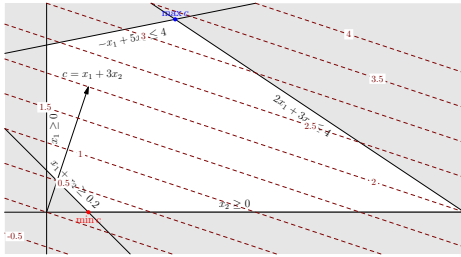


## Lesson 27: Use Linear Programmings



linear programming, applications

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## Going Shopping

- Suppose we have a number of food stuffs which we label with indices  $f \in \mathcal{F}$
- The price of food stuff  $f$  per kilogram we denote  $p_f$
- We are interested in buying a selection of foods  $x = (x_f | f \in \mathcal{F})$  where  $x_f$  is the quantity (in kg) of food  $f$
- We want to minimise the total price  $\sum_f p_f x_f = p \cdot x$
- However we want to ensure that the food has enough vitamins

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## Optimisation Problem

- We can write the food shopping problem as

$$\min_x p \cdot x \quad \text{subject to} \quad Ax \geq b \quad \text{and} \quad x \geq 0$$

- Note that the inequalities involving vectors means that each component must be satisfied, i.e.

$$\begin{aligned} Ax \geq b &\Rightarrow \forall v \in \mathcal{V} \quad \sum_{f \in \mathcal{F}} A_{vf} x_f \geq b_v \\ x \geq 0 &\Rightarrow \forall f \in \mathcal{F} \quad x_f \geq 0 \end{aligned}$$

- This is an example of a “linear program”

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## Constraints

- Each factory can only produce a certain overall tonnage of commodities

$$\sum_{c \in \mathcal{C}} x_{cf} \leq b_f \quad \forall f \in \mathcal{F}$$

where  $b_f$  is the maximum production capacity of factory  $f$

- The total demand for each commodity is  $d_c$  so

$$\sum_{f \in \mathcal{F}} x_{cf} = d_c \quad \forall c \in \mathcal{C}$$

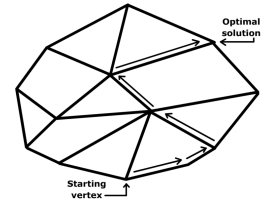
- We can only produce positive amounts, i.e.  $x_{cf} \geq 0$

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1. Examples
2. Linear Programs
3. Properties of Solution
4. Normal Form



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## Nutrition

- We consider the set of vitamins  $\mathcal{V}$
- Let  $A_{vf}$  be the quantity of vitamin  $v$  in food stuff  $f$
- Let  $b_v$  be the minimum daily requirement of vitamin  $v$
- We therefore require

$$\forall v \in \mathcal{V} \quad \sum_{f \in \mathcal{F}} A_{vf} x_f \geq b_v$$

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## Transportation

- We consider a set of factories  $\mathcal{F}$  producing a set of commodities  $\mathcal{C}$
- The amount of commodity  $c$  produced by factory  $f$  we denote by  $x_{cf}$
- The shipping cost of commodity  $c$  from factory  $f$  to the retailer of  $c$  we denote by  $p_{cf}$
- We want to choose  $x_{cf}$  to minimise the transportation costs

$$\sum_{c \in \mathcal{C}, f \in \mathcal{F}} p_{cf} x_{cf}$$

- However, we have constraints. . .

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## Linear Program

- We can write the full problem as

$$\min_x \sum_{c \in \mathcal{C}, f \in \mathcal{F}} p_{cf} x_{cf}$$

subject to

$$\begin{aligned} \sum_{c \in \mathcal{C}} x_{cf} &\leq b_f \quad \forall f \in \mathcal{F} \\ \sum_{f \in \mathcal{F}} x_{cf} &= d_c \quad \forall c \in \mathcal{C} \\ x_{cf} &\geq 0 \quad \forall c \in \mathcal{C}, \quad \forall f \in \mathcal{F} \end{aligned}$$

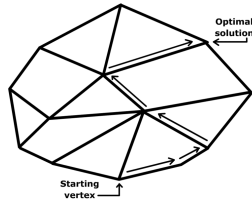
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## Outline

