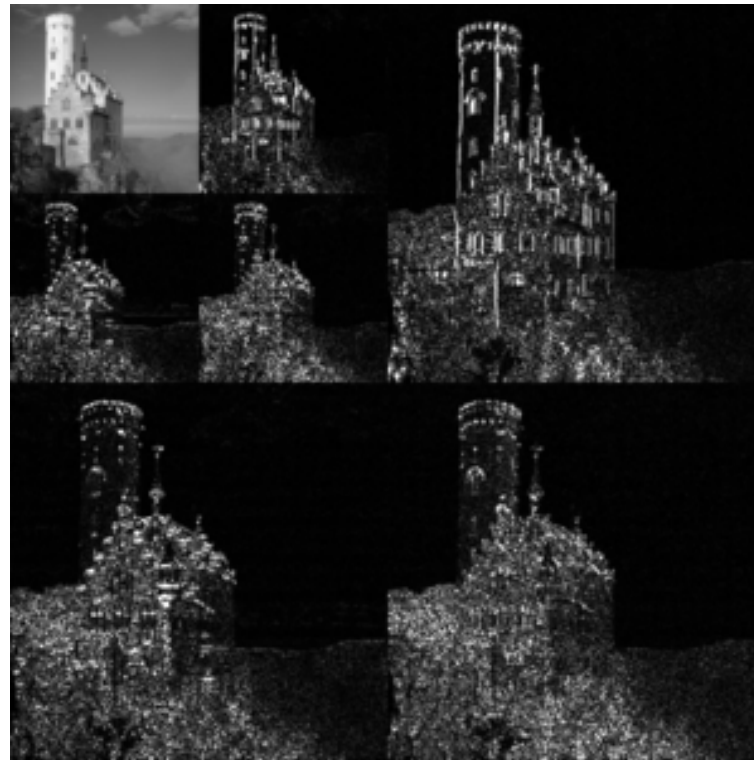


Algorithms and Analysis

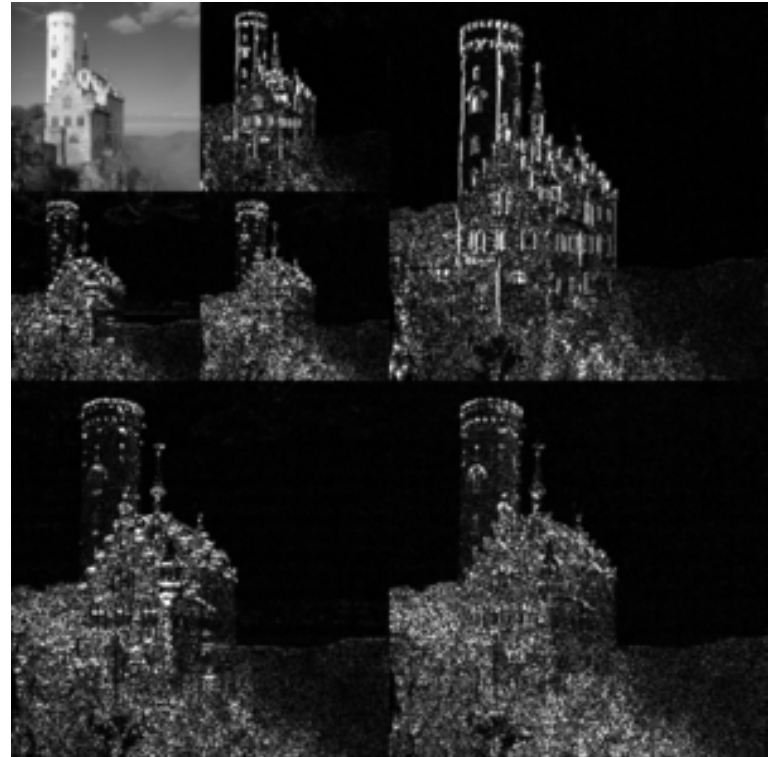
Lesson 24: *Use Smart Encoding!*



File compression, Huffman codes, wavelets

Outline

1. **Huffman codes**
2. Wavelets



File Compression

- File compression comes in two varieties
 - ★ Exact compression (e.g. zip used on text files)
 - ★ Lossy compression (e.g. jpeg used on pictures—jpeg can also be loss-less or exact)■
- Good exact compression (also known as entropy encodings) can give a compression ratio around 25%■
- Lossy compression can give a compression ratio from 10-1%■
- Important for saving space, but lossy compression can also be used for noise reduction■
- Even used for plagiarism detection!■

Entropy Encoding

- Exact encodings use the principle of using short words for frequently occurring sequences (symbols) and longer words for sequences that occur less often■
- Claude Shannon showed that for an alphabet of n symbols where the probability of symbol i occurring is p_i no code exists which can transmit information in less than

$$-\sum_{i=1}^n p_i \log_2(p_i) \text{ bits}$$

asymptotically this compression can be achieved■

- Different encoding schemes differ in the way they identify symbols of the alphabet■—this is rather specialist and we won't go into this■

Huffman Coding

- Given a sequence of symbols and their probabilities of occurrence, Huffman code provides a way of coding up the information■
- It is an example of a **greedy** strategy that happens to be optimal■
- Like many greedy strategies it is easily implemented using a priority queue■
- It is used in the UNIX compress program and in the exact part of JPEG■
- The idea is to assign short codes to commonly used symbols■

Symbol Frequency

- We start from an alphabet describing the original document
 - ★ This might be the set of characters
 - ★ For an image it might be the set of pixel values
 - ★ It might be pairs of pixel values
- We compute the number of occurrences of each symbol

