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TOPOLOGY FROM AN ALGEBRAIC VIEWPOINT

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THIS DOCUMENT IS an elaboration on a set of lecture notes delivered at Harvard in the Spring 2017 term (Math 231b), covering various aspects of homotopy theory under the standing assumption that the reader has had some prior exposure to ordinary co/homology.[1] It serves as a companion to the two main sources from which the material was drawn: Switzer's *Algebraic Topology: Homotopy and Homology*[2] and Mosher and Tangora's *Cohomology Operations and Applications in Homotopy Theory*.[3] The course is meant to highlight four items, presented in turn (though, of course, there is plenty of interplay):

1

2

3

1. We take a particularly abstract and algebraic perspective on the homotopy theory of spaces. Rather than dealing with spaces as geometric objects, our motif is to design our model of homotopy theory so that the algebraic structures encountered in the reader's prior exposure to algebraic topology become inherently available. This pushes us to explore methods by which we can *decompose* homotopy types, as well as how to use that to our advantage, since we cannot rely on geometry to do the work for us.

2. With such a model of homotopy types in hand, we explore how familiar invariants (e.g., homotopy, ordinary homology) come about in this particular framework, as well as what properties they enjoy.

3. We use these properties, together with categorical existence theorems, to construct new such invariants that enjoy a similar flavor and which are computable by similar means. In particular, this leads to the stable homotopy theory of spectra.

4. We use these tools to effect computations, entirely algebraic in origin, but with geometric interpretations.

This final goal is our true goal—we are not out to set up theory for its own sake, but in order to compute quantities already of interest and to motivate interest in heretofore unimagined quantities. Because of this centrality, we highlight a particular computation to come and its flavor when presented in this framework. We begin with the n–sphere S^n. Its homotopy groups are notoriously difficult and important to compute; in fact, we will show during our exploration of (1) that their nontriviality is essentially why homotopy theory itself is nontrivial. In spite of—or because of—their difficulty, we would like to be able to compute as much as possible about them. The Hurewicz theorem from (2) describes a link between homotopy and homology: for a space X and $m > 1$, when $\pi_{<m}X = 0$ then there is a natural isomorphism $\pi_m(X) \cong H_m(X;\mathbb{Z})$. Using our prior knowledge of the homology of the sphere, this garners us one of its homotopy groups: $\pi_n S^n \cong \mathbb{Z}$. One of the decompositions studied in (1) cleaves the homotopy groups of S^n into two pieces: there is a "kernel

Most any simply-connected space with known cohomology groups will do.

sequence”

$$S^n(n,\infty) \to S^n \to S^n(-\infty,n],$$

where these new spaces $S^n(n,\infty)$ and $S^n(-\infty,n]$ have the properties

$$\pi_* K(\pi_n S^n, n) = \begin{cases} \pi_n S^n & \text{if } * = n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\pi_* S^n[n+1,\infty) = \begin{cases} \pi_* S^n & \text{if } * \geq n+1, \\ 0 & \text{otherwise}. \end{cases}$$

The Serre spectral sequence, a tool from (4), interrelates the homology of the three terms of an exact sequence of spaces: here, it consumes $H_* S^n$ and $H_* S^n(-\infty,n]$—the first of which is simple, the second of which we must set aside as a computation to be done—and it produces from them $H_* S^n(n,\infty)$. After computing $H_* S^n(n,\infty)$, one can then apply the Hurewicz theorem to it to gain access to

$$H_{n+1} S^n(n,\infty) \cong \pi_{n+1} S^n(n,\infty) \cong \pi_{n+1} S^n,$$

and repeat.

Improve title.

We have a limit of 35-37 lectures.

I also gave a bunch of homework exercises that I’d prefer to be solved inline in the notes: for instance, facts about localizations, or the minimal models portion of unstable rational homotopy theory.

Emphasize the prevalence of moduli problems in homotopy theory.

Spectral sequences, early and often.

Emphasize when categorical constructions are used in a “wrong way” fashion.

Jun Hou gave a couple of lectures about the hammock localization midway through this.

Admit that these notes do not exhaustively cover their references.

Mention that this class won an award!

Contents

1 Homotopy Types and Exact Sequences

Two overall goals in this text are:

1. To treat topological objects as if they were algebraic, dissecting and assembling them by methods analogous to those we use with, e.g., modules.

2. To express commonly encountered spaces using small amounts of information. The data of a circle or of projective space, considered as raw topological spaces, is extremely substantial, and it would be preferable to be able to name and manipulate these spaces at less cost—ideally, with finite expression.

There are several routes to effect these goals.

- One can make a re-definition of a "space" which is inherently algebraic (or combinatorial). This eases the import of algebraic operations into this new context, but it requires one to link the chosen alternative definition with a traditional notion of a space.

For example, *simplicial sets* and *Kan complexes* are two such combinatorial options. Though we won't use them, *varieties* are a popular purely algebraic notion of "space".

- Alternatively, one can begin with some of the traditional tools of algebraic topology, since we are already certain that these capture interesting topological behavior. By supposing that these tools are "shadows" of—or perhaps are themselves—the desired operations which act directly on spaces, we examine their formal properties to see what manner of algebra we are dealing with. This leaves open how well these methods achieve the goal of "finite expression", which we investigate through excessive computation.

Each approach is interesting in its own right, but we will focus on this last approach. The tools we develop in this chapter and the language in which we cast them will become the fundamental methods of dissection and assembly used in all later chapters.

1.1 The categories of spaces, pointed spaces, and pairs

We first set up the basics of category theory and introduce a few example categories, which amounts to setting the stage on which the rest of the course will play out. Since one of our express intentions is do very little geometry or "point-set" topology, much of this Lecture will concern itself with boldly asserting such facts recast in category-theoretic language, so that they can be guiltlessly referred to later.

We begin with some basic operations that one may perform on spaces. The categorical perspective is to understand objects in a category not by intrinsic presentations, like how the points in a space are arranged, but by their extrinsic effects, like how they relate to other objects in the category. With this as a guiding principle, we give categorically-minded interpretations of some common operations performed on spaces:

(?, 0.2)

Products Let X, Y be spaces. Their *product* $X \times Y$ is a space such that the set $\mathsf{Spaces}(T, X \times Y)$ of continuous maps $T \to X \times Y$ bijects with the set $\mathsf{Spaces}(T, X) \times \mathsf{Spaces}(T, Y)$ of pairs (f_X, f_Y) of continuous maps $f_X \colon T \to X$ and $f_Y \colon Y \to T$. A useful mnemonic is that products pull out on the right:

$$\mathsf{Spaces}(T, X \times Y) \cong \mathsf{Spaces}(T, X) \times \mathsf{Spaces}(T, Y).$$

Coproducts The *disjoint union* $X \sqcup Y$ is a space with the property

$$\mathsf{Spaces}(X \sqcup Y, T) \cong \mathsf{Spaces}(X, T) \times \mathsf{Spaces}(Y, T).$$

For this reason, the disjoint union is sometimes called the *coproduct*, since it pulls out on the left to a product.

(?, 0.1)

Gluing Let X be a space with a *decomposition* $X = \bigcup_j A_j$ into closed subsets. Then the set $\mathsf{Spaces}(X, Y)$ bijects with the set

This also holds when each A_j is open in X.

$$\{(f_j \in \mathsf{Spaces}(A_j, Y)) : f_j|_{A_j \cap A_k} = f_k|_{A_j \cap A_k}\}$$

of morphisms on the various components which agree on the overlaps.

Remark about opens and unique extension to closures? Of course, this requires a separability hypothesis on the target.

Quotients For $\sim$ an equivalence relation on X, there is a space $X/\sim$ such that the set $\mathsf{Spaces}(X/\sim, Y)$ bijects with the subset

(?, 0.3)

$$\{f \in \mathsf{Spaces}(X, Y) \mid x \sim x' \Rightarrow f(x) = f(x')\}$$

of those continuous maps which are constant on the partition given by $\sim$. As a special case, let $A \subseteq X$ be nonempty, and define X/A by extending the total relation on A by the identity relation on X (i.e., the associated partition consists of singletons and A). Equivalently, maps in $\mathsf{Spaces}(X/A, T)$ correspond exactly to those maps in $\mathsf{Spaces}(X, T)$ which are constant on (nonempty) A. In diagrammatic terms, this becomes:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
* \arrow[r] \arrow[drr, bend right] & X/A \arrow[dr, dotted, "\exists!"] & \\
& & T
\end{tikzcd}$$

Adopting this diagram instead as the *definition*, this yields $X/\varnothing = X \sqcup *$, which authors sometimes posit as an unmoored "convention".

We mention one further construction which we will be very keen to make use of, and we offset it from those above because of its relative fragility.

(?, 0.9–11)

Function spaces/exponential objects For X, Y spaces, there is a *function space* Y^X whose underlying set is $\mathsf{Spaces}(X,Y)$. This space upgrades the bijections of mapping sets above into homeomorphisms of mapping spaces. If X is locally compact, then the evaluation map $ev\colon Y^X \times X \to Y$ is continuous. If X and Z are locally compact as well as Hausdorff, then the currying function $Y^{Z\times X} \to (Y^Z)^X$ is a homeomorphism.

For a fixed space Y, the coproduct and gluing properties characterize the functor $\mathsf{Spaces}(-,Y)$ *represented* by Y as a *sheaf*, i.e., a functor $\mathsf{Spaces}^{\mathrm{op}} \to \mathsf{Sets}$ which preserves limits (for certain covers). Unfortunately, the fragility of function spaces stymies the converse: not all sheaves on Spaces (or even on locally compact, Hausdorff spaces) are representable. Our affection for representable functors will grow substantially in the coming sections, and we will find relief in Section 3.1, which gives a representability criterion adapted to homotopy theory.

We will perpetually arrange to be in the situation where our function-spaces are well-behaved.

Remark 1.1.1. In each of these cases, the named property uniquely determines the space involved. For instance, if Q is some unknown space so that $\mathsf{Spaces}(Q,T)$ naturally bijects with the subset

$$\{f \in \mathsf{Spaces}(X,T) \mid x \sim x' \Rightarrow f(x) = f(x')\},$$

then there is a homeomorphism $Q \xrightarrow{\cong} X/\sim$.

These constructions also interact well with each other in most cases of geometric interest. For example:

(?, 0.4)

Lemma 1.1.2. *If Y is locally compact (e.g., if $Y = [0,1]$), then there is a homeomorphism*

$$\frac{X \times Y}{\sim \times \mathrm{id}} \xrightarrow{\cong} \left(\frac{X}{\sim}\right) \times Y. \qquad \square$$

(?, 0.8)

Corollary 1.1.3. *If $A \subseteq X$ is closed and $H(a,t) = H(a',t)$ for all $a, a' \in A$ and $t \in I$ (i.e., H is a homotopy relative to A), then it factors as*

$$X \times I \to X/A \times I \to Y. \qquad \square$$

Relative homotopy provides a convenient segue into two other categories of interest: the category of relative pairs and the category of pointed spaces.

Definition 1.1.4. The category of *relative pairs* has objects given by inclusions $A \subseteq X$, and a morphism $(A \subseteq X) \to (B \subseteq Y)$ between two such pairs is given by a continuous map $f: X \to Y$ with the property $f(A) \subseteq B$. These pairs and their maps appear naturally when discussing relative homology or relative homotopy.

There is no reason one must stop at a single inclusion. Chains of inclusions—even infinite chains—are interesting, and they are generally referred to as *filtered spaces*.

Definition 1.1.5. The category $\mathsf{Spaces}_{*/}$ of *pointed spaces* [1]has objects given by $\{x_0\} \subseteq X$ for some choice of singleton $\{x_0\}$, and a morphism $(\{x_0\} \subseteq X) \to (\{y_0\} \subseteq Y)$ between two such pointed spaces is a map $f\colon X \to Y$ with the property $f(x_0) = y_0$. Pointed spaces and their maps arise when defining reduced homology or homotopy groups (e.g., the fundamental group). Pointed spaces form a full subcategory of relative pairs: they are the special case where the privileged subset is a singleton.

(?, 0.12)

[1] This notation is to be read as "spaces under the singleton" or "spaces with a specified map from the singleton". Namely, the map $* \to X$ selects x_0.

These categories admit all of the same categorical constructions as Spaces: they have products, coproducts, and quotients, their morphisms glue, and one can build function objects when the spaces involved carry appropriate adjectives. In many cases, if one forgets about the privileged subset and considers just the underlying object of Spaces, the construction even agrees with that in Spaces.

This observation is owed to some adjunctions: a pair and a pointed space both forget to a space, a space becomes pointed by adding a disjoint basepoint, and a pair becomes a pointed space by quotienting the privileged subspace to a point. Each of these functors admits an adjoint, which helps pin down the formulas below.

Coproducts The coproduct of pairs is given by

$$(A \subseteq X) \sqcup (B \subseteq Y) = (A \sqcup B) \subseteq (X \sqcup Y).$$

Products The product is given by

$$(A \subseteq X) \times (B \subseteq Y) = (((A \times Y) \cup (X \times B)) \subseteq X \times Y),$$

but the privileged subspace has become more complicated.

Function spaces / exponential objects One can also define a function object

$$((B \subseteq Y)^{(A \subseteq A)} \subseteq (B \subseteq Y)^{(A \subseteq X)}),$$

itself a relative pair which we abbreviate to $(B \subseteq Y)^{(A \subseteq X)}$ for sanity's sake. Function objects again satisfy an analogue of currying:

(?, Proposition 0.13)

$$\begin{aligned}(B \subseteq Y)^{((Z \times A) \cup (C \times X)) \subseteq Z \times X} &= (B \subseteq Y)^{(C \subseteq Z) \times (A \subseteq X)} \\ &= \left((B \subseteq Y)^{(C \subseteq Z)}\right)^{(A \subseteq X)}.\end{aligned}$$

In trying to find analogues of these claims in pointed spaces, there is the following snag: the coproduct of two pointed spaces *in the category of relative pairs* is given by

$$(\{x_0\} \subseteq X) \sqcup (\{y_0\} \subseteq Y) = (\{x_0, y_0\} \subseteq (X \sqcup Y)),$$

but this coproduct has escaped the subcategory $\mathsf{Spaces}_{*/}$. One can forcefully correct this by quotienting the privileged subspace so that it becomes a point—and, in fact, this recovers the correct coproduct in pointed spaces:

Convert the margin note about adjunctions into the main text? One could then also comment on currying as an example of an adjunction.

Definition 1.1.6. For two pointed spaces $\{x_0\} \subseteq X$ and $\{y_0\} \subseteq Y$, their *wedge sum* $X \vee Y$ is the coproduct in the category of pointed spaces:

$$\mathsf{Spaces}_{*/}(X \vee Y, T) = \mathsf{Spaces}_{*/}(X, T) \times \mathsf{Spaces}_{*/}(Y, T).$$

In terms of the pair, it is given by

$$X \vee Y = \frac{\{x_0, y_0\} \subseteq (X \sqcup Y)}{\{x_0, y_0\}}.$$

The relationship between the products in relative pairs and in pointed spaces is much looser: the product of pointed spaces is given by

$$(\{x_0\} \subseteq X) \times (\{y_0\} \subseteq Y) = (\{(x_0, y_0)\} \subseteq (X \times Y)),$$

where the complexity in the privileged subspace has seemingly evaporated. However, we find ourselves in a further awkward position when we investigate the currying law for the function object for pointed spaces:

For the reader who knows about limits: we didn't have to quotient this time because of the chirality of the limits involved. Namely, coproducts and quotients are both colimits, but a product is a limit.

$$\begin{aligned}(\{y_0\} \subseteq Y)^{((Z\times\{x_0\})\cup(\{z_0\}\times X))\subseteq Z\times X} &= (\{y_0\} \subseteq Y)^{(\{z_0\}\subseteq Z)\times(\{x_0\}\subseteq X)} \\ &= \left((\{y_0\} \subseteq Y)^{(\{z_0\}\subseteq Z)}\right)^{(\{x_0\}\subseteq X)}.\end{aligned}$$

In particular, the exponent on the far left is *not* the product in pointed spaces—indeed, it isn't a pointed space at all. We can, again, force it to become pointed:

(?, 2.4)

Definition 1.1.7. The *smash product* of two pointed spaces $\{x_0\} \subseteq X$ and $\{z_0\} \subseteq Z$ is given by

$$(\{x_0\} \subseteq X) \wedge (\{z_0\} \subseteq Z) := \frac{X \times Z}{(X \times \{z_0\}) \cup (\{x_0\} \times Z)} = \frac{X \times Z}{X \vee Z}.$$

This gives a *monoidal structure* on $\mathsf{Spaces}_{*/}$ which is *not* the Cartesian one.

Define a monoidal structure?

That is: the one given by the ordinary product.

Corollary 1.1.8. *For pointed spaces X, Y, and Z which are appropriately nice, the currying law for function objects takes the form*

$$Y^{Z\wedge X} \cong (Y^Z)^X, \qquad \mathsf{Spaces}_{*/}(Z \wedge X, Y) \cong \mathsf{Spaces}_{*/}(X, Y^Z). \quad \square$$

Lemma 1.1.9. *This product is related to the product on unpointed spaces by*

$$X_+ \wedge Y_+ \cong (X \times Y)_+,$$

where $(-)_+ \colon \mathsf{Spaces} \to \mathsf{Spaces}_{*/}$ denotes the functor which adds a disjoint basepoint. $\square$

(?, 0.7)

Definition 1.1.10. Finally, we introduce the *homotopy category of (pointed) spaces*, $h\mathsf{Spaces}_{*/}$, with the "h" denoting "homotopy". Its objects are the same as that of $\mathsf{Spaces}_{*/}$ but whose morphism sets are given by the quotient

$$h\mathsf{Spaces}_{*/}(X, Y) = \frac{\mathsf{Spaces}_{*/}(X, Y)}{f \sim g},$$

where $f \sim g$ indicates that there is a homotopy $H \colon X \wedge I_+ \to Y$ between f and g. The objects are this category are often referred to as *homotopy* *types*. The reduced morphism sets mean that many maps in Spaces acquire inverses when considered as maps in hSpaces, e.g., the inclusion $* \to \mathbb{R}$. All of the basic invariants of algebraic topology descend to this quotient category: homotopy points and reduced homology groups behave the same on homotopic maps. Because we will work in this category so often, we will abbreviate this mapping set to

$$h\mathsf{Spaces}_{*/}(X,Y) = [X,Y].$$

1.2 Perspectives on the fundamental group

With categorical definitions in place, we are in position to investigate the traditional tools of algebraic topology in these terms. In this section, we treat the fundamental group, though we begin with an even simpler invariant.

(?, 2.1)

Definition 1.2.1. Write $I = [0,1]$ for the closed unit interval. The *pathspace* PX of a space X is given by $PX = X^I$. We write $\pi_0(X)$ for the set of path-components of X, i.e., the quotient of X where x is related to x' whenever there exists $\gamma \in X^I$ with

$$\gamma(0) = x, \qquad\qquad \gamma(1) = x'.$$

Remark 1.2.2. The mapping set given in Definition 1.1.10 is equivalent to

$$[X,Y] = \pi_0 Y^X.$$

By specializing X to the 0–sphere $S^0 = \{\pm 1\} \subset \mathbb{R}$, we have

$$Y^{S^0} = (Y, y_0)^{(\{\pm 1\}, -1)} = Y^{\{1\}} = Y,$$

and hence

$$\pi_0 Y = \pi_0 Y^{S^0} = [S^0, Y].$$

Recall now that the *fundamental group* is given by

$$\pi_1(Y) := \{\text{homotopy classes of pointed loops in } X\} = [S^1, Y].$$

We can use adjunctions to extend this to several equivalent definitions:

$$\pi_1(Y) = [S^1, Y] = [S^0 \wedge S^1, Y] = [S^0, Y^{(S^1)}] = \pi_0 Y^{S^1}.$$

One might wonder what properties of X and Y make $[X,Y]$ into a group, since we know that $[S^1, -]$ and $[S^0, (-)^{S^1}]$ are group-valued. In keeping with our investigation so far, we will be particularly interested in a *categorical* answer to this question, and to this end it will be helpful to have a categorical definition of a group.

(?, pg. 14)

Definition 1.2.3. A *group* is a pointed set G with maps $\mu\colon G \times G \to G$ and $\chi\colon G \to G$ which make the following diagrams commute:

Associativity:
$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G, \end{tikzcd}$$

Unitality:
$$\begin{tikzcd} G \arrow[r, "\eta \times \mathrm{id}"] \arrow[dr, equal] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\mathrm{id} \times \eta"'] \arrow[dl, equal] \\ & G, & \end{tikzcd}$$

Invertibility:
$$\begin{tikzcd} G \arrow[r, "\chi \times \mathrm{id}"] \arrow[d, "0"] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\mathrm{id} \times \chi"'] \arrow[d, "0"] \\ * \arrow[r, "\eta"] & G & *. \arrow[l, "\eta"'] \end{tikzcd}$$

More generally, a *group object* in a pointed category C is a pointed object G such that $G^{\times 2}$ and $G^{\times 0}$ exist, equipped with maps μ and χ satisfying the same diagrams.

Remark on non-Cartesian monoidal structures?

(?, Definition 2.9)

Definition 1.2.4. An *H–group K* is group object in $h\mathsf{Spaces}_{*/}$, i.e., a pointed space with maps μ and χ satisfying the group diagrams up to homotopy.

Lemma 1.2.5. *If G is a group object in a category C, then $\mathsf{C}(-,G)$ is a functor from C^{op} to Groups.*

Definition of op?

Proof. Recall that the defining property of the product $X \times Y$ is

$$\mathsf{C}(T, X \times Y) \cong \mathsf{C}(T,X) \times \mathsf{C}(T,Y).$$

For a fixed test object T, the functor $\mathsf{C}(T,-)$ applied to the multiplication μ on G gives

$$\mathsf{C}(T,G) \times \mathsf{C}(T,G) \cong \mathsf{C}(T, G \times G) \xrightarrow{\mu_*} \mathsf{C}(T,G),$$

and in this way μ_* becomes a multiplication on $\mathsf{C}(T,G)$. The inversion χ on G similarly induces an inversion χ_* on $\mathsf{C}(T,G)$, and together these make the group object diagrams commute. □

For a pair of maps
$$f, g \colon T \to G,$$
it can be helpful to factor
$$(f,g) \colon T \to G \times G$$
through the diagonal as in
$$T \xrightarrow{\Delta} T \times T \xrightarrow{f \times g} G \times G,$$
as this lets one leverage the bifunctoriality of $\times$. This doesn't come up directly here, but it does arise in applications.

Remark 1.2.6. In fact, this is biconditional: if a representable functor factors through Groups, then the representing object inherits the structure of a group object. This is a consequence of the *Yoneda lemma*: any natural transformation between representable functors is uniquely induced by pushforward along a map between the representing objects.

(?, Proposition 2.14)

Corollary 1.2.7. *For K an H–group, $[-,K]$ is valued in groups.* □

We use L in Section 1.3.

(?, Examples 2.15.ii)

Example 1.2.8. The usual verification that the fundamental group of a space is a group can be viewed as giving an H–space structure on $Y^{S^1} =: \Omega Y$. The multiplication map μ is given by rescaling and concatenating two loops, and the inversion map χ is given by reversing a loop. Hence, not only is the fundamental group $\pi_1 Y = \pi_0 \Omega Y = [S^0, \Omega Y]$ a group, but actually $[X, \Omega Y]$ is naturally a group for any X.

I remmeber $[Y,X]$ being convenient notation in past contexts. Review this whole doc to see if it would fit here too.

What about the other formulation? We also have that $\pi_1 Y = [S^1, Y]$ is naturally group-valued as Y varies. From this, one might expect the root cause does not reside in the Y-dependent object "ΩY" but rather in S^1 alone. In order to address this by the same tactic, we must confront the structure on X which makes $[X, Y]$ a group. Using the identity $[X \vee X, Y] \cong [X, Y] \times [X, Y]$, we are led to the following definition:

(?, Definition 2.16)

Definition 1.2.9. A *cogroup object* in a pointed category C is a pointed object K such that $K^{\vee 2}$ and $K^{\vee 0}$ exist, equipped with maps μ' and χ' so that the following diagrams commute:

Coassociativity:
$$\begin{tikzcd}
K \vee K \vee K & K \vee K \arrow[l, "\mu' \vee \mathrm{id}"'] \\
K \vee K \arrow[u, "\mathrm{id} \vee \mu'"'] & K, \arrow[l, "\mu'"'] \arrow[u, "\mu'"']
\end{tikzcd}$$

Counitality:
$$\begin{tikzcd}
K & K \vee K \arrow[l, "0 \vee \mathrm{id}"'] \arrow[r, "\mathrm{id} \vee 0"] & K \\
& K, \arrow[ul, equal] \arrow[u, "\mu'"'] \arrow[ur, equal] &
\end{tikzcd}$$

Coinvertibility:
$$\begin{tikzcd}
K & K \vee K \arrow[l, "\chi' \vee \mathrm{id}"'] \arrow[r, "\mathrm{id} \vee \chi'"] & K \\
* \arrow[u, "\eta"'] & K \arrow[l, "0"'] \arrow[u, "\mu'"'] \arrow[r, "0"'] & *. \arrow[u, "\eta"']
\end{tikzcd}$$

An *H-cogroup* K is then a cogroup object in $h\mathsf{Spaces}_{*/}$.

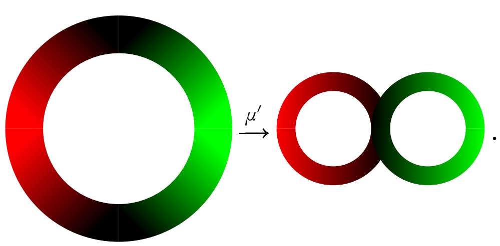
(?, Proposition 2.21)
Again, the Yoneda lemma for *corepresentable* functors gives a converse to this claim.

Corollary 1.2.10. *For K an H-cogroup, $[K, -]$ is valued in groups.* □

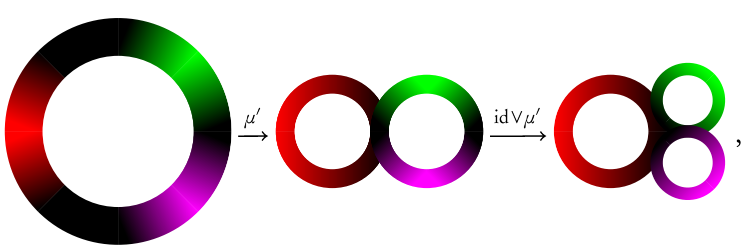
Example 1.2.11. S^1 is an H-cogroup. We depict its structure maps by shading the loops in $S^1 \vee \cdots \vee S^1$ different colors, then pulling back that shading along the indicated map. For example, the comultiplication map μ' is given by quotienting together the basepoint and its antipode, and the corresponding color map looks like

These would be improved by explicitly tagging the basepoint, which must remain black.

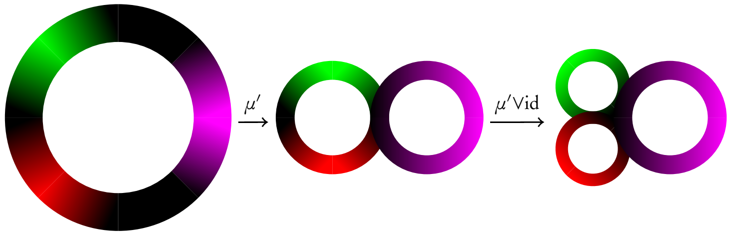
Generally, these pictures are buggy.



The left arm of the coassociativity axiom has color map



while the right arm has color map



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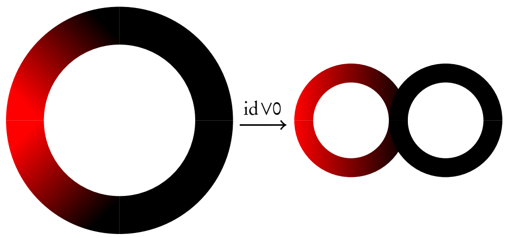
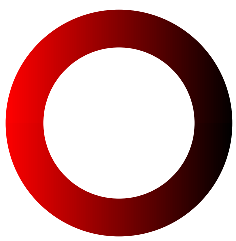
and coassociativity is checked by noting that the pulled-back color maps can be rotated into one another. Similarly, counitality is checked by noting that the color map



is a rescaling of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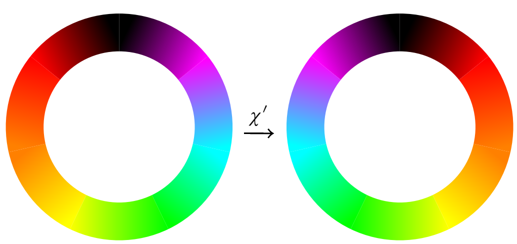
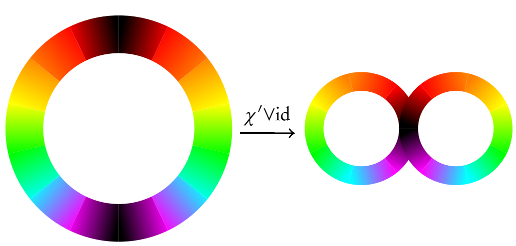
The coinversion map χ' acts as

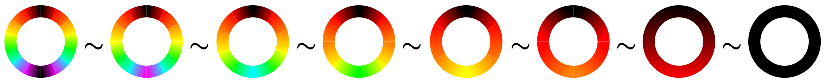


,

and coinvertibility is checked by noting that the pulled-back color map



is homotopic to the constant map through



Example 1.2.12. In fact, $S^1 \wedge X =: \Sigma X$ is an H–cogroup for *any* X, using the same pinching and reversing technique along the suspension coordinate. (?, 2.22)

Lemma 1.2.13. *The adjunction* $[\Sigma X, Y] \cong [X, \Omega Y]$ *is an isomorphism of groups.* (?, Proposition 2.23)

Proof sketch. This is a matter of writing out the formulas for

$$\Sigma X \xrightarrow{\mu'} \Sigma X \vee \Sigma X \xrightarrow{f' \vee g'} Y \vee Y \xrightarrow{\Delta'} Y$$

and

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \Omega Y \times \Omega Y \xrightarrow{\mu} Y. \qquad \square$$

1.3 *Higher homotopy groups*

Just as we used the exponential adjunction to give a family of equivalent definitions of the fundamental group, there is a similar family for higher homotopy groups, which we explore today. The key input is the following:

Definition 1.3.1. The n^{th} *homotopy group* of a pointed space X is defined by $\pi_n X = [S^n, X]$. (?, 3.1)

Lemma 1.3.2. *For all* $n \geq 0$, *there is a homeomorphism* $S^1 \wedge S^n \cong S^{n+1}$. (?, Lemma 2.27)

Proof. First, the *reduced cone* is defined by $CX = I \wedge X$, where I is again the closed unit interval. There is a homeomorphism

$$CS^n = I \wedge S^n \cong D^{n+1},$$

given by linear interpolation from the sphere to its basepoint. Now, decompose S^1 into the disjoint union of its closed upper- and lower-hemispheres, quotiented together along the equator. Applying the interchange lemmata from Section 1.1, this decomposes $S^1 \wedge S^n$ into two copies of $I \wedge S^n \cong D^{n+1}$, quotiented together along their boundaries. This is homeomorphic to the decomposition of S^{n+1} into its upper- and lower-hemispheres. $\square$

Corollary 1.3.3. *The following sets all give equivalent definitions of* $\pi_n X$:

$$\pi_n X = [\Sigma^n(S^0), X] = [\Sigma^{n-1} S^0, \Omega X] = \cdots = [S^0, \Omega^n X]. \qquad \square$$

We've quietly asserted that $\pi_n X$ is a group, but *which* group structure we mean is not immediately clear: in light of Corollary 1.3.3, there appear to be many different multiplications on the homotopy mapping sets, each coming from any one of the applications of Σ or of Ω. We tame this complexity by showing that each of these choices gives the same multiplication.

(?, Proposition 2.24)

Lemma 1.3.4 (Eckmann–Hilton)**.** *Let S be a set with two products $\circ$ and $*$ which share a unit e and which satisfy the distributive law:*

$$(x * x') \circ (y * y') = (x \circ y) * (x' \circ y').$$

Then the products agree, and both are associative and commutative.

Proof. To show commutativity, we cleverly redistribute:

We don't show associativity!

$$\begin{aligned} x \circ y &= (x * e) \circ (e * y) = (x \circ e) * (e \circ y) = x * y, \\ x \circ y &= (e * x) \circ (y * e) = (e \circ y) * (x \circ e) = y * x. \end{aligned}$$ □

(?, Proposition 2.25)

Corollary 1.3.5. *Let K be an H-cogroup and L an H-group. The two multiplications so-induced on the mapping set $[K, L]$ are equal and commutative.*

Proof. Consider the following diagram:

$$\begin{tikzcd} K \times K \arrow[r] & (K \vee K) \times (K \vee K) \arrow[r, "(f \vee f') \times (g \vee g')"] & (L \vee L) \times (L \vee L) \arrow[r] & L \times L \arrow[d] \\ K \arrow[u] \arrow[d] & & & L \\ K \vee K \arrow[r] & (K \times K) \vee (K \times K) \arrow[r, "(f \times g) \vee (f' \times g')"] & (L \times L) \vee (L \times L) \arrow[r] & L \vee L. \arrow[u] \end{tikzcd}$$

The top composite corresponds to $(f +_K f') +_L (g +_K g')$, and the bottom composite corresponds to $(f +_L g) +_K (f' +_L g')$. If these were to agree, we could deduce the claim from Lemma 1.3.4. We approach this in stages, breaking the diagram into smaller ones which more clearly commute.

The diagonal map $\Delta \colon K \to K \times K$ replicates its input onto both coordinates, as in $x \mapsto (x, x)$. Writing $\mu'(k) = (k_1, k_2)$, we can apply Δ either before or after the comultiplication map μ' on K, and get the same answer up to a twist:

$$\begin{aligned} (\Delta \vee \Delta) \circ \mu'(k) &= (\Delta \times \Delta)(k_1, k_2) = (k_1, k_1, k_2, k_2), \\ (\mu' \times \mu') \circ \Delta(k) &= (\mu' \times \mu')(k, k) = (k_1, k_2, k_1, k_2). \end{aligned}$$

Additionally, because $\mu' \colon K \to K \vee K$ targets the wedge sum, it is always the case that at least one of k_1 or k_2 equals the basepoint. From these two considerations, it follows that the left-hand portion of the following diagram commutes:

$$\begin{tikzcd}
K \times K \arrow[r] & (K \vee K) \times (K \vee K) \arrow[r] & (L \vee L) \times (L \vee L) \arrow[r] & L \times L \arrow[d] \\
K \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c} (K \times *) \times (K \times *) \\ \cup \\ (* \times K) \times (* \times K) \end{array} \arrow[u] \arrow[d, "\mathrm{id} \times T \times \mathrm{id}"] & & L \\
K \vee K \arrow[r] & (K \times K) \vee (K \times K) \arrow[r] & (L \times L) \vee (L \times L) \arrow[r] & L \vee L, \arrow[u]
\end{tikzcd}$$

where T denotes the twist map. Now, we apply (f, f', g, g') on top and (f, g, f', g') on bottom. Again, the diagram commutes up to transposition of the two middle coordinates.

$$\begin{tikzcd}
K \times K \arrow[r] & (K \vee K) \times (K \vee K) \arrow[r] & (L \vee L) \times (L \vee L) \arrow[r] & L \times L \arrow[d] \\
K \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c} (K \times *) \times (K \times *) \\ \cup \\ (* \times K) \times (* \times K) \end{array} \arrow[u] \arrow[d, "\mathrm{id} \times T \times \mathrm{id}"] \arrow[r] & \begin{array}{c} (L \times *) \times (L \times *) \\ \cup \\ (* \times L) \times (* \times L) \end{array} \arrow[u] \arrow[d, "\mathrm{id} \times T \times \mathrm{id}"] & L \\
K \vee K \arrow[r] & (K \times K) \vee (K \times K) \arrow[r] & (L \times L) \vee (L \times L) \arrow[r] & L \vee L. \arrow[u]
\end{tikzcd}$$

Finally, we make use of the specific subspace we'd highlighted in the middle. In (either half of) this subspace, two of the coordinates are constrained to lie at the basepoint. Because of this, we can apply the unit axioms for the multiplication map and for the fold map to conclude that they produce the same value. This amounts to the commutativity of the final two rectangles:

$$\begin{tikzcd}
K \times K \arrow[r] & (K \vee K) \times (K \vee K) \arrow[r] & (L \vee L) \times (L \vee L) \arrow[r] & L \times L \arrow[d] \\
K \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c} (K \times *) \times (K \times *) \\ \cup \\ (* \times K) \times (* \times K) \end{array} \arrow[u] \arrow[d, "\mathrm{id} \times T \times \mathrm{id}"] \arrow[r] & \begin{array}{c} (L \times *) \times (L \times *) \\ \cup \\ (* \times L) \times (* \times L) \end{array} \arrow[u] \arrow[d, "\mathrm{id} \times T \times \mathrm{id}"] \arrow[r] & L \\
K \vee K \arrow[r] & (K \times K) \vee (K \times K) \arrow[r] & (L \times L) \vee (L \times L) \arrow[r] & L \vee L. \arrow[u]
\end{tikzcd}$$

Since each of the squares commutes, the outer composites are equal. □

(?, Proposition 2.26)

Corollary 1.3.6. *For $n \geq 2$, $\pi_n X$ has only one multiplication, and it is commutative.* □

1.4 *Exact sequences in* Spaces

An *extremely* common device in algebraic topology is the exact sequence:

(?, 2.29)

Definition 1.4.1. A composable pair of maps of groups

$$N \xrightarrow{f} G \xrightarrow{g} H$$

is called *exact* when $\operatorname{im} f = \ker g$. More weakly, a sequence of pointed sets $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* if $\operatorname{im} f = g^{-1}(*)$.

