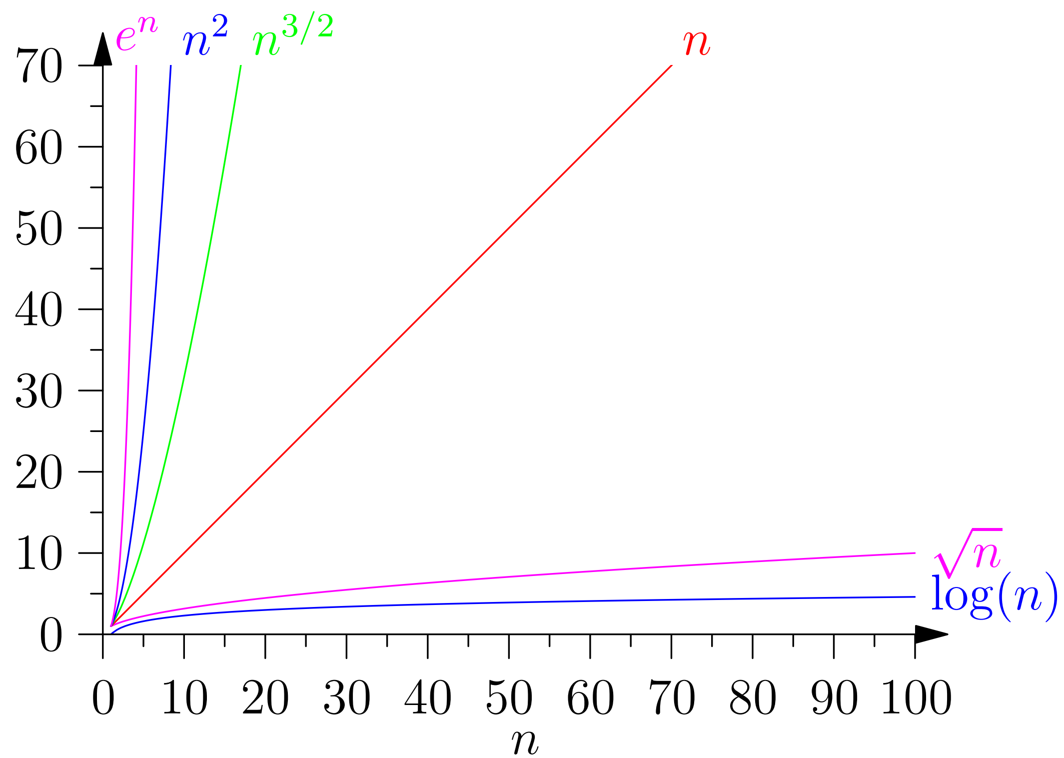
Algorithms and Analysis

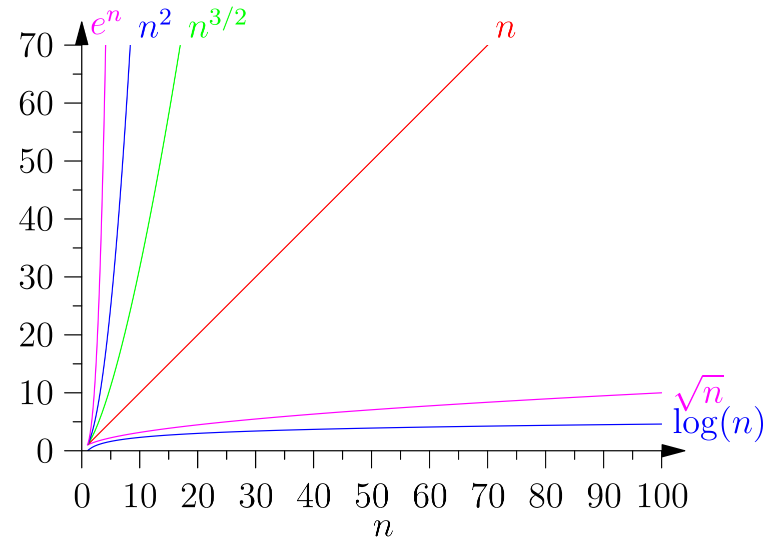
Lesson 30: *Understand Time Complexity*



Theta, Big-O, little-o, Big-Omega, little-omega

Outline

1. **Time Complexity Classes**
   - Theta—Θ
   - Big O
   - Little o
   - Big Omega—Ω
   - Little omega—ω
2. Computing Time Complexity



Recap

- We have seen many algorithms taking times of order 1, $\log(n)$, n, $n\log(n)$, n^2, etc

- Sometimes these are worst time, average time or best time results

- We have lots of different notations, e.g. $O(1)$, $\Theta(\log(n))$, $\Omega(n^2)$, etc.

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Complexity Class Sets

- The correct way to think about complexity classes is in terms of sets

- Suppose we have an algorithm which takes an input of size n and computes an output in $f(n)$ operations

- E.g. $f(n) = 4n^2 + 2n + 3$

- We can partition all run times into sets by considering only the leading order term and ignoring the constant term

- We denote these sets by $\Theta(g(n))$
  - ⋆ $4n^2 + 2n + 3 \in \Theta(n^2)$
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Defining $\Theta(g(n))$

- A function $f(n) \in \Theta(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \qquad\qquad 0 < c < \infty$$

- E.g.

$$\lim_{n\to\infty} \frac{4n^2 + 2n + 3}{n^2} = 4$$
$$\lim_{n\to\infty} \frac{5n\log(n) + 3n + 2}{n\log(n)} = 5$$

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- Nevertheless, ignoring the constant is often essential to make analysis of algorithms doable

Ordering Complexity Classes

- We can define the relation $\Theta(f(n)) < \Theta(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

- Informally if algorithm A has time complexity $\Theta(f(n))$ and algorithm B has time complexity $\Theta(g(n))$ then if $\Theta(f(n)) < \Theta(g(n))$ algorithm A is faster for sufficiently large n

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The Complexity Line

- We can order all complexity classes. E.g.

$$\Theta(1) < \Theta(\log(n)) < \Theta(\sqrt{n}) < \Theta(n) < \Theta(n^2)$$

- We can depict this as a complexity line

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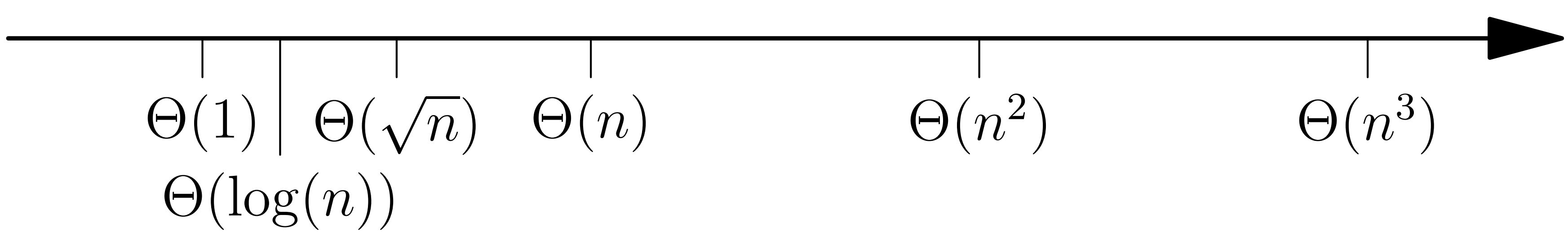

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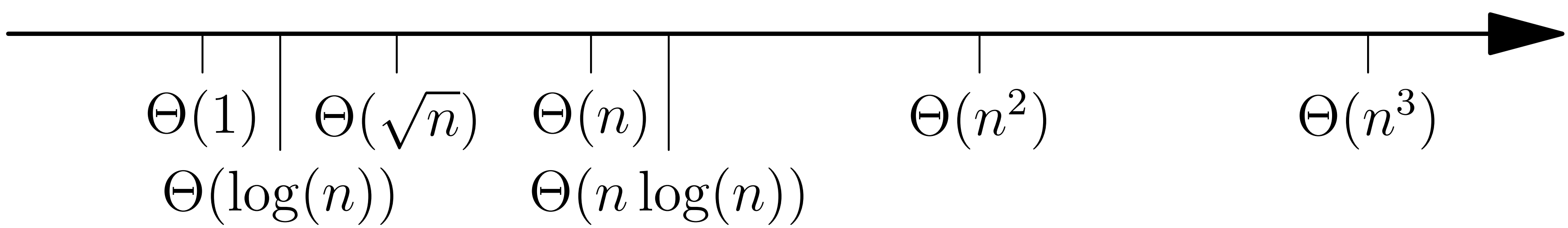

