Algorithms and Analysis

Lesson 8: *Make Friends with Trees*

Binary trees, binary search trees, sets, tree iterators

Outline

1. **Trees**
2. Binary Trees
   - Implementing Binary Trees
3. Binary Search Trees
   - Definition
   - Implementing a Set
4. Tree Iterators

Trees

- Trees are one of the major ways of structuring data
- They are used in a vast number of data structures
  - ⋆ Binary search trees
  - ⋆ B-trees
  - ⋆ splay trees
  - ⋆ heaps
  - ⋆ tries
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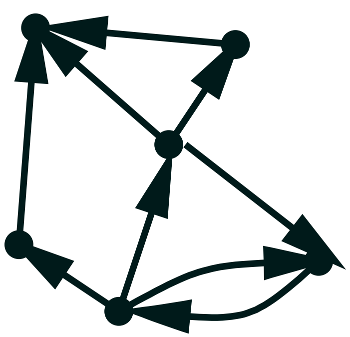
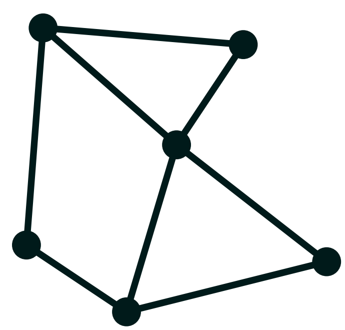
Defining Trees

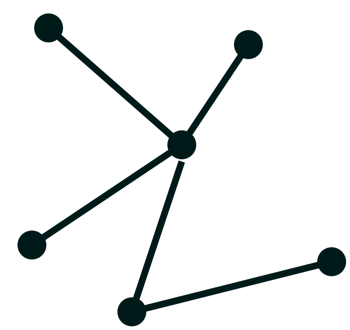
- Mathematically a tree is an **acyclic undirected graph**
  - ⋆ **graph**: a structure consisting of **nodes** or **vertices** joined by **edges**
  - ⋆ **undirected**: the edges goes both ways
  - ⋆ **acyclic**: there are no cycles in the graph



graph



undirected graph



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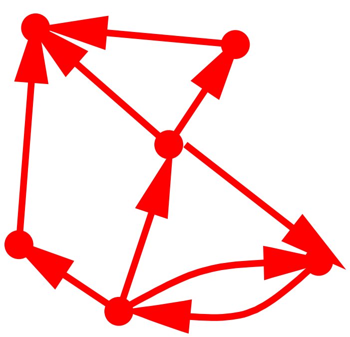

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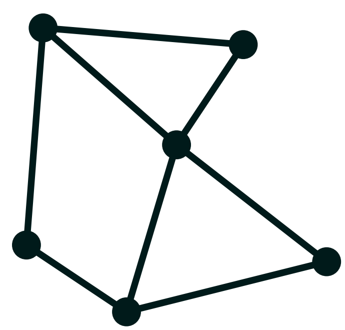

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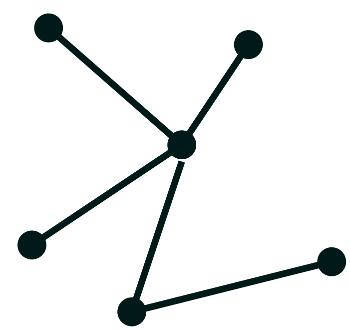

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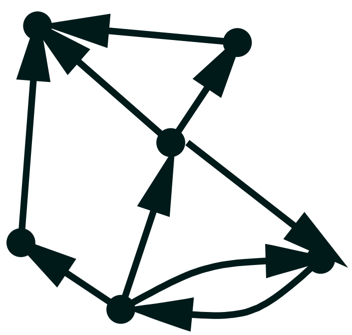

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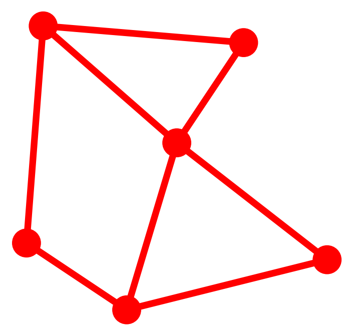

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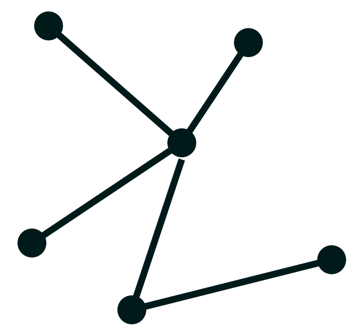

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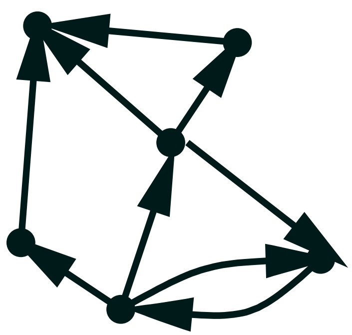

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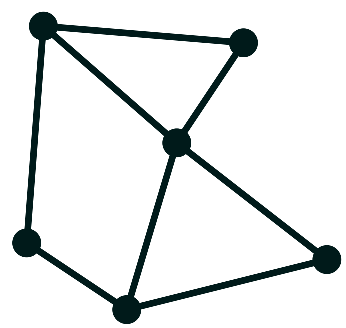

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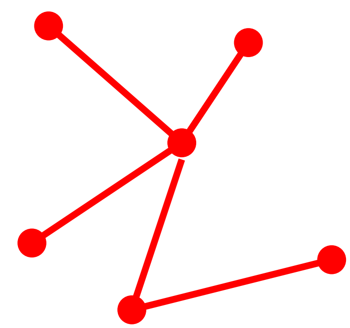
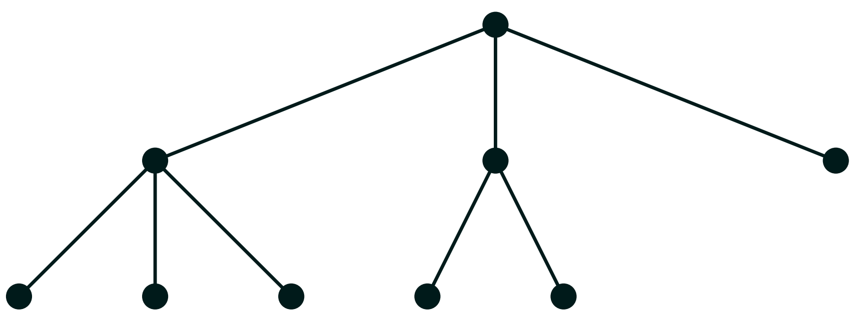


tree = acyclic undirected graph

Borrowing from Nature

- We often impose an ordering on the nodes (or a direction on the edges)—known as a rooted tree
- Borrowing from nature, we recognise one node as the **root** node
- Nodes have **children** nodes living beneath them
- Each child has a **parent** node above them except the root
- Nodes with no children are **leaf** nodes



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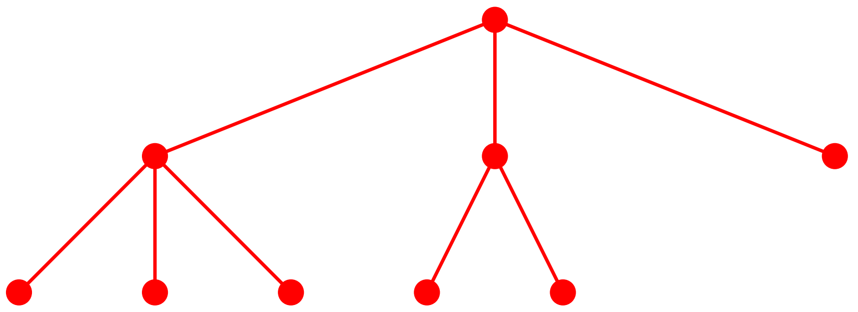

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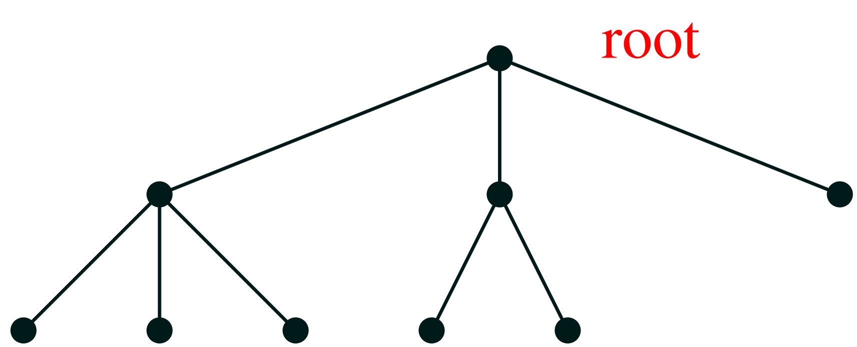

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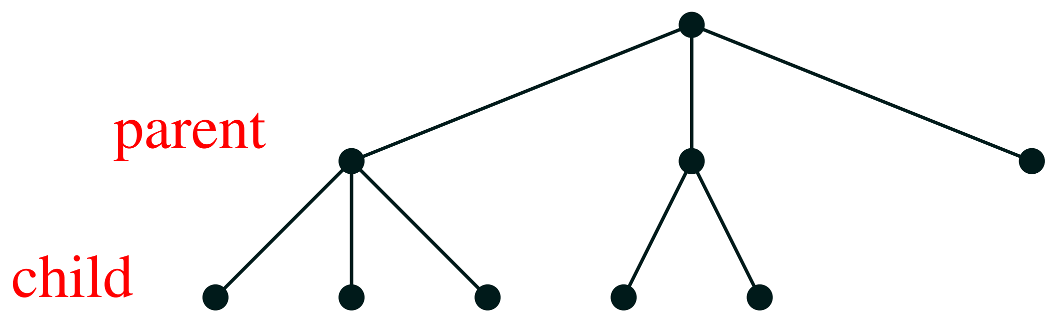

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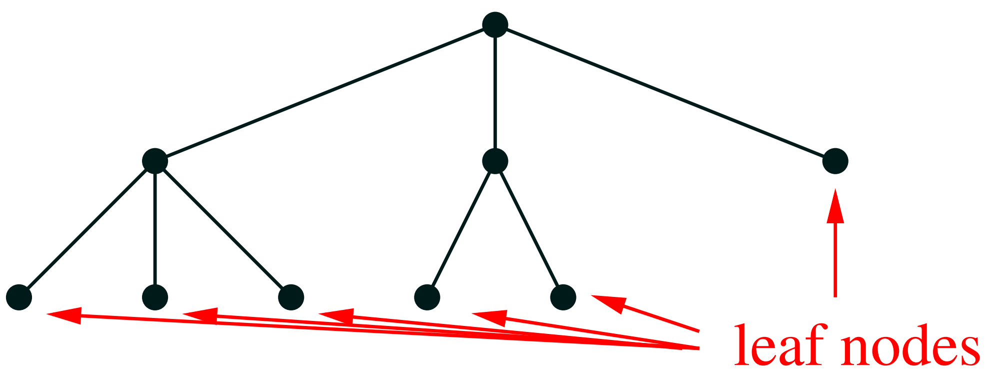

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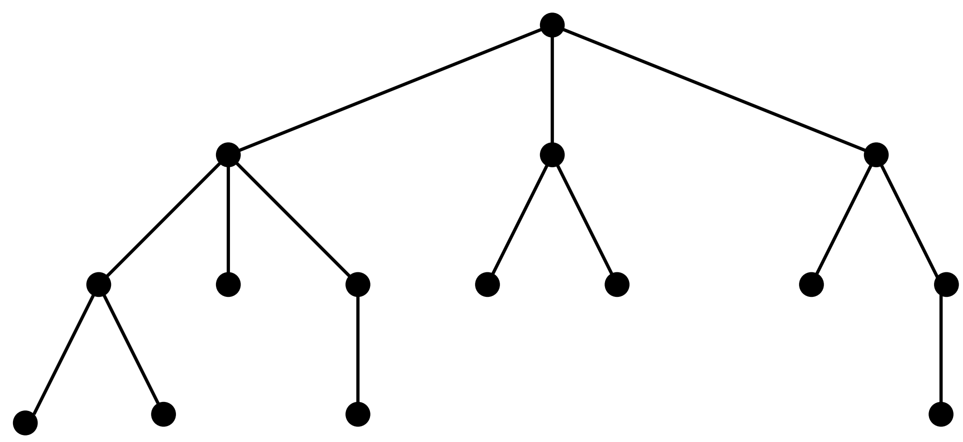
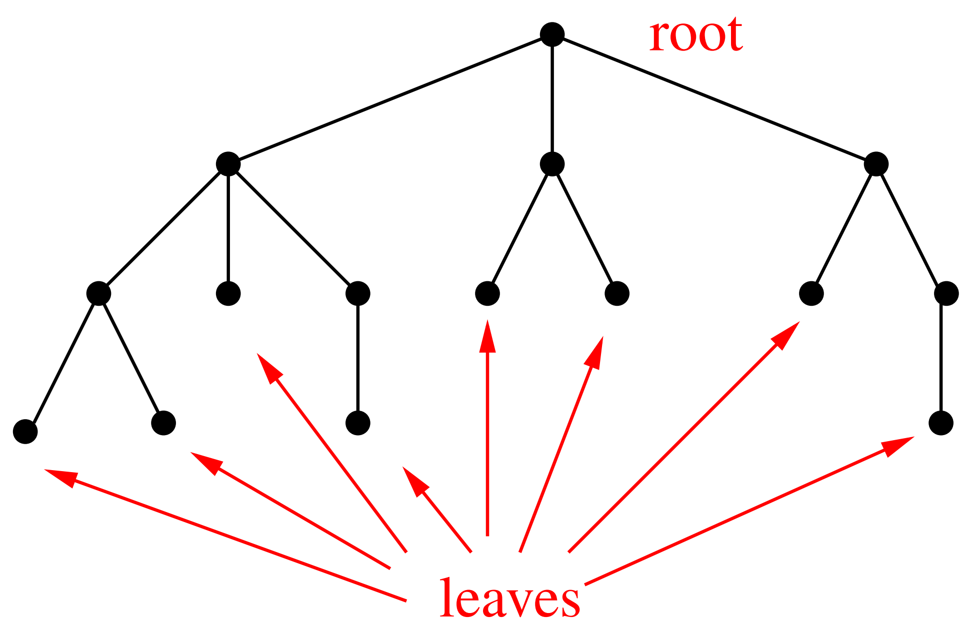


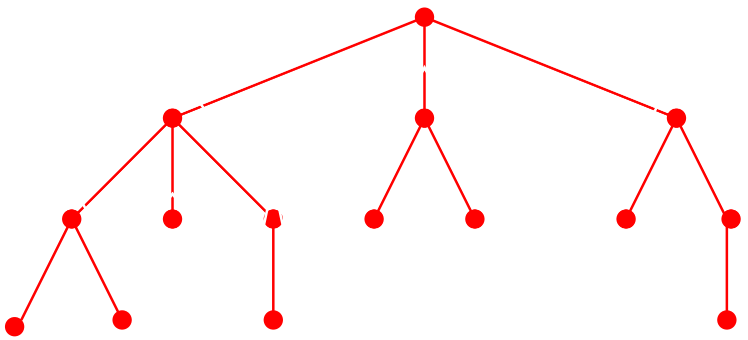
Spot the Error

- One small biological inconsistency
- Yep!, computer scientists draw there trees upside down
  - ⋆ root at the top
  - ⋆ leaves at the bottom



Subtrees

- We can think of the tree made up of **subtrees**



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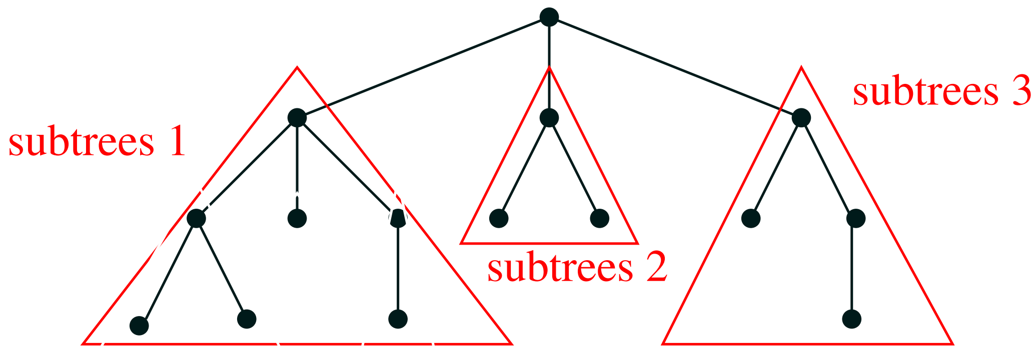

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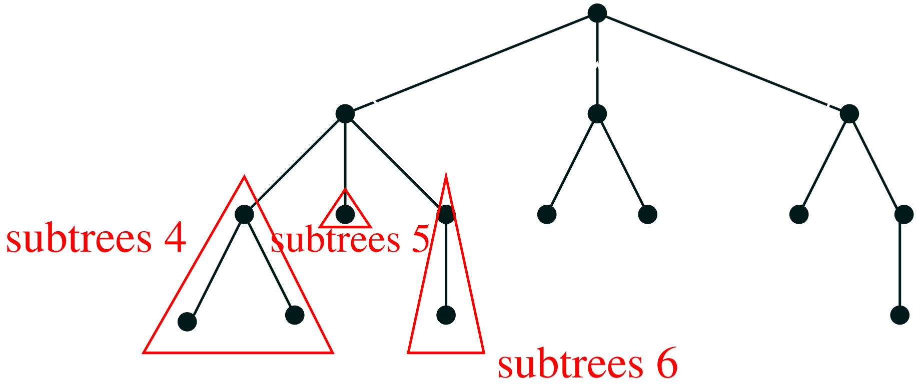
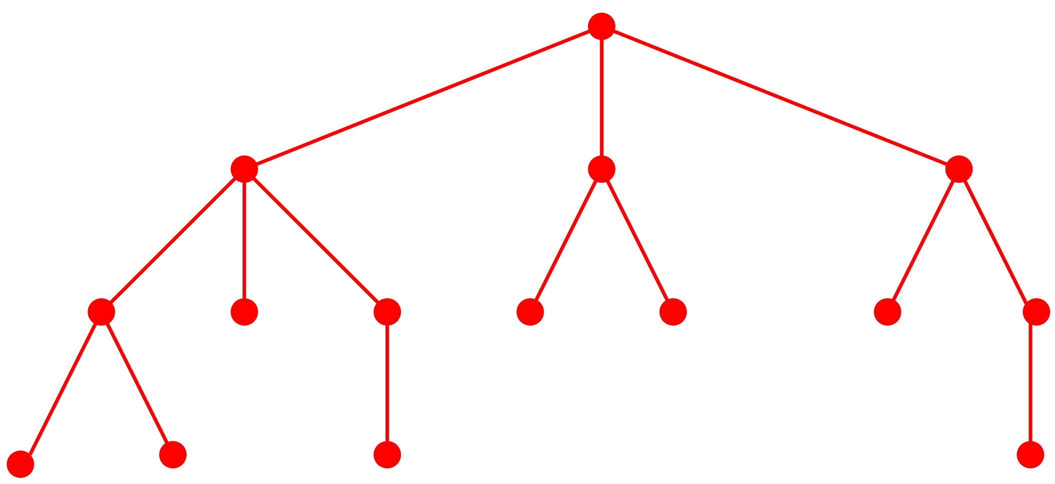


