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TOPOLOGY FROM AN ALGEBRAIC VIEWPOINT

DECEMBER 8, 2019

THIS DOCUMENT IS an elaboration on a set of lecture notes delivered at Harvard in the Spring 2017 term (Math 231b), covering various aspects of homotopy theory under the standing assumption that the reader has had some prior exposure to ordinary co/homology.[1] It serves as a companion to the two main sources from which the material was drawn: Switzer's *Algebraic Topology: Homotopy and Homology*[2] and Mosher and Tangora's *Cohomology Operations and Applications in Homotopy Theory*.[3] The course is meant to highlight four items, presented in turn (though, of course, there is plenty of interplay):

[1] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002

[2] Robert M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)]

[3] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, Publishers, New York-London, 1968

1. We take a particularly abstract and algebraic perspective on the homotopy theory of spaces. Rather than dealing with spaces as geometric objects, our motif is to design our model of homotopy theory so that the algebraic structures encountered in the reader's prior exposure to algebraic topology become inherently available. This pushes us to explore methods by which we can *decompose* homotopy types, as well as how to use that to our advantage, since we cannot rely on geometry to do the work for us.

2. With such a model of homotopy types in hand, we explore how familiar invariants (e.g., homotopy, ordinary homology) come about in this particular framework, as well as what properties they enjoy.

3. We use these properties, together with categorical existence theorems, to construct new such invariants that enjoy a similar flavor and which are computable by similar means. In particular, this leads to the stable homotopy theory of spectra.

4. We use these tools to effect computations, entirely algebraic in origin, but with geometric interpretations.

This final goal is our true goal—we are not out to set up theory for its own sake, but in order to compute quantities already of interest and to motivate interest in entirely new quantities. Because of this centrality, we highlight a particular computation to come and its flavor when presented in this framework. We begin with the n–sphere S^n.[4] Its homotopy groups are notoriously difficult and important to compute; in fact, we will show during our exploration of (1) that their nontriviality is essentially why homotopy theory itself is nontrivial. In spite of—or because of—their difficulty, we would like to be able to compute as much as possible about them. The Hurewicz theorem from (2) describes a link between homotopy and homology: for a space X and $m > 1$, when $\pi_{<m}X = 0$ then there is a natural isomorphism $\pi_m(X) \cong H_m(X;\mathbb{Z})$. Using our prior knowledge of the homology of the sphere, this garners us one of its homotopy groups: $\pi_n S^n \cong \mathbb{Z}$. One of the decompositions studied in (1) cleaves the homotopy

[4] Most any simply-connected space with known cohomology groups will do.

groups of S^n into two pieces: there is an "exact sequence"

$$S^n[n+1,\infty) \to S^n \to K(\pi_n S^n, n),$$

where these new spaces $S^n[n+1,\infty)$ and $K(\pi_n S^n, n)$ have the properties

$$\pi_* K(\pi_n S^n, n) = \begin{cases} \pi_n S^n & \text{if } * = n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\pi_* S^n[n+1,\infty) = \begin{cases} \pi_* S^n & \text{if } * \geq n+1, \\ 0 & \text{otherwise.} \end{cases}$$

The Serre spectral sequence, a tool from (4), interrelates the homology of the three terms of an exact sequence of spaces: here, it consumes $H_* S^n$ and $H_* K(\pi_n S^n, n)$—the first of which is simple, the second of which we must set aside as a computation to be done—and it produces from them $H_* S^n[n+1,\infty)$. After computing $H_* S^n[n+1,\infty)$, one can then apply the Hurewicz theorem to it to gain access to

$$H_{n+1} S^n[n+1,\infty) \cong \pi_{n+1} S^n[n+1,\infty) \cong \pi_{n+1} S^n,$$

and repeat.

Improve title.

We have a limit of 35-37 lectures.

One of the stated goals of the course was to introduce exotic cohomology theories, which we did none of. Should we? K-theory?

I also gave a bunch of homework exercises that I'd prefer to be solved inline in the notes: for instance, facts about localizations, or the minimal models portion of unstable rational homotopy theory.

Emphasize the prevalence of moduli problems in homotopy theory.

Spectral sequences, early and often.

Emphasize when categorical constructions are used in a "wrong way" fashion.

Jun Hou gave a couple of lectures about the hammock localization midway through this.

Admit that these notes do not exhaustively cover their references.

Mention that this class won an award.

Contents

1
Homotopy Types and Exact Sequences

Inject introduction

1.1 The categories of spaces, pointed spaces, and pairs

Today we set up the basics of category theory and introduce a few basic examples, which amounts to setting the stage on which the rest of the course will play out. One of our express intentions in this course is to try to get away with doing very little geometry or "point-set" topology, which means a lot of this Lecture will concern itself with boldly asserting such facts in category-theoretic language, so that they can be guiltlessly referred to later.

We begin with some basic operations that one may perform on spaces. The categorical perspective is to understand objects in a category not by intrinsic presentations, like how the points in a space are arranged, but how they relate to other objects in the category, like how a given space can be continuously mapped into another. With this as a guiding principle, we give categorically-minded interpretations of some common operations performed on spaces: [4, 0.2]

Products Let X, Y be spaces. Their *product* $X \times Y$ is a space such that the set $\mathsf{Spaces}(T, X \times Y)$ of continuous maps $T \to X \times Y$ bijects with with the set $\mathsf{Spaces}(T, X) \times \mathsf{Spaces}(T, Y)$ of pairs (f_X, f_Y) of continuous maps $f_X \colon T \to X$ and $f_Y \colon Y \to T$. A useful mnemonic is that products pull out on the right:

$$\mathsf{Spaces}(T, X \times Y) \cong \mathsf{Spaces}(T, X) \times \mathsf{Spaces}(T, Y).$$

The bijection in question is induced by a pair of maps $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ which save the indicated coordinate and drop the other: given a map $f \colon T \to X \times Y$, post-composing with either of these maps gives the two values $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$ in the pair.[1]

[1] There's an amusing circularity to this data: if you assume the bijection to begin with, then you can recover the maps π_X and π_Y by starting with the identity $\mathrm{id}_{X \times Y} \in \mathsf{Spaces}(X \times Y, X \times Y)$ and following it across the bijection to the pair $(\pi_X, \pi_Y) \in \mathsf{Spaces}(X \times Y, X) \times \mathsf{Spaces}(X \times Y, Y)$. This sort of gymnastic forms the foothills of the *Yoneda Lemma*.

Coproducts The *disjoint union* $X \sqcup Y$ is a space with the property

$$\mathsf{Spaces}(X \sqcup Y, T) \cong \mathsf{Spaces}(X, T) \times \mathsf{Spaces}(Y, T).$$

For this reason, the disjoint union is sometimes called the *coproduct*, since it pulls out to a product on the left.

[4, 0.1]

Gluing Let X be a space with a *decomposition* $X = \bigcup_j A_j$ into closed subsets. Then the set $\mathsf{Spaces}(X, Y)$ bijects with the set

$$\{(f_j \in \mathsf{Spaces}(A_j, Y)) : f_j|_{A_j \cap A_k} = f_k|_{A_j \cap A_k}\}$$

of morphisms on the various components which agree on the overlaps.

[4, 0.3]

Quotients For $\sim$ an equivalence relation on X, there is a space $X/\sim$ such that the set $\mathsf{Spaces}(X/\sim, Y)$ bijects with the subset

$$\{f \in \mathsf{Spaces}(X, Y) \mid x \sim x' \Rightarrow f(x) = f(x')\}$$

of those continuous maps which are constant on the partition components of $\sim$. As a special case, let $A \subseteq X$ be nonempty, and define X/A by extending the total relation on A by the identity relation on X (i.e., the associated partition consists of singletons and A). [2]

[2] As an edge case, we set $X/\emptyset = X \cup \{*\}$.

Explain the edge case: it's because it's a pushout.

We mention one further construction which we will be very keen to make use of, and we offset it from those above because of its relative fragility.

[4, 0.9–11]

Function spaces/exponential objects For X, Y spaces, there is a *function spaces* Y^X whose underlying set is $\mathsf{Spaces}(X, Y)$. If X is locally compact, then the evaluation map $ev\colon Y^X \times X \to Y$ is continuous. If X and Z are locally compact as well as Hausdorff, then the currying function $Y^{Z\times X} \to (Y^Z)^X$ is a homeomorphism.

We will perpetually arrange to be in the situation where our function-spaces are well-behaved.

Remark 1.1.1. In each of these cases, the property named uniquely determines the space involved: if Q is some unknown space so that $\mathsf{Spaces}(Q, T)$ naturally bijects with the subset

$$\{f \in \mathsf{Spaces}(X, T) \mid x \sim x' \Rightarrow f(x) = f(x')\},$$

then there is a homeomorphism $Q \xrightarrow{\cong} X/\sim$. [3]

[3] This, too, is fundamentally the Yoneda Lemma.

Remark 1.1.2. Together, the coproduct and gluing properties characterize the functor $\mathsf{Spaces}(T, -)$ as a *sheaf*.

Find time later to characterize a sheaf as a continuous functor off of the opposite category.

These constructions also interact well in most cases of geometric interest:

[4, 0.4]

Lemma 1.1.3. *If Y is locally compact, then there is a homeomorphism*

$$\frac{X \times Y}{\sim \times \mathrm{id}} \xrightarrow{\cong} \left(\frac{X}{\sim}\right) \times Y. \quad \square$$

Remark 1.1.4. Throughout the course, we will quietly apply Lemma 1.1.3 in the case of $Y = I := [0,1]$ as we "fatten" various quotient constructions, as it allows us to elect whether to apply the quotient operation before or after we perform the fattening.

Lemma 1.1.3 also encodes *relative homotopy* in terms of a quotient:

Corollary 1.1.5. *If $A \subseteq X$ is closed and $H(a,t) = H(a',t)$ for all $a, a' \in A$ and $t \in I$, then it factors as $X \times I \to X/A \times I \to Y$.* □ [4, 0.8]

Relative homotopy provides a convenient segue into two other categories of interest: the category of relative pairs and the category of pointed spaces.

Definition 1.1.6. The category of *relative pairs* has objects given by inclusions $A \subseteq X$, and a morphism $(A \subseteq X) \to (B \subseteq Y)$ between two such pairs is given by a continuous map $f \in \mathsf{Spaces}(X,Y)$ which has the property $f(A) \subseteq B$. [4] These pairs and their maps appear naturally when discussing relative homology or relative homotopy.

[4] There is no reason one must stop at a single inclusion. Chains of inclusions—even infinite chains—are interesting, and they are generally referred to as *filtered spaces*.

Definition 1.1.7. The category of *pointed spaces*, which we denote by $\mathsf{Spaces}_{*/}$, has objects given by $\{x_0\} \subseteq X$ for some choice of singleton $\{x_0\}$, and a morphism $(\{x_0\} \subseteq X) \to (\{y_0\} \subseteq Y)$ between two such pointed spaces is given by a map $f \in \mathsf{Spaces}(X,Y)$ with the property $f(x_0) = y_0$. Pointed spaces and their maps arise when defining reduced homology or homotopy groups (e.g., the fundamental group). This presentation also makes plain that pointed spaces form a full subcategory of relative pairs: they are the special case where the privileged subset is a singleton. [4, 0.12]

These categories admit all of the same categorical constructions as Spaces: they have products, coproducts, and quotients, they glue, and one can build function objects. In many case, if one forgets about the privileged subset and considers just the underlying object of Spaces, the construction even agrees with the one in Spaces. For instance, the coproduct of pairs is given by

$$(A \subseteq X) \sqcup (B \subseteq Y) = (A \sqcup B) \subseteq (X \sqcup Y).$$

The product also has this property:

$$(A \subseteq X) \times (B \subseteq Y) = (((A \times Y) \cup (X \times B)) \subseteq X \times Y),$$

but the privileged subspace has become more complicated. One can also define a function object

$$((B \subseteq Y)^{(A \subseteq A)} \subseteq (B \subseteq Y)^{(A \subseteq X)}),$$

itself a relative pair which we abbreviate to $(B \subseteq Y)^{(A \subseteq X)}$ for sanity's sake. It satisfies an analogue of currying: [4, Proposition 0.13]

$$\begin{aligned}(B \subseteq Y)^{((Z \times A) \cup (C \times X)) \subseteq Z \times X} &= (B \subseteq Y)^{(C \subseteq Z) \times (A \subseteq X)} \\ &= \left((B \subseteq Y)^{(C \subseteq Z)}\right)^{(A \subseteq X)}.\end{aligned}$$

In trying to find analogues of these claims in pointed spaces, there is the following snag: the coproduct of two pointed spaces *in the category of relative pairs* is given by

$$(\{x_0\} \subseteq X) \sqcup (\{y_0\} \subseteq Y) = (\{x_0, y_0\} \subseteq (X \sqcup Y)),$$

but this coproduct has escaped the subcategory $\mathsf{Spaces}_{*/}$. One can forcefully correct this by quotienting the privileged subspace so that it becomes a point—and, in fact, this recovers the correct coproduct in pointed spaces:

Definition 1.1.8. For two pointed spaces $\{x_0\} \subseteq X$ and $\{y_0\} \subseteq Y$, their *wedge sum* $X \vee Y$ is the coproduct in the category of pointed spaces:

$$\mathsf{Spaces}_{*/}(X \vee Y, T) = \mathsf{Spaces}_{*/}(X, T) \times \mathsf{Spaces}_{*/}(Y, T).$$

In terms of the pair, it is given by

$$X \vee Y = \frac{\{x_0, y_0\} \subseteq (X \sqcup Y)}{\{x_0, y_0\}}.$$

The relationship between the products in relative pairs and in pointed spaces is much looser: the product of pointed spaces is given by

$$(\{x_0\} \subseteq X) \times (\{y_0\} \subseteq Y) = (\{(x_0, y_0)\} \subseteq (X \times Y)),$$

where the complexity in the privileged subspace has seemingly evaporated. [5] However, we find ourselves in a further awkward position when we investigate the currying law for the function object for pointed spaces:

[5] The reason why we didn't have to quotient this time is related to the chirality of the limits constructions involved: coproducts and quotients are both colimits, but a product is a limit.

$$\begin{aligned}(\{y_0\} \subseteq Y)^{((Z \times \{x_0\}) \cup (\{z_0\} \times X)) \subseteq Z \times X} &= (\{y_0\} \subseteq Y)^{(\{z_0\} \subseteq Z) \times (\{x_0\} \subseteq X)} \\ &= \left((\{y_0\} \subseteq Y)^{(\{z_0\} \subseteq Z)}\right)^{(\{x_0\} \subseteq X)}.\end{aligned}$$

In particular, the exponent on the far left is *not* the product in pointed spaces—indeed, it isn't a pointed space at all. We can, again, force it to become one:

[4, 2.4]

Definition 1.1.9. The *smash product* of two pointed spaces $\{x_0\} \subseteq X$ and $\{z_0\} \subseteq Z$ is given by

$$(\{x_0\} \subseteq X) \wedge (\{z_0\} \subseteq Z) := \frac{X \times Z}{(X \times \{z_0\}) \cup (\{x_0\} \times Z)} = \frac{X \times Z}{X \vee Z}.$$

This gives a *monoidal structure* on $\mathsf{Spaces}_{*/}$ which is *not* the Cartesian one. [6]

[6] That is: the one given by the ordinary product.

Corollary 1.1.10. *For pointed spaces X, Y, and Z, the currying law for function objects takes the form*

$$Y^{Z \wedge X} \cong (Y^Z)^X, \qquad \mathsf{Spaces}_{*/}(Z \wedge X, Y) \cong \mathsf{Spaces}_{*/}(X, Y^Z).$$

[4, 0.7]

Definition 1.1.11. Finally, we introduce the *homotopy category of (pointed) spaces*, $h\mathsf{Spaces}_{*/}$, whose objects are the same as that of $\mathsf{Spaces}_{*/}$ but whose morphism sets are given by quotients

$$h\mathsf{Spaces}_{*/}(X,Y) = \frac{\mathsf{Spaces}_{*/}(X,Y)}{f \sim g \text{ when there is a homotopy } H\colon X \wedge I_+ \to Y \text{ between them}}.$$

Because we will work in this category so often, we will abbreviate this mapping set by

$$h\mathsf{Spaces}_{*/}(X,Y) = [X,Y].$$

Introduce the notation X_+, and perhaps the adjunctions between the different categories.

1.2 *Perspectives on the fundamental group*

[4, 2.1]

Definition 1.2.1. The *pathspace* PX of X is given by $PX = X^I$, $I = [0,1]$ the closed interval. We write $\pi_0(X)$ for the set of path-components of X, i.e., the quotient of X by the relation $\sim$, where $x \sim x'$ when there exists $\gamma \in X^I$ such that $\gamma(0) = x$, $\gamma(1) = x'$.

Remark 1.2.2. The mapping set given in Definition 1.1.11 can be equivalently described by $[X,Y] = \pi_0 Y^X$. By specializing to $X = *$, we also have $\pi_0 Y = [S^0, Y]$ for $S^0 = (\{\pm 1\}, 1) = *_+$.

We can use the adjunctions from last time to give several equivalent definitions of the fundamental group:

$$\begin{aligned}\pi_1(Y) &:= \{\text{homotopy classes of pointed loops in } X\} \\ &= [S^1, Y] = [S^0 \wedge S^1, Y] = [S^0, Y^{(S^1)}] = \pi_0 Y^{S^1}.\end{aligned}$$

One might wonder what properties of X and Y make $[X,Y]$ into a group, since we know that $[S^1, -]$ and $[S^0, (-)^{S^1}]$ are group-valued. As we intend to explore this essentially categorical question using yet more category theory, it will be helpful to have a categorical definition of a group.

[4, pg. 14]

Definition 1.2.3. A *group* is a pointed set G with pointed maps $\mu\colon G \times G \to G$ and $\chi\colon G \to G$ which make the following diagrams commute:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta\times\mathrm{id}"] \arrow[dr, equal] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\mathrm{id}\times\eta"'] \arrow[dl, equal] \\ & G, & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\chi\times\mathrm{id}"] \arrow[d, "0"] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\mathrm{id}\times\chi"'] \arrow[d, "0"] \\ * \arrow[r, "\eta"] & G & *. \arrow[l, "\eta"'] \end{tikzcd}$$

Lemma 1.2.4. *If G is a group object in a category C, then $\mathsf{C}(-,G)$ is a functor from C^{op} to Groups.* [7]

[7] In fact, this is biconditional: if a representable functor factors through Groups, then the representing object inherits the structure of a group object.

I think this lecture is a better place to introduce Yoneda than the previous one.

Proof. Recall that the defining property of the product $X \times Y$ is

$$\mathsf{C}(T, X \times Y) \cong \mathsf{C}(T, X) \times \mathsf{C}(T, Y).$$

For a fixed test object T, the functor $\mathsf{C}(T, -)$ applied to the multiplication μ on G gives

$$\mathsf{C}(T, G) \times \mathsf{C}(T, G) \cong \mathsf{C}(T, G \times G) \xrightarrow{\mu_*} \mathsf{C}(T, G),$$

and in this way μ_* becomes a multiplication on $\mathsf{C}(T, G)$. The inversion χ on G similarly induces an inversion χ_* on $\mathsf{C}(T, G)$, and together these make the group object diagrams commute. [8] $\square$

[8] For a pair of maps $f, g \colon X \to K$, it is sometimes helpful to factor $(f, g) \colon X \to K \times K$ through the diagonal on X, as in $X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} K \times K$, as this lets on leverage the bifunctoriality of $\times$.

