

Follow the Gradient

The power of differentiation

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- The big idea: optimisation by following gradients
- Recap: what are gradients and how do we find them?
- Recap: Singular Value Decomposition and its applications
- Example: Computing SVD using gradients - The Netflix Challenge

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 - How do we select those parameters?
- In deep learning/differentiable programming we typically define an objective function that we *minimise* (or *maximise*) with respect to those parameters
- This implies that we're looking for points at which the gradient of the objective function is zero w.r.t the parameters

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 - First order methods, second order methods, subgradient methods...
- With deep learning we're primarily interested in first-order methods¹.
 - Primarily using variants of gradient descent: a function $F(\mathbf{x})$ has a minima² (or a saddle-point) at a point $\mathbf{x} = \mathbf{a}$ where \mathbf{a} is given by applying $\mathbf{a}_{n+1} = \mathbf{a}_n - \alpha \nabla F(\mathbf{a}_n)$ until convergence from some initial point \mathbf{a}_0 .

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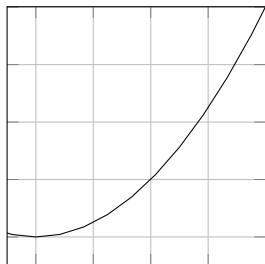
²not necessarily global or unique

Recap: what are gradients and how do we find them?

The derivative in 1D

- Recall that the gradient of a straight line is

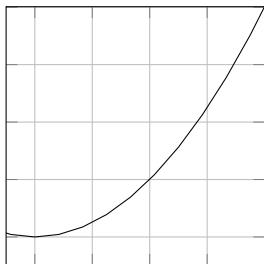
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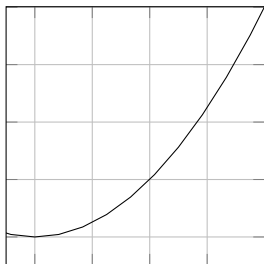
- Recall that the gradient of a straight line is $\frac{\Delta y}{\Delta x}$.
- For an arbitrary real-valued function, $f(a)$, we can approximate the derivative, $f'(a)$ using the gradient of the *secant line* defined by $(a, f(a))$ and a point a small distance, h , away $(a + h, f(a + h))$: $f'(a) \approx \frac{f(a+h) - f(a)}{h}$.



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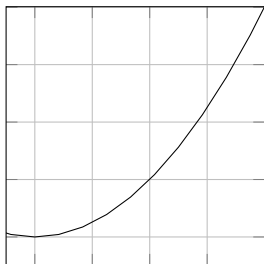
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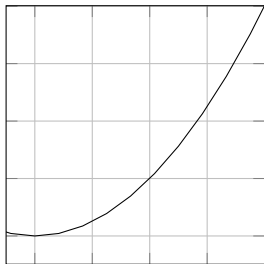
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- This expression is 'Newton's Difference Quotient'.
- As h becomes smaller, the approximated derivative becomes more accurate.
- If we take the limit as $h \rightarrow 0$, then we have an exact expression for the derivative:
$$\frac{df}{da} = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}.$$



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The derivative of $y = x^2$ from first principles

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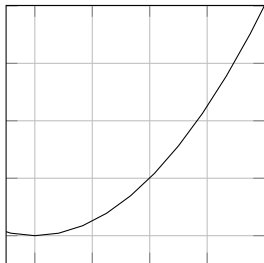
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Aside: numerical approximation of the derivative

- For numerical computation of derivatives it is better to use a “centralised” definition of the derivative:

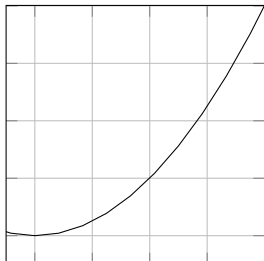
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 - The bit inside the limit is known as the *symmetric difference quotient*

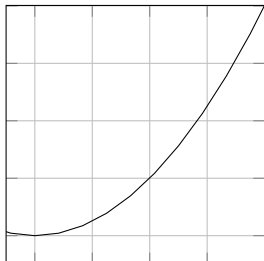


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- For small values of h this has less error than the standard one-sided difference quotient.