Level of Nodes

- It is useful to label different levels of the tree
- We take the **level** of a node in a tree as its distance from the root
- We take the **height** of a tree to be the number of levels



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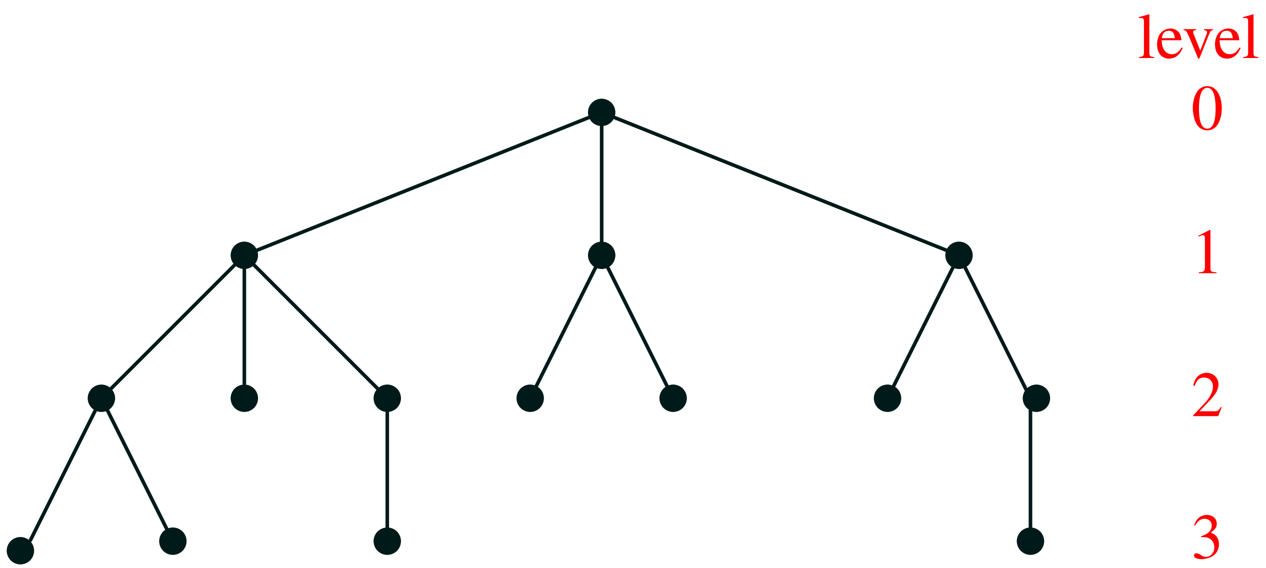

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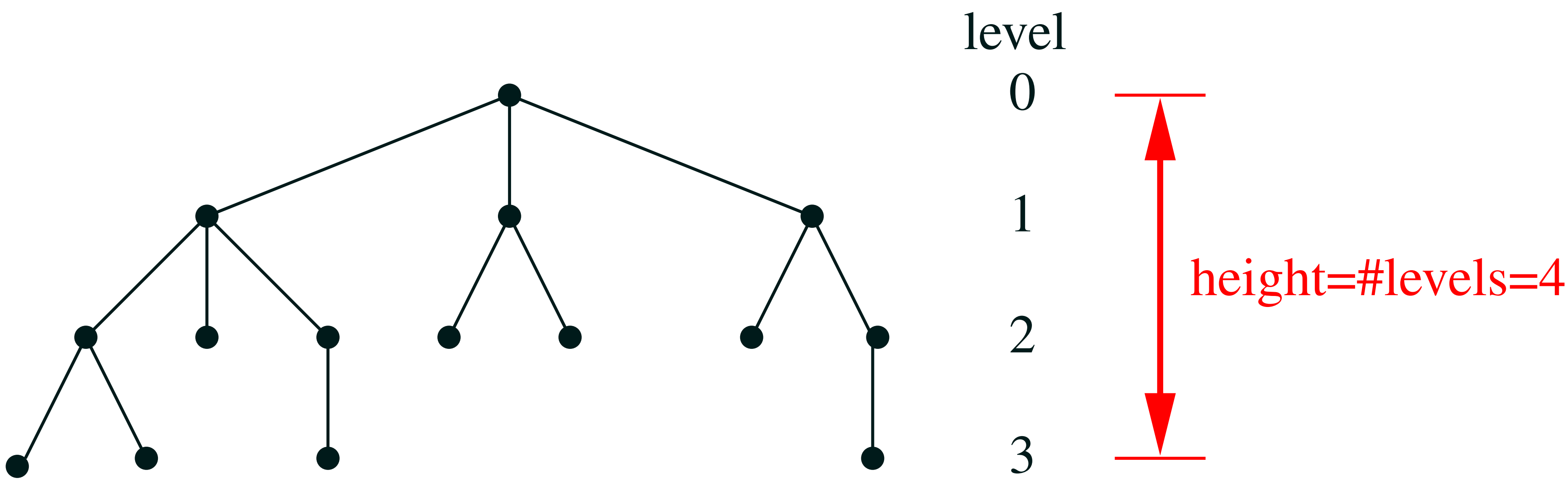


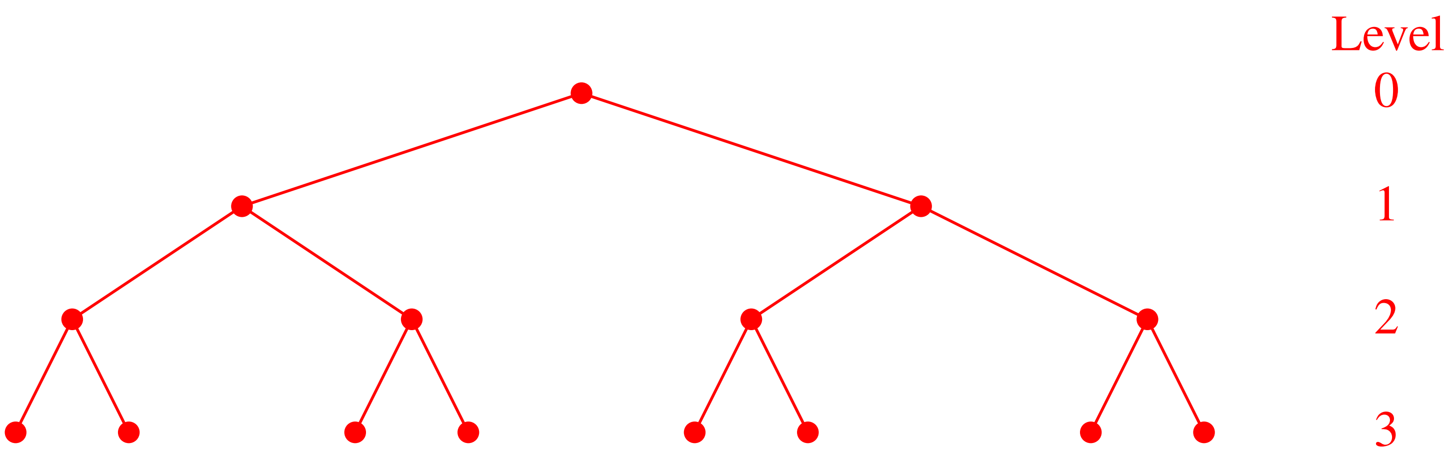
Outline

1. Trees
2. **Binary Trees**
   - Implementing Binary Trees
3. Binary Search Trees
   - Definition
   - Implementing a Set
4. Tree Iterators

Binary Trees

- A **binary tree** is a tree where each node can have zero, one or two children
- The total number of possible nodes at level l is 2^l
- The total number of possible nodes of a tree of height h is

$$1 + 2 + \cdots + 2^{h-1} = 2^h - 1$$



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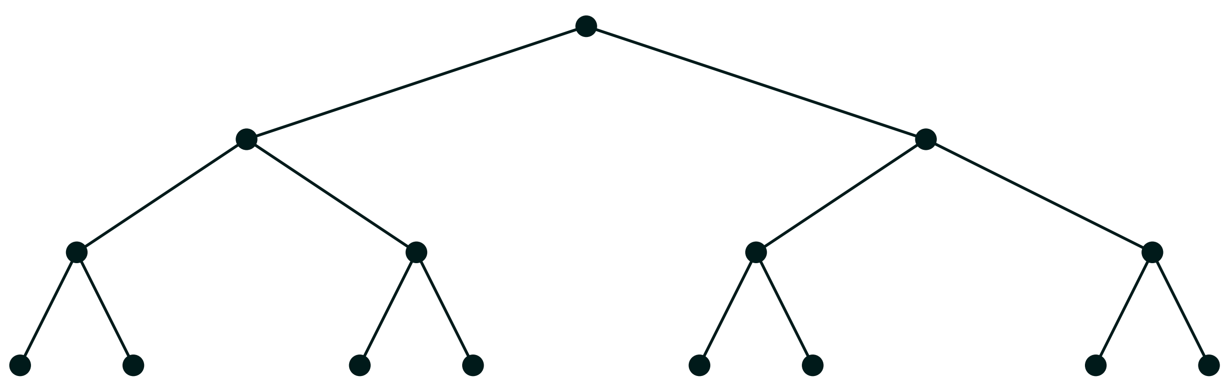

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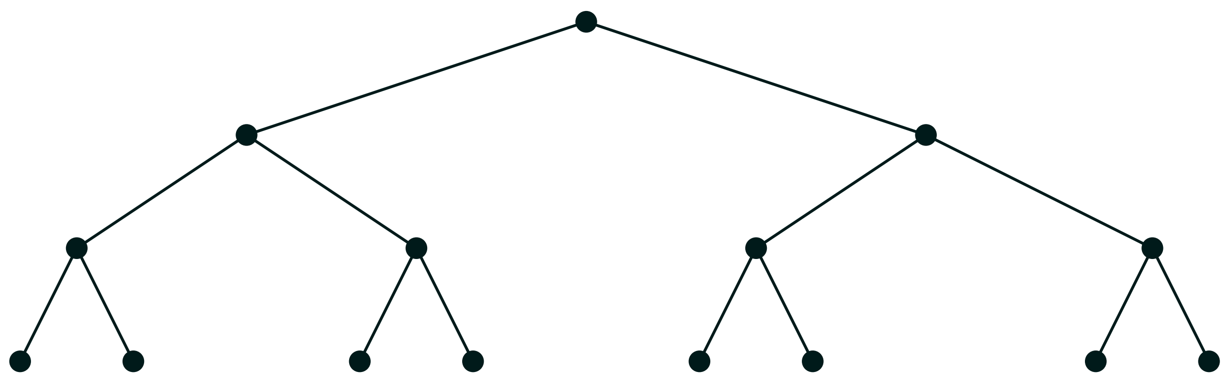

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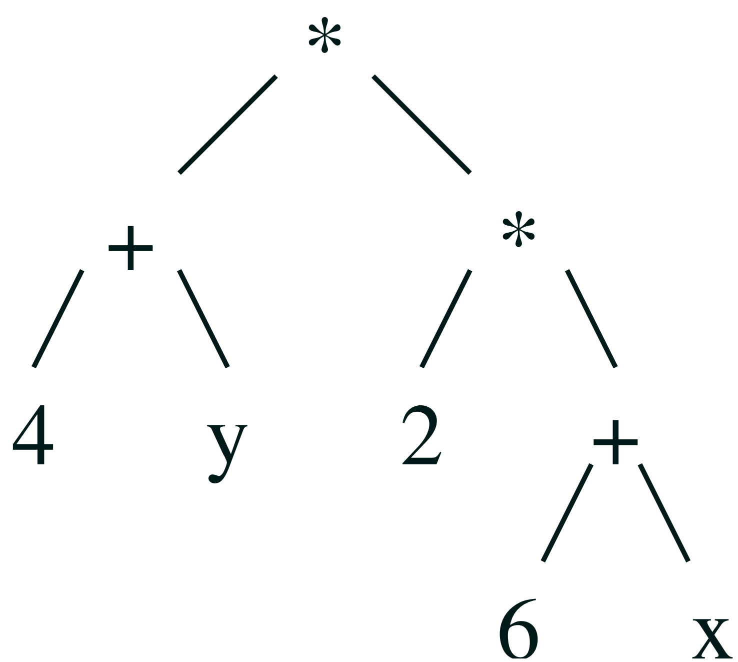
Uses of Binary Trees

- Binary trees have a huge number of applications
- For example, they are used as **expression trees** to represent formulae



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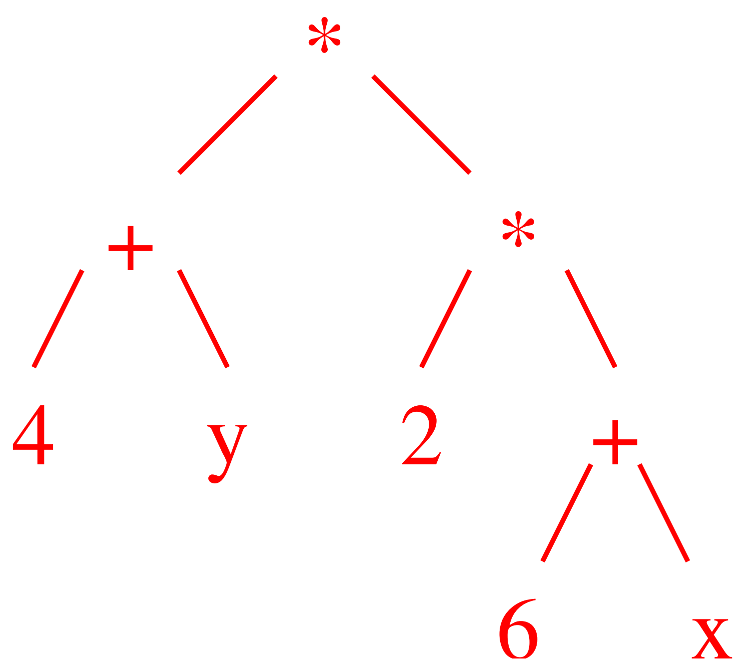

$$(4+y) * (2 * (6+x))$$

Implementation

- We wish to build a generic binary tree class with each node housing an element
- Again we use a `Node<T>` class as the building block for our data structure—in this case a node of the tree
- The `Node<T>` class will contain a pointer to left and right children
- To help navigate the tree each node will contain a pointer to its parent

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C++ Code

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template <typename T>
class binary_tree {
private:
  class Node {
  public:
    T element;
    Node* parent;
    Node* left = 0;
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    Node(const T& value, Node* parent_node) {
      element = value;
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  unsigned no_elements = 0;
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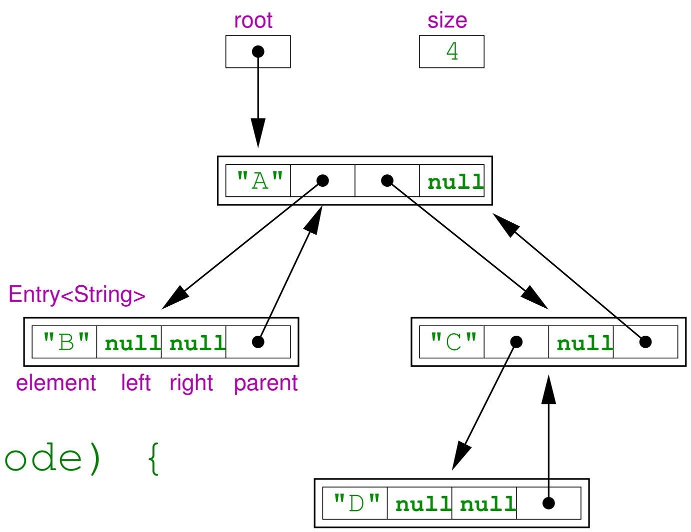

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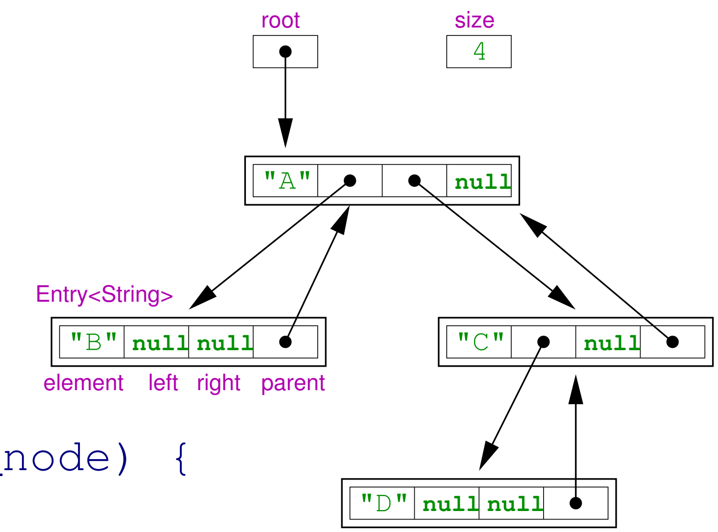