[4, Definition 2.9]

Definition 1.2.5. An H–*group* K is a pointed space with maps μ and χ satisfying the group diagrams up to homotopy (i.e., as diagrams in $h\mathsf{Spaces}_{*/}$).

[4, Proposition 2.14]

Corollary 1.2.6. *The functor* $[-, K]$ *is valued in groups.* $\square$

[4, Examples 2.15.ii]

Example 1.2.7. The usual verification that the fundamental group of a space is indeed a group can be viewed as giving an H–space structure on $Y^{S^1} =: \Omega Y$. The multiplication map μ is given by rescaling and concatenating two loops, and the inversion map χ is given by running a loop backward. Hence, not only is the fundamental group $\pi_1 Y = \pi_0 \Omega Y = [S^0, \Omega Y]$ a group, but actually $[X, \Omega Y]$ is a group for any choice of X.

What about the other formulation? We also have $\pi_1 Y = [S^1, Y]$, which is a group-valued functor even when Y is freely varying, and so one might suspect the magic does not reside in the Y–dependent object "ΩY" but rather in S^1 alone. In order to address this by the same tactic, we must confront the structure required on X in order to make $[X, Y]$ into a group. Using the identity $[X \vee X, Y] \cong [X, Y] \times [X, Y]$, we are led to the following definition:

[4, Definition 2.16]

Definition 1.2.8. An H–*cogroup* K has pointed maps $\mu' \colon K \to K \vee K$ and $\chi' \colon K \to K$ which make the following diagrams commute in the homotopy category:

$$\begin{tikzcd}
K \vee K \vee K & K \vee K \arrow[l, "\mu' \vee \mathrm{id}"'] \\
K \vee K \arrow[u, "\mathrm{id} \vee \mu'"] & K, \arrow[l, "\mu'"'] \arrow[u, "\mu'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
K & K \vee K \arrow[l, "0 \vee \mathrm{id}"'] \arrow[r, "\mathrm{id} \vee 0"] & K \\
& K, \arrow[u, "\mu'"'] \arrow[ul, equal] \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
K & K \vee K \arrow[l, "\chi \vee \mathrm{id}"'] \arrow[r, "\mathrm{id} \vee \chi"] & K \\
* \arrow[u, "\eta"'] & K \arrow[l, "0"'] \arrow[r, "0"'] \arrow[u, "\mu"'] & *. \arrow[u, "\eta"']
\end{tikzcd}$$

[4, Proposition 2.21]

Corollary 1.2.9. *The functor* $[H, -]$ *is valued in groups.* $\square$

Example 1.2.10. S^1 is an H–cogroup.

Draw pictures of μ', χ'.

Example 1.2.11. In fact, $S^1 \wedge X =: \Sigma X$ is an H–cogroup for *any* X. [4, 2.22]

Draw pictures of μ', χ'.

Lemma 1.2.12. *The adjunction* $[\Sigma X, Y] \cong [X, \Omega Y]$ *is an isomorphism of groups.* [4, Proposition 2.23]

Proof sketch. This is a matter of writing out the formulas for

$$\Sigma X \xrightarrow{\mu'} \Sigma X \vee \Sigma X \xrightarrow{f' \vee g'} Y \vee Y \xrightarrow{\Delta'} Y$$

and

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \Omega Y \times \Omega Y \xrightarrow{\mu} Y. \qquad \square$$

1.3 Higher homotopy groups

Just as we used the exponential adjunction to give a few equivalent definitions of the fundamental group, higher homotopy groups have a similar bunch of definitions.

Lemma 1.3.1. *For all* $n \geq 0$, *there is a homeomorphism* $S^1 \wedge S^n \cong S^{n+1}$. [4, Lemma 2.27]

Proof.

Carve S^1 into two halves, the smash against each of which gives a kind of hemisphere: "linearly interpolate between S^n + the basepoint of S^n and project to the lower hemisphere".

$\square$

Definition 1.3.2. The n^{th} *homotopy group* of a pointed space X is defined by $\pi_n X = [S^n, X]$. Equivalently, one may use any of [4, 3.1]

$$\pi_n X = [\Sigma^n(S^0), X] = [\Sigma^{n-1} S^0, \Omega X] = \cdots = [S^0, \Omega^n X].$$

We've quietly asserted that $\pi_n X$ is a group, but *which* group structure we mean is not immediately clear: there appear to be many different multiplications on the homotopy mapping sets, each coming from any one of the applications of Σ or of Ω. Today we tame this complexity by showing that each of these choices gives the same multiplication.

Lemma 1.3.3 (Eckmann–Hilton)**.** *Let S be a set with two products $\circ$ and $*$ which share a unit e and which satisfy* [4, Proposition 2.24]

$$(x * x') \circ (y * y') = (x \circ y) * (x' \circ y').$$

Then $\circ = *$ and both are associative and commutative.

Proof. We cleverly redistribute:

$$x \circ y = (x * e) \circ (e * y) = (x \circ e) * (e \circ y) = x * y,$$
$$x \circ y = (e * x) \circ (y * e) = (e \circ y) * (x \circ e) = y * x. \qquad \square$$

[4, Proposition 2.25]

Corollary 1.3.4. *Let K be an H–cogroup and L an H–group. The two multiplications so-induced on the mapping set $[K,L]$ are equal and commutative.*

Proof. We want the following diagram to commute:

$$\begin{tikzcd}
K \times K \arrow[r] & (K \vee K) \times (K \vee K) \arrow[r, "(f \vee f') \times (g \vee g')"] & (L \vee L) \times (L \vee L) \arrow[r] & L \times L \arrow[d] \\
K \arrow[u] \arrow[d] & & & L \\
K \vee K \arrow[r] & (K \times K) \vee (K \times K) \arrow[r, "(f \times g) \vee (f' \times g')"] & (L \times L) \vee (L \times L) \arrow[r] & L \vee L. \arrow[u]
\end{tikzcd}$$

The top composite corresponds to $(f +_K f') +_L (g +_K g')$, and the bottom composite corresponds to $(f +_L g) +_K (f' +_L g')$. Hence, if these were to agree, we could apply Lemma 1.3.3.

The diagonal map $\Delta \colon K \to K \times K$ replicates its input onto both coordinates, as in $x \mapsto (x,x)$. Writing $\mu'(k) = (k_1, k_2)$, we can apply Δ either before or after the comultiplication map μ' on K, and get the same answer up to a twist:

$$\begin{aligned}
(\Delta \vee \Delta) \circ \mu'(k) &= (\Delta \times \Delta)(k_1, k_2) = (k_1, k_1, k_2, k_2), \\
(\mu' \times \mu') \circ \Delta(k) &= (\mu' \times \mu')(k,k) = (k_1, k_2, k_1, k_2).
\end{aligned}$$

Additionally, because $\mu' \colon K \to K \vee K$ targets the wedge sum, it is always the case that at least one of k_1 or k_2 equals the basepoint. From these two considerations, it follows that the left-hand portion of the following diagram commutes:

$$\begin{tikzcd}
K \times K \arrow[r] & (K \vee K) \times (K \vee K) \arrow[r] & (L \vee L) \times (L \vee L) \arrow[r] & L \times L \arrow[d] \\
K \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c} (K \times *) \times (K \times *) \\ \cup \\ (* \times K) \times (* \times K) \end{array} \arrow[u] \arrow[d, "\mathrm{id} \times T \times \mathrm{id}"] & & L \\
K \vee K \arrow[r] & (K \times K) \vee (K \times K) \arrow[r] & (L \times L) \vee (L \times L) \arrow[r] & L \vee L. \arrow[u]
\end{tikzcd}$$

Now, we apply (f, f', g, g') on top and (f, g, f', g') on bottom. Again, the diagram commutes up to transposition of the two middle coordinates.

$$\begin{tikzcd}
K\times K \arrow[r] & (K\vee K)\times(K\vee K) \arrow[r] & (L\vee L)\times(L\vee L) \arrow[r] & L\times L \arrow[d] \\
K \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c}(K\times *)\times(K\times *)\\ \cup \\ (*\times K)\times(*\times K)\end{array} \arrow[u] \arrow[d, "\mathrm{id}\times T\times\mathrm{id}"] \arrow[r] & \begin{array}{c}(L\times *)\times(L\times *)\\ \cup \\ (*\times L)\times(*\times L)\end{array} \arrow[u] \arrow[d, "\mathrm{id}\times T\times\mathrm{id}"] & L \\
K\vee K \arrow[r] & (K\times K)\vee(K\times K) \arrow[r] & (L\times L)\vee(L\times L) \arrow[r] & L\vee L. \arrow[u]
\end{tikzcd}$$

Finally, we make use of the specific subspace we'd highlighted in the middle. In (either half of) this subspace, two of the coordinates are constrained to lie at the basepoint. Because of this, we can apply the unit axioms for multiplication and for fold to conclude that they produce the same value. This amounts to the commutativity of the final two rectangles:

$$\begin{tikzcd}
K\times K \arrow[r] & (K\vee K)\times(K\vee K) \arrow[r] & (L\vee L)\times(L\vee L) \arrow[r] & L\times L \arrow[d] \\
K \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c}(K\times *)\times(K\times *)\\ \cup \\ (*\times K)\times(*\times K)\end{array} \arrow[u] \arrow[d, "\mathrm{id}\times T\times\mathrm{id}"] \arrow[r] & \begin{array}{c}(L\times *)\times(L\times *)\\ \cup \\ (*\times L)\times(*\times L)\end{array} \arrow[u] \arrow[d, "\mathrm{id}\times T\times\mathrm{id}"] \arrow[r] & L \\
K\vee K \arrow[r] & (K\times K)\vee(K\times K) \arrow[r] & (L\times L)\vee(L\times L) \arrow[r] & L\vee L. \arrow[u]
\end{tikzcd}$$

Since each of the squares commutes, the two outer composites are equal.

□ [4, Proposition 2.26]

Corollary 1.3.5. *For $n \geq 2$, $\pi_n X = [\Sigma^{n-1} S^0, \Omega X]$ has only one multiplication and it is commutative.* □

1.4 Exact sequences in Spaces

An *extremely* common device in algebraic topology is the exact sequence: [4, 2.29]

Definition 1.4.1. A pair of maps of groups

$$N \xrightarrow{f} G \xrightarrow{g} H$$

is called *exact* when $\operatorname{im} f = \ker g$. More weakly, a sequence of pointed sets $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* if $\operatorname{im} f = g^{-1}(*)$.

Last time, we put structure onto a space so that $[K,-]$ or $[-,L]$ became valued in groups. Today we are after something similar: we would like to study when continuous maps $A \to B \to C$ induce exact sequences of the functors they co/represent.

[4, Definitions 2.30 and 2.49]

Definition 1.4.2. If the sequence of pointed sets

$$[-,A] \to [-,B] \to [-,C]$$

is exact, then we say that the underlying sequence of spaces is *exact*. If the sequence of pointed sets

$$[A,-] \leftarrow [B,-] \leftarrow [C,-]$$

is exact, then we say that the underlying sequence of spaces is *coexact*.[9]

At first brush, one might imagine that such sequences are somewhat rare, or that they at least require some nice properties of A, B, or f.[10] In fact, these sequences are extremely plentiful in homotopy theory.

[9] These more commonly go by the names *fiber* and *cofiber* sequences of spaces. We will work later to justify these alternative names.

[10] This scarcity is true for maps of groups: $f\colon N \to G$ can only participate in an exact sequence when the image of f is a normal subgroup. It is a good exercise to check that these constructions for spaces do not violate this.

[4, Proposition 2.35]

Lemma 1.4.3. *Any map $f\colon X \to Y$ extends to a coexact sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Proof. We first define the *cone* of X by $CX = X \wedge I$, and we use this to set

$$Z = Y \cup_f CX = \frac{Y \sqcup CX}{f(x) \sim (x,1)}.$$

We claim that the inclusion $g\colon Y \to Z$ induces the desired exact sequence on mapping sets. To check this, consider a test space T, the induced sequence

Define "mapping cone".

$$[Y \cup_f CX, T] \to [Y,T] \to [X,T],$$

and a function $\varphi\colon Y \to T$. If $\varphi \circ f\colon X \to T$ is null, then a choice of null-homotopy defines a map $CX \to T$ agreeing with f on the edge of I. The gluing operation then determines a map $Z \to T$ which restricts to φ. Conversely, a map $\tilde{\varphi}\colon Z \to T$ which restricts to φ can itself be restricted to CX. This gives the required null-homotopy of $\varphi \circ f$. $\square$

The remarkable lack of hypotheses in Lemma 1.4.3 mean that coexact sequences can be extended indefinitely to the right:

$$X \xrightarrow{f} Y \xrightarrow{g} Y \cup_f CX \to (Y \cup_f CX) \cup_g CY \to \cdots.$$

The presentations of these spaces given by Lemma 1.4.3 quickly become unwieldy, but the following alternative presentation remains quite tractible:

[4, Lemma 2.37]

Lemma 1.4.4. *There is a homotopy equivalence*

$$(Y \cup_f CX) \cup_g CY \to ((Y \cup_f CX) \cup_g CY)/CY. \quad \square$$

[4, Proposition 2.38]

Lemma 1.4.5. *For $A \subseteq X$ a subspace, there is a homeomorphism*

$$(X \cup_i CA)/CA \cong X/A. \quad \square$$

[4, Lemma 2.40]

Corollary 1.4.6. *The infinite coexact sequence takes the form*

$$X \xrightarrow{f} Y \to Y \cup_f CX \to \Sigma X \xrightarrow{-f} \Sigma Y \to \Sigma(Y \cup_f CX) \to \cdots .$$

Proof. We need only identify the next two terms after Z and the map between them. Once that is in hand, we need only note that suspension commutes with the cone and quotient operations used to define Z.

is the word “suspension” defined?

To see the claim for the first term after Z, we apply the two Lemmas in turn:

$$\begin{aligned}
(Y \cup_f CX) \cup_g CY &\simeq ((Y \cup_f CX) \cup_g CY)/CY && \text{(Lemma 1.4.4)}\\
&\cong (Y \cup_f CX)/Y && \text{(Lemma 1.4.5)}\\
&\cong CX/X \cong \Sigma X
\end{aligned}$$

Identically, the second term after Z is described by

Is $CX/X \cong \Sigma X$ obvious?

$$((Y \cup_f CX) \cup_g CY) \cup_{g'} C(Y \cup_f CX) \simeq \Sigma Y.$$

The coexact sequence thus takes the desired form. ☐

Coupling this to our results from last time, this gives altogether an exact sequence

$$\begin{tikzcd}
{[X,T]} & {[Y,T]} \arrow[l] & {[\overbrace{Y \cup_f CX}^{Z}, T]} \arrow[l] & & \text{ptd. sets} \\
{[\Sigma X,T]} \arrow[urr, bend left] & {[\Sigma Y,T]} \arrow[l] & {[\Sigma Z,T]} \arrow[l] & & \text{groups} \\
{[\Sigma^2 X,T]} \arrow[urr, bend left] & {[\Sigma^2 Y,T]} \arrow[l] & {[\Sigma^2 Z,T]} \arrow[l] & \cdots \arrow[l] & \text{ab. groups}
\end{tikzcd}$$

Exact sequences are best behaved on abelian groups, but not all of the above are abelian groups—or even groups! What can be said about the edges, where at least one term is a(n abelian) group?

Construction 1.4.7. Pinching the middle of the cone CX gives a map

$$Y \cup_f CX \to (Y \cup_f CX) \vee \Sigma X,$$

which gives an action

$$[Z,T] \times [\Sigma X, T] \to [Z,T].$$

Lemma 1.4.8 (2.42–48). *The map* $[Z,T] \to [Y,T]$ *is invariant under this action, and on orbits it is an injection.*

Proof sketch. We indicate the construction underlying injectivity. Supposing that $f_1, f_2 \colon Z \to T$ restrict to the same map on Y, so that they in particular agree on X but might differ on CX. Glue them together to get a map $d(f_1, f_2) \colon \Sigma X = CX \cup_X CX \to T$. One can show that $f_1 = d(f_1, f_2) \cdot f_2$. ☐

[4, Proposition 2.54]

There are also dual results for exact sequences: any map of spaces participates in an (infinite) exact sequence.

Lemma 1.4.9. *Any map* $f \colon X \to Y$ *extends to an exact sequence*

$$P \to X \xrightarrow{f} Y,$$

where P is given by

$$P = \{(x, \gamma) \in X \times PY \mid f(x) = \gamma(1)\}. \quad ☐$$

Proof sketch. Again, the construction of P rests on building into it the data of a null-homotopy. Suppose that $\theta \colon T \to Y$ is a null-homotopic map. A null-homotopy of θ is the same as a map $T \wedge I = CT \to Y$ restricting to θ, which is in turn the same as a map $T \to PY = Y^I$ restricting to θ. To model a map $\varphi\colon T \to X$ for which $\theta = f \circ \varphi$ becomes null-homotopic, we attach PY along X as in the statement. □

[4, Proposition 2.58]

Lemma 1.4.10. *Iterating this gives*

$$\cdots \to \Omega^2 X \to \Omega P_f \to \Omega X \to \Omega Y \to P_f \to X \xrightarrow{f} Y. \quad \square$$

In particular, by applying $\pi_0(-) = [S^0, -]$ and employing the definition $\pi_n X = \pi_0 \Omega^n X$, we get an exact sequence of homotopy groups:

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 P \to \pi_1 X \to \pi_1 Y \to \pi_0 P \to \pi_0 X \to \pi_0 Y.$$

One of our goals in this class will be to understand what understanding π_* earns us, how it changes as X changes, and how X can be effectively dissected to build up knowledge of π_* along the way. This long exact sequence will be one of our main tools for doing so.

1.5 Relative homotopy groups

The construction P from the previous lecture is a little mysterious. To close out this Chapter, we will explore it in two lights: today and a bit later on.

Let us return to the setting of pairs (X, A, x_0) of topological spaces, and let's use $i\colon A \to X$ to denote the inclusion of the preferred subspace.

[4, Definition 3.8]

Lemma 1.5.1. *The exact continuation P of i is given by the function object*

$$P(i) = (X, A, x_0)^{(I, \partial I, 0)}. \quad \square$$

Inspired by this, we consider the adjunction juggle:

[4, pg. 38]

Definition 1.5.2. We define the n^{th} *relative homotopy group* of the pair (X, A) by

$$\pi_n(X, A) := [(D^n, S^{n-1}), (X, A)] \cong [S^{n-1}, P] = \pi_{n-1} P.$$

That is, $\pi_n(X, A)$ consists of n–disk maps into X with boundary ∂D^n lying in A.

[4, Proposition 3.9]

Corollary 1.5.3. *There is a long exact sequence*

$$\cdots \to \pi_{n+1}(X, A) \to \pi_n A \to \pi_n X \to \pi_n(X, A) \to \pi_{n-1} A \to \cdots. \quad \square$$

These groups track the discrepancy between $\pi_* A$ and $\pi_* X$. This observation moves us to remark on when there is *no* discrepancy:

[4, 3.12–13 and 3.17]

Definition 1.5.4. A pair $A \subseteq X$ is *n–connected* if $\pi_{\leq n}(X, A) = 0$. Equivalently, the map $\pi_* A \to \pi_* X$ is an isomorphism for $* < n$ and an epimorphism for $* = n$. The pair $A \subseteq X$ is a *weak equivalence* if it is ∞–connected.

