

Further Mathematics and Algorithms

Lesson 10: *Make Friends with Trees*



Binary trees, binary search trees, sets, tree iterators

Outline

1. **Trees**
2. Binary Trees
 - Implementing Binary Trees
3. Binary Search Trees
 - Definition
 - Implementing a Set
4. Tree Iterators

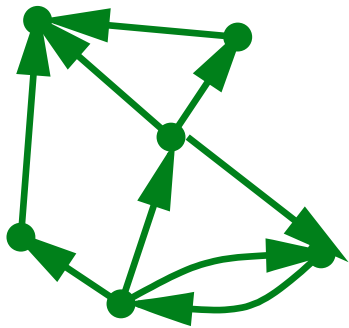


Trees

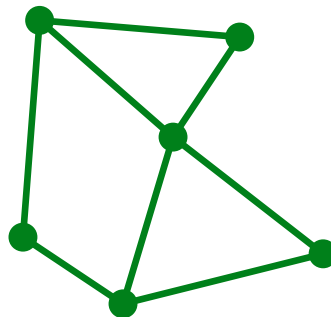
- Trees are one of the major ways of structuring data■
- They are used in a vast number of data structures
 - ★ Binary search trees
 - ★ B-trees
 - ★ splay trees
 - ★ heaps
 - ★ tries
 - ★ suffix trees■
- We shall cover most of these■

Defining Trees

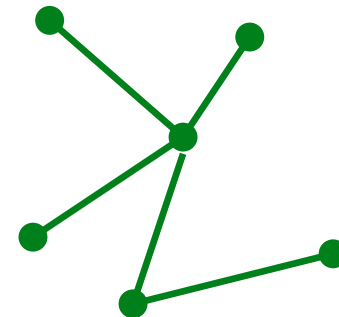
- Mathematically a tree is an **acyclic undirected graph**
 - ★ **graph**: a structure consisting of **nodes** or **vertices** joined by **edges**
 - ★ **undirected**: the edges goes both ways
 - ★ **acyclic**: there are no cycles in the graph



