Algorithms and Analysis

Lesson 10: *Keep Trees Balanced*

AVL trees, red-black trees, TreeSet, TreeMap

Outline

1. **Deletion**
2. Balancing Trees
   - Rotations
3. AVL
4. Red-Black Trees
   - TreeSet
   - TreeMap

Recap

- Binary search trees are commonly used to store data because we need to only look down one branch to find any element
- We saw how to implement many methods of the binary search tree
  - ⋆ `find`
  - ⋆ `insert`
  - ⋆ `successor` (in outline)
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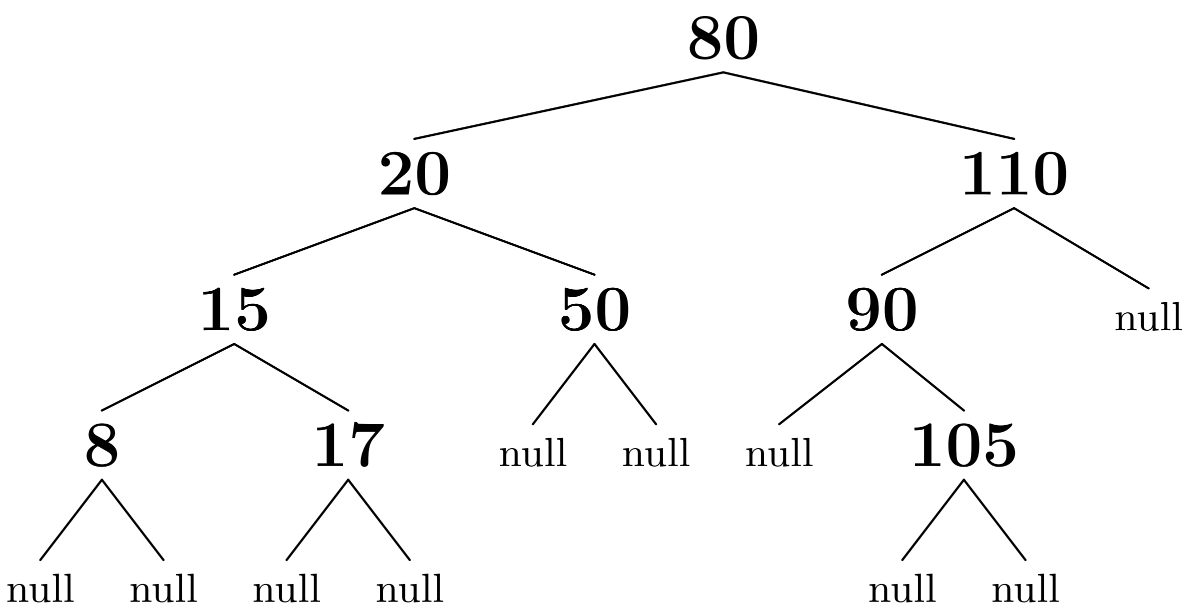
- It is relatively easy if the element is a leaf node (e.g. 50)
- It is not so hard if the node has one child (e.g. 20)



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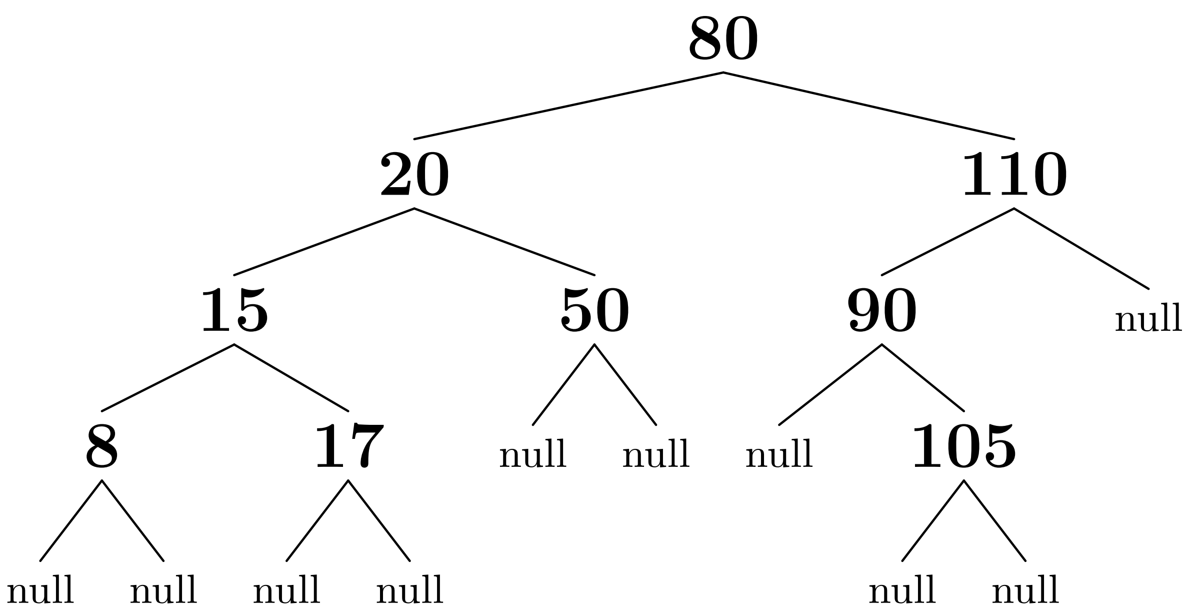

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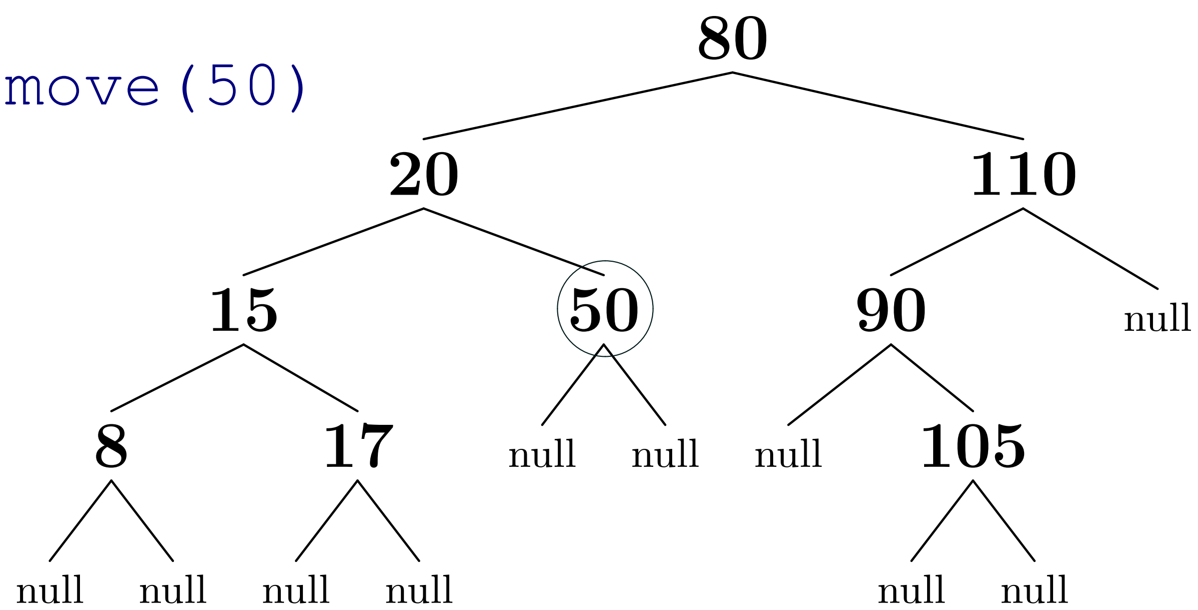
`remove(50)`



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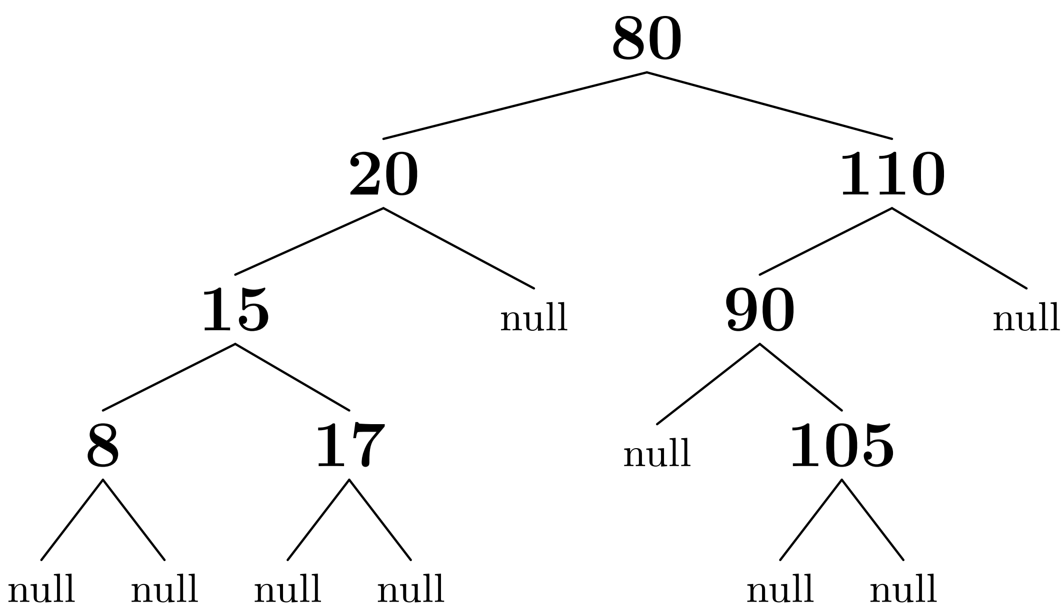

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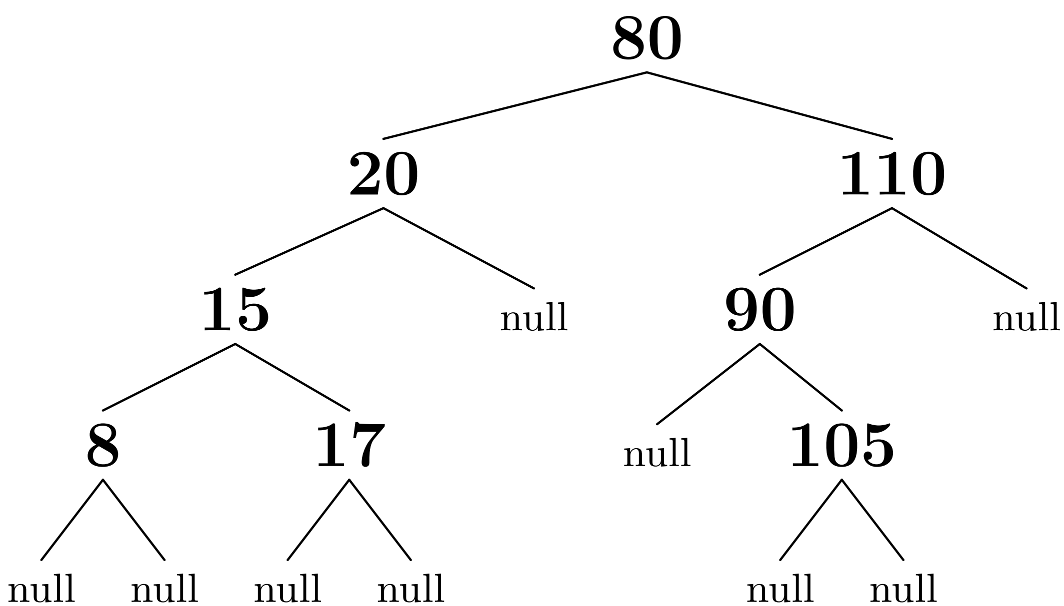

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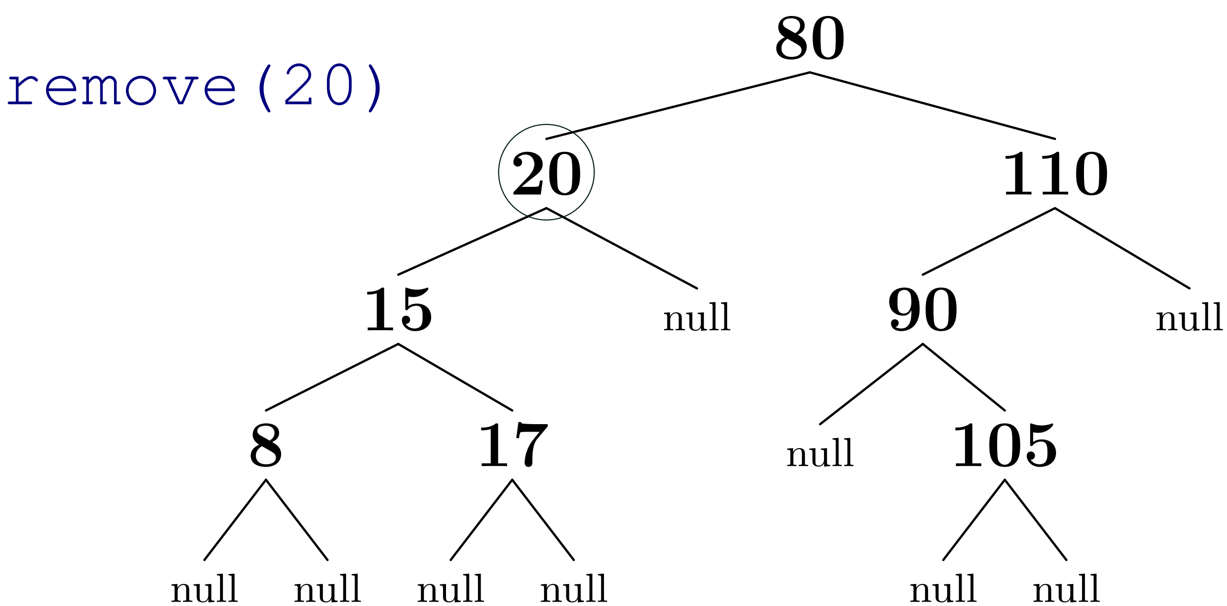

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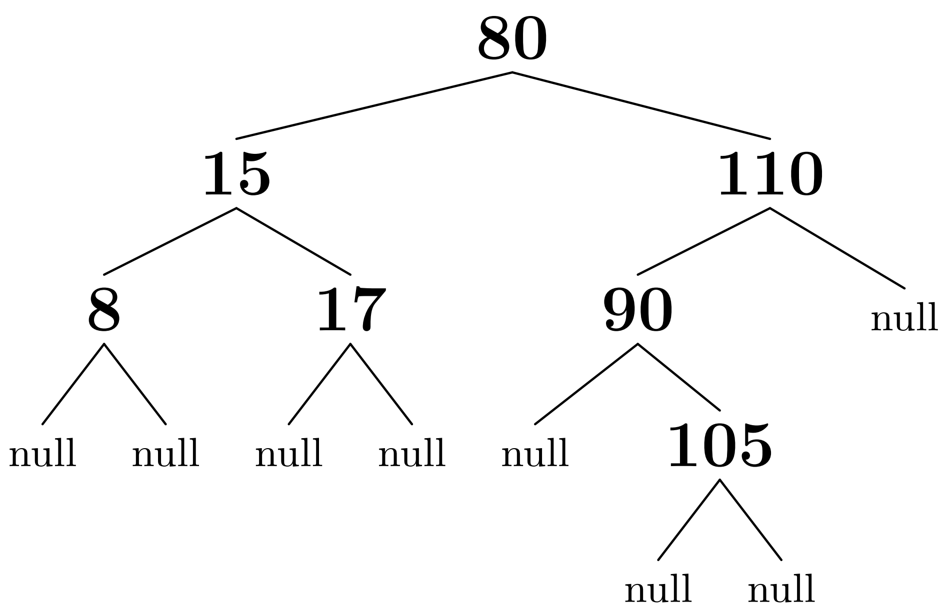

Code to remove Node n

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if (n->left==nullptr && n->right==nullptr) {
    if (n == n->parent->left)
        n->parent->left = nullptr;
    else
        n->parent->right = nullptr;
} else if (n->right==nullptr) {
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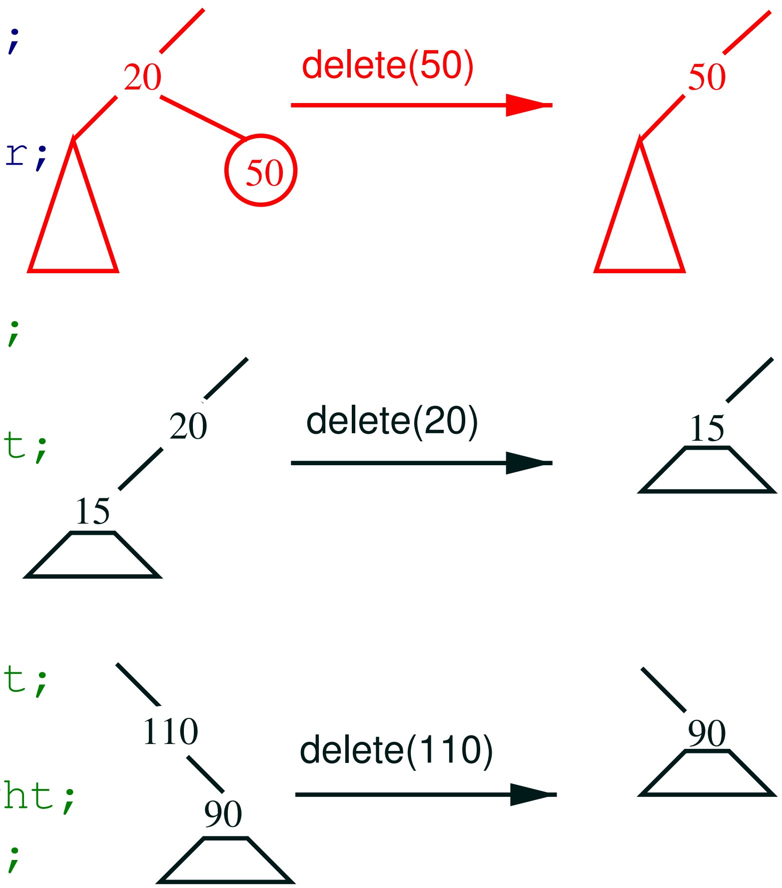

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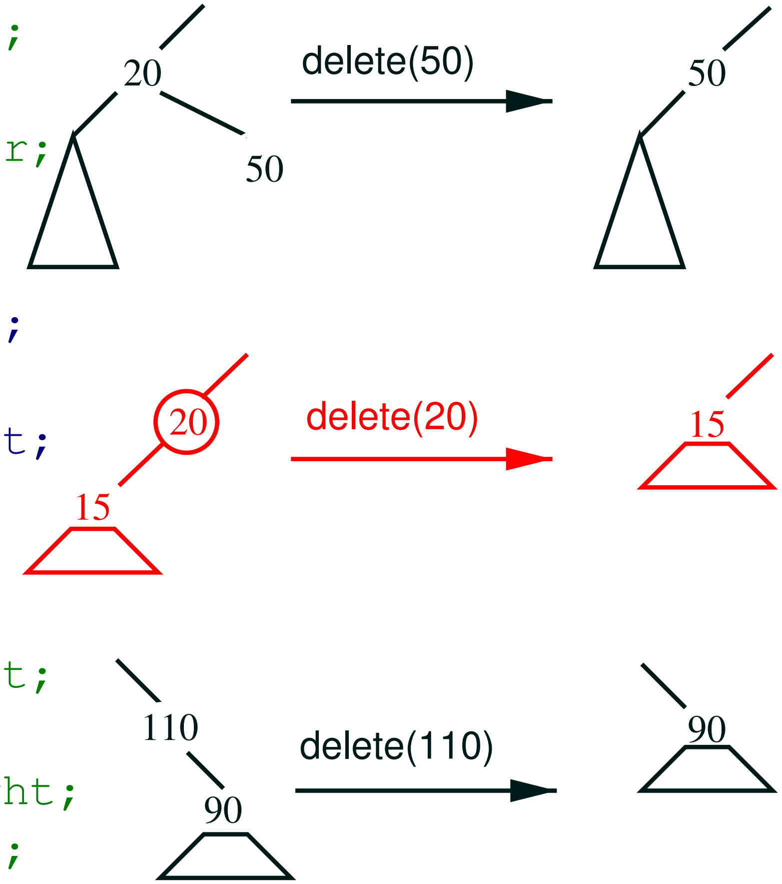

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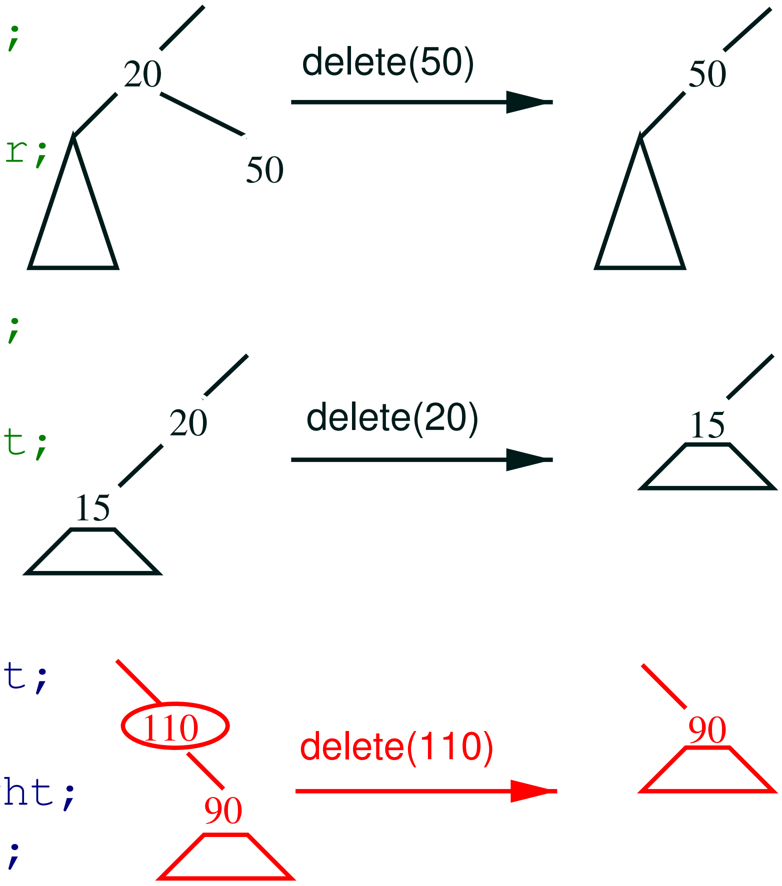

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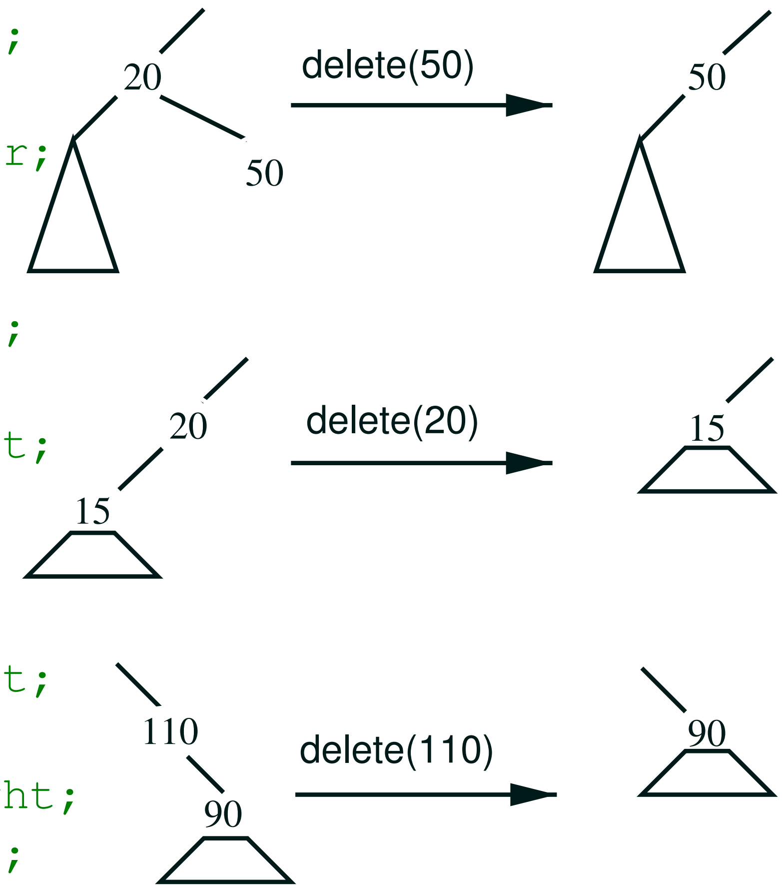

