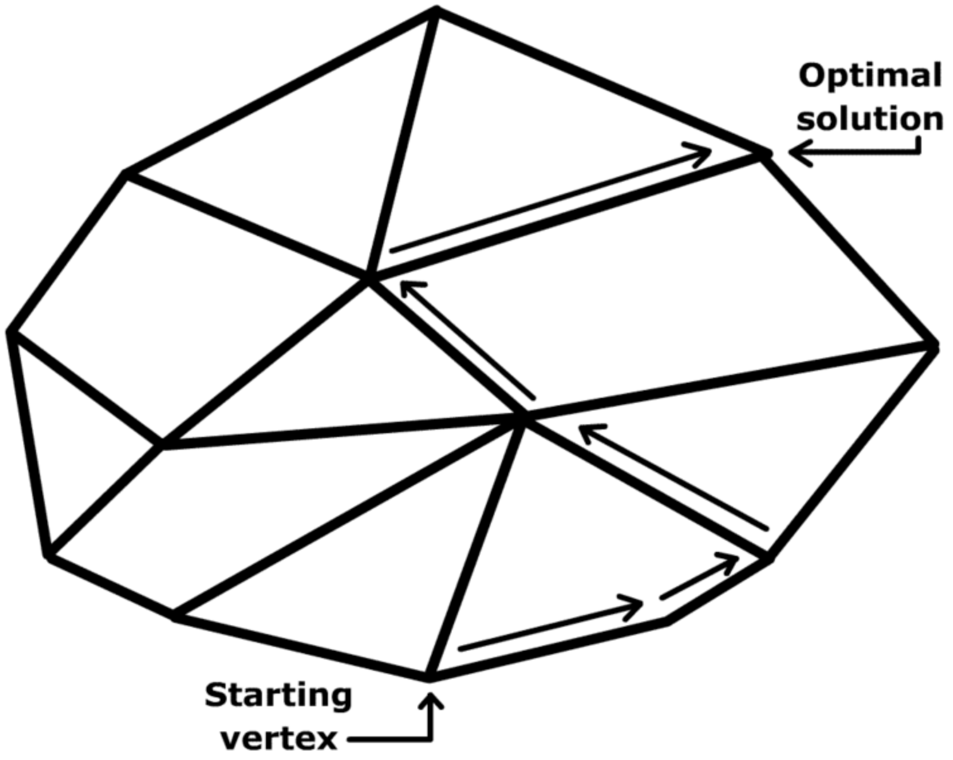
Algorithms and Analysis

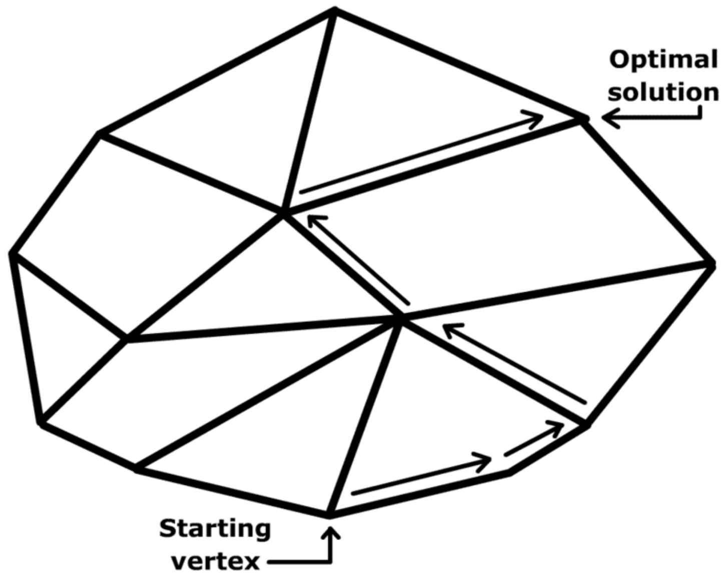
Lesson 28: *Solving Linear Programs*



linear programming, simplex methods, iterative search

Outline

1. **Recap**
2. Basic Feasible Solutions
3. Simplex Method
4. Classic LP Problems



Recap

- Linear programs are problems that can be formulated as follows

$$\min_{\boldsymbol{x}} \boldsymbol{c} \cdot \boldsymbol{x}$$

subject to

$$\mathbf{A}^{\leq}\boldsymbol{x} \leq \boldsymbol{b}^{\leq}, \quad \mathbf{A}^{\geq}\boldsymbol{x} \geq \boldsymbol{b}^{\geq}, \quad \mathbf{A}^{=}\boldsymbol{x} = \boldsymbol{b}^{=}, \quad \boldsymbol{x} \geq \mathbf{0}$$

- Where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$
- $\mathbf{A}^*$ are matrices and we interpret the inequalities to mean

$$\forall k \qquad \sum_{j=1}^{n} A^{\leq}_{kj} x_j \leq b^{\leq}_{k}$$

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Optima and Vertices

- Because the objective function is linear $(\boldsymbol{c} \cdot \boldsymbol{x})$ there is a direction where the objective is always improving
- Thus, the optima cannot lie in the interior of the search space
- When we meet a constraint that limits the direction we can move, but we can still move along the constraint
- We then meet another constraint which restricts the direction we can move by two degrees of freedom
- Eventually, we will reach n constraints which defines a vertex of the feasible region and is optimal

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Transforming Linear Programs

- We can always transform an inequality constraint into an equality constraint by adding slack variables
- E.g.

$$\begin{array}{lcll} \boldsymbol{a}_1 \cdot \boldsymbol{x} \geq 0 & \Rightarrow & \boldsymbol{a}_1 \cdot \boldsymbol{x} - z_1 = 0 & z_1 \geq 0 \\ \boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 0 & \Rightarrow & \boldsymbol{a}_2 \cdot \boldsymbol{x} + z_2 = 0 & z_2 \geq 0 \end{array}$$

  z_1 (the excess) and z_2 (the deficit) are known as slack variables

- A linear program with just equality constraints and non-negativity constraints is said to be in normal form

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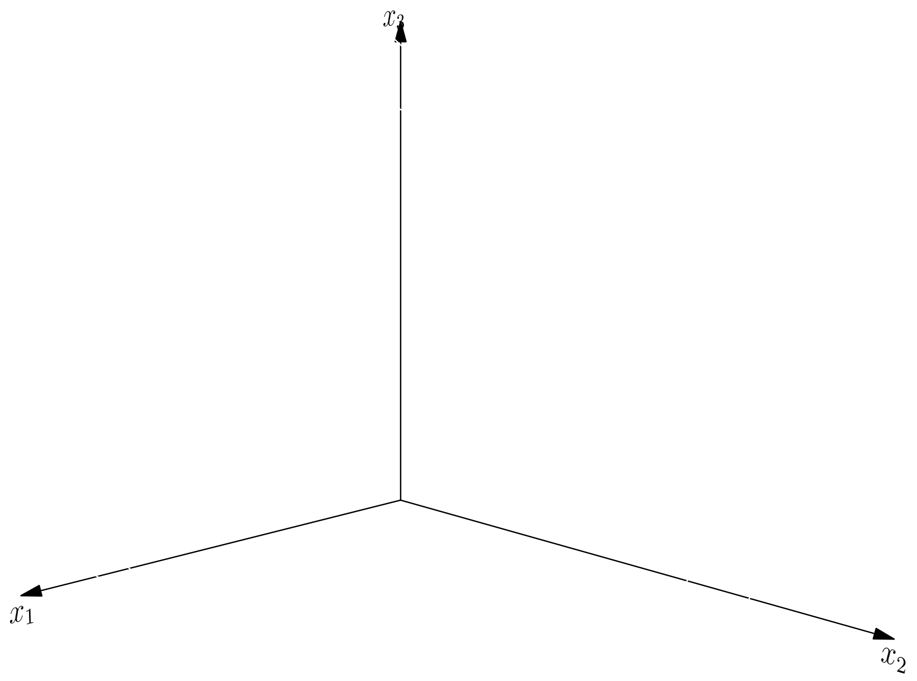
Solving Linear Programming