graph



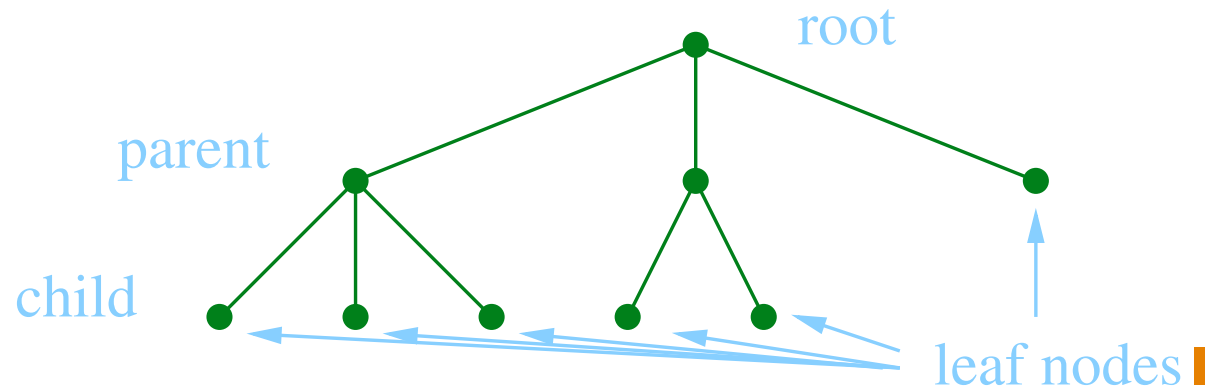
undirected graph



tree = acyclic undirected graph

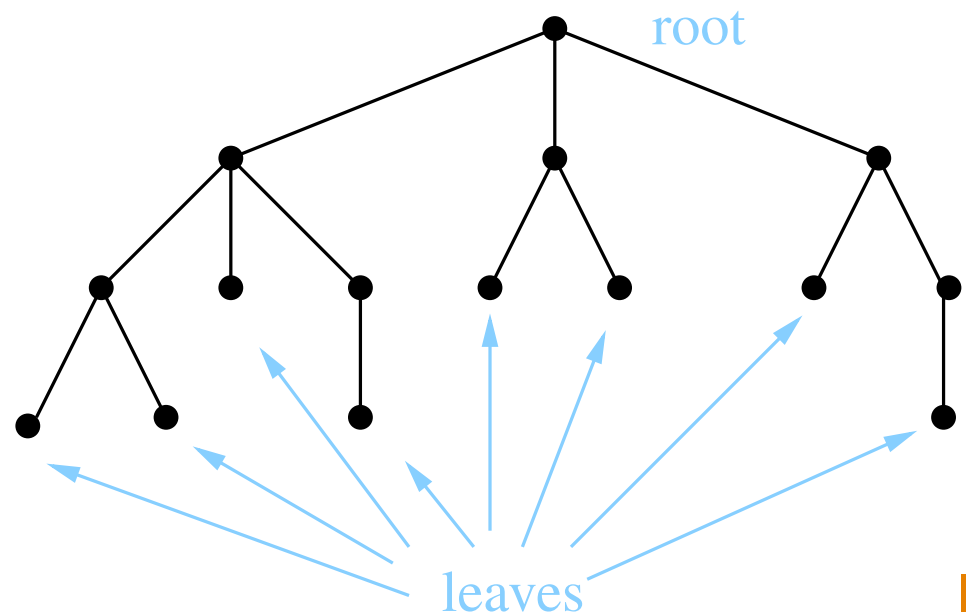
Borrowing from Nature

- We often impose an ordering on the nodes (or a direction on the edges) — known as a rooted tree
- Borrowing from nature, we recognise one node as the **root** node
- Nodes have **children** nodes living beneath them
- Each child has a **parent** node above them except the root
- Nodes with no children are **leaf** nodes