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- If an element has two children then
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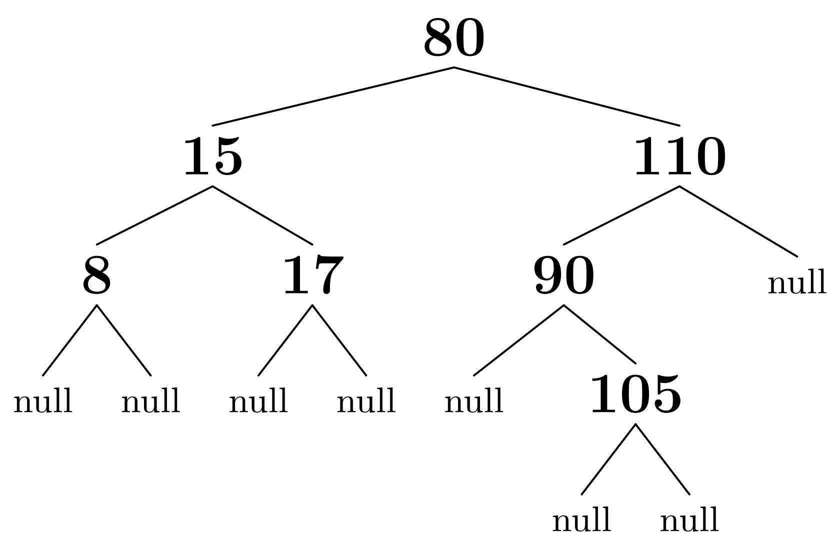

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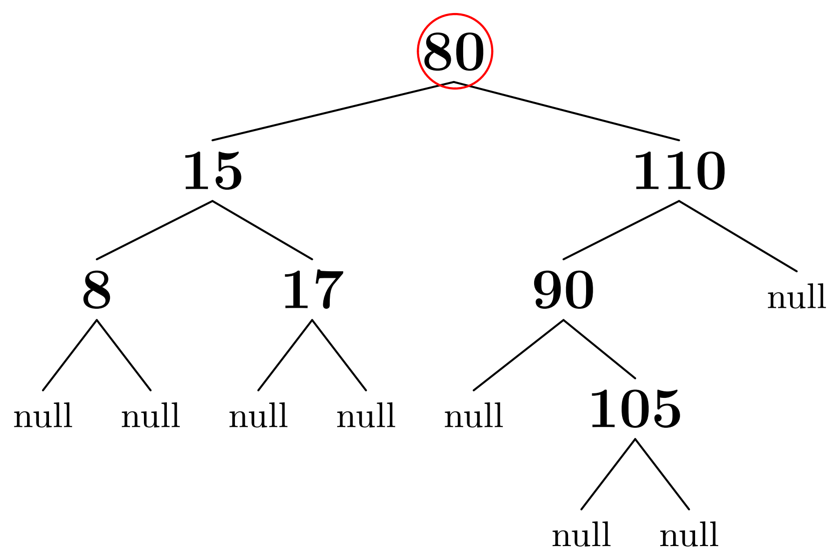

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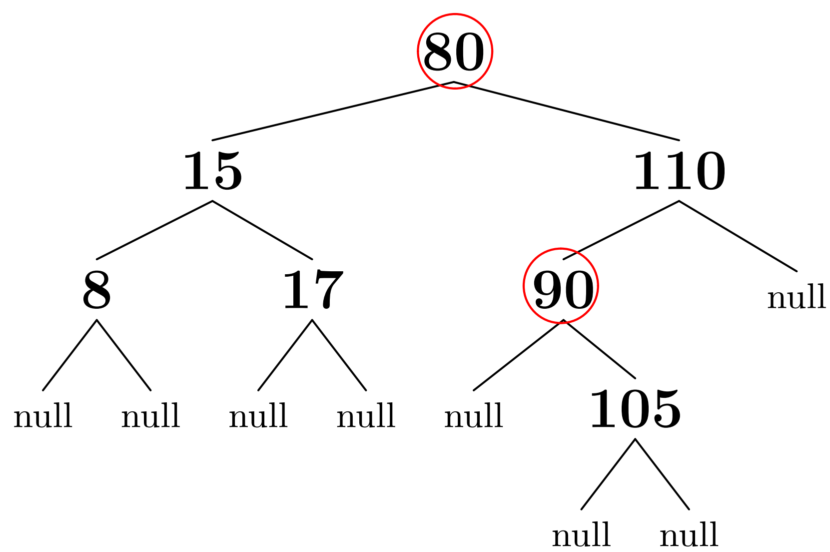

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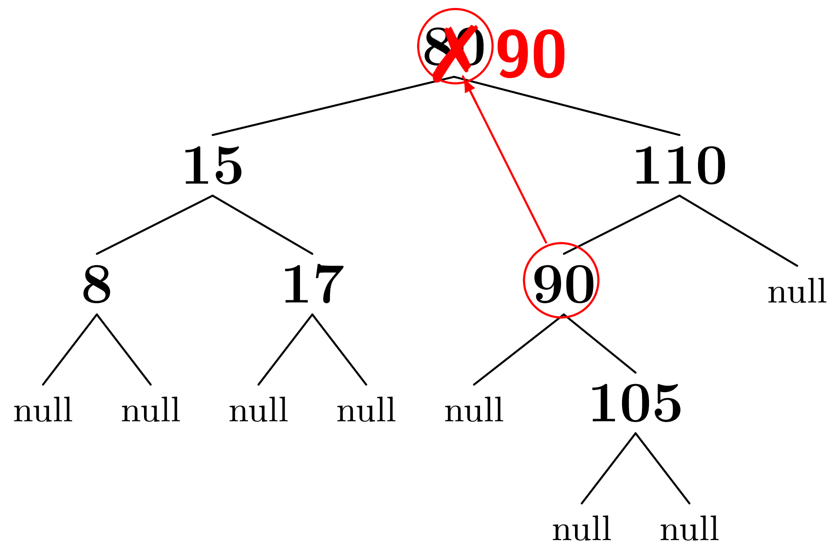

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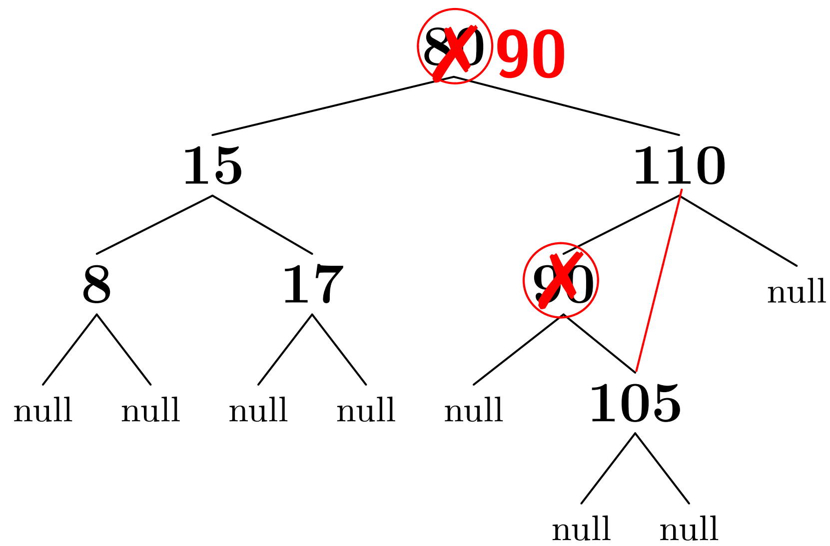

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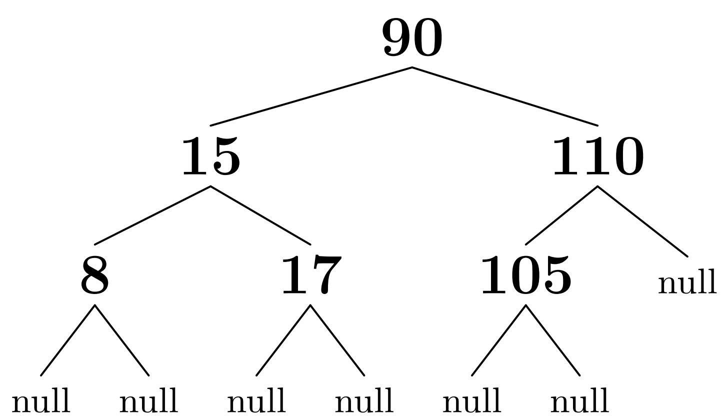

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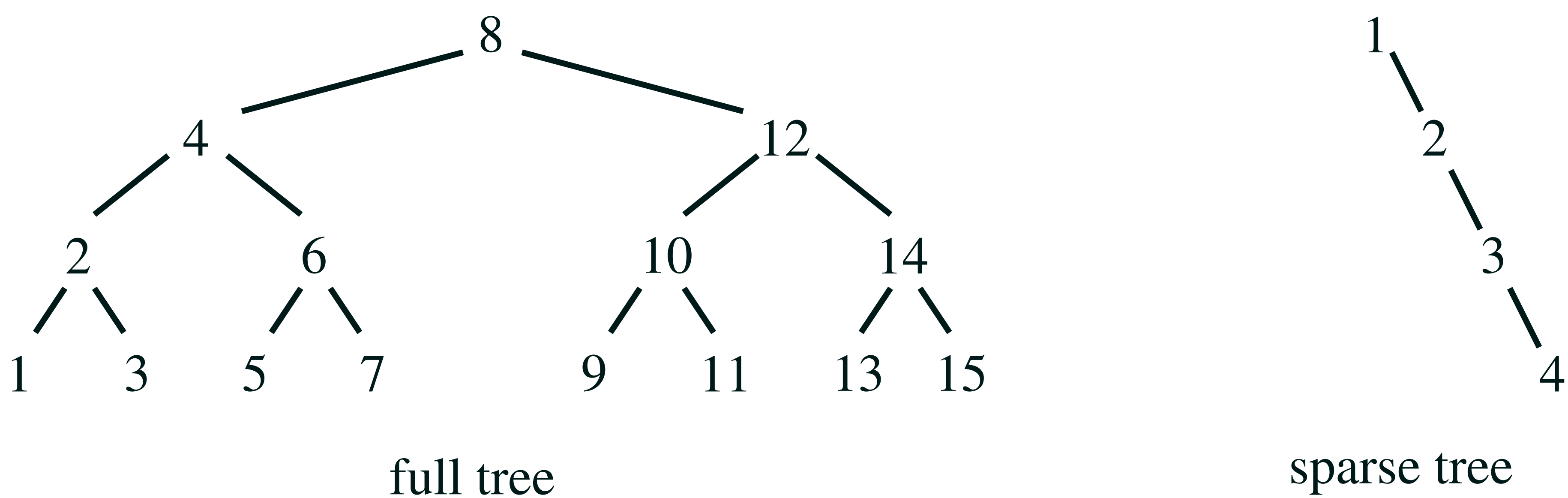
Why Balance Trees

- The number of comparisons to access an element depends on the depth of the node
- The average depth of the node depends on the shape of the tree



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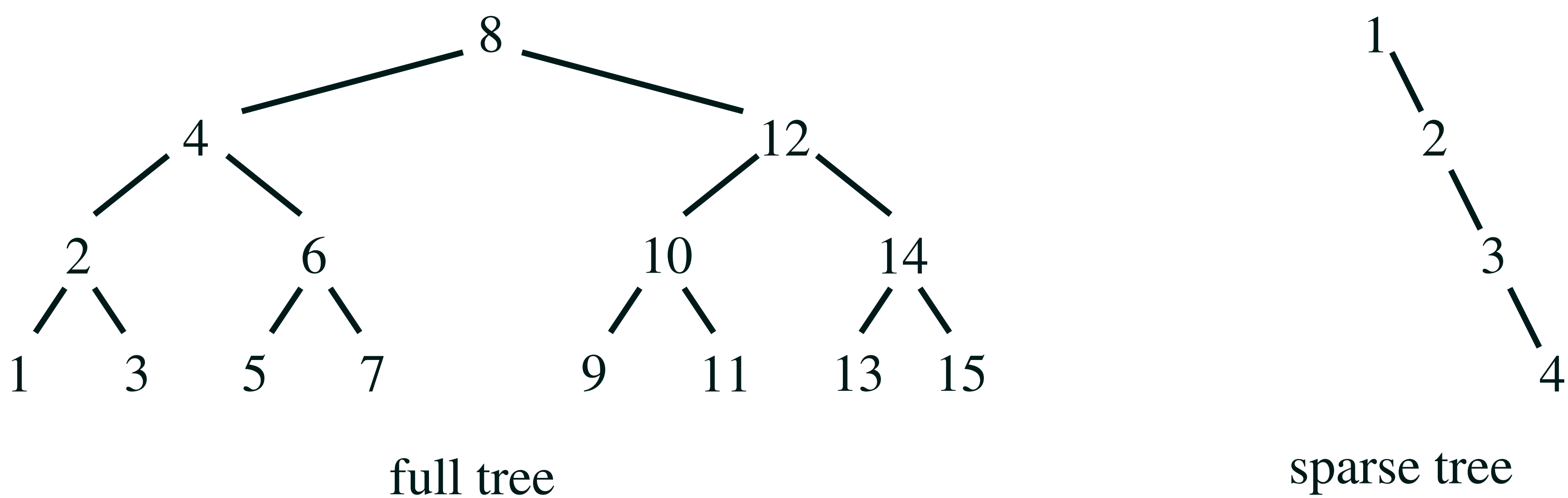

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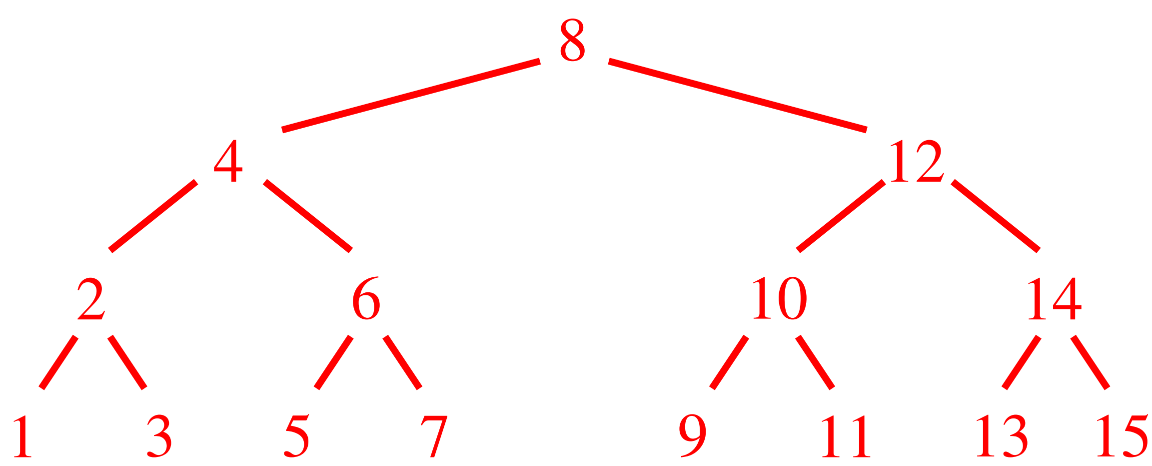
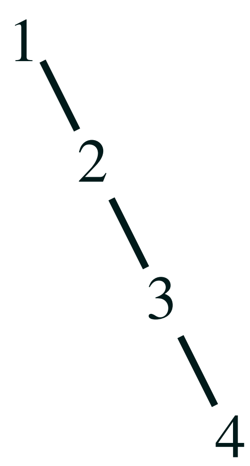


full tree



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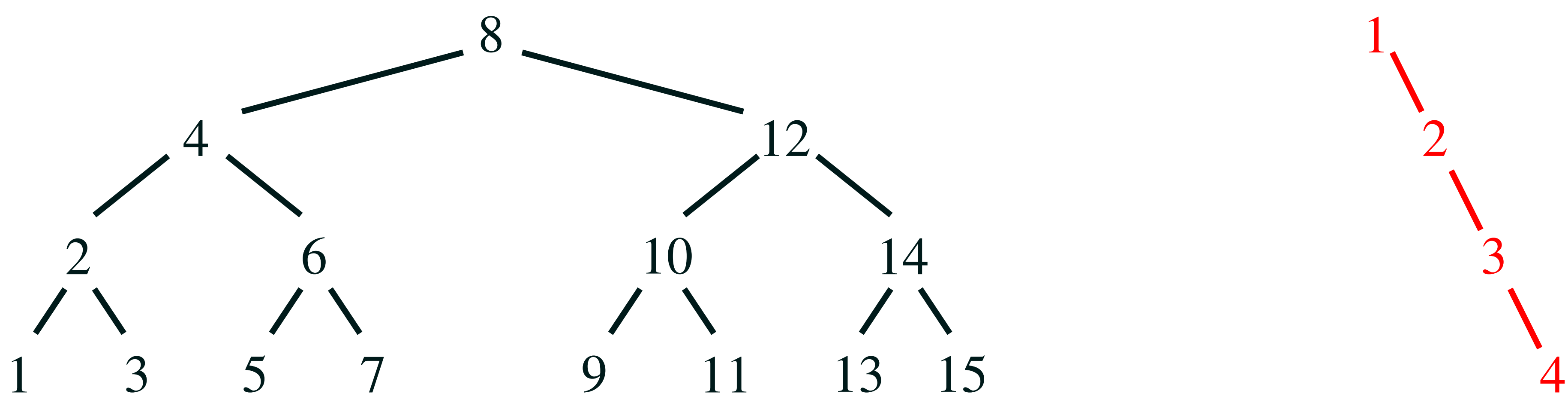

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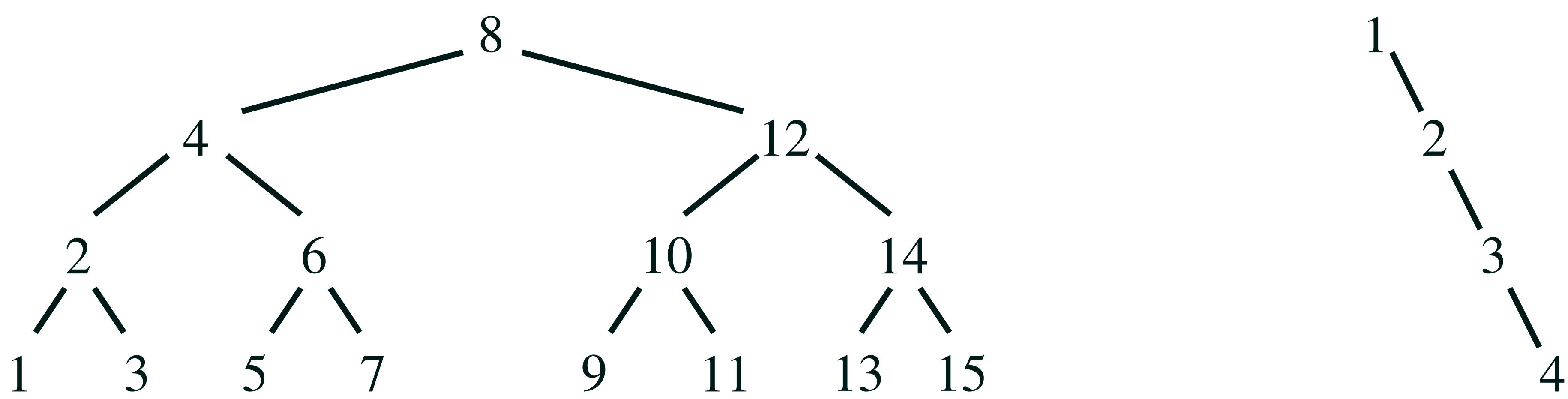

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Time Complexity

- In the best situation (a full tree) the number of elements in a tree is $n = \Theta(2^l)$ the depth is l so that the maximum depth is $\log_2(n)$
- It turns out for random sequences the average depth is $\Theta(\log(n))$
- In the worst case (when the tree is effectively a linked list), the average depth is $\Theta(n)$
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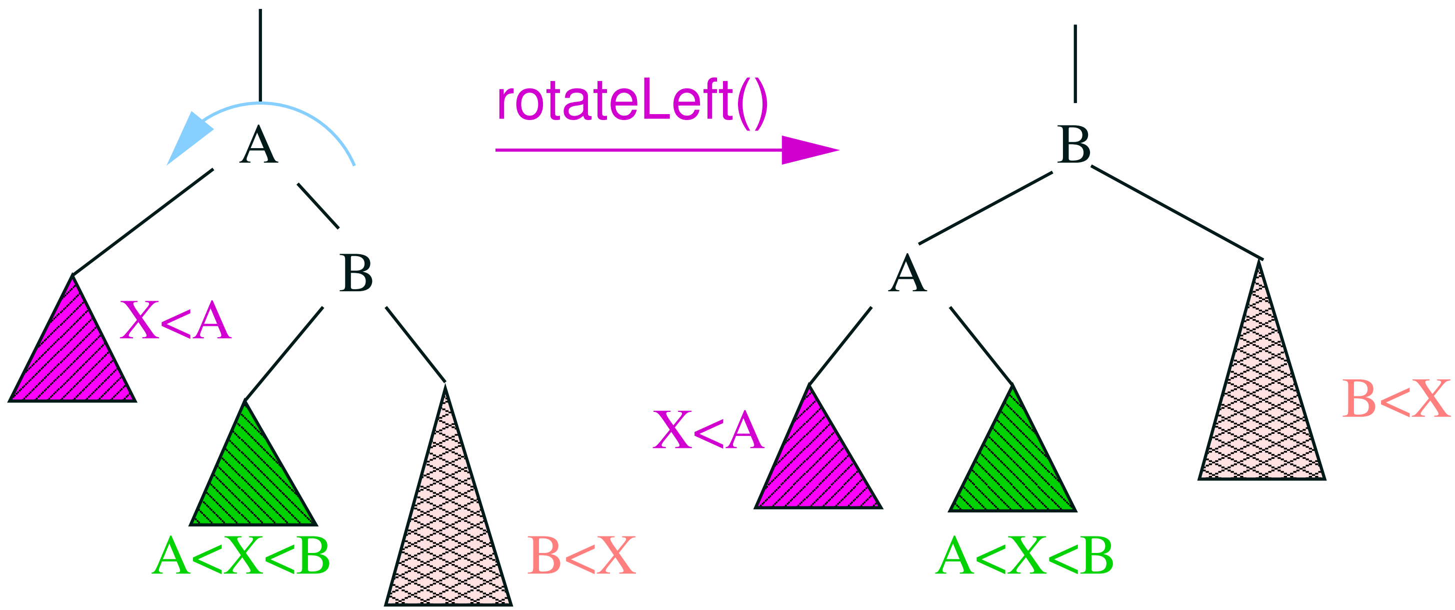
Rotations

- To avoid unbalanced trees we would like to modify the shape
- This is possible as the shape of the tree is not uniquely defined (e.g. we could make any node the root)
- We can change the shape of a tree using **rotations**
- E.g. left rotation