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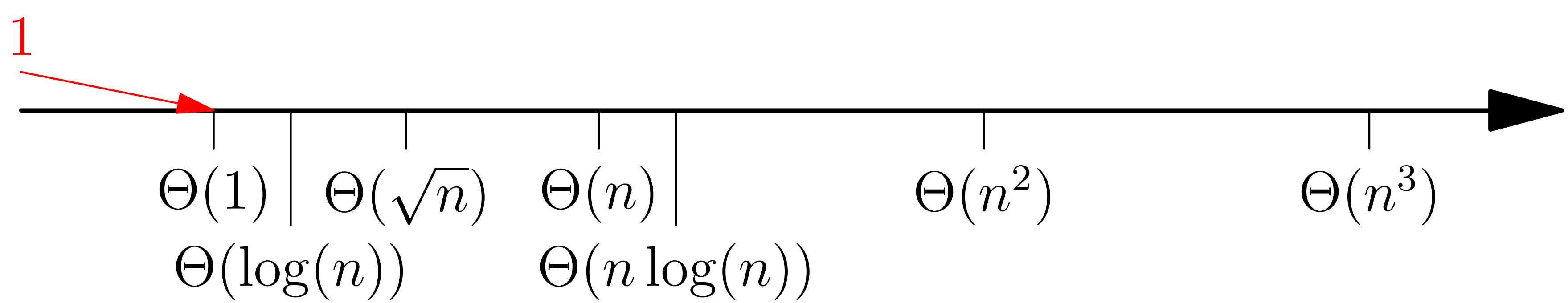

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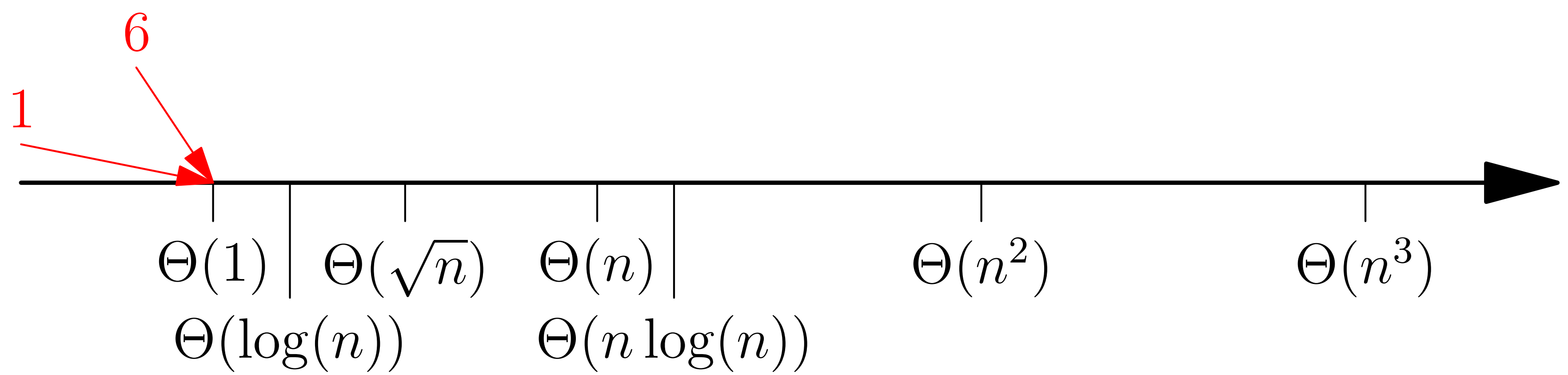

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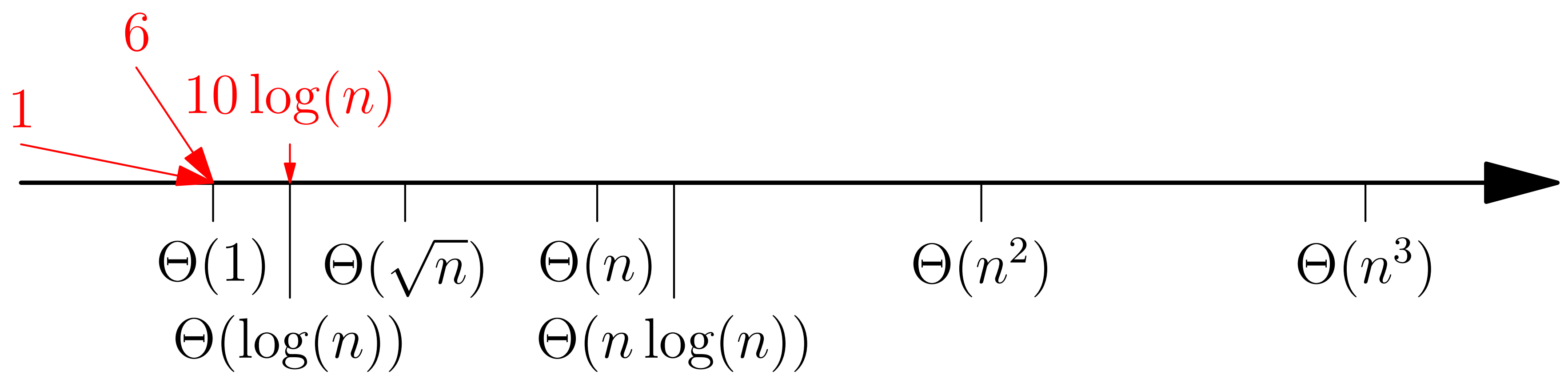

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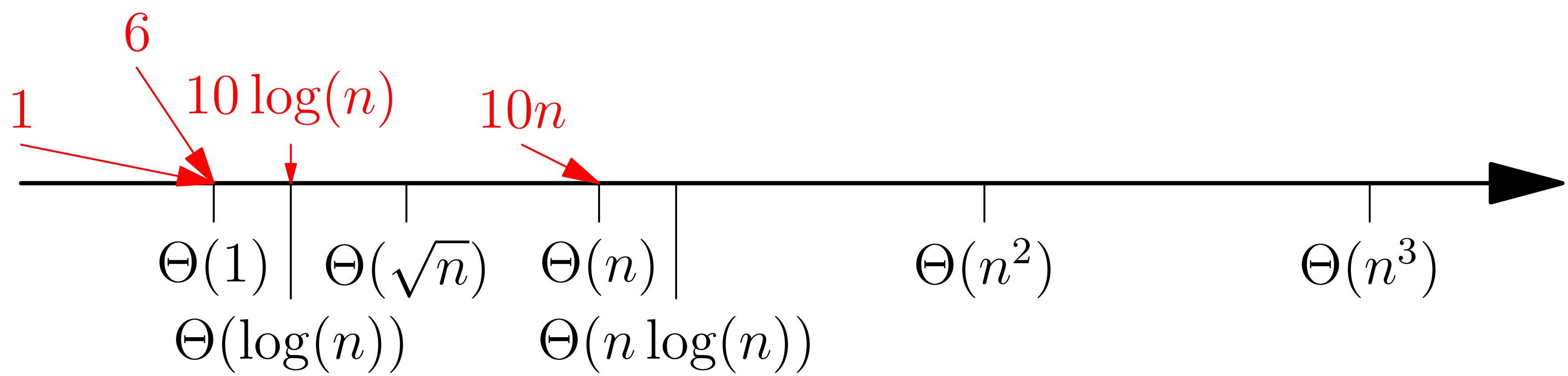

