Formal Geometry and Bordism Operations

Lecture notes

Eric Peterson

Compiled: June 30, 2017

Git hash: fef70f7(*)

"Let us be glad we don't work in algebraic geometry."

J. F. Adams [Ada78, Section 2.1]

Acknowledgements

This book owes an incredible amount to an incredible number of incredible people. Understanding the research program summarized here has been one of the primary pursuits—if not *the* primary pursuit—of my young academic career. It has been an unmistakable and enormous privilege not only to be granted the time and freedom to learn about these beautiful ideas, but to also be able to do so directly from their progenitors. I owe a very large debt to each of Matthew Ando, Michael Hopkins, and Neil Strickland, for having shown me such individual attention and care, as well as for having worked out the tail of this long story. Matt, in particular, is the person who got me into higher mathematics, and I feel that for a decade now I have been paying forward the good will and deep friendliness that he showed me during my time at Urbana–Champaign. Mike and Neil are not far behind. Mike has been my supervisor in one sense or another for years running, in which role he has been continually encouraging and giving. Among other things, Neil shared with me a note of his that eventually blossomed into my thesis problem, which is an awfully nice gift to have given.

Less directly in ideas but no less directly in stewardship, I also owe a very large debt to my Ph.D. adviser, Constantin Teleman. By the time I arrived at UC–Berkeley, I was already too soaked through with homotopy theory to develop a flavor for his sort of mathematical physics, and he nonetheless endeavored to meet me where I was. It was Constantin who encouraged me to put special attention into making these ideas accessible, speaking understandably, and highlighting the connections with nearby fields. He emphasized that mathematics done in isolation, rather than in maximal connection to other people, is mathematics wasted. He has very much contributed to my passion for communication and clarity, which—in addition to the literal mathematics presented here—is the main goal of this text.[1] It is up to the reader to determine whether I have actually succeeded at this, and any failure of mine in this regard can't possibly be visited upon him.

More broadly, the topology community has been very supportive of me as I

[1] I have a clear memory of delivering a grad student seminar talk during my first year, where my mathematical sibling, George Melvin, asked me why $\widehat{\mathbb{G}}_m$ was called the multiplicative formal group. I looked at him, looked at the board, and cautiously offered, "...because of the mixed term in the group law?" Silly as it seems, this book has been shaped in no small way by striving to correct for this single highly inarticulate incident, where it was revealed that I did not understand the original context of these tools that topologists were borrowing. Although everyone starts learning from somewhere and not-knowing something is no cause for lasting embarrassment, it is certainly still helpful to receive pushes in the direction of intellectual responsibility. Thanks, George.

have learned about, digested, and sometimes erroneously recapitulated the ideas of chromatic homotopy theory, tolerating me as a very loud and public learner. Haynes Miller, Doug Ravenel, and Steve Wilson (the *BP* Mafia [Hop08]) have all been invaluable resources: they have answered my questions tirelessly, they are each charming and friendly, they are prolific and meticulous authors, and they literally invented this subfield of mathematics. Jack Morava has played no smaller a role in both the discovery of chromatic homotopy theory and my own personal education. It has been an incredible treat to know him and to have received pushes from him at critical moments. Nat Stapleton and Charles Rezk also deserve special mention: power operations were among the last things that I managed to understand while writing this book, and it is an enormous credit first to their intelligence that they are so comfortable with something so bottomlessly complicated and second to their inexhaustible patience that they walked me through understanding this material time and time again, in the hopes that I would someday get it.

The bulk of this book began as a set of lecture notes for a topics course[2] that I was invited to teach at Harvard University in the spring term of 2016, and I would like to thank the department for the opportunity and for the very enriching time that I spent there. The *germ* of these notes, however, took root at the workshop *Flavors of Cohomology*, organized and hosted by Hisham Sati in June 2015. In particular, this was the first time that I tried to push the idea of a "context" on someone else, which—for better or worse—has grown into the backbone of this book. The book also draws on feedback from lecturing in the *In-Formal Groups Seminar*, which took place during the MSRI semester program in homotopy theory in the spring term of 2014, attended primarily by David Carchedi, Achim Krause, Matthew Pancia, and Sean Tilson. Finally, my thoughts about the material in this book and its presentation were further refined by many, many, many conversations with other students at UC–Berkeley, primarily: my undergraduate readers Hood Chatham and Geoffrey Lee, the visiting student Catherine Ray, and my officemates Aaron Mazel-Gee and Kevin Wray—who, poor guys, put up with listening to me for years on end.

This book also draws on a lot of unpublished material. The topology community gets some flak for this reluctance to publish certain documents, but I think it is to our credit that they are made available anyway, essentially without redaction. Reference materials of this sort which have influenced this book include: Matthew

[2]MATH 278 (159627), Spring 2016.

Ando's *Dieudonné Crystals Associated to Lubin–Tate Formal Groups*; Michael Hopkins's *Complex Oriented Cohomology Theories and the Language of Stacks*; Jacob Lurie's *Chromatic Homotopy (252x)*; Haynes Miller's *Notes on Cobordism*; Charles Rezk's *Elliptic Cohomology and Elliptic Curves*, as well as his *Notes on the Hopkins–Miller Theorem*, and his *Supplementary Notes for Math 512*; Neil Strickland's *Formal Schemes for K–Theory Spaces* as well as his *Functorial Philosophy for Formal Phenomena*; the Hopf archive preprint version of the Ando–Hopkins–Strickland article *Elliptic Spectra, the Witten Genus, and the Theorem of the Cube*[3]; and the unpublished Ando–Hopkins–Strickland article *Elliptic Cohomology of $BO\langle 8\rangle$ in Positive Characteristic*, recovered from the mists of time by Gerd Laures[4]. I would not have understood the material presented here without access to these resources, nevermind the supplementary guidance.

In addition to their inquisitive presences in the lecture hall, the students who took the Harvard topics course under me often contributed directly to the notes. These contributors are: Eva Belmont, Hood Chatham (especially his marvelous spectral sequence package, `sseqpages`), Dexter Chua (an outside consultant who helped translate the picture in Figure 3.3 from a scribble on a scrap of paper into something of professional caliber), Arun Debray (a student at UT–Austin), Jun Hou Fung, Jeremy Hahn (especially the material in Case Study 2 and Appendix A.4), Mauro Porta, Krishanu Sankar, Danny Shi, and Allen Yuan (especially, again, Appendix A.4, which I might have never tried to understand without his insistence that I speak about it and his further help in preparing that talk).

More broadly, the following people contributed to the course just by attending, where I have highlighted those who additionally survived to the end of the semester: Colin Aitken, Adam Al-Natsheh, **Eva Belmont**, Jason Bland, Dorin Boger, **Lukas Brantner**, **Christian Carrick**, **Hood Chatham**, David Corwin, **Jun Hou Fung**, **Meng Guo**, **Jeremy Hahn**, Changho Han, Chi-Yun Hsu, **Erick Knight**, Benjamin Landon, Gabriel Long, Yusheng Luo, Jake Marcinek, **Jake McNamara**, **Akhil Mathew**, Max Menzies, Morgan Opie, Alexander Perry, Mauro Porta, **Krishanu Sankar**, **Jay Shah**, Ananth Shankar, **Danny Shi**, Koji Shimizu, Geoffrey

[3]This earlier version contains a lot of information that didn't make it to publication, as the referee (perhaps rightly) found it too dense to make heads or tails of. Once the reader already has the sketch of the argument established, however, the original version is a truly invaluable resource to go back and re-read.

[4]There is a bit of a funny story here: none of the authors of this article could find their own preprint, but one of their old graduate students had held on to a paper copy. In their defense, two decades had passed—but that in turn only makes Gerd's organizational skills more heroic.

Smith, Hunter Spinik, Philip Tynan, Yi Xie, David Yang, Zijian Yao, Lynnelle Ye, Chenglong Yu, **Allen Yuan**, Adrian Zahariuc, Yifei Zhao, Rong Zhou, and Yihang Zhu. Their energy and enthusiasm were overwhelming—I felt duty-bound to keep telling them things they didn't already know, and despite my best efforts to keep out ahead I also felt like they were constantly nipping at my heels. As I've gone through my notes during the editing process, it has been astonishing to see how reliably they asked exactly the right question at exactly the right time, often despite my own confusion. They're a very bright group. Of the highlighted names, Erick Knight deserves special mention: he was an arithmetic geometer living among the rest of us topologists, and he attentively listened to me butcher his native field without once making me feel self-conscious.

Additionally, various others have contributed in this way or that during the long production of this book, from repairing typos to long conversations and between. Such helpful people include: Tobias Barthel, Jon Beardsley, Martin Bendersky, Sanath Devalapurkar, Ben Gadoua, Mike Hill, Johan Konter, Achim Krause, Akhil Mathew, Pedro Mendes de Araujo, Denis Nardin, Justin Noel, Sune Precht Reeh, Andrew Senger, and Sean Tilson.

Acknowledge the publisher

Lastly, but by far most importantly, this book—and, frankly, *I*—would not have made it out of the gates without Samrita Dhindsa's love, support, and patience. She made living in Boston a completely different experience: a balanced life instead of being quickly and totally overwhelmed by work, a lively circle of friends instead of what would have been a much smaller world, new experiences instead of worn-through ones, and love throughout. Without her compassion, tenderness, and understanding I would not be half the open and vibrant person that I am today, and I would know so much less of myself. It is awe-inspiring to think about, and it is a pleasure and an honor to acknowledge her like this and to dedicate this book to her.

Thanks to my many friends here, and thanks also to Thomas Dunning especially. Thanks, everyone. Theveryone.

–Eric

Contents

I would like as few section titles as possible to involve people's names.

Make sure you use either id or 1 everywhere to denote the identity morphism.

You're not very consistent about using $\widehat{\mathbb{G}}$ or Γ to denote an arbitrary formal group. It seems like you use one or the other based on your preference of whether it has finite height or not.

Double check that you're careful about choosing consistent names for your objects: $S = \operatorname{Spec} R$ is the base scheme, that sort of thing.

Be consistent about $\mathcal{O}_X$ vs $\mathcal{O}(X)$, and similarly with $\mathcal{I}_D$ versus $\mathcal{I}(D)$.

Be consistent about "$\mathbb{S}$" versus "$\mathbb{S}^0$" for the standard sphere spectrum.

We should use the convention that $\varphi \in \mathcal{M}_{\mathbf{fgl}}(T)$ corresponds to $x +_\varphi y \in T[\![x,y]\!]$ throughout.

Clark's use of MinionPro font is really nice (cf. `https://dl.dropboxusercontent.com/u/1741495/teaching/2017spring.18.917/gamma-blurb.pdf`). Generally, we should make an effort to pick a nice font. http://web.willbenton.com/writing/2008/better-latex has more info about converting FF fonts for use in LaTeX, including Minion.

Be consistent about hyphens / en-dashes connecting "$A/E/H_\infty$" to "ring".

Use E_∞RingSpectra everywhere appropriate.

Use * and not $^{\vee}$ everywhere for linear dual.

See whether /tag instead of a right column makes step-by-step-justified align* look better.

Chapter 0

Introduction

The goal of this book is to communicate a certain *Weltanschauung* uncovered in pieces by many different people working in bordism theory, and the goal just for this introduction is to tell a story about one theorem where it is especially apparent.

To begin, we will define a homology theory called "bordism homology". Recall that the singular homology of a space X comes about by probing X with simplices: beginning with the collection of continuous maps $\sigma\colon \Delta^n \to X$, we take the free $\mathbb{Z}$–module on each of these sets and construct a chain complex

$$\cdots \xrightarrow{\partial} \mathbb{Z}\{\Delta^n \to X\} \xrightarrow{\partial} \mathbb{Z}\{\Delta^{n-1} \to X\} \xrightarrow{\partial} \cdots .$$

Bordism homology is constructed analogously, but using manifolds Z as the probes instead of simplices: [1] Mention a word or two about inverses in footnote?

$$\begin{aligned} \cdots &\xrightarrow{\partial} \{Z^n \to X \mid Z^n \text{ a compact } n\text{–manifold}\} \\ &\xrightarrow{\partial} \{Z^{n-1} \to X \mid Z^{n-1} \text{ a compact } (n-1)\text{–manifold}\} \\ &\xrightarrow{\partial} \cdots . \end{aligned}$$

Lemma 0.0.1 ([Koc78, Section 4])**.** *This forms a chain complex of monoids under disjoint union of manifolds, and its homology is written $MO_*(X)$. These are naturally abelian groups, and moreover they satisfy the axioms of a generalized homology theory.* $\square$

[1] One doesn't need to take the free abelian group on anything, since the disjoint union of two manifolds is already a (disconnected) manifold, whereas the disjoint union of two simplices is not a simplex.

In fact, we can define a bordism theory MG for any suitable family of structure groups $G(n) \to O(n)$. The coefficient ring of MG, or its value $MG_*(*)$ on a point, gives the ring of G–bordism classes, and generally $MG_*(Y)$ gives a kind of "bordism in families over the space Y". There are comparison morphisms for the most ordinary kinds of bordism, given by replacing a chain of manifolds with an equivalent simplicial chain:

$$MO \to H\mathbb{Z}/2, \qquad\qquad MSO \to H\mathbb{Z}.$$

In both cases, we can evaluate on a point to get ring maps, called "genera":

$$MO_*(*) \to \mathbb{Z}/2, \qquad\qquad MSO_*(*) \to \mathbb{Z},$$

neither of which is very interesting, since they're both zero in positive degrees.

However, having maps of homology theories (rather than just maps of coefficient rings) is considerably more data than just the genus. For instance, we can use it to extract a theory of integration as follows. Consider the following special case of oriented bordism, where we evaluate MSO_* on an infinite loopspace:

$$\begin{aligned} MSO_n K(\mathbb{Z}, n) &= \{\text{oriented } n\text{–manifolds mapping to } K(\mathbb{Z}, n)\} / \sim \\ &= \left\{ \begin{array}{c} \text{oriented } n\text{–manifolds } Z \\ \text{with a specified class } \omega \in H^n(Z; \mathbb{Z}) \end{array} \right\} \Big/ \sim . \end{aligned}$$

Associated to such a representative (Z, ω), the yoga of stable homotopy theory then allows us to build a composite

$$\begin{aligned} \mathbb{S} &\xrightarrow{(Z,\omega)} MSO \wedge (\mathbb{S}^{-n} \wedge \Sigma^\infty_+ K(\mathbb{Z}, n)) \\ &\xrightarrow{\text{colim}} MSO \wedge H\mathbb{Z} \\ &\xrightarrow{\varphi \wedge 1} H\mathbb{Z} \wedge H\mathbb{Z} \\ &\xrightarrow{\mu} H\mathbb{Z}, \end{aligned}$$

where φ is the orientation map. Altogether, this composite gives us an element of $\pi_0 H\mathbb{Z}$, i.e., an integer.

I used to think that we got a generalized Stokes's theorem too, but now I'm not sure. Stokes's theorem is the statement that the chain and cochain differentials are adjoint: $\langle d\omega, \sigma\rangle = \langle \omega, \partial\sigma\rangle$, where the pairing is the integration pairing. It would be neat to interpret this in generality, but it might be a stretch.

Lemma 0.0.2. *The integer obtained by the above process is $\int_Z \omega$.* □

Cite me: Where is this proven?.

This definition of $\int_Z \omega$ via stable homotopy theory is pretty nice, in the sense that many theorems accompany it for free. It is also very general: given a ring map off of any bordism spectrum, a similar sequence of steps will furnish us with an integral tailored to that situation.

Now take $G = e$ to be the trivial structure group, which is the bordism theory of framed manifolds, i.e., those with stably trivial normal bundle. In this case, the Pontryagin–Thom construction gives an equivalence $\mathbb{S} \xrightarrow{\simeq} Me$. It is thus possible (and some people have indeed taken up this viewpoint) that stable homotopy theory can be investigated solely through the lens of framed bordism. We will prefer to view this the other way: the sphere spectrum $\mathbb{S}$ often appears to us as a natural object, and we will occasionally replace it by Me, the framed bordism spectrum. For example, given a ring spectrum E with unit map $\mathbb{S} \to E$, we can reconsider this as a ring map

$$Me \xrightarrow{\simeq} \mathbb{S} \to E.$$

Following along the lines of the previous paragraph, we learn that any ring spectrum E is automatically equipped with a theory of integration for framed manifolds.

Sometimes, as in the examples above, this unit map factors:

$$\mathbb{S} \simeq Me \to MO \to H\mathbb{Z}/2.$$

This is a witness to the overdeterminacy of $H\mathbb{Z}/2$'s integral for framed bordism: if the framed manifold is pushed all the way down to an unoriented manifold, there is still enough residual data to define the integral.[2] Given any ring spectrum E, we can ask the analogous question: If we filter O by a system of structure groups, through what stage does the unit map $Me \to E$ factor? For instance, the map

$$\mathbb{S} = Me \to MSO \to H\mathbb{Z}$$

considered above does *not* factor further through MO—an orientation is *required* to define the integral of an integer–valued cohomology class. Recognizing $SO \to O$ as the 0^{th} Postnikov–Whitehead truncation of O, we are inspired to use the rest of the Postnikov filtration as our filtration of structure groups. Here is a diagram of this filtration and some interesting minimally-factored integration theories related to it, circa 1970:

---

[2] It's literally more information than this: even unframeable unoriented manifolds acquire a compatible integral.

$$\begin{tikzcd}
Me \arrow[r] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr] & \cdots \arrow[r] & MString \arrow[r] & MSpin \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\
 & & & kO & H\mathbb{Z} & H\mathbb{Z}/2.
\end{tikzcd}$$

This is the situation homotopy theorists found themselves in some decades ago, when Ochanine and Witten proved the following mysterious theorem using analytical and physical methods:

Theorem 0.0.3 (Ochanine [Och87], Witten [Wit87, Wit88])**.** *There is a map of rings*

$$\sigma : MSpin_* \to \mathbb{C}((q)).$$

Moreover, if Z is a Spin manifold such that half its first Pontryagin class vanishes—that is, if Z lifts to a String–manifold—then $\sigma(Z)$ lands in the subring $MF \subseteq \mathbb{Z}[\![q]\!]$ of modular forms with integral coefficients. $\square$

However, neither party gave indication that their result should be valid "in families" (in our sense), and no theory of integration was formally produced (in our sense). From the perspective of the homotopy theorist, it wasn't even totally clear what such a claim would mean: to give a topological enrichment of these theorems would mean finding a ring spectrum E such that $E_*(*)$ had something to do with modular forms.

Around the same time, Landweber, Ravenel, and Stong began studying "elliptic cohomology" for independent reasons; sometime much earlier, Morava had constructed an object "K^{Tate}" associated to the Tate elliptic curve; and a decade later Ando, Hopkins, and Strickland put all these together in the following theorem:

Cite me: Landweber–Ravenel–Stong, Morava's *Forms of K–theory*, and Ando–Hopkins–Strickland.

Theorem 0.0.4 (Ando–Hopkins–Strickland)**.** *If E is an "elliptic cohomology theory", then there is a canonical map of homotopy ring spectra $MString \to E$ called the σ–orientation (for E). Additionally, there is an elliptic spectrum K^{Tate} whose σ–orientation gives Witten's genus $MString_* \to K^{\mathrm{Tate}}_*$.* $\square$

We now come to the motivation for this text: the homotopical σ–orientation was actually first constructed using formal geometry. The original proof of Ando–Hopkins–Strickland begins with a reduction to maps of the form

$$MU[6,\infty) \to E.$$

They then work to show that in especially good cases they can complete the missing arrow in the diagram

$$\begin{tikzcd} MU[6,\infty) \arrow[r] \arrow[dr] & MString \arrow[d, dotted] \\ & E. \end{tikzcd}$$

Leaving aside the extension problem for the moment, their main theorem is the following description of the cohomology ring $E^*MU[6,\infty)$:

Theorem 0.0.5 (Ando–Hopkins–Strickland [AHS01], cf. Singer [Sin68] and Stong [Sto63])**.** *For E an even–periodic cohomology theory,*

$$\operatorname{Spec} E_*MU[6,\infty) \cong C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0)),$$

where "$C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0))$" is a certain scheme. When E is taken to be elliptic, so that there is a specified isomorphism $\widehat{\mathbb{G}}_E \cong C_0^\wedge$ for C an elliptic curve, the theory of elliptic curves furnishes the scheme with a canonical point. Hence, there is a preferred class $MU[6,\infty) \to E$, natural in the choice of elliptic E. □

Our real goal is to understand theorems like this last one, where algebraic geometry asserts some real control over something squarely in the domain of homotopy theory, and we will work through a sequence of case studies where this perspective shines through most brightly. In particular, rather than taking an optimal route to the Ando–Hopkins–Strickland result, we will use it as a gravitational slingshot to cover many anciliary topics which are also governed by the technology of formal geometry. We will begin by working through Thom's calculation of the homotopy of *MO*, which holds the simultaneous attractive features of being approachable while revealing essentially all of the structural complexity of the general situation, so that we know what to expect later on. Having seen that through, we'll then venture on to other examples: the complex bordism ring, structure theorems for finite spectra, unstable cooperations, and, finally, the σ–orientation and its extensions. Again, the overriding theme of the text will be that algebraic geometry is a good organizing principle that gives us one avenue of insight into how homotopy theory functions: it allows us to organize "operations" of various sorts between spectra derived from bordism theories.

We should also mention that we will specifically *not* discuss the following aspects of this story:

- Analytic techniques will be completely omitted. Much of modern research stemming from the above problem is an attempt to extend index theory across

Witten's genus, or to find a "geometric cochains" model of certain elliptic cohomology theories. These often mean heavy analytic work, and we will strictly confine ourselves to the domain of homotopy theory.

- As sort of a sub-point (and despite the motivation provided in this Introduction), we will also mostly avoid manifold geometry. Again, much of the contemporary research about *tmf* is an attempt to find a geometric model, so that geometric techniques can be imported—including equivariance and the geometry of quantum field theories, to name two.

- In a different direction, our focus will not linger on actually computing bordism rings MX_*, nor will we consider geometric constructions on manifolds and their behavior after imaging into the bordism ring. This is also the source of active research: the structure of the symplectic bordism ring remains, to large extent, mysterious, and what we do understand of it comes through a mix of formal geometry and raw manifold geometry. This could be a topic that fits logically into this document, were it not for time limitations and the author's inexpertise.

- The geometry of E_∞ rings will also be avoided, at least to the extent possible. Such objects become inescapable by the conclusion of our story, but there are better resources from which to learn about E_∞ rings, and the pre–E_∞ story is not told so often these days. So, we will focus on the unstructured part, relegate the E_∞ rings to Appendix A, and leave their details to other authors.

- There will be plenty of places where we will avoid making statements in maximum generality or with maximum thoroughness. The story we are interested in telling draws from a blend of many others from different subfields of mathematics, many of which have their own dedicated textbooks. Sometimes this will mean avoiding stating the most beautiful theorem in a subfield in favor of a theorem we will find more useful. Other times this will mean abbreviating someone else's general definition to one more specialized to the task at hand. In any case, we will give references to other sources where you can find these cast in starring roles.

Finally, we must mention that there are several good companions to these notes. Essentially none of the material here is original—it's almost all cribbed either from published or unpublished sources—but the source documents are quite scattered and individually dense. We will make a point to cite useful references as we go. One document stands out above all others, though: Neil Strickland's *Functorial Philosophy for Formal Phenomena* [Strb]. These lecture notes can basically be viewed as an attempt to make it through this paper in the span of a semester.

0.1 Conventions

Throughout this book, we use the following conventions:

- $\mathsf{C}(X, Y)$ will denote the mapping object of arrows $X \to Y$ in a category C. If C is an ∞–category, this will often be interpreted as a mapping *space*. If C has a self-enrichment, we will often write $\underline{\mathsf{C}}(X, Y)$ (or, e.g., $\underline{\mathrm{Aut}}(X)$) to distinguish the internal mapping object from $\mathsf{C}(X, Y)$ the classical mapping set. As an exception to this uniform notation, we will sometimes abbreviate $\underline{\mathsf{Spaces}}(X, Y)$ to $F(X, Y)$, and similarly we will sometimes abbreviate $\underline{\mathsf{Spectra}}(X, Y)$ to $F(X, Y)$.

- Following Lurie, for an object $X \in \mathsf{C}$ we will write $\mathsf{C}_{/X}$ for the slice category of objects *over* X and $\mathsf{C}_{X/}$ for the slice category of objects *under* X.

- For a ring spectrum E, we will write $E_* = \pi_* E$ for its coefficient ring, $E^* = \pi_{-*} E$ for its coefficient ring with the opposite grading, and $E_0 = E^0 = \pi_0 E$ for the 0^{th} degree component of its coefficient ring. In particular, this allows us to make sense of expressions like "$E^*[\![x]\!]$", which we interpret as

$$E^*[\![x]\!] = (E^*)[\![x]\!] = (\pi_{-*}E)[\![x]\!].$$

- For a space or spectrum, we will write $X[n, \infty) \to X$ for the upward n^{th} Postnikov truncation over X and $X \to X(-\infty, n)$ for the downward n^{th} Postnikov truncation under X. There is thus a natural fiber sequence

$$X[n, \infty) \to X \to X(-\infty, n).$$

This notation extends naturally to objects like $X(a, b)$ or $X[a, b]$, where $-\infty \leq a < b < \infty$ denote the (closed or open) endpoints of any interval.

This notation is visible in Greenlees–May, but probably doesn't originate there.

Case Study 1

Unoriented bordism

A simple observation about the bordism ring $MO_*(*)$ (or $MO_*(X)$ more generally, for any space X) is that it consists entirely of 2–torsion: any chain $Z \to X$ can be bulked out to a constant cylinder $Z \times I \to X$, which has as its boundary the chain $2 \cdot (Z \to X)$. Accordingly, $MO_*(X)$ is always an $\mathbb{F}_2$–vector space. Our goal in this Case Study is to arrive at two remarkable calculations: first, in Corollary 1.5.7 we will make an explicit calculation of this $\mathbb{F}_2$–vector space in the case of the bordism homology of a point, and second, in Lemma 1.5.8 we will show that there is a natural isomorphism

$$MO_*(X) = H\mathbb{F}_{2*}(X) \otimes_{\mathbb{F}_2} MO_*(*).$$

Our goal in discussing these results in the first Case Study of the book is to take the opportunity to introduce several key concepts that will serve us throughout. First and foremost, we will require a definition of bordism spectrum that we can manipulate computationally, using just the tools of abstract homotopy theory. Once that is established, we immediately begin to bring algebraic geometry into the mix: the main idea is that the cohomology ring of a space is better viewed as a scheme (with plenty of extra structure), and the homology groups of a spectrum are better viewed as representation for a certain elaborate algebraic group. This data actually finds familiar expression in homotopy theory: we show that a form of group cohomology for this representation forms the input to the classical Adams spectral sequence, which classically takes the form

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}(Y)) \Rightarrow \pi_*(Y),$$

converging for certain very nice spectra Y—including, for instance, $Y = MO$. In particular, we can bring the tools from the preceding discussion to bear on the homology and cohomology of MO, where we make an explicit calculation of its representation structure. Finding that it is suitably free, we thereby gain control of the Adams spectral sequence, finish the computation, and prove the desired result.

Our *real* goal in this Case Study, however, is to introduce one of the main themes of the entire text: there is some governing algebro-geometric object, the formal group $\mathbb{RP}^{\infty}_{H\mathbb{F}_2}$, which exerts an extraordinary amount of control over everything in sight. We will endeavor to rephrase as much of this classical computation as possible so as to highlight its connection to this central object, and we will use this as motivation in future Case Studies to pursue similar objects, which will lead us down much deeper and more rewarding rabbit holes. The counterbalance to this is that, at least for now, we will not introduce concepts or theorems in their maximum generality. (For an obvious example, everything in this Case Study will be done relative to $\operatorname{Spec}\mathbb{F}_2$.) Essentially everything mentioned in this Case Study will be re-examined more thoroughly in future Case Studies, so the reader is advised to look to those for the more expansive set of results.

1.1 Thom spectra and the Thom isomorphism

Our goal is a sequence of theorems about the unoriented bordism spectrum MO. We will begin by recalling a definition of the spectrum MO using just abstract homotopy theory, because it involves ideas that will be useful to us throughout the semester and because we cannot compute effectively with the chain-level definition given in the Introduction.

Definition 1.1.1. For a spherical bundle $S^{n-1} \to \xi \to X$, its *Thom space* is given by the cofiber

$$\xi \to X \xrightarrow{\text{cofiber}} T_n(\xi).$$

"Proof" of definition. There is a more classical construction of the Thom space: take the associated disk bundle by gluing an n–disk fiberwise, and add a point at infinity by collapsing ξ:

What is this $\sqcup'$ notation?

$$T_n(\xi) = (\xi \sqcup'_{S^{n-1}} D^n)^+.$$

To compare this with the cofiber definition, recall that the thickening of ξ to an n–disk bundle is the same as taking the mapping cylinder on $\xi \to X$. Since the inclusion into the mapping cylinder is now a cofibration, the quotient by this subspace agrees with both the cofiber of the map and the introduction of a point at infinity. $\square$

Before proceeding, here are two important examples:

Example 1.1.2. If $\xi = S^{n-1} \times X$ is the trivial bundle, then $T_n(\xi) = S^n \wedge (X_+)$. This is supposed to indicate what Thom spaces are "doing": if you feed in the trivial bundle then you get the suspension out, so if you feed in a twisted bundle you should think of it as a *twisted suspension*.

Example 1.1.3. Let ξ be the tautological S^0–bundle over the unpointed space $\mathbb{RP}^\infty = BO(1)$. Because ξ has contractible total space, $EO(1)$, the cofiber degenerates and it follows that $T_1(\xi) = \mathbb{RP}^\infty$, a pointed space.[1] More generally, arguing by cells shows that the Thom space for the tautological bundle over $\mathbb{RP}^n$ is $\mathbb{RP}^{n+1}$.

Now we catalog a bunch of useful properties of the Thom space functor. Firstly, recall that a spherical bundle over X is the same data as a map $X \to BGL_1 S^{n-1}$, where $GL_1 S^{n-1}$ is the subspace of $F(S^{n-1}, S^{n-1})$ expressed by the pullback of spaces

Cite me: Give a reference for this general construction of classifying spaces for fibrations..

$$\begin{tikzcd}
GL_1 S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Spaces}(S^{n-1}, S^{n-1}) \arrow[d] & \\
h\mathsf{Spaces}(S^{n-1}, S^{n-1})^\times \arrow[r] & h\mathsf{Spaces}(S^{n-1}, S^{n-1}) \arrow[r, equal] & \pi_0 \mathsf{Spaces}(S^{n-1}, S^{n-1}).
\end{tikzcd}$$

Lemma 1.1.4. *The construction T_n can be viewed as a functor from the slice category over $BGL_1 S^{n-1}$ to* Spaces. *Maps of slices*

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "f^*\xi"'] & & X \arrow[dl, "\xi"] \\
& BGL_1 S^{n-1} &
\end{tikzcd}$$

induce maps $T_n(f^*\xi) \to T_n(\xi)$, and T_n is suitably homotopy-invariant. □

Next, the spherical subbundle of a vector bundle gives a common source of spherical bundles. The action of $O(n)$ on $\mathbb{R}^n$ preserves the unit sphere, and hence gives a map $O(n) \to GL_1 S^{n-1}$. These are maps of topological groups, and the block–inclusion maps $i^n \colon O(n) \to O(n+1)$ commute with the suspension map $GL_1 S^{n-1} \to GL_1 S^n$. In fact, much more can be said:

[1]If you already know what's coming, this should comport with the Thom isomorphism: $x \cdot H\mathbb{F}_2^* \mathbb{RP}^\infty \cong \widetilde{H\mathbb{F}_2}^{*+1} \mathbb{RP}^\infty$.

Lemma 1.1.5. *The block–sum maps $O(n) \times O(m) \to O(n+m)$ are compatible with the join maps $GL_1 S^{n-1} \times GL_1 S^{m-1} \to GL_1 S^{n+m-1}$.* □

Again taking a cue from K–theory, we take the colimit as n grows large, using the maps

$$\begin{tikzcd} BGL_1 S^{n-1} \arrow[r, equal] \arrow[d] & BGL_1 S^{n-1} \times * \arrow[r, "\mathrm{id} \times \mathrm{triv}"] \arrow[d] & BGL_1 S^{n-1} \times BGL_1 S^0 \arrow[r, "*"] \arrow[d] & BGL_1 S^n, \arrow[d] \\ BO(n) \arrow[r, equal] & BO(n) \times * \arrow[r, "\mathrm{id} \times \mathrm{triv}"] & BO(n) \times BO(1) \arrow[r, "\oplus"] & BO(n+1). \end{tikzcd}$$

Should you justify “group” rather than “space”?

Corollary 1.1.6. *The operations of block–sum and topological join imbue the colimiting spaces BO and $BGL_1\mathbb{S}$ with the structure of H–groups . Moreover, the colimiting map*

$$J_{\mathbb{R}} \colon BO \to BGL_1\mathbb{S},$$

called the J–homomorphism, is a morphism of H–groups. □

Does this calculation need justification?

Finally, we can ask about the compatibility of Thom constructions with all of this. In order to properly phrase the question, we need a version of the construction which operates on stable spherical bundles, i.e., whose source is the slice category over $BGL_1\mathbb{S}$. By calculating

$$T_{n+1}(\xi * \mathrm{triv}) \simeq \Sigma T_n(\xi),$$

we are inspired to make the following definition:

Definition 1.1.7. For ξ an S^{n-1}–bundle, we define the *Thom spectrum* of ξ to be

$$T(\xi) := \Sigma^{-n}\Sigma^{\infty} T_n(\xi).$$

By filtering the base space by compact subspaces, this begets a functor

$$T \colon \mathsf{Spaces}_{/BGL_1\mathbb{S}} \to \mathsf{Spectra}.$$

Lemma 1.1.8. *T is monoidal: it carries external fiberwise joins to smash products of Thom spectra. Correspondingly, $T \circ J_{\mathbb{R}}$ carries external direct sums of stable vector bundles to smash products of Thom spectra.* □

Definition 1.1.9. The spectrum MO arises as the universal example of all these constructions, strung together:

$$MO := T(J_{\mathbb{R}}) = \operatorname*{colim}_n T(J^n_{\mathbb{R}}) = \operatorname*{colim}_n \Sigma^{-n} T_n J^n_{\mathbb{R}}.$$

There should be a here (to Pontryagin, ly) saying that we re- as defined on the

The spectrum MO has several remarkable properties. The most basic such property is that it is a ring spectrum, and this follows immediately from $J_{\mathbb{R}}$ being a homomorphism of H–spaces. Much more excitingly, we can also deduce the presence of Thom isomorphisms just from the properties stated thus far. That $J_{\mathbb{R}}$ is a homomorphism means that the following square commutes:

$$\begin{tikzcd} BO \times BO \arrow[r, "\sigma", "\simeq"'] \arrow[drr, bend right] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}} \times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\ & BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}. \end{tikzcd}$$

We have extended this square very slightly by a certain shearing map σ defined by $\sigma(x, y) = (xy^{-1}, y)$. It is evident that σ is a homotopy equivalence, since just as we can de-scale the first coordinate by y we can re-scale by it. We can calculate directly the behavior of the long composite:

σ *almost* shows up in giving a categorical definition of a G–torsor. I wish I understood this, but I always get tangled up.

$$J_{\mathbb{R}} \circ \mu \circ \sigma(x, y) = J_{\mathbb{R}} \circ \mu(xy^{-1}, y) = J_{\mathbb{R}}(xy^{-1}y) = J_{\mathbb{R}}(x).$$

It follows that the second coordinate plays no role, and that the bundle classified by the long composite can be written as $J_{\mathbb{R}} \times 0$.[2] We are now in a position to see the Thom isomorphism:

Lemma 1.1.10 (Thom isomorphism, universal example, cf. [Mah79])**.** *As MO–modules,*

$$MO \wedge MO \simeq MO \wedge \Sigma^\infty_+ BO.$$

Proof. Stringing together the naturality properties of the Thom functor outlined above, we can thus make the following calculation:

$$\begin{aligned} T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}})) &\simeq T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}}) \circ \sigma) && \text{(homotopy invariance)} \\ &\simeq T(J_{\mathbb{R}} \times 0) && \text{(constructed lift)} \\ &\simeq T(J_{\mathbb{R}}) \wedge T(0) && \text{(monoidality)} \\ &\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(Example 1.1.2)} \\ T(J_{\mathbb{R}}) \wedge T(J_{\mathbb{R}}) &\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(monoidality)} \\ MO \wedge MO &\simeq MO \wedge \Sigma^\infty_+ BO. && \text{(definition of } MO\text{)} \end{aligned}$$

[2] This factorization does *not* commute with the rest of the diagram, just with the little lifting triangle it forms.

The equivalence is one of MO–modules because the MO–module structure of both sides comes from smashing with MO on the left. □

From here, the general version of Thom's theorem follows quickly:

Definition 1.1.11. A map $\varphi\colon MO \to E$ of homotopy ring spectra is called an *orientation* of E (by MO).[3]

Theorem 1.1.12 (Thom isomorphism). *Let $\xi\colon X \to BO$ classify a vector bundle and let $\varphi\colon MO \to E$ be a map of ring spectra. Then there is an equivalence of E–modules*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X.$$

Modifications to above proof. To accommodate X rather than BO as the base, we redefine $\sigma\colon BO \times X \to BO \times X$ by

$$\sigma(x, y) = \sigma(x\xi(y)^{-1}, y).$$

Follow the same proof as before with the diagram

$$\begin{tikzcd}
BO \times X \arrow[r, "\sigma", "\simeq"'] \arrow[drrr, bend right] & BO \times X \arrow[r, "\mathrm{id}\times\xi", "\simeq"'] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}}\times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
& & BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}$$

This gives an equivalence $\theta_{MO}\colon MO \wedge T(\xi) \to MO \wedge \Sigma^\infty_+ X$. To introduce E, note that there is a diagram

$$\begin{tikzcd}
E \wedge T(\xi) \arrow[d, "\eta_{MO}\wedge\mathrm{id}\wedge\mathrm{id}=f"] & E \wedge \Sigma^\infty_+ X \arrow[d, "\eta_{MO}\wedge\mathrm{id}\wedge\mathrm{id}"] \\
MO \wedge E \wedge T(\xi) \arrow[r, "\theta_{MO}\wedge E"] \arrow[d, "(\mu\circ(\varphi\wedge\mathrm{id}))\wedge\mathrm{id}=g"] & MO \wedge E \wedge \Sigma^\infty_+ X \arrow[d, "(\mu\circ(\varphi\wedge\mathrm{id}))\wedge\mathrm{id}=h"] \\
E \wedge T(\xi) \arrow[r, "\theta_E"] & E \wedge \Sigma^\infty_+ X
\end{tikzcd}$$

[3] Later, we will refer to analogous ring spectrum maps $MU \to E$ off of the complex bordism spectrum as *complex-orientations* of E. However, calling ring maps $MO \to E$ "unoriented-orientations" is rightfully considered distasteful.

The bottom arrow θ_E exists by applying the action map to both sides and pushing the map $\theta_{MO} \wedge E$ down. Since θ_{MO} is an equivalence, it has an inverse α_{MO}. Therefore, the middle map has inverse $\alpha_{MO} \wedge E$, and we can similarly push this down to a map α_E, which we now want to show is the inverse to θ_E. From here it is a simple diagram chase: we have renamed three of the maps in the diagram to f, g, and h for brevity. Noting that $g \circ f$ is the identity map because of the unit axiom, we conclude

$$\begin{aligned}
g \circ f &\simeq g \circ (\alpha_{MO} \wedge E) \circ (\theta_{MO} \wedge E) \circ f \\
&\simeq \alpha_E \circ h \circ (\theta_{MO} \wedge E) \circ f && \text{(action map)} \\
&\simeq \alpha_E \circ \theta_E \circ g \circ f && \text{(action map)} \\
&\simeq \alpha_E \circ \theta_E.
\end{aligned}$$

It follows that α_E gives an inverse to θ_E. □

Remark 1.1.13. One of the tentpoles of the theory of Thom spectra is that Theorem 1.1.12 has a kind of converse: if a ring spectrum E has suitably natural and multiplicative Thom isomorphisms for Thom spectra formed from real vector bundles, then one can define an essentially unique ring map $MO \to E$ realizing these isomorphisms via the machinery of Theorem 1.1.12.

Remark 1.1.14. There is also a cohomological version of the Thom isomorphism. Suppose that E is a ring spectrum under MO and let ξ be the spherical bundle of a vector bundle on a space X. The spectrum $F(\Sigma^\infty_+ X, E)$ is a ring spectrum under E (hence under MO), so there is a Thom isomorphism as well as an evaluation map

$$F(\Sigma^\infty_+ X, E) \wedge T(\xi) \xrightarrow{\simeq} F(\Sigma^\infty_+ X, E) \wedge \Sigma^\infty_+ X \xrightarrow{\text{eval}} E.$$

Passing through the exponential adjunction, the map

$$F(\Sigma^\infty_+ X, E) \xrightarrow{\simeq} F(T(\xi), E)$$

can be seen to give the cohomological Thom isomorphism

$$E^* X \cong E^* T(\xi).$$

Example 1.1.15. We will close out this section by using this to actually make a calculation. Recall from Example 1.1.3 that $T(\mathcal{L} \downarrow \mathbb{R}P^n) = \mathbb{R}P^{n+1}$. Because MO is a connective spectrum, the truncation map

$$MO \to MO(-\infty, 0] = H\pi_0 MO = H\mathbb{F}_2$$

is a map of ring spectra [May77, Lemma II.2.12]. Hence, we can apply the Thom isomorphism theorem to the mod–2 homology of Thom complexes coming from real vector bundles:

$$\begin{aligned} \pi_*(H\mathbb{F}_2 \wedge T(\mathcal{L}-1)) &\cong \pi_*(H\mathbb{F}_2 \wedge T(0)) && \text{(Thom isomorphism)} \\ \pi_*(H\mathbb{F}_2 \wedge \Sigma^{-1}\Sigma^\infty \mathbb{R}\mathrm{P}^{n+1}) &\cong \pi_*(H\mathbb{F}_2 \wedge \Sigma^\infty_+ \mathbb{R}\mathrm{P}^n) && \text{(Example 1.1.3)} \\ \widetilde{H\mathbb{F}}_{2*+1}\mathbb{R}\mathrm{P}^{n+1} &\cong H\mathbb{F}_{2*}\mathbb{R}\mathrm{P}^n. && \text{(generalized homology)} \end{aligned}$$

This powers an induction that shows that $H\mathbb{F}_{2*}\mathbb{R}\mathrm{P}^\infty$ has a single class in every degree. The cohomological version of the Thom isomorphism in Remark 1.1.14, together with the $H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^n$–module structure of $H\mathbb{F}_2^* T(\mathcal{L}-1)$, also gives the ring structure:

$$H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^n = \mathbb{F}_2[x]/x^{n+1}.$$

1.2 Cohomology rings and affine schemes

An abbreviated summary of this book is that we are going to put "Spec" in front of rings appearing in algebraic topology and see what happens. Before actually doing any algebraic topology, we should recall what this means on the level of algebra. The core idea is to replace an $\mathbb{F}_2$–algebra R by the functor it corepresents, which we will denote by $\operatorname{Spec} R$. For any other "test $\mathbb{F}_2$–algebra" T, we set

$$(\operatorname{Spec} R)(T) := \mathsf{Algebras}_{\mathbb{F}_2/}(R, T) \cong \mathsf{Schemes}_{/\mathbb{F}_2}(\operatorname{Spec} T, \operatorname{Spec} R).$$

More generally, we have the following definition:

Definition 1.2.1. An *affine* $\mathbb{F}_2$*–scheme* is a functor $X : \mathsf{Algebras}_{\mathbb{F}_2/} \to \mathsf{Sets}$ which is (noncanonically) isomorphic to $\operatorname{Spec} R$ for some $\mathbb{F}_2$–algebra R. Given such an isomorphism, we will refer to $\operatorname{Spec} R \to X$ as a *parameter* for X and its inverse $X \to \operatorname{Spec} R$ as a *coordinate* for X.

Lemma 1.2.2. *There is an equivalence of categories*

$$\operatorname{Spec} : \mathsf{Algebras}^{\mathrm{op}}_{\mathbb{F}_2/} \to \mathsf{AffineSchemes}_{/\mathbb{F}_2}. \qquad \square$$

The centerpiece of thinking about rings in this way, for us and for now, is to translate between a presentation of R as a quotient of a free algebra and a presentation of $(\operatorname{Spec} R)(T)$ as selecting tuples of elements in T subject to certain conditions. Consider the following example:

Example 1.2.3. Set $R_n = \mathbb{F}_2[x_1, \ldots, x_n]$. Elements of $(\operatorname{Spec} R_n)(T)$ can be identified with n–tuples of elements of T, since a function in

$$(\operatorname{Spec} R_n)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x_1, \ldots, x_n], T),$$

is entirely determined by where the x_j are sent. Consider also what happens when we impose a relation by passing to $R_n^J = \mathbb{F}_2[x_1, \ldots, x_n]/(x_k^{j_k+1})$: a function in

$$(\operatorname{Spec} R_n^J)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x_1, \ldots, x_n]/(x_k^{j_k+1}), T)$$

is again determined by where the x_j are sent, but now x_j can only be sent to an element which is nilpotent of order $j_k + 1$. These schemes are both important enough that we give them special names:

$$\mathbb{A}^n := \operatorname{Spec} \mathbb{F}_2[x_1, \ldots, x_n], \qquad \mathbb{A}^{n,J} := \operatorname{Spec} \mathbb{F}_2[x_1, \ldots, x_n]/(x_k^{j_k+1}).$$

The functor $\mathbb{A}^n$ is called *affine n–space*—reasonable, since the value $\mathbb{A}^n(T)$ is isomorphic to T^n. In particular, we refer to $\mathbb{A}^1$ as the *affine line*. Note that the quotient map $R_1 \to R_1^{(j)}$ induces an inclusion $\mathbb{A}^{1,(j)} \to \mathbb{A}^1$ and that $\mathbb{A}^{1,(0)}$ is a constant functor:

$$\mathbb{A}^{1,(0)}(T) = \{f : \mathbb{F}_2[x] \to T \mid f(x) = 0\}.$$

Accordingly, we declare "$\mathbb{A}^{1,(0)}$" to be the *origin on the affine line* and $\mathbb{A}^{1,(j)}$ to be the $(n+1)^{st}$ *order (nilpotent) neighborhood of the origin* in the affine line.

We can also use this language to re-express another common object arising in algebraic topology: the Hopf algebra, which appears when taking the mod–2 cohomology of an H–group. In addition to the usual ring structure on cohomology groups, the H–group multiplication, unit, and inversion maps additionally induce a diagonal map Δ, an augmentation map ε, and an antipode χ respectively. Running through the axioms, one quickly checks the following:

Lemma 1.2.4. *For a Hopf $\mathbb{F}_2$–algebra R, the functor* $\operatorname{Spec} R$ *is naturally valued in groups. Such functors are called group scheme. Conversely, a choice of group structure on* $\operatorname{Spec} R$ *endows R with the structure of a Hopf algebra.*

Proof sketch. This is a matter of recognizing the product in $\mathsf{Algebras}_{\mathbb{F}_2/}^{\mathrm{op}}$ as the tensor product, then using the Yoneda lemma to transfer structure around. □

Example 1.2.5. The functor $\mathbb{A}^1$ introduced above is naturally valued in groups: since $\mathbb{A}^1(T) \cong T$, we can use the addition on T to make it into an abelian group. When considering $\mathbb{A}^1$ with this group scheme structure, we notate it as $\mathbb{G}_a$. Applying the Yoneda lemma, one deduces the following formulas for the Hopf algebra structure maps:

$$\begin{aligned} \mathbb{G}_a \times \mathbb{G}_a &\xrightarrow{\mu} \mathbb{G}_a & x_1 + x_2 &\leftarrow\!\shortmid x, \\ \mathbb{G}_a &\xrightarrow{\chi} \mathbb{G}_a & -x &\leftarrow\!\shortmid x, \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \mathbb{G}_a & 0 &\leftarrow\!\shortmid x. \end{aligned}$$

As an example of how to reason this out, consider the following diagram:

$$\begin{tikzcd} \mathbb{G}_a \times \mathbb{G}_a \arrow[rr, "\mu"] & & \mathbb{G}_a \\ \operatorname{Spec} \mathbb{F}_2[x_1] \times \operatorname{Spec} \mathbb{F}_2[x_2] \arrow[u, "x_1 \simeq\ x_2 \simeq"] \arrow[d, equal] & & \\ \operatorname{Spec} \mathbb{F}_2[x_1, x_2] \arrow[rr, "\Delta^*"] \arrow[uurr, bend right, "x_1+x_2"] & & \operatorname{Spec} \mathbb{F}_2[x]. \arrow[uu, "x", "\simeq"'] \end{tikzcd}$$

It follows that the bottom map of affine schemes is induced by the algebra map

$$\mathbb{F}_2[x] \xrightarrow{\Delta} \mathbb{F}_2[x_1, x_2], \qquad x \mapsto x_1 + x_2.$$

Remark 1.2.6. In fact, $\mathbb{A}^1$ is naturally valued in *rings*. It models the inverse functor to Spec in the equivalence of categories above, i.e., the elements of a ring R always form a complete collection of $\mathbb{A}^1$–valued functions on some affine scheme Spec R.

Example 1.2.7. We define the *multiplicative group scheme* by

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{F}_2[x, y]/(xy - 1).$$

Its value $\mathbb{G}_m(T)$ on a test algebra T is the set of pairs (x, y) such that y is a multiplicative inverse to x, and hence $\mathbb{G}_m$ is valued in groups. Applying the Yoneda lemma, we deduce the following formulas for the Hopf algebra structure maps:

$$\begin{aligned} \mathbb{G}_m \times \mathbb{G}_m &\xrightarrow{\mu} \mathbb{G}_m & x_1 \otimes x_2 &\leftarrow\!\shortmid x \\ & & y_1 \otimes y_2 &\leftarrow\!\shortmid y, \\ \mathbb{G}_m &\xrightarrow{\chi} \mathbb{G}_m & (y, x) &\leftarrow\!\shortmid (x, y), \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \mathbb{G}_m & 1 &\leftarrow\!\shortmid x, y. \end{aligned}$$

Remark 1.2.8. As presented above, the multiplicative group comes with a natural inclusion $\mathbb{G}_m \to \mathbb{A}^2$. Specifically, the subset $\mathbb{G}_m \subseteq \mathbb{A}^2$ consists of pairs (x, y) in the graph of the hyperbola $y = 1/x$. However, the element x also gives an $\mathbb{A}^1$–valued function $x\colon \mathbb{G}_m \to \mathbb{A}^1$, and because multiplicative inverses in a ring are unique, we see that this map too is an inclusion. These two inclusions have rather different properties relative to their ambient spaces, and we will think harder about these essential differences later on.

Example 1.2.9 (cf. Example 4.4.12). This example showcases the complications that algebraic geometry introduces to this situation, and is meant as discouragement from thinking of the theory of affine group schemes as a strong analogue of the theory of linear complex Lie groups. We set $\alpha_2 = \operatorname{Spec} \mathbb{F}_2[x]/(x^2)$, with group scheme structure given by

$$\begin{aligned} \alpha_2 \times \alpha_2 &\xrightarrow{\mu} \alpha_2 & x_1 + x_2 &\mapsfrom x, \\ \alpha_2 &\xrightarrow{\chi} \alpha_2 & -x &\mapsfrom x, \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \alpha_2 & 0 &\mapsfrom x. \end{aligned}$$

This group scheme has several interesting properties, which we will merely state for now, reserving their proofs for Example 4.4.12.

1. α_2 has the same underlying structure ring as $\mu_2 := \mathbb{G}_m[2]$, the 2–torsion points of $\mathbb{G}_m$, but is not isomorphic to it. (For instance, $\underline{\mathsf{GroupSchemes}}(\mu_2, \mu_2)$ gives the constant group scheme $\underline{\mathbb{Z}/2}$, but $\underline{\mathsf{GroupSchemes}}(\alpha_2, \mu_2) = \alpha_2$.)

2. There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.

3. If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence

$$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$

   However, this short exact sequence does not split (even after base change).

4. The subgroups of $\alpha_2 \times \alpha_2$ of order 2 are parameterized by the scheme $\mathbb{P}^1$, i.e., for R an $\mathbb{F}_2$-algebra the subgroup schemes of $\alpha_2 \times \alpha_2$ of order two *which are defined over* R are parameterized by the set $\mathbb{P}^1(R)$.

We now turn to a different class of examples, which will wind up being the key players in our upcoming topological story. To begin, consider the colimit of the

Jeremy had some motivation for this, that quite generally one wants to consider ind-systems of compact objects. Why does one want this? Is it better motivation than just dropping into it?

sets $\operatorname{colim}_{j\to\infty} \mathbb{A}^{1,(j)}(T)$, which is of use in algebra: it is the collection of nilpotent elements in T. These kinds of conditions which are "unbounded in j" appear frequently enough that we are moved to give these functors a name too:

Erick points out that the morphisms in this system should be infintesimal thickenings. Is there a functor-of-points way to recognize such things, without reaching all the way up to talking about ideals? (He also thinks that we should allow finite type in addition to finite, but I don't think I want that. I've been wrong before, though.)

Definition 1.2.10. An *affine formal scheme* is an ind-system of finite affine schemes.[4] The morphisms between two formal schemes are computed by

$$\mathsf{FormalSchemes}(\{X_\alpha\}, \{Y_\beta\}) = \lim_\alpha \operatorname*{colim}_\beta \mathsf{Schemes}(X_\alpha, Y_\beta).$$

Given affine charts $X_\alpha = \operatorname{Spec} R_\alpha$, we will glibly suppress the system from the notation and write

$$\operatorname{Spf} R := \{\operatorname{Spec} R_\alpha\}.$$

Example 1.2.11. The individual schemes $\mathbb{A}^{1,(j)}$ do not support group structures. After all, the sum of two elements which are nilpotent of order $j+1$ can only be guaranteed to be nilpotent of order $2j+1$. It follows that the entire ind-system $\{\mathbb{A}^{1,(j)}\} =: \widehat{\mathbb{A}}^1$ supports a group structure, even though none of its constituent pieces do. We call such an object a *formal group scheme*, and this particular formal group scheme we denote by $\widehat{\mathbb{G}}_a$.

Example 1.2.12. Similarly, one can define the scheme $\mathbb{G}_m[j]$ of elements of unipotent order j:

$$\mathbb{G}_m[j] = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy-1, x^j-1)} \subseteq \mathbb{G}_m.$$

These *are* all group schemes, and they nest together in a complicated way: there is an inclusion of $\mathbb{G}_m[j]$ into $\mathbb{G}_m[jk]$. There is also a second filtration along the lines of the one considered in Example 1.2.11:

$$\mathbb{G}_m^{(j)} = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy-1, (x-1)^j)}.$$

[4] This has the effect of formally adjoining colimits of filtered diagrams to the category of finite affine schemes. In fact, a functor $X\colon \mathsf{Algebras} \to \mathsf{Sets}$ which preserves finite limits is a formal scheme exactly when there exists a family of maps $X_i \to X$ from a set of affine schemes X_i such that the induced map

$$\coprod_i X_i(T) \to X(T)$$

is jointly surjective for all test algebras T [Str99b, Proposition 4.6].

These schemes form a sequential system, but they are only occasionally group schemes. Specifically, $\mathbb{G}_m^{(2^k)}$ is a group scheme, in which case $\mathbb{G}_m^{(2^k)} \cong \mathbb{G}_m[2^k]$.[5] We define $\widehat{\mathbb{G}}_m$ using this common subsystem:

$$\widehat{\mathbb{G}}_m := \{\mathbb{G}_m^{(2^k)}\}_{k=0}^{\infty}.$$

Let us now consider the example that we closed with last time, where we calculated $H\mathbb{F}_2^*(\mathbb{RP}^n) = \mathbb{F}_2[x]/(x^{n+1})$. Putting "Spec" in front of this, we could reinterpret this calculation as

$$\operatorname{Spec} H\mathbb{F}_2^*(\mathbb{RP}^n) \cong \mathbb{A}^{1,(n)}.$$

This is so useful that we will give it a notation all of its own:

Definition 1.2.13. Let X be a finite cell complex, so that $H\mathbb{F}_2^*(X)$ is a ring which is finite–dimensional as an $\mathbb{F}_2$–vector space. We will write

$$X_{H\mathbb{F}_2} = \operatorname{Spec} H\mathbb{F}_2^* X$$

for the corresponding finite affine scheme.

Example 1.2.14. Putting together the discussions from this time and last time, in the new notation we have calculated

$$\mathbb{RP}^n_{H\mathbb{F}_2} \cong \mathbb{A}^{1,(n)}.$$

So far, this example just restates what we already knew in a mildly different language. Our driving goal for the next section is to incorporate as much information as we have about these cohomology rings $H\mathbb{F}_2^*(\mathbb{RP}^n)$ into this description, which will result in us giving a more "precise" name for this object. Along the way, we will discover why X had to be a *finite* complex and how to think about more general X. For now, though, we will content ourselves with investigating the Hopf algebra structure on $H\mathbb{F}_2^*\mathbb{RP}^\infty$, the cohomology of an infinite complex.

Example 1.2.15. Recall that $\mathbb{RP}^\infty$ is an H–space in two equivalent ways:

1. There is an identification $\mathbb{RP}^\infty \simeq K(\mathbb{F}_2, 1)$, and the H–space structure is induced by the sum on cohomology.

2. There is an identification $\mathbb{RP}^\infty \simeq BO(1)$, and the H–space structure is induced by the tensor product of real line bundles.

I thought we came up instructive third examp where to find the H–sp ture.

In either case, this induces a Hopf algebra diagonal

$$H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{RP}^\infty$$

which we would like to analyze. This map is determined by where it sends the class x, and because it must respect gradings it must be of the form $\Delta x = ax_1 + bx_2$ for some constants $a, b \in \mathbb{F}_2$. Furthermore, because it belongs to a Hopf algebra structure, it must satisfy the unitality axiom

$$\begin{tikzcd} H\mathbb{F}_2^*\mathbb{RP}^\infty & H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \arrow[l, "\begin{pmatrix}\varepsilon\otimes\mathrm{id}\\ \mathrm{id}\otimes\varepsilon\end{pmatrix}"'] & H\mathbb{F}_2^*\mathbb{RP}^\infty. \arrow[l, "\Delta"'] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}$$

and hence it takes the form

$$\Delta(x) = x_1 + x_2.$$

Noticing that this is exactly the diagonal map in Example 1.2.5, we tentatively identify "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" with the additive group. This is extremely suggestive but does not take into account the fact that $\mathbb{RP}^\infty$ is an infinite complex, so we have not yet allowed ourselves to write "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$". In light of the rest of the material discussed in this section, we have left open a very particular point: it is not clear if we should use the name "$\mathbb{G}_a$" or "$\widehat{\mathbb{G}}_a$". We will straighten this out in the subsequent Lecture.

1.3 The Steenrod algebra

We left off in the previous Lecture with an ominous finiteness condition in Definition 1.2.13, and we produced a pair of reasonable guesses as to what "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" could mean in Example 1.2.15. We will decide which of the two guesses is reasonable by rigidifying the target category so as to incorporate the following extra structures:

1. Cohomology rings are *graded*, and maps of spaces respect this grading.

[5] Additionally, the *only* values of j for which $\mathbb{G}_m[j]$ is an infinitesimal thickening of $\mathbb{G}_m[1]$ are those of the form $j = 2^k$.

2. Cohomology rings receive an action of the Steenrod algebra, and maps of spaces respect this action.
3. Both of these are made somewhat more complicated when taking the cohomology of an infinite complex.
4. (Cohomology rings for more elaborate cohomology theories are only skew-commutative, but "Spec" requires a commutative input.)

In this Lecture, we will fix all these deficiencies of $X_{H\mathbb{F}_2}$ except for #4, which does not matter with mod–2 coefficients but which will be something of a bugbear throughout the rest of the book.

We will begin by considering the grading on $H\mathbb{F}_2^* X$, where X is a finite complex. In algebraic geometry, the following standard construction is used to track gradings:[6]

Definition 1.3.1 ([Str99b, Definition 2.95])**.** A $\mathbb{Z}$–grading on a ring R is a system of additive subgroups R_k of R satisfying $R = \bigoplus_k R_k$, $1 \in R_0$, and $R_j R_k \subseteq R_{j+k}$. Additionally, a map $f \colon R \to S$ of graded rings is said to *respect the grading* if $f(R_k) \subseteq S_k$.[7]

Lemma 1.3.2 ([Str99b, Proposition 2.96])**.** *A graded ring R is equivalent data to an affine scheme* $\operatorname{Spec} R$ *with an action by $\mathbb{G}_m$. Additionally, a map $R \to S$ is homogeneous exactly when the induced map* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is $\mathbb{G}_m$–equivariant.*

Proof. A $\mathbb{G}_m$–action on $\operatorname{Spec} R$ is equivalent data to a coaction map

$$\alpha^* : R \to R \otimes \mathbb{F}_2[x^{\pm}].$$

Define R_k to be those points in r satisfying $\alpha^*(r) = r \otimes x^k$. It is clear that we have $1 \in R_0$ and that $R_j R_k \subseteq R_{j+k}$. To see that $R = \bigoplus_k R_k$, note that every tensor can be written as a sum of pure tensors. Conversely, given a graded ring R, define the coaction map on R_k by

$$(r_k \in R_k) \mapsto x^k r_k$$

and extend linearly. $\square$

[6]Strickland gives an alternative formalism for tracking gradings [Strb, Sections 11 and 14] called a *polarization*, which amounts to choosing a trivialization $\pi_2 E \cong \pi_0 E$ and considering the isomorphisms $\pi_{2n} E \cong (\pi_2 E)^{\otimes_{\pi_0 E} n}$.

[7]The terminology "$\mathbb{Z}$–filtering" might be more appropriate, but this is the language commonly used.

This notion from algebraic geometry is somewhat different from what we are used to in algebraic topology, essentially because the algebraic topologist's "cohomology ring" is not *really* a ring at all—one is only allowed to consider sums of homogeneous degree elements. This restriction stems directly from the provenance of cohomology rings: recall that

$$H\mathbb{F}_2^n X := \pi_{-n} F(\Sigma^\infty_+ X, H\mathbb{F}_2).$$

One can only form sums internal to a *particular* homotopy group, using the cogroup structure on $\mathbb{S}^{-n}$. On the other hand, the most basic ring of algebraic geometry is the polynomial ring, and hence their notion is adapted to handle, for instance, the potential degree drop when taking the difference of two (nonhomogeneous) polynomials of the same degree.

We can modify our perspective very slightly to arrive at the algebraic geometers', by replacing $H\mathbb{F}_2$ with the periodified spectrum

$$H\mathbb{F}_2 P = \bigvee_{j=-\infty}^{\infty} \Sigma^j H\mathbb{F}_2.$$

This spectrum becomes a ring in the homotopy category by using the factorwise-defined multiplication maps

$$\Sigma^j H\mathbb{F}_2 \wedge \Sigma^k H\mathbb{F}_2 \simeq \Sigma^{j+k}(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \xrightarrow{\Sigma^{j+k}\mu} \Sigma^{j+k} H\mathbb{F}_2.$$

This spectrum has the property that $H\mathbb{F}_2 P^0(X)$ is isomorphic to $\bigoplus_n H\mathbb{F}_2^n(X)$ as ungraded rings, but now we can make topological sense of the sum of two classes which used to live in different $H\mathbb{F}_2$–degrees. At this point we can manually craft the desired coaction map α^* from Lemma 1.3.2, but we will shortly find that algebraic topology gifts us with it on its own.

Our route to finding this internally occurring α^* is by turning to the next supplementary structure: the action of the Steenrod algebra. Naively approached, this does not fit into the framework we have been sketching so far: the Steenrod algebra arises as the homotopy endomorphisms of $H\mathbb{F}_2$ and so is a *noncommutative* algebra. In turn, the action map

$$\begin{tikzcd}
\mathcal{A}^* \otimes H\mathbb{F}_2^* X \arrow[r] \arrow[d, equal] & H\mathbb{F}_2^* X \arrow[d, equal] \\
{[H\mathbb{F}_2, H\mathbb{F}_2]_* \otimes [X, H\mathbb{F}_2]_*} \arrow[r, "\circ"] & {[X, H\mathbb{F}_2]}
\end{tikzcd}$$

will be difficult to squeeze into any kind of algebro-geometric framework. Milnor was the first person to see a way around this, with two crucial observations. First, the Steenrod algebra is a Hopf algebra[8], using the map

$$[H\mathbb{F}_2, H\mathbb{F}_2]_* \xrightarrow{\mu^*} [H\mathbb{F}_2 \wedge H\mathbb{F}_2, H\mathbb{F}_2]_* \cong [H\mathbb{F}_2, H\mathbb{F}_2]_* \otimes [H\mathbb{F}_2, H\mathbb{F}_2]_*$$

as the diagonal. This Hopf algebra structure is actually cocommutative—this is a rephrasing of the symmetry of the Cartan formula:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y).$$

It follows that the linear-algebraic dual of the Steenrod algebra $\mathcal{A}_*$ is a commutative ring, and hence $\operatorname{Spec} \mathcal{A}_*$ would make a reasonable algebro-geometric object.

Second, we want to identify the role of $\mathcal{A}_*$ in acting on $H\mathbb{F}_2^* X$. By assuming that X is a finite complex, we can write it as the Spanier–Whitehead dual $X = DY$ of some other finite complex Y. Starting with the action map on $H\mathbb{F}_2^* Y$:

$$\mathcal{A}^* \otimes H\mathbb{F}_2^* Y \to H\mathbb{F}_2^* Y$$

we take the $\mathbb{F}_2$–linear dual to get a coaction map

$$\mathcal{A}_* \otimes H\mathbb{F}_{2*} Y \leftarrow H\mathbb{F}_{2*} Y,$$

then use $X = DY$ to return to cohomology

$$\mathcal{A}_* \otimes H\mathbb{F}_2^* X \xleftarrow{\lambda^*} H\mathbb{F}_2^* X.$$

Finally, we re-interpret this as an action map

$$\operatorname{Spec} \mathcal{A}_* \times X_{H\mathbb{F}_2} \xrightarrow{\alpha} X_{H\mathbb{F}_2}.$$

Having produced the action map α, we are now moved to study α as well as the structure group $\operatorname{Spec} \mathcal{A}_*$ itself. Milnor works out the Hopf algebra structure of $\mathcal{A}_*$ by defining elements $\xi_j \in \mathcal{A}_*$ dual to the j^{th} *Milnor primitive*, which are defined by $Q_0 = \mathrm{id}$, by $Q_1 = \mathrm{Sq}^1$, and thereafter by the iterated commutator $Q_j = [Q_{j-1}, \mathrm{Sq}^{2^{j-1}}]$. Taking $X = \mathbb{R}\mathrm{P}^n$ and $x \in H\mathbb{F}_2^1(\mathbb{R}\mathrm{P}^n)$ the generator, he then

[8]The construction of both the Hopf algebra diagonal here and the coaction map below is somewhat ad hoc. We will give a more robust presentation in Lecture 3.1.

observes two things: first, $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} x = x^{2^j}$, and second, any other admissible sequence of squares applied to x vanishes for degree reasons. It follows that $Q_j x = x^{2^j}$ is the only family of operators that act nontrivially on x, and from this we deduce the formula

$$\lambda^*(x) = \sum_{j=0}^{\lfloor \log_2 n \rfloor} x^{2^j} \otimes \xi_j \quad (\text{in } H\mathbb{F}_2^* \mathbb{R}\mathrm{P}^n).$$

Noticing that we can take the limit $n \to \infty$ to get a well-defined infinite sum, he then makes the following calculation, stable in n:

$$\begin{aligned}
(\lambda^* \otimes \mathrm{id}) \circ \lambda^*(x) &= (\mathrm{id} \otimes \Delta) \circ \lambda^*(x) && \text{(coassociativity)} \\
(\lambda^* \otimes \mathrm{id}) \left(\sum_{j=0}^{\infty} x^{2^j} \otimes \xi_j \right) &= \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^i} \otimes \xi_i \right)^{2^j} \otimes \xi_j &= && \text{(ring homomorphism)} \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= && \text{(characteristic 2).}
\end{aligned}$$

Then, turning to the right-hand side:

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= (\mathrm{id} \otimes \Delta) \left(\sum_{m=0}^{\infty} x^{2^m} \otimes \xi_m \right) \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= \sum_{m=0}^{\infty} x^{2^m} \otimes \Delta(\xi_m),
\end{aligned}$$

from which it follows that

$$\Delta \xi_m = \sum_{i+j=m} \xi_i^{2^j} \otimes \xi_j.$$

Finally, Milnor shows that this is the complete story:

Theorem 1.3.3 (Milnor [Mil58, Theorem 2], [MT68, Chapter 6])**.** *There is an isomorphism*

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_j, \ldots].$$

Flippant proof. There is at least a map $\mathbb{F}_2[\xi_1, \xi_2, \ldots] \to \mathcal{A}_*$ given by the definition of the elements ξ_j above. This map is injective, since these elements are distinguished by how they coact on $H\mathbb{F}_2^* \mathbb{R}\mathrm{P}^\infty$. Then, since these rings are of graded finite type, Milnor can conclude his argument by counting how many elements he has produced, comparing against how many Adem and Cartan found (which we will do ourselves in Lecture 4.1), and noting that he has exactly enough. □

We are now in a position to uncover the desired map α^* from earlier. In order to retell Milnor's story with $H\mathbb{F}_2 P$ in place of $H\mathbb{F}_2$, note that there is a topological construction involving $H\mathbb{F}_2$ from which $\mathcal{A}_*$ emerges:

$$\mathcal{A}_* := \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2).$$

Performing substitution on this formula gives the periodified dual Steenrod algebra:

$$\mathcal{A}P_0 := \pi_0(H\mathbb{F}_2 P \wedge H\mathbb{F}_2 P) = H\mathbb{F}_2 P_0(H\mathbb{F}_2 P) = \mathcal{A}_*[\xi_0^{\pm}].$$

Lemma 1.3.4 ([Goe08, Formula 3.4, Remark 3.14]). *Projecting to the quotient Hopf algebra $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^{\pm}]$ gives exactly the coaction map α^*.*

Calculation. Starting with an auxiliary cohomology class $x \in H\mathbb{F}_2^n(X)$, we produce a homogenized cohomology class $x \cdot u^n \in H\mathbb{F}_2 P^0(X)$. Under the coaction map, this is sent to

This could be clearer. Sean wasn't even sure what we were trying to calculate—he believed immediately that ξ_0 did its job.

$$\begin{array}{ccccc} H\mathbb{F}_2 P^0(X) & \xrightarrow{\alpha^*} & H\mathbb{F}_2 P^0(X) \otimes \mathcal{A}P_0 & \longrightarrow & H\mathbb{F}_2 P^0(X) \otimes \mathbb{F}_2[\xi_0^{\pm}] \\ x \cdot u^n & \longmapsto & & & x \cdot u^n \otimes \xi_0^n. \end{array}$$

Applying Lemma 1.3.2 to this coaction thus selects the original degree n classes. □

Early on in this discussion, trading the language "graded map" for "$\mathbb{G}_m$–equivariant map" did not seem to have much of an effect on our mathematics. The thrust of this Lemma is that "Steenrod–equivariant map" already includes "$\mathbb{G}_m$–equivariant map", which is a visible gain in brevity. To study the rest of the content of Steenrod equivariance algebro–geometrically, we need only identify what the series $\lambda^*(x)$ embodies. Note that this necessarily involves some creativity, and the only justification we can supply will be moral, borne out over time, as our narrative encompasses more and more phenomena. With that caveat in mind, here is one such description. Recall the map induced by the H–space multiplication

$$H\mathbb{F}_2 P^0 \mathbb{RP}^\infty \otimes H\mathbb{F}_2 P^0 \mathbb{RP}^\infty \leftarrow H\mathbb{F}_2 P^0 \mathbb{RP}^\infty.$$

Since this map comes from a map of spaces, it is equivariant for the Steenrod coaction, and since the action on the left is furthermore diagonal, we deduce the formula

$$\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2).$$

Lemma 1.3.5. *The series $\lambda^*(x) = \sum_{j=0}^\infty x^{2^j} \otimes \xi_j$ is the universal example of a series satisfying $\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2)$. The set $(\operatorname{Spec} \mathcal{A}P_0)(T)$ is identified with the set of power series f with coefficients in the $\mathbb{F}_2$–algebra T satisfying*

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Proof. Given a point $f \in (\operatorname{Spec} \mathcal{A}P_0)(T)$, we extract such a series by setting

$$\lambda_f^*(x) = \sum_{j=0}^\infty f(\xi_j) x^{2^j} \in T[\![x]\!].$$

Conversely, any series $\lambda(x)$ satisfying this homomorphism property must have nonzero terms appearing only in integer powers of 2, and hence we can construct a point f by declaring that f sends ξ_j to the $(2^j)^{\text{th}}$ coefficient of λ. □

We close our discussion by codifying what Milnor did when he stabilized against n. Each $\mathbb{RP}^n_{H\mathbb{F}_2 P}$ is a finite affine scheme, and to make sense of the object $\mathbb{RP}^\infty_{H\mathbb{F}_2 P}$ Milnor's technique was to consider the ind-system $\{\mathbb{RP}^n_{H\mathbb{F}_2 P}\}_{n=0}^\infty$ of finite affine schemes. We will record this as our technique to handle general infinite complexes:

Definition 1.3.6 (cf. Definition 2.1.13)**.** When X is an infinite complex, filter it by its subskeleta $X^{(n)}$ and define $X_{H\mathbb{F}_2 P}$ to be the ind-system $\{X^{(n)}_{H\mathbb{F}_2 P}\}_{n=0}^\infty$ of finite schemes.

This choice to follow Milnor resolves our uncertainty about the topological example from last time:

Example 1.3.7 (cf. Examples 1.2.11 and 1.2.15). Write $\widehat{\mathbb{G}}_a$ for the ind-system $\mathbb{A}^{1,(n)}$ with the group scheme structure given in Example 1.2.15. That this group scheme structure filters in this way is a simultaneous reflection of two facts:

1. Algebraic: The set $\widehat{\mathbb{G}}_a(T)$ consists of all nilpotent elements in T. The sum of two nilpotent elements of orders n and m is guaranteed to itself be nilpotent with order at most $n+m$.

2. Topological: There is a factorization of the multiplication map on $\mathbb{RP}^\infty$ as $\mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{n+m}$ purely for dimensional reasons.

As group schemes, we have thus calculated

$$\mathbb{RP}^\infty_{H\mathbb{F}_2P} \cong \widehat{\mathbb{G}}_a.$$

Example 1.3.8. Given the appearance of a homomorphism condition in Lemma 1.3.5, we would like to connect $\operatorname{Spec} \mathcal{A}P_0$ with $\widehat{\mathbb{G}}_a$ more directly. Toward this, we define a "hom functor"[9] for two formal schemes:

$$\underline{\mathsf{FormalSchemes}}(X,Y)(T) = \left\{ (u,f) \,\middle|\, \begin{array}{c} u : \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2, \\ f : u^*X \to u^*Y \end{array} \right\}.$$

Restricting attention to homomorphisms, we see that a proper name for $\operatorname{Spec} \mathcal{A}P_0$ is

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

To check this, consider a point $g \in (\operatorname{Spec} \mathcal{A}P_0)(T)$ for an $\mathbb{F}_2$–algebra T. The $\mathbb{F}_2$–algebra structure of T (which is uniquely determined by a property of T) gives rise to a map $u\colon \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2$. The rest of the data of g gives rise to a power series in $T[\![x]\!]$ as in the proof of Lemma 1.3.5, which can be re-interpreted as an automorphism $g\colon u^*\widehat{\mathbb{G}}_a \to u^*\widehat{\mathbb{G}}_a$ of formal group schemes.[10]

Remark 1.3.9. The projection $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^\pm]$ is split as Hopf algebras, and hence there is a decomposition

$$\underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a \cong \mathbb{G}_m \times \underline{\operatorname{Aut}}_1\, \widehat{\mathbb{G}}_a,$$

where $\underline{\operatorname{Aut}}_1\, \widehat{\mathbb{G}}_a$ consists of those automorphisms with leading coefficient ξ_0 exactly equal to 1. This can be read to mean that the "interesting" part of the Steenrod algebra, $\underline{\operatorname{Aut}}_1\, \widehat{\mathbb{G}}_a$, consists of stable operations, in the sense that their action is independent of the degree–tracking mechanism.

[9]We are careful to say "functor" here because it is *not* generally another scheme.

[10]This description, too, is sensitive to the difference between $\widehat{\mathbb{G}}_a$ and $\mathbb{G}_a$. The scheme $\underline{\operatorname{End}}\, \mathbb{G}_a$ is populated by *polynomials* satisfying a homomorphism condition, and essentially none of them have inverses.

Example 1.3.10. Remembering the slogan

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a$$

also makes it easy to recall the structure formulas for the dual Steenrod algebra. For instance, consider the antipode map, which has the effect on $\underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a$ of sending a power series to its compositional inverse. That is:

$$\sum_{j=0}^{\infty} \chi(\xi_j) \left(\sum_{k=0}^{\infty} \xi_k x^{2^k} \right)^{2^j} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \chi(\xi_j) \xi_k^{2^j} x^{2^{j+k}} = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} \chi(\xi_j) \xi_k^{2^j} \right) x^{2^n} = 1,$$

from which we can extract formulas like

$$\chi(\xi_0) = \xi_0^{-1}, \qquad \chi(\xi_1) = \xi_0^{-3} \xi_1, \qquad \chi(\xi_2) = \xi_0^{-7} \xi_1^3 + \xi_0^{-5} \xi_2, \qquad \ldots.$$

Remark 1.3.11. Our interest in $\operatorname{colim}_n H\mathbb{F}_2 P^0 \mathbb{RP}^n$ is the culmination of quite a lot of definitions in basic algebraic topology, quietly lurking in the background. For instance, infinite CW-complexes are *defined* to have the weak topology coming from their skeleta, so that a continuous map off of such an infinite complex is continuous if and only if it is the colimit of a compatible system of maps. The cohomology of an infinite complex comes with a Milnor sequence, and if we arrange the situation so as not to encounter a $\lim^1$ term, it is exactly equal to the limit of the cohomology groups of the finite stages—but because all maps between complexes filter through the finite stages, all the induced maps on cohomology are necessarily continuous for the adic topology. This is exactly the phenomenon we are capturing (or, indeed, enforcing in algebra) when we track the system of finite schemes $\{\mathbb{RP}^n_{H\mathbb{F}_2 P}\}_{n=0}^{\infty}$.

In summary, the formula $\mathbb{RP}^\infty_{H\mathbb{F}_2 P} \cong \widehat{\mathbb{G}}_a$ is meant to point out that this language of formal schemes has an extremely good compression ratio—you can fit a lot of information into a very tiny space. This formula simultaneously encodes the cohomology ring of $\mathbb{RP}^\infty$ as the formal scheme, its diagonal as the group scheme structure, and the coaction of the dual Steenrod algebra by the identification with $\underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a$. As a separate wonder, it is also remarkable that the single cohomological calculation $\mathbb{RP}^\infty_{H\mathbb{F}_2 P}$ exerts such enormous control over mod–2 cohomology itself (e.g., the entire structure of the dual Steenrod algebra). We will eventually give a concrete reason for this in Lecture 4.1, but it will also turn out to be a surprisingly common occurrence even in many situations where such a direct link is not available—this is, already, one of the mysteries of the subject.

1.4 Hopf algebra cohomology

In this section, we will focus on an important classical tool: the Adams spectral sequence. We are going to study this in greater earnest later on, so we will avoid giving a satisfying construction in this Lecture. But, even without a construction, it is instructive to see how this comes about from a moral perspective.

Cite me: I first saw this presentation from Matt Ando. He must have learned it from someone. I'd like to know who to attribute this to.

Remark 1.4.1. Throughout this Lecture, we will work with graded homology groups, rather than with periodified cohomology as was the case in Lecture 1.3. This choice will remain mysterious for now, but we can at least reassure ourselves that it carries the same data as we were studying previously. Referring to our discussion of the construction of the coaction map, we see that without taking Spanier–Whitehead duals we already have an analogous coaction map on homology:

$$H\mathbb{F}_{2*}X \to H\mathbb{F}_{2*}X \otimes \mathcal{A}_*.$$

Additionally, building on the discussion in Remark 1.3.9, the splitting of the Hopf algebra shows that we are free to work gradedly or work with the periodified version of mod–2 homology, while still retaining the rest of the framework.

With this caveat out of the way, begin by considering the following three self-maps of the stable sphere:

$$\mathbb{S}^0 \xrightarrow{0} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{1} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{2} \mathbb{S}^0.$$

If we apply mod–2 homology to each line, the induced maps are

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2, \qquad \mathbb{F}_2 \xrightarrow{1} \mathbb{F}_2, \qquad \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2.$$

We see that mod–2 homology can immediately distinguish between the null map and the identity map just by its behavior on morphisms, but it cannot distinguish between the null map and the multiplication-by-2 map. To try to distinguish between these two, we use the only other tool available to us: homology theories send cofiber sequences to long exact sequences, and moreover the data of a map f and the data of the inclusion map $\mathbb{S}^0 \to C(f)$ into its cone are equivalent in the stable category. So, we trade our maps 0 and 2 for the following cofiber sequences:

$$\mathbb{S}^0 \longrightarrow C(0) \longrightarrow \mathbb{S}^1, \qquad \mathbb{S}^0 \longrightarrow C(2) \longrightarrow \mathbb{S}^1.$$

The homology groups of these spectra $C(0)$ and $C(2)$ are more complicated than just that of $\mathbb{S}^0$, and we will draw them according to the following conventions: each "$\bullet$" in the row labelled "$[j]$" indicates an $\mathbb{F}_2$–summand in the j^{th} $H\mathbb{F}_2$–homology of the spectrum. Applying homology to these cofiber sequences and drawing the results, these again appear to be identical:

$$\begin{tikzcd}[row sep=large]
{[1]} & & \bullet \arrow[r] & \bullet & & & \bullet \arrow[r] & \bullet \\
{[0]} & \bullet \arrow[r] & \bullet & & & \bullet \arrow[r] & \bullet &
\end{tikzcd}$$

$$H\mathbb{F}_{2*}\mathbb{S}^0 \to H\mathbb{F}_{2*}C(0) \to H\mathbb{F}_{2*}\mathbb{S}^1, \qquad H\mathbb{F}_{2*}\mathbb{S}^0 \to H\mathbb{F}_{2*}C(2) \to H\mathbb{F}_{2*}\mathbb{S}^1,$$

However, if we enrich our picture with the data we discussed in Lecture 1.3, we can finally see the difference. Recall the topological equivalences

$$C(0) \simeq \mathbb{S}^0 \vee \mathbb{S}^1, \quad C(2) \simeq \Sigma^{-1}\Sigma^\infty \mathbb{R}\mathrm{P}^2.$$

In the two cases, the coaction map λ_* is given by

$$\begin{array}{ll}
\lambda_* : H\mathbb{F}_{2*}C(0) \to H\mathbb{F}_{2*}C(0) \otimes \mathcal{A}_* & \lambda_* : H\mathbb{F}_{2*}C(2) \to H\mathbb{F}_{2*}C(2) \otimes \mathcal{A}_* \\
\qquad \lambda^* : e_0 \mapsto e_0 \otimes 1 & \qquad \lambda^* : e_0 \mapsto e_0 \otimes 1 + e_1 \otimes \xi_1 \\
\qquad \lambda^* : e_1 \mapsto e_1 \otimes 1, & \qquad \lambda^* : e_1 \mapsto e_1 \otimes 1.
\end{array}$$

We use a vertical line to indicate the nontrivial coaction involving ξ_1:

$$\begin{tikzcd}[row sep=large]
{[1]} & & \bullet \arrow[r] & \bullet & & & \bullet \arrow[r] \arrow[d, dash, "\xi_1"] & \bullet \\
{[0]} & \bullet \arrow[r] & \bullet & & & \bullet \arrow[r] & \bullet &
\end{tikzcd}$$

$$H\mathbb{F}_{2*}\mathbb{S}^0 \to H\mathbb{F}_{2*}C(0) \to H\mathbb{F}_{2*}\mathbb{S}^1, \qquad H\mathbb{F}_{2*}\mathbb{S}^0 \to H\mathbb{F}_{2*}C(2) \to H\mathbb{F}_{2*}\mathbb{S}^1.$$

We can now see what trading maps for cofiber sequences has bought us: mod–2 homology can distinguish the defining sequences for $C(0)$ and $C(2)$ by considering their induced extensions of comodules over $\mathcal{A}_*$. The Adams spectral sequence bundles this thought process into a single machine:

Can this be phrased so as to indicate how this works for longer extensions? I've never tried to think about even what happens for $C(4)$.

Theorem 1.4.2 ([Rav86, Definition 2.1.8, Lemma 2.1.16], [MT68, Chapter 18]). *There is a convergent spectral sequence of signature*

$$\operatorname{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow (\pi_* \mathbb{S}^0)^\wedge_2. \qquad \square$$

In effect, this asserts that the above process is *exhaustive*: every element of $(\pi_* \mathbb{S}^0)^\wedge_2$ can be detected and distinguished by some representative class of extensions of comodules for the dual Steenrod algebra. Mildly more generally, if X is a bounded-below spectrum, then there is even a spectral sequence of signature

$$\operatorname{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X) \Rightarrow \pi_* X^\wedge_2.$$

We could now work through the construction of the Adams spectral sequence, but it will fit more nicely into a story later on in Lecture 3.1. Before moving on to other pursuits, however, we will record the following utility Lemma. It is believable based on the above discussion, and we will need to use it before we get around to examining the guts of the construction.

Lemma 1.4.3 (cf. Remark 3.1.16). *The 0–line of the Adams spectral sequence consists of exactly those elements visible to the Hurewicz homomorphism.* $\square$

For the rest of the section, we will focus on the algebraic input "$\operatorname{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$", which will require us to grapple with the homological algebra of comodules for a Hopf algebra. To start that discussion, it's both reassuring and instructive to see that homological algebra can, in fact, even be done with comodules. In the usual development of homological algebra for *modules*, the key observations are the existence of projective and injective modules, and there is something at work similar here.

Remark 1.4.4 ([Rav86, Appendix A1]). Much of the results below do not rely on working with a Hopf algebra over the field $k = \mathbb{F}_2$. In fact, k can usually be taken to be a ring rather than a field. More generally, the theory goes through in the context of comodules over flat Hopf algebroids, cf. also Lemma 3.1.15.

Lemma 1.4.5 ([Rav86, Definition A1.2.1]). *Let A be a Hopf k–algebra, let M be an A–comodule, and let N be a k–module. There is a cofree adjunction:*

$$\mathsf{Comodules}_A(M, N \otimes_k A) \cong \mathsf{Modules}_k(M, N),$$

where $N \otimes_k A$ is given the structure of an A–comodule by the coaction map

$$N \otimes_k A \xrightarrow{\mathrm{id} \otimes \Delta} N \otimes_k (A \otimes_k A) = (N \otimes_k A) \otimes_k A.$$

Proof. Given a map $f\colon M \to N$ of k–modules, we can build the composite

$$M \xrightarrow{\psi_M} M \otimes_k A \xrightarrow{f \otimes \mathrm{id}_A} N \otimes_k A.$$

Alternatively, given a map $g\colon M \to N \otimes_k A$ of A–comodules, we build the composite

$$M \xrightarrow{g} N \otimes_k A \xrightarrow{\mathrm{id}_N \otimes \varepsilon} N \otimes_k k = N. \qquad \square$$

Corollary 1.4.6 ([Rav86, Lemma A1.2.2]). *The category* $\mathsf{Comodules}_A$ *has enough injectives. Namely, if M is an A–comodule and $M \to I$ is an inclusion of k–modules into an injective k–module I, then $M \to I \otimes_k A$ is an injective A–comodule under M.* $\square$

Remark 1.4.7. In our case, M itself is always k–injective, so there's already an injective map $\psi_M : M \to M \otimes A$: the coaction map. The assertion that this map is coassociative is identical to saying that it is a map of comodules.

Satisfied that "Ext" at least makes sense, we're free to pursue more conceptual ends. Recall from algebraic geometry that a module M over a ring R is equivalent data to quasicoherent sheaf $\widetilde{M}$ over $\operatorname{Spec} R$. We now give a definition of "quasicoherent sheaf" that fits with our functorial perspective:

Definition 1.4.8 ([Hov02, Definition 1.1], [Str99b, Definition 2.42]). A presheaf (of modules) over a scheme X is an assignment $\mathcal{F}\colon X(T) \to \mathsf{Modules}_T$, satisfying a kind of functoriality in T: for each map $f\colon T \to T'$, there is a compatible choice of natural transformation (between the *not necessarily equal composites*)

$$\begin{tikzcd}
X(T) \arrow[r, "\mathcal{F}(T)"] \arrow[d, "X(f)"'] \arrow[dr, phantom, "\Swarrow \tau(f)"] & \mathsf{Modules}_T \arrow[d, "- \otimes_T T'"] \\
X(T') \arrow[r, "\mathcal{F}(T')"'] & \mathsf{Modules}_{T'}.
\end{tikzcd}$$

(We think of the image of a particular point $t\colon \operatorname{Spec} T \to X$ in $\mathsf{Modules}_T$ as the module of "sections over t".) Such a presheaf is said to be a *quasicoherent sheaf* when these natural transformations are all natural isomorphisms.

Lemma 1.4.9 ([Str99b, Proposition 2.47]). *An R–module M gives rise to a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ by the rule*

$$(\operatorname{Spec} T \to \operatorname{Spec} R) \mapsto M \otimes_R T.$$

Conversely, every quasicoherent sheaf over an affine scheme arises in this way. □

The tensoring operation appearing in the definition of a presheaf appears more generally as an operation on the category of sheaves.

Definition 1.4.10. A map $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ induces maps $f^* \dashv f_*$ of categories of quasicoherent sheaves. At the level of modules, these are given by

Jay was frustrated with which adjoint I put on top (and perhaps which went on which side). Apparently there's some convention, which I should look up and obey.

$$\begin{tikzcd}
\mathsf{QCoh}_{\operatorname{Spec} R} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \mathsf{QCoh}_{\operatorname{Spec} S} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\
\mathsf{Modules}_R \arrow[r, shift left, "M \mapsto M \otimes_R S"] & \mathsf{Modules}_S. \arrow[l, shift left, "N \leftarrow\!\shortmid N"]
\end{tikzcd}$$

One of the main uses of these operations is to define the cohomology of a sheaf. Let $\pi\colon X \to \operatorname{Spec} k$ be a scheme over $\operatorname{Spec} k$, k a field, and let $\mathcal{F}$ be a sheaf over X. The adjunction above induces a derived adjunction

Why doesn't π^* need to be left-derived?

$$\operatorname{Ext}_X(\pi^* k, \mathcal{F}) \cong \operatorname{Ext}_{\operatorname{Spec} k}(k, R\pi_* \mathcal{F}),$$

which is used to translate the *definition* of sheaf cohomology to that of the cohomology of the derived pushforward $R\pi_*\mathcal{F}$, itself interpretable as a mere complex of k–modules. This pattern is very general: the sense of "cohomology" relevant to a situation is often accessed by taking the derived pushforward to a suitably terminal object.[11] To invent a notion of cohomology for comodules over a Hopf algebra, we are thus moved to produce push and pull functors for a map of Hopf algebras, and this is best motivated by another example.

[11] This perspective often falls under the heading of "six–functor formalism".

Example 1.4.11. A common source of Hopf algebras is through group–rings: given a group G, we can define the Hopf k–algebra $k[G]$ consisting of formal k–linear combinations of elements of G. This Hopf algebra is commutative exactly when G is abelian, and $k[G]$–modules are naturally equivalent to k–linear G–representations. Dually, the ring k^G of k–valued functions on G is always commutative, using pointwise multiplication of functions, and it is *cocommutative* exactly when G is abelian. If G is finite, then k^G and $k[G]$ are k–linear dual Hopf algebras, and hence finite–dimensional k^G–comodules are naturally equivalent to finite–dimensional k–linear G–representations.[12]

A map of groups $f\colon G \to H$ induces a map $k^f\colon k^H \to k^G$ of Hopf algebras, and it is reasonable to expect that the induced push and pull maps of comodules mimic those of G– and H–representations. Namely, given an H–representation M, we can produce a corresponding G–representation by precomposition with f. However, given a G–representation N, two features may have to be corrected to extract an H–representation:

1. If f is not surjective, we must decide what to do with the extra elements in H.

2. If f is not injective—say, $f(g_1) = f(g_2)$—then we must force the behavior of the extracted H–representation to agree on $f(g_1)$ and $f(g_2)$, even if g_1 and g_2 act differently on N. In the extreme case of $f\colon G \to 1$, we expect to recover the fixed points of N, since this pushforward computes $H^0_{\mathrm{gp}}(G; N)$.

These concerns, together with the definition of a tensor product as a coequalizer, motivate the following:

Definition 1.4.12. Given A–comodules M and N, their cotensor product is the k-module defined by the equalizer

$$M \mathbin{\square}_A N \to M \otimes_k N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} M \otimes_k A \otimes_k N.$$

[12] There is a variation on this equivalence that uses fewer dualities and which is instructive to expand. The Hopf algebra $k^G = \prod_{g \in G} k$ is the ring of functions on the constant group scheme G, and its k–points $(\operatorname{Spec} k^G)(k)$ biject with points in G. Namely, given $g \in G$ we can form a projection map $g\colon k^G \to k$ and hence a composite $M \to M \otimes_k k^G \xrightarrow{\mathrm{id} \otimes g} M \otimes_k k \cong M$. Collectively, this determines a map $G \times M \to M$ witnessing M as a G–representation. In the other direction, if G is finite then we can construct a map $M \to M \otimes_k \prod_{g \in G} k$ sending $m \in M$ to $g \cdot m$ in the g^{th} labeled component of the target.

Lemma 1.4.13. *Given a map $f\colon A \to B$ of Hopf k–algebras, the induced adjunction $f^* \dashv f_*$ is given at the level of comodules by*

$$\begin{tikzcd} \mathsf{QCoh}_{\operatorname{Spec} k /\!/ \operatorname{Spec} A} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \mathsf{QCoh}_{\operatorname{Spec} k /\!/ \operatorname{Spec} B} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\ \mathsf{Comodules}_A \arrow[r, shift left, "M \mapsto M"] & \mathsf{Comodules}_B . \arrow[l, shift left, "N \Box_B A \leftarrow\!\shortmid N"] \end{tikzcd} \qquad \Box$$

Remark 1.4.14. In Lecture 3.1 (and Definition 3.1.13 specifically), we will explain the notation "$\operatorname{Spec} k /\!/ \operatorname{Spec} A$" used above. For now, suffice it to say that there again exists a functor-of-points notion of "quasicoherent sheaf" associated to a Hopf k–algebra A, and such sheaves are equivalent to A–comodules.

As an example application, cotensoring gives rise to a concise description of what it means to be a comodule map:

Lemma 1.4.15 ([Rav86, Lemma A1.1.6b])**.** *Let M and N be A–comodules with M projective as a k–module. Then there is an equivalence*

$$\mathsf{Comodules}_A(M, N) = \mathsf{Modules}_k(M, k) \Box_A N. \qquad \Box$$

From this, we can deduce a connection between the push–pull flavor of comodule cohomology described above and the input to the Adams spectral sequence.

Corollary 1.4.16. *Let $N = N' \otimes_k A$ be a cofree comodule. Then $N \Box_A k = N'$.*

Proof. Picking $M = k$, we have

$$\begin{aligned} \mathsf{Modules}_k(k, N') &= \mathsf{Comodules}_A(k, N) \\ &= \mathsf{Modules}_k(k, k) \Box_A N \\ &= k \Box_A N. \end{aligned} \qquad \Box$$

Corollary 1.4.17. *There is an isomorphism*

$$\mathsf{Comodules}_A(k, N) = \mathsf{Modules}_k(k, k) \Box_A N = k \Box_A N$$

and hence

$$\operatorname{Ext}_A(k, N) \cong \operatorname{Cotor}_A(k, N) \, (= H^* R\pi_* N) \, .$$

Proof. Resolve N using the cofree modules described above, then apply either functor $\mathsf{Comodules}_A(k, -)$ or $k \,\square_A\, -$. In both cases, you get the same complex. $\square$

Example 1.4.18. In Lecture 1.3, we identified $\mathcal{A}_*$ with the ring of functions on the group scheme $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ of strict automorphisms of $\widehat{\mathbb{G}}_a$, which is defined by the kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

The punchline is that this is analogous to Example 1.4.11 above: $\mathrm{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$ is thought of as "the derived fixed points" of "$G = \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$" on the "$G$–module" $H\mathbb{F}_{2*}X$.

We now give several examples to get a sense of how the Adams spectral sequence behaves.

Example 1.4.19. Consider the degenerate case $X = H\mathbb{F}_2$. Then $H\mathbb{F}_{2*}(H\mathbb{F}_2) = \mathcal{A}_*$ is a cofree comodule, and hence Cotor is concentrated on the 0–line:

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}(H\mathbb{F}_2)) = \mathbb{F}_2.$$

The Adams spectral sequence collapses to show the wholly unsurprising equality $\pi_* H\mathbb{F}_2 = \mathbb{F}_2$, and indeed this is the element in the image of the Hurewicz map $\pi_* H\mathbb{F}_2 \to H\mathbb{F}_{2*}H\mathbb{F}_2$.

Example 1.4.20. In the slightly less degenerate case of $X = H\mathbb{Z}$, one can calculate

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}H\mathbb{Z}) \cong \mathrm{Cotor}^{*,*}_{\Lambda[\xi_1]}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0].$$

This spectral sequence collapses, and the additive extensions cause it to converge to $\mathbb{Z}_2^\wedge$ in degree 0 and 0 in all other degrees. The governing element ξ_1 is precisely the 2–*adic Bockstein*, which mediates the difference between trivial and nontrivial extensions of $\mathbb{Z}/2^j$ by $\mathbb{Z}/2$.

Example 1.4.21. Next, we consider the more computationally serious case of $X = kO$, the connective real K–theory spectrum. The main input we need is the structure of $H\mathbb{F}_{2*}kO$ as an $\mathcal{A}_*$–comodule, so that we can compute

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}kO) \Rightarrow \pi_* kO_2^\wedge.$$

There is a slick trick for doing this: by working in the category of kO–modules rather than in all spectra, we can construct a relative Adams spectral sequence

$$\mathrm{Cotor}^{*,*}_{\pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2}(\mathbb{F}_2, \pi_* H\mathbb{F}_2 \wedge_{kO} (kO \wedge H\mathbb{F}_2)) \Rightarrow \pi_*(kO \wedge H\mathbb{F}_2).$$

The second argument is easy to identify:

$$\pi_* H\mathbb{F}_2 \wedge_{kO} (kO \wedge H\mathbb{F}_2) = \pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 = \mathcal{A}_*.$$

The Hopf algebra requires further input. Consider the following trio of cofiber sequences[13]:

$$\Sigma kO \xrightarrow{\cdot\eta} kO \to kU, \qquad \Sigma^2 kU \xrightarrow{\cdot\beta} kU \to H\mathbb{Z}, \qquad H\mathbb{Z} \xrightarrow{\cdot 2} H\mathbb{Z} \to H\mathbb{F}_2.$$

These combine to give a resolution of $H\mathbb{F}_2$ via an iterated cofiber of free kO–modules, with Poincaré series

$$((1+t^2)+t^3(1+t^2)) + t((1+t^2)+t^3(1+t^2)) = 1+t+t^2+2t^3+t^4+t^5+t^6.$$

Repeatedly using the identity $kO \wedge_{kO} H\mathbb{F}_2 \simeq H\mathbb{F}_2$ gives a small presentation of the Hopf algebra $\pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2$: it is a commutative Hopf algebra over $\mathbb{F}_2$ with the above Poincaré series. The Borel–Milnor–Moore [MM65, Theorem 7.11] classification of commutative Hopf algebras over $\mathbb{F}_2$ shows that the algebra structure is either

$$\frac{\mathbb{F}_2[a,b,c]}{(a^2=0, b^2=0, c^2=0)} \quad \text{or} \quad \frac{\mathbb{F}_2[a,b,c]}{(a^2=b, b^2=0, c^2=0)}$$

for $|a|=1$, $|b|=2$, and $|c|=3$. By knowing that the natural map $\mathcal{A} \to \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2$ winds up inducing an isomorphism $\pi_{*\le 2}\mathbb{S} \to \pi_{*\le 2} kO$, we conclude that we are in the latter case, which gives a presentation of the Hopf algebra as a whole:

$$\begin{tikzcd}
\pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 \arrow[r] \arrow[d, equal] & \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2 \arrow[d, equal] \\
\mathcal{A}_* \arrow[r] & \dfrac{\mathbb{F}_2[\xi_1, \xi_2, \xi_3, \xi_4, \ldots]}{(\xi_1^4, \xi_2^2), (\xi_n \mid n \ge 3)}.
\end{tikzcd}$$

This Hopf algebra is commonly denoted $\mathcal{A}(1)_*$, and its corresponding subgroup scheme $\operatorname{Spec} \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2 \subseteq \underline{\mathrm{Aut}}_1 \widehat{\mathbb{G}}_a$ admits easy memorization: it is the subscheme of automorphisms of the form $x + \xi_1 x^2 + \xi_2 x^4$, with exactly the additional

[13] This first sequence, known as the Wood cofiber sequence, is a consequence of a very simple form of Bott periodicity [Har80, Section 5]: there is a fiber sequence of infinite-loopspaces $O/U \to BO \to BU$, and $\underline{kO}_1 = O/U$.

relations imposed on ξ_1 and ξ_2 so that this set is stable under composition and inversion.[14,15]

Formal geometry aside, this kO–based Adams spectral sequence collapses, giving an isomorphism

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathcal{A}_*) = \mathcal{A}_* /\!/ \mathcal{A}(1)_* \cong H\mathbb{F}_{2*}kO.$$

In turn, the original Adams spectral sequence takes the form

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathcal{A} /\!/ \mathcal{A}(1)_*) \cong \mathrm{Cotor}^{*,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_* kO.$$

This spectral sequence is also collapsing, and we provide a picture of it in Figure 1.1. In particular, eight-fold real Bott periodicity can be quickly read off from this picture.

Example 1.4.22. At the other extreme, we can pick the extremely nondegenerate case $X = \mathbb{S}$, where $\underline{\mathrm{Aut}}_1 \widehat{\mathbb{G}}_a$ acts maximally nonfreely on $\widetilde{\mathbb{F}_2}$. The resulting spectral sequence is pictured through a range in Figure 1.2. It is worth remarking that some of the stable stems receive identifiable names in this language: for instance, the elements of $H^1_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); \widetilde{\mathbb{F}_2})$ are exactly the 1–cocycles

$$h_j \colon \left(f = x + \sum_{j=1}^{\infty} \xi_j x^{2^j}\right) \mapsto \xi_j.$$

The cocycle h_j transforms in the $\mathbb{G}_m$–character $z \mapsto z^{2^j}$, hence h_1 is a name for the lone element in the spectral sequence contributing to $\pi_1 \mathbb{S}$, which we also know to be η. In general, the elements h_j selecting the power series coefficients are called the *Hopf invariant* 1 *elements*, and their survival or demise in the Adams spectral sequence is directly related to the problem of putting H–space structures on spheres.

Cite me: VFoS problem.

Jon asked: spectral sequences coming from π_* of a Tot tower increase Tot degree. ANSS differentials decrease degree: they run against the multiplicative structure in pictures. What's going on with this? I think this is a duality effect: working with the Steenrod algebra versus its dual.

[14] A similar analysis shows that $H\mathbb{F}_{2*}H\mathbb{Z}$ corepresents the subscheme of automorphisms of the form $x + \xi_1 x^2$ which are stable under composition and inversion.

[15] There is also an accidental isomorphism of this Hopf algebra with $\mathbb{F}_2^{D_4}$, where D_4 is the dihedral group with 8 elements.

1.5 The unoriented bordism ring

Our goal in this section is to use our results so far to make a calculation of $\pi_* MO$, the unoriented bordism ring. Our approach is the same as in the examples at the end of the previous section: we will want to use the Adams spectral sequence of signature

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)^{\sim}) \Rightarrow \pi_* MO,$$

which requires understanding $H\mathbb{F}_2 P_0(MO)$ as a comodule for the dual Steenrod algebra.

Our first step toward this is the following calculation:

Lemma 1.5.1 ([Swi02, Theorem 16.17]). *The natural map*

$$\mathrm{Sym}\, \widetilde{H\mathbb{F}_2 P_0}(BO(1)) \to H\mathbb{F}_2 P_0(BO).$$

induces a map

$$\mathrm{Sym}\, \widetilde{H\mathbb{F}_2 P_0}(BO(1)) = \frac{\mathrm{Sym}\, H\mathbb{F}_2 P_0(BO(1))}{\beta_0 = 0} \xrightarrow{\simeq} H\mathbb{F}_2 P_0(BO)$$

which is an isomorphism of Hopf algebras and of comodules for the dual Steenrod algebra.

Proof. This follows from a combination of standard facts about Stiefel–Whitney classes. First, these classes generate the cohomology ring $H\mathbb{F}_2^* BO(n)$: **Cite me:** You could cite these standard facts..

$$H\mathbb{F}_2^* BO(n) \cong \mathbb{F}_2[\![w_1, \ldots, w_n]\!].$$

Second, the total Stiefel–Whitney class is exponential, in the sense of ... which you haven't defined.

$$w(V \oplus W) = w(V) \cdot w(W).$$

From this, it follows that the natural map

$$H\mathbb{F}_2^* BO(n) \xrightarrow{\bigoplus_{j=1}^n \mathcal{L}_j} H\mathbb{F}_2^* BO(1)^{\times n} \cong (H\mathbb{F}_2^* BO(1))^{\otimes n}$$

is the inclusion of the symmetric polynomials, by calculating the total Stiefel–Whitney class

$$w\left(\bigoplus_{j=1}^n \mathcal{L}_j\right) = \prod_{j=1}^n (1 + w_1(\mathcal{L}_j)) = \sum_{j=0}^n \sigma_j(w_1(\mathcal{L}_1), \ldots, w_1(\mathcal{L}_n)) t^j.$$

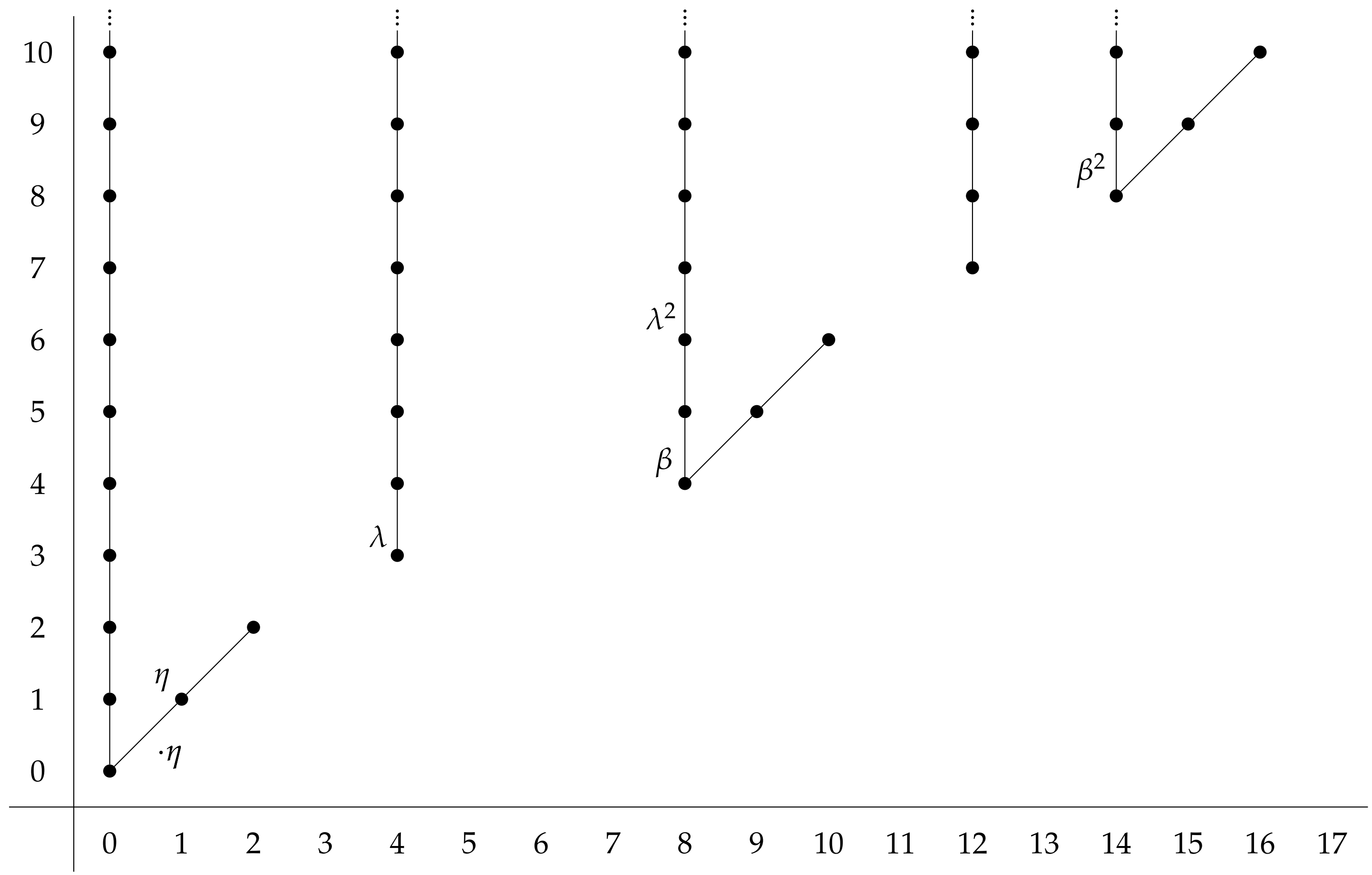


Figure 1.1: The $H\mathbb{F}_2$–Adams spectral sequence for kO, which collapses at the second page. North and north-east lines denote multiplication by 2 and by η.

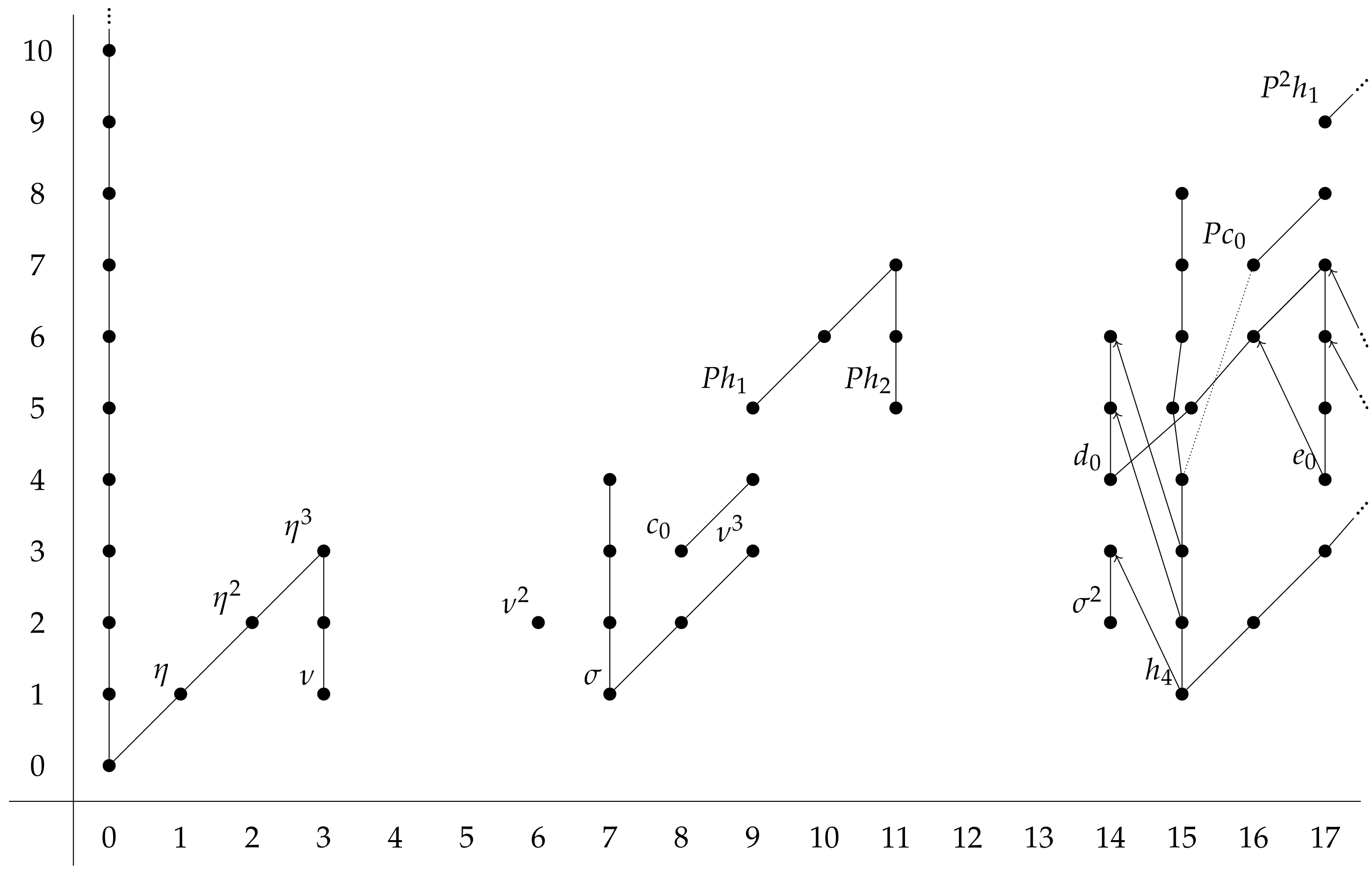


Figure 1.2: A small piece of the $H\mathbb{F}_2$–Adams spectral sequence for the sphere, beginning at the second page [Rav78, pg. 412]. North and north-east lines denote multiplication by 2 and by η, north-west lines denote d_2– and d_3–differentials.

