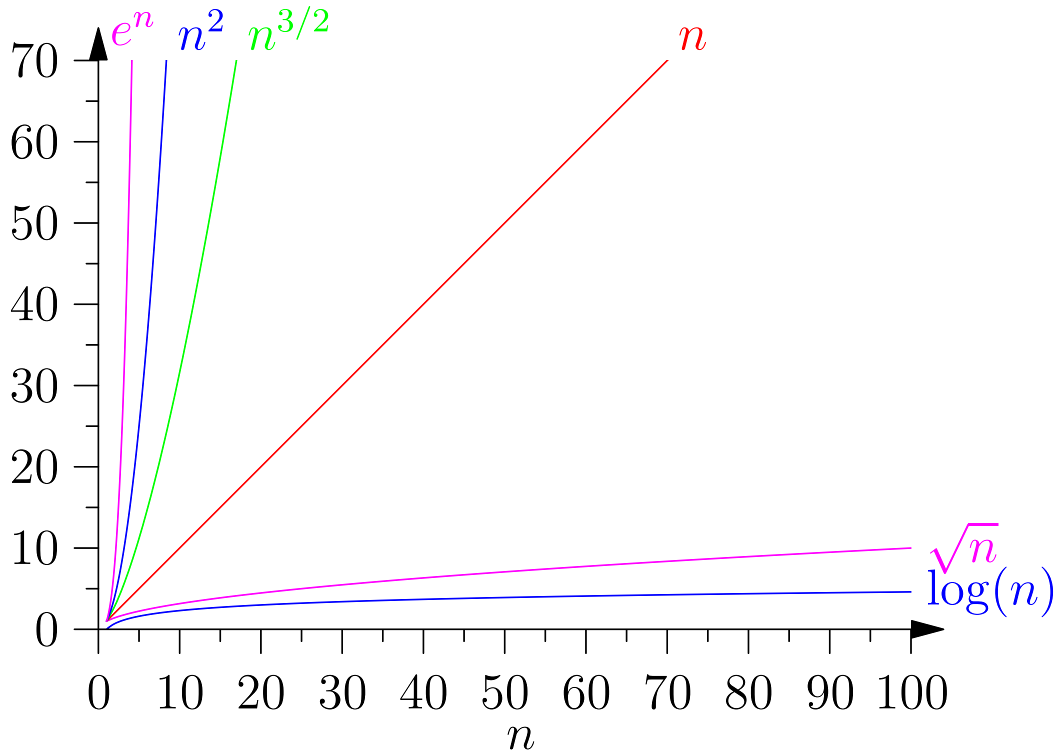
Algorithms and Analysis

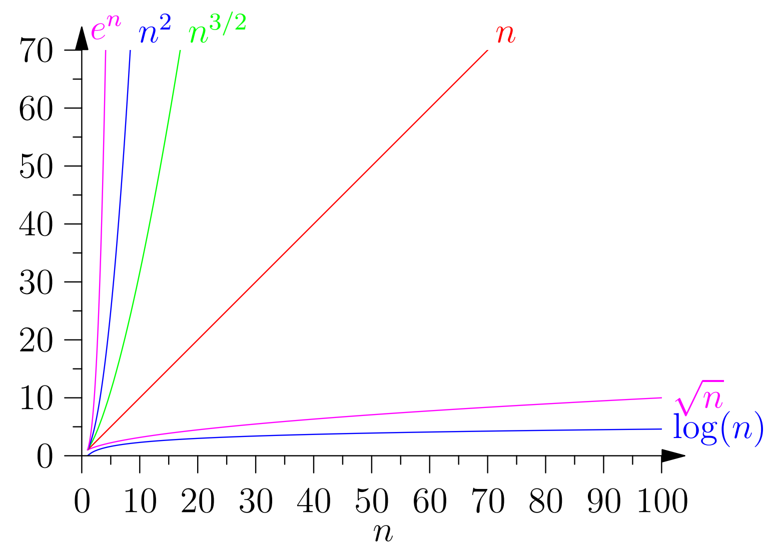
Lesson 31: *Understand Time Complexity*



Theta, Big-O, little-o, Big-Omega, little-omega

Outline

1. **Time Complexity Classes**
    - Theta—Θ
    - Big O
    - Little o
    - Big Omega—Ω
    - Little omega—ω
2. Computing Time Complexity



Recap

- We have seen many algorithms taking times of order 1, $\log(n)$, n, $n\log(n)$, n^2, etc
- Sometimes these are worst time, average time or best time results
- We have lots of different notations, e.g. $O(1)$, $\Theta(\log(n))$, $\Omega(n^2)$, etc.
- What does it all mean?