Last time, we put structures onto spaces K and L so that $[K, -]$ and $[-, L]$ became valued in groups. Today we are after something similar: we would like to study when continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ induce exact sequences of the functors they co/represent.

(?, Definitions 2.30 and 2.49)

Definition 1.4.2. If the sequence of pointed sets

$$[-, X] \xrightarrow{f_*} [-, Y] \xrightarrow{g_*} [-, Z]$$

is exact, then we say that the underlying sequence of spaces is *exact*. If the sequence of pointed sets

$$[X, -] \xleftarrow{f^*} [Y, -] \xleftarrow{g^*} [Z, -]$$

is exact, then we say that the underlying sequence of spaces is *coexact*.

These more commonly go by the names *fiber* and *cofiber* sequences of spaces.

At first brush, one might imagine that such sequences are somewhat rare, or that they at least require some nice properties of the spaces X, Y, Z or of the maps f, g. In fact, these sequences are bountiful in homotopy theory—a first sign that our approach to the subject is a sound one.

This scarcity is true for maps of groups: $f\colon H \to G$ can only participate in an exact sequence when $\operatorname{im} f$ is normal. It is a good exercise to check that the following constructions for spaces do not violate this.

(?, Proposition 2.35)

Lemma 1.4.3. *Any map $f\colon X \to Y$ extends to a coexact sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Proof. Recall the *cone* of X, $CX = X \wedge I$. We use this construction to define C_f, the *mapping cone* of f:

$$C_f := Y \cup_f CX = \frac{Y \sqcup CX}{f(x) \sim (x, 1)}.$$

We claim that C_f plays the role of Z, and the inclusion $g\colon Y \to C_f$ induces the desired exact sequence on mapping sets.

To check this, consider a test space T, the induced sequence

$$[Y \cup_f CX, T] \to [Y, T] \to [X, T],$$

and a function $\varphi\colon Y \to T$. Note that the exponential adjunction gives a bijection between maps $CX \to T$ and maps $X \to T$ together with a choice

The mapping cone is presented by the following pushout diagram in $\mathsf{Spaces}_{*/}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & CX \arrow[d] \\ Y \arrow[r] & C_f. \end{tikzcd}$$

While this is *not* a pushout diagram in the homotopy category $h\mathsf{Spaces}_{*/}$, there is an equivalence $CX \simeq *$, so that the diagram has the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & * \arrow[d] \\ Y \arrow[r] & C_f, \end{tikzcd}$$

whence the name "cofiber". The relationship between co/limits in $\mathsf{Spaces}_{*/}$ and interesting phenomena in the homotopy category is quite delicate. A rule of thumb for the uninitiated is that colimits of loopless diagrams of inclusions are well-behaved. A proper treatment of this interplay is given by the theory of *homotopy co/limits*.

of null-homotopy. Hence, if $\varphi \circ f\colon X \to T$ is null, then this gives a map $CX \to T$ which agrees with f on the edge of I, and the gluing operation determines a map $Z \to T$ which restricts to φ on Y. Conversely, a map $\widetilde{\varphi}\colon Z \to T$ which restricts to φ on Y can itself be restricted to CX, and this gives the required null-homotopy of $\varphi \circ f$. $\square$

Mention moduli objects?—moduli of maps + homotopies?

The remarkable lack of hypotheses in Lemma 1.4.3 mean that coexact sequences can be extended indefinitely to the right:

$$X \xrightarrow{f} Y \xrightarrow{g} Y \cup_f CX \to (Y \cup_f CX) \cup_g CY \to \cdots.$$

The naive presentations of these spaces quickly become unwieldy, but the following alternative presentations are quite tractible:

(?, Lemma 2.37)

Lemma 1.4.4. *There is a homotopy equivalence*

$$(Y \cup_f CX) \cup_g CY \to ((Y \cup_f CX) \cup_g CY)/CY. \qquad \square$$

(?, Proposition 2.38)

Lemma 1.4.5. *For $A \subseteq X$ a subspace, there is a homeomorphism*

$$(X \cup_i CA)/CA \cong X/A. \qquad \square$$

(?, Lemma 2.40)

Corollary 1.4.6. *The infinite coexact sequence takes the form*

$$X \xrightarrow{f} Y \to Y \cup_f CX \to \Sigma X \xrightarrow{-f} \Sigma Y \to \Sigma(Y \cup_f CX) \to \cdots,$$

where $\Sigma X = S^1 \wedge X$ is the suspension of X.

Proof. We need only identify the next two terms after Z and the map between them. Once that is in hand, we need only note that suspension commutes with the cone and quotient operations used to define Z. To see the claim for the first term after Z, we apply the two Lemmas in turn:

$$\begin{aligned}
(Y \cup_f CX) \cup_g CY &\simeq ((Y \cup_f CX) \cup_g CY)/CY && \text{(Lemma 1.4.4)} \\
&\cong (Y \cup_f CX)/Y && \text{(Lemma 1.4.5)} \\
&\cong CX/X \cong \Sigma X. && (I/\{0,1\} \cong S^1)
\end{aligned}$$

Identically, the second term after Z is described by

$$((Y \cup_f CX) \cup_g CY) \cup_{g'} C(Y \cup_f CX) \simeq \Sigma Y.$$

The coexact sequence thus takes the desired form. $\square$

Coupling this to our results from last time, this gives an exact sequence

Exact sequences are best behaved on abelian groups, but not all of these are abelian groups—or even groups! What can be said about the edges, where at least one term is a(n abelian) group?

$$\begin{tikzcd}
{[X,T]} & {[Y,T]} \arrow[l] & {[\underbrace{Y \cup_f CX}_{Z}, T]} \arrow[l] & & \text{ptd. sets} \\
{[\Sigma X,T]} \arrow[urr, bend left] & {[\Sigma Y,T]} \arrow[l] & {[\Sigma Z,T]} \arrow[l] & & \text{groups} \\
{[\Sigma^2 X,T]} \arrow[urr, bend left] & {[\Sigma^2 Y,T]} \arrow[l] & {[\Sigma^2 Z,T]} \arrow[l] & \cdots \arrow[l] & \text{ab. groups}
\end{tikzcd}$$

Construction 1.4.7. Pinching the middle of the cone CX gives a map

$$Y \cup_f CX \to (Y \cup_f CX) \vee \Sigma X,$$

which gives an action

$$[Z,T] \times [\Sigma X,T] \to [Z,T].$$

Lemma 1.4.8. *(?, 2.42–48) The map $[Z,T] \to [Y,T]$ is invariant under this action, and on orbits it is an injection.* □

There are also dual results for exact sequences: any map of spaces participates in an (infinite) exact sequence extending to the left.

Remove numbering from margin.

Lemma 1.4.9. *Any map $f\colon X \to Y$ extends to an exact sequence*

$$P_f \to X \xrightarrow{f} Y,$$

where P_f is given by

$$P_f = \{(x,\gamma) \in X \times PY \mid f(x) = \gamma(1)\}.$$

(?, Proposition 2.54) Again, P_f appears as a pullback in $\mathsf{Spaces}_{*/}$.

$$\begin{tikzcd}
P_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
PY \arrow[r] & Y
\end{tikzcd}$$

Again, this square is not a pullback in $h\mathsf{Spaces}_{*/}$, but $PY = Y^I$ is contractible, earning this object the name *homotopy fiber*. (Note that while the pointed function space Y^I is contractible, the unpointed function space of maps from I to Y is weakly equivalent to Y.)

Proof sketch. Again, the construction of P_f rests on building into it the data of a null-homotopy. Suppose that $\theta\colon T \to Y$ is a null-homotopic map. A null-homotopy of θ is the same as a map $T \wedge I = CT \to Y$ restricting to θ, which is in turn the same as a map $T \to PY = Y^I$ restricting to θ. To model a map $\varphi\colon T \to X$ for which $\theta = f \circ \varphi$ becomes null-homotopic, we attach PY along X as in the statement. □

Another opportunity for a description in terms of a moduli problem.

Lemma 1.4.10. *Iterating this gives*

$$\cdots \to \Omega^2 X \to \Omega P_f \to \Omega X \to \Omega Y \to P_f \to X \xrightarrow{f} Y. \qquad \square$$

(?, Proposition 2.58)

In particular, by applying $\pi_0(-) = [S^0, -]$ and employing the definition $\pi_n X = \pi_0 \Omega^n X$, we get an exact sequence of homotopy groups:

Move this into the next section as motivation?

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 P_f \to \pi_1 X \to \pi_1 Y \to \pi_0 P_f \to \pi_0 X \to \pi_0 Y.$$

One of our goals in this class will be to understand what understanding π_* earns us, how it changes as X changes, and how X can be effectively dissected to build up knowledge of π_* along the way. This long exact sequence will be one of our main tools for doing so.

1.5 Relative homotopy groups

The construction P_f from the previous lecture is a little mysterious. To close out this Chapter, we will explore it in two lights: today and a bit later

Motivate this with $H^*(X,A) = H^*(X \cup CA)$. P_f seems poorly adapted to H^* but well-adapted to π_*.

"Two lights": bad, awkward

on.

Let us return to the setting of pairs (X, A, x_0) of topological spaces, and let's use $i\colon A \to X$ to denote the inclusion of the preferred subspace. The following is a direct rephrasing of the definition of P_i from the previous section:

(?, Definition 3.8)

Lemma 1.5.1. *The exact continuation P_i of i is given by the function object*

$$P_i = (X, A, x_0)^{(I, \partial I, 0)}. \qquad \square$$

Inspired by this, we consider the adjunction juggle:

(?, pg. 38)

Definition 1.5.2. For a pair $(A \subseteq X)$, we define the n^{th} *relative homotopy group* to be

$$\pi_n(A \subseteq X) := [(S^{n-1} \subseteq D^n), (A \subseteq X)] \cong [S^{n-1}, P_i] = \pi_{n-1} P_i.$$

Note that this means that the first relative group is a *set*, the second a *group*, and third and above *abelian groups*—one off from the usual sequence of properties.

That is, $\pi_n(A \subseteq X)$ consists of n–disk maps into X with boundary ∂D^n lying in A, considered up to relative homotopy.

(?, Proposition 3.9)

Corollary 1.5.3. *There is a long exact sequence*

Describe these maps in terms of restrictions.

$$\cdots \to \pi_{n+1}(A \subseteq X) \to \pi_n A \to \pi_n X \to \pi_n(A \subseteq X) \to \pi_{n-1} A \to \cdots. \qquad \square$$

More qualitatively, these groups track the discrepancy between $\pi_* A$ and $\pi_* X$. This observation moves us to remark on when there is *no* discrepancy:

(?, 3.12–13 and 3.17)

Definition 1.5.4. A pair $A \subseteq X$ is *n–connected* if $\pi_{\le n}(A \subseteq X) = 0$. Equivalently, the map $\pi_{<n} A \to \pi_{<n} X$ is an isomorphism and $\pi_n A \to \pi_n X$ is an epimorphism. The pair $A \subseteq X$ is a *weak equivalence* if it is ∞–connected.

Usually "weak equivalence" refers to a map. Immediately get in the habit; there's nothing special about pairs here.

(?, 3.21)

Remark 1.5.5. As a simple structural application of Corollary 1.5.3, consider an inclusion $i\colon A \subseteq X$ which admits a retraction $r\colon X \to A$. The induced map $\pi_* A \to \pi_* X$ is then an inclusion, and the boundary map $\pi_*(X, A) \to \pi_{*-1}(A)$ is zero. It follows that there are split short exact sequences

That is: $r i \simeq \mathrm{id}_A$.

$$\begin{tikzcd} 0 \arrow[r] & \pi_n A \arrow[r] & \pi_n X \arrow[l, bend right] \arrow[r] & \pi_n(A \subseteq X) \arrow[r] & 0. \end{tikzcd}$$

For $n \ge 3$, all the groups are abelian, from which we deduce that $\pi_n X$ splits as a sum of $\pi_n A$ and $\pi_n(A \subseteq X)$. At $n = 2$, we learn naively that $\pi_2 X$ is a semidirect product of $\pi_2 A$ and $\pi_2(X, A)$—but this forces $\pi_2(A \subseteq X)$ to be abelian has well. Hence, in this situation we have

$$\pi_{\ge 2} X = \pi_{\ge 2} A \oplus \pi_{\ge 2}(A \subseteq X).$$

(?, 3.15)

Remark 1.5.6. We can extend these definitions from an inclusion $i\colon A \to X$ to a generic map $f\colon Y \to X$ using the *mapping cylinder*:

Forward reference to an analogous statement for the P construction.

$$M_f := (Y \times I) \cup_f X.$$

This space receives a map $X \to M_f$ which is a weak equivalence. It also receives two maps $Y \to M_f$, one along the "free" end of the cylinder $Y \times I$ and one along the "attached" end. Along the free end, the map from Y to M_f is an inclusion, via which $(Y \subseteq M_f)$ can be thought of as a pair. Along the attached end, the map from Y to M_f factors through X, via which it agrees with f. The situation is summarized in the following diagram:

It is a good exercise to check that it is a weak equivalence.

$$\begin{tikzcd}
Y \times \{1\} \arrow[d] \arrow[dr, hook] & M_f \arrow[d, equal] & \\
Y \times I \arrow[r] & (Y \times I) \cup_f X & X. \arrow[l, "\simeq"'] \\
Y \times \{0\} \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

This belongs earlier, maybe even just the start of this section.

WE PAUSE TO EMPHASIZE something that the reader might have dismissed as serendipity: by switching the notation $\pi_n(X, A) := \pi_n(A \subseteq X)$, Corollary 1.5.3 looks very much like a corresponding theorem about relative *homology* groups: it announces an exact sequence

$$\cdots \to \pi_{n+1}(X,A) \to \pi_n A \to \pi_n X \to \pi_n(X,A) \to \pi_{n-1}A \to \cdots.$$

It is extremely productive to see how far this analogy can be pushed: what theorems about homology can be replicated for homotopy? When a theorem for homology fails for homotopy, is it at least partially recoverable? What is the discrepancy?

Emphasize that homology is computable because it has these theorems, and homotopy has them too!

As a warm-up to this program, recall that the relative homology groups of a pair of inclusions $x_0 \in B \subseteq A \subseteq X$ can be interrelated. We will see whether the same can be said of relative homotopy groups. To this end, consider the long exact sequences associated to the pairs (X, A), (X, B), and (A, B), which arrange into the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \pi_{n+1}(X,A) \arrow[rr] \arrow[dr] & & \pi_n(A,B) \arrow[rr] \arrow[dr] & & \pi_{n-1}B \arrow[rr] \arrow[dr] & & \pi_{n-1}X \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[ur] \arrow[dr] & & \pi_n A \arrow[ur] \arrow[dr] & & \pi_n(X,B) \arrow[ur] \arrow[dr] & & \pi_{n-1}A \arrow[ur] \arrow[dr] & & \cdots \\
\cdots \arrow[r] & \pi_n B \arrow[rr] \arrow[ur] & & \pi_n X \arrow[rr] \arrow[ur] & & \pi_n(X,A) \arrow[rr] \arrow[ur] & & \pi_{n-1}(A,B) \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

To help with readability, we have colored the long exact sequences associated to (X, A), (A, B), and (X, B). To complete the symmetric pattern, we have additionally included maps $\pi_n(A, B) \to \pi_n(X, B)$ and $\pi_n(X, B) \to \pi_n(X, A)$ granted to us by naturality, as well as maps $\pi_n(X, A) \to \pi_{n-1}(A, B)$ granted by completing the relevant triangle. Tracing through the middle, we see that these new maps form an interesting sequence, highlighted in red:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, tail] & \pi_{n+1}(X,A) \arrow[rr, red] \arrow[dr] & & \pi_n(A,B) \arrow[rr] \arrow[dr, red] & & \pi_{n-1}B \arrow[rr] \arrow[dr] & & \pi_{n-1}X \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[ur, red] \arrow[dr] & & \pi_n A \arrow[ur] \arrow[dr] & & \pi_n(X,B) \arrow[ur] \arrow[dr, red] & & \pi_{n-1}A \arrow[ur] \arrow[dr] & & \cdots \\
\cdots \arrow[r] & \pi_n B \arrow[rr] \arrow[ur] & & \pi_n X \arrow[rr] \arrow[ur] & & \pi_n(X,A) \arrow[rr, red] \arrow[ur] & & \pi_{n-1}(A,B) \arrow[r, tail] \arrow[ur, red] & \cdots
\end{tikzcd}$$

Lemma 1.5.7. *The highlighted sequence is exact.*

(?, Theorem 3.20)

Proof sketch. Checking that the composites are zero is easy enough. We describe how the lifting condition is proved for the left-most map pictured. Suppose that we have an element $x \in \pi_n(A,B)$ mapping to zero in $\pi_n(X,B)$. Its image in $\pi_{n-1}B$ is then also zero, so that we may lift it to $y \in \pi_n A$. Define $z \in \pi_n X$ to be the image of y; since y is zero by the time it makes it to $\pi_n(X,B)$, so must z be. We may then also lift z to $w \in \pi_n B$. Pushing forward w to $y' \in \pi_n A$ gives a second element in the same group, and the difference of the two elements $y - y'$ pushes forward to zero in $\pi_n X$. Hence, we can lift the difference to an element $\tilde{x}$ in $\pi_{n+1}(X,A)$. One checks that this gives the desired lift of x: the image of $\tilde{x}$ in $\pi_n(A,B)$ factors through its image $y - y'$ in $\pi_n(A)$, and of those operands the first is sent to x, the second to 0. $\square$

$$\begin{tikzcd}[column sep=tiny]
\nexists \arrow[r, tail] & \tilde{x} \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \arrow[rr, mapsto] & & 0 \arrow[r, tail] & \cdots \\
\nexists \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{matrix} y - y' \\ y', y \end{matrix} \arrow[ur, mapsto] \arrow[dr, mapsto] & & 0 \arrow[ur, mapsto] \arrow[dr, mapsto] & 0 \arrow[ul, mapsto] & & \cdots \\
\nexists \arrow[r, tail] & w \arrow[rr, mapsto] \arrow[ur, mapsto] & & z \arrow[r, mapsto] \arrow[ur, mapsto] & 0 \arrow[rr, mapsto] & & 0 \arrow[r, tail] & \cdots
\end{tikzcd}$$

1.6 Fibrations

In contrast to the small and polite model for the coexact continuation C_f, the presentation of the exact continuation P_f as a function space feels large and unwieldly. In this section and the next, we explore a more familiar context in which spaces *like* P_f arise, as well as conditions under which these model exact continuations.

For instance, if $f: X \to Y$ is a map of spaces of dimension d_X and d_Y, then the dimension of C_f is no more than d_Y and $1 + d_X$. On the other hand, P_f is firmly infinite-dimensional. One might ask whether this is inescapable, and we will resolve this much later in Chapter 5.

To get off the ground, recall the whole point of the design of P_f was its participation in the long exact sequence

Lemma 1.4.10

$$\cdots \to \pi_2 Y \to \pi_1 P_f \to \pi_1 X \to \pi_1 Y \to \pi_0 P_f \to \pi_0 X \to \pi_0 Y.$$

Inasmuch as the exact sequence is truly what we're after, we might consider alternative situations in which such sequences arise.

(?, Theorem 4.1)

Example 1.6.1. For each $n \geq 0$, there is an isomorphism

$$\pi_n(X \times Y) = [S^n, X \times Y] \cong [S^n, X] \times [S^n, Y] = \pi_n(X) \times \pi_n(Y).$$

Rearranged as an exact sequence, the maps

$$\begin{tikzcd} Y \arrow[r, "i_Y"'] & X \times Y \arrow[l, bend right, "\pi_Y"'] \arrow[r, "\pi_X"'] & X \arrow[l, bend right, "i_X"'] \end{tikzcd}$$

induce a split-exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "0"] & \pi_n Y \arrow[r] & \pi_n(X \times Y) \arrow[r] \arrow[l, bend right] & \pi_n X \arrow[r, "0"] \arrow[l, bend right] & \cdots . \end{tikzcd}$$

We would like to axiomatize the pieces of the geometry of the product that we need to induce exact sequences on π_*. The main features that we used were the projection and retraction maps, which we used to lift maps $S^n \to X$ and $S^n \to Y$ to maps $S^n \to X \times Y$, and conversely to produce from a map $S^n \to X \times Y$ the projections $S^n \to X$ and $S^n \to Y$. We will ultimately ask for "one half" of this data (i.e., one projection and one inclusion), together with a requirement that they satisfy a very weak variant of this recovery property.

(?, Definition 4.2)

Definition 1.6.2. A map $p \colon E \to B$ has the *homotopy lifting property* with respect to a space T when for all solid diagrams

$p \colon X \to Y$?

"E" for Espace (fr.), "B" for Base.

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "- \times 0"'] & E \arrow[d, "p"] \\ T \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & B \end{tikzcd}$$

there exists a dashed diagonal lift $\widetilde{H}$. That is, homotopies in B lift to homotopies in E. A *fibration* has the HLP for (at least) the disks D^n. The *fiber* of a fibration is $F = p^{-1}(b_0) \subseteq E$.

Example 1.6.3. The projection $\pi_X \colon X \times Y \to X$ is a fibration with fiber Y.

(?, Proposition 4.3)

Example 1.6.4. For $PX = X^I$ the pointed path space, evaluation $ev \colon PX \to X$ gives a fibration with fiber ΩX.

For the remainder of the lecture, we will work to justify that this definition yields the intended long exact sequence of homotopy groups. The first step is a version of the homotopy lifting property for pairs and its relation to the original definition above.

(?, Proposition 4.5)

Lemma 1.6.5. *Consider a fibration $p \colon E \to B$, as well as a preferred subset $B' \subseteq B$ of the base and the subset $E' := p^{-1}(B') \subseteq E$ of the total space which lies over it. If p has the HLP for $X \times I$, then $p' \colon (E, E')^{(I, \partial I)} \to (B, B')^{(I, \partial I)}$ has the HLP for X.*

Proof. Let us begin with an HLP diagram for p', as in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & (E, E')^{(I, \partial I)} \arrow[d, "p'"] \\ X \times I \arrow[r, "H"] & (B, B')^{(I, \partial I)} \end{tikzcd}$$

for which we're meant to produce a lift

$$\widetilde{H} \colon X \times I \to (E, E)^{(I, \partial I)}.$$

We may remove the function spaces by applying de-exponentiating:

$$\begin{tikzcd} X \times (I \vee I) \arrow[r, "f''"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "H'"] & B. \end{tikzcd}$$

The homeomorphism $h\colon I \vee I \cong I$ extends to a homeomorphism h' as in

$$\begin{tikzcd} I \vee I \arrow[r, "h{,}\cong"] \arrow[d, hook] & I \arrow[d, hook] \\ I \times I \arrow[r, "h'{,}\cong"] & I \times I, \end{tikzcd}$$

which can be used to smooth out the top-left corner:

$$\begin{tikzcd} X \times I \arrow[r, "f'' \circ h^{-1}"] \arrow[d, "- \times 0"] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "H' \circ h^{-1}"] \arrow[ur, dotted, "\widetilde{H}'"] & B. \end{tikzcd}$$

Applying the HLP for p gives $\widetilde{H'}$. Reversing the application of h' and the exponential adjunction ultimately yields the desired $\widetilde{H}$ in the Lemma statement. □

(?, Theorem 4.6)

Corollary 1.6.6. *For $p\colon E \to B$ a fibration with subspaces $B' \subseteq B$ and $E' := p^{-1}(B') \subseteq E$, the natural map $\pi_{\geq 1}(E, E') \to \pi_{\geq 1}(B, B')$ is an isomorphism.*

Proof. Begin with a class $\omega\colon (I, \partial I) \to (B, B')$ in $\pi_1(B, B')$, and consider the lifting square induced by $p\colon E \to B$, without considering B' and E':

$$\begin{tikzcd} * \arrow[r, "e_0"] \arrow[d] & E \arrow[d, "p"] \\ * \times I \arrow[r, "\omega"] \arrow[ur, "\exists \widetilde{\omega}"] & B. \end{tikzcd}$$

Since $\omega(1) \in B'$ and $E' := p^{-1}(B')$, the lift $\widetilde{\omega}$ can be considered as a map $\widetilde{\omega}\colon (I, \partial I) \to (E, E')$. Hence, $\pi_1(E, E') \to \pi_1(B, B')$ is a surjection.

For injectivity on π_1, suppose that $\omega_1, \omega_2\colon (I, \partial I) \to E$ are two relative homotopy classes and that $H\colon I \times I \to B$ is a homotopy which relates them in B. We form the homotopy lifting diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & E \arrow[d] \\ I \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & B, \end{tikzcd}$$

where $U = \partial(I \times I) \setminus (I \times \{1\})$ is a ⊔–shaped figure, the top arrow acts by ω_1 and ω_2 on the two legs, and it carries the bottom of the figure to the basepoint. The same procedure as in the second half of Lemma 1.6.5 produces a filler $\widetilde{H}$, which witnesses equality in $\pi_1(E, E')$.

Specifically, we avail ourselves of a pointed self-homeomorphism of $I \times I$ which restricts to a homeomorphism of U with $I \times \{0\}$.

To dispatch the other homotopy groups, we use the equivalences

It is confusing to put the subspace on the left in this notation for a pair but on the right in the usual relative homology / homotopy notation. Be consistent, use $\supseteq$?

$$\begin{tikzcd}
\pi_m(E,E') \arrow[r, equal] \arrow[d] & \pi_{m-1}(E' \subseteq E)^{(\partial I \subseteq I)} \arrow[r, equal] \arrow[d] & \pi_{m-1}((E' \subseteq E)^{(\partial I \subseteq I)}, (E')^I) \arrow[d] \\
\pi_m(B,B') \arrow[r, equal] & \pi_{m-1}(B' \subseteq B)^{(\partial I \subseteq I)} \arrow[r, equal] & \pi_{m-1}((B' \subseteq B)^{(\partial I \subseteq I)}, (B')^I).
\end{tikzcd}$$

The first equivalence is definitional, and the second follows from contractibility of pathspaces. Lemma 1.6.5 entails that the map

$$((E' \subseteq E)^{(\partial I, I)}, (E')^I) \to ((B' \subseteq B)^{(\partial I, I)}, (B')^I)$$

inducing the rightmost vertical map is a fibration. Taking as an inductive hypothesis that it is an isomorphism on π_{m-1} entails the inductive conclusion that the leftmost vertical map is an isomorphism. □

(?, 4.7)

Corollary 1.6.7. *There is a long exact sequence*

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots .$$

Proof. Set $B' = \{b_0\}$ in the above, and identify these terms respectively with

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n(E,F) \to \pi_{n-1} F \to \cdots . \qquad \square$$

(?, Corollary 4.8)

Corollary 1.6.8. *If E is contractible (e.g., as in the fibration $\mathrm{ev}\colon PX \to X$), then the going-around map $\pi_{n+1} B \to \pi_n F$ is an isomorphism.* □

Didn't this already hold for P_f? Why is this here?

1.7 *Fiber bundles and examples*

Add non-example: the S-shaped graph "fibering" over the line.

Fibrations are all over classical geometry. Most commonly, they take the form of a *fiber bundle*, which is a less homotopically-minded take on the important properties of the Cartesian product.

(?, Definition 4.9)

Definition 1.7.1. A *fiber bundle with fiber F* is a map $p\colon E \to B$ such that B admits an open cover $\{U_\alpha\}_\alpha$ with local homeomorphisms $\varphi_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$, each participating in a commuting triangle

$$\begin{tikzcd}
U_\alpha \times F \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\mathrm{proj}"'] & & p^{-1}(U_\alpha) \arrow[dl, "p"] \\
& U_\alpha . &
\end{tikzcd}$$

(?, Proposition 4.10)
If B is paracompact, then p is actually a strong fibration.

Lemma 1.7.2. *Every fiber bundle is a fibration.*

Proof sketch. Fix an open cover as in Definition 1.7.1. Given a map

$$f\colon D^n \times I \to B$$

to be lifted, subdivide the compact space $D^n \times I$ so finely that its pieces are each contained in one member of the cover. Once this is arranged, one can build the desired lift purely locally. □

The most common source of these "local projections" are the *homogeneous spaces* of Lie theory: a quotient map of a Lie group by a subgroup has everywhere-isomorphic fibers, and in good cases the coset space can be arranged to admit a suitable open cover. We spend most of today contemplating this case.

Briefly, a fiber product is locally and compatibly a product space. This definition *must* be given in Spaces and not hSpaces: the S-shaped space $S = ([-1,0] \times \{0\}) \cup (\{0\} \times I) \cup (I \times \{1\})$ projects on its first coordinate to $[-1,1]$ with contractible fibers, but it is neither a fiber bundle nor a fibration.

(?, Theorem 4.13)

We do not assume normality!

Lemma 1.7.3. *Take $H \le G$ to be a closed subgroup of a topological group. Suppose that the identity coset $H \in G/H$ has an open neighborhood U with a section $U \xrightarrow{s} G \xrightarrow{p} G/H$. The map p is then a fiber bundle with fiber H.*

Proof. Left-multiplying by $g \in G$, the condition at H begets a local section at $gH \in G/H$. Such a section extends to a local homeomorphism as in the definition of a fiber bundle via

May need to discuss this in terms of G–bundles to meet requirements of next section.

$$U_{gH} \times H \xrightarrow{s_g \times 1} G \times G \xrightarrow{\mu} G.$$

Using these opens to cover the base shows p to be a fiber bundle. □

(?, Examples 4.14.1)

Example 1.7.4 (Stiefel manifolds). Consider the subgroup $O(n) \subseteq O(n+k)$ of block matrices

$$\left\{ \left[\begin{array}{c|c} O(n) & 0 \\ \hline 0 & I \end{array} \right] \right\} \le O(n+k).$$

The quotient is the space of orthonormal k–frames in $\mathbb{R}^{n+k}$. To construct a local section, note that there is an open neighborhood U of the standard frame $(e_{n+1}, \ldots, e_{n+k})$ consisting of those $(u_{n+1}, \ldots, u_{n+k})$ satisfying the condition

For example, $O(n)/O(n-1)$ is homeomorphic to S^{n-1}.

$$\det \left[e_1 \mid \cdots \mid e_n \mid u_{n+1} \mid \cdots \mid u_{n+k} \right] \ne 0.$$

On U, the local section is defined by applying Gram–Schmidt to the block matrix of columns $[e_1 \mid \cdots \mid e_n \mid u_{n+1} \mid \cdots \mid u_{n+k}]$ to produce an element of $O(n+k)$.

(?, Examples 4.14.2)

Example 1.7.5 (Grassmannians). The further quotient by

$$\left\{ \left[\begin{array}{c|c} O(n) & 0 \\ \hline 0 & O(k) \end{array} \right] \right\} \le O(n+k)$$

gives the space of k–dimensional subspaces in $\mathbb{R}^{n+k}$. Consider the open neighborhood U of $(e_{n+1}, \ldots, e_{n+k})$ consisting of those subspaces W which trivially intersect the subspace $\langle e_1, \ldots, e_n \rangle$. By projecting $e_{n+1}, \ldots, e_{n+k}$ into W and applying Gram–Schmidt, this produces an orthonormal k–frame, i.e., a section landing in $O(n+k)/O(n)$. This can be used to construct a triple of fiber bundles

$$\begin{tikzcd} & O(n) \arrow[dl] \arrow[dr] & \\ \frac{O(n+k)}{O(k)} \arrow[rr] & & \frac{O(n+k)}{O(n)\times O(k)}, \end{tikzcd}$$

even though the bottom map is not a quotient from a group.

Add a concrete example for Grassmannians, like projective space.

Example 1.7.6. These linear algebraic examples do not rest on $\mathbb{R}$. One can build analogues of these examples with $U(n)$ and $\mathbb{C}^n$, or even with $Sp(n)$ and $\mathbb{H}^n$.

(?, Examples 4.14.4, 5, 7)
(?, Examples 4.14.3)

Example 1.7.7. The sequence

$$SO(n) \to O(n) \xrightarrow{\det} O(1)$$

admits a section

$$\pm 1 \mapsto \left(\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & I \end{array} \right),$$

which witnesses it as a fiber bundle.

This base is extremely unexciting. Should probably do Spin too.

Example 1.7.8. Applying Corollary 1.6.7 to Example 1.7.7 yields

$$\pi_{\geq 1} SO(n) \cong \pi_{\geq 1} O(n).$$

(?, Proposition 4.15)

This isomorphism of homotopy groups is attractive enough that we give fiber bundles with discrete fibers a special name:

(?, Definition 4.16)

Definition 1.7.9. A fiber bundle with discrete fiber is called a *covering space*.

(?, Examples 4.18.1)

Example 1.7.10. The quotient sequence

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

witnesses $\mathbb{R}$ as a cover of S^1 with discrete fiber $\mathbb{Z}$. Thinking of $S^1 \subseteq \mathbb{C}$ as the complex numbers of unit norm, a local section is given by

$$z \mapsto \frac{1}{2\pi i} \log z.$$

Corollary 1.6.7 gives an exact sequence of pointed sets

$$0 \to \pi_1 S^1 \to \pi_0 \mathbb{Z} \to 0.$$

A discrete fiber has sufficiently simple structure that one can often automatically construct fiber bundles as quotients by group actions, without needing to manufacture a local section to guarantee sane behavior. We also record this easier situation.

(?, 4.19)

Definition 1.7.11. A discrete group G acts *properly discontinuously* on a space X when…

1. …each point $x \in X$ admits a neighborhood U_x so that gU_x never intersects U_x for non-identity g.

2. …for all points $x, y \in X$ in different orbits, there are neighborhoods U_x, U_y so that gU_x never meets U_y.

(?, Proposition 4.20)

Lemma 1.7.12. *If G acts properly discontinuously on X, then $p: X \to X/G$ is a covering (and X/G is Hausdorff).*

Proof. For $[x] \in X/G$, choose a neighborhood U_x guaranteed by (1), and let $p(U_x)$ be the neighborhood of $[x]$ in the base. The disjointness property allows us to define the local section

$$p^{-1}(p(U_x)) \to G \times p(U_x). \qquad \square$$

From here *up* to "A discrete fiber..." can be demoted to a condensed remark.

(?, Remarks 4.21.i)

Remark 1.7.13. If $\pi_0 X = 0$, then in the short exact sequence of pointed sets

$$0 \to \pi_1 X \to \pi_1 X/G \to \pi_0 G \to 0$$

the last map is actually a group homomorphism.

(?, Remarks 4.21.iv)

Example 1.7.14. Since $\mathbb{Z} \le \mathbb{R}$ is a closed subgroup, $\mathbb{Z}$ acting on $\mathbb{R}$ by $1 \cdot x = x + 1$ is properly discontinuous. Since $\mathbb{R}$ is furthermore contractible, we conclude

$$\pi_n S^1 = \begin{cases} \mathbb{Z} & \text{when } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(?, Remarks 4.21.ii)

Remark 1.7.15. A covering $p: E \to B$ is *normal* if $p_*(\pi_1 E) \subseteq \pi_1 B$ is normal. All normal covers arise as quotients by a G–action on E.

(?, Remarks 4.21.iii)

Remark 1.7.16. For X a Hausdorff space, acted on by a finite group G without fixed points, the G–action is automatically properly discontinuous.

Mention the π_1 action somewhere in all of this. Specifically, G acting on $\pi_1 X/G$.

I have scrawled "mention that this is a *group* quotient!", but I don't know in reference to what.

1.A Vector bundles

Theory of infinitesimal embeddings
  G–bundles
  Associated bundle construction for $\mathrm{Aut}\, F$ and $h\,\mathrm{Aut}\, F$
  Prop 11.44 states that $O(n)$ is a deformation retraction of $GL_n(\mathbb{R})$

2
(De)composition of Homotopy Types

We are now sufficiently armed with both a supply of algebraic invariants extracted from spaces (e.g., homotopy groups) as well as fundamental tools for forming new spaces (viz., the exact and coexact extensions of a map). To set the stage for the work in this Chapter, consider the following Lemma:

Lemma 2.0.1. *Given two maps $f, g: X \to Y$, a homotopy $H: f \sim g$ begets a homotopy equivalence $C_f \xrightarrow{\simeq} C_g$.*

Find me a citation.

Construction. Subdivide CX into an outer X–shaped cylinder (corresponding to the cone coordinate range $[1/2, 1]_+$) and an inner X–shaped disk (corresponding to the range $[0, 1/2]$). The homotopy equivalence is given by gluing the identity map on Y, the homotopy H on the cylinder, and the homeomorphism $X \wedge [0, 1/2] \cong CX$ on the disk. □

Equivalently, the cone construction descends to $h\mathsf{Spaces}_{*/}$, where it becomes an operation on maps of *homotopy types*. Since we are not used to working directly with homotopy types, this prompts new questions:

Emphasize that we weren't already using this?

1. How can a homotopy type be constructed out of pieces with very simple invariants? What data is needed to specify the stages of such a construction?

2. How can a prespecified homotopy type be decomposed into pieces with simpler invariants? Are there canonical such decompositions?

An attempt to understand the first question will lead us to the notion of a *CW–complex*, which is a particularly nice sort of space built out of homotopical data and coexact extensions. An attempt to understand the second question will lead us to the notion of a *Postnikov tower*, which siphons off layers of a space into *Eilenberg–Mac Lane spaces* via exact extensions. Ultimately, we will marry these two methods together to produce a concise tool for computing the collection of maps from one homotopy type to another, known as *obstruction theory*.