[4, 3.15]

Remark 1.5.5. We can extend these definitions to a generic $f\colon Y \to X$ using the *mapping cylinder*:

$$M_f := (Y \times I) \cup_f X.$$

This space receives a map $X \to M_f$ which is a weak equivalence. It also receives two maps $Y \to M_f$, one along the "free" end of the cylinder $Y \times I$ and one along the "attached" end. Along the free end, the map from Y to M_f is an inclusion, so that (M_f, Y) can be thought of as a pair. Along the attached end, the map from Y to M_f factors through X, where it is shown to agree with f. The situation is summarized in the following diagram:

$$\begin{tikzcd}
Y \times \{1\} \arrow[d] \arrow[dr, hook] & M_f \arrow[d, equal] & \\
Y \times I \arrow[r] & (Y \times I) \cup_f X & X. \arrow[l, "\simeq"'] \\
Y \times \{0\} \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

WE PAUSE TO EMPHASIZE something that the reader might have dismissed as serendipity: by switching notation from $(A \subseteq X)$ to $\pi_n(X, A)$, Corollary 1.5.3 looks very much like a corresponding theorem about relative *homology* groups. It is extremely productive to see how far this analogy can be pushed: what theorems about homology can be replicated for homotopy?—and, when a theorem for homology fails for homotopy, is it partially recoverable?

As a warm-up to this program, recall that the relative homology groups of a pair of inclusions $x_0 \in B \subseteq A \subseteq X$ can be interrelated. We will consider whether the same can be said of relative homotopy groups. To this end, consider the long exact sequences associated to the pairs (X, A), (X, B), and (A, B), which arrange into the following commutative diagram: To help

$$\begin{tikzcd}
\cdots & \pi_{n+1}(X,A) \arrow[rr] \arrow[dr, blue] & & \pi_n(A,B) \arrow[rr, orange] \arrow[dr] & & \pi_{n-1}B \arrow[rr, red] \arrow[dr, orange] & & \pi_{n-1}X & \cdots \\
\cdots & & \pi_n A \arrow[ur, orange] \arrow[dr, blue] & & \pi_n(X,B) \arrow[ur, red] \arrow[dr] & & \pi_{n-1}A \arrow[ur, blue] \arrow[dr, orange] & & \cdots \\
\cdots & \pi_n B \arrow[rr, red] \arrow[ur, orange] & & \pi_n X \arrow[rr, blue] \arrow[ur, red] & & \pi_n(X,A) \arrow[rr] \arrow[ur, blue] & & \pi_{n-1}(A,B) & \cdots
\end{tikzcd}$$

with readiability, we have colored the long exact sequences associated to (X, A), (A, B), and (X, B). To complete the symmetric pattern, we have additionally included maps $\pi_n(A, B) \to \pi_n(X, B)$ and $\pi_n(X, B) \to \pi_n(X, A)$ granted to us by naturality, as well as maps $\pi_n(X, A) \to \pi_{n-1}(A, B)$ granted by completing the relevant triangle. Tracing through the middle, we see that these new maps form an interesting sequence, highlighted in red:

$$\begin{tikzcd}[column sep=small]
\cdots & \pi_{n+1}(X,A) \arrow[rr] \arrow[dr] & & \pi_n(A,B) \arrow[rr] \arrow[dr] & & \pi_{n-1}B \arrow[rr] \arrow[dr] & & \pi_{n-1}X & \cdots \\
\cdots & & \pi_n A \arrow[ur] \arrow[dr] & & \pi_n(X,B) \arrow[ur] \arrow[dr] & & \pi_{n-1}A \arrow[ur] \arrow[dr] & & \cdots \\
\cdots & \pi_n B \arrow[rr] \arrow[ur] & & \pi_n X \arrow[rr] \arrow[ur] & & \pi_n(X,A) \arrow[rr] \arrow[ur] & & \pi_{n-1}(A,B) & \cdots
\end{tikzcd}$$

[4, Theorem 3.20]

Lemma 1.5.6. *This sequence is exact.*

Proof sketch. Checking that the composites are zero is easy enough. We describe how the lifting condition is proved for the left-most map pictured. Suppose that we have an element $x \in \pi_n(A,B)$ mapping to zero in $\pi_n(X,B)$. Its image in $\pi_{n-1}B$ is then also zero, so that we may lift it to $y \in \pi_n A$. Define $z \in \pi_n X$ to be the image of y; since y is zero by the time it makes it to $\pi_n(X,B)$, so must z be. We may then also lift z to $w \in \pi_n B$. Pushing forward w to $y' \in \pi_n A$ gives a second element in the same group, and the difference of the two elements $y - y'$ pushes forward to zero in $\pi_n X$. Hence, we can lift the difference to $\pi_{n+1}(X,A)$. This ultimately gives the desired lift of x. □

Overlay this diagram chase as a series of diagrams.

Remark 1.5.7. As a simple structural application, consider an inclusion $i\colon A \subseteq X$ which admits a retraction $r\colon X \to A$.[11] The induced map $\pi_* A \to \pi_* X$ is then an inclusion, and the boundary map $\pi_*(X,A) \to \pi_{*-1}(A)$ is zero. It follows that there are short exact sequences

[4, 3.21]

[11] That is: $ri \simeq \mathrm{id}_A$.

$$\begin{tikzcd}
0 \arrow[r] & \pi_n A \arrow[r] & \pi_n X \arrow[l, bend right] \arrow[r] & \pi_n(X,A) \arrow[r] & 0.
\end{tikzcd}$$

For $n \geq 3$, all the groups are abelian, from which we deduce that $\pi_n X$ splits as a sum of $\pi_n A$ and $\pi_n(X,A)$. At $n = 2$, we learn naively that $\pi_2 X$ is a semidirect product of $\pi_2 A$ and $\pi_2(X,A)$—but this forces $\pi_2(X,A)$ to be abelian has well. Hence, in this situation we have

$$\pi_{\geq 2} X = \pi_{\geq 2} A \oplus \pi_{\geq 2}(X,A).$$

1.6 Fibrations

We promised to explain the name "co/fiber". Now is the time to do it.

Relative to the small and polite model for the continuation of the coexact sequence, the function space P feels large and unwieldy. In this section we explore a more familiar context in which spaces *like* P arise, as well as conditions that they are weak models for exact continuations.

To get off the ground, recall the whole point of the design of P was its participation in the long exact sequence

Lemma 1.4.10

$$\cdots \to \pi_2 X \to \pi_1 P \to \pi_1 X \to \pi_1 Y \to \pi_0 P \to \pi_0 X \to \pi_0 Y.$$

Inasmuch as the exact sequence is truly what we're after, we might consider alternative constructions in which such sequences arise.

[4, Theorem 4.1]

Example 1.6.1. For each $n \geq 0$, there is an isomorphism

$$\pi_n(X \times Y) = [S^n, X \times Y] \cong = [S^n, X] \times [S^n, Y] \cong \pi_n(X) \times \pi_n(Y).$$

Rearranged as an exact sequence, the maps

$$\begin{tikzcd} Y \arrow[r, "i_Y"'] & X \times Y \arrow[l, bend right, "\pi_Y"'] \arrow[r, "\pi_X"'] & X \arrow[l, bend right, "i_X"'] \end{tikzcd}$$

induce a split-exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "0"] & \pi_n Y \arrow[r] & \pi_n(X \times Y) \arrow[l, bend right] \arrow[r] & \pi_n X \arrow[l, bend right] \arrow[r, "0"] & \cdots. \end{tikzcd}$$

We would like to axiomatize the part of the geometry of the Cartesian product that we need to induce exact sequences on π_*. Its main features that we used were the projection and retraction maps, which combine to let us recover a map $S^n \to X \times Y$ from the projections $S^n \to X$ and $S^n \to Y$. We will ultimately ask for "one half" of this data (i.e., one projection and one inclusion), together with a requirement that they satisfy a variant of this recovery property.

[4, Definition 4.2]

"E" for Espace (fr.), "B" for Base.

Definition 1.6.2. A map $p \colon E \to B$ has the *homotopy lifting property* with respect to a space X when for all solid diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "- \times 0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & B \end{tikzcd}$$

there exists a dashed diagonal lift $\widetilde{H}$. That is, homotopies in B lift to homotopies in E. A *fibration* has the HLP for all spaces. A *weak fibration* has the HLP for at least the disks D^n. The *fiber* of a fibration is $F = p^{-1}(b_0) \subseteq E$.

Example 1.6.3. The projection $\pi_X \colon X \times Y \to X$ forms a fibration with fiber Y.

[4, Proposition 4.3]

Example 1.6.4. The evaluation map $ev \colon PX \to X$ form a fibration with fiber ΩX.

For the remainder of the lecture, we will work to justify that this definition yields the intended long exact sequence of homotopy groups. The first step is a version of the homotopy lifting property for pairs and its relation to the original definition above.

[4, Proposition 4.5]

Lemma 1.6.5. *Consider a fibration $p \colon E \to B$, as well as a preferred subset $B' \subseteq B$ of the base and the subset $E' := p^{-1}(B') \subseteq E$ of the total space lying over it. If p has the HLP for $X \times I$, then $p' \colon (E, E')^{(I, \partial I)} \to (B, B')^{(I, \partial I)}$ has the HLP for X.*

Proof. Let us begin with an HLP diagram for p', as in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & (E,E')^{(I,\partial I)} \arrow[d, "p'"] \\ X \times I \arrow[r, "H"] & (B,B')^{(I,\partial I)} \end{tikzcd}.$$

We may remove the function spaces by applying the exponentiation adjunction:

$$\begin{tikzcd} X \times (I \vee I) \arrow[r, "f''"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "H'"] & B. \end{tikzcd}$$

The homeomorphism $h\colon I \vee I \cong I$ extends to a homeomorphism h' as in

$$\begin{tikzcd} I \vee I \arrow[r, "{h,\cong}"] \arrow[d, hook] & I \arrow[d, hook] \\ I \times I \arrow[r, "{h',\cong}"] & I \times I, \end{tikzcd}$$

which can be used to smooth out the top-left corner:

$$\begin{tikzcd} X \times I \arrow[r, "f'' \circ h^{-1}"] \arrow[d, "- \times 0"] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "H' \circ h^{-1}"] \arrow[ur, dotted, "\widetilde{H}'"] & B. \end{tikzcd}$$

Applying the HLP for p gives $\widetilde{H'}$. Reversing the application of h' and the exponential adjunction ultimately yields the desired $\widetilde{H}$ in the Lemma statement. □

[4, Theorem 4.6]

Corollary 1.6.6. *If p as above is a weak fibration, then the natural map*

$$\pi_n(E,E') \to \pi_n(B,B')$$

is an isomorphism for all $n \geq 1$.

Proof. Begin with a class $\omega\colon (I,\partial I) \to (B,B')$ in $\pi_1(B,B')$, and consider the diagram

$$\begin{tikzcd} * \arrow[r, "e_0"] \arrow[d] & E \arrow[d, "p"] \\ * \times I \arrow[r, "\omega"] \arrow[ur, dotted, "\widetilde{\omega}"] & B. \end{tikzcd}$$

Since $\omega(1) \in B'$ and $E' := p^{-1}(B')$, the class $\widetilde{\omega}$ can be considered as a map $\widetilde{\omega}\colon (I,\partial I) \to (E,E')$. Hence, $\pi_1(E,E') \to \pi_1(B,B')$ a surjection.

For injectivity, suppose that $\omega_1, \omega_2\colon (I,\partial I) \to E$ are two relative homotopy classes and that $H\colon I \times I \to B$ is a homotopy connecting them in B. We form the homotopy lifting diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & E \arrow[d] \\ I \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & B, \end{tikzcd}$$

where $U = \partial I \setminus (I \times \{1\})$ is a ⊔–shaped figure, the top arrow acts by ω_1 and ω_2 on the two legs, and it carries the bottom of the figure to the basepoint. The same procedure as in Lemma 1.6.5 produces a filler $\widetilde{H}$, which witnesses equality in $\pi_1(E, E')$. Injectivity follows.

For $\pi_{>1}$, one uses relative pathspaces to induct up. □

This is cryptic.

[4, 4.7]

Corollary 1.6.7. *There is a long exact sequence*

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots.$$

Proof. Set $B' = \{b_0\}$ in the above, and identify these terms respectively with

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n(E, F) \to \pi_{n-1} F \to \cdots. \qquad \square$$

[4, Corollary 4.8]

Corollary 1.6.8. *If E is contractible (e.g., as in the fibration $ev\colon PX \to X$), then the going-around map $\pi_{n+1} B \to \pi_n F$ is an isomorphism.* □

1.7 Fiber bundles and examples

Add non-example: the S-shaped graph "fibering" over the line.

Fibrations are all over classical geometry. Most commonly, they take the form of a *fiber bundle*, which is a less homotopically-minded take on the important properties of the Cartesian product.

[4, Definition 4.9]

Definition 1.7.1. A *fiber bundle* is a pair of maps $F \subseteq E \xrightarrow{p} B$ such that B has an open cover $\{U_\alpha\}_\alpha$ admitting local homeomorphisms $\varphi_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$ which satisfy $p\varphi_\alpha = p|_{U_\alpha}$.

[4, Proposition 4.10]
If B is paracompact, then p is actually a fibration.

Lemma 1.7.2. *Every fiber bundle is a weak fibration.*

Proof sketch. Fix an open cover as in Definition 1.7.1. Given a map $f\colon D^n \times I \to B$, subdivide $D^n \times I$ so finely that its pieces are each contained in one member of the cover. Once this is arranged, one can build the desired lift purely locally. □

The most common source of these "local projections" come from Lie theory: a quotient of a Lie group by a subgroup has everywhere-isomorphic fibers, and in good cases the coset space can be arranged to admit one of these covers. We spend most of today making this precise.

[4, Theorem 4.13]
We do not assume normality!

Lemma 1.7.3. *Take $H \le G$ to be a closed subgroup of a topological group. Suppose that the identity coset $H \in G/H$ has an open neighborhood U with a section $U \xrightarrow{s} G \xrightarrow{p} G/H$. The map p is then a fiber bundle with fiber H.*

Proof. Left-multiplying by $g \in G$, the condition at H begets sections near all $gH \in G/H$. The sections together become the data of a fiber bundle by

$$U_{gH} \times H \xrightarrow{s_g \times 1} G \times G \xrightarrow{\mu} G. \qquad \square$$

Example 1.7.4 (Stiefel manifolds). Consider the subgroup $O(n) \subseteq O(n+k)$ of block matrices [4, Examples 4.14.1]

$$\left\{\left[\begin{array}{c|c} O(n) & 0 \\ \hline 0 & I \end{array}\right]\right\} \subseteq O(n+k).$$

The quotient is the space of orthonormal k–frames in $\mathbb{R}^{n+k}$. To construct a local section, note there is an open neighborhood U of the standard frame $(e_{n+1}, \ldots, e_{n+k})$ consisting of those $(u_{n+1}, \ldots, u_{n+k})$ satisfying the condition

For example, $O(n)/O(n-1)$ is homeomorphic to S^{n-1}.

$$\det\left[e_1 \mid \cdots \mid e_n \mid u_{n+1} \mid \cdots \mid u_{n+k}\right] \neq 0.$$

On U, the local section is defined by applying Gram–Schmidt to the block matrix of columns $[e_1 \mid \cdots \mid e_n \mid u_{n+1} \mid \cdots \mid u_{n+k}]$ to produce an element of $O(n+k)$.

Example 1.7.5 (Grassmannians). The further quotient by [4, Examples 4.14.2]

$$\left\{\left[\begin{array}{c|c} O(n) & 0 \\ \hline 0 & O(k) \end{array}\right]\right\} \leq O(n+k)$$

gives the space of k–dimensional subspaces in $\mathbb{R}^{n+k}$. Consider the open neighborhood U of $(e_{n+1}, \ldots, e_{n+k})$ consisting of those subspaces W which trivially intersect the subspace $\langle e_1, \ldots, e_n \rangle$. By projecting $e_{n+1}, \ldots, e_{n+k}$ into W and applying Gram–Schmidt, this produces an orthonormal k–frame, i.e., a section landing in $O(n+k)/O(n)$. This can ultimately be used to construct a fiber bundle

$$O(n) \to \frac{O(n+k)}{O(k)} \to \frac{O(n+k)}{O(n) \times O(k)},$$

even though the middle term is not a group.

Example 1.7.6. These linear algebraic examples do not rest on properties of the reals. One can build analogues of these examples with $U(n)$ and $\mathbb{C}^n$, or with $Sp(n)$ and $\mathbb{H}^n$. [4, Examples 4.14.4, 5, 7]

Example 1.7.7. The sequence [4, Examples 4.14.3]

$$SO(n) \to O(n) \xrightarrow{\det} O(1)$$

admits a section

$$\pm 1 \mapsto \left(\begin{array}{c|c} \pm 1 & 0 \\ 0 & I \end{array}\right),$$

which witnesses it as a fiber bundle.

Example 1.7.8. Applying the long exact sequence of homotopy for a fiber bundle to Example 1.7.7 shows [4, Proposition 4.15]

$$\pi_{\geq 1} SO(m) \cong \pi_{\geq 1} O(n).$$

This isomorphism of homotopy groups is attractive enough that we give fiber bundles with discrete fibers a special name:

[4, Definition 4.16]

Definition 1.7.9. A *covering* of B is a fiber bundle with discrete fiber.

[4, Examples 4.18.1]

Example 1.7.10. The quotient sequence

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

witnesses $\mathbb{R}$ as a cover of S^1 with discrete fiber $\mathbb{Z}$. The local section in this case is given by

$$z \mapsto \frac{1}{2\pi i} \log z.$$

A discrete fiber has sufficiently simple structure that one can often automatically construct fiber bundles as quotients by group actions, without needing to manufacture a local section to guarantee sane behavior. As our final task for today, we record this easier situation.

[4, 4.19]

Definition 1.7.11. A discrete group G acts *properly discontinuously* on X when…

1. …for all points $x \in X$ there is a neighborhood U_x so that gU_x never intersects U_x for non-identity g.
2. …for all points $x, y \in X$ in different orbits, there are neighborhoods U_x, U_y so that gU_x never meets U_y.

[4, Proposition 4.20]

Lemma 1.7.12. *If G acts properly discontinuously on X, then $G \to X \xrightarrow{p} X/G$ is a covering (and X/G is Hausdorff).*

Proof. For $[x] \in X/G$, choose a neighborhood U_x guaranteed by (1), and let $p(U_x)$ be the neighborhood of $[x]$ in the base. We using the unicity in (1) to define the local section

$$p^{-1}(p(U_x)) \to G \times p(U_x). \qquad \square$$

[4, Remarks 4.21.i]

Remark 1.7.13. If $\pi_0 X = 0$, then in the short exact sequence of pointed sets

$$0 \to \pi_1 X \to \pi_1 X/G \to \pi_0 G \to 0$$

the last map is actually a group homomorphism.

[4, Remarks 4.21.iv]

Example 1.7.14. $\mathbb{Z}$ acting on $\mathbb{R}$ by $1 \cdot x = x + 1$ is properly discontinuous. Since $\mathbb{R}$ is contractible, we may conclude

$$\pi_n S^1 = \begin{cases} \mathbb{Z} & \text{when } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

[4, Remarks 4.21.ii]

Remark 1.7.15. A covering $p \colon E \to B$ is *regular* if $p_*(\pi_1 E) \subseteq \pi_1 B$ is normal. All regular covers arise as quotients by a G–action.