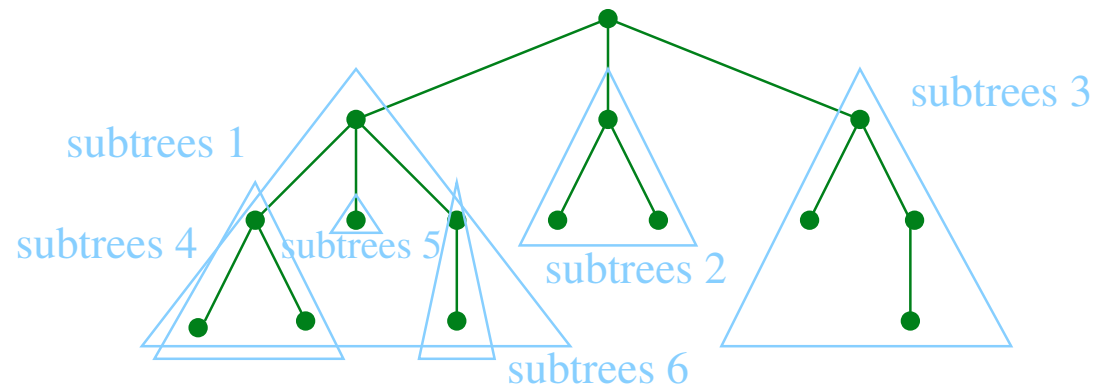
Spot the Error

- One small biological inconsistency■
- Yep!, computer scientists draw there trees upside down■
 - ★ root at the top
 - ★ leaves at the bottom■



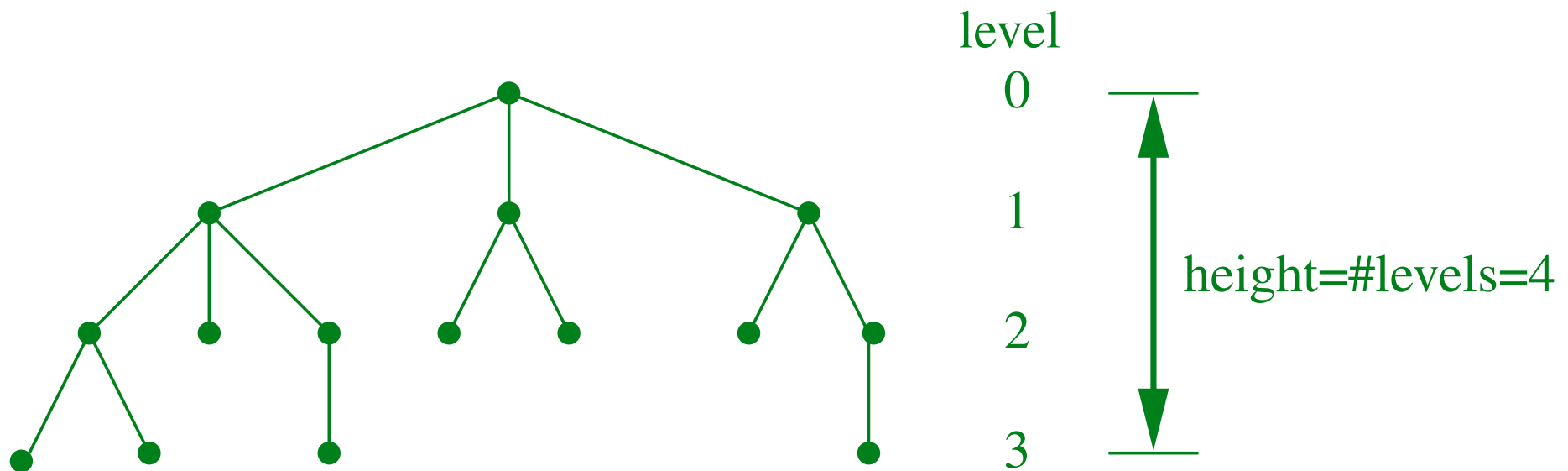
Subtrees

- We can think of the tree made up of **subtrees**



Level of Nodes

- It is useful to label different levels of the tree
- We take the **level** of a node in a tree as its distance from the root
- We take the **height** of a tree to be the number of levels