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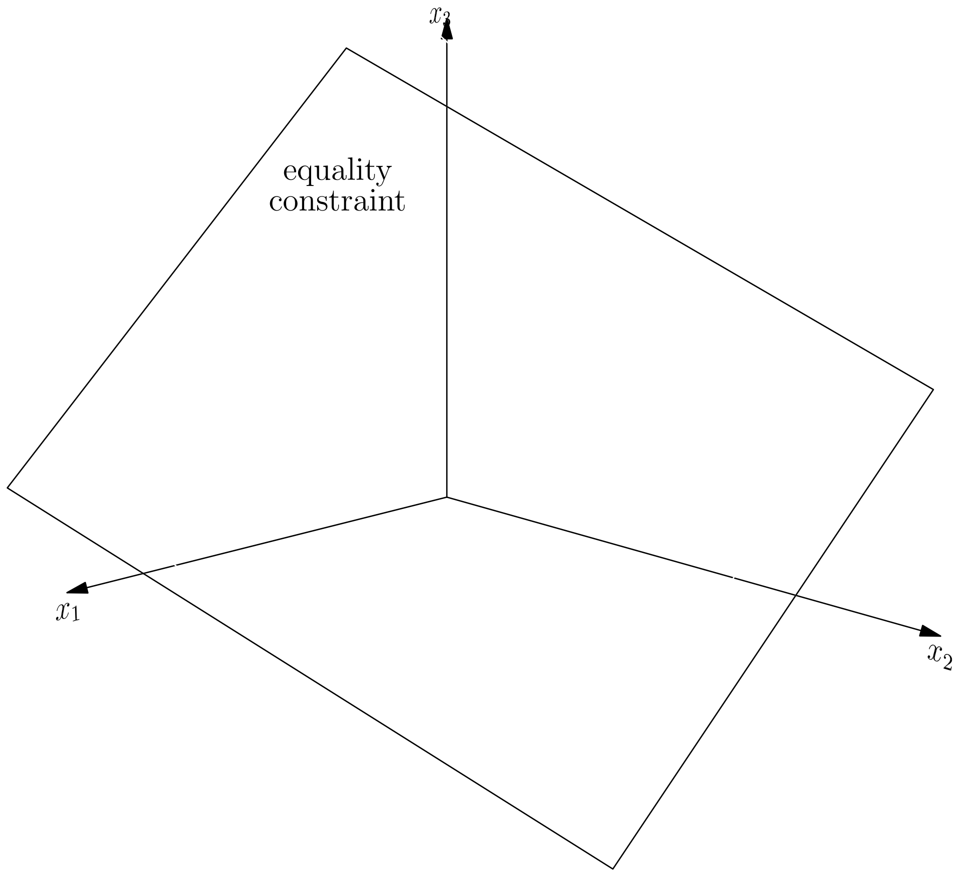

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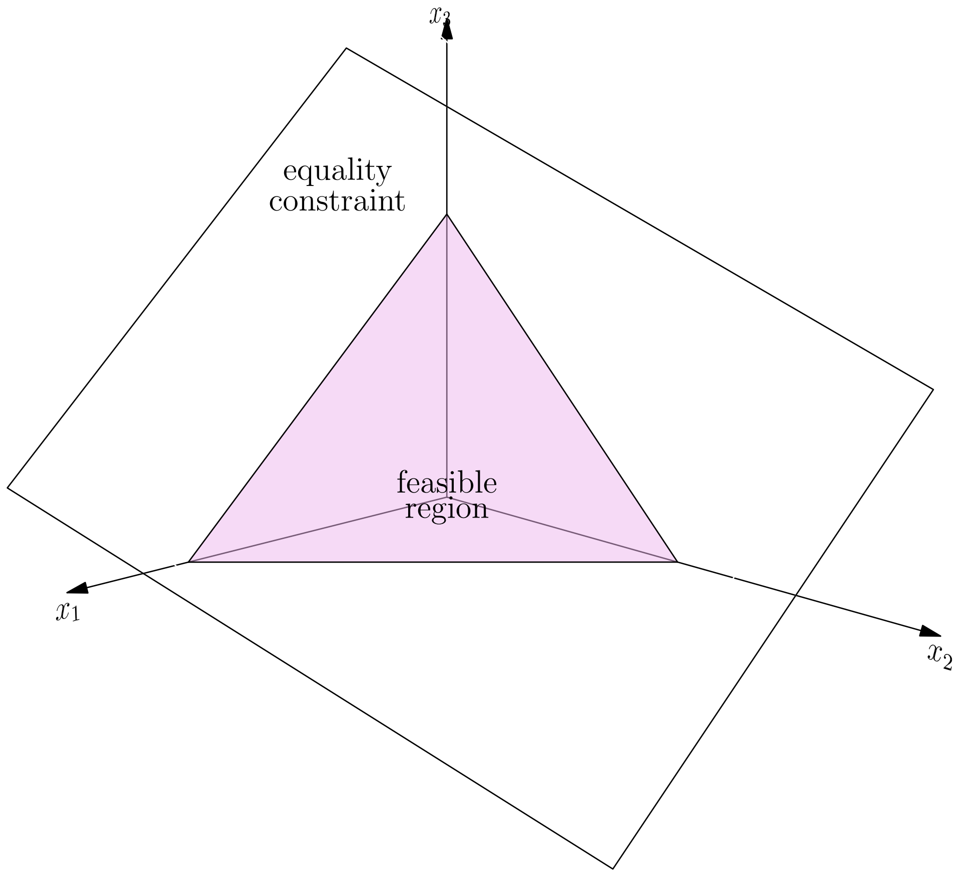

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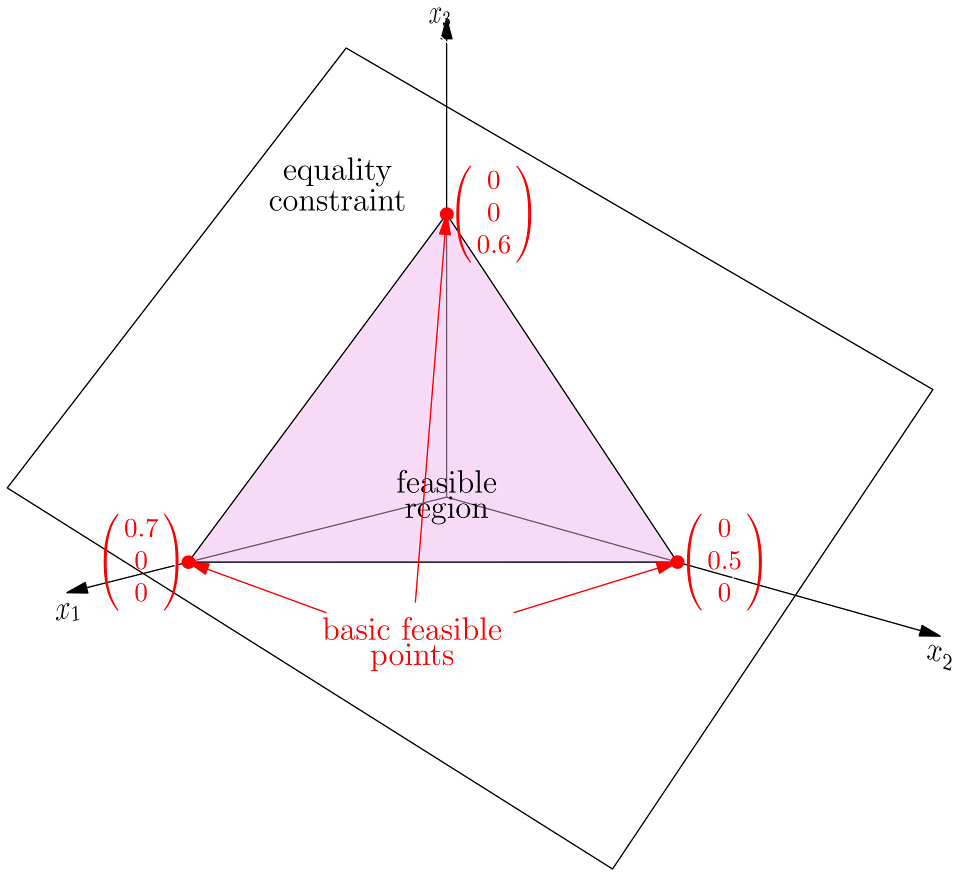
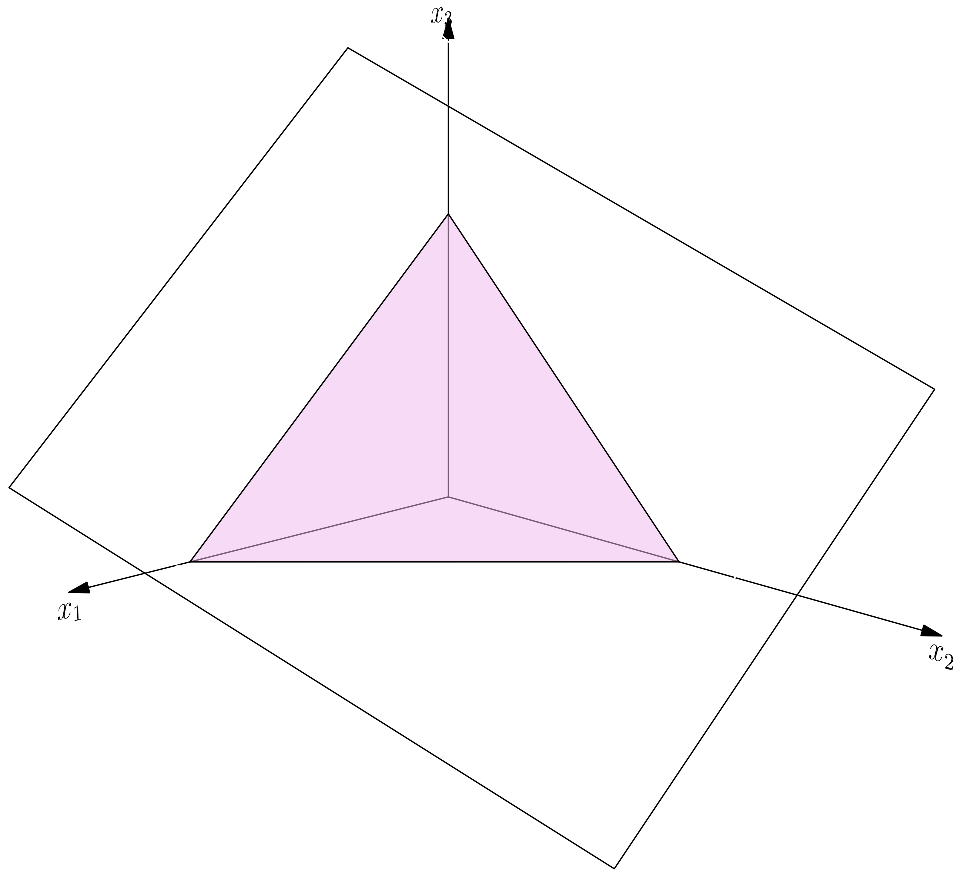

x_3
equality constraint
$\begin{pmatrix} 0 \\ 0 \\ 0.6 \end{pmatrix}$
feasible region
$\begin{pmatrix} 0.7 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix}$
x_1
x_2
basic feasible points