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Aside: numerical approximation of the derivative

- If you are going to use difference quotients to estimate derivatives you need to be aware of potential rounding errors due to floating point representations.
 - Calculating derivatives this way using less than 64-bit precision is rarely going to be useful. (Numbers are not represented exactly, so even if h is represented exactly, $x + h$ will probably not be)
 - You need to pick an appropriate h - too small and the subtraction will have a large rounding error!

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Derivatives of deeper functions

- Deep learning is all about optimising deeper functions; functions that are compositions of other functions
 - e.g. $z = f \circ g(x) = f(g(x))$

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- Deep learning is all about optimising deeper functions; functions that are compositions of other functions
 - e.g. $z = f \circ g(x) = f(g(x))$
- The chain rule of calculus tells us how to differentiate compositions of functions:
 - $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$

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Example: differentiating $z = x^4$

Note that this is a silly example that just serves to demonstrate the principle!

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Equivalently, from first principles:

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$$\frac{dz}{dx} = \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h}$$

$$\frac{dz}{dx} = \lim_{h \rightarrow 0} \frac{h^4 + 4h^3x + 6h^2x^2 + 4hx^3 + x^4 - x^4}{h}$$

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 - Equivalently, $\mathbf{y}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{y}(t+h) - \mathbf{y}(t)}{h}$ if the limit exists.

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Functions of multiple variables: partial differentiation

- What if the function we're trying to deal with has multiple variables³ (e.g. $f(x, y) = x^2 + xy + y^2$)?
 - This expression has a pair of *partial derivatives*, $\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = x + 2y$, computed by differentiating with respect to each variable x and y whilst holding the other(s) constant.

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$$\frac{\partial f}{\partial x_i}(a_1, \dots, a_n) = \lim_{h \rightarrow 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{h}.$$

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- In the case of a vector-valued multivariate function, the partial derivatives form a matrix called the **Jacobian**.

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 - **How will we find the gradients of these?**

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The chain rule for vectors

Suppose that $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, g maps from \mathbb{R}^m to \mathbb{R}^n and f maps from \mathbb{R}^n to \mathbb{R} .

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If $\mathbf{y} = g(\mathbf{x})$ and $z = f(\mathbf{y})$, then

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Equivalently, in vector notation:

$$\nabla_{\mathbf{x}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right)^\top \nabla_{\mathbf{y}} z$$

where $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ is the $n \times m$ Jacobian matrix of g .

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 - For all index tuples i , $(\nabla_{\mathbf{X}} z)_i$ gives $\frac{\partial z}{\partial \mathbf{x}_i}$.
 - Thus, if $\mathbf{Y} = g(\mathbf{X})$ and $z = f(\mathbf{Y})$ then $\nabla_{\mathbf{X}} z = \sum_j (\nabla_{\mathbf{X}} \mathbf{Y}_j) \frac{\partial z}{\partial \mathbf{Y}_j}$.

Recap: what are gradients and how do we find them?

Example: $\nabla_{\mathbf{W}} f(\mathbf{X}\mathbf{W})$

- Let $\mathbf{D} = \mathbf{X}\mathbf{W}$ where the rows of $\mathbf{X} \in \mathbb{R}^{n \times m}$ contain some fixed *features*, and $\mathbf{W} \in \mathbb{R}^{m \times h}$ is a matrix of weights.
- Also let $\mathcal{L} = f(\mathbf{D})$ be some scalar function of \mathbf{D} that we wish to minimise.

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- Also let $\mathcal{L} = f(\mathbf{D})$ be some scalar function of \mathbf{D} that we wish to minimise.
- What are the derivatives of \mathcal{L} with respect to the weights \mathbf{W} ?

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Example: $\nabla_w f(\mathbf{XW})$

- Start by considering a specific weight, W_{uv} : $\frac{\partial \mathcal{L}}{\partial W_{uv}} = \sum_{i,j} \frac{\partial \mathcal{L}}{\partial D_{ij}} \frac{\partial D_{ij}}{\partial W_{uv}}$.

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$$\therefore \frac{\partial D_{iv}}{\partial W_{uv}} = X_{iu}$$

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- We can then see that if we want this for all values of \mathbf{W} it simply generalises to: $\frac{\partial \mathcal{L}}{\partial \mathbf{W}} = \mathbf{X}^\top \frac{\partial \mathcal{L}}{\partial \mathbf{D}}$.