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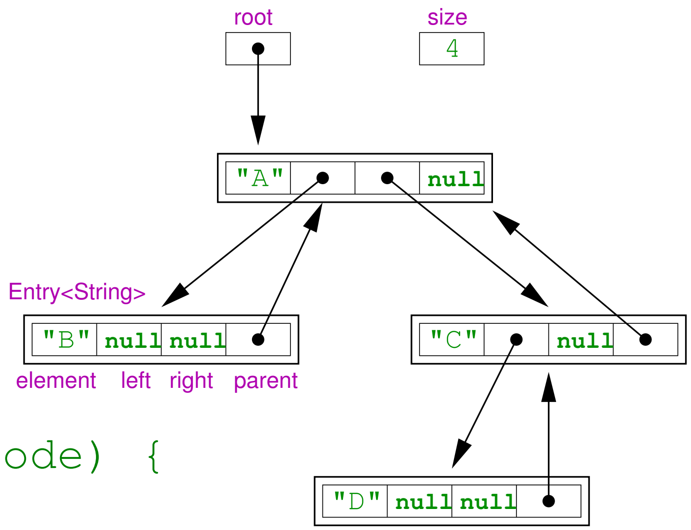

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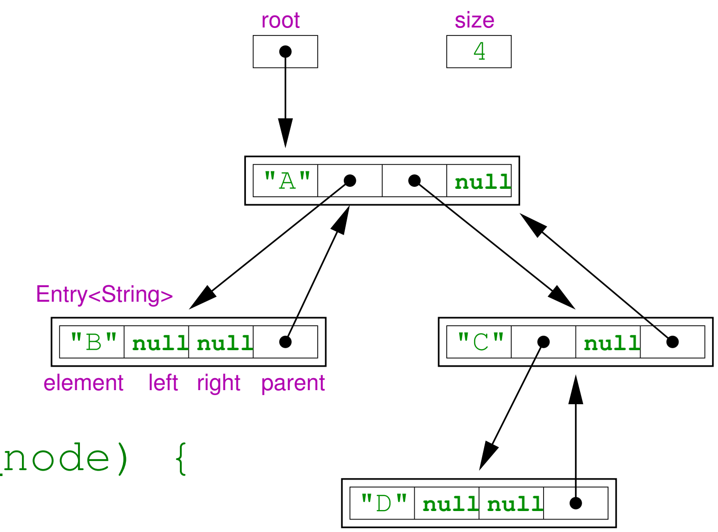

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Binary Search Trees

- We will concentrate on one of the most important binary trees, namely the **binary search tree**
- The binary search tree keeps the elements ordered
- We can define a binary search tree recursively
  1. Each element in the left subtree is less than the root element
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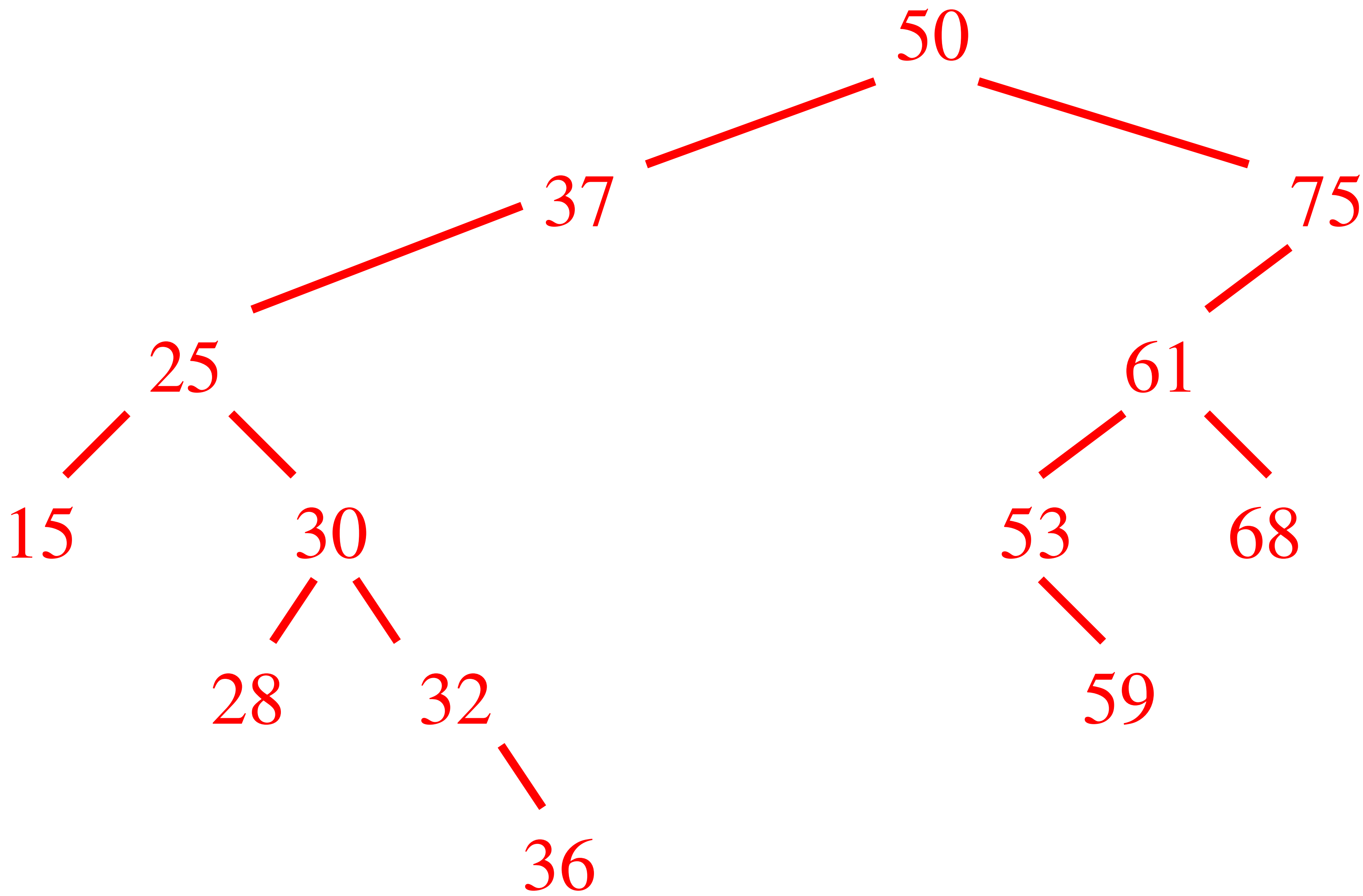
Example Binary Search Tree



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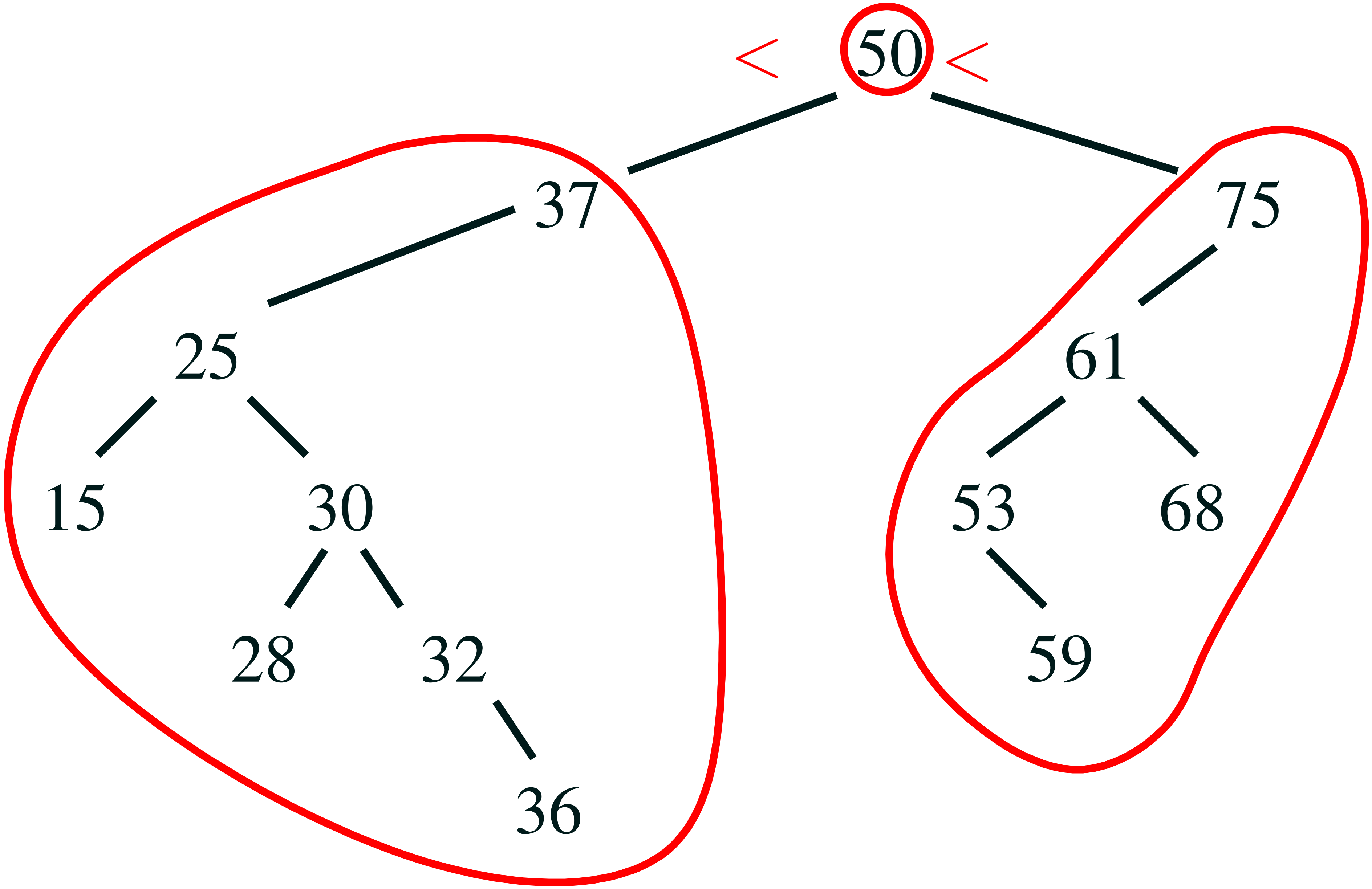

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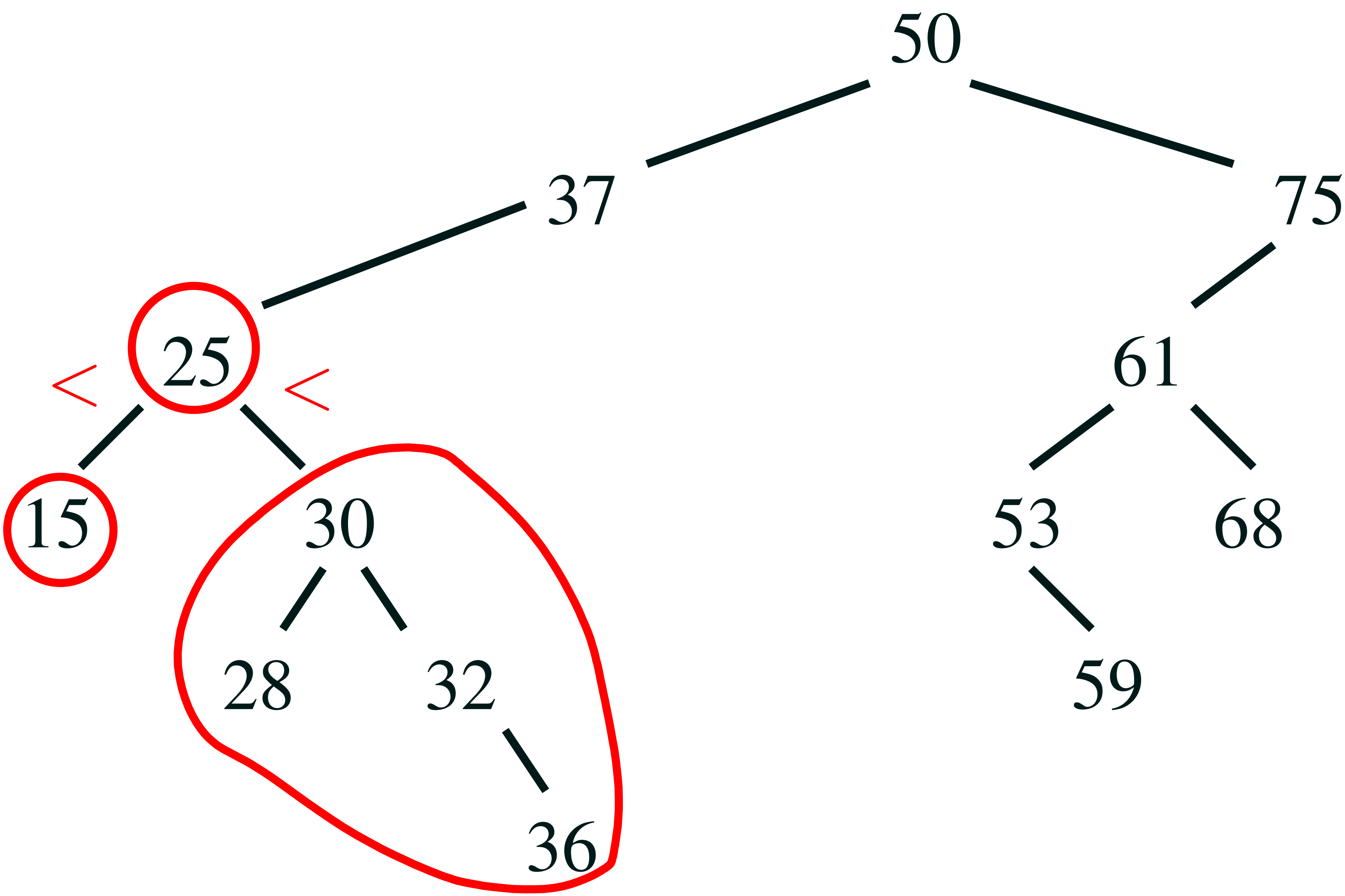

Searching A Binary Search Tree

- Searching a binary search tree is easy
- Start at the root
- Compare with element
  - ⋆ If less than element go left
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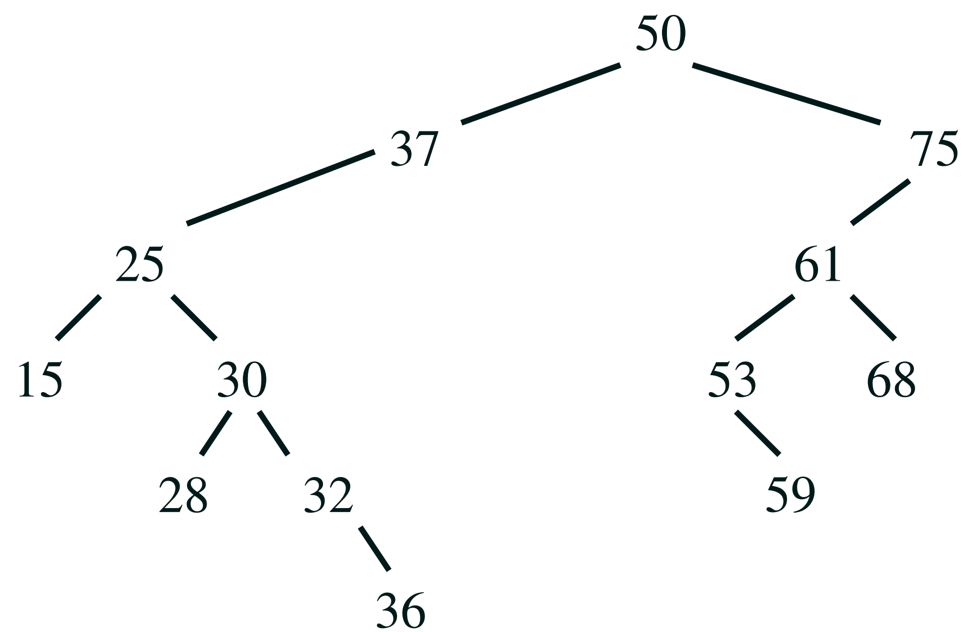
  - ⋆ If equal to element found



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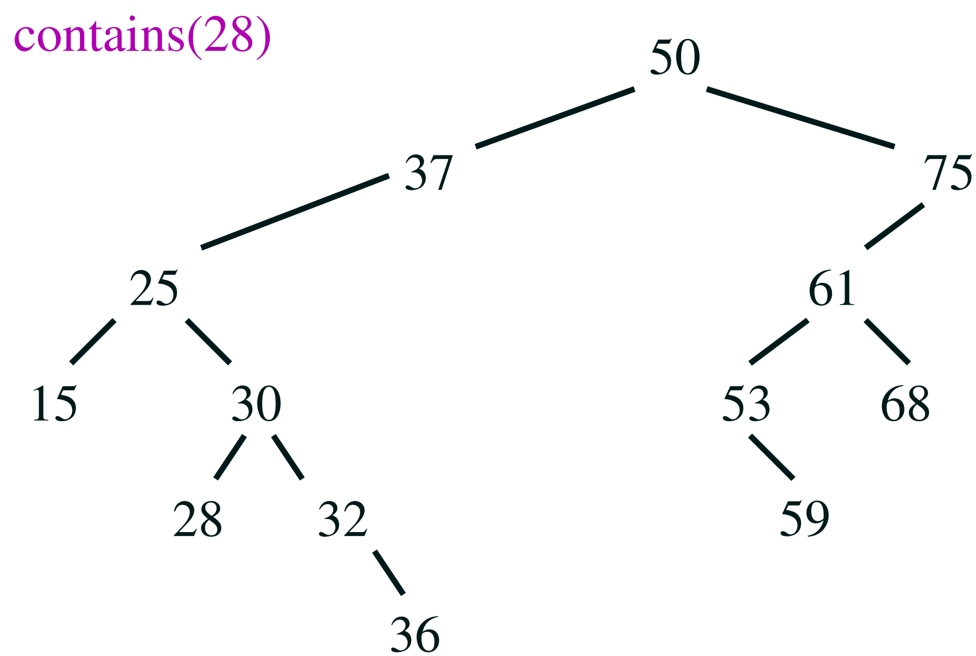

