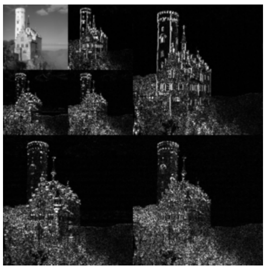
Algorithms and Analysis

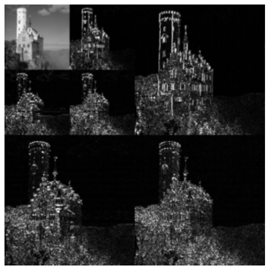
Lesson 24: *Use Smart Encoding!*



File compression, Huffman codes, wavelets

Outline

1. **Huffman codes**
2. Wavelets



File Compression

- File compression comes in two varieties
  - Exact compression (e.g. zip used on text files)
  - Lossy compression (e.g. jpeg used on pictures—jpeg can also be loss-less or exact)
- Good exact compression (also known as entropy encodings) can give a compression ratio around 25%
- Lossy compression can give a compression ratio from 10-1%
- Important for saving space, but lossy compression can also be used for noise reduction
- Even used for plagiarism detection!

Entropy Encoding

- Exact encodings use the principle of using short words for frequently occurring sequences (symbols) and longer words for sequences that occur less often
- Claude Shannon showed that for an alphabet of n symbols where the probability of symbol i occurring is p_i no code exists which can transmit information in less than

$$-\sum_{i=1}^{n} p_i \log_2(p_i) \text{ bits}$$

  asymptotically this compression can be achieved
- Different encoding schemes differ in the way they identify symbols of the alphabet—this is rather specialist and we won't go into this

Huffman Coding

- Given a sequence of symbols and their probabilities of occurance, Huffman code provides a way of coding up the information
- It is an example of a **greedy** strategy that happens to be optimal
- Like many greedy strategies it is easily implemented using a priority queue
- It is used in the UNIX compress program and in the exact part of JPEG
- The idea is to assign short codes to commonly used symbols

Symbol Frequency

- We start from an alphabet describing the original document
  - This might be the set of characters
  - For an image it might be the set of pixel values
  - It might be pairs of pixel values
- We compute the number of occurrences of each symbol

Symbol	# Occurrences
a	145
b	67
⋮	⋮

Encoding

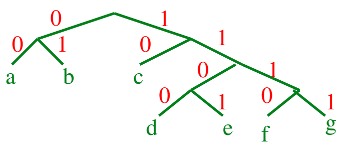
- We want to assign a code to each symbol
- To save space we want to assign short codes to frequently used symbols
- There is a problem: decoding
- If we assigned a code

$$e \to 0 \qquad a \to 1 \qquad r \to 01$$
$$o \to 10 \qquad i \to 11 \qquad t \to 000$$

  etc. we could compress a document very efficiently but we could never decode it uniquely

Huffman Trees

- Once again tree come to the rescue!
- We assign each symbol to a leaf of a binary tree
- We use the position of the branch as an encoding of the symbol



a → 00
b → 01
c → 10
d → 1100
e → 1101
f → 1110
g → 1111

```
01001111111111011100
LR LLRRRR RRRR RRLR RRLL
 b   a     g    g    e    d
```

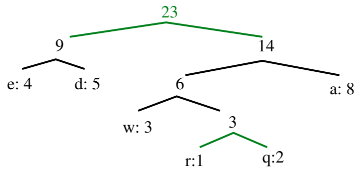
- The decoding is unique

Generating the Huffman Tree

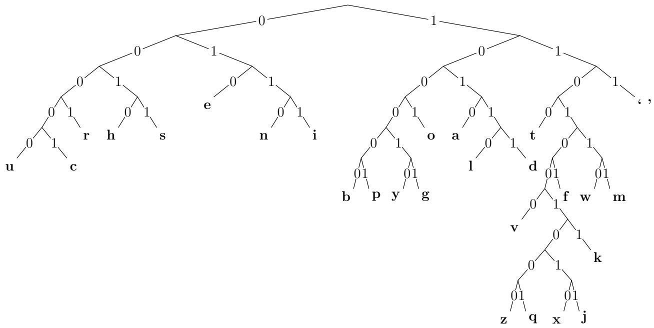
- We are left with the problem of constructing the Huffman tree such that frequently occurring letters have short codes
- A greedy approach is to iteratively build a tree by
  1. combine the two most infrequent symbols into a subtree
  2. Add their scores and treat them as a single symbol

aaaeedwqqadewwaaddreaad → 1111110000011001011101111...

e: 00
d: 01
w: 100
r: 1010
q:1011
a: 11



English Letters



the quick brown fox jumps over the lazy dog 344 bits
1100001001011111010010010000001110000111010011111000000001100111011001101111101
0110011101001010111110100101100000110111100001001111110011101000010000111110000
10010111101101010110100100010001011110111100110001 1 211 bits

