

Algorithms and Analysis

Lesson 10: *Keep Trees Balanced*



AVL trees, red-black trees, TreeSet, TreeMap

Outline

1. **Deletion**
2. Balancing Trees
 - Rotations
3. AVL
4. Red-Black Trees
 - TreeSet
 - TreeMap



Recap

- Binary search trees are commonly used to store data because we need to only look down one branch to find any element
- We saw how to implement many methods of the binary search tree
 - ★ find
 - ★ insert
 - ★ successor (in outline)
- One method we missed was remove

Recap

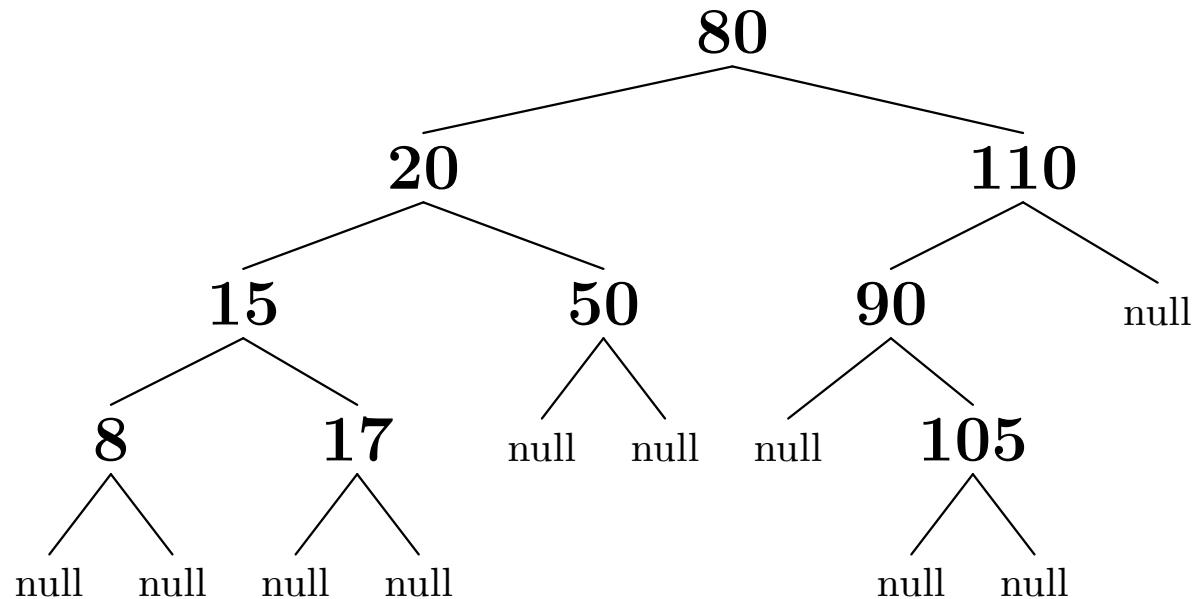
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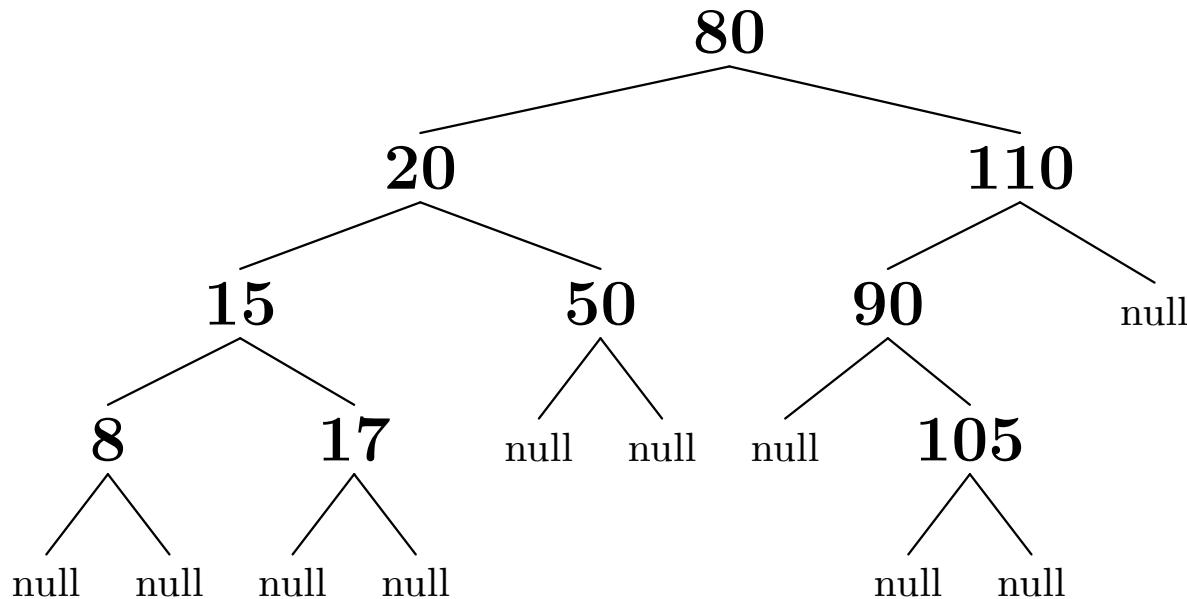
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- Suppose we want to delete some elements from a tree
- It is relatively easy if the element is a leaf node (e.g. 50)
- It is not so hard if the node has one child (e.g. 20)