2.1 *CW complexes*

Our first framework for studying these kinds of questions is that of a *cell structure* on a space X. Briefly, such a structure is a presentation of X given by inductively "attaching n–disks".

(?, Definition 5.10)

Definition 2.1.1. Consider a space Y and a continuous map

$$f\colon \bigvee_\alpha S_\alpha^{n-1} \to Y.$$

We say that the space

$$C_f = Y \cup_f \bigvee_\alpha C S_\alpha^{n-1}$$

is *formed from Y by attaching n–cells* (along f).

This definition touches on both of our intentions from the introduction to Chapter 1:

1. As a coexact extension, C_f participates in an exact sequence

$$\cdots \leftarrow \left[\bigvee_\alpha S_\alpha^{n-1}, T\right] \leftarrow [Y, T] \leftarrow [C_f, T] \leftarrow \cdots.$$

   The condition that $g\colon Y \to T$ lift to $\tilde{g}\colon C_f \to T$ is equivalent to the condition that $g_* f_\alpha$ is null-homotopic for each α and for each T. In particular, we can take for g the inclusion $Y \to C_f$ and for $\tilde{g}$ the identity on C_f to conclude that f_α becomes null when pushed forward to $\pi_{n-1} C_f$. In these senses, C_f behaves as a kind of quotient by the homotopy elements f_α.

2. Many commonly considered geometric spaces can be constructed by iteratively applying this operation. For example, one can construct a torus by taking $Y = S^1 \vee S^1$ and $f = \iota_1 \iota_2 \iota_1^{-1} \iota_2^{-1}$, where ι_1 and ι_2 are the identity maps on the two wedge-summands of Y.

   This *must* be accompanied by a picture.

   Let us codify the entire inductive process that we mean:

(?, Definitions 5.1–3)

Definition 2.1.2. We define a *CW–structure* on a space X to be a choice of sequence of spaces X^n satisfying the following properties. The head of the bi-infinite sequence is fixed as $X^{-\infty} = \cdots = X^{-1} = \{x_0\}$. Otherwise, X^n, called the *n–skeleton*, must be formed from X^{n-1} by attaching n–cells. Lastly, X must be the union of the X^n (with the weak topology). If X admits a CW–structure, then we say that it is a *CW–complex*.

If A is a CW-complex, then setting $X^{-\infty} = \cdots = X^{-1} = A$ gives rise to a relative CW-structure (X, A).

Some topological facts about CW–complexes that might interest you:

- Every CW–complex is Hausdorff.
- Every CW–complex is the disjoint union of the interiors of its cells.
- Each cell has only finitely many immediate faces.
- More generally, any compact subset has this property.

(?, Examples 5.4)

Example 2.1.3. Some common spaces come with natural CW–structures:

- S^n can be equipped with a CW–structure with only one n–cell, with boundary attached to the basepoint $\{x_0\}$.

Picture

- More elaborately, S^n can be equipped with an inductive CW–structure: one can form S^n from S^{n-1} by attaching two n–cells, both along the identity map $S^{n-1} \to S^{n-1}$, which serve as the upper- and lower-hemispheres of S^n.

- The second collection of CW–structures are *compatible*, in the sense that the m–skeleton of S^{n-1} agrees with the m–skeleton of S^n for all $m < n$. Using this, the union of these CW–structures puts a CW–structure on their colimit S^∞.

Picture

- Each projective space $\mathbb{R}\mathrm{P}^n$ (resp. $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$) can be endowed with a CW–structure where the k–skeleton is given by $\mathbb{R}\mathrm{P}^k$ (resp. $\mathbb{C}\mathrm{P}^k$, $\mathbb{H}\mathrm{P}^k$), and one attaches $\mathbb{R}^n \cong (CS^{n-1})^\circ$ (resp. $\mathbb{C}^n \cong (CS^{2n-1})^\circ$, $\mathbb{H}^n \cong (CS^{4n-1})^\circ$) along $\mathbb{R}\mathrm{P}^{n-1}$ (resp. $\mathbb{C}\mathrm{P}^{n-1}$, $\mathbb{H}\mathrm{P}^{n-1}$) as the 0^{th} affine chart to its complement.

- These CW–structures are also mutually compatible, so that one may form CW–structures on their respective colimits $\mathbb{R}\mathrm{P}^\infty$, $\mathbb{C}\mathrm{P}^\infty$, and $\mathbb{H}\mathrm{P}^\infty$.

The inductive presentation means we can attack problems cell-by-cell. For instance, we have the following simple applications of the gluing operation for topological spaces:

(?, Proposition 5.5)

Corollary 2.1.4. *Let X be a space equipped with a CW–structure as above. A function $f : X \to Y$ is continuous if and only if the restriction*

$$\bigvee_{n=0}^{\infty} \bigvee_{\alpha \in A_n} CS_\alpha^{n-1} \to X \to Y$$

to the ensemble of n–cells is continuous. □

Since each CS^{n-1} has a simple structure as a topological space, this is often an easier condition to manage.

It is also often possible to transfer CW–structures along other common topological operations.

(?, pg. 71)

Lemma 2.1.5. *If X carries a CW–structure and Y carries a finite CW–structure, then $X \times Y$ carries an induced CW–structure.*

If X and Y are infinite, then the weak topology on $X \times Y$ may not agree with the product of the weak topologies. We mostly concede this point and pick the one we want (usually the former).

Proof sketch. Repeatedly use the homeomorphism $D^n \times D^m \cong D^{n+m}$. □

(?, Exercise 5.14)

Lemma 2.1.6. *If X carries a CW–structure and $A \subseteq X$ is a subcomplex, then X/A receives an induced CW–structure.* □

(?, Proposition 5.6)

Corollary 2.1.7. *By consequence, homotopies and relative homotopies can also be constructed inductively over cells.* □

These cell decompositions imbue maps from some common spaces with moduli-theoretic interpretations.

Example 2.1.8. The set of maps $S^n \to Y$ agrees with the set of maps $D^n \to Y$ which send ∂D^n to y_0.

"Not compelling"

Example 2.1.9. Given $S^{n-1} \to Y$ and two choices of null-homotopies

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & Y \\ CS^{n-1}, \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

we can form a difference class $S^n \to Y$. Conversely, a map $S^n \to Y$ can be considered as the difference class associated to the null-homotopies of the restriction to the equator witnessed by the action of the map on two hemispheres.

Example 2.1.10. The space $\mathbb{R}\mathrm{P}^2$ also participates in such a description. Given a class $\omega \colon S^1 \to Y$ and a null-homotopy of its double

Do we ever discuss coherent null-homotopy of a C_2-action?

$$\begin{tikzcd} S^1 \arrow[r, "2\omega"] \arrow[d] & Y \\ D^2, \arrow[ur, "H"] & \end{tikzcd}$$

we can form a map $\mathbb{R}\mathrm{P}^2 \to Y$. Conversely, a map $\mathbb{R}\mathrm{P}^2 \to Y$ restricts to a class on $\mathbb{R}\mathrm{P}^1$ whose double carries a specified null-homotopy.

We close out today by recording a technical topological result that will be absolutely crucial in underpinning the results in the next two Lectures. Although we do not intend to offer careful proofs of those results, so that it is not *necessary* to state this technical result either, we feel that a topology course would be remiss without it and that it gives some intuition as to why these nice results for CW–complexes are possible.

(?, Proposition 6.8)

Lemma 2.1.11 (Simplicial approximation). *Let A be a CW–complex, and let $X = A \cup_g e^n$ consist of A with a single n–cell attached. Take (K, L) to be a* finite *simplicial pair, and consider a continuous map of pairs $f \colon (|K|, |L|) \to (X, A)$. There exists a subdivision (K', L') of (K, L) and a map $f' \colon (|K'|, |L'|) \to (X, A)$ with the following properties:*

1. *The two maps agree on A:*

$$f|_{f^{-1}(A)} = f'|_{f^{-1}(A)}.$$

2. *The two maps are homotopic while fixing their behavior on A:*

$$f \simeq_{rel\, f^{-1}(A)} f'.$$

3. *For each simplex $\sigma \in K'$, if $f'(|\sigma|)$ meets the interior $\mathring{e}^n$ of the new cell in X, then $f'(|\sigma|) \subseteq \mathring{e}^n$ is in fact contained in the new cell* and *$f'|_{|\sigma|}$ restricted to this simplex is a linear map.* $\square$

2.2 The homotopy theory of CW complexes I

In the previous section, we discussed CW–structures as placed on a preordained topological space X. One can also take the perspective that the CW–structure is the primary data, and the colimit X simply is whatever it is. From this second viewpoint, the most important property of CW–complexes which makes them suitable for use in homotopy theory is that if the attaching maps are perturbed within their homotopy class, the resulting CW–complex is homotopy equivalent to the original CW–complex.

This is the most basic property that one could ask of CW–complexes, but in fact they are remarkably well-behaved homotopically. Our goal today is to get *familiar* with some of these features and to use them to compute $\pi_{*\leq n} S^n$.

But not prove!

Lemma 2.2.1. *For (X,A) a relative CW-complex, the relative pair $(X,(X,A)^n)$ formed from the n–skeleton is n–connected.* □

(?, Theorem 6.10)
cf. Definition 1.5.4

Corollary 2.2.2. *The inclusion $X^n \to X$ is n–connected.* □

(?, Theorem 6.11)

Corollary 2.2.3. $\pi_{<n} S^n = 0$.

(?, Corollary 6.12)

Proof. We deploy the cell structure on S^n with $(S^n)^{n-1} = \{s_0\}$ from Example 2.1.3. The long exact sequence of relative homotopy groups takes the form

Corollary 1.5.3

$$\cdots \to \pi_k\{s_0\} \to \pi_k S^n \to \pi_k(S^n,\{s_0\}) \to \pi_{k-1}\{s_0\} \to \cdots.$$

The outer terms are always zero, as the homotopy groups of a singleton space. The relative term $\pi_k(S^n,\{s_0\})$ vanishes for $k < n$, using Corollary 2.2.2. Altogether, this shows the same of the non-relative groups $\pi_{<n} S^n$. □

Good candidate for the ⊊ notation.

Lemma 2.2.4. *If (X,A) is n–connected, then there exists an equivalence of pairs $(X,A) \sim (X',A')$ whose n–skeleton is $(X',A')^n = A'$.* □

(?, Proposition 6.13)

This is a kind of converse to the previous Lemma.
Taking $A = *$, this says that if $\pi_{<n} X = 0$, then there is a CW–model of X with no cells below dimension n (?, Corollary 6.14).

Corollary 2.2.5. *For X an n–connected CW–complex and Y an m–connected CW–complex, the CW–complex $X \wedge Y$ is $(n+m+1)$–connected.*

(?, Corollary 6.15)

Proof. The cells in $X \times Y$ take the form $* \times *$, $* \times e^j_\beta$, $e^i_\alpha \times *$, and $e^i_\alpha \times e^j_\beta$. All the first three classes lie within $X \vee Y$, so the first nontrivial surviving cell in $X \wedge Y$ lies in dimension at least $n+m+2$. □

This cell structure argument also feeds into the claim that the suspension of a CW complex has predictable cell structure and predictable attaching maps.

The following properties look like theorems for homology—but with a bound imposed, depending on the connectivities of the spaces involved.

Theorem 2.2.6 (Homotopy excision). *Let $A,B \subseteq X$ be a cover of X such that $(A,A\cap B)$ is an n–connected pair and $(B,A\cap B)$ is an m–connected pair. The natural double inclusion $\pi_*(A,A\cap B) \to \pi_*(X,B)$ is an isomorphism for $* < n+m$ and an epimorphism for $* = n+m$.* □

(?, Theorem 6.21)

(?, Corollary 6.22)

Corollary 2.2.7. *Suppose that (X,A) is an n–connected pair and that A is m–connected. Then the natural map $\pi_*(X,A) \to \pi_* X/A$ is an isomorphism for $1 < * \le n+m$ and an epimorphism at $n+m+1$.*

Proof. Consider the space $X \cup_A CA$, as well as the subspaces CA and X inside of it. Noting that $CA \cap X = A$ as subspaces of $X \cup_A CA$, Theorem 2.2.6 then gives the desired conclusion for the natural map

$$\pi_*(X,A) \to \pi_*(X \cup_A CA, CA).$$

The right-hand group can be made non-relative by passing along the projection, as in

$$(X \cup_A CA, CA) \xrightarrow{\simeq} (X \cup_A CA/CA, *),$$

and apply Lemma 1.4.5 to produce a weak equivalence

$$X/A \xrightarrow{\simeq} X \cup_A CA/CA. \qquad \square$$

There should be some Lemma recording that working relative to a contractible subspace is the same as quotienting out the subspace. This is 6.6 in Switzer.

Rmk: Comparison map between P_f and C_f.

Corollary 2.2.8 (Freudenthal suspension theorem)**.** *For an n–connected CW–complex X, the natural map*

$$\begin{array}{ccc} \pi_{*+1}(CX,X) & \longrightarrow & \pi_{*+1}(CX/X) \\ \| & & \| \\ \pi_* X & \longrightarrow & \pi_{*+1}\Sigma X \end{array}$$

is an isomorphism for $* \le 2n$ and an epimorphism for $* = 2n+1$. $\square$

(?, Theorem 6.26)
The left-hand vertical map is given by restriction to the boundary. Its participation in the long exact sequence of relative homotopy groups shows it to be an equivalence, since CX is contractible.

Example 2.2.9. Consider the following fibration:

$$\begin{array}{ccccc} \mathbb{C}^\times & \longrightarrow & \mathbb{C}^n \setminus 0 & \longrightarrow & \mathbb{CP}^{n-1} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ S^1 & \longrightarrow & S^{2n-1} & \longrightarrow & \mathbb{CP}^{n-1}. \end{array}$$

(?, Theorem 6.28)
The top row presents this fibration as a fiber bundle. To see that it is a fiber bundle, we rely on Example 1.7.5 and the sequence

$$U(1) \to \frac{U(n)}{U(n-1)} \to \frac{U(n)}{U(n-1)\times U(1)}.$$

Since S^{2n-1} is $(2n-2)$–connected, we conclude $\pi_{*+1}\mathbb{CP}^{n-1} \cong \pi_* S^1$ for $* \le 2(n-1)$. Using Example 1.7.14, we may conclude

For $n \ge 2$? ≥ 3?

$$\pi_* \mathbb{CP}^n \cong \begin{cases} \mathbb{Z} & \text{when } * = 2, \\ 0 & \text{when } * < 2n-1 \text{ and } * \ne 2, \\ ??? & \text{otherwise.} \end{cases}$$

We also know that $(\mathbb{CP}^{n-1}, \mathbb{CP}^1)$ is 2–connected, from which we may conclude

$$\pi_2 S^2 \cong \pi_2 \mathbb{CP}^1 \cong \pi_2 \mathbb{CP}^{n-1} \cong \pi_1 S^1 \cong \mathbb{Z},$$

even though this is outside of the range directly accessible by the fibration for $\mathbb{CP}^1$ alone. This then feeds into Freudenthal as applied to the sphere, which for $n \ge 2$ gives

$$\pi_n S^n \cong \pi_{n+1} S^{n+1}.$$

Hence, we ultimately conclude $\pi_n S^n \cong \mathbb{Z}$ for all $n \ge 1$.

"Comment on π_1 and Freudenthal"

2.3 The homotopy theory of CW complexes II

We continue our recounting of the pleasant homotopical properties of CW–complexes. The following technical lemma appeared in our study of the relative homotopy long exact sequence of a pair (Y, B):

Did it? Where? Citation? Doesn't seem to match Prop 3.14? Just delete it? I dunno.

(?, Proposition 3.14)

Lemma 2.3.1. *Given an n–equivalence (Y, B), for all squares*

Backreference.

$$\begin{tikzcd} B \arrow[r] & Y \\ S^{n-1} \arrow[u, "\omega|_{S^{n-1}}"] \arrow[r] & D^n \arrow[u, "\omega"'] \arrow[ul, dotted, "\omega'"] \end{tikzcd}$$

there exists a dashed filler for which the top-right triangle commutes up to homotopy. □

Add a 2–cell from Y to filler arrow.

This can be augmented in two ways: first, by extending it to cover more complicated sub-complexes, and second, by using it to govern the behavior of maps.

(?, Theorem 6.30)

Lemma 2.3.2. *If $f : Z \to Y$ is an n–equivalence and* $\dim(X, A) \le n$*, then for each solid square*

$$\begin{tikzcd} Z \arrow[r, "f"] & Y \\ A \arrow[u, "g"] \arrow[r] & X \arrow[u, "h"'] \arrow[ul, dotted] \end{tikzcd}$$

there exists a dashed filler such that the top-right triangle commutes up to homotopy. □

Add a 2–cell from Y to filler arrow.

(?, Theorem 6.31)

Corollary 2.3.3. *Given an n–equivalence $f : Z \to Y$ and a CW–complex X of dimension at most n, the natural map*

$$f_* : [X, Z] \to [X, Y]$$

is surjective. If the dimension of X is strictly less than n, then f_* is an isomorphism. □

These results ultimately culminate in the following:

(?, Theorem 6.32)

Corollary 2.3.4 (Whitehead)**.** *A weak equivalence $f : Z \to Y$ of CW-complexes is a homotopy equivalence.* □

There, finally, is justification that relative homotopy groups truly do measure the discrepancy between two spaces: if the discrepancy vanishes, then the two spaces are equivalent in the homotopy category.

Provided that those spaces are sufficiently nice!

Filtering a CW-complex by its skeleta and applying the Lemma yields a second thread of useful results.

(?, Definition 6.34)

Definition 2.3.5. A map $f : X \to Y$ of CW–complexes X and Y is said to be *cellular* if it carries the k^{th} skeleton of X to the k^{th} skeleton of Y:

$$f(X^k) \subseteq Y^k.$$

(?, Proposition 6.35)

Corollary 2.3.6. *Every map $f: (X, A) \to (Y, B)$ of CW-complexes is homotopic (relative to A) to a cellular map, and homotopies between cellular maps can themselves be made cellular.* □

Together, these results grant us serious control over how the homotopy groups of a CW–complex change as its skeleta are built up—in the qualitative sense of *which* groups change. This can be deployed to manufacture certain interesting spaces, called Eilenberg–Mac Lane spaces, on which we spend the rest of today.

(?, Proposition 6.36)

Corollary 2.3.7. *Let X be an n–connected CW–complex, and let Y be an m–connected CW–complex. The natural map $X \vee Y \to X \times Y$ induces an isomorphism in homotopy through a range:*

$$\pi_{\leq n+m}(X \vee Y) \xrightarrow{\cong} \pi_{\leq n+m}(X \times Y)$$

Proof. In Corollary 2.2.5, we argued that the cellular structure induced on $X \times Y$ was completely contained in $X \vee Y$ through a range:

$$(X \times Y, X \vee Y)^{n+m+1} = (X \vee Y)^{n+m+1}.$$

The relative homotopy groups of the pair thus vanish through this range, and the long exact sequence yields the desired statement. □

Consider making the above its own Lemma in Section 2.2.

(?, Corollary 6.37)
Also, $\pi_1 \bigvee_\alpha S^1_\alpha \cong \ast_\alpha \pi_1 S^1_\alpha$ and $\pi_0 \bigvee_\alpha S^0_\alpha \cong \bigvee \pi_0 S^0_\alpha$.

Corollary 2.3.8. *For $n \geq 2$,*

$$\pi_n\left(\bigvee_\alpha S^n_\alpha\right) \cong \bigoplus_\alpha \pi_n S^n_\alpha. \quad \square$$

(?, Theorem 6.39.i)

Lemma 2.3.9. *For any abelian group A and index $n \geq 2$, there exists a CW–complex $K(A, n)$ whose homotopy groups satisfy*

$$\pi_* K(A, n) = \begin{cases} A & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Select a presentation

$$0 \to \mathbb{Z}^J \xrightarrow{g} \mathbb{Z}^I \to A \to 0.$$

We begin by building a coexact sequence of $(n-1)$–connected spaces whose behavior on π_n encodes the chosen presentation. Begin by modeling the middle node as $\bigvee_I S^n$. Combining Example 2.2.9 and Corollary 2.3.8 gives

Explain one of the preceding results as saying that coexact sequences of spaces behave like exact sequences through a range!!

$$\pi_n\left(\bigvee_I S^n\right) \cong \bigoplus_I \mathbb{Z}.$$

The J–sized set of elements selected by g gives a J–sized set of maps $S^n \to \bigvee_I S^n$, and hence a single map $\tilde{g}: \bigvee_J S^n \to \bigvee_I S^n$ which induces g on π_n.

The cone on g gives a complex X_n with $\pi_{*<n} X_n = 0$ and $\pi_n X_n = A$. We inductively form X_{n+j+1} from X_{n+j} by killing the homotopy in degree $n+j+1$ by coning off any homotopy classes we find. Using the long exact sequence of a relative pair, we see that this coning operation never disturbs the homotopy groups at or below $n+j$, so the colimit indeed provides $K(A,n)$. □

We're not there yet, but X_n is sometimes called a *Moore space* (of dimension n, for the group A), because $H_n(X_n;\mathbb{Z}) \cong A$.

Is this a compactness argument? Has this come up?

Lemma 2.3.10. *Let X be a space satisfying $\pi_{<n} X = 0$, and let Y be a space satisfying $\pi_{>n} Y = 0$. Then homotopy classes $[X, Y]$ biject with homomorphisms $\pi_n X \to \pi_n Y$.*

(?, Theorem 6.39.ii)

X is sometimes said to be n–*connective* and Y to be n–*coconnective*.

Proof. Using Lemma 2.2.4, there exists a CW model of X with $X^{n-1} = *$. The rest of the presentation of X looks like

$$\begin{tikzcd}
\bigvee_I S^n \arrow[r, equal] & X^n \arrow[r] & X^{n+1} \arrow[r] & X^{n+2} \arrow[r] & \cdots \arrow[r] & X \\
& \bigvee_J S^n \arrow[u] & \bigvee S^{n+1} \arrow[u] & \bigvee S^{n+2} \arrow[u] & \cdots, \arrow[u]
\end{tikzcd}$$

where each angle is a cofiber sequence. Using this presentation, we can inductively study the available maps into Y. We begin with

$$[X^n, Y] = \left[\bigvee_I S^n, Y\right] = \bigoplus_I \pi_n Y.$$

To extend along the $(n+1)$–skeleton, the precomposite

$$\bigvee_J S^n \to \bigvee_I S^n \to Y$$

must vanish, and the unicity of each such extension is measured by $[\Sigma \bigvee_J S^n, Y] = 0$, so that we have an exact sequence

$$\begin{tikzcd}
{[X^{n+1}, Y]} \arrow[r] \arrow[d, equal] & {[X^n, Y]} \arrow[r] \arrow[d, equal] & {\left[\bigvee_J S^n, Y\right]} \arrow[d, equal] \\
\mathsf{Groups}(\pi_n X, \pi_n Y) \arrow[r] & \bigoplus_I \pi_n Y \arrow[r] & \bigoplus_J \pi_n Y.
\end{tikzcd}$$

For all higher stages, the obstruction to extension vanishes (because $\pi_{n+k} Y = 0$) and the obstruction to unicity vanishes (because $\pi_{n+k+1} Y = 0$). □

For a generic CW–complex Y, without assumptions on its $\pi_{>n}$, the relative group $\pi_n(Y^n, Y^{n-1})$ is free on generators $\gamma \cdot [f_\alpha^n]$, where $\gamma \in \pi_1 X^{n-1}$, and f_α^n is the characteristic map of an n–cell.

Have we used the term "characteristic map"?

Corollary 2.3.11. *$K(A,n)$ is independent of choice of presentation.*

(?, Corollary 6.42)

Proof. Consider two models X, Y for $K(A,n)$. Both X and Y satisfy the conditions of the Lemma, so that the identity map $\mathrm{id}\colon A \to A$ lifts to a map $\widetilde{\mathrm{id}}\colon X \to Y$. This map $\widetilde{\mathrm{id}}$ of CW–complexes is an isomorphism on homotopy groups, hence Whitehead's theorem witnesses it as a homotopy equivalence. □

Remark 2.3.12. Although we have managed to fully prescribe the homotopy groups of $K(A,n)$, the proof of Lemma 2.3.9 suggests that we have poor understanding of its co/homology. Though we don't have evidence for it yet, the spheres enjoy a kind of converse: they have fully prescribed co/homology, but we have a poor understanding of their homotopy. At present, this trade is a curiosity, but by the end of this Chapter we will find a reason to pursue a better understanding of the co/homology of $K(A,n)$.

(?, Remark 6.45)

Remark 2.3.13. Using Corollary 1.6.8, we may conclude that $\Omega K(A,n)$ has the same homotopy groups as $K(A,n-1)$. The Corollary shows that it, in fact, does model $K(A,n-1)$ via a map

$$\Omega K(A,n) \xrightarrow{\simeq} K(A,n-1).$$

However, the adjoint map $\Sigma K(A,n-1) \to K(A,n)$ is more mysterious. It can be taken to be an inclusion on $(n+1)$–skeleta, but otherwise it appears little can be said about it at this point. We will meet it again.

(?, Exercise 6.49)

Remark 2.3.14. For *all* spaces X, there exists a CW-complex $\widetilde{X} \to X$ such that the map is a weak equivalence.

This might be out of place. Trying to justify it probably reveals where it belongs.

2.4 *Spectral sequences*

Our treatment here deviates from Switzer's, which can be found on (?, pg. 336–340). Our approach is closer to that of Boardman (?), though he is much more technically intense (and, of course, more rigorous).

The argument we gave in the proof of Lemma 2.3.10 felt quite serendipitous and fragile. Had our hypotheses been even slightly weaker, we would have quickly found it very difficult to keep track of all the interacting groups involved. However, this kind of scenario is extremely common, where a space X of interest is constructed from an infinite sequence of steps and where we hope to compute some invariant, like $\pi_*(X)$, from knowledge of the relative invariants, like $\pi_*(X_n, X_{n-1})$. It is so common, in fact, that topologists have worked to codify the formal properties of this scenario, which goes by the name of a *spectral sequence*. Today we recount a mild specialization of this framework that will suit us well.

As with cells in a CW–complex.

More generally, one might discuss *exact couples*.

We begin by arranging into a single "jigsaw diagram" the exact sequences associated to each inclusion:

$$\begin{tikzcd}
0 \arrow[r] & \pi_* X_1 \arrow[rr] & & \pi_* X_2 \arrow[rr] \arrow[dll] & & \cdots \arrow[rr] \arrow[dll] & & \pi_* X_n \arrow[rr] \arrow[dll] & & \pi_* X_{n+1} \arrow[rr] \arrow[dll] & & \cdots \arrow[r] \arrow[dll] & \pi_* X \\
& \pi_*(X_2,X_1) \arrow[u] & & \pi_*(X_3,X_2) \arrow[u] & & \cdots \arrow[u] & & \pi_*(X_{n+1},X_n) \arrow[u] & & \pi_*(X_{n+2},X_{n+1}) \arrow[u] & & \cdots .
\end{tikzcd}$$

Each triangle is a "rolled up" exact sequence and each red arrow shifts degree by one, e.g., $\pi_*(X_{n+1}, X_n) \to \pi_{*-1} X_n$. It is common to refer to the bottom row as the "front" of the spectral sequence and to the top row as the "rear".

Let us study the problem of recovering $\pi_* X$ only by probing information about the groups $\pi_*(X_{n+1}, X_n)$ on the bottom row. Our first obser-

If we permitted ourselves access to the top row of $\pi_* X_n$, we could take a colimit and be done.

vation is that since each sphere is compact, every class $x \in \pi_* X$ must lift to some $\widetilde{x} \in \pi_* X_n$. However, since this is not a relative group, we are not permitted to make direct use of it, and we should instead consider the image of this class $x_n \in \pi_*(X_n, X_{n-1})$ in the bottom row. Our second observation is that at exactly the *minimal* such n to which (a nonzero class) x lifts to $\widetilde{x} \in \pi_* X_n$, it pushes down to give a nonzero class $x_n \in \pi_*(X_n, X_{n-1})$.

This should be a named lemma.

This explains some of the classes in $\pi_*(X_n, X_{n-1})$. How can we discern the classes that come from these minimal preimages? What use are those classes that don't?

Let us thus begin from the other vantage point by selecting a class $x_n \in \pi_*(X_n, X_{n-1})$ and asking when there is a class $\widetilde{x} \in \pi_* X_n$ of which it is the image.

$$\begin{tikzcd}
 & & & \color{red}\partial(x_n) & \color{red}(\exists\, \widetilde{x}???) \arrow[dddl, dotted, mapsto, red] & \\
\cdots \arrow[r] & \pi_* X_{n-2} \arrow[r] & \pi_* X_{n-1} \arrow[r] & \pi_* X_n \arrow[r] \arrow[dl] & \pi_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & \pi_*(X_{n-1}, X_{n-2}) \arrow[u] & \pi_*(X_n, X_{n-1}) \arrow[u] & \pi_*(X_{n+1}, X_n) \arrow[u] & \pi_*(X_{n+2}, X_{n+1}) \arrow[u] & \\
 & & & \color{red}x_n \, x_n \arrow[uuu, mapsto, bend left, red] & &
\end{tikzcd}$$

The existence of $\widetilde{x}$ is determined by whether $\partial(x_n)$ is nonzero—but, since it is in the rear of the spectral sequence, we cannot ask about it directly. However, we can examine its image $d_1(x_n) \in \pi_*(X_n, X_{n-1})$ in the front:

$$\begin{tikzcd}
 & & & \color{red}\partial(x_n) \arrow[dddl, red] & \color{red}(\exists\, \widetilde{x}???) \arrow[dddl, dotted, mapsto, red] & \\
\cdots \arrow[r] & \pi_* X_{n-2} \arrow[r] & \pi_* X_{n-1} \arrow[r] & \pi_* X_n \arrow[r] \arrow[dl] & \pi_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & \pi_*(X_{n-1}, X_{n-2}) \arrow[u] & \pi_*(X_n, X_{n-1}) \arrow[u] & \pi_*(X_{n+1}, X_n) \arrow[u] & \pi_*(X_{n+2}, X_{n+1}) \arrow[u] & \\
 & & \color{red}d_1(x_n) & \color{red}x_n \arrow[uuu, mapsto, bend left, red] & &
\end{tikzcd}$$

If $d_1(x_n)$ is not zero, then surely $\partial(x_n)$ is also not zero, which settles the question definitively that a lift $\widetilde{x}$ cannot exist. On the other hand, if it is zero, then it could be the case that $\partial(x_n)$ is zero (so that $\widetilde{x}$ exists) or that $\partial(x_n)$ is nonzero and merely in the kernel of the map $\pi_* X_n \to \pi_*(X_n, X_{n-1})$. In the second case, we can use exactness to build a preimage $i_n^{-1}\partial(x_n) \in \pi_* X_{n-1}$, as in:
Again, this yields a new class in the rear, which we push to the front to produce another class $d_2(x)$.

$$\begin{tikzcd}
 & & i_n^{-1}\partial(x_n) \arrow[r, mapsto] \arrow[dddl, mapsto] & \partial(x_n) \arrow[dddl, mapsto] & (\exists\, \tilde{x}???) \arrow[dddl, mapsto, dotted] & \\
\cdots \arrow[r] & \pi_* X_{n-2} \arrow[r] & \pi_* X_{n-1} \arrow[r] & \pi_* X_n \arrow[r] \arrow[dl] & \pi_* X_{n+1} \arrow[r] \arrow[dl] & \cdots \\
 & \pi_*(X_{n-1}, X_{n-2}) \arrow[u] & \pi_*(X_n, X_{n-1}) \arrow[u] & \pi_*(X_{n+1}, X_n) \arrow[u] & \pi_*(X_{n+2}, X_{n+1}) \arrow[u] & \\
 & d_2(x_n) & d_1(x) = 0 & x_n \arrow[uuu, mapsto, bend left] & &
\end{tikzcd}$$

We make a series of claims:

Citations!

1.

Induction over preimages.

2. This assignment d_2 is well-defined up to the image of d_1, and hence it determines a function $d_2 \colon H_*(\pi_*(X_*, X_{*-1}); d_1) \to H_{*-1}(\pi_*(X_{*-2}, X_{*-3}); d_1)$.

Generic r?

3. Since $X_0 = *$, eventually the preimage is guaranteed to be zero, hence $\partial(x_n) = 0$, and hence $\tilde{x}$ exists.

4. The surviving elements in a spectral sequence are the associated graded of a filtration of $\pi_* X$ (by minimal lift degree).

Introduce some terminology, especially *pages*.

Remark 2.4.1. This story can be retold with many variations. Some introduce no further complexity:

- In place of exact sequences of spaces and relative homotopy groups, one can use coexact sequences of spaces and a homology functor.

Others introduce substantial complexity:

- The spectral sequence can be "infinite to the left" (e.g., when applying a *cohomology* functor to a filtration by coexact sequences).
- The filtration can be bi-infinite, dropping the assumption that $X_0 = *$.
- If one does not introduce assumptions to avoid this, since homotopy groups of low orders are not abelian groups, their homological algebra is substantially more complicated (especially when coupled to the above variants).

The essential complexity introduced by the first two points is that such spectral sequences need not *stabilize*. In order to handle this, one has to incorporate taking inverse limits of the subquotients of $H_* A_*$, which can destroy some of the exactness in the third Claim or the argument used to justify the second Claim.

Spectral sequences have the simultaneous pleasant features of being valuable computational tools while also being sufficiently rigid that one can use them to prove theorems without computing anything. The following result is an example of their rigidity:

Lemma 2.4.2. *A map of spectral sequences is a family of homomorphisms, one per page, from the front of the source to the front of the target, which additionally commute with all of the d_r maps. If a map of spectral sequences is an isomorphism on the r^{th} page for any r, it is an isomorphism forever after.* □

Example 2.4.3. Consider the filtration of a CW-complex X by its skeleta. The filtration stages participate in coexact sequences with spheres of a fixed dimension, from which we conclude

$$\widetilde{H}_*(X_n, X_{n-1}; A) = \widetilde{H}_*\left(\bigvee_\alpha S^n_\alpha; A\right) = \bigoplus_\alpha \Sigma^n A.$$

The map d_1 associated to the spectral sequence is exactly the cellular differential, as can be seen by drawing the defining diagram:

$$\begin{tikzcd}
\widetilde{H}_* X_{n_1} \arrow[r] \arrow[d, phantom] & \widetilde{H}_* X_n \arrow[dl] \\
\widetilde{H}_* \bigvee_\beta S^{n-1}_\beta \arrow[u] & \widetilde{H}_* \bigvee_\alpha S^n_\alpha . \arrow[l, "d_1"] \arrow[u]
\end{tikzcd}$$

It follows that the E_2–page is given by $H_*(H_*(X_*, X_{*-1}); d_1) = \widetilde{H}^{\mathrm{cell}}_*(X; A)$. All higher differentials are zero because

$$d_r \colon \bigoplus_\alpha \Sigma^n A \xrightarrow{[-1]} \bigoplus_\gamma \Sigma^{n-r} A$$

has the wrong degree. We say that the spectral sequence *collapses at* E_2, and this is a proof of that cellular homology computes homology. Define "E_2".

Corollary 2.4.4. *More generally, if E and F satisfy Eilenberg–Steenrod and $E_*(S^n) \to F_*(S^n)$ is an isomorphism for all n, then $E_*(X) \to F_*(X)$ is an isomorphism for all CW complexes X.* □ Define.

Example 2.4.5. The Eilenberg–Mac Lane spaces play another important role in homotopy theory which we will more seriously explore in Chapter 3. Fixing an abelian group A and letting n vary, the functors

$$X \mapsto [X, K(A, n)]$$

satisfy the Eilenberg–Steenrod axioms: they carry wedge sums to products, and they convert coexact sequences of spaces to long exact sequences of groups. Moreover, there is a natural transformation

$$\begin{aligned}
[X, K(A, n)] &\to H^n(X; A), \\
f &\mapsto f^* \iota_n,
\end{aligned}$$

where $\iota_n \in H^n(K(A,n);A)$ corresponds under the universal coefficients theorem to the identity map

$$H_n(K(A,n);\mathbb{Z}) \cong A \xrightarrow{\mathrm{id}} A.$$

It then follows from Corollary 2.4.4 that this natural transformation is actually a natural *isomorphism*.

2.5 Obstruction theory

Today we exploit the machinery of spectral sequences to organize and extend some of our results in the realm of *obstruction theory*. Obstruction theory is generally concerned with starting with a map

$$A \longrightarrow B$$

and extending it to a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, dotted] & B \\ X \arrow[r, dotted] \arrow[ur, dotted] & Y \arrow[u] \end{tikzcd}$$

where $A \to X$ and $Y \to B$ are prescribed, any where one of the dashed maps might be the goal. We will focus our efforts by setting $A = B = *$, so that we will equivalently describe a method to compute $\pi_0 Y^X = [X,Y]$ from the data of X and of Y separately.

The "relative" cases where A and B are nonzero we leave to the interested reader.

Recall one the following Lemma from our study of the homotopy-theoretical properties of CW–complexes:

Lemma 2.3.10

Lemma 2.5.1. *For Y an n–connective CW–complex and Z an n–coconnective CW–complex, the natural map $[Y,Z] \to \mathsf{Groups}(\pi_n Y, \pi_n Z)$ is an isomorphism.* $\square$

Recall that n–connective is a synonym for $(n-1)$–connected, and n–coconnective for $\pi_{>n} Z = 0$.

This Lemma falls squarely into the realm of obstruction theory: it gives an algebraic description of what maps between certain homotopy types exist. Its proof also relied on spectral-sequence-style machinery, so it seems ripe for us to generalize now.

