigma^{|v_\alpha|} H\mathbb{F}_2$$

is a weak equivalence. □

Remark 1.5.9. Just using that $\pi_* MO$ is connective and $\pi_0 MO = \mathbb{F}_2$, we can produce a ring spectrum map $MO \to H\mathbb{F}_2$. What we have learned is that this map has a splitting: MO is also an $H\mathbb{F}_2$–algebra.

Remark 1.5.10. We are also in a position to understand the stable cooperations $MO_* MO$. We may rewrite this as

Make sure you got this right.

$$\begin{aligned} MO_* MO = \pi_* MO \wedge MO &= \pi_* MO \wedge_{H\mathbb{F}_2} (H\mathbb{F}_2 \wedge MO) \\ &\Leftarrow \operatorname{Tor}^{\mathbb{F}_2}_{*,*}(MO_*, H\mathbb{F}_{2*}MO) = MO_* \otimes_{\mathbb{F}_2} H\mathbb{F}_{2*}MO. \end{aligned}$$

Hence, a point in Spec $MO_* MO$ consists of a pair of points in Spec MO_* and Spec $H\mathbb{F}_{2*}MO$, which we have already identified individually as formal group laws with vanishing 2–series and formal group laws with logarithms respectively. This description can be amplified to capture all of the structure maps: a formal group law with vanishing 2–series admits a logarithm, which indicates how the composition and conjugation maps of Definition 3.1.14 behave.

Case Study 2

Complex bordism

Having totally dissected unoriented bordism, we can now turn our attention to other sorts of bordism theories, and there are many available: oriented, *Spin*, *String*, complex, . . . —the list continues. We would like to replicate the results from Case Study 1 for these other cases, but upon even a brief inspection we quickly see that only one of the bordism theories mentioned supports this program. Specifically, the space $\mathbb{R}P^\infty = BO(1)$ was a key player in the unoriented bordism story, and the only other similar ground object is $\mathbb{C}P^\infty = BU(1)$ in complex bordism. This informs our choice to spend this Case Study focused on it. To begin, the contents of Lecture 1.1 can be replicated essentially *mutatis mutandis*, resulting in the following theorems:

Theorem 2.0.1 (cf. Lemma 1.1.5 and surrounding discussion)**.** *There is a map of infinite–loopspaces*

$$J_{\mathbb{C}} \colon BU \to BGL_1\mathbb{S}$$

called the complex J–homomorphism.

Definition 2.0.2 (cf. Definition 1.1.9)**.** The associated Thom spectrum is written "MU" and called *complex bordism*. A map $MU \to E$ of ring spectra is said to be a *complex orientation of* E.

Something I've seen more than once is an equivalence $MU(k) \simeq BU(k)/BU(k-1)$. It's not immediately obvious to me where this comes from. Where does it come from? Is it helpful to think about?

Theorem 2.0.3 (cf. Theorem 1.1.12)**.** *For a complex vector bundle ξ on a space X and a complex-oriented ring spectrum E, there is a natural equivalence*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X. \quad \square$$

Corollary 2.0.4 (cf. Example 1.1.15)**.** *In particular, for a complex-oriented ring spectrum E it follows that $E^*\mathbb{C}P^\infty$ is isomorphic to a one–dimensional power series ring.* $\square$

Maybe I'm confused about grading issues, but I thought E^*CP^∞ was a polynomial ring and EP^0CP^∞ is the power series ring?

Also, this is a nice argument. Usually this computation proceeds through the AHSS. Can this method be adapted to spaces other than CP^∞?

We would like to then review the results of Lecture 1.3 and conclude (by reinterpreting Corollary 2.0.4) that $\mathbb{C}P^\infty_E$ gives a 1–dimensional formal group over $\operatorname{Spec} E_*$. In order to make this statement honestly, however, we are first required to describe more responsibly the algebraic geometry we outlined in Lecture 1.2. Specifically, the characteristic 2 nature of the unoriented bordism ring was a major simplifying feature which made it wholly amenable to study by $H\mathbb{F}_2$. In turn, $H\mathbb{F}_2$ has many nice properties—for example, it has a duality between homology and cohomology, and it supports a Künneth isomorphism—and these are reflected in the extremely simple algebraic geometry of $\operatorname{Spec}\mathbb{F}_2$. By contrast, the complex bordism ring is considerably more complicated, not least because it is a characteristic 0 ring, and more generally we have essentially no control over the behavior of the coefficient ring E_* of some other complex–oriented theory. Nonetheless, once the background theory and construction of "X_E" are taken care of in Lecture 2.1, we indeed find that $\mathbb{C}\mathrm{P}^\infty_E$ is a 1–dimensional formal group over $\operatorname{Spec} E_*$.

However, where we could explicitly calculate $\mathbb{R}\mathrm{P}^\infty_{H\mathbb{F}_2}$ to be $\widehat{\mathbb{G}}_a$, we again have little control over what formal group $\mathbb{C}\mathrm{P}^\infty_E$ could possibly be. In the universal case, $\mathbb{C}\mathrm{P}^\infty_{MU}$ comes equipped with a natural coordinate, and this induces a map

$$\operatorname{Spec} MU_* \to \mathcal{M}_{\mathbf{fgl}}$$

from the spectrum associated to the coefficient ring of complex bordism to the moduli of formal group laws. The conclusion of this Case Study in Corollary 2.6.11 (modulo an algebraic result, shown in the next Case Study as Theorem 3.2.2) states that this map is an isomorphism, so that $\mathbb{C}\mathrm{P}^\infty_{MU}$ is the universal—i.e., maximally complicated—formal group. Our route for proving this passes through the foothills of the theory of "p^{th} power operations", which simultaneously encode many possible natural transformations from MU–cohomology to itself glommed together in a large sum, one term of which is the literal p^{th} power. Remarkably, the identity operation also appears in this family of operations, and the rest of the operations are in some sense controlled by this naturally occuring formal group law. A careful analysis of this sum begets the inductive proof in Corollary 2.6.6 that $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ is surjective.

The execution of this proof requires some understanding of cohomology operations for complex-oriented cohomology theories generally. Stable such operations correspond to homotopy classes $MU \to E$, i.e., elements of $E^0 MU$, which correspond via the Thom isomorphism to elements of $E^0 BU$. This object is the repository of E–characteristic classes for complex vector bundles, which we describe in terms of divisors on formal curves. This amounts to a description of the formal schemes $BU(n)_E$, which underpins our understanding of the whole story and which significantly informs our study of connective orientations in Case Study 5.

2.1 Calculus on formal varieties

In light of the introduction, we see that it would be prudent to develop some of the theory of formal schemes and formal varieties outside of the context of $\mathbb{F}_2$–algebras. However, writing down a list of definitions and checking that they have good enough properties is not especially enlightening or fun. Instead, it will be informative to understand where these objects come from in algebraic geometry, so that we can carry the accompanying geometric intuition along with us as we maneuver our way back toward homotopy theory and bordism. Our overarching goal in this Lecture is to develop a notion of calculus (and analytic expansions in particular) in the context of affine schemes. The place to begin is with a definition of cotangent and tangent spaces, as well as some supporting vocabulary.

Definition 2.1.1 (cf. Definition 1.2.1)**.** For an R–algebra A, the functor $\operatorname{Spec} A \colon \mathsf{Algebras}_R \to \mathsf{Sets}$ defined by

$$(\operatorname{Spec} A)(T) := \mathsf{Algebras}_R(A, T)$$

is called the *spectrum of* A. A functor X which is naturally isomorphic to to $\operatorname{Spec} A$ for some A is called an *affine (R–)scheme*, and $A = \mathcal{O}_{\operatorname{Spec} A}$ is called its *ring of functions*. A subfunctor $Y \subseteq X$ is said to be a *closed*[1] *subscheme* when an identification[2] $X \cong \operatorname{Spec} A$ induces a further identification

$$\begin{tikzcd} Y \arrow[r] & X \\ \operatorname{Spec}(R/I) \arrow[u, "\simeq"'] \arrow[r] & \operatorname{Spec} R. \arrow[u, "\simeq"'] \end{tikzcd}$$

Definition 2.1.2. Take $S = \operatorname{Spec} R$ to be our base scheme, let $X = \operatorname{Spec} A$ be an affine scheme over S, and consider an S–point $x \colon S \to X$ of X. The point x is automatically closed, so that x is presented as $\operatorname{Spec} A/I \to \operatorname{Spec} A$ for some ideal I. The *cotangent space* $T_x^* X$ is defined by the quotient R–module

$$T_x^* X := I/I^2,$$

consisting of functions vanishing at x as considered up to first order. Examples of these include the linear parts of curves passing through x, so we additionally define the *tangent space* $T_x X$ by

$$T_x X = \mathsf{Schemes}_{\operatorname{Spec} R/}(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X),$$

i.e., maps $\operatorname{Spec} R[\varepsilon]/\varepsilon^2 \to X$ which restrict to $x \colon S \to X$ upon setting $\varepsilon = 0$.

Remark 2.1.3. In the situation above, there is a natural map $T_x X \to \mathsf{Modules}_R(T_x^* X, R)$. A map $\mathcal{O}_X \to R[\varepsilon]/\varepsilon^2$ induces a map $I \to (\varepsilon)$ and hence a map

$$I/I^2 \to (\varepsilon)/(\varepsilon^2) \cong R,$$

i.e., a point in $T_x^* X$.

[1]The word "closed" is meant to suggest properties of these inclusions: in suitable senses, they are closed under finite unions and arbitrary intersections.

[2]This property is independent of choice of chart.

Harkening back to Example 1.3.8, the definition of the R–module tangent space begs promotion to an S–scheme.

Lemma 2.1.4. *There is an affine scheme $T_x X$ defined by*

$$(T_x X)(T) := \left\{ (u, f) \,\middle|\, \begin{array}{c} u\colon\ \operatorname{Spec} T \to S, \\ f \in T_{u^* x} u^* X \end{array} \right\}.$$

Proof sketch. We specialize an argument of Strickland [Str99b, Proposition 2.94] to the case at hand.[3] We start by seeking an R–algebra B such that R–algebra maps $B \to T$ biject with pairs of maps $u\colon\ R \to T$ and T–algebra maps

$$f\colon\ A \otimes_R T \to R[\varepsilon]/\varepsilon^2 \otimes_R T.$$

Such maps f biject with R–algebra maps

$$A \to R[\varepsilon]/\varepsilon^2 \otimes_R T.$$

Seriously consider just writing out what you mean here in symbols.

Noting that $R[\varepsilon]/(\varepsilon)^2$ is free and finite–dimensional as an R–module, we forget from R–algebra maps down to just R–module maps, use R–linear duality to move it to the domain, promote it back to an R–algebra by forming the symmetric algebra, then finally try to pick out the maps of interest by imposing a quotient. By expanding Strickland's formulas, we arrive at the equation

$$\underline{\mathsf{Schemes}}_S(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X) = \operatorname{Spec} A\{1, da \mid a \in A\} \Big/ \left(\begin{array}{c} dr = 0 \text{ for } r \in R, \\ d(a_1 a_2) = da_1 \cdot a_2 + a_1 \cdot da_2 \end{array} \right).$$

To extract the scheme $T_x X$ from this, we construct the pullback

$$T_x X := \underline{\mathsf{Schemes}}_S(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X) \times_X S,$$

where the structure maps are given on the left by setting $\varepsilon = 0$ and on the right using the point x. Expanding the formulas again shows that the coordinate ring of this affine scheme is given by

$$\mathcal{O}_{T_x X} = A/I^2 \cong R \oplus T_x^* X. \qquad \square$$

Definition 2.1.5. The ring of functions appearing in the proof above fits into an exact sequence

$$0 \to \Omega_{A/R} \to A\{1, da \mid a \in A\} \Big/ \left(\begin{array}{c} dr = 0 \text{ for } r \in R, \\ d(a_1 a_2) = da_1 \cdot a_2 + a_1 \cdot da_2 \end{array} \right) \xrightarrow{da=0} A\{1\} \to 0.$$

The kernel $\Omega_{A/R}$ is called the module of *Kähler differentials* (of A, relative to R). The map $d\colon\ R \to \Omega^1_{A/R}$ is the universal R–linear derivation into an A–module, i.e.,

$$\mathsf{Derivations}_R(A, M) = \mathsf{Modules}_A(\Omega^1_{A/R}, M).$$

[3]Strickland also shows that mapping schemes between formal schemes exist considerably more generally [Str99b, Theorem 4.69]. The source either has to be "finite" in some sense, in which case the proof proceeds along the lines presented here, or it has to be *coalgebraic*, which is an important technical tool that we discuss much later in Definition 5.1.6.

The upshot of this calculation is that $\operatorname{Spec} A/I^2$ is a natural place to study the linear behavior of functions on X near x. We have also set the definitions up so that we can easily generalize to higher-order approximations:

Definition 2.1.6. More generally, the n^{th} *jet space* of X at x, or the n^{th} *order neighborhood* of x in X, is defined by

$$\underline{\mathsf{Schemes}}_S(\operatorname{Spec} R[\varepsilon]/\varepsilon^{n+1}, X) \times_X S \cong \operatorname{Spec} A/I^{n+1}.$$

Each jet space has an inclusion from the one before, modeled by the closed subscheme $\operatorname{Spec} A/I^n \to \operatorname{Spec} A/I^{n+1}$.

In order to study analytic expansions of functions, we bundle these jet spaces together into a single object embodying formal expansions in X at x:

Definition 2.1.7. Fix a scheme S. A *formal S–scheme* $X = \{X_\alpha\}_\alpha$ is an ind-system of S–schemes X_α.[4] Given a closed subscheme Y of an affine S–scheme X, we define the n^{th} *order neighborhood of Y in X* to be the scheme $\operatorname{Spec} R/I^{n+1}$. The *formal neighborhood of Y in X* is then defined to be the formal scheme

$$X^\wedge_Y := \operatorname{Spf} R^\wedge_I := \left\{\operatorname{Spec} R/I \to \operatorname{Spec} R/I^2 \to \operatorname{Spec} R/I^3 \to \cdots\right\}.$$

In the case that $Y = S$, this specializes to the system of jet spaces as in Definition 2.1.6.

Although we will make use of these definitions generally, the following ur-example captures the most geometrically-intuitive situation.

Example 2.1.8. Picking the affine scheme $X = \operatorname{Spec} R[x_1, \ldots, x_n] = \mathbb{A}^n$ and the point $x = (x_1 = 0, \ldots, x_n = 0)$ gives a formal scheme known as *formal affine n–space*, given explicitly by

$$\widehat{\mathbb{A}}^n = \operatorname{Spf} R[\![x_1, \ldots, x_n]\!].$$

Evaluated on a test algebra T, $\widehat{\mathbb{A}}^1(T)$ yields the ideal of nilpotent elements in T and $\widehat{\mathbb{A}}^n(T)$ its n–fold Cartesian power.

Lemma 2.1.9. *Pointed maps $\widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^m$ naturally biject with m–tuples of n–variate power series with no constant term.* □

[4] This definition, owing to Strickland [Str99b, Definition 4.1], is somewhat idiosyncratic. Its generality gives it good categorical properties, but it is somewhat disconnected from the formal schemes familiar to algebraic geometers, which primarily arise through linearly topologized rings **Find a citation for this style of definition**. For functor-of-points definitions that hang more tightly with the classical definition, the reader is directed toward Strickland's solid formal schemes [Str99b, Section 4.2] or to Beilinson and Drinfel'd [BD, Section 7.11.1].

The preceding Lemma shows how formal varieties are especially nice, because maps between them can be boiled down to statements about power series.[5] In particular, this allows local theorems from analytic differential geometry to be imported, including a version of the inverse function theorem.

Cite me: This is 3.1.8 in the Crystals notes..

Theorem 2.1.10. *A pointed map $f \colon V \to W$ of finite–dimensional formal varieties is an isomorphism if and only if the induced map $T_0 f \colon T_0 V \to T_0 W$ is an isomorphism of R–modules.*

Wait a second, have you introduced this terminology yet?

Consider deleting this proof? Who cares?

Proof. First, reduce to the case where $V \cong \widehat{\mathbb{A}}^n$ and $W \cong \widehat{\mathbb{A}}^n$ have the same dimension, and select charts for both. Then, $T_0 f$ is a matrix of dimension $n \times n$. If $T_0 f$ fails to be invertible, we are done, and if it is invertible, we replace f by $f \circ (T_0 f)^{-1}$ so that $T_0 f$ is the identity matrix.

We now construct the inverse function by induction on degree. Set $g^{(1)}$ to be the identity function, so that f and $g^{(1)}$ are mutual inverses when restricted to the first-order neighborhood. So, suppose that $g^{(r-1)}$ has been constructed, and consider its interaction with f on the r^{th} order neighborhood:

$$g_i^{(r-1)}(f(x)) = x_i + \sum_{|J|=r} c_J x_1^{J_1} \cdots x_n^{J_n} + o(r+1).$$

By adding in the correction term

$$g_i^{(r)} = g_i^{(r-1)} - \sum_{|J|=r} c_J x_1^{J_1} \cdots x_n^{J_n},$$

we have $g_i^{(r)}(f(x)) = x_i + o(r)$. $\square$

Part of the point of the geometric language is to divorce abstract rings (e.g., $E^0\mathbb{C}\mathrm{P}^\infty$) from concrete presentations (e.g., $E^0[\![x]\!]$), so we additionally reserve some vocabulary for the property of being isomorphic to $\widehat{\mathbb{A}}^n$:

Definition 2.1.11. A *formal affine variety* (of dimension n) is a formal scheme V which is (noncanonically) isomorphic to $\widehat{\mathbb{A}}^n$. The two maps in an isomorphism pair

$$V \xrightarrow{\simeq} \widehat{\mathbb{A}}^n, \quad V \xleftarrow{\simeq} \widehat{\mathbb{A}}^n$$

are called a *coordinate (system)* and a *parameter (system)* respectively. Finally, an S–point $x \colon S \to X$ is called *formally smooth* when $X^\wedge_x$ gives a formal variety.

Work this in, comparing property with structure.

Get rid of this obnoxious auxiliary module M and just call $I \otimes_R M$ by its real name: I^n.

Theorem 2.1.12. *Let R be a Noetherian ring and $F \colon \mathsf{Algebras}_{R/} \to \mathsf{Sets}$ be a functor such that*

[5] In some sense, this Lemma is a full explanation for why anyone would even think to involve formal geometry in algebraic topology (nevermind how useful the program has been in the long run). Calculations in algebraic topology have long been expressed in terms of power series rings, and with this Lemma we are provided geometric interpretations for such statements.

$F(R) = 0$, F takes surjective maps to surjective maps, and there is a fixed finite free R–module M such that F carries square-zero extensions of R–algebras $I \to B \to B'$ to product sequences

$$I \otimes_R M \to G(B) \to G(B').$$

Then, $V \cong \widehat{\mathbb{A}}^n$, where $n = \dim M$.

Proof. □

Cite me: This is 9.6.4 in the Crystals notes.

Corollary 2.1.13. *An S–point $x\colon S \to X$ is formally smooth exactly when for any nilpotent thickenings $S \to \operatorname{Spec} B \to \operatorname{Spec} B'$ and any solid diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "x"'] & \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} B' \arrow[dl, dotted] \\ & X & \end{tikzcd}$$

there exists a dotted arrow extending the diagram.

Proof. □

Prove me.

Below this, things are figured out.

With all this algebraic geometry in hand, we now return to our original motivation: extracting formal schemes from the rings appearing in algebraic topology.

Definition 2.1.14 (cf. Definition 1.3.6)**.** Let E be an even-periodic ring spectrum, and let X be a CW–space. Because X is compactly generated, it can be written as the colimit of its compact subspaces $X^{(\alpha)}$, and we set[6]

$$X_E := \operatorname{Spf} E^0 X := \{\operatorname{Spec} E^0 X^{(\alpha)}\}_\alpha.$$

Consider the example of $\mathbb{C}\mathrm{P}^\infty_E$ for E a complex-oriented cohomology theory. We saw in Corollary 2.0.4 that the complex-orientation determines an isomorphism $\mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^1$ (i.e., an isomorphism $E^0\mathbb{C}\mathrm{P}^\infty \cong E^0[\![x]\!]$). However, the object "$E^0\mathbb{C}\mathrm{P}^\infty$" is something that exists independent of the orientation map $MU \to E$, and the language of Definition 2.1.11 allows us to make the distinction between the property and the data:

[6] The careful reader will immediately notice that the rings in the pro-system underlying Definition 2.1.14 run the risk of not being even-concentrated. We are thus required to make the following technical compromise: for any pro-isomorphic system of even E^*–algebras $\{R_\beta \otimes_{E^0} E^*\} \cong \{E^0 X^{(\alpha)}\}_\alpha$ we set

$$X_E := \{\operatorname{Spec} R_\beta\}_\beta,$$

and otherwise we leave X_E undefined. For example, the technical condition of Definition 2.1.14 is satisfied if there exists a cofinal subsystem of $\{X^{(\alpha)}\}_\alpha$ with $E^* X^{(\alpha)}$ even–concentrated. This follows, for instance, from $H\mathbb{Z}_* X$ being free and even [Str99b, Definition 8.15, Proposition 8.17].

Lemma 2.1.15. *A cohomology theory E is complex orientable (i.e., it is able to receive a ring map from MU) precisely when $\mathbb{C}\mathrm{P}^\infty_E$ is a formal curve (i.e., it is a formal variety of dimension 1). A choice of orientation $MU \to E$ determines a coordinate $\mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^1$.* $\square$

As in Example 1.3.7, the formal scheme $\mathbb{C}\mathrm{P}^\infty_E$ has additional structure: it is a group. We close today with some remarks about such objects.

Definition 2.1.16. A formal group is a formal variety endowed with an abelian group structure.[7] If E is a complex-orientable cohomology theory, then $\mathbb{C}\mathrm{P}^\infty_E$ naturally forms a (1–dimensional) formal group using the map classifying the tensor product of line bundles.

Remark 2.1.17. As with formal schemes, formal groups can arise as formal completions of an algebraic group at its identity point. It turns out that there are many more formal groups than come from this procedure, a phenomenon that is of keen interest to stable homotopy theorists—see Appendix A.4.

We give the following Corollary as an example of how nice the structure theory of formal varieties is.

Corollary 2.1.18. *As with physical groups, the formal group addition map on $\widehat{\mathbb{G}}$ determines the inverse law.*

Proof. Consider the shearing map

$$\begin{gathered}\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \xrightarrow{\sigma} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}}, \\ (x, y) \mapsto (x, x+y).\end{gathered}$$

The induced map $T_0\sigma$ on tangent spaces is evidently invertible, so by Theorem 2.1.10 there is an inverse map $(x, y) \mapsto (x, y-x)$. Setting $y = 0$ and projecting to the second factor gives the inversion map. $\square$

Definition 2.1.19. Let $\widehat{\mathbb{G}}$ be a formal group. In the presence of a coordinate $\varphi\colon\ \widehat{\mathbb{G}} \cong \widehat{\mathbb{A}}^n$, the addition law on $\widehat{\mathbb{G}}$ begets a map

Again, the vertical arrows should be arrows, not equal signs?

$$\begin{tikzcd} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \arrow[r] \arrow[d, equal] & \widehat{\mathbb{G}} \arrow[d, equal] \\ \widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n \arrow[r] & \widehat{\mathbb{A}}^n, \end{tikzcd}$$

and hence a n-tuple of $(2n)$-variate power series "$+_\varphi$", satisfying

$$\begin{aligned} \underline{x} +_\varphi \underline{y} &= \underline{y} +_\varphi \underline{x}, && \text{(commutativity)} \\ \underline{x} +_\varphi \underline{0} &= \underline{x}, && \text{(unitality)} \\ \underline{x} +_\varphi (\underline{y} +_\varphi \underline{z}) &= (\underline{x} +_\varphi \underline{y}) +_\varphi \underline{z}. && \text{(associativity)} \end{aligned}$$

[7] Formal groups in dimension 1 are automatically commutative if and only if the ground ring has no elements which are simultaneously nilpotent and torsion [Haz12, Theorem I.6.1].

Such a tuple $+_\varphi$ is called a *formal group law*, and it is the concrete data associated for a formal group.

Let's now consider two examples of E which are complex-orientable and describe these invariants for them.

Example 2.1.20. There is an isomorphism $\mathbb{CP}^\infty_{H\mathbb{Z}P} \cong \widehat{\mathbb{G}}_a$. This follows from reasoning identical to that given in Example 1.3.7.

Example 2.1.21. There is also an isomorphism $\mathbb{CP}^\infty_{KU} \cong \widehat{\mathbb{G}}_m$. A reasonablechoice of first Chern class is given by the natural topological map

This is not a good word and not good reasoning.

$$c_1 \colon \Sigma^{-2}\Sigma^\infty\mathbb{CP}^\infty \xrightarrow{1-\beta\mathcal{L}} KU,$$

and a formula for the first Chern class of the tensor product is thus

$$\begin{aligned} c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) &= 1 - \beta(\mathcal{L}_1 \otimes \mathcal{L}_2) \\ &= -\beta^{-1}\left((1-\beta\mathcal{L}_1)\cdot(1-\beta\mathcal{L}_2)\right) + (1-\beta\mathcal{L}_1) + (1-\beta\mathcal{L}_2) \\ &= c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - \beta^{-1}c_1(\mathcal{L}_1)c_1(\mathcal{L}_2). \end{aligned}$$

In this coordinate on $\mathbb{CP}^\infty_{KU}$, the group law is then $F(x_1, x_2) = x_1 + x_2 - \beta^{-1}x_1x_2$. Using the coordinate function $1 - t$, this is also the coordinate that arises on the formal completion of $\mathbb{G}_m$ at $t = 1$:

$$\begin{aligned} x_1(t_1) +_{\mathbb{G}_m} x_2(t_2) &= 1 - (1-t_1)(1-t_2) \\ &= t_1 + t_2 - t_1t_2. \end{aligned}$$

As an application of all these tools, we will show that the rational theory of formal groups is highly degenerate: every rational formal group is isomorphic to $\widehat{\mathbb{G}}_a$. Suppose now that R is a $\mathbb{Q}$–algebra and that $A = R[\![x]\!]$ is the coordinatized ring of functions on a formal line over R. What's special about this rational curve case is that differentiation gives an isomorphism between the Kähler differentials $\Omega^1_{A/R}$ and the ideal (x) of functions vanishing at the origin (i.e., the ideal sheaf selecting the closed subscheme $0\colon \operatorname{Spec} R \to \operatorname{Spf} A$). Its inverse is formal integration:

$$\int \colon \left(\sum_{j=0}^\infty c_j x^j\right) dx \mapsto \sum_{j=0}^\infty \frac{c_j}{j+1} x^{j+1}.$$

Maybe move the definition of invariant differentials out to its own environment.

Theorem 2.1.22. *For R a $\mathbb{Q}$–algebra, there is a canonical isomorphism of formal groups*

$$\log \colon \widehat{\mathbb{G}} \to T_0\widehat{\mathbb{G}} \otimes \widehat{\mathbb{G}}_a.$$

Proof. Taking a cue from classical Lie theory, we attempt to use integration to define exponential and logarithm functions for a given formal group law F. This is typically accomplished by studying invariant differentials: a 1–form $\omega \in \Omega^1_{A/R}$ is said to be *invariant (under F)* when $\omega = T_y^* \omega$ for all translations $T_y(x) = x +_F y$. In terms of a coordinate $\omega = f(x)dx$, this condition becomes

Consider rewriting this to use x_1 and x_2 rather than x and y.

$$f(x)dx = f(y +_F x)d(y +_F x) = f(y +_F x)\frac{\partial(y +_F x)}{\partial x}dx.$$

Restricting to the origin by setting $x = 0$, we deduce the condition

$$f(0) = f(y) \cdot \left.\frac{\partial(y +_F x)}{\partial x}\right|_{x=0}.$$

Since R is a $\mathbb{Q}$–algebra, integrating against y yields

$$\log_F(y) = \int f(y)\, dy = f(0) \int \left(\left.\frac{\partial(y +_F x)}{\partial x}\right|_{x=0} \right)^{-1} dy.$$

To see that the series $\log_F$ has the claimed homomorphism property, note that

$$\frac{\partial \log_F(y +_F x)}{\partial x}dx = f(y +_F x)d(y +_F x) = f(x)dx = \frac{\partial \log_F(x)}{\partial x}dx,$$

so $\log_F(y +_F x)$ and $\log_F(x)$ differ by a constant. Checking at $y = 0$ shows that the constant is $\log_F(x)$, hence

$$\log_F(x +_F y) = \log_F(x) + \log_F(y).$$

The choice of boundary value $f(0)$ corresponds to the choice of vector in $T_0\widehat{\mathbb{G}}$. $\square$

Example 2.1.23. Consider the formal group law $x_1(t_1) +_{\widehat{\mathbb{G}}_m} x_2(t_2) = t_1 + t_2 - t_1 t_2$ studied in Example 2.1.21. Its associated rational logarithm is computed as

$$\log_{\widehat{\mathbb{G}}_m}(t_2) = f(0) \cdot \int \frac{1}{1 - t_2} dt_2 = -f(0)\log(1 - t_2) = f(0)\log(x_2),$$

where "$\log(x_2)$" refers to the classical natural logarithm of x_2.

2.2 Divisors on formal curves

I think this section is a little short. Maybe we could expand the proof of the comparison between Cartier and Weil divisors at the end? Or we would include a proof of the universality of the divisor scheme? We could talk about divisors and line bundles on complex curves, culminating in the observation that principal divisors appear as subsheaves of the sheaf of meromorphic functions and that every divisor is locally principal. Something to give

We continue to develop vocabulary and accompanying machinery used to give algebro-geometric reinterpretations of the results in the introduction to this Case Study. In the previous section we deployed the language of formal schemes to recast Corollary 2.0.4 in geometric terms, and we now turn towards reencoding Theorem 2.0.3. In Definition 1.4.8

and Lemma 1.4.9 we discussed a general correspondence between R–modules and quasicoherent sheaves over $\operatorname{Spec} R$, and the isomorphism of 1–dimensional E_*X–modules appearing in Theorem 2.0.3 moves us to study sheaves over X_E which are 1–dimensional—i.e., line bundles. In fact, for the purposes of Theorem 2.0.3, we will find that it suffices to understand the basics of the geometric theory of line bundles *just over formal curves*. This is our goal in this Lecture, and we leave the applications to algebraic topology aside for later. For the rest of this section we fix the following three pieces of data: a base formal scheme S, a formal curve C over S, and a distinguished point $\zeta\colon\ S \to C$ on C.

To begin, we will be interested in a very particular sort of line bundle over C: for any function f on C which is not a zero-divisor, the subsheaf $\mathcal{I}_f = f \cdot \mathcal{O}_C$ of functions on C which are divisible by f form a 1–dimensional $\mathcal{O}_C$–submodule of the ring of functions $\mathcal{O}_C$ itself—i.e., a line bundle on C. By interpreting $\mathcal{I}_f$ as an ideal sheaf, this gives rise to a second interpretation of this data in terms of a closed subscheme

$$\operatorname{Spec} \mathcal{O}_C / f \subseteq C,$$

which we will refer to as the *divisor* associated to $\mathcal{I}_f$. In general these can be somewhat pathological, so we specialize further to an extremely nice situation:

Definition 2.2.1 ([Str99b, Section 5.1])**.** An *effective Weil divisor* D on a formal curve C is a closed subscheme of C whose structure map $D \to S$ presents D as finite and free. We say that the *rank* of D is n when its ring of functions $\mathcal{O}_D$ is free of rank n over $\mathcal{O}_S$.

Lemma 2.2.2 ([Str99b, Proposition 5.2], see also [Str99b, Example 2.10])**.** *There is a scheme* $\operatorname{Div}_n^+ C$ *of effective Weil divisors of rank n. It is a formal variety of dimension n. In fact, a coordinate x on C determines an isomorphism* $\operatorname{Div}_n^+ C \cong \widehat{\mathbb{A}}^n$ *where a divisor D is associated to a monic polynomial $f_D(x)$ with nilpotent lower-order coefficients.*

Proof sketch. To pin down the functor we wish to analyze, we make the definition

$$\operatorname{Div}_n^+(C)(R) = \left\{ (a, D) \middle| \begin{array}{c} a : \operatorname{Spec} R \to S, \\ D \text{ is an effective divisor on } C \times_S \operatorname{Spec} R \end{array} \right\}.$$

To show that this is a formal variety, we pursue the final claim and select a coordinate x on C, as well as a point $(a, D) \in \operatorname{Div}_n^+(C)(T)$. The coordinate presents $C \times_S \operatorname{Spec} T$ as

$$C \times_X \operatorname{Spec} T = \operatorname{Spf} T[\![x]\!],$$

and the characteristic polynomial $f_D(x)$ of x in $\mathcal{O}_D$ presents D as the closed subscheme

$$D = \operatorname{Spf} R[\![x]\!]/(f_D(x))$$

for $f_D(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ monic. Additionally, for any prime ideal $\mathfrak{p} \in R$ we can form the field $R_{\mathfrak{p}}/\mathfrak{p}$, over which the module $\mathcal{O}_D \otimes_R R_{\mathfrak{p}}/\mathfrak{p}$ must still be of rank n. It follows that

$$f_D(x) \otimes_R R_{\mathfrak{p}}/\mathfrak{p} \equiv x^n,$$

hence that each a_j lies in the intersection of all prime ideals of R, hence that each a_j is nilpotent.

In turn, this means that the polynomial f_D is selected by a map $\operatorname{Spec} R \to \widehat{\mathbb{A}}^n$. Conversely, given such a map, we can form the polynomial $f_D(x)$ and the divisor D. $\square$

Remark 2.2.3. This Lemma effectively connects several simple dots: especially nice polynomials $f_D(x) \in \mathcal{O}_C$, their vanishing loci $D \subseteq C$, and the ideal sheaves $\mathcal{I}_D$ of functions divisible by f—i.e., functions with a partially prescribed vanishing set. Basic operations on polynomials affect their vanishing loci in predictable ways, and these operations are also reflected on the level of divisor schemes. For instance, there is a unioning map

$$\begin{aligned} \operatorname{Div}_n^+ C \times \operatorname{Div}_m^+ C &\to \operatorname{Div}_{n+m}^+ C, \\ (D_1, D_2) &\mapsto D_1 \sqcup D_2. \end{aligned}$$

At the level of ideal sheaves, we use their 1–dimensionality to produce the formula

$$\mathcal{I}_{D_1 \sqcup D_2} = \mathcal{I}_{D_1} \otimes_{\mathcal{O}_C} \mathcal{I}_{D_2}.$$

Under a choice of coordinate x, the map at the level of polynomials is given by

$$(f_{D_1}, f_{D_2}) \mapsto f_{D_1} \cdot f_{D_2}.$$

Next, note that there is a canonical isomorphism $C \to \operatorname{Div}_1^+ C$. Iterating the above addition map gives the vertical map in the following triangle:

$$\begin{tikzcd} & C^{\times n} \arrow[dl] \arrow[d, "\sqcup"] \\ C^{\times n}_{\Sigma_n} \arrow[r, dotted, "\cong"] & \operatorname{Div}_n^+ C. \end{tikzcd}$$

Lemma 2.2.4. *The object $C^{\times n}_{\Sigma_n}$ exists as a formal variety, it factors the iterated addition map, and the dotted arrow is an isomorphism.*

Proof. The first assertion is a consequence of Newton's theorem on symmetric polynomials: the subring of symmetric polynomials in $R[x_1, \ldots, x_n]$ is itself polynomial on generators

$$\sigma_j(x_1, \ldots, x_n) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = j}} x_{S_1} \cdots x_{S_j},$$

and hence

$$R[\sigma_1, \ldots, \sigma_n] \subseteq R[x_1, \ldots, x_n]$$

gives an affine model of $C^{\times n} \to C^{\times n}_{\Sigma_n}$. Picking a coordinate on C allows us to import this fact into formal geometry to deduce the existence of $C^{\times n}_{\Sigma_n}$. The factorization then follows by noting that the iterated $\sqcup$ map is symmetric. Finally, Remark 2.2.3 shows that the horizontal map pulls the coordinate a_j back to σ_j, so the third assertion follows. $\square$

Remark 2.2.5. The map $C^{\times n} \to C^{\times n}_{\Sigma_n}$ is an example of a map of schemes which surjective *as a map of sheaves*. This is somewhat subtle: for any given test ring T, it is not necessarily the case that $C^{\times n}(T) \to C^{\times n}_{\Sigma_n}(T)$ is surjective on T–points—this amounts to the claim that not every polynomial can be written as a product of linear factors. However, for a fixed point $f \in C^{\times n}_{\Sigma_n}(T)$, we are guaranteed a flat covering $T \to \prod_j T_j$ such that there are individual lifts $\widetilde{f}_j$ of f over each T_j.

Add an example here. Remember that the coefficients are supposed to be nilpotents.

Now we use the pointing $\zeta\colon S \to C$ to interrelate divisor schemes of varying ranks. Together with the $\sqcup$ operation, ζ gives a composite

$$\begin{array}{ccccccc} \operatorname{Div}_n^+ C & \longrightarrow & C \times \operatorname{Div}_n^+ C & \longrightarrow & \operatorname{Div}_1^+ C \times \operatorname{Div}_n^+ C & \longrightarrow & \operatorname{Div}_{n+1}^+ C, \\ D & \longmapsto & (\zeta, D) & \longmapsto & ([\zeta], D) & \longmapsto & [\zeta] \sqcup D. \end{array}$$

Definition 2.2.6. We define the following variants of "stable divisor schemes":

$$\begin{aligned} \operatorname{Div}^+ C &= \coprod_{n \geq 0} \operatorname{Div}_n^+ C, \\ \operatorname{Div}_n C &= \operatorname{colim}\left(\operatorname{Div}_n^+ C \xrightarrow{[\zeta]+-} \operatorname{Div}_{n+1}^+ C \xrightarrow{[\zeta]+-} \cdots \right), \\ \operatorname{Div} C &= \operatorname{colim}\left(\operatorname{Div}^+ C \xrightarrow{[\zeta]+-} \operatorname{Div}^+ C \xrightarrow{[\zeta]+-} \cdots \right) \\ &\cong \coprod_{n \in \mathbb{Z}} \operatorname{Div}_n C. \end{aligned}$$

Theorem 2.2.7 (cf. Corollary 5.1.10)**.** *The scheme* $\operatorname{Div}^+ C$ *models the free formal monoid on the unpointed formal curve* C*. The scheme* $\operatorname{Div} C$ *models the free formal group on the unpointed formal curve* C*. The scheme* $\operatorname{Div}_0 C$ simultaneously *models the free formal monoid and the free formal group on the* pointed *formal curve* C*.* □

We will postpone the proof of this Theorem until later, once we've developed a theory of coalgebraic formal schemes.

Remark 2.2.8. Given $q\colon C \to C'$ a map of formal curves over S and $D \subseteq C$ a divisor on C, the composite $D \to C \to C'$ is also a divisor, denoted $q_* D$. Theorem 2.2.7 gives a second construction of $q_* D$ in the stable case, using the composite

Is there a proof here? It's still finite and free over the base, but is it still a closed subscheme? Does this come out of some kind of finite $\Longrightarrow$ compact argument?

$$C \xrightarrow{q} C' \cong \operatorname{Div}_1^+ C' \to \operatorname{Div} C'.$$

Since the target of this map is a formal group scheme, universality induces a map

$$q_*\colon \operatorname{Div} C \to \operatorname{Div} C'.$$

On the other hand, for a general q the pullback $D \times_{C'} C$ of a divisor $D \subseteq C'$ will not be a divisor on C. It is possible to impose conditions on q so that this is so, and in this case q is called an *isogeny*. We will return to this in the future.

Put in a forward reference about isogenies when it exists.

Our final goal for the section is to broaden this discussion to line bundles on formal curves generally, using this nice case as a model. To begin, we need some vocabulary that connects the general case to the one studied above.

This and the following Lemma aren't great citations.

Definition 2.2.9 (cf. [Vak15, Section 14.2])**.** Suppose that $\mathcal{L}$ is a line bundle on C and select a section u of $\mathcal{L}$. There is a largest closed subscheme $D \subseteq C$ where the condition $u|_D = 0$ is satisfied. If D is a divisor, u is said to be *divisorial* and $D = \operatorname{div} u$.

Lemma 2.2.10 (cf. [Vak15, Exercise 14.2.E])**.** *A divisorial section u of a line bundle $\mathcal{L}$ induces an isomorphism $\mathcal{L} \cong \mathcal{I}_D$.* □

Line bundles which admit divisorial sections are thus those that arise through our construction above. However, in the classical situation, such line bundles account for roughly "half" of the available line bundles: line bundles are also used to house meromorphic functions with prescribed zeroes *and poles*, and we have not encountered such sections yet.

Definition 2.2.11 ([Str99b, Definition 5.20 and Proposition 5.26])**.** The ring of meromorphic functions on C, $\mathcal{M}_C$, is obtained by inverting all coordinates in $\mathcal{O}_C$.[8] Additionally, this can be augmented to a scheme $\operatorname{Mer}(C, \mathbb{G}_m)$ of meromorphic functions on C by

$$\operatorname{Mer}(C, \mathbb{G}_m)(R) := \left\{ (u, f) \,\middle|\, \begin{array}{l} u : \operatorname{Spec} R \to S, \\ f \in \mathcal{M}^{\times}_{C \times_S \operatorname{Spec} R} \end{array} \right\}.$$

Thinking of a meromorphic function as the formal expansion of a rational function, we are moved to study the monoidality of divisoriality.

Lemma 2.2.12. *If u_1 and u_2 are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then $u_1 \otimes u_2$ is a divisorial section of $\mathcal{L}_1 \otimes \mathcal{L}_2$ and $\operatorname{div}(u_1 \otimes u_2) = \operatorname{div} u_1 + \operatorname{div} u_2$.* □

Definition 2.2.13. A *meromorphic divisorial section* of a line bundle $\mathcal{L}$ is a decompositon $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ together with an expression of the form u_+/u_-, where u_+ and u_- are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We set $\operatorname{div}(u_+/u_-) = \operatorname{div} u_+ - \operatorname{div} u_-$.

In the case of a formal curve, the fundamental theorem is that meromorphic functions (or "Cartier divisors"), line bundles, and stable Weil divisors all essentially agree. A particular meromorphic function spans a 1–dimensional $\mathcal{O}_C$–submodule sheaf of $\mathcal{M}_C$, and hence it determines a line bundle. Conversely, a line bundle is determined by local gluing data, which is exactly the data of a meromorphic function. However, it is clear that there is some overdeterminacy in this first operation: scaling a meromorphic function by a nowhere vanishing entire function will not modify the submodule sheaf. Additionally, the function div gives an assignment from meromorphic functions to stable Weil divisors which is also insensitive to rescaling by a nowhere vanishing function.

Theorem 2.2.14 ([Str99b, Proposition 5.26])**.** *In the case of a formal curve C, there is a short exact sequence of formal groups*

$$0 \to \underline{\mathsf{FormalSchemes}}(C, \mathbb{G}_m) \to \operatorname{Mer}(C, \mathbb{G}_m) \to \operatorname{Div}(C) \to 0. \quad \square$$

[8] In fact, it suffices to invert any single one [Str99b, Lemma 5.21].

2.3 Line bundles associated to Thom spectra

Today we will exploit all of the algebraic geometry we set up yesterday to deduce a load of topological results.

Definition 2.3.1. Let E be a complex-orientable theory and let $V \to X$ be a complex vector bundle over a space X. According to Theorem 2.0.3, the cohomology of the Thom spectrum $E^*T(V)$ forms a 1–dimensional E^*X–module. Using Lemma 1.4.9, we construct a line bundle over X_E

$$\mathbb{L}(V) := \widetilde{E^*T(V)},$$

called the *Thom sheaf* of V.

Remark 2.3.2. One of the main utilities of this definition is that it only uses the *property* that E is complex-orientable, and it begets only the *property* that $\mathbb{L}(V)$ is a line bundle.

This construction enjoys many properties already established.

Corollary 2.3.3. *A vector bundle V over Y and a map $f\colon X \to Y$ induce an isomorphism*

$$\mathbb{L}(f^*V) \cong (f_E)^*\mathbb{L}(V).$$

There is also is a canonical isomorphism

$$\mathbb{L}(V \oplus W) = \mathbb{L}(V) \otimes \mathbb{L}(W).$$

Finally, this property can then be used to extend the definition of $\mathbb{L}(V)$ to virtual bundles:

$$\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}.$$

Proof. The first claim is justified by Lemma 1.1.4, the second by Lemma 1.1.8, and the last is a direct consequence of the first two. $\square$

We use these properties to work the following Example, which connects Thom sheaves with the major players from Lecture 2.1.

Example 2.3.4 ([AHS04, Section 8]). Take $\mathcal{L}$ to be the canonical line bundle over $\mathbb{C}P^\infty$. Using the same mode of argument as in Example 1.1.3, the zero-section

Various people have been uncomfortable about whether the grading matters here, whether E is periodified, ... **Actually, something is almost definitely wrong:** the functor T is defined to give the reduced Thom spectrum. Shit.

$$\Sigma^\infty\mathbb{C}P^\infty \xrightarrow{\simeq} T(\mathcal{L})$$

gives an identification

$$E^0\mathbb{C}P^\infty \supseteq \widetilde{E}^0\mathbb{C}P^\infty \xleftarrow{\cong} E^0T(\mathcal{L})$$

of $E^0T(\mathcal{L})$ with the augmentation ideal in $E^0\mathbb{C}P^\infty$. At the level of Thom sheaves, this gives an isomorphism

$$\mathcal{I}(0) \xleftarrow{\cong} \mathbb{L}(\mathcal{L})$$

of $\mathbb{L}(\mathcal{L})$ with the sheaf of functions vanishing at the origin of $\mathbb{C}\mathrm{P}^\infty_E$. Pulling $\mathcal{L}$ back along

$$0\colon * \to \mathbb{C}\mathrm{P}^\infty$$

gives a line bundle over the one-point space, which on Thom spectra gives the inclusion

$$\Sigma^\infty \mathbb{C}\mathrm{P}^1 \to \Sigma^\infty \mathbb{C}\mathrm{P}^\infty.$$

Stringing many results together, we can now calculate: [9]

$$\begin{aligned}
\widetilde{\pi_2 E} &\cong \widetilde{E^0 \mathbb{C}\mathrm{P}^1} && (S^2 \simeq \mathbb{C}\mathrm{P}^1)\\
&\cong \mathbb{L}(0^*\mathcal{L}) && \text{(Definition 2.3.1)}\\
&\cong 0^*\mathbb{L}(\mathcal{L}) && \text{(Corollary 2.3.3)}\\
&\cong 0^*\mathcal{I}(0) && \text{(preceding calculation)}\\
&\cong \mathcal{I}(0)/(\mathcal{I}(0)\cdot\mathcal{I}(0)) && \text{(definition of } 0^* \text{ from Definition 1.4.10)}\\
&\cong T_0^*\mathbb{C}\mathrm{P}^\infty_E && \text{(Definition 2.1.2)}\\
&\cong \omega_{\mathbb{C}\mathrm{P}^\infty_E}, && \text{(proof of Theorem 2.1.22)}
\end{aligned}$$

where $\omega_{\mathbb{C}\mathrm{P}^\infty_E}$ denotes the sheaf of invariant differentials on $\mathbb{C}\mathrm{P}^\infty_E$. Consequently, if $k\cdot\varepsilon$ is the trivial bundle of dimension k over a point, then

$$\widetilde{\pi_{2k}E} \cong \mathbb{L}(k\cdot\varepsilon) \cong \mathbb{L}(k\cdot 0^*\mathcal{L}) \cong \mathbb{L}(0^*\mathcal{L})^{\otimes k} \cong \omega^{\otimes k}_{\mathbb{C}\mathrm{P}^\infty_E}.$$

Finally, given an E–algebra $f\colon E \to F$ (e.g., $F = E^{X_+}$), then we have

$$\widetilde{\pi_{2k}F} \cong f_E^*\omega^{\otimes k}_{\mathbb{C}\mathrm{P}^\infty_E}.$$

Outside of this Example, it is difficult to find line bundles $\mathbb{L}(V)$ which we can analyze so directly. In order to get a handle on on $\mathbb{L}(V)$ in general, we now seek to strengthen this bond between line bundles and vector bundles by finding inside of algebraic topology the alternative presentations of line bundles given in Lecture 2.2. In particular, we would like a topological construction on vector bundles which produces divisors—i.e., finite schemes over X_E. This has the scent of a certain familiar topological construction called projectivization, and we now work to justify the relationship.

[9] The identification with $\mathcal{I}(0)/\mathcal{I}(0)\cdot\mathcal{I}(0)$ below deserves further explanation. The functor 0^* is right-exact, so sends the short exact sequence

$$0 \to \mathcal{I}(0)^2 \to \mathcal{I}(0) \to \mathcal{I}(0)/\mathcal{I}(0)^2 \to 0$$

to a right-exact sequence, and we need only check that the map $0^*\mathcal{I}(0)^2 \to 0^*\mathcal{I}(0)$ is zero. This is the statement that a function vanishing to second order also has vanishing first derivative.

Definition 2.3.5. Let V be a complex vector bundle of rank n over a base X. Define $\mathbb{P}(V)$, the *projectivization of* V, to be the $\mathbb{C}\mathrm{P}^{n-1}$–bundle over X whose fiber of $x \in X$ is the space of complex lines in the original fiber $V|_x$.

Theorem 2.3.6. *Take E to be* complex-oriented. *The E–cohomology of $\mathbb{P}(V)$ is given by the formula*

$$E^*\mathbb{P}(V) \cong E^*(X)[\![t]\!]/c(V)$$

for a certain monic polynomial

$$c(V) = t^n - c_1(V)t^{n-1} + c_2(V)t^{n-2} - \cdots + (-1)^n c_n(V).$$

Proof. We fit all of the fibrations we have into a single diagram:

$$\begin{tikzcd}
 & \mathbb{C}^\times \arrow[dd, equal] \arrow[dr] & & & \\
\mathbb{C}^n \arrow[dd] & & \mathbb{C}^n \setminus \{0\} \arrow[ll] \arrow[r] \arrow[dd] & \mathbb{C}\mathrm{P}^{n-1} \arrow[r] \arrow[dd] & \mathbb{C}\mathrm{P}^\infty \arrow[dd, equal] \\
 & \mathbb{C}^\times \arrow[dr] & & & \\
V \arrow[d] & & V \setminus \zeta \arrow[ll] \arrow[r] \arrow[d] & \mathbb{P}(V) \arrow[r] \arrow[d, "\pi"] & \mathbb{C}\mathrm{P}^\infty \arrow[d] \\
X \arrow[u, dotted, bend left, "\zeta"] \arrow[rr, equal] & & X \arrow[r, equal] & X \arrow[r] & *.
\end{tikzcd}$$

We read this diagram as follows: on the far left, there's the vector bundle we began with, as well as its zero-section ζ. Deleting the zero-section gives the second bundle, a $\mathbb{C}^n \setminus \{0\}$–bundle over X. Its quotient by the scaling $\mathbb{C}^\times$–action gives the third bundle, a $\mathbb{C}\mathrm{P}^{n-1}$–bundle over X. Additionally, the quotient map $\mathbb{C}^n \setminus \{0\} \to \mathbb{C}\mathrm{P}^{n-1}$ is itself a $\mathbb{C}^\times$–bundle, and this induces the structure of a $\mathbb{C}^\times$–bundle on the quotient map $V \setminus \zeta \to \mathbb{P}(V)$. Thinking of these as complex line bundles, they are classified by a map to $\mathbb{C}\mathrm{P}^\infty$, which can itself be thought of as the last vertical fibration, fibering over a point.

Note that the map between these two last fibers is surjective on E–cohomology. It follows that the Serre spectral sequence for the third vertical fibration is degenerate, since all the classes in the fiber must survive.[10] We thus conclude that $E^*\mathbb{P}(V)$ is a free $E^*(X)$–module on the classes $\{1, t, t^2, \ldots, t^{n-1}\}$ spanning $E^*\mathbb{C}\mathrm{P}^{n-1}$, where t encodes the chosen complex-orientation of E. To understand the ring structure, we need only compute $t^{n-1} \cdot t$, which must be able to be written in terms of the classes which are lower in t–degree:

$$t^n = c_1(V)t^{n-1} - c_2(V)t^{n-2} + \cdots + (-1)^{n-1}c_n(V)$$

for some classes $c_j(V) \in E^*X$. The main claim follows. □

[10] This is called the Leray–Hirsch theorem.

In coordinate-free language, we have the following Corollary:

Corollary 2.3.7 (Theorem 2.3.6 redux)**.** *Take E to be* complex-orientable. *The map*

$$\mathbb{P}(V)_E \to X_E \times \mathbb{C}\mathrm{P}^\infty_E$$

is a closed inclusion of X_E–schemes, and the structure map $\mathbb{P}(V)_E \to X_E$ is free and finite of rank n. It follows that $\mathbb{P}(V)_E$ is a divisor on $\mathbb{C}\mathrm{P}^\infty_E$ considered over X_E, i.e.,

$$\mathbb{P}(V)_E \in \left(\mathrm{Div}_n^+(\mathbb{C}\mathrm{P}^\infty_E)\right)(X_E). \quad \square$$

Does the multiplicativity need a proof?

Definition 2.3.8. The classes $c_j(V)$ of Theorem 2.3.6 are called the *Chern classes* of V (with respect to the complex-orientation t of E). They are visibly natural with respect to pullback of bundles, and the Chern polynomial $c(-)$ is multiplicative:

$$c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2).$$

The next major theorems concerning projectivization are the following:

Corollary 2.3.9. *The sub-bundle of $\pi^*(V)$ consisting of vectors $(v, (\ell, x))$ such that v lies along the line ℓ splits off a canonical line bundle.* $\square$

Corollary 2.3.10 ("Splitting principle" / "Complex–oriented descent")**.** *Associated to any n–dimensional complex vector bundle V over a base X, there is a canonical map $i_V\colon Y_V \to X$ such that $(i_V)_E\colon (Y_V)_E \to X_E$ is finite and faithfully flat, and there is a canonical splitting into complex line bundles:*

$$i_V^*(V) \cong \bigoplus_{i=1}^n \mathcal{L}_i. \quad \square$$

This last Corollary is extremely important. Its essential contents is to say that any question about characteristic classes can be checked for sums of line bundles. Specifically, because of the injectivity of i_V^*, any relationship among the characteristic classes deduced in E^*Y_V must already be true in the ring E^*X. The following theorem is a consequence of this principle:

Theorem 2.3.11. *Again take E to be complex-oriented. The coset fibration*

$$U(n-1) \to U(n) \to S^{2n-1}$$

deloops to a spherical fibration

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

The associated Serre spectral sequence

$$E_2^{*,*} = H^*(BU(n); E^*S^{2n-1}) \Rightarrow E^*BU(n-1)$$

degenerates at E_{2n} and induces an isomorphism

$$E^* BU(n) \cong E^*[\![\sigma_1, \ldots, \sigma_n]\!].$$

Now, let $V\colon X \to BU(n)$ classify a vector bundle V. Then the coefficient c_j in the polynomial $c(V)$ is selected by σ_j:

$$c_j(V) = V^*(\sigma_j).$$

Proof sketch. The first part is a standard calculation. To prove the relation between the Chern classes and the σ_j, the splitting principle states that we can factor complete the map $V\colon X \to BU(n)$ to a square

$$\begin{tikzcd} Y_V \arrow[r, dotted, "\bigoplus_{i=1}^n \mathcal{L}_i"] \arrow[d, "f_V"] & BU(1)^{\times n} \arrow[d, "\oplus"] \\ X \arrow[r, "V"] & BU(n). \end{tikzcd}$$

The equation $c_j(f_V^* V) = V^*(\sigma_j)$ can be checked in $E^* Y_V$. □

We now see that not only does $\mathbb{P}(V)_E$ produce a point of $\operatorname{Div}_n^+(\mathbb{CP}_E^\infty)$, but actually the scheme $\operatorname{Div}_n^+(\mathbb{CP}_E^\infty)$ itself appears internally to topology:

Corollary 2.3.12. [11] *For a complex orientable cohomology theory E, there is an isomorphism*

$$BU(n)_E \cong \operatorname{Div}_n^+ \mathbb{CP}_E^\infty,$$

so that maps $V\colon X \to BU(n)$ are transported to divisors $\mathbb{P}(V)_E \subseteq \mathbb{CP}_E^\infty \times X_E$. Selecting a particular complex orientation of E begets two isomorphisms

$$BU(n)_E \cong \widehat{\mathbb{A}}^n, \qquad \operatorname{Div}_n^+ \mathbb{CP}_E^\infty \cong \widehat{\mathbb{A}}^n,$$

and these are compatible with the centered isomorphism above. □

This description has two remarkable features. One is its "faithfulness": this isomorphism of formal schemes means that the entire theory of characteristic classes is captured by the behavior of the divisor scheme. The other aspect is its coherence with topological operations we find on $BU(n)$. For instance, the Whitney sum map translates as follows:

Lemma 2.3.13. *The sum map*

$$BU(n) \times BU(m) \xrightarrow{\oplus} BU(n+m)$$

induces on Chern polynomials the identity

$$c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2).$$

[11] See [Str99b, Proposition 8.31] for a proof that recasts Theorem 2.3.11 itself in coordinate-free terms.

Space	classifies	Scheme	classifies
$BU(n)$	vector bundles of rank n	$\operatorname{Div}_n^+ \mathbb{C}P^\infty_E$	effective Weil divisors of rank n
$\coprod_n BU(n)$	unstable vector bundles	$\operatorname{Div}^+ \mathbb{C}P^\infty_E$	semiring of effective divisors
$BU \times \mathbb{Z}$	stable virtual bundles	$\operatorname{Div} \mathbb{C}P^\infty_E$	ring of stable Weil divisors
$BU \times \{0\}$	stable virtual bundles of rank 0	$\operatorname{Div}_0 \mathbb{C}P^\infty_E$	ideal of stable divisors of rank 0

Figure 2.1: Different notions of vector bundles and their associated divisors

In terms of divisors, this means

$$\mathbb{P}(V_1 \oplus V_2)_E = \mathbb{P}(V_1)_E \sqcup \mathbb{P}(V_2)_E,$$

and hence there is an induced square

$$\begin{tikzcd} BU(n)_E \times BU(m)_E \arrow[r, "\oplus"] \arrow[d, equal] & BU(n+m) \arrow[d, equal] \\ \operatorname{Div}_n^+ \mathbb{C}P^\infty_E \times \operatorname{Div}_m^+ \mathbb{C}P^\infty_E \arrow[r, "\sqcup"] & \operatorname{Div}_{n+m}^+ \mathbb{C}P^\infty_E. \end{tikzcd} \quad \square$$

The following is a consequence of combining this Lemma with the splitting principle:

Corollary 2.3.14. *The map $Y_E \xrightarrow{f_V} X_E$ pulls $\mathbb{P}(V)_E$ back to give*

$$Y_E \times_{X_E} \mathbb{P}(V)_E \cong \coprod_{i=1}^n \mathbb{P}(\mathcal{L}_i)_E.$$

Interpretation. This says that the splitting principle is a topological enhancement of the claim that a divisor can be base-changed along a finite flat map where it splits as a sum of points. □

The other constructions from Lecture 2.2 are also easily matched up with topological counterparts:

Corollary 2.3.15. *There are natural isomorphisms $BU_E \cong \operatorname{Div}_0 \mathbb{C}P^\infty_E$ and $(BU \times \mathbb{Z})_E \cong \operatorname{Div} \mathbb{C}P^\infty_E$. Additionally, $(BU \times \mathbb{Z})_E$ is the free formal group on the curve $\mathbb{C}P^\infty_E$.* □

Corollary 2.3.16. *There is a commutative diagram*

$$\begin{tikzcd} BU(n)_E \times BU(m)_E \arrow[r, "\otimes"] \arrow[d, equal] & BU(nm)_E \arrow[d, equal] \\ \operatorname{Div}_n^+ \mathbb{C}P^\infty_E \times \operatorname{Div}_m^+ \mathbb{C}P^\infty_E \arrow[r, "\cdot"] & \operatorname{Div}_{nm}^+ \mathbb{C}P^\infty_E, \end{tikzcd}$$

where the bottom map acts by

$$(D_1, D_2 \subseteq \mathbb{C}P^\infty_E \times X_E) \mapsto (D_1 \times D_2 \subseteq \mathbb{C}P^\infty_E \times \mathbb{C}P^\infty_E \xrightarrow{\mu} \mathbb{C}P^\infty_E),$$

and μ is the map induced by the tensor product map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$.

Proof. By the splitting principle, it is enough to check this on sums of line bundles. A sum of line bundles corresponds to a totally decomposed divisor, so we consider the case of a pair of such divisors $\bigsqcup_{i=1}^n \{a_i\}$ and $\bigsqcup_{j=1}^m \{b_j\}$. Referring to Definition 2.1.16, the map acts by

$$\left(\bigsqcup_{i=1}^n \{a_i\}\right)\left(\bigsqcup_{j=1}^m \{b_j\}\right) = \bigsqcup_{i,j} \{\mu_{\mathbb{C}P^\infty_E}(a_i, b_j)\}. \qquad \square$$

Finally, we can connect our analysis of the divisors coming from topological vector bundles with the line bundles studied at the start of the section.

Lemma 2.3.17. *Let $\zeta : X_E \to X_E \times \mathbb{C}P^\infty_E$ denote the pointing of the formal curve $\mathbb{C}P^\infty_E$, and let $\mathcal{I}(\mathbb{P}(V)_E)$ denote the ideal sheaf on $X_E \times \mathbb{C}P^\infty_E$ associated to the divisor subscheme $\mathbb{P}(V)_E$. There is a natural isomorphism of sheaves over X_E:*

In FPFP, Neil has a $-D$ rather than a D.

$$\zeta^* \mathcal{I}(\mathbb{P}(V)_E) \cong \mathbb{L}(V).$$

Proof sketch. In terms of a complex-oriented E and Theorem 2.3.6, the effect of pulling back along the zero section is to set $t = 0$, which collapses the Chern polynomial to just the top class $c_n(V)$. This element, called *the Euler class of V*, provides the E^*X–module generator of $E^*T(V)$—or, equivalently, the trivializing section of $\mathbb{L}(V)$. $\square$

Theorem 2.3.18 (cf. Theorem 5.2.2). *A trivialization $t\colon \mathbb{L}(\mathcal{L} - 1) \cong \mathcal{O}_{\mathbb{C}P^\infty_E}$ of the Thom sheaf associated to the canonical bundle induces a ring map $MU \to E$.*

Do you have the right spectrum here: *MU* versus *MUP*?

Proof. Suppose that V is a rank n vector bundle over X, and let $f\colon Y \to X$ be the space guaranteed by the splitting principle to provide an isomorphism $f^*V \cong \bigoplus_{j=1}^n \mathcal{L}_j$. The chosen trivialization t then pulls back to give a trivialization of $\mathcal{I}(\mathbb{P}(f^*V)_E)$, and by finite flatness this descends to also give a trivialization of $\mathcal{I}(\mathbb{P}(V)_E)$. Pulling back along the zero section gives a trivialization of $\mathbb{L}(V)$. Then note that the system of trivializations produced this way is multiplicative, as a consequence of $\mathbb{P}(V_1 \oplus V_2)_E \cong \mathbb{P}(V_1)_E \sqcup \mathbb{P}(V_2)_E$. In the universal examples, this gives a sequence of compatible maps $MU(n) \to E$ which assemble on the colimit $n \to \infty$ to give the desired map of ring spectra. $\square$

2.4 Power operations for complex bordism

Is this the best name?

Our eventual goal, like in Case Study 1, is to give an algebro-geometric description of $MU_*(*)$ and of the cooperations MU_*MU. It is possible to approach this the same way as in Lecture 1.5, using the Adams spectral sequence [Qui69, Theorem 2]. However, $MU_*(*)$ is an integral algebra and so we cannot make do with working out the mod–2 Adams spectral sequence alone—we would at least have to work out the mod–p Adams spectral sequence for every p. At odd primes p, there is the following unfortunate theorem:

Theorem 2.4.1. *There is an isomorphism*

$$H\mathbb{F}_{p*}H\mathbb{F}_p \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \tau_2, \ldots]$$

with $|\xi_j| = 2p^j - 2$ *and* $|\tau_j| = 2p^j - 1$. □

There are odd–dimensional classes in this algebra, and the *graded-commutativity* of the dual mod–p Steenrod algebra means that these classes anti-commute. This prohibits us from writing "Spec($H\mathbb{F}_{p*}H\mathbb{F}_p$)" and from trusting "Spec($H\mathbb{F}_pP_0H\mathbb{F}_p$)" to do a good enough job. This is the first time we have encountered Hindrance #4 from Lecture 1.3 in the wild.

Because of this, we will not feel any guilt for taking a completely alternative approach to this calculation. This other method, due to Quillen, has as its keystone a completely different kind of cohomology operation called a *power operation*. These operations are quite technical to describe, but at their core is taking the n^{th} power of a cohomology class—and hence they have a frustrating lack of properties, including failures to be additive and to be stable. Our goal in this Lecture is just to define these cohomology operations, specialized to the particular setting we will need for Quillen's proof. Unlike with Steenrod operations, their algebro-geometric interpretation will not be immediately accessible to us, and indeed their eventual reinterpretation in these terms will be one of our more hard-won pursuits in this book (cf. Case Study 6).

Power operations arise not just from taking the n^{th} power of a cohomology class but from also remarking on the natural symmetry of that operation. We record this symmetry using the following technical apparatus:

Lemma 2.4.2 ([Lurb, Examples 6.1.4.2 and 6.1.6.2])**.** *Given a spectrum E, its n–fold smash power forms a Σ_n–spectrum, in the sense that there is a natural diagram $* // \Sigma_n \to \mathsf{Spectra}$ which selects $E^{\wedge n}$ on objects.* □

In turn, a cohomology class $f\colon \Sigma^\infty_+ X \to E$ gives rise to a morphism of Σ_n–spectra

$$f^{\wedge n}\colon (\Sigma^\infty_+ X)^{\wedge n} \to E^{\wedge n}.$$

The homotopy colimit of such a diagram is called the *homotopy orbits* of the spectrum, and this gives a natural diagram

$$\begin{tikzcd} (\Sigma^\infty_+ X)^{\wedge n} \arrow[r, "f^{\wedge n}"] \arrow[d] & E^{\wedge n} \arrow[r, "\mu"] \arrow[d] & E \\ (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \arrow[r, "f^{\wedge n}_{h\Sigma_n}"] & E^{\wedge n}_{h\Sigma_n} \arrow[ur, dotted, "\mu_n"] & \end{tikzcd}.$$

A ring spectrum E equipped with the indicated factorizations μ_n of the multiplication map which are furthermore compatible with taking products and iterated powers is called an H_∞ *ring spectrum* [BMMS86, Definition I.3.1]. The composite

$$P^{\Sigma_n}_{\mathrm{ext}} f \colon (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \xrightarrow{f^{\wedge n}_{h\Sigma_n}} E^{\wedge n}_{h\Sigma_n} \xrightarrow{\mu_n} E$$

is called the *external* Σ_n*–power operation* applied to f, and the restriction to the diagonal subspace

$$P^{\Sigma_n} f \colon \Sigma^\infty_+ X \wedge \Sigma^\infty_+ B\Sigma_n \simeq (\Sigma^\infty_+ X)_{h\Sigma_n} \xrightarrow{\Delta_{h\Sigma_n}} (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \xrightarrow{P^{\Sigma_n}_{\mathrm{ext}} f} E$$

is called the *(internal)* Σ_n*–power operation* applied to f. We can consider it as a cohomology class lying in

$$P^{\Sigma_n} f \in E^0(X \times B\Sigma_n).$$

This is pretty much the definition of an H_∞ ring spectrum, but this is not enough to extract power operations on positive-degree classes. There, you want compatible systems of maps like $(\Sigma^j E)^{\wedge n}_{h\Sigma_n} \to \Sigma^{nj} E$. In general, you get such systems only for all j which are divisible by some fixed d, and the resulting mess is called an H^d_∞–structure. For example, KU has an H^2_∞–structure and KO has an H^8_∞–structure. My suspicion with Nat is that power operation maps come from applying the power operation functor to the identity class in $[E, E]$, and that the H^d_∞–structures on these two spectra come from applying the power operation functor to the autoequivalence supplied by the relevant Bott element. Meanwhile, Jeremy keeps telling us that this really has to do with representation spheres: the n–fold smash power of S^j ought to be called $S^{j\rho}$ for ρ the permutation representation of Σ_n. Being able to identify $S^{j\rho}_{h\Sigma_n}$ with S^{jn} is somehow the main input to an H^n_∞–structure.

Remark 2.4.3 ([BMMS86, Definition IV.7.1]). This construction is sometimes referred to as the *total power operation*, since restriction along various classes in $MU_* BC_n$, thought of as functionals on $MU^* BC_n$, give rise to truly internal cohomology operations on $MU^* X$. For instance, restriction to the basepoint in BC_n gives the n–fold cup product f^n. More generally, a class $\sigma \in MU_* BC_n$ gives rise to a *restricted power operation* using the Kronecker pairing:

$$\Sigma^\infty_+ X \wedge \mathbb{S} \xrightarrow{1 \wedge \sigma} \Sigma^\infty_+ X \wedge \Sigma^\infty_+ BC_n \wedge MU \xrightarrow{P^n f \wedge 1} MU \wedge MU \xrightarrow{\mu} MU.$$

Some properties of this construction are immediately visible—for instance, it is multiplicative:

$$P^{\Sigma_n}(f \cdot g) = P^{\Sigma_n}(f) \cdot P^{\Sigma_n}(g),$$

and restriction of $P^{\Sigma_n}(f)$ to the basepoint in $\Sigma^\infty_+ B\Sigma_n$ yields the cup power class f^n. In order to state any further properties, we will need to make some extraneous observations. First, note that any map of groups $\varphi \colon G \to \Sigma_n$ gives a variation on this construction by restriction of diagrams

$$\begin{tikzcd} *//G \arrow[r, "\varphi"] & *//\Sigma_n \arrow[r, bend left, "(\Sigma^\infty_+ X)^{\wedge n}"] \arrow[r, bend right, "E^{\wedge n}"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Spectra}. \end{tikzcd}$$

This construction is useful when studying composites of power operations: the group $\Sigma_n \wr \Sigma_k$ acts naturally on $(E^{\wedge k})^{\wedge n}$, and indeed there is an equivalence

$$P^{\Sigma_n} \circ P^{\Sigma_k} = P^{\Sigma_n \wr \Sigma_k}.$$

In order to understand these modified power operations more generally, we are motivated to study such maps φ more seriously. Some basic constructions are summarized in the following definition:

Definition 2.4.4 ([May96, Sections XI.3 and XXV.3]). Let $\varphi\colon G \to H$ be an inclusion of finite groups and let F be an H–spectrum. There is a natural map of homotopy colimits $\varphi_*\colon F_{hG} \to F_{hH}$ which induces a *restriction map* on cohomology:

Cite me: Find a blanket reference for all these cohomology maps. The Alaska notes don't speak in terms of fancy diagram categories. Maybe there's a Barwick reference?.

$$\mathrm{Res}_G^H\colon E^0 F_{hH} \to E^0 F_{hG}.$$

The spectrum $(H/G) \times F$ considered as a G–spectrum with the diagonal G–action has the property $((H/G) \times F)_{hH} = F_{hG}$, and the G–equivariant averaging map

$$F \xrightarrow{\sum\limits_{[h]\in H/G} [h] \times (-)} (H/G) \times F$$

passes on homotopy orbits to the *additive norm map* $N_G^H\colon F_{hH} \to F_{hG}$, which again induces a map on cohomology classes

$$\mathrm{Tr}_G^H\colon E^0 F_{hG} \to E^0 F_{hH}$$

called the *transfer map*. The composite $\varphi_* \circ N_G^H$ acts by multiplication by the index $|H/G|$, and hence this is also true of $\mathrm{Res}_G^H \mathrm{Tr}_G^H$.

The restriction and transfer maps appear prominently in the following formula, which measures the failure of the power operation construction to be additive:

Lemma 2.4.5 ([BMMS86, Corollary II.1.6], [AHS04, Proposition A.5, Equation 3.6]). *For cohomology classes $f, g \in E^0 X$, there is a formula*[12]

$$P^{\Sigma_n}(f+g) = \sum_{i+j=n} \mathrm{Tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \left(P^{\Sigma_i}(f) \cdot P^{\Sigma_j}(g)\right). \quad \square$$

To produce binomial formulas for the modified power operations, we use the following Lemma:

Nat indicated that there is some kind of transitivity condition here that I am numb to.

Lemma 2.4.6 ([Ada78, p. 109-110], [HKR00, Section 6.5]). *Let G_1, G_2 be subgroups of H, and consider the homotopy pullback diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & *//G_1 \arrow[d] \\ *//G_2 \arrow[r] & *//H. \end{tikzcd}$$

Given any identification $P \simeq \coprod_K (*//K)$, there is a push-pull interchange formula

$$\mathrm{Tr}_H^{G_1} \mathrm{Res}_H^{G_2} = \sum_K \mathrm{Res}_K^{G_1} \mathrm{Tr}_K^{G_2}. \quad \square$$

[12] This should be compared with the classical binomial formula $\frac{1}{n!}(x+y)^n = \sum_{i+j=n} \frac{1}{i!j!} x^i y^j$.

Corollary 2.4.7. *For any subgroup $G \leq \Sigma_n$, there is a congruence*

$$P^G(f+g) \equiv P^G(f) + P^G(g) \quad (\text{mod } \textit{transfers from proper subgroups of } G).$$

Proof. Note that P^G can be defined by means of restriction: $P^G = \operatorname{Res}^{\Sigma_n}_G P^{\Sigma_n}$. We can hence reuse the previous binomial formula:

$$P^G(f+g) = \operatorname{Res}^{\Sigma_n}_G P^{\Sigma_n}(f+g) = \sum_{i+j=n} \operatorname{Res}^{\Sigma_n}_G \operatorname{Tr}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \left(P^{\Sigma_i}(f) \cdot P^{\Sigma_j}(g) \right).$$

In the cases $i = 0$ or $j = 0$, the transfer map is the identity operation, and we recover $P^G(g)$ and $P^G(f)$ respectively. In all the other terms, the interchange lemma lets us pull the transfer to the outside. $\square$

Since the only operations we understand so far are stable operations, which are in particular additive, we are moved to find a target for the power operation P^G which kills the ideal generated by the proper transfers yet which remains computable.

Definition 2.4.8 ([Lurb, Construction 6.1.6.4, Definition 6.1.6.24])**.** Again let F be an H–spectrum, and define its *homotopy fixed points* to be the homotopy limit spectrum F^{hH}. The norm map $F_{hH} \to F$ witnesses the trivial H–spectrum F_{hH} as a constant cone over the H–spectrum F, and hence gives a natural factorization $F_{hH} \to F^{hH} \to F$ of the norm. The cofiber of this first map is denoted F^{tH} and is called the *Tate spectrum* of F. This gives rise to a notion of Tate power operation via

$$\begin{array}{ccccccc} \pi_0 E^{\Sigma^\infty_+ X} & \xrightarrow{P^G} & \pi_0 E^{(\Sigma^\infty_+ X)_{hG}} & = & \pi_0 (E^{\Sigma^\infty_+ X})^{hG} & \longrightarrow & \pi_0 (E^{\Sigma^\infty_+ X})^{tG}, \\ f & \longmapsto & P^G f & & \longmapsto & & P^G_{\mathrm{Tate}} f. \end{array}$$

Lemma 2.4.9. *The Tate power operation is additive.*

Proof. The image of the map $\pi_0(E^{\Sigma^\infty_+ X})_{hG} \to \pi_0(E^{\Sigma^\infty_+ X})^{hG}$ is the kernel of the projection to the Tate object, and this image contains all transfers. $\square$

This isn't super clear to me.

Hence, the Tate power operation is at least additive, but there is no reason to think that it is especially computable. The miracle is that in the case where G is the transitive cyclic subgroup C_n, the target is completely computable.

Lemma 2.4.10. *The assignment $\pi_0 E^{\Sigma^\infty_+ X} \mapsto \pi_0 (E^{(\Sigma^\infty_+ X)^{\wedge n}})^{tC_n}$ is a cohomology theory.*

Proof. The Eilenberg–Steenrod axioms are clear except for the cofiber sequence axiom, which boils down to checking that the assignment $X \mapsto (X^{\wedge n})^{tC_n}$ preserves cofiber sequences of finite complexes. A cofiber sequence $X \to Y \to Y/X$ of pointed spaces is equivalent data to the diagram

Cite me: I learned this proof from Tyler Lawson. He suggested looking at Greenlees–May or Lunøe-Nielsen–Rognes. This is Prop 2.2.3 of DAG XIII. He also said that Charles has referred him to this point of view before.

$$X \to Y,$$

which has colimit Y and which admits a filtration by distance from the initial node with filtration quotients:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \| & & \downarrow \\ X & & Y/X. \end{array}$$

By taking the n–fold Cartesian power of the diagram $X \to Y$, we produce a diagram shaped like an n–dimensional hypercube with colimit $Y^{\wedge n}$ and which again admits a filtration by distance from the initial node. The colimits of these partial diagrams give a Σ_n–equivariant filtration of $Y^{\wedge n}$ as:

$$\begin{array}{ccccccccc} F_0 & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_{n-1} & \longrightarrow & Y^{\wedge n} \\ \| & & \downarrow & & & & \downarrow & & \downarrow \\ X^{\wedge n} & & \bigvee X^{\wedge(n-1)} \wedge (Y/X) & & \cdots & & \bigvee X \wedge (Y/X)^{\wedge(n-1)} & & (Y/X)^{\wedge n}. \end{array}$$

We now apply $(-)^{tC_n}$ to this diagram. The Tate construction carries cofiber sequences of C_n–spectra to cofiber sequences of spectra, so this is again a filtration diagram. In the intermediate filtration quotients, the C_n–action is given by freely permuting wedge factors (i.e., these spectra are induced up from spectra with trivial actions), from which it follows that the Tate construction vanishes on these nodes. Hence, the diagram postcomposed with the Tate construction takes the form

$$\begin{array}{ccccccccc} F_0^{tC_n} & \xrightarrow{\simeq} & F_1^{tC_n} & \xrightarrow{\simeq} & \cdots & \xrightarrow{\simeq} & F_{n-1}^{tC_n} & \longrightarrow & (Y^{\wedge n})^{tC_n} \\ \| & & \downarrow & & & & \downarrow & & \downarrow \\ (X^{\wedge n})^{tC_n} & & * & & \cdots & & * & & ((Y/X)^{\wedge n})^{tC_n}. \end{array}$$

Eliminating the intermediate filtration stages with empty filtration quotients, we see that this filtration is equivalent data to a cofiber sequence

$$(X^{\wedge n})^{tC_n} \to (Y^{\wedge n})^{tC_n} \to ((Y/X)^{\wedge n})^{tC_n}. \qquad \square$$

The effect of this Lemma is twofold. For one, the Tate operation is not only additive, it is even *stable*. Secondly, it suffices to understand the behavior of passing to the Tate construction in the case of $X = *$, i.e., the effect of the map $E^{hC_n} \to E^{tC_n}$. Since we are intending to make a computation, it will at this point be convenient to also specialize to our case of interest of where $E = MU$.[13]

Theorem 2.4.11. *There is an isomorphism*

$$\pi_* MU^{tC_n} = MU^* BC_n[x^{-1}],$$

where x is the restriction to $MU^2 BC_n$ of the canonical class $x \in MU^2(\mathbb{C}\mathrm{P}^\infty)$.

[13]In fact, everything we say here will be valid for any complex-oriented ring spectrum E.

Proof. Consider the C_n–equivariant cofiber sequence

$$S(\mathbb{C}^m)_+ \to S^0 \to S^{\mathbb{C}^m},$$

where $S(\mathbb{C}^m)$ is the unit sphere inside of $\mathbb{C}^m$ and $S^{\mathbb{C}^m}$ is the one-point compactification of the C_n–representation $\mathbb{C}^m$. A key fact is that $S(\mathbb{C}^m)$ admits a C_n–equivariant cell decomposition by free cells, natural with respect to the inclusions as m increases. This buys us several facts:

1. The following Tate objects vanish: $(MU \wedge S(\mathbb{C}^m)_+)^{tC_n} \simeq *$. As in the proof of Lemma 2.4.10, this is because the Tate construction vanishes on free C_n–cells.

2. We can use $\operatorname{colim}_{m\to\infty} S(\mathbb{C}^m)_+$ as a model for EC_n, so that
$$MU_{hC_n} = \left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)_{hC_n}.$$

3. Coupling these two facts together, we get
$$MU_{hC_n} = \left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)_{hC_n} = \left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)^{hC_n}.$$

4. Pulling the fixed points functor out, this gives
$$\begin{aligned} MU^{tC_n} &= \operatorname{cofib}(MU_{hC_n} \to MU^{hC_n}) \\ &= \operatorname{cofib}\left(\left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)^{hC_n} \to MU^{hC_n}\right) \\ &= \left(\operatorname{cofib}\left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right) \to MU \wedge S^0\right)^{hC_n} \\ &= \left(\operatorname*{colim}_{m\to\infty} MU \wedge S^{\mathbb{C}^m}\right)^{hC_n}. \end{aligned}$$

This last formula puts us in a position to calculate. The Thom isomorphism for MU gives an identification $MU \wedge S^{\mathbb{C}} \simeq \Sigma^2 MU$ as C_n–spectra, and the map

$$(MU \wedge S^0)^{hC_n} \to (MU \wedge S^{\mathbb{C}})^{hC_n}$$

can be identified with multiplication by the Thom class:

$$MU^{\Sigma^\infty BC_n} \xrightarrow{x\cdot -} (\Sigma^2 MU)^{\Sigma^\infty BC_n}.$$

In all, this gives

$$MU^{tC_n} \simeq \operatorname*{colim}_{m\to\infty} (MU \wedge S^{\mathbb{C}^m})^{hC_n} \simeq \operatorname*{colim}_{m\to\infty} (\Sigma^{2m} MU)^{BC_n} \simeq MU^{BC_n}[x^{-1}]. \qquad \square$$

To apply this machinery in our case, we need only equip MU with the factorized multiplication maps appearing in the definition of an H_∞ ring spectrum.

Definition 2.4.12 ([Rud98, Definition VII.7.4])**.** Suppose that $\xi\colon X \to BU(k)$ presents a complex vector bundle of rank k on X. The n–fold direct sum of this bundle gives a new bundle

$$X^{\times n} \xrightarrow{\xi^{\oplus n}} BU(k)^{\times n} \xrightarrow{\oplus} BU(n \cdot k)$$

Make it clear that you're also taking the quotient of the total space.

of rank nk on which the cyclic group C_n acts. By taking the (homotopy) C_n–quotient, we produce a vector bundle $\xi(n)$ on $X^{\times n}_{hC_n}$ participating in the diagram

$$\begin{tikzcd}
X^{\times n} \arrow[r, "\xi^{\oplus n}"] \arrow[d] & BU(k)^{\times n} \arrow[r] \arrow[d] & BU(nk) \\
X^{\times n}_{hC_n} \arrow[r, "\xi(n)"] & BU(k)^{\times n}_{hC_n}. \arrow[ur, dotted, "\mu"] &
\end{tikzcd}$$

The universal case gives the map μ.

This citation is actually for Thom spaces, where he picks up a factor of $S^n_{hC_n}$. This might be important to get right for the future, when we're doing unstable / degree–sensitive things. **Jeremy warned me that this is more serious than I wanted to admit. Compare carefully with Rudyak.**

Lemma 2.4.13 ([Rud98, Equation VII.7.3])**.** *There is an isomorphism of Thom spectra*

$$T(\xi(n)) \simeq (T\xi)^{\wedge n}_{hC_n}.$$

Proof. This proof is mostly a matter of having had the idea to write down the Lemma to begin with. From here, we string basic properties together:

$$\begin{aligned}
T(\xi(n)) &= T(\xi^{\oplus n}_{hC_n}) && \text{(definition)} \\
&= T(\xi^{\oplus n})_{hC_n} && \text{(colimits commute with colimits)} \\
&= T(\xi)^{\wedge n}_{hC_n}. && \text{(T is monoidal: Lemma 1.1.8)}
\end{aligned}$$

□

Applying the Lemma to the universal case produces a factorization

$$MU(k)^{\wedge n} \to MU(k)^{\wedge n}_{hC_n} \to MU(nk)$$

of the unstable multiplication map, and hence a stable factorization

$$MU^{\wedge n} \to MU^{\wedge n}_{hC_n} \xrightarrow{\mu} MU.$$

Remark 2.4.14 ([Qui71, Equations 3.10-11]). The picture Quillen paints of all this is considerably different from ours. He begins by giving a different presentation of the complex cobordism groups of a manifold M: a complex orientation of a smooth map $Z \to M$ is a factorization

$$Z \xrightarrow{i} E \xrightarrow{\pi} M$$

through a complex vector bundle $\pi\colon E \to M$ by an embedding i, as well as a complex structure on the normal bundle ν_i. Up to suitable notions of stability (in the dimension of E) and homotopy equivalence (involving, in particular, isotopies of different embeddings i), these quotient to give cobordism classes of maps complex-oriented maps $Z \to M$. The collection of cobordism classes over M of codimension q over is isomorphic to $MU^q(M)$ [Qui71, Proposition 1.2]. Quillen's definition of the power operations is then given in terms of this geometric model: a representative $f\colon Z \to M$ of a cobordism class gives rise to another complex-oriented map $f^{\times n}\colon Z^{\times n} \to M^{\times n}$, and he defines $P^{C_n}_{\mathrm{ext}}(f)$ to be the postcomposition with $M^{\times n} \to M^{\times n}_{hC_n}$. All the properties of his construction must therefore be explored through the lens of groups acting on manifolds.

Example 2.4.15. The chain model for $H\mathbb{F}_2$–homology is actually also rigid enough to define power operations, and somewhat curiously these operations automatically turn out to be additive, without passing to the Tate construction. In particular, this means that they are recognizable in terms of classical Steenrod operations. Specifically, the C_2–construction

Cite me: Where can you find Steenrod squares defined like this? The VFoS notes, but that's not a great reference..

$$\{\Sigma^n \Sigma^\infty_+ X \xrightarrow{f} H\mathbb{F}_2\} \xrightarrow{P^{C_2}} \{\Sigma^{2n}\Sigma^\infty_+ X \wedge \Sigma^\infty_+ BC_2 \xrightarrow{P^{C_2}(f)} H\mathbb{F}_2\}$$

gives a class in $H\mathbb{F}_2^{2n-*}(X) \otimes H\mathbb{F}_2^*(\mathbb{RP}^\infty)$, which decomposes as

$$P^{C_2}(f) = \sum_{j=0}^{2n} \mathrm{Sq}^{2n-j}(f) \otimes x^j.$$

The Adem relations can be extracted by studying the wreath product $\Sigma_2 \wr \Sigma_2$ and the compositional identity for power operations.

Remark 2.4.16 ([BMMS86, Theorems III.4.1-3, Remark III.4.4]). Since the failure of the power operations to be additive was a consequence of the binomial formula, it is somewhat intuitive that modulo 2, where $(x+y)^2 \equiv x^2 + y^2$, that this operation becomes stable. In fact, more than this is true: for instance, if an H_∞ ring spectrum E has $\pi_0 E = \mathbb{F}_p$, it must be the case that E is an $H\mathbb{F}_p$–algebra. This fact also gives an inexplicit means to recover Lemma 1.5.8.

2.5 Explicitly stabilizing the cyclic power operations for MU

Having thus demonstrated that the Tate variant of the cyclic power operation decomposes as a sum of stable operations, we are motivated to understand the available such stable operations in complex bordism. This follows quickly from our discussions in the previous few Lectures. We learned in Corollary 2.3.12 that for any complex-oriented cohomology theory E we have the calculation

$$E^* BU \cong E^*[\![\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots]\!],$$

and we gave a rich interpretation of this in terms of divisor schemes:

$$BU_E \cong \operatorname{Div}_0 \mathbb{C}P^\infty_E.$$

We would like to leverage the Thom isomorphism to gain a description of E^*MU generally and MU^*MU specifically. However, the former is *not* a ring, and although the latter is a ring its multiplication is exceedingly complicated[14], which means that our extremely compact algebraic description of E^*BU in Corollary 2.3.12 will be of limited use. Instead, we will have to content ourselves with an E_*–module basis of E^*MU.

Definition 2.5.1. Take $MU \to E$ to be a complex-oriented ring spectrum, which presents E^*BU as the subalgebra of symmetric functions inside of an infinite–dimensional polynomial algebra:

$$E^*BU \subseteq E^*BU(1)^{\times\infty} \cong E^*[\![x_1, x_2, \ldots]\!].$$

For any nonnegative multi-index $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finitely many entries nonzero, there is an associated *monomial symmetric function* b_α, which is the sum of those monomials whose exponent lists contain exactly α_j many instances of j.[15] We then set $s_\alpha \in E^*MU$ to be the image of b_α under the Thom isomorphism of E_*–modules

$$E^*MU \cong E^*BU.$$

It is called the α^{th} *Landweber–Novikov operation* with respect to the orientation $MU \to E$.

Definition 2.5.2. In the case of the identity orientation $MU \xrightarrow{\mathrm{id}} MU$, the resulting system of Chern classes is called the *Conner–Floyd–Chern classes* and the associated system of cohomology operations is called the *Landweber–Novikov operations* without further qualification.

Remark 2.5.3. For a vector bundle V and a complex-oriented cohomology theory E, we define the *total symmetric Chern class* of V by the sum

$$c_{\mathbf{t}}(V) = \sum_\alpha c_\alpha(V)\mathbf{t}^\alpha.$$

In the case of a line bundle $\mathcal{L}$ with first Chern class $c_1(\mathcal{L}) = x$, this degenerates to the sum

$$c_{\mathbf{t}}(\mathcal{L}) = \sum_{j=0}^{\infty} x^j t_j.$$

[14] For a space X, E^*X has a ring structure because X has a diagonal, and MU does not have a diagonal. In the special case of $E = MU$, there is a ring product coming from endomorphism composition.

[15] For example, $\alpha = (1, 2, 0, 0, \ldots)$ corresponds to the sum

$$b_\alpha = \sum_i \sum_{j \ne i} \sum_{\substack{k \ne i \\ k > j}} x_i x_j^2 x_k^2 = x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2^2 x_3 + \cdots .$$

For a direct sum $U = V \oplus W$, the total symmetric Chern class satisfies a Cartan formula:

$$c_{\mathbf{t}}(U) = c_{\mathbf{t}}(V \oplus W) = c_{\mathbf{t}}(V) \cdot c_{\mathbf{t}}(W).$$

Again specializing to line bundles $\mathcal{L}$ and $\mathcal{H}$ with first Chern classes $c_1(\mathcal{L}) = x$ and $c_1(\mathcal{H}) = y$, this gives

$$\begin{aligned} c_{\mathbf{t}}(U) = c_{\mathbf{t}}(\mathcal{L} \oplus \mathcal{H}) &= \left(\sum_{j=0}^{\infty} x^j t_j\right) \cdot \left(\sum_{k=0}^{\infty} y^k t_k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x^j y^k t_j t_k \\ &= 1 + (x+y)t_1 + (xy)t_1^2 + (x^2+y^2)t_2 + (xy^2 + x^2y)t_1 t_2 + \cdots \\ &= 1 + c_1(U)t_1 + c_2(U)t_1^2 + (c_1^2(U) - 2c_2(U))t_2 + (c_1(U)c_2(U))t_1 t_2 + \cdots, \end{aligned}$$

where we have expanded out some of the pieces of the total symmetric Chern class in polynomials in the usual Chern classes.