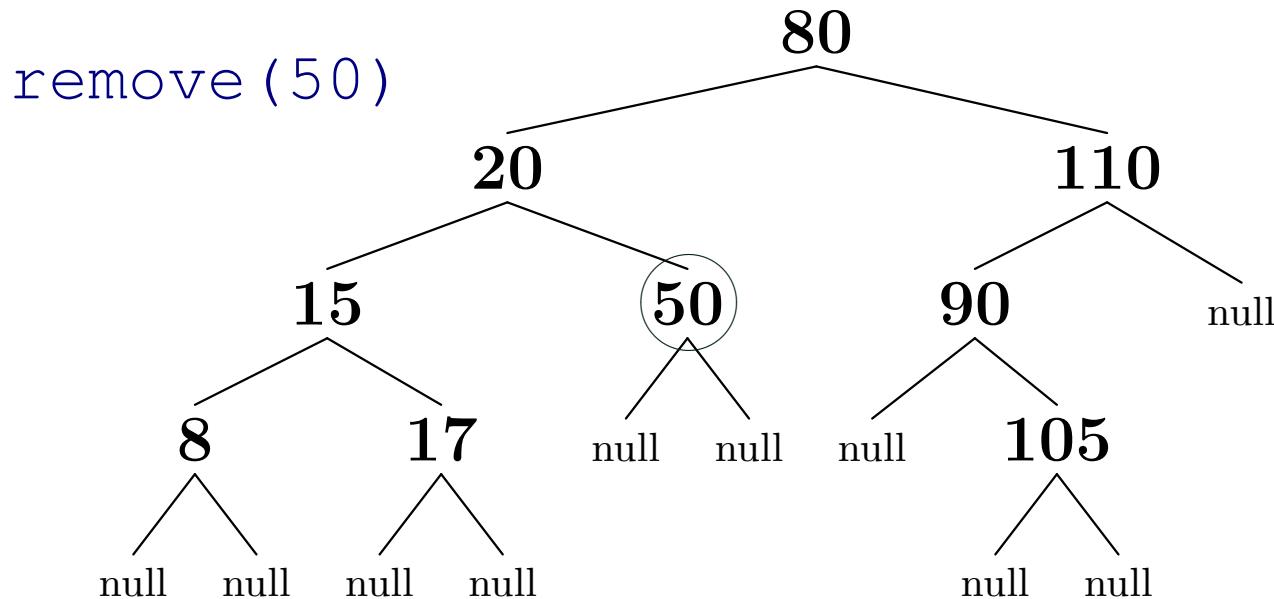
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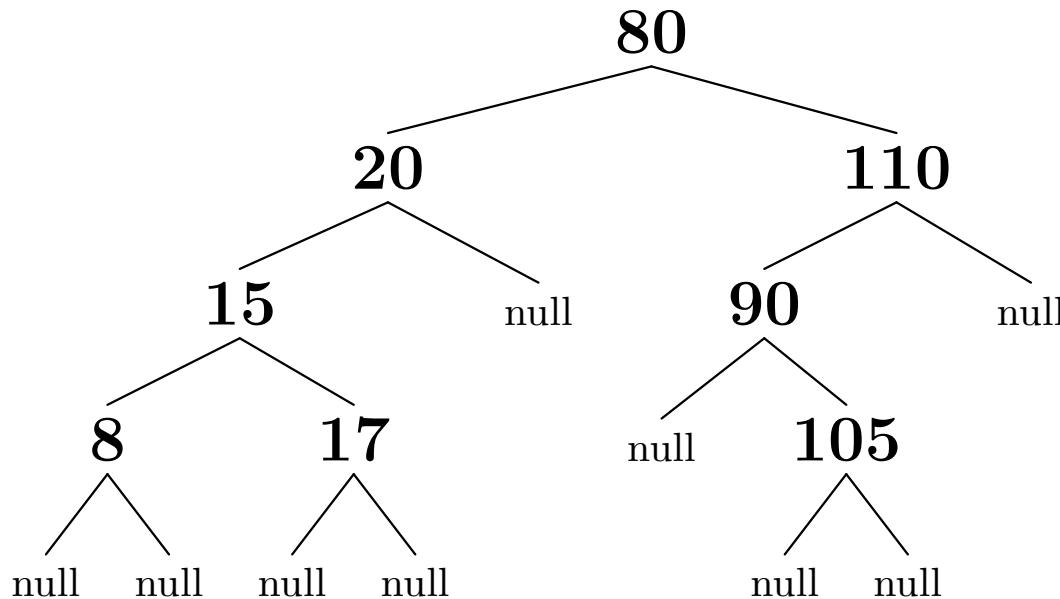
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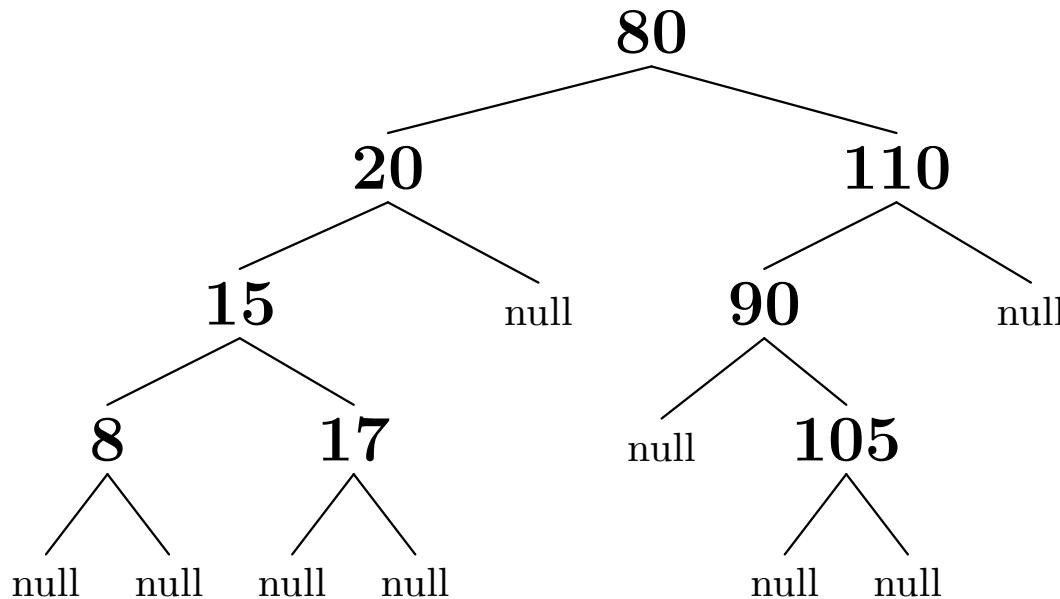
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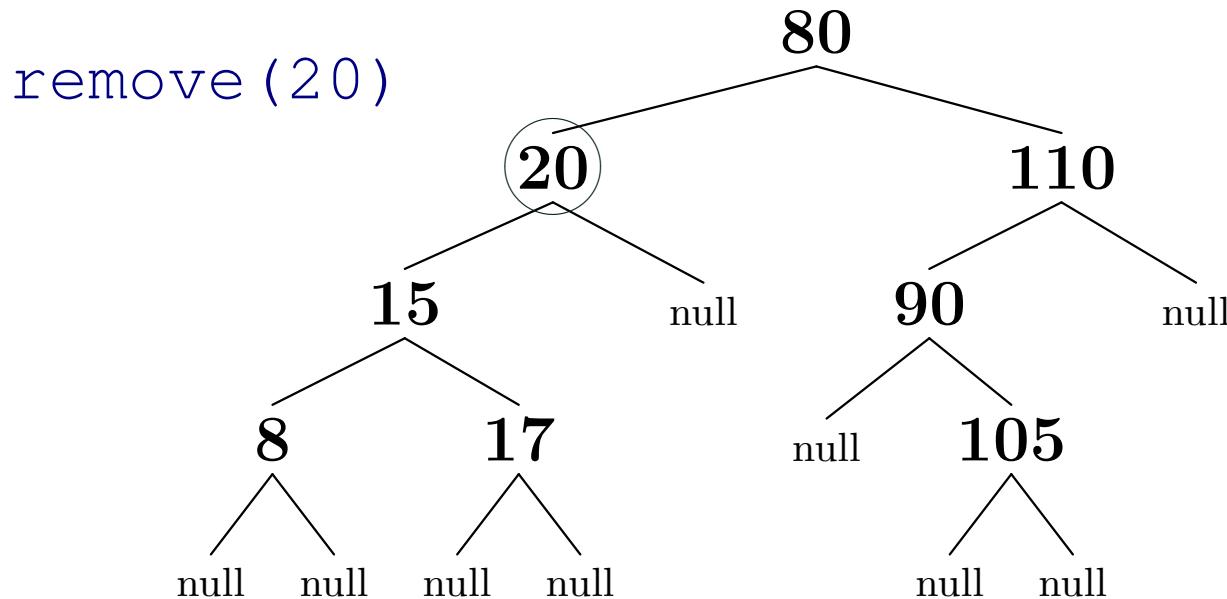
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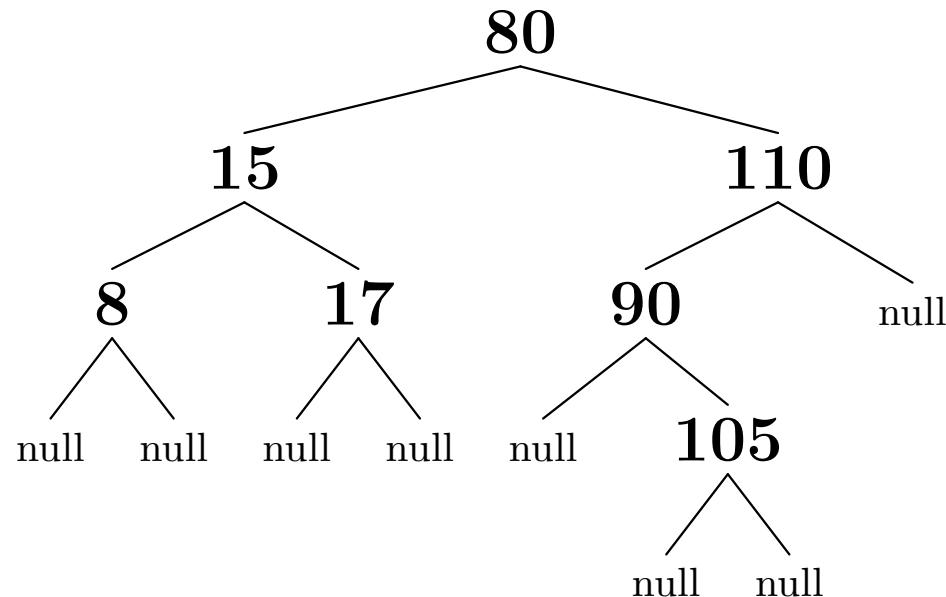
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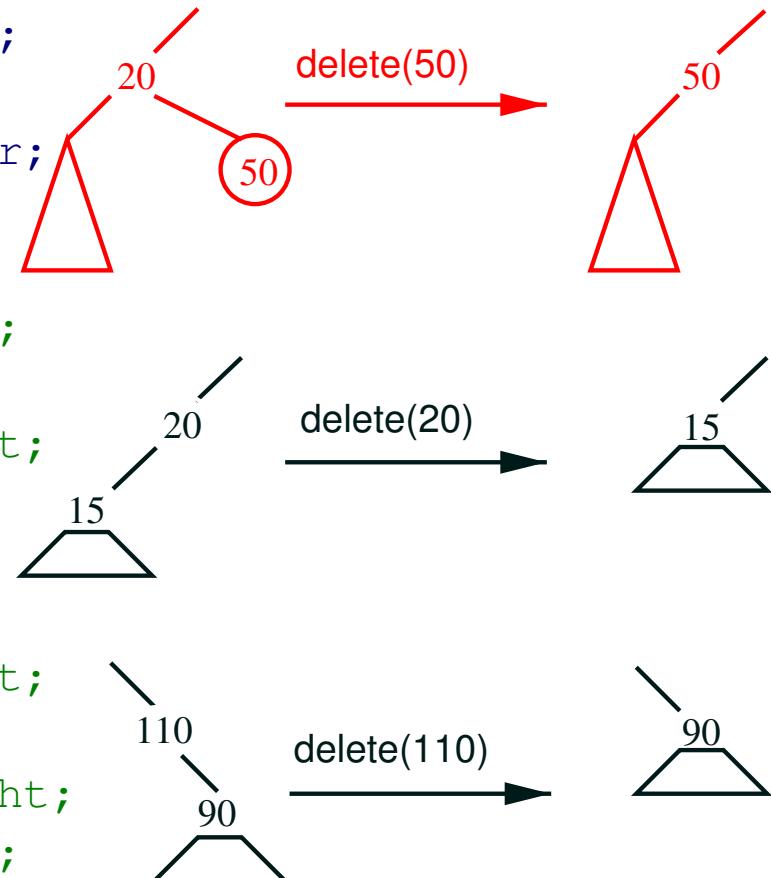
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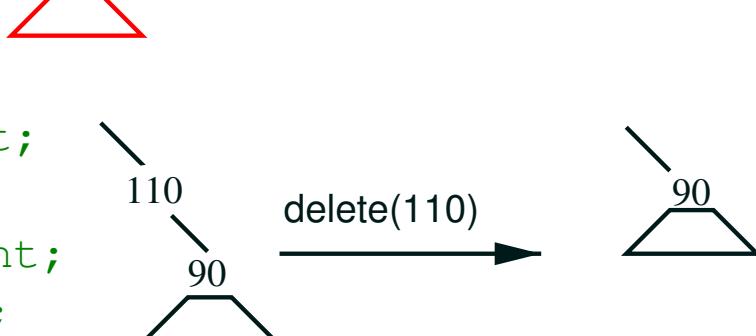
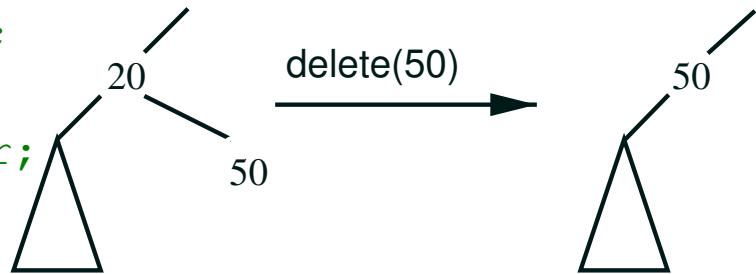
Code to remove Node n

```
if (n->left==nullptr && n->right==nullptr) {  
    if (n == n->parent->left)  
        n->parent->left = nullptr;  
    else  
        n->parent->right = nullptr;  
} else if (n->right==nullptr) {  
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    else  
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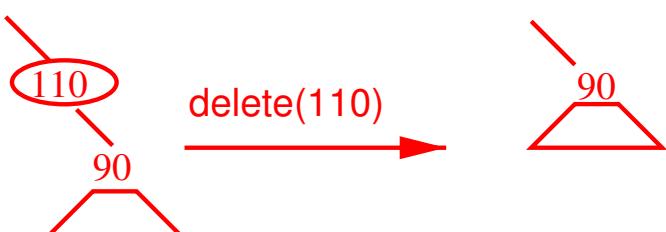
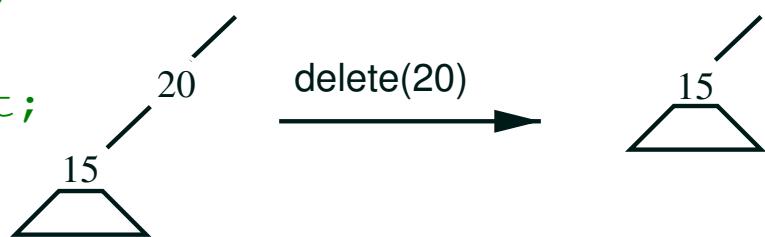
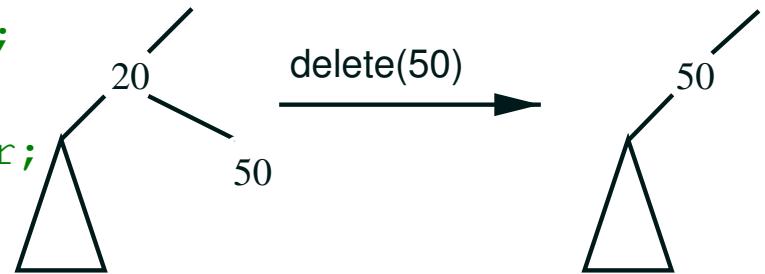
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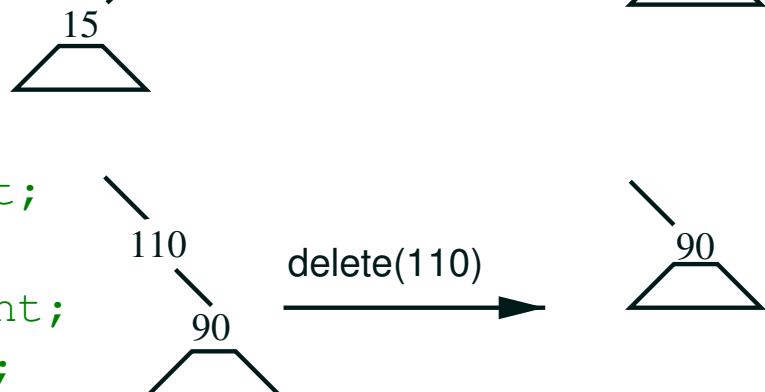
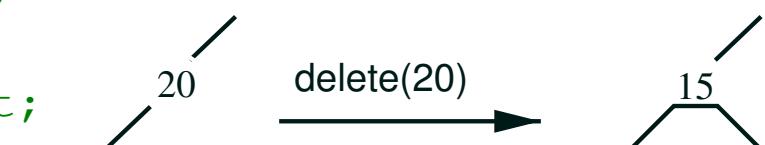
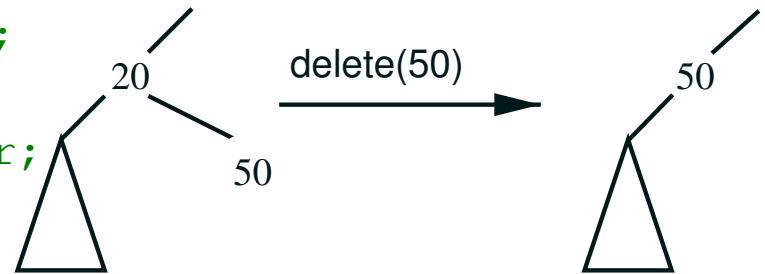
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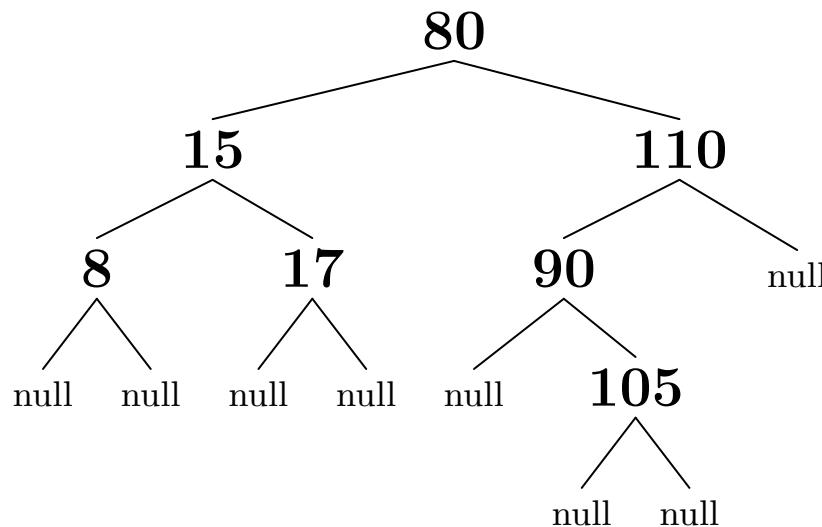
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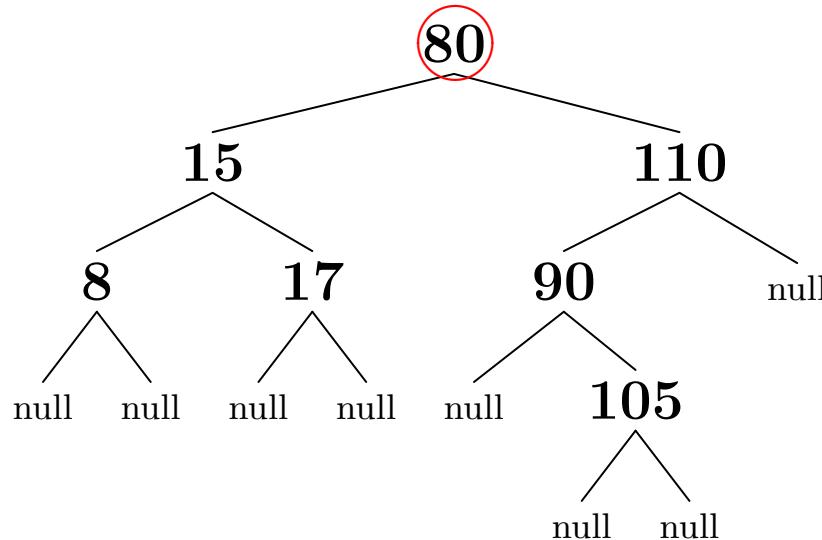
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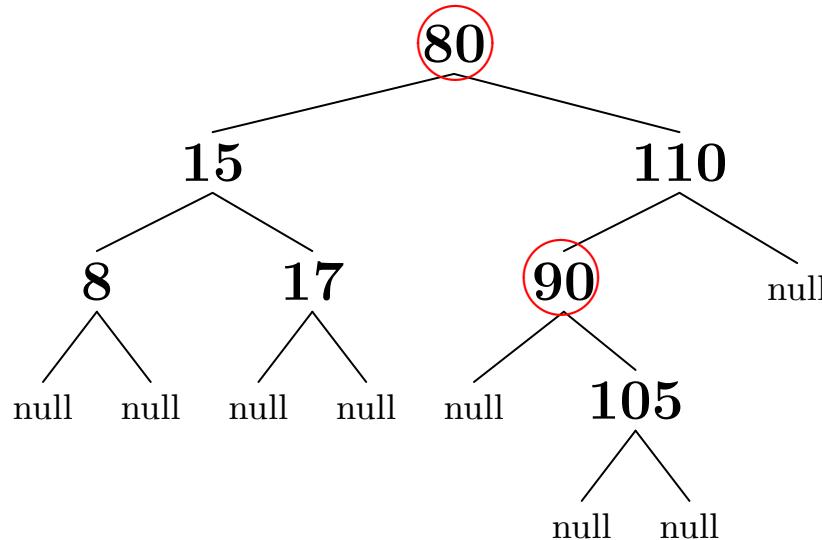
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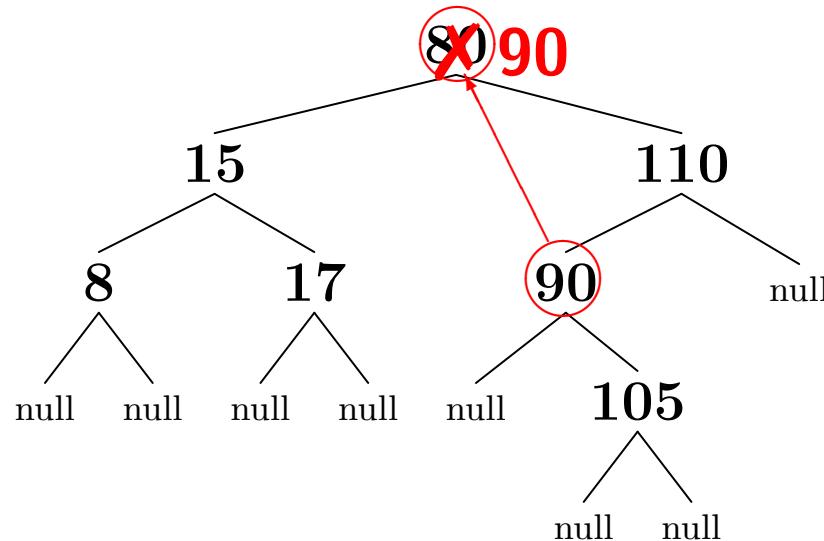
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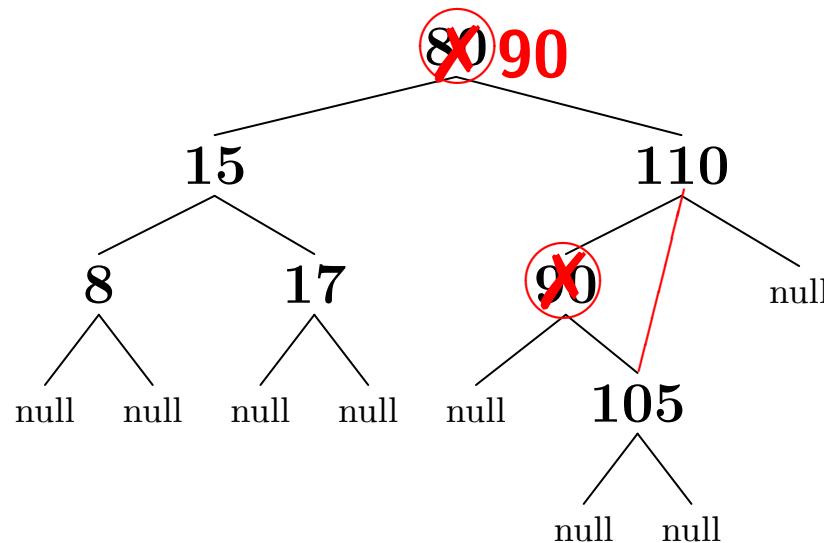
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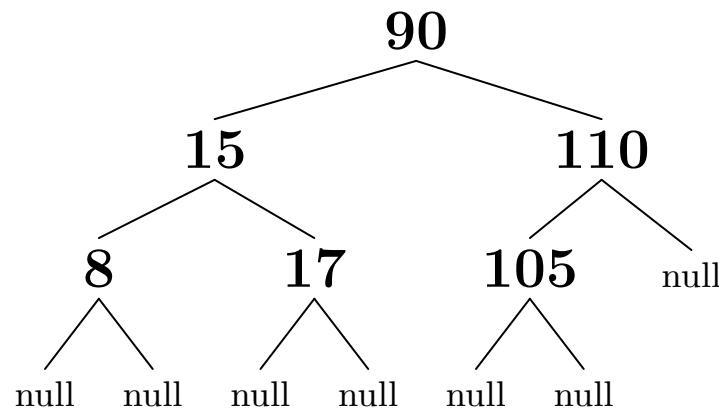
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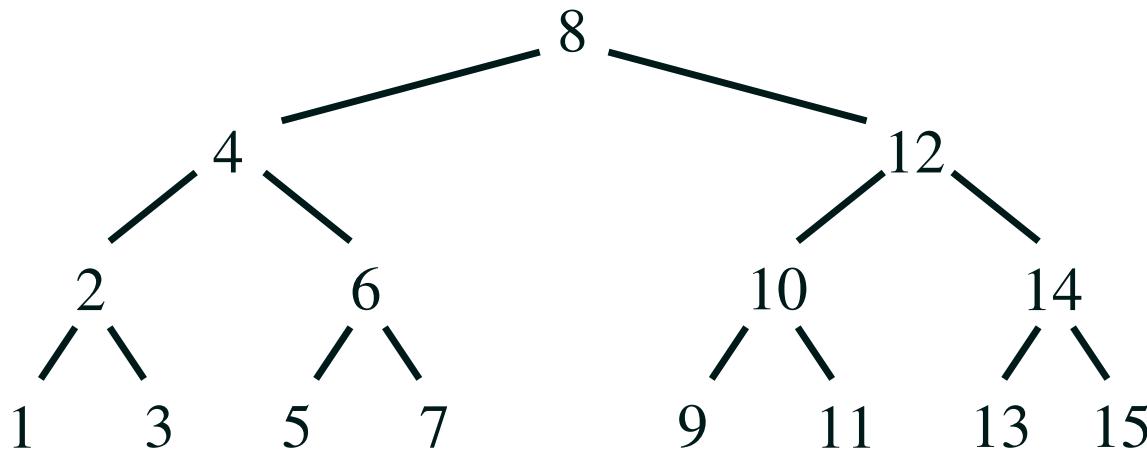
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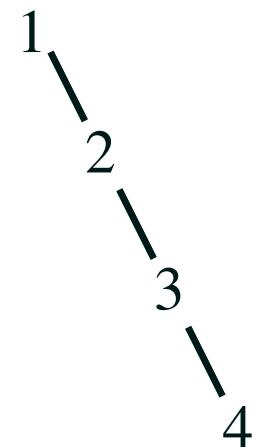


Why Balance Trees

- The number of comparisons to access an element depends on the depth of the node
- The average depth of the node depends on the shape of the tree



full tree

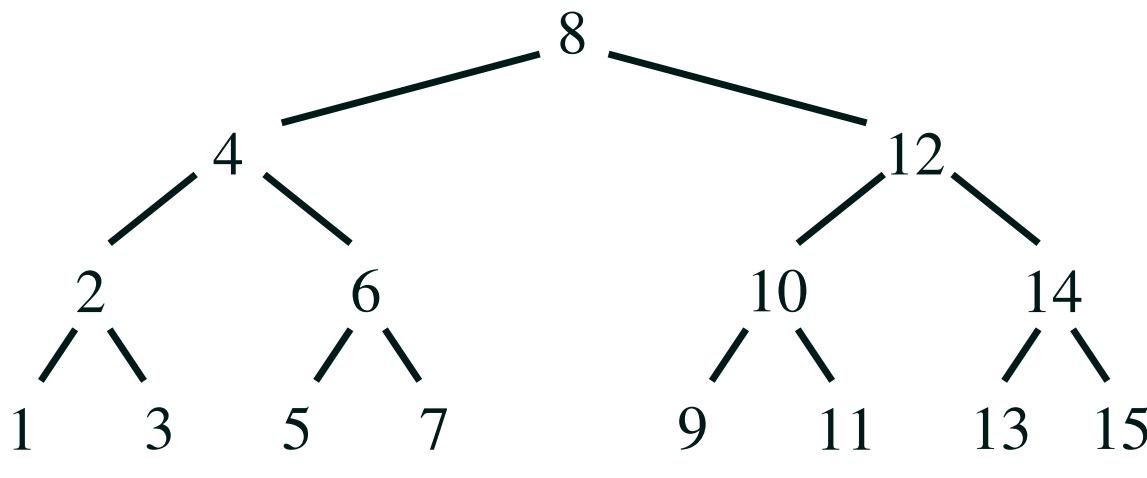


sparse tree

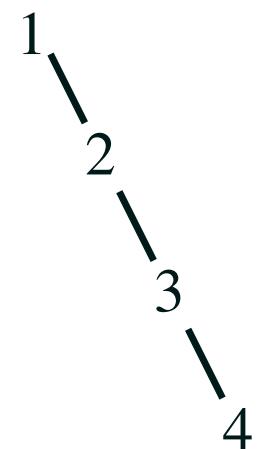
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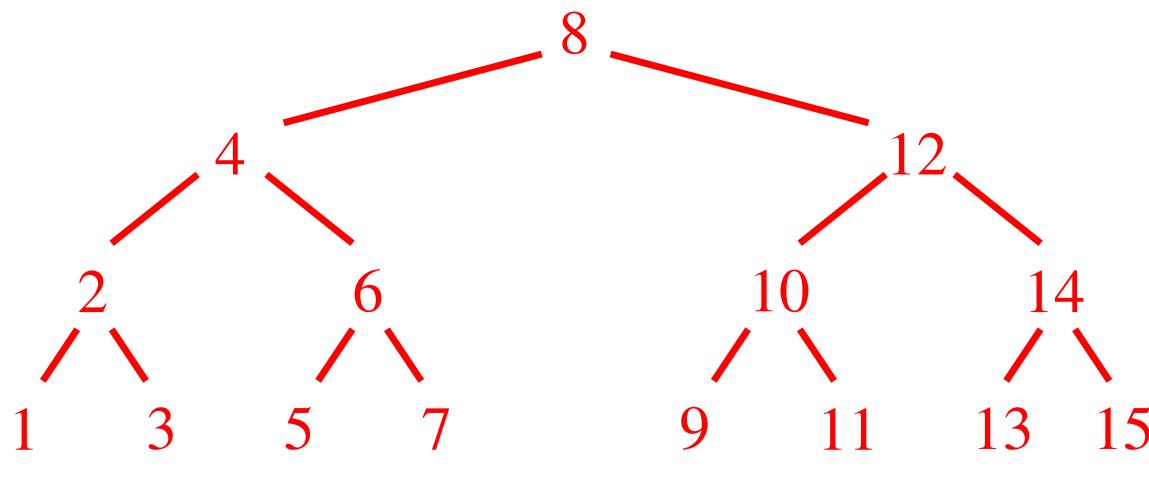


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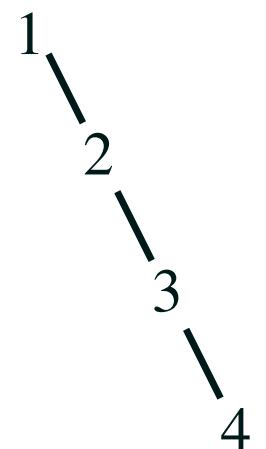
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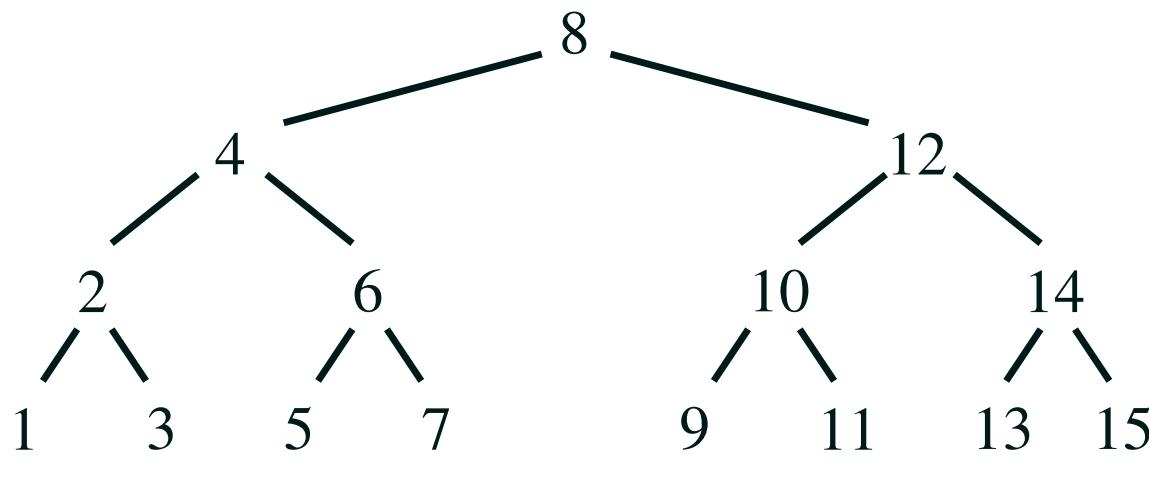


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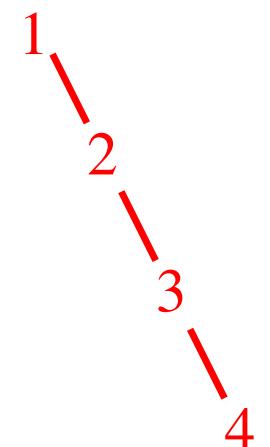
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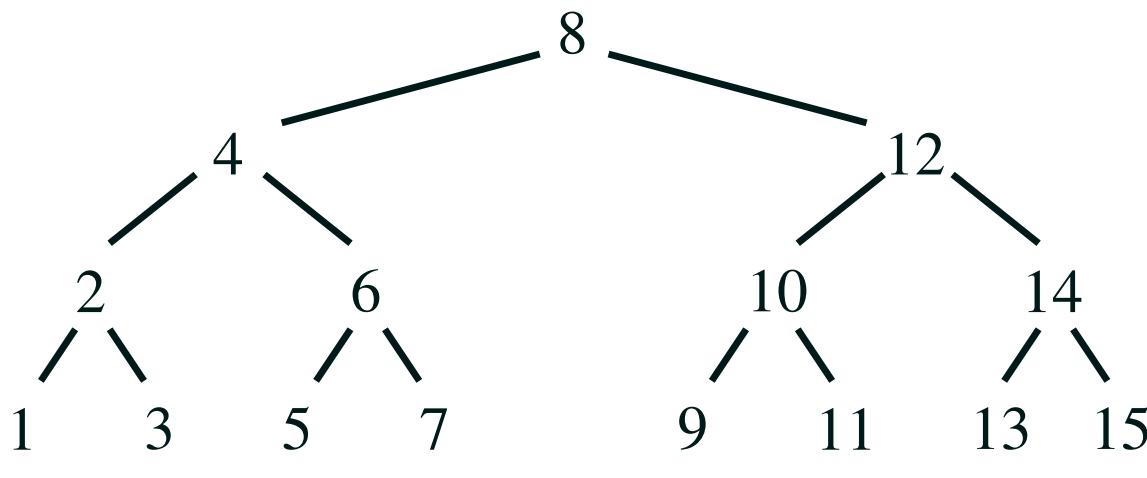


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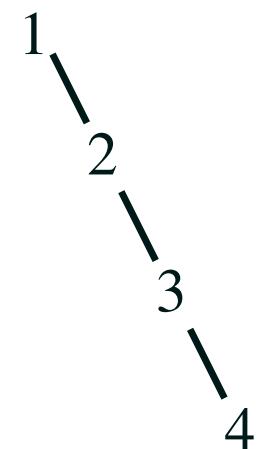
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Time Complexity

- In the best situation (a full tree) the number of elements in a tree is $n = \Theta(2^l)$ the depth is l so that the maximum depth is $\log_2(n)$
- It turns out for random sequences the average depth is $\Theta(\log(n))$
- In the worst case (when the tree is effectively a linked list), the average depth is $\Theta(n)$
- Unfortunately, the worst case happens when the elements are added *in order* (not a rare event)

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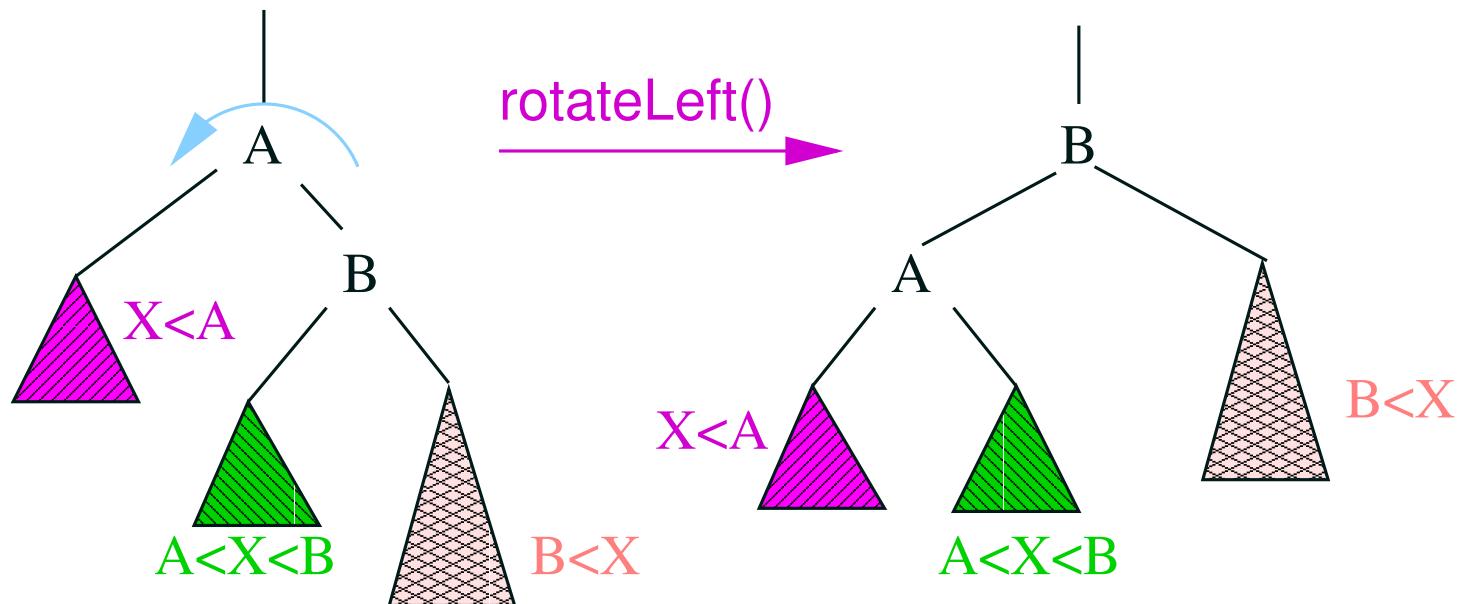
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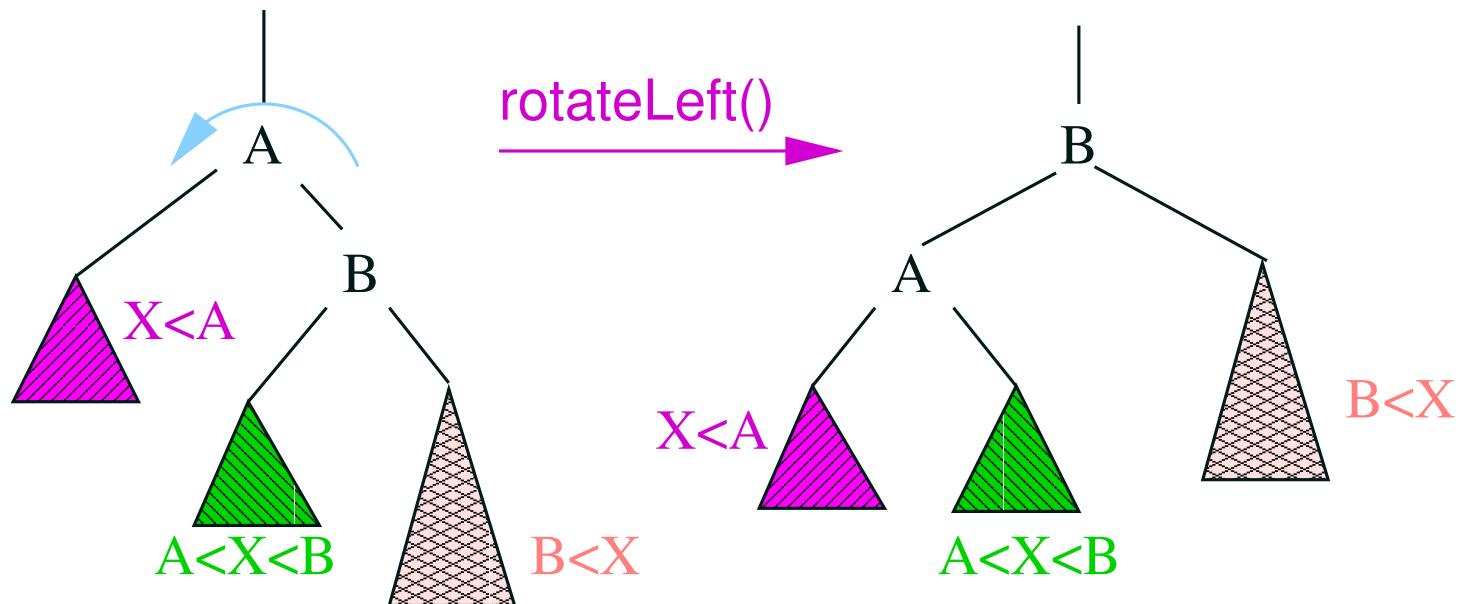
Rotations

- To avoid unbalanced trees we would like to modify the shape
- This is possible as the shape of the tree is not uniquely defined (e.g. we could make any node the root)
- We can change the shape of a tree using **rotations**
- E.g. left rotation



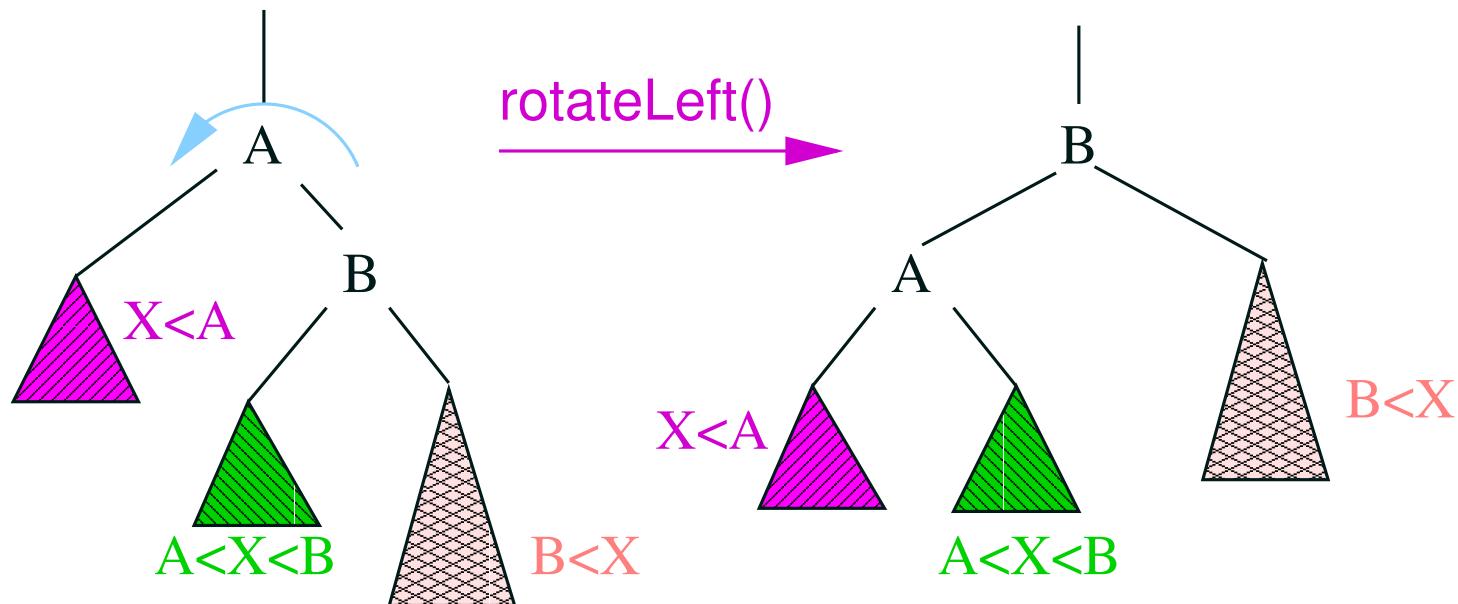
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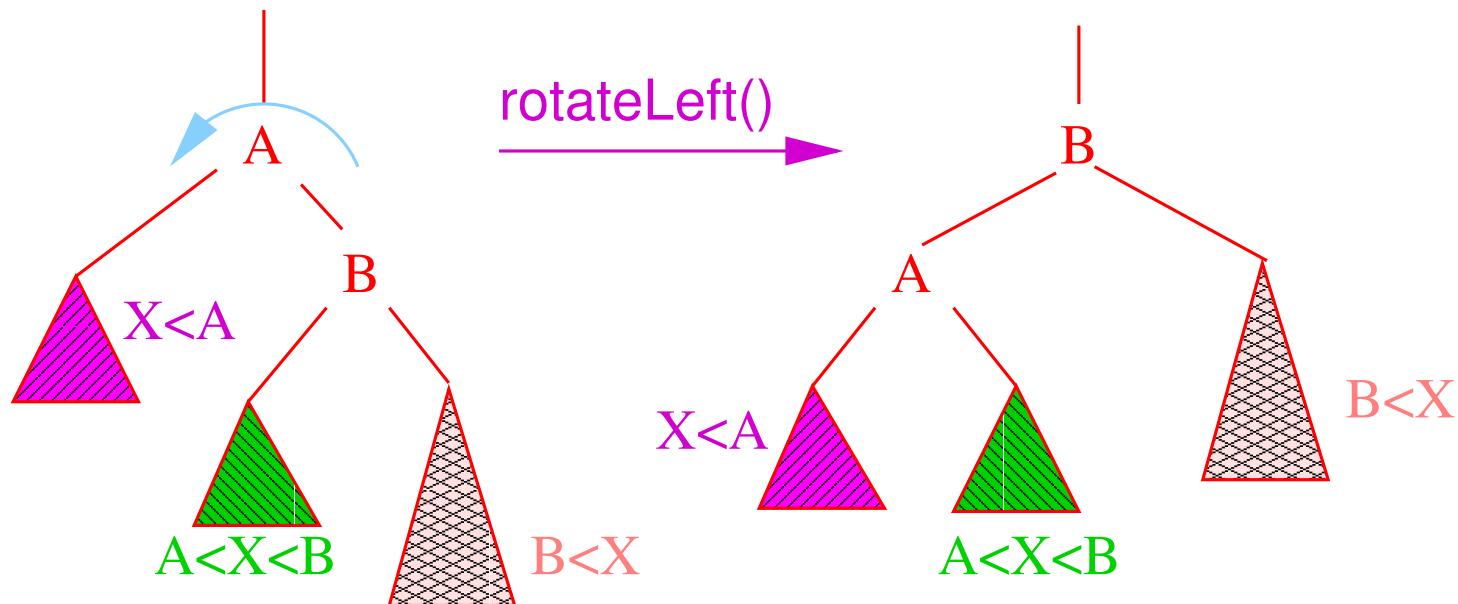
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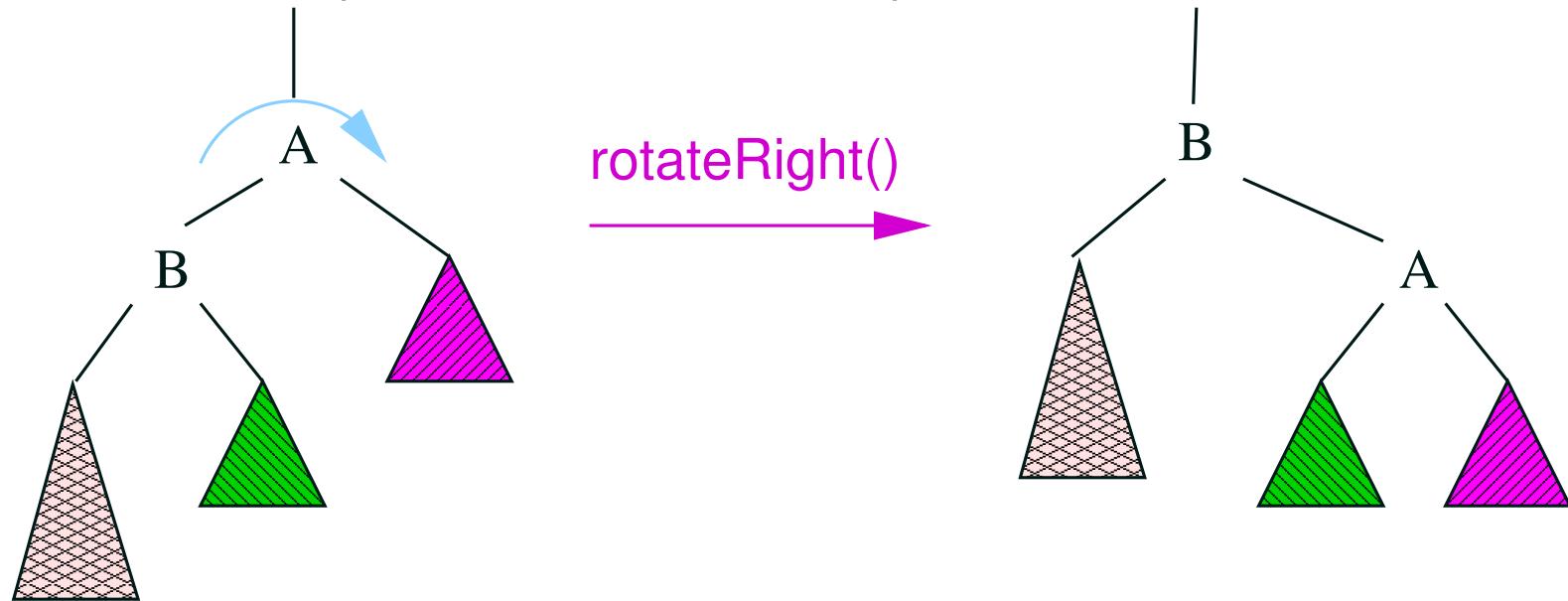
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Types of Rotations

- We can get by with 4 types of rotations

- ★ Left rotation (as above)
- ★ Right rotation (symmetric to above)

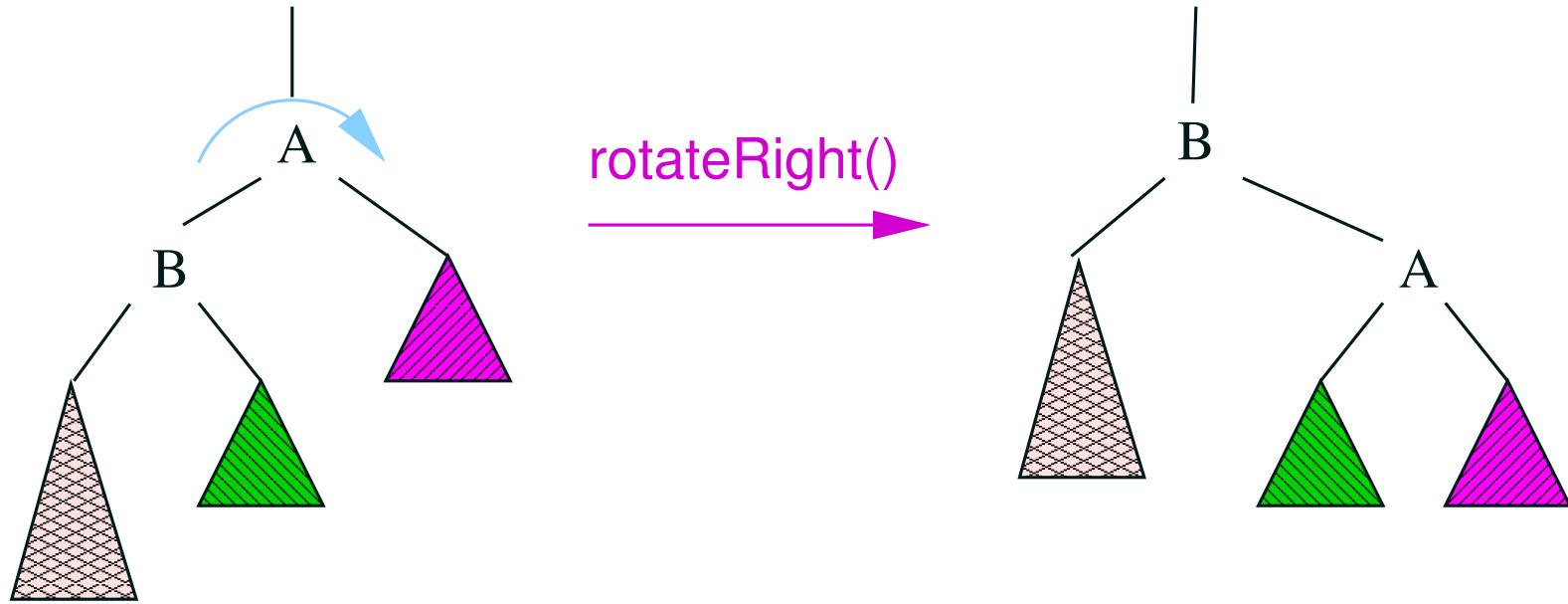


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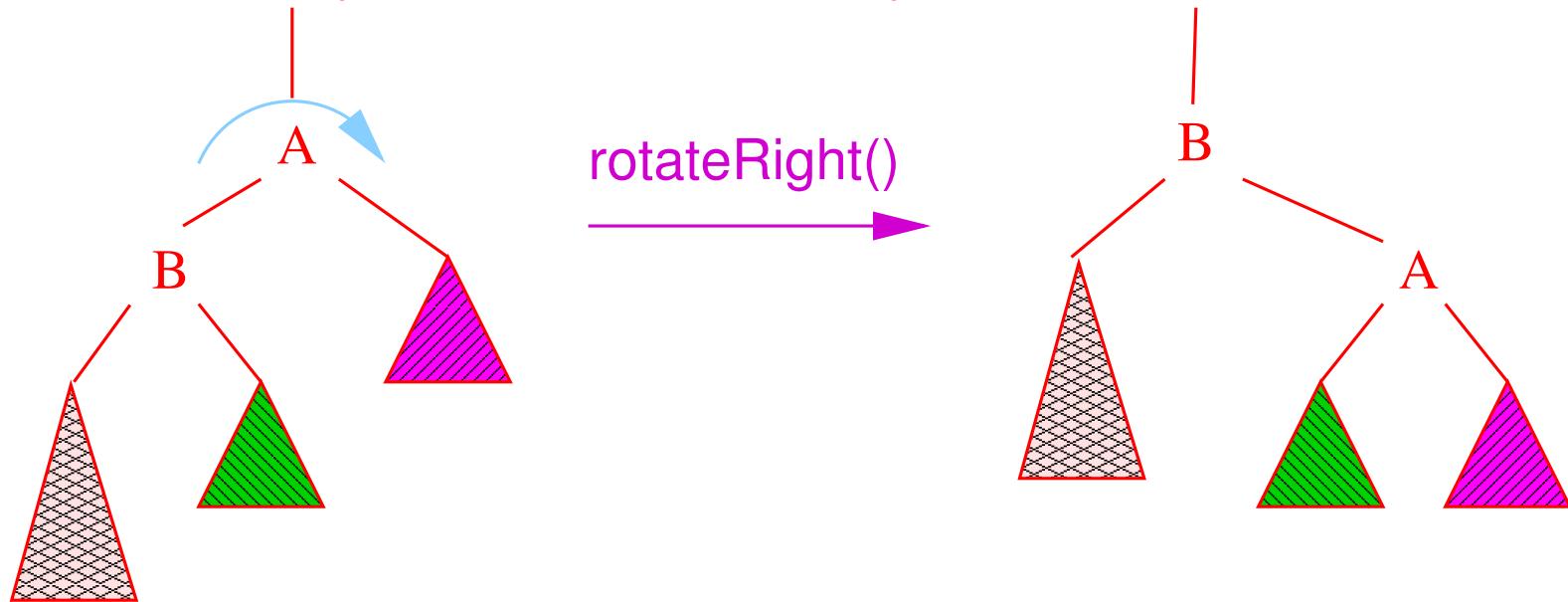


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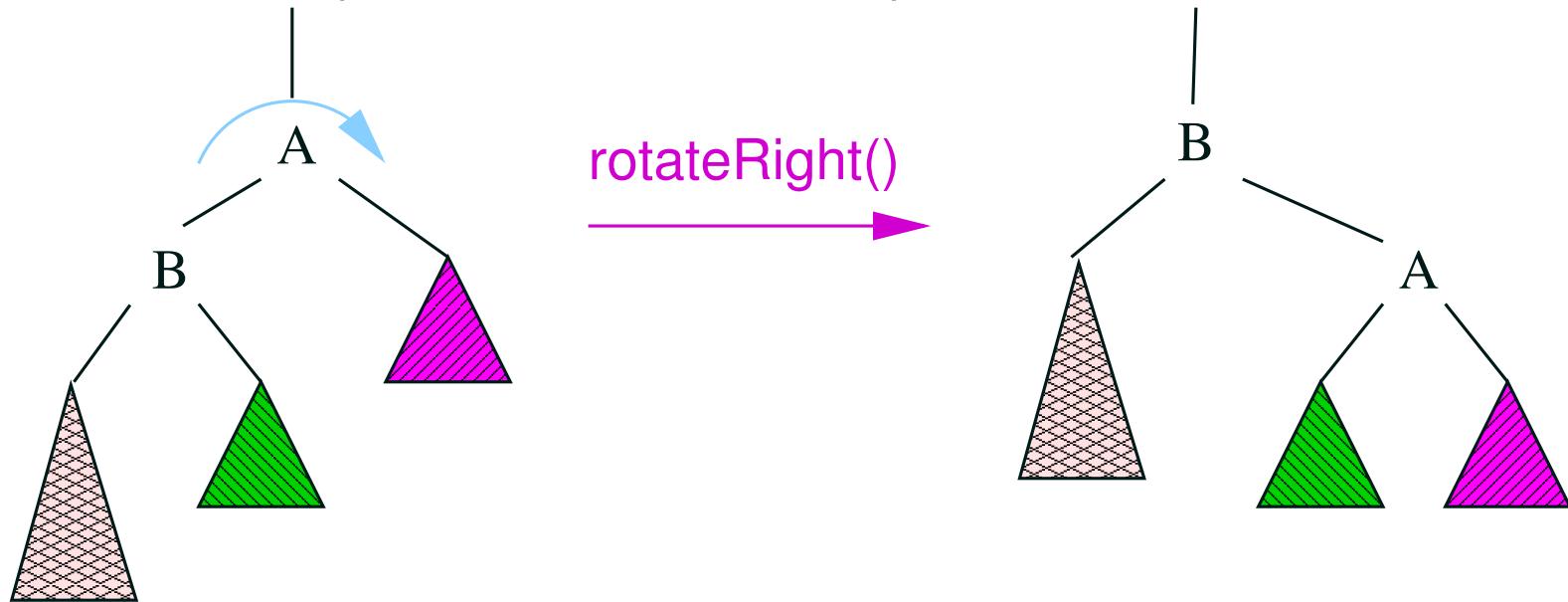


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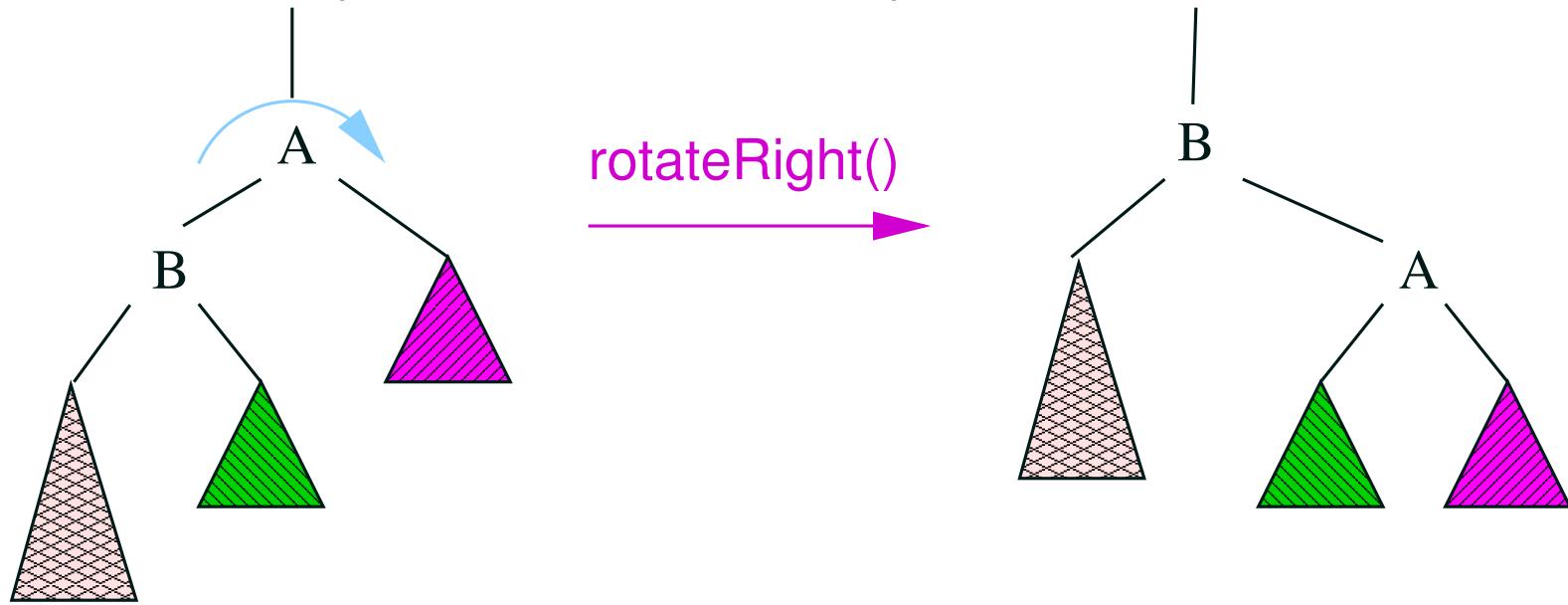


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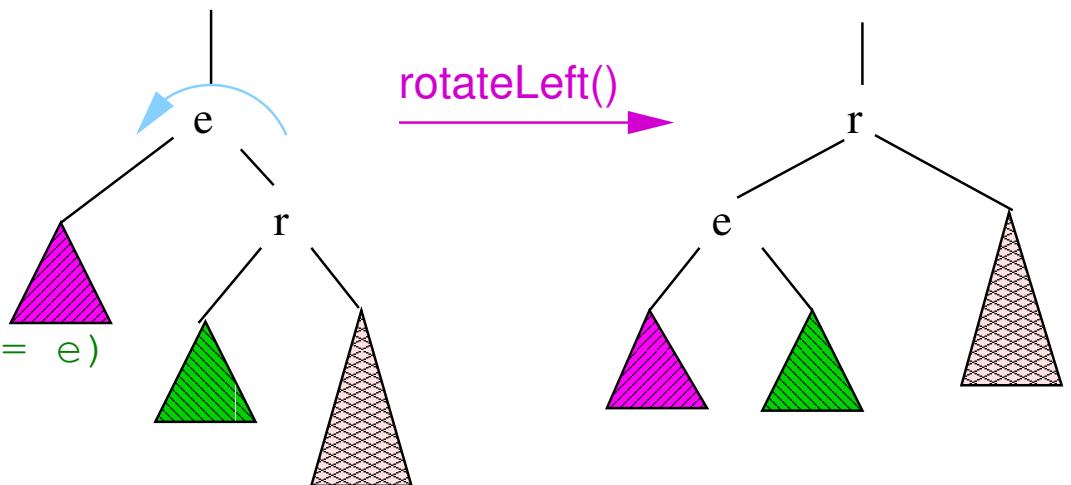
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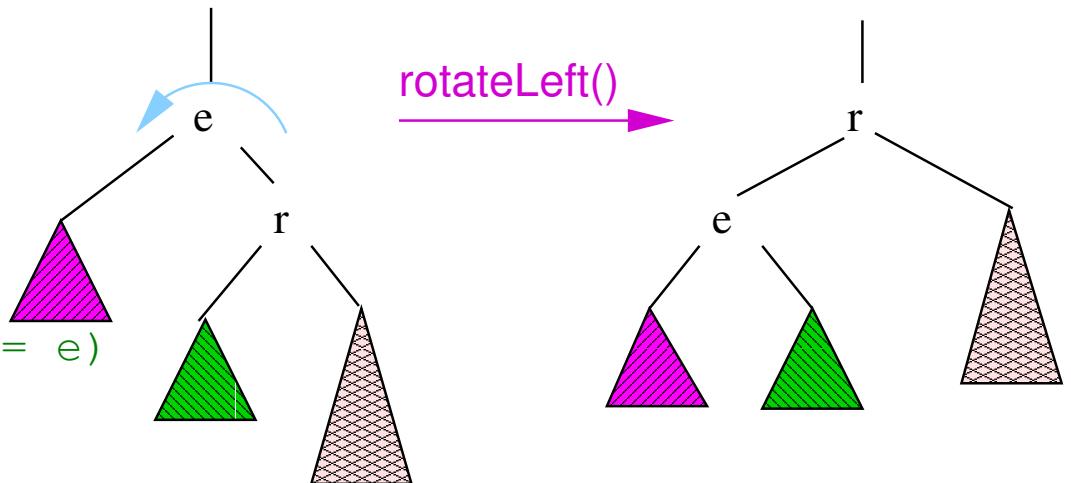
Coding Rotations

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void rotateLeft(Node* e)
{
    Node* r = e->right;
    e->right = r->left;
    if (r->left != nullptr)
        r->left->parent = e;
    r->parent = e->parent;
    if (e->parent == nullptr)
        root = r;
    else if (e->parent->left == e)
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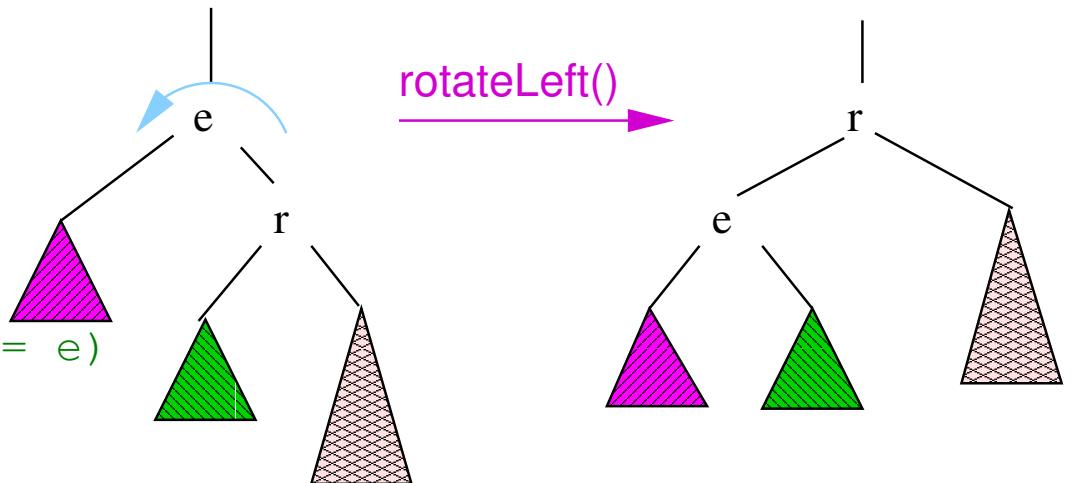
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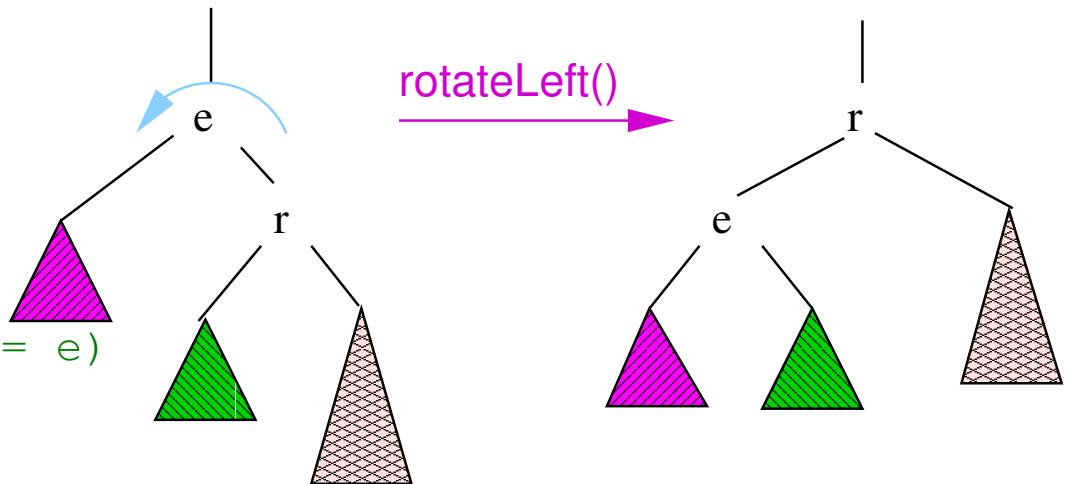
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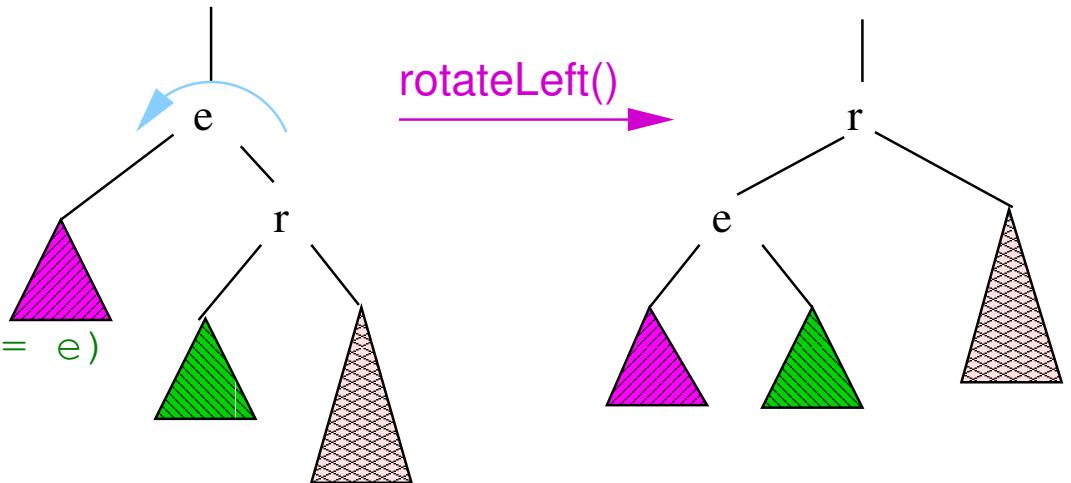
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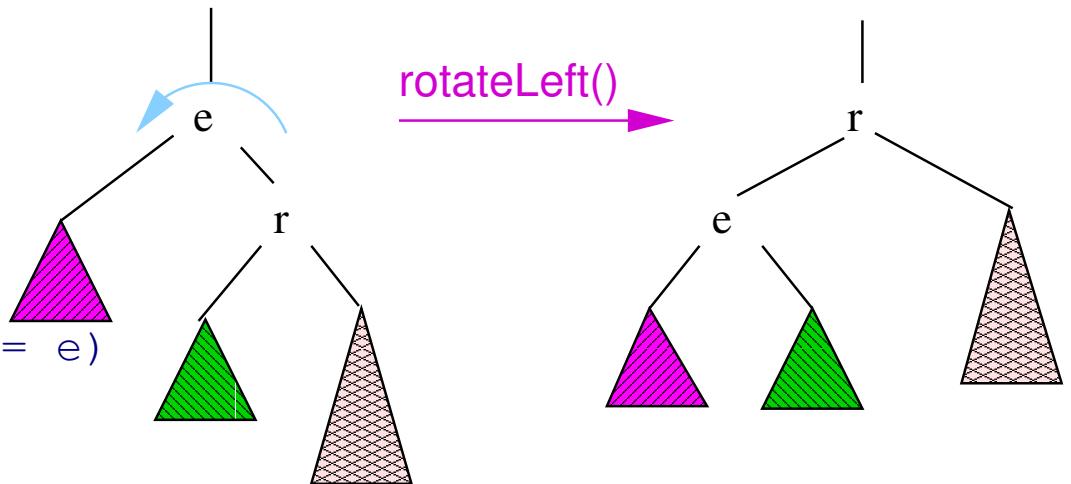
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    r->parent = e->parent;
    if (e->parent == nullptr)
        root = r;
    else if (e->parent->left == e)
        e->parent->left = r;
    else
        e->parent->right = r;
    r->left = e;
    e->parent = r;
}
```