In the moment, however, we only needed the Lemma as part of a program to construct and analyze certain spaces called $K(A,n)$. It was the key ingredient in showing that the presentation of A used in their construction was purely auxiliary, and that $K(A,n)$ itself was a well-defined object. It has another application: the construction of *Postnikov towers*.

Corollary 2.5.2. *If Y is n–connective, then there is a canonical map*

$$Y \to K(\pi_n Y, n)$$

induced by the identity on π_n. $\square$

Corollary 2.5.3. *For Y an n–connective space, the exact extension of the canonical map $Y \to K(\pi_n Y, n)$ is called the $(n+1)$–upward-truncation of Y, denoted $Y(n,\infty)$ or $Y[n+1,\infty)$. It has the properties*

$$\pi_* Y(n,\infty) = \begin{cases} \pi_* Y & \text{if } * > n, \\ 0 & \text{otherwise} \end{cases}$$

and $\pi_* Y(n,\infty) \to \pi_* Y$ is an isomorphism for $* > n$.

In fact, it is the universal n–connected space over Y: any map in from any other such n–connected space factors through $Y(n,\infty)$.

Proof. Use the long exact sequence of relative homotopy. □

Definition 2.5.4. Starting with a 0–connected space and repeatedly applying Corollary 2.5.3 leads to the *Postnikov tower*,

$$\begin{tikzcd} Y \arrow[d] & Y(1,\infty) \arrow[l] \arrow[d] & Y(2,\infty) \arrow[l] \arrow[d] & \cdots \arrow[l] & Y(n,\infty) \arrow[l] \arrow[d] & \cdots \arrow[l] & * \arrow[l] \\ K(\pi_1 Y, 1) & K(\pi_2 Y, 2) & K(\pi_3 Y, 3) & & K(\pi_{n+1} Y, n+1), & & \end{tikzcd}$$

which is a diagram of interlocking exact sequences.

This situation is ripe for a spectral sequence. We apply $\pi_* F(X, -)$, where X is some fixed test space. The functor $F(X,-)$ preserves exact sequences, π_* turns them into long exact sequences, and Example 2.4.5 shows that $\pi_* F(X, K(\pi_{n+1} Y, n+1))$ has a recognizable form. Bundling these results together gives:

Simply applying π_* to this diagram gives an extremely boring spectral sequence!

Definition 2.5.5. The *unstable Atiyah–Hirzebruch spectral sequence* has signature

$$\begin{aligned} E^1_{m,n} &= \pi_m F(X, K(\pi_{n+1} Y, n+1)) \\ &= \widetilde{H}^{n-m+1}(X; \pi_{n+1} Y) \Rightarrow \pi_m F(X, Y). \end{aligned}$$

Picture!

As previously remarked, for arbitrary X and Y the objects π_0 and π_1 of $F(X,Y)$ are *sets* and *groups* respectively. This situation is called a *fringed spectral sequence*, which is considerably more obnoxious. Ensure $X = \Sigma^2 X'$ or $Y = \Omega^2 Y'$ for an easy way to avoid this situation.

This spectral sequence captures and generalizes several facts seen so far.

Remark 2.5.6. If $\pi_{<n} X = 0$ and $\pi_{>n} Y = 0$, we have $H^{<n}(X; \text{any}) = 0$ and $H^{\text{any}}(X; \pi_{>n} Y) = 0$. This puts a single nonvanishing group in the region of the spectral sequence contributing to $\pi_0 Y^X$:

$$H^n(X; \pi_n Y) \cong \mathsf{AbGps}(H_n X, \pi_n Y) \cong \mathsf{AbGps}(\pi_n X, \pi_n Y).$$

A good example of vanishing obstruction theory is that any space with even homology and homotopy admits the unital multiplication of an H–space.

Picture!

Since this group cannot be the source or target of any differentials, we conclude Lemma 2.3.10:

$$\mathsf{AbGps}(\pi_n X, \pi_n Y) \cong F(X, Y).$$

Example 2.5.7. By taking $Y = K(A,n)$ in Definition 2.5.5, the spectral sequences collapses to give

$$\pi_m F(X, K(A,n)) \cong \widetilde{H}^{n-m}(X; A),$$

and hence we relearn that ordinary cohomology is representable:

$$\widetilde{H}^n(X;A) \cong [X, K(A,n)].$$

Remark 2.5.8. The relative version of this spectral sequence also recovers Lemma 2.3.2.

Even though we have constructed this very appealing machine, we must stop short of actually using it to do anything. Nothing comes for free: the price of organizing the information in the E_1–term of a spectral sequence in a pretty way means that the differentials will surely be difficult to understand. To see what can happen, consider the d_1–differential, which is induced by pushforward along the map k_n in

$$\begin{tikzcd}
Y[n,\infty) \arrow[d] & & Y(n,\infty) \arrow[ll] \arrow[d] \\
& \Omega K(\pi_n Y, n) \arrow[ur] \arrow[dr, bend left, "k_n"] & \\
K(\pi_n Y, n) & & K(\pi_{n+1} Y, n+1).
\end{tikzcd}$$

This map k_n is part of the homotopy data of Y called the n^{th} *k–invariant* of Y. As noted in Remark 2.3.13, we have a weak equivalence

$$\Omega K(\pi_n Y, n) \simeq K(\pi_n Y, n-1),$$

so that k_n can be considered as a map of Eilenberg–Mac Lane spaces:

$$k_n \colon K(\pi_n Y, n-1) \to K(\pi_{n+1} Y, n+1).$$

This induces a natural transformation

$$\begin{tikzcd}
{[-, K(\pi_n Y, n-1)]} \arrow[r, "(k_n)_*"] \arrow[d, equal] & {[-, K(\pi_{n+1} Y, n+1)]} \arrow[d, equal] \\
H^{n-1}(-;\pi_n Y) \arrow[r] & H^{n+1}(-;\pi_{n+1} Y).
\end{tikzcd}$$

This leaves us with some burning questions which we must answer before using this spectral sequence with any seriousness:

1. What do natural transformations of the cohomology functor to itself look like in general? How many are there? Is there a classification, or a general formula? We at least know, by Yoneda, that such transformations are in natural bijection with homotopy classes $[K(A,m), K(B,n)]$.

This is kind of the "dual problem" to trying to compute $[S^m, S^n]$, the basic coexact building blocks.

2. How can the transformation $(k_n)_*$ be discerned / extracted for a given space Y?

2.A A complicated example

This needs *lots* of work.

We haven't yet had an example of a spectral sequence in which we can compute. This is because at our stage it is impossible to find a spectral sequence that is easy, tangible, nontrivial, and well-motivated all at once. Today we will retreat to algebra in order to work an example that covers the first three attributes, with the promise that it will become well-motivated later on.

Let us consider the Hopf algebra

Define

$$\mathscr{A}(1)_* = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$$

whose comultiplication is determined by

$$\begin{aligned} \Delta\xi_1 &= 1 \mid \xi_1 + \xi_1 \mid 1, \\ \Delta\xi_2 &= 1 \mid \xi_2 + \xi_1 \mid \xi_1^2 + \xi_2 \mid 1. \end{aligned}$$

Topologists often write "|" in place of "⊗" to make lengthy formulas less exhausting to write. This practice is also where the name "co/*bar* complex" comes from.

Associated to this Hopf algebra there is a *(reduced) cobar complex*, given by

$$C^n(\mathscr{A}(1)_*) = \left\{ \cdots \to \overline{\mathscr{A}}(1)^{\otimes n} \xrightarrow{\delta^n} \overline{\mathscr{A}}(1)^{\otimes(n+1)} \to \cdots \right\},$$

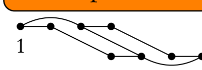
where $\overline{\mathscr{A}(1)}$ denotes the quotient of $\mathscr{A}(1)$ by the submodule spanned by the unit class. The differential δ^n is given by the alternating sum

$$\delta^n(x_1 \mid \cdots \mid x_n) = \sum_{j=1}^{n} (-1)^{j+1} \left(x_1 \mid \cdots \mid \Delta(x_j) \mid \cdots \mid x_n \right).$$

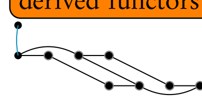
Credit Reuben Stern for these beautiful pictures!

Despite being only of modest size, the cohomology of this complex is still fairly difficult to compute. Before turning spectral sequences to bear on this problem, we use the theory of derived functors to perform a manual computation, so that the reader has a firmer sense of the effort that is being saved. The subalgebra $\mathscr{A}(1) \subset \mathscr{A}$ can be represented pictorially as in the margin. Each dot represents a class in $A(1)$; lines of horizontal distance 1 represent multiplication by Sq^1, and those of distance 2 represent multiplication by Sq^2. We will visually construct a free resolution of a single class of $\mathbb{F}_2$ by $A(1)$-modules by "chasing dots and lines" exhaustively, killing off all classes that are left over. In the first step, we map one copy of $A(1)$ to the single copy of $\mathbb{F}_2$: The classes not supporting a blue line are in the kernel of this map (circled in gray), and thus survive to the degree one part of the resolution.

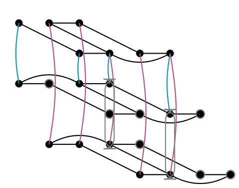
Honestly, these pictures should interrupt the text.



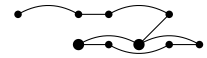
State why a free resolution suffices. What are we calculating derived functors of?



Now we turn to resolving the kernel. The kernel is generated by two classes—namely, the classes with no incoming lines from their left. Here, the vertical lines are drawn so that the maps are as $A(1)$ modules; i.e., they commute with Sq^1 and Sq^2.



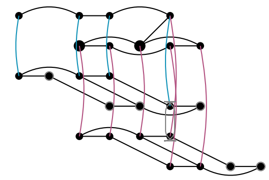
To move to the next stage of the resolution, we must calculate the kernel at this stage. If a class supporting a red line and one supporting a blue line

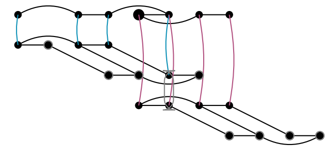


both hit the same class, their sum will be in the kernel. Thus, the diagram to resolve for degree two is as portrayed at right.

Following the same procedure of selecting dots which have no edges incoming from the left, we can produce a surjective map from a free module and hence the next stage of the resolution.

In fact, we have to follow this procedure twice more. Note that this kernel decomposes into a sum of modules, each of which we've seen before.

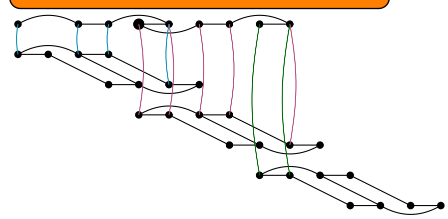




This is missing the gray circles.

Draw this kernel, gives names to the summands.

We can use this observation to extend this to an infinite free resolution by appropriately repeating the steps we have already taken. Since the cohomology of the cobar complex is meant to compute $\operatorname{Ext}^{*,*}_{\mathscr{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$, we can equivalently compute its cohomology by mapping our minimal free resolution into $\mathbb{F}_2$ and calculating its cohomology—which carries the trivial differential. In all, this produces the family of Ext–groups pictured at right.



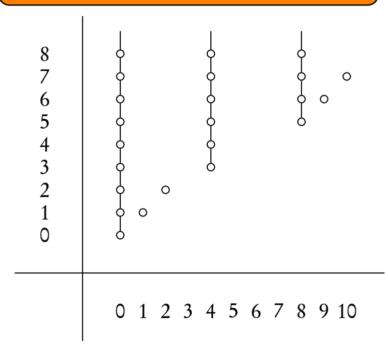
Why towers? These are groups, not a ring.

Now we consider a less manual method of computing these Ext groups. We will need a filtration to produce a spectral sequence, and we select the filtration given by powers of the augmentation ideal. This is a useful choice because the associated graded looks like the cobar complex for an exterior algebra, which has known homology: it is given by a polynomial algebra on homology-suspended shifts of the same classes.

Theorem 2.A.1 (May). *This gives a spectral sequence of algebras*

$$E_1^{*,*,*} \cong \mathbb{F}_2[h_{10}, h_{11}, h_{20}] \Rightarrow H^*(\mathscr{A}(1)_*),$$

where h_{ij} represents $\xi_i^{2^j}$ and has degree $|h_{ij}| = (1, 2^i - 1, 2^j(2^i - 1))$. □



(?)

Explain that this means there's a Leibniz law in addition to a product.

The first differential in this spectral sequence is tautological to compute:

Lemma 2.A.2. *d_1 is fully specified by*

$$d_1(h_{10}) = 0, \qquad d_1(h_{11}) = 0, \qquad d_1(h_{20}) = h_{10}h_{11}.$$

Proof. Recall that h_{ij} represents $\xi_i^{2^j}$. The first two equations then follow from $\delta^1(\xi_1^{2^j}) = 0$. To analyze h_{20}, note that the differential of ξ_2 in the normalized cobar complex is

$$\delta^1(\xi_2) = \Delta\xi_2 = \xi_1 \mid \xi_1^2.$$

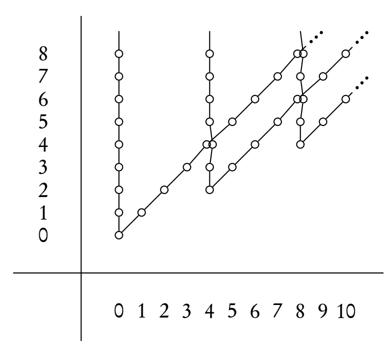
This is represented by $h_{10}h_{11}$. □

Give differential tridegree.

Since these form a multiplicative generating set, we can use the Leibniz law to capture the rest of d_1:

$$d_1(xy) = (d_1x)y + (-1)^{|x|}x(d_1y).$$

Lemma 2.A.3. $d(h_{20}^2) = h_{11}^3$.



Vertical lines indicate classes linked by multiplication by h_{10}, and diagonal lines indicate classes linked by multiplication by h_{11}.

Proof. The class h_{20}^2 is represented by $\xi_2 \mid \xi_2$, which has cobar differential

$$\delta(\xi_2 \mid \xi_2) = \xi_1 \mid \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1 \mid \xi_1^2,$$

which in turn represents on E_1

$$2h_{10}h_{11}h_{20} \equiv 0.$$

That this class vanishes in the spectral sequence indicates that we can produce a longer spectral sequence differential by adding a correcting term of higher filtration to the chosen representative $\xi_2 \mid \xi_2$. Namely, we need to find a preimage of $\xi_1 \mid \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1 \mid \xi_1^2$ along the cobar differential, up to higher still filtration. Some fussing shows that the following fits the bill:

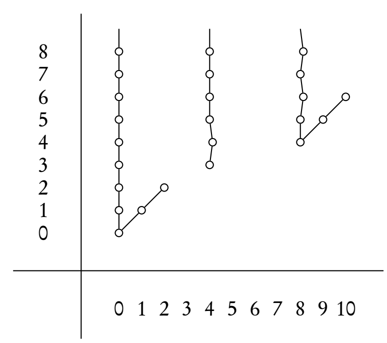
$$\xi_1\xi_2 \mid \xi_1^2 + \xi_1 \mid \xi_2\xi_1^2,$$

from which we compute

$$\begin{aligned} d(\xi_2 \mid \xi_2 + \xi_1\xi_2 \mid \xi_1^2 + \xi_1 \mid \xi_2\xi_1^2) = {} & (\xi_1 \mid \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1 \mid \xi_1^2) \\ & + (\xi_1^2 \mid \xi_1^2 + \xi_1 \mid \xi_2 + \xi_2 \mid \xi_1 + \xi_1 \mid \xi_1^3) \mid \xi_1^2 \\ & + \xi_1 \mid (\xi_2 \mid \xi_1^2 + \xi_1 \mid \xi_1^4 + \xi_1^2 \mid \xi_2 + \xi_1^3 \mid \xi_1^2) \\ = {} & \xi_1^2 \mid \xi_1^2 \mid \xi_1^2. \end{aligned}$$

This last class represents h_{11}^3, which is nonzero on the second page of the May spectral sequence. $\square$

This again is enough to determine the differential on a multiplicative generating set. More than this, this is the last differential in the spectral sequence, since the page is now too sparse to support any further differentials. In this way, the spectral sequence and its multiplicative structure have converted the computation the cohomology of an infinite complex, a seemingly intractible problem, into two relatively minor calculations in the complex.



Remark 2.A.4.

Expand remark about Leibniz rule into the Christianson-style presentation of the spectral sequence.

3
Representability

In this Chapter, we turn to one of the most unique features of homotopy theory: Brown's representability theorem. This theorem shows that many of the tools of algebraic topology can themselves be thought of as homotopy types in their own right. We have already encountered this idea with our most fundamental tools so far: homotopy groups are corepresented by spheres, and cohomology groups are represented by Eilenberg–Mac Lane spaces, and this gives a wild generalization of this pattern. As an application of this idea, we introduce the notions of the *localization* and *completion* of a space, imported from their corresponding algebraic effects on homotopy groups. We also introduce the notion of a *spectrum*, whose role is to capture the elaborate structure present in a co/homology theory beyond mere representability. Spectra are interesting enough objects in their own right, and will play a large enough role later on, that we work through a good amount of their theory here.

This used to live in the obstruction theory section, where it no longer makes sense because we don't know what infinite loopspaces are. It probably belongs somewhere in this chapter.

3.1 Brown representability

Early on, when we were getting used to the categorical approach to homotopy theory, we noted in ?? that $\mathsf{Spaces}(-, T)$ forms a *sheaf* on the category Spaces. This isn't our only example of such an object: cohomology functors can also be thought of as sheaves, as the Eilenberg–Steenrod axioms include the sheaf axioms as a subset. However, in the course of our study of the homotopy theory of CW–complexes, we discovered that these two examples are actually not separate: Example 2.5.7 gave a natural isomorphism

$$\widetilde{H}^n(X;A) \xrightarrow{\cong} [X, K(A,n)],$$

where $K(A,n)$ is an *Eilenberg–Mac Lane space* as in Lemma 2.3.9. Today we prove that this is not an accident: all sheaves in the homotopy category are representable.

(?, Theorem 9.12)

Theorem 3.1.1 (Brown). *Let $F\colon h\mathsf{Spaces}^{\mathrm{op}}_{\mathrm{conn},*/} \to \mathsf{Sets}_{*/}$ be a functor on the homotopy category of pointed, connected spaces which satisfies the following:*

Remark somewhere that phantom maps are not a concern if you work with space-valued lifts the whole while.

Wedge axiom: F converts wedges to products, as in

$$F\left(\bigvee_\alpha X_\alpha\right) \cong \prod_\alpha F(X_\alpha).$$

Gluing condition: Elements in the image of F glue. For a decomposition $X = A_1 \cup A_2$ and for elements $f_1 \in F(A_1)$, $f_2 \in F(A_2)$ which agree on the intersection, as in

$$f_1|_{A_1 \cap A_2} = f_2|_{A_1 \cap A_2},$$

then there exists a glued element $f \in F(X)$ which satisfies $f_1 = f|_{A_1}$ and $f_2 = f|_{A_2}$.

Remark that this isn't quite the sheaf axiom, since it's missing unicity.

There then exists a representing CW-complex Y and a universal element $u \in F(Y)$ such that the natural transformation

$$\begin{aligned}[T, Y] &\to F(T), \\ \varphi &\mapsto \varphi^*(u)\end{aligned}$$

is a natural bijection. Moreover, there is a compatible bijection between natural transformations $F \to F'$ between such functors and homotopy classes $Y \to Y'$ between their representing objects.

We construct this in stages.

(?, Definition 9.6)

Definition 3.1.2. An element $u \in Y$ is said to be *n–universal* if the associated natural transformation

$$\begin{aligned}[S^q, Y] &\to F(S^q) \\ \varphi &\mapsto \varphi^*(u)\end{aligned}$$

is surjective for $q \leq n$ and bijective for $q < n$.

It follows that Y will represent F for all CW–complexes of dimension less than n.

(?, Lemma 9.8)

Lemma 3.1.3. *If there exists an n–universal element on Y, then there exists a Y' carrying an $(n+1)$–universal element.*

Proof. Suppose we have an n–universal element u_n on a complex Y. From this, we would like to construct an $(n+1)$–universal element u_{n+1} on a complex Y'. We set about trying to "fix" $[S^n, Y] \to F(S^n)$, which might have too many elements to be a bijection, and $[S^{n+1}, Y] \to F(S^{n+1})$, which might be missing some elements to be a surjection. Note that because S^n is an H–cogroup for $n \geq 1$, the map $[S^n, Y] \to F(S^n)$ is actually a map of groups. It follows that if we merely ensure that this surjective map does not have a kernel, it will be an isomorphism.

This is exactly why we restricted attention to *connected* spaces: it gives us control over all the fibers of the map $[S^n, Y] \to F(S^n)$, rather than just the fiber over the constant map.

This inspires us to consider the defect sets

$$A = \{\alpha \in \pi_n Y \mid \alpha^* u_n = 0\}, \qquad L = F(S^{n+1}),$$

Why A and L? Seems like weird notation.

Here we are being a bit glib: perhaps some items in $F(S^{n+1})$ can be expressed as pullbacks of u_n, but there's no harm in adding more things to make sure we hit.

and form the mapping cone

$$\bigvee_{\alpha\in A} S^n_\alpha \xrightarrow{\alpha} Y \vee \bigvee_{\lambda\in L} S^{n+1}_\lambda \to Y'.$$

Applying F, we have

$$0 = \bigvee_\alpha \alpha^*(u_n) \leftarrow\!\shortmid u_n \vee \bigvee_\lambda \lambda,$$

hence we can lift it to an element $u_{n+1} \in F(Y')$. Since Y' is formed from Y using $(n+1)$–cells, it agrees with Y on $\pi_{<n}$, hence it is n–universal. As for $(n+1)$–universality, it is designed to fix the defect at π_n exactly, and the wedge over λ forces surjectivity at π_{n+1}. □

(?, Corollary 9.9, Lemma 9.11)

Lemma 3.1.4. *Let Y be a space with universal element u, let (X,A) be a CW-pair, let $v \in F(X)$ be a choice of element, and let $g\colon A \to Y$ be a cellular map which classifies $v|_A$. There then exists a cellular map which classifies v and which extends g.*

$$\begin{tikzcd}
 & & v' & & u \arrow[ll, mapsto] \\
v|_A \arrow[urr, equal] & & A \arrow[r, "g'"] \arrow[d, hook] & Y & \\
 & & X \arrow[ur, dotted, "g"] & & \\
v \arrow[uu, mapsto] \arrow[uurrrr, dotted, mapsto] & & & &
\end{tikzcd}$$

Proof idea. We define a "double mapping cylinder" T which consists of the space X, the space Y, and the space $A \times I$, so that the leading edge of $A \times I$ is sewn to its image in X and the trailing edge of $A \times I$ is sewn to its image under g in Y. This space has a decomposition into A_1, which consists of X and half of the cylinder, and A_2, which consists of Y and half of the cylinder. Since A_1 and A_2 are respectively homotopy equivalent to X and Y, we may respectively consider u and v as elements of $F(A_1)$ and $F(A_2)$. Definitionally, they agree on $A_1 \cap A_2$ (i.e., on the cylinder, which is equivalent to A), and hence they give rise to the glued element $w \in F(T)$. We can extend T to a CW pair (Y',T) with universal element u' restricting to w (and hence to u). We apply Whitehead's theorem to the induced weak equivalence $Y \to Y'$ to produce an inverse, and the composite gives the desired map: $X \to Y' \to Y$. □

This is the "homotopy pushout" of $X \xleftarrow{i} A \xrightarrow{g} Y$.

There's a picture here.

Just re-run Lemma 3.1.3, starting instead with T and w rather than a point and the trivial class.

Proof of Theorem 3.1.1. To get surjectivity, set $A = \{x_0\}$ and apply Lemma 3.1.4. To get injectivity, set $X' = X \times I$, $A' = X \times \partial I$, and apply Lemma 3.1.4 again. To get the statement about natural transformations, one need only chase the identity transformation through

$$[Y,Y] \xrightarrow{\cong} F(Y) \xrightarrow{T} F'(Y) \xleftarrow{\cong} [Y,Y']$$

to produce an element f. □

There is a useful companion result that works with functors F defined only on *finite* CW-complexes.

(?, Theorem 9.21)

Theorem 3.1.5 (Adams)**.** *If F is a functor to* groups *from* finite *CW-complexes satisfying the two conditions of Brown's theorem, then it is representable. Natural transformations induce maps of representing objects that are unique up to* weak homotopy*: if $t, t'\colon Y \to Z$ are two maps of representing objects inducing the maps t_* and t'_* on finite complexes, then any finite complex $f\colon F \to Y$ has $t \circ f \simeq t' \circ f$.* □

There's a margin note here that I threw out as unhelpful. See if it can be salvaged.

3.2 Arithmetic decomposition

Add citations from Sullivan.

What's the concrete punchline of this day? What question did we start with which we can now answer? What will we ask next?

In Definition 2.5.5, we deduced the existence of a spectral sequence with signature

$$\begin{aligned} E^1_{m,n} &= \pi_m F(X, K(\pi_{n+1} Y, n+1)) \\ &= \widetilde{H}^{n-m+1}(X; \pi_{n+1} Y) \Rightarrow \pi_m F(X, Y). \end{aligned}$$

This spectral sequence precisely encodes the claim that the available maps from X to Y are captured by certain cohomology groups, subject to an elaborate cancellation process by way of the spectral sequence's differentials. An interesting consequence of the foreground role of algebra is that different primes do not interact much: a p–primary component appearing in the spectral sequence can only participate in nonzero differentials which involve other p–primary components for the same prime p. This suggests that the homotopy types X and Y can themselves be pulled apart into pieces associated to the different arithmetic primes. Indeed, this turns out to be so.

The specific algebraic reconstruction theorem which we hope to recapture for spaces is:

Theorem 3.2.1. *For a finitely generated abelian group A, there is a pullback square*

$$\begin{array}{ccc} A & \longrightarrow & \widehat{A} \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes A & \longrightarrow & \mathbb{Q} \otimes \widehat{A}, \end{array}$$

where $\widehat{A} = \lim_{\substack{B \le A \\ B \text{ finite}}} A/B$ denotes the profinite completion. □

Is it possible to drop the finite generation hypotheses, at the price of working with complexes of abelian groups. For example, the profinite completion of $A = \mathbb{Z}/p^\infty$ turns out to be $\Sigma^{-1}\mathbb{Z}_p$, which presents A via the exact sequence

$$\Sigma^{-1}\mathbb{Z}_p \to \Sigma^{-1}\mathbb{Q}_p \xrightarrow{\partial} \mathbb{Z}/p^\infty.$$

Thus, our first goal will be to define analogues of rationalization and profinite completion for spaces, in such a way that in good cases we have $\pi_n \widehat{X} = \widehat{\pi_n X}$, $\pi_n \mathbb{Q} \otimes X = \mathbb{Q} \otimes \pi_n X$, and a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & \widehat{X} \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes X & \longrightarrow & \mathbb{Q} \otimes \widehat{X}. \end{array}$$

We begin with localization.

Definition 3.2.2. A simply connected homotopy type X is said to be *rational* when $\pi_* X$ is a rational vector space. The *rationalization* $\mathbb{Q} \otimes X$ is a universal rational space under X: any other map $X \to Y$ with Y a rational space admits a factorization

All of these definitions and constructions can also be made when inverting only a subset of all available primes, giving rise to p–localization.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{Q}\otimes X \arrow[dl, dotted] \\ & Y. & \end{tikzcd}$$

Lemma 3.2.3. *The following are equivalent:*

Indications on proof. I think this is where mod-$\mathscr{C}$ theory is actually valuable.

- *A map* $\ell\colon X \to X'$ *is rationalization.*
- *The map* $\ell_*\colon \pi_* X \to \pi_* X'$ *is rationalization.*
- *The map* $\ell_*\colon H_*(X;\mathbb{Z}) \to H_*(X';\mathbb{Z})$ *is rationalization.*

Definitionally, rational spaces are closed under exact sequences. This Lemma also shows that they are closed under coexact sequences.

Proof sketch. The first and second conditions are equivalent via Whitehead's theorem. The second and last conditions are equivalent via a modified form of the Hurewicz theorem. □

Probably flesh this out.

Theorem 3.2.4. *The rationalization of any simply connected CW-complex* X *exists.*

Proof. We produce two distinct constructions.

First, using $\pi_n S^n \cong \mathbb{Z}$ for $n \geq 1$, the rationalization $\mathbb{Q}\otimes S^n$ is simple to produce:

Flesh this out.

$$\mathbb{Q}\otimes S^n = \operatorname{colim}\left(\cdots \to S^n \xrightarrow{p} S^n \to \cdots\right).$$

Second, we can also rationalize the Postnikov tower of X. □

Flesh this out too.

No rationalization (or localization) functor can preserve all co/exact extensions (?, pg. 40).

The construction of the profinite completion of a space is much more indirect: the profinite completion of the sphere is not obvious, and the example of $B\mathbb{Z}/p^\infty$ indicates that it is not clear how to modify the Postnikov tower. Meanwhile, when translating the definition of the profinite completion of an abelian group it is not clear what should play the role of a surjection onto a finite object. To alleviate this, note that the profinite completion of an abelian group can also be formed from a larger system: one takes the limit of *all* finite groups under A in the category of *all* groups under A.

Definition 3.2.5. Consider the representable functor $X(Y) = [Y,X]$ determined by X. The *profinite completion* $\widehat{X}$ of this functor is determined by

$$\widehat{X}(Y) = \lim [Y,F],$$

where F ranges over all spaces under X such that $\pi_* F$ is finite.

All of these definitions can constructions can also be made when targeting only finite p-groups, giving rise to p-adic completion. In fact, $\widehat{X} = \prod_p X_p$.

Lemma 3.2.6. *The functor* $\widehat{X}$ *satisfies the hypotheses for Brown representability as in Theorem 3.1.5.* □

Indicate proof?

Definition 3.2.7. The associated homotopy type, $\widehat{X}$, is the *profinite completion* of X.

Lemma 3.2.8. *For X a simply connected space of finite type,*

$$\pi_n \widehat{X} = \widehat{\pi_n X}, \qquad H_n(\widehat{X};\mathbb{Z}) = \widehat{H_n(X;\mathbb{Z})}. \quad \square$$

Lemma 3.2.9. *If X is of finite type, Y is p-complete, and $H_*(X;\mathbb{F}_p) \to H_*(Y;\mathbb{F}_p)$ is an isomorphism, then $X_p \to Y$ is a weak equivalence.* $\square$

Proof? Also, corollary statement that $H_*(X;\mathbb{F}_p) \to$

Theorem 3.2.10. *A simply connected space X can be recovered as the homotopy pullback*

Is there a proof of this that doesn't use the SSS or Serre finiteness?

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ \mathbb{Q} \otimes X \arrow[r] & \mathbb{Q} \otimes \widehat{X}. \end{tikzcd} \quad \square$$

3.3 Spectra

Our discussion in Section 3.1 was motivated by our observation in Lemma 2.3.9 that there is a natural isomorphism

$$H^n(X;A) \xrightarrow{\cong} [X, K(A,n)],$$

which we generalized to any functor satisfying the wedge and Mayer–Vietoris axioms. These axioms are most of what it means to be a cohomology theory. The remaining axiom, which we have not yet discussed, is the following:

Definition 3.3.1 (Suspension axiom)**.** There is a natural isomorphism

$$\widetilde{H}^n(X) \xrightarrow{\cong} \widetilde{H}^{n+1}(\Sigma X).$$

It's pleasing to write this identity as

$$\Sigma \widetilde{H}^*(X) \cong \widetilde{H}^*(\Sigma X).$$

Its role is an interesting one, and it is best understood in the context of π_*, which we have shown in Corollary 2.2.8 to *partially* have this property. An interesting feature of Corollary 2.2.8 is that the range in which suspension invariance holds improves the more times you suspend: if $\pi_* X \to \pi_* \Sigma X$ is an isomorphism through degree $2n$, then $\pi_* \Sigma X \to \pi_* \Sigma^2 X$ is an isomorphism through degree $2(n+1)$, and so on. It follows that we can associate to X its *stable homotopy groups*, given by $\operatorname{colim}_n \pi_{*+n} \Sigma^n X$. A second interesting feature of Corollary 2.2.8 is that a spectral sequence argument shows it to generalize away from spheres to CW-complexes of bounded dimension:

The fibration appearing in Example 2.2.9 for $n = 2$ has the form

$$S^1 \to S^3 \to S^3.$$

From this we can conclude $S^3 \simeq S^2[3,\infty)$, and hence $\pi_3 S^2 \cong \mathbb{Z}$. This gives a concrete counterexample to any extension of Freudenthal beyond the advertised range, as $0 \cong \pi_2 S^1 \not\cong \pi_3 S^2 \cong \mathbb{Z}$.

Corollary 3.3.2 (of Corollary 2.2.8)**.** *Let X be an s-connected CW-complex, and let Y be a CW-complex of dimension t. Then*

Find me a citation. I don't seem to be in Switzer.

$$F(Y,X) \to F(\Sigma Y, \Sigma X)$$

is a $(2s-t)$-equivalence. $\square$

Expand this argument. Are we looking at a map of Federer spectral sequences for $[-,X] \to [X,\Omega\Sigma X]$? I'm unsure.

It follows again that $\pi_m F(\Sigma^n Y, \Sigma^n X)$ is independent of n for $n \gg 0$. This spurs us to make the following categorical definition:

Definition 3.3.3. Let $h\mathsf{SuspensionSpectra}$ denote the category which has an object $\Sigma^\infty X$ for every pointed space X and whose morphism sets are given by the formula

$$[\Sigma^\infty Y, \Sigma^\infty X] := \operatorname{colim}_n [\Sigma^n Y, \Sigma^n X].$$

(?, Example 8.2)

A funny consequence of this definition is that $[\Sigma^\infty Y, \Sigma^\infty X]$ (and, later, $[E, E']$ generally) is *always* an abelian group, since one can always take at least 2 suspensions to be involved.

One need not leave Spaces to understand this new category:

$$\begin{aligned}
[\Sigma^\infty Y, \Sigma^\infty X] &:= \operatorname{colim}_n [\Sigma^n Y, \Sigma^n X] \\
&= \operatorname{colim}_n [Y, \Omega^n \Sigma^n X] \\
&= [Y, \operatorname{colim}_n \Omega^n \Sigma^n X] =: [Y, QX].
\end{aligned}$$

The stable homotopy groups then appear as honest homotopy groups:

$$\pi_n \Sigma^\infty X \cong \pi_n QX.$$

The map $h\mathsf{SuspensionSpectra} \to h\mathsf{Spaces}$ produced in this way is commonly denoted by Ω^∞.

The functor Q gives rise to a host of stable invariants: fixing a space X, the family of functors

$$X^{-*}(-) := [-, Q\Sigma^* X]$$

satisfy the wedge and Mayer–Vietoris axioms (because the functors are representable), and they are additionally *suspension invariant* (because $\Omega Q \Sigma \Sigma^n X = Q \Sigma^n X$). Unfortunately, not all cohomology theories arise in this way: there is generally no space X so that $QX \simeq K(A, n)$. This failure, however, is interesting on its own, measureable, and surmountable. Consider the space $QK(A, n)$: Corollary 2.2.8 shows that its homotopy is given by

Be clear about what fails and how partial the failure is, since, e.g., $K(\mathbb{Q}, n) = QK(\mathbb{Q}, n)$.

$$\pi_* K(A, n) \cong \begin{cases} 0 & \text{when } * < n, \\ A & \text{when } * = n, \\ 0 & \text{when } n < * \le 2n, \\ ??? & \text{otherwise.} \end{cases}$$

As n increases, the range through which $K(A, n) \to QK(A, n)$ is an equivalence grows like $2n$. In particular, $K(A, n+1) \to QK(A, n+1)$ is a $2(n+1)$–equivalence, so that $K(A, n) = \Omega K(A, n+1) \to \Omega QK(A, n+1)$ is a $(2n+1)$–equivalence — an improvement over the $2n$–equivalence $K(A, n) \to QK(A, n)$ by one. If we were to repeat this trick and take a formal colimit along the maps $\Sigma K(A, n) \to K(A, n+1)$ adjoint to $K(A, n) \xrightarrow{\simeq} \Omega K(A, n+1)$, then we could formally write

$$K(A, n) = \Omega^\infty \left(\operatorname{colim}_m \Sigma^{-m} \Sigma^\infty K(A, n+m) \right).$$

We use this formal expression to justify the following:

Is there a good reason to use Σ^{-1} here rather than Ω?

Definition 3.3.4. A *spectrum* (up to homotopy) is an ind-diagram of formal desuspensions of suspension spectra.

Example 3.3.5. Suspension spectra themselves qualify as spectra: $\Sigma^\infty X$ is trivially an ind-diagram. The *sphere spectrum* is the special case of $\mathbb{S} = \Sigma^\infty S^0$.

Example 3.3.6. The *Eilenberg–Mac Lane spectrum* is given by the formula above:

$$HA = \operatorname{colim}_n \Sigma^{-n}\Sigma^\infty K(A,n).$$

Example 3.3.7. The functor $X \mapsto (\pi_* \Sigma^\infty X) \otimes \mathbb{Z}_{(p)}$ is exact, hence has an associated spectrum $\mathbb{S}_{(p)}$, the *p-local sphere spectrum*.

Example 3.3.8. The functor $X \mapsto \mathsf{AbGps}(\pi_0^s X, \mathbb{Q}/\mathbb{Z})$ is exact, hence has an associated spectrum $\mathbb{I}$. This is called the *Brown–Comenetz dualizing object*.