Dually, the homological map

$$(H\mathbb{F}_{2*}BO(1))^{\otimes n} \to H\mathbb{F}_{2*}BO(n)$$

is surjective, modeling the quotient from the tensor product to the symmetric tensor product. Stabilizing as $n \to \infty$, we recover the statement of the Lemma. □

With this in hand, we now turn to the homotopy ring $H\mathbb{F}_2 P_0 MO$. There are two equivalences that we might consider employing. We have the Thom isomorphism:

$$\begin{array}{ccc} H\mathbb{F}_2P_0(BO(1)) & \Longleftrightarrow & H\mathbb{F}_2P_0(MO(1)) \\ \beta_j, j \geq 0 & \longmapsto & \beta'_j, j \geq 0, \end{array}$$

and we also have the equivalence induced by the topological map in Example 1.1.3:

$$\begin{array}{ccc} \widetilde{H\mathbb{F}_2P}_0(BO(1)) & \Longleftrightarrow & H\mathbb{F}_2P_0(\Sigma MO(1)) \\ \beta_j, j \geq 1 & \longmapsto & \beta'_{j-1}, j \geq 1. \end{array}$$

We will use them both in turn.

Corollary 1.5.2 ([Ada95, Section I.3], [Hop, Proposition 6.2]). *There is an isomorphism*

$$H\mathbb{F}_2P_0(MO) \cong \frac{\operatorname{Sym} H\mathbb{F}_2P_0MO(1)}{b'_0 = 1}.$$

Proof. The block sum maps

$$BO(n) \times BO(m) \to BO(n+m)$$

Thomify to give compatible maps

$$MO(n) \wedge MO(m) \to MO(n+m).$$

Taking the colimit, this gives a ring structure on MO compatible with that on $\Sigma^\infty_+ BO$ and compatible with the Thom isomorphism. □

We now seek to understand the utility of the scheme $\operatorname{Spec} H\mathbb{F}_2P_0(MO)$, as well as its action of $\underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a)$. The first of these tasks comes from untangling some of the topological dualities we've been using thus far.

Lemma 1.5.3. *The following square commutes:*

$$\begin{tikzcd} \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[r, equal] & \mathsf{Spectra}(MO, H\mathbb{F}_2P) \\ \mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[u, hook] \arrow[r, equal] & \mathsf{RingSpectra}(MO, H\mathbb{F}_2P). \arrow[u, hook] \end{tikzcd}$$

Proof. The top isomorphism asserts only that $\mathbb{F}_2$–cohomology and $\mathbb{F}_2$–homology are linearly dual to one another. The second follows immediately from investigating the effect of the ring homomorphism diagrams in the bottom-right corner in terms of the subset they select in the top-left. □

Corollary 1.5.4. *There is a bijection between homotopy classes of ring maps $MO \to H\mathbb{F}_2P$ and homotopy classes of factorizations*

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r] \arrow[dr] & MO(1) \arrow[d, dotted] \\ & H\mathbb{F}_2P. \end{tikzcd}$$

Proof. We extend the square in the Lemma 1.5.3 using the following diagram:

$$\begin{tikzcd} \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO(1)), \mathbb{F}_2) & \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[l] \\ \{f \colon H\mathbb{F}_2P_0(MO(1)) \to \mathbb{F}_2 \mid f(\beta_0') = 1\} \arrow[u, hook] \arrow[r, equal] & \mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2), \arrow[u, hook] \end{tikzcd}$$

where the equality at bottom follows from the universal property of $H\mathbb{F}_2P_0(MO)$ in $\mathbb{F}_2$–algebras expressed in Corollary 1.5.2. Noting that β_0' is induced by the topological map $\mathbb{S}^0 \to MO(1)$, the condition $f(\beta_0') = 1$ is exactly the condition expressed in the statement of the Corollary. □

Corollary 1.5.5. *There is an $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$–equivariant isomorphism of schemes*

Cite me: I wish I could find this in the literature.

$$\operatorname{Spec} H\mathbb{F}_2P_0(MO) \cong \operatorname{Coord}_1(\mathbb{RP}^\infty_{H\mathbb{F}_2P}),$$

where the latter is the subscheme of functions $\mathbb{RP}^\infty_{H\mathbb{F}_2P} \to \widehat{\mathbb{A}}^1$ which are coordinates (i.e., which are isomorphisms *of formal schemes—or, equivalently, which restrict to the canonical identification of tangent spaces $\mathbb{RP}^1_{H\mathbb{F}_2P} = \widehat{\mathbb{A}}^{1,(1)}$).*

Proof. The conclusion of the previous Corollary is that the $\mathbb{F}_2$–points of $\operatorname{Spec} H\mathbb{F}_2P_0(MO)$ biject with classes $H\mathbb{F}_2P^0MO(1) \cong \widetilde{H\mathbb{F}_2P}^0\mathbb{RP}^\infty$ satisfying the condition that they give an isomorphism $\mathbb{RP}^\infty_{H\mathbb{F}_2P}$. Because $H\mathbb{F}_2P_0(MO)$ is a polynomial algebra, this holds in general: for $u\colon \mathbb{F}_2 \to T$ an $\mathbb{F}_2$–algebra, the T–points of $\operatorname{Spec} H\mathbb{F}_2P_0(MO)$ will biject with coordinates on $u^*\mathbb{RP}^\infty_{H\mathbb{F}_2P}$. The isomorphism of schemes follows, though we have not yet discussed equivarience.

To compute the action of $\underline{\operatorname{Aut}}\,\widehat{\mathbb{G}}_a$, we turn to the map in Example 1.1.3:

$$\Sigma^\infty BO(1) \xrightarrow{c,\simeq} \Sigma MO(1).$$

Writing $\beta(t) = \sum_{j=0}^\infty \beta_j t^j$ and $\xi(t) = \sum_{k=0}^\infty \xi_k t^{2^k}$, the dual Steenrod coaction on $H\mathbb{F}_2P_0BO(1)$ is encoded by the formula

$$\sum_{j=0}^\infty \psi(\beta_j)t^j = \psi(\beta(t)) = \beta(\xi(t)) = \sum_{j=0}^\infty \beta_j \left(\sum_{k=0}^\infty \xi_k t^{2^k}\right)^j.$$

Because $c_*(\beta_j) = \beta'_{j-1}$, this translates to the formula $\psi(\beta'(t)) = \beta'(\xi(t))$, where

$$\beta'(t) = \sum_{j=0}^\infty \beta'_j t^{j+1}.$$

Passing from $H\mathbb{F}_2P_0(MO(1))$ to $H\mathbb{F}_2P_0(MO) \cong \operatorname{Sym} H\mathbb{F}_2P_0(MO(1))/(\beta'_0 = 1)$, this is precisely the formula for precomposing a coordinate with a strict automorphism—i.e., a point in $\operatorname{Aut}_1(\widehat{\mathbb{G}}_a)$ acts on a point in $\operatorname{Coord}(\mathbb{RP}^\infty_{H\mathbb{F}_2P})$ in the way claimed. □

We are now ready to analyze the group cohomology of $\underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a)$ with coefficients in the comodule $H\mathbb{F}_2P_0(MO)$. This is the last piece of input we need to assess the Adams spectral sequence computing π_*MO.

Theorem 1.5.6 ([Str06, Theorem 12.2], [Mit83, Proposition 2.1])**.** *The action of* $\underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a)$ *on* $\operatorname{Coord}_1(\widehat{\mathbb{G}}_a)$ *is free:*

$$\operatorname{Coord}_1(\widehat{\mathbb{G}}_a) \cong \operatorname{Spec} \mathbb{F}_2[b_j \mid j \neq 2^k - 1] \times \underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a).$$

Proof. Recall, again, that $\underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a)$ is defined by the (split) kernel sequence

$$0 \to \underline{\operatorname{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Consider a point $f \in \mathrm{Coord}_1(\widehat{\mathbb{G}}_a)(R)$, which in terms of the standard coordinate can be expressed as

$$f(x) = \sum_{j=1}^{\infty} b_{j-1} x^j,$$

where $b_0 = 1$. Decompose this series as $f(x) = f_2(x) + f_{\mathrm{rest}}(x)$, with

$$f_2(x) = \sum_{k=0}^{\infty} b_{2^k-1} x^{2^k}, \qquad f_{\mathrm{rest}}(x) = \sum_{j \neq 2^k} b_{j-1} x^j.$$

Because we assumed $b_0 = 1$ and f_2 is concentrated in power–of–2 degrees, it follows that f_2 gives a point $f_2 \in \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$. We can use it to de-scale and get a new coordinate $g(x) = f_2^{-1}(f(x))$, which has an analogous decomposition into series $g_2(x)$ and $g_{\mathrm{rest}}(x)$. Finally, note that $g_2(x) = x$ and that f_2 is the unique point in $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$ that has this property. □

Corollary 1.5.7 ([Str06, Remark 12.3])**.** *$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1, j \geq 1]$ with $|b_j| = j$.*

Proof. Set $M = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]$, and write $\mathcal{A}P_0'$ for the ring of functions on $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$. It follows from Corollary 1.4.16 applied to Theorem 1.5.6 that the $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$–cohomology of $H\mathbb{F}_2 P_0(MO)$ has amplitude 0:

$$\begin{aligned}
\mathrm{Cotor}^{*,*}_{\mathcal{A}P_0'}(\mathbb{F}_2, H\mathbb{F}_2 P_0(MO)) &= \mathrm{Cotor}^{*,*}_{\mathcal{A}P_0'}(\mathbb{F}_2, \mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\
&= \mathbb{F}_2 \,\square_{\mathcal{A}_*} (\mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\
&= \mathbb{F}_2 \otimes_{\mathbb{F}_2} M = M.
\end{aligned}$$

Since the Adams spectral sequence

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)) \Rightarrow \pi_* MO$$

is concentrated on the 0–line, it collapses. Using the residual $\mathbb{G}_m$–action to infer the grading, we deduce

$$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]. \qquad \square$$

This is pretty remarkable: some statement about manifold geometry came down to understanding how we could reparametrize a certain formal group, itself a (fairly simple) purely algebraic problem. The connection between these two problems seems fairly miraculous: we needed a small object, $\mathbb{RP}^\infty$, which controlled the whole story; we needed to be able to compute everything about it; and we needed various other "generation" or "freeness" results to work out in our favor. It is not obvious that we will get this lucky twice, should we try to reapply these ideas to other cases. Nevertheless, trying to push our luck as far as possible is the main thrust of the rest of the book. We could close this section with this accomplishment, but there are two easy consequences of this calculation that are worth recording before we leave.

Lemma 1.5.8. *MO splits as a wedge of shifts of $H\mathbb{F}_2$.*

Proof. Referring to Lemma 1.4.3, we find that the Hurewicz map induces a π_*–injection $MO \to H\mathbb{F}_2 \wedge MO$. Pick an $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha$ for $\pi_* MO$ and extend it to an $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha \cup \{w_\beta\}_\beta$ for $\pi_* H\mathbb{F}_2 \wedge MO$. Altogether, this larger basis can be represented as a single map

$$\bigvee_\alpha \Sigma^{|v_\alpha|}\mathbb{S} \vee \bigvee_\beta \Sigma^{|w_\beta|}\mathbb{S} \xrightarrow{\bigvee_\alpha v_\alpha \vee \bigvee_\beta w_\beta} H\mathbb{F}_2 \wedge MO.$$

Smashing through with $H\mathbb{F}_2$ gives an equivalence

$$\bigvee_\alpha \Sigma^{|v_\alpha|}H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{|w_\beta|}H\mathbb{F}_2 \xrightarrow{\simeq} H\mathbb{F}_2 \wedge MO.$$

The composite map

$$MO \to H\mathbb{F}_2 \wedge MO \xleftarrow{\simeq} \bigvee_\alpha \Sigma^{|v_\alpha|}H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{|w_\beta|}H\mathbb{F}_2 \to \bigvee_\alpha \Sigma^{|v_\alpha|}H\mathbb{F}_2$$

is a weak equivalence. □

Remark 1.5.9. Just using that $\pi_* MO$ is connective and $\pi_0 MO = \mathbb{F}_2$, we can produce a ring spectrum map $MO \to H\mathbb{F}_2$. What we have learned is that this map has a splitting: MO is also an $H\mathbb{F}_2$–algebra.

Remark 1.5.10. We are also in a position to understand the stable cooperations $MO_* MO$. We may rewrite this as

$$\begin{aligned} MO_*MO &= \pi_* MO \wedge MO = \pi_* MO \wedge_{H\mathbb{F}_2} (H\mathbb{F}_2 \wedge MO) \\ &\Leftarrow \operatorname{Tor}^{\mathbb{F}_2}_{*,*}(MO_*, H\mathbb{F}_{2*}MO) = MO_* \otimes_{\mathbb{F}_2} H\mathbb{F}_{2*}MO. \end{aligned}$$

Hence, a point in Spec $MO_* MO$ consists of a pair of points in Spec MO_* and Spec $H\mathbb{F}_{2*} MO$, which we have already identified respectively as formal group laws with vanishing 2–series, a property, and formal group laws with specified logarithms, data. This description can be amplified to capture all of the structure maps: a formal group law with vanishing 2–series admits a logarithm, which indicates how the composition and conjugation maps of Definition 3.1.13 behave.

A referee suggested mentioning the resolution of Steenrod's realization problem.

Case Study 2

Complex bordism

Having totally dissected unoriented bordism, we can now turn our attention to other sorts of bordism theories, and there are many available: oriented, *Spin*, *String*, complex, ...—the list continues. We would like to replicate the results from Case Study 1 for these other cases, but upon even a brief inspection we quickly see that only one of the bordism theories mentioned supports this program. Specifically, the space $\mathbb{RP}^\infty = BO(1)$ was a key player in the unoriented bordism story, and the only other similar ground object is $\mathbb{CP}^\infty = BU(1)$ in complex bordism. This informs our choice to spend this Case Study focused on it. To begin, the contents of Lecture 1.1 can be replicated essentially *mutatis mutandis*, resulting in the following theorems:

Theorem 2.0.1 (cf. Lemma 1.1.5 and surrounding discussion)**.** *There is a map of infinite–loopspaces*

$$J_{\mathbb{C}} \colon BU \to BGL_1\mathbb{S}$$

called the complex J–homomorphism. □

Definition 2.0.2 (cf. Definition 1.1.9)**.** The associated Thom spectrum is written "MU" and called *complex bordism*. A map $MU \to E$ of ring spectra is said to be a *complex orientation of E*.

Theorem 2.0.3 (cf. Theorem 1.1.12)**.** *For a complex vector bundle ξ on a space X and a complex-oriented ring spectrum E, there is a natural equivalence*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X. \qquad \square$$

Corollary 2.0.4 (cf. Example 1.1.15)**.** *In particular, for a complex-oriented ring spectrum E it follows that $E^*\mathbb{C}\mathrm{P}^\infty$ is isomorphic to a one–dimensional power series ring.* □

We would like to then review the results of Lecture 1.3 and conclude (by reinterpreting Corollary 2.0.4) that $\mathbb{C}\mathrm{P}^\infty_E$ gives a 1–dimensional formal group over $\operatorname{Spec} E_*$. In order to make this statement honestly, however, we are first required to describe more responsibly the algebraic geometry we outlined in Lecture 1.2. Specifically, the characteristic 2 nature of the unoriented bordism ring was a major simplifying feature which made it wholly amenable to study by $H\mathbb{F}_2$. In turn, $H\mathbb{F}_2$ has many nice properties—for example, it has a duality between homology and cohomology, and it supports a Künneth isomorphism—and these are reflected in the extremely simple algebraic geometry of $\operatorname{Spec} \mathbb{F}_2$. By contrast, the complex bordism ring is considerably more complicated, not least because it is a characteristic 0 ring, and more generally we have essentially no control over the behavior of the coefficient ring E_* of some other complex–oriented theory. Nonetheless, once the background theory and construction of "X_E" are taken care of in Lecture 2.1, we indeed find that $\mathbb{C}\mathrm{P}^\infty_E$ is a 1–dimensional formal group over $\operatorname{Spec} E_*$.

However, where we could explicitly calculate $\mathbb{R}\mathrm{P}^\infty_{H\mathbb{F}_2}$ to be $\widehat{\mathbb{G}}_a$, we again have little control over what formal group $\mathbb{C}\mathrm{P}^\infty_E$ could possibly be. In the universal case, $\mathbb{C}\mathrm{P}^\infty_{MU}$ comes equipped with a natural coordinate, and this induces a map

$$\operatorname{Spec} MU_* \to \mathcal{M}_{\mathbf{fgl}}$$

from the spectrum associated to the coefficient ring of complex bordism to the moduli of formal group laws. The conclusion of this Case Study in Corollary 2.6.11 (modulo an algebraic result, shown in the next Case Study as Theorem 3.2.2) states that this map is an isomorphism, so that $\mathbb{C}\mathrm{P}^\infty_{MU}$ is the universal—i.e., maximally complicated—formal group. Our route for proving this passes through the foothills of the theory of "p^{th} power operations", which simultaneously encode many possible natural transformations from MU–cohomology to itself glommed together in a large sum, one term of which is the literal p^{th} power. Remarkably, the identity operation also appears in this family of operations, and the rest of the operations are in some sense controlled by this naturally occuring formal group law. A careful analysis of this sum begets the inductive proof in Corollary 2.6.6 that $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ is surjective.

The execution of this proof requires some understanding of cohomology operations for complex-oriented cohomology theories generally. Stable such operations correspond to homotopy classes $MU \to E$, i.e., elements of $E^0 MU$, which correspond via the Thom isomorphism to elements of $E^0 BU$. This object is the repository of E–characteristic classes for complex vector bundles, which we describe in terms of divisors on formal curves. This amounts to a description of the formal schemes $BU(n)_E$, which underpins our understanding of the whole story and which significantly informs our study of connective orientations in Case Study 5.

2.1 Calculus on formal varieties

In light of the introduction, we see that it would be prudent to develop some of the theory of formal schemes and formal varieties outside of the context of $\mathbb{F}_2$–algebras. However, writing down a list of definitions and checking that they have good enough properties is not especially enlightening or fun. Instead, it will be informative to understand where these objects come from in algebraic geometry, so that we can carry the accompanying geometric intuition along with us as we maneuver our way back toward homotopy theory and bordism. Our overarching goal in this Lecture is to develop a notion of calculus (and analytic expansions in particular) in the context of affine schemes. The place to begin is with definitions of cotangent and tangent spaces, as well as some supporting vocabulary.

Definition 2.1.1 (cf. Definition 1.2.1)**.** For an R–algebra A, the functor $\operatorname{Spec} A \colon \mathsf{Algebras}_R \to \mathsf{Sets}$ defined by

$$(\operatorname{Spec} A)(T) := \mathsf{Algebras}_R(A, T)$$

is called the *spectrum of A*. A functor X which is naturally isomorphic to some $\operatorname{Spec} A$ is called an *affine (R–)scheme*, and $A = \mathcal{O}_{\operatorname{Spec} A}$ is called its *ring of functions*. A subfunctor $Y \subseteq X$ is said to be a *closed*[1] *subscheme* when an identification[2] $X \cong \operatorname{Spec} A$ induces a further identification

[1]The word “closed” is meant to suggest properties of these inclusions: in suitable senses, they are closed under finite unions and arbitrary intersections. The complementary concept of “open” is harder to describe: open subschemes of affine schemes are merely “covered” by finitely many affines, which requires a discussion of coverings, which we remit to the actual algebraic geometers [Strb, Definition 8.1].

[2]This property is independent of choice of chart.

$$\begin{tikzcd} Y \arrow[r] & X \\ \operatorname{Spec}(R/I) \arrow[r] \arrow[u, "\simeq"'] & \operatorname{Spec} R. \arrow[u, "\simeq"'] \end{tikzcd}$$

Definition 2.1.2. Take $S = \operatorname{Spec} R$ to be our base scheme, let $X = \operatorname{Spec} A$ be an affine scheme over S, and consider an S–point $x \colon S \to X$ of X. The point x is automatically closed, so that x is presented as $\operatorname{Spec} A/I \to \operatorname{Spec} A$ for some ideal I. The *cotangent space* $T_x^* X$ is defined by the quotient R–module

$$T_x^* X := I/I^2,$$

consisting of functions vanishing at x as considered up to first order. Examples of these include the linear parts of curves passing through x, so we additionally define the *tangent space* $T_x X$ by

$$T_x X = \mathsf{Schemes}_{\operatorname{Spec} R/}(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X),$$

i.e., maps $\operatorname{Spec} R[\varepsilon]/\varepsilon^2 \to X$ which restrict to $x \colon S \to X$ upon setting $\varepsilon = 0$.

Remark 2.1.3. In the situation above, there is a natural map $T_x X \to \mathsf{Modules}_R(T_x^* X, R)$. A map $\mathcal{O}_X \to R[\varepsilon]/\varepsilon^2$ induces a map $I \to (\varepsilon)$ and hence a map

$$I/I^2 \to (\varepsilon)/(\varepsilon^2) \cong R,$$

i.e., a point in $T_x^* X$.

Harkening back to Example 1.3.8, the definition of the R–module tangent space begs promotion to an S–scheme.

Lemma 2.1.4. *There is an affine scheme $T_x X$ defined by*

$$(T_x X)(T) := \left\{ (u, f) \;\middle|\; \begin{array}{c} u \colon \operatorname{Spec} T \to S, \\ f \in T_{u^* x} u^* X \end{array} \right\}.$$

Proof sketch. We specialize an argument of Strickland [Str99b, Proposition 2.94] to the case at hand.[3] We start by seeking an R–algebra B such that R–algebra maps $B \to T$ biject with pairs of maps $u \colon R \to T$ and T–algebra maps

$$f \colon A \otimes_R T \to R[\varepsilon]/\varepsilon^2 \otimes_R T.$$

[3]Strickland also shows that mapping schemes between formal schemes exist considerably more generally [Str99b, Theorem 4.69]. The source either has to be “finite” in some sense, in which case the proof proceeds along the lines presented here, or it has to be *coalgebraic*, which is an important technical tool that we discuss much later in Definition 5.1.6.

Such maps f biject with R–algebra maps

$$A \to R[\varepsilon]/\varepsilon^2 \otimes_R T.$$

Using the sequence of inclusions

$$\begin{aligned}\mathsf{Algebras}_{R/}(A, R[\varepsilon]/\varepsilon^2 \otimes_R T) &\subseteq \mathsf{Modules}_R(A, R[\varepsilon]/\varepsilon^2 \otimes_R T) \\ &\cong \mathsf{Modules}_R(A \otimes_R (R[\varepsilon]/\varepsilon^2)^*, T) \\ &\cong \mathsf{Algebras}_{R/}(\mathrm{Sym}_R(A \otimes_R (R[\varepsilon]/\varepsilon^2)^*), T),\end{aligned}$$

we see that we can pick out the original mapping set by passing to a quotient of the domain. After some thought, we arrive at the equation

$$\underline{\mathsf{Schemes}}_{/S}(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X) = \operatorname{Spec} A\{1, da \mid a \in A\} \Big/ \left(\begin{array}{c} dr = 0 \text{ for } r \in R, \\ d(a_1 a_2) = da_1 \cdot a_2 + a_1 \cdot da_2 \end{array} \right).$$

To extract the scheme $T_x X$ from this, we construct the pullback

$$T_x X := \underline{\mathsf{Schemes}}_S(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X) \times_X S,$$

where the structure maps are given on the left by setting $\varepsilon = 0$ and on the right using the point x. Expanding the formulas again shows that the coordinate ring of this affine scheme is given by

$$\mathcal{O}_{T_x X} = A/I^2 \cong R \oplus T_x^* X. \qquad \square$$

Definition 2.1.5. The ring of functions appearing in the proof above fits into an exact sequence

$$0 \to \Omega_{A/R} \to A\{1, da \mid a \in A\} \Big/ \left(\begin{array}{c} dr = 0 \text{ for } r \in R, \\ d(a_1 a_2) = da_1 \cdot a_2 + a_1 \cdot da_2 \end{array} \right) \xrightarrow{da=0} A\{1\} \to 0.$$

The kernel $\Omega_{A/R}$ is called the module of *Kähler differentials* (of A, relative to R). The map $d\colon A \to \Omega^1_{A/R}$ is the universal R–linear derivation into an A–module, i.e.,

$$\mathsf{Derivations}_R(A, M) = \mathsf{Modules}_A(\Omega^1_{A/R}, M).$$

The upshot of this calculation is that $\operatorname{Spec} A/I^2$ is a natural place to study the linear behavior of functions on X near x. We have also set the definitions up so that we can easily generalize to higher-order approximations:

Definition 2.1.6. More generally, the n^{th} *jet space* of X at x, or the n^{th} *order neighborhood* of x in X, is defined by

$$\underline{\mathsf{Schemes}}_S(\operatorname{Spec} R[\varepsilon]/\varepsilon^{n+1}, X) \times_X S \cong \operatorname{Spec} A/I^{n+1}.$$

Each jet space has an inclusion from the one before, modeled by the closed subscheme $\operatorname{Spec} A/I^n \to \operatorname{Spec} A/I^{n+1}$.

In order to study analytic expansions of functions, we bundle these jet spaces together into a single object embodying formal expansions in X at x:

Definition 2.1.7. Fix a scheme S. A *formal S–scheme* $X = \{X_\alpha\}_\alpha$ is an ind-system of S–schemes X_α.[4] Given a closed subscheme Y of an affine S–scheme X, we define the n^{th} *order neighborhood of Y in X* to be the scheme $\operatorname{Spec} R/I^{n+1}$. The *formal neighborhood of Y in X* is then defined to be the formal scheme

$$X^\wedge_Y := \operatorname{Spf} R^\wedge_I := \left\{\operatorname{Spec} R/I \to \operatorname{Spec} R/I^2 \to \operatorname{Spec} R/I^3 \to \cdots\right\}.$$

In the case that $Y = S$, this specializes to the system of jet spaces as in Definition 2.1.6.

Although we will make use of these definitions generally, the following urexample captures the most geometrically-intuitive situation.

Example 2.1.8. Picking the affine scheme $X = \operatorname{Spec} R[x_1, \ldots, x_n] = \mathbb{A}^n$ and the point $x = (x_1 = 0, \ldots, x_n = 0)$ gives a formal scheme known as *formal affine n–space*, given explicitly by

$$\widehat{\mathbb{A}}^n = \operatorname{Spf} R[\![x_1, \ldots, x_n]\!].$$

Evaluated on a test algebra T, $\widehat{\mathbb{A}}^1(T)$ yields the ideal of nilpotent elements in T and $\widehat{\mathbb{A}}^n(T)$ its n–fold Cartesian power. Pointed maps $\widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^m$ naturally biject with m–tuples of n–variate power series with no constant term.[5]

[4] This definition, owing to Strickland [Str99b, Definition 4.1], is somewhat idiosyncratic. Its generality gives it good categorical properties, but it is somewhat disconnected from the formal schemes familiar to algebraic geometers, which primarily arise through linearly topologized rings [Har77, pg. 194]. For functor-of-points definitions that hang more tightly with the classical definition, the reader is directed toward Strickland's solid formal schemes [Str99b, Section 4.2] or to Beilinson and Drinfel'd [BD, Section 7.11.1].

[5] In some sense, this Lemma is a full explanation for why anyone would even think to involve formal geometry in algebraic topology (nevermind how useful the program has been in the long run). Calculations in algebraic topology have long been expressed in terms of power series rings, and with this Lemma we are provided geometric interpretations for such statements.

Part of the point of the geometric language is to divorce abstract rings (e.g., $E^0\mathbb{C}\mathrm{P}^\infty$) from concrete presentations (e.g., $E^0[\![x]\!]$), so we additionally reserve some vocabulary for the property of being isomorphic to $\widehat{\mathbb{A}}^n$:

Definition 2.1.9. A *formal affine variety* (of dimension n) is a formal scheme V which is (noncanonically) isomorphic to $\widehat{\mathbb{A}}^n$. The two maps in an isomorphism pair

$$V \xrightarrow{\simeq} \widehat{\mathbb{A}}^n, \quad V \xleftarrow{\simeq} \widehat{\mathbb{A}}^n$$

are called a *coordinate (system)* and a *parameter (system)* respectively. Finally, an S–point $x\colon S \to X$ is called *formally smooth* when $X^\wedge_x$ gives a formal variety.

This definition allows local theorems from analytic differential geometry to be imported in coordinate-free language. For instance, there is the following version of the inverse function theorem:

Theorem 2.1.10. *A pointed map $f\colon V \to W$ of finite–dimensional formal varieties is an isomorphism if and only if the induced map $T_0 f\colon T_0 V \to T_0 W$ is an isomorphism of R–modules.* $\square$

Coordinate-free theorems are only really useful if we can verify their hypotheses by coordinate-free methods as well. The following two results are indispensible in this regard:

Theorem 2.1.11. *Let R be a Noetherian ring and $F\colon \mathsf{Algebras}_{R/} \to \mathsf{Sets}_{*/}$ be a functor such that $F(R) = *$, F takes surjective maps to surjective maps, and there is a fixed finite free R–module M such that F carries square-zero extensions of Noetherian R–algebras $I \to B \to B'$ to product sequences*

$$* \to I \otimes_R M \to F(B) \to F(B') \to *.$$

Then, a basis $M \cong R^n$ determines an isomorphism $F \cong \widehat{\mathbb{A}}^n$.

Proof sketch. In the motivating case where $F \cong \widehat{\mathbb{A}}^n$ is given, we can define M to be

$$M := F(R[\varepsilon]/\varepsilon^2) = (\varepsilon)^{\times n} = R\{\varepsilon_1, \ldots, \varepsilon_n\}.$$

In fact, this is always the case: the square-zero extension

$$(\varepsilon) \to R[\varepsilon]/\varepsilon^2 \to R$$

induces a product sequence and hence an isomorphism

$$* \to (\varepsilon) \otimes_R M \xrightarrow{\cong} F(R[\varepsilon]/\varepsilon^2) \to * \to *.$$

A choice of basis $M \cong R^{\times n}$ thus induces an isomorphism

$$F(R[\varepsilon]/\varepsilon^2) = (\varepsilon) \otimes_R M = M \cong R^{\times n} = \widehat{\mathbb{A}}^n(R[\varepsilon]/\varepsilon^2).$$

Lastly, induction shows that if the maps between the outer terms of the set-theoretic product sequence exist, then so must the middle:

$$\begin{tikzcd}
* \arrow[r] & I \otimes_R M \arrow[r] \arrow[d, equal] & F(B) \arrow[r] & F(B') \arrow[r] & * \\
* \arrow[r] & I \otimes_R M \arrow[r] & \widehat{\mathbb{A}}^n(B) \arrow[r] \arrow[u, dotted, "\simeq"] & \widehat{\mathbb{A}}^n(B') \arrow[r] \arrow[u, "\simeq"] & *.
\end{tikzcd}$$

□

I think this is right.

Corollary 2.1.12. *An S–point $x\colon S \to X$ of a Noetherian scheme is formally smooth exactly when $T_x X$ is a free S–module and for any nilpotent thickenings $S \to \operatorname{Spec} B \to \operatorname{Spec} B'$ and any solid diagram*

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, "x"'] & \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} B' \arrow[dl, dotted] \\
& X &
\end{tikzcd}$$

there exists a dotted arrow extending the diagram. □

With all this algebraic geometry in hand, we now return to our original motivation: extracting formal schemes from the rings appearing in algebraic topology.

Definition 2.1.13 (cf. Definition 1.3.6)**.** Let E be an even-periodic ring spectrum, and let X be a CW–space. Because X is compactly generated, it can be written as the colimit of its compact subspaces $X^{(\alpha)}$, and we set[6,7]

$$X_E := \operatorname{Spf} E^0 X := \{\operatorname{Spec} E^0 X^{(\alpha)}\}_\alpha.$$

Consider the example of $\mathbb{C}\mathrm{P}^\infty_E$ for E a complex-oriented cohomology theory. We saw in Corollary 2.0.4 that the complex-orientation determines an isomorphism $\mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^1$ (i.e., an isomorphism $E^0\mathbb{C}\mathrm{P}^\infty \cong E^0[\![x]\!]$). However, the object "$E^0\mathbb{C}\mathrm{P}^\infty$" is something that exists independent of the orientation map $MU \to E$, and the language of Definition 2.1.9 allows us to make the distinction between the property and the data:

Lemma 2.1.14. *A cohomology theory E is complex orientable (i.e., it is able to receive a ring map from MU) precisely when $\mathbb{C}\mathrm{P}^\infty_E$ is a formal curve (i.e., it is a formal variety of dimension 1). A choice of orientation $MU \to E$ determines a coordinate $\mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^1$.* □

As in Example 1.3.7, the formal scheme $\mathbb{C}\mathrm{P}^\infty_E$ has additional structure: it is a group. We close today with some remarks about such objects.

Definition 2.1.15. A formal group is a formal variety endowed with an abelian group structure.[8] If E is a complex-orientable cohomology theory, then $\mathbb{C}\mathrm{P}^\infty_E$ naturally forms a (1–dimensional) formal group using the map classifying the tensor product of line bundles.

Remark 2.1.16. As with formal schemes, formal groups can arise as formal completions of an algebraic group at its identity point. It turns out that there are many more formal groups than come from this procedure, a phenomenon that is of keen interest to stable homotopy theorists—see Appendix B.

[6]The careful reader will immediately notice that the rings in the pro-system underlying Definition 2.1.13 run the risk of not being even-concentrated. We are thus required to make the following technical compromise: for any pro-isomorphic system of even E^*–algebras $\{R_\beta \otimes_{E^0} E^*\} \cong \{E^0 X^{(\alpha)}\}_\alpha$ we set

$$X_E := \{\operatorname{Spec} R_\beta\}_\beta,$$

and otherwise we leave X_E undefined. For example, the technical condition of Definition 2.1.13 is satisfied if there exists a cofinal subsystem of $\{X^{(\alpha)}\}_\alpha$ with $E^* X^{(\alpha)}$ even–concentrated. This follows, for instance, from $H\mathbb{Z}_* X$ being free and even [Str99b, Definition 8.15, Proposition 8.17].

[7]In cases of "large" spaces and cohomology theories, the technical points underlying this definition are necessary: BU_{KU} is an instructive example, as it is *not* the formal scheme associated to $KU^0(BU)$ by any adic topology.

[8]Formal groups in dimension 1 are automatically commutative if and only if the ground ring has no elements which are simultaneously nilpotent and torsion [Haz12, Theorem I.6.1].

We give the following Corollary as an example of how nice the structure theory of formal varieties is.

Corollary 2.1.17. *As with physical groups, the formal group addition map on $\widehat{\mathbb{G}}$ determines the inverse law.*

Proof. Consider the shearing map

$$\begin{aligned} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} &\xrightarrow{\sigma} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}}, \\ (x, y) &\mapsto (x, x + y). \end{aligned}$$

The induced map $T_0\sigma$ on tangent spaces is evidently invertible, so by Theorem 2.1.10 there is an inverse map $(x, y) \mapsto (x, y - x)$. Setting $y = 0$ and projecting to the second factor gives the inversion map. $\square$

Definition 2.1.18. Let $\widehat{\mathbb{G}}$ be a formal group. In the presence of a coordinate $\varphi\colon \widehat{\mathbb{G}} \cong \widehat{\mathbb{A}}^n$, the addition law on $\widehat{\mathbb{G}}$ begets a map

$$\begin{tikzcd} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \arrow[r] \arrow[d, "\simeq"'] & \widehat{\mathbb{G}} \arrow[d, "\simeq"] \\ \widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n \arrow[r] & \widehat{\mathbb{A}}^n, \end{tikzcd}$$

and hence a n–tuple of $(2n)$–variate power series "$+_\varphi$", satisfying

$$\begin{aligned} \underline{x} +_\varphi \underline{y} &= \underline{y} +_\varphi \underline{x}, && \text{(commutativity)} \\ \underline{x} +_\varphi \underline{0} &= \underline{x}, && \text{(unitality)} \\ \underline{x} +_\varphi (\underline{y} +_\varphi \underline{z}) &= (\underline{x} +_\varphi \underline{y}) +_\varphi \underline{z}. && \text{(associativity)} \end{aligned}$$

Such a series $+_\varphi$ is called a *formal group law*, and it is the concrete data associated to a formal group.

Let's now consider two examples of E which are complex-orientable and describe these invariants for them.

Example 2.1.19. There is an isomorphism $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{Z}P} \cong \widehat{\mathbb{G}}_a$. This follows from reasoning identical to that given in Example 1.3.7.

Example 2.1.20. There is also an isomorphism $\mathbb{CP}^\infty_{KU} \cong \widehat{\mathbb{G}}_m$. The standard choice of first Chern class is given by the topological map

$$c_1\colon \Sigma^{-2}\Sigma^\infty \mathbb{CP}^\infty \xrightarrow{1-\beta\mathcal{L}} KU,$$

and a formula for the first Chern class of the tensor product is thus

$$\begin{aligned} c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) &= 1 - \beta(\mathcal{L}_1 \otimes \mathcal{L}_2) \\ &= -\beta^{-1}\left((1-\beta\mathcal{L}_1)\cdot(1-\beta\mathcal{L}_2)\right) + (1-\beta\mathcal{L}_1) + (1-\beta\mathcal{L}_2) \\ &= c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - \beta^{-1}c_1(\mathcal{L}_1)c_1(\mathcal{L}_2). \end{aligned}$$

In this coordinate on $\mathbb{CP}^\infty_{KU}$, the group law is then $x_1 +_{\mathbb{CP}^\infty_{KU}} x_2 = x_1 + x_2 - \beta^{-1}x_1x_2$. Using the coordinate function $1-t$, this is also the coordinate that arises on the formal completion of $\mathbb{G}_m$ at $t = 1$:

$$\begin{aligned} x_1(t_1) +_{\mathbb{G}_m} x_2(t_2) &= 1 - (1-t_1)(1-t_2) \\ &= t_1 + t_2 - t_1t_2. \end{aligned}$$

As an application of all these tools, we will show that the *rational* theory of formal groups is highly degenerate: every rational formal group is isomorphic to $\widehat{\mathbb{G}}_a$. Suppose now that R is a $\mathbb{Q}$–algebra and that $A = R[\![x]\!]$ is the coordinatized ring of functions on a formal line over R. What's special about this rational curve case is that differentiation gives an isomorphism between the Kähler differentials $\Omega^1_{A/R}$ and the ideal (x) of functions vanishing at the origin (i.e., the ideal sheaf selecting the closed subscheme $0\colon\ \operatorname{Spec} R \to \operatorname{Spf} A$). Its inverse is formal integration:

$$\int\colon\ \left(\sum_{j=0}^\infty c_j x^j\right) dx \mapsto \sum_{j=0}^\infty \frac{c_j}{j+1} x^{j+1}.$$

In pursuit of the construction of a *logarithm* for a formal group $\widehat{\mathbb{G}}$ over R, we now take a cue from classical Lie theory:

Definition 2.1.21. A 1–form $\omega \in \Omega^1_{A/R}$ is said to be *invariant (under a group law $+_\varphi$)* when $\omega = T_y^*\omega$ for all translations $T_y(x) = x +_\varphi y$.

Theorem 2.1.22. *For R a $\mathbb{Q}$–algebra, there is a canonical isomorphism of formal groups*

$$\log\colon\ \widehat{\mathbb{G}} \to T_0\widehat{\mathbb{G}} \otimes \widehat{\mathbb{G}}_a.$$

Proof. In terms of a coordinate $\omega = f(x)dx$, the definition above becomes

$$f(x)dx = f(y +_\varphi x)d(y +_\varphi x) = f(y +_\varphi x)\frac{\partial(y +_\varphi x)}{\partial x}dx.$$

Restricting to the origin by setting $x = 0$, this yields the condition

$$f(0) = f(y) \cdot \left.\frac{\partial(y +_\varphi x)}{\partial x}\right|_{x=0}.$$

Since R is a $\mathbb{Q}$–algebra, integrating against y yields

$$\log_\varphi(y) := \int f(y)\, dy = f(0) \int \left(\left.\frac{\partial(y +_\varphi x)}{\partial x}\right|_{x=0} \right)^{-1} dy.$$

To see that the series $\log_\varphi$ has the claimed homomorphism property, note that

$$\frac{\partial \log_\varphi(y +_\varphi x)}{\partial x}dx = f(y +_\varphi x)d(y +_\varphi x) = f(x)dx = \frac{\partial \log_\varphi(x)}{\partial x}dx,$$

so $\log_\varphi(y +_\varphi x)$ and $\log_\varphi(x)$ differ by a constant. Checking at $x = 0$ shows that the constant is $\log_\varphi(y)$, hence

$$\log_\varphi(x +_\varphi y) = \log_\varphi(x) + \log_\varphi(y).$$

The choice of boundary value $f(0)$ corresponds to the choice of vector in $T_0\widehat{\mathbb{G}}$. □

Example 2.1.23. Consider the formal group law $x_1(t_1) +_{\widehat{\mathbb{G}}_m} x_2(t_2) = t_1 + t_2 - t_1 t_2$ studied in Example 2.1.20. Its associated rational logarithm is computed as

$$\log_{\widehat{\mathbb{G}}_m}(t_2) = f(0) \cdot \int \frac{1}{1 - t_2} dt_2 = -f(0) \log(1 - t_2) = -f(0) \log(x_2),$$

where "$\log(x_2)$" refers to Napier's classical natural logarithm of x_2.

2.2 Divisors on formal curves

We continue to develop vocabulary and accompanying machinery used to give algebro-geometric reinterpretations of the results in the introduction to this Case Study. In the previous section we deployed the language of formal schemes to recast Corollary 2.0.4 in geometric terms, and we now turn towards reencoding Theorem 2.0.3. In Definition 1.4.8 and Lemma 1.4.9 we discussed a general correspondence between R–modules and quasicoherent sheaves over $\operatorname{Spec} R$, and the isomorphism of 1–dimensional E^*X–modules appearing in Theorem 2.0.3 moves us to study sheaves over X_E which are 1–dimensional—i.e., line bundles. In fact, for the purposes of Theorem 2.0.3, we will find that it suffices to understand the basics of the geometric theory of line bundles *just over formal curves*. This is our goal in this Lecture, and we leave the applications to algebraic topology aside for later. For the rest of this section we fix the following three pieces of data: a base formal scheme S, a formal curve C over S, and a distinguished point $\zeta\colon S \to C$ on C.

To begin, we will be interested in a very particular sort of line bundle over C: for any function f on C which is not a zero-divisor, the subsheaf $\mathcal{I}_f = f \cdot \mathcal{O}_C$ of functions on C which are divisible by f form a 1–dimensional $\mathcal{O}_C$–submodule of the ring of functions $\mathcal{O}_C$ itself—i.e., a line bundle on C. By interpreting $\mathcal{I}_f$ as an ideal sheaf, this gives rise to a second interpretation of this data in terms of a closed subscheme

$$\operatorname{Spec} \mathcal{O}_C / f \subseteq C,$$

which we will refer to as the *divisor* associated to $\mathcal{I}_f$. In general these can be somewhat pathological, so we specialize further to an extremely nice situation:

Definition 2.2.1 ([Str99b, Section 5.1])**.** An *effective Weil divisor* D on a formal curve C is a closed subscheme of C whose structure map $D \to S$ presents D as finite and free. We say that the *rank* of D is n when its ring of functions $\mathcal{O}_D$ is free of rank n over $\mathcal{O}_S$.

Lemma 2.2.2 ([Str99b, Proposition 5.2], see also [Str99b, Example 2.10])**.** *There is a scheme $\operatorname{Div}_n^+ C$ of effective Weil divisors of rank n. It is a formal variety of dimension n. In fact, a coordinate x on C determines an isomorphism $\operatorname{Div}_n^+ C \cong \widehat{\mathbb{A}}^n$ where a divisor D is associated to a monic polynomial $f_D(x)$ with nilpotent lower-order coefficients.*

Proof sketch. To pin down the functor we wish to analyze, we make the definition

$$\operatorname{Div}_n^+(C)(R) = \left\{ (a, D) \,\middle|\, \begin{array}{c} a : \operatorname{Spec} R \to S, \\ D \text{ is an effective divisor on } C \times_S \operatorname{Spec} R \end{array} \right\}.$$

To show that this is a formal variety, we pursue the final claim and select a coordinate x on C, as well as a point $(a, D) \in \operatorname{Div}_n^+(C)(T)$. The coordinate presents

$C \times_S \operatorname{Spec} T$ as

$$C \times_X \operatorname{Spec} T = \operatorname{Spf} T[\![x]\!],$$

and the characteristic polynomial $f_D(x)$ of x in $\mathcal{O}_D$ presents D as the closed subscheme

$$D = \operatorname{Spf} R[\![x]\!]/(f_D(x))$$

for $f_D(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ monic. Additionally, for any prime ideal $\mathfrak{p} \in R$ we can form the field $R_{\mathfrak{p}}/\mathfrak{p}$, over which the module $\mathcal{O}_D \otimes_R R_{\mathfrak{p}}/\mathfrak{p}$ must still be of rank n. It follows that

$$f_D(x) \otimes_R R_{\mathfrak{p}}/\mathfrak{p} \equiv x^n,$$

hence that each a_j lies in the intersection of all prime ideals of R, hence that each a_j is nilpotent.

In turn, this means that the polynomial f_D is selected by a map $\operatorname{Spec} R \to \widehat{\mathbb{A}}^n$. Conversely, given such a map, we can form the polynomial $f_D(x)$ and the divisor D. $\square$

Remark 2.2.3. This Lemma effectively connects several simple dots: especially nice polynomials $f_D(x) \in \mathcal{O}_C$, their vanishing loci $D \subseteq C$, and the ideal sheaves $\mathcal{I}_D$ of functions divisible by f—i.e., functions with a partially prescribed vanishing set. Basic operations on polynomials affect their vanishing loci in predictable ways, and these operations are also reflected on the level of divisor schemes. For instance, there is a unioning map

$$\begin{aligned} \operatorname{Div}_n^+ C \times \operatorname{Div}_m^+ C &\to \operatorname{Div}_{n+m}^+ C, \\ (D_1, D_2) &\mapsto D_1 \sqcup D_2. \end{aligned}$$

At the level of ideal sheaves, we use their 1–dimensionality to produce the formula

$$\mathcal{I}_{D_1 \sqcup D_2} = \mathcal{I}_{D_1} \otimes_{\mathcal{O}_C} \mathcal{I}_{D_2}.$$

Under a choice of coordinate x, the map at the level of polynomials is given by

$$(f_{D_1}, f_{D_2}) \mapsto f_{D_1} \cdot f_{D_2}.$$

Next, note that there is a canonical isomorphism $C \xrightarrow{\cong} \operatorname{Div}_1^+ C$. Iterating the above addition map gives the vertical map in the following triangle:

$$\begin{tikzcd} & C^{\times n} \arrow[dl] \arrow[d, "\sqcup"] \\ C^{\times n}_{\Sigma_n} \arrow[r, dotted, "\cong"] & \operatorname{Div}_n^+ C. \end{tikzcd}$$

Lemma 2.2.4. *The object $C^{\times n}_{\Sigma_n}$ exists as a formal variety, it factors the iterated addition map, and the dotted arrow is an isomorphism.*

Proof. The first assertion is a consequence of Newton's theorem on symmetric polynomials: the subring of symmetric polynomials in $R[x_1, \ldots, x_n]$ is itself polynomial on generators

$$\sigma_j(x_1, \ldots, x_n) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=j}} x_{S_1} \cdots x_{S_j},$$

and hence

$$R[\sigma_1, \ldots, \sigma_n] \subseteq R[x_1, \ldots, x_n]$$

gives an affine model of the quotient map $\widehat{\mathbb{A}}^n \to (\widehat{\mathbb{A}}^n)_{\Sigma_n}$. Picking a coordinate on C allows us to import this fact into formal geometry to deduce the existence of $C^{\times n}_{\Sigma_n}$. The factorization then follows by noting that the iterated $\sqcup$ map is symmetric. Finally, Remark 2.2.3 shows that the horizontal map pulls the coordinate a_j back to σ_j, so the third assertion follows. □

Remark 2.2.5. The map $C^{\times n} \to C^{\times n}_{\Sigma_n}$ is an example of a map of schemes which is surjective *as a map of sheaves*. This is somewhat subtle: for any given test ring T, it is not necessarily the case that $C^{\times n}(T) \to C^{\times n}_{\Sigma_n}(T)$ is surjective on T–points. However, for a fixed point $f \in C^{\times n}_{\Sigma_n}(T)$, we are guaranteed a flat covering $T \to \prod_j T_j$ such that there are individual lifts $\widetilde{f}_j$ of f over each T_j.[9]

Now we use the pointing $\zeta\colon S \to C$ to interrelate divisor schemes of varying ranks. Together with the $\sqcup$ operation, ζ gives a composite

$$\begin{array}{ccccccc} \operatorname{Div}_n^+ C & \longrightarrow & C \times \operatorname{Div}_n^+ C & \longrightarrow & \operatorname{Div}_1^+ C \times \operatorname{Div}_n^+ C & \longrightarrow & \operatorname{Div}_{n+1}^+ C, \\ D & \longmapsto & (\zeta, D) & \longmapsto & (\langle\zeta\rangle, D) & \longmapsto & \langle\zeta\rangle \sqcup D. \end{array}$$

[9]This amounts to the claim that not every polynomial can be written as a product of linear factors. For instance, the divisor on $C = \operatorname{Spf} \mathbb{R}[\![x]\!]$ defined by the equation $x^2 + 1$ splits as $(x+i)(x-i)$ after base-change along the flat cover $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

Definition 2.2.6. We define the following variants of "stable divisor schemes":

$$\begin{aligned}
\operatorname{Div}^+ C &= \coprod_{n \geq 0} \operatorname{Div}_n^+ C, \\
\operatorname{Div}_n C &= \operatorname{colim}\left(\operatorname{Div}_n^+ C \xrightarrow{\langle\zeta\rangle +-} \operatorname{Div}_{n+1}^+ C \xrightarrow{\langle\zeta\rangle +-} \cdots\right), \\
\operatorname{Div} C &= \operatorname{colim}\left(\operatorname{Div}^+ C \xrightarrow{\langle\zeta\rangle +-} \operatorname{Div}^+ C \xrightarrow{\langle\zeta\rangle +-} \cdots\right) \\
&\cong \coprod_{n \in \mathbb{Z}} \operatorname{Div}_n C.
\end{aligned}$$

The first of these constructions is very suggestive: it looks like the free commutative monoid formed on a set, and we might hope that the construction in formal schemes enjoys a similar universal property. In fact, all three constructions have universal properties:

Theorem 2.2.7 (cf. Corollary 5.1.10)**.** *The scheme* $\operatorname{Div}^+ C$ *models the free formal monoid on the unpointed formal curve C. The scheme* $\operatorname{Div} C$ *models the free formal group on the unpointed formal curve C. The scheme* $\operatorname{Div}_0 C$ simultaneously *models the free formal monoid and the free formal group on the* pointed *formal curve C.* □

We will postpone the proof of this Theorem until later, once we've developed a theory of coalgebraic formal schemes.

Remark 2.2.8. Given $q\colon C \to C'$ a map of formal curves over S and $D \subseteq C$ a divisor on C, the construction of $\operatorname{Div}_n^+ C$ as a symmetric space as in Lemma 2.2.4 shows that there is a corresponding divisor $q_* D$ on C'. This can be thought of in some other ways—for instance, this map at the level of sheaves [Har77, Ch. IV, Exercise 2.6] is given by

$$\det q_* \mathcal{I}_D \cong (\det q_* \mathcal{O}_C) \otimes \mathcal{I}_{q_* D}.$$

We can also use Theorem 2.2.7 in the stable case: the composite map

$$C \xrightarrow{q} C' \cong \operatorname{Div}_1^+ C' \to \operatorname{Div} C'$$

targets a formal group scheme, and hence universality induces a map

$$q_*\colon \operatorname{Div} C \to \operatorname{Div} C'.$$

On the other hand, for a general q the pullback $D \times_{C'} C$ of a divisor $D \subseteq C'$ will not be a divisor on C. It is possible to impose conditions on q so that this is so, and in this case q is called an *isogeny*. We will return to this in Appendix A.2.

Our final goal for the section is to broaden this discussion to line bundles on formal curves generally, using this nice case as a model. The main classical theorem is that line bundles, sometimes referred to as *Cartier divisors*, arise as the group-completion and sheafification of zero-loci of polynomials, referred to (as above) as *Weil divisors*. In the case of a *formal* curve, sheafification has little effect, and so we seek to exactly connect line bundles on a formal curve with formal differences of Weil divisors. To begin, we need some vocabulary that connects the general case to the one studied above.

Definition 2.2.9 (cf. [Vak15, Section 14.2])**.** Suppose that $\mathcal{L}$ is a line bundle on C and select a section u of $\mathcal{L}$. There is a largest closed subscheme $D \subseteq C$ where the condition $u|_D = 0$ is satisfied. If D is a divisor, u is said to be *divisorial* and we write $\operatorname{div} u := D$.

Cite me: This and the following Lemma aren't great citations.

Lemma 2.2.10 (cf. [Vak15, Exercise 14.2.E])**.** *A divisorial section u of a line bundle $\mathcal{L}$ induces an isomorphism $\mathcal{L} \cong \mathcal{I}_D$.* □

Line bundles which admit divisorial sections are therefore those that arise through our construction above, i.e., those which are controlled by the zero locus of a polynomial. In keeping with the classical inspiration, we expect generic line bundles to be controlled by the zeroes *and poles* of a rational function, and so we introduce the following class of functions:

Definition 2.2.11 ([Str99b, Definition 5.20 and Proposition 5.26])**.** The ring of meromorphic functions on C, $\mathcal{M}_C$, is obtained by inverting all coordinates in $\mathcal{O}_C$.[10] Additionally, this can be augmented to a scheme $\operatorname{Mer}(C, \mathbb{G}_m)$ of meromorphic functions on C by

$$\operatorname{Mer}(C, \mathbb{G}_m)(R) := \left\{ (u, f) \;\middle|\; \begin{array}{l} u : \operatorname{Spec} R \to S, \\ f \in \mathcal{M}^{\times}_{C \times_S \operatorname{Spec} R} \end{array} \right\}.$$

Thinking of a meromorphic function as the formal expansion of a rational function, we are moved to study the monoidality of Definition 2.2.9.

Lemma 2.2.12. *If u_1 and u_2 are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then $u_1 \otimes u_2$ is a divisorial section of $\mathcal{L}_1 \otimes \mathcal{L}_2$ and* $\operatorname{div}(u_1 \otimes u_2) = \operatorname{div} u_1 + \operatorname{div} u_2$. □

[10] In fact, it suffices to invert any single one [Str99b, Lemma 5.21].

Definition 2.2.13. A *meromorphic divisorial section* of a line bundle $\mathcal{L}$ is a decompositon $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ together with an expression of the form u_+/u_-, where u_+ and u_- are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We set $\operatorname{div}(u_+/u_-) = \operatorname{div} u_+ - \operatorname{div} u_-$.

In the case of a formal curve, the fundamental theorem is that meromorphic functions, line bundles, and stable Weil divisors all essentially agree. A particular meromorphic function spans a 1–dimensional $\mathcal{O}_C$–submodule sheaf of $\mathcal{M}_C$, and hence it determines a line bundle. Conversely, a line bundle is determined by local gluing data, which is exactly the data of a meromorphic function. However, it is clear that there is some overdeterminacy in this presentation: scaling a meromorphic function by a nowhere vanishing entire function will not modify the submodule sheaf. We now make the observation that the function div gives an assignment from meromorphic functions to stable Weil divisors which is *also* insensitive to rescaling by a nowhere vanishing function. These inputs are arranged in the following theorem:

Theorem 2.2.14 ([Str99b, Proposition 5.26], [Strb, Proposition 33.4])**.** *In the case of a formal curve C, there is a short exact sequence of formal groups*

$$0 \to \underline{\mathsf{FormalSchemes}}(C, \mathbb{G}_m) \to \operatorname{Mer}(C, \mathbb{G}_m) \to \operatorname{Div}(C) \to 0. \qquad \square$$

2.3 Line bundles associated to Thom spectra

In this Lecture, we will exploit all of the algebraic geometry previous set up to deduce a load of topological results.

Definition 2.3.1. Let E be a complex-orientable theory and let $V \to X$ be a complex vector bundle over a space X. According to Theorem 2.0.3, the cohomology of the Thom spectrum $E^*T(V)$ forms a 1–dimensional E^*X–module. Using Lemma 1.4.9, we construct a line bundle over X_E

$$\mathbb{L}(V) := \widetilde{E^*T(V)},$$

called the *Thom sheaf* of V.

Remark 2.3.2. One of the main utilities of this definition is that it only uses the *property* that E is complex-orientable, and it begets only the *property* that $\mathbb{L}(V)$ is a line bundle.

This construction enjoys many properties already established.

Corollary 2.3.3. *A vector bundle V over Y and a map $f\colon X \to Y$ induce an isomorphism*

$$\mathbb{L}(f^*V) \cong (f_E)^*\mathbb{L}(V).$$

There is also is a canonical isomorphism

$$\mathbb{L}(V \oplus W) = \mathbb{L}(V) \otimes \mathbb{L}(W).$$

Finally, this property can then be used to extend the definition of $\mathbb{L}(V)$ to virtual bundles:

$$\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}.$$

Proof. The first claim is justified by Lemma 1.1.4, the second by Lemma 1.1.8, and the last is a direct consequence of the first two. □

We use these properties to work the following Example, which connects Thom sheaves with the major players from Lecture 2.1.

Example 2.3.4 ([AHS04, Section 8]). Take $\mathcal{L}$ to be the canonical line bundle over $\mathbb{C}P^\infty$. Using the same mode of argument as in Example 1.1.3, the zero-section

$$\Sigma^\infty \mathbb{C}P^\infty \xrightarrow{\simeq} T_2(\mathcal{L})$$

gives an identification

$$E^0\mathbb{C}P^\infty \supseteq \widetilde{E}^0\mathbb{C}P^\infty \xleftarrow{\simeq} E^0T_2(\mathcal{L})$$

of $E^0T_2(\mathcal{L})$ with the augmentation ideal in $E^0\mathbb{C}P^\infty$. At the level of Thom sheaves, this gives an isomorphism

$$\mathcal{I}(0) \xleftarrow{\simeq} \mathbb{L}(\mathcal{L})$$

of $\mathbb{L}(\mathcal{L})$ with the sheaf of functions vanishing at the origin of $\mathbb{C}P^\infty_E$. Pulling $\mathcal{L}$ back along

$$0\colon * \to \mathbb{C}P^\infty$$

gives a line bundle over the one-point space, which on Thom spectra gives the inclusion

$$\Sigma^\infty \mathbb{C}P^1 \to \Sigma^\infty \mathbb{C}P^\infty.$$

Stringing many results together, we can now calculate:[11]

$$\begin{aligned}
\widetilde{\pi_2 E} &\cong \widetilde{E^0 \mathbb{C}P^1} && (S^2 \simeq \mathbb{C}P^1)\\
&\cong \mathbb{L}(0^*\mathcal{L}) && \text{(Definition 2.3.1)}\\
&\cong 0^*\mathbb{L}(\mathcal{L}) && \text{(Corollary 2.3.3)}\\
&\cong 0^*\mathcal{I}(0) && \text{(preceding calculation)}\\
&\cong \mathcal{I}(0)/(\mathcal{I}(0)\cdot\mathcal{I}(0)) && \text{(definition of } 0^* \text{ from Definition 1.4.10)}\\
&\cong T_0^*\mathbb{C}P^\infty_E && \text{(Definition 2.1.2)}\\
&\cong \omega_{\mathbb{C}P^\infty_E}, && \text{(proof of Theorem 2.1.22)}
\end{aligned}$$

where $\omega_{\mathbb{C}P^\infty_E}$ denotes the sheaf of invariant differentials on $\mathbb{C}P^\infty_E$. Consequently, if $k \cdot \varepsilon$ is the trivial bundle of dimension k over a point, then

$$\widetilde{\pi_{2k} E} \cong \mathbb{L}(k\cdot\varepsilon) \cong \mathbb{L}(k \cdot 0^*\mathcal{L}) \cong \mathbb{L}(0^*\mathcal{L})^{\otimes k} \cong \omega^{\otimes k}_{\mathbb{C}P^\infty_E}.$$

Finally, given an E–algebra $f\colon E \to F$ (e.g., $F = E^{X_+}$), then we have

$$\widetilde{\pi_{2k} F} \cong f_E^* \omega^{\otimes k}_{\mathbb{C}P^\infty_E}.$$

Outside of this Example, it is difficult to find line bundles $\mathbb{L}(V)$ which we can analyze so directly. In order to get a handle on on $\mathbb{L}(V)$ in general, we now seek to strengthen this bond between line bundles and vector bundles by finding inside of algebraic topology the alternative presentations of line bundles given in Lecture 2.2. In particular, we would like a topological construction on vector bundles which produces divisors—i.e., finite schemes over X_E. This has the scent of a certain familiar topological construction called projectivization, and we now work to justify the relationship.

Definition 2.3.5. Let V be a complex vector bundle of rank n over a base X. Define $\mathbb{P}(V)$, the *projectivization of* V, to be the $\mathbb{C}P^{n-1}$–bundle over X whose fiber of $x \in X$ is the space of complex lines in the original fiber $V|_x$.

[11]The identification $0^*\mathcal{I}(0) = \mathcal{I}(0)/\mathcal{I}(0)\cdot\mathcal{I}(0)$ deserves further explanation. The functor 0^* is right-exact, so sends the short exact sequence

$$0 \to \mathcal{I}(0)^2 \to \mathcal{I}(0) \to \mathcal{I}(0)/\mathcal{I}(0)^2 \to 0$$

to a right-exact sequence, and we need only check that the map $0^*\mathcal{I}(0)^2 \to 0^*\mathcal{I}(0)$ is zero. This is the statement that a function vanishing to second order also has vanishing first derivative.

Theorem 2.3.6. *Take E to be* complex-oriented. *The E–cohomology of $\mathbb{P}(V)$ is given by the formula*

$$E^*\mathbb{P}(V) \cong E^*(X)[\![t]\!]/c(V)$$

for a certain monic polynomial

$$c(V) = t^n - c_1(V)t^{n-1} + c_2(V)t^{n-2} - \cdots + (-1)^n c_n(V).$$

Proof. We fit all of the fibrations we have into a single diagram:

$$\begin{tikzcd}
 & \mathbb{C}^\times \arrow[dr] \arrow[dd, equal] & & & \\
\mathbb{C}^n \arrow[dd] & & \mathbb{C}^n \setminus \{0\} \arrow[ll] \arrow[r] \arrow[dd] & \mathbb{CP}^{n-1} \arrow[r] \arrow[dd] & \mathbb{CP}^\infty \arrow[dd, equal] \\
 & \mathbb{C}^\times \arrow[dr] & & & \\
V \arrow[d] & & V \setminus \zeta \arrow[ll] \arrow[r] \arrow[d] & \mathbb{P}(V) \arrow[r] \arrow[d, "\pi"] & \mathbb{CP}^\infty \arrow[d] \\
X \arrow[u, bend left, dotted, "\zeta"] \arrow[rr, equal] & & X \arrow[r, equal] & X \arrow[r] & *.
\end{tikzcd}$$

We read this diagram as follows: on the far left, there's the vector bundle we began with, as well as its zero-section ζ. Deleting the zero-section gives the second bundle, a $\mathbb{C}^n \setminus \{0\}$–bundle over X. Its quotient by the scaling $\mathbb{C}^\times$–action gives the third bundle, a $\mathbb{CP}^{n-1}$–bundle over X. Additionally, the quotient map $\mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ is itself a $\mathbb{C}^\times$–bundle, and this induces the structure of a $\mathbb{C}^\times$–bundle on the quotient map $V \setminus \zeta \to \mathbb{P}(V)$. Thinking of these as complex line bundles, they are classified by a map to $\mathbb{CP}^\infty$, which can itself be thought of as the last vertical fibration, fibering over a point.

Note that the map between these two last fibers is surjective on E–cohomology. It follows that the Serre spectral sequence for the third vertical fibration is degenerate, since all the classes in the fiber must survive.[12] We thus conclude that $E^*\mathbb{P}(V)$ is a free $E^*(X)$–module on the classes $\{1, t, t^2, \ldots, t^{n-1}\}$ spanning $E^*\mathbb{CP}^{n-1}$, where t encodes the chosen complex-orientation of E. To understand the ring structure,

[12] This is called the Leray–Hirsch theorem.

we need only compute $t^{n-1} \cdot t$, which must be able to be written in terms of the classes which are lower in t–degree:

$$t^n = c_1(V)t^{n-1} - c_2(V)t^{n-2} + \cdots + (-1)^{n-1}c_n(V)$$

for some classes $c_j(V) \in E^*X$. The main claim follows. □

In coordinate-free language, we have the following Corollary:

Corollary 2.3.7 (Theorem 2.3.6 redux)**.** *Take E to be* complex-orientable. *The map*

$$\mathbb{P}(V)_E \to X_E \times \mathbb{C}\mathrm{P}^\infty_E$$

is a closed inclusion of X_E–schemes, and the structure map $\mathbb{P}(V)_E \to X_E$ is free and finite of rank n. It follows that $\mathbb{P}(V)_E$ is a divisor on $\mathbb{C}\mathrm{P}^\infty_E$ considered over X_E, i.e.,

$$\mathbb{P}(V)_E \in \left(\mathrm{Div}^+_n(\mathbb{C}\mathrm{P}^\infty_E)\right)(X_E). \qquad \square$$

Definition 2.3.8. The classes $c_j(V)$ of Theorem 2.3.6 are called the *Chern classes* of V (with respect to the complex-orientation t of E), and the polynomial $c(V) = \sum_{j=0}^n (-1)^{n-j} t^j c_{n-j}(V)$ is called the *Chern polynomial*.

The next major theorems concerning projectivization are the following:

Corollary 2.3.9. *The sub-bundle of $\pi^*(V)$ consisting of vectors $(v, (\ell, x))$ such that v lies along the line ℓ splits off a canonical line bundle.* □

Corollary 2.3.10 ("Splitting principle" / "Complex–oriented descent")**.** *Associated to any n–dimensional complex vector bundle V over a base X, there is a canonical map $i_V \colon Y_V \to X$ such that $(i_V)_E \colon (Y_V)_E \to X_E$ is finite and faithfully flat, and there is a canonical splitting into complex line bundles:*

$$i_V^*(V) \cong \bigoplus_{i=1}^n \mathcal{L}_i. \qquad \square$$

This last Corollary is extremely important. Its essential contents is to say that any question about characteristic classes can be checked for sums of line bundles. Specifically, because of the injectivity of i_V^*, any relationship among the characteristic classes deduced in E^*Y_V must already be true in the ring E^*X. The following theorem is a consequence of this principle:

Theorem 2.3.11 ([Swi02, Theorem 16.2 and 16.10]). *Again take E to be complex-oriented. The coset fibration*

$$U(n-1) \to U(n) \to S^{2n-1}$$

deloops to a spherical fibration

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

The associated Serre spectral sequence

$$E_2^{*,*} = H^*(BU(n); E^*S^{2n-1}) \Rightarrow E^*BU(n-1)$$

degenerates at E_{2n} and induces an isomorphism

$$E^*BU(n) \cong E^*[\![\sigma_1, \ldots, \sigma_n]\!].$$

Now, let $V\colon X \to BU(n)$ classify a vector bundle V. Then the coefficient c_j in the polynomial $c(V)$ is selected by σ_j:

$$c_j(V) = V^*(\sigma_j).$$

Proof sketch. The first part is a standard calculation. To prove the relation between the Chern classes and the σ_j, the splitting principle states that we can complete the map $V\colon X \to BU(n)$ to a square

$$\begin{tikzcd} Y_V \arrow[r, dotted, "\bigoplus_{i=1}^n \mathcal{L}_i"] \arrow[d, "f_V"] & BU(1)^{\times n} \arrow[d, "\oplus"] \\ X \arrow[r, "V"] & BU(n). \end{tikzcd}$$

The equation $c_j(f_V^* V) = V^*(\sigma_j)$ can be checked in E^*Y_V. □

We now see that not only does $\mathbb{P}(V)_E$ produce a point of $\mathrm{Div}_n^+(\mathbb{C}\mathrm{P}_E^\infty)$, but actually the scheme $\mathrm{Div}_n^+(\mathbb{C}\mathrm{P}_E^\infty)$ itself appears internally to topology:

Corollary 2.3.12. [13] *For a complex orientable cohomology theory E, there is an isomorphism*

$$BU(n)_E \cong \mathrm{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty,$$

[13] See [Str99b, Proposition 8.31] for a proof that recasts Theorem 2.3.11 itself in coordinate-free terms.

so that maps $V\colon X \to BU(n)$ are transported to divisors $\mathbb{P}(V)_E \subseteq \mathbb{C}\mathrm{P}^\infty_E \times X_E$. Selecting a particular complex orientation of E begets two isomorphisms

$$BU(n)_E \cong \widehat{\mathbb{A}}^n, \qquad \operatorname{Div}_n^+ \mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^n,$$

and these are compatible with the centered isomorphism above. □

This description has two remarkable features. One is its "faithfulness": this isomorphism of formal schemes means that the entire theory of characteristic classes is captured by the behavior of the divisor scheme. The other aspect is its coherence with topological operations we find on $BU(n)$. For instance, the Whitney sum map translates as follows:

Lemma 2.3.13. *There is a commuting square*

$$\begin{tikzcd} BU(n)_E \times BU(m)_E \arrow[r, "\oplus"] \arrow[d, equal] & BU(n+m) \arrow[d, equal] \\ \operatorname{Div}_n^+ \mathbb{C}\mathrm{P}^\infty_E \times \operatorname{Div}_m^+ \mathbb{C}\mathrm{P}^\infty_E \arrow[r, "\sqcup"] & \operatorname{Div}_{n+m}^+ \mathbb{C}\mathrm{P}^\infty_E. \end{tikzcd}$$

Proof. The sum map

$$BU(n) \times BU(m) \xrightarrow{\oplus} BU(n+m)$$

induces on Chern polynomials the identity [Swi02, Theorem 16.2.d]

$$c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2).$$

In terms of divisors, this means

$$\mathbb{P}(V_1 \oplus V_2)_E = \mathbb{P}(V_1)_E \sqcup \mathbb{P}(V_2)_E.$$ □

The following is a consequence of combining this Lemma with the splitting principle:

Corollary 2.3.14. *The map $Y_E \xrightarrow{f_V} X_E$ pulls $\mathbb{P}(V)_E$ back to give*

$$Y_E \times_{X_E} \mathbb{P}(V)_E \cong \bigsqcup_{i=1}^n \mathbb{P}(\mathcal{L}_i)_E.$$

Spaces		Schemes	
Object	Classifies	Object	Classifies
$BU(n)$	vector bundles of rank n	$\mathrm{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty$	effective divisors of rank n
$\coprod_n BU(n)$	all vector bundles	$\mathrm{Div}^+ \mathbb{C}\mathrm{P}_E^\infty$	all effective divisors
$BU \times \mathbb{Z}$	stable virtual bundles	$\mathrm{Div}\, \mathbb{C}\mathrm{P}_E^\infty$	stable Weil divisors
$BU \times \{0\}$	stable virtual bundles of rank 0	$\mathrm{Div}_0 \mathbb{C}\mathrm{P}_E^\infty$	stable divisors of rank 0

Figure 2.1: Different notions of vector bundles and their associated divisors

Interpretation. This says that the splitting principle is a topological enhancement of the claim that a divisor can be base-changed along a finite flat map where it splits as a sum of points. □

The other constructions from Lecture 2.2 are also easily matched up with topological counterparts:

Corollary 2.3.15. *There are natural isomorphisms* $BU_E \cong \mathrm{Div}_0 \mathbb{C}\mathrm{P}_E^\infty$ *and* $(BU \times \mathbb{Z})_E \cong \mathrm{Div}\, \mathbb{C}\mathrm{P}_E^\infty$. *Additionally,* $(BU \times \mathbb{Z})_E$ *is the free formal group on the curve* $\mathbb{C}\mathrm{P}_E^\infty$. □

Corollary 2.3.16. *There is a commutative diagram*

$$\begin{array}{ccc} BU(n)_E \times BU(m)_E & \xrightarrow{\otimes} & BU(nm)_E \\ \| & & \| \\ \mathrm{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty \times \mathrm{Div}_m^+ \mathbb{C}\mathrm{P}_E^\infty & \xrightarrow{\cdot} & \mathrm{Div}_{nm}^+ \mathbb{C}\mathrm{P}_E^\infty, \end{array}$$

where the bottom map acts by

$$(D_1, D_2 \subseteq \mathbb{C}\mathrm{P}_E^\infty \times X_E) \mapsto (D_1 \times D_2 \subseteq \mathbb{C}\mathrm{P}_E^\infty \times \mathbb{C}\mathrm{P}_E^\infty \xrightarrow{\mu} \mathbb{C}\mathrm{P}_E^\infty),$$

and μ *is the map induced by the tensor product map* $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$.

Proof. By the splitting principle, it is enough to check this on sums of line bundles. A sum of line bundles corresponds to a totally decomposed divisor, so we consider the case of a pair of such divisors $\bigsqcup_{i=1}^n \{a_i\}$ and $\bigsqcup_{j=1}^m \{b_j\}$. Referring to Definition 2.1.15, the map acts by

$$\left(\bigsqcup_{i=1}^{n}\{a_i\}\right)\left(\bigsqcup_{j=1}^{m}\{b_j\}\right) = \bigsqcup_{i,j}\{\mu_{\mathbb{C}P_E^\infty}(a_i, b_j)\}. \qquad \square$$

Finally, we can connect our analysis of the divisors coming from topological vector bundles with the line bundles studied at the start of the section.

Lemma 2.3.17. *Let $\zeta : X_E \to X_E \times \mathbb{C}P_E^\infty$ denote the pointing of the formal curve $\mathbb{C}P_E^\infty$, and let $\mathcal{I}(\mathbb{P}(V)_E)$ denote the ideal sheaf on $X_E \times \mathbb{C}P_E^\infty$ associated to the divisor subscheme $\mathbb{P}(V)_E$. There is a natural isomorphism of sheaves over X_E:*

In FPFP, Neil has a $-D$ rather than a D.

$$\zeta^*\mathcal{I}(\mathbb{P}(V)_E) \cong \mathbb{L}(V).$$

Proof sketch. In terms of a complex-oriented E and Theorem 2.3.6, the effect of pulling back along the zero section is to set $t = 0$, which collapses the Chern polynomial to just the top class $c_n(V)$. This element, called *the Euler class of V*, provides the E^*X–module generator of $E^*T(V)$—or, equivalently, the trivializing section of $\mathbb{L}(V)$. $\square$

Theorem 2.3.18 (cf. Theorem 5.2.1)**.** *A trivialization $t\colon \mathbb{L}(\mathcal{L}) \cong \mathcal{O}_{\mathbb{C}P_E^\infty}$ of the Thom sheaf associated to the canonical bundle induces a ring map $MUP \to E$.*

Proof. Suppose that V is a rank n vector bundle over X, and let $f\colon Y \to X$ be the space guaranteed by the splitting principle to provide an isomorphism $f^*V \cong \bigoplus_{j=1}^n \mathcal{L}_j$. The chosen trivialization t then pulls back to give a trivialization of $\mathcal{I}(\mathbb{P}(f^*V)_E)$, and by finite flatness this descends to also give a trivialization of $\mathcal{I}(\mathbb{P}(V)_E)$. Pulling back along the zero section gives a trivialization of $\mathbb{L}(V)$. Then note that the system of trivializations produced this way is multiplicative, as a consequence of $\mathbb{P}(V_1 \oplus V_2)_E \cong \mathbb{P}(V_1)_E \sqcup \mathbb{P}(V_2)_E$. In the universal examples, this gives a sequence of compatible maps $MU(n) \to E$ which assemble on the colimit $n \to \infty$ to give the desired map of ring spectra. $\square$

2.4 Power operations for complex bordism

Our eventual goal, like in Case Study 1, is to give an algebro-geometric description of $MU_*(*)$ and of the cooperations MU_*MU. It is possible to approach this the same way as in Lecture 1.5, using the Adams spectral sequence ([Qui69, Theorem 2], [Lura, Lecture 9]). However, $MU_*(*)$ is an integral algebra and so we cannot make do with working out the mod–2 Adams spectral sequence alone—we would at least have to work out the mod–p Adams spectral sequence for every p. At odd primes p, there is the following unfortunate theorem:

Theorem 2.4.1. *There is an isomorphism*

$$H\mathbb{F}_{p*}H\mathbb{F}_p \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \tau_2, \ldots]$$

with $|\xi_j| = 2p^j - 2$ *and* $|\tau_j| = 2p^j - 1$. □

There are odd–dimensional classes in this algebra, and the *graded-commutativity* of the dual mod–p Steenrod algebra means that these classes anti-commute. This prohibits us both from writing "$\operatorname{Spec}(H\mathbb{F}_{p*}H\mathbb{F}_p)$" and from trusting "$\operatorname{Spec}(H\mathbb{F}_p P_0 H\mathbb{F}_p)$" to do a good enough job. This is the first time we have encountered Hindrance #4 from Lecture 1.3 in the wild.

Because of this, we will not feel any guilt for taking a completely alternative approach to this calculation. This other method, due to Quillen, has as its keystone a completely different kind of cohomology operation called a *power operation*. These operations are quite technical to describe, but at their core is taking the n^{th} power of a cohomology class—and hence they have a frustrating lack of properties, including failures to be additive and to be stable. Our goal in this Lecture is just to define these cohomology operations, specialized to the particular setting we will need for Quillen's proof. Unlike with Steenrod squares, their algebro-geometric interpretation will not be immediately accessible to us, and indeed their eventual reinterpretation in these terms are one of the more hard-won pursuits in this subfield (cf. Appendix A.2).

Power operations arise not just from taking the n^{th} power of a cohomology class but from also remarking on the natural symmetry of that operation. We record this symmetry using the following technical apparatus:

Lemma 2.4.2 ([Lurb, Examples 6.1.4.2 and 6.1.6.2])**.** *Given a spectrum E, its n–fold smash power forms a Σ_n–spectrum, in the sense that there is a natural diagram $*//\Sigma_n \to$* $\mathsf{Spectra}$ *of ∞–categories which selects $E^{\wedge n}$ on objects.* □

In turn, a cohomology class $f\colon \Sigma^\infty_+ X \to E$ gives rise to a morphism of Σ_n–spectra

$$f^{\wedge n}\colon (\Sigma^\infty_+ X)^{\wedge n} \to E^{\wedge n}.$$

The homotopy colimit of such a diagram is called the *homotopy orbits* of the spectrum, and this gives a natural diagram

$$\begin{tikzcd} (\Sigma^\infty_+ X)^{\wedge n} \arrow[r, "f^{\wedge n}"] \arrow[d] & E^{\wedge n} \arrow[r, "\mu"] \arrow[d] & E \\ (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \arrow[r, "f^{\wedge n}_{h\Sigma_n}"] & E^{\wedge n}_{h\Sigma_n}. \arrow[ur, dotted, "\mu_n"'] \end{tikzcd}$$

A ring spectrum E equipped with a suite of factorizations μ_n satisfying compatibility laws embodying term collection (i.e., $x^a \cdot x^b = x^{a+b}$) and iterated exponentiation (i.e., $(x^a)^b = x^{ab}$) is called an H_∞*–ring spectrum* [BMMS86, Definition I.3.1]. The composite

$$P^{\Sigma_n}_{\text{ext}} f \colon (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \xrightarrow{f^{\wedge n}_{h\Sigma_n}} (\mathbb{S}^q \wedge E)^{\wedge n}_{h\Sigma_n} \xrightarrow{\mu_{n,q}} \mathbb{S}^{nq} \wedge E$$

is called the *external (total)* Σ_n*–power operation* applied to f, and the restriction to the diagonal subspace

$$P^{\Sigma_n} f \colon \Sigma^\infty_+ X \wedge \Sigma^\infty_+ B\Sigma_n \simeq (\Sigma^\infty_+ X)_{h\Sigma_n} \xrightarrow{\Delta_{h\Sigma_n}} (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \xrightarrow{P^{\Sigma_n}_{\text{ext}} f} E$$

is called the *(internal total)* Σ_n*–power operation* applied to f. We can consider it as a cohomology class lying in

$$P^{\Sigma_n} f \in E^0(X \times B\Sigma_n).$$

Cohomology classes $f \in E^{-q}(X)$ lying in degrees $q \neq 0$ require more care. Representing such a class as a map $f \colon \Sigma^\infty_+ X \to \mathbb{S}^q \wedge E$, the analogous diagram is

$$\begin{tikzcd} (\Sigma^\infty_+ X)^{\wedge n} \arrow[r, "f^{\wedge n}"] \arrow[d] & (\mathbb{S}^q \wedge E)^{\wedge n} \arrow[r, "\mu"] \arrow[d] & \mathbb{S}^{nq} \wedge E \\ (\Sigma^\infty_+ X)^{\wedge n}_{h\Sigma_n} \arrow[r, "f^{\wedge n}_{h\Sigma_n}"] & (\mathbb{S}^q \wedge E)^{\wedge n}_{h\Sigma_n} \arrow[r, equal] & \mathbb{S}^{q\rho}_{h\Sigma_n} \wedge E^{\wedge n}_{h\Sigma_n}, \arrow[u, dotted, "\mu_{n,q}"] \end{tikzcd}$$

where $\mathbb{S}^\rho$ is the representation sphere for the permutation representation of Σ_n. Such a system of factorizations is called an H^1_∞–ring structure on E—but in practice these are very uncommon (and in fact can only appear on $H\mathbb{F}_2$–algebras [BMMS86, Section VII.6.1]), and instead a subsystem of factorizations for every $q \equiv 0 \pmod d$ is called an H^d_∞–ring structure [BMMS86, Definition I.4.3]. In the same way, these give rise to external and internal power operations on cohomology classes of negative degree (or at least those which are divisible by d). We will mostly concern ourselves with the theory of H_∞ ring spectra, but we will eventually work with an H_∞^2 ring spectrum in the intended application.

Remark 2.4.3 ([BMMS86, Definition IV.7.1]). The optional adjective "total" is a reference to the following variant of this construction. By choosing a class in $E_* B\Sigma_n$, thought of as a functional on $E^* B\Sigma_n$, the Kronecker pairing of the total power operation with this fixed class gives rise to a truly internal cohomology operation on $E^* X$ as in the following composite:

$$\Sigma^\infty_+ X \wedge \mathbb{S} \xrightarrow{1 \wedge \sigma} \Sigma^\infty_+ X \wedge \Sigma^\infty_+ B\Sigma_n \wedge E \xrightarrow{P^{\Sigma_n} f \wedge 1} E \wedge E \xrightarrow{\mu} E.$$

For instance, restriction to the basepoint in $B\Sigma_n$ gives the n–fold cup product f^n.

Some properties of this construction are immediately visible—for instance, it is multiplicative:

$$P^{\Sigma_n}(f \cdot g) = P^{\Sigma_n}(f) \cdot P^{\Sigma_n}(g),$$

and restriction of $P^{\Sigma_n}(f)$ to the basepoint in $\Sigma^\infty_+ B\Sigma_n$ yields the cup power class f^n. In order to state any further properties, we will need to make some extraneous observations. First, note that any map of groups $\varphi\colon G \to \Sigma_n$ gives a variation on this construction by restriction of diagrams

$$\begin{tikzcd} * // G \arrow[r, "\varphi"] & * // \Sigma_n \arrow[r, bend left, "(\Sigma^\infty_+ X)^{\wedge n}"] \arrow[r, bend right, "E^{\wedge n}"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Spectra}. \end{tikzcd}$$

This construction is useful when studying composites of power operations: the group $\Sigma_n \wr \Sigma_k$ acts naturally on $(E^{\wedge k})^{\wedge n}$, and indeed there is an equivalence

$$P^{\Sigma_n} \circ P^{\Sigma_k} = P^{\Sigma_n \wr \Sigma_k}.$$

In order to understand these modified power operations more generally, we are motivated to study such maps φ more seriously. Some basic constructions are summarized in the following definition:

Definition 2.4.4 ([May96, Sections XI.3 and XXV.3]). Let $\varphi\colon G \to H$ be an inclusion

Cite me: Find a blanket reference for all these cohomology maps. The Alaska notes don't speak in terms of fancy diagram categories. Maybe there's a Barwick reference?.

of finite groups and let F be an H–spectrum. There is a natural map of homotopy colimits $\varphi_*\colon F_{hG} \to F_{hH}$ which induces a *restriction map* on cohomology:

$$\mathrm{Res}_G^H\colon E^0 F_{hH} \to E^0 F_{hG}.$$

The spectrum $(H/G) \times F$ considered as a G–spectrum with the diagonal G–action has the property $((H/G) \times F)_{hH} = F_{hG}$, and the G–equivariant averaging map

$$F \xrightarrow{\sum_{[h] \in H/G} [h] \times (-)} (H/G) \times F$$

passes on homotopy orbits to the *additive norm map* $N_G^H\colon F_{hH} \to F_{hG}$, which again induces a map on cohomology classes

$$\mathrm{Tr}_G^H\colon E^0 F_{hG} \to E^0 F_{hH}$$

called the *transfer map*. The composite $\varphi_* \circ N_G^H$ acts by multiplication by the index $|H/G|$, and hence this is also true of $\mathrm{Res}_G^H \mathrm{Tr}_G^H$.

The restriction and transfer maps appear prominently in the following formula, which measures the failure of the power operation construction to be additive:

Lemma 2.4.5 ([BMMS86, Corollary II.1.6], [AHS04, Proposition A.5, Equation 3.6])**.** *For cohomology classes $f, g \in E^0 X$, there is a formula*[14]

$$P^{\Sigma_n}(f+g) = \sum_{i+j=n} \mathrm{Tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_n} \left(P^{\Sigma_i}(f) \cdot P^{\Sigma_j}(g) \right). \qquad \square$$

To produce binomial formulas for the modified power operations, we use the following Lemma:

Nat indicated that there is some kind of transitivity condition here that I am numb to.

Lemma 2.4.6 ([Ada78, p. 109-110], [HKR00, Section 6.5])**.** *Let G_1, G_2 be subgroups of H, and consider the homotopy pullback diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * // G_1 \arrow[d] \\ * // G_2 \arrow[r] & * // H. \end{tikzcd}$$

[14] This should be compared with the classical binomial formula $\frac{1}{n!}(x+y)^n = \sum_{i+j=n} \frac{1}{i!j!} x^i y^j$.

Given any identification $P \simeq \coprod_K (*//K)$, there is a push-pull interchange formula

$$\mathrm{Res}^{G_1}_H \mathrm{Tr}^{G_2}_H = \sum_K \mathrm{Tr}^{G_1}_K \mathrm{Res}^{G_2}_K. \qquad \square$$

Corollary 2.4.7. *For any subgroup $G \le \Sigma_n$, there is a congruence*

$$P^G(f+g) \equiv P^G(f) + P^G(g) \pmod{\textit{transfers from proper subgroups of } G}.$$

Proof. Note that P^G can be defined by means of restriction: $P^G = \mathrm{Res}^{\Sigma_n}_G P^{\Sigma_n}$. We can hence reuse the previous binomial formula:

$$P^G(f+g) = \mathrm{Res}^{\Sigma_n}_G P^{\Sigma_n}(f+g) = \sum_{i+j=n} \mathrm{Res}^{\Sigma_n}_G \mathrm{Tr}^{\Sigma_n}_{\Sigma_i \times \Sigma_j} \left(P^{\Sigma_i}(f) \cdot P^{\Sigma_j}(g)\right).$$

In the cases $i = 0$ or $j = 0$, the transfer map is the identity operation, and we recover $P^G(g)$ and $P^G(f)$ respectively. In all the other terms, the interchange lemma lets us pull the transfer to the outside. $\square$

Since the only operations we understand so far are stable operations, which are in particular additive, we are moved to find a target for the power operation P^G which kills the ideal generated by the proper transfers (i.e., the intermediate terms in the binomial formula) yet which remains computable.

Definition 2.4.8 ([Lurb, Construction 6.1.6.4, Definition 6.1.6.24])**.** Again let F be an G–spectrum, and define its *homotopy fixed points* to be the homotopy limit spectrum F^{hG}. The norm map $F_{hG} \to F$ associated to the subgroup $1 \le G$ witnesses the trivial G–spectrum F_{hG} as a constant cone over the G–spectrum F, and hence gives a natural factorization $F_{hG} \to F^{hG} \to F$ of the norm. The cofiber of this first map is denoted F^{tG} and is called the *Tate spectrum* of G. This gives rise to a notion of Tate power operation via

$$\begin{tikzcd} \pi_0 E^{\Sigma^\infty_+ X} \arrow[r, "P^G"] & \pi_0 E^{(\Sigma^\infty_+ X)_{hG}} \arrow[r, equal] & \pi_0 (E^{\Sigma^\infty_+ X})^{hG} \arrow[r] & \pi_0 (E^{\Sigma^\infty_+ X})^{tG}, \\ f \arrow[r, mapsto] & P^G f \arrow[rr, mapsto] & & P^G_{\mathrm{Tate}} f. \end{tikzcd}$$

Lemma 2.4.9. *In the case $G = C_p$, the Tate power operation is additive.*

Proof. The image of the map $\pi_0(E^{\Sigma^\infty_+ X})_{hC_p} \to \pi_0(E^{\Sigma^\infty_+ X})^{hC_p}$ is the kernel of the projection to the Tate object. Since the only proper subgroup of C_p is the trivial subgroup, this image contains all transfers.[15] $\square$ Maybe find a reference for Justin's construction.

[15]For a general group G, the definition of "Tate construction" must be modified to kill all

The real miracle is that these cyclic Tate power operations are not only additive, but they are even *completely* computable.

Lemma 2.4.10. *The assignment* $X \mapsto \pi_0(E^{(\Sigma^\infty_+ X)^{\wedge p}})^{tC_p}$ *is a cohomology theory.*

Cite me: I learned this proof from Tyler Lawson. He suggested looking at Greenlees–May or Lunøe-Nielsen–Rognes. This is Prop 2.2.3 of DAG XIII. He also said that Charles has referred him to this point of view before.

Proof. The Eilenberg–Steenrod axioms are clear except for the cofiber sequence axiom, which we will boil down to checking that the assignment $X \mapsto (X^{\wedge p})^{tC_p}$ preserves cofiber sequences of finite complexes. A cofiber sequence $X \to Y \to Y/X$ of pointed spaces is equivalent data to the diagram

$$X \to Y,$$

which has colimit Y and which admits a filtration by distance from the initial node with filtration quotients:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \| & & \downarrow \\ X & & Y/X. \end{array}$$

By taking the p–fold Cartesian power of the diagram $X \to Y$, we produce a diagram shaped like an p–dimensional hypercube with colimit $Y^{\wedge p}$ and which again admits a filtration by distance from the initial node. The colimits of these partial diagrams give a C_p–equivariant filtration of $Y^{\wedge p}$ as:

$$\begin{array}{ccccccccc} F_0 & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_{p-1} & \longrightarrow & Y^{\wedge p} \\ \| & & \downarrow & & & & \downarrow & & \downarrow \\ X^{\wedge p} & & \bigvee X^{\wedge (p-1)} \wedge (Y/X) & & \cdots & & \bigvee X \wedge (Y/X)^{\wedge (p-1)} & & (Y/X)^{\wedge p}. \end{array}$$

We now apply $(-)^{tC_p}$ to this diagram. The Tate construction carries cofiber sequences of C_p–spectra to cofiber sequences of spectra, so this is again a filtration diagram. In the intermediate filtration quotients, the C_p–action is given by freely cycling wedge factors (i.e., these spectra are induced up from spectra with trivial actions), from which it follows that the Tate construction vanishes on these nodes. Hence, the diagram postcomposed with the Tate construction takes the form

intermediate transfers, and not just those which transfer up from the trivial subgroup. Such a topological construction is possible, but it requires understanding *families* of subgroups, which is more equivariant homotopy theory than we otherwise need.

$$\begin{tikzcd} F_0^{tC_p} \arrow[r, "\simeq"] \arrow[d, equal] & F_1^{tC_p} \arrow[r, "\simeq"] \arrow[d] & \cdots \arrow[r, "\simeq"] & F_{p-1}^{tC_p} \arrow[r] \arrow[d] & (Y^{\wedge p})^{tC_p} \arrow[d] \\ (X^{\wedge p})^{tC_p} & * & \cdots & * & ((Y/X)^{\wedge p})^{tC_p}. \end{tikzcd}$$

Eliminating the intermediate filtration stages with empty filtration quotients, we see that this filtration is equivalent data to a cofiber sequence

$$(X^{\wedge p})^{tC_p} \to (Y^{\wedge p})^{tC_p} \to ((Y/X)^{\wedge p})^{tC_p}.$$

Repeating this proof inside of E–module spectra gives the desired result. □

The effect of this Lemma is twofold. For one, the Tate operation is not only additive, it is even *stable*. Secondly, it suffices to understand the behavior of passing to the Tate construction in the case of $X = *$, i.e., the effect of the map $E^{hC_n} \to E^{tC_n}$. Since we are intending to make a computation, it will at this point be convenient to also specialize to our case of interest of where $E = MU$.[16]

Theorem 2.4.11. *There is an isomorphism* Cite me: Credit Jeremy?.

$$\pi_* MU^{tC_n} = MU^* BC_n[x^{-1}],$$

where x is the restriction to $MU^2 BC_n$ of the canonical class $x \in MU^2(\mathbb{C}\mathrm{P}^\infty)$.

Proof. Consider the C_n–equivariant cofiber sequence

$$S(\mathbb{C}^m)_+ \to S^0 \to S^{\mathbb{C}^m},$$

where $S(\mathbb{C}^m)$ is the unit sphere inside of $\mathbb{C}^m$ and $S^{\mathbb{C}^m}$ is the one-point compactification of the C_n–representation $\mathbb{C}^m$. A key fact is that $S(\mathbb{C}^m)$ admits a C_n–equivariant cell decomposition by free cells, natural with respect to the inclusions as m increases. This buys us several facts:

1. The following Tate objects vanish: $(MU \wedge S(\mathbb{C}^m)_+)^{tC_n} \simeq *$. As in the proof of Lemma 2.4.10, this is because the Tate construction vanishes on free C_n–cells.

2. We can use $\operatorname{colim}_{m\to\infty} S(\mathbb{C}^m)_+$ as a model for EC_n, so that

$$MU_{hC_n} = \left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)_{hC_n}.$$

[16] Everything we say here will actually be valid for any number n, not just a prime, as well as any ring spectrum E under MU.

3. Coupling these two facts together, we get
$$MU_{hC_n} = \left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)_{hC_n} = \left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)^{hC_n}.$$

4. Pulling the fixed points functor out, this gives
$$\begin{aligned} MU^{tC_n} &= \operatorname{cofib}(MU_{hC_n} \to MU^{hC_n}) \\ &= \operatorname{cofib}\left(\left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right)^{hC_n} \to MU^{hC_n}\right) \\ &= \left(\operatorname{cofib}\left(\operatorname*{colim}_{m\to\infty} MU \wedge S(\mathbb{C}^m)_+\right) \to MU \wedge S^0\right)^{hC_n} \\ &= \left(\operatorname*{colim}_{m\to\infty} MU \wedge S^{\mathbb{C}^m}\right)^{hC_n}. \end{aligned}$$

This last formula puts us in a position to calculate. The Thom isomorphism for MU gives an identification $MU \wedge S^{\mathbb{C}} \simeq \Sigma^2 MU$ as C_n–spectra, and the map
$$(MU \wedge S^0)^{hC_n} \to (MU \wedge S^{\mathbb{C}})^{hC_n}$$
can be identified with multiplication by the Thom class:
$$MU^{\Sigma^\infty BC_n} \xrightarrow{x \cdot -} (\Sigma^2 MU)^{\Sigma^\infty BC_n}.$$
In all, this gives
$$MU^{tC_n} \simeq \operatorname*{colim}_{m\to\infty}\left((MU \wedge S^{\mathbb{C}^m})^{hC_n}\right) \simeq \operatorname*{colim}_{m\to\infty}\left((\Sigma^{2m} MU)^{BC_n}\right) \simeq MU^{BC_n}[x^{-1}]. \quad \square$$

To apply this machinery in our case, we need only equip MU with the factorized multiplication maps appearing in the definition of an H_∞ ring spectrum. (In fact, we will note that this enriches to an H_∞^2 ring spectrum structure in the next Lecture.)

Definition 2.4.12 ([Rud98, Definition VII.7.4])**.** Suppose that $\xi\colon X \to BU(k)$ presents a complex vector bundle of rank k on X. The n–fold direct sum of this bundle gives a new bundle
$$X^{\times n} \xrightarrow{\xi^{\oplus n}} BU(k)^{\times n} \xrightarrow{\oplus} BU(n \cdot k)$$
of rank nk on which the cyclic group C_n acts. By taking the (homotopy) C_n–quotient of the fiber, total, and base spaces, we produce a vector bundle $\xi(n)$ on $X^{\times n}_{hC_n}$ participating in the diagram

$$\begin{tikzcd}
X^{\times n} \arrow[r, "\xi^{\oplus n}"] \arrow[d] & BU(k)^{\times n} \arrow[r, "\oplus"] \arrow[d] & BU(nk) \\
X^{\times n}_{hC_n} \arrow[r, "\xi(n)"] & BU(k)^{\times n}_{hC_n}. \arrow[ur, dotted, "\widetilde{\mu}_n"] &
\end{tikzcd}$$

The universal case gives the map $\widetilde{\mu}_n$.

Lemma 2.4.13 ([Rud98, Equation VII.7.3])**.** *There is an isomorphism of Thom spectra*

$$T(\xi(n)) \simeq (T\xi)^{\wedge n}_{hC_n}.$$

This citation is actually for Thom spaces, where he picks up a factor of $S^n_{hC_n}$. This might be important to get right for the future, when we're doing unstable / degree–sensitive things. **Jeremy warned me that this is more serious than I wanted to admit. Compare carefully with Rudyak.**

Proof. This proof is mostly a matter of having had the idea to write down the Lemma to begin with. From here, we string basic properties together:

$$\begin{aligned}
T(\xi(n)) &= T(\xi^{\oplus n}_{hC_n}) && \text{(definition)} \\
&= T(\xi^{\oplus n})_{hC_n} && \text{(colimits commute with colimits)} \\
&= T(\xi)^{\wedge n}_{hC_n}. && \text{(T is monoidal: Lemma 1.1.8)}
\end{aligned}$$
□

Applying the Lemma to the universal case produces a factorization

$$MU(k)^{\wedge n} \to MU(k)^{\wedge n}_{hC_n} \to MU(nk)$$

of the unstable multiplication map, and hence a stable factorization

$$MU^{\wedge n} \to MU^{\wedge n}_{hC_n} \xrightarrow{\mu_n} MU.$$

In fact, this enriches to an H^2_∞ ring spectrum structure: by applying Lemma 2.4.13 in reverse, the basic construction is

$$\begin{aligned}
\mu_{2,2} \colon \mathbb{S}^2_{h\Sigma_2} \wedge MU^{\wedge 2}_{h\Sigma_2} &\xrightarrow{1 \wedge \mu_2} \mathbb{S}^2_{h\Sigma_2} \wedge MU \\
&\simeq T(1_{\mathbb{R}}(2) \downarrow *) \wedge MU && \text{(Lemma 2.4.13,} \\
& && \text{+ “$1_{\mathbb{R}}$” the trivial real bundle)} \\
&\simeq T(1_{\mathbb{C}} \downarrow *) \wedge MU && \text{(MU–Thom isomorphism)} \\
&\simeq \mathbb{S}^2 \wedge MU. && \text{(Thom spectrum of $1_{\mathbb{C}}$)}
\end{aligned}$$

Our goal in the next Lecture will be to gain a computational understanding of $\mu_{n,2d}$.

Remark 2.4.14 ([Qui71, Equations 3.10-11]). The picture Quillen paints of all this is considerably different from ours. He begins by giving a different presentation of the complex cobordism groups of a manifold M: a complex orientation of a smooth map $Z \to M$ is a factorization

Cite me: Cite Christia... dergrad thesis?.

$$Z \xrightarrow{i} E \xrightarrow{\pi} M$$

through a complex vector bundle $\pi\colon E \to M$ by an embedding i, as well as a complex structure on the normal bundle ν_i. Up to suitable notions of stability (in the dimension of E) and homotopy equivalence (involving, in particular, isotopies of different embeddings i), these quotient to give cobordism classes of maps complex-oriented maps $Z \to M$. The collection of cobordism classes over M of codimension q over is isomorphic to $MU^q(M)$ [Qui71, Proposition 1.2]. Quillen's definition of the power operations is then given in terms of this geometric model: a representative $f\colon Z \to M$ of a cobordism class gives rise to another complex-oriented map $f^{\times n}\colon Z^{\times n} \to M^{\times n}$, and he defines $P^{C_n}_{\mathrm{ext}}(f)$ to be the postcomposition with $M^{\times n} \to M^{\times n}_{hC_n}$ (after taking care that the target isn't typically a manifold). All the properties of his construction must therefore be explored through the lens of groups acting on manifolds.

Cite me: Where can you find Steenrod squares defined like this? The VFoS notes, but that's not a great reference..

Example 2.4.15. The chain model for $H\mathbb{F}_2$–homology is actually also rigid enough to define power operations, and somewhat curiously these operations automatically turn out to be additive, without passing to the Tate construction. This means that they are recognizable in terms of classical Steenrod operations: the C_2–construction

$$\{\Sigma^n \Sigma^\infty_+ X \xrightarrow{f} H\mathbb{F}_2\} \xrightarrow{P^{C_2}} \{\Sigma^{2n}\Sigma^\infty_+ X \wedge \Sigma^\infty_+ BC_2 \xrightarrow{P^{C_2}(f)} H\mathbb{F}_2\}$$

gives a class in $H\mathbb{F}_2^{2n-*}(X) \otimes H\mathbb{F}_2^*(\mathbb{RP}^\infty)$, which decomposes as

$$P^{C_2}(f) = \sum_{j=0}^{2n} \mathrm{Sq}^{2n-j}(f) \otimes x^j.$$

The Adem relations can be extracted by studying the wreath product $\Sigma_2 \wr \Sigma_2$ and the compositional identity for power operations.

Remark 2.4.16 ([BMMS86, Theorems III.4.1-3, Remark III.4.4]). Since the failure of the power operations to be additive was a consequence of the binomial formula, it is somewhat intuitive that modulo 2, where $(x+y)^2 \equiv x^2 + y^2$, that this operation becomes stable. In fact, more than this is true: for instance, if an H_∞ ring spectrum E has $\pi_0 E = \mathbb{F}_p$, it must be the case that E is an $H\mathbb{F}_p$–algebra. This fact also gives an inexplicit means to recover Lemma 1.5.8, after noting that the same methods used to endow MU with an H_∞ ring spectrum structure do the same for MO.

2.5 Explicitly stabilizing cyclic MU–power operations

Having thus demonstrated that the Tate variant of the cyclic power operation decomposes as a sum of stable operations, we are motivated to understand the available such stable operations in complex bordism. This follows quickly from our discussions in the previous few Lectures. We learned in Corollary 2.3.12 that for any complex-oriented cohomology theory E we have the calculation

$$E^*BU \cong E^*[\![\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots]\!],$$

and we gave a rich interpretation of this in terms of divisor schemes:

$$BU_E \cong \operatorname{Div}_0 \mathbb{C}\mathrm{P}^\infty_E.$$

We would like to leverage the Thom isomorphism to gain a description of E^*MU generally and MU^*MU specifically. However, the former is *not* a ring, and although the latter is a ring its multiplication is exceedingly complicated[17], which means that our extremely compact algebraic description of E^*BU in Corollary 2.3.12 will be of limited use. Instead, we will have to content ourselves with an E_*–module basis of E^*MU.

Definition 2.5.1. Take $MU \to E$ to be a complex-oriented ring spectrum, which presents E^*BU as the subalgebra of symmetric functions inside of an infinite–dimensional polynomial algebra:

$$E^*BU \subseteq E^*BU(1)^{\times\infty} \cong E^*[\![x_1, x_2, \ldots]\!].$$

For any nonnegative multi-index $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finitely many entries nonzero, there is an associated *monomial symmetric function* b_α, which is the sum

[17]For a space X, E^*X has a ring structure because X has a diagonal, and MU does not have a diagonal. In the special case of $E = MU$, there is a ring product coming from endomorphism composition.

of those monomials whose exponent lists contain exactly α_j many instances of j.[18] We then set $s_\alpha \in E^*MU$ to be the image of b_α under the Thom isomorphism of E_*–modules

$$E^*MU \cong E^*BU.$$

It is called the α^{th} *Landweber–Novikov operation* with respect to the orientation $MU \to E$.

Definition 2.5.2. In the case of the identity orientation $MU \xrightarrow{\mathrm{id}} MU$, the resulting classes are called the *Conner–Floyd–Chern classes* and the associated cohomology operations are called the *Landweber–Novikov operations* (without further qualification).

Remark 2.5.3. For a vector bundle V and a complex-oriented cohomology theory E, we define the *total symmetric Chern class* of V by the sum

$$c_{\mathbf{t}}(V) = \sum_\alpha c_\alpha(V)\mathbf{t}^\alpha.$$

In the case of a line bundle $\mathcal{L}$ with first Chern class $c_1(\mathcal{L}) = x$, this degenerates to the sum

$$c_{\mathbf{t}}(\mathcal{L}) = \sum_{j=0}^{\infty} x^j t_j.$$

For a direct sum $U = V \oplus W$, the total symmetric Chern class satisfies a Cartan formula:

$$c_{\mathbf{t}}(U) = c_{\mathbf{t}}(V \oplus W) = c_{\mathbf{t}}(V) \cdot c_{\mathbf{t}}(W).$$

Again specializing to line bundles $\mathcal{L}$ and $\mathcal{H}$ with first Chern classes $c_1(\mathcal{L}) = x$ and $c_1(\mathcal{H}) = y$, this gives

$$\begin{aligned} c_{\mathbf{t}}(U) = c_{\mathbf{t}}(\mathcal{L} \oplus \mathcal{H}) &= \left(\sum_{j=0}^{\infty} x^j t_j\right) \cdot \left(\sum_{k=0}^{\infty} y^k t_k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x^j y^k t_j t_k \\ &= 1 + (x+y)t_1 + (xy)t_1^2 + (x^2+y^2)t_2 + (xy^2 + x^2y)t_1t_2 + \cdots \\ &= 1 + c_1(U)t_1 + c_2(U)t_1^2 + (c_1^2(U) - 2c_2(U))t_2 + (c_1(U)c_2(U))t_1t_2 + \cdots, \end{aligned}$$

[18] For example, $\alpha = (1, 2, 0, 0, \ldots)$ corresponds to the sum

$$b_\alpha = \sum_i \sum_{j \neq i} \sum_{\substack{k \neq i \\ k > j}} x_i x_j^2 x_k^2 = x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2^2 x_3 + \cdots.$$

where we have expanded out some of the pieces of the total symmetric Chern class in polynomials in the usual Chern classes.

Definition 2.5.4 ([Ada95, Theorem I.5.1])**.** Take the orientation to be $MU \xrightarrow{\mathrm{id}} MU$, so that we are considering MU^*MU and the Landweber–Novikov operations arising from the Conner–Floyd–Chern classes. These account for the *stable* operations in MU–cohomology, analogous to the Steenrod operations for $H\mathbb{F}_2$. They satisfy the following properties:

Should you include Quillen's "norm" perspective on these operations (cf. between equations 2.2 and 2.3)? You were just almost-but-not-quite talking about isogenies, and you're going to want to talk about norm constructions eventually...

- s_0 is the identity.
- s_α is natural: $s_\alpha(f^*x) = f^*(s_\alpha x)$.
- s_α is stable: $s_\alpha(\sigma x) = \sigma(s_\alpha x)$.
- s_α is additive: $s_\alpha(x+y) = s_\alpha(x) + s_\alpha(y)$.
- s_α satisfies a Cartan formula. Define

  $$s_{\mathbf{t}}(x) := \sum_\alpha s_\alpha(x)\mathbf{t}^\alpha := \sum_\alpha s_\alpha(x) \cdot t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \cdots \in MU^*(X)[\![t_1, t_2, \ldots]\!]$$

  for an infinite sequence of indeterminates $t_1, t_2, \ldots$. Then

  $$s_{\mathbf{t}}(xy) = s_{\mathbf{t}}(x) \cdot s_{\mathbf{t}}(y).$$

- Let $\xi \colon X \to BU(n)$ classify a vector bundle and let φ denote the Thom isomorphism

  $$\varphi \colon MU^*X \to MU^*T(\xi).$$

  Then the Chern classes of ξ are related to the Landweber–Novikov operations on the Thom spectrum by the formula

  $$\varphi(c_\alpha(\xi)) = s_\alpha(\varphi(1)).$$

Having now set up an encompassing theory of stable operations, we now seek to give a formula for the cyclic Tate power operation in this framework. In order to approach this, we initially set our sights on the too-lofty goal of computing the total power operation $P^{C_n}(f)$ for $f \in MU^{2q}(X)$ an MU–cohomology class on a finite complex X, without necessarily passing to the Tate construction. Because of the definition $MU = \operatorname{colim}_k MU(k)$ and because P^{C_n} is natural under pullback, it will suffice for us to study the effect of P^{C_n} on the universal classes

$$u_m \colon MU(m) \to MU,$$

after using the canonical MU–Thom isomorphism to reinterpret them as classes on a suspension spectrum. We begin with the canonical orientation itself:

$$u_1 = x \in h\mathsf{Spectra}(MU(1), MU) \cong MU^2\mathbb{C}\mathrm{P}^\infty.$$

In order to understand the effect $P^{C_n}(x)$ of the power operation on x, we recall a different interpretation of x: it is also the 1st Conner–Floyd–Chern class of the tautological bundle $\mathcal{L}$ on $\mathbb{C}\mathrm{P}^\infty$, i.e.,

$$x \colon MU(1) \to MU$$

is the Thomification of the block inclusion

$$\mathcal{L} \colon BU(1) \to BU.$$

The H^2_∞ ring spectrum construction defining $P^{C_n}_{\mathrm{ext}}(x)$ thus fits into the following diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ BU(1) \arrow[r] \arrow[d, "\Delta"] & \Sigma^\infty_+ BU(1) \wedge \Sigma^\infty_+ BC_n \arrow[d, "\Delta_{hC_n}"] & \\
(\Sigma^\infty_+ BU(1))^{\wedge n} \arrow[r] \arrow[d, "c_1^{\wedge n}"] \arrow[rr, bend right, "\mathcal{L}^{\oplus n}"'] & \Sigma^\infty_+ BU(1)^{\wedge n}_{hC_n} \arrow[r, "\mathcal{L}(n)"] \arrow[d] \arrow[dr, "P^{C_n}_{\mathrm{ext}}"] & \Sigma^\infty_+ BU(n) \arrow[d, "c_n"] \\
(\Sigma^2 MU)^{\wedge n} \arrow[r] & (\Sigma^2 MU)^{\wedge n}_{hC_n} \arrow[r] & \Sigma^{2n} MU.
\end{tikzcd}$$

The commutativity of the widest rectangle (i.e., the justification for the name "c_n" on the right-most vertical arrow) comes from the Cartan formula for Chern classes: because $\mathcal{L}^{\oplus n}$ splits as the sum of n line bundles, $c_n(\mathcal{L}^{\oplus n})$ is computed as the product of the 1st Chern classes of those line bundles. Second, the commutativity of the right-most square is not trivial: it is a specific consequence of how the multiplicative structure on MU arises from the direct sum of vector bundles.[19] The

[19] In general, any notion of first Chern class $\Sigma^\infty_+ BU(1) \to \Sigma^2 E$ gives rise to a *noncommuting* diagram of this same shape. The two composites $\Sigma^\infty_+ BU(1)^{\wedge n}_{hC_n} \to \Sigma^{2n} E$ need not agree, since $\mathcal{L}(n)$ has no *a priori* reason to be compatible with the factorization appearing in the H^2_∞–structure. They turn out to be related nonetheless, and their exact relation (as well as a procedure for making them agree) is the subject of Appendix A.2.

commutativities of the other two squares comes from the natural transformation from a C_n–space to its homotopy orbit space.