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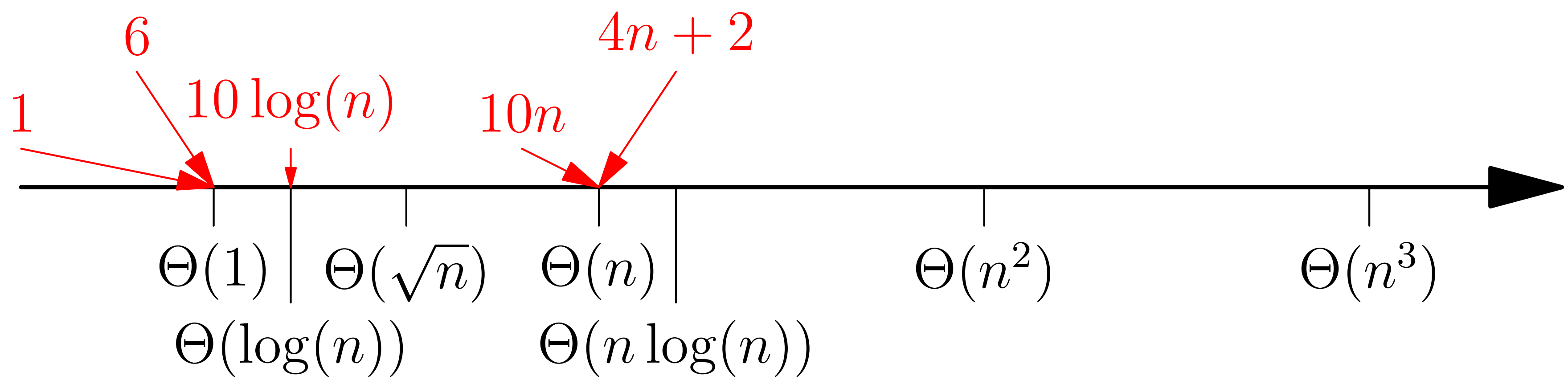

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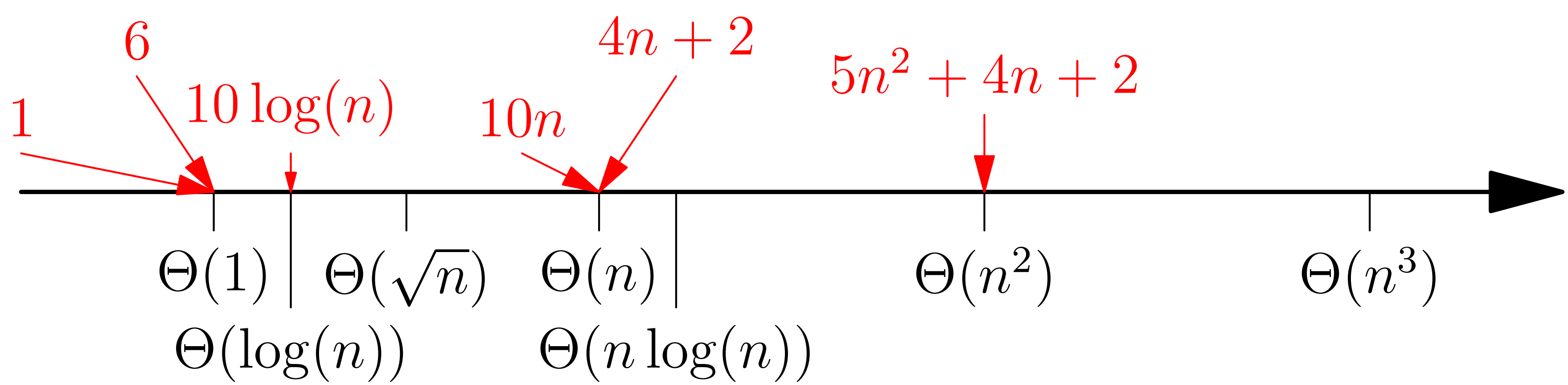

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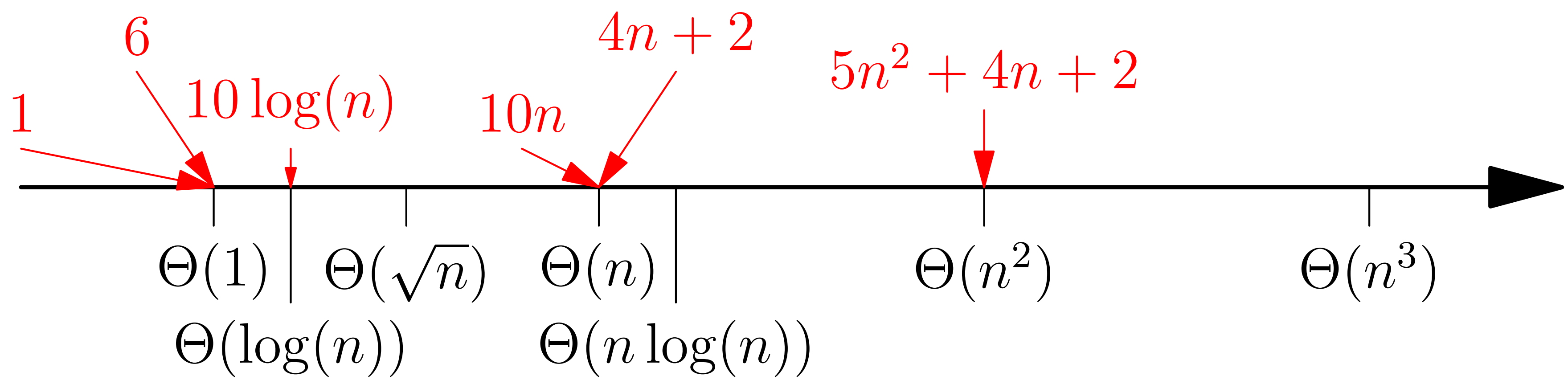

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- The run time of many algorithms depends on the input
- In this case we can define different time complexities
  - ⋆ Worst case time complexity (the longest time an algorithm will take)
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- Every algorithm will have a Θ complexity class for the worst, average and best time complexity

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Unknown Time Complexity

- Algorithms are often rather complicated and knowing the exact time complexity (for either worst, average or best cases) might not be known

- In reality it will have some run time (e.g. $f(n) = 3n^2\log(n) + 2n^2 - n + 3$) and will belong to a Θ time complexity set (e.g. $\Theta(n^2\log(n))$) but we might not be able to calculate it

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Big-O

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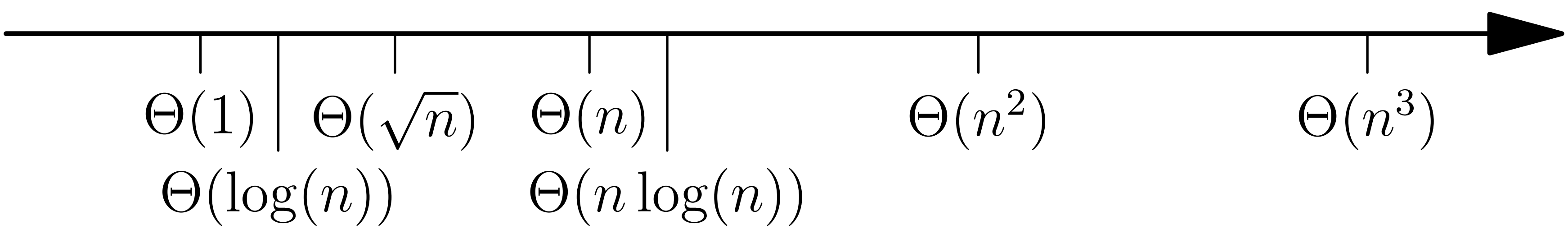
- If an algorithm is $O(g(n))$ then its time complexity is no more than $\Theta(g(n))$



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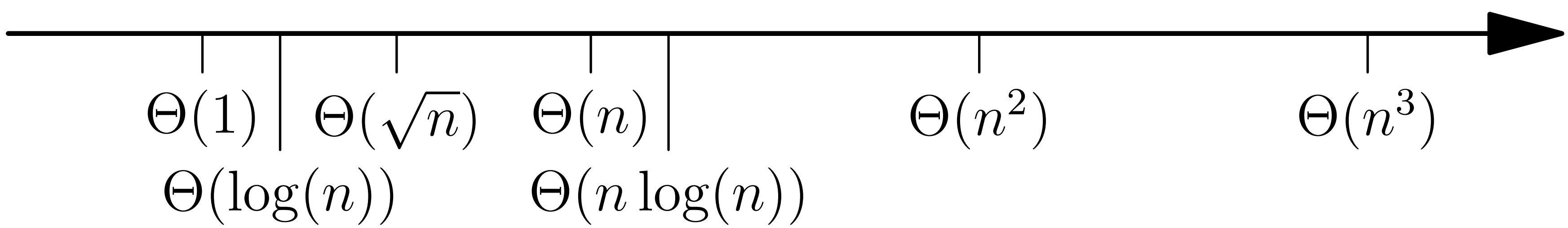

