Algorithms and Analysis

Lesson 13: *Analyse!*

Pseudo code, binary search, insertion sort, selection sort, lower bound complexity

Outline

1. **Algorithm Analysis**
2. Search
3. Simple Sort
   - Insertion Sort
   - Selection Sort
4. Lower Bound

Algorithm Analysis

- We've covered most of the basic data structures
- The rest of the course is going to focus more on algorithms
- We will look predominantly at
  - Searching
  - Sorting
  - Graph Algorithms
- Emphasise general solution strategies

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Code and Pseudo Code

- C++ code is often difficult to read—there are often programming details we don't care abour
- It contains details such as throwing exception which are repetitive and often depends on who you are writing the code for
- Algorithms are not language dependent (data structures are a bit more language dependent)
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Pseudo Code

- There is no standard for pseudo code
- The commands are not too dissimilar to C++
- The one strange convention is that assignments use an arrow $\leftarrow$
- Arrays are written in bold $\boldsymbol{a}$ with elements a_i
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Dumb Search

```
DUMBSEARCH(a, x)
{
  /* search array a = (a_1, ... a_n)  */
  /* for x return true */
  /* if  successful  else  false */
  for i←1 to n
    if (a_i = x)
      return true
    endif
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bool search(T a[], T x)
{
  for (int i=0; i<n; i++) {
    if (a[i] == x)
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56	26	62	60	53	53	77	91	60	41

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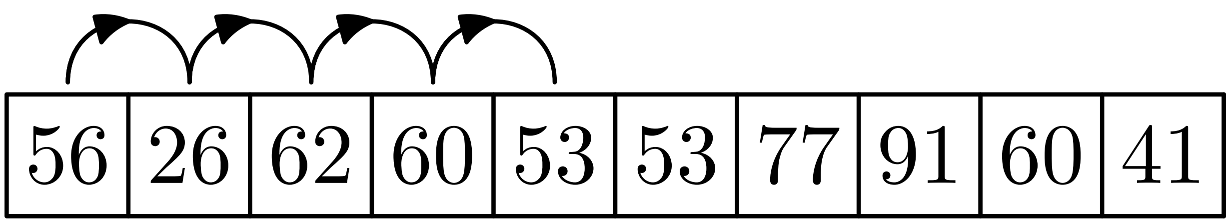
find(53) ⟶ true



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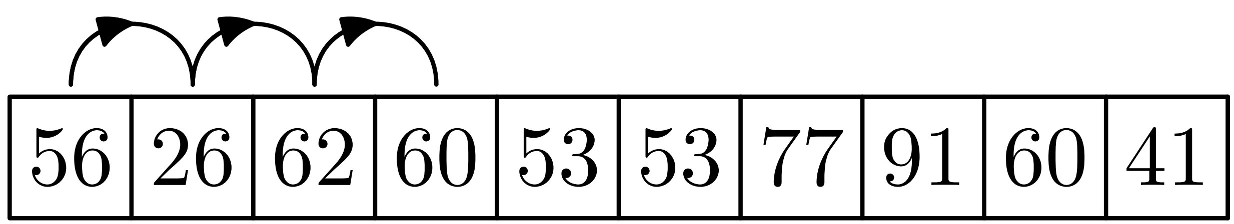
$$\text{find}(60) \longrightarrow \text{true}$$



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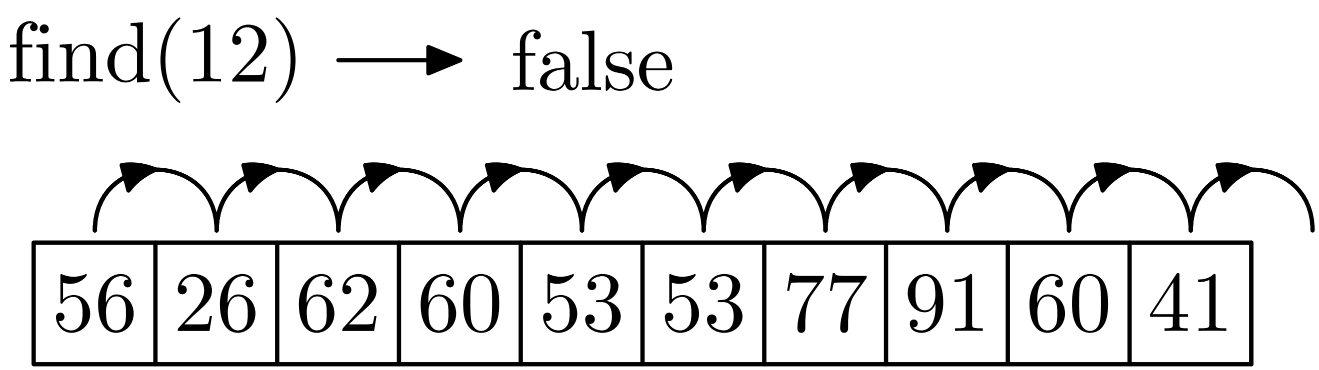

Time Complexity

- Worst case:
  - ⋆ The worst case for a successful search is when the element is in the last location in the array
  - ⋆ This takes n comparisons: worst case is $\Theta(n)$
- Best case:
  - ⋆ The best case is when the element is in the first location
  - ⋆ This takes 1 comparison: best case is $\Theta(1)$
- Average case:
  - ⋆ Assume every location is equally likely to hold the key

$$\frac{1+2+\ldots+n}{n} = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2}$$

- For an unsuccessful search n comparison are necessary

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Binary Search

- If the array is ordered we can do better
- At each step we bisect the array

```
BinarySearch(a, x)
{
  low ←1
  high ←n
  while (low ≤ high)
    mid ←⌊(low + high)/2⌋
    if x > a_mid
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```

⋆ Based on a **divide-and-conquer** strategy

⋆ We check the middle of the array

$$\underbrace{a_1, a_2, \cdots, a_{m-1},}_{x<a_m} \overbrace{a_m}^{x=a_m}, \underbrace{a_{m+1}, \cdots a_n}_{x>a_m}$$

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Binary Search in Action

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95

Binary Search in Action

$\textsc{BinarySearch}(\mathbf{a}, 27)$

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
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Binary Search in Action

$\textsc{BinarySearch}(\mathbf{a}, 27)$

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BinarySearch($\mathbf{a}$, 27)

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Binary Search in Action

BinarySearch($\mathbf{a}$, 20)

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
low									mid										high

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14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
low								high											

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14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
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Binary Search in Action

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14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
		low	high																

Binary Search in Action

BinarySearch($\mathbf{a}$, 20)

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
		low mid	high																

Binary Search in Action

BinarySearch($\mathbf{a}$, 20)

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
	high	low																	

Binary Search in Action

$\textsc{BinarySearch}(\mathbf{a}, 20)$ not found

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
	high	low																	

Binary Search in Action

BinarySearch($\mathbf{a}$, 84)

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95

low high

Binary Search in Action

$\textsc{BinarySearch}(\mathbf{a}, 84)$

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
low									mid										high

Binary Search in Action

BinarySearch($\mathbf{a}$, 84)

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
										low									high

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BinarySearch($\mathbf{a}$, 84)

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95
										low				mid					high

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Binary Search in Action

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low high

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Binary Search in Action

BinarySearch($\mathbf{a}$, 84) found

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																low mid high			

Binary Search in Action

BinarySearch($\mathbf{a}$, 99)

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Binary Search in Action

BinarySearch($\mathbf{a}$, 99)

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Binary Search in Action

BinarySearch($\mathbf{a}$, 99) not found

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high low

Analysis

- We count the number of comparisons (counting each `if/else if` statement as a single comparison)

- Let $C(n)$ be the number of comparisons needed to search in an array of size n

- After one comparison we are left (in the worst case) with having to search an array not larger than $\lfloor n/2 \rfloor$, thus

$$C(n) < C(\lfloor n/2 \rfloor) + 1$$

- We've seen this relation before (lesson on Recursion)

- Easy to show $C(n) < \lfloor \log_2(n) \rfloor + 1 = O(\log(n))$

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Outline

1. Algorithm Analysis
2. Search
3. **Simple Sort**
   - Insertion Sort
   - Selection Sort
4. Lower Bound

Sort Characteristics

- Sort is one of the best studied algorithms. We care about stability, space and time complexity
- A sort algorithm is said to be **stable** if it does not change the order of elements that have the same value
- Space Complexity. Sort is said to be
  - ⋆ **In-place** if the memory used is $O(1)$
- Time Complexity. In particular we are interested in
  - ⋆ Worst case
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Insertion Sort