Outline

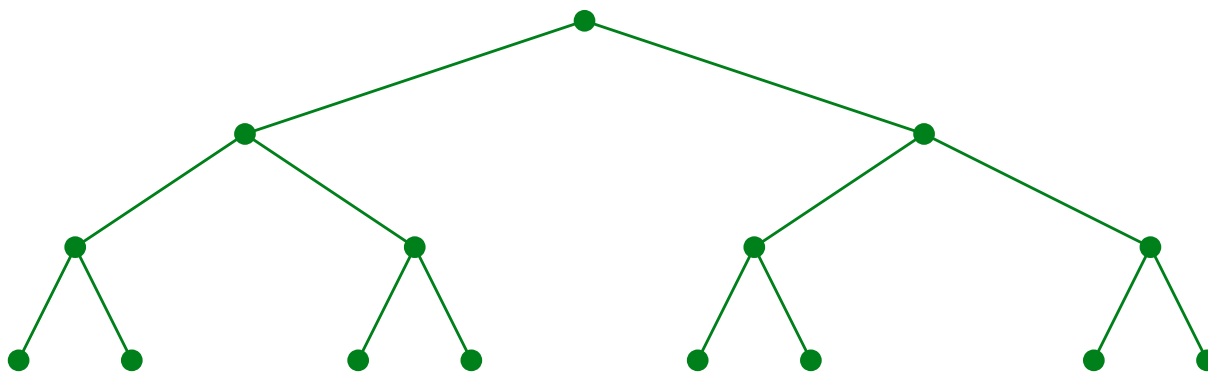
1. Trees
2. **Binary Trees**
 - Implementing Binary Trees
3. Binary Search Trees
 - Definition
 - Implementing a Set
4. Tree Iterators



Binary Trees

- A **binary tree** is a tree where each node can have zero, one or two children
- The total number of possible nodes at level l is 2^l
- The total number of possible nodes of a tree of height h is

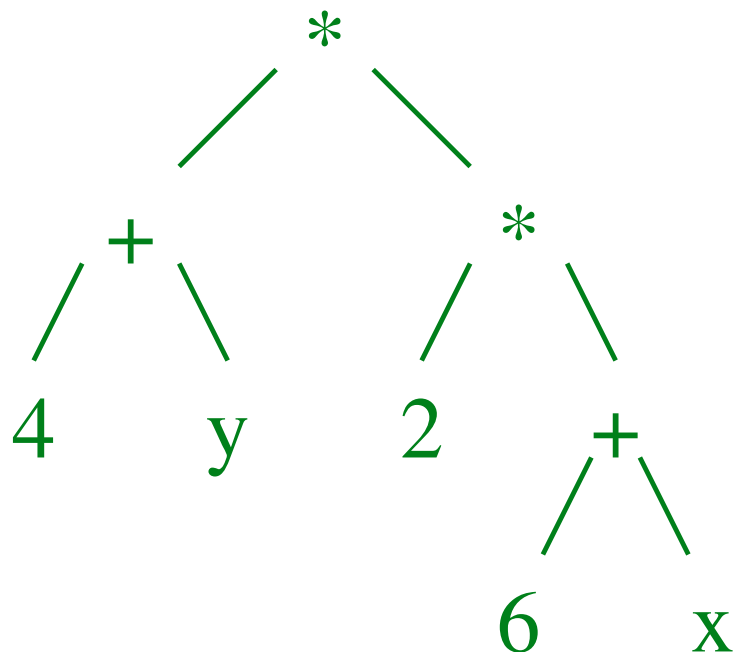
$$1 + 2 + \dots + 2^{h-1} = 2^h - 1$$



Level	# Nodes
0	1
1	2
2	4
3	8
	<hr/>
	15
	<hr/>

Uses of Binary Trees

- Binary trees have a huge number of applications
- For example, they are used as **expression trees** to represent formulae