Definition 2.5.4 ([Ada95, Theorem I.5.1])**.** todoShould you include Quillen's "norm" perspective on these operations (cf. between equations 2.2 and 2.3)? You were just almost-but-not-quite talking about isogenies, and you're going to want to talk about norm constructions eventually... Take the orientation to be $MU \xrightarrow{\mathrm{id}} MU$, so that we are considering MU^*MU and the Landweber–Novikov operations arising from the Conner–Floyd–Chern classes. These account for the *stable* operations in MU–cohomology, analogous to the Steenrod operations for $H\mathbb{F}_2$. They satisfy the following properties:

- s_0 is the identity.
- s_α is natural: $s_\alpha(f^*x) = f^*(s_\alpha x)$.
- s_α is stable: $s_\alpha(\sigma x) = \sigma(s_\alpha x)$.
- s_α is additive: $s_\alpha(x+y) = s_\alpha(x) + s_\alpha(y)$.
- s_α satisfies a Cartan formula. Define

  $$s_{\mathbf{t}}(x) := \sum_\alpha s_\alpha(x)\mathbf{t}^\alpha := \sum_\alpha s_\alpha(x) \cdot t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \cdots \in MU^*(X)[\![t_1, t_2, \ldots]\!]$$

  for an infinite sequence of indeterminates $t_1, t_2, \ldots$. Then

  $$s_{\mathbf{t}}(xy) = s_{\mathbf{t}}(x) \cdot s_{\mathbf{t}}(y).$$

- Let $\xi \colon X \to BU(n)$ classify a vector bundle and let φ denote the Thom isomorphism

  $$\varphi \colon MU^*X \to MU^*T(\xi).$$

  Then the Chern classes of ξ are related to the Landweber–Novikov operations on the Thom spectrum by the formula

  $$\varphi(c_\alpha(\xi)) = s_\alpha(\varphi(1)).$$

Having now set up an encompassing theory of stable operations, we now seek to give a formula for the cyclic Tate power operation in this framework. In order to approach this, we initially set our sights on the too-lofty goal of computing $P^{C_n}(f)$ for $f \in MU^{2q}(X)$ an MU–cohomology class in a finite complex X. Because of the definition $MU = \operatorname{colim}_k MU(k)$ and because P^{C_n} is natural under pullback, it will suffice for us to study the effect of P^{C_n} on the universal classes

Do you need to put suspensions in here so that this is a map off of a suspension spectrum?

$$u_m \colon MU(m) \to MU,$$

beginning with the canonical orientation

$$u_1 = x \in h\mathsf{Spectra}(MU(1), MU) \cong MU^2\mathbb{C}\mathrm{P}^\infty.$$

In order to understand the effect $P^{C_n}(x)$ of the power operation on x, we recall a different interpretation of x: it is also the 1st Conner–Floyd–Chern class of the tautological bundle $\mathcal{L}$ on $\mathbb{C}\mathrm{P}^\infty$, i.e.,

$$x \colon MU(1) \to MU$$

is the Thomification of the block inclusion

$$\mathcal{L} \colon BU(1) \to BU.$$

The construction defining $P^{C_n}_{\mathrm{ext}}(x)$ thus fits into the following diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ BU(1) \arrow[r] \arrow[d, "\Delta"] & \Sigma^\infty_+ BU(1) \wedge \Sigma^\infty_+ BC_n \arrow[d, "\Delta_{hC_n}"] & \\
(\Sigma^\infty_+ BU(1))^{\wedge n} \arrow[r] \arrow[d, "c_1^{\wedge n}"] \arrow[rr, bend right, "\mathcal{L}^{\oplus n}"'] & \Sigma^\infty_+ BU(1)^{\wedge n}_{hC_n} \arrow[r, "\mathcal{L}(n)"] \arrow[d] \arrow[dr, "P^{C_n}_{\mathrm{ext}}"] & \Sigma^\infty_+ BU(n) \arrow[d, "c_n"] \\
(\Sigma^2 MU)^{\wedge n} \arrow[r] & (\Sigma^2 MU)^{\wedge n}_{hC_n} \arrow[r] & \Sigma^{2n} MU.
\end{tikzcd}$$

The commutativity of the widest rectangle (i.e., the justification for the name "c_n" on the right-most vertical arrow) comes from the Cartan formula for Chern classes: because $\mathcal{L}^{\oplus n}$ splits as the sum of n line bundles, $c_n(\mathcal{L}^{\oplus n})$ is computed as the product of the 1st Chern classes of those line bundles. Second, the commutativity of the right-most square is not trivial: it is a specific consequence of how the multiplicative structure on MU arises from the direct sum of vector bundles.[16] The commutativities of the other two squares comes from the natural transformation from a C_n–space to its homotopy orbit space.

[16] In general, any notion of first Chern class $\Sigma^\infty_+ BU(1) \to \Sigma^2 E$ gives rise to a *noncommuting* diagram of this same shape. The two composites $\Sigma^\infty_+ BU(1)^{\wedge n}_{hC_n} \to \Sigma^{2n} E$ need not agree, since $\mathcal{L}(n)$ has no *a priori* reason to be compatible with the factorization appearing in the H^2_∞–structure. They turn out to be related nonetheless, and their exact relation (as well as a procedure for making them agree) is the subject of **PUT REFERENCE TO CHAPTER 6**.

Hence, the the internal cyclic power operation $P^{C_n}(x)$ is defined by the composite

$$\Sigma^\infty_+ BU(1) \wedge \Sigma^\infty_+ BC_n \simeq \Sigma^\infty_+ BU(1)_{hC_n} \xrightarrow{\Delta_{hC_n}} (\Sigma^\infty_+ BU(1))^{\wedge n}_{hC_n} \to \Sigma^{2n} MU^{\wedge n}_{hC_n} \to \Sigma^{2n} MU,$$

which is to say

$$P^{C_n}(x) = c_n(\Delta^*_{hC_n}\mathcal{L}(n)).$$

We have thus reduced to computing a particular Conner–Floyd–Chern class of a particular bundle.

Our next move is to realize that we have not lost information by passing from the bundle $\Delta^*\mathcal{L}^{\oplus n}$ to the bundle $\Delta^*_{hC_n}\mathcal{L}(n)$.

Theorem 2.5.5. *There is a natural bijection between G–equivariant vector bundles over a base X on which G acts trivially and vector bundles on $X \times BG$.*

Proof. This is the exponential adjunction

$$\begin{tikzcd} \mathsf{Spaces}(X \times BG, BU) \arrow[r, shift left] & \mathsf{Spaces}(* /\!/ G, \mathsf{Spaces}(X, BU)), \arrow[l, shift left] \\ V_{hG} & V. \arrow[l, mapsto, shift left] \end{tikzcd}$$

$$V_{hG} \leftarrow\!\!\!\mapsto V,\qquad V_{hG} \mapsto V$$

The right-hand side consists of G–equivariant vector bundles over the G–trivial base X, and the left-hand side consists of vector bundles over $X \times BG$. □

This isn't much of a proof.

Corollary 2.5.6. *Under this bijection, the vector bundle $\Delta^*_{hC_n}\mathcal{L}(n)$ on $BU(1) \times BC_n$ corresponds to the C_n–equivariant vector bundle $\Delta^*\mathcal{L}^{\oplus n}$ on $BU(1)$.* □

We thus proceed to analyze $\Delta^*_{hC_n}\mathcal{L}(n)$ by studying the C_n–equivariant bundle $\Delta^*\mathcal{L}^{\oplus n}$ instead. The C_n–action is given by permutation of the factors, and hence we have an identification

$$\Delta^*\mathcal{L}^{\oplus n} \cong \mathcal{L} \otimes \pi^*\rho,$$

where ρ is the permutation representation of C_n (considered as a vector bundle over a point) and $\pi\colon BU(1) \to *$ is the constant map. The permutation representation for the abelian group C_n, also known as its regular representation, is accessible by character theory. The generating character $\chi\colon U(1)[n] \to U(1)$ gives a decomposition

$$\rho \cong \bigoplus_{j=0}^{n-1} \chi^{\otimes j}.$$

Applying this to our situation, we get a sequence of isomorphisms of C_n–equivariant vector bundles

$$\Delta^*\mathcal{L}^{\oplus n} \cong \mathcal{L} \otimes \pi^*\rho \cong \mathcal{L} \otimes \bigoplus_{j=0}^{n-1} \pi^*\chi^{\otimes j} \cong \bigoplus_{j=0}^{n-1} \mathcal{L} \otimes \pi^*\chi^{\otimes j}.$$

Applying Theorem 2.5.5, we recast this as a calculation of the bundle $\Delta^*_{hC_n}\mathcal{L}(n)$:

$$\Delta^*_{hC_n}\mathcal{L}(n) = \bigoplus_{j=0}^{n-1} \pi_1^*\mathcal{L} \otimes \pi_2^*\eta^{\otimes j},$$

where η is the bundle classified by $\eta\colon BU(1)[n] \to BU(1)$ and π_1, π_2 are the two projections off of $BU(1) \times BC_n$.

We now use this to access $c_n(\Delta^*_{hC_n}\mathcal{L}(n))$. As the top Chern class of this n–dimensional vector bundle, we think of this as a calculation of its Euler class, which lets us lean on multiplicativity:

$$\begin{aligned} P^{C_n}(x) = c_n(\Delta^*_{hC_n}\mathcal{L}(n)) &= e\left(\bigoplus_{j=0}^{n-1} \pi_1^*\mathcal{L} \otimes \pi_2^*\eta^{\otimes j}\right) = \prod_{j=0}^{n-1} e\left(\pi_1^*\mathcal{L} \otimes \pi_2^*\eta^{\otimes j}\right) \\ &= \prod_{j=0}^{n-1} c_1\left(\pi_1^*\mathcal{L} \otimes \pi_2^*\eta^{\otimes j}\right) = \prod_{j=0}^{n-1}(x +_{MU} [j]_{MU}(t)). \end{aligned}$$

Here x is still the 1st Conner–Floyd–Chern class of $\mathcal{L}$ and t is the Euler class of η. We now try to make sense of this product expression for $c_n(\Delta^*_{hC_n}\mathcal{L}(n))$ by expanding it in powers of x and identifying its component pieces.

Lemma 2.5.7. *There is a series expansion*

$$P^{C_n}(x) = \prod_{j=0}^{n-1}(x +_{MU} [j]_{MU}(t)) = w + \sum_{j=1}^{\infty} a_j(t)x^j,$$

where $a_j(t)$ is a series with coefficients in the subring $C \subseteq MU_*$ spanned by the coefficients of the natural MU–formal group law. The leading term

$$w = e(\rho) = \prod_{j=0}^{n-1} e(\eta^{\otimes j}) = \prod_{j=0}^{n-1}[j]_{MU}(e(\eta)) = (n-1)!t^{n-1} + \sum_{j \geq n} b_j t^j$$

is the Euler class of the reduced permutation representation, and, again, the elements b_j lie in the subring C. □

This is about as much information as we can hope to extract in the 1st universal case. We thus return to our original goal: understanding the action of P^{C_n} on each of the canonical classes

$$u_m\colon MU(m) \to MU.$$

Our best hope to approach this is to use the splitting principle, so we rewrite the formula in Lemma 2.5.7 in a form amenable to direct sums:

$$P^{C_n}(x) = \sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x), \qquad a_\alpha(t) = \prod_{j=0}^{\infty} a_j(t)^{\alpha_j}.$$

Corollary 2.5.8. *There is the universal formula*

$$P^{C_n}(u_m) = \sum_{|\alpha| \le m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m).$$

Proof. This follows directly from the splitting principle and the Cartan formula:

$$\begin{aligned} P^{C_n}(u_m) &= \overbrace{P^{C_n}(x_1) \cdot \cdots \cdot P^{C_n}(x_m)}^{\text{each of the } m \text{ factors}} \\ &= \left(\sum_{|\alpha| \le 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x_1) \right) \cdot \cdots \cdot \left(\sum_{|\alpha| \le 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x_m) \right) \\ &= \sum_{|\alpha| \le m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m). \end{aligned}$$

□

We will use this to power the following conclusion about cohomology classes on in general, starting with an observation about the fundamental class of a sphere:

Lemma 2.5.9 (cf. [Rud98, Corollary VII.7.14]). *For $f \in MU^{2q}(X)$ a cohomology class in a finite complex X, there is the suspension relation*

$$P^{C_n}(\sigma^{2m} f) = w^m \sigma^{2m} P^{C_n}(f).$$

Proof. We calculate P^{C_n} applied to the fundamental class

$$S^{2m} \xrightarrow{\iota_{2m}} T_m BU(m) \simeq \Sigma^{2m} MU(m) \xrightarrow{\Sigma^{2m} u_m} \Sigma^{2m} MU$$

by restricting the universal formula:

$$P^{C_n}(\iota_{2m}^* u_m) = \iota_{2m}^* P^{C_n}(u_m) = \iota_{2m}^* \left(\sum_{|\alpha| \le m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m) \right) = w^m \iota_{2m},$$

since $s_\alpha(\iota_{2m}) = 0$ for any nonzero α, as the cohomology of S^{2m} is too sparse. Because $\sigma^{2m} f = \iota_{2m} \wedge f$, we conclude the proof by multiplicativity of P^{C_n}. □

Theorem 2.5.10 (cf. [Qui71, Proposition 3.17], [Rud98, Corollary VII.7.14]). *Let X be a finite pointed space and let f be a cohomology class*

$$f \in \widetilde{MU}^{2q}(X).$$

For $m \gg 0$, there is a formula

$$w^m P^{C_n}(f) = \sum_{|\alpha| \le m+q} w^{q+m-|\alpha|} a_\alpha(t) s_\alpha(f),$$

with t, w, and $a_\alpha(t)$ as defined above.[17]

Proof. We take m large enough so that f is represented by an unstable map

$$g\colon \Sigma^{2m} X \to T_{m+q} BU(m+q),$$

in the sense that g intertwines f with the universal class u_{m+q} by the formula

$$g^* u_{m+q} = \sigma^{2m} f.$$

We use Lemma 2.5.9 and naturality to conclude

$$\begin{aligned} w^m \sigma^{2m} P^{C_n}(f) &= P^{C_n}(\sigma^{2m} f) = P^{C_n}(g^* u_{m+q}) = g^* P^{C_n}(u_{m+q}) \\ &= g^* \left(\sum_{|\alpha| \le m+q} w^{m+q-|\alpha|} a_\alpha(t) s_\alpha(u_{m+q}) \right) \\ &= \sum_{|\alpha| \le m+q} w^{m+q-|\alpha|} a_\alpha(t) \sigma^{2m} s_\alpha(f). \end{aligned}$$

□

Our conclusion, then, is that P^{C_n} is *almost* naturally expressible in terms of the Landweber–Novikov operations, where the "*almost*" is controlled by some w–torsion. Additionally, the various factors in this statement—including w itself—are controlled by the formal group law "$+_{MU}$" associated to the tautological complex orientation of MU and the subring C. In the next Lecture, we will discover the surprising fact that we only need to multiply by a *single* w and the equally surprising consequences this entails for MU_* itself.

The previous lecture calls the Euler class "x", which does not match the notation here. Also, the conclusion of this lecture doesn't mention the connection to the Tate construction. Also, the ligament of this lecture doesn't point out that we're trying to compute the total power operation and *not* its Tate projection, but our strategy ends up most naturally giving us information about the Tate projection and we could phrase it in a Corollary describing $P^{C_n}_{\mathrm{Tate}}(f) = w^{-m} \cdots$.

2.6 The complex bordism ring

With Theorem 2.5.10 in hand, we will now deduce Quillen's major structural theorem about MU_*. We will preserve the notation used in Lemma 2.5.7 and Theorem 2.5.10:

- ρ is the reduced regular representation of C_n, which coincides with its reduced permutation representation, and $w = e(\rho)$ is its Euler class.
- $\eta\colon BU(1)[n] \to BU(1)$ is the line bundle associated to a generating character for C_n, and $t = e(\eta)$ its Euler class.
- C is the subring of MU_* generated by the coefficients of the formal group law associated to the identity complex–orientation.

[17] The reader comparing with Quillen's paper will notice various apparent discrepancies between the statements of our Theorem and of his. These are notational: he grades his cohomology functor homologically, which occasionally causes our q to match his $-q$, so that his n is comparable to our $m - q$. **GET THIS WARNING RIGHT**

In the course of working out the main Theorem, we will want to make use of some properties of the class t.

Lemma 2.6.1 (cf. [RW80, Theorem 5.7], [HL, Proposition 2.4.4], [Rud98, Theorem VII.7.9])**.** *There is an isomorphism of formal groups*[18]

$$BU(1)[n]_{MU} \cong BU(1)_{MU}[n].$$

Proof. Consider the pullback diagram of spherical fibrations:

Put in a pullback corner here.

$$\begin{tikzcd} U(1) \arrow[r] \arrow[d, equal] & BU(1)[n] \arrow[r] \arrow[d] & BU(1) \arrow[d, "n"] \\ U(1) \arrow[r] & EU(1) \arrow[r] & BU(1). \end{tikzcd}$$

The Euler class of the first bundle is the class x, and it pulls back along $n\colon BU(1) \to BU(1)$ to give the Euler class $[n](x)$ of the second bundle. The induced long exact sequence[19] takes the form

$$\begin{tikzcd} & MU^*BU(1) \arrow[dl] & \\ MU^*(BU(1)[n]) \arrow[rr, "\partial"'] & & MU^{*+2}BU(1) \arrow[ul, "-\smile[n](x)"'] \end{tikzcd}$$

where x is the coordinate on $BU(1)_{MU}$. Because $[n]_{MU}(x) = nx + \cdots$ and because $\widetilde{MU}^2\mathbb{C}P^1$ has no n–torsion, the right diagonal map is injective and hence $\partial = 0$. This therefore gives a short exact sequence of Hopf algebras, which we can reinterpret as a short exact sequence of group schemes

$$0 \to BU(1)[n]_{MU} \to BU(1)_{MU} \xrightarrow{n} BU(1)_{MU} \to 0. \qquad \square$$

Corollary 2.6.2. *The Künneth map*

$$MU^*(X) \otimes_{MU^*} MU^*(BU(1)[n]) \to MU^*(X \times BU(1)[n])$$

is an isomorphism. In terms of coordinate rings, this gives isomorphisms

$$(X \times BU(1)[n])_{MU} \cong X_{MU} \times BU(1)[n]_{MU} \cong X_{MU} \times BU(1)_{MU}[n].$$

Proof. This follows from the evenness of $MU^*(BU(1)[n])$. $\square$

[18] In algebraic language, $MU^*BU(1)[n] \cong MU^*BU(1)/[n](x)$, where $x \in MU^2BU(1)$ is the canonical coordinate and $[n](x)$ is the n–series.

[19] This sequence is known as the Gysin sequence. It arises as the exact couple for the Serre spectral sequence for the spherical fibration.

Corollary 2.6.3 ([Qui71, Proposition 4.4]). *Write*

$$\langle n \rangle_{MU}(x) = \frac{[n]_{MU}(x)}{x}.$$

If $\omega \in MU^*BU(1)[n]$ satisfies $t \cdot \omega = 0$, then there exists a class y with $\omega = y \cdot \langle n \rangle_{MU}(t)$.

Proof. By Lemma 2.6.1 we know $MU^*BU(1)[n] \cong MU^*[\![t]\!]/[n]_{MU}(t)$, so the kernel of multiplication by t is exactly $\langle n \rangle_{MU}(t)$.[20] □

In all, we learn that the Euler class $t = e(\eta)$ corresponds to the restriction of the coordinate x along the closed inclusion

$$BU(1)[n]_{MU} \cong BU(1)_{MU}[n] \to BU(1)_{MU}.$$

We now turn to the main Theorem.

Theorem 2.6.4 ([Qui71, Theorem 5.1]). *If X has the homotopy type of a finite complex, then*

$$MU^*(X) = C \cdot \sum_{q \geq 0} MU^q(X),$$
$$\widetilde{MU}^*(X) = C \cdot \sum_{q > 0} MU^q(X).$$

Remark 2.6.5. In what follows, the reader should carefully remember the degree conventions stemming from the formula

$$MU^*X = \pi_{-*}F(\Sigma^\infty_+ X, MU).$$

The homotopy ring MU_* appears in the *negative* degrees of $MU^*(*)$, but the fundamental class of S^m appears in the *positive* degree $MU^m(S^m)$.

Proof of Theorem 2.6.4. We can immediately reduce the claim in two ways. First, it is true if and only if it is also true for reduced cohomology. Second, because $MU^{2*+1}(*) = 0$, we can restrict attention just to $MU^{2*}(X)$, since we can then handle the odd-degree parts of $MU^*(X)$ by suspending X once. Defining

$$R^{2*} := C \cdot \sum_{q > 0} MU^{2q}(X),$$

we can thus focus on the claim

$$\widetilde{MU}^{2*}(X) \stackrel{?}{=} C \cdot \sum_{q > 0} MU^{2q}(X).$$

[20]Quillen considers this as coming from the Gysin sequence for $S^1 \to S(\eta) \to BU(1)[n]$, which has Euler class t.

Noting that the claim is trivially true for all positive values of $*$, we will show this by working p–locally and inducting on the value of "$-*$".

Suppose that

$$R^{-2j}_{(p)} = \widetilde{MU}^{-2j}(X)_{(p)}$$

for $j < q$ and consider $x \in \widetilde{MU}^{-2q}(X)$. Then, for $m \gg 0$, we have

$$w^m P^p(x) = \sum_{|\alpha| \le m-q} w^{m-q-|\alpha|} a(t)^\alpha s_\alpha x = w^{m-q} x + \sum_{\substack{|\alpha| \le m-q \\ \alpha \ne 0}} w^{m-q-|\alpha|} a(t)^\alpha s_\alpha x.$$

Recall that w is a power series in t with coefficients in C and leading term $(p-1)! \cdot t^{p-1}$, so that $t^{p-1} = w \cdot \theta(t)$ for some multiplicatively invertible series $\theta(t)$ with coefficients in C. Since s_α raises degree, we have $s_\alpha x \in R$ by the inductive hypothesis, and we may thus collect all those terms into a series $\psi_x(t) \in R_{(p)}[\![T]\!]$ to write

I think you confused t and w in this equation. Shouldn't it be more like $(m-q)(k-1)$?

$$t^{m-q}(w^q P^p(x) - x) = \psi_x(t).$$

Suppose $m > q$ is the least integer for which we can write such an equation—we will show $m = q+1$ in a moment. Using **??**, we find that restricting this equation along the inclusion $i \colon X \to X \times BU(1)[p]$ sets $t = 0$ and yields $\psi_x(0) = 0$. It follows that $\psi_x(t) = t\varphi_x(t)$ is at least once t–divisible, and thus

$$t(t^{m-q-1}(w^q Px - x) - \varphi_x(t)) = 0.$$

Appealing to Corollary 2.6.3, we produce a class $y \in \widetilde{MU}^{-2q+2(m-1)}(X)$ with

$$t^{m-q-1}(w^q P^p(x) - x) = \varphi_x(x) + y\langle p\rangle(t).$$

If $m > q+1$, then $y \in R_{(p)}$ for degree reasons and hence the right-hand side gives a series expansion contradicting our minimality hypothesis. So, $m = q+1$, and the outer factor of t^{m-q-1} is not present in the last expression.[21] Restricting along i again to set $w = t = 0$ and $P^p(x) = x^p$, we obtain the equation

$$\left.\begin{array}{rl} -x & \text{if } q > 0 \\ x^p - x & \text{if } q = 0 \end{array}\right\} = \varphi_x(0) + py.$$

In the first case, where $q > 0$, it follows that $MU^{-2q}(X) \subseteq R^{-2q} + pMU^{-2q}(X)$, and since $MU^{-2q}(X)$ has finite order torsion[22] it follows that $MU^{-2q}(X) = R^{-2q}$. In the other

[21] One can interpret the proof thus far as giving a bound on the amount of w–torsion needed to get the stability relation described in Theorem 2.5.10. Our answer is quite surprising: we have found that we need just a single w, which isn't much stability at all!

[22] This is a consequence of X having finitely many cells, MU having finitely many cells in each degree, and each homotopy group of the stable sphere being finitely generated.

case, x can be rewritten as a sum of elements in R^0, elements in $p\widetilde{MU}^0(X)$, and elements in $(\widetilde{MU}^0(X))^p$. Since the ideal $\widetilde{MU}^0(X)$ is nilpotent, it again follows that $\widetilde{MU}^0(X) = R^0$, concluding the induction. □

Corollary 2.6.6. *The coefficients of the formal group law generate MU_*.*

Proof. This is the case $X = *$. □

Remark 2.6.7. This proof actually also goes through for MO as well. In that case, it's even easier, since the equation $2 = 0$ in $\pi_0 MO$ causes much of the algebra to collapse. The proof does not extend further to cases like MSO or MSp, as explained in the introduction to this Case Study: these bordism theories do not have associated formal group laws, and so we lose the control we had in Lecture 2.5.

Take $\mathcal{M}_{\mathbf{fgl}}$ to be the moduli of formal group laws. Since a formal group law is a power series satisfying some algebraic identities, this moduli object is an affine scheme with coordinate ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. A rephrasing of Corollary 2.6.6 is that the natural map

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$$

is *surjective*. This is reason enough to start studying $\mathcal{M}_{\mathbf{fgl}}$ in earnest, which we take up in the next Case Study—but while we're here, if we anachronistically assume one algebraic fact about $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ we can prove that the natural map is actually an *isomorphism*. The place to start is with the following topological observation about mixing complex–orientations:

Lemma 2.6.8 ([Ada95, Lemma 6.3 and Corollary 6.5])**.** *Let $\varphi\colon MU \to E$ be a complex–oriented ring spectrum and consider the two orientations on $E \wedge MU$ given by*

$$\mathbb{S} \wedge MU \xrightarrow{\eta_E \wedge 1} E \wedge MU, \qquad\qquad MU \wedge \mathbb{S} \xrightarrow{\varphi \wedge \eta_{MU}} E \wedge MU.$$

The two induced coordinates x^E and x^{MU} on $\mathbb{CP}^\infty_{E \wedge MU}$ are related by the formulas

$$x^{MU} = \sum_{j=0}^{\infty} b_j^E (x^E)^{j+1} =: g(x^E),$$
$$g^{-1}(x^{MU} +_{MU} y^{MU}) = g^{-1}(x^E) +_E g^{-1}(y^E).$$

where $E_* MU \cong \frac{\mathrm{Sym}_{E_*} E_*\{\beta_1, \beta_2, \beta_3, \ldots\}}{\beta_1 = 1} \cong E_*[b_1, b_2, \ldots]$, as in Lemma 1.5.1, Corollary 1.5.2, and Corollary 2.0.4.

Proof. The second formula is a direct consequence of the first. The first formula comes from taking the module generators $\beta_{j+1} \in E_{2(j+1)}\mathbb{CP}^\infty = E_{2j}MU(1)$ and pushing them forward to get the algebra generators $b_j \in E_{2j}MU$. Then, the triangle

$$\begin{tikzcd}[\mathbb{C}\mathrm{P}^\infty, MU] \arrow[rr] \arrow[dr] & & [\mathbb{C}\mathrm{P}^\infty, E \wedge MU] \arrow[dl, "\cong"] \\ & \mathsf{Modules}_{E_*}(E_*\mathbb{C}\mathrm{P}^\infty, E_*MU) & \end{tikzcd}$$

allows us to pair x^{MU} with $(x^E)^{j+1}$ to determine the coefficients of the series. □

Corollary 2.6.9 ([Ada95, Corollary 6.6]). *In particular, for the orientation $MU \to H\mathbb{Z}$ we have*

$$x_1 +_{MU} x_2 = \exp^H(\log^H(x_1) + \log^H(x_2)),$$

where $\exp^H(x) = \sum_{j=0}^{\infty} b_j x^{j+1}$. □

However, one also notes that $H\mathbb{Z}_*MU = \mathbb{Z}[b_1, b_2, \ldots]$ carries the universal example of a formal group law with a logarithm—this observation is independent of any knowledge about the coefficient ring MU_*. It turns out that this brings us one step away from understanding MU_*:

Theorem 2.6.10 (To be proven as Theorem 3.2.2). *There is a ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ carrying the universal formal group law, and it is free: it is a polynomial ring over $\mathbb{Z}$ in countably many generators.* □

Corollary 2.6.11. *The natural map $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ classifying the formal group law on MU_* is an isomorphism.*

Proof. We proved in Corollary 2.6.6 that this map is surjective. We also proved in Theorem 2.1.22 that every rational formal group law has a logarithm, i.e., the long composite on the second row

$$\begin{tikzcd} \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \arrow[r] \arrow[d] & MU_* \arrow[r] \arrow[d] & (H\mathbb{Z}_*MU) \arrow[d] \\ \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \arrow[r] & MU_* \otimes \mathbb{Q} \arrow[r, "\simeq"] & (H\mathbb{Z}_*MU) \otimes \mathbb{Q} \end{tikzcd}$$

is an isomorphism. It follows from Theorem 2.6.10 that the left-most vertical map is injective, hence the top-left horizontal map is injective, hence it is an isomorphism. □

Corollary 2.6.12. *The ring $\pi_*(MU \wedge MU)$ carries the universal example of two strictly isomorphic formal group laws. Additionally, the ring $\pi_0(MUP \wedge MUP)$ carries the universal example of two isomorphic formal group laws.*

Proof. Combine Lemma 2.6.8 and Corollary 2.6.11. □

Case Study 3

Finite spectra

Andy Senger correctly points out that "stalkwise" is the wrong word to use in all this (if we mean to be working in the Zariski topology, which surely we must). The stalks are selected by maps from certain local rings; E_Γ selects the formal neighborhood of the special point inside of this; and K_Γ selects the special point itself. Is "fiber-wise" enough of a weasel word to get out of this? In any case, make sure you clean up all instances of the word "stalk".

Our goal in this Case Study is to thoroughly examine one of the techniques from Case Study 1 that has not yet resurfaced: the idea that $H\mathbb{F}_2$–homology takes values in quasicoherent sheaves over some algebro-geometric object encoding the coaction of the dual Steenrod Hopf algebra. We will find that this situation is quite generic: associated to mildly nice ring spectra E, we will construct a very rich algebro-geometric object $\mathcal{M}_E$, called its context, such that E–homology sends spaces X to sheaves $\mathcal{M}_E(X)$ over $\mathcal{M}_E$. In still nicer situations, the difference between the E_*–module $E_*(X)$ and the sheaf $\mathcal{M}_E(X)$ tracks exactly the analogue of the action of the dual Steenrod algebra, called the *Hopf algebroid of stable E–homology cooperations*. From this perspective, we will reinterpret Quillen's Corollary 2.6.11 as giving a presentation

$$\mathcal{M}_{MUP} \xrightarrow{\simeq} \mathcal{M}_{\mathbf{fg}},$$

where $\mathcal{M}_{\mathbf{fg}}$ is the *moduli of formal groups*. This indicates a program for studying periodic complex bordism, which we will spend the rest of this introduction outlining.

Abstractly, one can hope to study any sheaf, including $\mathcal{M}_E(X)$, by analyzing its stalks. The main utility of Quillen's theorem is that it gives us access to a concrete model of the context $\mathcal{M}_{MUP}$, so that we can determine where to even look for those stalks. However, even this is not really enough to get off the ground: the stalks of some sheaf can exhibit nearly arbitrary behavior. In particular, there is little reason to expect the stalks of $\mathcal{M}_E(X)$ to vary nicely with X. Accordingly, given a map f in the diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "f"] \arrow[dr] & \mathcal{M}_{\mathbf{fgl}} \arrow[r, equal] \arrow[d] & \mathcal{M}_{MUP}[0] \arrow[r, equal] \arrow[d] & \operatorname{Spec} MUP_0 \\ & \mathcal{M}_{\mathbf{fg}} \arrow[r, equal] & \mathcal{M}_{MUP}, & \end{tikzcd}$$

life would be easiest if the R–module determined by $f^*\mathcal{M}_{MUP}(X)$ were itself the value of a homology theory $R_0(X) = MUP_0 X \otimes_{MUP_0} R$—this is exactly what it would mean for $R_0(X)$ to "vary nicely with X". Of course, this is unreasonable to expect in general: homology theories are functors which convert cofiber sequences of spectra to long exact sequences of groups, but base–change from $\mathcal{M}_{\mathbf{fg}}$ to $\operatorname{Spec} R$ preserves exact sequences exactly when the diagonal arrow is *flat*. However, if flatness is satisfied, this gives the following theorem:

Theorem 3.0.1 (Landweber)**.** *Given such a diagram where the diagonal arrow is flat, the functor*

$$R_0(X) := MUP_0(X) \otimes_{MUP_0} R$$

is a homology theory.

In the course of proving this theorem, Landweber additionally devised a method to recognize flat maps. Recall that a map $f\colon Y \to X$ of schemes is flat exactly when for any closed subscheme $i\colon A \to X$ with ideal sheaf $\mathcal{I}$ there is an exact sequence

$$0 \to f^*\mathcal{I} \to f^*\mathcal{O}_X \to f^* i_* \mathcal{O}_A \to 0.$$

Landweber classified the closed subobjects of $\mathcal{M}_{\mathbf{fg}}$, thereby giving a precise list of conditions needed to check maps for flatness.

This appears to be a moot point, however, as it is unreasonable to expect this idea to apply to computing stalks: the inclusion of a geometric point is flat only in highly degenerate cases. We will see that this can be repaired: the inclusion of the formal completion of a subobject is flat in friendly situations, and so we naturally become interested in the infinitesimal deformation spaces of the geometric points Γ on $\mathcal{M}_{\mathbf{fg}}$. If we can analyze those, then Landweber's theorem will produce homology theories called *Morava E_Γ–theories*. Moreover, if we find that these deformation spaces are *smooth*, it will follow that their deformation rings support regular sequences. In this excellent case, by taking the regular quotient we will be able to recover *Morava K_Γ–theory*, a *homology theory*, which plays the role[1] of computing the stalk of $\mathcal{M}_{MUP}(X)$ at Γ.[2]

We have thus assembled a task list:

- Describe the open and closed subobjects of $\mathcal{M}_{\mathbf{fg}}$.
- Describe the geometric points of $\mathcal{M}_{\mathbf{fg}}$.
- Analyze their infinitesimal deformation spaces.

[1]To be clear: $K_\Gamma(X)$ may not actually compute the literal stalk of $\mathcal{M}_{MUP}(X)$ at Γ, since the homotopical operation of quotienting out the regular sequence is potentially sensitive to torsion sections of $\mathcal{M}_{MUP}(X)$.

[2]Incidentally, this program has no content when applied to $\mathcal{M}_{H\mathbb{F}_2}$, as $\operatorname{Spec} \mathbb{F}_2$ is simply too small.

These will occupy our attention for the first half of this Case Study. In the second half, we will exploit these homology theories E_Γ and K_Γ, as well as their connection to $\mathcal{M}_{\mathbf{fg}}$ and to MU, to make various structural statements about the category Spectra. These homology theories are especially well-suited to understanding the subcategory $\mathsf{Spectra}^{\mathrm{fin}}$ of finite spectra, and we will recount several important statements in that setting. Together with these homology theories, these celebrated results (collectively called the nilpotence and periodicity theorems) form the basis of *chromatic homotopy theory*. In fact, our *real* goal in this Case Study is to give an introduction to the chromatic perspective that remains in line with our algebro-geometrically heavy narrative.

3.1 Descent and the context of a spectrum

In Lecture 1.4 we took for granted the $H\mathbb{F}_2$–Adams spectral sequence, which had the form

You are sloppy about EP_0 versus E_* in this lecture. Pretty sure you mean to choose EP_0 and be done with it.

$$E_2^{*,*} = H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a); \widetilde{H\mathbb{F}_2 P_0 X}) \Rightarrow \pi_* X_2^\wedge,$$

where we had already established some yoga by which we could identify the dual Steenrod coaction on $H\mathbb{F}_2 P_0 X$ with an action of $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$ on its associated quasicoherent sheaf over $\operatorname{Spec} \mathbb{F}_2$. Our goal in this Lecture is to revise this tool to work for other ring spectra E and target spectra X, eventually arriving at a spectral sequence with signature

$$E_2^{*,*} = H^*(\mathcal{M}_E; \mathcal{M}_E(X)) \Rightarrow \pi_* X.$$

In particular, we will encounter along the way the object "$\mathcal{M}_E$" envisioned in the introduction to this Case Study.

At a maximum level of vagueness, we are seeking a process by which its homotopy $\pi_* X$ can be recovered from the E–homology groups $E_* X$. Generally speaking, spectral sequences arise from taking homotopy groups of a topological version of this same recovery process—i.e., recovering the spectrum X from the spectrum $E \wedge X$. Recognizing that X can be thought of as an $\mathbb{S}$–module and $E \wedge X$ can be thought of as its base change to an E–module, we are inspired to double back and consider as inspiration an algebraic analogue of the same situation. Given a ring map $f\colon R \to S$ and an S–module N, Grothendieck's framework of *(faithfully flat) descent* addresses the following questions:

1. When is there an R–module M such that $N \cong S \otimes_R M = f^* M$?

2. What extra data can be placed on N, called *descent data*, so that the category of descent data for N is equivalent to the category of R–modules under the map f^*?

3. What conditions can be placed on f so that the category of descent data for any given module is always contractible, called *effectivity*?

Suppose that we begin with an R–module M and set $N = f^*M$, so that we are certain *a priori* that the answer to the first question is positive. The S–module N has a special property, arising from f being a ring map: there is a canonical isomorphism of $(S \otimes_R S)$–modules

Get this right.

$$\varphi\colon S \otimes_R N = (f \otimes 1)^* N = ((f \otimes 1) \circ f)^* M \cong ((1 \otimes f) \circ f)^* M = (1 \otimes f)^* N = N \otimes_R S,$$
$$s_1 \otimes (s_2 \otimes m) \mapsto (s_1 \otimes m) \otimes s_2.$$

In fact, this isomorphism is compatible with further shuffles, in the sense that the following diagram commutes:[3]

$$\begin{tikzcd}
N \otimes_R S \otimes_R S \arrow[rr, "\varphi_{13}", "\simeq"'] \arrow[dr, "\varphi_{12}", "\simeq"'] & & S \otimes_R S \otimes_R N \\
& S \otimes_R N \otimes_R S, \arrow[ur, "\simeq", "\varphi_{23}"'] &
\end{tikzcd}$$

where φ_{ij} denotes applying φ to the i^{th} and j^{th} coordinates.

Definition 3.1.1. An S–module N equipped with such an isomorphism $\varphi\colon S \otimes_R N \to N \otimes_R S$ which causes the triangle to commute is called a *descent datum for f*.

Descent data admit two equivalent reformulations, both of which are useful to note.

Cite me: Find a citation for the canonical coring..

Remark 3.1.2. The ring $C = S \otimes_R S$ admits the structure of an S–coring: we can use the map f to produce a relative diagonal map

$$\Delta\colon S \otimes_R S \cong S \otimes_R R \otimes_R S \xrightarrow{1 \otimes f \otimes 1} S \otimes_R S \otimes_R S \cong (S \otimes_R S) \otimes_S (S \otimes_R S).$$

The descent datum φ on an S–module N is equivalent to a C–coaction map. The S–linearity of the coaction map is encoded by a square

$$\begin{tikzcd}
S \otimes_R N \arrow[r, "1 \otimes \psi"] \arrow[d] \arrow[drr, "\varphi"] & S \otimes_R N \otimes_S (S \otimes_R S) \arrow[d] & \\
N \arrow[r, "\psi"] & N \otimes_S (S \otimes_R S) \arrow[r, equal] & N \otimes_R S,
\end{tikzcd}$$

and the long composite gives the descent datum φ. Conversely, given a descent datum φ we can restrict it to get a coaction map by

$$\psi\colon N = R \otimes_R N \xrightarrow{f \otimes 1} S \otimes_R N \xrightarrow{\psi} N \otimes_R S.$$

The coassociativity condition on the comodule is equivalent under this correspondence to the commutativity of the triangle associated to φ.

[3]The commutativity of this triangle shows that any number of shuffles also commutes.

Remark 3.1.3 ([Hov02, Theorem A]). Alternatively, descent data also arise naturally as sheaves on simplicial schemes. Associated to the map $f\colon\ \operatorname{Spec} S \to \operatorname{Spec} R$, we can form a Čech complex

$$\mathcal{D}_f := \left\{ \operatorname{Spec} S \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{matrix} \operatorname{Spec} S \\ \times_{\operatorname{Spec} R} \\ \operatorname{Spec} S \end{matrix} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{matrix} \operatorname{Spec} S \\ \times_{\operatorname{Spec} R} \\ \operatorname{Spec} S \\ \times_{\operatorname{Spec} R} \\ \operatorname{Spec} S \end{matrix} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots \right\},$$

which factors the map f as

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r, "\mathrm{sk}^0"] \arrow[rr, bend left, "f"] & \mathcal{D}_f \arrow[r, "c"] & \operatorname{Spec} R. \end{tikzcd}$$

A quasicoherent (and Cartesian [Sta14, Tag 09VK]) sheaf $\mathcal{F}$ over a simplicial scheme X is a sequence of quasicoherent sheaves $\mathcal{F}[n]$ on $X[n]$ as well as, for each map $\sigma\colon\ [m] \to [n]$ in the simplicial indexing category inducing a map $X(\sigma)\colon\ X[n] \to X[m]$, a natural choice of isomorphism of sheaves

$$\mathcal{F}(\sigma)^*\colon\ X(\sigma)^*\mathcal{F}[m] \to \mathcal{F}[n].$$

In particular, a pullback $c^*\widetilde{M}$ gives such a quasicoherent sheaf on $\mathcal{D}_f$. By restricting attention to the first three levels we find exactly the structure of the descent datum described before. Additionally, we have a natural *Segal isomorphism*

$$\mathcal{D}_f[1]^{\times_{\mathcal{D}_f[0]}(n)} \xrightarrow{\simeq} \mathcal{D}_f[n] \qquad (\text{cf. } S \otimes_R S \otimes_R S \cong (S \otimes_R S) \otimes_S (S \otimes_R S) \text{ at } n = 2),$$

which shows that any descent datum (including those not arising, a priori, from a pullback) can be naturally extended to a full quasicoherent sheaf on $\mathcal{D}_f$.

The following Theorem is the culmination of a typical first investigation of descent:[4]

Theorem 3.1.4 (Grothendieck)**.** *If $f\colon\ R \to S$ is faithfully flat, then the natural assignments*

Cite me: Actually give an original citation for f.f. descent? Or just reference Vakil.

$$\begin{tikzcd} \mathsf{QCoh}(\operatorname{Spec} R) \arrow[r, bend left, "c^*"] & \mathsf{QCoh}(\mathcal{D}_f) \arrow[l, bend left, "\lim"] \end{tikzcd}$$

form an equivalence of categories.

[4]For details and additional context, see Vistoli [Vis05, Section 4.2.1]. The story in the context of Hopf algebroids is also spelled out in detail by Miller [Milb].

Jumping off point. The basic observation in this case is that $0 \to R \to S \to S \otimes_R S$ is an exact sequence of R–modules.[5] This makes much of the homological algebra involved work out. $\square$

Without the flatness hypothesis, this Theorem fails dramatically and immediately. For instance, the inclusion of the closed point

$$f\colon \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$$

fails to distinguish the $\mathbb{Z}$–modules $\mathbb{Z}$ and $\mathbb{Z}/p$. Remarkably, this can be to large extent repaired by reintroducing homotopy theory and passing to derived categories—for instance, the complexes $Lf^*\widetilde{\mathbb{Z}}$ and $Lf^*\widetilde{\mathbb{Z}/p}$ become distinct as objects of $D(\operatorname{Spec} \mathbb{F}_p)$. Our preceding discussion of descent in Remark 3.1.3 can be quickly revised for this new homotopical setting, provided we remember to derive not just the categories of sheaves but also their underlying geometric objects. Our approach is informed by the following result:

Does a correct statement need boundedness conditions?

Lemma 3.1.5 ([EKMM97, Theorem IV.2.4])**.** *There is an equivalence of ∞–categories between $D(\operatorname{Spec} R)$ and $\mathsf{Modules}_{HR}$.* $\square$

Hence, given a map of rings $f\colon R \to S$, we redefine the derived descent object to be the cosimplicial ring spectrum

$$\mathcal{D}_{Hf} := \left\{ \begin{array}{ccccccc} & & & & & \longrightarrow & \\ & & & \longrightarrow & HS & \longleftarrow & \\ & \longrightarrow & HS & \longleftarrow & \wedge_{HR} & \longrightarrow & \\ HS & \longleftarrow & \wedge_{HR} & \longrightarrow & HS & \longleftarrow & \cdots \\ & \longrightarrow & HS & \longleftarrow & \wedge_{HR} & \longrightarrow & \\ & & & \longrightarrow & HS & \longleftarrow & \\ & & & & & \longrightarrow & \end{array} \right\},$$

and note that an R–module M gives rise to a cosimplicial left–$\mathcal{D}_{Hf}$–module which we denote $\mathcal{D}_{Hf}(HM)$. The totalization of this cosimplicial module gives rise to an HR–module receiving a natural map from M, and we can ask for an analogue of Theorem 3.1.4.

Lemma 3.1.6. *For $f\colon \mathbb{Z} \to \mathbb{F}_p$ and M a connective complex of $\mathbb{Z}$–modules, the totalization* $\operatorname{Tot} \mathcal{D}_{Hf}(HM)$ *recovers the p–completion of M.*

Proof sketch. The Hurewicz map $H\mathbb{Z} \to H\mathbb{F}_p$ kills $(p) \subseteq \pi_0 H\mathbb{Z}$, and we further calculate

$$H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p \simeq H\mathbb{F}_p \vee \Sigma H\mathbb{F}_p$$

to be connective. Combining these facts shows that the filtration of $\mathcal{D}_{Hf}(HM)$ gives the p–adic filtration of the homotopy groups $\pi_* HM$. If $\pi_* HM$ is already p–complete, then the reassembly map $HM \to \operatorname{Tot} \mathcal{D}_{Hf}(HM)$ is a weak equivalence. $\square$

[5] In the language of Example 1.4.18, this says that R itself appears as the cofixed points $S\square_{S\otimes_R S}R$.

We are now close enough to our original situation that we can make the last leap: rather than studying a map $Hf\colon HR \to HS$, we instead have the unit map $\eta\colon \mathbb{S} \to E$ associated to some ring spectrum E. Fixing a target spectrum X, we define the analogue of the descent object:

Definition 3.1.7. The *descent object* for X along $\eta\colon \mathbb{S} \to E$ is the cosimplicial spectrum

$$\mathcal{D}_E(X) := \left\{ \begin{array}{ccccccc}
 & & & & & \longrightarrow & \\
 & & & \longrightarrow & E & \longleftarrow & \\
 & \xrightarrow{\eta_L} & E & \longleftarrow & \wedge & \longrightarrow & \\
E & \xleftarrow{\mu} & \wedge & \xrightarrow{\Delta} & E & \longleftarrow & \\
\wedge & \xrightarrow{\eta_R} & E & \longleftarrow & \wedge & \longrightarrow & \cdots \\
X & & \wedge & \longrightarrow & E & \longleftarrow & \\
 & & X & & \wedge & \longrightarrow & \\
 & & & & X & &
\end{array} \right\}.$$

Lemma 3.1.8 ([Lurb, Theorem 4.4.2.8.ii])**.** *If E is an A_∞–ring spectrum, then $\mathcal{D}_E(X)$ can be considered as a cosimplicial object in the ∞–category of* Spectra. □

Definition 3.1.9. The *E–nilpotent completion* of X is the totalization of this cosimplicial spectrum:

$$X^\wedge_E := \operatorname{Tot} \mathcal{D}_E(X).$$

It receives a natural map $X \to X^\wedge_E$, the analogue of the natural map of R–modules $M \to c_* c^* M$ considered in Theorem 3.1.4.

Remark 3.1.10 ([Rav84, Theorem 1.12], [Bou79]). Ravenel proves the following generalization of Lemma 3.1.6. Let E be a connective ring spectrum, let J be the set of primes complementary to those primes p for which E_* is uniquely p–divisible, and let X be a connective spectrum. If each element of E_* has finite order, then $X^\wedge_E = X^\wedge_J$ gives the arithmetic completion of X—which we reinterpret as $\mathbb{S}^\wedge_J \to E$ being of effective descent. Otherwise, if Is this right? E_* has elements of infinite order, then $X^\wedge_E = X_{(J)}$ gives the arithmetic localization—which we reinterpret as saying that $\mathbb{S}_{(J)} \to E$ is of effective descent. Sorting out more encompassing conditions on maps $f\colon R \to S$ of E_∞–rings for which descent holds is a subject of serious study [Lurc, Appendix D].

Remark 3.1.11. Even for connective ring spectra E, the Bousfield localization $L_E X$ does *not* have to recover an arithmetic localization of X if X is not connective. Take $E = H\mathbb{Z}$ and $X = KU$, which Snaith's theorem[6] presents as

$$X = KU = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}],$$

[6] You can also work this example without knowing Snaith's theorem. All you really need to know is that $\Sigma^\infty_+\mathbb{C}\mathrm{P}^\infty[\beta^{-1}] \to KU$ is a map of ring spectra, so that $\pi_0\Sigma^\infty_+\mathbb{C}\mathrm{P}^\infty[\beta^{-1}]$ can't be a rational group since $\pi_0 KU = \mathbb{Z}$.

where $\beta\colon \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is the Bott element. This gives

$$H\mathbb{Z}_* KU = H\mathbb{Z}_*(\mathbb{C}P^\infty[\beta^{-1}]) = (H\mathbb{Z}_*\mathbb{C}P^\infty)[b_1^{-1}].$$

We can identify the pieces in turn: Example 2.1.20 shows $\mathbb{C}P^\infty_{H\mathbb{Z}} = \widehat{\mathbb{G}}_a$, so the dual Hopf algebra $\mathcal{O}^*_{\widehat{\mathbb{G}}_a} = H\mathbb{Z}_*\mathbb{C}P^\infty$ is a divided polynomial algebra on the class b_1. Inverting b_1 then gives

$$(H\mathbb{Z}_*\mathbb{C}P^\infty)[b_1^{-1}] = \Gamma[b_1][b_1^{-1}] = \mathbb{Q}[b_1^{\pm}],$$

so that, in particular, there is a weak equivalence $H\mathbb{Z} \wedge KU \to H\mathbb{Q} \wedge KU$. The cofiber

$$KU \to KU \otimes \mathbb{Q} \to KU \otimes \mathbb{Q}/\mathbb{Z}$$

is thus a nonzero $H\mathbb{Z}$–acyclic spectrum.

Finally, we can interrelate these algebraic and topological notions of descent by studying the coskeletal filtration spectral sequence for $\pi_* X^\wedge_E$, which we define to be the *E–Adams spectral sequence* for X. Applying the homotopy groups functor to the cosimplicial ring spectrum $\mathcal{D}_E$ gives a cosimplicial ring $\pi_*\mathcal{D}_E$, which we would like to connect with an algebraic descent object of the sort considered in Remark 3.1.3. In order to make this happen, we need two niceness conditions on E:

Definition 3.1.12. A ring spectrum E satisfies **CH**, the **C**ommutativity **H**ypothesis, when the ring $\pi_* E^{\wedge j}$ is commutative for all $j \geq 1$. In this case, we can form the simplicial scheme

$$\mathcal{M}_E = \operatorname{Spec} \pi_* \mathcal{D}_E,$$

called the *context* of E.

Definition 3.1.13. A ring spectrum E satisfies **FH**, the **F**latness **H**ypothesis, when the right-unit map $E_* \to E_*E$ is flat.[7] In this case, the Segal map

$$(E_*E)^{\otimes_{E_*} j} \otimes_{E_*} E_*X \to \pi_*(E^{\wedge(j+1)} \wedge X) = \pi_*\mathcal{D}_E(X)[j]$$

is an isomorphism for all X. In geometric language, this says that $\mathcal{M}_E$ is valued in simplicial sets equivalent to nerves of groupoids and that

$$\mathcal{M}_E(X) := \widetilde{\pi_*\mathcal{D}_E(X)}$$

forms a Cartesian quasicoherent sheaf over $\mathcal{M}_E$. In this sense, we have constructed a factorization

[7]If E is a commutative ring spectrum, then this is equivalent to asking that the left-unit map is a flat map of E_*–modules.

$$\begin{tikzcd} \mathsf{Spectra} \arrow[rr, "E_*(-)"] \arrow[dr, "\mathcal{M}_E(-)"'] & & \mathsf{Modules}_{E_*} \\ & \mathsf{QCoh}(\mathcal{M}_E). \arrow[ur, "(-)[0]"'] & \end{tikzcd}$$

While **CH** and **FH** are enough to guarantee that $\mathcal{M}_E$ and $\mathcal{M}_E(X)$ are well-behaved, they still do not exactly connect us with Remark 3.1.3. The main difference is that the ring of homology cooperations for E

$$E_*E = \pi_*(E \wedge E) = \pi_*\mathcal{D}_E[1]$$

is only distantly related to the tensor product $E_* \otimes_{\pi_*\mathbb{S}} E_*$ (or even $\mathrm{Tor}^{\pi_*\mathbb{S}}_{*,*}(E_*, E_*)$). This is a trade we are eager to make, as the latter groups are typically miserably behaved, whereas E_*E is typically fairly nice. In order to take advantage of this, we enlarge our definition to match:

Definition 3.1.14. Let A and Γ be commutative rings with associated affine schemes $X_0 = \operatorname{Spec} A$, $X_1 = \operatorname{Spec} \Gamma$. A *Hopf algebroid* consists of the pair (A, Γ) together with structure maps

Fix the spacing around the super-scripts.

$$\begin{aligned} \eta_L &\colon A \to \Gamma, & s &\colon X_1 \to X_0, \\ \eta_R &\colon A \to \Gamma, & t &\colon X_1 \to X_0, \\ \Delta &\colon \Gamma \to \Gamma^{\eta_R} \otimes_A^{\eta_L} \Gamma, & \circ &\colon X_1{}^t \times^s_{X_0} X_1 \to X_1, \\ \chi &\colon \Gamma \to \Gamma, & (-)^{-1} &\colon X_1 \to X_1, \end{aligned}$$

such that (X_0, X_1) forms a groupoid scheme. An (A, Γ)–*comodule* is an A–module equipped with a Γ–comodule structure, and such a comodule is equivalent to a Cartesian quasicoherent sheaf on the nerve of (X_0, X_1).

Example 3.1.15. A Hopf k–algebra H gives a Hopf algebroid (k, H). The scheme of objects $\operatorname{Spec} k$ in the groupoid scheme is the constant scheme 0.

Lemma 3.1.16. *For E an A_∞–ring spectrum satisfying **CH** and **FH**, the E_2–page of its Adams spectral sequence can be identified as*

$$\begin{aligned} E_2^{*,*} &= \mathrm{Cotor}^{*,*}_{E_*E}(E_*, E_*X) \\ &\cong H^*(\mathcal{M}_E; \mathcal{M}_E(X) \otimes \omega^{\otimes *}) \oplus H^*(\mathcal{M}_E; \mathcal{M}_E(\Sigma X) \otimes \omega^{\otimes *})[1] \Rightarrow \pi_* X^\wedge_E. \end{aligned}$$

Proof sketch. The homological algebra of Hopf algebras from Lecture 1.4 can be lifted almost verbatim, allowing us to define resolutions suitable for computing derived functors [Rav86, Definition A1.2.3]. This includes the cobar resolution [Rav86, Definition A1.2.11], which shows that the associated graded for the coskeletal filtration of $\mathcal{D}_E(X)$ is a complex computing the derived functors claimed in the Lemma statement. □

Remark 3.1.17. The sphere spectrum fails to satisfy **CH**, so the above results do not apply to it, but the $\mathbb{S}$–Adams spectral sequence is particularly degenerate: it consists of $\pi_* X$, concentrated on the 0–line. For any other ring spectrum E, the unit map $\mathbb{S} \to E$ induces a map of Adams spectral sequences whose image on the 0–line are those maps of comodules induced by applying E–homology to a homotopy element of X—i.e., the Hurewicz image of E.

Remark 3.1.18. In Lemma 3.1.6, we discussed translating from the algebra descent picture to a homotopical one, and a crucial point was how thorough we had to be: we transferred not just to the derived category $D(\operatorname{Spec} R)$ but we also replaced the base ring R with its homotopical incarnation HR. In Definition 3.1.14, we have not been as thorough as possible: both X_0 and X_1 are schemes and hence satisfy a sheaf condition individually, but the functor (X_0, X_1), thought of as valued in homotopy 1–types, does not necessarily satisfy a homotopy sheaf condition. Enforcing this descent condition results in the *associated stack* [Hop, Definition 8.13], denoted

$$\operatorname{Spec} A /\!/ \operatorname{Spec} \Gamma = X_0 /\!/ X_1.$$

Remarkably, this does not change the category of Cartesian quasicoherent sheaves—it is still equivalent to the category of (A, Γ)–comodules [Hop, Proposition 11.6]. However, several different Hopf algebroids (even those with maps between them inducing natural equivalences of groupoid schemes, as studied by Hovey [Hov02, Theorem D]) can give the same associated stack, resulting in surprising equivalences of comodule categories.[8] For the most part, it will not be especially relevant to us whether we are considering the groupoid scheme or its associated stack, so we will not draw much of a distinction.

Maybe this is irresponsible and we should be careful not to be sloppy.

Example 3.1.19. Most of the homology theories we will discuss have these **CH** and **FH** properties. For an easy example, $H\mathbb{F}_2 P$ certainly has this property: there is only one possible algebraic map $\mathbb{F}_2 \to \mathcal{A}_*$, so **FH** is necessarily satisfied. This grants us access to a description of the context for $H\mathbb{F}_2$:

$$\mathcal{M}_{H\mathbb{F}_2 P} = \operatorname{Spec} \mathbb{F}_2 /\!/ \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

Example 3.1.20. The context for MUP is considerably more complicated, but Quillen's theorem can be equivalently stated as giving a description of it. Quillen's theorem on its face gives an equivalence $\operatorname{Spec} MUP_0 \cong \mathcal{M}_{\mathbf{fgl}}$, but in Lemma 2.6.8 we also gave a description of $\operatorname{Spec} MUP_0 MUP$: it is the moduli of pairs of formal group laws equipped with an invertible power series intertwining them. Altogether, this presents $\mathcal{M}_{MUP}$ as the moduli of formal groups:

$$\mathcal{M}_{MUP} \simeq \mathcal{M}_{\mathbf{fg}} := \mathcal{M}_{\mathbf{fgl}} /\!/ \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}},$$

[8] We will employ one of these surprising equivalences in Remark 3.3.17.

where $\mathcal{M}_{\mathbf{ps}} = \underline{\mathrm{End}}(\widehat{\mathbb{A}}^1)$ is the moduli of self-maps of the affine line (i.e., of power series) and $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}$ is the multiplicative subgroup of invertible such maps. We include a picture of the p–localized Adams E_2–page in Figure 3.1 and Figure 3.2. In view of Remark 3.1.18, there is an important subtlety about the stack $\mathcal{M}_{\mathbf{fg}}$: an R–point is a functor on affines over $\operatorname{Spec} R$ which is locally isomorphic to a formal group, but whose local isomorphism *may not patch* to give a global isomorphism. This does not agree, a priori, with the definition of formal group given in Definition 2.1.16, where the isomorphism witnessing a functor as a formal variety was expected to be global. We will address this further in Lemma 3.2.7 below.

Sidewaysfigure ANSS2 could use some label re-alignment. Sidewaysfigure ANSS3 has a missing feature: $\beta_3\alpha_1$ has a 3–division.

Example 3.1.21. The context for MOP, by contrast, is reasonably simple. Corollary 1.5.7 shows that the scheme $\operatorname{Spec} MOP_0$ classifies formal group laws over $\mathbb{F}_2$ which admit logarithms, so that $\mathcal{M}_{MOP}$ consists of the groupoid of formal group laws with logarithms and isomorphisms between them. This admits a natural deformation-retraction to the moduli consisting just of $\widehat{\mathbb{G}}_a$ and its automorphisms, expressing the redundancy in $MOP_0(X)$ encoded in the splitting of Lemma 1.5.8.

Remark 3.1.22. The algebraic moduli $\mathcal{M}_{MU} = (\operatorname{Spec} MU_*, \operatorname{Spec} MU_*MU)$ and the topological moduli $(MU, MU \wedge MU)$ are quite different. An orientation $MU \to E$ selects a coordinate on the formal group $\mathbb{C}\mathrm{P}^\infty_E$, but $\mathbb{C}\mathrm{P}^\infty_E$ itself exists independently of the orientation. Hence, while $\mathcal{M}_{MU}(E_*)$ can have many connected components corresponding to *distinct formal groups* on the coefficient ring E_*, the groupoid $\mathsf{RingSpectra}(\mathcal{D}_{MU}, E)$ has only one connected component corresponding to the formal group $\mathbb{C}\mathrm{P}^\infty_E$ intrinsic to E.[9]

Remark 3.1.23. If E is a complex-oriented ring spectrum, then the simplicial sheaf $\mathcal{M}_{MU}(E)$ has an extra degeneracy, which causes the MU–based Adams spectral sequence for E to degenerate. In this sense, the "stackiness" of $\mathcal{M}_{MU}(E)$ is exactly a measure of the failure of E to be orientable.

Cite me: pg 5 of From Spectra To Stacks by Hopkins in the TMF volume.

You can build an Adams resolution in the absence of an A_∞ structure too, you just miss the descent picture.

3.2 The structure of $\mathcal{M}_{\mathbf{fg}}$ I: The affine cover

Think about what sorts of simplicial sheaves you really want. They seem like they should be valued in something like quasicategories: the stable operations are valued in space-like simplicial sets, the isogenies pile is a sheaf of *categories*, then unstable operations generically have some other weird structure...

In Definition 3.1.14 we gave a factorization

$$\begin{tikzcd} \mathsf{Spectra} \arrow[rr, "MUP_0(-)"] \arrow[dr, "\mathcal{M}_{MUP}(-)"'] & & \mathsf{Modules}_{MUP_0} \\ & \mathsf{QCoh}(\mathcal{M}_{MUP}), \arrow[ur, "(-)[0]"'] & \end{tikzcd}$$

We should expand the comparison of a Cartesian q.c. sheaf on an affinely presented stack with a module plus structural data.

[9] The reader ought to compare this with the situation in explicit local class field theory, where a local number field has a preferred formal group attached to it.

Cite me: Pridham's article *Presenting higher stacks as simplicial schemes* seems like a good reference? Maybe

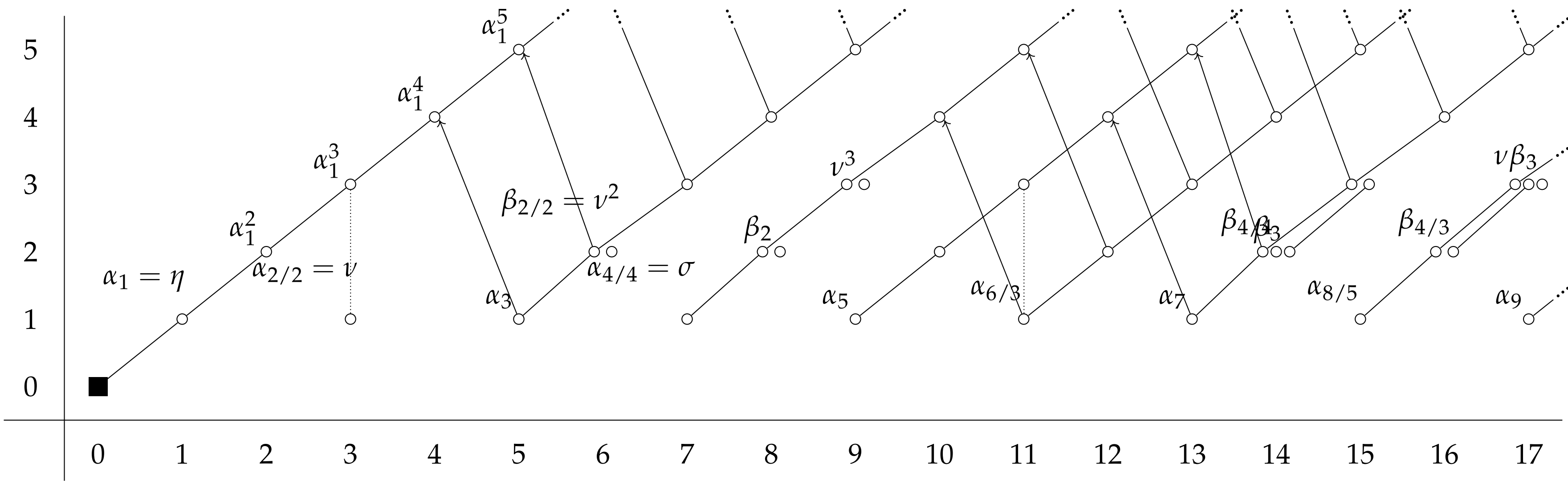

Figure 3.1: A small piece of the $MU_{(2)}$–Adams spectral sequence for the sphere, beginning at the second page [Rav78, pg. 429]. North-east lines denote multiplication by $\eta = \alpha_1$, north-west lines denote d_3–differentials, and vertical dotted lines indicate additive extensions. Elements are labeled according to the conventions of Remark 3.6.21, and in particular $\alpha_{i/j}$ is 2^j–torsion.

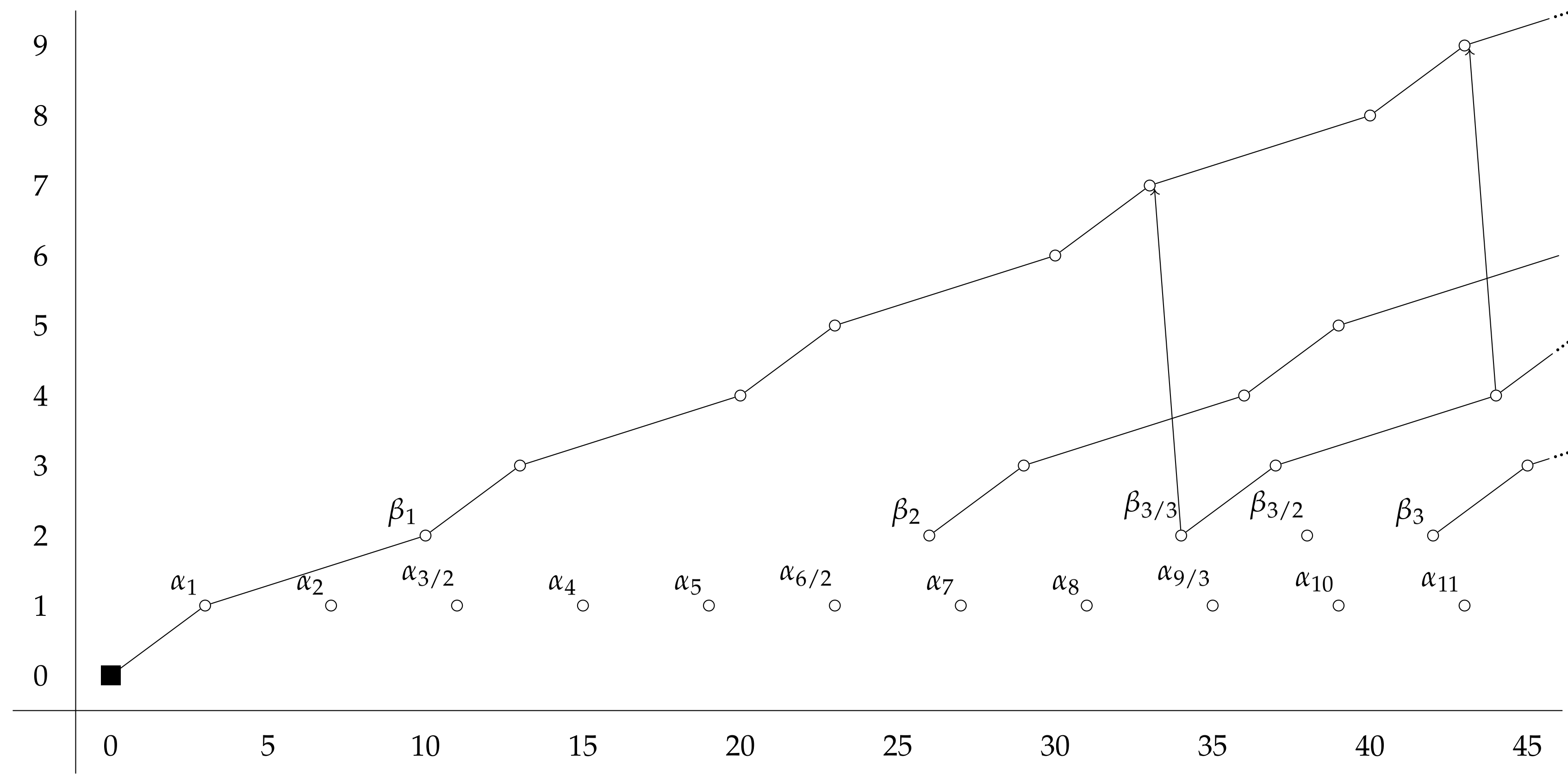

Figure 3.2: A small piece of the $MU_{(3)}$–Adams spectral sequence for the sphere, beginning at the second page [Rav86, Figure 1.2.19]. North-east lines denote multiplication by α_1 or by β_1/α_1, and north-west lines denote d_5–differentials. Elements are labeled according to the conventions of Remark 3.6.21, and in particular $\alpha_{i/j}$ is 3^j–torsion.

and in Example 3.1.20 we established an equivalence

$$\varphi\colon \mathcal{M}_{MUP} \xrightarrow{\simeq} \mathcal{M}_{\mathbf{fg}}.$$

Our program, as outlined in the introduction, is to analyze this functor $\mathcal{M}_{MUP}(-)$ by postcomposing it with φ^* and studying the resulting sheaf over $\mathcal{M}_{\mathbf{fg}}$. In order to perform such an analysis, we will want a firm grip on the geometry of the stack $\mathcal{M}_{\mathbf{fg}}$, and in this Lecture we begin by studying the scheme $\mathcal{M}_{\mathbf{fgl}}$ as well as the natural covering map

$$\mathcal{M}_{\mathbf{fgl}} \to \mathcal{M}_{\mathbf{fg}}.$$

Definition 3.2.1. There is an affine scheme $\mathcal{M}_{\mathbf{fgl}}$ classifying formal group laws. Begin with the scheme classifying *all* bivariate power series:

$$\begin{aligned} \operatorname{Spec} \mathbb{Z}[a_{ij} \mid i,j \geq 0] &\leftrightarrow \{\text{bivariate power series}\}\,, \\ f \in \operatorname{Spec} \mathbb{Z}[a_{ij} \mid i,j \geq 0](R) &\leftrightarrow \sum_{i,j \geq 0} f(a_{ij}) x^i y^j. \end{aligned}$$

Then, $\mathcal{M}_{\mathbf{fgl}}$ is the closed subscheme selected by the formal group law axioms in Definition 2.1.19.

This presentation of $\mathcal{M}_{\mathbf{fgl}}$ as a subscheme appears to be extremely complicated in that its ideal is generated by many hard-to-describe elements, but $\mathcal{M}_{\mathbf{fgl}}$ itself is actually not complicated at all. We will prove the following:

Theorem 3.2.2 ([Laz55, Théorème II])**.** *There is a noncanonical isomorphism*

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \cong \mathbb{Z}[b_n \mid 1 \leq n < \infty] =: L. \quad \square$$

Proof. Let $L = \mathbb{Z}[b_0, b_1, b_2, \ldots]/(b_0 - 1)$ be the universal ring supporting an exponential

$$\exp(x) := \sum_{j=0}^{\infty} b_j x^{j+1}$$

with compositional inverse

$$\log(x) := \sum_{j=0}^{\infty} m_j x^{j+1}.$$

They induce a formal group law on L by the conjugation formula

$$x +_! y = \exp(\log(x) + \log(y)),$$

which is in turn classified by a map $u\colon \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to L$.[10] Modulo decomposables, we compute

$$\begin{aligned} x &= \exp(\log(x)) \\ &= x + \sum_{n=1}^{\infty} m_n x^{n+1} + \sum_{n=1}^{\infty} b_n \left(x + \sum_{j=1}^{\infty} m_j x^{j+1} \right)^{n+1} \\ &\equiv x + \sum_{n=1}^{\infty} m_n x^{n+1} + \sum_{n=1}^{\infty} b_n x^{n+1} \pmod{\text{decomposables}}, \end{aligned}$$

hence $b_n \equiv -m_n \pmod{\text{decomposables}}$. Using this, we then compute

$$\begin{aligned} x +_! y &= \exp(\log(x) + \log(y)) \\ &= \left((x+y) + \sum_{n=1}^{\infty} m_n (x^{n+1} + y^{n+1}) \right) + \sum_{n=1}^{\infty} b_n \left((x+y) + \sum_{j=1}^{\infty} m_j (x^{j+1} + y^{j+1}) \right)^{n+1} \\ &\equiv x + y + \sum_{n=1}^{\infty} -b_n (x^{n+1} + y^{n+1}) + \sum_{n=1}^{\infty} b_n (x+y)^{n+1} \pmod{\text{decomposables}} \\ &= x + y + \sum_{n=1}^{\infty} b_n ((x+y)^{n+1} - x^{n+1} - y^{n+1}), \end{aligned}$$

hence

$$u(a_{i(n-i)}) \equiv \binom{n}{i} b_{n-1} \pmod{\text{decomposables}}.$$

It follows that the map Qu on degree $2n$ has image the subgroup T_{2n} generated by $d_{n+1}b_n$, where $d_{n+1} = \gcd\left(\binom{n+1}{k} \middle| 0 < k < n+1 \right)$. Lemma 3.2.3 below provides a canonical splitting of Qu, and we couple it to the freeness of L to *choose* an algebra splitting

$$L \xrightarrow{v} \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \xrightarrow{u} L.$$

The map uv is injective, so v is injective. Furthermore, Qv is designed to be surjective, so v itself is surjective and hence an isomorphism. $\square$

Recall that we have yet to prove the following Lemma:

Lemma 3.2.3. *There is a canonical splitting $T_{2n} \to (Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n}$.*

Definition 3.2.4. In order to prove the missing Lemma 3.2.3, it will be useful to study the series $+_\varphi$ "up to degree n", i.e., modulo $(x, y)^{n+1}$. Such a truncated series satisfying the

[10]This is *not* the universal formal group law. We will soon see that some formal group laws do not admit logarithms.

analogues of the formal group law axioms is called a *formal n–bud*.[11] We will additionally be moved to study the difference between a formal n–bud and a formal $(n+1)$–bud extending it. The simplest case of this is when the formal n–bud is just the additive law $x +_\varphi y = x + y$, in which case any extension to an $(n+1)$–bud has the form $x + y + f(x, y)$ for $f(x, y)$ a homogeneous polynomial of degree n. Symmetry of the group law requires $f(x, y)$ to be symmetric, and associativity of the group law requires $f(x, y)$ to satisfy the equation

$$f(x, y) - f(t + x, y) + f(t, x + y) - f(t, x) = 0.$$

Such a polynomial is called a *symmetric* 2*–cocycle* (of degree n).[12]

Reduction of Lemma 3.2.3 to Lemma 3.2.5. We now show that the following conditions are equivalent:

1. (Lemma 3.2.5) Symmetric 2–cocycles that are homogeneous polynomials of degree n are spanned by
$$c_n = \frac{1}{d_n} \cdot ((x + y)^n - x^n - y^n),$$
where $d_n = \gcd\left(\binom{n}{k} \middle| 0 < k < n\right)$.
2. For F is an n–bud, the set of $(n+1)$–buds extending F form a torsor under addition for $R \otimes c_n$.
3. Any homomorphism $(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n} \to A$ factors through the map $(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n} \to T_{2n}$.
4. (Lemma 3.2.3) There is a canonical splitting $T_{2n} \to (Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n}$.

To verify that Claims 1 and 2 are equivalent, suppose that $x +_\varphi y$ is some $(n+1)$–bud and that $x +'_\varphi y$ is some $(n+1)$–bud such that

$$(x +'_\varphi y) = (x +_\varphi y) + f(x, y)$$

where $f(x, y)$ is homogeneous of degree $(n+1)$. Symmetry of $x +'_\varphi y$ enforces symmetry of f, and from associativity we calculate

Typeset me better.

$$\begin{aligned}
(x +'_\varphi y) +'_\varphi z &= x +'_\varphi (y +'_\varphi z) \\
(x +_\varphi y + f(x, y)) +'_\varphi z &= x +'_\varphi (y +_\varphi z + f(y, z)) \\
(x +_\varphi y + f(x, y)) +_\varphi z + f(x +_\varphi y + f(x, y), z) &= x +_\varphi (y +_\varphi z + f(y, z)) + f(x, y +_\varphi z + f(y, z)) \\
(x +_\varphi y) +_\varphi z + f(x, y) + f(x + y, z) &\equiv x +_\varphi (y +_\varphi z) + f(y, z) + f(x, y + z) \quad (\text{mod } (x, y)^{n+2}
\end{aligned}$$

[11]A formal n–bud determines a "multiplication" $(\widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1)^{(n)} \to \widehat{\mathbb{A}}^{1,(n)}$. Note that this does *not* belong to a group object, since $(\widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1)^{(n)} \not\cong \widehat{\mathbb{A}}^{1,(n)} \times \widehat{\mathbb{A}}^{1,(n)}$. This is the observation that the ideals $(x, y)^{n+1}$ and (x^{n+1}, y^{n+1}) are distinct.

[12]We will justify the "2–cocycle" terminology in the course of the proof of Lemma 3.2.5.

Conversely, given such an $f(x, y)$, the formal $(n+1)$–bud $+'_\varphi$ formed by translating $+_\varphi$ by f is again a formal $(n+1)$–bud extending the same formal n–bud.

To see that Claim 2 is equivalent to Claim 3, note that a group map

$$(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n} \to A$$

is equivalent data to a ring map

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to Z \oplus A$$

with the prescribed behavior on $(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n}$ and which sends all other indecomposables to 0.. This shows that such a homomorphism of groups determines an extension of the n–bud $\widehat{\mathbb{G}}_a$ to an $(n+1)$–bud, which takes the form of a 2–cocycle with coefficients in A, and hence factors through T_{2n}.

Finally, Claim 4 is the universal case of Claim 3. $\square$

We will now verify Claim 1 computationally, completing the proof of Lemma 3.2.3 (and hence Theorem 3.2.2).

Lemma 3.2.5 (Symmetric 2–cocycle lemma [Laz55, Lemme 3], cf. [Hop, Theorem 3.1])**.** *Symmetric* 2*–cocycles that are homogeneous polynomials of degree n are spanned by*

$$c_n = \frac{1}{d_n} \cdot ((x+y)^n - x^n - y^n),$$

where $d_n = \gcd\left(\binom{n}{k} \middle| 0 < k < n\right)$.

Proof. We begin with a reduction of the sorts of rings over which we must consider the possible symmetric 2–cocycles. First, notice that only the additive group structure of the ring matters: the symmetric 2–cocycle condition does not involve any ring multiplication. Second, it suffices to show the Lemma over a finitely generated abelian group, as a particular polynomial has finitely many terms and hence involves finitely many coefficients. Noticing that the Lemma is true for $A \oplus B$ if and only if it's true for A and for B, we couple these facts to the structure theorem for finitely generated abelian groups to reduce to the cases $\mathbb{Z}$ and $\mathbb{Z}/p^r$. From here, we can reduce to the prime fields: if $A \leq B$ is a subgroup and the Lemma is true for B, it's true for A, so we will be able to deduce the case of $\mathbb{Z}$ from the case of $\mathbb{Q}$. Lastly, we can also reduce from $\mathbb{Z}/p^r$ to $\mathbb{Z}/p$ using an inductive Bockstein-style argument over the extensions

$$(p^{r-1})/(p^r) \to \mathbb{Z}/p^r \to \mathbb{Z}/p^{r-1}$$

and noticing that $(p^{r-1})/(p^r) \cong \mathbb{Z}/p$ as abelian groups. Hence, we can now freely assume that our ground object is a prime field.

We now ground ourselves by fitting symmetric 2–cocycles into a more general homological framework, hoping that we can use such a thing to power a computation. For a formal group scheme $\widehat{\mathbb{G}}$, we can form a simplicial scheme $B\widehat{\mathbb{G}}$ in the usual way:

$$B\widehat{\mathbb{G}} := \left\{ \begin{array}{ccccccc} & & & & * & \longleftarrow & \\ & & * & \longleftarrow & \times & \longrightarrow & \\ * & \longleftarrow & \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \\ \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \times & \longrightarrow & \cdots \\ * & \longleftarrow & \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \\ & & * & \longleftarrow & \times & \longrightarrow & \\ & & & & * & \longleftarrow & \end{array} \right\}.$$

By applying the functor $\underline{\mathsf{FormalSchemes}}(-, \widehat{\mathbb{G}}_a)(k)$, we get a cosimplicial abelian group stemming from the group scheme structure on $\widehat{\mathbb{G}}_a$, and this gives a cochain complex of which we can take the cohomology. In the case $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$, the 2–cocycles in this cochain complex are *precisely* the things we've been calling 2–cocycles[13], so we are interested in computing H^2. First, we can quickly compute B^2, since C^1 is so small:

$$d^1(x^k) = d_k c_k.$$

Secondly, one may think of this complex as a resolution computing various[14] derived functors

$$\mathrm{Cotor}_{\mathcal{O}_{\widehat{\mathbb{G}}}}(k, k) \cong \mathrm{Ext}_{\mathcal{O}_{\widehat{\mathbb{G}}}}(k, k) \cong \mathrm{Tor}_{\mathcal{O}^*_{\widehat{\mathbb{G}}}}(k, k).$$

Do you use * or $^\vee$ later for linear dual?

We are now going to compute these last groups using a more efficient complex.

$\mathbb{Q}$: There is a free $\mathbb{Q}[t]$–module resolution

$$\begin{array}{ccccccc} & & \mathbb{Q} & & & & \\ & & \uparrow & & & & \\ 0 & \longleftarrow & \mathbb{Q}[t] & \xleftarrow[\cdot t]{} & \mathbb{Q}[t] & \longleftarrow & 0, \end{array}$$

to which we apply $(-) \otimes_{\mathbb{Q}[t]} \mathbb{Q}$ to calculate

$$H^* \underline{\mathsf{FormalSchemes}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{when } * = 0, \\ \mathbb{Q} & \text{when } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This means that every 2–cocycle is a coboundary, symmetric or not.

[13] They aren't obligated to be symmetric or of homogeneous degree, though.

[14] Refer back to Corollary 1.4.17.

$\mathbb{F}_p$: Now we are computing Ext over a free commutative $\mathbb{F}_p$–algebra on one generator with divided powers. Such an algebra splits as a tensor of truncated polynomial algebras, and again computing a minimal free resolution results in the calculation

$$H^* \underline{\mathsf{FormalSchemes}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{F}_p) = \begin{cases} \frac{\mathbb{F}_p[\alpha_k | k \geq 0]}{\alpha_k^2 = 0} \otimes \mathbb{F}_p[\beta_k \mid k \geq 0] & \text{when } p > 2, \\ \mathbb{F}_2[\alpha_k \mid k \geq 0] & \text{when } p = 2, \end{cases}$$

with $\alpha_k \in H^1$ and $\beta_k \in H^2$. Now that we know what to look for, we can find representatives of each of these classes:

- The class α_k can be represented by x^{p^k}, as this is a minimally divisible monomial of degree p^k satisfying the 1–cocycle condition

$$x^{p^k} - (x+y)^{p^k} + y^{p^k} = 0.$$

- The 2–cohomology is concentrated in degrees of the form p^k and $p^j + p^k$, corresponding to β_k and $\alpha_j \alpha_k$. Since c_{p^k} is a 2–cocycle of the correct degree and not a 2–coboundary (cf. $d^1(x^{p^k}) = d_{p^k} c_{p^k}$, and $p \mid d_{p^k}$), we can use it as a representative for β_k. (Additionally, the asymmetric class $\alpha_k \alpha_j$ is represented by $x^{p^k} y^{p^j}$.)
- Similarly, in the case $p = 2$ the exceptional class α_{k-1}^2 is represented by $c_{2^k}(x, y)$, as this is a 2–cocycle in the correct degree which is not a 2–coboundary.

Given how few 2–coboundaries and 2–cohomology classes there are, we conclude that $c_n(x, y)$ and $x^{p^a} y^{p^b}$ give a basis for *all* of the 2–cocycles. Of these it is easy to select the symmetric ones, which agrees with our expected conclusion. □

The most important consequence of Theorem 3.2.2 is *smoothness*:

Corollary 3.2.6. *Given a formal group law F over a ring R and a surjective ring map $f\colon S \to R$, there exists a formal group law $\widetilde{F}$ over S with*

$$F = f^* \widetilde{F}.$$

Proof. Identify F with the classifying map $\operatorname{Spec} R \to \mathcal{M}_{\mathbf{fgl}}$. Employ an isomorphism

$$\varphi\colon \mathcal{M}_{\mathbf{fgl}} \to \operatorname{Spec} L$$

afforded by Theorem 3.2.2, so that $\varphi \circ F$ is selected by a sequence of elements $r_n = \varphi^* F^*(t_n) \in R$. Each of these admit preimages s_n through f, and we determine a map

$$\widetilde{\varphi \circ F}\colon \operatorname{Spec} S \to \operatorname{Spec} L$$

by the formula $\widetilde{\varphi \circ F}^*(t_j) = s_j$ and freeness of L. Since φ is an isomorphism, this determines a map $\widetilde{F} = \varphi^{-1} \circ \widetilde{\varphi \circ F}$ factoring F. □

In order to employ Corollary 3.2.6 effectively, we need to know when a map $\operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$ classifying a formal group can be lifted to a triangle

This isn't super well stated, but it's at least here to be smoothed out later.

$$\begin{tikzcd} & \mathcal{M}_{\mathbf{fgl}} \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & \mathcal{M}_{\mathbf{fg}}, \end{tikzcd}$$

so that a surjective map of rings $\operatorname{Spec} R \to \operatorname{Spec} S$ can then be completed to a second diagram

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r, dotted] \arrow[dr, dotted] & \mathcal{M}_{\mathbf{fgl}} \arrow[d] \\ \operatorname{Spec} R \arrow[u] \arrow[r] \arrow[ur] & \mathcal{M}_{\mathbf{fg}}. \end{tikzcd}$$

Lemma 3.2.7 ([Lura, Proposition 11.7])**.** *A map $\widehat{\mathbb{G}} \colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$ lifts to $\mathcal{M}_{\mathbf{fgl}}$ exactly when the Lie algebra $T_0\widehat{\mathbb{G}}$ of $\widehat{\mathbb{G}}$ is isomorphic to R.*

Proof. Certainly if $\widehat{\mathbb{G}}$ admits a global coordinate, then $T_0\widehat{\mathbb{G}} \cong R$. Conversely, the formal group $\widehat{\mathbb{G}}$ is certainly locally isomorphic to $\widehat{\mathbb{A}}^1$ by a covering $i_\alpha \colon X_\alpha \to \operatorname{Spec} R$ and isomorphisms φ_α—but, *a priori*, these isomorphisms may not glue, precisely corresponding to the nontriviality of the Čech 1–cocycle

$$[\varphi_\alpha] \in \check{H}^1(\operatorname{Spec} R; \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}).$$

The group scheme $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}$ is populated by T–points of the form

$$\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}(T) = \left\{ t_0 x + t_1 x^2 + t_2 x^3 + \cdots \mid t_j \in T, t_0 \in T^\times \right\},$$

and it is admits a filtration by the closed subschemes

$$\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd},\geq N}(T) = \left\{ 1 \cdot x + t_N x^{N+1} + t_{N+1} x^{N+2} + \cdots \mid t_j \in T \right\}.$$

The associated graded of this filtration is $\mathbb{G}_m \times \mathbb{G}_a^{\times \infty}$, and hence the filtration spectral sequence shows

$$\check{H}^1(\operatorname{Spec} R; \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \xrightarrow{\simeq} \check{H}^1(\operatorname{Spec} R; \mathbb{G}_m),$$

as $\check{H}^1(\operatorname{Spec} R; \mathbb{G}_a) = 0$ for all affine schemes. Finally, given a choice of trivialization $T_0\widehat{\mathbb{G}} \cong R$, this induces compatible trivializations of $T_0 i_\alpha^* \widehat{\mathbb{G}}$, which we can use to rescale the isomorphisms φ_α so that their image in $\check{H}^1(\operatorname{Spec} R; \mathbb{G}_m)$ vanishes, and hence $[\varphi_\alpha]$ is induced from a class in

$$\check{H}^1(\operatorname{Spec} R; \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd},\geq 1}).$$

This obstruction group vanishes. $\square$

Remark 3.2.8. Incidentally, a choice of trivialization of $T_0\widehat{\mathbb{G}}$ exactly resolves the indeterminacy of $\log'(0)$ in Theorem 2.1.22.

Remark 3.2.9. The subgroup scheme $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd},\geq 1}$ is often referred to in the literature as the group of *strict isomorphisms*. There is an associated moduli of formal groups identified only up to strict isomorphism, which sits in a fiber sequence

$$\mathbb{G}_m \to \mathcal{M}_{\mathbf{fgl}} // \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd},\geq 1} \to \mathcal{M}_{\mathbf{fg}}.$$

These appeared earlier in this Lecture as well: in the proof of Theorem 3.2.2, we constructed over L the universal formal group law equipped with a *strict* exponential map. Give a comparison with $\mathcal{M}_{MU}$ vs $\mathcal{M}_{MUP}$.

3.3 The structure of $\mathcal{M}_{\mathbf{fg}}$ II: Large scales

We now turn to understanding the geometry of the quotient stack $\mathcal{M}_{\mathbf{fg}}$ itself, armed with two important tools: Theorem 2.1.22 and Corollary 3.2.6. We begin with a rephrasing of the former:

Theorem 3.3.1 (cf. Theorem 2.1.22)**.** *Let k be any field of characteristic 0. Then $\widehat{\mathbb{G}}_a$ describes a unique map*

$$\operatorname{Spec} k \xrightarrow{\simeq} \mathcal{M}_{\mathbf{fg}}. \quad \square$$

One of our overarching tasks from the introduction to this Case Study is to enhance this to a classification of *all* of the geometric points of $\mathcal{M}_{\mathbf{fg}}$, including those where k is a field of positive characteristic p:

$$\widehat{\mathbb{G}} \colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}.$$

We proved this Theorem in the characteristic 0 case by solving a certain differential equation, which necessitated integrating a power series, and integration is what we expect to fail in characteristic p. The following definition tracks *where* it fails:

Definition 3.3.2. Let $+_\varphi$ be a formal group law over a $\mathbb{Z}_{(p)}$–algebra. Let n be the largest degree such that there exists a formal power series ℓ with

$$\ell(x +_\varphi y) = \ell(x) + \ell(y) \pmod{(x,y)^n},$$

i.e., ℓ is a logarithm for the $(n-1)$–bud determined by $+_\varphi$. The *p–height of $+_\varphi$* is defined to be $\log_p(n)$.

This turns out to be a crucial invariant of a formal group law, admitting many other interesting presentations. In this Lecture, investigation of this definition will lead us to a classification of the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, another of our overarching tasks. As a first step, we would like to show that this value is well-behaved in various senses, including the following:

Lemma 3.3.3 (cf. [Lura, Proposition 13.6]). *Over a field of positive characteristic p, the p–height of a formal group law is always an integer (or ∞). (That is, the radius of convergence of the logarithmic differential equation is either ∞ or p^d for some nonnegative natural d.)*

We will have to develop some machinery to get there. First, we note that this definition really depends on the formal group rather than the formal group law.

Lemma 3.3.4. *The height of a formal group law is an isomorphism invariant, i.e., it descends to give a function*

$$\mathrm{ht}\colon \pi_0\mathcal{M}_{\mathbf{fg}}(T) \to \mathbb{N} \cup \{\infty\}$$

for any test $\mathbb{Z}_{(p)}$–algebra T.

Proof. The series ℓ is a partial logarithm for the formal group law φ, i.e., an isomorphism between the formal group defined by φ and the additive group. Since isomorphisms compose, this statement follows. □

With this in mind, we look for a more standard form for formal group laws, where Lemma 3.3.3 will hopefully be obvious. The most blindly optimistic standard form is as follows:

Definition 3.3.5 (cf. [Haz12, Proposition 15.2.4]). Suppose that a formal group law $+_\varphi$ does have a logarithm. We say that its logarithm is *p–typical* when it takes the form

$$\log_\varphi(x) = \sum_{j=0}^{\infty} \ell_j x^{p^j}.$$

Lemma 3.3.6 ([Haz12, Theorem 15.2.9]). *Every formal group law $+_\varphi$ over a $\mathbb{Z}_{(p)}$–algebra with a logarithm $\log_\varphi$ is naturally isomorphic to one whose logarithm is p–typical, called the p–typification of $+_\varphi$.*

Proof. Let $\widehat{\mathbb{G}}$ be the formal group associated to $+_\varphi$, and denote its inherited parameter by

$$g_0\colon \widehat{\mathbb{A}}^1 \xrightarrow{\cong} \widehat{\mathbb{G}},$$

so that the composite

$$\widehat{\mathbb{A}}^1 \xrightarrow{g_0} \widehat{\mathbb{G}} \xrightarrow{\log} \widehat{\mathbb{G}}_a \xrightarrow{x} \widehat{\mathbb{A}}^1$$

expresses $\log_\varphi = \log \circ g_0$ as the power series

$$\log_\varphi(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Our goal is to perturb this coordinate to a new coordinate g_∞ which couples with the logarithm in the same way to give a series expansion of the form

$$\log(g_\infty(x)) = \sum_{n=0}^{\infty} a_{p^n} x^{p^n}.$$

To do this, we introduce four operators on functions[15] $\widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$:

- Given $r \in R$, we can define a *homothety* by rescaling the coordinate by r:

$$\log(\theta_r g_0) = \log(g_0(rx)) = \sum_{n=1}^{\infty} (a_n r^n) x^n.$$

- For $\ell \in \mathbb{Z}$, we can define a shift operator (or *Verschiebung*) by

$$\log(V_\ell g_0(x)) = \log(g(x^\ell)) = \sum_{n=1}^{\infty} a_n x^{n\ell}.$$

- Given an $\ell \in \mathbb{Z}_{(p)}$, we define the *$\ell$–series* by[16]

$$\log([\ell](g_0(x))) = \ell \log(g_0(x)) = \sum_{n=1}^{\infty} \ell a_n x^n.$$

- For $\ell \in \mathbb{Z}$, we can define a *Frobenius operator*[17] by

$$\log(F_\ell g_0(x)) = \log\left(\sum_{j=1}^{\ell}{}_{\widehat{\mathbb{G}}}\, g_0(\zeta_\ell^j x^{1/\ell})\right),$$

  where ζ_ℓ is a primitive ℓ^{th} root of unity. Because this formula is Galois–invariant in choice of primitive root, it actually expands to a series which lies over the ground ring (without requiring an extension by ζ_ℓ). But, by pulling the logarithm through and noting

$$\sum_{j=1}^{\ell} \zeta_\ell^{jn} = \begin{cases} \ell & \text{if } \ell \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

  we can explicitly compute the behavior of F_ℓ:

$$\log(F_\ell g_0(x)) = \sum_{n=1}^{\infty} \ell a_{n\ell} x^n.$$

[15] Unfortunately, it is standard in the literature to call these operators on "curves", which does not fit well with our previous use of the term in Case Study 2.