Hence, the the internal cyclic power operation $P^{C_n}(x)$ applied to the canonical coordinate x is defined by the composite

$$\Sigma^\infty_+ BU(1) \wedge \Sigma^\infty_+ BC_n \simeq \Sigma^\infty_+ BU(1)_{hC_n} \xrightarrow{\Delta_{hC_n}} (\Sigma^\infty_+ BU(1))^{\wedge n}_{hC_n} \to \Sigma^{2n} MU^{\wedge n}_{hC_n} \to \Sigma^{2n} MU,$$

which is to say

$$P^{C_n}(x) = c_n(\Delta^*_{hC_n} \mathcal{L}(n)).$$

We have thus reduced to computing a particular Conner–Floyd–Chern class of a particular bundle.

Our next move is to realize that we have not lost information by passing from the bundle $\Delta^* \mathcal{L}^{\oplus n}$ to the bundle $\Delta^*_{hC_n} \mathcal{L}(n)$.

Theorem 2.5.5. *There is a natural bijection between G–equivariant vector bundles over a base X on which G acts trivially and vector bundles on $X \times BG$.*

Construction. This is the exponential adjunction

$$\begin{tikzcd} \mathsf{Spaces}(X \times BG, BU) \arrow[r, shift left] & \mathsf{Spaces}(* /\!/ G, \mathsf{Spaces}(X, BU)), \arrow[l, shift left] \\ V_{hG} \arrow[r, mapsto, shift left] & V. \arrow[l, mapsto, shift left] \end{tikzcd}$$

The right-hand side consists of G–equivariant vector bundles over the G–trivial base X, and the left-hand side consists of vector bundles over $X \times BG$. $\square$ This isn't much of a proof.

Corollary 2.5.6. *Under this bijection, the vector bundle $\Delta^*_{hC_n} \mathcal{L}(n)$ on $BU(1) \times BC_n$ corresponds to the C_n–equivariant vector bundle $\Delta^* \mathcal{L}^{\oplus n}$ on $BU(1)$.* $\square$

We thus proceed to analyze $\Delta^*_{hC_n} \mathcal{L}(n)$ by studying the C_n–equivariant bundle $\Delta^* \mathcal{L}^{\oplus n}$ instead. The C_n–action is given by permutation of the factors, and hence we have an identification

$$\Delta^* \mathcal{L}^{\oplus n} \cong \mathcal{L} \otimes \pi^* \rho,$$

where ρ is the permutation representation of C_n (considered as a vector bundle over a point) and $\pi \colon BU(1) \to *$ is the constant map. The permutation representation for the abelian group C_n, also known as its regular representation, is accessible by character theory. The generating character $\chi\colon U(1)[n] \to U(1)$ gives a decomposition

$$\rho \cong \bigoplus_{j=0}^{n-1} \chi^{\otimes j}.$$

Applying this to our situation, we get a sequence of isomorphisms of C_n–equivariant vector bundles

$$\Delta^* \mathcal{L}^{\oplus n} \cong \mathcal{L} \otimes \pi^* \rho \cong \mathcal{L} \otimes \bigoplus_{j=0}^{n-1} \pi^* \chi^{\otimes j} \cong \bigoplus_{j=0}^{n-1} \mathcal{L} \otimes \pi^* \chi^{\otimes j}.$$

Applying Theorem 2.5.5, we recast this as a calculation of the bundle $\Delta^*_{hC_n} \mathcal{L}(n)$:

$$\Delta^*_{hC_n} \mathcal{L}(n) = \bigoplus_{j=0}^{n-1} \pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j},$$

where η is the bundle classified by $\eta\colon BU(1)[n] \to BU(1)$ and π_1, π_2 are the two projections off of $BU(1) \times BC_n$.

We can use this to access $c_n(\Delta^*_{hC_n} \mathcal{L}(n))$. As the top Chern class of this n–dimensional vector bundle, we think of this as a calculation of its Euler class, which lets us lean on multiplicativity:

$$\begin{aligned} P^{C_n}(x) = c_n(\Delta^*_{hC_n} \mathcal{L}(n)) &= e\left(\bigoplus_{j=0}^{n-1} \pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j}\right) = \prod_{j=0}^{n-1} e\left(\pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j}\right) \\ &= \prod_{j=0}^{n-1} c_1\left(\pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j}\right) = \prod_{j=0}^{n-1} (x +_{MU} [j]_{MU}(t)). \end{aligned}$$

Here x is still the 1st Conner–Floyd–Chern class of $\mathcal{L}$ and t is the Euler class of η. We now try to make sense of this product expression for $c_n(\Delta^*_{hC_n} \mathcal{L}(n))$ by expanding it in powers of x and identifying its component pieces.

Lemma 2.5.7. *There is a series expansion*

$$P^{C_n}(x) = \prod_{j=0}^{n-1} (x +_{MU} [j]_{MU}(t)) = w + \sum_{j=1}^{\infty} a_j(t) x^j,$$

where $a_j(t)$ is a series with coefficients in the subring $C \subseteq MU_*$ spanned by the coefficients of the natural MU–formal group law. The leading term

$$w = e(\rho) = \prod_{j=0}^{n-1} e(\eta^{\otimes j}) = \prod_{j=0}^{n-1} [j]_{MU}(e(\eta)) = (n-1)!t^{n-1} + \sum_{j \geq n} b_j t^j$$

is the Euler class of the reduced permutation representation, and, again, the elements b_j lie in the subring C. □

This is about as much information as we can hope to extract in the 1st universal case. We thus return to our original goal: understanding the action of P^{C_n} on each of the canonical classes

$$u_m \colon MU(m) \to MU.$$

We approach this via the splitting principle by first rewriting the formula in Lemma 2.5.7 in a form amenable to direct sums:

$$P^{C_n}(x) = \sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x), \qquad a_\alpha(t) = \prod_{j=0}^{\infty} a_j(t)^{\alpha_j}.$$

Corollary 2.5.8. *There is the universal formula*

$$P^{C_n}(u_m) = \sum_{|\alpha| \leq m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m).$$

Proof. This follows directly from the splitting principle and the Cartan formula:

$$\begin{aligned} P^{C_n}(u_m) &= \overbrace{P^{C_n}(x_1) \cdot \cdots \cdot P^{C_n}(x_m)}^{\text{each of the } m \text{ factors}} \\ &= \left(\sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x_1) \right) \cdot \cdots \cdot \left(\sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x_m) \right) \\ &= \sum_{|\alpha| \leq m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m). \end{aligned}$$

□

We will use this to power the following conclusion about cohomology classes on in general, starting with an observation about the fundamental class of a sphere:

Lemma 2.5.9 (cf. [Rud98, Corollary VII.7.14])**.** *For $f \in MU^{2q}(X)$ a cohomology class in a finite complex X, there is the suspension relation*

$$P^{C_n}(\sigma^{2m} f) = w^m \sigma^{2m} P^{C_n}(f).$$

Proof. We calculate P^{C_n} applied to the fundamental class

$$S^{2m} \xrightarrow{\iota_{2m}} T_m BU(m) \simeq \Sigma^{2m} MU(m) \xrightarrow{\Sigma^{2m} u_m} \Sigma^{2m} MU$$

by restricting the universal formula:

$$P^{C_n}(\iota_{2m}^* u_m) = \iota_{2m}^* P^{C_n}(u_m) = \iota_{2m}^* \left(\sum_{|\alpha| \le m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m) \right) = w^m \iota_{2m},$$

since $s_\alpha(\iota_{2m}) = 0$ for any nonzero α, as the cohomology of S^{2m} is too sparse. Because $\sigma^{2m} f = \iota_{2m} \wedge f$, we conclude the proof by multiplicativity of P^{C_n}. □

Theorem 2.5.10 (cf. [Qui71, Proposition 3.17], [Rud98, Corollary VII.7.14])**.** *Let X be a finite pointed space and let f be a cohomology class*

$$f \in \widetilde{MU}^{2q}(X).$$

For $m \gg 0$, there is a formula

$$w^m P^{C_n}(f) = \sum_{|\alpha| \le m+q} w^{q+m-|\alpha|} a_\alpha(t) s_\alpha(f),$$

with t, w, and $a_\alpha(t)$ as defined above.[20]

Proof. We take m large enough so that f is represented by an unstable map

$$g \colon \Sigma^{2m} X \to T_{m+q} BU(m+q),$$

in the sense that g intertwines f with the universal class u_{m+q} by the formula

$$g^* u_{m+q} = \sigma^{2m} f.$$

We use Lemma 2.5.9 and naturality to conclude

$$\begin{aligned} w^m \sigma^{2m} P^{C_n}(f) &= P^{C_n}(\sigma^{2m} f) = P^{C_n}(g^* u_{m+q}) = g^* P^{C_n}(u_{m+q}) \\ &= g^* \left(\sum_{|\alpha| \le m+q} w^{m+q-|\alpha|} a_\alpha(t) s_\alpha(u_{m+q}) \right) \\ &= \sum_{|\alpha| \le m+q} w^{m+q-|\alpha|} a_\alpha(t) \sigma^{2m} s_\alpha(f). \end{aligned}$$

□

[20] The reader comparing with Quillen's paper will notice various apparent discrepancies between the statements of our Theorem and of his. These are notational: he grades his cohomology functor homologically, which occasionally causes our q to match his $-q$, so that his n is comparable to our $m - q$.

Our conclusion, then, is that P^{C_n} is *almost* naturally expressible in terms of the Landweber–Novikov operations, where the "*almost*" is controlled by some w–torsion. Our discussion of the Tate construction in the previous Lecture shows that this is, in some sense, a generic phenomenon, and indeed the above Theorem can be divided by w^m to recover a statement about the cyclic Tate power operation. However, we have learned the additional information that the various factors in this statement—including w itself—are controlled by the formal group law "$+_{MU}$" associated to the tautological complex orientation of MU and the subring C. This is *not* generic behavior. In the next Lecture, we will discover the surprising fact that we only need to multiply by a *single* w—also highly non-generic—and the equally surprising consequences this entails for MU_* itself.

2.6 The complex bordism ring

With Theorem 2.5.10 in hand, we will now deduce Quillen's major structural theorem about MU_*. We will preserve the notation used in Lemma 2.5.7 and Theorem 2.5.10:

- ρ is the reduced regular representation of C_n, which coincides with its reduced permutation representation, and $w = e(\rho)$ is its Euler class.
- $\eta\colon BU(1)[n] \to BU(1)$ is the line bundle associated to a generating character for C_n, and $t = e(\eta)$ its Euler class.
- C is the subring of MU_* generated by the coefficients of the formal group law associated to the identity complex–orientation.

In the course of working out the main Theorem, we will want to make use of some properties of the class t.

Lemma 2.6.1 (cf. [RW80, Theorem 5.7], [HL, Proposition 2.4.4], [Rud98, Theorem VII.7.9])**.** *There is an isomorphism of formal groups*[21]

$$BU(1)[n]_{MU} \cong BU(1)_{MU}[n].$$

Proof. Consider the pullback diagram of spherical fibrations:

[21] In algebraic language, $MU^*BU(1)[n] \cong MU^*BU(1)/[n](x)$, where $x \in MU^2BU(1)$ is the canonical coordinate and $[n](x)$ is the n–series.

$$\begin{tikzcd} U(1) \arrow[r] \arrow[d, equal] & BU(1)[n] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & BU(1) \arrow[d, "n"] \\ U(1) \arrow[r] & EU(1) \arrow[r] & BU(1). \end{tikzcd}$$

The Euler class of the first bundle is the class x, and it pulls back along $n\colon BU(1) \to BU(1)$ to give the Euler class $[n](x)$ of the second bundle. The induced long exact sequence[22] takes the form

$$\begin{tikzcd} & MU^*BU(1) \arrow[dl] & \\ MU^*(BU(1)[n]) \arrow[rr, "\partial"'] & & MU^{*+2}BU(1) \arrow[ul, "- \smile [n](x)"'] \end{tikzcd}$$

where x is the coordinate on $BU(1)_{MU}$. Because $[n]_{MU}(x) = nx + \cdots$ and because $\widetilde{MU}^2\mathbb{C}P^1$ has no n–torsion , the right diagonal map is injective and hence $\partial = 0$. This therefore gives a short exact sequence of Hopf algebras, which we can reinterpret as a short exact sequence of group schemes

Is this really enough? I'm confused again.

$$0 \to BU(1)[n]_{MU} \to BU(1)_{MU} \xrightarrow{n} BU(1)_{MU} \to 0. \qquad \square$$

Corollary 2.6.2. *The Künneth map*

$$MU^*(X) \otimes_{MU^*} MU^*(BU(1)[n]) \to MU^*(X \times BU(1)[n])$$

is an isomorphism. In terms of coordinate rings, this gives isomorphisms

$$(X \times BU(1)[n])_{MU} \cong X_{MU} \times BU(1)[n]_{MU} \cong X_{MU} \times BU(1)_{MU}[n].$$

Proof. This follows from the evenness of $MU^*(BU(1)[n])$. $\square$

Corollary 2.6.3 ([Qui71, Proposition 4.4])**.** *Write*

$$\langle n \rangle_{MU}(x) = \frac{[n]_{MU}(x)}{x}.$$

If $\omega \in MU^*BU(1)[n]$ satisfies $t \cdot \omega = 0$, then there exists a class y with $\omega = y \cdot \langle n \rangle_{MU}(t)$.

[22] This sequence is known as the Gysin sequence. It arises as the exact couple for the Serre spectral sequence for the spherical fibration.

Proof. By Lemma 2.6.1 we know $MU^*BU(1)[n] \cong MU^*[\![t]\!]/[n]_{MU}(t)$, so the kernel of multiplication by t is exactly $\langle n\rangle_{MU}(t)$.[23] □

In all, we learn that the Euler class $t = e(\eta)$ corresponds to the restriction of the coordinate x along the closed inclusion

$$BU(1)[n]_{MU} \cong BU(1)_{MU}[n] \to BU(1)_{MU}.$$

We now turn to the main Theorem.

Theorem 2.6.4 ([Qui71, Theorem 5.1])**.** *If X has the homotopy type of a finite complex, then*

$$MU^*(X) = C \cdot \sum_{q \geq 0} MU^q(X),$$
$$\widetilde{MU}^*(X) = C \cdot \sum_{q > 0} MU^q(X).$$

Remark 2.6.5. In what follows, the reader should carefully remember the degree conventions stemming from the formula

$$MU^*X = \pi_{-*}F(\Sigma^\infty_+ X, MU).$$

The homotopy ring MU_* appears in the *negative* degrees of $MU^*(*)$, but the fundamental class of S^m appears in the *positive* degree $MU^m(S^m)$.

Proof of Theorem 2.6.4. We can immediately reduce the claim in two ways. First, it is true if and only if it is also true for reduced cohomology. Second, because $MU^{2*+1}(*) = 0$,[24] we can restrict attention just to $MU^{2*}(X)$, since we can then handle the odd-degree parts of $MU^*(X)$ by suspending X once. Defining

$$R^{2*} := C \cdot \sum_{q>0} MU^{2q}(X),$$

we can thus focus on the claim

$$\widetilde{MU}^{2*}(X) \stackrel{?}{=} C \cdot \sum_{q>0} MU^{2q}(X).$$

[23] Quillen considers this as coming from the Gysin sequence for $S^1 \to S(\eta) \to BU(1)[n]$, which has Euler class t.

[24] This is a direct consequence of the *geometric* definition of U–structured bordism, essentially because of the equivalence $X_{2n} = U(n) = X_{2n+1}$, where the map $BX_{2n+1} \to BGL_1 S^{2n}$ classifies the tautological bundle on $BU(n)$ plus a trivial bundle.

Noting that the claim is trivially true for all positive values of $*$, we will show this by working p–locally and inducting on the value of "$-*$".

Suppose that

$$R^{-2j}_{(p)} = \widetilde{MU}^{-2j}(X)_{(p)}$$

for $j < q$ and consider $x \in \widetilde{MU}^{-2q}(X)$. Then, for $m \gg 0$, we have

$$w^m P^{C_p}(x) = \sum_{|\alpha| \le m-q} w^{m-q-|\alpha|} a(t)^\alpha s_\alpha x = w^{m-q} x + \sum_{\substack{|\alpha| \le m-q \\ \alpha \neq 0}} w^{m-q-|\alpha|} a(t)^\alpha s_\alpha x.$$

Recall that w is a power series in t with coefficients in C and leading term $(p-1)! \cdot t^{p-1}$, so that $t^{p-1} = w \cdot \theta(t)$ for some multiplicatively invertible series $\theta(t)$ with coefficients in C. Since s_α raises degree, we have $s_\alpha x \in R$ by the inductive hypothesis, and we may thus collect all those terms (as well as many factors of $\theta(t)^{-1}$) into a series $\psi_x(t) \in R_{(p)}[\![t]\!]$ to write

$$t^N(w^q P^{C_p}(x) - x) = \psi_x(t),$$

where $N = (m-q)(p-1)$.

Consider the set of possible integers N for which we can write such an equation—we know that $N = (m-q)(p-1)$ works, but we now also consider values of N which are not multiples of $(p-1)$. We aim to conclude that $N = 1$ is the minimum of this set, so suppose that N is the minimum such value. Using Remark 2.4.3, we find that restricting this equation along the inclusion $i\colon X \to X \times BU(1)[p]$ sets $t = 0$ and yields $\psi_x(0) = 0$. It follows that $\psi_x(t) = t\varphi_x(t)$ is at least once t–divisible, and thus

$$t(t^{N-1}(w^q P^{C_p} x - x) - \varphi_x(t)) = 0.$$

Appealing to Corollary 2.6.3, we produce a class $y \in \widetilde{MU}^{-2q+2(N-1)}(X)$ with

$$t^{N-1}(w^q P^{C_p}(x) - x) = \varphi_x(x) + y\langle p\rangle(t).$$

If $N > 1$, then $y \in R_{(p)}$ for degree reasons and hence the right-hand side gives a series expansion contradicting our minimality hypothesis. So, $N = 1$, the class y lies in the critical degree $-2q$, and the outer factor of t^{N-1} is not present in the last expression.[25] Restricting along i again to set $w = t = 0$ and $P^{C_p}(x) = x^p$, we obtain the equation

$$\left.\begin{array}{rl} -x & \text{if } q > 0 \\ x^p - x & \text{if } q = 0 \end{array}\right\} = \varphi_x(0) + py.$$

In the first case, where $q > 0$, it follows that $MU^{-2q}(X) \subseteq R^{-2q} + pMU^{-2q}(X)$, and since $MU^{-2q}(X)$ has finite order torsion[26] it follows that $MU^{-2q}(X) = R^{-2q}$. In the other case, x can be rewritten as a sum of elements in R^0, elements in $p\widetilde{MU}^0(X)$, and elements in $(\widetilde{MU}^0(X))^p$. Since the ideal $\widetilde{MU}^0(X)$ is nilpotent, it again follows that $\widetilde{MU}^0(X) = R^0$, concluding the induction. □

Corollary 2.6.6 ([Qui71, Corollary 5.2])**.** *The coefficients of the formal group law generate MU_*.*

Proof. This is the case of Theorem 2.6.4 where we set $X = *$. □

Remark 2.6.7. This proof actually also goes through for MO as well. In that case, it's even easier, since the equation $2 = 0$ in $\pi_0 MO$ causes much of the algebra to collapse. The proof does not extend further to cases like MSO or MSp, as explained in the introduction to this Case Study: these bordism theories do not have associated formal group laws, and so we lose the control we had in Lecture 2.5.

Take $\mathcal{M}_{\mathbf{fgl}}$ to be the moduli of formal group laws. Since a formal group law is a power series satisfying some algebraic identities, this moduli object is an affine scheme with coordinate ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. A rephrasing of Corollary 2.6.6 is that the natural map

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$$

is *surjective*. This is reason enough to start studying $\mathcal{M}_{\mathbf{fgl}}$ in earnest, which we take up in the next Case Study—but while we're here, if we anachronistically assume one algebraic fact about $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ we can prove that the natural map is actually an *isomorphism*. The place to start is with the following topological observation about mixing complex–orientations:

[25] One can interpret the proof thus far as giving a bound on the amount of w–torsion needed to get the stability relation described in Theorem 2.5.10. Our answer is quite surprising: we have found that we need just a single w (indeed, a single t), which isn't much stability at all!

[26] This is a consequence of X having finitely many cells, MU having finitely many cells in each degree, and each homotopy group of the stable sphere being finitely generated.

Lemma 2.6.8 ([Ada95, Lemma 6.3 and Corollary 6.5])**.** *Let $\varphi\colon MU \to E$ be a complex–oriented ring spectrum and consider the two orientations on $E \wedge MU$ given by*

$$\mathbb{S} \wedge MU \xrightarrow{\eta_E \wedge 1} E \wedge MU, \qquad MU \wedge \mathbb{S} \xrightarrow{\varphi \wedge \eta_{MU}} E \wedge MU.$$

The two induced coordinates x^E and x^{MU} on $\mathbb{C}\mathrm{P}^\infty_{E\wedge MU}$ are related by the formulas

$$x^{MU} = \sum_{j=0}^{\infty} b_j^E (x^E)^{j+1} =: g(x^E),$$
$$g^{-1}(x^{MU} +_{MU} y^{MU}) = g^{-1}(x^E) +_E g^{-1}(y^E).$$

where $E_*MU \cong \frac{\mathrm{Sym}_{E_*} E_*\{\beta_1,\beta_2,\beta_3,\ldots\}}{\beta_1=1} \cong E_*[b_1, b_2, \ldots]$, as in Lemma 1.5.1, Corollary 1.5.2, and Corollary 2.0.4.

Proof. The second formula is a direct consequence of the first. The first formula comes from taking the module generators $\beta_{j+1} \in E_{2(j+1)}\mathbb{C}\mathrm{P}^\infty = E_{2j}MU(1)$ and pushing them forward to get the algebra generators $b_j \in E_{2j}MU$. Then, the triangle

$$\begin{tikzcd}
{[\mathbb{C}\mathrm{P}^\infty, MU]} \arrow[rr] \arrow[dr] & & {[\mathbb{C}\mathrm{P}^\infty, E \wedge MU]} \arrow[dl, "\cong"] \\
& \mathsf{Modules}_{E_*}(E_*\mathbb{C}\mathrm{P}^\infty, E_*MU) &
\end{tikzcd}$$

allows us to pair x^{MU} with $(x^E)^{j+1}$ to determine the coefficients of the series. □

Corollary 2.6.9 ([Ada95, Corollary 6.6])**.** *In particular, for the orientation $MU \to H\mathbb{Z}$ we have*

$$x_1 +_{MU} x_2 = \exp^H(\log^H(x_1) + \log^H(x_2)),$$

where $\exp^H(x) = \sum_{j=0}^{\infty} b_j x^{j+1}$. □

However, one also notes that $H\mathbb{Z}_*MU = \mathbb{Z}[b_1, b_2, \ldots]$ carries the universal example of a formal group law with a logarithm—this observation follows directly from the Thom isomorphism, and so is independent of any knowledge about the coefficient ring MU_*. It turns out that this brings us one step away from understanding MU_*:

Theorem 2.6.10 (To be proven as Theorem 3.2.2)**.** *The ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ carrying the universal formal group law is free: it is isomorphic to a polynomial ring over $\mathbb{Z}$ in countably many generators.* □

Corollary 2.6.11. *The natural map $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ classifying the formal group law on MU_* is an isomorphism.*

Proof. We proved in Corollary 2.6.6 that this map is surjective. We also proved in Theorem 2.1.22 that every rational formal group law has a logarithm, i.e., the long composite on the second row

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \arrow[r] \arrow[d] & MU_* \arrow[r] \arrow[d] & (H\mathbb{Z}_* MU) \arrow[d] \\
\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \arrow[r] & MU_* \otimes \mathbb{Q} \arrow[r, "\simeq"] & (H\mathbb{Z}_* MU) \otimes \mathbb{Q}
\end{tikzcd}$$

is an isomorphism. It follows from Theorem 2.6.10 that the left-most vertical map is injective, hence the top-left horizontal map is injective, hence it is an isomorphism. □

Corollary 2.6.12. *The ring $\pi_*(MU \wedge MU)$ carries the universal example of two strictly isomorphic formal group laws. Additionally, the ring $\pi_0(MUP \wedge MUP)$ carries the universal example of two isomorphic formal group laws.*

Proof. Combine Lemma 2.6.8 and Corollary 2.6.11. □

Case Study 3

Finite spectra

Our goal in this Case Study is to thoroughly examine one of the techniques from Case Study 1 that has not yet resurfaced: the idea that $H\mathbb{F}_2$–homology takes values in quasicoherent sheaves over some algebro-geometric object encoding the coaction of the dual Steenrod Hopf algebra. We will find that this situation is quite generic: associated to mildly nice ring spectra E, we will construct a very rich algebro-geometric object $\mathcal{M}_E$, called its context, such that E–homology sends spaces X to sheaves $\mathcal{M}_E(X)$ over $\mathcal{M}_E$. In still nicer situations, the difference between the E_*–module $E_*(X)$ and the sheaf $\mathcal{M}_E(X)$ tracks exactly the analogue of the action of the dual Steenrod algebra, called the *Hopf algebroid of stable E–homology cooperations*. From this perspective, we will reinterpret Quillen's Corollary 2.6.11 as giving a presentation

$$\mathcal{M}_{MUP} \xrightarrow{\simeq} \mathcal{M}_{\mathbf{fg}},$$

where $\mathcal{M}_{\mathbf{fg}}$ is the *moduli of formal groups*. This indicates a program for studying periodic complex bordism, which we will spend the rest of this introduction outlining.

Abstractly, one can hope to study any sheaf, including $\mathcal{M}_E(X)$, by analyzing its stalks, or relatedly (with some luck) by analyzing its geometric fibers. The main utility of Quillen's theorem is that it gives us access to a concrete model of the context $\mathcal{M}_{MUP}$, so that we can determine where to even look for those fibers. However, even this is not really enough to get off the ground: the stalks of some sheaf can exhibit nearly arbitrary behavior. In particular, there is little reason to expect the stalks of $\mathcal{M}_E(X)$ to vary nicely with X. Accordingly, given a map f in the diagram

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "f"] \arrow[dr] & \mathcal{M}_{\mathbf{fgl}} \arrow[r, equal] \arrow[d] & \mathcal{M}_{MUP}[0] \arrow[r, equal] \arrow[d] & \operatorname{Spec} MUP_0 \\
& \mathcal{M}_{\mathbf{fg}} \arrow[r, equal] & \mathcal{M}_{MUP},
\end{tikzcd}$$

life would be easiest if the R–module determined by $f^*\mathcal{M}_{MUP}(X)$ were itself the value of a homology theory $R_0(X) = MUP_0X \otimes_{MUP_0} R$—this is exactly what it would mean for $R_0(X)$ to "vary nicely with X". Of course, this is unreasonable to expect in general: homology theories are functors which convert cofiber sequences of spectra to long exact sequences of groups, but base–change from $\mathcal{M}_{\mathbf{fg}}$ to Spec R preserves exact sequences exactly when the diagonal arrow is *flat*. However, if flatness is satisfied, this gives the following theorem:

Theorem 3.0.1 (Landweber)**.** *Given such a diagram where the diagonal arrow is flat, the functor*

$$R_0(X) := MUP_0(X) \otimes_{MUP_0} R$$

is a 2*–periodic homology theory.*

In the course of proving this theorem, Landweber additionally devised a method to recognize flat maps. Recall that a map $f\colon Y \to X$ of schemes is flat exactly when for any closed subscheme $i\colon A \to X$ with ideal sheaf $\mathcal{I}$ there is an exact sequence

$$0 \to f^*\mathcal{I} \to f^*\mathcal{O}_X \to f^*i_*\mathcal{O}_A \to 0.$$

Landweber classified the closed subobjects of $\mathcal{M}_{\mathbf{fg}}$, thereby giving a precise list of conditions needed to check maps for flatness.

This appears to be a moot point, however, as it is unreasonable to expect this idea to apply to computing geometric fibers: the inclusion of a geometric point is flat only in highly degenerate cases. We will see that this can be repaired: the inclusion of the formal completion of a subobject is flat in friendly situations, and so we naturally become interested in the infinitesimal deformation spaces of the geometric points Γ on $\mathcal{M}_{\mathbf{fg}}$. If we can analyze those, then Landweber's theorem will produce homology theories called *Morava E_Γ–theories*. Moreover, if we find that these deformation spaces are *smooth*, it will follow that their deformation rings support regular sequences. In this excellent case, by taking the regular quotient we will be able to recover *Morava K_Γ–theory*, a *homology theory*, which plays the role[1] of computing the stalk of $\mathcal{M}_{MUP}(X)$ at Γ.[2]

We have thus assembled a task list:

- Describe the open and closed subobjects of $\mathcal{M}_{\mathbf{fg}}$.
- Describe the geometric points of $\mathcal{M}_{\mathbf{fg}}$.
- Analyze their infinitesimal deformation spaces.

These will occupy our attention for the first half of this Case Study. In the second half, we will exploit these homology theories E_Γ and K_Γ, as well as their connection to $\mathcal{M}_{\mathbf{fg}}$ and to MU, to make various structural statements about the category Spectra. These homology theories are especially well-suited to understanding the subcategory $\mathsf{Spectra}^{\mathrm{fin}}$ of finite spectra, and we will recount several important statements in that setting. Together with these homology theories, these celebrated results (collectively called the nilpotence and periodicity theorems) form the basis of *chromatic homotopy theory*. In fact, our *real* goal in this Case Study is to give an introduction to the chromatic perspective that remains in line with our algebro-geometrically heavy narrative.

3.1 Descent and the context of a spectrum

In Lecture 1.4 we took for granted the $H\mathbb{F}_2$–Adams spectral sequence, which had the form

$$E_2^{*,*} = H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); \widetilde{H\mathbb{F}_2 P_0 X}) \Rightarrow \pi_* X_2^\wedge,$$

where we had already established some yoga by which we could identify the dual Steenrod coaction on $H\mathbb{F}_2 P_0 X$ with an action of $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$ on its associated quasicoherent sheaf over $\operatorname{Spec}\mathbb{F}_2$. Our goal in this Lecture is to revise this tool to work for other ring spectra E and target spectra X, eventually arriving at a spectral sequence with signature

$$E_2^{*,*} = H^*(\mathcal{M}_E; \mathcal{M}_E(X)) \Rightarrow \pi_* X.$$

[1]To be clear: $K_\Gamma(X)$ may not actually compute the literal stalk of $\mathcal{M}_{MUP}(X)$ at Γ, since the homotopical operation of quotienting out the regular sequence is potentially sensitive to torsion sections of $\mathcal{M}_{MUP}(X)$.

[2]Incidentally, this program has no content when applied to $\mathcal{M}_{H\mathbb{F}_2}$, as $\operatorname{Spec}\mathbb{F}_2$ is simply too small.

In particular, we will encounter along the way the object "$\mathcal{M}_E$" envisioned in the introduction to this Case Study.

At a maximum level of vagueness, we are seeking a process by which its homotopy $\pi_* X$ can be recovered from the E–homology groups $E_* X$. Generally speaking, spectral sequences arise from taking homotopy groups of a topological version of this same recovery process—i.e., recovering the spectrum X from the spectrum $E \wedge X$. Recognizing that X can be thought of as an $\mathbb{S}$–module and $E \wedge X$ can be thought of as its base change to an E–module, we are inspired to double back and consider as inspiration an algebraic analogue of the same situation. Given a ring map $f\colon R \to S$ and an S–module N, Grothendieck's framework of *(faithfully flat) descent* addresses the following questions:

1. When is there an R–module M such that $N \cong S \otimes_R M = f^* M$?

2. What extra data can be placed on N, called *descent data*, so that the category of descent data for N is equivalent to the category of R–modules under the map f^*?

3. What conditions can be placed on f so that the category of descent data for any given module is always contractible, called *effectivity*?

Suppose that we begin with an R–module M and set $N = f^* M$, so that we are certain *a priori* that the answer to the first question is positive. The S–module N has a special property, arising from f being a ring map: there is a canonical isomorphism of $(S \otimes_R S)$–modules

$$\begin{aligned} \varphi\colon S \otimes_R N &= \\ (f \otimes 1)^* N &= \\ ((f \otimes 1) \circ f)^* M &\cong ((1 \otimes f) \circ f)^* M \qquad s_1 \otimes (s_2 \otimes m) \mapsto (s_1 \otimes m) \otimes s_2. \\ &= (1 \otimes f)^* N \\ &= N \otimes_R S, \end{aligned}$$

In fact, this isomorphism is compatible with further shuffles, in the sense that the following diagram commutes:[3]

[3]The commutativity of this triangle shows that any number of shuffles also commutes.

$$\begin{tikzcd}
N \otimes_R S \otimes_R S \arrow[rr, "\varphi_{13}", "\simeq"'] \arrow[dr, "\varphi_{12}", "\simeq"'] & & S \otimes_R S \otimes_R N \\
& S \otimes_R N \otimes_R S, \arrow[ur, "\simeq", "\varphi_{23}"'] &
\end{tikzcd}$$

where φ_{ij} denotes applying φ to the i^{th} and j^{th} coordinates.

Definition 3.1.1. An S–module N equipped with such an isomorphism $\varphi\colon S \otimes_R N \to N \otimes_R S$ which causes the triangle to commute is called a *descent datum for f*.

Descent data admit two equivalent reformulations, both of which are useful to note.

Remark 3.1.2 ([Ami59]). The ring $C = S \otimes_R S$ admits the structure of an S–coring: we can use the map f to produce a relative diagonal map

$$\Delta\colon S \otimes_R S \cong S \otimes_R R \otimes_R S \xrightarrow{1 \otimes f \otimes 1} S \otimes_R S \otimes_R S \cong (S \otimes_R S) \otimes_S (S \otimes_R S).$$

The descent datum φ on an S–module N is equivalent to a C–coaction map. The S–linearity of the coaction map is encoded by a square

$$\begin{tikzcd}
S \otimes_R N \arrow[r, "1 \otimes \psi"] \arrow[d] \arrow[drr, "\varphi"] & S \otimes_R N \otimes_S (S \otimes_R S) \arrow[d] & \\
N \arrow[r, "\psi"] & N \otimes_S (S \otimes_R S) \arrow[r, equal] & N \otimes_R S,
\end{tikzcd}$$

and the long composite gives the descent datum φ. Conversely, given a descent datum φ we can restrict it to get a coaction map by

$$\psi\colon N = R \otimes_R N \xrightarrow{f \otimes 1} S \otimes_R N \xrightarrow{\psi} N \otimes_R S.$$

The coassociativity condition on the comodule is equivalent under this correspondence to the commutativity of the triangle associated to φ.

Remark 3.1.3 ([Hov02, Theorem A]). Alternatively, descent data also arise naturally as sheaves on simplicial schemes. Associated to the map $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$, we can form a Čech complex

$$\mathcal{D}_f := \left\{ \operatorname{Spec} S \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{matrix} \operatorname{Spec} S \\ \times_{\operatorname{Spec} R} \\ \operatorname{Spec} S \end{matrix} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{matrix} \operatorname{Spec} S \\ \times_{\operatorname{Spec} R} \\ \operatorname{Spec} S \\ \times_{\operatorname{Spec} R} \\ \operatorname{Spec} S \end{matrix} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots \right\},$$

which factors the map f as

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r, "\mathrm{sk}^0"] \arrow[rr, bend left, "f"] & \mathcal{D}_f \arrow[r, "c"] & \operatorname{Spec} R. \end{tikzcd}$$

A quasicoherent (and Cartesian [Sta14, Tag 09VK]) sheaf $\mathcal{F}$ over a simplicial scheme X is a sequence of quasicoherent sheaves $\mathcal{F}[n]$ on $X[n]$ as well as, for each map $\sigma\colon [m] \to [n]$ in the simplicial indexing category inducing a map $X(\sigma)\colon X[n] \to X[m]$, a natural choice of isomorphism of sheaves

$$\mathcal{F}(\sigma)^*\colon X(\sigma)^*\mathcal{F}[m] \to \mathcal{F}[n].$$

In particular, a pullback $c^*\widetilde{M}$ gives such a quasicoherent sheaf on $\mathcal{D}_f$. By restricting attention to the first three levels we find exactly the structure of the descent datum described before. Additionally, we have a natural *Segal isomorphism*

$$\mathcal{D}_f[1]^{\times_{\mathcal{D}_f[0]}(n)} \xrightarrow{\simeq} \mathcal{D}_f[n] \quad (\text{cf. } S \otimes_R S \otimes_R S \cong (S \otimes_R S) \otimes_S (S \otimes_R S) \text{ at } n = 2),$$

which shows that any descent datum (including those not arising, a priori, from a pullback) can be naturally extended to a full quasicoherent sheaf on $\mathcal{D}_f$.

The following Theorem is the culmination of a typical first investigation of descent:[4]

Cite me: Actually give an original citation for f.f. descent? Or just reference Vakil.

Theorem 3.1.4 (Grothendieck)**.** *If $f\colon R \to S$ is faithfully flat, the natural assignments*

[4]For details and additional context, see Vistoli [Vis05, Section 4.2.1]. The story in the context of Hopf algebroids is also spelled out in detail by Miller [Milb].

$$\begin{tikzcd} \mathsf{QCoh}(\operatorname{Spec} R) \arrow[r, bend left, "c^*"] & \mathsf{QCoh}(\mathcal{D}_f) \arrow[l, bend left, "\lim"] \end{tikzcd}$$

form an equivalence of categories.

Jumping off point. The basic observation in this case is that $0 \to R \to S \to S \otimes_R S$ is an exact sequence of R–modules.[5] This makes much of the homological algebra involved work out. $\square$

Without the flatness hypothesis, this Theorem fails dramatically and immediately. For instance, the inclusion of the closed point

$$f\colon\ \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$$

fails to distinguish the $\mathbb{Z}$–modules $\mathbb{Z}$ and $\mathbb{Z}/p$. Remarkably, this can be to large extent repaired by reintroducing homotopy theory and passing to derived categories—for instance, the complexes $Lf^*\widetilde{\mathbb{Z}}$ and $Lf^*\widetilde{\mathbb{Z}/p}$ become distinct as objects of $D(\operatorname{Spec} \mathbb{F}_p)$. Our preceding discussion of descent in Remark 3.1.3 can be quickly revised for this new homotopical setting, provided we remember to derive not just the categories of sheaves but also their underlying geometric objects. Our approach is informed by the following result:

Lemma 3.1.5 ([EKMM97, Theorem IV.2.4])**.** *There is an equivalence of ∞–categories between $D(\operatorname{Spec} R)$ and $\mathsf{Modules}_{HR}$.* $\square$

Hence, given a map of rings $f\colon R \to S$, we redefine the derived descent object to be the cosimplicial ring spectrum

$$\mathcal{D}_{Hf} := \left\{ \begin{array}{ccccccc} & & & & & \longrightarrow & \\ & & & \longrightarrow & HS & \longleftarrow & \\ & \longrightarrow & HS & \longleftarrow & \wedge_{HR} & \longrightarrow & \\ HS & \longleftarrow & \wedge_{HR} & \longrightarrow & HS & \longleftarrow & \cdots \\ & \longrightarrow & HS & \longleftarrow & \wedge_{HR} & \longrightarrow & \\ & & & \longrightarrow & HS & \longleftarrow & \\ & & & & & \longrightarrow & \end{array} \right\},$$

[5]In the language of Example 1.4.18, this says that R itself appears as the cofixed points $S \,\square_{S \otimes_R S}\, R$.

and note that an R–module M gives rise to a cosimplicial left–$\mathcal{D}_{Hf}$–module which we denote $\mathcal{D}_{Hf}(HM)$. The totalization of this cosimplicial module gives rise to an HR–module receiving a natural map from M, and we can ask for an analogue of Theorem 3.1.4.

Lemma 3.1.6. *For $f\colon \mathbb{Z} \to \mathbb{F}_p$ and M a connective complex of $\mathbb{Z}$–modules, the totalization* $\operatorname{Tot} \mathcal{D}_{Hf}(HM)$ *recovers the p–completion of M.*

Proof sketch. The Hurewicz map $H\mathbb{Z} \to H\mathbb{F}_p$ kills $(p) \subseteq \pi_0 H\mathbb{Z}$, and we further calculate

$$H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p \simeq H\mathbb{F}_p \vee \Sigma H\mathbb{F}_p$$

to be connective. Combining these facts shows that the filtration of $\mathcal{D}_{Hf}(HM)$ gives the p–adic filtration of the homotopy groups $\pi_* HM$. If $\pi_* HM$ is already p–complete, then the reassembly map $HM \to \operatorname{Tot} \mathcal{D}_{Hf}(HM)$ is a weak equivalence. $\square$

We are now close enough to our original situation that we can make the last leap: rather than studying a map $Hf\colon HR \to HS$, we instead have the unit map $\eta\colon \mathbb{S} \to E$ associated to some ring spectrum E. Fixing a target spectrum X, we define the analogue of the descent object:

Definition 3.1.7. The *descent object* for X along $\eta\colon \mathbb{S} \to E$ is the cosimplicial spectrum

$$\mathcal{D}_E(X) := \left\{ \begin{array}{ccccccc} & & & & & \longrightarrow & \\ & & & \longrightarrow & E & \longleftarrow & \\ & \xrightarrow{\eta_L} & E & \longleftarrow & \wedge & \longrightarrow & \\ E & \xleftarrow{\mu} & \wedge & \xrightarrow{\Delta} & E & \longleftarrow & \\ \wedge & \xrightarrow{\eta_R} & E & \longleftarrow & \wedge & \longrightarrow & \cdots \\ X & & \wedge & \longrightarrow & E & \longleftarrow & \\ & & X & & \wedge & \longrightarrow & \\ & & & & X & & \end{array} \right\}.$$

Lemma 3.1.8 ([Lurb, Theorem 4.4.2.8.ii])**.** *If E is an A_∞–ring spectrum, then $\mathcal{D}_E(X)$ can be considered as a cosimplicial object in the ∞–category of* $\mathsf{Spectra}$. $\square$

Definition 3.1.9. The *E–nilpotent completion* of X is the totalization of this cosimplicial spectrum:

$$X^\wedge_E := \operatorname{Tot} \mathcal{D}_E(X).$$

It receives a natural map $X \to X^\wedge_E$, the analogue of the natural map of R–modules $M \to c_* c^* M$ considered in Theorem 3.1.4.

Remark 3.1.10 ([Rav84, Theorem 1.12], [Bou79]). Ravenel proves the following generalization of Lemma 3.1.6. Let E be a connective ring spectrum, let J be the set of primes complementary to those primes p for which E_* is uniquely p–divisible, and let X be a connective spectrum.[6] If each element of E_* has finite order, then $X^\wedge_E = X^\wedge_J$ gives the arithmetic completion of X—which we reinterpret as $\mathbb{S}^\wedge_J \to E$ being of effective descent for connective objects. Otherwise, if E_* has elements of infinite order, then $X^\wedge_E = X_{(J)}$ gives the arithmetic localization—which we reinterpret as saying that $\mathbb{S}_{(J)} \to E$ is of effective descent. Sorting out more encompassing conditions on maps $f\colon R \to S$ of E_∞–rings for which descent holds is a subject of serious study [Lurc, Appendix D].

Finally, we can interrelate these algebraic and topological notions of descent by studying the coskeletal filtration spectral sequence for $\pi_* X^\wedge_E$, which we define to be the *E–Adams spectral sequence* for X. Applying the homotopy groups functor to the cosimplicial ring spectrum $\mathcal{D}_E$ gives a cosimplicial ring $\pi_* \mathcal{D}_E$, which we would like to connect with an algebraic descent object of the sort considered in Remark 3.1.3. In order to make this happen, we need two niceness conditions on E:

Definition 3.1.11. An even-periodic ring spectrum E satisfies **CH**, the **C**ommutativity **H**ypothesis, when the ring $\pi_* E^{\wedge j}$ is commutative for all $j \geq 1$. In this case, we can form the simplicial scheme

$$\mathcal{M}_E = \operatorname{Spec} \pi_0 \mathcal{D}_E,$$

called the *context* of E.

[6] Even for connective ring spectra E, the Bousfield localization $L_E X$ does *not* have to recover an arithmetic localization of X if X is not connective. Take $E = H\mathbb{Z}$ and $X = KU$, which Snaith's theorem presents as $X = KU = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}]$, where $\beta\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ is the Bott element. This gives $H\mathbb{Z}_* KU = H\mathbb{Z}_*(\mathbb{C}\mathrm{P}^\infty[\beta^{-1}]) = (H\mathbb{Z}_* \mathbb{C}\mathrm{P}^\infty)[b_1^{-1}]$. We can identify the pieces in turn: Example 2.1.19 shows $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{Z}} = \widehat{\mathbb{G}}_a$, so the dual Hopf algebra $(\mathcal{O}_{\widehat{\mathbb{G}}_a})^* = H\mathbb{Z}_* \mathbb{C}\mathrm{P}^\infty$ is a divided polynomial algebra on the class b_1. Inverting b_1 then gives $(H\mathbb{Z}_* \mathbb{C}\mathrm{P}^\infty)[b_1^{-1}] = \Gamma[b_1][b_1^{-1}] = \mathbb{Q}[b_1^{\pm}]$, so that, in particular, there is a weak equivalence $H\mathbb{Z} \wedge KU \to H\mathbb{Q} \wedge KU$. The cofiber $KU \to KU \otimes \mathbb{Q} \to KU \otimes \mathbb{Q}/\mathbb{Z}$ is thus a nonzero $H\mathbb{Z}$–acyclic spectrum. You can also work this example without knowing Snaith's theorem: all you really need to know is that $\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}] \to KU$ is a map of ring spectra, so that $\pi_0 \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}]$ can't be a rational group since $\pi_0 KU = \mathbb{Z}$.

Definition 3.1.12. An even-periodic ring spectrum E satisfies **FH**, the **F**latness **H**ypothesis, when the right-unit map $E_0 \to E_0E$ is flat.[7] In this case, the Segal map

$$(E_0E)^{\otimes_{E_0} j} \otimes_{E_0} E_0X \to \pi_0(E^{\wedge(j+1)} \wedge X) = \pi_0\mathcal{D}_E(X)[j]$$

is an isomorphism for all X. In geometric language, this says that $\mathcal{M}_E$ is valued in simplicial sets equivalent to nerves of groupoids and that

$$\mathcal{M}_E(X) := \pi_0\widetilde{\mathcal{D}_E(X)}$$

forms a Cartesian quasicoherent sheaf over $\mathcal{M}_E$. In this sense, we have constructed a factorization

$$\begin{tikzcd} \mathsf{Spectra} \arrow[rr, "E_0(-)"] \arrow[dr, "\mathcal{M}_E(-)"'] & & \mathsf{Modules}_{E_0} \\ & \mathsf{QCoh}(\mathcal{M}_E). \arrow[ur, "(-)[0]"'] & \end{tikzcd}$$

While **CH** and **FH** are enough to guarantee that $\mathcal{M}_E$ and $\mathcal{M}_E(X)$ are well-behaved, they still do not exactly connect us with Remark 3.1.3. The main difference is that the ring of homology cooperations for E

$$E_0E = \pi_0(E \wedge E) = \pi_0\mathcal{D}_E[1]$$

is only distantly related to the tensor product $E_* \otimes_{\pi_*\mathbb{S}} E_*$ (or even $\mathrm{Tor}^{\pi_*\mathbb{S}}_{*,*}(E_*, E_*)$). This is a trade we are eager to make, as the latter groups are typically miserably behaved, whereas E_0E is typically fairly nice. In order to take advantage of this, we enlarge our definition to match:

Definition 3.1.13. Let A and Γ be commutative rings with associated affine schemes $X_0 = \operatorname{Spec} A$, $X_1 = \operatorname{Spec} \Gamma$. A *Hopf algebroid* consists of the pair (A, Γ) together with structure maps

$$\begin{aligned} \eta_L\colon\ & A \to \Gamma, & s\colon\ & X_1 \to X_0, \\ \eta_R\colon\ & A \to \Gamma, & t\colon\ & X_1 \to X_0, \\ \chi\colon\ & \Gamma \to \Gamma, & (-)^{-1}\colon\ & X_1 \to X_1, \\ \Delta\colon\ & \Gamma \to \Gamma \, {}^{\eta_R}\!\underset{A}{\otimes}{}^{\eta_L} \Gamma, & \circ\colon\ & X_1 \, {}^{t}\!\underset{X_0}{\times}{}^{s} X_1 \to X_1, \end{aligned}$$

[7]If E is a commutative ring spectrum, then this is equivalent to asking that the left-unit map is a flat map of E_0–modules.

such that (X_0, X_1) forms a groupoid scheme. An (A, Γ)*–comodule* is an A–module equipped with a Γ–comodule structure, and such a comodule is equivalent to a Cartesian quasicoherent sheaf on the nerve of (X_0, X_1).

Example 3.1.14. A Hopf k–algebra H gives a Hopf algebroid (k, H). The scheme of objects $\operatorname{Spec} k$ in the groupoid scheme is the constant scheme 0.

Lemma 3.1.15. *For E an A_∞–ring spectrum satisfying* **CH** *and* **FH***, the E_2–page of its Adams spectral sequence can be identified as*

$$\begin{aligned} E_2^{*,*} &= \operatorname{Cotor}^{*,*}_{E_*E}(E_*, E_*X) \\ &\cong H^*(\mathcal{M}_E; \mathcal{M}_E(X) \otimes \omega^{\otimes *}) \oplus H^*(\mathcal{M}_E; \mathcal{M}_E(\Sigma X) \otimes \omega^{\otimes *})[1] \Rightarrow \pi_* X^\wedge_E. \end{aligned}$$

Proof sketch. The homological algebra of Hopf algebras from Lecture 1.4 can be lifted almost verbatim, allowing us to define resolutions suitable for computing derived functors [Rav86, Definition A1.2.3]. This includes the cobar resolution [Rav86, Definition A1.2.11], which shows that the associated graded for the coskeletal filtration of $\mathcal{D}_E(X)$ is a complex computing the derived functors claimed in the Lemma statement. □

Remark 3.1.16. The sphere spectrum fails to satisfy **CH**, so the above results do not apply to it, but the $\mathbb{S}$–Adams spectral sequence is particularly degenerate: it consists of $\pi_* X$, concentrated on the 0–line. For any other ring spectrum E, the unit map $\mathbb{S} \to E$ induces a map of Adams spectral sequences whose image on the 0–line are those maps of comodules induced by applying E–homology to a homotopy element of X—i.e., the Hurewicz image of E.

Remark 3.1.17. In Lemma 3.1.6, we discussed translating from the algebra descent picture to a homotopical one, and a crucial point was how thorough we had to be: we transferred not just to the derived category $D(\operatorname{Spec} R)$ but we also replaced the base ring R with its homotopical incarnation HR. In Definition 3.1.13, we have not been as thorough as possible: both X_0 and X_1 are schemes and hence satisfy a sheaf condition individually, but the functor (X_0, X_1), thought of as valued in homotopy 1–types, does not necessarily satisfy a homotopy sheaf condition. Enforcing this descent condition results in the *associated stack* [Hop, Definition 8.13], denoted

$$\operatorname{Spec} A // \operatorname{Spec} \Gamma = X_0 // X_1.$$

Remarkably, this does not change the category of Cartesian quasicoherent sheaves—it is still equivalent to the category of (A, Γ)–comodules [Hop, Proposition 11.6].

However, several different Hopf algebroids (such as those with maps between them inducing natural equivalences of groupoid schemes, as studied by Hovey [Hov02, Theorem D], but also some with *no* such zig-zag) can give the same associated stack, resulting in surprising equivalences of comodule categories.[8] For the most part, it will not be especially relevant to us whether we are considering the groupoid scheme or its associated stack, so we will not draw much of a distinction. For the most part, the associated stack is theoretically preferable, but the groupoid scheme is easier to think about.[9]

Example 3.1.18. Most of the homology theories we will discuss have these **CH** and **FH** properties. For an easy example, $H\mathbb{F}_2P$ certainly has this property: there is only one possible algebraic map $\mathbb{F}_2 \to \mathcal{A}_*$, so **FH** is necessarily satisfied. This grants us access to a description of the context for $H\mathbb{F}_2$:

$$\mathcal{M}_{H\mathbb{F}_2P} = \operatorname{Spec} \mathbb{F}_2 // \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

Example 3.1.19. The context for MUP is considerably more complicated, but Quillen's theorem can be equivalently stated as giving a description of it. Quillen's theorem on its face gives an equivalence $\operatorname{Spec} MUP_0 \cong \mathcal{M}_{\mathbf{fgl}}$, but in Lemma 2.6.8 we also gave a description of $\operatorname{Spec} MUP_0 MUP$: it is the moduli of pairs of formal group laws equipped with an invertible power series intertwining them. Altogether, this presents $\mathcal{M}_{MUP}$ as the moduli of formal groups:

$$\mathcal{M}_{MUP} \simeq \mathcal{M}_{\mathbf{fg}} := \mathcal{M}_{\mathbf{fgl}} // \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}},$$

where $\mathcal{M}_{\mathbf{ps}} = \underline{\operatorname{End}}(\widehat{\mathbb{A}}^1)$ is the moduli of self-maps of the affine line (i.e., of power series) and $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}$ is the multiplicative subgroup of invertible such maps. We include a picture of the p–localized Adams E_2–page in Figure 3.1 and Figure 3.2. In view of Remark 3.1.17, there is an important subtlety about the stack $\mathcal{M}_{\mathbf{fg}}$: an R–point is a functor on affines over $\operatorname{Spec} R$ which is locally isomorphic to a formal group, but whose local isomorphism *may not patch* to give a global isomorphism. This does not agree, a priori, with the definition of formal group given in Definition 2.1.15, where the isomorphism witnessing a functor as a formal variety was expected to be global. We will address this further in Lemma 3.2.7 below.

[8] We will employ one of these surprising equivalences in Remark 3.3.17.

[9] Constructing the correct derived category of comodules also has subtle associated homotopical issues. Hovey gives a good reference for this in the case of a stack associated to a Hopf algebroid [Hov04].

Example 3.1.20. The context for MOP, by contrast, is reasonably simple. Corollary 1.5.7 shows that the scheme $\operatorname{Spec} MOP_0$ classifies formal group laws over $\mathbb{F}_2$ which admit logarithms, so that $\mathcal{M}_{MOP}$ consists of the groupoid of formal group laws with logarithms and isomorphisms between them. This admits a natural deformation-retraction to the moduli consisting just of $\widehat{\mathbb{G}}_a$ and its automorphisms, expressing the redundancy in $MOP_0(X)$ encoded in the splitting of Lemma 1.5.8.

Remark 3.1.21. The algebraic moduli $\mathcal{M}_{MU} = (\operatorname{Spec} MU_*, \operatorname{Spec} MU_*MU)$ and the topological moduli $(MU, MU \wedge MU)$ are quite different. An orientation $MU \to E$ selects a coordinate on the formal group $\mathbb{C}\mathrm{P}^\infty_E$, but $\mathbb{C}\mathrm{P}^\infty_E$ itself exists independently of the orientation. Hence, while $\mathcal{M}_{MU}(E_*)$ can have many connected components corresponding to *distinct formal groups* on the coefficient ring E_*, the groupoid $\mathsf{RingSpectra}(\mathcal{D}_{MU}, E)$ has only one connected component corresponding to the formal group $\mathbb{C}\mathrm{P}^\infty_E$ intrinsic to E.[10,11]

Remark 3.1.22. If E is a complex-oriented ring spectrum, then the simplicial sheaf $\mathcal{M}_{MU}(E)$ has an extra degeneracy, which causes the MU–based Adams spectral sequence for E to degenerate. In this sense, the "stackiness" of $\mathcal{M}_{MU}(E)$ is exactly a measure of the failure of E to be orientable.

Cite me: pg 5 of From Spectra To Stacks by Hopkins in the TMF volume.

Remark 3.1.23. It is also possible to construct an Adams spectral sequence by iteratively smashing with the fiber sequence $\overline{E} \to \mathbb{S} \to E$ to form the tower

$$\begin{tikzcd}
\mathbb{S} \wedge X \arrow[d] & \overline{E} \wedge X \arrow[l] \arrow[d] & \overline{E}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E \wedge X & E \wedge \overline{E} \wedge X & E \wedge \overline{E}^{\wedge 2} \wedge X & \cdots .
\end{tikzcd}$$

This presentation makes the connection to descent much more opaque, but it does not require E to be an A_∞–ring spectrum.

Remark 3.1.24. With contexts in mind, the pro-spectrum $DX_+ = \{F(X_\alpha, \mathbb{S})\}_\alpha$ gains our attention as a kind of universal example in the following sense. Because each X_α is a compact object, base-change along the unit map $\eta \colon \mathbb{S} \to E$ can be computed by the following formula:

$$\eta^* DX_+ = E \wedge \{F(X_\alpha, \mathbb{S})\}_\alpha = \{E \wedge F(X_\alpha, \mathbb{S})\}_\alpha = \{F(X_\alpha, E)\}_\alpha.$$

[10] The reader ought to compare this with the situation in explicit local class field theory, where a local number field has a preferred formal group attached to it.

[11] The precocious student might ask what functor MU represents as an E_∞ ring spectrum. To date, this functor has not been algebraically recognized.

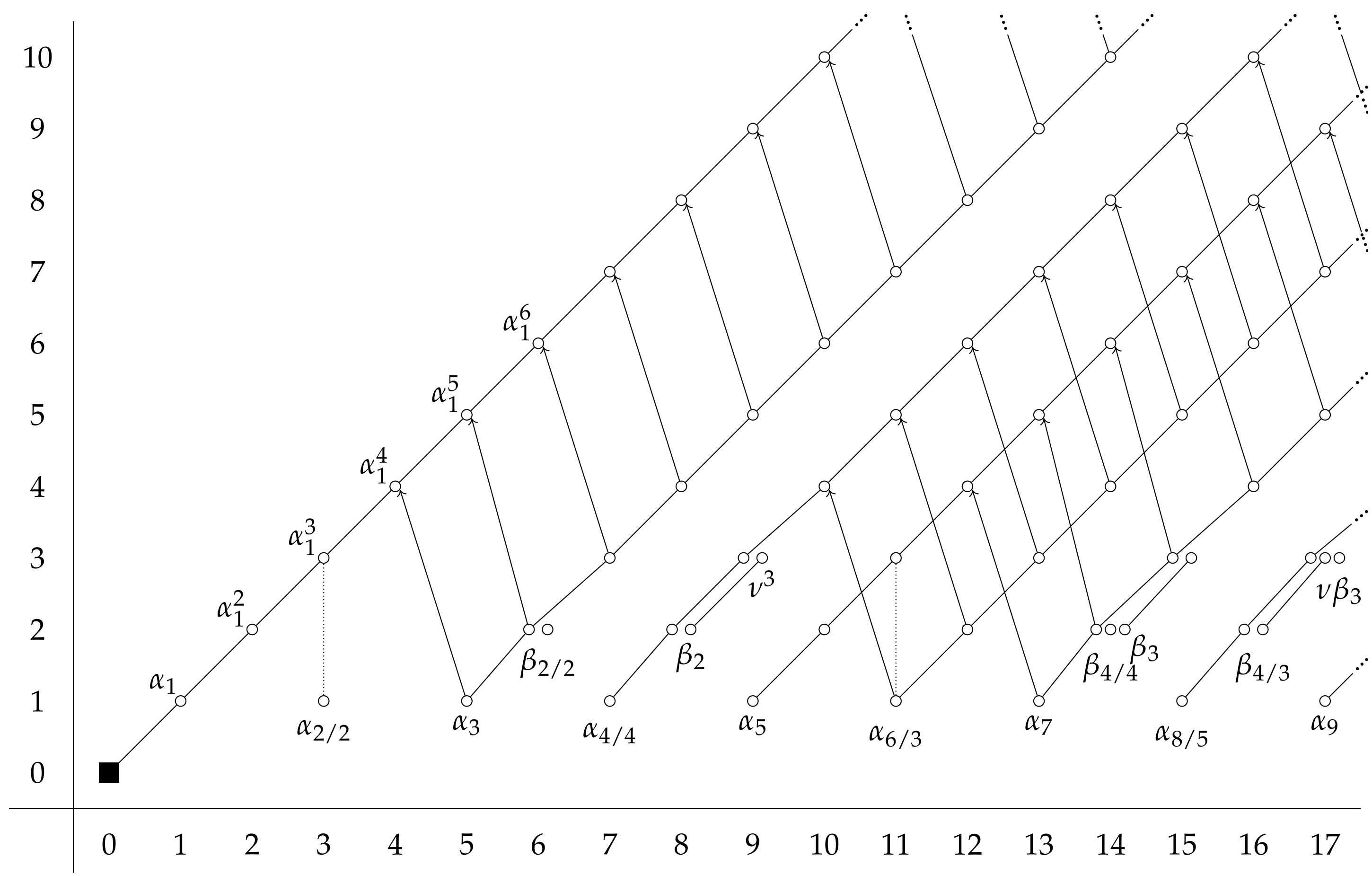


Figure 3.1: A small piece of the $MU_{(2)}$–Adams spectral sequence for the sphere, beginning at the second page [Rav78, pg. 429]. North-east lines denote multiplication by $\eta = \alpha_1$, north-west lines denote d_3–differentials, and vertical dotted lines indicate additive extensions. Elements are labeled according to the conventions of Remark 3.6.22, and in particular $\alpha_{i/j}$ is 2^j–torsion.

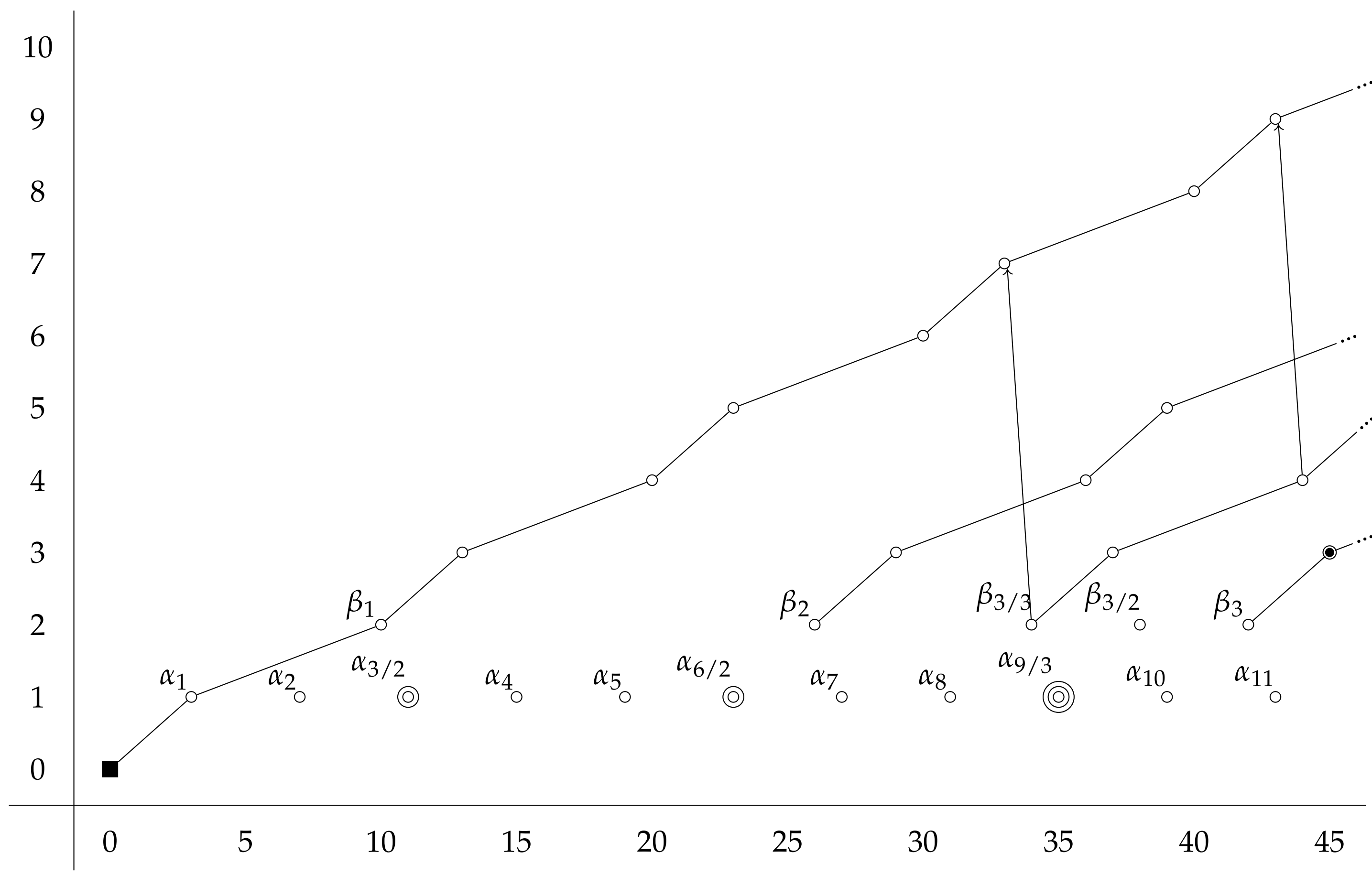


Figure 3.2: A small piece of the $MU_{(3)}$–Adams spectral sequence for the sphere, beginning at the second page [Rav86, Figure 1.2.19]. North-east lines denote multiplication by α_1 or by β_1/α_1, and north-west lines denote d_5–differentials. Elements are labeled according to the conventions of Remark 3.6.22, and in particular $\alpha_{i/j}$ is 3^j–torsion.

Applying the functor $\mathrm{Spf} \circ \pi_0$ to this pro-system yields the formal scheme X_E considered in Definition 2.1.13. One of the themes of this book will be to think of the objects X and DX_+ as spectral incarnations of some formal group construction, which base-change along η to specialize to classical algebro-geometric constructions applied to the formal group associated to E.

Cite me: Pridham's article *Presenting higher stacks as simplicial schemes* seems like a good reference? Maybe some Toen things are appropriate? I don't really know where this simplicial scheme stuff is written down..

Say what open, closed, flat maps of simplicial schemes are?

3.2 The structure of $\mathcal{M}_{\mathbf{fg}}$ I: The affine cover

In Definition 3.1.13 we gave a factorization

$$\begin{tikzcd}
\mathsf{Spectra} \arrow[rr, "MUP_0(-)"] \arrow[dr, "\mathcal{M}_{MUP}(-)"'] & & \mathsf{Modules}_{MUP_0} \\
& \mathsf{QCoh}(\mathcal{M}_{MUP}), \arrow[ur, "(-)[0]"'] &
\end{tikzcd}$$

and in Example 3.1.19 we established an equivalence

$$\varphi \colon \mathcal{M}_{MUP} \xrightarrow{\simeq} \mathcal{M}_{\mathbf{fg}}.$$

Our program, as outlined in the introduction, is to analyze this functor $\mathcal{M}_{MUP}(-)$ by postcomposing it with φ^* and studying the resulting sheaf over $\mathcal{M}_{\mathbf{fg}}$. In order to perform such an analysis, we will want a firm grip on the geometry of the stack $\mathcal{M}_{\mathbf{fg}}$, and in this Lecture we begin by studying the scheme $\mathcal{M}_{\mathbf{fgl}}$ as well as the natural covering map

$$\mathcal{M}_{\mathbf{fgl}} \to \mathcal{M}_{\mathbf{fg}}.$$

Additionally, Example 3.1.19 was a consequence of Corollary 2.6.11, which relied on the unproven result stated as Theorem 2.6.10, which we will now prove in this section as Theorem 3.2.2.

Definition 3.2.1. There is an affine scheme $\mathcal{M}_{\mathbf{fgl}}$ classifying formal group laws. Begin with the scheme classifying *all* bivariate power series:

$$\operatorname{Spec} \mathbb{Z}[a_{ij} \mid i, j \geq 0] \leftrightarrow \{\text{bivariate power series}\},$$
$$f \in \operatorname{Spec} \mathbb{Z}[a_{ij} \mid i, j \geq 0](R) \leftrightarrow \sum_{i,j \geq 0} f(a_{ij}) x^i y^j.$$

Then, $\mathcal{M}_{\mathbf{fgl}}$ is the closed subscheme selected by the formal group law axioms in Definition 2.1.18.

This presentation of $\mathcal{M}_{\mathbf{fgl}}$ as a subscheme appears to be extremely complicated in that its ideal is generated by many hard-to-describe elements, but $\mathcal{M}_{\mathbf{fgl}}$ itself is actually not complicated at all. We will prove the following:

Theorem 3.2.2 ([Laz55, Théorème II])**.** *There is a noncanonical isomorphism*

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \cong \mathbb{Z}[b_n \mid 1 \le n < \infty] =: L. \qquad \square \quad \square$$

Proof. Let $L = \mathbb{Z}[b_0, b_1, b_2, \ldots]/(b_0 - 1)$ be the universal ring supporting an exponential and a logarithm:[12]

$$\exp(x) := \sum_{j=0}^{\infty} b_j x^{j+1}, \qquad \log(x) := \sum_{j=0}^{\infty} m_j x^{j+1}.$$

They induce a formal group law on L by the conjugation formula

$$x +_! y = \exp(\log(x) + \log(y)),$$

which is in turn classified by a map $u\colon \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to L$.[13] Modulo decomposables, we compute

$$\begin{aligned} x &= \exp(\log(x)) \\ &= x + \sum_{n=1}^{\infty} m_n x^{n+1} + \sum_{n=1}^{\infty} b_n \left(x + \sum_{j=1}^{\infty} m_j x^{j+1} \right)^{n+1} \\ &\equiv x + \sum_{n=1}^{\infty} m_n x^{n+1} + \sum_{n=1}^{\infty} b_n x^{n+1} \pmod{\text{decomposables}}, \end{aligned}$$

hence $b_n \equiv -m_n \pmod{\text{decomposables}}$. Using this, we then compute

$$\begin{aligned} x +_! y &= \exp(\log(x) + \log(y)) \\ &= \left((x+y) + \sum_{n=1}^{\infty} m_n (x^{n+1} + y^{n+1}) \right) + \sum_{n=1}^{\infty} b_n \left((x+y) + \sum_{j=1}^{\infty} m_j (x^{j+1} + y^{j+1}) \right)^{n+1} \\ &\equiv x + y + \sum_{n=1}^{\infty} -b_n (x^{n+1} + y^{n+1}) + \sum_{n=1}^{\infty} b_n (x+y)^{n+1} \pmod{\text{decomposables}} \\ &= x + y + \sum_{n=1}^{\infty} b_n ((x+y)^{n+1} - x^{n+1} - y^{n+1}), \end{aligned}$$

[12] In the context of complex-oriented cohomology theories, this is called the *Miščenko logarithm*, given by the formula $\log_\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi[\mathbb{C}\mathrm{P}^n]}{n+1} x^{n+1}$.

[13] This is *not* the universal formal group law. We will soon see that some formal group laws do not admit logarithms.

hence
$$u(a_{i(n-i)}) \equiv \binom{n}{i} b_{n-1} \quad \text{(mod decomposables)}.$$

It follows that the map Qu on degree $2n$ has image the subgroup T_{2n} generated by $d_{n+1}b_n$, where $d_{n+1} = \gcd\left(\binom{n+1}{k} \middle| 0 < k < n+1\right)$. Lemma 3.2.3 below provides a canonical splitting of Qu, and we couple it to the freeness of L to *choose* an algebra splitting
$$L \xrightarrow{v} \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \xrightarrow{u} L.$$

The map uv is injective, so v is injective. Furthermore, Qv is designed to be surjective, so v itself is surjective and hence an isomorphism. □

Recall that we have yet to prove the following Lemma:

Lemma 3.2.3. *There is a canonical splitting $T_{2n} \to (Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n}$.*

Definition 3.2.4. In order to prove the missing Lemma 3.2.3, it will be useful to study the series $+_\varphi$ "up to degree n", i.e., modulo $(x, y)^{n+1}$. Such a truncated series satisfying the analogues of the formal group law axioms is called a *formal n–bud*.[14] We will additionally be moved to study the difference between a formal n–bud and a formal $(n+1)$–bud extending it. The simplest case of this is when the formal n–bud is just the additive law $x +_\varphi y = x + y$, in which case any extension to an $(n+1)$–bud has the form $x + y + f(x, y)$ for $f(x, y)$ a homogeneous polynomial of degree n. Symmetry of the group law requires $f(x, y)$ to be symmetric, and associativity of the group law requires $f(x, y)$ to satisfy the equation
$$f(x, y) - f(t + x, y) + f(t, x + y) - f(t, x) = 0.$$

Such a polynomial is called a *symmetric* 2*–cocycle* (of degree n).[15]

Reduction of Lemma 3.2.3 to Lemma 3.2.5. We now show that the following conditions are equivalent:

[14] A formal n–bud determines a "multiplication" $(\widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1)^{(n)} \to \widehat{\mathbb{A}}^{1,(n)}$. Note that this does *not* belong to a group object, since $(\widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1)^{(n)} \not\cong \widehat{\mathbb{A}}^{1,(n)} \times \widehat{\mathbb{A}}^{1,(n)}$. This is the observation that the ideals $(x, y)^{n+1}$ and (x^{n+1}, y^{n+1}) are distinct.

[15] We will justify the "2–cocycle" terminology in the course of the proof of Lemma 3.2.5.

1. (Lemma 3.2.5) Symmetric 2–cocycles that are homogeneous polynomials of degree n are spanned by

$$c_n = \frac{1}{d_n} \cdot ((x+y)^n - x^n - y^n),$$

where $d_n = \gcd\left(\binom{n}{k} \middle| 0 < k < n\right)$.

2. For F is an n–bud, the set of $(n+1)$–buds extending F form a torsor under addition for $R \otimes c_n$.

3. Any homomorphism $(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n} \to A$ factors through $(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n} \to T_{2n}$.

4. (Lemma 3.2.3) There is a canonical splitting $T_{2n} \to (Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n}$.

To verify that Claims 1 and 2 are equivalent, suppose that $x +_\varphi y$ is some $(n+1)$–bud and that $x +'_\varphi y$ is some $(n+1)$–bud such that

$$(x +'_\varphi y) = (x +_\varphi y) + f(x,y)$$

where $f(x,y)$ is homogeneous of degree $(n+1)$. Symmetry of $x +'_\varphi y$ enforces symmetry of f, and from associativity we calculate

$$\begin{aligned} x +'_\varphi (y +'_\varphi z) &= x +'_\varphi (y +_\varphi z + f(y,z)) \\ &= x +_\varphi (y +_\varphi z + f(y,z)) + f(x, y +_\varphi z + f(y,z)) \\ &\equiv x +_\varphi (y +_\varphi z) + f(y,z) + f(x, y+z) \pmod{(x,y)^{n+2}}, \\ (x +'_\varphi y) +'_\varphi z &= (x +_\varphi y + f(x,y)) +'_\varphi z \\ &= (x +_\varphi y + f(x,y)) +_\varphi z + f(x +_\varphi y + f(x,y), z) \\ &\equiv (x +_\varphi y) +_\varphi z + f(x,y) + f(x+y, z) \pmod{(x,y)^{n+2}}, \end{aligned}$$

resulting in the 2–cocycle condition on f. Conversely, given such a 2–cocycle $f(x,y)$, the formal $(n+1)$–bud $+'_\varphi$ formed by translating $+_\varphi$ by f is again a formal $(n+1)$–bud extending the same formal n–bud.

To see that Claim 2 is equivalent to Claim 3, note that a group map

$$(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n} \to A$$

is equivalent data to a ring map

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to Z \oplus A$$

with the prescribed behavior on $(Q\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}})_{2n}$ and which sends all other indecomposables to 0.. This shows that such a homomorphism of groups determines an extension of the n–bud $\widehat{\mathbb{G}}_a$ to an $(n+1)$–bud, which takes the form of a 2–cocycle with coefficients in A, and hence factors through T_{2n}.

Finally, Claim 4 is the universal case of Claim 3. □

We will now verify Claim 1 computationally, completing the proof of Lemma 3.2.3 (and hence Theorem 3.2.2).

Lemma 3.2.5 (Symmetric 2–cocycle lemma [Laz55, Lemme 3], cf. [Hop, Theorem 3.1])**.** *Symmetric 2–cocycles that are homogeneous polynomials of degree n are spanned by*

$$c_n = \frac{1}{d_n} \cdot ((x+y)^n - x^n - y^n),$$

where $d_n = \gcd\left(\binom{n}{k} \middle| 0 < k < n\right)$.

Proof. We begin with a reduction of the sorts of rings over which we must consider the possible symmetric 2–cocycles. First, notice that only the additive group structure of the ring matters: the symmetric 2–cocycle condition does not involve any ring multiplication. Second, it suffices to show the Lemma over a finitely generated abelian group, as a particular polynomial has finitely many terms and hence involves finitely many coefficients. Noticing that the Lemma is true for $A \oplus B$ if and only if it's true for A and for B, we couple these facts to the structure theorem for finitely generated abelian groups to reduce to the cases $\mathbb{Z}$ and $\mathbb{Z}/p^r$. From here, we can reduce to the prime fields: if $A \leq B$ is a subgroup and the Lemma is true for B, it's true for A, so we will be able to deduce the case of $\mathbb{Z}$ from the case of $\mathbb{Q}$. Lastly, we can also reduce from $\mathbb{Z}/p^r$ to $\mathbb{Z}/p$ using an inductive Bockstein-style argument over the extensions

$$(p^{r-1})/(p^r) \to \mathbb{Z}/p^r \to \mathbb{Z}/p^{r-1}$$

and noticing that $(p^{r-1})/(p^r) \cong \mathbb{Z}/p$ as abelian groups. Hence, we can now freely assume that our ground object is a prime field.

We now ground ourselves by fitting symmetric 2–cocycles into a more general homological framework, hoping that we can use such a machine to power a computation. For a formal group scheme $\widehat{\mathbb{G}}$, we can form a simplicial scheme $B\widehat{\mathbb{G}}$

in the usual way:

$$B\widehat{\mathbb{G}} := \left\{ \begin{array}{ccccccc} & & & & * & \longleftarrow & \\ & & * & \longleftarrow & \times & \longrightarrow & \\ * & \longleftarrow & \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \\ \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \times & \longrightarrow & \cdots \\ * & \longleftarrow & \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \\ & & * & \longleftarrow & \times & \longrightarrow & \\ & & & & * & \longleftarrow & \end{array} \right\}.$$

By applying the functor $\underline{\mathsf{FormalSchemes}}(-, \widehat{\mathbb{G}}_a)(k)$, we get a cosimplicial abelian group stemming from the group scheme structure on $\widehat{\mathbb{G}}_a$, and this gives a cochain complex of which we can take the cohomology. In the case $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$, the 2–cocycles in this cochain complex are *precisely* what we've been calling 2–cocycles[16], so we are interested in computing H^2. First, we can quickly compute B^2, since C^1 is so small:

$$d^1(x^k) = d_k c_k.$$

Secondly, one may think of this complex as a resolution computing various[17] derived functors

$$\mathrm{Cotor}_{\mathcal{O}_{\widehat{\mathbb{G}}}}(k, k) \cong \mathrm{Ext}_{\mathcal{O}_{\widehat{\mathbb{G}}}}(k, k) \cong \mathrm{Tor}_{\mathcal{O}^*_{\widehat{\mathbb{G}}}}(k, k).$$

We are now going to compute these last groups using a more efficient complex.

$\mathbb{Q}$: There is a free $\mathbb{Q}[t]$–module resolution

$$\begin{array}{ccccccc} & & \mathbb{Q} & & & & \\ & & \uparrow & & & & \\ 0 & \longleftarrow & \mathbb{Q}[t] & \xleftarrow[\cdot t]{} & \mathbb{Q}[t] & \longleftarrow & 0, \end{array}$$

to which we apply $(-) \otimes_{\mathbb{Q}[t]} \mathbb{Q}$ to calculate

$$H^* \, \underline{\mathsf{FormalSchemes}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{when } * = 0, \\ \mathbb{Q} & \text{when } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

[16]They aren't obligated to be symmetric or of homogeneous degree, though.
[17]Refer back to Corollary 1.4.17.

This means that every 2–cocycle is a coboundary, symmetric or not.

$\mathbb{F}_p$: Now we are computing Ext over a free commutative $\mathbb{F}_p$–algebra on one generator with divided powers. Such an algebra splits as a tensor of truncated polynomial algebras, and again computing a minimal free resolution results in the calculation

$$H^* \underline{\mathsf{FormalSchemes}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{F}_p) = \begin{cases} \frac{\mathbb{F}_p[\alpha_k \mid k \geq 0]}{\alpha_k^2 = 0} \otimes \mathbb{F}_p[\beta_k \mid k \geq 0] & \text{when } p > 2, \\ \mathbb{F}_2[\alpha_k \mid k \geq 0] & \text{when } p = 2, \end{cases}$$

with $\alpha_k \in H^1$ and $\beta_k \in H^2$. Now that we know what to look for, we can find representatives of each of these classes:

- The class α_k can be represented by x^{p^k}, as this is a minimally divisible monomial of degree p^k satisfying the 1–cocycle condition

$$x^{p^k} - (x+y)^{p^k} + y^{p^k} = 0.$$

- The 2–cohomology is concentrated in degrees of the form p^k and $p^j + p^k$, corresponding to β_k and $\alpha_j \alpha_k$. Since c_{p^k} is a 2–cocycle of the correct degree and not a 2–coboundary (cf. $d^1(x^{p^k}) = d_{p^k} c_{p^k}$, and $p \mid d_{p^k}$), we can use it as a representative for β_k. (Additionally, the asymmetric class $\alpha_k \alpha_j$ is represented by $x^{p^k} y^{p^j}$.)
- Similarly, in the case $p = 2$ the exceptional class α_{k-1}^2 is represented by $c_{2^k}(x, y)$, as this is a 2–cocycle in the correct degree which is not a 2–coboundary.

Given how few 2–coboundaries and 2–cohomology classes there are, we conclude that $c_n(x, y)$ and $x^{p^a} y^{p^b}$ give a basis for *all* of the 2–cocycles. Of these it is easy to select the symmetric ones, which agrees with our expected conclusion. $\square$

The most important consequence of Theorem 3.2.2 is *smoothness*:

Corollary 3.2.6. *Given a formal group law F over a ring R and a surjective ring map $f\colon S \to R$, there exists a formal group law $\widetilde{F}$ over S with*

$$F = f^* \widetilde{F}.$$

Proof. Identify F with the classifying map $\operatorname{Spec} R \to \mathcal{M}_{\mathbf{fgl}}$. Employ an isomorphism

$$\varphi\colon \mathcal{M}_{\mathbf{fgl}} \to \operatorname{Spec} L$$

afforded by Theorem 3.2.2, so that $\varphi \circ F$ is selected by a sequence of elements $r_n = \varphi^* F^*(t_n) \in R$. Each of these admit preimages s_n through f, and we determine a map

$$\widetilde{\varphi \circ F}\colon \operatorname{Spec} S \to \operatorname{Spec} L$$

by the formula $\widetilde{\varphi \circ F}^*(t_j) = s_j$ and freeness of L. Since φ is an isomorphism, this determines a map $\widetilde{F} = \varphi^{-1} \circ \widetilde{\varphi \circ F}$ factoring F. $\square$

In order to employ Corollary 3.2.6 effectively, we need to know when a map $\operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$ classifying a formal group can be lifted to a triangle

This isn't super well stated, but it's at least here to be smoothed out later.

$$\begin{tikzcd}
 & \mathcal{M}_{\mathbf{fgl}} \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & \mathcal{M}_{\mathbf{fg}},
\end{tikzcd}$$

so that a surjective map of rings $\operatorname{Spec} R \to \operatorname{Spec} S$ can then be completed to a second diagram

$$\begin{tikzcd}
\operatorname{Spec} S \arrow[r, dotted] \arrow[dr, dotted] & \mathcal{M}_{\mathbf{fgl}} \arrow[d] \\
\operatorname{Spec} R \arrow[u] \arrow[r] \arrow[ur] & \mathcal{M}_{\mathbf{fg}}.
\end{tikzcd}$$

Lemma 3.2.7 ([Lura, Proposition 11.7])**.** *A map $\widehat{\mathbb{G}}\colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$ lifts to $\mathcal{M}_{\mathbf{fgl}}$ exactly when the Lie algebra $T_0\widehat{\mathbb{G}}$ of $\widehat{\mathbb{G}}$ is isomorphic to R.*

Proof. Certainly if $\widehat{\mathbb{G}}$ admits a global coordinate, then $T_0\widehat{\mathbb{G}} \cong R$. Conversely, the formal group $\widehat{\mathbb{G}}$ is certainly locally isomorphic to $\widehat{\mathbb{A}}^1$ by a covering $i_\alpha\colon X_\alpha \to \operatorname{Spec} R$ and isomorphisms φ_α—but, *a priori*, these isomorphisms may not glue, precisely corresponding to the nontriviality of the Čech 1–cocycle

$$[\varphi_\alpha] \in \check{H}^1(\operatorname{Spec} R; \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}).$$

The group scheme $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}$ is populated by T–points of the form

$$\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}(T) = \left\{ t_0 x + t_1 x^2 + t_2 x^3 + \cdots \mid t_j \in T, t_0 \in T^\times \right\},$$

and it is admits a filtration by the closed subschemes

$$\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}, \geq N}(T) = \left\{ 1 \cdot x + t_N x^{N+1} + t_{N+1} x^{N+2} + \cdots \mid t_j \in T \right\}.$$

The associated graded of this filtration is $\mathbb{G}_m \times \mathbb{G}_a^{\times \infty}$, and hence the filtration spectral sequence shows

$$\check{H}^1(\operatorname{Spec} R; \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \xrightarrow{\simeq} \check{H}^1(\operatorname{Spec} R; \mathbb{G}_m),$$

as $\check{H}^1(\operatorname{Spec} R; \mathbb{G}_a) = 0$ for all affine schemes. Finally, given a choice[18] of trivialization $T_0 \widehat{\mathbb{G}} \cong R$, this induces compatible trivializations of $T_0 i_\alpha^* \widehat{\mathbb{G}}$, which we can use to rescale the isomorphisms φ_α so that their image in $\check{H}^1(\operatorname{Spec} R; \mathbb{G}_m)$ vanishes, and hence $[\varphi_\alpha]$ is induced from a class in

$$\check{H}^1(\operatorname{Spec} R; \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}, \geq 1}).$$

This obstruction group vanishes. □

Remark 3.2.8. The subgroup scheme $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}, \geq 1}$ is often referred to in the literature as the group of *strict isomorphisms*. There is an associated moduli of formal groups identified only up to strict isomorphism, which sits in a fiber sequence

$$\mathbb{G}_m \to \mathcal{M}_{\mathbf{fgl}} // \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}, \geq 1} \to \mathcal{M}_{\mathbf{fg}}.$$

These appeared earlier in this Lecture as well: in the proof of Theorem 3.2.2, we constructed over L the universal formal group law equipped with a *strict* exponential map. The moduli of formal group laws modulo strict isomorphisms appears as the context associated to the graded version, rather than even-periodic version, of the story told so far—i.e., as a sort of non-periodic context $\mathcal{M}_{MU}$.

[18]Incidentally, a choice of trivialization of $T_0 \widehat{\mathbb{G}}$ exactly resolves the indeterminacy of $\log'(0)$ in Theorem 2.1.22.

3.3 The structure of $\mathcal{M}_{\mathbf{fg}}$ II: Large scales

We now turn to understanding the geometry of the quotient stack $\mathcal{M}_{\mathbf{fg}}$ itself, armed with two important tools: Theorem 2.1.22 and Corollary 3.2.6. We begin with a rephrasing of the former:

Theorem 3.3.1 (cf. Theorem 2.1.22)**.** *Let k be any field of characteristic* 0*. Then $\widehat{\mathbb{G}}_a$ describes a unique map*

$$\operatorname{Spec} k \xrightarrow{\simeq} \mathcal{M}_{\mathbf{fg}}. \qquad \square$$

One of our overarching tasks from the introduction to this Case Study is to enhance this to a classification of *all* of the geometric points of $\mathcal{M}_{\mathbf{fg}}$, including those where k is a field of positive characteristic p:

$$\widehat{\mathbb{G}}\colon\ \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}.$$

We proved this Theorem in the characteristic 0 case by solving a certain differential equation, which necessitated integrating a power series, and integration is what we expect to fail in characteristic p. The following definition tracks *where* it fails:

Definition 3.3.2. Let $+_\varphi$ be a formal group law over a $\mathbb{Z}_{(p)}$–algebra. Let n be the largest degree such that there exists a formal power series ℓ with

$$\ell(x +_\varphi y) = \ell(x) + \ell(y) \pmod{(x, y)^n},$$

i.e., ℓ is a logarithm for the $(n-1)$–bud determined by $+_\varphi$. The *p–height of* $+_\varphi$ is defined to be $\log_p(n)$.

This turns out to be a crucial invariant of a formal group law, admitting many other interesting presentations. In this Lecture, investigation of this definition will lead us to a classification of the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, another of our overarching tasks. As a first step, we would like to show that this value is well-behaved in various senses, including the following:

Lemma 3.3.3 (cf. [Lura, Proposition 13.6])**.** *Over a field of positive characteristic p, the p–height of a formal group law is always an integer (or ∞). (That is, the radius of convergence of the logarithmic differential equation is either ∞ or p^d for some nonnegative natural d.)*

We will have to develop some machinery to get there. First, we note that this definition really depends on the formal group rather than the formal group law.

Lemma 3.3.4. *The height of a formal group law is an isomorphism invariant, i.e., it descends to give a function*

$$\mathrm{ht}\colon \pi_0\mathcal{M}_{\mathbf{fg}}(T) \to \mathbb{N} \cup \{\infty\}$$

for any test $\mathbb{Z}_{(p)}$–algebra T.

Proof. The series ℓ is a partial logarithm for the formal group law φ, i.e., an isomorphism between the formal group defined by φ and the additive group. Since isomorphisms compose, this statement follows. □

With this in mind, we look for a more standard form for formal group laws, where Lemma 3.3.3 will hopefully be obvious. The most blindly optimistic standard form is as follows:

Definition 3.3.5 (cf. [Haz12, Proposition 15.2.4])**.** Suppose that a formal group law $+_\varphi$ does have a logarithm. We say that its logarithm is *p–typical* when it takes the form

$$\log_\varphi(x) = \sum_{j=0}^{\infty} \ell_j x^{p^j}.$$

Lemma 3.3.6 ([Haz12, Theorem 15.2.9])**.** *Every formal group law $+_\varphi$ over a $\mathbb{Z}_{(p)}$–algebra with a logarithm $\log_\varphi$ is naturally isomorphic to one whose logarithm is p–typical, called the p–typification of $+_\varphi$.*

Proof. Let $\widehat{\mathbb{G}}$ be the formal group associated to $+_\varphi$, and denote its inherited parameter by

$$g_0\colon \widehat{\mathbb{A}}^1 \xrightarrow{\cong} \widehat{\mathbb{G}},$$

so that the composite

$$\widehat{\mathbb{A}}^1 \xrightarrow{g_0} \widehat{\mathbb{G}} \xrightarrow{\log} \widehat{\mathbb{G}}_a \xrightarrow{x} \widehat{\mathbb{A}}^1$$

expresses $\log_\varphi = \log \circ g_0$ as the power series

$$\log_\varphi(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Our goal is to perturb this coordinate to a new coordinate g_∞ which couples with the logarithm in the same way to give a series expansion of the form

$$\log(g_\infty(x)) = \sum_{n=0}^{\infty} a_{p^n} x^{p^n}.$$

To do this, we introduce four operators on functions[19] $\widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$:

- Given $r \in R$, we can define a *homothety* by rescaling the coordinate by r:
$$\log(\theta_r g_0) = \log(g_0(rx)) = \sum_{n=1}^{\infty}(a_n r^n)x^n.$$
- For $\ell \in \mathbb{Z}$, we can define a shift operator (or *Verschiebung*) by
$$\log(V_\ell g_0(x)) = \log(g(x^\ell)) = \sum_{n=1}^{\infty} a_n x^{n\ell}.$$
- Given an $\ell \in \mathbb{Z}_{(p)}$, we define the *$\ell$–series* by[20]
$$\log([\ell](g_0(x))) = \ell \log(g_0(x)) = \sum_{n=1}^{\infty} \ell a_n x^n.$$
- For $\ell \in \mathbb{Z}$, we can define a *Frobenius operator*[21] by
$$\log(F_\ell g_0(x)) = \log\left(\sum_{j=1}^{\ell}{}_{\widehat{\mathbb{G}}}\, g_0(\zeta_\ell^j x^{1/\ell})\right),$$
where ζ_ℓ is a primitive ℓ^{th} root of unity. Because this formula is Galois–invariant in choice of primitive root, it actually expands to a series which lies over the ground ring (without requiring an extension by ζ_ℓ). But, by pulling the logarithm through and noting
$$\sum_{j=1}^{\ell} \zeta_\ell^{jn} = \begin{cases} \ell & \text{if } \ell \mid n, \\ 0 & \text{otherwise,} \end{cases}$$
we can explicitly compute the behavior of F_ℓ:
$$\log(F_\ell g_0(x)) = \sum_{n=1}^{\infty} \ell a_{n\ell} x^n.$$

[19] Unfortunately, it is standard in the literature to call these operators on "curves", which does not fit well with our previous use of the term in Case Study 2.

[20] Note that for $\ell \in \mathbb{Z}$, this agrees with $[\ell](g_0(x)) = \overbrace{g_0(x) +_{\widehat{\mathbb{G}}} \cdots +_{\widehat{\mathbb{G}}} g_0(x)}^{\ell \text{ times}}$.

[21] There are other definitions of the Frobenius operator which are less mysterious but less explicit. For instance, it also arises from applying the Verschiebung to the character group (or "Cartier dual") of $\widehat{\mathbb{G}}$.

Stringing these together, for $p \nmid \ell$ we have

$$\log([1/\ell]V_\ell F_\ell g_0(x)) = \sum_{n=1}^{\infty} a_{n\ell} x^{n\ell}.$$

Hence, we can iterate over primes $\ell \neq p$, and for two adjacent such primes $\ell' > \ell$ we consider the perturbation

$$g_{\ell'} = g_\ell -_{\widehat{\mathbb{G}}} [1/\ell]V_\ell F_\ell g_\ell.$$

Each of these differences gives a parameter according to Theorem 2.1.10, and the first possible nonzero term appears in degree ℓ, hence the coefficients stabilize linearly in ℓ. Passing to the limit thus gives a new parameter g_∞ on the same formal group $\widehat{\mathbb{G}}$, but now with a p–typical logarithm. □

Of course, the whole idea of "height" is that not every formal group law supports a logarithm. Because of this, we would like to re-express p–typicality in more general terms. Our foothold for this is the following computation of the p–series of a formal group law with p–typical logarithm:

Lemma 3.3.7 ([Ara73, Section 4]). *For a formal group $+_\varphi$ with a logarithm $\log_\varphi$, the logarithm is p–typical if and only if there are elements v_d with*

$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_d x^{p^d} +_\varphi \cdots .$$

Proof sketch. Suppose first that $\log_\varphi$ is p–typical. We can then compare the two series

$$\begin{aligned} \log_\varphi(px) &= px + \cdots, \\ \log_\varphi([p]_\varphi(x)) &= p\log_\varphi(x) = px + \cdots . \end{aligned}$$

The difference is concentrated in degrees of the form p^d, beginning in degree p, so we can find an element v_1 such that

$$p\log_\varphi(x) - (\log_\varphi(px) + \log_\varphi(v_1 x^p))$$

is also concentrated in degrees of the form p^d but now starts in degree p^2. Iterating this gives the equation

$$p\log_\varphi(x) = \log_\varphi(px) + \log_\varphi(v_1 x^p) + \log_\varphi(v_2 x^{p^2}) + \cdots ,$$

at which point we can use formal properties of the logarithm to deduce

$$\begin{aligned} \log_\varphi [p]_\varphi(x) &= \log_\varphi \left(px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots\right), \\ [p]_\varphi(x) &= px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots . \end{aligned}$$

In the other direction, the logarithm coefficients can be recursively recovered from the coefficients v_d for a formal group law with p–typical p–series, using a similar manipulation:

$$\begin{aligned} p \log_\varphi(x) &= \log_\varphi\left([p]_\varphi(x)\right) \\ p \sum_{n=0}^\infty m_n x^n &= \log_\varphi \left({\sum_{d=0}^\infty}_\varphi v_d x^{p^d} \right) = \log_\varphi(px) + \sum_{d=1}^\infty \log_\varphi\left(v_d x^{p^d}\right), \end{aligned}$$

which is only soluable if $\log_\varphi$ is concentrated in degrees of the form p^d. In that case, we can push this slightly further:

$$\sum_{d=0}^\infty p m_{p^d} x^{p^d} = \sum_{d=0}^\infty \sum_{j=0}^\infty m_{p^j} v_d^{p^j} x^{p^{d+j}} = \sum_{n=0}^\infty \left(\sum_{k=0}^n m_{p^k} v_{n-k}^{p^k} \right) x^{p^n},$$

implicitly taking $m_1 = 1$ and $v_0 = p$. □

This result portends much of what is to come. We now set our definition of p–typical to correspond to the manipulations we were making in the course of proving Lemma 3.3.6.

Definition 3.3.8. A parameter $g\colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$ of a formal group is said to be *p–typical* when $F_\ell g = 0$ for all $p \nmid \ell$.

Corollary 3.3.9 (cf. Lemma 3.3.6)**.** *Every formal group law $+_\varphi$ is naturally isomorphic to a p–typical one.* □

Lemma 3.3.10 ([Ara73, Section 4], cf. Lemma 3.3.7)**.** *If $+_\varphi$ is a p–typical formal group law, then there are elements v_d with*

$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_d x^{p^d} +_\varphi \cdots .$$

Proof. As before, let $\widehat{\mathbb{G}}$ denote the formal group associated to $+_\varphi$ and let $g\colon \widehat{\mathbb{A}}^1 \to G$ denote the induced p–typical coordinate. Any auxiliary function $h\colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$ can be expressed in the form

$$h = \sum_{m=0}^{\infty}{}_{\widehat{\mathbb{G}}} V_m \theta_{a_m} g.$$

We will show that if h is p–typical (i.e., $F_\ell h = 0$ for $p \nmid \ell$) then $a_m = 0$ for every $m \neq p^d$.[22] Suppose instead that we can find a smallest index $m = rp^d$ with $p \nmid r$, $r \neq 1$, and $a_m \neq 0$. We can then write

$$\begin{aligned} F_\ell \left(h -_{\widehat{\mathbb{G}}} \sum_{j=0}^{d}{}_{\widehat{\mathbb{G}}} V_{p^j} \theta_{a_{p^j}} g \right) &= F_\ell(V_m \theta_{a_m} g + \cdots) \\ &= r V_{p^d} \theta_{a_m} g + \cdots \neq 0. \end{aligned}$$

Since p–typical curves are closed under difference, h could not have been p–typical.

Finally, we specialize to the case $h = [p]_{\widehat{\mathbb{G}}}(g)$. Since F_ℓ and $[p]$ commute, $[p]$ is p–typical, hence has an expression of the desired form. $\square$

Proof of Lemma 3.3.3. Replace the formal group law by its p–typification. Using the formulas from Lemma 3.3.7, we see that the height of a p–typical formal group law over a field of characteristic p coincides with the appearance of the first nonzero coefficient in its p–series. $\square$

Lemma 3.3.7 shows that the p–series of a formal group law with p–typical logarithm contains exactly as much information as the logarithm itself (and hence fully determines the formal group law). We would again like to show that “all” of the data of a p–typical group law is found in its p–series, even if it does not have a logarithm to mediate the two. The following important theorem makes this thought precise.

Theorem 3.3.11 (cf. [Mila, Proposition 5.1], [Rav86, Theorem A2.2.3], and the proof of [Hop, Proposition 19.10])**.** *The Kudo–Araki map determined by Lemma 3.3.10*

$$\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots] \xrightarrow{v} \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$$

is an isomorphism.

[22]The converse so this claim also holds: since $F_\ell F_p = F_p F_\ell$ for $p \nmid \ell$, we can commute F_ℓ through the sum expression (which is absent any non-commuting terms by hypothesis), where it then kills g to give $F_\ell h = 0$.

Proof. Begin with a universal group law over the ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. This group law p–typifies by Corollary 3.3.9 to a second group law which is selected by a map $\varepsilon\colon \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. The following diagram includes the image factorization of ε, as well as its rationalization and the map v:

$$\begin{tikzcd}
 & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \arrow[rr] \arrow[dr, "\varepsilon"] \arrow[dd, "s"'] & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \arrow[dr, "\varepsilon"] \arrow[dd, "s"] & \\
 & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \arrow[rr] & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \\
\mathbb{Z}_{(p)}[v_1, \ldots, v_d, \ldots] \arrow[r, "v"] & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} \arrow[ur, "i"] \arrow[rr] & & \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} \otimes \mathbb{Q}. \arrow[ur, "i"] &
\end{tikzcd}$$

We immediately deduce that all the horizontal arrows are injections: in Theorem 3.2.2 we calculated $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ to be torsion-free; $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$ is a subring of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$, hence it is also torsion-free; and Lemma 3.3.7 shows that $(i \circ v)(v_n)$ agrees with pm_{p^n} in the module of indecomposables $Q(\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q})$.

To complete the proof, we need to show that v is surjective, which will follow from calculating the indecomposables in $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$ and checking that Qv is surjective. Since s is surjective, the map Qs on indecomposables is surjective as well, and its effect can be calculated rationally. Since $(Q\varepsilon)(m_n) = 0$ for $n \neq p^d$, we have that $Q(\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}})$ is generated by $s(b_{p^d-1})$ under an isomorphism as in Theorem 3.2.2. It follows that Qi injects, hence Qv must surject by the calculation of $Q(i \circ v)(v_n)$ above. □

Corollary 3.3.12. *If $[p]_\varphi(x) = [p]_\varphi(x)$ for two p–typical formal group laws $+_\varphi$ and $+_\psi$, then $+_\varphi$ and $+_\psi$ are themselves equal.* □

Corollary 3.3.13. *For any sequence of coefficients $v_j \in R$ in a $\mathbb{Z}_{(p)}$–algebra R, there is a unique p–typical formal group law $+_\varphi$ with*

$$[p]_\varphi = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_d x^{p^d} +_\varphi \cdots . \qquad \square$$

Finally, we exploit these results to make deductions about the geometry of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. There is an inclusion of groupoid–valued sheaves from p–typical formal group laws with p–typical isomorphisms to all formal group laws with all isomorphisms. Corollary 3.3.9 can be viewed as presenting this inclusion as a deformation retraction, witnessing a natural *equivalence* of groupoids. It follows from Remark 3.1.17 that they both present the same stack. The central utility of this equivalence is that the Kudo–Araki moduli of p–typical formal group laws is a considerably smaller algebra than $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$, resulting in a less noisy picture of the Hopf algebroid.

Our final goal in this Lecture is to exploit this refined presentation in the study of invariant functions.

Definition 3.3.14 ([Goe08, Lemma 2.28]). Let (X_0, X_1) be the groupoid scheme associated to a Hopf algebroid (A, Γ). A function $f\colon X_0 \to \mathbb{A}^1$ is said to be *invariant* when it is stable under isomorphism, i.e., when there is a diagram

$$\begin{tikzcd} X_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \arrow[dr, "s^* f = t^* f"] & \\ X_0 \arrow[r, "f"] & \mathbb{A}^1. \end{tikzcd}$$

(In terms of Hopf algebroids, the corresponding element $a \in A$ satisfies $\eta_L(a) = \eta_R(a)$.) Correspondingly, a closed subscheme $A \subseteq X_0$ determined by the simultaneous vanishing of functions f_α is said to be *invariant* when the vanishing condition is invariant—i.e., a point lies in the simultaneous vanishing locus if and only if its entire orbit under X_1 also lies in the simultaneous vanishing locus. (In terms of Hopf algebroids, the corresponding ideal $I \subseteq A$ satisfies $\eta_L(I) = \eta_R(I)$.) Finally, a *closed substack* is a substack determined by an invariant ideal of X_0.

We now have the language to describe all of the closed substacks of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$—or, equivalently, to discern all of the invariant ideals of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$.

Corollary 3.3.15 ([Wil82, Theorem 4.6 and Lemmas 4.7-8]). *The ideal $I_d = (p, v_1, \ldots, v_{d-1})$ is invariant under the action of* strict *formal group law isomorphisms for all d. It determines the closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d}$ of formal group laws of p–height at least d.*

Proof. Recall from Theorem 3.3.11 the Kudo–Araki isomorphism

$$\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}} \xrightarrow{\simeq} \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots] =: \operatorname{Spec} V,$$

and let $+_L$ denote the associated universal p–typical formal group law with p–series

$$[p]_L(x) = px +_L v_1 x^p +_L v_2 x^{p^2} +_L \cdots +_L v_d x^{p^d} +_L \cdots .$$

Over $\operatorname{Spec} V[t_1, t_2, \ldots]$, we can form a second group law $+_R$ by conjugating $+_L$ by the universal p–typical coordinate transformation $g(x) = \sum_{j=0}^{\infty}{}_L t_j x^{p^j}$. The corresponding p–series

$$[p]_R(x) = \sum_{d=0}^{\infty}{}_R \eta_R(v_d) x^{p^d}$$

determines the η_R map of the Hopf algebroid $(V, V[t_1, t_2, \ldots])$ presenting the moduli of p–typical formal group laws and p–typical isomorphisms. We cannot hope to compute $\eta_R(v_d)$ explicitly, but modulo p we can apply Freshman's Dream to the expansion of

$$[p]_L(g(x)) = g([p]_R(x))$$

to discern some information:

$$\sum_{\substack{i \geq 0 \\ j > 0}}{}_L t_i \eta_R(v_j)^{p^i} \equiv \sum_{\substack{i > 0 \\ j \geq 0}}{}_L v_i t_j^{p^i} \pmod{p}.$$

This is still inexplicit, since $+_L$ is a very complicated operation, but we can see $\eta_R(v_d) \equiv v_d \pmod{I_d}$. It follows that I_d is invariant for each d. Additionally, the closed substack this determines are those formal groups admitting local p–typical coordinates for which $v_{\leq d} = 0$, guaranteeing that the height of the associated formal group is at least d. □

What is *much* harder to prove is the following:

Theorem 3.3.16 ([Lan75, Corollary 2.4 and Proposition 2.5], cf. [Wil82, Theorem 4.9])**.** *The unique closed reduced substack of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ of codimension d is selected by the invariant prime ideal $I_d \subseteq \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$.*

Proof sketch. We want to show that if I is an invariant prime ideal, then $I = I_d$ for some d. To begin, note that $v_0 = p$ is the only invariant function on $\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}$, hence I must either be trivial or contain p. Then, inductively assume that $I_d \subseteq I$. If this is not an equality, we want to show that $I_{d+1} \subseteq I$ is forced. Take $y \in I \setminus I_d$; if we could show

$$\eta_R(y) = a v_d^j t^K + \text{higher order terms}$$

for nonzero $a \in \mathbb{Z}_{(p)}$, we could proceed by primality to show that $v_d \in I$ and hence $I_{d+1} \subseteq I$. This is possible (and, indeed, this is how the full proof goes), but it requires serious bookkeeping. □

Remark 3.3.17. The complementary open substack of dimension d is harder to describe.From first principles, we can say only that it is the locus where the coordinate functions $p, v_1, \ldots, v_d$ do not *all simultaneously vanish*. It turns out that:

1. On a cover, at least one of these coordinates can be taken to be invertible.

2. Once one of them is invertible, a coordinate change on the formal group law can be used to make v_d (and perhaps others in the list) invertible. Hence, we can use $v_d^{-1}\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$ as a coordinate chart.

3. Over a further base extension and a further coordinate change, the higher coefficients v_{d+k} can be taken to be zero. Hence, we can also use $v_d^{-1}\mathbb{Z}_{(p)}[v_1, \ldots, v_d]$ as a coordinate chart.

Remark 3.3.18 (cf. [Str06, Section 12] and [Lura, Remark 13.9]). Specialize now to the case of a field k of characteristic p. Since the additive group law has vanishing p–series and is p–typical, a consequence of Corollary 3.3.12 is that *every* p–typical group law with vanishing p–series is exactly equal to $\widehat{\mathbb{G}}_a$, and in fact any formal group law with vanishing p–series p–typifies exactly to $\widehat{\mathbb{G}}_a$. This connects several ideas we have seen so far: the presentation of formal group laws with logarithms in Theorem 1.5.6, the presentation of the context $\mathcal{M}_{MOP}$ in Example 3.1.20, and the Hurewicz image of MU_* in $H\mathbb{F}_{p*}MU$ in Corollary 2.6.9.

Remark 3.3.19. It's worth pointing out how strange all of this is. In Euclidean geometry, open subspaces are always top-dimensional, and closed subspaces can drop dimension. Here, proper open substacks of every dimension appear, and every nonempty closed substack is ∞–dimensional (albeit of positive codimension).

Remark 3.3.20. The results of this section have several alternative forms in the literature. For instance, $[p]_\varphi(x)$ can also be expressed as

$$[p]_\varphi(x) = px + v_1 x^p + v_2 x^{p^2} + \cdots + v_d x^{p^d} + \cdots,$$

and this also determines a presentation of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}$. These other elements v_d, called *Hazewinkel coordinates*, differ substantially from the Kudo–Araki coordinates favored here, although they are equally "canonical". Different coordinate patches are useful for accomplishing different tasks, and the reader would be wise to remain flexible.[23]

[23] In particular, it is a largely open question whether there is a (partial) coordinate patch that is compatible with the results of Appendix A.2.

Remark 3.3.21 ([Haz12, Section 17.5]). The p–typification operation often gives "unusual" results. For instance, we will examine the standard multiplicative formal group law of Example 2.1.23, its rational logarithm, and its rational exponential:

$$x +_{\widehat{\mathbb{G}}_m^{\mathrm{std}}} y = x + y - xy, \quad \log_{\widehat{\mathbb{G}}_m^{\mathrm{std}}}(x) = -\log(1-x), \quad \exp_{\widehat{\mathbb{G}}_m^{\mathrm{std}}}(x) = 1 - \exp(-x).$$

By Lemma 3.3.6, we see that the p–typification of this rational logarithm takes the form

$$\log_{\widehat{\mathbb{G}}_m^{p\text{-typ}}}(x) = \sum_{j=0}^{\infty} \frac{x^{p^j}}{p^j}.$$

We can couple this to the standard exponential of the rational multiplicative group

$$\begin{tikzcd} \widehat{\mathbb{A}}^1 \arrow[r, "\varepsilon x"] \arrow[rr, bend left, "\log^{p\text{-typ}}"] & \widehat{\mathbb{G}}_m \arrow[r, "\log"] & \widehat{\mathbb{G}}_a \arrow[r, "\exp"] \arrow[rr, bend left, "\exp^{\mathrm{std}}"] & \widehat{\mathbb{G}}_m \arrow[r, "x"] & \widehat{\mathbb{A}}^1 \end{tikzcd}$$

to produce the coordinate change from Corollary 3.3.9:

$$1 - \exp\left(-\sum_{j=0}^{\infty} \frac{x^{p^j}}{p^j}\right) = 1 - E_p(-x).$$

This series $E_p(x)$ is known as the *Artin–Hasse exponential*, and it has the miraculous property that it is a series lying in $\mathbb{Z}_{(p)}[\![x]\!] \subseteq \mathbb{Q}[\![x]\!]$, as it is a change of coordinate series on $\widehat{\mathbb{G}}_m$ over $\operatorname{Spec} \mathbb{Z}_{(p)}$.

3.4 The structure of $\mathcal{M}_{\mathbf{fg}}$ III: Small scales

In the previous two Lectures, we analyzed the structure of $\mathcal{M}_{\mathbf{fg}}$ as a whole: first we studied the cover

$$\mathcal{M}_{\mathbf{fgl}} \to \mathcal{M}_{\mathbf{fg}},$$

and then we turned to the stratification described by the height function

$$\mathrm{ht}\colon \pi_0 \mathcal{M}_{\mathbf{fg}}(T \text{ a } \mathbb{Z}_{(p)}\text{–algebra}) \to \mathbb{N} \cup \{\infty\}.$$

In this Lecture, we will concern ourselves with the small scale behaviors of $\mathcal{M}_{\mathbf{fg}}$: its geometric points and their local neighborhoods.[24] To begin, we have all the tools in place to perform an outright classification of the geometric points.

[24]For an alternative perspective on much of this material, see [Str06, Section 18], where the presentation connects rather tightly with our Lecture 4.4.

Theorem 3.4.1 ([Laz55, Théorème IV]). *Let $\bar{k}$ be an algebraically closed field of positive characteristic p. The height map*

$$\mathrm{ht}\colon\ \pi_0\mathcal{M}_{\mathbf{fg}}(\bar{k}) \to \mathbb{N}_{>0} \cup \{\infty\}$$

is a bijection.