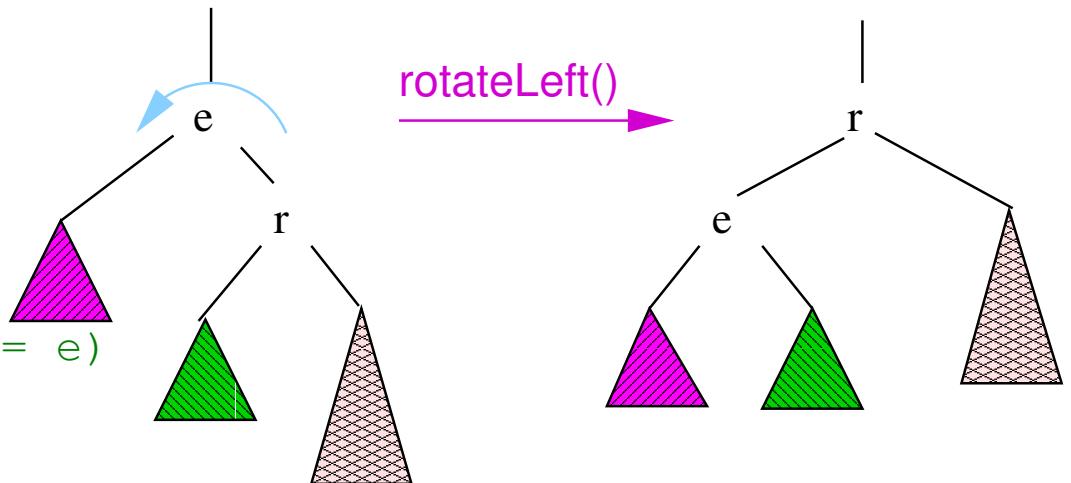
Coding Rotations

```
void rotateLeft(Node* e)
{
    Node* r = e->right;
    e->right = r->left;
    if (r->left != nullptr)
        r->left->parent = e;
    r->parent = e->parent;
    if (e->parent == nullptr)
        root = r;
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        e->parent->left = r;
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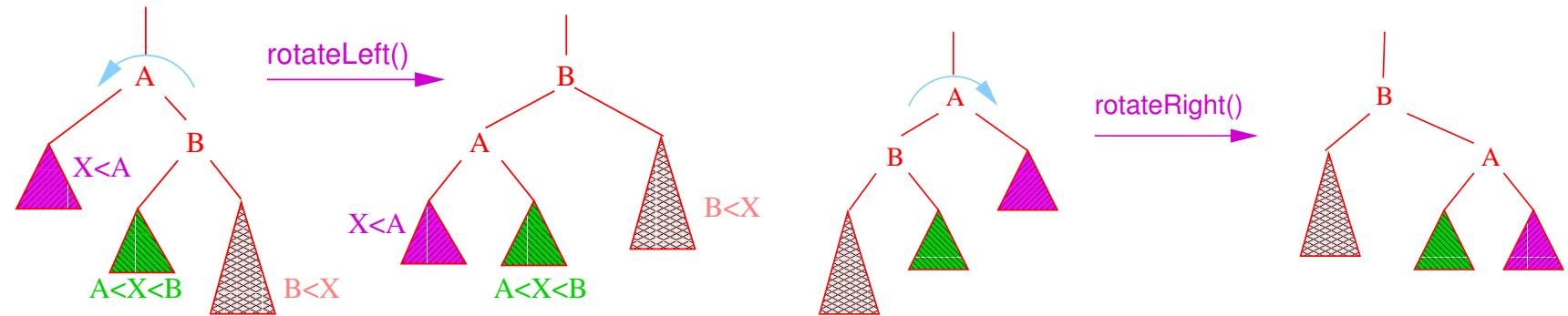
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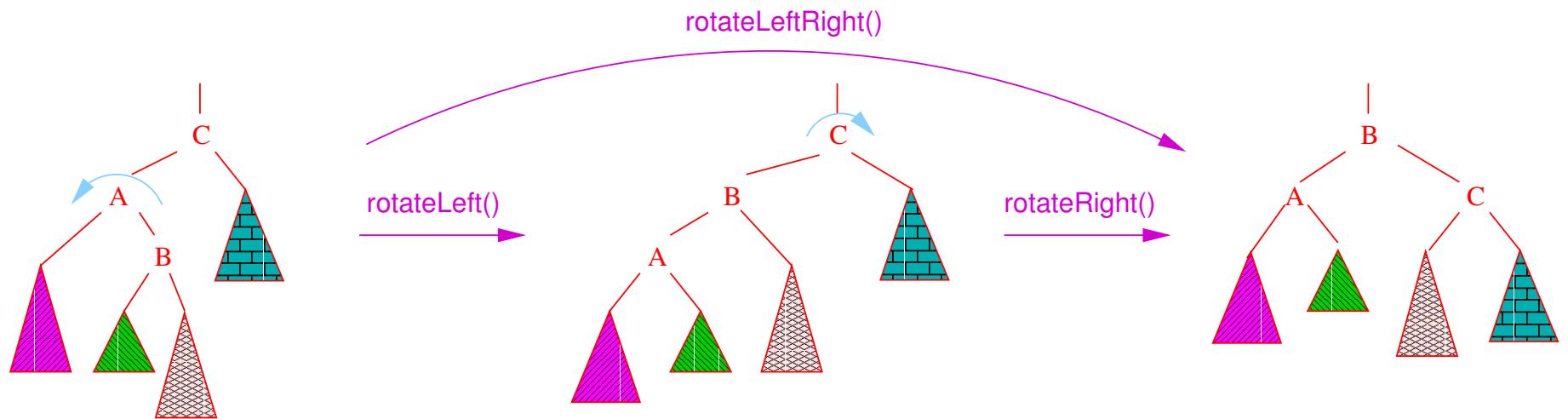
When Single Rotations Work

- Single rotations balance the tree when the unbalanced subtree is on the outside



Double Rotations

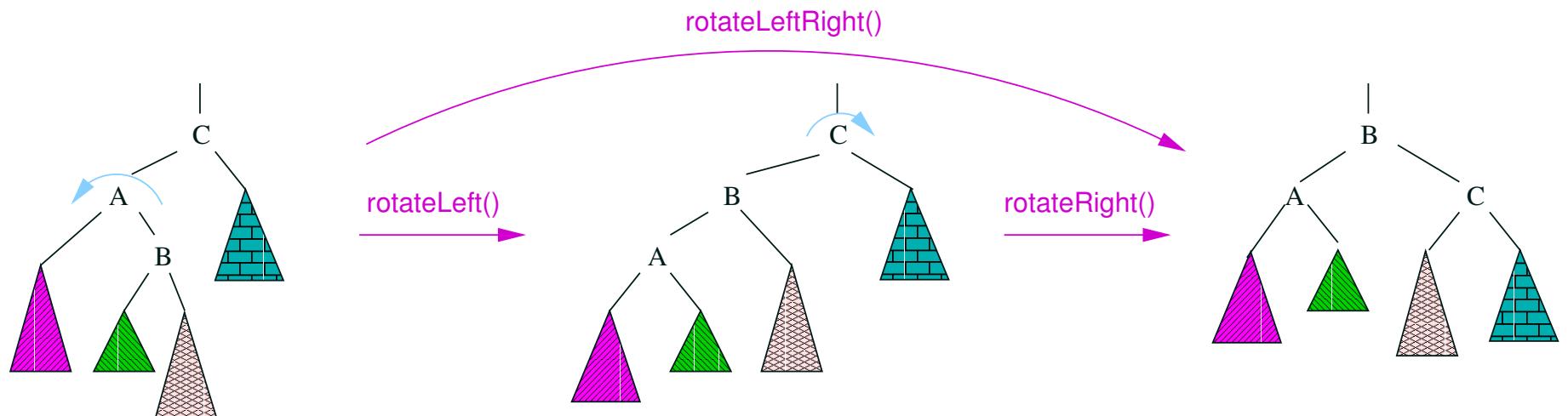
- If the unbalanced subtree is on the inside we need a double rotation