This is *covariant*. What is the correct statement for covariant Brown representation? This is 14.35 in Switzer. It's actually clear with spectra: replace $\pi_n E \wedge X$ with $\pi_n F(DX, E)$ for finite X and appeal to the finite form of Brown. This absolutely should appear in the section on Spectra and homology theories—but it needs SW duals and $\wedge$, so this isn't easy.

K-theory? Bordism?

Given all this, one might be motivated by a need for concreteness to pursue a point-set model for spectra, maps of spectra, and homotopies among maps, whose homotopy category recovers $h\mathsf{Spectra}$. There are many such models available, with competing strengths and deficiencies. We reproduce an easy one here, due to Lima, Boardman, and Vogt.

Less perjoratively: a desire to build a bridge between these ideas and geometry.

Definition 3.3.9. A *spectrum* is a collection $\{E_n\}_n$ of CW–complexes together with cellular maps $i_n \colon \Sigma E_n \to E_{n+1}$ which are homeomorphisms onto their images.

(?, Definition 8.1)

This is not so restrictive: given a suitable notion of homotopy equivalence, one may use the mapping cylinder construction to make the subcomplex and homeomorphism conditions apply.

As indicated by the "ind–system" appearing in the abstract definition, maps between spectra are not quite given by levelwise maps which commute with the inclusions i_n. Instead, one asks for maps to only be defined *eventually*.

For instance, we know $\pi_n S^n \cong \mathbb{Z}$ for $n \ge 1$, but $\pi_0 S^0 = \{\pm 1\}$. If we were to define maps of spectra as such commuting sequences, then we would get $\pi_0 \Sigma^\infty \mathbb{S} = \{\pm 1\}$—the *wrong* answer.

Definition 3.3.10. A *subspectrum* $F \subseteq E$ is a sequence of subcomplexes of E_n, forming a spectrum by restriction. It is *cofinal* when every cell $e^m_\alpha \subseteq E_n$ has $\Sigma^{j_\alpha} e^m_\alpha \subseteq F_{n+j_\alpha}$—it eventually appears in F. A map $E \to E'$ is required only to be defined on a cofinal $F \subseteq E$, and two maps are equal if they agree on a mutually cofinal subspectrum. Finally, two maps of spectra are *homotopic* if there is a common cofinal subspectrum F' and a map $F' \wedge I_+ \to E'$ witnessing the homotopy.

(?, Definitions 8.9, 8.10, 8.12, 8.15)

In particular, the inclusion of a cofinal subspectrum is equivalent to the identity map.

The advantage of having a model available is that we can use it to lift some familiar constructions from Spaces.

(?, 8.17, 8.18)

Lemma 3.3.11. *Spectra have wedge sums and mapping cones, both given level-wise.* □

This, together with our knowledge of Spaces generally, is enough to copy the proof of Whitehead:

Which we did not give for Spaces either!

(?, Corollary 8.24)

Theorem 3.3.12. *If a map $f \colon E \to E'$ induces a weak equivalence, it is a homotopy equivalence.* □

(?, Theorem 8.26)

Corollary 3.3.13. *The spectra $\{E_n \wedge S^1\}_n$ and $\{E_{n+1}\}_n$ are equivalent, and so the spectrum $\{E_{n-1} \wedge S^1\}_n = \Omega\Sigma E$ is equivalent to E.* □

3.4 Co/homology theories from spectra

We defined spectra in such a way that a cohomology theory gives rise to a spectrum by extracting the representing objects $E^n(-) = [-, E_n]$ and building from them the inductive system

$$E := \operatorname{colim}_n \Sigma^{-n}\Sigma^\infty E_n.$$

Our definition was lax enough, though, that the converse is not quite as clear: do spectra precipitate cohomology theories? If so, how tight is the correspondence between the two?

This should probably be wholly reorganized.

Definition 3.4.1. For a spectrum E, we define its *associated (reduced) co/homology theories* as follows:

(?, 8.33)

- $\widetilde{E}^n(X) = [\Sigma^\infty X, \Sigma^n E]$.

"I guess this defines Ω^∞ through representability."

- $\widetilde{E}_n(X) = \pi_n(E \wedge X)$, where the smash product $(E \wedge X)_n = E_n \wedge X$ is induced up from Spaces.

In order to see that these are co/homology functors, it's useful to record

Lemma 3.4.2. *For X a pointed space and E a generic spectrum,*

$$[\Sigma^\infty X, E] = \operatorname{colim}_{n,m}[\Sigma^m X, \Sigma^{m-n} E_n].$$

In general, homotopy classes of maps of spectra are presented by π_0 of a kind of pro-ind-space.
If one uses Brown representability of a cohomology functor to manufacture the spaces E_n in the definition of E, then suspension invariance makes the system in the Lemma *constant*. This is called an "Ω–spectrum".

Proof. One couples the formula for suspension spectra

$$[\Sigma^\infty X, \Sigma^\infty Y] = \operatorname{colim}_m[\Sigma^m X, \Sigma^m Y]$$

to a presentation of E:

$$E = \operatorname{colim}_n \Sigma^{-n}\Sigma^\infty E_n. \qquad \square$$

How does this formula work with m and n both tending to ∞? Is "constant" in the margin note a clue?

Why can we pull the colimit out of the target?

To feel confident in our definition, we should check that these functors indeed satisfy the Eilenberg–Steenrod axioms.

define?

1. We've built in suspension invariance:

$$\widetilde{E}_{n+1}(\Sigma X) \cong [S^{n+1}, E \wedge \Sigma X] \cong [S^n, E \wedge X] \cong \widetilde{E}_n(X),$$
$$\widetilde{E}^{n+1}(\Sigma X) \cong [\Sigma^\infty \Sigma X, \Sigma^{n+1} E] \cong [\Sigma^\infty X, \Sigma^n E] \cong \widetilde{E}^n(X).$$

2. A coexact sequence

$$A \xrightarrow{i} X \to X \cup_i CA$$

of pointed spaces induces a coexact sequence of spectra

Do we need coexact = exact for spectra?

$$E \wedge A \to E \wedge X \to E \wedge (X \cup_i CA) = (E \wedge X) \cup_i C(E \wedge A),$$

so we get the desired long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(E \wedge A) \arrow[r] \arrow[d, equal] & \pi_n(E \wedge X) \arrow[r] \arrow[d, equal] & \pi_n(E \wedge C(i)) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \widetilde{E}_n(A) \arrow[r] & \widetilde{E}_n(X) \arrow[r] & \widetilde{E}_n(X,A) \arrow[r] & \cdots .
\end{tikzcd}$$

For the analogous fact in cohomology, the sequence of suspension spectra

$$\Sigma^\infty A \to \Sigma^\infty X \to \Sigma^\infty (X \cup_i CA)$$

is coexact, so mapping into E makes it exact.

We haven't talked about function spectra... so is this really fair? Perhaps we only need $\wedge$ and representability, which we do have.

3. The cohomological wedge axiom is easy: by pulling coproducts out on the left to products, we get

$$\left[\bigvee_\alpha \Sigma^\infty X_\alpha, E\right] = \prod_\alpha [\Sigma^\infty X_\alpha, E].$$

Homology is harder and requires a filtration trick. We know that our homology functor satisfies the *finite* wedge axiom by appeal to the Mayer–Vietoris axiom. Smash products also commute with colimits, hence one may check

(?, Lemma 8.34)

$$E \wedge \operatorname{colim}_{\substack{S \subseteq A \\ S \text{ finite}}} \bigvee_{\alpha \in S} X_\alpha \cong \operatorname{colim}_S E \wedge \bigvee_{\alpha \in S} X_\alpha \cong \operatorname{colim}_S \bigvee_{\alpha \in S} E \wedge X_\alpha \cong \bigvee_\alpha E \wedge X_\alpha.$$

From this, the wedge axiom follows.

This feels clumsy. Is it really necessary? This must have been some garbled $\prod^\infty$ vs $\coprod^\infty$...? This is a *compactness* argument.

Remark 3.4.3. The Mayer–Vietoris axiom amounts to the assertion that co/homology functors commute with finite (homotopy) colimits, and the wedge axiom adds a special case on top of that. Cohomology *does not* commute with general colimits. Instead, there is a *Milnor sequence*:

(?, Propositions 7.66 and 8.37)

$$0 \to R^1 \lim_\alpha E^{n-1} X_\alpha \to E^n(\operatorname{colim}_\alpha X_\alpha) \to \lim_\alpha E^n(X_\alpha) \to 0.$$

Satisfied that we have indeed produced co/homology theories, we can investigate whether these assignments are mutual inverses. To compare objects, this mostly comes down to Whitehead's theorem for spectra. Using Brown representability for natural transformations, we can lift maps of cohomology theories up to maps of spectra: a map $E^* \xrightarrow{f} F^*$ of cohomology functors induces a compatible family of maps $E_n \xrightarrow{f_n} F_n$ and hence a map $\widetilde{f} \colon E \to F$ of spectra. From this, one sees that a natural isomorphism of cohomology theories induces a weak equivalence of spectra, and conversely. However, Brown representability falls short of giving a *functorial* correspondence: the same natural transformation of cohomology theories can be induced by multiple homotopy-inequivalent maps of spectra. This difference is called a "phantom". More precisely,

It's even reasonable to extend the definition of cohomology to $F^0(E) = [E, F]$, of which $\widetilde{f}$ is then an element.

Brown's result shows that the construction

$$\begin{aligned} \mathsf{Spectra} &\to \mathsf{CohomologyTheories} \\ E &\mapsto [\Sigma^\infty -, \Sigma^* E] \end{aligned}$$

is full and bijective on isomorphism classes.

(?, Theorem 10.25)

Theorem 3.4.4 (Hurewicz). *There is a map $\mathbb{S} \to H\mathbb{Z}$ which has 0-connected fiber. By consequence, the difference between $\mathbb{S}_*(X)$ and $H\mathbb{Z}_*(X)$ begins one degree above the bottommost cell in X. By consequence, for X n-connected and $n \geq 1$, $\pi_n X \cong \pi_n^s X \cong H\mathbb{Z}_n X$.* □

Where does this belong?

3.5 *The smash product*

For all our discussion of homotopy and homology *groups*, we have not yet found a framework for the cohomology *ring* of a space. The following observation is key:

Definition 3.5.1. A *ring* is a (commutative, unital) monoid in $\mathsf{AbelianGroups}$ under the $\otimes$-product.

It is particularly important that one does *not* use the Cartesian / categorical $\times$.

Our discussion around Lemma 1.2.5 then indicates a way forward: since we have constructed an object HR which represents ordinary cohomology with coefficients in R, a monoidal structure on $H^*(-;R)$ should induce some manner of monoid structure on HR. In order to make sense of this, we need a monoidal structure on $h\mathsf{Spectra}$ which is compatible with the other monoidal structures in play, in the sense that the following pair of functors should be made monoidal:

Ideally, it would even be visibly related to the original monoidal structure on R.

$$\mathsf{AbelianGroups} \xrightarrow{H} \mathsf{Spectra} \xleftarrow{\Sigma^\infty} \mathsf{Spaces}_{*/}.$$

We have already introduced the operation

$$\Sigma^\infty X \wedge \Sigma^\infty Y := \Sigma^\infty (X \wedge Y)$$

for two pointed spaces X and Y. Since $\mathsf{Spectra}$ is suitably generated by the image of Σ^∞, it should seem likely that this will pin down any putative monoidal structure.

For inspiration as to how to define the smash product in general, recall that we have also already defined $E \wedge \Sigma^\infty X$ for a generic spectrum E and a pointed space X. Given a presentation $E = \{\Sigma^{n_j} \Sigma^\infty E_j\}_j$, we set

$$E \wedge \Sigma^\infty X := \{\Sigma^{n_j} \Sigma^\infty (E_j \wedge X)\}_j.$$

That is, we commuted the smash product through the ind-system, where we reduced to the case of the smash product of suspension spectra. In the fully general case of $E \wedge F$, we may also choose a presentation of the second spectrum F as $F = \{\Sigma^{m_k} \Sigma^\infty F_k\}_k$, and then we set

$$(E \wedge F)_{j,k} := \{\Sigma^{n_j + m_k} \Sigma^\infty E_j \wedge F_k\},$$

another ind-system.

Remark 3.5.2. To define the smash product in terms of Boardman and Vogt's concrete model given in Definition 3.3.9, we must convert this doubly-indexed ind–system into a sequential system. One option is to select any cofinal subsystem, but this destroys the associativity of the product (and often destroys the commutativity). A superior option is to "sum over possible choices": we set $(E \wedge F)_n$ to be the colimit of the diagram under the n^{th} antidiagonal (after replacing the maps by cofibrations).

(?, pg. 254–267)

From here, the main task is to show that this definition is sufficiently insensitive to the choice of presentation: given a pair of weakly equivalent presentation, one must show that this induces a weak equivalence after smashing. This is possible, and hence one learns:

(?, Theorem 13.40)

Theorem 3.5.3. *In the homotopy category, this determines a symmetric monoidal product,* $\wedge$, *on* $\mathsf{Spectra}$. □

Warning: This product is *not* especially nice before passing to the homotopy category. It turns out that this is unavoidable. (?, Definition 13.50)

Definition 3.5.4. A *ring spectrum* is a spectrum E equipped with a multiplication map $\mu\colon E \wedge E \to E$ and a unit map $\eta\colon \mathbb{S} \to E$ making E into a (unital) monoid object in $h\mathsf{Spectra}$.

Example 3.5.5. The cup product maps

The statement here applies to any ring-valued theory.

$$H^n(-;R) \times H^m(-;R) \xrightarrow{\smile} H^{n+m}(-;R)$$

induce maps

$$K(R,n) \wedge K(R,m) \xrightarrow{\smile} K(R,n+m)$$

which altogether induce a product

$$HR \wedge HR \to HR.$$

Example 3.5.6. The sphere spectrum, $\mathbb{S}$, is the monoidal unit and hence also a ring.

The inverse construction is now straightforward. Suppose that we have a pair of cohomology classes $\omega_n \in E^n(X)$ and $\omega_m \in E^m(X)$, for which we choose representatives $\omega_n\colon \Sigma^\infty X \to \Sigma^n E$ and $\omega_m\colon \Sigma^\infty X \to \Sigma^m E$. Given a multiplication $\mu\colon E \wedge E \to E$, we define the product $\omega_n \smile \omega_m$ like so:

This same product can be used to make E^*X (and E_*X) into E_*–modules.

$$X \xrightarrow{\Delta} X \wedge X \xrightarrow{\omega_n \wedge \omega_m} \Sigma^n E \wedge \Sigma^m E \simeq \Sigma^{n+m}(E \wedge E) \xrightarrow{\mu} \Sigma^{n+m} E.$$

Remark 3.5.7. One can check that $- \wedge E$ preserves colimits. By positing an exponential adjunction, one can use Brown representability on the putative formula

$$F(E_1, E_2)^*(X) = \text{“}\pi_0 F(E_1 \wedge X, E_2)\text{”} = \mathsf{Spectra}(E_1 \wedge X, E_2)$$

to define a notion of *function spectrum*. As with other objects extracted from Brown's result, this definition does not have excellent functoriality properties. However, there is a version of the adjoint functor theorem that also applies to give a fully functorial statement—but this is beyond our current technology.

Of course, one can also give a direct definition.

Ring spectra induce a useful duality pairing between their associated co/homology theories.

(?, pg. 281)

Definition 3.5.8. Given co/homology classes

$$(\sigma\colon \mathbb{S}^n \to E \wedge \Sigma^\infty X) \in E_n X,$$
$$(\omega\colon \mathbb{S}^m \wedge \Sigma^\infty X \to E) \in E^m(X),$$

we define their pairing to be

$$\langle \omega, \sigma \rangle\colon \mathbb{S}^{n+m} \xrightarrow{\Sigma^m \sigma} E \wedge \mathbb{S}^m \wedge \Sigma^\infty X \xrightarrow{1\wedge\omega} E \wedge E \xrightarrow{\mu} E.$$

(?, Proposition 13.62.i)

Lemma 3.5.9. *Under this pairing, the maps f^*, f_* induced by f are adjoint:*

$$\langle f^*\omega, \sigma \rangle = \langle \omega, f_*\sigma \rangle.$$

Proof. This is a consequence of the following diagram:

$$\begin{tikzcd}
\mathbb{S}^{n+m} \arrow[r, "\Sigma^m \sigma"] \arrow[dr, "\Sigma^m f_*\sigma"'] & E \wedge \mathbb{S}^m \wedge \Sigma^\infty X \arrow[d, "1\wedge 1\wedge f"] \arrow[dr, "1\wedge f^*\omega"] & & \\
& E \wedge \mathbb{S}^m \wedge \Sigma^\infty Y \arrow[r, "\omega"] & E \wedge E \arrow[r, "\mu"] & E.
\end{tikzcd}$$

□

3.A Grassmannians

4
Computations in the Steenrod Algebra

In Section 2.5, we closed with a discussion of *k–invariants* and an urgent question about how to classify cohomology operations. We have now assembled enough tools to address this program meaningfully. In this Chapter, we will compute various collections of cohomology operations, referred to in various forms as the *Steenrod algebra*. Along the way, we will introduce a hereto-unmentioned fundamental tool for the study of spaces: the Serre spectral sequence, which grants access to the cohomology of the terms in an exact sequence of spaces. In addition to its critical role in the development of the Steenrod algebra, we will use it to compute the cohomology of some other interesting spaces (e.g., loopspaces of spheres) and to reprove some other fundamental results (e.g., Freudenthal's theorem).

For the moment, we will consider these interesting and engaging calculations to be their own reward. Only in the subsequent Chapter will we explore some of their consequences.

4.1 G–bundles and fiber bundles

A major up-shot of representability is that the tools of algebraic topology can be turned on themselves. We have previously announced our intention to understand the collection of natural transformations

$$H^n(-;A) \to H^m(-;B).$$

By appealing to representability, this is not only equivalent to the collection of homotopy classes

$$K(A,n) \to K(m,B)$$

but also to the cohomology group

$$H^m(K(A,n);B).$$

If we can get a sufficiently explicit handle on $K(A,n)$, we can use such a presentation to finish our original analysis.

There is a particularly restrictive form of fiber bundle that appears very often in geometric contexts:

(?, Definition 11.1)

Definition 4.1.1. A *(real) vector bundle (of rank k)* over a base B is a fiber bundle $p\colon E \to B$ with fiber $\mathbb{R}^k$ and whose transition maps are linear functions.

Example 4.1.2. The co/tangent bundles of a manifold are vector bundles.

This constraint on the transition maps admits a universal form:

(?, Definition 11.4)

Definition 4.1.3. A *G–bundle* is a fiber bundle $p\colon E \to B$ where G acts on E (and trivially on B, and the map p is equivariant), the identifications $\varphi_U\colon p^{-1}(U) \cong G \times B$ are equivariant, and the compatibilities $\varphi_U|_{U\cap V} = \varphi_V|_{U\cap V}$ are equivariant too.

(?, 11.21)

Remark 4.1.4. This construction is universal in the following sense: if G acts on an auxiliary space F, one can extract from a G–bundle $p\colon E \to B$ an F–fiber bundle by

$$E' = (F \times E)/(fg, e) \sim (f, ge).$$

Conversely, a fiber bundle with fiber F has an associated $(\operatorname{Aut} F)$–bundle.

(?, Theorem 11.20, Proposition 11.22)

Example 4.1.5. Real vector bundles correspond with $\mathrm{GL}(\mathbb{R}^n)$–bundles under this construction. The maximal compact subgroup of $\mathrm{GL}(\mathbb{R}^n)$ is the orthogonal group $O(\mathbb{R}^n)$, and the equivariant retraction $O(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$ gives an equivalence between $O(\mathbb{R}^n)$–bundles and $\mathrm{GL}(\mathbb{R}^n)$–bundles (and hence with real vector bundles as well).

The local nature of the definition of a vector bundle gives rise to the following observation:

(?, Proposition 11.32)

Lemma 4.1.6. *The assignment $X \mapsto \{$isomorphism classes of G–bundles on $X\}$ satisfies the wedge axiom and Mayer–Vietoris.* □

(?, 11.33)

Corollary 4.1.7. *There is a homotopy type BG representing this functor.* □

Purely through abstract principles, one can make an interesting qualitative statement about this homotopy type:

(?, Proposition 11.27, Theorem 11.35)

Lemma 4.1.8. *Let $E \to B$ be a G–bundle with E n–connected. The classifying map $B \to BG$ is then an n–equivalence. It follows that the induced natural transformation $[-, B] \to [-, BG] \to k_G(-)$ is an equivalence on complexes of dimension $\leq n$.* □

Corollary 4.1.9. *The universal bundle EG classified by* id: $BG \to BG$ *has* contractible *total space. Conversely, G–bundle with contractible total space is a model for the universal such bundle.* □

(?, 11.43)

Remark 4.1.10. With Corollary 1.6.8 and Remark 2.3.13 in mind, we can now smell the connection between these ideas and our pursuit of Eilenberg–Mac Lane spaces. Namely, Corollary 1.6.8 shows that the natural map

$$G \to \Omega BG$$

is an equivalence, so that

$$K(A, n+1) \to BK(A, n)$$

is also an equivalence.

Making sense of this requires a model of $K(A,n)$ as an honest group, rather than as an H–group. We blithely assert to the reader that such a model exists—its precise form won't turn out to be important.

We now turn to the problem of producing a reliable model of BG, using Corollary 4.1.9 as a guide. Our route, as ever, will pass through some creative category theory.

The first-time reader will probably find it easier to conceptualize the following under the further condition that G be finite (e.g., $\mathbb{Z}/2$).

From here on, you owe the reader citations.

Definition 4.1.11. Let C be a category. Its *nerve* $N(\mathsf{C})$ is a simplicial set with 0–simplices the objects of C, 1–simplices the arrows of C, 2–simplices commuting triangles, 3–simplices commuting tetrahedra, ...

Equivalently: the n–simplices are given by length n chains of composable morphisms.

Remark 4.1.12. This construction is a very faithful encoding of a category: the original category and its composition law can be recovered from the 0–. 1–, and 2–simplices. It also translates categorical ideas to recognizable topological objects: for instance, functors become continuous maps and natural transformations become homotopies of maps.

Inject a comment about being able to extend this business to topologically enriched categories too.

Example 4.1.13. For G a group, we define two categories:

1. $G//G$ has objects $g \in G$ and maps $g \xrightarrow{h} gh$.
2. $*//G$ has one object $*$ and maps $* \xrightarrow{h} *$.

Lemma 4.1.14. *$G//G$ is contractible.*

Proof sketch. This amounts to showing that any "outer horn" (i.e., a chain of morphisms of length $n-1$ and a morphism with either the same ultimate source or same ultimate target) is "fillable" (i.e., there is a chain of morphisms of length n which extends the original chain and whose composite is the auxiliary morphism). This is so: given

$$g_1 \xrightarrow{g_2} g_1 g_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} g_1 \cdots g_n$$

and

$$g_1 \xrightarrow{h} h_{n+1},$$

we can set $g_{n+1} = g_n^{-1} \cdots g_1^{-1} h_{n+1}$ to get

$$g_1 \xrightarrow{g_2} g_1 g_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} g_1 \cdots g_n \xrightarrow{g_{n+1}} h_{n+1}. \qquad \square$$

Remark 4.1.15. The G–action on $G//G$ is *free*.

Corollary 4.1.16. *The quotient map $N(G//G) \to N(*//G)$ has fiber G, hence it models models $EG \to BG$.* □

Remark 4.1.17. This is a more conceptual statement than you might think. There are equivalences $G//G \to \{G\text{–torsors with a trivialization}\}$ and $*//G \to \{G\text{–torsors}\}$, and a map $X \to N(\{G\text{–torsors}\})$ for X a simplicial set assigns each point in X to a G–torsor, each path to a map of torsors, ... This *sounds* like it's building a G–bundle on X by specifying the fibers. The *Grothendieck construction* makes this precise.

This very concrete model for BG has one really excellent feature: is has a naturally occurring skeletal filtration (viz., by simplex dimension) with identifiable quotients:

$$\begin{tikzcd} BG^{(0)} \arrow[r] \arrow[d, equal] & BG^{(1)} \arrow[r] \arrow[d] & BG^{(2)} \arrow[r] \arrow[d] & \cdots \arrow[r] & BG^{(n-1)} \arrow[r] \arrow[d] & BG^{(n)} \arrow[r] \arrow[d] & \cdots \\ * & \Sigma G & (\Sigma G)^{\wedge 2} & \cdots & (\Sigma G)^{\wedge(n-1)} & (\Sigma G)^{\wedge n} & \cdots . \end{tikzcd}$$

If h is a homology theory with Künneth isomorphisms, this gives a spectral sequence

$$E^1_{*,*} = (\widetilde{h}_* \Sigma G)^{\otimes *} \Rightarrow h_* BG.$$

More than this, the d^1–differential in then identifiable:

$$d_1(g_1 \otimes \cdots \otimes g_n) = \sum_{j=2}^{n} (g_1 \otimes \cdots \otimes g_{j-1} g_j \otimes \cdots \otimes g_n).$$

This is a standard resolution appearing in homological algebra:

$$E^2_{*,*} = \mathrm{Tor}^{h_* G}_{*,*}(h_*, h_*).$$

This is a very common situation: some "fully derived" construction appearing in homotopy theory has behavior mediated by analogous homological algebra and a spectral sequence. Since $\mathrm{Tor}^{h_* G}(h_*, h_*) = \pi_*(h_* \otimes^{\mathbb{L}}_{h_* G} h_*)$, this leads one to think of BG as some kind of $* \times_G *$. This turns out to be fruitful.

4.2 *The Steenrod algebra: calculation*

Throughout today, you owe the reader citations.

Today we put the machinery of yesterday to work in the case of $H^m(K(\mathbb{F}_2, n); \mathbb{F}_2)$. Our method is *inductive*, and it ultimately rests on the following key observations:

Example 4.2.1. Tor algebras are generally remarkably computable: there is an algorithm, due to Tate, which forms a resolution of h_* by a DGA which is levelwise $(h_* G)$–free. To cover the algebras appearing in this computation, we will only need the following observations:

Cite

$$\mathrm{Tor}^{A \otimes B}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Tor}^{A}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Tor}^{B}_{*,*}(\mathbb{F}_2, \mathbb{F}_2), \quad \mathrm{Tor}^{\Lambda[x]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \bigotimes_{j=0}^{\infty} \Gamma[\sigma x],$$

where A and B are $\mathbb{F}_2$–algebras, $\Lambda[x]$ denotes an exterior $\mathbb{F}_2$–algebra generated by the lone element x, and $\Gamma[\sigma x]$ denotes a divided power $\mathbb{F}_2$–algebra generated by the suspension of the element x.

(?, Theorem 7.24)

Lemma 4.2.2. *The pairing* $K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, 1) \xrightarrow{\smile} K(\mathbb{F}_2, n+1)$ *induces a pairing "$\circ$" of spectral sequences*

$$\begin{array}{ccc} \mathrm{Tor}_{*,*}^{H_*(K(\mathbb{F}_2,n);\mathbb{F}_2)} \otimes H_*(K(\mathbb{F}_2,1);\mathbb{F}_2) & \xrightarrow{\circ} & \mathrm{Tor}_{*,*}^{H_*(K(\mathbb{F}_2,n+1);\mathbb{F}_2)} \\ \Downarrow & & \Downarrow \\ H_*(K(\mathbb{F}_2,n+1);\mathbb{F}_2) \otimes H_*(K(\mathbb{F}_2,1);\mathbb{F}_2) & \xrightarrow{\smile} & H_*(K(\mathbb{F}_2,n+2);\mathbb{F}_2) \end{array}$$

which converges to the cup product and which satisfies $d(x \circ y) = (dx) \smile y$. □

We begin with the base case:

(?, Proof of 8.11)

Lemma 4.2.3. *The spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}_{*,*}^{H_*(\mathbb{F}_2;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(K(\mathbb{F}_2,1);\mathbb{F}_2)$$

collapses to give

$$H_*(K(\mathbb{F}_2,1);\mathbb{F}_2) \cong \Gamma[\sigma a].$$

Proof. We analyze the input to the spectral sequence

$$E^2_{*,*} = \mathrm{Tor}_{*,*}^{H_*(\mathbb{F}_2;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(K(\mathbb{F}_2,1);\mathbb{F}_2).$$

The homology algebra $H_*(\mathbb{F}_2;\mathbb{F}_2)$ can be equivalently presented as

$$H_*(\mathbb{F}_2;\mathbb{F}_2) \cong \mathbb{F}_2[\underline{1}]/(\underline{1}^2 = 1) \cong \mathbb{F}_2[\underline{1}-1]/(\underline{1}-1)^2.$$

Since this algebra is exterior, we may compute

$$\mathrm{Tor}_{*,*}^{H_*(\mathbb{F}_2;\mathbb{F}_2)} \cong \Gamma[\sigma a],$$

for $a = \underline{1} - 1$. The homology groups of $K(\mathbb{F}_2, 1) \simeq \mathbb{RP}^\infty$ have one class in every degree. The bar spectral sequence also has one class in every degree, and there can therefore be no nonzero differentials, so that the spectral sequence collapses at E_2. To establish convention, we write

$$\Gamma[\sigma a] \cong \mathbb{F}_2[a_{(0)}, a_{(1)}, a_{(2)}, \ldots]/(a_{(j)}^2 = 0)$$

for the algebra generators. □

(?, Theorem 8.11)

Theorem 4.2.4. *The above Lemma generalizes in n to give*

$$H_*(K(\mathbb{F}_2, n);\mathbb{F}_2) \cong \mathbb{F}_2[a_{(j_1)} \circ \cdots \circ a_{(j_n)}]/(\textit{squares}).$$

Proof sketch. We proceed by induction, having shown the claim in the case $n = 0$. By assumption, $H_* K(\mathbb{F}_2, n)$ is a tensor product of exterior algebras, so the Künneth formula for Tor–algebras gives

$$\begin{aligned} \mathrm{Tor}_{*,*}^{H_* K(\mathbb{F}_2,n)}(\mathbb{F}_2,\mathbb{F}_2) &\cong \bigotimes_J \mathrm{Tor}_{*,*}^{\Lambda[a_{(j_1)} \smile \cdots \smile a_{(j_n)}]}(\mathbb{F}_2,\mathbb{F}_2) \\ &\cong \bigotimes_J \Gamma[a_{(j_1)} \smile \cdots \smile a_{(j_n)}]. \end{aligned}$$

One can show the identity

Wilson 8.16

$$(a_{(J)})_{(k)} \equiv a_{(J)} \smile a_{(k)} \quad \text{(mod decomposables)}.$$

It follows that there are no differentials, since the spectral sequence for $H_*(K(\mathbb{F}_2,1);\mathbb{F}_2)$ had none. □

This has a great many consequences.

(?, Theorem 18.14)
In terms of the operations Sq^n below, $H^*(K(\mathbb{F}_2,n);\mathbb{F}_2)$ is given by the algebra $\mathbb{F}_2[\mathrm{Sq}^I \iota_n \mid I_j \geq 2I_{j+1}, 2I_1 - I_+ < n]$.

Corollary 4.2.5. *On cohomology, we have the calculation*

$$H^*(K(\mathbb{F}_2,n+1);\mathbb{F}_2) \cong \mathbb{F}_2[a_{(j_1)} \circ \cdots \circ a_{(j_n)}].$$

Proof sketch. One employs that the dual of a primitively-generated divided-power Hopf algebra is a primitively-generated polynomial Hopf algebra. □

This finishes the task we set out for ourselves at the beginning of this excursion, but we can collect a bit more at the level of spectra.

(?, Theorem 18.20)

Corollary 4.2.6. *The dual Steenrod algebra is given by*

$$\mathscr{A}_* := H\mathbb{F}_{2*}H\mathbb{F}_2 \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_n, \ldots],$$

where $|\xi_n| = 2^n - 1$ *is represented by* $a_{(n)} \in H_{2^n-1}(\Sigma^{-1}K(\mathbb{F}_2,1);\mathbb{F}_2)$*.*

Proof. From our definition of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, we have

$$H\mathbb{F}_2 \wedge H\mathbb{F}_2 \simeq H\mathbb{F}_2 \wedge (\operatorname{colim}_n \Sigma^{-n}K(\mathbb{F}_2,n)) \simeq \operatorname{colim}(H\mathbb{F}_2 \wedge \Sigma^{-n}K(\mathbb{F}_2,n)),$$

so that applying π_m gives

$$(H\mathbb{F}_2)_m H\mathbb{F}_2 = \lim_n H_{m-n}(K(\mathbb{F}_2,n);\mathbb{F}_2).$$

The Theorem gives us access to these groups, provided we can describe the maps participating in the colimit. These maps turn out to be $- \smile a_{(0)}$, owing to the factorization

$$\begin{tikzcd} S^1 \wedge K(\mathbb{F}_2,n) \arrow[rr] \arrow[dr, "a_{(0)} \times \mathrm{id}"'] & & K(\mathbb{F}_2,n+1) \\ & K(\mathbb{F}_2,1) \wedge K(\mathbb{F}_2,n). \arrow[ur, "\smile"'] & \end{tikzcd}$$

□

The analogous formula for stable cohomology is harder: the answer as a coalgebra is encoded in the above, but to produce a description as a Hopf algebra we need to understand the comultiplication on $\mathscr{A}_*$. This is more complicated, so we merely quote the result:

Lemma 4.2.7. *The comuliplication on $\mathscr{A}_*$ is given by*

$$\Delta\xi_n = \sum_{j=0}^{n} \xi_j \otimes \xi_{n-j}^{2^j}.$$

The primitive elements of this algebra are $\xi_1^{2^j}$. □

(?, Theorem 18.20)
Many more such formulas can be read off, e.g., the Hopf algebra antipode.

Corollary 4.2.8. *The Steenrod algebra, $\mathscr{A}^* := H\mathbb{F}_2^* H\mathbb{F}_2$, is noncommutative and generated by elements Sq^{2^j} dual to $\xi_1^{2^j}$.* □

(?, pg. 451)

Remark 4.2.9. A lot can be computed about $\mathscr{A}^*$ by studying universal cases. For instance, $\Delta(x^2)^* = 1 \mid (x^2)^* + \xi_1 \mid (x)^* \in \mathscr{A}_* \otimes \widetilde{H}_* \mathbb{R}\mathrm{P}^\infty$ says $\mathrm{Sq}^0(x^2) = x^2$ and $\mathrm{Sq}^1(x) = x^2$ for $|x| = 1$. In fact, we have

The space-level versions of these calculations are called the *unstable (dual) Steenrod algebra*.

1. $\mathrm{Sq}^0(x) = x$.
2. $\mathrm{Sq}^{>|x|}(x) = 0$.
3. $\mathrm{Sq}^{|x|}(x) = x^2$.
4. $\mathrm{Sq}^n(x+y) = \mathrm{Sq}^n x + \mathrm{Sq}^n y$.
5. $\mathrm{Sq}^n(xy) = \sum_{n_1+n_2=n} \mathrm{Sq}^{n_1}(x) \cdot \mathrm{Sq}^{n_2} y$.
6. "The Adem relations", summarized by
   (a) $\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0$, and
   (b) $d(\mathrm{Sq}^n) = \mathrm{Sq}^{n-1}$ extends to a derivation.

   So, for instance,

$$\begin{aligned} 0 &= d^3(\mathrm{Sq}^5\mathrm{Sq}^3) \\ &= d(\mathrm{Sq}^3\mathrm{Sq}^3 + \mathrm{Sq}^5\mathrm{Sq}^1) \\ &= \mathrm{Sq}^2\mathrm{Sq}^3 + \mathrm{Sq}^3\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^5\mathrm{Sq}^0. \end{aligned}$$

Remark 4.2.10. By defining the Steenrod algebra as an endomorphism algebra and enunciating its structure, we have given ourselves a powerful tool. Namely, mod–2 cohomology is naturally valued in representations for this algebra, which puts serious constraints even on which algebras can be realized as the cohomology of a space. For example, we know

$$H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_1], \qquad H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x_2]$$

for classes x_1 and x_2 in degrees 1 and 2 respectively. However, $\mathbb{F}_2[x_3]$ for a class in degree 3 cannot be realized as the cohomology of a space, as it is precluded by the Adem relation

$$x_3^2 = \mathrm{Sq}^3 x_3 = \mathrm{Sq}^1\mathrm{Sq}^2 x_3 = \mathrm{Sq}^1(0) = 0.$$

4.3 The Serre spectral sequence

In this Lecture, we introduce one of the most important computational tools in algebraic topology. We have spent a good deal of time emphasizing the fundamental compatibilities between homotopy groups and exact sequences as well as between co/homology groups and coexact sequences. It turns out that a fibration over a CW–complex admits a filtration whose quotients have tractable co/homology, resulting in a spectral sequence governing the co/homology of an *exact* sequence. The ability to access this information is sufficiently exotic that it is typically well-worth the price of contending with a spectral sequence.

Definition 4.3.1. Fix a fibration over a base CW–complex B, as in

$$F \to E \xrightarrow{p} B.$$

By assumption, B comes with a skeletal filtration whose associated-graded is a wedge of spheres (viz., its cells). Pulling back the cellular filtration on B yields the *Serre filtration* on E:

$$E_n = p^{-1}(B^{(n)}).$$

The associated-graded of this filtration takes the form

$$E_n/E_{n-1} \simeq (B^{(n)}/B^{(n-1)}) \times F = \bigvee_\alpha S^n_\alpha \times F.$$

Applying a homology theory h_* gives the *Serre spectral sequence*.