Proof. Surjectivity follows from Corollary 3.3.13. Namely, the d^{th} *Honda formal group law* is the p–typical formal group law over k determined by

$$[p]_{\varphi_d}(x) = x^{p^d},$$

and it gives a preimage for d. To show injectivity, we must show that every p–typical formal group law φ over $\bar{k}$ is isomorphic to the appropriate Honda group law. Suppose that the p–series for φ begins

$$[p]_\varphi(x) = x^{p^d} +_\varphi ax^{p^{d+k}} + \cdots .$$

Then, we will construct a coordinate transformation $g(x) = \sum_{j=0}^{\infty}{}^{\varphi} b_j x^{p^j}$ satisfying

$$\begin{aligned}
g(x^{p^d}) &\equiv [p]_\varphi(g(x)) && \pmod{x^{p^{d+k}+1}} \\
\sum_{j=0}^{\infty}{}^{\varphi} b_j x^{p^{d+j}} &\equiv \left(\sum_{j=0}^{\infty}{}^{\varphi} b_j x^{p^j}\right)^{p^d} +_\varphi a \left(\sum_{j=0}^{\infty}{}^{\varphi} b_j x^{p^j}\right)^{p^{d+k}} && \pmod{x^{p^{d+k}+1}} \\
\sum_{j=0}^{\infty}{}^{\varphi} b_j x^{p^{d+j}} &\equiv \left(\sum_{j=0}^{\infty}{}^{(\mathrm{Frob}^d)^*\varphi} b_j^{p^d} x^{p^{d+j}}\right) + ax^{p^{d+k}} && \pmod{x^{p^{d+k}+1}}.
\end{aligned}$$

For g to be a coordinate transformation, we must have $b_0 = 1$, and because $\bar{k}$ is algebraically closed we can induct on j to solve for the other coefficients in the series. The coordinate for φ can thus be perturbed so that the term $x^{p^{d+k}}$ does not appear in the p–series, and inducting on d gives the result. □

Remark 3.4.2 ([Str06, Remark 11.2]). We can now see see that $\pi_0\mathcal{M}_{\mathbf{fg}}$, sometimes called the *coarse moduli of formal groups*, is not representable by a scheme. From Theorem 3.4.1, we see that there are infinitely many points in $\pi_0\mathcal{M}_{\mathbf{fg}}(\mathbb{F}_p)$. From Corollary 3.2.6, we see that these lift along the surjection $\mathbb{Z} \to \mathbb{F}_p$ to give infinitely many distinct points in $\pi_0\mathcal{M}_{\mathbf{fg}}(\mathbb{Z})$. On the other hand, by Theorem 3.3.1 there is a single $\mathbb{Q}$–point of the coarse moduli, whereas the $\mathbb{Z}$–points of a representable functor would inject into its $\mathbb{Q}$–points.

We now turn to understanding the infinitesimal neighborhoods of these geometric points. In general, for $p\colon \operatorname{Spec} k \to X$ a closed k–point of a scheme, we defined in Definition 2.1.6 and Definition 2.1.7 an infinitesimal neighborhood object $X_p^\wedge$ with a lifting property

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "p"] \arrow[d] & X_p^\wedge \arrow[d] \\ \operatorname{Spf} R \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

for any infinitesimal thickening $\operatorname{Spf} R$ of $\operatorname{Spec} k$. Thinking of X as representing a moduli problem, a typical choice for $\operatorname{Spf} R$ is $\widehat{\mathbb{A}}^1_k$, and a map $\widehat{\mathbb{A}}^1_k \to X$ extending p gives a series solution to the moduli problem which specializes at the origin to p. In turn, $X_p^\wedge$ is the smallest object through which all such maps factor, and so we think of it as classifying Taylor expansions of solutions passing through p.

For a formal group $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$, the definition is formally similar, but actually writing it out is made complicated by Remark 3.1.17. In particular, $p\colon \operatorname{Spec} k \to X$ may not lift directly through $\operatorname{Spf} R \to X$, but instead $\operatorname{Spec} R/\mathfrak{m} \to X$ may present p on a cover $i\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spec} k$.

Definition 3.4.3 ([Reza, Section 2.4], cf. [Str97, Section 6])**.** Define $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$, the *Lubin–Tate stack*, to be the groupoid-valued functor which on an infinitesimal thickening R of k has objects

$$\begin{tikzcd} & & \mathcal{M}_{\mathbf{fg}} & & \\ \operatorname{Spec} k \arrow[urr, bend left, "\Gamma"] & & \operatorname{Spec} R/\mathfrak{m} \arrow[ll, "i"'] \arrow[rr, "\pi"] \arrow[u, bend left=40, "i^*\Gamma"] \arrow[u, bend right=40, "\pi^*\widehat{\mathbb{G}}"'] \arrow[u, phantom, "\overset{\alpha}{\Longrightarrow}"] & & \operatorname{Spf} R, \arrow[ull, bend right, "\widehat{\mathbb{G}}"'] \end{tikzcd}$$

where i is an inclusion of k into the residue field $R/\mathfrak{m}$ and $\alpha\colon i^*\Gamma \to \pi^*\widehat{\mathbb{G}}$ is an isomorphism of formal groups. The morphisms in the groupoid are maps $f\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}'$ of formal groups over $\operatorname{Spf} R$ covering the identity on $i^*\Gamma$, called $\star$–*isomorphisms*.

Remark 3.4.4 (cf. [Rezb, Section 4.1]). The local formal group $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ always has trivializable Lie algebra, hence Lemma 3.2.7 shows that it always admits a presentation by a formal group law. In fact, any deformation $\widehat{\mathbb{G}} \colon \operatorname{Spf} R \to \mathcal{M}_{\mathbf{fg}}$ of Γ also has a trivializable Lie algebra, since projective modules (such as $T_0\widehat{\mathbb{G}}$) over local rings like R are automatically free (i.e., trivializable). It follows that the groupoid $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R)$ admits a presentation in terms of formal group *laws*. Starting with the pullback square of groupoids

$$\begin{tikzcd}
(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathcal{M}_{\mathbf{fg}}(R) \arrow[d] \\
\coprod_{i\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spec} k} \{\Gamma\} \arrow[r] & \coprod_{i\colon \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spec} k} \mathcal{M}_{\mathbf{fg}}(k) \arrow[r] & \mathcal{M}_{\mathbf{fg}}(R/\mathfrak{m})
\end{tikzcd}$$

and selecting formal group laws everywhere, the objects of the groupoid $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R)$ are given by diagrams

$$\begin{tikzcd}
(\widehat{\mathbb{A}}^1_k, +_\Gamma) \arrow[d] & (\widehat{\mathbb{A}}^1_{R/\mathfrak{m}}, +_{i^*\Gamma}) \arrow[l] \arrow[dr] \arrow[dl, phantom, "\llcorner", very near start] \arrow[rr, equal] & & (\widehat{\mathbb{A}}^1_{R/\mathfrak{m}}, +_{\pi^*\widehat{\mathbb{G}}}) \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & (\widehat{\mathbb{A}}^1_R, +_{\widehat{\mathbb{G}}}) \arrow[d] \\
\operatorname{Spec} k & & \operatorname{Spec} R/\mathfrak{m} \arrow[ll, "i"'] \arrow[rr, "\pi"] & & \operatorname{Spf} R,
\end{tikzcd}$$

where we have required an *equality* of formal group laws over the common pullback. A morphism in this groupoid is a formal group law isomorphism f over $\operatorname{Spf} R$ which reduces to the identity over $\operatorname{Spec} R/\mathfrak{m}$.

The main result about this infinitesimal space $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$ is due to Lubin and Tate:

Theorem 3.4.5 ([LT66, Theorem 3.1])**.** *Suppose that* $\operatorname{ht}\Gamma < \infty$ *for* Γ *a formal group over* k *a perfect field of positive characteristic* p*. The functor* $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$ *is valued in essentially discrete groupoids, and it is naturally equivalent to a smooth formal scheme over* $\mathbb{W}_p(k)$ *of dimension* $(\operatorname{ht}(\Gamma) - 1)$.

Remark 3.4.6 ([Zin84, Theorem 4.35]). The presence of the *p–local Witt ring* $\mathbb{W}_p(k)$ is explained by its universal property: for k as above and R an infinitesimal thickening of k, $\mathbb{W}_p(k)$ has the lifting property[25]

[25] Rings with such lifting properties are generally called *Cohen rings*. In the case that k is a perfect field of positive characteristic p, the Witt ring $\mathbb{W}_p(k)$ happens to model a Cohen ring for k.

$$\begin{tikzcd} \mathbb{W}_p(k) \arrow[r, dotted, "\exists!"] \arrow[d] & R \arrow[d] \\ k \arrow[r, "i"] & R/\mathfrak{m} \end{tikzcd}$$

For the finite perfect fields $k = \mathbb{F}_{p^d} = \mathbb{F}_p(\zeta_{p^d-1})$, the Witt ring can be computed to be $\mathbb{W}_p(\mathbb{F}_{p^d}) = \mathbb{Z}_p(\zeta_{p^d-1})$.

Remark 3.4.7. In light of Remark 3.4.4, we can also state Theorem 3.4.5 in terms of formal group laws and their $\star$–isomorphisms. For a group law $+_\Gamma$ over a perfect field k of positive characteristic, it claims that there exists a ring X, noncanonically isomorphic to $\mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!]$, as well as a certain group law $+_{\widetilde{\Gamma}}$ on this ring. The group law $+_{\widetilde{\Gamma}}$ has the following property: if $+_{\widehat{\mathbb{G}}}$ is a formal group law on an infinitesimal thickening $\operatorname{Spf} R$ of $\operatorname{Spec} k$ which reduces along $\pi\colon\ \operatorname{Spec} R/\mathfrak{m} \to \operatorname{Spf} R$ to $+_\Gamma$, then there is a unique ring map $f\colon X \to R$ such that $f^*(+_{\widetilde{\Gamma}})$ is $\star$–isomorphic to $\pi^*(+_{\widehat{\mathbb{G}}})$. Moreover, this $\star$–isomorphism is unique.

We will spend the rest of this Lecture working towards a proof of Theorem 3.4.5. We first consider a very particular sort of infinitesimal thickening: the square-zero extension $R = k[\varepsilon]/\varepsilon^2$ with pointing $\varepsilon = 0$. We are interested in two kinds of data over R: formal group laws $+_\Delta$ over R reducing to $+_\Gamma$ at the pointing, and formal group law automorphisms φ of $+_\Gamma$ which reduce to the identity automorphism at the pointing.

Lemma 3.4.8. *Define*

$$\Gamma_1 = \frac{\partial(x +_\Gamma y)}{\partial x}, \qquad\qquad \Gamma_2 = \frac{\partial(x +_\Gamma y)}{\partial y}.$$

Such automorphisms φ are determined by series ψ satisfying

$$0 = \Gamma_1(x,y)\psi(x) - \psi(x +_\Gamma y) + \psi(y)\Gamma_2(x,y).$$

Such formal group laws $+_\Delta$ are determined by bivariate series $\delta(x,y)$ satisfying

$$0 = \Gamma_1(x +_\Gamma y, z)\delta(x,y) - \delta(x, y +_\Gamma z) + \delta(x +_\Gamma y, z) - \delta(y,z)\Gamma_2(x, y +_\Gamma z).$$

Proof. Such an automorphism φ admits a series expansion

$$\varphi(x) = x + \varepsilon \cdot \psi(x).$$

Then, we take the homomorphism property

$$\begin{aligned}\varphi(x +_\Gamma y) &= \varphi(x) +_\Gamma \varphi(y) \\ (x +_\Gamma y) + \varepsilon \cdot \psi(x +_\Gamma y) &= (x + \varepsilon \cdot \psi(x)) +_\Gamma (y + \varepsilon \cdot \psi(y))\end{aligned}$$

and apply $\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0}$ to get

$$\psi(x +_\Gamma y) = \Gamma_1(x, y) \cdot \psi(x) + \Gamma_2(x, y) \cdot \psi(y).$$

Similarly, such a formal group law $+_\Delta$ admits a series expansion

$$x +_\Delta y = (x +_\Gamma y) + \varepsilon \cdot \delta(x, y).$$

Beginning with the associativity property

$$(x +_\Delta y) +_\Delta z = x +_\Delta (y +_\Delta z),$$

we compute $\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0}$ applied to both sides:

$$\begin{aligned}\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} ((x +_\Delta y) +_\Delta z) &= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} ((((x +_\Gamma y) + \varepsilon \cdot \delta(x, y)) +_\Gamma z) + \varepsilon \cdot \delta(x +_\Gamma y, z)) \\ &= \Gamma_1(x +_\Gamma y, z) \cdot \delta(x, y) + \delta(x +_\Gamma y, z),\end{aligned}$$

and similarly

$$\begin{aligned}\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (x +_\Delta (y +_\Delta z)) &= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} ((x +_\Gamma ((y +_\Gamma z) + \varepsilon \cdot \delta(y, z))) + \varepsilon \cdot \delta(x, y +_\Gamma z)) \\ &= \Gamma_2(x, y +_\Gamma z) \cdot \delta(y, z) + \delta(x, y +_\Gamma z).\end{aligned}$$

Equating these gives the condition in the Lemma statement. □

I would like a coordinate-independent model for this complex, which will hopefully make the following Corollary self-evident.

The key observation is that these two conditions appear as cocycle conditions for the first two levels of a natural cochain complex.

Definition 3.4.9 ([Laz97, Section 3])**.** The deformation complex[26] $\widehat{C}^*(+_\Gamma; k)$ is defined by

$$k \to k[\![x_1]\!] \to k[\![x_1, x_2]\!] \to k[\![x_1, x_2, x_3]\!] \to \cdots$$

[26]Pieces are of this complex are visible in work of Drinfel'd [Dri74, Section 4.A] and of Lubin–Tate [LT66], but neither actually assemble the whole complex.

with differential

$$(df)(x_1, \dots, x_{n+1}) = \Gamma_1\left(\sum_{i=1}^{n}{}_{\Gamma}\, x_i, x_{n+1}\right) \cdot f(x_1, \dots, x_n)$$
$$+ \sum_{i=1}^{n} (-1)^i \cdot f(x_1, \dots, x_i +_\Gamma x_{i+1}, \dots, x_{n+1})$$
$$+ (-1)^{n+1} \cdot \Gamma_2\left(x_1, \sum_{i=2}^{n+1}{}_{\Gamma}\, x_i\right) \cdot f(x_2, \dots, x_{n+1}),$$

where we have again written

$$\Gamma_1(x, y) = \frac{\partial(x +_\Gamma y)}{\partial x}, \qquad \Gamma_2(x, y) = \frac{\partial(x +_\Gamma y)}{\partial y}.$$

The complex even knits the information together intelligently:

Corollary 3.4.10 ([Laz97, p. 1320]). *Two extensions $+_\Delta$ and $+_{\Delta'}$ of $+_\Gamma$ to $k[\varepsilon]/\varepsilon^2$ are isomorphic if their corresponding 2–cocycles in $\widehat{Z}^2(+_\Gamma; k)$ differ by an element in $\widehat{B}^2(+_\Gamma; k)$.* □

Remarkably, we have already encountered this complex before:

Lemma 3.4.11 ([Laz97, p. 1320]). *Write $\widehat{\mathbb{G}}$ for the formal group associated to the group law $+_\Gamma$. The cochain complex $\widehat{C}^*(+_\Gamma; k)$ is quasi-isomorphic to the cohomology cochain complex considered in the proof of Lemma 3.2.5:*

$$\widehat{C}^*(+_\Gamma; k) \to \underline{\mathsf{FormalSchemes}}(B\widehat{\mathbb{G}}, \widehat{\mathbb{G}}_a)(k)$$
$$f \mapsto \Gamma_1\left(0, \sum_{i=1}^{n}{}_{\Gamma}\, x_i\right)^{-1} f(x_1, \dots, x_n). \quad \square$$

Two Lectures ago while proving Lemma 3.2.5, we computed the cohomology of this complex in the specific case of $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$. This is the one case where Lubin and Tate's theorem does *not* apply, since it requires $\operatorname{ht} \widehat{\mathbb{G}} < \infty$. Nonetheless, by filtering the multiplication on $\widehat{\mathbb{G}}$ by degree, we can use this specific calculation to get up to the general one we now seek.

Lemma 3.4.12. *Let $\widehat{\mathbb{G}}$ be a formal group of finite height d over a field k. Then $H^1(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a) = 0$ and $H^2(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a)$ is a free k–vector space of dimension $(d-1)$.*

d or $(d-1)$? There's β_0 through β_{d-1}.... Also compare this with Lemma 3.2.5.

Proof (after Hopkins). We select a p–typical coordinate on $\widehat{\mathbb{G}}$ of the form

$$x +_\varphi y = x + y + \text{unit} \cdot c_{p^d}(x, y) + \cdots,$$

where $c_{p^d}(x, y)$ is as in one of Lazard's symmetric 2–cocycles, as in Lemma 3.2.5. Filtering $\widehat{\mathbb{G}}$ by degree, the multiplication projects to $x +_\varphi y = x + y$ in the associated graded, and the resulting filtration spectral sequence has signature

$$[H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)]_* \Rightarrow H^*(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a),$$

where the second grading comes from the degree of the homogeneous polynomial representatives of classes in $H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)$.

Because Lemma 3.2.5 gives different calculations of $H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)$ for $p = 2$ and $p > 2$, we specialize to $p > 2$ for the remainder of the proof and leave the similar $p = 2$ case to the reader. For $p > 2$, Lemma 3.2.5 gives

$$[H^*(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a)]_* = \left[\frac{k[\alpha_j \mid j \geq 0]}{\alpha_j^2 = 0} \otimes k[\beta_j \mid j \geq 0] \right]_*,$$

where α_j is represented by x^{p^j} and β_j is represented by $c_{p^j}(x, y)$. To compute the differentials in this spectral sequence generally, one computes by hand the formula for the differential in the bar complex, working up to lowest nonzero degree. For instance, to compute $d(\alpha_j)$ we examine the series

$$(x +_\varphi y)^{p^j} - (x^{p^j} + y^{p^j}) = (\text{unit}) \cdot c_{p^{d+j}}(x, y) + \cdots,$$

where we used $c_{p^d}^{p^j} = c_{p^{j+d}}$, giving $d(\alpha_j) = \beta_{j+d}$. So, we see that nothing in the 1–column of the spectral sequence is a permanent cocycle and that there are d classes at the bottom of the 2–column of the spectral sequence which are not coboundaries. To conclude the Lemma statement, we need only to check that they are indeed permanent cocycles. To do this, we note that they are indeed realized as deformations, by noting

$$x +_{\text{univ}} y \cong x + y + v_j c_{p^j}(x, y) \pmod{v_1, \ldots, v_{j-1}, (x, y)^{p^j+1}}$$

where $+_{\text{univ}}$ is the Kudo–Araki universal p–typical law (cf. [LT66, Proposition 1.1]). □

Proof of the square-zero case of Theorem 3.4.5 using Remark 3.4.7. We will prove this inductively on the order of the infinitesimal neighborhood of $\operatorname{Spec} k = \operatorname{Spec} R/\mathfrak{m}$ in $\operatorname{Spf} R$:

$$\operatorname{Spec} R/\mathfrak{m} \xrightarrow{j_r} \operatorname{Spec} R/\mathfrak{m}^r \xrightarrow{i_r} \operatorname{Spf} R.$$

Suppose that we have demonstrated the Theorem for $+_{\widehat{\mathbb{G}}_{r-1}} = i_{r-1}^*(+_{\widehat{\mathbb{G}}})$, so that there is a map $\alpha_{r-1}\colon \mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!] \to R/\mathfrak{m}^{r-1}$ and a strict isomorphism $g_{r-1}\colon +_{\widehat{\mathbb{G}}_{r-1}} \to \alpha_{r-1}^* +_{\tilde{\Gamma}}$ of formal group laws. The exact sequence

$$0 \to \mathfrak{m}^{r-1}/\mathfrak{m}^r \to R/\mathfrak{m}^r \to R/\mathfrak{m}^{r-1} \to 0$$

exhibits $R/\mathfrak{m}^r$ as a square–zero extension of $R/\mathfrak{m}^{r-1}$ by $M = \mathfrak{m}^{r-1}/\mathfrak{m}^r$. Then, let β be *any* lift of α_{r-1} and h be *any* lift of g_{r-1} to R/I^r, and let A and B be the induced group laws

$$x +_A y = \beta^* \tilde{\varphi}, \qquad\qquad x +_B y = h\left(h^{-1}(x) +_{\widehat{\mathbb{G}}_r} h^{-1}(y)\right).$$

Since these both deform the group law $+_{\widehat{\mathbb{G}}_{r-1}}$, by Corollary 3.4.10 and Lemma 3.4.12 there exist $m_j \in M$ and $f(x) \in M[\![x]\!]$ satisfying

$$(x +_B y) - (x +_A y) = (df)(x, y) + \sum_{j=1}^{d-1} m_j c_{p^j}(x, y),$$

where $c_{p^j}(x, y)$ is the 2–cocycle associated to the cohomology 2–class β_j. The following definitions complete the induction:

$$g_r(x) = h(x) - f(x), \qquad\qquad \alpha_r(u_j) = \beta(u_j) + m_j. \qquad \square$$

Lemma 3.4.13 ([Lura, Reduction to Theorem 21.5])**.** *Theorem 3.4.5 is true in general if and only if it holds for the square-zero thickening $R = k[\varepsilon]/\varepsilon^2$.*

Proof. Of course, if Theorem 3.4.5 is true in general, then it holds for the special case of $R = k[\varepsilon]/\varepsilon^2$. Our task is to reduce the general case to this special case, and we proceed by inducting on the length of the Artinian ring R. If R has length 1, then $R = k$ and we are done. Otherwise, we can choose an element $r \in R$ which is annihilated by $\mathfrak{m}$, and we consider the relationship between $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R)$ and $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R/r)$. There is a pullback diagram

$$\begin{tikzcd}
(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R \times_{R/r} R) \arrow[r, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R) \arrow[d, "\pi"] \\
(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R) \arrow[r] & (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R/r).
\end{tikzcd}$$

First observing that $R \times_{R/r} R \cong k[\varepsilon]/\varepsilon^2 \times_k R$, we note that $G = (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(k[\varepsilon]/\varepsilon^2)$ forms a group, that the map G acts on $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R)$ through the map m, that π factors to give an embedding

$$\pi/G\colon (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R)/G \to (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R/r),$$

and that Corollary 3.2.6 shows that π/G is additionally surjective. From all this, we see that π presents $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R)$ as a principal homogeneous space for G over $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}(R/r)$. An identical argument shows that formal schemes with this same property are smooth, and hence there is an isomorphism

$$G \cong \mathsf{Algebras}_{k/}(\mathbb{W}_p(k)[\![u_1, \ldots, u_{n-1}]\!], k[\varepsilon]/\varepsilon^2). \qquad \square$$

Remark 3.4.14. Our calculation $H^1(\widehat{\mathbb{G}}_\varphi; \widehat{\mathbb{G}}_a) = 0$ shows that the automorphisms $\alpha\colon \Gamma \to \Gamma$ of the special fiber induce automorphisms of the entire Lubin–Tate stack by universality. Namely, for $\Gamma \to \widetilde{\Gamma}$ the universal deformation, the precomposite

$$\Gamma \xrightarrow{\alpha} \Gamma \to \widetilde{\Gamma}$$

presents $\widetilde{\Gamma}$ as a deformation of Γ in a different way, hence induces a map $\widetilde{\alpha}\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$, which by Theorem 3.4.5 is in turn induced by a map $\widetilde{\alpha}\colon (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma} \to (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$. The action is *highly* nontrivial in all but the most degenerate cases, and its study is of serious interest to homotopy theorists (cf. Lecture 3.6) and to arithmetic geometers (cf. Remark 4.4.23).

Remark 3.4.15. We also see that our analysis fails wildly for the case $\Gamma = \widehat{\mathbb{G}}_a$. The differential calculation in Lemma 3.4.12 is meant to give us an upper bound on the dimensions of $H^1(\Gamma; \widehat{\mathbb{G}}_a)$ and $H^2(\Gamma; \widehat{\mathbb{G}}_a)$, but this family of differentials is zero in the additive case. Accordingly, both of these vector spaces are infinite dimensional, completely prohibiting us from making any further assessment.

Having accomplished all our major goals, we close our algebraic analysis of $\mathcal{M}_{\mathbf{fg}}$ with Figure 3.3, a diagram summarizing our results.

Complete picture

Hood suggests including two pictures, one labeled with algebraic names and one labeled with topological names. This is a good idea.

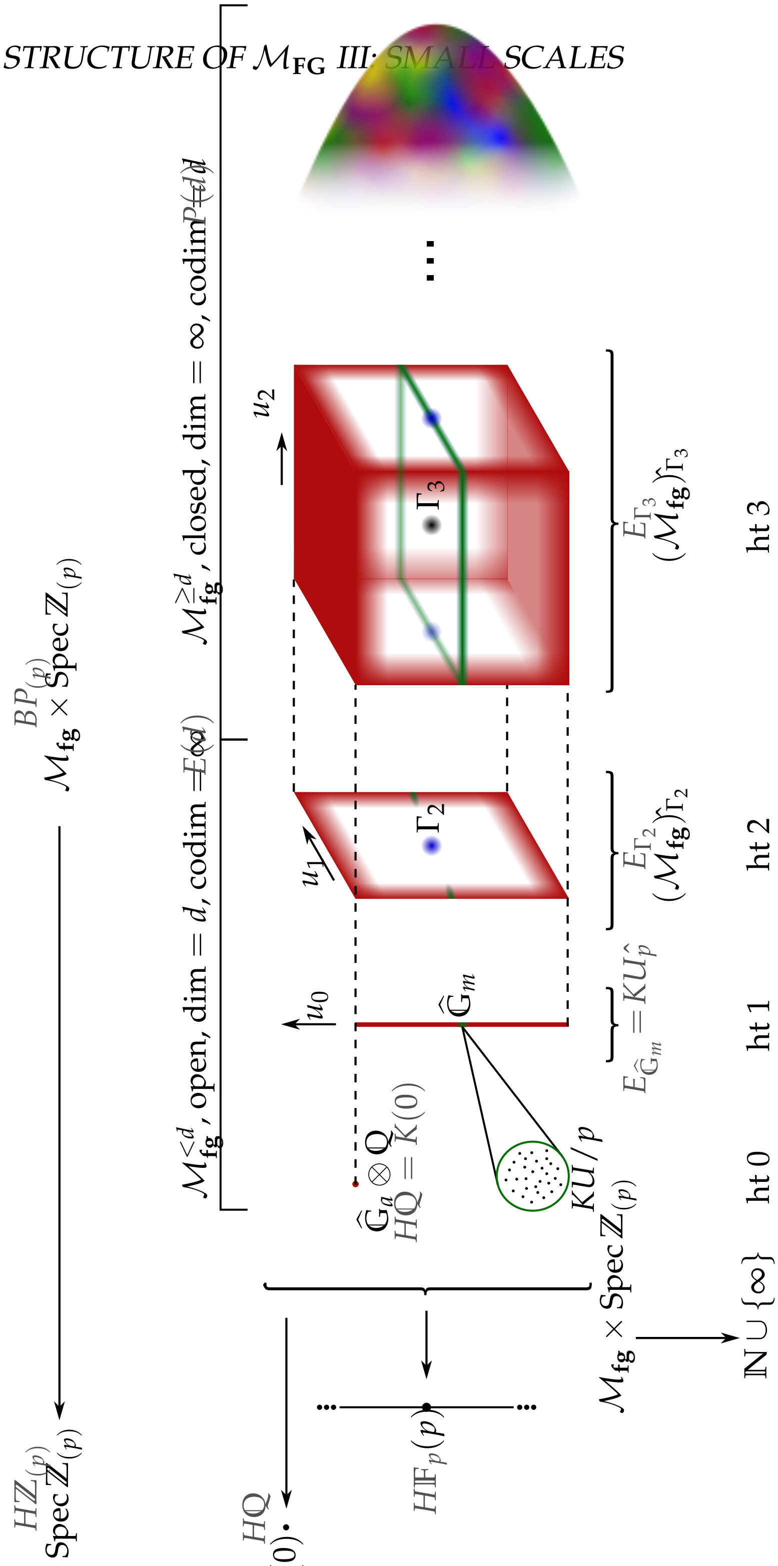


Figure 3.3: Portrait of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$.

3.5 Nilpotence and periodicity in finite spectra

With our analysis of $\mathcal{M}_{\mathbf{fg}}$ complete, our first goal in this Lecture is to finish the program sketched in the introduction to this Case Study by manufacturing those interesting homology theories connected to the functor $\mathcal{M}_{MU}(-)$. We begin by rephrasing our main tool, Theorem 3.0.1, in terms of algebraic conditions.

Theorem 3.5.1 ([Lan76, Corollary 2.7] and [Hop, Theorem 21.4 and Proposition 21.5], cf. Theorem 3.0.1)**.** *Let $\mathcal{F}$ a quasicoherent sheaf over $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$, thought of as a comodule M for the Kudo–Araki Hopf algebroid (cf. Theorem 3.3.11)*

$$(A, \Gamma) = (\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}, \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}[t_1, t_2, \ldots]).$$

If $(p, v_1, \ldots, v_d, \ldots)$ forms an infinite regular sequence on M, then

$$X \mapsto M \otimes_{\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}} MU_0(X)$$

determines a homology theory on finite *spectra X. Moreover, if $M/I_d = 0$ for some $d \gg 0$, then the same formula determines a homology theory on* all *spectra X.*

Proof. Following the discussion in the introduction, we note that a cofiber sequence

$$X' \to X \to X''$$

of spectra gives rise to an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{M}_{MU}(X') \arrow[r] \arrow[d, equal] & \mathcal{M}_{MU}(X) \arrow[r] \arrow[d, equal] & \mathcal{M}_{MU}(X'') \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathcal{N}' \arrow[r] & \mathcal{N} \arrow[r] & \mathcal{N}'' \arrow[r] & \cdots .
\end{tikzcd}$$

We thus see that we are essentially tasked with showing that $\mathcal{F}$ is flat, so that tensoring with $\mathcal{F}$ does not disturb the exactness of this sequence. In that case, we can then apply Brown representability to the composite functor $\mathcal{F} \otimes \mathcal{M}_{MU}(X)$.

Flatness of $\mathcal{F}$ is equivalent to $\operatorname{Tor}_1(\mathcal{F}, \mathcal{N}) = 0$ for an arbitrary auxiliary quasicoherent sheaf $\mathcal{N}$ (soon to be thought of as $\mathcal{M}_{MU}(X)$). By our regularity hypothesis, there is an exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{p} \mathcal{F} \to \mathcal{F}/(p) \to 0,$$

so applying $\operatorname{Tor}_*(-, \mathcal{N})$ gives an exact sequence

$$\operatorname{Tor}_2(\mathcal{F}/(p), \mathcal{N}) \to \operatorname{Tor}_1(\mathcal{F}, \mathcal{N}) \xrightarrow{p} \operatorname{Tor}_1(\mathcal{F}, \mathcal{N})$$

of Tor groups. The sequence gives the following sufficiency condition:

$$[\operatorname{Tor}_1(p^{-1}\mathcal{F}, \mathcal{N}) = 0 \quad \text{and} \quad \operatorname{Tor}_2(\mathcal{F}/(p), \mathcal{N}) = 0] \quad \Rightarrow \quad \operatorname{Tor}_1(\mathcal{F}, \mathcal{N}) = 0.$$

Similarly, the v_1–multiplication sequence gives another sufficiency condition:

$$[\operatorname{Tor}_2(v_1^{-1}\mathcal{F}/(p), \mathcal{N}) = 0 \quad \text{and} \quad \operatorname{Tor}_3(\mathcal{F}/I_2, \mathcal{N}) = 0] \quad \Rightarrow \quad \operatorname{Tor}_2(\mathcal{F}/(p), \mathcal{N}) = 0.$$

Continuing in this fashion, for some $D \gg 0$ we would like to show

$$\begin{aligned} &\operatorname{Tor}_{d+1}(v_d^{-1}\mathcal{F}/I_d, \mathcal{N}) = 0 && \text{(for each } d < D\text{)}, \\ &\operatorname{Tor}_{D+1}(\mathcal{F}/I_{D+1}, \mathcal{N}) = 0. \end{aligned}$$

The second condition is satisfied one of two ways, corresponding to our two auxiliary hypotheses and two conclusions in the Theorem statement:

- If $\mathcal{F}$ itself satisfies $\mathcal{F}/I_{D+1} = 0$, we are done.
- Writing $j_{D+1} \colon \mathcal{M}_{\mathbf{fg}}^{\geq D+1} \to \mathcal{M}_{\mathbf{fg}}$ for the inclusion of the prime closed substack, we can identify $\mathcal{N}/I_{D+1}$ with $j_{D+1*}j_{D+1}^*\mathcal{N}$. If $\mathcal{N}$ is coherent (for instance, in the case that $\mathcal{N} = \mathcal{M}_{MU}(X)$ for a *finite* complex X), then $j_{D+1}^*\mathcal{N}$ is free for large D and hence has vanishing Tor groups.

Remark 3.5.2. This construction always gives an MUP–module spectrum in the homotopy category. If $\mathcal{F}$ comes from the pushforward of the ring of functions along a flat map $\operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$, then the resulting spectrum is a ring spectrum, and flat sheaves on it become module spectra over this ring spectrum.

Remark 3.5.3 ([Hop, Equation 20.1]). If $f \colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$, then in the pullback

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{\mathbf{fgl}} \arrow[d] \\ \operatorname{Spec} R \arrow[r, "f"] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

the sheaf $f'^*\mathcal{M}_{MU}(X)$ gives a model for $(MU \wedge R)_0(X)$ *without first defining* R_0. We can use this to define $R_0(X)$ by suitably equalizing the maps $s, t\colon (\mathcal{M}_{\mathbf{fgl}} \times \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \to \mathcal{M}_{\mathbf{fgl}}$, but forming this equalizer may not be exact. By selecting a lift $\widetilde{f}\colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fgl}}$, this equalizer becomes *forked* (i.e., we gain an extra degeneracy in the simplicial object), and the equalizer is thus guaranteed to be exact.

We then turn to the first collection of conditions. They are *always* satisfied, but this requires an argument. We write $i_d\colon \mathcal{M}_{\mathbf{fg}}^{=d} \to \mathcal{M}_{\mathbf{fg}}$ for the inclusion of the substack of formal groups of height exactly d, which (following Remark 3.3.17) has a presentation by the Hopf algebroid

$$(v_d^{-1}A/I_d, \Gamma \otimes v_d^{-1}A/I_d).$$

We are trying to study the derived functors of

$$\mathcal{N} \mapsto (i_{d*}i_d^*\mathcal{F}) \otimes \mathcal{N} \cong i_{d*}(i_d^*\mathcal{F} \otimes i_d^*\mathcal{N}).$$

Since i_{d*} is exact, we are moved to study the composite functor spectral sequence for

$$\mathsf{QCoh}_{\mathcal{M}_{\mathbf{fg}}} \xrightarrow{i_d^*} \mathsf{QCoh}_{\mathcal{M}_{\mathbf{fg}}^{=d}} \xrightarrow{i_d^*\mathcal{F}\otimes -} \mathsf{QCoh}_{\mathcal{M}_{\mathbf{fg}}^{=d}}.$$

The second functor is actually exact: the geometric map

$$\Gamma_d\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}^{=d}$$

is a faithfully flat cover, and k–modules have no nontrivial Tor. Meanwhile, the first functor has at most d derived functors: i_d^* is modeled by tensoring with $v_d^{-1}A/I_d$, but A/I_d admits a Koszul resolution with d stages and $A/I_d \to v_d^{-1}A/I_d$ is exact. As Tor_{d+1} is beyond the length of this resolution, it is always zero. □

Definition 3.5.4. Coupling Theorem 3.5.1 to our understanding of $\mathcal{M}_{\mathbf{fg}}$, we produce many interesting homology theories, collectively referred to as *chromatic*[27] *homology theories*:

[27] The elements of Figure 3.1 and Figure 3.2 are related to each other by "v_d–multiplication" (cf. Remark 3.6.22), and families of such elements can be selected for by inverting v_d, i.e., by passing to the open substack $\mathcal{M}_{\mathbf{fg}}^{\leq d}$. The word "chromatic" here thus refers to an analogy: this localization selects certain periodic families of elements, like a bandpass filter selects certain frequencies out of a complicated radio signal.

- Recall that the moduli of p–typical group laws is affine, presented in Theorem 3.3.11 by
$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots].$$
Since the inclusion of p–typical group laws into all group laws induces an equivalence of stacks, it is in particular flat, and hence this formula determines a homology theory on finite spectra, called *Brown–Peterson homology*:
$$BPP_0(X) := MUP_0(X) \otimes_{MUP_0} BPP_0.$$

- A chart for the open substack $\mathcal{M}_{\mathbf{fg}}^{\le d}$ in terms of $\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}$ was given in Remark 3.3.17 by $\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d^{\pm}]$. Since open maps are in particular flat, it follows that there is a homology theory $E(d)P$, called *the d^{th} Johnson–Wilson homology*, defined on all spectra by
$$E(d)P_0(X) := MUP_0(X) \otimes_{MUP_0} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d^{\pm}].$$

- Similarly, for a formal group Γ of height $d < \infty$, we produced in Theorem 3.4.5 a chart $\operatorname{Spf} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!]$ for its deformation neighborhood. Since inclusions of deformation neighborhoods of substacks of Noetherian stacks are flat [Mat89], there is a corresponding homology theory E_Γ, called *the (discontinuous) Morava E–theory for* Γ, determined by
$$E_{\Gamma 0}(X) := MUP_0(X) \otimes_{MUP_0} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!][u^{\pm}].$$
In the case that $\Gamma = \Gamma_d$ is the Honda formal group of height d, the notation is often abbreviated from E_{Γ_d} to merely E_d.

- Since $(p, u_1, \ldots, u_{d-1})$ forms a regular sequence on $E_{\Gamma *}$, we can form the regular quotient at the level of spectra, using cofiber sequences
$$\begin{gathered} E_\Gamma \xrightarrow{p} E_\Gamma \to E_\Gamma/(p), \\ E_\Gamma/(p) \xrightarrow{u_1} E_\Gamma/(p) \to E_\Gamma/(p, u_1), \\ \vdots \\ E_\Gamma/I_{d-1} \xrightarrow{u_{d-1}} E_\Gamma/I_{d-1} \to E_\Gamma/I_d. \end{gathered}$$

This determines a spectrum $K_\Gamma = E_\Gamma / I_d$, and hence determines a homology theory called *the Morava K–theory for* Γ. In the case where Γ comes from the Honda p–typical formal group law (of height d), this spectrum is often written as $K(d)$. As an edge case, we also set $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$.[28]

- More delicately, there is a version of Morava E–theory which takes into account the formal topology on $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$, called *continuous Morava E–theory*. It is defined by the pro-system $\{E_\Gamma(X)/u^I\}$, where I ranges over multi-indices and the quotient is again given by cofiber sequences.

- There is also a homology theory associated to the closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d}$. Since $I_d = (p, v_1, \ldots, v_{d-1})$ is generated by a regular sequence on BPP_0, we can directly define the spectrum $P(d)P$ by a regular quotient:

$$P(d)P = BP/(p, v_1, \ldots, v_{d-1}).$$

This spectrum does have the property $P(d)P_0 = BPP_0/I_d$ on coefficient rings, but $P(d)P_0(X) = BPP_0(X)/I_d$ *only* when I_d forms a regular sequence on $BPP_0(X)$—which is reasonably rare among the cases of interest.

Remark 3.5.5. The trailing "P" in these names is to disambiguate them from similar less-periodic objects in the literature. Namely, BP is often taken to be a minimal wedge summand of $MU_{(p)}$, whereas $E(d)$, E_Γ, and $K(d)$ can all be taken to be $2(p^d - 1)$–periodic (for heights $0 < d < \infty$). The one exception to this minimality convention is E_Γ, which is *usually* taken to be 2–periodic already, so we do not attach a "P" to its name.

Example 3.5.6 (cf. Example 2.1.20). In the case $\Gamma = \widehat{\mathbb{G}}_m$, the resulting spectra are connected to complex K–theory:

$$E_{\widehat{\mathbb{G}}_m} \cong KU_p^\wedge, \qquad K_{\widehat{\mathbb{G}}_m} \cong KU/p, \qquad E(1)P \cong KU_{(p)}.$$

Remark 3.5.7 ([KLW04, Section 5.2], [Rav84, Corollaries 2.14 and 2.16], [Str99a, Theorem 2.13]). In general, the quotient of a ring spectrum by a homotopy element does not give another ring spectrum. The most typical example of this phenomenon is that $\mathbb{S}/2$ is not a ring spectrum, since its homotopy is not 2–torsion. Most of our constructions above do not suffer from this deficiency, with one exception:

[28] By Theorem 3.4.1 and Corollary 3.5.15 to follow, it often suffices to consider just these spectra $K(d)$ to make statements about all K_Γ. With more care, it even often suffices to consider $d \neq \infty$.

Morava K–theories at $p = 2$ are not commutative. Instead, there is a derivation $Q_d : K(d) \to \Sigma K(d)$ which tracks the commutativity by the relation

$$ab - ba = uQ_d(a)Q_d(b).$$

In particular, we find that $K(d)^*X$ is a commutative ring whenever $K(d)^1X = 0$, which is often the case.

Remark 3.5.8. Before turning to the other half of this Lecture, we feel obligated to include one out-of-place remark that the first-time reader is advised to skip. The pullback idea expressed in Remark 3.5.3 intermingles interestingly with the detection result expressed in Remark 3.1.22: there is a certain program, which is beyond the scope of this point of the book, which realizes spectra associated to certain nontrivial *stacks* equipped with maps to the moduli of formal groups. For instance, the non-complex-orientable spectrum KO is attached to a map

Cite me: Section 3 of From Spectra to Stacks.

$$\operatorname{Spec}\mathbb{Z}//C_2 \to \mathcal{M}_{\mathbf{fg}}$$

which selects KU with its complex-conjugation action. There is a sequence of non-complex-orientable ring spectra $X(n) = T(\Omega SU(n))$ which limit to $X(\infty) = MU$ and which carry "partial complex-orientations" that give a Thom isomorphism for $\mathbb{C}\mathrm{P}^{n-1}$, i.e., they come equipped with a specified formal group law modulo terms of degree n. We write $\mathcal{M}_{\mathbf{fg}}^{(n)}$ for the moduli of formal groups where the isomorphisms are required to act trivially on space of n–jets (so that, e.g., Remark 3.2.8 concerns $\mathcal{M}_{\mathbf{fg}}^{(1)}$). The pullback of stacks

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{\mathbf{fg}}^{(2)} \arrow[d] \\ \operatorname{Spec}\mathbb{Z}//C_2 \arrow[r] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

is actually affine, and this is a witness to the fact that $X(2) \wedge KO$ is a complex-orientable ring spectrum, even if neither of KO or $X(2)$ is.

Having constructed these chromatic homology theories, for the rest of this Lecture we pursue an example of a "fiberwise" analysis of a phenomenon in homotopy theory. First, recall the following classical theorem:

Theorem 3.5.9 ([Nis73], [BMMS86, Section II.2]). *Every homotopy class $\alpha \in \pi_{\geq 1}\mathbb{S}$ is nilpotent.* $\square$

People studying K–theory in the '70s discovered the following related phenomenon:

Theorem 3.5.10 ([Ada66, Theorem 12.1]). *Let $M_{2n}(p)$ denote the mod–p Moore spectrum with bottom cell in degree $2n$. Then there is an index n and a map $v : M_{2n}(p) \to M_0(p)$ such that $KU_* v$ acts by multiplication by the n^{th} power of the Bott class. The minimal such n is given by the formula*

$$n = \begin{cases} p-1 & \textit{when } p \geq 3, \\ 4 & \textit{when } p = 2. \end{cases} \qquad \square$$

In particular, the map v cannot be nilpotent, since a null-homotopic map induces the zero map in any homology theory. Just as we took the non-nilpotent endomorphism $p \in \pi_0 \operatorname{End} \mathbb{S}$ and coned it off, we can take the endomorphism $v \in \pi_{2p-2} \operatorname{End} M_0(p)$ and cone it off to form a new spectrum called $V(1)$.[29] One can ask, then, whether the pattern continues: does $V(1)$ have a non-nilpotent self-map, and can we cone it off to form a new such spectrum with a new such map? Can we then do that again, indefinitely? In order to study this question, we are motivated to find spectra satisfying the following condition:

Definition 3.5.11 ([HS98, Definition 4], cf. [DHS88, Theorem 1]). A ring spectrum E *detects nilpotence* if for any ring spectrum R the kernel of the Hurewicz homomorphism

$$R_* \eta_E \colon \pi_* R \to E_* R$$

consists of nilpotent elements. (In particular, such an E cannot send such a nontrivial self-map to zero.)

This question and surrounding issues formed the basis of Ravenel's nilpotence conjectures [Rav84, Section 10], which were resolved by by Devinatz, Hopkins, and Smith [DHS88, HS98]. One of their two main technical achievements was to demonstrate that we already have access to a nice homology theory which detects nilpotence:

Theorem 3.5.12 ([DHS88, Theorem 1.i]). *The spectrum MU detects nilpotence.* $\square$

[29] The spectrum $V(1)$ is actually defined to be a finite spectrum with $BP_* V(1) \cong BP_*/(p, v_1)$. At $p = 2$ this spectrum doesn't exist and this is a misnomer. More generally, at odd primes p Nave shows that $V((p+1)/2)$ doesn't exist [Nav10, Theorem 1.3].

This is a very hard theorem, and we will not attempt to prove it.[30] However, taking this as input, they are easily able to show several other interesting structural results about finite spectra. For instance, they also show that the MU is the universal object which detects nilpotence, in the sense that any other ring spectrum can have this property checked stalkwise on $\mathcal{M}_{MU}$.

Corollary 3.5.13 ([HS98, Theorem 3])**.** *A ring spectrum E detects nilpotence if and only if for all $0 \leq d \leq \infty$ and for all primes p, $K(d)_* E \neq 0$ (i.e., the support of $\mathcal{M}_{MU}(E)$ is not a proper substack of $\mathcal{M}_{MU}$).*

Proof. If $K(d)_* E = 0$ for some d, then the non-nilpotent unit map $\mathbb{S} \to K(d)$ lies in the kernel of the Hurewicz homomorphism for E, so E fails to detect nilpotence.

In the other direction, suppose that for every d we have $K(d)_* E \neq 0$. Because $K(d)_*$ is a field, it follows by picking a basis of $K(d)_* E$ that $K(d) \wedge E$ is a nonempty wedge of suspensions of $K(d)$. So, for $\alpha \in \pi_* R$, if $E_* \alpha = 0$ then $(K(d) \wedge E)_* \alpha = 0$ and hence $K(d)_* \alpha = 0$. So, we need to show that if $K(d)_* \alpha = 0$ for all n and all p then α is nilpotent. Taking Theorem 3.5.12 as given, it would suffice to show merely that $MU_* \alpha$ is nilpotent. This is equivalent to showing that the ring spectrum $MU \wedge R[\alpha^{-1}]$ is contractible or that the unit map is null:

$$\mathbb{S} \to MU \wedge R[\alpha^{-1}].$$

A nontrivial result of Johnson and Wilson shows that if $MU_* X = 0$, then for any d we have $K([0, d])_* X = 0$ and $P(d+1)_* X = 0$.[31] Taking $X = R[\alpha^{-1}]$, we have

[30] In particular, this is a very hard *geometric* theorem. Its proof is phrased homotopically, but it comes down to very concrete, computational facts about such geometric objects as double-loopspaces of spheres. (One unfulfilled daydream of this book is to re-encode these computations in the language of formal geometry.) One way of understanding the contents of the Theorem is that the spectral sequences pictured in Figure 3.1 and Figure 3.2 have *asymptotically flat* vanishing curves, so that powers of any particular element eventually escape the populated region of the spectral sequence. (In fact, this vanishing curve is conjectured to be asymptotically equivalent to a square root function, but little about this is known.) On the other hand, you can see from the E_2 page of the 2–primary spectral sequence that such a vanishing line is *not* initially present, due to the nonnilpotence of η—and as a consequence, there is no analogue of the nilpotence theorems for $\mathsf{QCoh}(\mathcal{M}_{\mathbf{fg}})$ itself. This is another sense in which it is a fundamentally geometric fact. It is also not even true at the E_∞ page of Figure 1.2, where the "image of J pattern" (the name for the "shark fins" near the main diagonal) give a family of permanent cycles along a line of slope 1.

[31] Specifically, it is immediate that $MU_* X = 0$ forces $P(d+1)_* X = 0$ and $v_{d'}^{-1} P(d')_*(X) = 0$ for all $d' < d$. What's nontrivial is showing that $v_{d'}^{-1} P(d')_*(X) = 0$ if and only if $K(d')_*(X) = 0$ [Rav84, Theorem 2.1.a], [JW75, Section 3].

assumed all of these are zero except for $P(d+1)$. But $\operatorname{colim}_d P(d+1) \simeq H\mathbb{F}_p \simeq K(\infty)$, and $\mathbb{S} \to K(\infty) \wedge R[\alpha^{-1}]$ is assumed to be null as well. By compactness of $\mathbb{S}$, that null-homotopy factors through some finite stage $P(d+1) \wedge R[\alpha^{-1}]$ with $d \gg 0$. □

Corollary 3.5.13 has the following consequence, which speaks to the primacy of both the chromatic program and these results.

Definition 3.5.14. A ring spectrum R is a *field spectrum* when every R–module (in the homotopy category) splits as a wedge of suspensions of R. (Equivalently, R is a field spectrum when it has Künneth isomorphisms.)

Corollary 3.5.15. *Every field spectrum R splits as a wedge of Morava's $K(d)$ theories.*

Proof. It is easy to check (as mentioned in the proof of Corollary 3.5.13) that $K(d)$ is a field spectrum.

Now, consider an arbitrary field spectrum R. Set $E = \bigvee_{\text{primes } p} \bigvee_{d \in [0,\infty]} K(d)$, so that E detects nilpotence. The class 1 in the field spectrum R is non-nilpotent, so it survives when paired with some K–theory $K(d)$, and hence $R \wedge K(d)$ is not contractible. Because both R and $K(d)$ are field spectra, the smash product of the two simultaneously decomposes into a wedge of $K(d)$s and a wedge of Rs. So, R is a retract of a wedge of $K(d)$s, and picking a basis for its image on homotopy shows that it is a sub-wedge of $K(d)$s. □

Remark 3.5.16. In the 2–periodic setting we've become accustomed to, the analogue of Corollary 3.5.15 is that every 2–periodic field spectrum splits as a wedge of suspensions of $K(d)P$.

Remark 3.5.17. In service of Example 3.5.6, the geometric definition of MU given in Lemma 0.0.1, the edge cases of $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, and the claimed primacy of these methods, we might wonder if there is any geometric interpretation of the field theories $K(d)$ for $0 < d < \infty$. To date, this is a completely open question and the subject of intense research.

We're now well-situated to address Ravenel's question about finite spectra and periodic self-maps. The key observation is that spectra admitting such self-maps are closed under some natural operations, leading to the following definition:

Definition 3.5.18. A full subcategory of a triangulated category (e.g., the homotopy category of p–local finite spectra) is *thick* if...

- ...it is closed under isomorphisms and retracts.
- ...it has a 2-out-of-3 property for cofiber sequences.

Examples of thick subcategories include:

- The category C_d of p–local finite spectra which are $K(d-1)$–acyclic. (For instance, if $d = 1$, the condition of $K(0)$–acyclicity is that the spectrum have purely torsion homotopy groups.) These are called "finite spectra of type at least d".
- The category D_d of p–local finite spectra F for which there is a self-map $v : \Sigma^N F \to F$, $N \gg 0$ which induces multiplication by a unit in $K(d)$–homology and which is nilpotent in $K(\neq d)$–homology. These are called "finite spectra admitting v_d–self–maps".

The categories C_d are the ones we are interested in analyzing, and we hope to identify these putative spaces $V(d)$ inside of them. Ravenel shows the following foothold interrelating the C_d:

Lemma 3.5.19 ([Rav84, Theorem 2.11])**.** *For X a finite complex, there is a bound*

$$\dim K(d-2)_* X \leq \dim K(d-1)_* X.$$

In particular, there is an inclusion $\mathsf{C}_{d-1} \subseteq \mathsf{C}_d$.

Proof sketch. One should compare this with the statement that the stalk dimension of a coherent sheaf is upper semi-continuous. In fact, this analogy gives the essentials of Ravenel's proof: one considers the ring spectrum $v_d^{-1}BP/I_{d-1}$, which admits two maps

$$\begin{tikzcd} & v_{d-1}^{-1}(v_d^{-1}BP/I_{d-1}) \\ v_d^{-1}BP/I_{d-1} \arrow[ur] \arrow[dr] & \\ & (v_d^{-1}BP/I_{d-1})/v_{d-1}. \end{tikzcd}$$

Studying the relevant Tor spectral sequences gives the result. □

Hopkins and Smith are able to use their local nilpotence detection result, Corollary 3.5.13, to completely understand the behavior not only of the thick subcategories C_d but of *all* thick subcategories of $\mathsf{Spectra}^{\mathrm{fin}}_{(p)}$. In particular, this connects the C_d with the D_d, as we will see.

Theorem 3.5.20 ([HS98, Theorem 7])**.** *Any thick subcategory C of the category of p–local finite spectra must be C_d for some d.*

Proof. Since C_d are nested by Lemma 3.5.19 and they form an exhaustive filtration (i.e., $\mathsf{C}_\infty = 0$), it is thus sufficient to show that any object $X \in \mathsf{C}$ with $X \in \mathsf{C}_d$ induces an inclusion $\mathsf{C}_d \subseteq \mathsf{C}$. Write R for the endomorphism ring spectrum $R = F(X, X)$, and write F for the fiber of its unit map:

$$F \xrightarrow{f} \mathbb{S} \xrightarrow{\eta_R} R.$$

Finally, let $Y \in \mathsf{C}_d$ be *any* finite spectrum of type at least d. Our goal is to demonstrate $Y \in \mathsf{C}$.

Now consider applying $K(n)$–homology (for *arbitrary* n) to the map

$$1 \wedge f \colon Y \wedge F \to Y \wedge \mathbb{S}.$$

The induced map is always zero:

- In the case that $K(n)_*X$ is nonzero, then $K(n)_*\eta_R$ is injective because $K(n)_*$ is a graded field, and so $K(n)_*f$ is zero.
- In the case that $K(n)_*X$ is zero, then $n \leq d$ and, because of the bound on type, $K(n)_*Y$ is zero as well.

By a small variant of local nilpotence detection (Corollary 3.5.13, [HS98, Corollary 2.5]), it follows for $j \gg 0$ that

$$Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}$$

is null-homotopic. Hence, one can calculate the cofiber to be

$$\mathrm{cofib}\left(Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}\right) \simeq Y \wedge \mathrm{cofib}\, f^{\wedge j} \simeq Y \vee (Y \wedge \Sigma F^{\wedge j}),$$

so that Y is a retract of this cofiber.

We now work to show that this smash product lies in the thick subcategory C of interest. First, note that it suffices to show that $\operatorname{cofib} f^{\wedge j}$ on its own lies in C: a finite spectrum (such as Y or F) can be expressed as a finite gluing diagram of spheres, and smashing this through with $\operatorname{cofib} f^{\wedge j}$ then expresses $Y \wedge \operatorname{cofib} f^{\wedge j}$ as the iterated cofiber of maps with source and target in C. With that in mind, we will in fact show that $\operatorname{cofib} f^{\wedge k}$ lies in C for all $k \geq 1$. Consider the following smash version of the octahedral axiom: the factorization

$$F \wedge F^{\wedge(k-1)} \xrightarrow{1 \wedge f^{\wedge(k-1)}} F \wedge \mathbb{S}^{\wedge(k-1)} \xrightarrow{f \wedge 1} \mathbb{S} \wedge \mathbb{S}^{\wedge(k-1)}$$

begets a cofiber sequence

$$F \wedge \operatorname{cofib} f^{\wedge(k-1)} \to \operatorname{cofib} f^{\wedge k} \to \operatorname{cofib} f \wedge \mathbb{S}^{\wedge(k-1)}.$$

Noting that the base case $\operatorname{cofib}(f) = R = X \wedge DX$ lies in C, we can inductively use the 2-out-of-3 property on the octahedral cofiber sequence to see that $\operatorname{cofib}(f^{\wedge k})$ lies in C for all k. It follows in particular that $Y \wedge \operatorname{cofib}(f^{\wedge j})$ lies in C, and using the retraction Y belongs to C as well. □

Theorem 3.5.21 ([HS98, Theorem 4.11, Section 5, and Theorem 9])**.** *A p–local finite spectrum is $K(d-1)$–acyclic exactly when it admits a v_d–self–map. Additionally, the inclusion $\mathsf{C}_d \subsetneq \mathsf{C}_{d-1}$ is proper.*

Executive summary of proof. Given the classification of thick subcategories, if a property is closed under thickness then one need only exhibit a single spectrum with the property to know that all the spectra in the thick subcategory it generates also all have that property. Inductively, they manually construct finite spectra $M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}})$ for sufficiently large[32] indices i_* which admit a self-map v governed by a commuting square

$$\begin{array}{ccc} BP_* M_{|v_d| i_d}(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{v} & BP_* M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) \\ \| & & \| \\ \Sigma^{|v_d| i_d} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{-\cdot v_d^{i_d}} & BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}). \end{array}$$

These maps are guaranteed by very careful study of Adams spectral sequences. □

[32]We ran into the asymptotic condition $I \gg 0$ earlier, when we asserted that there is no root of the 2–local v_1–self–map $v\colon M_8(2) \to M_0(2)$.

They therefore conclude the strongest possible positive response to Ravenel's conjectures. Not only can we continue the sequence

$$\mathbb{S},\ \mathbb{S}/p,\ \mathbb{S}/(p,v),\ \ldots,$$

but in fact *any* finite spectrum admits an (essentially unique) interesting periodic self-map. This is maybe the most remarkable of the statements: although Nishida's theorem initially led us to think of periodic self-maps as rare, they are in fact ubiquitous. Additionally, we learned that the shift[33] of this self-map is determined by the first nonvanishing $K(d)$–homology, giving an effective detection tool. Finally, all such periodicity shifts arise: for any d, there is a spectrum admitting a v_d–self–map but not a v_{d-1}–self–map.

3.6 Chromatic disassembly

In this Lecture, we will couple the ideas of Lecture 3.1 to the homology theories and structure theorems described in Lecture 3.5. In particular, we have not yet exhausted Theorem 3.5.20, and for inspiration about how to utilize it, we will begin with an algebraic analogue of the situation considered thus far.

For a ring R, the full derived category $D(\operatorname{Spec} R)$ and the derived category of perfect complexes $D^{\mathrm{perf}}(\operatorname{Spec} R)$ form examples of triangulated categories analogous to $\mathsf{Spectra}$ and $\mathsf{Spectra}^{\mathrm{fin}}$. By interpreting an R–module as a quasicoherent sheaf over $\operatorname{Spec} R$, we can use them to probe for structure of $\operatorname{Spec} R$—for instance, we can test whether $\widetilde{M}$ is supported over some closed subscheme $\operatorname{Spec} R/I$ by restricting the sheaf, which amounts algebraically to asking whether M is annihilated by I. In the reverse, we can also try to discern what "closed subscheme" should mean in some arbitrary triangulated category by codifying the properties of the subcategory of $D(\operatorname{Spec} R)$ supported away from $\operatorname{Spec} R$. The key observation is this subcategory is closed under tensoring modules: if M is annihilated by I, then $M \otimes_R N$ is also annihilated by I.

Definition 3.6.1 ([Bal10, Definition 1.3])**.** Let C be a triangulated $\otimes$–category C. A thick subcategory $\mathsf{C}' \subseteq \mathsf{C}$ is…

- …a $\otimes$–*ideal* when $x \in \mathsf{C}'$ forces $x \otimes y \in \mathsf{C}'$ for any $y \in \mathsf{C}$.

[33] This is sometimes referred to as the "wavelength" in the chromatic analogy.

- ... a *prime* $\otimes$*–ideal* when $x \otimes y \in \mathsf{C}'$ also forces at least one of $x \in \mathsf{C}'$ or $y \in \mathsf{C}'$.

Finally, define the *spectrum* of C to be its collection of prime $\otimes$–ideals. For any $x \in \mathsf{C}$ we define a basic open $U(x) = \{\mathsf{C}' \mid x \in \mathsf{C}'\}$, which altogether give a basis for a topology on the spectrum.

The basic result about this definition is that it does not miss any further conditions:

Theorem 3.6.2 ([Bal10, Proposition 8.1]). *The spectrum of* $D^{\mathrm{perf}}(\operatorname{Spec} R)$ *is naturally homeomorphic to the Zariski spectrum of* R. □

Satisfied, we apply the definition to the more difficult case of Spectra.

Theorem 3.6.3 ([Bal10, Corollary 9.5]). *The spectrum of* $\mathsf{Spectra}^{\mathrm{fin}}_{(p)}$ *consists of the thick subcategories* C_d*, and* $\{\mathsf{C}_n\}_{n=0}^{d}$ *are its open sets.*

Proof. Using Theorem 3.5.21, we can characterize C_d as the kernel of $K(d-1)_*$. This shows it to be a prime $\otimes$–ideal:

$$K(d-1)_*(X \wedge Y) \cong K(d-1)_* X \otimes_{K(d-1)_*} K(d-1)_* Y$$

is zero exactly when at least one of X and Y is $K(d-1)$–acyclic. □

Corollary 3.6.4 (cf. Theorem 3.5.20 and Theorem 3.5.21). *The functor*

$$\mathcal{M}_{MU}(-)\colon \mathsf{Spectra}^{\mathrm{fin}} \to \mathsf{Coh}(\mathcal{M}_{MU})$$

induces[34] *a homeomorphism of the spectrum of* $\mathsf{Spectra}^{\mathrm{fin}}$ *to that of* $\mathcal{M}_{\mathbf{fg}}$. □

The construction as we have described it falls short of completely recovering Spec R, as we have constructed only a topological space rather than a locally ringed space (or anything otherwise equipped locally with algebraic data, as in our functor of points perspective). The approach taken by Balmer [Bal10, Section 6] is to use Tannakian reconstruction to extract a structure sheaf of local rings from the prime $\otimes$–ideal subcategories. We, however, are at least as interested in finite spectra as we are the ring spectrum $\mathbb{S}$, so we will take an approach that emphasizes module categories rather than local rings. Specifically, Bousfield's theory of homological localization allows us to lift the localization structure among open substacks of $\mathcal{M}_{\mathbf{fg}}$ to the category Spectra as follows:

[34] This has to be interpreted delicately, as the functor $\mathcal{M}_{MU}(-)$ is not (quite) a functor of triangulated categories [Mor07a, 2.4.2].

Theorem 3.6.5 ([Bou79], [Mar83, Theorem 7.7]). *Let $j\colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$ be a flat map, and let R_* denote the homology theory associated to it by Theorem 3.0.1. There is then a diagram*

$$\begin{tikzcd} \mathsf{Spectra}_R \arrow[rr, "\mathcal{M}_R(-)", "\text{conservative}"'] \arrow[d, shift left, "i"] & & \mathsf{QCoh}(\mathcal{M}_R) \arrow[d, shift left, "j_*"] \\ \mathsf{Spectra} \arrow[u, shift left, "L_R"] \arrow[urr, "\mathcal{M}_R(-)"] \arrow[rr, "\mathcal{M}_{MU}(-)"] & & \mathsf{QCoh}(\mathcal{M}_{MU}), \arrow[u, shift left, "j^*"] \end{tikzcd}$$

such that L_R is left-adjoint to i, j^* is left-adjoint to j_*, i and j_* are inclusions of full subcategories, L_R and j^* are idempotent, the red composites are all equal, and R_* is conservative on $\mathsf{Spectra}_R$.[35] $\square$

The idea, then, is that $\mathsf{Spectra}_R$ plays the topological role of the derived category of those sheaves supported on the image of the map j. In Definition 3.5.4, we identified several classes of interesting such maps j tied to the geometry of $\mathcal{M}_{\mathbf{fg}}$. We record these special cases now:

Definition 3.6.6. In the case that $R = E_\Gamma$ models the inclusion of the deformation space around the point Γ, we will denote the localizer by L_Γ. In the special case that $\Gamma = \Gamma_d$ is taken to be the Honda formal group, we further abbreviate the localizer by

$$\mathsf{Spectra} \xrightarrow{\widehat{L}_d} \mathsf{Spectra}_{\Gamma_d}.$$

In the case when $R = E(d)$ models the inclusion of the open complement of the unique closed substack of codimension d, we will denote the localizer by

$$\mathsf{Spectra} \xrightarrow{L_d} \mathsf{Spectra}_d = \mathsf{Spectra}_{\mathcal{M}_{\mathbf{fg}}^{\leq d}}.$$

These localizers have a number of nice properties linking them to algebraic models.

---

[35]The meat of this theorem is in overcoming set-theoretic difficulties in the construction of $\mathsf{Spectra}_R$. Bousfield accomplished this by describing a model structure on Spectra for which R–equivalences create the weak–equivalences.

Lemma 3.6.7. *There are natural factorizations*

$$\mathrm{id} \to L_d \to L_{d-1}, \qquad \mathrm{id} \to L_d \to \widehat{L}_d.$$

In particular, $L_d X = 0$ implies both $L_{d-1} X = 0$ and $\widehat{L}_d X = 0$.

Analogy to $j_* \vdash j^*$. The open substack of dimension d properly contains both the open substack of dimension $(d-1)$ and the infinitesimal deformation neighborhood of the geometric point of height d. The factorization is inclusions gives a factorization of pullback functors. □

Lemma 3.6.8 ([Rav92, Theorem 7.5.6], [Hov95, Proof of Lemma 2.3])**.** *There are equivalences*

$$L_d X \simeq (L_d \mathbb{S}) \wedge X, \qquad \widehat{L}_d X \simeq \lim_I \left(M_0(v^I) \wedge L_d X \right).$$

Analogy to $j_* \vdash j^*$. The first formula stems from j an open inclusion, which has $j^* M \simeq R \otimes M$ in the algebraic setting. The second formula can be compared to the inclusion j of the formal infinitesimal neighborhood of a closed subscheme, which has algebraic model $j^* M = \lim_j (R/I^j \otimes M)$.[36] □

Lemma 3.6.9. *Let k be a field of positive characteristic p, and let Γ and Γ' be two formal groups over k of differing heights $0 \le d, d', \le \infty$. Then $K_\Gamma \wedge K_{\Gamma'} \simeq 0$.*

Analogy to $j_* \vdash j^*$. The map classifying the formal group $\mathbb{C}\mathrm{P}^\infty_{K_\Gamma \wedge K_{\Gamma'}}$ simultaneously factors through the maps classifying the formal groups $\mathbb{C}\mathrm{P}^\infty_{K_\Gamma} = \Gamma$ and $\mathbb{C}\mathrm{P}^\infty_{K_{\Gamma'}} = \Gamma'$. By Lemma 3.3.4, such a formal group must simultaneously have heights d and d', which forces the homotopy ring to be the zero ring.[37] □

Lemma 3.6.10 ([Lura, Lemma 23.6])**.** *For $d > \operatorname{ht}\Gamma$, $\widehat{L}_\Gamma L_d \simeq 0$.*

Proof sketch. After a nontrivial reduction argument, this comes down to an identical fact: the formal group associated to $E(d) \wedge K_\Gamma$ must simultaneously be of heights at most d and exactly $\operatorname{ht}\Gamma > d$, which forces the spectrum to vanish. □

[36] In keeping with our discussion of continuous Morava E–theory, it is also possible to consider the object $\{(M_0(v^I) \wedge L_d X)\}_I$ itself as a pro-spectrum. This is interesting to explore: Davis and Lawson have shown that setting $X = \mathbb{S}$ gives an E_∞ pro-spectrum, even though none of the individual objects are E_∞ ring spectra themselves [DL14].

[37] Alternatively, Corollary 3.5.15 shows that $K_\Gamma \wedge K_{\Gamma'}$ simultaneously decomposes as a wedge of K_Γs and of $K_{\Gamma'}$, which forces both wedges to be empty.

Corollary 3.6.11. *$L_\Gamma E = 0$ for any coconnective E, and hence $L_\Gamma E = L_\Gamma(E[n,\infty))$ for* any *spectrum E and* any *index n.*[38]

Proof. Any coconnective spectrum can be expressed as the colimit of its truncations

$$\begin{tikzcd} E[n,n] \arrow[r] \arrow[d, equal] & E[n-1,n] \arrow[r] \arrow[d] & E[n-2,n] \arrow[r] \arrow[d] & \cdots \arrow[r, "\mathrm{colim}"] & E(-\infty,n] \\ \Sigma^n H\pi_n E & \Sigma^{n-1} H\pi_{n-1} E & \Sigma^{n-2} H\pi_{n-2} E & \cdots . \end{tikzcd}$$

Applying L_Γ preserves this colimit diagram, but the above argument shows that HA is L_Γ–acyclic for any abelian group A. This gives the statement about coconnective spectra, from which the general statement follows by considering the cofiber sequence

$$E[n,\infty) \to E \to E(-\infty,n).$$ □

Corollary 3.6.12 ([Lura, Proposition 23.5])**.** *There are a homotopy pullback squares*

$$\begin{tikzcd} L_d X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{L}_d X \arrow[d] \\ L_{d-1} X \arrow[r] & L_{d-1}\widehat{L}_d X, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} . \end{tikzcd}$$

Analogy to $j_* \vdash j^*$. For the left-hand square, the inclusion of the open substack of dimension $d-1$ into the one of dimension d has relatively closed complement the point of height d. Algebraically, this gives a Mayer-Vietoris sequence with analogous terms. The right-hand square is analogous to the adèlic decomposition of abelian groups.[39] □

Remark 3.6.13. Corollary 3.6.12 is maybe the most useful result discussed in this Lecture. It shows that a map to an L_d–local spectrum can be understood as a system of compatible maps to its $\widehat{L}_j$–localizations, $j \leq d$. In turn, any map into an $\widehat{L}_j$–local object factors through the $\widehat{L}_j$–localization of the source. Thus, if the source itself

[38] This property has the memorable slogan that Morava K–theories remember the "germ at ∞" of E.

[39] Whenever $L_B L_A = 0$, $L_{A\vee B}$ appears as the homotopy pullback of the cospan $L_A \to L_A L_B \leftarrow L_B$. Hence, this follows from Lemma 3.6.10, as well as the identification $L_{E(d-1)\vee K(d)} \simeq L_{E(d)}$.

has chromatic properties, this often puts *very* strong restrictions on how maps to the original target can behave.

These functors and their properties listed thus far give a tight analogy between certain local categories of spectra and sheaves supported on particular submoduli of formal groups, in a way that lifts the six-functors formalism of $j_* \vdash j^*$ to the level of spectra. With this analogy in hand, however, one is led to ask considerably more complicated questions whose proofs are not at all straightforward. For instance, a useful fact about coherent sheaves on $\mathcal{M}_{\mathbf{fg}}$ is that they are completely determined by their restrictions to all of the open submoduli. The analogous fact about finite spectra is referred to as *chromatic convergence*:

Theorem 3.6.14 ([Rav92, Theorem 7.5.7]). *The homotopy limit of the tower*

$$\cdots \to L_d F \to L_{d-1} F \to \cdots \to L_1 F \to L_0 F$$

recovers the p–local homotopy type of any finite spectrum F.[40,41,42] □

In addition to furthering the analogy, Theorem 3.6.14 suggests a method for analyzing the homotopy groups of spheres: we could study the homotopy groups of each $L_d\mathbb{S}$ and perform the reassembly process encoded by this inverse limit. Additionally, Corollary 3.6.12 shows that this process is inductive: $L_d\mathbb{S}$ can be understood in terms of the spectrum $L_{d-1}\mathbb{S}$, the spectrum $\widehat{L}_d\mathbb{S}$, and some gluing data in the form of $L_{d-1}\widehat{L}_d\mathbb{S}$. Hence, we become interested in the homotopy of $\widehat{L}_d\mathbb{S}$, which is the target of the E_d–Adams spectral sequence considered in Lecture 3.1.

Theorem 3.6.15 (Lemma 3.1.15, see also Example 2.3.4, Definition 3.1.9, and Definition 3.1.13). *The E_d–based*[43] *Adams spectral sequence for the sphere converges strongly to $\pi_*\widehat{L}_d\mathbb{S}$. Writing ω for the line bundle on $\mathcal{M}_{E_d}$ of invariant differentials, we have*

$$E_2^{*,*} = H^*(\mathcal{M}_{E_d}; \omega^{\otimes *}) \Rightarrow \pi_*\widehat{L}_d\mathbb{S}. \qquad \square$$

[40] Spectra satisfying this limit property are said to be *chromatically complete*, which is closely related to being *harmonic*, i.e., being local with respect to $\bigvee_{d=0}^{\infty} K(d)$. (I believe this a joke about "music of the spheres".) It is known that nice Thom spectra are harmonic [Kří94] (so, in particular, every suspension and finite spectrum), that every finite spectrum is chromatically complete, and that there exist some harmonic spectra which are not chromatically complete [Bar, Section 5.1].

[41] A consequence of the fact that p–local finite spectra are $\bigvee_{d<\infty} K(d)$–local, there are no nontrivial maps $H\mathbb{F}_p \to F$, and in particular the Spanier–Whitehead dual of $H\mathbb{F}_p$ is null.

[42] While we're talking about $\bigvee_{d<\infty} K(d)$, the cofiber of $\bigvee_{d<\infty} K(d) \to \prod_{d<\infty} K(d)$, where we are using the $2(p^d-1)$–periodic spectra, is concentrated in degree zero since all other homotopy degrees carry contributions from only finitely many factors. It follows that the cofiber is Eilenberg–Mac Lane—an unusual property.

[43] Although the $K(d)$–Adams spectral sequence more obviously targets $\widehat{L}_d\mathbb{S}$, we have chosen

The utility of this Theorem is in the identification of the stack $\mathcal{M}_{E_d} \cong (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma_d}$ from Definition 3.5.4. Our algebraic analysis from Theorem 3.4.5 and Remark 3.4.7 shows a further identification

$$\mathcal{M}_{E_{\Gamma_d}} = \left(\mathcal{M}_{\mathbf{fg}}\right)^{\wedge}_{\Gamma_d} \simeq \widehat{\mathbb{A}}^{d-1}_{\mathbb{W}(k)} /\!/ \underline{\mathrm{Aut}}(\Gamma_d).$$

This computation is thus boiled down to a calculation of the cohomology of the $\mathrm{Aut}(\Gamma_d)$–representations arising via Remark 3.4.14 as the global sections of the sheaves $\omega^{\otimes *}$ (cf. the discussion in Example 1.4.11 and Example 1.4.18).[44] We will later deduce the following polite description of $\mathrm{Aut}\,\Gamma_d$:

Theorem 3.6.16 (cf. Example 4.4.13)**.** *For Γ_d the Honda formal group law of height d over a perfect field k of positive characteristic p, we compute*

$$\mathrm{Aut}\,\Gamma_d \cong \left(\mathbb{W}_p(k)\langle S\rangle \middle/ \left(\begin{array}{c} Sw = w^{\varphi}S, \\ S^d = p \end{array} \right)\right)^{\times},$$

where φ denotes a lift of the Frobenius from k to $\mathbb{W}_p(k)$. □

Remark 3.6.17 ([Strb, Section 24], [DH95]). As a matter of emphasis, this Theorem does not give any description of the *representation* of $\mathrm{Aut}\,\Gamma_d$, which is very complicated (cf. Remark 4.4.23). Here are some basic facts about it:

- The center of $\mathrm{Aut}\,\Gamma_d$ is given by $\mathbb{Z}_p^{\times}$, the subgroup spanned by the multiplication–by–n maps for various n. This subgroup acts trivially on the deformation space.

---

to analyze the E_d–Adams spectral sequence above because $K(d)$ fails to satisfy **CH**. Starting with $BPP_0BPP \cong BPP_0[t_0^{\pm}, t_1, t_2, \ldots]$ from Definition 3.5.4 and Corollary 3.3.15, we can calculate $E(d)P_0E(d)P$ by base-changing this Hopf algebroid: $E(d)P_0E(d)P = E(d)P_0 \otimes_{BPP_0} BPP_0BPP \otimes_{BPP_0} E(d)P_0$, which is again free over $E(d)P_0$. Since $K(d)P$ is formed from $E(d)P$ by quotienting by a regular sequence, we calculate that $K(d)P_0E(d)P$ is free over $K(d)P_0$, generated by the same summands. However, when quotienting by the regular sequence *again* to form $K(d)P_*K(d)P$, the maps in the quotient sequences act by elements in $I_d = 0$, hence introduce Bocksteins. The end result is

$$K(d)P_*K(d)P = (K(d)P_* \otimes_{BPP_*} BPP_*BPP \otimes_{BPP_*} K(d)P_*) \otimes \Lambda[\tau_0, \ldots, \tau_{d-1}],$$

where τ_j in degree 1 controls the cofiber of $E(d)P \xrightarrow{v_j} E(d)P$. This pattern seems to be quite generic: whenever odd-primary information appears near the formal geometric picture, it comes from a poorly behaved homotopical quotient: some element was killed twice, say, or in some specific case a map was zero but is generically nonzero. So it is here.

[44]In fact, the stable *operations* of E_d take the form of the twisted group-ring $E_d^0E_d = E_d^0\langle\langle \mathrm{Aut}(\Gamma_d)\rangle\rangle$.

- More generally, one can show that the action of $\mathbb{W}_p(k)$ extends this action. It is the stabilizer of the *canonical lift*, which is the $\mathbb{W}_p(k)$–point of Lubin–Tate space given by sending the generators v_j to zero, $j \geq 1$.
- The formula for the action of $a = \sum_{j=0}^{n-1} a_j S^j$ on the tangent space is specified by the linear system

$$a^* \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} \equiv a_0 \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2}^p & \cdots & a_0^{p^n} \end{pmatrix}^{-1} \cdot \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} \pmod{\mathfrak{m}^2}.$$

- Plenty more information can be found in the work of Devinatz and Hopkins [DH95], which is best digested after Lecture 4.4.

Remark 3.6.18. The arithmetically-minded reader might recognize this description of $\operatorname{Aut}\Gamma_d$ as the group of units of a maximal order $\mathbf{o}_D$ in the division algebra D of Brauer–Hasse invariant $1/d$ over k—another glimpse of arithmetic geometry poking through to affect stable homotopy theory.[45]

Example 3.6.19 (Adams, [HMS94, Lemma 2.5]). In the case $d = 1$, the objects involved are small enough that we can compute them by hand. To begin, we have an isomorphism $\operatorname{Aut}(\Gamma_1) = \mathbb{Z}_p^\times$, and the action of this group on $\pi_* E_1 = \mathbb{Z}_p[u^\pm]$ is by $\gamma \cdot u^n \mapsto \gamma^n u^n$. At odd primes p, one computes[46]

$$H^s(\operatorname{Aut}(\Gamma_1); \pi_* E_1) = \begin{cases} \mathbb{Z}_p & \text{when } s = 0, \\ \bigoplus_{j=2(p-1)k} \mathbb{Z}_p\{u^j\}/(pku^j) & \text{when } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

[45] This finally explains our preference for using the letter "d" to represent the height of a formal group—the "d" (or, rather the "D") stands for "division algebra". The typical algebraic topologist writes "n" for the height, a trend set by Jack Morava, allegedly because the localizer "L_n" is a homophone of his wife's name, Ellen. It has been further alleged that the "E" in "Morava E–theory" is also an abbreviation for Ellen. With no disrespect meant to either of them, I find "d" to be a considerably better mnemonic and to be less likely to conflict with other indices. "E" is also prone to collisions (with spectral sequences, with operads, ...), but there is no compelling alternative.

[46] At odd primes, p is coprime to the order of the torsion part of $\mathbb{Z}_p^\times$. At $p = 2$, this is not true, so the representation has infinite cohomological dimension and there is plenty of room for differentials in the ensuing $E_{\widehat{\mathbb{G}}_m}$–Adams spectral sequence..

This, in turn, gives the calculation[47]

$$\pi_t \widehat{L}_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}_p & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = k|v_1| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

With this in hand, we can compute the homotopy of the rest of the fracture square:

$$\begin{tikzcd} \pi_* L_1 \mathbb{S} \arrow[r] \arrow[d] & \mathbb{Z}_p \oplus \bigoplus_{t=k|v_1|-1} \Sigma^t \mathbb{Z}_p/(pk) \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q}_p \oplus \Sigma^{-1}\mathbb{Q}_p, \end{tikzcd}$$

from which we deduce

$$\pi_t L_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}_{(p)} & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = k|v_1| - 1 \text{ and } t \neq 0, \\ \mathbb{Z}/p^\infty & \text{when } t = (0 \cdot |v_1| - 1) - 1 = -2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.6.20 ([Rezc, Example 7.18]). We can also give an explicit chromatic analysis of the homotopy element $\eta \in \pi_1 \mathbb{S}$ studied in Lecture 1.4. As before, consider the complex $\mathbb{C}\mathrm{P}^2 = \Sigma^2 C(\eta)$. We now consider the possibility that $\mathbb{C}\mathrm{P}^2$ splits as $\mathbb{S}^2 \vee \mathbb{S}^4$, in which case there would be a dotted retraction in the cofiber sequence

$$\begin{tikzcd} \mathbb{S}^2 \arrow[r] & \mathbb{C}\mathrm{P}^2 \arrow[l, bend right, dotted, "i"'] \arrow[r] & \mathbb{S}^4. \end{tikzcd}$$

If this were possible, we would also be able to detect the retraction after chromatic localization—so, for instance, we could consider the cohomology theory $E_{\widehat{\mathbb{G}}_m} = KU_p^\wedge$ from Example 3.5.6 and test this hypothesis in $\widehat{\mathbb{G}}_m$–local homotopy. Writing t for a coordinate on $\mathbb{C}\mathrm{P}^\infty_{KU_p^\wedge}$, this cofiber sequence gives a short exact sequence on $KU_p^\wedge$–cohomology:

[47]The groups $\pi_* \widehat{L}_1 \mathbb{S}$ are familiar to homotopy theorists: the Adams conjecture [Ada66] (and its solution) implies that the J–homomorphism $J_\mathbb{C} \colon BU \to BGL_1\mathbb{S}$ described in Corollary 1.1.6 and Theorem 2.0.1 selects exactly these elements for nonnegative t.

$$0 \longleftarrow (t)/(t)^2 \longleftarrow (t)/(t)^3 \longleftarrow (t)^2/(t)^3 \longleftarrow 0.$$

Because i is taken to be a retraction, the map i^* would satisfy $i^*(t) = t \pmod{t^2}$, so that $i^*(t) = t + at^2$ for some a. Additionally, i^* would be natural with respect to all cohomology operations on $KU_p^\wedge$. In particular, the element $(-1) \in \mathbb{Z}_p^\times \cong \operatorname{Aut}\widehat{\mathbb{G}}_m$ gives rise to an operation ψ^{-1}, which acts by the (-1)–series on the coordinate t. In the case that t is the coordinate considered in Example 2.1.20, this gives

$$[-1](t) = -\sum_{j=1}^{\infty} t^j = -t - t^2 \pmod{t^3}.$$

We thus compute:

$$\begin{aligned} \psi^{-1} \circ i(t) &= i \circ \psi^{-1}(t) \\ \psi^{-1}(t + at^2) &= i(-t) \\ (-t - t^2) + a(-t - t^2)^2 &= -(t + at^2) \\ -t + (a-1)t^2 &= -t - at^2, \end{aligned}$$

so that we would arrive at a contradiction if the equation $2a = 1$ were insoluable. Note that this has no solution in $\mathbb{Z}_2$, so that the attaching map η in $\mathbb{CP}^2$ is nontrivial in $\widehat{\mathbb{G}}_m$–local homotopy at the prime 2 (hence also in the global homotopy group $\pi_1\mathbb{S}$). For p odd, this equation *does* have a solution in $\mathbb{Z}_p$, and it furthermore turns out that $\eta = 0$ at odd primes. This problem also disappears if we require $i(t) = 2t + at^2$ instead, so that the above argument does not obstruct the triviality of 2η (and, indeed, Figure 1.2 shows that the relation $2\eta = 0$ holds in 2–adic homotopy).

Example 3.6.21 ([Rezc, Example 7.17 and Corollary 5.12]). Take k to be a perfect field of positive characteristic p, and take Γ over $\operatorname{Spec} k$ to be a finite height formal group with associated Morava E–theory E_Γ. By smashing the unit map $\mathbb{S} \to E_\Gamma$ with the mod–p Moore spectrum, we get an induced map of homotopy groups

$$h_{2n} \colon \pi_{2n} M_0(p) \to \pi_{2n} E_\Gamma.$$

We concluded as a consequence of Corollary 3.3.15 that there is an invariant section v_1 of $\omega^{\otimes(p-1)}$ on $\mathcal{M}_{\mathbf{fg}}^{\geq 1} \to \widehat{\mathbb{A}}^1$, and hence a preferred element of $\pi_{2(p-1)} E_\Gamma$ which is natural in the choice of Γ. One might hope that these elements are the image of an element in $\pi_{2(p-1)} M_0(p)$ under the Hurewicz map h, and this turns out to be true: this element is called $\alpha_{1/1}$. This element furthermore turns out to be p–torsion, meaning it extends to a map

$$\begin{tikzcd} \mathbb{S}^{2(p-1)} \arrow[r, "p"] \arrow[dr, "0"'] & \mathbb{S}^{2(p-1)} \arrow[r, "\text{cofib}"] \arrow[d, "\alpha_{1/1}"] & M_{2(p-1)}(p) \arrow[dl, dotted, "v"] \\ & M_0(p). & \end{tikzcd}$$

At odd primes, this turns out to be the v_1–self-map $v \colon M_{2(p-1)}(p) \to M_0(p)$ announced in Theorem 3.5.10 (cf. also [Ada66, Proposition 12.7]).

More generally, different powers v_1^j of the section v_1 also give rise to homotopy elements $\alpha_{j/1} \in \pi_{2(p-1)j} M_0(p)$. These have varying orders of divisibility, and we write $\alpha_{j/k}$ for the element satisfying $p^{k-1}\alpha_{j/k} = \alpha_{j/1}$. Compositionally, these maps satisfy the useful relation $\alpha^p_{p^{j-1}/j-1} = \alpha_{p^j/j}$. The other invariant functions described in Corollary 3.3.15 (e.g., v_d modulo I_d) also give rise to elements in $H^*(\mathcal{M}^{\geq d}_{\mathbf{fg}}; \omega^{\otimes *})$, which map to the BP–Adams E_2–term and which sometimes survive the spectral sequence to give to homotopy elements of the generalized Moore spectra $M_0(v^I)$. Homotopy elements arising in this way are collectively referred to as *Greek letter families* [MRW77, Section 3].

Is this relation right?

Remark 3.6.22. In the broader literature, the phrase "Greek letter elements" typically refers to the pushforward of the above elements to the homotopy groups of $\mathbb{S}$ by pinching to the top cell. This is somewhat obscuring: for instance, this significantly entangles how multiplication by $\alpha_{j/k}$ behaves. Finally, the incarnation of these elements in $\widehat{\mathbb{G}}_m$–local homotopy are exactly the elements witnessed by the invariant function $u^{2(p-1)k}$ in Example 3.6.19.

Danny was asking some interesting questions about the relationships between the t_i and the h_j elements of the Adams spectral sequence. You could try to explain some of this comparison and how you get the Hopf invariant one theorem out of it. This could also include a brief discussion of the (geometric and algebraic) chromatic spectral sequence(s).

Case Study 4

Unstable cooperations

In Lecture 3.1 (and more broadly in Case Study 3), we codified the structure of the stable E–cooperations acting on the E–homology of a spectrum X, attached to it the E–Adams spectral sequence which approximates the stable homotopy groups $\pi_* X$, and gave algebro-geometric descriptions of the stable cooperations for some typical spectra: $H\mathbb{F}_2$, MO, and MU. We will now pursue a variation on this theme, where we consider the E–homology of a *space* rather than of a generic spectrum. In this Case Study, we will examine the theory of cooperations that arises from this set-up, called the *unstable E–cooperations*. This broader collection of cooperations has considerably more intricate structure than their stable counterparts, requiring the introduction of a new notion of an unstable context. With that established, we will again find that E–homology assigns spaces to quasicoherent Cartesian sheaves over the unstable context, and we will again assemble an *unstable* E–Adams spectral sequence approximating the *unstable* homotopy groups of the input space, whose E^2–page in favorable situations is tracked by the cohomology of the sheaf over the unstable context.

Remarkably, these unstable contexts also admit algebro-geometric interpretations. In finding the right language for this, we introduce different subclasses of cooperations (e.g., *additive*), and we are also naturally led to consider *mixed cooperations* (as we did stably in Lemma 2.6.8) of the form $F_* \underline{E}_*$. The running theme is that when E and F are complex-orientable, there is a natural approximation map

$$\operatorname{Spec} Q^* F_* \underline{E}_* \to \underline{\mathsf{FormalGroups}}(\mathbb{C}\mathrm{P}^\infty_F, \mathbb{C}\mathrm{P}^\infty_E)$$

which is an isomorphism in every situation of interest. However, these isomorphisms do not appear to admit uniform proofs[1], so we instead investigate the following cases by hand:

- (Lecture 4.1:) For $F = E = H\mathbb{F}_2$, we compute the full unstable dual Steenrod algebra $H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_*$ by means of iterated bar spectral sequences. We then pass to the additive unstable cooperations, where we show by hand that this presents the endomorphism scheme $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)$. Finally, we pass to the stable additive cooperations, and we check that our results here are compatible with the isomorphism

$$\operatorname{Spec} H\mathbb{F}_{2*}H\mathbb{F}_2 \cong \underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$$

  presented in Lemma 1.3.5.

- (Lecture 4.3:) We next consider the case where $E = MU$ and where F is any complex-orientable theory. We begin with the case $F = H\mathbb{F}_p$, where we can again approach the problem using iterated bar spectral sequences. The resulting computation is sufficiently nice that we can use this special case of $F = H\mathbb{F}_p$ to deduce the further case of $F = H\mathbb{Z}_{(p)}$, then $F = H\mathbb{Z}$, then $F = MU$, and then finally F any complex-orientable theory.

- (Lecture 4.5:) Having been able to vary F as widely as possible in the previous case, we then turn to trying to vary E. This is considerably harder, since the infinite loopspaces $\underline{E}_*$ associated to E are extremely complicated and vary wildly under even "small" changes in E. However, in the special case of $F = H\mathbb{F}_p$, we have an incredibly powerful trick available to us: Dieudonné theory, discussed in Lecture 4.4, gives an equivalence of categories

$$D_* \colon \mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/} \to \mathsf{GradedDMods},$$

  which postcomposes with

$$\begin{aligned}\mathsf{Spectra} &\xrightarrow{\Omega^\infty} \mathsf{Loopspaces}\\ &\xrightarrow{H\mathbb{F}_{p*}} \mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}\\ &\xrightarrow{D_*} \mathsf{GradedDMods}\\ &\subseteq \mathsf{GradedModules}_{\mathrm{Cart}}\end{aligned}$$

[1] The best uniform result I can find is due to Butowiez and Turner [BT00, Theorem 3.12].

to give a homological functor. This means that the Dieudonné module associated to an infinite loopspace varies stably with the spectrum underlying the loopspace, which is enough leverage to the settle the case where E is any Landweber–flat theory.

- (Lecture 4.6:) Finally, we settle one further case not covered by any of our generic hypotheses above: $F = K_\Gamma$ and $E = H\mathbb{Z}/p^j$. Neither K_Γ nor $H\mathbb{Z}/p^j$ is Landweber–flat, but because K_Γ is a field spectrum and because the additive group law associated to $H\mathbb{Z}/p^j$ is so simple, we can still perform the requisite iterated bar spectral sequence calculation by hand.

This last case is actually our real goal, as we are about to return to the project outlined in the Introduction. In the language of Theorem 0.0.5, choosing Γ to be the formal completion of an elliptic curve at the identity section presents the spectra K_Γ and E_Γ of Lecture 3.5 as the most basic examples of *elliptic spectra*. The goal of that Theorem is to study $E_*BU[6,\infty)$ for E an elliptic spectrum, so when proving it in Case Study 5 we will be led to consider the fiber sequences

$$BSU \to BU \to \underline{H\mathbb{Z}}_2, \qquad \underline{H\mathbb{Z}}_3 \to BU[6,\infty) \to BSU,$$

which mediate the difference between E_*BU and $E_*BU[6,\infty)$ by means of $E_*\underline{H\mathbb{Z}}_2$ and $E_*\underline{H\mathbb{Z}}_3$. Thus, in our pursuit of $K_{\Gamma*}BU[6,\infty)$, we will want to have $K_{\Gamma*}\underline{H\mathbb{Z}}_*$ already in hand, as well as an algebro-geometric interpretation of it.

4.1 Unstable contexts and the Steenrod algebra

In this Lecture, our goal is to codify the study of unstable cooperations, beginning with an unstructured account of how they arise. Recall that for a ring map $f\colon R \to S$, in Lecture 3.1 we studied the problem of recovering an R–module from an S–module plus extra data. The intermediate category of extra data that we settled on was that of *descent data*, which we phrased most enduringly as a certain cosimplicial diagram. Stripping away the commutative algebra, the only categorical formality that went into this was the adjunction

$$\mathsf{Modules}_R \underset{\text{forget}}{\overset{-\otimes_R S}{\rightleftarrows}} \mathsf{Modules}_S,$$

or later on, when given a ring spectrum $\eta\colon \mathbb{S} \to E$, the adjunction

$$\mathsf{Spectra} = \mathsf{Modules}_{\mathbb{S}} \underset{\text{forget}}{\overset{-\wedge E}{\rightleftarrows}} \mathsf{Modules}_E.$$

The identification of $\mathsf{Modules}_S$ with T–algebras in $\mathsf{Modules}_R$ for the monad $T = \text{forget} \circ (- \otimes_R S)$ is the objective of *monadic descent* [Lurb, Theorem 4.7.4.5]. This categorical recasting is ignorant of some of the algebraic geometry we discovered next, but it is suitable for us now as we consider the composition with a second adjunction:

$$\mathsf{Spaces} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathsf{Spectra} = \mathsf{Modules}_{\mathbb{S}} \underset{\text{forget}}{\overset{-\wedge E}{\rightleftarrows}} \mathsf{Modules}_E.$$

We will write $E(-)$ for the induced monad on Spaces, given by the formula

$$E(X) = \Omega^\infty(E \wedge \Sigma^\infty X) = \operatorname*{colim}_{j \to \infty} \Omega^j(\underline{E}_j \wedge X),$$

where $\underline{E}_*$ are the constituent spaces in the Ω–spectrum of E. This space has the property $\pi_* E(X) = \widetilde{E}_{*\geq 0} X$. The monadic structure comes from the two evident natural transformations:

$$\eta\colon\ X \to \Omega^\infty\Sigma^\infty X \simeq \Omega^\infty(\mathbb{S} \wedge \Sigma^\infty X) \to \Omega^\infty(E \wedge \Sigma^\infty X) = E(X),$$

$$\begin{aligned} \mu\colon\ E(E(X)) &= \Omega^\infty(E \wedge \Sigma^\infty\Omega^\infty(E \wedge \Sigma^\infty X)) \\ &\to \Omega^\infty(E \wedge E \wedge \Sigma^\infty X) \to \Omega^\infty(E \wedge \Sigma^\infty X) = E(X). \end{aligned}$$

Just as in the stable situation, we can extract from this a cosimplicial space:

Definition 4.1.1. The *unstable descent object* is the cosimplicial space

$$\mathcal{UD}_E(X) := \left\{ \begin{array}{ccccccc} & & & & & \longrightarrow & \\ & & & \longrightarrow & E & \longleftarrow & \\ & \xrightarrow{\eta_L} & E & \longleftarrow & \circ & \longrightarrow & \\ E & \xleftarrow{\mu} & \circ & \xrightarrow{\Delta} & E & \longleftarrow & \\ \circ & \xrightarrow{\eta_R} & E & \longleftarrow & \circ & \longrightarrow & \cdots \\ X & & \circ & \longrightarrow & E & \longleftarrow & \\ & & X & & \circ & \longrightarrow & \\ & & & & X & & \end{array} \right\}.$$

Its totalization gives the *unstable E–completion of X*. The simplicial scheme

$$\mathcal{UM}_E = \operatorname{Spec} \pi_0 \mathcal{UD}_E(S^0)$$

forms the *unstable context* for E, and a space X gives rise to a quasicoherent sheaf

$$\mathcal{UM}_E(X) = (\pi_0 \mathcal{UD}_E(X))\tilde{}.$$

The remainder of this section will be spent trying to understand the spectral sequence associated to the coskeletal filtration of such an unstable descent object. In the stable situation, we recognized that in favorite situations the homotopy groups of the descent object formed a cosimplicial module over a certain cosimplicial ring—or, equivalently, a sheaf over a certain simplicial scheme. Furthermore, we found that the simplicial scheme itself had some arithmetic meaning, and that the E_2–page of the descent spectral sequence computed the cohomology of this sheaf. We will find analogues of all of these results in the unstable setting, listed above in order from least to most difficult.

To begin, we would like to recognize the cosimplicial abelian group $\pi_0 \mathcal{UD}_E(X)$ as a sort of comodule. In the stable case, this came from the smash product map $\mathbb{S}^0 \wedge X \to X$, as well as the lax monoidality of the functor $\mathcal{D}_E(-)$. However, to get a Segal condition by which we could identify the higher-dimensional objects in $\pi_0 \mathcal{D}_E(X)$, we had to introduce the condition **FH**.[2] The unstable situation has an analogous antecedent:

Definition 4.1.2 ([BCM78, Assumption 6.5])**.** A ring spectrum E is said to satisfy the **U**nstable **F**latness **H**ypothesis, or **UFH**, if $E_* \underline{E}_k$ is a free E_*–module for every value of k.[3]

Under this condition, we again turn to studying the structure of $\mathcal{UD}_E(S^0)$. If we had a Segal condition, we would expect the structure present to be determined by $\pi_0 \mathcal{UD}_E(S^0)[j]$ for $j \le 2$. The data at $j = 0$ is largely redundant:

$$\pi_0 E(S^0) = \pi_0 \underline{E}_0 = E_0$$

[2] In particular, **FH** caused the marked map in $E_0 X \xrightarrow{\eta_R} E_0(E \wedge X) \xleftarrow{\star} E_0 E \otimes_{E_0} E_0 X$ to become invertible.

[3] This helps us understand the following analogous zigzag:

$$\pi_0 E(X) \xrightarrow{\eta_R} \pi_0 E(E(X)) \xleftarrow{\mu \circ 1} \pi_0 E(E(E(X))) \xleftarrow{\text{compose}} \pi_m E(E(S^0)) \times \pi_0 E(X).$$

computes the coefficient ring of E. The data at $j = 1$ consists of the homology groups of the the infinite loopspace associated to E:

$$\pi_0 E(E(S^0)) = E_0\underline{E}_0.$$

There are three pieces of structure present here: the augmentation map $E_0\underline{E}_0 \to E_0(E) \to E_0$ and the left- and right-unit maps $E_0 \to E_0\underline{E}_0$. The assumption **UFH** gives us a foothold on the case $j = 2$: a choice of basis for $E_0\underline{E}_0$ gives an unstable isomorphism $E(E(S^0)) \simeq \prod_\ell \underline{E}_0$, so that $\pi_0\mathcal{UD}_E(S^0)[2]$ splits as a tensor product of terms of the form $E_0\underline{E}_0$. Here, we find a lot of structure: the addition of cohomology classes induces a map

$$*\colon\ E_0\underline{E}_0 \otimes_{E_0} E_0\underline{E}_0 \to E_0\underline{E}_0;$$

the multiplication induces a map

$$\circ\colon\ E_0\underline{E}_0 \otimes E_0\underline{E}_0 \to E_0\underline{E}_0;$$

these are compatible with the images of unit classes $0, 1 \in \pi_0 E$ under the left- and right-units specified above; there is an additive inverse map

$$\chi\colon\ E_0\underline{E}_0 \to E_0\underline{E}_0$$

compatible with the $*$–product and giving a skew-commutativity formula $a \circ b = \chi(b \circ a)$; there is a diagonal map

$$\Delta\colon\ E_0\underline{E}_0 \to E_0\underline{E}_0 \otimes_{E_0} E_0\underline{E}_0;$$

each $E_0\underline{E}_0$ becomes a Hopf algebra using χ, $*$, and Δ; and there is a distributivity condition pictured in Figure 4.1 intertwining $*$, $\circ$, and Δ.[4,5]

[4]Analogous structure also appears for aperiodic ring spectra E satisfying a graded version of **UFH**, and in that case tracking through the extra grading indices is actually helpful for deciphering what these maps “feel” like.

[5]There are a few more pieces of structure that Boardman–Johnson–Wilson call an *enriched Hopf ring*, which we omit from the definition because they do not affect the homological algebra of modules for a Hopf ring:

- An element $v \in \pi_0 E$ selects a connected component of $\underline{E}_0$, and there is an attached element $[v] \in E_0\underline{E}_0$.
- A cohomology operation $r\colon\ E^0(-) \to E^0(-)$ induces a map $\underline{E}_0 \to \underline{E}_0$ and hence a map on E–homology $r_*\colon\ E_0\underline{E}_0 \to E_0\underline{E}_0$.
- In particular, this gives a homology suspension element $e = (e_2)_*$, where $e_2\colon\ E^0(-) \to E^2(-) \cong E^0(\Sigma^{-2}-)$ is the map witnessing the 2–periodicity of E.

$$\begin{tikzcd}
A \otimes_R (A \otimes_R A) \arrow[r, "1 \otimes *"] \arrow[d, "\Delta \otimes (1 \otimes 1)"] & A \otimes_R A \arrow[ddddd, "\circ"] \\
(A \otimes_R A) \otimes_{R_*} (A \otimes_R A) \arrow[d, "\simeq"] & \\
(A \otimes_R A \otimes_R A \otimes_R A) \arrow[d, "1 \otimes \tau \otimes 1"] & \\
(A \otimes_R A \otimes_R A \otimes_R A) \arrow[d, "\circ \otimes \circ"] & \\
(A \otimes_R A) \arrow[r, "*"] & A.
\end{tikzcd}$$

Figure 4.1: The distributivity axiom for $*$ over $\circ$ in a Hopf ring.

Definition 4.1.3 ([BJW95, Summary 10.46]). A *Hopf ring* is a module equipped with the structure maps $+$, $-$, $\cdot$, $*$, $\circ$, Δ, and χ subject to the axioms declared above. A Hopf ring becomes an *enriched Hopf ring* when it is furthermore equippped with a right-unit, and augmentation, and maps

$$E_0 \underline{E}_0 \times (E_0 \underline{E}_0)^\vee \to E_0 \underline{E}_0$$

(determined by composition with the dual cohomology classes in the examples of interest).

Lemma 4.1.4. *If X is a space with $E_* X$ a free E_*–module, then $E_0 X$ furthermore forms a coalgebra for the comonad G associated to the enriched Hopf ring $\pi_0 \mathcal{U}\mathcal{D}_E(S^0)$.*

Proof. The proof is a matter of elucidating the last condition. For X satisfying this freeness condition, there is again a splitting $E(X) \simeq \prod_\ell \underline{E}_{n_\ell}$. We interpret this at the level of algebra by defining a functor

$$G \colon \mathsf{Modules}_{E_0}^{\mathrm{free}} \to \mathsf{Modules}_{E_0}$$

which sends E_0 to $E_0 \underline{E}_0$ and which splits over direct sums. The enriched Hopf ring structure of $\pi_0 \mathcal{U}\mathcal{D}_E(S^0)[\leq 2]$ endows G with a comonad structure, and the structure maps of $\pi_0 \mathcal{U}\mathcal{D}_E(X)[\leq 2]$ endow $E_0 X$ itself with the structure of a G–coalgebra. $\square$

Theorem 4.1.5 ([BCM78, Theorem 6.17]). *$\pi_0 \mathcal{U}\mathcal{D}_E(X)$ is the bar resolution for the free Hopf module comonad, and the E_2–page of the unstable descent spectral sequence is presented as*

$$E_2^s = L^s \mathsf{Coalgebras}_G(E_0, E_0 X). \qquad \square$$

At this point, it is instructive to work through an example to understand the kinds of objects we have constructed. At first appraisal, these objects appear to be so bottomlessly complicated that it must be a hopeless enterprise to actually compute even just the enriched Hopf ring associated to a spectrum E. In fact, the abundance of structure maps involved gives enough footholds that this is actually often feasible, provided we have sufficiently strong stomachs. Our example will be the aperiodic spectrum $E = H\mathbb{F}_2$, and the place to start is with a very old lemma:

Lemma 4.1.6. *If E is a spectrum with $\pi_{-1}E = 0$, then $\underline{E}_1 \simeq B\underline{E}_0$. Consequentially, if E is a connective spectrum then $\underline{E}_n = B^n \underline{E}_0$ for $n \geq 0$.* $\square$

This is useful to us because $B(-)$ comes with a natural skeletal filtration, which we can use to form a spectral sequence.

Lemma 4.1.7 ([Seg70, Proposition 3.2], [RW80, Theorem 2.1]). *Let G be a topological group. There is a convergent spectral sequence of algebras of signature*

$$E^1_{*,j} = F_*(\Sigma G)^{\wedge j} \Rightarrow F_* BG.$$

In the case that F has Künneth isomorphisms $\widetilde{F}_*((\Sigma G)^{\wedge j}) \cong \widetilde{F}_*(\Sigma G)^{\otimes_{F_*} j}$, the E^2–page is identifiable as

$$E^2_{*,*} \cong \operatorname{Tor}^{F_* G}_{*,*}(F_*, F_*)$$

and the spectral sequence is one of Hopf algebras. $\square$

Corollary 4.1.8. *If E is a connective spectrum and F has Künneth isomorphisms $\widetilde{F}_*(\underline{E}_j \wedge \underline{E}_j) \cong \widetilde{F}_*\underline{E}_j \otimes_{F_*} \widetilde{F}_*\underline{E}_j$ for all j, then there is a family of spectral sequences of Hopf algebras with signatures*

$$E^2_{*,*} \cong \operatorname{Tor}^{F_*\underline{E}_j}_{*,*}(F_*, F_*) \Rightarrow F_*\underline{E}_{j+1}. \qquad \square$$

That this spectral sequence is multiplicative for the $*$–product is useful enough, but the situation is actually much, much better than this:

Lemma 4.1.9 ([TW80, Equation 1.3], [RW80, Theorem 2.2]). *Denote by $E^r_{*,*}(F_*\underline{E}_j)$ the spectral sequence considered above whose E^2–term is constructed from* Tor *over $F_*\underline{E}_j$. There are maps*

$$E^r_{*,*}(F_*\underline{E}_j) \otimes_{F_*} F_*\underline{E}_m \to E^r_{*,*}(F_*\underline{E}_{j+m})$$

which agree with the map

$$F_*\underline{E}_{j+1} \otimes_{F_*} F_*\underline{E}_m \xrightarrow{\circ} F_*\underline{E}_{j+m+1}$$

on the E^∞–page and which satisfy

$$d^r(x \circ y) = (d^r x) \circ y. \qquad \square$$

This Lemma is obscenely useful: it means that differentials can be transported *between spectral sequences* for classes which can be decomposed as $\circ$–products. This means that the bottom spectral sequence (i.e., the case $j = 0$) exerts a large amount of control over the others—and this spectral sequence often turns out to be very computable.

We now turn to concrete computations for $E = H\mathbb{F}_2$ and $F = H\mathbb{F}_2$. To ground the induction, we will consider the first spectral sequence

$$\operatorname{Tor}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}B\mathbb{F}_2.$$

Using that $\mathbb{RP}^\infty$ gives a model for $B\mathbb{F}_2$, we use Example 1.1.15 to analyze the target of this spectral sequence: as an $\mathbb{F}_2$–module, we have already demonstrated an isomorphism

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \mathbb{F}_2\{a_j \mid j \geq 0\}.$$

Using our further computation in Example 1.2.15, we can also give a presentation of the Hopf algebra structure on $H\mathbb{F}_{2*}B\mathbb{F}_2$: it is dual to the primitively-generated polynomial algebra on a single class, so forms a divided power algebra on a single class which we will denote by $a_{()}$. In characteristic 2, this decomposes as

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \Gamma[a_{()}] \cong \bigotimes_{j=0}^{\infty} \mathbb{F}_2[a_{(j)}]/a_{(j)}^2,$$

where we have written $a_{(j)}$ for $a_{()}^{[2^j]}$ in the divided power structure.

Corollary 4.1.10. *This* Tor *spectral sequence collapses at the E^2–page.*

Proof. As an algebra, the homology $H\mathbb{F}_{2*}(\mathbb{F}_2)$ of the discrete space $\mathbb{F}_2$ is presented by a group ring, which we can identify with a truncated polynomial algebra:

$$H\mathbb{F}_{2*}(\mathbb{F}_2) \cong \mathbb{F}_2[\mathbb{F}_2] \cong \mathbb{F}_2[[1]]/([1]^{*2} - [0]) \cong \mathbb{F}_2[[1]-[0]]/([1]-[0])^{*2}.$$

The Tor–algebra of this is then divided power on a single class:

$$\operatorname{Tor}_{*,*}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = \Gamma[a_{()}].$$

In order for the two computations to agree, there can therefore be no differentials in the spectral sequence. □

We now summarize the rest of the induction:

Theorem 4.1.11. *$H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2}}_t$ is the exterior $*$–algebra on the t–fold $\circ$–products of the generators $a_{(j)} \in H\mathbb{F}_{2*}B\mathbb{F}_2$.*

Proof. Noting that the case $t = 0$ is what was proved above, make the inductive assumption that this is true for some fixed value of $t \geq 0$. The Tor groups of the associated bar spectral sequence

$$\operatorname{Tor}_{*,*}^{H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2}}_t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2}}_{t+1}$$

form a divided power algebra generated by the same t–fold $\circ$–products. An analogue of another Ravenel–Wilson lemma ([RW80, Lemma 9.5], [Wil82, Claim 8.16]) gives a congruence

$$(a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]} \equiv a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})} \quad (\text{mod } *\text{–decomposables}).$$

It follows from Lemma 4.1.9 that the differentials vanish:

$$\begin{aligned} d((a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]}) &\equiv d(a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})}) && (\text{mod } *\text{–decomposables}) \\ &= a_{(j_1)} \circ d(a_{(j_2)} \circ \cdots \circ a_{(j_{t+1})}) && (\text{Lemma 4.1.9}) \\ &= 0. && (\text{inductive hyp.}) \end{aligned}$$

Hence, the spectral sequence collapses. To see that there are no multiplicative extensions, note that the only potentially undetermined multiplications occur as $*$–squares of exterior classes. However, the $*$–squaring map is induced by the topological map

$$\underline{H\mathbb{F}_{2}}_t \xrightarrow{\cdot 2} \underline{H\mathbb{F}_{2}}_t,$$

which is already null on the level of spaces. It follows that there are no extensions and the induction holds. □

Corollary 4.1.12. *It follows that $\circ$–product induces an isomorphism*

$$H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2*}} \xleftarrow{\simeq} \bigoplus_{t=0}^{\infty}(H_*(\mathbb{RP}^\infty;\mathbb{F}_2))^{\wedge t},$$

where $(-)^{\wedge t}$ denotes the t^{th} exterior power in the category of Hopf algebras. □

Remark 4.1.13 ([Wil82, Theorems 8.5 and 8.11]). The odd–primary analogue of this result appears in Wilson's book, where again the bar spectral sequences are collapsing. The end result is

$$H\mathbb{F}_{p*}\underline{H\mathbb{F}_{p*}} \cong \frac{\bigotimes_{I,J}\mathbb{F}_p[e_1\circ\alpha_I\circ\beta_J, \alpha_I\circ\beta_J]}{(e_1\circ\alpha_I\circ\beta_J)^{*2}=0, (\alpha_I\circ\beta_J)^{*p}=0, e_1\circ e_1=\beta_1},$$

where $e_1 \in (H\mathbb{F}_p)_1\underline{H\mathbb{F}_p}_1$ is the homology suspension element, $\alpha_{(j)} \in (H\mathbb{F}_p)_{2p^j}\underline{H\mathbb{F}_p}_1$ are the analogues of the elements considered above, and $\beta_{(j)} \in (H\mathbb{F}_p)_{2p^j}\mathbb{CP}^\infty$ are the algebra generators of the Hopf algebra dual of the ring of functions on the formal group $\mathbb{CP}^\infty_{H\mathbb{F}_p}$ associated to $H\mathbb{F}_p$ by its natural complex orientation. In particular, the Hopf ring is *free* on these Hopf algebras, subject to the single interesting relation $e_1\circ e_1 = \beta_{(0)}$, essentially stemming from the equivalence $S^1\wedge S^1 \simeq \mathbb{CP}^1$.