```
leftRotation(C.left);  
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Outline

1. Deletion
2. Balancing Trees
 - Rotations
3. **AVL**
4. Red-Black Trees
 - TreeSet
 - TreeMap



Balancing Trees

- There are different strategies for using rotations for balancing trees
- The three most popular are
 - ★ AVL-trees
 - ★ Red-black trees
 - ★ Splay trees
- They differ in the criteria they use for doing rotations

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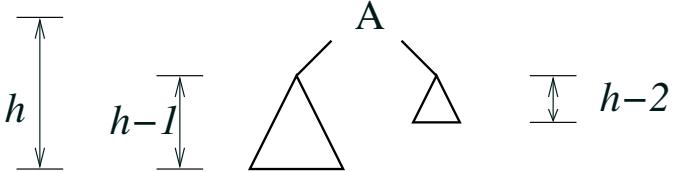
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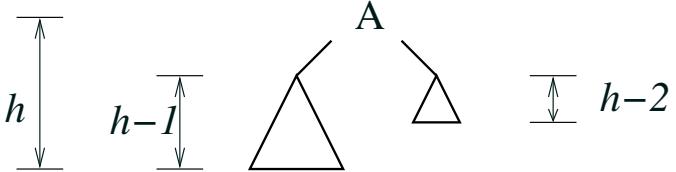
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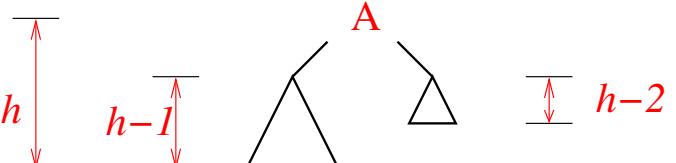
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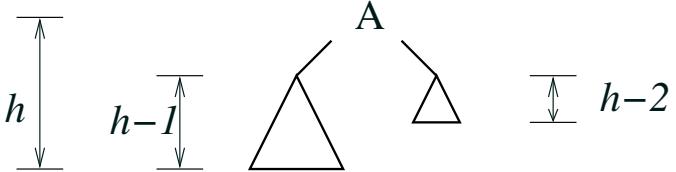
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add(97)

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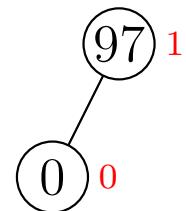
add(0)

(97) 0

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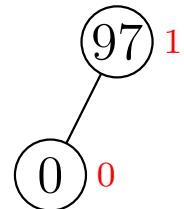


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add(77)

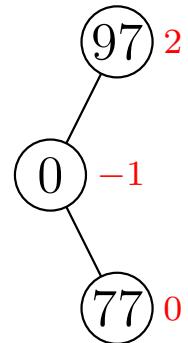


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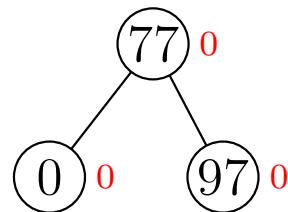
RotateLeftRight



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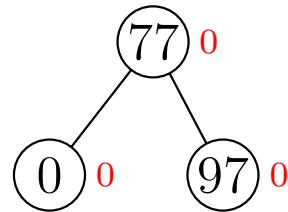


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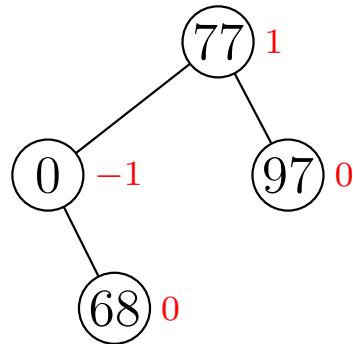
add(68)