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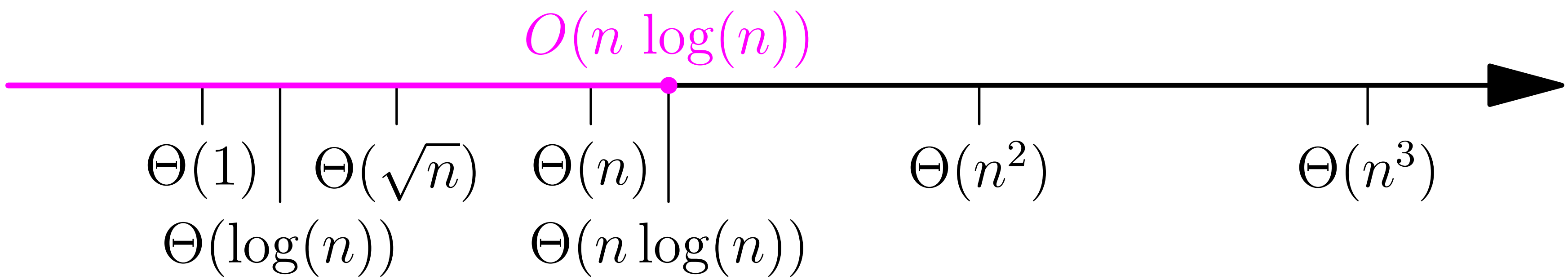

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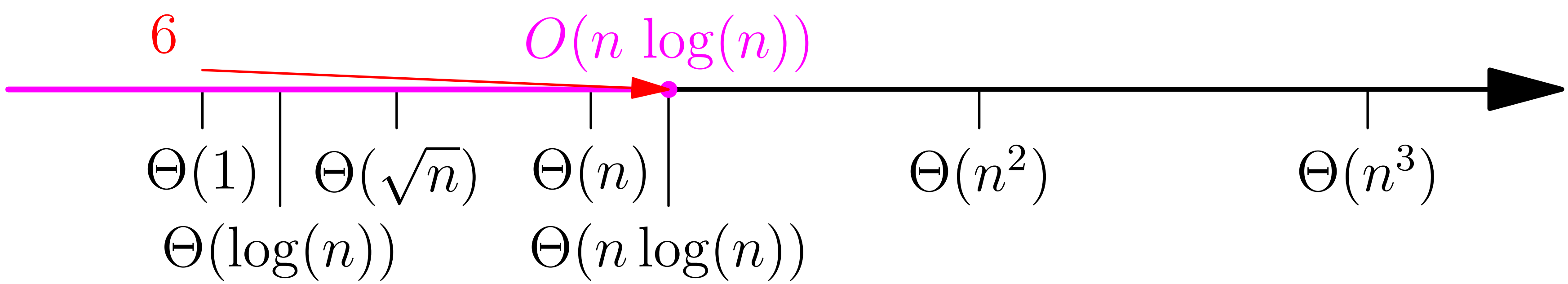

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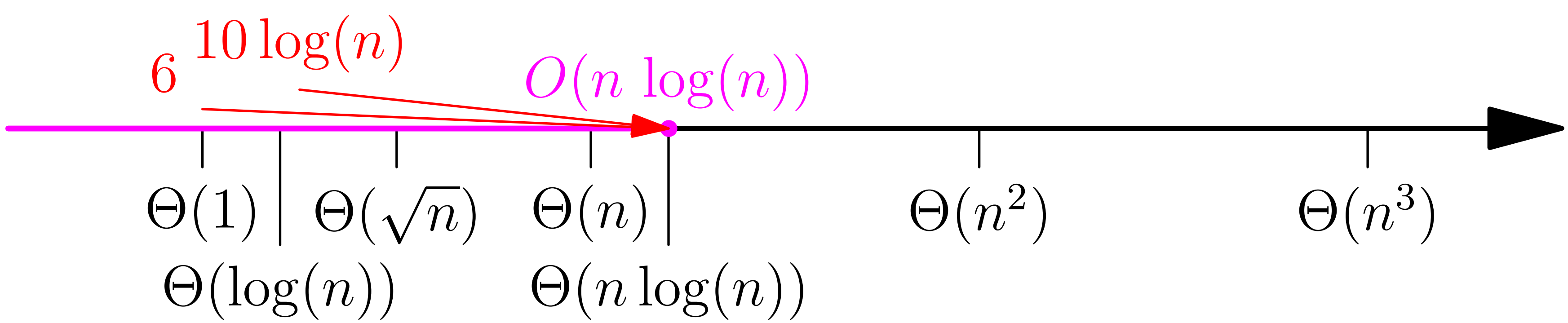

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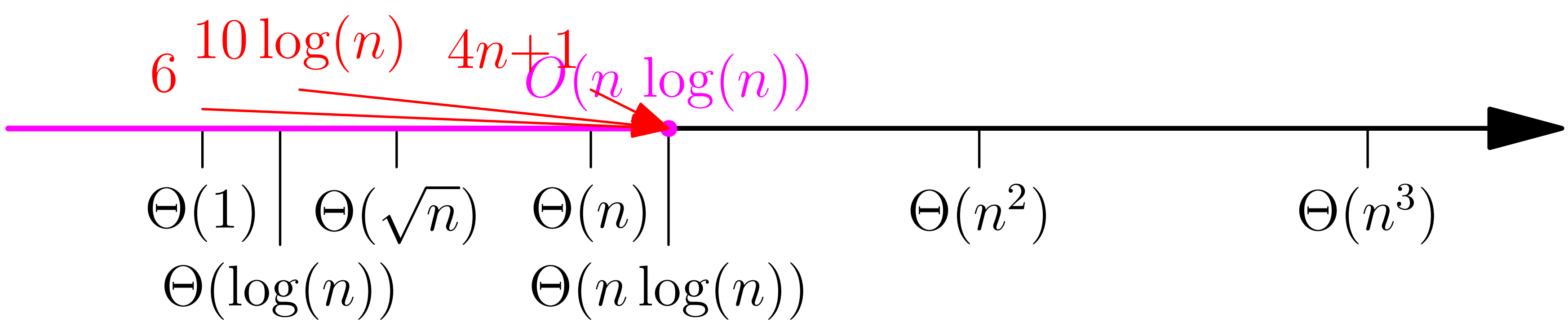

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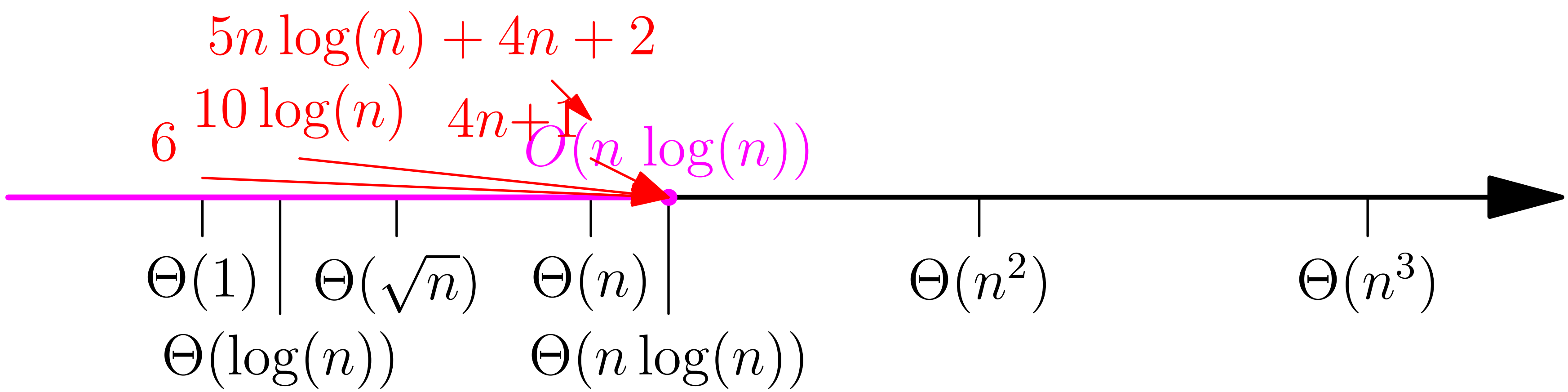