Implementing Huffman Encoding

- To implement Huffman encoding you need
  1. A class to build Huffman trees by combining subtrees
  2. A way to find the least frequently used symbols or symbol combinations
- Priority queues are ideal for this application
- They allow you to find the least frequently used symbols (`removeMin`) and to add new symbols (`add`)
- To decode you follow the Huffman tree

Greedy Strategy

- Huffman encoding is an example of a **Greedy solution pattern**
- That is we look for local optimality (i.e. we combine the two least frequently used symbols)
- In this case, we obtain global optimality (i.e. the Huffman tree obtained gives an optimal Huffman code)
- There are a number of important problems where greedy algorithms lead to global optimality (we saw this earlier)
- For these algorithms priority queues commonly are used for implementing the algorithm

Advanced Techniques

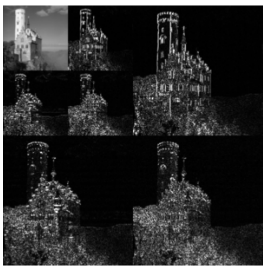
- Huffman code is optimal given the frequency of symbols
- However, there is considerable art in identifying which 'symbols' to use
- Advanced compression algorithms (LZ78, LZW Lempel-Ziv-Welch) build dictionaries of sequences seen in the files—they tend to be rather specialised
- Some recent algorithms (e.g. Burrows-Wheeler) transform the file in such a way that similar symbols are mapped to adjacent sites—depends on the generating mechanism of the language

File Compression and Plagiarism Detection

- One way of spotting plagiarism is to compare the compressed lengths of two files and the length of the compressed file when the two files are concatenated first
- If the files have the same structure the concatenated version can often be significantly reduced
- Also used in identifying closeness of species in constructing phylogenetic trees

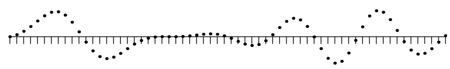
Outline

1. Huffman codes
2. **Wavelets**



Signals and Energies

- We consider compressing a signal $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$



- We can define the "energy" as the squared deviations

$$E = \sum_{i=1}^{n} x_i^2$$

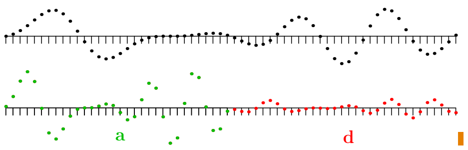
- Our strategy in lossy compression is to transmit as much "energy" in as few bits as possible
- There are different strategies to achieve good compress

Wavelets

- With wavelets we try to re-represent the signal so as to squeeze as much energy as possible into fewer bits
- The easiest way to do this is with Haar wavelets

$$a_i = \frac{x_{2i} + x_{2i+1}}{\sqrt{2}} \qquad d_i = \frac{x_{2i} - x_{2i+1}}{\sqrt{2}}$$

- Define new signal $(a_0, a_1, a_2, \ldots, a_{n/2-1}, d_0, d_1, \ldots, d_{n/2-1})$



Carrier and Difference Signals

- The terms $a_i = (x_{2i} + x_{2i+1})/\sqrt{2}$ takes the "average" of the signal, but compresses it in half the space
- The terms $d_i = (x_{2i} - x_{2i+1})/\sqrt{2}$ takes the difference and is small if the signal does not change much
- The energy is conserved since

$$a_i^2 + d_i^2 = \left(\frac{x_{2i} + x_{2i+1}}{\sqrt{2}}\right)^2 + \left(\frac{x_{2i} - x_{2i+1}}{\sqrt{2}}\right)^2$$
$$= \frac{x_{2i}^2 + 2x_{2i}x_{2i+1} + x_{2i+1}^2 + x_{2i}^2 - 2x_{2i}x_{2i+1} + x_{2i+1}^2}{2} = x_{2i}^2 + x_{2i+1}^2$$

- Attempt to push all the energy into the carrier signal, a_i

Inverse Transform

- The wavelet transform can be easily reversed

$$a_i = \frac{x_{2i} + x_{2i+1}}{\sqrt{2}} \qquad d_i = \frac{x_{2i} - x_{2i+1}}{\sqrt{2}}$$
$$x_{2i} = \frac{a_i + d_i}{\sqrt{2}} \qquad x_{2i+1} = \frac{a_i - d_i}{\sqrt{2}}$$

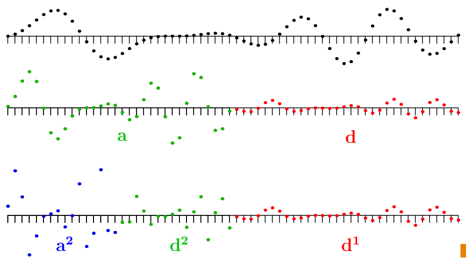
- Can compute transform using vectors (wavelets)

$$a_i = \boldsymbol{V_i} \cdot \boldsymbol{x} \qquad d_i = \boldsymbol{W_i} \cdot \boldsymbol{x}$$

- These vectors are orthogonal to each other ($\boldsymbol{V_i} \cdot \boldsymbol{V_j} = 0$, $\boldsymbol{V_i} \cdot \boldsymbol{W_j} = 0$, etc.)