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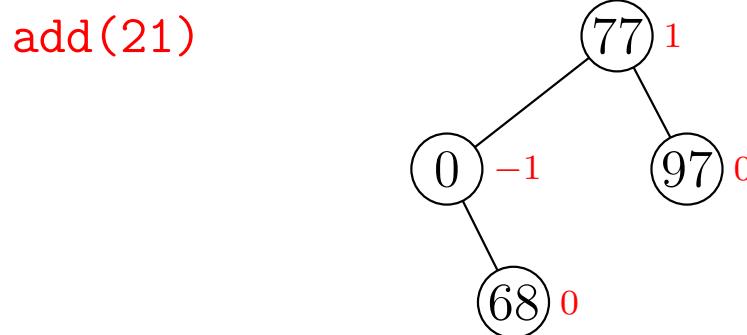
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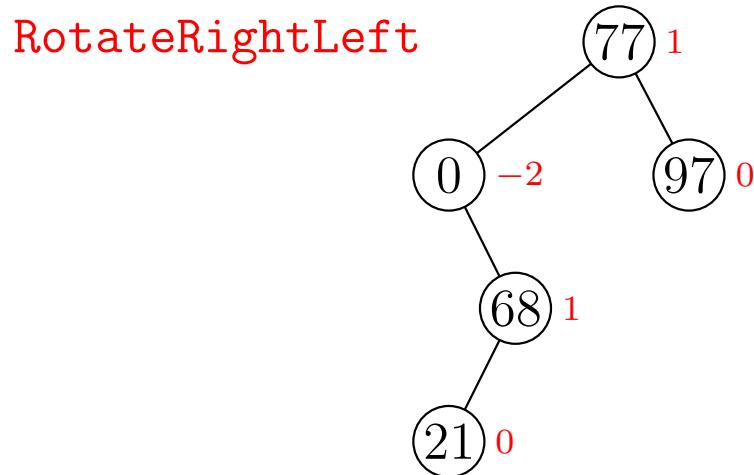
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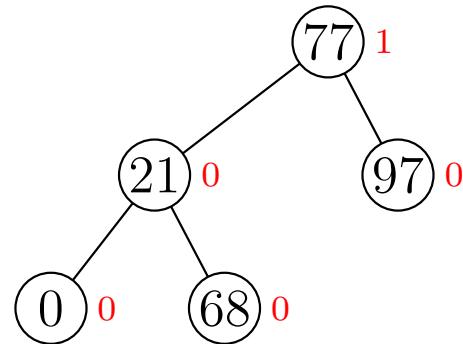
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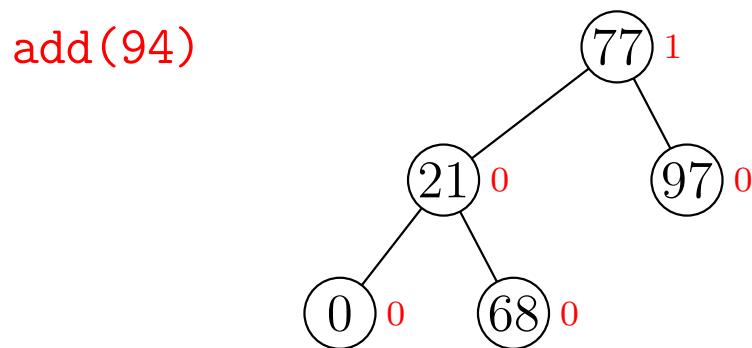
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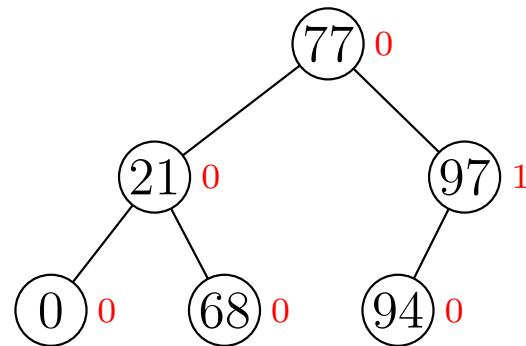
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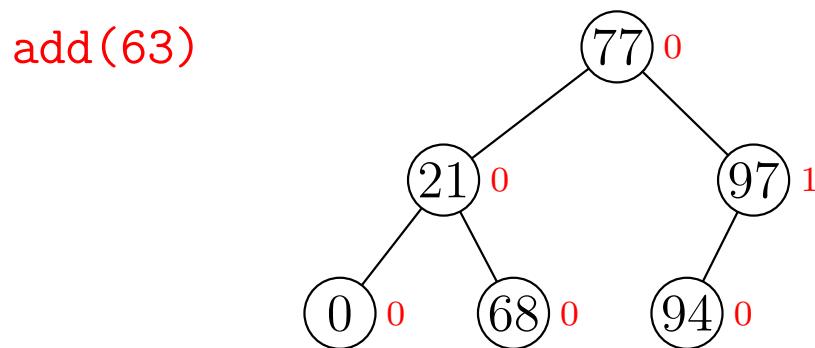
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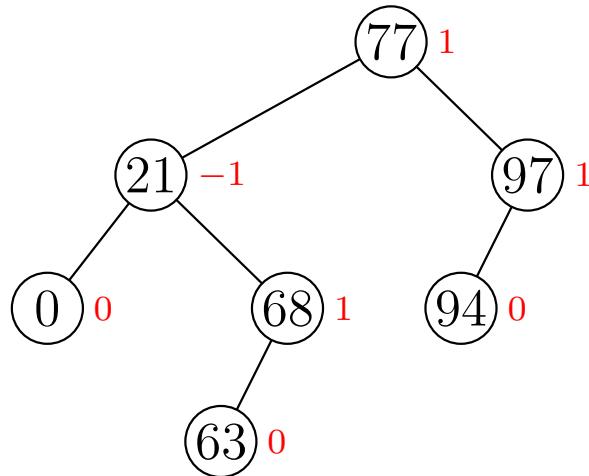
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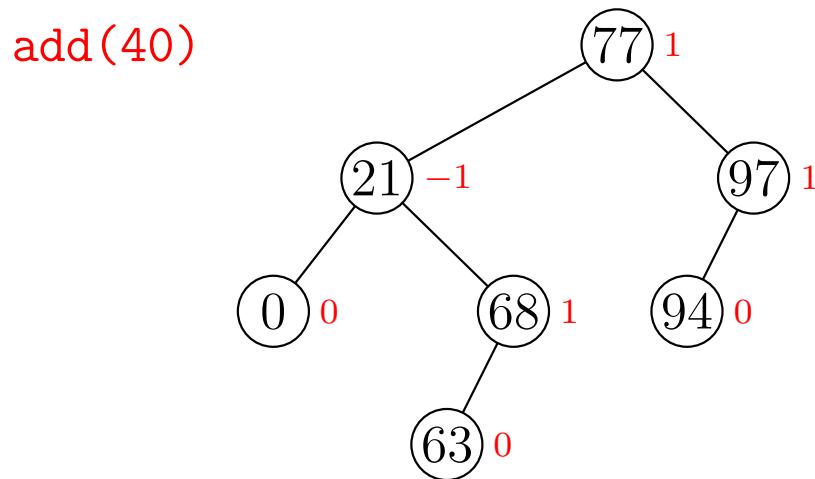
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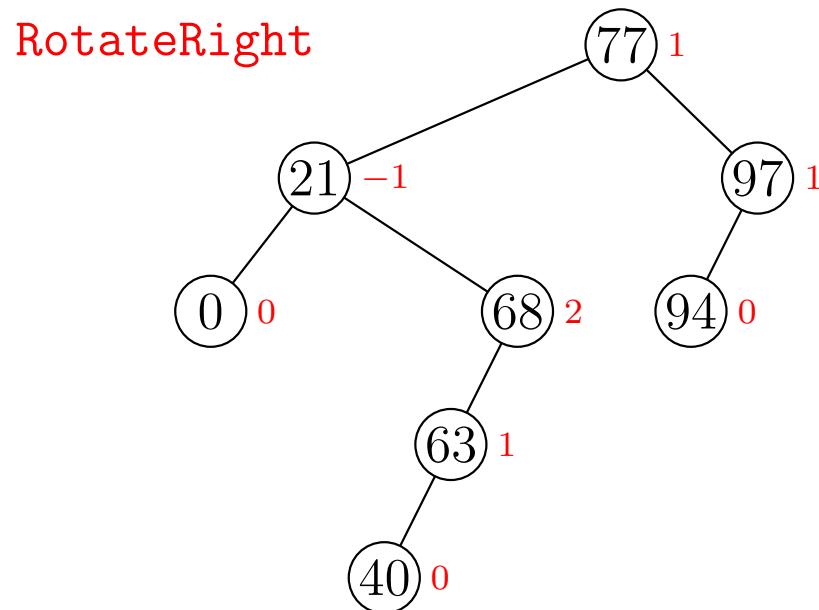
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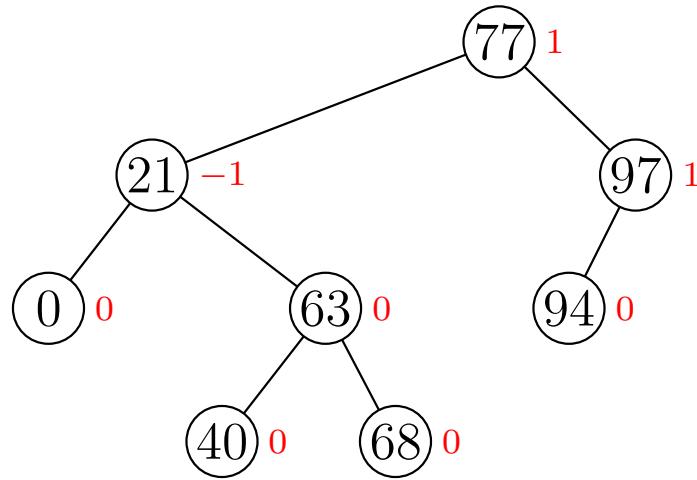
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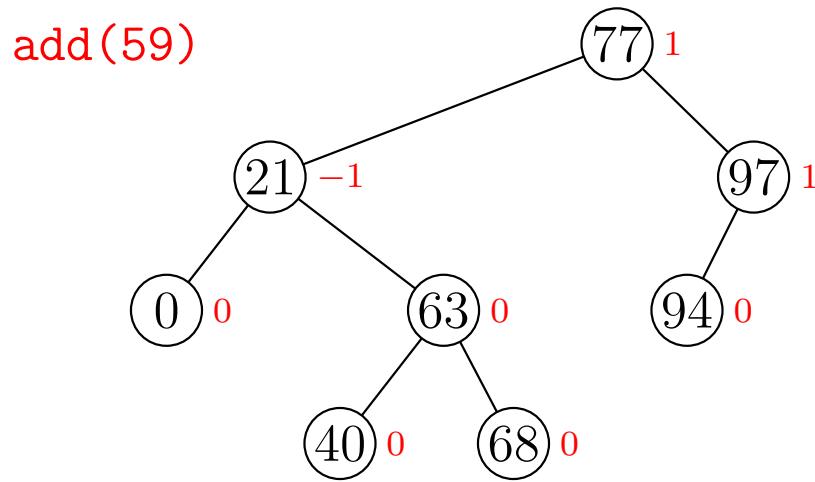
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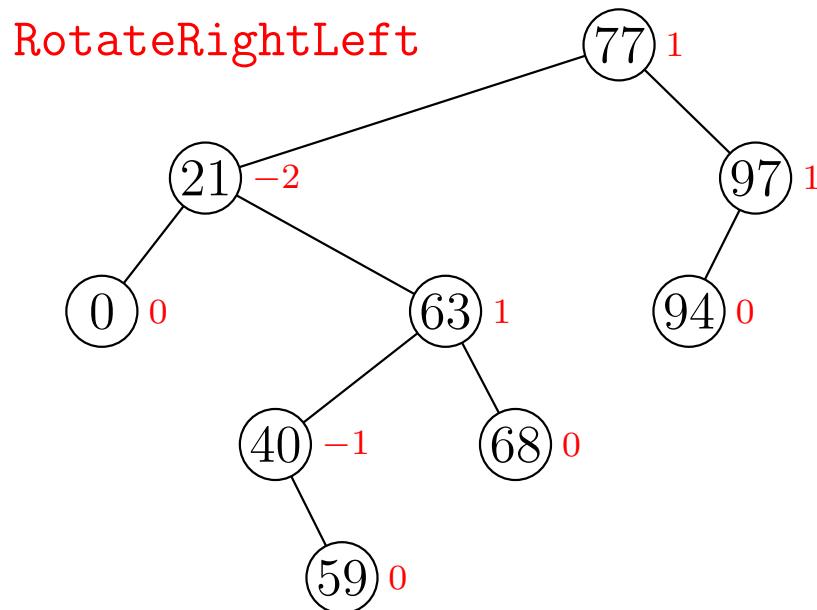
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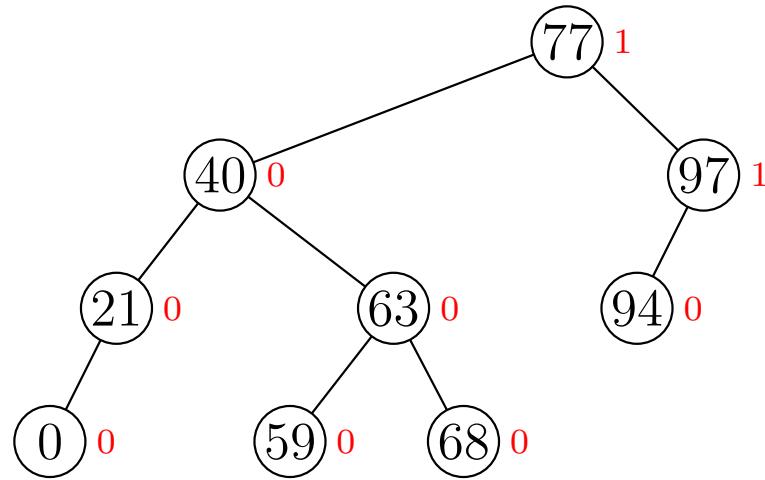
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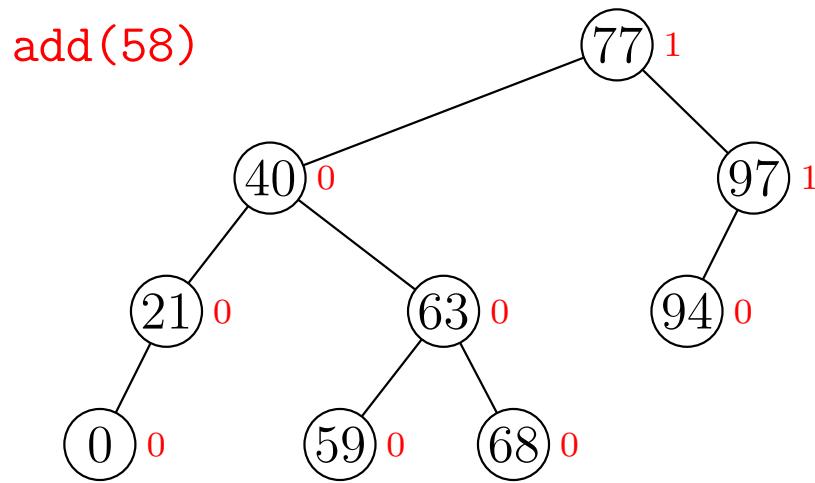
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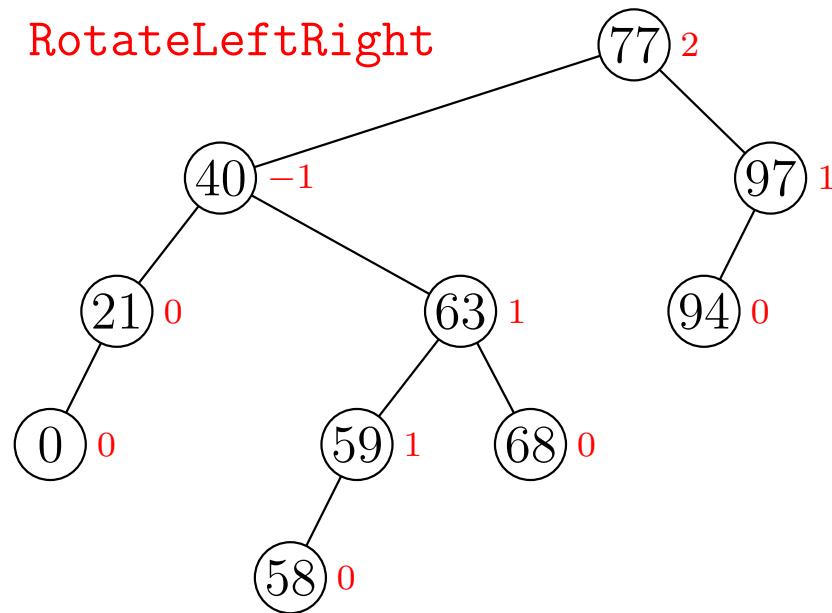
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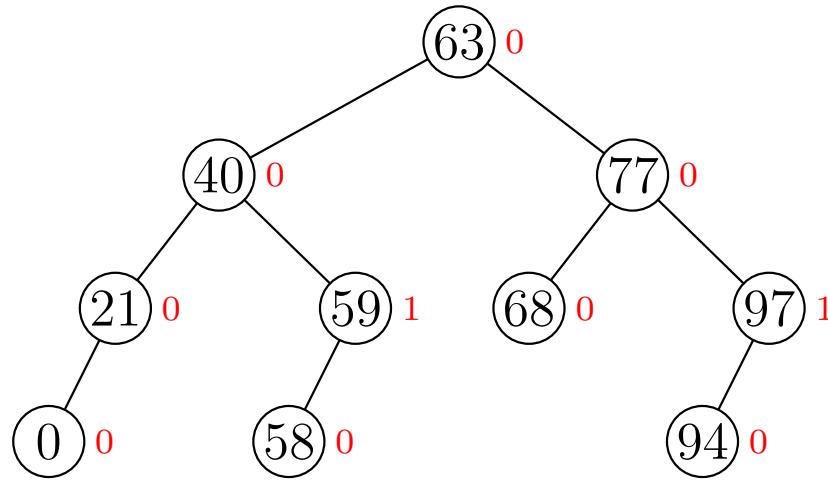
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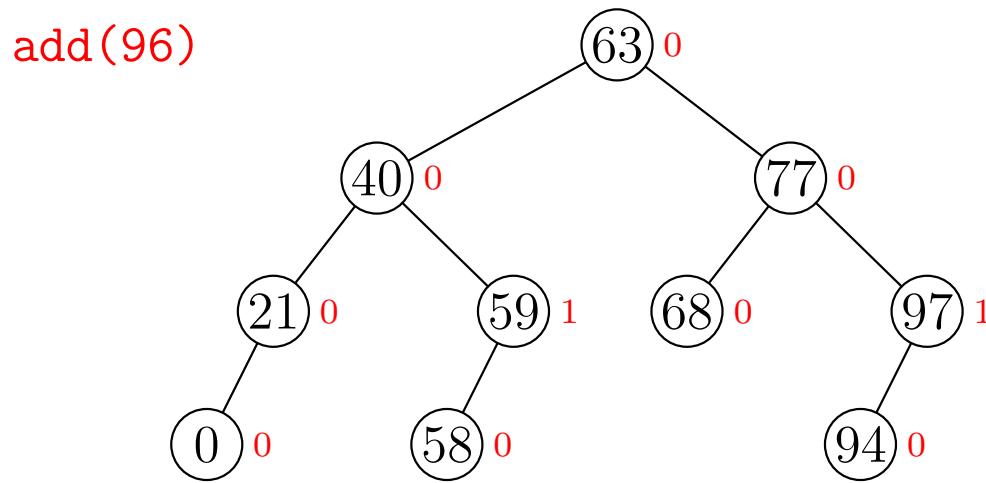
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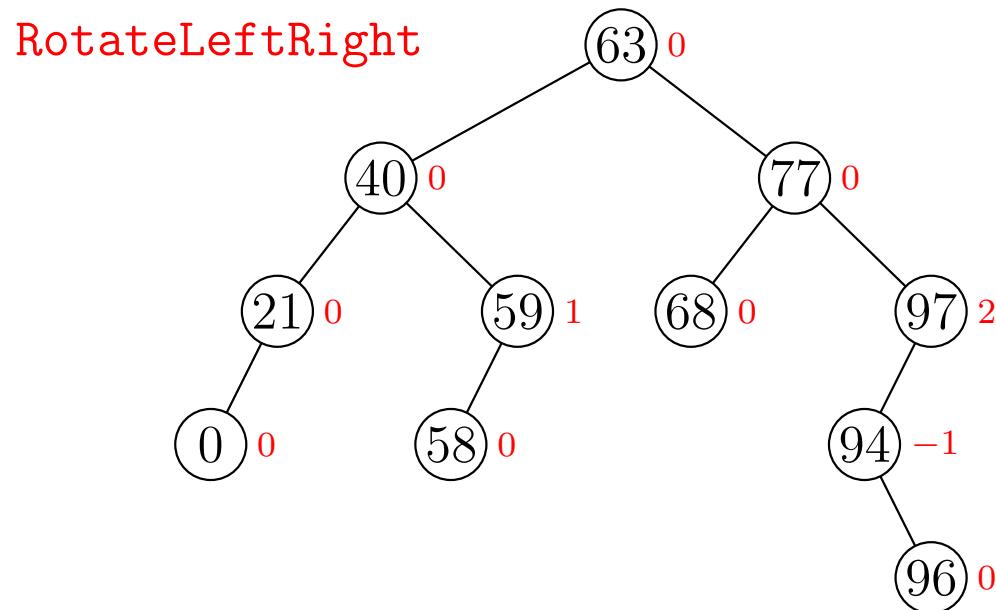
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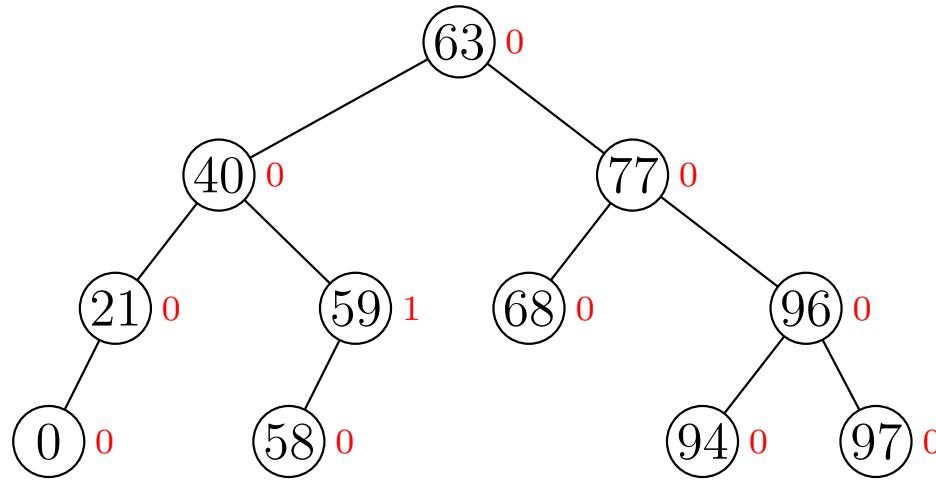