It is now instructive to try to relate this unstable computation to the stable one from Lecture 1.3 (and, particularly, its algebro-geometric interpretation in Lemma 1.3.5). Consider the situation of cohomology operations: each stable operation consists of a family of unstable operations intertwined by suspensions, each of which is additive and takes 0 to 0. In terms of an element $\psi \in E^*\underline{E}_j$, such an unstable operation takes 0 to 0 exactly when it lies in the augmentation ideal, and it is additive exactly when it satisfies Hopf algebra primitivity:

$$\Delta^*\psi^* = (\psi\otimes\psi)^*\Delta^*.$$

Definition 4.1.14. In the setting of unstable homology cooperations, we define an *additive unstable operation* to be one which lies in the $*$–indecomposable quotient $Q_*E_*\underline{E}_j$.

We now apply this philosophy to our example:

Corollary 4.1.15 (cf. Theorem 1.3.3, [Wil82, Theorem 8.15])**.** $\mathcal{A}_* = \mathbb{F}_2[\xi_0,\xi_1,\xi_2,\ldots][\xi_0^{-1}]$.

Proof. First, we compute $*$–indecomposable quotient of the unstable dual Steenrod algebra to be

$$\begin{aligned} Q^* H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_* &\cong \mathbb{F}_2\left\{a_I \middle| I \text{ a multi-index}\right\} \\ &= \mathbb{F}_2\left\{a_{(I_0)} \circ a_{(I_1)} \circ \cdots \circ a_{(I_n)} \middle| I = (I_0, \ldots, I_n) \text{ a multi-index}\right\}. \\ &\cong \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots], \end{aligned}$$

where we have translated to our previous notation by writing ξ_j for $a_{(j)}$ and juxtaposition for $\circ$–product. From here, sequences of additive unstable cooperations which are intertwined by suspension are exactly elements of the sequential colimit that inverts the homology suspension element. We have already explicitly identified this element as $a_{(0)} = \xi_0$, and this yields the claim. □

Our last goal in this Lecture is to sketch a foothold that this example has furnished us with for the algebro-geometric interpretation of unstable cooperations. First, we should remark that it has been shown that there is no manifestation of the homology of a space as any kind of classical comodule [BJW95, Theorem 9.4], so we are unstable to directly pursue an analogue of Definition 3.1.13 presenting the homology of a space as a Cartesian quasicoherent sheaf over some object. This no-go result is quite believable from the perspective of cohomology operations: we have calculated in the case of $E = H\mathbb{F}_2$ that a generic unstable cohomology operation takes the form

$$x \mapsto \sum_{S \text{ a set of multi-indices}} \left(c_S \cdot \prod_{I \in S} \mathrm{Sq}^I(x)\right).$$

This inherently uses the multiplicative structure on $H\mathbb{F}_2^*(X)$, and the proof of the result of Boardman, Johnson, and Wilson rests entirely on the observation that decomposable elements cannot be mapped to indecomposable elements by maps of algebras, but maps of modules have no such control.[6]

However, exactly this complaint is eliminated by passing to the additive unstable cooperations: all the product terms in the above formula vanish, and the homology of a space does indeed have the structure of a comodule for this Hopf algebra. Still in the setting of our running example $E = H\mathbb{F}_2$, this makes $H\mathbb{F}_{2*}(X)$

[6]It is probably still possible to treat this carefully enough to cast the whole of unstable operations (and, in particular, the comonad G) into algebro-geometric language.

into a Cartesian quasicoherent sheaf for the simplicial scheme

$$\mathcal{UM}_{H\mathbb{F}_2 P} = \operatorname{Spec} \mathbb{F}_2 // \operatorname{Spec} Q^* H\mathbb{F}_2 P_0 \underline{H\mathbb{F}_2 P}_0.$$

In this specific example, we can even identify what this simplicial scheme is: using Lemma 1.3.5, we have already made the identification

$$\begin{aligned} \operatorname{Spec} H\mathbb{F}_2 P_0 H\mathbb{F}_2 P &\cong \underline{\mathrm{Aut}}\, \widehat{\mathbb{G}}_a \\ (f\colon \mathbb{F}_2[\xi_0^{\pm}, \xi_1, \ldots] \to R) &\mapsto \left(x \mapsto \sum_{j=0}^{\infty} f(\xi_j) x^{2^j} \right), \end{aligned}$$

and the computation above presents $\operatorname{Spec} \mathcal{A}P_0$ as the open subscheme of $\operatorname{Spec} \mathbb{F}_2[\xi_0, \xi_1, \ldots]$ determined by the invertibility of ξ_0.[7] Hence, the more general target is

$$\begin{aligned} \operatorname{Spec} Q^* H\mathbb{F}_2 P_0 \underline{H\mathbb{F}_2 P}_0 &\cong \underline{\mathrm{End}}\, \widehat{\mathbb{G}}_a \\ (f\colon \mathbb{F}_2[\xi_0, \xi_1, \ldots] \to R) &\mapsto \left(x \mapsto \sum_{j=0}^{\infty} f(\xi_j) x^{2^j} \right). \end{aligned}$$

Some of the complexity here was eliminated by the smallness of $\operatorname{Spec} H\mathbb{F}_2 P_0$. For a general ring spectrum E, we also have to account for $\operatorname{Spec} E_0$, but the end result is similar to that of Definition 3.1.13:

Lemma 4.1.16. *For a ring spectrum E satisfying **UFH**, the additive unstable cooperations form rings of functions on the objects and morphisms of a category scheme $\mathcal{UM}_E$, and the E–homology of a space X forms a Cartesian quasicoherent sheaf $\mathcal{UM}_E(X)$ over its nerve.* □

Although it seems like we have lost a lot of information in passing to $*$–indecomposables, in many cases this is actually enough to recover everything.

Definition 4.1.17 ([BCM78, Assumptions 7.1 and 7.7])**.** We say that a ring spectrum E satisfying **UFH** furthermore satisfies the **U**nstable **G**eneration **H**ypothesis, or **UGH**, when the following conditions all hold:

1. The module of primitives $PE_0\underline{E}_0$ is E_0–free.

[7] Note that $Q^* H\mathbb{F}_{2*} \underline{H\mathbb{F}_{2*}}$ does *not* form a Hopf algebra, essentially because it is missing a version of χ that inverts $\circ$–multiplication. It is remarkable that inverting the homology suspension element automatically produces such an antipode.

2. The composite $PE_0\underline{E}_0 \to E_0\underline{E}_0 \to E_0E$ is injective.

3. $E_0\underline{E}_0 \to SPE_0\underline{E}_0$ is an isomorphism, where S is the cofree (nonunital) coalgebra functor.

Lemma 4.1.18 ([BCM78, Lemma 7.5]). *Let E be a ring spectrum satisfying **UGH**, and let G be the comonad from Lemma 4.1.4. The composite functor $U = PG$ extends from a functor on free E_0–modules to all E_0–modules by using 2–stage free resolutions and enforcing exactness, and the result is a comonad. Coalgebras for this comonad are exactly comodules for the Hopf algebra of additive unstable cooperations.* □

Corollary 4.1.19 ([BCM78, Remark 7.8]). *If E satisfies **UGH** and X is a space with $E_0X = SN$ for some connective free E_0–module N, then the unstable E–Adams E_2–term is computed by*

$$E_2^s = \operatorname{Ext}^s_{\mathsf{Coalgebras}_U}(E_0, PE_0X).$$

Proof. Under **UGH**, we have a factorization

$$\mathsf{Coalgebras}_G(E_0, -) = \mathsf{Comodules}_U(E_0, P(-))$$

and the injective objects intertwine to give a composite functor spectral sequence

$$E_2^{r,s} = \operatorname{Ext}^r_{\mathsf{Coalgebras}_U}(E_0, R^s_{\mathsf{Coalgberas}_G}P(M)) \Rightarrow \operatorname{Ext}^{r+s}_{\mathsf{Coalgebras}_G}(E_0, M).$$

If $M = E_0X = SN$ for some connective free E_0–module N, then $R^{q>0}_{\mathsf{Coalgebras}_G}P(E_0X) = 0$, the composite functor spectral sequence collapses, and $\operatorname{Ext}^s_{\mathsf{Coalgebras}_U}(E_0, PE_0X)$ computes the unstable Adams E_2–term as claimed. □

Remark 4.1.20 (cf. Remark 1.3.9). As in the stable case, the Hopf ring associated to a ring spectrum satisfying **UFH** splits off a factor of $B\mathbb{N}$ that tracks the grading, and the passage to the stable context in particular covers the localization $B\mathbb{N} \to B\mathbb{G}_m$.

Remark 4.1.21 ([Bau14]). Tilman Bauer has studied some of the algebraic geometry associated to unstable cohomology operations, which he gave a model for in terms of *formal plethories*.

4.2 The algebra of mixed unstable cooperations

For simplicity, we return to the stable setting of Lecture 3.1 for a moment. For an arbitrary spectrum X and ring spectrum E, the completion $X^\wedge_E$ is typically a quite poor approximation to X itself. Though this can be partially mediated by placing hypotheses on X, the approximation can always be improved by "enlarging" the cohomology theory involved—namely, selecting a second ring spectrum F and forming the completion $X^\wedge_{E\vee F}$ at the wedge. This has the following factorization property

$$\begin{tikzcd} & & X^\wedge_E \\ X \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & X^\wedge_{E\vee F} \arrow[ur] \arrow[dr] & \\ & & X^\wedge_F, \end{tikzcd}$$

so that homotopy classes visible in either of $X^\wedge_E$ or $X^\wedge_F$ are therefore also visible in the homotopy of $X^\wedge_{E\vee F}$. Now consider the descent object $\mathcal{D}_{E\vee F}(X)$ and its layers $\mathcal{D}_{E\vee F}(X)[n]$:

$$\begin{aligned} \mathcal{UD}_{E\vee F}(X)[n] &= (E\vee F)^{\wedge(n+1)}\wedge(X) \\ &\simeq (E^{\wedge(n+1)}\wedge X)\vee(F^{\wedge(n+1)}\wedge X)\vee \bigvee_{\substack{i+j=n+1\\ i\neq 0\neq j}} (E^{\wedge i}\wedge F^{\wedge j}\wedge X)^{\vee\binom{n}{i,j}}. \end{aligned}$$

In the edge cases of $i=0$ or $j=0$, we can identify the descent objects $\mathcal{D}_E(X)$ and $\mathcal{D}_F(X)$ as sub-cosimplicial objects of $\mathcal{D}_{E\vee F}(X)$. The role of the cross-terms at the end of the expression is to prevent the completion at $E\vee F$ from double-counting the parts of X already simultaneously visible to the completions at E and at F—i.e., the cross-terms handle communication between E and F.[8]

There is a similar (but algebraically murkier) story for the unstable descent object formed at a wedge of two ring spectra. Let X now be a space, and consider the first two layers of $\mathcal{UD}_{E\vee F}(X)$:

$$\begin{aligned} &\mathcal{UD}_{E\vee F}(X)[0] = (E\vee F)(X) = E(X)\times F(X), \\ &\mathcal{UD}_{E\vee F}(X)[1] = (E\vee F)(E(X)\times F(X)) = E(E(X)\times F(X))\times F(E(X)\times F(X)). \end{aligned}$$

Consider just first factor, $E(E(X)\times F(X))$. The homotopy of this object receives a bilinear map from $\pi_*E(E(X))\times\pi_*E(F(X))$, and if E has Künneth isomorphisms

[8]From the perspective of spectral shemes, you might think of the descent object for $E\vee F$ as that coming from the joint cover $\{\mathbb{S}\to E, \mathbb{S}\to F\}$, and these cross-terms correspond to the scheme-theoretic intersection of E and F over $\mathbb{S}$.

then the induced map off of the tensor product is an equivalence. Again, we can identify the $E(E(X))$ part of this expression as belonging to $\mathcal{UD}_E(X)[1]$, and there is a cross-term $E(F(X))$ accounting for the shared information with F. The other term also contains information present in $\mathcal{UD}_F(X)[1]$ and a cross-term $F(E(X))$ accounting for shared information with E. In order to understand how these cross-terms affect the reconstruction process, it is useful to identify what they are at the level of the unstable context:

$$\mathcal{O}(\mathcal{UM}_{E \vee F}(S^n)[1]) \leftarrow \pi_* F(E(S^n)) = F_* \underline{E}_n,$$

and as n ranges these again form a Hopf ring.

Definition 4.2.1. We will refer to $F_*(\underline{E}_*)$ as the *Hopf ring of mixed unstable cooperations* (from F to E) or the *topological Hopf ring* (from F to E).

We thus set about trying to understand the Hopf rings $F_*(\underline{E}_*)$ in general. In our computational example in Lecture 4.1, we found that the topological Hopf ring $H\mathbb{F}_{2*}(\underline{H\mathbb{F}_2}_*)$ modeled endomorphisms of the additive formal group after passing to a suitable quotient, and we will take this as inspiration to construct an algebraic model, or "expected answer", approximating the topological Hopf ring.

We approach this problem in stages. To start, note that homotopy elements both of F and of E can be used to contribute elements to the topological Hopf ring: an element $f \in F_n$ begets a family of natural elements $f_m \in F_n \underline{E}_m$, and an element $e \in E^n = \pi_0 \underline{E}_n$ begets an element $[e] \in F_0 \underline{E}_n$ by Hurewicz. The interaction of these rings F_* and E^* is captured in the following definition:

Definition 4.2.2 ([RW80, pg. 706])**.** Let R and S be graded rings. The *Hopf ring–ring* $R[S]$ forms a Hopf ring over R: as an R–module, it is free and generated by symbols $[s]$ for $s \in S$, and the ring structure on S is promoted up a level to become the Hopf ring operations. Explicitly, the Hopf ring structure maps $*$, $\circ$, χ, and Δ are determined by the formulas

$$\begin{aligned} R[S] \otimes_R R[S] &\xrightarrow{*} R[S] & [s] * [s'] &= [s + s'], \\ R[S] \otimes_R R[S] &\xrightarrow{\circ} R[S] & [s] \circ [s'] &= [s \cdot s'], \\ R[S] &\xrightarrow{\chi} R[S] & \chi[s] &= [-s], \\ R[S] &\xrightarrow{\Delta} R[S] \otimes_R R[S] & \Delta[s] &= [s] \otimes [s]. \end{aligned}$$

Lemma 4.2.3 ([RW80, pg. 706]). *There are natural maps of Hopf rings*

$$F_*[E^*] \to F_*(\underline{E}_*) \to F_*[E^*]$$

augmenting the topological Hopf ring over the Hopf ring–ring. □

Supposing that E and F are complex-orientable, we now seek to involve their formal groups. The construction we are about to undertake is a variation on the proof of Lemma 2.1.4, which is itself a variation of a more general result in the theory of formal schemes:

Lemma 4.2.4 ([Str99b, Proposition 2.94]). *Let X and Y be schemes over $S = \operatorname{Spec} R$, such that $\mathcal{O}_X$ forms a finite and free R–module. There is then a mapping scheme M, such that points $f \in M(A)$ naturally biject with maps $f\colon X \times_S \operatorname{Spec} A \to Y \times_S \operatorname{Spec} A$ of A–schemes.* □

The mode of proof of this result is to form the symmetric R–algebra on the R–module $\mathcal{O}_Y \otimes_R \mathcal{O}_X^*$, then quotient by the relations encoding multiplicativity of functions. These are the same steps we will take to form a Hopf ring embodying homomorphisms of formal groups $\mathbb{C}\mathrm{P}^\infty_F \to \mathbb{C}\mathrm{P}^\infty_E$.

Definition 4.2.5 (cf. [RW77, Equation 1.17]). Given a R–coalgebra A and an S–algebra B, we form the *free relative Hopf $R[S]$–ring* $A_{R[S]}[B]$ generated under the Hopf ring operations by symbols $a[b]$ for $a \in A$, $b \in B$, according to the atomic rules

$$\begin{aligned}
A_{R[S]}[B] \otimes_R A_{R[S]}[B] &\xrightarrow{*} A_{R[S]}[B] & a[b] * a'[b'] &= (a * a')[b + b'], \\
A_{R[S]}[B] \otimes_R A_{R[S]}[B] &\xrightarrow{\circ} A_{R[S]}[B] & a[b] \circ a'[b'] &= (a \circ a')[bb'], \\
A_{R[S]}[B] &\xrightarrow{\chi} A_{R[S]}[B] & \chi(a[b]) &= (\chi a)[b] = a[-b], \\
A_{R[S]}[B] &\xrightarrow{\Delta} A_{R[S]}[B] & \Delta(a[b]) &= \sum_j (a'_j[b] \otimes a''_j[b]),
\end{aligned}$$

where $\Delta(a) = \sum_j a'_j \otimes_{F_*} a''_j$. There are two additional families of relations we might impose:

1. For $a \in A$ and $b', b'' \in B$, we devise the relation

$$\sum_j (a'_j[b'] \circ a''_j[b'']) = a[b'b'']$$

as an analogue of the multiplicativity relation imposed in Lemma 4.2.4.

2. For $a \in A$, we devise the relation

$$a[\eta(1)] = \varepsilon(a)$$

as an analogue of the unitality relation imposed in Lemma 4.2.4.

3. Assume further that the coalgebra A is coaugmented and that the algebra B is augmented. We can then consider the pointedness relation

$$(\eta(1))[b] = [\varepsilon(b)].$$

4. Finally, assume the entire structure of a Hopf R–algebnra on A and of a Hopf S–algebra on B. For each $a', a'' \in A$ and $b \in B$ with diagonal $\Delta b = \sum_j b'_j \otimes b''_j$, we can consider the homomorphism relation

$$(a'a'')[b] = \mathop{\Huge *}_{j} (a'[b'_j] \circ a''[b''_j]).$$

We denote the result of imposing all of these relations on $A_{R[S]}[B]$ as $A^{\circlearrowleft}_{R[S]}[B]$.

Lemma 4.2.6. *There is a natural map*

$$(F_*\mathbb{C}\mathrm{P}^\infty)^{\circlearrowleft}_{F_*[E^*]}[E^*\mathbb{C}\mathrm{P}^\infty] \to F_*(\underline{E}_*).$$

Proof. For any space X, we construct a Kronecker-type pairing

$$\langle -, - \rangle \colon F_n Z \times E^m Z \to F_n(\underline{E}_m)$$

as follows: given a class $f \in \pi_n F(X)$ and a class $e \colon X \to \underline{E}_m$, we can compose the two to produce an element $e_*(f) \in \pi_n F(\underline{E}_m)$. This pairing is "bilinear" in the following senses:

$$\langle a' + a'', b \rangle = \langle a', b \rangle + \langle a', b \rangle, \qquad \langle f \cdot a, b \rangle = f \cdot \langle a, b \rangle,$$
$$\langle a, b' + b'' \rangle = \langle a, b' \rangle * \langle a, b'' \rangle, \qquad \langle a, e \cdot b \rangle = [e] \circ \langle a, b \rangle.$$

Universality thus gives a map of Hopf rings $(F_*X)_{F_*[E^*]}[E^*X] \to F_*(\underline{E}_*)$. Specializing to $X = \mathbb{C}\mathrm{P}^\infty$, the factorization of this map through the indicated Hopf ring quotient follows the duality property of this enhanced Kronecker pairing. Namely, the four maps and their associated diagrams pictured in Figure 4.2 respectively witness the relations

$$\langle \Delta_* a, b' \otimes b'' \rangle = \langle a, \Delta^*(b' \otimes b'') \rangle, \quad \langle \mu_*(a' \otimes a''), b \rangle = \langle a' \otimes a'', \mu^* b \rangle,$$
$$\langle \varepsilon_* 1, b \rangle = \langle 1, \varepsilon^* b \rangle, \qquad \langle \eta_* 1, b \rangle = \langle 1, \eta^* b \rangle.$$

The Kronecker pairings are related to the Künneth isomorphisms for $F_*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$ and $E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty)$ by the product formula

$$\langle a' \otimes a'', b' \otimes b'' \rangle = \langle a', b' \rangle \circ \langle a'', b'' \rangle.$$

Hence, writing $\Delta_* a = \sum_j a'_j \otimes a''_j$ and $\mu^* b = \sum_j b'_j \otimes b''_j$, these relations become exactly the equations

$$\sum_j (a'_j[b'] \circ a''_j[b'']) = a[b'b''], \qquad (a'a'')[b] = \mathop{\ast}_j (a'[b'_j] \circ a''[b''_j]),$$
$$(\eta(1))[b] = [\varepsilon(b)], \qquad a[\eta(1)] = \varepsilon(a). \qquad \square$$

The main theme of this Case Study is that this induced map off of the quotient is very often an isomorphism (and, in turn, that the theory of formal groups also captures everything about the theory of unstable cooperations). Because we will be carrying this algebraic model around with us, we pause to give it a name.

Definition 4.2.7. For F and E ring spectra, we define their *algebraic Hopf ring* $\mathbb{A}(F, E)$ (or *algebraic approximation*) by

$$\mathbb{A}(F, E) = (F_* \mathbb{C}\mathrm{P}^\infty)^{\circlearrowleft}_{F_*[E^*]}[E^* \mathbb{C}\mathrm{P}^\infty].$$

Lemma 4.2.8 ([RW77, Theorem 3.8], [Wil82, Theorem 9.7])**.** *After choosing complex orientations of E and F, there is a natural isomorphism of Hopf rings*

$$\mathbb{A}(F, E) \cong \frac{(F_* \mathbb{C}\mathrm{P}^\infty)_{F_*[E^*]}[E^*]}{\beta(s +_F t) = \beta(s) +_{[E]} \beta(t)},$$

where the equation is of power series and the equality is imposed term-by-term on the Hopf ring. The formal sum $\beta(s)$ is given by $\beta(s) = \sum_j \beta_j x^j$, where β_j is dual to the j^{th} power of

$$(\Delta\colon \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \rightsquigarrow \left(\begin{tikzcd} & F(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \\ S^n \arrow[ur, "F(\Delta)_*\sigma"] \arrow[r, "\sigma"'] & F(\mathbb{C}\mathrm{P}^\infty) \arrow[u, "F(\Delta)"] \arrow[ur, "F(\Delta^*\omega)"'] & \end{tikzcd}\right),$$

$$(\mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty) \rightsquigarrow \left(\begin{tikzcd} S^n \arrow[r, "\sigma"] \arrow[dr, "F(\mu)_*\sigma"'] & F(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \arrow[d, "F(\mu)"] \arrow[dr, "F(\mu^*\omega)"] & \\ & F(\mathbb{C}\mathrm{P}^\infty) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \end{tikzcd}\right),$$

$$(\varepsilon\colon \mathbb{C}\mathrm{P}^\infty \to *) \rightsquigarrow \left(\begin{tikzcd} & F(*) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \\ S^n \arrow[ur, "F(\varepsilon)_*\sigma"] \arrow[r, "\sigma"'] & F(\mathbb{C}\mathrm{P}^\infty) \arrow[u, "F(\varepsilon)"] \arrow[ur, "F(\varepsilon^*\omega)"'] & \end{tikzcd}\right),$$

$$(\eta\colon * \to \mathbb{C}\mathrm{P}^\infty) \rightsquigarrow \left(\begin{tikzcd} S^n \arrow[r, "\sigma"] \arrow[dr, "F(\eta)_*\sigma"'] & F(*) \arrow[d, "F(\eta)"] \arrow[dr, "F(\eta^*\omega)"] & \\ & F(\mathbb{C}\mathrm{P}^\infty) \arrow[r, "F(\omega)"] & F(\underline{E}_m) \end{tikzcd}\right),$$

Figure 4.2: Four Kronecker pairing relations.

the chosen coordinate in $F^*\mathbb{C}\mathrm{P}^\infty$, and the formal group law expressions expand to

$$\beta(s +_F t) = \sum_n \beta_n \left(\sum_{i,j} a_{ij}^F s^i t^j \right)^n,$$

$$\beta(s) +_{[E]} \beta(t) = \mathop{\ast}_{i,j} \left([a_{ij}^E] \circ \left(\sum_k \beta_k s^k \right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right).$$

Proof sketch. The orientations of E and F beget classes $x^j \in E^{2j}\mathbb{C}\mathrm{P}^\infty$ and $\beta_k \in F_{2k}\mathbb{C}\mathrm{P}^\infty$, and hence classes $\beta_k[x^j] \in \mathbb{A}(F, E)$. The duality relations imposed on this Hopf ring give us three useful identities:

1. The relation
$$\beta_k[x^0] = \varepsilon(\beta_k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0 \end{cases}$$
eliminates all elements of this form except $\beta_0[x^0] = 1$.

2. The relation
$$\beta_k[x^{j+1}] = \sum_{k'+k''=k} \beta_{k'}[x^j] \circ \beta_{k''}[x]$$
lets us rewrite these terms as $\circ$–products of terms of lower j–degree and no larger k–degree.

3. The relation
$$\beta_0[x^j] = [\varepsilon(x^j)] = \begin{cases} [1] & \text{if } j = 0, \\ [0] & \text{if } j \neq 0 \end{cases}$$
couples to the above relation to give
$$\beta_k[x^{j+1}] = \sum_{\substack{k'+k''=k \\ k',k''\neq 0}} \beta_{k'}[x^j] \circ \beta_{k''}[x],$$
so that the rewrite is in terms of both lower j–degree *and* lower k–degree.

By consequence, the surviving terms are all sums of $\circ$–products of terms of the form $\beta_k[x]$, so that imposing these three relations produces a surjection

$$(F_*\mathbb{C}\mathrm{P}^\infty)_{F_*[E^*]}[E^*] \to \mathbb{A}(F, E).$$

The remaining assertion is a now a matter of imposing the fourth relation, i.e., of calculating the behavior of

$$\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{\mu} \mathbb{C}\mathrm{P}^\infty \xrightarrow{x} \underline{E}_2$$

in two different ways: using the effect of μ in F–homology and pushing forward in x, or using the effect of μ in E–cohomology and pushing forward along the Hurewicz map $\mathbb{S} \to F$. $\square$

Finally, we are able to explain the phenomenon uncovered by computation in Lecture 4.1, where we passed to the $*$–indecomposables to find the classical ring of functions on the endomorphism scheme of $\widehat{\mathbb{G}}_a$. Our explanation arises out of two parallel questions:

1. What functor $\operatorname{SpH} \mathcal{A}(F, E)$ does the Hopf ring $\mathcal{A}(F, E)$ corepresent when evaluated on another Hopf ring?
2. How does this functor interact with classical rings?

Towards the second, there is an embedding of Rings into $\mathsf{HopfRings}$, analogous to the square-zero extension functor sending an abelian group A to the augmented algebra $\mathbb{Z} \oplus A$ with trivial multiplication on A.

Definition 4.2.9. For a ring R, the $*$–square–zero Hopf ring iR has underlying abelian group $\mathbb{Z} \oplus R$. For an element $r \in R$, we write $[r] = (0, r)$ for the corresponding element in iR, and in this notation the Hopf ring structure maps are set by the formulas

$$\begin{array}{rll}
iR \otimes iR \xrightarrow{*} iR, & [r] * [r'] = 0, & [r] * 1 = [r], \\
iR \otimes iR \xrightarrow{\circ} iR, & [r] \circ [r'] = [rr'], & [r] \circ 1 = [r], \\
iR \xrightarrow{\chi} iR, & \chi[r] = [-r], & \chi(1) = 1, \\
iR \xrightarrow{\Delta} iR \otimes iR, & \Delta[r] = [r] \otimes 1 + 1 \otimes [r], & \Delta(1) = 1 \otimes 1, \\
iR \xrightarrow{\varepsilon} \mathbb{Z}, & \varepsilon([r]) = 1, & \varepsilon(1) = 1, \\
\mathbb{Z} \xrightarrow{\eta} iR, & \eta(1) = 1. &
\end{array}$$

Lemma 4.2.10. *We have $S \cong Q^* iS$, and moreover there is an adjunction*

$$(\operatorname{Spec} Q^* H)(S) = \mathsf{Rings}(Q^* H, S) = \mathsf{HopfRings}(H, iS) = (\operatorname{SpH} H)(iS). \qquad \square$$

Remark 4.2.11. More generally, if A is an augmented S–algebra and R is an auxiliary ring, there is a version of this square-zero extension construction that sends A to a square-zero Hopf ring augmented over $R[S]$. It admits a similar right adjoint to the category of R–algebras.

We are thus algebraically motivated to understand the affine scheme Spec $Q^* \mathcal{A}(F, E)$, as this is what SpH $\mathcal{A}(F, E)$ restricts to on the subcategory of classical rings. Note that the Hopf ring-ring $R[S]$ and the free relative Hopf ring $A_{R[S]}[B]$ both have an augmentation given by $[x] \mapsto 1$, so that the elements $\langle x \rangle = [x] - [0]$ form a generating set of the augmentation ideal.

Lemma 4.2.12. *In the $*$–indecomposable quotient, there are the formulas*

$$\langle x \rangle + \langle y \rangle = \langle x + y \rangle, \qquad \langle x \rangle \circ \langle y \rangle = \langle xy \rangle.$$

Proof. Modulo $*$–decomposables, we can write

$$0 \equiv \langle x \rangle * \langle y \rangle = [x] * [y] - [x] - [y] + [0] = \langle x + y \rangle - \langle x \rangle - \langle y \rangle.$$

We can also directly calculate

$$\langle x \rangle \circ \langle y \rangle = [xy] - [0] - [0] + [0] = \langle xy \rangle. \qquad \square$$

Corollary 4.2.13. *There is an isomorphism $Q^* R[S] \cong R \otimes S$.* $\square$

Corollary 4.2.14. *For complex-orientable F and E, there is a natural isomorphism*

$$\operatorname{Spec} Q^* \mathcal{A}(F, E) \cong \underline{\mathsf{FormalGroups}}(\mathbb{CP}^\infty_F, \mathbb{CP}^\infty_E).$$

Proof. This is a matter of calculating $Q^* \mathcal{A}(F, E)$, which is possible to do coordinate-freely, but it is at least as clear to just give in and pick coordinates. Doing this and using Lemma 4.2.12, we have

$$\mathop{\Huge *}_{i,j} \left([a^E_{ij}] \circ \left(\sum_k \beta_k s^k \right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right) \equiv \sum_{i,j} a^E_{ij} \left(\sum_k \beta_k s^k \right)^i \left(\sum_\ell \beta_\ell t^\ell \right)^j \text{ (in } Q^*\text{)},$$

from which it follows that

$$Q^* \mathcal{A}(F, E) = (F_* \otimes E_*)[\beta_0, \beta_1, \beta_2, \ldots] / (\beta(s +_F t) = \beta(s) +_E \beta(t)) \,. \qquad \square$$

Remark 4.2.15. Using the equivalence $\mathbb{C}\mathrm{P}^1 \simeq \mathbb{S}^2$, the homology suspension element e_2 is modeled by β_1. It follows immediately that $\operatorname{Spec}(Q^* \mathcal{A}\mathcal{A}(F,E))[e_2^{-1}]$ models the scheme $\underline{\mathsf{FormalGroups}}(\mathbb{C}\mathrm{P}^\infty_F, \mathbb{C}\mathrm{P}^\infty_E)^{\mathrm{gpd}}$ of isomorphisms.

Remark 4.2.16. In the unmixed case of $E = F$, as we saw in the computational example in Lecture 4.1, the algebraic Hopf ring $\mathcal{A}\mathcal{A}(E,E)$ picks up an extra diagonal corresponding to the composition of formal group endomorphisms of $\mathbb{C}\mathrm{P}^\infty_E$, and the resulting pair $(\operatorname{Spec} E_*, \underline{\mathrm{End}}(\mathbb{C}\mathrm{P}^\infty_E))$ forms a category scheme. These schemes act by pre- and post-composition on the mixed algebraic Hopf ring $\operatorname{Spec} Q^* \mathcal{A}\mathcal{A}(F,E)$, and these actions are compatible with the structure maps in the unstable context $\mathcal{U}\mathcal{M}_{E \vee F}$ described at the beginning of this Lecture. This description is also compatible with pulling back to the stable context $\mathcal{M}_{E \vee F}$: it is exactly the inclusion of the simplicial subobject consisting of the formal group isomorphisms and automorphisms.

4.3 Unstable cooperations for complex bordism

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Our theme for the rest of this Case Study is that the comparison map

$$\mathcal{A}\mathcal{A}(F,E) \to F_* \underline{E}_*$$

of Lemma 4.2.6 is often an isomorphism. In this Lecture, we begin by investigating the very modest and concrete setting of $F = H = H\mathbb{F}_p$ and $E = BP$, simply because it is the least complicated choice after the unstable Steenrod algebra: the spectrum H has Künneth isomorphisms, and the formal group law associated to BP has a very understandable role. We record our goal in the following Theorem statement:

Theorem 4.3.1 ([RW77, Theorem 4.2]). *The natural homomorphism*

$$\mathcal{A}\mathcal{A}(H, BP) \to H_* \underline{BP}_{2*}$$

is an isomorphism. (In particular, $H_* \underline{BP}_{2*}$ is even–concentrated.)

This is proved by a fairly elaborate counting argument: the rough idea is to show that the topological Hopf ring is polynomial, the comparison map is surjective, and the degrees arrange themselves so that the map then has no choice but to be an isomorphism. Our first move will thus be to produce an upper bound for the size of the source Hopf ring, so that surjectivity can be used to compare it with the size of the algebraic approximation.

Crucially, polynomiality will often let us consider the ring of $*$–indecomposables rather than the full Hopf ring. To begin, recall the following consequence of Corollary 4.2.13:

Corollary 4.3.2. *As an algebra under the $\circ$–product,*

$$Q^* H_*[BP^*] \cong \mathbb{F}_p[[v_n] - [0] \mid n \geq 1]. \qquad \square$$

From Lemma 4.2.8, we now know that $Q^* \mathbb{A}(H, BP)$ is generated by $[v_n] - [0]$ for $n \geq 1$ and $\beta_j \in H_{2j}\underline{BP}_2$, $j \geq 0$. In fact, p–typicality shows [RW77, Lemma 4.14] that it suffices to consider $\beta_{p^d} = \beta_{(d)}$ for $i \geq 0$. Altogether, this gives a secondary comparison map

$$A := \mathbb{F}_p[[v_n], \beta_{(d)} \mid n > 0, d \geq 0] \twoheadrightarrow Q^* \mathbb{A}(H, BP).$$

Although this map is onto it is not an isomorphism, as these elements are subject to the following relation:

Lemma 4.3.3 ([RW77, Lemma 3.14], [Wil82, Theorem 9.13])**.** *Write $I = ([p], [v_1], [v_2], \ldots)$, and work in $Q^* \mathbb{A}(H, BP)/I^{\circ 2} \circ Q^* \mathbb{A}(H, BP)$. For any n we have*

$$\sum_{i=1}^{n} [v_i] \circ \beta_{(n-i)}^{\circ p^i} \equiv 0.$$

Proof. Since the group law on $\mathbb{C}P^\infty_H$ is additive, the Ravenel–Wilson relation applied to the p–series[9] specializes to

$$[p]_{[BP]}(\beta(s)) = \beta(ps).$$

If we work over the square-zero part of BP_* to simplify its group law, we have the relation

$$[p]_{BP}(s) \equiv \sum_{j \geq 0} v_j s^{p^j} \pmod{(p, v_1, v_2, \ldots)^2},$$

[9]We are very sorry for the collision of $[p]_{BP}$ the p–series and $[p]$ the symbol in the Hopf ring induced from $p \in BP_0$. The p–series won't linger, and we will always differentiate them with a subscript.

which combines with the above to give

$$\beta_0 = [p]_{[BP]}(\beta(s)) \equiv \mathop{\ast}_{j\geq 0}([v_j] \circ \beta(s)^{\circ p^j}) \pmod{I^{\circ 2}}.$$

Passing to Q^*, we have $[p] \circ \beta(s) \equiv \beta_0$ and hence

$$0 \equiv \sum_{j>0} [v_j] \circ \beta(s)^{\circ p^j}.$$

The coefficient of s^{p^n} gives the identity claimed. □

Let r_n, the n^{th} relation, denote the same sum taken in A instead:

$$r_n := \sum_{i=1}^{n} [v_i] \circ \beta_{(n-i)}^{\circ p^i} \in A.$$

The Lemma then shows that the image of r_n in $Q^* \mathbb{A}(H, BP)$ is a $\circ$–decomposable element. Our goal is to show that enforcing these relations cuts A down to exactly the right size, and the easiest way to track the size of a quotient is for the quotient to be by a regular ideal.

Lemma 4.3.4 ([RW77, Lemma 4.15.b])**.** *The sequence $(r_1, r_2, \ldots)$ is regular in A.*

Proof. Our approach is intricate but standard. We seek to show that $J = (r_1, r_2, \ldots, r_n)$ is regular for every n, and we accomplish this by interpolation. Fixing a particular n, define the intermediate ideals

$$J_j = (r_n, r_{n-1}, \ldots, r_{n-j+1}),$$

as well as the intermediate rings

$$A_j = A/(\beta_{(0)}, \ldots, \beta_{(n-j-1)}), \qquad B_j = \beta_{(n-j)}^{-1} A_j.$$

Noting that $A_n = A$ and $J_n = J$, we will inductively show that J_j is a regular ideal of A_j. The case $j = 1$ is simple: J_1 is a nonzero principal ideal in a ring without zerodivisors, so it must be regular.

Assume the inductive result holds below some index j. In the quotient sequence

$$0 \to \Sigma^{|\beta_{(n-j)}|} A_j \xrightarrow{\beta_{(n-j)}} A_j \to A_{j-1} \to 0,$$

the degree shift in the multiplication map (and induction on degree) shows that if J_{j-1} is regular on A_{j-1}, then J_{j-1} is automatically regular on A_j. If we additionally prove that J_{j-1} is prime on A_j and that $r_{n-j+1} \neq 0$ in the quotient, then A_j / J_{j-1} would be an integral domain, multiplication by r_{n-j+1} would be injective, and we would be done. In the degree $|r_{n-j+1}|$ of interest, there is an isomorphism $(A)_{|r_{n-j+1}|} \cong (A_j / J_{j-1})_{|r_{n-j+1}|}$, and hence $r_{n-j+1} \neq 0$ as desired.

We thus turn to primality. Note first that J_{j-1} is automatically prime in B_j, since B_j is a polynomial $\mathbb{F}_p[\beta^{\pm}_{(n-j)}]$–algebra and each of the generators of J_{j-1} is one of these polynomial generators of B_j. Suppose for contradiction that J_{j-1} is not prime in A_j, as witnessed by some elements $x, y \notin J_{j-1}$ satisfying $xy \in J_{j-1}$. Since J_{j-1} *is* prime in B_j, (by perhaps trading x and y) there is some minimum $k > 0$ such that

$$\beta^{\circ k}_{(n-j)} \circ x \in J_{j-1}.$$

We may as well assume $k = 1$, which we can arrange by tucking the stray factors of $\beta_{(n-j)}$ into x. Invoking the generators of J_{j-1}, we thus have an equation

$$\beta_{(n-j)} \circ x = \sum_{i=1}^{j-1} a_i \circ r_{n-i+1}$$

with $a_i \in A_j$ not all divisible by $\beta_{(n-j)}$. In fact, by moving elements onto the left-hand side we can assume that if $a_i \neq 0$ then $a_i \notin J_{i-1}$. In A_{j-1}, this equation becomes

$$0 = \sum_{i=1}^{j-1} a_i \circ r_{n-i+1}$$

with a_i not all in J_{i-1}. This is the desired contradiction, since J_{j-1} is regular in A_{j-1} by inductive hypothesis. □

Corollary 4.3.5. *Set*

$$c_{i,j} = \dim_{\mathbb{F}_p} Q^* \mathbb{A}(H, BP)_{(2i,2j)}, \qquad d_{i,j} = \dim_{\mathbb{F}_p} \mathbb{F}_p[[v_n], b_{(0)}]_{2i,2j}.$$

Then $c_{i,j} \le d_{i,j}$ and $d_{i,j} = d_{i+2,j+2}$.

Proof. We have seen that $c_{i,j}$ is bounded by the $\mathbb{F}_p$–dimension of

$$\left[\mathbb{F}_p[[v_n], b_{(d)} \mid d \ge 0, n \ge 0] / (r_1, r_2, \ldots)\right]_{i,j}.$$

But, since this ideal is regular and $|r_j| = |b_{(j)}|$, this is the same value as $d_{i,j}$. The other relation among the $d_{i,j}$ follows from multiplication by $b_{(0)}$, with $|b_{(0)}| = (2, 2)$. □

We now turn to showing that this estimate is *sharp* and that the secondary comparison map is *onto*, and hence an isomorphism, using the bar spectral sequence. Recalling that the bar spectral sequence converges to a the homology of the *connective* delooping, let $\underline{BP}'_{2*}$ denote the connected component of $\underline{BP}_{2*}$ containing $[0_{2*}]$. We will then demonstrate the following theorem inductively:

Theorem 4.3.6 ([RW77, Induction 4.18]). *The following hold for all values of the induction index k:*

1. $Q^* H_{\le 2(k-1)}\underline{BP}'_{2*}$ *is generated by* $\circ$*–products of the* $[v_n]$ *and* $b_{(j)}$.
2. $H_{\le 2(k-1)}\underline{BP}'_{2*}$ *is isomorphic to a polynomial algebra in this range.*
3. *For* $0 < i \le 2(k-1)$, *we have* $d_{i,j} = \dim_{\mathbb{F}_p} Q^* H_i \underline{BP}_{2j}$.

Before addressing the Theorem, we show that this finishes our calculation:

Proof of Theorem 4.3.1, assuming Theorem 4.3.6 for all k. Recall that we are considering the natural map

$$A\!\!\!A(H, BP) \to H_*\underline{BP}_{2*}.$$

The first part of Theorem 4.3.6 shows that this map is a surjection. The third part of Theorem 4.3.6 together with our counting estimate shows that the induced map

$$Q^* A\!\!\!A(H, BP) \to Q^* H_*\underline{BP}_{2*}$$

is an isomorphism. Finally, the second part of Theorem 4.3.6 says that the original surjective map, before passing to $*$–indecomposables, targets a polynomial algebra and is an isomorphism on indecomposables, hence must be an isomorphism as a whole. □

Proof of Theorem 4.3.6. The infinite loopspaces in $\underline{BP}_{2*}$ are related by $\Omega^2\underline{BP}'_{2(*+1)} = \underline{BP}_{2*}$, so we will use two bar spectral sequences to extract information about $\underline{BP}'_{2(*+1)}$ from $\underline{BP}_{2*}$. Since we have assumed that $H_{\le 2(k-1)}\underline{BP}_{2*}$ is polynomial in the indicated triangular range near zero, we know that in the first spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{H_*\underline{BP}_{2*}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*\underline{BP}_{2*+1}$$

the E^2–page is, in the same range, exterior on generators in Tor–degree 1 and topological degree one higher than the generators in the polynomial algebra. Since differentials lower Tor–degree, the spectral sequence is multiplicative, and there are no classes on the 0–line, it collapses in the range $[0, 2k-1]$. Additionally, since all the classes are in odd topological degree, there are no algebra extension problems, and we conclude that $H_* \underline{BP}_{2*+1}$ is indeed exterior up through degree $(2k-1)$.

We now consider the second bar spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{H_* \underline{BP}_{2*+1}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_* \underline{BP}_{2(*+1)'}.$$

The Tor algebra of an exterior algebra is divided power on a class of topological dimension one higher. Since these classes are now all in even degrees, the spectral sequence collapses in the range $[0, 2k]$. Additionally, these primitive classes are related to the original generating classes by double suspension, i.e., by forming the $\circ$–product with $b_{(0)}$. This shows the first inductive claim on the *primitive classes* through degree $2k$, and we must argue further to deduce our generation result for $x^{[p^j]}$ of degree $2k$ with $j > 0$. By inductive assumption, we can write

$$x = [y] \circ b_{(0)}^{\circ I_0} \circ b_{(1)}^{\circ I_1} \circ \cdots ,$$

and one might be divinely inspired to consider the element

$$z := [y] \circ b_{(j)}^{\circ I_0} \circ b_{(j+1)}^{\circ I_1} \circ \cdots .$$

This element z isn't equal to $x^{[p^j]}$ on the nose, but the diagonal of the difference $z - x^{[p^j]}$ lies in lower filtration degree—i.e., it is primitive as far as the filtration is concerned—and so we are again done.

The remaining thing to do is to use the size bounds: the only way that the map

$$\mathbb{A}(H, BP) \to H_* \underline{BP}_{2*}$$

could be surjective is if there were multiplicative extensions in the spectral sequence joining $x^{[p]}$ to x^p. Granting this, we see that the module ranks of the algebra itself and of its indecomposables are exactly the right size to be a free (i.e., polynomial) algebra, and hence this must be the case. □

We have actually accomplished quite a lot in proving Theorem 4.3.1, as this forms the input to an Atiyah–Hirzebruch spectral sequence.

Corollary 4.3.7 ([RW77, Corollary 4.7]). *For any complex-orientable cohomology theory E, the natural approximation maps give isomorphisms of Hopf rings*[10]

$$\mathbb{A}(E, MU) \xrightarrow{\simeq} E_*\underline{MU}_{2*}, \qquad \mathbb{A}(E, BP) \xrightarrow{\simeq} E_*\underline{BP}_{2*}.$$

Proof. First, because $MU_{(p)}$ splits multiplicatively as a product of BPs, we deduce from Theorem 4.3.1 the case of $E = H\mathbb{F}_p$. Since $H\mathbb{F}_{p*}\underline{BP}_{2*}$ is even, it follows that $H\mathbb{Z}_{(p)*}\underline{BP}_{2*}$ is torsion–free on a lift of a basis, and similarly (working across primes) $H\mathbb{Z}_*\underline{MU}_{2*}$ is torsion–free on a simultaneous lift of basis. Next, using torsion–freeness, we conclude from an Atiyah–Hirzebruch spectral sequence that $MU_*\underline{MU}_{2*}$ is even and torsion–free itself, and moreover that the comparison is an isomorphism. Lastly, using naturality of Atiyah–Hirzebruch spectral sequences, given a complex–orientation $MU \to E$ we deduce that the spectral sequence

$$E_* \otimes H_*(\underline{MU}_{2*}; \mathbb{Z}) \cong E_* \otimes_{MU_*} MU_*\underline{MU}_{2*} \Rightarrow E_*\underline{MU}_{2*}$$

collapses, and similarly for the case of BP. □

This is an impressively broad claim: the loopspaces $\underline{MU}_{2*}$ are quite complicated, and that any general statement can be made about them is remarkable. That this fact follows from a calculation in $H\mathbb{F}_p$–homology and some niceness observations is meant to showcase the density of $\mathbb{C}\mathrm{P}^\infty_H \cong \widehat{\mathbb{G}}_a$ inside of $\mathcal{M}_{\mathbf{fg}}$.

Remark 4.3.8. The analysis of the first bar spectral sequence in the proof of Theorem 4.3.6 also gave us a description of $H_*\underline{BP}_{2*+1}$, which is not directly visible to $\mathbb{A}(H, BP)$. Namely, the Hopf ring $H_*\underline{BP}_*$ can be presented as

$$H_*\underline{BP}_* \xleftarrow{\simeq} \mathbb{A}(H, BP)[e]/(e^{\circ 2} = \beta_{(0)}),$$

with e of degree 1. Additionally, analyzing the cohomological bar spectral sequence (and noting that the dual of a divided power algebra is a polynomial algebra) shows that each $H_*\underline{BP}_{2*}$ forms a *bipolynomial Hopf algebra*—i.e., both it and its dual are polynomial algebras.

Remark 4.3.9 ([Cha82], [Wil82, Section 10]). There is an alternative proof, due to Chan, that $H_*\underline{BP}_{2j}$ forms a bipolynomial Hopf algebra for each choice of j that

[10] In the case $E = MU$, we actually have brushed against this before: the formulas leading to Lemma 2.5.7 look suspiciously like formal group homomorphisms with prescribed kernels. We explore this observation more seriously in Appendix A.2.

makes no reference to Hopf rings. It proceeds along very similar lines, as it also studies the iterated bar spectral sequence, but it proceeds entirely by counting: the elements in the spectral sequence are never given explicit names, and hence there is no real hope of understanding the functor SpH $H_* \underline{BP}_{2*}$ using these methods. By contrast, the Ravenel–Wilson method can be used to give an explicit enumeration of these classes [RW77, Section 5]. Our presentation here is something of a compromise.

Remark 4.3.10. The identification of the p–local and mod–p homology and cohomology of $\underline{BP}_{2k}$ as a bipolynomial Hopf algebra was first accomplished by Wilson in his PhD thesis [Wil73, Theorem 3.3]. He deduces quite a lot of interesting results from this observation. For instance, each bipolynomial Hopf algebra can be shown to split as a tensor product of indecomposable such [Wil73, Proposition 3.5], and this splitting is reflected by a splitting of $\underline{BP}_{2k}$ into a product of indecomposable H–spaces.

Remarkably, these indecomposable spaces can themselves be identified. For each n there is a ring spectrum $BP\langle n\rangle$ over BP with homotopy presented by the subalgebra $\pi_* BP\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$. This spectrum is *not* uniquely specified, a reflection of the algebraic failure of the ideal $(v_{n+1}, v_{n+2}, \ldots)$ to be invariant, and so this resists formal-geometric interpretation (cf., however, [LN12], [Str99a], ...). Nonetheless, using Steenrod module techniques Wilson shows [Wil75, Section 6] that every simply-connected p–local H–space with torsion-free homology and (p–local ordinary) homology splits into a product of spaces Y_k, and that $Y_k = \underline{BP\langle n\rangle}_k$ for $|v_n| < k(p-1) \leq |v_{n+1}|$.

In particular, the spaces $\underline{BP\langle n\rangle}_k$ in these bands *are* independent of choice of parent spectrum $BP\langle n\rangle$, and all p–local H–spaces satisfying these freeness properties are automatically infinite loopspaces—both extremely surprising results.

These bipolynomial algebras also play a critical role in the next section.

4.4 Dieudonné modules

Our goal in this Lecture is to give a compact presentation of what a formal group is based on the following observation: the category of commutative cocommutative Hopf algebras of finite type over a ground field k forms an abelian category. It follows abstractly that this category admits a presentation as the module category for some (possibly noncommutative) ring, but in fact this ring and the assignment from a group scheme to linear algebraic data can both be described explicitly. This is the subject of *Dieudonné theory*, and our goal is to give an overview of some of its main results, including three different presentations of the equivalence.[11]

In the first presentation, we follow notes by Weinstein [Wei11, Lecture 1]. Begin with a 1–dimensional formal group $\widehat{\mathbb{G}}$ over a ring A. Our first avenue into Dieudonné theory is to recall that we have previously been interested in the invariant differentials $\omega_{\widehat{\mathbb{G}}} \subseteq \Omega^1_{\widehat{\mathbb{G}}/A}$ on $\widehat{\mathbb{G}}$. As explored in Theorem 2.1.22, when A is a $\mathbb{Q}$–algebra such differentials give rise to logarithms through integration. On the other hand, if A has positive characteristic p then there is a potential obstruction to integrating terms with exponents of the form $-1 \pmod p$, and in Lecture 3.3 we used this to lead us to the notion of p–height. We now explore a third twist on this set-up. Recall that $\Omega^1_{\widehat{\mathbb{G}}/A}$ forms the first level of the *algebraic de Rham complex* $\Omega^*_{\widehat{\mathbb{G}}/A}$. The de Rham complex only uses the underlying formal variety of $\widehat{\mathbb{G}}$ and not its group structure, but the product map

$$\mu\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$$

and the two projection maps

$$\pi_1, \pi_2\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$$

induce maps

$$\mu^*, \pi_1^*, \pi_2^*\colon C^1_{dR}(\widehat{\mathbb{G}}/A) \to C^1_{dR}(\widehat{\mathbb{G}} \times \widehat{\mathbb{G}}/A).$$

The translation invariant differentials are exactly those in the kernel of $\mu^* - \pi_1^* - \pi_2^*$, as considered at the chain level. We can weaken this to request only that that difference be *exact*, or zero at the level of cohomology of the de Rham complex.

Definition 4.4.1. The *cohomologically translation invariant differentials* is the A–submodule $PH^1_{dR}(\widehat{\mathbb{G}}/A) \subseteq H^1_{dR}(\widehat{\mathbb{G}}/A)$ defined as the kernel of $\mu^* - \pi_1^* - \pi_2^*$.

Example 4.4.2. [Kat81, Lemma 5.1.2] Consider the case that A is torsion-free (or "$\mathbb{Z}$–flat", if you like), and set $K = A \otimes \mathbb{Q}$ so that $A \to K$ is an injection. In this case the differentiation map $xA[\![x]\!] \to A[\![x]\!]$ is an injection and integration of power series is possible in K, so we can re-express first the definition of H^1_{dR} and second the conditions on our algebraic differentials in the following diagram of exact rows:

[11] Emphasis on "*some of its results*". Dieudonné theory is an enormous subject with many interesting results both internal and connected to arithmetic geometry and the theory of abelian varieties. We will explore almost none of this.

$$\begin{tikzcd}
0 \arrow[r] & \left\{\begin{array}{c}\text{integrals}\\ \text{with } A\\ \text{coefficients}\end{array}\right\} \arrow[r] \arrow[d, equal] & \left\{\begin{array}{c}\text{all conceivable integrals}\\ \text{of differentials}\\ \text{defined over } A\end{array}\right\} \arrow[r] \arrow[d, equal] & \left\{\begin{array}{c}\text{missing}\\ \text{integrals}\end{array}\right\} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & xA[\![x]\!] \arrow[r] \arrow[d, equal] & \{f \in xK[\![x]\!] \mid df \in A[\![x]\!]dx\} \arrow[r, "d"] & H^1_{dR}(\widehat{\mathbb{G}}/A) \arrow[r] & 0 \\
0 \arrow[r] & xA[\![x]\!] \arrow[r] & \left\{ f \in xK[\![x]\!] \middle| \begin{array}{c} df \in A[\![x]\!]dx, \\ \delta f \in A[\![x, y]\!] \end{array} \right\} \arrow[r, "d"] \arrow[u] & PH^1_{dR}(\widehat{\mathbb{G}}/A) \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

where x is a coordinate on $\widehat{\mathbb{G}}$, and δ is defined by $\delta[\omega] = (\mu^* - \pi_1^* - \pi_2^*)[\omega]$.

The flatness condition is not satisfied when working over a perfect field of positive characteristic p—our favorite setting in Lecture 3.3 and Case Study 3 more generally. However, de Rham cohomology has the following remarkable lifting property (which we have written here after specializing to H^1_{dR}):

Theorem 4.4.3. *[Kat81, Key Lemma 5.1.3] Let A be a p–local torsion-free ring, and let $f_1(x), f_2(x) \in xA[\![x]\!]$ be power series without constant term. If $f_1 \equiv f_2 \pmod{p}$, then for any differential $\omega \in A[\![x]\!]dx$ the difference $f_1^*(\omega) - f_2^*(\omega)$ is exact.*

Proof. Write $\omega = dg$ for $g \in K[\![x]\!]$, and write $f_2 = f_1 + p\Delta$. Then

$$\begin{aligned}
\int (f_2^*\omega - f_1^*\omega) &= g(f_2) - g(f_1) = g(f_1 + p\Delta) - g(f_1) \\
&= \sum_{n=1}^{\infty} \frac{(p\Delta)^n}{n!} g^{(n)}(f_1).
\end{aligned}$$

Since $g' = \omega$ has coefficients in A, so does the iterated derivative $g^{(n)}$ for all n, and hence the fraction $p^n/n!$ lies in the $\mathbb{Z}_{(p)}$–algebra A. $\square$

Corollary 4.4.4 (H^1_{dR} is "crystalline")**.** *If $f_1, f_2\colon V \to V'$ are maps of pointed formal varieties which agree mod p, then they induce the same map on H^1_{dR}.* $\square$

Corollary 4.4.5 ([Kat81, Theorem 5.1.4])**.** *Any map $f\colon \widehat{\mathbb{G}}' \to \widehat{\mathbb{G}}$ of pointed varieties which is a group homomorphism mod p restricts to give a map $f^*\colon PH^1_{dR}(\widehat{\mathbb{G}}/A) \to PH^1_{dR}(\widehat{\mathbb{G}}'/A)$. Additionally, if f_1, f_2, and f_3 are three such maps of pointed varieties satisfying*

$$f_3 \equiv f_1 + f_2 \in \mathsf{FormalGroups}(\widehat{\mathbb{G}}'/p, \widehat{\mathbb{G}}/p),$$

then $f_3^* = f_1^* + f_2^*$ *as maps* $PH^1_{dR}(\widehat{\mathbb{G}}/A) \to PH^1_{dR}(\widehat{\mathbb{G}}'/A)$. □

In the case that k is a *perfect* field, the ring $\mathbb{W}_p(k)$ of p–typical Witt vectors on k is simultaneously torsion-free and universal among nilpotent thickenings of the residue field k. This emboldens us to make the following definition:[12]

Definition 4.4.6. [Kat81, Section 5.5] Let k be a perfect field of characteristic $p > 0$, and let $\widehat{\mathbb{G}}_0$ be a (1–dimensional) formal group over k. Then, choose a lift $\widehat{\mathbb{G}}$ of $\widehat{\mathbb{G}}_0$ to $\mathbb{W}_p(k)$, and define the *(contravariant) Dieudonné module* of $\widehat{\mathbb{G}}_0$ by $D^*(\widehat{\mathbb{G}}_0) := PH^1_{dR}(\widehat{\mathbb{G}}/\mathbb{W}_p(k))$.

Remark 4.4.7. This is independent of choice of lift up to coherent isomorphism. Given any other lift $\widehat{\mathbb{G}}'$ of $\widehat{\mathbb{G}}_0$ to $\mathbb{W}_p(k)$, we can find *some* power series—not necessarily a group homomorphism—covering the identity on $\widehat{\mathbb{G}}_0$. Corollary 4.4.4 then shows that this map induces a canonical isomorphism between the two potential definitions of $D^*(\widehat{\mathbb{G}}_0)$.

Note that the module $D^*(\widehat{\mathbb{G}}_0)$ carries some natural operations:

- Arithmetic: $D^*(\widehat{\mathbb{G}}_0)$ is naturally a $\mathbb{W}_p(k)$–module, with the action by ℓ corresponding to multiplication–by–ℓ internal to $\widehat{\mathbb{G}}_0$.

- Frobenius: The map $x \mapsto x^p$ is a homomorphism of formal groups modulo p, so it induces a φ–semilinear map $F\colon D^*(\widehat{\mathbb{G}}_0) \to D^*(\widehat{\mathbb{G}}_0)$. That is, $F(\alpha v) = \alpha^{\varphi} F(v)$, where φ is a lift of the Frobenius on k to $\mathbb{W}_p(k)$.

- Verschiebung: Inspired by Lemma 3.3.6, we might also seek a Verschiebung operator V satisfying $FV = p$. Our explicit formula for F lets us guess such a map:
$$V\colon \sum_{n=1}^{\infty} a_n x^n \mapsto p \sum_{n=1}^{\infty} a_{pn}^{\varphi^{-1}} x^n.$$
It satisfies $FV = p$ and anti-semilinearity: $aV(v) = V(a^{\varphi} v)$.

With this, we come to the main theorem of this Lecture:

[12] There is a better definition one might hope for, which instead assigns to each potential thickening and lift a "Dieudonne module", and then work to show that they all arise as base-changes of this universal one. This is possible and technically superior to the approach we are taking here [Kat81, Theorem 5.1.6], [Mes72, Chapter 4], [Gro74].

Theorem 4.4.8 ([Gro74, Théorème 4.2], [Dem86, Sections III.8-9])**.** *The functor D^* determines a contravariant equivalence of categories between smooth 1–dimensional formal groups $\widehat{\mathbb{G}}_0$ over k of finite p–height and Dieudonné modules, which are modules M over the Cartier–Dieudonné ring*

$$\mathrm{Cart}_p = \mathbb{W}_p(k)\langle F, V\rangle \Bigg/ \left(\begin{array}{c} FV = VF = p, \\ Fw = w^{\varphi}F, \\ wV = Vw^{\varphi} \end{array} \right)$$

which furthermore satisfy the following three technical conditions:

Are these the correct technical conditions for the *contravariant* Dieudonné module?

- *Finiteness: M is a finite-dimensional free $\mathbb{W}_p(k)$–module.*
- *Reduced: $M \cong \lim_r M/V^r M$.*
- *Uniform: $M/VM \to V^r M/V^{r+1}M$ is an isomorphism.* □

Remark 4.4.9. Several invariants of the formal group associated to a Dieudonné module can be read off from the functor D^*. For example, the $\mathbb{W}_p(k)$–rank of $D^*(\widehat{\mathbb{G}}_0)$ computes the height of $\widehat{\mathbb{G}}_0$. Additionally, the quotient $D^*(\widehat{\mathbb{G}}_0)/FD^*(\widehat{\mathbb{G}}_0)$ is canonically isomorphic to the cotangent space $T_0^*\widehat{\mathbb{G}}_0 \cong \omega_{\widehat{\mathbb{G}}_0}$.

Example 4.4.10. Consider $\widehat{\mathbb{G}}_0 = \widehat{\mathbb{G}}_m$. For x the usual coordinate, we have $[p](x) = x^p$, and hence the Frobenius F acts on $D^*(\widehat{\mathbb{G}}_m)$ by $Fx = px$. It follows that $Vx = x$ and $D^*(\widehat{\mathbb{G}}_m) \cong \mathbb{W}_p(k)\{x\}$ with this Cart_p–module structure.

Example 4.4.11. We also give a kind of non-example: $\widehat{\mathbb{G}}_a$ is *not* a finite height formal group, and its Dieudonné module is correspondingly strangely behaved:

$$D^*(\widehat{\mathbb{G}}_a) = \mathbb{F}_2\{x, Fx, F^2x, \ldots\}/(V = 0).$$

Example 4.4.12 (cf. Example 1.2.9). Dieudonné theory admits an extension to finite (flat) group schemes as well, and the torsion quotient of the Dieudonné module of a formal group agrees with the Dieudonné module associated to its torsion subscheme:

$$D^*(\widehat{\mathbb{G}}_0[p^j]) = D^*(\widehat{\mathbb{G}}_0)/p^j.$$

For example, this gives

$$D^*(\widehat{\mathbb{G}}_m[p]) = \mathbb{F}_p\{x\} \Bigg/ \left(\begin{array}{c} Fx = 0, \\ Vx = x \end{array} \right).$$

We extract the subgroup scheme α_2 as the finite Dieudonné quotient module $D^*(\alpha_2) = \mathbb{F}_2\{x\} \leftarrow D^*(\widehat{\mathbb{G}}_a)$ of the Dieudonné module associated to $\widehat{\mathbb{G}}_a$ above. We can now verify the four claims from Example 1.2.9:

- *The group scheme α_2 has the same underlying structure ring as $\mu_2 = \mathbb{G}_m[2]$ but is not isomorphic to it.* There are now several ways to see this, the simplest of which is that the Verschiebung operator acts nontrivially on $D^*(\mu_2)$ but wholly trivially on $D^*(\alpha_2)$.[13]

- *There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.* Suppose that G were such a group scheme, so that $D^*(G)/2$ would give $D^*(\alpha_2)$. It can't be the case that $D^*(G)$ has only 2–torsion, since then this quotient would be a null operation, so it must be the case that $D^*(G) = \mathbb{Z}/4\{x\}$. The action of both F and V on x must vanish after quotienting by 2, so it must be the case that $Fx = 2cx$ and $Vx = 2dx$ for some constants c and d—but this violates $FVx = 2x$.

- *If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence*

  $$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$

  However, this short exact sequence doesn't split (even after making a base change). This follows from calculating the action of F and V to get a short exact sequence of Dieudonné modules:

  $$0 \to \mathbb{F}_2\{Fx\}/(F,V) \to \mathbb{F}_2\{x, Fx\} \bigg/ \left(\begin{array}{c} F^2x = 0, \\ V = 0 \end{array} \right) \to \mathbb{F}_2\{x\}/(F,V) \to 0.$$

  The exact sequence is split as $\mathbb{F}_2$–modules, but not as Dieudonné modules.

- *The subgroups of $\alpha_2 \times \alpha_2$ of rank two are parameterized by $\mathbb{P}^1$.* The Dieudonné module of the product is quickly computed:

  $$D^*(\alpha_2 \times \alpha_2) = D^*(\alpha_2) \oplus D^*(\alpha_2) = \mathbb{F}_2\{x_1, x_2\} \bigg/ \left(\begin{array}{c} F = 0, \\ V = 0 \end{array} \right).$$

  An inclusion of a rank 2 subgroup scheme corresponds to a projection of this Dieudonné module onto a 1–dimensional quotient module, and the ways to choose the kernel of this projection encompass a $\mathbb{P}^1$.

[13] See also [Strb, Example 8.5].

Example 4.4.13. We can also use Dieudonné theory to compute the automorphism group of a fixed Honda formal group, which is information we wanted back in Lecture 3.6. Take Γ_d to be Honda formal group law of height d over $\mathbb{F}_{p^d}$, which has Dieudonné module

$$D^*(\Gamma_d) = \mathrm{Cart}_p/(F^d = p).$$

The endomorphism ring of a quotient module of its parent ring is canonically isomorphic to the module itself, giving

$$\operatorname{End}\Gamma_d \cong \mathbb{W}_p(\mathbb{F}_{p^d})\langle F\rangle \Big/ \left(\begin{array}{c} Fw = w^{\varphi}F, \\ F^d = p \end{array} \right)$$

and hence

$$\operatorname{Aut}\Gamma_d \cong \left(\mathbb{W}_p(\mathbb{F}_{p^d})\langle F\rangle \Big/ \left(\begin{array}{c} Fw = w^{\varphi}F, \\ F^d = p \end{array} \right) \right)^{\times}.$$

Remark 4.4.14 ([Kat81, Theorem 5.2.1]). There is also a relationship between this representation of the Dieudonné functor and the deformation theory of formal groups from Lecture 3.4: a class $[f(x)dx] \in D^*(\widehat{\mathbb{G}}_0)$ begets a class in $e(f) \in \operatorname{Ext}^1(\widehat{\mathbb{G}}, \widehat{\mathbb{G}}_a)$ given by the cobar 1–cocycle $f(x +_{\widehat{\mathbb{G}}} y) - f(x) - f(y)$. In fact, this assignment is surjective, and the additional information lost in the kernel is a trivialization of the Lie algebra extension

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\widehat{\mathbb{G}}_a) \arrow[r] & \mathrm{Lie}(E) \arrow[r] & \mathrm{Lie}(\widehat{\mathbb{G}}) \arrow[r] \arrow[l, bend right, dotted] & 0 \end{tikzcd}$$

associated to the group scheme extension E classified by $e(f)$.

Having gotten some feel for the behavior and the usefulness of the Dieudonné functor, we now turn our attention to some alternative presentations of it. In this next presentation we will not have to worry about lifts to $\mathbb{W}_p(k)$, so we take $\widehat{\mathbb{G}}$ itself to be a formal group over a perfect field k of positive characteristic p. Cartier's *functor of curves* is defined by the formula

$$C\widehat{\mathbb{G}} = \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}).$$

This is, again, a kind of relaxing of familiar data from Lie theory, taken from a different direction: rather than studying just the exponential curves, $C\widehat{\mathbb{G}}$ tracks all possible curves. In Lecture 3.3, we considered three kinds of operations on a given curve $\gamma\colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$:

- Homothety: given a scalar $a \in A$, we define $\theta_a \gamma(t) = \gamma(a \cdot t)$.

- Verschiebung: given an integer $n \geq 1$, we define $V_n \gamma(t) = \gamma(t^n)$.

- Arithmetic: given two curves γ_1 and γ_2, we can use the group law on $\widehat{\mathbb{G}}$ to define $\gamma_1 +_{\widehat{\mathbb{G}}} \gamma_2$. Moreover, given $\ell \in \mathbb{Z}$, the ℓ–fold sum in $\widehat{\mathbb{G}}$ gives an operator
$$\ell \cdot \gamma = \overbrace{\gamma +_{\widehat{\mathbb{G}}} \cdots +_{\widehat{\mathbb{G}}} \gamma}^{\ell \text{ times}}.$$
This extends to an action by $\ell \in \mathbb{W}_p(k)$.

- Frobenius: given an integer $n \geq 1$, we define
$$F_n \gamma(t) = \sum_{i=1}^{n}{}_{\widehat{\mathbb{G}}}\, \gamma(\zeta_n t^{1/n}),$$
where ζ_n is an n^{th} root of unity. (This formula is invariant under permuting the root of unity chosen, so determines a curve defined over the original ground ring by Galois descent.)

Definition 4.4.15 (cf. Definition 3.3.5, Definition 3.3.8)**.** A curve γ on a formal group is p–typical when $F_n \gamma = 0$ for $n \neq p^j$. Write $D_* \widehat{\mathbb{G}} \subseteq C\widehat{\mathbb{G}}$ for the subset of p–typical curves.

Lemma 4.4.16 ([Zin84, Equation 4.13])**.** *In the case that the base ring is p–local, $C\widehat{\mathbb{G}}$ splits as a sum of copies of $D_* \widehat{\mathbb{G}}$. There is a natural section $C\widehat{\mathbb{G}} \to D_* \widehat{\mathbb{G}}$ called p–typification, given by the same formula as in Lemma 3.3.6.* □

This construction also plays the role of a Dieudonné functor:

Theorem 4.4.17 ([Zin84, Theorem 3.5 and Theorem 3.28])**.** *The functor D_* determines a* covariant *equivalence of categories between smooth* 1*–dimensional formal groups over k of finite p–height and Dieudonné modules satisfying the three technical conditions of Theorem 4.4.8.* □

One of the main ingredients in the proof of Theorem 4.4.17 is a representability result, which comes out of the following interesting construction. The *Witt scheme* $\mathbb{W}$ represents power series of the form $\prod_{j=1}^{\infty}(1 - a_j x^j)$, and it carries the structure of a group scheme by re-factoring the product of two such power series. Using a rational exponential and logarithm, we also have

$$\begin{aligned}\prod_{j=1}^{\infty}(1-a_jx^j) &= \exp\log\prod_{j=1}^{\infty}(1-a_jx^j) \\ &= \exp\sum_{j=1}^{\infty}\left(-\sum_{k=1}^{\infty}\frac{1}{k}(a_jx^j)^k\right) \\ &= \exp\sum_{n=1}^{\infty}\frac{-t^n}{n}\sum_{m\mid n} m a_m^{n/m} =: \exp\sum_{n=1}^{\infty}\frac{-t^n}{n}w_n(a).\end{aligned}$$

These polynomials w_n, called *ghost polynomials*, describe an injective logarithmic map

$$(w_1, w_2, \ldots)\colon \mathbb{W} \to \mathbb{G}_a^{\infty},$$

the image of which is characterized by the property $w_n(a) \equiv w_{n/p}(a)^p \pmod{p^j}$ for p^j the maximum power of p dividing n. Over a $\mathbb{Z}_{(p)}$–algebra, there is a natural map on ghost components

$$t(w_*(a_*))_{n'p^j} = (w_*(a_{p^j})),$$

witnessing a decomposition $\mathbb{W} \times \operatorname{Spec}_{(p)} \cong \prod_{p\nmid n}\mathbb{W}_p$. Both of these objects are natural limits of their truncations to finitely many power series product terms, and hence they both admit natural *formal* objects, $\widehat{\mathbb{W}}$ and $\widehat{\mathbb{W}}_p$, by taking colimits of the formal completions of these truncations.

Lemma 4.4.18 ([Zin84, Chapter 3])**.** *There are natural correspondences*

$$\begin{aligned}\mathsf{FormalGroups}(\widehat{\mathbb{W}}, \widehat{\mathbb{G}}) &\cong \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) = C\widehat{\mathbb{G}}, \\ \mathsf{FormalGroups}(\widehat{\mathbb{W}}_p, \widehat{\mathbb{G}}) &\cong D_*(\widehat{\mathbb{G}}),\end{aligned}$$

where the universal curve is specified by

FINISH ME

□

This Lemma induces quite a lot of structure on $\widehat{\mathbb{W}}_p$: the Frobenius and Verschiebung operators on curves become operators acting on the formal Witt scheme, which on ghost components have the following behavior:

$$w_n(V_m a_*) = m w_{nm}(a_*), \qquad w_n(F_m a_*) = \begin{cases} w_{n/m}(a_*) & \text{if } m \mid n, \\ 0 & \text{otherwise.}\end{cases}$$

One can use this to give an explicit description of the inverse to the covariant Dieudonné functor whose existence is asserted by Theorem 4.4.17: the idea is that the evaluation map $\widehat{\mathbb{W}}_p \times C\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is surjective, and so we can recover $\widehat{\mathbb{G}}$ by imposing intertwining relations stemming from the evaluation map. Following this through results in the following presentation:

$$D_*{}^{-1}(M)(T) = (\widehat{\mathbb{W}}_p(T) \otimes_{\mathbb{W}_p(k)} M) \Big/ \left(\begin{array}{c} Va \otimes m = a \otimes Fm, \\ Fa \otimes m = a \otimes Vm \end{array} \right).$$

We now seek to compare our two Dieudonné functors. In some sense, the comparison is inspired by an integration pairing: $D_*(\widehat{\mathbb{G}})$ is populated by curves γ on $\widehat{\mathbb{G}}$ and $D^*(\widehat{\mathbb{G}})$ is populated by 1–forms ω on $\widehat{\mathbb{G}}$, which we would like to sew together to form

$$D_*(\widehat{\mathbb{G}}) \times D^*(\widehat{\mathbb{G}}) \xrightarrow{(\gamma,\omega)\mapsto \text{“}\int_\gamma \omega\text{”}} \mathbb{W}_p(k).$$

There is more to say here. Can you give more intuition about how these two presentations are related, for example from Lie theory? What curve does a cohomologically left-invariant form get sent to? Is “cohomologically-invariant” analogous to “p-typification”, perhaps along the lines of the “crystalline”-ness of H^1_{dR}? Can these primitives be furthermore related to the idea that taking primitive cooperations selects the additive ones? // Erick Knight suggested that there should be an alternative definition of covariant Dieudonné theory also expressed by the cohomology of some sheaf, and that this will give rise to the desired duality pairing with values in some Tate-twisted ground object, which is precisely what we are looking for. I wonder if the right thing to do is to form the bar complex on the de Rham sheaf...! // I talked to Mike about this, and his suggestion was to learn about Cartier’s “logarithmic modules”, which lift the p–typical curves on a formal group in positive characteristic to a preferred submodule of curves on a lift to mixed characteristic, and that once these two things (1–forms and logarithmic curves) are in the same place, the pairing I want is probably the residue map / the Serre duality pairing. He half-heartedly suggested Lazard’s book, especially VII.3.10 and V.7.3. (Also on the table of ideas: the Cartier dual is witnessed by the existence of a Poincaré bundle, which is a biextension. Maybe this leads somewhere?)

This can be made rigorous by noting the following basic but ultra-important fact about the Witt scheme:

Sure seems to me like $\widehat{C\mathbb{W}}_p$ is (canonically, even??) isomorphic to $\widehat{\mathbb{W}}_p$. It would be nice to sort this out.

Lemma 4.4.19. *There is a formal group scheme $\widehat{C\mathbb{W}}_p$ with the property*

$$\mathsf{FormalGroups}(\widehat{\mathbb{W}}_p, \widehat{\mathbb{G}})^* \cong \mathsf{FormalGroups}(\widehat{\mathbb{G}}, \widehat{C\mathbb{W}}_p). \qquad \square$$

Corollary 4.4.20 ([MM74, Section II.15], [Kat81, Equation 5.5.2])**.** *There is an isomorphism $(D_*\widehat{\mathbb{G}})^* \xrightarrow{\cong} D^*\widehat{\mathbb{G}}$.*

Construction. There is a canonical short exact sequence

$$0 \to \widehat{\mathbb{G}}_a \to \widehat{C\mathbb{W}}_p \xrightarrow{V} \widehat{C\mathbb{W}}_p \to 0,$$

and a co-curve $\gamma^* \colon \widehat{\mathbb{G}} \to \widehat{C\mathbb{W}}_p$ gives a pullback sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathbb{G}}_a & \longrightarrow & \widehat{\mathbb{W}}_p & \xrightarrow{V} & \widehat{\mathbb{W}}_p & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \gamma^*} & & \\ 0 & \longrightarrow & \widehat{\mathbb{G}}_a & \longrightarrow & E & \longrightarrow & \widehat{\mathbb{G}} & \longrightarrow & 0. \end{array}$$

This latter sequence is a rigidified extension of $\widehat{\mathbb{G}}$ by $\widehat{\mathbb{G}}_a$, as in Remark 4.4.14. The conclusion of Mazur and Messing is that this is an isomorphism. □

Remark 4.4.21 ([Strb, Remark 18.1]). A coordinate x on $\mathbb{CP}^\infty_E$ induces a sequence of isomorphisms

$$\mathsf{FormalGroups}(BU_E, \widehat{\mathbb{G}}) \cong \mathsf{FormalSchemes}(\mathbb{CP}^\infty_E, \widehat{\mathbb{G}}) \overset{x}{\cong} \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) = C\widehat{\mathbb{G}},$$

which presents E^*BU as the Witt Hopf algebra. However, this isomorphism is not especially interesting: for one, it is *highly* dependent upon the choice of coordinate, but also the far right-hand object has no dependence on $\mathbb{CP}^\infty_E$, and so the operations we have been studying—formal group auto- and endomorphisms of $\mathbb{CP}^\infty_E$, mainly—do not act, and this isomorphism cannot be equivariant in any useful sense.

Remark 4.4.22 ([Gro74, Chapitre VI]). The contravariant Dieudonné functor described above has a natural extension by choosing lifts over other pro-Artinian k–algebras, like the Lubin–Tate moduli stack of Definition 3.4.3. The resulting network of objects most naturally organizes into a sheaf over the *crystalline site*, but it is possible in this setting to re-express such a sheaf as a quasi-coherent sheaf over the Lubin–Tate stack which is equipped with a flat connection, and it is additionally acted upon by the familiar operators F and V.

Remark 4.4.23 (cf. [Mor85, Section 2.3]). Dieudonné theory gives rise to an important function called the *period map*. Although the crystalline nature of the cohomology group H^1_{dR} makes our definition of D^* invariant of choice of lift, the underlying chain complex is *not* invariant of choice of lift. In particular, the subsheaf of honestly invariant differentials $\omega_{\widehat{\mathbb{G}}}$ selects an interesting 1–dimensional vector subspace of $PH^1_{dR}(\widehat{\mathbb{G}})$ [Mor85, Section 2.3]. Thinking of $\widehat{\mathbb{G}}$ as a point in $(\mathcal{M}_{\mathbf{fg}})^\wedge_{\widehat{\mathbb{G}}_0}(\mathbb{W}_p(k))$, this observation gives rise to an interesting function

$$\begin{aligned} \pi_{GH}\colon (\mathcal{M}_{\mathbf{fg}})^\wedge_{\widehat{\mathbb{G}}_0}(\mathbb{W}_p(k)) &\to \mathbb{P}(D^*(\widehat{\mathbb{G}})), \\ \widehat{\mathbb{G}} &\mapsto [\omega_{\widehat{\mathbb{G}}} \subseteq D^*(\widehat{\mathbb{G}})]. \end{aligned}$$

This map has incredibly good properties. It is equivariant for the action of $\mathrm{Aut}\,\widehat{\mathbb{G}}_0$ [HG94b, Theorem 1], and with enough work one can use this to extract explicit (recursive) formulas expressing the action [DH95], [Strb, Section 24], [HG94a, Section 22], bringing some relief to the problem of Remark 3.6.17. Also, in a suitable context it becomes an étale morphism with identifiable fibers [HG94b, Theorem 1], [HG94a, Sections 23-4], which allows one to give an explicit formula for the dualizing sheaf [HG94b, Corollary 3] with direct applications to topology [HG94b, Theorem 6]:

$$\mathcal{M}_{E_\Gamma}(\Sigma^{-d^2-d}\mathbb{I}_{\mathbb{Q}/\mathbb{Z}}) = \Omega^{d-1}_{(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma} = \omega^{\otimes d}[\det].$$

In Figure 4.3 we sketch the period map at $n = 2$ and $p = 2$, which has the following list of properties [HG94a, Appendix 25], [Yu95]:

- The center of the $\mathbb{Z}_{p^2}$–points of Lubin–Tate space corresponds to the canonical lift, which is the formal group that further acquires an $\mathcal{O}_A$–module structure. It has π–series $[\pi](x) = \pi x + x^{q^2}$.
- There are three nontrivial points in $\widehat{\mathbb{G}}[2]$: α, β, and $\alpha + \beta$. Quotienting by them gives three points at order $1/(q + 1)$, the first bunch of "quasicanonical lifts", which have partial formal $\mathcal{O}_A$–module structures.
- The map π_{GH} sends the canonical lift to $0 = [1 : 0]$, sends the first order quasicanonical points to $\infty = [0 : 1]$, and alternates from there. The three branches of "directions to quotient" carve $\mathbb{P}^1$ up into three lobes. This is because π_{GH} is equivariant for *isogenies*, and quotienting by one of these order 2 subgroups is a lift of the Frobenius isogeny on the residue formal group.
- Out to order $1/q$, π_{GH} is injective.
- You can compute these orders using the Newton polygon associated to the π–series.
- At each quasicanonical point, you also also form three quotients: two of them make the situation "worse", and one of them makes the situation "better", which is a kind of witness to the identification $\widehat{\mathbb{G}}/\widehat{\mathbb{G}}[2] \cong \widehat{\mathbb{G}}$.
- The canonical Frobenius $F_{can} = \begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}$ first flips the two coordinates (and

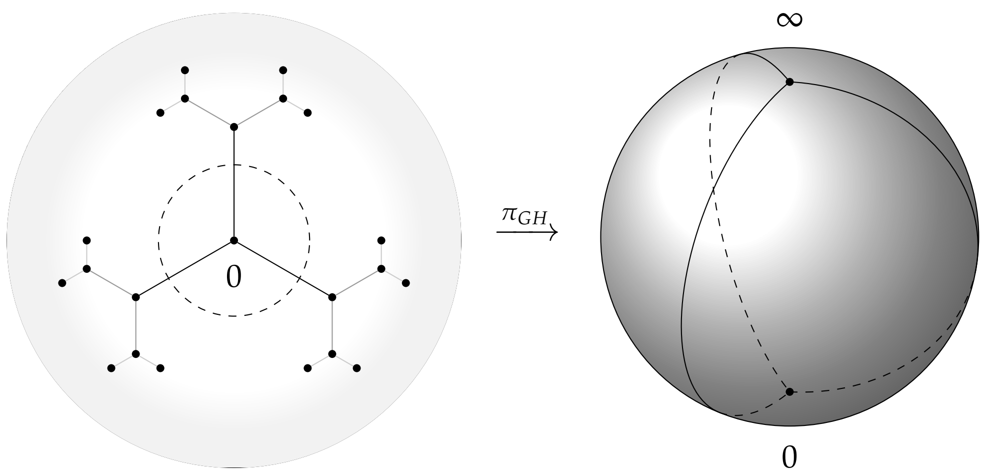


Figure 4.3: The period map at $n = 2$, $p = 2$

scales one by p), then flips them back (and scales the other by p), and after two flips scaling everything by p scales back down by homogeneous coordinates.

- The group $\mathbb{F}_4^\times$ acts by rotation on $\mathbb{P}^1$.
- These quasicanonical points are the ones with nontrivial stabilizers under the action by the Morava stabilizer group—all the other points belong to free orbits. The canonical lift has the largest stabilizer of all.

Remark 4.4.24. It is also possible to build versions of Dieudonné theory over still more exotic rings. The most successful such version is Zink's theory of Dieudonné displays [Zin02], which has found some application in algebraic topology [Law10].

4.5 Ordinary cooperations for Landweber flat theories

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Our goal in this Lecture is to put Dieudonné modules to work for us in algebraic topology. The executive summary of Dieudonné theory is that it gives a *linear* presentation of the theory of Hopf algebras. From the perspective of algebraic topology, functors sending cofiber sequences to exact sequences in a linear category are precisely the homology functors. Tying these two ideas together, if we can find a functor that sends exact sequences of spaces (or spectra) to exact sequences of Hopf algebras, we can post-compose it with a suitable version of the Dieudonné functor to get a homology functor and hence a *spectrum*.

To meet algebraic topology in its natural setting, it will be useful to also have a version of Dieudonné theory that is well-adapted to working with formal groups whose coordinate ring forms a *graded* Hopf algebra. Using as inspiration our previous identification

$$C(\widehat{\mathbb{G}}) = \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}) \cong \mathsf{FormalGroups}(\widehat{\mathbb{G}}, \widehat{C\mathbb{W}}_p) \cong \mathsf{HopfAlgebras}(\mathcal{O}_{\widehat{C\mathbb{W}}_p}, \mathcal{O}_{\widehat{\mathbb{G}}}),$$

we are thus moved to form graded versions of the Witt Hopf algebra. More precisely, the following theorem says that there are graded versions of the Witt Hopf algebra that give a sequence of projective generators for the category of connected graded Hopf algebras over $\mathbb{F}_p$:

Theorem 4.5.1 ([Sch70, Section 3.2], [GLM93, Proposition 1.6])**.** *Let $S(n)$ denote the free graded-commutative Hopf algebra over $\mathbb{F}_p$ on a single generator in degree $n > 0$. There is a projective cover $H(n) \twoheadrightarrow S(n)$, given by the formula*

- *If either of the following conditions hold...*
  - *$p = 2$ and $n = 2^m k$ for $2 \nmid k$ and $m > 0$,* or
  - *$p \neq 2$ and $n = 2p^m k$ for $p \nmid k$ and $m > 0$,*

  then $H(n) = \mathbb{F}_p[x_0, x_1, \ldots, x_k]$ with the Witt vector diagonal, i.e., the diagonal is arranged so that the elements $w_j = x_0^{p^j} + p x_1^{p^{j-1}} + \cdots + x_j$ are primitive.
- *Otherwise, $H(n) = S(n)$ is the identity.* □

Corollary 4.5.2. *The category $\mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}$ of finite–type graded connected Hopf $\mathbb{F}_p$–algebras is a full subcategory of modules over*

$$\bigoplus_{n,m} \mathsf{GradedHopfAlgebras}(H(n), H(m)).$$

Proof sketch. This is a general nonsense consequence of having found a set of projective generators. The functor presenting the inclusion is

$$M \mapsto \bigoplus_{n=0}^{\infty} \mathsf{GradedHopfAlgebras}(H(n), M),$$

and since this functor is corepresented its automorphisms are encoded by the indicated ring. □

We would also like to give a set of conditions, analogous to the technical conditions appearing in the previous two presentations, which select this full subcategory out from all possible modules over this endomorphism ring.

Definition 4.5.3 ([GLM93, pg. 116])**.** Let $\mathsf{GradedDMods}$ denote the category of graded abelian groups M equipped with maps $V\colon M_{pn} \to M_n$ and $F\colon M_n \to M_{pn}$ (where n is even if $p \neq 2$) satisfying

1. $M_{<1} = 0$.
2. If n is odd, then $pM_n = 0$.
3. The composites are controlled by $FV = p$ and $VF = p$.[14]

Remark 4.5.4. Suppose that n is even, written at odd primes in the form $n = 2p^m k$ with $p \nmid k$ or at $p = 2$ in the form $n = 2^m k$ with $2 \nmid k$ at $p = 2$. Then, combining the above relations, we get the torsion condition $p^{m+1} M_n = F^{m+1} V^{m+1} M_n = 0$.

Theorem 4.5.5 ([Sch70, Section 5], [GLM93, Theorem 1.11])**.** *The functor*

$$D_*\colon \mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/} \to \mathsf{GradedDMods},$$
$$D_*(H) = \bigoplus_n D_n(H) = \bigoplus_n \mathsf{GradedHopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}(H(n), H)$$

is an exact equivalence of categories. Moreover, $D_* H(n)$ is characterized by the equation

$$\mathsf{GradedDMods}(D_* H(n), M) = M_n.$$ □

Having produced our desired graded Dieudonné theory, we now need some topological input. We are by now well aware that the homology of an H–space forms a Hopf algebra, and the Serre spectral sequence for a fibration of H–spaces

$$F \to E \to B$$

takes the form

$$E_2^{*,*} = H^* B \otimes H^* F \Rightarrow H^* E.$$

The following result of Goerss–Lannes–Morel says that in the case that the fibration is one of *infinite loopspaces*, we have the exactness property we need:

[14] These are induced by the inclusion $H(n) \subseteq H(pn)$ and by the map $H(pn) \to H(n)$ sending x_n to x_{n-1}^p.

Theorem 4.5.6 ([GLM93, Lemma 2.8])**.** *Let $X \to Y \to Z$ be a cofiber sequence of spectra. Then, provided $n > 1$ satisfies $n \not\equiv \pm 1 \pmod{2p}$, there is an exact sequence*

$$D_n H_* \Omega^\infty X \to D_n H_* \Omega^\infty Y \to D_n H_* \Omega^\infty Z. \qquad \square$$

This Theorem is not especially easy to prove: one works very directly with unstable modules over the Steenrod algebra, the bar spectral sequence, and Postnikov decomposition of infinite loopspaces. We refer the reader to the paper directly, as we have been unable to find a useful improvement upon or even summary of the results presented there. Nonetheless, granting this Theorem, we use Brown representability to draw the following consequence:

Corollary 4.5.7 ([GLM93, Theorem 2.1, Remark 2.9])**.** *For $n > 1$ an integer satisfying $n \not\equiv \pm 1 \pmod{2p}$, there is a spectrum $B(n)$ satisfying*

$$B(n)_n X \cong D_n H_* \Omega^\infty X.$$

(As convention, when $n \equiv \pm 1 \pmod{2p}$ we set $B(n) := B(n-1)$, and $B(0) := \mathbb{S}^0$.) $\square$

Before exploiting this result to compute something about unstable cooperations, we will prove a sequence of small results making these spectra somewhat more tangible.

Lemma 4.5.8 ([GLM93, Lemma 3.2])**.** *The spectrum $B(n)$ is connective and p–complete.*

Proof. First, rearrange:

$$\pi_k B(n) = B(n)_n \mathbb{S}^{n-k} = D_n H_* \Omega^\infty \Sigma^\infty S^{n-k}.$$

If $k < 0$, n is below the connectivity of $\Omega^\infty \mathbb{S}^{n-k}$ and hence this vanishes. The second assertion follows from the observation that $H\mathbb{Z}_* B(n)$ is an $\mathbb{F}_p$–module, followed by an Adams spectral sequence argument. To see the assertion about being an $\mathbb{F}_p$–module, restrict to the case $n \not\equiv \pm 1 \pmod{2p}$ and calculate

$$\begin{aligned} H\mathbb{Z}_k B(n) &= B(n)_n \Sigma^{n-k} H\mathbb{Z} \\ &= D_n H_* K(\mathbb{Z}, n-k) \\ &= [H(n), H_* K(\mathbb{Z}, n-k)]_n \\ &= [Q^* H_* K(\mathbb{Z}, n-k)]_n. \end{aligned} \qquad \square$$

We can use a similar trick as in the second part of the proof to calculate the cohomology groups $H^*B(n)$:

Definition 4.5.9 ([GLM93, Example 3.6]). Let $G(n)$ be the free unstable $\mathcal{A}^*$–module on one generator of degree n, so that if M is an unstable $\mathcal{A}^*$–module then

$$\mathsf{Modules}_{\mathcal{A}^*}(G(n), M) = M_n.$$

This module admits a presentation as

$$G(n) = \begin{cases} \Sigma^n \mathcal{A}/\{\beta^\varepsilon P^i \mid 2pi + 2\varepsilon > n\}\mathcal{A} & \text{if } p > 2, \\ \Sigma^n \mathcal{A}/\{\mathrm{Sq}^i \mid 2i > n\}\mathcal{A} & \text{if } p = 2. \end{cases}$$

The Spanier–Whitehead dual of this right-module, $DG(n)$, is given by

$$\Sigma^n (DG(n))^* = \begin{cases} \mathcal{A}/\mathcal{A}\{\chi(\beta^\varepsilon P^i) \mid 2pi + 2\varepsilon > n\} & \text{if } p > 2, \\ \mathcal{A}/\mathcal{A}\{\chi \mathrm{Sq}^i \mid 2i > n\} & \text{if } p = 2. \end{cases}$$

Theorem 4.5.10 ([GLM93, Proof of Theorem 3.1]). *There is an isomorphism*

$$H^*B(n) \cong \Sigma^n (DG(n))^*.$$

Proof. We restrict attention to $n \not\equiv \pm 1 \pmod p$, where we can use Corollary 4.5.7 directly. Start, as before, by addressing the dual problem of computing the mod–p homology:

$$H_k B(n) = B(n)_n \Sigma^{n-k} H = D_n H_* K(\mathbb{F}_p, n-k).$$

The unstable module $G(n)$ also enjoys a universal property in the category of *stable* $\mathcal{A}^*$–modules, by passing to the maximal unstable quotient $\Omega^\infty M$ of a stable module M:

$$\mathsf{Modules}_{\mathcal{A}^*}(G(n), M) \cong [\Omega^\infty M]_n.$$

Hence, we can continue our computation:

$$\begin{aligned} H_k B(n) &= D_n H_* K(\mathbb{F}_p, n-k) \\ &= \mathsf{Modules}_{\mathcal{A}^*}(G(n), \Sigma^{n-k} \mathcal{A}_*) \\ &= \mathsf{Modules}_{\mathbb{F}_p}(G(n)_{n-k}, \mathbb{F}_p). \end{aligned}$$

We learn immediately that $H_*B(n)$ is finite. We would like to show, furthermore, that $H_*B(n)$ is the Spanier–Whitehead dual $\Sigma^n DG(n)$. It suffices to show

$$\mathsf{Modules}_{\mathcal{A}^*}(G(n), \Sigma^j \mathcal{A}_*) = \mathsf{Modules}_{\mathcal{A}^*}(\mathbb{F}_p, \Sigma^j \mathcal{A}_* \otimes H_*B(n))$$

for all values of j. This follows from calculating $B(n)_n \Sigma^{n+j} H$ using the same method. Finally, linear-algebraic duality and Definition 4.5.9 give the Theorem. □

Lastly, for a *space* X, we definitionally have that have H_*X forms an unstable module over the Steenrod algebra, i.e., $\Omega^\infty H_*X = H_*X$. This has the following direct sequence (with minor fuss at the bad indices $n \equiv \pm 1 \pmod p$):

Lemma 4.5.11 ([GLM93, Lemma 3.3])**.** *For X a space, there is a natural surjection*

$$B(n)_n X \to H_n X. \qquad \square$$

Let's now work toward using the $B(n)$ spectra to analyze the Hopf rings arising from unstable cooperations. Our intention is to prove the following:

Theorem 4.5.12. *For $F = H$ and E a Landweber flat homology theory, the comparison map*

$$\mathcal{A}\!\!\mathcal{A}(H, E) \to H_*\underline{E}_{2*}$$

is an isomorphism of Hopf rings.

We have previously computed that the comparison map

$$\mathcal{A}\!\!\mathcal{A}(H, BP) \to H_*\underline{BP}_{2*}$$

is an isomorphism. We will now reimagine this statement in terms of Dieudonné theory, but in order to do this we first have to reimagine some of Dieudonné theory itself, as our description of it is concerned with *Hopf algebras* rather than *Hopf rings*. A Hopf ring is not much structure on top of an (externally) graded system of (internally) graded Hopf algebras A_*: it is a multiplication map

$$\circ \colon A_* \boxtimes A_* \to A_*,$$

where "⊠" is a kind of graded tensor product of externally graded Hopf algebras [HT98, Proposition 2.6], [BL07, Definition 2.2], [Goe99, Section 5]. Since D_* gives an equivalence of categories between internally graded Hopf algebras and internally graded Dieudonné modules, we should be able to find an analogous formula for the tensor product of Dieudonné modules.

Definition 4.5.13. [Goe99, pg. 154] The naive tensor product $M \otimes N$ of Dieudonné modules M and N receives the structure of a $\mathbb{W}(k)[V]$–module, where $V(x \otimes y) = V(x) \otimes V(y)$. We define the *tensor product of Dieudonné modules*[15] by

$$M \boxtimes N = \frac{\mathbb{W}(k)[F,V]}{(VF = p)} \otimes_{\mathbb{W}(k)[V]} (M \otimes N) \Big/ \left(\begin{array}{c} 1 \otimes Fx \otimes y = F \otimes x \otimes Vy, \\ 1 \otimes x \otimes Fy = F \otimes Vx \otimes y \end{array} \right).$$

Lemma 4.5.14 ([Goe99, Corollary 8.14])**.** *The natural map*

$$D_*(M) \boxtimes D_*(N) \to D_*(M \boxtimes N)$$

is an isomorphism. □

Definition 4.5.15. For a ring R, a *Dieudonné R–algebra A_** is an externally graded Dieudonné module equipped with an R–action and a unital multiplication

$$\circ\colon A_* \boxtimes A_* \to A_*.$$

Example 4.5.16 ([Goe99, Proposition 10.2]). Inspired by Lemma 4.2.8 and our interest in $H_*\underline{E}_*$, for a complex-oriented homology theory E we define its *algebraic Dieudonné E_*–algebra* by

$$R_E = E_*[b_1, b_2, \ldots] / \left(\ b(s+t) = b(s) +_E b(t) \ \right),$$

where V is multiplicative, V fixes E_*, and V satisfies $Vb_{pj} = b_j$.[16] We also write $D_E = \{D_{2m} H_* \underline{E}_{2n}\}$ for the even part of the topological Dieudonné algebra, and these come with natural comparison maps

$$R_E \to D_E \to D_* H_* \underline{E}_{2*}.$$

Theorem 4.5.17 ([Goe99, Theorem 11.7])**.** *Restricting attention to the even parts, the maps*

$$R_E \to D_E \to D_* H_* \underline{E}_{2*}$$

are isomorphisms for E Landweber flat.

[15] This definition is specialized to $\mathbb{F}_p$, where we don't have to worry about Frobenius semilinearity.

[16] If E_* is torsion-free, then this determines the behavior of $F = \frac{1}{p} V$.

Proof. In Corollary 4.3.7, we showed that these maps are isomorphisms for $E = BP$. However, the right–hand object can be identified via Brown–Gitler juggling:

$$D_n H_* \underline{E}_{2j} = B(n)_n \Sigma^{2j} E = E_{2j+n} B(n).$$

It follows that if E is Landweber flat, then the middle– and right–terms are determined by change-of-base from the respective BP terms. Finally, the left term commutes with change-of-base by its algebraic definition, and the theorem follows. □

Remark 4.5.18. The proof of Theorem 4.5.17 originally given by Goerss [Goe99] involved a lot more work, essentially because he didn't want to assume Theorem 4.3.1 or Corollary 4.3.7. Instead, he used the fact that $\Sigma^\infty_+ \Omega^2 S^3$ is a regrading of the ring spectrum $\bigvee_n B(n)$, together with knowledge of $BP_* \Omega^2 S^3$. Since we have already done the hard work of proving Theorem 4.3.1, we are not obligated to pursue this other line of thought.

Remark 4.5.19 ([Goe99, Proposition 11.6, Remark 11.4]). The Dieudonné algebra framework also makes it easy to add in the odd part after the fact. Namely, suppose that E is a torsion–free ring spectrum and suppose that $E_* B(n)$ is even for all n. In this setting, we can verify the purely topological version of this statement: the map

$$D_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*$$

is an isomorphism. To see this, note that because

$$E_{2n-2k-1} B(2n) \to D_{2n} H_* \underline{E}_{2k+1}$$

is onto and $E_{2n-2k-1}B(2n)$ is assumed zero, the group $D_{2n} H_* \underline{E}_{2k+1}$ vanishes as well. A bar spectral sequence argument shows that $D_{2n+1} H_* \underline{E}_{2k+2}$ is also empty [Goe99, Lemma 11.5.1]. Hence, the map on even parts

$$(D_E[e]/(e^2 - b_1))_{*,2n} \to (D_* H_* \underline{E}_*)_{*,2n}$$

is an isomorphism, and we need only show that

$$D_* H_* \underline{E}_{2n} \xrightarrow{e \circ -} D_* H_* \underline{E}_{2n+1}$$

is an isomorphism as well. Since $e(Fx) = F(Ve \circ x) = 0$ and $D_* A / F D_* A \cong Q^* A$ for a Hopf algebra A, we see that e kills decomposables and suspends indecomposables:

$$e \circ D_* H_* \underline{E}_{2n} = \Sigma Q H_* \underline{E}_{2n}.$$

This is also what happens in the bar spectral sequence, and the claim follows. In light of Theorem 4.5.17, this means that for Landweber flat E, the comparison isomorphism can be augmented to a further isomorphism

$$R_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*.$$

Remark 4.5.20 ([HH95]). The results of this Lecture are accessed from a different perspective by Hopkins and Hunton, essentially by forming a tensor product of Hopf rings and showing that Landweber flatness induces a kind of flatness with respect to the Hopf ring tensor product as well.

4.6 Cooperations among geometric points on $\mathcal{M}_{\mathbf{fg}}$

Our discussion of unstable cooperations has touched on each of the families of chromatic homology theories described in Definition 3.5.4 except one: the Morava K– and E–theories. Our final goal before moving on to other subjects is to describe some of the mixed unstable cooperations for $(K_\Gamma)_* \underline{K_{\Gamma'}}_*$. In complete generality, this seems like a difficult problem: our algebraic model is rooted in formal group homomorphisms, and we have not proven any theorems about the moduli of such for arbitrary finite-height formal groups. However, the landscape brightens considerably in the case where we pick $\Gamma' = \widehat{\mathbb{G}}_a$, as this is the sort of calculation we considered in Lemma 3.4.11 and Lemma 3.4.12. In light of this, we specialize Γ' to $\widehat{\mathbb{G}}_a$ (and hence $K_{\Gamma'}$ to an Eilenberg–Mac Lane spectrum H), and we abbreviate K_Γ to just K.

As with all the other major results of this Case Study, our approach will rest on the bar spectral sequence

$$\operatorname{Tor}^{K_* \underline{H}_q}_{*,*}(K_*, K_*) \Rightarrow K_* \underline{H}_{q+1}.$$

The analysis of this spectral sequence was first accomplished by Ravenel and Wilson [RW80], but has since been re-envisioned by Hopkins and Lurie [HL, Section 2]. In order to give an effective analysis of this spectral sequence in line with the theme of this book, we will endeavor to give algebro-geometric interpretations of its input and its output, beginning with the case $q = 0$ and $H = H(S^1[p^j])$ for some $1 \leq j \leq \infty$. This task itself begins with giving just *algebraic* descriptions of the input and output. For $j < \infty$, we have essentially already computed the output by other means:

Theorem 4.6.1 ([RW80, Theorem 5.7], [HL, Proposition 2.4.4], cf. Lemma 2.6.1)**.** *There is an isomorphism*

$$BS^1[p^j]_K \cong BS^1_K[p^j].$$

Remarks on proof. We have already proven a very similar theorem as Lemma 2.6.1. The crux of that argument was to show that multiplication by the p–series in $MU^*\mathbb{C}\mathrm{P}^\infty$ was injective without knowing very much about the ground ring MU^*. Here our task is even easier: any p–series for Γ takes the form $[p](x) = cx^{p^d} + \cdots$ for c a unit. Such an element is certainly not a zero-divisor. □

Remark 4.6.2. In the proof of the homological statement dual to Theorem 4.6.1, there is a corresponding exact sequence of Hopf algebras

$$\begin{tikzcd} & K_*BS^1 \arrow[dr, "- \frown [p^j](x)"] & \\ K_*(BS^1[p^j]) \arrow[ur] & & K_*BS^1, \arrow[ll, "\partial"] \end{tikzcd}$$

where again $\partial = 0$ and hence $K_*(BS^1[p^j])$ is presented as the kernel of the map "cap with $[p^j](x)$". We will revisit this duality in the next Case Study.

With this in hand, the analysis of the bar spectral sequence proceeds very much analogously to the example of the unstable dual Steenrod algebra of Lecture 4.1. We will analyze what *must* happen in the bar spectral sequence

$$\mathrm{Tor}_{*,*}^{K_*S^1[p^j]}(K_*, K_*) \Rightarrow K_*BS^1[p^j]$$

in order to reach the conclusion of Theorem 4.6.1. In the input to this spectral sequence, the ground algebra is given by a noncanonical isomorphism

$$K_*\underline{H(S^1[p^j])}_0 \cong K_*\underline{H\mathbb{Z}/p^j}_0 = K_*[[1]]/([1]^{p^j} - 1) = K_*[[1]-[0]]/\langle [1]-[0]\rangle^{p^j}.$$

The Tor–algebra for this truncated polynomial algebra $K_*[a_\varnothing]/a_\varnothing^p$ is then given by the formula

$$\mathrm{Tor}_{*,*}^{K_*[a_\varnothing]/a_\varnothing^{p^j}}(K_*, K_*) = \Lambda[a'_\varnothing] \otimes \Gamma[a_\varnothing],$$

the combination of an exterior algebra and a divided power algebra.[17] We know which classes are supposed to survive this spectral sequence, and hence we know

[17] In Ravenel–Wilson [RW80, Lemma 6.6], the elements $a_\varnothing$ and $a'_\varnothing$ are identified as a *transpotence* and a *homology suspension* respectively.

where the differentials must be:

$$d(a_\varnothing)^{[p^{jd}]} = a'_\varnothing,$$
$$\Rightarrow d(a_\varnothing)^{[i+p^{jd}]} = a'_\varnothing \cdot a_\varnothing^{[i]}.$$

The spectral sequence collapses after this differential. In the case $1 < j < \infty$, there are some hidden multiplicative extensions in the spectral sequence, but these too are all determined by already knowing the multiplicative structure on $K_* \underline{H(S^1[p^j])}_1$.

However, the case of $j = \infty$ is a bit different, beginning with the following Lemma:

Lemma 4.6.3. *For $q \geq 1$, there is a p–adic equivalence of $K(\mathbb{Q}/\mathbb{Z}_{(p)}, q)$ with $K(\mathbb{Z}, q+1)$.*

Proof. This is a consequence of the fiber sequence

$$K(\mathbb{Q}, q) \to K(\mathbb{Q}/\mathbb{Z}_{(p)}, q) \to K(\mathbb{Z}_{(p)}, q+1).$$

The first term has vanishing mod–p homology, forcing the $H\mathbb{F}_p$–Serre spectral sequence of the fibration to collapse and for the edge homomorphism to be an isomorphism. Similarly, the map $K(\mathbb{Z}, q+1) \to K(\mathbb{Z}_{(p)}, q+1)$ is an equivalence on mod–p homology. □

Remark 4.6.4. Thinking of $K(\mathbb{Z}, q+1)$ as $B^q S^1$, one can also think of this theorem as giving a p–adic equivalence between $B^q(S^1[p^\infty])$ and $B^q S^1$—i.e., the prime–to–p parts of S^1 do not matter for p–adic homotopy theory.

We use this to continue the analysis of the case $q = 1$ and $j = \infty$, where the Lemma gives $B(S^1[p^\infty]) = \mathbb{C}\mathrm{P}^\infty$. The bar spectral sequence of interest then takes the form

$$\mathrm{Tor}_{*,*}^{K_* S^1}(K_*, K_*) \Rightarrow K_* \mathbb{C}\mathrm{P}^\infty.$$

The input algebra $K_* S^1$ is exterior on a single generator in odd degree, and so its Tor–algebra is linearly dual to a power series algebra on a single generator in even degree. Since all of its input is even, this spectral sequence collapses immediately.

There is a lot of structure visible in this collection of spectral sequences, as considered simultaneously. Without further inspection, the spectral sequence at $j = \infty$ records that $\mathbb{C}\mathrm{P}^\infty_K$ is a formal variety, and the spectral sequences at the finite values $1 \leq j < \infty$ encode in their differentials the behavior of the map $p^j \colon \Gamma \to \Gamma$ on functions—indeed, this appears to be the entire job of $a'_\varnothing$. Lastly, we notice that the E_∞ page of each finite-range spectral sequence includes into the spectral sequence at $j = \infty$, and moreover this filtration is exhaustive: every term in the $j = \infty$ spectral sequence appears at some $j < \infty$ stage. Since this last property is about the E_∞ pages, it is really a property of the formal group Γ, which we record in a definition:

Definition 4.6.5 ([Gro74, Definition 4.1]). A *p–divisible group*[18] over a field k is a system $\mathbb{G}_j$ of finite group schemes and inclusions $i^j_k \colon \mathbb{G}_k \to \mathbb{G}_j$ for $k < j$ which model the inclusions $\mathbb{G}_j[p^k] \subseteq \mathbb{G}_j$.

Definition 4.6.6. A p–divisible group is said to be *connected* when its constituent subgroups $\mathbb{G}_j$ are infinitesimal thickenings of $\operatorname{Spec} k$. An example of this is the sequence of torsion subgroups $\widehat{\mathbb{G}}[p^j]$ of a formal group $\widehat{\mathbb{G}}$. A *nonexample* of this is the sequence of constant group schemes $\widehat{\mathbb{G}}_j = S^1[p^j]$.

Lemma 4.6.7 ([Gro74, Section 6.7]). *Over a perfect field of positive characteristic p, a connected p–divisible group is equivalent to a smooth formal group of finite height.*

Correspondence. The maps in both directions are easy: a p–divisible group is sent to its colimit, and a formal group of finite height is sent to its system of p^j–torsion subgroups. In both directions there is something mild to check: that the colimit gives a formal variety, and that the system of p^j–torsion subgroups has the indicated exactness properties. □

We will soon see that these interrelations among the bar spectral sequences for the different Eilenberg–Mac Lane spaces, as well as the special behavior of the spectral sequence at $j = \infty$, are generic phenomena in q. We record the steps in our upcoming induction in the following Theorem:

Theorem 4.6.8 ([HL, Theorems 2.4.11–13]). *The following claims give a complete description of the Morava K–theory schemes associated to Eilenberg–Mac Lane spaces.*

1. *The formal scheme $(\underline{H(S^1[p^\infty])_q})_K$ is a formal variety of dimension $\binom{d-1}{q-1}$.*

2. *Suppose that $(\underline{H(S^1[p^\infty])_{q-1}})_K$ is a p–divisible formal group of height $\binom{d}{q-1}$ and dimension $\binom{d-1}{q-1}$, that $(\underline{H(S^1[p])_{q-1}})_K$ models its p–torsion, and that the cup product*

[18] Some like to call these *Barsotti–Tate groups*, which is probably the better name, since "p–divisible group" sounds like a property rather than all this data.

induces an isomorphism

$$\theta^{q-1}\colon \mathbb{Q}/\mathbb{Z}_{(p)} \otimes D(\underline{H(S^1[p^\infty])}_1)^{\wedge(q-1)} \to D(\underline{H(S^1[p^\infty])}_{q-1}),$$

where $D(G)$ denotes the Dieudonné module associated to $K_0(G)$ for $K_*(G) = K_0(G) \otimes_k K_*$ a p–divisible Hopf algebra. The same claims are then true with $q-1$ replaced everywhere by q.

3. *Consider the model $\mathbb{Q}/\mathbb{Z}_{(p)} \cong S^1[p^\infty]$ for the p–primary part of the circle group. Suppose that for each j, the short exact sequence of groups*

$$0 \to \frac{1/p^j \cdot \mathbb{Z}_{(p)}}{\mathbb{Z}_{(p)}} \to \frac{\mathbb{Q}}{\mathbb{Z}_{(p)}} \to \frac{\mathbb{Q}}{1/p^j \cdot \mathbb{Z}_{(p)}} \to 0$$

induces a short exact sequence of group-schemes upon applying $(\underline{H(-)}_{q-1})_K$. The sequence of group schemes under $(\underline{H(-)}_q)_K$ is then also short-exact.

Proof of Part 1. This claim turns out to be entirely algebraic and a matter of being able to compute $H^*(\mathbb{G};\widehat{\mathbb{G}}_a)$ for $\mathbb{G}$ a connected p–divisible group. This is expressed in the main algebraic result of Hopkins–Lurie:

Theorem 4.6.9 ([HL, Theorem 2.2.10])**.** *Let $\mathbb{G}$ be a p–divisible group over a perfect field k of positive characteristic p. There is then an isomorphism*

$$H^*(\mathbb{G};\widehat{\mathbb{G}}_a) \cong \operatorname{Sym}^*(\Sigma H^1(\mathbb{G}[p],\widehat{\mathbb{G}}_a)),$$

where "Σ" indicates that the classes are taken to lie in degree 2. □

Lemma 4.6.10 ([HL, Remark 2.2.5])**.** *If $\mathbb{G}$ is a connected p–divisible group of height d and of dimension n as a formal variety, then*

$$\operatorname{rank}\left(H^1(\mathbb{G}[p];\widehat{\mathbb{G}}_a)\right) = d - n. \qquad \square$$

Remark 4.6.11. In the case where $\mathbb{G} = \Gamma$ is the original height d formal group of dimension 1, this computes $H^*(\Gamma;\widehat{\mathbb{G}}_a)$ to be a power series algebra on $(d-1)$ generators. This is precisely the result we recorded by hand in Lemma 3.4.12.

Returning to the task at hand, we assume inductively that $(\underline{H(S^1[p^\infty])}_{q-1})_K$ is a connected p–divisible group of height $\binom{d}{q-1}$ and dimension $\binom{d-1}{q-1}$. Since the input to the bar spectral sequence is computed by formal group cohomology

([Laz97], [HL, Example 2.3.5], Proof of Lemma 3.2.5), it follows that the instance computing $K^*\underline{H(S^1[p^\infty])}_q$ has E_2–page an even-concentrated power series algebra of dimension

$$\binom{d}{q-1} - \binom{d-1}{q-1} = \binom{d-1}{q}.$$

The spectral sequence therefore collapses at this page, so that $(\underline{H(S^1[p^\infty])}_q)_K$ is a formal variety of the dimension claimed.