Solving Linear Programming



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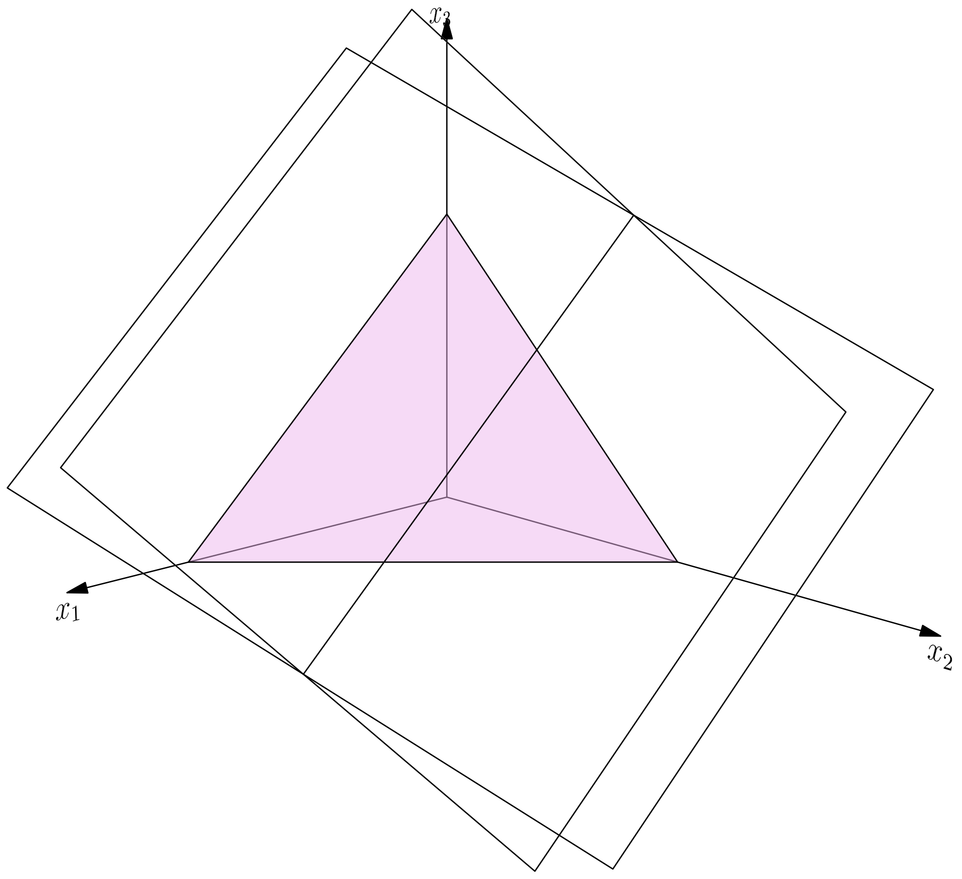

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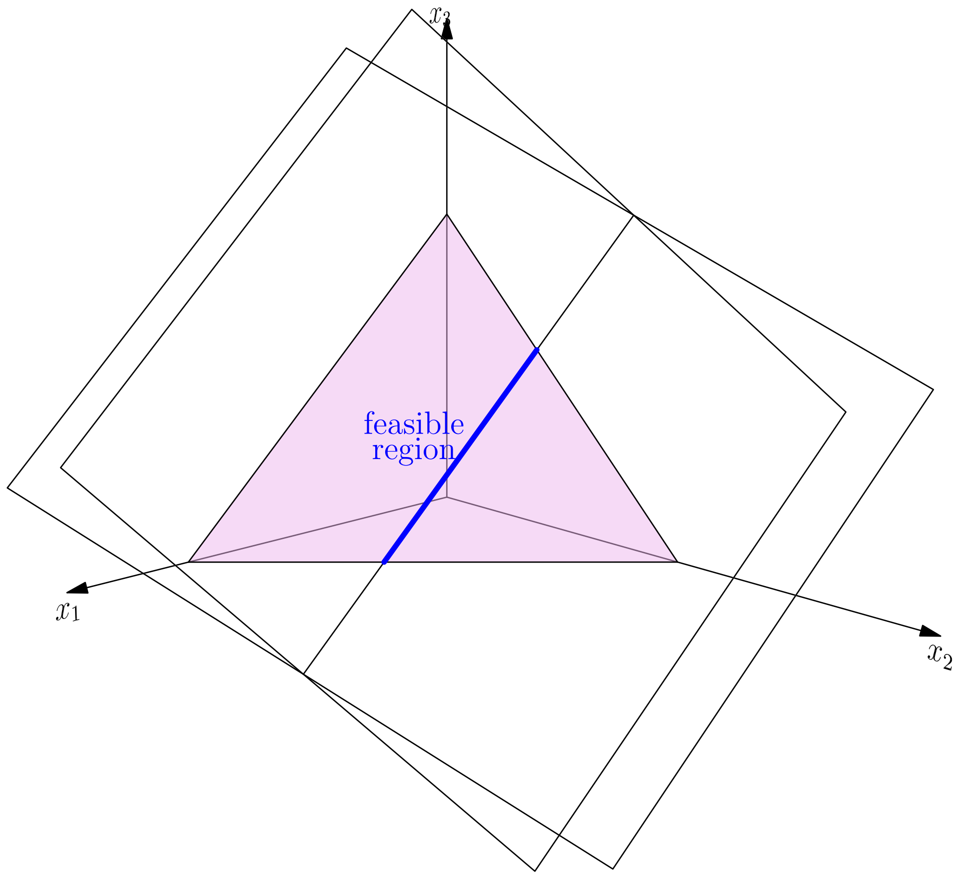