[16] Note that for $\ell \in \mathbb{Z}$, this agrees with $[\ell](g_0(x)) = \overbrace{g_0(x) +_{\widehat{\mathbb{G}}} \cdots +_{\widehat{\mathbb{G}}} g_0(x)}^{\ell \text{ times}}$.

[17] There are other definitions of the Frobenius operator which are less mysterious but less explicit. For instance, it also arises from applying the Verschiebung to the character group (or "Cartier dual") of $\widehat{\mathbb{G}}$.

Stringing these together, for $p \nmid \ell$ we have

$$\log([1/\ell]V_\ell F_\ell g_0(x)) = \sum_{n=1}^{\infty} a_{n\ell} x^{n\ell}.$$

Hence, we can iterate over primes $\ell \neq p$, and for two adjacent such primes $\ell' > \ell$ we consider the perturbation

$$g_{\ell'} = g_\ell -_{\widehat{\mathbb{G}}} [1/\ell]V_\ell F_\ell g_\ell.$$

Each of these differences gives a parameter according to Theorem 2.1.10, and the first possible nonzero term appears in degree ℓ, hence the coefficients stabilize linearly in ℓ. Passing to the limit thus gives a new parameter g_∞ on the same formal group $\widehat{\mathbb{G}}$, but now with a p–typical logarithm. □

Of course, the whole idea of "height" is that not every formal group law supports a logarithm. Because of this, we would like to re-express p–typicality in more general terms. Our foothold for this is the following computation of the p–series of a formal group law with p–typical logarithm:

Lemma 3.3.7. *For a formal group $+_\varphi$ with a logarithm $\log_\varphi$, the logarithm is p–typical if and only if there are elements v_d with*

$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_d x^{p^d} +_\varphi \cdots .$$

Proof sketch. Suppose first that $\log_\varphi$ is p–typical. We can then compare the two series

$$\begin{aligned} \log_\varphi(px) &= px + \cdots , \\ \log_\varphi([p]_\varphi(x)) &= p \log_\varphi(x) = px + \cdots . \end{aligned}$$

The difference is concentrated in degrees of the form p^d, beginning in degree p, so we can find an element v_1 such that

$$p \log_\varphi(x) - (\log_\varphi(px) + \log_\varphi(v_1 x^p))$$

is also concentrated in degrees of the form p^d but now starts in degree p^2. Iterating this gives the equation

$$p \log_\varphi(x) = \log_\varphi(px) + \log_\varphi(v_1 x^p) + \log_\varphi(v_2 x^{p^2}) + \cdots ,$$

at which point we can use formal properties of the logarithm to deduce

$$\begin{aligned} \log_\varphi [p]_\varphi(x) &= \log_\varphi \left(px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots \right), \\ [p]_\varphi(x) &= px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots . \end{aligned}$$

In the other direction, the logarithm coefficients can be recursively recovered from the coefficients v_d for a formal group law with p–typical p–series, using a similar manipulation:

$$\begin{aligned} p\log_\varphi(x) &= \log_\varphi\left([p]_\varphi(x)\right) \\ p\sum_{n=0}^\infty m_n x^n &= \log_\varphi\left(\sum_{d=0}^\infty {}_\varphi v_d x^{p^d}\right) = \log_\varphi(px) + \sum_{d=1}^\infty \log_\varphi\left(v_d x^{p^d}\right), \end{aligned}$$

which is only soluable if $\log_\varphi$ is concentrated in degrees of the form p^d. In that case, we can push this slightly further:

$$\sum_{d=0}^\infty p m_{p^d} x^{p^d} = \sum_{d=0}^\infty \sum_{j=0}^\infty m_{p^j} v_d^{p^j} x^{p^{d+j}} = \sum_{n=0}^\infty \left(\sum_{k=0}^n m_{p^k} v_{n-k}^{p^k}\right) x^{p^n},$$

implicitly taking $m_1 = 1$ and $v_0 = p$. □

This result portends much of what is to come. We now set our definition of p–typical to correspond to the manipulations we were making in the course of proving Lemma 3.3.6.

Definition 3.3.8. A parameter $g\colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$ of a formal group is said to be *p–typical* when $F_\ell g = 0$ for all $p \nmid \ell$.

Corollary 3.3.9 (cf. Lemma 3.3.6)**.** *Every formal group law $+_\varphi$ is naturally isomorphic to a p–typical one.* □

Lemma 3.3.10 (cf. Lemma 3.3.7)**.** *If $+_\varphi$ is a p–typical formal group law, then there are elements v_d with*

$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_d x^{p^d} +_\varphi \cdots.$$

Proof. As before, let $\widehat{\mathbb{G}}$ denote the formal group associated to $+_\varphi$ and let $g\colon \widehat{\mathbb{A}}^1 \to G$ denote the induced p–typical coordinate. Any auxiliary function $h\colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$ can be expressed in the form

Cite me: Is there a Kudo–Araki citation for this?.

$$h = \sum_{m=0}^\infty {}_{\widehat{\mathbb{G}}} V_m \theta_{a_m} g.$$

We will show that if h is p–typical (i.e., $F_\ell h = 0$ for $p \nmid \ell$) then $a_m = 0$ for every $m \neq p^d$.[18] Suppose instead that we can find a smallest index $m = rp^d$ with $p \nmid r$, $r \neq 1$, and $a_m \neq 0$. We can then write

$$\begin{aligned} F_\ell\left(h -_{\widehat{\mathbb{G}}} \sum_{j=0}^d {}_{\widehat{\mathbb{G}}} V_{p^j} \theta_{a_{p^j}} g\right) &= F_\ell(V_m \theta_{a_m} g + \cdots) \\ &= rV_{p^d}\theta_{a_m} g + \cdots \neq 0. \end{aligned}$$

[18]The converse so this claim also holds: since $F_\ell F_p = F_p F_\ell$ for $p \nmid \ell$, we can commute F_ℓ through the sum expression (which is absent any non-commuting terms by hypothesis), where it then kills g to give $F_\ell h = 0$.

Since p–typical curves are closed under difference, h could not have been p–typical.

Finally, we specialize to the case $h = [p]_{\widehat{\mathbb{G}}}(g)$. Since F_ℓ and $[p]$ commute, $[p]$ is p–typical, hence has an expression of the desired form. $\square$

Proof of Lemma 3.3.3. Replace the formal group law by its p–typification. Using the formulas from Lemma 3.3.7, we see that the height of a p–typical formal group law over a field of characteristic p coincides with the appearance of the first nonzero coefficient in its p–series. $\square$

Lemma 3.3.7 shows that the p–series of a formal group law with p–typical logarithm contains exactly as much information as the logarithm itself (and hence fully determines the formal group law). We would again like to show that "all" of the data of a p–typical group law is found in its p–series, even if it does not have a logarithm to mediate the two. The following important theorem makes this thought precise.

Theorem 3.3.11 (cf. [Mila, Proposition 5.1], [Rav86, Theorem A2.2.3], and the proof of [Hop, Proposition 19.10])**.** *The Kudo–Araki map determined by Lemma 3.3.10*

$$\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots] \xrightarrow{v} \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$$

is an isomorphism.

Is there a version of this proof that doesn't make a rational argument?

Proof. Begin with a universal group law over the ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. This group law p–typifies by Corollary 3.3.9 to a second group law which is selected by a map $\varepsilon\colon \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. The following diagram includes the image factorization of ε, as well as its rationalization and the map v:

$$\begin{tikzcd}
& \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \arrow[rr] \arrow[dr, "\varepsilon"] \arrow[dd, "s"'] & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \arrow[dr, "\varepsilon"] \arrow[dd, "s"] & \\
& & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \arrow[rr] & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \\
\mathbb{Z}_{(p)}[v_1, \ldots, v_d, \ldots] \arrow[r, "v"] & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} \arrow[ur, "i"] \arrow[rr] & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} \otimes \mathbb{Q}. \arrow[ur, "i"] &
\end{tikzcd}$$

We immediately deduce that all the horizontal arrows are injections: in Theorem 3.2.2 we calculated $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ to be torsion-free; $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$ is a subring of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$, hence it is also torsion-free; and Lemma 3.3.7 shows that $(i \circ v)(v_n)$ agrees with pm_{p^n} in the module of indecomposables $Q(\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q})$.

This is muddy.

To complete the proof, we need to show that v is surjective, which will follow from calculating the indecomposables in $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$ and checking that Qv is surjective. Since s is surjective, the map Qs on indecomposables is surjective as well, and its effect can largely be calculated rationally. Since $(Q\varepsilon)(m_n) = 0$ for $n \neq p^d$, we have that $Q(\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}})$ is generated by $s(b_{p^d-1})$ under an isomorphism as in Theorem 3.2.2. It follows that Qi injects, hence Qv must surject by the calculation of $Q(i \circ v)(v_n)$ above. $\square$

Corollary 3.3.12. *If $[p]_\varphi(x) = [p]_\psi(x)$ for two p–typical formal group laws $+_\varphi$ and $+_\psi$, then $+_\varphi$ and $+_\psi$ are themselves equal.* $\square$

Corollary 3.3.13. *For any sequence of coefficients $v_j \in R$ in a $\mathbb{Z}_{(p)}$–algebra R, there is a unique p–typical formal group law $+_\varphi$ with*

$$[p]_\varphi = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_d x^{p^d} +_\varphi \cdots . \quad \square$$

Finally, we exploit these results to make deductions about the geometry of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. There is an inclusion of groupoid–valued sheaves from p–typical formal group laws with p–typical isomorphisms to all formal group laws with all isomorphisms. Corollary 3.3.9 can be viewed as presenting this inclusion as a deformation retraction, witnessing a natural *equivalence* of groupoids. It follows from Remark 3.1.18 that they both present the same stack. The central utility of this equivalence is that the Kudo–Araki moduli of p–typical formal group laws is a considerably smaller algebra than $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$, resulting in a less noisy picture of the Hopf algebroid.

Our final goal in this Lecture is to exploit this refined presentation in the study of invariant functions.

Definition 3.3.14 ([Goe08, Lemma 2.28])**.** Let (X_0, X_1) be the groupoid scheme associated to a Hopf algebroid (A, Γ). A function $f\colon X_0 \to \mathbb{A}^1$ is said to be *invariant* when it is stable under isomorphism, i.e., when there is a diagram

$$\begin{tikzcd} X_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \arrow[dr, "s^* f = t^* f"] & \\ X_0 \arrow[r, "f"] & \mathbb{A}^1. \end{tikzcd}$$

(In terms of Hopf algebroids, the corresponding element $a \in A$ satisfies $\eta_L(a) = \eta_R(a)$.) Correspondingly, a closed subscheme $A \subseteq X_0$ determined by the simultaneous vanishing of functions f_α is said to be *invariant* when the vanishing condition is invariant—i.e., a point lies in the simultaneous vanishing locus if and only if its entire orbit under X_1 also lies in the simultaneous vanishing locus. (In terms of Hopf algebroids, the corresponding ideal $I \subseteq A$ satisfies $\eta_L(I) = \eta_R(I)$.) Finally, a *closed substack* is a substack determined by an invariant ideal of X_0.

We are in a good position to discern all of the closed substacks of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$—or, equivalently, to discern all of the invariant ideals of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$.

At some point you really should talk about open complements (of affine schemes and then of affinely presented stacks).

The geometric definition of invariant ideal is carelessly phrased.

You could be more careful about stating this: these functions v_d aren't really invariant modulo I_d, since

Corollary 3.3.15 ([Wil82, Theorem 4.6 and Lemmas 4.7-8])**.** *The ideal $I_d = (p, v_1, \ldots, v_{d-1})$ is invariant for all d. It determines the closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d}$ of formal group laws of p–height at least d.*

Proof. Recall from Theorem 3.3.11 the Kudo–Araki isomorphism

$$\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}} \xrightarrow{\simeq} \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots] =: \operatorname{Spec} V,$$

and let $+_L$ denote the associated universal p–typical formal group law with p–series

$$[p]_L(x) = px +_L v_1 x^p +_L v_2 x^{p^2} +_L \cdots +_L v_d x^{p^d} +_L \cdots .$$

Over $\operatorname{Spec} V[t_1, t_2, \ldots]$, we can form a second group law $+_R$ by conjugating $+_L$ by the universal p–typical coordinate transformation $g(x) = \sum_{j=0}^{\infty}{}_L t_j x^{p^j}$. The corresponding p–series

$$[p]_R(x) = \sum_{d=0}^{\infty}{}_R \eta_R(v_d) x^{p^d}$$

determines the η_R map of the Hopf algebroid $(V, V[t_1, t_2, \ldots])$ presenting the moduli of p–typical formal group laws and p–typical isomorphisms. We cannot hope to compute $\eta_R(v_d)$ explicitly, but modulo p we can apply Freshman's Dream to the expansion of

$$[p]_L(g(x)) = g([p]_R(x))$$

to discern some information:

$$\sum_{\substack{i \geq 0 \\ j > 0}}{}_L t_i \eta_R(v_j)^{p^i} \equiv \sum_{\substack{i > 0 \\ j \geq 0}}{}_L v_i t_j^{p^i} \pmod{p}.$$

This is still inexplicit, since $+_L$ is a very complicated operation, but we can see $\eta_R(v_d) \equiv v_d \pmod{I_d}$. It follows that I_d is invariant for each d. Additionally, the closed substack this determines are those formal groups admitting local p–typical coordinates for which $v_{\leq d} = 0$, guaranteeing that the height of the associated formal group is at least d. $\square$

What is *much* harder to prove is the following:

Theorem 3.3.16 ([Lan75, Corollary 2.4 and Proposition 2.5], cf. [Wil82, Theorem 4.9])**.** *The unique closed reduced substack of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ of codimension d is selected by the invariant prime ideal $I_d \subseteq \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$.*

Proof sketch. We want to show that if I is an invariant prime ideal, then $I = I_d$ for some d. To begin, note that $v_0 = p$ is the only invariant function on $\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}$, hence I must either be trivial or contain p. Then, inductively assume that $I_d \subseteq I$. If this is not an equality, we want to show that $I_{d+1} \subseteq I$ is forced. Take $y \in I \setminus I_d$; if we could show

$$\eta_R(y) = av_d^j t^K + \text{higher order terms}$$

for nonzero $a \in \mathbb{Z}_{(p)}$, we could proceed by primality to show that $v_d \in I$ and hence $I_{d+1} \subseteq I$. This is possible (and, indeed, this is how the full proof goes), but it requires serious bookkeeping. □

Remark 3.3.17. The complementary open substack of dimension d is harder to describe. From first principles, we can say only that it is the locus where the coordinate functions p, $v_1, \ldots, v_d$ do not *all simultaneously vanish*. It turns out that:

1. On a cover, at least one of these coordinates can be taken to be invertible.
2. Once one of them is invertible, a coordinate change on the formal group law can be used to make v_d (and perhaps others in the list) invertible. Hence, we can use $v_d^{-1}\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$ as a coordinate chart.
3. Over a further base extension and a further coordinate change, the higher coefficients v_{d+k} can be taken to be zero. Hence, we can also use $v_d^{-1}\mathbb{Z}_{(p)}[v_1, \ldots, v_d]$ as a coordinate chart.

Remark 3.3.18 (cf. [Str06, Section 12] and [Lura, Remark 13.9]). Specialize now to the case of a field k of characteristic p. Since the additive group law has vanishing p–series and is p–typical, a consequence of Corollary 3.3.12 is that *every* p–typical group law with vanishing p–series is exactly equal to $\widehat{\mathbb{G}}_a$, and in fact any formal group law with vanishing p–series p–typifies exactly to $\widehat{\mathbb{G}}_a$. This connects several ideas we have seen so far: the presentation of formal group laws with logarithms in Theorem 1.5.6, the presentation of the context $\mathcal{M}_{MOP}$ in Example 3.1.21, and the Hurewicz image of MU_* in $H\mathbb{F}_{p*}MU$ in Corollary 2.6.9.

Remark 3.3.19. It's worth pointing out how strange all of this is. In Euclidean geometry, open subspaces are always top-dimensional, and closed subspaces can drop dimension. Here, proper open substacks of every dimension appear, and every nonempty closed substack is ∞–dimensional (albeit of positive codimension).

Remark 3.3.20. The results of this section have several alternative forms in the literature. For instance, $[p]_\varphi(x)$ can also be expressed as

$$[p]_\varphi(x) = px + v_1 x^p + v_2 x^{p^2} + \cdots + v_d x^{p^d} + \cdots,$$

and this also determines a presentation of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$. These other elements v_d, called *Hazewinkel coordinates*, differ substantially from the Kudo–Araki coordinates favored here, although they are equally "canonical". Different coordinate patches are useful for accomplishing different tasks, and the reader would be wise to remain flexible.

The end of this remark could be beefed up by mentioning some of the stuff around Question 100 in the E–theory seminar notes.

Remark 3.3.21. The p–typification operation often gives “unusual” results. For instance, we will examine the standard multiplicative formal group law of Example 2.1.23, its rational logarithm, and its rational exponential:

$$x +_{\widehat{\mathbb{G}}_m^{\mathrm{std}}} y = x + y - xy, \qquad \log_{\widehat{\mathbb{G}}_m^{\mathrm{std}}}(x) = -\log(1-x), \qquad \exp_{\widehat{\mathbb{G}}_m^{\mathrm{std}}}(x) = 1 - \exp(-x).$$

Maybe add a reference to the Hazewinkel book section on Artin–Hasse exponentials.

By Lemma 3.3.6, we see that the p–typification of this rational logarithm takes the form

$$\log_{\widehat{\mathbb{G}}_m^{p\text{-typ}}}(x) = \sum_{j=0}^{\infty} \frac{x^{p^j}}{p^j}.$$

We can couple this to the standard exponential of the rational multiplicative group

$$\begin{tikzcd} \widehat{\mathbb{A}}^1 \arrow[r, "\varepsilon x"] \arrow[rr, bend left, "\log^{p\text{-typ}}"] & \widehat{\mathbb{G}}_m \arrow[r, "\log"] & \widehat{\mathbb{G}}_a \arrow[r, "\exp"] \arrow[rr, bend left, "\exp^{\mathrm{std}}"] & \widehat{\mathbb{G}}_m \arrow[r, "x"] & \widehat{\mathbb{A}}^1 \end{tikzcd}$$

to produce the coordinate change from Corollary 3.3.9:

$$1 - \exp\left(-\sum_{j=0}^{\infty} \frac{x^{p^j}}{p^j}\right) = 1 - E_p(-x).$$

This series $E_p(x)$ is known as the *Artin–Hasse exponential*, and it has the miraculous property that it is a series lying in $\mathbb{Z}_{(p)}[\![x]\!] \subseteq \mathbb{Q}[\![x]\!]$, as it is a change of coordinate series on $\widehat{\mathbb{G}}_m$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$.

I'm not sure where this goes—maybe it goes right here—but Allen told me how to give a nice proof of this fact. He didn't give me a citation for this, but the main point was that flatness can be checked locally, so the general formula for the pullback $\mathcal{M}_{(A_1,\Gamma_1)} \to \mathcal{M}_{(L,W)} \leftarrow \mathcal{M}_{(A_2,\Gamma_2)}$ being $\mathcal{M}_{(A_1 \otimes W \otimes A_2, \Gamma_1 \otimes W \otimes \Gamma_2)}$ specializes to compute the pullback of $\operatorname{Spec} A \to \mathcal{M}_{(L,W)} \leftarrow \operatorname{Spec} E(d)_*$ to be $\mathcal{M}_{(A \otimes W \otimes E(d)_*, A \otimes W \otimes E(d)_*)} = \operatorname{Spec}(A \otimes W \otimes E(d)_*)$. But the map $\operatorname{Spec} A \to \mathcal{M}_{(L,W)}$ factors through $\operatorname{Spec} L$, so it suffices just to check flatness for $\operatorname{Spec} L \otimes_L W \otimes_L E(d)_* \cong \operatorname{Spec} W \otimes_L E(d)_* \to \operatorname{Spec} E(d)_*$, which you finally do by hand.

Also, people seem to say things about the Mischenko logarithm rather than the invariant differential, but I wonder if we should phrase things in those terms.

3.4 The structure of $\mathcal{M}_{\mathbf{fg}}$ III: Small scales

In the previous two Lectures, we analyzed the structure of $\mathcal{M}_{\mathbf{fg}}$ as a whole: first we studied the cover

$$\mathcal{M}_{\mathbf{fgl}} \to \mathcal{M}_{\mathbf{fg}},$$

and then we turned to the stratification described by the height function

$$\mathrm{ht}\colon \pi_0 \mathcal{M}_{\mathbf{fg}}(T \text{ a } \mathbb{Z}_{(p)}\text{–algebra}) \to \mathbb{N} \cup \{\infty\}.$$

In this Lecture, we will concern ourselves with the small scale behaviors of $\mathcal{M}_{\mathbf{fg}}$: its geometric points and their local neighborhoods. To begin, we have all the tools in place to perform an outright classification of the geometric points.

Theorem 3.4.1 ([Laz55, Théorème IV]). *Let $\bar{k}$ be an algebraically closed field of positive characteristic p. The height map*

$$\mathrm{ht}\colon \pi_0 \mathcal{M}_{\mathbf{fg}}(\bar{k}) \to \mathbb{N}_{>0} \cup \{\infty\}$$

is a bijection.

Proof. Surjectivity follows from Corollary 3.3.13. Namely, the d^{th} *Honda formal group law* is the p–typical formal group law over k determined by

$$[p]_{\varphi_d}(x) = x^{p^d},$$

and it gives a preimage for d. To show injectivity, we must show that every p–typical formal group law φ over $\bar{k}$ is isomorphic to the appropriate Honda group law. Suppose that the p–series for φ begins

$$[p]_\varphi(x) = x^{p^d} + ax^{p^{d+k}} + \cdots .$$

Then, we will construct a coordinate transformation $g(x) = \sum_{j=1}^{\infty} b_j x^j$ satisfying This transformation needs to be p–typical.

$$\begin{aligned} g(x^{p^d}) &\equiv [p]_\varphi(g(x)) && \pmod{x^{p^{d+k}+1}} \\ \sum_{j=1}^{\infty} b_j x^{jp^d} &\equiv \sum_{j=1}^{\infty} b_j^{p^d} x^{jp^d} + \sum_{j=1}^{\infty} ab_j^{p^{d+k}} x^{jp^{d+k}} && \pmod{x^{p^{d+k}+1}}. \end{aligned}$$

For g to be a coordinate transformation, we must have $b_1 = 1$, which in the critical degree $x^{p^{d+k}}$ forces the relation

$$b_{p^k} = b_{p^k}^{p^d} + a.$$

Since $\bar{k}$ is algebraically closed, this relation is solvable, and thus the coordinate for φ can be perturbed so that the term $x^{p^{d+k}}$ does not appear in the p–series. Inducting on d gives the result. $\square$

Remark 3.4.2 ([Str06, Remark 11.2]). We can now see see that $\pi_0\mathcal{M}_{\mathbf{fg}}$, sometimes called the *coarse moduli of formal groups*, is not representable by a scheme. From Theorem 3.4.1, we see that there are infinitely many points in $\pi_0\mathcal{M}_{\mathbf{fg}}(\mathbb{F}_p)$. From Corollary 3.2.6, we see that these lift along the surjection $\mathbb{Z} \to \mathbb{F}_p$ to give infinitely many distinct points in $\pi_0\mathcal{M}_{\mathbf{fg}}(\mathbb{Z})$. On the other hand, by Theorem 3.3.1 there is a single $\mathbb{Q}$–point of the coarse moduli, whereas the $\mathbb{Z}$–points of a representable functor would inject into its $\mathbb{Q}$–points.

We now turn to understanding the infinitesimal neighborhoods of these geometric points. In general, for $p\colon \operatorname{Spec} k \to X$ a closed k–point of a scheme, we defined in Definition 2.1.6 and Definition 2.1.7 an infinitesimal neighborhood object $X_p^\wedge$ with a lifting property

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "p"] \arrow[d] & X_p^\wedge \arrow[d] \\ \operatorname{Spf} R \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

for any infinitesimal thickening $\operatorname{Spf} R$ of $\operatorname{Spec} k$. Thinking of X as representing a moduli problem, a typical choice for $\operatorname{Spf} R$ is $\widehat{\mathbb{A}}^1_k$, and a map $\widehat{\mathbb{A}}^1_k \to X$ extending p gives a series solution to the moduli problem which specializes at the origin to p. In turn, $X^\wedge_p$ is the smallest object through which all such maps factor, and so we think of it as classifying Taylor expansions of solutions passing through p.

For a formal group $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$, the definition is formally similar, but actually writing it out is made complicated by Remark 3.1.18. In particular, $p\colon \operatorname{Spec} k \to X$ may not lift directly through $\operatorname{Spf} R \to X$, but instead $\operatorname{Spec} R/\mathfrak{m} \to X$ may present p on a cover $i\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spec} k$.

Definition 3.4.3 ([Reza, Section 2.4], cf. [Str97, Section 6])**.** Define $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$, the *Lubin–Tate stack*, to be the groupoid-valued functor which on an infinitesimal thickening R of k has objects

$$\begin{tikzcd} & \mathcal{M}_{\mathbf{fg}} & \\ \operatorname{Spec} k \arrow[ur, bend left, "\Gamma"] & \operatorname{Spec} R/\mathfrak{m} \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[u, bend left=40, "i^*\Gamma"] \arrow[u, bend right=40, "\pi^*\widehat{\mathbb{G}}"'] \arrow[u, phantom, "\overset{\alpha}{\Longrightarrow}"] & \operatorname{Spf} R, \arrow[ul, bend right, "\widehat{\mathbb{G}}"'] \end{tikzcd}$$

where i is an inclusion of k into the residue field $R/\mathfrak{m}$ and $\alpha\colon i^*\Gamma \to \pi^*\widehat{\mathbb{G}}$ is an isomorphism of formal groups. The morphisms in the groupoid are maps $f\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}'$ of formal groups over $\operatorname{Spf} R$ covering the identity on $i^*\Gamma$, called $\star$*–isomorphisms*.

Remark 3.4.4 (cf. [Rezb, Section 4.1]). The local formal group $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ always has trivializable Lie algebra, hence Lemma 3.2.7 shows that it always admits a presentation by a formal group law. In fact, any deformation $\widehat{\mathbb{G}}\colon \operatorname{Spf} R \to \mathcal{M}_{\mathbf{fg}}$ of Γ also has a trivializable Lie algebra, since projective modules (such as $T_0\widehat{\mathbb{G}}$) over local rings like R are automatically free (i.e., trivializable). It follows that the groupoid $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma(R)$ admits a presentation in terms of formal group *laws*. Starting with the pullback square of groupoids

$$\begin{tikzcd} (\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma(R) \arrow[rr] \arrow[d] & & \mathcal{M}_{\mathbf{fg}}(B) \arrow[d] \\ \coprod\limits_{i\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spec} k} \{\Gamma\} \arrow[r] & \coprod\limits_{i\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spec} k} \mathcal{M}_{\mathbf{fg}}(k) \arrow[r] & \mathcal{M}_{\mathbf{fg}}(R/\mathfrak{m}) \end{tikzcd}$$

and selecting formal group laws everywhere, the objects of the groupoid $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma(R)$ are given by diagrams

$$\begin{tikzcd}
(\widehat{\mathbb{A}}^1_k, +_\Gamma) \arrow[d] & (\widehat{\mathbb{A}}^1_{R/\mathfrak{m}}, +_{i^*\Gamma}) \arrow[l] \arrow[dr] \arrow[rr, equal] & & (\widehat{\mathbb{A}}^1_{R/\mathfrak{m}}, +_{\pi^*\widehat{\mathbb{G}}}) \arrow[r] \arrow[dl] & (\widehat{\mathbb{A}}^1_R, +_{\widehat{\mathbb{G}}}) \arrow[d] \\
\operatorname{Spec} k & & \operatorname{Spec} R/\mathfrak{m} \arrow[ll, "i"'] \arrow[rr, "\pi"] & & \operatorname{Spf} R,
\end{tikzcd}$$

where we have required an *equality* of formal group laws over the common pullback. A morphism in this groupoid is a formal group law isomorphism f over $\operatorname{Spf} R$ which reduces to the identity over $\operatorname{Spec} R/\mathfrak{m}$.

The main result about this infinitesimal space $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$ is due to Lubin and Tate:

Theorem 3.4.5 ([LT66, Theorem 3.1])**.** *Suppose that* $\operatorname{ht}\Gamma < \infty$ *for* Γ *a formal group over* k *a perfect field of positive characteristic* p*. The functor* $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$ *is valued in essentially discrete groupoids, and it is naturally equivalent to a smooth formal scheme over* $\mathbb{W}_p(k)$ *of dimension* $(\operatorname{ht}(\Gamma) - 1)$.

Remark 3.4.6 ([**?**, Theorem 4.35]). The presence of the *p–local Witt ring* $\mathbb{W}_p(k)$ is explained by its universal property: for k as above and R an infinitesimal thickening of k, $\mathbb{W}_p(k)$ has the lifting property[19]

$$\begin{tikzcd}
\mathbb{W}_p(k) \arrow[r, dotted, "\exists!"] \arrow[d] & R \arrow[d] \\
k \arrow[r, "i"] & R/\mathfrak{m}
\end{tikzcd}$$

For the finite perfect fields $k = \mathbb{F}_{p^d} = \mathbb{F}_p(\zeta_{p^d-1})$, the Witt ring can be computed to be $\mathbb{W}_p(\mathbb{F}_{p^d}) = \mathbb{Z}_p(\zeta_{p^d-1})$.

Remark 3.4.7. In light of Remark 3.4.4, we can also state Theorem 3.4.5 in terms of formal group laws and their $\star$–isomorphisms. For a group law $+_\Gamma$ over a perfect field k of positive characteristic, it claims that there exists a ring X, noncanonically isomorphic to $\mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!]$, as well as a certain group law $+_{\widetilde{\Gamma}}$ on this ring. The group law $+_{\widetilde{\Gamma}}$ has the following property: if $+_{\widehat{\mathbb{G}}}$ is a formal group law on an infinitesimal thickening $\operatorname{Spf} R$ of $\operatorname{Spec} k$ which reduces along $\pi\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spf} R$ to $+_\Gamma$, then there is a unique ring map $f\colon X \to R$ such that $f^*(+_{\widetilde{\Gamma}})$ is $\star$–isomorphic to $\pi^*(+_{\widehat{\mathbb{G}}})$. Moreover, this $\star$–isomorphism is unique.

We will spend the rest of this Lecture working towards a proof of Theorem 3.4.5. We first consider a very particular sort of infinitesimal thickening: the square-zero extension $R = k[\varepsilon]/\varepsilon^2$ with pointing $\varepsilon = 0$. We are interested in two kinds of data over R: formal group laws $+_\Delta$ over R reducing to $+_\Gamma$ at the pointing, and formal group law automorphisms φ of $+_\Gamma$ which reduce to the identity automorphism at the pointing.

[19]Rings with such lifting properties are generally called *Cohen rings.* In the case that k is a perfect field of positive characteristic p, the Witt ring $\mathbb{W}_p(k)$ happens to model a Cohen ring for k.

Lemma 3.4.8. *Define*

$$\Gamma_1 = \frac{\partial(x +_\Gamma y)}{\partial x}, \qquad \Gamma_2 = \frac{\partial(x +_\Gamma y)}{\partial y}.$$

Such automorphisms φ are determined by series ψ satisfying

$$0 = \Gamma_1(x, y)\psi(x) - \psi(x +_\Gamma y) + \psi(y)\Gamma_2(x, y).$$

Such formal group laws $+_\Delta$ are determined by bivariate series $\delta(x, y)$ satisfying

$$0 = \Gamma_1(x +_\Gamma y, z)\delta(x, y) - \delta(x, y +_\Gamma z) + \delta(x +_\Gamma y, z) - \delta(y, z)\Gamma_2(x, y +_\Gamma z).$$

Proof. Such an automorphism φ admits a series expansion

$$\varphi(x) = x + \varepsilon \cdot \psi(x).$$

Then, we take the homomorphism property

$$\begin{aligned} \varphi(x +_\Gamma y) &= \varphi(x) +_\Gamma \varphi(y) \\ (x +_\Gamma y) + \varepsilon \cdot \psi(x +_\Gamma y) &= (x + \varepsilon \cdot \psi(x)) +_\Gamma (y + \varepsilon \cdot \psi(y)) \end{aligned}$$

and apply $\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0}$ to get

$$\psi(x +_\Gamma y) = \Gamma_1(x, y) \cdot \psi(x) + \Gamma_2(x, y) \cdot \psi(y).$$

Similarly, such a formal group law $+_\Delta$ admits a series expansion

$$x +_\Delta y = (x +_\Gamma y) + \varepsilon \cdot \delta(x, y).$$

Beginning with the associativity property

$$(x +_\Delta y) +_\Delta z = x +_\Delta (y +_\Delta z),$$

we compute $\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0}$ applied to both sides:

$$\begin{aligned} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} ((x +_\Delta y) +_\Delta z) &= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} ((((x +_\Gamma y) + \varepsilon \cdot \delta(x, y)) +_\Gamma z) + \varepsilon \cdot \delta(x +_\Gamma y, z)) \\ &= \Gamma_1(x +_\Gamma y, z) \cdot \delta(x, y) + \delta(x +_\Gamma y, z), \end{aligned}$$

and similarly

$$\begin{aligned} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (x +_\Delta (y +_\Delta z)) &= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} ((x +_\Gamma ((y +_\Gamma z) + \varepsilon \cdot \delta(y, z))) + \varepsilon \cdot \delta(x, y +_\Gamma z)) \\ &= \Gamma_2(x, y +_\Gamma z) \cdot \delta(y, z) + \delta(x, y +_\Gamma z). \end{aligned}$$

Equating these gives the condition in the Lemma statement. □

The key observation is that these two conditions appear as cocycle conditions for the first two levels of a natural cochain complex.

Can this be phrased geometrically?

Definition 3.4.9 ([Laz97, Section 3])**.** The deformation complex $\widehat{C}^*(+_\Gamma; k)$ is defined by

$$k \to k[\![x_1]\!] \to k[\![x_1, x_2]\!] \to k[\![x_1, x_2, x_3]\!] \to \cdots$$

with differential

$$\begin{aligned}(df)(x_1, \ldots, x_{n+1}) = \Gamma_1\left(\sum_{i=1}^{n}{}_\Gamma x_i, x_{n+1}\right) \cdot f(x_1, \ldots, x_n) \\ + \sum (-1)^i f(x_1, \ldots, x_i +_\Gamma x_{i+1}, \ldots, x_{n+1}) \\ + (-1)^{n+1}\left(\varphi_2\left(x_1, \sum_{i=2}^{n+1}{}_\Gamma x_i\right) \cdot f(x_2, \ldots, x_{n+1})\right),\end{aligned}$$

where we have again written

$$\Gamma_1(x, y) = \frac{\partial(x +_\Gamma y)}{\partial x}, \qquad \Gamma_2(x, y) = \frac{\partial(x +_\Gamma y)}{\partial y}.$$

The complex even knits the information together intelligently:

Corollary 3.4.10 ([Laz97, p. 1320])**.** *Two extensions $+_\Delta$ and $+_{\Delta'}$ of $+_\Gamma$ to $k[\varepsilon]/\varepsilon^2$ are isomorphic if their corresponding 2–cycles in $\widehat{Z}^2(+_\Gamma; k)$ differ by an element in $\widehat{B}^2(+_\Gamma; k)$.* □

Consider proving this.

Remarkably, we have already encountered this complex before:

Lemma 3.4.11 ([Laz97, p. 1320])**.** *Write $\widehat{\mathbb{G}}$ for the formal group associated to the group law $+_\Gamma$. The cochain complex $\widehat{C}^*(+_\Gamma; k)$ is quasi-isomorphic to the cohomology cochain complex considered in the proof of Lemma 3.2.5:*

$$\begin{aligned}\widehat{C}^*(+_\Gamma; k) &\to \underline{\mathsf{FormalSchemes}}(B\widehat{\mathbb{G}}, \widehat{\mathbb{G}}_a)(k) \\ f &\mapsto \Gamma_1\left(0, \sum_{i=1}^{n}{}_\Gamma x_i\right)^{-1} f(x_1, \ldots, x_n). \quad \square\end{aligned}$$

Two Lectures ago while proving Lemma 3.2.5, we computed the cohomology of this complex in the specific case of $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$. This is the one case where Lubin and Tate's theorem does *not* apply, since it requires $\operatorname{ht}\widehat{\mathbb{G}} < \infty$. Nonetheless, by filtering the multiplication on $\widehat{\mathbb{G}}$ by degree, we can use this specific calculation to get up to the general one we now seek.

Lemma 3.4.12. *Let $\widehat{\mathbb{G}}$ be a formal group of finite height d over a field k. Then $H^1(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a) = 0$ and $H^2(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a)$ is a free k–vector space of dimension $(d-1)$.*

d or $(d-1)$? There's β_0 through β_{d-1}.... Also compare this with Lemma 3.2.5.

Proof (after Hopkins). We select a p–typical coordinate on $\widehat{\mathbb{G}}$ of the form

$$x +_\varphi y = x + y + \text{unit} \cdot c_{p^d}(x, y) + \cdots ,$$

where $c_{p^d}(x, y)$ is as in one of Lazard's symmetric 2–cocycles, as in Lemma 3.2.5. Filtering $\widehat{\mathbb{G}}$ by degree, the multiplication projects to $x +_\varphi y = x + y$ in the associated graded, and the resulting filtration spectral sequence has signature

$$[H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)]_* \Rightarrow H^*(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a),$$

where the second grading comes from the degree of the homogeneous polynomial representatives of classes in $H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)$.

Because Lemma 3.2.5 gives different calculations of $H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)$ for $p = 2$ and $p > 2$, we specialize to $p > 2$ for the remainder of the proof and leave the similar $p = 2$ case to the reader. For $p > 2$, Lemma 3.2.5 gives

$$[H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)]_* = \left[\frac{k[\alpha_j \mid j \geq 0]}{\alpha_j^2 = 0} \otimes k[\beta_j \mid j \geq 0] \right]_* ,$$

where α_j is represented by x^{p^j} and β_j is represented by $c_{p^j}(x, y)$. To compute the differentials in this spectral sequence generally, one computes by hand the formula for the differential in the bar complex, working up to lowest nonzero degree. For instance, to compute $d(\alpha_j)$ we examine the series

$$(x +_\varphi y)^{p^j} - (x^{p^j} + y^{p^j}) = (\text{unit}) \cdot c_{p^{d+j}}(x, y) + \cdots ,$$

where we used $c_{p^d}^{p^j} = c_{p^{j+d}}$. So, we see that nothing in the 1–column of the spectral sequence is a permanent cocycle and that there are $d - 1$ things at the bottom of the 2–column of the spectral sequence which are not coboundaries. To conclude the Lemma statement, we need only to check that they are indeed permanent cocycles. To do this, we note that they are indeed realized as deformations, by noting

$$x +_{\text{univ}} y \cong x + y + v_j c_{p^j}(x, y) \pmod{v_1, \ldots, v_{j-1}, (x, y)^{p^j + 1}}$$

where $+_{\text{univ}}$ is the Kudo–Araki universal p–typical law (cf. [LT66, Proposition 1.1]). □

Proof of Theorem 3.4.5 using Remark 3.4.7. We will prove this inductively on the order of the infinitesimal neighborhood of $\operatorname{Spec} k = \operatorname{Spec} R/\mathfrak{m}$ in $\operatorname{Spf} R$:

$$\operatorname{Spec} R/\mathfrak{m} \xrightarrow{j_r} \operatorname{Spec} R/\mathfrak{m}^r \xrightarrow{i_r} \operatorname{Spf} R.$$

Suppose that we have demonstrated the Theorem for $+_{\widehat{\mathbb{G}}_{r-1}} = i_{r-1}^*(+_{\widehat{\mathbb{G}}})$, so that there is a map $\alpha_{r-1}\colon \mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!] \to R/\mathfrak{m}^{r-1}$ and a strict isomorphism $g_{r-1}\colon +_{\widehat{\mathbb{G}}_{r-1}} \to \alpha_{r-1}^* +_{\tilde{\Gamma}}$ of formal group laws. The exact sequence

$$0 \to \mathfrak{m}^{r-1}/\mathfrak{m}^r \to R/\mathfrak{m}^r \to R/\mathfrak{m}^{r-1} \to 0$$

exhibits $R/\mathfrak{m}^r$ as a square–zero extension of $R/\mathfrak{m}^{r-1}$ by $M = \mathfrak{m}^{r-1}/\mathfrak{m}^r$. Then, let β be *any* lift of α_{r-1} and h be *any* lift of g_{r-1} to R/I^r, and let A and B be the induced group laws

$$x +_A y = \beta^* \tilde{\varphi}, \qquad\qquad x +_B y = h\left(h^{-1}(x) +_{\widehat{\mathbb{G}}_r} h^{-1}(y)\right).$$

Since these both deform the group law $+_{\widehat{\mathbb{G}}_{r-1}}$, by Corollary 3.4.10 and Lemma 3.4.12 there exist $m_j \in M$ and $f(x) \in M[\![x]\!]$ satisfying

$$(x +_B y) - (x +_A y) = (df)(x, y) + \sum_{j=1}^{d-1} m_j c_{p^j}(x, y),$$

where $c_{p^j}(x, y)$ is the 2–cocycle associated to the cohomology 2–class β_j. The following definitions complete the induction:

$$g_r(x) = h(x) - f(x), \qquad\qquad \alpha_r(u_j) = \beta(u_j) + m_j. \qquad \square$$

Remark 3.4.13. Our calculation $H^1(\widehat{\mathbb{G}}_\varphi; \widehat{\mathbb{G}}_a) = 0$ shows that the automorphisms $\alpha\colon \Gamma \to \Gamma$ of the special fiber induce automorphisms of the entire Lubin–Tate stack by universality. Namely, for $\Gamma \to \widetilde{\Gamma}$ the universal deformation, the precomposite

$$\Gamma \xrightarrow{\alpha} \Gamma \to \widetilde{\Gamma}$$

presents $\widetilde{\Gamma}$ as a deformation of Γ in a different way, hence induces a map $\widetilde{\alpha}\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$, which by Theorem 3.4.5 is in turn induced by a map $\widetilde{\alpha}\colon (\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma \to (\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$. The action is *highly* nontrivial in all but the most degenerate cases, and its study is of serious interest to homotopy theorists (cf. Lecture 3.6) and to arithmetic geometers (cf. Appendix A.3).

Remark 3.4.14. We also see that our analysis fails wildly for the case $\Gamma = \widehat{\mathbb{G}}_a$. The differential calculation in Lemma 3.4.12 is meant to give us an upper bound on the dimensions of $H^1(\Gamma; \widehat{\mathbb{G}}_a)$ and $H^2(\Gamma; \widehat{\mathbb{G}}_a)$, but this family of differentials is zero in the additive case. Accordingly, both of these vector spaces are infinite dimensional, completely prohibiting us from making any further assessment.

Having accomplished all our major goals, we close our algebraic analysis of $\mathcal{M}_{\mathbf{fg}}$ with Figure 3.3, a diagram summarizing our results.

[Dri74, Section 4.A] also contains a Lubin–Tate cohomology theory, this time for formal A–modules. You should also cite [LT66].

Cite me: Neil's FG notes in the first half of section 18 talk about additive extensions

PICTURE GOES HERE.

Draw a picture of $\operatorname{Spec} \mathbb{Z}_{(p)}$ for the base object: a generic point (0) and a point (p) with a 1–dimensional arithmetic neighborhood. Draw some of the geometric points of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$: $\widehat{\mathbb{G}}_a \otimes \mathbb{Q}$, $\widehat{\mathbb{G}}_m$, something at height 2, something at height 3, ..., and $\widehat{\mathbb{G}}_a \otimes \mathbb{F}_p$ at height ∞. Draw formal neighborhoods of each finite height point: include the arithmetic deformation direction covering the arithmetic deformation of the base, plus the geometric deformation directions where available. Draw a crazy cloud around the infinite height point, indicating a poorly understood deformation theory. Draw braces around parts of the picture to indicate a closed and open substack. Make the braces right-align to indicate that dropping height is "normal" and raising height is "exceptional". Label the substacks with some basic properties: their co/dimensions, for example. Label the heights of the formal group, and indicate the behavior of the height function. Draw a "zoomed in" version of the height 1 geometric point, indicating the existence of many non-closed field points covering it, or "Forms of $\widehat{\mathbb{G}}_m$". Draw "attaching data" between the different formal neighborhoods, indicating that they are nontrivially connected to one another. The idea should be something like projective space, where the open height-dropping condition determines an "around the edges" map for the closed height-raising condition. Indicate (perhaps in a different color) the topological analogues of everything in the picture: $H\mathbb{Z}_{(p)}$, $H\mathbb{F}_p$, $H\mathbb{Q}$, K_Γ, $E(d)$, $P(d)$, E_Γ, $BP_{(p)}$, ... Include a legend: dots for geometric points, fuzz for deformation neighborhoods, ...

I drew an approximation to this picture by hand in `other resources`. I didn't get everything right, but I'd definitely like to use it as a template.

Figure 3.3: Portrait of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$.

3.5 Nilpotence and periodicity in finite spectra

With our analysis of $\mathcal{M}_{\mathbf{fg}}$ complete, our first goal in this Lecture is to finish the program sketched in the introduction to this Case Study by manufacturing those interesting homology theories connected to the functor $\mathcal{M}_{MU}(-)$. We begin by rephrasing our main tool, Theorem 3.0.1, in terms of algebraic conditions.

Theorem 3.5.1 ([Lan76, Corollary 2.7] and [Hop, Theorem 21.4 and Proposition 21.5], cf. Theorem 3.0.1)**.** *Let $\mathcal{F}$ a quasicoherent sheaf over $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$, thought of as a comodule M for the Kudo–Araki Hopf algebroid (cf. Theorem 3.3.11)*

$$(A, \Gamma) = (\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}, \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}[t_1, t_2, \ldots]).$$

If $(p, v_1, \ldots, v_d, \ldots)$ forms an infinite regular sequence on M, then

$$X \mapsto M \otimes_{\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}} MU_0(X)$$

determines a homology theory on finite *spectra X. Moreover, if $M/I_d = 0$ for some $d \gg 0$, then the same formula determines a homology theory on* all *spectra X.*

Proof. Following the discussion in the introduction, we note that a cofiber sequence

$$X' \to X \to X''$$

of spectra gives rise to an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{M}_{MU}(X') \arrow[r] \arrow[d, equal] & \mathcal{M}_{MU}(X) \arrow[r] \arrow[d, equal] & \mathcal{M}_{MU}(X'') \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathcal{N}' \arrow[r] & \mathcal{N} \arrow[r] & \mathcal{N}'' \arrow[r] & \cdots .
\end{tikzcd}$$

We thus see that we are essentially tasked with showing that $\mathcal{F}$ is flat, so that tensoring with $\mathcal{F}$ does not disturb the exactness of this sequence. In that case, we can then apply Brown representability to the composite functor $\mathcal{F} \otimes \mathcal{M}_{MU}(X)$.

Flatness of $\mathcal{F}$ is equivalent to $\operatorname{Tor}_1(\mathcal{F}, \mathcal{N}) = 0$ for an arbitrary auxiliary quasicoherent sheaf $\mathcal{N}$ (soon to be thought of as $\mathcal{M}_{MU}(X)$). By our regularity hypothesis, there is an exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{p} \mathcal{F} \to \mathcal{F}/(p) \to 0,$$

so applying $\operatorname{Tor}_*(-, \mathcal{N})$ gives an exact sequence

$$\operatorname{Tor}_2(\mathcal{F}/(p), \mathcal{N}) \to \operatorname{Tor}_1(\mathcal{F}, \mathcal{N}) \xrightarrow{p} \operatorname{Tor}_1(\mathcal{F}, \mathcal{N})$$

The bottom of COCTALOS page 68 has a better interpretation of what picking a formal group law lifting a flat map to $\mathcal{M}_{\mathbf{fg}}$ has to do with anything. It probably belongs in this Lecture.

Much of this section is written gradedly. I guess we still haven't decided whether this is the right presentation.

Danny has been nervous lately about the LEFT yielding module spectra and ring spectra. It would be good to write out how LEFT applied to a sheaf of rings gives a ring spectrum.

Akhil says some nice things about nilpotence and periodicity here: http://mathoverflow.r of-xn-spectra-to-formal-group-laws . The discussion of vanishing lines could be included here. The moral indication that nilpotence is the geometric link between stable homotopy and the moduli of formal groups also seems like an important point to be made here.

of Tor groups. The sequence gives the following sufficiency condition:

$$[\mathrm{Tor}_1(p^{-1}\mathcal{F},\mathcal{N})=0 \quad\text{and}\quad \mathrm{Tor}_2(\mathcal{F}/(p),\mathcal{N})=0] \quad\Rightarrow\quad \mathrm{Tor}_1(\mathcal{F},\mathcal{N})=0.$$

Similarly, the v_1–multiplication sequence gives another sufficiency condition:

$$[\mathrm{Tor}_2(v_1^{-1}\mathcal{F}/(p),\mathcal{N})=0 \quad\text{and}\quad \mathrm{Tor}_3(\mathcal{F}/I_2,\mathcal{N})=0] \quad\Rightarrow\quad \mathrm{Tor}_2(\mathcal{F}/(p),\mathcal{N})=0.$$

Continuing in this fashion, for some $D \gg 0$ we would like to show

$$\begin{aligned} \mathrm{Tor}_{d+1}(v_d^{-1}\mathcal{F}/I_d,\mathcal{N}) &= 0 & &\text{(for each } d < D\text{)},\\ \mathrm{Tor}_{D+1}(\mathcal{F}/I_{D+1},\mathcal{N}) &= 0. \end{aligned}$$

The second condition is satisfied one of two ways, corresponding to our two auxiliary hypotheses and two conclusions in the Theorem statement:

- If $\mathcal{F}$ itself satisfies $\mathcal{F}/I_{D+1}=0$, we are done.
- Writing $j_{D+1}\colon \mathcal{M}_{\mathbf{fg}}^{\geq D+1} \to \mathcal{M}_{\mathbf{fg}}$ for the inclusion of the prime closed substack, we can identify $\mathcal{N}/I_{D+1}$ with $j_{D+1*}j_{D+1}^*\mathcal{N}$. If $\mathcal{N}$ is coherent (for instance, in the case that $\mathcal{N}=\mathcal{M}_{MU}(X)$ for a *finite* complex X), then $j_{D+1}^*\mathcal{N}$ is free for large D and hence has vanishing Tor groups.

We then turn to the first collection of conditions. They are *always* satisfied, but this requires an argument. We write $i_d\colon \mathcal{M}_{\mathbf{fg}}^{=d} \to \mathcal{M}_{\mathbf{fg}}$ for the inclusion of the substack of formal groups of height exactly d, which (following Remark 3.3.17) has a presentation by the Hopf algebroid

$$(v_d^{-1}A/I_d, \Gamma\otimes v_d^{-1}A/I_d).$$

We are trying to study the derived functors of

$$\mathcal{N} \mapsto (i_{d*}i_d^*\mathcal{F})\otimes\mathcal{N} \cong i_{d*}(i_d^*\mathcal{F}\otimes i_d^*\mathcal{N}).$$

Since i_{d*} is exact, we are moved to study the composite functor spectral sequence for

$$\mathsf{QCoh}_{\mathcal{M}_{\mathbf{fg}}} \xrightarrow{i_d^*} \mathsf{QCoh}_{\mathcal{M}_{\mathbf{fg}}^{=d}} \xrightarrow{i_d^*\mathcal{F}\otimes -} \mathsf{QCoh}_{\mathcal{M}_{\mathbf{fg}}^{=d}}.$$

The second functor is actually exact: the geometric map

$$\Gamma_d\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}^{=d}$$

is a faithfully flat cover, and k–modules have no nontrivial Tor. Meanwhile, the first functor has at most d derived functors: i_d^* is modeled by tensoring with $v_d^{-1}A/I_d$, but A/I_d admits a Koszul resolution with d stages and $A/I_d \to v_d^{-1}A/I_d$ is exact. As Tor_{d+1} is beyond the length of this resolution, it is always zero. □

The word "chromatic" here deserves a footnote explaining its provenance.

Definition 3.5.2. Coupling Theorem 3.5.1 to our understanding of $\mathcal{M}_{\mathbf{fg}}$, we produce many interesting homology theories, collectively referred to as *chromatic homology theories*:

- Recall that the moduli of p–typical group laws is affine, presented in Theorem 3.3.11 by
$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots].$$
Since the inclusion of p–typical group laws into all group laws induces an equivalence of stacks, it is in particular flat, and hence this formula determines a homology theory on finite spectra, called *Brown–Peterson homology*:
$$BPP_0(X) := MUP_0(X) \otimes_{MUP_0} BPP_0.$$

- A chart for the open substack $\mathcal{M}_{\mathbf{fg}}^{\le d}$ in terms of $\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}$ was given in Remark 3.3.17 by $\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d^{\pm}]$. Since open maps are in particular flat, it follows that there is a homology theory $E(d)P$, called *the d^{th} Johnson–Wilson homology*, defined on all spectra by
$$E(d)P_0(X) := MUP_0(X) \otimes_{MUP_0} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d^{\pm}].$$

- Similarly, for a formal group Γ of height $d < \infty$, we produced in Theorem 3.4.5 a chart $\operatorname{Spf} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!]$ for its deformation neighborhood. Since inclusions of deformation neighborhoods of substacks of Noetherian stacks are flat, there is a corresponding homology theory E_Γ, called *the (discontinuous) Morava E–theory for* Γ, determined by

Cite me: Consider citing this.

$$E_{\Gamma 0}(X) := MUP_0(X) \otimes_{MUP_0} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!][u^{\pm}].$$
In the case that $\Gamma = \Gamma_d$ is the Honda formal group of height d, the notation is often abbreviated from E_{Γ_d} to merely E_d.

- Since $(p, u_1, \ldots, u_{d-1})$ forms a regular sequence on $E_{\Gamma *}$, we can form the regular quotient at the level of spectra, using cofiber sequences
$$\begin{aligned} E_\Gamma &\xrightarrow{p} E_\Gamma \to E_\Gamma/(p), \\ E_\Gamma/(p) &\xrightarrow{u_1} E_\Gamma/(p) \to E_\Gamma/(p, u_1), \\ &\vdots \\ E_\Gamma/I_{d-1} &\xrightarrow{u_{d-1}} E_\Gamma/I_{d-1} \to E_\Gamma/I_d. \end{aligned}$$
This determines a spectrum $K_\Gamma = E_\Gamma/I_d$, and hence determines a homology theory called *the Morava K–theory for* Γ. In the case where Γ comes from the Honda p–typical

formal group law (of height d), this spectrum is often written as $K(d)$. As an edge case, we also set $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$.[20]

- More delicately, there is a version of Morava E–theory which takes into account the formal topology on $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$, called *continuous Morava E–theory*. It is defined by the pro-system $\{E_\Gamma(X)/u^I\}$, where I ranges over multi-indices and the quotient is again given by cofiber sequences.

- There is also a homology theory associated to the closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d}$. Since $I_d = (p, v_1, \ldots, v_{d-1})$ is generated by a regular sequence on BPP_0, we can directly define the spectrum $P(d)P$ by a regular quotient:

$$P(d)P = BP/(p, v_1, \ldots, v_{d-1}).$$

  This spectrum does have the property $P(d)P_0 = BPP_0/I_d$ on coefficient rings, but $P(d)P_0(X) = BPP_0(X)/I_d$ *only* when I_d forms a regular sequence on $BPP_0(X)$—which is reasonably rare among the cases of interest.

Remark 3.5.3. The trailing "P" in these names is to disambiguate them from similar less-periodic objects in the literature. Namely, BP is often taken to be a minimal wedge summand of $MU_{(p)}$, whereas $E(d)$, E_Γ, and $K(d)$ can all be taken to be $2(p^d - 1)$–periodic (for heights $0 < d < \infty$). The one exception to this minimality convention is E_Γ, which is *usually* taken to be 2–periodic already, so we do not attach a "P" to its name.

Example 3.5.4 (cf. Example 2.1.21). In the case $\Gamma = \widehat{\mathbb{G}}_m$, the resulting spectra are connected to complex K–theory:

$$E_{\widehat{\mathbb{G}}_m} \cong KU_p^\wedge, \qquad K_{\widehat{\mathbb{G}}_m} \cong KU/p, \qquad E(1)P \cong KU_{(p)}.$$

Remark 3.5.5 ([KLW04, Section 5.2], [Rav84, Corollaries 2.14 and 2.16], [Str99a, Theorem 2.13]). In general, the quotient of a ring spectrum by a homotopy element does not give another ring spectrum. The most typical example of this phenomenon is that $\mathbb{S}/2$ is not a ring spectrum, since its homotopy is not 2–torsion. Most of our constructions above do not suffer from this deficiency, with one exception: Morava K–theories at $p = 2$ are not commutative. Instead, there is a derivation $Q_d : K(d) \to \Sigma K(d)$ which tracks the commutativity by the relation

He, in turn, cites Yosimura's *Universal coefficient sequences for cohomology theories of CW-spectra.*

$$ab - ba = uQ_d(a)Q_d(b).$$

In particular, we find that $K(d)^* X$ is a commutative ring whenever $K(d)^1 X = 0$, which is often the case.

[20] By Theorem 3.4.1 and Corollary 3.5.12 to follow, it often suffices to consider just these spectra $K(d)$ to make statements about all K_Γ. With more care, it even often suffices to consider $d \neq \infty$.

Having constructed these chromatic homology theories, for the rest of this Lecture we pursue an example of a "fiberwise" analysis of a phenomenon in homotopy theory. First, recall the following classical theorem:

Theorem 3.5.6 (Nishida)**.** *Every homotopy class $\alpha \in \pi_{\geq 1}\mathbb{S}$ is nilpotent.* □ Cite me: Nishida.

People studying K–theory in the '70s discovered the following related phenomenon:

Theorem 3.5.7 (Adams)**.** *Let $M_{2n}(p)$ denote the mod–p Moore spectrum with bottom cell in degree $2n$. Then there is an index n and a map $v : M_{2n}(p) \to M_0(p)$ such that $KU_* v$ acts by multiplication by the n^{th} power of the Bott class. The minimal such n is given by the formula* Cite me: Adams.

$$n = \begin{cases} p-1 & \text{when } p \geq 3, \\ 4 & \text{when } p = 2. \end{cases} \quad \square$$

In particular, the map v cannot be nilpotent, since a null-homotopic map induces the zero map in any homology theory. Just as we took the non-nilpotent endomorphism $p \in \pi_0 \operatorname{End} \mathbb{S}$ and coned it off, we can take the endomorphism $v \in \pi_{2p-2} \operatorname{End} M_0(p)$ and cone it off to form a new spectrum called $V(1)$.[21] One can ask, then, whether the pattern continues: does $V(1)$ have a non-nilpotent self-map, and can we cone it off to form a new such spectrum with a new such map? Can we then do that again, indefinitely? In order to study this question, we are motivated to find spectra satisfying the following condition:

Definition 3.5.8 ([HS98, Definition 4], cf. [DHS88, Theorem 1])**.** A ring spectrum E *detects nilpotence* if for any ring spectrum R the kernel of the Hurewicz homomorphism

$$R_*\eta_E \colon \pi_* R \to E_* R$$

consists of nilpotent elements. (In particular, such an E cannot send such a nontrivial self-map to zero.)

This question and surrounding issues formed the basis of Ravenel's nilpotence conjectures [Rav84, Section 10], which were resolved by by Devinatz, Hopkins, and Smith [DHS88, HS98]. One of their two main technical achievements was to demonstrate that we already have access to a nice homology theory which detects nilpotence:

Theorem 3.5.9 ([DHS88, Theorem 1.i])**.** *The spectrum MU detects nilpotence.* □

This is a very hard theorem, and we will not attempt to prove it. However, taking this as input, they are easily able to show several other interesting structural results about finite spectra. For instance, they also show that the MU is the universal object which detects nilpotence, in the sense that any other ring spectrum can have this property checked stalkwise on $\mathcal{M}_{MU}$.

[21] The spectrum $V(1)$ is actually defined to be a finite spectrum with $BP_* V(1) \cong BP_*/(p, v_1)$. At $p = 2$ this spectrum doesn't exist and this is a misnomer. More generally, at odd primes p Nave shows that $V((p+1)/2)$ doesn't exist [Nav10, Theorem 1.3].

Corollary 3.5.10 ([HS98, Theorem 3]). *A ring spectrum E detects nilpotence if and only if for all $0 \leq d \leq \infty$ and for all primes p, $K(d)_*E \neq 0$ (i.e., the support of $\mathcal{M}_{MU}(E)$ is not a proper substack of $\mathcal{M}_{MU}$).*

Proof. If $K(d)_*E = 0$ for some d, then the non-nilpotent unit map $\mathbb{S} \to K(d)$ lies in the kernel of the Hurewicz homomorphism for E, so E fails to detect nilpotence.

In the other direction, suppose that for every d we have $K(d)_*E \neq 0$. Because $K(d)_*$ is a field, it follows by picking a basis of $K(d)_*E$ that $K(d) \wedge E$ is a nonempty wedge of suspensions of $K(d)$. So, for $\alpha \in \pi_*R$, if $E_*\alpha = 0$ then $(K(d) \wedge E)_*\alpha = 0$ and hence $K(d)_*\alpha = 0$. So, we need to show that if $K(d)_*\alpha = 0$ for all n and all p then α is nilpotent. Taking Theorem 3.5.9 as given, it would suffice to show merely that $MU_*\alpha$ is nilpotent. This is equivalent to showing that the ring spectrum $MU \wedge R[\alpha^{-1}]$ is contractible or that the unit map is null:

$$\mathbb{S} \to MU \wedge R[\alpha^{-1}].$$

A nontrivial result of Johnson and Wilson shows that if $MU_*X = 0$, then for any d we have $K([0,d])_*X = 0$ and $P(d+1)_*X = 0$.[22] Taking $X = R[\alpha^{-1}]$, we have assumed all of these are zero except for $P(d+1)$. But $\operatorname{colim}_d P(d+1) \simeq H\mathbb{F}_p \simeq K(\infty)$, and $\mathbb{S} \to K(\infty) \wedge R[\alpha^{-1}]$ is assumed to be null as well. By compactness of $\mathbb{S}$, that null-homotopy factors through some finite stage $P(d+1) \wedge R[\alpha^{-1}]$ with $d \gg 0$. $\square$

Corollary 3.5.10 has the following consequence, which speaks to the primacy of both the chromatic program and these results.

Definition 3.5.11. A ring spectrum R is a *field spectrum* when every R–module (in the homotopy category) splits as a wedge of suspensions of R. (Equivalently, R is a field spectrum when it has Künneth isomorphisms.)

Corollary 3.5.12. *Every field spectrum R splits as a wedge of Morava's $K(d)$ theories.*

Proof. It is easy to check (as mentioned in the proof of Corollary 3.5.10) that $K(d)$ is a field spectrum.

Now, consider an arbitrary field spectrum R. Set $E = \bigvee_{\text{primes } p} \bigvee_{d \in [0,\infty]} K(d)$, so that E detects nilpotence. The class 1 in the field spectrum R is non-nilpotent, so it survives when paired with some K–theory $K(d)$, and hence $R \wedge K(d)$ is not contractible. Because both R and $K(d)$ are field spectra, the smash product of the two simultaneously decomposes into a wedge of $K(d)$s and a wedge of Rs. So, R is a retract of a wedge of $K(d)$s, and picking a basis for its image on homotopy shows that it is a sub-wedge of $K(d)$s. $\square$

[22] Specifically, it is immediate that $MU_*X = 0$ forces $P(d+1)_*X = 0$ and $v_{d'}^{-1}P(d')_*(X) = 0$ for all $d' < d$. What's nontrivial is showing that $v_{d'}^{-1}P(d')_*(X) = 0$ if and only if $K(d')_*(X) = 0$ [Rav84, Theorem 2.1.a], [JW75, Section 3].

Remark 3.5.13. In the 2–periodic setting we've become accustomed to, the analogue of Corollary 3.5.12 is that every 2–periodic field spectrum splits as a wedge of suspensions of $K(d)P$.

Remark 3.5.14. In service of Example 3.5.4, the geometric definition of MU given in Lemma 0.0.1, the edge cases of $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, and the claimed primacy of these methods, we might wonder if there is any geometric interpretation of the field theories $K(d)$ for $0 < d < \infty$. To date, this is a completely open question and the subject of intense research.

We're now well-situated to address Ravenel's question about finite spectra and periodic self-maps. The key observation is that spectra admitting such self-maps are closed under some natural operations, leading to the following definition:

Definition 3.5.15. A full subcategory of a triangulated category (e.g., the homotopy category of p–local finite spectra) is *thick* if…

- …it is closed under isomorphisms and retracts.
- …it has a 2-out-of-3 property for cofiber sequences.

Examples of thick subcategories include:

- The category C_d of p–local finite spectra which are $K(d-1)$–acyclic. (For instance, if $d = 1$, the condition of $K(0)$–acyclicity is that the spectrum have purely torsion homotopy groups.) These are called "finite spectra of type at least d".
- The category D_d of p–local finite spectra F for which there is a self-map $v : \Sigma^N F \to F$, $N \gg 0$ which induces multiplication by a unit in $K(d)$–homology and which is nilpotent in $K(\neq d)$–homology. These are called "finite spectra admitting v_d–self–maps".

Do you really need the nilpotence condition?

The categories C_d are the ones we are interested in analyzing, and we hope to identify these putative spaces $V(d)$ inside of them. Ravenel shows the following foothold interrelating the C_d:

Lemma 3.5.16 ([Rav84, Theorem 2.11])**.** *For X a finite complex, there is a bound*

$$\dim K(d-2)_* X \leq \dim K(d-1)_* X.$$

In particular, there is an inclusion $\mathsf{C}_{d-1} \subseteq \mathsf{C}_d$.

Proof sketch. One should compare this with the statement that the stalk dimension of a coherent sheaf is upper semi-continuous. In fact, this analogy gives the essentials of Ravenel's proof: one considers the ring spectrum $v_d^{-1} BP/I_{d-1}$, which admits two maps

$$\begin{tikzcd}[column sep=small]
 & v_{d-1}^{-1}(v_d^{-1}BP/I_{d-1}) \\
v_d^{-1}BP/I_{d-1} \arrow[ur] \arrow[dr] & \\
 & (v_d^{-1}BP/I_{d-1})/v_{d-1}.
\end{tikzcd}$$

Studying the relevant Tor spectral sequences gives the result. $\square$

Hopkins and Smith are able to use their local nilpotence detection result, Corollary 3.5.10, to completely understand the behavior not only of the thick subcategories C_d but of *all* thick subcategories of $\mathsf{Spectra}^{\mathrm{fin}}_{(p)}$. In particular, this connects the C_d with the D_d, as we will see.

Theorem 3.5.17 ([HS98, Theorem 7])**.** *Any thick subcategory* C *of the category of p–local finite spectra must be* C_d *for some d.*

Proof. Since C_d are nested by Lemma 3.5.16 and they form an exhaustive filtration (i.e., C_∞ = 0), it is thus sufficient to show that any object $X \in \mathsf{C}$ with $X \in \mathsf{C}_d$ induces an inclusion $\mathsf{C}_d \subseteq \mathsf{C}$. Write R for the endomorphism ring spectrum $R = F(X, X)$, and write F for the fiber of its unit map:

$$F \xrightarrow{f} \mathbb{S} \xrightarrow{\eta_R} R.$$

Finally, let $Y \in \mathsf{C}_d$ be *any* finite spectrum of type at least d. Our goal is to demonstrate $Y \in \mathsf{C}$.

Now consider applying $K(n)$–homology (for *arbitrary* n) to the map

$$1 \wedge f \colon\ Y \wedge F \to Y \wedge \mathbb{S}.$$

The induced map is always zero:

- In the case that $K(n)_*X$ is nonzero, then $K(n)_*\eta_R$ is injective because $K(n)_*$ is a graded field, and so $K(n)_*f$ is zero.
- In the case that $K(n)_*X$ is zero, then $n \leq d$ and, because of the bound on type, $K(n)_*Y$ is zero as well.

By a small variant of local nilpotence detection (Corollary 3.5.10, [HS98, Corollary 2.5]), it follows for $j \gg 0$ that

$$Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}$$

is null-homotopic. Hence, one can calculate the cofiber to be

$$\operatorname{cofib}\left(Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}\right) \simeq Y \wedge \operatorname{cofib} f^{\wedge j} \simeq Y \vee (Y \wedge \Sigma F^{\wedge j}),$$

so that Y is a retract of this cofiber.

We now work to show that this smash product lies in the thick subcategory C of interest. First, note that it suffices to show that $\operatorname{cofib} f^{\wedge j}$ on its own lies in C: a finite spectrum (such as Y or F) can be expressed as a finite gluing diagram of spheres, and smashing this through with $\operatorname{cofib} f^{\wedge j}$ then expresses $Y \wedge \operatorname{cofib} f^{\wedge j}$ as the iterated cofiber of maps with source and target in C. With that in mind, we will in fact show that $\operatorname{cofib} f^{\wedge k}$ lies in C for all $k \geq 1$. Consider the following smash version of the octahedral axiom: the factorization

$$F \wedge F^{\wedge(k-1)} \xrightarrow{1 \wedge f^{\wedge(k-1)}} F \wedge \mathbb{S}^{\wedge(k-1)} \xrightarrow{f \wedge 1} \mathbb{S} \wedge \mathbb{S}^{\wedge(k-1)}$$

begets a cofiber sequence

$$F \wedge \operatorname{cofib} f^{\wedge(k-1)} \to \operatorname{cofib} f^{\wedge k} \to \operatorname{cofib} f \wedge \mathbb{S}^{\wedge(k-1)}.$$

Noting that the base case $\operatorname{cofib}(f) = R = X \wedge DX$ lies in C, we can inductively use the 2-out-of-3 property on the octahedral cofiber sequence to see that $\operatorname{cofib}(f^{\wedge k})$ lies in C for all k. It follows in particular that $Y \wedge \operatorname{cofib}(f^{\wedge j})$ lies in C, and using the retraction Y belongs to C as well. □

Theorem 3.5.18 ([HS98, Theorem 9])**.** *A p–local finite spectrum is $K(d-1)$–acyclic exactly when it admits a v_d–self–map. Additionally, the inclusion $\mathsf{C}_d \subsetneq \mathsf{C}_{d-1}$ is proper.*

Executive summary of proof. Given the classification of thick subcategories, if a property is closed under thickness then one need only exhibit a single spectrum with the property to know that all the spectra in the thick subcategory it generates also all have that property. Inductively, they manually construct finite spectra $M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}})$ for sufficiently large[23] indices i_* which admit a self-map v governed by a commuting square

$$\begin{array}{ccc} BP_* M_{|v_d| i_d}(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{v} & BP_* M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) \\ \| & & \| \\ \Sigma^{|v_d| i_d} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{- \cdot v_d^{i_d}} & BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}). \end{array}$$

These maps are guaranteed by very careful study of Adams spectral sequences. □

This spectrum you've described is the one from section 5 of Nilpotence II paper as an example of a spectrum with few self maps, right? Hopkins and Smith construct a (possibly?) different spectrum X_n with v_n self maps in section 4 – or at least assert their existence anyway (theorem 4.11). I don't know whether it is easy to see that the generalized Moore spectra you've described have v_n self maps without already knowing the result in the statement of this theorem.

They therefore conclude the strongest possible positive response to Ravenel's conjectures. Not only can we continue the sequence

$$\mathbb{S},\ \mathbb{S}/p,\ \mathbb{S}/(p, v),\ \ldots,$$

[23] We ran into the asymptotic condition $I \gg 0$ earlier, when we asserted that there is no root of the 2–local v_1–self–map $v \colon M_8(2) \to M_0(2)$.

but in fact *any* finite spectrum admits an (essentially unique) interesting periodic self-map. This is maybe the most remarkable of the statements: although Nishida's theorem initially led us to think of periodic self-maps as rare, they are in fact ubiquitous. Additionally, we learned that the shift[24] of this self-map is determined by the first nonvanishing $K(d)$–homology, giving an effective detection tool. Finally, all such periodicity shifts arise: for any d, there is a spectrum admitting a v_d–self–map but not a v_{d-1}–self–map.

In either this section, near the definition of chromatic homology theories, or in the next section, where the acyclicity of $K(d) \wedge K(d')$ is stated, we should discuss $H\mathbb{F}_{p*}BP$ (perhaps in terms of $H\mathbb{F}_{p*}MU$ but certainly in terms of a quotient algebra of the dual Steenrod algebra), as well as the utility of the Adams spectral sequence in computing extraordinary homology theories from ordinary ones, and the role that Bocksteins play in those calculations (and, in particular, where all the τ_j operators went in the deformation from $H\mathbb{F}_p$ to BP).

3.6 Chromatic dissembly

In this Lecture, we will couple the ideas of Lecture 3.1 to the homology theories and structure theorems described in Lecture 3.5. In particular, we have not yet exhausted Theorem 3.5.17, and for inspiration about how to utilize it, we will begin with an algebraic analogue of the situation considered thus far.

For a ring R, the full derived category $D(\operatorname{Spec} R)$ and the derived category of perfect complexes $D^{\mathrm{perf}}(\operatorname{Spec} R)$ form examples of triangulated categories analogous to Spectra and Spectra$^{\mathrm{fin}}$. By interpreting an R–module as a quasicoherent sheaf over $\operatorname{Spec} R$, we can use them to probe for structure of $\operatorname{Spec} R$—for instance, we can test whether $\widetilde{M}$ is supported over some closed subscheme $\operatorname{Spec} R/I$ by restricting the sheaf, which amounts algebraically to asking whether M is annihilated by I. In the reverse, we can also try to discern what "closed subscheme" should mean in some arbitrary triangulated category by codifying the properties of the subcategory of $D(\operatorname{Spec} R)$ supported away from $\operatorname{Spec} R$. The key observation is this subcategory is closed under tensoring modules: if M is annihilated by I, then $M \otimes_R N$ is also annihilated by I.

Definition 3.6.1 ([Bal10, Definition 1.3])**.** Let C be a triangulated $\otimes$–category C. A thick subcategory $\mathsf{C}' \subseteq \mathsf{C}$ is...

- ...a $\otimes$*–ideal* when $x \in \mathsf{C}'$ forces $x \otimes y \in \mathsf{C}'$ for any $y \in \mathsf{C}$.
- ...a *prime* $\otimes$*–ideal* when $x \otimes y \in \mathsf{C}'$ also forces at least one of $x \in \mathsf{C}'$ or $y \in \mathsf{C}'$.

Finally, define the *spectrum* of C to be its collection of prime $\otimes$–ideals. For any $x \in \mathsf{C}$ we define a basic open $U(x) = \{\mathsf{C}' \mid x \in \mathsf{C}'\}$, which altogether give a basis for a topology on the spectrum.

Double check that you have the directionality of this right. Is U a basic open or a basic closed? It it full of things that contain x or that don't contain in x?

The basic result about this definition is that it does not miss any further conditions:

Theorem 3.6.2 ([Bal10, Proposition 8.1])**.** *The spectrum of $D^{\mathrm{perf}}(\operatorname{Spec} R)$ is naturally homeomorphic to the Zariski spectrum of R.* $\square$

[24]This is sometimes referred to as the "wavelength" in the chromatic analogy.

Satisfied, we apply the definition to the more difficult case of Spectra.

Theorem 3.6.3 ([Bal10, Corollary 9.5])**.** *The spectrum of* $\mathsf{Spectra}^{\mathrm{fin}}_{(p)}$ *consists of the thick subcategories* C_d*, and* $\{\mathsf{C}_n\}_{n=0}^{d}$ *are its open sets.*

Proof. Using Theorem 3.5.18, we can characterize C_d as the kernel of $K(d-1)_*$. This shows it to be a prime $\otimes$–ideal:

$$K(d-1)_*(X \wedge Y) \cong K(d-1)_*X \otimes_{K(d-1)_*} K(d-1)_*Y$$

is zero exactly when at least one of X and Y is $K(d-1)$–acyclic. □

Corollary 3.6.4 (cf. Theorem 3.5.17 and Theorem 3.5.18)**.** *The functor*

$$\mathcal{M}_{MU}(-)\colon \mathsf{Spectra}^{\mathrm{fin}} \to \mathsf{Coh}(\mathcal{M}_{MU})$$

induces[25] *a homeomorphism of the spectrum of* $\mathsf{Spectra}^{\mathrm{fin}}$ *to that of* $\mathcal{M}_{\mathbf{fg}}$*.* □

The construction as we have described it falls short of completely recovering $\operatorname{Spec} R$, as we have constructed only a topological space rather than a locally ringed space (or anything otherwise equipped locally with algebraic data, as in our functor of points perspective). The approach taken by Balmer [Bal10, Section 6] is to use Tannakian reconstruction to extract a structure sheaf of local rings from the prime $\otimes$–ideal subcategories. We, however, are at least as interested in finite spectra as we are the ring spectrum $\mathbb{S}$, so we will take an approach that emphasizes module categories rather than local rings. Specifically, Bousfield's theory of homological localization allows us to lift the localization structure among open substacks of $\mathcal{M}_{\mathbf{fg}}$ to the category Spectra as follows:

Theorem 3.6.5 ([Bou79], [Mar83, Theorem 7.7])**.** *Let* $j\colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$ *be a flat map, and let* R_* *denote the homology theory associated to it by Theorem 3.0.1. There is then a diagram*

$$\begin{tikzcd}
\mathsf{Spectra}_R \arrow[rr, "\mathcal{M}_R(-)", "\text{conservative}"'] \arrow[d, shift left, "i"] & & \mathsf{QCoh}(\mathcal{M}_R) \arrow[d, shift left, "j_*"] \\
\mathsf{Spectra} \arrow[u, shift left, "L_R"] \arrow[urr, "\mathcal{M}_R(-)"] \arrow[rr, "\mathcal{M}_{MU}(-)"] & & \mathsf{QCoh}(\mathcal{M}_{MU}), \arrow[u, shift left, "j^*"]
\end{tikzcd}$$

such that L_R *is left-adjoint to* i*,* j^* *is left-adjoint to* j_**,* i *and* j_* *are inclusions of full subcategories,* L_R *and* j^* *are idempotent, the red composites are all equal, and* R_* *is conservative on* $\mathsf{Spectra}_R$*.*[26] □

[25] This has to be interpreted delicately, as the functor $\mathcal{M}_{MU}(-)$ is not (quite) a functor of triangulated categories [Mor07, 2.4.2].

[26] The meat of this theorem is in overcoming set-theoretic difficulties in the construction of $\mathsf{Spectra}_R$. Bousfield accomplished this by describing a model structure on Spectra for which R–equivalences create the weak–equivalences.

The idea, then, is that $\mathsf{Spectra}_R$ plays the topological role of the derived category of those sheaves supported on the image of the map j. In Definition 3.5.2, we identified several classes of interesting such maps j tied to the geometry of $\mathcal{M}_{\mathbf{fg}}$. We record these special cases now:

Definition 3.6.6. In the case that $R = E_\Gamma$ models the inclusion of the deformation space around the point Γ, we will denote the localizer by L_Γ. In the special case that $\Gamma = \Gamma_d$ is taken to be the Honda formal group, we further abbreviate the localizer by

$$\mathsf{Spectra} \xrightarrow{\widehat{L}_d} \mathsf{Spectra}_{\Gamma_d}.$$

In the case when $R = E(d)$ models the inclusion of the open complement of the unique closed substack of codimension d, we will denote the localizer by

$$\mathsf{Spectra} \xrightarrow{L_d} \mathsf{Spectra}_d = \mathsf{Spectra}_{\mathcal{M}_{\mathbf{fg}}^{\leq d}}.$$

These localizers have a number of nice properties linking them to algebraic models.

Lemma 3.6.7. *There are natural factorizations*

$$\mathrm{id} \to L_d \to L_{d-1}, \qquad \mathrm{id} \to L_d \to \widehat{L}_d.$$

In particular, $L_d X = 0$ implies both $L_{d-1}X = 0$ and $\widehat{L}_d X = 0$.

Analogy to $j_* \vdash j^*$. The open substack of dimension d properly contains both the open substack of dimension $(d-1)$ and the infinitesimal deformation neighborhood of the geometric point of height d. The factorization is inclusions gives a factorization of pullback functors. $\square$

Lemma 3.6.8 ([Rav92, Theorem 7.5.6], [Hov95, Proof of Lemma 2.3])**.** *There are equivalences*

$$L_d X \simeq (L_d \mathbb{S}) \wedge X, \qquad \widehat{L}_d X \simeq \lim_I \left(M_0(v^I) \wedge L_d X \right).$$

Analogy to $j_* \vdash j^*$. The first formula stems from j an open inclusion, which has $j^*M \simeq R \otimes M$ in the algebraic setting. The second formula can be compared to the inclusion j of the formal infinitesimal neighborhood of a closed subscheme, which has algebraic model $j^*M = \lim_j (R/I^j \otimes M)$.[27] $\square$

Lemma 3.6.9. *Let k be a field of positive characteristic p, and let Γ and Γ' be two formal groups over k of differing heights $0 \leq d, d', \leq \infty$. Then $K_\Gamma \wedge K_{\Gamma'} \simeq 0$.*

[27] In keeping with our discussion of continuous Morava E–theory, it is also possible to consider the object $\{(M_0(v^I) \wedge L_d X)\}_I$ itself as a pro-spectrum. This is an interesting thing to do: Davis and Lawson have shown that setting $X = \mathbb{S}$ gives an E_∞ pro-spectrum, even though none of the individual objects are E_∞ ring spectra themselves [DL14].

Analogy to $j_* \vdash j^*$. The map classifying the formal group $\mathbb{C}P^\infty_{K_\Gamma \wedge K_{\Gamma'}}$ simultaneously factors through the maps classifying the formal groups $\mathbb{C}P^\infty_{K_\Gamma} = \Gamma$ and $\mathbb{C}P^\infty_{K_{\Gamma'}} = \Gamma'$. By Lemma 3.3.4, such a formal group must simultaneously have heights d and d', which forces the homotopy ring to be the zero ring.[28] □

Lemma 3.6.10 ([Lura, Lemma 23.6]). *For $d > \operatorname{ht}\Gamma$, $\widehat{L}_\Gamma L_d \simeq 0$.*

Proof sketch. After a nontrivial reduction argument, this comes down to an identical fact: the formal group associated to $E(d) \wedge K_\Gamma$ must simultaneously be of heights at most d and exactly $\operatorname{ht}\Gamma > d$, which forces the spectrum to vanish. □

Corollary 3.6.11. *$L_\Gamma E = 0$ for any coconnective E, and hence $L_\Gamma E = L_\Gamma(E[n, \infty))$ for* any *spectrum E and* any *index n.*[29]

Proof. Any coconnective spectrum can be expressed as the colimit of its truncations

$$\begin{tikzcd}
E[n,n] \arrow[r] \arrow[d, equal] & E[n-1,n] \arrow[r] \arrow[d] & E[n-2,n] \arrow[r] \arrow[d] & \cdots \arrow[r, "\operatorname{colim}"] & E(-\infty,n] \\
\Sigma^n H\pi_n E & \Sigma^{n-1} H\pi_{n-1} E & \Sigma^{n-2} H\pi_{n-2} E & \cdots .
\end{tikzcd}$$

Applying L_Γ preserves this colimit diagram, but the above argument shows that HA is L_Γ–acyclic for any abelian group A. This gives the statement about coconnective spectra, from which the general statement follows by considering the cofiber sequence

$$E[n, \infty) \to E \to E(-\infty, n).$$ □

Corollary 3.6.12 ([Lura, Proposition 23.5]). *There are a homotopy pullback squares*

$$\begin{tikzcd}
L_d X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{L}_d X \arrow[d] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p X_p^\wedge \arrow[d] \\
L_{d-1} X \arrow[r] & L_{d-1}\widehat{L}_d X, & X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} .
\end{tikzcd}$$

Analogy to $j_* \vdash j^*$. For the left-hand square, the inclusion of the open substack of dimension $d-1$ into the one of dimension d has relatively closed complement the point of height d. Algebraically, this gives a Mayer-Vietoris sequence with analogous terms. The right-hand square is analogous to the adèlic decomposition of abelian groups.[30] □

[28] Alternatively, Corollary 3.5.12 shows that $K_\Gamma \wedge K_{\Gamma'}$ simultaneously decomposes as a wedge of K_Γs and of $K_{\Gamma'}$, which forces both wedges to be empty.

[29] This property has the memorable slogan that Morava K–theories remember the "germ at ∞" of E.

[30] Whenever $L_B L_A = 0$, $L_{A \vee B}$ appears as the homotopy pullback of the cospan $L_A \to L_A L_B \leftarrow L_B$. Hence, this follows from Lemma 3.6.10, as well as the identification $L_{E(d-1) \vee K(d)} \simeq L_{E(d)}$.

Remark 3.6.13. Corollary 3.6.12 is maybe the most useful result discussed in this Lecture. It shows that a map to an L_d–local spectrum can be understood as a system of compatible maps to its $\widehat{L}_j$–localizations, $j \le d$. In turn, any map into an $\widehat{L}_j$–local object factors through the $\widehat{L}_j$–localization of the source. Thus, if the source itself has chromatic properties, this often puts *very* strong restrictions on how maps to the original target can behave.

These functors and their properties listed thus far give a tight analogy between certain local categories of spectra and sheaves supported on particular submoduli of formal groups, in a way that lifts the six-functors formalism of $j_* \vdash j^*$ to the level of spectra. With this analogy in hand, however, one is led to ask considerably more complicated questions whose proofs are not at all straightforward. For instance, a useful fact about coherent sheaves on $\mathcal{M}_{\mathbf{fg}}$ is that they are completely determined by their restrictions to all of the open submoduli. The analogous fact about finite spectra is referred to as *chromatic convergence*:

Theorem 3.6.14 ([Rav92, Theorem 7.5.7])**.** *The homotopy limit of the tower*

$$\cdots \to L_d F \to L_{d-1} F \to \cdots \to L_1 F \to L_0 F$$

recovers the p–local homotopy type of any finite spectrum F.[31] □

In addition to furthering the analogy, Theorem 3.6.14 suggests a method for analyzing the homotopy groups of spheres: we could study the homotopy groups of each $L_d\mathbb{S}$ and perform the reassembly process encoded by this inverse limit. Additionally, Corollary 3.6.12 shows that this process is inductive: $L_d\mathbb{S}$ can be understood in terms of the spectrum $L_{d-1}\mathbb{S}$, the spectrum $\widehat{L}_d\mathbb{S}$, and some gluing data in the form of $L_{d-1}\widehat{L}_d\mathbb{S}$. Hence, we become interested in the homotopy of $\widehat{L}_d\mathbb{S}$, which is the target of the E_d–Adams spectral sequence considered in Lecture 3.1.

Theorem 3.6.15 (Lemma 3.1.16, see also Example 2.3.4, Definition 3.1.9, and Definition 3.1.14)**.** *The E_d–based*[32] *Adams spectral sequence for the sphere converges strongly to $\pi_*\widehat{L}_d\mathbb{S}$. Writing ω for the line bundle on $\mathcal{M}_{E_d}$ of invariant differentials, we have*

$$E_2^{*,*} = H^*(\mathcal{M}_{E_d}; \omega^{\otimes *}) \Rightarrow \pi_*\widehat{L}_d\mathbb{S}. \quad \square$$

[31] Spectra satisfying this limit property are said to be *chromatically complete*, which is closely related to being *harmonic*, i.e., being local with respect to $\bigvee_{d=0}^{\infty} K(d)$. (I believe this a joke about "music of the spheres".) It is known that nice Thom spectra are harmonic [Kří94] (so, in particular, every suspension and finite spectrum), that every finite spectrum is chromatically complete, and that there exist some harmonic spectra which are not chromatically complete [Bar, Section 5.1].

[32] Although the $K(d)$–Adams spectral sequence more obviously targets $\widehat{L}_d\mathbb{S}$, we have chosen to analyze the E_d–Adams spectral sequence above because $K(d)$ fails to satisfy **CH**. Starting with $BPP_0BPP \cong BPP_0[t_0^{\pm}, t_1, t_2, \ldots]$ from Definition 3.5.2 and Corollary 3.3.15, we can calculate $E(d)P_0E(d)P$ by base-changing this Hopf algebroid: $E(d)P_0E(d)P = E(d)P_0 \otimes_{BPP_0} BPP_0BPP \otimes_{BPP_0} E(d)P_0$, which is again free over $E(d)P_0$. Since $K(d)P$ is formed from $E(d)P$ by quotienting by a regular sequence, we calculate that $K(d)P_0E(d)P$ is free over $K(d)P_0$, generated by the same summands. However, when quotienting by the regular sequence *again* to form $K(d)P_*K(d)P$, the maps in the quotient sequences act by elements in $I_d = 0$, hence introduce

The utility of this Theorem is in the identification of the stack $\mathcal{M}_{E_d} \cong (\mathcal{M}_{\mathbf{fg}})^\wedge_{\Gamma_d}$ from Definition 3.5.2. Our algebraic analysis from Theorem 3.4.5 and Remark 3.4.7 shows a further identification

$$\mathcal{M}_{E_{\Gamma_d}} = (\mathcal{M}_{\mathbf{fg}})^\wedge_{\Gamma_d} \simeq \widehat{\mathbb{A}}^{d-1}_{\mathbb{W}(k)} // \underline{\mathrm{Aut}}(\Gamma_d).$$

This computation is thus boiled down to a calculation of the cohomology of the $\mathrm{Aut}(\Gamma_d)$–representations arising via Remark 3.4.13 as the global sections of the sheaves $\omega^{\otimes *}$ (cf. the discussion in Example 1.4.11 and Example 1.4.18).[33] We will later deduce the following polite description of $\mathrm{Aut}\,\Gamma_d$:

Theorem 3.6.16 (cf. Example 4.4.13)**.** *For Γ_d the Honda formal group law of height d over a perfect field k of positive characteristic p, we compute*

$$\mathrm{Aut}\,\Gamma_d \cong \left(\mathbb{W}_p(k)\langle S\rangle \middle/ \left(\begin{array}{c} Sw = w^\varphi S, \\ S^d = p \end{array} \right)\right)^\times,$$

where φ denotes a lift of the Frobenius from k to $\mathbb{W}_p(k)$. □

Remark 3.6.17. As a matter of emphasis, this Theorem does not give any description of the *representation* of $\mathrm{Aut}\,\Gamma_d$, which is very complicated (cf. Appendix A.3). Nonetheless, the arithmetically-minded reader might recognize this description of $\mathrm{Aut}\,\Gamma_d$ as the group of units of a maximal order $\mathbf{o}_D$ in the division algebra D of Brauer–Hasse invariant $1/d$ over k—another glimpse of arithmetic geometry poking through to affect stable homotopy theory.[34]

Example 3.6.18 (Adams). In the case $d = 1$, the objects involved are small enough that we can compute them by hand. To begin, we have an isomorphism $\mathrm{Aut}(\Gamma_1) = \mathbb{Z}_p^\times$, and the action of this group on $\pi_* E_1 = \mathbb{Z}_p[u^\pm]$ is by $\gamma \cdot u^n \mapsto \gamma^n u^n$. At odd primes p, one computes[35]

Cite me: Find a citation.

$$H^s(\mathrm{Aut}(\Gamma_1); \pi_* E_1) = \begin{cases} \mathbb{Z}_p & \text{when } s = 0, \\ \bigoplus_{j=2(p-1)k} \mathbb{Z}_p\{u^j\}/(pku^j) & \text{when } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Bocksteins. The end result is

$$K(d)P_*K(d)P = (K(d)P_* \otimes_{BPP_*} BPP_*BPP \otimes_{BPP_*} K(d)P_*) \otimes \Lambda[\tau_0, \ldots, \tau_{d-1}],$$

where τ_j in degree 1 controls the cofiber of $E(d)P \xrightarrow{v_j} E(d)P$.

[33] In fact, the stable *operations* of E_d take the form of the twisted group-ring $E_d^0 E_d = E_d^0\langle\langle \mathrm{Aut}(\Gamma_d)\rangle\rangle$.

[34] This finally explains our preference for using the letter "d" to represent the height of a formal group—the "d" (or, rather the "D") stands for "division algebra".

[35] At odd primes, p is coprime to the order of the torsion part of $\mathbb{Z}_p^\times$. At $p = 2$, this is not true, so the representation has infinite cohomological dimension and there is plenty of room for differentials in the ensuing $E_{\widehat{\mathbb{G}}_m}$–Adams spectral sequence..