Proof of Part 2, with a gap. The other claims in Part 2 are formal after we check that θ^q is an isomorphism, since the p–power–torsion structure of $(\underline{H(S^1[p^\infty])}_q)_K$ can be read off from its Dieudonné module, as can its height. We introduce notation to analyze this statement: set M to be the Dieudonné module associated to $K_*\mathbb{C}\mathrm{P}^\infty$, i.e.,

$$M = D(\underline{H(S^1[p^\infty])}_1).$$

In the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{\wedge q} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge q} & \longrightarrow & \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \\ 0 & \longrightarrow & M^{\wedge q} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge q} & \longrightarrow & \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q} & \longrightarrow & 0 \end{array}$$

the middle map is an isomorphism. This forces V to be a surjective endomorphism of $M^{\wedge q} \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$, and the snake lemma shows that there is an isomorphism

$$\ker(V\colon\ \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q} \to \mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}) \cong \operatorname{coker}(V\colon\ M^{\wedge q} \to M^{\wedge q}).$$

Picking any coordinate x and considering it as an element in the curves model of the Dieudonné functor, we see that the right-hand side is spanned by elements $x \wedge V^{\wedge I}x$, and hence the left-hand side has k–vector–space dimension $\binom{d-1}{q}$.

By very carefully studying the bar spectral sequence, one can learn that θ^q induces a surjection[19]

$$\ker V|_{\mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}} \to \ker V|_{D(\underline{H(S^1[p^\infty])}_q)}.$$

[19] The proof of this is quite complicated, and it rests on a pairing between the spectral sequence for $q = 1$ and the spectral sequence at $q - 1$, mapping to the spectral sequence for q. Remarkably, this same pairing is the main tool that powers the original approach of Ravenel–Wilson [RW80]. There, their program is to fix $j = 1$ and inductively analyze q using this same pairing, then use these base cases to ground a strong induction on j and q, and then finally to fix q and take the limit as $j \to \infty$.

In fact, since these two have the same rank, $\theta^q|_{\ker V}$ is an isomorphism on these subspaces. This is enough to conclude that θ^q is an injection: since the action of V is locally nilpotent, if θ^q ever failed to be an injection then we could apply V enough times to get an example of a nontrivial element in $\ker V|_{\mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}}$ mapping to zero. Finally, to show that θ^q is surjective, we again use the local nilpotence of V to filter $\mathbb{Q}/\mathbb{Z}_{(p)} \otimes M^{\wedge q}$ by the subspaces $\ker V^\ell$, $\ell \geq 1$, and it is then possible (though we admit the proof) to use our understanding of $\ker V$ to form preimages.

Proof of Part 3, mostly omitted. This proof is quite complicated, but it is, in spirit, a generalization of the observation at $q = 1$ that the role of the odd-degree classes in the bar spectral sequence is to pair up with those classes in the image of the $[p^j]$–map. In fact, their main assertion is:

> We give notation for the following rings:
>
> $$A = K_0 \underline{H(S^1[p^\infty])}_{q-1}, \quad A' = K_0 \underline{H(S^1[p^j])}_{q-1}, \quad R = K_0 \underline{H(S^1[p^\infty])}_q.$$
>
> Let $x' \in E_2^{1,0}$ be an element, and let $y' \in \mathfrak{m}_R$ satisfy
>
> $$y' = \psi(x') \otimes v \in \operatorname{Ext}^2_A \otimes_k \pi_2 K \cong \mathfrak{m}_R / \mathfrak{m}_R^2.$$
>
> Suppose that the Hopf algebra homomorphism $[p^j]\colon R \to R$ carries y' to an element $y \in \mathfrak{m}_R^s$, and let $x \in E_2^{2s,2s-2}$ denote the image of y under the composite
>
> $$\mathfrak{m}_R^s / \mathfrak{m}_R^{s+1} \cong \operatorname{Ext}^{2s}_A \otimes_k \pi_{2s} K \to \operatorname{Ext}^{2s}_{A'} \otimes_k \pi_{2s} K = E_2^{2s,2s} \xrightarrow{-v^{-1}} E_2^{2s,2s-2}.$$
>
> Then x and x' survive to the $(2s-1)^{\text{th}}$ page of the bar spectral sequence, and there we have
>
> $$d_{2s-1} x' = x.$$

From here, it is a matter of *very* carefully pairing elements up (cf. [HL, pg. 60]). □

Remark 4.6.12. Theorem 4.6.8 admits a restatement purely in terms of Hopf algebras, although Dieudonné theory was essential in its proof. The cup product gives a natural map

$$\operatorname{Alt}^q K_* \underline{H(S^1[p^j])}_1 \to K_* \underline{H(S^1[p^j])}_q,$$

where "Alt" refers to the alternating Hopf algebra.[20] The main result of this section is that this map is an isomorphism for all q, and indeed that the map from the free alternating Hopf *ring* maps isomorphically to the topological Hopf ring.

Remark 4.6.13 ([HL, Section 3], [Hed], [Hed14]). Because $K^*\underline{H\mathbb{Z}/p^j}_q$ is even, you can hope to augment this to a calculation of $E^*\underline{H\mathbb{Z}/p^j}_q$ for $E = E_\Gamma$ the associated Morava E–theory. This is indeed possible, and the analogous formula is true at the level of Hopf algebras:

$$E_*\underline{H(S^1[p^j])}_q \cong \mathrm{Alt}^q\, E_*\underline{H(S^1[p^j])}_1.$$

However, the attendant algebraic geometry is quite complicated: you either need a form of Dieudonné theory that functions over $\mathcal{M}_{E_\Gamma}$ (and then attempt to drag the proof above through that setting), or you need to directly confront what "alternating power of a p–divisible group" means at the level of p–divisible groups (and forego all of the time-saving help afforded to you by Dieudonné theory).

Remark 4.6.14 (cf. Lemma 3.6.9). You'll notice that in $K_*\underline{H}_{q+1}$ if we let the q–index tend to ∞, we get the K–homology of a point. This is another way to see that the stable cooperations K_*H vanish, meaning that the *only* information present comes from unstable cooperations.

Remark 4.6.15. Although the method of starting with the $j = \infty$ case and deducing from it the $1 \le j < \infty$ cases is due to Hopkins and Lurie, the observation that the spectral sequence at $j = \infty$ is remarkably simple had already been made by Ravenel and Wilson [RW80, Theorem 12.3], [RWY98, Theorem 8.1.3].

Remark 4.6.16. Theorem 4.6.8 has a statement in the language of Lecture 4.2. Rather than forming the algebraic approximation $\mathbb{A}(K,H) = K_*(\mathbb{C}\mathrm{P}^\infty)^{\circlearrowleft}_{K_*[H_*]}[H_*\mathbb{C}\mathrm{P}^\infty]$ as usual, we form the modified version

$$\mathbb{A}_{p^j}(K,H) = K_*(\underline{H(S^1[p^j])}_1)^{\circlearrowleft}_{K_*[H_*]}[H_*\underline{H(S^1[p^j])}_1].$$

Using the same unstable Kronecker pairing, this supports a natural map to $K_*\underline{H}_*$, and a summary of this Lecture's results is that it is an isomorphism. In particular, this auxiliary approximation makes use of *odd*–dimensional Eilenberg–Mac Lane spaces.

[20] Note that this alternation condition becomes dramatically more complicated in the case that the formal group law and its formal group inverse series become more complicated than that of $\widehat{\mathbb{G}}_a$.

Case Study 5

The σ-orientation

By this point, we have seen a great many ways that algebraic geometry exerts control over the behavior of homotopy theory, stable and unstable. The goal in this Case Study is to explore a setting where algebraic geometry is itself tightly controlled: whereas the behavior of formal groups is quite open-ended, the behavior of *abelian varieties* is comparatively strict. We import this strictness into algebraic topology by stiudying complex-orientable cohomology theories E which have been tagged with an auxiliary abelian variety A via an isomorphism $\varphi\colon\ \mathbb{C}\mathrm{P}^\infty_E \cong A^\wedge_0$. In the case that A is an elliptic curve, this is our definition of an *elliptic cohomology theory*. The idea, then, is not that this puts serious constraints on the formal group $\mathbb{C}\mathrm{P}^\infty_E$ (although it does place some), but rather that the theory of abelian varieties endows A, and hence $A^\wedge_0$, with various bits of preferred data. This is the tack we take to construct a *canonical* $MU[6,\infty)$–orientation of E: for any complex-orientable E, we identify the collection of such ring maps with "Θ–structures on $\mathbb{C}\mathrm{P}^\infty_E$"; a basic theorem about abelian varieties endows the elliptic curve A with a canonical such structure; and altogether this yields the desired orientation for an elliptic spectrum.

Making the identification of $MU[6,\infty)$–orientations with Θ–structures requires real work, but many of the stepping stones are now in place. We begin with a technical section about especially nice formal schemes, called *coalgebraic*, and we use this to finally give the proof, announced back in Theorem 2.2.7, that the scheme of stable Weil divisors on a formal curve presents the free formal group on that curve. With that out of the way and with $MU[6,\infty)$ in mind as the eventual goal, we then summarize the behavior of the part of the Postnikov tower for complex bordism that we *do* understand—the cases of MUP and MU—and use this to make an analysis of MSU. In particular, we rely heavily on the results from Case Study 2 and Case Study 3 to understand the co/homological behaviors of $BU \times \mathbb{Z}$, BU, MUP, and MU.

The coherence of all of these statements gives us very explicit target theorems to aim for in our study of $MU[6, \infty)$, but we are forced to approach them from a different vantage point: whereas we can prove a splitting principle for SU–bundles, the analogous statement for $U[5, \infty)$–bundles does not appear to admit a direct proof. Consequently, the proofs of the other structure theorems for $BU[6, \infty)$ and $MU[6, \infty)$ are made considerably more complicated because we have to work with our splitting principle hands tied. Instead, our main tools are the results developed in Case Study 4, which give us direct access to the co/homology of the layers of the Postnikov tower. When the dust of all this settles, we will have arrived at a very satisfying and complete theory of $MU[6, \infty)$–orientations, applicable to an arbitrary complex-orientable cohomology theory.

The reader gifted with an exceptional attention span will recall from the Introduction that we were *really* interested in *MString*–orientations, and that our interest in $MU[6, \infty)$–orientations was itself only a stepping stone. We close this Case Study with an analysis of this last setting, where we finally yield and place more hypotheses on E—a necessity for gaining calculational access to co/homological behavior of objects like *BString*, which lie outside of the broader complex-orientable story.

We also give a short résumé on the theory of elliptic curves in Lecture 5.5, extracting the smallest possible subset of their theory that we will need here.

5.1 Coalgebraic formal schemes

We will now finally address an elephant that has been lingering in our metaphorical room: in the first third of this book we were primarily interested in the formal scheme associated to the *cohomology* of a space, but in the second third we were primarily interested in a construction converting the *homology* of a spectrum to a sheaf over a context. Our goal for today is to, when possible, put these two variances on even footing. Our motivation for putting this lingering discrepancy to rest is more technical than aesthetic: we have previously wanted access to certain colimits of formal schemes (e.g., in Theorem 2.2.7). While such colimits are generally forbidding, similarly to colimits of manifolds, we will in effect produce certain conditions under which they are accessible.

For E a ring spectrum and X a space, the diagonal map $\Delta\colon X \to X \times X$ induces a multiplication map on E–cohomology via the composite

$$E^*X \otimes_{E^*} E^*X \xrightarrow{\text{Künneth}} E^*(X \times X) \xrightarrow{E^*\Delta} E^*X.$$

Dually, applying E–homology, we have a pair of maps

$$E_*X \xrightarrow{E_*\Delta} E_*(X \times X) \xleftarrow{\text{Künneth}} E_*X \otimes_{E_*} E_*X,$$

where, remarkably, the Künneth map goes the wrong way to form a composite. In the case that that map is an isomorphism, the long composite induces the structure of an E_*–coalgebra on E_*X. In the most generous case that E is a field spectrum (in the sense of Corollary 3.5.15), the Künneth map is always invertible and, moreover, E^*X is functorially the linear dual of E_*X. This motivates us to consider the following purely algebraic construction:

Definition 5.1.1. Let C be a coalgebra over a field k. We define a functor

$$\operatorname{Sch} C\colon \mathsf{Algebras}_{k/} \to \mathsf{Sets},$$
$$T \mapsto \left\{ f \in C \otimes T \,\middle|\, \begin{array}{c} \Delta f = f \otimes f \in (C \otimes T) \otimes_T (C \otimes T), \\ \varepsilon f = 1 \end{array} \right\}.$$

Lemma 5.1.2. *For a field k and a k–algebra A which is finite–dimensional as a k–module, there is a natural isomorphism* $\operatorname{Spec} A \cong \operatorname{Sch} A^*$*.*

Proof sketch. A point $f \in (\operatorname{Sch} A^*)(T) \subseteq A^* \otimes T$ gives rise to a k–module map $f_*\colon A \to T$, which the extra conditions in the formation of $(\operatorname{Sch} C)(T)$ force to be a ring homomorphism. The finiteness assumption is present exactly so that A is its own double–dual, giving an inverse assignment. □

If we drop the finiteness assumption, then this comparison proof fails entirely. Indeed, the multiplication on A gives rise only to maps

$$A^* \to (A \otimes_k A)^* \leftarrow A^* \otimes_k A^*,$$

which is not enough to make A^* into a k–coalgebra. However, if we start instead with a k–coalgebra C of infinite dimension, the following result is very telling:

Lemma 5.1.3 ([Dem86, pg. 12], [Mic03, Appendix 5.3], [HL, Remark 1.1.8])**.** *For C a coalgebra over a field k, any finite–dimensional k–linear subspace of C can be finitely enlarged to a subcoalgebra of C. Accordingly, taking the colimit gives a canonical equivalence*

$$\operatorname{Ind}(\mathsf{Coalgebras}_k^{\mathrm{fin}}) \xrightarrow{\simeq} \mathsf{Coalgebras}_k.$$ □

This result allows us to leverage our duality Lemma pointwise: for an arbitrary k–coalgebra, we break it up into a lattice of finite k–coalgebras, and take their linear duals to get a reversed lattice of finite k–algebras. Altogether, this indicates that k–coalgebras generally want to model *formal schemes*.

Corollary 5.1.4. *For C a coalgebra over a field k expressed as a colimit $C = \operatorname{colim}_k C_k$ of finite subcoalgebras, there is an equivalence*

$$\operatorname{Sch} C \cong \{\operatorname{Spec} C_k^*\}_k.$$

This induces a covariant *equivalence of categories*

$$\mathsf{Coalgebras}_k \cong \mathsf{FormalSchemes}_{/k}.$$

This equivalence translates between the product of formal schemes, the tensor product of pro-algebras, and the tensor product of coalgebras. □

This covariant algebraic model for formal schemes is very useful. For instance, this equivalence makes the following calculation trivial:

Lemma 5.1.5 (cf. Lemma 1.5.1, Theorem 2.2.7, and Corollary 2.3.15)**.** *Select a coalgebra C over a field k together with a pointing $k \to C$. Write M for the coideal $M = C/k$. The free formal monoid on the pointed formal scheme* $\operatorname{Sch} k \to \operatorname{Sch} C$ *is given by*

$$F(\operatorname{Sch} k \to \operatorname{Sch} C) = \operatorname{Sch} \operatorname{Sym}^* M.$$

Writing $\Delta c = \sum_j \ell_j \otimes r_j$ for the diagonal on C, the diagonal on $\operatorname{Sym}^* C$ *is given by*

$$\Delta(c_1 \cdots c_n) = \sum_{j_1,\ldots,j_n} (\ell_{1,j_1} \cdots \ell_{n,j_n}) \otimes (r_{1,j_1} \cdots r_{n,j_n}). \qquad \square$$

It is unfortunate, then, that when working over a ring rather than a field Lemma 5.1.3 fails [Mic03, Appendix 5.3]. Nonetheless, it is possible to bake into the definitions the machinery needed to get a good-enough analogue of Corollary 5.1.4.

Definition 5.1.6 ([Str99b, Definition 4.58])**.** Let C be an R–coalgebra which is free as an R–module. A basis $\{x_j\}$ of C is said to be a *good basis* when any finite subcollection of $\{x_j\}$ can be finitely enlarged to a subcollection that spans a subcoalgebra. The coalgebra C is itself said to be *good* when it admits a good basis. A formal scheme X is said to be *coalgebraic* when it is isomorphic to $\operatorname{Sch} C$ for a good coalgebra C.

Example 5.1.7. The formal scheme $\widehat{\mathbb{A}}^n$ is coalgebraic. Beginning with the presentation

$$\widehat{\mathbb{A}}^n = \operatorname{Spf} R[\![x_1, \ldots, x_n]\!] = \operatorname*{colim}_J \operatorname{Spec} R[x_1, \ldots, x_n]/(x_1^{j_1}, \ldots, x_n^{j_n}),$$

write A_J for the algebra on the right-hand side. Each A_J is a free R–module, and we write

$$C_J = A_J^* = R\{\beta_K \mid K < J\}$$

for the dual coalgebra, with

$$\beta_K(x^L) = \begin{cases} 1 & \text{if } K = L, \\ 0 & \text{otherwise.} \end{cases}$$

The elements β_K form a good basis for the full coalgebra $C = \operatorname{colim}_J C_J$: any finite collection of them $\{\beta_K\}_{K \in \mathcal{K}}$ is contained inside any C_J satisfying $K < J$ for all $K \in \mathcal{K}$. As an additional consequence, all formal varieties are coalgebraic.

The main utility of this condition is that it gives us access to colimits of formal schemes:

Theorem 5.1.8 ([Str99b, Proposition 4.64]). *Suppose that $F \colon \mathsf{I} \to \mathsf{Coalgebras}_R$ is a colimit diagram of coalgebras such that each object in the diagram, including the colimit point, is a good coalgebra. Then*

$$\mathrm{Sch} \circ F \colon \mathsf{I} \to \mathsf{FormalSchemes}$$

is a colimit diagram of formal schemes. □

For an example of the sort of constructions that become available via this Theorem, we prove the following Corollary by analyzing the symmetric power of coalgebras:

Corollary 5.1.9 ([Str99b, Example 4.65 and Proposition 6.4]). *When a formal scheme X is coalgebraic, the symmetric power $X^{\times n}_{\Sigma_n}$ exists. In fact, $\coprod_{n \ge 0} X^{\times n}_{\Sigma n}$ models the free formal monoid on X. Given an additional pointing* Spec $R \to X$*, the colimit of the induced system*

$$\operatorname{colim}\left(\cdots \to X^{\times n}_{\Sigma_n} = \operatorname{Spec} R \times X^{\times n}_{\Sigma_n} \to X \times X^{\times n}_{\Sigma_n} \to X^{\times (n+1)}_{\Sigma_{n+1}} \to \cdots\right)$$

models the free formal monoid on the pointed formal scheme.

Proof sketch. The main points entirely mirror the case over a field: the symmetric power construction gives models for $X^{\times n}_{\Sigma_n}$, the symmetric algebra construction gives a model for the free formal monoid, and the stabilization against the pointing is modeled by inverting an element in the symmetric algebra. In each case, choosing a good basis for the coalgebra underlying X yields choices of good bases for the coalgebras arising from these constructions, essentially because their elements are crafted out of finite combinations of the elements of the original. □

In the specific case that $\operatorname{Spec} R \to X$ is a pointed formal *curve*, we can prove something more:

Corollary 5.1.10 ([Str99b, Proposition 6.12])**.** *For* $\operatorname{Spec} R \to X$ *a pointed formal curve, the free formal monoid is automatically an abelian group.*

Proof sketch. The main idea is that the coalgebra associated to a formal curve admits an increasing filtration F_k so that the reduced diagonal $\overline{\Delta} = \Delta - (1 \otimes \eta) - (\eta \otimes 1)$ reduces filtration degree:

$$\overline{\Delta}|_{F_k} \colon F_k \to \sum_{\substack{i,j>0 \\ i+j=k}} F_i \otimes F_j.$$

In turn, the symmetric algebra on the coalgebra associated to a formal curve inherits enough of this filtration that one can iteratively solve for a Hopf algebra antipode. □

We now reconnect this algebraic discussion with the algebraic topology that spurred it.

Lemma 5.1.11. *If E and X are such that $E_* X$ is an E_*–coalgebra and*

$$E^* X = \mathsf{Modules}_{E_*}(E_* X, E_*),$$

then there is an equivalence

$$\operatorname{Sch} E_* X \cong X_E.$$

Proof. We have defined X_E to have formal topology induced by the compactly generated topology of X, and this same topology can also be used to write $\operatorname{Sch} E_* X$ as the colimit of finite E_*–coalgebras. □

Example 5.1.12 (cf. Theorem 4.6.1 and Remark 4.6.2). For a Morava K–theory K_Γ associated to a formal group Γ of finite height, we have seen that there is an exact sequence of Hopf algebras

$$K^0_\Gamma(BS^1) \xrightarrow{[p^j]} K^0_\Gamma(BS^1) \to K^0_\Gamma(BS^1[p^j]),$$

presenting $(BS^1[p^j])_K$ as the p^j–torsion formal subscheme $BS^1_K[p^j]$. The Hopf algebra calculation also holds in K–homology, where there is instead the exact sequence

$$(K_\Gamma)_0 B(S^1[p^j]) \to (K_\Gamma)_0 BS^1 \xrightarrow{(-)^{*p^j}} (K_\Gamma)_0 BS^1$$

presenting $(K_\Gamma)_0 B(S^1[p^j])$ as the p^j–order $*$–nilpotence in the middle Hopf algebra. Applying Sch to this last line covariantly converts this second statement about Hopf algebras to the corresponding statement above about the associated formal schemes—i.e., the behavior of the homology Hopf algebra is a direct reflection of the behavior of the formal schemes.

The example above, where the space in question is an H–space, also spurs us to consider a certain "wrong-way" operation. We have seen that the algebra structure of the K–cohomology of a space and the coalgebra structure of the K–homology of the same space contain equivalent data: they both give rise to the same formal scheme. However, in the case of a commutative H–space, the K–homology and K–cohomology give *commutative and cocommutative Hopf algebras*. Hence, in addition to considering the coalgebraic formal scheme $\operatorname{Sch}(K_\Gamma)_0 B(S^1[p^j])$, we can also consider the affine scheme $\operatorname{Spec}(K_\Gamma)_0 B(S^1[p^j])$. This, too, should contain identical information, and this is the subject of Cartier duality.

Definition 5.1.13 ([Str99b, Sections 6.3–4]). The *Cartier dual* of a commutative finite group scheme G is defined by the formula

$$DG = \underline{\mathsf{GroupSchemes}}(G, \mathbb{G}_m),$$

itself a finite group scheme. More generally, the Cartier dual of a commutative *coalgebraic* formal group $\widehat{\mathbb{G}}$ can also be defined by

$$D\widehat{\mathbb{G}} = \underline{\mathsf{GroupSchemes}}(\widehat{\mathbb{G}}, \mathbb{G}_m).$$

Lemma 5.1.14 ([Str99b, Proposition 6.19]). *Let $\widehat{\mathbb{G}}$ be a coalgebraic commutative formal group over a formal scheme X, and write $\mathbb{H} = \operatorname{Spec} \mathcal{O}^*_{\widehat{\mathbb{G}}}$ for the group scheme associated to its dual Hopf algebra. Cartier duality then has the effects $D\widehat{\mathbb{G}} = \mathbb{H}$ and $D\mathbb{H} = \widehat{\mathbb{G}}$.*

Proof. We show that the first two objects, $D\widehat{\mathbb{G}}$ and $\mathbb{H}$, represent the same object. A point $(u, f) \in D\widehat{\mathbb{G}}(T)$ is specified by a pair of functions

$$\left(u\colon\ \operatorname{Spec} T \to X, f\colon\ u^*\widehat{\mathbb{G}} \to u^*(\mathbb{G}_m \times X)\right).$$

The map f is equivalent to a map of Hopf algebras $f^*\colon\ T[u^{\pm}] \to \mathcal{O}_{\widehat{\mathbb{G}}} \otimes_{\mathcal{O}_X} T$, which is determined by its value $f^*(u) \in \mathcal{O}_{\widehat{\mathbb{G}}} \otimes_{\mathcal{O}_X} T$, which must satisfy the two relations $\Delta(f^*u) = f^*u \otimes f^*u$ and $\varepsilon(f^*u) = 1$. Invoking linear duality, f^*u can also be considered as an element of $\mathsf{Modules}_{\mathcal{O}_X}(\mathcal{O}^*_{\widehat{\mathbb{G}}}, T)$, and the two relations on f^*u show that it lands in the subset

$$f^*u \in \mathsf{Algebras}_{\mathcal{O}_X/}(\mathcal{O}^*_{\widehat{\mathbb{G}}}, T) \subseteq \mathsf{Modules}_{\mathcal{O}_X}(\mathcal{O}^*_{\widehat{\mathbb{G}}}, T).$$

This assignment is invertible, and the proof is entirely similar for $D\mathbb{H} \cong \widehat{\mathbb{G}}$. □

Remark 5.1.15 ([Dem86, pg. 72]). The effect of Cartier duality on the Dieudonné module of a formal group is *also* described by linear duality. Hence, the covariant and contravariant Dieudonné modules described in Lecture 4.4 can be taken to be related by Cartier duality.

Remark 5.1.16. Cartier duality intertwines the homological and cohomological schemes assigned to a commutative H–space. When such a commutative H–space X has free and even E–homology, there is an isomorphism

$$D(\operatorname{Spf} E^0 X) = DX_E = \underline{\mathsf{GroupSchemes}}(X_E, \mathbb{G}_m) \cong \operatorname{Spec} E_0 X.$$

5.2 Special divisors and the special splitting principle

Starting today, after our extended interludes on chromatic homotopy theory and cooperations, we are going to return to thinking about bordism orientations directly. To begin, we will summarize the various perspectives already adopted in Case Study 2 when we were studying complex orientations of ring spectra.

1. (Definition 2.0.2:) A complex–orientation of E is, definitionally, a map $MUP \to E$ of ring spectra in the homotopy category.

2. (Theorem 2.3.18:) A complex–orientation of E is also equivalent to a multiplicative system of Thom isomorphisms for complex vector bundles. Such a

system is determined by its value on the universal line bundle $\mathcal{L}$ over $\mathbb{C}P^\infty$. We can also phrase this algebro-geometrically: such a Thom isomorphism is the data of a trivialization of the Thom sheaf $\mathbb{L}(\mathcal{L})$ over $\mathbb{C}P^\infty_E$.

3. (Lemma 2.6.8:) Ring spectrum maps $MUP \to E$ induce on E–homology maps $E_0 MUP \to E_0$ of E_0–algebras. This, too, can be phrased algebro-geometrically: these are elements of $(\operatorname{Spec} E_0 MUP)(E_0)$.

We can summarize our main result about these perspectives as follows:

Theorem 5.2.1 ([AHS01, Example 2.53]). *Take E to be* complex–orientable*. The functor*

$$\begin{aligned} &\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets}, \\ &(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \{\textit{trivializations of } u^*\mathbb{L}(\mathcal{L}) \textit{ over } u^*\mathbb{C}P^\infty_E\} \end{aligned}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MUP$*. Moreover, the E_0–points of this scheme biject with ring spectrum maps $MUP \to E$.*

Proof summary. The equivalence between (1) and (3)—i.e., between complex-orientations and E_0–points of $\operatorname{Spec} E_0 MUP$—follows from calculating that $E_0 MUP$ is a free E_0–module, so that there is a collapse in the universal coefficient theorem. Then, the equivalence between (1) and (2) follows from the splitting principle for complex line bundles, which says that the first Chern class of $\mathcal{L}$—i.e., a trivialization of $\mathbb{L}(\mathcal{L})$—determines the rest of the map $MUP \to E$. □

An analogous result holds for ring spectrum maps $MU \to E$ and the line bundle $1 - \mathcal{L}$, and it is proven in analogous way. In particular, we will want a version of the splitting principle for virtual vector bundles of virtual rank 0. Given a finite complex X and such a rank 0 virtual vector bundle, write $\tilde{V}\colon X \to BU$ for the classifying map. Because X is a finite complex, there exists an integer n so that $\tilde{V} = -(n \cdot 1 - V)$ for an honest rank n vector bundle V over X. Using Corollary 2.3.10, the splitting $f^*V \cong \bigoplus_{i=1}^n \mathcal{L}_i$ over Y gives a presentation of $\tilde{V}$ as

$$\tilde{V} = -(n \cdot 1 - V) = -\bigoplus_{i=1}^{n}(1 - \mathcal{L}_i).$$

Crucially, we have organized this sum *entirely in terms of bundles classified by BU*, as each bundle $1 - \mathcal{L}_i$ itself has the natural structure of a rank 0 virtual vector bundle. This version of the splitting principle, together with our extended discussion of formal geometry, begets the following analogue of the previous result:

Theorem 5.2.2 ([AHS01, Example 2.54], cf. also [AS01, Lemma 6.2]). *Take E to be complex–orientable. The functor*

$$\begin{aligned}&\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets},\\ &(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \{\textit{trivializations of } u^*\mathbb{L}(1-\mathcal{L}) \textit{ over } u^*\mathbb{CP}^\infty_E\}\end{aligned}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MU$. *Moreover, the E_0–points of this scheme biject with ring spectrum maps $MU \to E$.* □

In Lecture 2.3, we preferred to think of the cohomology of a Thom spectrum as a sheaf over the formal scheme associated to its base space. This extra structure has not evaporated in the homological context—it just takes a different form.

Lemma 5.2.3. *For $\xi\colon G \to BGL_1\mathbb{S}$ a group map, the Thom spectrum $T\xi$ is a $(\Sigma^\infty_+ G)$–cotorsor.*

Construction. The Thom isomorphism $T\xi \wedge T\xi \simeq T\xi \wedge \Sigma^\infty_+ G$ composes with the unit map $\mathbb{S} \to T\xi$ to give the *Thom diagonal*

$$T\xi \to T\xi \wedge \Sigma^\infty_+ G.$$ □

Applying $\operatorname{Spec} E_0(-)$, the Thom diagonal is translated into the structure of a free and transitive action map

$$\operatorname{Spec} E_0 T(\xi) \times \operatorname{Spec} E_0 G \to \operatorname{Spec} E_0 T(\xi).$$

This construction is natural in the formation of G or ξ, and so we are also moved to specialize to the cases of $G = \mathbb{Z} \times BU$ and $G = BU$ and to understand $\operatorname{Spec} E_0 G$ in those contexts. Again, this is a matter of chaining together results we have already proven:

$$\begin{aligned}\operatorname{Spec} E_0(\mathbb{Z} \times BU) &= D((\mathbb{Z} \times BU)_E) && \text{(Remark 5.1.16)}\\ &= D(\operatorname{Div} \mathbb{CP}^\infty_E) && \text{(Corollary 2.3.15)}\\ &= \underline{\mathsf{FormalGroups}}(\operatorname{Div} \mathbb{CP}^\infty_E, \mathbb{G}_m) && \text{(Definition 5.1.13)}\\ &= \underline{\mathsf{FormalSchemes}}(\mathbb{CP}^\infty_E, \mathbb{G}_m), && \text{(Corollary 5.1.9)}\end{aligned}$$

and similarly

$$\operatorname{Spec} E_0(BU) = \underline{\mathsf{FormalSchemes}}_{*/}(\mathbb{CP}^\infty_E, \mathbb{G}_m)$$

is the subscheme of those maps sending the identity point of $\mathbb{C}\mathrm{P}^\infty_E$ to the identity point of $\mathbb{G}_m$. Such functions can be identified with trivializations of the trivial sheaf over $\mathbb{C}\mathrm{P}^\infty_E$, and the action map induced by the Thom diagonal belongs to a commuting square

$$\begin{array}{ccc}
\begin{matrix}\operatorname{Spec} E_0 MU \\ \times \\ \operatorname{Spec} E_0 BU\end{matrix} & \longrightarrow & \operatorname{Spec} E_0 MU \\
\| & & \| \\
\begin{matrix}\{\mathrm{triv}^{\mathrm{ns}} \text{ of } \mathbb{L}(1-\mathcal{L}) \downarrow \mathbb{C}\mathrm{P}^\infty_E\} \\ \times \\ \{\mathrm{triv}^{\mathrm{ns}} \text{ of } 1 \downarrow \mathbb{C}\mathrm{P}^\infty_E\}\end{matrix} & \longrightarrow & \{\mathrm{triv}^{\mathrm{ns}} \text{ of } \mathbb{L}(1-\mathcal{L}) \otimes 1 \downarrow \mathbb{C}\mathrm{P}^\infty_E\}.
\end{array}$$

Remark 5.2.4 ([AHS01, Corollary 2.30 and Theorem 2.50]). The topological maps

$$BU \to \mathbb{Z} \times BU, \qquad MU \to MUP$$

induce recognizable algebro-geometric maps upon application of $\operatorname{Spec} E_0(-)$. The comparison map

$$(\operatorname{Spec} E_0(\mathbb{Z} \times BU))(T) \to \underline{\mathsf{FormalSchemes}}(\mathbb{C}\mathrm{P}^\infty_E, \mathbb{G}_m)$$

reads off the image of a map $f\colon E_0(\mathbb{Z} \times BU) \cong E_0[b_0^\pm, b_1, \ldots] \to T$ as the components of a function $\sum_j f(b_0^j b_j) x^j \in T \otimes \mathcal{O}_{\mathbb{C}\mathrm{P}^\infty_E}$, whereas the comparison map for BU reads off the image of a map $g\colon E_0(BU) \cong E_0[b_0^\pm, b_1, b_2, \ldots]/(b_0 = 1) \to T$ as the components of a function $\sum_j g(b_j) x^j$, effecting a normalizing division by b_0, itself the image of $\mathbb{C}\mathrm{P}^0_E \subseteq \mathbb{C}\mathrm{P}^\infty_E$ in $\mathbb{G}_m$. Geometrically, this gives the commuting square

$$\begin{array}{ccc}
\operatorname{Spec} E_0(\mathbb{Z} \times BU) & \longrightarrow & \operatorname{Spec} E_0(BU) \\
\| & & \| \\
\underline{\mathsf{FormalSchemes}}(\mathbb{C}\mathrm{P}^\infty_E, \mathbb{G}_m) & \longrightarrow & \underline{\mathsf{FormalSchemes}}_{*/}(\mathbb{C}\mathrm{P}^\infty_E, \mathbb{G}_m) \\
 & & \\
f(t) & \longmapsto & f(t)/f(0).
\end{array}$$

At the level of Thom spectra, these identifications are controlled by the Chern classes associated to these bundles, and the briefest way to summarize their relationship is this. The spaces $\mathbb{Z} \times BU$ and BU are the 0^{th} and 2^{nd} spaces in the Ω–spectrum for connective complex K–theory, and since connective complex K–theory is complex-orientable, we have $kU^*(\mathbb{CP}^\infty) = \mathbb{Z}[\beta][\![c_1]\!]$. Inside this ring there is the relation

$$\beta c_1 = (1 - \mathcal{L}).$$

Recognizing β as the restriction of the tautological bundle on $\mathbb{CP}^\infty$ to $S^2 \simeq \mathbb{CP}^1$ and employing Example 2.3.4, this says that the trivialization f of $\mathbb{L}(u^*\mathcal{L})$, corresponding to a point in $(\operatorname{Spec} E_0 MUP)(T)$ and to $(1 - \mathcal{L}) \in kU^0(\mathbb{CP}^\infty) = [\mathbb{CP}^\infty, \mathbb{Z} \times BU]$, is sent to the trivialization $f'(0)/f$ of $\mathbb{L}(u^*(1 - \mathcal{L}))$, corresponding to the induced point in $(\operatorname{Spec} E_0 MU)(T)$ and to $c_1 \in kU^2(\mathbb{CP}^\infty) = [\mathbb{CP}^\infty, BU]$.

This last remark indicates a direction of possible generalization to the other spaces in the Ω–spectrum for connective complex K–theory, which have the following polite description:

Lemma 5.2.5. *There is an equivalence*

$$\underline{kU}_{2k} = BU[2k, \infty).$$

Proof. Consider the element $\beta^k \in kU_* = \mathbb{Z}[\beta]$. The source of the induced map $\beta^k \colon \Sigma^{2k} kU \to kU$ is $2k$–connective, and hence there is a factorization

$$\Sigma^{2k} kU \to kU[2k, \infty) \to kU.$$

Then, the structure of the homotopy ring kU_* shows that this is an equivalence: every class of degree at least $2k$ can be uniquely written as a β^k–multiple.[1] Applying Ω^∞ gives the desired statement:

$$\underline{kU}_{2k} = \Omega^\infty \Sigma^{2k} kU \simeq \Omega^\infty kU[2k, \infty) = BU[2k, \infty). \qquad \square$$

The next space and Thom spectrum in the sequence are thus BSU and MSU. This case will be wholly amenable to analysis through methods we have developed so far, which is now our stated goal for the rest of this Lecture. Our jumping off point for that story will be, again, a partial extension of the splitting principle.

[1] Similarly, there is an equivalence $\underline{kO}_{8k} = BO[8k, \infty)$, and this *does not hold* for indices which are not precise multiples of 8.

Lemma 5.2.6. *Let X be a finite complex, and let $\tilde{V}\colon X \to BU$ classify a virtual vector bundle of rank 0 over X. Select a factorization $\tilde{\tilde{V}}\colon X \to BSU$ of $\tilde{V}$ through BSU. Then, there is a space $f\colon Y \to X$, where $f_E\colon Y_E \to X_E$ is finite and flat, as well as a collection of line bundles $\mathcal{H}_j$, $\mathcal{H}'_j$ expressing a BSU–internal decomposition*

$$\tilde{\tilde{V}} = -\bigoplus_{j=1}^{n}(1-\mathcal{H}_j)(1-\mathcal{H}'_j).$$

Proof. Begin by using Corollary 2.3.10 on V to get an equality of BU–bundles

$$\tilde{\tilde{V}} = V' + \mathcal{L}_1 + \mathcal{L}_2 - n\cdot 1.$$

Adding $(1-\mathcal{L}_1)(1-\mathcal{L}_2)$ to both sides, this gives

$$\begin{aligned}\tilde{\tilde{V}} + (1-\mathcal{L}_1)(1-\mathcal{L}_2) &= V' + \mathcal{L}_1 + \mathcal{L}_2 + (1-\mathcal{L}_1)(1-\mathcal{L}_2) - n\cdot 1\\ &= V' + \mathcal{L}_1\mathcal{L}_2 - (n-1)\cdot 1.\end{aligned}$$

By thinking of $(1-\mathcal{L}_j)$ as an element of $kU^2(Y) = [Y, BU]$, we see that the product element $(1-\mathcal{L}_1)(1-\mathcal{L}_2) \in kU^4(Y) = [Y, BSU]$ has the natural structure of a BSU–bundle and hence so does the sum on the left-hand side[2]. The right-hand side is the rank 0 virtualization of a rank $(n-1)$ vector bundle, hence succumbs to induction. Finally, because $SU(1)$ is the trivial group, there are no nontrivial complex line bundles with structure group $SU(1)$, grounding the induction. □

From this, we would like to directly conclude an equivalence between trivializations of the Thom sheaf $\mathbb{L}((\mathcal{L}_1-1)(\mathcal{L}_2-1)) \downarrow (\mathbb{C}\mathrm{P}^\infty)^{\times 2}_E$ and multiplicative maps $MSU \to E$, but we are not quite yet ready to do so. Certainly an MSU–orientation of E gives such a trivialization, but it is not clear that all possible trivializations of that universal Thom sheaf give consistent trivializations of other Thom sheaves—that is, the decomposition in Lemma 5.2.6 may admit unexpected symmetries which, in turn, place requirements on our universal trivialization so that these symmetric decompositions all result in the same restricted trivialization.[3]

[2] In the language of the previous Case Study, we are making use of a certain Hopf ring $\circ$–product on $\underline{kU}_{2*}$.

[3] By contrast, our splitting principle for ordinary complex vector bundles was completely deterministic, since a given isomorphism class of line bundles tautologically admits no other expression as an isomorphism class of line bundles.

Example 5.2.7. There is an equivalence of SU–bundles

$$(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) \cong (\mathcal{L}_2 - 1)(\mathcal{L}_1 - 1).$$

Correspondingly, the trivializations of $\mathbb{L}((\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1))$ which respect this twist are the *symmetric* sections.

Example 5.2.8. There is an equivalence of SU–bundles

$$(1 - 1)(\mathcal{L}_2 - 1) \cong 0.$$

Correspondingly, the trivializations of $\mathbb{L}((1 - \mathcal{L}_1)(1 - \mathcal{L}_2))$ which respect this degeneracy are the *rigid* sections, meaning they trivialize the Thom sheaf of the trivial bundle using the trivial section 1.

Example 5.2.9. There is another less obvious symmetry, inspired by our use of the product map

$$kU^2(Y) \otimes kU^2(Y) \to kU^4(Y)$$

in the course of the proof. There is also a product map

$$kU^2(Y) \otimes kU^0(Y) \times kU^2(Y) \to kU^4(Y).$$

Taking one of our splitting summands $(1 - \mathcal{L}_1)(1 - \mathcal{L}_2)$ and acting by some line bundle $\mathcal{H} \in kU^0(Y)$ gives

$$\begin{aligned} (1 - \mathcal{L}_1)\mathcal{H}(1 - \mathcal{L}_2) &= (1 - \mathcal{L}_1)\mathcal{H}(1 - \mathcal{L}_2) \\ (\mathcal{H} - \mathcal{L}_1\mathcal{H})(1 - \mathcal{L}_2) &= (1 - \mathcal{L}_1)(\mathcal{H} - \mathcal{H}\mathcal{L}_2) \\ (1 - \mathcal{L}_1\mathcal{H})(1 - \mathcal{L}_2) - (1 - \mathcal{H})(1 - \mathcal{L}_2) &= (1 - \mathcal{L}_1)(1 - \mathcal{H}\mathcal{L}_2) - (1 - \mathcal{L}_1)(1 - \mathcal{H}). \end{aligned}$$

This "kU^0–linearity" is sometimes called a "2–cocycle condition", in reference to the similarity with the formula in Definition 3.2.4.

We would like to show that these observations suffice, as in the following version of Theorem 5.2.1 and Theorem 5.2.2:

Theorem 5.2.10 ([AHS01, Theorem 2.50]). *The functor*

$$\{\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0\} \to \left\{ \begin{array}{c} \textit{trivializations of } u^*\mathbb{L}((1 - \mathcal{L}_1)(1 - \mathcal{L}_2)) \\ \textit{over } u^*(\mathbb{C}\mathrm{P}^\infty)^{\times 2}_E \textit{ which are} \\ \textit{symmetric, rigid, and } kU^0\textit{–linear} \end{array} \right\}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MSU$*. Moreover, the* E_0*–points of this scheme biject with ring spectrum maps* $MSU \to E$*.*

In pursuit of this, we will show rather manually that BSU_E represents an object subject to exactly such symmetries, hence Spec E_0BSU represents the scheme of such symmetric functions, and finally conclude that Spec E_0MSU represents the scheme of such symmetric trivializations. The place to begin is with a Serre spectral sequence:

Lemma 5.2.11 ([AS01, Lemma 6.1], cf. also [AS01, Proposition 6.5]). *The Postnikov fibration*

$$BSU \to BU \xrightarrow{B\det} BU(1)$$

induces a short exact sequence of Hopf algebras

$$E^0BSU \leftarrow E^0BU \xleftarrow{c_1 \leftarrow c_1} E^0BU(1). \qquad \square$$

An equivalent statement is that there is a short exact sequence of formal group schemes

$$\begin{array}{ccccc} BSU_E & \longrightarrow & BU_E & \xrightarrow{B\det} & BU(1)_E \\ \| & & \| & & \| \\ \mathrm{SDiv}_0\,\mathbb{C}\mathrm{P}^\infty_E & \longrightarrow & \mathrm{Div}_0\,\mathbb{C}\mathrm{P}^\infty_E & \xrightarrow{\mathrm{sum}} & \mathbb{C}\mathrm{P}^\infty_E, \end{array}$$

where the scheme "$\mathrm{SDiv}_0\,\mathbb{C}\mathrm{P}^\infty_E$" of *special divisors* is defined to parametrized those divisors which vanish under the summation map. However, whereas the map $BU(1)_E \to BU_E$ has an identifiable universal property—it presents BU_E as the universal formal group on the pointed curve $BU(1)_E$—the description of BSU_E as a scheme of special divisors does not bear much immediate resemblance to a free object on the special divisor $(1-[a])(1-[b])$ classified by

$$(\mathbb{C}\mathrm{P}^\infty)^{\times 2}_E \xrightarrow{(1-\mathcal{L}_1)(1-\mathcal{L}_2)_E} BSU_E \to BU_E = \mathrm{Div}_0\,\mathbb{C}\mathrm{P}^\infty_E.$$

Our task is thus exactly to justify this statement.

Definition 5.2.12. If it exists, let $C_2\widehat{\mathbb{G}}$ denote the symmetric square of $\mathrm{Div}_0\,\widehat{\mathbb{G}}$, thought of as a module over the ring scheme $\mathrm{Div}\,\widehat{\mathbb{G}}$. This scheme has the property that a formal group homomorphism $\varphi\colon C_2\widehat{\mathbb{G}} \to H$ is equivalent data to a symmetric function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying a rigidity condition ($\psi(x,0) = 0$) and a 2–cocycle condition as in Example 5.2.9.

Theorem 5.2.13 (Ando–Hopkins–Strickland, unpublished)**.** $\mathrm{SDiv}_0 \widehat{\mathbb{G}}$ *is a model for* $C_2\widehat{\mathbb{G}}$.

Proof sketch. Consider the map

$$\begin{aligned} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} &\to \mathrm{Div}_0 \widehat{\mathbb{G}}, \\ (a, b) &\mapsto (1 - [a])(1 - [b]) \end{aligned}$$

for which there is a factorization of formal schemes

$$\begin{tikzcd} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \arrow[d, dotted] \arrow[dr] & & \\ F \arrow[r, "\ker"] & \mathrm{Div}_0 \widehat{\mathbb{G}} \arrow[r, "\sigma"] & \widehat{\mathbb{G}} \end{tikzcd}$$

because

$$\sigma((1 - [a])(1 - [b])) = (a + b) - a - b + 0 = 0.$$

One can check that a homomorphism $\varphi\colon F \to H$ pulls back to a function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying the properties of Definition 5.2.12:

- To check rigidity, we have

$$\psi(a, 0) = \varphi((1 - [a])(1 - [0])) = \varphi((1 - [a])(1 - 1)) = \varphi(0) = 0.$$

- To check symmetry, we have

$$\psi(a, b) = \varphi((1 - [a])(1 - [b])) = \varphi((1 - [b])(1 - [a])) = \psi(b, a).$$

- To check kU^0–linearity, we have

$$\begin{aligned} \psi(ac, b) - \psi(c, b) &= \varphi((1 - [a][c])(1 - [b])) - \varphi((1 - [c])(1 - [b])) \\ &= \varphi((1 - [a][c])(1 - [b]) - (1 - [c])(1 - [b])) \\ &= \varphi((1 - [a])(1 - [c][b]) - (1 - [a])(1 - [c])) \\ &= \varphi((1 - [a])(1 - [c][b])) - \varphi((1 - [a])(1 - [c])) \\ &= \psi(a, cb) - \psi(a, c). \end{aligned}$$

The other direction is more obnoxious, so we give only a sketch. Begin by selecting a function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$, then mimic the construction in Lemma 5.2.6. Expanding the definition of $\operatorname{Div}_0 \widehat{\mathbb{G}}$, we are moved to consider the object $\widehat{\mathbb{G}}^{\times k}$, where we define a map

$$\widehat{\mathbb{G}}^{\times k} \to H,$$
$$(a_1, \ldots, a_k) \mapsto -\sum_{j=2}^{k} \psi\left(\sum_{i=1}^{j-1} a_i, a_j\right).$$

This gives a compatible system of symmetric maps, and hence altogether this gives a map $\tilde{\varphi}\colon \operatorname{Div}_0 \widehat{\mathbb{G}} \to H$ off of the colimit. In general, this map is not a homomorphism, but it is a homomorphism when restricted to

$$\varphi\colon F \to \operatorname{Div}_0 \widehat{\mathbb{G}} \xrightarrow{\tilde{\varphi}} H.$$

Finally, one checks that any homomorphism $F \to H$ of formal groups restricting to the zero map $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ was already the zero map, and this gives the desired identification of F with the universal property of $C_2\widehat{\mathbb{G}}$. □

Building the homomorphism seems boring, but possibly checking that it's zero is interesting — this is kind of what was confounding us from just using the topological *SU*–splitting principle outright. Somehow working in algebra must make this more evident, and if it's so evident then we should write it out.

Corollary 5.2.14. *There is an isomorphism*

$$\operatorname{Spec} E_0 BSU = \left\{ \begin{array}{c} \text{functions } f\colon u^*(\mathbb{C}\mathrm{P}^\infty_E)^{\times 2} \to \mathbb{G}_m \\ \text{which are symmetric, rigid, and } kU^0\text{–linear} \end{array} \right\}.$$

Proof. Follow the sequence of isomorphisms

$$\begin{aligned} \operatorname{Spec} E_0 BSU &= D(BSU_E) && \text{(Remark 5.1.16)} \\ &= D(\operatorname{SDiv}_0 \mathbb{C}\mathrm{P}^\infty_E) && \text{(Lemma 5.2.11)} \\ &= D(C_2 \mathbb{C}\mathrm{P}^\infty_E) && \text{(Theorem 5.2.13)} \\ &= \underline{\mathsf{FormalGroups}}(C_2 \mathbb{C}\mathrm{P}^\infty_E, \mathbb{G}_m), && \text{(Definition 5.1.13)} \end{aligned}$$

and then use the universal property in Definition 5.2.12. □

In order to lift this analysis to $\operatorname{Spec} E_0 MSU$, we again appeal to the torsor structure. At this point, it will finally be useful to introduce some notation:

Definition 5.2.15 ([AHS01, Definition 2.39])**.** For a sheaf $\mathcal{L}$ over a formal group $\widehat{\mathbb{G}}$, we introduce the schemes

$$\begin{aligned} C^0(\widehat{\mathbb{G}}_E; \mathcal{L})(T) &= \{\text{triv}^{\text{ns}} \text{ of } u^*\mathcal{L} \downarrow u^*\widehat{\mathbb{G}}\}, \\ C^1(\widehat{\mathbb{G}}_E; \mathcal{L})(T) &= \left\{\text{triv}^{\text{ns}} \text{ of } u^*\left(\frac{e^*\mathcal{L}}{\mathcal{L}}\right) \downarrow u^*\widehat{\mathbb{G}} \text{ which are rigid}\right\} \\ C^2(\widehat{\mathbb{G}}_E; \mathcal{L})(T) &= \left\{ \begin{array}{c} \text{triv}^{\text{ns}} \text{ of } u^*\left(\frac{e^*\mathcal{L}\otimes\mu^*\mathcal{L}}{\pi_1^*\mathcal{L}\otimes\pi_2^*\mathcal{L}}\right) \downarrow u^*\widehat{\mathbb{G}}^{\times 2} \\ \text{which are rigid, symmetric, and } kU^0\text{–linear} \end{array} \right\}. \end{aligned}$$

Thus far, we have established the following families of isomorphisms:

$$\begin{aligned} \text{(Cohomological formal schemes:)} \qquad (\mathbb{Z}\times BU)_E &\cong C_0\mathbb{CP}^\infty_E, \\ BU_E &\cong C_1\mathbb{CP}^\infty_E, \\ BSU_E &\cong C_2\mathbb{CP}^\infty_E, \\ \text{(Homological schemes:)} \qquad \operatorname{Spec} E_0(\mathbb{Z}\times BU) &\cong C^0(\mathbb{CP}^\infty_E; \mathbb{G}_m), \\ \operatorname{Spec} E_0(BU) &\cong C^1(\mathbb{CP}^\infty_E; \mathbb{G}_m), \\ \operatorname{Spec} E_0(BSU) &\cong C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m), \\ \text{(Orientation schemes:)} \qquad \operatorname{Spec} E_0(MUP) &\cong C^0(\mathbb{CP}^\infty_E; \mathcal{I}(0)), \\ \operatorname{Spec} E_0(MU) &\cong C^1(\mathbb{CP}^\infty_E; \mathcal{I}(0)), \end{aligned}$$

where we have abusively abbreviated the sheaf of functions on $\mathbb{CP}^\infty_E$ to $\mathbb{G}_m$. In order to fill in the missing piece, we exploit the torsor structure on Thom spectra discussed earlier.

Lemma 5.2.16 ([AHS01, Theorem 2.50])**.** *There is a system of compatible maps*

$$\begin{tikzcd} \operatorname{Spec} E_0 BSU \times \operatorname{Spec} E_0 MSU \arrow[r] \arrow[d] & \operatorname{Spec} E_0 MSU \arrow[d] \\ C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \times C^2(\mathbb{CP}^\infty_E; \mathcal{I}(0)) \arrow[r] & C^2(\mathbb{CP}^\infty_E; \mathcal{I}(0)), \end{tikzcd}$$

where the horizontal maps are the action maps defining torsors, and the vertical maps are those described above.

Proof sketch. Recall the isomorphism $T(\mathcal{L} \downarrow \mathbb{CP}^\infty) \simeq \Sigma^\infty\mathbb{CP}^\infty$. The main point of this claim is that the Thom diagonal for $MU[2k, \infty)$ restricts to a very familiar diagonal:

$$\begin{array}{ccc} (\Sigma^\infty \mathbb{C}P^\infty)^{\wedge k} & \xrightarrow{\Delta} & (\Sigma^\infty \mathbb{C}P^\infty)^{\wedge k} \wedge \Sigma^\infty_+ (\mathbb{C}P^\infty)^{\times k} \\ \downarrow & & \downarrow \\ MU[2k,\infty) & \xrightarrow{\Delta} & MU[2k,\infty) \times BU[2k,\infty). \end{array}$$

The diagonal at the level of $(\mathbb{C}P^\infty)^{\times k}$ is responsible for the cup product, so that the classes in the cohomology of projective space which induce the maps

$$\operatorname{Spec} E_0 MU[2k,\infty) \to C^k(\mathbb{C}P^\infty_E; \mathcal{I}(0)), \quad \operatorname{Spec} E_0 BU[2k,\infty) \to C^k(\mathbb{C}P^\infty_E; \mathbb{G}_m)$$

literally multiply together to give the description of the action. This multiplication of sections is exactly the action claimed in the model. □

Proof of Theorem 5.2.10. The claim of this Theorem is that the map

$$\operatorname{Spec} E_0 MSU \to C^2(\mathbb{C}P^\infty_E; \mathcal{I}(0))$$

studied above is an isomorphism. Any map of torsors over a fixed base is automatically an isomorphism. □

Remark 5.2.17 ([AS01, Lemma 6.4]). We can also analyze the map $\operatorname{Spec} E_0 BSU \to \operatorname{Spec} E_0 BU$ in terms of these models of functions to $\mathbb{G}_m$. Again, the analysis passes through a computation in connective K–theory, using the identification

$$kU^*(\mathbb{C}P^\infty)^{\times 2} = \mathbb{Z}[\beta][\![x_1, x_2]\!],$$

where $x_1 = \pi_1^* x$ and $x_2 = \pi_2^* x$ are the Chern classes associated to the tautological bundle pulled back along projections to the first and second factors

$$\pi_1 \colon (\mathbb{C}P^\infty)^{\times 2} \to \mathbb{C}P^\infty \times *, \qquad \pi_2 \colon (\mathbb{C}P^\infty)^{\times 2} \to * \times \mathbb{C}P^\infty,$$

Inside of this ring, we have the equations

$$\begin{aligned} \beta^2 x_1 x_2 &= (1 - \mathcal{L}_1)(1 - \mathcal{L}_2) \\ &= (1 - \mathcal{L}_1) - (1 - \mathcal{L}_1\mathcal{L}_2) + (1 - \mathcal{L}_2) \\ &= \beta\left(\pi_1^*(x) - \mu^*(x) + \pi_2^*(x)\right), \end{aligned}$$

where $\mu \colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the tensor product map. Since the orientation schemes are governed as torsors over these base schemes, we automatically get a description

$$\begin{aligned} \operatorname{Spec} E_0 MU &\longrightarrow \operatorname{Spec} E_0 MSU, \\ f(x) &\longmapsto \frac{f(x_1)\cdot f(x_2)}{f(x_1 +_{\mathbb{CP}^\infty_E} x_2)} \end{aligned}$$

as a section of

$$\pi_1^* \left(\frac{e^*\mathcal{I}(0)}{\mathcal{I}(0)} \right) \otimes \pi_2^* \left(\frac{e^*\mathcal{I}(0)}{\mathcal{I}(0)} \right) \otimes \left(\mu^* \left(\frac{e^*\mathcal{I}(0)}{\mathcal{I}(0)} \right) \right)^{-1} = \frac{e^*\mathcal{I}(0) \otimes \mu^*\mathcal{I}(0)}{\pi_1^*\mathcal{I}(0) \otimes \pi_2^*\mathcal{I}(0)}.$$

Remark 5.2.18 ([AHS01, Remark 2.32]). The published proofs of Ando, Hopkins, and Strickland differ substantially from the account given here. The primary difference is that "$C_2\widehat{\mathbb{G}}$" does not even get mention, essentially because it is a fair amount of technical work to show that such a scheme even exists (especially in the case to come of $BU[6,\infty)$). On the other hand, it is very easy to demonstrate the existence of its Cartier dual: this is a scheme parametrizing certain bivariate power series subject to certain algebraic conditions, hence exists for the same reason that $\mathcal{M}_{\mathbf{fgl}}$ existed (cf. Definition 3.2.1). The compromise for this is that they then have to analyze the scheme $\operatorname{Spec} E_0 BSU$ directly, through considerably more computational avenues. This is not too high of a price: the analysis of the $BU[6,\infty)$ case turns out to be primarily computational anyhow, so this manner of approach is inevitable.

Remark 5.2.19. Our definition of the scheme $C_2\widehat{\mathbb{G}}$ was by the formula

$$C_2\widehat{\mathbb{G}} = \operatorname{Sym}^2_{\operatorname{Div}\widehat{\mathbb{G}}} \operatorname{Div}_0 \widehat{\mathbb{G}},$$

where we are thinking of of $\operatorname{Div}_0 \widehat{\mathbb{G}} \subseteq \operatorname{Div} \widehat{\mathbb{G}}$ as the augmentation ideal inside of an augmented ring. The formal schemes $\operatorname{Div} \widehat{\mathbb{G}}$ and $\operatorname{Div}_0 \widehat{\mathbb{G}}$ are the formal schemes associated by E–theory to the infinite loopspaces underlying kU and $\Sigma^2 kU$ respectively. Remarking that BSU is the infinite loopspace underlying $\Sigma^4 kU$, we arrive at the analogous topological formula

$$\Sigma^4 kU = (\Sigma^2 kU) \wedge_{kU} (\Sigma^2 kU).$$

It should be possible to give an example of a complex-oriented theory which receives an MSU orientation which *does not* factor the complex orientation but *does* (*must*, really) factor the unit? Even if one can find an example of this, I think it will be somewhat artificial: the sequence of group schemes

$$0 \to BSU_E \to BU_E \to BU(1)_E \to 0$$

is short exact, *and* it has a splitting on the level of formal schemes. The splitting is what gives you the isomorphism on points $BSU_E(T) \times BU(1)_E(T) \cong BU_E(T)$. On the other hand, because the splitting *isn't* a map of formal groups, it doesn't survive to the Cartier dual short exact sequence

$$0 \leftarrow BSU^E \leftarrow BU^E \leftarrow BU(1)^E \leftarrow 0,$$

so this will come down to exhibiting a test ring $T = E_*$ for which $BU^E(T) \to BSU^E(T)$, despite being induced by an fppf-

5.3 Chromatic analysis of $BU[6,\infty)$

We now embark on an analysis of $MU[6,\infty)$–orientations in earnest. As in the case of MSU, it is fruitful to first study the behavior of vector bundles with structure map lifted through $\underline{kU}_6 = BU[6,\infty)$ and to analyze the schemes $BU[6,\infty)_E$ and $\operatorname{Spec} E_0 BU[6,\infty)$. In the previous case, we studied a particular bundle

$$\Pi_2 \colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{(1-\mathcal{L}_1)(1-\mathcal{L}_2)} BSU,$$

which controlled much of the geometry through our splitting principle for BSU–bundles, recorded as Lemma 5.2.6. Analogously, we can construct a naturally occurring such bundle as the product

$$\Pi_3 \colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{(1-\mathcal{L}_1)(1-\mathcal{L}_2)(1-\mathcal{L}_3)} BU[6,\infty),$$

but the proof of Lemma 5.2.6 falls apart almost immediately—there does not appear to be a splitting principle for bundles lifted through $BU[6,\infty)$. This is quite worrying, and it dampens our optimism across the board: about the behavior of Π_3 exerting enough control over $BU[6,\infty)$, about the existence of "$C_3\widehat{\mathbb{G}}$", and about $C_3\mathbb{C}\mathrm{P}^\infty_E$ serving as a good model for $BU[6,\infty)_E$.

Nevertheless, we will show that this algebraic model is still accurate by complete topological calculation. Our approach is divided between two fronts.

1. If we specialize to a particularly nice cohomology theory—such as $E = E_\Gamma$ a Morava E–theory—then we can use our extensive body of knowledge about finite height formal groups and their relationship to algebraic topology in order to force nice behavior into the story. This should be thought of as an exploratory step: if there is a general statement to be found, it will be visible in this particularly algebro-geometric setting, where we can maybe compute fully enough to determine what it is.

2. If we specialize to a particularly simple formal group—such as $\widehat{\mathbb{G}}_a$ and its associated cohomology theory $H\mathbb{F}_p$—then we can use our talent for performing computations in algebraic topology to completely exhaust the problem. This should be thought of as the "actual" proof: as in Lecture 4.3, we will show that successfully transferring the guess result from Morava E–theory to the setting of ordinary cohomology entails the result for *any* complex-orientable cohomology theory.

In this Lecture, we will pursue the first avenue. We begin by setting Γ to be a formal group of finite p–height of a field k of positive characteristic p, and we let

$E = E_\Gamma$ denote the associated Morava E–theory. Our main technical tool will be the Postnikov fibration

$$\underline{H\mathbb{Z}}_3 \to BU[6,\infty) \to BSU,$$

and our main goals are to construct a model sequence of formal schemes, then show that E–theory is well-behaved enough that the formal schemes it constructs exactly match the model.

In the previous setting of MSU, we gained indirect access to the algebraic model $C_2\widehat{\mathbb{G}}$ by separately proving that it was modeled by $\mathrm{SDiv}_0\,\widehat{\mathbb{G}}$ and that this had a good comparison map to BSU_E. This time, since we do not have access to $C_3\widehat{\mathbb{G}}$ or anything like it, we proceed by much more indirect means, along the lines of Remark 5.2.18: we know that $C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ exists as an affine scheme, since we can explicitly construct it as a closed subscheme of the scheme of trivariate power series, and so we seek a map

$$\operatorname{Spec} E_0BU[6,\infty) \to C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m)$$

that does not pass through any intermediate cohomological construction. Our main tool for accomplishing this is as follows:

Definition 5.3.1. A map $f\colon X \to Y$ of spaces induces a map $f_E\colon X_E \to Y_E$ of formal schemes. In the case that Y is a commutative H–space and Y_E is connected, we can construct a map according to the composite

$$\begin{tikzcd}
X_E \times \underline{\mathsf{GroupSchemes}}(Y_E,\mathbb{G}_m) \arrow[rrr, dotted] \arrow[d, equal] & & & \widehat{\mathbb{A}}^1 \\
X_E \times \underline{\mathsf{FormalGroups}}(Y_E,\widehat{\mathbb{G}}_m) \arrow[r, "f_E\times 1"] & Y_E \times \underline{\mathsf{FormalGroups}}(Y_E,\widehat{\mathbb{G}}_m) \arrow[rr, "\mathrm{ev}"] & & \widehat{\mathbb{G}}_m. \arrow[u, "\simeq"']
\end{tikzcd}$$

This is called *the adjoint map*, and we write $\widehat{f}$ for any of the above versions of this map, whether valued in $\widehat{\mathbb{G}}_m$, $\mathbb{G}_m$, or $\widehat{\mathbb{A}}^1$. It encodes equivalent information to the E_0–linear map

$$E_0 \to E_0Y \widehat{\otimes}_{E_0} E^0X$$

dual to the map $E_0X \to E_0Y$ induced on E–homology.

Remark 5.3.2. This construction converts many properties of f into corresponding properties of this adjoint element. For instance:

- It is natural in the source: for $f\colon X \to Y$ and $g\colon W \to X$, we have
$$\widehat{fg} = \widehat{f} \circ (g_E \times \mathrm{id}_{Y_E})\colon W_E \times D(Y_E) \to \mathbb{G}_m.$$
- It is natural in the target: for $f\colon X \to Y$ and $h\colon Y \to Z$ a map of H–spaces, we have
$$\widehat{hf} = \widehat{f} \circ (\mathrm{id}_{X_E} \times D(h_E))\colon X_E \times D(Z_E) \to \mathbb{G}_m.$$
- It converts sums of classes to products of maps to $\mathbb{G}_m$.

Example 5.3.3. Recall the vector bundle Π_2 lifted through BSU, defined at the top of this Lecture and of great interest to us in Lecture 5.2. The adjoint to the classifying map of Π_2 is a map of formal schemes
$$\widehat{\Pi}_2\colon (\mathbb{C}\mathrm{P}^\infty_E)^{\times 2} \times \operatorname{Spec} E_0 BSU \to \mathbb{G}_m,$$
which passes through the exponential adjunction to become a map
$$\operatorname{Spec} E_0 BSU \to \underline{\mathsf{FormalSchemes}}((\mathbb{C}\mathrm{P}^\infty_E)^{\times 2}, \mathbb{G}_m).$$
Because the adjoint construction preserves properties of the class Π_2, we learn that this map factors through the closed subscheme
$$\begin{tikzcd} \operatorname{Spec} E_0 BSU \arrow[r, dotted] \arrow[rr, bend left] & C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m) \arrow[r] & \underline{\mathsf{FormalSchemes}}((\mathbb{C}\mathrm{P}^\infty_E)^{\times 2}, \mathbb{G}_m) \end{tikzcd}$$
of symmetric, rigid functions satisfying kU^0–linearity. By careful manipulation of divisors in Theorem 5.2.13, we showed that $BSU_E \cong \mathrm{SDiv}_0\, \mathbb{C}\mathrm{P}^\infty_E$, which on applying Cartier duality showed the factorized map $\operatorname{Spec} E_0 BSU \to C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$ to be an isomorphism.

Example 5.3.4. Similarly, we have defined a cohomology class
$$\Pi_3 = (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)(\mathcal{L}_3 - 1) \in kU^6(\mathbb{C}\mathrm{P}^\infty)^{\times 3} = [(\mathbb{C}\mathrm{P}^\infty)^{\times 3}, BU[6,\infty)].$$
As above, its adjoint induces a map (which we abusively also denote by $\widehat{\Pi}_3$)
$$\widehat{\Pi}_3\colon \operatorname{Spec} E_0 BU[6,\infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m),$$
where $C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$ is the scheme of $\mathbb{G}_m$–valued trivariate functions on $\mathbb{C}\mathrm{P}^\infty_E$ satisfying symmetry, rigidity, and kU^0–linearity.[4]

[4] If $C_3 \mathbb{C}\mathrm{P}^\infty_E := \mathrm{Sym}^3_{\mathrm{Div}\, \mathbb{C}\mathrm{P}^\infty_E} \mathrm{Div}_0\, \mathbb{C}\mathrm{P}^\infty_E$ were to exist, this scheme $C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$ would be its Cartier dual.

We also have the following analogue of the compatibility results Remark 5.2.4 and Remark 5.2.17 of the previous section:

Lemma 5.3.5 ([AS01, Lemma 7.1], [AHS01, Proposition 2.27, Corollary 2.30]). *There is a commutative square*

$$\begin{tikzcd}
\operatorname{Spec} E_0 BSU \arrow[r] \arrow[d, "\widehat{\Pi}_2"] & \operatorname{Spec} E_0 BU[6,\infty) \arrow[d, "\widehat{\Pi}_3"] \\
C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m),
\end{tikzcd}$$

where the map[5] *δ is specified by*

$$\delta(f)(x_1, x_2, x_3) := \frac{f(x_1, x_3) f(x_2, x_3)}{f(x_1 +_E x_2, x_3)}.$$

Proof. As in the proofs of Remark 5.2.4 and Remark 5.2.17, this is checked by performing a calculation in kU–cohomology of projective space, where we have the relation

$$\begin{aligned}
\Pi_3 &= (1 - \mathcal{L}_1)(1 - \mathcal{L}_2)(1 - \mathcal{L}_3) \\
&= ((1 - \mathcal{L}_1) - (1 - \mathcal{L}_1\mathcal{L}_2) + (1 - \mathcal{L}_2))(1 - \mathcal{L}_3) \\
&= ((\pi_1 \times 1)^* - (\mu \times 1)^* + (\pi_2 \times 1)^*)((1 - \mathcal{L}_1)(1 - \mathcal{L}_3)) \\
&= ((\pi_1 \times 1)^* - (\mu \times 1)^* + (\pi_2 \times 1)^*)\Pi_2. \qquad \square
\end{aligned}$$

Thus far, we have constructed the solid maps in the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} E_0 BSU \arrow[r] \arrow[d, "\cong"', "\widehat{\Pi}_2"] & \operatorname{Spec} E_0 BU[6,\infty) \arrow[r] \arrow[d, "\widehat{\Pi}_3"] & \operatorname{Spec} E_0 \underline{H\mathbb{Z}}_3 \arrow[d, "\cong"'] \\
C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, dotted, "e"] & \underline{\mathsf{FormalGroups}}((\mathbb{CP}^\infty_E)^{\wedge 2}, \widehat{\mathbb{G}}_m),
\end{tikzcd}$$

where $\widehat{\mathbb{G}}^{\wedge n}$ denotes the exterior n^{th} power of $\widehat{\mathbb{G}}$, the left-most vertical map is an isomorphism by Corollary 5.2.14, and right-most vertical map is an isomorphism

[5] Despite its name and its formula, this map δ does not really belong to a cochain complex from our perspective. *All* of the functions we are considering, no matter how many inputs they take, are always subject to a 2–cocycle condition.

by Remark 4.6.13. We would like to enrich this diagram to an isomorphism of short exact sequences, and to do so we need to finish constructing the sequences themselves—we need a horizontal map e making the diagram commute.

The idea behind the construction of e is to pretend that $\widehat{\Pi}_3$ is an isomorphism, so that we could completely detect e by comparing the image of the identity point on $\operatorname{Spec} E_0 BU[6,\infty)$ through $\widehat{\Pi}_3$ to the image of the same identity point through the maps

$$\operatorname{Spec} E_0 BU[6,\infty) \to \operatorname{Spec} E_0 \underline{H\mathbb{Z}}_3 \to \mathsf{FormalGroups}((\mathbb{CP}^\infty_E)^{\wedge 2}, \mathbb{G}_m).$$

Using our calculation that $(\mathbb{CP}^\infty_E)^{\wedge 2}$ is a p–divisible group, we see that we can further restrict attention to the torsion subgroups $(\mathbb{CP}^\infty_E)^{\wedge 2}[p^j] = (BS^1[p^j]_E)^{\wedge 2}$ which filter it, corresponding to analyzing the bundle classified by the restriction

$$BS^1[p^j]^{\wedge 2} \xrightarrow{\mu} \underline{HS^1[p^j]}_2 \xrightarrow{\beta_j} \underline{H\mathbb{Z}}_3 \xrightarrow{\gamma} \underline{kU}_6.$$

Using the abbreviation $B_j = BS^1[p^j]$, our summary goal is to find an express description of a map d making the following square commute:

$$\begin{tikzcd}
B_j \wedge B_j \arrow[r] \arrow[d, "\beta_j\mu(\alpha\wedge\alpha)"] & \mathbb{CP}^\infty \wedge \mathbb{CP}^\infty \arrow[d, dotted, "d"] \\
\Sigma^3 H\mathbb{Z} \arrow[r, "\gamma"] & \Sigma^6 kU,
\end{tikzcd}$$

where we have quietly replaced spaces by their suspension spectra, and where $\beta_j\mu(\alpha \wedge \alpha)$ denotes the composite

$$B_j^{\wedge 2} \xrightarrow{\alpha\wedge\alpha} (\Sigma H\mathbb{Z}/p^j)^{\wedge 2} \xrightarrow{\mu} \Sigma^2 H\mathbb{Z}/p^j \xrightarrow{\beta_j} \Sigma^3 H\mathbb{Z}.$$

Our strategy is to extend this putative square to a map of putative cofiber sequences

$$\begin{tikzcd}
(\mathbb{CP}^\infty)^{\wedge 2}/B_j^{\wedge 2} \arrow[r, "\Delta"] \arrow[d, dotted, "f"] & \Sigma B_j \wedge B_j \arrow[r] \arrow[d, "\beta_j\mu(\alpha\wedge\alpha)"] & \Sigma\mathbb{CP}^\infty \wedge \mathbb{CP}^\infty \arrow[r] \arrow[d, dotted, "d"] & \Sigma(\mathbb{CP}^\infty)^{\wedge 2}/(B_j)^{\wedge 2} \arrow[d, dotted, "f"] \\
\Sigma^4 kU \arrow[r, "\sigma"] & \Sigma^4 H\mathbb{Z} \arrow[r, "\gamma"] & \Sigma^7 kU \arrow[r] & \Sigma^5 kU,
\end{tikzcd}$$

and thereby trade the task of constructing d for the task of constructing f. This is a gain because $\sigma\colon kU \to H\mathbb{Z}$, the standard kU–orientation of $H\mathbb{Z}$, is a considerably simpler map to understand than γ.

Lemma 5.3.6 ([AS01, Section 5]). *Make the definitions*

- $x\colon \mathbb{C}\mathrm{P}^\infty \to \Sigma^2 kU$ *is the kU–Euler class for* $(1-\mathcal{L})$.
- $u\colon T(\mathcal{L}^{\otimes p^j}) \to kU^2$ *is the kU–Thom class for* $T(\mathcal{L}^{\otimes p^j}) = \mathbb{C}\mathrm{P}^\infty/B_j$.
- A_1 *is the projection* $\frac{\mathbb{C}\mathrm{P}^\infty \wedge \mathbb{C}\mathrm{P}^\infty}{B_j \wedge B_j} \to \frac{\mathbb{C}\mathrm{P}^\infty \wedge \mathbb{C}\mathrm{P}^\infty}{B_j \wedge \mathbb{C}\mathrm{P}^\infty} = (\mathbb{C}\mathrm{P}^\infty/B_j) \wedge \mathbb{C}\mathrm{P}^\infty = T(\mathcal{L}^{\otimes p^j}) \wedge \mathbb{C}\mathrm{P}^\infty$.
- *Similarly,* A_2 *is the swapped projection* $(\mathbb{C}\mathrm{P}^\infty)^{\wedge 2}/B_j^{\wedge 2} \to \mathbb{C}\mathrm{P}^\infty \wedge T(\mathcal{L}^{\otimes p^j})$.

Setting $f = \mu(u \wedge x)A_1 - \mu(x \wedge u)A_2$ *gives the desired commuting square:*

$$\sigma \circ f = \beta_j \mu(\alpha \wedge \alpha) \circ \Delta.$$

Proof. The idea is to gain control of the cohomology group $H\mathbb{Z}^4((\mathbb{C}\mathrm{P}^\infty)^{\wedge 2}, B_j^{\wedge 2})$ by Mayer-Vietoris, which is rendered complicated by our simultaneous use of the cofiber sequence

$$B_j \to \mathbb{C}\mathrm{P}^\infty \to T(\mathcal{L}^{\otimes p^j})$$

in *two* factors of a smash product. Toward this end, consider the maps

$$B_1\colon B_j \wedge T(\mathcal{L}^{\otimes p^j}) \to (\mathbb{C}\mathrm{P}^\infty)^{\wedge 2}/B_j^{\wedge 2}, \qquad B_2\colon T(\mathcal{L}^{\otimes p^j}) \wedge B_j \to (\mathbb{C}\mathrm{P}^\infty)^{\wedge 2}/B_j^{\wedge 2},$$

which have cofibers A_1 and A_2 respectively. Direct calculation [AS01, Lemma 5.6] shows that $(\ker B_1^*) \cap (\ker B_2^*)$ is torsion-free, so if we can identify $B_1^*(\beta_j \mu \circ \Delta)$ and $B_2^*(\beta_j \mu \circ \Delta)$, we will be most of the way there. We pick B_1 to consider, as B_2 is similar, and we start computing, beginning with

$$B_1^*(\beta_j \mu(\alpha \wedge \alpha) \circ \Delta) = \beta_j \mu(\alpha \wedge \alpha) \circ \Delta B_1.$$

Writing $\delta\colon T(\mathcal{L}^{\otimes p^j}) \to \Sigma B_j$ for the going-around map in that cofiber sequence, we have

$$\begin{tikzcd} \Sigma B_j^{\wedge 2} & B_j \wedge T(\mathcal{L}^{\otimes p^j}) \arrow[l, "1\wedge\delta"'] \arrow[d, "B_1"] \\ & (\mathbb{C}\mathrm{P}^\infty)^{\wedge 2}/B_j^{\wedge 2}, \arrow[ul, "\Delta"] \end{tikzcd}$$

and hence

$$\begin{aligned} \beta_j \mu(\alpha \wedge \alpha) \circ \Delta B_1 &= \beta_j \mu(\alpha \wedge \alpha) \circ (1_B \wedge \delta) \\ &= \beta_j \mu(\alpha \wedge \alpha\delta). \end{aligned}$$

The maps α and δ appear in the following map of cofiber sequences:

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "\alpha"] & P \arrow[r, "q"] \arrow[d, "y"] & T \arrow[r, "\delta"] \arrow[d, "w"] & \Sigma B \arrow[d, "\alpha"] \\ \Sigma H\mathbb{Z}/p^j \arrow[r, "\beta_j"] & \Sigma^2 H\mathbb{Z} \arrow[r, "p^j"] & \Sigma^2 H\mathbb{Z} \arrow[r, "\rho"] & \Sigma^2 H\mathbb{Z}/p^j, \end{tikzcd}$$

where y is the standard Euler class in $H^2\mathbb{C}\mathrm{P}^\infty$ and the first block commutes because the bottom row is the stabilization of the top row; w is the Thom class associated to $T(\mathcal{L}^{\otimes p^j})$ and the middle block commutes because it witnesses the $H\mathbb{Z}$–analogue of the statements expressed by Lemma 2.6.1 and Theorem 4.6.1; and the last block commutes because $[B, \Sigma^2 H\mathbb{Z}] = 0$ and because the other two do. In particular, an application of the right-most block gives

$$\beta_j\mu(\alpha \wedge \alpha\delta) = \beta_j\mu(\alpha \wedge \rho w).$$

Using the fact that β_j is the cofiber of the ring map ρ, there is a juggle

$$\beta_j\mu(\alpha \wedge \rho w) = \mu(\beta_j\alpha \wedge w),$$

and then we use the first block in the above map of cofiber sequences to conclude

$$\mu(\beta_j\alpha \wedge w) = \mu(yj \wedge w).$$

Finally, we can use this to guess a formula for our desired map f: we set

$$f = \mu(u \wedge x)A_1 + \mu(x \wedge u)A_2,$$

because, for instance,

$$\begin{aligned} B_1^*(\sigma f) &= \sigma(\mu(u \wedge x)A_1 + \mu(x \wedge u)A_2)B_1 \\ &= \sigma\mu(x \wedge u)(j \wedge \mathrm{id}_T), \end{aligned}$$

where we used $A_1B_1 = 0$ and $A_2B_1 = (j \wedge \mathrm{id}_T)$, a calculation similar to the calculation involving δ earlier in the proof. Then, because $\sigma\colon kU \to H\mathbb{Z}$ sends Euler classes to Euler classes, we have

$$\begin{aligned} \sigma\mu(x \wedge u)(j \wedge \mathrm{id}_T) &= \mu(y \wedge w)(j \wedge \mathrm{id}_T) \\ &= \mu(yj \wedge w). \end{aligned}$$

Hence, we have crafted a class f with $\sigma f - \beta_j \mu(\alpha \wedge \alpha) \in (\ker B_1^*) \cap (\ker B_2^*)$.

What remains is to show that this class is torsion, hence identically zero. Half of this is obvious: $p^j \beta_j \mu(\alpha \wedge \alpha) = 0$, since $p^j \beta_j = 0$ on its own. For $p^j \sigma f$, we make an explicit calculation:

$$\begin{aligned} p^j \sigma f &= p^j (\mu(w \wedge y) A_1 - \mu(y \wedge w) A_2) \\ &= \mu(w \wedge p^j y) A_1 - \mu(p^j y \wedge w) A_2 \\ &= \mu(w \wedge q^* w) A_1 - \mu(q^* w \wedge w) A_2 \\ &= \mu(w \wedge w) \circ ((1 \wedge q) A_1 - (q \wedge 1) A_2) = 0. \end{aligned}$$ □

The upshot of all of this is that we have our desired calculation of the map e:

Corollary 5.3.7 ([AS01, Lemma 5.4 and Subsection "Modelling $d_n(L_1, L_2)$"]). *There is a commuting triangle*

$$\begin{tikzcd} (\Sigma^\infty BS^1[p^j])^{\wedge 2} \arrow[d, "\beta_j"] \arrow[dr, "d_j"] & \\ \underline{H\mathbb{Z}_3} \arrow[r, "\gamma"] & \underline{kU_6}, \end{tikzcd}$$

where d_j classifies the bundle

$$d_j = \sum_{k=1}^{p^j - 1} \left((1 - \mathcal{L}_1)(1 - \mathcal{L}_1^{\otimes k})(1 - \mathcal{L}_2) - (1 - \mathcal{L}_1)(1 - \mathcal{L}_2^{\otimes k})(1 - \mathcal{L}_2) \right).$$

Proof. We return to our putative map of cofiber sequences, and in particular to the right-most block

$$\begin{tikzcd} \mathbb{C}\mathrm{P}^\infty \wedge \mathbb{C}\mathrm{P}^\infty \arrow[r, "r"] \arrow[d, dotted, "d"] & (\mathbb{C}\mathrm{P}^\infty)^{\wedge 2} / B_j^{\wedge 2} \arrow[d, "f"] \\ \Sigma^6 kU \arrow[r, "\beta"] & \Sigma^4 kU. \end{tikzcd}$$

This expresses d in terms of f in the cohomology ring $kU^*(\mathbb{C}\mathrm{P}^\infty)^{\times 2}$, a by-now familiar situation. Namely, we have

$$\begin{aligned} \beta d &= (\mu(u \wedge x) A_1 - \mu(x \wedge u) A_2) r \\ &= \mu(u \wedge x)(q \wedge \mathrm{id}_{\mathbb{C}\mathrm{P}^\infty}) - \mu(x \wedge u)(\mathrm{id}_{\mathbb{C}\mathrm{P}^\infty} \wedge q) \\ &= \mu(q^* u \wedge x) - \mu(x \wedge q^* u). \end{aligned}$$

At this point, we need to make an actual identification: u is a Thom class associated to the line bundle $\mathcal{L}^{\oplus p^j}$, hence q^*u is its associated Euler class, which we compute in terms of x to be $q^*u = [p^j]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x)$, where the n–series on $\mathbb{C}\mathrm{P}^\infty_{kU}$ expressed in terms of the coordinate x is given by $[n]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x) = \beta^{-1}(1-(1-\beta x)^n)$. We use this formula to continue the calculation:

$$\begin{aligned}\mu(q^*u \wedge x) - \mu(x \wedge q^*u) &= [p^j]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x_1)\cdot x_2 - x_1 \cdot [p^j]_{\mathbb{C}\mathrm{P}^\infty_{kU}}(x_2) \\ &= x_1x_2\left(\frac{1-(1-\beta x_1)^{p^j}}{\beta x_1} - \frac{1-(1-\beta x_2)^{p^j}}{\beta x_2}\right) \\ &= \sum_{k=1}^{p^j-1}(x_1[k](x_1)x_2 - x_1[k](x_2)x_2). \qquad \square\end{aligned}$$

We take this as inspiration for an algebraic definition:

Definition 5.3.8. Let $\widehat{\mathbb{G}}$ be a connected p–divisible group of dimension 1. Given a point $f \in C^3(\widehat{\mathbb{G}};\mathbb{G}_m)(T)$, we construct the function

$$\begin{aligned} e_{p^j}(f)\colon\ & \widehat{\mathbb{G}}[p^j]^{\wedge 2} \to \mathbb{G}_m, \\ e_{p^j}(f)\colon\ & (x_1,x_2) \mapsto \prod_{k=1}^{p^j}\frac{f(x_1,kx_1,x_2)}{f(x_1,kx_2,x_2)}.\end{aligned}$$

As j ranges, this assembles into a map

$$e\colon\ C^3(\widehat{\mathbb{G}};\mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}^{\wedge 2},\mathbb{G}_m),$$

called the *Weil pairing* associated to f.

By design, the map e participates in a commuting square with $\operatorname{Spec} E_0BU[6,\infty) \to \operatorname{Spec} E_0\underline{H\mathbb{Z}}_3$, so that this fills out the map of sequences we were considering before we got involved in this analysis of vector bundles. What remains, then, is to assemble enough exactness results to apply the 5–lemma.

Lemma 5.3.9 ([AS01, Lemma 7.2])**.** *For $\widehat{\mathbb{G}}$ a connected p–divisible group of dimension 1, the map $\delta\colon\ C^2(\widehat{\mathbb{G}};\mathbb{G}_m) \to C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ is injective.*

Proof. Being finite height means that the multiplication-by-p map of $\widehat{\mathbb{G}}$ is fppf–surjective. The kernel of δ consists of symmetric, biexponential maps $\widehat{\mathbb{G}}^{\times 2} \to \mathbb{G}_m$.[6] By restricting such a map f to

$$f\colon \widehat{\mathbb{G}}[p^j] \times \widehat{\mathbb{G}} \to \mathbb{G}_m,$$

we can calculate

$$f(x, p^j y) = f(p^j x, y) = f(0, y) = 1.$$

But since p^j is surjective on $\widehat{\mathbb{G}}$, every point on the right-hand side can be so written (after perhaps passing to a flat cover of the base), so at every left-hand stage the map is trivial. Finally, $\widehat{\mathbb{G}} = \operatorname{colim}_j \widehat{\mathbb{G}}[p^j]$, so this filtration is exhaustive and we conclude that the kernel is trivial. □

Lemma 5.3.10 ([AS01, Lemma 7.3]). *More generally, the following sequence is exact*

$$0 \to C^2(\widehat{\mathbb{G}}; \mathbb{G}_m) \xrightarrow{\delta} C^3(\widehat{\mathbb{G}}; \mathbb{G}_m) \xrightarrow{e} \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}^{\wedge 2}, \mathbb{G}_m).$$

Remarks on proof. The previous Lemma demonstrates exactness at the first node. Showing that $e \circ \delta = 0$ is simple enough, but constructing preimages of $\ker \delta$ through e is hard work. The main tool, again, is p–divisibility: given a point $(g_1, g_2) \in \widehat{\mathbb{G}}[p^j]^{\wedge 2}$, over some flat base extension we can find g_2' satisfying $p^j g_2' = g_2$. With significant effort, the assignment $(g_1, g_2) \mapsto \{e_{p^j}(f)(g_1, g_2')^{-1}\}$ as j ranges can be shown to be independent of the choices g_2' and which, if $e(f) = 1$, determines an element of $C^2(\widehat{\mathbb{G}}; \mathbb{G}_m)$. □

Luckily, the remaining bit of topology is very easy:

Lemma 5.3.11 ([AS01, Lemma 7.5]). *The top row of the main diagram is a short exact sequence of group schemes.*

Proof. Consider the sequence of homology Hopf algebras, before applying Spec. Since the integral homology of BSU and the E–homology of $\underline{H\mathbb{Z}}_3$ are both free and even, the Atiyah–Hirzebruch spectral sequence for $E_*BU[6, \infty)$ collapses to their tensor product over E_*. □

---

[6] The condition $f \in \ker \delta$ gives $f(x, y+z) = f(x,y)f(x,z)$, so that the kU^0–linearity condition becomes redundant:

$$\frac{f(x,y)f(t,x+y)}{f(t+x,y)f(t,x)} = \frac{f(x,y)[f(t,x)f(t,y)]}{[f(t,y)f(x,y)]f(t,x)} = 1.$$

Corollary 5.3.12 ([AS01, Theorem 1.4]). *The map*

$$\widehat{\Pi}_3 \colon \operatorname{Spec} E_0 BU[6,\infty) \to C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m)$$

is an isomorphism.

Proof. This is now a direct consequence of the 5–lemma. □

Remark 5.3.13 (cf. Theorem 5.4.1). We will soon show that $H_* BU[6,\infty)$ is also free and even. The proof of Lemma 5.3.11 thus also shows that the E–theory of $\underline{kU}_8$ fits into a similar short exact sequence.

Remark 5.3.14 ([AS01, Corollary 7.6]). The topological input to the 5–lemma also gave us a purely algebraic result for free: the map e is a *surjective* map of group schemes.

5.4 Analysis of $BU[6,\infty)$ at infinite height

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Motivated by our success at analyzing the schemes $\operatorname{Spec}(E_\Gamma)_0 BU[6,\infty)$ associated to $BU[6,\infty)$ through Morava E–theory, we move on to considering the scheme constructed via ordinary homology. As usual, we expect this to be harder: the formal group associated to ordinary homology is not p–divisible, and this causes many sequences which are short exact from the perspective of Morava E–theory to go awry. Instead, we will have to examine the problem more directly—luckily, the extremely polite formal group law associated to $\widehat{\mathbb{G}}_a$ makes computations accessible. We also expect the reward to be greater: as in Corollary 4.3.7, we will be able to use a successful analysis of the ordinary homology scheme to give a description of the complex-orientable homology schemes, no matter what complex-orientable homology theory we use.

As in the p–divisible case, our framework comes in the form of a map of sequences

$$\begin{array}{ccccc}
\operatorname{Spec} H_* BSU & \longrightarrow & \operatorname{Spec} H_* BU[6,\infty) & \longrightarrow & \text{“}\operatorname{Spec} H_* \underline{H\mathbb{Z}}_3\text{”} \\
\downarrow & & \downarrow & & \downarrow \\
C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) & \longrightarrow & C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) & \longrightarrow & \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m).
\end{array}$$

Our task, as then, is to discern as much about these nodes as possible, as well as any exactness properties of the two sequences.[7]

We begin with the topological sequence. The Serre spectral sequence

$$E_2^{*,*} = H^*BSU \otimes H^*\underline{H\mathbb{Z}}_3 \Rightarrow H^*BU[6,\infty)$$

gives us easy access to the middle node, and we will recount the case of $p = 2$ in detail. In this case, the spectral sequence has E_2–page

$$E_2^{*,*} = H\mathbb{F}_2^*BSU \otimes H\mathbb{F}_2^*\underline{H\mathbb{Z}}_3 \cong \mathbb{F}_2[c_2, c_3, \ldots] \otimes \mathbb{F}_2\left[\mathrm{Sq}^I \iota_3 \,\middle|\, \begin{array}{c} I_j \geq 2I_{j+1}, \\ 2I_1 - I_+ > 1 \end{array}\right].$$

Because the target is 6–connective, we must have the transgressive differential $d_4\iota_3 = c_2$, which via the Kudo transgression theorem spurs the much larger family of differentials

$$d_{4+I_+}\,\mathrm{Sq}^I \iota_3 = \mathrm{Sq}^I c_2.$$

This necessitates understanding the action of the Steenrod operations on the cohomology of BSU, which is due to Wu [May99, Section 23.6]:

Can these formulas be read off from the divisorial calculation? Maybe not, since it's easy to read off the Milnor primitives but harder to see the Steenrod squares.

$$\mathrm{Sq}^{2^j} \cdots \mathrm{Sq}^4 \mathrm{Sq}^2 c_2 \equiv c_{1+2^j} \pmod{\text{decomposables}}.$$

Accounting for the squares of classes left behind, this culminates in the following calculation:

This spectral sequence can be drawn in using Hood's package.

Theorem 5.4.1. *There is an isomorphism*

$$H\mathbb{F}_2^*BU[6,\infty) \cong \frac{H\mathbb{F}_2^*BU}{(c_j \mid j \neq 2^k + 1, j \geq 3)} \otimes F_2[\iota_3^2, (\mathrm{Sq}^2 \iota_3)^2, \ldots]. \qquad \square$$

Remark 5.4.2 ([Sin68, Sto63]). More generally, there is an isomorphism

$$H\mathbb{F}_2^*\underline{kU}_{2k} \cong \frac{H\mathbb{F}_2^*BU}{(c_j \mid \sigma_2(j-1) < k-1)} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-1}],$$

where σ_2 is the dyadic digital sum and "Op" denotes the Steenrod–Hopf–subalgebra of $H\mathbb{F}_2^*\underline{H\mathbb{Z}}_{2k-1}$ generated by the indicated class. Stong specialized to $p = 2$ and carefully applied the Serre spectral sequence to the fibrations

$$\underline{kU}_{2(k+1)} \to \underline{kU}_{2k} \to \underline{H\mathbb{Z}}_{2k}.$$

[7] The quotes indicate that the right-hand topological node does not even make sense: $H^*\underline{H\mathbb{Z}}_3$ is not even-concentrated, and we do not understand the algebraic geometry of spaces whose homology is not even-concentrated. This is quite troubling—but we will press on for now.

Singer worked at an arbitrary prime and used the Eilenberg–Moore spectral sequence for the fibrations

$$\underline{H\mathbb{Z}}_{2k-1} \to \underline{kU}_{2(k+1)} \to \underline{kU}_{2k}.$$

Both used considerable knowledge of the interaction of these spectral sequences with the Steenrod algebra.

Remark 5.4.3. These methods and results generalize directly to odd primes. The necessary modifications come from understanding the unstable mod–p Steenrod algebra, using analogues of Wu's formulas [Sha77], and employing Singer's Eilenberg–Moore calculation. Again, $H^*BU[6,\infty)$ is presented as a quotient by H^*BU by certain Chern classes whose indices satisfy a p–adic digital sum condition, tensored up with the subalgebra of $H^*\underline{H\mathbb{Z}}_3$ generated by a certain element.

From the edge homomorphisms in Theorem 5.4.1, we can already see that the sequence of formal group schemes

$$\text{“}(\underline{H\mathbb{Z}}_3)_{HP}\text{”} \to BU[6,\infty)_{HP} \to BSU_{HP}$$

is neither left-exact nor right-exact. This seems bleak.

Ever the optimists, we turn to the algebra. We begin by reusing a strategy previously employed in Lemma 3.4.12: first perform a tangent space calculation

$$T_0C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m) \cong C^k(\widehat{\mathbb{G}}_a;\widehat{\mathbb{G}}_a),$$

then study the behavior of the different tangent directions to determine the full object $C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$. As a warm-up to the case $k=3$ of interest, we will first consider the case $k=2$. We have already performed the tangent space calculation:

Corollary 5.4.4 (cf. Lemma 3.2.5)**.** *The unique symmetric additive* 2*–cocycle of homogeneous degree n has the form*

$$c_n(x,y) = \begin{cases} (x+y)^n - x^n - y^n & \text{if } n \neq p^j, \\ \frac{1}{p}\left((x+y)^n - x^n - y^n\right) & \text{if } n = p^j. \end{cases} \qquad \square$$

Our goal, then, is to select such an $f_+ \in C^2(\widehat{\mathbb{G}}_a;\widehat{\mathbb{G}}_a)$ and study the minimal conditions needed on a symbol a to produce a point in $C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ of the form $1 + af_+ + \cdots$. Since $c_n = \frac{1}{d_n}\delta(x^n)$ is itself produced by an additive formula, life would be easiest if we had access to an exponential, so that we could build

$$\text{“}\delta_{(\widehat{\mathbb{G}}_a;\mathbb{G}_m)} \exp(a_n x^n)^{1/d_n} = \exp(\delta_{(\widehat{\mathbb{G}}_a;\widehat{\mathbb{G}}_a)} a_n x^n/d_n) = \exp(a_n c_n).\text{”}$$

However, the existence of an exponential series is equivalent to requiring that a_n carry a divided-power structure, which turns out not to be minimal. In fact, we can show that *no* conditions on a_n are required *at all*.

Definition 5.4.5 (cf. Remark 3.3.21). The *Artin–Hasse exponential* is the power series

$$E_p(t) = \exp\left(\sum_{j=0}^{\infty} \frac{t^{p^j}}{p^j}\right) \in \mathbb{Z}_{(p)}[\![t]\!].$$

Lemma 5.4.6 ([AHS01, Proposition 3.9]). *Write* $\delta_{(\widehat{\mathbb{G}}_a,\mathbb{G}_m)} \colon C^1(\widehat{\mathbb{G}}_a;\mathbb{G}_m) \to C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ *and*

$$d_n = \begin{cases} 1 & \text{if } n = p^j, \\ 0 & \text{otherwise.} \end{cases}$$

The class $g_n(x,y) = \delta_{(\widehat{\mathbb{G}}_a,\mathbb{G}_m)} E_p(a_n x^n)^{1/p^{d_n}}$ *is a series in* $\mathbb{F}_p[a_n][\![x,y]\!]$ *and presents a point in* $C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ *reducing to* $a_n c_n \in C^2(\widehat{\mathbb{G}}_a;\widehat{\mathbb{G}}_a)$ *on tangent spaces.*

Proof. Recall from Remark 3.3.21 that E_p has coefficients in $\mathbb{Z}_{(p)}$, and hence it can be reduced to a series with coefficients in $\mathbb{F}_p$. With this in mind, we make the calculation

$$\begin{aligned} g_n(x,y) &= \delta_{(\widehat{\mathbb{G}}_a,\mathbb{G}_m)} E_p(a_n x^n)^{1/p^{d_n}} \\ &= \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} \delta_{(\widehat{\mathbb{G}}_a,\widehat{\mathbb{G}}_a)} x^{np^j}}{p^{j+d_n}}\right) \\ &= \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} c_{np^j}(x,y)}{p^j}\right). \end{aligned}$$

As claimed, the leading term is exactly $a_n c_n$, this series is symmetric, and since it is in the image of $\delta_{(\widehat{\mathbb{G}}_a,\mathbb{G}_m)}$ it is certainly a 2–cocycle. Finally, the integrality properties of E_p mean that g_n has coefficients in $\mathbb{Z}_{(p)}[a_n]$. □

Letting n range, this culminates in the following calculation:

Lemma 5.4.7 ([AHS01, Equation 3.7]). *The map*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_n \mid n \geq 2] \xrightarrow{\prod_{n\geq 2} g_n} C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}$$

is an isomorphism. □

The trivariate case $k = 3$ is similar, with one important new wrinkle: over an $\mathbb{F}_p$–algebra there is an equality $c_n^p = c_{pn}$, but this relation does not generalize to trivariate 2–cocycles. For instance, consider the following example at $p = 2$:

$$\frac{1}{2}\delta(c_6) = x^2y^2z^2 + x^4yz + xy^4z + xyz^4, \qquad \left(\frac{1}{2}\delta c_3\right)^2 = x^2y^2z^2.$$

The following Lemma states that this is the only new feature:

Lemma 5.4.8 ([AHS01, Proposition 3.20, Proposition A.12]). *The p–primary residue of the scheme of trivariate symmetric 2–cocycles is presented by*

$$\operatorname{Spec} \mathbb{F}_p[a_d \mid d \geq 3] \times \operatorname{Spec} \mathbb{F}_p[b_d \mid d = p^j(1+p^k)] \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a) \times \operatorname{Spec} \mathbb{F}_p. \quad \square$$

Similar juggling of the Artin–Hasse exponential yields the following multiplicative classification:

Theorem 5.4.9 ([AHS01, Proposition 3.28]). *There is an isomorphism*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_d \mid d \geq 3, d \neq 1 + p^t] \times \operatorname{Spec} \Gamma[b_{1+p^t}] \to C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}.$$

Proof sketch. The main claim is that the Artin–Hasse exponential trick used in the case $k = 2$ works here as well, provided $d \neq 1 + p^t$ so that taking an appropriate p^{th} root works out. They then show that the remaining exceptional cases extend to multiplicative cocycles only when the p^{th} power of the leading coefficient vanishes. Finally, a rational calculation shows how to bind these truncated generators together into a divided power algebra. $\square$

It is now time to clear up our confusion about the right-hand topological node by pursuing a link between $H_* \underline{H\mathbb{Z}}_3$ and the algebraic model $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m)$. Analyzing the edge homomorphism from our governing Serre spectral sequence shows that the map

$$H^* BU[6,\infty) \to H^* \underline{H\mathbb{Z}}_3$$

factors through the subalgebra $A^* \subseteq H^* \underline{H\mathbb{Z}}_3$ generated by the *squares* of the polynomial generators. Accordingly, we aim to replace the right-hand node of the topological sequence with $\operatorname{Spec} A_*$ outright.

Lemma 5.4.10 ([AHS01, Lemma 3.36, Proposition 4.13, Lemma 4.11]). *The scheme* $\operatorname{Spec} A_*$ *models* $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m)$ *by an isomorphism λ commuting with $e \circ \hat{\Gamma}_3$.*

Proof sketch. We can describe the $\mathbb{F}_p$–scheme $\underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m)$ completely explicitly:

$$\begin{tikzcd} (a_{mn})_{m,n} \arrow[r, mapsto] & \prod_{m<n} \operatorname{texp}\left(a_{mn}(x^{p^m}y^{p^n} - x^{p^n}y^{p^m})\right) \\ \operatorname{Spec} \mathbb{F}_p[a_{mn} \mid m < n]/(a_{mn}^p) \arrow[r, "\cong"] & \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \mathbb{G}_m), \end{tikzcd}$$

where $\operatorname{texp}(t) = \sum_{j=0}^{p-1} t^j/j!$ is the truncated exponential series. It is easy to check that this ring of functions agrees with A^*, and it requires hard work (although not much creativity) to check the remainder of the statement: that $e \circ \hat{\Pi}_3$ factors through $\operatorname{Spec} A_*$ and that the factorization is an isomorphism. $\square$

We have now finally assembled our map of sequences,

$$\begin{tikzcd} \operatorname{Spec} H_*BSU \arrow[r] \arrow[d, "\cong"', "\hat{\Pi}_2"] & \operatorname{Spec} H_*BU[6,\infty) \arrow[r] \arrow[d, "\hat{\Pi}_3"] & \operatorname{Spec} A^* \arrow[r] \arrow[d, "\cong"', "\lambda"] & 0 \\ C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "e"] & \mathsf{Weil}(\widehat{\mathbb{G}}_a) \arrow[r] & 0 \end{tikzcd}$$

which we have shown to be exact at all the indicated nodes. (The exactness of the topological sequence follows from the Serre spectral sequence analysis. The exactness of the bottom sequence follows from it receiving a map from the top exact sequence, where the left-hand vertical map is an isomorphism.) Our calculations now pay off:

Corollary 5.4.11 ([AHS01, Corollary 4.14])**.** *The map $\hat{\Pi}_3$ is an isomorphism:*

$$\hat{\Pi}_3 \colon \operatorname{Spec} H_*BU[6,\infty) \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m).$$

Proof sketch. We don't actually have to compute much about the middle map. Because the squares in the map of sequences commute and the sequences themselves are exact as indicated, we at least learn that $\hat{\Pi}_3$ is an epimorphism on group schemes, hence a monomorphism on rings of functions. But, since both source and target are affine schemes of graded finite type with equal Poincaré series in each case, this monomorphism is an isomorphism. $\square$

Corollary 5.4.12 ([AHS01, Theorem 2.31])**.** *The map $\hat{\Pi}_3$ is an isomorphism for any complex-orientable E.*

Proof sketch. This follows much along the lines of Corollary 4.3.7. The evenness of the topological calculation at $E = H\mathbb{F}_p$ shows that the statement holds for $H\mathbb{Z}_p^\wedge$ and $H\mathbb{Z}_{(p)}$, and since p is arbitrary we conclude it for $H\mathbb{Z}$ as well. We thus learn that the statement holds for $E = MUP$ using a tangent space argument, and then an Atiyah–Hirzebruch argument gives the statement for any complex-oriented E. $\square$

Remark 5.4.13. This argument does *not* extend to a claim that we have an isomorphism of topological and algebraic exact sequences for any choice of complex-orientable homology theory E. Our trick of replacing $H_*\underline{H\mathbb{Z}_3}$ by A_* has no generic analogue.

Our analysis of $\operatorname{Spec} E_*BU[6,\infty)$ forms input to two related pursuits: the homology scheme $\operatorname{Spec} E_*MU[6,\infty)$ arising in the theory of Thom spectra, and the object $BU[6,\infty)_E$ predual to $\operatorname{Spec} E_0BU[6,\infty)$. The analysis of the Thom spectrum is completely analogous to the analysis performed at the end of Lecture 5.2, and so we merely state the relevant results.

Definition 5.4.14. For a formal group $\widehat{\mathbb{G}}$, define maps $\mu_{ij}\colon \widehat{\mathbb{G}}^{\times 3} \to \widehat{\mathbb{G}}$ which multiply the i^{th} and j^{th} factors, discarding the remaining factor. For a line bundle $\mathcal{L}$ over $\widehat{\mathbb{G}}$, we define the scheme $C^3(\widehat{\mathbb{G}};\mathcal{L})$ by

$$C^3(\widehat{\mathbb{G}};\mathcal{L})(T) = \left\{ \begin{array}{l} \text{triv}^{\text{ns}} \text{ of } u^*\left(\frac{e^*\mathcal{L}\otimes(\mu_{12}^*\mathcal{L}\otimes\mu_{13}^*\mathcal{L}\otimes\mu_{23}^*\mathcal{L})}{(\pi_1^*\mathcal{L}\otimes\pi_2^*\mathcal{L}\otimes\pi_3^*\mathcal{L})\otimes\mu_{\text{all}}^*\mathcal{L}}\right) \downarrow u^*\widehat{\mathbb{G}}^{\times 3} \\ \text{which are rigid, symmetric, and } kU^0\text{–linear} \end{array} \right\}.$$

Lemma 5.4.15 ([AHS01, Theorem 2.50])**.** *There is a system of compatible maps*

$$\begin{tikzcd} \operatorname{Spec} E_0BU[6,\infty) \times \operatorname{Spec} E_0MU[6,\infty) \arrow[r] \arrow[d, equal] & \operatorname{Spec} E_0MU[6,\infty) \arrow[d] \\ C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m) \times C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0)) \arrow[r] & C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0)), \end{tikzcd}$$

where the horizontal maps are the action maps defining torsors and the vertical maps are those induced by $\widehat{\Pi}_3$. $\square$

Corollary 5.4.16. *Take E to be* complex–orientable*. The functor $C^3(\mathbb{CP}^\infty_E;\mathcal{I}(0))$ is isomorphic to the affine scheme* $\operatorname{Spec} E_0MU[6,\infty)$*. Moreover, the E_0–points of this scheme biject with ring spectrum maps $MU[6,\infty) \to E$.* $\square$

Lemma 5.4.17. *The ring map $MU[6,\infty) \to MSU$ is modeled by the map*

$$\begin{aligned} C^2(\mathbb{CP}^\infty_E; \mathcal{I}(0)) &\xrightarrow{\delta} C^3(\mathbb{CP}^\infty_E; \mathcal{I}(0)), \\ s \in \Theta^2\mathcal{I}(0) &\mapsto \frac{\mu_{12}^* s}{\mu_1^* s \otimes \mu_2^* s} \in \Theta^1\Theta^2\mathcal{I}(0) \cong \Theta^3\mathcal{I}(0). \quad \square \end{aligned}$$

Remark 5.4.18. The rational conclusion of this analysis admits a very mild reformulation: there is always a natural $MU[6,\infty)$–orientation of a rational spectrum E given by the composite

$$MU[6,\infty) \to MU[6,\infty) \otimes \mathbb{Q} \to H\mathbb{Q} = \mathbb{S} \otimes \mathbb{Q} \xrightarrow{\eta_E} E \otimes \mathbb{Q} = E.$$

This canonical point turns the $C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m)$–torsor structure of $C^3(\mathbb{CP}^\infty_E; \mathcal{I}(0))$ into an isomorphism $C^3(\mathbb{CP}^\infty_E; \mathcal{I}(0)) \cong C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m)$. In sum, an arbitrary $MU[6,\infty)$–orientation of E is witnessed by a symmetric rigid trivariate power series satisfying kU^0–linearity, called its *characteristic series*. There are similar theories of characteristic series for rational orientations by MSU, MU, and MUP; in the latter two cases this recovers the *Hirzebruch series*, which associated to an orientation $\varphi\colon MUP \to \mathbb{Q} \otimes E$ is the difference of trivializations $x / \exp_\varphi(x)$. Conversely, any such series gives rise to a characteristic class by the formula

$$K_\varphi(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) = \prod_{j=1}^n \frac{c_1(\mathcal{L}_j)}{\exp_\varphi(c_1(\mathcal{L}_j))},$$

where c_1 is the first Chern class associated to the canonical rational orientation.

Our second task is to analyze the cohomology formal scheme associated to $BU[6,\infty)$, and we begin with the choice $E = H$.

Lemma 5.4.19 (Ando–Hopkins–Strickland, unpublished)**.** *$DC^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ are all formal varieties for $k \leq 3$.*

Proof. We know that $\mathcal{O}C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ are all free $\mathbb{Z}$–modules of graded finite rank in the range $k \leq 3$, so we may write

$$\mathcal{O}(DC^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)) \cong (\mathcal{O}C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m))^*.$$

Our task is to show that this Hopf algebra $\mathcal{O}(C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m))^*$ is a power series ring.

Specialize, for the moment, to the case of $k = 2$. It will suffice to show that it is a power series ring modulo p for every prime p. Such graded connected finite-type Hopf algebras over $\mathbb{F}_p$ were classified by Borel (and exposited by Milnor–Moore [MM65, Theorem 7.11]) as either polynomial or truncated polynomial. These two cases are distinguished by the Frobenius operation: the Frobenius on a polynomial ring is injective, whereas the Frobenius on a truncated polynomial ring is not. It is therefore equivalent to show that the *Verschiebung* on the original ring $\mathcal{O}(C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m)) \otimes \mathbb{F}_p$ is *surjective*. Recalling the calculation $c_n^p = c_{np}$ at the level of bivariate 2–cocycles, we compute

$$p^* a_n = a_{np}^p,$$

and since $Fa_{np} = a_{np}^p$ and $FV = p^*$, we learn

$$V(a_{np}) = a_n.$$

Essentially the same proof handles the cases $k = 1$ and $k = 0$.

The case $k = 3$ requires a small modification, to cope with the two classes of trivariate 2–cocycles. On the polynomial tensor factor of $\mathcal{O}(C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m))$ we can reuse the same Verschiebung argument to see that its dual Hopf algebra is polynomial. For the other fact, the dual of the divided power tensor factor is, without any further argument, always a primitively generated polynomial algebra. $\square$

Theorem 5.4.20 (Ando–Hopkins–Strickland, unpublished)**.** *The scheme $C_3\mathbb{CP}^\infty_E$ exists, and it is modeled by $BU[6,\infty)_E$.*

Proof sketch. Let $\widehat{\mathbb{G}}$ be an arbitrary formal group. Note first that if $C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ is coalgebraic, then $C_3\widehat{\mathbb{G}}$ exists and is its Cartier dual: the diagram presenting $\mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ as a reflexive coequalizer of free Hopf algebras is also the diagram meant to present $C_3\widehat{\mathbb{G}}$ as a coalgebraic formal scheme. So, if the coequalizing Hopf algebra has a good basis, it will follow from Theorem 5.1.8 that the resulting diagram is a colimit diagram in formal schemes, with $C_3\widehat{\mathbb{G}}$ sitting at the cone point. It will additionally follow that the isomorphism

$$\operatorname{Spec} E_0 BU[6,\infty) \xrightarrow{\cong} C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m)$$

of Corollary 5.4.12 will re-dualize to an isomorphism

$$BU[6,\infty)_E \xleftarrow{\cong} C_3\mathbb{CP}^\infty_E.$$

So, we reduce to checking that $\mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ admits a good basis. By a base change argument, it suffices to take $\widehat{\mathbb{G}}$ to be the universal formal group over the Lazard ring, and we thus set about finding a nice basis for that.