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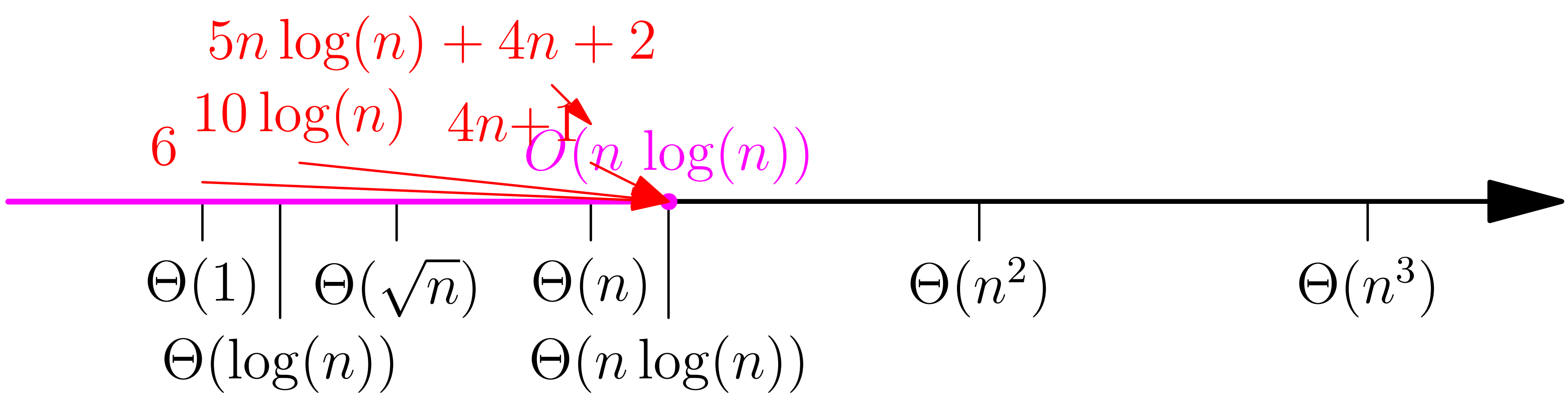

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Upper Bounding Time Complexity

- Consider a program

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for(int i=0; i<n; i++)  {
  // do something
  if (/* some condition */) {
    for (int j=0; j<n; j++) {
      // do other stuff
    }
  }
}
// clean up
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- If the `if` statements is never true this is a $\Theta(n)$ algorithm if it is always true it is a $\Theta(n^2)$ algorithm
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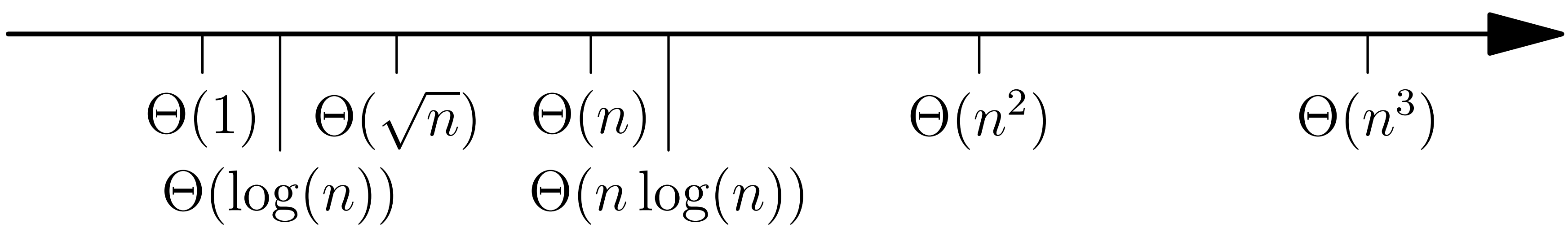
Little-o

- Sometimes we want to say that the time complexity for an algorithm $\Theta(f(n))$ is **strictly less** than a known time complexity $\Theta(g(n))$



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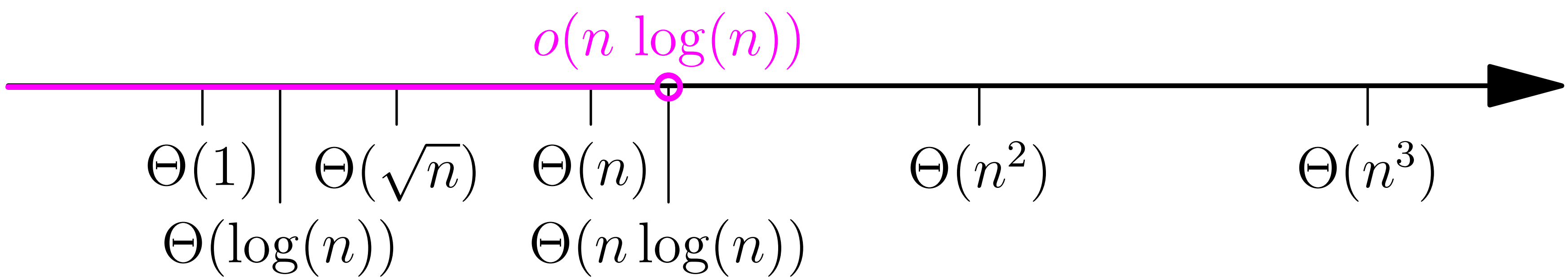

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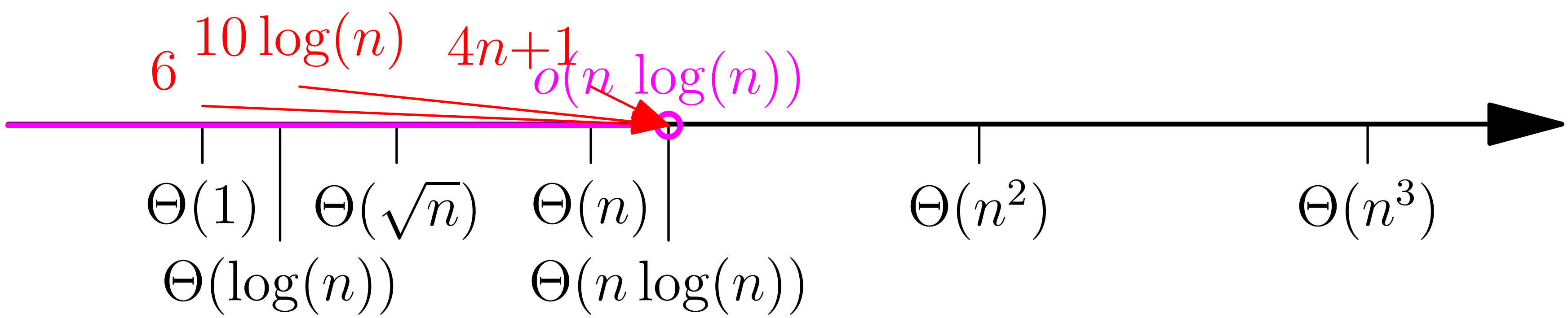

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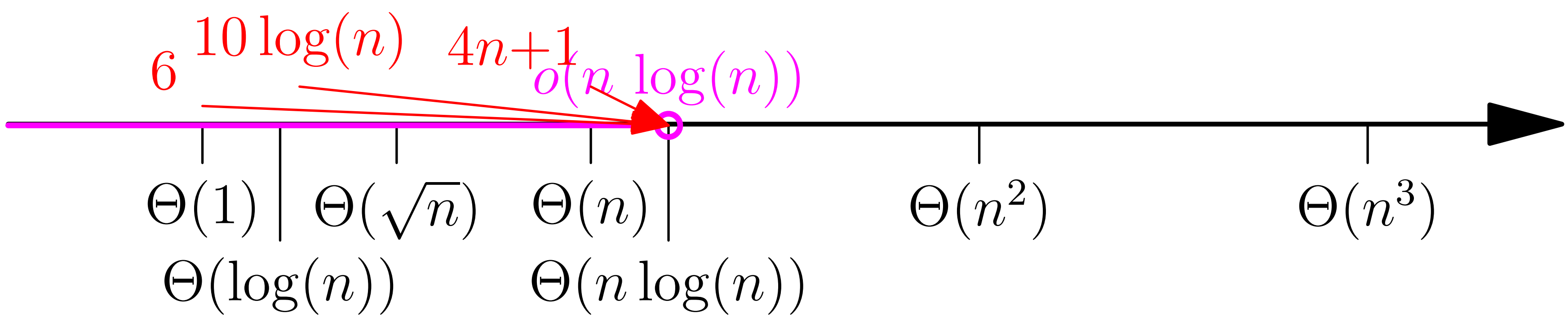