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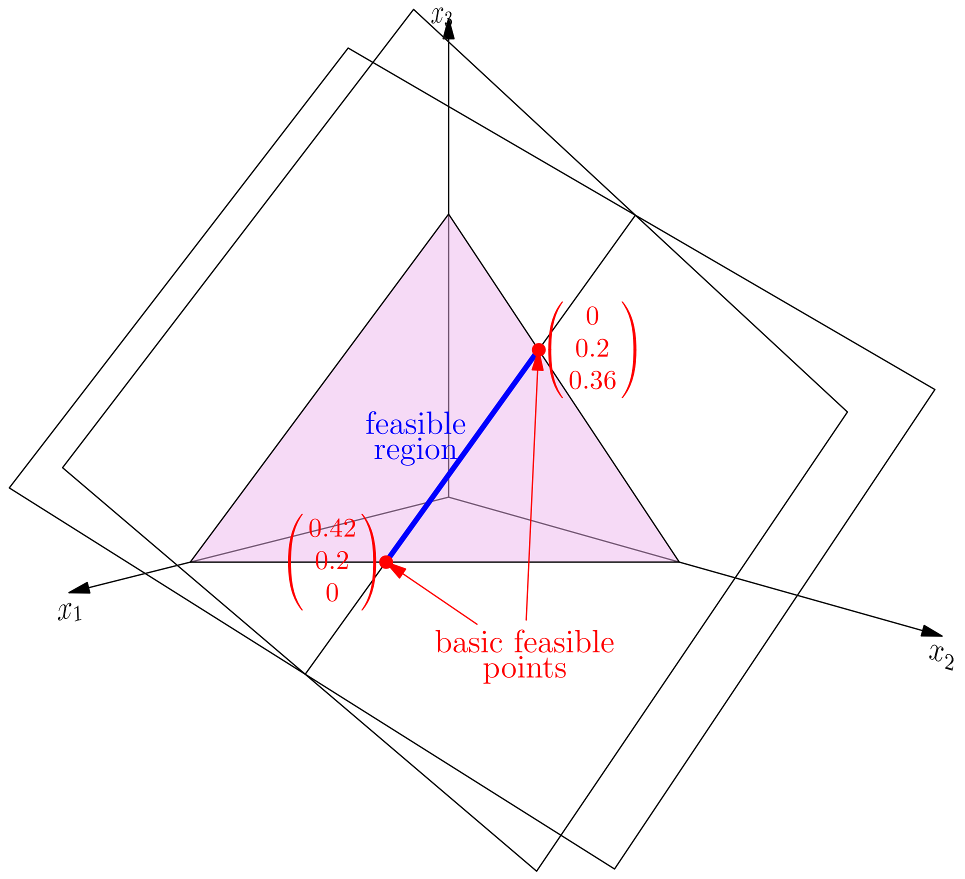
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x_1
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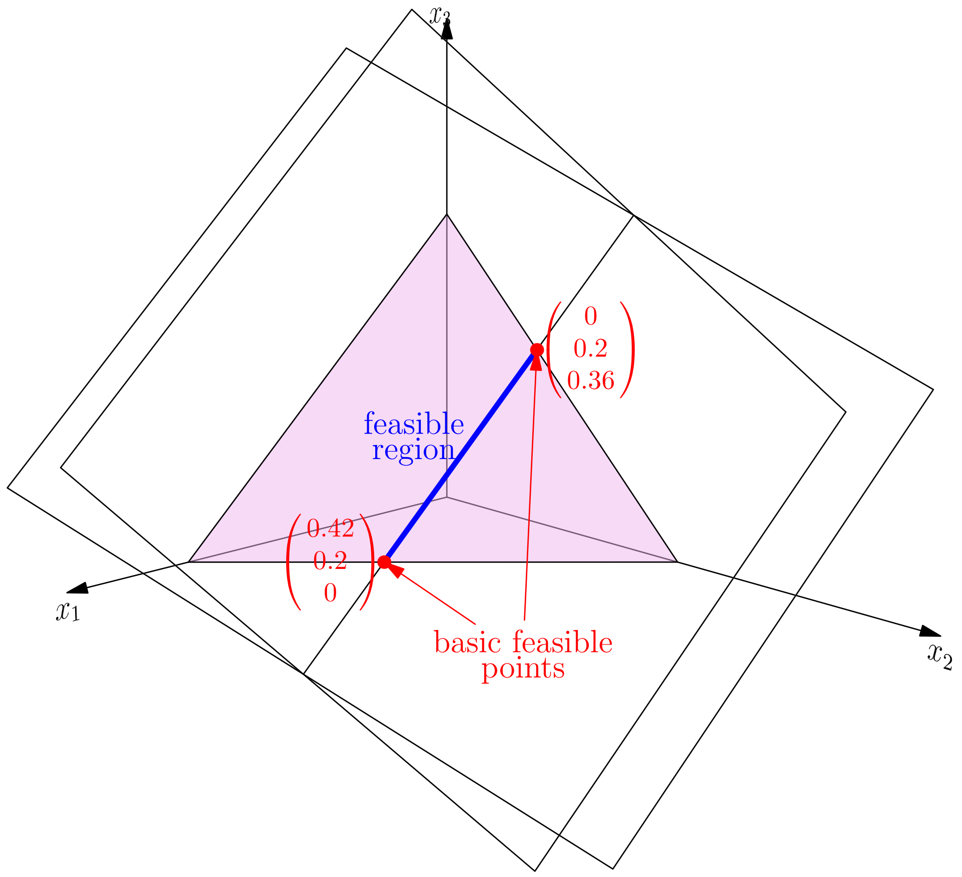

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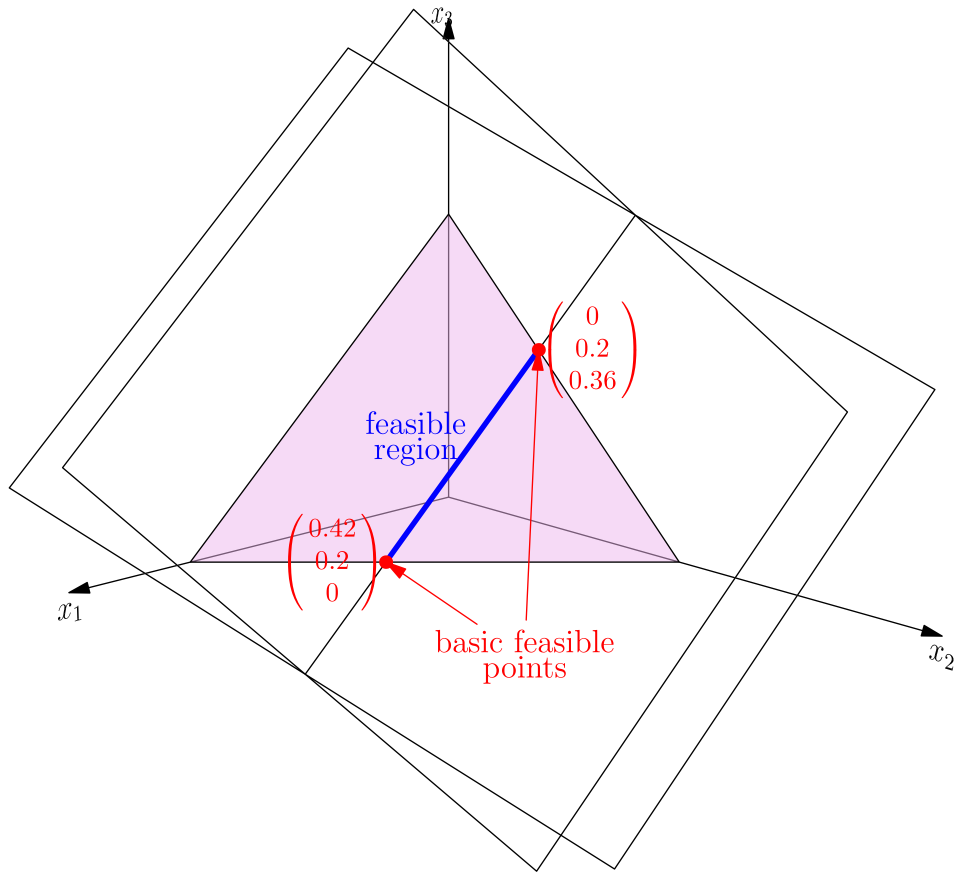

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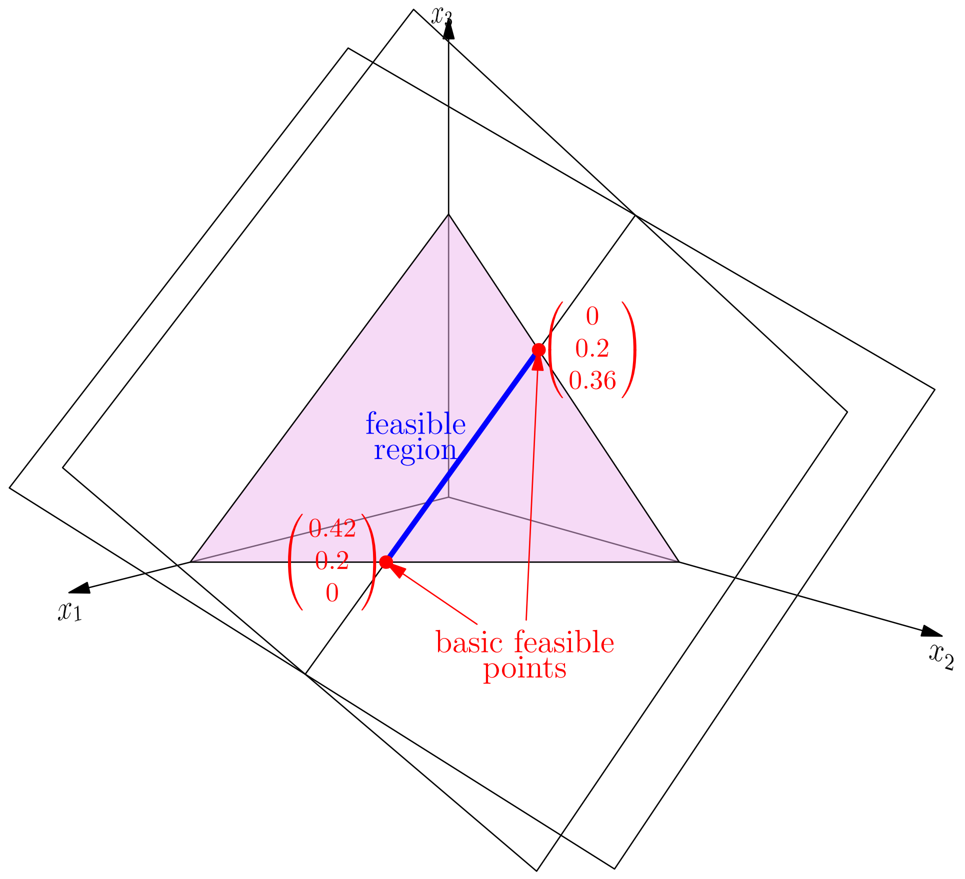

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- Simplex algorithm organises iterative search for global solutions