[4, Remarks 4.21.iii]

Remark 1.7.16. For G finite, X Hausdorff, and no fixed points, the G–action is properly discontinuous.

1.A The action of π_1

I remember not being convinced of the value of this lecture.

Before continuing our study of P_f, there is one other "old" fact about homotopy groups we should investigate: their dependence on $x_0 \in X$. You might even remember that paths γ in X induce isomorphisms $\pi_1(X;\gamma(0)) \to \pi_1(X;\gamma(1))$. This fits into a framework we have considered already. We begin with the algebraic thing we are trying to model.

Definition 1.A.1. For G, A groups, an action $\alpha\colon G \times A \to A$ is *compatible* when $g(a_1a_2) = (ga_1)(ga_2)$, i.e., the following commutes

$$\begin{tikzcd}
G \times A \times A \arrow[r, "\Delta \times \mathrm{id} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"] & G \times G \times A \times A \arrow[r, "\simeq"] & G \times A \times G \times A \arrow[r, "\alpha \times \alpha"] & A \times A \arrow[d, "\mu"] \\
G \times A \arrow[rrr, "\alpha"] & & & A.
\end{tikzcd}$$

Example 1.A.2. G acts compatibly on itself by conjugation.

Example 1.A.3. For A abelian (i.e., a $\mathbb{Z}$–module), this is equivalent to a $\mathbb{Z}[G]$–module structure.

The theorem we want to prove is that $\pi_1 X$ acts compatibly on $\pi_n X$, $n \geq 1$.

Definition 1.A.4. An H–cogroup K acts *compatibly* on an H–cogroup L when the diagram dual to the one above commutes.

Corollary 1.A.5. *For such H–cogroups, the action of $[K, T]$ on $[L, T]$ is compatible.* □

Lemma 1.A.6. *The following defines a compatible coaction of S^1 on S^n:*

Picture!

Proof.

Picture!

[12] □

[12] The proof is a little more obnoxious when $n = 1$.

Remark 1.A.7. There is also a relative version of this story: S^1 has a compatible coaction on (CS^n, S^n), hence $\pi_1 A$ acts compatibly on $\pi_n(X, A)$ for $n \geq 2$.

Lemma 1.A.8. *The action satisfies various naturalities:*

1. $(X, A) \to (Y, B)$ *induces*

$$\begin{tikzcd}
\pi_1 X \times \pi_n X \arrow[r] \arrow[d] & \pi_n X \arrow[d] & \pi_1 A \times \pi_n(X, A) \arrow[r] \arrow[d] & \pi_n(X, A) \arrow[d] \\
\pi_1 Y \times \pi_1 Y \arrow[r] & \pi_n Y, & \pi_1 B \times \pi_n(Y, B) \arrow[r] & \pi_n(Y, B).
\end{tikzcd}$$

2. *The following diagrams commute:*

$$\begin{array}{ccc} \pi_1 A \times \pi_n A & \longrightarrow & \pi_n A \\ \downarrow & & \downarrow \\ \pi_1 X \times \pi_n X & \longrightarrow & \pi_n X, \end{array} \qquad \begin{array}{ccc} \pi_1 A \times \pi_n(X,A) & \longrightarrow & \pi_n(X,A) \\ \downarrow & & \downarrow \\ \pi_1 A \times \pi_{n-1} A & \longrightarrow & \pi_{n-1} A. \end{array}$$

3. *The action of $\pi_1 X$ on itself is by conjugation: $\gamma \cdot \alpha = \gamma\alpha\gamma^{-1}$.* □

Remark 1.A.9. A weak equivalence $f\colon Y \to X$ thus induces an isomorphism of $\mathbb{Z}[\pi_1 X]$–modules, in addition to just abelian groups, a strictly stronger condition. There is much more structure; this is the start of the study of Π–algebras.

Remark 1.A.10. There is a messy version of this that lets us encode the change of basepoint maps from the intro. The main conclusion is that $\pi_n(X, x_0) \cong \pi_n(X, x_0')$ *unnaturally* if $\pi_0 X = *$ and *naturally* if $\pi_1 X = 1$.

See Switzer pg. 47–49

Remark 1.A.11. Take discrete G to act properly discontinuously on X. If $\pi_1 X = 0$, then $\pi_1 X/G \cong \pi_0 G$, hence we get a G–action on $\pi_{\geq 2}(X/G)$. In fact, $\pi_n X \to \pi_n(X/G)$ respects this action.

2
(De)composition of Homotopy Types

Inject introduction

Sufficiently armed with an ample supply of algebraic invariants extracted from homotopy types (e.g., homotopy groups), we now turn to the study of operations on homotopy types. This can be approached from two complementary perspectives. Suppose that we have a "target" homotopy type X in mind, which carries potentially complicated invariants.

1. How can X be decomposed into pieces with simpler invariants?

2. How can X be constructed by pieces with very simple invariants?

2.1 CW complexes

Our first framework for studying these kinds of questions is that of a *cell structure* on X. Informally, such a structure is a kind of presentation of X given by inductively "attaching n–cells".

[4, Definition 5.10]

Definition 2.1.1. Consider a space Y and a continuous map $g\colon \bigvee_\alpha S^{n-1}_\alpha \to Y$. We say that $Y \cup_g \bigvee_\alpha CS^{n-1}$ is *formed from* Y *by attaching* n*–cells* (along g).

There are several motivations for considering this definition. Mostly directly, it falls under the second heading: many commonly considered geometric spaces can be constructed by iteratively applying this operation. Less directly, it also falls under the first heading: the homotopy class $g \in \pi_{n-1}Y$ is given an explicit null-homotopy in $Y \cup_g CS^{n-1}$, so that g vanishes when pushed forward into $Y \cup_g CS^{n-1}$.

We here codify the entire inductive process that we mean:

[4, Definitions 5.1–3]

Definition 2.1.2. We define a *CW–structure* on a space X to be a choice of sequence of spaces X_n satisfying certain properties. The head of the bi-infinite sequence is fixed as $X^{-\infty} = \cdots = X^{-1} = \{x_0\}$. Otherwise, X^n, called the *n–skeleton*, must be formed from X^{n-1} by attaching n–cells. Lastly, X must be the union of the X^n (with the weak topology). If X admits a

If A is a CW-complex, then setting $X^{-\infty} = \cdots = X^{-1} = A$ gives rise to a relative CW-structure (X, A).

CW–structure, then we say that it is a *CW–complex*.

[4, Examples 5.4]

Some topological facts about CW–complexes that might interest you:

- Every CW–complex is Hausdorff.
- Every CW–complex is the disjoint union of the interiors of its cells.
- Each cell has only finitely many immediate faces.
- More generally, any compact subset has this property.

Example 2.1.3. Some common spaces come with the following natural CW–structures:

- S^n can be equipped with a CW–structure with only an n–cells, with boundary attached to the basepoint $\{x_0\}$.

Picture

- More elaborately, S^n can be equipped with an inductive CW–structure: one can form S^n from S^{n-1} by attaching two n–cells, both along the identity map $S^{n-1} \to S^{n-1}$, which serve as the upper- and lower-hemispheres of S^n.

Picture

- The second collection of CW–structures are *compatible*, in the sense that the m–skeleton of S^{n-1} agrees with the m–skeleton of S^n for all $m < n$. Using this, the union of these CW–structures puts a CW–structure on their colimit S^∞.
- Each projective space $\mathbb{RP}^n$ (resp. $\mathbb{CP}^n$, $\mathbb{HP}^n$) can be endowed with a CW–structure where the k–skeleton is given by $\mathbb{RP}^k$ (resp. $\mathbb{CP}^k$, $\mathbb{HP}^k$), and one attaches $\mathbb{R}^n \cong CS^{n-1}$ (resp. $\mathbb{C}^n \cong CS^{2n-1}$, $\mathbb{H}^n \cong CS^{4n-1}$) along $\mathbb{RP}^{n-1}$ (resp. $\mathbb{CP}^{n-1}$, $\mathbb{HP}^{n-1}$) as the 0^{th} affine chart to its complement.
- These CW–structures are also mutually compatible, so that one may form CW–structures on their respective colimits $\mathbb{RP}^\infty$, $\mathbb{CP}^\infty$, and $\mathbb{HP}^\infty$.

The inductive presentation means we can attack problems cell-by-cell. For instance, we have the following simple applications of the gluing operation for topological spaces:

[4, Proposition 5.5]

Corollary 2.1.4. *Let X be a space equipped with a CW–structure as above. A function $f : X \to Y$ is continuous if and only if the restriction*

$$\bigvee_n \bigvee_{\alpha \in A_n} CS^{n-1}_\alpha \to X \to Y$$

to the ensemble of n–cells is continuous. □

Since each CS^{n-1} has a simple structure as a topological space, this is often an easier condition to manage.

It is also often possible to transfer CW–structures along other common topological operations.

[4, pg. 71]

Lemma 2.1.5. *If X carries a CW–structure and Y carries a* finite *CW–structure, then $X \times Y$ carries an induced CW–structure.*

If X and Y are infinite, then the weak topology on $X \times Y$ may not agree with the product of the weak topologies. We mostly concede this point and pick the one we want (usually the former).

Proof sketch. Repeatedly use the homeomorphism $D^n \times D^m \cong D^{n+m}$. □

[4, Exercise 5.14]

Lemma 2.1.6. *If X carries a CW–structure and $A \subseteq X$ is a subcomplex, then X/A carries an induced CW–structure.* □

[4, Proposition 5.6]

Corollary 2.1.7. *By consequence, homotopies and relative homotopies can also be constructed inductively over cells.* □

These cell decompositions imbue maps from some common spaces with moduli-theoretic interpretations.

Example 2.1.8. The set of maps $S^n \to Y$ agrees with the set of maps $D^n \to Y$ which send ∂D^n to y_0.

Example 2.1.9. Given $S^{n-1} \to Y$ and two choices of null-homotopies

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & Y \\ CS^{n-1}, \arrow[ur, shift left] \arrow[ur, shift right] \end{tikzcd}$$

we can form a difference class $S^n \to Y$. Conversely, a map $S^n \to Y$ can be considered as the difference class associated to the null-homotopies of the restriction to the equator witnessed by the action of the map on two hemispheres.

Example 2.1.10. The space $\mathbb{R}P^2$ also participates in such a description. Given a class $\omega\colon S^1 \to Y$ and a null-homotopy of its double

$$\begin{tikzcd} S^1 \arrow[r, "2\omega"] \arrow[d] & Y \\ D^2, \arrow[ur, "H"] \end{tikzcd}$$

we can form a map $\mathbb{R}P^2 \to Y$. Conversely, a map $\mathbb{R}P^2 \to Y$ restricts to a class on $\mathbb{R}P^1$ whose double carries a specified null-homotopy.

We close out today by recording a technical topological result that will be absolutely crucial in underpinning the results in the next two Lectures. Although we do not intend to offer careful proofs of those results, so that it is not *necessary* to state this technical result either, we feel that a topology lecture would be remiss without it and that it gives some intuition as to why these nice results for CW–complexes are possible.

[4, Proposition 6.8]

Lemma 2.1.11 (Simplicial approximation)**.** *Let A be a CW–complex, and let $X = A \cup_g e^n$ consist of A with a single n–cell attached. Take (K, L) to be a* finite *simplicial pair, and consider a continuous map of pairs $f\colon (|K|, |L|) \to (X, A)$. There exists a subdivision (K', L') of (K, L) and a map $f'\colon (|K'|, |L'|) \to (X, A)$ with the following properties:*

1. *The two maps agree on A:*

$$f|_{f^{-1}(A)} = f'|_{f^{-1}(A)}.$$

2. *The two maps are homotopic while fixing their behavior on A:*

$$f \simeq_{rel\, f^{-1}(A)} f'.$$

3. *For each simplex $\sigma \in K'$, if $f'(|\sigma|)$ meets the interior $\mathring{e}^n$ of the new cell in X, then $f'(|\sigma|) \subseteq \mathring{e}^n$ is in fact contained in the new cell* and $f'|_{|\sigma|}$ *restricted to this simplex is a linear map.* □

2.2 The homotopy theory of CW complexes I

In the previous section, we discussed CW–structures as placed on a preordained topological space X. One can also take the perspective that the CW–structure is the primary data, and the colimit X simply is whatever it is. From this second viewpoint, the most important property of CW–complexes which makes them suitable for use in homotopy theory is that if the attaching maps g are perturbed within their homotopy class to some $g' \sim g$, the resulting CW–complex X' is homotopy equivalent to the original CW–complex X. As ever, this is easiest to argue cell-by-cell.

Lemma 2.2.1. *Given two maps $g_1, g_2 \colon S^{n-1} \to A$, consider the associated cell complexes $X_1 = CS^{n=1} \cup_{g_1} A$ and $X_2 = CS^{n-1} \cup_{g_2} A$. A homotopy $H \colon g_1 \sim g_2$ begets a homotopy equivalence $X_1 \xrightarrow{\simeq} X_2$.*

Find me a citation.

Construction. Subdivide CS^{n-1} into an outer annulus (corresponding to the cone coordinate range $[1/2, 1]$) and an inner disk (corresponding to the cone coordinate range $[0, 1/2)$). The homotopy equivalence is then given by gluing the identity map on A, the homotopy H on the outer annulus, and the homeomorphism $S^{n-1} \wedge [0, 1/2] \cong CS^{n-1}$ on the inner disk. □

This is the most basic property that one could ask of CW–complexes, but in fact they are remarkably well-behaved homotopically. Our goal today is to get *familiar* with some of these features and to use them to compute $\pi_{*\le n} S^n$.

But not prove!

Lemma 2.2.2. *For (X, A) a relative CW-complex, the relative pair $(X, (X, A)^n)$ formed from the n–skeleton is n–connected.* □

[4, Theorem 6.10]
cf. Definition 1.5.4

Corollary 2.2.3. *The inclusion $X^n \to X$ is n–connected.* □

[4, Theorem 6.11]

Corollary 2.2.4. $\pi_{<n} S^n = 0$.

[4, Corollary 6.12]

Proof. We deploy the cell structure on S^n with $(S^n)^{n-1} = \{s_0\}$. The long exact sequence of relative homotopy groups takes the form

Corollary 1.5.3

$$\cdots \to \pi_k\{s_0\} \to \pi_k S^n \to \pi_k(S^n, \{s_0\}) \to \pi_{k-1}\{s_0\} \to \cdots .$$

The outer terms are always zero, as the homotopy groups of a singleton space. The relative term $\pi_k(S^n, \{s_0\})$ vanishes for $k < n$, using Corollary 2.2.3. Altogether, this shows the same of the non-relative groups $\pi_{<n} S^n$. □

Lemma 2.2.5. *If (X, A) is n–connected, then there exists an equivalence $(X, A) \sim (X', A')$ with $(X', A')^n = A'$.* □

[4, Proposition 6.13]

This is a kind of converse to the previous Lemma.
Taking $A = *$, this says that if $\pi_{<n} X = 0$, then there is a CW–model of X with no cells below dimension n [4, Corollary 6.14].

Corollary 2.2.6. *For X an n-connected CW-complex and Y an m-connected CW-complex, the CW-complex $X \wedge Y$ is $(n+m+1)$-connected.*

[4, Corollary 6.15]

Proof. The cells in $X \times Y$ take the form $* \times *$, $* \times e^j_\beta$, $e^i_\alpha \times *$, and $e^i_\alpha \times e^j_\beta$. All the first three classes lie within $X \vee Y$, so the first nontrivial surviving cell in $X \wedge Y$ lies in dimension at least $n+m+2$. □

This cell structure argument also feeds into the claim that the suspension of a CW complex has predictable cell structure and predictable attaching maps.

Some of these miraculous properties again look like theorems we've previously seen for homology—but with a bound imposed, depending on the connectivities of the spaces involved.

Theorem 2.2.7 (Homotopy excision)**.** *Let $A, B \subseteq X$ be subspaces of X such that $(A, A \cap B)$ is an n-connected pair and $(B, A \cap B)$ is an m-connected pair. The natural double inclusion $\pi_*(A, A \cap B) \to \pi_*(X, B)$ is an isomorphism for $* < n+m$ and an epimorphism for $* = n+m$.* □

[4, Theorem 6.21]

Corollary 2.2.8. *Suppose that (X, A) is an n-connected pair and that A is m-connected. Then the natural map $\pi_*(X, A) \to \pi_* X/A$ is an isomorphism for $1 < * \leq n+m$ and an epimorphism at $n+m+1$.*

[4, Corollary 6.22]

Proof. Consider the space $X \cup_A CA$, as well as the subspaces CA and X inside of it. Noting that $CA \cap X = A$ as subspaces of $X \cup_A CA$, Theorem 2.2.7 then gives the desired conclusion for the natural map

$$\pi_*(X, A) \to \pi_*(X \cup_A CA, CA).$$

The right-hand group can be made non-relative by passing along the projection, as in

$$(X \cup_A CA, CA) \xrightarrow{\simeq} (X \cup_A CA/CA, *).$$

To conclude, apply Lemma 1.4.5 to produce a weak equivalence

$$X/A \xrightarrow{\simeq} X \cup_A CA/CA.$$ □

There should be some Lemma recording that working relative to a contractible subspace is the same as quotienting out the subspace. This is 6.6 in Switzer.

Corollary 2.2.9 (Freudenthal suspension theorem)**.** *For an n-connected CW-complex X, the natural map*

[4, Theorem 6.26]

The left-hand vertical map is given by restriction to the boundary. Its participation in the long exact sequence of relative homotopy groups shows it to be an equivalence, since CX is contractible.

$$\begin{tikzcd} \pi_{*+1}(CX, X) \arrow[r] \arrow[d, equal] & \pi_{*+1}(CX/X) \arrow[d, equal] \\ \pi_* X \arrow[r] & \pi_{*+1}\Sigma X \end{tikzcd}$$

is an isomorphism for $* \leq 2n$ and an epimorphism for $* = 2n+1$. □

Example 2.2.10. Consider the following fibration:

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r] \arrow[d, "\simeq"] & \mathbb{C}^n \setminus 0 \arrow[r] \arrow[d, "\simeq"] & \mathbb{CP}^{n-1} \arrow[d, "\simeq"] \\ S^1 \arrow[r] & S^{2n-1} \arrow[r] & \mathbb{CP}^{n-1}. \end{tikzcd}$$

[4, Theorem 6.28]

The top row presents this fibration as a fiber bundle. To see that it is a fiber bundle, we rely on Example 1.7.5 and the sequence

$$U(1) \to \frac{U(n)}{U(n-1)} \to \frac{U(n)}{U(n-1) \times U(1)}.$$

Since S^{2n-1} is $(2n-2)$–connected, we conclude $\pi_{*+1}\mathbb{C}\mathrm{P}^{n-1} \cong \pi_* S^1$ for $* \le 2(n-1)$. Using Example 1.7.14, we may conclude

$$\pi_* \mathbb{C}\mathrm{P}^n \cong \begin{cases} \mathbb{Z} & \text{when } * = 2, \\ 0 & \text{when } * < 2n-1 \text{ and } * \neq 2, \\ ??? & \text{otherwise.} \end{cases}$$

We also know that $(\mathbb{C}\mathrm{P}^{n-1}, \mathbb{C}\mathrm{P}^1)$ is 2–connected, from which we may conclude $\pi_2 \mathbb{C}\mathrm{P}^1 \cong \pi_2 \mathbb{C}\mathrm{P}^{n-1} \cong \pi_1 S^1 \cong \mathbb{Z}$, even though this is outside of the range directly accessible by the fibration alone. This then feeds into Freudenthal as applied to the sphere, which for $n \ge 2$ gives

$$\pi_n S^n \cong \pi_{n+1} S^{n+1}.$$

Hence, we ultimately conclude $\pi_n S^n \cong \mathbb{Z}$ for all $n \ge 1$.