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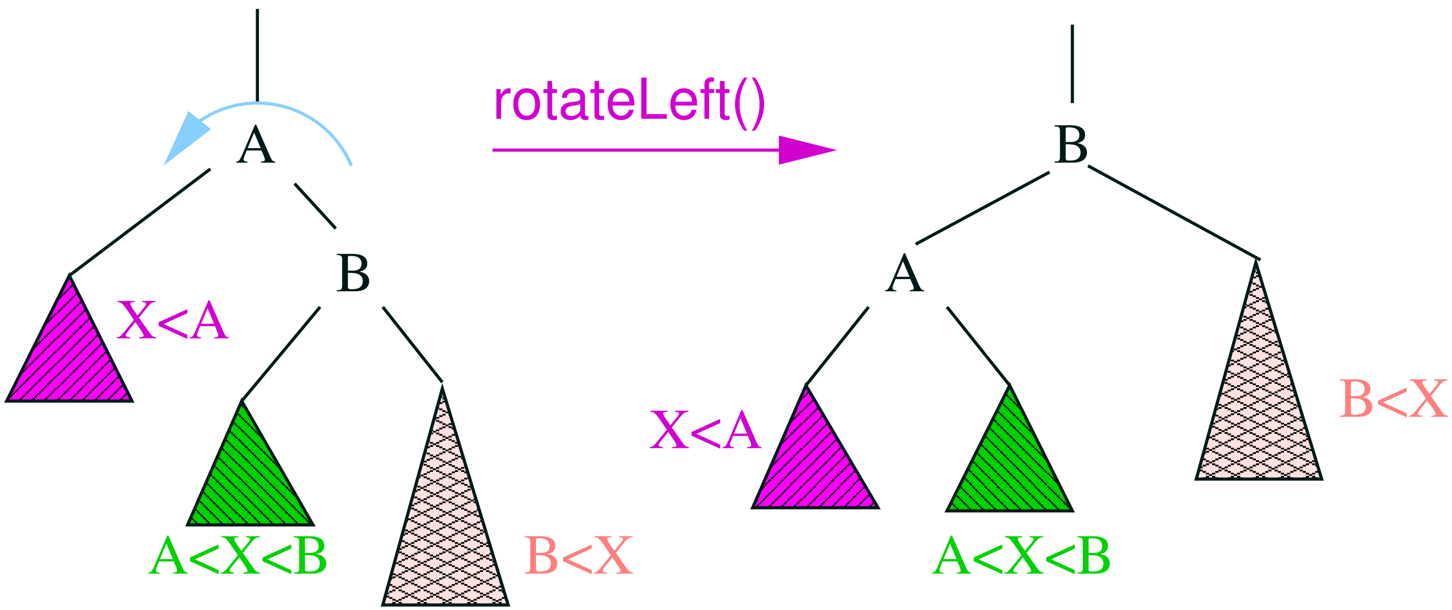

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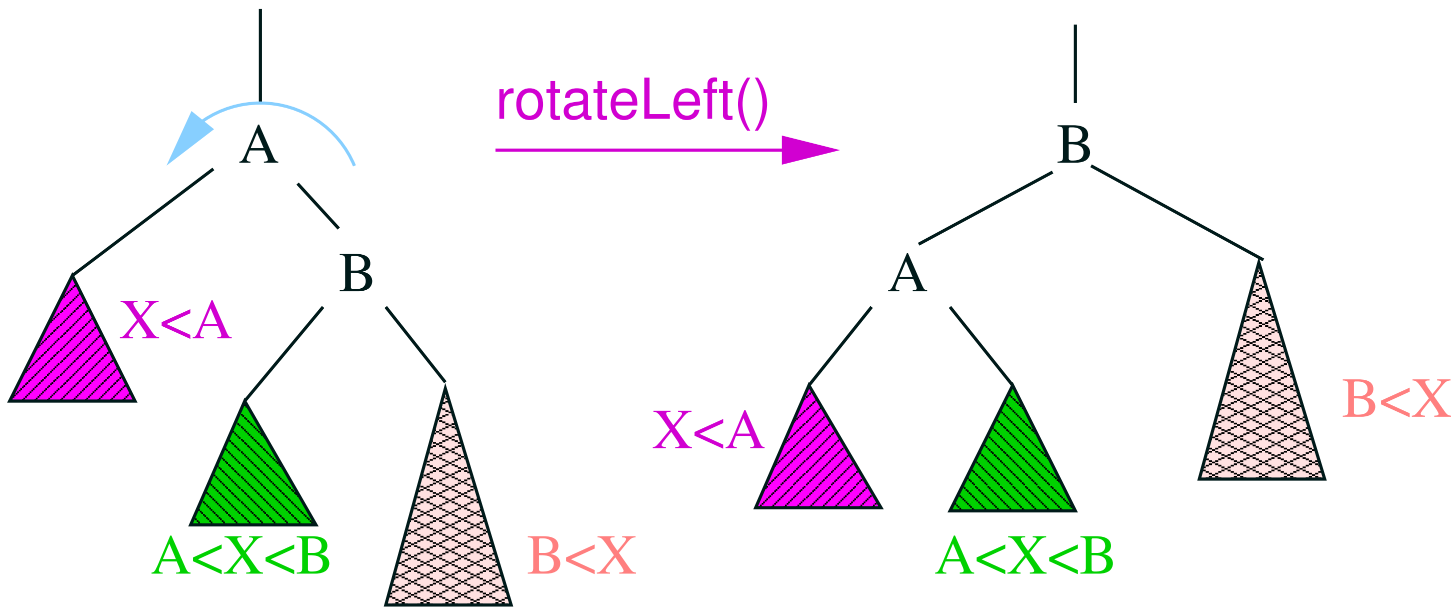

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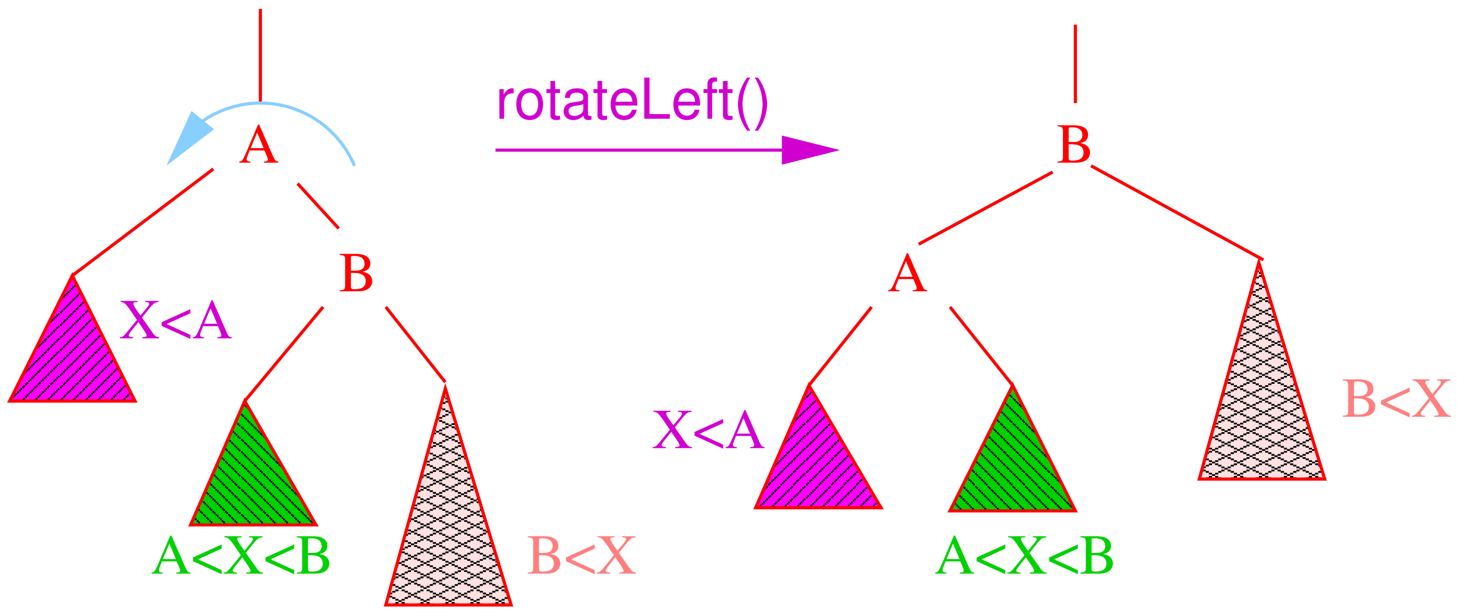
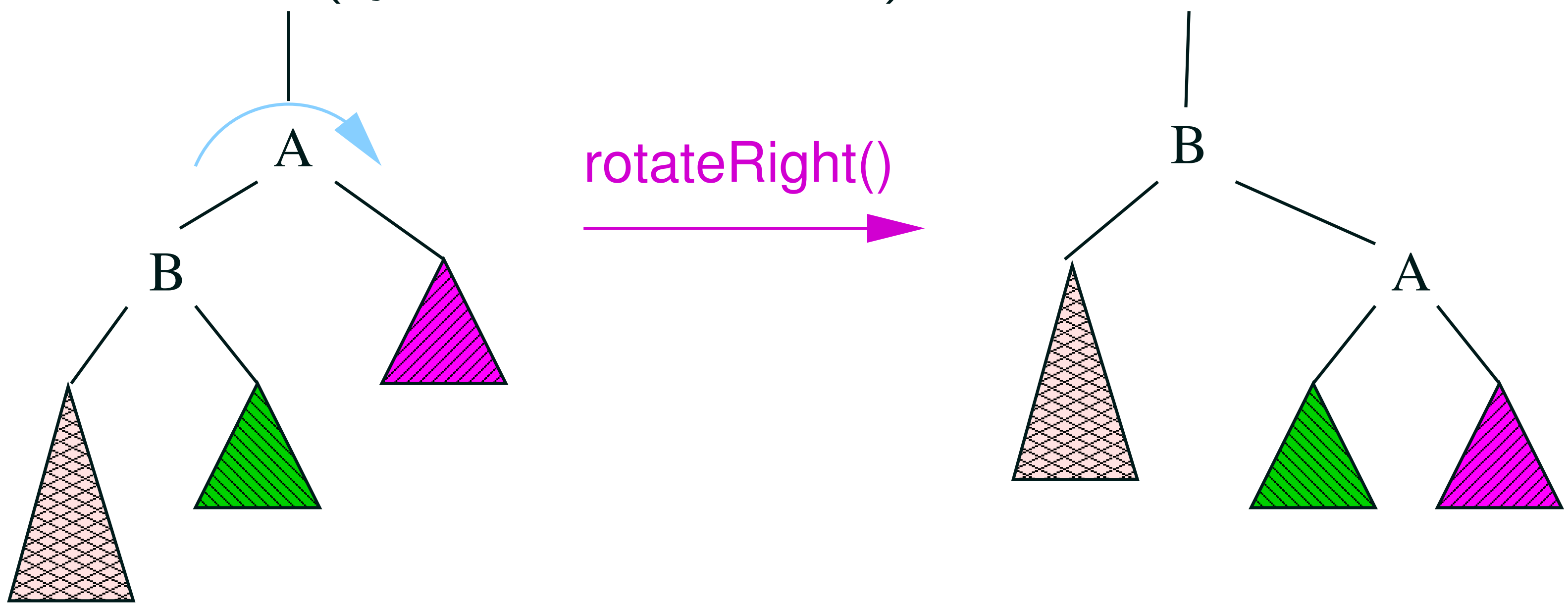


Types of Rotations

- We can get by with 4 types of rotations
  - ⋆ Left rotation (as above)
  - ⋆ Right rotation (symmetric to above)



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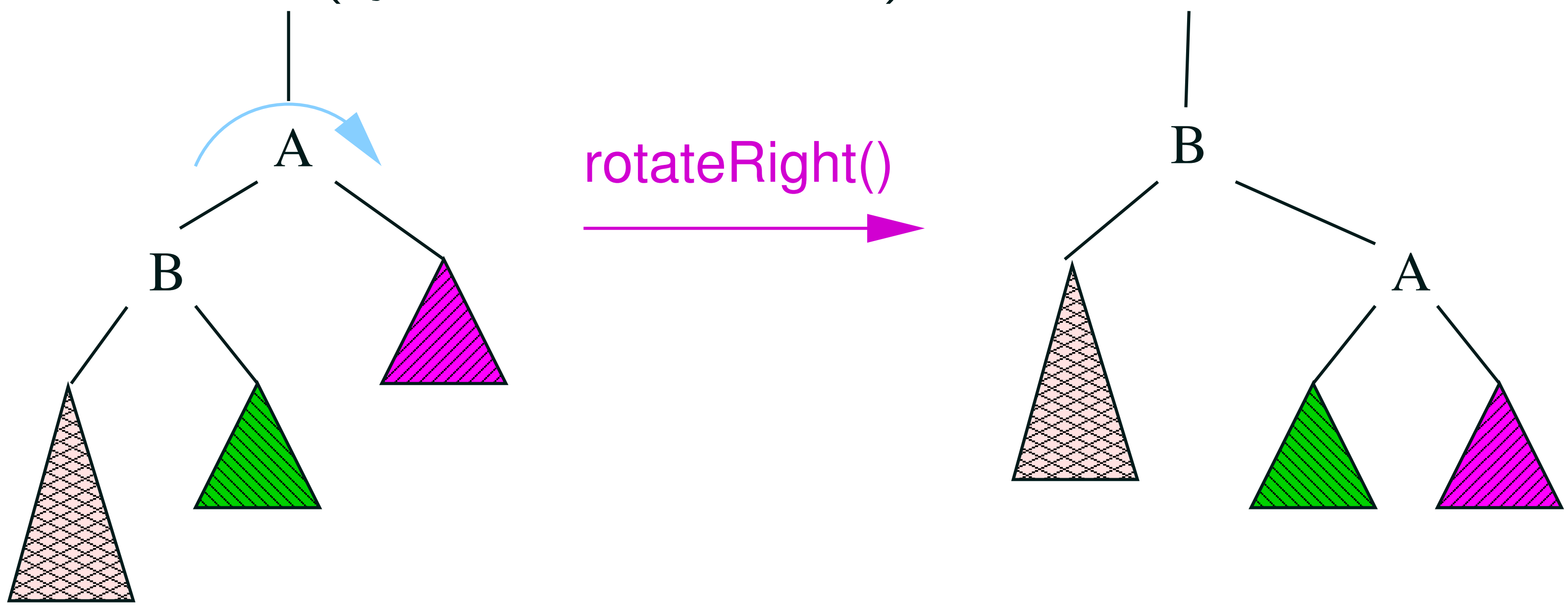

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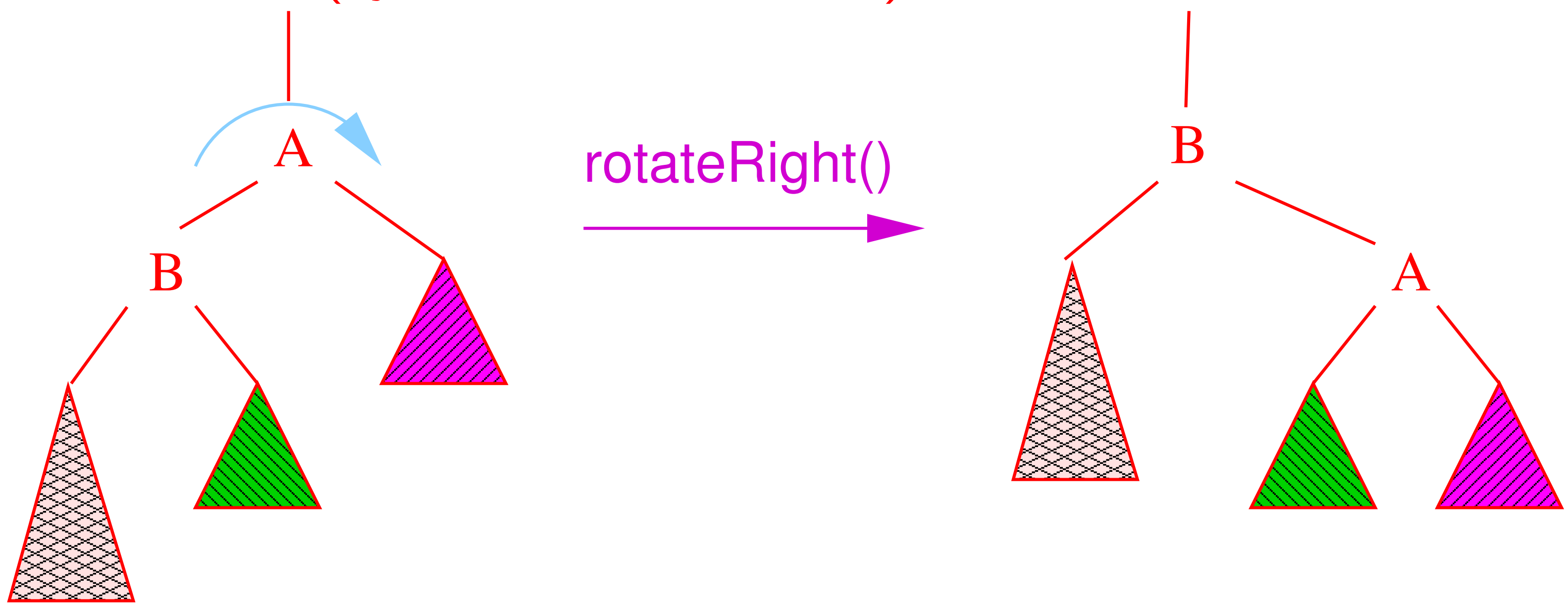

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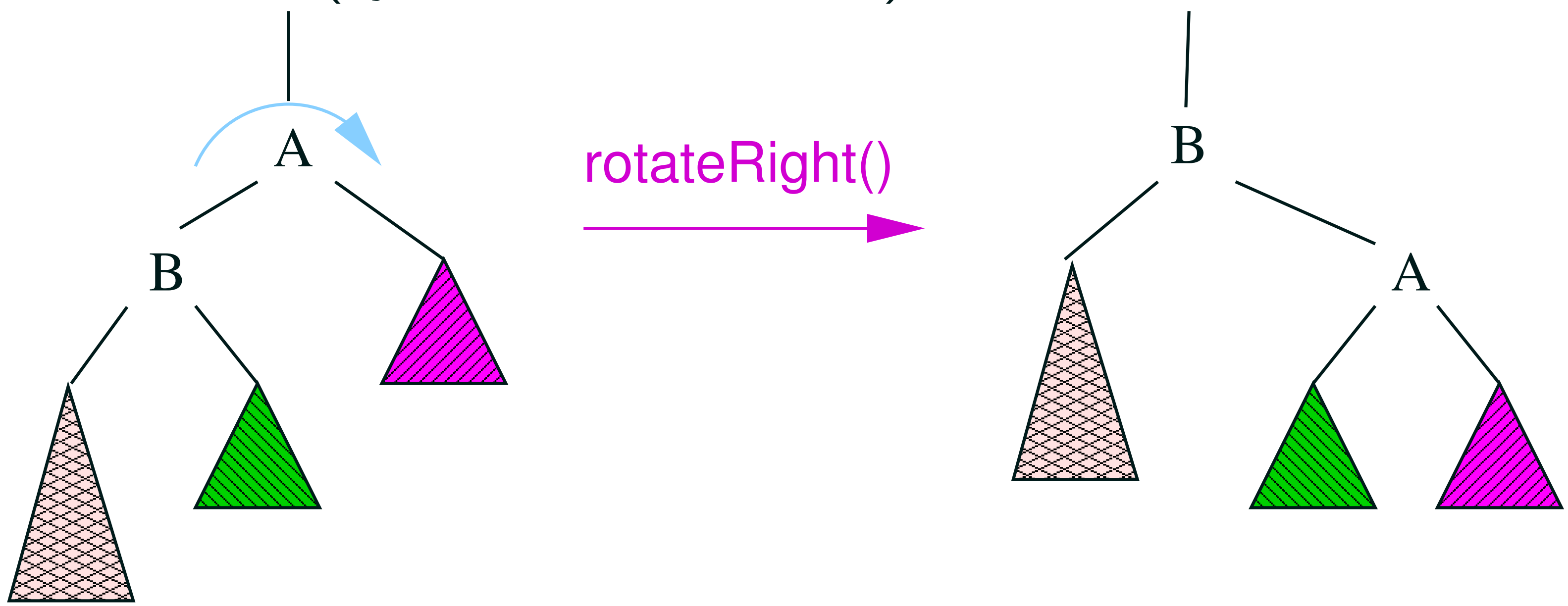

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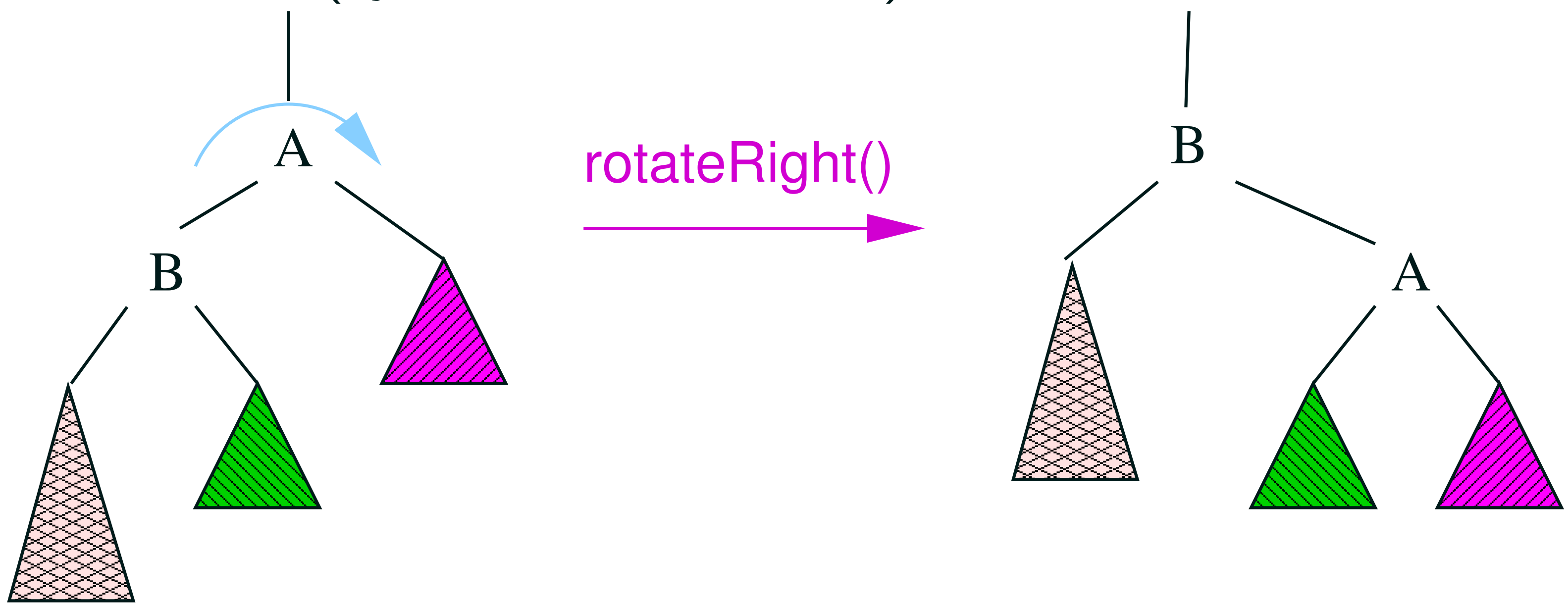