This, in turn, gives the calculation[36]

$$\pi_t \widehat{L}_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}_p & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = k|v_1| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

With this in hand, we can compute the homotopy of the rest of the fracture square:

$$\begin{tikzcd} \pi_* L_1 \mathbb{S} \arrow[r] \arrow[d] & \mathbb{Z}_p \oplus \bigoplus_{t=k|v_1|-1} \Sigma^t \mathbb{Z}_p/(pk) \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q}_p \oplus \Sigma^{-1}\mathbb{Q}_p, \end{tikzcd}$$

from which we deduce

$$\pi_t L_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}_{(p)} & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = k|v_1| - 1 \text{ and } t \neq 0, \\ \mathbb{Z}/p^\infty & \text{when } t = (0 \cdot |v_1| - 1) - 1 = -2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.6.19 ([Rezc, Example 7.18]). We can also give an explicit chromatic analysis of the homotopy element $\eta \in \pi_1 \mathbb{S}$ studied in Lecture 1.4. As before, consider the complex $\mathbb{C}\mathrm{P}^2 = \Sigma^2 C(\eta)$. We now consider the possibility that $\mathbb{C}\mathrm{P}^2$ splits as $\mathbb{S}^2 \vee \mathbb{S}^4$, in which case there would be a dotted retraction in the cofiber sequence

$$\begin{tikzcd} \mathbb{S}^2 \arrow[r] & \mathbb{C}\mathrm{P}^2 \arrow[r] \arrow[l, bend right, dotted, "i"'] & \mathbb{S}^4. \end{tikzcd}$$

If this were possible, we would also be able to detect the retraction after chromatic localization—so, for instance, we could consider the cohomology theory $E_{\widehat{\mathbb{G}}_m} = KU_p^\wedge$ from Example 3.5.4 and test this hypothesis in $\widehat{\mathbb{G}}_m$–local homotopy. Writing t for a coordinate on $\mathbb{C}\mathrm{P}^\infty_{KU_p^\wedge}$, this cofiber sequence gives a short exact sequence on $KU_p^\wedge$–cohomology:

$$\begin{tikzcd} 0 & (t)/(t)^2 \arrow[l] \arrow[r, bend left, dotted, "i^*"] & (t)/(t)^3 \arrow[l] & (t)^2/(t)^3 \arrow[l] & 0. \arrow[l] \end{tikzcd}$$

Because i is taken to be a retraction, the map i^* would satisfy $i^*(t) = t \pmod{t^2}$, so that $i^*(t) = t + at^2$ for some a. Additionally, i^* would be natural with respect to all cohomology

[36]The groups $\pi_* \widehat{L}_1 \mathbb{S}$ are familiar to homotopy theorists: the Adams conjecture [Ada66] (and its solution) implies that the J–homomorphism $J_{\mathbb{C}} \colon BU \to BGL_1\mathbb{S}$ described in Corollary 1.1.6 and Theorem 2.0.1 selects exactly these elements for nonnegative t.

operations on $KU_p^\wedge$. In particular, the element $(-1) \in \mathbb{Z}_p^\times \cong \operatorname{Aut}\widehat{\mathbb{G}}_m$ gives rise to an operation ψ^{-1}, which acts by the (-1)–series on the coordinate t. In the case that t is the coordinate considered in Example 2.1.21, this gives

$$[-1](t) = -\sum_{j=1}^{\infty} t^j = -t - t^2 \pmod{t^3}.$$

We thus compute:

$$\begin{aligned} \psi^{-1} \circ i(t) &= i \circ \psi^{-1}(t) \\ \psi^{-1}(t + at^2) &= i(-t) \\ (-t - t^2) + a(-t - t^2)^2 &= -(t + at^2) \\ -t + (a-1)t^2 &= -t - at^2, \end{aligned}$$

so that we would arrive at a contradiction if the equation $2a = 1$ were insoluable. Note that this has no solution in $\mathbb{Z}_2$, so that the attaching map η in $\mathbb{C}\mathrm{P}^2$ is nontrivial in $\widehat{\mathbb{G}}_m$–local homotopy at the prime 2 (hence also in the global homotopy group $\pi_1\mathbb{S}$). For p odd, this equation *does* have a solution in $\mathbb{Z}_p$, and it furthermore turns out that $\eta = 0$ at odd primes. This problem also disappears if we require $i(t) = 2t + at^2$ instead, so that the above argument does not obstruct the triviality of 2η (and, indeed, Figure 1.2 shows that the relation $2\eta = 0$ holds in 2–adic homotopy).

Example 3.6.20 ([Rezc, Example 7.17 and Corollary 5.12]). Take k to be a perfect field of positive characteristic p, and take Γ over $\operatorname{Spec} k$ to be a finite height formal group with associated Morava E–theory E_Γ. By smashing the unit map $\mathbb{S} \to E_\Gamma$ with the mod–p Moore spectrum, we get an induced map of homotopy groups

$$h_{2n} \colon \pi_{2n} M_0(p) \to \pi_{2n} E_\Gamma.$$

We concluded as a consequence of Corollary 3.3.15 that there is an invariant section v_1 of $\omega^{\otimes(p-1)}$ on $\mathcal{M}_{\mathbf{fg}}^{\geq 1} \to \widehat{\mathbb{A}}^1$, and hence a preferred element of $\pi_{2(p-1)} E_\Gamma$ which is natural in the choice of Γ. One might hope that these elements are the image of an element in $\pi_{2(p-1)} M_0(p)$ under the Hurewicz map h, and this turns out to be true: this element is called $\alpha_{1/1}$. This element furthermore turns out to be p–torsion, meaning it extends to a map

$$\begin{tikzcd} \mathbb{S}^{2(p-1)} \arrow[r, "p"] \arrow[dr, "0"'] & \mathbb{S}^{2(p-1)} \arrow[r, "\text{cofib}"] \arrow[d, "\alpha_{1/1}"] & M_{2(p-1)}(p) \arrow[dl, dotted, "v"] \\ & M_0(p). & \end{tikzcd}$$

At odd primes, this turns out to be the v_1–self-map $v\colon M_{2(p-1)}(p) \to M_0(p)$ announced in Theorem 3.5.7.

More generally, different powers v_1^j of the section v_1 also give rise to homotopy elements $\alpha_{j/1} \in \pi_{2(p-1)j} M_0(p)$. These have varying orders of divisibility, and we write $\alpha_{j/k}$ for the element satisfying $p^{k-1}\alpha_{j/k} = \alpha_{j/1}$. Compositionally, these maps satisfy the useful relation $\alpha^p_{p^{j-1}/j-1} = \alpha_{p^j/j}$. The other invariant functions described in Corollary 3.3.15 (e.g., v_d modulo I_d) also give rise to elements in $H^*(\mathcal{M}_{\mathbf{fg}}^{\geq d}; \omega^{\otimes *})$, which map to the BP–Adams E_2–term and which sometimes survive the spectral sequence to give to homotopy elements of the generalized Moore spectra $M_0(v^I)$. Homotopy elements arising in this way are collectively referred to as *Greek letter families* [MRW77, Section 3].

Is this relation right?

Remark 3.6.21. In the broader literature, the phrase "Greek letter elements" typically refers to the pushforward of the above elements to the homotopy groups of $\mathbb{S}$ by pinching to the top cell. This is somewhat obscuring: for instance, this significantly entangles how multiplication by $\alpha_{j/k}$ behaves. Finally, the incarnation of these element in $\widehat{\mathbb{G}}_m$–local homotopy are exactly the elements witnessed by the invariant function $u^{2(p-1)k}$ in Example 3.6.18.

Another fun fact is that because p–local finite spectra F are $\bigvee_{d<\infty} K(d)$–local, there are no nontrivial maps $H\mathbb{F}_p \to F$ (and, in particular, the Spanier–Whitehead dual of $H\mathbb{F}_p$ is null). (In fact, the Hovey paper *Bousfield Localization Functors and Hopkins' Chromatic Splitting Conjecture* shows that they are local for any properly infinite collection of Morava K–theories.)

Danny was asking some interesting questions about the relationships between the t_i and the h_j elements of the Adams spectral sequence. You could try to explain some of this comparison and how you get the Hopf invariant one theorem out of it.

Danny was also asking about the role of the $T(j)$ spectra, which are pretty cool. They (and their utility as described in *From Spectra to Stacks*) might live in a Remark somewhere.

Case Study 4

Unstable cooperations

In Lecture 3.1 (and more broadly in Case Study 3), we codified the structure of the stable E–cooperations acting on the E–homology of a spectrum X, attached to it the E–Adams spectral sequence which approximates the stable homotopy groups $\pi_* X$, and gave algebro-geometric descriptions of the stable cooperations for some typical spectra: $H\mathbb{F}_2$, MO, and MU. We will now pursue a variation on this theme, where we consider the E–homology of a *space* rather than of a generic spectrum. In this Case Study, we will examine the theory of cooperations that arises from this set-up, called the *unstable E–cooperations*. This broader collection of cooperations has considerably more intricate structure than their stable counterparts, requiring the introduction of a new notion of an unstable context. With that established, we will again find that E–homology assigns spaces to quasicoherent Cartesian sheaves over the unstable context, and we will again assemble an *unstable E–Adams* spectral sequence approximating the *unstable* homotopy groups of the input space, whose E^2–page in favorable situations is tracked by the cohomology of the sheaf over the unstable context.

Remarkably, these unstable contexts also admit algebro-geometric interpretations. In finding the right language for this, we introduce different subclasses of cooperations (e.g., *additive*), and we are also naturally led to consider *mixed cooperations* (as we did stably in Lemma 2.6.8) of the form $F_* \underline{E}_*$. The running theme is that when E and F are complex-orientable, there is a natural approximation map

$$\operatorname{Spec} Q^* F_* \underline{E}_* \to \underline{\mathsf{FormalGroups}}(\mathbb{C}\mathrm{P}^\infty_F, \mathbb{C}\mathrm{P}^\infty_E)$$

which is an isomorphism in every situation of interxest. However, these isomorphisms do not appear to admit uniform proofs[1], so we instead investigate the following cases by hand:

You will probably have to re-reference this list.

- (Lecture 4.1:) For $F = E = H\mathbb{F}_2$, we compute the full unstable dual Steenrod algebra $H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_*$ by means of iterated bar spectral sequences. We then pass to the additive

[1] The best uniform result I can find is due to Butowiez and Turner [BT00, Theorem 3.12].

unstable cooperations, where we show by hand that this presents the homomorphism scheme $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)$. Finally, we pass to the stable additive cooperations, and we check that our results here are compatible with the isomorphism

$$\operatorname{Spec} H\mathbb{F}_{2*}H\mathbb{F}_2 \cong \underline{\operatorname{Aut}}\,\widehat{\mathbb{G}}_a$$

presented in Lemma 1.3.5.

- (Lecture 4.3:) We next consider the case where $E = MU$ and where F is any complex-orientable theory. We begin with the case $F = H\mathbb{F}_p$, where we can again approach the problem using iterated bar spectral sequences. The resulting computation is sufficiently nice that we can use this special case of $F = H\mathbb{F}_p$ to deduce the further case of $F = H\mathbb{Z}_{(p)}$, then $F = H\mathbb{Z}$, then $F = MU$, and then finally F any complex-orientable theory.

- (Lecture 4.5:) Having been able to vary F as widely as possible in the previous case, we then turn to trying to vary E. This is considerably harder, since the infinite loopspaces $\underline{E}_*$ associated to E are extremely complicated and vary wildly under even "small" changes in E. However, in the special case of $F = H\mathbb{F}_p$, we have an incredibly powerful trick available to us: Dieudonné theory, discussed in Lecture 4.4, gives an equivalence of categories

$$D_* \colon \mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/} \to \mathsf{GradedDMods},$$

which postcomposes with

$$\mathsf{Spectra} \xrightarrow{\Omega^\infty} \mathsf{Loopspaces} \xrightarrow{H\mathbb{F}_{p*}} \mathsf{HopfAlgebra}^{>0,\mathrm{fin}}_{\mathbb{F}_p/} \xrightarrow{D_*} \mathsf{GradedDMods} \subseteq \mathsf{Modules}_{\mathsf{Cart}}$$

to give a homological functor. This means that the Dieudonné module associated to an infinite loopspace varies stably with the spectrum underlying the loopspace, which is enough leverage to the settle the case where E is any Landweber–flat theory.

- (Lecture 4.6:) Finally, we settle one further case not covered by any of our generic hypotheses above: $F = K_\Gamma$ and $E = H\mathbb{Z}/p^j$. Neither K_Γ nor $H\mathbb{Z}/p^j$ is Landweber–flat, but because K_Γ is a field spectrum and because the additive group law associated to $H\mathbb{Z}/p^j$ is so simple, we can still perform the requisite iterated bar spectral sequence calculation by hand.

This last case is actually our real goal, as we are about to return to the project outlined in the Introduction. In the language of Theorem 0.0.5, choosing Γ to be the formal completion of an elliptic curve at the identity section presents the spectra K_Γ and E_Γ of Lecture 3.5 as the most basic examples of *elliptic spectra*. The goal of that Theorem is to study $E_*BU[6,\infty)$

for E an elliptic spectrum, so when proving it in Case Study 5 we will be led to consider the fiber sequences

$$BSU \to BU \to \underline{H\mathbb{Z}}_2, \qquad \underline{H\mathbb{Z}}_3 \to BU[6,\infty) \to BSU,$$

which mediate the difference between E_*BU and $E_*BU[6,\infty)$ by means of $E_*\underline{H\mathbb{Z}}_2$ and $E_*\underline{H\mathbb{Z}}_3$. Thus, in our pursuit of $K_{\Gamma *}BU[6,\infty)$, we will want to have $K_{\Gamma *}\underline{H\mathbb{Z}}_*$ already in hand, as well as an algebro-geometric interpretation of it.

4.1 Unstable contexts and the unstable Steenrod algebra

In this Lecture, our goal is to codify the study of unstable cooperations, beginning with an unstructured account of how they arise. Recall that for a ring map $f\colon R \to S$, in Lecture 3.1 we studied the problem of recovering an R–module from an S–module plus extra data. The intermediate category of extra data that we settled on was that of *descent data*, which we phrased most enduringly as a certain cosimplicial diagram. Stripping away the commutative algebra, the only categorical formality that went into this was the adjunction

We should spend a paragraph arguing about why we don't care about odd unstable cooperations—in either sense, as neither index in $E_*\underline{E}_*$ is allowed to be odd. This can be done in a few ways: for instance, if we only care about spaces with even-concentrated homology, then none of the odd cooperations act nontrivially, so we can discard them all. There's also the bar spectral sequence explanation (though this is a bit ad hoc): the odd information can be recovered from the even information, and it tends to be a large exterior algebra. There's also the Bockstein explanation.

$$\mathsf{Modules}_R \underset{\text{forget}}{\overset{-\otimes_R S}{\rightleftarrows}} \mathsf{Modules}_S,$$

or later on, when given a ring spectrum $\eta\colon \mathbb{S} \to E$, the adjunction

$$\mathsf{Spectra} = \mathsf{Modules}_{\mathbb{S}} \underset{\text{forget}}{\overset{-\wedge E}{\rightleftarrows}} \mathsf{Modules}_E.$$

The identification of $\mathsf{Modules}_S$ with T–algebras in $\mathsf{Modules}_R$ for the monad $T = \text{forget} \circ (- \otimes_R S)$ is the objective of *monadic descent*. This categorical recasting is ignorant of some of the algebraic geometry we discovered next, but it is suitable for us now as we consider the composition with a second adjunction:

Cite me: give a reference to Barr–Beck here.

$$\mathsf{Spaces} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathsf{Spectra} = \mathsf{Modules}_{\mathbb{S}} \underset{\text{forget}}{\overset{-\wedge E}{\rightleftarrows}} \mathsf{Modules}_E.$$

We will write $E(-)$ for the induced monad on Spaces, given by the formula

$$E(X) = \Omega^\infty(E \wedge \Sigma^\infty X) = \operatorname*{colim}_{j \to \infty} \Omega^j(\underline{E}_j \wedge X),$$

where $\underline{E}_*$ are the constituent spaces in the Ω–spectrum of E. This space has the property $\pi_*E(X) = \widetilde{E}_{*\geq 0}X$. The monadic structure comes from the two evident natural transformations:

$$\eta\colon X \to \Omega^\infty\Sigma^\infty X \simeq \Omega^\infty(\mathbb{S} \wedge \Sigma^\infty X) \to \Omega^\infty(E \wedge \Sigma^\infty X) = E(X),$$

$$\begin{aligned} \mu\colon\ E(E(X)) &= \Omega^\infty(E \wedge \Sigma^\infty \Omega^\infty(E \wedge \Sigma^\infty X)) \\ &\to \Omega^\infty(E \wedge E \wedge \Sigma^\infty X) \to \Omega^\infty(E \wedge \Sigma^\infty X) = E(X). \end{aligned}$$

Just as in the stable situation, we can extract from this a cosimplicial space:

Definition 4.1.1. The *unstable descent object* is the cosimplicial space

$$\mathcal{UD}_E(X) := \left\{ \begin{array}{ccccccc} & & & & & \longrightarrow & \\ & & & \longrightarrow & E & \longleftarrow & \\ & \xrightarrow{\eta_L} & E & \longleftarrow & \circ & \longrightarrow & \\ E & \xleftarrow{\mu} & \circ & \xrightarrow{\Delta} & E & \longleftarrow & \\ \circ & \xrightarrow{\eta_R} & E & \longleftarrow & \circ & \longrightarrow & \cdots \\ X & & \circ & \longrightarrow & E & \longleftarrow & \\ & & X & & \circ & \longrightarrow & \\ & & & & X & & \end{array} \right\}.$$

Its totalization gives the *unstable E–completion of X*.

The remainder of this section will be spent trying to understand the spectral sequence associated to the coskeletal filtration of such an unstable descent object. In the stable situation, we recognized that in favorite situations the homotopy groups of the descent object formed a cosimplicial module over a certain cosimplicial ring—or, equivalently, a sheaf over a certain simplicial scheme. Furthermore, we found that the simplicial scheme itself had some arithmetic meaning, and that the E_2–page of the descent spectral sequence computed the cohomology of this sheaf. We will find analogues of all of these results in the unstable setting, listed above in order from least to most difficult.

To begin, we would like to recognize the cosimplicial abelian group $\pi_*\mathcal{UD}_E(X)$ as a sort of comodule. In the stable case, this came from the smash product map $\mathbb{S}^0 \wedge X \to X$, as well as the lax monoidality of the functor $\mathcal{D}_E(-)$. However, to get a Segal condition by which we could identify the higher-dimensional objects in $\pi_*\mathcal{D}_E(X)$, we had to introduce the condition **FH**.[2] The unstable situation has an analogous antecedent:

Definition 4.1.2 ([BCM78, Assumption 6.5])**.** A ring spectrum E is said to satisfy the **U**nstable **F**latness **H**ypothesis, or **UFH**, if $E_*\underline{E}_k$ is a free E_*–module for every value of k.[3]

Under this condition, we again turn to studying the structure of $\mathcal{UD}_E(S^k)$ as k ranges. If we had a Segal condition, we would expect the structure present to be determined by

[2]In particular, **FH** caused the marked map in $E_*X \xrightarrow{\eta_R} E_*(E \wedge X) \xleftarrow{\star} E_*E \otimes_{E_*} E_*X$ to become invertible.

[3]This helps us understand the following analogous zigzag:

$$\pi_m E(X) \xrightarrow{\eta_R} \pi_m E(E(X)) \xleftarrow{\mu \circ 1} \pi_m E(E(E(X))) \xleftarrow{\text{compose}} \pi_m E(E(S^n)) \times \pi_n E(X).$$

$\pi_* \mathcal{UD}_E(S^*)[j]$ for $j \leq 2$. The data at $j = 0$ is largely redundant:

$$\pi_* E(S^k) = \pi_* \underline{E}_k = E_{*-k}$$

is a shifted copy of the coefficient ring of E for each k. The data at $j = 1$ consists of the homology groups of the spaces in the Ω–spectrum for E:

$$\pi_* E(E(S^k)) = E_* \underline{E}_k.$$

There are three pieces of structure present here: the augmentation map $E_* \underline{E}_k \to E_*(\Sigma^k E) \to E_{*-k}$, the left unit map $E_{*-k} \to E_{*-k}\underline{E}_0$, and the right unit map $E_{*-k} \to E_* \underline{E}_k$. The assumption **UFH** gives us a foothold on the case $j = 2$: a choice of basis for $E_* \underline{E}_k$ gives an unstable isomorphism $E(E(S^k)) \simeq \prod_\ell \underline{E}_{n_\ell}$, so that $\pi_* \mathcal{UD}_E(S^k)[2]$ splits as a tensor product of terms of the form $E_* \underline{E}_{n_\ell}$. Here, we find a lot of structure: the addition of cohomology classes induces a map

Get this right.

$$*\colon\ E_* \underline{E}_k \otimes_{E_*} E_* \underline{E}_k \to E_* \underline{E}_k;$$

the multiplication induces a map

$$\circ\colon\ E_* \underline{E}_k \otimes E_* \underline{E}_{k'} \to E_* \underline{E}_{k+k'};$$

these are compatible with the images of unit classes $0, 1 \in \pi_0 E$ under the left- and right-units specified above; there is an additive inverse map

$$\chi\colon\ E_* \underline{E}_k \to E_* \underline{E}_k$$

compatible with the $*$–product; there is a diagonal map

$$\Delta\colon\ E_* \underline{E}_k \to E_* \underline{E}_k \otimes_{E_*} E_* \underline{E}_k;$$

each $E_* \underline{E}_k$ becomes a Hopf algebra using χ, $*$, and Δ; and there is a distributivity condition pictured in Figure 4.1 intertwining $*$, $\circ$, and Δ.

Definition 4.1.3 ([BJW95, Summary 10.46])**.** A *Hopf ring* is a bigraded module equipped with the structure maps $+, -, \cdot, *, \circ, \Delta$, and χ subject to the axioms declared above. A Hopf ring becomes an *enriched Hopf ring* when it is furthermore equippped with a right-unit, and augmentation, and maps

$$E_* \underline{E}_k \times (E_j \underline{E}_k)^\vee \to E_* \underline{E}_j$$

(determined by composition with the dual cohomology classes in the examples of interest).

Make sure you have all of BJW's structure here: "These are Hopf rings plus $[v] \in E_0 \underline{E}_*$, the maps $r_*\colon E_* \underline{E}_j \to E_* \underline{E}_k$ for $r\colon E^k \underline{E}_j$ a cohomology operation, and the "augmentation" (cf. BJW 10.42) $E_* \underline{E}_k \to Q(E)^k_* \to E_* E \to E$." In particular, I do not understand this last piece, so I should make sure it's in here.)

Lemma 4.1.4. *If X is a space with $E_* X$ a free E_*–module, then $E_* X$ furthermore forms a coalgebra for the comonad G associated to the enriched Hopf ring $\pi_* \mathcal{UD}_E(S^k)$.*

I'm uncomfortable with all this graded stuff happening here. I'd like to be able to make statements in terms of periodic cohomology theories, and I'd like to use the stable suspension operation to make things degree 0 as needed. What I'd *really* like is to make the unstable Adams spectral sequence converge to a quasicoherent sheaf over $B\mathbb{G}_m$ (or

$$\begin{tikzcd}
A_{s,t} \otimes_{R_*} (A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[r, "1\otimes *"] \arrow[d, "\Delta\otimes(1\otimes 1)"] & A_{s,t} \otimes_{R_*} A_{s'+s'',t'} \arrow[dddd, "\circ"] \\
\left(\bigoplus_{s_1+s_2=s} A_{s_1,t} \otimes_{R_*} A_{s_2,t}\right) \otimes_{R_*} (A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[d, "\simeq"] & \\
\bigoplus_{s_1+s_2=s} (A_{s_1,t} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[d, "1\otimes\tau\otimes 1"] & \\
\bigoplus_{s_1+s_2=s} (A_{s_1,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s'',t'}) \arrow[d, "\circ\otimes\circ"] & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1+s',t+t'} \otimes_{R_*} A_{s_2+s'',t+t'}\right) \arrow[r, "*"] & A_{s+s'+s'',t+t'}.
\end{tikzcd}$$

Figure 4.1: The distributivity axiom for $*$ over $\circ$ in a Hopf ring.

Proof. The proof is a matter of elucidating the last condition. For X satisfying this freeness condition, there is again a splitting $E(X) \simeq \prod_\ell \underline{E}_{n_\ell}$. We interpret this at the level of algebra by defining a functor

$$G\colon \mathsf{Modules}^{\mathrm{free}}_{E_*} \to \mathsf{Modules}_{E_*}$$

which sends $\Sigma^k E_*$ to $E_* \underline{E}_k$ and which splits over direct sums. The enriched Hopf ring structure of $\pi_* \mathcal{UD}_E(S^*)[\leq 2]$ endows G with a comonad structure, and the structure maps of $\pi_* \mathcal{UD}_E(X)[\leq 2]$ endow $E_* X$ itself with the structure of a G–coalgebra. $\square$

To be clear about this ground category: the functor $G(M) = R_*\Omega^\infty R[M]$ determines a comonad on the category of free R_*–modules, and we refer to coalgebras for this comonad.

Theorem 4.1.5 ([BCM78, Theorem 6.17])**.** *$\pi_* \mathcal{UD}_E(\Sigma^k X)$ is the bar resolution for the free Hopf module comonad, and the E_2–page of the unstable descent spectral sequence is presented as*

$$E_2^{s,t} = L^s \mathsf{Coalgebras}_G(E_* S^t, E_* X). \quad \square$$

At this point, it is instructive to work through an example to understand the kinds of objects we have constructed. At first appraisal, these objects appear to be so bottomlessly complicated that it must be a hopeless enterprise to actually compute even just the enriched Hopf ring associated to a spectrum E. In fact, the abundance of structure maps involved gives enough footholds that this is actually often feasible, provided we have sufficiently strong stomachs. Our example will be $E = H\mathbb{F}_2$, and the place to start is with a very old lemma:

Lemma 4.1.6. *If E is a spectrum with $\pi_{-1} E = 0$, then $\underline{E}_1 \simeq B\underline{E}_0$. Consequentially, if E is a connective spectrum then $\underline{E}_n = B^n \underline{E}_0$ for $n \geq 0$.* $\square$

This is useful to us because $B(-)$ comes with a natural skeletal filtration, which we can use to form a spectral sequence.

Lemma 4.1.7 ([Seg70, Proposition 3.2], [RW80, Theorem 2.1])**.** *Let G be a topological group. There is a convergent spectral sequence of algebras of signature*

$$E^1_{*,j} = F_*(\Sigma G)^{\wedge j} \Rightarrow F_* BG.$$

In the case that F has Künneth isomorphisms $F_*((\Sigma G)^{\wedge j}) \cong F_*(\Sigma G)^{\otimes_{F_*} j}$, the E^2–page is identifiable as

$$E^2_{*,*} \cong \operatorname{Tor}^{F_*G}_{*,*}(F_*, F_*)$$

and the spectral sequence is one of Hopf algebras. □

Corollary 4.1.8. *If E is a connective spectrum and F has Künneth isomorphisms $F_*(\underline{E}_j \wedge \underline{E}_j) \cong F_*\underline{E}_j \otimes_{F_*} F_*\underline{E}_j$ for all j, then there is a family of spectral sequences of Hopf algebras with signatures*

Are you getting reduced and unreduced right in all these Künneth assumptions?

$$E^2_{*,*} \cong \operatorname{Tor}^{F_*\underline{E}_j}_{*,*}(F_*, F_*) \Rightarrow F_*\underline{E}_{j+1}. \quad \square$$

That this spectral sequence is multiplicative for the $*$–product is useful enough, but the situation is actually much, much better than this:

Lemma 4.1.9 ([TW80, Equation 1.3], [RW80, Theorem 2.2])**.** *Denote by $E^r_{*,*}(F_*\underline{E}_j)$ the spectral sequence considered above whose E^2–term is constructed from* Tor *over $F_*\underline{E}_j$. There are maps*

$$E^r_{*,*}(F_*\underline{E}_j) \otimes_{F_*} F_*\underline{E}_m \to E^r_{*,*}(F_*\underline{E}_{j+m})$$

which agree with the map

$$F_*\underline{E}_{j+1} \otimes_{F_*} F_*\underline{E}_m \xrightarrow{\circ} F_*\underline{E}_{j+m+1}$$

on the E^∞–page and which satisfy

$$d^r(x \circ y) = (d^r x) \circ y. \quad \square$$

This Lemma is obscenely useful: it means that differentials can be transported *between spectral sequences* for classes which can be decomposed as $\circ$–products. This means that the bottom spectral sequence (i.e., the case $j = 0$) exerts a large amount of control over the others—and this spectral sequence often turns out to be very computable.

We now turn to concrete computations for $E = H\mathbb{F}_2$ and $F = H\mathbb{F}_2$. To ground the induction, we will consider the first spectral sequence

$$\operatorname{Tor}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}B\mathbb{F}_2.$$

Using that $\mathbb{RP}^\infty$ gives a model for $B\mathbb{F}_2$, we use Example 1.1.15 to analyze the target of this spectral sequence: as an $\mathbb{F}_2$–module, we have already demonstrated an isomorphism

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \mathbb{F}_2\{a_j \mid j \geq 0\}.$$

Using our further computation in Example 1.2.15, we can also give a presentation of the Hopf algebra structure on $H\mathbb{F}_{2*}B\mathbb{F}_2$: it is dual to the primitively-generated polynomial algebra on a single class, so forms a divided power algebra on a single class which we will denote by $a_{()}$. In characteristic 2, this decomposes as

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \Gamma[a_{()}] \cong \bigotimes_{j=0}^{\infty} \mathbb{F}_2[a_{(j)}]/a_{(j)}^2,$$

where we have written $a_{(j)}$ for $a_{()}^{[2^j]}$ in the divided power structure.

Corollary 4.1.10. *This* Tor *spectral sequence collapses at the E^2–page.*

Proof. As an algebra, the homology $H\mathbb{F}_{2*}(\mathbb{F}_2)$ of the discrete space $\mathbb{F}_2$ is presented by a group ring, which we can identify with a truncated polynomial algebra:

$$H\mathbb{F}_{2*}(\mathbb{F}_2) \cong \mathbb{F}_2[\mathbb{F}_2] \cong \mathbb{F}_2[[1]]/([1]^{*2} - [0]) \cong \mathbb{F}_2[[1]-[0]]/([1]-[0])^{*2}.$$

The Tor–algebra of this is then divided power on a single class:

$$\operatorname{Tor}_{*,*}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = \Gamma[a_{()}].$$

In order for the two computations to agree, there can therefore be no differentials in the spectral sequence. □

We now summarize the rest of the induction:

Theorem 4.1.11. *$H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_t$ is the exterior $*$–algebra on the t–fold $\circ$–products of the generators $a_{(j)} \in H\mathbb{F}_{2*}B\mathbb{F}_2$.*

Proof. Noting that the case $t = 0$ is what was proved above, make the inductive assumption that this is true for some fixed value of $t \geq 0$. The Tor groups of the associated bar spectral sequence

$$\operatorname{Tor}_{*,*}^{H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_{t+1}$$

form a divided power algebra generated by the same t–fold $\circ$–products. An analogue of another Ravenel–Wilson lemma ([RW80, Lemma 9.5], [Wil82, Claim 8.16]) gives a congruence

$$(a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]} \equiv a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})} \quad (\text{mod } *\text{–decomposables}).$$

It follows from Lemma 4.1.9 that the differentials vanish:

$$\begin{aligned} d((a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]}) &\equiv d(a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})}) && (\text{mod } *\text{–decomposables}) \\ &= a_{(j_1)} \circ d(a_{(j_2)} \circ \cdots \circ a_{(j_{t+1})}) && (\text{Lemma 4.1.9}) \\ &= 0. && (\text{inductive hyp.}) \end{aligned}$$

Hence, the spectral sequence collapses. To see that there are no multiplicative extensions, note that the only potentially undetermined multiplications occur as $*$–squares of exterior classes. However, the $*$–squaring map is induced by the topological map

$$\underline{H\mathbb{F}_2}_t \xrightarrow{\cdot 2} \underline{H\mathbb{F}_2}_t,$$

which is already null on the level of spaces. It follows that there are no extensions and the induction holds. □

Corollary 4.1.12. *It follows that $\circ$–product induces an isomorphism*

$$H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_* \xleftarrow{\simeq} \bigoplus_{t=0}^{\infty}(H_*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2))^{\wedge t},$$

where $(-)^{\wedge t}$ denotes the t^{th} exterior power in the category of Hopf algebras. □

Remark 4.1.13 ([Wil82, Theorems 8.5 and 8.11]). The odd–primary analogue of this result appears in Wilson's book, where again the bar spectral sequences are collapsing. The end result is

$$H\mathbb{F}_{p*}\underline{H\mathbb{F}_p}_* \cong \frac{\bigotimes_{I,J}\mathbb{F}_p[e_1\circ\alpha_I\circ\beta_J,\alpha_I\circ\beta_J]}{(e_1\circ\alpha_I\circ\beta_J)^{*2}=0,(\alpha_I\circ\beta_J)^{*p}=0,e_1\circ e_1=\beta_1},$$

where $e_1 \in (H\mathbb{F}_p)_1\underline{H\mathbb{F}_p}_1$ is the homology suspension element, $\alpha_{(j)} \in (H\mathbb{F}_p)_{2p^j}\underline{H\mathbb{F}_p}_1$ are the analogues of the elements considered above, and $\beta_{(j)} \in (H\mathbb{F}_p)_{2p^j}\mathbb{C}\mathrm{P}^\infty$ are the algebra generators of the Hopf algebra dual of the ring of functions on the formal group $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{F}_p}$ associated to $H\mathbb{F}_p$ by its natural complex orientation. In particular, the Hopf ring is *free* on these Hopf algebras, subject to the single interesting relation $e_1\circ e_1=\beta_{(0)}$, essentially stemming from the equivalence $S^1\wedge S^1\simeq\mathbb{C}\mathrm{P}^1$.

It is now instructive to try to relate this unstable computation to the stable one from Lecture 1.3 (and, particularly, its algebro-geometric interpretation in Lemma 1.3.5). Consider the situation of cohomology operations: each stable operation consists of a family of unstable operations intertwined by suspensions, each of which is additive and takes 0 to 0. In terms of an element $\psi \in E^*\underline{E}_j$, such an unstable operation takes 0 to 0 exactly when it lies in the augmentation ideal, and it is additive exactly when it satisfies Hopf algebra primitivity:

$$\Delta^*\psi^* = (\psi\otimes\psi)^*\Delta^*.$$

Definition 4.1.14. In the setting of unstable homology cooperations, we define an *additive unstable operation* to be one which lies in the $*$–indecomposable quotient $Q_*E_*\underline{E}_j$.

We now apply this philosophy to our example:

Corollary 4.1.15 (cf. Theorem 1.3.3, [Wil82, Theorem 8.15])**.** $\mathcal{A}_* = \mathbb{F}_2[\xi_0,\xi_1,\xi_2,\ldots][\xi_0^{-1}]$.

Proof. First, we compute $*$–indecomposable quotient of the unstable dual Steenrod algebra to be

$$\begin{aligned} Q^* H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2*}} &\cong \mathbb{F}_2 \{a_I | I \text{ a multi-index}\} \\ &= \mathbb{F}_2 \left\{ a_{(I_0)} \circ a_{(I_1)} \circ \cdots \circ a_{(I_n)} \middle| I = (I_0, \ldots, I_n) \text{ a multi-index} \right\}. \\ &\cong \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots], \end{aligned}$$

where we have translated to our previous notation by writing ξ_j for $a_{(j)}$ and juxtaposition for $\circ$–product. From here, sequences of additive unstable cooperations which are intertwined by suspension are exactly elements of the sequential colimit that inverts the homology suspension element. We have already explicitly identified this element as $a_{(0)} = \xi_0$, and this yields the claim. $\square$

Our last goal in this Lecture is to sketch a foothold that this example has furnished us with for the algebro-geometric interpretation of unstable cooperations. First, we should remark that it has been shown that there is no manifestation of the homology of a space as any kind of classical comodule [BJW95, Theorem 9.4], so we are unstable to directly pursue an analogue of Definition 3.1.14 presenting the homology of a space as a Cartesian quasicoherent sheaf over some object. This no-go result is quite believable from the perspective of cohomology operations: we have calculated in the case of $E = H\mathbb{F}_2$ that a generic unstable cohomology operation takes the form

$$x \mapsto \sum_{S \text{ a set of multi-indices}} \left(c_S \cdot \prod_{I \in S} \mathrm{Sq}^I(x) \right).$$

This inherently uses the multiplicative structure on $H\mathbb{F}_2^*(X)$, and the proof of the result of Boardman, Johnson, and Wilson rests entirely on the observation that decomposable elements cannot be mapped to indecomposable elements by maps of algebras, but maps of modules have no such control.[4]

However, exactly this complaint is eliminated by passing to the additive unstable cooperations: all the product terms in the above formula vanish, and the homology of a space does indeed have the structure of a comodule for this Hopf algebra. Still in the setting of our running example $E = H\mathbb{F}_2$, this makes $H\mathbb{F}_{2*}(X)$ into a Cartesian quasicoherent sheaf for the simplicial scheme

$$\mathcal{UM}_{H\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2 // \operatorname{Spec} Q^* H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2*}}.$$

In this specific example, we can even identify what this simplicial scheme is: using

[4] It is probably still possible to treat this carefully enough to cast the whole of unstable operations (and, in particular, the comonad G) into algebro-geometric language.

Lemma 1.3.5, we have already made the identification

$$\operatorname{Spec} H\mathbb{F}_{2*}H\mathbb{F}_2 \cong \underline{\operatorname{Aut}}\,\widehat{\mathbb{G}}_a$$
$$(f \colon \mathbb{F}_2[\xi_0^{\pm}, \xi_1, \ldots] \to R) \mapsto \left(x \mapsto \sum_{j=0}^{\infty} f(\xi_j)x^{2^j}\right),$$

and the computation above presents $\operatorname{Spec} \mathcal{A}_*$ as the open subscheme of $\operatorname{Spec} \mathbb{F}_2[\xi_0, \xi_1, \ldots]$ determined by the invertibility of ξ_0. Hence, the more general target is

$$\operatorname{Spec} Q^* H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_* \cong \underline{\operatorname{End}}\,\widehat{\mathbb{G}}_a$$
$$(f \colon \mathbb{F}_2[\xi_0, \xi_1, \ldots] \to R) \mapsto \left(x \mapsto \sum_{j=0}^{\infty} f(\xi_j)x^{2^j}\right).$$

Some of the complexity here was eliminated by the smallness of $\operatorname{Spec} H\mathbb{F}_{2*}$. For a general ring spectrum E, we also have to account for $\operatorname{Spec} E_*$, but the end result is similar to that of Definition 3.1.14:

Consider elaborating on how Q^* does not give a Hopf algebroid because it's missing a version of χ for $\circ$–multiplication. It's kind of remarkable that inverting the homology suspension element automatically produces such a χ°.

Lemma 4.1.16. *For a ring spectrum E satisfying **UFH**, the additive unstable cooperations form rings of functions on the objects and morphisms of a category scheme $\mathcal{UM}_E$, and the E–homology of a space X forms a Cartesian quasicoherent sheaf $\mathcal{UM}_E(X)$ over its nerve.* □

Although it seems like we have lost a lot of information in passing to $*$–indecomposables, it is a classical observation that in many cases this is actually enough to recover everything.

Definition 4.1.17 ([BCM78, Assumptions 7.1 and 7.7])**.** We say that a ring spectrum E satisfying **UFH** furthermore satisfies the **U**nstable **G**eneration **H**ypothesis, or **UGH**, when the following conditions all hold:

Make sure you got primitives vs indecomposables right. Remember that you have to apply primitives to a *comodule* in a moment, so maybe indecomposables don't even make sense. If that's the case, you need to work out what the relationship is to the algebro-geometric story.

1. $Q^*E_*\underline{E}_j$ is E_*–free.
2. The following composite $Q^*E_*\underline{E}_j \to E_*\underline{E}_j \to E_*E$ is injective.
3. $E_*\underline{E}_j \to SQ^*E_*\underline{E}_j$ is an isomorphism, where S is the cofree (nonunital) coalgebra functor.

Lemma 4.1.18 ([BCM78, Lemma 7.5])**.** *Let E be a ring spectrum satisfying **UGH**, and let G be the comonad from Lemma 4.1.4. The composite functor $U = Q^*G$ extends from a functor on free E_*–modules to all E_*–modules by using 2–stage free resolutions and enforcing exactness, and the result is a comonad. Coalgebras for this comonad are exactly comodules for the Hopf algebra of additive unstable cooperations.* □

Corollary 4.1.19 ([BCM78, Remark 7.8])**.** *If E satisfies **UGH** and X is a space with $E_*X = SN$ for some connective free E_*–module N, then the unstable E–Adams E_2–term is computed by*

$$E_2^{s,t} = \operatorname{Ext}^s_{\mathsf{Coalgebras}_U}(E_*S^t, Q^*E_*X).$$

Proof. Under **UGH**, we have a factorization

$$\mathsf{Coalgebras}_G(E_* S^t, -) = \mathsf{Comodules}_U(E_* S^t, Q^*(-))$$

and the injective objects intertwine to give a composite functor spectral sequence

$$E_2^{r,s} = \operatorname{Ext}^r_{\mathsf{Coalgebras}_U}(E_* S^t, R^s_{\mathsf{Coalgberas}_G} Q^*(M)) \Rightarrow \operatorname{Ext}^{r+s}_{\mathsf{Coalgebras}_G}(E_* S^t, M).$$

If $M = E_* X = SN$ for some connective free E_*–module N, then $R^{q>0}_{\mathsf{Coalgebras}_G} Q^*(E_* X) = 0$, the composite functor spectral sequence collapses, and $\operatorname{Ext}^s_{\mathsf{Coalgebras}_U}(E_* S^t, Q^* E_* X)$ computes the unstable Adams E_2–term as claimed. $\square$

Reinterpret this Ext group as some kind of sheaf cohomology. Don't forget the goal of having a residual $\mathbb{G}_m$–action (or whatever) to distinguish the grading.

BJW: lemma 2.10, questions 2.22 seems relevant

Define what the homology suspension element e is. The point is that the equivalence $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$ is adjoint to a map $\Sigma \underline{E}_n \to \underline{E}_{n+1}$, and the effect of this map on F–homology is $\circ$–ing with e.

Neil wrote this to tell Chris Schommer–Pries how to compute $H_* K(\mathbb{Z}, 3)$ (http://mathoverflow.net/a/216041/1094), which I think should be included as an exercise here: Given a complex oriented cohomology theory E, one can define a formal scheme of Weil pairings on the associated formal group, as explained in the paper "Weil pairings and Morava K-theory" by Matthew Ando and me. If we let R_E denote the ring of functions on this scheme, then there is a natural map $R_E \to E^* K(\mathbb{Z}, 3)$. This is an isomorphism if E is Morava K-theory or Morava E-theory. I think that it is also an isomorphism for $E = MU$ or $E = kU$ but not for $E = H$. However, there is a natural short exact sequence

$$kU^* K(\mathbb{Z}, 3)/v \to H^* K(\mathbb{Z}, 3) \to \operatorname{ann}(v, kU^* K(\mathbb{Z}, 3))$$

(where v is the standard generator of $\pi_2 kU = kU^{-2}(\text{point})$). I think that this is probably an effective way to understand $H^* K(\mathbb{Z}, 3)$. Some other things that are going on in the background here: - There is a fibration $K(\mathbb{Q}/\mathbb{Z}, 2) \to K(\mathbb{Z}, 3) \to K(\mathbb{Q}, 3)$. Here $K(\mathbb{Q}, 3)$ is the rationalisation of S^3 and is not so hard to understand. - $K(\mathbb{Q}/\mathbb{Z}, 2)$ is the colimit of the spaces $K(\mathbb{Z}/n, 2)$. - One can understand $K(\mathbb{Z}/n, 2)$ using the multiplication map $K(\mathbb{Z}/n, 1) \times K(\mathbb{Z}/n, 1) \to K(\mathbb{Z}/n, 2)$.

Tilman Bauer is proving results about the homological algebra of these things using weird plethory structures. He deserves a mention here as part of the developing body of literature.

4.2 Mixed unstable cooperations and their algebraic model

For simplicity, we return to the stable setting of Lecture 3.1 for a moment. For an arbitrary spectrum X and ring spectrum E, the completion $X^\wedge_E$ is typically a quite poor approximation to X itself. Though this can be partially mediated by placing hypotheses on X, the approximation can always be improved by "enlarging" the cohomology theory involved—namely, selecting a second ring spectrum F and forming the completion $X^\wedge_{E \vee F}$ at the wedge. This has the following factorization property

$$\begin{tikzcd}
 & & X^\wedge_E \\
X \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & X^\wedge_{E \vee F} \arrow[ur] \arrow[dr] & \\
 & & X^\wedge_F,
\end{tikzcd}$$

so that homotopy classes visible in either of $X^\wedge_E$ or $X^\wedge_F$ are therefore also visible in the homotopy of $X^\wedge_{E\vee F}$. Now consider the descent object $\mathcal{D}_{E\vee F}(X)$ and its layers $\mathcal{D}_{E\vee F}(X)[n]$:

$$\begin{aligned}\mathcal{UD}_{E\vee F}(X)[n] &= (E\vee F)^{\wedge(n+1)}\wedge(X)\\ &\simeq (E^{\wedge(n+1)}\wedge X)\vee(F^{\wedge(n+1)}\wedge X)\vee\bigvee_{\substack{i+j=n+1\\ i\neq 0\neq j}}(E^{\wedge i}\wedge F^{\wedge j}\wedge X)^{\vee\binom{n}{i,j}}.\end{aligned}$$

In the edge cases of $i=0$ or $j=0$, we can identify the descent objects $\mathcal{D}_E(X)$ and $\mathcal{D}_F(X)$ as sub-cosimplicial objects of $\mathcal{D}_{E\vee F}(X)$. The role of the cross-terms at the end of the expression is to prevent the completion at $E\vee F$ from double-counting the parts of X already simultaneously visible to the completions at E and at F—i.e., the cross-terms handle communication between E and F.[5]

There is a similar (but algebraically murkier) story for the unstable descent object formed at a wedge of two ring spectra. Let X now be a space, and consider the first two layers of $\mathcal{UD}_{E\vee F}(X)$:

$$\begin{aligned}&\mathcal{UD}_{E\vee F}(X)[0] = (E\vee F)(X) = E(X)\times F(X),\\ &\mathcal{UD}_{E\vee F}(X)[1] = (E\vee F)(E(X)\times F(X)) = E(E(X)\times F(X))\times F(E(X)\times F(X)).\end{aligned}$$

Consider just first factor, $E(E(X)\times F(X))$. The homotopy of this object receives a bilinear map from $\pi_*E(E(X))\times\pi_*E(F(X))$, and if E has Künneth isomorphisms then the induced map off of the tensor product is an equivalence. Again, we can identify the $E(E(X))$ part of this expression as belonging to $\mathcal{UD}_E(X)[1]$, and there is a cross-term $E(F(X))$ accounting for the shared information with F. The other term also contains information present in $\mathcal{UD}_F(X)[1]$ and a cross-term $F(E(X))$ accounting for shared information with E. In order to understand how these cross-terms affect the reconstruction process, it is useful to identify what they are at the level of the unstable context:

$$\mathcal{O}(\mathcal{UM}_{E\vee F}(S^n)[1]) \leftarrow \pi_*F(E(S^n)) = F_*\underline{E}_n,$$

and as n ranges these again form a Hopf ring.

Definition 4.2.1. We will refer to $F_*(\underline{E}_*)$ as the *Hopf ring of mixed unstable cooperations* (from F to E) or the *topological Hopf ring* (from F to E).

We thus set about trying to understand the Hopf rings $F_*(\underline{E}_*)$ in general. In our computational example in Lecture 4.1, we found that the topological Hopf ring $H\mathbb{F}_{2*}(\underline{H\mathbb{F}_2}_*)$ modeled endomorphisms of the additive formal group after passing to a suitable quotient,

[5]From the perspective of spectral shemes, you might think of the descent object for $E\vee F$ as that coming from the joint cover $\{\mathbb{S}\to E, \mathbb{S}\to F\}$, and these cross-terms correspond to the scheme-theoretic intersection of E and F over $\mathbb{S}$.

and we will take this as inspiration to construct an algebraic model, or "expected answer", approximating the topological Hopf ring.

We approach this problem in stages. To start, note that homotopy elements both of F and of E can be used to contribute elements to the topological Hopf ring: an element $f \in F_n$ begets a family of natural elements $f_m \in F_n \underline{E}_m$, and an element $e \in E^n = \pi_0 \underline{E}_n$ begets an element $[e] \in F_0 \underline{E}_n$ by Hurewicz. The interaction of these rings F_* and E^* is captured in the following definition:

Definition 4.2.2 ([RW80, pg. 706])**.** Let R and S be graded rings. The *Hopf ring–ring* $R[S]$ forms a Hopf ring over R: as an R–module, it is free and generated by symbols $[s]$ for $s \in S$, and the ring structure on S is promoted up a level to become the Hopf ring operations. Explicitly, the Hopf ring structure maps $*$, $\circ$, χ, and Δ are determined by the formulas

$$\begin{aligned} R[S] \otimes_R R[S] &\xrightarrow{*} R[S] & [s] * [s'] &= [s + s'], \\ R[S] \otimes_R R[S] &\xrightarrow{\circ} R[S] & [s] \circ [s'] &= [s \cdot s'], \\ R[S] &\xrightarrow{\chi} R[S] & \chi[s] &= [-s], \\ R[S] &\xrightarrow{\Delta} R[S] \otimes_R R[S] & \Delta[s] &= [s] \otimes [s]. \end{aligned}$$

Lemma 4.2.3 ([RW80, pg. 706])**.** *There are natural maps of Hopf rings*

$$F_*[E^*] \to F_*(\underline{E}_*) \to F_*[E^*]$$

augmenting the topological Hopf ring over the Hopf ring–ring. □

Supposing that E and F are complex-orientable, we now seek to involve their formal groups. The construction we are about to undertake is a variation on the proof of Lemma 2.1.4, which is itself a variation of a more general result in the theory of formal schemes:

Lemma 4.2.4 ([Str99b, Proposition 2.94])**.** *Let X and Y be schemes over $S = \operatorname{Spec} R$, such that $\mathcal{O}_X$ forms a finite and free R–module. There is then a mapping scheme M, such that points $f \in M(A)$ naturally biject with maps $f \colon X \times_S \operatorname{Spec} A \to Y \times_S \operatorname{Spec} A$ of A–schemes.* □

The mode of proof of this result is to form the symmetric R–algebra on the R–module $\mathcal{O}_Y \otimes_R \mathcal{O}_X^*$, then quotient by the relations encoding multiplicativity of functions. These are the same steps we will take to form a Hopf ring embodying homomorphisms of formal groups $\mathbb{C}P^\infty_F \to \mathbb{C}P^\infty_E$.

Definition 4.2.5 (cf. [RW77, Equation 1.17])**.** Given a R–coalgebra A and an S–algebra B, we form the *free relative Hopf $R[S]$–ring $A_{R[S]}[B]$* generated under the Hopf ring operations by symbols $a[b]$ for $a \in A$, $b \in B$, according to the atomic rules

$$\begin{aligned} A_{R[S]}[B] \otimes_R A_{R[S]}[B] &\xrightarrow{*} A_{R[S]}[B] & a[b] * a'[b'] &= (a * a')[b + b'], \\ A_{R[S]}[B] \otimes_R A_{R[S]}[B] &\xrightarrow{\circ} A_{R[S]}[B] & a[b] \circ a'[b'] &= (a \circ a')[bb'], \\ A_{R[S]}[B] &\xrightarrow{\chi} A_{R[S]}[B] & \chi(a[b]) &= (\chi a)[b] = a[-b], \\ A_{R[S]}[B] &\xrightarrow{\Delta} A_{R[S]}[B] & \Delta(a[b]) &= \sum_j (a'_j[b] \otimes a''_j[b]), \end{aligned}$$

where $\Delta(a) = \sum_j a'_j \otimes_{F_*} a''_j$. There are two additional families of relations we might impose:

1. For $a \in A$ and $b', b'' \in B$, we devise the relation
$$\sum_j (a'_j[b'] \circ a''_j[b'']) = a[b'b'']$$
as an analogue of the multiplicativity relation imposed in Lemma 4.2.4.

2. For $a \in A$, we devise the relation
$$a[\eta(1)] = \varepsilon(a)$$
as an analogue of the unitality relation imposed in Lemma 4.2.4.

3. Assume further that the coalgebra A is coaugmented and that the algebra B is augmented. We can then consider the pointedness relation
$$(\eta(1))[b] = [\varepsilon(b)].$$

4. Finally, assume the entire structure of a Hopf R–algebnra on A and of a Hopf S–algebra on B. For each $a', a'' \in A$ and $b \in B$ with diagonal $\Delta b = \sum_j b'_j \otimes b''_j$, we can consider the homomorphism relation
$$(a'a'')[b] = \mathop{\ast}_j (a'[b'_j] \circ a''[b''_j]).$$

We denote the result of imposing all of these relations on $A_{R[S]}[B]$ as $A^{\circlearrowleft}_{R[S]}[B]$.

Lemma 4.2.6. *There is a natural map*

$$(F_*\mathbb{C}\mathrm{P}^\infty)^{\circlearrowleft}_{F_*[E^*]}[E^*\mathbb{C}\mathrm{P}^\infty] \to F_*(\underline{E}_*).$$

Proof. For any space X, we construct a Kronecker-type pairing

$$\langle -, - \rangle \colon\ F_n Z \times E^m Z \to F_n(\underline{E}_m)$$

as follows: given a class $f \in \pi_n F(X)$ and a class $e\colon\ X \to \underline{E}_m$, we can compose the two to produce an element $e_*(f) \in \pi_n F(\underline{E}_m)$. This pairing is "bilinear" in the following senses:

$$\begin{aligned} \langle a' + a'', b \rangle &= \langle a', b \rangle + \langle a', b \rangle, & \langle f \cdot a, b \rangle &= f \cdot \langle a, b \rangle, \\ \langle a, b' + b'' \rangle &= \langle a, b' \rangle * \langle a, b'' \rangle, & \langle a, e \cdot b \rangle &= [e] \circ \langle a, b \rangle. \end{aligned}$$

Universality thus gives a map of Hopf rings $(F_* X)_{F_*[E^*]}[E^* X] \to F_*(\underline{E}_*)$. Specializing to $X = \mathbb{C}\mathrm{P}^\infty$, the factorization of this map through the indicated Hopf ring quotient follows the duality property of this enhanced Kronecker pairing. Namely, the following four maps induce four commutative diagrams:

$$(\Delta\colon\ \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \rightsquigarrow \left(\begin{tikzcd} & F(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \\ S^n \arrow[ur, "F(\Delta)_*\sigma"] \arrow[r, "\sigma"] & F(\mathbb{C}\mathrm{P}^\infty) \arrow[u, "F(\Delta)"] \arrow[ur, "F(\Delta^*\omega)"'] & \end{tikzcd} \right),$$

$$(\mu\colon\ \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty) \rightsquigarrow \left(\begin{tikzcd} S^n \arrow[r, "\sigma"] \arrow[dr, "F(\mu)_*\sigma"'] & F(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \arrow[d, "F(\mu)"] \arrow[dr, "F(\mu^*\omega)"] & \\ & F(\mathbb{C}\mathrm{P}^\infty) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \end{tikzcd} \right),$$

$$(\varepsilon\colon\ \mathbb{C}\mathrm{P}^\infty \to *) \rightsquigarrow \left(\begin{tikzcd} & F(*) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \\ S^n \arrow[ur, "F(\varepsilon)_*\sigma"] \arrow[r, "\sigma"] & F(\mathbb{C}\mathrm{P}^\infty) \arrow[u, "F(\varepsilon)"] \arrow[ur, "F(\varepsilon^*\omega)"'] & \end{tikzcd} \right),$$

$$(\eta\colon\ * \to \mathbb{C}\mathrm{P}^\infty) \rightsquigarrow \left(\begin{tikzcd} S^n \arrow[r, "\sigma"] \arrow[dr, "F(\eta)_*\sigma"'] & F(*) \arrow[d, "F(\eta)"] \arrow[dr, "F(\eta^*\omega)"] & \\ & F(\mathbb{C}\mathrm{P}^\infty) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \end{tikzcd} \right),$$

and these diagrams respectively witness the relations

$$\begin{aligned} \langle \Delta_* a, b' \otimes b'' \rangle &= \langle a, \Delta^*(b' \otimes b'') \rangle, & \langle \mu_*(a' \otimes a''), b \rangle &= \langle a' \otimes a'', \mu^* b \rangle, \\ \langle \varepsilon_* 1, b \rangle &= \langle 1, \varepsilon^* b \rangle, & \langle \eta_* 1, b \rangle &= \langle 1, \eta^* b \rangle. \end{aligned}$$

The Kronecker pairings are related to the Künneth isomorphisms for $F_*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$ and $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$ by the product formula

$$\langle a' \otimes a'', b' \otimes b'' \rangle = \langle a', b' \rangle \circ \langle a'', b'' \rangle.$$

Hence, writing $\Delta_* a = \sum_j a'_j \otimes a''_j$ and $\mu^* b = \sum_j b'_j \otimes b''_j$, these relations become exactly the equations

$$\sum_j (a'_j[b'] \circ a''_j[b'']) = a[b'b''], \qquad (a'a'')[b] = \mathop{\ast}_j (a'[b'_j] \circ a''[b''_j]),$$
$$(\eta(1))[b] = [\varepsilon(b)], \qquad a[\eta(1)] = \varepsilon(a). \qquad \square$$

The main theme of this Case Study is that this induced map off of the quotient is very often an isomorphism (and, in turn, that the theory of formal groups also captures everything about the theory of unstable cooperations). Because we will be carrying this algebraic model around with us, we pause to give it a name.

Definition 4.2.7. For F and E ring spectra, we define their *algebraic Hopf ring* $\mathbb{A}(F, E)$ (or *algebraic approximation*) by

$$\mathbb{A}(F, E) = (F_* \mathbb{C}\mathrm{P}^\infty)^{\circlearrowleft}_{F_*[E^*]}[E^* \mathbb{C}\mathrm{P}^\infty].$$

Lemma 4.2.8 ([RW77, Theorem 3.8], [Wil82, Theorem 9.7])**.** *After choosing complex orientations of E and F, there is a natural isomorphism of Hopf rings*

$$\mathbb{A}(F, E) \cong \frac{(F_* \mathbb{C}\mathrm{P}^\infty)_{F_*[E^*]}[E^*]}{\beta(s +_F t) = \beta(s) +_{[E]} \beta(t)},$$

where the equation is of power series and the equality is imposed term-by-term on the Hopf ring. The formal sum $\beta(s)$ is given by $\beta(s) = \sum_j \beta_j x^j$, where β_j is dual to the j^{th} power of the chosen coordinate in $F^* \mathbb{C}\mathrm{P}^\infty$, and the formal group law expressions expand to

$$\beta(s +_F t) = \sum_n \beta_n \left(\sum_{i,j} a^F_{ij} s^i t^j \right)^n,$$
$$\beta(s) +_{[E]} \beta(t) = \mathop{\ast}_{i,j} \left([a^E_{ij}] \circ \left(\sum_k \beta_k s^k \right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right).$$

Proof sketch. The orientations of E and F beget classes $x^j \in E^{2j} \mathbb{C}\mathrm{P}^\infty$ and $\beta_k \in F_{2k} \mathbb{C}\mathrm{P}^\infty$, and hence classes $\beta_k[x^j] \in \mathbb{A}(F, E)$. The duality relations imposed on this Hopf ring give us three useful identities:

1. The relation
$$\beta_k[x^0] = \varepsilon(\beta_k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0 \end{cases}$$
eliminates all elements of this form except $\beta_0[x^0] = 1$.

2. The relation
$$\beta_k[x^{j+1}] = \sum_{k'+k''=k} \beta_{k'}[x^j] \circ \beta_{k''}[x]$$
lets us rewrite these terms as $\circ$–products of terms of lower j–degree and no larger k–degree.

3. The relation
$$\beta_0[x^j] = [\varepsilon(x^j)] = \begin{cases} [1] & \text{if } j = 0, \\ [0] & \text{if } j \neq 0 \end{cases}$$
couples to the above relation to give
$$\beta_k[x^{j+1}] = \sum_{\substack{k'+k''=k \\ k',k''\neq 0}} \beta_{k'}[x^j] \circ \beta_{k''}[x],$$
so that the rewrite is in terms of both lower j–degree *and* lower k–degree.

By consequence, the surviving terms are all sums of $\circ$–products of terms of the form $\beta_k[x]$, so that imposing these three relations produces a surjection
$$(F_*\mathbb{C}\mathrm{P}^\infty)_{F_*[E^*]}[E^*] \to \mathbb{A}(F, E).$$
The remaining assertion is a now a matter of imposing the fourth relation, i.e., of calculating the behavior of
$$\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{\mu} \mathbb{C}\mathrm{P}^\infty \xrightarrow{x} \underline{E}_2$$
in two different ways: using the effect of μ in F–homology and pushing forward in x, or using the effect of μ in E–cohomology and pushing forward along the Hurewicz map $\mathbb{S} \to F$. □

Finally, we are able to explain the phenomenon uncovered by computation in Lecture 4.1, where we passed to the $*$–indecomposables to find the classical ring of functions on the endomorphism scheme of $\widehat{\mathbb{G}}_a$. Our explanation arises out of two parallel questions:

1. What functor $\operatorname{SpH} \mathbb{A}(F, E)$ does the Hopf ring $\mathbb{A}(F, E)$ corepresent when evaluated on another Hopf ring?

2. How does this functor interact with classical rings?

Towards the second, there is an embedding of Rings into $\mathsf{HopfRings}$, analogous to the square-zero extension functor sending an abelian group A to the augmented algebra $\mathbb{Z} \oplus A$ with trivial multiplication on A.

There ought to be a version of this relative to $R[S]$, and it ought to play a role when mapping into augmented Hopf $R[S]$–rings.

Definition 4.2.9. For a ring R, the $*$–square–zero Hopf ring iR has underlying abelian group $\mathbb{Z} \oplus R$. For an element $r \in R$, we write $[r] = (0, r)$ for the corresponding element in iR, and in this notation the Hopf ring structure maps are set by the formulas

$$\begin{aligned} iR \otimes iR &\xrightarrow{*} iR, & [r] * [r'] &= 0, & [r] * 1 &= [r], \\ iR \otimes iR &\xrightarrow{\circ} iR, & [r] \circ [r'] &= [rr'], & [r] \circ 1 &= [r], \\ iR &\xrightarrow{\chi} iR, & \chi[r] &= [-r], & \chi(1) &= 1, \\ iR &\xrightarrow{\Delta} iR \otimes iR, & \Delta[r] &= [r] \otimes 1 + 1 \otimes [r], & \Delta(1) &= 1 \otimes 1, \\ iR &\xrightarrow{\varepsilon} \mathbb{Z}, & \varepsilon([r]) &= 1, & \varepsilon(1) &= 1, \\ \mathbb{Z} &\xrightarrow{\eta} iR, & \eta(1) &= 1. \end{aligned}$$

Lemma 4.2.10. *We have $R \cong Q^* iR$, and moreover there is an adjunction*

$$(\operatorname{Spec} Q^* S)(R) = \mathsf{Rings}(Q^* S, R) = \mathsf{HopfRings}(S, iR) = (\operatorname{SpH} S)(iR). \quad \square$$

We are thus algebraically motivated to understand the affine scheme $\operatorname{Spec} Q^* \mathcal{A}(F, E)$, as this is what $\operatorname{SpH} \mathcal{A}(F, E)$ restricts to on the subcategory of classical rings. Note that the Hopf ring-ring $R[S]$ and the free relative Hopf ring $A_{R[S]}[B]$ both have an augmentation given by $[x] \mapsto 1$, so that the elements $\langle x \rangle = [x] - [0]$ form a generating set of the augmentation ideal.

Lemma 4.2.11. *In the $*$–indecomposable quotient, there are the formulas*

$$\langle x \rangle + \langle y \rangle = \langle x + y \rangle, \qquad \langle x \rangle \circ \langle y \rangle = \langle xy \rangle.$$

Proof. Modulo $*$–decomposables, we can write

$$0 \equiv \langle x \rangle * \langle y \rangle = [x] * [y] - [x] - [y] + [0] = \langle x + y \rangle - \langle x \rangle - \langle y \rangle.$$

We can also directly calculate

$$\langle x \rangle \circ \langle y \rangle = [xy] - [0] - [0] + [0] = \langle xy \rangle. \qquad \square$$

Corollary 4.2.12. *There is an isomorphism $Q^* R[S] \cong R \otimes S$.* $\square$

Corollary 4.2.13. *For complex-orientable F and E, there is a natural isomorphism*

$$\operatorname{Spec} Q^* \mathcal{A}(F, E) \cong \underline{\mathsf{FormalGroups}}(\mathbb{C}\mathrm{P}^\infty_F, \mathbb{C}\mathrm{P}^\infty_E).$$

Proof. This is a matter of calculating $Q^* \mathbb{A}(F, E)$, which is possible to do coordinate-freely, but it is at least as clear to just give in and pick coordinates. Doing this and using Lemma 4.2.11, we have

$$\mathop{\ast}_{i,j} \left([a_{ij}^E] \circ \left(\sum_k \beta_k s^k \right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right) \equiv \sum_{i,j} a_{ij}^E \left(\sum_k \beta_k s^k \right)^i \left(\sum_\ell \beta_\ell t^\ell \right)^j \text{ (in } Q^*\text{)},$$

from which it follows that

$$Q^* \mathbb{A}(F, E) = (F_* \otimes E_*)[\beta_0, \beta_1, \beta_2, \ldots]/(\beta(s +_F t) = \beta(s) +_E \beta(t)) . \qquad \square$$

You could also include the odd part of the approximation, with $e \circ e = \beta_1$, and from that calculate the algebraic model of the stabilization.

Remark 4.2.14. In the unmixed case of $E = F$, as we saw in the computational example in Lecture 4.1, the algebraic Hopf ring $\mathbb{A}(E, E)$ picks up an extra diagonal corresponding to the composition of formal group endomorphisms of $\mathbb{C}P^\infty_E$, and the resulting pair $(\operatorname{Spec} E_*, \underline{\operatorname{End}}(\mathbb{C}P^\infty_E))$ forms a category scheme. These schemes act by pre- and post-composition on the mixed algebraic Hopf ring $\operatorname{Spec} Q^* \mathbb{A}(F, E)$, and these actions are compatible with the structure maps in the unstable context $\mathcal{UM}_{E \vee F}$ described at the beginning of this Lecture. This description is also compatible with pulling back to the stable context $\mathcal{M}_{E \vee F}$: it is exactly the inclusion of the simplicial subobject consisting of the formal group isomorphisms and automorphisms.

Section III.11 of Wilson's *Primer* has a synopsis of how additive unstable operations should be treated.

You should talk about how $\mathbb{A}(F, E)$ only hopes to grab the even information, since the simultaneous even-periodicity and complex-orientability of F and E force everything in $F_* \mathbb{C}P^\infty$ and $E^* \mathbb{C}P^\infty$ to lie in even degree. *However*, I think there is a variation on this that touches the Morava K–theory case, where $\mathbb{C}P^\infty$ is replaced by $\underline{HC}_{p^\infty 1}$ and the $\circ$–product is that induced by the Pontryagin pairing. That degree shift probably fixes some things?

4.3 Unstable cooperations for complex bordism

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Our theme for the rest of this Case Study is that the comparison map

$$\mathbb{A}(F, E) \to F_* \underline{E}_*$$

of Lemma 4.2.6 is often an isomorphism. In this Lecture, we begin by investigating the very modest and concrete setting of $F = H = H\mathbb{F}_p$ and $E = BP$, simply because it is the least complicated choice after the unstable Steenrod algebra: the spectrum H has Künneth isomorphisms, and the formal group law associated to BP has a very understandable role. We record our goal in the following Theorem statement:

Theorem 4.3.1 ([RW77, Theorem 4.2])**.** *The natural homomorphism*

$$\mathbb{A}(H, BP) \to H_* \underline{BP}_{2*}$$

is an isomorphism. (In particular, $H_* \underline{BP}_{2*}$ is even–concentrated.)

This is proved by a fairly elaborate counting argument: the rough idea is to show that the topological Hopf ring is polynomial, the comparison map is surjective, and the degrees arrange themselves so that the map then has no choice but to be an isomorphism. Our first move will thus be to produce an upper bound for the size of the source Hopf ring, so that surjectivity can be used to compare it with the size of the algebraic approximation.

Crucially, polynomiality will often let us consider the ring of $*$–indecomposables rather than the full Hopf ring. To begin, recall the following consequence of Corollary 4.2.12:

Corollary 4.3.2. *As an algebra under the $\circ$–product,*

$$Q^* H_*[BP^*] \cong \mathbb{F}_p[[v_n] - [0] \mid n \geq 1]. \quad \square$$

From Lemma 4.2.8, we now know that $Q^* \mathbb{A}(H, BP)$ is generated by $[v_n] - [0]$ for $n \geq 1$ and $\beta_j \in H_{2j}\underline{BP}_2$, $j \geq 0$. In fact, p–typicality shows [RW77, Lemma 4.14] that it suffices to consider $\beta_{p^d} = \beta_{(d)}$ for $i \geq 0$. Altogether, this gives a secondary comparison map

$$A := \mathbb{F}_p[[v_n], \beta_{(d)} \mid n > 0, d \geq 0] \twoheadrightarrow Q^* \mathbb{A}(H, BP).$$

Although this map is onto it is not an isomorphism, as these elements are subject to the following relation:

Lemma 4.3.3 ([RW77, Lemma 3.14], [Wil82, Theorem 9.13])**.** *Write $I = ([p], [v_1], [v_2], \ldots)$, and work in $Q^* \mathbb{A}(H, BP) / I^{\circ 2} \circ Q^* \mathbb{A}(H, BP)$. For any n we have*

$$\sum_{i=1}^{n} [v_i] \circ \beta_{(n-i)}^{\circ p^i} \equiv 0.$$

Proof. Since the group law on $\mathbb{C}P^\infty_H$ is additive, the Ravenel–Wilson relation applied to the p–series[6] specializes to

$$[p]_{[BP]}(\beta(s)) = \beta(ps).$$

If we work over the square-zero part of BP_* to simplify its group law, we have the relation

$$[p]_{BP}(s) \equiv \sum_{j \geq 0} v_j s^{p^j} \pmod{(p, v_1, v_2, \ldots)^2},$$

which combines with the above to give

$$\beta_0 = [p]_{[BP]}(\beta(s)) \equiv \mathop{\Huge *}_{j \geq 0} ([v_j] \circ \beta(s)^{\circ p^j}) \pmod{I^{\circ 2}}.$$

Passing to Q^*, we have $[p] \circ \beta(s) \equiv \beta_0$ and hence

$$0 \equiv \sum_{j > 0} [v_j] \circ \beta(s)^{\circ p^j}.$$

The coefficient of s^{p^n} gives the identity claimed. $\square$

[6]We are very sorry for the collision of $[p]_{BP}$ the p–series and $[p]$ the symbol in the Hopf ring induced from $p \in BP_0$. The p–series won't linger, and we will always differentiate them with a subscript.

Let r_n, the n^{th} relation, denote the same sum taken in A instead:

$$r_n := \sum_{i=1}^{n} [v_i] \circ \beta_{(n-i)}^{\circ p^i} \in A.$$

The Lemma then shows that the image of r_n in $Q^* \mathbb{A}(H, BP)$ is a $\circ$–decomposable element. Our goal is to show that enforcing these relations cuts A down to exactly the right size, and the easiest way to track the size of a quotient is for the quotient to be by a regular ideal.

Lemma 4.3.4 ([RW77, Lemma 4.15.b])**.** *The sequence $(r_1, r_2, \ldots)$ is regular in A.*

Proof. Our approach is intricate but standard. We seek to show that $J = (r_1, r_2, \ldots, r_n)$ is regular for every n, and we accomplish this by interpolation. Fixing a particular n, define the intermediate ideals

$$J_j = (r_n, r_{n-1}, \ldots, r_{n-j+1}),$$

as well as the intermediate rings

$$A_j = A/(\beta_{(0)}, \ldots, \beta_{(n-j-1)}), \qquad\qquad B_j = \beta_{(n-j)}^{-1} A_j.$$

Noting that $A_n = A$ and $J_n = J$, we will inductively show that J_j is a regular ideal of A_j. The case $j = 1$ is simple: J_1 is a nonzero principal ideal in a ring without zerodivisors, so it must be regular.

Assume the inductive result holds below some index j. In the quotient sequence

$$0 \to \Sigma^{|\beta_{(n-j)}|} A_j \xrightarrow{\beta_{(n-j)}} A_j \to A_{j-1} \to 0,$$

the degree shift in the multiplication map (and induction on degree) shows that if J_{j-1} is regular on A_{j-1}, then J_{j-1} is automatically regular on A_j. If we additionally prove that J_{j-1} is prime on A_j and that $r_{n-j+1} \neq 0$ in the quotient, then A_j/A_{j-1} is an integral domain and multiplication by r_{n-j+1} would be injective and we would be done. In the degree $|r_{n-j+1}|$ of interest, there is an isomorphism $(A)_{|r_{n-j+1}|} \cong (A_j/J_{j-1})_{|r_{n-j+1}|}$, and hence $r_{n-j+1} \neq 0$ as desired.

We thus turn to primality. Note first that J_{j-1} is automatically prime in B_j, since B_j is a polynomial $\mathbb{F}_p[\beta_{(n-j)}^{\pm}]$–algebra and each of the generators of J_{j-1} is one of these polynomial generators of B_j. Suppose for contradiction that J_{j-1} is not prime in A_j, as witnessed by some elements $x, y \notin J_{j-1}$ satisfying $xy \in J_{j-1}$. Since J_{j-1} *is* prime in B_j, (by perhaps trading x and y) there is some minimum $k > 0$ such that

$$\beta_{(n-j)}^{\circ k} \circ x \in J_{j-1}.$$

We may as well assume $k = 1$, which we can arrange by tucking the stray factors of $\beta_{(n-j)}$ into x. Invoking the generators of J_{j-1}, we thus have an equation

$$\beta_{(n-j)} \circ x = \sum_{i=1}^{j-1} a_i \circ r_{n-i+1}$$

with $a_i \in A_j$ not all divisible by $\beta_{(n-j)}$. In fact, by moving elements onto the left-hand side we can assume that if $a_i \neq 0$ then $a_i \notin J_{i-1}$. In A_{j-1}, this equation becomes

$$0 = \sum_{i=1}^{j-1} a_i \circ r_{n-i+1}$$

with a_i not all in J_{i-1}. This is the desired contradiction, since J_{j-1} is regular in A_{j-1} by inductive hypothesis. $\square$

Corollary 4.3.5. *Set*

$$c_{i,j} = \dim_{\mathbb{F}_p} Q^* \mathbb{A}(H, BP)_{(2i,2j)}, \qquad\qquad d_{i,j} = \dim_{\mathbb{F}_p} \mathbb{F}_p[[v_n], b_{(0)}]_{2i,2j}.$$

Then $c_{i,j} \leq d_{i,j}$ *and* $d_{i,j} = d_{i+2,j+2}$.

Proof. We have seen that $c_{i,j}$ is bounded by the $\mathbb{F}_p$–dimension of

$$\left[\mathbb{F}_p[[v_n], b_{(d)} \mid d \geq 0, n \geq 0]/(r_1, r_2, \ldots)\right]_{i,j}.$$

But, since this ideal is regular and $|r_j| = |b_{(j)}|$, this is the same value as $d_{i,j}$. The other relation among the $d_{i,j}$ follows from multiplication by $b_{(0)}$, with $|b_{(0)}| = (2, 2)$. $\square$

We now turn to showing that this estimate is *sharp* and that the secondary comparison map is *onto*, and hence an isomorphism, using the bar spectral sequence. Recalling that the bar spectral sequence converges to a the homology of the *connective* delooping, let $\underline{BP}'_{2*}$ denote the connected component of $\underline{BP}_{2*}$ containing $[0_{2*}]$. We will then demonstrate the following theorem inductively:

Theorem 4.3.6 ([RW77, Induction 4.18])**.** *The following hold for all values of the induction index k:*

1. $Q^* H_{\leq 2(k-1)} \underline{BP}'_{2*}$ *is generated by* $\circ$*–products of the* $[v_n]$ *and* $b_{(j)}$.
2. $H_{\leq 2(k-1)} \underline{BP}'_{2*}$ *is isomorphic to a polynomial algebra in this range.*
3. *For* $0 < i \leq 2(k-1)$*, we have* $d_{i,j} = \dim_{\mathbb{F}_p} Q^* H_i \underline{BP}_{2j}$.

Before addressing the Theorem, we show that this finishes our calculation:

Proof of Theorem 4.3.1, assuming Theorem 4.3.6 for all k. Recall that we are considering the natural map

$$\mathbb{A}(H, BP) \to H_*\underline{BP}_{2*}.$$

The first part of Theorem 4.3.6 shows that this map is a surjection. The third part of Theorem 4.3.6 together with our counting estimate shows that the induced map

$$Q^*\mathbb{A}(H, BP) \to Q^*H_*\underline{BP}_{2*}$$

is an isomorphism. Finally, the second part of Theorem 4.3.6 says that the original surjective map, before passing to $*$–indecomposables, targets a polynomial algebra and is an isomorphism on indecomposables, hence must be an isomorphism as a whole. $\square$

Proof of Theorem 4.3.6. The infinite loopspaces in $\underline{BP}_{2*}$ are related by $\Omega^2\underline{BP}'_{2(*+1)} = \underline{BP}_{2*}$, so we will use two bar spectral sequences to extract information about $\underline{BP}'_{2(*+1)}$ from $\underline{BP}_{2*}$. Since we have assumed that $H_{\le 2(k-1)}\underline{BP}_{2*}$ is polynomial in the indicated triangular range near zero, we know that in the first spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{H_*\underline{BP}_{2*}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*\underline{BP}_{2*+1}$$

the E^2–page is, in the same range, exterior on generators in Tor–degree 1 and topological degree one higher than the generators in the polynomial algebra. Since differentials lower Tor–degree, the spectral sequence is multiplicative, and there are no classes on the 0–line, it collapses in the range $[0, 2k-1]$. Additionally, since all the classes are in odd topological degree, there are no algebra extension problems, and we conclude that $H_*\underline{BP}_{2*+1}$ is indeed exterior up through degree $(2k-1)$.

We now consider the second bar spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{H_*\underline{BP}_{2*+1}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*\underline{BP}_{2(*+1)'}.$$

The Tor algebra of an exterior algebra is divided power on a class of topological dimension one higher. Since these classes are now all in even degrees, the spectral sequence collapses in the range $[0, 2k]$. Additionally, these primitive classes are related to the original generating classes by double suspension, i.e., by forming the $\circ$–product with $b_{(0)}$. This shows the first inductive claim on the *primitive classes* through degree $2k$, and we must argue further to deduce our generation result for $x^{[p^j]}$ of degree $2k$ with $j > 0$. By inductive assumption, we can write

$$x = [y] \circ b_{(0)}^{\circ I_0} \circ b_{(1)}^{\circ I_1} \circ \cdots,$$

and one might be divinely inspired to consider the elemtn

$$z := [y] \circ b_{(j)}^{\circ I_0} \circ b_{(j+1)}^{\circ I_1} \circ \cdots.$$

This element z isn't equal to $x^{[p^j]}$ on the nose, but the diagonal of the difference $z - x^{[p^j]}$ lies in lower filtration degree—i.e., it is primitive as far as the filtration is concerned—and so we are again done.

The remaining thing to do is to use the size bounds: the only way that the map

$$\mathbb{A}(H, BP) \to H_* \underline{BP}_{2*}$$

could be surjective is if there were multiplicative extensions in the spectral sequence joining $x^{[p]}$ to x^p. Granting this, we see that the module ranks of the algebra itself and of its indecomposables are exactly the right size to be a free (i.e., polynomial) algebra, and hence this must be the case. □

We have actually accomplished quite a lot in proving Theorem 4.3.1, as this forms the input to an Atiyah–Hirzebruch spectral sequence.

Corollary 4.3.7 ([RW77, Corollary 4.7])**.** *For any complex-orientable cohomology theory E, the natural approximation maps give isomorphisms of Hopf rings*

$$\mathbb{A}(E, MU) \xrightarrow{\simeq} E_* \underline{MU}_{2*}, \qquad \mathbb{A}(E, BP) \xrightarrow{\simeq} E_* \underline{BP}_{2*}.$$

Proof. First, because $MU_{(p)}$ splits multiplicatively as a product of BPs, we deduce from Theorem 4.3.1 the case of $E = H\mathbb{F}_p$. Since $H\mathbb{F}_{p*}\underline{BP}_{2*}$ is even, it follows that $H\mathbb{Z}_{(p)*}\underline{BP}_{2*}$ is torsion–free on a lift of a basis, and similarly (working across primes) $H\mathbb{Z}_* \underline{MU}_{2*}$ is torsion–free on a simultaneous lift of basis. Next, using torsion–freeness, we conclude from an Atiyah–Hirzebruch spectral sequence that $MU_* \underline{MU}_{2*}$ is even and torsion–free itself, and moreover that the comparison is an isomorphism. Lastly, using naturality of Atiyah–Hirzebruch spectral sequences, given a complex–orientation $MU \to E$ we deduce that the spectral sequence

$$E_* \otimes H_*(\underline{MU}_{2*}; \mathbb{Z}) \cong E_* \otimes_{MU_*} MU_* \underline{MU}_{2*} \Rightarrow E_* \underline{MU}_{2*}$$

collapses, and similarly for the case of BP. □

This is an impressively broad claim: the loopspaces $\underline{MU}_{2*}$ are quite complicated, and that any general statement can be made about them is remarkable. That this fact follows from a calculation in $H\mathbb{F}_p$–homology and some niceness observations is meant to showcase the density of $\mathbb{CP}^\infty_H \cong \widehat{\mathbb{G}}_a$ inside of $\mathcal{M}_{\mathbf{fg}}$.

Remark 4.3.8. The analysis of the first bar spectral sequence in the proof of Theorem 4.3.6 also gave us a description of $H_* \underline{BP}_{2*+1}$, which is not directly visible to $\mathbb{A}(H, BP)$. Namely, the Hopf ring $H_* \underline{BP}_*$ can be presented as

$$H_* \underline{BP}_* \xleftarrow{\simeq} \mathbb{A}(H, BP)[e]/(e^{\circ 2} = \beta_{(0)}),$$

with e of degree 1. Additionally, analyzing the cohomological bar spectral sequence (and noting that the dual of a divided power algebra is a polynomial algebra) shows that each $H_* \underline{BP}_{2*}$ forms a *bipolynomial Hopf algebra*—i.e., both it and its dual are polynomial algebras.

Remark 4.3.9 ([Cha82], [Wil82, Section 10]). There is an alternative proof, due to Chan, that $H_* \underline{BP}_{2j}$ forms a bipolynomial Hopf algebra for each choice of j that makes no reference to Hopf rings. It proceeds along very similar lines, as it also studies the iterated bar spectral sequence, but it proceeds entirely by counting: the elements in the spectral sequence are never given explicit names, and hence there is no real hope of understanding the functor $\operatorname{SpH} H_* \underline{BP}_{2*}$ using these methods. By contrast, the Ravenel–Wilson method can be used to give an explicit enumeration of these classes [RW77, Section 5]. Our presentation here is something of a compromise.

Remark 4.3.10. The identification of the p–local and mod–p homology and cohomology of $\underline{BP}_{2k}$ as a bipolynomial Hopf algebra was first accomplished by Wilson in his PhD thesis [Wil73, Theorem 3.3]. He deduces quite a lot of interesting results from this observation. For instance, each bipolynomial Hopf algebra can be shown to split as a tensor product of indecomposable such [Wil73, Proposition 3.5], and this splitting is reflected by a splitting of $\underline{BP}_{2k}$ into a product of indecomposable H–spaces.

Remarkably, these indecomposable spaces can themselves be identified. For each n there is a ring spectrum $BP\langle n\rangle$ over BP with homotopy presented by the subalgebra $\pi_* BP\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$. This spectrum is *not* uniquely specified, a reflection of the algebraic failure of the ideal $(v_{n+1}, v_{n+2}, \ldots)$ to be invariant, and so this resists formal-geometric interpretation (cf., however, [LN12], [Str99a], …). Nonetheless, using Steenrod module techniques Wilson shows [Wil75, Section 6] that every simply-connected p–local H–space with torsion-free homology and (p–local ordinary) homology splits into a product of spaces Y_k, and that $Y_k = \underline{BP\langle n\rangle}_k$ for $|v_n| < k(p-1) \le |v_{n+1}|$.

In particular, the spaces $\underline{BP\langle n\rangle}_k$ in these bands *are* independent of choice of parent spectrum $BP\langle n\rangle$, and all p–local H–spaces satisfying these freeness properties are automatically infinite loopspaces—both extremely surprising results.

These bipolynomial algebras also play a critical role in the next section.

Theorem 6.1 of R–W *The Hopf ring for complex bordism* sounds like something related to Quillen's elementary proof. Since power operations give, in particular, unstable cohomology operations, we should add a remark here saying "You know, we've seen these before... see Ch 6."

4.4 Dieudonné modules

Our goal in this Lecture is to give a compact presentation of what a formal group is based on the following observation: the category of commutative cocommutative Hopf algebras of finite type over a ground field k forms an abelian category. It follows abstractly that this category admits a presentation as the module category for some (possibly noncommutative) ring, but in fact this ring and the assignment from a group scheme to linear algebraic data can both be described explicitly. This is the subject of *Dieudonné theory*, and our goal is to give an overview of some of its main results, including three different presentations of the equivalence.[7]

Begin with a 1–dimensional formal group $\widehat{\mathbb{G}}$ over a ring A. Our first avenue into Dieudonné theory is to recall that we have previously been interested in the invariant differentials $\omega_{\widehat{\mathbb{G}}} \subseteq \Omega^1_{\widehat{\mathbb{G}}/A}$ on $\widehat{\mathbb{G}}$. As explored in Theorem 2.1.22, when A is a $\mathbb{Q}$–algebra such differentials give rise to logarithms through integration. On the other hand, if A has positive characteristic p then there is a potential obstruction to integrating terms with exponents of the form $-1 \pmod p$, and in Lecture 3.3 we used this to lead us to the notion of p–height. We now explore a third twist on this set-up. Recall that $\Omega^1_{\widehat{\mathbb{G}}/A}$ forms the first level of the *algebraic de Rham complex* $\Omega^*_{\widehat{\mathbb{G}}/A}$. The de Rham complex only uses the underlying formal variety of $\widehat{\mathbb{G}}$ and not its group structure, but the product map

Cite me: Weinstein's geometry of Lubin–Tate spaces notes.

$$\mu\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$$

and the two projection maps

$$\pi_1, \pi_2\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$$

induce maps

$$\mu^*, \pi_1^*, \pi_2^*\colon C^1_{dR}(\widehat{\mathbb{G}}/A) \to C^1_{dR}(\widehat{\mathbb{G}} \times \widehat{\mathbb{G}}/A).$$

The translation invariant differentials are exactly those in the kernel of $\mu^* - \pi_1^* - \pi_2^*$, as considered at the chain level. We can weaken this to request only that that difference be *exact*, or zero at the level of cohomology of the de Rham complex.

Definition 4.4.1. The *cohomologically translation invariant differentials* is the A–submodule $PH^1_{dR}(\widehat{\mathbb{G}}/A) \subseteq H^1_{dR}(\widehat{\mathbb{G}}/A)$ defined as the kernel of $\mu^* - \pi_1^* - \pi_2^*$.

Example 4.4.2. [Kat81, Lemma 5.1.2] Consider the case that A is torsion-free (or "$\mathbb{Z}$–flat", if you like), and set $K = A \otimes \mathbb{Q}$ so that $A \to K$ is an injection. In this case the differentiation map $xA[\![x]\!] \to A[\![x]\!]$ is an injection and integration of power series is possible in K, so we can re-express first the definition of H^1_{dR} and second the conditions on our algebraic differentials in the following diagram of exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \left\{ \begin{array}{c} \text{integrals with} \\ A \text{ coefficients} \end{array} \right\} & \to & \left\{ \begin{array}{c} \text{all conceivable integrals} \\ \text{of differentials defined over } A \end{array} \right\} & \to & \{\text{missing integrals}\} & \to & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & xA[\![x]\!] & \longrightarrow & \{f \in xK[\![x]\!] \mid df \in A[\![x]\!]dx\} & \xrightarrow{d} & H^1_{dR}(\widehat{\mathbb{G}}/A) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & xA[\![x]\!] & \longrightarrow & \left\{ f \in xK[\![x]\!] \middle| \begin{array}{c} df \in A[\![x]\!]dx, \\ \delta f \in A[\![x,y]\!] \end{array} \right\} & \xrightarrow{d} & PH^1_{dR}(\widehat{\mathbb{G}}/A) & \longrightarrow & 0,
\end{array}$$

[7] Emphasis on "*some of its results*". Dieudonné theory is an enormous subject with many interesting results both internal and connected to arithmetic geometry and the theory of abelian varieties. We will explore almost none of this.

where x is a coordinate on $\widehat{\mathbb{G}}$, and δ is defined by $\delta[\omega] = (\mu^* - \pi_1^* - \pi_2^*)[\omega]$.

The flatness condition is not satisfied when working over a perfect field of positive characteristic p—our favorite setting in Lecture 3.3 and Case Study 3 more generally. However, de Rham cohomology has the following remarkable lifting property (which we have written here after specializing to H^1_{dR}):

Theorem 4.4.3. *[Kat81, Key Lemma 5.1.3] Let A be a p–local torsion-free ring, and let $f_1(x), f_2(x) \in xA[\![x]\!]$ be power series without constant term. If $f_1 \equiv f_2 \pmod{p}$, then for any differential $\omega \in A[\![x]\!]dx$ the difference $f_1^*(\omega) - f_2^*(\omega)$ is exact.*

Proof. Write $\omega = dg$ for $g \in K[\![x]\!]$, and write $f_2 = f_1 + p\Delta$. Then

$$\begin{aligned}\int (f_2^*\omega - f_1^*\omega) = g(f_2) - g(f_1) &= g(f_1 + p\Delta) - g(f_1) \\ &= \sum_{n=1}^{\infty} \frac{(p\Delta)^n}{n!} g^{(n)}(f_1).\end{aligned}$$

Since $g' = \omega$ has coefficients in A, so does the iterated derivative $g^{(n)}$ for all n, and hence the fraction $p^n/n!$ lies in the $\mathbb{Z}_{(p)}$–algebra A. □

Corollary 4.4.4 (H^1_{dR} is "crystalline"). *If $f_1, f_2 \colon V \to V'$ are maps of pointed formal varieties which agree mod p, then they induce the same map on H^1_{dR}.* □

Corollary 4.4.5 ([Kat81, Theorem 5.1.4]). *Any map $f \colon \widehat{\mathbb{G}}' \to \widehat{\mathbb{G}}$ of pointed varieties which is a group homomorphism mod p restricts to give a map $f^* \colon PH^1_{dR}(\widehat{\mathbb{G}}/A) \to PH^1_{dR}(\widehat{\mathbb{G}}'/A)$. Additionally, if f_1, f_2, and f_3 are three such maps of pointed varieties satisfying*

$$f_3 \equiv f_1 + f_2 \in \mathsf{FormalGroups}(\widehat{\mathbb{G}}'/p, \widehat{\mathbb{G}}/p),$$

then $f_3^* = f_1^* + f_2^*$ as maps $PH^1_{dR}(\widehat{\mathbb{G}}/A) \to PH^1_{dR}(\widehat{\mathbb{G}}'/A)$. □

In the case that k is a *perfect* field, the ring $\mathbb{W}_p(k)$ of p–typical Witt vectors on k is simultaneously torsion-free and universal among nilpotent thickenings of the residue field k. This emboldens us to make the following definition:[8]

Definition 4.4.6. [Kat81, Section 5.5] Let k be a perfect field of characteristic $p > 0$, and let $\widehat{\mathbb{G}}_0$ be a (1–dimensional) formal group over k. Then, choose a lift $\widehat{\mathbb{G}}$ of $\widehat{\mathbb{G}}_0$ to $\mathbb{W}_p(k)$, and define the *(contravariant) Dieudonné module* of $\widehat{\mathbb{G}}_0$ by $D^*(\widehat{\mathbb{G}}_0) := PH^1_{dR}(\widehat{\mathbb{G}}/\mathbb{W}_p(k))$.

[8]There is a better definition one might hope for, which instead assigns to each potential thickening and lift a "Dieudonne module", and then work to show that they all arise as base-changes of this universal one. This is possible and technically superior to the approach we are taking here [Kat81, Theorem 5.1.6], [Mes72, Chapter 4], [Gro74].

Remark 4.4.7. This is independent of choice of lift up to coherent isomorphism. Given any other lift $\widehat{\mathbb{G}}'$ of $\widehat{\mathbb{G}}_0$ to $\mathbb{W}_p(k)$, we can find *some* power series—not necessarily a group homomorphism—covering the identity on $\widehat{\mathbb{G}}_0$. Corollary 4.4.4 then shows that this map induces a canonical isomorphism between the two potential definitions of $D^*(\widehat{\mathbb{G}}_0)$.

Note that the module $D^*(\widehat{\mathbb{G}}_0)$ carries some natural operations:

- Arithmetic: $D^*(\widehat{\mathbb{G}}_0)$ is naturally a $\mathbb{W}_p(k)$–module, with the action by ℓ corresponding to multiplication–by–ℓ internal to $\widehat{\mathbb{G}}_0$.

- Frobenius: The map $x \mapsto x^p$ is a homomorphism of formal groups modulo p, so it induces a φ–semilinear map $F\colon D^*(\widehat{\mathbb{G}}_0) \to D^*(\widehat{\mathbb{G}}_0)$. That is, $F(\alpha v) = \alpha^\varphi F(v)$, where φ is a lift of the Frobenius on k to $\mathbb{W}_p(k)$.

- Verschiebung: Inspired by Lemma 3.3.6, we might also seek a Verschiebung operator V satisfying $FV = p$. Our explicit formula for F lets us guess such a map:

$$V\colon \sum_{n=1}^{\infty} a_n x^n \mapsto p \sum_{n=1}^{\infty} a_{pn}^{\varphi^{-1}} x^n.$$

  It satisfies $FV = p$ and anti-semilinearity: $aV(v) = V(a^\varphi v)$.