Theorem 4.3.2 (Serre). *If $\pi_1 B$ acts trivially on $h_* F$, then $d^1 = d^{\mathrm{cell}}$, i.e., there is an identification as chain complexes*

$$E^1_{*,*} = h_*(E_*, E_{*-1}) = C^{\mathrm{cell}}_*(B) \otimes h_*(F).$$

For instance, $\pi_1 B$ acts trivially on $h_* F$ if $\pi_1 B = 0$. There is a version of this without the hypothesis on $\pi_1 B$, where "local coefficients" / "twisted homology" are used. We won't need it.

The E^2–page is then given by

$$E^2_{p,q} = H_p(B; h_q F) \Rightarrow h_{p+q}(E)$$

If h_* is a field, then $H_*(B; h_* F) \cong H_*(B; h_*) \otimes_{h_*} h_* F$.

with differential

$$d^r : E^r_{p,q} \to E^r_{p-r-1,q+r}. \quad \square$$

(?, Example 15.32)

Example 4.3.3. It is remarkable how much this automates. To see what we mean, we consider as an example the fibration

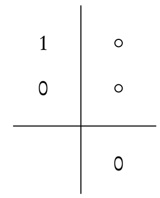
$$S^1 \to S^{1+2\cdot 2} \to \mathbb{C}\mathrm{P}^2,$$

and suppose for the sake of argument that we do not know the homology of $\mathbb{C}\mathrm{P}^2$. The associated spectral sequence has E^2–page given by

$$E^2_{p,q} = H_p(\mathbb{C}\mathrm{P}^2; H_q S^1),$$

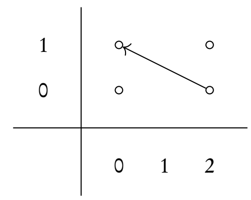
which mostly consists of groups that we do not know, since we're ignorant of the homology of $\mathbb{C}P^2$. However, it is easy enough to see that $\mathbb{C}P^2$ is connected, so we learn

$$H_0(\mathbb{C}P^2; H_q S^1) \cong \begin{cases} \mathbb{Z} & \text{when } q = 0, \\ \mathbb{Z} & \text{when } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

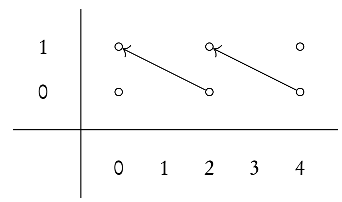


From this, we start to paint the spectral sequence.

This cannot be the whole story: this spectral sequence is meant to converge to $H_*(S^5)$, and the class in position $(0,1)$ would contribute a class to $H_1 S^5$ if it were not to participate in a differential. Since $H_1 S^5 = 0$, we know that such a differential must exist. There is only one option: the earliest page on which a differential could appear is E^2, and any later page would require the differential to exit the first quadrant, hence there must exist a class in degree $(2,0)$ which sources as the source of our sought-after differential.



Of course, a class in degree $(2,0)$ must come from a class in $H_2(\mathbb{C}P^2; H_0 S^1)$, which is then also present in $H_2(\mathbb{C}P^2; H_1 S^1)$ and hence also contributes a class to the spectral sequence in degree $(2,1)$. This new class suffers from the same problem: if it were to survive, it would contribute a nonzero summand to $H_3 S^5$, which is not permitted to exist. The solution is the same as well: there must exist a class in degree $(4,0)$ which participates in a differential.



With the addition of this new class, we are no longer in contradiction: its survival would contribute a nonzero summand to $H_5 S^5$, which we *require*, and so it cannot participate in a differential. By indirect consequence, we have achieved a calculation of $H_* \mathbb{C}P^2$, by examining the bottom row of the E^2–page:

There is a little more to explore here for the reader to be fully convinced that there are no features of the spectral sequence missing from our description: no classes in odd degrees, no extra classes in the degrees we've considered, no classes above the row $q = 1, \ldots$. The reader is highly encouraged to chase through these thought experiments.

$$E^2_{p,0} \cong H_*(\mathbb{C}P^2; H_0 S^1) \cong H_*(\mathbb{C}P^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{when } * = 0, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

There is nothing particular about the dimension of $\mathbb{C}P^2$ in this example: one can just as well compute the homology of $\mathbb{C}P^n$, or even of $\mathbb{C}P^\infty$, using the same methods.

4.4 *The Serre spectral sequence in cohomology*

Let us now exploit the naturality of the Serre spectral sequence to put it to real work. Namely, naturality gives rise to the following critical observations:

Part of the fun of spectral sequences in algebraic topology, as opposed to their appearances in other fields, is that we don't stop at filtering our problems away into differentials—we then compute the differentials.

Lemma 4.4.1. *The filtration used to form the Serre spectral sequence is* multiplicative, *in the sense that for $F \to E \to B$ and $F' \to E' \to B'$ fiber sequences, we have $E_n \times E'_m \subseteq (E \times E')_{n+m}$, inducing a pairing $E_r(E) \otimes E_r(E') \to E_r(E \otimes E')$.* □

(?, Remark 15.4)
Everything about this fact is nearly automatic, except the Leibniz law, which comes out of thinking about the cell structure on $(D^n, \partial D^n) \times (D^m, \partial D^m)$.

Corollary 4.4.2. *The cohomological Serre spectral sequence has a multiplication:*

$$E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}.$$

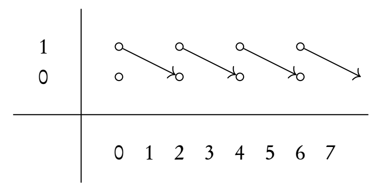
This product restricts on the edges to the cup products on H^*F and H^*B, and it converges to cup on h^*E. If x generates $E_2^{p,0}$ and y generates $E_2^{0,q}$, then xy generates $E_2^{p,q}$. Additionally, it satisfies a Leibniz law for the differentials:

$$d_r(xy) = (d_r x)y + (-1)^{|x|}x(d_r y). \quad \square$$

(?, Example 15.32)

Example 4.4.3. As a jumping-off point, let's consider the interplay between Corollary 4.4.2 and Example 4.3.3, where we considered the exact sequence

$$S^1 \to * \to K(\mathbb{Z}, 2).$$

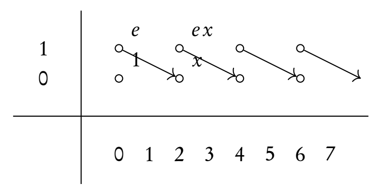


Simply by repeating the argument made there for the cohomological Serre spectral sequence, we can produce a full description of the integral cohomology spectral sequence as a system of abelian groups.

We will now use Corollary 4.4.2 to deduce the structure of $H^*(K(\mathbb{Z}, 2); \mathbb{Z})$ as a *ring*. For covenience, we begin by naming some of the generators in the lower-left corner:

- We use 1 to name the generator of $H^0(K(\mathbb{Z}, 2); H^0(S^1; \mathbb{Z})) \cong H^0(K(\mathbb{Z}, 2); \mathbb{Z})$.
- We use e to name the generator of $H^0(K(\mathbb{Z}, 2); H^1(S^1; \mathbb{Z})) \cong H^1(S^1; \mathbb{Z})$.
- We use x to name the generator of $H^2(K(\mathbb{Z}, 2); H^0(S^1; \mathbb{Z})) \cong H^2(K(\mathbb{Z}, 2); \mathbb{Z})$.

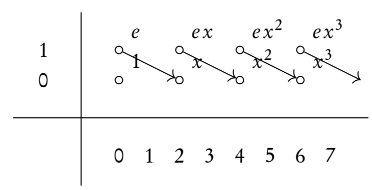
By consequence, we learn that the term in $E_2^{2,1}$ is generated by the product class ex. We include these names in the portrait of the spectral sequence:



Corollary 4.4.2 also provides a tool for computing the differential on such product classes as ex:

$$\begin{aligned} d_2(ex) &= d_2(e)x + (-1)^{|e|} e d_2(x) \\ &= x \cdot x + (-1) \cdot e \cdot 0 = x^2. \end{aligned}$$

Since we have already separately argued that $d_2(ex)$ generates $E_2^{4,0}$, this calculation with the Leibniz rule informs us that x^2 generates $E_2^{4,0}$. In general, $d_2(ex^m) = x^{m+1}$, so that we can give similar names to all of the classes in the bottom row: This totally describes the product structure on the E_2–page of the Serre spectral sequence. Corollary 4.4.2 guarantees that this restricts to the cup product structure on $H^*(K(\mathbb{Z}, 2); \mathbb{Z})$, so that we ultimately learn



$$H^*(K(\mathbb{Z}, 2); \mathbb{Z}) \cong \mathbb{Z}[x].$$

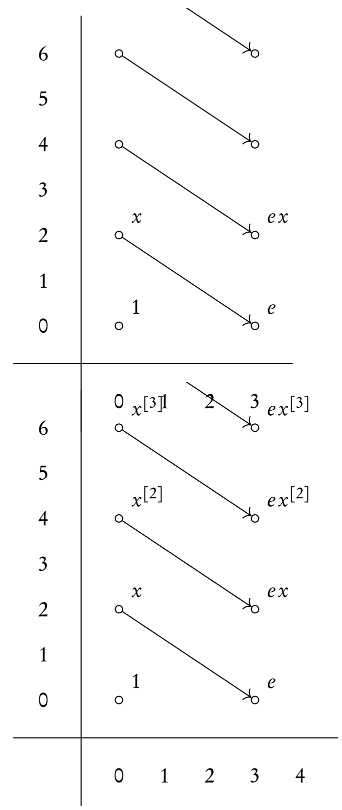
(?, Exercise 15.36)

Example 4.4.4. We now turn to a slightly more complex example: the exact sequence

$$\Omega S^3 \to * \to S^3.$$

Again, by arguing as in Example 4.3.3, we can produce a description of the cohomological Serre spectral sequence as a system of abelian groups. Beginning to argue the same as in Example 4.4.3, we give names to some of the generators in the lower-left corner:

- We use 1 to name the generator of $H^0(S^3;H^0(\Omega S^3;\mathbb{Z}))\cong H^0(S^3;\mathbb{Z})$.
- We use e to name the generator of $H^0(S^3;H^1(\Omega S^3;\mathbb{Z}))\cong H^3(S^3;\mathbb{Z})$.
- We use x to name the generator of $H^2(S^3;H^0(\Omega S^3;\mathbb{Z}))\cong H^2(\Omega S^3;\mathbb{Z})$.
- By consequence, ex names the generator of $E_2^{3,2}=H^2(S^3;H^3(\Omega S^3;\mathbb{Z}))$.



(?, Exercise 15.36)

At this point, however, the analysis diverges from that of Example 4.4.3. The Leibniz law for the differential gives

$$\begin{aligned} d_3(x^2) &= d_3(x)\cdot x+(-1)^{|x|}x\cdot d_3(x) \\ &= e\cdot x+x\cdot e=2ex. \end{aligned}$$

However, we know that d_3 is an isomorphism which maps the generator of $E_3^{0,4}$ to the generator of $E_3^{3,2}$. The class x^2 therefore cannot generate $E_3^{0,4}$—instead, there is a class $x^{[2]}$ satisfying $2x^{[2]}=x^2$ which generates it. Continuing this process, we find that $E_3^{0,2n}$ is generated by a class $x^{[n]}$ satisfying $n!x^{[n]}=x^n$. This ring is called a *divided power algebra*, and one usually writes

$$H^*(\Omega S^3;\mathbb{Z})\cong\Gamma_{\mathbb{Z}}[x],$$

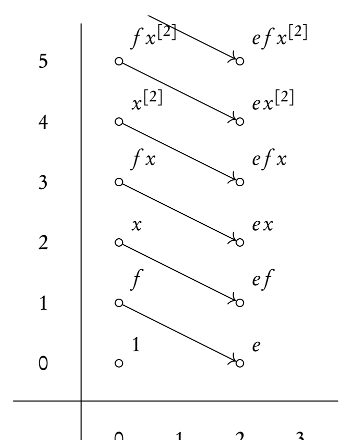
where x has degree 2.

Example 4.4.5. Let us now consider the superficially similar exact sequence

$$\Omega S^2\to *\to S^2.$$

Again, the analysis proceeds identically to that in Example 4.4.3 up until we begin to trace the behavior of classes under the multiplication map. The first difference to note is that any choice of generator f of the torsion-free group $E_2^{0,1}\cong H^1(\Omega S^2;\mathbb{Z})$ satisfies $f\cdot f=-f\cdot f$, hence is 2–torsion, hence must be zero as an element of $E_2^{0,2}$. Instead, the generator $x\in E_2^{0,2}$ satisfying $d_2(x)=ef$ in the spectral sequence is nonetheless algebraically independent of f in the ring $H^*(\Omega S^2;\mathbb{Z})$. The square of this new generator x is not required to be 2–torsion, and an identical calculation to the previous case shows that $x^{[2]}$ generates $E_2^{0,4}$. Ultimately, we learn

$$H^*(\Omega S^2;\mathbb{Z})\cong\Gamma[x]\otimes\Lambda[f],$$



(?, Exercise 15.36)

where $\Lambda[f]$ denotes the exterior algebra on the class f, the degree of f is 1, and the degree of x is 2.

Remark 4.4.6. In fact, this dichotomy between ΩS^{2n} and ΩS^{2n+1} continues:

$$H^*(\Omega S^{2n+1};\mathbb{Z})\cong\Gamma_{\mathbb{Z}}[x],\qquad H^*(\Omega S^{2n};\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[f]\otimes\Gamma_{\mathbb{Z}}[x].$$

In the first case, x has degree $2n$. In the second case, x has degree $4n-2$ and f has degree $2n-1$.

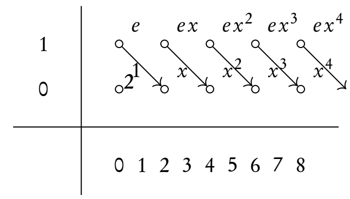
Example 4.4.7. The Serre spectral sequence of a spherical fibration is commonly called its *Gysin sequence*. The long exact sequence of infinite loop spaces

$$K(\mathbb{Z},1) \to K(\mathbb{Z}/2,1) \to K(\mathbb{Z},2) \xrightarrow{2} K(\mathbb{Z},2)$$

gives rise to a pullback square of fibrations:

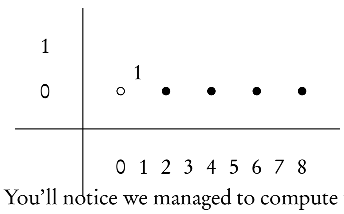
$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, equal] & \mathbb{R}\mathrm{P}^\infty \arrow[r] \arrow[d] & BS^1 \arrow[d, "\cdot 2"] \\ S^1 \arrow[r] & * \arrow[r] & BS^1. \end{tikzcd}$$

(?, 15.30)



In turn, the naturality of the Serre spectral sequence from Example 4.4.3 shows that the differential in the Serre spectral sequence for the top fibration has d_2–differential given by $d_2(e) = 2x$. From this, we learn

$$H^*\mathbb{R}\mathrm{P}^\infty \cong \mathbb{Z}[x]/(2x).$$



You'll notice we managed to compute the cohomology of the non-simply-connected space $\mathbb{R}\mathrm{P}^\infty$ by finding it in a position other than the base. The Serre spectral sequence theorem we stated does *not* apply to $C_2 \to * \to BC_2$.

(?, Remark 15.5)

4.5 *The cohomology of Eilenberg–Mac Lane spaces*

Remark 4.5.1. The construction is *natural* against maps of fiber sequences (by using cellular base maps), inducing maps of spectral sequences. There are edge homomorphisms induced by

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[r] \arrow[d] & * \arrow[d] \\ F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, equal] \\ * \arrow[r] & B \arrow[r] & B \end{tikzcd}$$

which converge to the behavior of $h_*F \to h_*E \to h_*B$.

Are there filtration effects?

(?, Theorem 15.46)

Theorem 4.5.2 (Freudenthal)**.** *For X an n–connected space, the unit map $X \to \Omega\Sigma X$ is an n–equivalence.*

Proof. Consider the pathspace fibration

$$\Omega\Sigma X \to P\Sigma X \to \Sigma X$$

and its associated Serre spectral sequence. Because this spectral sequence is converging to the homology of a contractible space, every class present must participate in a differential. Because of the vanishing regions, the differential

inject picture

$$d^r : H_r\Sigma X \xrightarrow{\cong} H_{r-1}\Omega\Sigma X,$$

which runs from the bottom row to the left column, must be an isomorphism for $r \le 2n+2$. Note also that we can reidentify the source of d^r by

$$H_{r-1}X \xrightarrow{\sigma,\cong} H_r\Sigma X \xrightarrow{d^r,\cong} H_{r-1}\Omega\Sigma X.$$

Now we need only identify the behavior of this map with the unit.

The unit participates in the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ \Omega\Sigma X \arrow[r] & P\Sigma X \arrow[r] & \Sigma X, \end{tikzcd}$$

where the top row is a coexact sequence and the bottom row is exact. These maps induce the following diagram of relative homology groups:

$$\begin{tikzcd} \widetilde{H}_r X \arrow[d] & H_{r+1}(CX, X) \arrow[l, "\partial"', "\cong"] \arrow[r, "\cong"] \arrow[d] & \widetilde{H}_{r+1}(\Sigma X) \arrow[d, equal] \\ \widetilde{H}_r(\Omega\Sigma X) & H_{r+1}(P\Sigma X, \Omega\Sigma X) \arrow[l, "\partial"] \arrow[r] & H_{r+1}(\Sigma X), \arrow[ll, bend left, "\text{transgressive } d^r"] \end{tikzcd}$$

where the bottom curved arrow is the transgressive differential of the Serre spectral sequence. Since the outer arrows (excepting the leftmost vertical arrow) are all isomorphisms and the diagram commutes, the leftmost vertical arrow is an isomorphism as well—the desired conclusion. □

Remark 4.5.3. More generally, suppose that $F \to E \to B$ is an exact sequence in which each space is n–connected. Then the natural map f in the diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d, equal] & E \arrow[r] \arrow[d, equal] & B \\ F \arrow[r, "j"] & E \arrow[r] & C(j) \arrow[u, "f"] \end{tikzcd}$$

is a $2n$–equivalence.

Make sure you got the bounds right here.

(?, pg. 447)

Example 4.5.4. Having sufficiently warmed up, we now turn to the analysis of a spectral sequence which *does not collapse on any page*: the mod–2 cohomology Serre spectral sequence for the exact sequence

$$K(\mathbb{F}_2, 1) \to * \to K(\mathbb{F}_2, 2).$$

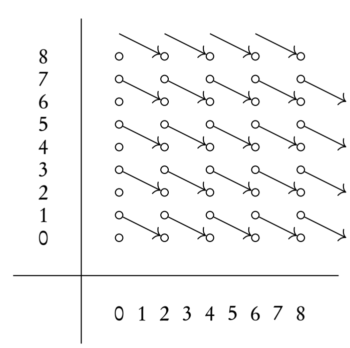
In Lemma 4.2.3, we previously computed

$$H^*(K(\mathbb{F}_2, 1); \mathbb{F}_2) \cong \mathbb{F}_2[x],$$

where $|x| = 1$. This information fills out the left-most column of the spectral sequence with a polynomial tower of classes.

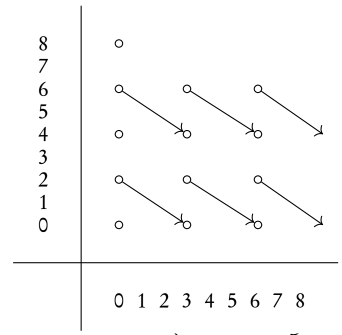
Because the spectral sequence is converging to the cohomology of a point, we must find differentials which clear this tower. The first class x requires the presence of a class $y_2 \in E_2^{2,0}$ with $d_2(x) = y$. In fact, one can show that all the classes $x^m y_2^n$ exist, and the Leibniz law gives

$$d_2(x^m y_2^n) = m x^{m-1} y_2^n \equiv \begin{cases} x^{m-1} y_2^n & \text{when } m \text{ is odd,} \\ 0 & \text{when } m \text{ is even.} \end{cases}$$



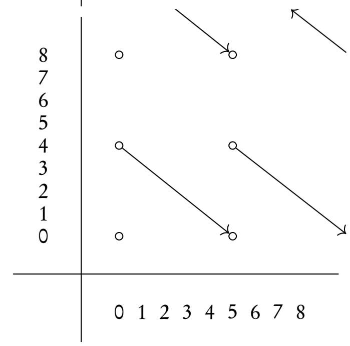
This yields "ladders" of differentials connecting odd rows to even rows.

When these differentials clear, the subring $\mathbb{F}_2[x^2]$ still remains in the left-most column. The first opportunity to clear the class x^2 is to introduce a class y_3 with $d_3(x^2) = y_3$. Again, all classes of the form $x^{2m}y_3^n$ exist and participate in ladders of differentials.



This pattern continues: each class x^{2^j} requires a class y_{1+2^j} participating in a differential $d_{1+2^k}(x^{2^j}) = y_{1+2^j}$. Altogether, these assemble into an infinite-dimensional polynomial algebra on the bottom row of the spectral sequence:

$$H^*(K(\mathbb{F}_2,2);\mathbb{F}_2) \cong \mathbb{F}_2[y_2, y_3, y_5, y_9, \ldots].$$

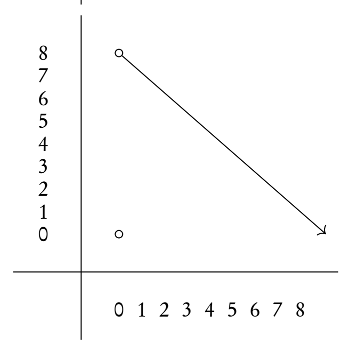


Example 4.5.5. Similarly, one can use the exact sequence

$$K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$$

to calculate $H^*(K(\mathbb{Z},3);\mathbb{F}_2)$. The main difference from the previous example is that, since $H^*(K(\mathbb{Z},2);\mathbb{F}_2)$ is even-concentrated, there is no action on the E_2–page of the spectral sequence. Accounting for this, we find

$$H^*(K(\mathbb{Z},3);\mathbb{F}_2) \cong \mathbb{F}_2[y_3, y_5, y_9, \ldots] = \mathbb{F}_2[y_{1+2^j} \mid j \geq 1].$$



Example 4.5.6. The elegance of Example 4.5.4 is something special to cohomology with field coefficients. Let us consider what happens with Serre spectral sequece associated to cohomology with integral coefficients for the exact sequence

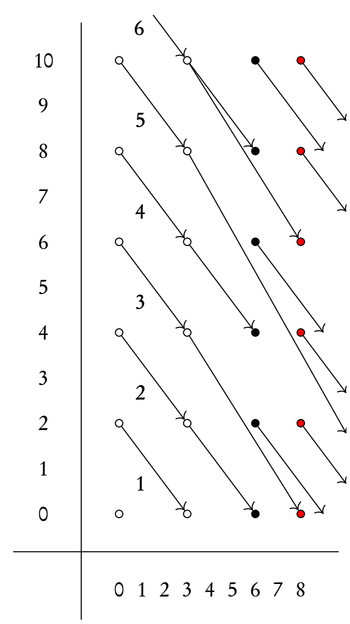
$$K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3).$$

Writing $H^*(K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z}[x]$, the same reasoning as before dictates the presence of a class y wtih $d_2 x = y$. The Leibniz law then gives

$$d_2 x^n = n x^{n-1} y,$$

where now all of these targets are nonzero. This differential is injective, so all the classes on the leftmost column die. In the third column, it leaves residue classes $\langle x^{n-1}y\rangle \cong C_n$. This page is also the last chance for the class xy to die, so the class y^2 must exist, which automatically satisfies $\langle y^2\rangle = C_2$. Similarly, y^n exists for all n, and each is 2–torsion. These classes participate in ladders of differentials, so that all of the classes $x^n y^m$, $m \geq 2$, are killed. At $m = 2$, what is left is

$$\langle x^{n-1}y\rangle = \begin{cases} C_n & \text{if } n \text{ odd}, \\ C_{n/2} & \text{if } n \text{ even}. \end{cases}$$



Continuing, there must exist a class z_3 with $d_5(x^2y) = z_3$. There are no mixed products $y^2 z_3$, but all other products are present. By considering their degrees, one sees that some must participate in short differentials before E_5. There's no end in sight.

This sseq has a TODO in it: a differential is unjustified, and I think the $\mathbb{Z}/4$ class isn't fully treated.

This all feels a little ass-backwards / convoluted. We can calculate the un/stable homology cooperations. We asserted something about the stable cohomology operations. We can prove Kudo's theorem. I guess here we're going to deduce something about unstable cohomology operations.

In Section 4.2, we gave a calculation of $H_*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ as n ranged, as well as a calculation of $H_*(H\mathbb{F}_2; \mathbb{F}_2)$ and an assertion about its dual, the Steenrod algebra. In Section 4.5, we gave a separate calculation of $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$. We now tie these two calculations together in the form of *Kudo's theorem*.

(?, pg. 80)

Definition 4.5.7. When we were investigating Theorem 4.5.2, we came across *transgressive differentials*. Namely, in the Serre spectral sequence for an exact sequence

The transgression already arose. Move this definition up?

$$F \to E \to B,$$

the edge-to-edge differential

$$d_n \colon H^0(B; H^n F) \to H^{n+1}(B; H^0 F)$$

is called the *transgression*. Unwinding the definition of the Serre differential, one finds that it is given by the maps

There might be an extra condition about other differentials exiting the column needing to be zero…?

$$(B, b_0) \xleftarrow{p} (E, F) \xrightarrow{j} (E \cup CF, CF) \xrightarrow{i} (\Sigma F, *).$$

Given an actual pair of classes $f \in H^n F$ and $b \in H^{n+1} B$, we say that f *transgresses* to b when

$$j^* i^* f = p^* b.$$

Equivalently,

$$d_n f = b$$

in the Serre spectral sequence.

(?, Proposition 8.1)

Lemma 4.5.8 (Kudo transgression)**.** *If f transgresses to b, then* $\mathrm{Sq}^m f$ *transgresses to* $\mathrm{Sq}^m b$.

Proof. We rely on naturality of the squaring operations:

$$j^* i^* \mathrm{Sq}^m f = \mathrm{Sq}^m j^* i^* f = \mathrm{Sq}^m p^* b = p^* \mathrm{Sq}^m b. \qquad \square$$

This is a powerful computational tool, in the same way that the Leibniz law is powerful.

(?, pg. 86–88)

Example 4.5.9. We will use it in a more mundane way first, to give better names to the generators $y_{1+2^j} \in H^*(K(\mathbb{F}_2, 2); \mathbb{F}_2)$. These generators were introduced in order to pair with $x^{2^j} \in H^*(K(\mathbb{F}_2, 1); \mathbb{F}_2)$. Since $|x| = 1$, we may rewrite x^{2^j} as

$$x^{2^j} = \mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 x.$$

All of the differentials sourced from the classes x^{2^j} are transgressive, hence we learn

$$\begin{aligned} y_{1+2^j} &= d_{1+2^j} x^{2^j} \\ &= d_{1+2^j} \mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 x \\ &= \mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 d_2 x && \text{(Lemma 4.5.8)} \\ &= \mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 y_2. \end{aligned}$$

The class y_2 is more commonly called ι_2, as in Example 2.4.5, and so we learn

$$H^*(K(\mathbb{F}_2, 2); \mathbb{F}_2) \cong \mathbb{F}_2[\iota_2, \mathrm{Sq}^1 \iota_2 \mathrm{Sq}^2 \mathrm{Sq}^1 \iota_2, \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 \iota_2, \ldots].$$

In fact, this generalizes:

(?, Theorem 18.14)
A different way of phrasing this Theorem is that $H^*(K(\mathbb{F}_2, q); \mathbb{F}_2)$ is the free commutative algebra on the free "unstable" Steenrod module on the class ι_q in degree q. Similarly, $H^*(K(\mathbb{Z}, q); \mathbb{F}_2)$ is free on a class in degree q, subject to the lone constraint $\mathrm{Sq}^1 \iota_q = 0$.

Theorem 4.5.10. *The Steenrod action on the polynomial ring $H^*(K(\mathbb{F}_2, q); \mathbb{F}_2)$ is given by*

$$H^*(K(\mathbb{F}_2, q); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Sq}^I \iota_q \mid I_j \geq 2(I_{j+1}), 2I_1 - I_+ < q].$$

For the polynomial ring $H^*(K(\mathbb{Z}, q); \mathbb{F}_2)$, it is given by

$$H^*(K(\mathbb{Z}, q); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Sq}^I \iota_q \mid I_j \geq 2(I_{j+1}), 2I_1 - I_+ < q, I_{\mathit{final}} \neq 1]. \quad \square$$

4.6 Bocksteins

Before turning out attention to the application of these calculations, we consider one last fundamental matter: the relationship between co/homology with integral coefficients and with mod–2 coefficients. Consider the universal coefficient sequence:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{n-1}(X; \mathbb{Z}), \mathbb{F}_2) \to H^n(X; \mathbb{F}_2) \to \mathsf{Groups}(H_n(X; \mathbb{Z}), \mathbb{F}_2) \to 0.$$

A torsion-free, indivisible class in $H_n(X; \mathbb{Z})$ will contribute a lone class in $H^n(X; \mathbb{F}_2)$. On the other hand, a 2–torsion class will contribute both a class in $H^n(X; \mathbb{F}_2)$ and $H^{n+1}(X; \mathbb{F}_2)$. In fact, so will a 2^j–torsion class for any $j \geq 1$. In the reverse, this prompts a host of questions: given a class in $H^n(X; \mathbb{F}_2)$, when does it belong to a torsion-free homology class or a 2^j–torsion class—and, if the latter, what is the value of j?

With this problem in mind, note that the exact sequences

$$H\mathbb{Z}/2 \to H\mathbb{Z}/2^{j+1} \to H\mathbb{Z}/2^j.$$

string together as j ranges to give a filtered object

$$\begin{tikzcd} \cdots \arrow[r] & H\mathbb{Z}/2^{j+1} \arrow[r] & H\mathbb{Z}/2^j \arrow[r] & \cdots \arrow[r] & H\mathbb{Z}/2^2 \arrow[r] & H\mathbb{Z}/2 \\ & H\mathbb{Z}/2 \arrow[u] & H\mathbb{Z}/2 \arrow[u] & \cdots & H\mathbb{Z}/2 \arrow[u] & H\mathbb{Z}/2 \arrow[u, equal] \end{tikzcd}$$

whose inverse limit is $H\mathbb{Z}_2$.

(?, pg. 60–62)

Definition 4.6.1. Mapping in a space X gives rise to the *Bockstein spectral sequence*, with signature

$$H^*(X;\mathbb{F}_2)\otimes\mathbb{F}_2[b]\Rightarrow H^*(X;\mathbb{Z}_2).$$

Up to a shift in filtration, it is equivalent to repeatedly use the exact sequence $H\mathbb{Z}_2 \xrightarrow{2} H\mathbb{Z}_2 \to H\mathbb{F}_2$. This alternative makes certain "homogeneity" properties of the spectral sequence clearer.

If X is connective and of finite type, this spectral sequence converges. It is convention to write its n^{th} differential and β_n, and β_n is b–linear: if $\beta_n(x)=b^n y$ holds, then $\beta_n(b^m x)=b^{n+m}y$ holds as well.

(?, Proposition 18.12.b),(?, pg. 104)

Lemma 4.6.2. *The first Bockstein differential is given by the formula*

$$\beta_1(x)=b\mathrm{Sq}^1(x).$$

Proof. The first differential is a stable, additive map

$$\beta_1\colon H^n(X;\mathbb{F}_2)\to b\cdot H^{n+1}(X;\mathbb{F}_2)$$

which is natural in X—and hence qualifies as a cohomology operation. Since there is only one available nonzero cohomology operation which shifts degree by 1, we deduce the claimed formula. □

Definition 4.6.3. In fact, the other differentials β_n in the Bockstein spectral sequence also behave like cohomology operations, except that they are only defined on classes for which $\beta_{<n}$ vanishes, and even then they are only determined up to the images of $\beta_{<n}$. Such conditionally-defined maps are termed *secondary cohomology operations*.

Unfortunately, these operations are difficult to compute directly without substantially more foundational work. However, the following observation powers a calculation tool which gives some limited access to them:

Corollary 4.6.4. *Transgressive differentials in a Serre spectral sequence respect the Bockstein operations.*

Proof. Having identified the Bockstein operations as kinds of cohomology operations, one repeats the proof of Lemma 4.5.8. □

(?, Theorem 11.1)

Theorem 4.6.5. *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be an exact sequence, let $u\in H^n(F;\mathbb{F}_2)$ transgress to a class v, and suppose there exists a class w with $\beta_i w=v$. Then $\beta_{i+1}p^*u$ is defined and*

$$j^*\beta_{i+1}p^*w=\beta_1 u.$$

Proof. Definitionally, the fundamental class $\iota_n\in H^n(K(\mathbb{Z}/2^i,n);\mathbb{F}_2)$ is connected to the unique nonzero class $\beta_i\iota_n\in H^{n+1}(K(\mathbb{Z}/2^i,n);\mathbb{F}_2)$ in the next degree. With this in mind, consider the Serre spectral sequence associated to the fibration

$$K(\mathbb{Z}/2,n)\to K(\mathbb{Z}/2^{i+1},n)\to K(\mathbb{Z}/2^i,n).$$

The fundamental class associated to $K(\mathbb{Z}/2,n)$ is not allowed to persist in the spectral sequence, hence must participate in the transgressive differential

$$d_{n+1}(\iota_n^{\mathbb{Z}/2}) = \beta_n \iota_n^{\mathbb{Z}/2^i}.$$

In turn, the only classes available to become $\iota_n^{\mathbb{Z}/2^{i+1}}$ and $\beta_{i+1}\iota_n^{\mathbb{Z}/2^{i+1}}$ respectively are $\iota_n^{\mathbb{Z}/2^i}$ and $\beta_1 \iota_n^{\mathbb{Z}/2}$. Setting

$$u = \iota_n^{\mathbb{Z}/2}, \qquad v = \beta_n \iota_n^{\mathbb{Z}/2^i}, \qquad w = \iota_n^{\mathbb{Z}/2^i},$$

we have proven the claim in this example.

The claim for a generic fibration then comes by constructing a map of fibrations

$$\begin{tikzcd}
F \arrow[r] \arrow[d, dotted] & E \arrow[r] \arrow[d, dotted] & B \arrow[d, dotted] \\
K(\mathbb{Z}/2,n) \arrow[r] & K(\mathbb{Z}/2^{i+1},n) \arrow[r] & K(\mathbb{Z}/2^i,n).
\end{tikzcd}$$

The existence of a map $B \to K(\mathbb{Z}/2^i,n)$ amounts to a choice of cohomology class $w \in H^n(B;\mathbb{F}_2)$ such that $\beta_i w$ is defined, which is part of the hypothesis. In order to extend this to a map $E \to K(\mathbb{Z}/2^{i+1},n)$, we need only show that

$$E \xrightarrow{p} B \xrightarrow{\tilde{w}} K(\mathbb{Z}/2^i,n) \xrightarrow{\tilde{\beta}_i} K(\mathbb{Z}/2,n+1)$$

is null. This composite calculates $p^*\beta_i w$, and since $\beta_i w$ is the target of a transgressive differential, it is indeed null after pulling back along p. This also grants the extension $F \to K(\mathbb{Z}/2,n)$. Although this last map may not be equal to u, it is at least some class which also transgresses to v, which is enough to deduce the claim. □

There's an idea for an extension here.

4.A K-theory

Since we're talking about G-bundles, I think it would be smart to give a brief treatment of complex K-theory, even if it doesn't directly factor into our future calculations.

We can also calculate hella cohomology groups.

Theorem 4.A.1. $H^*U(n) \cong \Lambda[e_1, \ldots, e_n]$. $H^*BU(n) \cong \mathbb{Z}[x_1, \ldots, x_n]$.

Proof. We have fiber sequences $U(n-1) \to U(n) \to S^{2n-1}$ and $U(n) \to * \to BU(n)$ (and, for that matter, $S^{2n-1} \to BU(n-1) \to BU(n)$). has no room for differentials on indecomposables, hence no differentials at all. Also no room for multiplicative extensions: no even classes in lower filtration to connect with. attaches a polynomial class to each of the old exterior classes. □

Insert $U(n)$ picture

Insert $BU(n)$ picture.