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Coding Rotations

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{
  Node* r = e->right;
  e->right = r->left;
  if (r->left != nullptr)
     r->left->parent = e;
  r->parent = e->parent;
  if (e->parent == nullptr)
     root = r;
  else if (e->parent->left == e)
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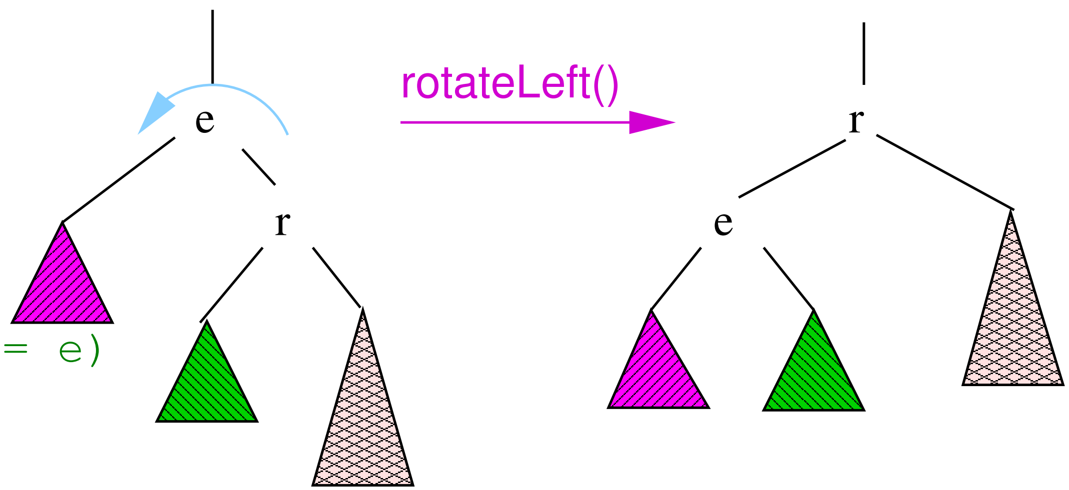

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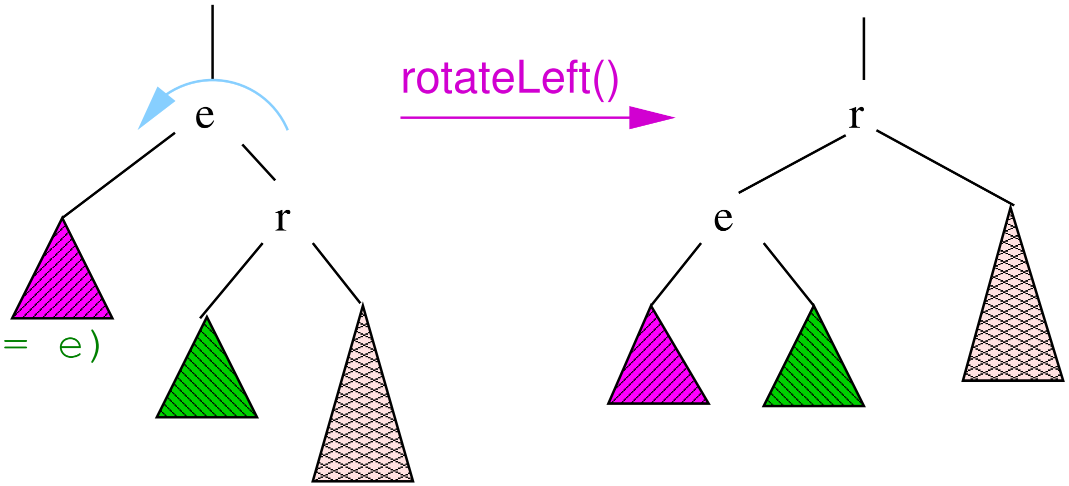

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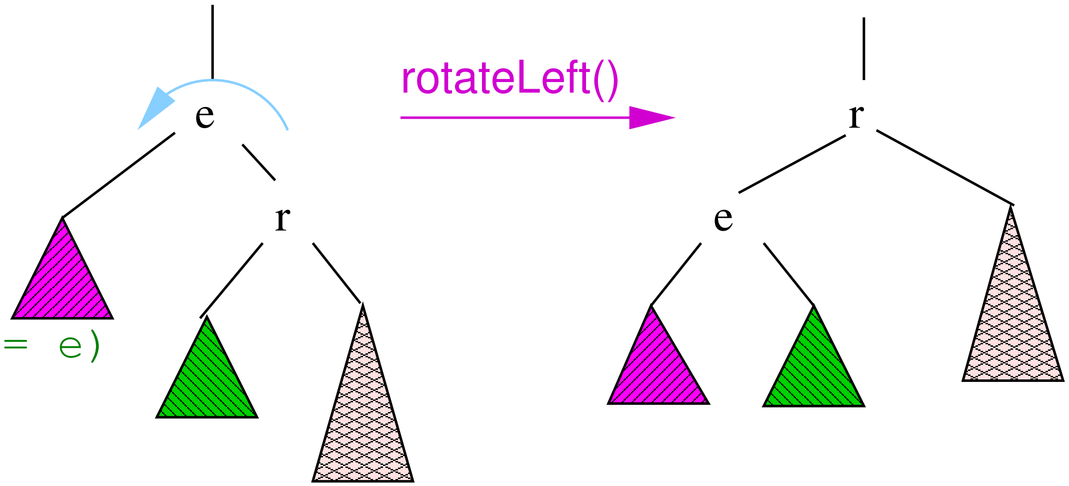

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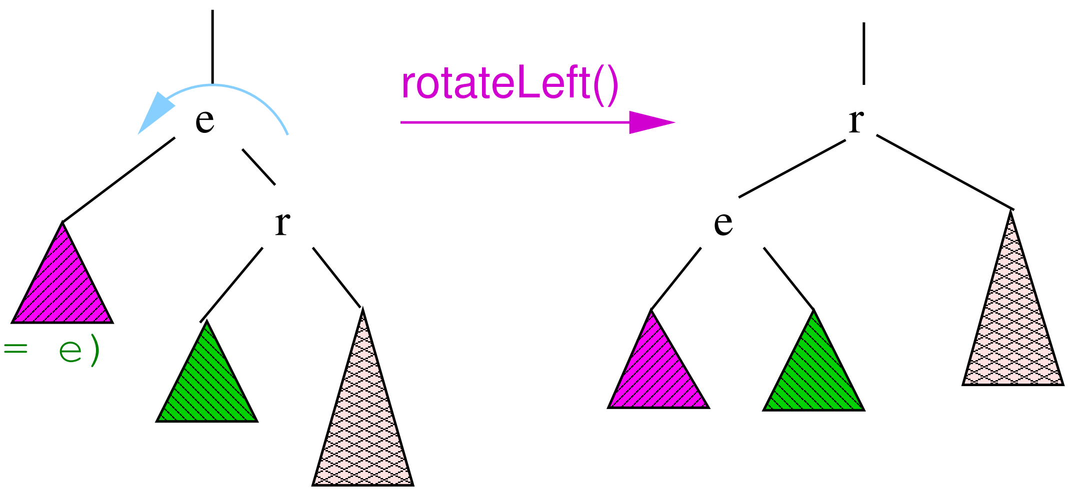

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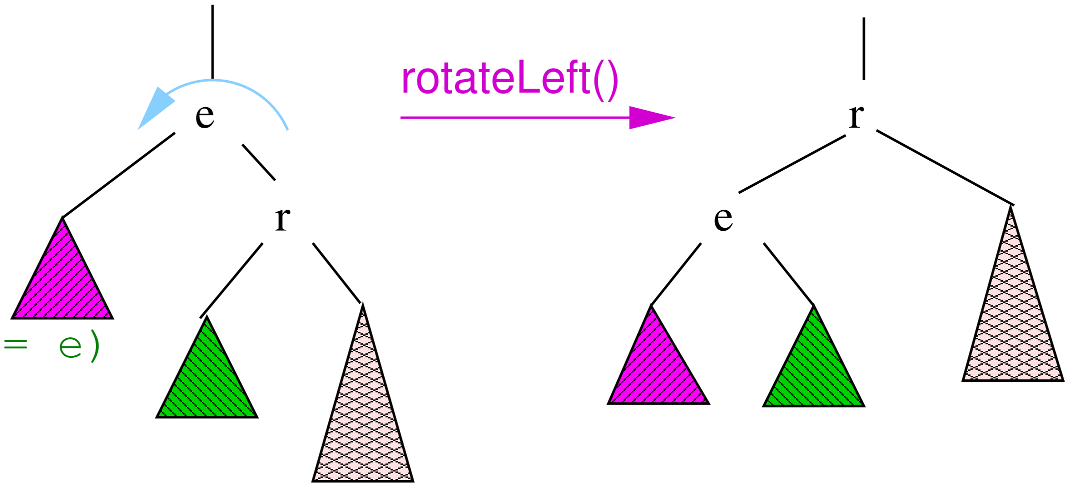