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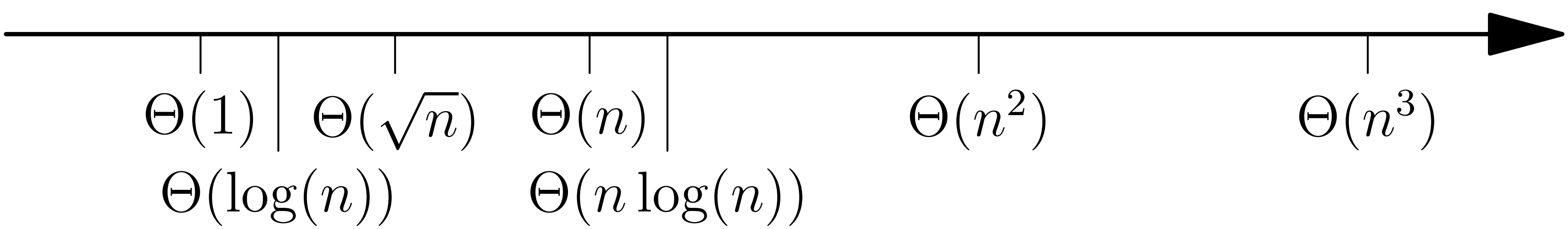

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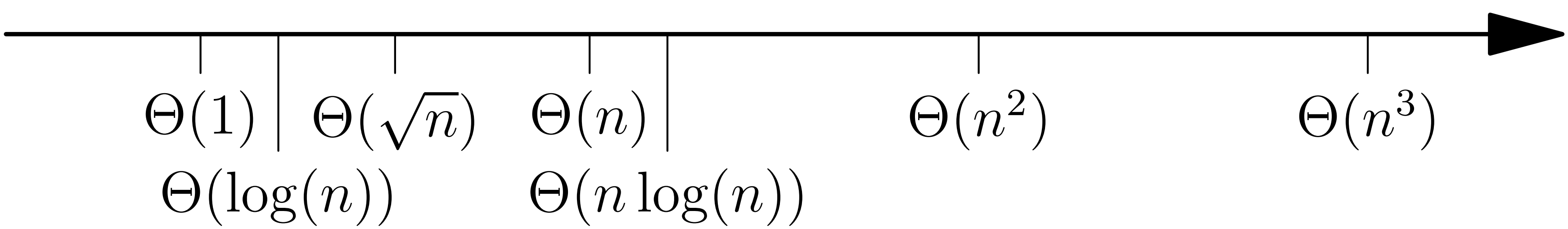

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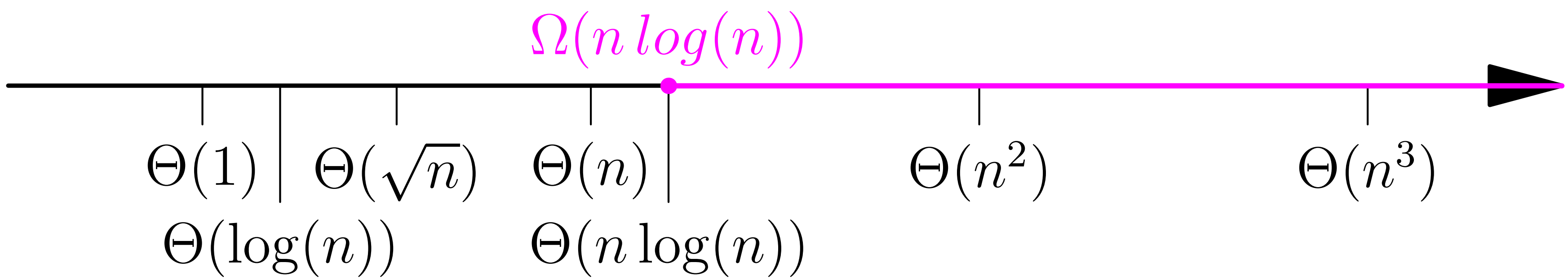

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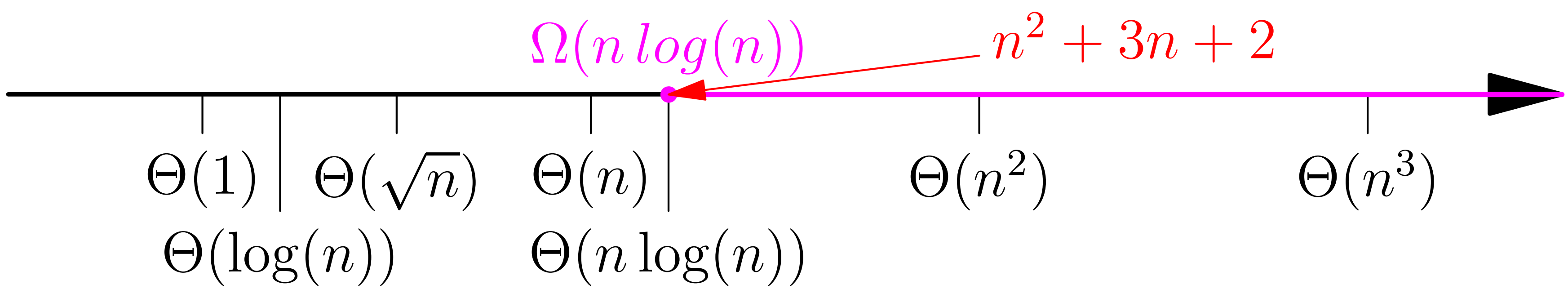

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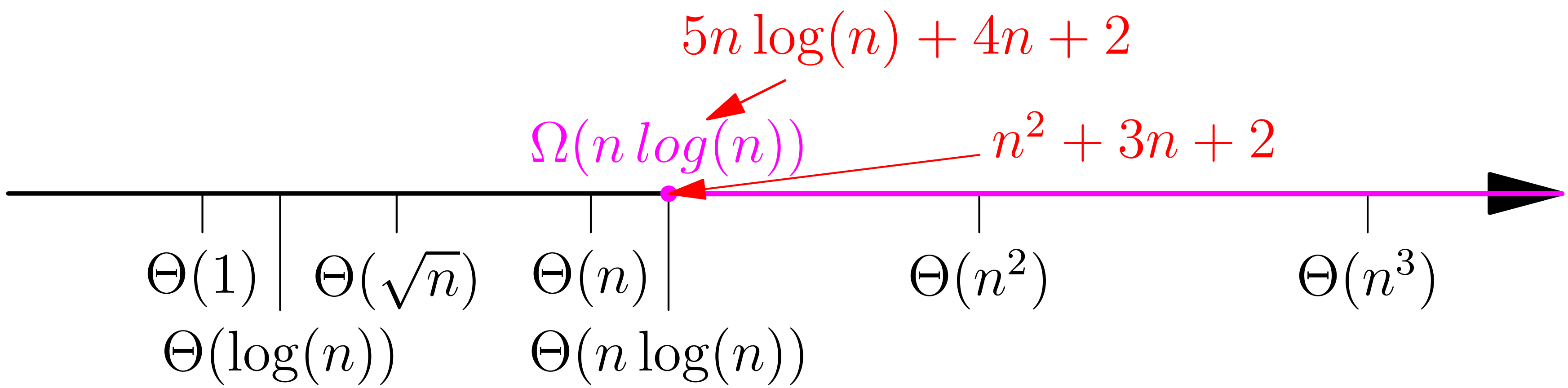