2.3 *The homotopy theory of CW complexes II*

We continue our study of the pleasant homotopical properties of CW–complexes. The following technical lemma appeared in our study of the relative homotopy long exact sequence of a pair (Y, B):

[4, Proposition 3.14]

Lemma 2.3.1. *For all solid squares*

Backreference.

$$\begin{tikzcd} B \arrow[r] & Y \\ S^{n-1} \arrow[u, "\omega|_{S^{n-1}}"] \arrow[r] & D^n \arrow[u, "\omega"'] \arrow[ul, dotted, "\omega'"] \end{tikzcd}$$

there exists a dashed filler for which the top-right triangle commutes up to homotopy. □

Add a 2–cell from Y to filler arrow.

This can be augmented in two ways: first, by extending it to cover more complicated sub-complexes, and second, by using it to govern the behavior of maps.

[4, Theorem 6.30]

Lemma 2.3.2. *If $f\colon Z \to Y$ is an n–equivalence and* $\dim(X, A) \le n$, *then for each solid square*

$$\begin{tikzcd} Z \arrow[r, "f"] & Y \\ A \arrow[u, "g"] \arrow[r] & X \arrow[u, "h"'] \arrow[ul, dotted] \end{tikzcd}$$

there exists a dashed filler such that the top-right triangle commutes up to homotopy. □

Add a 2–cell from Y to filler arrow.

[4, Theorem 6.31]

Corollary 2.3.3. *Given an n–equivalence $f\colon Z \to Y$ and a CW–complex X of dimension at most n, the natural map*

$$f_*\colon [X, Z] \to [X, Y]$$

is surjective. If the dimension of X is exactly n, f_* is an isomorphism. □

These results ultimately culminate in the following:

[4, Theorem 6.32]

Corollary 2.3.4 (Whitehead)**.** *A weak equivalence $f\colon Z \to Y$ of CW-complexes is a homotopy equivalence.* □

Here, finally, is justification that relative homotopy groups truly do measure the discrepancy between two spaces: if the discrepancy vanishes, then the two spaces are equivalent in the homotopy category.

Provided that those spaces are sufficiently nice!

Filtering a CW-complex by its skeleta and applying the Lemma yields the second thread of useful results.

[4, Definition 6.34]

Definition 2.3.5. A map $f\colon X \to Y$ of CW–complexes X and Y is said to be *cellular* if it carries the k^{th} skeleton of X to the k^{th} skeleton of Y:

$$f(X^k) \subseteq Y^k.$$

[4, Proposition 6.35]

Corollary 2.3.6. *Every map $f\colon (X, A) \to (Y, B)$ of CW-complexes is homotopic (relative to A) to a cellular map, and homotopies between cellular maps admit cellular replacements.* □

Together, these results grant us serious control over how the homotopy groups of a CW–complex change as its skeleta are built up—in the qualitative sense of *which* groups change. This can be deployed to manufacture certain interesting spaces, called Eilenberg–Mac Lane spaces, which we spend the rest of today doing.

[4, Proposition 6.36]

Corollary 2.3.7. *Let X be an n–connected CW–complex, and let Y be an m–connected CW–complex. The natural map $X \vee Y \to X \times Y$ induces an isomorphism in homotopy through a range:*

$$\pi_{\le n+m}(X \vee Y) \xrightarrow{\cong} \pi_{\le n+m}(X \times Y)$$

Proof. Last time, we argued that the cellular structure induced on $X \times Y$ was completely contained in $X \vee Y$ through a range:

$$(X \times Y, X \vee Y)^{n+m+1} = (X \vee Y)^{n+m+1}.$$

The relative homotopy groups of the pair thus vanish through this range, and the long exact sequence yields the desired statement. □

[4, Corollary 6.37]
Also, $\pi_1 \bigvee_\alpha S^1_\alpha \cong \mathop{\ast}_\alpha \pi_1 S^1_\alpha$.

Corollary 2.3.8. *For $n \ge 2$,*

$$\pi_n\Big(\bigvee_\alpha S^n_\alpha\Big) \cong \bigoplus_\alpha \pi_n S^n_\alpha. \quad \square$$

[4, Theorem 6.39.i]

Lemma 2.3.9. *For any abelian group A and index $n \ge 2$, there exists a CW–complex $K(A, n)$ whose homotopy groups satisfy*

$$\pi_* K(A, n) = \begin{cases} A & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Select a presentation

$$0 \to \mathbb{Z}^J \xrightarrow{g} \mathbb{Z}^I \to A \to 0.$$

We intend to build a coexact sequence of $(n-1)$–connected spaces whose behavior on π_n encodes the chosen presentation. Begin by modeling the middle node as $\bigvee_I S^n$. Combining Example 2.2.10 and Corollary 2.3.8 gives

Explain one of the preceding results as saying that coexact sequences of spaces behave like exact sequences through a range.

$$\pi_n\left(\bigvee_I S^n\right) \cong \bigoplus_I \mathbb{Z}.$$

The J–sized set of elements selected by g gives a J–sized set of maps $S^n \to \bigvee_I S^n$, and hence a single map $\tilde{g}\colon \bigvee_J S^n \to \bigvee_I S^n$ which induces g on π_n. The cone on g gives a complex X_n with $\pi_{*<n} X_n = 0$ and $\pi_n X_n = A$. We inductively form X_{n+j+1} from X_{n+j} by killing the homotopy in degree $n+j+1$ by coning off any homotopy classes we find. Using the long exact sequence of a relative pair, we see that this coning operation never disturbs the homotopy groups at or below $n+j$, so the colimit indeed provides $K(A,n)$. □

We're not there yet, but X_n is sometimes called a *Moore space* (of dimension n, for the group A), because $H_n(X_n;\mathbb{Z}) \cong A$.

Lemma 2.3.10. *Let X be a space satisfying $\pi_{<n} X = 0$, and let Y be a space satisfying $\pi_{>n} Y = 0$. Then homotopy classes $[X,Y]$ biject with homomorphisms $\pi_n X \to \pi_n Y$.*

[4, Theorem 6.39.ii]

X is sometimes said to be n*–connective* and Y to be n*–coconnective*.

Proof. Using Lemma 2.2.5, there exists a CW model of X with $X^{n-1} = *$. The rest of the presentation of X looks like:

$$\begin{tikzcd}
\bigvee_I S^n \arrow[r, equal] & X^n \arrow[r] & X^{n+1} \arrow[r] & X^{n+2} \arrow[r] & \cdots \arrow[r] & X \\
& \bigvee_J S^n \arrow[u] & \bigvee S^{n+1} \arrow[u] & \bigvee S^{n+2} \arrow[u] & \cdots . \arrow[u]
\end{tikzcd}$$

Using this presentation, we can inductively study the available maps into Y. We begin with

$$[X^n, Y] = \left[\bigvee_I S^n, Y\right] = \bigoplus_I \pi_n Y.$$

To extend along the $(n+1)$–skeleton, the precomposite

$$\bigvee_J S^n \to \bigvee_I S^n \to Y$$

must vanish, and the unicity of each such extension is measured by $[\Sigma \bigvee_J S^n, Y] = 0$, so that we have an exact sequence

$$\begin{tikzcd}
{[X^{n+1}, Y]} \arrow[r] \arrow[d, equal] & {[X^n, Y]} \arrow[r] \arrow[d, equal] & {\left[\bigvee_J S^n, Y\right]} \arrow[d, equal] \\
\mathsf{Groups}(\pi_n X, \pi_n Y) \arrow[r] & \bigoplus_I \pi_n Y \arrow[r] & \bigoplus_J \pi_n Y.
\end{tikzcd}$$

For all higher stages, both the obstruction to extension vanishes (because $\pi_{n+k} Y = 0$) in addition to the unicity (because $\pi_{n+k+1} Y = 0$). □

For a generic CW–complex Y, without assumptions on its $\pi_{>n}$, the relative group $\pi_n(Y^n, Y^{n-1})$ is free on generators $\gamma \cdot [f_\alpha^n]$, where $\gamma \in \pi_1 X^{n-1}$, and f_α^n is the characteristic map of an n–cell. [4, Corollary 6.42]

Corollary 2.3.11. *$K(A,n)$ is independent of choice of presentation.*

Proof. Consider two models X, Y for $K(A,n)$. Both X and Y satisfy the conditions of the Lemma, so that the identity map id: $A \to A$ lifts to a map $\widetilde{\mathrm{id}}\colon X \to Y$. This map $\widetilde{\mathrm{id}}$ of CW–complexes is an isomorphism on homotopy groups, hence Whitehead's theorem witnesses it as a homotopy equivalence. □

[4, Remark 6.45]

Remark 2.3.12. Using Corollary 1.6.8, we may conclude that $\Omega K(A,n)$ has the same homotopy groups as $K(A,n-1)$. The Corollary shows that it, in fact, does model $K(A,n-1)$ via a map

$$\Omega K(A,n) \xrightarrow{\simeq} K(A,n-1).$$

However, the adjoint map $\Sigma K(A,n-1) \to K(A,n)$ is more mysterious. It can be taken to be an inclusion on $(n+1)$–skeleta, but otherwise it appears little can be said about it at this point. We will meet it again.

[4, Exercise 6.49]

Remark 2.3.13. For *all* spaces X, there exists a CW-complex $\widetilde{X} \to X$ such that the map is a weak equivalence.

This might be out of place. Trying to justify it probably reveals where it belongs.

2.4 *Spectral sequences*

Our treatment here deviates from Switzer's, which can be found on [4, pg. 336–340]. Our approach is closer to that of Boardman [1], though he is much more technically intense (and, of course, more rigorous).

The argument we gave in the proof of Lemma 2.3.10 felt quite serendipitous and fragile. Had our hypotheses been even slightly weaker, we would have quickly found it very difficult to keep track of all the interacting groups involved. However, this kind of scenario is extremely common, where a space X of interest is constructed from an infinite sequence of steps and where we hope to compute some invariant, like $\pi_*(X)$, from knowledge of the relative invariants, like $\pi_*(X_n, X_{n-1})$. It is so common, in fact, that topologists have worked to codify the formal properties of this scenario, which goes by the name of a *spectral sequence*. Today we recount a mild specialization of this framework that will suit us well.

As with cells in a CW–complex.

More generally, one might discuss *exact couples*.

We begin by arranging into a single diagram the exact sequences associated to each inclusion:

$$\begin{tikzcd}
0 \arrow[r] & \pi_* X_1 \arrow[r] & \pi_* X_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] \arrow[dl] & \pi_* X_n \arrow[r] \arrow[dl] & \pi_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] \arrow[dl] & \pi_* X \\
& \pi_*(X_2, X_1) \arrow[u, red] & \pi_*(X_3, X_2) \arrow[u, red] & \cdots \arrow[u, red] & \pi_*(X_{n+1}, X_n) \arrow[u, red] & \pi_*(X_{n+2}, X_{n+1}) \arrow[u, red] & \cdots,
\end{tikzcd}$$

where each triangle is a "rolled up" exact sequence and each red arrow shifts degree by one, e.g., $\pi_*(X_{n+1}, X_n) \to \pi_{*-1} X_n$. Let us study in earnest the problem of recovering $\pi_* X$ only by probing information about the groups $\pi_*(X_{n+1}, X_n)$ on the bottom row. Our first observation is that since each sphere is compact, every class $x \in \pi_* X$ must lift to some $\widetilde{x} \in \pi_* X_n$. However, since this is not a relative group, we are not permitted to make direct use of it, and we should instead consider the image of this class $x_n \in \pi_*(X_n, X_{n-1})$ in the bottom row. Our second observation is that at exactly the *minimal* such n to which (a nonzero class) x lifts to $\widetilde{x} \in \pi_* X_n$, it pushes down to give a nonzero class $x_n \in \pi_*(X_n, X_{n-1})$. This explains some of the classes in $\pi_*(X_n, X_{n-1})$. How can we discern the classes that come from these minimal preimages? What use are those classes that don't?

If we permitted ourselves access to the top row of $\pi_* X_n$, we could take a colimit and be done.

Let us thus begin from the other vantage point by selecting a class $x_n \in \pi_*(X_n, X_{n-1})$ and asking when there is a class $\widetilde{x} \in \pi_* X_n$ of which it is the image.

$$\begin{tikzcd}
 & & {\color{red}\partial(x_n)} \arrow[dddl, mapsto, red] & {\color{red}(\exists\, \widetilde{x}???)} \arrow[dddl, mapsto, dotted, red] & \\
\cdots \arrow[r] & \pi_* X_{n-1} \arrow[r] & \pi_* X_n \arrow[r] \arrow[dl] & \pi_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & \pi_*(X_n, X_{n-1}) \arrow[u] & \pi_*(X_{n+1}, X_n) \arrow[u] & \pi_*(X_{n+2}, X_{n+1}) \arrow[u] & \\
 & {\color{red}d_1(x_n)} & {\color{red}x_n} \arrow[uuu, mapsto, bend left, red] & &
\end{tikzcd}$$

From x_n, we can manufacture two more classes: $\partial(x_n) \in \pi_* X_n$ and $d_1(x_n) \in \pi_*(X_n, X_{n-1})$. The existence of $\widetilde{x}$ is determined by whether $\partial(x_n)$ is nonzero—but, since it is not on the bottom row, we cannot ask about it directly. We are allowed to probe $d_1(x_n)$ directly. If it is not zero, then surely $\partial(x_n)$ is also not zero, which settles the question definitively that a lift $\widetilde{x}$ cannot exist. On the other hand, if it is zero, then it could be the case that $\partial(x_n)$ is zero (so that $\widetilde{x}$ exists) or that $\partial(x_n)$ is nonzero and merely in the kernel of the map $\pi_* X_n \to \pi_*(X_n, X_{n-1})$. In the second case, we can use exactness to build a preimage $i_n^{-1}\partial(x_n) \in \pi_* X_{n-1}$, as in:

$$\begin{tikzcd}
 & & {\color{red}i_n^{-1}\partial(x_n)} \arrow[r, mapsto, red] \arrow[dddl, mapsto, red] & {\color{red}\partial(x_n)} \arrow[dddl, mapsto, red] & {\color{red}(\exists\, \widetilde{x}???)} \arrow[dddl, mapsto, dotted, red] & \\
\cdots \arrow[r] & \pi_* X_{n-2} \arrow[r] & \pi_* X_{n-1} \arrow[r] & \pi_* X_n \arrow[r] \arrow[dl] & \pi_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & \pi_*(X_{n-1}, X_{n-2}) \arrow[u] & \pi_*(X_n, X_{n-1}) \arrow[u] & \pi_*(X_{n+1}, X_n) \arrow[u] & \pi_*(X_{n+2}, X_{n+1}) \arrow[u] & \\
 & {\color{red}d_2(x_n)} & {\color{red}0} & {\color{red}x_n} \arrow[uuu, mapsto, bend left, red] & &
\end{tikzcd}$$

We make a series of claims:

1. This assignment d_2 is well-defined up to the image of d_1, and hence it determines a function $d_2\colon H_*(\pi_*(X_*, X_{*-1}); d_1) \to H_{*-1}(\pi_*(X_{*-2}, X_{*-3}); d_1)$.

2. This process continues step-by-step exactly as written. Since $X_0 = *$, eventually the preimage is guaranteed to be zero, hence $\partial(x_n) = 0$, and hence $\tilde{x}$ exists.

3. The surviving elements in a spectral sequence are the associated graded of a filtration of $\pi_* X$ (by minimal lift degree).

Introduce some terminology, especially *pages*.

Remark 2.4.1. This story can be retold with many variations. Some introduce no further complexity:

- In place of exact sequences of spaces and relative homotopy groups, one can use coexact sequences of spaces and a homology functor.

Are there other simple variants?

Others introduce substantial complexity:

- The filtration can be bi-infinite, dropping the assumption that $X_0 = *$.
- The spectral sequence can be "infinite to the left" (e.g., when applying a *cohomology* functor to a filtration by coexact sequences).
- If one does not introduce assumptions to avoid this, since homotopy groups of low orders are not abelian groups, their homological algebra is substantially more complicated (especially when coupled to the above variants).

The essential complexity introduced by the first two points is that such spectral sequences need not *stabilize*. In order to handle this, one has to incorporate taking inverse limits of the subquotients of $H_* A_*$, which can destroy some of the exactness in the third Claim or the argument used to justify the second Claim.

Spectral sequences have the simultaneous pleasant features of being valuable computational tools while also being sufficiently rigid that one can use them to prove theorems without computing anything. The following result is an example of their rigidity:

Lemma 2.4.2. *A map of spectral sequences is a family of homomorphisms from the "bottom row" of one to the "bottom row" of the other which additionally commute with all of the d_r maps. If a map of spectral sequences is an isomorphism on the r^{th} page for any r, it is an isomorphism forever after. Their targets are also isomorphic by the same map.* □

Example 2.4.3. Consider the filtration of a CW-complex X by its skeleta. The filtration stages participate in coexact sequences with spheres of a fixed dimension, from which we conclude

$$\widetilde{H}_*(X_n, X_{n-1}; A) = \widetilde{H}_*\left(\bigvee_\alpha S^n_\alpha; A\right) = \bigoplus_\alpha \Sigma^n A.$$

The map d_1 associated to the spectral sequence is exactly the cellular differential, as can be seen by drawing the defining diagram:

$$\begin{tikzcd} \widetilde{H}_* X_{n_1} \arrow[r] & \widetilde{H}_* X_n \arrow[dl] \\ \widetilde{H}_* \bigvee_\beta S^{n-1}_\beta \arrow[u] & \widetilde{H}_* \bigvee_\alpha S^n_\alpha . \arrow[l, "d_1"] \arrow[u] \end{tikzcd}$$

It follows that the E_2–page is given by $H_*(H_*(X_*, X_{*-1}); d_1) = \widetilde{H}^{\text{cell}}_*(X;A)$. All higher differentials are zero because

$$d_r \colon \bigoplus_\alpha \Sigma^n A \xrightarrow{[-1]} \bigoplus_\gamma \Sigma^{n-r} A$$

has the wrong degree. We say that the spectral sequence *collapses at E_2*, and this is a proof of that cellular homology computes homology.

Corollary 2.4.4. *More genrally, if E and F satisfy Eilenberg–Steenrod and $E_*(S^n) \to F_*(S^n)$ is an isomorphism for all n, then $E_*(X) \to F_*(X)$ is an isomorphism for all CW complexes X.* □

Example 2.4.5. The Eilenberg–Mac Lane spaces play another important role in homotopy theory which we will more seriously explore in Chapter 3. Fixing an abelian group A and letting n vary, the functors

$$X \mapsto [X, K(A,n)]$$

satisfy the Eilenberg–Steenrod axioms: they carry wedge sums to products, and they convert coexact sequences of spaces to long exact sequences of groups. Moreover, there is a natural transformation

$$\begin{aligned} [X, K(A,n)] &\to H^n(X;A), \\ f &\mapsto f^* \iota_n, \end{aligned}$$

where $\iota_n \in H^n(K(A,n);A)$ represents the identity map

$$H_n(K(A,n);\mathbb{Z}) \cong A \xrightarrow{\text{id}} A.$$

It then follows from Corollary 2.4.4 that this natural transformation is actually a natural *isomorphism*.