Corollary 4.A.2. *Associated to each complex vector bundle V/X, we have defined a sequence of classes $c_n(V) \in H^{2n}(X;\mathbb{Z})$, the Chern classes of V.*

Proof. Associated to V/X is a $U(n)$-bundle ξ/X, classified by a map $f\colon X \to BU(n)$. This induces a map $f^*\colon H^*BU(n) \to H^*X$, along which we send teh classes x_j, $j \le n$. □

Theorem 4.A.3. *For each $U(n)$-bundle ξ over a CW complex X there are unique element $c_j(\xi) \in H^{2j}(X)$ depending only on the isomorphism class of ξ such that*

1. *For a map $f\colon Y \to X$, $c_j(f^*\xi) = f^*c_j(\xi)$.*

2. *$c_0(\xi) = 1$ for all ξ.*

3. *For γ the tautological bundle on $\mathbb{C}\mathrm{P}^n$, $c_1(\gamma) = x_1$.*

4. *For ξ a $U(n)$-bundle and ζ a $U(m)$-bundle on X, $c_k(\xi \oplus \zeta) = \sum_{i+j=k}(c_i(\xi) \cdot c_j(\zeta))$.*

Lemma 4.A.4. *For ξ a $\mathbb{C}^n$-bundle on X, there exists a space $f\colon Y \to X$ over X such that*

1. *$f^*\colon H^*X \to H^*Y$ is an injection.*

2. *$f^*(\xi) = \xi' \oplus \eta$, where η is a line bundle on Y.*

Proof. Set $Y = \mathbb{P}(\xi)$ to be the fiberwise projectivization of ξ; this is a $\mathbb{C}\mathrm{P}^{n-1}$-bundle on X. The pullback $f^*\xi$ has a natural subbundle η of those vectors in $f^*\xi$ which lie in the line chosen in $\mathbb{P}(\xi)$. All of our data thus far assembles into the following diagram:

$$\begin{tikzcd}
 & \mathbb{C}^\times \arrow[dr] \arrow[dd, equal] & & & \\
\mathbb{C}^n \arrow[dd] & & \mathbb{C}^n \setminus 0 \arrow[ll] \arrow[r] \arrow[dd] & \mathbb{C}\mathrm{P}^{n-1} \arrow[r] \arrow[dd] & \mathbb{C}\mathrm{P}^\infty \arrow[dd, equal] \\
 & \mathbb{C}^\times \arrow[dr] & & & \\
\xi \arrow[d] & & \xi \setminus \text{zero} \arrow[ll] \arrow[r] \arrow[d] & \mathbb{P}(\xi) \arrow[r, "f"] \arrow[d] & \mathbb{C}\mathrm{P}^\infty \arrow[d] \\
X \arrow[rr, equal] & & X \arrow[r, equal] & X \arrow[r] & *.
\end{tikzcd}$$

The Serre spectral sequence for $\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{P}(\xi) \xrightarrow{f} X$ degenerates, since $H^*\mathbb{C}\mathrm{P}^\infty \to H^*\mathbb{C}\mathrm{P}^{n-1}$ is onto. The edge homomorphism $H^*X \to H^*\mathbb{P}(\xi)$ is thus an inclusion. □

Remark 4.A.5. In $H^*\mathbb{P}(\xi)$, there is a potential multiplicative extension for $x^{n-1} \cdot x$, i.e., a relation $x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^n b_n = 0$. We will show that these b_*s model the c_*s from the theorem and the x_*s from last time.

Proof of Theorem. The first three points are automatic for the b_*s. To get unicity, apply the construction twice: first to split ξ into $\xi' \oplus \eta$, then to compute the Chern classes of ξ' and η, and find $c_i(\xi) = c_j(\xi') + c_1(\eta) \cdot c_{j-1}(\xi')$. To get the claim about sums of bundles, note that $\mathbb{P}(\zeta)$, $\mathbb{P}(\xi)$ are subspaces of $\mathbb{P}(\zeta \oplus \xi)$ such that $\mathbb{P}(\zeta)$ is a deformation retract of $\mathbb{P}(\zeta \oplus \xi) \setminus \mathbb{P}(\xi)$ and vice versa. Form the sums $b_\zeta = \sum_{j=0}^m (-1)^j b_j(\zeta) y^{m-j}$ and $b_\xi = \sum_{j=0}^n (-1)^j b_j(\xi) y^{n-j}$ *as elements of* $H^*\mathbb{P}(\zeta \oplus \xi)$. Then $b_\zeta|_{\mathbb{P}(\zeta)} = 0$ and $b_\xi|_{\mathbb{P}(\xi)} = 0$, so a Mayer-Vietoris argument says $b_\zeta \cdot b_\xi = 0$ in $H^*\mathbb{P}(\zeta \oplus \xi)$. But $b_{\zeta \oplus \xi}$ is the *unique* monic polynomial with this property (of degree $n+m$). □

To get the claim about the b_*s and the $x_j \in H^{2j}BU(n)$, consider the maps $(\mathbb{C}\mathrm{P}^\infty)^{\times n} \to BU(n)$ classifying the universal $\mathbb{C}^n$–bundle with a decomposition into n lines. This participates in a map of long exact sequences

$$\begin{tikzcd}
H^{*+2n-1}BU(n-1) \arrow[r] \arrow[d] & H^*BU(n) \arrow[r, "\cdot x_n"] \arrow[d] & H^{*+2n}BU(n) \arrow[r, "(1)"] \arrow[d] & H^{*+2n}BU(n-1) \arrow[r] \arrow[d, "(2)"] & H^{*+1}BU(n) \arrow[d] \\
0 \arrow[r] & H^*(\mathbb{C}\mathrm{P}^\infty)^{\times n} \arrow[r] & H^{*+2n}(\mathbb{C}\mathrm{P}^\infty)^{\times n} \arrow[r, "(3)"] & H^{*+2n}(\mathbb{C}\mathrm{P}^\infty)^{\times (n-1)} \arrow[r] & 0.
\end{tikzcd}$$

The top is the Gysin sequence. Assuming the theorem at $n-1$, the map (2) sends x_j to $b_j(\eta^{\oplus(n-1)}) = \sigma_j(\pi_1^* x_1, \ldots, \pi_{n-1} x_1)$, an elementary symmetric function. In general, the x_j land in the symmetric functions, and x_n lands in the kernel of (3), hence is a constant multiple of σ_n. Actually, the vertical maps are all *ring* maps, which puts a huge restriction on their behavior: x_n must be sent to σ_n on the nose.

5

Computations in the Homotopy Groups of Spheres

At last, we come to the program outlined in the introduction to this book, which employs all of the technology developed so far.

Our goal now is to perform some computatins in the homotopy groups of spheres, themselves interesting for their role in determining what attaching maps are available during the cellular construction of homotopy types. Our access to these groups is indirect and manifold. In the first method, known as *Serre's method*, we will essentially construct the Postnikov decomposition of a sphere by using the Serre spectral sequence and our intimate knowledge of the Steenrod algebra to deduce how its k–invariants *must* fit together. In the second method, we will assemble various stable phenomena to produce a simplification of the computation that is valid for spheres of large dimension, known as the *Adams spectral sequence*. In the third method, we will use certain exact sequences among the spheres themselves, called *EHP sequences*, to propagate the knowledge garnered in the first two attempts to otherwise unreachable areas.

By the conclusion of this chapter, we will have computed $\pi_{n+k} S^n$ for all $n \geq 1$ and for $0 \leq k \leq 7$, though our methods could certainly be pushed farther by the interested reader.

5.1 Serre's method

We now use these techniques to start computing homotopy groups. Recall that when we were constructing Eilenberg–Mac Lane spaces, we started with the sphere, with known homology and unknown homotopy, and we repeatedly applied the Hurewicz theorem to identify a layer of homotopy, used coexact sequences to remove it, and ultimately produced a space with minimal homotopy. It turns out that the *Eckmann–Hilton dual* to this approach is also possible, as motivated by the following observation:

Remark 5.1.1. Consider a cohomology class $\omega \in H^n(X;A)$, represented by a map $\omega \colon X \to K(A,n)$. In the spectral sequence for

$$K(A, n-1) \to * \to K(A,n),$$

(?, Formula 11.1)
If a cohomology class is *spherical*, one can take the cofiber to kill the class. This makes more obvious use of the Eilenberg–Steenrod axioms. However, not all cohomology classes are spherical, and even for a spherical class a choice of spherical representative is potentially unnatural.

we know that the fundamental class transgresses. Now consider the pulled back fibration, P_ω:

$$\begin{tikzcd}
K(A,n-1) \arrow[r] \arrow[d, equal] & P_\omega \arrow[r] \arrow[d] & X \arrow[d, "\omega"] \\
K(A,n-1) \arrow[r] & * \arrow[r] & K(A,n).
\end{tikzcd}$$

Because the fundamental class transgresses in the universal case, it also transgresses in this spectral sequence:

$$d_{n+1}\iota_n = \omega.$$

Aside from killing ω, this operation has somewhat wild behavior. First, note that the entire Steenrod module spanned by ω is also destroyed by the ensuing pattern of differentials:

$$d_{n+1+I_+}\mathrm{Sq}^I\iota_n = \mathrm{Sq}^I\omega.$$

However, if $\mathrm{Sq}^I\omega = 0$, this leaves $\mathrm{Sq}^I\iota_n$ as a survivor of the spectral sequence. These surviving classes can participate in nontrivial extensions of the Steenrod action.

With this technique in mind, the rough idea in the section is to perform the reverse of the construction of Eilenberg–Mac Lane spaces from Lemma 2.3.9. We start with an Eilenberg–Mac Lane space, which has *both* known homotopy and known co/homology. We will then repeatedly apply the exact sequence technique from Remark 5.1.1 to excise the bottom layer of cohomology (which gives some information about homotopy via Hurewicz), ultimately producing a space with minimal homology. In order to make full use of Remark 5.1.1, we will work with mod–2 homology rather than integral homology—which means, according to Lemma 3.2.9, that we will produce a model for $(S^n)_2$.

(?, pg. 114–118)

Let's get started, beginning with $X_n = K(\mathbb{Z}, n)$. This space models a downward Postnikov trunction of S^n:

$$S^n \to S^n[0,n] = K(\mathbb{Z}, n).$$

The mod–2 cohomology of $S^n[0,n]$ matches that of S^n through degree $n+1$, but there is a class $\mathrm{Sq}^2\iota_n \in H^{n+2}(S^n[0,n]; \mathbb{F}_2)$ which is not present in $H^{n+2}(S^n; \mathbb{F}_2)$. To correct this discrepancy, we apply Remark 5.1.1 to this class.

This class in H^{n+2} indicates a discrepancy in π_{n+1}. The difference in degree arises from the torsion shift in the universal coefficient theorem.

$$\begin{tikzcd}
 & K(\mathbb{Z}/2, n+1) \arrow[r, equal] \arrow[d] & K(\mathbb{Z}/2, n+1) \arrow[d] \\
 & S^n[0,n+1] \arrow[r] \arrow[d] & * \arrow[d] \\
S^n \arrow[r] \arrow[ur, dotted] & K(\mathbb{Z}, n) \arrow[r, "\mathrm{Sq}^2\iota_n"'] & K(\mathbb{Z}/2, n+2).
\end{tikzcd}$$

We would like to continue this process by identifying the next discrepancy between the mod–2 cohomologies of S^n and $S^n[0,n+1]$—which means we must compute the cohomology of $S^n[0,n+1]$. Remark 5.1.1 gives us access to this via a Serre spectral sequence with *lots* of differentials.

Important Note. In order to simplify the analysis of this spectral sequence, we make the assumption $n \gg 0$. This assumption eliminates the inequality condition in Theorem 4.5.10 and pushes all product terms into the high degree $2n \gg n$.

With this established, one works through the applications of Lemma 4.5.8 to produce the following family of transgressive differentials:

$$\begin{array}{lrcl}
 & H^{n+k-1}(K(\mathbb{Z}/2,n+1);\mathbb{F}_2) & \xrightarrow{d} & H^{n+k}(S^n[0,n];\mathbb{F}_2) \\
k=0: & & & \iota_n \\
k=2: & \iota_{n+1} & \mapsto & \mathrm{Sq}^2\iota_n, \\
k=3: & \mathrm{Sq}^1\iota_{n+1} & \mapsto & \mathrm{Sq}^1\mathrm{Sq}^2\iota_n, \\
k=4: & \mathrm{Sq}^2\iota_{n+1}, & & \mathrm{Sq}^4\iota_n, \\
k=5: & \mathrm{Sq}^1\mathrm{Sq}^2\iota_{n+1}, & & \\
 & \mathrm{Sq}^2\mathrm{Sq}^1\iota_{n+1} & \mapsto & \mathrm{Sq}^5\iota_n, \\
k=6: & \mathrm{Sq}^3\mathrm{Sq}^1\iota_{n+1}, & & \\
 & \mathrm{Sq}^4\iota_{n+1} & \mapsto & \mathrm{Sq}^4\mathrm{Sq}^2\iota_n, \\
k=7: & \mathrm{Sq}^5\iota_{n+1}+\mathrm{Sq}^4\mathrm{Sq}^1\iota_{n+1}, & & \mathrm{Sq}^7\iota_n, \\
 & \mathrm{Sq}^5\iota_{n+1} & \mapsto & \mathrm{Sq}^5\mathrm{Sq}^2\iota_n, \\
k=8: & \mathrm{Sq}^5\mathrm{Sq}^1\iota_{n+1}, \mathrm{Sq}^4\mathrm{Sq}^2\iota_{n+1}, & & \\
 & \mathrm{Sq}^6\iota_{n+1} & \mapsto & \cdots
\end{array}$$

The classes colored red do not participate in differentials and so survive to the E_∞–page. This table thus has enough information to read off $H^{\le n+7}(S^n[0,n+1];\mathbb{F}_2)$, as well as much of its Steenrod action.

The first abberration from $H^{\le n+7}(S^n;\mathbb{F}_2)$ is now

$$\mathrm{Sq}^2\iota_{n+1} \in H^{n+3}(S^n[0,n+1];\mathbb{F}_2).$$

We proceed to use Remark 5.1.1 to delete this class. This yields a new Serre

$$\begin{tikzcd}
 & K(\mathbb{Z}/2,n+2) \arrow[r, equal] \arrow[d] & K(\mathbb{Z}/2,n+2) \arrow[d] \\
 & S^n[0,n+2] \arrow[r] \arrow[d] & * \arrow[d] \\
S^n \arrow[r] \arrow[ur, dotted] & S^n[0,n+1] \arrow[r, "\mathrm{Sq}^2\iota_{n+1}"'] & K(\mathbb{Z}/2,n+3).
\end{tikzcd}$$

spectral sequence to work through:

$$
\begin{array}{ll}
& H^{n+k-1}(K(\mathbb{Z}/2, n+2); \mathbb{F}_2) \xrightarrow{d} H^{n+k}(S^n[0, n+1]; \mathbb{F}_2) \\
k = 0: & \iota_n, \\
k = 3: & \iota_{n+2} \mapsto \mathrm{Sq}^2 \iota_{n+1}, \\
k = 4: & \mathrm{Sq}^4 \iota_n, \\
& \mathrm{Sq}^1 \iota_{n+2} \mapsto \mathrm{Sq}^3 \iota_{n+1}, \\
k = 5: & \mathrm{Sq}^2 \iota_{n+2} \mapsto \mathrm{Sq}^3 \mathrm{Sq}^1 \iota_{n+1} \\
k = 6: & \mathrm{Sq}^3 \iota_{n+2}, \quad \mathrm{Sq}^6 \iota_n, \\
& \mathrm{Sq}^2 \mathrm{Sq}^1 \iota_{n+2} \mapsto (\mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1) \iota_n \\
k = 7: & \mathrm{Sq}^7 \iota_n, \\
& \mathrm{Sq}^3 \mathrm{Sq}^1 \iota_{n+2} \mapsto \mathrm{Sq}^5 \mathrm{Sq}^1 \iota_n, \\
& \mathrm{Sq}^4 \iota_{n+2} \mapsto \mathrm{Sq}^4 \mathrm{Sq}^2 \iota_n, \\
k = 8: & (\mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1) \iota_{n+2}, \\
& \mathrm{Sq}^5 \iota_{n+2} \mapsto \cdots.
\end{array}
$$

The first abberation from $H^{\leq n+7}(S^n; \mathbb{F}_2)$ is now

$$\mathrm{Sq}^4 \iota_n \in H^{n+4}(S^n[0, n+2]; \mathbb{F}_2).$$

We would like to apply Remark 5.1.1 to the map $\mathrm{Sq}^4 \iota_n \colon S^n[0, n+2] \to K(\mathbb{Z}/2, n+4)$, but there is no class $\mathrm{Sq}^1 \mathrm{Sq}^4 \iota_n \in H^*(S^n[0, n+2]; \mathbb{F}_2)$, so that $\mathrm{Sq}^1 \iota_{n+3}$ will contribute another class in the same degree. This signifies that we should use $K(\mathbb{Z}/2^j, n+4)$ instead for some $j > 1$, according to which j satisfies

$$\beta_j \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^3 \iota_{n+2}.$$

To make this calculation, we appeal repeatedly to Theorem 4.6.5, which says that

The Lemma only works up to filtration.

$$\beta_j \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^1 (\mathrm{Sq}^2 \iota_{n+2})$$

would hold if

$$\beta_{j-1} \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^1 (\mathrm{Sq}^2 \mathrm{Sq}^1 \iota_{n+1}) = d(\mathrm{Sq}^2 \iota_{n+2})$$

were true, where d denotes a transgressive differential. In turn, this left equality would hold if

$$\beta_{j-2} \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^1 (\mathrm{Sq}^4 \iota_n) = d(\mathrm{Sq}^2 \mathrm{Sq}^1 \iota_{n+1})$$

were to hold, where d is now the transgressive differential associated to the *previous* exact sequence. This can be made to hold by taking $j = 3$, so that

$$\beta_3 \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^3 \iota_{n+2}.$$

From this, we calculate the following table of differentials:

$$\begin{array}{ll} & H^{n+k-1}(K(\mathbb{Z}/2^3, n+3);\mathbb{F}_2) \xrightarrow{d} H^{n+k}(S^n[0,n+2];\mathbb{F}_2) \\ k=0: & \iota_n, \\ k=4: & \iota_{n+3} \mapsto \mathrm{Sq}^4\iota_n, \\ k=5: & \beta_3\iota_{n+3} \mapsto \mathrm{Sq}^3\iota_{n+2}, \\ k=6: & \mathrm{Sq}^2\iota_{n+3} \mapsto \mathrm{Sq}^6\iota_n, \\ k=7: & \mathrm{Sq}^3\iota_{n+3} \mapsto \mathrm{Sq}^7\iota_n, \\ & \mathrm{Sq}^2\beta_3\iota_{n+3} \mapsto (\mathrm{Sq}^5+\mathrm{Sq}^4\mathrm{Sq}^1)\iota_{n+2}, \\ k=8: & \mathrm{Sq}^4\iota_{n+3}, \\ & \mathrm{Sq}^3\beta_3\iota_{n+3} \mapsto \cdots. \end{array}$$

From this we conclude that the next abberrant class lies in degree 7.

(?, Theorem 12.2)

Looking back at the Postnikov layers we have constructed, we see that we have computed

Use booktabs

n	0	1	2	3	4	5
$\pi_n\mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^3$	0	0

We could, presumably, keep going.

Actually, there is an indeterminacy at $k=14$ which *cannot* be resolved with the methods here. We will meet this again. (?, pg. 123–127)

Remark 5.1.2. For contrast, we also consider the computation in very low degrees for the fibration

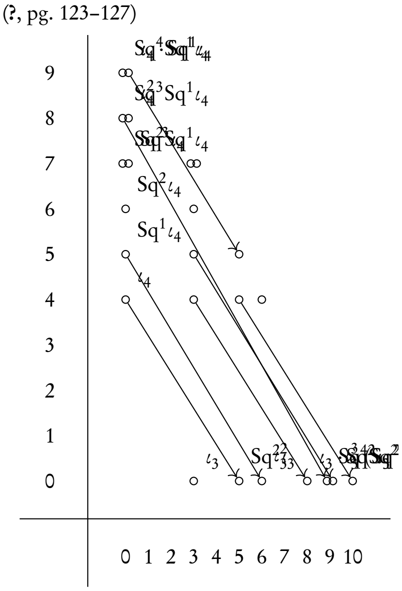
$$K(\mathbb{Z}/2,4) \to S^3[0,4] \to S^3[0,3],$$

where the pertinent differentials coming from Lemma 4.5.8 are

$$d_5\iota_4 = \mathrm{Sq}^2\iota_3, \qquad d_6\mathrm{Sq}^1\iota_4 = \iota_3^2, \qquad d_9\mathrm{Sq}^4\iota_4 = \mathrm{Sq}^4\mathrm{Sq}^2\iota_3.$$

The surviving classes are

$$\iota_3, \quad \mathrm{Sq}^2\iota_4, \quad \mathrm{Sq}^3\iota_4, \quad \mathrm{Sq}^2\mathrm{Sq}^1\iota_4, \quad \mathrm{Sq}^3\mathrm{Sq}^1\iota_4, \quad \mathrm{Sq}^4\mathrm{Sq}^1\iota_4, \quad \iota_3\otimes\mathrm{Sq}^2\iota_4.$$



This is quite different from the $n \gg 0$ calculation: we have already felt the effects both of the inequalities and the products appearing in Theorem 4.5.10.

Continuing in this manner, the sufficiently intrepid computationalist can produce the table

n	0	1	2	3
$\pi_{n+3}S^3$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$,

which showcases the unstable phenomenon $\pi_6 S^3 \ncong \pi_3\mathbb{S}$.

There are some classes in the bulk in my notes that are neither circled nor participate in differentials. They are fringe classes, though...?

5.2 *The Adams spectral sequence*

In the previous section, we made an innocuous-seeming observation: the map

$$H^*(H\mathbb{F}_2;\mathbb{F}_2) \to H^{*+n}(K(\mathbb{F}_2,n);\mathbb{F}_2)$$

is an isomorphism through degree n. We limited our calculation to this range in order to skirt the excess inequality and the presence of product classes in $H^{*+n}(K(\mathbb{F}_2, n); \mathbb{F}_2)$, two difficult features. In fact, this restriction skirted much more: even the Serre spectral sequences were quite simple, populated only with transgressive differentials. This is a consequence of Remark 4.5.3: in the range considered, the exact sequences

For contrast, consider the nontransgressive differential in Remark 5.1.2.

$$K(\pi_{n+m} S^n, n+m) \to S^n[0, n+m] \to S^n[0, n+m)$$

can instead be thought of as coexact sequences of spectra

$$\Sigma^{n+m} H(\pi_{n+m} S^n) \to \mathbb{S}^n[0, n+m] \to \mathbb{S}^n[0, n+m),$$

and the associated Serre spectral sequences can instead be thought of as exact sequences:

$$\cdots \leftarrow H^*(H(\pi_{n+m} S^n); \mathbb{F}_2) \leftarrow H^*(\mathbb{S}^n[0, n+m]; \mathbb{F}_2) \leftarrow H^*(\mathbb{S}^n[0, n+m); \mathbb{F}_2) \leftarrow \cdots.$$

The pervasiveness of Lemma 4.5.8 in governing the behavior of the Serre spectral sequence is abbreviated stably to this long exact sequence being one of Steenrod modules.

Our goal in this section is to pursue this construction directly within the stable category. Our intention is essentially the same: we will strip a spectrum of its homotopy by iteratively applying the stable Hurewicz map. The main difference is that, stably, we are permitted to do this using coexact sequences.

Strictly speaking, this is opposite to our approach using Serre's method, where we were stripping the cohomology of $K(\mathbb{Z}, n)$ to match those of S^n, rather than stripping the homotopy of S^n to match $K(\mathbb{Z}, n)$.

Lemma 5.2.1. *Take X to be $(n-1)$–connected with $\pi_n X$ finite exponent and 2–torsion. Consider the exact extension*

$$X' \to \mathbb{S} \wedge X \to H\mathbb{F}_2 \wedge X.$$

Then $\pi_n X' < \pi_n X$.

Proof. Various aspects of the Hurewicz theorem tell us

$$\pi_{<n}(H\mathbb{F}_2 \wedge X) = 0, \qquad \operatorname{im}(\pi_n X \to \pi_n H\mathbb{F}_2 \wedge X) = (\pi_n X)/2,$$

and the map $\pi_{n+1} X \to H_{n+1} X$ is onto. Altogether, this shows X' to be $(n-1)$–connected and that there is an inclusion $\pi_n X' < \pi_n X$. □

(?, Proposition 19.12)

Theorem 5.2.2. *Let $\overline{H} \to \mathbb{S} \xrightarrow{\eta} H$ be the exact extension of the unit map. For X a connected spectrum with $\pi_* X$ 2–torsion and degreewise finite exponent, the associated co-filtered object*

$$\begin{tikzcd}
\mathbb{S} \wedge X \arrow[d] & \overline{H} \wedge X \arrow[l] \arrow[d] & \overline{H}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
H \wedge X & H \wedge \overline{H} \wedge X & H \wedge \overline{H}^{\wedge 2} \wedge X &
\end{tikzcd}$$

has contractible limit, hence gives a spectral sequence which strongly converges to $\pi_* X$.

For X a connective spectrum of finite type, this converges to the homotopy of the 2–adic completion, $\pi_* X_2$.

It is also helpful to note that $H \wedge \overline{H}$ has this same finiteness property.

Proof. This spectral sequence is exactly the Lemma, repeatedly applied. □

This construction is easier to think about after taking $\mathbb{F}_2$–duals. Then $E^1_{*,t} = (\pi_* H \wedge \overline{H}^{\wedge t} \wedge X)^* = (\overline{\mathscr{A}^*})^{\otimes t} \wedge H\mathbb{F}_2^* X$. The differential exactly tracks $\mathscr{A}^*$–module maps $H\mathbb{F}_2^* X \to \mathbb{F}_2$, and a little homological algebra reveals:

Lemma 5.2.3. $E^2_{*,*} = \operatorname{Ext}^{\mathscr{A}^*}_{*,*}(H\mathbb{F}_2^* X, \mathbb{F}_2)$. □

Remark 5.2.4.

Explain the relationship between the Bockstein lemma and h_0–extensions.

Work through a minimal free resolution?

5.3 *The May spectral sequence*

Our goal in this section is to learn a systematic method for computing in the E_2–page for the Adams spectral sequence for the sphere, due to Peter May. The idea is that connected, graded, finite-type Hopf algebras admit filtration (essentially by "word length") which *trivialize* their multiplication and comultiplication. This gives a spectral sequence beginning with the cohomology of a bouquet of exterior algebras.

It is in this section that we make good on explaining the spectral sequence previously considered in Section 2.A.

(?)

Theorem 5.3.1 (May)**.** *There is a spectral sequence* of algebras *with* $E_1^{*,*,*} \cong \mathbb{F}_2[h_{ij} \mid i \geq 1, j \geq 0]$ *converging to the Adams E_2–term for the sphere. The element h_{ij} is represented by $\xi_i^{2^j}$, hence*

$$d_1 h_{ij} = \sum_{k=1}^{i-1} h_{kj} h_{(i-k)(k+j)}. \quad \square$$

In order to produce the E_2–page out through degree 15, we will need to study the following generators:

name	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{20}	h_{21}	h_{22}	h_{30}	h_{31}	h_{40}
horiz. degree	0	1	3	7	15	2	5	11	6	13	14.

All differentials are linear against h_{1j}, and the first differential is linear against h_{ij}^2. Applying the Theorem, it is determined on a generating set by

$$d(h_{20}) = h_{10}h_{11}, \qquad d(h_{21}) = h_{11}h_{12}, \qquad d(h_{22}) = h_{12}h_{13},$$
$$d(h_{30}) = h_{10}h_{21} + h_{20}h_{12}, \qquad d(h_{31}) = h_{11}h_{22} + h_{21}h_{13},$$
$$d(h_{40}) = h_{10}h_{31} + h_{20}h_{22} + h_{30}h_{13}.$$

We now give a flavor of this computation, working column by column:

0. The zeroth column is populated with powers of h_{10}, which persist.

1. The first column is populated by $h_{10}^j h_{11}$, and $d(h_{20}) = h_{10}h_{11}$ removes all but the first one.

2. The second column is populated by $h_{10}^j h_{11}^2$, and again only h_{11}^2 survives.

3. After clearing the differential $d(h_{10}^j h_{11}^2 h_{20}) = h_{10}^{1+j} h_{11}^3$, the third column is populated by h_{11}^3 and $h_{10}^j h_{12}$. At this point, we will invoke a useful theorem of Nakamura:

   Theorem 5.3.2 (Nakamura). *There are operators* Sq^n *acting on the May spectral sequence, satisfying…*

   (a) $\mathrm{Sq}^0 h_{ij} = h_{i(j+1)}$.

   (b) $\mathrm{Sq}^1 h_{ij} = h_{ij}^2$.

   (c) Sq^n *is linear.*

   (d) $\mathrm{Sq}^n(x \cdot y) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y)$.

   (e) $\mathrm{Sq}^n(d_? x) = d_? \mathrm{Sq}^n(x)$, *where "?" is as small as possible so that both sides are nonzero.*

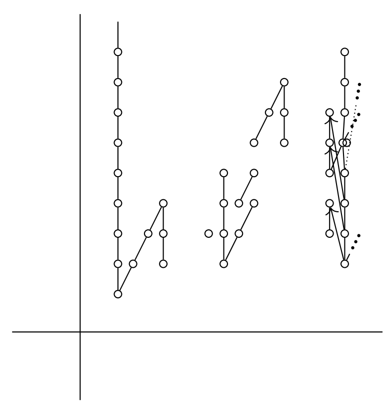
(?)
Nakamura actually proves these items with various bounds on the size of x and y. His bound is more than enough to pursue the calculation through our range of interest, but the Theorem appears to hold even without a bound.

   We can then compute

$$\begin{aligned} d(h_{20}^2) &= d(\mathrm{Sq}^1 h_{20}) = \mathrm{Sq}^1 d(h_{20}) = \mathrm{Sq}^1(h_{10}h_{11}) \\ &= \mathrm{Sq}^1(h_{10})\mathrm{Sq}^0(h_{11}) + \mathrm{Sq}^0(h_{10})\mathrm{Sq}^1(h_{11}) \\ &= h_{11}^3 + h_{10}^2 h_{12}. \end{aligned}$$

   This differential identifies the classes $h_{11}^3 = h_{10}^2 h_{12}$ and it deletes all classes of the form $h_{10}^{2+j} h_{12}$, $j \geq 1$. The only remaining class in this column is therefore $h_{11}^3 = h_{10}^2 h_{12}$.

4. The differentials considered so far eliminate all classes in the fourth column other than $h_{10}^j h_{11} h_{12}$, which is exactly the target of $d(h_{10}^j h_{21})$. No classes survive in this column at all.



5. Similarly, the differentials considered so far eliminate all classes in the fifth column other than $h_{10}h_{21} + h_{20}h_{12}$, itself composed of two classes targeting the same class in the fourth column:

$$d(h_{10}h_{21}) = h_{10}(h_{11}h_{12}) = (h_{10}h_{11})h_{12} = d(h_{20}h_{12}).$$

   This class is exactly the target of $d(h_{30})$, so that no classes survive.

This is already enough to recover the range we worked through with Serre's method. We include at right a picture of the spectral sequence through degree 15.

Remark 5.3.3. We have additionally included a pair of Adams differentials

$$d_2(h_{14}) = h_{13}^2 h_{10}, \qquad d_3(h_{10}^{1+j} h_{14}) = h_{10}^{1+j} d_0.$$

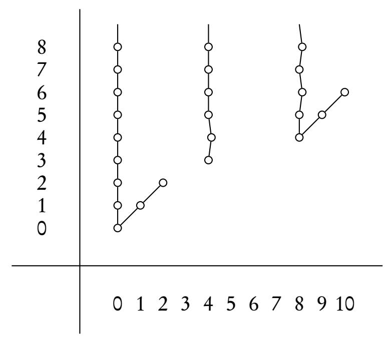
Resolving the presence of absence of these differentials manifests in Serre's method as an indeterminacy in the Serre spectral sequence at that stage. These differentials also have a geometric origin: their presence is equivalent to the nonexistence of a normed real division algebra of dimension 16. In fact, the Adams differentials $d_2(h_{1j}) = h_{10}h_{1(j-1)}^2$ prohibit the existence of such division algebras of dimension 2^j for $j \geq 4$. This differential was originally deduced by Adams through an intensive study of secondary cohomology operations. Later, Adams found a second, shorter proof using Adams spectral sequences based on other cohomology theories (viz., K–theory and complex bordism). A third proof was produced by Bruner using power operations—a kind of spiritual cousin to Nakamura's theorem above. These are all well beyond the technology presented in these notes, though they are each logical subjects for the reader to study next.

(?)

(?)

(?, Corollary VI.1.5)

Remark 5.3.4. There is a version of the May spectral sequence converging to the Adams E_2–page for $\pi_* kO_2^\wedge$, beginning with just $\mathbb{F}_2[h_{10}, h_{11}, h_{20}]$. This was the computation we preformed in Section 2.A, recalled at right. Granting that this is indeed the Adams E_2–page for the homotopy of $kO_2^\wedge$, we can read off its homotopy groups:



Note: they're periodic!

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_n kO_2^\wedge$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_2$	0	0	0.

More than this, we can use the algebra structure of the May E_∞–page to present it as a ring:

$$\pi_* kO_2^\wedge = \mathbb{Z}_2[h_{11}, b'_{20}, h_{20}^4] \Big/ \!\!\! \Big/ \left(\begin{matrix} 2h_{11} = 0, & h_{11}^3 = 0, \\ b'_{20}h_{11} = 0, & (b'_{20})^2 = 4 \cdot h_{20}^4 \end{matrix} \right)$$

5.4 *Hopf invariants and EHP fiber sequences*

Our goal today is to construct some important maps, called *Hopf invariants*, and to identify their fibers in one extremely important case.

The main ingredient is the following reflection of algebra in topology:

(?, Proposition 4.2.2)

Theorem 5.4.1 (James). *For X connected, there is an equivalence*

$$\Sigma(\Omega\Sigma X) \simeq \Sigma\left(\bigvee_{j=1}^{\infty} X^{\wedge j}\right). \quad \square$$

Though we will not recount it, the proof of this is surprisingly easy. The idea is to show it rational homology and in mod–p homology for all p, then use Hurewicz. The whole business is modeled on the tensor algebra functor $T(M) = \bigoplus_j M^{\otimes j}$.

This splitting is highly nontrivial, and the inverse map that James's theorem guarantees is full of interesting information. For instance, the map

(?, pg. 122–124)

$$\Sigma\Omega\Sigma X \xrightarrow{\simeq} \Sigma\left(\bigvee_j X^{\wedge j}\right) \to \Sigma X^{\wedge 2}$$

is adjoint to a map

$$\Omega\Sigma X \xrightarrow{h} \Omega\Sigma X^{\wedge 2}$$

called the *Hopf invariant*.

This construction is important, for instance, in the vector fields on spheres problem: an H–space structure on a sphere gives rise to an interesting map

$$\begin{tikzcd} CS^n \times S^n \arrow[rr] & & CS^n & \\ & S^n \times S^n \arrow[ul] \arrow[dl] \arrow[rr, "\mu"] & & S^n \arrow[ul] \arrow[dl] \\ S^n \times CS^n \arrow[rr] & & CS^n, & \end{tikzcd}$$

hence a map

$$\begin{tikzcd} S^n * S^n \arrow[r, "H\mu"] \arrow[d, equal] & \Sigma S^n \arrow[d, equal] \\ S^{2n+1} \arrow[r, "H\mu"] & S^{n+1} \end{tikzcd}$$

which interacts beautifully with h.

Theorem 5.4.2. *For $X = S^n$, there is a 2–local exact sequence*

$$S^n \to \Omega\Sigma S^n \xrightarrow{h} \Omega\Sigma S^{2n}.$$

The statement of this theorem requires that we actually do localization, not rationalization, back in that section.

Proof. We will use the associated Serre spectral sequence for integral cohomology. Recall that in Remark 4.4.6 we computed

$$H^*\Omega S^{2n+1} \cong \Gamma[x_{2n}], \qquad H^*\Omega S^{2n} \cong \Lambda[e_{2n-1}] \otimes \Gamma[x_{4n-2}].$$

We thus know the cohomology of the base and of the total space, and we know a basic relation among them by the edge homomorphism of Remark 4.5.1. If we can show the edge homomorphism is injective, then the spectral sequence will have to collapse.

Or, equivalently, if we can show that the edge homomorphism is surjective in homology.

(?, Corollary 4.4.3)

Since the cohomology changes based on the parity of n, we break into those two cases. Let's begin with $n = 2m+1$. In this case, we have

$$\begin{tikzcd} H^*(\Omega\Sigma S^{2n};\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^*(\Omega\Sigma S^n;\mathbb{Z}) \arrow[d, equal] \\ \Gamma[x_{2n}] \arrow[r] & \Gamma[y_{2n}] \otimes \Lambda[e]. \end{tikzcd}$$

If we can show $x_{2n} \mapsto y_{2n}$, the algebra structure will finish the job. This follows from the splitting itself: h started life as

$$\Sigma\Omega\Sigma S^n \to \Sigma(S^n)^{\wedge 2},$$

which is an isomorphism in cohomology degree $2n+1$, so that h has this property in cohomology degree $2n$. Since the Serre spectral sequence then collapses, in order for its E_2–page to have the correct form, the cohomology of the fiber F must be

$$H^*F \cong \Sigma^n\mathbb{Z}.$$

Since F is simply connected, we thus learn $F \simeq S^n$.

In general, the identification of the fiber is harder because the cohomology of X is not so sparse.

(?, Proposition 4.4.4)

Next, we turn to the case where $n = 2m$ is even. This time the same map acts by

$$\Gamma[x_{2n}] \to \Gamma[y_n],$$
$$x_{2n} \mapsto \frac{1}{2}y_n^2.$$

The algebra structure then gives

$$x_{2n}^{[k]} = \frac{1}{k!}x_{2n}^k \mapsto \frac{1}{k!}\left(\frac{1}{2}y_n^2\right)^k = \frac{(2k)!}{2^k k!}y_n^{[2k]},$$

which is a unit 2–*locally*. From this, running the Serre spectral sequence

Why is this being a unit relevant?

with 2–local coefficients, we learn

$$H^*(F;\mathbb{Z}_{(2)}) \cong \Sigma^n \mathbb{Z}_{(2)}.$$

From this, we conclude $F_{(2)} \simeq S^n_{(2)}$. □

Remark 5.4.3. In fact, we can even identify the exact extension as a familiar *map*. Simply by connectivity, the Freudenthal map

$$S^n \xrightarrow{e} \Omega\Sigma S^n$$

becomes null when postcomposed with

$$\Omega\Sigma S^n \to \Omega\Sigma S^{2n}.$$

However, we also know by Freudenthal's theorem that the map e is a cohomology isomorphism in degree n. It follows that its factorization

$$S^n \xrightarrow{\simeq} F \to \Omega\Sigma S^n$$

is a homotopy equivalence.