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Complexity Class Sets

- The correct way to think about complexity classes is in terms of sets

- Suppose we have an algorithm which takes an input of size n and computes an output in $f(n)$ operations

- E.g. $f(n) = 4n^2 + 2n + 3$

- We can partition all run times into sets by considering only the leading order term and ignoring the constant term

- We denote these sets by $\Theta(g(n))$
  - ⋆ $4n^2 + 2n + 3 \in \Theta(n^2)$
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Defining $\Theta(g(n))$

- A function $f(n) \in \Theta(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \qquad\qquad 0 < c < \infty$$

- E.g.

$$\lim_{n\to\infty} \frac{4n^2 + 2n + 3}{n^2} = 4$$

$$\lim_{n\to\infty} \frac{5n\log(n) + 3n + 2}{n\log(n)} = 5$$

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- However asymptotically (i.e. for sufficiently large n) an order $n\log(n)$ algorithm will always run faster than an order n^2 algorithm even when they are run on different machines
- The constant is important in practice (if there are two algorithms A and B that are both $n\log(n)$, but algorithm A runs twice as fast as algorithm B, which one should you use?)
- Nevertheless, ignoring the constant is often essential to make analysis of algorithms doable

Ordering Complexity Classes

- We can define the relation $\Theta(f(n)) < \Theta(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

- Informally if algorithm A has time complexity $\Theta(f(n))$ and algorithm B has time complexity $\Theta(g(n))$ then if $\Theta(f(n)) < \Theta(g(n))$ algorithm A is faster for sufficiently large n

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The Complexity Line

- We can order all complexity classes. E.g.

$$\Theta(1) < \Theta(\log(n)) < \Theta(\sqrt{n}) < \Theta(n) < \Theta(n^2)$$

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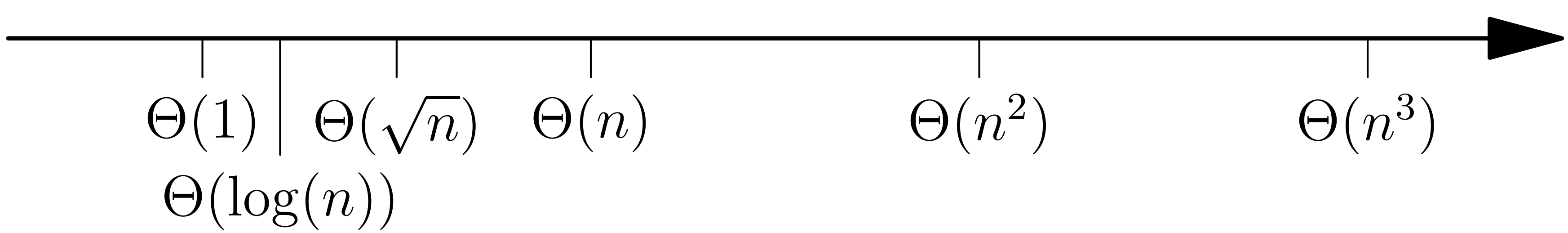

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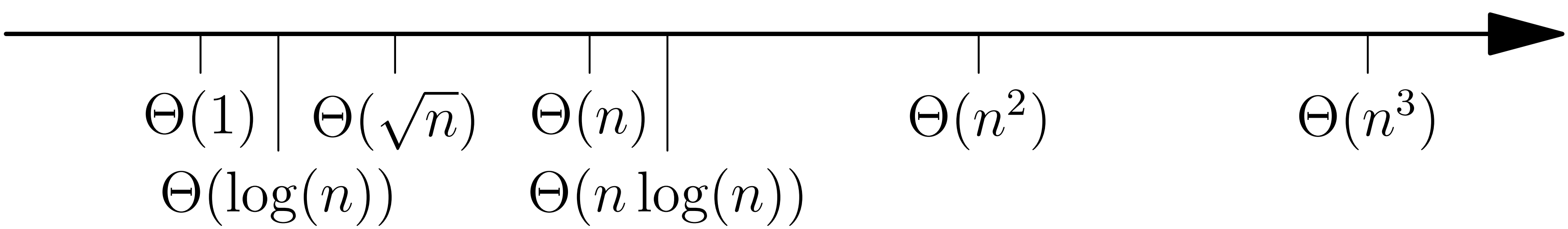