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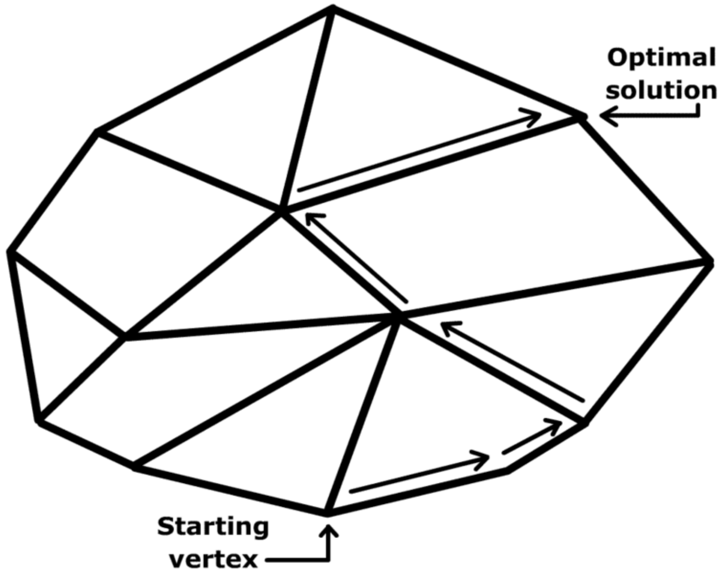

Basic Feasible Solution

- A *basic feasible solution* or *basic feasible point* is a solution that lies at a vertex of the feasible space
- To solve a linear program we will start at a basic feasible point and move to the neighbour which best improves the objective function
- When we cannot find a better solution we are at the optimal solution
- This is an example of an iterative improvement algorithm which gives an optimal solution

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Constraints

- There are two types of constraints
  1. n non-negativity constraints $x_i \geq 0$
  2. m additional constraints, which we can take to be equalities $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
- Note that some of the variables might be slack variables
- We consider the case when there are more variables than additional constraints, i.e. $n > m$
- This is usually be the case, but. . .
- If this isn't true it turns out you can consider an equivalent problem (dual problem) where you have a variable for each constraint and a constraint for each variable

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Basic Variable

- In total we have n equality and m non-negativity constraints
- n constraints must be satisfied to be at a vertex of feasible region
- So at least $n-m$ of the non-negativity constraints are satisfied (i.e. $x_i = 0$)
- The $n-m$ variables that are zero are said to be **non-basic variables**
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Initial Basic Feasible Solution

- One of the tricky bits of tackling a linear program is to find an initial feasible solution
- We do this in **phase one** of the simplex program
- To do this for each additional constraint we add a new **auxiliary variable** ξ_k, e.g.

$$\forall\, k \in \{1, 2, \ldots, m\} \qquad \xi_k + \sum_i A_{ki} x_i = b_k \geq 0$$

- We then can find a basic feasible solution by setting $x_i = 0$ so

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Eliminating Auxiliary Variables

- In phase one we run a simplex algorithm with an auxiliary cost function

$$\min f_{\mathsf{aux}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{k=1}^{m} \xi_k$$

- This should find a solution where all the $\xi_k = 0$
- If no solution exists it means there is no feasible solution and we're finished
- If there is a solution then we can eliminate the auxiliary variables and we have a feasible solution

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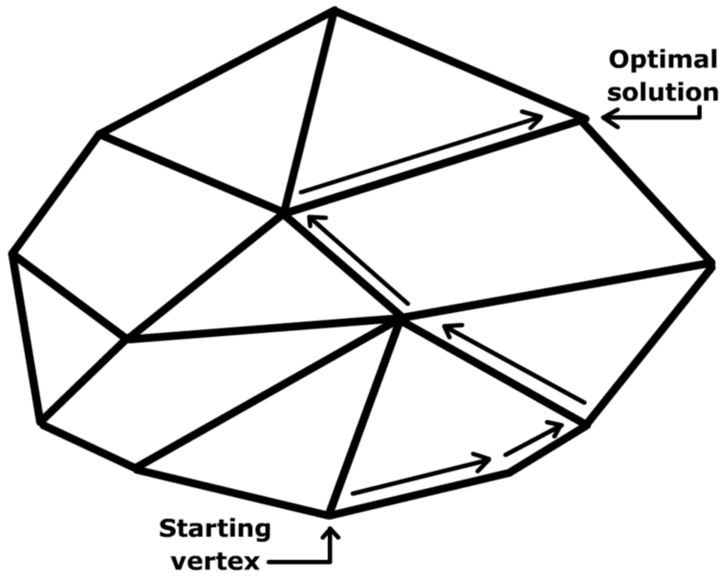

Phase Two

- In phase two we now have an initial basic feasible solution (with $n - m$ zero variables)
- We then run the simplex algorithm on the original objective function $f(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x}$
- That is we move to a neighbouring vertex which gives the best increase in the objective function
- To help organise this search we write the objective function and constraints in a **restricted normal form** and then build a **tableau** showing the *basic variables* and the *non-basic variables*

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Restricted Normal Form

- To perform the moves between vertices it helps to represent the problem in a **restricted normal form**
- Starting from a basic feasible point we have a constraint for each basic (non-zero) variable
- We write the constraints as an equality between basic and non-basic (zero valued) variables
- Similarly we write the objective function in terms of non-basic variables
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Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

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$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$
$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$
$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$
$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

$$x_1 = 0.2, \quad x_3 = 0, \quad f(\mathbf{x}) = -7$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \ f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$
$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$
$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$
$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

$$x_4 = 0.12, \quad x_3 = 0, \quad f(\mathbf{x}) = -7.1$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

$$x_5 = 0.12, \quad x_3 = 0, \quad f(\mathbf{x}) = -7.6$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

$x_7 = 0.6, \quad x_2 = 0, \quad f(\mathbf{x}) = -3.2$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

where

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9 \geq 0$$

$$x_3 = 1 - 5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9 \geq 0$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$$

Best pivot: $x_7 = 0.6$, $x_2 = 0$, $f(\mathbf{x}) = -3.2$ swap x_2 with x_7

		x_1	x_4	x_5	x_7	x_8	x_9
$f(\mathbf{x})$	-8	5	7	3	8	-6	-3
x_2	3	6	6	-5	-5	5	-5
x_3	1	-5	-8	-8	4	-8	-5
x_6	8	-8	-7	-2	-5	-3	4

Tableau

Swap x_2 with x_7

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9$$

$$\Rightarrow \quad x_7 = 0.6 + 1.2x_1 + 1.2x_4 - x_5 - 0.2x_2 + x_8 - x_9$$

Tableau

Swap x_2 with x_7

$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9$

$\Rightarrow$ $x_7 = 0.6 + 1.2x_1 + 1.2x_4 - x_5 - 0.2x_2 + x_8 - x_9$

$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$

$\Rightarrow$ $\max_{\mathbf{x}} \; f(\mathbf{x}) = -3.2 + 15x_1 + 17x_4 - 5x_5 - 1.6x_2 + 2x_8 - 11x_9$

Tableau

Swap x_2 with x_7

$$x_2 = 3 + 6x_1 + 6x_4 - 5x_5 - 5x_7 + 5x_8 - 5x_9$$

$$\Rightarrow \quad x_7 = 0.6 + 1.2x_1 + 1.2x_4 - x_5 - 0.2x_2 + x_8 - x_9$$

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -8 + 5x_1 + 7x_4 + 3x_5 + 8x_7 - 6x_8 - 3x_9$$

$$\Rightarrow \quad \max_{\mathbf{x}} \; f(\mathbf{x}) = -3.2 + 15x_1 + 17x_4 - 5x_5 - 1.6x_2 + 2x_8 - 11x_9$$

$$x_3 = -5x_1 - 8x_4 - 8x_5 + 4x_7 - 8x_8 - 5x_9$$

$$\Rightarrow \quad x_3 = 3.4 - 0.2x_1 - 3.2x_4 - 12x_5 - 0.8x_2 - 4x_8 - 9x_9$$

$$x_6 = 8 - 8x_1 - 7x_4 - 2x_5 - 5x_7 - 3x_8 + 4x_9$$

$$\Rightarrow \quad x_6 = 5 - 14x_1 - 13x_4 + 3x_5 + x_2 - 8x_8 + 9x_9$$

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -3.2 + 15x_1 + 17x_4 - 5x_5 - 1.6x_2 + 2x_8 - 11x_9$$

where

$$x_7 = 0.6 + 1.2x_1 + 1.2x_4 - x_5 - 0.2x_2 + x_8 - x_9 \geq 0$$

$$x_3 = 3.4 - 0.2x_1 - 3.2x_4 - 12x_5 - 0.8x_2 - 4x_8 - 9x_9 \geq 0$$

$$x_6 = 5 - 14x_1 - 13x_4 + 3x_5 + x_2 - 8x_8 + 9x_9 \geq 0$$

$$x_1 = x_4 = x_5 = x_2 = x_8 = x_9 = 0$$

		x_1	x_4	x_5	x_2	x_8	x_9
$f(\mathbf{x})$	-3.2	15	17	-5	-1.6	2	-11
x_7	0.6	1.2	1.2	-1	-0.2	1	-1
x_3	3.4	-0.2	-3.2	-12	-0.8	-4	-9
x_6	5	-14	-13	3	1	-8	9