With this, we come to the main theorem of this Lecture:

Theorem 4.4.8 ([Gro74, Théorème 4.2], [Dem86, Sections III.8-9])**.** *The functor D^* determines a contravariant equivalence of categories between smooth 1–dimensional formal groups $\widehat{\mathbb{G}}_0$ over k of finite p–height and Dieudonné modules, which are modules M over the Cartier–Dieudonné ring*

$$\mathrm{Cart}_p = \mathbb{W}_p(k)\langle F, V\rangle \Bigg/ \left(\begin{array}{c} FV = VF = p, \\ Fw = w^\varphi F, \\ wV = Vw^\varphi \end{array}\right)$$

which furthermore satisfy the following three technical conditions:

Are these the correct technical conditions for the *contravariant* Dieudonné module?

- *Finiteness: M is a finite-dimensional free $\mathbb{W}_p(k)$–module.*

- *Reduced: $M \cong \lim_r M/V^r M$.*

- *Uniform: $M/VM \to V^r M/V^{r+1} M$ is an isomorphism.* □

Remark 4.4.9. Several invariants of the formal group associated to a Dieudonné module can be read off from the functor D^*. For example, the $\mathbb{W}_p(k)$–rank of $D^*(\widehat{\mathbb{G}}_0)$ computes the height of $\widehat{\mathbb{G}}_0$. Additionally, the quotient $D^*(\widehat{\mathbb{G}}_0)/FD^*(\widehat{\mathbb{G}}_0)$ is canonically isomorphic to the cotangent space $T_0^*\widehat{\mathbb{G}}_0 \cong \omega_{\widehat{\mathbb{G}}_0}$.

Example 4.4.10. Consider $\widehat{\mathbb{G}}_0 = \widehat{\mathbb{G}}_m$. For x the usual coordinate, we have $[p](x) = x^p$, and hence the Frobenius F acts on $D^*(\widehat{\mathbb{G}}_m)$ by $Fx = px$. It follows that $Vx = x$ and $D^*(\widehat{\mathbb{G}}_m) \cong \mathbb{W}_p(k)\{x\}$ with this Cart_p–module structure.

Example 4.4.11. We also give a kind of non-example: $\widehat{\mathbb{G}}_a$ is *not* a finite height formal group, and its Dieudonné module is correspondingly strangely behaved:

$$D^*(\widehat{\mathbb{G}}_a) = \mathbb{F}_2\{x, Fx, F^2x, \ldots\}/(V = 0).$$

Example 4.4.12 (cf. Example 1.2.9). Dieudonné theory admits an extension to finite (flat) group schemes as well, and the torsion quotient of the Dieudonné module of a formal group agrees with the Dieudonné module associated to its torsion subscheme:

$$D^*(\widehat{\mathbb{G}}_0[p^j]) = D^*(\widehat{\mathbb{G}}_0)/p^j.$$

For example, this gives

$$D^*(\widehat{\mathbb{G}}_m[p]) = \mathbb{F}_p\{x\} \Big/ \left(\begin{array}{c} Fx = 0, \\ Vx = x \end{array} \right).$$

We extract the subgroup scheme α_2 as the finite Dieudonné quotient module $D^*(\alpha_2) = \mathbb{F}_2\{x\} \leftarrow D^*(\widehat{\mathbb{G}}_a)$ of the Dieudonné module associated to $\widehat{\mathbb{G}}_a$ above. We can now verify the four claims from Example 1.2.9:

- *The group scheme α_2 has the same underlying structure ring as $\mu_2 = \mathbb{G}_m[2]$ but is not isomorphic to it.* There are now several ways to see this, the simplest of which is that the Verschiebung operator acts nontrivially on $D^*(\mu_2)$ but wholly trivially on $D^*(\alpha_2)$.

- *There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.* Suppose that G were such a group scheme, so that $D^*(G)/2$ would give $D^*(\alpha_2)$. It can't be the case that $D^*(G)$ has only 2–torsion, since then this quotient would be a null operation, so it must be the case that $D^*(G) = \mathbb{Z}/4\{x\}$. The action of both F and V on x must vanish after quotienting by 2, so it must be the case that $Fx = 2cx$ and $Vx = 2dx$ for some constants c and d—but this violates $FVx = 2x$.

- *If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence*

  $$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$

  However, this short exact sequence doesn't split (even after making a base change). This follows from calculating the action of F and V to get a short exact sequence of Dieudonné modules:

  $$0 \to \mathbb{F}_2\{Fx\} \Big/ \left(\begin{array}{c} F = 0, \\ V = 0 \end{array} \right) \to \mathbb{F}_2\{x, Fx\} \Big/ \left(\begin{array}{c} F^2x = 0, \\ V = 0 \end{array} \right) \to \mathbb{F}_2\{x\} \Big/ \left(\begin{array}{c} F = 0, \\ V = 0 \end{array} \right) \to 0.$$

  The exact sequence is split as $\mathbb{F}_2$–modules, but not as Dieudonné modules.

- *The subgroups of $\alpha_2 \times \alpha_2$ of rank two are parameterized by $\mathbb{P}^1$.* The Dieudonné module of the product is quickly computed:

$$D^*(\alpha_2 \times \alpha_2) = D^*(\alpha_2) \oplus D^*(\alpha_2) = \mathbb{F}_2\{x_1, x_2\} \Big/ \left(\begin{array}{c} F = 0, \\ V = 0 \end{array} \right).$$

  An inclusion of a rank 2 subgroup scheme corresponds to a projection of this Dieudonné module onto a 1–dimensional quotient module, and the ways to choose the kernel of this projection encompass a $\mathbb{P}^1$.

Example 4.4.13. We can also use Dieudonné theory to compute the automorphism group of a fixed Honda formal group, which is information we wanted back in Lecture 3.6. Take Γ_d to be Honda formal group law of height d over $\mathbb{F}_{p^d}$, which has Dieudonné module

$$D^*(\Gamma_d) = \mathrm{Cart}_p/(F^d = p).$$

The endomorphism ring of a quotient module of its parent ring is canonically isomorphic to the module itself, giving

$$\mathrm{End}\,\Gamma_d \cong \mathbb{W}_p(\mathbb{F}_{p^d})\langle F\rangle \Big/ \left(\begin{array}{c} Fw = w^\varphi F, \\ F^d = p \end{array} \right)$$

and hence

$$\mathrm{Aut}\,\Gamma_d \cong \left(\mathbb{W}_p(\mathbb{F}_{p^d})\langle F\rangle \Big/ \left(\begin{array}{c} Fw = w^\varphi F, \\ F^d = p \end{array} \right) \right)^\times.$$

Remark 4.4.14 ([Kat81, Theorem 5.2.1]). There is also a relationship between this representation of the Dieudonné functor and the deformation theory of formal groups from Lecture 3.4: a class $[f(x)dx] \in D^*(\widehat{\mathbb{G}}_0)$ begets a class in $e(f) \in \mathrm{Ext}^1(\widehat{\mathbb{G}}, \widehat{\mathbb{G}}_a)$ given by the cobar 1–cocycle $f(x +_{\widehat{\mathbb{G}}} y) - f(x) - f(y)$. In fact, this assignment is surjective, and the additional information lost in the kernel is a trivialization of the Lie algebra extension

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\widehat{\mathbb{G}}_a) \arrow[r] & \mathrm{Lie}(E) \arrow[r] & \mathrm{Lie}(\widehat{\mathbb{G}}) \arrow[r] \arrow[l, bend right, dotted] & 0 \end{tikzcd}$$

associated to the group scheme extension E classified by $e(f)$.

Having gotten some feel for the behavior and the usefulness of the Dieudonné functor, we now turn our attention to some alternative presentations of it. In this next presentation we will not have to worry about lifts to $\mathbb{W}_p(k)$, so we take $\widehat{\mathbb{G}}$ itself to be a formal group over a perfect field k of positive characteristic p. Cartier's *functor of curves* is defined by the formula

$$C\widehat{\mathbb{G}} = \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}).$$

This is, again, a kind of relaxing of familiar data from Lie theory, taken from a different direction: rather than studying just the exponential curves, $C\widehat{\mathbb{G}}$ tracks all possible curves. In Lecture 3.3, we considered three kinds of operations on a given curve $\gamma \colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$:

- Homothety: given a scalar $a \in A$, we define $\theta_a \gamma(t) = \gamma(a \cdot t)$.
- Verschiebung: given an integer $n \geq 1$, we define $V_n \gamma(t) = \gamma(t^n)$.
- Arithmetic: given two curves γ_1 and γ_2, we can use the group law on $\widehat{\mathbb{G}}$ to define $\gamma_1 +_{\widehat{\mathbb{G}}} \gamma_2$. Moreover, given $\ell \in \mathbb{Z}$, the ℓ–fold sum in $\widehat{\mathbb{G}}$ gives an operator

$$\ell \cdot \gamma = \overbrace{\gamma +_{\widehat{\mathbb{G}}} \cdots +_{\widehat{\mathbb{G}}} \gamma}^{\ell \text{ times}}.$$

  This extends to an action by $\ell \in \mathbb{W}_p(k)$.
- Frobenius: given an integer $n \geq 1$, we define

$$F_n \gamma(t) = \sum_{i=1}^{n}{}_{\widehat{\mathbb{G}}} \gamma(\zeta_n t^{1/n}),$$

  where ζ_n is an n^{th} root of unity. (This formula is invariant under permuting the root of unity chosen, so determines a curve defined over the original ground ring by Galois descent.)

Definition 4.4.15 (cf. Definition 3.3.5, Definition 3.3.8)**.** A curve γ on a formal group is p–typical when $F_n \gamma = 0$ for $n \neq p^j$. Write $D_* \widehat{\mathbb{G}} \subseteq C\widehat{\mathbb{G}}$ for the subset of p–typical curves.

Lemma 4.4.16 ([Zin84, Equation 4.13])**.** *In the case that the base ring is p–local, $C\widehat{\mathbb{G}}$ splits as a sum of copies of $D_*\widehat{\mathbb{G}}$. There is a natural section $C\widehat{\mathbb{G}} \to D_*\widehat{\mathbb{G}}$ called p–typification, given by the same formula as in Lemma 3.3.6.* □

This construction also plays the role of a Dieudonné functor:

Theorem 4.4.17. *[Zin84, Theorem 3.5 and Theorem 3.28] The functor D_* determines a* covariant *equivalence of categories between smooth 1–dimensional formal groups over k of finite p–height and Dieudonné modules satisfying the three technical conditions of Theorem 4.4.8.* □

One of the main ingredients in the proof of Theorem 4.4.17 is a representability result, which comes out of the following interesting construction. The *Witt scheme* $\mathbb{W}$ represents power series of the form $\prod_{j=1}^{\infty}(1 - a_j x^j)$, and it carries the structure of a group scheme by re-factoring the product of two such power series. Using a rational exponential and logarithm, we also have

$$\prod_{j=1}^{\infty}(1 - a_j x^j) = \exp \log \prod_{j=1}^{\infty}(1 - a_j x^j) = \exp \sum_{j=1}^{\infty} \left(- \sum_{k=1}^{\infty} \frac{1}{k} (a_j x^j)^k \right) = \exp \sum_{n=1}^{\infty} \frac{-t^n}{n} \sum_{m|n} m a_m^{n/m}.$$

These polynomials w_n, called *ghost polynomials*, describe an injective logarithmic map

$$(w_1, w_2, \ldots)\colon \mathbb{W} \to \mathbb{G}_a^{\infty},$$

the image of which is characterized by the property $w_n(a) \equiv w_{n/p}(a)^p \pmod{p^j}$ for p^j the maximum power of p dividing n. Over a $\mathbb{Z}_{(p)}$–algebra, there is a natural map on ghost components

$$t(w_*(a_*))_{n'p^j} = (w_*(a_{p^j})),$$

witnessing a decomposition $\mathbb{W} \times \operatorname{Spec}_{(p)} \cong \prod_{p \nmid n} \mathbb{W}_p$. Both of these objects are natural limits of their truncations to finitely many power series product terms, and hence they both admit natural *formal* objects, $\widehat{\mathbb{W}}$ and $\widehat{\mathbb{W}}_p$, by taking colimits of the formal completions of these truncations.

Lemma 4.4.18 ([Zin84, Chapter 3]). *There are natural correspondences*

$$\begin{aligned} \mathsf{FormalGroups}(\widehat{\mathbb{W}}, \widehat{\mathbb{G}}) &\cong \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) = C\widehat{\mathbb{G}}, \\ \mathsf{FormalGroups}(\widehat{\mathbb{W}}_p, \widehat{\mathbb{G}}) &\cong D_*(\widehat{\mathbb{G}}), \end{aligned}$$

where the universal curve is specified by

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□

This Lemma induces quite a lot of structure on $\widehat{\mathbb{W}}_p$: the Frobenius and Verschiebung operators on curves become operators acting on the formal Witt scheme, which on ghost components have the following behavior:

$$w_n(V_m a_*) = m w_{nm}(a_*), \qquad w_n(F_m a_*) = \begin{cases} x_{n/m} & \text{if } m \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

One can use this to give an explicit description of the inverse to the covariant Dieudonné functor whose existence is asserted by Theorem 4.4.17: the idea is that the evaluation map $\widehat{\mathbb{W}}_p \times C\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is surjective, and so we can recover $\widehat{\mathbb{G}}$ by imposing intertwining relations stemming from the evaluation map. Following this through results in the following presentation:

$$D_*{}^{-1}(M)(T) = (\widehat{W}_p(T) \otimes_{\mathbb{W}_p(k)} M) \Big/ \begin{pmatrix} Va \otimes m = a \otimes Fm, \\ Fa \otimes m = a \otimes Vm \end{pmatrix}.$$

We now seek to compare our two Dieudonné functors. In some sense, the comparison is inspired by an integration pairing: $D_*(\widehat{\mathbb{G}})$ is populated by curves γ on $\widehat{\mathbb{G}}$ and $D^*(\widehat{\mathbb{G}})$ is populated by 1–forms ω on $\widehat{\mathbb{G}}$, which we would like to sew together to form

$$D_*(\widehat{\mathbb{G}}) \times D^*(\widehat{\mathbb{G}}) \xrightarrow{(\gamma,\omega) \mapsto \text{“}\int_\gamma \omega\text{”}} \mathbb{W}_p(k).$$

There is more to say here. Erick Knight suggested that there should be an alternative definition of covariant Dieudonné theory also expressed by the cohomology of some sheaf, and that this will give rise to the desired duality pairing with values in some Tate-twisted ground object, which is precisely what we are looking for.

This can be made rigorous by noting the following basic but ultra-important fact about the Witt scheme:

Lemma 4.4.19. *There is a formal group scheme $\widehat{C\mathbb{W}}_p$ with the property*

Sure seems to me like $\widehat{C\mathbb{W}}_p$ is (canonically, even??) isomorphic to $\widehat{\mathbb{W}}_p$. It would be nice to sort this out.

$$\mathsf{FormalGroups}(\widehat{\mathbb{W}}_p, \widehat{\mathbb{G}})^* \cong \mathsf{FormalGroups}(\widehat{\mathbb{G}}, \widehat{C\mathbb{W}}_p). \quad \square$$

Corollary 4.4.20 ([MM74, Section II.15], [Kat81, Equation 5.5.2])**.** *There is an isomorphism $(D_*\widehat{\mathbb{G}})^* \xrightarrow{\cong} D^*\widehat{\mathbb{G}}$.*

Construction. There is a canonical short exact sequence

$$0 \to \widehat{\mathbb{G}}_a \to \widehat{C\mathbb{W}}_p \xrightarrow{V} \widehat{C\mathbb{W}}_p \to 0,$$

and a co-curve $\gamma^* \colon \widehat{\mathbb{G}} \to \widehat{C\mathbb{W}}_p$ gives a pullback sequence

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathbb{G}}_a \arrow[r] & \widehat{\mathbb{W}}_p \arrow[r, "V"] & \widehat{\mathbb{W}}_p \arrow[r] & 0 \\
0 \arrow[r] & \widehat{\mathbb{G}}_a \arrow[r] \arrow[u, equal] & E \arrow[r] \arrow[u] & \widehat{\mathbb{G}} \arrow[r] \arrow[u, "\gamma^*"] & 0.
\end{tikzcd}$$

This latter sequence is a rigidified extension of $\widehat{\mathbb{G}}$ by $\widehat{\mathbb{G}}_a$, as in Remark 4.4.14. The conclusion of Mazur and Messing is that this is an isomorphism. $\square$

Can you give more intuition about how these two presentations are related, for example from Lie theory? What curve does a cohomologically left-invariant form get sent to? Is "cohomologically-invariant" analogous to "p-typification", perhaps along the lines of the "crystalline"-ness of H^1_{dR}? Can these primitives be furthermore related to the idea that taking primitive cooperations selects the additive ones?

Remark 4.4.21 ([Str, Remark 18.1]). A coordinate x on $\mathbb{C}\mathrm{P}^\infty_E$ induces a sequence of isomorphisms

$$\mathsf{FormalGroups}(BU_E, \widehat{\mathbb{G}}) \cong \mathsf{FormalSchemes}(\mathbb{C}\mathrm{P}^\infty_E, \widehat{\mathbb{G}}) \overset{x}{\cong} \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) = C\widehat{\mathbb{G}},$$

which presents E^*BU as the Witt Hopf algebra. However, this isomorphism is not especially interesting: for one, it is *highly* dependent upon the choice of coordinate, but also the far right-hand object has no dependence on $\mathbb{C}\mathrm{P}^\infty_E$, and so the operations we have been studying—formal group auto- and endomorphisms of $\mathbb{C}\mathrm{P}^\infty_E$, mainly—do not act, and this isomorphism cannot be equivariant in any useful sense.

It would be nice to talk a little bit about connected and non-connected Hopf algebras and how Dieudonné theory extends across them. This is something Goerss takes the time to compensate for and it's something that Jacob bothers with in the Hopkins–Lurie manuscript (it's the difference between DM and DM_+ in the language of 1.4.15).

Remark 4.4.22 ([Gro74, Chapitre VI]). The contravariant Dieudonné functor described above has a natural extension by choosing lifts over other pro-Artinian k–algebras, like the Lubin–Tate moduli stack of Definition 3.4.3. The resulting network of objects most naturally organizes into a sheaf over the *crystalline site*, but it is possible in this setting to re-express such a sheaf as a quasi-coherent sheaf over the Lubin–Tate stack which is equipped with a flat connection, and it is additionally acted upon by the familiar operators F and V.

Remark 4.4.23. Dieudonné theory gives rise to an important function called the *period map*. Although the crystalline nature of the cohomology group H^1_{dR} makes our definition of D^* invariant of choice of lift, the underlying chain complex is *not* invariant of choice of lift. In turn, the subsheaf of honestly invariant differentials $\omega_{\widehat{\mathbb{G}}}$ selects an interesting 1–dimensional vector subspace of $PH^1_{dR}(\widehat{\mathbb{G}})$. Thinking of $\widehat{\mathbb{G}}$ as a point in $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\widehat{\mathbb{G}}_0}(\mathbb{W}_p(k))$, this observation gives rise to an interesting function

$$\begin{aligned}\pi_{GH}\colon (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\widehat{\mathbb{G}}_0}(\mathbb{W}_p(k)) &\to \mathbb{P}(D^*(\widehat{\mathbb{G}})),\\ \widehat{\mathbb{G}} &\mapsto [\omega_{\widehat{\mathbb{G}}} \subseteq \mathbb{P}(D^*(\widehat{\mathbb{G}}))].\end{aligned}$$

This map has incredibly good properties. It is equivariant for the action of $\operatorname{Aut}\widehat{\mathbb{G}}_0$ [HG94b, Theorem 1], and with enough work one can use this to extract explicit (recursive) formulas expressing the action [DH95], [Str, Section 24], [HG94a, Section 22], bringing some relief to the problem of Remark 3.6.17. Also, in a suitable context it becomes an étale morphism with identifiable fibers [HG94b, Theorem 1], [HG94a, Sections 23-4].

Remark 4.4.24. It is also possible to build versions of Dieudonné theory over still more exotic rings. The most successful such version is Zink's theory of Dieudonné displays [Zin02], which has found some application in algebraic topology [Law10].

4.5 Ordinary cooperations for Landweber flat theories

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Our goal in this Lecture is to put Dieudonné modules to work for us in algebraic topology. The executive summary of Dieudonné theory is that it gives a *linear* presentation of the theory of Hopf algebras. From the perspective of algebraic topology, functors sending cofiber sequences to exact sequences in a linear category are precisely the homology functors. Tying these two ideas together, if we can find a functor that sends exact sequences of spaces (or spectra) to exact sequences of Hopf algebras, we can post-compose it with a suitable version of the Dieudonné functor to get a homology functor and hence a *spectrum*.

To meet algebraic topology in its natural setting, it will be useful to also have a version of Dieudonné theory that is well-adapted to working with formal groups whose coordinate ring forms a *graded* Hopf algebra. Using as inspiration our previous identification

$$C(\widehat{\mathbb{G}}) = \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) \cong \mathsf{FormalGroups}(\widehat{\mathbb{G}}, \widehat{C\mathbb{W}}_p) \cong \mathsf{HopfAlgebras}(\mathcal{O}_{\widehat{C\mathbb{W}}_p}, \mathcal{O}_{\widehat{\mathbb{G}}}),$$

we are thus moved to form graded versions of the Witt Hopf algebra. More precisely, the following theorem says that there are graded versions of the Witt Hopf algebra that give a sequence of projective generators for the category of connected graded Hopf algebras over $\mathbb{F}_p$:

Theorem 4.5.1 ([Sch70, Section 3.2], [GLM93, Proposition 1.6]). *Let $S(n)$ denote the free graded-commutative Hopf algebra over $\mathbb{F}_p$ on a single generator in degree $n > 0$. There is a projective cover $H(n) \twoheadrightarrow S(n)$, given by the formula*

- *If either of the following conditions hold...*
  - *$p = 2$ and $n = 2^m k$ for $2 \nmid k$ and $m > 0$,* or
  - *$p \neq 2$ and $n = 2p^m k$ for $p \nmid k$ and $m > 0$,*

  then $H(n) = \mathbb{F}_p[x_0, x_1, \ldots, x_k]$ with the Witt vector diagonal, i.e., the diagonal is arranged so that the elements $w_j = x_0^{p^j} + px_1^{p^{j-1}} + \cdots + x_j$ are primitive.
- *Otherwise, $H(n) = S(n)$ is the identity.* □

Corollary 4.5.2. *The category* $\mathsf{GradedHopfAlgebras}_{\mathbb{F}_p/}^{>0,\mathrm{fin}}$ *of finite–type graded connected Hopf $\mathbb{F}_p$–algebras is a full subcategory of modules over*

$$\bigoplus_{n,m} \mathsf{GradedHopfAlgebras}(H(n), H(m)).$$

Proof sketch. This is a general nonsense consequence of having found a set of projective generators. The functor presenting the inclusion is

$$M \mapsto \bigoplus_{n=0}^{\infty} \mathsf{GradedHopfAlgebras}(H(n), M),$$

and since this functor is corepresented its automorphisms are encoded by the indicated ring. □

We would also like to give a set of conditions, analogous to the technical conditions appearing in the previous two presentations, which select this full subcategory out from all possible modules over this endomorphism ring.

Definition 4.5.3 ([GLM93, pg. 116]). Let $\mathsf{GradedDMods}$ denote the category of graded abelian groups M equipped with maps $V\colon M_{pn} \to M_n$ and $F\colon M_n \to M_{pn}$ (where n is even if $p \neq 2$) satisfying

1. $M_{<1} = 0$.
2. If n is odd, then $pM_n = 0$.
3. The composites are controlled by $FV = p$ and $VF = p$.[9]

[9] These are induced by the inclusion $H(n) \subseteq H(pn)$ and by the map $H(pn) \to H(n)$ sending x_n to x_{n-1}^p.

Remark 4.5.4. Suppose that n is even, written at odd primes in the form $n = 2p^m k$ with $p \nmid k$ or at $p = 2$ in the form $n = 2^m k$ with $2 \nmid k$ at $p = 2$. Then, combining the above relations, we get the torsion condition $p^{m+1} M_n = F^{m+1} V^{m+1} M_n = 0$.

Theorem 4.5.5 ([Sch70, Section 5], [GLM93, Theorem 1.11]). *The functor*

$$D_* \colon \mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/} \to \mathsf{GradedDMods},$$
$$D_*(H) = \bigoplus_n D_n(H) = \bigoplus_n \mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}(H(n), H)$$

is an exact equivalence of categories. Moreover, $D_* H(n)$ is characterized by the equation

$$\mathsf{GradedDMods}(D_* H(n), M) = M_n. \quad \square$$

Having produced our desired graded Dieudonné theory, we now need some topological input. We are by now well aware that the homology of an H–space forms a Hopf algebra, and the Serre spectral sequence for a fibration of H–spaces

$$F \to E \to B$$

takes the form

$$E_2^{*,*} = H^* B \otimes H^* F \Rightarrow H^* E.$$

The following result of Goerss–Lannes–Morel says that in the case that the fibration is one of *infinite loopspaces*, we have the exactness property we need:

Theorem 4.5.6 ([GLM93, Lemma 2.8]). *Let $X \to Y \to Z$ be a cofiber sequence of spectra. Then, provided $n > 1$ satisfies $n \not\equiv \pm 1 \pmod{2p}$, there is an exact sequence*

$$D_n H_* \Omega^\infty X \to D_n H_* \Omega^\infty Y \to D_n H_* \Omega^\infty Z. \quad \square$$

This Theorem is not especially easy to prove: one works very directly with unstable modules over the Steenrod algebra, the bar spectral sequence, and Postnikov decomposition of infinite loopspaces. We refer the reader to the paper directly, as we have been unable to find a useful improvement upon or even summary of the results presented there. Nonetheless, granting this Theorem, we use Brown representability to draw the following consequence:

Corollary 4.5.7 ([GLM93, Theorem 2.1, Remark 2.9]). *For $n > 1$ an integer satisfying $n \not\equiv \pm 1 \pmod{2p}$, there is a spectrum $B(n)$ satisfying*

$$B(n)_n X \cong D_n H_* \Omega^\infty X.$$

(As convention, when $n \equiv \pm 1 \pmod{2p}$ we set $B(n) := B(n-1)$, and $B(0) := \mathbb{S}^0$.) $\square$

Before exploiting this result to compute something about unstable cooperations, we will prove a sequence of small results making these spectra somewhat more tangible.

Lemma 4.5.8 ([GLM93, Lemma 3.2]). *The spectrum $B(n)$ is connective and p–complete.*

Proof. First, rearrange:

$$\pi_k B(n) = B(n)_n \mathbb{S}^{n-k} = D_n H_* \Omega^\infty \Sigma^\infty S^{n-k}.$$

If $k < 0$, n is below the connectivity of $\Omega^\infty \mathbb{S}^{n-k}$ and hence this vanishes. The second assertion follows from the observation that $H\mathbb{Z}_* B(n)$ is an $\mathbb{F}_p$–module, followed by an Adams spectral sequence argument. To see the assertion about being an $\mathbb{F}_p$–module, restrict to the case $n \not\equiv \pm 1 \pmod{2p}$ and calculate

$$\begin{aligned} H\mathbb{Z}_k B(n) &= B(n)_n \Sigma^{n-k} H\mathbb{Z} \\ &= D_n H_* K(\mathbb{Z}, n-k) \\ &= [H(n), H_* K(\mathbb{Z}, n-k)]_n \\ &= [Q^* H_* K(\mathbb{Z}, n-k)]_n. \end{aligned}$$

□

We can use a similar trick as in the second part of the proof to calculate the cohomology groups $H^* B(n)$:

Definition 4.5.9 ([GLM93, Example 3.6]). Let $G(n)$ be the free unstable $\mathcal{A}^*$–module on one generator of degree n, so that if M is an unstable $\mathcal{A}^*$–module then

$$\mathsf{Modules}_{\mathcal{A}^*}(G(n), M) = M_n.$$

This module admits a presentation as

$$G(n) = \begin{cases} \Sigma^n \mathcal{A} / \{\beta^\varepsilon P^i \mid 2pi + 2\varepsilon > n\} \mathcal{A} & \text{if } p > 2, \\ \Sigma^n \mathcal{A} / \{\mathrm{Sq}^i \mid 2i > n\} \mathcal{A} & \text{if } p = 2. \end{cases}$$

The Spanier–Whitehead dual of this right-module, $DG(n)$, is given by

$$\Sigma^n (DG(n))^* = \begin{cases} \mathcal{A} / \mathcal{A}\{\chi(\beta^\varepsilon P^i) \mid 2pi + 2\varepsilon > n\} & \text{if } p > 2, \\ \mathcal{A} / \mathcal{A}\{\chi \mathrm{Sq}^i \mid 2i > n\} & \text{if } p = 2. \end{cases}$$

Theorem 4.5.10 ([GLM93, Proof of Theorem 3.1]). *There is an isomorphism*

$$H^* B(n) \cong \Sigma^n (DG(n))^*.$$

Proof. We restrict attention to $n \not\equiv \pm 1 \pmod{p}$, where we can use Corollary 4.5.7 directly. Start, as before, by addressing the dual problem of computing the mod–p homology:

$$H_k B(n) = B(n)_n \Sigma^{n-k} H = D_n H_* K(\mathbb{F}_p, n-k).$$

The unstable module $G(n)$ also enjoys a universal property in the category of *stable* $\mathcal{A}^*$–modules, by passing to the maximal unstable quotient $\Omega^\infty M$ of a stable module M:

$$\mathsf{Modules}_{\mathcal{A}^*}(G(n), M) \cong [\Omega^\infty M]_n.$$

Hence, we can continue our computation:

$$\begin{aligned} H_k B(n) &= D_n H_* K(\mathbb{F}_p, n-k) \\ &= \mathsf{Modules}_{\mathcal{A}^*}(G(n), \Sigma^{n-k}\mathcal{A}_*) \\ &= \mathsf{Modules}_{\mathbb{F}_p}(G(n)_{n-k}, \mathbb{F}_p). \end{aligned}$$

We learn immediately that $H_*B(n)$ is finite. We would like to show, furthermore, that $H_*B(n)$ is the Spanier–Whitehead dual $\Sigma^n DG(n)$. It suffices to show

$$\mathsf{Modules}_{\mathcal{A}^*}(G(n), \Sigma^j \mathcal{A}_*) = \mathsf{Modules}_{\mathcal{A}^*}(\mathbb{F}_p, \Sigma^j \mathcal{A}_* \otimes H_*B(n))$$

for all values of j. This follows from calculating $B(n)_n \Sigma^{n+j} H$ using the same method. Finally, linear-algebraic duality and Definition 4.5.9 give the Theorem. $\square$

Lastly, for a *space* X, we definitionally have that have H_*X forms an unstable module over the Steenrod algebra, i.e., $\Omega^\infty H_*X = H_*X$. This has the following direct sequence (with minor fuss at the bad indices $n \equiv \pm 1 \pmod p$):

Lemma 4.5.11 ([GLM93, Lemma 3.3]). *For X a space, there is a natural surjection*

$$B(n)_n X \to H_n X. \quad \square$$

Let's now work toward using the $B(n)$ spectra to analyze the Hopf rings arising from unstable cooperations. Our intention is to prove the following:

Theorem 4.5.12. *For $F = H$ and E a Landweber flat homology theory, the comparison map*

$$\mathbb{A}(H, E) \to H_* \underline{E}_{2*}$$

is an isomorphism of Hopf rings.

We have previously computed that the comparison map

$$\mathbb{A}(H, BP) \to H_* \underline{BP}_{2*}$$

is an isomorphism. We will now reimagine this statement in terms of Dieudonné theory, but in order to do this we first have to reimagine some of Dieudonné theory itself, as our description of it is concerned with *Hopf algebras* rather than *Hopf rings*. A Hopf ring is not much structure on top of an (externally) graded system of (internally) graded Hopf algebras A_*: it is a multiplication map

$$\circ\colon A_* \boxtimes A_* \to A_*,$$

where "$\boxtimes$" is a kind of graded tensor product of externally graded Hopf algebras [HT98, Proposition 2.6], [BL07, Definition 2.2], [Goe99, Section 5]. Since D_* gives an equivalence of categories between internally graded Hopf algebras and internally graded Dieudonné modules, we should be able to find an analogous formula for the tensor product of Dieudonné modules.

Definition 4.5.13. [Goe99, pg. 154] The naive tensor product $M \otimes N$ of Dieudonné modules M and N receives the structure of a $\mathbb{W}(k)[V]$–module, where $V(x \otimes y) = V(x) \otimes V(y)$. We define the *tensor product of Dieudonné modules*[10] by

$$M \boxtimes N = \frac{\mathbb{W}(k)[F,V]}{(VF = p)} \otimes_{\mathbb{W}(k)[V]} (M \otimes N) \Big/ \left(\begin{array}{c} 1 \otimes Fx \otimes y = F \otimes x \otimes Vy, \\ 1 \otimes x \otimes Fy = F \otimes Vx \otimes y \end{array} \right).$$

Lemma 4.5.14 ([Goe99, Corollary 8.14]). *The natural map*

$$D_*(M) \boxtimes D_*(N) \to D_*(M \boxtimes N)$$

is an isomorphism. □

Definition 4.5.15. For a ring R, a *Dieudonné R–algebra A_** is an externally graded Dieudonné module equipped with an R–action and a unital multiplication

$$\circ\colon\ A_* \boxtimes A_* \to A_*.$$

Example 4.5.16 ([Goe99, Proposition 10.2]). Inspired by Lemma 4.2.8 and our interest in $H_*\underline{E}_*$, for a complex-oriented homology theory E we define its *algebraic Dieudonné E_*–algebra* by

$$R_E = E_*[b_1, b_2, \ldots] / \left(\ b(s+t) = b(s) +_E b(t)\ \right),$$

where V is multiplicative, V fixes E_*, and V satisfies $Vb_{pj} = b_j$.[11] We also write $D_E = \{D_{2m}H_*\underline{E}_{2n}\}$ for the even part of the topological Dieudonné algebra, and these come with natural comparison maps

$$R_E \to D_E \to D_*H_*\underline{E}_{2*}.$$

Theorem 4.5.17 ([Goe99, Theorem 11.7]). *Restricting attention to the even parts, the maps*

$$R_E \to D_E \to D_*H_*\underline{E}_{2*}$$

are isomorphisms for E Landweber flat.

[10] This definition is specialized to $\mathbb{F}_p$, where we don't have to worry about Frobenius semi-linearity.
[11] If E_* is torsion-free, then this determines the behavior of $F = \frac{1}{p}V$.

Proof. In Corollary 4.3.7, we showed that these maps are isomorphisms for $E = BP$. However, the right–hand object can be identified via Brown–Gitler juggling:

$$D_n H_* \underline{E}_{2j} = B(n)_n \Sigma^{2j} E = E_{2j+n} B(n).$$

It follows that if E is Landweber flat, then the middle– and right–terms are determined by change-of-base from the respective BP terms. Finally, the left term commutes with change-of-base by its algebraic definition, and the theorem follows. $\square$

Remark 4.5.18. The proof of Theorem 4.5.17 originally given by Goerss [Goe99] involved a lot more work, essentially because he didn't want to assume Theorem 4.3.1 or Corollary 4.3.7. Instead, he used the fact that $\Sigma^\infty_+ \Omega^2 S^3$ is a regrading of the ring spectrum $\bigvee_n B(n)$, together with knowledge of $BP_* \Omega^2 S^3$. Since we have already done the hard work of proving Theorem 4.3.1, we are not obligated to pursue this other line of thought.

Remark 4.5.19 ([Goe99, Proposition 11.6, Remark 11.4]). The Dieudonné algebra framework also makes it easy to add in the odd part after the fact. Namely, suppose that E is a torsion–free ring spectrum and suppose that $E_* B(n)$ is even for all n. In this setting, we can verify the purely topological version of this statement: the map

$$D_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*$$

is an isomorphism. To see this, note that because

$$E_{2n-2k-1} B(2n) \to D_{2n} H_* \underline{E}_{2k+1}$$

is onto and $E_{2n-2k-1} B(2n)$ is assumed zero, the group $D_{2n} H_* \underline{E}_{2k+1}$ vanishes as well. A bar spectral sequence argument shows that $D_{2n+1} H_* \underline{E}_{2k+2}$ is also empty [Goe99, Lemma 11.5.1]. Hence, the map on even parts

$$(D_E[e]/(e^2 - b_1))_{*,2n} \to (D_* H_* \underline{E}_*)_{*,2n}$$

is an isomorphism, and we need only show that

$$D_* H_* \underline{E}_{2n} \xrightarrow{e \circ -} D_* H_* \underline{E}_{2n+1}$$

is an isomorphism as well. Since $e(Fx) = F(Ve \circ x) = 0$ and $D_* A / F D_* A \cong Q^* A$ for a Hopf algebra A, we see that e kills decomposables and suspends indecomposables:

$$e \circ D_* H_* \underline{E}_{2n} = \Sigma Q H_* \underline{E}_{2n}.$$

This is also what happens in the bar spectral sequence, and the claim follows. In light of Theorem 4.5.17, this means that for Landweber flat E, the comparison isomorphism can be augmented to a further isomorphism

$$R_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*.$$

Remark 4.5.20 ([HH95]). The results of this Lecture are accessed from a different perspective by Hopkins and Hunton, essentially by forming a tensor product of Hopf rings and showing that Landweber flatness induces a kind of flatness with respect to the Hopf ring tensor product as well.

Jeremy asked whether there was a connection between Goerss's original proof and the free E_2–algebra with p killed which we keep dancing around this semester. I don't know, and it's a good question.

4.6 Cooperations among geometric points on $\mathcal{M}_{\mathbf{fg}}$

Our discussion of unstable cooperations has touched on each of the families of chromatic homology theories described in Definition 3.5.2 except one: the Morava K– and E–theories. Our final goal before moving on to other subjects is to describe some of the mixed unstable cooperations for $(K_\Gamma)_*\underline{K_{\Gamma'}}_*$. In complete generality, this seems like a difficult problem: our algebraic model is rooted in formal group homomorphisms, and we have not proven any theorems about the moduli of such for arbitrary finite-height formal groups. However, the landscape brightens considerably in the case where we pick $\Gamma' = \widehat{\mathbb{G}}_a$, as this is the sort of calculation we considered in Lemma 3.4.11 and Lemma 3.4.12. In light of this, we specialize Γ' to $\widehat{\mathbb{G}}_a$ (and hence $K_{\Gamma'}$ to an Eilenberg–Mac Lane spectrum H), and we abbreviate K_Γ to just K.

As with all the other major results of this Case Study, our approach will rest on the bar spectral sequence

$$\operatorname{Tor}_{*,*}^{K_*\underline{H}_q}(K_*, K_*) \Rightarrow K_*\underline{H}_{q+1}.$$

The analysis of this spectral sequence was first accomplished by Ravenel and Wilson [RW80], but has since been re-envisioned by Hopkins and Lurie [HL, Section 2]. In order to give an effective analysis of this spectral sequence in line with the theme of this book, we will endeavor to give algebro-geometric interpretations of its input and its output, beginning with the case $q = 0$ and $H = H(S^1[p^j])$ for some $1 \le j \le \infty$. This task itself begins with giving just *algebraic* descriptions of the input and output. For $j < \infty$, we have essentially already computed the output by other means:

Theorem 4.6.1 ([RW80, Theorem 5.7], [HL, Proposition 2.4.4], cf. Lemma 2.6.1)**.** *There is an isomorphism*

$$BS^1[p^j]_K \cong BS^1_K[p^j].$$

Remarks on proof. We have already proven a very similar theorem as Lemma 2.6.1. The crux of that argument was to show that multiplication by the p–series in $MU^*\mathbb{C}\mathrm{P}^\infty$ was injective without knowing very much about the ground ring MU^*. Here our task is even easier: any p–series for Γ takes the form $[p](x) = cx^{p^d} + \cdots$ for c a unit. Such an element is certainly not a zero-divisor. □

Remark 4.6.2. In the proof of the homological statement dual to Theorem 4.6.1, there is a corresponding exact sequence of Hopf algebras

$$\begin{tikzcd} & K_*BS^1 \arrow[dr, "-\frown [p^j](x)"] & \\ K_*(BS^1[p^j]) \arrow[ur] & & K_*BS^1, \arrow[ll, "\partial"] \end{tikzcd}$$

where again $\partial = 0$ and hence $K_*(BS^1[p^j])$ is presented as the kernel of the map "cap with $[p^j](x)$". We will revisit this duality in the next Case Study.

With this in hand, the analysis of the bar spectral sequence proceeds very much analogously to the example of the unstable dual Steenrod algebra of Lecture 4.1. We will analyze what *must* happen in the bar spectral sequence

$$\operatorname{Tor}_{*,*}^{K_*S^1[p^j]}(K_*, K_*) \Rightarrow K_*BS^1[p^j]$$

in order to reach the conclusion of Theorem 4.6.1. In the input to this spectral sequence, the ground algebra is given by a noncanonical isomorphism

$$K_*\underline{H(S^1[p^j])}_0 \cong K_*\underline{H\mathbb{Z}/p^j}_0 = K_*[[1]]/([1]^{p^j} - 1) = K_*[[1]-[0]]/\langle [1]-[0]\rangle^{p^j}.$$

The Tor–algebra for this truncated polynomial algebra $K_*[a_\varnothing]/a_\varnothing^p$ is then given by the formula

$$\operatorname{Tor}_{*,*}^{K_*[a_\varnothing]/a_\varnothing^{p^j}}(K_*, K_*) = \Lambda[a'_\varnothing] \otimes \Gamma[a_\varnothing],$$

the combination of an exterior algebra and a divided power algebra.[12] We know which classes are supposed to survive this spectral sequence, and hence we know where the differentials must be:

$$\begin{aligned} d(a_\varnothing)^{[p^{jd}]} &= a'_\varnothing, \\ \Rightarrow d(a_\varnothing)^{[i+p^{jd}]} &= a'_\varnothing \cdot a_\varnothing^{[i]}. \end{aligned}$$

The spectral sequence collapses after this differential. In the case $1 < j < \infty$, there are some hidden multiplicative extensions in the spectral sequence, but these too are all determined by already knowing the multiplicative structure on $K_*\underline{H(S^1[p^j])}_1$.

However, the case of $j = \infty$ is a bit different, beginning with the following Lemma:

Lemma 4.6.3. *For $q \geq 1$, there is a p–adic equivalence of $K(\mathbb{Q}/\mathbb{Z}_{(p)}, q)$ with $K(\mathbb{Z}, q+1)$.*

Proof. This is a consequence of the fiber sequence

$$K(\mathbb{Q}, q) \to K(\mathbb{Q}/\mathbb{Z}_{(p)}, q) \to K(\mathbb{Z}_{(p)}, q+1).$$

The first term has vanishing mod–p homology, forcing the $H\mathbb{F}_p$–Serre spectral sequence of the fibration to collapse and for the edge homomorphism to be an isomorphism. Similarly, the map $K(\mathbb{Z}, q+1) \to K(\mathbb{Z}_{(p)}, q+1)$ is an equivalence on mod–p homology. □

[12] In Ravenel–Wilson [RW80, Lemma 6.6], the elements $a_\varnothing$ and $a'_\varnothing$ are identified as a *transpotence* and a *homology suspension* respectively.

Remark 4.6.4. Thinking of $K(\mathbb{Z}, q+1)$ as $B^q S^1$, one can also think of this theorem as giving a p–adic equivalence between $B^q(S^1[p^\infty])$ and $B^q S^1$—i.e., the prime–to–p parts of S^1 do not matter for p–adic homotopy theory.

We use this to continue the analysis of the case $q = 1$ and $j = \infty$, where the Lemma gives $B(S^1[p^\infty]) = \mathbb{C}\mathrm{P}^\infty$. The bar spectral sequence of interest then takes the form

$$\mathrm{Tor}_{*,*}^{K_* S^1}(K_*, K_*) \Rightarrow K_* \mathbb{C}\mathrm{P}^\infty.$$

The input algebra $K_* S^1$ is exterior on a single generator in odd degree, and so its Tor–algebra is linearly dual to a power series algebra on a single generator in even degree. Since all of its input is even, this spectral sequence collapses immediately.

There is a lot of structure visible in this collection of spectral sequences, as considered simultaneously. Without further inspection, the spectral sequence at $j = \infty$ records that $\mathbb{C}\mathrm{P}^\infty_K$ is a formal variety, and the spectral sequences at the finite values $1 \le j < \infty$ encode in their differentials the behavior of the map $p^j \colon \Gamma \to \Gamma$ on functions—indeed, this appears to be the entire job of $a'_\varnothing$. Lastly, we notice that the E_∞ page of each finite-range spectral sequence includes into the spectral sequence at $j = \infty$, and moreover this filtration is exhaustive: every term in the $j = \infty$ spectral sequence appears at some $j < \infty$ stage. Since this last property is about the E_∞ pages, it is really a property of the formal group Γ, which we record in a definition:

Cite me: You can find a definition for this too.

Definition 4.6.5. A *p–divisible group*[13] over a field k is a system $\mathbb{G}_j$ of finite group schemes and inclusions $i_j \colon \mathbb{G}_j \to \mathbb{G}_{j+1}$ which participate in maps of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{G}_1 \arrow[r] \arrow[d] & \mathbb{G}_j \arrow[r, "p"] \arrow[d] & \mathbb{G}_j \arrow[d] \\ 0 \arrow[r] & \mathbb{G}_1 \arrow[r] & \mathbb{G}_{j+1} \arrow[r] & \mathbb{G}_{j+1}. \end{tikzcd}$$

In particular, this identifies $\mathbb{G}_j$ as the subgroup of p^j–torsion of the system $\mathbb{G}$. A p–divisible group is said to be *connected* when its constituent subgroups $\mathbb{G}_j$ are infinitesimal thickenings of $\operatorname{Spec} k$.

Cite me: Hazewinkel presumably.

Lemma 4.6.6. *Over a perfect field of positive characteristic p, a connected p–divisible group is equivalent to a smooth formal group of finite height.*

Correspondence. The maps in both directions are easy: a p–divisible group is sent to its colimit, and a formal group of finite height is sent to its system of p^j–torsion subgroups. In both directions there is something mild to check: that the colimit gives a formal variety, and that the system of p^j–torsion subgroups has the indicated exactness properties. □

[13]Some like to call these *Barsotti–Tate groups*, which is probably the better name, since "p–divisible group" sounds like a property rather than all this data.

We will soon see that these interrelations among the bar spectral sequences for the different Eilenberg–Mac Lane spaces, as well as the special behavior of the spectral sequence at $j = \infty$, are generic phenomena in q. We record the steps in our upcoming induction in the following Theorem:

Theorem 4.6.7 ([HL, Theorems 2.4.11–13]). *The following claims give a complete description of the Morava K–theory schemes associated to Eilenberg–Mac Lane spaces.*

1. *The formal scheme $(\underline{H(S^1[p^\infty])}_q)_K$ is a formal variety of dimension $\binom{d-1}{q-1}$.*

2. *Suppose that $(\underline{H(S^1[p^\infty])}_{q-1})_K$ is a p–divisible formal group of height $\binom{d}{q-1}$ and dimension $\binom{d-1}{q-1}$, that $(\underline{H(S^1[p])}_{q-1})_K$ models its p–torsion, and that the cup product induces an isomorphism*

$$\theta^{q-1} \colon \mathbb{Q}/\mathbb{Z}_{(p)} \otimes D(\underline{H(S^1[p^\infty])}_1)^{\wedge(q-1)} \to D(\underline{H(S^1[p^\infty])}_{q-1}),$$

where $D(G)$ denotes the Dieudonné module associated to $K_0(G)$ for $K_*(G) = K_0(G) \otimes_k K_*$ a p–divisible Hopf algebra. The same claims are then true with $q-1$ replaced everywhere by q.

3. *Consider the model $\mathbb{Q}/\mathbb{Z}_{(p)} \cong S^1[p^\infty]$ for the p–primary part of the circle group. Suppose that for each j, the short exact sequence of groups*

$$0 \to \frac{1/p^j \cdot \mathbb{Z}_{(p)}}{\mathbb{Z}_{(p)}} \to \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}} \to \frac{\mathbb{Q}}{1/p^j \cdot \mathbb{Z}_{(p)}} \to 0$$

induces a short exact sequence of group-schemes upon applying $(\underline{H(-)}_{q-1})_K$. The sequence of group schemes under $(\underline{H(-)}_q)_K$ is then also short-exact.

Look into whether you *really* need this funny system (cf. Notation 2.4.3) and this tensor with $\mathbb{Q}/\mathbb{Z}_{(p)}$.

Proof of Part 1. This claim turns out to be entirely algebraic and a matter of being able to compute $H^*(\mathbb{G}; \widehat{\mathbb{G}}_a)$ for $\mathbb{G}$ a connected p–divisible group. This is expressed in the main algebraic result of Hopkins–Lurie:

Theorem 4.6.8 ([HL, Theorem 2.2.10]). *Let $\mathbb{G}$ be a p–divisible group over a perfect field k of positive characteristic p. There is then an isomorphism*

$$H^*(\mathbb{G}; \widehat{\mathbb{G}}_a) \cong \mathrm{Sym}^*(\Sigma H^1(\mathbb{G}[p], \widehat{\mathbb{G}}_a)),$$

where "Σ" indicates that the classes are taken to lie in degree 2. □

Lemma 4.6.9 ([HL, Remark 2.2.5]). *If $\mathbb{G}$ is a connected p–divisible group of height d and of dimension n as a formal variety, then*

$$\mathrm{rank}\left(H^1(\mathbb{G}[p]; \widehat{\mathbb{G}}_a)\right) = d - n. \quad \square$$

Remark 4.6.10. In the case where $\mathbb{G} = \Gamma$ is the original height d formal group of dimension 1, this computes $H^*(\Gamma; \widehat{\mathbb{G}}_a)$ to be a power series algebra on $(d-1)$ generators. This is precisely the result we recorded by hand in Lemma 3.4.12.

Returning to the task at hand, we assume inductively that $(\underline{H(S^1[p^\infty])}_{q-1})_K$ is a connected p–divisible group of height $\binom{d}{q-1}$ and dimension $\binom{d-1}{q-1}$. Since the input to the bar spectral sequence is computed by formal group cohomology ([Laz97], [HL, Example 2.3.5], Proof of Lemma 3.2.5), it follows that the instance computing $K^*\underline{H(S^1[p^\infty])}_q$ has E_2–page an even-concentrated power series algebra of dimension

$$\binom{d}{q-1} - \binom{d-1}{q-1} = \binom{d-1}{q}.$$

The spectral sequence therefore collapses at this page, so that $(\underline{H(S^1[p^\infty])}_q)_K$ is a formal variety of the dimension claimed.

Proof of Part 2, with a gap. The other claims in Part 2 are formal after we check that θ^q is an isomorphism, since the p–power–torsion structure of $(\underline{H(S^1[p^\infty])}_q)_K$ can be read off from its Dieudonné module, as can its height. We introduce notation to analyze this statement: set M to be the Dieudonné module associated to $K_*\mathbb{C}\mathrm{P}^\infty$, i.e.,

$$M = D(\underline{H(S^1[p^\infty])}_1).$$

In the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\wedge q} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge q} & \longrightarrow & \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q} & \longrightarrow & 0 \\
 & & \downarrow V & & \downarrow V & & \downarrow V & & \\
0 & \longrightarrow & M^{\wedge q} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge q} & \longrightarrow & \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q} & \longrightarrow & 0
\end{array}$$

the middle map is an isomorphism. This forces V to be a surjective endomorphism of $M^{\wedge q} \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$, and the snake lemma shows that there is an isomorphism

$$\ker(V\colon\ \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q} \to \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}) \cong \operatorname{coker}(V\colon\ M^{\wedge q} \to M^{\wedge q}).$$

Picking any coordinate x and considering it as an element in the curves model of the Dieudonné functor, we see that the right-hand side is spanned by elements $x \wedge V^{\wedge I}x$, and hence the left-hand side has k–vector–space dimension $\binom{d-1}{q}$.

By very carefully studying the bar spectral sequence, one can learn that θ^q induces a surjection[14]

$$\ker V|_{\mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}} \to \ker V|_{D(\underline{H(S^1[p^\infty])}_q)}.$$

[14] The proof of this is quite complicated, and it rests on a pairing between the spectral sequence for $q = 1$ and the spectral sequence at $q - 1$, mapping to the spectral sequence for q. Remarkably, this same pairing is the main tool that powers the original approach of Ravenel–Wilson [RW80]. There, their program is to fix $j = 1$ and inductively analyze q using this same pairing, then use these base cases to ground a strong induction on j and q, and then finally to fix q and take the limit as $j \to \infty$.

In fact, since these two have the same rank, $\theta^q|_{\ker V}$ is an isomorphism on these subspaces. This is enough to conclude that θ^q is an injection: since the action of V is locally nilpotent, if θ^q ever failed to be an injection then we could apply V enough times to get an example of a nontrivial element in $\ker V|_{\mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}}$ mapping to zero. Finally, to show that θ^q is surjective, we again use the local nilpotence of V to filter $\mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}$ by the subspaces $\ker V^\ell$, $\ell \geq 1$, and it is then possible (though we admit the proof) to use our understanding of $\ker V$ to form preimages.

Proof of Part 3, mostly omitted. This proof is quite complicated, but it is, in spirit, a generalization of the observation at $q = 1$ that the role of the odd-degree classes in the bar spectral sequence is to pair up with those classes in the image of the $[p^j]$–map. In fact, their main assertion is:

> We give notation for the following rings:
>
> $$A = K_0 \underline{H(S^1[p^\infty])}_{q-1}, \qquad A' = K_0 \underline{H(S^1[p^j])}_{q-1}, \qquad R = K_0 \underline{H(S^1[p^\infty])}_{q}.$$
>
> Let $x' \in E_2^{1,0}$ be an element, and let $y' \in \mathfrak{m}_R$ satisfy
>
> $$y' = \psi(x') \otimes v \in \mathrm{Ext}^2_A \otimes_k \pi_2 K \cong \mathfrak{m}_R / \mathfrak{m}_R^2.$$
>
> Suppose that the Hopf algebra homomorphism $[p^j] \colon R \to R$ carries y' to an element $y \in \mathfrak{m}_R^s$, and let $x \in E_2^{2s,2s-2}$ denote the image of y under the composite
>
> $$\mathfrak{m}_R^s / \mathfrak{m}_R^{s+1} \cong \mathrm{Ext}^{2s}_A \otimes_k \pi_{2s} K \to \mathrm{Ext}^{2s}_{A'} \otimes_k \pi_{2s} K = E_2^{2s,2s} \xrightarrow{-v^{-1}} E_2^{2s,2s-2}.$$
>
> Then x and x' survive to the $(2s-1)^{\text{th}}$ page of the bar spectral sequence, and there we have
>
> $$d_{2s-1} x' = x.$$

From here, it is a matter of *very* carefully pairing elements up (cf. [HL, pg. 60]). □

Remark 4.6.11. Theorem 4.6.7 admits a restatement purely in terms of Hopf algebras, although Dieudonné theory was essential in its proof. The cup product gives a natural map

$$\mathrm{Alt}^q\, K_* \underline{H(S^1[p^j])}_1 \to K_* \underline{H(S^1[p^j])}_q,$$

where "Alt" refers to the alternating Hopf algebra.[15] The main result of this section is that this map is an isomorphism for all q, and indeed that the map from the free alternating Hopf *ring* maps isomorphically to the topological Hopf ring.

[15] Note that this alternation condition becomes dramatically more complicated in the case that the formal group law and its formal group inverse series become more complicated than that of $\widehat{\mathbb{G}}_a$.

Remark 4.6.12 ([HL, Section 3], [Hed], [Hed14]). Because $K^* \underline{H\mathbb{Z}/p^j}_q$ is even, you can hope to augment this to a calculation of $E^* \underline{H\mathbb{Z}/p^j}_q$ for $E = E_\Gamma$ the associated Morava E–theory. This is indeed possible, and the analogous formula is true at the level of Hopf algebras:

$$E_* \underline{H(S^1[p^j])}_q \cong \operatorname{Alt}^q E_* \underline{H(S^1[p^j])}_1.$$

However, the attendant algebraic geometry is quite complicated: you either need a form of Dieudonné theory that functions over $\mathcal{M}_{E_\Gamma}$ (and then attempt to drag the proof above through that setting), or you need to directly confront what "alternating power of a p–divisible group" means at the level of p–divisible groups (and forego all of the time-saving help afforded to you by Dieudonné theory).

Remark 4.6.13 (cf. Lemma 3.6.9). You'll notice that in $K_* \underline{H}_{q+1}$ if we let the q–index tend to ∞, we get the K–homology of a point. This is another way to see that the stable cooperations $K_* H$ vanish, meaning that the *only* information present comes from unstable cooperations.

Remark 4.6.14. Although the method of starting with the $j = \infty$ case and deducing from it the $1 \le j < \infty$ cases is due to Hopkins and Lurie, the observation that the spectral sequence at $j = \infty$ is remarkably simple had already been made by Ravenel and Wilson [RW80, Theorem 12.3], [RWY98, Theorem 8.1.3].

Remark 4.6.15. Theorem 4.6.7 has a statement in the language of Lecture 4.2. Rather than forming the algebraic approximation $\mathcal{A}(K, H) = K_*(\mathbb{C}\mathrm{P}^\infty)^{\circlearrowleft}_{K_*[H_*]}[H_* \mathbb{C}\mathrm{P}^\infty]$ as usual, we form the modified version

$$\mathcal{A}_{p^j}(K, H) = K_*(\underline{H(S^1[p^j])}_1)^{\circlearrowleft}_{K_*[H_*]}[H_* \underline{H(S^1[p^j])}_1].$$

Using the same unstable Kronecker pairing, this supports a natural map to $K_* \underline{H}_*$, and a summary of this Lecture's results is that it is an isomorphism.

Is this really right? I want this remark to explain the presence of odd-degree E–M spaces in this calculation, and I don't think I've exactly done that.

Maybe talk about some consequences: the Hopkins–Ravenel–Wilson results on finite Postnikov towers and so on?

I was thinking that this would give a counterexample to the idea that the additive unstable cooperations always present the functions on the scheme of homomorphisms, but now I see that this example works too. As lazy evidence, I think counting the ranks of $Q^* K(d)_* \underline{H\mathbb{Z}/p}_*$ and $\underline{\mathsf{FormalGps}}(\Gamma_d, \widehat{\mathbb{G}}_a)$ (using Dieudonné theory, or using Callan's tangent space trick) gives the same number. More seriously, I think if you write out the scheme of homo-

Case Study 5

The σ-orientation

By this point, we have seen a great many ways that algebraic geometry exerts control over the behavior of homotopy theory, stable and unstable. The goal in this Case Study is to explore a setting where algebraic geometry is itself tightly controlled: whereas the behavior of formal groups is quite open-ended, the behavior of *abelian varieties* is comparatively strict. We import this strictness into algebraic topology by stiudying complex-orientable cohomology theories E which have been tagged with an auxiliary abelian variety A via an isomorphism $\varphi\colon \mathbb{C}\mathrm{P}^\infty_E \cong A^\wedge_0$. In the case that A is an elliptic curve, this is our definition of an *elliptic cohomology theory*. The idea, then, is not that this puts serious constraints on the formal group $\mathbb{C}\mathrm{P}^\infty_E$ (although it does place some), but rather that the theory of abelian varieties endows A, and hence $A^\wedge_0$, with various bits of preferred data. This is the tack we take to construct a *canonical* $MU[6,\infty)$–orientation of E: for any complex-orientable E, we identify the collection of such ring maps with "θ–structures on $\mathbb{C}\mathrm{P}^\infty_E$"; a basic theorem about abelian varieties endows the elliptic curve A with a canonical such structure; and altogether this yields the desired orientation for an elliptic spectrum.

Making the identification of $MU[6,\infty)$–orientations with θ–structures requires real work, but many of the stepping stones are now in place. We begin with a technical section about especially nice formal schemes, called *coalgebraic*, and we use this to finally give the proof, announced back in Theorem 2.2.7, that the scheme of stable Weil divisors on a formal curve presents the free formal group on that curve. With that out of the way and with $MU[6,\infty)$ in mind as the eventual goal, we then summarize the behavior of the part of the Postnikov tower for complex bordism that we *do* understand—the cases of MUP and MU—and use this to make an analysis of MSU. In particular, we rely heavily on the results from Case Study 2 and Case Study 3 to understand the co/homological behaviors of $BU \times \mathbb{Z}$, BU, MUP, and MU.

The coherence of all of these statements gives us very explicit target theorems to aim for in our study of $MU[6,\infty)$, but we are forced to approach them from a different vantage point: whereas we can prove a splitting principle for SU–bundles, the analogous statement for $U[5,\infty)$–bundles does not appear to admit a direct proof. Consequently, the proofs of the other structure theorems for $BU[6, \infty)$ and $MU[6, \infty)$ are made considerably more complicated because we have to work with our splitting principle hands tied. This time, our main tools are the results developed in Case Study 4, which give us direct access to the co/homology of the layers of the Postnikov tower. When the dust of all this settles, we will have arrived at a very satisfying and complete theory of $MU[6, \infty)$–orientations, applicable to an arbitrary complex-orientable cohomology theory.

The reader gifted with an exceptional attention span will recall from Case Study 0 that we were *really* interested in *MString*–orientations, and that our interest in $MU[6, \infty)$–orientations was itself only a stepping stone. We close this Case Study with an analysis of this last setting, where we finally yield and place more hypotheses on E—a necessity for gaining calculational access to co/homological behavior of objects like *BString*, which lie outside of the broader complex-orientable story.

We also give a short résumé on the theory of elliptic curves in Lecture 5.5, extracting the smallest possible subset of their theory that we will need here.

Part of the theme of this chapter should be to use the homomorphism from topological vector bundles to algebraic line bundles—Neil's $\mathbb{L}$ construction—as inspiration for what to do, given suitable algebraic background.

Jack has a paper called *The motivic Thom isomorphism* in the Elliptic Cohomology LMS volume where he discusses some pretty interesting perspectives on genera. Could be worth mentioning as a "further reading" sort of thing, at least.

5.1 Coalgebraic formal schemes

We will now finally address an elephant that has been lingering in our metaphorical room: in the first third of this book we were primarily interested in thje formal scheme associated to the *cohomology* of a space, but in the second this we were primarily interested in a construction converting the *homology* of a spectrum to a sheaf over a context. Our goal for today is to, when possible, put these two variances on even footing. Our motivation for putting this lingering discrepancy to rest is more technical than aesthetic: we have previously wanted access to certain colimits of formal schemes (e.g., in Theorem 2.2.7). While such colimits are generally forbidding, similarly to colimits of manifolds, we will produce certain conditions under which they are accessible.

For E a ring spectrum and X a space, the diagonal map $\Delta\colon X \to X \times X$ induces a multiplication map on E–cohomology via the composite

$$E^* X \otimes_{E^*} E^* X \xrightarrow{\text{Künneth}} E^*(X \times X) \xrightarrow{E^*\Delta} E^* X.$$

Dually, applying E–homology, we have a pair of maps

$$E_* X \xrightarrow{E_*\Delta} E_*(X \times X) \xleftarrow{\text{Künneth}} E_* X \otimes_{E_*} E_* X,$$

where, remarkably, the Künneth map goes the wrong way to form a composite. In the case that that map is an isomorphism, the long composite induces the structure of an E_*–coalgebra on $E_* X$. In the most generous case that E is a field spectrum (in the sense of Corollary 3.5.12), the Künneth map is always invertible and, moreover, $E^* X$ is functorially the linear dual of $E_* X$. This motivates us to consider the following purely algebraic construction:

Definition 5.1.1. Let C be a coalgebra over a field k. We define a functor

$$\begin{aligned} \operatorname{Sch} C\colon \mathsf{Algebras}_{k/} &\to \mathsf{Sets}, \\ T &\mapsto \left\{ f \in C \otimes T \;\middle|\; \begin{array}{c} \Delta f = f \otimes f \in (C \otimes T) \otimes_T (C \otimes T), \\ \varepsilon f = 1 \end{array} \right\}. \end{aligned}$$

Lemma 5.1.2. *For a k–algebra A which is finite–dimensional as a k–module, there is a natural isomorphism* $\operatorname{Spec} A \cong \operatorname{Sch} A^*$.

Proof sketch. A point $f \in (\operatorname{Sch} A^*)(T) \subseteq A^* \otimes T$ gives rise to a k–module map $f_*\colon A \to T$, which the extra conditions in the formation of $(\operatorname{Sch} C)(T)$ force to be a ring homomorphism. The finiteness assumption is present exactly so that A is its own double–dual, giving an inverse assignment. □

If we drop the finiteness assumption, then this comparison proof fails entirely. Indeed, even picking a predual A of C, the multiplication on A gives rise only to maps

$$A^* \to (A \otimes_k A)^* \leftarrow A^* \otimes_k A^*,$$

which is not enough to make A^* into a k–coalgebra. However, if we start instead with a k–coalgebra C of infinite dimension, the following result is very telling:

Lemma 5.1.3 ([Dem86, pg. 12], [Mic03, Appendix 5.3], [HL, Remark 1.1.8])**.** *For C a coalgebra over a field k, any finite–dimensional k–linear subspace of C can be finitely enlarged to a subcoalgebra of C. Accordingly, taking the colimit gives a canonical equivalence*

$$\operatorname{Ind}(\mathsf{Coalgebras}_k^{\mathrm{fin}}) \xrightarrow{\simeq} \mathsf{Coalgebras}_k. \quad \square$$

This result allows us to leverage our duality Lemma pointwise: for an arbitrary k–coalgebra, we break it up into a lattice of finite k–coalgebras, and take their linear duals to get a reversed lattice of finite k–algebras. Altogether, this indicates that k–coalgebras generally want to model *formal schemes*.

Corollary 5.1.4. *For C a coalgebra over a field k expressed as a colimit $C = \operatorname{colim}_k C_k$ of finite subcoalgebras, there is an equivalence*

$$\operatorname{Sch} C \cong \{\operatorname{Spec} C_k^*\}_k.$$

This induces a covariant *equivalence of categories*

$$\mathsf{Coalgebras}_k \cong \mathsf{FormalSchemes}_{/k}.$$

This equivalence translates between the product of formal schemes, the tensor product of pro-algebras, and the tensor product of coalgebras. □

This covariant algebraic model for formal schemes is very useful. For instance, this equivalence makes the following calculation trivial:

Lemma 5.1.5 (cf. Lemma 1.5.1, Theorem 2.2.7, and Corollary 2.3.15)**.** *Select a coalgebra C over a field k together with a pointing $k \to C$. Write M for the coideal $M = C/k$. The free formal monoid on the pointed formal scheme* $\operatorname{Sch} k \to \operatorname{Sch} C$ *is given by*

$$F(\operatorname{Sch} k \to \operatorname{Sch} C) = \operatorname{Sch} \operatorname{Sym}^* M.$$

Writing $\Delta c = \sum_j \ell_j \otimes r_j$ for the diagonal on C, the diagonal on $\operatorname{Sym}^* C$ *is given by*

$$\Delta(c_1 \cdots c_n) = \sum_{j_1, \ldots, j_n} (\ell_{1,j_1} \cdots \ell_{n,j_n}) \otimes (r_{1,j_1} \cdots r_{n,j_n}). \quad \square$$

It is unfortunate, then, that when working over an object more general than a field Lemma 5.1.3 fails. Nonetheless, it is possible to bake into the definitions the machinery needed to get a good-enough analogue of Corollary 5.1.4.

Cite me: Include (a reference to) an example..

Definition 5.1.6. Let C be an R–coalgebra which is free as an R–module. A basis $\{x_j\}$ of C is said to be a *good basis* when any finite subcollection of $\{x_j\}$ can be finitely enlarged to a subcollection that spans a subcoalgebra. The coalgebra C is itself said to be *good* when it admits a good basis. A formal scheme X is said to be *coalgebraic* when it is isomorphic to $\operatorname{Sch} C$ for a good coalgebra C.

Example 5.1.7. The formal scheme $\widehat{\mathbb{A}}^n$ is coalgebraic. Beginning with the presentation

$$\widehat{\mathbb{A}}^n = \operatorname{Spf} R[\![x_1, \ldots, x_n]\!] = \operatorname*{colim}_J \operatorname{Spec} R[x_1, \ldots, x_n]/(x_1^{j_1}, \ldots, x_n^{j_n}),$$

write A_J for the algebra on the right-hand side. Each A_J is a free R–module, and we write

$$C_J = A_J^* = R\{\beta_K \mid K < J\}$$

for the dual coalgebra, with

$$\beta_K(x^L) = \begin{cases} 1 & \text{if } K = L, \\ 0 & \text{otherwise.} \end{cases}$$

The elements β_K form a good basis for the full coalgebra $C = \operatorname{colim}_J C_J$: any finite collection of them $\{\beta_K\}_{K \in \mathcal{K}}$ is contained inside any C_J satisfying $K < J$ for all $K \in \mathcal{K}$. As an additional consequence, all formal varieties are coalgebraic.

The main utility of this condition is that it gives us access to colimits of formal schemes:

Theorem 5.1.8 ([Str99b, Proposition 4.64])**.** *Suppose that $F \colon \mathsf{I} \to \mathsf{Coalgebras}_R$ is a colimit diagram of coalgebras such that each object in the diagram, including the colimit point, is a good coalgebra. Then*

$$\operatorname{Sch} \circ F \colon \mathsf{I} \to \mathsf{FormalSchemes}$$

is a colimit diagram of formal schemes. $\square$

For an example of the sort of constructions that become available via this Theorem, we prove the following Corollary by analyzing the symmetric power of coalgebras:

Corollary 5.1.9 ([Str99b, Example 4.65 and Proposition 6.4])**.** *When a formal scheme X is coalgebraic, the symmetric power $X^{\times n}_{\Sigma_n}$ exists. In fact, $\coprod_{n \ge 0} X^{\times n}_{\Sigma n}$ models the free formal monoid on X. Given an additional pointing* $\operatorname{Spec} R \to X$*, the colimit of the induced system*

$$\operatorname{colim}\left(\cdots \to X^{\times n}_{\Sigma_n} = \operatorname{Spec} R \times X^{\times n}_{\Sigma_n} \to X \times X^{\times n}_{\Sigma_n} \to X^{\times(n+1)}_{\Sigma_{n+1}} \to \cdots\right)$$

models the free formal monoid on the pointed formal scheme.

Proof sketch. The main points entirely mirror the case over a field: the symmetric power construction gives models for $X^{\times n}_{\Sigma_n}$, the symmetric algebra construction gives a model for the free formal monoid, and the stabilization against the pointing is modeled by inverting an element in the symmetric algebra. In each case, choosing a good basis for the coalgebra underlying X yields choices of good bases for the coalgebras arising from these constructions, essentially because their elements are crafted out of finite combinations of the elements of the original. □

In the specific case that $\operatorname{Spec} R \to X$ is a pointed formal *curve*, we can prove something more:

Corollary 5.1.10 ([Str99b, Proposition 6.12])**.** *For* $\operatorname{Spec} R \to X$ *a pointed formal curve, the free formal monoid is automatically an abelian group.*

Proof sketch. The main idea is that the coalgebra associated to a formal curve admits an increasing filtration F_k so that the reduced diagonal $\overline{\Delta} = \Delta - (1 \otimes \eta) - (\eta \otimes 1)$ reduces filtration degree:

$$\overline{\Delta}|_{F_k} \colon F_k \to \sum_{\substack{i,j>0 \\ i+j=k}} F_i \otimes F_j.$$

In turn, the symmetric algebra on the coalgebra associated to a formal curve inherits enough of this filtration that one can iteratively solve for a Hopf algebra antipode. □

We now reconnect this algebraic discussion with the algebraic topology that spurred it.

Lemma 5.1.11. *If E and X are such that E_*X is an E_*–coalgebra and*

$$E^*X = \mathsf{Modules}_{E_*}(E_*X, E_*),$$

then there is an equivalence

$$\operatorname{Sch} E_*X \cong X_E.$$

Proof. We have defined X_E to have formal topology induced by the compactly generated topology of X, and this same topology can also be used to write $\operatorname{Sch} E_*X$ as the colimit of finite E_*–coalgebras. □

Example 5.1.12 (cf. Theorem 4.6.1 and Remark 4.6.2). For a Morava K–theory K_Γ associated to a formal group Γ of finite height, we have seen that there is an exact sequence of Hopf algebras

$$K^0_\Gamma(BS^1) \xrightarrow{[p^j]} K^0_\Gamma(BS^1) \to K^0_\Gamma(BS^1[p^j]),$$

presenting $(BS^1[p^j])_K$ as the p^j–torsion formal subscheme $BS^1_K[p^j]$. The Hopf algebra calculation also holds in K–homology, where there is instead the exact sequence

$$(K_\Gamma)_0 B(S^1[p^j]) \to (K_\Gamma)_0 BS^1 \xrightarrow{(-)^{*p^j}} (K_\Gamma)_0 BS^1$$

presenting $(K_\Gamma)_0 B(S^1[p^j])$ as the p^j–order $*$–nilpotence in the middle Hopf algebra. Applying Sch to this last line covariantly converts this second statement about Hopf algebras to the corresponding statement above about the associated formal schemes—i.e., the behavior of the homology coalgebra is a direct reflection of the behavior of the formal schemes.

The example above also spurs us to consider an intermediate operation. We have seen that the algebra structure of the K–cohomology of a space and the coalgebra structure of the K–homology of the same space contain equivalent data: they both give rise to the same formal scheme. However, in the case at hand, BS^1 and $BS^1[p^j]$ are commutative H–spaces and hence give rise to *commutative and cocommutative Hopf algebras* on both K–cohomology and K–homology. Hence, in addition to considering the coalgebraic formal scheme $\mathrm{Sch}(K_\Gamma)_0 B(S^1[p^j])$, we can also consider the affine scheme $\mathrm{Spec}(K_\Gamma)_0 B(S^1[p^j])$. This, too, should contain identical information, and this is the subject of Cartier duality.

Definition 5.1.13 ([Str99b, Sections 6.3–4]). The *Cartier dual* of a commutative finite group scheme G is defined by the formula

$$DG = \underline{\mathsf{GroupSchemes}}(G, \mathbb{G}_m),$$

itself a finite group scheme. More generally, the Cartier dual of a commutative *coalgebraic* formal group $\widehat{\mathbb{G}}$ can also be defined by

$$D\widehat{\mathbb{G}} = \underline{\mathsf{GroupSchemes}}(\widehat{\mathbb{G}}, \mathbb{G}_m).$$

Lemma 5.1.14 ([Str99b, Proposition 6.19]). *Let $\widehat{\mathbb{G}}$ be a coalgebraic commutative formal group over a formal scheme X, and write $\mathbb{H} = \operatorname{Spec} \mathcal{O}^*_{\widehat{\mathbb{G}}}$ for the group scheme associated to its dual Hopf algebra. Cartier duality then has the effects $D\widehat{\mathbb{G}} = \mathbb{H}$ and $D\mathbb{H} = \widehat{\mathbb{G}}$.*

Proof. We show that the first two objects, $D\widehat{\mathbb{G}}$ and $\mathbb{H}$, represent the same object. A point $(u, f) \in D\widehat{\mathbb{G}}(T)$ is specified by a pair of functions

$$\Big(u\colon \operatorname{Spec} T \to X, f\colon u^*\widehat{\mathbb{G}} \to u^*(\mathbb{G}_m \times X)\Big).$$

The map f is equivalent to a map of Hopf algebras $f^*\colon T[u^{\pm}] \to \mathcal{O}_{\widehat{\mathbb{G}}} \otimes_{\mathcal{O}_X} T$, which is determined by its value $f^*(u) \in \mathcal{O}_{\widehat{\mathbb{G}}} \otimes_{\mathcal{O}_X} T$, which must satisfy the two relations $\Delta(f^*u) = f^*u \otimes f^*u$ and $\varepsilon(f^*u) = 1$. Invoking linear duality, f^*u can also be considered as an element of $\mathsf{Modules}_{\mathcal{O}_X}(\mathcal{O}^*_{\widehat{\mathbb{G}}}, T)$, and the two relations on f^*u show that it lands in the subset

$$f^*u \in \mathsf{Algebras}_{\mathcal{O}_X/}(\mathcal{O}^*_{\widehat{\mathbb{G}}}, T) \subseteq \mathsf{Modules}_{\mathcal{O}_X}(\mathcal{O}^*_{\widehat{\mathbb{G}}}, T).$$

This assignment is invertible, and the proof is entirely similar for $D\mathbb{H} \cong \widehat{\mathbb{G}}$. $\square$

Remark 5.1.15. The effect of Cartier duality on the Dieudonné module of a formal group is *also* described by linear duality. Hence, the covariant and contravariant Dieudonné modules described in Lecture 4.4 can be taken to be related by Cartier duality.

Remark 5.1.16. Cartier duality intertwines the homological and cohomological schemes assigned to a commutative H–space. When such a commutative H–space X has free and even E–homology, there is an isomorphism

$$D(\operatorname{Spf} E^0 X) = DX_E = \underline{\mathsf{GroupSchemes}}(X_E, \mathbb{G}_m) \cong \operatorname{Spec} E_0 X.$$

5.2 Special divisors and the special splitting principle

Starting today, after our extended interludes on chromatic homotopy theory and cooperations, we are going to return to thinking about bordism orientations directly. To begin, we will summarize the various perspectives already adopted in Case Study 2 when we were studying complex orientations of ring spectra.

These references aren't totally right: #3 is just off, and then there's some confusion about *MU* versus *MUP*.

1. (Definition 2.0.2:) A complex–orientation of E is, definitionally, a map $MUP \to E$ of ring spectra in the homotopy category.

2. (Theorem 2.3.18:) A complex–orientation of E is also equivalent to a multiplicative system of Thom isomorphisms for complex vector bundles. Such a system is determined by its value on the universal line bundle $\mathcal{L}$ over $\mathbb{C}\mathrm{P}^\infty$. We can also phrase this algebro-geometrically: such a Thom isomorphism is the data of a trivialization of the Thom sheaf $\mathbb{L}(\mathcal{L})$ over $\mathbb{C}\mathrm{P}^\infty_E$.

3. (Lemma 2.6.8:) Ring spectrum maps $MUP \to E$ induce on E–homology maps $E_0 MUP \to E_0$ of E_0–algebras. This, too, can be phrased algebro-geometrically: these are elements of $(\operatorname{Spec} E_0 MUP)(E_0)$.

We can summarize our main result about these perspectives as follows:

Theorem 5.2.1 ([AHS01, Example 2.53])**.** *Take E to be* complex–orientable*. The functor*

$$\begin{aligned} &\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets}, \\ &(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \{\textit{trivializations of } u^*\mathbb{L}(\mathcal{L}) \textit{ over } u^*\mathbb{C}\mathrm{P}^\infty_E\} \end{aligned}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MUP$*. Moreover, the* E_0*–points of this scheme biject with ring spectrum maps* $MUP \to E$*.*

Proof summary. The equivalence between (1) and (3)—i.e., between complex-orientations and E_0–points of $\operatorname{Spec} E_0 MUP$—follows from calculating that $E_0 MUP$ is a free E_0–module, so that there is a collapse in the universal coefficient theorem. Then, the equivalence between (1) and (2) follows from the splitting principle for complex line bundles, which says that the first Chern class of $\mathcal{L}$—i.e., a trivialization of $\mathbb{L}(\mathcal{L})$—determines the rest of the map $MUP \to E$. □

An analogous result holds for ring spectrum maps $MU \to E$ and the line bundle $1 - \mathcal{L}$, and it is proven in analogous way. In particular, we will want a version of the splitting principle for virtual vector bundles of virtual rank 0. Given a finite complex X and such a rank 0 virtual vector bundle, write $\tilde{V} \colon X \to BU$ for the classifying map. Because X is a finite complex, there exists an integer n so that $\tilde{V} = -(n \cdot 1 - V)$ for an honest rank n vector bundle V over X. Using Corollary 2.3.10, the splitting $f^* V \cong \bigoplus_{i=1}^n \mathcal{L}_i$ over Y gives a presentation of $\tilde{V}$ as

$$\tilde{V} = -(n \cdot 1 - V) = -\bigoplus_{i=1}^{n} (1 - \mathcal{L}_i).$$

Crucially, we have organized this sum *entirely in terms of bundles classified by* BU, as each bundle $1 - \mathcal{L}_i$ itself has the natural structure of a rank 0 virtual vector bundle. This version of the splitting principle, together with our extended discussion of formal geometry, begets the following analogue of the previous result:

Theorem 5.2.2 ([AHS01, Example 2.54], cf. also [AS01, Lemma 6.2])**.** *Take* E *to be complex–orientable. The functor*

$$\begin{aligned} &\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets}, \\ &(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \{\textit{trivializations of } u^* \mathbb{L}(1 - \mathcal{L}) \textit{ over } u^* \mathbb{C}\mathrm{P}^\infty_E\} \end{aligned}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MU$*. Moreover, the* E_0*–points of this scheme biject with ring spectrum maps* $MU \to E$*.* □

In Lecture 2.3, we preferred to think of the cohomology of a Thom spectrum as a sheaf over the formal scheme associated to its base space. This extra structure has not evaporated in the homological context—it just takes a different form.

Lemma 5.2.3. *For* $\xi \colon G \to BGL_1\mathbb{S}$ *a group map, the Thom spectrum* $T\xi$ *is a* $(\Sigma^\infty_+ G)$*–cotorsor.*

Construction. The Thom isomorphism $T\xi \wedge T\xi \simeq T\xi \wedge \Sigma^\infty_+ G$ composes with the unit map $\mathbb{S} \to T\xi$ to give the *Thom diagonal*

$$T\xi \to T\xi \wedge \Sigma^\infty_+ G. \qquad \square$$

Applying $\operatorname{Spec} E_0(-)$, the Thom diagonal is translated into the structure of a free and transitive action map

$$\operatorname{Spec} E_0 T(\xi) \times \operatorname{Spec} E_0 G \to \operatorname{Spec} E_0 T(\xi).$$

This construction is natural in the formation of G or ξ, and so we are also moved to specialize to the cases of $G = \mathbb{Z} \times BU$ and $G = BU$ and to understand $\operatorname{Spec} E_0 G$ in those contexts. Again, this is a matter of chaining together results we have already proven:

$$\begin{aligned}
\operatorname{Spec} E_0(\mathbb{Z} \times BU) &= D((\mathbb{Z} \times BU)_E) && \text{(Remark 5.1.16)} \\
&= D(\operatorname{Div} \mathbb{CP}^\infty_E) && \text{(Corollary 2.3.15)} \\
&= \underline{\mathsf{FormalGroups}}(\operatorname{Div} \mathbb{CP}^\infty_E, \mathbb{G}_m) && \text{(Definition 5.1.13)} \\
&= \underline{\mathsf{FormalSchemes}}(\mathbb{CP}^\infty_E, \mathbb{G}_m), && \text{(Corollary 5.1.9)}
\end{aligned}$$

and similarly

$$\operatorname{Spec} E_0(BU) = \underline{\mathsf{FormalSchemes}}_{*/}(\mathbb{CP}^\infty_E, \mathbb{G}_m)$$

is the subscheme of those maps sending the identity point of $\mathbb{CP}^\infty_E$ to the identity point of $\mathbb{G}_m$. Such functions can be identified with trivializations of the trivial sheaf over $\mathbb{CP}^\infty_E$, and the action map induced by the Thom diagonal belongs to a commuting square

$$\begin{tikzcd}
\operatorname{Spec} E_0 MU \times \operatorname{Spec} E_0 BU \arrow[r] \arrow[d, equal] & \operatorname{Spec} E_0 MU \arrow[d, equal] \\
\{\text{triv}^{\text{ns}} \text{ of } \mathbb{L}(1 - \mathcal{L})\} \times \{\text{triv}^{\text{ns}} \text{ of } 1\} \arrow[r] & \{\text{triv}^{\text{ns}} \text{ of } \mathbb{L}(1 - \mathcal{L}) \otimes 1\}.
\end{tikzcd}$$

Remark 5.2.4. The topological maps

Despite some effort, this remark is still quite unclear.

$$BU \to \mathbb{Z} \times BU, \qquad\qquad MU \to MUP$$

induce recognizable algebro-geometric maps upon application of $\operatorname{Spec} E_0(-)$, which are baked into our identifications. The comparison map

$$(\operatorname{Spec} E_0(\mathbb{Z} \times BU))(T) \to \underline{\mathsf{FormalSchemes}}(\mathbb{CP}^\infty_E, \mathbb{G}_m)$$

reads off the image of a map $f\colon E_0(\mathbb{Z} \times BU) \cong E_0[b_0^\pm, b_1, \ldots] \to T$ as the components of a function $\sum_j f(b_0^j b_j) x^j \in T \otimes \mathcal{O}_{\mathbb{CP}^\infty_E}$, whereas the comparison map for BU reads off the image of a map $g\colon E_0(BU) \cong E_0[b_0^\pm, b_1, b_2, \ldots]/(b_0 = 1) \to T$ as the components of a function $\sum_j g(b_j)$, effecting a normalizing division by b_0, itself the image of $\mathbb{CP}^0_E \subseteq \mathbb{CP}^\infty_E$ in $\mathbb{G}_m$. Geometrically, this gives the commuting square

$$\begin{array}{ccc} \operatorname{Spec} E_0(\mathbb{Z} \times BU) & \longrightarrow & \operatorname{Spec} E_0(BU) \\ \| & & \| \\ \underline{\mathsf{FormalSchemes}}(\mathbb{C}\mathrm{P}^\infty_E, \mathbb{G}_m) & \longrightarrow & \underline{\mathsf{FormalSchemes}}_{*/}(\mathbb{C}\mathrm{P}^\infty_E, \mathbb{G}_m) \\ & & \\ f(t) & \longmapsto & f(t)/f(0). \end{array}$$

At the level of Thom spectra, these identifications are controlled by the Chern classes associated to these bundles, and the briefest way to summarize their relationship is this. The spaces $\mathbb{Z} \times BU$ and BU are the 0^{th} and 2^{nd} spaces in the Ω–spectrum for connective complex K–theory, and since connective complex K–theory is complex-orientable, we have $kU^*(\mathbb{C}\mathrm{P}^\infty) = \mathbb{Z}[\beta][\![c_1]\!]$. Inside this ring there is the relation

$$\beta c_1 = (1 - \mathcal{L}).$$

Recognizing β as the restriction of the tautological bundle on $\mathbb{C}\mathrm{P}^\infty$ to $S^2 \simeq \mathbb{C}\mathrm{P}^1$ and employing Example 2.3.4, this says that the trivialization f of $\mathbb{L}(u^*\mathcal{L})$, corresponding to a point in $(\operatorname{Spec} E_0 MUP)(T)$ and to $(1 - \mathcal{L}) \in kU^0(\mathbb{C}\mathrm{P}^\infty) = [\mathbb{C}\mathrm{P}^\infty, \mathbb{Z} \times BU]$, is sent to the trivialization $f'(0)/f$ of $\mathbb{L}(u^*(1 - \mathcal{L}))$, corresponding to the induced point in $(\operatorname{Spec} E_0 MU)(T)$ and to $c_1 \in kU^2(\mathbb{C}\mathrm{P}^\infty) = [\mathbb{C}\mathrm{P}^\infty, BU]$.

This last remark indicates a direction of possible generalization to the other spaces in the Ω–spectrum for connective complex K–theory, which have the following polite description:

Lemma 5.2.5. *There is an equivalence*

$$\underline{kU}_{2k} = BU[2k, \infty).$$

Proof. Consider the element $\beta^k \in kU_* = \mathbb{Z}[\beta]$. The source of the induced map $\beta^k \colon \Sigma^{2k} kU \to kU$ is $2k$–connective, and hence there is a factorization

$$\Sigma^{2k} kU \to kU[2k, \infty) \to kU.$$

Then, the structure of the homotopy ring kU_* shows that this is an equivalence: every class of degree at least $2k$ can be uniquely written as a β^k–multiple.[1] Applying Ω^∞ gives the desired statement:

$$\underline{kU}_{2k} = \Omega^\infty \Sigma^{2k} kU \simeq \Omega^\infty kU[2k, \infty) = BU[2k, \infty). \qquad \square$$

[1] Similarly, there is an equivalence $\underline{kO}_{8k} = BO[8k, \infty)$, and this *does not hold* for indices which are not precise multiples of 8.

The next space and Thom spectrum in the sequence are thus BSU and MSU. This case will be wholly amenable to analysis through methods we have developed so far, which is now our stated goal for the rest of this Lecture. Our jumping off point for that story will be, again, a partial extension of the splitting principle.

Lemma 5.2.6. *Let X be a finite complex, and let $\tilde{V}\colon X \to BU$ classify a virtual vector bundle of rank 0 over X. Select a factorization $\tilde{\tilde{V}}\colon X \to BSU$ of $\tilde{V}$ through BSU. Then, there is a space $f\colon Y \to X$, where $f_E\colon Y_E \to X_E$ is finite and flat, as well as a collection of line bundles $\mathcal{H}_j$, $\mathcal{H}'_j$ expressing a BSU–internal decomposition*

$$\tilde{\tilde{V}} = -\bigoplus_{j=1}^{n}(1-\mathcal{H}_j)(1-\mathcal{H}'_j).$$

Proof. Begin by using Corollary 2.3.10 on V to get an equality of BU–bundles

$$\tilde{\tilde{V}} = V' + \mathcal{L}_1 + \mathcal{L}_2 - n\cdot 1.$$

Adding $(1-\mathcal{L}_1)(1-\mathcal{L}_2)$ to both sides, this gives

$$\begin{aligned}\tilde{\tilde{V}} + (1-\mathcal{L}_1)(1-\mathcal{L}_2) &= V' + \mathcal{L}_1 + \mathcal{L}_2 + (1-\mathcal{L}_1)(1-\mathcal{L}_2) - n\cdot 1 \\ &= V' + \mathcal{L}_1\mathcal{L}_2 - (n-1)\cdot 1.\end{aligned}$$

By thinking of $(1-\mathcal{L}_j)$ as an element of $kU^2(Y) = [Y, BU]$, we see that the product element $(1-\mathcal{L}_1)(1-\mathcal{L}_2) \in kU^4(Y) = [Y, BSU]$ has the natural structure of a BSU–bundle and hence so does the sum on the left-hand side[2]. The right-hand side is the rank 0 virtualization of a rank $(n-1)$ vector bundle, hence succumbs to induction. Finally, because $SU(1)$ is the trivial group, there are no nontrivial complex line bundles with structure group $SU(1)$, grounding the induction. □

From this, we would like to directly conclude an equivalence between trivializations of the Thom sheaf $\mathbb{L}((\mathcal{L}_1-1)(\mathcal{L}_2-1)) \downarrow (\mathbb{C}\mathrm{P}^\infty)_E^{\times 2}$ and multiplicative maps $MSU \to E$, but we are not quite yet ready to do so. Certainly an MSU–orientation of E gives such a trivialization, but it is not clear that all possible trivializations of that universal Thom sheaf give consistent trivializations of other Thom sheaves—that is, the decomposition in Lemma 5.2.6 may admit unexpected symmetries which, in turn, place requirements on our universal trivialization so that these symmetric decompositions all result in the same restricted trivialization.[3]

[2]In the language of the previous Case Study, we are making use of a certain Hopf ring $\circ$–product on $\underline{kU}_{2*}$.

[3]By contrast, our splitting principle for ordinary complex vector bundles was completely deterministic, since a given line bundle admits no other expression as a line bundle.

Example 5.2.7. There is an equivalence of SU–bundles

$$(1-\mathcal{L}_1)(1-\mathcal{L}_2) \cong (1-\mathcal{L}_2)(1-\mathcal{L}_1).$$

Correspondingly, the trivializations of $\mathbb{L}((1-\mathcal{L}_1)(1-\mathcal{L}_2))$ which respect this twist are the *symmetric* sections.

Example 5.2.8. There is an equivalence of SU–bundles

$$(1-1)(1-\mathcal{L}_2) \cong 0.$$

Correspondingly, the trivializations of $\mathbb{L}((1-\mathcal{L}_1)(1-\mathcal{L}_2))$ which respect this degeneracy are the *rigid* sections, meaning they trivialize the Thom sheaf of the trivial bundle using the trivial section 1.

Example 5.2.9. There is another less obvious symmetry, inspired by our use of the product map

$$kU^2(Y) \otimes kU^2(Y) \to kU^4(Y)$$

in the course of the proof. There is also a product map

$$kU^2(Y) \otimes kU^0(Y) \times kU^2(Y) \to kU^4(Y).$$

Taking one of our splitting summands $(1-\mathcal{L}_1)(1-\mathcal{L}_2)$ and acting by some line bundle $\mathcal{H} \in kU^0(Y)$ gives

$$\begin{aligned}
(1-\mathcal{L}_1)\mathcal{H}(1-\mathcal{L}_2) &= (1-\mathcal{L}_1)\mathcal{H}(1-\mathcal{L}_2) \\
(\mathcal{H}-\mathcal{L}_1\mathcal{H})(1-\mathcal{L}_2) &= (1-\mathcal{L}_1)(\mathcal{H}-\mathcal{H}\mathcal{L}_2) \\
(1-\mathcal{L}_1\mathcal{H})(1-\mathcal{L}_2)-(1-\mathcal{H})(1-\mathcal{L}_2) &= (1-\mathcal{L}_1)(1-\mathcal{H}\mathcal{L}_2)-(1-\mathcal{L}_1)(1-\mathcal{H}).
\end{aligned}$$

This "kU^0–linearity" is sometimes called a "2–cocycle condition", in reference to the similarity with the formula in Definition 3.2.4.