- In insertion sort we keep a subsequence of elements on the left in sorted order
- This subsequence is increased by *inserting* the next element into its correct position

```
InsertionSort(a)
{
 for i←2 to n
   v←a_i
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66	37	23	69	74	90	39	84	69	50

sorted unsorted

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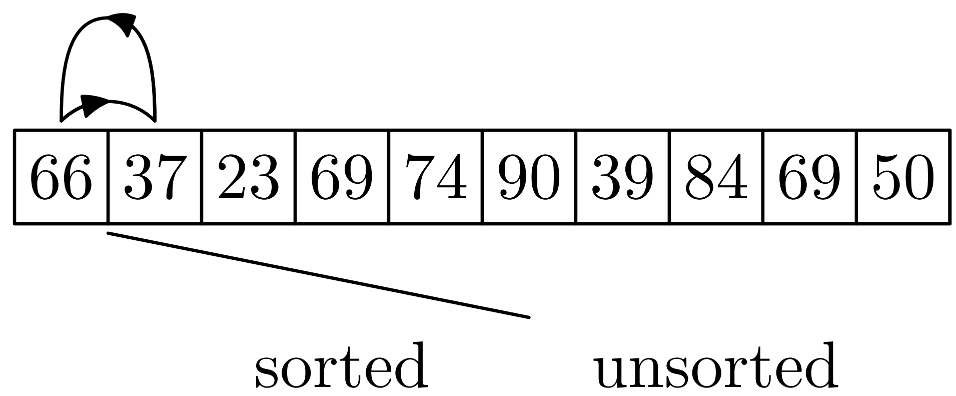

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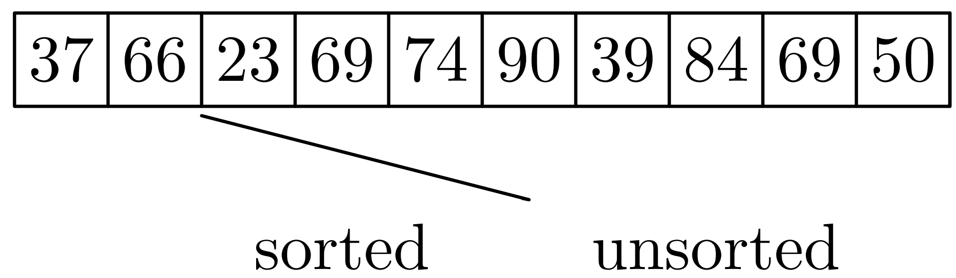

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  for i $\leftarrow$ 2 **to** n
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    while j $\geq$ 1 **and** $a_j >$ v
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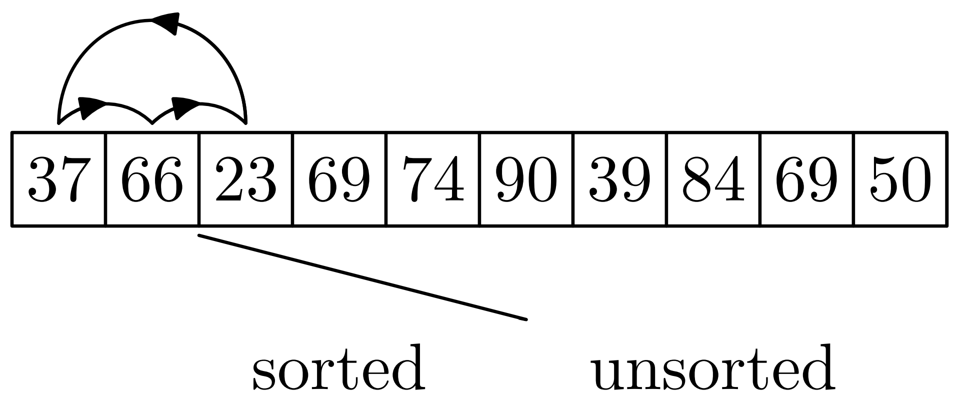

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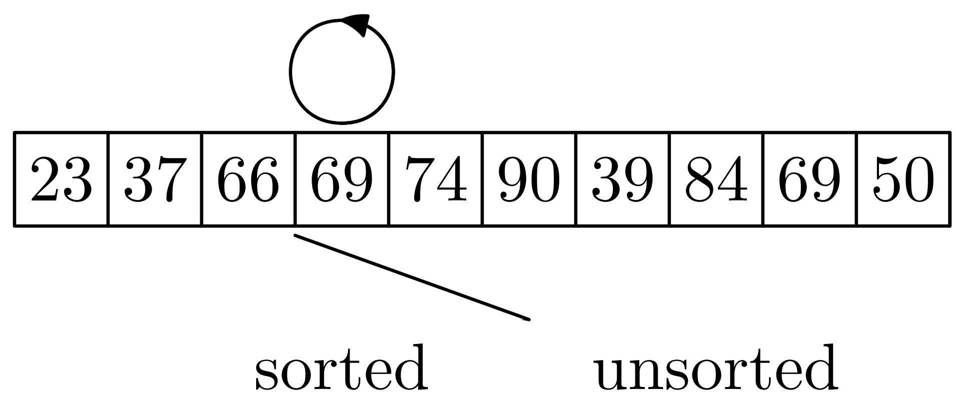

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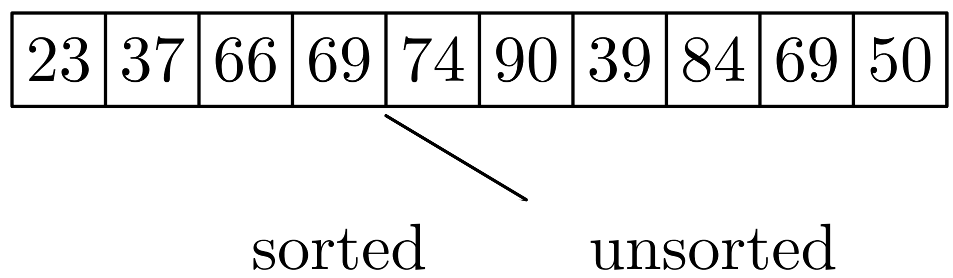

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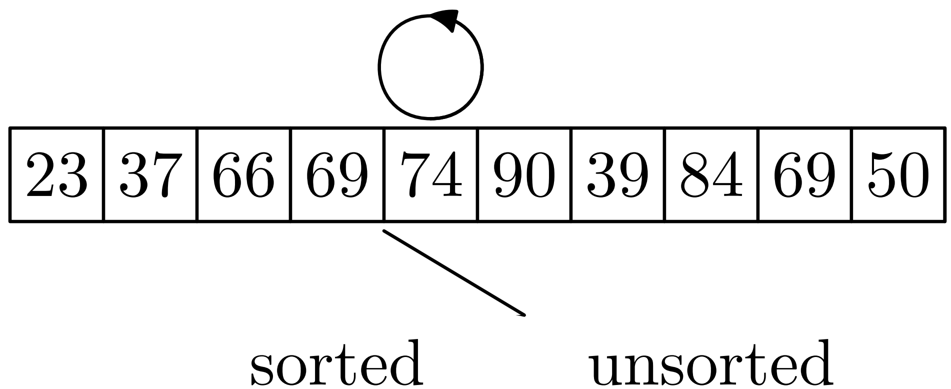

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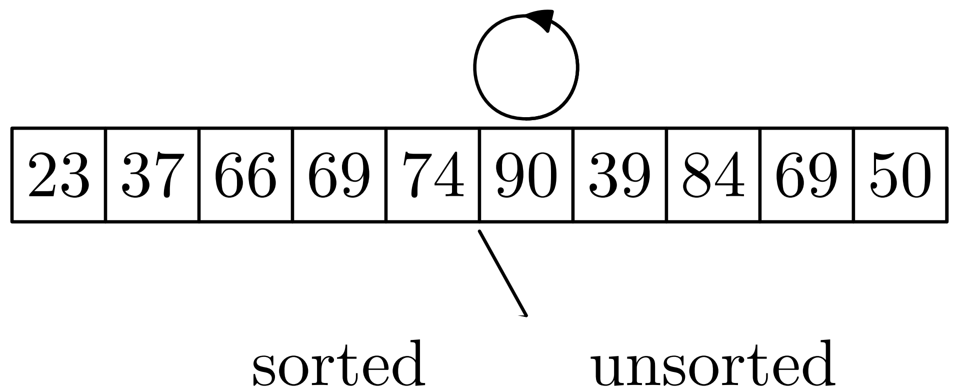

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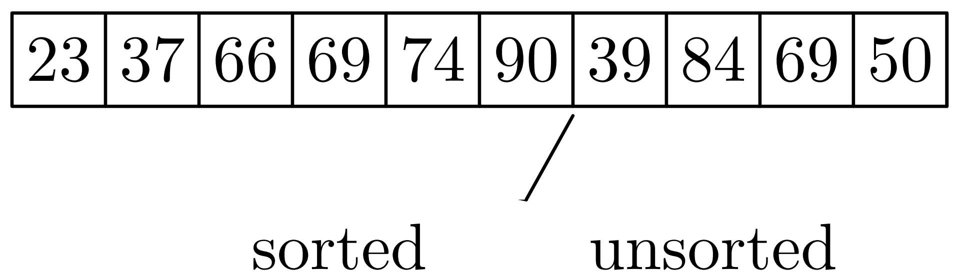

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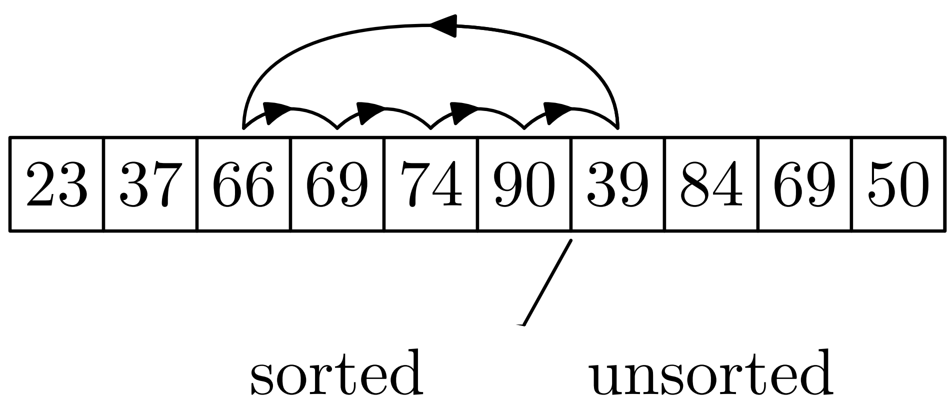

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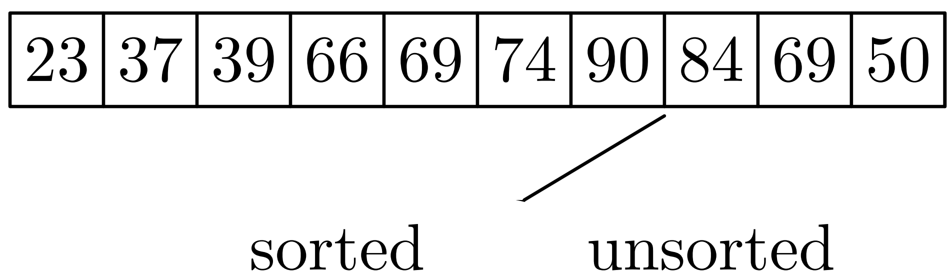

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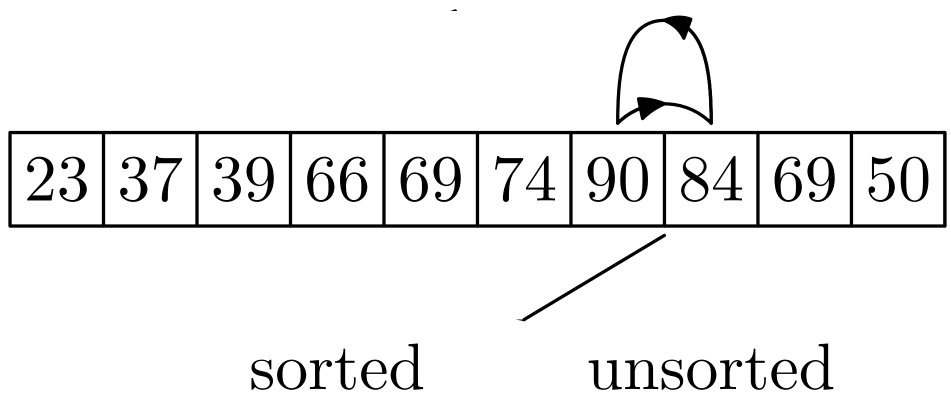

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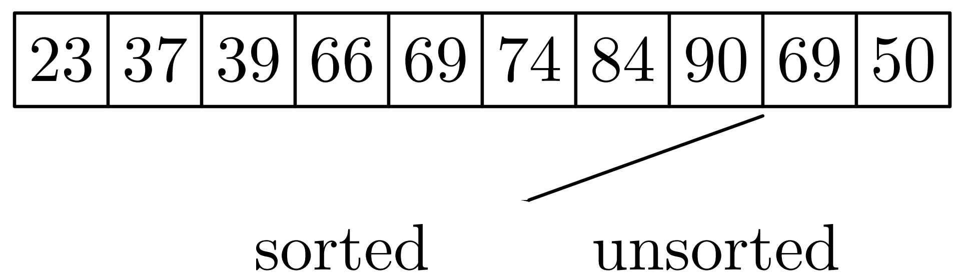

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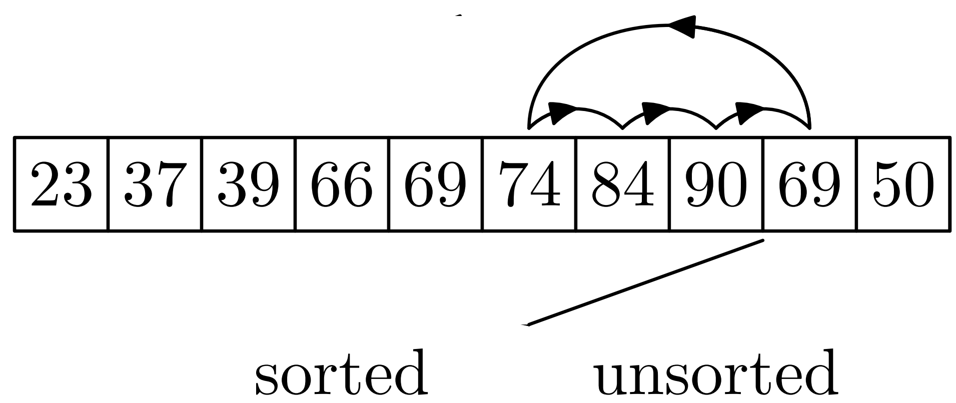

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    endwhile
    a_{j+1}←v
  endfor
}
```