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`balanceFactor = {`

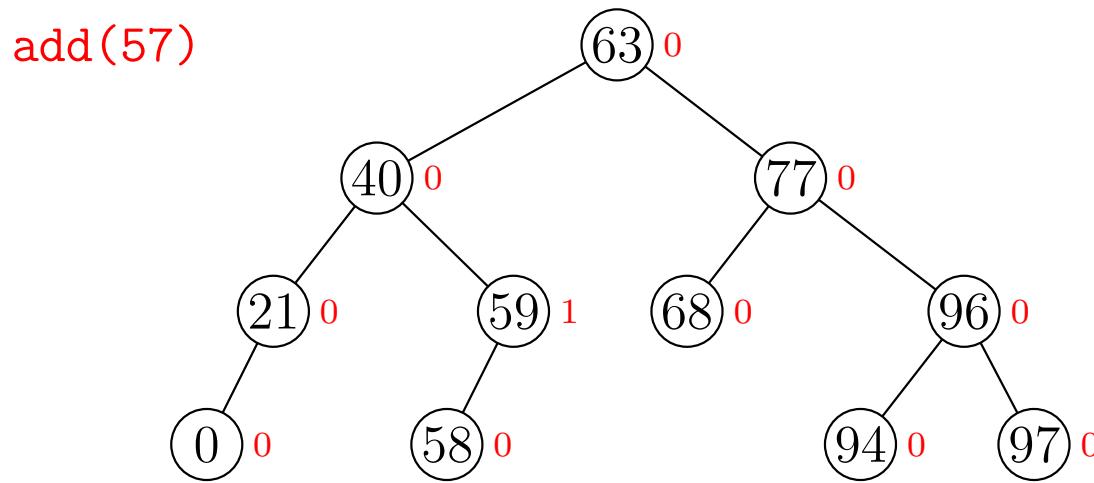
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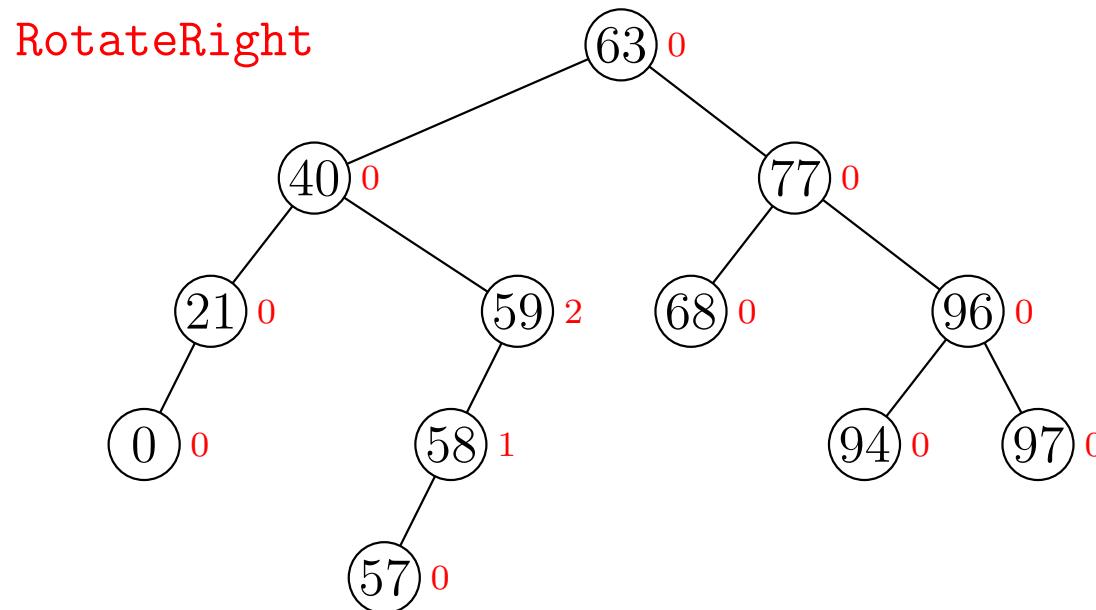
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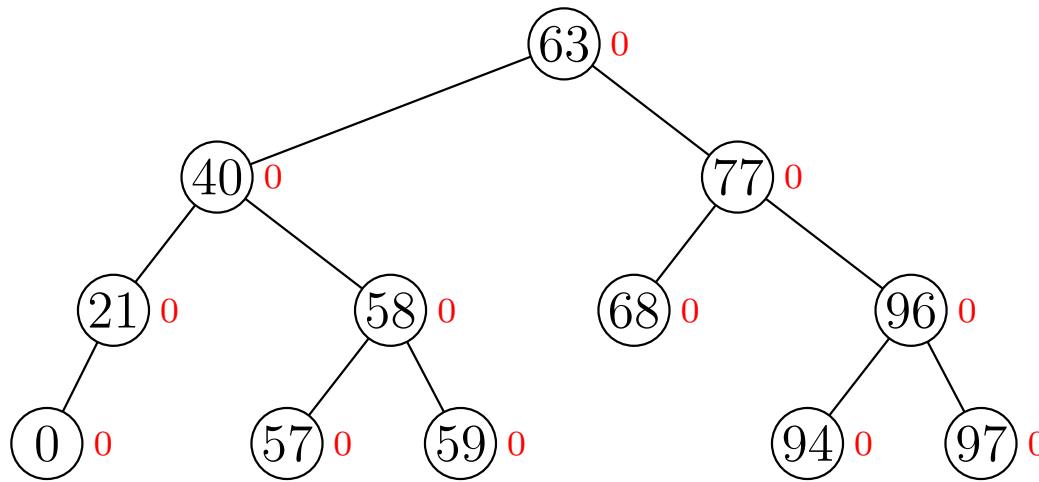
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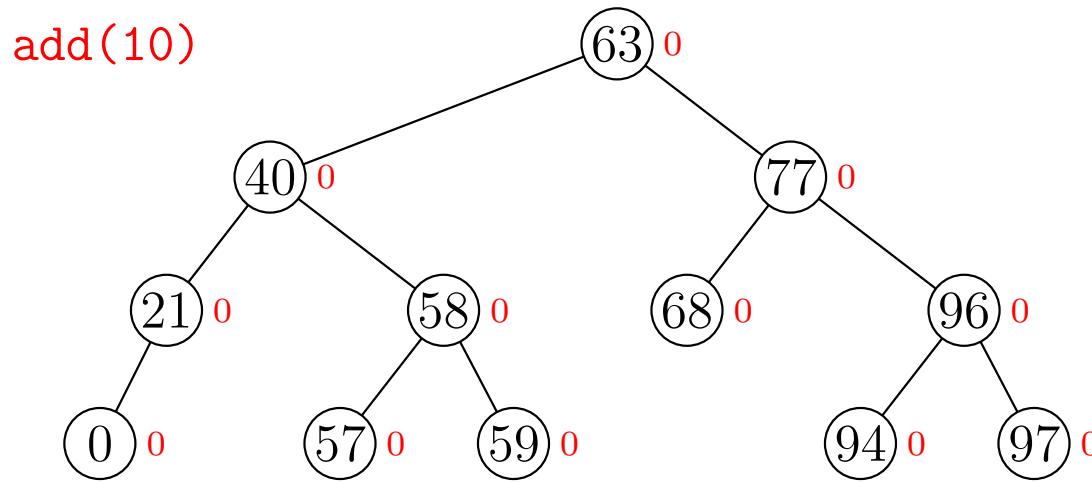
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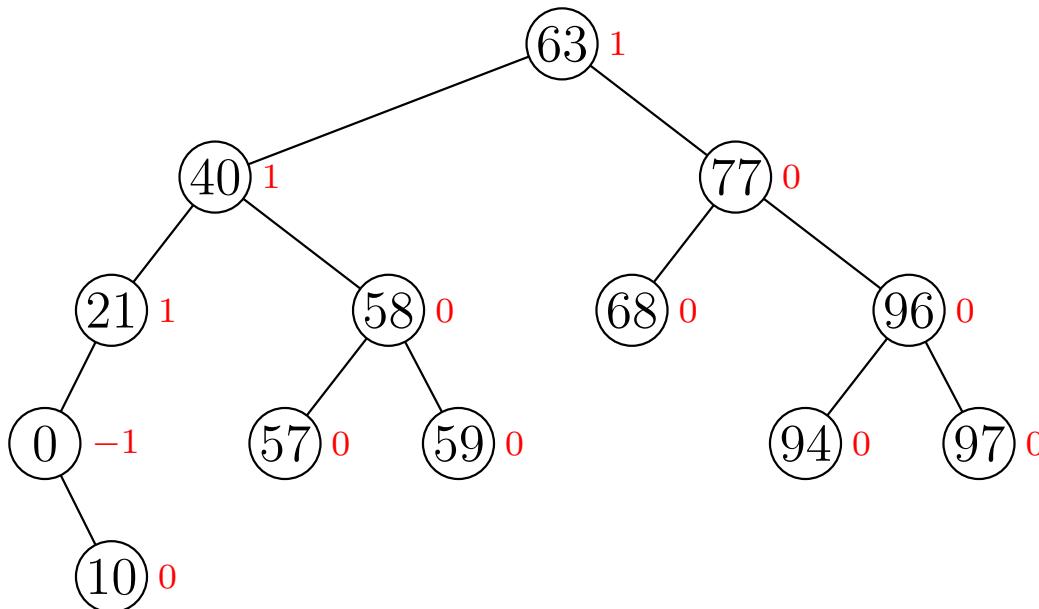
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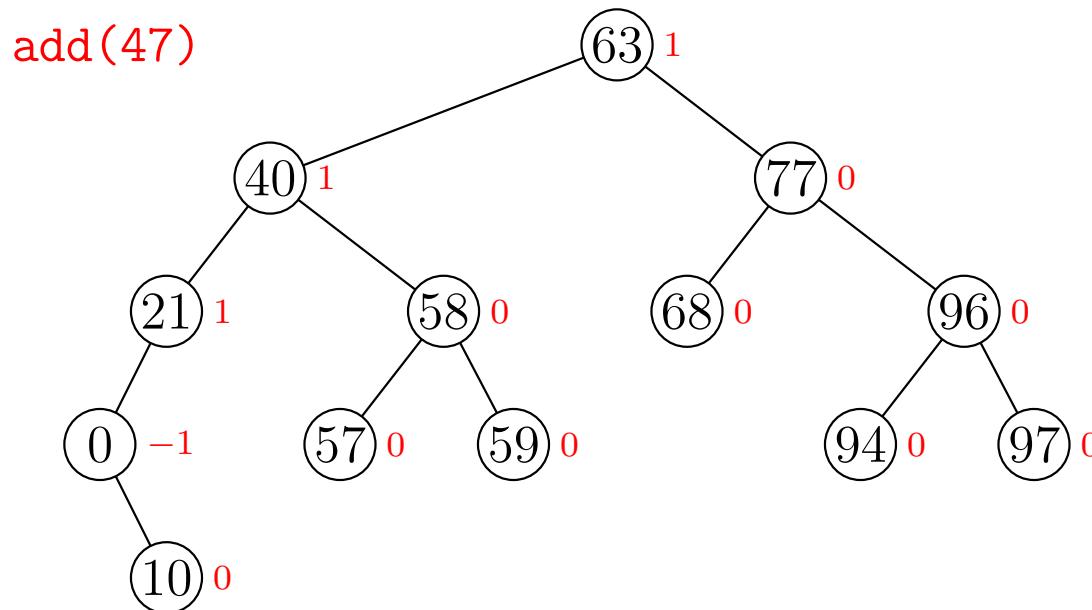
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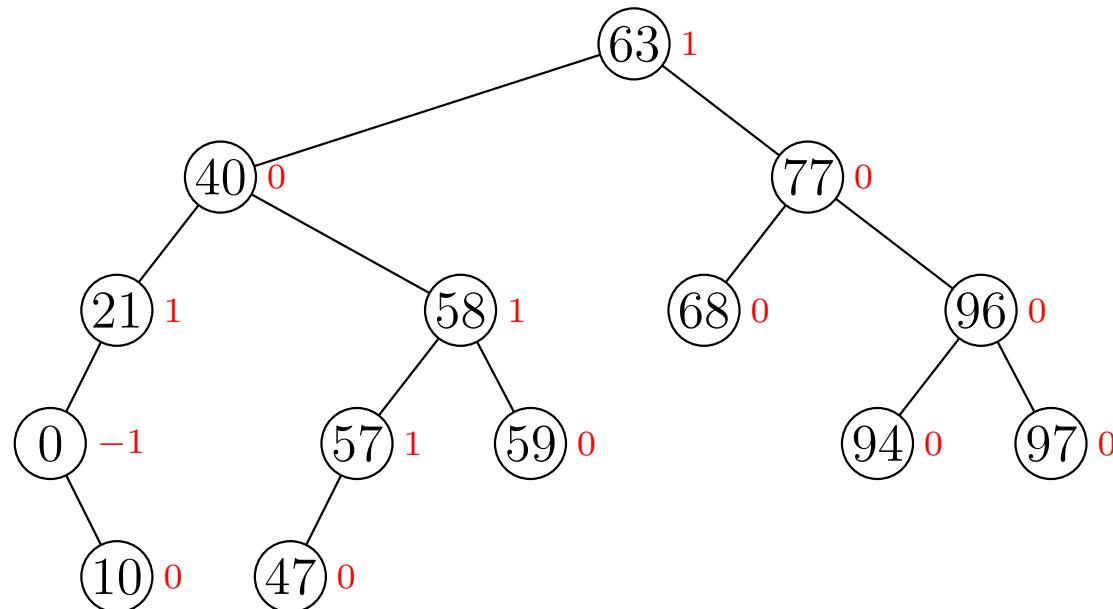
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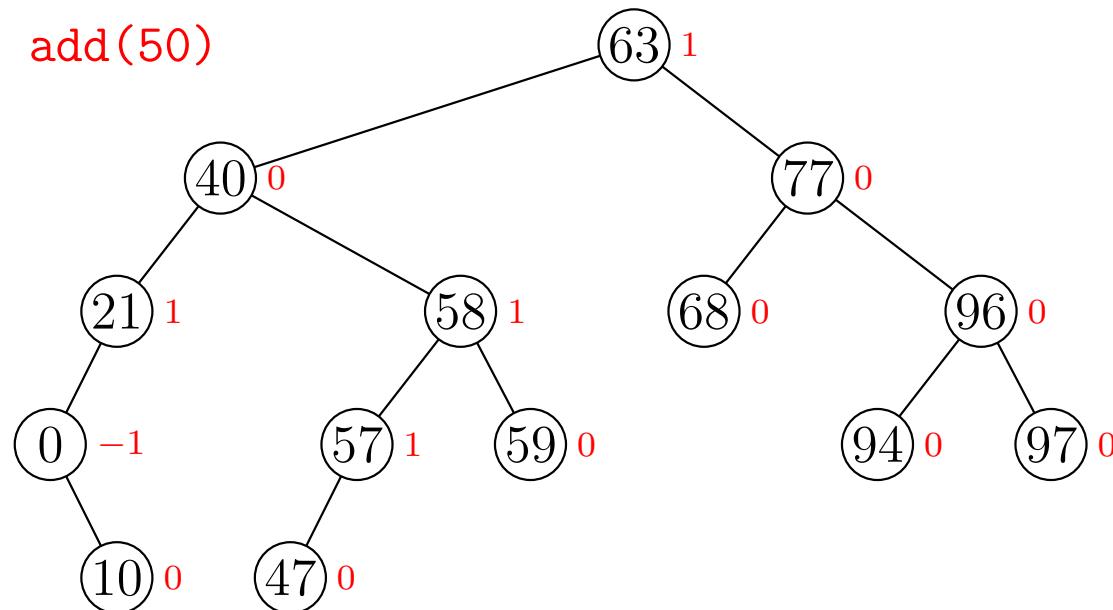
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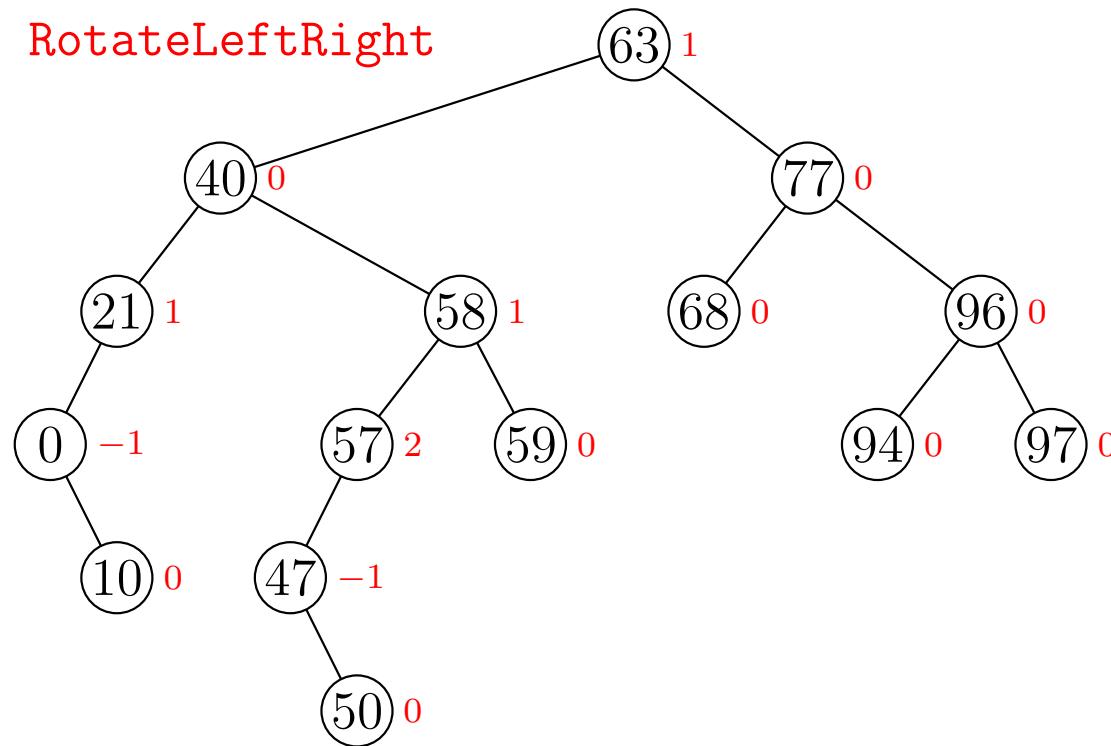
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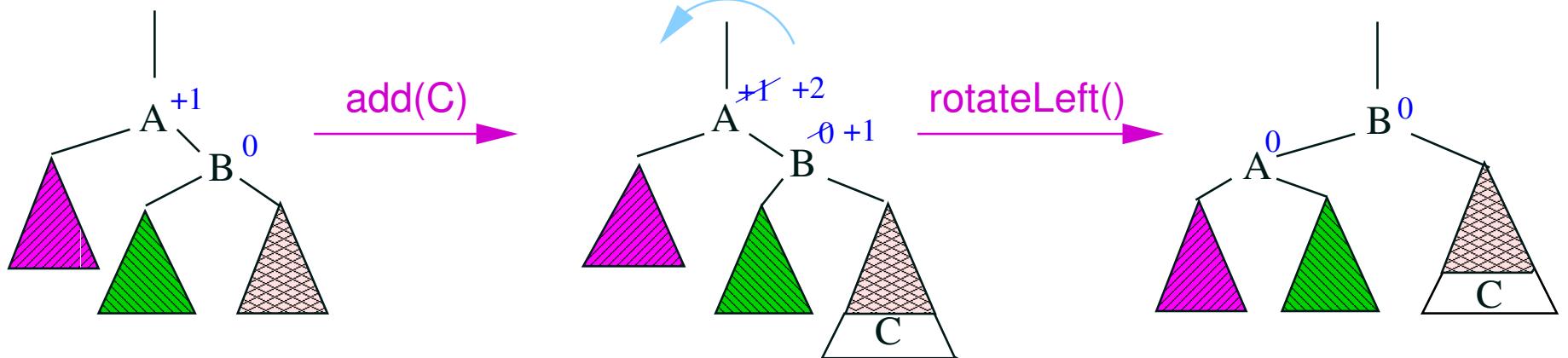
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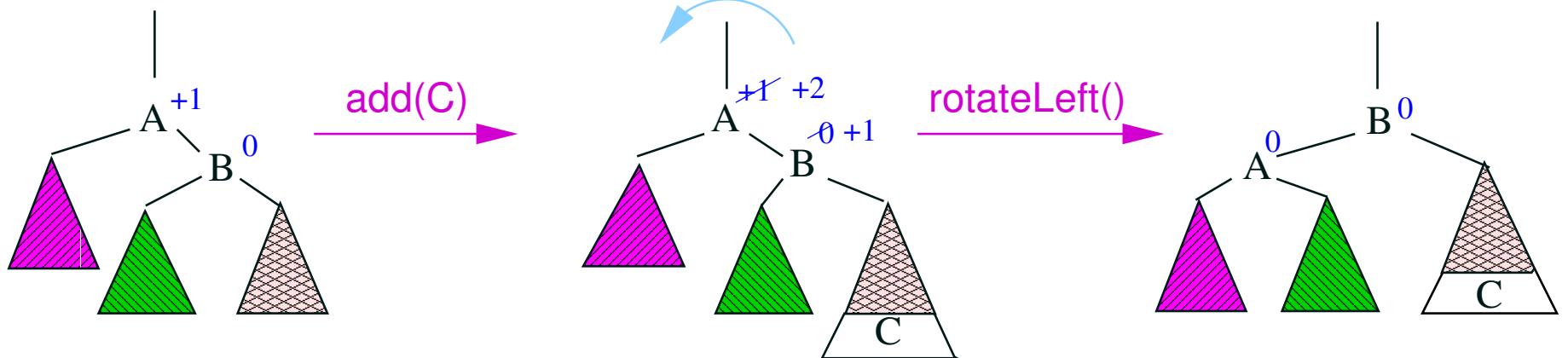
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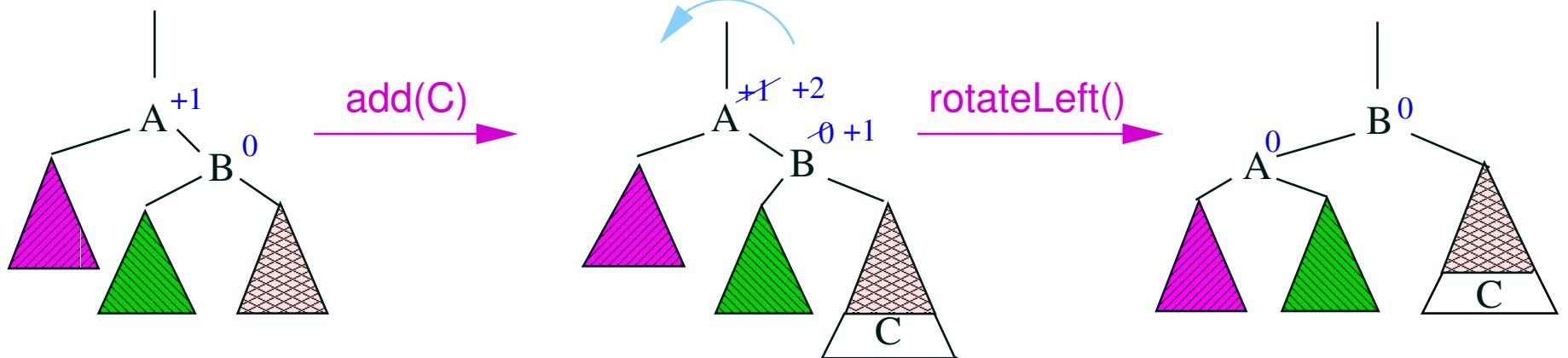
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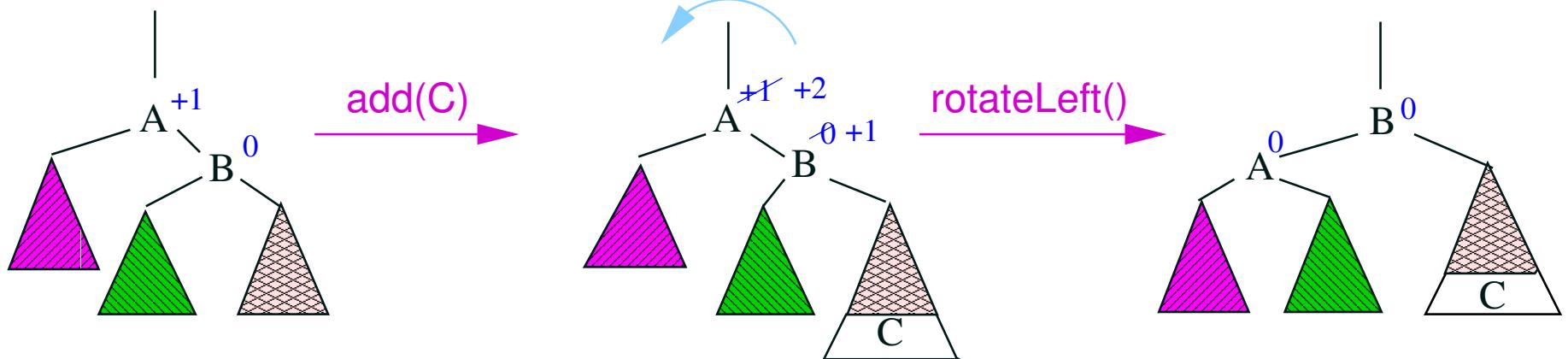
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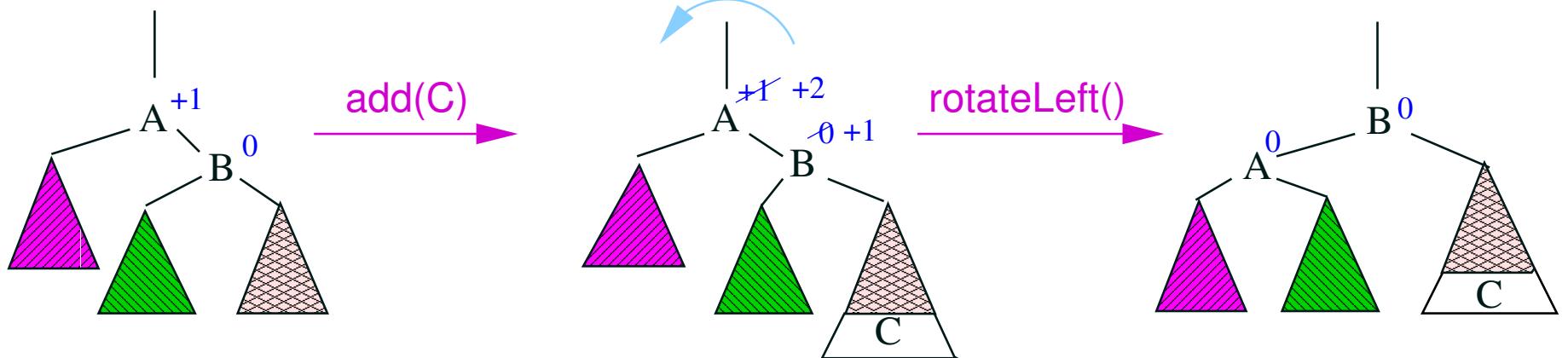
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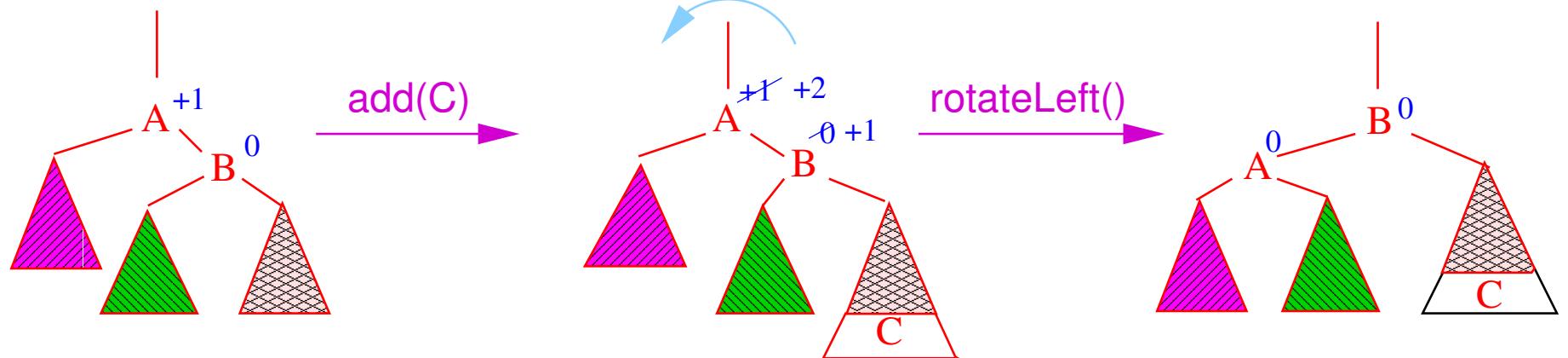
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- AVL deletions are similar to AVL insertions
- One difference is that after performing a rotation the tree may still not satisfy the AVL criteria so higher levels need to be examined
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- The height of an average AVL tree is $1.44 \log_2(n)$
- The height of an average binary search tree is $2.1 \log_2(n)$
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- Search is, of course, quicker