We would like to show that these observations suffice, as in the following version of Theorem 5.2.1 and Theorem 5.2.2:

Theorem 5.2.10. *The functor*

$$\{\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0\} \to \left\{ \begin{array}{c} \textit{trivializations of } u^*\mathbb{L}((1-\mathcal{L}_1)(1-\mathcal{L}_2)) \textit{ over } u^*(\mathbb{C}\mathrm{P}^\infty)^{\times 2}_E \\ \textit{which are symmetric, rigid, and } kU^0\textit{–linear} \end{array} \right\}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MSU$*. Moreover, the* E_0*–points of this scheme biject with ring spectrum maps* $MSU \to E$*.*

In pursuit of this, we will show rather manually that BSU_E represents an object subject to exactly such symmetries, hence $\operatorname{Spec} E_0 BSU$ represents the scheme of such symmetric functions, and hence conclude that $\operatorname{Spec} E_0 MSU$ represents the scheme of such symmetric trivializations. The place to begin is with a Serre spectral sequence:

Lemma 5.2.11 ([AS01, Lemma 6.1], cf. also [AS01, Proposition 6.5]). *The Postnikov fibration*

$$BSU \to BU \xrightarrow{B\det} BU(1)$$

induces a short exact sequence of Hopf algebras

$$E^0 BSU \leftarrow E^0 BU \xleftarrow{c_1 \leftarrow c_1} E^0 BU(1). \quad \square$$

An equivalent statement is that there is a short exact sequence of formal group schemes

$$\begin{tikzcd} BSU_E \arrow[r] \arrow[d, equal] & BU_E \arrow[r, "B\det"] \arrow[d, equal] & BU(1)_E \arrow[d, equal] \\ \operatorname{SDiv}_0 \mathbb{CP}^\infty_E \arrow[r] & \operatorname{Div}_0 \mathbb{CP}^\infty_E \arrow[r, "\mathrm{sum}"] & \mathbb{CP}^\infty_E, \end{tikzcd}$$

where the scheme "$\operatorname{SDiv}_0 \mathbb{CP}^\infty_E$" of *special divisors* is defined to parametrized those divisors which vanish under the summation map. However, whereas the map $BU(1)_E \to BU_E$ has an identifiable universal property—it presents BU_E as the universal formal group on the pointed curve $BU(1)_E$—the description of BSU_E as a scheme of special divisors does not bear much immediate resemblance to a free object on the special divisor $(1 - [a])(1 - [b])$ classified by

$$(\mathbb{CP}^\infty)^{\times 2}_E \xrightarrow{(1-\mathcal{L}_1)(1-\mathcal{L}_2)_E} BSU_E \to BU_E = \operatorname{Div}_0 \mathbb{CP}^\infty_E.$$

Our task is exactly to justify this statement.

Definition 5.2.12. If it exists, let $C_2\widehat{\mathbb{G}}$ denote the symmetric square of $\operatorname{Div}_0 \widehat{\mathbb{G}}$, thought of as a module over $\operatorname{Div} \widehat{\mathbb{G}}$. This scheme has the property that a formal group homomorphism $\varphi\colon C_2\widehat{\mathbb{G}} \to H$ is equivalent data to a symmetric function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying a rigidity condition ($\psi(x, 0) = 0$) and a 2–cocycle condition as in Example 5.2.9.

Theorem 5.2.13 (Ando–Hopkins–Strickland, unpublished). $\operatorname{SDiv}_0 \widehat{\mathbb{G}}$ *is a model for* $C_2\widehat{\mathbb{G}}$.

Cite me: This is Prop 3.2 of the AHS preprint or Prop 2.13 of Strickland's FSKS preprint.

Proof sketch. Consider the map

$$\begin{aligned} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} &\to \operatorname{Div}_0 \widehat{\mathbb{G}}, \\ (a, b) &\mapsto (1 - [a])(1 - [b]) \end{aligned}$$

for which there is a factorization of formal schemes

$$\begin{tikzcd} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \arrow[d, dotted] \arrow[dr] & & \\ F \arrow[r, "\ker"] & \operatorname{Div}_0 \widehat{\mathbb{G}} \arrow[r, "\sigma"] & \widehat{\mathbb{G}} \end{tikzcd}$$

because

$$\sigma((1-[a])(1-[b])) = (a+b) - a - b + 0 = 0.$$

One can check that a homomorphism $\varphi\colon F \to H$ pulls back to a function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying the properties of Definition 5.2.12:

- To check rigidity, we have

$$\psi(a,0) = \varphi((1-[a])(1-[0])) = \varphi((1-[a])(1-1)) = \varphi(0) = 0.$$

- To check symmetry, we have

$$\psi(a,b) = \varphi((1-[a])(1-[b])) = \varphi((1-[b])(1-[a])) = \psi(b,a).$$

- To check kU^0–linearity, we have

$$\begin{aligned}\psi(ac,b) - \psi(c,b) &= \varphi((1-[a][c])(1-[b])) - \varphi((1-[c])(1-[b])) \\ &= \varphi((1-[a][c])(1-[b]) - (1-[c])(1-[b])) \\ &= \varphi((1-[a])(1-[c][b]) - (1-[a])(1-[c])) \\ &= \varphi((1-[a])(1-[c][b])) - \varphi((1-[a])(1-[c])) \\ &= \psi(a,cb) - \psi(a,c).\end{aligned}$$

The other direction is more obnoxious, so we give only a sketch. Begin by selecting a function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$, then mimic the construction in Lemma 5.2.6. Expanding the definition of $\operatorname{Div}_0 \widehat{\mathbb{G}}$, we are moved to consider the object $\widehat{\mathbb{G}}^{\times k}$, where we define a map

$$\begin{gathered}\widehat{\mathbb{G}}^{\times k} \to H, \\ (a_1, \ldots, a_k) \mapsto -\sum_{j=2}^{k} \psi\left(\sum_{i=1}^{j-1} a_i, a_j\right).\end{gathered}$$

This gives a compatible system of symmetric maps, and hence altogether this gives a map $\tilde{\varphi}\colon\ \operatorname{Div}_0 \widehat{\mathbb{G}} \to H$ off of the colimit. In general, this map is not a homomorphism, but it is a homomorphism when restricted to

$$\varphi\colon\ F \to \operatorname{Div}_0 \widehat{\mathbb{G}} \xrightarrow{\tilde{\varphi}} H.$$

Finally, one checks that any homomorphism $F \to H$ of formal groups restricting to the zero map $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ was already the zero map, and this gives the desired identification of F with the universal property of $C_2\widehat{\mathbb{G}}$. □

Building the homomorphism seems boring, but possibly checking that it's zero is interesting — this is kind of what was confounding us from just using the topological SU–splitting principle outright. Somehow working in algebra must make this more evident, and if it's so evident then we should write it out.

Corollary 5.2.14. *There is an isomorphism*

$$\operatorname{Spec} E_0 BSU = \left\{ \begin{array}{c} \text{functions } f\colon\ u^*(\mathbb{CP}^\infty_E)^{\times 2} \to \mathbb{G}_m \\ \text{which are symmetric, rigid, and } kU^0\text{–linear} \end{array} \right\}.$$

Proof. Follow the sequence of isomorphisms

$$\begin{aligned}
\operatorname{Spec} E_0 BSU &= D(BSU_E) && \text{(Remark 5.1.16)}\\
&= D(\mathrm{SDiv}_0\, \mathbb{C}P^\infty_E) && \text{(Lemma 5.2.11)}\\
&= D(C_2 \mathbb{C}P^\infty_E) && \text{(Theorem 5.2.13)}\\
&= \underline{\mathsf{FormalGroups}}(C_2 \mathbb{C}P^\infty_E, \mathbb{G}_m), && \text{(Definition 5.1.13)}
\end{aligned}$$

and then use the universal property in Definition 5.2.12. □

In order to lift this analysis to $\operatorname{Spec} E_0 MSU$, we again appeal to the torsor structure. At this point, it will finally be useful to introduce some notation:

Definition 5.2.15 ([AHS01, Definition 2.39]). For a sheaf $\mathcal{L}$ over a formal group $\widehat{\mathbb{G}}$, we introduce the schemes

You could motivate these using Cartier duality.

$$\begin{aligned}
C^0(\widehat{\mathbb{G}}_E; \mathcal{L})(T) &= \{\text{triv}^{\text{ns}} \text{ of } u^*\mathcal{L} \downarrow u^*\widehat{\mathbb{G}}\},\\
C^1(\widehat{\mathbb{G}}_E; \mathcal{L})(T) &= \left\{\text{triv}^{\text{ns}} \text{ of } u^*\left(\frac{e^*\mathcal{L}}{\mathcal{L}}\right) \downarrow u^*\widehat{\mathbb{G}} \text{ which are rigid}\right\}\\
C^2(\widehat{\mathbb{G}}_E; \mathcal{L})(T) &= \left\{\begin{array}{c}\text{triv}^{\text{ns}} \text{ of } u^*\left(\frac{e^*\mathcal{L}\otimes\mu^*\mathcal{L}}{\pi_1^*\mathcal{L}\otimes\pi_2^*\mathcal{L}}\right) \downarrow u^*\widehat{\mathbb{G}}^{\times 2}\\ \text{which are rigid, symmetric, and } kU^0\text{–linear}\end{array}\right\}.
\end{aligned}$$

Thus far, we have established the following isomorphisms:

You could include the cohomological schemes.

$$\begin{aligned}
\operatorname{Spec} E_0(\mathbb{Z}\times BU) &\cong C^0(\mathbb{C}P^\infty_E; \mathbb{G}_m), & \operatorname{Spec} E_0(MUP) &\cong C^0(\mathbb{C}P^\infty_E; \mathcal{I}(0)),\\
\operatorname{Spec} E_0(BU) &\cong C^1(\mathbb{C}P^\infty_E; \mathbb{G}_m), & \operatorname{Spec} E_0(MU) &\cong C^1(\mathbb{C}P^\infty_E; \mathcal{I}(0)),\\
\operatorname{Spec} E_0(BSU) &\cong C^2(\mathbb{C}P^\infty_E; \mathbb{G}_m), & &
\end{aligned}$$

where we have abusively abbreviated the sheaf of functions on $\mathbb{C}P^\infty_E$ to $\mathbb{G}_m$.

In order to fill in the missing piece, we exploit the torsor structure on Thom spectra discussed earlier.

Lemma 5.2.16 ([AHS01, Theorem 2.50]). *There is a system of compatible maps*

$$\begin{tikzcd}
\operatorname{Spec} E_0 BSU \times \operatorname{Spec} E_0 MSU \arrow[r] \arrow[d, equal] \arrow[d] & \operatorname{Spec} E_0 MSU \arrow[d] \\
C^2(\mathbb{C}P^\infty_E; \mathbb{G}_m) \times C^2(\mathbb{C}P^\infty_E; \mathcal{I}(0)) \arrow[r] & C^2(\mathbb{C}P^\infty_E; \mathcal{I}(0)),
\end{tikzcd}$$

where the horizontal maps are the action maps defining torsors, and the vertical maps are those described above.

Proof sketch. Recall the isomorphism $T(\mathcal{L} \downarrow \mathbb{C}P^\infty) \simeq \Sigma^\infty \mathbb{C}P^\infty$. The main point of this claim is that the Thom diagonal for $MU[2k, \infty)$ restricts to a very familiar diagonal:

$$\begin{tikzcd}
(\Sigma^\infty \mathbb{C}\mathrm{P}^\infty)^{\wedge k} \arrow[r, "\Delta"] \arrow[d] & (\Sigma^\infty \mathbb{C}\mathrm{P}^\infty)^{\wedge k} \wedge \Sigma^\infty_+ (\mathbb{C}\mathrm{P}^\infty)^{\times k} \arrow[d] \\
MU[2k,\infty) \arrow[r, "\Delta"] & MU[2k,\infty) \times BU[2k,\infty).
\end{tikzcd}$$

(The diagonal at the level of $(\mathbb{C}\mathrm{P}^\infty)^{\times k}$ is responsible for the cup product, so that the classes in the cohomology of projective space which induce the maps

$$\operatorname{Spec} E_0 MU[2k,\infty) \to C^k(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0)), \qquad \operatorname{Spec} E_0 BU[2k,\infty) \to C^k(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$$

literally multiply together to give the description of the action. This multiplication of sections is exactly the action claimed in the model. $\square$

Proof of Theorem 5.2.10. The claim of this Theorem is that the map

$$\operatorname{Spec} E_0 MSU \to C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0))$$

studied above is an isomorphism. Any map of torsors over a fixed base is automatically an isomorphism. $\square$

Remark 5.2.17 ([AS01, Lemma 6.4]). We can also analyze the map $\operatorname{Spec} E_0 BSU \to \operatorname{Spec} E_0 BU$ in terms of these models of functions to $\mathbb{G}_m$. Again, the analysis passes through a computation in connective K–theory, using the identification

$$kU^*(\mathbb{C}\mathrm{P}^\infty)^{\times 2} = \mathbb{Z}[\beta][\![x_1, x_2]\!],$$

where $x_1 = \pi_1^* x$ and $x_2 = \pi_2^* x$ are the Chern classes associated to the tautological bundle pulled back along projections to the first and second factors

$$\pi_1 \colon (\mathbb{C}\mathrm{P}^\infty)^{\times 2} \to \mathbb{C}\mathrm{P}^\infty \times *, \qquad \pi_2 \colon (\mathbb{C}\mathrm{P}^\infty)^{\times 2} \to * \times \mathbb{C}\mathrm{P}^\infty,$$

Inside of this ring, we have the equations

$$\begin{aligned}
\beta^2 x_1 x_2 &= (1 - \mathcal{L}_1)(1 - \mathcal{L}_2) \\
&= (1 - \mathcal{L}_1) - (1 - \mathcal{L}_1 \mathcal{L}_2) + (1 - \mathcal{L}_2) \\
&= \beta \left(\pi_1^*(x) - \mu^*(x) + \pi_2^*(x)\right),
\end{aligned}$$

where $\mu \colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ is the tensor product map. Since the orientation schemes are governed as torsors over these base schemes, we automatically get a description

$$\begin{aligned}
\operatorname{Spec} E_0 MU &\longrightarrow \operatorname{Spec} E_0 MSU, \\
f(x) &\longmapsto \tfrac{f(x_1) \cdot f(x_2)}{f(x_1 +_{\mathbb{C}\mathrm{P}^\infty_E} x_2)}
\end{aligned}$$

There's an obnoxious sign here error here.

as a section of

$$\pi_1^*\left(\frac{e^*\mathcal{I}(0)}{\mathcal{I}(0)}\right) \otimes \pi_2^*\left(\frac{e^*\mathcal{I}(0)}{\mathcal{I}(0)}\right) \otimes \left(\mu^*\left(\frac{e^*\mathcal{I}(0)}{\mathcal{I}(0)}\right)\right)^{-1} = \frac{e^*\mathcal{I}(0) \otimes \mu^*\mathcal{I}(0)}{\pi_1^*\mathcal{I}(0) \otimes \pi_2^*\mathcal{I}(0)}.$$

Remark 5.2.18 ([AHS01, Remark 2.32]). The published proofs of Ando, Hopkins, and Strickland differ substantially from the account given here. The primary difference is that "$C_2\widehat{\mathbb{G}}$" does not even get mention, essentially because it is a fair amount of technical work to show that such a scheme even exists (especially in the case to come of $BU[6,\infty)$). On the other hand, it is very easy to demonstrate the existence of its Cartier dual: this is a scheme parametrizing certain bivariate power series subject to certain algebraic conditions, hence exists for the same reason that $\mathcal{M}_{\mathbf{fgl}}$ existed (cf. Definition 3.2.1). The compromise for this is that they then have to analyze the scheme $\operatorname{Spec} E_0BSU$ directly, through considerably more computational avenues. This is not too high of a price: the analysis of the $BU[6,\infty)$ case turns out to be primarily computational anyhow, so this manner of approach is inevitable.

Remark 5.2.19. The scheme $C_2\widehat{\mathbb{G}}$ might also be written as

$$C_2\widehat{\mathbb{G}} = \operatorname{Sym}^2_{\operatorname{Div}\widehat{\mathbb{G}}} \operatorname{Div}_0 \widehat{\mathbb{G}},$$

where we are thinking of of $\operatorname{Div}_0 \widehat{\mathbb{G}} \subseteq \operatorname{Div} \widehat{\mathbb{G}}$ as the augmentation ideal inside of an augmented ring.

5.3 Chromatic analysis of $BU[6,\infty)$

It should be possible to give an example of a complex-oriented theory which receives an MSU orientation which *does not* factor the complex orientation but *does* (*must*, really) factor the unit? Even if one can find an example of this, I think it will be somewhat artificial: the sequence of group schemes

$$0 \to BSU_E \to BU_E \to$$

is short exact, *and* it has a splitting on the level of formal schemes. The splitting is what gives you the isomorphism on points $BSU_E(T) \times BU(1)_E(T) \cong BU_E(T)$. On the other hand, because the splitting *isn't* a map of formal groups, it doesn't survive to the Cartier dual short exact sequence

$$0 \leftarrow BSU^E \leftarrow BU^E \leftarrow$$

so this will come down to exhibiting a test ring $T = E_*$ for which $BU^E(T) \to BSU^E(T)$, despite being induced by an fppf-surjective map of sheaves, is not surjective on

We now embark on an analysis of $MU[6,\infty)$–orientations in earnest. As in the case of MSU, it is fruitful to first study the behavior of vector bundles with structure map lifted through $\underline{kU}_6 = BU[6,\infty)$ and to analyze the schemes $BU[6,\infty)_E$ and $\operatorname{Spec} E_0BU[6,\infty)$. In the previous case, we studied a particular bundle

$$\Pi_2\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{(1-\mathcal{L}_1)(1-\mathcal{L}_2)} BSU,$$

which controlled much of the geometry through our splitting principle for BSU–bundles, recorded as Lemma 5.2.6. Analogously, we can construct a naturally occurring such bundle as the product

$$\Pi_3\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{(1-\mathcal{L}_1)(1-\mathcal{L}_2)(1-\mathcal{L}_3)} BU[6,\infty),$$

but the proof of Lemma 5.2.6 falls apart almost immediately—there does not appear to be a splitting principle for bundles lifted through $BU[6,\infty)$. This is quite worrying, and it dampens our optimism across the board: about the behavior of Π_3 exerting enough control over $BU[6,\infty)$, about the existence of "$C_3\widehat{\mathbb{G}}$", and about $C_3\mathbb{C}\mathrm{P}^\infty_E$ serving as a good model for $BU[6,\infty)_E$.

Nevertheless, we will show that this algebraic model is still precise by exhaustive topological calculation. Our approach is divided between two fronts.

1. If we specialize to a particularly nice cohomology theory—such as $E = E_\Gamma$ a Morava E–theory—then we can use our extensive body of knowledge about finite height formal groups and their relationship to algebraic topology in order to force nice behavior into the story.

2. If we specialize to a particularly simple formal group—such as $\widehat{\mathbb{G}}_a$ and its associated cohomology theory $H\mathbb{F}_p$—then we can use our talent for performing computations in algebraic topology to completely exhaust the problem.

In this Lecture, we will pursue the first avenue. We begin by setting Γ to be a formal group of finite p–height of a field k of positive characteristic p, and we let $E = E_\Gamma$ denote the associated Morava E–theory. Our main tool will be the Postnikov fibration

$$\underline{H\mathbb{Z}_3} \to BU[6,\infty) \to BSU,$$

and our main goals are to construct a model sequence of formal schemes, then show that E–theory is well-behaved enough that the formal schemes it constructs exactly match the model.

In the previous setting of MSU, we gained indirect access to the algebraic model $C_2\widehat{\mathbb{G}}$ by separately proving that it was modeled by $\mathrm{SDiv}_0\,\widehat{\mathbb{G}}$ and that this had a good comparison map to BSU_E. This time, since we do not have access to $C_3\widehat{\mathbb{G}}$ or anything like it, we proceed by much more indirect means, along the lines of Remark 5.2.18: we know that $C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ exists as an affine scheme, since we can explicitly construct it as a closed subscheme of the scheme of trivariate power series, and so we seek a map

$$\operatorname{Spec} E_0BU[6,\infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathbb{G}_m)$$

that does not pass through any intermediate cohomological construction. Our main tool for accomplishing this is as follows:

Definition 5.3.1. A map $f\colon X \to Y$ of spaces induces a map $f_E\colon X_E \to Y_E$ of formal schemes. In the case that Y is a commutative H–space and Y_E is connected, we can construct a map according to the composite

$$\begin{tikzcd} X_E \times \underline{\mathsf{GroupSchemes}}(Y_E,\mathbb{G}_m) \arrow[rr, dotted] \arrow[d, equal] & & \widehat{\mathbb{A}}^1 \\ X_E \times \underline{\mathsf{FormalGroups}}(Y_E,\widehat{\mathbb{G}}_m) \arrow[r, "f_E\times 1"] & Y_E \times \underline{\mathsf{FormalGroups}}(Y_E,\widehat{\mathbb{G}}_m) \arrow[r, "\mathrm{ev}"] & \widehat{\mathbb{G}}_m. \arrow[u, "\simeq"'] \end{tikzcd}$$

This is called *the adjoint map*, and we write $\hat{f}$ for any of the above versions of this map, whether valued in $\widehat{\mathbb{G}}_m$, $\mathbb{G}_m$, or $\widehat{\mathbb{A}}^1$. It encodes equivalent information to the map of E_0–linear map

$$E_0 \to E_0 Y \widehat{\otimes}_{E_0} E^0 X$$

dual to the map $E_0 X \to E_0 Y$ induced on E–homology.

Remark 5.3.2. This construction converts many properties of f into corresponding properties of this adjoint element. For instance:

- It is natural in the source: for $f\colon X \to Y$ and $g\colon W \to X$, we have
$$\widehat{fg} = \hat{f} \circ (g_E \times \mathrm{id}_{Y_E})\colon W_E \times D(Y_E) \to \mathbb{G}_m.$$
- It is natural in the target: for $f\colon X \to Y$ and $h\colon Y \to Z$ a map of H–spaces, we have
$$\widehat{hf} = \hat{f} \circ (\mathrm{id}_{X_E} \times D(h_E))\colon X_E \times D(Z_E) \to \mathbb{G}_m.$$
- It converts sums of classes to products of maps to $\mathbb{G}_m$.

Example 5.3.3. Recall the vector bundle Π_2, defined at the top of this Lecture but of great interest to us in Lecture 5.2. The adjoint to the classifying map of Π_2 is a map of formal schemes

$$\hat{\Pi}_2\colon (\mathbb{C}\mathrm{P}^\infty_E)^{\times 2} \times \operatorname{Spec} E_0 BSU \to \mathbb{G}_m,$$

which passes through the exponential adjunction to become a map

$$\operatorname{Spec} E_0 BSU \to \underline{\mathsf{FormalSchemes}}((\mathbb{C}\mathrm{P}^\infty_E)^{\times 2}, \mathbb{G}_m).$$

Because the adjoint construction preserves properties of the class Π_2, we learn that this map factors through the closed subscheme

$$\begin{tikzcd} \operatorname{Spec} E_0 BSU \arrow[r, dotted] \arrow[rr, bend left] & C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m) \arrow[r] & \underline{\mathsf{FormalSchemes}}((\mathbb{C}\mathrm{P}^\infty_E)^{\times 2}, \mathbb{G}_m) \end{tikzcd}$$

of symmetric, rigid functions satisfying kU^0–linearity. By careful manipulation of divisors in Theorem 5.2.13, we showed that $BSU_E \cong \mathrm{SDiv}_0 \mathbb{C}\mathrm{P}^\infty_E$, which on applying Cartier duality showed the factorized map $\operatorname{Spec} E_0 BSU \to C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$ to be an isomorphism.

Example 5.3.4. Similarly, we have defined a cohomology class

$$\Pi_3 = (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)(\mathcal{L}_3 - 1) \in kU^6(\mathbb{C}\mathrm{P}^\infty)^{\times 3} = [(\mathbb{C}\mathrm{P}^\infty)^{\times 3}, BU[6, \infty)].$$

As above, its adjoint induces a map (which we abusively also denote by $\widehat{\Pi_3}$)

$$\widehat{\Pi}_3\colon \operatorname{Spec} E_0 BU[6, \infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m),$$

where $C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$ is the scheme of $\mathbb{G}_m$–valued trivariate functions on $\mathbb{C}\mathrm{P}^\infty_E$ satisfying symmetry, rigidity, and kU^0–linearity.[4]

[4] If $C_3 \mathbb{C}\mathrm{P}^\infty_E := \mathrm{Sym}^3_{\mathrm{Div}\, \mathbb{C}\mathrm{P}^\infty_E} \mathrm{Div}_0 \mathbb{C}\mathrm{P}^\infty_E$ were to exist, this scheme $C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$ would be its Cartier dual.

We also have the following analogue of the compatibility results Remark 5.2.4 and Remark 5.2.17 of the previous section:

Lemma 5.3.5 ([AS01, Lemma 7.1], [AHS01, Proposition 2.27, Corollary 2.30])**.** *There is a commutative square*

$$\begin{tikzcd} \operatorname{Spec} E_0 BSU \arrow[r] \arrow[d, "\widehat{\Pi}_2"] & \operatorname{Spec} E_0 BU[6,\infty) \arrow[d, "\widehat{\Pi}_3"] \\ C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m), \end{tikzcd}$$

where the map[5] *δ is specified by*

$$\delta(f)(x_1, x_2, x_3) := \frac{f(x_1, x_3) f(x_2, x_3)}{f(x_1 +_E x_2, x_3)}.$$

Proof. As in the proofs of Remark 5.2.4 and Remark 5.2.17, this is checked by performing a calculation in kU–cohomology of projective space, where we have the relation

$$\begin{aligned} \Pi_3 &= (1 - \mathcal{L}_1)(1 - \mathcal{L}_2)(1 - \mathcal{L}_3) \\ &= ((1 - \mathcal{L}_1) - (1 - \mathcal{L}_1\mathcal{L}_2) + (1 - \mathcal{L}_2))(1 - \mathcal{L}_3) \\ &= ((\pi_1 \times 1)^* - (\mu \times 1)^* + (\pi_2 \times 1)^*)((1 - \mathcal{L}_1)(1 - \mathcal{L}_3)) \\ &= ((\pi_1 \times 1)^* - (\mu \times 1)^* + (\pi_2 \times 1)^*)\Pi_2. \end{aligned}$$

□

Thus far, we have constructed the solid maps in the following commutative diagram:

$$\begin{tikzcd} \operatorname{Spec} E_0 BSU \arrow[r] \arrow[d, "\cong"', "\widehat{\Pi}_2"] & \operatorname{Spec} E_0 BU[6,\infty) \arrow[r] \arrow[d, "\widehat{\Pi}_3"] & \operatorname{Spec} E_0 \underline{H\mathbb{Z}}_3 \arrow[d, "\cong"'] \\ C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, dotted, "e"] & \underline{\mathsf{FormalGroups}}((\mathbb{CP}^\infty_E)^{\wedge 2}, \widehat{\mathbb{G}}_m), \end{tikzcd}$$

where the left-most vertical map is an isomorphism by Corollary 5.2.14 and right-most vertical map is an isomorphism by Remark 4.6.12. We would like to enrich this diagram to an isomorphism of short exact sequences, and to do so we need to finish constructing the sequences themselves—we need a horizontal map e making the diagram commute.

The idea behind the construction of e is to pretend that $\widehat{\Pi}_3$ is an isomorphism, so that we could completely detect e by comparing the image of the identity point on $\operatorname{Spec} E_0 BU[6,\infty)$ through $\widehat{\Pi}_3$ to the image of the same identity point through the maps

$$\operatorname{Spec} E_0 BU[6,\infty) \to \operatorname{Spec} E_0 \underline{H\mathbb{Z}}_3 \to \mathsf{FormalGroups}((\mathbb{CP}^\infty_E)^{\wedge 2}, \mathbb{G}_m).$$

[5] Despite its name and its formula, this map δ does not really belong to a cochain complex from our perspective. *All* of the functions we are considering, no matter how many inputs they take, are always subject to a 2–cocycle condition.

Using our calculation that $(\mathbb{C}P^\infty_E)^{\wedge 2}$ is a p–divisible group, we see that we can further restrict attention to the torsion subgroups $(\mathbb{C}P^\infty_E)^{\wedge 2}[p^j] = (BS^1[p^j]_E)^{\wedge 2}$ which filter it, corresponding to analyzing the bundle classified by the restriction

$$BS^1[p^j]^{\wedge 2} \xrightarrow{\mu} \underline{HS^1[p^j]}_2 \xrightarrow{\beta} \underline{H\mathbb{Z}}_3 \xrightarrow{\gamma} \underline{kU}_6.$$

Using the abbreviation $B_j = BS^1[p^j]$, our summary goal is to find an express description of a map d making the following square commute:

$$\begin{tikzcd} B_j \wedge B_j \arrow[r] \arrow[d, "\beta\mu(a\wedge a)"] & \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \arrow[d, dotted, "d"] \\ \Sigma^3 H\mathbb{Z} \arrow[r, "\gamma"] & \Sigma^6 kU, \end{tikzcd}$$

where we have quietly replaced spaces by their suspension spectra, and where $\beta\mu(a \wedge a)$ denotes the composite

$$B_j^{\wedge 2} \xrightarrow{a\wedge a} (\Sigma H\mathbb{Z}/n)^{\wedge 2} \xrightarrow{\mu} \Sigma^2 H\mathbb{Z}/n \xrightarrow{\beta} \Sigma^3 H\mathbb{Z}.$$

Our strategy is to extend this putative square to a map of putative cofiber sequences

$$\begin{tikzcd} (\mathbb{C}P^\infty)^{\wedge 2}/B_j^{\wedge 2} \arrow[r, "\Delta"] \arrow[d, dotted, "f"] & \Sigma B_j \wedge B_j \arrow[r] \arrow[d, "\beta\mu(a\wedge a)"] & \Sigma\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \arrow[r] \arrow[d, dotted, "d"] & \Sigma(\mathbb{C}P^\infty)^{\wedge 2}/(B_j)^{\wedge 2} \arrow[d, dotted, "f"] \\ \Sigma^4 kU \arrow[r, "\sigma"] & \Sigma^4 H\mathbb{Z} \arrow[r, "\gamma"] & \Sigma^7 kU \arrow[r] & \Sigma^5 kU, \end{tikzcd}$$

and thereby trade the task of constructing d for the task of constructing f. This is a gain because $\sigma\colon kU \to H\mathbb{Z}$, the standard kU–orientation of $H\mathbb{Z}$, is a considerably simpler map to understand than γ.

Lemma 5.3.6 ([AS01, Section 5]). *Make the definitions*

- $x\colon \mathbb{C}P^\infty \to \Sigma^2 kU$ *is the* kU*–Euler class for* $(1 - \mathcal{L})$.
- $u\colon T(\mathcal{L}^{\otimes p^j}) \to kU^2$ *is the* kU*–Thom class for* $T(\mathcal{L}^{\otimes p^j}) = \mathbb{C}P^\infty/B_j$.
- A_1 *is the projection* $\frac{\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty}{B_j \wedge B_j} \to \frac{\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty}{B_j \wedge \mathbb{C}P^\infty} = (\mathbb{C}P^\infty/B_j) \wedge \mathbb{C}P^\infty = T(\mathcal{L}^{\otimes p^j}) \wedge \mathbb{C}P^\infty$.
- *Similarly,* A_2 *is the swapped projection* $(\mathbb{C}P^\infty)^{\wedge 2}/B_j^{\wedge 2} \to \mathbb{C}P^\infty \wedge T(\mathcal{L}^{\otimes p^j})$.

Setting $f = \mu(u \wedge x)A_1 - \mu(x \wedge u)A_2$ *gives the desired commuting square:*

$$\sigma \circ f = \beta\mu(a \wedge a) \circ \Delta.$$

Proof. The idea is to gain control of the cohomology group $H\mathbb{Z}^4((\mathbb{C}P^\infty)^{\wedge 2}, B_j^{\wedge 2})$ by Mayer-Vietoris, which is rendered complicated by our simultaneous use of the cofiber sequence

$$B_j \to \mathbb{C}P^\infty \to T(\mathcal{L}^{\otimes p^j})$$

in *two* factors of a smash product. Toward this end, consider the maps

$$B_1\colon\ B_j \wedge T(\mathcal{L}^{\otimes p^j}) \to (\mathbb{C}P^\infty)^{\wedge 2}/B_j^{\wedge 2}, \qquad B_2\colon\ T(\mathcal{L}^{\otimes p^j}) \wedge B_j \to (\mathbb{C}P^\infty)^{\wedge 2}/B_j^{\wedge 2},$$

which have cofibers A_1 and A_2 respectively. Direct calculation [AS01, Lemma 5.6] shows that $(\ker B_1^*) \cap (\ker B_2^*)$ is torsion-free, so if we can identify $B_1^*(\beta\mu \circ \Delta)$ and $B_2^*(\beta\mu \circ \Delta)$, we will be most of the way there. We pick B_1 to consider, as B_2 is similar, and we start computing, beginning with

$$B_1^*(\beta\mu(a \wedge a) \circ \Delta) = \beta\mu(a \wedge a) \circ \Delta B_1.$$

Writing $\delta\colon\ T(\mathcal{L}^{\otimes p^j}) \to \Sigma B_j$ for the going-around map in that cofiber sequence, we have

$$\begin{tikzcd}
\Sigma B_j^{\wedge 2} & B_j \wedge T(\mathcal{L}^{\otimes p^j}) \arrow[l, "\beta\wedge\delta"'] \arrow[d, "B_1"] \\
 & (\mathbb{C}P^\infty)^{\wedge 2}/B_j^{\wedge 2}, \arrow[ul, "\Delta"]
\end{tikzcd}$$

and hence

$$\begin{aligned}
\beta\mu(a \wedge a) \circ \Delta B_1 &= \beta\mu(a \wedge a) \circ (1_B \wedge \delta) \\
&= \beta\mu(a \wedge a\delta).
\end{aligned}$$

The maps a and δ appear in the following map of cofiber sequences:

$$\begin{tikzcd}
B \arrow[r, "j"] \arrow[d, "a"] & P \arrow[r, "q"] \arrow[d, "y"] & T \arrow[r, "\delta"] \arrow[d, "w"] & \Sigma B \arrow[d, "a"] \\
\Sigma H\mathbb{Z}/p^j \arrow[r, "\beta"] & \Sigma^2 H\mathbb{Z} \arrow[r, "p^j"] & \Sigma^2 H\mathbb{Z} \arrow[r, "\rho"] & \Sigma^2 H\mathbb{Z}/p^j,
\end{tikzcd}$$

where y is the standard Euler class in $H^2\mathbb{C}P^\infty$ and the first block commutes because the bottom row is the stabilization of the top row; w is the Thom class associated to $T(\mathcal{L}^{\otimes p^j})$ and the middle block commutes because it witnesses the $H\mathbb{Z}$–analogue of the statements expressed by Lemma 2.6.1 and Theorem 4.6.1; and the last block commutes because the other two do. In particular, an application of the right-most block gives

$$\beta\mu(a \wedge a\delta) = \beta\mu(a \wedge \rho w).$$

Using the fact that β is the cofiber of the ring map ρ, there is a juggle

$$\beta\mu(a \wedge \rho w) = \mu(\beta a \wedge w),$$

and then we use the first block in the above map of cofiber sequences to conclude

$$\mu(\beta a \wedge w) = \mu(yj \wedge w).$$

Finally, we can use this to guess a formula for our desired map f: we set

$$f = \mu(u \wedge x)A_1 + \mu(x \wedge u)A_2,$$

because, for instance,

$$\begin{aligned} B_1^*(\sigma f) &= \sigma(\mu(u \wedge x)A_1 + \mu(x \wedge u)A_2)B_1 \\ &= \sigma\mu(x \wedge u)(j \wedge \mathrm{id}_T), \end{aligned}$$

where we used $A_1B_1 = 0$ and $A_2B_1 = (j \wedge \mathrm{id}_T)$, a calculation similar to the calculation involving δ earlier in the proof. Then, because $\sigma\colon kU \to H\mathbb{Z}$ sends Euler classes to Euler classes, we have

$$\begin{aligned} \sigma\mu(x \wedge u)(j \wedge \mathrm{id}_T) &= \mu(y \wedge w)(j \wedge \mathrm{id}_T) \\ &= \mu(yj \wedge w). \end{aligned}$$

Hence, we have crafted a class f with $\sigma f - \beta\mu(a \wedge a) \in (\ker B_1^*) \cap (\ker B_2^*)$.

What remains is to show that this class is torsion, hence identically zero. Half of this is obvious: $p^j\beta\mu(a \wedge a) = 0$, since $p^j\beta = 0$ on its own. For $p^j\sigma f$, we make an explicit calculation:

$$\begin{aligned} p^j\sigma f &= p^j(\mu(w \wedge y)A_1 - \mu(y \wedge w)A_2) \\ &= \mu(w \wedge p^j y)A_1 - \mu(p^j y \wedge w)A_2 \\ &= \mu(w \wedge q^* w)A_1 - \mu(q^* w \wedge w)A_2 \\ &= \mu(w \wedge w) \circ ((1 \wedge q)A_1 - (q \wedge 1)A_2) = 0. \end{aligned}$$ □

The upshot of all of this is that we have our desired calculation of the map e:

Corollary 5.3.7 ([AS01, Lemma 5.4 and Subsection "Modelling $d_n(L_1, L_2)$"]). *There is a commuting triangle*

$$\begin{tikzcd} (\Sigma^\infty BS^1[p^j])^{\wedge 2} \arrow[d, "\beta"] \arrow[dr, "d_j"] & \\ \underline{H\mathbb{Z}}_3 \arrow[r, "\gamma"] & \underline{kU}_6, \end{tikzcd}$$

where d_j classifies the bundle

$$d_j = \sum_{k=1}^{p^j-1} \left((1-\mathcal{L}_1)(1-\mathcal{L}_1^{\otimes k})(1-\mathcal{L}_2) - (1-\mathcal{L}_1)(1-\mathcal{L}_2^{\otimes k})(1-\mathcal{L}_2) \right).$$

Proof. We return to our putative map of cofiber sequences, and in particular to the right-most block

$$\begin{tikzcd} \mathbb{C}\mathrm{P}^\infty \wedge \mathbb{C}\mathrm{P}^\infty \arrow[r, "r"] \arrow[d, dotted, "d"] & (\mathbb{C}\mathrm{P}^\infty)^{\wedge 2}/B_j^{\wedge 2} \arrow[d, "f"] \\ \Sigma^6 kU \arrow[r] & \Sigma^4 kU. \end{tikzcd}$$

This expresses d in terms of f in the cohomology ring $kU^*(\mathbb{C}\mathrm{P}^\infty)^{\times 2}$, a by-now familiar situation. Namely, we have

$$\begin{aligned} \beta d &= (\mu(u \wedge x)A_1 - \mu(x \wedge u)A_2)r \\ &= \mu(u \wedge x)(q \wedge \mathrm{id}_{\mathbb{C}\mathrm{P}^\infty}) - \mu(x \wedge u)(\mathrm{id}_{\mathbb{C}\mathrm{P}^\infty} \wedge q) \\ &= \mu(q^*u \wedge x) - \mu(x \wedge q^*u). \end{aligned}$$

At this point, we need to make an actual identification: u is a Thom class associated to the line bundle $\mathcal{L}^{\oplus p^j}$, hence q^*u is its associated Euler class, which we compute in terms of x to be $q^*u = [p^j]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x)$, where the n–series on $\mathbb{C}\mathrm{P}^\infty_{kU}$ expressed in terms of the coordinate x is given by $[n]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x) = \beta^{-1}(1-(1-\beta x)^n)$. We use this formula to continue the calculation:

$$\begin{aligned} \mu(q^*u \wedge x) - \mu(x \wedge q^*u) &= [p^j]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x_1) \cdot x_2 - x_1 \cdot [p^j]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x_2) \\ &= x_1 x_2 \left(\frac{1-(1-\beta x_1)^{p^j}}{\beta x_1} - \frac{1-(1-\beta x_2)^{p^j}}{\beta x_2} \right) \\ &= \sum_{k=1}^{p^j-1} (x_1[k](x_1)x_2 - x_1[k](x_2)x_2). \end{aligned}$$

□

Definition 5.3.8. Let $\widehat{\mathbb{G}}$ be a connected p–divisible group of dimension 1. Given a point $f \in C^3(\widehat{\mathbb{G}}; \mathbb{G}_m)(T)$, we construct the function

$$\begin{gathered} e_{p^j}(f)\colon\ \widehat{\mathbb{G}}[p^j]^{\wedge 2} \to \mathbb{G}_m, \\ e_{p^j}(f)(x_1, x_2) = \prod_{k=1}^{p^j} \frac{f(x_1, kx_1, x_2)}{f(x_1, kx_2, x_2)}. \end{gathered}$$

As j ranges, this assembles into a map

$$e\colon\ C^3(\widehat{\mathbb{G}}; \mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}^{\wedge 2}, \mathbb{G}_m),$$

called the *Weil pairing* associated to f.

By design, the map e participates in a commuting square with $\operatorname{Spec} E_0 BU[6,\infty) \to \operatorname{Spec} E_0 \underline{H\mathbb{Z}}_3$, so that this fills out the map of sequences we were considering before we got involved in this analysis of vector bundles. What remains, then, is to assemble enough exactness results to apply the 5–lemma.

Lemma 5.3.9 ([AS01, Lemma 7.2])**.** *For $\widehat{\mathbb{G}}$ a connected p–divisible group of dimension 1, the map $\delta\colon C^2(\widehat{\mathbb{G}};\mathbb{G}_m) \to C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ is injective.*

Proof. Being finite height means that the multiplication-by-p map of $\widehat{\mathbb{G}}$ is fppf–surjective. The kernel of δ consists of symmetric, biexponential maps $\widehat{\mathbb{G}}^{\times 2} \to \mathbb{G}_m$.[6] By restricting such a map f to

$$f\colon \widehat{\mathbb{G}}[p^j] \times \widehat{\mathbb{G}} \to \mathbb{G}_m,$$

we can calculate

$$f(x, p^j y) = f(p^j x, y) = f(0, y) = 1.$$

But since p^j is surjective on $\widehat{\mathbb{G}}$, every point on the right-hand side can be so written (after perhaps passing to a flat cover of the base), so at every left-hand stage the map is trivial. Finally, $\widehat{\mathbb{G}} = \operatorname{colim}_j \widehat{\mathbb{G}}[p^j]$, so this filtration is exhaustive and we conclude that the kernel is trivial. □

Lemma 5.3.10 ([AS01, Lemma 7.3])**.** *More generally, the following sequence is exact*

$$0 \to C^2(\widehat{\mathbb{G}};\mathbb{G}_m) \xrightarrow{\delta} C^3(\widehat{\mathbb{G}};\mathbb{G}_m) \xrightarrow{e} \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}^{\wedge 2}, \mathbb{G}_m).$$

Remarks on proof. The previous Lemma demonstrates exactness at the first node. Showing that $e \circ \delta = 0$ is simple enough, but constructing preimages of $\ker \delta$ through e is hard work. The main tool, again, is p–divisibility: given a point $(g_1, g_2) \in \widehat{\mathbb{G}}[p^j]^{\wedge 2}$, over some flat base extension we can find g_2' satisfying $p^j g_2' = g_2$. With significant effort, the assignment $(g_1, g_2) \mapsto \{e_{p^j}(f)(g_1, g_2')^{-1}\}$ as j ranges can be shown to be independent of the choices g_2' and which, if $e(f) = 1$, determines an element of $C^2(\widehat{\mathbb{G}};\mathbb{G}_m)$. □

Luckily, the remaining bit of topology is very easy:

Lemma 5.3.11 ([AS01, Lemma 7.5])**.** *The top row of the main diagram is a short exact sequence of group schemes.*

Proof. Consider the sequence of homology Hopf algebras, before applying Spec. Since the integral homology of BSU and the E–homology of $\underline{H\mathbb{Z}}_3$ are both free and even, the Atiyah–Hirzebruch spectral sequence for $E_* BU[6,\infty)$ collapses to their tensor product over E_*. □

[6] The condition $f \in \ker \delta$ gives $f(x, y+z) = f(x,y) f(x,z)$, so that the kU^0–linearity condition becomes redundant:

$$\frac{f(x,y) f(t, x+y)}{f(t+x, y) f(t,x)} = \frac{f(x,y)[f(t,x) f(t,y)]}{[f(t,y) f(x,y)] f(t,x)} = 1.$$

Corollary 5.3.12 ([AS01, Theorem 1.4]). *The map*

$$\Pi_3\colon \operatorname{Spec} E_0 BU[6,\infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$$

is an isomorphism.

Proof. This is now a direct consequence of the 5–lemma. □

Remark 5.3.13 (cf. Theorem 5.4.1). We will soon show that $H_* BU[6,\infty)$ is also free and even. The proof of Lemma 5.3.11 thus also shows that the E–theory of $\underline{kU}_8$ fits into a similar short exact sequence.

Remark 5.3.14 ([AS01, Corollary 7.6]). The topological input to the 5–lemma also gave us a purely algebraic result for free: the map e is a surjective map of group schemes.

5.4 Analysis of $BU[6,\infty)$ at infinite height

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Motivated by our success at analyzing the schemes $\operatorname{Spec}(E_\Gamma)_0 BU[6,\infty)$ associated to $BU[6,\infty)$ through Morava E–theory, we move on to considering the scheme constructed via ordinary homology. As usual, we expect this to be harder: the formal group associated to ordinary homology is not p–divisible, and this causes many sequences which are short exact from the perspective of Morava E–theory to go awry. Instead, we will have to examine the problem more directly—luckily, the extremely polite formal group law associated to $\widehat{\mathbb{G}}_a$ makes computations accessible. We also expect the reward to be greater: as in Corollary 4.3.7, we will be able to use a successful analysis of the ordinary homology scheme to give a description of the complex-orientable homology schemes, no matter what complex-orientable homology theory we use.

As in the p–divisible case, our framework comes in the form of a map of sequences

$$\begin{array}{ccccc}
\operatorname{Spec} H_* BSU & \longrightarrow & \operatorname{Spec} H_* BU[6,\infty) & \longrightarrow & \text{“}\operatorname{Spec} H_* \underline{H\mathbb{Z}}_3\text{”} \\
\downarrow & & \downarrow & & \downarrow \\
C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) & \longrightarrow & C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) & \longrightarrow & \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m).
\end{array}$$

Our task, as then, is to discern as much about these nodes as possible, as well as any exactness properties of the two sequences.[7]

We begin with the topological sequence. The Serre spectral sequence

$$E_2^{*,*} = H^* BSU \otimes H^* \underline{H\mathbb{Z}}_3 \Rightarrow H^* BU[6,\infty)$$

[7]The quotes indicate that the right-hand topological node does not even make sense: $H^* \underline{H\mathbb{Z}}_3$ is not even-concentrated, and we do not understand the algebraic geometry of spaces whose homology is not even-concentrated. This is quite troubling—but we will press on for now.

gives us easy access to the middle node, and we will recount the case of $p = 2$ in detail. In this case, the spectral sequence has E_2–page

$$E_2^{*,*} = H\mathbb{F}_2^* BSU \otimes H\mathbb{F}_2^* \underline{H\mathbb{Z}}_3 \cong \mathbb{F}_2[c_2, c_3, \ldots] \otimes \mathbb{F}_2 \left[\mathrm{Sq}^I \iota_3 \,\middle|\, \begin{array}{c} I_j \geq 2I_{j+1}, \\ 2I_1 - I_+ > 1 \end{array} \right].$$

Because the target is 6–connective, we must have the transgressive differential $d_4 \iota_3 = c_2$, which via the Kudo transgression theorem spurs the much larger family of differentials

$$d_{4+I_+} \mathrm{Sq}^I \iota_3 = \mathrm{Sq}^I c_2.$$

This necessitates understanding the action of the Steenrod operations on the cohomology of BSU, which is due to Wu [May99, Section 23.6]:

Can these formulas be read off from the divisorial calculation? Maybe not, since it's easy to read off the Milnor primitives but harder to see the Steenrod squares.

$$\mathrm{Sq}^{2^j} \cdots \mathrm{Sq}^4 \mathrm{Sq}^2 c_2 \equiv c_{1+2^j} \pmod{\text{decomposables}}.$$

Accounting for the squares of classes left behind, this culminates in the following calculation:

This spectral sequence can be drawn in using Hood's package.

Theorem 5.4.1. *There is an isomorphism*

$$H\mathbb{F}_2^* BU[6, \infty) \cong \frac{H\mathbb{F}_2^* BU}{(c_j \mid j \neq 2^k + 1, j \geq 3)} \otimes F_2[\iota_3^2, (\mathrm{Sq}^2 \iota_3)^2, \ldots]. \quad \square$$

Remark 5.4.2 ([Sin68, Sto63]). More generally, there is an isomorphism

$$H\mathbb{F}_2^* \underline{kU}_{2k} \cong \frac{H\mathbb{F}_2^* BU}{(c_j \mid \sigma_2(j-1) < k-1)} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-1}],$$

where σ_2 is the dyadic digital sum and "Op" denotes the subalgebra of $H\mathbb{F}_2^* \underline{H\mathbb{Z}}_{2k-1}$ generated by the indicated class. Stong specialized to $p = 2$ and carefully applied the Serre spectral sequence to the fibrations

$$\underline{kU}_{2(k+1)} \to \underline{kU}_{2k} \to \underline{H\mathbb{Z}}_{2k}.$$

Singer worked at an arbitrary prime and used the Eilenberg–Moore spectral sequence for the fibrations

$$\underline{H\mathbb{Z}}_{2k-1} \to \underline{kU}_{2(k+1)} \to \underline{kU}_{2k}.$$

Both used considerable knowledge of the interaction of these spectral sequences with the Steenrod algebra.

Remark 5.4.3. These methods and results generalize directly to odd primes. The necessary modifications come from understanding the unstable mod–p Steenrod algebra, using the analogues of Wu's formulas due to Shay [Sha77], and employing Singer's Eilenberg–Moore calculation. Again, $H^* BU[6, \infty)$ is presented as a quotient by $H^* BU$ by certain Chern classes whose indices satisfy a p–adic sum condition, tensored up with the subalgebra of $H^* \underline{H\mathbb{Z}}_3$ generated by a certain element.

From the edge homomorphisms in Theorem 5.4.1, we can already see that the sequence of formal group schemes

$$\text{“}(\underline{H\mathbb{Z}}_3)_{HP}\text{”} \to BU[6,\infty)_{HP} \to BSU_{HP}$$

is neither left-exact nor right-exact. This seems bleak.

Ever the optimists, we turn to the algebra. We begin by reusing a strategy previously employed in Lemma 3.4.12: first perform a tangent space calculation

$$T_0 C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \cong C^k(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a),$$

then study the behavior of the different tangent directions to determine the full object $C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$. As a warm-up to the case $k = 3$ of interest, we will first consider the case $k = 2$. We have already performed the tangent space calculation:

Corollary 5.4.4 (cf. Lemma 3.2.5). *The unique symmetric* 2*–cocycle of homogeneous degree n has the form*

$$c_n(x,y) = \begin{cases} (x+y)^n - x^n - y^n & \text{if } n \neq p^j, \\ \frac{1}{p}\left((x+y)^n - x^n - y^n\right) & \text{if } n = p^j. \end{cases} \qquad \square$$

Our goal, then, is to select such an $f_+ \in C^2(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)$ and study the minimal conditions needed on a symbol a to produce a point in $C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ of the form $1 + af_+ + \cdots$. Since $c_n = \frac{1}{d_n}\delta(x^n)$ is itself produced by an additive formula, life would be easiest if we had access to an exponential, so that we could build

$$\text{“}\delta \exp(a_n x^n)^{1/d_n} = \exp(\delta a_n x^n / d_n) = \exp(a_n c_n).\text{”}$$

However, the existence of an exponential series is equivalent to requiring that a_n carry a divided-power structure, which turns out not to be minimal. In fact, we can show that *no* conditions on a_n are required *at all*.

Definition 5.4.5 (cf. Remark 3.3.21). The *Artin–Hasse exponential* is the power series

$$E_p(t) = \exp\left(\sum_{j=0}^{\infty} \frac{t^{p^j}}{p^j}\right) \in \mathbb{Z}_{(p)}[\![t]\!].$$

Lemma 5.4.6. *Write $\delta_{\widehat{\mathbb{G}}_a,\mathbb{G}_m} \colon C^1(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \to C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ and*

$$d_n = \begin{cases} 1 & \text{if } n = p^j, \\ 0 & \text{otherwise.} \end{cases}$$

The class $g_n(x,y) = \delta_{\widehat{\mathbb{G}}_a,\mathbb{G}_m} E_p(a_n x^n)^{1/p^{d_n}}$ is a series in $\mathbb{F}_p[\![a_n]\!][\![x,y]\!]$ and presents a point in $C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ reducing to $a_n c_n \in C^2(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)$ on tangent spaces.

Proof. Recall from Remark 3.3.21 that E_p has coefficients in $\mathbb{Z}_{(p)}$, and hence it can be reduced to a series with coefficients in $\mathbb{F}_p$. We then make the calculation

$$g_n(x,y) = \delta_{\widehat{\mathbb{G}}_a,\mathbb{G}_m} E_p(a_n x^n)^{1/p^{d_n}} = \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} \delta_{\widehat{\mathbb{G}}_a,\widehat{\mathbb{G}}_a} x^{np^j}}{p^{j+d_n}}\right) = \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} c_{np^j}(x,y)}{p^j}\right).$$

As claimed, the leading term is exactly $a_n c_n$, this series is symmetric, and since it is in the image of $\delta_{\widehat{\mathbb{G}}_a,\mathbb{G}_m}$ it is certainly a 2–cocycle. □

Letting n range, this culminates in the following calculation:

Lemma 5.4.7 ([AHS01, Proposition 3.9])**.** *The map*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_n \mid n \geq 2] \xrightarrow{\prod_{n\geq 2} g_2} C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}$$

is an isomorphism. □

The case $k = 3$ is similar, with one important new wrinkle: over an $\mathbb{F}_p$–algebra there is an equality $c_n^p = c_{pn}$, but this relation does not generalize to trivariate 2–cocycles. For instance, consider the following example at $p = 2$:

$$\frac{1}{2}\delta(c_6) = x^2y^2z^2 + x^4yz + xy^4z + xyz^4, \qquad \left(\frac{1}{2}\delta c_3\right)^2 = x^2y^2z^2.$$

The following Lemma states that this is the only new feature of the trivariate setting:

Lemma 5.4.8 ([AHS01, Proposition 3.20, Proposition A.12])**.** *The p–primary residue of the scheme of trivariate symmetric 2–cocycles is presented by*

$$\operatorname{Spec} \mathbb{F}_p[a_d \mid d \geq 3] \times \operatorname{Spec} \mathbb{F}_p[b_d \mid d = p^j(1+p^k)] \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a) \times \operatorname{Spec} \mathbb{F}_p. \quad \square$$

Similar juggling of the Artin–Hasse exponential yields the following multiplicative classification:

Theorem 5.4.9 ([AHS01, Proposition 3.28])**.** *There is an isomorphism*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_d \mid d \geq 3, d \neq 1+p^t] \times \operatorname{Spec} \Gamma[b_{1+p^t}] \to C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}.$$

Proof sketch. The main claim is that the Artin–Hasse exponential trick used in the case $k = 2$ works here as well, provided $d \neq 1 + p^t$ so that taking an appropriate p^{th} root works out. They then show that the remaining exceptional cases extend to multiplicative cocycles only when the p^{th} power of the leading coefficient vanishes. Finally, a rational calculation shows how to bind these truncated generators together into a divided power algebra. □

It is now time to clear up our confusion about the right-hand topological node by pursuing a link between $H_* \underline{H\mathbb{Z}_3}$ and the algebraic model $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m)$. Analyzing the edge homomorphism from our governing Serre spectral sequence shows that the map

$$H^* BU[6,\infty) \to H^* \underline{H\mathbb{Z}_3}$$

factors through the subalgebra $A^* \subseteq H^* \underline{H\mathbb{Z}_3}$ generated by the *squares* of the polynomial generators. Accordingly, we aim to replace the right-hand node of the topological sequence with $\operatorname{Spec} A_*$ outright.

Lemma 5.4.10 ([AHS01, Lemma 3.36, Proposition 4.13, Lemma 4.11])**.** *The scheme* $\operatorname{Spec} A_*$ *models* $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m)$ *by an isomorphism* λ *commuting with* $e \circ \hat{\Pi}_3$.

Proof sketch. We can describe the $\mathbb{F}_p$–scheme $\mathsf{FormalGroups}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m)$ completely explicitly:

$$\begin{array}{ccc} (a_{mn})_{m,n} & \longmapsto & \prod_{m<n} \operatorname{texp}\left(a_{mn}(x^{p^m} y^{p^n} - x^{p^n} y^{p^m})\right) \\ \\ \operatorname{Spec} \mathbb{F}_p[a_{mn} \mid m < n]/(a_{mn}^p) & \xrightarrow{\cong} & \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m), \end{array}$$

where $\operatorname{texp}(t) = \sum_{j=0}^{p-1} t^j/j!$ is the truncated exponential series. It is easy to check that this ring of functions agrees with A^*, and it requires hard work (although not much creativity) to check the remainder of the statement: that $e \circ \hat{\Pi}_3$ factors through $\operatorname{Spec} A_*$ and that the factorization is an isomorphism. $\square$

We have now finally assembled our map of sequences,

$$\begin{tikzcd}
\operatorname{Spec} H_* BSU \arrow[r] \arrow[d, "\hat{\Pi}_2", "\cong"'] & \operatorname{Spec} H_* BU[6,\infty) \arrow[r] \arrow[d, "\hat{\Pi}_3"] & \operatorname{Spec} A^* \arrow[r] \arrow[d, "\lambda", "\cong"'] & 0 \\
C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "e"] & \mathsf{Weil}(\widehat{\mathbb{G}}_a) \arrow[r] & 0
\end{tikzcd}$$

which we have shown to be exact at all the indicated nodes. (The exactness of the topological sequence follows from the Serre spectral sequence analysis. The exactness of the bottom sequence follows from it receiving a map from the top exact sequence, where the left-hand vertical map is an isomorphism.) Our calculations now pay off:

Corollary 5.4.11. *The map* $\hat{\Pi}_3$ *is an isomorphism:*

$$\hat{\Pi}_3 \colon \operatorname{Spec} H_* BU[6,\infty) \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m).$$

Proof sketch. We don't actually have to compute much about the middle map. Because the squares in the map of sequences commute and the sequences themselves are exact as indicated, we at least learn that $\hat{\Pi}_3$ is an epimorphism on rings of functions. But, since both source and target are affine schemes of graded finite type with equal Poincaré series in each case, our epimorphism is an isomorphism. $\square$

Epi or mono? Group schemes or Hopf algebras of functions?

Corollary 5.4.12 ([AHS01, Theorem 2.31]). *The map $\hat{\Pi}_3$ is an isomorphism for any complex-orientable E.*

Proof sketch. This follows much along the lines of Corollary 4.3.7. The evenness of the topological calculation at $E = H\mathbb{F}_p$ shows that the statement holds for $H\mathbb{Z}_p^\wedge$ and $H\mathbb{Z}_{(p)}$, and since p is arbitrary we conclude it for $H\mathbb{Z}$ as well. We thus learn that the statement holds for $E = MUP$ using a tangent space argument, and then an Atiyah–Hirzebruch argument gives the statement for any complex-oriented E. $\square$

Remark 5.4.13. This argument does *not* extend to a claim that we have an isomorphism of topological and algebraic exact sequences for any choice complex-orientable homology theory E. Our trick of replacing $H_*\underline{H\mathbb{Z}}_3$ by A_* has no generic analogue.

Our analysis of $\operatorname{Spec} E_*BU[6,\infty)$ forms input to two related pursuits: the homology scheme $\operatorname{Spec} E_*MU[6,\infty)$ arising in the theory of Thom spectra, and the object $BU[6,\infty)_E$ predual to $\operatorname{Spec} E_0BU[6,\infty)$. The analysis of the Thom spectrum is completely analogous to the analysis performed at the end of Lecture 5.2, and so we merely state the relevant results.

Definition 5.4.14. For a formal group $\widehat{\mathbb{G}}$, define maps $\mu_{ij}\colon \widehat{\mathbb{G}}^{\times 3} \to \widehat{\mathbb{G}}$ which multiply the i^{th} and j^{th} factors, discarding the remaining factor. For a line bundle $\mathcal{L}$ over $\widehat{\mathbb{G}}$, we define the scheme $C^3(\widehat{\mathbb{G}};\mathcal{L})$ by

$$C^3(\widehat{\mathbb{G}};\mathcal{L})(T) = \left\{ \begin{array}{l} \text{triv}^{\text{ns}} \text{ of } u^*\left(\frac{e^*\mathcal{L}\otimes(\mu_{12}^*\mathcal{L}\otimes\mu_{13}^*\mathcal{L}\otimes\mu_{23}^*\mathcal{L})}{(\pi_1^*\mathcal{L}\otimes\pi_2^*\mathcal{L}\otimes\pi_3^*\mathcal{L})\otimes\mu_{\text{all}}^*\mathcal{L}}\right)\downarrow u^*\widehat{\mathbb{G}}^{\times 3} \\ \text{which are rigid, symmetric, and } kU^0\text{–linear} \end{array} \right\}.$$

Lemma 5.4.15 ([AHS01, Theorem 2.50]). *There is a system of compatible maps*

$$\begin{array}{ccc} \operatorname{Spec} E_0BU[6,\infty) \times \operatorname{Spec} E_0MU[6,\infty) & \longrightarrow & \operatorname{Spec} E_0MU[6,\infty) \\ \| \qquad\qquad\qquad \downarrow & & \downarrow \\ C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m) \times C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0)) & \longrightarrow & C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0)), \end{array}$$

where the horizontal maps are the action maps defining torsors and the vertical maps are those induced by $\widehat{\Pi}_3$. $\square$

Corollary 5.4.16. *Take E to be* complex–orientable. *The functor $C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0))$ is isomorphic to the affine scheme* $\operatorname{Spec} E_0MU[6,\infty)$. *Moreover, the E_0–points of this scheme biject with ring spectrum maps $MU[6,\infty) \to E$.* $\square$

Lemma 5.4.17. *The ring map $MU[6,\infty) \to MSU$ is controlled on orientations by the map*

$$C^2(\mathbb{CP}^\infty_E;\mathcal{I}(0)) \xrightarrow{\delta} C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0)). \quad \square$$

Remark 5.4.18. The rational conclusion of this analysis admits a very mild reformulation: there is always a natural $MU[6,\infty)$–orientation of a rational spectrum E given by the composite

$$MU[6,\infty) \to MU[6,\infty) \otimes \mathbb{Q} \to H\mathbb{Q} = \mathbb{S} \otimes \mathbb{Q} \xrightarrow{\eta_E} E \otimes \mathbb{Q} = E.$$

This canonical point turns the $C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m)$–torsor structure of $C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0))$ into an isomorphism $C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0)) \cong C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m)$. In sum, an arbitrary $MU[6,\infty)$–orientation of E is witnessed by a symmetric rigid trivariate power series satisfying kU^0–linearity.

Our second task is to analyze the cohomology formal scheme associated to $BU[6,\infty)$, and we begin with the choice $E = H$.

Lemma 5.4.19 (Ando–Hopkins–Strickland, unpublished)**.** *$DC^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ are all formal varieties for $k \le 3$.*

Proof. We know that $\mathcal{O}C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ are all free $\mathbb{Z}$–modules of graded finite rank in the range $k \le 3$, so we may write

$$\mathcal{O}(DC^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)) \cong (\mathcal{O}C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m))^*.$$

Our task is to show that this Hopf algebra $\mathcal{O}(C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m))^*$ is a power series ring.

Specialize, for the moment, to the case of $k = 2$. It will suffice to show that it is a power series ring modulo p for every prime p. Such graded connected finite-type Hopf algebras over $\mathbb{F}_p$ were classified by Borel (and exposited by Milnor–Moore [MM65, Theorem 7.11]) as either polynomial or truncated polynomial. These two cases are distinguished by the Frobenius operation: the Frobenius on a polynomial ring is injective, whereas the Frobenius on a truncated polynomial ring is not. It is therefore equivalent to show that the *Verschiebung* on the original ring $\mathcal{O}(C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m)) \otimes \mathbb{F}_p$ is *surjective*. Recalling the calculation $c_n^p = c_{np}$ at the level of bivariate 2–cocycles, we compute

$$p^* a_n = a_{np}^p,$$

and since $Fa_{np} = a_{np}^p$ and $FV = p^*$, we learn

$$V(a_{np}) = a_n.$$

Essentially the same proof handles the cases $k = 1$ and $k = 0$.

The case $k = 3$ requires a small modification, to cope with the two classes of trivariate 2–cocycles. On the polynomial tensor factor of $\mathcal{O}(C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m))$ we can reuse the same Verschiebung argument to see that its dual Hopf algebra is polynomial. For the other fact, the dual of the divided power tensor factor is, without any further argument, always a primitively generated polynomial algebra. ∎

Theorem 5.4.20 (Ando–Hopkins–Strickland, unpublished)**.** *The scheme $C_3\mathbb{CP}^\infty_E$ exists, and it is modeled by $BU[6,\infty)_E$.*

Proof sketch. Let $\widehat{\mathbb{G}}$ be an arbitrary formal group. Note first that if $C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ is coalgebraic, then $C_3\widehat{\mathbb{G}}$ exists and is its Cartier dual: the diagram presenting $\mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ as a reflexive coequalizer of free Hopf algebras is also the diagram meant to present $C_3\widehat{\mathbb{G}}$ as a coalgebraic formal scheme. So, if the coequalizing Hopf algebra has a good basis, it will follow from Theorem 5.1.8 that the resulting diagram is a colimit diagram in formal schemes, with $C_3\widehat{\mathbb{G}}$ sitting at the cone point. It will additionally follow that the isomorphism

$$\operatorname{Spec} E_0 BU[6,\infty) \xrightarrow{\cong} C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m)$$

of Corollary 5.4.12 will re-dualize to an isomorphism

$$BU[6,\infty)_E \xleftarrow{\cong} C_3\mathbb{CP}^\infty_E.$$

So, we reduce to checking that $\mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ admits a good basis. By a base change argument, it suffices to take $\widehat{\mathbb{G}}$ to be the universal formal group over the Lazard ring, and we thus set about finding a nice basis for that.

We hope to gain control (as in Corollary 4.3.7 or Corollary 5.4.12) of this situation using our strong knowledge or $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$. We know from Lemma 5.4.19 that $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ is a free abelian group, and we know from Theorem 3.2.2 that $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$ is as well. By picking a $\mathbb{Z}$–basis $\mathbb{Z}\{\beta_j\}_j$ of $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ and considering the specialization map from $\widehat{\mathbb{G}}$ over $\mathcal{M}_{\mathbf{fgl}}$ to $\widehat{\mathbb{G}}_a$ over $\operatorname{Spec}\mathbb{Z}$, we choose a map α of $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$–modules

$$\begin{tikzcd}
\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})\{\tilde\beta_j\}_j \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m) \arrow[d] \\
\mathbb{Z}\{\beta_j\}_j \arrow[r, "\cong"] & \mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m).
\end{tikzcd}$$

By induction on degree, one sees that α is surjective, and since the source and target are abelian groups of graded finite rank *and the source is free*, we need only check that they have the same rational Poincaré series to conclude that α is an isomorphism. Over $\operatorname{Spec}\mathbb{Q}$ we can use the logarithm to construct an isomorphism

$$\operatorname{Spec}\mathbb{Q}\times(\mathcal{M}_{\mathbf{fgl}}\times C^k(\widehat{\mathbb{G}};\mathbb{G}_m)) \to \operatorname{Spec}\mathbb{Q}\times(\mathcal{M}_{\mathbf{fgl}}\times C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)),$$

hence the Poincaré series agree, hence $\alpha\otimes\mathbb{Q}$ is an isomorphism, and finally α is too.

Lastly, one checks that this basis gives us access to the desired collection of good subcoalgebras: these are indexed on an integer d, spanned by those basis vectors of degree at most d. □

5.5 Elliptic curves and θ–functions

The goal of this Lecture is to give the briefest possible summary of the theory of elliptic curves that covers the topics necessary to us in the coming sections. Accordingly, we won't cover many topics that a sane introduction to elliptic curves would make a point to cover, and—perhaps worse—we will hardly prove anything. We will, however, discover a place where "$C_3\widehat{\mathbb{G}}$" appears internally to the theory of elliptic curves, and I hope nonetheless that this will give the non-arithmetic reader a foothold on the "elliptic" part of "elliptic cohomology".

To begin, recall that an elliptic curve in the complex setting is a torus, and it admits a presentation by selecting a lattice Λ of full rank in $\mathbb{C}$ and forming the quotient

$$\mathbb{C} \xrightarrow{\pi_\Lambda} \mathbb{C}/\Lambda =: E_\Lambda.$$

A meromorphic function f on E_Λ pulls back to give a meromorphic function $\pi_\Lambda^* f$ on $\mathbb{C}$ which satisfies a periodicity constraint in the form of the functional equation

$$\pi_\Lambda^* f(z + \Lambda) = \pi_\Lambda^* f(z).$$

It follows immediately that there are no holomorphic such functions, save the constants—such a function would be bounded, and Liouville's theorem would apply. It is, however, possible to build the following meromorphic special function, which has poles of order 2 at the lattice points and satisfies the periodicity constraints:

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Its derivative is also a meromorphic function satisfying the periodicity constraint:

$$\wp'_\Lambda(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}.$$

In fact, these two functions generate all other meromorphic functions on E_Λ, in the sense that the subsheaf spanned by the algebra generators $\wp_\Lambda$ and $\wp'_\Lambda$ is exactly $\pi_\Lambda^* \mathcal{M}_{E_\Lambda}$. This algebra is subject to the following relation, in the form of a differential equation:

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda),$$

for some special values $g_2(\Lambda), g_3(\Lambda) \in \mathbb{C}$. Accordingly, writing $C \subseteq \mathbb{C}P^2$ for the projective curve $wy^2 = 4x^3 - g_2(\Lambda)w^2x - g_3(\Lambda)w^3$, there is an analytic group isomorphism

$$\begin{aligned} E_\Lambda &\to C, \\ z \pmod{\Lambda} &\mapsto [1 : \wp_\Lambda(z) : \wp'_\Lambda(z)]. \end{aligned}$$

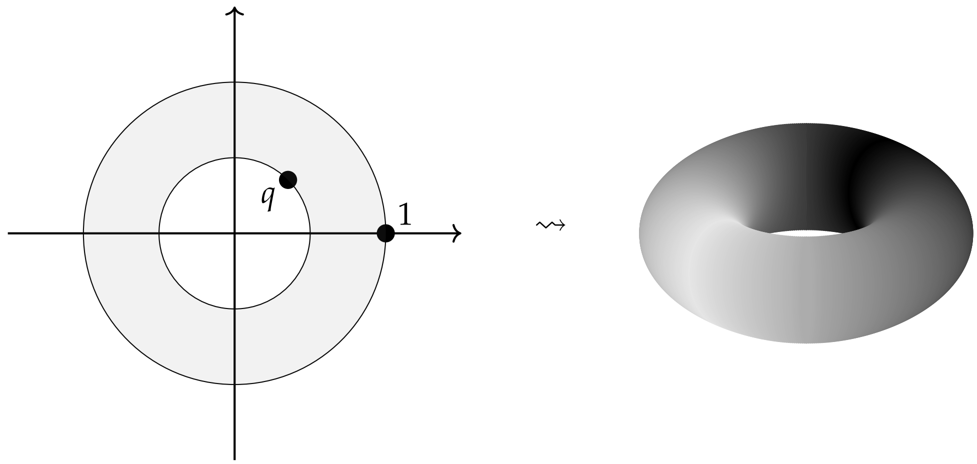

Figure 5.1: Presentation of an elliptic curve as the quotient of an annulus.

This is sometimes referred to as the *Weierstrass presentation* of E_Λ.

There is a second standard embedding of a complex elliptic curve into projective space, using *θ–functions*, which are most naturally expressed with an alternative basic presentation of an elliptic curve. Select a lattice Λ and a basis for it, and rescale the lattice so that the basis takes the form $\{1, \tau\}$ with τ in the upper half-plane. Then, the normalized exponential function $\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto \exp(2\pi i z)$ has $1 \cdot \mathbb{Z}$ as its kernel. Setting $q = \exp(2\pi i \tau)$ to account for the missing component of the kernel of π_Λ, we get a second presentation of E_Λ as $\mathbb{C}^\times / q^{\mathbb{Z}}$.

Definition 5.5.1. The basic θ–function associated to E_Λ is defined by

$$\theta_q(u) = \prod_{m \geq 1} (1 - q^m)(1 + q^{m-\frac{1}{2}} u)(1 + q^{m - \frac{1}{2}} u^{-1}) = \sum_{n \in \mathbb{Z}} u^n q^{\frac{1}{2} n^2}.$$

Given two rational numbers $0 \leq a, b \leq 1$, we can also shift the zero-set of θ_q in the 1 and q directions by the fractions a and b, giving translated θ–functions:

$$\theta_q^{a,b}(u) = q^{\frac{a^2}{2}} \cdot u^a \cdot \exp(2\pi i a b) \theta_q(u q^a \exp(2\pi i b)).$$

The basic θ–function vanishes on the set $\{\exp(2\pi i(\frac{1}{2} m + \frac{\tau}{2} n))\}$, i.e., at the center of the fundamental annulus. Since it has no poles, it cannot descend to give a function on $\mathbb{C}^\times / q^{\mathbb{Z}}$, and its failure to descend is witnessed by its imperfect periodicity relation:

$$\theta_q(qu) = u^{-1} q^{\frac{-1}{2}} \theta_q(u).$$

Lemma 5.5.2 ([Hus04, Proposition 10.2.6])**.** *For any $N > 0$, define $V_q[N]$ to be the space of functions $f \colon \mathbb{C}^\times \to \mathbb{C}$ satisfying*

$$f(qu) = u^{-N} q^{-N^2/2} f(u).$$

Then, $V_q[N]$ has $\mathbb{C}$–dimension N^2, and the functions $\theta_q^{a,b}$ give a basis as a and b range over rational numbers with denominator N. □

Even though these functions do not themselves descend to $\mathbb{C}^\times/q^{\mathbb{Z}}$, we can collectively use them to construct a map to complex projective space, where the quasi-periodicity relations will mutually cancel in homogeneous coordinates.

Theorem 5.5.3 ([Hus04, Proposition 10.3.2])**.** *Consider the map*

$$\begin{aligned} \mathbb{C}/(N\cdot\Lambda) &\xrightarrow{f_{(N)}} \mathbb{P}^{N^2-1}(\mathbb{C}), \\ z &\mapsto [\cdots : \theta_q^{i/N,j/N}(z) : \cdots]. \end{aligned}$$

For $N > 1$, this map is an embedding. □

Example 5.5.4. Let us expand this in the case of $N = 2$. The four functions involved are labeled $\theta_q^{0,0}$, $\theta_q^{0,1/2}$, $\theta_q^{1/2,0}$, and $\theta_q^{1/2,1/2}$, and we record their zero loci in the following table:

Function	Zero locus
$\theta_q^{0,0}$	$q^{\mathbb{Z}}\cdot q^{1/2}\cdot i$
$\theta_q^{0,1/2}$	$q^{\mathbb{Z}}\cdot q^{1/2}$
$\theta_q^{1/2,0}$	$q^{\mathbb{Z}}\cdot i$
$\theta_q^{1/2,1/2}$	$q^{\mathbb{Z}}$.

The image of $f_{(2)}$ in $\mathbb{P}^{2^2-1}(\mathbb{C})$ is cut out by the equations

$$A^2x_0^2 = B^2x_1^2 + C^2x_2^2, \qquad A^2x_3^2 = C^2x_1^2 - B^2x_2^2,$$

where

$$x_0 = \theta_q^{0,0}(u^2), \qquad x_1 = \theta_q^{0,1/2}(u^2), \qquad x_2 = \theta_q^{1/2,0}(u^2), \qquad x_3 = \theta_q^{1/2,1/2}(u^2)$$

and

$$A = \theta_q^{0,0}(0) = \sum_n q^{n^2}, \quad B = \theta_q^{0,1/2}(0) = \sum_n (-1)^n q^{n^2}, \quad C = \theta_q^{1/2,0}(0) = \sum_n q^{(n+1/2)^2},$$

upon which there is the additional "Jacobi" relation

$$A^4 = B^4 + C^4.$$

Remark 5.5.5. This embedding of E_Λ as an intersection of quadric surfaces in $\mathbb{CP}^3$ is quite different from the Weierstrass embedding. Nonetheless, the embeddings are analytically related. Namely, there is an equality

$$\frac{d^2}{dz^2}\log\theta_q(u) = \wp_\Lambda(z).$$

Separately, Weierstrass considered a function σ_Λ, defined by

$$\sigma_\Lambda(z) = z \prod_{\omega \in \Lambda \setminus 0} \left(1 - \frac{z}{\omega}\right) \cdot \exp\left[\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right],$$

which also has the property that its second logarithmic derivative is $\wp$ and so is "basically $\theta_q^{1/2,1/2}$". In fact, any elliptic function can be written in the form

$$c \cdot \prod_{i=1}^{n} \frac{\sigma_\Lambda(z - a_i)}{\sigma_\Lambda(z - b_i)}.$$

The θ–functions version of the story has two main successes: it continues to function in algebraic geometry, without invoking transcendental functions, and in fact there is a version of this story for an *arbitrary* abelian variety. It turns out that all abelian varieties are projective, and the theorem sitting at the heart of this claim is

Corollary 5.5.6 ("Theorem of the Cube", [Mil86, Corollary I.6.4 and Theorem I.7.1])**.** *Let A be an abelian variety, let $p_i : A \times A \times A \to A$ be the projection onto the i^{th} factor, and let $p_{ij} = p_i +_A p_j$, $p_{ijk} = p_i +_A p_j +_A p_k$. Then for any invertible sheaf $\mathcal{L}$ on A, the sheaf*

$$\Theta^3(\mathcal{L}) := \frac{p_{123}^*\mathcal{L} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}}{p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_{31}^*\mathcal{L} \otimes p_\varnothing^*\mathcal{L}} = \bigotimes_{I \subseteq \{1,2,3\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}$$

on $A \times A \times A$ is trivial. If $\mathcal{L}$ is rigid (i.e., it has a specified trivialization at the identity point of A), then $\Theta^3(\mathcal{L})$ is canonically trivialized by a section $s(A; \mathcal{L})$. □

Remark 5.5.7. One way to read the Theorem of the Cube is that a weight zero divisor on an abelian variety is principal (i.e., it is the zeroes and poles of a meromorphic function) if and only if its nodes sum to zero. The meromorphic function you construct this way is unique up to scale, so if you impose a normalization condition at the identity point, you get a unique such function. Altogether, this gives a pairing between $(A \times A)^*$ and A, which can be reconsidered as a multiplication map $A \times A \to A$.

Remark 5.5.8. The section $s(A; \mathcal{L})$ satisfies three familiar properties:

- It is symmetric: pulling back $\Theta^3\mathcal{L}$ along a shuffle automorphism of A^3 yields $\Theta^3\mathcal{L}$ again, and the pullback of the section $s(A; \mathcal{L})$ along this shuffle agrees with the original $s(A; \mathcal{L})$ across this identification.

- It is rigid: by restricting to $* \times A \times A$, the tensor factors in $\Theta^3\mathcal{L}$ cancel out to give the trivial bundle over $A \times A$. The restriction of the section $s(A; \mathcal{L})$ to this pullback bundle agrees with the extension of the rigidifying section.

- It satisfies a 2–cocycle condition: in general, we define

$$\Theta^k \mathcal{L} := \bigotimes_{I \subseteq \{1,\ldots,k\}} (p_I^* \mathcal{L})^{(-1)^{|I|-1}}.$$

In fact, $\Theta^{k+1}\mathcal{L}$ can be written as a pullback of $\Theta^k\mathcal{L}$:

$$\Theta^{k+1}\mathcal{L} = \frac{(p_{12} \times \mathrm{id}_{A^{k-1}})^* \mathcal{L}}{(p_1 \times \mathrm{id}_{A^{k-1}})^* \mathcal{L} \otimes (p_2 \times \mathrm{id}_{A^{k-1}})^* \mathcal{L}},$$

and pulling back a section s along this map gives a new section

$$(\delta s)(x_0, x_1, \ldots, x_k) := \frac{s(x_0 +_A x_1, x_2, \ldots, x_k)}{s(x_0, x_2, \ldots, x_k) \cdot s(x_1, x_2, \ldots, x_k)}.$$

Performing this operation on the first and second factors yields the defining equation of a 2–cocycle.

Remark 5.5.9. The proof of projectivity arising from this method rests on choosing a line bundle on A, extracting from it a very ample line bundle, and then constructing some generating global sections to get an embedding into $\mathbb{P}(\mathcal{L}^{\oplus n})$ [Har77, Remark II.7.8.2]. Mumford [Mum66] showed that a choice of "θ–structure" on $(A, \mathcal{L})$, which is only slightly more data, gives a canonical choice of generating global sections as well as a canonical identification of $\mathbb{P}(\mathcal{L}^{\oplus n})$ with a *fixed* projective space. This is suitable for studying how these equations change as one considers different points in the moduli of abelian varieties [Mum67a, Mum67b].

Remark 5.5.10 ([Bre83, Section 4]). Breen presented a relative version of this story that applies to arbitrary *commutative group schemes*, where the basic objects are a choice of line bundle $\mathcal{L}$ over a commutative group scheme A, a choice of trivialization of $\Theta^3\mathcal{L}$, and and epimorphism $\pi\colon A' \to A$ that trivializes $\mathcal{L}$.

Finally, we remark that the function

$$e\colon C^3(\widehat{\mathbb{G}}; \mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \widehat{\mathbb{G}}_m)$$

considered in Lecture 5.3 also manifests in the theory of abelian varieties. Let A be an abelian variety equipped with a line bundle $\mathcal{L}$. Suppose that s is a symmetric, rigid section of $\Theta^3\mathcal{L}$, sometimes called a *cubical structure on* $\mathcal{L}$. Using the identification $(p_{12} - p_1 - p_2)^*\Theta^2\mathcal{L} = \Theta^3\mathcal{L}$, this induces the structure of a *symmetric biextension* on $\Theta^2\mathcal{L}$ via the multiplication maps

$$(\Theta^2\mathcal{L})_{x,y} \otimes (\Theta^2\mathcal{L})_{x',y} \to (\Theta^2\mathcal{L})_{x+x',y}, \qquad (\Theta^2\mathcal{L})_{x,y} \otimes (\Theta^2\mathcal{L})_{x,y'} \to (\Theta^2\mathcal{L})_{x,y+y'}.$$

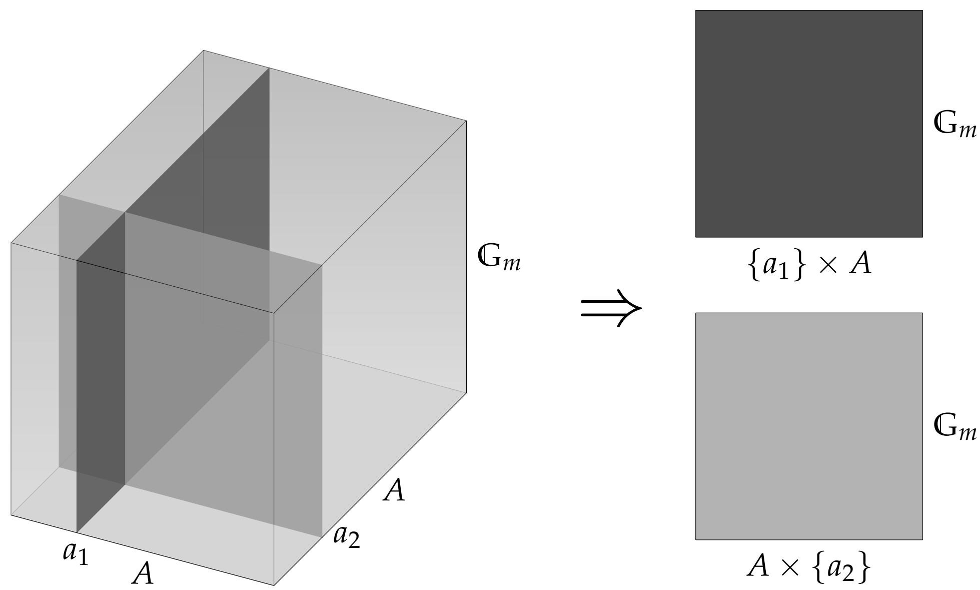

Figure 5.2: Extensions contained in a biextension.

Definition 5.5.11. There is a canonical piece of gluing data on this biextension, in the form of an isomorphism of pullback bundles

$$e_{p^j} \colon (p^j \times 1)^* \mathcal{L}|_{A[p^j] \times A[p^j]} \cong (1 \times p^j)^* \mathcal{L}|_{A[p^j] \times A[p^j]},$$
$$(\ell, x, y) \mapsto \left(\ell \cdot \prod_{k=1}^{p^j - 1} \frac{s(x, [k]x, y)}{s(x, [k]y, y)} \right).$$

This function e_{p^j} is called the *$(p^j)^{th}$ Weil pairing*.

Remark 5.5.12. In the case that A is an elliptic curve, this agrees with the usual definition of its "Weil pairing". In the case of a *complex* elliptic curve $\mathbb{C}/(1, \tau)$, this degenerates further to the assignment

$$\left(\frac{a}{n}, \frac{b}{n}\tau \right) \mapsto \exp\left(-2\pi i \frac{ab}{n} \right).$$

5.6 Modular forms from $MU[6, \infty)$–manifolds

We now actually leverage the arithmetic geometry in Corollary 5.5.6 by placing ourselves in a situation where algebraic topology is directly linked to abelian varieties.

Definition 5.6.1. An *elliptic spectrum* consists of a even–periodic ring spectrum E, a (generalized) elliptic curve C over $\operatorname{Spec} E_0$, and a fixed isomorphism

$$\varphi\colon\ C_0^\wedge \xrightarrow{\cong} \mathbb{CP}^\infty_E.$$

A map among such spectra consists of a map of ring spectra $f\colon\ E \to E'$ together with a specified isomorphism of elliptic curves $\psi\colon\ f^*C \to C'$.[8]

Remark 5.6.2. We have chosen to consider *isomorphisms* of elliptic curve rather than general homomorphisms because this is what algebraic topology suggests that we do. After all, the mixed cooperations of complex-oriented ring spectra are modeled by the isomorphisms of the associated formal groups. In the next Case Study, we will develop a theory (with an attendant notion of a "context") which incorporates isogenies of elliptic curves in addition to isomorphisms.

Coupling Definition 5.6.1 to Corollary 5.4.16 and Corollary 5.5.6, we conclude the following:

Corollary 5.6.3. *An elliptic spectrum (E, C, φ) receives a canonical map of ring spectra*

$$MU[6, \infty) \to E.$$

This map is natural in choice of elliptic spectrum: if $(E, C, \varphi) \to (E', C', \varphi')$ is a map of elliptic spectra, then the triangle

$$\begin{tikzcd} & MU[6,\infty) \arrow[dl] \arrow[dr] & \\ E \arrow[rr] & & E' \end{tikzcd}$$

commutes. □

Example 5.6.4. Our basic example of an elliptic curve was $E_\Lambda = \mathbb{C}/\Lambda$, with Λ a complex lattice. The projection $\mathbb{C} \to \mathbb{C}/\Lambda$ has a local inverse which defines an isomorphism of formal groups

$$\varphi\colon\ (E_\Lambda)_0^\wedge \xrightarrow{\cong} \widehat{\mathbb{G}}_a \times \operatorname{Spec} \mathbb{C},$$

as well as an isomorphism of cotangent spaces

$$\begin{tikzcd} T^0(E_\Lambda)_0^\wedge \arrow[r, "\cong"] & \mathbb{C} \arrow[d, equal] \\ T^0(\widehat{\mathbb{G}}_a \times \operatorname{Spec} \mathbb{C}) \arrow[u, "\cong"'] \arrow[r, "\cong"] & \mathbb{C}. \end{tikzcd}$$

[8]Elliptic curves and cohomology theories can be brought much closer together still, as in Lurie's framework [Lurd, Sections 4 and 5.3].

Accordingly, we define an elliptic spectrum $HE_\Lambda P$ whose underlying ring spectrum is $H\mathbb{C}P$ and whose associated elliptic curve and isomorphism are E_Λ and φ. This spectrum receives a natural map

$$MU[6,\infty) \to HE_\Lambda P,$$

which to a bordism class $M \in MU[6,\infty)_{2n}$ assigns an element $\Phi_\Lambda(M) \cdot u_\Lambda^n \in HE_\Lambda P_{2n}$, where u_Λ is the canonical element of $\pi_2 HE_\Lambda P = \Gamma(\omega_{\mathbb{C}\mathrm{P}^\infty_{HE_\Lambda P}})$ and $\Phi_\Lambda(M) \in \mathbb{C}$ is some complex number.

Example 5.6.5. The naturality of the $MU[6,\infty)$–orientation moves us to consider more than one elliptic spectrum at a time. If Λ' is another lattice with $\Lambda' = \lambda \cdot \Lambda$, then the multiplication map $\lambda\colon \mathbb{C} \to \mathbb{C}$ descends to an isomorphism $E_\Lambda \to E_{\Lambda'}$ and hence a map of elliptic spectra $HE_{\Lambda'}P \to HE_\Lambda P$ acting by $u_{\Lambda'} \mapsto \lambda u_\Lambda$. The commuting triangle in Corollary 5.6.3 then begets the *modularity relation*

$$\Phi(M; \lambda \cdot \Lambda) = \lambda^{-n} \Phi(M; \Lambda).$$

Example 5.6.6. This equation leads us to consider all curves E_Λ simultaneously—or, equivalently, to consider modular forms. The lattice Λ can be put into a standard form, by picking a basis and scaling it so that one vector lies at 1 and the other vector lies in the upper half-plane. This gives a cover

$$\mathfrak{h} \to \mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec} \mathbb{C}$$

which is well-behaved (i.e., unramified) away from the special points i and $e^{2\pi i/6}$. A *complex modular form of weight k* is an analytic function $\mathfrak{h} \to \mathbb{C}$ which satisfies a certain decay condition and which is quasi-periodic for the action of $SL_2(\mathbb{Z})$, i.e.,[9]

$$f\left(M; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(M; \tau).$$

Using these ideas, we construct a cohomology theory $H\mathcal{O}_\mathfrak{h}P$, where $\mathcal{O}_\mathfrak{h}$ is the ring of complex-analytic functions on the upper half-plane. The $\mathfrak{h}$–parametrized family of elliptic curves

$$\mathfrak{h} \times \mathbb{C}/(1,\tau) \to \mathfrak{h},$$

together with the logarithm, present $H\mathcal{O}_\mathfrak{h}P$ as an elliptic spectrum $H\mathfrak{h}P$. The canonical map $\Phi\colon MU[6,\infty) \to H\mathfrak{h}P$ specializes at a point to give the functions $\Phi(-;\Lambda)$ considered above, and hence $\Phi(M)$ is itself a complex modular form of weight k.

In fact, this totalized map Φ is a ghost of Ochanine and Witten's modular genus from Theorem 0.0.3, as a bordism class in $MU[6,\infty)_{2n}$ is, in particular, a bordism class in $MString_{2n}$. However, they know more about this function than we can presently see: for

[9] That is, for the action of change of basis vectors.

instance, they claim that it has an integral q–expansion. In terms of the modular form, its q–expansion is given by building the Taylor expansion "at ∞" (using that unspoken decay condition). In order to use our topological methods, it would be nice to have an elliptic spectrum embodying these q–expansions in the same way that $H\mathfrak{h}P$ embodied holomorphic functions, together with a comparison map that trades a modular form for its q–expansion. The main ideas leading to such a spectrum come from considering the behavior of E_Λ as τ tends to $i \cdot \infty$.

Definition 5.6.7. Note that as $\tau \to i \cdot \infty$, the parameter $q = \exp(2\pi i \tau)$ tends to 0. In the multiplicative model of Lecture 5.5, we considered D' the punctured complex disk with associated family of elliptic curves

$$C'_{\mathrm{an}} = \mathbb{C}^\times \times D'/(u,q) \sim (qu,q).$$

The fiber of C' over a particular point $q \in D'$ is the curve $\mathbb{C}^\times/q^{\mathbb{Z}}$. The Weierstrass equations give an embedding of C'_{an} into $D' \times \mathbb{C}P^2$ described by

$$wy^2 + wxy = x^3 - 5\alpha_3 w^2 x + -\frac{5\alpha_3 + 7\alpha_5}{12} w^3$$

Do these have weights?

for certain functionsα_3 and α_5 of q. At $q = 0$, this curve collapses to the twisted cubic

$$wy^2 + wxy = x^3,$$

and over the whole open unit disc D we call this extended family C_{an}.

Now let $A \subseteq \mathbb{Z}[\![q]\!]$ be the subring of power series which converge absolutely on the open unit disk. It turns out that the coefficients of the Weierstrass cubic (i.e., $5\alpha_3$ and $\frac{1}{12}(5\alpha_3 + 7\alpha_5)$) lie in A, so it determines a generalized elliptic curve C over $\operatorname{Spec} A$, and C_{an} is the curve given by base-change from A to the ring of holomorphic functions on D.The *Tate curve* is the intermediate family C_{Tate} over the intermediate base $D_{\mathrm{Tate}} = \operatorname{Spec} \mathbb{Z}[\![q]\!]$, as base-changed from A.