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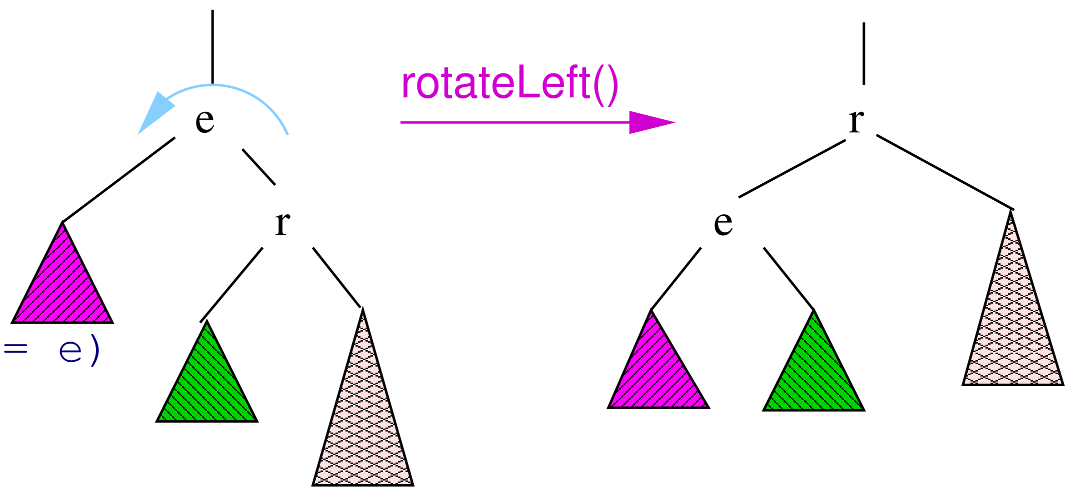
  Node* r = e->right;
  e->right = r->left;
  if (r->left != nullptr)
     r->left->parent = e;
  r->parent = e->parent;
  if (e->parent == nullptr)
     root = r;
  else if (e->parent->left == e)
     e->parent->left = r;
  else
     e->parent->right = r;
  r->left = e;
  e->parent = r;
}
```



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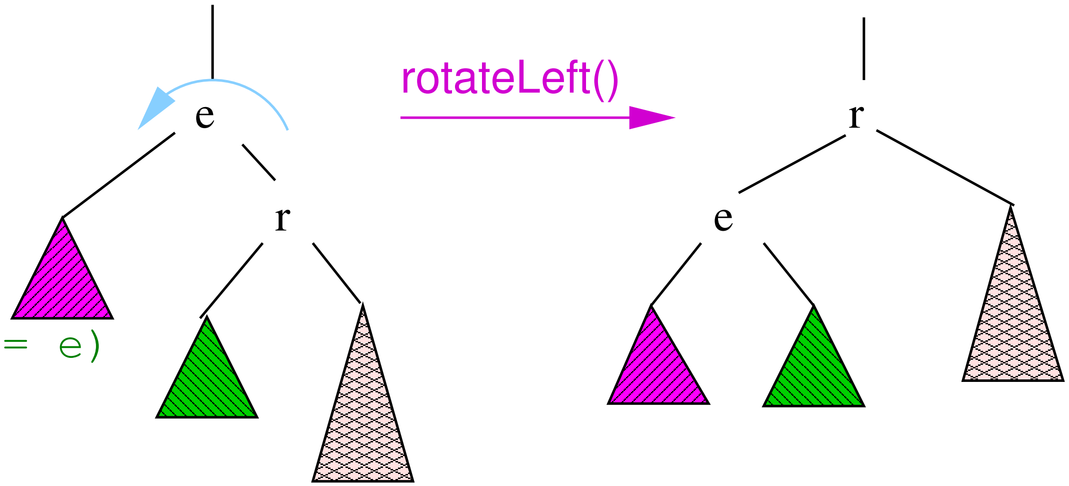
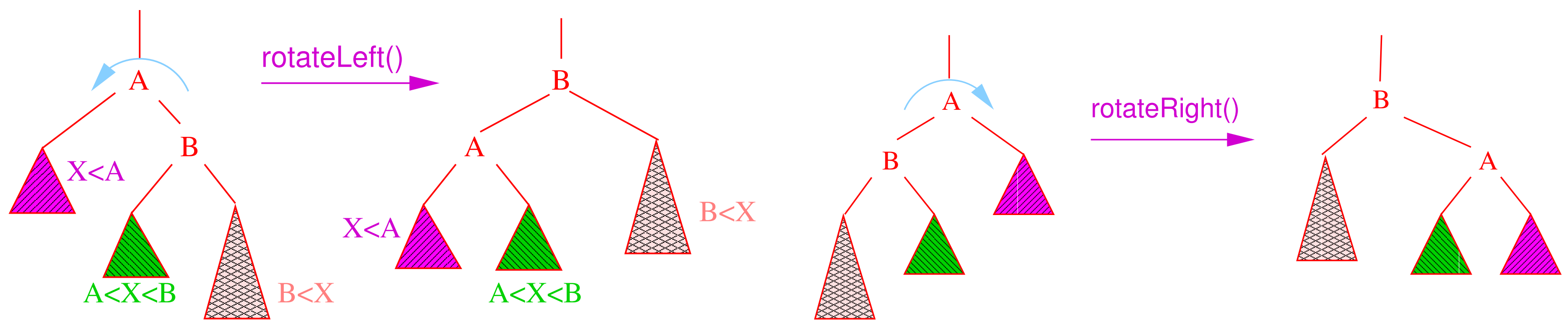


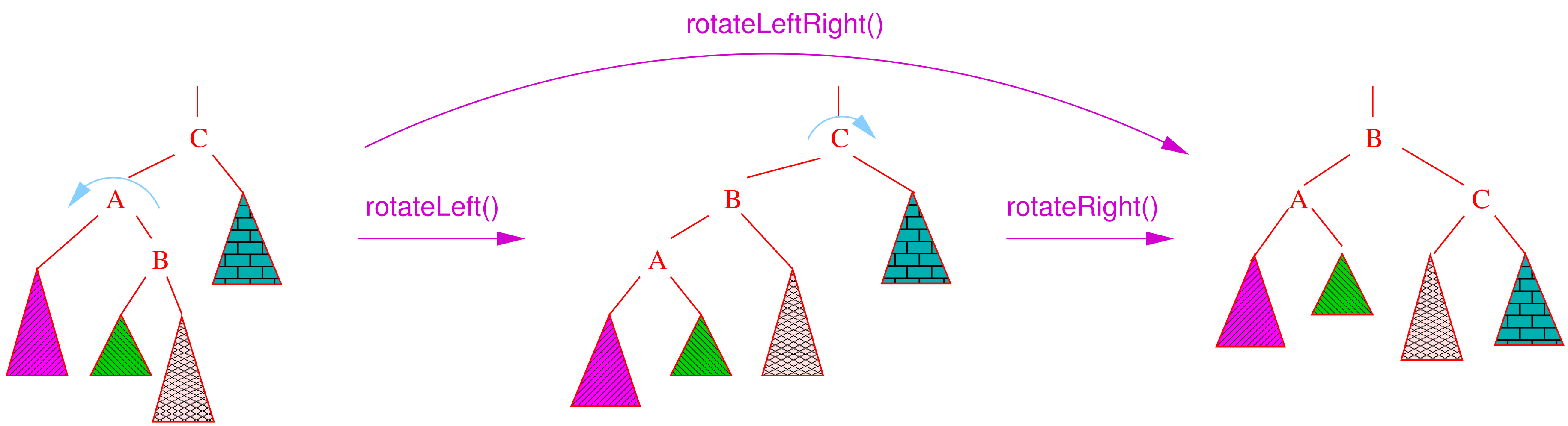
When Single Rotations Work

- Single rotations balance the tree when the unbalanced subtree is on the outside



Double Rotations

- If the unbalanced subtree is on the inside we need a double rotation

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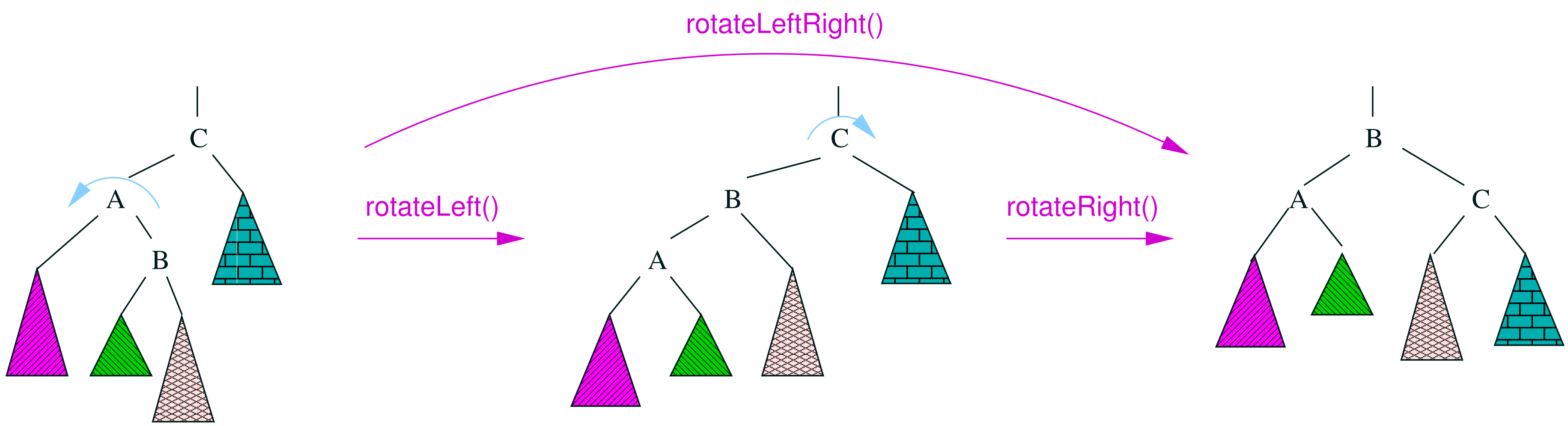


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Outline

1. Deletion
2. Balancing Trees
   - Rotations
3. **AVL**
4. Red-Black Trees
   - TreeSet
   - TreeMap

Balancing Trees

- There are different strategies for using rotations for balancing trees

- The three most popular are
  - ⋆ AVL-trees
  - ⋆ Red-black trees
  - ⋆ Splay trees

- They differ in the criteria they use for doing rotations

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- In AVL trees

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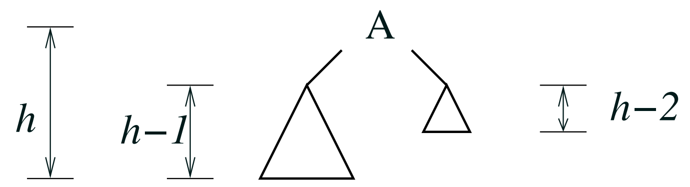
Minimum Number of Nodes

- Let $m(h)$ be the minimum number of nodes in a tree of height h
- This has to be made up of two subtrees: one of height $h-1$; and, in the worst case, one of height $h-2$
- Thus, the least number of nodes in a tree of height h is

$$m(h) = m(h-1) + m(h-2) + 1$$



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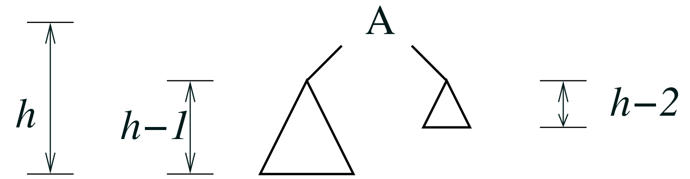

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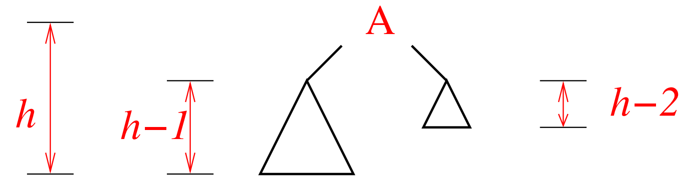

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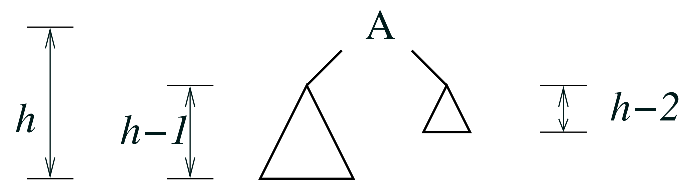

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- We have $m(h) = m(h-1) + m(h-2) + 1$ with $m(1) = 1$, $m(2) = 2$
- This gives us a sequence $1, 2, 4, 7, 12, \cdots$
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- $m(h) = m(h-1) + m(h-2) + 1$ with $m(1) = 1$, $m(2) = 2$
- We can prove by inductions, $m(h) \geq (3/2)^{h-1}$
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`add(97)`

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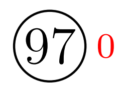