1. Examples
2. **Linear Programs**
3. Properties of Solution
4. Normal Form



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## Maximising

- We can also maximise rather than minimise
- Whether we want to maximise or minimise will depend on the application
- Note that

$$\max_x c \cdot x \equiv \min_x (-c) \cdot x$$

- We can thus always reformulate a maximisation problem as a minimisation problem and vice versa

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## Key Features

- There are three key features of linear programs
  1. The cost (objective function) is linear in  $x_i$  ( $c \cdot x$ )
  2. The constraints are linear in  $x_i$  (e.g.  $A_1 x \leq b_1$ )
  3. The component of  $x$  are non-negative (i.e.  $x_i \geq 0$ )
- These are very special features, very often they don't apply, but a surprising large number of problems can be formulated as linear programming problems

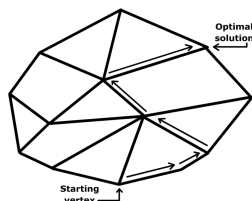
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## Outline

1. Examples
2. Linear Programs
3. **Properties of Solution**
4. Normal Form



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## General Linear Programs

- Linear programs are problems that can be formulated as follows

$$\begin{aligned} & \min_x c \cdot x \\ & \text{subject to} \\ & A^{\leq} x \leq b^{\leq}, \quad A^{\geq} x \geq b^{\geq}, \quad A^{\doteq} x = b^{\doteq}, \quad x \geq 0 \end{aligned}$$

- Note in the previous example it was convenient to use two indices  $c$  and  $f$  to denote the components  $x_{cf}$ , however, it still has this structure

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## Linear Program Applications

- A huge number of problems can be mapped to linear programming problems
- Or modelled as linear (even when they're not, e.g. oil extraction)
- Realistic problems might have many more constraints and large number of variables
- State of the art solvers can deal with problems with hundreds of thousands or even millions of variables

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## History

- Linear programming was "invented" by Leonid Kantorovich in 1939 to help Soviet Russia maximise its production
- It was kept secret during the war, but was finally made public in 1947 when George Dantzig published the **simplex method** which still today is a standard method for solving linear programs
- John von Neumann developed the idea of duality (you can turn a maximisation problem for a set of variables  $x$  into a minimisation problem for a dual set of variables  $\lambda$  associated with each constraint)
- von Neumann used this idea as the basis for "game theory"

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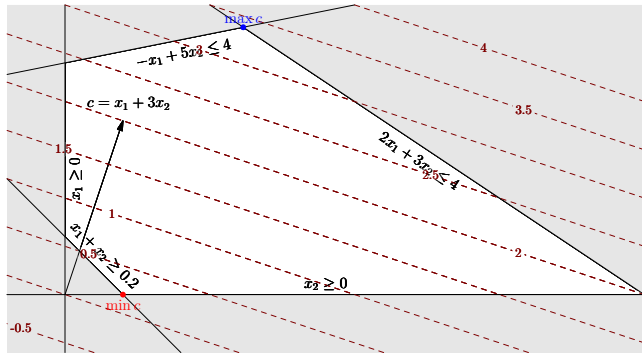
## Structure of Linear Programs

- Before we go into the details of solving linear programs its useful to consider the structure of the solutions
- The set of  $x$  that satisfy all the constraints is known as the set of **feasible solutions**
- The set of feasible solutions may be empty in which case it is impossible to satisfy all the constraints
- This is rather disappointing, but usually doesn't happen if we have formulated a sensible problem

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- The space of feasible solutions is a polyhedra or polytope
- The maximum or minimum solution will always lie at a vertex of the polytope
- Our solution policy will be to start at a vertex and move to a neighbouring vertex that gives the best improvement in cost
- When this isn't possible then we are finished
- However, there is still a lot of work to realise this solution strategy