Cite me: Find a reference for this. You may be able to look in Morava's Section 5.

The singular fiber at $q = 0$ prompts us to enlarge our notion of elliptic curve slightly.

Definition 5.6.8 ([AHS01, Definitions B.1-2]). A *Weierstrass curve* is *any* curve of the form

$$C(a_1, a_2, a_3, a_4, a_6) := \left\{ ([x:y:w], s) \in \mathbb{P}^2 \times S \;\middle|\; \begin{array}{c} y^2 w + a_1(s) xyw + a_3(s) yw^2 = \\ x^3 + a_2(s) x^2 w + a_4(s) x w^2 + a_6(s) w^3 \end{array} \right\}.$$

A *generalized elliptic curve* over S is a scheme C equipped with maps

$$S \xrightarrow{0} C \xrightarrow{\pi} S$$

such that C is Zariski–locally isomorphic to a system of Weierstrass curves (in a way preserving 0 and π).[10,11]

Remark 5.6.9 ([AHS01, pg. 670]). The singularities of a degenerate Weierstrass equation always occur outside of a formal neighborhood of the marked identity point, which in fact still carries the structure of a formal group. The formal group associated to the twisted cubic is the formal multiplicative group (indeed, the smooth locus of the twisted cubic *is the multiplicative group*), and the isomorphism making the identification extends a family of such isomorphisms φ over the nonsingular part of the Tate curve.

Definition 5.6.10 ([Mor89, Section 5], [AHS01, Section 2.7]). The elliptic spectrum K_{Tate}, called *Tate K–theory*, has as its underlying spectrum $KU[\![q]\!]$. The associated elliptic curve is C_{Tate}, and the isomorphism $\mathbb{C}\mathrm{P}^\infty_{KU[\![q]\!]} \cong (C_{\mathrm{Tate}})^\wedge_0$ is φ from Remark 5.6.9.

The trade for the breadth of this definition is that theorems pulled from the study of abelian varities have to be shown to extend uniquely to those generalized elliptic curves which are not smooth curves.

Theorem 5.6.11 ([AHS01, Propositions 2.57 and B.25]). *For a generalized elliptic curve C, there is a canonical*[12] *trivialization s of $\Theta^3\mathcal{I}(0)$ which is compatible with change of base and with isomorphisms. If C is a smooth elliptic curve, then s agrees with that of Corollary 5.5.6.* □

Corollary 5.6.12. *The trivializing section s associated to C_{Tate} is given by $\delta^{\circ 3}\tilde{\theta}$, where $\tilde{\theta}_q$ is a slight modification of the classical θ–function:*

$$\tilde{\theta}_q(u) = (1-u)\prod_{n>0}(1-q^n u)(1-q^n u^{-1}), \qquad \tilde{\theta}_q(qu) = -u^{-1}\tilde{\theta}_q(u).$$

Proof. Even though θ is not a function on C_{Tate} because of its quasiperiodicity, it does trivialize both $\pi^*\mathcal{I}(0)$ for $\pi\colon \mathbb{C}^\times \times D \to C_{\mathrm{Tate}}$ and $\mathcal{I}(0)$ for $(C_{\mathrm{Tate}})^\wedge_0$. Moreover, the quasiperiodicities in the factors in the formula defining $\delta^3\tilde{\theta}|_{(C_{\mathrm{Tate}})^\wedge_0}$ cancel each other out, and the resulting function *does* descend to give a trivialization of $\Theta^3\mathcal{I}(0)$. By the unicity clause in Theorem 5.6.11, it must give a formula expressing s. □

Definition 5.6.13. The induced map

$$\sigma_{\mathrm{Tate}}\colon MU[6,\infty) \to K_{\mathrm{Tate}}$$

is called the *complex σ–orientation*.

[10] An elliptic curve in the usual sense turns out to be a generalized elliptic curve which is smooth, i.e., the discriminant of the Weierstrass equations is a unit.

[11] Unfortunately, "generalized elliptic curve" already means something in number theory, but Ando, Hopkins, and Strickland reused this phrase for this definition in their published article. In a number theorist's language, these are "stable curves of genus 1 with specified section in the smooth locus". No adjective other than "generalized" seems to be much better: singular, for instance, evokes the right idea but is also already taken.

[12] Canonical, with a unique continuous extension from the smooth bulk of the moduli of generalized elliptic curves, but *not* actually unique over the singular locus.

Corollary 5.6.14. *Let $M \in \pi_{2n} MU[6,\infty)$ be a bordism class. The q–expansion of Witten's modular form $\Phi(M)$ has integral coefficients.*

Proof. The span of elliptic spectra equipped with $MU[6,\infty)$–orientations

$$\begin{tikzcd} & MU[6,\infty) \arrow[dl, "\sigma_{\mathrm{Tate}}"'] \arrow[d] \arrow[dr, "\Phi"] & \\ K_{\mathrm{Tate}} \arrow[r] & K_{\mathrm{Tate}} \otimes \mathbb{C} & H\mathfrak{h}P \arrow[l] \end{tikzcd}$$

models q–expansion. The arrow $K_{\mathrm{Tate}} \to K_{\mathrm{Tate}} \otimes \mathbb{C}$ is injective on homotopy, which shows that the q–expansion of $\Phi(M)$ lands in the subring of integral power series. $\square$

We can use the formula $\sigma_{\mathrm{Tate}} = \delta^3 \tilde{\theta}$ appearing in Corollary 5.6.12 to explicitly understand the genus associated to σ_{Tate} by passing to homotopy groups. To begin, the appearances of the map δ in Remark 5.2.4, Remark 5.2.17, and Lemma 5.4.17 show that σ_{Tate} belongs to the commutative triangle

$$\begin{tikzcd} MU[6,\infty) \arrow[r, "\delta"] \arrow[drrr, "\sigma_{\mathrm{Tate}}"'] & MSU \arrow[r, "\delta"] & MU \arrow[r, "\delta"] & MUP \arrow[d, "\tilde{\theta}"] \\ & & & KU[\![q]\!]. \end{tikzcd}$$

We will analyze this triangle by comparing $\tilde{\theta}$ to the usual MUP–orientation of KU, which selects the coordinate $f(u) = 1 - u$ on the formal completion of $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[u^{\pm}]$. Appealing to Remark 5.2.4, the induced MU–orientation

$$MU \xrightarrow{\delta} MUP \xrightarrow{\mathrm{Td}} KU$$

sends f to the rigid section δf of

$$\Theta^1 \mathcal{I}(0) = \mathcal{I}(0)_0 \otimes \mathcal{I}(0)^{-1} \cong \omega \otimes \mathcal{I}(0)$$

given by

$$\delta f = \frac{1}{1-u}\left(-\frac{du}{u}\right).$$

The difference between $\delta\,\mathrm{Td}$ and $\delta\tilde{\theta}$ is expressed by an element $\psi \in C^1(\widehat{C}_{\mathrm{Tate}}; \mathbb{G}_m)$, given explicitly by the quotient formula

$$\psi = \left(\frac{\mathrm{Td}(1)}{\mathrm{Td}(u)}\right)^{-1} \cdot \frac{\theta_q(1)}{\theta_q(u)} = \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}.$$

This gives a re-expression of $\delta\tilde{\theta}$ as the composite

$$\delta\theta \colon MU \xrightarrow{\eta_R} MU \wedge MU \simeq MU \wedge BU_+ \xrightarrow{\delta(1-u)\wedge\psi} K_{\mathrm{Tate}},$$

and hence its effect on a line bundle is determined by the evaluation of this characteristic series:

$$\psi(1-\mathcal{L}) = \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n\mathcal{L})(1-q^n\mathcal{L}^{-1})}.$$

Its effect on vector bundles in general is determined by the splitting principle and an exponential law, which after some computation gives the generic formula

$$\psi(\dim V \cdot 1 - V) = \bigotimes_{n\geq 1}\bigoplus_{j\geq 0} \mathrm{Sym}^j(\dim V \cdot 1 - V \otimes_{\mathbb{R}} \mathbb{C})q^{jn} =: \bigotimes_{n\geq 1} \mathrm{Sym}_{q^n}(-\bar{V}_{\mathbb{C}}).$$

Finally, the map $(\eta_R)_* \colon MU_* \to \pi_*(MU \wedge \Sigma^\infty_+ BU)$ sends a manifold M with stable normal bundle ν to the pair (M, ν), so we at last compute

$$\begin{aligned}\sigma_{\mathrm{Tate}}(M \in \pi_{2n} MU[6,\infty)) &= (\delta(1-u) \wedge \theta')(M,\nu) \\ &=: \mathrm{Td}\left(M; \bigotimes_{n\geq 1} \mathrm{Sym}_{q^n}(\bar{\tau}_{\mathbb{C}})\right).\end{aligned}$$

This is exactly Witten's formula for his genus, as applied to complex manifolds with first two Chern classes trivialized.

Remark 5.6.15 ([Reza, Section 1.5]). Witten defines his characteristic series for *oriented* manifolds by the formula

$$\begin{aligned}K_{\mathrm{Witten}}(x) &= \exp\left(\sum_{k=2}^{\infty} 2G_{2k}(\tau) \cdot \frac{x^{2k}}{(2k)!}\right) \\ &= \frac{x/2}{\sinh(x/2)} \cdot \left(\prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}\right) \cdot e^{-G_2(\tau)x^2},\end{aligned}$$

where G_{2k} is the $2k^{\mathrm{th}}$ Eisenstein series. Noting that G_2 is *not* a modular form, the condition that $p_1(M)/2$ vanish is precisely the condition that G_2 contribute nothing to the sum, so that the remainder *is* a modular form.

5.7 Chromatic *Spin* and *String* orientations

This is under construction. The stuff here is correct, but it didn't turn into a super coherent lecture, and I have an idea for something better to say.

A different approach to this is to work through the $p \neq 2$ case, since you can totally, correctly work it out and see the Σ–structure relation get imposed. That's kind of attractive. Too bad you had this thought after the class was complete. Maybe *start* this section with the odd-primary case. —— We might also try to understand the extension problem of orientations across $MU \to MSO$ for spectra which are local away from 2. It seems like some juggling of the complex-conjugation idempotents could give you access to BSO information in terms of BU, which would be satisfying. Compare with Neil's response at `http://mathoverflow.net/questions/123958/a-formal-group-law-over-oriented-bordism`. Hood also points out that the usual "$\tanh^{-1}$" equation for the L–genus has an expansion in terms of the usual logarithm: it's an averaging between the positive and negative logarithmic series. Hood also points out that the mysterious series $x/(1-e^{-x})$ occurs as $\exp_{\widehat{\mathbb{G}}_a} / \exp_{\widehat{\mathbb{G}}_m}$. Also, Section 3 of Ando–Hopkins–Rezk sets up a bunch of the rational Hirzebruch genera stuff in a way that extends to $MO\langle 8\rangle$…which I guess now appears in Appendix A.1.

In the previous Lecture, we proved that elliptic spectra receive canonical $MU[6,\infty)$–orientations, that complex elliptic spectra collectively give rise to a genus valued in modular forms, and that the q–expansions of these modular forms are integral. However, the original Theorem 0.0.3 of Ochanine and Witten claimed to describe a genus on *Spin*– and *String*–manifolds, which we have only managed to approximate with our study of $MU[6,\infty)$–orientations. Our last goal for this Case Study is to show that the chromatic formal schemes associated to spaces like *BString* are somewhat accessible, and so chromatically-amenable elliptic spectra receive canonical *MString*–orientations.

Fix a formal group Γ of finite height d, and write $K = K_\Gamma$ for the associated Morava K–theory. We will start with the more modest goal of understanding the bottom few layers of the Postnikov tower for $\underline{kO}_0 \simeq BO \times \mathbb{Z}$, which have the names

$$BO[2,\infty) := BSO, \qquad BO[4,\infty) := BSpin, \qquad BO[8,\infty) := BString.$$

The := should be =: perhaps?

Remark 5.7.1. Unlike kU, there is *not* an equivalence

$$\underline{kO}_n \not\simeq (BO \times \mathbb{Z})[n,\infty),$$

unless n happens to take the form $n = 8k$ for a nonnegative integer k. The reasoning for this stray equivalence is similar to that for kU: the homotopy ring of kO has a polynomial factor of degree 8, and the other elements lie in a band of dimensions smaller than 8. Otherwise, other things happen—for instance,

$$\underline{kO}_1 \simeq O/U, \qquad \underline{kO}_6 \simeq Spin/SU.$$

Remark 5.7.2 ([KLW04, Section 5.2]). We may as well take the ground field of our Morava K–theory to have characteristic $p = 2$, since at odd characteristics there is little distinction between kO and kU, owing to the fiber sequence

$$\Sigma kO \xrightarrow{\cdot\eta = 0} kO \to kU.$$

The way you wrote this fiber sequence doesn't show why $p = 2$ is special.

However, this reveals a disadvantage of Morava K–theory that will finally cause us real consternation: Morava K–theories at the prime 2 are not commutative ring spectra. Accordingly, $(K_\Gamma)_* G$ for a commutative H–group G may fail to give a commutative algebra. Luckily, Remark 3.5.5 tells us that if $(K_\Gamma)_* G$ happens to be even-concentrated, then the obstructions to commutativity identically vanish. So, we can be somewhat indelicate about this noncommutativity issue, provided that we continually check that the algebras we are forming are even-concentrated.

In order to get off the ground, we will need the following Lemma about the behavior of the Atiyah–Hirzebruch spectral sequence for a Morava K–theory:

Lemma 5.7.3 ([Yag80, Lemma 2.1])**.** *Let k_Γ be the connective cover of the Morava K–theory K_Γ. In the Atiyah–Hirzebruch spectral sequence*

Does this Lemma admit a coordinate-free statement? They probably aren't all identically controlled by Q_d, but rather by Q_d plus decomposables.

$$E_2^{*,*} = Hk^* X \otimes_k k_\Gamma^* \Rightarrow k_\Gamma^* X,$$

the differentials are given by

$$d_r(x) = \begin{cases} 0 & \text{if } r \le 2(p^d - 1), \\ \lambda Q_d x \otimes v_d & \text{if } r = 2(p^d-1)+1 \end{cases}$$

where $\lambda \neq 0$ and Q_d is the d^{th} Milnor primitive. □

Corollary 5.7.4 ([RWY98, Section 2.5], [KLW04, Equation 3.1])**.** *There is a bi-Cartesian square of coalgebraic formal schemes*

$$\begin{tikzcd}
& \mathrm{Div}_0 \overline{\mathbb{G}}[2] \arrow[r] \arrow[dl] & \mathrm{Div}_0 \widehat{\mathbb{G}}[2] \arrow[dl] \\
\mathrm{Div}_0 \overline{\mathbb{G}} \arrow[r] & BO_K. &
\end{tikzcd}$$

Proof. We apply Lemma 5.7.3 to the analysis of the spectral sequence of Hopf algebras

$$H\mathbb{F}_{2*}BO \otimes (K_\Gamma)_* \Rightarrow (K_\Gamma)_* BO.$$

We have $H\mathbb{F}_{2*}BO \cong \mathbb{F}_2[b_1, b_2, \ldots]$ and

Cite me: I don't know where these Milnor primitives are calculated. I guess we could have done them using formal geometry.

$$Q_d b_{2^{d+1}+2j} = b_{2j+1},$$

from which it follows that all the odd generators are killed, all their squares survive , and only the even generators of low degree are permanent cycles. This results in a decomposition

Only the squares of even generators survive.

$$(K_\Gamma)_* BO \cong (K_\Gamma)_*[b_2, b_4, b_{2^{d+1}-2}] \underset{(K_\Gamma)[b_{2j}^2 | j < 2^d]}{\otimes} (K_\Gamma)_*[b_{2j}^2],$$

and so we are tasked with assigning names to the coalgebraic formal schemes appearing in this formula.

The left-hand factor is the free Hopf algebra on the coalgebra determined by the 2–torsion in the formal group Γ . The right-hand factor is the free Hopf algebra on the formal curve $\overline{\mathbb{G}} := \mathbb{HP}_K^\infty$, using the isogeny

You write Γ but also $\widehat{\mathbb{G}}$.

$$\begin{aligned} \mathbb{HP}_K^\infty &\to \mathbb{CP}_K^\infty \\ y &\mapsto x \cdot [-1](x) \end{aligned}$$

induced by desymplectification. Because $\mathcal{H}^\times$ is not commutative, $\overline{\mathbb{G}}$ is not a formal group, but we pull back the multiplication-by-2 isogeny from $\widehat{\mathbb{G}}$ to $\overline{\mathbb{G}}$ and define the subscheme $\overline{\mathbb{G}}[2]$ of points mapping to zero. □

It would be nice if you could put in a bit more detail about why these formal schemes correspond to the coalgebras from above.

Suppose that the Postnikov sections

$$X(n,\infty) \to X[n,\infty) \to X[n,n]$$

induce short exact sequences of formal groups. A presentation of X_K as a bi-Cartesian square then acquires value via the following algebraic Lemma:

Lemma 5.7.5. *Consider the cube of formal group schemes constructed by taking pointwise fibers of the composite to G:*

$$\begin{tikzcd}
 & A' \arrow[rr] \arrow[dl] \arrow[dd] & & B' \arrow[dl] \arrow[dd] & \\
C' \arrow[rr] \arrow[dd] & & D' \arrow[dd] & & \\
 & A \arrow[rr] \arrow[dl] & & B \arrow[dl] & \\
C \arrow[rr] & & D \arrow[rr] & & G.
\end{tikzcd}$$

If the bottom face is bi-Cartesian, then so is the top. □

Corollary 5.7.6. *There is a bi-Cartesian square*

$$\begin{tikzcd}
 & \mathrm{Div}_0\,\overline{\mathbb{G}}[2] \arrow[rr] \arrow[dl] & & \mathrm{SDiv}_0\,\widehat{\mathbb{G}}[2] \arrow[dl] \\
\mathrm{Div}_0\,\overline{\mathbb{G}} \arrow[rr] & & BSO_K. &
\end{tikzcd}$$

Proof sketch. The fibration $BSO \to BO \to BO(1)$ gives a short exact sequence of Hopf algebras , so using Corollary 5.7.4 we are in the situation of Lemma 5.7.5. To compute the pointwise kernels, begin by considering the commuting square of Postnikov sections

after applying K-homology – what is the argument for this? Do I need to look at some spectral sequence?

$$\begin{tikzcd}
BO_K \arrow[r] \arrow[d] & BU_K \arrow[d] \\
\widehat{\mathbb{G}}[2] \arrow[r] & \widehat{\mathbb{G}}.
\end{tikzcd}$$

Both horizontal maps are injections. Since $\mathrm{Div}\,\overline{\mathbb{G}} \to BU_K \cong \mathrm{Div}_0\,\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is null, the composite $\mathrm{Div}_0\,\overline{\mathbb{G}} \to \widehat{\mathbb{G}}[2]$ is null. Similarly, the composite $\mathrm{Div}\,\widehat{\mathbb{G}}[2] \to \widehat{\mathbb{G}}[2]$ acts by summation, and its kernel is $\mathrm{SDiv}_0\,\widehat{\mathbb{G}}[2]$. □

This diagram seems to be missing $\mathrm{Div}_0\,\bar{G}$ and $\mathrm{Div}\,\bar{G}$. Also, you are identifying $K(\mathbb{Z}/2.1)_K$ with $\widehat{\mathbb{G}}[2]$. You argue using BU, but it also seems that whatever argument you are using to show that the right vertical arrows have null composite should also just work for BO.

From here, the computation gets harder.

Corollary 5.7.7 ([KLW04, Section 5.3]). *There is a bi-Cartesian square*

$$\begin{tikzcd}
 & \mathrm{Div}_0 \overline{\mathbb{G}}[2] \arrow[rr] \arrow[dl] & & \ker \omega \arrow[dl] \\
\mathrm{Div}_0 \overline{\mathbb{G}} \arrow[rr] & & BSpin_K,
\end{tikzcd}$$

where $\omega\colon C_2\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}[2]^{\wedge 2}$ *is the map* $([a]-[0])([b]-[0]) \mapsto a \wedge b$.

Proof. This goes similarly to Corollary 5.7.6, once you know that the Postnikov section induces a short exact sequence of formal groups. The composite $\mathrm{Div}\,\overline{\mathbb{G}} \to \widehat{\mathbb{G}}[2]^{\wedge 2}$ is shown to be zero using an identical technique. To identify the behavior on the other factor, we need the following diagram of exact sequences of Hopf algebras from Kitchloo, Laures, and Wilson [KLW04, Theorem 6.4]:

Cite me: Cite the relevant part of KLW.

Really? This is what I have written down in my notes, but I can't expand it out.

$$\begin{tikzcd}
 & & & K_* \arrow[d] \\
 & K_* \arrow[d] & & K_*K(\mathbb{Z},3) \arrow[d] \\
K_* \arrow[r] & K_*BSpin \arrow[r] \arrow[d] & K_*BSU \arrow[r, "\tau"] \arrow[d, equal] & K_*BU[6,\infty) \arrow[d, "\delta"] \\
K_* \arrow[r] & K_*BSO \arrow[r, "i"] \arrow[d] & K_*BSU \arrow[r, "1-\xi"] & K_*BSU \arrow[d] \\
 & K_*K(C_2,2) \arrow[d] \arrow[uurr, dotted, bend right] & & K_* \\
 & K_*, & &
\end{tikzcd}$$

where $\tau\colon C_2\widehat{\mathbb{G}} \to C_3\widehat{\mathbb{G}}$ is specified at the level of formal schemes by

$$\tau\colon ([a]-[0])([b]-[0]) \mapsto ([a]-[0])([b]-[0])([-a-b]-[0]).$$

Since $(1-\xi)\circ i = 0$, we have that $\delta \circ \tau \circ i = 0$ and hence that $\tau \circ i$ lifts to $K_*K(\mathbb{Z},3)$. Identifying $\mathrm{SDiv}_0\,\widehat{\mathbb{G}}[2]$ with $C_2\widehat{\mathbb{G}}[2]$, we check that the composites

$$C_2\widehat{\mathbb{G}}[2] \xrightarrow{\omega} \widehat{\mathbb{G}}[2]^{\wedge 2} \xrightarrow{\varepsilon} C_3\widehat{\mathbb{G}}$$

and

$$C_2\widehat{\mathbb{G}}[2] \to C_2\widehat{\mathbb{G}} \xrightarrow{\tau} C_3\widehat{\mathbb{G}}$$

agree. For a point $[a,b] \in C_2\widehat{\mathbb{G}}$, this is the claim

$$\begin{aligned} 0 &= \varepsilon(a \wedge b) - \tau[a,b] \\ &= [a,a,b] - [b,a,b] - [-a-b,a,b] \\ &= [a,a,b] - [b+a,a,b] + [b,a+a,b] - [b,a,b], \end{aligned}$$

and this is forced null in $C_3\widehat{\mathbb{G}}$, as it looks like a 2–cocycle shuffle.

Where is $\widehat{\mathbb{G}}[2]$^2 in the diagram? What is ξ? What is ε? Is the dotted arrow the Bockstein? In the diagram you wrote $K_*K(C_2,2)$, but I think we already have too many C_2's. I'm also confused about the appearances of all the symmetric powers C_2 and C_3. Where do they appear in the diagram?

I'm a little fuzzy on the coherence of this with the Bockstein: this computes the lift of $\tau \circ f$ into $K(\mathbb{Z},3)_K$, and it does happen to factor through the subscheme $K(\mathbb{Z}/2,2)_K$ determined by the Bockstein. However, I don't immediately see why this agrees with the bottom Postnikov section of BSO: that's a map off of BSO and this is a rotated map into $BU[6,\infty)$, so it's not an immediate consequence of naturality. It has to do with rotating the Wood cofiber sequence just right, and in particular where the horizontal sequences come from: they're stitched-together from two consecutive Wood cofiber sequences.

□

Ideally, we would use this presentation of $BSpin_K$ to say something about $MSpin$–orientations. I just realized I don't know how, though!

Remark 5.7.8. This computation becomes almost unfeasible for *BString*, but we will sketch two approaches. One is that the sequence

$$\underline{HC_2}2 \to \underline{HC_{2^\infty}}2 \to BString \to BSpin \to \underline{HC_{2^\infty}}4 \xrightarrow{2} \underline{HC_{2^\infty}}4$$

induces an exact sequence of group schemes. The other avenue of access is the pair of fiber sequences

$$\underline{H\mathbb{Z}}_3 \to \widetilde{BSpin} \to BSpin, \qquad BString \to \widetilde{BSpin} \to \underline{HC_2}3,$$

formed by considering the pullback of the corner

$$BSpin \to \underline{H\mathbb{Z}}_4 \leftarrow \underline{HC_2}3.$$

Both of these fibrations induce short exact sequences of Hopf algebras.

However, since we are specifically interested in *MString*–orientations, there is an alternative approach that avoids describing the formal scheme $BString_K$. Again appealing to results of Kitchloo, Laures, and Wilson, we find that the sequence

$$K_*Spin/SU \to K_*BU[6,\infty) \to K_*BString$$

Cite me: KLW.

Cite me: Theorem 2.3.5.vi of KLW.

is exact and right-exact. The kernel of the map $K_*Spin/SU \to K_*BU[6,\infty)$ is a Hopf algebra they call "CK_3", where

$$CK_j = \bigoplus_{k=j}^{\infty} K_*K(\mathbb{Z}/2,k).$$

But I have not checked!

More than that, KLW even say where the polynomial and nonpolynomial parts of K_*Spin/SU land inside of $K_*BU[6,\infty)$. I thinkthat this means that $K_*BU[6,\infty)$ is a flat K_*Spin/SU–module at heights $d \leq 2$.

Applying the Thom spectrum functor to the fiber square gives the pushout diagram

$$
\begin{tikzcd}
\Sigma^\infty_+ Spin/SU \arrow[r] \arrow[d] & MU[6,\infty) \arrow[d] \\
\mathbb{S} \arrow[r] & MString,
\end{tikzcd}
$$

or, equivalently, an equivalence

$$MString \simeq MU[6,\infty) \wedge_{\Sigma^\infty_+ Spin/SU} \mathbb{S}.$$

This in turn gives a Tor spectral sequence of signature

It seems that this approach, which I presume is the one that ultimately works, is independent of all the earlier work involving *BSpin*.

$$\mathrm{Tor}^{K_* Spin/SU}_{*,*}(K_* MU[6,\infty), K_*) \Rightarrow K_* MString.$$

So, under the flatness hypothesis above, there are no higher Tor terms so the spectral sequence collapses to give

$$K_* MString \cong K_* MU[6,\infty) // K_* Spin/SU.$$

So, what remains to be shown is that $K_* Spin/SU$ picks out the correct extra relation for Σ–structures . Then, we need a density argument to show that this handles all of the at-a-point cases of elliptic cohomology.

You have not introduced Σ–structures yet.

Remark 5.7.9. However, the spectra $HE_\Lambda P$ do not qualify as "chromatically amenable" from the perspective of this last argument, and so we lose access to our genus valued in modular forms. Additionally, K_{Tate} does not qualify, essentially because it is integral rather than p–adic. The methods described here give rise to putative p–adic q–expansions of modular forms, but we have not been able to check that they satisfy the modularity condition, nor that they assemble into a single, integral object. Amazingly, such theorems are achievable, and they are the impetus for the study of topological modular forms and algebraic geometry done with E_∞–rings.

Mike thinks that $K(2)_* Spin/SU \to K(2)_* BU[6,\infty)$ follows from this being a map of co/commutative Hopf algebras which is injective on indecomposables. I don't know how to prove this hypothesis condition and I also don't know how to prove the reduction. Bousfield's paper *On λ–rings and the K–theory of infinite loopspaces* in Theorem 10.8 gives a description of flat Hopf algebras in the setting of periodic K–theory (or, rather, a reference). It looks like his conclusion is that flat maps are *precisely* the inclusions of Hopf algebras. If we could adapt something like this to our situation, we would immediately learn that $K(2)_* Spin/SU \to K(2)_* BU[6,\infty)$ is flat from Kitchloo–Laures–Wilson.

Some other things that might belong in this chapter

The cubical structure on a singular (generalized) elliptic curve is not unique, but (published) AHS has an argument showing that the unicity of the choice on the nonsingular "bulk" extends to a unique choice on the "boundary" of the compactified moduli too.

There's also the work of Ando–French–Ganter on factorized / iterated Θ structures and how they give rise to the "two–variable Jacobi genus".

Expand this last sentence. Point to the Appendix, or cite the *TMF* book, or idk.

The Atiyah–Bott–Shapiro orientation and the fibration $BSU \to BSpin$. This is possibly somewhere around Theorem 2.3.5.iv in KLW, the last fibration in 2.3.2 at $k = -2$, and sections 5.3 and 5.13

Case Study 6

Power operations

I wish this had a better title.

Write an introduction for me.

Baker's POWER OPERATIONS AND COACTIONS IN HIGHLY COMMUTATIVE HOMOLOGY THEORIES seems like a nice place to learn about these things. He advertises some interaction with the traditional context story, which is appealing, and he mostly treats the case of ordinary homology, which we probably ought to spend a section on.

This is completely under construction.

There should be a context-based presentation of this chapter's material too. What do contexts for structured ring spectra look like? Why would you consider them—what object are you trying to approximate? How do you guess that the algebraic model is reasonable until you're aware of something like Strickland's theorem?

Since you spend so much time talking about descent in other parts of these notes, maybe you should also read the end of the AHS H_∞ paper where they claim to recast their results in the usual language of descent.

Conversation with Nat on 2/9 suggests taking the following route in this chapter: contexts for E_∞ mapping spaces in general; Subgroups and level structures; the Drinfel'd ring and the universal level structure; the isogenies pile; power operations and Adams operations, after Ando (naturally indexed vs indexed on subgroups; have a look at the Screenshot you took on this day); comparison of comodules M for the isogenies pile with the action of $M_n(\mathbb{Z}_p)$ on $M \otimes_{E_n^*} D_\infty$ (this is a modern result due to Tomer, Tobi, Lukas, and Nat); H_∞ MU–orientations and Matt's thesis; the analogous results for Θ^k–structures. In particular, leave character theory, p–divisible groups, and rational phenomena for spillover at the end of the year. They aren't strictly necessary to telling the story; you just need to know a little about the Drinfel'd ring to construct Matt's maps. (If you have time, though, the point is that the rationalized Drinfel'd ring carries the universal level structure which is also an isomorphism.)

Charles's Sections 2.10,12 of the Felix Klein notes has a nice, compact exposition of power operations for $K(n)$–local E_∞–ring spectra (using, in particular, $R^{B\Sigma_m} \simeq R \wedge_E E^{B\Sigma_m}$) as well as a discussion of "descent for isogenies" generally and Koszul-ality in Section 3.

The stuff around 4.3.1-2 of Matt's published thesis talks about H_∞–maps being determined by their values on $*$ and $\mathbb{C}P^\infty$, which is an interesting result. You might also compare with Butowiez–Turner.

Work in height 1 (and height 2??) examples through this? K–theory is pretty accessible, and the height 2 examples are somewhat understood (Charles, Yifei), and they're both relevant for the elliptic *MString* story. (There's also the pile of elliptic curves with isogenies...)

Nat warns that the very end of Matt's thesis uses character theory for S^1, which you have to be very careful about to pull off correctly. (S^1 is not a finite group, but in certain contexts it can be approximated by its torsion subgroups...)

Yifei warned me that Matt's "there exists a unique coordinate..." Lemma is specifically about lifting the *Honda* formal group law over $\mathbb{F}_q$. If you want to do this with elliptic cohomology or something, then you need a stronger statement (and it's clear what this statement should be, but no one has proven it).

Don Davis has a *Handbook of Algebraic Topology* chapter on unstable v_1–periodic homotopy of spheres.

—— Here are various notes from conversations with Nat, recorded and garbled well after they happened. ——

This MO conversation looks interesting: http://chat.stackexcha

We could try to understand Matt's thesis's Section 4.2. It identifies the action of the internal power operation on E_n using the internal theory of quotient isogenies to the

Jeremy pointed out that $\beta \in E_* QS^n$ comes up when considering a class $\alpha \in \pi_n R$ for a commutative E–algebra, which we promote to a class $E \wedge QS^n \to R$, and then precompose with the homotopy class β.

Lubin–Tate deformation problem (2.5.4). Conditions 1 and 3 of 4.2.1 are easy to verify: they are 4.2.3 (evaluate on a point) and 4.2.4 (the power operation does raise things to a power) respectively. Condition 2 takes more work, and it's about identifying the divisor associated to the isogeny granted by Condition 1. It's worked out in 4.2.5, which is not very hard, and 4.2.6, which shows that *the* Thom class associated to a vector bundle is sent under a power operation to *some* Thom class. 4.2.5 then uses that the quotient of *some* Thom classes has to be a unit in the underlying ring.

(Q: Can 4.2.5 be phrased about two coordinates on the same formal group, rather than two presentations of the same divisor? There's a comparison between functions on the quotient with invariant functions on the original group—and perhaps with functions invariant by pulling back along the isogeny?)

Prop 8.3 in "Character of the Total Power Operation" provides an algebro-geometric proof of something in AHS04, using the fact that for R a nice complete local ring and G, G' p–divisible groups over R, there is an injection

$$\operatorname{Isog}_R(G, G') \hookrightarrow \operatorname{Isog}_{R/\mathfrak{m}}(G, G').$$

Nat thinks that using the power operation internal to MU is what gets you Lubin's product formula for the quotient (cf. the calculation in Quillen's theorem), and using the power operation internal to E–theory gives you *something* called ψ^H, which you separately calculate on Euler classes. The point (cf. Ando's thesis's Theorem 1) is to pick a coordinate so that these coincide. (Lubin–Tate theory and Lubin's theory of isogenies says that they always coincide up to unique $\star$–isomorphism—after all, automorphisms (and isogenies) don't deform—and the point is that for particular coordinates on particular formal groups, you can take the $\star$–isomorphisms to all be the identity.)

Ando's thesis only deals with power operations internal to E–theory starting in Section 4. Before then, he shows that the pushforward of the power operations internal to MU can be lifted through maps on E–theory (although these maps may not be topologically induced). It's not clear to me what the value of this is—if you're constructing the operations on the E–theory side, then surely you're going to construct them so that they're on-the-nose equal to the MU–operations?

The meaty part of AHS04 is Theorem 6.1, that the necessary condition is sufficient. It falls into steps: first, we can restrict attention to Σ_p, and even inside of there we can restrict attention to C_p. Then, the two directions around the H_∞ square give two trivializations (cf. 4.2.6 of Ando's thesis) $g_{cl(ockwise)}$ and $g_{c(ounter)c(lockwise)}$ of $\Theta^k\mathcal{I}(0)$. The fact that they're both trivializations means there's an equation $g_{cl} = rg_{cc}$ for $r \in E^0 D_{C_p} BU[2k, \infty)^\times$. Then, he wants to study the map

$$E^0 D_{C_p} BU[2k, \infty)_+ \xrightarrow{\Delta^* \times i^*} E^0 BC_p^* \times BU[2k, \infty) \times E^0 BU[2k, \infty)^{\times p},$$

which they know to be an injection by work of McClure, but for some reason they can restrict attention to just the left-hand factor. The left-hand factor is the ring of functions on

$\underline{\mathsf{FormalGps}}(A, \widehat{\mathbb{G}}_F) \times BU[2k, \infty)_E$, and they can further restrict attention to level structures, where there are only two: the injective one and the null map. They then check these two cases by hand, and it follows that $r = 0$, so the two ways of navigating the diagram agree at the level of topology.

(Section 8 of Hopkins–Lawson has an injectivity proof that smells similar to the above injectivity trick with McClure's map.)

Just working in the case $k = 1$ (or $k = 0$), which is supposed to recover the "classical" results of Ando's thesis, we can try to recursively expand the various arguments and definitions. The counterclockwise map appears to be the easy one, and it's discussed around 4.11. The clockwise map appears to be the hard one, and it's discussed in 3.21. For $\chi_\ell = \chi_\ell \times \widehat{\mathbb{G}}$ given by

$$T \times \widehat{\mathbb{G}} \xrightarrow{\chi_\ell} \underline{\mathrm{Hom}}(A, \widehat{\mathbb{G}}) \times \widehat{\mathbb{G}},$$

the main content of 3.21 is an equality

$$\chi_\ell^* s_{cl} = \psi_\ell^{\mathcal{L}}(s_g) = (\psi_\ell^{\widehat{\mathbb{G}}/E})^* (\psi_\ell^E)^* s_g,$$

where ψ_ℓ^E is defined in 3.9, $\psi_\ell^{\widehat{\mathbb{G}}/E}$ is defined in 3.14 and the preceding remarks, s_g is the section describing the source coordinate (cf. part 2 of 3.21), and $\psi_\ell^{\mathcal{L}}$ is described between the paragraph before 3.16 and Definition 3.20. Trying to rewrite $\psi_\ell^{\mathcal{L}}$ into the form required for 3.21 requires pushing through 8.11 and 10.15.

We spent a lot of time just writing out the definitions of things, trying to get them straight in the universal case (which AHS04 wants to avoid for some reason—maybe they didn't yet have a good form of Strickland's theorem?). It was helpful in the moment, but hard to read now.

All of this rests, most importantly, on how a quotient of the Lubin–Tate universal deformation by a subgroup still gives a Lubin–Tate universal deformation. This is Section 12.3 of AHS04, and it's Section 9 of Neil's Finite Subgroups paper. (Nat says there's something to look out for in here. Watch where they say they have E_0–algebra maps versus ring maps.)

6.1 E_∞ ring spectra and their contexts

Mike has suggested looking at the paper *The K–theory localization of an unstable sphere*, by Mahowald and Thompson. In it, they manually construct a resolution of S^{2n+1} suitable for computing the unstable Adams spectral sequence for K–theory, but the resolution that they build is also exactly what you would use to compute the mapping spectral sequence for $E_\infty(K^{S^{2n-1}}, K)$. Additionally, because the unstable K–theoretic operations are exhausted by the power operations, these two spectral sequences converge to the same target.
Purely in terms of the E_∞ version, one can consider the composition of spectral sequences

$$\mathrm{Ext}_{\mathbb{Z}[\theta]}(\mathbb{Z}, \mathrm{Der}_{K_*-alg}(K^*X, K^*)) \Rightarrow \mathrm{Der}_{K_*-Dyer-Lashof-alg}(K^*X, K^*) \Rightarrow E_\infty(\widehat{\mathbb{S}^0}^X, K_p^\wedge)$$

and

$$E_\infty(\widehat{\mathbb{S}^0}^X, K_p^\wedge)^{h\mathbb{Z}_p^*} = E_\infty(\widehat{\mathbb{S}^0}^X, \widehat{\mathbb{S}^0})$$

where the first spectral sequence is a composition spectral sequence for derivations in K_*–algebras and then derivations respecting the Mandell's θ–operation. If X is an odd sphere, then K^*X has no derivations and this composite spectral sequence collapses, making the composition possible.
This is also related to recent work of Behrens–Rezk on the Bousfield–Kuhn functor...

Another unpublished theorem of Hopkins and Lurie is that the natural map $Y = F(*, Y) \to E_\infty(E_n^Y, E_n)$ is an equivalence when Y is a finite Postnikov tower in the range of degrees that E_n can see.

6.2 Subgroups and level structures

Something that these notes routinely fail to do is to lead into the algebraic geometry in a believable way. "Today we're going to talk about isogenies"—and then, lo' and behold, isogenies appear the next day in algebraic topology. This book would read much better if it showed how these structures were guessed to exist to begin with.

Here's a definition of an isogeny. Weierstrass preparation can be phrased as saying that a Weierstrass map is a coordinate change and a standard isogeny.

Cite me: Definition 5.17 of FSFG.

Definition 6.2.1. Take C and D to be formal curves over X. A map $f : C \to D$ is an *isogeny* when the induced map $C \to C \times_X D$ exhibits C as a divisor on $C \times_X D$ as D–schemes.

Be sure to compare this definition with the usual one for formal groups: surjections with finite cokernel. These are easy to come up with lots of examples for! (Don't ditch this definition, though, since it's the one that lets you prove something about Weierstrass preparation geometrically.)

In fact, every map in positive characteristic can be factored as a coordinate change and an isogeny, which is a weak form of preparation.

Lubin's finite quotients of formal groups. (Interaction with the Lubin–Tate moduli problem? Or does this belong in the next day?)

Write out isogenies of the additive formal group, note that you just get the unstable Steenrod algebra again. This is a remarkable accident.

Push and pull maps for divisor schemes

Moduli of subgroup divisors

The Drinfel'd moduli ring, level structures

—-

Cite me: Prop 6.2 of HKR.

Lemma 6.2.2. *The following conditions on a homomorphism*

$$\varphi : \Lambda_r^* \to F[p^r](R)$$

are equivalent:

1. *For all $\alpha \neq 0$ in Λ_r^*, $\varphi(\alpha)$ is a unit (resp., not a zero-divisor).*
2. *The Hopf algebra homomorphism*

$$R[\![x]\!]/[p^r](x) \to R^{\Lambda_r^*}$$

is an isomorphism (resp., a monomorphism). □

Cite me: Shortly after Prop 6.2 of HKR. Section 7?.

Lemma 6.2.3. *Let $\mathcal{L}_r(R)$ be the set of all group homomorphism*

$$\varphi : \Lambda_r^* \to F[p^r](R)$$

satisfying either of the conditions 1 or 2 above. This functor is representable by a ring

$$L_r(E^*) := S^{-1}E^*(B\Lambda_r)$$

that is finite and faithfully flat over $p^{-1}E^*$. (Here S is generated by the $\varphi(\alpha)$ with $\alpha \neq 0$, $\varphi : \Lambda_r^* \to F[p^r](E^*B\Lambda_r)$ the canonical map.)

—

Section 2: complete local rings

"Galois" means $R \to S$ a finite extension of integral domains has R as the fixed subring for $\mathrm{Aut}_R(S)$ and S is free over R. Galois extension of rings implies the extension of fraction fields is Galois. The converse holds for finite (finitely generated as a module) dominant (kernel of f is nilpotent) maps of smooth (regular local ring) schemes.

Section 3: basic facts about formal groups

definition of height

Section 4: basic facts about divisors

Since $x -_F a \doteq x - a$, you can treat the divisor $[a]$ (defined in a coordinate by the ideal sheaf generated by $x - x(a)$) as generated just by $x - a$.

Lemma 6.2.4. *Let D and D' be two divisors on $\widehat{\mathbb{G}}$ over X. There is then a closed subscheme $Y < X$ such that for any map $a : Z \to X$ we have $a^*D \le a^*D'$ if and only if a factors through Y.* □

Cite me: Prop 4.6 of Finite Subgroups.

Section 5: quotient by a finite sbgp is again a fml gp

Definition 6.2.5. A *finite subgroup* of $\widehat{\mathbb{G}}$ will mean a divisor K on $\widehat{\mathbb{G}}$ which is also a subgroup scheme. Let $\mathcal{O}_{\widehat{\mathbb{G}}/K}$ be the equalizer

$$\mathcal{O}_{\widehat{\mathbb{G}}/K} \longrightarrow \mathcal{O}_{\widehat{\mathbb{G}}} \underset{\pi^*}{\overset{\mu^*}{\rightrightarrows}} \mathcal{O}_K \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{\mathbb{G}}}.$$

Lemma 6.2.6. *Write $y = N_\pi \mu^* x \in \mathcal{O}_{\widehat{\mathbb{G}}}$.*[1] *Then $y \equiv x^{p^m} \pmod{\mathfrak{m}_X}$ and $\mathcal{O}_{\widehat{\mathbb{G}}/K} = \mathcal{O}_X[\![y]\!]$. Moreover, the projection $\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}/K$ is the categorical cokernel of $K \to \widehat{\mathbb{G}}$. This all commutes with base change: given $f : Y \to X$ we have $f^*\widehat{\mathbb{G}}/f^*K = f^*(\widehat{\mathbb{G}}/K)$.* □

Cite me: Theorem 5.3 of Finite Subgroups.

Expand this out in the case of a subgroup scheme given by a sum of point divisors.

Section 6: coordinate-free lubin-tate theory

nothing you haven't already seen. in fact, most of it is done in coordinates, with only passing reference to the decoordinatization.

cf. also Prop 2.2.2 of Matt's thesis

Section 7: level–A structures: smooth, finite, flat

As discussed long ago, for finite abelian p–groups there's a scheme

$$\underline{\mathsf{FormalGroups}}(A, \widehat{\mathbb{G}})(Y) = \underline{\mathsf{Groups}}(A, \widehat{\mathbb{G}}(Y)).$$

Be careful to distinguish the physical group A from the associated *constant group scheme.*

If $\widehat{\mathbb{G}}$ were a discrete group, we could decompose this as

$$\text{“}\underline{\mathsf{FormalGroups}}(A, \widehat{\mathbb{G}}) = \coprod_{B \le A} \mathrm{Mono}(A/B, \widehat{\mathbb{G}})\text{”}$$

along the different kernel types of homomorphisms, but Mono does not exist as a scheme.Lev

Come up with a really compelling example. You had one when you were talking to Danny and Jeremy. Probably you got it *from* Jeremy.

[1]Remember that if $f : X \to Y$ is a finite flat map, then $N_f : \mathcal{O}_X \to \mathcal{O}_Y$ is the nonadditive map sending u to the determinant of multiplication by u, considered as an $\mathcal{O}_Y$–linear endomorphism of $\mathcal{O}_X$.

structures approximate this as best one can be approximating $\widehat{\mathbb{G}}$ by something essentially discrete: an étale group scheme.

For a map $\varphi : A \to \widehat{\mathbb{G}}(Y)$, we write $[\varphi A] = \sum_{a \in A}[\varphi(a)]$. We also write $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, so that $\Lambda[p^m] = (\mathbb{Z}/p^m)^{\times n}$. Note

$$|\mathsf{AbelianGroups}(A, \Lambda)| = |A|^n = \operatorname{rank}\left(\underline{\mathsf{FormalGroups}}(A, \widehat{\mathbb{G}}) \to X\right).$$

Definition 6.2.7. A *level–A structure* on $\widehat{\mathbb{G}}$ over an X–scheme Y is a map $\varphi : A \to \widehat{\mathbb{G}}(Y)$ such that $[\varphi A[p]] \leq G[p]$ as divisors. A *level–m structure* means a level–$\Lambda[p^m]$ structure.

Cite me: Prop 7.2-4 of Finite Subgroups.

Lemma 6.2.8. *The functor from schemes over X to sets given by*

$$Y \mapsto \{\textit{level–}A \textit{ structures on } \widehat{\mathbb{G}} \textit{ over } Y\}$$

is represented by a finite flat scheme $\mathrm{Level}(A, \widehat{\mathbb{G}})$ *over X. It is contravariantly functorial for monomorphisms of abelian groups. Also, if $\varphi : A \to \widehat{\mathbb{G}}$ is a level structure then $[\varphi A]$ is a subgroup divisor and $[\varphi A[p^k]] < \widehat{\mathbb{G}}[p^k]$ for all k. In fact, if $A = \Lambda[p^m]$ then $[\varphi A] = \widehat{\mathbb{G}}[p^m]$.* □

I can't imagine proving this. It's worth noting that it's proven by considering just the universal case, which we know to be smooth.

In Section 26 of FPFP Neil says there's a decomposition into irreducible components

$$\mathrm{Hom}(A, \widehat{\mathbb{G}}) = \mathrm{Hom}(A, \widehat{\mathbb{G}}_{\mathrm{red}}) = \bigcup_B \mathrm{Level}(A/B, \widehat{\mathbb{G}})$$

and this $\bigcup$ turns into a $\coprod$ after inverting p. He also mentions this as motivation in Finite Subgroups, but he doesn't appear to prove it?

Section 8: maps among level–A schemes, their Galois behavior

Cite me: Theorem 8.1 of Finite Subgroups.

Theorem 6.2.9. *Let A, B be finite abelian p–groups of rank at most n, and let $u : A \to B$ be a monomorphism. Then:*

1. $$\mathsf{FormalSchemes}_X(\mathrm{Level}(B, \widehat{\mathbb{G}}), \mathrm{Level}(A, \widehat{\mathbb{G}})) = \mathrm{Mono}(A, B).$$
2. *Such homomorphisms are detected by the behavior at the generic point.*
3. *The map $u^! : \mathrm{Level}(B, \widehat{\mathbb{G}}) \to \mathrm{Level}(A, \widehat{\mathbb{G}})$ is finite and flat.*
4. *If $B \simeq \Lambda[p^m]$, then $u^!$ is a Galois covering.*
5. *The torsion subgroup of $\widehat{\mathbb{G}}(\mathrm{Level}(A, \widehat{\mathbb{G}}))$ is A.* □

Section 9: epimorphisms of groups become maps of level schemes, quotients by level structures

Let $\widehat{\mathbb{G}}_0$ be a formal group of height n over $X_0 = \operatorname{Spec} \kappa$. For every m, the divisor $p^m[0]$ is a subgroup of $\widehat{\mathbb{G}}_0$. We write $\widehat{\mathbb{G}}_0\langle p^m \rangle$ for the quotient group $\widehat{\mathbb{G}}_0/p^m[0]$ and $\widehat{\mathbb{G}}\langle m \rangle \to X\langle m \rangle$ for the universal deformation of $\widehat{\mathbb{G}}_0\langle m \rangle \to X_0$. Note that $\widehat{\mathbb{G}}_0[p] = p^n[0]$, which induces isomorphisms $\widehat{\mathbb{G}}_0\langle m+n \rangle \to \widehat{\mathbb{G}}_0\langle m \rangle$, and we use this to make as many identifications as we can.

Cite me: 9.1 of Finite Subgroups.

Lemma 6.2.10. *Let $u : A \to B$ be an epimorphism of abelian p–groups wit kernel $|\ker(u)| = p^\ell$. Then u induces a map*

$$u_! : \mathrm{Level}(A, \widehat{\mathbb{G}}\langle m\rangle) \to \mathrm{Level}(B, \widehat{\mathbb{G}}\langle m+\ell\rangle).$$

Also, if $A = \Lambda[p^m]$, then $u_!$ is a Galois covering with Galois group

$$\Gamma = \{\alpha \in \mathrm{Aut}(A) \mid u\alpha = u\}. \quad \square$$

Corollary 6.2.11. *In particular, the map $A \to 0$ induces a map*

Cite me: Interstitial text between 9.1 and 9.2 of Finite Subgroups.

$$0_! : \mathrm{Level}(A, \widehat{\mathbb{G}}\langle m\rangle) \to \mathrm{Level}(0, \widehat{\mathbb{G}}\langle m+\ell\rangle) = X\langle m+\ell\rangle$$

which extracts quotient formal groups from level structures. In the case $A = \Lambda[p^\ell]$, $0_!$ is just the projection $0^!$. $\square$

Section 10: moduli of subgroup schemes

Theorem 6.2.12. *The functor*

Cite me: Theorem 10.1 of Finite Subgroups.

$$Y \mapsto \{\textit{subgroups of } \widehat{\mathbb{G}} \times_X Y \textit{ of degree } p^m\}$$

is represented by a finite flat scheme $\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}})$ over X of degree $|\mathrm{Sub}_{p^m}(\Lambda)|$. The formation commutes with base change. $\square$

We can at least give the construction: let D be the universal divisor defined over $Y = \mathrm{Div}_{p^m}(\widehat{\mathbb{G}})$ with equation $f_D(x) = \sum_{k=0}^{p^m} c_k x^k$. There are unique elements $a_{ij} \in \mathcal{O}_Y$ such that

$$f(x +_F y) = \sum_{i,j=0}^{p^m-1} a_{ij} x^i y^j \pmod{f(x), f(y)}.$$

Define

$$\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}}) = \mathrm{Spf}\, \mathcal{O}_Y / (c_0, a_{ij} \mid 0 \le i, j < p^m).$$

Finiteness, flatness, and rank counting are what take real work, starting with an arithmetic fracture square.

Section 13: deformation theory of isogenies

Definition 6.2.13. Suppose we have a morphism of formal groups

$$\begin{array}{ccc} \widehat{\mathbb{G}}_0 & \xrightarrow{q_0} & \widehat{\mathbb{G}}'_0 \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{f_0} & X'_0 \end{array}$$

such that the induced map $\widehat{\mathbb{G}}_0 \to f_0^*\widehat{\mathbb{G}}'_0$ is an isogeny of degree p^m. By a deformation of q_0 we mean a prism

$$\begin{tikzcd}
\mathbb{H} \arrow[dr, "q"] \arrow[dd] & & \mathbb{H}_0 \arrow[ll] \arrow[rr] \arrow[dr] \arrow[dd] & & \widehat{\mathbb{G}}_0 \arrow[dr, "q_0"] \arrow[dd] & \\
& \mathbb{H}' \arrow[dd] & & \mathbb{H}'_0 \arrow[ll] \arrow[rr] \arrow[dd] & & \widehat{\mathbb{G}}'_0 \arrow[dd] \\
Y \arrow[dr, "1"] & & Y_0 \arrow[ll] \arrow[rr] \arrow[dr, "1"] & & X_0 \arrow[dr, "f_0"] & \\
& Y & & Y_0 \arrow[ll] \arrow[rr] & & X'_0,
\end{tikzcd}$$

where the middle face is the pullback of the left face, the back-right and front-right faces are pullbacks, so that q is also an isogeny of degree p^m.

Let $\widehat{\mathbb{G}}/X$ be the universal deformation of $\widehat{\mathbb{G}}_0$, let $a : \operatorname{Sub}_{p^m}(\widehat{\mathbb{G}}) \to X$ be the usual projection, and let $K < a^*\widehat{\mathbb{G}}$ be the universal example of a subgroup of degree p^m. As $\operatorname{Sub}_{p^m}(\widehat{\mathbb{G}})$ is a closed subscheme of $\operatorname{Div}_{p^m}(\widehat{\mathbb{G}})$ and $\operatorname{Div}_{p^m}(\widehat{\mathbb{G}})_0 = X_0$, we see that $\operatorname{Sub}_{p^m}(\widehat{\mathbb{G}})_0 = X_0$. There is a unique subgroup of order p^m of $\widehat{\mathbb{G}}_0$ defined over X_0, viz. the divisor $p^m[0] = \operatorname{Spf} \mathcal{O}_{\widehat{\mathbb{G}}_0}/x^{p^m}$. In particular, $K_0 = p^m[0] = \ker(q_0)$. It follows that there is a pullback diagram as shown below:

$$\begin{tikzcd}
(a^*\widehat{\mathbb{G}}/K)_0 \arrow[r, "\simeq"] \arrow[d] & \widehat{\mathbb{G}}_0/p^m[0] \arrow[r, "\overline{q}_0{,}\simeq"] \arrow[d] & \widehat{\mathbb{G}}'_0 \arrow[d] \\
\operatorname{Sub}_{p^m}(\widehat{\mathbb{G}})_0 \arrow[r, "a_0{,}\simeq"] & X_0 \arrow[r, "f_0{,}\simeq"] & X'_0.
\end{tikzcd}$$

We see that $a^*\widehat{\mathbb{G}} \to a^*\widehat{\mathbb{G}}/K$ is a deformation of q_0, and it is terminal in the category of such.

Now let $\widehat{\mathbb{G}}'/X'$ be the universal deformation of $\widehat{\mathbb{G}}'_0/X'_0$. The above construction also exhibits $a^*\widehat{\mathbb{G}}/K$ as a deformation of $\widehat{\mathbb{G}}'_0$, so it is classified by a map $b : \operatorname{Sub}_{p^m}(\widehat{\mathbb{G}}) \to X'$ extending the map $b_0 = f_0 \circ a_0 : \operatorname{Sub}_{p^m}(\widehat{\mathbb{G}})_0 \to X'_0$.

Cite me: Prop 13.1 of Finite Subgroups, *hard.*

Theorem 6.2.14. *b is finite and flat of degree $|\operatorname{Sub}_{p^m}(\Lambda)|$.* □

Cf. Matt's thesis's Prop 2.5.1: Φ is a formal group over $\mathbb{F}_p$, F a lift of Φ to E_n, H a finite subgroup of $F(D_k)$, then F/H is a lift of Φ to D_k. (This is because the quotient map to F/H reduces to $t \mapsto t^{p^r}$ for some r over $\mathbb{F}_p$, which is an endomorphism of Φ, so the quotient map over the residue field doesn't do anything!) See also Prop 2.5.4, where he characterizes all isogenies of this sort as arising from this construction.

Section 14: connections to AT

Neil's *Finite Subgroups of Formal Groups* has (in addition to lots of results) a section 14 where he talks about the action of a generalized Hecke algebra on the E–theory of a space.

Let a and b be two points of X, with fibers $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_b$, and let $q : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_b$ be an isogeny. Then there's an induced map $(Z_E)_a \to (Z_E)_b$, functorial in q and natural in Z. "Certain Ext groups over this Hecke algebra form the input to spectral sequences that compute homotopy groups of spaces of maps of strictly commutative ring spectra, for example." **This sounds like the beginning of an answer to my context question.**

Section 11: flags of controlled rank ascending to $\widehat{\mathbb{G}}[p]$ and a map $\mathrm{Level}(1, \widehat{\mathbb{G}}) \to \mathrm{Flag}(\lambda, \widehat{\mathbb{G}})$. Section 12: the orbit scheme $\mathrm{Type}(A, \widehat{\mathbb{G}}) = \mathrm{Level}(A, \widehat{\mathbb{G}}) / \mathrm{Aut}(A)$: smooth, finite, flat Section 15: formulas for computation Section 16: examples

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Theorem 6.2.15. *Let R be a complete local domain with positive residue characteristic p, and let F be a formal group of finite height d over R. If $\mathcal{O}$ is the ring of integers in the algebraic closure of the fraction field of R, then $F(\mathcal{O})[p^k] \cong (\mathbb{Z}/p^k)^d$ and $F(\mathcal{O})_{\mathrm{tors}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^d$.* □

Cite me: See Theorem 2.4.1 of Ando's thesis, though he just cites other people.

Section 20 of FPFP is about "full sets of points" and the comparison with the cohomology of the flag variety of a vector bundle.

Talk with Nat:

- Definitions in terms of divisors.
- Equalizer diagram for quotients by finite subgroups.
- The image of a level structure ℓ is a subgroup divisor.
- The schemes classifying subgroups and level structures (which are hard and easy respectively, and which have hard and easy connections to topology respectively).
- It's easy to give explicit examples of the behavior of level structures based on cyclic groups.
- Galois actions on the rings of level structures.

6.3 The Drinfel'd ring and the universal level structure

Talk with Nat:

- Recall the Lubin–Tate moduli problem.
- Show that quotients of deformations by finite subgroups give deformations again.
- Define the Drinfel'd ring.
- As an E^0–algebra, it carries the universal level structure.

- As an ind–(complete local ring), it corepresents deformations (by precomposition with the map $E^0 \to D_n$) *equipped with level structures*.
- Describe the action by $GL_n(\mathbb{Z}_p)$. (Hint at the action by $M_{n\times n}(\mathbb{Z}_p)$ with det $\neq 0$.)
- Describe the isogenies pile and its relation to all this? (This doesn't really fit precisely, but it may be good to put here, on an algebraic day.)

6.4 Descending coordinates along level structures

It's not clear to me what theorems about level structures and so forth are best included on this day and which belong back in the lecture above. We should be able to split things apart into stuff desired for character theory and stuff desired for descent.

Ando's Theorem 3.4.4: Let D_j be the ring extension of E_n which trivializes the p^j–torsion subgroup of $\widehat{\mathbb{G}}_{E_n}$. Let H be a finite subgroup of $\widehat{\mathbb{G}}_{E_n}(D_k)$. There is an unstable transformation of ring-valued functors

$$E_n X \xrightarrow{\Psi^H} D_j \otimes E_n X,$$

and if F is an Ando coordinate then for any line bundle $\mathcal{L} \to X$ there is a formula

$$\psi^H(e\mathcal{L}) = \prod_{h\in H}(h +_F e\mathcal{L}) \in D_j \otimes E_n(X).$$

D_j is Galois over E_n with Galois group $GL_n(\mathbb{Z}/p^j)$. If ρ is a collection of finite subgroups weighted by elements of E_n which is stable under the action of the Galois group, then Ψ^ρ descends to take values in just E_n. (For example, the entire subgroup has this property.)

This is built by a character map. Take $H \subseteq F(D_j)[p^j]$ to be a finite subgroup again; then there is a map

$$\chi^H : E_n(D_{H^*}X) \to D_j \otimes E_n(X),$$

where D_{H^*} denotes the extended power construction on X using the Pontryagin dual of H. This composes to give an operation

$$Q^H : MU^{2*}(X) \xrightarrow{P_{H^*}} MU^{2|H|*}(D_{H^*}X) \to E_n^{2|H|*}(D_{H^*}X) \xrightarrow{\chi^H} D_j \otimes E_n^{2|H|*}(X).$$

Then Q^H is a ring homomorphism with effects

$$Q^H F^{MU} = F/H, \qquad\qquad Q^H(e_{MU}\mathcal{L}) = \prod_{h\in H} h +_F e\mathcal{L}.$$

Then we need to factor $Q^H : MU(X) \to D_j \otimes E_n(X)$ across the orienting map $MU \to E_n$. Since E_n is Landweber flat and Q^H is a ring map, it suffices to do this for the one–point space, i.e., to construct a ring homomorphism

$$\Psi^H : E_n \to D_j$$

so that $\Psi^H = \Psi^H(*) \otimes Q^H$. The first condition above then translates to $\Psi^H F^{MU} = F/H$.

Cite me: Theorem 2.5.7 of Matt's thesis.

Theorem 6.4.1. *For each $\star$–isomorphism class of lift F of Φ to E_n, there is a unique choice of coordinate x on F, lifting the preferred coordinate on Φ, such that $\alpha_*^H F_x = F_x/H$, or equivalently that $l_H^x = f_H^x$, for all finite subgroups H. (These morphisms are arranged in the following diagram:)*

$$\begin{tikzcd}
F_x \arrow[r, "f_H^x"] \arrow[rr, bend left, "l_H^x"] & F_x/H \arrow[r, "g_H^x"] & \alpha_*^H F_x \\
F \arrow[u, "x"] \arrow[r] & F/H \arrow[u, "x_H"] \arrow[r, "g_H"] & \alpha_*^H F, \arrow[u, "\alpha_* x"']
\end{tikzcd}$$

where $\alpha_H : E_n \to D_k$ is the unique ring homomorphism such that there is a $\star$–isomorphism $g_H : F/H \to \alpha_*^H F$. ∎

This is some serious work, and I don't think we'll prove it. The main point is that $\alpha_*^p F_x = F_x/p$ can be reimagined as $f_p^x(t) = [p]_{F_x}(t)$, and this already is enough to determine what x is by descending along the power of the maximal ideal in E_n, the length of a full level structure, and pieces of a smaller level structure inside of the full one. It really is a long argument.

Section 2.7 of Matt's thesis works the example of a normalized coordinate for $\widehat{\mathbb{G}}_m$. It's *not* the p–typical coordinate. It *is* the standard one! Cool.

Lemma 6.4.2. $P_r(x+y) = \sum_{j=0}^r \operatorname{Tr}_{j,r}^{MU} d^*(P_j x \times P_{r-j} y)$.

This expresses the non-additivity of the power operations on MU. It's apparently needed in the proof that Q^H acts as it should on Euler classes. It involves transfer formulas, which may mean we need to work that section of HKR into that day.

Cite me: Lemma 3.2.7 of Matt's thesis, BMMS86 page 25, AHS H_∞ appendix.

Proof. Represent x and y by maps

$$U \xrightarrow{f} X, \qquad V \xrightarrow{g} Y.$$

Then $P_r(x+y)$ is represented by

$$D_r(U \sqcup V) \xrightarrow{D_r(f \sqcup g)} D_r X.$$

There is a decomposition

$$D_r(U \sqcup V) = \coprod_{j=0}^r E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (U^j \times V^{r-j}),$$

and on the j factor the map $D_r(f \sqcup g)$ restricts to

$$\begin{tikzcd}
E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (U^j \times V^{r-j}) \arrow[r, "E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (f^j \times g^{r-j})"] \arrow[d] & E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} X^r \arrow[d] \\
D_r(U \sqcup V) \arrow[r, "D_r(f \sqcup g)"] & D_r X,
\end{tikzcd}$$

where the vertical maps are projections. The counterclockwise composite represents the j summand of $P_r(x+y)$ coming from the decomposition above; the clockwise composite represents the class $\operatorname{Tr}_{j,r}^{MU} d^*(P_j x \times P_{r-j} y)$. ∎

There's also this useful naturality Lemma for power operations and Euler classes: $P_\pi(eV) = e(D_\pi V \to D_\pi X)$. Does that come up in the Quillen chapter? Maybe it should.

Cite me: Prop 3.2.10 of Matt's thesis, see also p. 42 of Quillen.

Lemma 6.4.3. *Write $\Delta : B\pi \times X \to D_\pi X$ and let $\mathcal{L}$ be a complex line bundle on X.*

$$\Delta^* P_\pi(e\mathcal{L}) = \prod_{u \in \pi^*} \left(e \left(\begin{array}{c} E\pi \times_u \mathbb{C} \\ \downarrow \\ B\pi \end{array} \right) +_{MU} e(\mathcal{L}) \right).$$

I need to already know: Matt claims that 3.2.10, the above Lemma, is the beating heart of the paper. Look how similar it looks to the formal group law quotient formula! That's why an expanded formula must be included in the previous days, not just Neil's geometric scribblings.

Matt in and before Theorem 3.3.2 describes the ring D_k as the *image* of the localization map $E_n(B\Lambda_k) \to S^{-1}E_n(B\Lambda_k)$ rather than as the whole target. Why?? He cites HKR for this, but the citation is meaningless because the theorem numbering scheme is so old. Ah, comparing with Lemma 3.3.3 yields a clue: D_k has a universal property as it sits under E_n, rather than under $E_n(B\Lambda_k)$...

Now, suppose that we pass down to the k^{th} Drinfel'd ring, so that the p^k–torsion in the formal group is presented as a discrete group $\Lambda^*[p^k]$. Pick such a subgroup $H \subseteq \Lambda^*[p^k]$ with $|H| = r$, and consider also the dual map $\pi : \Lambda[p^k] \to H^*$. We define the character map associated to H to be the composite

$$\chi^H \colon E_n(D_{H^*}X) \xrightarrow{\Delta^*} E_n(BH^*) \otimes_{E_n} E_n(X) \xrightarrow{\chi\pi \otimes 1} D_k \otimes_{E_n} E_n(X) =: D_k(X).$$

This definition is set up so that

$$\chi^H \left(e \left(\begin{array}{c} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{array} \right) \right).$$

In the presence of a coordinate x, this sews together to give a cohomology operation:

$$\begin{aligned} Q^H \colon MU^{2*}(X) &\xrightarrow{P_G^{MU}} MU^{2r*}(D_{H^*}X) \\ &\xrightarrow{\Delta^*} MU^{2r*}(BH^* \times X) \\ &\xrightarrow{t_x} E_n(BH^* \times X) \\ &\xrightarrow{\simeq} E_n BH^* \otimes_{E_n} E_n X \\ &\xrightarrow{\chi^H \otimes 1} D_k X. \end{aligned}$$

It turns out that Q^H is a ring homomorphism (cf. careful manipulation of HKR's Theorem C, which may not be worth it to write out, but it seems like the main manipulation is the last line of Proof of Theorem 3.3.8 on pg. 466), so each choice of H (and x) determines a new coordinate on D_k.

Theorem 6.4.4. *The effect of Q^H on Euler classes is*

$$Q^H e_{MU}\mathcal{L} = f_H^x e_x \mathcal{L} \in D_k(X),$$

and its effect on coefficients is

$$Q_*^H F_{MU} = F_x / H.$$

Proof. We chase through results established so far:

$$\begin{aligned}
Q^H(e_{MU}\mathcal{L}) &= (\chi^H \otimes 1) \circ t_x \circ \Delta^* \circ P_G^{MU}(e_{MU}\mathcal{L}) \\
&= (\chi^H \otimes 1) \circ t_x \left(\prod_{u \in H^* * = H} e_{MU} \begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} +_{MU} e_{MU}\mathcal{L} \right) \\
&= (\chi^H \otimes 1) \left(\prod_{u \in H} e_{E_n} \begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} +_{F_x} e_{E_n}\mathcal{L} \right) \\
&= \prod_{u \in H} (\varphi_{univ}(u) +_{F_x} e_{E_n}\mathcal{L}) = f_H^x(e_{E_n}\mathcal{L}).
\end{aligned}$$

Then, "since D_k is a domain, F_x/H is completely determined by the functional equation"

$$f_H^x(F_x(t_1, t_2)) = F_x/H(f_H^x(t_1, f_H^x(t_2))).$$

Take t_1 and t_2 to be the Euler classes of the two tautological bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, so that

$$\begin{aligned}
Q^H(e_{MU}\mathcal{L}_1 +_{MU} e_{MU}\mathcal{L}_2) &= Q^H \left(e_{MU} \begin{pmatrix} \mathcal{L}_1 \otimes \mathcal{L}_2 \\ \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{pmatrix} \right) \\
&= f_H^x \left(e_{E_n} \begin{pmatrix} \mathcal{L}_1 \otimes \mathcal{L}_2 \\ \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{pmatrix} \right) = f_H^x(t_1 +_{F_x} t_2).
\end{aligned}$$

On the other hand, Q^H is a ring homomorphism, so we can also split it over the sum first:

$$\begin{aligned}
Q^H(e_{MU}\mathcal{L}_1 +_{MU} e_{MU}\mathcal{L}_2) &= Q^H(e_{MU}\mathcal{L}_1) +_{Q_*^H F^{MU}} Q^H(e_{MU}\mathcal{L}_2) \\
&= f_H^x(t_1) +_{Q_*^H F^{MU}} f_H^x(t_2),
\end{aligned}$$

hence $f_H^x(t_1) +_{Q_*^H F^{MU}} f_H^x(t_2) = f_H^x(t_1 +_{F_x} t_2)$ and $Q_*^H F^{MU} = F_x/H$. □

Finally, we would like to produce a factorization

$$MU \xrightarrow{\Psi^H} E_n \to D_k$$

of the long natural transformation Q^H. Since E_n was built by Landweber flatness, it suffices to do this on coefficient rings, i.e., when applying the functors in the diagram to the one-point space. On a point, our calculations above show that Ψ^H exists exactly when $\alpha_*^H F_x = F_x/H$. We did this algebraic calculation earlier: given any coordinate, there is a unique coordinate P that is $\star$–isomorphic to it and through which the operations Q^H

factor to give ring operations Ψ^H for all subgroups $H \subseteq \Lambda_k^* = F_P(D_k)[p^k]$. This solves the problem of giving the operations the right *source*.

Leave a remark in here about this: McClure in BMMS works along similar lines to show that the Quillen idempotent is not H_∞, but he doesn't get any positive results (and, in particular, he can't complete his analysis as we do because he doesn't have access to the BP–homology of finite groups and to HKR character theory). One wonders whether the stuff here does say something about BP as the height tends toward ∞. So far as I know, no one has written much about this. Surely it remains a bee in Matt's bonnet.

Now we focus on giving the operations the right *target*. This is considerably easier. The group $\mathrm{Aut}(\Lambda_k^*)$ acts on the set of subgroups of Λ_k^*, and we define a ring Op^k by the fixed points of $\mathrm{Aut}(\Lambda_k^*)$ acting on the polynomial ring $E_n[\text{subgroups of } \Lambda_k^*]$. Note that $Op^k \subseteq Op^{k+1}$, and define $Op = \mathrm{colim}_k\, Op^k$, which consists of elements $\rho = \sum_{i \in I} a_i \prod_{H \in \alpha_i} H$, I a finite set, $a_i \in E_n$, and α_i are certain $\mathrm{Aut}(\Lambda_k^*)$–stable lists of subgroups of Λ_k^*, $k \gg 0$, with possible repetitions. For such a ρ, we define the associated operations

$$Q^\rho \colon\ MU^{2*}(X) \xrightarrow{\sum_{i \in I} a_i \prod_{H \in \alpha_i} Q^H} D_k(X),$$
$$\Psi^\rho \colon\ E_n(X) \xrightarrow{\sum_{i \in I} a_i \prod_{H \in \alpha_i} \Psi^H} D_k(X).$$

The theorem is that these actually land in $E_n(X)$, as they definitely land in $D_k^{\mathrm{Aut}(\Lambda_k^*)} \otimes_{E_n} E_n(X)$, and Galois descent for level structures says that left–hand factor is just E_n.

Matt runs the example of the subgroups $\widehat{\mathbb{G}}_m[p^j]$ in p–adic K–theory and he compares it with some Hopf ring analysis of $E_n E_{n*}$ due to Wilson

6.5 The moduli of subgroup divisors

Following... the original? Following Nat?

Continuing on from the above, if we expected E_n to be E_∞ (or even H_∞) so that it had power operations, then we would want to understand $E_n B\Sigma_{p^j}$ and match that with the operations we see.

—

There are union maps

$$B\Sigma_j \times B\Sigma_k \to B\Sigma_{j+k},$$

stable transfer maps

$$B\Sigma_{j+k} \to B\Sigma_j \times B\Sigma_k,$$

and diagonal maps

$$B\Sigma_j \to B\Sigma_j \times B\Sigma_j.$$

These induce a coproduct ψ as well as products $\times$ and $\bullet$ on $E^0 \mathbb{P}\mathbb{S}^0$, where $\mathbb{P}\mathbb{S}^0 = \coprod_{j=0}^\infty B\Sigma_j$ is the free E_∞–ring on $\mathbb{S}^0$. This is a Hopf ring, and under $\times$ alone it is a formal power series ring. The $\times$–indecomposables (which, I guess, are analogues of considering additive unstable cooperations) are

$$Q^\times E^0 \mathbb{P}\mathbb{S}^0 = \prod_{k \geq 0} \left(E^0 B\Sigma_{p^k} / \operatorname{tr} E^0 B\Sigma_{p^{k-1}}^p \right),$$

where the k^{th} factor in the product is naturally isomorphic to $\mathcal{O}_{\mathrm{Sub}_{p^k}(\widehat{\mathbb{G}})}$. The primitives are also accessible as the kernel of the dual restriction map.

Theorem 3.2 shows that $E^0 B\Sigma_k$ is free over E^0, Noetherian, and of rank controlled by generalized binomial coefficients. Prop 3.4 is the only place where work gets done, and it's all in terms of K–theory and HKR characters.

There's actually an extra coproduct, coming from applying D to the fold map $S^0 \vee S^0 \to S^0$.

The main content of Prop 5.1 (due to Kashiwabara) is that $K_0 \mathbb{P}\mathbb{S}^0$ injects into $K_0 \underline{BP}_0$. Grading $K_0 \mathbb{P}\mathbb{S}^0$ using the k–index in $B\Sigma_k$, you can see that it's of graded finite type, so we need only know it has no nilpotent elements to see that $K_0 \mathbb{P}\mathbb{S}^0$ is $*$–polynomial. This follows from our computation that $K_0 \underline{BP}_0$ is a tensor of power series and Laurent series rings. Corollary 5.2 is about $K_0 QS^0$, which is the group completion of $K_0 \mathbb{P}\mathbb{S}^0$, so it's the tensor of $K_0 \mathbb{P}\mathbb{S}^0$ with a graded field.

Prop 5.6, using a double bar spectral sequence method, shows that $K^0 QS^2$ is a formal power series algebra. Tracking the spectral sequences through, you'll find that $Q^\times K^0 QS^0$ agrees with $PK^0 QS^2$. (You'll also notice that $K^0 QS^2$ only has one product on it, cf. Remark 5.4.)

Snaith's theorem says $\Sigma^\infty QX = \Sigma^\infty \mathbb{P}X$ for connected spaces X. You can also see (just after Theorem 6.2) the nice equivalences

$$\mathbb{P}_k S^2 \simeq B\Sigma_k^{V_k} \simeq \mathbb{P}_k (S^0)^{V_k},$$

where superscript denotes Thom complex. So, for a complex-orientable cohomology theory, you can learn about $\mathbb{P}_k S^0$ from $\mathbb{P}_k S^2$. In particular, we finally learn that $E^0 \mathbb{P} S^0$ is a formal power series $\times$–algebra (once checking that the Thom isomorphism is a ring map). (We already knew the homological version of this claim.)

Section 8 has a nice discussion about indecomposables and primitives, to help move back and forth between homology and cohomology. It probably helps most with the dimension count argument below that we aren't going to get into.

Start again with $D_{p^k} S^2 \simeq B\Sigma_{p^k}^{V_{p^k}}$. We can associate to this a divisor $\mathbb{D}(V_{p^k})$ on $(B\Sigma_{p^k})_E$, which we know little about, but it is classified by a map to $\mathrm{Div}_{p^k} \mathbb{C}\mathrm{P}^\infty_E$. This receives a closed inclusion from $\mathrm{Sub}_{p^k} \mathbb{C}\mathrm{P}^\infty_E$, so their pullback Z_k is the largest subscheme of $(B\Sigma_{p^k})_E$ over which $\mathbb{D}(V_{p^k})$ is a subgroup divisor.

$$\begin{tikzcd}
H_k \arrow[rr] \arrow[dd] & & \mathbb{D}(V_{p^k}) \arrow[dd] & & \\
& Z_k \arrow[rr, crossing over] \arrow[dr] & & \mathrm{Sub}_{p^k} \mathbb{C}\mathrm{P}^\infty_E \arrow[dr] & \\
\mathrm{Spf}\, E^0 B\Sigma_{p^k}/\mathrm{tr} \arrow[rr] \arrow[ur, dotted] & & (B\Sigma_{p^k})_E \arrow[rr] & & \mathrm{Div}_{p^k} \mathbb{C}\mathrm{P}^\infty_E
\end{tikzcd}$$

We will show the existence of the dashed map, implying that the restricted divisor H_k is a subgroup divisor on $Y_k = \operatorname{Spf} E^0 B\Sigma_{p^k}/\mathrm{tr}$.

(Prop 9.1:) This proof falls into two parts: first we construct a family of maps to $(B\Sigma_{p^k})_E$ on whose image $\mathbb{D}(V_{p^k})$ restricts to a subgroup divisor, and then we show that the union of their images is exactly Y_k. Let A be an abelian p–subgroup of Σ_{p^k} that acts transitively on $\{1, \ldots, p^k\}$ (i.e., it is not boosted from some transfer). The restriction of V_{p^k} to A is the regular representation, which splits as a sum of characters $V_{p^k}|_A = \bigoplus_{\mathcal{L} \in A^*} \mathcal{L}$. Identifying $BA_E = \underline{\mathsf{FormalGroups}}(A^*, \mathbb{C}\mathrm{P}^\infty_E)$, $\mathbb{D}(V_{p^k})$ restricts all the way to $\sum_{\mathcal{L} \in A^*}[\varphi(\mathcal{L})]$, with $\varphi : A^* \to$"$\Gamma(\operatorname{Hom}(A^*, \widehat{\mathbb{G}}), \widehat{\mathbb{G}})$". In Finite Subgroups of Formal Groups (see Props 22 and 32), we learned that the restriction of $\mathbb{D}(V_{p^k})$ further to $\operatorname{Level}(A^*, \mathbb{C}\mathrm{P}^\infty_E)$ is a subgroup divisor. So, our collection of maps are those of the form

$$\operatorname{Level}(A^*, \mathbb{C}\mathrm{P}^\infty_E) \to \underline{\mathsf{FormalGroups}}(A^*, \mathbb{C}\mathrm{P}^\infty_E) = BA_E \to (B\Sigma_{p^k})_E.$$

Here, finally, is where we have to do some real work involving Chern classes and commutative algebra, so I'm inclined to skip it in the lectures. Finally, you do a dimension count to see that Z_k and $\operatorname{Spf} E^0 B\Sigma_{p^k}/\mathrm{tr}$ have the same dimension (which requires checking enough commutative algebra to see that "dimension" even makes sense), and so you show the map is injective and you're done.

—

Here's Neil's proof of the joint images claim. It seems like a clear enough use of character theory that we should include it, if we can make character theory itself clear.

Recall from [18, Theorem 23] that $\operatorname{Level}(A^*, \widehat{\mathbb{G}})$ is a smooth scheme, and thus that $D(A) = \mathcal{O}_{\operatorname{Level}(A^*, \widehat{\mathbb{G}})}$ is an integral domain. Using [18, Proposition 26], we see that when $\mathcal{L} \in A^*$ is nontrivial, we have $\varphi(\mathcal{L}) \neq 0$ as sections of $\widehat{\mathbb{G}}$ over $\operatorname{Level}(A^*, \widehat{\mathbb{G}})$, and thus $e(\mathcal{L}) = x(\varphi(\mathcal{L})) \neq 0$ in $D(A)$. It follows that that $c_{p^k} = \prod_{\mathcal{L} \neq 1} e(\mathcal{L})$ is not a zero-divisor in $D(A)$. On the other hand, if A' is an Abelian p-subgroup of Σ_{p^k} which does not act transitively on $\{1, \ldots, p^k\}$, then the restriction of $V_{p^k}1$ to A' has a trivial summand, and thus c_{p^k} maps to zero in $D(A')$. Next, we recall the version of generalised character theory described in [8, Appendix A].

$$p^{-1}E^0 BG = \left(\prod_A p^{-1} D(A) \right)^G$$

where A runs over all Abelian p-subgroups of G. As $\overline{R}_k = E^0(B\Sigma_{p^k})/ann(c_{p^k})$ and everything in sight is torsion-free, we see that $p^1\overline{R}_k$ is the quotient of $p^1 E^0 B\Sigma_{p^k}$ by the annihilator of the image of c_{p^k}. Using our analysis of the images of c_{p^k} in the rings $D(A)$, we conclude that

$$p^{-1}\overline{R}_k = \left(\prod_A p^1 D(A) \right)^{\Sigma_{p^k}},$$

where the product is now over all transitive Abelian p-subgroups. This implies that for such A, the map $E^0 B\Sigma_{p^k} \to D(A)$ factors through $\overline{R}_k$, and that the resulting maps $\overline{R}_k \to D(A)$ are jointly injective. This means that $Y_k = \operatorname{Spf} \overline{R}_k$ is the union of the images of the corresponding schemes $\mathrm{Level}(A^*, \widehat{\mathbb{G}})$, as required.

6.6 Interaction with Θ–structures

The Ando–Hopkins–Strickland result that the σ–orientation is an H_∞–map

The main classical point is that an $MU\langle 0\rangle$–orientation is H_∞ when the following diagram commutes for every choice of A:

$$\begin{tikzcd}
(BA^* \times \mathbb{C}\mathrm{P}^\infty)^{V_{reg} \otimes \mathcal{L}} \arrow[r] & D_n MU\langle 0\rangle \arrow[r] \arrow[d] & D_n E \arrow[d] \\
& MU\langle 0\rangle \arrow[r] & E
\end{tikzcd}$$

(This is equivalent to the condition given in the section on Matt's thesis. In fact, maybe I should try writing this so that Matt's thesis uses the same language?) If you write out what this means, you'll see that a given coordinate on E pulls back to give two elements in the E–cohomology of that Thom spectrum (or: sections of the Thom sheaf), and the orientation is H_∞ when they coincide.

Similarly, an $MU\langle 6\rangle$–orientation corresponds to a section of the sheaf of cubical structures on a certain Thom sheaf. Using the H_∞ structures on $MU\langle 6\rangle$ and on E give two sections of the pulled back sheaf of cubical structures, and the H_∞ condition is that they agree for all choices of group A.

Then you also need to check that the σ–orientation actually satisfies this.

The AHS document really restrictions attention to E_2. Is there a version of this story that gives non-supersingular orientations too, or even the K_{Tate} orientation? I can't tell if the restriction in AHS's exposition comes from not knowing that K_{Tate} has an E_∞ structure or if it comes from a restriction on the formal group. (At one point it looks like they only need to know that p is regular on $\pi_0 E$, cf. 16.5...)

Section 3.1: Intrinsic description of the isogenies story for an H_∞ *complex orientable* ring spectrum, without mention of a specific orientation / coordinate. This is nice: it means that a complex orientation has to be a coordinate which is compatible with the descent picture already extant on the level of formal groups, which is indeed the conclusion of Matt's thesis.

Section 3.2: They define an abelian group indexed extended power construction

$$D_A(X) = \mathcal{L}(U^{A^*}, U) \wedge_{A^*} X^{(A^*)},$$

where $\mathcal{L}(U^{A^*}, U)$ is the space of linear isometries from the $A^{*\mathrm{th}}$ power of a universe U down to itself. Yuck. Then, given a level structure $(i\colon \operatorname{Spf} R \to S_E, \ell\colon A_{\operatorname{Spf} R} \to i^*\widehat{\mathbb{G}})$, they construct a map

$$\psi^E_\ell\colon \pi_0 E \xrightarrow{D_A} \pi_0 \mathsf{Spectra}(D_A S^0, E) = \pi_0 E^{BA^*_+} \to \mathcal{O}((BA^*)_E) \xrightarrow{\chi_\ell} R,$$

where χ_ℓ is the map classifying the homomorphism ℓ. This is a continuous map of rings: it's clearly multiplicative, it's additive up to transfers (but those vanish for an abelian group), and it's continuous by an argument in Lemma 3.10. (You don't actually need an abelian group here; you can work in the scheme of subgroups—i.e., in the cohomology of $B\Sigma_k$ modulo transfers—and this will still work.) This construction is natural in H_∞ maps $f\colon E \to F$:

$$\begin{tikzcd}
i^*S_F \arrow[dr, dotted, "\psi_\ell^{F/E}"] \arrow[drr, bend left, "\psi_\ell^F"] \arrow[ddr, bend right] & & \\
& \operatorname{Spf} R \times_{i,S_E,S_f} S_F \arrow[r, "\psi_\ell^F"] \arrow[d] & S_F \arrow[d, "S_f"] \\
& \operatorname{Spf} R \arrow[r, "\psi_\ell^E"] & S_E,
\end{tikzcd}$$

begetting the relative map $\psi_\ell^{F/E}\colon i^*S_F \to (\psi_\ell^E)^*S_F$ as indicated. For example, take $F = E^{\mathbb{CP}^\infty_+}$, so that $\widehat{\mathbb{G}} = S_F$, giving the (group) map

$$\psi_\ell^{\widehat{\mathbb{G}}/E}\colon i^*\widehat{\mathbb{G}} \to (\psi_\ell^E)^*\widehat{\mathbb{G}}.$$

One of the immediate goals is to show that this is an isogeny. A different construction we can do is take V to be a virtual bundle over X and set $F = E^{X_+}$. Given $m \in \pi_0\mathsf{Spectra}(X^V, E)$ applying the construction of D_A above gives an element

$$\psi_\ell^V(m) \in R \mathop{\widehat{\otimes}}_{\chi_\ell, \hat{\pi}_0 E^{BA^*_+}} \hat{\pi}_0\mathsf{Spectra}((BA^* \times X)^{V_{reg}\otimes V}, E).$$

This map is additive and also $\psi_\ell^V(xm) = \psi_\ell^F(x)\psi_\ell^V(m)$, so we can interpret this as a map

$$\psi_\ell^V\colon (\psi_\ell^F)^*\mathbb{L}(V) \to \chi_\ell^*\mathbb{L}(V_{reg} \otimes V)$$

of line bundles over $i^*S_F = i^*X_E$.

Cite me: Lemma 3.19 of AHS H_∞.

Lemma 6.6.1. *The map ψ_ℓ^V has the following properties:*

1. *If m trivializes $\mathbb{L}(V)$ then $\psi_\ell^V(m)$ trivializes $\chi_\ell^*\mathbb{L}(V_{reg} \otimes V)$.*
2. $\psi_\ell^{V_1 \oplus V_2} = \psi_\ell^{V_1} \otimes \psi_\ell^{V_2}$.
3. *For $f\colon Y \to X$ a map, $\psi_\ell^{f^*V} = f^*\psi_\ell^V$.* □

In particular, we can apply this to $X = \mathbb{CP}^\infty$ and $\mathbb{L}(\mathcal{L} - 1) = \mathcal{I}(0)$. Then 8.11 gives

$$\psi_\ell^{\mathcal{L}-1}\colon (\psi_\ell^F)^*\mathcal{I}_{\widehat{\mathbb{G}}}(0) \to \chi_\ell^*\mathbb{L}(V_{reg} \otimes (\mathcal{L} - 1)) = \mathcal{I}_{i^*\widehat{\mathbb{G}}}(\ell).$$

Cite me: Prop 3.21.

Theorem 6.6.2. *The map $\psi_\ell^{\widehat{\mathbb{G}}/E}\colon i^*\widehat{\mathbb{G}} \to (\psi_\ell^E)^*\widehat{\mathbb{G}}$ of 3.15 is an isogeny with kernel $[\ell(A)]$. Using $\psi_\ell^{\widehat{\mathbb{G}}/E}$ to make the identification*

$$(\psi_\ell^{\widehat{\mathbb{G}}/E})^*\mathcal{I}_{(\psi_\ell^E)^*\widehat{\mathbb{G}}}(0) \cong \mathcal{I}_{i^*\widehat{\mathbb{G}}}(\ell),$$

the map $\psi_\ell^{\mathcal{L}-1}$ sends a coordinate x on $\widehat{\mathbb{G}}$ to the trivialization $(\psi_\ell^{\widehat{\mathbb{G}}/E})^*(\psi_\ell^E)^*x$ of $\mathcal{I}_{i^*\widehat{\mathbb{G}}}(\ell)$. □

3.24 might be interesting.

So far, it seems like the point is that the identity map on $MU\langle 0\rangle$ classifies a section of the ideal sheaf at zero of the universal formal group which is compatible with descent for level structures, so any H_∞ map out of $MU\langle 0\rangle$ classifies not just a section of the ideal sheaf at zero of whatever other formal group but does so in a way that is, again, compatible with descent for level structures.

Theorem 6.6.3. *Let $g : MU\langle 0\rangle \to E$ be a homotopy multiplicative map, and let $s = s_g$ be the corresponding trivialization of $\mathcal{I}_{\widehat{\mathbb{G}}}(0)$. If the map g is H_∞, then for any level structure $\ell : A \to i^*\widehat{\mathbb{G}}$ the section s satisfies the identity*

Cite me: Prop 4.13.

The discussion leading up to this theorem seems interesting, especially equations 4.10,12.

$$N_{\psi_\ell^{\widehat{\mathbb{G}}/E}} i^*s = (\psi_\ell^E)^*s,$$

in which the isogeny $\psi_\ell^{\widehat{\mathbb{G}}/E}$ has been used to make the identification

$$N_{\psi_\ell^{\widehat{\mathbb{G}}/E}} i^*\mathcal{I}_{\widehat{\mathbb{G}}}(0) \cong \mathcal{I}_{(\psi_\ell^E)^*\widehat{\mathbb{G}}}(0). \quad \square$$

Lemma 6.6.4. *For V a vector bundle on a space X and V_{reg} the (vector bundle over BA^* induced from) the regular representation on A, there is an isomorphism of sheaves over $(BA^* \times X)_E$*

Cite me: Eqn 5.3, generalizes Quillen's splitting formula.

$$\mathbb{L}(V_{reg} \otimes V) \cong \bigotimes_{a\in A} \widetilde{T}_a \mathbb{L}(V).$$

Eqn 5.4 claims to use 5.3 but seems to be using something about the behavior of the norm map on line bundles vs the translated sum of divisors appearing in 5.3.

The beginning of the proof of 6.1 appears to be a simplification of some of the descent arguments appearing in the algebraic parts of Matt's thesis's main calculations. On the other hand, I can't even read what the McClure reference in 6.1 is doing. What's Δ^*??

Lemma 6.6.5. *Take $\pi_0 E$ to be a complete local ring and $\widehat{\mathbb{G}}_E$ to be of finite height. If $B^* \subset A^*$ is a proper subgroup, then the following composite map of $\pi_0 E$–modules is zero:*

Cite me: Prop 7.5.

$$\pi_0 E^{BB^*_+} \xrightarrow{transfer} \pi_0 E^{BA^*_+} \xrightarrow{\chi_\ell} \mathcal{O}(T).$$

Proof. It suffices to consider the tautological level structure over $\mathrm{Level}(A, \widehat{\mathbb{G}})$. We may take A to be a p–group, and indeed for now we set $A = \mathbb{Z}/p$, $B = 0$. For $t \in \pi_0 E^{\mathbb{C}\mathrm{P}^\infty_+}$ a coordinate with formal group law F, we have

$$\pi_0 E^{BA^*_+} \cong \pi_0 E[\![t]\!]/[p]_F(t)$$

and $\tau : \pi_0 E^{BB^*_+} = \pi_0 E \to \pi_0 E^{BA^*_+}$ is given by $\tau(1) = \langle p\rangle_F(t)$, where $\langle p\rangle_F(t) = [p]_F(t)/t$ is the "reduced p–series". The result then follows from the isomorphism $\mathcal{O}(\mathrm{Level}(\mathbb{Z}/p, \widehat{\mathbb{G}}_E)) \cong \pi_0 E[\![t]\!]/\langle p\rangle_F(t)$. The result then follows in general by induction: B^* can be taken to be a *maximal* proper subgroup of A^*, with cokernel $\mathbb{Z}/p$. □

Example 6.6.6. Let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group with coordinate x so that the group law is

$$x +_{\widehat{\mathbb{G}}_m} y = x + y - xy, \quad [p](x) = 1 - (1-x)^p.$$

The monomorphism $\mathbb{Z}/p \to \widehat{\mathbb{G}}_m(\mathbb{Z}[\![y]\!]/[p](y))$ given by $j \mapsto [j](y)$ becomes the zero map under the base change

$$\begin{aligned} \mathbb{Z}[\![y]\!]/[p](y) &\to \mathbb{Z}/p, \\ y &\mapsto 0. \end{aligned}$$

Cite me: Corollary 9.21, Prop 9.17.