We hope to gain control (as in Corollary 4.3.7 or Corollary 5.4.12) of this situation using our strong knowledge of $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$. We know from Lemma 5.4.19 that $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ is a free abelian group, and we know from Theorem 3.2.2 that $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$ is as well. By picking a $\mathbb{Z}$–basis $\mathbb{Z}\{\beta_j\}_j$ of $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$ and considering the specialization map from $\widehat{\mathbb{G}}$ over $\mathcal{M}_{\mathbf{fgl}}$ to $\widehat{\mathbb{G}}_a$ over $\operatorname{Spec}\mathbb{Z}$, we choose a map α of $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$–modules

$$\begin{tikzcd} \mathcal{O}(\mathcal{M}_{\mathbf{fgl}})\{\tilde{\beta}_j\}_j \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m) \arrow[d] \\ \mathbb{Z}\{\beta_j\}_j \arrow[r, "\cong"] & \mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m). \end{tikzcd}$$

By induction on degree, one sees that α is surjective, and since the source and target are abelian groups of graded finite rank *and the source is free*, we need only check that they have the same rational Poincaré series to conclude that α is an isomorphism. Over $\operatorname{Spec}\mathbb{Q}$ we can use the logarithm to construct an isomorphism

$$\operatorname{Spec}\mathbb{Q} \times (\mathcal{M}_{\mathbf{fgl}} \times C^k(\widehat{\mathbb{G}};\mathbb{G}_m)) \to \operatorname{Spec}\mathbb{Q} \times (\mathcal{M}_{\mathbf{fgl}} \times C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)),$$

hence the Poincaré series agree, hence $\alpha \otimes \mathbb{Q}$ is an isomorphism, and finally α is too.

Lastly, one checks that this basis gives us access to the desired collection of good subcoalgebras: these are indexed on an integer d, spanned by those basis vectors of degree at most d. $\square$

5.5 Modular forms and $MU[6,\infty)$–manifolds

The first goal of this Lecture is to give the briefest possible summary of the theory of elliptic curves that covers the topics necessary to us in the coming sections. Accordingly, we won't cover many topics that a sane introduction to elliptic curves would make a point to cover, and—perhaps worse—we will hardly prove anything. We will, however, discover a place where "$C_3\widehat{\mathbb{G}}$" appears internally to the theory of elliptic curves, and I hope nonetheless that this will give the arithmetically disinclined reader a foothold on the "elliptic" part of "elliptic cohomology".

To begin, recall that an elliptic curve in the complex setting is a torus, and it admits a presentation by selecting a lattice Λ of full rank in $\mathbb{C}$ and forming the quotient

$$\mathbb{C} \xrightarrow{\pi_\Lambda} \mathbb{C}/\Lambda =: E_\Lambda.$$

A meromorphic function f on E_Λ pulls back to give a meromorphic function $\pi_\Lambda^* f$ on $\mathbb{C}$ which satisfies a periodicity constraint in the form of the functional equation

$$\pi_\Lambda^* f(z+\Lambda) = \pi_\Lambda^* f(z).$$

It follows immediately that there are no holomorphic such functions, save the constants—such a function would be bounded, and Liouville's theorem would apply. It is, however, possible to build the following meromorphic special function, which has poles of order 2 at the lattice points and satisfies the periodicity constraints:

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Its derivative is also a meromorphic function satisfying the periodicity constraint:

$$\wp'_\Lambda(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}.$$

In fact, these two functions generate all other meromorphic functions on E_Λ, in the sense that the subsheaf spanned by the algebra generators $\wp_\Lambda$ and $\wp'_\Lambda$ is exactly $\pi_\Lambda^* \mathcal{M}_{E_\Lambda}$. This algebra is subject to the following relation, in the form of a differential equation:

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - 60G_4(\Lambda)\wp_\Lambda(z) - 140G_6(\Lambda),$$

for some special values $G_4(\Lambda), G_6(\Lambda) \in \mathbb{C}$. Accordingly, writing $C \subseteq \mathbb{C}P^2$ for the projective curve $wy^2 = 4x^3 - G_4(\Lambda)w^2x - G_6(\Lambda)w^3$, there is an analytic group isomorphism

$$\begin{aligned} E_\Lambda &\to C, \\ z \pmod{\Lambda} &\mapsto [1 : \wp_\Lambda(z) : \wp'_\Lambda(z)]. \end{aligned}$$

This is sometimes referred to as the *Weierstrass presentation* of E_Λ.

Remark 5.5.1. Before proceeding, the values $G_4(\Lambda)$ and $G_6(\Lambda)$ are themselves interesting when considered as functions of the lattice. Expanding out the relation above gives an explicit formula for the $(2k)^{th}$ *Eisenstein series*

$$G_{2k}(\Lambda) = \sum_{\ell \in \Lambda} \frac{1}{\ell^{2k}}, \quad G_{2k}(\lambda\Lambda) = \lambda^{-2k} G_{2k}(\Lambda).$$

A function on the space of lattices which satisfies such a homogeneity condition is referred to as a *modular form* of weight $2k$, and they appear naturally as global sections over the moduli of elliptic curves on the $2k^{\text{th}}$ tensor power of the sheaf of invariant differentials. Already, one can deduce that the ring of complex-analytic modular forms has the form $\mathbb{C}[G_4, G_6]$, but it is actually a theorem of Deligne that an analogous theorem is true integrally: there exist modular forms c_4, c_6, and Δ such that

$$H^0(\mathcal{M}_{\mathbf{ell}}; \omega^{\otimes *}) \cong \mathbb{Z}[c_4, c_6, \Delta^{\pm}]/(c_4^3 - c_6^2 - 2^6 3^3 \Delta).$$

There is a second standard embedding of a complex elliptic curve into projective space, using θ*–functions*, which are most naturally expressed with an alternative basic presentation of an elliptic curve. Select a lattice Λ and a basis for it, and rescale the lattice so that the basis takes the form $\{1, \tau\}$ with τ in the upper half-plane. Then, the normalized exponential function $\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto \exp(2\pi i z)$ has $1 \cdot \mathbb{Z}$ as its kernel. Setting $q = \exp(2\pi i \tau)$ to account for the missing component of the kernel of π_Λ, we get a second presentation of E_Λ as $\mathbb{C}^\times / q^{\mathbb{Z}}$, as pictured in Figure 5.1.

Should this picture include q^2 and q^{-1}?

Definition 5.5.2. The basic θ–function associated to E_Λ is defined by

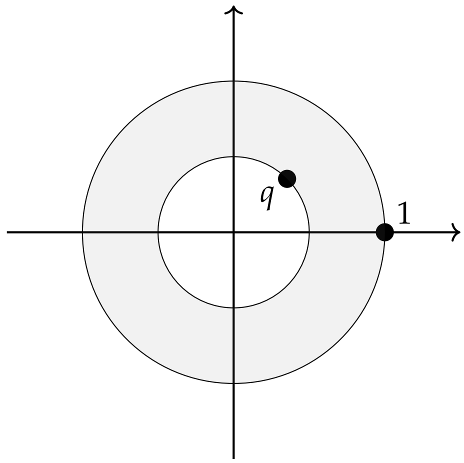
$$\theta_q(u) = \prod_{m \geq 1} (1 - q^m)(1 + q^{m-\frac{1}{2}} u)(1 + q^{m - \frac{1}{2}} u^{-1}) = \sum_{n \in \mathbb{Z}} u^n q^{\frac{1}{2} n^2}.$$

Given two rational numbers $0 \leq a, b \leq 1$, we can also shift the zero-set of θ_q in the 1 and q directions by the fractions a and b, giving translated θ–functions:

$$\theta_q^{a,b}(u) = \left(q^{\frac{a^2}{2}} \cdot u^a \cdot \exp(2\pi i a b) \right) \cdot \theta_q(u q^a \exp(2\pi i b)).$$

The basic θ–function vanishes on the set $\{\exp(2\pi i(\frac{1}{2} m + \frac{\tau}{2} n))\}$, i.e., at the center of the fundamental annulus. Since it has no poles, it cannot descend to give a function on $\mathbb{C}^\times / q^{\mathbb{Z}}$, and its failure to descend is witnessed by its imperfect periodicity relation:

$$\theta_q(qu) = u^{-1} q^{\frac{-1}{2}} \theta_q(u).$$



$\rightsquigarrow$

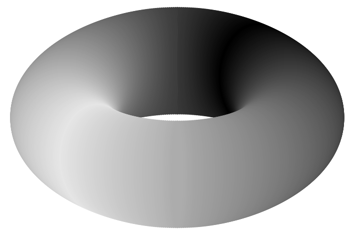


Figure 5.1: Presentation of an elliptic curve as the quotient of an annulus.

Lemma 5.5.3 ([Hus04, Proposition 10.2.6]). *For any $N > 0$, define $V_q[N]$ to be the space of functions $f\colon \mathbb{C}^\times \to \mathbb{C}$ satisfying*

$$f(qu) = u^{-N} q^{-N^2/2} f(u).$$

Then, $V_q[N]$ has $\mathbb{C}$–dimension N^2, and the functions $\theta_q^{a,b}$ give a basis as a and b range over rational numbers with denominator N. □

Even though these functions do not themselves descend to $\mathbb{C}^\times/q^{\mathbb{Z}}$, we can collectively use them to construct a map to complex projective space, where the quasi-periodicity relations will mutually cancel in homogeneous coordinates.

Theorem 5.5.4 ([Hus04, Proposition 10.3.2]). *Consider the map*

$$\begin{aligned} \mathbb{C}/(N \cdot \Lambda) &\xrightarrow{f_{(N)}} \mathbb{P}^{N^2-1}(\mathbb{C}), \\ z &\mapsto [\cdots : \theta_q^{i/N, j/N}(z) : \cdots]. \end{aligned}$$

For $N > 1$, this map is an embedding. □

Example 5.5.5. Let us expand this in the case of $N = 2$. The four functions involved are labeled $\theta_q^{0,0}$, $\theta_q^{0,1/2}$, $\theta_q^{1/2,0}$, and $\theta_q^{1/2,1/2}$, and we record their zero loci in Figure 5.2. The image of $f_{(2)}$ in $\mathbb{P}^{2^2-1}(\mathbb{C})$ is cut out by the equations

$$A^2 x_0^2 = B^2 x_1^2 + C^2 x_2^2, \qquad A^2 x_3^2 = C^2 x_1^2 - B^2 x_2^2,$$

Function	Zero locus
$\theta_q^{0,0}$	$q^{\mathbb{Z}} \cdot q^{1/2} \cdot i$
$\theta_q^{0,1/2}$	$q^{\mathbb{Z}} \cdot q^{1/2}$
$\theta_q^{1/2,0}$	$q^{\mathbb{Z}} \cdot i$
$\theta_q^{1/2,1/2}$	$q^{\mathbb{Z}}$.

Figure 5.2: Standard θ–functions and their offsets

where

$$x_0 = \theta_q^{0,0}(u^2), \qquad x_1 = \theta_q^{0,1/2}(u^2), \qquad x_2 = \theta_q^{1/2,0}(u^2), \qquad x_3 = \theta_q^{1/2,1/2}(u^2)$$

and

$$A = \theta_q^{0,0}(0) = \sum_n q^{n^2}, \quad B = \theta_q^{0,1/2}(0) = \sum_n (-1)^n q^{n^2}, \quad C = \theta_q^{1/2,0}(0) = \sum_n q^{(n+1/2)^2},$$

upon which there is the additional "Jacobi" relation

$$A^4 = B^4 + C^4.$$

Remark 5.5.6. This embedding of E_Λ as an intersection of quadric surfaces in $\mathbb{C}\mathrm{P}^3$ is quite different from the Weierstrass embedding. Nonetheless, the embeddings are analytically related. Namely, there is an equality

$$\frac{d^2}{dz^2} \log \theta_q(u) = \wp_\Lambda(z).$$

Separately, Weierstrass considered a function σ_Λ, defined by

$$\sigma_\Lambda(z) = z \prod_{\omega \in \Lambda \setminus 0} \left(1 - \frac{z}{\omega}\right) \cdot \exp\left[\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right],$$

which also has the property that its second logarithmic derivative is $\wp$ and so is "basically $\theta_q^{1/2,1/2}$". In fact, any elliptic function can be written in the form

$$c \cdot \prod_{i=1}^n \frac{\sigma_\Lambda(z - a_i)}{\sigma_\Lambda(z - b_i)}.$$

The θ–functions version of the story has two main successes: it continues to make sense in algebraic geometry, without invoking transcendental functions, and in fact there is a version of this story for an *arbitrary* abelian variety. It turns out that all abelian varieties are projective, and the theorem sitting at the heart of this claim is

Corollary 5.5.7 ("Theorem of the Cube", [Mil86, Corollary I.6.4 and Theorem I.7.1])**.** *Let A be an abelian variety, let $p_i : A \times A \times A \to A$ be the projection onto the i^{th} factor, and let $p_{ij} = p_i +_A p_j$, $p_{ijk} = p_i +_A p_j +_A p_k$. Then for any invertible sheaf $\mathcal{L}$ on A, the sheaf*

$$\Theta^3(\mathcal{L}) := \frac{p_{123}^*\mathcal{L} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}}{p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_{31}^*\mathcal{L} \otimes p_\varnothing^*\mathcal{L}} = \bigotimes_{I \subseteq \{1,2,3\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}$$

on $A \times A \times A$ is trivial. If $\mathcal{L}$ is rigid (i.e., it has a specified trivialization at the identity point of A), then $\Theta^3(\mathcal{L})$ is canonically *trivialized by a section $s(A; \mathcal{L})$.* □

Remark 5.5.8. One way to read the Theorem of the Cube is that a weight zero divisor on an abelian variety is principal (i.e., it is the zeroes and poles of a meromorphic function) if and only if its nodes sum to zero. The meromorphic function you construct this way is unique up to scale, so if you impose a normalization condition at the identity point, you get a unique such function. Altogether, this gives a pairing between $(A \times A)^*$ and A, which can be reconsidered as a multiplication map $A \times A \to A$.

Remark 5.5.9. The section $s(A; \mathcal{L})$ satisfies three familiar properties:

- It is symmetric: pulling back $\Theta^3\mathcal{L}$ along a shuffle automorphism of A^3 yields $\Theta^3\mathcal{L}$ again, and the pullback of the section $s(A; \mathcal{L})$ along this shuffle agrees with the original $s(A; \mathcal{L})$ across this identification.

- It is rigid: by restricting to $* \times A \times A$, the tensor factors in $\Theta^3\mathcal{L}$ cancel out to give the trivial bundle over $A \times A$. The restriction of the section $s(A; \mathcal{L})$ to this pullback bundle agrees with the extension of the rigidifying section.

- It satisfies a 2–cocycle condition: in general, we define

$$\Theta^k\mathcal{L} := \bigotimes_{I \subseteq \{1,\ldots,k\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}.$$

In fact, $\Theta^{k+1}\mathcal{L}$ can be written as a pullback of $\Theta^k\mathcal{L}$:

$$\Theta^{k+1}\mathcal{L} = \frac{(p_{12} \times \mathrm{id}_{A^{k-1}})^*\mathcal{L}}{(p_1 \times \mathrm{id}_{A^{k-1}})^*\mathcal{L} \otimes (p_2 \times \mathrm{id}_{A^{k-1}})^*\mathcal{L}},$$

and pulling back a section s along this map gives a new section

$$(\delta s)(x_0, x_1, \ldots, x_k) := \frac{s(x_0 +_A x_1, x_2, \ldots, x_k)}{s(x_0, x_2, \ldots, x_k) \cdot s(x_1, x_2, \ldots, x_k)}.$$

Performing this operation on the first and second factors yields the defining equation of a 2–cocycle.

Remark 5.5.10. The proof of projectivity arising from this method rests on choosing a line bundle on A, extracting from it a very ample line bundle, and then constructing some generating global sections to get an embedding into $\mathbb{P}(\mathcal{L}^{\oplus n})$ [Har77, Remark II.7.8.2]. Mumford [Mum66] showed that a choice of "θ–structure" on $(A, \mathcal{L})$, which is only slightly more data, gives a canonical choice of generating global sections as well as a canonical identification of $\mathbb{P}(\mathcal{L}^{\oplus n})$ with a *fixed* projective space. This is suitable for studying how these equations change as one considers different points in the moduli of abelian varieties [Mum67a, Mum67b].

Remark 5.5.11 ([Bre83, Section 4]). Breen presented a relative version of this story that applies to arbitrary *commutative group schemes*, where the basic objects are a choice of line bundle $\mathcal{L}$ over a commutative group scheme A, a choice of trivialization of $\Theta^3\mathcal{L}$, and an epimorphism $\pi\colon A' \to A$ that trivializes $\mathcal{L}$.

Finally, we remark that the function

$$e\colon C^3(\widehat{\mathbb{G}}; \mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}_a^{\wedge 2}, \widehat{\mathbb{G}}_m)$$

considered in Lecture 5.3 also manifests in the theory of abelian varieties. Let A be an abelian variety equipped with a line bundle $\mathcal{L}$. Suppose that s is a symmetric, rigid section of $\Theta^3\mathcal{L}$, sometimes called a *cubical structure on* $\mathcal{L}$. Using the identification $(p_{12} - p_1 - p_2)^*\Theta^2\mathcal{L} = \Theta^3\mathcal{L}$, this induces the structure of a *symmetric biextension* on $\Theta^2\mathcal{L}$ via the multiplication maps

$$(\Theta^2\mathcal{L})_{x,y} \otimes (\Theta^2\mathcal{L})_{x',y} \to (\Theta^2\mathcal{L})_{x+x',y}, \quad (\Theta^2\mathcal{L})_{x,y} \otimes (\Theta^2\mathcal{L})_{x,y'} \to (\Theta^2\mathcal{L})_{x,y+y'}.$$

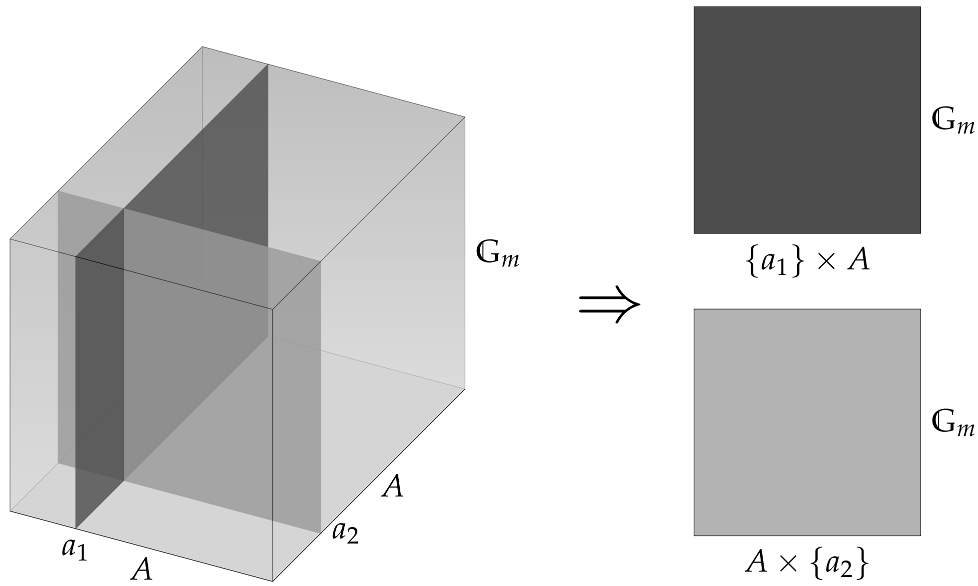


Figure 5.3: Extensions contained in a biextension.

Definition 5.5.12. There is a canonical piece of gluing data on this biextension, in the form of an isomorphism of pullback bundles

$$\begin{aligned} e_{p^j} \colon (p^j \times 1)^* \mathcal{L}|_{A[p^j] \times A[p^j]} &\cong (1 \times p^j)^* \mathcal{L}|_{A[p^j] \times A[p^j]}, \\ (\ell, x, y) &\mapsto \left(\ell \cdot \prod_{k=1}^{p^j - 1} \frac{s(x, [k]x, y)}{s(x, [k]y, y)} \right). \end{aligned}$$

This function e_{p^j} is called the *$(p^j)^{th}$ Weil pairing*.

Remark 5.5.13. In the case that A is an elliptic curve, this agrees with the usual definition of its "Weil pairing". In the case of a *complex* elliptic curve $\mathbb{C}/(1, \tau)$, this degenerates further to the assignment

$$\left(\frac{a}{n}, \frac{b}{n}\tau \right) \mapsto \exp\left(-2\pi i \frac{ab}{n} \right).$$

We now actually leverage this arithmetic geometry by placing ourselves in a situation where algebraic topology is directly linked to abelian varieties.

Definition 5.5.14. An *elliptic spectrum* consists of a even–periodic ring spectrum E, a (generalized) elliptic curve C over Spec E_0, and a fixed isomorphism

$$\varphi\colon\ C_0^{\wedge} \xrightarrow{\cong} \mathbb{C}\mathrm{P}_E^{\infty}.$$

A map among such spectra consists of a map of ring spectra $f\colon\ E \to E'$ together with a specified isomorphism of elliptic curves $\psi\colon\ f^*C \to C'$.[8]

Remark 5.5.15. Our preference for *isomorphisms* of elliptic curves rather than general homomorphisms is prompted by our study of stable operations: we have seen that a stable operation between complex-oriented ring spectra can only ever give rise to an isomorphism of formal group laws (incorporating a suitable base change). More broadly, we have also found that the collection of possible isomorphisms of formal group laws organizes into the *stable context*, a very important object in our study. In the next Case Study, we will develop a theory (with an attendant notion of context) which incorporates *isogenies* of elliptic curves in addition to isomorphisms.

Coupling Definition 5.5.14 to Corollary 5.4.16 and Corollary 5.5.7, we conclude the following:

Corollary 5.5.16. *An elliptic spectrum (E, C, φ) receives a canonical map of ring spectra*

$$MU[6,\infty) \to E.$$

This map is natural in choice of elliptic spectrum: if $(E, C, \varphi) \to (E', C', \varphi')$ is a map of elliptic spectra, then the triangle

$$\begin{tikzcd} & MU[6,\infty) \arrow[dl] \arrow[dr] & \\ E \arrow[rr] & & E' \end{tikzcd}$$

commutes. □

Example 5.5.17. Our basic example of an elliptic curve was $E_\Lambda = \mathbb{C}/\Lambda$, with Λ a complex lattice. The projection $\mathbb{C} \to \mathbb{C}/\Lambda$ has a local inverse which defines an isomorphism of formal groups

$$\varphi\colon\ (E_\Lambda)_0^{\wedge} \xrightarrow{\cong} \widehat{\mathbb{G}}_a \times \operatorname{Spec} \mathbb{C},$$

as well as an isomorphism of cotangent spaces

[8] Elliptic curves and cohomology theories can be brought much closer together still, as in Lurie's framework [Lurd, Sections 4 and 5.3].

$$\begin{tikzcd} T^0(E_\Lambda)^\wedge_0 \arrow[r, "\cong"] & \mathbb{C} \arrow[d, equal] \\ T^0(\widehat{\mathbb{G}}_a \times \operatorname{Spec} \mathbb{C}) \arrow[u, "\cong"] \arrow[r, "\cong"] & \mathbb{C}. \end{tikzcd}$$

Accordingly, we define an elliptic spectrum $HE_\Lambda P$ whose underlying ring spectrum is $H\mathbb{C}P$ and whose associated elliptic curve and isomorphism are E_Λ and φ. This spectrum receives a natural map

$$MU[6,\infty) \to HE_\Lambda P,$$

which to a bordism class $M \in MU[6,\infty)_{2n}$ assigns an element $\Phi_\Lambda(M) \cdot u_\Lambda^n \in HE_\Lambda P_{2n}$, where u_Λ is the canonical section of $\pi_2 HE_\Lambda P = \omega_{\mathbb{CP}^\infty_{HE_\Lambda P}}$ and $\Phi_\Lambda(M) \in \mathbb{C}$ is some resulting complex number.

Example 5.5.18. The naturality of the $MU[6,\infty)$–orientation moves us to consider more than one elliptic spectrum at a time. If Λ' is another lattice with $\Lambda' = \lambda \cdot \Lambda$, then the multiplication map $\lambda \colon \mathbb{C} \to \mathbb{C}$ descends to an isomorphism $E_\Lambda \to E_{\Lambda'}$ and hence a map of elliptic spectra $HE_{\Lambda'}P \to HE_\Lambda P$ acting by $u_{\Lambda'} \mapsto \lambda u_\Lambda$. The commuting triangle in Corollary 5.5.16 then begets the *modularity relation*

$$\Phi(M; \lambda \cdot \Lambda) = \lambda^{-n} \Phi(M; \Lambda).$$

Example 5.5.19. This equation leads us to consider all curves E_Λ simultaneously—or, equivalently, to consider modular forms. The lattice Λ can be put into a standard form, by picking a basis and scaling it so that one vector lies at 1 and the other vector lies in the upper half-plane. This gives a cover

$$\mathfrak{h} \to \mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec} \mathbb{C}$$

which is well-behaved (i.e., unramified) away from the special points i and $e^{2\pi i/6}$. A *complex modular form of weight n* is an analytic function $\mathfrak{h} \to \mathbb{C}$ which satisfies a certain decay condition and which is quasi-periodic for the action of $SL_2(\mathbb{Z})$, i.e.,[9]

$$f\left(M; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(M; \tau).$$

[9] That is, for the action of change of basis vectors.

Using these ideas, we construct a cohomology theory $H\mathcal{O}_{\mathfrak{h}}P$, where $\mathcal{O}_{\mathfrak{h}}$ is the ring of complex-analytic functions on the upper half-plane. The $\mathfrak{h}$–parametrized family of elliptic curves

$$\mathfrak{h} \times \mathbb{C}/(1, \tau) \to \mathfrak{h},$$

together with the logarithm, present $H\mathcal{O}_{\mathfrak{h}}P$ as an elliptic spectrum $H\mathfrak{h}P$. The canonical map $\Phi\colon MU[6, \infty) \to H\mathfrak{h}P$ specializes at a point to give the functions $\Phi(-; \Lambda)$ considered above, and hence $\Phi(M) \in u^k \cdot \mathcal{O}_{\mathfrak{h}}$ is itself a complex modular form of weight k.

In fact, this totalized map Φ is a ghost of Ochanine and Witten's modular genus from Theorem 0.0.3, as a bordism class in $MU[6, \infty)_{2n}$ is, in particular, a bordism class in $MString_{2n}$. However, they know more about this function than we can presently see: for instance, they claim that it has an integral q–expansion. In terms of the modular form, its q–expansion is given by building the Taylor expansion "at ∞" (using that unspoken decay condition). In order to use our topological methods, it would be nice to have an elliptic spectrum embodying these q–expansions in the same way that $H\mathfrak{h}P$ embodied holomorphic functions, together with a comparison map that trades a modular form for its q–expansion. The main ideas leading to such a spectrum come from considering the behavior of E_Λ as τ tends to $i \cdot \infty$.

Definition 5.5.20. Note that as τ tends to $i \cdot \infty$, the parameter $q = \exp(2\pi i \tau)$ tends to 0. In the multiplicative model of Lecture 5.5, we considered the punctured complex disk D' with its associated family of elliptic curves

$$C'_{\mathrm{an}} = \mathbb{C}^\times \times D'/(u, q) \sim (qu, q).$$

The fiber of C' over a particular point $q \in D'$ is the curve $\mathbb{C}^\times/q^{\mathbb{Z}}$. The Weierstrass equations give an embedding of C'_{an} into $D' \times \mathbb{CP}^2$ described by

$$wy^2 + wxy = x^3 - 5\alpha_3 w^2 x + -\frac{5\alpha_3 + 7\alpha_5}{12} w^3$$

for certain functions α_3 and α_5 of q.[10] At $q = 0$, this curve collapses to the twisted cubic

$$wy^2 + wxy = x^3,$$

and over the whole open unit disc D we call this extended family C_{an}.

[10]Indeed, certain modular forms: α_3 has weight 6 and α_5 has weight 10.

Now let $A \subseteq \mathbb{Z}[\![q]\!]$ be the subring of power series which converge absolutely on the open unit disk. It turns out that the coefficients of the Weierstrass cubic (i.e., $5\alpha_3$ and $\frac{1}{12}(5\alpha_3 + 7\alpha_5)$) lie in A, so it determines a generalized elliptic curve C over $\operatorname{Spec} A$, and C_{an} is the curve given by base-change from A to the ring of holomorphic functions on D [Mor89, Section 5]. The *Tate curve* is the intermediate family C_{Tate} over the intermediate base $D_{\mathrm{Tate}} = \operatorname{Spec} \mathbb{Z}[\![q]\!]$, as base-changed from A.

The singular fiber at $q = 0$ prompts us to enlarge our notion of elliptic curve slightly.

Definition 5.5.21 ([AHS01, Definitions B.1-2])**.** A *Weierstrass curve* is *any* curve of the form

$$C(a_1, a_2, a_3, a_4, a_6) := \left\{ [x : y : w] \in \mathbb{P}^2 \,\middle|\, \begin{array}{c} y^2 w + a_1 xyw + a_3 yw^2 = \\ x^3 + a_2 x^2 w + a_4 xw^2 + a_6 w^3 \end{array} \right\}.$$

A *generalized elliptic curve* over S is a scheme C equipped with maps

$$S \xrightarrow{0} C \xrightarrow{\pi} S$$

such that C is Zariski–locally isomorphic to a system of Weierstrass curves (in a way preserving 0 and π).[11,12]

Remark 5.5.22 ([AHS01, pg. 670]). The singularities of a degenerate Weierstrass equation always occur outside of a formal neighborhood of the marked identity point, which in fact still carries the structure of a formal group. The formal group associated to the twisted cubic is the formal multiplicative group (indeed, the smooth locus of the twisted cubic *is the multiplicative group*), and the isomorphism making the identification extends a family of such isomorphisms φ over the nonsingular part of the Tate curve.

[11] An elliptic curve in the usual sense turns out to be a generalized elliptic curve which is smooth, i.e., the discriminant of the Weierstrass equations is a unit.

[12] Unfortunately, "generalized elliptic curve" already means something in number theory, but Ando, Hopkins, and Strickland reused this phrase for this definition in their published article. In a number theorist's language, these are "stable curves of genus 1 with specified section in the smooth locus". No adjective other than "generalized" seems to be much better: singular, for instance, evokes the right idea but is also already taken.

Definition 5.5.23 ([Mor89, Section 5], [AHS01, Section 2.7])**.** The generalized elliptic spectrum K_{Tate}, called *Tate K–theory*, has as its underlying spectrum $KU[\![q]\!]$. The associated generalized elliptic curve is C_{Tate}, and the isomorphism $\mathbb{C}\mathrm{P}^{\infty}_{KU[\![q]\!]} \cong (C_{\mathrm{Tate}})^{\wedge}_{0}$ is φ from Remark 5.5.22.

The trade for the breadth of this definition is that theorems pulled from the study of abelian varities have to be shown to extend uniquely to those generalized elliptic curves which are not smooth curves.

Theorem 5.5.24 ([AHS01, Propositions 2.57 and B.25])**.** *For a generalized elliptic curve C, there is a canonical*[13] *trivialization s of $\Theta^3\mathcal{I}(0)$ which is compatible with change of base and with isomorphisms. If C is a smooth elliptic curve, then s agrees with that of Corollary 5.5.7.* □

Corollary 5.5.25. *The trivializing section s associated to C_{Tate} is given by $\delta^{\circ 3}\tilde{\theta}$, where $\tilde{\theta}_q$ is a slight modification of the classical θ–function:*

$$\tilde{\theta}_q(u) = (1-u)\prod_{n>0}(1-q^n u)(1-q^n u^{-1}), \qquad \tilde{\theta}_q(qu) = -u^{-1}\tilde{\theta}_q(u).$$

Proof. Even though $\tilde{\theta}$ is not a function on C_{Tate} because of its quasiperiodicity, it does trivialize both $\pi^*\mathcal{I}(0)$ for $\pi\colon \mathbb{C}^\times \times D \to C_{\mathrm{Tate}}$ and $\mathcal{I}(0)$ for $(C_{\mathrm{Tate}})^{\wedge}_{0}$. Moreover, the quasiperiodicities in the factors in the formula defining $\delta^3\tilde{\theta}|_{(C_{\mathrm{Tate}})^{\wedge}_{0}}$ cancel each other out, and the resulting function *does* descend to give a trivialization of $\Theta^3\mathcal{I}(0)$. By the unicity and continuity clauses in Theorem 5.5.24, it must give a formula expressing s. □

Definition 5.5.26. The induced map

$$\sigma_{\mathrm{Tate}}\colon MU[6,\infty) \to K_{\mathrm{Tate}}$$

is called the *complex σ–orientation*.

Corollary 5.5.27. *Let $M \in \pi_{2n}MU[6,\infty)$ be a bordism class. The q–expansion of Witten's modular form $\Phi(M)$ has integral coefficients.*

Proof. The span of elliptic spectra equipped with $MU[6,\infty)$–orientations

[13] Canonical, with a unique continuous extension from the smooth bulk of the moduli of generalized elliptic curves, but *not* actually unique over the singular locus.

$$\begin{tikzcd}
 & MU[6,\infty) \arrow[dl, "\sigma_{\mathrm{Tate}}"'] \arrow[d] \arrow[dr, "\Phi"] & \\
K_{\mathrm{Tate}} \arrow[r] & K_{\mathrm{Tate}} \otimes \mathbb{C} & H\mathfrak{h}P \arrow[l]
\end{tikzcd}$$

models q–expansion. The arrow $K_{\mathrm{Tate}} \to K_{\mathrm{Tate}} \otimes \mathbb{C}$ is injective on homotopy, which shows that the q–expansion of $\Phi(M)$ lands in the subring of integral power series. $\square$

We can use the formula $\sigma_{\mathrm{Tate}} = \delta^3\tilde{\theta}$ appearing in Corollary 5.5.25 to explicitly understand the genus associated to σ_{Tate} by passing to homotopy groups. To begin, the appearances of the map δ in Remark 5.2.4, Remark 5.2.17, and Lemma 5.4.17 show that σ_{Tate} belongs to the commutative triangle

$$\begin{tikzcd}
MU[6,\infty) \arrow[r, "\delta"] \arrow[rrrd, "\sigma_{\mathrm{Tate}}"'] & MSU \arrow[r, "\delta"] & MU \arrow[r, "\delta"] & MUP \arrow[d, "\tilde{\theta}"] \\
 & & & KU[\![q]\!].
\end{tikzcd}$$

We will analyze this triangle by comparing $\tilde{\theta}$ to the usual MUP–orientation of KU, which selects the coordinate $f(u) = 1 - u$ on the formal completion of $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[u^{\pm}]$. Appealing to Remark 5.2.4, the induced MU–orientation

$$MU \xrightarrow{\delta} MUP \xrightarrow{\mathrm{Td}} KU$$

sends f to the rigid section δf of

$$\Theta^1\mathcal{I}(0) = \mathcal{I}(0)_0 \otimes \mathcal{I}(0)^{-1} \cong \omega \otimes \mathcal{I}(0)$$

given by

$$\delta f = \frac{1}{1-u}\left(-\frac{du}{u}\right).$$

The difference between $\delta\,\mathrm{Td}$ and $\delta\tilde{\theta}$ is expressed by an element $\psi \in C^1(\widehat{C}_{\mathrm{Tate}}; \mathbb{G}_m)$, given explicitly by the quotient formula

$$\psi = \left(\frac{\mathrm{Td}(1)}{\mathrm{Td}(u)}\right)^{-1} \cdot \frac{\tilde{\theta}_q(1)}{\tilde{\theta}_q(u)} = \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}.$$

This gives a re-expression of $\delta\tilde{\theta}$ as the composite

$$\delta\tilde{\theta}\colon MU \xrightarrow{\eta_R} MU \wedge MU \simeq MU \wedge BU_+ \xrightarrow{\delta \operatorname{Td} \wedge \psi} K_{\text{Tate}},$$

and hence its effect on a line bundle is determined by the evaluation of this characteristic series:

$$\psi(1-\mathcal{L}) = \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n\mathcal{L})(1-q^n\mathcal{L}^{-1})}.$$

Its effect on vector bundles in general is determined by the splitting principle and an exponential law, which after some computation [AHS01, Section 2.7] gives the generic formula

$$\psi(\dim V \cdot 1 - V) = \bigotimes_{n\geq 1} \bigoplus_{j\geq 0} \operatorname{Sym}^j(\dim V \cdot 1 - V \otimes_{\mathbb{R}} \mathbb{C}) q^{jn} =: \bigotimes_{n\geq 1} \operatorname{Sym}_{q^n}(-\bar{V}_{\mathbb{C}}).$$

Finally, the map $(\eta_R)_*\colon\ MU_* \to \pi_*(MU \wedge \Sigma^\infty_+ BU)$ sends a manifold M with stable normal bundle ν to the pair (M, ν), so we at last compute

$$\begin{aligned}\sigma_{\text{Tate}}(M \in \pi_{2n} MU[6,\infty)) &= (\delta \operatorname{Td} \wedge \theta')(M, \nu) \\ &=: \operatorname{Td}\left(M; \bigotimes_{n\geq 1} \operatorname{Sym}_{q^n}(\bar{\tau}_{\mathbb{C}})\right).\end{aligned}$$

This is exactly Witten's formula for his genus, as applied to complex manifolds with first two Chern classes trivialized.

Remark 5.5.28 ([Reza, Section 1.5]). Witten defines his characteristic series for *oriented* manifolds by the formula

$$\begin{aligned}K_{\text{Witten}}(x) &= \exp\left(\sum_{k=2}^{\infty} 2G_{2k}(\tau) \cdot \frac{x^{2k}}{(2k)!}\right) \\ &= \frac{x/2}{\sinh(x/2)} \cdot \left(\prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}\right) \cdot e^{-G_2(\tau)x^2},\end{aligned}$$

where G_{2k} is the $2k^{\text{th}}$ Eisenstein series. Noting that G_2 is *not* a modular form, the condition that $p_1(M)/2$ vanish is precisely the condition that G_2 contribute nothing to the sum, so that the remainder *is* a modular form.

Remark 5.5.29 ([AM01]). Another location where this series ψ appears is in the theory of Tate p–divisible groups of Ando and Morava (and, in some sense, in the sense of Katz and Mazur). For a formal group $\widehat{\mathbb{G}}$ over $\operatorname{Spf} R$ with group law $+_\varphi$, they consider the Weierstrass product

$$\Theta_\varphi(x;q) = x \cdot \prod_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \frac{(x +_\varphi [k]_\varphi(q))}{[k]_\varphi(q)} \in R[\![x,q]\!],$$

which plays the role of a kind of θ–function for $\widehat{\mathbb{G}}$. They also connect this series the quotient function mapping to the group scheme $\widehat{\mathbb{G}}/q^{\mathbb{Z}}$ (in the sense of Definition A.2.22), which they further identify as the universal extension of $\widehat{\mathbb{G}}$ by a constant p–divisible group of dimension 1. In the presense of a global object $\mathbb{G}$ specializing to $\widehat{\mathbb{G}}$ (as in the case of $\widehat{\mathbb{G}}_a$ or $\widehat{\mathbb{G}}_m$), they show how to modify this product to an analytically convergent power series, recovering the characteristic series for the $\widehat{A}$–genus and the L–genus. The higher height analogues of these results seem very mysterious and very interesting.[14]

Remark 5.5.30 ([AFG08]). Ando, French, and Ganter have given a construction that convert $MU[2k,\infty)$–orientations of a spectrum E to $MU[2(k-1),\infty)$–orientations of the pro-spectrum $E^{\mathbb{C}\mathrm{P}^\infty_{-\infty}}$. Performing this operation to the σ–orientation of K^{Tate} gives the "two–variable Jacobi genus", which assigns SU–manifolds to certain classes in $\pi_0(K^{\mathrm{Tate}})^{\mathbb{C}\mathrm{P}^\infty_{-\infty}} = \mathbb{Z}[\![q]\!]((y))$ connected to meromorphic Jacobi forms.

5.6 Chromatic *Spin* and *String* orientations

We now turn to understanding *MString*–orientations in turns of $MU[6,\infty)$–orientations. We will approach this in successive passes, keeping the desired picture sharp the entire time but introducing generality slowly. Unfortunately, most of the original source material [HAS99, Stra] for this has not been published, and hence it is difficult to give references.[15] We have gone to some lengths to prepare the reader for the sorts of arguments that appear here: they are not so different from the arguments appearing elsewhere in this chapter.

We begin at the maximally simple situation, where 2 is inverted. In this case,

[14] Morava is generally full of interesting ideas about genera—see, for instance, [Mor07b].

[15] Versions of some of this have appeared in work of Kitchloo and Laures [KL02].

the Wood cofiber sequence

$$\Sigma kO \xrightarrow{\eta} kO \xrightarrow{c} kU \xrightarrow{\lambda} \Sigma^2 kO$$

becomes split, since η is a 2–torsion element. By considering different underlying infinite-loopspaces, this gives a number of identifications:

$$\begin{aligned}
\underline{(-)_0}: &\qquad BO \times \mathbb{Z} \longrightarrow BU \times \mathbb{Z} \longrightarrow \underline{kO_2}, \\
\underline{(-)_2}: &\qquad \underline{kO_2} \longrightarrow BU \longrightarrow \underline{kO_4}, \\
\underline{(-)_4}: &\qquad \underline{kO_4} \longrightarrow BSU \longrightarrow \underline{kO_6}, \\
\underline{(-)_6}: &\qquad \underline{kO_6} \longrightarrow BU[6,\infty) \longrightarrow BString.
\end{aligned}$$

Next, by recognizing $c\colon kO \to kU$ as the complexification map, we note that it lies in fixed points for complex-conjugation on kU. The other gain that inverting 2 gets us is an idempotent selecting this fixed point subspectrum: there are a pair of orthogonal idempotents

$$P_+ = \frac{1+\xi}{2}, \qquad\qquad P_- = \frac{1-\xi}{2},$$

which reidentify the splitting $kU \simeq kO \vee \Sigma^2 kO$ in various equivalent ways:

$$kU \simeq kO \vee \Sigma^2 kO \simeq \operatorname{im} P_- \vee \operatorname{im} P_+ \simeq \ker P_+ \vee \ker P_- \simeq \left(\frac{kU}{\operatorname{im} P_+}\right) \vee \left(\frac{kU}{\operatorname{im} P_-}\right).$$

This last identification immediately gives access to the following result:

Lemma 5.6.1. *For E a complex-orientable ring spectrum with $1/2 \in \pi_0 E$, we have*

$$\begin{aligned}
BString_E &= C_3(\mathbb{CP}^\infty_E)/([a,b,c] = -[-a,-b,-c]), \\
\operatorname{Spec} E_0 BString &= \left\{ f \in C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m) \,\middle|\, f(a,b,c) = \frac{1}{f(-a,-b,-c)} \right\} \subseteq C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m).
\end{aligned}$$

Proof sketch. This is a matter of translating the splittings above across Corollary 5.4.16. The complex-conjugation map ξ acts on $C_k(\widehat{\mathbb{G}})$ according to

$$\xi[a_1, \ldots, a_n] = (-1)^n [-a_1, \ldots, -a_n],$$

which encodes $BString_E$ as the quotient of $BU[6,\infty)_E$ by $\operatorname{im} P_-$. Finally, the claim about the homological scheme follows from the description of the cohomological formal scheme by Cartier duality. $\square$

The repeated appearances of the terms in the above splittings suggest that the composite

$$\tau\colon\ kU \xrightarrow{\lambda} \Sigma^2 kO \xrightarrow{\Sigma^2 c} \Sigma^2 kU$$

of the maps in two adjacent cofiber sequences itself plays an interesting role. This is the map that encodes surjecting onto one factor in the preceding splitting, then reincluding it into the next splitting.

Lemma 5.6.2. *At the level of formal schemes, the map τ acts by*

$$\begin{aligned} \tau\colon\ C_k(\widehat{\mathbb{G}}) &\to C_{k+1}(\widehat{\mathbb{G}}) \\ [a_1, \ldots, a_k] &\mapsto [a_1, \ldots, a_k, -(a_1 + \cdots + a_k)]. \end{aligned}$$

Proof. We are in pursuit of the following calculation in $kU^*(\mathbb{C}\mathrm{P}^\infty)^{\times(k+1)}$ encoding complexification after decomplexification:

$$\begin{aligned} \beta^k x_1 \cdots x_k + \beta^k \overline{x_1} \cdots \overline{x_k} &= (1 - \mathcal{L}_1) \cdots (1 - \mathcal{L}_k) + (1 - \overline{\mathcal{L}}_1) \cdots (1 - \overline{\mathcal{L}}_k) \\ &= (1 - \mathcal{L}_1) \cdots (1 - \mathcal{L}_k) + \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_k (\mathcal{L}_1 - 1) \cdots (\mathcal{L}_k - 1) \\ &= (1 - \mathcal{L}_1) \cdots (1 - \mathcal{L}_k)(1 - (-1)^{k+1} \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_k) \\ &= \beta^{k+1} x_1 \cdots x_k \cdot \xi^* \mu^*(x_1, \ldots, x_k). \end{aligned}$$ □

Still in the case where E is local away from 2, we can use this alternative description of the inclusion of the P_- factor to deduce an alternative description of the formal scheme associated to *BString*:

Corollary 5.6.3. *For E a complex-orientable ring spectrum with $1/2 \in \pi_0 E$, we have*

$$\begin{aligned} BString_E &= C_3(\mathbb{C}\mathrm{P}^\infty_E)/([a, b, -a-b] = 0), \\ \operatorname{Spec} E_0 BString &= \left\{ f \in C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m) \,\middle|\, f(a, b, -a-b) = 1 \right\} \subseteq C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m). \end{aligned}$$ □

Definition 5.6.4. We denote this last functor by $\Sigma^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$, and more generally if $\mathcal{L}$ is a line bundle on $\mathbb{C}\mathrm{P}^\infty_E$ then $\Sigma^3(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{L})$ will denote the subscheme of $C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{L})$ of those trivializations which restrict to the rigidifying trivialization of $\Theta^3\mathcal{L}$ on the subscheme of $(\mathbb{C}\mathrm{P}^\infty)^{\times 3}$ specified by $[a, b, -a-b]$. Such trivializations are referred to as Σ–*structures on* $\mathcal{L}$, following Breen [Bre83, Section 5].

Corollary 5.6.5. *For E a complex-orientable ring spectrum with $1/2 \in \pi_0 E$, the functor $\Sigma^3(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0))$ is isomorphic to the affine scheme* $\operatorname{Spec} E_0 MString$. *Moreover, the E_0–points of this scheme biject with ring spectrum maps $MString \to E$.* □

Remark 5.6.6 ([Hir95, Section 26.1]). One of the most pleasant features of the case where 2 is inverted is that real orientations can not only be identified, but the projection idempotents mean that they can be *crafted* from complex orientations, and the idempotents have a recognizable effect on the traditional mechanisms for specifying orientations. For instance, the Conner–Floyd orientation on K–theory is specified by the classical logarithm, and the associated real orientation formed by projection to the positive idempotent has logarithm

$$\tanh^{-1}(x) = \frac{-\ln(1-x) + \ln(1+x)}{2},$$

i.e., the average of the logarithm and its conjugate. The resulting orientation is commonly referred to as the *Atiyah–Bott–Shapiro orientation*, whose associated genus is the $\widehat{A}$*–genus*. Similar techniques give access to connective real orientations associated to preexisting connective complex orientations.

We now turn to the *much* more complicated setting where 2 is not invertible, where we again aim to identify *MString*–orientations of a complex-orientable cohomology theory with certain Σ–structures. This is not possible in much generality, a situation which is hinted by the analogous lower-order case: complex-orientability is flatly *not enough* to conclude anything about whether a cohomology theory admits an *MO*–orientation, which we saw in Lemma 1.5.8 to be actually equivalent to admitting an $H\mathbb{F}_2$–algebra structure. In order to get a handle on the task in front of us, consider the following diagram of fiber sequences of infinite loopspaces:

$$\begin{tikzcd}
& Spin/SU \arrow[rr] \arrow[dl, equal] \arrow[dd, equal] & & BU[6,\infty) \arrow[rr] \arrow[dl, equal] \arrow[dd] & & BString \arrow[dl, equal] \arrow[dd] \\
\underline{kO[8,\infty)}_{-2} \arrow[rr] \arrow[dd, equal] & & \underline{kU[8,\infty)}_{-2} \arrow[rr] \arrow[dd] & & \underline{(\Sigma^2 kO)[8,\infty)}_{-2} \arrow[dd] & \\
& Spin/SU \arrow[rr] \arrow[dl, equal] & & BSU \arrow[rr] \arrow[dl, equal] & & BSpin \arrow[dl, equal] \\
\underline{kO[6,\infty)}_{-2} \arrow[rr] & & \underline{kU[6,\infty)}_{-2} \arrow[rr] & & \underline{(\Sigma^2 kO)[6,\infty)}_{-2}. &
\end{tikzcd}$$

Our program is to analyze the chromatic homology of $Spin/SU$ as well as the maps to $BU[6,\infty)$ and BSU. We hope to show that the scheme associated to $Spin/SU$ selects exactly the relations defining Σ–structures *and* that the map is flat, so that the associated bar spectral sequences

$$\operatorname{Tor}_{*,*}^{E_*Spin/SU}(E_*, E_*BSU) \Rightarrow E_*BSpin, \quad \operatorname{Tor}_{*,*}^{E_*Spin/SU}(E_*, E_*BU[6,\infty)) \Rightarrow E_*BString$$

collapse to give short exact sequences of Hopf algebras.

We embark on this project with an analysis of the natural bundles classified by the topological objects, so that we can guess the relevant algebraic model. We have the following complexification and decomplexification maps:

$$\begin{tikzcd} & & \underline{kU}_6 \arrow[d, "\delta"] \\ \underline{kU}_4 \arrow[r, "j"] & \underline{kO}_6 \arrow[r, "i"] \arrow[ur, "\lambda"] & \underline{kU}_4. \end{tikzcd}$$

Lemma 5.6.7. *Let E be a complex-orientable cohomology theory. The map j induces an injection*

$$\operatorname{Spec} E_0(Spin/SU) \to \Sigma^2(\mathbb{CP}^\infty_E; \mathbb{G}_m).$$

Proof, with some details omitted. Recall that our reinterpretation of Corollary 5.2.14 in Example 5.3.3 passed through the adjoint map $\widehat{\Pi}_2$ of the natural product bundle $\Pi_2\colon (\mathbb{CP}^\infty)^{\times 2} \to BSU$. We extend the map Π_2 in two directions:

$$\mathbb{CP}^\infty \xrightarrow{(1-\mathcal{L})(1-\overline{\mathcal{L}})} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\Pi_2} BSU \xrightarrow{j} Spin/SU.$$

Checking that this composite is zero on E–homology will give a factorization

$$\begin{tikzcd} \operatorname{Spec} E_0 Spin/SU \arrow[r, "E_0 j"] \arrow[d, dotted] & \operatorname{Spec} E_0 BSU \arrow[d, equal] \\ \Sigma^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r] & C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m), \end{tikzcd}$$

since $\Sigma^2(\mathbb{CP}^\infty_E; \mathbb{G}_m)$ is defined to be the closed subscheme of $C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m)$ of those functions satisfying $f(x, -x) = 1$.

The ordinary homology of $Spin/SU$ is free and even, so it suffices to check that this map is null in the case $E = H\mathbb{Z}_{(2)}$, since one can then conclude the general case by the Atiyah–Hirzebruch spectral sequence. Manual calculation in a Serre spectral sequence shows gives the calculation $H\mathbb{Z}_{(2)*}(Spin/SU) \cong \Gamma_{\mathbb{Z}_{(2)}}[a_{2n+1} \mid n \geq 1]$ as well as that the homological maps induced by i and j are respectively injective and surjective. In particular, we need only check that the above composite is zero on

$H\mathbb{Z}_{(2)}$–homology after postcomposition with $H_*(i)$. The SU–bundle classified by postcomposition with i is

$$(1-\mathcal{L})(1-\overline{\mathcal{L}}) - (1-\overline{\mathcal{L}})(1-\mathcal{L}),$$

which is itself trivial, hence the classifying map is null and so must be the map on homology.

We are left with showing that the factorized map is an isomorphism, and again appealing to Atiyah–Hirzebruch spectral sequences it suffices to show this in the case $E = H\mathbb{F}_2$. The diagram considered above extends as follows:

$$\begin{tikzcd} \operatorname{Spec} H\mathbb{F}_2 P_0 BSU \arrow[r, "H\mathbb{F}_2 P_0 i"] \arrow[d, equal] & \operatorname{Spec} H\mathbb{F}_2 P_0 Spin/SU \arrow[r, "H\mathbb{F}_2 P_0 j"] \arrow[d, dotted] & \operatorname{Spec} H\mathbb{F}_2 P_0 BSU \arrow[d, equal] \\ C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "\lambda_2"] & \Sigma^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r] & C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m), \end{tikzcd}$$

where

$$\lambda_2(f)\colon\ (x,y) \mapsto \frac{f(x,y)}{f(-_{\widehat{\mathbb{G}}}x, -_{\widehat{\mathbb{G}}}y)}.$$

Since the bottom-right map is definitionally injective and the outer rectangle commutes, it follows that the left-hand square commutes. The Serre spectral sequence calculation in integral homology indicated above shows that the top-right map is an injection, so the dotted comparison map is automatically an injection. □

Lemma 5.6.8. *The above comparison map also belongs to a commuting diagram*

$$\begin{tikzcd} \operatorname{Spec} E_0 BU[6,\infty) \arrow[r] \arrow[d, equal] & \operatorname{Spec} E_0 Spin/SU \arrow[r] \arrow[d] & \operatorname{Spec} E_0 BSU \arrow[d] \\ C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\lambda_3"] & \Sigma^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r] & C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m), \end{tikzcd}$$

where

$$\lambda_3(f)\colon\ (x,y) \mapsto f(x,y,-_{\widehat{\mathbb{G}}}(x +_{\widehat{\mathbb{G}}} y)).$$

Proof. This is again a matter of checking that the outer rectangle commutes. □

This is as much as we can discern without delving into the analysis of the algebraic model. Our primary concern in that respect is to show that the algebraic map has the desired flatness property—we will actually quickly show that our comparison map is an isomorphism in our pursuit of this. Our main tool for addressing flatness is the following theorem, strongly related to the Milnor–Moore classification of Hopf algebras over a field of positive characteristic:

Lemma 5.6.9. *Let k be a field, and let G and H be group schemes over $\operatorname{Spec} k$. A map $f\colon G \to H$ of groups is faithfully flat if and only if for every test k–algebra T and T–point $a \in H(T)$ there is a faithfully flat T–algebra S and an S–point $b \in G(S)$ covering a.* □

Cite me: Gerd may have an article with a version of this forthcoming, and HAS cites [?, III, Section 3, n. 7], cf. also FPFP Example 12.10.

Based on this Lemma, we see that if we had a sufficiently strong understanding of the possible points of $\Sigma^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$, we could manually check this condition by constructing preimages for these points. This becomes a manageable task after we note the following reductions:

1. Because of the equation $\lambda_2 = \delta \circ \lambda_3$ and because δ is surjective, checking that λ_2 is (faithfully) flat will automatically entail that λ_3 is (faithfully) flat.

2. We do not actually need *faithful* flatness to control the bar spectral sequence, but merely flatness. Accordingly, note that a map $f\colon G \to H$ of commutative affine group schemes is flat if and only if the map $f^\circ\colon G^\circ \to H^\circ$ on connected components of the identity is flat. Hence, we can reduce to checking (faithful) flatness on the identity component if necessary. In fact, because $E_0 BSU$ is polynomial, we have $(\operatorname{Spec} E_0 BSU)^\circ = \operatorname{Spec} E_0 BSU$.

3. This last condition can be reduced further: a map of affine algebraic group schemes is flat if and only if the induced map on tangent spaces is surjective. Although $E_0 BSU$ is *infinite* polynomial, and hence not algebraic, this technicality can be overcome by means of the grading and the same condition applies.

We thus set about studying the points of $\Sigma^2(\widehat{\mathbb{G}}; \mathbb{G}_m)$ up to first order. As in Lecture 5.4, we begin with the special case $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$ and understand the generic case in terms of perturbation.

Lemma 5.6.10. *The map*

$$\operatorname{Spec} \Gamma_{\mathbb{Z}_{(2)}}[a_{2n+1} \mid n \geq 1] \to \Sigma^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(2)}$$

classifying the product

$$\prod_{n \geq 1} \exp(a_{2n+1} c_{2n+1})$$

is an isomorphism. In turn, there is an isomorphism

$$\Sigma^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{F}_2 \cong \operatorname{Spec} \mathbb{F}_2 \left[a_{2n+1}^{[2^j]} \middle| n \geq 1, j \geq 0 \right] \Big/ \left(\left(a_{2n+1}^{[2^j]} \right)^2 = 0 \right).$$

Proof sketch. This is a combination of three observations:

1. For $\widehat{\mathbb{G}}$ arbitrary and $f \in \Sigma^2(\widehat{\mathbb{G}}; \mathbb{G}_m)$ which expands in a coordinate to $1 + bc_{2^m} + \cdots$, then $b = 0$ as an element of $B[1/2]$ and of $B/2$.
2. For any $f \in \Sigma^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ with expansion $1 + bc_n + \cdots$, $b^2 \equiv 0 \pmod 2$.
3. Modulo 2, the putative universal product series above decomposes further as

$$\prod_{n \geq 1} \left(\prod_{j \geq 0} (1 + a_{2n+1}^{[2^j]} \cdot c_{(2n+1)2^j}) \right). \qquad \square$$

Corollary 5.6.11. *The injective comparison map*

$$\operatorname{Spec} E_0 Spin/SU \to \Sigma^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$$

of Lemma 5.6.7 is an isomorphism.

Proof. The same Serre spectral sequence calculation as in the proof of Lemma 5.6.7 shows that the source and target of the factorization have the same Poincaré series, and we are done. $\square$

From here, the work gets considerably more technical. In particular, we will only be able to conclude flatness in the case $\operatorname{ht} \widehat{\mathbb{G}} \leq 2$, and then only through explicit calculation. The results cleave into the following three pieces:

Lemma 5.6.12 (Nonexistence lemma)**.** *Let $\widehat{\mathbb{G}}$ be a formal group of height 1 or 2 over an $\mathbb{F}_2$–algebra. For $f \in \Sigma^2(\widehat{\mathbb{G}}; \mathbb{G}_m)$ of the form $1 + ac_n + \cdots$, ...*

1. *... if $n = 2^m$ then $a = 0$.*
2. *... if $n > 3$, n is odd, and $n \neq 2^m - (2^d - 1)$, then $a = 0$.*
3. *... if $d = 2$ and $n = 3$, then $a^2 - a = 0$.*

Proof. We address the claims in turn.

1. This is a generic claim that does not rely on the height of $\widehat{\mathbb{G}}$. The 2–cocycle c_{2^m} takes the form $c_{2^m}(x, y) = x^{2^{m-1}} y^{2^{m-1}}$, so that $f(x, -_{\widehat{\mathbb{G}}} x) = 1 + a x^{2^m} + \cdots$. The expected value is $f(x, -_{\widehat{\mathbb{G}}} x) = 1$, hence $a = 0$.[16]

2. This is again an explicit calculation of the leading term. Since $\widehat{\mathbb{G}}$ is of height d, it admits a coordinate where the negation series takes the form
$$[-1]_{\widehat{\mathbb{G}}}(x) = -x + cx^{2^d} + \cdots$$
for some c not a zero divisor. In the case that n is as described, one can then calculate
$$c_n(-_{\widehat{\mathbb{G}}} x, -_{\widehat{\mathbb{G}}} y) + c_n(x, y) = c_{n+(2^d-1)}(x, y) + \cdots,$$
from which we make a trio of calculations:
$$\begin{aligned} \lambda_2 f &= f^2, && \text{(uses } f \in \Sigma^2(\widehat{\mathbb{G}}; \mathbb{G}_m)) \\ \lambda_2 f &= 1 + a c_{n+(2^d-1)}(x, y) + \cdots, && \text{(above observation)} \\ f^2 &= 1 + a^2 c_{2n}(x, y) + \cdots. && \text{(characteristic 2)} \end{aligned}$$
For $n \geq 2^d$, this forces $a = 0$.

3. In the case $d = 2$ of the previous statement, this leaves one case open: $n = 3$. Equating the two sides then gives $a^2 = a$. □

Lemma 5.6.13 (Generic existence lemma). *Let $f = 1 + ac_n + \cdots \in C^2(\widehat{\mathbb{G}}; \mathbb{G}_m)$. If $2n + 1 \geq 3$ is not of the form $2^r - (2^d - 1)$ for any r, then $\lambda_2 f = 1 + ac_{n+2^d-1} + \cdots$.*

Proof sketch. This is also a consequence of the same manipulation of the negation series. □

Lemma 5.6.14 (Special existence lemma). *There exists a function $f \in C^2(\widehat{\mathbb{G}}; \mathbb{G}_m)$ with $\lambda_2 f = 1 + (a^2 + \varepsilon a + \delta) c_{2^r+(2^d-1)}$.*

Indication of proof. This, finally, makes explicit use of height 1 and 2 formal group laws. The starting point is to set $g = \delta_{(\widehat{\mathbb{G}}_a, \mathbb{G}_m)} E_2(ax^{2^{r-1}+(2^d-1)})$ and then to carefully cancel low-order terms by multiplying in other known Σ^2–structures. □

[16] One can show by almost identical calculation that $a = 0$ in this case if 2 is invertible.

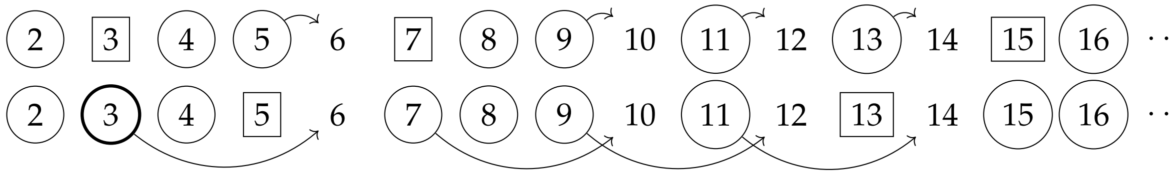


Figure 5.4: Application of the three Lemmas in a small range. Circles denote prohibitions of Σ^2–structures by Lemma 5.6.12, arrows denote Σ^2–structures constructed by Lemma 5.6.13, and squares denote exceptional Σ^2–structures constructed by Lemma 5.6.14.

Corollary 5.6.15. *For $\widehat{\mathbb{G}}$ a formal group over $\operatorname{Spec} \mathbb{F}_2$ with $\operatorname{ht} \widehat{\mathbb{G}} \leq 2$, the maps λ_2 and λ_3 are flat.*

Proof. This is a culmination of the above calculations. As argued at the beginning of our algebraic analysis, it suffices to show just that λ_2° is faithfully flat to make the cocnclusion. Flatness follows by showing that the map is surjective on tangent spaces, which is exactly the content of Lemma 5.6.12, Lemma 5.6.13, and Lemma 5.6.14. $\square$

Theorem 5.6.16. *Let $\widehat{\mathbb{G}}$ be a formal group over a perfect field of characteristic 2, and assume $\operatorname{ht} \widehat{\mathbb{G}} \leq 2$. For E the associated Morava K–theory or E–theory, there are isomorphisms*

$$\operatorname{Spec} E_0 MString \xrightarrow{\cong} \Sigma^3(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0)), \qquad \operatorname{Spec} E_0 MSpin \xrightarrow{\cong} C^2_{\mathrm{is}}(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0)),$$

where $C^2_{\mathrm{is}}(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0))$ is the subscheme of $C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0))$ consisting of those inverse-symmetric functions satisfying $f(x,y)/f(-_{\widehat{\mathbb{G}}}x, -_{\widehat{\mathbb{G}}}y) = 1$. The E_0–points of these schemes biject with ring spectrum maps $MString \to E$ and $MSpin \to E$ respectively.[17] $\square$

Proof. Our goal all along was to use the algebraic model to govern the bar spectral sequences

$$\operatorname{Tor}_{*,*}^{E_*Spin/SU}(E_*, E_*BSU) \Rightarrow E_*BSpin, \quad \operatorname{Tor}_{*,*}^{E_*Spin/SU}(E_*, E_*BU[6,\infty)) \Rightarrow E_*BString.$$

Our conclusion from Corollary 5.6.15 is that they are both concentrated on the 0–line and isomorphic to the respective Hopf algebra quotients

$$E_0BSU/\!/(E_0Spin/SU) \cong E_0BSpin, \qquad E_0BU[6,\infty)/\!/(E_0Spin/SU) \cong E_0BString.$$

[17] This is sufficient to conclude the existence of the Atiyah–Bott–Shapiro orientation in the homotopy category.

Our algebraic models furthermore explicitly identify these quotients on homological schemes as those subschemes on which the stated algebraic identities hold. The final claim of the theorem follows from even-ness and the universal coefficient spectral sequence. □

Corollary 5.6.17. *If E is a finite height Morava K– or E–theory considered as an elliptic spectrum, the complex σ–orientation of Corollary 5.5.16 lifts uniquely to a ring map $MString \to E$.*

Proof. An *immediate* consequence of the canonical cubical structure associated to an abelian variety is that it automatically satisfies the extra condition required to be a Σ^3–structure. □

Remark 5.6.18 ([Hop95, Theorem 7.2]). More generally, the σ–orientation associated to a elliptic spectrum which either has $1/2 \in \pi_0 E$ or which has $\pi_* E$ torsion–free is supposed to lift through *MString*. Flatness in this setting is supposed to be approached via a fiber-by-fiber criterion, but here the grading tricked used above are less visibly helpful in lifting the classical algebraic result to the periodic one we require to make this work. Rather than pursue this, we will give a sketch of the construction of the *String*–orientation of *tmf* in Appendix A.4, which automatically gives this much stronger result.

Remark 5.6.19 ([KLW04, Stra]). A the cohomology formal schemes of a number of other infinite loopspaces related to real K–theory admit reasonable descriptions, often even independent of height. A routinely useful result in this arena is due to Yagita [Yag80, Lemma 2.1]: for k_Γ the connective cover of the Morava K–theory K_Γ, in the Atiyah–Hirzebruch spectral sequence

$$E_2^{*,*} = Hk^* X \otimes_k k_\Gamma^* \Rightarrow k_\Gamma^* X,$$

the differentials are given by

$$d_r(x) = \begin{cases} 0 & \text{if } r \le 2(p^d - 1), \\ \lambda Q_d x \otimes v_d & \text{if } r = 2(p^d - 1) + 1 \end{cases}$$

where $\lambda \neq 0$ and Q_d is the d^{th} Milnor primitive. For instance, this shows that $K_* BO$ decomposes as

$$K_* BO \cong K_*[b_2, b_4, b_{2^{d+1}-2}] \underset{K_*[b_{2j}^2 | j < 2^d]}{\otimes} K_*[b_{2j}^2].$$

This, coupled to a theorem governing the result of the double bar spectral sequence, powers most of the results of Kitchloo, Laures, and Wilson [KLW04, Section 4]. Their stronger results on the connective covers of $BO \times \mathbb{Z}$ are summarized in Figure 5.5. The remaining formal scheme $BString_K$, our prized object, is harder to access by these means: the sequence $(\underline{HS^1}_2)_K \to BString_K \to BSpin_K$ is exact in the middle, but neither left- nor right-exact [KLW04, pg. 234], causing significant headache. Satisfyingly, their methods also tell us that our analysis fails at higher heights: the formal scheme $Spin/SU_K$ contains $\coprod_{j \geq 3}(\underline{HS^1[2]}_j)_K$ as a subscheme, and it accounts for the kernel of the map to $BSpin_K$. Once $\operatorname{ht}\Gamma \geq 3$ is satisfied, this kernel is nonzero.

Remark 5.6.20. There are variations on this analysis that remain to be sorted out. For instance, there is a naturally occurring orientation $MU \to MSO$, whose associated genus factors through the quotient of the Lazard ring by the relation $[-1](x) = -x$. This factored map is injective and it becomes an isomorphism after inverting 2—but, without inverting 2, MSO_* itself is populated by plenty of 2–torsion. It would be nice to have available an interpretation of such real orientations in terms of formal group laws.

Mention all this work of Gerd. He has a bunch of papers that are all clustered around these topics.

$$
\begin{tikzcd}
& \mathrm{Div}_0\,\overline{\Gamma}[2] \arrow[rr] \arrow[dl] \arrow[dd, equal] & & \mathrm{Div}_0\,\overline{\Gamma} \arrow[dl] \arrow[dd, equal] & \\
\ker\omega \arrow[rr] \arrow[dd] & & BSpin_K \arrow[dd] & & \\
& \mathrm{Div}_0\,\overline{\Gamma}[2] \arrow[rr] \arrow[dl] \arrow[dd, equal] & & \mathrm{Div}_0\,\overline{\Gamma} \arrow[dl] \arrow[dd, equal] & \\
\mathrm{SDiv}_0\,\Gamma[2] \arrow[rr] \arrow[dd] & & BSO_K \arrow[rr, "\omega"] \arrow[dd] & & \Gamma[2]^{\wedge 2} \\
& \mathrm{Div}_0\,\overline{\Gamma}[2] \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Div}_0\,\overline{\Gamma} \arrow[dl] \arrow[dd] & \\
\mathrm{Div}_0\,\Gamma[2] \arrow[rr] \arrow[dd] & & BO_K \arrow[rr, "\sigma"] \arrow[dd] & & \Gamma[2] \\
& \mathrm{Div}\,\overline{\Gamma}[2] \arrow[rr] \arrow[dl] & & \mathrm{Div}\,\overline{\Gamma} \arrow[dl] & \\
\mathrm{Div}\,\Gamma[2] \arrow[rr] & & (BO \times \mathbb{Z})_K \arrow[rr] & & \underline{\mathbb{Z}}.
\end{tikzcd}
$$

Figure 5.5: Presentations of the Morava K–theoretic cohomological formal schemes associated to connective covers of $BO \times \mathbb{Z}$. Each level in the prism is a bi-Cartesian square in the category of formal group schemes, and each level is the fiber of the maps off of the previous level to the Postnikov section. The formal curve $\overline{\Gamma}$ is $\mathbb{HP}^\infty_K$, which is the fiber of $\sigma\colon\ \mathrm{Div}_2^+\,\Gamma \to \Gamma$, and $\overline{\Gamma}[2]$ is the fiber of the lift of the endo-isogeny $2\colon\ \Gamma \to \Gamma$ to $\overline{\Gamma}$. The map σ vanishes on $\mathrm{Div}_0\,\overline{\Gamma}$ and acts by summation on $\mathrm{Div}_0\,\Gamma[2]$. The map ω vanishes on $\mathrm{Div}_0\,\overline{\Gamma}$ and acts on $\mathrm{SDiv}_0\,\Gamma[2] \cong C_2\Gamma[2]$ by $[a,b] \mapsto a \wedge b$.

Appendix A

Power operations

Our goal in this Appendix is to give a tour of the interaction of the σ–orientation with a topic of modern research, the theory of E_∞ ring spectra, in a manner consistent with the rest of the topics in this book. Because the theory of E_∞ ring spectra (in particular: their algebraic geometry) is still very much developing, we have no hope of stating results in their maximum strength or giving a completely clear picture—as of this writing, the maximum strength is unknown and the picture is still resolving. Although E_∞ ring spectra themselves were introduced decades ago, we will even avoid giving a proper definition of them here, instead referring to the original work of May and collaborators [EKMM97] and the more recent work of Lurie [Lurb, Chapter 7] for a proper treatment. In acknowledgement of this underwhelming level of rigor, we have downgraded our discussion from a Case Study to an Appendix.

As far as we are concerned, E_∞ ring spectra arise in order to solve the following problem: given two ring spectra R and S in the homotopy category, the set of homotopy classes of ring maps $\mathsf{RingSpectra}(R, S)$ forms a subset of the set of all homotopy classes $[R, S] = \pi_0\mathsf{Spectra}(R, S)$, selected by a homomorphism condition. There is no meaningful way to enrich this to a *space* of ring spectrum maps from R to S, which inhibits us from understanding an obstruction theory for ring spectra, i.e., approximating R by "nearby" ring spectra R' in a way that relates $\mathsf{RingSpectra}(R, S)$ and $\mathsf{RingSpectra}(R', S)$ by a fiber sequence.

The extra data that accomplishes this mapping space feat turns out to be an explicit naming of the homotopies controlling the associativity and commutativity of the ring spectrum multiplication, which are subject to highly intricate compatibility conditions.[1,2] Again, rather than spell this out, it suffices for our purposes to say that there is such a notion of a structured ring spectrum that begets a mapping space between two such. Additionally, we record the following omnibus theorem as indication that this program overlaps with the one we have been describing already:

Theorem A.0.1. *The following are examples of E_∞ ring spectra:*

- *([May77, Section VIII.1]) The classical K–theories KU and KO.*
- *([May77, Section VIII.1]) The Eilenberg–Mac Lane spectra HR.*
- *([GH04, Corollary 7.6–7]) The Morava E–theories E_Γ and their fixed point spectra.*[3]
- *([May77, Section IV.3]) The Thom spectra arising from the J–homomorphism, including MO, MSO, MSpin, MString, MU, MSU, and $MU[6, \infty)$.*
- Cite me: Behrens. *(, cf. Appendix A.3) The spectra TMF, Tmf, and tmf.* □

The forgetful map from E_∞ rings down to ring spectra in the homotopy category factors through an intermediate category, that of H_∞ ring spectra, which captures the extra factorizations expressing these associativity and commutativity relations. Specifically, recall the following definition from the discussion in Lecture 2.4:

Definition A.0.2 ([BMMS86, Definition I.3.1], cf. Lecture 2.4)**.** An H_∞ ring spectrum is a ring spectrum E equipped with factorizations μ_n as in

$$\begin{tikzcd} E^{\wedge n} \arrow[r, "\mu"] \arrow[d] & E \\ E^{\wedge n}_{h\Sigma_n}, \arrow[ur, dashed, "\mu_n"'] & \end{tikzcd}$$

which are subject to compatibilities induced by the inclusions $\Sigma_n \times \Sigma_m \subseteq \Sigma_{n+m}$ and the inclusions $\Sigma_n \wr \Sigma_m \subseteq \Sigma_{nm}$.

[1] This is rather analogous to the extra data required on a *space*, beyond just a multiplication, which allows one to use the bar construction to assemble a delooping.

[2] The high degree of intricacy accomplishes this goal of constructing mapping spaces, but it interacts strangely with the classical notion of a ring spectrum in the homotopy category: there are ring spectrum maps that admit *no* enrichment to an E_∞ map, and there are ring spectrum maps that admit *multiple* enrichments to E_∞ maps.

[3] Notably, the Morava K–theories are *not* E_∞ rings at finite heights, in view of Remark 2.4.16.

Lemma A.0.3. *Each E_∞ ring spectrum gives rise to an H_∞ ring spectrum in the homotopy category.* $\square$

We care about this secondary definition because our results thus far have all concerned the cohomology of spaces, which is, at its core, a calculation at the level of *homotopy classes*. This is therefore as much of the E_∞ structure as one could hope would interact with our analyses in the preceding Case Studies.

In Lecture 2.4, Lecture 2.5, and Lecture 2.6, we introduced an H_∞ ring structure on MU and used it to make a calculation of the coefficient ring MU_*. Our primary goal in this Appendix is to introduce an H_∞ ring structure on certain chromatically interesting spectra, including Morava's theories E_Γ, and to describe the compatibility laws arising from intertwining these two H_∞ structures. The culminating result is as follows:

Theorem A.0.4 (cf. Corollary A.2.36)**.** *An orientation $MU[6, \infty) \to E_\Gamma$ is H_∞ if and only if the induced cubical structure is "norm-coherent" (cf. Definition A.2.31).* $\square$

Before addressing this, we discuss in Appendix A.1 an important phenomenon: after deleting certain forms of torsion, the Morava E_Γ–homology of a finite spectrum can be well–approximated by its Morava $E_{\Gamma'}$–homology where Γ' satisfies $\operatorname{ht}\Gamma' < \operatorname{ht}\Gamma$. This is interesting in its own right, and we will quickly see that the precise form of the approximation bears directly on the study of power operations. Finally, with the homotopy category exposed, we give a summary of the known results about E_∞ orientations themselves in Appendix A.3, where we summarize the construction of a spectrum *TMF*, and in Appendix A.4, where we summarize what goes into giving it a *String*–orientation.

A.1 Rational phenomena: chromatic character theory

Nat is going to revamp arXiv:1308.1414 for use here.

Some other references: Morava's *Local fields* paper; Theorem 2.6 of Greenlees–Strickland; work of Stapleton and Schlank–Stapleton; Nat's little monologue to me about the classical case, where the word "character" comes from; try to cover the action by $GL_n(\mathbb{Z}_p)$ (and hint at the action by $M_{n\times n}(\mathbb{Z}_p)$ with $\det \neq 0$)

A.2 Orientations and power operations

Our introduction of E_∞ rings also automatically introduces a few interesting accompanying functors:

$$\begin{tikzcd}
\mathsf{Spaces} \arrow[rrr, bend left, "E^{(-)_+}"] & \mathsf{Modules}_E \arrow[r, shift left, "\mathbb{P}_E"] \arrow[d, shift left] & E_\infty\mathsf{RingSpectra}_E \arrow[l, shift left] \arrow[d, shift left] \\
& \mathsf{Spectra} \arrow[r, shift left, "\mathbb{P}"] \arrow[u, shift left, "(-)\wedge E"] & E_\infty\mathsf{RingSpectra}. \arrow[l, shift left] \arrow[u, shift left, "(-)\wedge E"]
\end{tikzcd}$$

The first functor sends a space X to its spectrum of E–cochains $E^{\Sigma^\infty_+ X}$, and the other two functors form a free/forgetful monad resolving a mapping space in $E_\infty\mathsf{RingSpectra}_E$ by a sequence of mapping spaces in $\mathsf{Modules}_E$. These kinds of functors are familiar to us from the discussion of contexts in Lecture 3.1 and Lecture 4.1, and the recipe applied in those situations gives an analogous story here. First, there is a natural map

$$\mathsf{Spaces}(*, X) \to E_\infty\mathsf{RingSpectra}_E(E^{X_+}, E),$$

which one hopes is an equivalence under (often very strong) hypotheses on E and on X.[4] Second, the adjunction gives a mechanism for resolving E^{X_+}, which feeds into a spectral sequence computing this right-hand mapping space. The functors $\mathbb{P}$ and $\mathbb{P}_E$ can be given by explicit formulas:

$$\mathbb{P}(X) = \bigvee_{j=0}^{\infty} X^{\wedge j}_{h\Sigma_j}, \qquad \mathbb{P}_E(M) = \bigvee_{j=0}^{\infty} M^{\wedge_E j}_{h\Sigma_j}.$$

Finally, we expect the homotopy groups of this resolution to form a quasicoherent sheaf over a suitable E_∞ *context*, which arises as the simplicial scheme associated to this resolution in the case where X is a point. In this case, we can explicitly name some of the terms in this resolution: the bottom two stages take the form

[4]It is an unpublished theorem of Hopkins and Lurie that if X is a space admitting a finite Postnikov system with at most $\operatorname{ht}\Gamma$ stages and involving only finite groups, then the natural map $F(*, X) \to E_\infty\mathsf{RingSpectra}_{E_\Gamma/}(E_\Gamma^{X_+}, E_\Gamma)$ is an equivalence.

$$\begin{tikzcd}
E_\infty\mathsf{RingSpectra}_E(E, E) & & \\
E_\infty\mathsf{RingSpectra}_E(\mathbb{P}_E(E), E) \arrow[u] & E_\infty\mathsf{RingSpectra}_E(\mathbb{P}_E^2(E), E) \arrow[l, shift left] \arrow[l, shift right] & \cdots . \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

The available adjunctions give a more explicit presentation of these terms:

$$\begin{aligned}
E_\infty\mathsf{RingSpectra}_E(\mathbb{P}_E(E), E) &\simeq \mathsf{Modules}_E(E, E) \\
&\simeq \mathsf{Spectra}(\mathbb{S}, E),
\end{aligned}$$

which on homotopy groups computes the coefficient ring of E, and

$$\begin{aligned}
E_\infty\mathsf{RingSpectra}_E(\mathbb{P}_E^2(E), E) &\simeq \mathsf{Modules}_E(\mathbb{P}_E(E), E) \\
&\simeq \mathsf{Modules}_E\left(\bigvee_{j=0}^{\infty} E_{h\Sigma_j}^{\wedge_E j}, E\right) \\
&\simeq \mathsf{Spectra}\left(\bigvee_{j=0}^{\infty} \mathbb{S}_{h\Sigma_j}^{\wedge j}, E\right) \\
&\simeq \prod_{j=0}^{\infty} \mathsf{Spectra}\left(B\Sigma_j, E\right),
\end{aligned}$$

which on homotopy groups is made up of a product of the cohomology rings $E^*(B\Sigma_j)$. The higher terms track the compositional behavior of these summands.

Remark A.2.1 ([Bou75, MT92, Bou96, Dav95]). One of the first places these ideas appear in the literature is in work of Mahowald and Thompson. Bousfield defined an unstable local homotopy type associated to a (simply-connected) space and a homology theory. In the case of the space S^{2n-1} and p–adic K–theory, Mahowald and Thompson calculated that $L_K S^{2n-1}$ appears as the homotopy fiber

$$L_K S^{2n-1} \to L_K \Omega^\infty \Sigma^\infty S^{2n-1} \to L_K \Omega^\infty \Sigma^\infty ((S^{2n-1})_{h\Sigma_p}^{\wedge p}),$$

which is an abbreviated form of the monadic resolution described above.

Remark A.2.2. In general, if $E^* B\Sigma_j$ is sufficiently nice, then the the E^2–page of the monadic descent spectral sequence computes the derived functors of derivations, taken in a suitable category of monad-algebras for the monad specifying the behavior of power operations.

Cite me: This should be in...Charles's work?.

Strickland's theorems

We thus set out to understand the formal schemes constituting the E_∞ context associated to Morava E–theory.[5] As described in Lecture 4.1 and Lecture 4.2, the correct language for these phenomena are schemes defined on Hopf rings, together with the adjunction between classical rings and Hopf rings consisting of the $*$–square–zero extension functor and the $*$–indecomposables functor. The rings $E^0 B\Sigma_j$ assemble into a Hopf ring using the following structure:

- The $*$–product comes from the stable transfer maps $B\Sigma_{i+j} \to B\Sigma_i \times B\Sigma_j$.
- The $\circ$–product comes from the diagonal maps $B\Sigma_j \to B\Sigma_j \times B\Sigma_j$.
- The diagonal comes from the block-inclusion maps $B\Sigma_i \times B\Sigma_j \to B\Sigma_{i+j}$.

Definition A.2.3. Accordingly, we set the *natural E_∞ context* to be

$$E_\infty \mathcal{M}_{E_\Gamma} = \operatorname{SpH} E^0 B\Sigma_*.$$

The effect of this functor on classical rings is given by Lemma 4.2.10: $E_\infty \mathcal{M}_{E_\Gamma}(T) = \mathsf{Algebras}_{E^0_\Gamma}(Q^* E^0 B\Sigma_*, T)$, where

$$Q^* E^0 B\Sigma_* = \frac{E^0 B\Sigma_*}{\operatorname{im}\left(\operatorname{Tr}^{\Sigma_{*_1+*_2}}_{\Sigma_{*_1}\times\Sigma_{*_2}} \colon E^0 B\Sigma_{*_1} \times E^0 B\Sigma_{*_2} \to E^0 B\Sigma_{*_1+*_2}\right)}.$$

The ideal appearing in this equation is called the *transfer ideal*, written I_{Tr}.

Remark A.2.4. In terms of the descent spectral sequence described in the previous subsection, all of the $*$–decomposables are in the image of the d^1–differential, so already do not contribute to the E^2–page of the spectral sequence.

We dissect this Hopf ring spectrum by considering the j^{th} graded piece of $E_\infty \mathcal{M}_{E_\Gamma}$ as restricted to classical rings, i.e., by understanding the formal schemes $\operatorname{Spec}(E^0 B\Sigma_j / I_{\mathrm{Tr}})$ for individual indices j.

Example A.2.5. To gain a foothold, it is helpful to further specialize to a particular case—say, $j = p$. In light of the results of Appendix A.1, we might begin by analyzing the (maximal) abelian subgroups of Σ_p, of which the only p–locally interesting one is the transitive subgroup $C_p \subseteq \Sigma_p$. In Theorem 4.6.1, we calculated $BS^1[p]_E = \mathbb{C}\mathrm{P}^\infty_E[p]$, and we now make the further observation that the regular representation map $\rho\colon BS^1[p] \to BU(p)$ induces the following map on cohomological formal schemes:

[5] Much of the analysis for the case of $H\mathbb{F}_2$ can be read off from a pleasant paper of Baker [Bak15].

$$\begin{tikzcd} BS^1[p]_E \arrow[r, "\rho_E"] \arrow[d, equal] & BU(p)_E \arrow[d, equal] \\ \underline{\mathsf{FormalGroups}}(\mathbb{Z}/p, \mathbb{C}P^\infty_E) \arrow[r] & \operatorname{Div}_p^+ \mathbb{C}P^\infty_E, \end{tikzcd}$$

where the bottom arrow sends such a homomorphism to its image divisor. This map belongs to a larger diagram of schemes:

$$\begin{tikzcd} \operatorname{Spf} E^0 B\Sigma_p / I_{\mathrm{Tr}} \arrow[r] & (B\Sigma_p)_E \arrow[r] & BU(p)_E \\ \operatorname{Spf} E^0 BC_p / I_{\mathrm{Tr}} \arrow[r] \arrow[u] & (BC_p)_E. \arrow[u] \arrow[ur] \end{tikzcd}$$

The effect of killing the transfer ideal in $(BC_p)_E$ is to force the image divisor to be a subdivisor of $\mathbb{C}P^\infty_E[p]$ (i.e., the zero homomorphism is disallowed), and this Explain why.
subscheme of homomorphisms is written $\operatorname{Level}(\mathbb{Z}/p, \mathbb{C}P^\infty_E)$. Finally, passing to $B\Sigma_p$ from BC_p exactly destroys the choice of generator of $\mathbb{Z}/p$, i.e., it encodes passing from the homomorphism to the image divisor. This winds up giving an isomorphism

$$\operatorname{Spf} E^0 B\Sigma_p / I_{\mathrm{Tr}} \cong \operatorname{Sub}_p \mathbb{C}P^\infty_E,$$

where Sub_p denotes the subscheme of Div_p^+ consisting of those effective Weil divisors of rank p which are subgroup divisors.

The broad features of this example hold for a general index j.

Definition A.2.6. The *abelian E_∞ context* is formed by considering the inclusions

$$\bigvee_{\substack{A \le \Sigma_j \\ A \text{ abelian}}} BA \to B\Sigma_j.$$

A consequence of Appendix A.1 is that this map is *injective* on Morava E–cohomology, so that we can understand the natural E_∞ context in terms of this larger object. A benefit to this auxiliary context is that we can already predict its behavior in Morava E–theory: the cohomological formal scheme associated to an abelian group can be presented as an internal scheme of group homomorphisms, just as above. Using this auxiliary context for reference, Strickland has proven the following results:

Theorem A.2.7 ([Str98, Theorem 1.1])**.** *There is an isomorphism*

$$\operatorname{Spf} E^0 B\Sigma_j / I_{\mathrm{Tr}} \cong \operatorname{Sub}_j \mathbb{CP}^\infty_E,$$

where Sub_j *denotes the subscheme of* Div_j^+ *consisting of those effective Weil divisors of rank j which are subgroup divisors.*[6] □

Theorem A.2.8 ([Strb, pg. 45])**.** *For a finite abelian group A, there is a diagram*

$$\begin{tikzcd}
\operatorname{Spf} E^0 BA \Big/ \left(\sum_{\substack{a \in A \\ a \neq 0}} \operatorname{ann}(a)\right) \arrow[r] \arrow[d, equal] & \operatorname{Spf} E^0 BA / I_{\mathrm{Tr}} \arrow[r] & \operatorname{Spf} E^0 BA \arrow[d, equal] \\
\operatorname{Level}(A^*, \mathbb{CP}^\infty_E) \arrow[rr] & & \underline{\mathsf{FormalGroups}}(A^*, \mathbb{CP}^\infty_E),
\end{tikzcd}$$

where $\operatorname{Level}(A^*, \mathbb{CP}^\infty_E)$ *denotes the subscheme of* $\underline{\mathsf{FormalGroups}}(A^*, \mathbb{CP}^\infty_E)$ *subject to the condition that $A^*[n]$ forms a subdivisor of $\mathbb{CP}^\infty_E[n]$.*[7] □

Remark A.2.9. An important piece of intuition about the schemes $\operatorname{Level}(A^*, \mathbb{CP}^\infty_E)$ is that they form a kind of replacement for the nonexistent "scheme of monomorphisms". Specifically, the p–series for a Lubin–Tate universal deformation group is only once x–divisible, and hence the divisor $\widehat{\mathbb{G}}[p]$ only contains the divisor $[0]$ with multiplicity one.[8] This excludes noninjective morphisms in this case. On the other hand, the only subgroups of the formal group restricted to the special fiber are of the form $p^m \cdot [0]$. In particular, any level structure on $\widehat{\mathbb{G}}$ restricts to a morphism with this image divisor at the special fiber, and hence functoriality considerations force us to count these—which are *not* images of monomorphisms—as level structures as well.

Remark A.2.10 ([Str97]). The schemes $\operatorname{Sub}_j \widehat{\mathbb{G}}$ and $\operatorname{Level}(A, \widehat{\mathbb{G}})$ are known to possess many very pleasant algebraic properties: they are finite and free of predictable rank, they have Galois descent properties, the schemes $\operatorname{Level}(A, \widehat{\mathbb{G}})$ are all reduced, there are important decompositions coming from presenting a subgroup scheme as a flag of smaller subgroups, ... Indeed, these algebraic results form important ingredients to the proof of the connection with homotopy theory [Str98, Section 9].

Cite me: Power operations are Koszul paper.

Remark A.2.11. Rezk has shown that the E_∞ descent object for Morava E–theory

[6] If j is not a power of p, this is the terminal scheme.
[7] If $A[p] \cong (\mathbb{Z}/p)^{\times k}$ has $k > \operatorname{ht}\Gamma$, then this is the terminal scheme.
[8] This kind of reasoning applies to domains of characteristic 0 generally.

is, in a certain sense, of finite length. Specifically, there is a subobject of the descent object which consists levelwise of those flags of formal subgroups of $\widehat{\mathbb{G}}_E$ whose composition is contained in $\widehat{\mathbb{G}}[p]$, and the Koszul condition entails that this inclusion induces a weak equivalence of derived categories.

Isogenies and the Lubin–Tate moduli

In this subsection, we seek a comparison of the natural E_∞ context and the unstable context considered in Lecture 4.1. Our model for the unstable context in Lecture 4.2 focuses on the effect of unstable operations on the cohomology of $\mathbb{C}P^\infty$, as summarized in the following result:

Lemma A.2.12 (mild extension of Theorem 4.5.12 along the lines of Corollary 4.3.7)**.** *There is an isomorphism*

$$\operatorname{Spec} Q^* \pi_* L_\Gamma(E \wedge E) \cong \underline{\mathsf{FormalGroups}}(\mathbb{C}P^\infty_E, \mathbb{C}P^\infty_E). \quad \square$$

In order to form a comparison map between these two contexts, we will want algebraic constructions that trade a subgroup divisor (i.e., a point in the natural E_∞ context) for a formal group endomorphism (i.e., a point in the unstable context). It will be useful to phrase our ideas in the language of *isogenies*.

Definition A.2.13 ([Str99b, Definition 5.17])**.** Take C and D to be formal curves over X. A map $f\colon C \to C'$ is an *isogeny* (of degree d) when the induced map $C \to C \times_X C'$ exhibits C as a divisor (of rank d) on $C \times_X C'$ as C'–schemes.

Remark A.2.14 (cf. Remark 2.2.8). In this case, a divisor D on C' gives rise to a divisor f^*D on C by scheme-theoretic pullback:

$$\begin{tikzcd} f^*D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d] \\ C \arrow[r, "f"] & C', \end{tikzcd}$$

altogether inducing a map

$$f^*\colon \operatorname{Div}^+_n C' \to \operatorname{Div}^+_{nd} C.$$

This map interacts with pushforward by $f_* f^* D = d \cdot D$, where d is the degree of the isogeny.

The usual source of examples of isogenies are polynomial maps between curves. In fact, this is close to the general case, and the following result is the source of much intuition:

Lemma A.2.15 (Weierstrass preparation, [Str99b, Section 5.2]). *Let R be a complete local ring. Every degree d isogeny $f\colon \widehat{\mathbb{A}}^1_R \to \widehat{\mathbb{A}}^1_R$ admits a unique factorization as a coordinate change and a monic polynomial of degree d.* $\square$

In the case of formal groups over a perfect field of positive characteristic, this reduces to two familiar structural results:

Corollary A.2.16. *Every nonzero map of formal groups over a perfect field of positive charactersitic can be factored as an iterate of Frobenius and a coordinate change.*[9] $\square$

Corollary A.2.17. *A map of formal groups over a complete local ring with a perfect positive-characteristic residue field is an isogeny if and only if the kernel subscheme of the map is a divisor.* $\square$

This last result forms the headwaters of the connection we are seeking: isogenies are exactly the class of formal group homomorphisms whose kernels form subgroup divisors. Again, we are seeking an assignment in the opposite direction, some special collection of endoisogenies naturally attached to prescribed kernel divisors. As a first approximation to this goal, we drop the *endo-* and aim to construct just *isogenies* with this kernel property, a candidate for which is a theory of *quotient groups*.

Definition A.2.18. For $K \subseteq \widehat{\mathbb{G}}$ be a subgroup divisor, we define the quotient group $\widehat{\mathbb{G}}/K$ to be the formal scheme whose ring of functions is the equalizer

$$\mathcal{O}_{\widehat{\mathbb{G}}/K} \longrightarrow \mathcal{O}_{\widehat{\mathbb{G}}} \underset{1\otimes\eta^*}{\overset{\mu^*}{\rightrightarrows}} \mathcal{O}_{\widehat{\mathbb{G}}} \otimes \mathcal{O}_K.$$

Lemma A.2.19 ([Str97, Theorem 5.3.2-3]). *The functor $\widehat{\mathbb{G}}/K$ is again a 1–dimensional smooth commutative formal group.* $\square$

The inclusion of rings of functions determines an isogeny $q\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}/K$ of degree $|K|$. In this particular case, the induced pullback map q^* of divisor schemes has an especially easy formulation:

[9]Incidentally, the Frobenius iterate appearing in the Weierstrass factorization of the multiplication-by-p isogeny $p\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is another definition of the height of $\widehat{\mathbb{G}}$.

Lemma A.2.20. *Pullback along the isogeny $q\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}/K$ is computed by*

$$q^*\colon D \mapsto D * K,$$

where $*\colon \operatorname{Div}_n^+ \widehat{\mathbb{G}} \times \operatorname{Div}_d^+ \widehat{\mathbb{G}} \to \operatorname{Div}_{nd}^+ \widehat{\mathbb{G}}$ is the convolution product of divisors on formal groups as described in Corollary 2.3.16. $\square$

In the case that K is specified by a level structure, this admits a further refinement:

Corollary A.2.21. *Let $\ell\colon A \to \widehat{\mathbb{G}}$ be a level structure parametrizing a subgroup divisor K. The divisor pullback map can then be computed by the expansion*

$$q^*D = \sum_{a \in A} \tau^*_{\ell(a)} D,$$

where $\tau_g\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is the translation by g map. $\square$

Definition A.2.22. This second construction can be upgraded to an assignment from *functions* on $\widehat{\mathbb{G}}$ to *functions* on $\widehat{\mathbb{G}}/K$, rather than just the ideals that they generate. Specifically, we define the *norm* of $\varphi \in \mathcal{O}_{\widehat{\mathbb{G}}}$ along a level structure $\ell\colon A \to \widehat{\mathbb{G}}$ by the formula

$$N_\ell \varphi = \prod_{a \in A} \tau^*_{\ell(a)} \varphi.$$

An often-useful property of this norm construction is that if φ is a coordinate on $\widehat{\mathbb{G}}$, then $N_\ell \varphi$ is a coordinate on $\widehat{\mathbb{G}}/K$ [Str97, Theorem 5.3.1].

Remark A.2.23. In general, the pullback map q^* admits the following description: an isogeny $q\colon C \to C'$ gives a presentation of $\mathcal{O}_C$ as a finite free $\mathcal{O}_{C'}$–module. A function $\varphi \in \mathcal{O}_C$ therefore begets a linear endomorphism $\varphi \cdot (-) \in \mathrm{GL}_{\mathcal{O}_{C'}}(\mathcal{O}_C)$, and the determinant of this map gives an element of $q^*\varphi \in \mathcal{O}_{C'}$. Letting φ range, this gives a multiplicative (but not typically additive) map $q^*\colon \mathcal{O}_C \to \mathcal{O}_{C'}$. If a divisor D is specified as the zero-locus of a function φ_D, the divisor q^*D is specified as the zero-locus of $q^*\varphi_D$.

Our last technical remark is that this definition of quotient does, indeed, have the suggested universal property:

Lemma A.2.24 (Third isomorphism theorem for formal groups, [Str97, Theorem 5.3.4])**.** *If $f\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ is any isogeny with kernel divisor K, then there is a uniquely specified commuting triangle*

$$\begin{tikzcd} & \widehat{\mathbb{G}} \arrow[dl, "q"'] \arrow[dr, "f"] & \\ \widehat{\mathbb{G}}/K \arrow[rr, "g", "\simeq"'] & & \widehat{\mathbb{H}}. \end{tikzcd}$$

□

We now use this Lemma, along with properties of the Lubin–Tate moduli problem, to associate endo-isogenies to subgroup divisors. To begin, consider the multiplication–by–p endo-isogeny of a Lubin–Tate group $\widehat{\mathbb{G}}$. Since $\widehat{\mathbb{G}}$ is finite height, this map is an isogeny and the Lemma above gives rise to an isomorphism

$$\begin{tikzcd} & \widehat{\mathbb{G}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \widehat{\mathbb{G}}/\widehat{\mathbb{G}}[p] \arrow[rr, "g", "\simeq"'] & & \widehat{\mathbb{G}}. \end{tikzcd}$$

In particular, this diagram shows that the quotient map $\widehat{\mathbb{G}}/\widehat{\mathbb{G}}[p]$ is *again* a universal deformation, as witnessed by a preferred isomorphism to $\widehat{\mathbb{G}}$. For a generic subgroup divisor $K \leq \widehat{\mathbb{G}}$, we have access to the isogeny q_K using the methods described above, but it is the magic of the Lubin–Tate moduli that furnishes us with replacements for p and for g. Notice first that the problem simplifies dramatically for the formal group at the special fiber: all subgroups of the special fiber formal group are of the form $p^j \cdot [0]$, and hence we can always use the Frobenius map to complete the desired triangle:

$$\begin{tikzcd} & \Gamma \arrow[dl, "q_{p^j[0]}"'] \arrow[dr, "\mathrm{Frob}^j"] & \\ \Gamma/(p^j \cdot [0]) \arrow[rr, "g_{p^j[0]}", "\simeq"'] & & (\varphi^j)^*\Gamma, \end{tikzcd}$$

where $\varphi\colon k \to k$ is the Frobenius on coefficients and $\mathrm{Frob}\colon \Gamma \to \varphi^*\Gamma$ is the "geometric Frobenius", specified at the level of formal group *laws* by the equation $\mathrm{Frob}(x) = x^p$ and

$$(x +_\Gamma y)^p = x^p +_{\varphi^*\Gamma} y^p.$$

Lemma A.2.25 ([AHS04, Section 12.3]). *Let G be an infinitesimal deformation of a finite height formal group Γ to a complete local ring R. Associated to a subgroup divisor $K \leq G$, there is a commuting triangle*

$$\begin{tikzcd} & G \arrow[dl, "q_K"'] \arrow[dr, "P_K"] & \\ G/K \arrow[rr, "g_K", "\simeq"'] & & \psi_H^*\widehat{\mathbb{G}} \end{tikzcd}$$

for a map $\psi_K \colon \operatorname{Spf} R \to (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$ *and* $\widehat{\mathbb{G}}$ *the Lubin–Tate universal deformation of* Γ.

Proof. The main content of the deformation theory of finite height formal groups, recounted in Lecture 3.4, is that there is a natural correspondence between the following two kinds of deformation data:

$$\left\{\begin{tikzcd} \Gamma \arrow[d] & i^*\Gamma \arrow[l] \arrow[r, equal, "\alpha"] \arrow[dr] & j^*G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G \arrow[d] \\ \operatorname{Spec} k & & \operatorname{Spec} R/\mathfrak{m} \arrow[ll, "i"'] \arrow[r, "j"] & \operatorname{Spf} R \end{tikzcd}\right\} \leftrightarrow \left\{\begin{tikzcd} G \arrow[r, equal, "\beta"] \arrow[dr] & \psi^*\widehat{\mathbb{G}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{\mathbb{G}} \arrow[d] \\ & \operatorname{Spf} R \arrow[r, "\psi"] & (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma} \end{tikzcd}\right\}.$$

Accordingly, to construct the map ψ_K from the Lemma statement, we need only exhibit G/K as belonging to a natural diagram of the sort at left. Using the fact that finite subgroups of formal groups over a *field* are always of the form $p^j \cdot [0]$, this is exactly what the Frobenius discussion above accomplishes:

$$\begin{tikzcd} \Gamma \arrow[d] & (\varphi^j)^*\Gamma \arrow[l] \arrow[r, equal, "g_{p^j[0]}"] \arrow[dr] & (j^*G)/p^j \cdot [0] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G/K \arrow[d] \\ \operatorname{Spec} k & & \operatorname{Spec} k \arrow[ll, "\varphi^j"'] \arrow[r, "j"] & \operatorname{Spf} R. \end{tikzcd}$$

Transferring this to a diagram of the sort at right, this gives the map ψ_K and the isomorphism g_K, and the map P_K is constructed as their composite. $\square$

Applying this Lemma to the universal case gives the following result:

Corollary A.2.26. *There is a unique sequence of maps*

$$P^{\Sigma_{p^k}} \colon \widehat{\mathbb{G}} \times \operatorname{Sub}_{p^k} \widehat{\mathbb{G}} \to \psi^*_{\Sigma_{p^k}} \widehat{\mathbb{G}}$$

determined by the following properties:

1. *Restricting to any point in* $\operatorname{Sub}_{p^k} \widehat{\mathbb{G}}$ *gives a group homomorphism with that kernel.*

2. *(Deformation of Frobenius:) At the special fiber, $P^{\Sigma_{p^k}}$ reduces to the k^{th} Frobenius iterate.* □ Prove me?

Amazingly, this is exactly the behavior of the power operation for E–theory when applied to $\mathbb{C}P^\infty$. The Σ_{p^k}–power operation map in Morava E–theory

$$E^0\mathbb{C}P^\infty \otimes E^0 B\Sigma_{p^k} \leftarrow E^0\mathbb{C}P^\infty$$

becomes the following map of formal schemes:

$$\begin{tikzcd}
\operatorname{Spf}(E^0\mathbb{C}P^\infty \otimes E^0 B\Sigma_{p^k} / I_{\mathrm{Tr}}) \arrow[r] \arrow[d, equal] & \operatorname{Spf} E^0\mathbb{C}P^\infty \arrow[d, equal] \\
\mathbb{C}P^\infty_E \times \operatorname{Sub}_{p^k} \mathbb{C}P^\infty_E \arrow[r, "P^{\Sigma_{p^k}}"] & \mathbb{C}P^\infty_E,
\end{tikzcd}$$

and this becomes an E^0–algebra map (i.e., a map over $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$) when the bottom map is factored as

$$\mathbb{C}P^\infty_E \times \operatorname{Sub}_{p^k} \mathbb{C}P^\infty_E \xrightarrow{P^{\Sigma_{p^k}}} \psi^*_{\Sigma_{p^k}} \mathbb{C}P^\infty_E \to \mathbb{C}P^\infty_E.$$

This map has exactly the prescribed kernel, and by its very nature as a *power operation* it reduces to the Frobenius on the special fiber. Finally, this identification is definitionally compatible with the map sending a power operation to its constituent sum of unstable operations, i.e., the map from the natural E_∞ context to the unstable context.

Remark A.2.27 ([CCO14, 1.4.2.3]). There is a useful result about isogenies that more topologists out to be aware of, although it doesn't tie in directly to any of our exposition here. Nonetheless, we record its statement: let R be a (nice) complete local ring, and let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{G}}'$ be two p–divisible groups over R. There is an injection

$$\operatorname{Isog}_R(\widehat{\mathbb{G}}, \widehat{\mathbb{G}}') \hookrightarrow \operatorname{Isog}_{R/\mathfrak{m}}(\widehat{\mathbb{G}}, \widehat{\mathbb{G}}').$$

H_∞ orientations

Having worked through enough of the underlying algebra, we now return to our intended topological application of studying H_∞ orientations of Morava E–theory by MUP. There are two reduction theorems, due to McClure, that lighten our workload from an infinite number of conditions to check to merely two conditions:

Theorem A.2.28 ([BMMS86, Proposition VIII.7.2]). *Let E and F be H_∞ ring spectra with F p–local, and let $f\colon E \to F$ be a map of ring spectra in the homotopy category. Then f is furthermore an H_∞ ring map if and only if the following equation is satisfied:*

$$f \circ P_E^{\Sigma_p} = P_F^{\Sigma_p} \circ f_{h\Sigma_p}^{\wedge p}. \qquad \square$$

Theorem A.2.29. *Let E be an E_∞ ring spectrum and let X be a space such that $E^{\Sigma_+^\infty X}$ is a wedge of copies of E. The map*

Cite me: Find a Lawson citation for this last claim..

$$\iota^* \otimes \Delta^*\colon\ \widetilde{E}^* X_{hG}^{\wedge G} \to \widetilde{E}^* X^{\wedge G} \oplus \widetilde{E}^*(X \wedge BG_+)$$

is then injective.[10] $\square$

Corollary A.2.30 ([AHS04, pg. 271], [BMMS86, Proposition VII.7.2]). *A ring map $x\colon MUP \to E_\Gamma$ is H_∞ if and only if the internal power operations commute:*

$$x \circ \mu_{MUP}^{C_p} \circ \Delta = \mu_{E_\Gamma}^{C_p} \circ x_{hC_p}^{\wedge p} \circ \Delta \in E_\Gamma^0 T(\mathcal{L} \otimes \rho \downarrow \mathbb{C}\mathrm{P}^\infty \times BC_p). \qquad \square$$

We now expand the algebraic condition that this last Corollary encodes. The power operation for MUP was described in Lemma 2.5.7, where we found that it applies the norm construction to f for the universal $\mathbb{Z}/p$–level structure on $\widehat{\mathbb{G}}$. The cyclic power operation on Morava E–theory was determined in the previous section to act by pullback along the deformation of Frobenius isogeny associated to the same universal level structure. This condition is important enough to warrant a name:

Definition A.2.31. A coordinate $\varphi\colon\ \widehat{\mathbb{G}} \to \widehat{\mathbb{A}}^1$ on a Lubin–Tate group $\widehat{\mathbb{G}}$ is said to be *norm-coherent* when for all subgroups $K \subseteq \widehat{\mathbb{G}}$, we have that $N_K\varphi$ and $\psi_K^*\varphi$ are related by the isomorphism g_K.

Corollary A.2.32. *The subset of those orientations which are H_∞ correspond exactly to those coordinates which are norm-coherent, and it suffices to check the norm-coherence condition just for the universal $\mathbb{Z}/p$–level structure.* $\square$

[10] The actual statement of McClure's result [BMMS86, Proposition VIII.7.3] has several additional hypotheses: $\pi_* E$ is taken to be even-concentrated and free over $\mathbb{Z}_{(p)}$, X has homology free abelian in even dimensions and zero in odd dimensions, and X and E are both taken to be finite type. Although this is the theorem cited in the source material [And95, Section 4] [AHS04, Proof of Proposition 6.1], the version that has the weak hypotheses that we require only appeared in print much later.

This characterization is already somewhat interesting, but it will only be truly interesting once we have found examples of such coordinates. Our actual main result is that such coordinates are remarkably (and perhaps unintuitively) common. In order to set up the statement and proof of this result, we consider the following somewhat more general situation:

Definition A.2.33. A line bundle $\mathcal{L}$ on $\widehat{\mathbb{G}}$ together with a level structure $\ell \colon A \to \widehat{\mathbb{G}}$ induces a line bundle $N_\ell \mathcal{L}$ on $N_\ell \widehat{\mathbb{G}}$ according to the formula

$$N_\ell \mathcal{L} = \bigotimes_{a \in A} \tau^*_{\ell(a)} \mathcal{L}.$$

This line bundle interacts with the norm-coherence triangles well: $N_\ell \mathcal{L} = g_\ell^* \psi_\ell^* \mathcal{L}$. However, the operations on individual sections can be different: a section $s \in \Gamma(\mathcal{L} \downarrow \widehat{\mathbb{G}})$ is said to be *norm-coherent* when for any choice of level structure we have $N_\ell s = g_\ell^* \psi_\ell^* s$.

Example A.2.34. In particular, functions $\varphi \colon \widehat{\mathbb{G}} \to \widehat{\mathbb{A}}^1$ can be thought of as sections of the trivial line bundle $\mathcal{O}_{\widehat{\mathbb{G}}}$, and coordinates can be thought of as trivializations of the same trivial line bundle. We also remarked in Definition A.2.22 that the property of being a coordinate is stable under the operations on both sides of the norm-coherence equation.[11]

Cite me: Ando's thesis and also Peterson-Stapleton.

Theorem A.2.35. *Let $\mathcal{L}$ be a line bundle on a Lubin–Tate formal group $\widehat{\mathbb{G}}$, and let s_0 be a section of $\mathcal{L}_0$, the restriction of $\mathcal{L}$ to the formal group at the special fiber. Then there exists exactly one norm-coherent section s of $\mathcal{L}$ itself which restricts to s_0.*

Proof sketch. The first observation is that the norm-coherence condition can be made sense of after reducing modulo any power of the maximal ideal in the Lubin–Tate ring, and that the resulting condition is trivially satisfied in the case where the entire maximal ideal is killed. We then address the problem inductively: given a norm-coherent section modulo $\mathfrak{m}^j$, we seek a norm-coherent section modulo $\mathfrak{m}^{j+1}$ extending this. By picking *any* lift extending this, we can test the norm-coherence condition and produce an error-term that measures its failure to hold. In the specific case of the canonical subgroup $\widehat{\mathbb{G}}[p] \leq \widehat{\mathbb{G}}$, this error term is contained in the Lubin–Tate ring, and so we can modify our lift by subtracting off this error term. This perturbation has no effect on the "norm" side of the norm-coherence

[11]This is reflected in topology as the assertion that the two power operations give two Thom classes for the regular representation, which must therefore differ by a unit.

condition, and it has a linear effect on the "$\psi_{[p]}$" side, so it cancels with the error term to give a section which is norm-coherent *only when tested against the canonical subgroup*.

One can already show that the unicity clause applies: there is only one section s of $\mathcal{L}$ which reduces to s_0 and which is norm-coherent for the canonical subgroup $\widehat{\mathbb{G}}[p]$. In order to conclude that s is truly norm-coherent, one shows that $N_\ell s$ satisfies its own $[p]$–norm-coherence condition (for the new canonical subgroup $(\widehat{\mathbb{G}}/\ell)[p] \leq \widehat{\mathbb{G}}/\ell$), forcing it to agree with $\psi_\ell^* s$. □

Corollary A.2.36. *Every orientation $s_0 \colon MUP \to K_\Gamma$ extends uniquely to a diagram*

$$\begin{tikzcd} MUP \arrow[d, "s"] \arrow[dr, "s_0"] & \\ E_\Gamma \arrow[r] & K_\Gamma, \end{tikzcd}$$

where s is an H_∞ ring map. □

Example A.2.37 ([And95, Section 2.7], cf. Remark 3.3.21). The usual coordinate on $\widehat{\mathbb{G}}_m$ with $x +_{\widehat{\mathbb{G}}_m} y = x + y - xy$ satisfies the norm-coherence condition. We compute directly in the case of the canonical subgroup:

$$\begin{aligned} (p = 2) \qquad\qquad N_{[2]}(x) &= x(x +_{\widehat{\mathbb{G}}_m} 2) \\ &= x(x + 2 - 2x) \\ &= 2x - x^2 = 2^*(x), \\ (p > 2) \qquad\qquad N_{[p]}(x) &= \prod_{j=0}^{p-1} (x +_{\widehat{\mathbb{G}}_m} (1 - \zeta^j)) \\ &= \prod_{j=0}^{p-1} (x + (1 - \zeta^j) - x(1 - \zeta^j)) \\ &= \prod_{j=0}^{p-1} (1 - \zeta^j(1 - x)) \\ &= (1 - (1 - x)^p) = p^*(x). \end{aligned}$$

This is exactly the computation that $g_{[p]}^* \psi_{[p]}^*(x)$ and $N_{[p]}(x)$ agree for this choice of $x \in \mathcal{O}_{\widehat{\mathbb{G}}_m}$.

The generality with which we approached the proof of Theorem A.2.35 not only clarifies which operations apply to the objects under consideration[12], but it also applies naturally to the other kinds of orientations discussed in Case Study 5.