Symbol	# Occurrences
a	145
b	67
⋮	⋮

Encoding

- We want to assign a code to each symbol■
- To save space we want to assign short codes to frequently used symbols■
- There is a problem:■decoding■
- If we assigned a code

$$e \rightarrow 0$$

$$a \rightarrow 1$$

$$r \rightarrow 01$$

$$o \rightarrow 10$$

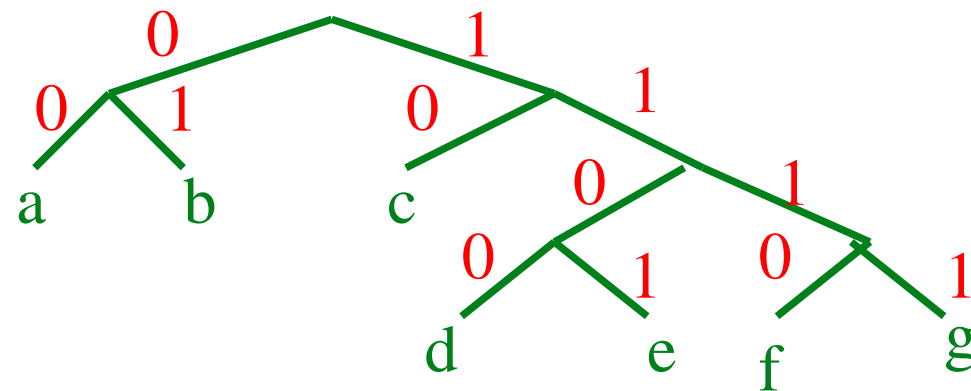
$$i \rightarrow 11$$

$$t \rightarrow 000$$

etc. we could compress a document very efficiently but we could never decode it uniquely■

Huffman Trees

- Once again tree come to the rescue!■
- We assign each symbol to a leaf of a binary tree■
- We use the position of the branch as an encoding of the symbol■



a	→	00
b	→	01
c	→	10
d	→	1100
e	→	1101
f	→	1110
g	→	1111

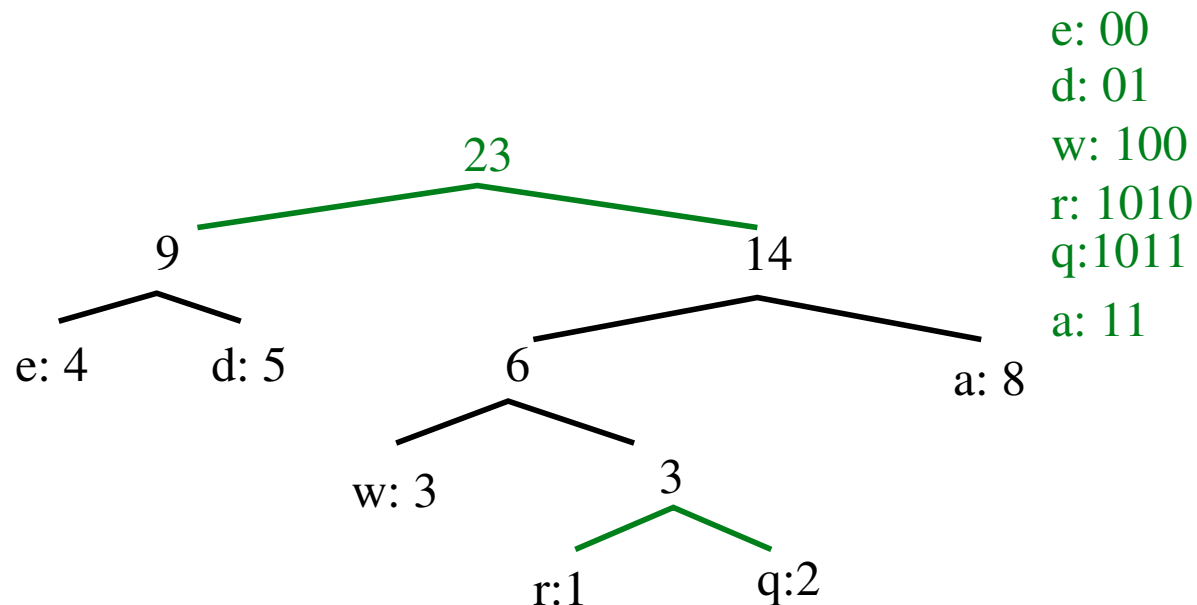
01001111111111011100
LR LLRRRRRRRRRRRLRRLL
b a g g e d