$(4+y) * (2 * (6+x))$

Implementation

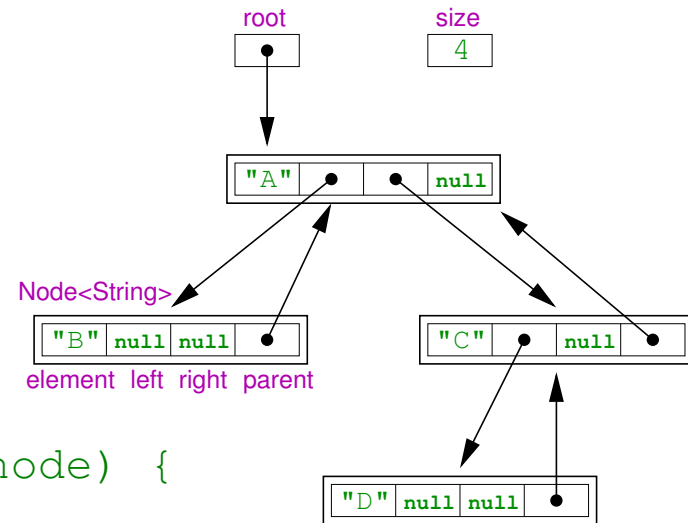
- We wish to build a generic binary tree class with each node housing an element■
- Again we use a `Node<T>` class as the building block for our data structure—in this case a node of the tree■
- The `Node<T>` class will contain a pointer to left and right children■
- To help navigate the tree each node will contain a pointer to its parent■

C++ Code

```
template <typename T>
class binary_tree {
private:
    class Node {
    public:
        T element;
        Node* parent;
        Node* left = 0;
        Node* right = 0;

        Node(const T& value, Node* parent_node) {
            element = value;
            parent = parent_node;
        }
    };
};
```

```
unsigned no_elements = 0;
Node* root = 0;
```



Outline

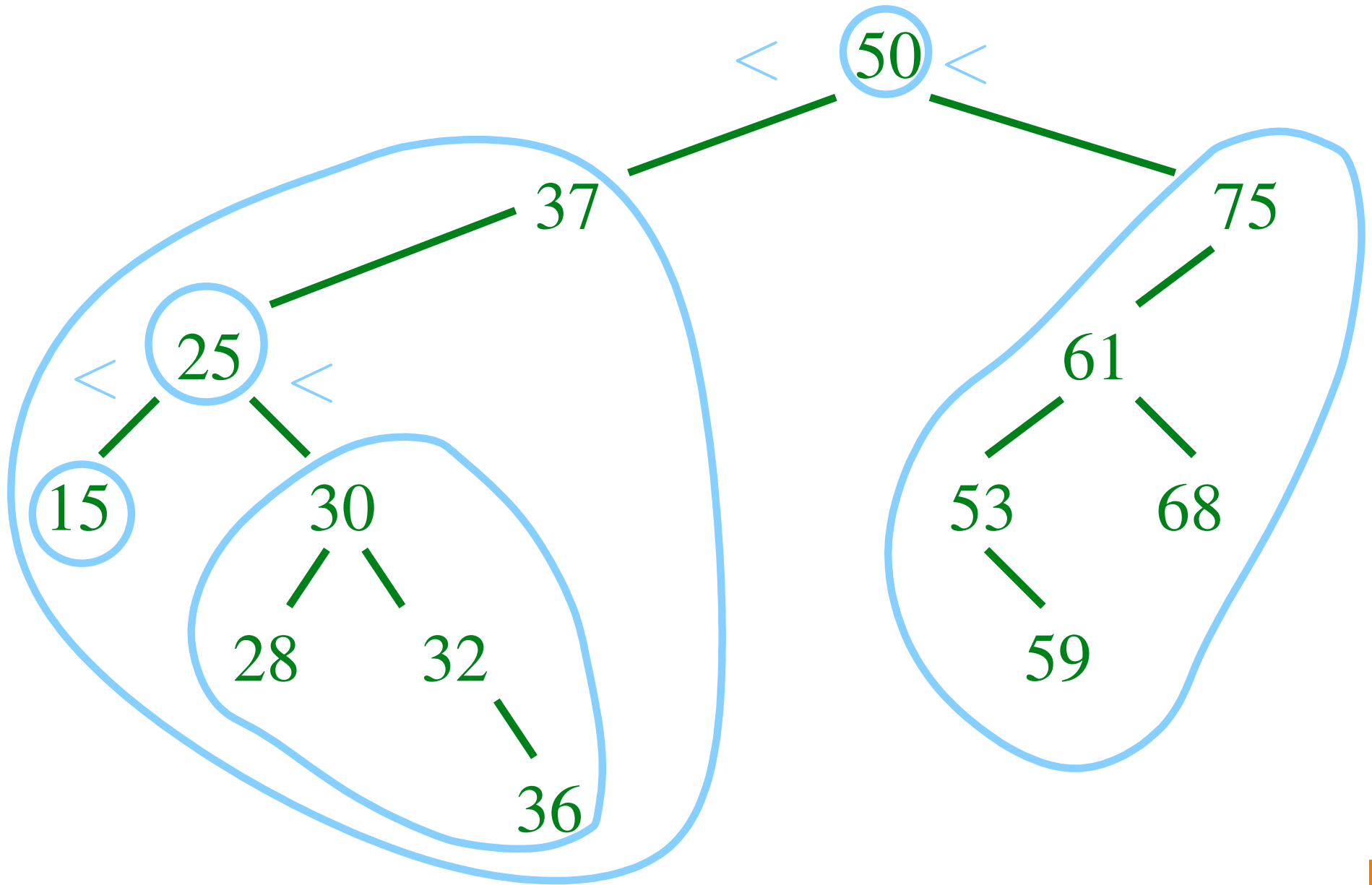
1. Trees
2. Binary Trees
 - Implementing Binary Trees
3. **Binary Search Trees**
 - Definition
 - Implementing a Set
4. Tree Iterators