The map "e" is usually called this as an abbreviation of "Einhängung", German for "suspend". The continuation of this fiber sequence is

$$\cdots \to \Omega S^n \xrightarrow{e} \Omega^2 S^{n+1} \xrightarrow{h} \Omega^2 S^{2n+1} \xrightarrow{p} S^n \xrightarrow{e} \ldots,$$

where "p" is short for "Whitehead product". The enterprising unstable homotopy theorist can read more in Neisendorfer's book.

5.5 Calculations in the EHP spectral sequence

In this section, we will implicitly 2–localize everything.

Last time, we produced an exact sequence

$$S^n \to \Omega S^{2n+1} \to \Omega S^{n+1},$$

and Lemma 1.4.10 gives rise to exact sequences

$$\Omega^m S^n \to \Omega^{m+1} S^{2n+1} \to \Omega^{m+1} S^{n+1}$$

for all $m \geq 0$. Judicious choice of m allows us to knit these exact sequences together into a filtered object:

$$\begin{array}{ccccccccccccc}
\Omega^1 S^1 & \longrightarrow & \Omega^2 S^2 & \longrightarrow & \Omega^3 S^3 & \longrightarrow & \cdots & \longrightarrow & \Omega^{n-1} S^{n-1} & \longrightarrow & \Omega^n S^n & \longrightarrow \cdots \longrightarrow & \Omega^\infty \mathbb{S} \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\Omega^1 S^1 & & \Omega^2 S^3 & & \Omega^3 S^5 & & \cdots & & \Omega^{n-1} S^{2n-3} & & \Omega^n S^{2n-1} & \cdots. &
\end{array}$$

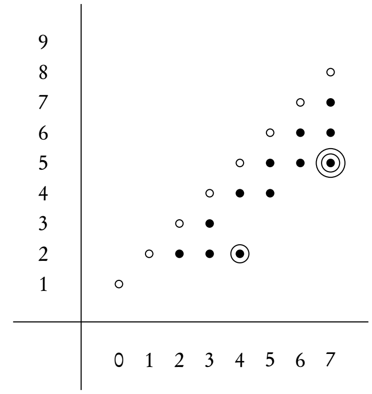
(?, Proposition 1.5.7)

Definition 5.5.1. Taking homotopy groups then gives the *EHP spectral sequence* of signature

$$E^1_{s,t} = \pi_s \Omega^t S^{2t-1} \cong \pi_{s+t} S^{2t-1} \Rightarrow \pi_* \mathbb{S}.$$

This type signature may seem ridiculous: it takes as input all of the unstable homotopy groups of spheres, and it computes the stable ones—which, by Corollary 2.2.8 we must have already known. We will see that the utility of the EHP spectral sequence ultimately lies *how* it performs its computation with only *partially* specified inputs, rather than the impossible situation of perfect information.

Indeed, for all our work, we can't fill out very much of this spectral sequence: we know a few stable groups and we know $\pi_{\le 6} S^3$. These known groups are tabulated at right. The j^{th} column of the E_∞–page of the spectral sequence forms the associated graded of a filtration on $\pi_j \mathbb{S}$. These facts together are enough to make a small observation: the $s = 1$ column is empty aside from a torsion-free group in position $(1,2)$, hence this group must receive a differential if it's to compute the expected $\pi_1 \mathbb{S} \cong \mathbb{Z}/2$. This is actually part of a family of differentials:



(?, Proposition 1.5.13)

Lemma 5.5.2. *The differentials*

$$d^1 \colon E^1_{s,s+1} \to E^1_{s-1,s}$$

along the main diagonal are given by multiplication by $1 + (-1)^s$.

Proof. These differentials have a geometric origin. There is a map

$$J_n \colon O(n) \to \Omega^n S^n$$

given by restricting attention to the unit sphere and suspending once. These maps assemble into a map of exact sequences:

The suspension here is crucial: the behavior of a point in $O(n)$ on the unit sphere need not fix any particular point, so does not map to the pointed loopspace without first passing to the suspension.

$$\begin{tikzcd}
\Omega^{n-1} S^{n-1} \arrow[r] & \Omega^n S^n \arrow[r] & \Omega^n S^{2n-1} \\
O(n-1) \arrow[r] \arrow[u, "J_{n-1}"] & O(n) \arrow[r] \arrow[u, "J_n"] & S^{n-1}. \arrow[u]
\end{tikzcd}$$

This collection of exact sequences can also be stitched together to form a filtered object, whose associated spectral sequence takes the form

$$E^1_{s,t} = \pi_s S^{t-1} \Rightarrow \pi_s O(\infty).$$

The behavior of this map and of the exact sequence involving the orthogonal groups is complex, but one may check geometrically that the map $S^{n-1} \to \Omega^n S^{2n-1}$ selects the bottom cell.

Actually, the right-hand square only commutes after looping once, but it commutes as-is after applying π_*.

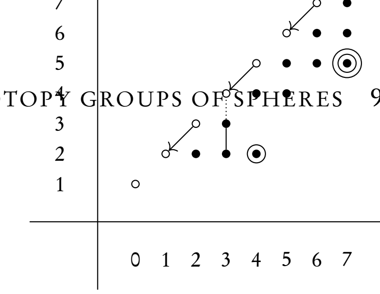
To avoid the complexity of the orthogonal groups, we may restrict to an even simpler situation: there is a map

$$\mathbb{R}\mathrm{P}^{n-1} \to O(n)$$

which sends a line in $\mathbb{R}^n$ to the reflection along that line. These maps also arrange into a commutative diagram

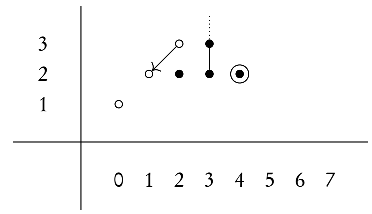
$$\begin{tikzcd}
O(n-1) \arrow[r] & O(n) \arrow[r] & S^{n-1} \\
\mathbb{R}\mathrm{P}^{n-2} \arrow[r] \arrow[u] & \mathbb{R}\mathrm{P}^{n-1} \arrow[r] \arrow[u] & S^{n-1}, \arrow[u, equal]
\end{tikzcd}$$

but this time the bottom row is a *coexact* sequence. The functorial Hurewicz isomorphism $\pi_{n-1} S^{n-1} \cong H_{n-1}(S^{n-1}; \mathbb{Z})$ means that, for the purposes of understanding this bottommost group, we can study the behavior of the exact sequence in homology. This family of coexact sequences assembles



into the cellular filtration of $\mathbb{R}\mathrm{P}^\infty$, and the associated spectral sequence is that of Example 2.4.3. The differential in $C_*^{\mathrm{cell}}(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ is given by the claimed formula. □

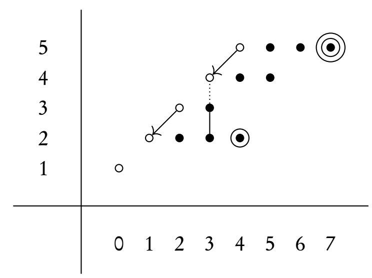
The term $E^1_{2,2}$ we know from many sources: $\pi_4 S^3$ lies in the stable range; we have computed $\pi_4 S^3$ using Serre's method; if the EHPSS converges to $\mathbb{Z}/2$ in that column, then *at least* a $\mathbb{Z}/2$ must be present; and a fourth method which we now describe. Rather than studying the full filtered object $\operatorname{colim}_n \Omega^n S^n$, we can stop at any finite stage. This produces a spectral sequence converging to $\pi_* \Omega^n S^n$, and its E^1–page looks like a horizontal trunction of the full EHP spectral sequence, with the bottom n rows surviving. For example, we have produced the truncation converging to $\pi_{*+3} S^3$. The first four columns compute



$$\pi_3 S^3 \cong \mathbb{Z}, \qquad \pi_4 S^3 \cong \mathbb{Z}/2, \qquad \pi_5 S^3 \cong \mathbb{Z}/2, \qquad \pi_6 S^3 \cong \mathbb{Z}/4,$$

confirming the unstable calculation asserted at the end of Section 5.1.

In order to calculate any further in this spectral sequence—that is, in order to access the $s = 4$ column—we must understand the unknown group $E^1_{3,4} \cong \pi_7 S^5$. This, too, can be accessed by truncation, where it is found to be $\mathbb{Z}/2$.



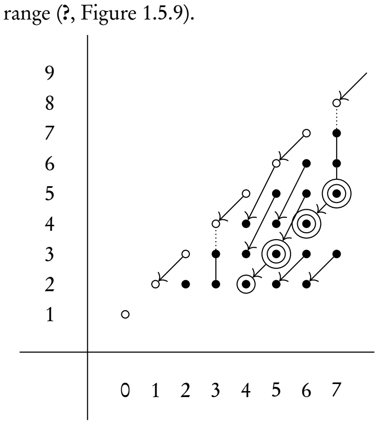
These kinds of observations power significant inductive computation. They also power inductive proofs of *qualitative* results, as in the following:

Corollary 5.5.3. *The rational homotopy groups of spheres are given by*

$$\pi_* S^{2n+1} \otimes \mathbb{Q} \cong \Sigma^{2n+1}\mathbb{Q}, \qquad \pi_* S^{2n} \otimes \mathbb{Q} \cong \Sigma^{2n}\mathbb{Q} \oplus \Sigma^{4n-1}\mathbb{Q}.$$

Proof. We begin by studying the torsion-free groups on the main diagonal. After the E_2–page, all those groups other than $E^1_{0,1}$ get replaced by torsion groups: either 0 or $\mathbb{Z}/2$. When truncating the spectral sequence to study the homotopy groups of the even-dimensional sphere S^{2n}, the torsion-free group in position $E^1_{2n-1,2n}$ is no longer able to receive a differential, hence survives to contribute a torsion-free term to $\pi_{4n-1} S^{2n}$. This phenomenon does not occur for odd-dimensional spheres: the truncation in that case deletes only pairs of terms on the main diagonal.

We also include a picture of the fully specified EHP spectral sequence in this range (?, Figure 1.5.9).



We must additionally argue that the region below the main diagonal is populated only by finite torsion groups. Using the truncation method, each term $E^1_{s,t} = \pi_{s+t} S^{2t-1}$ is determined by the terms $E^1_{\leq t-s-1, \leq 2t-1}$—that is, the distance of a term from the main diagonal determines what vertical strip of the spectral sequence is needed to understand it. The march of the diagonal line down-and-right outpaces the march of the vertical line to the right, powering the inductive claim that if the corner of the E^∞–page is finite and torsion (apart from $E^\infty_{0,1}$), then the entire E^∞–page is finite and torsion (apart from $E^\infty_{0,1}$). □

5.A Bott periodicity

5.2 Homework #1

Problem 5.2.1. In class, I used the mysterious phrase "limit condition" twice. Given a functor $F\colon \mathsf{J} \to \mathsf{C}$, thought of as a J–shaped diagram in C, we define a *cone* of F to be a constant functor $x\colon \mathsf{J} \to \mathsf{C}$ together with a natural transformation $x \to F$. A *limit* of F is a terminal object in the category of cones.

1. Expand the definition of natural transformation and constant functor to reveal that a cone is equivalent to the data of an object $x \in \mathsf{C}$ together with maps $f_j\colon x \to F(j)$ for each object $j \in J$ such that for any map $g : j \to j'$ in the diagram there is a commuting triangle

$$\begin{tikzcd} & x \arrow[dl, "f_j"'] \arrow[dr, "f_{j'}"] & \\ F(j) \arrow[rr, "F(g)"] & & F(j'). \end{tikzcd}$$

2. Now expand the definition of limit to see that a limit, expressed as an object ℓ together with maps h_j, has the property that any other cone point x and its maps f_j factor uniquely through a map $x \to \ell$.

3. Show that the product $X \times Y$ is the limit of the diagram with objects X and Y and no non-identity arrows.

4. Show that the equalizer E of a pair of functions $X \rightrightarrows Y$ of sets is indeed the limit of that diagram.

Problem 5.2.2. A functor $G\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Sets}$ is called *representable* when there exists an object Y and a natural isomorphism

$$G \xrightarrow{\simeq} \mathsf{Sets}(-, Y).$$

1. From a morphism $t\colon Y \to Y'$ of representing objects, construct a natural transformation $t_*\colon G \to G'$ of the functors they represent.

2. From a natural transformation $G \to G'$ of represented functors, construct a morphism $Y \to Y'$ of the representing objects.

3. Show also that your assignments respect composition of natural transformations and of morphisms.

4. Show that your assignments are mutual inverses, i.e., a natural transformation of representable functors is exactly the same information as a morphism of representing objects.

Congratulations! You have proved the Yoneda lemma: the functor

$$\mathsf{C} \to \mathsf{Categories}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sets})$$

describes a fully faithful embedding.

Problem 5.2.3. Explain convincingly why the usual recipe for forming a group structure on $\pi_1(X, x_0)$ does not apply to the set of relative homotopy classes $\pi_1(I, \partial I)$.

Problem 5.2.4. Let $p\colon E \to B$ be a map and consider

$$Z = \{(e, \gamma) \in E \times B^I : p(e) = \gamma(0)\} \subseteq E \times B^I.$$

A *path lifting function* for p is a map $\lambda\colon Z \to E^I$ with $\lambda(e, \gamma)(0) = e$ and $p \circ \lambda(e, \gamma) = \gamma$.

1. Show that p is a fibration if and only if there is a path lifting function λ for p.
2. Let $p\colon E \to B$ be a fibration with fiber F, and let P_p be the pathspace construction p described in class. Given a path lifting function $\lambda\colon Z \to E^I$ for p, define maps

$$\begin{aligned} g\colon F &\to P_p, & f &\mapsto (f, \omega_0), \\ h\colon P_p &\to F, & (e, \gamma) &\mapsto [\lambda(e, \gamma^{-1})](1), \end{aligned}$$

   where "γ^{-1}" denotes the path γ run backwards. Show that g and h present the two halves of a homotopy equivalence.

Problem 5.2.5. Show that if (X, A) is a relative CW-complex, then X/A is a CW-complex. Given CW-complexes X and Y, use this to concoct appropriate conditions so that $X \wedge Y$ is a CW-complex.

Problem 5.2.6. Suppose (X, A) is a relative CW-complex and $p\colon E \to B$ is a weak fibration. Show that for any map $f\colon X \to E$ and homotopies $F\colon X \times I \to B$, $H\colon A \times I \to E$ with $F_0 = p \circ f$, $H_0 = f|_A$, and $p \circ H = F|_{A \times I}$ there is a homotopy $G\colon X \times I \to E$ lifting F with $G/(A \times I) = H$, $G_0 = f$, and $p \circ G = F$. Diagrammatically, these conditions are summarized as

$$\begin{tikzcd} (X \times \{0\}) \cup (A \times I) \arrow[r, "f \cup H"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dotted, "G"] & B. \end{tikzcd}$$

Problem 5.2.7. Suppose X is obtained from A by attending n–cells $\{e^n_\beta \mid \beta \in B\}$. Show that $X/A \cong \bigvee_{\beta \in B} S^n_\beta$, and that the homeomorphism can be chosen so that the diagram

$$\begin{tikzcd} & (S^n, *) \arrow[r, "i_\beta"] & (\bigvee_\beta S^n_\beta, *) \arrow[dd, "\cong"] \\ (D^n, S^{n-1}) \arrow[ur, "p'"] \arrow[dr, "f_\beta"'] & & \\ & (X, A) \arrow[r, "p"] & (X/A, *) \end{tikzcd}$$

commutes, where f_β is the characteristic map of e^n_β.

5.3 Homework #2

Task 5.3.1. Read Chapter 5 to see a "proper" definition of a CW-structure on a pre-existing space.

Task 5.3.2. Skim through Chapter 6 and look at all the proofs we skipped. Try reading a few. Then try reading a few more. Move on to the rest of the problem set whenever you like.

Problem 5.3.3. Show that if (X, A) is a relative CW-complex, then X/A is a CW-complex. Given CW-complexes X and Y, use this to concoct appropriate conditions so that $X \wedge Y$ is a CW-complex.

Problem 5.3.4. Suppose (X, A) is a relative CW-complex and $p\colon E \to B$ is a weak fibration. Show that for any map $f\colon X \to E$ and homotopies $F\colon X \times I \to B$, $H\colon A \times I \to E$ with $F_0 = p \circ f$, $H_0 = f|_A$, and $p \circ H = F|_{A \times I}$ there is a homotopy $G\colon X \times I \to E$ lifting F with $G|_{A\times I} = H$, $G_0 = f$, and $p \circ G = F$. Diagrammatically, these conditions are summarized as

$$\begin{tikzcd}
(X \times \{0\}) \cup (A \times I) \arrow[r, "f \cup H"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "F"] \arrow[ur, dotted, "G"] & B.
\end{tikzcd}$$

Problem 5.3.5. Suppose X is obtained from A by attending n–cells $\{e^n_\beta \mid \beta \in B\}$. Show that $X/A \cong \bigvee_{\beta \in B} S^n_\beta$, and that the homeomorphism can be chosen so that the diagram

$$\begin{tikzcd}
 & (S^n, *) \arrow[r, "i_\beta"] & (\bigvee_\beta S^n_\beta, *) \arrow[dd, "\cong"] \\
(D^n, S^{n-1}) \arrow[ur, "p'"] \arrow[dr, "f_\beta"'] & & \\
 & (X, A) \arrow[r, "p"] & (X/A, *)
\end{tikzcd}$$

commutes, where f_β is the characteristic map of e^n_β.

Problem 5.3.6. Show that if $f\colon X \to Y$ is a cellular map of CW-complexes, then $Y \cup_f CX$ is naturally a CW-complex.

Problem 5.3.7. Justify some of our ad hoc constructions from class by proving the following: let C be a category with finite products and a zero object and let $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Groups}$ be a group-valued functor. Show that if F is represented by an object Y by a natural transformation $t\colon \mathsf{C}(-, Y) \to F$, then Y carries a group structure which causes $\mathsf{C}(-, Y)$ to be group-valued and the comparison natural isomorphism t to respect the group structure.

5.4 Homework #3

Problem 5.4.1. Consider a diagram of three inverse systems of abelian groups

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & A_n \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d] & B_n \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "h_{n+1}"] \arrow[d] & C_n \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots,
\end{tikzcd}$$

such that every column forms a short exact sequence.

1. Show that the limit of, say, (A_n) can be described by the kernel sequence

$$\lim A_n \xrightarrow{\ker} \prod_n A_n \xrightarrow{\prod_n \mathrm{id} - \prod_n f_n} \prod_n A_n.$$

2. The map on limits $\lim A_n \to \lim B_n \to \lim C_n$ no longer need be short-exact. Define $\lim^1 A_n$ to be the *cokernel* of the map described above, and show that there is instead an exact sequence of the form

$$0 \to \lim A_n \to \lim B_n \to \lim C_n \to \lim^1 A_n \to \lim^1 B_n \to \lim^1 C_n \to 0.$$

Problem 5.4.2. Consider a tower of fibrations

$$\cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_n \to \cdots.$$

Show that there is a short exact sequence, called the *Milnor sequence*, given by

$$0 \to \lim_n{}^1 \left(\pi_{m+1} X_n\right) \to \pi_m \left(\lim_n X_n\right) \to \lim_n (\pi_m X_n) \to 0.$$

(Hint: find a model for the limit of the tower of fibrations analogous to the one for the colimit of a tower of inclusions presented in Switzer 7.53. This model is itself inspired by rewriting the tower as an endomorphism of an infinite product, and filtering this model by "near the start" and "near the end" of the endomorphism.)

Problem 5.4.3. Produce the Milnor short exact sequence for a generalized cohomology theory E applied to an increasing union of spaces X_n:

$$0 \to \lim_n{}^1 (E^{m-1} X_n) \to E^m (\mathrm{colim}_n X_n) \to \lim_n (E^m X_n) \to 0.$$

This problem can also be done independently of Problem 2, by directly using the endomorphism-cylinder construction presented in Switzer and applying cohomology to that, rather than taking the time to figure out what its dual looks like and remembering that cohomology appears as the homotopy of a certain spectral object.

Problem 5.4.4. 1. Use this to calculate the integral cohomology of "the circle with p inverted". This space is given by infinitely iterating the mapping cylinder construction on the p-fold covering

$$S^1 \xrightarrow{p} S^1.$$

(That is: the first stages of this look like $S^1 \cup_p (S^1 \times I)$, then $(S^1 \cup_p (S^1 \times I)) \cup_p (S^1 \times I), \ldots$.)

If you haven't seen this construction before, you should check that for a ring element $r \in R$ and an R-module M,

$$\operatorname{colim}\left(M \xrightarrow{r} M \xrightarrow{r} M \xrightarrow{r} \cdots\right) = M[r^{-1}].$$

2. Compare your answer with the *homology* of this same space and analyze the behavior of the universal coefficient sequence.

Task 5.4.5. Read pages 158–163 of Switzer, which describe the representability of sheaf-like functors defined only on *finite* CW-complexes. (In particular, this makes fairly intensive use of the understanding of inverse limits which you have just developed.)

Task 5.4.6. Strongly consider reading pages 346–351 of Switzer, which actually goes through the identification of the Serre E^2 term. It's very tedious, but it's worth seeing once. Alternatively, you could these course notes, which give a much prettier description of the Serre spectral sequence in terms of a double complex: `http://math.mit.edu/classes/18.906/spr09/sss.pdf` . As trade, you then have to additionally work out how such a spectral sequence arises as a filtration spectral sequence.

Problem 5.4.7. As in class, define $\mathbb{S}_{(p)}$ to be the spectrum representing $X \mapsto h\mathsf{Spectra}(\Sigma^\infty X, \mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

1. Describe the homology functor associated to $\mathbb{S}_{(p)}$. (Hint: restrict attention to finite complexes X, where $DX = F(X, \mathbb{S})$ defines an involutive dual.)
2. Demonstrate $E_{(p)} \simeq E \wedge \mathbb{S}_{(p)}$ for any spectrum E. In particular, this gives $\mathbb{S}_{(p)} \wedge \mathbb{S}_{(p)} \simeq \mathbb{S}_{(p)}$.
3. Define $\pi_{n,(p)} E = [\mathbb{S}^n_{(p)}, E]$. Show $\pi_{n,(p)} E_{(p)} = \pi_n E_{(p)}$.
4. Conclude the more general adjunction
$$h\mathsf{Spectra}(F, E_{(p)}) \cong h\mathsf{Spectra}(F_{(p)}, E_{(p)}).$$
5. Finally, we can also give a concrete construction of $\mathbb{S}_{(p)}$. Consider the infinite directed system
$$\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \cdots.$$
Form the mapping telescope T associated to this system and check that this gives a model for $\mathbb{S}_{(p)}$ in $h\mathsf{Spectra}$.

5.5 Homework #4

Problem 5.5.1. Prove the transgressive differential lemma from class. Let $F \xrightarrow{i} E \to B$ be a fibration, and let

$$B \xleftarrow{\pi} C(i) \xrightarrow{\delta} \Sigma F$$

by the naturally induced maps. Show that the following situations are equivalent:

- A class $x \in H_n B$ has $d_{<n}(x) = 0$ and $d_n(x) = y$ for some class $y \in H_{n-1} F$ (up to some indeterminacy).
- There is a class $\tau(x) \in H_n C(i)$ with $\delta_* \tau x = y$ and $\pi_* \tau x = x$.

Problem 5.5.2. The *(2-adic) Bockstein spectral sequence* is the filtration spectral sequence arising from the diagram

$$\begin{array}{ccccccccc} \mathbb{Z}_2^\wedge & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & \cdots & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2. \end{array}$$

Applying $H^*(X;-)$ to this diagram of coefficients gives a spectral sequence of signature

$$E_1^{*,*} = \bigoplus_* H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[w] \otimes H^*(X;\mathbb{F}_2) \Rightarrow H^*(X;\mathbb{Z}_2^\wedge),$$

where the E_1–page consists of many duplicated copies of $H^*(X;\mathbb{F}_2)$, which we can think of as tagged by monomials in w.

There is (of course) also a homological version of this construction, which you should also be aware of.

1. Show that the differentials in this spectral sequence are "w–linear", i.e., $d_r^{\mathrm{BSS}}(w^k x) = w^k d_r^{\mathrm{BSS}}(x)$.
2. Show that a torsion-free class $x \in H^*(X;\mathbb{Z}_2^\wedge)$ is in $\ker d_r^{\mathrm{BSS}}$ on all pages E_r and never in $\operatorname{im} d_r^{\mathrm{BSS}}$. Demonstrate that this condition is equivalent to the corresponding class in the spectral sequence being w–torsion–free.
3. More generally, show that the order of w–torsion of a class on the E_∞ page of the spectral sequence is identical to the 2–primary torsion order of the corresponding cohomology class in $H^*(X;\mathbb{Z}_2^\wedge)$.
4. Show that d_1^{BSS} in this spectral sequence is computed by the Steenrod square Sq^1.

Problem 5.5.3. Use this spectral sequence to make a calculation of $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}_2^\wedge)$ from the calculation of $H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ given in class. You will want to know the following mysterious formula: for any class $x \in H^{\mathrm{even}}(X;\mathbb{F}_2)$ where $d_r^{\mathrm{BSS}}(x)$ is defined, we have

This is Proposition 6.8 of May's *A general algebraic approach to Steenrod operations.*

$$d_r^{\mathrm{BSS}}(x^2) = \begin{cases} \mathrm{Sq}^1(x)\cdot x + \mathrm{Sq}^{|x|}\mathrm{Sq}^1(x) & \text{for } r = 2, \\ d_{r-1}^{\mathrm{BSS}}(x)\cdot x & \text{for } r > 2. \end{cases}$$

Problem 5.5.4. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fiber sequence, let $u \in H^n(F;\mathbb{F}_2)$ be class that transgresses to $\tau(u) \in H^{n+1}(B;\mathbb{F}_2)$, and suppose that for some integer $i \geq 1$ there is a Bockstein differential $d_i^{\mathrm{BSS}} v = \tau(u)$. Show that $d_{i+1}^{\mathrm{BSS}} p^* v$ is then defined and that $j^* d_{i+1}^{\mathrm{BSS}} p^*(v) = d_1^{\mathrm{BSS}}(u)$, where again d_1^{BSS} is the first Bockstein differential.

I haven't actually tried to work this out. You might find it helpful to know that there's a re-indexing of the Bockstein spectral sequence, where you instead use the inverse system $\{\mathbb{Z}/(2^j)\}_{j=1}^\infty$ and identify all the *fibers* of these maps as $\mathbb{Z}/2$—or maybe not.

Problem 5.5.5. **I may not have done a good job of stating this problem. If you run into issues with solving this, please email me so that I can fix whatever mistakes I've made. (The algebra extensions in part 2 seem particularly fishy. . . .)** In this problem, you will reinvent one of the main results of unstable rational homotopy. For a simply connected space X, we inductively define its *rationalization* to be a space $\mathbb{Q} \otimes X$ under X as follows: given a Postnikov fibration

$$K(\pi_n X, n) \to X[0, n] \to X[0, n),$$

and the rationalization map $X[0, n) \to (\mathbb{Q} \otimes X)[0, n)$, we construct a corresponding Postnikov fibration for $\mathbb{Q} \otimes X$ as the back face in

$$\begin{tikzcd}
 & K(\mathbb{Q} \otimes \pi_n X, n) \arrow[rr, equal] \arrow[dd] & & K(\mathbb{Q} \otimes \pi_n X, n) \arrow[dd] \\
K(\pi_n X, n) \arrow[ur] \arrow[rr, equal] \arrow[dd] & & K(\pi_n X, n) \arrow[ur] \arrow[dd] & \\
 & (\mathbb{Q} \otimes X)[0, n] \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
X[0, n] \arrow[ur] \arrow[rr] \arrow[dd] & & * \arrow[ur] \arrow[dd] & \\
 & (\mathbb{Q} \otimes X)[0, n) \arrow[rr] & & K(\mathbb{Q} \otimes \pi_n X, n+1) \\
X[0, n) \arrow[ur] \arrow[rr] & & K(\pi_n X, n+1). \arrow[ur] &
\end{tikzcd}$$

Here the nodes $X[0, n]$ and $(\mathbb{Q} \otimes X)[0, n]$ are *defined* as the total spaces of the pullback fibrations, and the map between them is induced by the universal map of fibrations. We set $\mathbb{Q} \otimes X$ to be the homotopy inverse limit

$$\mathbb{Q} \otimes X = \lim_n (\mathbb{Q} \otimes X)[0, n],$$

which has the factorization property

$$\begin{tikzcd}
\pi_* X \arrow[r] \arrow[rr, bend left] & \mathbb{Q} \otimes \pi_* X \arrow[r, "\simeq"] & \pi_*(\mathbb{Q} \otimes X).
\end{tikzcd}$$

Now, justify the following claims:

1. The rational cohomology of rational Eilenberg–Mac Lane spaces is given by
$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[x_n] & \text{if } n \text{ is even,} \\ \mathbb{Q}[x_n]/x_n^2 & \text{if } n \text{ is odd.} \end{cases}$$

2. The cohomology $H^*(X(n, \infty); \mathbb{Q})$ *as well as its ring structure* are completely determined by the cohomology ring $H^*(X[n, \infty); \mathbb{Q})$.

3. The map $X \to \mathbb{Q} \otimes X$ is an isomorphism on rational cohomology.

4. The Postnikov fibrations $K(\mathbb{Q}\otimes\pi_n X, n) \to (\mathbb{Q}\otimes X)[0,n] \to (\mathbb{Q}\otimes X)[0,n)$ give a model for $C^*(X;\mathbb{Q})$ whose underlying graded-commutative algebra is *free* and which uses the minimal number of algebra generators.

5. Any rational commutative differential-graded algebra A^* with $A^0 = \mathbb{Q}$ and $A^1 = 0$ inductively receives a quasi-isomorphism from a Sullivan model.

6. There is a sequence of Postnikov sections $X[0,n) \to K(\pi_n X, n+1)$, hence a space X, whose Sullivan model is the one associated to A^*.

7. Given a Sullivan model for $C^*(X;\mathbb{Q})$, its indecomposables compute the rational homotopy groups of X.

8. The rational homotopy groups of S^n, $n > 1$, are given by

$$\mathbb{Q}\otimes\pi_* S^n = \begin{cases} \Sigma^n \mathbb{Q} & \text{if } n \text{ is odd,} \\ \Sigma^n\mathbb{Q}\oplus\Sigma^{2n-1}\mathbb{Q} & \text{if } n \text{ is even.} \end{cases}$$

Such a presentation of the rational cochain complex is called a *Sullivan minimal model*. It may please you to check that two such models are related by a chain homotopy equivalence.

In fact, this happens in a natural way: a map $A^* \to B^*$ of cDGAs induces a map of their Sullivan models.

It's fun / instructive to see the natural algorithm for this *fail* in the case of $C^*(S^1 \vee S^1;\mathbb{Q})$.

Problem 5.5.6. 1. The tensor product of line bundles induces a map

$$BU(1)\times BU(1) \xrightarrow{\otimes} BU(1)$$

on the object $BU(1)$ representing the functor $X \mapsto \{\text{iso-classes of line bundles on } X\}$. Describe the behavior of this map in ordinary cohomology with $\mathbb{Z}$ coefficients.

2. In general, the tensor product of vector bundles induces a similar map

$$BU(n)\times BU(m) \xrightarrow{\otimes} BU(nm).$$

Describe the behavior of this map in ordinary cohomology as well.

Problem 5.5.7. The dual Steenrod algebra is a *Hopf algebra*, meaning that it not only has a multiplication map but also a diagonal map $\Delta\colon \mathscr{A}_* \to \mathscr{A}_*\otimes\mathscr{A}_*$ and an antipode map $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$. In class, we deduced a formula for Δ, and we showed that as an algebra the dual Steenrod algebra forms a polynomial ring. The antipode fits into the commutative diagram

$$\begin{tikzcd}
 & \mathscr{A}\otimes\mathscr{A} \arrow[rr, "\chi\otimes 1"] & & \mathscr{A}\otimes\mathscr{A} \arrow[dr, "\mu"] & \\
\mathscr{A} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{F}_2 \arrow[rr, "\eta"] & & \mathscr{A} \\
 & \mathscr{A}\otimes\mathscr{A} \arrow[rr, "1\otimes\chi"] & & \mathscr{A}\otimes\mathscr{A}, \arrow[ur, "\mu"'] &
\end{tikzcd}$$

along with the algebra unit η and counit ε. Use all this to give a recursive formula for the behavior of χ.

5.6 Homework #5

Problem 5.6.1. Suppose you believe in complex Bott periodicity, so that the homotopy groups of $BU(n)$ have the form $\pi_{\mathrm{odd}} BU(n) = 0$ and $\pi_{\mathrm{even}} BU(n) = \mathbb{Z}$ in the range $[0, 2n]$. Set $n = 3$ and describe the action of the Steenrod algebra on $H^*(BU(3); \mathbb{F}_2)$. Then try $n = 4$. Then $n = 5$. Stop once you get sick of the exercise.

Problem 5.6.2. Return to the picture of the Adams spectral sequence computing $\pi_* ko$ described in class. At a glance, it appears that there could be a potential differential $d_r h_1 = h_0^{r+1}$. *Without* assuming Bott periodicity, argue why this differential cannot occur. (Hint: $h_0 h_1 = 0$.)

Problem 5.6.3. Compute the first several terms (until you get tired) of a free resolution of $\mathbb{F}_2$ as a module over the Steenrod algebra. (To check your answer, you can find a considerable chunk of such a resolution on page 85 of this PDF: `https://www.math.cornell.edu/ hatcher/AT/ATch5.pdf`.) Once you have the resolution, use it to compute Ext and compare your answer with the part of the Adams spectral sequence drawn in class.

Problem 5.6.4. Let $E(1)$ denote the exterior $\mathbb{F}_2$–algebra on two generators e_1 and e_3, of degrees 1 and 3 respectively. Calculate $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$.

Task 5.6.5. Try to read Section 8 of Steve Wilson's *Brown–Peterson Homology: An Introduction and Sampler*. He gives a calculation of the mod–p Steenrod algebra there—try to convert it into a calculation of the mod–2 Steenrod algebra, which simplifies his discussion considerably. (The space he calls $\underline{K}_q$ is what we are calling $K(\mathbb{F}_p, q)$.) (You shouldn't need any of the meat of the previous 7 sections to read this—except for the definition of a "Hopf ring", which is at the start of Section 7.)

Problem 5.6.6. Figure out both the statement and the proof of the 5–Lemma and the Snake Lemma in mod–$\mathscr{C}$ homological algebra.

5.7 Where to go from here (Oh, the places you'll go!)

You now know enough algebraic topology to be dangerous. (Anyone who hears "spectrum" and doesn't flinch is dangerous.) However, we're sitll a ways from the forefront of the field, and I wanted to point out some current pedagogical landmarks, c. 2017 (and tailored to my preferences, of course).

A lot of what we learned in this class feeds directly into the *vector fields on spheres* problem, a major accomplishment of 1970's topology, and among the first conceptual problems solved by broad-reaching computational invention. This is totally a "next step" for this class.

Stable homotopy houses a lot of geometry through *bordism homology*, where formal sums of singular simplices are replaced by maps in from manifolds with boundary. Even the bordism homology of a point is of great interest—it houses a kind of homotopical intersection theory for manifolds.

Algebraic K–theory is a generic tool that captures a lot of geometric information about any kind of context: algebraic geometry, manifolds, groups, ... It's kind of bottomless, and being able to compute anything about it is often met with cheers and wild career success.

Homological stability refers to the phenomenon that "things", like symmetry groups, often occur in families, like $\{\Sigma_m\}_{m=1}^{\infty}$, and that these families have highly compatible homologies. This comes in many shapes and sizes and generally has geometric content.

Equivariant homotopy theory is of geometric interest because symmetric groups appear everywhere in geometric contexts, and requiring homotopy theory to be mindful of it makes homotopy theory itself considerably more geometry.

Homotopy theory has surprising applications in algebraic geomtery. Artin–Mazur gives access to a homotopy type associated to a site on a scheme, and e.g. the *étale homotopy type* carries a remarkable amount of information about the scheme. Meanwhile, *motivic homotopy theory* is a modern blend of homotopical techniques with algebro-geometric ones, with large implications for both sides: Voevodsky used this to settle the Milnor/Bloch–Kato conjectures, and Isaksen has used this to push $\pi_*\mathbb{S}$ computations considerably further.

Goodwillie calculus describes a natural sequence of stable invariants to almost any homotopical functor, and it turns out that these invariants generalize / unify a bunch of unrelated classical invariants from practically every corner of homotopy theory.

Spectral algebra(ic geometry) is a re-encoding of classical algebra into spectra. We talked about "ring spectra" in this class, but it turns out that asking for, e.g., associativity requires real heavy machinery. The rewards are great: the theory of "modules" associated to such a spectrum is very rich and eases a lot of arguments, and the extra operations impressed on such a spectrum encode useful (and often classically relevant) data. (One can then try "doing algebraic geometry" in this setting, which appears to be a hole without bottom.)

There is an older program in homotopy theory, called *chromatic homotopy theory*, which continues to give the tightest results on the "global" behavior of homotopy theory be comparing it to particular (highly complex) algebraic models. You can get a lot of intuition very quickly by learning some of this. Its genesis was in understanding certain periodic phenomena in the Adams spectral sequence—hence is often taught from a highly computational point of view, if that's your thing.