- The decoding is unique■

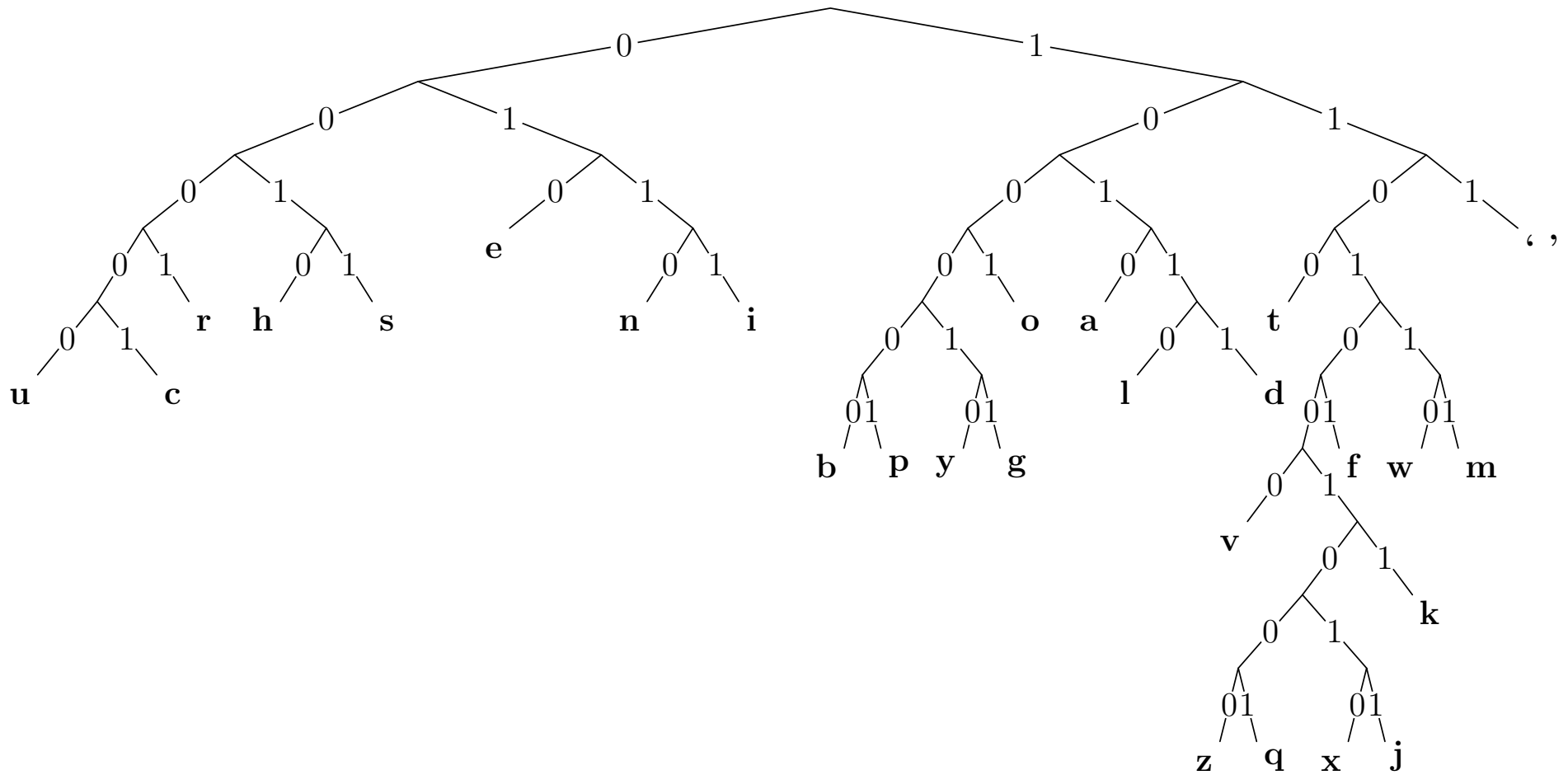
Generating the Huffman Tree

- We are left with the problem of constructing the Huffman tree such that frequently occurring letters have short codes
- A greedy approach is to iteratively build a tree by
 1. combine the two most infrequent symbols into a subtree
 2. Add their scores and treat them as a single symbol

aaeedwqqadewwaaddreaad → 1111110000011001011101111...



English Letters



the quick brown fox jumps over the lazy dog

344 bits

11000010010111110100100100000011100001110100111111000000001100111011001101111101
01100111010010101111101001011000001101111000010011111100111010000100001111110000
100101111011010101101001000100010111101111001100011 211 bits

Implementing Huffman Encoding

- To implement Huffman encoding you need
 1. A class to build Huffman trees by combining subtrees■
 2. A way to find the least frequently used symbols or symbol combinations■
- Priority queues are ideal for this application■
- They allow you to find the least frequently used symbols (`removeMin`) and to add new symbols (`add`)■
- To decode you follow the Huffman tree■

Greedy Strategy

- Huffman encoding is an example of a **Greedy solution pattern**■
- That is we look for local optimality (i.e. we combine the two least frequently used symbols)■
- In this case, we obtain global optimality (i.e. the Huffman tree obtained gives an optimal Huffman code)■
- There are a number of important problems where greedy algorithms lead to global optimality (we saw this earlier)■
- For these algorithms priority queues commonly are used for implementing the algorithm■

Advanced Techniques

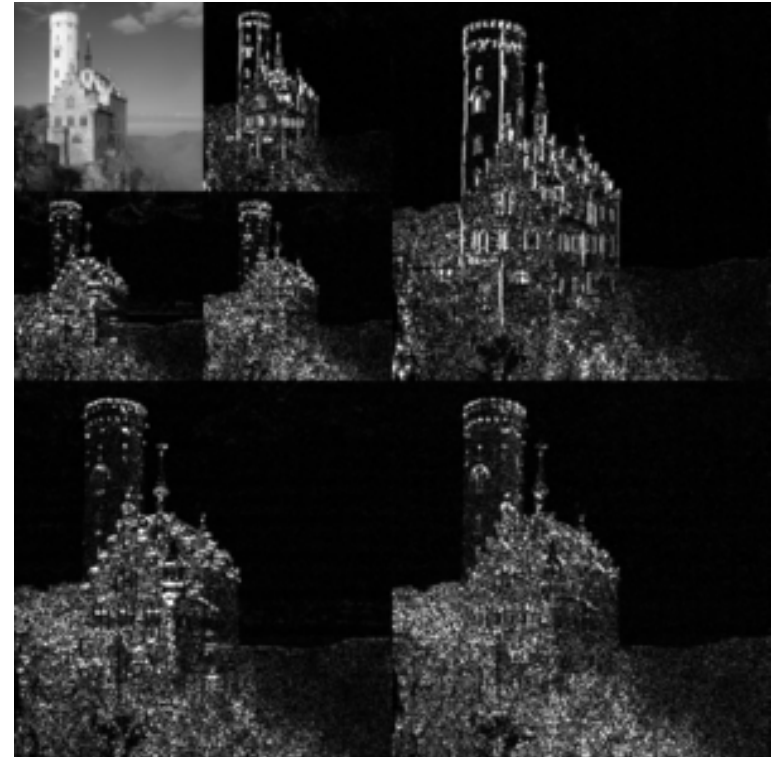
- Huffman code is optimal given the frequency of symbols■
- However, there is considerable art in identifying which 'symbols' to use■
- Advanced compression algorithms (LZ78, LZW Lempel-Ziv-Welch) build dictionaries of sequences seen in the files—they tend to be rather specialised■
- Some recent algorithms (e.g. Burrows-Wheeler) transform the file in such a way that similar symbols are mapped to adjacent sites—depends on the generating mechanism of the language■