23	37	39	66	69	69	74	84	90	50

sorted unsorted

Insertion Sort

- In insertion sort we keep a subsequence of elements on the left in sorted order
- This subsequence is increased by *inserting* the next element into its correct position

```
INSERTIONSORT(a)
{
  for i←2 to n
    v←a_i
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    while j ≥ 1 and a_j>v
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    endwhile
    a_{j+1}←v
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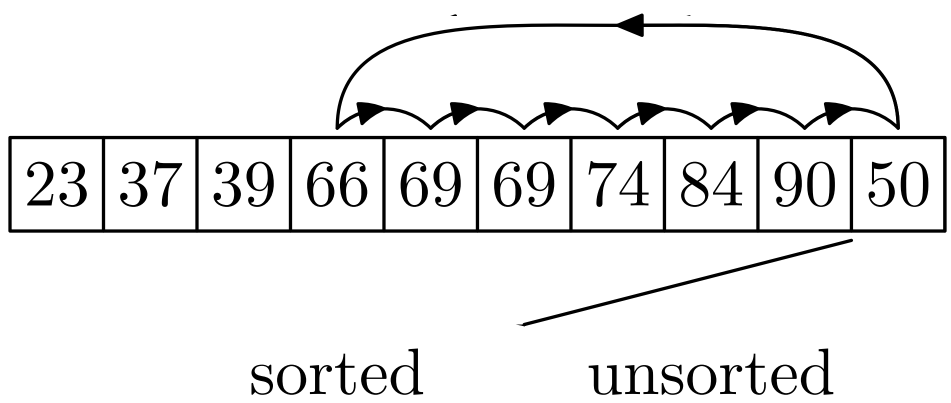

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sorted unsorted

Properties of Insertion Sort

- Insertion sort is **stable**. We only swap the ordering of two elements if one is strictly less than the other

- It is **in-place**

- Worst time complexity
  - ⋆ Occurs when the array is in inverse order
  - ⋆ Every element has to be moved to front of the array
  - ⋆ Number of comparisons for an array of size $C_w(n)$

$$C_w(n) = \sum_{i=2}^{n} (i-1) = 1 + 2 + \cdots n - 1 = \frac{n(n-1)}{2} \in \Theta(n^2)$$

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Time Complexity

- Average Time Complexity
  - ⋆ On average we can expect that each new element being sorted moves half the way down sorted list
  - ⋆ This gives us an average time complexity, $C_a(n)$ of half the worst time

$$C_a(n) = \frac{n(n-1)}{4} \in \Theta(n^2)$$

- Best Time Complexity
  - ⋆ This occurs if the array is already sorted
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Selection Sort

- A more direct **brute force** method is to find the least element iteratively