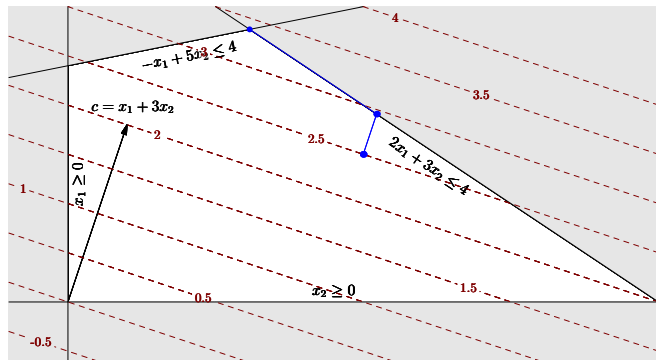
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## Optimal Solution

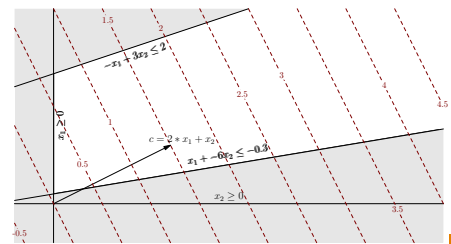
## Unbounded Solutions



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- But usually this would not happen because of the problem definition

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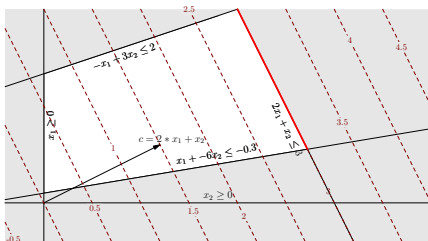
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## Multiple Solutions

## Outline

- You can also get multiple solutions if a constraint is orthogonal to the objective function



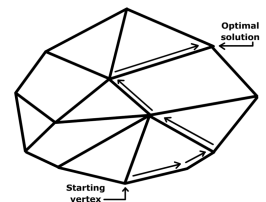
- Nevertheless the optimal will be at a vertex

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1. Examples
2. Linear Programs
3. Properties of Solution
4. **Normal Form**



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## Converting Linear Programs

## Slack Variables

- Solving full linear programs is difficult
- However, it is much easier to solve linear programs in **normal form**
- This is basically a form where we get rid of all inequalities and rewriting the equalities
- Fortunately its rather easy to convert linear programs to normal form

- We can change an inequality into an equality by introducing a new "slack" variable

- E.g.

$$\begin{aligned} a_1 \cdot x &\geq 0 &\Rightarrow & a_1 \cdot x - z_1 = 0 & z_1 &\geq 0 \\ a_2 \cdot x &\leq 0 &\Rightarrow & a_2 \cdot x + z_2 = 0 & z_2 &\geq 0 \end{aligned}$$

$z_1$  (the excess) and  $z_2$  (the deficit) are known as slack variables

- We eliminate inequalities at the expense of increasing the number of variables
- We can treat the slack variables on an equivalent footing to the normal variables (they just provide a different way of describing the original problem)

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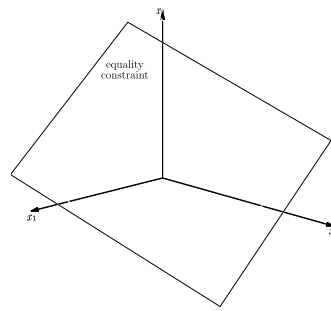
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## Normal Form

- A linear program with only equality constraints is said to be in **normal form**
- We will find in the next lecture that this is a convenient form for solving linear programs
- An equality constraint restricts the solutions to a subspace (some lower dimensional space)

## Solving Linear Programming



- The basic feasible points for LP problems with  $n$  variables and  $m$  constraints have at least  $n - m$  zero variables
- Typical number of basic feasible solutions is  $\binom{n}{m} \geq \left(\frac{n}{m}\right)^m$
- Simplex algorithm organises iterative search for global solutions

## Lessons

- There are a huge number of problems that can be set up as linear programs
- They are particularly useful in resource allocation where the resources are all positive
- The solution to linear programming problems is at the vertex of the feasible space (intersection of constraints)
- We can search for solutions by moving from vertex to vertex
- We can transform inequality constraints to equality constraints using slack variables