File Compression and Plagiarism Detection

- One way of spotting plagiarism is to compare the compressed lengths of two files and the length of the compressed file when the two files are concatenated first■
- If the files have the same structure the concatenated version can often be significantly reduced■
- Also used in identifying closeness of species in constructing phylogenetic trees■

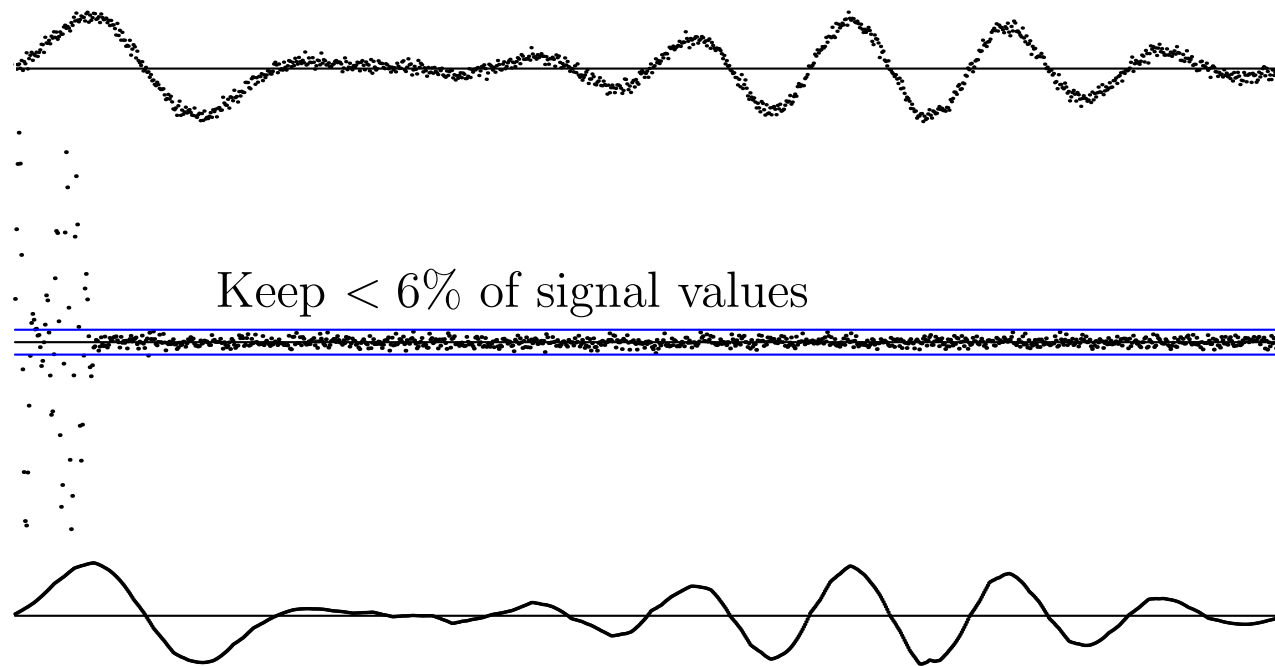
Outline

1. Huffman codes
2. **Wavelets**



Signals and Energies

- We consider compressing a signal $x = (x_0, x_1, \dots, x_{n-1})$ ■



- We can define the “energy” as the squared deviations

$$E = \sum_{i=1}^n x_i^2$$

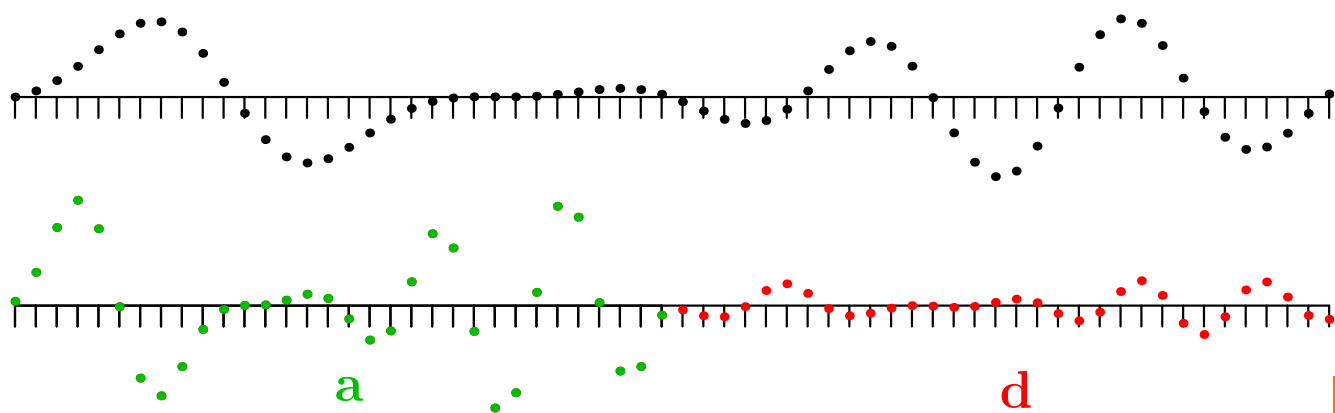
- Our strategy in lossy compression is to transmit as much “energy” in as few bits as possible
- There are different strategies to achieve good compress

Wavelets

- With wavelets we try to re-represent the signal so as to squeeze as much energy as possible into fewer bits
- The easiest way to do this is with Haar wavelets

$$a_i = \frac{x_{2i} + x_{2i+1}}{\sqrt{2}} \qquad d_i = \frac{x_{2i} - x_{2i+1}}{\sqrt{2}}$$

- Define new signal $(a_0, a_1, a_2, \dots, a_{n/2-1}, d_0, d_1, \dots, d_{n/2-1})$