- We can make this an in-place method by swapping the least element with the first element, the second least element with the second element, etc.

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SelectionSort(a)
{
  for i←1 to n-1
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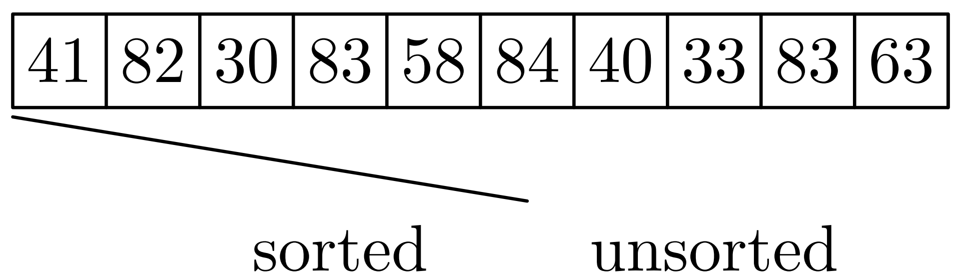

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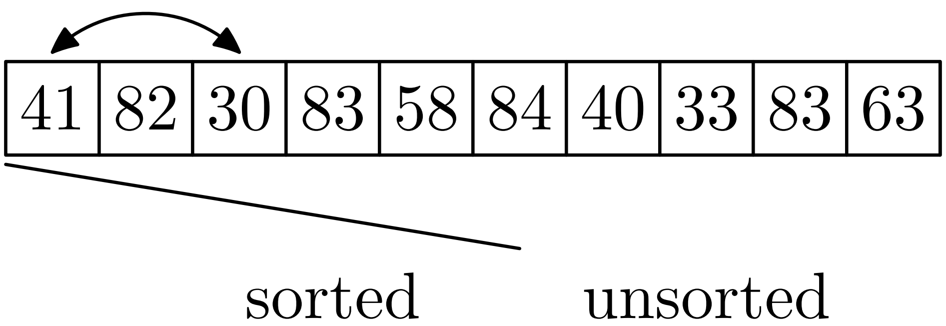

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30	82	41	83	58	84	40	33	83	63

sorted unsorted

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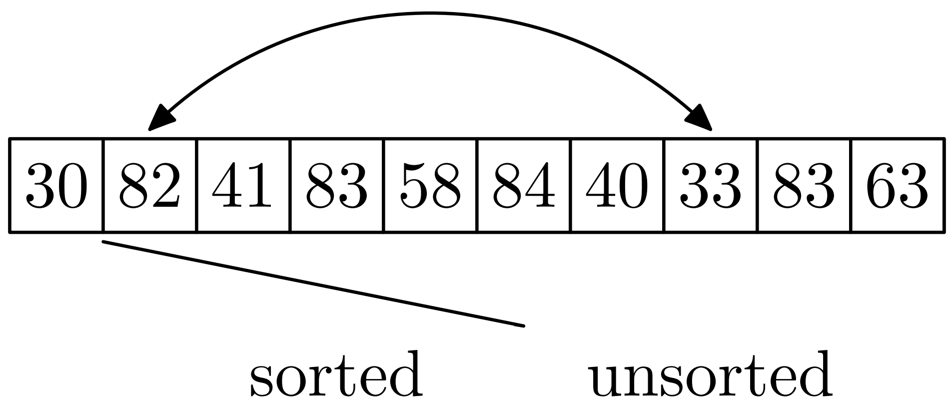

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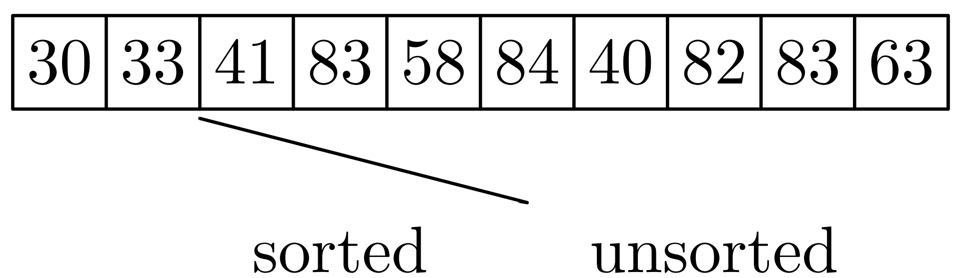

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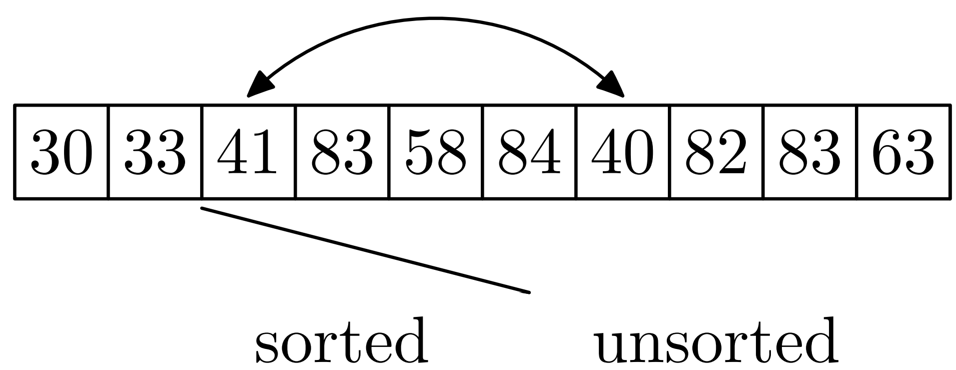

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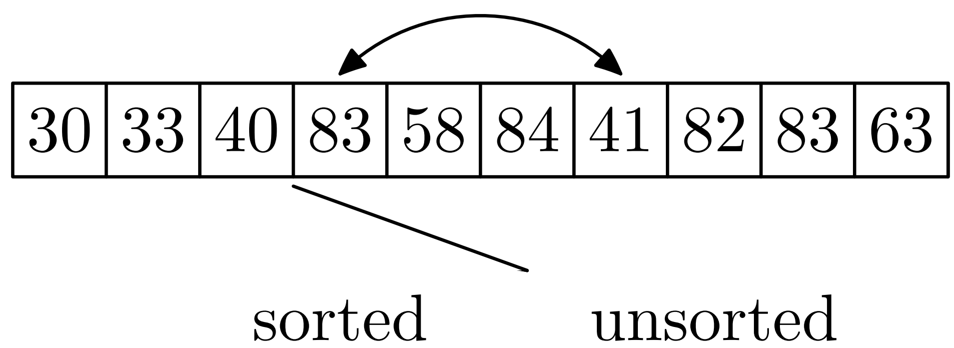

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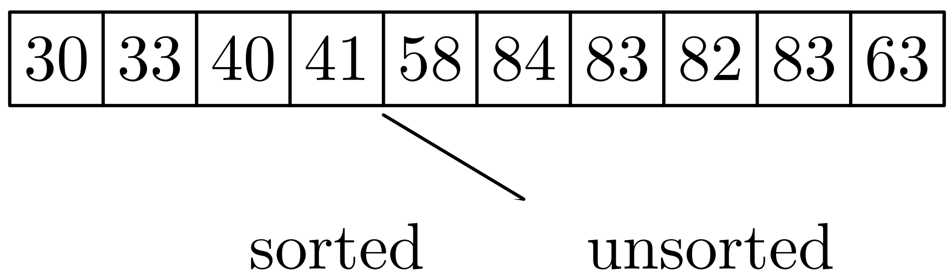

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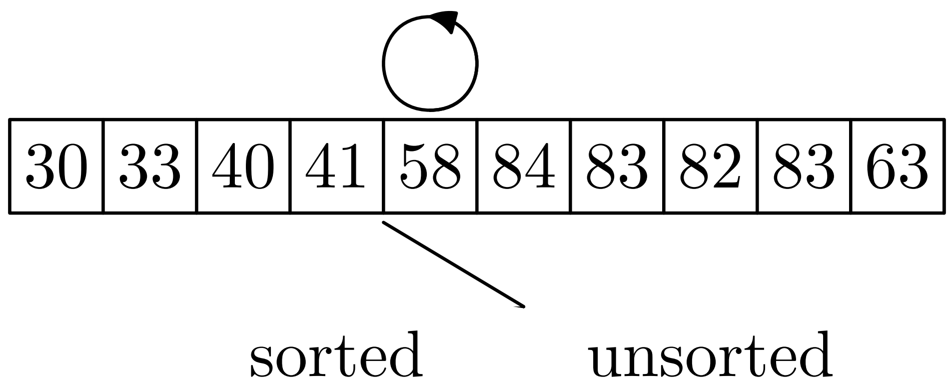

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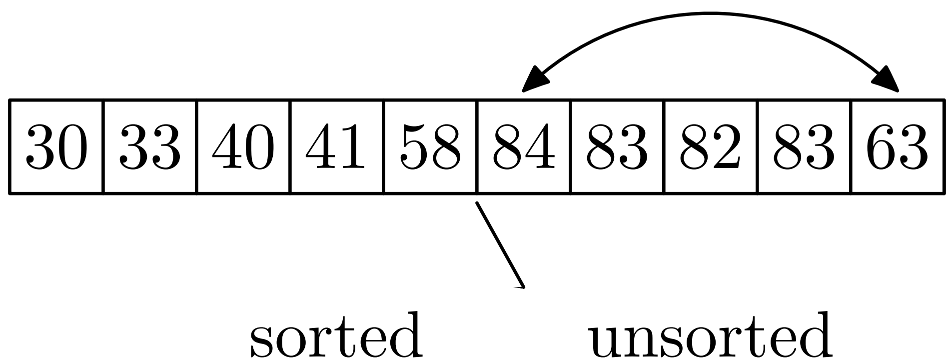

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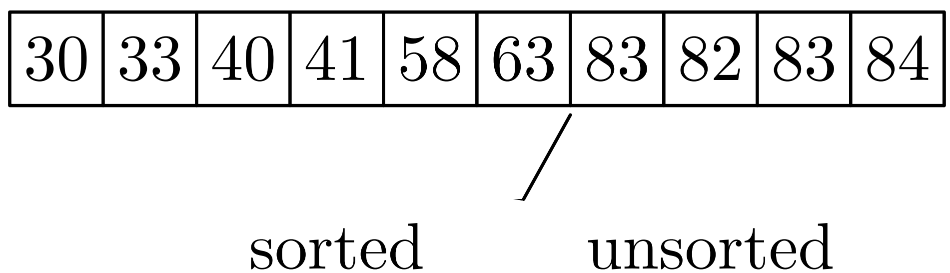

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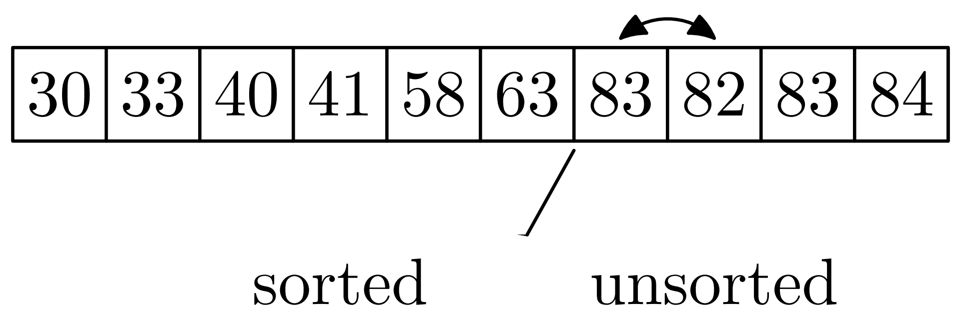

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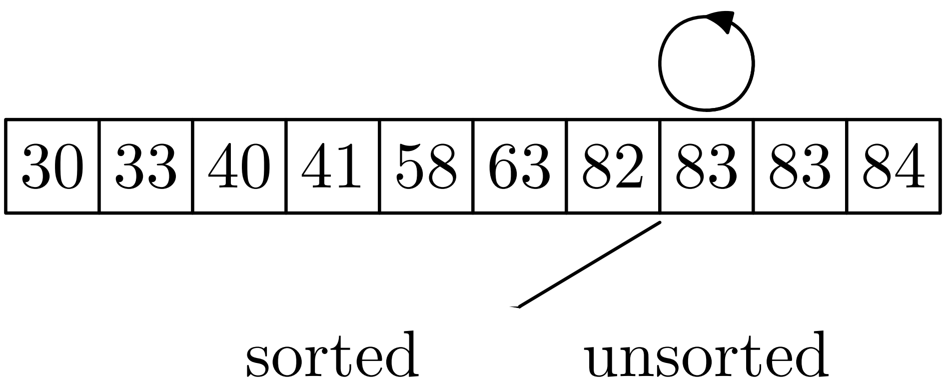

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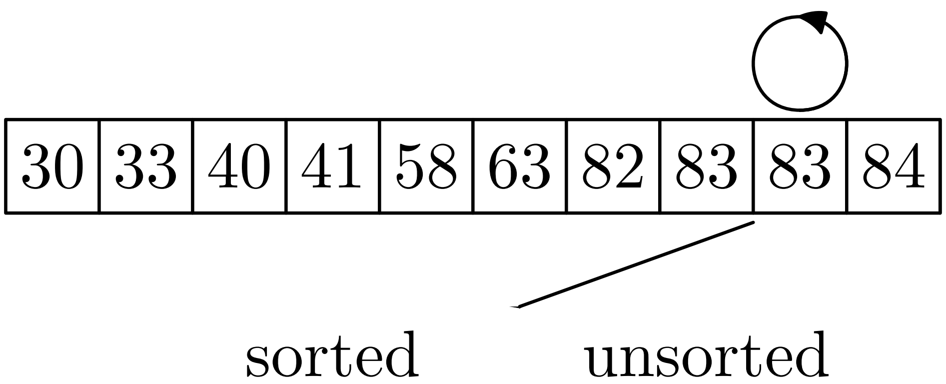

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      if a_j<a_min
        min←j
      end if
    end for
    swap a_i  and a_min
  end for
}
```