Lower Bounding Time Complexity

- Returning to the program

```
// define stuff
for(int i=0; i<n; i++)  {
  // do something
  if (/* some condition */) {
    for (int j=0; j<n; j++) {
      // do other stuff
    }
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}
// clean up
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- We might not know how frequently the `if` statement is true, but we know in all cases the first `for` loop iterates over n

- Thus we know this algorithm is in $\Omega(n)$

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- When we are given an algorithm to analyse we want to compute $\Theta(n)$
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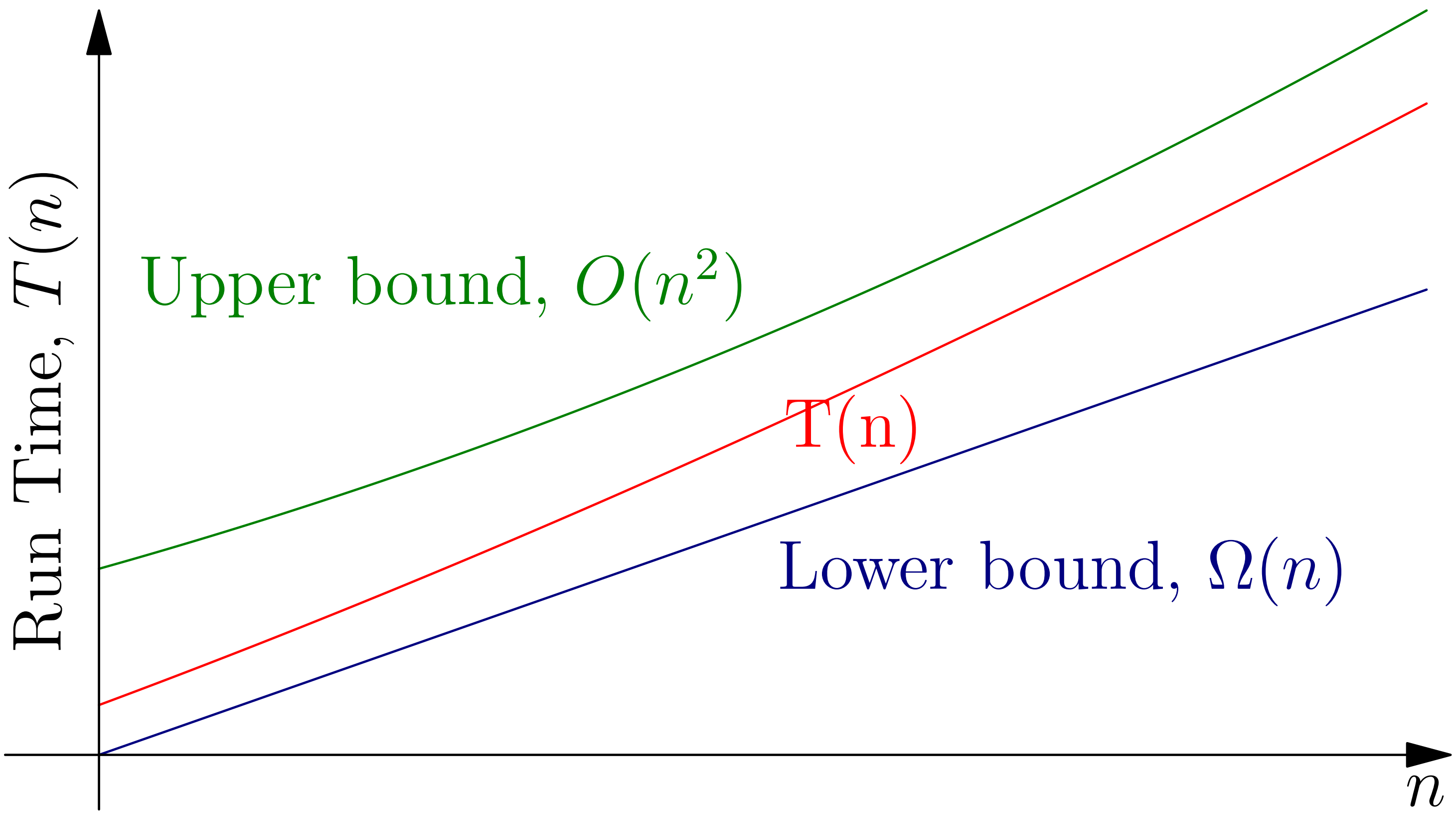

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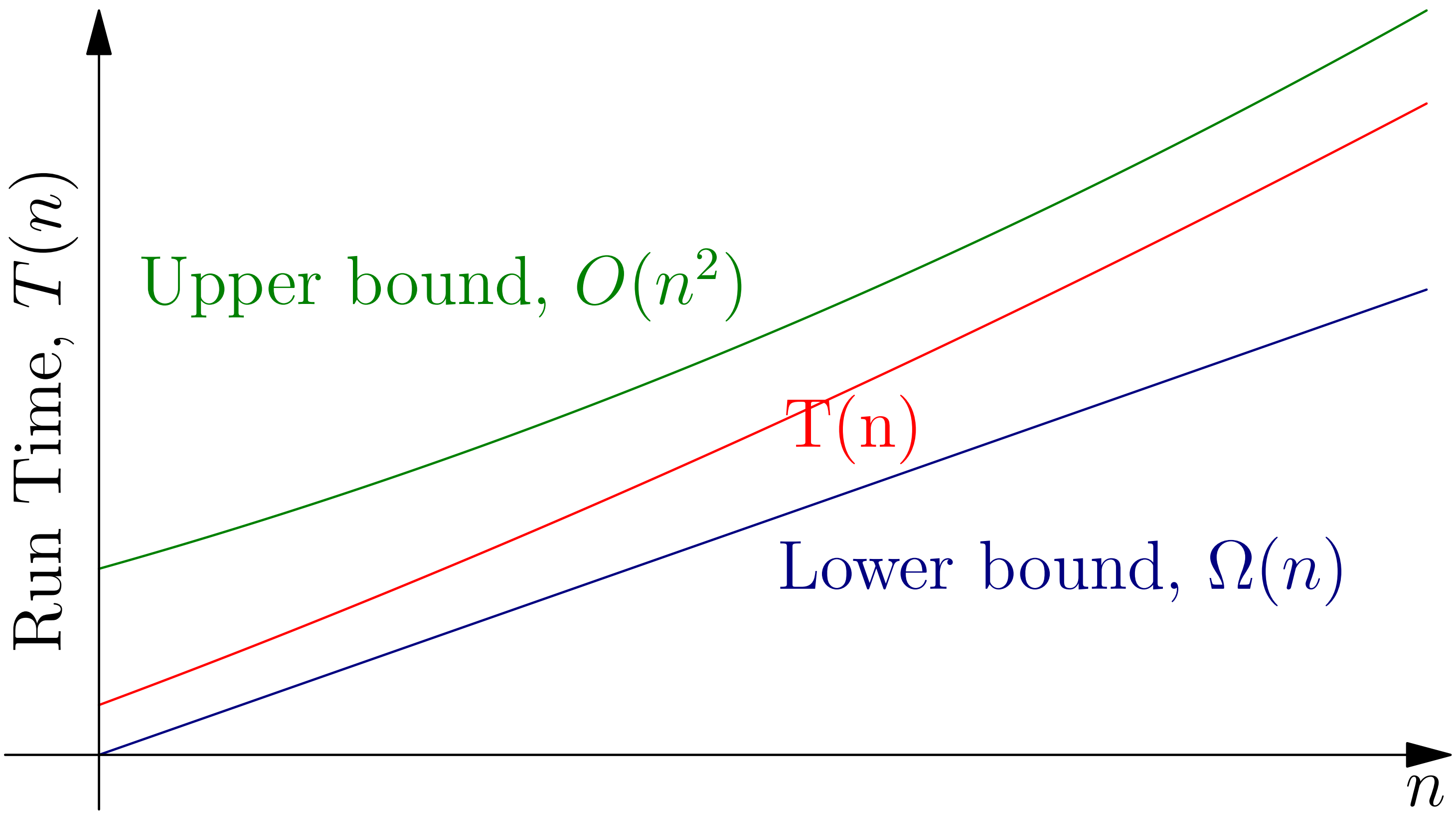

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- If we know an algorithm is
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Meaning of Time Complexity

- Insertion sort has time complexity $\Theta(n^2)$
- Because it consists of two `for` loops
- It takes 2 seconds to sort $100\,000$ items
- How long does it take to sort $1\,000\,000$ items?

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- n increases by 10, time complexity increases by $10^2 = 100$
- Time taken is approximately 200 seconds or around 3.5 minutes

Exponential Time Complexity

- When we talk about exponential time complexity we usually mean that

$$\log(T(n)) \in \Theta(n)$$

- This is true if
  - $T(n) = 2^n$
  - $T(n) = 6.1\,\mathrm{e}^{0.003n}$
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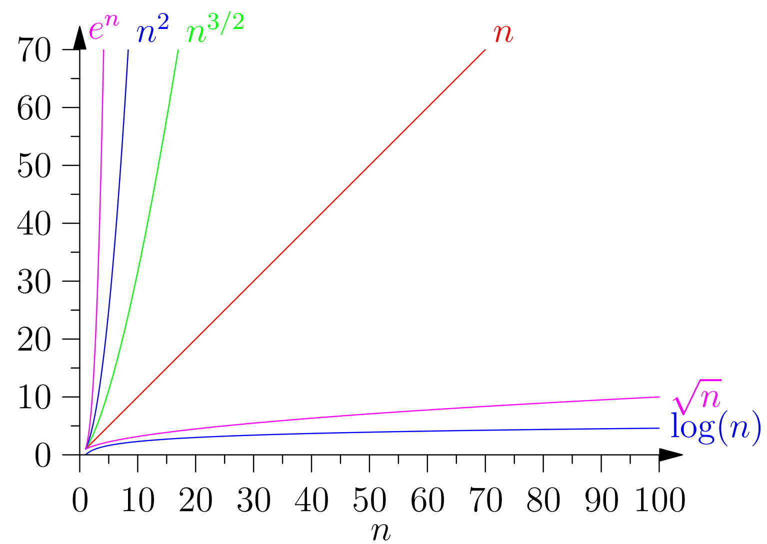
Outline

1. Time Complexity Classes
   - Theta—Θ
   - Big O
   - Little o
   - Big Omega—Ω
   - Little omega—ω
2. **Computing Time Complexity**



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- Answer $\Theta(n^2)$

Recursion

- Determining time complexity is harder when we use recursion
- Consider Euclid's algorithm for determining the greatest common divisor