Carrier and Difference Signals

- The terms $a_i = (x_{2i} + x_{2i+1})/\sqrt{2}$ takes the “average” of the signal, but compresses it in half the space■
- The terms $d_i = (x_{2i} - x_{2i+1})/\sqrt{2}$ takes the difference and is small if the signal does not change much■
- The energy is conserved since

$$\begin{aligned} a_i^2 + d_i^2 &= \left(\frac{x_{2i} + x_{2i+1}}{\sqrt{2}} \right)^2 + \left(\frac{x_{2i} - x_{2i+1}}{\sqrt{2}} \right)^2 \\ &= \frac{x_{2i}^2 + 2x_{2i}x_{2i+1} + x_{2i+1}^2 + x_{2i}^2 - 2x_{2i}x_{2i+1} + x_{2i+1}^2}{2} = x_{2i}^2 + x_{2i+1}^2 \quad \blacksquare \end{aligned}$$

- Attempt to push all the energy into the carrier signal, a_i ■

Inverse Transform

- The wavelet transform can be easily reversed

$$\begin{aligned} a_i &= \frac{x_{2i} + x_{2i+1}}{\sqrt{2}} & d_i &= \frac{x_{2i} - x_{2i+1}}{\sqrt{2}} \\ x_{2i} &= \frac{a_i + d_i}{\sqrt{2}} & x_{2i+1} &= \frac{a_i - d_i}{\sqrt{2}} \end{aligned}$$

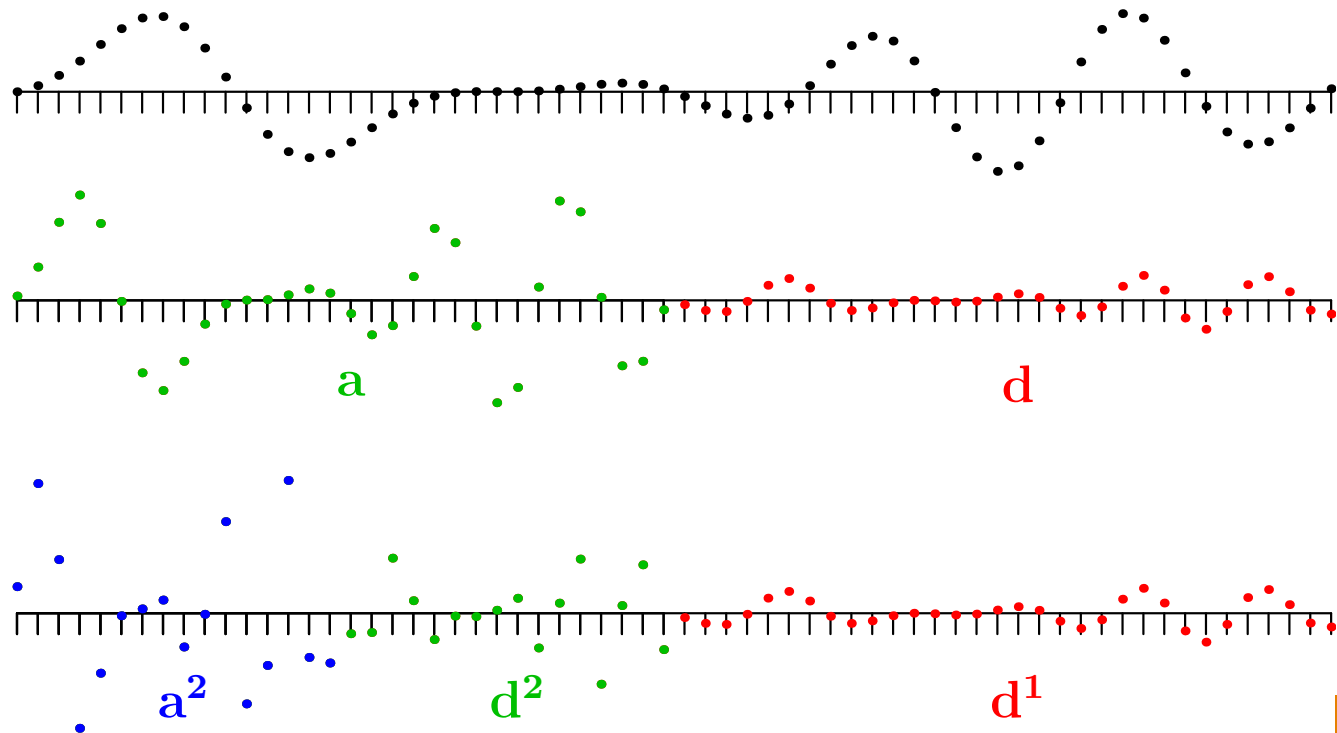
- Can compute transform using vectors (wavelets)

$$a_i = \mathbf{V}_i \cdot \mathbf{x} \qquad d_i = \mathbf{W}_i \cdot \mathbf{x}$$

- These vectors are orthogonal to each other ($\mathbf{V}_i \cdot \mathbf{V}_j = 0$, $\mathbf{V}_i \cdot \mathbf{W}_j = 0$, etc.)

And So On. . .