2.5 Obstruction theory

Today we exploit the machinery of spectral sequences to organize and extend some of our results in the realm of *obstruction theory*. Obstruction theory is generally concerned with starting with a map

$$A \longrightarrow B$$

and extending it to a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, dotted] & B \\ X \arrow[r, dotted] \arrow[ur, dotted] & Y \arrow[u] \end{tikzcd}$$

where any one of the dashed maps might be the goal. We will focus our efforts by setting $A = B = *$, so that we will equivalently describe a method to compute $\pi_0 Y^X$ from the data of X and of Y separately.

The "relative" cases where A and B are nonzero we leave to the interested reader.

Recall one the following Lemma from our study of the homotopy-theoretical properties of CW–complexes:

Lemma 2.3.10

Lemma 2.5.1. *For Y an n–connective CW–complex and Z an n–coconnective CW–complex, the natural map $[Y, Z] \to \mathsf{Groups}(\pi_n Y, \pi_n Z)$ is an isomorphism.* □

Recall that n–connective is a synonym for $(n-1)$–connected, and n–coconnective for $\pi_{>n} Z = 0$.

This Lemma falls squarely into the realm of obstruction theory: it gives an algebraic description of what maps between certain homotopy types exist. Its proof also relied on spectral-sequence-style machinery, so it seems ripe for us to generalize now.

In the moment, however, we only needed the Lemma as part of a program to construct and analyze certain spaces called $K(A, n)$. It was the key ingredient in showing that the presentation of A used in their construction was purely auxiliary, and that $K(A, n)$ itself was a well-defined object. It has another application: the construction of *Postnikov towers*.

Corollary 2.5.2. *If Y is n–connective, then there is a canonical map*

$$Y \to K(\pi_n Y, n)$$

induced by the identity on π_n. □

Corollary 2.5.3. *For Y an n–connective space, the exact extension of the canonical map $Y \to K(\pi_n Y, n)$ is called the $(n+1)$–upward-truncation of Y, denoted $Y(n, \infty)$ or $Y[n+1, \infty)$. It has the properties*

$$\pi_* Y(n, \infty) = \begin{cases} \pi_* Y & \text{if } * > n, \\ 0 & \text{otherwise} \end{cases}$$

and $\pi_* Y(n, \infty) \to \pi_* Y$ is an isomorphism for $* > n$.

In fact, it is the universal n–connected space over Y: any map in from any other such n–connected space factors through $Y(n, \infty)$.

Proof. This is entirely an application of the long exact sequence of relative homotopy. □

Definition 2.5.4. Starting with a 0–connected space and repeatedly applying Corollary 2.5.3 leads to the *Postnikov tower*,

which is a diagram of interlocking exact sequences.

$$\begin{tikzcd} Y \arrow[d] & Y(1,\infty) \arrow[l] \arrow[d] & Y(2,\infty) \arrow[l] \arrow[d] & \cdots \arrow[l] & Y(n,\infty) \arrow[l] \arrow[d] & \cdots \arrow[l] & * \arrow[l] \\ K(\pi_1 Y,1) & K(\pi_2 Y,2) & K(\pi_3 Y,3) & & K(\pi_{n+1} Y, n+1), \end{tikzcd}$$

This situation is ripe for a spectral sequence. We apply $\pi_* F(X,-)$, where X is some fixed test space. The functor $F(X,-)$ preserves exact sequences, π_* turns them into long exact sequences, and Example 2.4.5 shows that $\pi_* F(X, K(\pi_{n+1} Y, n+1))$ has a recognizable form. Bundling these results together gives:

Simply applying π_* to this diagram gives an extremely boring spectral sequence!

Definition 2.5.5. *Federer's spectral sequence* has signature

$$\begin{aligned} E^1_{m,n} &= \pi_m F(X, K(\pi_{n+1} Y, n+1)) \\ &= \widetilde{H}^{n-m+1}(X; \pi_{n+1} Y) \Rightarrow \pi_m F(X,Y). \end{aligned}$$

As previously remarked, for arbitrary X and Y the objects π_0 and π_1 of $F(X,Y)$ are *sets* and *groups* respectively. This situation is called a *fringed spectral sequence*, which is considerably more obnoxious. Ensure $X = \Sigma^2 X'$ or $Y = \Omega^2 Y'$ for an easy way to avoid this situation.

Remark 2.5.6. If $\pi_{<n} X = 0$ and $\pi_{>n} Y = 0$, we have $H^{<n}(X; \text{any}) = 0$ and $H^{\text{any}}(X; \pi_{>n} Y) = 0$. This puts a single nonvanishing group in the region of the spectral sequence contributing to $\pi_0 Y^X$:

$$H^n(X; \pi_n Y) \cong \mathsf{AbGps}(H_n X, \pi_n Y) \cong \mathsf{AbGps}(\pi_n X, \pi_n Y).$$

Since this group cannot be the source or target of any differentials, we conclude Lemma 2.3.10:

$$\mathsf{AbGps}(\pi_n X, \pi_n Y) \cong F(X,Y).$$

Corollary 2.5.7. *Ordinary cohomology is representable:*

$$\widetilde{H}^n(X;A) \cong [X, K(A,n)].$$

Proof. Take $Y = K(A,n)$ in Definition 2.5.5. The spectral sequences collapses to give

$$\pi_m F(X, K(A,n)) \cong \widetilde{H}^{n-m}(X;A). \qquad \square$$

Remark 2.5.8. The relative version of this spectral sequence also recovers Lemma 2.3.2.

Even though we have constructed this very appealing machine, we must stop short of actually using it to do anything. Nothing comes for free: the price of organizing the information in the E_1–term of a spectral sequence in a pretty way means that the differentials will surely be difficult to understand. To see what can happen, consider the d_1–differential, which is induced by pushforward along the map k_n in

$$\begin{tikzcd} Y[n,\infty) \arrow[dd] & & Y(n,\infty) \arrow[ll] \arrow[dd] \\ & \Omega K(\pi_n Y, n) \arrow[ur] \arrow[dr, bend left, "k_n"] & \\ K(\pi_n Y, n) & & K(\pi_{n+1} Y, n+1). \end{tikzcd}$$

This map is part of the homotopy data of Y called the n^{th} *k–invariant* of Y. As noted in Remark 2.3.12, we have a weak equivalence

Indeed, it appears naturally when constructing $Y = X^{S^0}$ from S^0 using obstruction theory. Also, the "k" is presumably the same "K" appearing in "$K(A,n)$".

$$\Omega K(\pi_n Y, n) \simeq K(\pi_n Y, n-1),$$

so that k_n can be considered as a map of Eilenberg–Mac Lane spaces:

$$k_n \colon K(\pi_n Y, n-1) \to K(\pi_{n+1} Y, n+1).$$

This induces a natural transformation

$$\begin{array}{ccc} [-, K(\pi_n Y, n-1)] & \xrightarrow{(k_n)_*} & [-, K(\pi_{n+1} Y, n+1)] \\ \| & & \| \\ H^{n-1}(-; \pi_n Y) & \longrightarrow & H^{n+1}(-; \pi_{n+1} Y). \end{array}$$

This leaves us with some burning questions which we must answer before using this spectral sequence with any seriousness:

1. What do natural transformations of the cohomology functor look like in general? How many are there? Is there a classification, or a general formula?

This is kind of the "dual problem" to trying to compute $[S^m, S^n]$.

2. How can they be discerned / extracted for some space Y?

2.6 *A complicated example*

We haven't yet had an example of a spectral sequence in which we can compute. This is because at our stage it is impossible to find a spectral sequence that is easy, tangible, nontrivial, and well-motivated all at once. Today we will retreat to algebra in order to work an example that covers the first three attributes, with the promise that it will become well-motivated later on.

This lecture is vastly underbaked.

$\mathscr{A}(1)_* = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$ with $\overline{\Delta}\xi_1 = 0$ and $\overline{\Delta}\xi_2 = \xi_1 \mid \xi_1^2$. The *(reduced) cobar complex* associated to $\mathscr{A}(1_*$ is given by $C^n(\mathscr{A}(1)_*) = \overline{\mathscr{A}}(1)^{\otimes n}$ with differential an alternating sum of $\overline{\Delta}$s.

One can filter this complex using powers of the augmentation ideal, which on the associated graded looks like the cobar complex for an exterior algebra, which has homology given by a polynomial algebra on homology-suspended shifts of the same classes.

Theorem 2.6.1 (May)**.** *This gives a spectral sequence of algebras*

$$E_1^{*,*,*} \cong \mathbb{F}_2[h_{10}, h_{11}, h_{20}] \Rightarrow H^*(\mathscr{A}(1)_*). \quad \square$$

We can even compute differentials in this spectral sequence.

Lemma 2.6.2. $d(h_{20}) = h_{10} h_{11}$.

Proof. h_{20} represents ξ_2, which has differential $d(\xi_2) = \Delta\xi_2 = \xi_1 \mid \xi_1^2$, which is represented by $h_{10}h_{11}$. □

Lemma 2.6.3. $d(h_{20}^2) = h_{11}^3$.

Proof. h_{20}^2 represents $\xi_2 \mid \xi_2$, which has differential

$$d(\xi_2 \mid \xi_2) = \xi_1 \mid \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1 \mid \xi_1^2,$$

which is represented by $2h_{10}h_{11}h_{20} \equiv 0$. By finding a preimage of this class along the cobar differential, we can perturb the representative cocycle and get a longer May differential:

$$\begin{aligned} d(\xi_2 \mid \xi_2 + \xi_1\xi_2 \mid \xi_1^2 + \xi_1 \mid \xi_2\xi_1^2) = {} & (\xi_1 \mid \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1 \mid \xi_1^2) \\ & + (\xi_1^2 \mid \xi_1^2 + \xi_1 \mid \xi_2 + \xi_2 \mid \xi_1 + \xi_1 \mid \xi_1^3) \mid \xi_1^2 \\ & + \xi_1 \mid (\xi_2 \mid \xi_1^2 + \xi_1 \mid \xi_1^4 + \xi_1^2 \mid \xi_2 + \xi_1^3 \mid \xi_1^2) \\ = {} & \xi_1^2 \mid \xi_1^2 \mid \xi_1^2, \end{aligned}$$

which is represented by h_{11}^3. □

This is the last differential in the spectral sequence.

Remark 2.6.4.

Expand remark about Leibniz rule into the Christianson-style presentation of the spectral sequence.

Include pictures of these SS pages.

3
Representability

This used to live in the obstruction theory section, where it no longer makes sense because we don't know what infinite loopspaces are. It probably belongs somewhere in this chapter.

3.1 Brown representability

Early on, when we were getting used to the categorical approach to homotopy theory, we noted in Remark 1.1.2 that $\mathsf{Spaces}(-, T)$ forms a *sheaf* on the category Spaces. This isn't our only example of such an object: cohomology functors can also be thought of as sheaves, as the Eilenberg–Steenrod axioms include the sheaf axioms as a subset. However, in the course of our study of the homotopy theory of CW–complexes, we discovered that these two examples are actually not separate: Corollary 2.5.7 gave a natural isomorphism

$$\widetilde{H}^n(X; A) \xrightarrow{\cong} [X, K(A, n)],$$

where $K(A, n)$ is an *Eilenberg–Mac Lane space* as in Lemma 2.3.9. Today we prove that this is not an accident: all sheaves in the homotopy category are representable.

[4, Theorem 9.12]

Theorem 3.1.1 (Brown). *Let $F\colon h\mathsf{Spaces}^{\mathrm{op}}_{\mathrm{conn},*/} \to \mathsf{Sets}_{*/}$ be a functor on pointed, connected spaces which satisfies the following pair of axioms:*

Wedge axiom: F converts wedges to products, as in

$$F(\bigvee_\alpha X_\alpha) \cong \prod_\alpha F(X_\alpha).$$

Gluing condition: Elements in the image of F glue. For a decomposition $X = A_1 \cup A_2$ and for elements $f_1 \in F(A_1)$, $f_2 \in F(A_2)$ which agree on the intersection, as in

$$f_1|_{A_1 \cap A_2} = f_2|_{A_1 \cap A_2},$$

then there exists a glued element $f \in F(X)$ which satisfies $f_1 = f|_{A_1}$ and $f_2 = f|_{A_2}$.

There then exists a representing CW-complex Y and a universal element $u \in F(Y)$ such that the natural transformation

$$\begin{aligned}[T, Y] &\to F(T), \\ \varphi &\mapsto \varphi^*(u)\end{aligned}$$

is a natural bijection. Moreover, there is a compatible bijection between natural transformations $F \to F'$ between such functors and homotopy classes $Y \to Y'$ between their representing objects.

We construct this in stages.

[4, Definition 9.6]

Definition 3.1.2. An element $u \in Y$ is said to be *n–universal* if the associated natural transformation

$$\begin{aligned}[S^q, Y] &\to F(S^q) \\ \varphi &\mapsto \varphi^*(u)\end{aligned}$$

is surjective for $q \leq n$ and bijective for $q < n$.

It follows that Y will represent F for all CW–complexes of dimension less than n.

[4, Lemma 9.8]

Lemma 3.1.3. *If there exists an n–universal element, then there exists an $(n+1)$–universal element.*

Proof. Suppose we have an n–universal element u_n on a complex Y. From this, we would like to construct an $(n+1)$–universal element u_{n+1} on a complex Y'. We set about trying to "fix" $[S^n, Y] \to F(S^n)$, which might have too many elements to be a bijection, and $[S^{n+1}, Y] \to F(S^{n+1})$, which might be missing some elements to be a bijection. Note that because S^n is an H–cogroup for $n \geq 1$, the map $[S^n, Y] \to F(S^n)$ is actually a map of groups. It follows that if we merely ensure that this surjective map does not have a kernel, it will be an isomorphism.

This is exactly why we restricted attention to *connected* spaces: it gives us control over all the fibers of the map $[S^n, Y] \to F(S^n)$, rather than just the fiber over the constant map.

This inspires us to consider the defect sets

$$A = \{\alpha \in \pi_n Y \mid \alpha^* u_n = 0\}, \qquad L = F(S^{n+1}),$$

and form the mapping cone

Here we are being a bit glib: perhaps some items in $F(S^{n+1})$ can be expressed as pullbacks of u_n, but there's no harm in adding more things to make sure we hit.

$$\bigvee_{\alpha \in A} S^n_\alpha \xrightarrow{\alpha} Y \vee \bigvee_{\lambda \in L} S^{n+1}_\lambda \to Y'.$$

Applying F, we have

$$0 = \bigvee_\alpha \alpha^*(u_n) \leftarrow\!\shortmid u_n \vee \bigvee_\lambda \lambda,$$

hence we can lift it to an element $u_{n+1} \in F(Y')$. Since Y' is formed from Y using $(n+1)$–cells, it agrees with Y on $\pi_{<n}$, hence it is n–universal. As for $(n+1)$–universality, it is designed to fix the defect at π_n exactly, and the wedge over λ forces surjectivity at π_{n+1}. $\square$

[4, Corollary 9.9, Lemma 9.11]

Lemma 3.1.4. *Let Y be a space with universal element u, let (X, A) be a CW-pair, let $v \in F(X)$ be a choice of element, and let $g: A \to Y$ be a cellular map which classifyies $v|_A$. There then exists a cellular map which classifies v and which extends g.*

Proof idea. We define a "double mapping cylinder" T which consists of the space X, the space Y, and the space $A \times I$, so that the leading edge of $A \times I$ is sewn to its image in X and the trailing edge of $A \times I$ is sewn to its image under g in Y. This space has a decomposition into A_1, which consists of X and half of the cylinder, and A_2, which consists of Y and half of the cylinder. Since A_1 and A_2 are respectively homotopy equivalent to X and Y, we may respectively consider u and v as elements of $F(A_1)$ and $F(A_2)$. Definitionally, they agree on $A_1 \cap A_2$ (i.e., on the cylinder, which is equivalent to A), and hence they give rise to the glued element $w \in F(T)$. We can extend T to a CW pair (Y', T) with universal element u' restricting to w (and hence to u). We apply Whitehead's theorem to the induced weak equivalence $Y \to Y'$ to produce an inverse, and the composite gives the desired map: $X \to Y' \to Y$. □

This is the "homotopy pushout" of $X \xleftarrow{i} A \xrightarrow{g} Y$.

There's a picture here.

Just re-run Lemma 3.1.3, starting instead with T and w rather than a point and the trivial class.

Proof of Theorem 3.1.1. To get surjectivity, set $A = \{x_0\}$ and apply Lemma 3.1.4. To get injectivity, set $X' = X \times I$, $A' = X \times \partial I$, and apply Lemma 3.1.4 again. To get the statement about natural transformations, one need only chase id through

$$[Y, Y] \xrightarrow{\cong} F(Y) \xrightarrow{T} F'(Y) \xleftarrow{\cong} [Y, Y']$$

to produce an element f. □

There is a useful companion result that works with functors F defined only on *finite* CW-complexes.

[4, Theorem 9.21]

Theorem 3.1.5 (Adams). *If F is a functor to* groups *from* finite *CW-complexes satisfying the two conditions of Brown's theorem, then it is representable. Natural transformations induce maps of representing objects that are unique up to* weak homotopy*: restricting along an incoming map from* any *finite complex gives two homotopic maps.* □

A key trick is to define $\widehat{F}(X) = \lim_\alpha F(X_\alpha)$. This modification of F satisfies the wedge axiom on the nose, it satisfies only a weak form of Mayer-Vietoris, but it gains the fact that the usual projection is an isomorphism:

$$F(\lim_\alpha X_\alpha) \to \lim_\alpha F(X_\alpha).$$

3.2 Spectra

Our discussion last time was motivated by our observation in Lemma 2.3.9 that there is a natural isomorphism

$$H^n(X; A) \xrightarrow{\cong} [X, K(A, n)],$$

which we generalized to any functor satisfying the wedge and Mayer–Vietoris axioms. These axioms are most of what it means to be a cohomology theory. The remaining axiom, which we have not yet discussed, is the following:

Definition 3.2.1 (Suspension axiom)**.** There is a natural isomorphism

$$\widetilde{H}^n(X) \xrightarrow{\cong} \widetilde{H}^{n+1}(\Sigma X).$$

It's pleasing to write this identity as $\Sigma\widetilde{H}^*(X) \cong \widetilde{H}^*(\Sigma X)$.

Its role is an interesting one, and it is best understood in the context of π_*, which we have shown in Corollary 2.2.9 to *partially* have this property. An interesting feature of Corollary 2.2.9 is that the range in which suspension invariance holds improves the more times you suspend: if $\pi_* X \to \pi_* \Sigma X$ is an isomorphism through degree $2n$, then $\pi_* \Sigma X \to \pi_* \Sigma^2 X$ is an isomorphism through degree $2(n+1)$, and so on. It follows that we can associate to X its *stable homotopy groups*, given by $\operatorname{colim}_n \pi_{*+n} \Sigma^n X$. A second interesting feature of Corollary 2.2.9 is that a spectral sequence argument shows it to generalize away from spheres to CW–complexes of bounded dimension:

The fibration appearing in Example 2.2.10 for $n = 2$ has the form

$$S^1 \to S^3 \to S^3.$$

From this we can conclude $S^3 \simeq S^2[3,\infty)$, and hence $\pi_3 S^2 \cong \mathbb{Z}$. This gives a concrete counterexample to any extension of Freudenthal beyond the advertised range, as $0 \cong \pi_2 S^1 \not\cong \pi_3 S^2 \cong \mathbb{Z}$.