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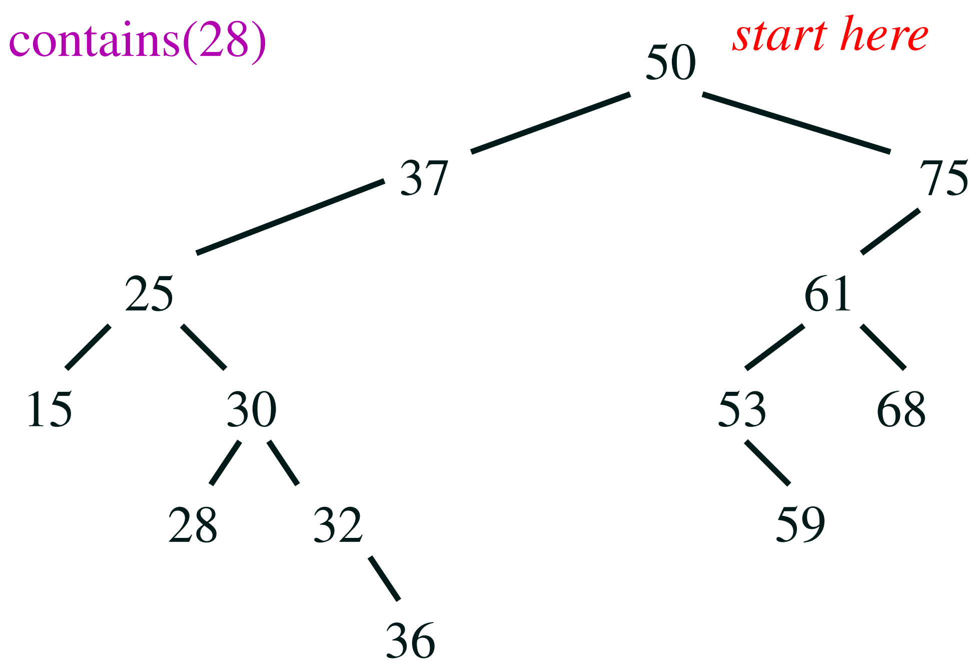

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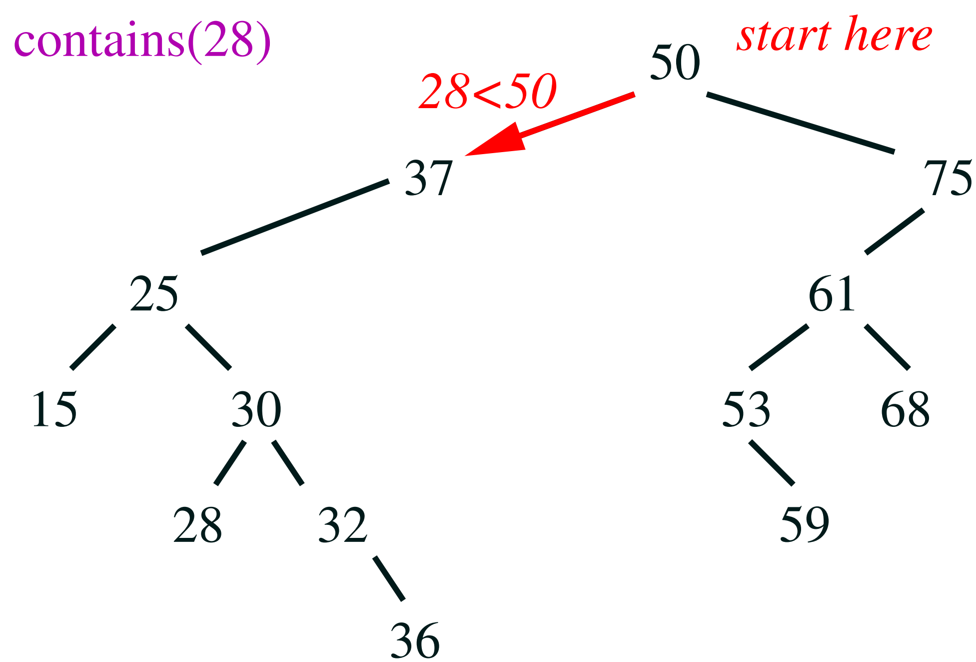

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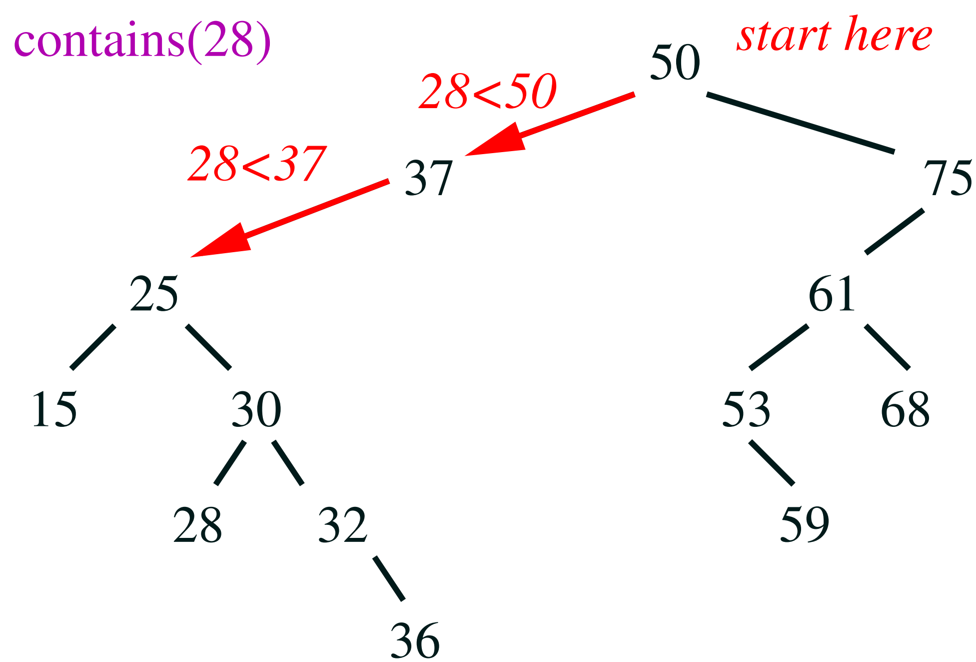

Searching A Binary Search Tree

- Searching a binary search tree is easy
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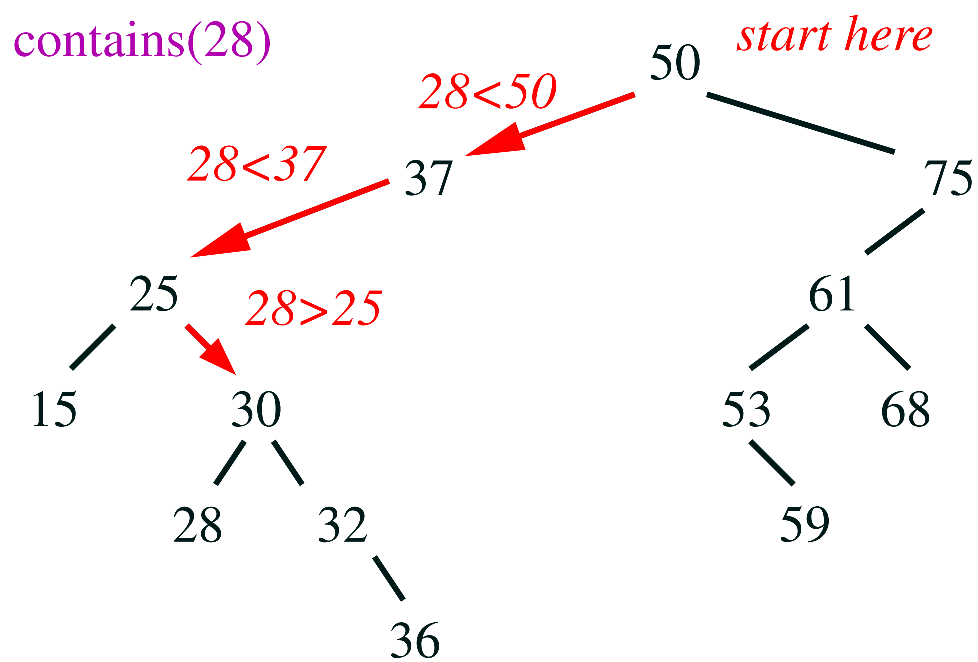

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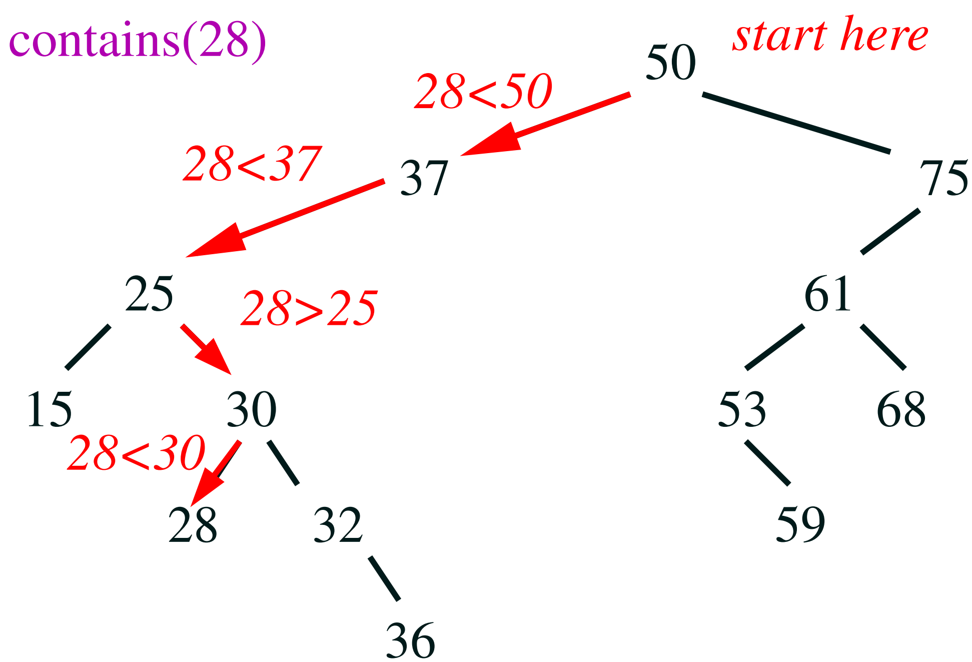

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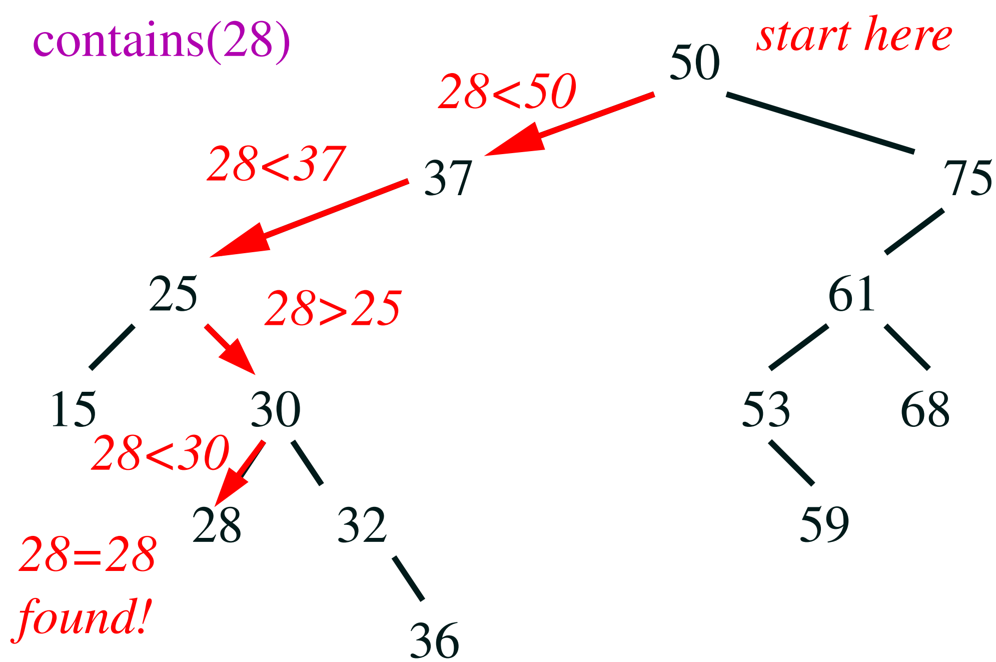
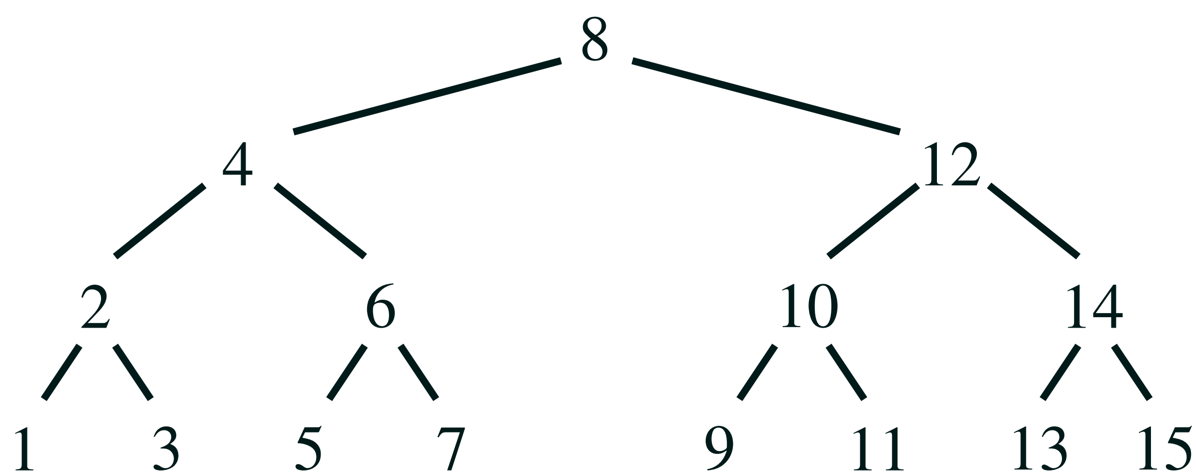
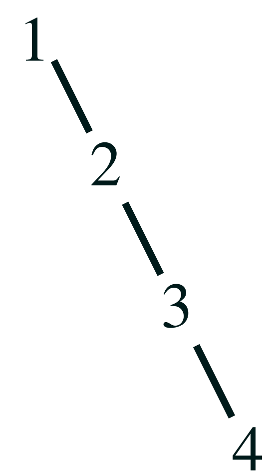


Speed of Search

- The number of comparisons necessary to find an element in a binary tree depends on the level of the node in the tree
- The worst case number of comparisons is therefore the height of the tree
- This depends on the density of the tree



full tree



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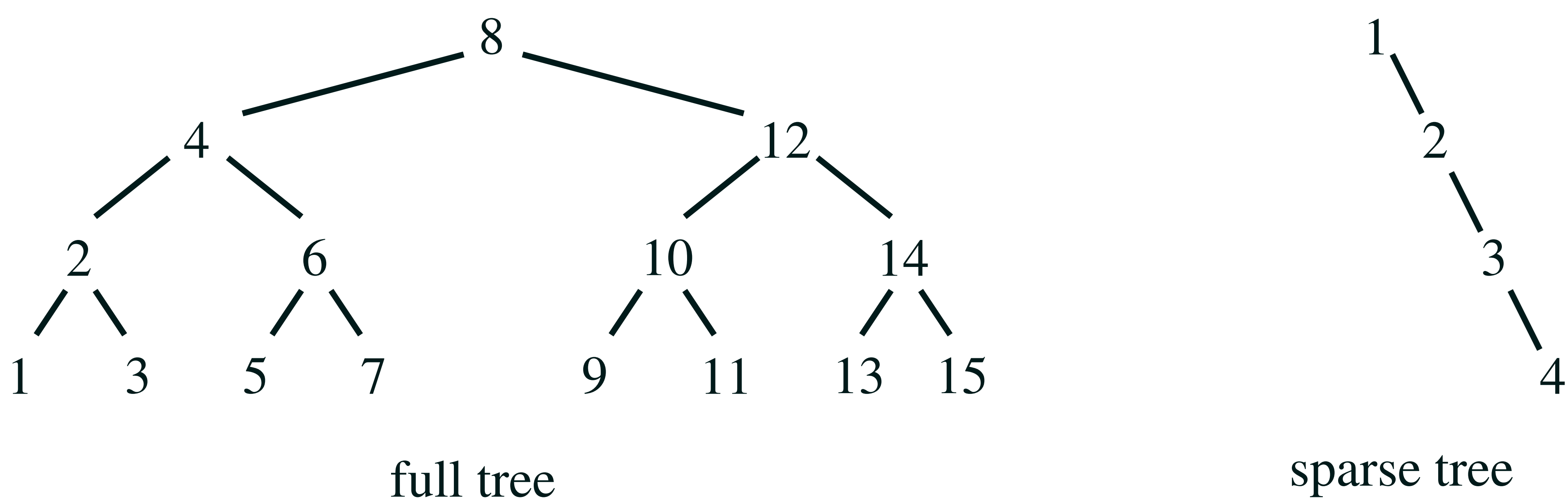