- We can repeat the process again to concentrate the energy further
- We apply the Haar transform just to the carry part
 $a = (a_0, a_1, \dots, a_{n/2-1})$



Daubechies Wavelets

- Ingrid Daubechies suggested a host of wavelets which do better than Haar for smooth signals■
- The simplest is Daub4 defined by

$$a_i = c_0x_{2i} + c_1x_{2i+1} + c_2x_{2i+2} + c_3x_{2i+3}$$

$$d_i = c_3x_{2i} - c_2x_{2i+1} + c_1x_{2i+2} - c_0x_{2i+3}$$

$$c_0 = \frac{1 + \sqrt{3}}{4\sqrt{2}} \quad c_1 = \frac{3 + \sqrt{3}}{4\sqrt{2}} \quad c_2 = \frac{3 - \sqrt{3}}{4\sqrt{2}} \quad c_3 = \frac{1 - \sqrt{3}}{4\sqrt{2}} \quad \blacksquare$$

- Again conserves energy

$$\sum_{i=1}^{n/2} a_i^2 + b_i^2 = \sum_{i=1}^n x_i^2 \quad \blacksquare$$

Properties of Daub4

- Similar to the Haar transform

$$c_0 + c_1 + c_2 + c_3 = \sqrt{2}, \quad c_3 - c_2 + c_1 - c_0 = 0$$

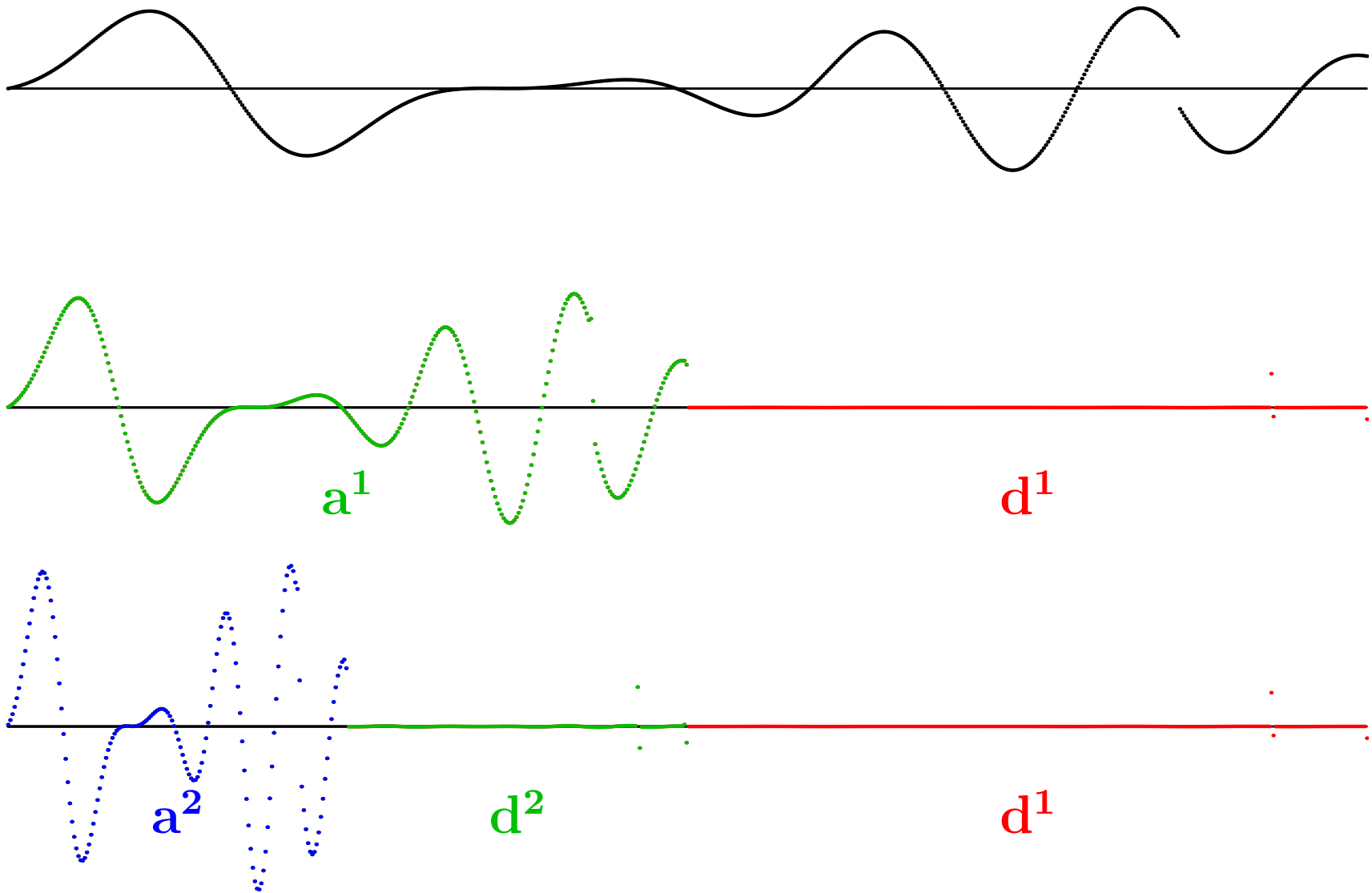
so the carrier signal (a_i) is approximately $\sqrt{2}$ times the original and the difference part (d_i) is equal to 0 for a flat signal, x ■