Corollary 3.2.2 (of Corollary 2.2.9)**.** *Let X be an s–connected CW–complex, and let Y be a CW–complex of dimension t. Then*

Find me a citation. I don't seem to be in Switzer.

$$F(Y, X) \to F(\Sigma Y, \Sigma X)$$

is a $(2s - t)$–equivalence. □

It follows again that $\pi_m F(\Sigma^n Y, \Sigma^n X)$ is independent of n for $n \gg 0$. This spurs us to make the following categorical definition:

[4, Example 8.2]

Definition 3.2.3. Let hSuspensionSpectra denote the category which has an object $\Sigma^\infty X$ for every pointed space X and whose morphism sets are given by the formula

$$[\Sigma^\infty Y, \Sigma^\infty X] := \operatorname{colim}_n [\Sigma^n Y, \Sigma^n X].$$

A funny consequence of this definition is that $[\Sigma^\infty Y, \Sigma^\infty X]$ (and, later, $[E, E']$ generally) is *always* an abelian group, since one can always take at least 2 suspensions to be involved.

One need not leave Spaces to understand this new category:

$$\begin{aligned}
[\Sigma^\infty Y, \Sigma^\infty X] &:= \operatorname{colim}_n [\Sigma^n Y, \Sigma^n X] \\
&= \operatorname{colim}_n [Y, \Omega^n \Sigma^n X] \\
&= [Y, \operatorname{colim}_n \Omega^n \Sigma^n X] =: [Y, QX].
\end{aligned}$$

The stable homotopy groups then appear as honest homotopy groups:

$$\pi_n \Sigma^\infty X \cong \pi_n QX.$$

The map hSuspensionSpectra $\to h$Spaces produced in this way is commonly denoted by Ω^∞.

The functor Q gives rise to a host of stable invariants: fixing a space X, the family of functors

$$X^{-*}(-) := [-, Q\Sigma^* X]$$

satisfy the wedge and Mayer–Vietoris axioms (because the functors are representable) and suspension invariance (because $\Omega Q \Sigma^{n+1} X = Q \Sigma^n X$). Unfortunately, not all cohomology theories arise in this way: there is generally no space X so that $QX \simeq K(A, n)$. This failure, however, is interesting on its own and measureable. Consider the space $QK(A,n)$: Corollary 2.2.9 shows that its homotopy is given by

$$\pi_* K(A,n) \cong \begin{cases} 0 & \text{when } * < n, \\ A & \text{when } * = n, \\ 0 & \text{when } n < * \le 2n, \\ ??? & \text{otherwise.} \end{cases}$$

As n increases, the range through which $K(A,n) \to QK(A,n)$ is an equivalence grows like $2n$. If we were to permit ourselves to take the colimit in n *and* to shift spaces downward by desuspension, then we could write

$$K(A,n) = \Omega^\infty \left(\operatorname{colim}_n \Sigma^{-n}\Sigma^\infty K(A,n)\right).$$

Here the colimit is taken along the maps $\Sigma K(A,n) \to K(A,n+1)$ adjoint to $K(A,n) \to \Omega K(A,n+1)$.

Definition 3.2.4. A *spectrum* (up to homotopy) is an ind-diagram of formal desuspensions of suspension spectra.

Example 3.2.5. Suspension spectra themselves qualify as spectra: $\Sigma^\infty X$ is trivially an ind-diagram. The *sphere spectrum* is the special case of $\mathbb{S} = \Sigma^\infty S^0$.

Example 3.2.6. The *Eilenberg–Mac Lane spectrum* is given by the formula above:

$$HA = \operatorname{colim}_n \Sigma^{-n}\Sigma^\infty K(A,n).$$

Example 3.2.7. The functor $X \mapsto (\pi_*\Sigma^\infty X) \otimes \mathbb{Z}_{(p)}$ is exact, hence has an associated spectrum $\mathbb{S}_{(p)}$, the *p–local sphere spectrum*.

This is *covariant*. What is the correct statement for covariant Brown representation? This is 14.35 in Switzer. It's actually clear with spectra: replace $\pi_n E \wedge X$ with $\pi_n F(DX, E)$ for finite X and appeal to the finite form of Brown. This absolutely should appear in the section on Spectra and homology theories.

Example 3.2.8. The functor $X \mapsto \mathsf{AbGps}(\pi_0^s X, \mathbb{Q}/\mathbb{Z})$ is exact, hence has an associated spectrum $\mathbb{I}$. This is called the *Brown–Comenetz dualizing object*.

Given all this, one might be motivated by a need for concreteness to pursue a point-set model for spectra, maps of spectra, and homotopies among maps, whose homotopy category recovers $h\mathsf{Spectra}$. There are many such models available, with competing strengths and deficiencies. We reproduce one here, due to Boardman and Vogt.

Less perjoratively: a desire to build a bridge between these ideas and geometry.

Definition 3.2.9. A *spectrum* is a collection $\{E_n\}_n$ of CW–complexes together with cellular maps $i_n : \Sigma E_n \to E_{n+1}$ which are homeomorphisms onto their images.

[4, Definition 8.1]

This is not so restrictive: given a suitable notion of homotopy equivalence, one may use the mapping cylinder construction to make the subcomplex and homeomorphism conditions apply.

As indicated by the "ind–system" appearing in the abstract definition, maps between spectra are not quite given by levelwise maps which commute with the inclusions i_n. Instead, one asks for maps to only be defined *eventually*.

For instance, we know $\pi_n S^n \cong \mathbb{Z}$ for $n \ge 1$, but $\pi_0 S^0 = \{\pm 1\}$. If we were to define maps of spectra as such commuting sequences, then we would get $\pi_0 \Sigma^\infty \mathbb{S} = \{\pm 1\}$—the *wrong* answer.

Definition 3.2.10. A *subspectrum* $F \subseteq E$ is a sequence of subcomplexes of E_n, forming a spectrum by restriction. It is *cofinal* when every cell $e^m_\alpha \subseteq E_n$ has $\Sigma^{j_\alpha} e^m_\alpha \subseteq F_{n+j_\alpha}$—it eventually appears in F. A map $E \to E'$ is required only to be defined on a cofinal $F \subseteq E$, and two maps are equal if they agree on a mutually cofinal subspectrum. Finally, two maps of spectra are *homotopic* if there is a common cofinal subspectrum F' and a map $F' \wedge I_+ \to E'$ witnessing the homotopy.

[4, Definitions 8.9, 8.10, 8.12, 8.15]

In particular, the inclusion of a cofinal subspectrum is equivalent to the identity map.

The advantage of having a model available is that we can use it to lift some familiar constructions from Spaces.

[4, 8.17, 8.18]

Lemma 3.2.11. *Spectra have wedge sums and mapping cones, both given level-wise.* □

This, together with our knowledge of Spaces generally, is enough to copy the proof of Whitehead:

Which we did not give for Spaces either!

[4, Corollary 8.24]

Theorem 3.2.12. *If a map $f : E \to E'$ induces a weak equivalence, it is a homotopy equivalence.* □

[4, Theorem 8.26]

Corollary 3.2.13. *The spectra $\{E_n \wedge S^1\}_n$ and $\{E_{n+1}\}_n$ are equivalent, and so the spectrum $\{E_{n-1} \wedge S^1\}_n$ is equivalent to E.* □

3.3 Co/homology theories from spectra

We defined spectra in such a way that a cohomology theory gives rise to a spectrum by extracting the representing objects $E^n(-) = [-, E_n]$ and building from them the inductive system

$$E := \operatorname{colim}_n \Sigma^{-n} \Sigma^\infty E_n.$$

Our definition was lax enough, though, that the converse is not quite as clear: do spectra precipitate cohomology theories? If so, how tight is the correspondence between the two?

[4, 8.33]

Definition 3.3.1. For a spectrum E, we define its *associated (reduced) co/homology theories* as follows:

- $\widetilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E]$.
- $\widetilde{E}_n(X) = \pi_n(E \wedge X)$, where the smash product $(E \wedge X)_n = E_n \wedge X$ is induced up from Spaces.

In order to see that these are co/homology functors, it's useful to record

In general, homotopy classes of maps of spectra are presented by π_0 of a kind of pro-ind-space.

Lemma 3.3.2. *For X a pointed space and E a generic spectrum,*

$$[\Sigma^\infty X, E] = \operatorname{colim}_{n,m} [\Sigma^m X, \Sigma^{m-n} E_n].$$

If one uses Brown representability of a cohomology functor to manufacture the spaces E_n in the definition of E, then suspension invariance makes the system in the Lemma *constant*. This is called an "Ω–spectrum".

Proof. One couples the formula for suspension spectra

$$[\Sigma^\infty X, \Sigma^\infty Y] = \operatorname{colim}_m [\Sigma^m X, \Sigma^m Y]$$

to a presentation of E:

$$E = \operatorname{colim}_n \Sigma^{-n}\Sigma^\infty E_n. \qquad \square$$

To feel confident in our definition, we should check that these functors indeed satisfy the Eilenberg–Steenrod axioms.

1. We've built in suspension invariance:

$$\widetilde{E}_{n+1}(\Sigma X) \cong [S^{n+1}, E \wedge \Sigma X] \cong [S^n, E \wedge X] \cong \widetilde{E}_n(X),$$
$$\widetilde{E}^{n+1}(\Sigma X) \cong [\Sigma^\infty \Sigma X, \Sigma^{n+1} E] \cong [\Sigma^\infty X, \Sigma^n E] \cong \widetilde{E}^n(X).$$

2. A coexact sequence

$$A \xrightarrow{i} X \to X \cup_i CA$$

of pointed spaces induces a coexact sequence of spectra

$$E \wedge A \to E \wedge X \to E \wedge (X \cup_i CA) = (E \wedge X) \cup_i C(E \wedge A),$$

so we get the desired long exact sequence

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n E \wedge A & \longrightarrow & \pi_n E \wedge X & \longrightarrow & \pi_n E \wedge C(i) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & \widetilde{E}_n(A) & \longrightarrow & \widetilde{E}_n(X) & \longrightarrow & \widetilde{E}_n(X, A) & \longrightarrow & \cdots. \end{array}$$

For the analogous fact in cohomology, the sequence of suspension spectra

$$\Sigma^\infty A \to \Sigma^\infty X \to \Sigma^\infty (X \cup_i CA)$$

is coexact, so mapping into E makes it exact.

We haven't talked about function spectra... so is this really fair?

3. The cohomological wedge axiom is easy: by pulling coproducts out on the left to products, we get

$$\left[\bigvee_\alpha \Sigma^\infty X_\alpha, E\right] = \prod_\alpha [\Sigma^\infty X_\alpha, E].$$

Homology is harder and requires a filtration trick. We know that our homology functor satisfies the *finite* wedge axiom by appeal to the Mayer–Vietoris axiom. Smash products also commute with colimits, hence one may check

[4, Lemma 8.34]

$$E \wedge \operatorname{colim}_{\substack{S \subseteq A \\ S \text{ finite}}} \bigvee_{\alpha \in S} X_\alpha \cong \operatorname{colim}_S E \wedge \bigvee_{\alpha \in S} X_\alpha \cong \operatorname{colim}_S \bigvee_{\alpha \in S} E \wedge X_\alpha \cong \bigvee_\alpha E \wedge X_\alpha.$$

From this, the wedge axiom follows.

This feels clumsy. Is it really necessary?

Remark 3.3.3. The Mayer–Vietoris axiom amounts to the assertion that co/homology functors commute with finite (homotopy) colimits, and the wedge axiom adds a special case on top of that. Cohomology *does not* commute with general colimits. Instead, there is a *Milnor sequence*:

[4, Propositions 7.66 and 8.37]

$$0 \to R^1 \lim_\alpha E^{n-1} X_\alpha \to E^n(\operatorname{colim}_\alpha X_\alpha) \to \lim_\alpha E^n(X_\alpha) \to 0.$$

Satisfied that we have indeed produced co/homology theories, we can investigate whether these assignments are mutual inverses. To compare objects, this mostly comes down to Whitehead's theorem for spectra. Using Brown representability for natural transformations, we can lift maps of cohomology theories up to maps of spectra: a map $E^* \xrightarrow{f} F^*$ induces a unique sequence of maps $E_n \xrightarrow{f_n} F_n$ and hence a map $\tilde{f}\colon E \to F$ of spectra. From this, one sees that a natural isomorphism of cohomology theories induces a weak equivalence of spectra, and conversely. However, Brown representability falls short of giving a *functorial* correspondence: the same natural transformation of cohomology theories can be induced by multiple homotopy-inequivalent maps of spectra. More precisely, Brown's result shows that the construction

It's even reasonable to extend the definition of cohomology to $F^0(E) = [E, F]$, of which $\tilde{f}$ is then an element.

$$\begin{aligned} \mathsf{Spectra} &\to \mathsf{CohomologyTheories} \\ E &\mapsto [\Sigma^\infty -, \Sigma^* E] \end{aligned}$$

is full and bijective on isomorphism classes.

[4, Theorem 10.25]

Theorem 3.3.4 (Hurewicz). *There is a map $\mathbb{S} \to H\mathbb{Z}$ which has 0–connected fiber. By consequence, the difference between $\mathbb{S}_*(X)$ and $H\mathbb{Z}_*(X)$ begins one degree above the bottommost cell in X. By consequence, for X n–connected and $n \geq 1$, $\pi_n X \cong \pi_n^s X \cong H\mathbb{Z}_n X$.* □

Where does this belong?

3.4 *The smash product*

For all our discussion of homotopy and homology *groups*, we have not yet found a framework for the cohomology *ring* of a space. The following observation is key:

Definition 3.4.1. A *ring* is a (commutative, unital) monoid in $\mathsf{AbelianGroups}$ under the ⊗–product.

It is particularly important that one does *not* use the Cartesian / categorical ×.

Our discussion around Lemma 1.2.4 then indicates a way forward: since we have constructed an object HR which represents ordinary cohomology with coefficients in R, a monoidal structure on $H^*(-;R)$ should induce a monoidal structure on HR. In order to make sense of this, we need a monoidal structure on $h\mathsf{Spectra}$ which is compatible with the other monoidal structures in play, in the sense that the following pair of functors should be made monoidal:

Ideally, it would even be visibly related to the original monoidal structure on R.

$$\mathsf{AbelianGroups} \xrightarrow{H} \mathsf{Spectra} \xleftarrow{\Sigma^\infty} \mathsf{Spaces}_{*/}.$$

We have already introduced the operation

$$\Sigma^\infty X \wedge \Sigma^\infty Y := \Sigma^\infty(X \wedge Y)$$

for two pointed spaces X and Y. Since Spectra is suitably generated by the image of Σ^∞, it should seem likely that this will pin down any putative monoidal structure.

For inspiration as to how to define the smash product in general, recall that we have also already defined $E \wedge \Sigma^\infty X$ for a generic spectrum E and a pointed space X. Given a presentation $E = \{\Sigma^{n_j}\Sigma^\infty E_j\}_j$, we set

$$E \wedge \Sigma^\infty X := \{\Sigma^{n_j}\Sigma^\infty(E_j \wedge X)\}_j.$$

That is, we commuted the smash product through the ind–system, where we reduced to the case of the smash product of suspension spectra. In the fully general case of $E \wedge F$, we may also choose a presentation of the second spectrum F as $F = \{\Sigma^{m_k}\Sigma^\infty F_k\}_k$, and then we set

$$(E \wedge F)_{j,k} := \{\Sigma^{n_j+m_k}\Sigma^\infty E_j \wedge F_k\},$$

another ind–system.

[4, pg. 254–267]

Remark 3.4.2. To define the smash product in terms of Boardman and Vogt's concrete model, we must convert this doubly-indexed ind–system into a sequential system. One option is to select any cofinal subsystem, but this destroys the associativity of the product (and often destroys the commutativity). A superior option is to "sum over possible choices": we set $(E \wedge F)_n$ to be the colimit of the diagram under the n^{th} antidiagonal (after replacing the maps by cofibrations).

From here, the main task is to show that this definition is sufficiently insensitive to the choice of presentation: given a pair of weakly equivalent presentation, one must show that this induces a weak equivalence after smashing. This is possible, and hence one learns:

[4, Theorem 13.40]

Theorem 3.4.3. *In the homotopy category, this determines a symmetric monoidal product, $\wedge$, on* Spectra. □

Warning: This product is *not* especially nice before passing to the homotopy category. It turns out that this is unavoidable. [4, Definition 13.50]

Definition 3.4.4. A *ring spectrum* is a spectrum E equipped with a multiplication map $\mu\colon E \wedge E \to E$ and a unit map $\eta\colon \mathbb{S} \to E$ making E into a (unital) monoid object in hSpectra.

The statement here applies to any ring-valued theory.

Corollary 3.4.5. *The cup product maps*

$$H^n(-;R) \times H^m(-;R) \xrightarrow{\smile} H^{n+m}(-;R)$$

induce maps

$$K(R,n) \wedge K(R,m) \xrightarrow{\smile} K(R,n+m)$$

which altogether induce a product

$$HR \wedge HR \to HR. \quad \square$$

Example 3.4.6. The sphere spectrum, $\mathbb{S}$, is the monoidal unit and hence also a ring.

The inverse construction is now straightforward. Suppose that we have a pair of cohomology classes $\omega_n \in E^n(X)$ and $\omega_m \in E^m(X)$, for which we choose representatives $\omega_n \colon \Sigma^\infty X \to \Sigma^n E$ and $\omega_m \colon \Sigma^\infty X \to \Sigma^m E$. Given a multiplication $\mu \colon E \wedge E \to E$, we define the product $\omega_n \smile \omega_m$ like so:

This same product can be used to make E^*X (and E_*X) into E_*-modules.

$$X \xrightarrow{\Delta} X \wedge X \xrightarrow{\omega_n \wedge \omega_m} \Sigma^n E \wedge \Sigma^m E \simeq \Sigma^{n+m}(E \wedge E) \xrightarrow{\mu} \Sigma^{n+m} E.$$

Remark 3.4.7. One can check that $- \wedge E$ preserves colimits. By positing an exponential adjunction, one can use Brown representability on the putative formula

$$F(E_1, E_2)^*(X) = \text{“}\pi_0 F(E_1 \wedge X, E_2)\text{”} = \mathsf{Spectra}(E_1 \wedge X, E_2)$$

to define a notion of *function spectrum*. As with other objects extracted from Brown's result, this definition does not have excellent functoriality properties. However, there is a version of the adjoint functor theorem that also applies to give a fully functorial statement—but this is beyond our current technology.

Of course, one can also give a direct definition.

Ring spectra induce a useful duality pairing between their associated co/homology theories.