Outline

1. Deletion
2. Balancing Trees
 - Rotations
3. AVL
4. Red-Black Trees
 - TreeSet
 - TreeMap

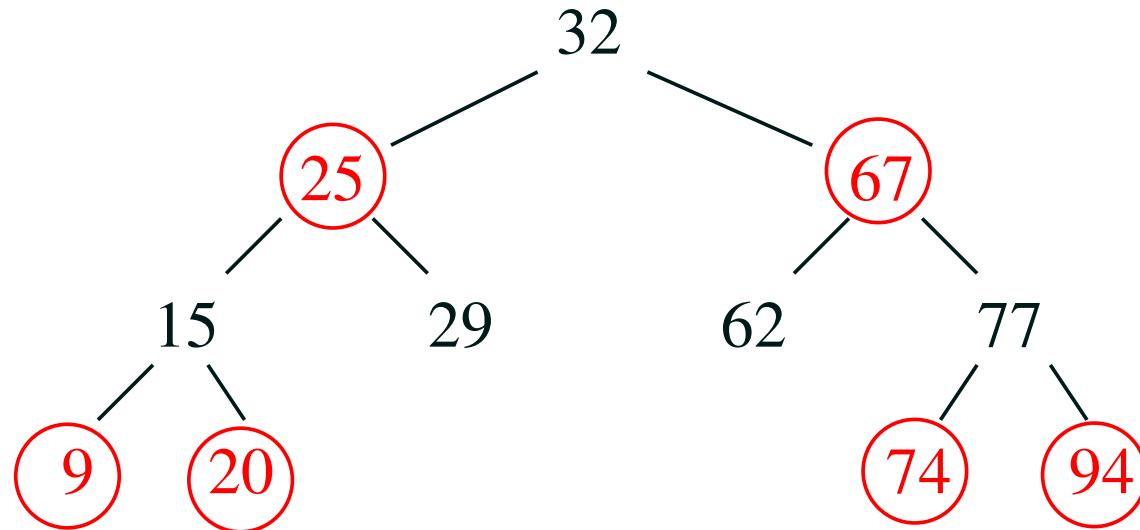


Red-Black Trees

- Red-black trees are another strategy for balancing trees
- Nodes are either *red* or *black*
- Two rules are imposed

Red Rule: the children of a red node must be black

Black Rule: the number of black elements must be the same in all paths from the root to elements with no children or with one child

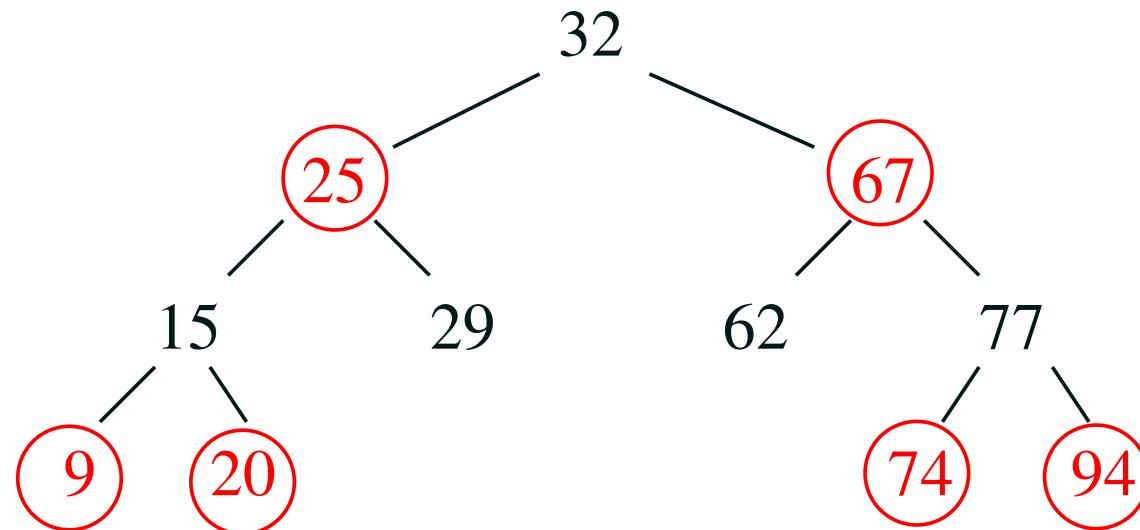


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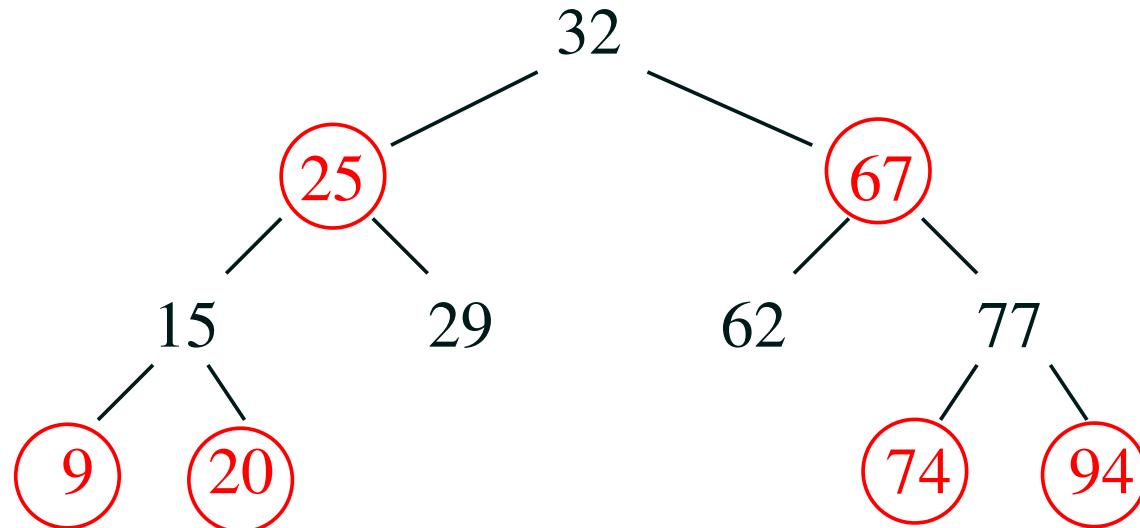


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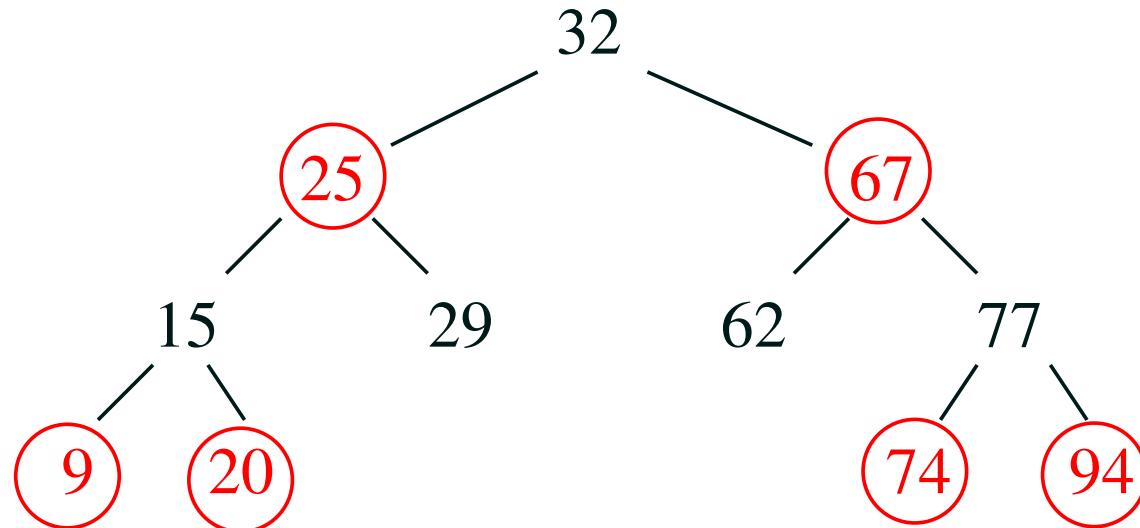


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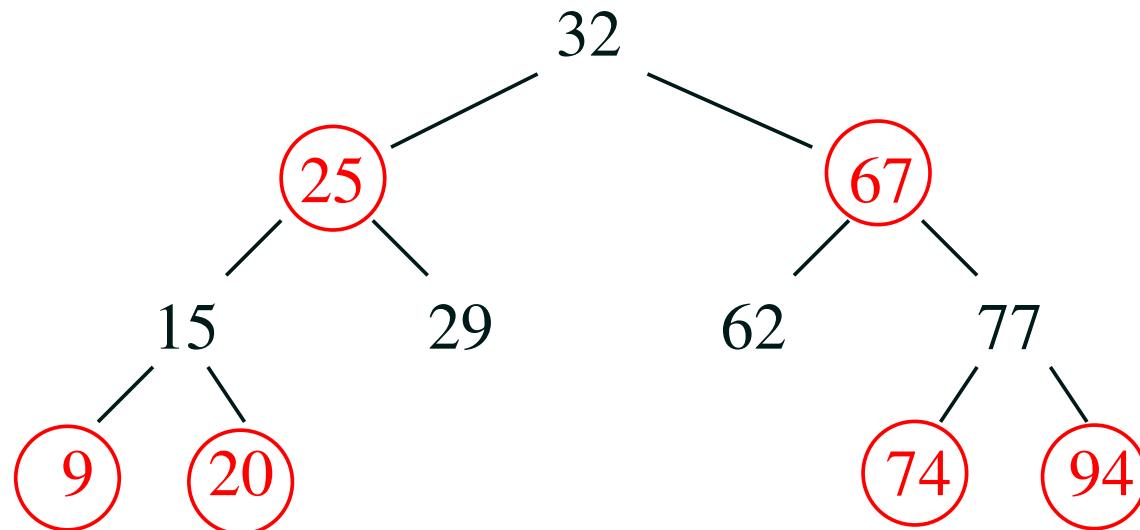


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Restructuring

- When inserting a new element we first find its position
- If it is the root we colour it black otherwise we colour it red
- If its parent is red we must either relabel or restructure the tree