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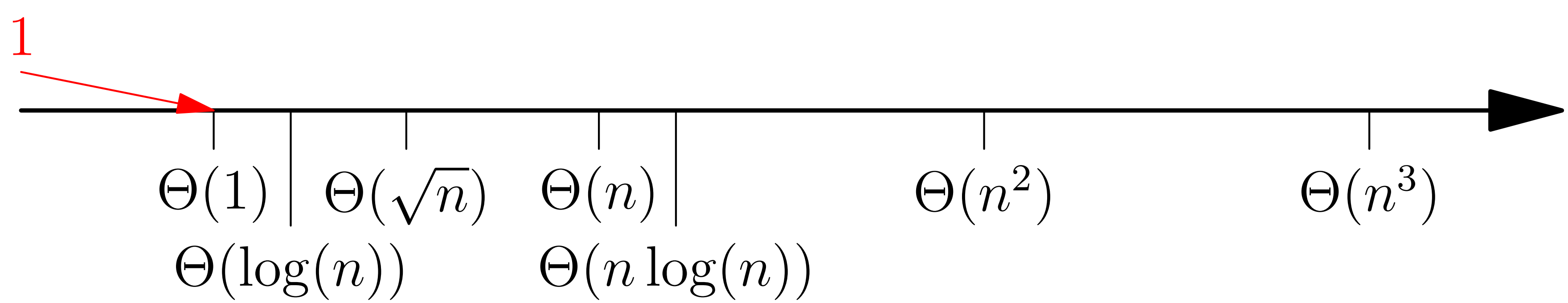

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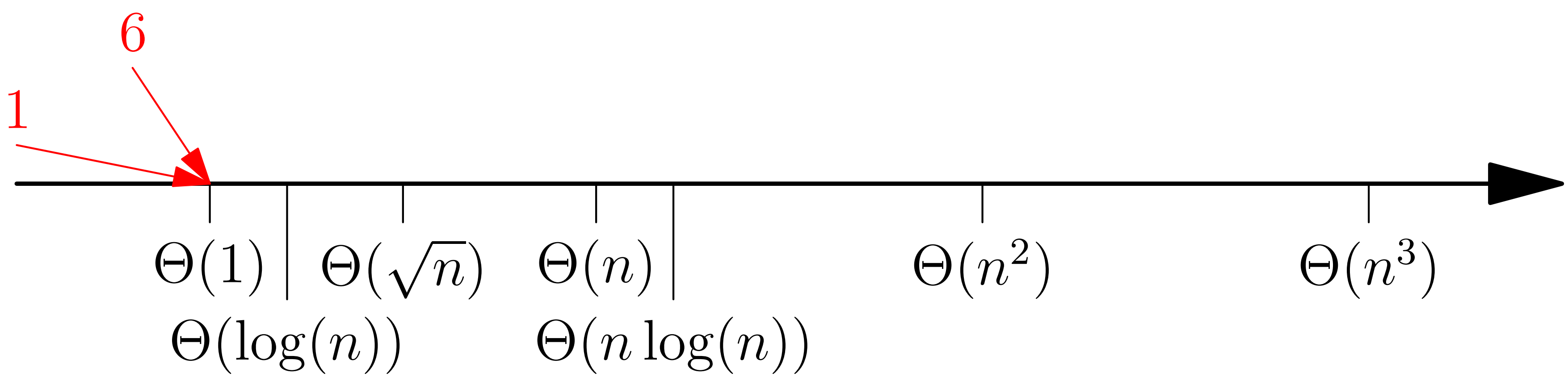

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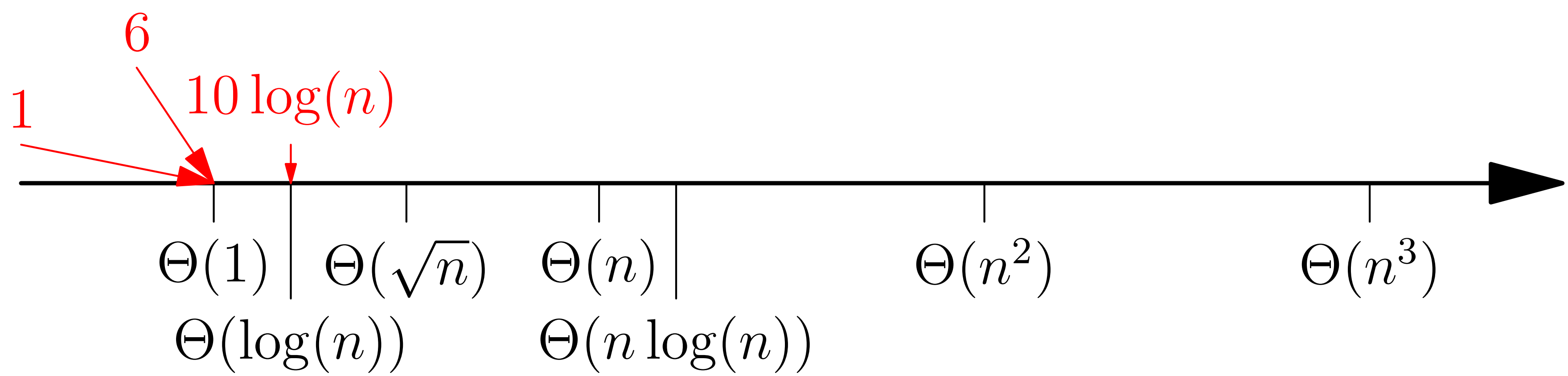