30	33	40	41	58	63	82	83	83	84

sorted unsorted

Analysis of Selection Sort

- Selection sort is in-place
- It isn't stable

$$\boxed{7_A \mid 7_B \mid 2} \longrightarrow \boxed{2 \mid 7_B \mid 7_A}$$

- Selection sort always requires $n(n-1)/2$ comparisons so has the same worst case, but worse average case and best case complexity as insertion sort
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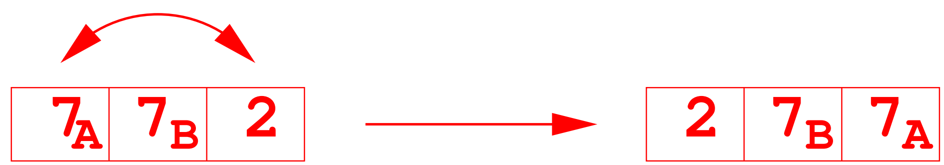

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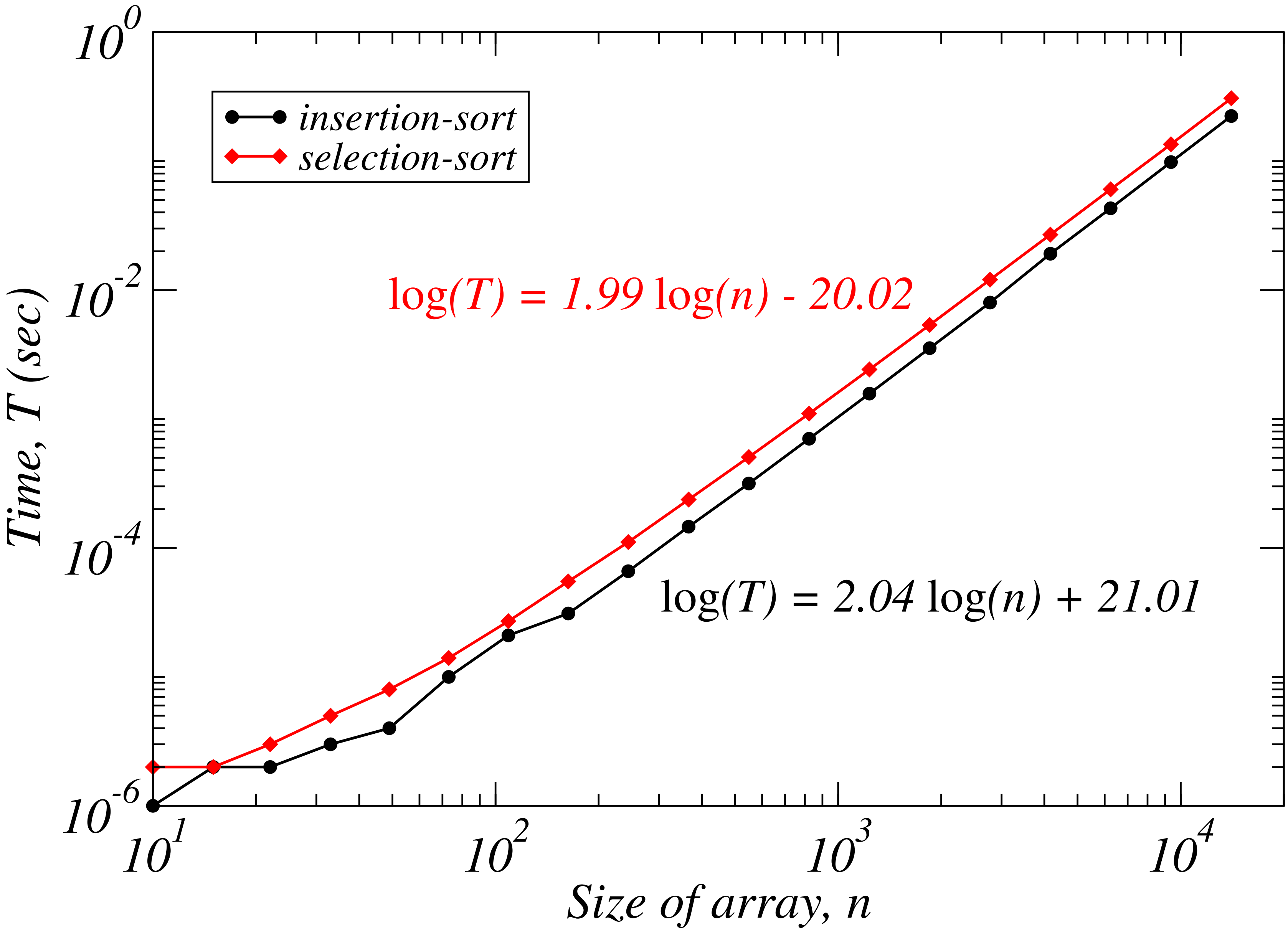
Insertion versus Selection Sort



Bubble Sort

- There are many other simple sort strategies
- One popular one is bubble sort—keep on swapping neighbours until the array is sorted
- It is stable and in-place
- This again has $O(n^2)$ complexity
- This isn't bad for a simple sort, but it does do more work than insertion sort and selection sort
- Apart from its name it just doesn't have anything going for it

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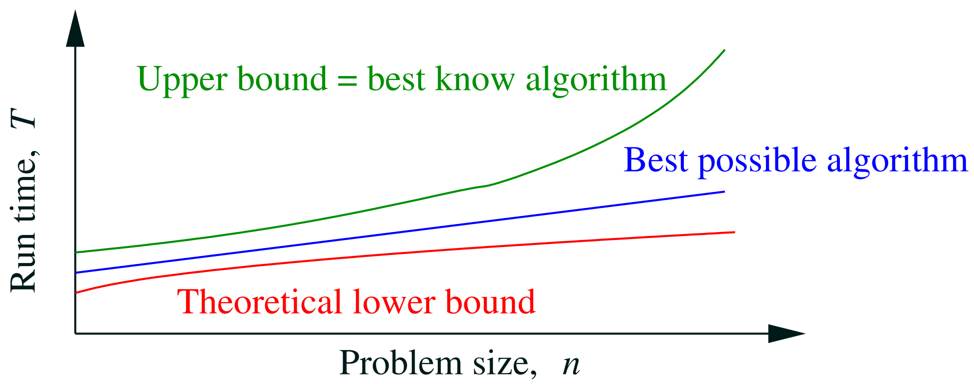
Outline

1. Algorithm Analysis
2. Search
3. Simple Sort
   - Insertion Sort
   - Selection Sort
4. **Lower Bound**

How Well Can You Do?