Binary Search Trees

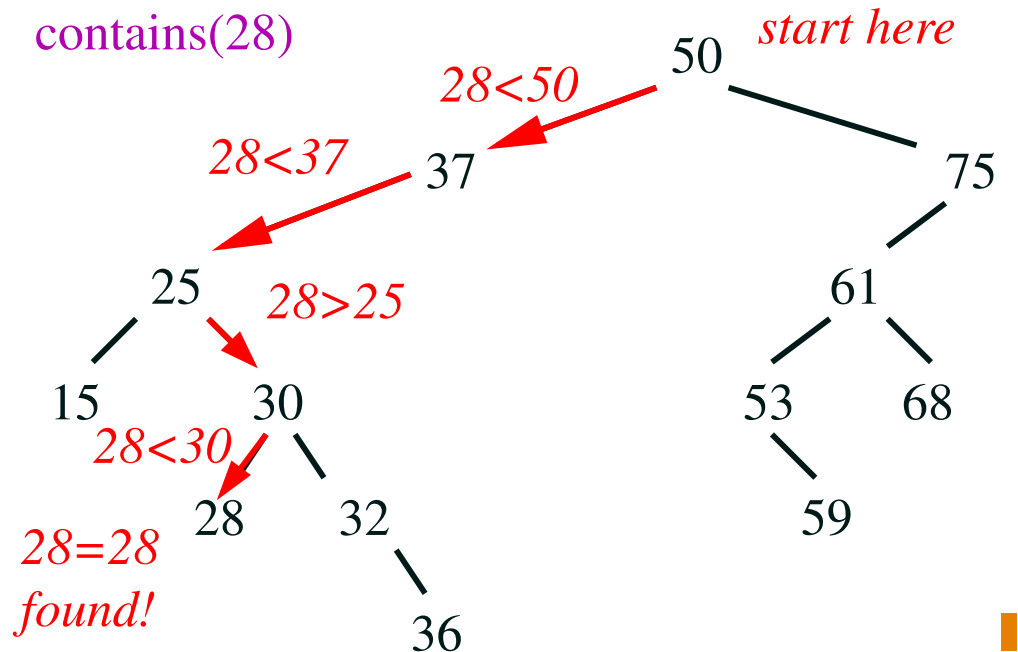
- We will concentrate on one of the most important binary trees, namely the **binary search tree**■
- The binary search tree keeps the elements ordered■
- We can define a binary search tree recursively■
 1. Each element in the left subtree is less than the root element■
 2. Each element in the right subtree is greater than the root element■
 3. Both left and right subtrees are binary search trees■