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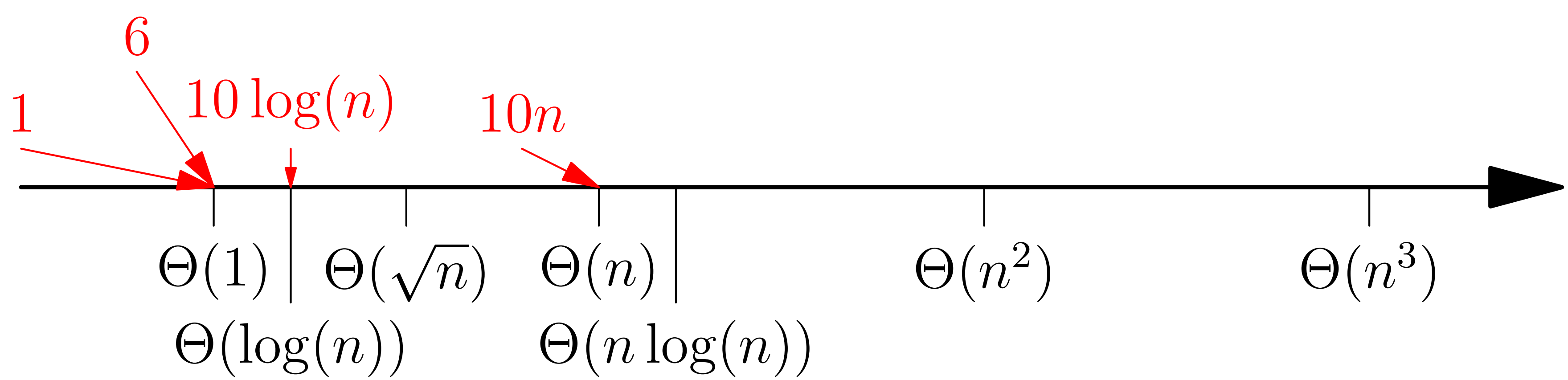

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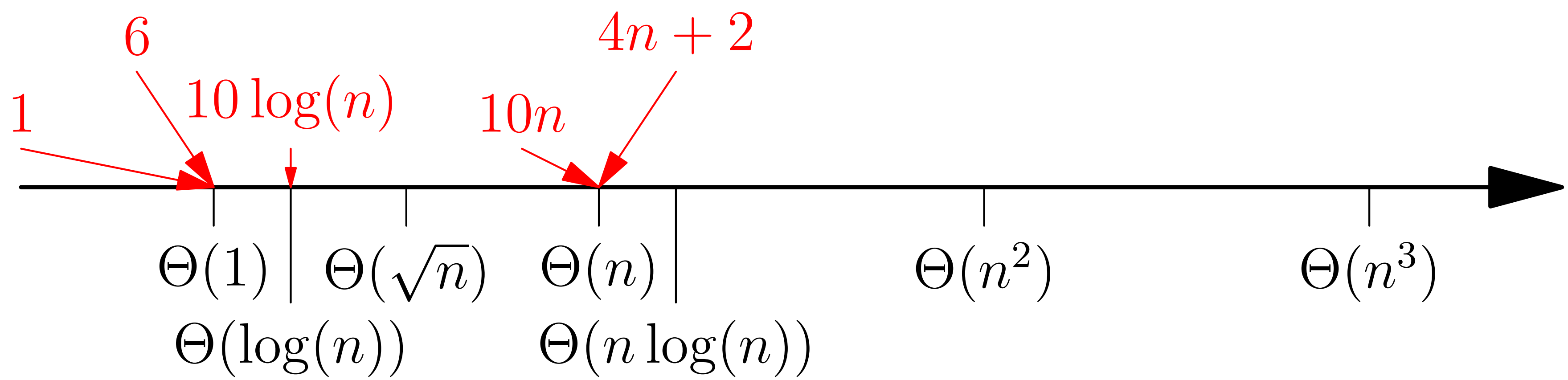