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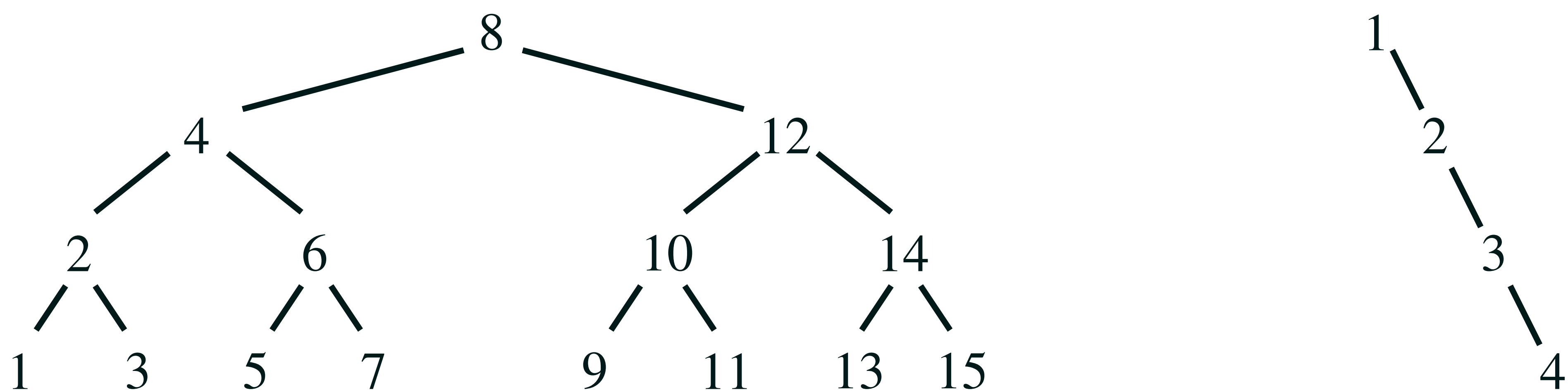

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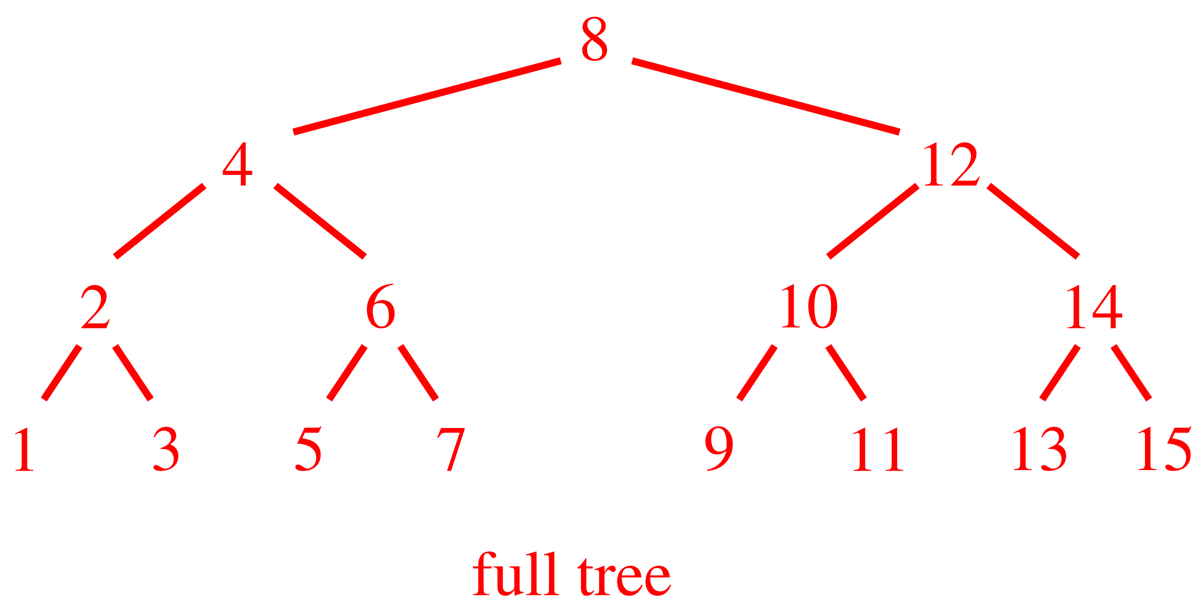

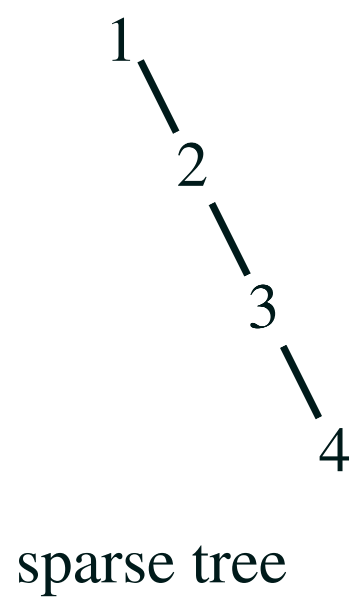

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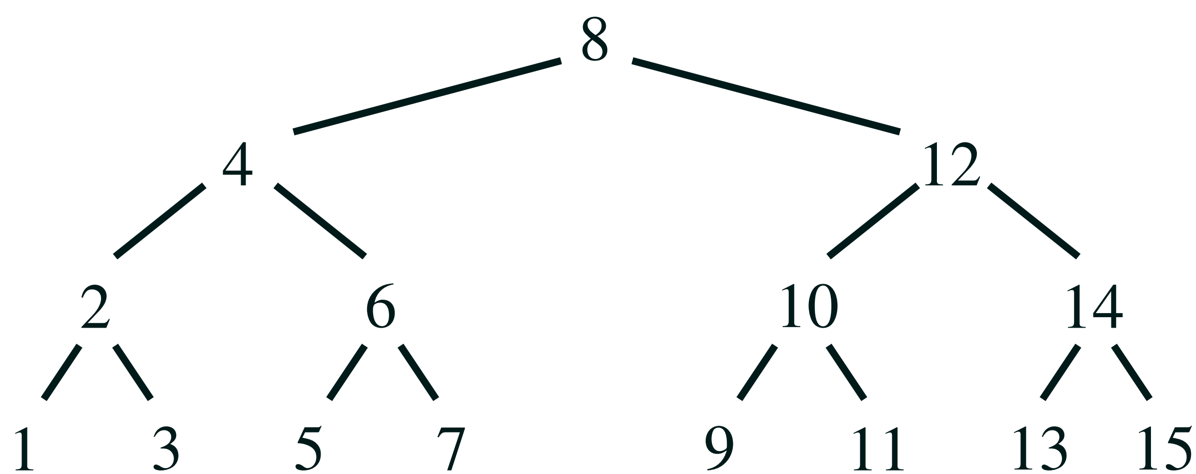

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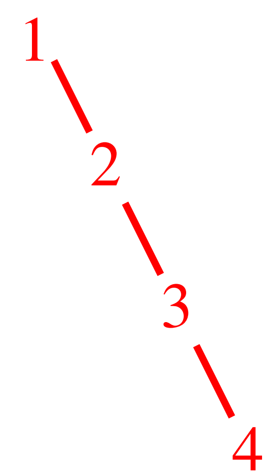

sparse tree

Implementing a Set

- A set is a fundamental **abstract data type**
- It is a collection of things with no repetition and no order
- Ironically because order doesn't matter we can order the elements

$$\{1,3,5,5,3,4\} = \{5,3,4,1\} = \{1,3,4,5\}$$

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Fitting In

- The standard template library provides a class `std:set<T>`
- This contains many functions like
  - Constructors
  - `size()`
  - `insert(T o)`
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Comparable

- To sort any objects they must be comparable
- In the STL the set implementation has a second template parameter: `std::set<T, Compare = less<T> >`
- by default this is defined to be `less<T>` (which is a function already defined for most common types) which you can define
- If you have a set of complex objects you will have to define `Compare`

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bool MyCompare(MyObject left, MyObject right) {
  return something
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mySet = set<MyObject, MyCompare>;
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Find an Element

- One of the core operations of a binary tree is to find a node

```
iterator find(const T& element) {
  Node* current = root;
  while (current!=0) {
    if (current->element == element) {
      return iterator(current);
    }
    if (element < current->element) {
      current = current->left;
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      current = current->right;
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Add an Element