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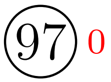
`add(0)`



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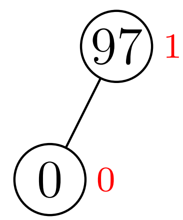

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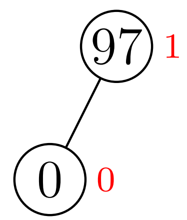
`add(77)`



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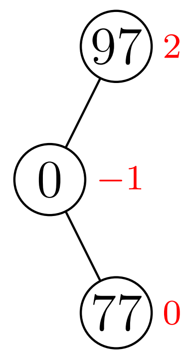
RotateLeftRight



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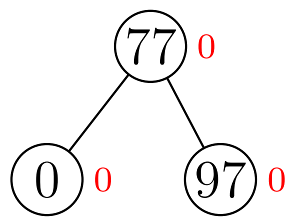

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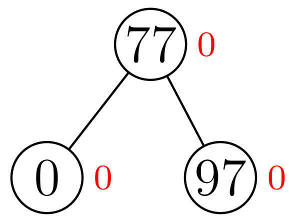
add(68)



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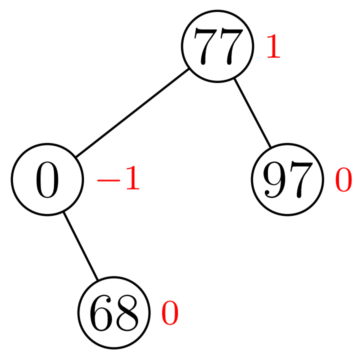

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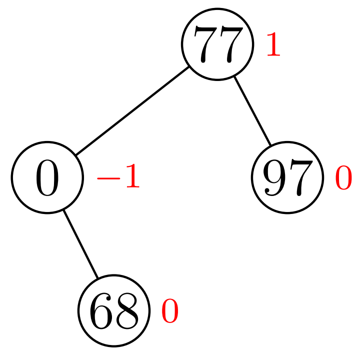
add(21)



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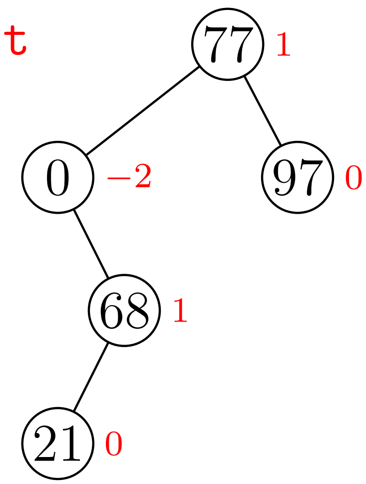

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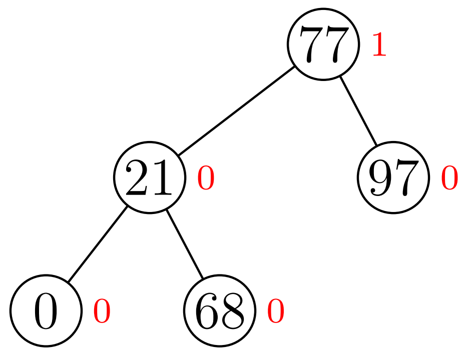

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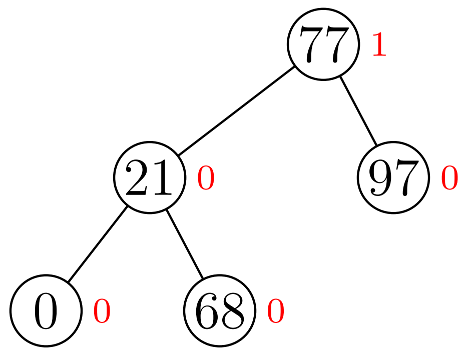

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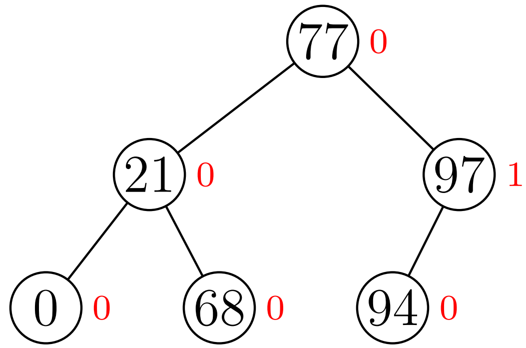

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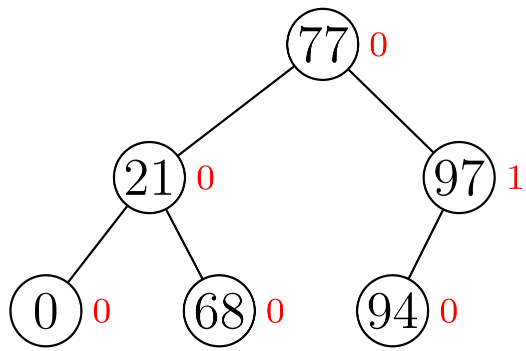

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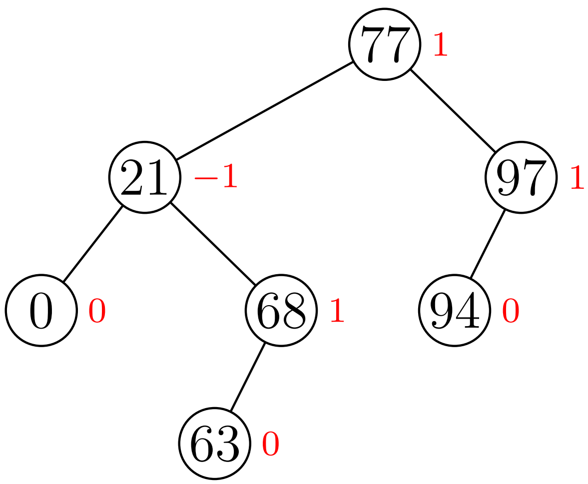

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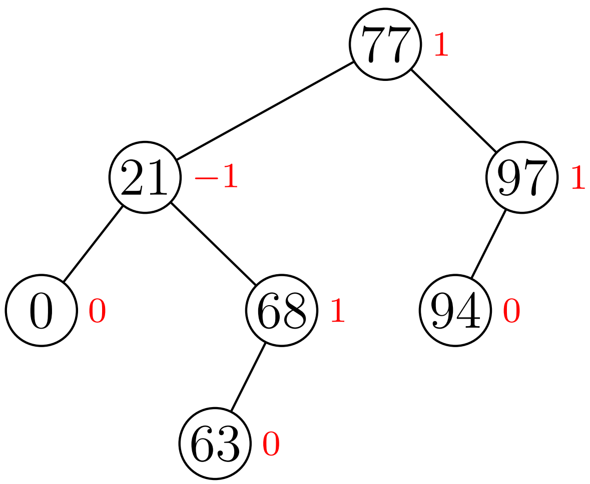
add(40)



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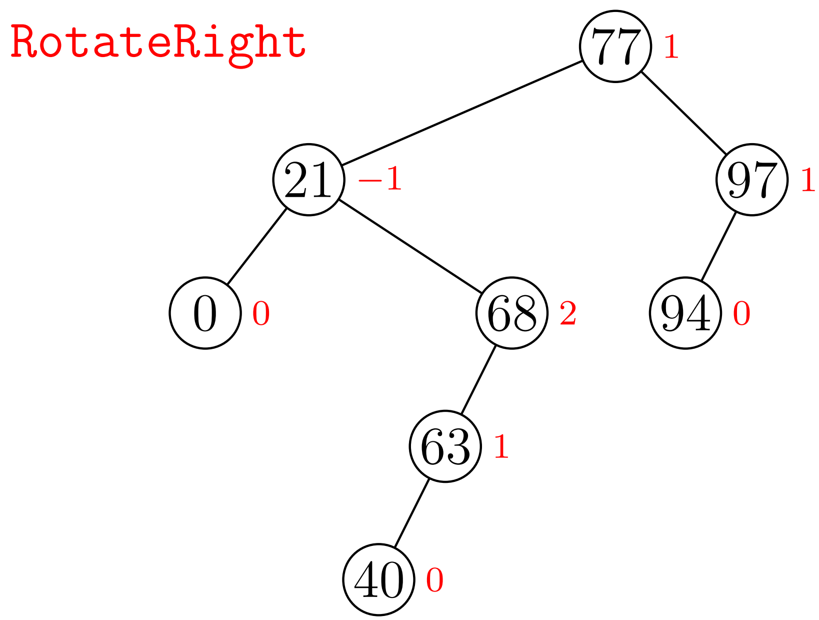

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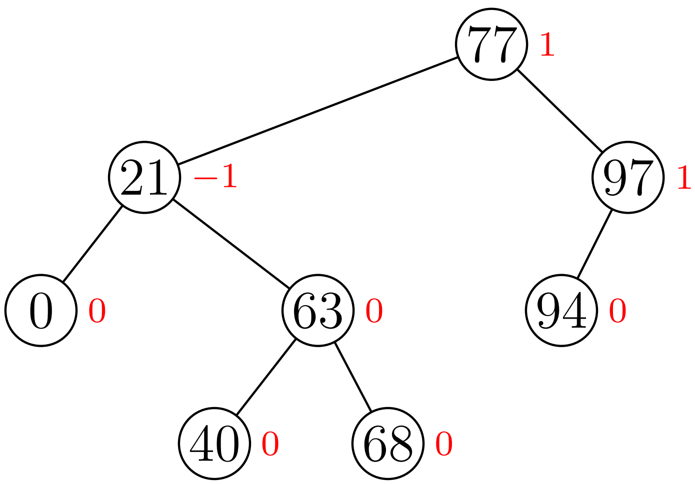

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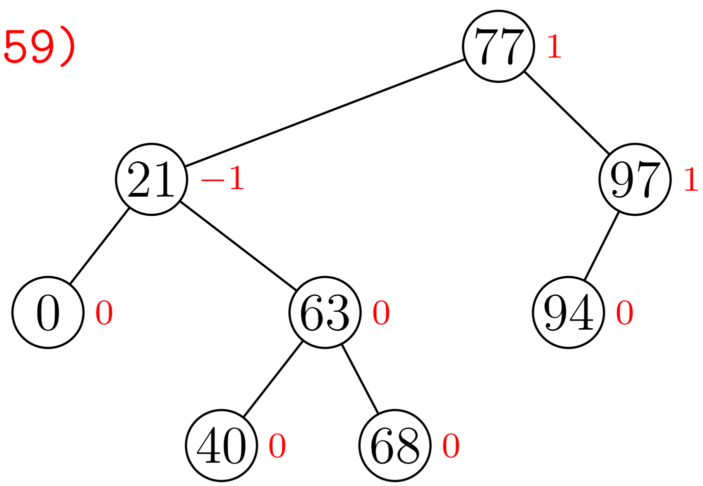
add(59)



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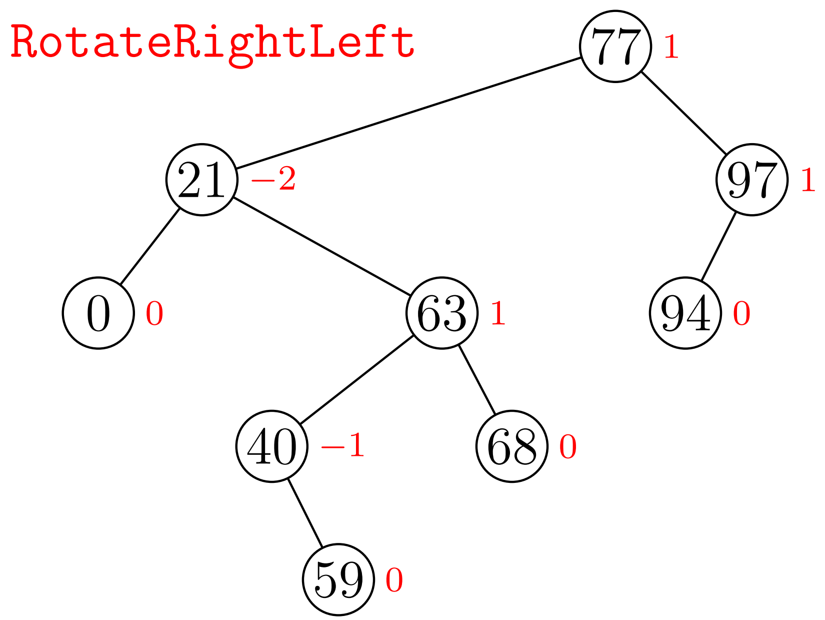

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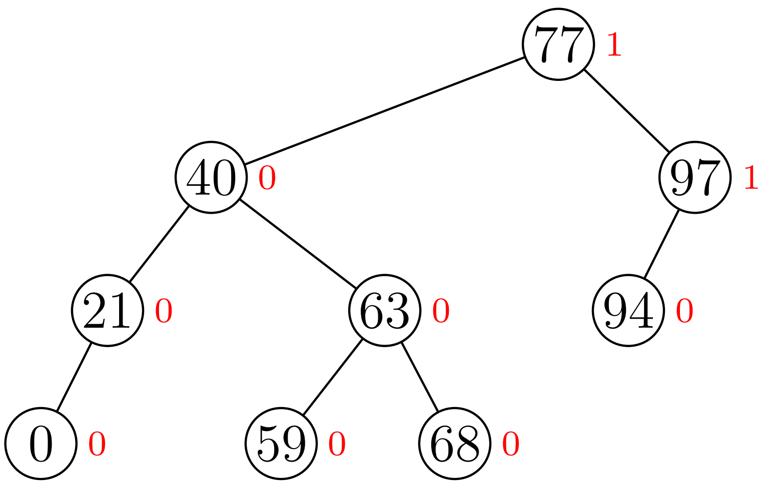

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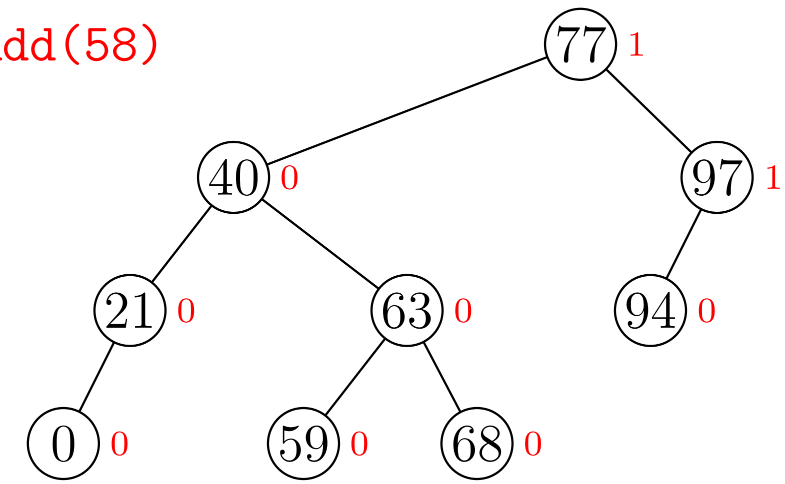

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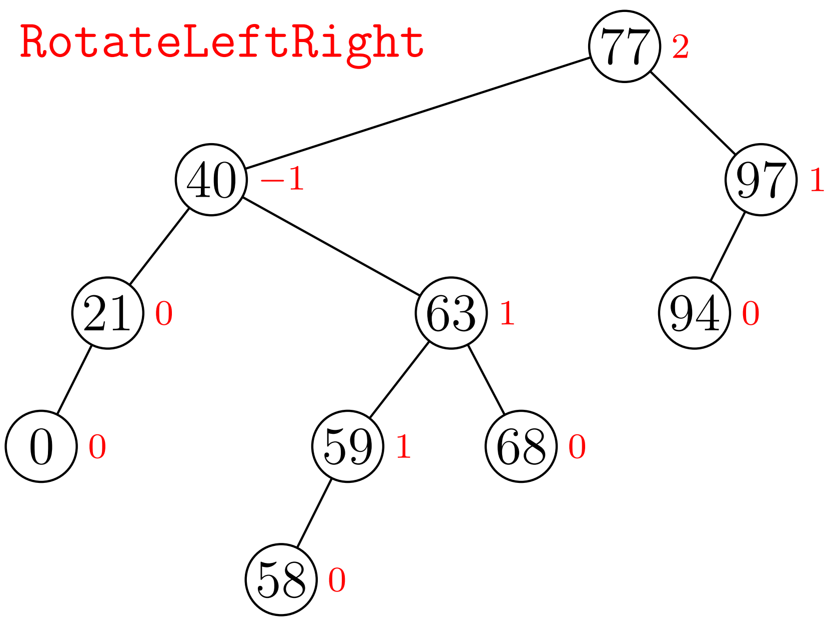

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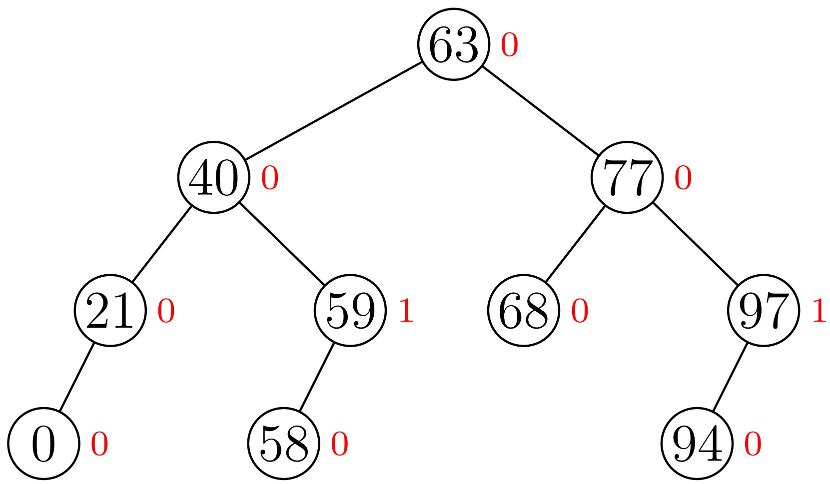

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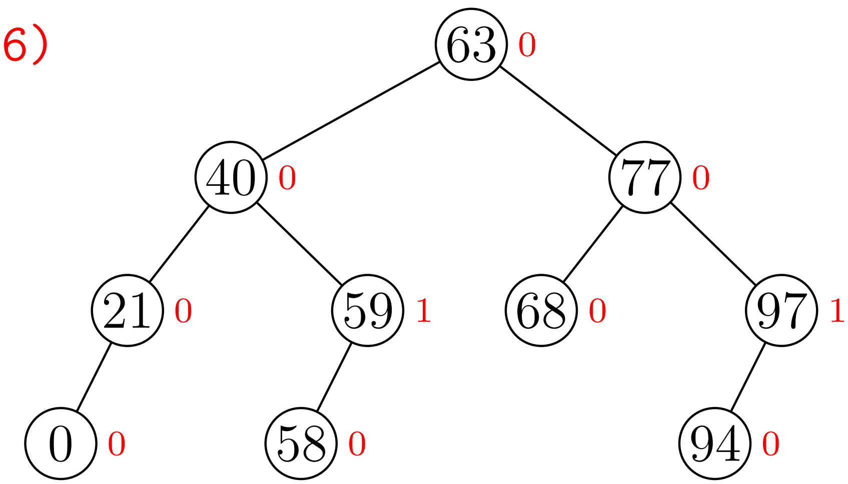

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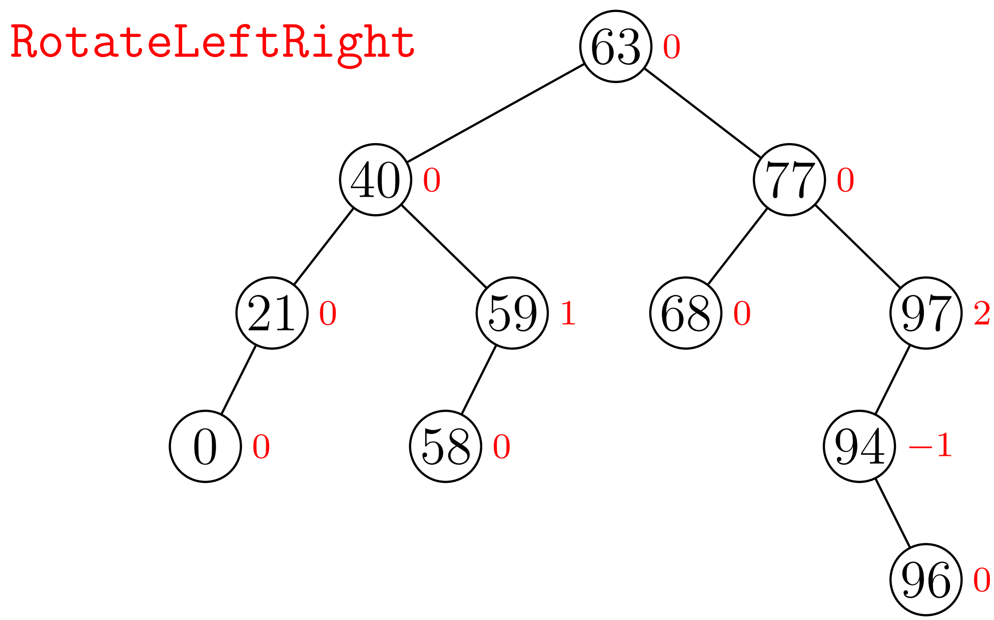

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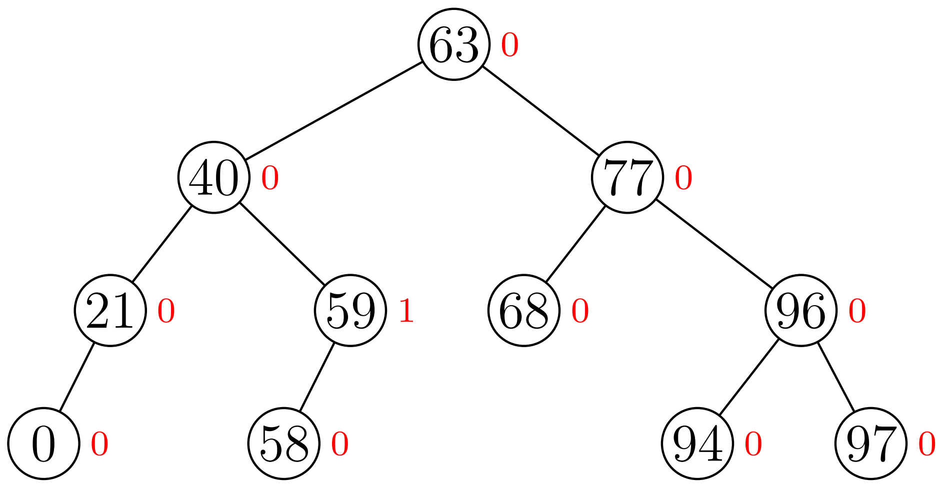