- However in addition

$$0c_3 - 1c_2 + 2c_1 - 3c_0 = 0$$

so the difference part (d_i) is equal to 0 for any linear signal, x ■

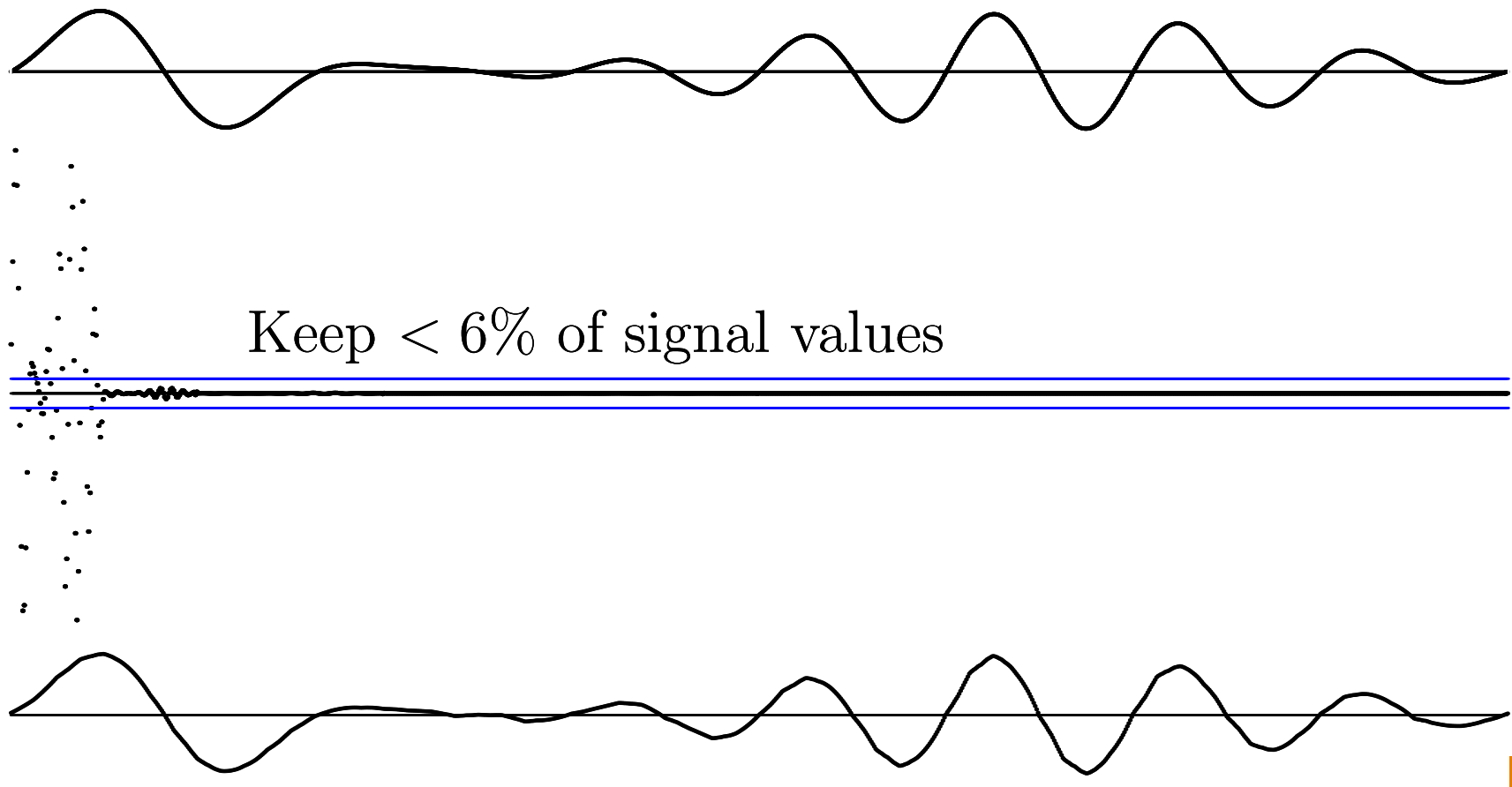
Daub4



Signal Compression

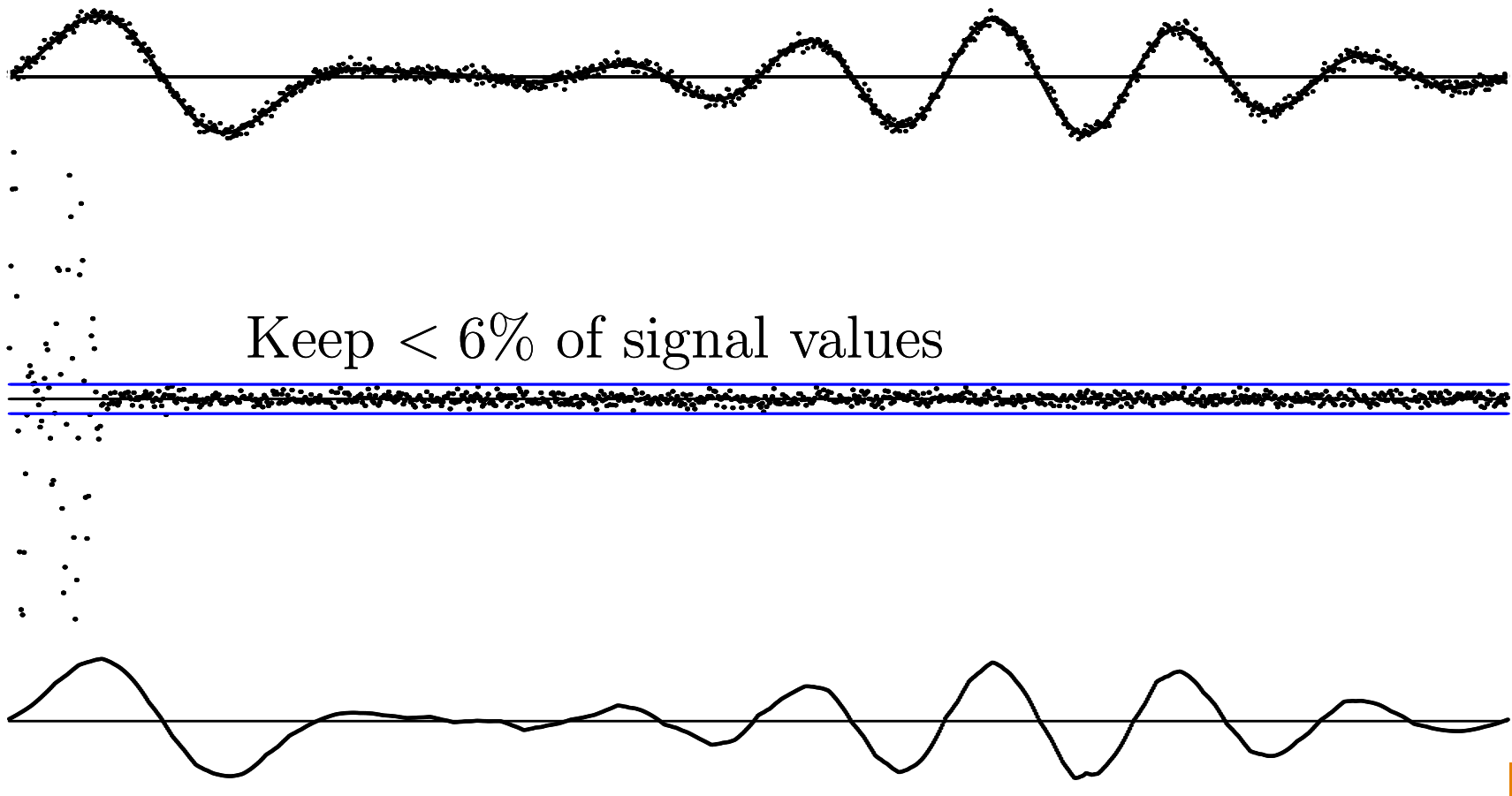
- To compress the signal we can set all components of the transformed signal whose magnitude lies below a threshold to 0■
- We transmit the non-zero magnitude together with a binary mask showing the position of the non-zero magnitude■
- We can reduce the accuracy (number of decimal places) of the non-zero magnitudes (quantisation)—this is repaired on inverting transform■
- We can compress the binary mask using Huffman encoding or other scheme■