Theorem A.2.38. *Orientations $MU[6,\infty) \to E_\Gamma$ which are H_∞ correspond to norm-coherent cubical structures. If $\operatorname{ht}\Gamma \leq 2$, orientations $MString \to E_\Gamma$ which are H_∞ correspond to norm-coherent Σ–structures.* □

Example A.2.39 ([AHS04, Sections 15-16]). Let C_0 be an elliptic curve over a perfect field of positive characteristic. The Serre–Tate theorem says that the infinitesimal deformation theory of C_0 is naturally isomorphic to the infinitesimal deformation theory of its p–divisible group $C_0[p^\infty]$, and we let C be the universally deformed elliptic curve. Our discussion of norm-coherence for Lubin–Tate groups can be repeated almost verbatim for elliptic curves, and we note further that level structures on $\widehat{C}$ inject into level structures on C.

We can apply these observations to $E_{\widehat{C}}$, the Morava E–theory for the formal group $\widehat{C}$, considered as an elliptic spectrum. The natural orientation $MU[6,\infty) \to E_{\widehat{C}}$ from Corollary 5.5.16 is determined by the natural cubical structure on $\mathcal{I}(0)$, which by Corollary 5.5.7 is *uniquely* specified by the elliptic curve. Our main new observation, then, is that the N_ℓ and ψ_ℓ^* constructions both convert this to a cubical structure on C/ℓ, and hence are forced to agree by unicity. In turn, it follows that the σ–orientation is a map of H_∞ rings.

Power operations in Tate K–theory vs this story?

A.3 The spectrum of modular forms

The introduction of the geometry of E_∞ ring spectra has borne out a second version of the σ–orientation, summarized as a map of E_∞ ring spectra

$$\sigma\colon\ MString \to tmf.$$

This map has *extremely* good properties, not only owing to it being a map of structured ring spectra but to the object *tmf* itself, not heretofore discussed. This is one variant of the spectrum of *topological modular forms*, which comes about from

[12] This is meant in contrast to Ando's original proofs [And95, Section 2.6], where he deals only with coordinates and uses uncomfortable composition operations.

the following daydream: according to Corollary 5.5.16 and Corollary 5.6.17, every elliptic spectrum is naturally oriented, and the system of orientations should give rise to a *String*–orientation of the homotopy limit over all elliptic spectra, itself a kind of "universal" elliptic spectrum. There are several delicate points to this: the limit of a diagram of rings need not be a complex-orientable ring spectrum (cf. the C_2–equivariant spectrum KU); this diagram is very large; and the diagram exists only in the homotopy category and contains loops, so "homotopy limit" is not automatically defined without finding a lift to a more structured context.

The goal of this Lecture is to sketch out both the ingredients and the recipe for constructing this object. We aren't going to prove the main theorem in full detail, as the details are so thick that they do not admit a more reasonable presentation than what Beherns has already given . The idea is to make use of the obstruction theory **Cite me:** Behrens. that E_∞ rings garner us, and to "work locally" on a particular class of examples where the obstruction theory is well-behaved, using these to carefully exhaust the problem. We begin with a precise definition of what "locally" means in this setting:

Definition A.3.1 ([Lur09, Definition 6.2.2.6, Section 6.5])**.** A D–*valued sheaf* on a site C is a functor $F\colon \mathsf{C} \to \mathsf{D}$ that converts Čech diagrams to homotopy limit diagrams.

Definition A.3.2 ([Goe10, Remark 4.1])**.** Let $f\colon \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a flat, representable morphism of stacks. A *topological enrichment* of $\mathcal{N}$ is a sheaf of E_∞ ring spectra $\mathcal{O}$ on $\mathcal{N}$ such that[13]

$$\pi_n \circ \mathcal{O} \cong \begin{cases} f^*\omega^{\otimes k} & \text{if } n = 2k \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Fix a fixed map $f\colon \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$, this has its own associated moduli problem of topological enrichments of f.

Our goal, then, is to outline the proof of the following theorem:

Theorem A.3.3 (Goerss–Hopkins–Miller [GH04]; Behrens; Lurie [Lurd])**.** *The moduli of topological enrichments of $\mathcal{M}_{\mathbf{ell}}$, the moduli of elliptic curves, is contractible.* **Cite me:** Behrens from the tmf volume; Hopkins–Miller from the tmf volume.

The moduli of elliptic curves

In order to give a coherent strategy for proving Theorem A.3.3, we need to know something about the moduli of elliptic curves itself. Recalling from Lecture 5.5

[13] In particular, $\pi_0\mathcal{O}$ recovers the structure sheaf of $\mathcal{N}$.

the idea of a Weierstrass presentation of an elliptic curve, we define a general Weierstrass curve to be a projective curve specified by an equation of the form

$$C_a := \{y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\},$$

the universal one of which is defined over

$$\mathcal{M}_{\mathbf{Weier}} = \operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}],$$

where the invertibility of the function Δ guarantees that these curves are nonsingular. The point at infinity in the set of projective solutions gives the curve a canonical marked point and hence the structure of an abelian variety. The fraction y/x gives a coordinate in a neighborhood of the point at infinity, and hence Taylor expansion in this coordinate describes a map

$$\mathcal{M}_{\mathbf{Weier}} \to \mathcal{M}_{\mathbf{fgl}}.$$

Just as several formal group laws give the same formal group, several Weierstrass curves present the same elliptic curve, which are related by transformations of the form

$$\begin{aligned} f_{\lambda,s,r,t} \colon C_a &\to C_{a'}, \\ x &\mapsto \lambda^2 x + r, \\ y &\mapsto \lambda^3 y + sx + t, \end{aligned}$$

the universal one of which is defined over

$$\mathcal{M}_{\mathbf{Weier.trans.}} = \mathcal{M}_{\mathbf{Weier}} \times \operatorname{Spec} \mathbb{Z}[\lambda^{\pm}, r, s, t].$$

This structure map is the groupoid component map in a groupoid scheme structure on $\mathcal{M}_{\mathbf{ell}} = (\mathcal{M}_{\mathbf{Weier}}, \mathcal{M}_{\mathbf{Weier.trans.}})$, along which we have a descended map

$$\mathcal{M}_{\mathbf{ell}} \to \mathcal{M}_{\mathbf{fg}}.$$

With the moduli of elliptic curves now specified, we will construct a topological enrichment of $\mathcal{M}_{\mathbf{ell}}$ by doing so locally, then gluing the resulting local definitions together along common subsets to generate a sheaf on the entire object. We first divide $\mathcal{M}_{\mathbf{ell}}$ up over primes by passing to the p–completion, then we further divide the p–complete moduli itself into two regions via the following result:

Lemma A.3.4. *The p–divisible group of an elliptic curve is either formal of height 2 (called the supersingular case) or an extension of an étale p–divisible group of height 1 by a formal group of height 1 (called the ordinary case).*

Proof. The category of Dieudonné modules with quasi-isogenies inverted becomes semisimple, and the simple components all take the form

$$M_{m,n} = \mathrm{Cart}_{\overline{\mathbb{F}}_p} / (V^m = F^n),$$

for m and n copromise. An abelian variety is always isogenous to its dual, and hence in this semisimple category the Dieudonné module associated to an abelian variety decomposes into a Cartier self-symmetric sum of generators. An abelian variety of dimension d has p–divisible group of height $2d$, and Cartier duality on these simple components obeys the formula $DM_{m,n} = M_{n,m}$, from which it follows that the only possibilities for the quasi-isogenous components of a p–divisible group associated to an elliptic curve are $M_{1,1}$ and $M_{1,0} \oplus M_{0,1}$. $\square$

The names supersingular and ordinary are partially explained by the following result, which says that ordinary curves form the generic case and that supersingular curves are comparatively very rare:

Lemma A.3.5. *The supersingular locus of $\mathcal{M}_{\mathbf{ell}}$ is 0–dimensional. In fact, it is the zero-locus of a polynomial of degree $\lfloor (p-1)/12 \rfloor + \{0, 1, 2\}$.* $\square$

We write $i\colon \mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}} \to \mathcal{M}_{\mathbf{ell}}$ for the open inclusion of the ordinary locus. We then plan to recover a topological enrichment by constructing the pieces of the following pullback:

$$\begin{tikzcd}
 & \mathcal{O}_{\mathrm{top}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}_{\mathrm{top}})^{\wedge}_{\mathcal{M}_{\mathbf{ell}}^{\mathrm{ss}}} \arrow[r, equal] \arrow[d] & \mathcal{O}^{\mathrm{ss}} \\
\mathcal{O}^{\mathrm{ord}} \arrow[r, equal] & i_* i^* \mathcal{O}_{\mathrm{top}} \arrow[r] & i_* i^* \left((\mathcal{O}_{\mathrm{top}})^{\wedge}_{\mathcal{M}_{\mathbf{ell}}^{\mathrm{ss}}} \right). &
\end{tikzcd}$$

This decomposition is compatible with the perspective on homotopy theory taken up in the rest of this textbook: this decomposition is an instantiation of the chromatic fracture square. The top-right node forms the $\widehat{L}_2$–local component, the bottom-left forms the $\widehat{L}_1$–local component, and the bottom-right is the gluing data: the $\widehat{L}_1$–localization of the $\widehat{L}_2$–local component.

The supersingular locus

Our task in this section is to define $\mathcal{O}^{\mathrm{ss}}$ on $\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}}$, the supersingular part of the topological enrichment, and it suffices to specify its behavior on formal étale affines. Since the moduli is itself 0–dimensional, these are exactly the affine covers of the deformation spaces of the supersingular curves in the larger moduli $\mathcal{M}_{\mathbf{ell}}$. The following arithmetic result gives us a crucial reduction:

Theorem A.3.6 (Serre–Tate [Del81, Appendix 1])**.** *The map $\mathcal{M}_{\mathbf{ell}} \to \mathcal{M}_{\mathbf{pdiv}}(2)$ is formally étale, where $\mathcal{M}_{\mathbf{pdiv}}(2)$ is the moduli of p–divisible groups of height 2.*[14] $\square$

Lemma A.3.7. *The deformation theory of a connected p–divisible group of height d as a p–divisible group is isomorphic to the deformation theory of the associated formal group of height d as a formal group.* $\square$

This reduces us to finding a topological enrichment for $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\widehat{C}}$, i.e., a version of Morava E–theory. A *very* extravagant application of the resolution tools for E_∞ ring spectra yields the following theorem, essentially owing to the very nice (i.e., formally smooth) deformation space and very nice (i.e., formally smooth) space of operations:

Theorem A.3.8 (Goerss–Hopkins–Miller [GH04, Corollary 7.6–7])**.** *Let Γ be a finite height formal group over a perfect field. The moduli of topological enrichments of $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$ is homotopy equivalent to $B\operatorname{Aut}\Gamma$. An element γ of π_1 of this moduli based at a specific realization E_Γ gives a cohomology operation $\psi^\gamma\colon E_\Gamma \to E_\Gamma$ whose behavior on $\mathbb{C}\mathrm{P}^\infty_{E_\Gamma}$ is to induce the automorphism γ.* $\square$

Now we use the reduction above to extract from this a topological enrichment of $\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}}$: the enrichment sheaf arises as the pullback of the Goerss–Hopkins–Miller sheaf along the Serre–Tate map

$$\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}} = \coprod_{\text{s.s. } C} (\mathcal{M}_{\mathbf{ell}})^{\wedge}_{C} \xrightarrow{\text{f.é.}} \coprod_{\text{s.s. } C} (\mathcal{M}_{\mathbf{pdiv}}(2))^{\wedge}_{C[p^\infty]} \xleftarrow{\cong} \coprod_{\text{s.s. } C} (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\widehat{C}}.$$

Remark A.3.9. This buys more than just a bouquet of Morava E–theories, or even the global sections

$$\mathcal{O}^{\mathrm{ss}}\left(\widehat{\mathcal{M}^{\mathrm{ss}}_{\mathbf{ell}}}\right) = \prod_{\text{supersingular } C} E_{\widehat{C}}^{h\operatorname{Aut} C}.$$

[14]In general, the Serre–Tate theorem states that $\mathcal{M}^d_{\mathbf{ab}} \to \mathcal{M}_{\mathbf{pdiv}}(2d)$ is formally étale.

For instance, the moduli $\mathcal{M}^{ss}_{\mathbf{ell}}(N)$ of supersingular elliptic curves C equipped with a level–N structure[15] forms an étale cover of $\mathcal{M}^{ss}_{\mathbf{ell}}$ whenever $p \nmid N$, and hence this sheaf produces a spectrum $TMF(N)^{ss} = \mathcal{O}^{ss}(\mathcal{M}^{ss}_{\mathbf{ell}}(N))$ satisfying $(TMF(N)^{ss})^{hGL_2(\mathbb{Z}/N)} \simeq TMF^{ss}$.

The ordinary locus

We now turn to the ordinary locus, which constitutes the bulk of the problem: remember that the supersingular locus was essentially discrete, and we are setting out to construct a sheaf, which means that we will be manufacturing a *lot* of spectra. The main tool for analyzing this situation is a specialization of the obstruction theory of Goerss–Hopkins–Miller. First, note that completed p–adic K–homology (i.e., continuous Morava E–theory for $\widehat{\mathbb{G}}_m$) carries an action by $\operatorname{Aut} \widehat{\mathbb{G}}_m$, and using the results of Appendix A.2 this extends to an action by $\operatorname{End} \widehat{\mathbb{G}}_m$ using the p^{th} power operation. In turn, the completed p–adic K–homology of an E_∞ ring spectrum carries an action by $\operatorname{End} \widehat{\mathbb{G}}_m$, which is sometimes referred to as the structure of a *θ–algebra*. To be more explicit:

Theorem A.3.10 (McClure)**.** *The homotopy of a $K(1)$–local commutative K_p–algebra spectrum R, such as $\widehat{L}_1(K_p \wedge E)$, carries an extra family of ring operations ψ^k indexed on $k \in \mathbb{Z}_p$, as well as a ring map θ,[16] such that*

$$\psi^1(x) = x, \qquad \psi^k(\psi^{k'} x) = \psi^{kk'}(x),$$
$$\psi^p(x) = x^p + p\theta(x), \qquad \psi^k(x) \cdot \psi^{k'}(x) = \psi^{k+k'}(x). \qquad \square$$

Goerss–Hopkins–Miller obstruction theory reverses this information flow by seeking answers to the questions:

- Given a θ–algebra A, what is the moduli of E_∞ rings whose completed p–adic K–homology is isomorphic to A, called a *realization* of A?[17]
- Given a map $f\colon A \to B$ of θ–algebras, as well as specified realizations R and S of A and B respectively, what is the moduli of maps $R \to S$ of E_∞ rings which realize to f?

[15] A *level–N structure* is a specified isomorphism $C[N] \cong (\mathbb{Z}/N)^{\times 2}$, i.e., a choice of basis for the N–torsion.

[16] If E_* is torsion-free, then the last condition means that θ is redundant.

[17] Note that the homotopy of E itself can be recovered from that of $R = \widehat{L}_1(K_p \wedge E)$ by taking fixed points for the $\mathbb{Z}_p^\times$–action, i.e., by an Adams spectral sequence.

Theorem A.3.11 (Goerss–Hopkins, $K(1)$–locally)**.** *Given a map of θ–algebras $f\colon A_* \to B_*$, the following André–Quillen cohomology groups (internal to θ–algebras) measure various obstructions:*

moduli problem	*existence*	*uniqueness*
a model E for A	$H_\theta^{s\geq 3}(A_*, \Omega^{s-2}A_*)$	$H_\theta^{s\geq 2}(A_*, \Omega^{s-1}A_*)$
a map $E \to F$ of models	$H_\theta^{s\geq 2}(A_*, \Omega^{s-1}B_*)$	$H_\theta^{s\geq 1}(A_*, \Omega^{s}B_*)$.

Finally, given such a map f, there is a spectral sequence computing the homotopy groups of the E_∞ mapping space:

$$E_2^{s,t} = H_\theta^s(A_*, \Omega^{-t}B_*) \Rightarrow \pi_{-s-t}(E_\infty(E,F), f). \qquad \square$$

Remark A.3.12. These enhanced André–Quillen cohomology groups can be computed using a Grothendieck-type spectral sequence, intertwining classical André–Quillen cohomology groups for commutative rings with the extra task of checking compatibility with the θ–algebra structure. In practice, this means that if the underlying ring of a θ–algebra is especially nice, it is immediately guaranteed that the relevant obstruction groups vanish.

In order to apply this theorem, we need a guess as to what θ–algebra should correspond to the completed p–adic K–theory of TMF. The discussion in Remark 3.5.3, Definition 3.5.4, and Remark 3.5.8 provide the foothold we need. We expect the θ–algebra to appear as the corner in the following pullback square:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Spf} W_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \operatorname{Spf} V_\infty^\wedge \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \operatorname{Spf} \mathbb{Z}_p \arrow[d, "\text{f.é.}"] \\
\cdots \arrow[r] & \mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}(p^1) \arrow[r] & \mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}} \arrow[r] & \mathcal{M}_{\mathbf{fg}}.
\end{tikzcd}$$

Defined via these moduli, $V_\infty^\wedge$ has a natural structure as a solution to a moduli problem itself: it parameterizes pairs $(C, \eta\colon \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C})$ of ordinary elliptic curves and markings of their associated formal groups. It also carries a natural interpretation as a θ–algebra: the interesting operation ψ^p acts by

$$\psi^p\colon (C, \eta\colon \widehat{\mathbb{G}}_m \xrightarrow{\cong} \widehat{C}) \mapsto \left(\begin{tikzcd}
\widehat{\mathbb{G}}_m[p] \arrow[r] \arrow[d] & \widehat{\mathbb{G}}_m \arrow[r, "p"] \arrow[d, "\eta"] & \widehat{\mathbb{G}}_m \arrow[d, dotted, "\eta^{(p)}"] \\
C[p] \arrow[r] & C \arrow[r, "p"] & C^{(p)}
\end{tikzcd}\right).$$

Unfortunately, this θ–algebra is not nice enough to apply the Goerss–Hopkins–Miller theorem. In order to fix this, it becomes convenient to work at $p \geq 5$ for simplicity, and we then pass to a slightly more rigid moduli: we introduce a formal $(\mathbb{Z}/p)$–level structure, i.e., an isomorphism $\mathbb{Z}/p \cong (\widehat{C})[p]$. This étale $(\mathbb{Z}/p)^\times$–cover of $\mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}$ has the following exceptional property:

Lemma A.3.13 (Igusa)**.** *For $p > 5$, the moduli $\mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}(p)$ is* affine. □ Cite me: Igusa.

Corollary A.3.14. *The associated θ–algebra W_1 has vanishing Goerss–Hopkins–Miller obstruction groups, hence realizes uniquely to an ordinary E_∞ ring spectrum $TMF(p)^{\mathrm{ord}}$, and the action of $(\mathbb{Z}/p)^\times$ on the level structure enhances to a coherent $(\mathbb{Z}/p)^\times$–action on $TMF(p)^{\mathrm{ord}}$.* □

We define TMF^{ord}, our candidate for $\Gamma(\mathcal{O}^{\mathrm{ord}})$, to be the $(\mathbb{Z}/p)^\times$–fixed points of $TMF(p)^{\mathrm{ord}}$, and indeed its p–adic K–theory is $V_\infty^\wedge$. More than this, it turns out that the θ–algebra associated to any formal étale affine open of $\mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}$ has a unique realization *as an algebra under $TMF(p)^{\mathrm{ord}}$*, and maps between such also lift uniquely. Altogether, this gives us the desired sheaf $\mathcal{O}^{\mathrm{ord}}$—and it shows that the potential complexity introduced by working with sheaves in an ∞–category does not arise in this case.

Remark A.3.15. This approach is also a common strategy: first find a topological enrichment of an affine cover of your stack of interest, then descend it to the stack itself.

Gluing data

The last thing we have to do to construct the pullback square is to manufacture a map of sheaves

$$i_* i^* \mathcal{O}_{\mathrm{top}} \to i_* i^* \left((\mathcal{O}_{\mathrm{top}})^\wedge_{\mathcal{M}_{\mathbf{ell}}^{\mathrm{ss}}} \right).$$

This is rather similar to the construction of $\mathcal{O}^{\mathrm{ord}}$ itself: we construct a candidate map $TMF^{\mathrm{ord}} \to (TMF^{\mathrm{ss}})^{\mathrm{ord}} =: \widehat{L}_1(TMF^{\mathrm{ss}})$ of global sections, and then we use this to control the map of sheaves using relative Goerss–Hopkins obstruction theory. The main results that marry algebra to topology are the following two facts about $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$. The first is that $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$ counts as an elliptic spectrum:

Lemma A.3.16. *There is an elliptic curve C^{alg} over an affine* $\mathrm{Spf}((V_\infty^\wedge)^{\mathrm{ss}})$ *such that $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$ is an elliptic spectrum for this curve.*

Remarks on proof. This comes down to *algebraization*: in certain cases involving formal schemes, one can guarantee the existence of extensions of the following form:

$$\begin{tikzcd} \operatorname{Spf} A \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r, dotted, "\exists"] & Y. \end{tikzcd}$$

Such a theorem appears here when studying the homotopy ring of $\widehat{L}_1 E_{\widehat{C}}$, which can be calculated to be

$$\pi_0 \widehat{L}_1 E_{\widehat{C}} = (\pi_0 E_{\widehat{C}})[u_1^{-1}]^{\wedge}_p,$$

which is no longer easily viewed as the ring of functions on a formal scheme. However, if the classifying map $\operatorname{Spec} \pi_0 E_{\widehat{C}} \to \mathcal{M}_{\mathbf{ell}}$ is first algebraized, these operations of localization and completion can be performed on ordinary affine schemes. This has its own wrinkle: algebraization is hard to understand, for one, but we are also briefly obligated to replace $X = \mathcal{M}_{\mathbf{ell}}$ by a certain compactified moduli $Y = \overline{\mathcal{M}_{\mathbf{ell}}}$ of cubic curves with nodal singularities allowed. □

This specification of a map $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}}) \to \mathcal{M}_{\mathbf{ell}}$ gives two candidates for a θ–algebra structure on the p–adic K–theory of TMF^{ss}: there is the θ–algebra structure coming from transfer of structure along the map of schemes, and there is the θ–algebra structure coming from the shear fact that $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$ is an E_∞ ring spectrum, and hence topology simply imbues it with such algebraic structure.

Theorem A.3.17. *The natural θ–algebra structure on $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}})$ induced by the map $\operatorname{Spf}((V_\infty^\wedge)^{\mathrm{ss}}) \to \operatorname{Spf} V_\infty^\wedge$ agrees with the Goerss–Hopkins–Miller θ–algebra structure on $K_p(TMF^{\mathrm{ss}})$.* □

This is to be read as a recognition theorem for the θ–algebra structure on the topological object $(TMF^{\mathrm{ss}})^{\mathrm{ord}}$: it matches the algebraic model. Once this is established, the Goerss–Hopkins–Miller obstructions can be shown to vanish after introducing a suitable level structure; it follows that the above map lifts to a $(\mathbb{Z}/p)^\times$–equivariant map of the E_∞ rings of the global sections over the moduli with level structure; this descends to a map of global sections over the original module after taking $(\mathbb{Z}/p)^\times$–fixed points; and one finally produces the map of sheaves by further applications of relative obstruction theory.

Remark A.3.18. Arithmetic fracture is dealt with similarly, but it is *far* simpler. Because $\mathbb{Q} \otimes \mathit{TMF}$ has a smooth $\mathbb{Q}$–algebra as its homotopy, the obstructions governing the version of Goerss–Hopkins–Miller for commutative $H\mathbb{Q}$–algebras vanish, letting us lift algebraic results into homotopy theory wholesale.

Variations on these results

Remark A.3.19. At the prime 3, the proof of Igusa's theorem needs amplification, but the statement remains the same and the rest of the story goes through smoothly.

Remark A.3.20. At the prime 2, two further things go wrong: one must pass to the Igusa cover $\mathcal{M}_{\mathbf{ell}}^{\mathrm{ord}}(4)$ before it becomes affine, but then the Galois group of this cover is C_2, which has infinite cohomological dimension at 2. Appealing to the equivalence $KO = KU^{hC_2}$, one works with 2–adic *real* K–theory instead, which somehow pre-computes the Galois action.

Remark A.3.21. There is another way to construct $\mathit{TMF}^{\mathrm{ord}}$ at low primes, given by a complex consisting of two E_∞ cells attached to $\mathbb{S}$. The way this is done, essentially, is by constructing a complex whose p–adic K–theory matches the expected value: first it must have the right dimension, and then the action of θ must be corrected.

Remark A.3.22 ([Law09, Section 7]). There is an analogous (and much easier) picture for the moduli of forms of the multiplicative group: any ordered pair of puncture points in $\mathbb{A}^1$ can be used to give $\mathbb{P}^1$ the unique structure of a group with identity at ∞, and the associated formal group is classified by a map $\mathcal{M}_{\mathbb{G}_{\mathbf{m}}} \to \mathcal{M}_{\mathbf{fg}}$; there is an equivalence $\mathcal{M}_{\mathbb{G}_{\mathbf{m}}} \simeq BC_2$; and KU forms the global sections of a topological enhancement of $\operatorname{Spec} \mathbb{Z} \to \mathcal{M}_{\mathbf{fg}}$ which descends using the complex-conjugation action to $BC_2 \to \mathcal{M}_{\mathbf{fg}}$.

Remark A.3.23. With some effort, the construction of $\mathcal{O}_{\mathrm{top}}$ outlined here extends to the compactified moduli $\overline{\mathcal{M}_{\mathbf{ell}}}$ where Weierstrass curves with nodal singularities are allowed, i.e., where Δ is *not* inverted (as in $y^2 + xy = x^3$). The resulting global sections yields a spectrum Tmf, which is *not* a periodic ring spectrum. The connective truncation of that spectrum is denoted tmf, and it arises as the global sections of a topological enrichment of a stack of generalized cubics, i.e., where cuspidal singularities are also allowed (as in $y^2 = x^3$).

Remark A.3.24 ([Sto12]). A topological enrichment can be thought of as an enhancement of a classical algebro-geometric object to a *spectral* (or *derived*) one. This opens the door for exploring all sorts of phenomena: for instance, there is a very interesting manifestation of Serre duality on $\mathcal{M}_{\mathbf{ell}}$ in this enhanced setting, whose exploration is due to Stojanoska.

Descent on homotopy

One of the main upsides of producing a topological enrichment is that it is naturally equipped with a spectral sequence computing the homotopy of its global sections, coming from recovering $\mathcal{O}(\mathcal{N})$ as the homotopy limit of finer and finer covers of $\mathcal{N}$.

Lemma A.3.25. *For $\mathcal{O}$ a topological enrichment of an appropriate map $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$, there is a spectral sequence*

$$E_2^{s,t} = H^s(\mathcal{N}; \pi_t \circ \mathcal{O}) \Rightarrow \pi_{t-s}\mathcal{O}(\mathcal{N}). \qquad \square$$

Lemma A.3.26. *This spectral sequence is isomorphic to the MU–Adams spectral sequence for $\mathcal{O}(\mathcal{N})$.*

Main observation. Consider the Čech complex associated to the affine cover $\mathcal{M}_{\mathbf{Weier}} \to \mathcal{M}_{\mathbf{ell}}$. We claim that the complex making up the E_1–term of the descent spectral sequence is isomorphic to the complex making up the E_1–term of the MU–Adams spectral sequence. To illustrate, we compute the first two terms of each and compare them.

1. Consider the pullback diagram of stacks

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{\mathbf{fgl}} \arrow[d] \\ \mathcal{M}_{\mathbf{ell}} \arrow[r] & \mathcal{M}_{\mathbf{fg}}. \end{tikzcd}$$

   In the same stroke, this is also the pullback diagram computing Spec MU_*TMF.

2. Now consider the pair of cubes of iterated pullbacks pictured in Figure A.1. These compute the pullback of the cube in two different ways, producing an isomorphism $\mathcal{M}_{\mathbf{Weier.trans.}} \cong (\mathcal{M}_{\mathbf{fg}} \times \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \times_{\mathcal{M}_{\mathbf{fgl}}} \mathcal{M}_{\mathbf{Weier}}$.

n. The general case is similar, but requires stomaching iterated pullbacks in n–cubes. $\square$

$$
\begin{tikzcd}
& (\mathcal{M}_{\mathbf{fgl}} \times \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}) \times_{\mathcal{M}_{\mathbf{fgl}}} \mathcal{M}_{\mathbf{Weier}} \arrow[rr] \arrow[dl] & & \mathcal{M}_{\mathbf{Weier}} \arrow[dl] \arrow[dd] \\
\mathcal{M}_{\mathbf{fgl}} \times \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}} \arrow[rr] \arrow[dd] & & \mathcal{M}_{\mathbf{fgl}} \arrow[dd] & \\
& & & \mathcal{M}_{\mathbf{ell}} \arrow[dl] \\
\mathcal{M}_{\mathbf{fgl}} \arrow[rr] & & \mathcal{M}_{\mathbf{fg}}, &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{M}_{\mathbf{Weier.trans.}} \arrow[rr] \arrow[dd] & & \mathcal{M}_{\mathbf{Weier}} \arrow[dl] \arrow[dd] \\
& & \mathcal{M}_{\mathbf{fgl}} \arrow[dd] & \\
& \mathcal{M}_{\mathbf{Weier}} \arrow[rr] \arrow[dl] & & \mathcal{M}_{\mathbf{ell}} \arrow[dl] \\
\mathcal{M}_{\mathbf{fgl}} \arrow[rr] & & \mathcal{M}_{\mathbf{fg}}. &
\end{tikzcd}
$$

Figure A.1: Two expressions of the same cubical pullback.

Example A.3.27. We now appeal to basic results about $\mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec} \mathbb{Z}[1/6]$ to compute $\pi_* TMF[1/6]$ using these methods.[18] After inverting 2 and 3, we can use scaling and translation transformations to complete both the cube and the square, replacing an arbitrary Weierstrass curve with a *unique* one of the form $y^2 = x^3 + c_4 x + c_6$. This exhausts the morphisms in the groupoid $\mathcal{M}_{\mathbf{ell}}$: the map

$$\operatorname{Spec} \mathbb{Z}[c_4, c_6, \Delta^{-1}][1/6] \to \mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec} \mathbb{Z}[1/6]$$

is an equivalence of stacks (cf. Remark 5.5.1). Since the quasicoherent sheaf cohomology of affines is always amplitude 0, this spectral sequence is concentrated on the 0–line, and we recover

$$\pi_* TMF[1/6] \cong \mathbb{Z}[c_4, c_6, \Delta^{-1}][1/6].$$

A.4 Orientations by E_∞ maps

We now recount a more modern take on the story of the σ–orientation which passes directly through the algebra of E_∞ ring spectra. Though technically intensive, our

[18] This is admittedly a rather elaborate way of recovering the homotopy of the *complex-orientable* ring spectrum $TMF[1/6]$.

reward for grappling with this will be the modularity of the *String*–orientation, enriching Corollary 5.5.27 to the real setting. Luckily, most of the basic ideas are classically familiar, centering on a particular functor

$$gl_1 \colon E_\infty\mathsf{RingSpectra} \to \mathsf{Spectra}.$$

This functor derives its name from two compatible sources: for one, its underlying infinite loopspace is the construction GL_1 described in Lecture 1.1; and secondly, it participates in an adjunction

$$\begin{tikzcd} \mathsf{ConnectiveSpectra} \arrow[r, shift left, "\Sigma^\infty_+\Omega^\infty"] & E_\infty\mathsf{RingSpectra} \arrow[l, shift left, "gl_1"] \end{tikzcd}$$

analogous to the adjunction between the group of units and the group-ring constructions in classical algebra. Its relevance to us is its participation in the theory of highly structured Thom spectra. Let $j \colon g \to gl_1\mathbb{S}$ be a map of connective spectra, begetting a map $J \colon G \to GL_1\mathbb{S}$ of infinite loopspaces, where we have written $G = \Omega^\infty g$.

Cite me: May? or ABGHR?.

Lemma A.4.1. *The Thom spectrum of the map BJ is presented by the pushout of E_∞ rings*[19]

$$\begin{tikzcd} \Sigma^\infty_+ GL_1\mathbb{S} \arrow[r, "\Omega^\infty\Sigma j"] \arrow[d] & \Sigma^\infty_+\Omega^\infty gl_1\mathbb{S}/g \arrow[d] \\ \mathbb{S} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & MG. \end{tikzcd}$$

□

Cite me: May, but also ABGHR.

Corollary A.4.2. *There is a natural equivalence between the space of null-homotopies of the composite*

$$g \xrightarrow{j} gl_1\mathbb{S} \xrightarrow{gl_1\eta_R} gl_1 R$$

and the space of E_∞ ring maps $MG \to R$, where MG is the Thom spectrum of the stable spherical bundle classified by J.

Proof. Applying the mapping space functor $E_\infty(-, R)$ to the pushout diagram in the Lemma, we have a pullback diagram of mapping spaces:

[19]This is a kind of "twisted group-ring" construction.

$$\begin{tikzcd}
E_\infty(\Sigma^\infty_+ GL_1 \mathbb{S}, R) & E_\infty(\Sigma^\infty_+ \Omega^\infty gl_1 \mathbb{S}/g, R) \arrow[l] \\
E_\infty(\mathbb{S}, R) \arrow[u] & E_\infty(MG, R). \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

We can reidentify each of the three terms to get

$$\begin{tikzcd}
\mathsf{Spectra}(gl_1 \mathbb{S}, gl_1 R) & \mathsf{Spectra}(gl_1 \mathbb{S}/g, gl_1 R) \arrow[l] \\
\{gl_1 \eta_R\} \arrow[u] & E_\infty(MG, R), \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

hence $E_\infty(MG, R)$ appears as the fiber at $gl_1 \eta_R$ of the restriction map, which coincides with the space of nullhomotopies as claimed. $\square$

Corollary A.4.3. *The mapping set $E_\infty(Mj, R)$ is nonempty if and only if $gl_1 \eta_R \circ j$ is null-homotopic. If this is the case, then $E_\infty(Mj, R)$ is a torsor for $[\Sigma g, gl_1 R]$.* $\square$ Cite me: AHR.

Ando, Hopkins, and Rezk have used this presentation to understand the mapping space $E_\infty(MString, tmf)$. In this Lecture, we will use this same technology to understand the mapping space $E_\infty(MSpin, KO_{(p)})$, which proceeds along entirely similar lines but is a *considerably* simpler computation.[20] The approach to this computation is to mix the presentation above with chromatic fracture applied to the target:

$$\begin{tikzcd}
MSpin \arrow[r, dotted] \arrow[rr, bend left] \arrow[dr] \arrow[drr] & KO_{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & KO_p \arrow[d] \\
& \mathbb{Q} \otimes KO \arrow[r] & \mathbb{Q} \otimes KO_p.
\end{tikzcd}$$

So, we seek a pair of E_∞ ring maps into the rationalization and the p–completion of KO which agree on the p–local adèles, which involves understanding not just the mapping spaces but also the pushforward maps between them.

[20] Ando, Hopkins, and Rezk also do $E_\infty(MSpin, KO)$ as a warm-up computation [AHR, Section 7], and we are further p–localizing that result so as not to have to think about arithmetic fracture. Working arithmetically globally should be an easy exercise for the reader.

Rational orientations

We begin with the two rational nodes in the pullback diagram. As a first approximation to our goal, consider the problem of giving a complex orientation $MU \to \mathbb{Q} \otimes R$ of a rational ring spectrum $\mathbb{Q} \otimes R$. There is an automatic such orientation granted by

$$\begin{tikzcd} MU \arrow[r, dotted, "D"] \arrow[dr] & \mathbb{Q} \otimes R \\ \mathbb{S} \arrow[u] \arrow[r] \arrow[ur] & H\mathbb{Q} \arrow[u] \end{tikzcd}$$

constructed out of the unit map $\mathbb{S} \to MU$, the unit map $\mathbb{S} \to \mathbb{Q} \otimes R$, the rationalization map $S \to \mathbb{Q} \otimes \mathbb{S} \cong H\mathbb{Q}$, and the standard additive orientation $MU \to H\mathbb{Q}$ of an Eilenberg–Mac Lane spectrum. When $E_\infty(MU, T)$ is nonempty, it is a torsor for $[bu, gl_1 T]$, and since we have a preferred orientation D we thus have isomorphisms

$$\pi_0 E_\infty(MU, \mathbb{Q} \otimes R) \xleftarrow{\cong} [bu, gl_1 \mathbb{Q} \otimes R] \xleftarrow{\cong} [bu, \mathbb{Q} \otimes gl_1 R] \xrightarrow{\cong} [\mathbb{Q} \otimes bu, \mathbb{Q} \otimes gl_1 R],$$

the last of which is specified by a sequence of rational numbers $(t_{2k})_{k \geq 1}$. The role played by the sequence (t_{2k}) is to perturb the Thom class.

Lemma A.4.4. *Write x for the Thom class of $\mathcal{L}$ on $\mathbb{C}\mathrm{P}^\infty$ in $(\mathbb{Q} \otimes R)$–cohomology as furnished by the automatic orientation D. The Thom class associated to some other orientation of $\mathbb{Q} \otimes R$ is tracked by a difference series $x / \exp_F(x)$, and the sequence (t_k) above is expressed by $x / \exp_F(x) = \exp(\sum_k t_k / k! \cdot x^k)$.*

Proof sketch. Let $v^k \colon S^{2k} \to BU$ be the k^{th} power of the class $\mathcal{L}$, so that it comes from a restriction

$$S^{2k} \to (\mathbb{C}\mathrm{P}^\infty)^{\wedge k} \xrightarrow{\mathcal{L}^{\boxtimes k}} BU.$$

The Thom class for this bundle comes from the top Chern class, which is the top coefficient in the product of total Chern classes applied to the individual bundles. Following the usual formulas shows the map v^k to behave on homotopy by multiplication by $(-1)^k t_k$. □

Now we move away from MU. There are three directions for generalization: connective orientations, real orientations, and non-complex targets.

1. Rationally, the analysis of Ando–Hopkins–Strickland identifies $[BU\langle 2k\rangle, \mathbb{Q} \otimes R]$ with k–variate symmetric multiplicative 2–cocycles over R, every one of which arises as δ^1 repeatedly applied to a univariate series. In homotopy theoretic terms, this means that every $MU\langle 2k\rangle$–orientation of a rational spectrum factors through an MU–orientation.

2. The cofiber sequence $kO \to kU \to \Sigma^2 kO$ splits rationally, using the idempotents $\frac{1\pm\chi}{2}$ on kU. Accordingly, MU–orientations of rational spectra that factor through MSO–orientations have an invariance property under χ: $-[-1](x) = x$, corresponding to the idempotent factor $+$. This pattern continues for the characteristic series of connective orientations.

3. This same cofiber sequence and idempotent splitting also tells us that rational KU–cohomology classes in the image of KO–cohomology are χ–invariant, i.e., they belong to the $-$ factor.

Our main example is the usual orientation $MU \to KU$ that selects the formal group law $x + y - xy$. This is associated to the difference Thom class $x/(e^x - 1) = x/\exp_{\widehat{\mathbb{G}}_m}(x)$. To make this difference $[-1]$–invariant (and hence give a complex-orientation of KO), we use the averaged exponential class $(e^{x/2} - 1) - (e^{-x/2} - 1)$.[21] In turn, we use the Lemma to calculate the behavior on homotopy of the associated orientation:[22]

$$\frac{x}{e^{x/2} - e^{-x/2}} = \exp\left(-\sum_{k=2}^{\infty} \frac{B_k}{k} \cdot \frac{x^k}{k!}\right).$$

Finally, we calculate the effect of the orientation on the second half of the factorization

$$MSU \to MSpin \to KO,$$

again using the relevant idempotent, which has the effect of halving the coefficients in the characteristic series: $-\frac{B_k}{2k}$.[23]

This discussion accounts for both $E_\infty(MSpin, \mathbb{Q} \otimes KO)$ and $E_\infty(MSpin, \mathbb{Q} \otimes KO_p)$: the set of rational characteristic series includes into the set of adèlic characteristic series as the subset with rational coefficients.

[21] Incidentally, this is equal to $2\sinh(x/2)$.

[22] This comes out of applying $d\log$ to the fraction.

[23] While we're here, you might want to observe that elements in $[bu, gl_1 R]$ push forward to elements in $[bu, gl_1 \mathbb{Q} \otimes R]$ which do not disturb the denominators of the elements t_k. (On the other hand, the "Miller invariant" associated to a rational ring spectrum is *zero*, because arbitrary elements in $[bu, gl_1 \mathbb{Q} \otimes R]$ can completely destroy the denominators.)

Finite place orientations

We want now to understand $E_\infty(MSpin, KO_p)$ and its map to $E_\infty(MSpin, \mathbb{Q} \otimes KO_p)$. Here's the initial set-up:

$$\begin{tikzcd} spin \arrow[r, "j"] & gl_1\mathbb{S} \arrow[r] \arrow[dr, "gl_1\eta_{KO_p}"'] & Cj \arrow[d, dotted, "A"'] \\ & & gl_1 KO_p. \end{tikzcd}$$

We are looking to understand the space of filler diagrams A (i.e., vertical maps with choice of homotopy of the precomposite to $gl_1\eta_{KO_p}$). Notice first that there is a natural cofiber sequence to be placed on the bottom row:

I don't like the placing of this A. I want it to indicate a choice of filler.

$$\begin{tikzcd} spin \arrow[r, "j"] \arrow[dr, red] & gl_1\mathbb{S} \arrow[r] \arrow[d, red] \arrow[dr, "gl_1\eta_{KO_p}"'] & Cj \arrow[d, dotted, "A"'] \\ & \Sigma^{-1}\mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p \arrow[r] & gl_1 KO_p \arrow[r] & \mathbb{Q} \otimes gl_1 KO_p \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p. \end{tikzcd}$$

There is a canonical red vertical lift of $gl_1\eta_{KO_p}$ since $gl_1\mathbb{S}$ is a torsion spectrum, and this precomposes with j to give another vertical map. Notice now that selecting a filler triangle A gives a commuting square with choice of homotopy and that $[gl_1\mathbb{S}, \mathbb{Q} \otimes gl_1 KO_p] = 0$, and hence we would get a natural map (and natural homotopy) off of the homotopy cofibers:

$$\begin{tikzcd} spin \arrow[r, "j"] \arrow[dr] & gl_1\mathbb{S} \arrow[r] \arrow[d] \arrow[dr, "gl_1\eta_{KO_p}"'] & Cj \arrow[r] \arrow[d, dotted, "B"] & bspin \arrow[r] \arrow[d, dotted, "C"] \arrow[dr] & bgl_1\mathbb{S} \arrow[d] \\ & \Sigma^{-1}\mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p \arrow[r] & gl_1 KO_p \arrow[r] & \mathbb{Q} \otimes gl_1 KO_p \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p, \end{tikzcd}$$

where C is a map making the triangle it belongs to commute. This all gives a function assigning A to B and A to C (and, in fact, the latter assignment factors through the former).

In order to show nonconstructively that the set of As is nonempty, we might try to discern that $gl_1\eta_{KO_p} \circ j \in [spin, gl_1 KO_p]$ is zero by demonstrating something about the mapping set $[spin, gl_1 KO_p]$ itself. We proceed by a sequence of quite improbable steps, beginning with the following Theorem original to Ando–Hopkins–Rezk:

Theorem A.4.5 ([AHR, Theorem 4.11]). *Let R be a $E(d)$–local E_∞ ring spectrum, and set F to be the fiber*

$$F \to gl_1 R \to L_d gl_1 R.$$

Then $\pi_* F$ is torsion and F satisfies the coconnectivity condition $F \simeq F(-\infty, d]$. □

It follows that $gl_1 KO_p \to L_1 gl_1 KO_p$ is a 1–connected map, and hence

$$[spin, gl_1 KO_p] = [spin, L_1 gl_1 KO_p].$$

In fact, we can even pass to the $K(1)$–localization, if we digress for a moment to introduce Rezk's logarithmic cohomology operation.

Lemma A.4.6 ([Kuh08, Theorem 1.1]). *For each $d \geq 1$ there is a functor $\Phi_d \colon \mathsf{Spaces}_{*/} \to \mathsf{Spectra}$ which commutes with finite limits, is insensitive to upward truncation, and which evaluates on infinite loopspaces to give $\Phi_d(\Omega^\infty X) = \widehat{L}_d X$.*[24,25] □

Definition A.4.7 ([Rez06, Section 3]). The natural equivalence $(GL_1 R)[1, \infty) \to (\Omega^\infty R)[1, \infty)$ gives rise to a map ℓ as in the diagram

$$\begin{tikzcd}
& \Phi_d(GL_1 R)[1,\infty) \arrow[r, "\simeq"] \arrow[d, equal] & \Phi_d(\Omega^\infty R)[1,\infty) \arrow[d, equal] \\
gl_1 R \arrow[r] \arrow[rr, bend left, "\ell_d"] & \widehat{L}_d gl_1 R \arrow[r, "\simeq"] & \widehat{L}_d R.
\end{tikzcd}$$

Remark A.4.8. Applying the logarithm to the corners in the height 1 chromatic fracture square yields the following identification:

$$\begin{tikzcd}
& L_1 gl_1 R \arrow[rr] \arrow[dd] \arrow[dl, equal] \arrow[drr, phantom, "\lrcorner", very near start] & & \widehat{L}_1 R \arrow[dd] \\
L_1 gl_1 R \arrow[rr] \arrow[dd] \arrow[drr, phantom, "\lrcorner", very near start] & & \widehat{L}_1 gl_1 R \arrow[ur, "\ell_1"] \arrow[dd] & \\
& \widehat{L}_0 R \arrow[rr] & & \widehat{L}_0 \widehat{L}_1 R \\
\widehat{L}_0 gl_1 R \arrow[rr] \arrow[ur, "\ell_0"] & & \widehat{L}_0 \widehat{L}_1 gl_1 R. \arrow[ur, "\widehat{L}_0 \ell_1"] &
\end{tikzcd}$$

---

[24]Importantly, Φ_d does *not* care about the actual infinite loopspace structure on $\Omega^\infty X$, just that it has *some* lift to a spectrum X.

[25]There is also a version of this theorem for $d = 0$, but since rational localization has no periodic behavior the results as not nearly as striking.

The front and back faces are connected by logarithms of *different* heights—or, equivalently, the bottom horizontal arrow of the back face is *twisted* from the usual chromatic fracture presentation of $L_1 R$. The identification of this map is the usual sticking point in this approach.

Theorem A.4.9 ([Rez06, Theorem 1.9])**.** *For R a $K(1)$–local E_∞ ring with $\pi_0 R$ torsion–free, the map $\pi_0 \ell_1 \colon \pi_0 R^\times \to \pi_0 R$ is given by the formula*[26]

$$\ell_1(x) = \frac{1}{p} \log\left(\frac{x^p}{\psi^p x}\right) = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k} \left(\frac{\theta(x)}{x^p}\right)^k. \qquad \square$$

Corollary A.4.10. *The natural map $L_1 gl_1 KO_p \to \widehat{L}_1 gl_1 KO_p$ is a connective equivalence.*

Proof. We specialize the above square to $R = KO_p$:

$$\begin{tikzcd}
 & & & & KO_p \arrow[dd] \\
L_1 gl_1 KO_p \arrow[rrr] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & & & \widehat{L}_1 gl_1 KO_p \arrow[ur, "\ell_1"] \arrow[dd] & \\
 & L_0 KO_p[4,\infty) \arrow[rrr] & & & L_0 KO_p \\
L_0 gl_1 KO_p \arrow[ur, "\ell_0"] \arrow[rrr] & & & L_0 \widehat{L}_1 gl_1 KO_p. \arrow[ur, "\ell_1"] &
\end{tikzcd}$$

The behavior of the back horizontal map is determined by Rezk's formula for the logarithm. It acts by some nonzero number in every positive degree, hence the fiber has the form $\prod_{k=-\infty}^{0} \Sigma^{4k-1} H\mathbb{Q}$. Since the front face is a fiber square, this is also a calculation of the fiber of the map in the Lemma statement.[27] $\square$

As a consequence, we have identifications

$$gl_1 \eta_{KO_p} \circ j \in [spin, gl_1 KO_p] \cong [spin, L_1 gl_1 KO_p] \cong [spin, \widehat{L}_1 gl_1 KO_p].$$

[26] The analogue of this formula for E_Γ (but not an arbitrary $K(d)$–local E_∞ ring spectrum) is given in [Rez06, Subsection 1.10].

[27] As a corollary of this same method, the Rezk logarithm for $R = KU_p^\wedge$ gives an equivalence $gl_1 KU_p^\wedge[3,\infty) \to KU_p^\wedge[3,\infty)$. This was previously known by nonconstructive methods to Adams and Priddy [AP76, Corollary 1.4].

A direct application of the Rezk logarithm replaces $\widehat{L}_1 gl_1 KO_p$ with KO_p, and the $K(1)$–localization of *spin* recovers $\Sigma^{-1}KO_p$. Altogether, this identifies $gl_1\eta_{KO_p} \circ j$ with a point in the mapping set $[\Sigma^{-1}KO_p, KO_p]$—and we mark this as a point where we would like to understand the space of KO–operations.

We claim also that the kernel of the assignment $A \mapsto C$ is easy to understand: two fillers A are related by an element of $[bspin, gl_1 KO_p]$, and their corresponding Cs are related by the corresponding element of $[bspin, \mathbb{Q} \otimes gl_1 KO_p]$. This set is rational, hence factors through the rationalization of $[bspin, gl_1 KO_p]$ where it must already be null, and hence it is a torsion element of $[bspin, gl_1 KO_p]$. Meanwhile, the same argument as above identifies

$$[bspin, gl_1 KO_p] = [KO_p, KO_p],$$

which we again mark as a point where we would like to understand the space of KO–operations. In particular, if we were to find the group of degree-preserving KO–operations to be torsion-free, then the assignment $A \mapsto C$ would be *injective*.

We would like to understand the behavior of C on homotopy based on some data about A. This serves two purposes: there is the necessary condition that the triangle formed by C and the canonical map $bspin \to \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p$ commute, and then also the composite

$$bspin \xrightarrow{C} \mathbb{Q} \otimes gl_1 KO_p \to (gl_1(\mathbb{Q} \otimes KO_p))[1, \infty)$$

describes the map into the adèlic component. In order to gain access to C, first notice that we can postcompose B with the localization map off of $gl_1 KO_p$ as in Figure A.2.[28] This gives a new map $B' \colon KO_p \to KO_p$—another reason to understand KO–operations.

We are now in a position to compute the action of C on a homotopy class in $\pi_* bspin$ by chasing through the following steps:

1. We push such a class forward to $\widehat{L}_1 bspin \simeq KO_p$ along the localization map.
2. We then pull it back to $\widehat{L}_1 Cj \simeq KO_p$ along $KO_p \xrightarrow{1-\psi^c} KO_p$, which acts by multiplication by $(1 - c^k)$ on π_{4k}.
3. We push it down along B' to $\widehat{L}_1 gl_1 KO_p \simeq KO_p$, which acts by an unknown factor.

[28] Importantly, and differently from what every source says, this isn't a map of cofiber sequences and so the back second vertical map does not have to exist.

4. We include it into the rational component of $\mathbb{Q} \otimes \widehat{L}_1 gl_1 KO_p$, using the fact that $\pi_* \widehat{L}_1 gl_1 KO_p$ is torsion–free.

5. Finally, we pull it back to $\mathbb{Q} \otimes gl_1 KO_p$ along the logarithm ℓ_1, which acts by multiplication by $(1 - p^{k-1})$ using Rezk's $K(1)$–local formula.[29]

The effect of this sequence of steps is

$$t_{4k} = (1 - c^k)^{-1} b_{4k} (1 - p^{k-1})^{-1},$$

where t_{4k} and b_{4k} are the effects on π_{4k} of the maps C and B' respectively. In the course of this proof, we are using the fact that division in the ring $\mathbb{Z}_p$ is unique when it is possible—the more responsible-looking equation to write is

$$b_{4k} = (1 - c^k) t_{4k} (1 - p^{k-1}).$$

Now, finally, the diagonal map $bspin \to \mathbb{Q}/\mathbb{Z} \otimes gl_1 KO_p$ becomes relevant. To check the commutativity of the triangle with C, we need only compare the results of the composite on homotopy since the map C targets a rational spectrum and hence is determined its effect on homotopy. The following invariance property makes this map accessible:[30]

Theorem A.4.11 ([AHR, Proposition 3.15 and Corollary 3.16]). *For any A_∞ orientation $\varphi\colon MU \to R$ of an A_∞ ring spectrum R, the denominators of the characteristic series associated to $\mathbb{Q} \otimes \varphi$ compute the behavior of the map $\pi_* BU \to \mathbb{Q}/\mathbb{Z} \otimes GL_1 R$.* □

Corollary A.4.12. *The numbers t_{4k} describing the effect of C satisfy the congruences*

$$t_{4k} \equiv -\frac{B_k}{2k} \pmod{\mathbb{Z}}.$$

Proof sketch. The Todd orientation $MU \to KU$ is known to be A_∞ [EKMM97, Theorem V.4.1], and the characteristic series of the Todd orientation has coefficients B_k. The extra division by 2 is picked up by studying the map $\pi_* BSU \to \pi_* BSpin$ and the map $\pi_* KO \to \pi_* KU$. □

[29] The formula for the logarithm in nonzero degrees comes from thinking of the logarithm as a *natural transformation* and applying it to the mapping set $\ell\colon gl_1 KO^0(S^{2n}) \to KO^0(S^{2n})$.

[30] It is also possible to compute the effect of this map on homotopy using the S^1–transfer. This is the subject of a paper by Miller [Mil82], after which the Miller invariant is named, and also the subject of further research by Baker and company [BCG+88].

$$
\begin{tikzcd}
 & & \widehat{L}_1\mathbb{S} \arrow[rr] \arrow[d, equal] & & KO_p \arrow[rr, "1-\psi^c"] \arrow[d, equal] & & KO_p \arrow[d, equal] & \\
 & & \widehat{L}_1 gl_1\mathbb{S} \arrow[rr] & & \widehat{L}_1 Cj \arrow[rr] \arrow[dd, dotted, "B'"] & & \widehat{L}_1 bspin & \\
spin \arrow[r, "j"] \arrow[ddr] & gl_1\mathbb{S} \arrow[ur] \arrow[rr] \arrow[dd] \arrow[ddrr, "gl_1\eta_{KO_p}"'] & & Cj \arrow[ur] \arrow[rr] \arrow[dd, dotted, "A"', "B"] & & bspin \arrow[ur] \arrow[rr] \arrow[dd, dotted, "C"] \arrow[ddrr, bend left] & & bgl_1\mathbb{S} \arrow[dd] \\
 & & & & \widehat{L}_1 gl_1 KO_p \arrow[rr] & & \mathbb{Q}\otimes \widehat{L}_1 gl_1 KO_p & \\
 & \Sigma^{-1}\mathbb{Q}/\mathbb{Z}\otimes gl_1 KO_p \arrow[rr] & & gl_1 KO_p \arrow[ur] \arrow[rr] & & \mathbb{Q}\otimes gl_1 KO_p \arrow[ur] \arrow[rr] & & \mathbb{Q}/\mathbb{Z}\otimes gl_1 KO_p .
\end{tikzcd}
$$

Figure A.2: A diagram showing the interconnections among the main components of the p–primary part of the Ando–Hopkins–Rezk argument.

We have thus identified the legal fillers C as those sequences of rational numbers t_{4k} satisfying conditions:

1. t_{4k} has the correct denominators: for $k \geq 1$, $t_{4k} \equiv -B_k/(2k) \pmod{\mathbb{Z}}$.
2. b_{4k} is the effect on homotopy of some map $B' \colon KO_p \to KO_p$.

Stable *KO* operations

We have identified three points where we want to understand the collection of stable *KO* operations. Although much of the main text of this book has been concerned with this sort of subject, this does not appear to be so immediately accessible: we want operations rather than cooperations, and *KO* is *not* a complex-orientable ring spectrum. It is close to one, though, and we gain access to it through familiar approximation.

The easy initial calculation is $K^\vee K = \mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, the ring of $\mathbb{Z}_p$–valued functions[31] on $\mathbb{Z}_p^\times$ which are continuous for the adic topologies on the domain and the target. This comes out of the stable cooperations of Landweber flat homology theories discussed in Definition 3.5.4, where we showed that E_Γ has cooperations given by the ring of functions on the pro-étale group scheme $\operatorname{Aut}\Gamma$. For $\Gamma = \widehat{\mathbb{G}}_m$, this group scheme $\operatorname{Aut}\widehat{\mathbb{G}}_m$ is constant at $\mathbb{Z}_p^\times$, so that $K^\vee K$ is the ring of $\mathbb{Z}_p$–valued functions on $\mathbb{Z}_p^\times$. Turning to cohomology, it follows by the universal coefficient spectral sequence that $K^0K = \operatorname{Hom}(\mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p)$ and that $K^1K = 0$. These correspondences behave as follows:

1. The Kronecker pairing
$$\mathbb{S}^0 \xrightarrow{c} K \wedge K \xrightarrow{1 \wedge f} K \wedge K \xrightarrow{\mu} K$$
is computed by the evaulation pairing
$$(c \in K^\vee K, f \in K^0 K) \mapsto f(c).$$
2. The stable operation ψ^λ attached to $[\lambda] \in \operatorname{Aut}\widehat{\mathbb{G}}_m$ is evaluation at λ.
3. The stable cooperation $v^{-k} \wedge v^k \in \pi_0 K \wedge K$ corresponds to the polynomial function $x \mapsto x^k$, as justified by the computation
$$\mathrm{ev}_\lambda(v^{-k} \wedge v^k) = \frac{\psi^\lambda v^k}{v^k} = \frac{\lambda^k v^k}{v^k} = \lambda^k.$$

[31]Not homomorphisms!

These last two facts mean that the behavior of a stable operation on homotopy is identical information to the values of a functional f on the standard polynomial functions x^k. We record this algebraic model as follows:

Lemma A.4.13. *For any $N \ge 0$, the assignment*

$$\operatorname{Hom}(\mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p) \xrightarrow{(f(x \mapsto x^k))_k} \prod_{k \ge N} \mathbb{Z}_p$$

is injective. A sequence (x_k) is said to be a Kümmer sequence *when it lies in this image.*[32] $\square$

Remark A.4.14. An interesting feature of the Lemma is the auxiliary index N, which is *not* part of the property of being Kümmer. In p–adic geometry, this is reflected by the p–adic convergence of the sequence

$$d + (p-1)p^r \xrightarrow{r \to \infty} d,$$

and hence the continuous reconstruction property

$$x_d = \lim_{r \to \infty} x_{d+(p-1)p^r}.$$

In homotopy theory, this is reflected by the reconstruction property $K \wedge K[2k, \infty) \simeq K \wedge K$.

Remark A.4.15. With this computation in hand, the p–local operations $KU_{(p)} \wedge KU_{(p)}$ can be recovered from arithmetic fracture, as can the global operations $KU \wedge KU$. The answer is quite similar: $\pi_0 KU \wedge KU$ is populated by rational polynomials which evaluate to integers on all integer inputs, called *numerical polynomials*.

We now pass from KU to KO. To begin, use the Tate trick

$$\begin{aligned}
K \wedge KO &\simeq K \wedge (K^{hC_2}) && (KO \text{ is a homotopy fixed point spectrum}) \\
&\simeq K \wedge (K_{hC_2}) && (\text{Tate objects vanish } K(1)\text{–locally}) \\
&\simeq (K \wedge K)_{hC_2} && (\text{homotopy colimits pull past smash products}) \\
&\simeq (K \wedge K)^{hC_2}, && (\text{Tate objects vanish } K(1)\text{–locally})
\end{aligned}$$

so that $\pi_0 K \wedge KO = \mathrm{cts}(\mathbb{Z}_p^\times / C_2, \mathbb{Z}_p)$. Taking fixed points again, we then also have $\pi_* KO \wedge KO = \mathrm{cts}(\mathbb{Z}_p^\times / C_2, KO_*)$, and KO^*KO is the KO_*–linear dual. It follows that $[\Sigma^{-1}KO, KO] = 0$ and that $[KO, KO] = \operatorname{Hom}(\mathrm{cts}(\mathbb{Z}_p^\times / C_2, \mathbb{Z}_p), \mathbb{Z}_p)$ is torsion-free, which account for our outstanding claims.

[32] A bit more explicitly: (x_k) is Kümmer when for all $h(x) = \sum_{k=N}^n a_k x^k \in \mathbb{Q}[x]$ we have $\sum_{k=N}^m a_k x_k \in \mathbb{Z}_p$.

Mazur's construction of Kubota–Leopoldt p–adic L–functions

Having learned enough about KO–operations to justify the program enacted in the previous subsections, we now need to show that there exist sequences of p–adic integers satisfying those criteria.

Theorem A.4.16 (Mazur). *For any auxiliary $c \in \mathbb{Z}_p^\times$, there is a functional f_c satisfying*[33,34,35]

$$f_c(x^{k \ge 1}) = \frac{-B_k}{k}(1 - p^{k-1})(1 - c^k).$$

This Theorem is stated in exactly the generality it was originally proven, and so uou might wonder why Mazur had already proven *exactly* what we needed. To understand his program, recall these two facts about ζ:

1. Except for a real Euler factor, ζ is basically the Mellin transform of the measure $\frac{dx}{e^x-1}$ (i.e., its sequence of moments):
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{dx}{e^x - 1}.$$
2. For any $k \in \mathbb{Z}_{>0}$, $\zeta(1-k) = -B_k/k$, where $\frac{t}{e^t-1} = \sum_{k=0}^\infty B_k \frac{t^k}{k!}$.

Mazur's idea was to build a p–adic ζ–function by investigating similar p–adic integrals, beginning with certain finitary approximations to this one. To begin, a Bernoulli polynomial for $k \in \mathbb{Z}_{>0}$ is

$$\sum_{k=0}^\infty B_k(x) \frac{t^k}{k!} = \frac{te^{tx}}{e^t - 1}.$$

These polynomials beget Bernoulli distributions according to the rule

$$\begin{aligned} \mathbb{Z}/p^n\mathbb{Z} &\xrightarrow{E_k} \mathbb{Q} \subseteq \mathbb{Q}_p \\ x \in [0, p^n) &\mapsto k^{-1} p^{n(k-1)} B_k(xp^{-n}). \end{aligned}$$

33

$$\text{Explicitly, } f_c(h) = \int_{\mathbb{Z}_p^\times} h(x) d\mu_c = \lim_{r \to \infty} \frac{1}{p^r} \sum_{\substack{0 \le i < p^r \\ p \nmid i}} \int_i^{ci} \frac{h(t)}{t} dt.$$

34 With considerable effort, this output can be halved [AHR, Section 10.3].

35 It also satisfies the normalizing property $\int_{\mathbb{Z}_p^\times} d\mu_c = \frac{1}{p}\log(c^{p-1})$.

A distribution in general is a function on $\mathbb{Z}_p$ such that its value at any node in the p–adic tree is equal to the sum of the values of its immediate children, and the p–adic integral of a locally constant function with respect to such a distribution is defined by their convolution. For example, the constant function 1 factors through $\mathbb{Z}/p$, hence

$$\int_{\mathbb{Z}_p} dE_k = \overbrace{\frac{1}{k}\sum_{a=0}^{p-1} B_k\left(\frac{a}{p}\right) = \frac{B_k(0)}{k}}^{\text{non-obvious}} = \frac{B_k}{k}.$$

However, this distribution is not a *measure*, in the sense that it is not bounded and hence does not extend to a functional on all continuous functions (rather than just locally constant ones). The standard fix for this is called *regularization*: pick $c \in \mathbb{Z}$ with $p \nmid c$, and set $E_{k,c}(x) = E_k(x) - c^k E_k(c^{-1}x)$. This is a measure, and for $k \geq 1$ it has total volume given by

$$\int_{\mathbb{Z}_p} dE_{k,c} = \int_{\mathbb{Z}_p} dE_k - c^k \int_{\mathbb{Z}_p} dE_k(c^{-1}x) = \frac{B_k}{k}(1 - c^k).$$

These measures interrelate: $E_{k,c} = x^{k-1}E_{1,c}$, and hence the single measure $E_{1,c}$ has all of these values as moments. We would like to perform p–adic interpolation in k to remove the restriction $k \geq 1$, but this is not naively possible: if $k = 0$, say, then we naively have $E_{0,c} = x^{-1}E_{1,c}$, which will not make sense whenever $x \in p\mathbb{Z}_p$. This is most easily solved by restricting x to lie in $\mathbb{Z}_p^\times$, which has a predictable effect for $k \in \mathbb{Z}_{>0}$:

$$\begin{aligned}
\int_{\mathbb{Z}_p^\times} x^{k-1} dE_{1,c} &= \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} - \int_{p\mathbb{Z}_p} x^{k-1} dE_{1,c} \\
&= \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} - p^{k-1} \int_{\mathbb{Z}_p} x^{k-1} dE_{1,c} \\
&= \frac{B_k}{k}(1 - c^k)(1 - p^{k-1}).
\end{aligned}$$

Hence, the Mellin transform of the measure $dE_{1,c}$ on $\mathbb{Z}_p^\times$ gives a sort of p–adic interpolation of the ζ–function.

It also has *exactly* the properties we need to guarantee the existence of an E_∞ orientation $MSpin \to KO$. It is remarkable that the three factors in

$$\int_{\mathbb{Z}_p^\times} x^{k-1} dE_{1,c} = \frac{B_k}{k}(1 - c^k)(1 - p^{k-1})$$

have discernable provenances in the two fields. In stable homotopy theory these arise respectively in the characteristic series of the orientation $MU \to KU$, in the finite Adams resolution for the $K(1)$–local sphere, and in the Rezk logarithm. In p–adic analytic number theory, they arise as the special values of the ζ–function, the regularization to make it a measure, and the restriction to perform p–adic interpolation. It is completely mysterious how or if these operations correspond.

Remark A.4.17. These Bernoulli sequences are *not* the only sequences satisfying these reconstruction properties—in fact, there are infinitely many, and an explicit presentation of them is available [SN14]. Sprang leaves open whether there is a way to single out the Bernoulli solution among the rest, and it seems plausible that this is the only solution with a "reasonable" growth rate (as measured in $\mathbb{R}$). It would also be great if this "real place" condition had something to do with a smooth cohomology theory like differential real K–theory.

Footnotes on the *tmf* case

The case of the orientation $MString \to tmf$ has all of the same trappings, but its order of complexity is $(-)^{3/2}$ of the above case, essentially because the height 1 chromatic fracture *square* gets replaced by the height 2 chromatic fracture *cube*. (There is also the issue of the more complicated coefficient ring tmf_* over KO_*.) Many of the steps remain the same:

1. Begin with a rational orientation, which is basically the Witten genus valued in holomorphic expansions of modular forms.

2. Analyze the homotopy type of $\widehat{L}_1 tmf$ and compare it to that of KO. This lets us use another universal coefficient theorem to lift our description of KO^*KO as KO^*–valued measures to $\widehat{L}_1 tmf^* KO$ as $\widehat{L}_1 tmf^*$–valued measures.

3. The homotopy type of $\widehat{L}_2 tmf$ is "naively irrelevant" in the chromatic fracture square: maps $bSpin \to \widehat{L}_2 tmf$ factor through $\widehat{L}_2 bSpin = \widehat{L}_2 KO_p = 0$.

4. However, the logarithm's presence in the chromatic fracture square $\widehat{L}_1 tmf \to \widehat{L}_1 \widehat{L}_2 tmf$ has a real effect that must be understood. This is not easy: the height 2 logarithm is not so accessible, so this requires a real understanding of power operations in tmf.

5. You also have to calculate the Miller invariant associated to tmf. In the case of $E_\infty(MSpin, KO)$, one uses the A_∞ orientation $MU \to KU$, as well

as an understanding of the maps $\pi_* BU \to \pi_* BO$ and $\pi_* KO \to \pi_* KU$. The case of $E_\infty(MString, tmf)$ is similar: one constructs an A_∞ lift of the σ–orientation $MU[6,\infty) \to K^{\mathrm{Tate}}$, as well as an understanding of the maps $\pi_* BU[6,\infty) \to \pi_* BString$ and $\pi_* tmf \to \pi_* K^{\mathrm{Tate}}$.

I think the conclusion here is that you have to be a q–expansion of a modular form (of a particular weight) with constant term a Bernoulli number and every other coefficient integral. Mike told me that this (or something like this) fully determines these generalized Eisenstein series; that's nice.

6. Finally, you have to ramp up the algebraic part of the calculation by identifying the analogues of the Mazur moments in $\pi_* \widehat{L}_1 tmf$. These turn out to be normalized Eisenstein series.

Mike said that there's some kind of Bousfield–Kuhn argument you can make to use the H_∞ orientation to detect the generalized Eisenstein congruences. I can't figure out what it would be.

Remark A.4.18. The presence of such interesting arithmetic invariants (Bernoulli numbers, Bernoulli polynomials, generalized Eisenstein series, ...) hiding in the Miller invariant and its analogues is very striking. One wonders what the analogous values are (or perhaps the values stemming from the iterated S^1–transfer of Baker et al [BCG$^+$88]) associated to a Morava E_Γ.

Remark A.4.19. Some more open questions about this can be found in

Mike has some open questions about the end of this analysis (and in particular about the fiber of the Atkin map that appears in the $K(1)$–local analysis of *tmf*, an analogue of the chromatic splitting fiber) at the end of his talk notes *The String orientation of tmf*. Some of that should be copied here.

KU is known to have a unique E_∞ structure by work of Baker–Richter. Is this also true of K^{Tate}? If so, it lends a lot of credibility to this Miller invariant calculation and its relation to *tmf*.

I think it's possible to show, as a side-example, that the total exterior power operation $\lambda^q \colon K \to K[\![q]\!]$ is an E_∞ map where $K[\![q]\!]$ is K^{Tate} and *not*, e.g., $K^{\mathbb{CP}^\infty}$.

Appendix B

Loose ends

I'd like to spend a couple of days talking about ways the picture in this class can be extended, finally, some actually unanswered questions that naturally arise. The following two section titles are totally made up and probably won't last.

Also, write a broad-scale introduction to this appendix.

Higher orientations

TAF and friends

The $\alpha_{1/1}$ argument: Prop 2.3.2 of Hovey's v_n–elements of ring spectra

Equivariance

This is tied up with the theory of power operations in a way I've never really thought about. Seems complicated.

You should also mention the "rigidity" of the elliptic genus, which is about an S^1–equivariant version.

Index theorems

Connections with analysis

The Stolz–Teichner program

—–

Bousfield's work on the K–theory of infinite loopspaces [Bou96] and Morava K–theoretic analogues of the results of Lecture 4.5

Ask Mike (and Jacob?) if there are analogues of these results for kO which explain Mahowald's generalized K–theoretic Brown–Gitler spectra. 3/29: I did ask Mike, he said he didn't know. I also asked Paul, and he said this seemed unreasonable, since kO isn't valued in co/commutative Hopf algebras. This is a fair point: one would need to invent an "analogue" of Dieudonné theory for kO, in the sense that some category it takes values in would have to be identified as abelian, where the category is rigid enough that it often sends fiber sequences to exact sequences in the category.

Constructing sheaves of spectra on $\mathcal{M}_{\mathbf{fg}}$: the no-go results for E_∞ and A_∞ rings on the flat site. There's a little MO discussion about it here: `http://chat.stackexchange.co`

Contexts for structured ring spectra

Difficulty in computing $\mathbb{S}_d \circlearrowleft E_d^*$. (Gross–Hopkins and the period map.)

Barry's p–adic measures

Fixed point spectra and e.g. $L_{K(2)} tmf$.

Blueshift, A–M–S, and the relationship to A–F–G?

Does E_n receive an E_∞ orientation? Does BP? (Johnson–Noel says BP usually does not. A recent preprint of Lawson says BP is not even E_∞ at $p = 2$!!!)

p–divisible groups and transchromatic phenomena

Remark 12.13 of published H_∞ AHS says their obstruction framework agrees with the E_∞ obstruction framework (if you take everything in sight to have E_∞ structures). This is almost certainly related to the discussion at the end of Matt's thesis about the MU–orientation of E_d.

Section 12.4 compares doing H_∞ descent with doing E_∞ descent and shows that they're the same (in the case of interest?).

Hovey's paper on v_n–periodic elements in ring spectra. He has a nice (and thorough!) exposition on why one should be interested in bordism spectra and their splittings: for instance, a careful analysis of $MSpin$ will inexorably lead one toward studying KO. It would be nice if studying $MString$ (and potentially higher analogues) would lead one toward non-completed, non-connective versions of EO_n. Talk about BoP, for instance.

Matt's short resolutions of chromatically localized MU.

Nilpotency and vanishing curves in the (MU–)Adams spectral sequence. The *non*-nilpotency of η in the MU–Adams spectral sequence. Mathew–Meier type theorems about horizontal vanishing lines and the Tate construction (and related results about the TMF spectral sequence and the Johnson–Wilson theories).

Additive degeneration and $kO \neq kU^{hC_2}$.

Hopkins's program for p–adic interpolation and Salchian L–functions.

Hopkins's analytic Weierstrass products in the Bockstein spectral sequence:

$$v_2^{(n)} = v_2^{p^n} \prod_{j=0}^{n-2} \left(1 - \left(\frac{v_1^{p+1}}{v_2}\right)^{p^{n-1}-p^j}\right) \pmod{v_1^{p^n+p^{n-1}}}.$$

In general, Mike has formulas like these for the 0–line and for the elements $v_{n+1}h_n$ on the 1–line.

Remark B.0.1 (Yanovski–Schlank). Set S_{small} to be the category generated under finite colimits (including actions by finite groups, say) by π–finite spaces. A *generalized homotopy cardinality function* is a function $\chi\colon \mathsf{S}_{\mathrm{small}} \to \mathbb{Q}$ satisfying

1. Homotopy invariance: if $A \simeq B$, then $\chi(A) = \chi(B)$.

2. Normalization: $\chi(*) = 1$.

3. Additivity: $\chi(A \cup_C B) = \chi(A) + \chi(B) - \chi(C)$.

4. Multiplicativity: $\chi(A \times B) = \chi(A) \cdot \chi(B)$.

For example, $\chi_{n,p}(X) = \dim K^0(X) - \dim K^1(X)$. In the case $n = 0$ or $n = \infty$, this recovers the Euler characteristic, and in the intermediate cases we have computations like $\chi_{n,p}(\underline{H\mathbb{Z}/p}_m) = p^{\binom{n}{m}}$. One can show that the function $n \mapsto \chi_{n,p}(X)$ extends uniquely to a dyadic analytic function $\widehat{L}_{X,p} \colon \mathbb{Z}_2^\wedge \to \mathbb{Z}_2^\wedge$.

We also have the Baez–Dolan function

$$|X| = \sum_{x_0 \in \pi_0 X} \prod_{n=1}^{\infty} |\pi_n(X, x_0)|^{(-1)^n} \in \mathbb{Q}_{\geq 0},$$

and you can convince yourself of things like $|B\mathbb{Z}/p| = 1/p$ and $|M_g| = 2 - 2g$. This is not a global homotopy cardinality function, since $BC_p \wedge BC_q = *$. However, restricted to p–local objects in S_{small}, it appears as $\widehat{L}_{X,p}(-1)$. For instance, this recovers

$$\widehat{L}_{\underline{H\mathbb{Z}/p}_m,p}(-1) = p^{\binom{-1}{m}} = p^{(-1)^m} = |\underline{H\mathbb{Z}/p}_m|.$$

Agnes and everyone else on the chromatic splitting conjecture.

Remark B.0.2. It is completely unclear why MU plays such an important mediating role between geometry (i.e., the stable category) and algebra (i.e., sheaves on the moduli of formal groups). Given a general ring spectrum R and thick prime $\otimes$–ideals C_α of perfect R–modules, one ask the analogous two questions:

1. Is it possible to find an R–algebra S whose context functor induces a homeomorphism of Balmer spectra $\operatorname{Spec}(\mathsf{Modules}_R^{\mathrm{perf}}) \to \operatorname{Spec}(\mathsf{QCoh}(\mathcal{M}_{S/R}))$?

2. Are there complementary localizers $L_\alpha \colon \mathsf{Modules}_R \to \mathsf{Modules}_{R,(\alpha)}$? Can they be presented via Bousfield's framework as homological localizations for auxiliary S–algebra spectra S_α? Do the contexts $\mathcal{M}_{S_\alpha}$ admit compatible localizers with $\mathcal{M}_S$?

For $R = \mathbb{S}$, this is the role that the R–algebra $S = MU$ and the S–algebras $S_d = E(d)$ play. Finding these spectra feels like striking gold, and it is unclear how to produce analogous spectra in general.

Mathews's work on Galois descent shows that the fixed point map $\mathsf{Modules}_{E_\Gamma, \mathrm{Aut}\,\Gamma}^{\mathrm{complete}} \to \mathsf{Spectra}_\Gamma$ is an equivalence of categories.

One can ask the same question from the geometric direction: why bordism? Why should these spectra have these nice flatness properties? Why should they have recognizable computational properties? Why bordism?

Remark B.0.3. The homotopy of $\widehat{L}_2\mathbb{S}$ is also known, by work of Shimomura and collaborators [Shi86, SY94, SY95] (but see also the reorganization by Behrens [Beh12]). It is *exceedingly* complicated, and it is an open problem to find an expression of it which admits human digestion. Behrens has pursued a program encoding this problem in terms of modular forms [Beh09, Beh06, Beh07], and Hopkins has proposed a program involving L–functions [Str92], motivated by which Hovey and Strickland have shown a kind of continuity result for among the groups [HS99, Section 14].

Remark B.0.4. There are also "finitary" flavors of chromatic localization available, which are typically less robust but more computable. They assemble into a diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & L_d^{\mathrm{fin}} E \arrow[r] \arrow[d] & L_d E \arrow[d] \\ L_{X(d)} E \arrow[r] & \widehat{L}_d^{\mathrm{fin}} E \arrow[r] & \widehat{L}_d E, \end{tikzcd}$$

where $X(d)$ is a finite complex of type exactly d, v is a v_d–self-map of $X(d)$, $T(d) = X(d)[v^{-1}]$ is the localizing telescope, $\widehat{L}_d^{\mathrm{fin}}$ is Bousfield localization with respect to $T(d)$ (which can be shown to be independent of choice of $X(d)$ and of v), and L_d^{fin} denotes localization with respect to the class of *finite* $E(d)$–acyclics. Much is known about these functors: for instance, $L_{X(d)} L_d = \widehat{L}_d$, there is a chromatic fracture square relating L_d^{fin} to $\widehat{L}_{\le d}^{\mathrm{fin}}$, and $L_d^{\mathrm{fin}} E \simeq L_d E$ if and only if $\widehat{L}_{\le d}^{\mathrm{fin}} E \simeq \widehat{L}_{\le d} E$. One major question about these functors remains open, corresponding the last unsettled nilpotence and periodicity conjecture of Ravenel [Rav84, Conjecture 10.5]: is the map $\widehat{L}_d^{\mathrm{fin}} E \to \widehat{L}_d E$ an equivalence? Multiple proofs and disproofs have been offered, but the literature remains unsettled.

Cite me: Find some proofs and disproofs..

Remark B.0.5. Writing M_d for the fiber in the sequence $M_d \to L_d \to L_{d-1}$, the filtration spectral sequence associated to the tower in Theorem 3.6.14 is called the *geometric chromatic spectral sequence*, which has the form $\pi_* M_* \mathbb{S} \Rightarrow \pi_* \mathbb{S}_{(p)}$. The two forms of filtration data $M_d X$ and $\widehat{L}_d X$ are actually functorially equivalent to one another:

$$\widehat{L}_d M_d \simeq \widehat{L}_d, \qquad\qquad M_d \widehat{L}_d \simeq M_d,$$

but they have fairly distinct properties. For instance, M_d is smashing whereas $\widehat{L}_d$ is not, M_d is not part of an adjoint pair whereas $\widehat{L}_d$ is, and the analogue of Lemma 3.6.8 for M_d is "backwards":

Cite me: I forget who this is due to.

$$M_d X \simeq \operatorname{colim}_I \left(M^0(v^I) \wedge L_d X \right).$$

The spectrum $M_d X$ also relates to the chromatic fracture square for X:

$$\begin{tikzcd}
M_d X \arrow[r, equal] \arrow[d] & M_d X \arrow[d] \\
L_d X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{L}_d X \arrow[d] \\
L_{d-1} X \arrow[r] & L_{d-1} \widehat{L}_d X.
\end{tikzcd}$$

From this, we see that there is a fiber sequence $M_d X \to \widehat{L}_d X \to L_{d-1} \widehat{L}_d X$.

The case $d = 1$ gives the prototypical example of the difference between these two presentations of the "exact height d data", where the sequence becomes:

$$\operatorname{colim}_j (M^0(p^j) \wedge L_1 X) \to \lim_j (M_0(p^j) \wedge L_1 X) \to \left(\lim_j (M_0(p^j) \wedge L_1 X) \right)_{\mathbb{Q}}.$$

If, for instance, $\pi_0 L_1 X = \mathbb{Z}_{(p)}$, then the long exact sequence of homotopy groups associated to this fiber sequence gives

$$\begin{tikzcd}
\pi_0 \widehat{L}_1 X \arrow[r] \arrow[d, equal] & \pi_0 L_0 \widehat{L}_1 X \arrow[r] \arrow[d, equal] & \pi_{-1} M_1 X \arrow[d, equal] \\
\mathbb{Z}_p^\wedge \arrow[r] & \mathbb{Q}_p \arrow[r] & \mathbb{Z}/p^\infty.
\end{tikzcd}$$

Coupling this to Example 3.6.19, we compute

$$\pi_t M_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}/p^\infty & \text{when } t = -1, \\ \mathbb{Z}_p/(pk) & \text{when } t = k|v_1| - 1 \text{ and } t =\neq 0, \\ \mathbb{Z}/p^\infty & \text{when } t = (0 \cdot |v_1| - 1) - 1 = -2, \\ 0 & \text{otherwise.} \end{cases}$$

This is a model for what happens generally when passing from $\pi_* \widehat{L}_d X$ to $\pi_* M_d X$: the v_j–torsion–free groups get converted to infinitely v_j–divisible groups, with some dimension shifts.[1]

Theorem B.0.6 (Unpublished work of Hopkins and Lurie)**.** *Let F_d denote the discrepancy spectrum for E_d. There is a natural equivalence of infinite loopspaces $\Omega^\infty F_d \simeq \Sigma^d \mathbb{I}_{\mathbb{Q}/\mathbb{Z}}$.* □

free E_∞–orientations off of MU

At the top fo page 10 Neil talks in FPFP very briefly about Greenlees–May and local homology.

Comparison of comodules M for the isogenies pile with the action of $M_n(\mathbb{Z}_p)$ on $M \otimes_{E_n^*} D_\infty$ (this is a modern result due to Tomer, Tobi, Lukas, and Nat). This is basically Nat's rational claim: start with a sheaf on the isogenies pile. Tensor everything with $\mathbb{Q}$. That turns this thing into a rational algebra under the Drinfel'd ring together with an equivariant action of $GL_n \mathbb{Q}_p$.

Leave a remark in here about this: McClure in BMMS works along similar lines to show that the Quillen idempotent is not H_∞, but he doesn't get any positive results (and, in particular, he can't complete his analysis as we do because he doesn't have access to the BP–homology of finite groups and to HKR character theory). One wonders whether the stuff here does say something about BP as the height tends toward ∞. So far as I know, no one has written much about this. Surely it remains a bee in Matt's bonnet.

Sections 5.3-4 of Hopkins's ICM address *Algebraic Topology and Modular Forms* has a discussion of what η and ν have to do with tmf, as well as the construction of some interesting "topological θ–series" in the elliptic cohomology of certain Thom complexes.

Jacob wrote me an email giving a very slightly fuller sketch of what the DAG perspective on the σ–orientation is. Interestingly, it boils down to a fact from projective geometry: there just aren't that many line bundles on projective varieties. This forces a couple of things to become equal, and in a suitable setting they even become canonically equal. The email has no subject line, which will make it hard to find, but you should include a summary of it (which is dependent on whatever's written in the *Survey* paper) all the same.

[1] A height 2 example of this same phenomenon is visible in Behrens's paper [Beh12, Section 7].

Bibliography

[Ada66] J. F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1966. 3.5.10, 47, 3.6.21

[Ada78] John Frank Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978. (document), 2.4.6

[Ada95] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original. 1.5.2, 2.5.4, 2.6.8, 2.6.9

[AFG08] Matthew Ando, Christopher P. French, and Nora Ganter. The Jacobi orientation and the two-variable elliptic genus. *Algebr. Geom. Topol.*, 8(1):493–539, 2008. 5.5.30

[AHR] Matthew Ando, Michael J. Hopkins, and Charles Rezk. Multiplicative orientations of *ko*–theory and the spectrum of topological modular forms. `http://www.math.illinois.edu/~mando/papers/koandtmf.pdf`, Accessed: 2016-11-16. 20, A.4.5, A.4.11, 34

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. *Invent. Math.*, 146(3):595–687, 2001. 0.0.5, 5.2.1, 5.2.2, 5.2.4, 5.2.10, 5.2.15, 5.2.16, 5.2.18, 5.3.5, 5.4.6, 5.4.7, 5.4.8, 5.4.9, 5.4.10, 5.4.11, 5.4.12, 5.4.15, 5.5.21, 5.5.22, 5.5.23, 5.5.24, 5.5

[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland. The sigma orientation is an H_∞ map. *Amer. J. Math.*, 126(2):247–334, 2004. 2.3.4, 2.4.5, A.2.25, A.2.30, 10, A.2.39

[AM01] Matthew Ando and Jack Morava. A renormalized Riemann-Roch formula and the Thom isomorphism for the free loop space. In *Topology, geometry, and algebra: interactions and new directions (Stanford, CA, 1999)*, volume 279 of *Contemp. Math.*, pages 11–36. Amer. Math. Soc., Providence, RI, 2001. 5.5.29

[Ami59] S. A. Amitsur. Simple algebras and cohomology groups of arbitrary fields. *Trans. Amer. Math. Soc.*, 90:73–112, 1959. 3.1.2

[And95] Matthew Ando. Isogenies of formal group laws and power operations in the cohomology theories E_n. *Duke Math. J.*, 79(2):423–485, 1995. 10, A.2.37, 12

[AP76] J. F. Adams and S. B. Priddy. Uniqueness of BSO. *Math. Proc. Cambridge Philos. Soc.*, 80(3):475–509, 1976. 27

[Ara73] Shôrô Araki. *Typical formal groups in complex cobordism and K-theory*. Kinokuniya Book-Store Co., Ltd., Tokyo, 1973. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 6. 3.3.7, 3.3.10

[AS01] M. Ando and N. P. Strickland. Weil pairings and Morava K-theory. *Topology*, 40(1):127–156, 2001. 5.2.2, 5.2.11, 5.2.17, 5.3.5, 5.3.6, 5.3, 5.3.7, 5.3.9, 5.3.10, 5.3.11, 5.3.12, 5.3.14

[Bak15] Andrew Baker. Power operations and coactions in highly commutative homology theories. *Publ. Res. Inst. Math. Sci.*, 51(2):237–272, 2015. 5

[Bal10] Paul Balmer. Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010. 3.6.1, 3.6.2, 3.6.3, 3.6

[Bar] Tobias Barthel. Chromatic completion. `http://people.mpim-bonn.mpg.de/tbarthel/CC.pdf`, Accessed: 2016-10-26. 40

[Bau14] Tilman Bauer. Formal plethories. *Adv. Math.*, 254:497–569, 2014. 4.1.21

[BCG+88] A. Baker, D. Carlisle, B. Gray, S. Hilditch, N. Ray, and R. Wood. On the iterated complex transfer. *Math. Z.*, 199(2):191–207, 1988. 30, A.4.18

[BCM78] M. Bendersky, E. B. Curtis, and H. R. Miller. The unstable Adams spectral sequence for generalized homology. *Topology*, 17(3):229–248, 1978. 4.1.2, 4.1.5, 4.1.17, 4.1.18, 4.1.19

[BD] A. A. Beilinson and V. Drinfeld. *Quantization of Hitchin's integrable system and Hecke eigensheaves*. Preprint available online at http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf. 4

[Beh06] Mark Behrens. A modular description of the $K(2)$-local sphere at the prime 3. *Topology*, 45(2):343–402, 2006. B.0.3

[Beh07] Mark Behrens. Buildings, elliptic curves, and the $K(2)$-local sphere. *Amer. J. Math.*, 129(6):1513–1563, 2007. B.0.3

[Beh09] Mark Behrens. Congruences between modular forms given by the divided β family in homotopy theory. *Geom. Topol.*, 13(1):319–357, 2009. B.0.3

[Beh12] Mark Behrens. The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited. *Adv. Math.*, 230(2):458–492, 2012. B.0.3, 1

[BJW95] J. Michael Boardman, David Copeland Johnson, and W. Stephen Wilson. Unstable operations in generalized cohomology. In *Handbook of algebraic topology*, pages 687–828. North-Holland, Amsterdam, 1995. 4.1.3, 4.1

[BL07] Victor Buchstaber and Andrey Lazarev. Dieudonné modules and p-divisible groups associated with Morava K-theory of Eilenberg-Mac Lane spaces. *Algebr. Geom. Topol.*, 7:529–564, 2007. 4.5

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. 2.4, 2.4.3, 2.4.5, 2.4.16, 3.5.9, A.0.2, A.2.28, A.2.30, 10

[Bou75] A. K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14:133–150, 1975. A.2.1

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979. 3.1.10, 3.6.5

[Bou96] A. K. Bousfield. On λ-rings and the K-theory of infinite loop spaces. *K-Theory*, 10(1):1–30, 1996. A.2.1, B

[Bre83] Lawrence Breen. *Fonctions thêta et théorème du cube*, volume 980 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983. 5.5.11, 5.6.4

[BT00] Jean-Yves Butowiez and Paul Turner. Unstable multiplicative cohomology operations. *Q. J. Math.*, 51(4):437–449, 2000. 1

[CCO14] Ching-Li Chai, Brian Conrad, and Frans Oort. *Complex multiplication and lifting problems*, volume 195 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014. A.2.27

[Cha82] Ken Chan. A simple proof that the unstable (co)-homology of the Brown-Peterson spectrum is torsion free. *J. Pure Appl. Algebra*, 26(2):155–157, 1982. 4.3.9

[Dav95] Donald M. Davis. Computing v_1-periodic homotopy groups of spheres and some compact Lie groups. In *Handbook of algebraic topology*, pages 993–1048. North-Holland, Amsterdam, 1995. A.2.1

[Del81] P. Deligne. Cristaux ordinaires et coordonnées canoniques. In *Algebraic surfaces (Orsay, 1976–78)*, volume 868 of *Lecture Notes in Math.*, pages 80–137. Springer, Berlin-New York, 1981. With the collaboration of L. Illusie, With an appendix by Nicholas M. Katz. A.3.6

[Dem86] Michel Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1972 original. 4.4.8, 5.1.3, 5.1.15

[DH95] Ethan S. Devinatz and Michael J. Hopkins. The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts. *Amer. J. Math.*, 117(3):669–710, 1995. 3.6.17, 4.4.23

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988. 3.5.11, 3.5, 3.5.12

[DL14] Daniel G. Davis and Tyler Lawson. Commutative ring objects in pro-categories and generalized Moore spectra. *Geom. Topol.*, 18(1):103–140, 2014. 36

[Dri74] V. G. Drinfel′d. Elliptic modules. *Mat. Sb. (N.S.)*, 94(136):594–627, 656, 1974. 26

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. 3.1.5, A, A.4

[GH04] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004. A.0.1, A.3.3, A.3.8

[GLM93] P. Goerss, J. Lannes, and F. Morel. Hopf algebras, Witt vectors, and Brown-Gitler spectra. In *Algebraic topology (Oaxtepec, 1991)*, volume 146 of *Contemp. Math.*, pages 111–128. Amer. Math. Soc., Providence, RI, 1993. 4.5.1, 4.5.3, 4.5.5, 4.5.6, 4.5.7, 4.5.8, 4.5.9, 4.5.10, 4.5.11

[Goe99] Paul G. Goerss. Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 115–174. Amer. Math. Soc., Providence, RI, 1999. 4.5, 4.5.13, 4.5.14, 4.5.16, 4.5.17, 4.5.18, 4.5.19

[Goe08] Paul G. Goerss. Quasi-coherent sheaves on the moduli stack of formal groups. 2008. 1.3.4, 3.3.14

[Goe10] Paul G. Goerss. Topological modular forms [after Hopkins, Miller and Lurie]. *Astérisque*, (332):Exp. No. 1005, viii, 221–255, 2010. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. A.3.2

[Gro74] Alexandre Grothendieck. *Groupes de Barsotti-Tate et cristaux de Dieudonné*. Les Presses de l'Université de Montréal, Montreal, Que., 1974. Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970). 12, 4.4.8, 4.4.22, 4.6.5, 4.6.7

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. 4, 2.2.8, 5.5.10

[Har80] Bruno Harris. Bott periodicity via simplicial spaces. *J. Algebra*, 62(2):450–454, 1980. 13

[HAS99] Michael Hopkins, Matthew Ando, and Neil Strickland. Elliptic cohomology of $BO\langle 8\rangle$ in positive charactersitic. 1999. Unpublished preprint. 5.6

[Haz12] Michiel Hazewinkel. *Formal groups and applications*. AMS Chelsea Publishing, Providence, RI, 2012. Corrected reprint of the 1978 original. 8, 3.3.5, 3.3.6, 3.3.21

[Hed] S. Mohammad Hadi Hedayatzadeh. Exterior powers of Lubin–Tate groups. `http://hedayatzadeh.info/Home_files/EPLTG.pdf`, Accessed: 2015-01-09. 4.6.13

[Hed14] S. Mohammad Hadi Hedayatzadeh. Exterior powers of π-divisible modules over fields. *J. Number Theory*, 138:119–174, 2014. 4.6.13

[HG94a] M. J. Hopkins and B. H. Gross. Equivariant vector bundles on the Lubin-Tate moduli space. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 23–88. Amer. Math. Soc., Providence, RI, 1994. 4.4.23

[HG94b] M. J. Hopkins and B. H. Gross. The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):76–86, 1994. 4.4.23

[HH95] M. J. Hopkins and J. R. Hunton. On the structure of spaces representing a Landweber exact cohomology theory. *Topology*, 34(1):29–36, 1995. 4.5.20

[Hir95] Friedrich Hirzebruch. *Topological methods in algebraic geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition. 5.6.6

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594 (electronic), 2000. 2.4.6

[HL] Michael J. Hopkins and Jacob Lurie. Ambidexterity in $K(n)$-local stable homotopy theory. `http://www.math.harvard.edu/~lurie/papers/Ambidexterity.pdf`, Accessed: 2015-01-09. 2.6.1, 4.6, 4.6.1, 4.6.8, 4.6.9, 4.6.10, 4.6, 4.6.13, 5.1.3

[HMS94] Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky. Constructions of elements in Picard groups. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 89–126. Amer. Math. Soc., Providence, RI, 1994. 3.6.19

[Hop] Michael J. Hopkins. Complex oriented cohomology theories and the language of stacks. *Unpublished*. 1.5.2, 3.1.17, 3.2.5, 3.3.11, 3.5.1, 3.5.3

[Hop95] Michael J. Hopkins. Topological modular forms, the Witten genus, and the theorem of the cube. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 554–565. Birkhäuser, Basel, 1995. 5.6.18

[Hop08] Michael J. Hopkins. The mathematical work of Douglas C. Ravenel. *Homology Homotopy Appl.*, 10(3):1–13, 2008. (document)

[Hov95] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 225–250. Amer. Math. Soc., Providence, RI, 1995. 3.6.8

[Hov02] Mark Hovey. Morita theory for Hopf algebroids and presheaves of groupoids. *Amer. J. Math.*, 124(6):1289–1318, 2002. 1.4.8, 3.1.3, 3.1.17

[Hov04] Mark Hovey. Homotopy theory of comodules over a Hopf algebroid. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 261–304. Amer. Math. Soc., Providence, RI, 2004. 9

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998. 3.5.11, 3.5, 3.5.13, 3.5.20, 3.5, 3.5.21

[HS99] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999. B.0.3

[HT98] John R. Hunton and Paul R. Turner. Coalgebraic algebra. *J. Pure Appl. Algebra*, 129(3):297–313, 1998. 4.5

[Hus04] Dale Husemöller. *Elliptic curves*, volume 111 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2004. With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen. 5.5.3, 5.5.4

[JW75] David Copeland Johnson and W. Stephen Wilson. *BP* operations and Morava's extraordinary K-theories. *Math. Z.*, 144(1):55–75, 1975. 31

[Kat81] Nicholas M. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Inst. Fundamental Res., Bombay, 1981. 4.4.2, 4.4.3, 4.4.5, 4.4.6, 12, 4.4.14, 4.4.20

[KL02] Nitu Kitchloo and Gerd Laures. Real structures and Morava K-theories. *K-Theory*, 25(3):201–214, 2002. 15

[KLW04] Nitu Kitchloo, Gerd Laures, and W. Stephen Wilson. The Morava K-theory of spaces related to BO. *Adv. Math.*, 189(1):192–236, 2004. 3.5.7, 5.6.19

[Koc78] Stanley O. Kochman. A chain functor for bordism. *Trans. Amer. Math. Soc.*, 239:167–196, 1978. 0.0.1

[Kří94] Igor Kříž. All complex Thom spectra are harmonic. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 127–134. Amer. Math. Soc., Providence, RI, 1994. 40

[Kuh08] Nicholas J. Kuhn. A guide to telescopic functors. *Homology Homotopy Appl.*, 10(3):291–319, 2008. A.4.6

[Lan75] Peter S. Landweber. Invariant regular ideals in Brown-Peterson homology. *Duke Math. J.*, 42(3):499–505, 1975. 3.3.16

[Lan76] Peter S. Landweber. Homological properties of comodules over $MU_*(MU)$ and $\mathrm{BP}_*(\mathrm{BP})$. *Amer. J. Math.*, 98(3):591–610, 1976. 3.5.1

[Law09] Tyler Lawson. An overview of abelian varieties in homotopy theory. In *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 179–214. Geom. Topol. Publ., Coventry, 2009. A.3.22

[Law10] Tyler Lawson. Structured ring spectra and displays. *Geom. Topol.*, 14(2):1111–1127, 2010. 4.4.24

[Laz55] Michel Lazard. Sur les groupes de Lie formels à un paramètre. *Bull. Soc. Math. France*, 83:251–274, 1955. 3.2.2, 3.2.5, 3.4.1

[Laz97] A. Lazarev. Deformations of formal groups and stable homotopy theory. *Topology*, 36(6):1317–1331, 1997. 3.4.9, 3.4.10, 3.4.11, 4.6

[LN12] Tyler Lawson and Niko Naumann. Commutativity conditions for truncated Brown-Peterson spectra of height 2. *J. Topol.*, 5(1):137–168, 2012. 4.3.10

[LT66] Jonathan Lubin and John Tate. Formal moduli for one-parameter formal Lie groups. *Bull. Soc. Math. France*, 94:49–59, 1966. 3.4.5, 26, 3.4

[Lura] Jacob Lurie. *Chromatic Homotopy*. `http://www.math.harvard.edu/~lurie/252x.html`, Accessed: 2016-09-22. 2.4, 3.2.7, 3.3.3, 3.3.18, 3.4.13, 3.6.10, 3.6.12

[Lurb] Jacob Lurie. *Higher Algebra*. `http://www.math.harvard.edu/~lurie/papers/HA.pdf`, Accessed: 2016-09-22. 2.4.2, 2.4.8, 3.1.8, 4.1, A

[Lurc] Jacob Lurie. *Spectral Algebraic Geometry*. `http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`, Accessed: 2016-09-22. 3.1.10

[Lurd] Jacob Lurie. A survey of elliptic cohomology. `http://www.math.harvard.edu/~lurie/papers/survey.pdf`, Accessed: 2017-03-04. 8, A.3.3

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. A.3.1

[Mah79] Mark Mahowald. Ring spectra which are Thom complexes. *Duke Math. J.*, 46(3):549–559, 1979. 1.1.10

[Mar83] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category. 3.6.5

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid. 3.5.4

[May77] J. Peter May. *E_∞ ring spaces and E_∞ ring spectra*. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. 1.1.15, A.0.1

[May96] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. 2.4.4

[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. 5.4

[Mes72] William Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin-New York, 1972. 12

[Mic03] Walter Michaelis. Coassociative coalgebras. In *Handbook of algebra, Vol. 3*, pages 587–788. North-Holland, Amsterdam, 2003. 5.1.3, 5.1

[Mila] Haynes Miller. *Notes on Cobordism*. `www-math.mit.edu/~hrm/papers/cobordism.pdf`, Accessed: 2016-10-04. 3.3.11

[Milb] Haynes Miller. Sheaves, gradings, and the exact functor theorem. *Unpublished*. 4

[Mil58] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958. 1.3.3

[Mil82] Haynes Miller. Universal Bernoulli numbers and the S^1-transfer. In *Current trends in algebraic topology, Part 2 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 437–449. Amer. Math. Soc., Providence, RI, 1982. 30

[Mil86] J. S. Milne. Abelian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York, 1986. 5.5.7

[Mit83] Stephen A. Mitchell. Power series methods in unoriented cobordism. In *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of *Contemp. Math.*, pages 247–254. Amer. Math. Soc., Providence, RI, 1983. 1.5.6

[MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965. 1.4.21, 5.4

[MM74] B. Mazur and William Messing. *Universal extensions and one dimensional crystalline cohomology*. Lecture Notes in Mathematics, Vol. 370. Springer-Verlag, Berlin-New York, 1974. 4.4.20

[Mor85] Jack Morava. Noetherian localisations of categories of cobordism comodules. *Ann. of Math. (2)*, 121(1):1–39, 1985. 4.4.23

[Mor89] Jack Morava. Forms of K-theory. *Math. Z.*, 201(3):401–428, 1989. 5.5.20, 5.5.23

[Mor07a] Jack Morava. Complex cobordism in algebraic topology. 2007. 34

[Mor07b] Jack Morava. The motivic Thom isomorphism. In *Elliptic cohomology*, volume 342 of *London Math. Soc. Lecture Note Ser.*, pages 265–285. Cambridge Univ. Press, Cambridge, 2007. 14

[MRW77] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977. 3.6.21

[MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, Publishers, New York-London, 1968. 1.3.3, 1.4.2

[MT92] Mark Mahowald and Robert D. Thompson. The K-theory localization of an unstable sphere. *Topology*, 31(1):133–141, 1992. A.2.1

[Mum66] D. Mumford. On the equations defining abelian varieties. I. *Invent. Math.*, 1:287–354, 1966. 5.5.10

[Mum67a] D. Mumford. On the equations defining abelian varieties. II. *Invent. Math.*, 3:75–135, 1967. 5.5.10

[Mum67b] D. Mumford. On the equations defining abelian varieties. III. *Invent. Math.*, 3:215–244, 1967. 5.5.10

[Nav10] Lee S. Nave. The Smith-Toda complex $V((p+1)/2)$ does not exist. *Ann. of Math. (2)*, 171(1):491–509, 2010. 29

[Nis73] Goro Nishida. The nilpotency of elements of the stable homotopy groups of spheres. *J. Math. Soc. Japan*, 25:707–732, 1973. 3.5.9

[Och87] Serge Ochanine. Sur les genres multiplicatifs définis par des intégrales elliptiques. *Topology*, 26(2):143–151, 1987. 0.0.3

[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969. 2.4

[Qui71] Daniel Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Math.*, 7:29–56 (1971), 1971. 2.4.14, 2.4.14, 2.5.10, 2.6.3, 2.6.4, 2.6.6

[Rav78] Douglas C. Ravenel. A novice's guide to the Adams-Novikov spectral sequence. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 404–475. Springer, Berlin, 1978. 1.2, 3.1

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984. 3.1.10, 3.5.7, 3.5, 31, 3.5.19, B.0.4

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. 1.4.2, 1.4.4, 1.4.5, 1.4.6, 1.4.15, 3.1, 3.2, 3.3.11

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith. 3.6.8, 3.6.14

[Reza] Charles Rezk. Elliptic cohomology and elliptic curves (Felix Klein lectures, bonn 2015). `http://www.math.uiuc.edu/~rezk/felix-klein-lectures-notes.pdf`, Accessed: 2016-10-07. 3.4.3, 5.5.28

[Rezb] Charles Rezk. Notes on the Hopkins–Miller theorem. `www.math.uiuc.edu/~rezk/hopkins-miller-thm.pdf`, Accessed: 2016-10-07. 3.4.4

[Rezc] Charles Rezk. Supplementary notes for math 512. `www.math.uiuc.edu/~rezk/512-spr2001-notes.pdf`, Accessed: 2016-10-28. 3.6.20, 3.6.21

[Rez06] Charles Rezk. The units of a ring spectrum and a logarithmic cohomology operation. *J. Amer. Math. Soc.*, 19(4):969–1014, 2006. A.4.7, A.4.9, 26

[Rud98] Yuli B. Rudyak. *On Thom spectra, orientability, and cobordism*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller. 2.4.12, 2.4.13, 2.5.9, 2.5.10, 2.6.1

[RW80] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980. 2.6.1, 4.1.7, 4.1.9, 4.1, 4.2.2, 4.2.3, 4.6, 4.6.1, 17, 19, 4.6.15

[RW77] Douglas C. Ravenel and W. Stephen Wilson. The Hopf ring for complex cobordism. *J. Pure Appl. Algebra*, 9(3):241–280, 1976/77. 4.2.5, 4.2.8, 4.3.1, 4.3, 4.3.3, 4.3.4, 4.3.6, 4.3.7, 4.3.9

[RWY98] Douglas C. Ravenel, W. Stephen Wilson, and Nobuaki Yagita. Brown-Peterson cohomology from Morava K-theory. *K-Theory*, 15(2):147–199, 1998. 4.6.15

[Sch70] Colette Schoeller. Étude de la catégorie des algèbres de Hopf commutatives connexes sur un corps. *Manuscripta Math.*, 3:133–155, 1970. 4.5.1, 4.5.5

[Seg70] Graeme Segal. Cohomology of topological groups. In *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, pages 377–387. Academic Press, London, 1970. 4.1.7

[Sha77] P. Brian Shay. mod p Wu formulas for the Steenrod algebra and the Dyer-Lashof algebra. *Proc. Amer. Math. Soc.*, 63(2):339–347, 1977. 5.4.3

[Shi86] Katsumi Shimomura. On the Adams-Novikov spectral sequence and products of β-elements. *Hiroshima Math. J.*, 16(1):209–224, 1986. B.0.3

[Sin68] William M. Singer. Connective fiberings over BU and U. *Topology*, 7:271–303, 1968. 0.0.5, 5.4.2

[SN14] Johannes Sprang and Niko Naumann. p–adic interpolation and multiplicative orientations of ko and tmf. 2014. A.4.17

[Sta14] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2014. 3.1.3

[Sto63] Robert E. Stong. Determination of $H^*(\mathrm{BO}(k,\cdots,\infty),Z_2)$ and $H^*(\mathrm{BU}(k,\cdots,\infty),Z_2)$. *Trans. Amer. Math. Soc.*, 107:526–544, 1963. 0.0.5, 5.4.2

[Sto12] Vesna Stojanoska. Duality for topological modular forms. *Doc. Math.*, 17:271–311, 2012. A.3.24

[Stra] Neil Strickland. Formal schemes for K–theory spaces. Unpublished preprint. 5.6, 5.6.19

[Strb] Neil P. Strickland. Functorial philosophy for formal phenomena. Unpublished. `http://hopf.math.purdue.edu/Strickland/fpfp.pdf`, Accessed: 2014-01-09. 0, 6, 1, 2.2.14, 3.6.17, 13, 4.4.21, 4.4.23, A.2.8

[Str92] N. P. Strickland. On the p-adic interpolation of stable homotopy groups. In *Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990)*, volume 176 of *London Math. Soc. Lecture Note Ser.*, pages 45–54. Cambridge Univ. Press, Cambridge, 1992. B.0.3

[Str97] Neil P. Strickland. Finite subgroups of formal groups. *J. Pure Appl. Algebra*, 121(2):161–208, 1997. 3.4.3, A.2.10, A.2.19, A.2.22, A.2.24

[Str98] N. P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998. A.2.7, A.2.10

[Str99a] N. P. Strickland. Products on MU-modules. *Trans. Amer. Math. Soc.*, 351(7):2569–2606, 1999. 3.5.7, 4.3.10

[Str99b] Neil P. Strickland. Formal schemes and formal groups. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 263–352. Amer. Math. Soc., Providence, RI, 1999. 4, 1.3.1, 1.3.2, 1.4.8, 1.4.9, 2.1, 3, 4, 6, 2.2.1, 2.2.2, 2.2.11, 10, 2.2.14, 13, 4.2.4, 5.1.6, 5.1.8, 5.1.9, 5.1.10, 5.1.13, 5.1.14, A.2.13, A.2.15

[Str06] Neil P. Strickland. Formal groups. `https://neil-strickland.staff.shef.ac.uk/courses/formalgroups/fg.pdf`, 2006. 1.5.6, 1.5.7, 3.3.18, 24, 3.4.2

[Swi02] Robert M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)]. 1.5.1, 2.3.11, 2.3

[SY94] Katsumi Shimomura and Atsuko Yabe. On the chromatic E_1-term $H^*M_0^2$. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 217–228. Amer. Math. Soc., Providence, RI, 1994. B.0.3

[SY95] Katsumi Shimomura and Atsuko Yabe. The homotopy groups $\pi_*(L_2S^0)$. *Topology*, 34(2):261–289, 1995. B.0.3

[TW80] Robert W. Thomason and W. Stephen Wilson. Hopf rings in the bar spectral sequence. *Quart. J. Math. Oxford Ser. (2)*, 31(124):507–511, 1980. 4.1.9

[Vak15] Ravi Vakil. *The rising sea: Foundations of algebraic geometry*. 2015. Preprint available at http://math.stanford.edu/~vakil/216blog/FOAGdec2915public.pdf. 2.2.9, 2.2.10

[Vis05] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005. 4

[Wei11] Jared Weinstein. The geometry of Lubin–Tate spaces. `http://math.bu.edu/people/jsweinst/FRGLecture.pdf`, 2011. 4.4

[Wil73] W. Stephen Wilson. The Ω-spectrum for Brown-Peterson cohomology. I. *Comment. Math. Helv.*, 48:45–55; corrigendum, ibid. 48 (1973), 194, 1973. 4.3.10

[Wil75] W. Stephen Wilson. The Ω-spectrum for Brown-Peterson cohomology. II. *Amer. J. Math.*, 97:101–123, 1975. 4.3.10

[Wil82] W. Stephen Wilson. *Brown-Peterson homology: an introduction and sampler*, volume 48 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1982. 3.3.15, 3.3.16, 4.1, 4.1.13, 4.1.15, 4.2.8, 4.3.3, 4.3.9

[Wit87] Edward Witten. Elliptic genera and quantum field theory. *Comm. Math. Phys.*, 109(4):525–536, 1987. 0.0.3

[Wit88] Edward Witten. The index of the Dirac operator in loop space. In *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, volume 1326 of *Lecture Notes in Math.*, pages 161–181. Springer, Berlin, 1988. 0.0.3

[Yag80] Nobuaki Yagita. On the Steenrod algebra of Morava K-theory. *J. London Math. Soc. (2)*, 22(3):423–438, 1980. 5.6.19

[Yu95] Jiu-Kang Yu. On the moduli of quasi-canonical liftings. *Compositio Math.*, 96(3):293–321, 1995. 4.4.23

[Zin84] Thomas Zink. *Cartiertheorie kommutativer formaler Gruppen*, volume 68 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984. With English, French and Russian summaries. 3.4.6, 4.4.16, 4.4.17, 4.4.18

[Zin02] Thomas Zink. The display of a formal p-divisible group. *Astérisque*, (278):127–248, 2002. Cohomologies p-adiques et applications arithmétiques, I. 4.4.24

Index

Number of to-dos used: 104

Material for lecture

Mike's 1995 announcement is a nice read. The end of section 3, with $\tau \mapsto 1/\tau$, is mysterious.

Akhil Mathew has notes from an algebraic geometry class (https://math.berkeley.edu/~amathew/23
) where lectures 3–5 address the theorem of the cube.

The Hattori–Stong theorem states that $MU_* \to K_*MU$ has image a direct summand. Maybe this also deserves mention. (Baker claims in *Combinatorial and Arithmetic Identities Based on Formal Group Laws* that it has a direct algebraic proof given by Araki in *Typical formal groups in complex cobordism and K-theory*.)

+ Warning: noncontinuous maps of high-dimensional formal affine spaces. + Description of the Lubin–Tate tower and the local Langlands correspondence + Uniqueness of $\mathcal{O}_K$-module structure in characteristic zero

Ideas

1. Matt's calculation of E_∞–orientations of $K(1)$–local spectra using the short free resolution of MU in the $K(1)$–local category

2. Sinkinson's calculation and $MBP\langle m\rangle$–orientations

Akhil wrote a couple of blog posts about Ochanine's theorem: `https://amathew.wordpress.co`
and `https://amathew.wordpress.com/2012/05/31/the-other-direction-of-ochaines`
Mentioning a more precise result might lend to a more beefy introduction.