- Given a problem we would like to know what is the time complexity of the best possible program
- Usually there is no way of knowing this
- We can get an upper bound—if we know the time complexity of any algorithm that solves the problem we have an upper bound
- Lower bounds are far trickier
- A lower bound of $f(n)$ is a guarantee that we spend at least $f(n)$ operations to solve the problem



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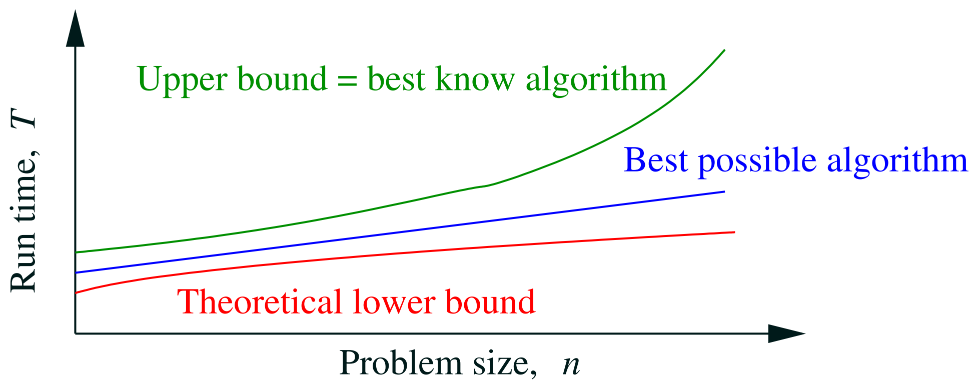

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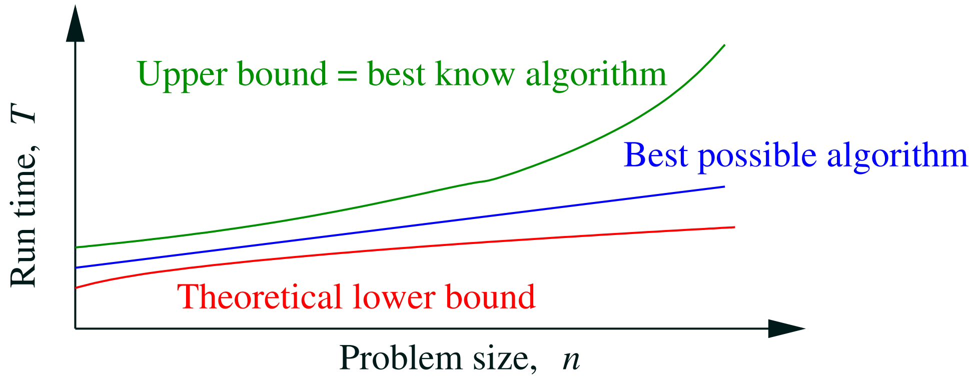

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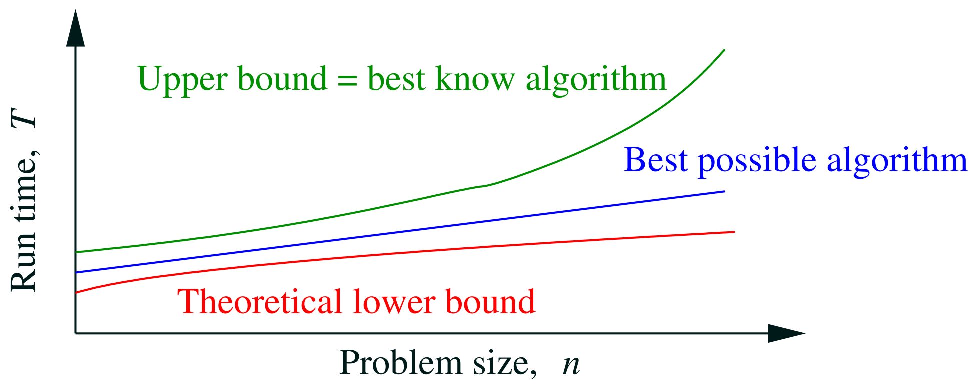

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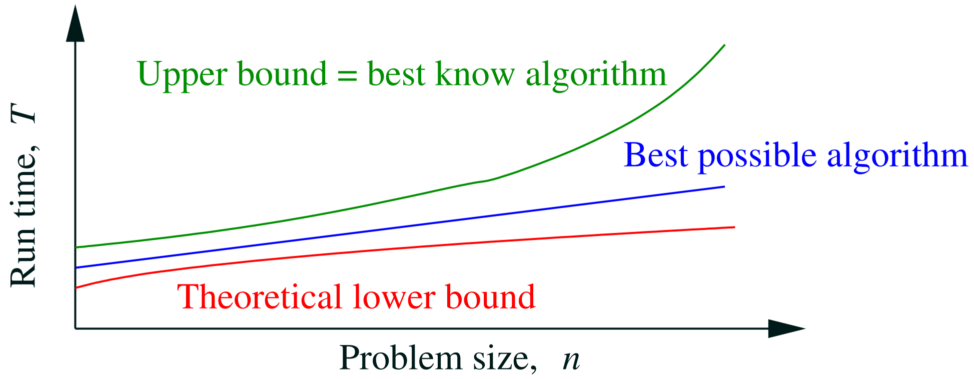

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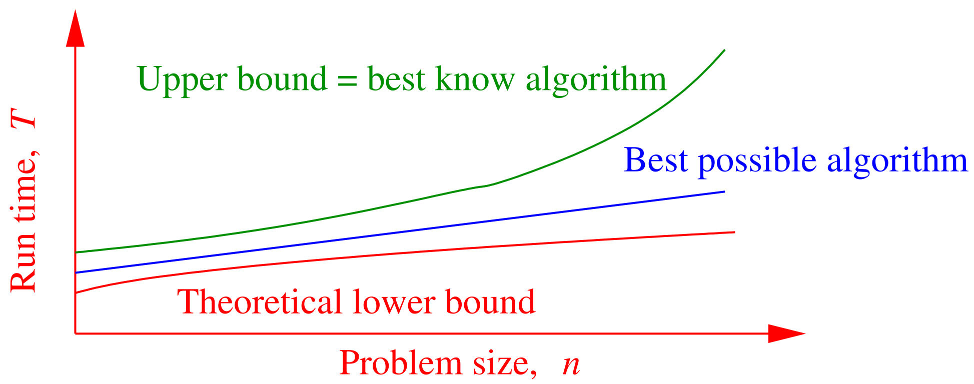

Decision Trees

- Decision trees are a way to visualise (at least, in principle) many algorithms
- They will eventually give us a lower bound on the time complexity of sort using binary decisions
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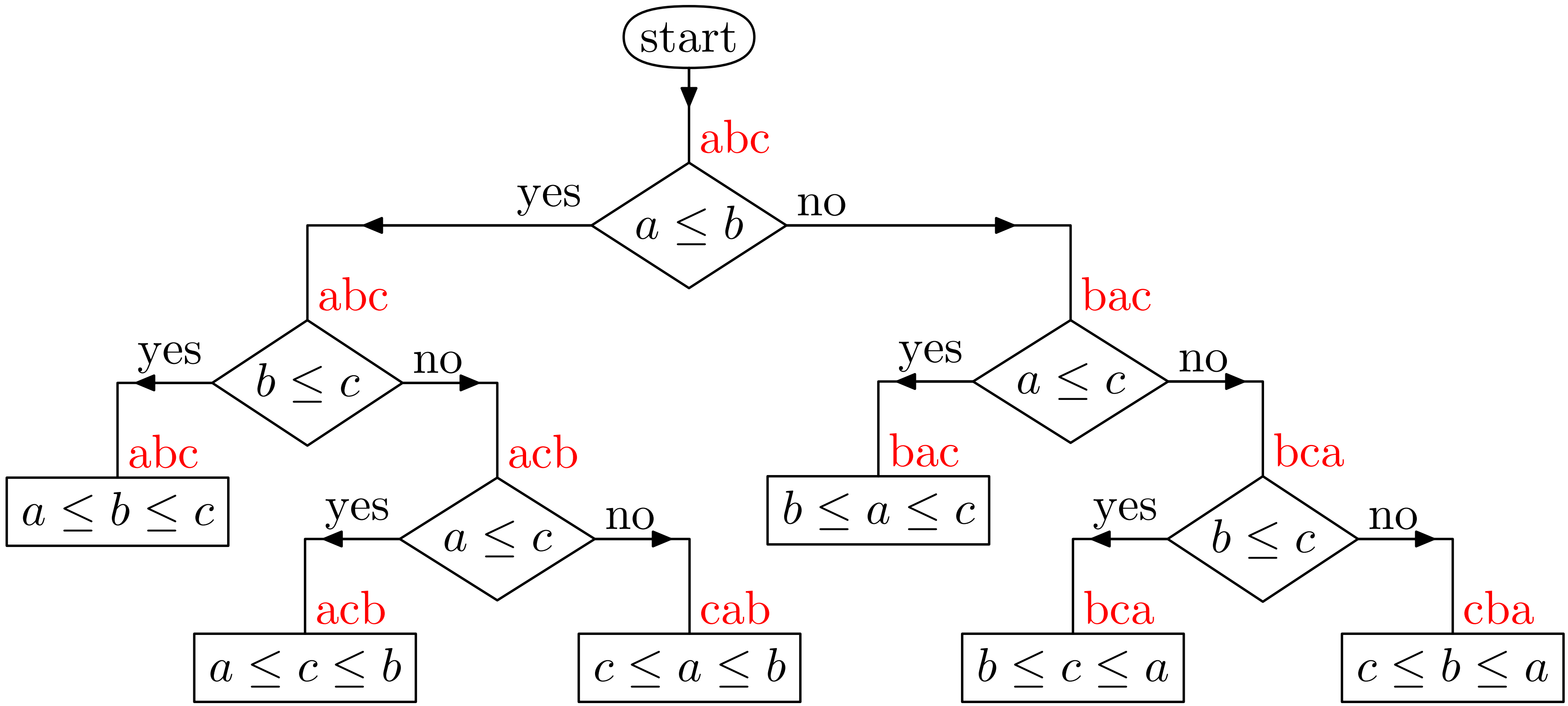