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 - **The gradient of the loss with respect to a parameter tells you how much the loss will change with a small perturbation to that parameter.**

Recap: Singular Value Decomposition and its applications

Let's now change direction — we're going to look at an early success story resulting from using some differentiation and the Singular Value Decomposition (SVD).

For complex \mathbf{A} :

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$$

where \mathbf{V}^* is the *conjugate transpose* of \mathbf{V} .

For real \mathbf{A} :

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

Recap: Singular Value Decomposition and its applications

- SVD has many uses:
 - Computing the Eigendecomposition:
 - Eigenvectors of $\mathbf{A}\mathbf{A}^\top$ are columns of \mathbf{U} ,
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 - and the non-zero values of $\mathbf{\Sigma}$ are the square roots of the non-zero eigenvalues of both $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top\mathbf{A}$.

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 - Low-rank approximation and matrix completion
 - if you take the ρ columns of \mathbf{U} , and the ρ rows of \mathbf{V}^\top corresponding to the ρ largest singular values, you can form the matrix $\mathbf{A}_\rho = \mathbf{U}_\rho\mathbf{\Sigma}_\rho\mathbf{V}_\rho^\top$ which will be the *best* rank- ρ approximation of the original \mathbf{A} in terms of the Frobenius norm.

Example: Computing SVD using gradients - The Netflix Challenge

- There are many standard ways of computing the SVD:
 - e.g. 'Power iteration', or 'Arnoldi iteration' or 'Lanczos algorithm' coupled with the 'Gram-Schmidt process' for orthonormalisation

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 - e.g. computing the SVD of a sparse matrix with 17770 rows, 480189 columns and 100480507 non-zero entries!
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- OK, so what can you do?
 - The 'Simon Funk' solution: realise that there is a really simple (and quick) way to compute the SVD by following gradients...

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Deriving a gradient-descent solution to SVD

- One of the definitions of rank- ρ SVD of a matrix \mathbf{A} is that it minimises reconstruction error in terms of the Frobenius norm.

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- Without loss of generality we can write SVD as a 2-matrix decomposition $\mathbf{A} = \hat{\mathbf{U}}\hat{\mathbf{V}}^T$ by rolling in the square roots of Σ to both $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$: $\hat{\mathbf{U}} = \mathbf{U}\Sigma^{0.5}$ and $\hat{\mathbf{V}}^T = \Sigma^{0.5}\mathbf{V}^T$.

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- Then we can define the decomposition as finding $\min_{\hat{\mathbf{U}}, \hat{\mathbf{V}}} (\|\mathbf{A} - \hat{\mathbf{U}}\hat{\mathbf{V}}^T\|_F^2)$

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Deriving a gradient-descent solution to SVD

Start by expanding our optimisation problem:

$$\begin{aligned}\min_{\hat{\mathbf{U}}, \hat{\mathbf{V}}}(\|\mathbf{A} - \hat{\mathbf{U}}\hat{\mathbf{V}}^T\|_F^2) &= \min_{\hat{\mathbf{U}}, \hat{\mathbf{V}}}(\sum_r \sum_c (A_{rc} - \hat{U}_r \hat{V}_c)^2) \\ &= \min_{\hat{\mathbf{U}}, \hat{\mathbf{V}}}(\sum_r \sum_c (A_{rc} - \sum_{p=1}^{\rho} \hat{U}_{rp} \hat{V}_{cp})^2)\end{aligned}$$

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Let $e_{rc} = A_{rc} - \sum_{p=1}^{\rho} \hat{U}_{rp} \hat{V}_{cp}$ denote the error. Then, our problem becomes:

$$\text{Minimise } J = \sum_r \sum_c e_{rc}^2$$

We can then differentiate with respect to specific variables \hat{U}_{rq} and \hat{V}_{cq}

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and use this as the basis for a gradient descent algorithm:

$$\begin{aligned}\hat{U}_{rq} &\leftarrow \hat{U}_{rq} + \lambda \sum_r \sum_c \hat{V}_{cq} e_{rc} \\ \hat{V}_{cq} &\leftarrow \hat{V}_{cq} + \lambda \sum_r \sum_c \hat{U}_{rq} e_{rc}\end{aligned}$$

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- A stochastic version of this algorithm (updates on one single item of \mathbf{A} at a time) helped win the Netflix Challenge competition in 2009.

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- It was both *fast* and *memory efficient*