```
pair<iterator, bool> insert(const T& element) {
  if (no_elements==0) {
    root = new Node(element, 0);
    ++no_elements;
    return pair<iterator, bool>(iterator(root), true);
  }
  Node* parent = 0;
  Node* current = root;
  while(current != 0) {
    if (current->element == element) {
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    parent = current;
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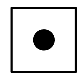
  }
```

Tree in Action

add(84)



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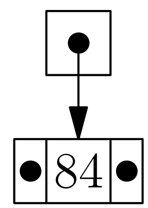
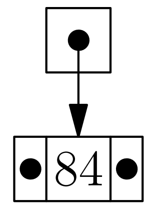


Tree in Action

add(39)



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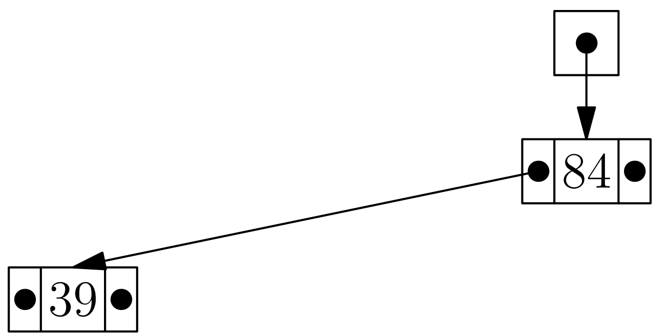

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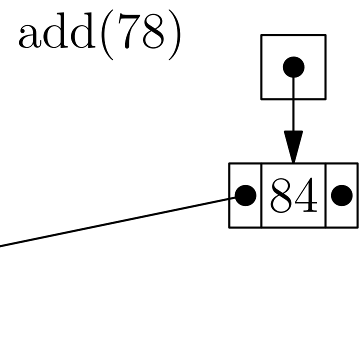

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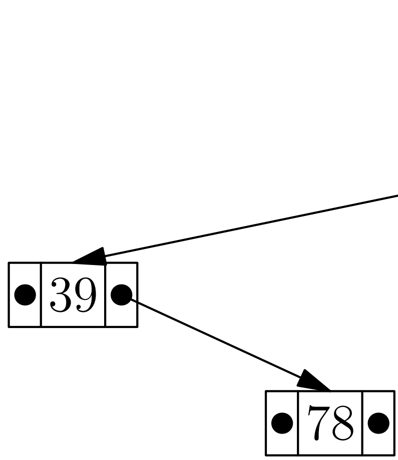

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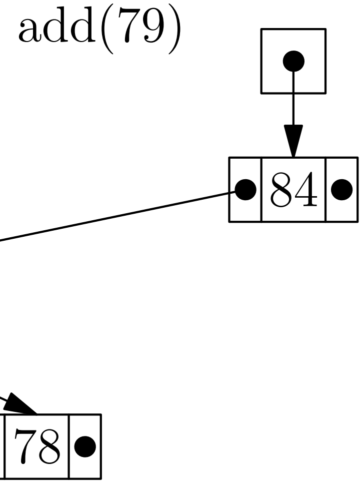

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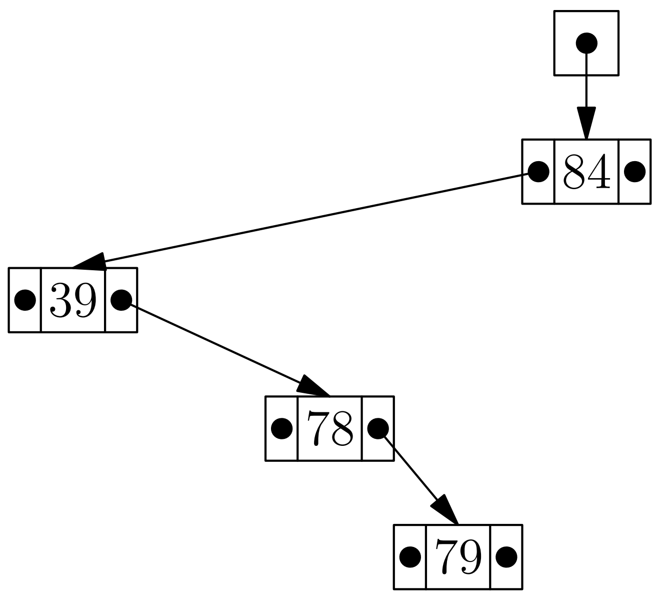

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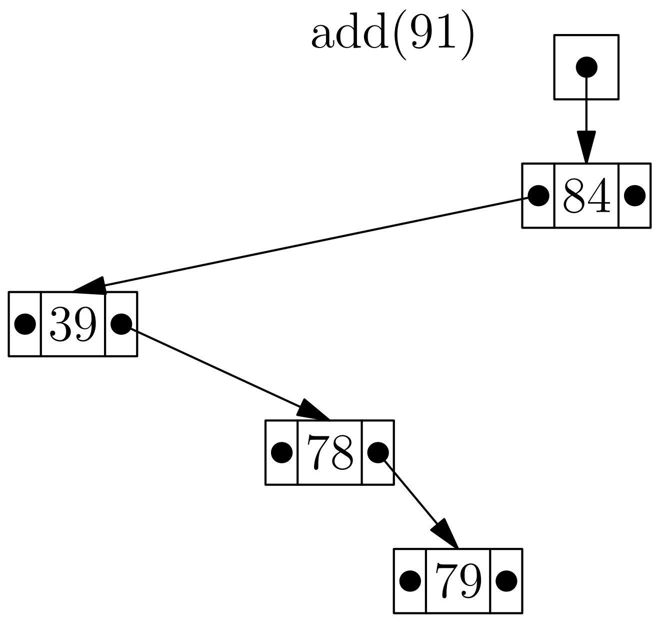

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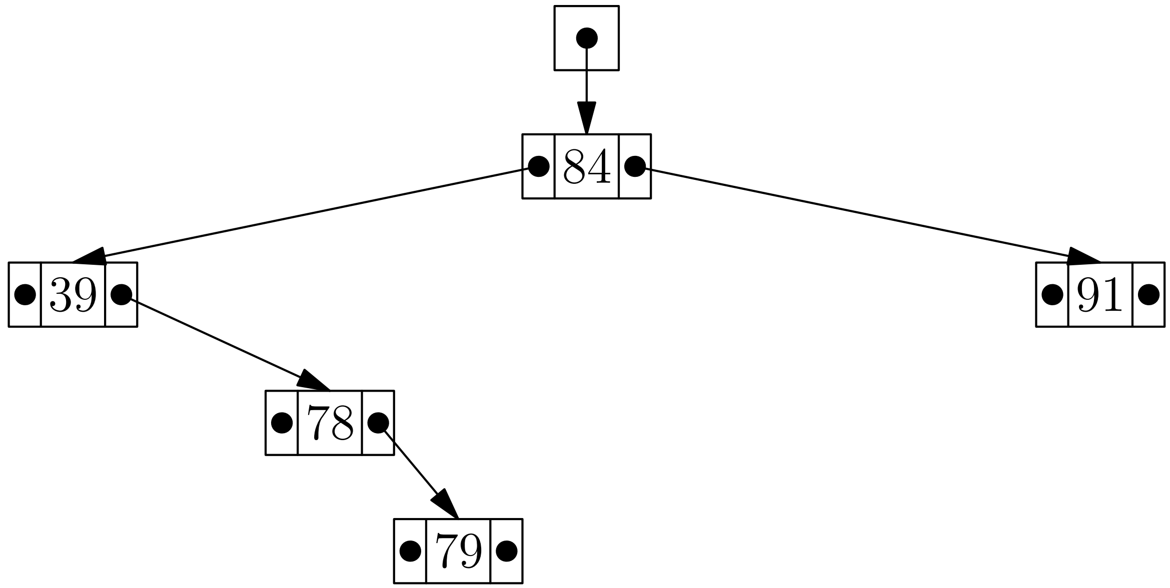

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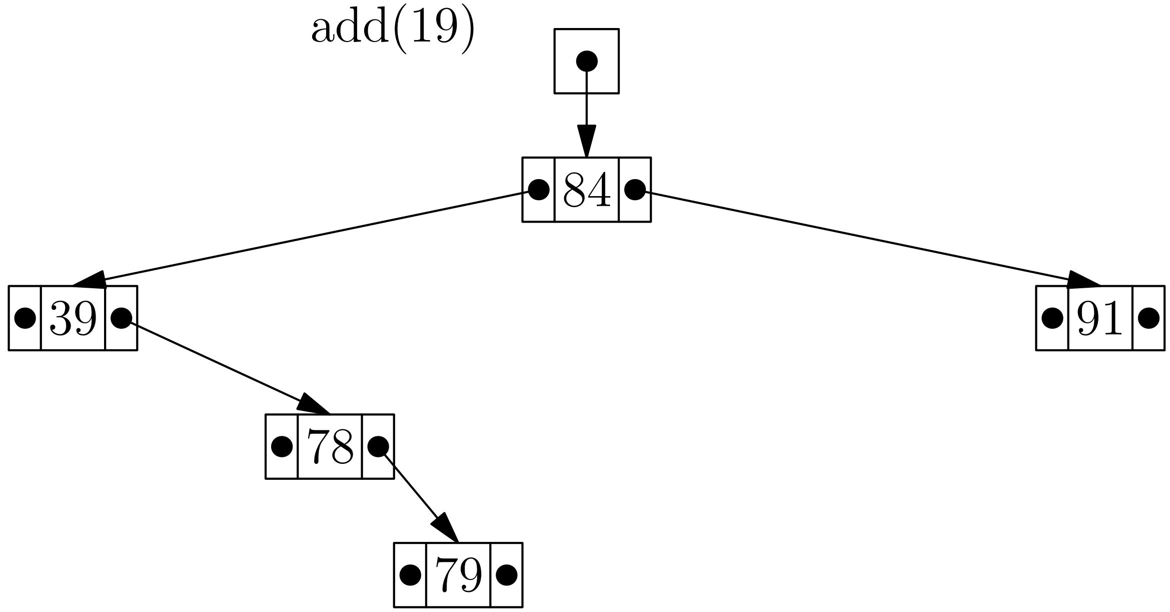

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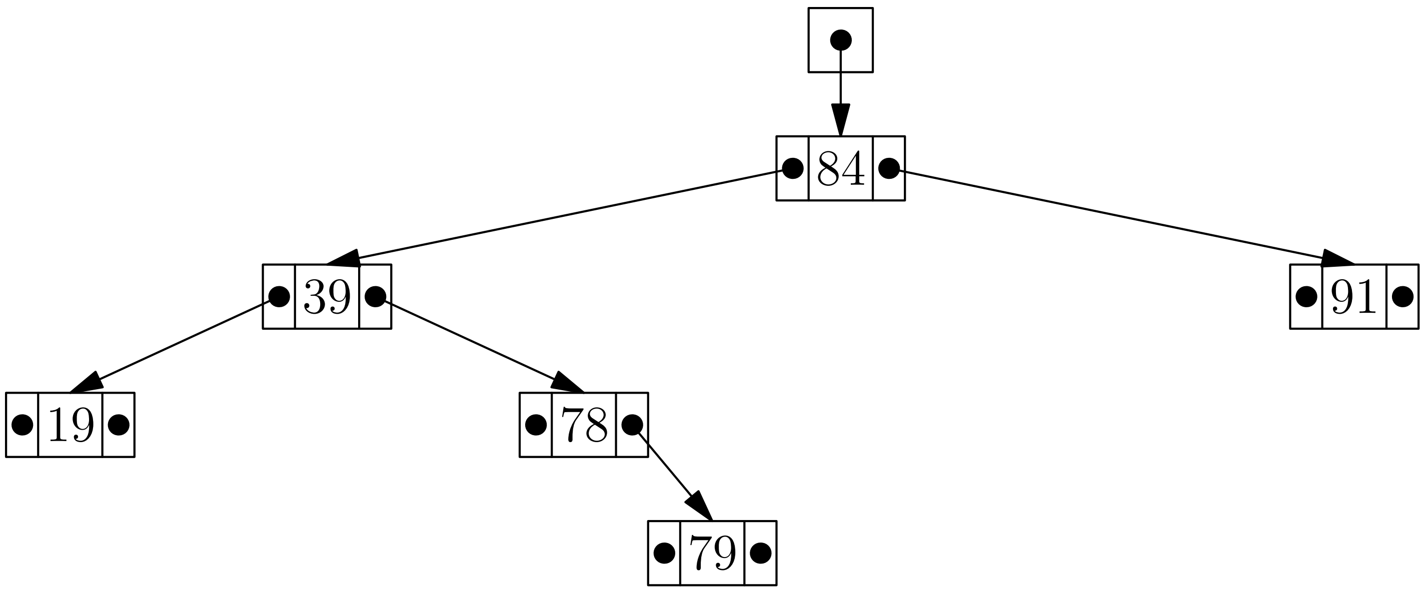