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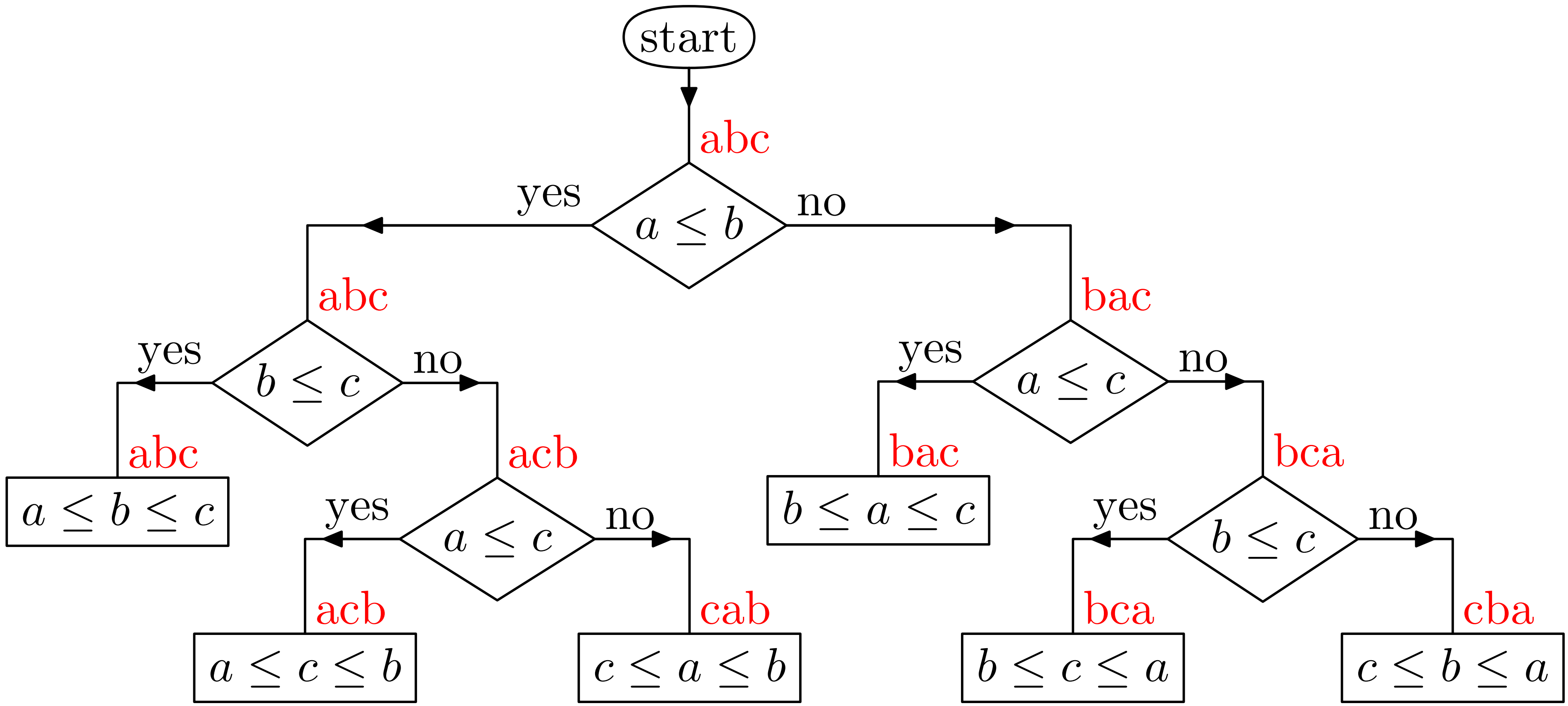

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Decision Trees and Time Complexity

- The time taken to complete the task is the depth of the tree at which we finish (i.e. the leaf nodes)

- We can thus read of the time complexity
  - ⋆ worst case time: depth of the deepest of leaf
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- Different sort strategies will have different decision trees

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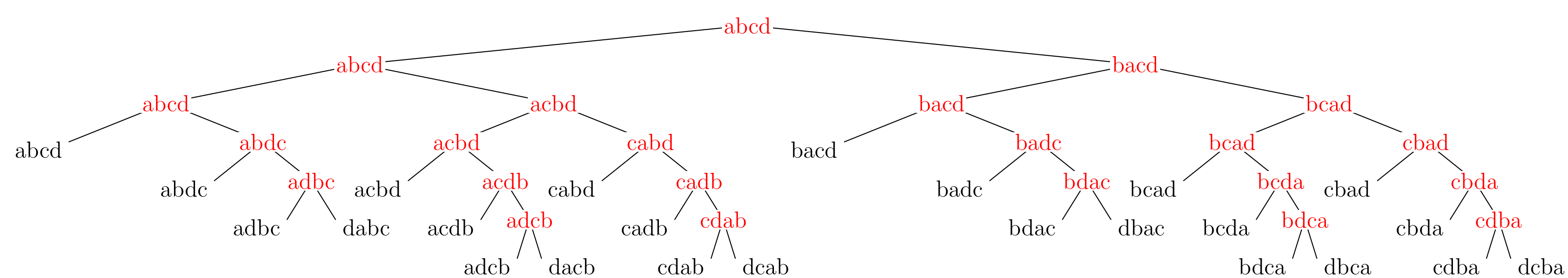

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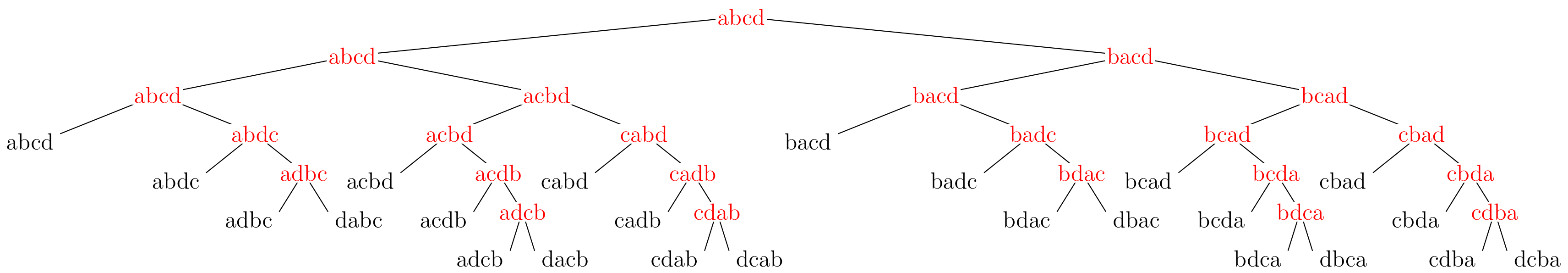

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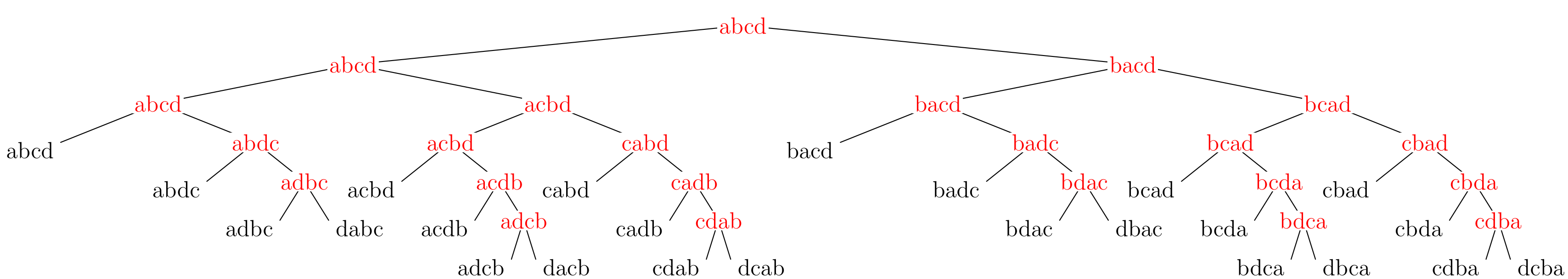

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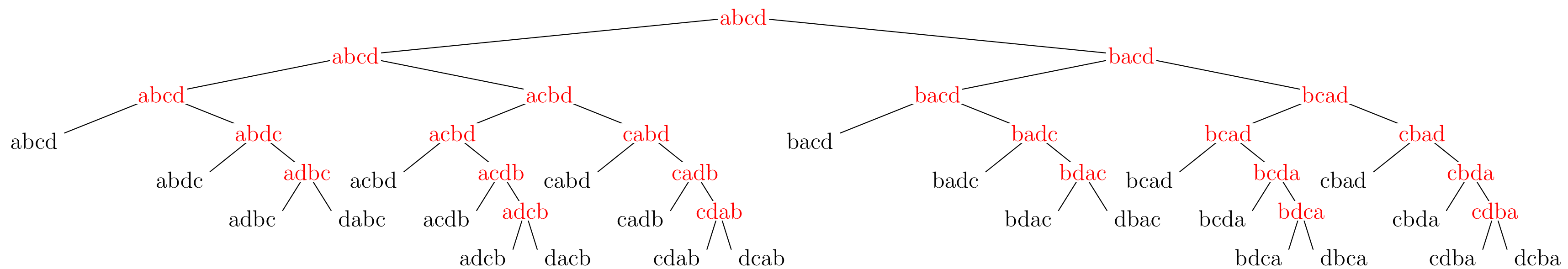