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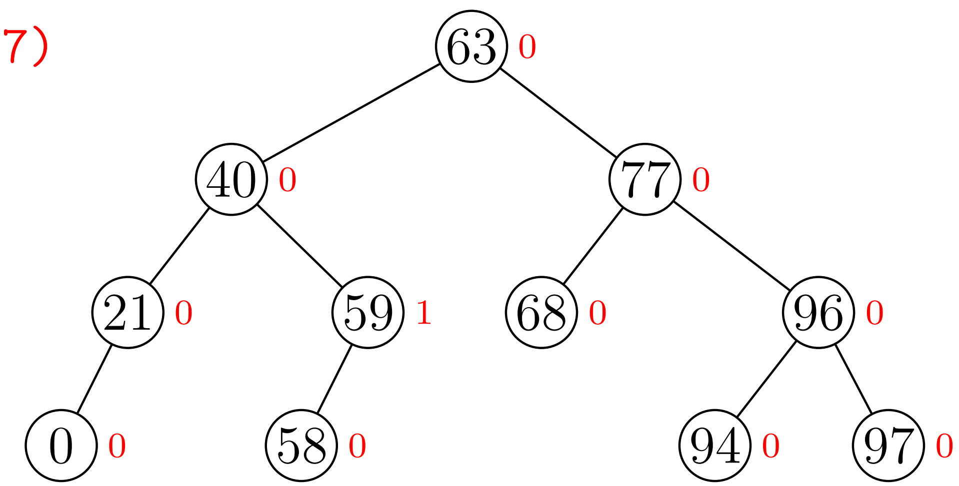

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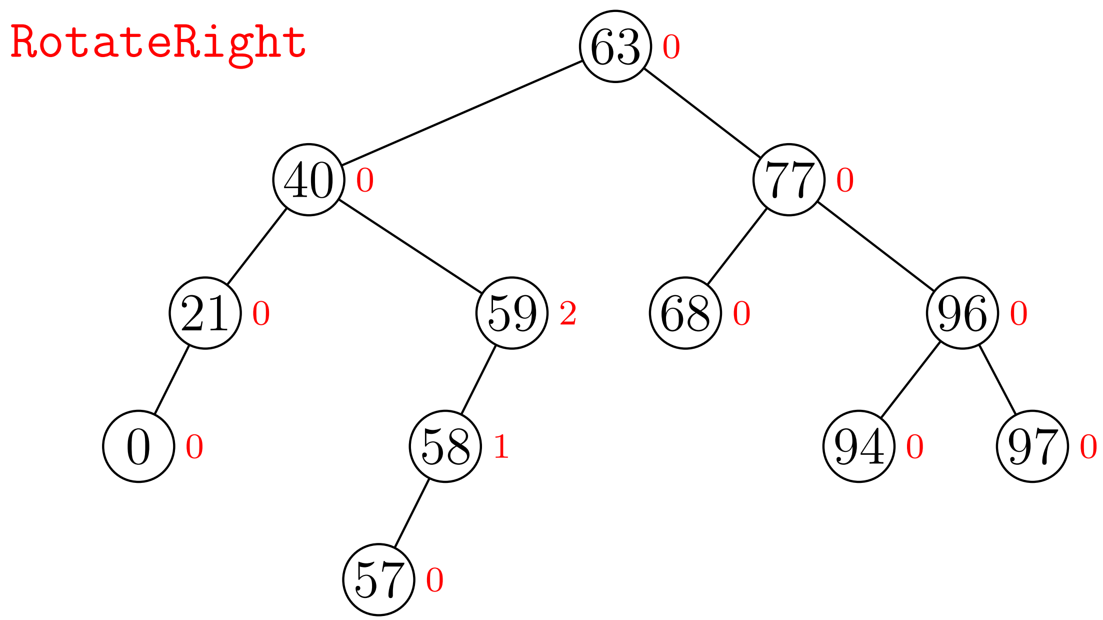

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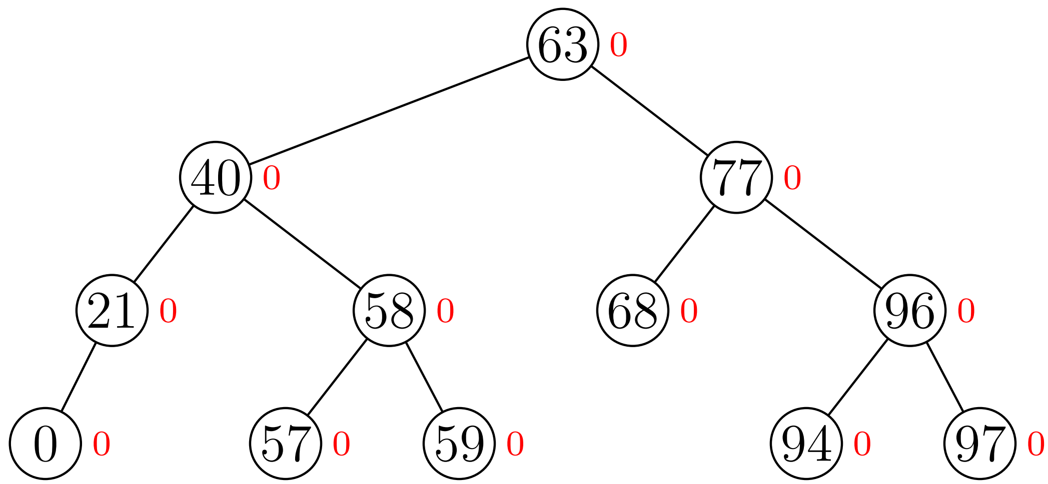

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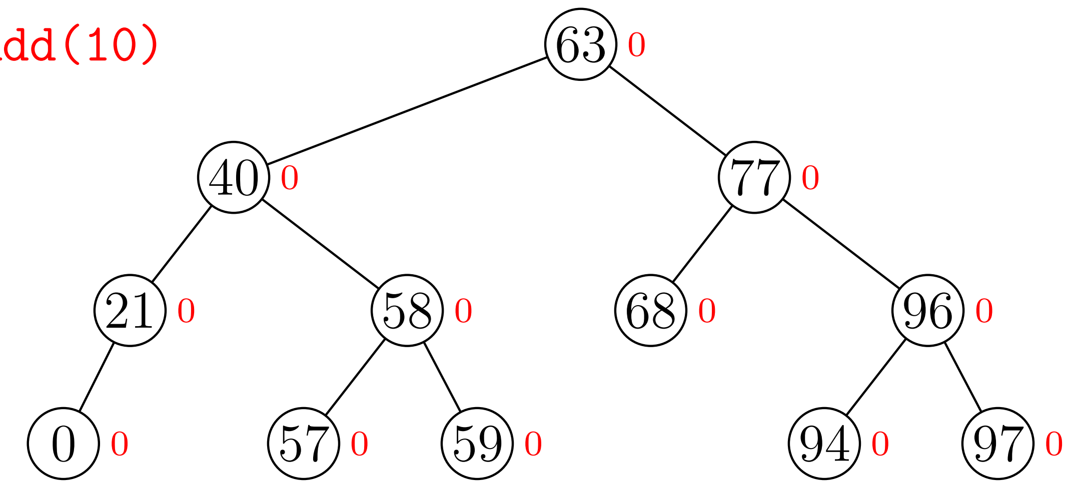
add(10)



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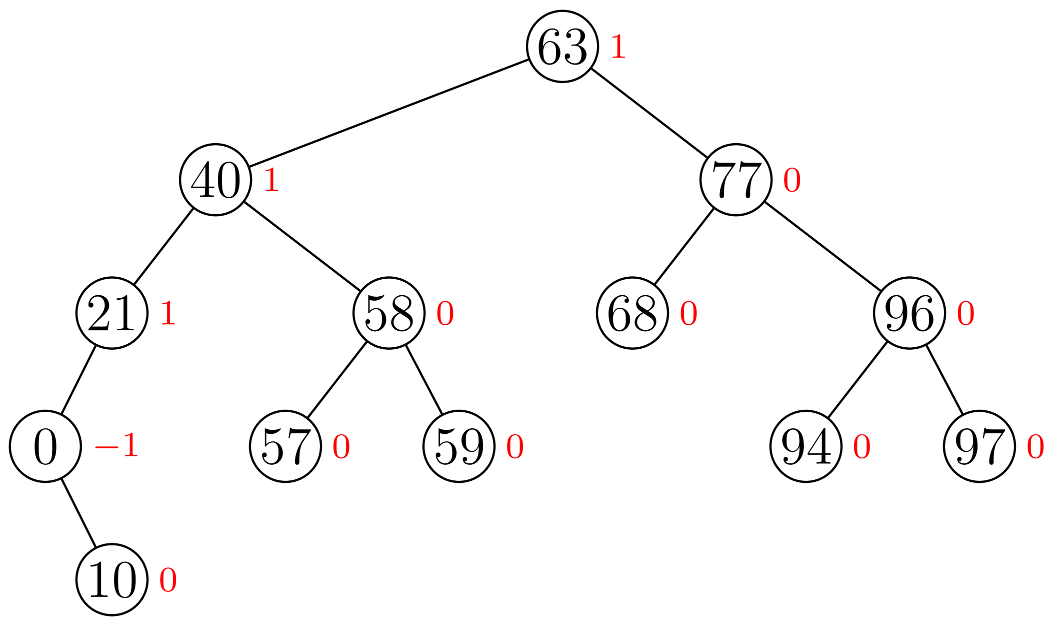

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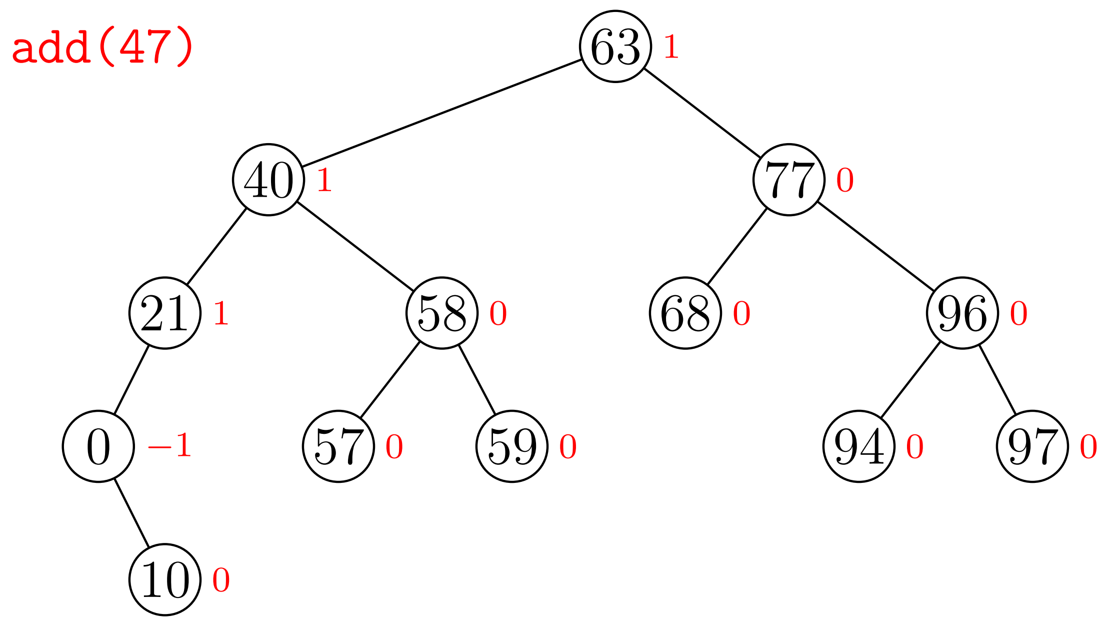

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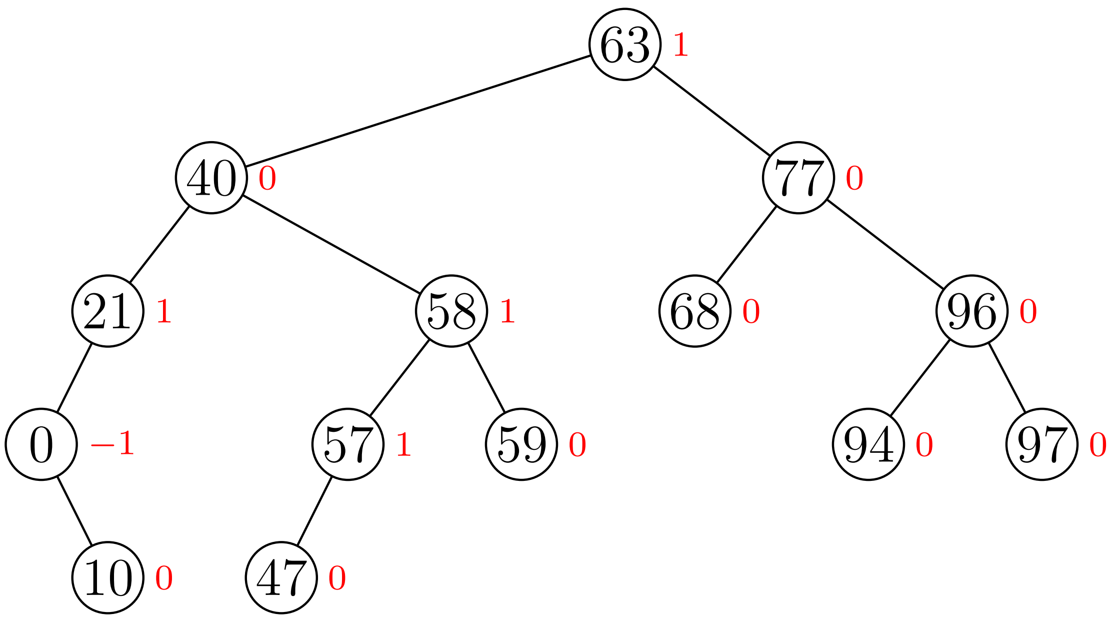

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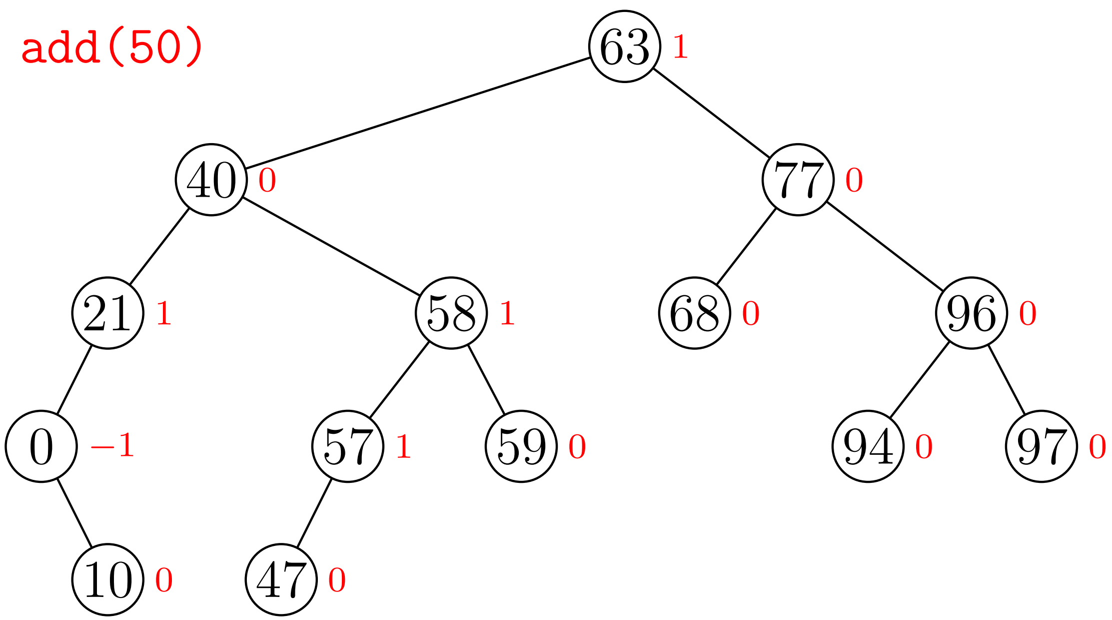

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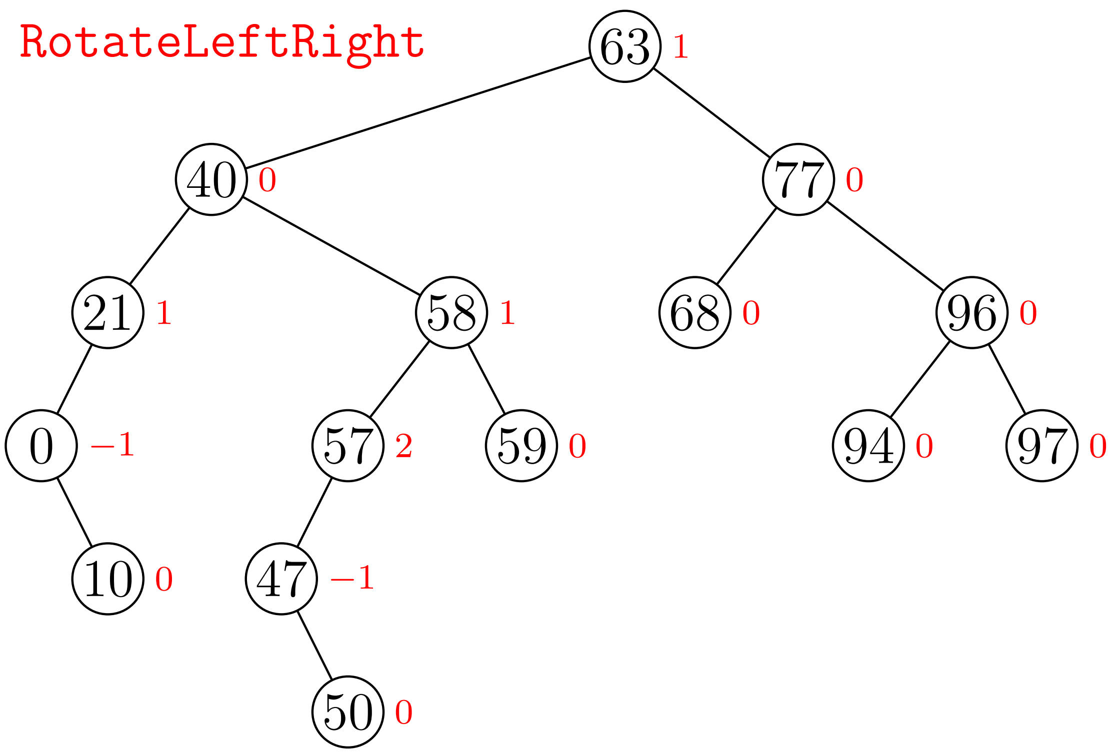
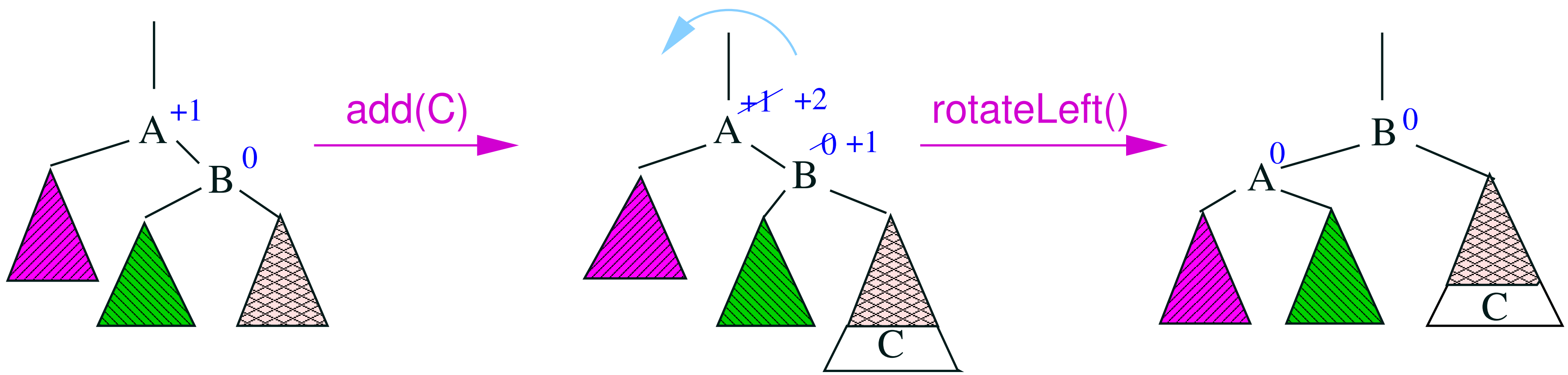


Balancing AVL Trees

- When `adding` an element to an AVL tree
  - ⋆ Find the location where it is to be inserted
  - ⋆ Iterate up through the parents re-adjusting the `balanceFactor`
  - ⋆ **If** the balance factor exceeds ± 1 `then` re-balance the tree and stop
  - ⋆ `else if` the balance factor goes to zero **then** stop



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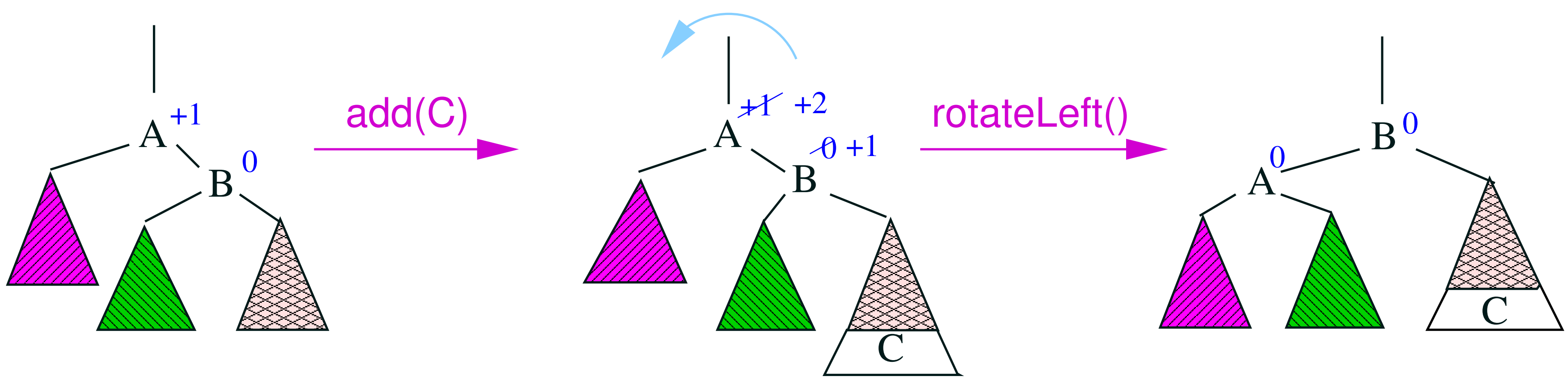

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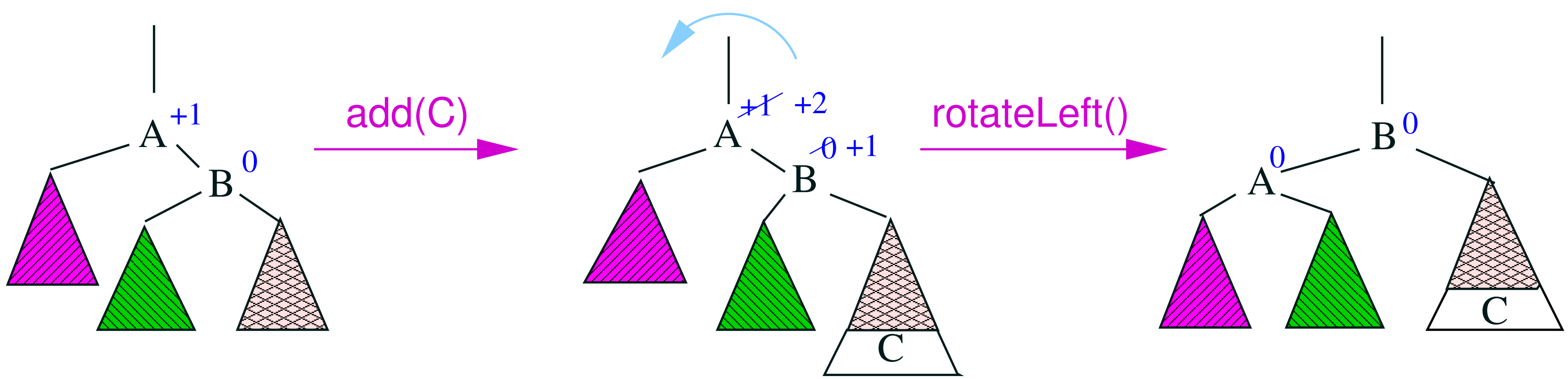

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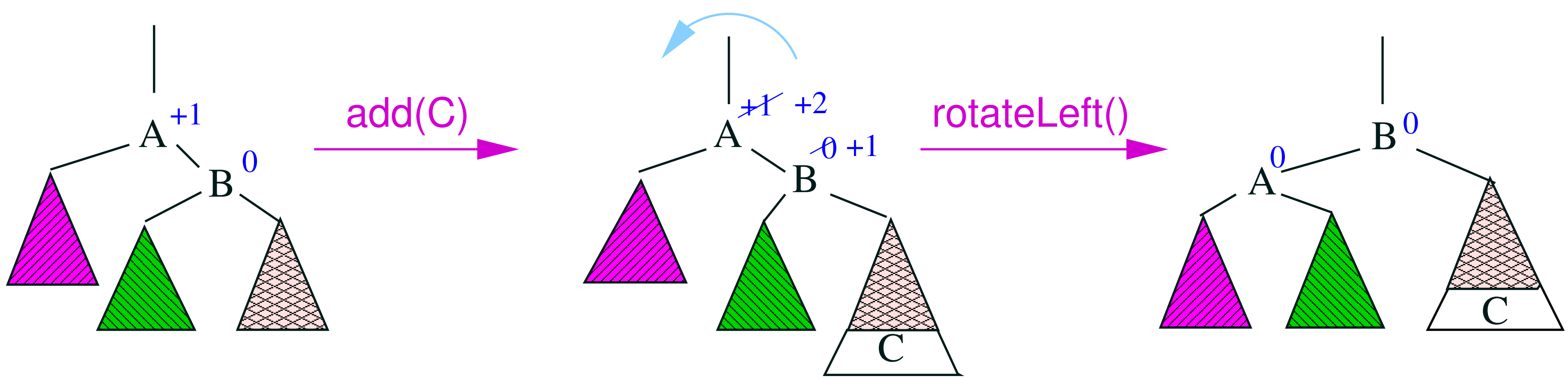

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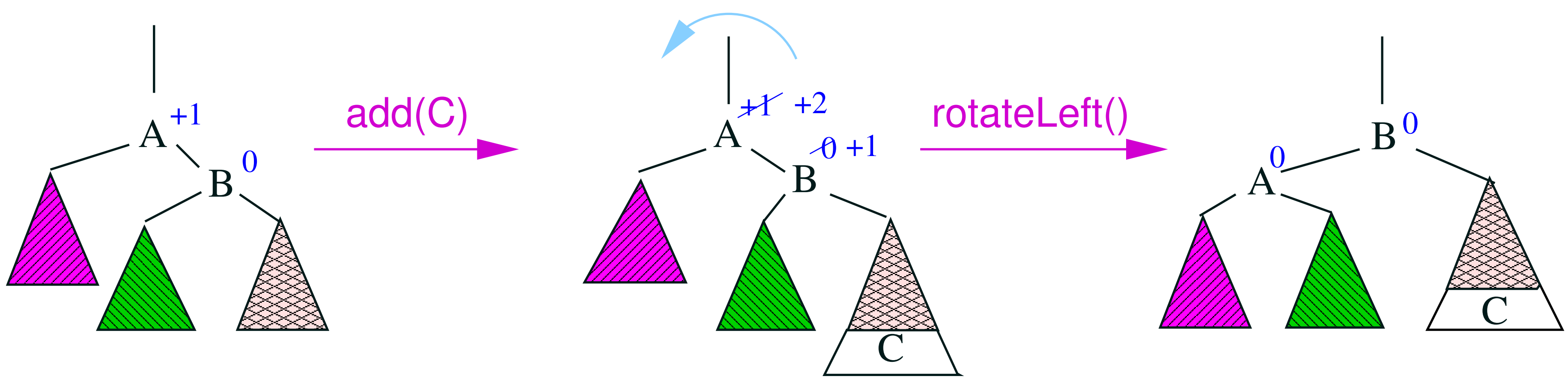

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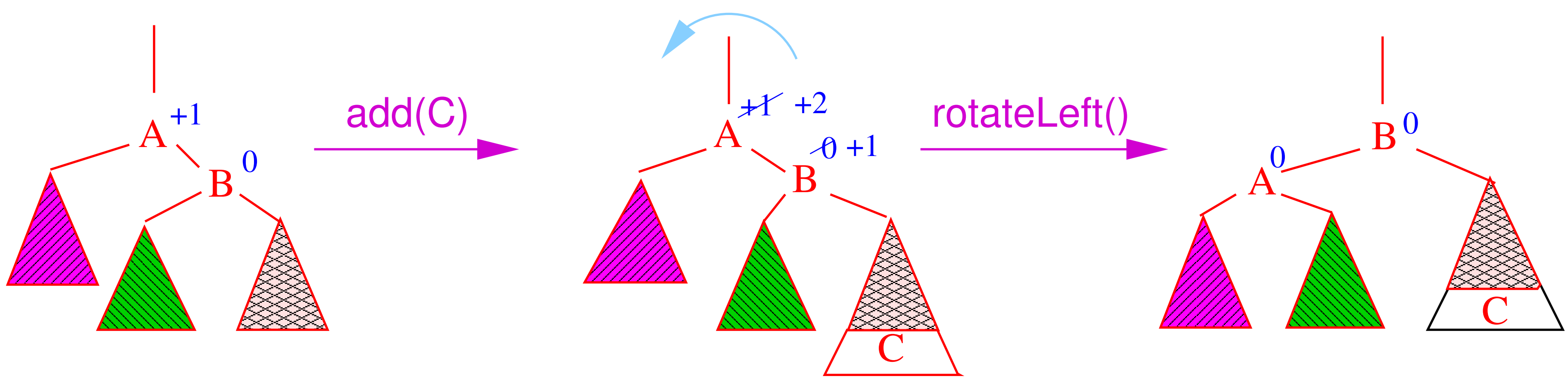

AVL Deletions

- AVL deletions are similar to AVL insertions
- One difference is that after performing a rotation the tree may still not satisfy the AVL criteria so higher levels need to be examined
- In the worst case $\Theta(\log(n))$ rotations may be necessary
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- Insertion, deletion and search in AVL trees are, at worst, $\Theta(\log(n))$
- The height of an average AVL tree is $1.44 \log_2(n)$
- The height of an average binary search tree is $2.1 \log_2(n)$
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Outline

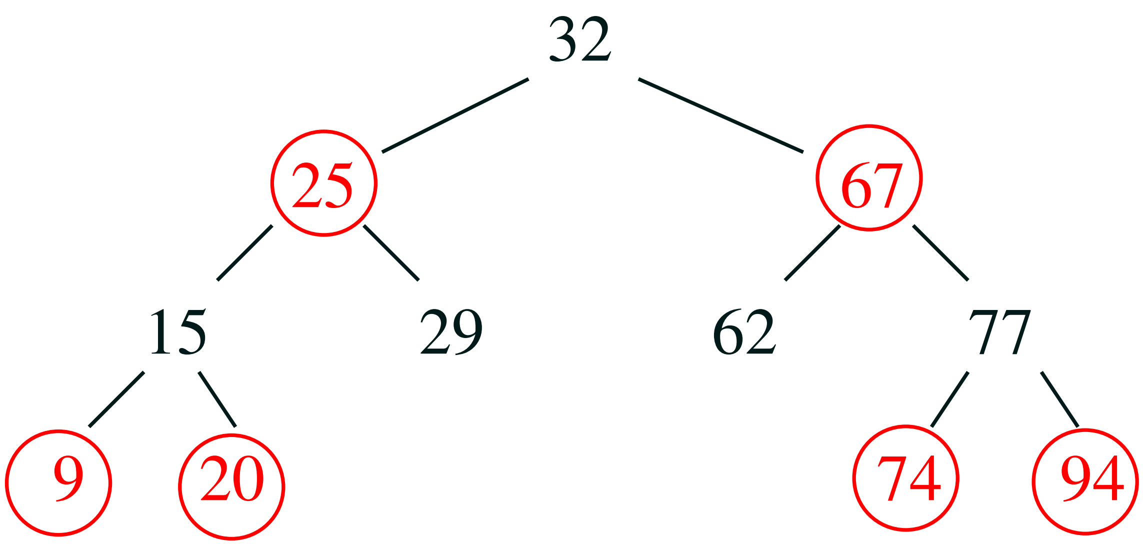
1. Deletion
2. Balancing Trees
   - Rotations
3. AVL
4. **Red-Black Trees**
   - TreeSet
   - TreeMap

Red-Black Trees

- Red-black trees are another strategy for balancing trees
- Nodes are either *red* or *black*
- Two rules are imposed