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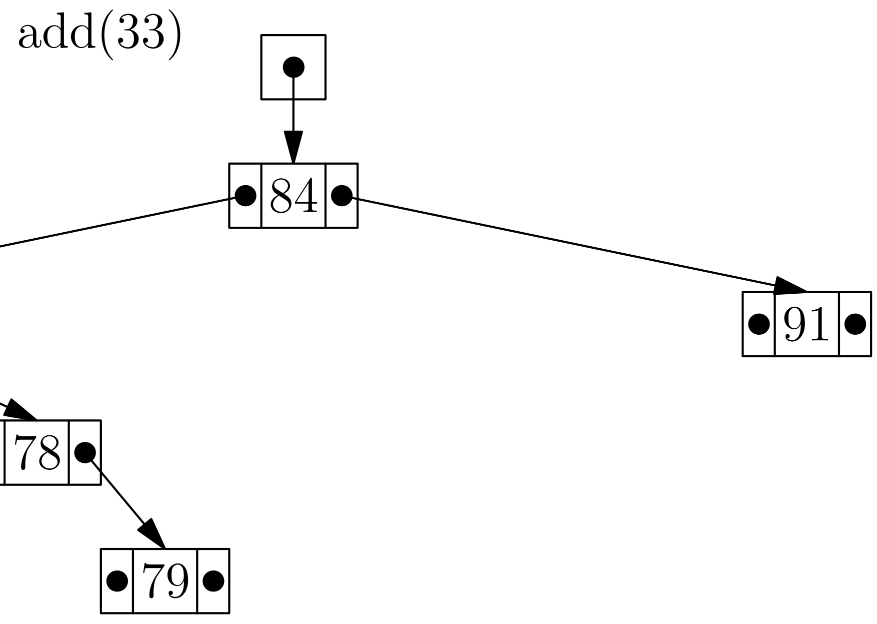

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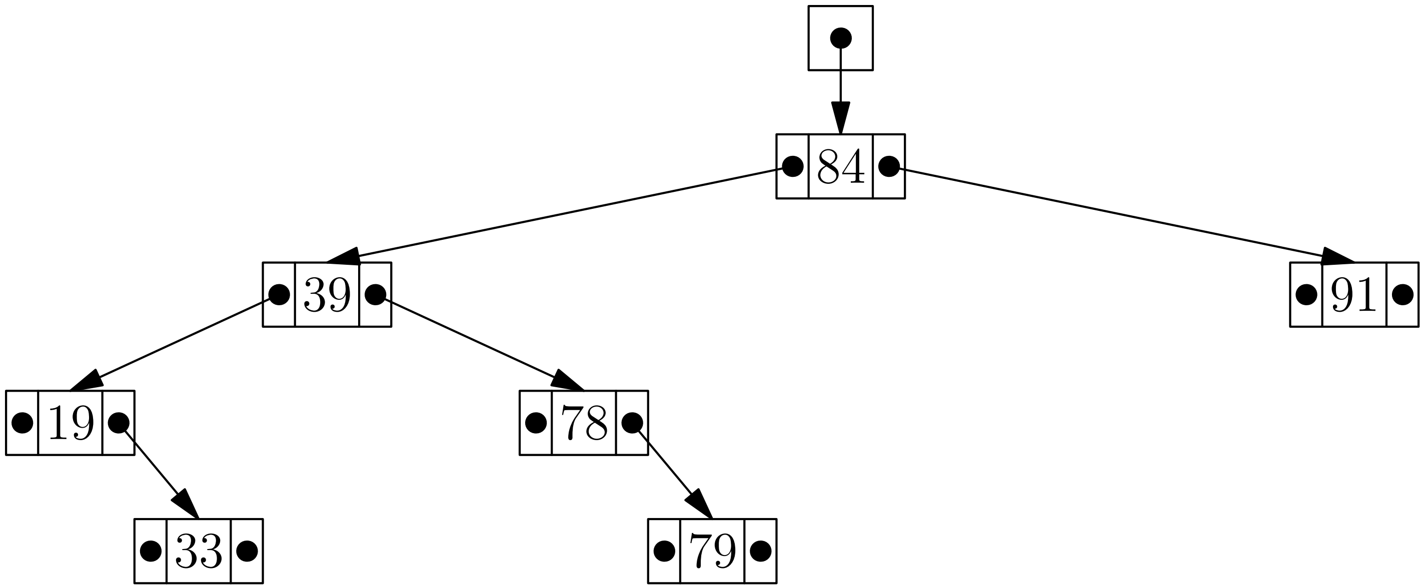

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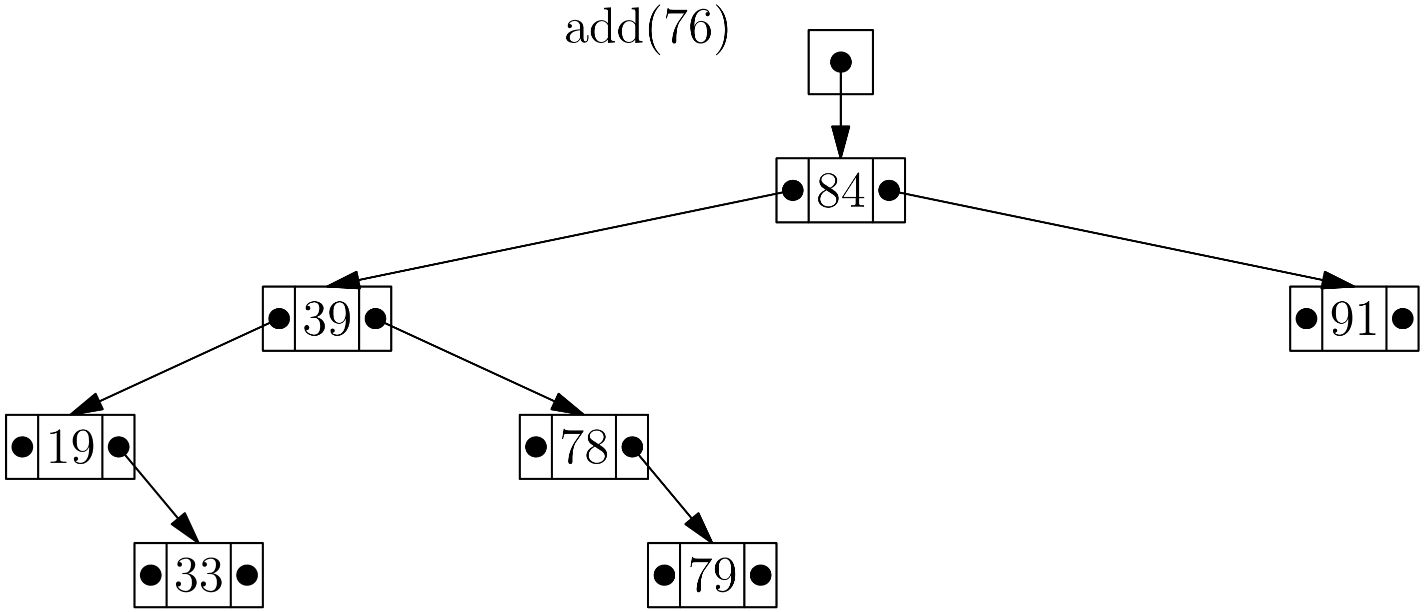

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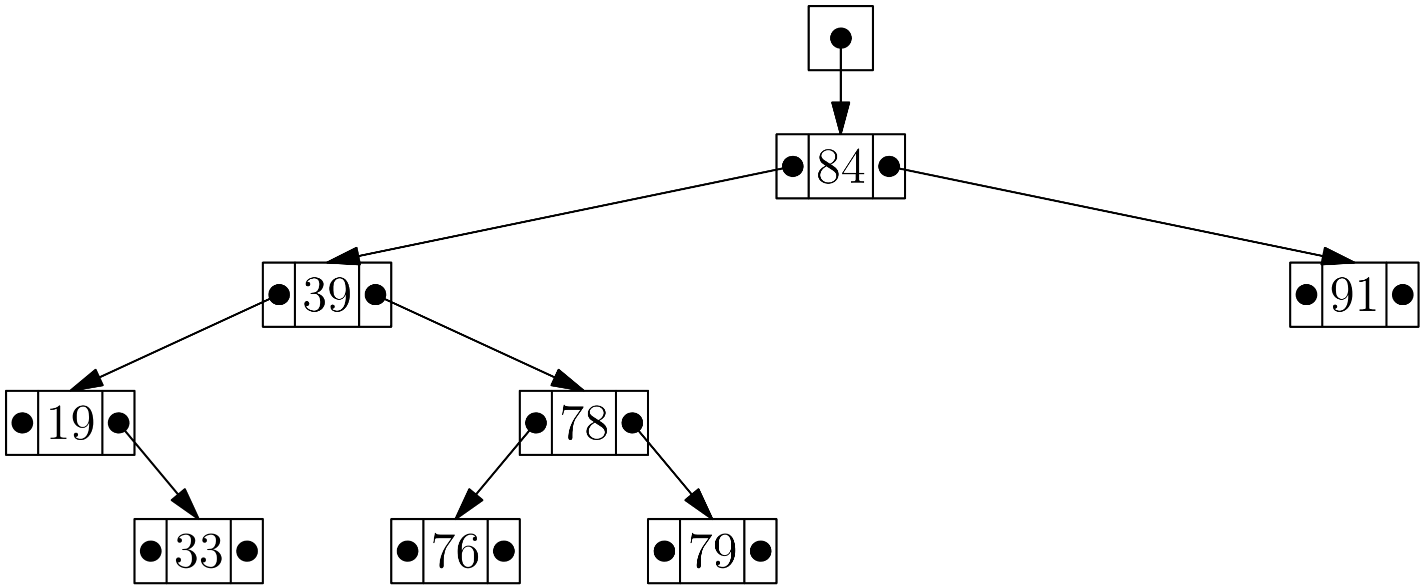

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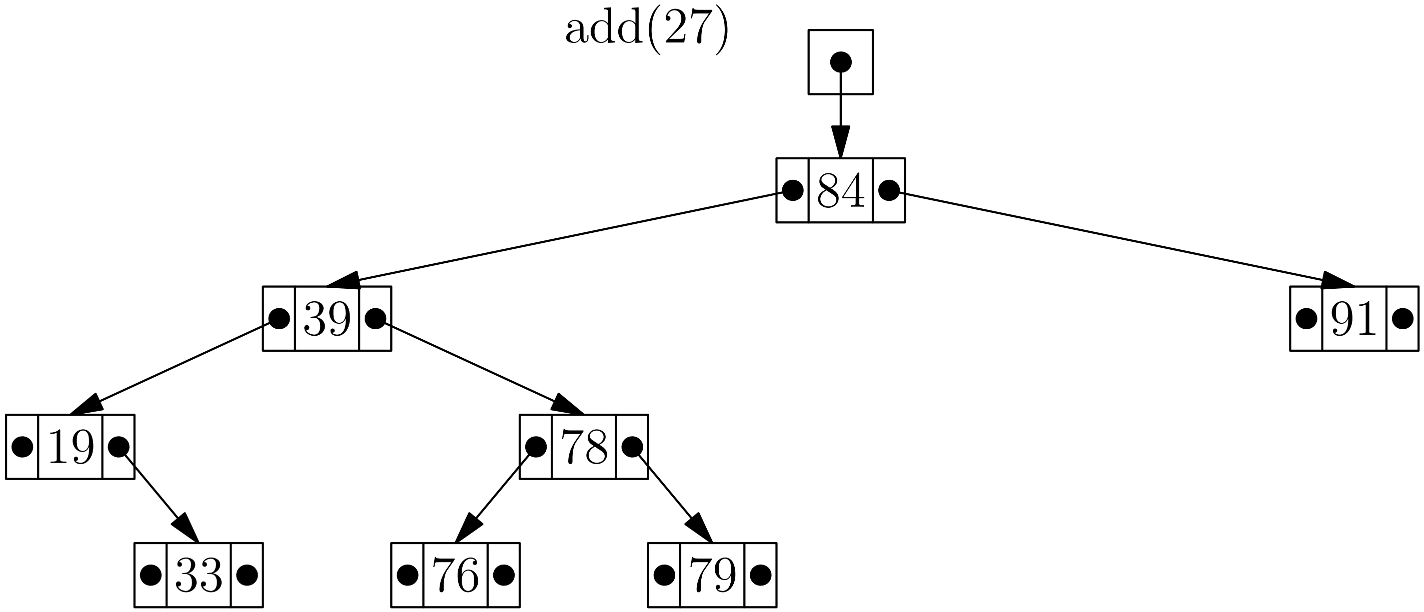

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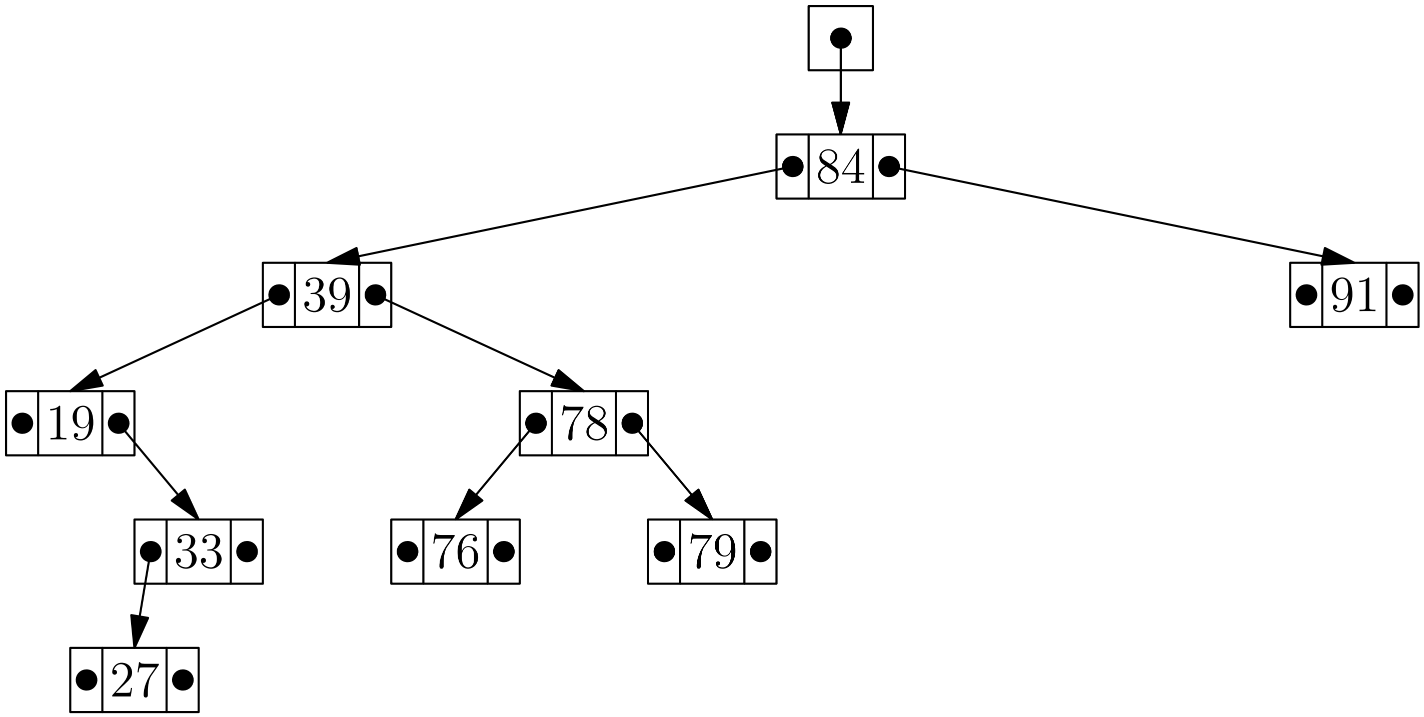

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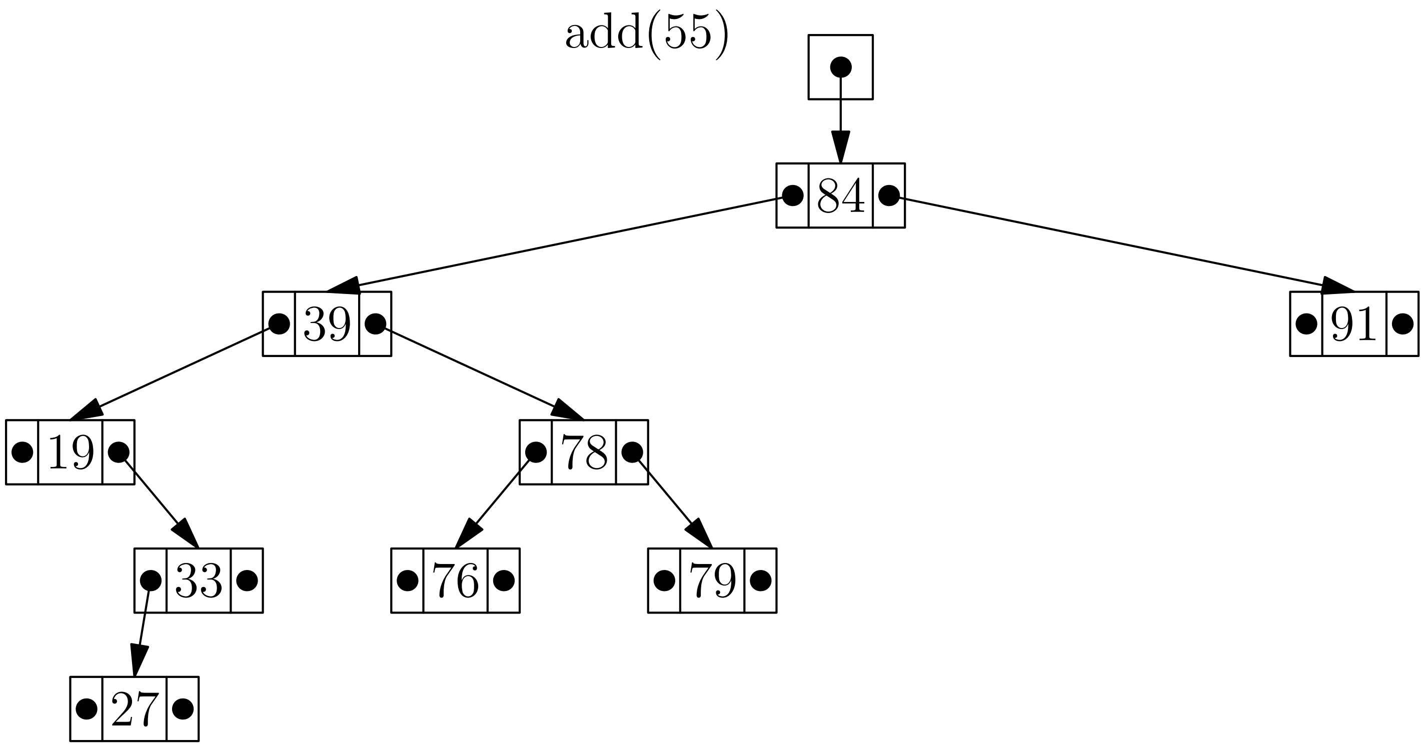

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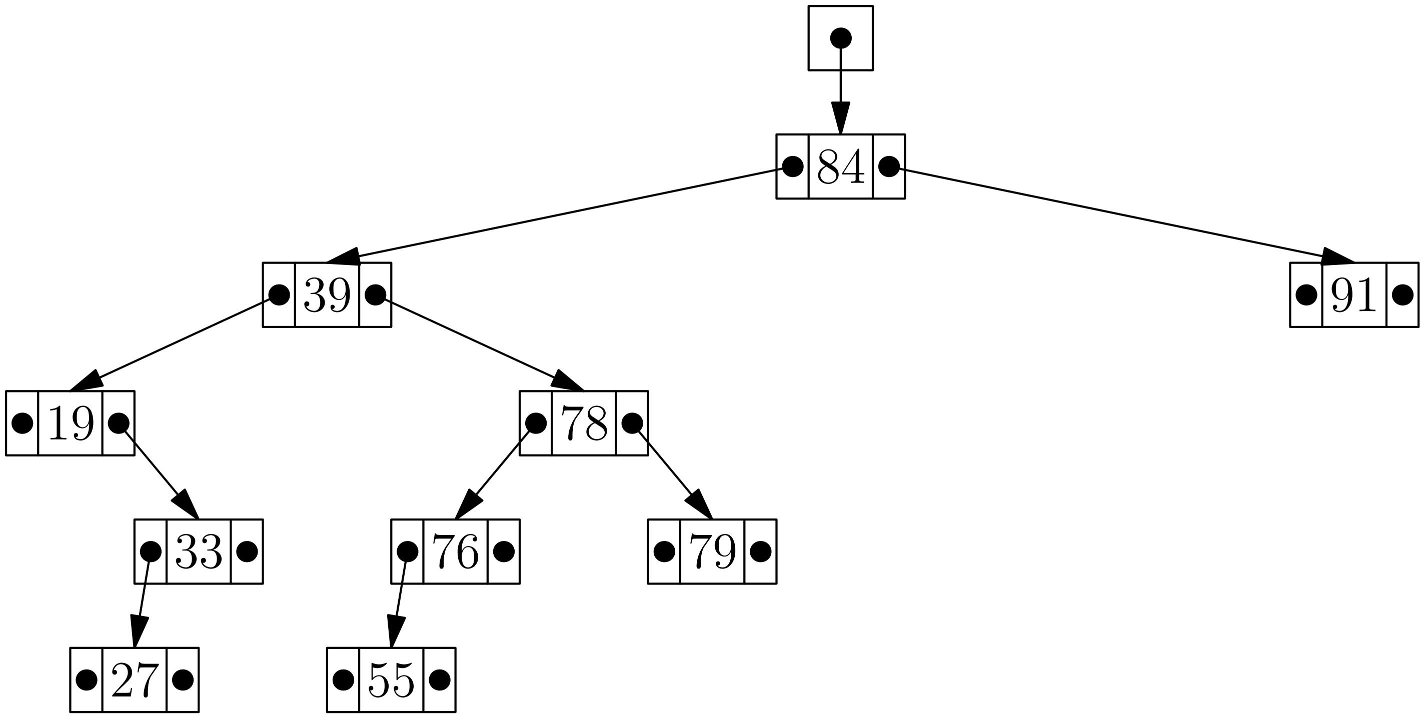

Shape of Tree

- The structure of the tree depends on the order in which we add elements to it
- Suppose we add

  To be, or not to be: that is the question:
  Whether 'tis nobler in the mind to suffer
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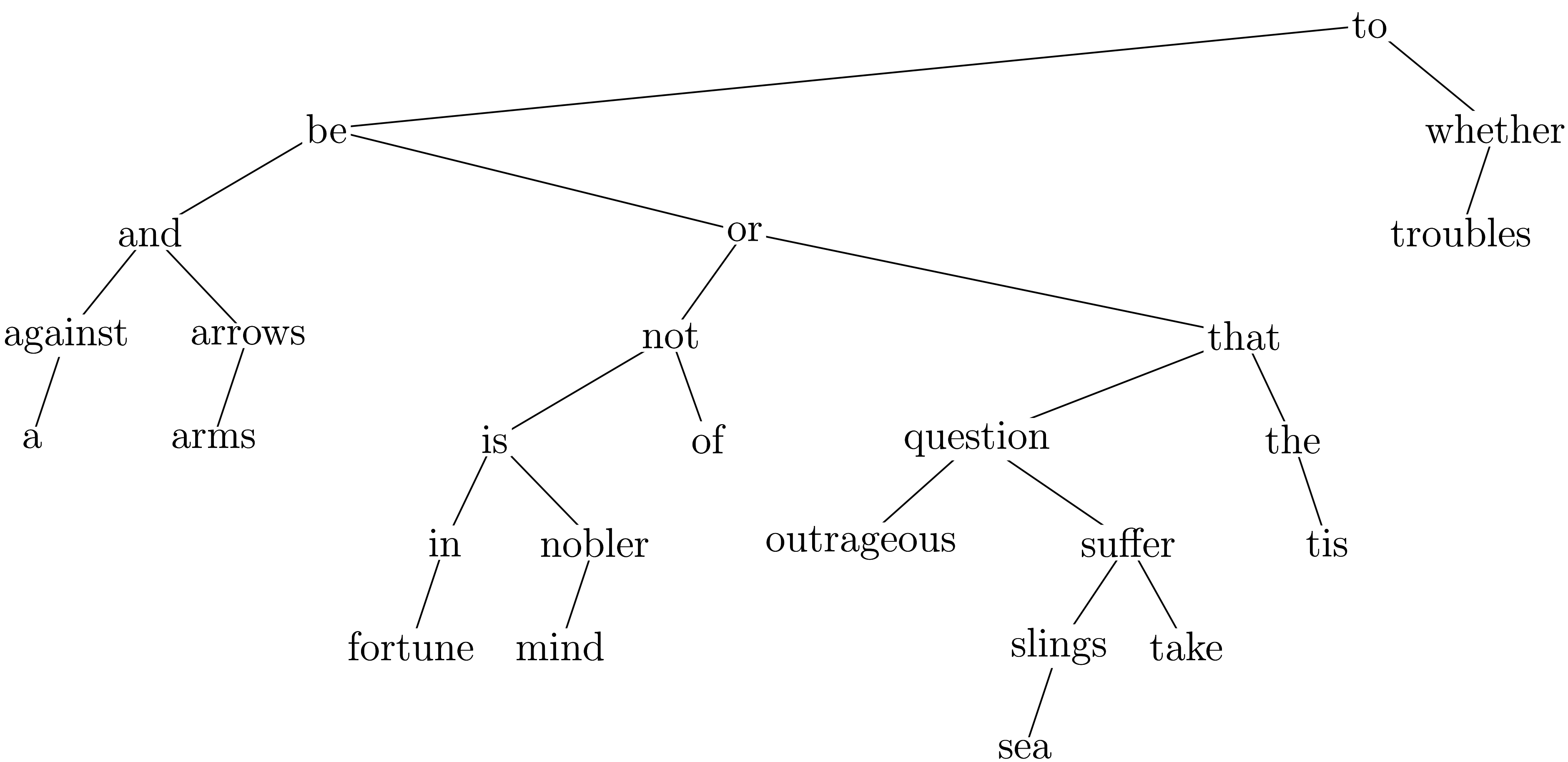
Hamlet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Outline

1. Trees
2. Binary Trees
   - Implementing Binary Trees
3. Binary Search Trees
   - Definition
   - Implementing a Set
4. **Tree Iterators**

Tree Iterators

- As with most container classes it is very useful to define iterators
- `begin()` should return a "pointer" to the start of the tree
- `end()` provides a "pointer" past the end
- `operator*()` returns the element
- `opeator++()` increments the "pointer"
- `operator!=(lhs, rhs)` is used to compare iterators