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -3.2 + 15x_1 + 17x_4 - 5x_5 - 1.6x_2 + 2x_8 - 11x_9$$

where

$$x_7 = 0.6 + 1.2x_1 + 1.2x_4 - x_5 - 0.2x_2 + x_8 - x_9 \geq 0$$
$$x_3 = 3.4 - 0.2x_1 - 3.2x_4 - 12x_5 - 0.8x_2 - 4x_8 - 9x_9 \geq 0$$
$$x_6 = 5 - 14x_1 - 13x_4 + 3x_5 + x_2 - 8x_8 + 9x_9 \geq 0$$
$$x_1 = x_4 = x_5 = x_2 = x_8 = x_9 = 0$$

Best pivot: $x_4 = 0.38$, $x_6 = 0$, $f(\mathbf{x}) = 3.2$ swap x_6 with x_4

		x_1	x_4	x_5	x_2	x_8	x_9
$f(\mathbf{x})$	-3.2	15	17	-5	-1.6	2	-11
x_7	0.6	1.2	1.2	-1	-0.2	1	-1
x_3	3.4	-0.2	-3.2	-12	-0.8	-4	-9
x_6	5	-14	-13	3	1	-8	9

Tableau

Swap x_6 with x_4

$x_6 = 5 - 14x_1 - 13x_4 + 3x_5 + x_2 - 8x_8 + 9x_9$

$\Rightarrow$ $x_4 = 0.38 - 1.1x_1 - 0.077x_6 + 0.23x_5 + 0.077x_2 - 0.62x_8 + 0.69x_9$

Tableau

Swap x_6 with x_4

$$x_6 = 5 - 14x_1 - 13x_4 + 3x_5 + x_2 - 8x_8 + 9x_9$$

$$\Rightarrow \quad x_4 = 0.38 - 1.1x_1 - 0.077x_6 + 0.23x_5 + 0.077x_2 - 0.62x_8 + 0.69x_9$$

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = -3.2 + 15x_1 + 17x_4 - 5x_5 - 1.6x_2 + 2x_8 - 11x_9$$

$$\Rightarrow \quad \max_{\mathbf{x}} \; f(\mathbf{x}) = 3.2 - 3.3x_1 - 1.3x_6 - 1.2x_5 - 0.32x_2 - 8.2x_8 + 0.49x_9$$

Tableau

Swap x_6 with x_4

$x_6 = 5 - 14x_1 - 13x_4 + 3x_5 + x_2 - 8x_8 + 9x_9$

$\Rightarrow$ $x_4 = 0.38 - 1.1x_1 - 0.077x_6 + 0.23x_5 + 0.077x_2 - 0.62x_8 + 0.69x_9$

$\max_{\mathbf{x}} \; f(\mathbf{x}) = -3.2 + 15x_1 + 17x_4 - 5x_5 - 1.6x_2 + 2x_8 - 11x_9$

$\Rightarrow$ $\max_{\mathbf{x}} \; f(\mathbf{x}) = 3.2 - 3.3x_1 - 1.3x_6 - 1.2x_5 - 0.32x_2 - 8.2x_8 + 0.49x_9$

$x_7 = 0.6 + 1.2x_1 + 1.2x_4 - x_5 - 0.2x_2 + x_8 - x_9$

$\Rightarrow$ $x_7 = 1.1 - 0.092x_1 - 0.092x_6 - 0.72x_5 - 0.11x_2 + 0.26x_8 - 0.17x_9$

$x_3 = 3.4 - 0.2x_1 - 3.2x_4 - 12x_5 - 0.8x_2 - 4x_8 - 9x_9$

$\Rightarrow$ $x_3 = 2.2 + 3.2x_1 + 0.25x_6 - 13x_5 - 1x_2 - 2x_8 - 11x_9$

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = 3.2 - 3.3x_1 - 1.3x_6 - 1.2x_5 - 0.32x_2 - 8.2x_8 + 0.49x_9$$

where

$$x_7 = 1.1 - 0.092x_1 - 0.092x_6 - 0.72x_5 - 0.11x_2 + 0.26x_8 - 0.17x_9 \geq 0$$

$$x_3 = 2.2 + 3.2x_1 + 0.25x_6 - 13x_5 - 1x_2 - 2x_8 - 11x_9 \geq 0$$

$$x_4 = 0.38 - 1.1x_1 - 0.077x_6 + 0.23x_5 + 0.077x_2 - 0.62x_8 + 0.69x_9 \geq 0$$

$$x_1 = x_6 = x_5 = x_2 = x_8 = x_9 = 0$$

		x_1	x_6	x_5	x_2	x_8	x_9
$f(\mathbf{x})$	3.2	-3.3	-1.3	-1.2	-0.32	-8.2	0.49
x_7	1.1	-0.092	-0.092	-0.72	-0.11	0.26	-0.17
x_3	2.2	3.2	0.25	-13	-1	-2	-11
x_4	0.38	-1.1	-0.077	0.23	0.077	-0.62	0.69

Tableau

$$\max_{\mathbf{x}} \ f(\mathbf{x}) = 3.2 - 3.3x_1 - 1.3x_6 - 1.2x_5 - 0.32x_2 - 8.2x_8 + 0.49x_9$$

where

$$x_7 = 1.1 - 0.092x_1 - 0.092x_6 - 0.72x_5 - 0.11x_2 + 0.26x_8 - 0.17x_9 \geq 0$$

$$x_3 = 2.2 + 3.2x_1 + 0.25x_6 - 13x_5 - 1x_2 - 2x_8 - 11x_9 \geq 0$$

$$x_4 = 0.38 - 1.1x_1 - 0.077x_6 + 0.23x_5 + 0.077x_2 - 0.62x_8 + 0.69x_9 \geq 0$$

$$x_1 = x_6 = x_5 = x_2 = x_8 = x_9 = 0$$

Best pivot: $x_9 = 0.19$, $x_3 = 0$, $f(\mathbf{x}) = 3.3$ swap x_3 with x_9

		x_1	x_6	x_5	x_2	x_8	x_9
$f(\mathbf{x})$	3.2	-3.3	-1.3	-1.2	-0.32	-8.2	0.49
x_7	1.1	-0.092	-0.092	-0.72	-0.11	0.26	-0.17
x_3	2.2	3.2	0.25	-13	-1	-2	-11
x_4	0.38	-1.1	-0.077	0.23	0.077	-0.62	0.69

Tableau

Swap x_3 with x_9

$x_3 = 2.2 + 3.2x_1 + 0.25x_6 - 13x_5 - 1x_2 - 2x_8 - 11x_9$

$\Rightarrow$ $x_9 = 0.19 + 0.29x_1 + 0.022x_6 - 1.1x_5 - 0.093x_2 - 0.18x_8 - 0.089x_3$

Tableau

Swap x_3 with x_9

$$x_3 = 2.2 + 3.2x_1 + 0.25x_6 - 13x_5 - 1x_2 - 2x_8 - 11x_9$$

$$\Rightarrow \quad x_9 = 0.19 + 0.29x_1 + 0.022x_6 - 1.1x_5 - 0.093x_2 - 0.18x_8 - 0.089x_3$$

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = 3.2 - 3.3x_1 - 1.3x_6 - 1.2x_5 - 0.32x_2 - 8.2x_8 + 0.49x_9$$

$$\Rightarrow \quad \max_{\mathbf{x}} \; f(\mathbf{x}) = 3.3 - 3.1x_1 - 1.3x_6 - 1.7x_5 - 0.37x_2 - 8.3x_8 - 0.044x_3$$

Tableau

Swap x_3 with x_9

$$x_3 = 2.2 + 3.2x_1 + 0.25x_6 - 13x_5 - 1x_2 - 2x_8 - 11x_9$$

$$\Rightarrow \quad x_9 = 0.19 + 0.29x_1 + 0.022x_6 - 1.1x_5 - 0.093x_2 - 0.18x_8 - 0.089x_3$$

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = 3.2 - 3.3x_1 - 1.3x_6 - 1.2x_5 - 0.32x_2 - 8.2x_8 + 0.49x_9$$

$$\Rightarrow \quad \max_{\mathbf{x}} \; f(\mathbf{x}) = 3.3 - 3.1x_1 - 1.3x_6 - 1.7x_5 - 0.37x_2 - 8.3x_8 - 0.044x_3$$

$$x_7 = 1.1 - 0.092x_1 - 0.092x_6 - 0.72x_5 - 0.11x_2 + 0.26x_8 - 0.17x_9$$

$$\Rightarrow \quad x_7 = 1 - 0.14x_1 - 0.096x_6 - 0.53x_5 - 0.092x_2 + 0.29x_8 + 0.015x_3$$

$$x_4 = 0.38 - 1.1x_1 - 0.077x_6 + 0.23x_5 + 0.077x_2 - 0.62x_8 + 0.69x_9$$

$$\Rightarrow \quad x_4 = 0.52 - 0.88x_1 - 0.062x_6 - 0.56x_5 + 0.012x_2 - 0.74x_8 - 0.062x_3$$