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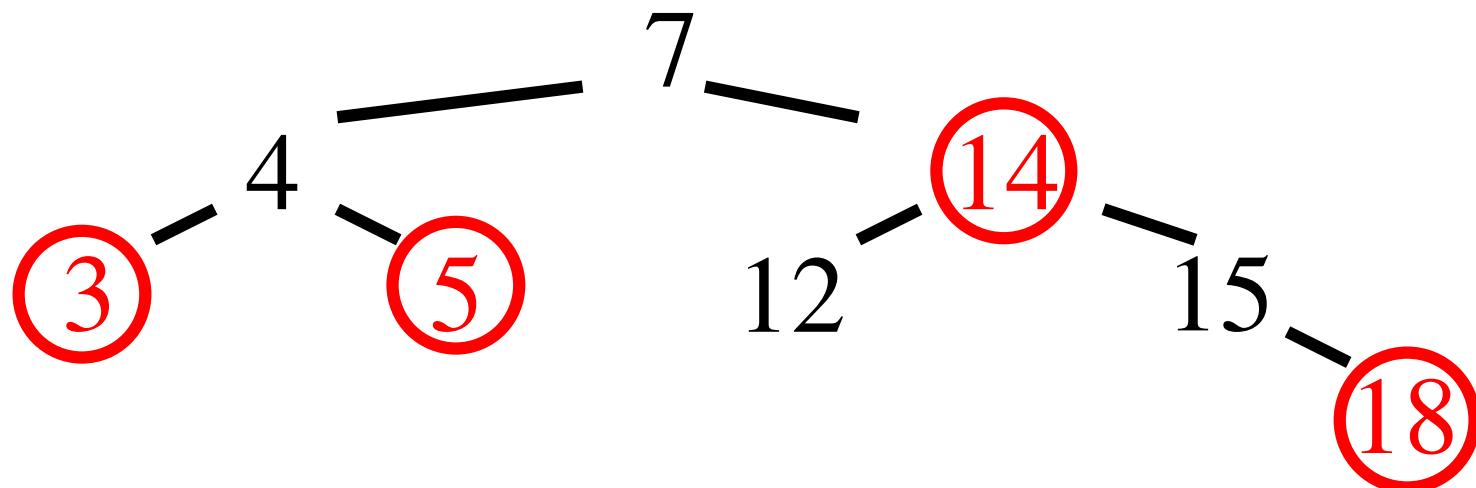
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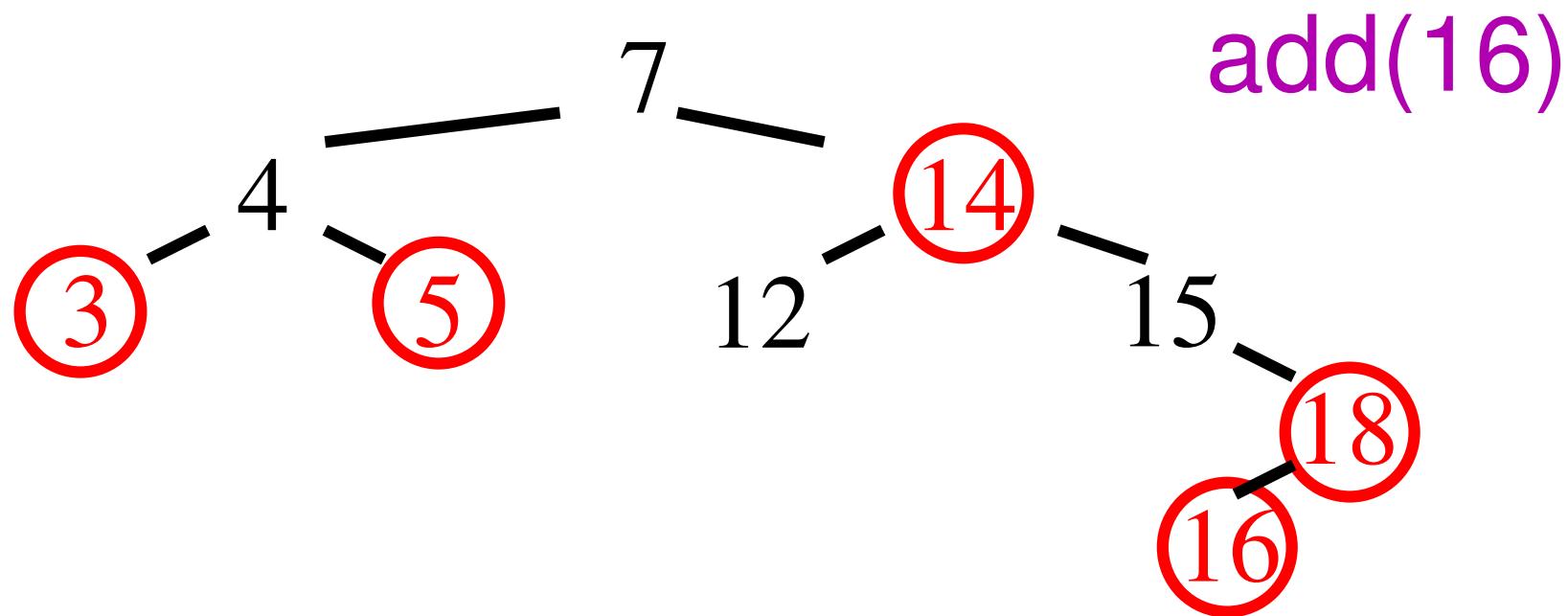
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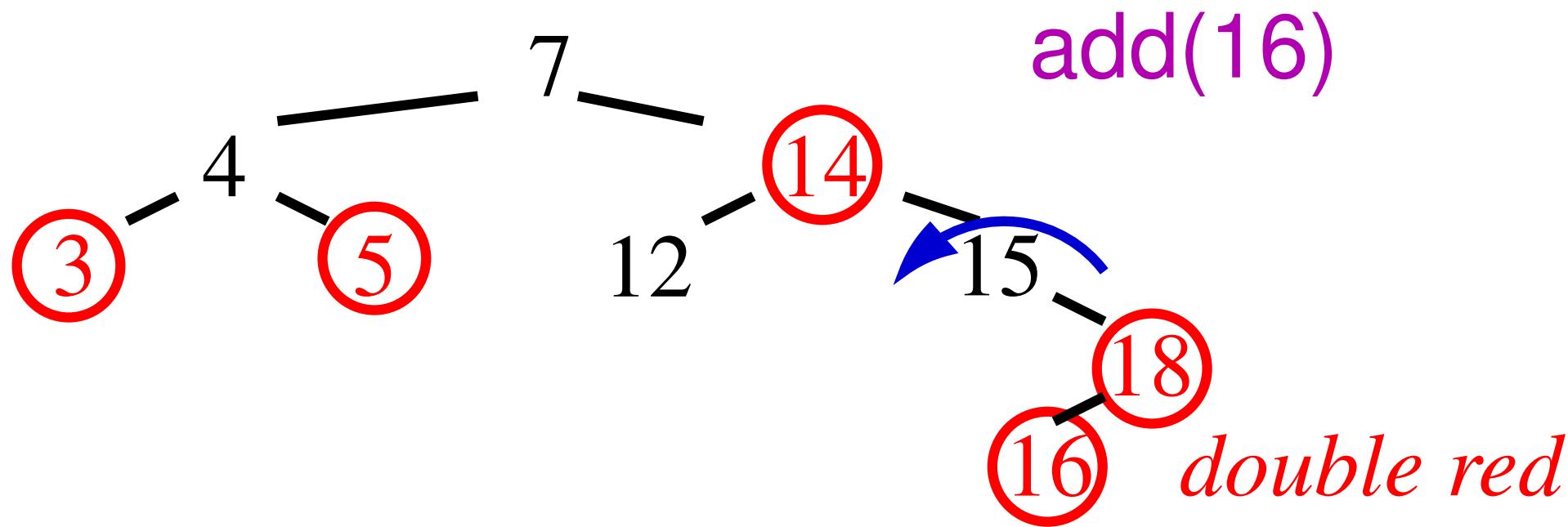
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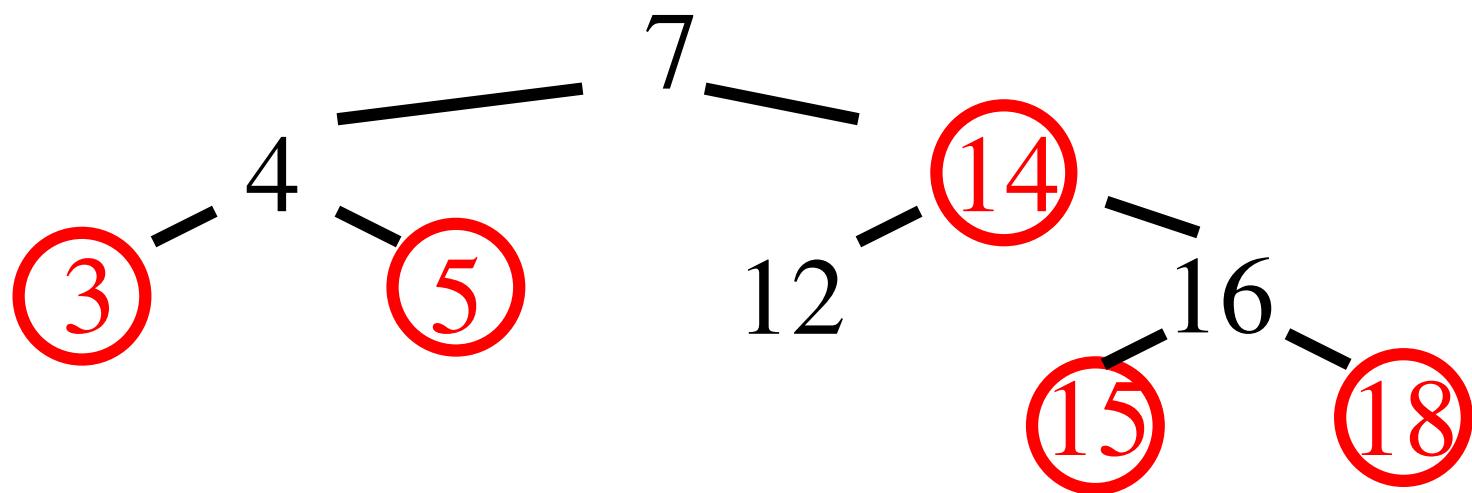
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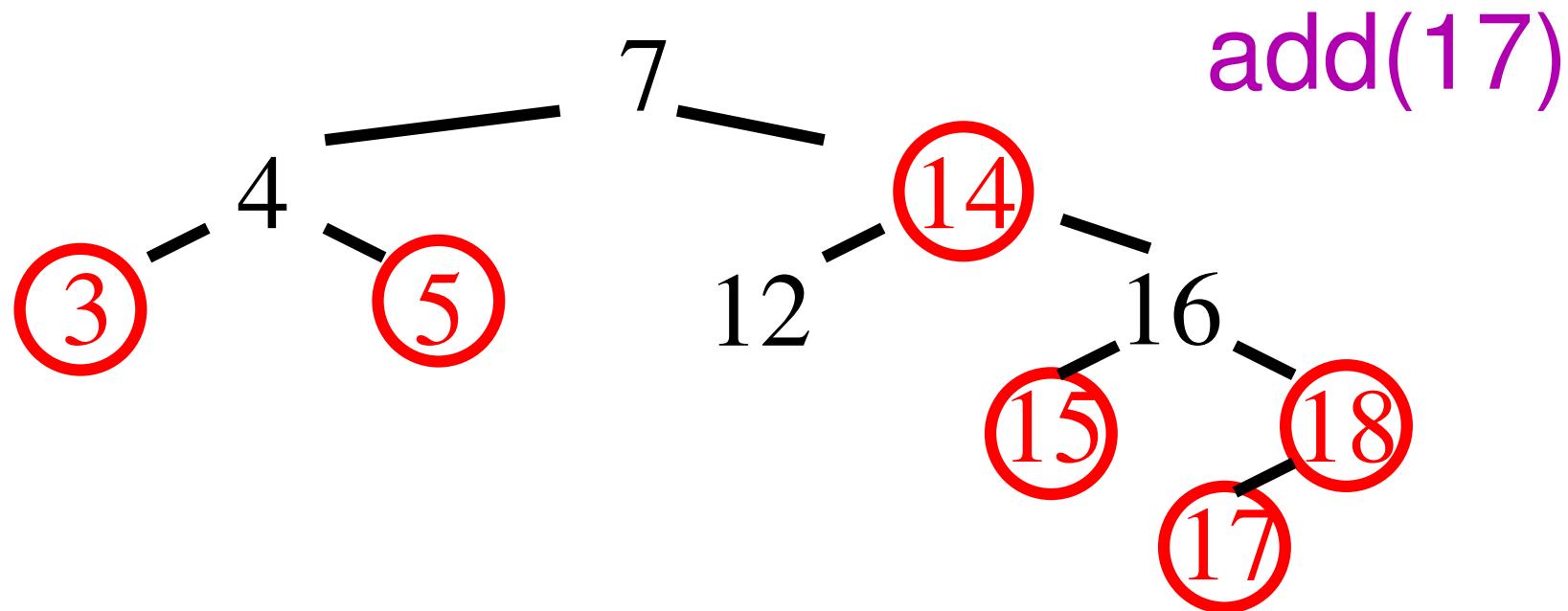
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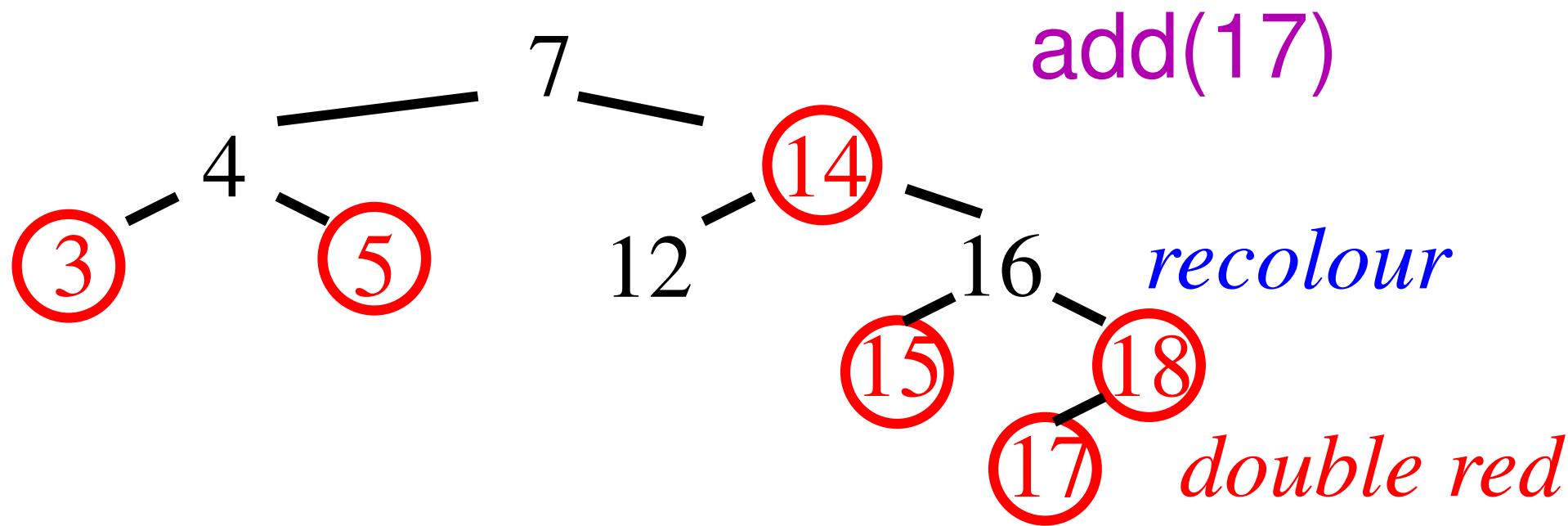
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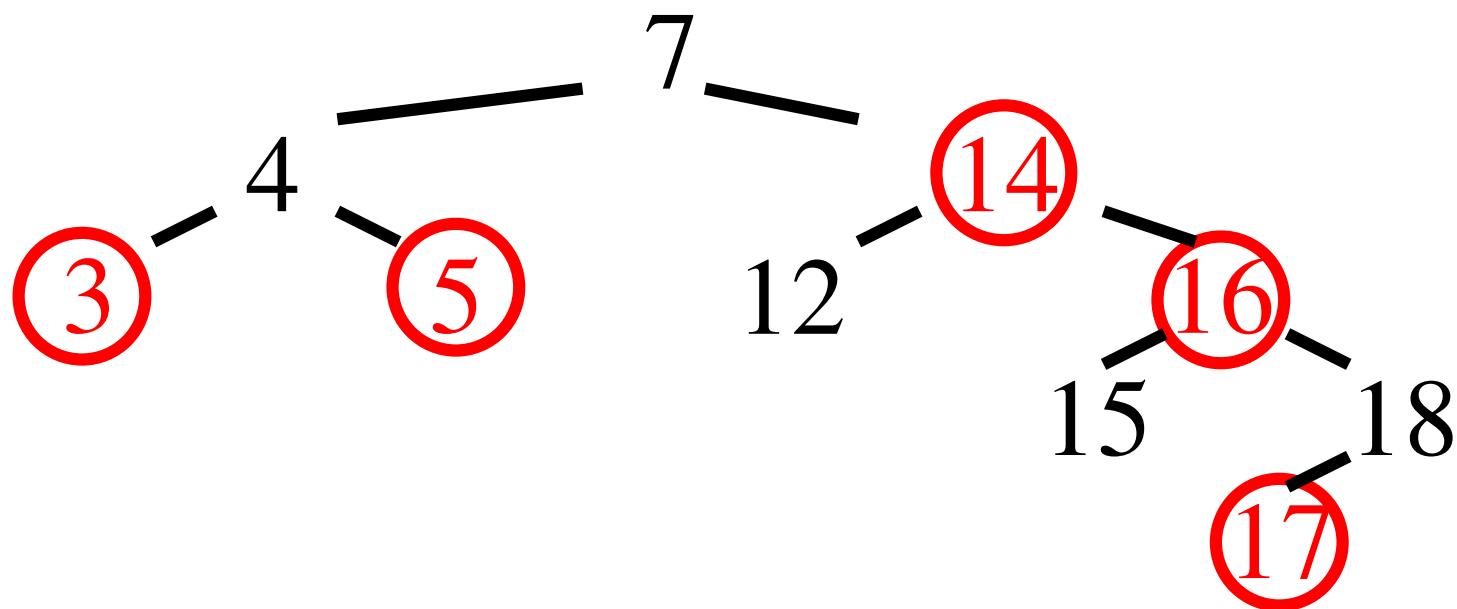
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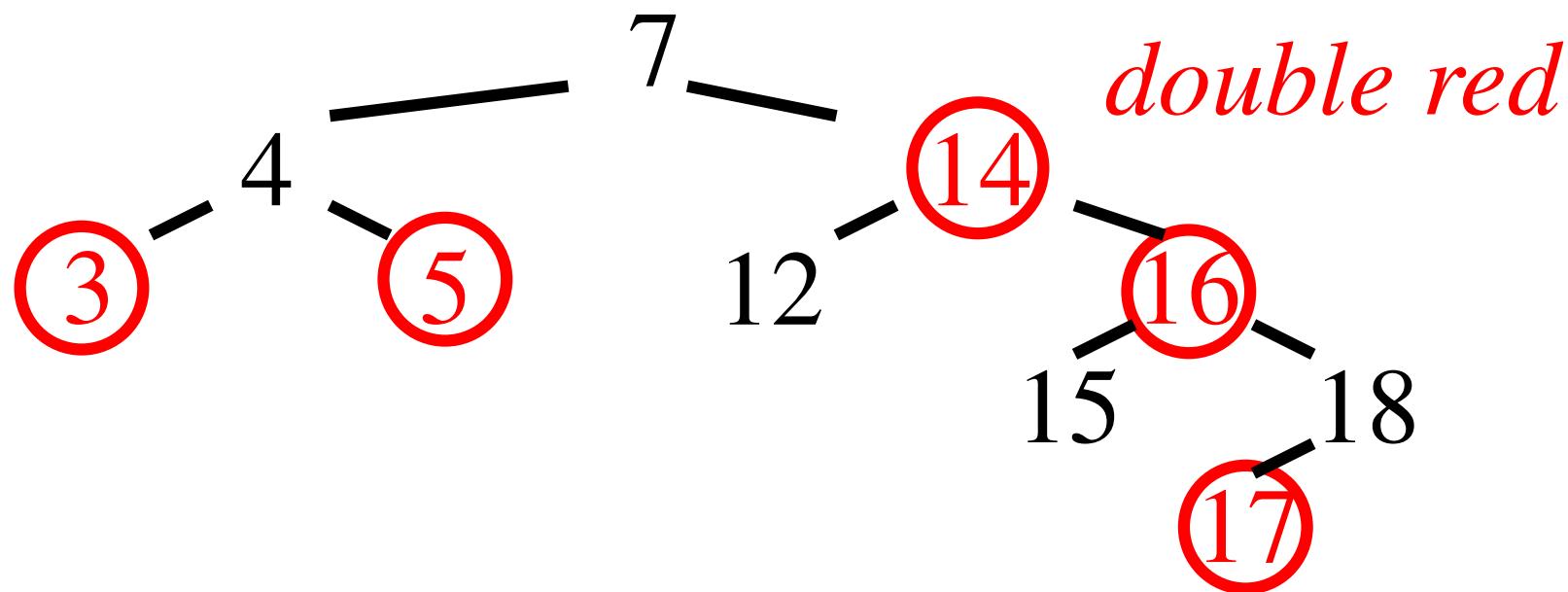
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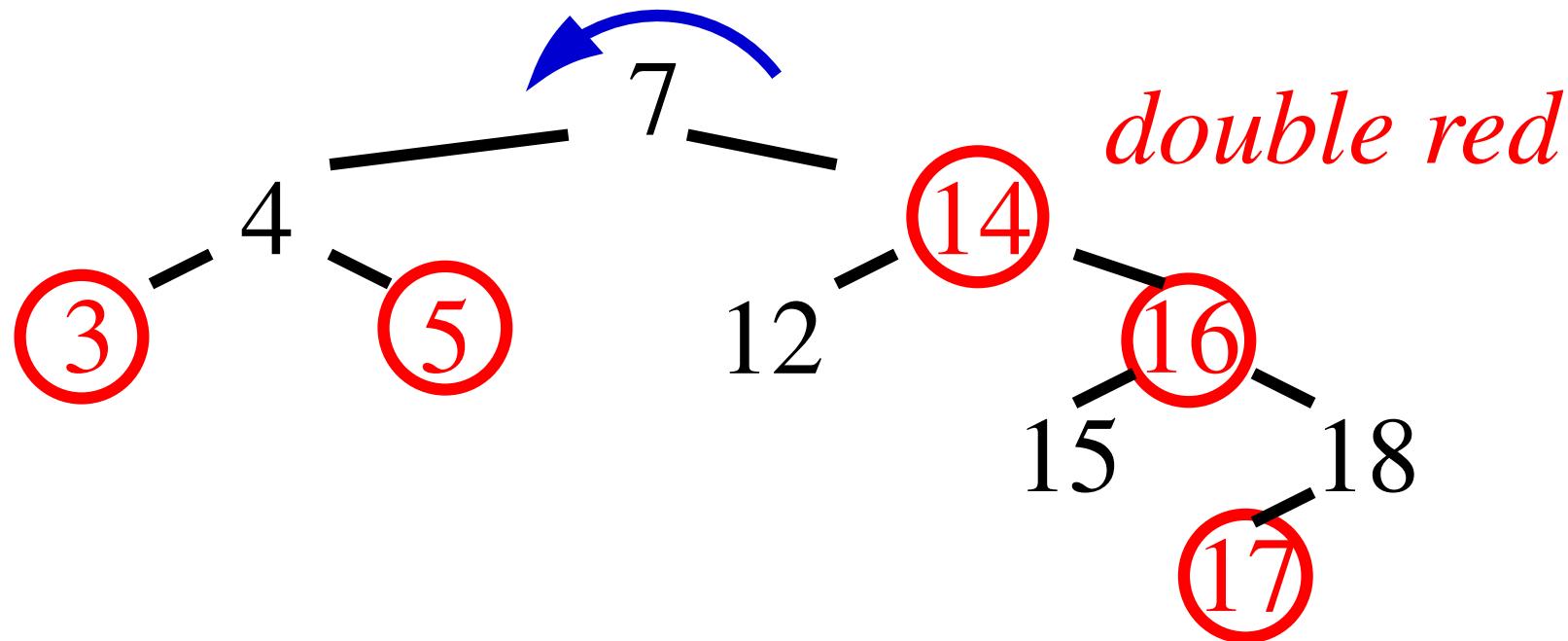
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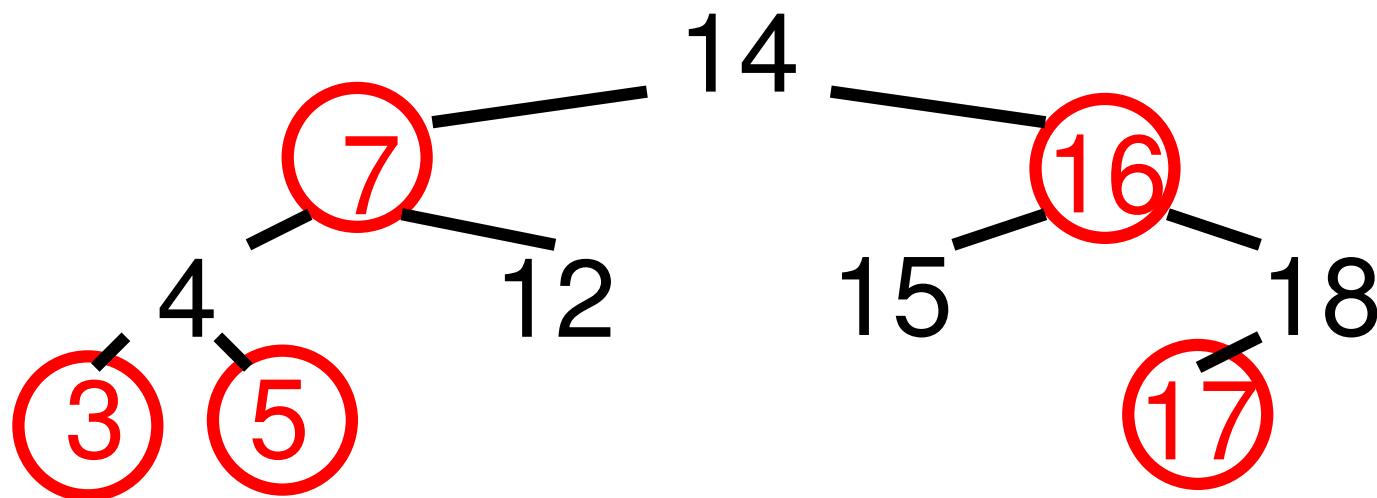
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Performance of Red-Black Trees

- Red-black trees are slightly more complicated to code than AVL trees
- Red-black trees tend to be slightly less compact than AVL trees
- However, insertion and deletion run slightly quicker
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Set

- The standard template library (STL) has a class `std::set<T>`
- It also has a `std::unordered_set<T>` class (which uses a hash table covered later)
- As well as `std::multiset<T>` that implements a multiset (i.e. a set, but with repetitions)
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Maps

- One major abstract data type (ADT) we have not encountered is the map class
- The map class `std::map<Key, V>` contain key-value pairs `pair<Key, V>`
 - ★ The first element of type Key is the **key**
 - ★ The second element of type V is the **value**
- Maps work as content addressable arrays

```
map<string, int> students;
student["John Smith"] = 89;
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cout << students["John Smith"];
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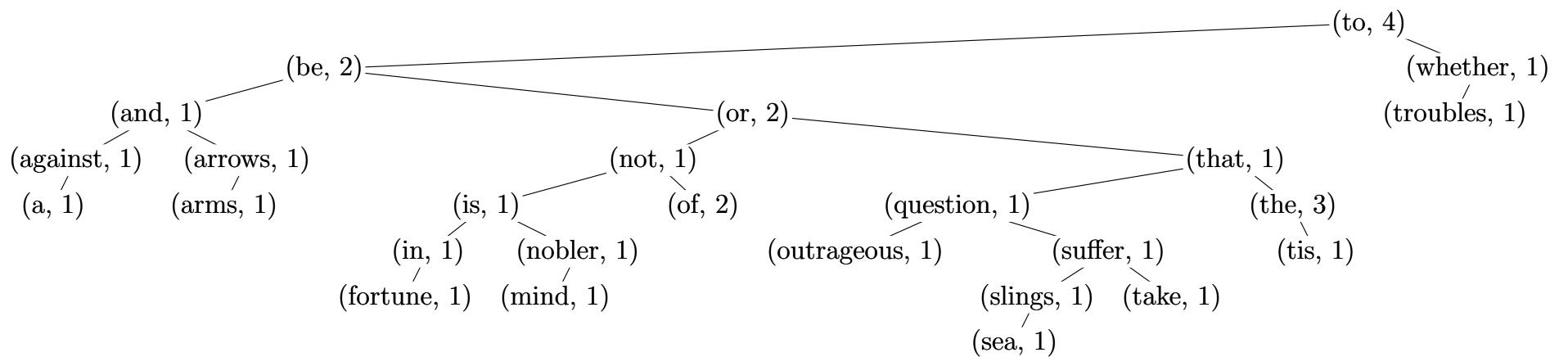
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Implementing a Map

- Maps can be implemented using a set by making each node hold a pair<K, V> objects

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class pair<K, V>
{
    public:
        K first;
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- We can count words using the key for words and value to count

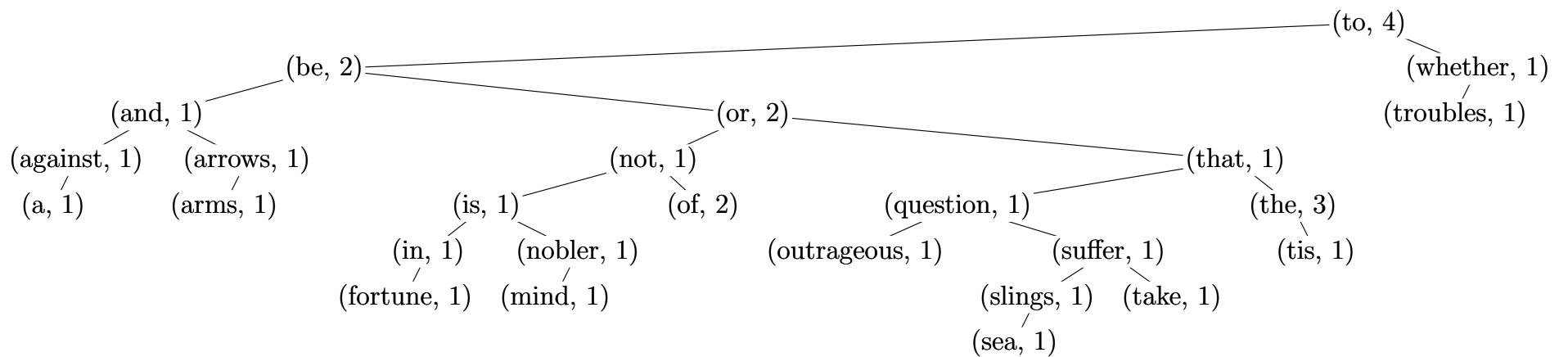


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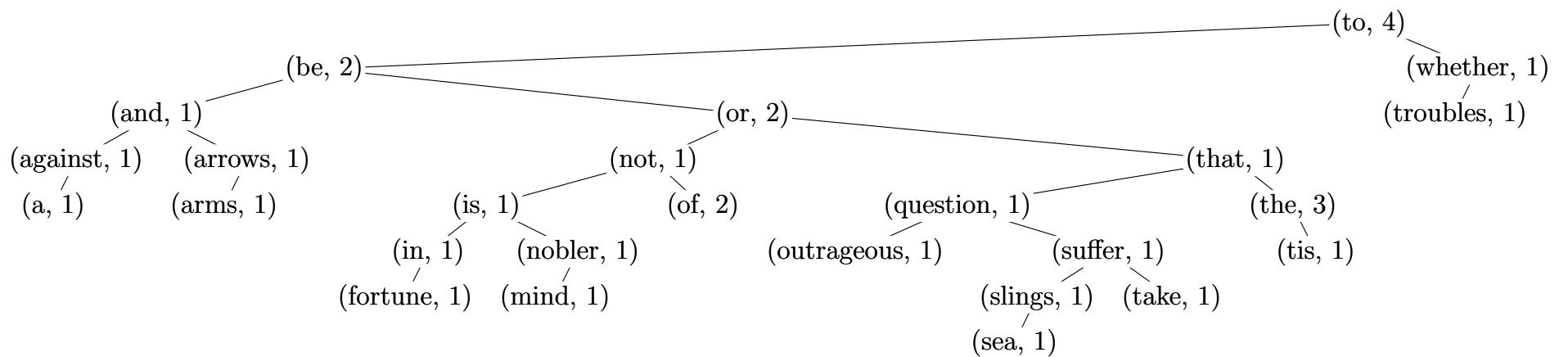


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Lessons

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- Balanced trees are achieved by performing rotations
- There are different strategies for deciding when to rotate including
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