```
set<int> mySet;
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for(auto pt=mySet.begin(), pt!=mySet.end(), ++pt) {
  cout << *pt;
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for(auto pt=mySet.begin(), pt!=mySet.end(), ++pt) {
  cout << *pt;
}
```

C++ Code

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class binary_tree {

public:
  class iterator {
  private:
    Node* current;

  public:
    iterator(Node* node) {current=node;}
    T operator*() const {return current->element;}
    iterator operator++() {
      current = successor(current);
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Successor

- To find the successor we first start in the left most branch
- We follow two rules

1. **If** right child exist **then** move right once and then move as far left as possible
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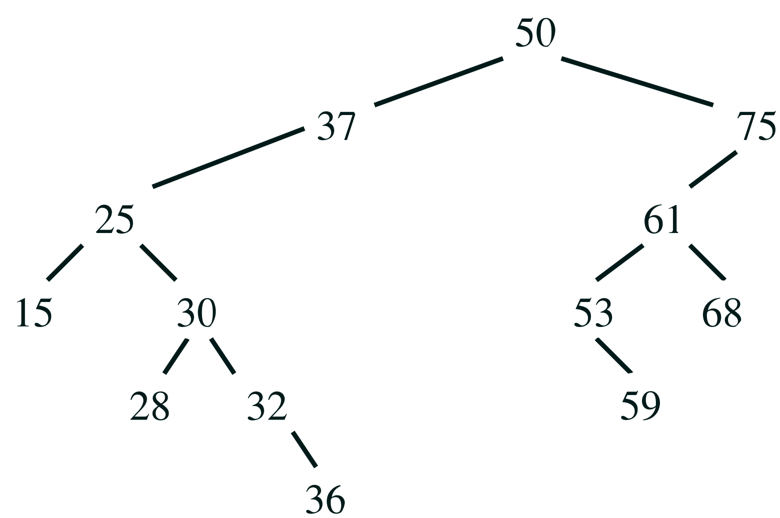

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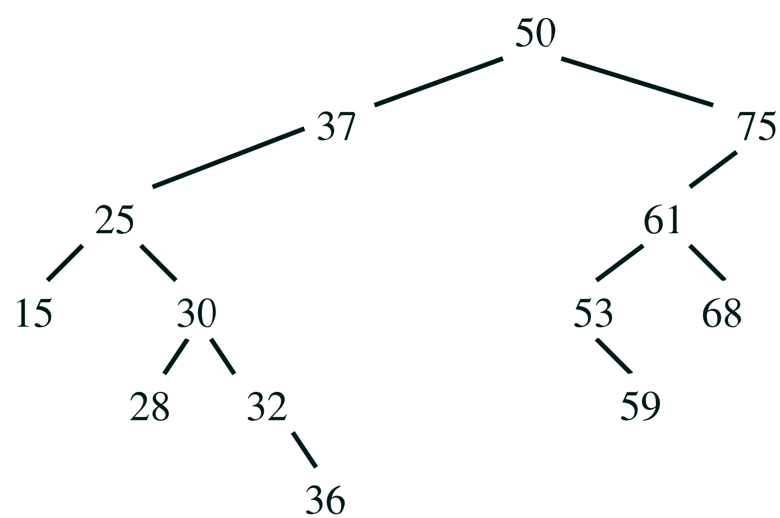

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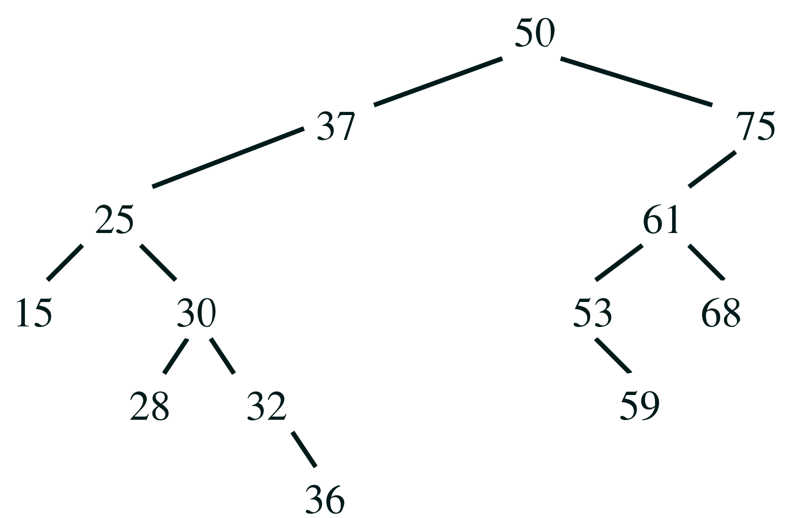

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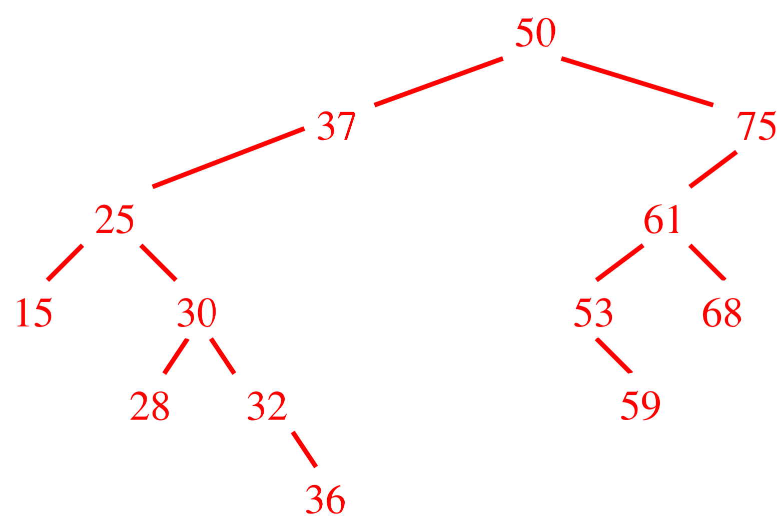

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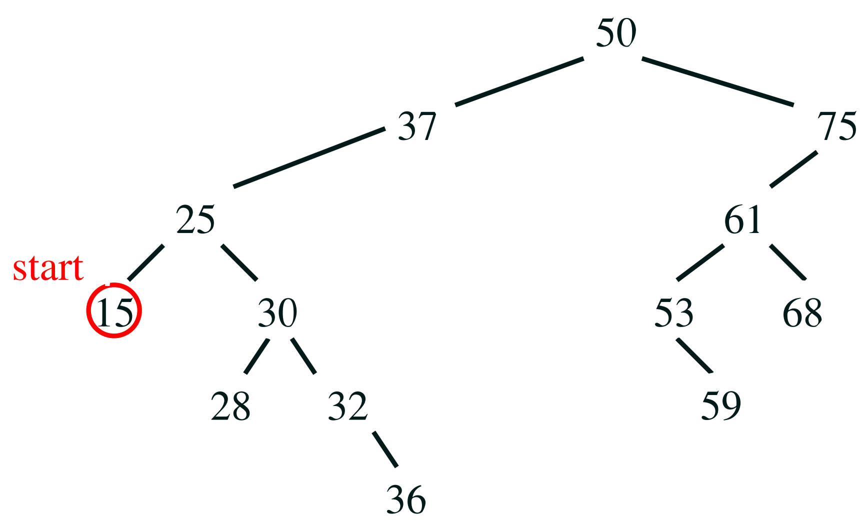

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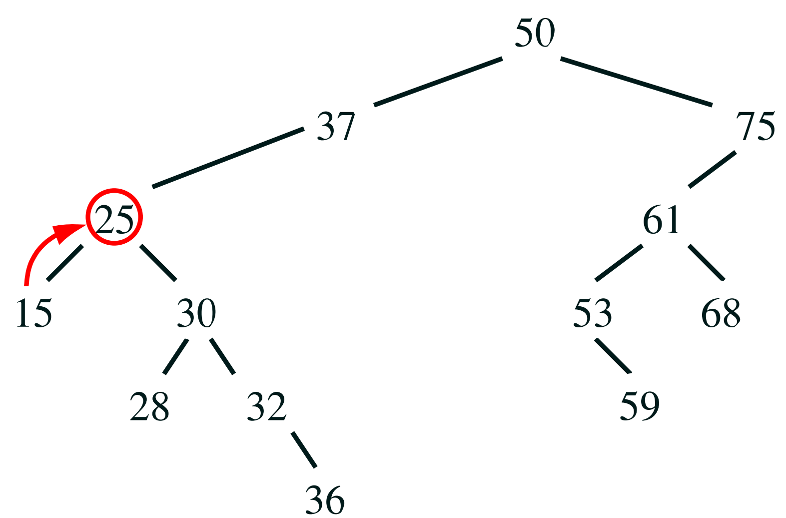

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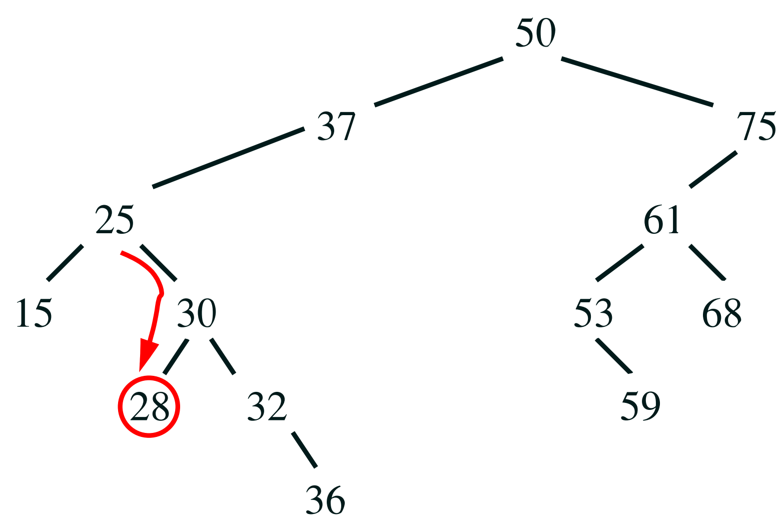

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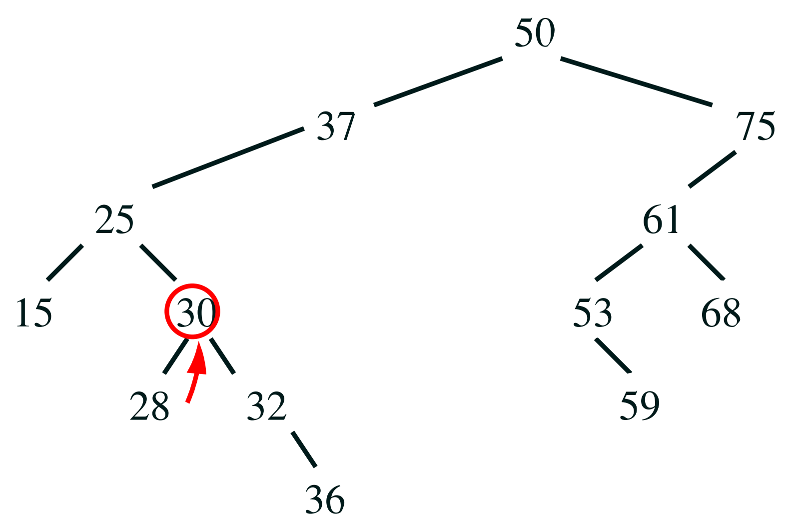

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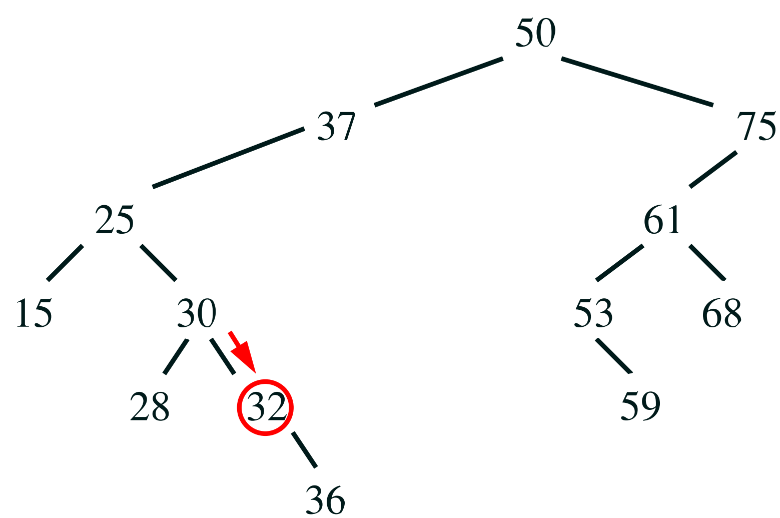

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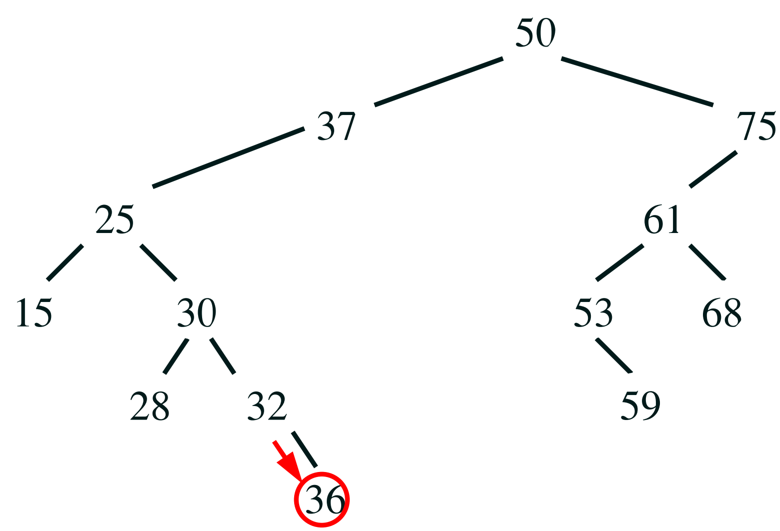

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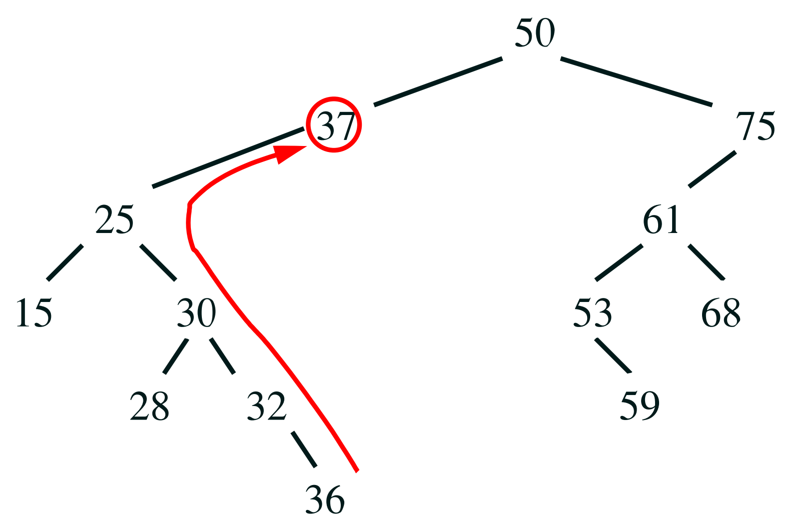

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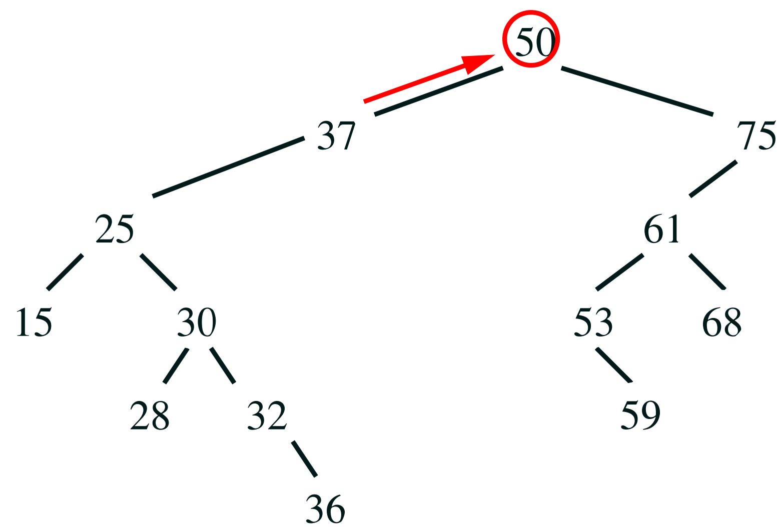

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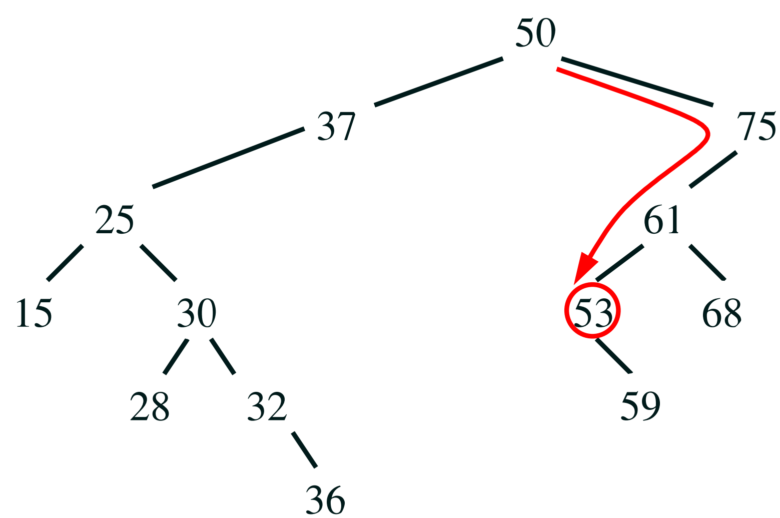

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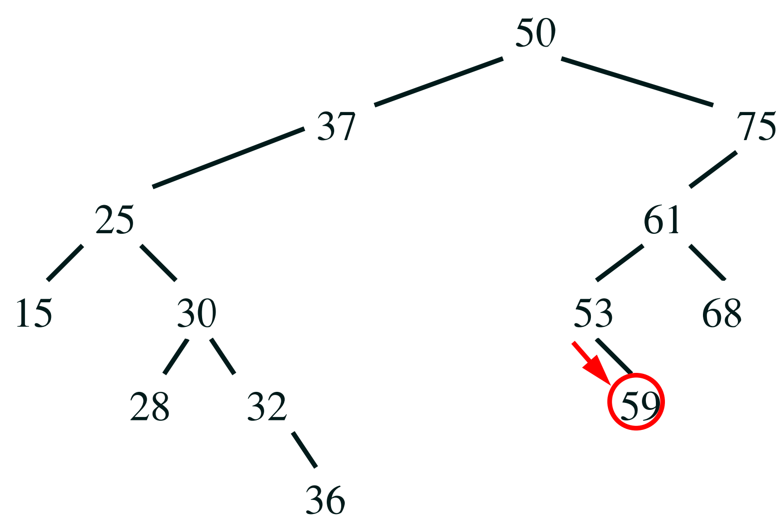

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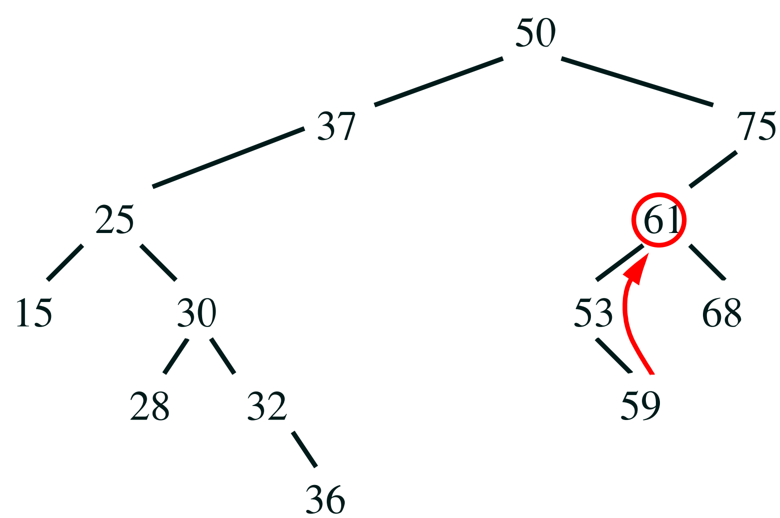

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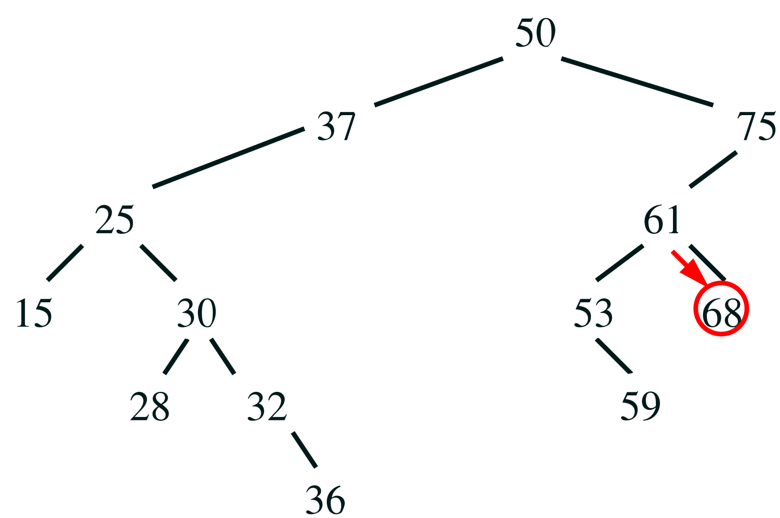

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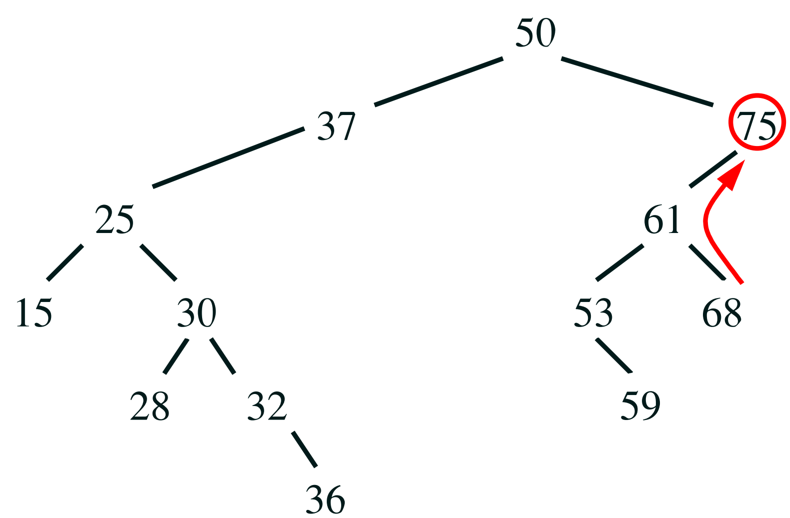

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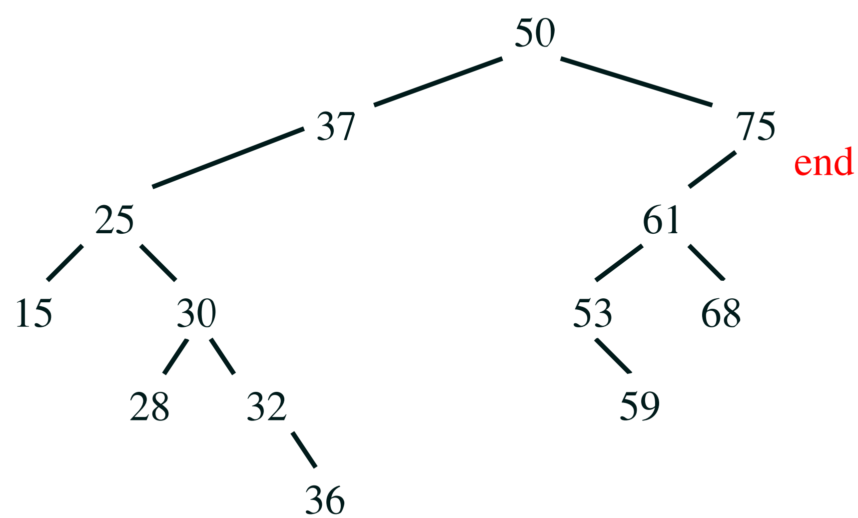

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- Conceptually they are quite simple
- However, there are a lot of details that need to be understood
- Coding even simple trees needs great care
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