Tableau

$$\max_{\mathbf{x}} \ f(\mathbf{x}) = 3.3 - 3.1x_1 - 1.3x_6 - 1.7x_5 - 0.37x_2 - 8.3x_8 - 0.044x_3$$

where

$$x_7 = 1 - 0.14x_1 - 0.096x_6 - 0.53x_5 - 0.092x_2 + 0.29x_8 + 0.015x_3 \geq 0$$

$$x_9 = 0.19 + 0.29x_1 + 0.022x_6 - 1.1x_5 - 0.093x_2 - 0.18x_8 - 0.089x_3 \geq 0$$

$$x_4 = 0.52 - 0.88x_1 - 0.062x_6 - 0.56x_5 + 0.012x_2 - 0.74x_8 - 0.062x_3 \geq 0$$

$$x_1 = x_6 = x_5 = x_2 = x_8 = x_3 = 0$$

		x_1	x_6	x_5	x_2	x_8	x_3
$f(\mathbf{x})$	3.3	-3.1	-1.3	-1.7	-0.37	-8.3	-0.044
x_7	1	-0.14	-0.096	-0.53	-0.092	0.29	0.015
x_9	0.19	0.29	0.022	-1.1	-0.093	-0.18	-0.089
x_4	0.52	-0.88	-0.062	-0.56	0.012	-0.74	-0.062

Tableau

$$\max_{\mathbf{x}} \; f(\mathbf{x}) = 3.3 - 3.1x_1 - 1.3x_6 - 1.7x_5 - 0.37x_2 - 8.3x_8 - 0.044x_3$$

where

$$x_7 = 1 - 0.14x_1 - 0.096x_6 - 0.53x_5 - 0.092x_2 + 0.29x_8 + 0.015x_3 \geq 0$$

$$x_9 = 0.19 + 0.29x_1 + 0.022x_6 - 1.1x_5 - 0.093x_2 - 0.18x_8 - 0.089x_3 \geq 0$$

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Awkward Problems

- If there are any column with all entries positive then this variable can be increase forever—this is a signal that the linear programming problem is unbounded
- You can also find that a basic variable becomes zero—this is known as a degenerate feasible vector
- It can by removed by exchanging variables on the left of the inequality with variables on the right
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High Performance Solvers

- Although the tableau method is the “classic solver” it doesn’t cut the mustard for large scale problems
- The simplex update can also be viewed as solving a linear set of equations which is facilitated by performing an LU-decomposition
- However, the constraints are often very sparse so good solvers try to take advantage of the sparsity
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Time Complexity of Simplex

- The time complexity of the updates is $O(n^2)$
- The critical question is how many updates are necessary
- It turns out that typically this is $O(n)$ making the simplex algorithm $O(n^3)$
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Interior Point Method

- An alternative to the simplex method is the interior point method which always remains in the feasible region, away from the constraints
- These method iterate towards the constraints and are provably polynomial
- For small linear programming problems they are out-performed in practice by the simplex method
- On large and very large problems they seem to perform as well if not better than the simplex method
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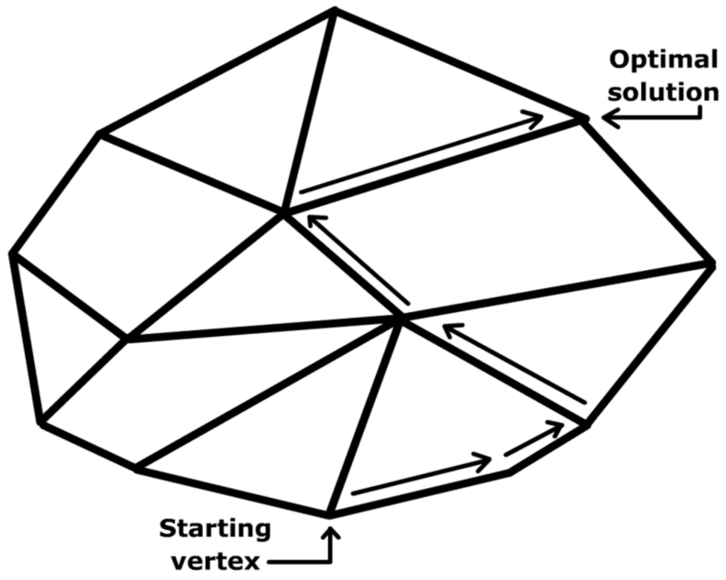
Outline

1. Recap
2. Basic Feasible Solutions
3. Simplex Method
4. **Classic LP Problems**



LP Problems

- Any problem that can be set up as a linear program can be solved in polynomial time
- One way is just to feed it to a LP-solver
- Sometimes the problems are important enough and have such a distinctive formulation that faster specialised algorithms have been developed
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Maximum Flow

- In maximum flow we consider a directed graph representing a network of pipes
- We choose one vertex as the source and a second vertex as a sink
- Each edge has a flow capacity that cannot be exceeded
- The problem is to maximise the flow between source of sink
- This can be used to model the flow of a fluid, parts in an assembly line, current in an electrical circuit or packets through a communication network

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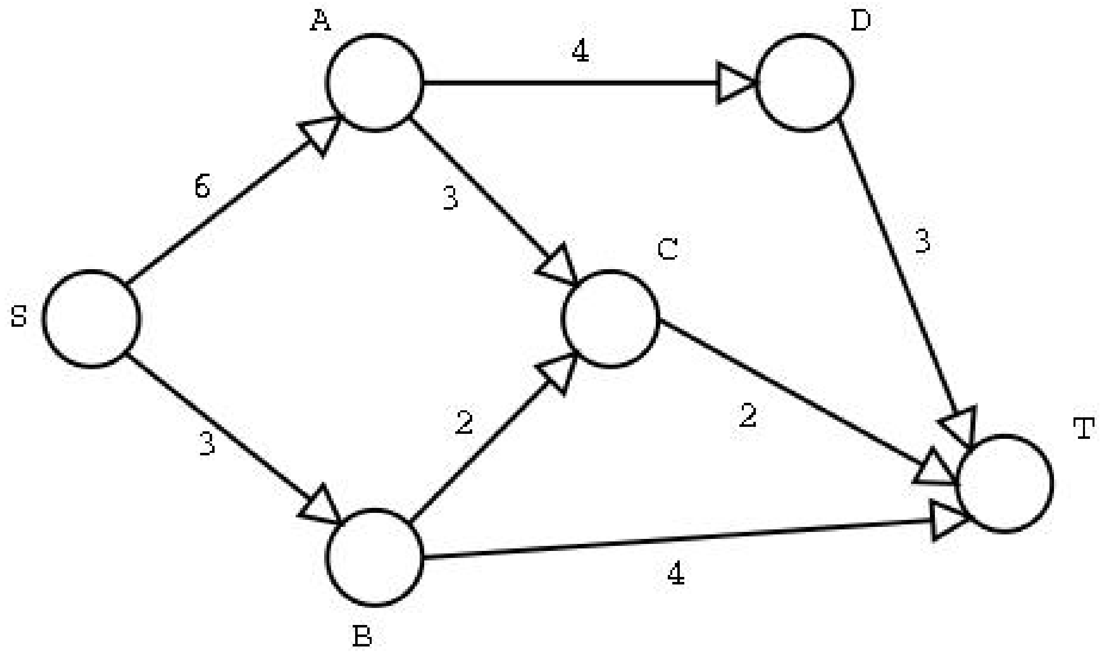
Example

- Consider a firm that has to ship haggis from Edinburgh to Southampton
- The shipping firm transports this in crates which it sends through intermediate cities
- The number of crates is limited by the size of the lorries it uses



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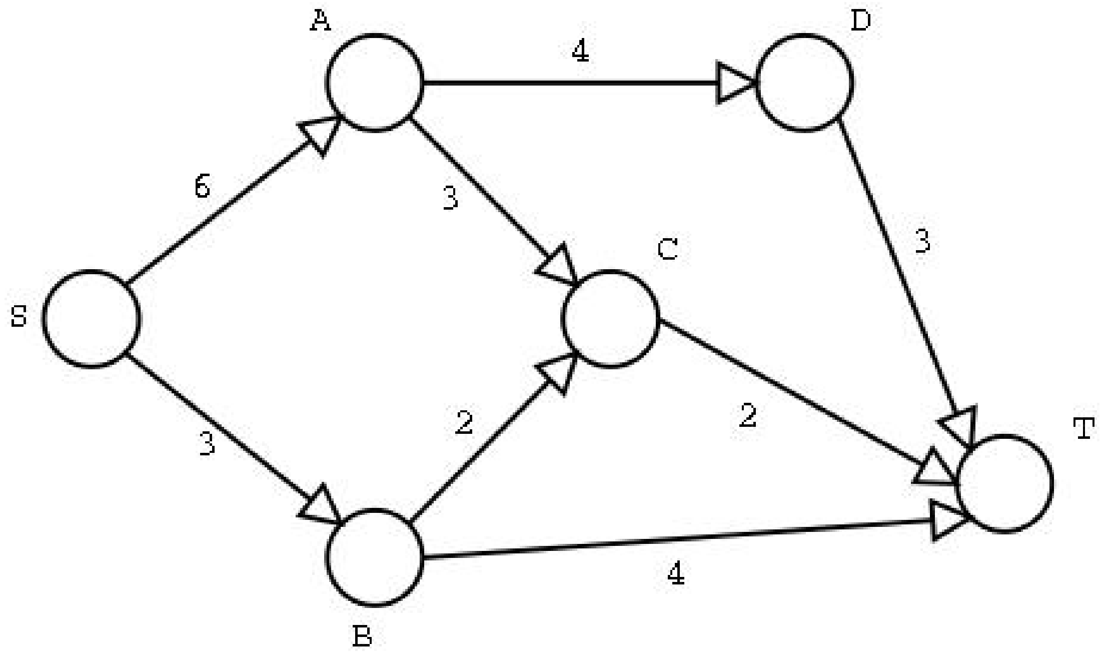