```
public static
long gcd(long m, long n)
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  while(n!=0) {
    long rem = m%n;
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Example of gcd

- Example of Euclid's algorithm `gcd(1989,1590)`
- Sequence of remainders is 399, 393, 6, 3, 0
- The greatest common divisor is 3
- How long does it take compute `gcd(n,m)` with $n > m$
- This is subtle as could depend in a complex way on the pair n and m

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- An observation which makes the analysis relatively simple is that the remainder is reduced by at least 2 after two iterations
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  - ⋆ Using the recursion (assuming $m, n < 0$)

$$\mathrm{gcd}(m,n) = \mathrm{gcd}(n, \mathrm{rem}(m,n)) = \mathrm{gcd}(\mathrm{rem}(m,n), \mathrm{rem}(n, \mathrm{rem}(m,n)))$$

  - ⋆ The proof follows by showing that $\mathrm{rem}(n, \mathrm{rem}(m,n)) < n/2$
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  - ⋆ Note that $T(1) = 1$

$$T(n) < T(2^{-1}n) + 2 < T(2^{-2}n) + 4 < \cdots < T(2^{-t}n) + 2t$$

  - ⋆ Choose $t = \lceil \log_2(n) \rceil$
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  - ⋆ $T(n) < 1 + 2t = 1 + 2\lceil \log_2(n) \rceil \in O(\log(n))$

- A huge calculation shows the the average number of iterations is about $(12 \log(2) \log(n))/\pi^2 + 1.47$

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- Consider the following program to compute the probability of relative primes for all numbers up to n

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public static double probRelPrime(n)
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  int rel=0, tot=0;
  for(int i=1; i<=n; i++)
      for(int j=i+1; j<=n; j++) {
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- How could we provide empirical support for this calculation?

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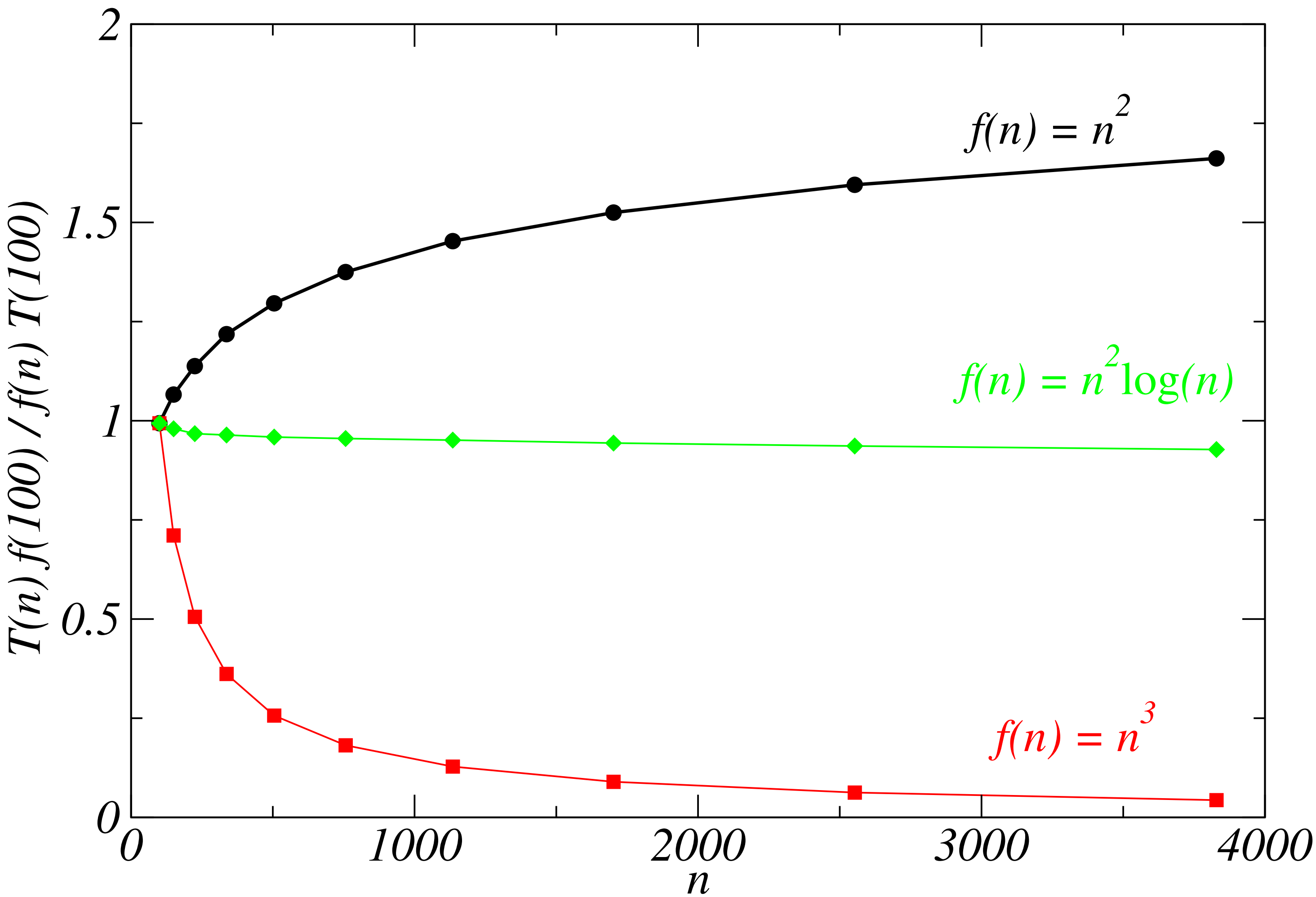
Testing Hypothesis

- We can test our hypothesis by scaling the run time by the complexity



Conclusions

- You should understand the difference between Θ, O,o, Ω and ω
- You need to be able to compute time complexity by loop counting
- To compute time complexity for recursive functions you need to be able to obtain recurrence equations
- You should be able to solve simple recurrence equations and sum up simple series
- You should be able to prove more complicated results using proof by induction

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- Thank you for attending the course