Daub6



Noise Reduction

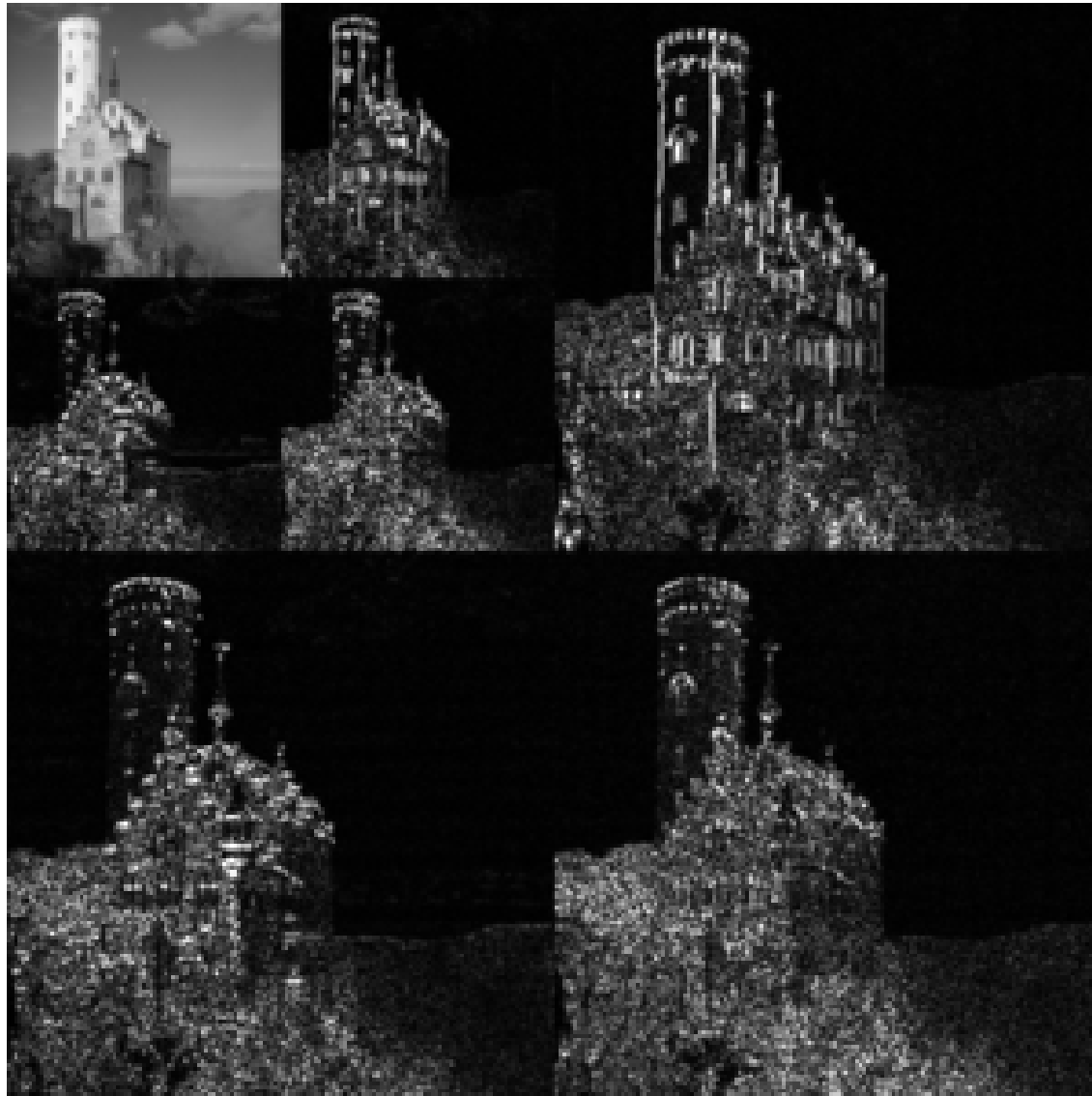
- Can also be used in noise reduction



Other Wavelets

- Can use high-order wavelets which captures more energy in the carrier signal, e.g. Daub10 or Daub20■
- Many other wavelets capture other properties (e.g. Coiflets capture properties of a continuous signal sampled at discrete points)■
- Efficiency of wavelets depend on how well the capture underlying properties of signals■
- Can also construct 2-d wavelets for image compression (jpeg-2000)■

2-D Wavelets



Summary

- File compression is an important task in its own right■
- Files may either be compressed losslessly or lossily■
- Lossy compression is typically much more efficient (e.g. an order of magnitude smaller)■
- Huffman encoding often lies at the lowest level in many compression algorithms■
- Wavelets illustrate a strategy of changing the representation to concentrate the energy of a signal■