Remark 6.6.7. If R is a domain of characteristic 0, then a level structure over R actually induces a monomorphism on points.

Cite me: Prop 9.24.

One of the reduction steps in Prop 6.1 is handled by 9.24, which is in turn equivalent to a basic case of an HKR theorem, so should be stated on that day (or in the algebraic day).

Fill this.

Lemma 6.6.8. *The natural map*

$$\mathcal{O}(\underline{\mathsf{FormalGroups}}(\mathbb{Z}/p, \widehat{\mathbb{G}})) \to R \times \mathcal{O}(\mathsf{Level}(\mathbb{Z}/p, \widehat{\mathbb{G}}))$$

is injective.

Proof. □

I left off at Section 10.

—— Descent along level structures, simplicially (Section 11) ——

Actually, this section appears *not* to be about FGps, and instead it's about the *coarse moduli quotient* to the functor of formal groups, which is not locally representable. I'm a little confused about this—I intend to ask Mike what's going on.

Write $\mathsf{Level}(A) \to \mathsf{FGps}$ for the parameter space of a formal group equipped with a level–A structure, together with its structure map (to the *coarse moduli of formal groups!!!*). We define a sequence of schemes by: $\mathsf{Level}_0 = \mathsf{FGps}$, $\mathsf{Level}_1 = \coprod_{A_0} \mathsf{Level}(A_0)$ for finite abelian groups A_0, and most generally

$$\mathsf{Level}_n = \coprod_{0 = A_n \subseteq \cdots \subseteq A_0} \mathsf{Level}(A_0).$$

There are two maps $\mathsf{Level}_1 \to \mathsf{Level}_0$. One is the structural one, where we simply peel off the formal group and forget the level structure. The other comes from the quotient map: $\ell \colon A \to \widehat{\mathbb{G}}$ yields a quotient isogeny $q \colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}/\ell$, and we take the second map $\mathsf{Level}_1 \to \mathsf{Level}_0$ to send ℓ to $\widehat{\mathbb{G}}/\ell$. Then, consider the following Lemma:

Cite me: AHS Lemma 11.3.

Lemma 6.6.9. *For $\ell \colon A \to \widehat{\mathbb{G}}$ a level structure and $B \subseteq A$ a subgroup, the induced map $\ell|_B \colon B \to \widehat{\mathbb{G}}$ is a level structure and the quotient $\widehat{\mathbb{G}}/\ell|_B$ receives a level structure $\ell' \colon A/B \to \widehat{\mathbb{G}}/\ell|_B$.* □

This gives us enough compatibility among quotients to use the two maps above to assemble the Level_* schemes into a simplicial object. Most face maps just omit a subgroup, except for the last face map, since the zero subgroup is not permitted to be omitted.

Instead, the last face map sends the string of subgroups $0 = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_0$ and level structure $\ell\colon A_0 \to \widehat{\mathbb{G}}$ to the quotient string $0 = A_{n-1}/A_{n-1} \subseteq \cdots \subseteq A_0/A_{n-1}$ and quotient level structure $\ell\colon A_0/A_{n-1} \to \widehat{\mathbb{G}}/\ell|_{A_{n-1}}$. The degeneracy maps come from lengthening one of these strings by an identity inclusion.

Definition 6.6.10. Let $\widehat{\mathbb{G}}\colon F \to \mathrm{FGps}$ be a functor over formal groups, and define schemes $\mathrm{Level}(A, F) = \mathrm{Level}(A) \times_{\widehat{\mathbb{G}}} F$ and $\mathrm{Level}_n(F) = \mathrm{Level}_n \times_{\widehat{\mathbb{G}}} F$. Then, *descent data for level structures on F* is the structure of a simplicial scheme on $\mathrm{Level}_*(F)$, together with a morphism of simplicial schemes $\mathrm{Level}_*(F) \to \mathrm{Level}_*$. It is enough to specify a map $d_1\colon\ \mathrm{Level}_1(F) \to F$, use that to build the simplicial scheme structure as in the above Lemma, and assert that the following square commutes:

Cite me: Definition 11.10, Remark 11.11.

$$\begin{tikzcd}
\mathrm{Level}_1(F) \arrow[r] \arrow[d, "d_1"] & \mathrm{Level}_1 \arrow[d, "d_1"] \\
F \arrow[r] & \mathrm{FGps}.
\end{tikzcd}$$

Example 6.6.11. Let $\widehat{\mathbb{G}}\colon S \to \mathrm{FGps}$ be a formal group of finite height over a p–local formal scheme S. The functor $\mathrm{Level}(A, \widehat{\mathbb{G}})$ is exactly the functor defined in Section 9 (see above), and in particular it is represented by an S–scheme. The maps ψ_ℓ and f_ℓ from Definition 3.1 amount to giving a map $d_1\colon\ \mathrm{Level}_1(\widehat{\mathbb{G}}) \to S$ and an isogeny $q\colon d_0^*\widehat{\mathbb{G}} \to d_1^*\widehat{\mathbb{G}}$ whose kernel on $\mathrm{Level}(A, \widehat{\mathbb{G}})$ is A. The other conditions on Definition 3.1 exactly ensure that $(\mathrm{Level}_*(\widehat{\mathbb{G}}), d_*, s_*)$ is a simplicial functor and over $\mathrm{Level}_2(\widehat{\mathbb{G}})$ the relevant hexagonal diagram commutes:

$$\begin{tikzcd}
& d_0^* d_0^* \widehat{\mathbb{G}} \arrow[dl, equal] \arrow[dr, "d_0^* q"] & \\
d_1^* d_0^* \widehat{\mathbb{G}} \arrow[d, "d_1^* q"] & & d_0^* d_1^* \widehat{\mathbb{G}} \arrow[d, equal] \\
d_1^* d_1^* \widehat{\mathbb{G}} \arrow[dr, equal] & & d_2^* d_0^* \widehat{\mathbb{G}} \arrow[dl, "d_2^* q"] \\
& d_2^* d_1^* \widehat{\mathbb{G}}. &
\end{tikzcd}$$

Example 6.6.12. We now further package this into a single object. Let $\widehat{\mathbb{G}}$ be the functor over $\underline{\mathrm{FGps}}$ whose value on R is the set of pullback diagrams

$$\begin{tikzcd}
\widehat{\mathbb{G}}' \arrow[r, "f"] \arrow[d] & \widehat{\mathbb{G}} \arrow[d] \\
\operatorname{Spf} R \arrow[r, "i"] & S
\end{tikzcd}$$

such that the map $\widehat{\mathbb{G}}' \to i^*\widehat{\mathbb{G}}$ induced by f is a homomorphism (hence isomorphism) of formal groups over $\operatorname{Spf} R$. For a finite abelian group A, write $\mathrm{Level}(A, \underline{\widehat{\mathbb{G}}})(R)$ for the set of diagrams

$$\begin{tikzcd} A_{\operatorname{Spf} R} \arrow[r, "\ell"] \arrow[dr] & \widehat{\mathbb{G}}' \arrow[r, "f"] \arrow[d] & \widehat{\mathbb{G}} \arrow[d] \\ & \operatorname{Spf} R \arrow[r, "i"] & S \end{tikzcd}$$

where the square forms a point in $\underline{\widehat{\mathbb{G}}}(R)$ and ℓ is a level–A structure. Giving a map of functors $d_1\colon\ \mathrm{Level}_1(\underline{\widehat{\mathbb{G}}}) \to \underline{\widehat{\mathbb{G}}}$ making the above square commute is to give a pullback diagram

$$\begin{tikzcd} \widehat{\mathbb{G}}/\ell \arrow[r] \arrow[d] & \widehat{\mathbb{G}} \arrow[d] \\ \mathrm{Level}_1(\widehat{\mathbb{G}}) \arrow[r] & S, \end{tikzcd}$$

or equivalently a map of formal schemes $\mathrm{Level}_1(\widehat{\mathbb{G}}) \to S$ and an isogeny $q\colon\ d_0^*\widehat{\mathbb{G}} d_1^*\widehat{\mathbb{G}}$ whose kernel on $\mathrm{Level}(A, \widehat{\mathbb{G}})$ is A. Therefore, descent data for level structures on the formal group $\widehat{\mathbb{G}}$ (in the sense of Section 3) are equivalent to descent data for level structures on the functor $\underline{\widehat{\mathbb{G}}}$.

—— Section 12: Descent for level structures on Lubin–Tate groups ——

Let k be perfect of positive characteristic p, and let Γ be a formal group of finite height over k. Recall that this induces a relative Frobenius

$$\begin{tikzcd} \Gamma \arrow[r, "F"] \arrow[rr, bend left, "\varphi_\Gamma"] \arrow[dr] & \varphi_k^*\Gamma \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ & \operatorname{Spec} k \arrow[r, "\varphi_k"] & \operatorname{Spec} k. \end{tikzcd}$$

The map F is an isogeny of degree p, with kernel the divisor $p \cdot [0]$. Recall also that a deformation H of Γ to T induces a map $\underline{H} \to \mathrm{Def}(\Gamma)$, and there is a universal such $\widehat{\mathbb{G}}$ over the ground scheme $S \cong \operatorname{Spf} \mathbb{W}(k)[\![u_1, \ldots, u_{d-1}]\!]$ such that $\underline{\widehat{\mathbb{G}}} \to \mathrm{Def}(\Gamma)$ is an isomorphism of functors over FGps.

Now consider a point in $\mathrm{Level}(A, \operatorname{Def}\Gamma)$:

$$\begin{tikzcd} A_T \arrow[r, "\ell"] \arrow[dr] & H \arrow[d] & H_0 \arrow[l] \arrow[r, "f"] \arrow[d] & \Gamma \arrow[d] \\ & T & T_0 \arrow[l] \arrow[r, "j"] & \operatorname{Spec} k. \end{tikzcd}$$

The level structure ℓ gives rise to a quotient isogeny $q\colon H \to H'$. Since A is sent to 0 in $\mathcal{O}_{T_0}$, there is a canonical map $\bar{q}$ fitting into the diagram

$$\begin{tikzcd}
H \arrow[rr, "q"] \arrow[dr] & & H' \arrow[dl] & & & & & \\
& & & H_0 \arrow[ulll] \arrow[r] \arrow[rr, bend left] \arrow[ddr] & H_0' \arrow[ull] \arrow[r, dotted, "\bar{q}"] \arrow[dd] & (\varphi^r)^* H_0 \arrow[r] \arrow[ddl] & H_0 \arrow[r, "f"] \arrow[dd] & \Gamma \arrow[dd] \\
& T & & & & & & \\
& & & & T_0 \arrow[ulll] \arrow[rr, "\varphi^r"] & & T_0 \arrow[r, "j"] & \operatorname{Spec} k.
\end{tikzcd}$$

The map $\bar{q}$ combines with the rest of the maps to exhibit H' as a deformation of Γ, and hence we get a natural transformation

$$d_1\colon \operatorname{Level}_1(\operatorname{Def}(\Gamma)) \to \operatorname{Def}(\Gamma).$$

Since $\varphi^r \varphi^s = \varphi^{r+s}$, this gives descent data for level structures on $\operatorname{Def}(\Gamma)$. Identifying this functor with $\underline{\widehat{\mathbb{G}}}$ using Lubin–Tate theory, we equivalently have shown the existence of descent data for level structures on $\underline{\widehat{\mathbb{G}}}$.

Incidentally, the descent data constructed here is also the descent data that would come from the structure of an E_∞–orientation on the Morava E–theory E_d, essentially because the divisor associated to the kernel of the relative Frobenius on the special fiber is forced to be $p[0]$, and everything is dictated by how the deformation theory *has* to go (and the fact that the topological operations we're studying induce deformation-theoretic-describable operations on algebra).

——Section 15: Level structures on elliptic curves, and the relation to the σ–orientation / the corresponding section of the Θ^3–sheaf——

Tyler's argument

There's an important injectivity result used by Ando and Ando–Hopkins–Strickland (though Matt blames it on Hopkins and Strickland both times) about the injectivity of a certain p^{th} power map. They cite the McClure chapter of BMMS, but McClure's proof requires finite type hypotheses on the cohomology theory involved, which Morava E–theory does not satisfy. There is a similar proof in the recent paper of Hopkins–Lawson, and so Nat and I wrote to Tyler about whether there was a common generalization of the two theorems that would give a good replacement argument. Here is his reply:

—

Here are my current thought processes, which may be a bit messy at present. Fix a space X and take $X^{(p)}$ for its smash power, as McClure does.

Let's write $M = F(\Sigma^\infty X^{(p)}, E)$ for the function spectrum which is now C_p–equivariant, and $N = F(\Sigma^\infty X, E)$. Let's assume that E has an E_∞ multiplication and that X is nice in the following sense: E^X is a wedge of copies of E (unshifted). This is satisfied when E is E–theory and X is finite type with $\mathbb{Z}_{(p)}$-homology only in even degrees.

We get two maps:

$$M^{hC_p} \to M$$

This will realize our "forgetful" map $E^*(DX) \to E^*(X^{(p)})$.

$$M^{hC_p} \to N^{hC_p}$$

This will realize the "other" map $E^*(DX) \to E^*((BC_p)_+ \wedge X)$.

We want to prove that these are jointly monomorphisms.

The assumptions on X actually imply that $E^{X^{(p)}} = (E^X)^{(p)}$ where the latter smash is taken over E. This decomposes, C_p–equivariantly, into a wedge of copies of E with trivial action and a bunch of regular representations $E[C_p]$. Since M is E–dual to this, we find that the map $M^{hC_p} \to M$ is a monomorphism on all the $E[C_p]$ components; on the E parts with trivial action it decomposes as a product of projections $E_*[\![x]\!]/[p](x) \to E_*$. The kernel of this consists of the multiples of x. So if we want to prove a monomorphism, all we have to do now is show that these multiples of x map monomorphically into the homotopy of N^{hC_p}.

I now want to consider the composite to the Tate spectra

$$M^{hC_p} \to N^{hC_p} \to N^{tC_p}$$

or equivalently

$$M^{hC_p} \to M^{tC_p} \to N^{tC_p}.$$

The first composition shows that, if we can show that this composite is a monomorphism on the multiples of x, we will be done. The second composition has, as its first map, inverting x, and it's a monomorphism on the desired classes. So we just have to check that the second map preserves that.

This has the following benefit: instead of being born out of the unstable diagonal map $X \to X^{(p)}$, the constructions

$$M^{tC_p} = F(X^{(p)}, E)^{tC_p}$$

and

$$N^{tC_p} = F(X, E)^{tC_p}$$

take cofiber sequences in (finite) X to fiber sequences of spectra. I think that this means that, instead of being functions of the unstable diagonal map on X, they are constructions that only require knowledge of the *stable* homotopy type of X. I believe in fact that, by checking the case $X = S^0$, we then find that the map $M^{tC_p} \to N^{tC_p}$ is an equivalence for any finite X, and that should hopefully be enough to buy us a monomorphism on any of the X's that we're describing.

I don't understand this. I guess this has been a recurring theme in the Thursday seminar, and also in Chapters 2 and 6 of these notes (in some guise). I can ask Mike to explain it to me.

Other stuff that goes in this chapter

Dyer–Lashof operations, the Steenrod operations, and isogenies of the formal additive group

Cite me: See Neil's *Steenrod algebra* note, maybe? Talk to Mike?.

Another augmentation to the notion of a context: working not just with E_*X but with $E_*(X \times BG)$ for finite G.

Charles's *The congruence criterion* paper codifies the Hecke algebra picture Neil is talking about, and in particular it talks about sheaves over the pile of isogenies.

If we're going to talk about that Hecke algebra, then maybe we can also talk about the period map, since one of the main points of it is that it's equivariant for that action.

Section 3.7 of Matt's thesis also seems to deal with the context question: he gives a character-theoretic description of the total power operation, which ties the behavior of the total power operation to a formula of type "decomposition into subgroups". Worth reading.

The rational story: start with a sheaf on the isogenies pile. Tensor everything with $\mathbb{Q}$. That turns this thing into a rational algebra under the Drinfel'd ring together with an equivariant action of $GL_n\mathbb{Q}_p$.

This is Nat's claim. Check back with him about how this is visible.

Matt's Section 4 talks about the E_∞ structure on E_n and compatibility with his power operations. It's not clear how this doesn't immediately follow from the stuff he proves in Section 3, but I think I'm just running out of stream in reading this thesis. One of the neat features of this later section is that it relies on calculations in $E_n D_\pi \underline{MU}_{2*}$, which is an interesting way to mix operations coming from instability and from an H_∞–structure. This is yet another clue about what the relevant picture of a context should look like. He often cites VIII.7 of BMMS.

Mike says that Mahowald–Thompson analyzed $L_{K(n)}\Omega S^{2n+1}$ by writing down some clever finite resolution. The resolution that they produce by hand is actually exactly what you would get if you tried to understand the mapping spectral sequence for $E_\infty(E_n^{\Omega S^{2n+1}}, E_n)$.

Mike also says that a consequence of the unpublished Hopkins–Lurie ambidexterity follow-up is that the comparison map $\mathsf{Spaces}(*, Y) \to E_\infty(E_n^Y, E_n^*)$ is an equivalence if Y is a finite Postnikov tower living in the range of degrees visible to Morava E–theory.

The final chapter of Matt's thesis has never really been published, where he investigates power operations on elliptic cohomology theories. That might belong in this chapter as an example of the techinques, since we've already defined elliptic cohomology theories.

Barry's description of the image of

$$E_\infty(A, B) \to \mathsf{Spaces}(\Omega^\infty A, \Omega^\infty B)$$

for $K(1)$–local A and B using p–adic moments is pretty digestable. That might belong in here, or at least it could be referenced. (I guess it didn't ever get published??)

Appendix A

Loose ends

I'd like to spend a couple of days talking about ways the picture in this class can be extended, finally, some actually unanswered questions that naturally arise. The following two section titles are totally made up and probably won't last.

Also, write a broad-scale introduction to this appendix.

A.1 Orientations by E_∞ maps

What follows is some scaffolding for giving an explicit understanding the *Spin* orientation of *KO* by an E_∞ map, as well as some notes on how the *String* orientation of *tmf* plays out. It is not sewn together, or chronologically sound, but I wanted to get the brain-dump down before it's lost.

A more modern take on the story of the σ–orientation passes through the algebraic geometry of E_∞–ring spectra, which are the homotopically coherent analogues of commutative ring objects that one finds in the ∞–category Spectra. Essentially all of the ring spectra discussed in this book have incarnations as E_∞–rings:

Theorem A.1.1. *The classical K–theories KU and KO, the Eilenberg–Mac Lane spectra HR, the Morava E–theories E_Γ, their fixed point spectra, the Thom spectra arising from the J–homomorphism (including MO, MSO, MSpin, MString, MU, MSU, and $MU[6,\infty)$), the spectra TMF, Tmf, and tmf are all E_∞–ring spectra.*[1] □

Cite me: this is a mega-theorem.

A great deal of classical commutative algebra can be lifted into this new setting, including a particular functor

$$gl_1\colon\ E_\infty\mathsf{RingSpectra} \to \mathsf{Spectra}.$$

This functor derives its name from two compatible sources: for one, its underlying infinite loopspace is the construction GL_1 described in Lecture 1.1; and secondly, it participates in an adjunction

$$\begin{tikzcd} \mathsf{ConnectiveSpectra} \arrow[r, shift left, "\Sigma^\infty_+\Omega^\infty"] & E_\infty\mathsf{RingSpectra} \arrow[l, shift left, "gl_1"] \end{tikzcd}$$

[1] Notably, the Morava K–theories are *not* E_∞ rings at finite heights.

analogous to the adjunction between the group of units and the group-ring constructions in classical algebra. Its relevance to us is its participation in the theory of highly structured Thom spectra. Let $j\colon g \to gl_1\mathbb{S}$ be a map of connective spectra, begetting an infinite loopmap $J\colon G \to GL_1\mathbb{S}$, where we have written $G = \Omega^\infty g$.

Cite me: May? or ABGHR?.

Lemma A.1.2. *The Thom spectrum of the map BJ is presented by the pushout of E_∞ rings*

$$\begin{tikzcd} \Sigma^\infty_+ GL_1\mathbb{S} \arrow[r, "\Omega^\infty\Sigma j"] \arrow[d] & \Sigma^\infty_+\Omega^\infty gl_1\mathbb{S}/g \arrow[d] \\ \mathbb{S} \arrow[r] & MG. \end{tikzcd} \qquad \square$$

Cite me: May, but also ABGHR.

Corollary A.1.3. *There is a natural equivalence between the space of null-homotopies of the composite*

$$g \xrightarrow{j} gl_1\mathbb{S} \xrightarrow{gl_1\eta_R} gl_1R$$

and the space of E_∞ ring maps $MG \to R$, where MG is the Thom spectrum of the stable spherical bundle classified by J.

Proof. Applying the mapping space functor $E_\infty(-, R)$ to the pushout diagram in the Lemma, we have a pullback diagram of mapping spaces:

$$\begin{tikzcd} E_\infty(\Sigma^\infty_+ GL_1\mathbb{S}, R) & E_\infty(\Sigma^\infty_+\Omega^\infty gl_1\mathbb{S}/g, R) \arrow[l] \\ E_\infty(\mathbb{S}, R) \arrow[u] & E_\infty(MG, R). \arrow[l] \arrow[u] \end{tikzcd}$$

We can reidentify each of the three terms to get

$$\begin{tikzcd} \mathsf{Spectra}(gl_1\mathbb{S}, gl_1R) & \mathsf{Spectra}(gl_1\mathbb{S}/g, gl_1R) \arrow[l] \\ \{gl_1\eta_R\} \arrow[u] & E_\infty(MG, R), \arrow[l] \arrow[u] \end{tikzcd}$$

hence $E_\infty(MG, R)$ appears as the fiber at $gl_1\eta_R$ of the restriction map, which coincides with the space of nullhomotopies as claimed. $\square$

Question: Is it helpful to observe that $\mathbb{C}P^\infty$ rationally *multiplicatively* generates bSU, or something like that?

There is the following important logarithm utility calculation:

Theorem A.1.4 ([AHR, Theorem 4.11])**.** *Let R be an E_∞ ring spectrum satisfying $R = L_dR$, and set F to be the fiber*

$$F \to gl_1R \to L_n gl_1R.$$

Then π_*F is torsion and F satisfies the coconnectivity condition $F \simeq F(-\infty, d]$. $\square$

from which it follows that $gl_1 KO_p \to KO_p$ is a 1–connected map and $gl_1 tmf_p \to tmf_p$ is a 2–connected map. So, for the purposes of studying *String* orientations, it is safe to follow the logarithm and consider its target as the spectrum under $gl_1\mathbb{S}$:

Theorem A.1.5 (Unpublished work of Hopkins and Lurie)**.** *Let F_d denote the discrepancy spectrum for E_d. There is a natural equivalence of infinite loopspaces $\Omega^\infty F_d \simeq \Sigma^d \mathbb{I}_{\mathbb{Q}/\mathbb{Z}}$.* □

Example 5.6.6 is an inspiration for considering *tmf* as well.

A.1.1 Other things to mention back here

The *tmf* outline is to work by chromatic fracture, comparing $tmf = L_2 tmf$ to its K–localizations. There are no maps from $bString = kO[8, \infty)$ into $L_{K(2)} tmf$, since $L_{K(2)} kO[8, \infty) = L_{K(2)} kO = 0$. So, all of the work is in the $K(1)$–local case, either into $L_{K(1)} tmf$ or $L_{K(1)} L_{K(2)} tmf$. One of these is supposed to look like the p–adic K–theory case (though I'm not sure exactly how) (cf. the introduction to the lesser known Ando–Hopkins–Rezk, where they talk about mapping into a K–algebra). The other is supposed to get involved through power operations and the logarithm. The algebraic picture is supposed to be Katz's monodromy of the p–adic moduli of elliptic curves, though it is also not clear to me how.

The "moments of measures" picture is pretty nice inside of stable homotopy theory, and it has a nice pedigree (as outlined in the lesser-known AHR). There they also say that the same picture is crucial in the classical analysis of p–adic modular forms, which is worth understanding and describing too.

Haynes's *Universal Bernoulli Numbers* paper is about describing the effect of the S^1–transfer in the MU–Adams spectral sequence. The notion is that any convergent spectral sequence can be used to calculate it, and inside of a complex-orientable setting (as for a connected integral complex-orientable spectrum), one can get control over the basics of the Adams spectral sequence by understanding the coaction of MU_*MU on $MU_*\mathbb{C}P^\infty_{-\infty}$, which is the cofiber of some other (related) transfer. Unsurprisingly, given all the references, he ends up playing a similar trick to what appears in AHR: he rationalizes MU and studies the difference series $x / \exp_{MU}(x)$ (which specializes in the case of $MU \to KU$ to $x/(1 - e^{-x})$, which is supposed to be where the original Bernoulli numbers appear in the ABS orientation arise).

The Hattori–Stong theorem states that $MU_* \to K_*MU$ has image a direct summand. Maybe this also deserves mention. (Baker claims in *Combinatorial and Arithmetic Identities Based on Formal Group Laws* that it has a direct algebraic proof given by Araki in *Typical formal groups in complex cobordism and K-theory*.)

Theorem A.1.6. *The integral K–theory cooperations $K_0 K$ are parametrized by rational Laurent polynomials that induce continuous functions $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ for every p.*

Proof. Consider the p–local homotopy pullback square

$$\begin{array}{ccc} (K \wedge K)_{(p)} & \longrightarrow & (K \wedge K)^{\wedge}_p \\ \downarrow & & \downarrow \\ (K \wedge K) \otimes \mathbb{Q} & \longrightarrow & (K \wedge K)^{\wedge}_p \otimes \mathbb{Q}. \end{array}$$

We can identify the homotopy of the three nodes in the cospan as follows. First, $(K \wedge K) \otimes \mathbb{Q}$ is equivalent to $H\mathbb{Q}P \wedge H\mathbb{Q}P$, which has $\pi_0(H\mathbb{Q}P \wedge H\mathbb{Q}P) \cong \mathbb{Q}[x^{\pm}]$. Second, $(K \wedge K)^{\wedge}_p = L_{\widehat{\mathbb{G}}_m}(E_{\widehat{\mathbb{G}}_m} \wedge E_{\widehat{\mathbb{G}}_m})$ is a ring of continuous cooperations for a Morava E–theory, so that we have a calculation

$$\pi_0 L_{\widehat{\mathbb{G}}_m}(E_{\widehat{\mathbb{G}}_m} \wedge E_{\widehat{\mathbb{G}}_m}) = \mathcal{O}_{\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_m} = \mathcal{O}_{\underline{(\mathrm{Aut}\,\widehat{\mathbb{G}}_m)(\mathbb{F}_p)}} = \mathsf{TopologicalGroups}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

Their common target is $\mathsf{TopologicalGroups}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \otimes \mathbb{Q}$, into which they both include. Letting p range yields the result. $\square$

Corollary A.1.7. *The set of such functions $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ found in K_0K are dense in all continuous functions $\mathbb{Z}_p^\times \to \mathbb{Z}_p$.* $\square$

Now consider the map $K[2k, \infty) \wedge K$. Rationally, this induces the inclusion $x^k\mathbb{Q}[x] \to \mathbb{Q}[x^{\pm}]$, but p–adically we have the equivalence

$$L_{\widehat{\mathbb{G}}_m}(K[2n, \infty) \wedge K) \simeq L_{\widehat{\mathbb{G}}_m}(E_{\widehat{\mathbb{G}}_m} \wedge E_{\widehat{\mathbb{G}}_m}),$$

which shows that polynomials satisfying the above condition which are *also* highly x–divisible are *still* dense in the set of all continuous functions.

The operations on p–adic K–theory are dual to this story:

$$E^0_{\widehat{\mathbb{G}}_m} E_{\widehat{\mathbb{G}}_m} \cong \mathsf{TopologicalModules}_{\mathbb{Z}_p}(\mathsf{TopologicalGroups}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p).$$

The right-hand set we think of as measures on $\mathbb{Z}_p$–valued functions with domain $\mathbb{Z}_p^\times$, so that we can combine the two to get

$$\int_{\mathbb{Z}_p^\times} (f \in E^\vee_{\widehat{\mathbb{G}}_m} E_{\widehat{\mathbb{G}}_m}) d(\alpha \in E^0_{\widehat{\mathbb{G}}_m} E_{\widehat{\mathbb{G}}_m}) \mapsto \alpha(f).$$

This is the Kronecker pairing. The Adams operation ψ^λ corresponds to the Dirac measure of evaluating at λ. The Bott element $w^n \otimes w^{-n} \in \pi_0 K \wedge K$ pairs with the Dirac measure to give

$$\int_{\mathbb{Z}_p^\times} (w^n \otimes w^{-n}) d\psi^\lambda = \frac{\psi^\lambda(w^n)}{w^n} = \lambda^n,$$

so that $(w^n \otimes w^{-n})$ corresponds to $(\lambda \mapsto \lambda^n) \in \mathsf{TopologicalGroups}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$.

Theorem A.1.8. *The assignment*

$$\begin{aligned} \mathsf{TopologicalModules}_{\mathbb{Z}_p}(\mathsf{TopologicalGroups}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p) &\to \prod_{n=-\infty}^{\infty} \mathbb{Q}_p, \\ \alpha &\mapsto \left(\int_{\mathbb{Z}_p^\times} x^n d\alpha \right)_{n=-\infty}^{\infty} \end{aligned}$$

is an injection. Its image is the set of sequences satisfying the generalized Kümmer congruences, i.e., plugging them into a numerical polynomial gives integral output.

Proof. That this map is valued in Kümmer sequences is easy: integrating a numerical polynomial is the same as doing the substitution in the Kümmer condition, and by assumption integrating against $d\alpha$ gives integral output. Conversely, being able to define a measure's behavior on numerical polynomials is enough to recover the entire measure, since the functions represented by numerical polynomials is dense in all continuous functions. □

If $\{z_n\}_{n=N}^{\infty}$ is a truncated Kümmer sequence, the rest of the sequence can be recovered by the formula

$$z_d = \lim_{r \to \infty} z_{d+(p-1)p^r}.$$

Theorem A.1.9 (Mazur, [AHR, Section 10.2])**.** *For any $c \in \mathbb{Z}_p^\times$, there is a p–adic measure μ_c satisfying*

$$\int_{\mathbb{Z}_p^\times} x^{n \geq 1} d\mu_c = -\frac{B_n}{n}(1 - p^{n-1})(1 - c^n), \qquad \int_{\mathbb{Z}_p^\times} d\mu_c = \frac{1}{p} \log c^{p-1}. \quad \square$$

If $E_\infty(MG, R)$ is nonempty, then it is a torsor for $E_\infty(\Sigma_+^\infty BG, R) = \mathsf{Spectra}(bg, gl_1 R)$.

Theorem A.1.10. *For R an L_d–local E_∞ ring spectrum, the natural map*

$$gl_1 R \to L_n gl_1 R$$

has d–coconnective cofiber: $\pi_{* \geq d} F = 0$. □

Lemma A.1.11. *The natural map $L_1 gl_1 KO_p \to \widehat{L}_1 gl_1 KO_p$ is a connective equivalence.*

Proof. This map appears in the chromatic fracture square for $L_1 gl_1 KO_p$, which we draw below, incorporating the logarithm:

$$\begin{tikzcd}
gl_1 KO_p \arrow[dr] & & & & KO_p \arrow[dd] \\
& L_1 gl_1 KO_p \arrow[rr] \arrow[dd] & & \widehat{L}_1 gl_1 KO_p \arrow[ur, "\ell_1"] \arrow[dd] & \\
& & L_0 KO_p[4,\infty) \arrow[rr] & & L_0 KO_p \\
& L_0 gl_1 KO_p \arrow[rr] \arrow[ur, "\ell_0"] & & L_0 \widehat{L}_1 gl_1 KO_p. \arrow[ur, "\ell_1"] &
\end{tikzcd}$$

The behavior of the back horizontal map is determined by Rezk's formula for the logarithm. **It acts by some nonzero number in every positive degree,** hence the fiber has the form $\prod_{k=-\infty}^{0} \Sigma^{4k-1} H\mathbb{Q}$. Since the front face is a fiber square, this is also a calculation of the fiber of the map in the Lemma statement. $\square$

Devinatz–Hopkins's continuous fixed points paper shows $E_n^\vee E_n^{hG}$ for a closed subgroup $G \leq \widehat{\mathbb{G}}_n$ gives

$$E_n^\vee E_n^{hG} = \mathsf{Cts}(\widehat{\mathbb{G}}_n, E_{n*})^G$$

by approximating it by $U_j G$ for some sequence of open subgroups U_j with $\bigcap_j U_j = 1$. In the case $n = 1$ and $G = \{\pm 1\}$, I think this is approximating G by things that have a dense $\mathbb{Z}$ inside them, and homotopy fixed points against $\mathbb{Z}$ is something that commutes with co/limits, and then you pray that this matches the original thing. From there, you can use continuous duality and a second fixed point sequence to get what you want. This seems really excessive, though—I really imagine there's a slicker way.

Allen pointed out that the formula for the logarithm in other degrees comes from thinking of the logarithm as a natural transformation and applying it to the mapping set

$$\ell\colon\ gl_1 KO^0(S^{2n}) \to KO^0(S^{2n}).$$

In particular, they compute $\ell\colon\ x \mapsto (1 - p^{n-1})(x)$.

A.1.2 Jeremy's talk

Analogy between classical $\mathbb{Z}[-]$, $GL_1(-)$ and $\mathbb{S}[-]$, $GL_1(-)$, $gl_1(-)$.

$\pi_* gl_1 R$ and $\pi_* R$.

MString as a twisted group ring: $f\colon\ X \to \Sigma gl_1 \mathbb{S}$ gives Mf an E_∞ ring with $Mf \to R$ bijective with null-homotopies of $X \to gl_1 R$.

$E_\infty(Mf, R) \neq \emptyset$ implies it's a torsor for $[\Sigma X, \Sigma gl_1 R]$.

Exercise: for $F\colon\ \mathsf{C} \to \mathsf{D}$ lax monoidal, $\operatorname{colim} F$ is an algebra and $(\operatorname{colim} F) \to A$ are in bijection with lax monoidal lifts $\mathsf{C} \to \mathsf{D}_{/A} \to \mathsf{D}$.

Prop: A *String*–orientation of *tmf* exists iff $String \to gl_1 tmf$ is null, and the collection of such is then a torsor over $[bString, gl_1 tmf]$. Similarly for *Spin* and *kO*.

Prop: If R is an L_n–local E_∞ ring spectrum, then $gl_1 R[n+2,\infty) \xrightarrow{\simeq} L_n gl_1 R[n+2,\infty)$. The connectivities of *Spin* and *String* let us replace $gl_1 KO$ and $gl_1 tmf$ by $L_1 gl_1 KO$ and $L_2 gl_1 tmf$. Chromatic fracture then reduces to $L_{K(0,1)} gl_1 KO$ and $L_{K(0,1.2)} gl_1 tmf$.

Prop (Kuhn's telescoping localization survey): $\Phi_n\colon \mathsf{Spaces}_* \to \mathsf{Spectra}$ commutes with finite limits, insensitive to upward truncation, and $\Phi_n(\Omega^\infty X) = L_{K(n)} X$. This gives a map $(gl_1 R \to) L_{K(n)} gl_1 R \xrightarrow{\simeq} L_{K(n)} R$, called the Rezk logarithm.

The skew in the logarithmic fracture square.

Thm: For R $K(1)$–local and E_∞ with $\pi_0 R$ torsion–free, then $\ell\colon \pi_0 R^\times \to \pi_0 R$ is

$$\ell(x) = \frac{1}{p} \log\left(\frac{x^p}{\psi x}\right) = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k} \left(\frac{\theta(x)}{x^p}\right)^k.$$

This formula comes from the "total transfer":

$$(\Sigma^\infty_+ B\Sigma_k \to) \Sigma^\infty \Omega^\infty \mathbb{S} \to \mathbb{S} \to R,$$

as well as a correspondence between additive transfers on $gl_1 R$ with power operations on R:

$$\begin{tikzcd} \Omega^\infty \mathbb{S} \arrow[rr, bend right, dotted] \arrow[rrr, bend right, dotted, "P(x)"'] & S^0 \arrow[l] \arrow[r, "1+x"] & \Omega^\infty gl_1 R \arrow[r] & \Omega^\infty R. \end{tikzcd}$$

Capping with a homology class in $R_0^\wedge(\Omega^\infty \mathbb{S})$ brings us back down to $R^0(\mathbb{S})$, and the logarithm is given by capping with one fixed class: $\lambda_R \in R_0^\wedge(\Omega^\infty \mathbb{S})$, which is the Hurewicz image of $\lambda_{\mathbb{S}} \in \pi_0 L_{K(n)} \Sigma^\infty \Omega^\infty \mathbb{S}$, which comes from $\Omega^\infty \mathbb{S} \to \Omega^\infty \Sigma^\infty \Omega^\infty \mathbb{S}$.

Thm: For $\tau\colon R_0^\wedge \Omega^\infty \mathbb{S} \to R_0^\wedge \mathbb{S}$ and $\circ\colon (R_0^\wedge \Omega^\infty \mathbb{S})^{\widehat{\otimes} 2} \to R_0^\wedge \Omega^\infty \mathbb{S}$, λ_R is uniquely characterized by $\tau(\lambda_R) = 1$ and $x \circ \lambda_R = \tau(x) \cdot \lambda_R$.

Rezk then guesses a formula in the case $R = E_n$. In the special case $n = 1$, you can get an arbitrary $K(1)$–local R by the fiber sequence

$$L_{K(1)} \Sigma^\infty \Omega^\infty \mathbb{S} \to L_{K(1)}(KU \wedge \Sigma^\infty \Omega^\infty \mathbb{S}) \xrightarrow{\psi^\ell - 1} L_{K(1)}(KU \wedge \Sigma^\infty \Omega^\infty \mathbb{S}).$$

(Looking back on this, it's not clear how this helps. Is there supposed to be an R in here?)

Finally, the formula (for $* > 0$) gives an equivalence $gl_1 KU_p^\wedge[3,\infty) \to KU_p^\wedge[3,\infty)$, which was known nonconstructively previously by Adams–Priddy.

Jeremy told me to think about the S^1–transfer business this way: start by thinking of $\mathbb{C}P^\infty_+$ as the homotopy orbits of a point with trivial S^1–action, so that $\mathbb{S}^0$ is the homotopy orbits of ES^1. The transfer is an equivariant stable map $pt \to ES^1_+$ which is dual to the restriction map $ES^1_+ \to pt$. Calculating the cofiber of the transfer is equivalent to calculating the fiber of the restriction, dualizing, and taking homotopy orbits. Since we're stable, calculating the fiber of the restriction is equivalent to calculating the cofiber, which is geometrically-visibly a representation sphere. The homotopy orbits of that are the colimit over BS^1 of a certain spherical bundle, i.e., a certain Thom spectrum.

A.1.3 Juvitop talk sketch

These talk notes are meant to pick up from Jeremy's Juvitop talk, where he introduced the functor gl_1 and the Rezk logarithm.

We are going to study $E_\infty(MSpin, KO)$. At full strength this proceeds via arithmetic fracture, but for sanity's sake we are going to consider just the chromatic fracture square:

$$\begin{tikzcd}
MSpin \arrow[r] & KO_{(p)} \arrow[r] \arrow[d] & KO_p \arrow[d] \\
& \mathbb{Q} \otimes KO \arrow[r] & \mathbb{Q} \otimes KO_p.
\end{tikzcd}$$

We seek a pair of E_∞ ring maps into the rationalization and the finite completion of KO which agree on the adèles.

Rational orientations

We begin with the two rational nodes in the pullback diagram. As a first approximation to our goal, consider the problem of giving a complex orientation $MU \to \mathbb{Q} \otimes R$ of a rational ring spectrum $\mathbb{Q} \otimes R$. There is an automatic such orientation granted by

$$\begin{tikzcd}
MU \arrow[r, dotted, "D"] \arrow[dr] & \mathbb{Q} \otimes R \\
\mathbb{S} \arrow[u] \arrow[r] \arrow[ur] & H\mathbb{Q}. \arrow[u]
\end{tikzcd}$$

By Jeremy's talk, when $E_\infty(MU, T)$ is nonempty it is a torsor for $[bu, gl_1 T]$, and since we have a preferred orientation D we thus have isomorphisms

$$\pi_0 E_\infty(MU, \mathbb{Q} \otimes R) \xleftarrow{\cong} [bu, gl_1 \mathbb{Q} \otimes R] \xleftarrow{\cong} [bu, \mathbb{Q} \otimes gl_1 R] \xrightarrow{\cong} [\mathbb{Q} \otimes bu, \mathbb{Q} \otimes gl_1 R],$$

the last of which is specified by a sequence of rational numbers $(t_{2k})_{k \geq 1}$. The role played by the sequence (t_{2k}) is to perturb the Thom class.

Proposition A.1.12. *Write x for the Thom class of $\mathcal{L}$ on $\mathbb{C}\mathrm{P}^\infty$ in $\mathbb{Q} \otimes R$–cohomology as furnished by the dumb orientation D. The Thom class associated to some other orientation of $\mathbb{Q} \otimes R$ is tracked by a difference series $x/\exp_F(x)$, and the sequence (t_k) above is expressed by $x/\exp_F(x) = \exp(\sum_k t_k/k! \cdot x^k)$.*

This is confused.

Proof. Let $v^k \colon S^{2k} \to BU$ be the k^{th} power of the class $\mathcal{L}$, so that it comes from a restriction

$$S^{2k} \to (\mathbb{C}\mathrm{P}^\infty)^{\wedge k} \xrightarrow{\mathcal{L}^{\boxtimes k}} BU.$$

The Thom class for this bundle comes from the top Chern class, which is the top coefficient in the product of total Chern classes applied to the individual bundles. Following the usual formulas shows the map v^k to behave on homotopy by multiplication by $(-1)^k t_k$. $\square$

Now we move away from MU. There are three directions to be concerned about: connective orientations, real orientations, and non-complex targets.

1. Rationally, the analysis of Ando–Hopkins–Strickland identifies $[BU\langle 2k\rangle, \mathbb{Q} \otimes R]$ with k–variate symmetric multiplicative 2–cocycles over R, every one of which arises as δ^1 repeatedly applied to a univariate series. In homotopy theoretic terms, this means that every $MU\langle 2k\rangle$–orientation of a rational spectrum factors through an MU–orientation.

2. The cofiber sequence $kO \to kU \to \Sigma^2 kO$ splits rationally, using the idempotents $\frac{1\pm\chi}{2}$ on kU. Accordingly, MU–orientations of rational spectra that factor through MSO–orientations have an invariance property under χ: $-[-1](x) = x$, corresponding to the idempotent factor $+$. This pattern continues for the characteristic series of connective orientations.

3. This same cofiber sequence and idempotent splitting also tells us that rational KU–cohomology classes in the image of KO–cohomology are χ–invariant, i.e., they belong to the $-$ factor.

Our main example is the usual orientation $MU \to KU$ that selects the formal group law $x + y - xy$. This is associated to the difference Thom class $x/(e^x - 1) = x/\exp_{\widehat{\mathbb{G}}_m}(x)$. To make this difference $[-1]$–invariant (and hence give a complex-orientation of KO), we use the averaged exponential class $(e^{x/2} - 1) - (e^{-x/2} - 1)$.[2] In turn, we use the Proposition to calculate the behavior on homotopy of the associated orientation:[3]

$$\frac{x}{e^{x/2} - e^{-x/2}} = \exp\left(-\sum_{k=2}^{\infty} \frac{B_k}{k} \cdot \frac{x^k}{k!}\right).$$

Finally, we calculate the effect of the orientation on the second half of the factorization

$$MSU \to MSpin \to KO,$$

[2]Incidentally, this is equal to $2\sinh(x/2)$.

[3]This comes out of applying $d\log$ to the fraction.

again using the relevant idempotent, which has the effect of halving the coefficients in the characteristic series: $-\frac{B_k}{2k}$.[4]

This discussion accounts for both $E_\infty(MSpin, \mathbb{Q} \otimes KO)$ and $E_\infty(MSpin, \mathbb{Q} \otimes KO_p)$: the set of rational characteristic series includes into the set of adèlic characteristic series as the subset with rational coefficients.

Finite place orientations

We want now to understand $E_\infty(MSpin, KO_p)$ and its map to $E_\infty(MSpin, \mathbb{Q} \otimes KO_p)$. Here's the initial set-up:

$$\begin{tikzcd}
spin \arrow[r, "j" description] & gl_1\mathbb{S} \arrow[r] \arrow[dr, "gl_1\eta_R"'] & Cj \arrow[d, dotted, "A"'] \\
& & gl_1 KO_p.
\end{tikzcd}$$

We are looking to understand the space of filler diagrams A (i.e., vertical maps with choice of homotopy of the precomposite to $gl_1\eta_{KO_p}$). Notice first that there is a natural cofiber sequence to be placed on the bottom row:

$$\begin{tikzcd}
spin \arrow[r, "j" description] \arrow[dr, red] & gl_1\mathbb{S} \arrow[r] \arrow[d, red] \arrow[dr, "gl_1\eta_{KO_p}"'] & Cj \arrow[d, dotted, "A"'] & \\
& \Sigma^{-1}\mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p \arrow[r] & gl_1 KO_p \arrow[r] & \mathbb{Q} \otimes gl_1 KO_p \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p.
\end{tikzcd}$$

There is a canonical red vertical lift of $gl_1\eta_{KO_p}$ since $gl_1\mathbb{S}$ is a torsion spectrum, and this precomposes with j to give another vertical map. Notice now that selecting a filler triangle A gives a commuting square with choice of homotopy and that $[gl_1\mathbb{S}, \mathbb{Q} \otimes gl_1 KO_p] = 0$, and hence we would get a natural map (and natural homotopy) off of the homotopy cofibers:

$$\begin{tikzcd}
spin \arrow[r, "j" description] \arrow[dr] & gl_1\mathbb{S} \arrow[r] \arrow[d] \arrow[dr, "gl_1\eta_{KO_p}"'] & Cj \arrow[r] \arrow[d, dotted, "A"', "B"] & bspin \arrow[r] \arrow[d, dotted, "C"] \arrow[dr] & bgl_1\mathbb{S} \arrow[d] \\
& \Sigma^{-1}\mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p \arrow[r] & gl_1 KO_p \arrow[r] & \mathbb{Q} \otimes gl_1 KO_p \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p,
\end{tikzcd}$$

[4]While we're here, you might want to observe that elements in $[bu, gl_1 R]$ push forward to elements in $[bu, gl_1\mathbb{Q} \otimes R]$ which do not disturb the denominators of the elements t_k. (On the other hand, the "Miller invariant" associated to a rational ring spectrum is *zero*, because arbitrary elements in $[bu, gl_1\mathbb{Q} \otimes R]$ can completely destroy the denominators.)

where C is a map making the triangle it belongs to commute. This all gives a function assigning A to B and A to C (and, in fact, the latter assignment factors through the former).

In order to show nonconstructively that the set of As is nonempty, we might try to compute

$$\begin{aligned} gl_1\eta_{KO_p} \circ j \in [spin, gl_1KO_p] &= [spin, L_1 gl_1 KO_p] && \text{(using discrepancy spectrum Lemm} \\ &= [spin, L_{K(1)} gl_1 KO_p] && \text{(controlling lower chromatic (i.e., rational) layer} \\ &= [\Sigma^{-1}KO_p, L_{K(1)} gl_1 KO_p] && (spin \to \Sigma^{-1}KO_p \text{ models } L_{K(1} \\ &= [\Sigma^{-1}KO_p, KO_p] && \text{(the Rezk logarithm} \end{aligned}$$

We claim also that the kernel of the map $A \mapsto C$ is easy to understand: two fillers A are related by an element of $[bspin, gl_1KO_p]$, and their corresponding Cs are related by the corresponding element of $[bspin, \mathbb{Q} \otimes gl_1KO_p]$. This set is rational, hence factors through the rationalization of $[bspin, gl_1KO_p]$ where it must already be null, and hence it is a torsion element of $[bspin, gl_1KO_p]$. Meanwhile, the same argument as above identifies

$$[bspin, gl_1KO_p] = [KO_p, KO_p].$$

The map C is the rational orientation induced by postcomposition, so it describes the map into the adèlic component. We would like to understand the behavior of C on homotopy based on some data about A. First notice that we can postcompose B with the localization map off of gl_1KO_p as follows:[5]

$$\begin{tikzcd}
 & & & L_{K(1)}Cj \arrow[rr] \arrow[dd, dotted, "B'"] & & L_{K(1)}bspin & \\
spin \arrow[r, "j"] \arrow[ddr] & gl_1\mathbb{S} \arrow[r] \arrow[dd] \arrow[ddr, "gl_1\eta_{KO_p}"'] & Cj \arrow[ur] \arrow[rr] \arrow[dd, dotted, "A"', "B"] & & bspin \arrow[ur] \arrow[rr] \arrow[dd, dotted, "C"] \arrow[ddrr, bend left] & & bgl_1\mathbb{S} \arrow[dd] \\
 & & & L_{K(1)}gl_1KO_p \arrow[rr, "C"] & & \mathbb{Q} \otimes L_{K(1)}gl_1KO_p & \\
 & \Sigma^{-1}\mathbb{Q}/\mathbb{Z} \otimes gl_1KO_p \arrow[r] & gl_1KO_p \arrow[ur] \arrow[rr] & & \mathbb{Q} \otimes gl_1KO_p \arrow[ur] \arrow[rr] & & \mathbb{Q}/\mathbb{Z} \otimes gl_1
\end{tikzcd}$$

This gives a new map B'. Now we are in a position to compute. Given a homotopy class in $\pi_* bspin$, we want to understand the action of C on it. We push it forward to $L_{K(1)}bspin \simeq KO_p$, pull it back to $L_{K(1)}Cj \simeq KO_p$ along $KO_p \xrightarrow{1-\psi^c} KO_p$, push it down along B' to $L_{K(1)}gl_1KO_p \simeq KO_p$, include it into the rational component of $\mathbb{Q} \otimes \mathcal{L}_{K(1)}gl_1KO_p$,

[5]Importantly, and differently from what every source says, this isn't a map of cofiber sequences and so the back second vertical map does not have to exist.

and pull it back to $\mathbb{Q} \otimes gl_1 KO_p$ along the logarithmic map. The effect of this sequence of steps is

$$t_{4k}(C) = (1 - c^k)^{-1} b_{4k}(B')(1 - p^{k-1})^{-1},$$

where b_k is the sequence of effects on homotopy of the map $B' \colon KO_p \to KO_p$.

Now, finally, the diagonal map $bspin \to \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p$ becomes relevant. To check the commutativity of the triangle with C, we need only compare the results of the composite on homotopy since the map C targets a rational spectrum and hence is determined its effect on homotopy. That effect is kind of hard to compute: a priori, it amounts to understanding something about the stable S^1–transfer, but a posteriori you can cheat and use the following theorem.

Theorem A.1.13. *If $f \colon bspin \to gl_1(\mathbb{Q} \otimes R)$ comes from an integral lift, then the following two composites are equal:*

$$bspin \to bgl_1 \mathbb{S} \xleftarrow{\simeq} \mathbb{Q}/\mathbb{Z} \otimes gl_1 \mathbb{S} \to \mathbb{Q}/\mathbb{Z} \otimes gl_1 R,$$

$$\begin{tikzcd}
bspin \arrow[d, "f"] \arrow[dr, "\exists!"] \arrow[drrrr] & & & & \\
gl_1(\mathbb{Q} \otimes R) & gl_1(\mathbb{Q} \otimes R)[4,\infty) \arrow[l] \arrow[r, "\simeq"] & \mathbb{Q} \otimes R[4,\infty) & \mathbb{Q} \otimes gl_1 R[4,\infty) \arrow[l, "\simeq"'] \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes gl_1 R[4,\infty
\end{tikzcd}$$

We already have in mind a preferred map $f \colon bspin \to gl_1 \mathbb{Q} \otimes KO_p$ from the previous section, and so we're going to cheat and use it to conclude the behavior of the Miller invariant in stable homotopy from its characteristic series.

We thus identify the legal fillers C as those sequences of rational numbers $t_k(C)$ satisfying conditions:

1. For $4 \nmid k$, $t_k = 0$.
2. t_{4k} has the correct denominators: for $k \geq 1$, $t_{4k} \equiv -B_k/(2k) \pmod{\mathbb{Z}}$.
3. b_{4k} is the effect on homotopy of some map $B' \colon KO_p \to KO_p$.

Stable *KO* operations

We have identified three points where we want to understand the collection of stable KO operations. The easy calculation is $K^\vee K = \operatorname{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, which comes out of Landweber flat stable cooperations: E_Γ has cooperations given by the ring of functions on the pro-étale group scheme $\operatorname{Aut}\Gamma$, but at $\Gamma = \widehat{\mathbb{G}}_m$ this group scheme is constant at $\mathbb{Z}_p^\times$ (and hence $K^\vee K$ is the ring of $\mathbb{Z}_p$–valued functions on $\mathbb{Z}_p^\times$). Turning to cohomology, it follows by the universal coefficient spectral sequence that $K^0 K = \operatorname{Hom}(\operatorname{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p)$. We have some claims about how these isomorphisms behave:

1. The Kronecker pairing
$$\mathbb{S}^0 \xrightarrow{c} K \wedge K \xrightarrow{1 \wedge f} K \wedge K \xrightarrow{\mu} K$$
is computed by the evaluation pairing
$$(c \in K^\vee K, f \in K^0 K) \mapsto f(c).$$

2. The stable operation ψ^λ attached to $[\lambda] \in \operatorname{Aut} \widehat{\mathbb{G}}_m$ is evaluation at λ.

3. The stable cooperation $v^{-k} \wedge v^k \in \pi_0 K \wedge K$ corresponds to the polynomial function $x \mapsto x^k$, as justified by the computation
$$\mathrm{ev}_\lambda(v^{-k} \wedge v^k) = \frac{\psi^\lambda v^k}{v^k} = \frac{\lambda^k v^k}{v^k} = \lambda^k.$$

These last two facts mean that the question of what a stable operation does on homotopy is identical to the value a function f takes on the standard stable cooperations $v^{-k} \wedge v^k$, via the Kronecker pairing. We give a name to this:

Proposition A.1.14. $\operatorname{Hom}(\operatorname{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p) \xrightarrow{(f(x \mapsto x^k))_k} \prod_{k \geq N} \mathbb{Z}_p$ *is injective. A sequence* (x_k) *is said to be a* Kümmer sequence *when it lies in this image.*[6]

An interesting feature of the Proposition is the auxiliary index N. In p–adic geometry, this is reflected by the p–adic convergence of the sequence
$$d + (p-1)p^r \xrightarrow{r \to \infty} d,$$
and hence the continuous reconstruction property
$$x_d = \lim_{r \to \infty} x_{d+(p-1)p^r}.$$
Meanwhile, topologically, we have $K \wedge K[2k, \infty) \simeq K \wedge K$.

To get KO into this picture, use the Tate trick
$$K \wedge KO \simeq K \wedge (K^{hC_2}) \simeq K \wedge (K_{hC_2}) \simeq (K \wedge K)_{hC_2} \simeq (K \wedge K)^{hC_2},$$
so that $\pi_0 K \wedge KO = \operatorname{cts}(\mathbb{Z}_p^\times / C_2, \mathbb{Z}_p)$. Taking fixed points again, $KO \wedge KO$ is valued in the same valued in KO_*, and $KO^* KO$ is the linear dual. It follows that $[\Sigma^{-1} KO, KO] = 0$ and that $[KO, KO] = \operatorname{Hom}(\operatorname{cts}(\mathbb{Z}_p^\times / C_2, \mathbb{Z}_p), \mathbb{Z}_p)$ is torsion-free, two of our outstanding claims.

[6] You can be a little more explicit about this: for all $h(x) = \sum_{k=n}^N a_k x^k \in A(n)$ we have $\sum_{k=n}^N a_k x_k \in \mathbb{Z}_p$.

Mazur's construction of Kubota–Leopoldt p–adic L–functions

Now need to show that there exist sequences of p–adic integers satisfying the criteria from the preceding summary.

Theorem A.1.15 (Mazur). *For any auxiliary $c \in \mathbb{Z}_p^\times$, there is a homomorphism f_c such that $f_c(x^{k\geq 1}) = \frac{-B_k}{k}(1-p^{k-1})(1-c^k)$ (and $\int_{\mathbb{Z}_p^\times} d\mu_c = \frac{1}{p}\log(c^{p-1})$).*[7] *(With considerable effort, this can be halved.)*

You might wonder why Mazur had already proven exactly the Theorem we needed. To understand his program, recall these two facts about ζ:

1. ζ is basically the Mellin transform of the measure $\frac{dx}{e^x-1}$ (i.e., its sequence of moments):

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{dx}{e^x - 1}$$

(except for the real Euler factor $\Gamma(s)$).

2. For any $k \in \mathbb{Z}_{>0}$, $\zeta(1-k) = -B_k/k$, where $\frac{t}{e^t-1} = \sum_{k=0}^\infty B_k \frac{t^k}{k!}$.

The idea is to build a p–adic ζ–function by means of similar p–adic integrals that occur as finite approximations to this one. A Bernoulli polynomial for $k \in \mathbb{Z}_{>0}$ is

$$\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} = \frac{te^{tx}}{e^t - 1},$$

and these polynomials beget Bernoulli distributions according to the rule

$$\begin{aligned} \mathbb{Z}/p^n\mathbb{Z} &\xrightarrow{E_k} \mathbb{Q} \subseteq \mathbb{Q}_p \\ x \in [0, p^n) &\mapsto k^{-1}p^{n(k-1)}B_k(xp^{-n}). \end{aligned}$$

A distribution in general is a function on $\mathbb{Z}_p$ such that its value at any node in the p–adic tree is equal to the sum of the values of its immediate children. The p–adic integral of a locally constant function with respect to such a distribution is defined by their convolution. For example, the constant function 1 factors through $\mathbb{Z}/p$, hence

$$\int_{\mathbb{Z}_p} dE_k = \overbrace{\frac{1}{k}\sum_{a=0}^{p-1} B_k\left(\frac{a}{p}\right) = \frac{B_k(0)}{k}}^{\text{non-obvious}} = \frac{B_k}{k}.$$

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Explicitly, $f_c(h) = \int_{\mathbb{Z}_p^\times} h(x)d\mu_c = \lim_{r\to\infty} \frac{1}{p^r} \sum_{\substack{0\leq i<p^r \\ p\nmid i}} \int_i^{ci} \frac{h(t)}{t}dt.$

However, this distribution is not a *measure*, in the sense that it is not bounded and hence does not extend to a functional on all continuous functions (rather than just locally constant ones). The standard fix for this is called *regularization*: pick $c \in \mathbb{Z}$ with $p \nmid c$, and set $E_{k,c}(x) = E_k(x) - c^k E_k(c^{-1}x)$. This is a measure, and for $k \geq 1$ it has

$$\int_{\mathbb{Z}_p} dE_{k,c} = \int_{\mathbb{Z}_p} dE_k - c^k \int_{\mathbb{Z}_p} dE_k(c^{-1}x) = \frac{B_k}{k}(1 - c^k).$$

These measures interrelate: $E_{k,c} = x^{k-1}E_{1,c}$, and hence the single measure $E_{1,c}$ has all of these values as moments. We would like to perform p–adic interpolation in k to remove the restriction $k \geq 1$, but this is not naively possible: if $x = 0$, say, then we have x^{-1}, which will not make sense if $x \in p\mathbb{Z}_p$. This is most easily solved by restricting x to lie in $\mathbb{Z}_p^\times$, which has a predictable effect for $k \in \mathbb{Z}_{>0}$:

$$\int_{\mathbb{Z}_p^\times} x^{k-1} dE_{1,c} = \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} - \int_{p\mathbb{Z}_p} x^{k-1} dE_{1,c} = \int_{\mathbb{Z}_p^\times} x^{k-1} dE_{1,c} - p^{k-1} \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} = \frac{B_k}{k}(1 - c^k)(1 - p$$

Hence, the Mellin transform of the measure $dE_{1,c}$ on $\mathbb{Z}_p^\times$ gives a sort of p–adic interpolation of the ζ–function.

It also has exactly the properties we need to guarantee the existence of an E_∞ orientation $MSpin \to KO$. It is remarkable that the three factors in

$$\int_{\mathbb{Z}_p^\times} x^{k-1} dE_{1,c} = \frac{B_k}{k}(1 - c^k)(1 - p^{k-1})$$

have discernable provenances in the two fields. In stable homotopy theory these arise respectively in the characteristic series of the orientation $MU \to KU$, in the finite Adams resolution for the $K(1)$–local sphere, and in the Rezk logarithm. In p–adic analytic number theory, they arise as the special values of the ζ–function, the regularization to make it a measure, and the restriction to perform p–adic interpolation. It is completely mysterious how or if these operations correspond.

Footnotes on the *tmf* case

The case of the orientation $MString \to tmf$ has all of the same trappings, but its order of complexity is $(-)^{3/2}$ whatever this one was for you, since the height 1 chromatic fracture square gets replaced by the height 2 chromatic fracture cube. Many of the steps remain the same:

1. Begin with a rational orientation, which is basically the Witten genus valued in holomorphic expansions of modular forms.

2. Analyze the homotopy type of $L_{K(1)}tmf$ and compare it to that of KO. This lets us use another universal coefficient theorem to lift our description of KO^*KO as KO^*–valued measures to $L_{K(1)}tmf^*KO$ as $L_{K(1)}tmf^*$–valued measures.

3. The homotopy type of $L_{K(2)}tmf$ is "naively irrelevant" in the chromatic fracture square: maps $bSpin \to L_{K(2)}tmf$ factor through $L_{K(2)}bSpin = L_{K(2)}KO_p = 0$.
4. However, the logarithm's presence in the chromatic fracture square $L_{K(1)}tmf \to L_{K(1)}L_{K(2)}tmf$ has a real effect that must be understood. This is not easy: the height 2 logarithm is not so accessible.
5. You also have to ramp up the algebraic part of the calculation by identifying the analogues of the Mazur moments in $\pi_* L_{K(1)}tmf$. These turn out to be normalized Eisenstein series.

These Bernoulli sequences are *not* the only sequences satisfying these reconstruction properties, but Mike thinks that they may be the only ones with a "reasonable" growth rate (as measured in $\mathbb{R}$). It would also be great if someone could figure this out. It would also be great if this "real place" condition had something to do with a smooth cohomology theory like differential real K–theory.

The transfer map from a dimension n compact Lie group lies in Novikov filtration n, which accounts for why the Miller invariant map $S^1 \wedge \mathbb{C}P^\infty \to u \to gl_1\mathbb{S} \to \Sigma^{-1}\mathbb{Q}/\mathbb{Z} \otimes gl_1 R$ (which is secretly the S^1–transfer) consists of torsion classes. One wonders how the n–torus transfer behaves on stable homotopy, e.g., what classes it selects in $\pi_* gl_1 E_\Gamma$ for Γ a height n formal group. (We might also be interested in the behavior of the lower-height transfers and their values in the Lubin–Tate ring.)

It's not clear to me how the Miller invariant is computed in the $L_{K(1)}tmf$ case. It seems like their game is to use the cover $L_{K(1)}tmf \to K^{\mathrm{Tate}}$ and the classical σ–orientation there, then deduce the effect on homotopy in $L_{K(1)}tmf$ by some kind of injectivity statement. I don't understand how K^{Tate} is already understood to be an E_∞ ring, though, or how the σ–orientation $MU[6,\infty) \to K^{\mathrm{Tate}}$ is understood to be even an A_∞–orientation (which I think you need in order to compute the Miller invariant). There's an interesting accompanying thought: you can probably also construct $K[\![q]\!]$ as an E_∞ ring with underlying homotopy ring spectrum the same as K^{Tate} but which carries an E_∞ MU–orientation that factors as $MU \to K \to K[\![q]\!]$. In turn, its Miller invariant is probably no more complicated than that of K. It's remarkable that the E_∞ ring structure already carries so much number theory internally, without reference to the existence of an E_∞ orientation.

Mike has some open questions about the end of this analysis (and in particular about the fiber of the Atkin map that appears in the $K(1)$–local analysis of *tmf*, an analogue of the chromatic splitting fiber) at the end of his talk notes *The String orientation of tmf*. Some of that should be copied here.

The modularity of the *MString* orientation

E_∞ orientations by *MString*

tmf, *TMF*, and *Tmf* in terms of $\mathcal{M}_{\mathbf{ell}}$

Thom spectra and ∞–categories

The Bousfield–Kuhn functor and the Rezk logarithm free E_∞–orientations off of MU

Section 5.3 of Lurie's Survey talks about a derived algebro-geometric perspective on the σ–orientation.

A.2 Rational phenomena: character theory for Lubin–Tate spectra

There's a sufficient amount of reliance on character theory in Matt's thesis that we should talk about it. You should write that action and then backtrack here to see what you need for it.

See Morava's *Local fields* paper

Remark A.2.1. Theorem 2.6 of Greenlees–Strickland for a nice transchromatic perspective. See also work of Stapleton and Schlank–Stapleton, of course.

Flesh this out.

Theorem A.2.2. *Let E be any complex-oriented cohomology theory. Take G to be a finite group and let* Ab_G *be the full subcategory of the orbit category of G built out of abelian subgroups of G. Finally, let X be a finite G–CW complex. Then, each of the natural maps*

Cite me: Theorem A.

$$E^*(EG \times_G X) \to \lim_{A \in \mathsf{Ab}_G} E^*(EG \times_A X) \to \int_{A \in \mathsf{Ab}_G} E^*(BA \times X^A)$$

becomes an isomorphism after inverting the order of G. In particular, there is an isomorphism

$$\frac{1}{|G|} E^* BG \to \lim_{A \in \mathsf{Ab}_G} \frac{1}{|G|} E^* BA. \quad \square$$

This is an analogue of Artin's theorem:

Theorem A.2.3. *There is an isomorphism*

$$\frac{1}{|G|} R(G) \to \lim_{C \in \mathsf{Cyclic}_G} \frac{1}{|G|} R(C). \quad \square$$

HKR intro material connecting Theorem A to character theory:

Recall that classical characters for finite groups are defined in the following situation: take $L = \mathbb{Q}^{\mathrm{ab}}$ to be the smallest characteristic 0 field containing all roots of unity, and for a finite group G let $Cl(G; L)$ be the ring of class functions on G with values in L. The units in the profinite integers $\widehat{\mathbb{Z}}$ act on L as the Galois group over $\mathbb{Q}$, and since $G = \mathsf{Groups}(\widehat{\mathbb{Z}}, G)$ they also act naturally on G. Together, this gives a conjugation action on $Cl(G; L)$: for $\varphi \in \widehat{\mathbb{Z}}$, $g \in G$, and $\chi \in Cl(G; L)$, one sets

$$(\varphi \cdot \chi)(g) = \varphi(\chi(\varphi^{-1}(g))).$$

The character map is a ring homomorphism

$$\chi : R(G) \to Cl(G;L)^{\hat{\mathbb{Z}}},$$

and this induces isomorphisms

$$\chi : L \otimes R(G) \xrightarrow{\simeq} Cl(G;L)$$

and even

$$\chi : \mathbb{Q} \otimes R(G) \xrightarrow{\simeq} Cl(G;L)^{\hat{\mathbb{Z}}}.$$

Now take $E = E_\Gamma$ to be a Morava E–theory of finite height $d = \mathrm{ht}(\Gamma)$. Take $E^*(B\mathbb{Z}_p^d)$ to be topologized by $B(\mathbb{Z}/p^j)^d$. A character $\alpha : \mathbb{Z}_p^d \to S^1$ will induce a map $\alpha^* : E^*\mathbb{C}\mathrm{P}^\infty \to E^*B\mathbb{Z}_p^d$. We define $L(E^*) = S^{-1}E^*(B\mathbb{Z}_p^d)$, where S is the set of images of a coordinate on $\mathbb{C}\mathrm{P}_E^\infty$ under α^* for nonzero characters α. Note that this ring inherits an $\mathrm{Aut}(\mathbb{Z}_p^d)$ action by E^*–algebra maps.

The analogue of $Cl(G;L)$ will be $Cl_{d,p}(G;L(E^*))$, defined to be the ring of functions $\chi : G_{d,p} \to L(E^*)$ stable under G–orbits. Noting that

$$G_{d,p} = \mathrm{Hom}(\mathbb{Z}_p^d, G),$$

one sees that $\mathrm{Aut}(\mathbb{Z}_p^d)$ acts on $G_{d,p}$ and thus on $Cl_{d,p}(G;L(E^*))$ as a ring of E^*–algebra maps: given $\varphi \in \mathrm{Aut}(\mathbb{Z}_p^d)$, $\alpha \in G_{d,p}$, and $\chi \in Cl_{d,p}(G;L(E^*))$ one lets

$$(\varphi \cdot \chi)(\alpha) = \varphi(\chi(\varphi^{-1}(\alpha))).$$

Now we introduce a finite G–CW complex X. Let

$$\mathrm{Fix}_{d,p}(G,X) = \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)} X^{\mathrm{im}\,\alpha}.$$

This space has commuting actions of G and $\mathrm{Aut}(\mathbb{Z}_p^d)$. We set

$$Cl_{d,p}(G,X;L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{d,p}(G,X))^G,$$

which is again an E^*–algebra acted on by $\mathrm{Aut}(Z_p^d)$. We define the character map "componentwise": a homomorphism $\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)$ induces

$$E^*(EG \times_G X) \to E^*(B\mathbb{Z}_p^d) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}) \to L(E^*) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}).$$

Taking the direct sum over α, this assembles into a map

$$\chi_{d,p}^G : E^*(EG \times_G X) \to Cl_{d,p}(G,X;L(E^*))^{\mathrm{Aut}(Z_p^d)}.$$

Nat taught you how to say all these things with p–adic tori, which was *much* clearer.

Cite me: Theorem C.

Checking this invariant ring claim is easiest done by comparing the functors the two things corepresent.

Theorem A.2.4. *The invariant ring is $L(E^*)^{\mathrm{Aut}(\mathbb{Z}_p^d)} = p^{-1}E^*$, and $L(E^*)$ is faithfully flat over $p^{-1}E^*$. The character map $\chi_{d,p}^G$ induces isomorphisms*

$$\chi_{d,p}^G \colon L(E^*) \otimes_{E^*} E^*(EG \times_G X) \xrightarrow{\simeq} Cl_{d,p}(G, X; L(E^*)),$$
$$\chi_{d,p}^G \colon p^{-1}E^*(EG \times_G X) \xrightarrow{\simeq} Cl_{d,p}(G, X; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}.$$

In particular, when $X = *$, there are isomorphisms

$$\chi_{d,p}^G \colon L(E^*) \otimes_{E^*} E^*(BG) \xrightarrow{\simeq} Cl_{d,p}(G; L(E^*)),$$
$$\chi_{d,p}^G \colon p^{-1}E^*(BG) \xrightarrow{\simeq} Cl_{d,p}(G; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}. \quad \square$$

Jack gives an interpretation of this in terms of formal $\mathcal{O}_L$–modules.

I also have this summary of Nat's of the classical case:

It's not easy to decipher if you weren't there for the conversation, but here's my take on it. First, the map we wrote down today was the non-equivariant chern character: it mapped non-equivariant $KU \otimes \mathbb{Q}$ to non-equivariant $H\mathbb{Q}$, periodified. The first line on Nat's board points out that if you use this map on Borel-equivariant cohomology, you get nothing interesting: $K^0(BG)$ is interesting, but $H\mathbb{Q}^*(BG) = H\mathbb{Q}^*(*)$ collapses for finite G. So, you have to do something more impressive than just directly marry these two constructions to get something interesting.

That bottom row is Nat's suggestion of what "more interesting" could mean. (Not really his, of course, but I don't know who did this first. Chern, I suppose.) For an integer n, there's an evaluation map of (forgive me) topological stacks

$$*//(\mathbb{Z}/n) \times \mathrm{Hom}(*//(\mathbb{Z}/n), *//G) \xrightarrow{\mathrm{ev}} *//G$$

which upon applying a global-equivariant theory like K_G gives

$$K_{\mathbb{Z}/n}(*) \otimes K_G\Big(\coprod_{\text{conjugacy classes of } g \text{ in } G} *\Big) \xleftarrow{ev^*} K_G(*).$$

Now, apply the genuine G-equivariant Chern character to the K_G factor to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) \leftarrow K_{\mathbb{Z}/n}(*) \otimes K_G(\coprod *),$$

where the coproduct is again taken over conjugacy classes in G. Now, compute $K_{\mathbb{Z}/n}(*) = R(\mathbb{Z}/n) = \mathbb{Z}[x]/(x^n - 1)$, and insert this calculation to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) = \mathbb{Q}(\zeta_n) \otimes \Big(\bigoplus_{\text{conjugacy classes}} \mathbb{Q}\Big),$$

where ζ_n is an n^{th} root of unity. As n grows large, this selects sort of the part of the complex numbers $\mathbb{C}$ that the character theory of finite groups cares about, and so following all the composites we've built a map

$$K_G(*) \to \mathbb{C} \otimes \Big(\bigoplus_{\text{conjugacy classes}} \mathbb{C}\Big).$$

The claim, finally, is that this map sends a G-representation (thought of as a point in $K_G(*)$) to its class function decomposition.

—

A.3 The period map

pp. 42-43 of FPFP has some easy-to-state results about this collected.

Kohlhaase [Koh13] references Yu [Yu95] for having closed formulas generalizing the Hopkins–Gross example section.

Section 2.3 (pg. 35–36) of Morava's *Noetherian Localisations* gives a sketch of the period map (in terms of curves, I guess).

Describe Dieudonné crystals and the Tapis de Cartier.

Show how Dieudonné crystals are used to give formulas for the action of the stabilizer group [DH95].

Give a sketch explanation of the Gross–Hopkins period map [Wei11].

Draw the picture of the period map at $n = 2$, $p = 2$. The main reference for this, except for the literal picture, is [HG94a, Appendix 25].

- The center of the $\mathbb{Z}_{p^2}$–points of Lubin–Tate space corresponds to the canonical lift, which is the formal group that further acquires an $\mathcal{O}_A$–module structure. It has π–series $[\pi](x) = \pi x + x^{q^2}$.
- There are three nontrivial points in $\widehat{\mathbb{G}}[2]$: α, β, and $\alpha + \beta$. Quotienting by them gives three points at order $1/(q+1)$, the first bunch of "quasicanonical lifts", which have partial formal $\mathcal{O}_A$–module structures.
- At each quasicanonical point, you also also form three quotients: two of them make the situation "worse", and one of them makes the situation "better". This has to do with the identification $\widehat{\mathbb{G}}/\widehat{\mathbb{G}}[2] \cong \widehat{\mathbb{G}}$.
- Computing these orders has to do with the Newton polygon associated to the π–series.
- The canonical Frobenius $F_{can} = \begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}$ first flips the two coordinates (and scales one by p), then flips them back (and scales the other by p), and after two flips scaling everything by p scales back down by homogeneous coordinates.
- Out to order $1/q$, π_{GH} is injective.
- The group $\mathbb{F}_4^\times$ should act by rotation on $\mathbb{P}^1$.

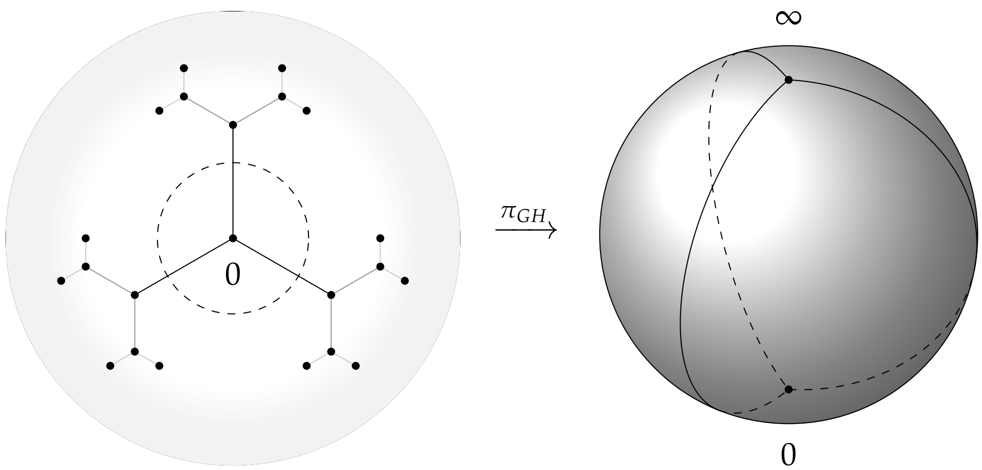

Figure A.1: The period map at $n = 2$, $p = 2$

- The map π_{GH} sends the canonical lift to $0 = [1 : 0]$, sends the first order quasicanonical points to $\infty = [0 : 1]$, and alternates from there. The three branches of "directions to quotient" carve $\mathbb{P}^1$ up into three lobes. This is because π_{GH} is equivariant for *isogenies*, and quotienting by one of these order 2 subgroups is a lift of the Frobenius isogeny on the residue formal group.

- These quasicanonical points are the ones with nontrivial stabilizers under the action by the Morava stabilizer group—all the other points belong to free orbits. (The canonical lift has the largest stabilizer of all.)

Theorem A.3.1. *The sheaf $\mathcal{M}_{E_\Gamma}(\mathbb{I}_{\mathbb{Q}/\mathbb{Z}})$ is the dualizing sheaf on $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$.* □

Cite me: [HG94b, Str00].

Make sure you get this right.

A.4 Knowns and unknowns

Higher orientations

TAF and friends

The $\alpha_{1/1}$ argument: Prop 2.3.2 of Hovey's v_n–elements of ring spectra

Equivariance

This is tied up with the theory of power operations in a way I've never really thought about. Seems complicated.

You should also mention the "rigidity" of the elliptic genus, which is about an S^1–equivariant version.

Index theorems

Connections with analysis

The Stolz–Teichner program

—

Bousfield's work on the K–theory of infinite loopspaces [Bou96] and Morava K–theoretic analogues of the results of Lecture 4.5

Ask Mike (and Jacob?) if there are analogues of these results for kO which explain Mahowald's generalized K–theoretic Brown–Gitler spectra. 3/29: I did ask Mike, he said he didn't know. I also asked Paul, and he said this seemed unreasonable, since kO isn't valued in co/commutative Hopf algebras. This is a fair point: one would need to invent an "analogue" of Dieudonné theory for kO, in the sense that some category it takes values in would have to be identified as abelian, where the category is rigid enough that it often sends fiber sequences to exact sequences in the category.

Constructing sheaves of spectra on $\mathcal{M}_{\mathbf{fg}}$: the no-go results for E_∞ and A_∞ rings on the flat site. There's a little MO discussion about it here: `http://chat.stackexchange.com/transcri`

Contexts for structured ring spectra

Difficulty in computing $\mathbb{S}_d \circlearrowleft E_d^*$. (Gross–Hopkins and the period map.)

Barry's p–adic measures

Fixed point spectra and e.g. $L_{K(2)} tmf$.

Blueshift, A–M–S, and the relationship to A–F–G?

Does E_n receive an E_∞ orientation? Does BP? (Johnson–Noel says BP usually does not. A recent preprint of Lawson says BP is not even E_∞ at $p = 2$!!!)

p–divisible groups and transchromatic phenomena

Remark 12.13 of published H_∞ AHS says their obstruction framework agrees with the E_∞ obstruction framework (if you take everything in sight to have E_∞ structures). This is almost certainly related to the discussion at the end of Matt's thesis about the MU–orientation of E_d.

Section 12.4 compares doing H_∞ descent with doing E_∞ descent and shows that they're the same (in the case of interest?).

Hovey's paper on v_n–periodic elements in ring spectra. He has a nice (and thorough!) exposition on why one should be interested in bordism spectra and their splittings: for instance, a careful analysis of *MSpin* will inexorably lead one toward studying KO. It would be nice if studying *MString* (and potentially higher analogues) would lead one toward non-completed, non-connective versions of EO_n. Talk about BoP, for instance.

Matt's short resolutions of chromatically localized MU.

Nilpotency and vanishing curves in the (MU–)Adams spectral sequence. The *non*-nilpotency of η in the MU–Adams spectral sequence. Mathew–Meier type theorems about horizontal vanishing lines and the Tate construction (and related results about the TMF spectral sequence and the Johnson–Wilson theories).

Additive degeneration and $kO \neq kU^{hC_2}$.

Remark A.4.1. It is completely unclear why MU plays such an important mediating role between geometry (i.e., the stable category) and algebra (i.e., sheaves on the moduli of formal groups). Given a general ring spectrum R and thick prime $\otimes$–ideals C_α of perfect R–modules, one ask the analogous two questions:

1. Is it possible to find an R–algebra S whose context functor induces a homeomorphism of Balmer spectra $\operatorname{Spec}(\mathsf{Modules}_R^{\mathrm{perf}}) \to \operatorname{Spec}(\mathsf{QCoh}(\mathcal{M}_{S/R}))$?

2. Are there complementary localizers $L_\alpha \colon \mathsf{Modules}_R \to \mathsf{Modules}_{R,(\alpha)}$? Can they be

presented via Bousfield's framework as homological localizations for auxiliary S–algebra spectra S_α? Do the contexts $\mathcal{M}_{S_\alpha}$ admit compatible localizers with $\mathcal{M}_S$?

For $R = \mathbb{S}$, this is the role that the R–algebra $S = MU$ and the S–algebras $S_d = E(d)$ play. Finding these spectra feels like striking gold, and it is unclear how to produce analogous spectra in general.

Mathews's work on Galois descent shows that the fixed point map $\mathsf{Modules}^{\mathrm{complete}}_{E_\Gamma, \operatorname{Aut}\Gamma} \to \mathsf{Spectra}_\Gamma$ is an equivalence of categories.

One can ask the same question from the geometric direction: why bordism? Why should these spectra have these nice flatness properties? Why should they have recognizable computational properties? Why bordism?

Remark A.4.2. The homotopy of $\widehat{L}_2\mathbb{S}$ is also known, by work of Shimomura and collaborators [Shi86, SY94, SY95] (but see also the reorganization by Behrens [Beh12]). It is *exceedingly* complicated, and it is an open problem to find an expression of it which admits human digestion. Behrens has pursued a program encoding this problem in terms of modular forms [Beh09, Beh06, Beh07], and Hopkins has proposed a program involving L–functions [Str92], motivated by which Hovey and Strickland have shown a kind of continuity result for among the groups [HS99, Section 14].

Remark A.4.3. There are also "finitary" flavors of chromatic localization available, which are typically less robust but more computable. They assemble into a diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & L_d^{\mathrm{fin}} E \arrow[r] \arrow[d] & L_d E \arrow[d] \\ L_{X(d)} E \arrow[r] & \widehat{L}_d^{\mathrm{fin}} E \arrow[r] & \widehat{L}_d E, \end{tikzcd}$$

where $X(d)$ is a finite complex of type exactly d, v is a v_d–self-map of $X(d)$, $T(d) = X(d)[v^{-1}]$ is the localizing telescope, $\widehat{L}_d^{\mathrm{fin}}$ is Bousfield localization with respect to $T(d)$ (which can be shown to be independent of choice of $X(d)$ and of v), and L_d^{fin} denotes localization with respect to the class of *finite* $E(d)$–acyclics. Many things about these functors are known: for instance, $L_{X(d)} L_d = \widehat{L}_d$, there is a chromatic fracture square relating L_d^{fin} to $\widehat{L}_{\le d}^{\mathrm{fin}}$, and $L_d^{\mathrm{fin}} E \simeq L_d E$ if and only if $\widehat{L}_{\le d}^{\mathrm{fin}} E \simeq \widehat{L}_{\le d} E$. One major question about these functors remains open, corresponding the last unsettled nilpotence and periodicity conjecture of Ravenel [Rav84, Conjecture 10.5]: is the map $\widehat{L}_d^{\mathrm{fin}} E \to \widehat{L}_d E$ an equivalence? Multiple proofs and disproofs have been offered, but the literature remains unsettled.

Cite me: Find some proofs and disproofs..

Remark A.4.4. Writing M_d for the fiber in the sequence $M_d \to L_d \to L_{d-1}$, the filtration spectral sequence associated to the tower in Theorem 3.6.14 is called the *geometric chromatic spectral sequence*, which has the form $\pi_* M_* \mathbb{S} \Rightarrow \pi_* \mathbb{S}_{(p)}$. The two forms of filtration data $M_d X$ and $\widehat{L}_d X$ are actually functorially equivalent to one another:

$$\widehat{L}_d M_d \simeq \widehat{L}_d, \qquad\qquad M_d \widehat{L}_d \simeq M_d,$$

but they have fairly distinct properties. For instance, M_d is smashing whereas $\widehat{L}_d$ is not, M_d is not part of an adjoint pair whereas $\widehat{L}_d$ is, and the analogue of Lemma 3.6.8 for M_d is

Cite me: I forget who this is due to.

"backwards":

$$M_d X \simeq \operatorname*{colim}_I \left(M^0(v^I) \wedge L_d X \right).$$

The spectrum $M_d X$ also relates to the chromatic fracture square for X:

$$\begin{tikzcd}
M_d X \arrow[r, equal] \arrow[d] & M_d X \arrow[d] \\
L_d X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{L}_d X \arrow[d] \\
L_{d-1} X \arrow[r] & L_{d-1} \widehat{L}_d X.
\end{tikzcd}$$

From this, we see that there is a fiber sequence $M_d X \to \widehat{L}_d X \to L_{d-1} \widehat{L}_d X$.

The case $d = 1$ gives the prototypical example of the difference between these two presentations of the "exact height d data", where the sequence becomes:

$$\operatorname*{colim}_j (M^0(p^j) \wedge L_1 X) \to \lim_j (M_0(p^j) \wedge L_1 X) \to \left(\lim_j (M_0(p^j) \wedge L_1 X) \right)_{\mathbb{Q}}.$$

If, for instance, $\pi_0 L_1 X = \mathbb{Z}_{(p)}$, then the long exact sequence of homotopy groups associated to this fiber sequence gives

$$\begin{tikzcd}
\pi_0 \widehat{L}_1 X \arrow[r] \arrow[d, equal] & \pi_0 L_0 \widehat{L}_1 X \arrow[r] \arrow[d, equal] & \pi_{-1} M_1 X \arrow[d, equal] \\
\mathbb{Z}_p^{\wedge} \arrow[r] & \mathbb{Q}_p \arrow[r] & \mathbb{Z}/p^{\infty}.
\end{tikzcd}$$

Coupling this to Example 3.6.18, we compute

$$\pi_t M_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}/p^{\infty} & \text{when } t = -1, \\ \mathbb{Z}_p/(pk) & \text{when } t = k|v_1| - 1 \text{ and } t = \neq 0, \\ \mathbb{Z}/p^{\infty} & \text{when } t = (0 \cdot |v_1| - 1) - 1 = -2, \\ 0 & \text{otherwise.} \end{cases}$$

This is a model for what happens generally when passing from $\pi_* \widehat{L}_d X$ to $\pi_* M_d X$: the v_j–torsion–free groups get converted to infinitely v_j–divisible groups, with some dimension shifts.[8]

[8] A height 2 example of this same phenomenon is visible in Behrens's paper [Beh12, Section 7].

Sections 5.3-4 of Hopkins's ICM address *Algebraic Topology and Modular Forms* has a discussion of what η and ν have to do with *tmf*, as well as the construction of some interesting "topological θ–series" in the elliptic cohomology of certain Thom complexes.

Jacob wrote me an email giving a very slightly fuller sketch of what the DAG perspective on the σ–orientation is. Interestingly, it boils down to a fact from projective geometry: there just aren't that many line bundles on projective varieties. This forces a couple of things to become equal, and in a suitable setting they even become canonically equal. The email has no subject line, which will make it hard to find, but you should include a summary of it (which is dependent on whatever's written in the *Survey* paper) all the same.

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Number of to-dos used: 319

Material for lecture

Mike's 1995 announcement is a nice read. There are many snippets you could pull out of it for use here. "$H\mathbb{Q}$ serves as the target for the Todd genus, but actually the Todd genus of a manifold is an integer and it turns out that KU refines the Todd genus." The end of section 3, with $\tau \mapsto 1/\tau$, is mysterious. In section 4, Mike claims that there's a $BU[6,\infty)$–structured splitting principle into sums of things of the form $(1-\mathcal{L}_1)(1-\mathcal{L}_2)(1-\mathcal{L}_3)$. He then says that one expects the characteristic series of a $BU[6,\infty)$–genus to be a series of 3 variables, which is nice intuition. Could mention that Θ^k is a kind of k^{th} difference operator, so that things in the kernel of Θ^k are "k^{th} order polynomials". (More than this, the theorem of the cube is reasonable from this perspective, since Θ^3 kills "quadratic things" and the topological object $H^2(-;\mathbb{Z})$ classifying line bundles is indeed "quadratic".) If the bundle admits a symmetry operation, then the fiber over $(x, y, -x-y)$ is canonically trivialized, so a Σ–structure on a symmetric line bundle is a Θ^3–structure that restricts to the identity on these canonical parts. Mike claims (Theorem 6.2) that if $1/2 \in E^0(*)$ or if E is $K(n)$–local, $n \le 2$, then $BString^E$ is the parameter space of Σ–structures on the sheaf of functions vanishing at the identity on G_E. The map $MString \to KO_{\text{Tate}}$ actually factors through *MSpin*, so even though this produces the right q–series, you really need to know that *MString* factors through *tmf* and *MSpin doesn't* to deduce the modularity for *String*–manifolds. (You can prove modularity separately for $BU[6,\infty)$–manifolds, though, by essentially the same technique: refer to the rest of the (complex!) moduli of elliptic curves, which exist as $MU[6,\infty)$–spectra.)

Generally: if X is a space, then $X_{H\mathbb{F}_2}$ is a scheme with an $\operatorname{Aut}\hat{\hat{\mathbb{G}}}_a$–action. If X is a spectrum (so it fails to have a diagonal map) then $(H\mathbb{F}_2)_*X$ is just an $\mathbb{F}_2$–module, also with an $\operatorname{Aut}\hat{\hat{\mathbb{G}}}_a$–action.

The cohomology of a qc sheaf pushed forward from a scheme to a stack along a cover agrees with just the cohomology over the scheme. (In the case of $* \to *//G$, this probably uses the cospan $* \to *//G \leftarrow *$ with pullback G...)

Akhil Mathew has notes from an algebraic geometry class (https://math.berkeley.edu/~amathew/23) where lectures 3–5 address the theorem of the cube.

Equivalences of various sorts of cohomologies: Ext in modules and quasicoherent cohomology (goodness. Hartshorne, I suppose); Ext in comodules and quasicoherent cohomology on stacks (COCTALOS Lemma 12.4); quasicoherent cohomology on simplicial schemes (Stacks project 09VK).

Make clear the distinction between E_n and $\widehat{E(n)}$. Maybe explain the Devinatz–Hopkins remark that $r : \widehat{E(n)} \to E_n$ is an inclusion of fixed points and as such does not classify the versal formal group law.

when describing Quillen's model, he makes a lot of use of Gysin maps and Thom / Euler classes. at this point, maybe you can introduce what a Thom sheaf / Thom class is for a pointed formal curve?

Lemma A.4.5 ([Str99b, Proposition 4.6]). *A functor $X\colon \mathsf{Algebras} \to \mathsf{Sets}$ is a formal scheme exactly when*

1. *X preserves finite limits.*
2. *There exists a family of maps $X_i \to X$ from a set of S–schemes X_i such that the induced map*

$$\coprod_i X_i(T) \to X(T)$$

is jointly surjective for all test algebras T. □

MUP happens to be the Thom spectrum of $BU \times \mathbb{Z}$.

+ Warning: noncontinuous maps of high-dimensional formal affine spaces. + Lemma and proof: homomorphisms $F \to G$ of $\mathbb{F}_p$-FGLs factor as $F \to G' \to G$, where $G' \to G$ is a Frobenius isogeny and $F \to G'$ is invertible. + Plausibility argument for square-zero deformations being classified by "$\mathrm{Ext}^1(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a)$" + Honda's theorem about ζ–functions as manufacturing integral genera. + Definition of forms of a module, map to Galois cohomology + Computation of the Galois cohomology for: $H\mathbb{F}_p$, MU/p, KU/p + Computation of the Galois cohomology for $\widehat{\mathbb{G}}_m$, explicit description of the invariant via the ζ–function + Morava's sheaf over $L_1(\mathbb{Z}_p^{nr})$, Gamma-equivariance and transitivity, ConnerâĂŤFloyd + Identification of L_1/Γ with $\mathcal{M}_{\mathbf{fg}}^{\leq 1}$, connection to LEFT. + Description of the Lubin–Tate tower and the local Langlands correspondence + Uniqueness of $\mathcal{O}_K$-module structure in characteristic zero

Ideas

1. Statement of Lurie's characterization of *TMF*, using this to determine a map from *MString* by AHR
2. Matt's calculation of E_∞–orientations of $K(1)$–local spectra using the short free resolution of MU in the $K(1)$–local category

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3. *Forms of K–theory*, Elliptic spectra, Tate K–theory, *TMF*

4. Sinkinson's calculation and $MBP\langle m \rangle$–orientations

5. The Serre–Tate theorem

6. The fundamental domain of π_{GH}

Resources

Barry Walker's thesis

Morava's *Forms of K–theory*

Akhil wrote a couple of blog posts about Ochanine's theorem: `https://amathew.wordpress.com` and `https://amathew.wordpress.com/2012/05/31/the-other-direction-of-ochaines-` Mentioning a more precise result might lend to a more beefy introduction.