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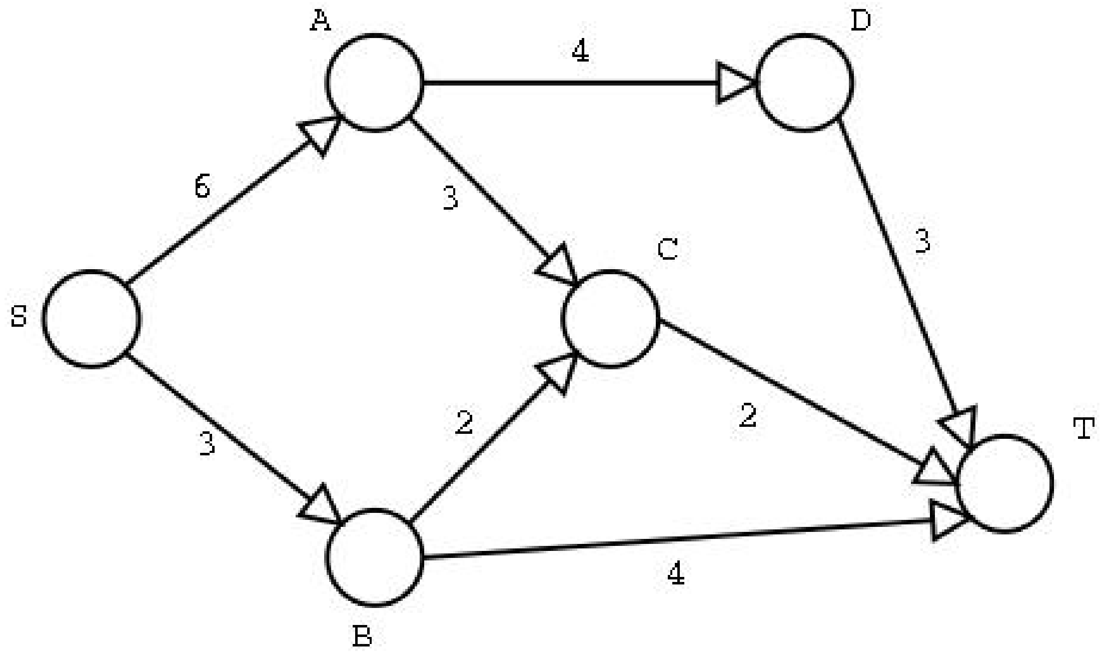

Flow

- We are given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each edge has a capacity $c(i,j)$
- We define the flow from i to j as $f(i,j)$ with $0 \leq f(i,j) \leq c(i,j)$
- For all vertices except the source (s) and sink (t) we assume

$$\forall i \in \mathcal{V}/\{s,t\} \qquad \sum_{j \in \mathcal{V} | (i,j) \in \mathcal{E}} f(i,j) = \sum_{j \in \mathcal{V} | (i,j) \in \mathcal{E}} f(j,i)$$

(i.e. no flow is lost from source to sink)

- We want to maximise the flow from the source

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Solving Maximum Flow

- As set up we have a linear objective function with linear constraints
- We can therefore solve this problem with a LP-solver
- (Note the solution will typically involve a fraction flow)
- However, this is such a classic problem with a distinctive structure that we can solve it more quickly with other algorithms
- The classic algorithm is the Ford-Fulkerson method with run time $O(|\mathcal{E}| \times f_{\max})$ where $f_{\max}$ is the maximum flow, although we won't cover this in the course

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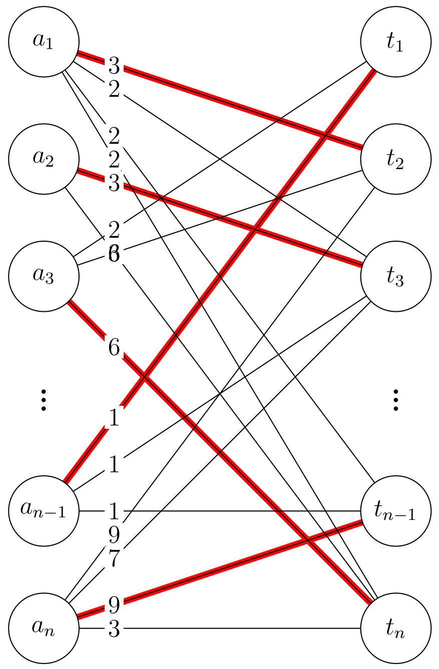
Linear Assignment

- We are given a set of n agents, $\mathcal{A}$, and n tasks, $\mathcal{T}$
- Each agent has a cost associated with performing a task $c(a,t)$
- We want to assign an agent to one task so as to minimise the total cost
- Consider a taxi firm with taxi's at 5 different locations and 5 requests to fulfil. The cost is the distance to the clients. Which taxi should go to which client?



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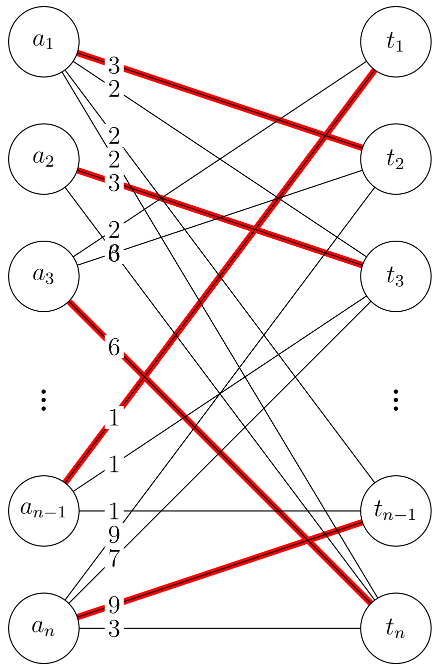

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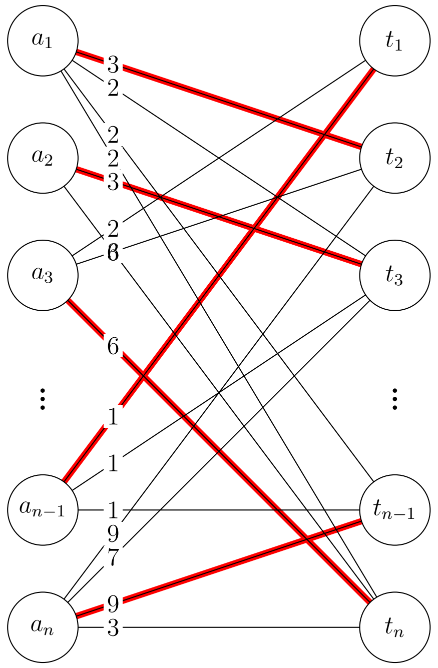

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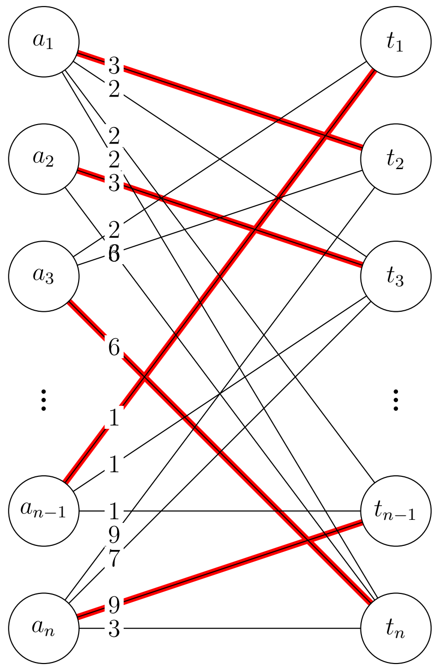

LA as LP

- The linear assignment problem can be set as a linear programming problem

$$\min_{\boldsymbol{x}} \sum_{a\in\mathcal{A},t\in\mathcal{T}} c(a,t)x_{a,t}$$

subject to

$$\begin{aligned}
\forall a \in \mathcal{A} &\qquad \sum_{t\in\mathcal{T}} x_{a,t} = 1 \\
\forall t \in \mathcal{T} &\qquad \sum_{a\in\mathcal{A}} x_{a,t} = 1 \\
\forall (a,t) \in (\mathcal{A},\mathcal{T}) &\qquad x_{a,t} \geq 0
\end{aligned}$$

Hungarian Algorithm

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- The most efficient algorithm is the Hungarian algorithms
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