Example Binary Search Tree



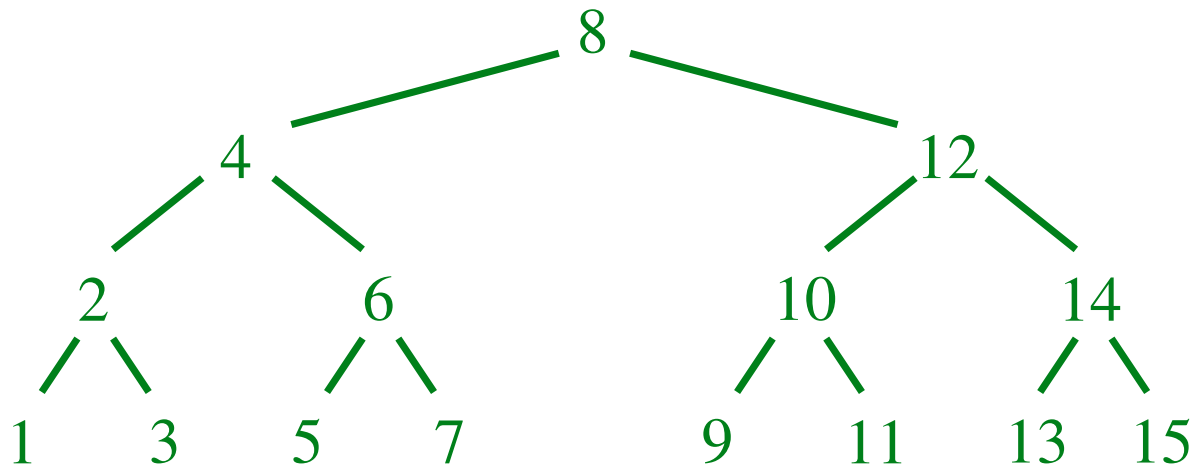
Searching A Binary Search Tree

- Searching a binary search tree is easy ■
- Start at the root ■
- Compare with element ■
 - ★ If less than element go left ■
 - ★ If greater than element go right ■
 - ★ If equal to element found ■

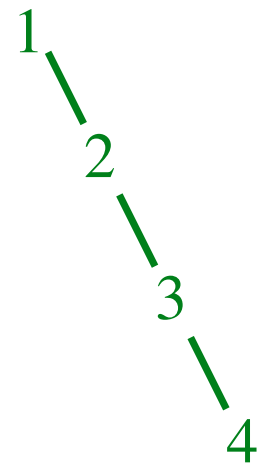


Speed of Search

- The number of comparisons necessary to find an element in a binary tree depends on the level of the node in the tree■
- The worst case number of comparisons is therefore the height of the tree■
- This depends on the density of the tree■



full tree



sparse tree

Implementing a Set

- A set is a fundamental **abstract data type**■
- It is a collection of things with no repetition and no order■
- Ironically because order doesn't matter we can order the elements

$$\{1, 3, 5, 5, 3, 4\} = \{5, 3, 4, 1\} = \{1, 3, 4, 5\}■$$

- This allows rapid search—a feature we care about■
- Binary trees are one of the efficient ways of implementing a set■

Fitting In

- The standard template library provides a class `std::set<T>`
- This contains many functions like
 - ★ Constructors
 - ★ `size()`
 - ★ `insert(T o)`
 - ★ `find(Object o)`
 - ★ `erase(Object o)`
 - ★ `begin()` and `end()`

Comparable

- To sort any objects they must be comparable■
- In the STL the set implementation has a second template parameter: `std::set<T, Compare = less<T> >`■
- by default this is defined to be `less<T>` (which is a function already defined for most common types) which you can define■
- If you have a set of complex objects you will have to define
Compare

```
bool MyCompare(MyObject left, MyObject right) {  
    return something  
}
```

```
mySet = set<MyObject, MyCompare>;
```



Find an Element

- One of the core operations of a binary tree is to find a node

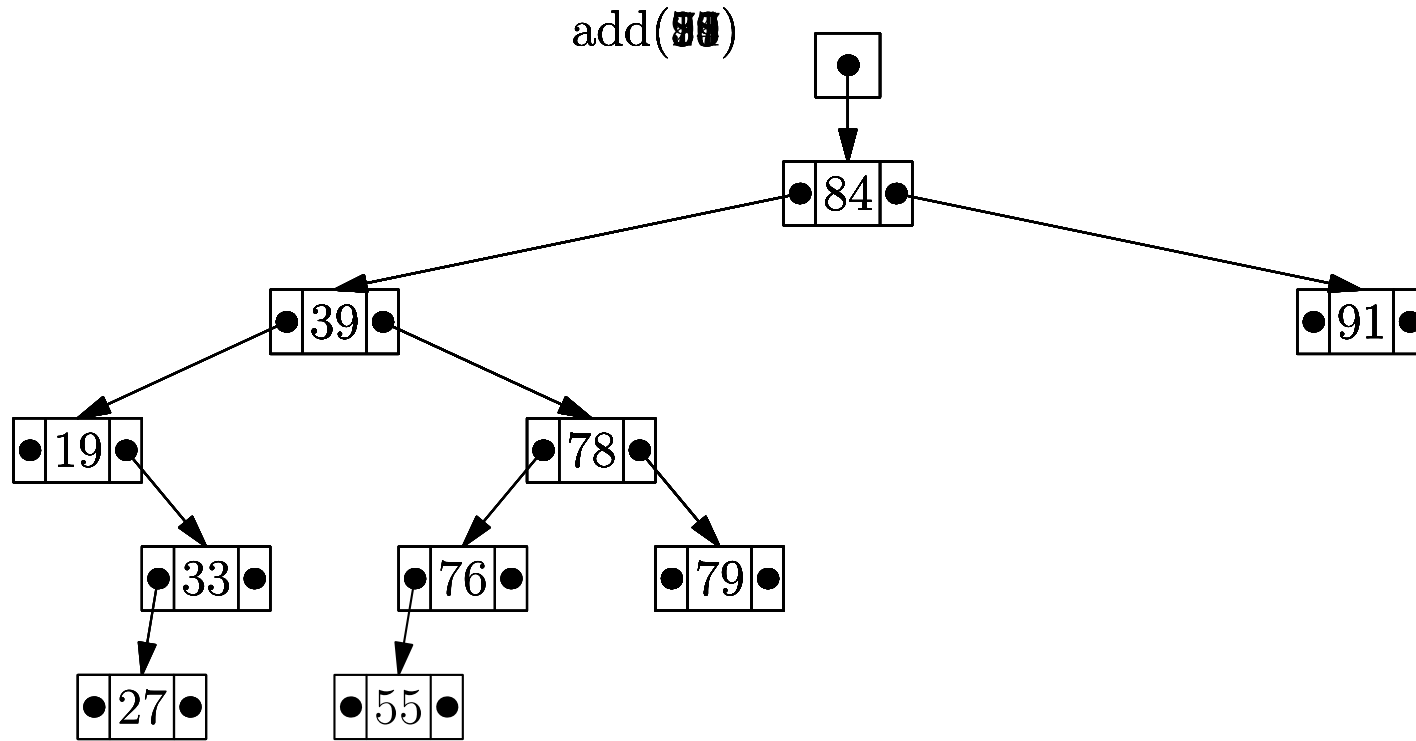
```
iterator find(const T& element) {  
    Node* current = root;  
    while (current!=0) {  
        if (current->element == element) {  
            return iterator(current);  
        }  
        if (element < current->element) {  
            current = current->left;  
        } else {  
            current = current->right;  
        }  
    }  
    return iterator(0);  
}
```