  Red Rule: the children of a red node must be black

  Black Rule: Every path from a node to its descendant leaves must have the same number of black nodes



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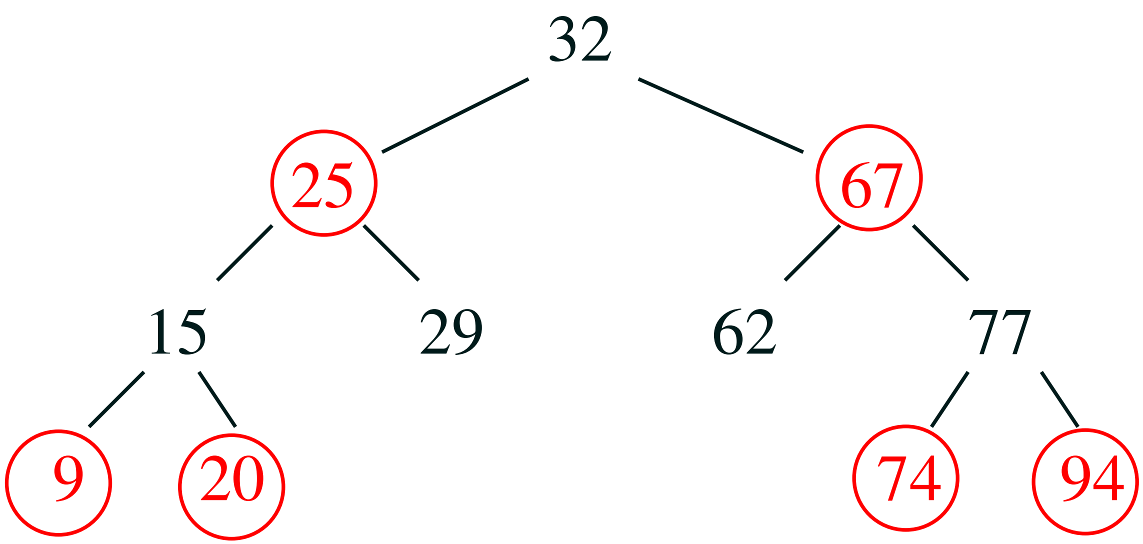

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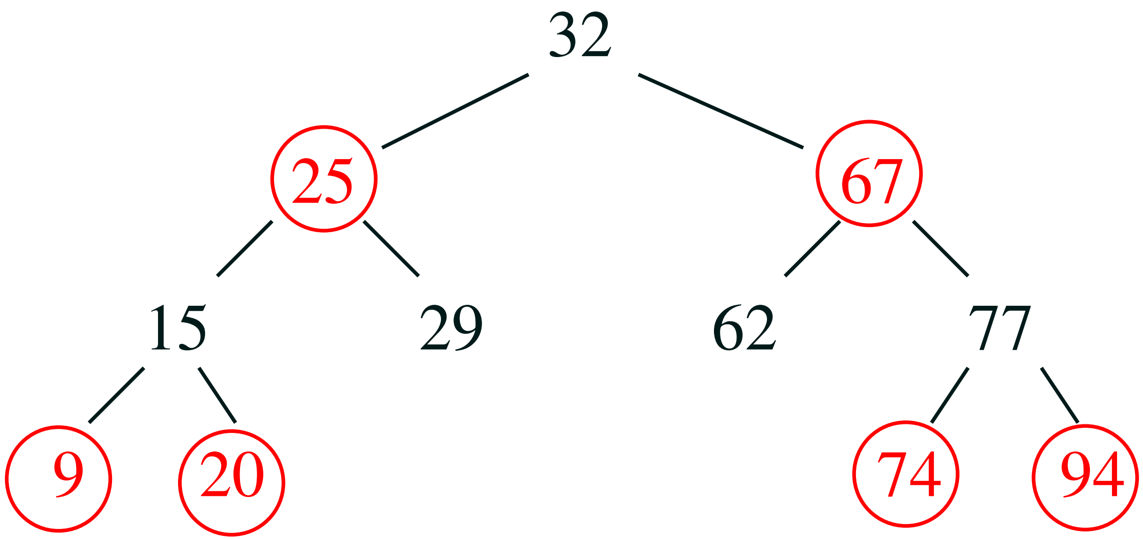

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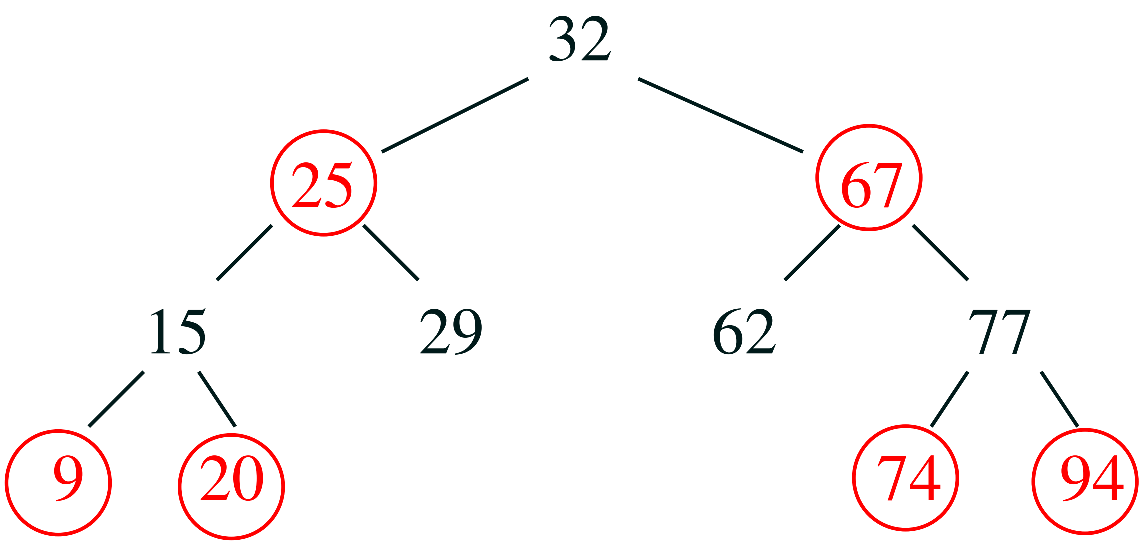

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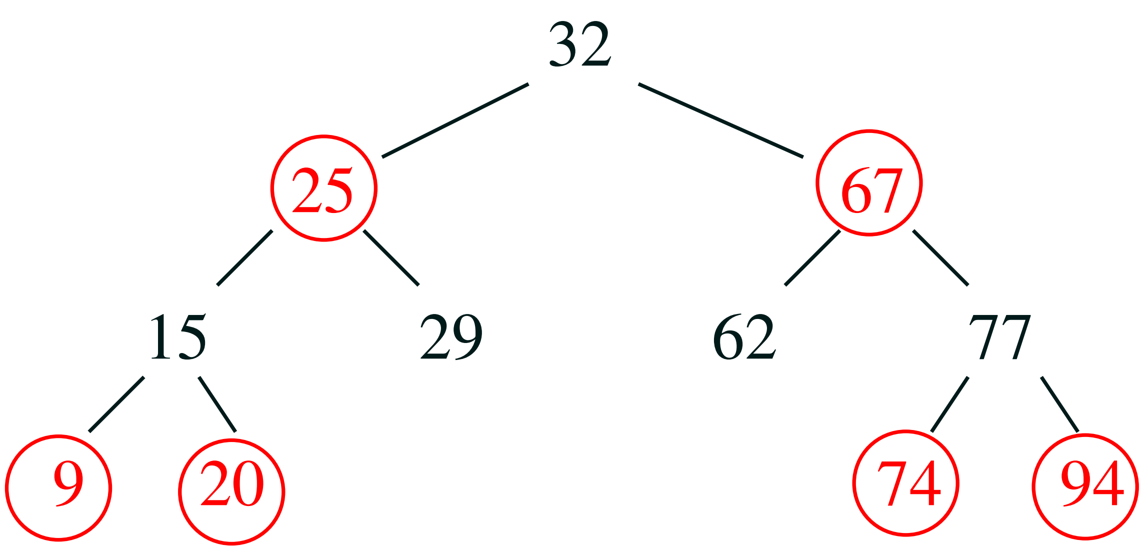

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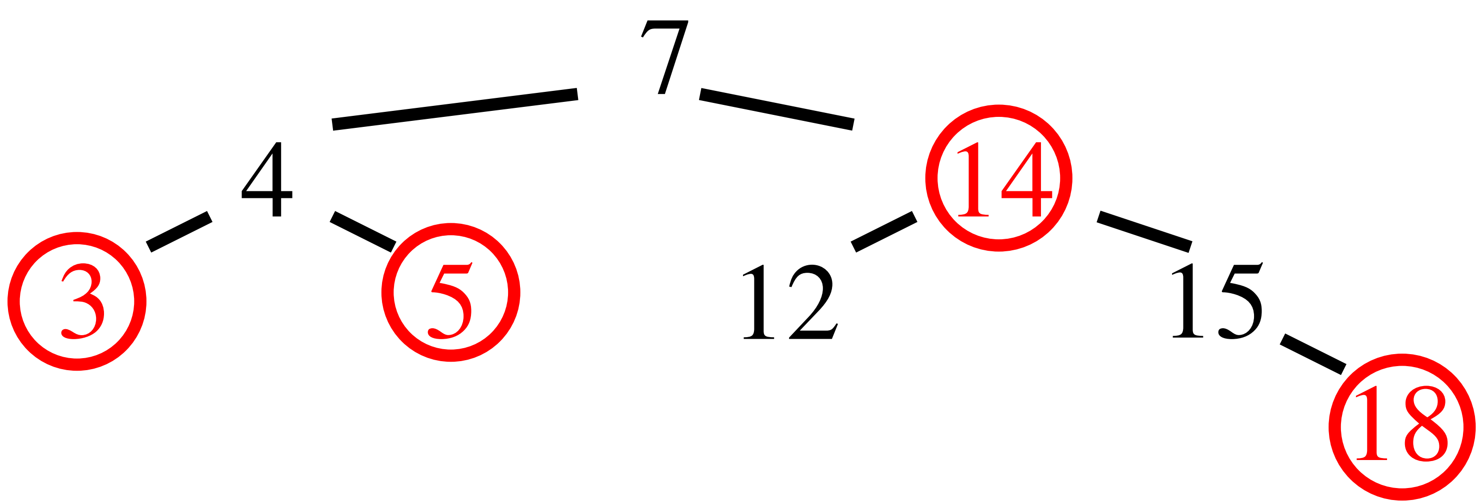

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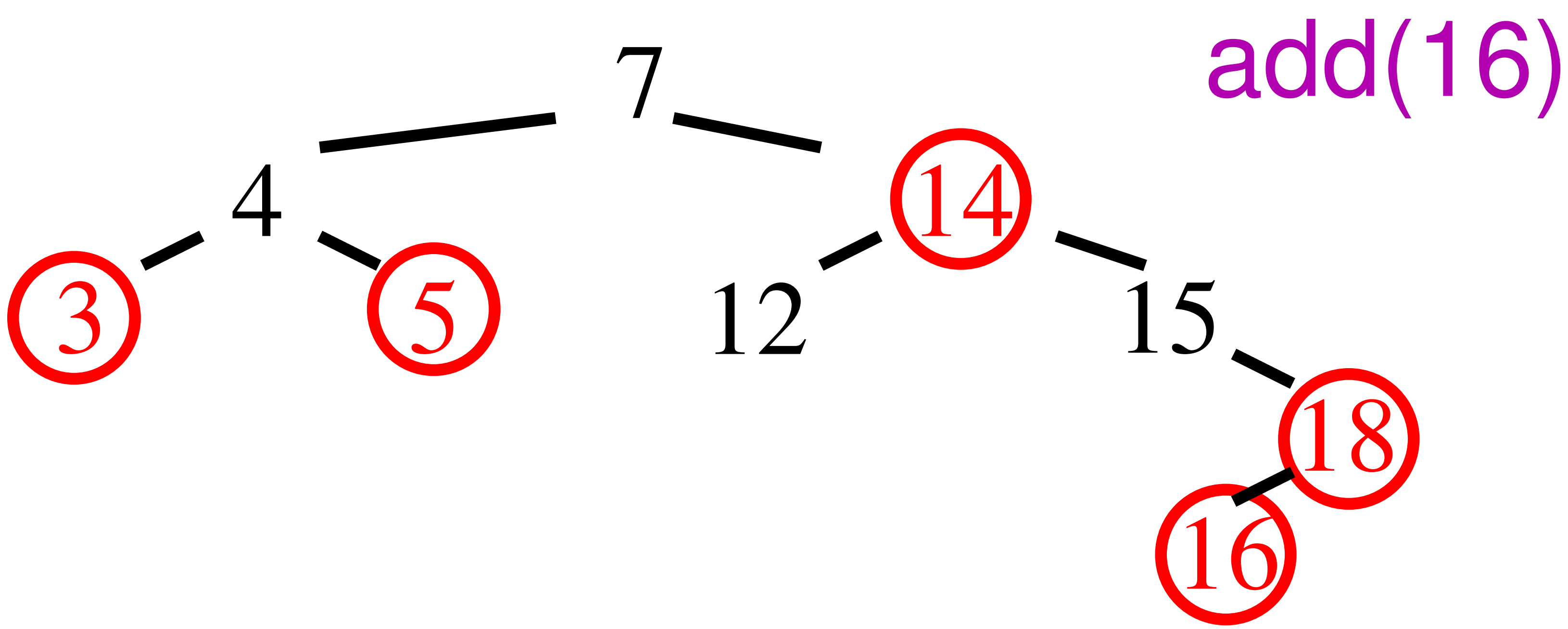

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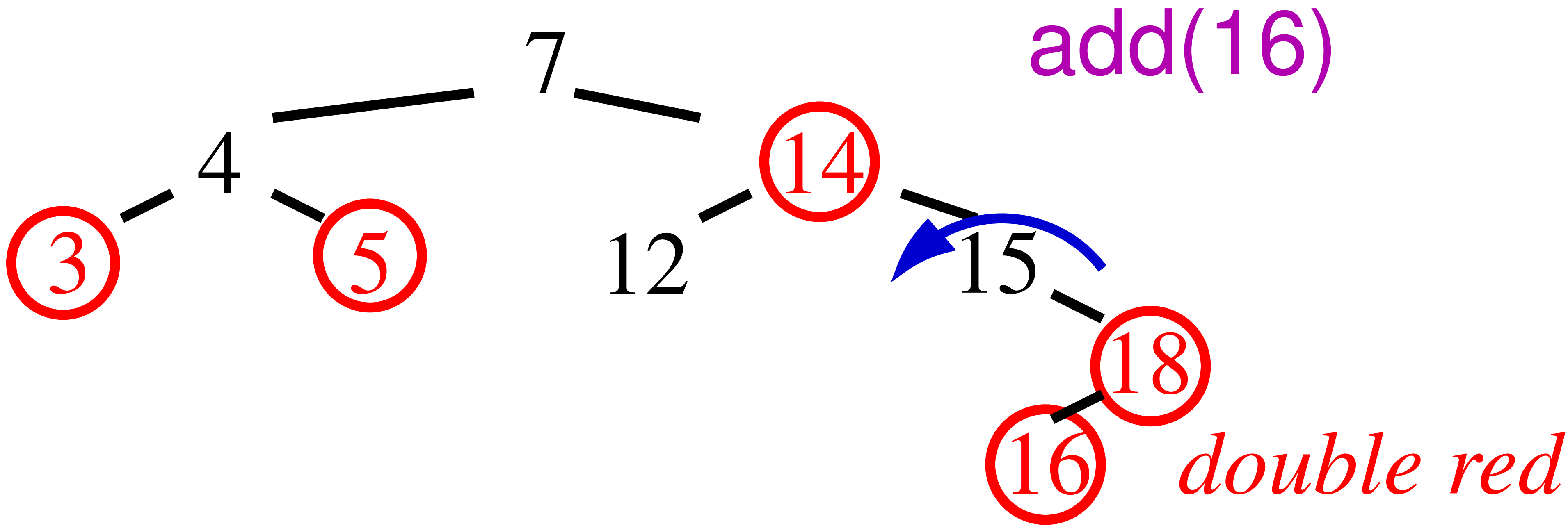

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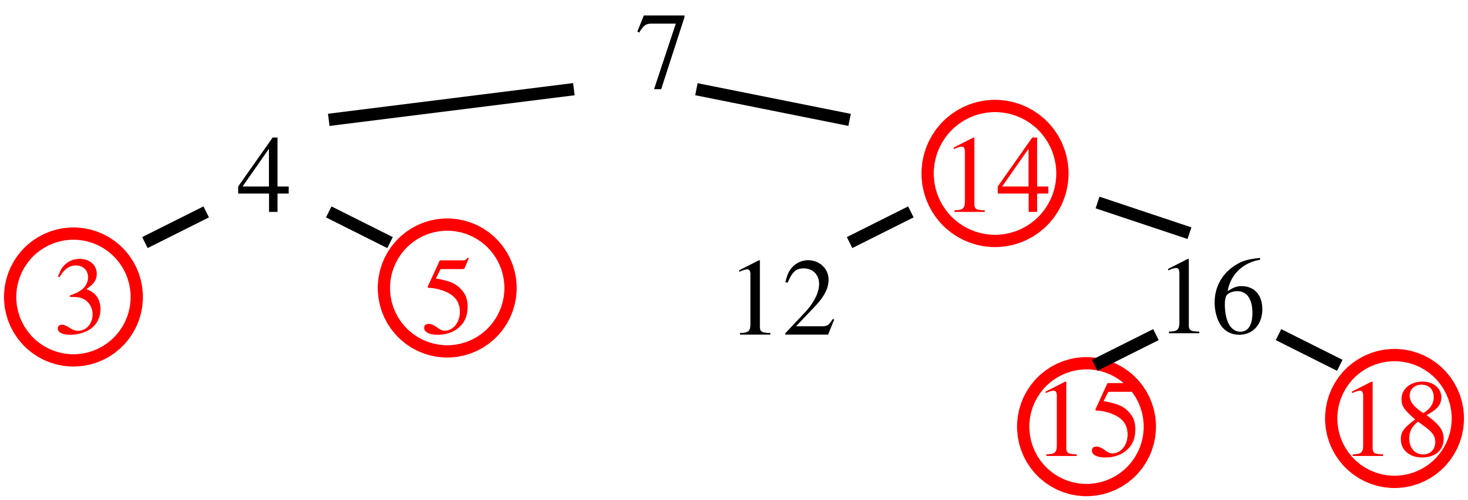

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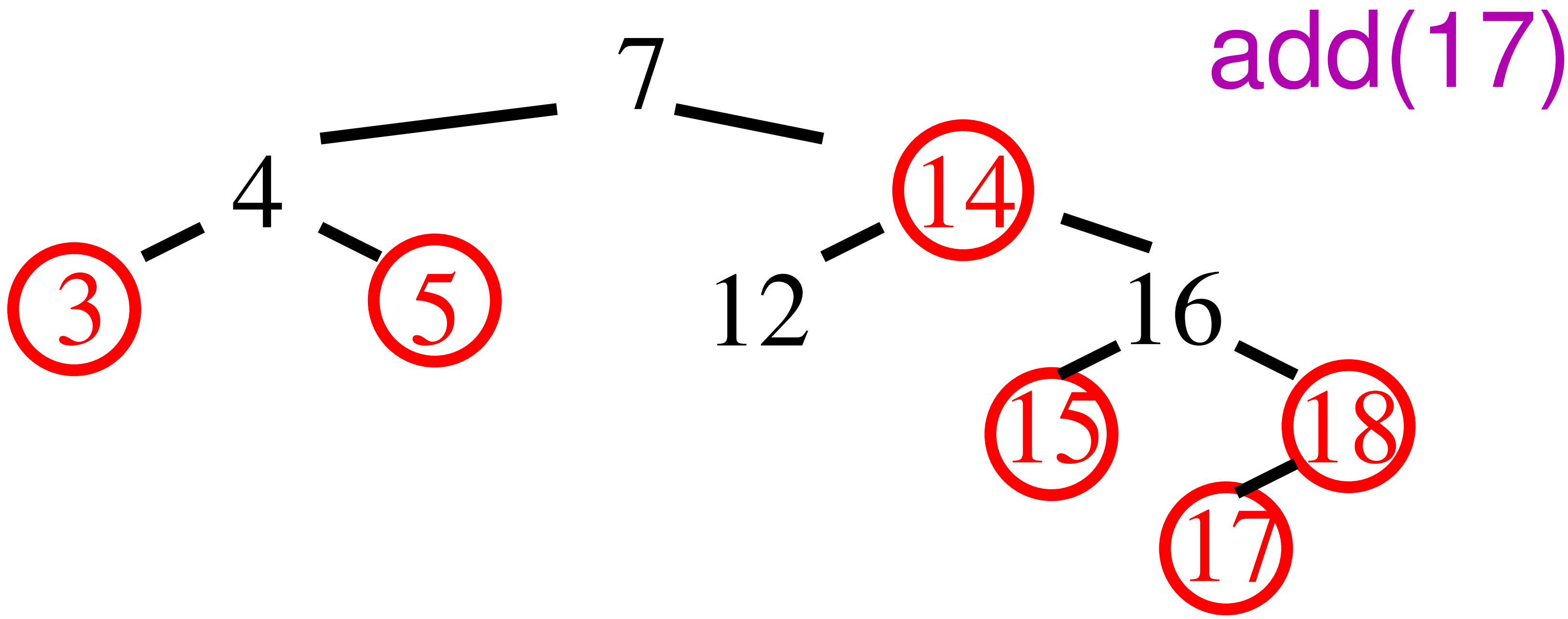

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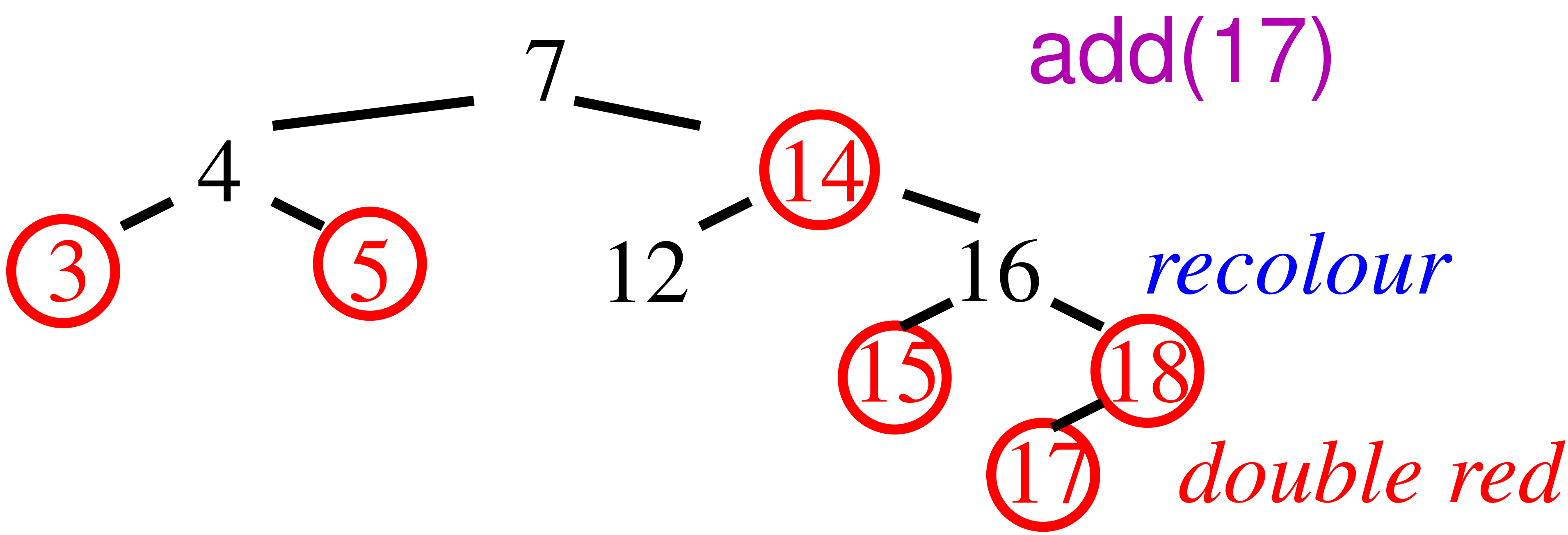

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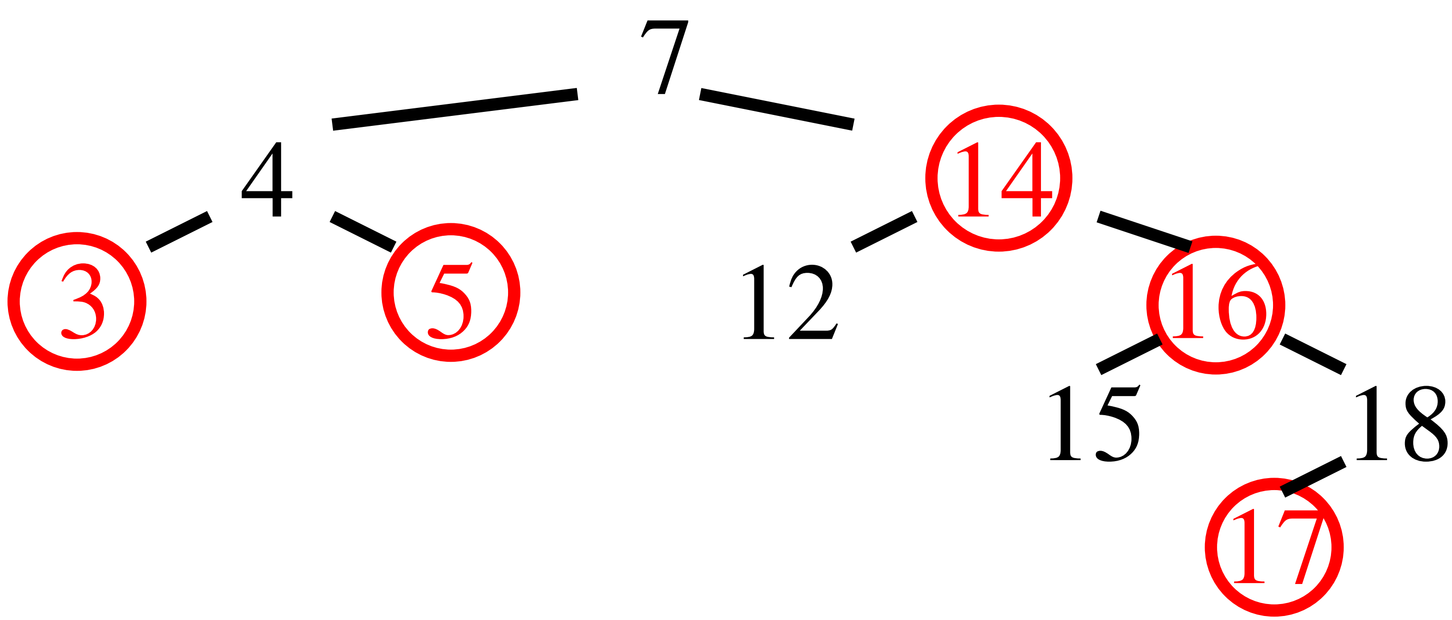

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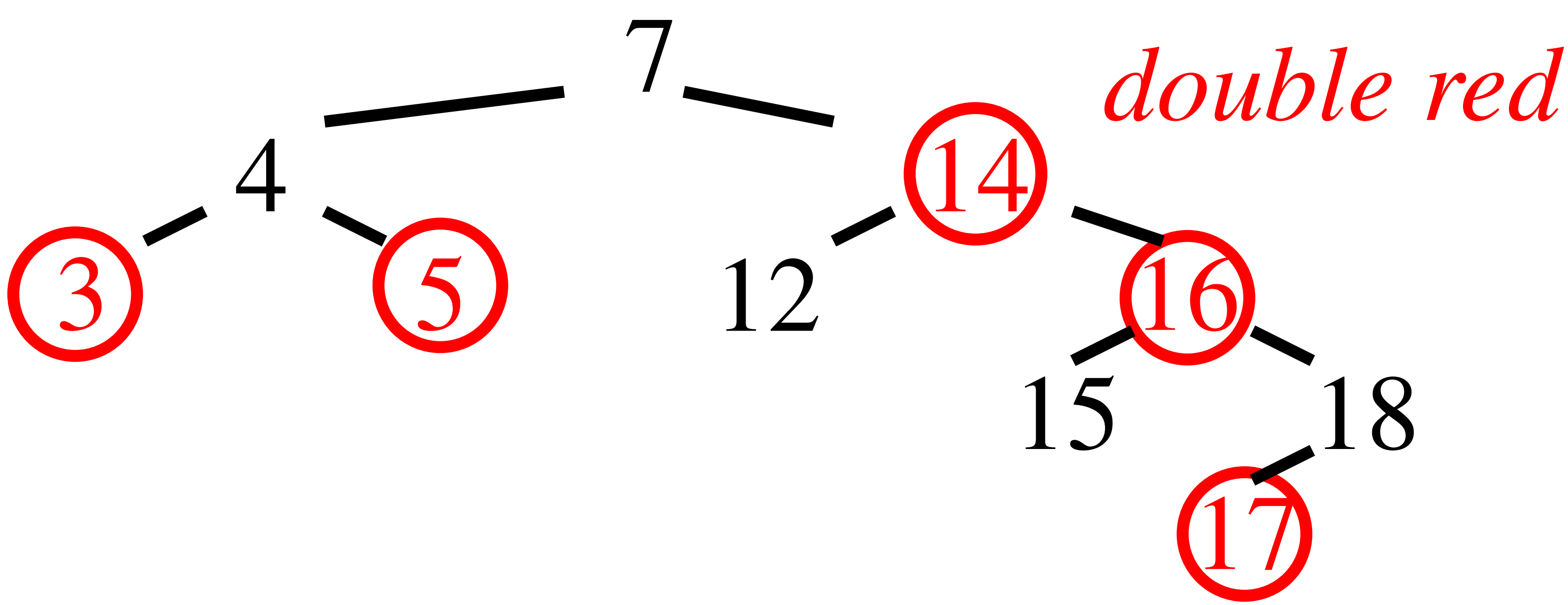

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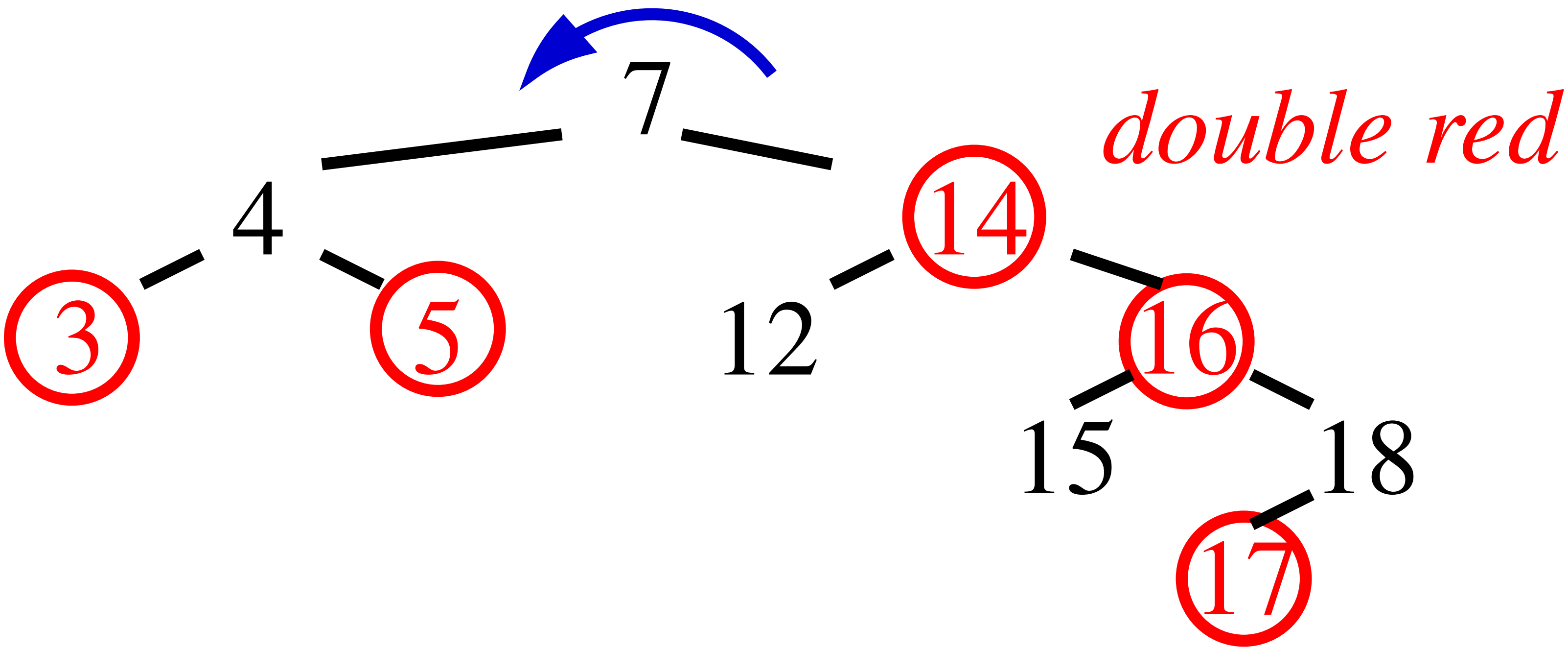

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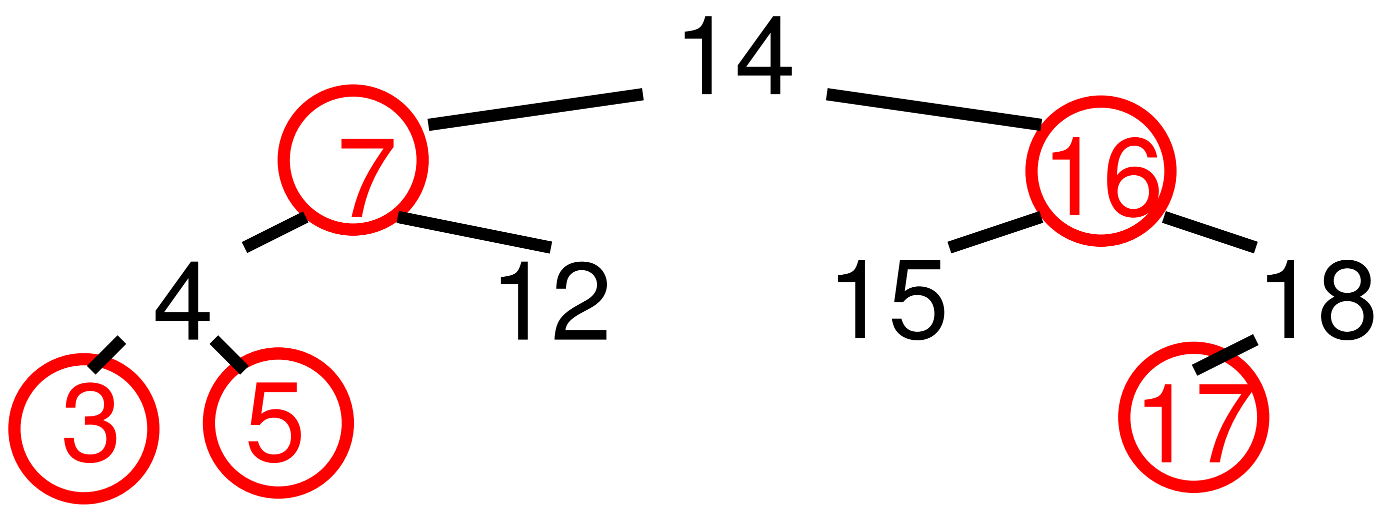

Performance of Red-Black Trees

- Red-black trees are slightly more complicated to code than AVL trees
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Maps

- One major abstract data type (ADT) we have not encountered is the map class
- The map class `std:map<Key,V>` contain key-value pairs `pair<Key,V>`
  - ⋆ The first element of type `Key` is the **key**
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- Maps work as content addressable arrays

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map<string, int> students;
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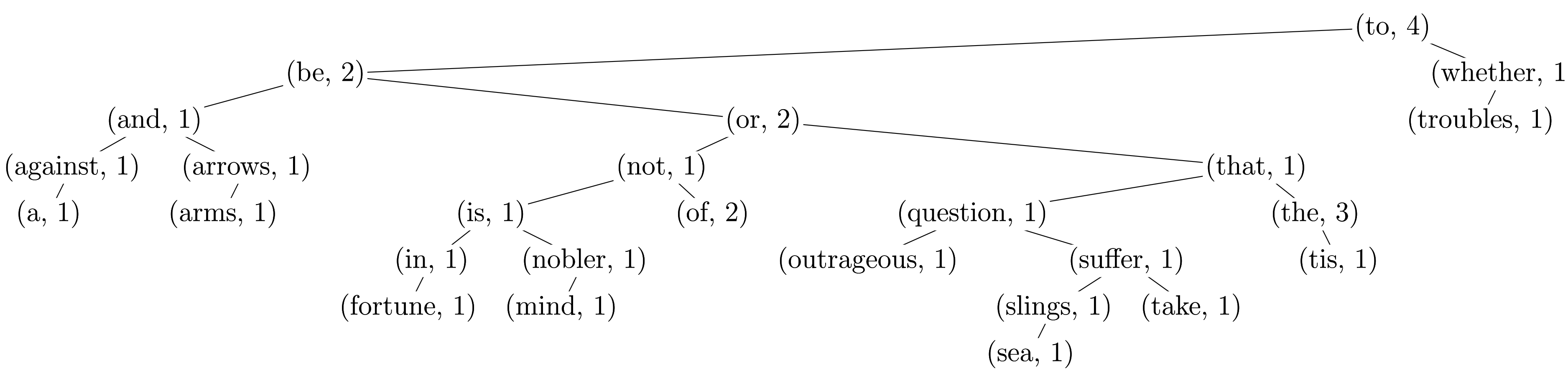
- We can count words using the key for words and value to count



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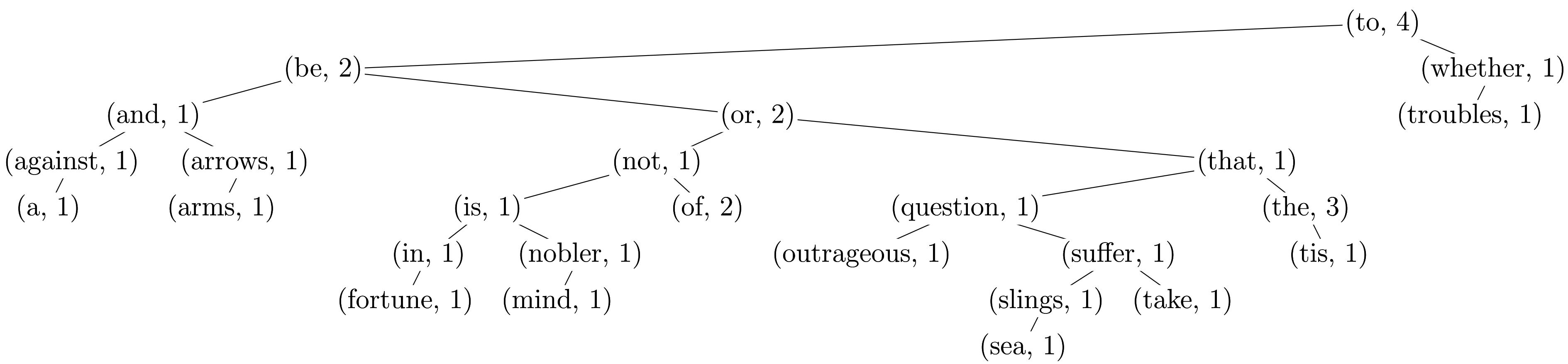

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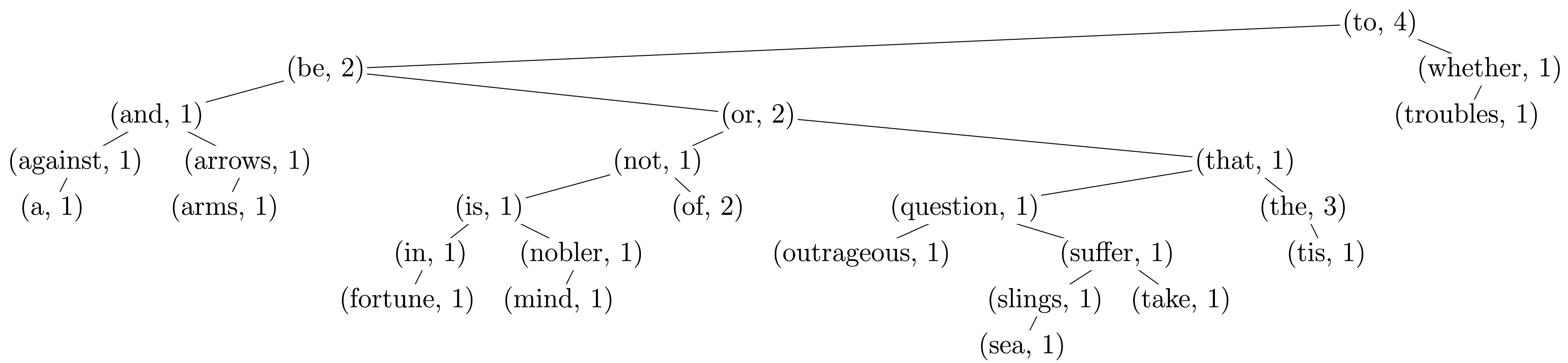

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- There are different strategies for deciding when to rotate including
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