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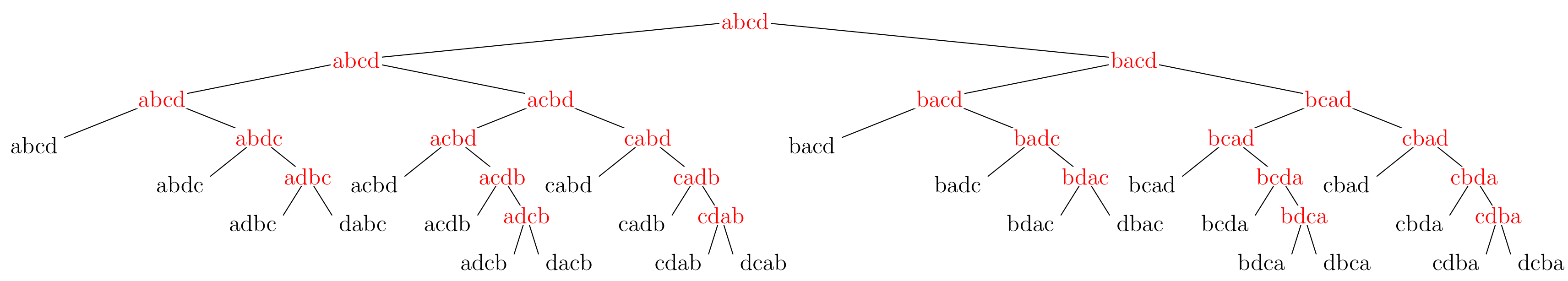

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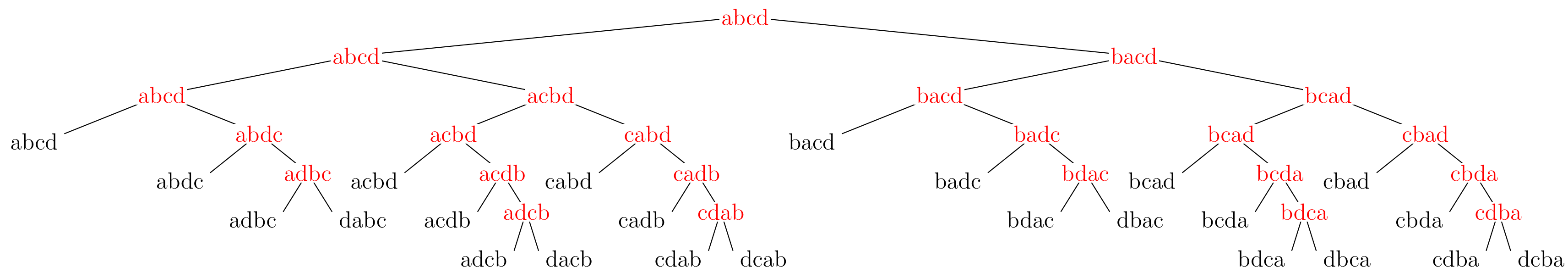

Requirements of Correct Sort

- Any sort based on binary comparisons must have a leaf of the tree for every possible way of sorting the list

- The array $[a, b, c]$ must be arranged differently for all combinations

$$[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]$$

- That is they must go through a different path of the decision tree

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Minimum Number of Leaves

- There must be, at least, one leaf node of the decision tree for each possible permutation of the list
- How many permutations are there of a list of size n?
- Start with a sequence $(a_1, a_2, \ldots, a_n)$
- To create a new permutation we can choose any member of the list as the first element
- We can choose any of the remaining $n-1$ elements of the list as the second element of the permutation
- The total number of permutation is $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$

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Lower Bound Time Complexity for Sorting

- Any sort algorithm using binary comparisons must have a decision tree with at least $n!$ leaf nodes
- This will be a binary tree with some depth d
- The number of leaves at depth d is 2^d
- Thus the smallest depth tree must have a depth d such that $2^d \geq n!$
- That is, the depth of the decision tree satisfies $d \geq \log_2(n!)$
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How Big is $\log_2(n!)$

- We showed in the second lecture that

$$\left(\frac{n}{2}\right)^{n/2} < n! < n^n$$

- It is not too difficult to show that asymptotically (i.e. as $n \to \infty$) that $n!$ approaches $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$—this is known as **Stirling's approximation**
- Thus

$$\begin{aligned}\log_2(n!) &\approx n\log_2(n) - n\log_2(e) + \frac{\log_2(n)}{2} + \frac{\log_2(2\pi)}{2} \\ &= \Theta(n\log_2(n))\end{aligned}$$

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Complexity of Sorting

- We therefore have a lower bound on the time complexity of $\Omega(n \log(n))$
- This is true for any sort using binary comparisons
- We will see in the next lecture there exists algorithms with time complexity $O(n \log(n))$
- This means our lower bound is tight—i.e. it is the true cost of the best algorithm
- Having a lower bound we know we are not going to obtain a substantially faster algorithm

Complexity of Sorting

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- This is true for any sort using binary comparisons

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- Analysis is important: without it we don't know if we have a good algorithm or whether we should try to find a more efficient one
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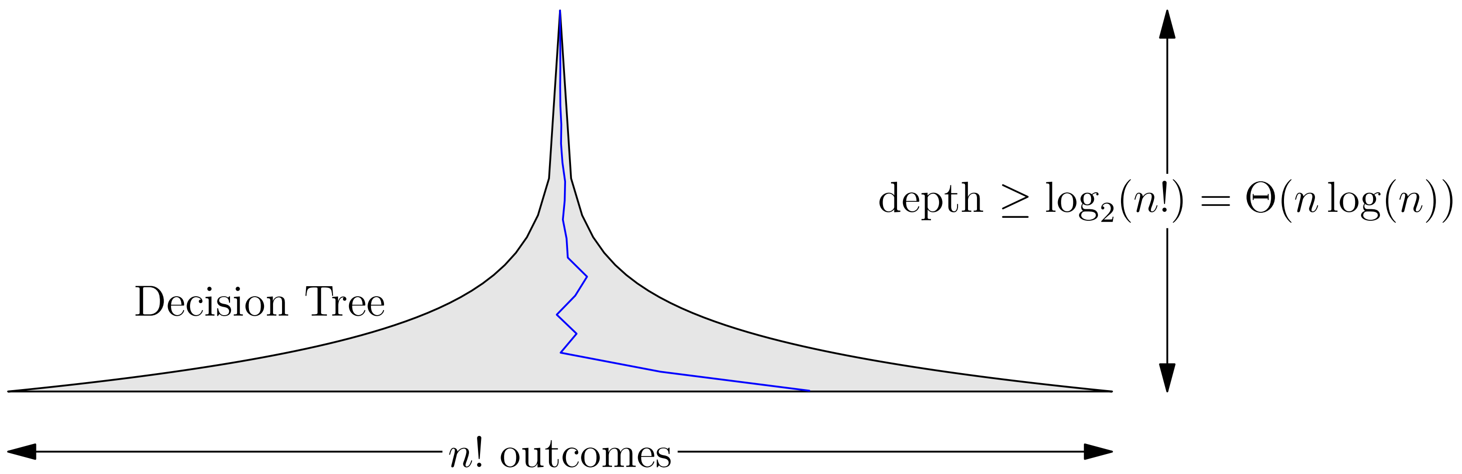

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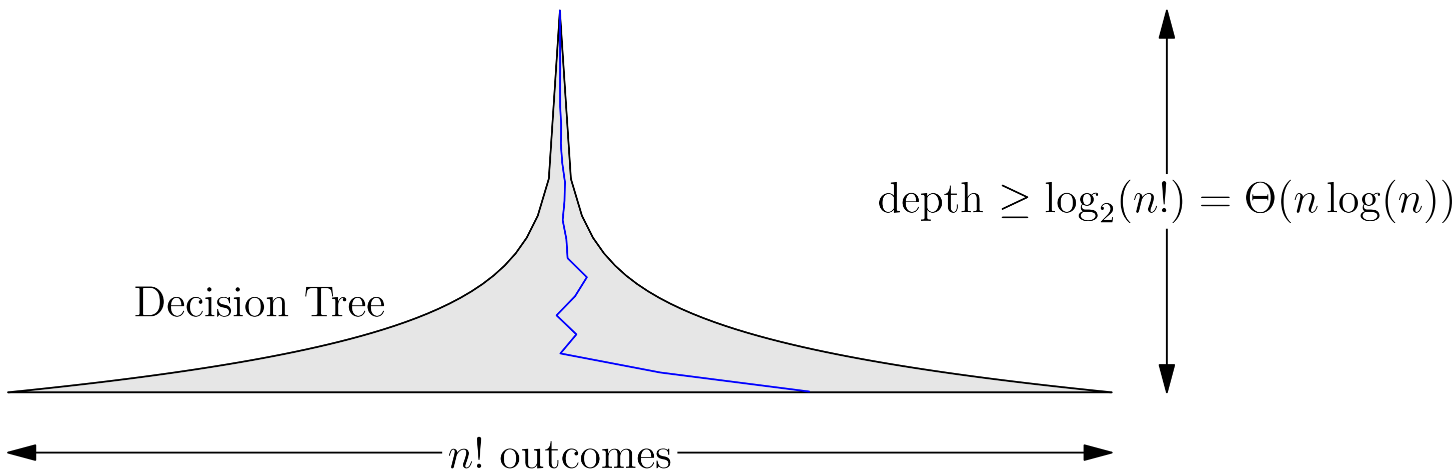

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