Add an Element

```
pair<iterator, bool> insert(const T& element) {  
    if (no_elements==0) {  
        root = new Node(element, 0);  
        ++no_elements;  
        return pair<iterator, bool>(iterator(root), true);  
    }  
    Node* parent = 0;  
    Node* current = root;  
    while(current != 0) {  
        if (current->element == element) {  
            return pair<iterator, bool>(iterator(0), false);  
        }  
        parent = current;  
        if (element < current->element) {  
            current = current->left;  
        } else {  
            current = current->right;  
        }  
    }  
}
```

```
current = new Node(element, parent);  
if (element < parent->element) {  
    parent->left = current;  
} else {  
    parent->right = current;  
}  
++no_elements;  
return pair<iterator, bool>(iterator(current), true);  
}
```


Tree in Action



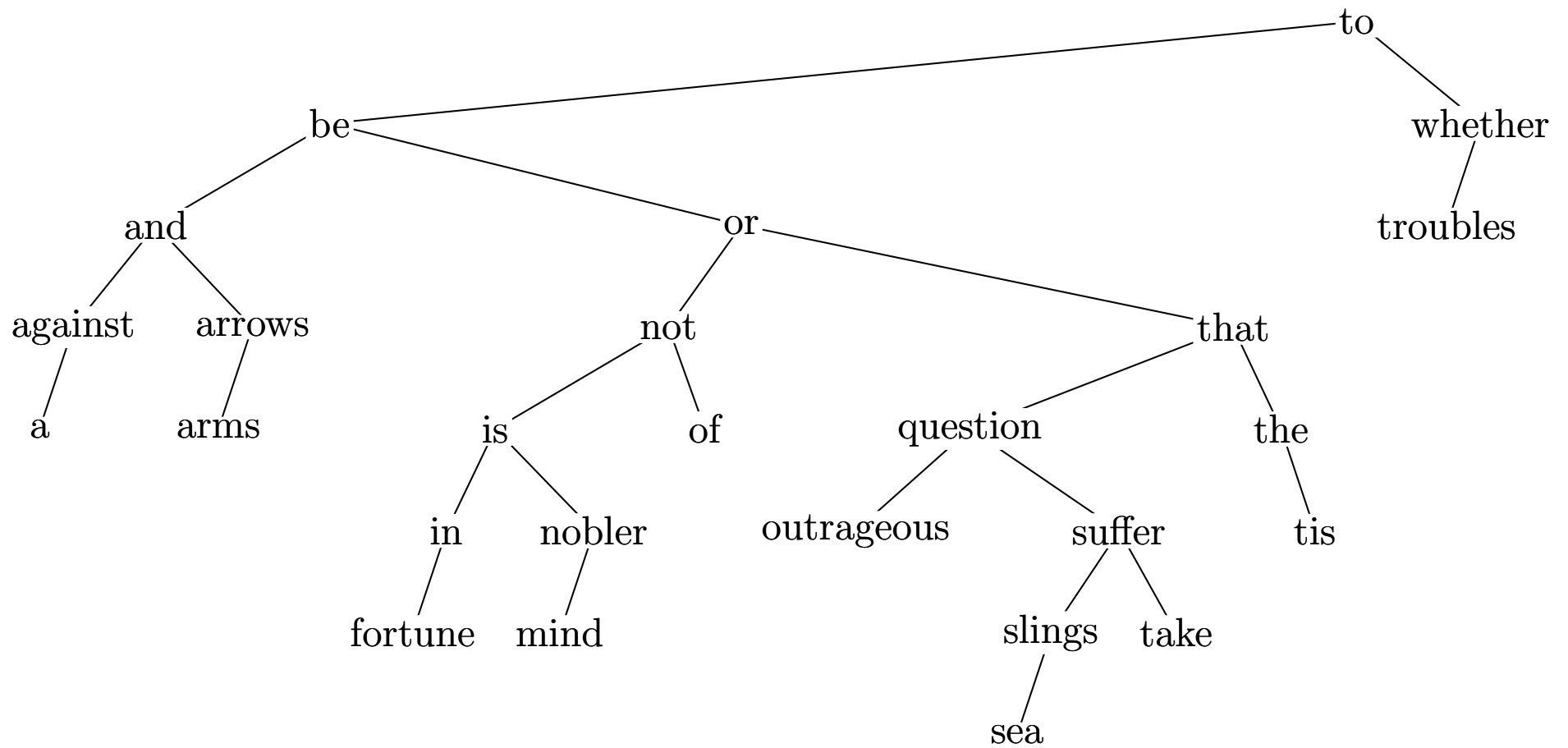
Shape of Tree

- The structure of the tree depends on the order in which we add elements to it■
- Suppose we add

*To be, or not to be: that is the question:
Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous fortune,
Or to take arms against a sea of troubles,■*

- Ignoring punctuation we get the following tree■

Hamlet



Outline

1. Trees
2. Binary Trees
 - Implementing Binary Trees
3. Binary Search Trees
 - Definition
 - Implementing a Set
4. **Tree Iterators**



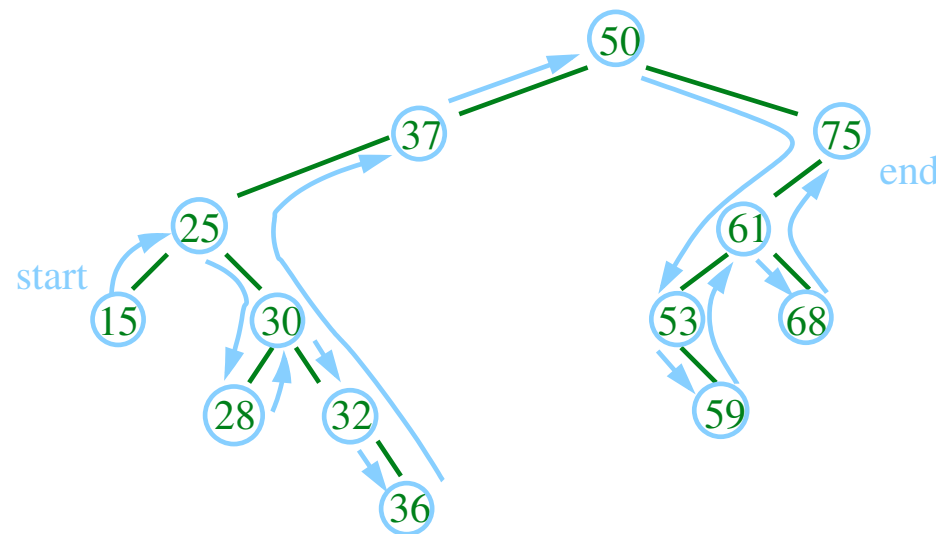
Tree Iterators

- As with most container classes it is very useful to define iterators
- `begin()` should return a “pointer” to the start of the tree
- `end()` provides a “pointer” past the end
- `operator*()` returns the element
- `operator++()` increments the “pointer”
- `operator!=(lhs, rhs)` is used to compare iterators

```
set<int> mySet;  
...  
for(auto pt=mySet.begin(), pt!=mySet.end(), ++pt) {  
    cout << *pt;  
}
```

Successor

- To find the successor we first start in the left most branch ■
- We follow two rules■
 1. **If** right child exist **then** move right once and then move as far left as possible ■
 2. **else** go *up* to the left as far as possible and then move up right ■



{15■25■28■30■32■36■37■50■53■59■61■68■75}■

Lessons

- Trees and particularly binary trees are one of the most important tools of a computer scientist■
- Conceptually they are quite simple■
- However, there are a lot of details that need to be understood■
- Coding even simple trees needs great care■
- As we will see things get more complicated■