[4, pg. 281]

Definition 3.4.8. Given co/homology classes

$$(\sigma \colon \mathbb{S}^n \to E \wedge \Sigma^\infty X) \in E_n X,$$
$$(\omega \colon \mathbb{S}^m \wedge \Sigma^\infty X \to E) \in E^m(X),$$

we define their pairing to be

$$\langle \omega, \sigma \rangle \colon \mathbb{S}^{n+m} \xrightarrow{\Sigma^m \sigma} E \wedge \mathbb{S}^m \wedge \Sigma^\infty X \xrightarrow{1 \wedge \omega} E \wedge E \xrightarrow{\mu} E.$$

[4, Proposition 13.62.i]

Lemma 3.4.9. *Under this pairing, the maps f^*, f_* induced by f are adjoint:*

$$\langle f^* \omega, \sigma \rangle = \langle \omega, f_* \sigma \rangle.$$

Proof. This is a consequence of the following diagram:

$$\begin{tikzcd}
\mathbb{S}^{n+m} \arrow[r, "\Sigma^m \sigma"] \arrow[dr, "\Sigma^m f_* \sigma"'] & E \wedge \mathbb{S}^m \wedge \Sigma^\infty X \arrow[d, "1 \wedge 1 \wedge f"] \arrow[dr, "1 \wedge f^* \omega"] & & \\
& E \wedge \mathbb{S}^m \wedge \Sigma^\infty Y \arrow[r, "\omega"] & E \wedge E \arrow[r, "\mu"] & E.
\end{tikzcd}$$

∎

3.5 G-bundles and fiber bundles

A major up-shot of representability is that the tools of algebraic topology can be turned on themselves. We have previously announced our intention to understand the collection of natural transformations

$$H^n(-; A) \to H^m(-; B).$$

By appealing to representability, this is not only equivalent to the collection of homotopy classes

$$K(A,n) \to K(m,B)$$

but also to the cohomology group

$$H^m(K(A,n);B).$$

If we can get a sufficiently explicit handle on $K(A,n)$, we can use such a presentation to finish our original analysis.

There is a particularly restrictive form of fiber bundle that appears very often in geometric contexts:

[4, Definition 11.1]

Definition 3.5.1. A *(real) vector bundle (of rank k)* over a base B is a fiber bundle $p\colon E \to B$ with fiber $\mathbb{R}^k$ and whose transition maps are linear functions.

Example 3.5.2. The co/tangent bundles of a manifold are vector bundles.

This constraint on the transition maps admits a universal form:

[4, Definition 11.4]

Definition 3.5.3. A *G–bundle* is a fiber bundle $p\colon E \to B$ where G acts on E (and trivially on B, and the map p is equivariant), the identifications $\varphi_U\colon p^{-1}(U) \cong G \times B$ are equivariant, and the compatibilities $\varphi_U|_{U\cap V} = \varphi_V|_{U\cap V}$ are equivariant too.

[4, 11.21]

Remark 3.5.4. This construction is universal in the following sense: if G acts on an auxiliary space F, one can extract from a G–bundle $p\colon E \to B$ an F–fiber bundle by

$$E' = (F \times E)/(fg,e) \sim (f,ge).$$

Conversely, a fiber bundle with fiber F has an associated $(\operatorname{Aut} F)$–bundle.

[4, Theorem 11.20, Proposition 11.22]

Example 3.5.5. Real vector bundles correspond with $\mathrm{GL}(\mathbb{R}^n)$–bundles under this construction. The maximal compact subgroup of $\mathrm{GL}(\mathbb{R}^n)$ is the orthogonal group $O(\mathbb{R}^n)$, and the equivariant retraction $O(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$ gives an equivalence between $O(\mathbb{R}^n)$–bundles and $\mathrm{GL}(\mathbb{R}^n)$–bundles (and hence with real vector bundles as well).

The local nature of the definition of a vector bundle gives rise to the following observation:

[4, Proposition 11.32]

Lemma 3.5.6. *The assignment $X \mapsto \{$isomorphism classes of G–bundles on $X\}$ satisfies the wedge axiom and Mayer–Vietoris.* □

[4, 11.33]

Corollary 3.5.7. *There is a homotopy type BG representing this functor.* □

Purely through abstract principles, one can make an interesting qualitative statement about this homotopy type:

[4, Proposition 11.27, Theorem 11.35]

Lemma 3.5.8. *Let $E \to B$ be a G–bundle with E n–connected. The classifying map $B \to BG$ is then an n–equivalence. It follows that the induced natural transformation $[-, B] \to [-, BG] \to k_G(-)$ is an equivalence on complexes of dimension $\leq n$.* □

Corollary 3.5.9. *The universal bundle EG classified by* id: $BG \to BG$ *has* contractible *total space. Conversely, G–bundle with contractible total space is a model for the universal such bundle.* □

[4, 11.43]

Remark 3.5.10. With Corollary 1.6.8 and Remark 2.3.12 in mind, we can now smell the connection between these ideas and our pursuit of Eilenberg–Mac Lane spaces. Namely, Corollary 1.6.8 shows that the natural map

$$G \to \Omega BG$$

is an equivalence, so that

$$K(A, n+1) \to BK(A, n)$$

is also an equivalence.

Making sense of this requires a model of $K(A, n)$ as an honest group, rather than as an H–group. We blithely assert to the reader that such a model exists—its precise form won't turn out to be important.

We now turn to the problem of producing a reliable model of BG, using Corollary 3.5.9 as a guide. Our route, as ever, will pass through some creative category theory.

The first-time reader will probably find it easier to conceptualize the following under the further condition that G be finite (e.g., $\mathbb{Z}/2$).

From here on, you owe the reader citations.

Definition 3.5.11. Let C be a category. Its *nerve* $N(\mathsf{C})$ is a simplicial set with 0–simplices the objects of C, 1–simplices the arrows of C, 2–simplices commuting triangles, 3–simplices commuting tetrahedra, ...

Equivalently: the n–simplices are given by length n chains of composable morphisms.

Remark 3.5.12. This construction is a very faithful encoding of a category: the original category and its composition law can be recovered from the 0–. 1–, and 2–simplices. It also translates categorical ideas to recognizable topological objects: for instance, functors become continuous maps and natural transformations become homotopies of maps.

Inject a comment about being able to extend this business to topologically enriched categories too.

Example 3.5.13. For G a group, we define two categories:

1. $G//G$ has objects $g \in G$ and maps $g \xrightarrow{h} gh$.

2. $*//G$ has one object $*$ and maps $* \xrightarrow{h} *$.

Lemma 3.5.14. *$G//G$ is contractible.*

Proof sketch. This amounts to showing that any "outer horn" (i.e., a chain of morphisms of length $n-1$ and a morphism with either the same ultimate source or same ultimate target) is “fillable” (i.e., there is a chain of morphisms of length n which extends the original chain and whose composite is the auxiliary morphism). This is so: given

$$g_1 \xrightarrow{g_2} g_1 g_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} g_1 \cdots g_n$$

and

$$g_1 \xrightarrow{h} h_{n+1},$$

we can set $g_{n+1} = g_n^{-1} \cdots g_1^{-1} h_{n+1}$ to get

$$g_1 \xrightarrow{g_2} g_1 g_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} g_1 \cdots g_n \xrightarrow{g_{n+1}} h_{n+1}. \qquad \square$$

Remark 3.5.15. The G–action on $G/\!/G$ is *free.*

Corollary 3.5.16. *The quotient map $N(G/\!/G) \to N(*/\!/G)$ has fiber G, hence it models models $EG \to BG$.* $\square$

Remark 3.5.17. This is a more conceptual statement than you might think. There are equivalences $G/\!/G \to \{G\text{–torsors with a trivialization}\}$ and $*/\!/G \to \{G\text{–torsors}\}$, and a map $X \to N(\{G\text{–torsors}\})$ for X a simplicial set assigns each point in X to a G–torsor, each path to a map of torsors, ... This *sounds* like it’s building a G–bundle on X by specifying the fibers. The *Grothendieck construction* makes this precise.

This very concrete model for BG has one really excellent feature: is has a naturally occurring skeletal filtration (viz., by simplex dimension) with identifiable quotients:

$$\begin{tikzcd} BG^{(0)} \arrow[r] \arrow[d, equal] & BG^{(1)} \arrow[r] \arrow[d] & BG^{(2)} \arrow[r] \arrow[d] & \cdots \arrow[r] & BG^{(n-1)} \arrow[r] \arrow[d] & BG^{(n)} \arrow[r] \arrow[d] & \cdots \\ * & \Sigma G & (\Sigma G)^{\wedge 2} & \cdots & (\Sigma G)^{\wedge(n-1)} & (\Sigma G)^{\wedge n} & \cdots . \end{tikzcd}$$

If h is a homology theory with Künneth isomorphisms, this gives a spectral sequence

$$E^1_{*,*} = (\widetilde{h}_* \Sigma G)^{\otimes *} \Rightarrow h_* BG.$$

More than this, the d^1–differential in then identifiable:

$$d_1(g_1 \otimes \cdots \otimes g_n) = \sum_{j=2}^{n} (g_1 \otimes \cdots \otimes g_{j-1} g_j \otimes \cdots \otimes g_n).$$

This is a standard resolution appearing in homological algebra:

$$E^2_{*,*} = \operatorname{Tor}^{h_* G}_{*,*}(h_*, h_*).$$

This is a very common situation: some “fully derived” construction appearing in homotopy theory has behavior mediated by analogous homological algebra and a spectral sequence. Since $\operatorname{Tor}^{h_* G}(h_*, h_*) = \pi_*(h_* \otimes^{\mathbb{L}}_{h_* G} h_*)$, this leads one to think of BG as some kind of $* \times_G *$. This turns out to be fruitful.

3.6 The Steenrod algebra: calculation

Today we put the machinery of yesterday to work in the case of $H^m(K(\mathbb{F}_2, n); \mathbb{F}_2)$. Our method is *inductive*, and it ultimately rests on the following key observations:

Throughout today, you owe the reader citations.

Example 3.6.1. Tor algebras are generally remarkably computable: there is an algorithm, due to Tate, which forms a resolution of h_* by a DGA which is levelwise $(h_* G)$–free. To cover the algebras appearing in this computation, we will only need the following observations:

$$\mathrm{Tor}^{A\otimes B}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Tor}^{A}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Tor}^{B}_{*,*}(\mathbb{F}_2, \mathbb{F}_2), \quad \mathrm{Tor}^{\Lambda[x]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \bigotimes_{j=0}^{\infty} \Gamma[\sigma x],$$

where A and B are $\mathbb{F}_2$–algebras, $\Lambda[x]$ denotes an exterior $\mathbb{F}_2$–algebra generated by the lone element x, and $\Gamma[\sigma x]$ denotes a divided power $\mathbb{F}_2$–algebra generated by the suspension of the element x.

Lemma 3.6.2. *The pairing $K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, 1) \xrightarrow{\smile} K(\mathbb{F}_2, n+1)$ induces a pairing "$\circ$" of spectral sequences*

$$\begin{array}{ccc} \mathrm{Tor}^{H_*(K(\mathbb{F}_2,n);\mathbb{F}_2)}_{*,*} \otimes H_*(K(\mathbb{F}_2,1);\mathbb{F}_2) & \xrightarrow{\circ} & \mathrm{Tor}^{H_*(K(\mathbb{F}_2,n+1);\mathbb{F}_2)}_{*,*} \\ \Downarrow & & \Downarrow \\ H_*(K(\mathbb{F}_2,n+1);\mathbb{F}_2) \otimes H_*(K(\mathbb{F}_2,1);\mathbb{F}_2) & \xrightarrow{\smile} & H_*(K(\mathbb{F}_2,n+2);\mathbb{F}_2) \end{array}$$

which converges to the cup product and which satisfies $d(x \circ y) = (dx) \smile y$. □

We begin with the base case:

Lemma 3.6.3. *The spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}^{H_*(\mathbb{F}_2;\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(K(\mathbb{F}_2, 1); \mathbb{F}_2)$$

collapses to give

$$H_*(K(\mathbb{F}_2, 1); \mathbb{F}_2) \cong \Gamma[\sigma a].$$

Proof. We analyze the input to the spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{H_*(\mathbb{F}_2;\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(K(\mathbb{F}_2, 1); \mathbb{F}_2).$$

The homology algebra $H_*(\mathbb{F}_2; \mathbb{F}_2)$ can be equivalently presented as

$$H_*(\mathbb{F}_2; \mathbb{F}_2) \cong \mathbb{F}_2[\underline{1}]/(\underline{1}^2 = 1) \cong \mathbb{F}_2[\underline{1} - 1]/(\underline{1} - 1)^2.$$

Since this algebra is exterior, we may compute

$$\mathrm{Tor}^{H_*(\mathbb{F}_2;\mathbb{F}_2)}_{*,*} \cong \Gamma[\sigma a],$$

for $a = \underline{1} - 1$. The homology groups of $K(\mathbb{F}_2, 1) \simeq \mathbb{R}\mathrm{P}^\infty$ have one class in every degree. The bar spectral sequence also has one class in every degree, and there can therefore be no nonzero differentials, so that the spectral sequence collapses at E_2. To establish convention, we write

$$\Gamma[\sigma a] \cong \mathbb{F}_2[a_{(0)}, a_{(1)}, a_{(2)}, \ldots]/(a_{(j)}^2 = 0)$$

for the algebra generators. □

Theorem 3.6.4. *The above Lemma generalizes in n to give*

$$H_*(K(\mathbb{F}_2, n); \mathbb{F}_2) \cong \mathbb{F}_2[a_{(j_1)} \circ \cdots \circ a_{(j_n)}]/(\textit{squares}).$$

Proof sketch. We proceed by induction, having shown the claim in the case $n = 0$. By assumption, $H_* K(\mathbb{F}_2, n)$ is a tensor product of exterior algebras, so the Künneth formula for Tor–algebras gives

$$\begin{aligned}\operatorname{Tor}_{*,*}^{H_*K(\mathbb{F}_2,n)}(\mathbb{F}_2, \mathbb{F}_2) &\cong \bigotimes_J \operatorname{Tor}_{*,*}^{\Lambda[a_{(j_1)} \smile \cdots \smile a_{(j_n)}]}(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong \bigotimes_J \Gamma[a_{(j_1)} \smile \cdots \smile a_{(j_n)}].\end{aligned}$$

One can show the identity Wilson 8.16

$$(a_{(J)})_{(k)} \equiv a_{(J)} \smile a_{(k)} \quad \text{(mod decomposables)}.$$

It follows that there are no differentials, since the spectral sequence for $H_*(K(\mathbb{F}_2, 1); \mathbb{F}_2)$ had none. □

This has a great many consequences.

[4, Theorem 18.14]
In terms of the operations Sq^n below, $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ is given by the algebra $\mathbb{F}_2[\mathrm{Sq}^I \iota_n \mid I_j \ge 2I_{j+1}, 2I_1 - I_+ < n]$.

Corollary 3.6.5. *On cohomology, we have the calculation*

$$H^*(K(\mathbb{F}_2, n+1); \mathbb{F}_2) \cong \mathbb{F}_2[a_{(j_1)} \circ \cdots \circ a_{(j_n)}].$$

Proof sketch. One employs that the dual of a primitively-generated divided-power Hopf algebra is a primitively-generated polynomial Hopf algebra. □

This finishes the task we set out for ourselves at the beginning of this excursion, but we can collect a bit more at the level of spectra.

[4, Theorem 18.20]

Corollary 3.6.6. *The dual Steenrod algebra is given by*

$$\mathscr{A}_* := H\mathbb{F}_{2*} H\mathbb{F}_2 \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_n, \ldots],$$

where $|\xi_n| = 2^n - 1$ *is represented by* $a_{(n)} \in H_{2^n-1}(\Sigma^{-1} K(\mathbb{F}_2, 1); \mathbb{F}_2)$.

Proof. From our definition of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, we have

$$H\mathbb{F}_2 \wedge H\mathbb{F}_2 \simeq H\mathbb{F}_2 \wedge (\operatorname{colim}_n \Sigma^{-n} K(\mathbb{F}_2, n)) \simeq \operatorname{colim}(H\mathbb{F}_2 \wedge \Sigma^{-n} K(\mathbb{F}_2, n)),$$

so that applying π_m gives

$$(H\mathbb{F}_2)_m H\mathbb{F}_2 = \lim_n H_{m-n}(K(\mathbb{F}_2, n); \mathbb{F}_2).$$

The Theorem gives us access to these groups, provided we can describe the maps participating in the colimit. These maps turn out to be $- \smile a_{(0)}$, owing to the factorization

$$\begin{tikzcd}
S^1 \wedge K(\mathbb{F}_2, n) \arrow[rr] \arrow[dr, "a_{(0)} \times \mathrm{id}"'] & & K(\mathbb{F}_2, n+1) \\
& K(\mathbb{F}_2, 1) \wedge K(\mathbb{F}_2, n). \arrow[ur, "\smile"'] &
\end{tikzcd}$$

□

The analogous formula for stable cohomology is harder: the answer as a coalgebra is encoded in the above, but to produce a description as a Hopf algebra we need to understand the comultiplication on $\mathscr{A}_*$. This is more complicated, so we merely quote the result:

Lemma 3.6.7. *The comuliplication on $\mathscr{A}_*$ is given by*

$$\Delta \xi_n = \sum_{j=0}^{n} \xi_j \otimes \xi_{n-j}^{2^j}.$$

The primitive elements of this algebra are $\xi_1^{2^j}$. □

Lots more formulas like this can be read off. Maybe on your homework?

Corollary 3.6.8 ([4, pg. 451])**.** *The Steenrod algebra, $\mathscr{A}^* := H\mathbb{F}_2^* H\mathbb{F}_2$, is noncommutative and generated by elements Sq^{2^j} dual to $\xi_1^{2^j}$.* □

The space-level versions of these calculations are called the *unstable (dual) Steenrod algebra*.

How is Sq^n defined in general? Dual to ξ_1^n? I don't think so.

Remark 3.6.9. A lot can be computed about $\mathscr{A}^*$ by studying universal cases. For instance, $\Delta(x^2)^* = 1 \mid (x^2)^* + \xi_1 \mid (x)^* \in \mathscr{A}_* \otimes \widetilde{H}_* \mathbb{R}\mathrm{P}^\infty$ says $\mathrm{Sq}^0(x^2) = x^2$ and $\mathrm{Sq}^1(x) = x^2$ for $|x| = 1$. In fact, we have

1. $\mathrm{Sq}^0(x) = x$.
2. $\mathrm{Sq}^{>|x|}(x) = 0$.
3. $\mathrm{Sq}^{|x|}(x) = x^2$.
4. $\mathrm{Sq}^n(x+y) = \mathrm{Sq}^n x + \mathrm{Sq}^n y$.
5. $\mathrm{Sq}^n(xy) = \sum_{n_1+n_2=n} \mathrm{Sq}^{n_1}(x) \cdot \mathrm{Sq}^{n_2} y$.
6. "The Adem relations", summarized by
   (a) $\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0$, and
   (b) $d(\mathrm{Sq}^n) = \mathrm{Sq}^{n-1}$ extends to a derivation.

   So, for instance,

$$\begin{aligned}
0 &= d^3(\mathrm{Sq}^5\mathrm{Sq}^3) \\
&= d(\mathrm{Sq}^3\mathrm{Sq}^3 + \mathrm{Sq}^5\mathrm{Sq}^1) \\
&= \mathrm{Sq}^2\mathrm{Sq}^3 + \mathrm{Sq}^3\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^5\mathrm{Sq}^0.
\end{aligned}$$

Emphasize that the cohomology of spaces and spectra have natural actions of these endomorphism algebras.

3.A K–theory

Since we're talking about G–bundles, I think it would be smart to give a brief treatment of complex K–theory, even if it doesn't directly factor into our future calculations.

4
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