And So On. . .

- We can repeat the process again to concentrate the energy further
- We apply the Haar transform just to the carry part $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n/2-1})$



Daubechies Wavelets

- Ingrid Daubechies suggested a host of wavelets which do better than Haar for smooth signals
- The simplest is Daub4 defined by

$$a_i = c_0 x_{2i} + c_1 x_{2i+1} + c_2 x_{2i+2} + c_3 x_{2i+3}$$
$$d_i = c_3 x_{2i} - c_2 x_{2i+1} + c_1 x_{2i+2} - c_0 x_{2i+3}$$

$$c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}} \qquad c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}} \qquad c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}} \qquad c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

- Again conserves energy

$$\sum_{i=1}^{n/2} a_i^2 + b_i^2 = \sum_{i=1}^{n} x_i^2$$

Properties of Daub4

- Similar to the Haar transform

$$c_0 + c_1 + c_2 + c_3 = \sqrt{2}, \qquad c_3 - c_2 + c_1 - c_0 = 0$$

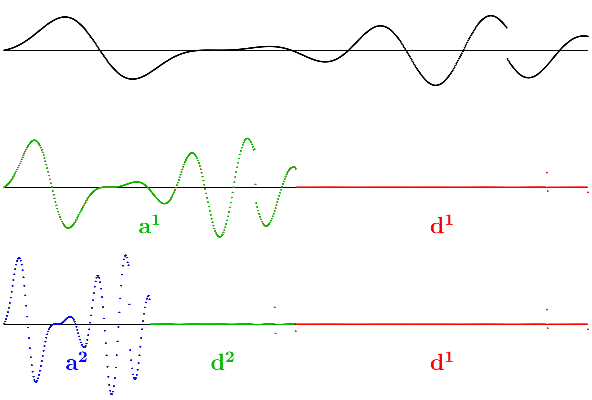
so the carrier signal (a_i) is approximately $\sqrt{2}$ times the original and the difference part (d_i) is equal to 0 for a flat signal, $\boldsymbol{x}$

- However in addition

$$0c_3 - 1c_2 + 2c_1 - 3c_0 = 0$$

so the difference part (d_i) is equal to 0 for any linear signal, $\boldsymbol{x}$

Daub4

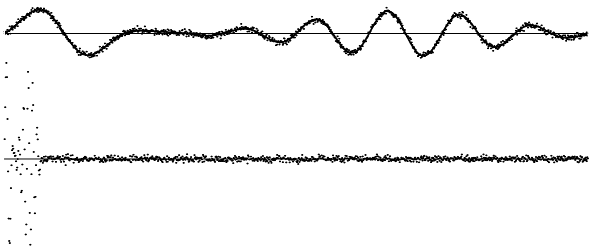


Signal Compression

- To compress the signal we can set all components of the transformed signal whose magnitude lies below a threshold to 0
- We transmit the non-zero magnitude together with a binary mask showing the position of the non-zero magnitude
- We can reduce the accuracy (number of decimal places) of the non-zero magnitudes (quantisation)—this is repaired on inverting transform
- We can compress the binary mask using Huffman encoding or other scheme

Noise Reduction

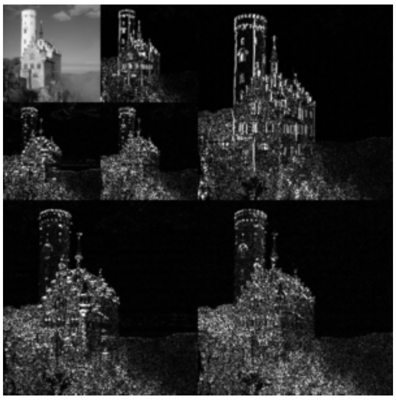
- Can also be used in noise reduction



Other Wavelets

- Can use high-order wavelets which captures more energy in the carrier signal, e.g. Daub10 or Daub20
- Many other wavelets capture other properties (e.g. Coiflets capture properties of a continuous signal sampled at discrete points)
- Efficiency of wavelets depend on how well the capture underlying properties of signals
- Can also construct 2-d wavelets for image compression (jpeg-2000)

2-D Wavelets



Summary

- File compression is an important task in its own right
- Files may either be compressed losslessly or lossily
- Lossy compression is typically much more efficient (e.g. an order of magnitude smaller)
- Huffman encoding often lies at the lowest level in many compression algorithms
- Wavelets illustrate a strategy of changing the representation to concentrate the energy of a signal