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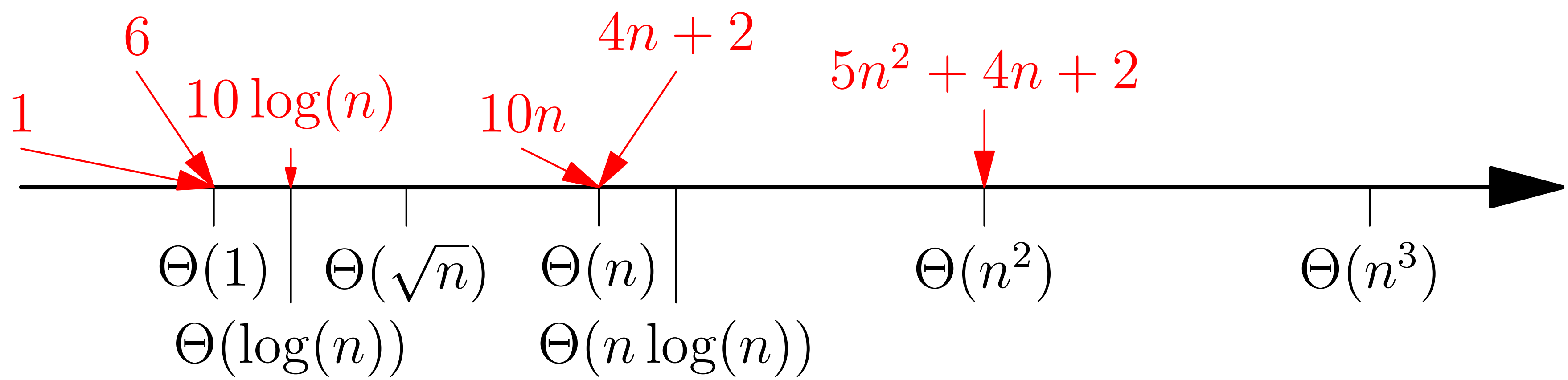

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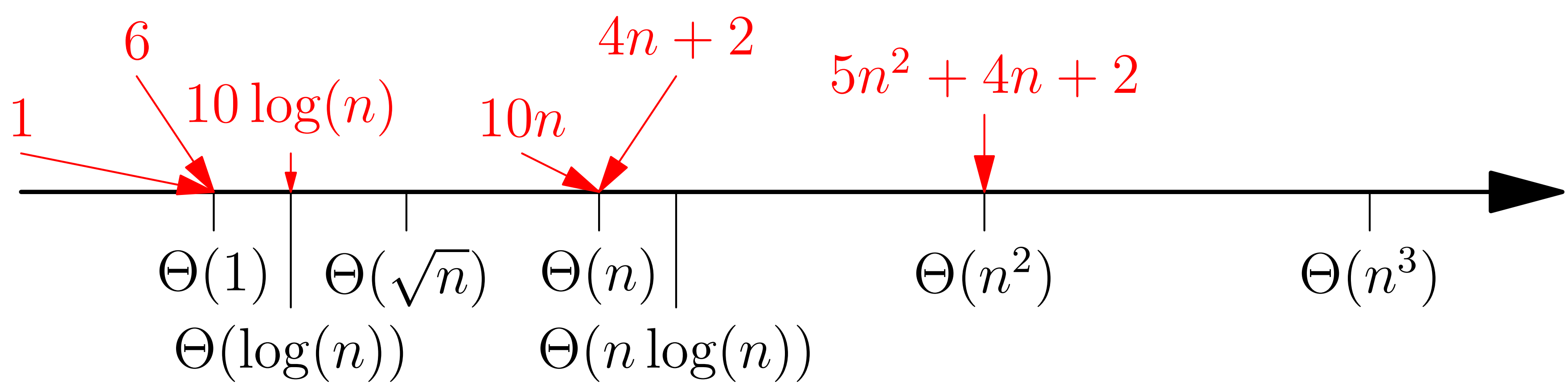

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- The run time of many algorithms depends on the input
- In this case we can define different time complexities
  - ⋆ Worst case time complexity (the longest time an algorithm will take)
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- Every algorithm will have a Θ complexity class for the worst, average and best time complexity

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Unknown Time Complexity

- Algorithms are often rather complicated and knowing the exact time complexity (for either worst, average or best cases) might not be known
- In reality it will have some run time (e.g. $f(n) = 3n^2 \log(n) + 2n^2 - n + 3$) and will belong to a Θ time complexity set (e.g. $\Theta(n^2 \log(n))$) but we might not be able to calculate it
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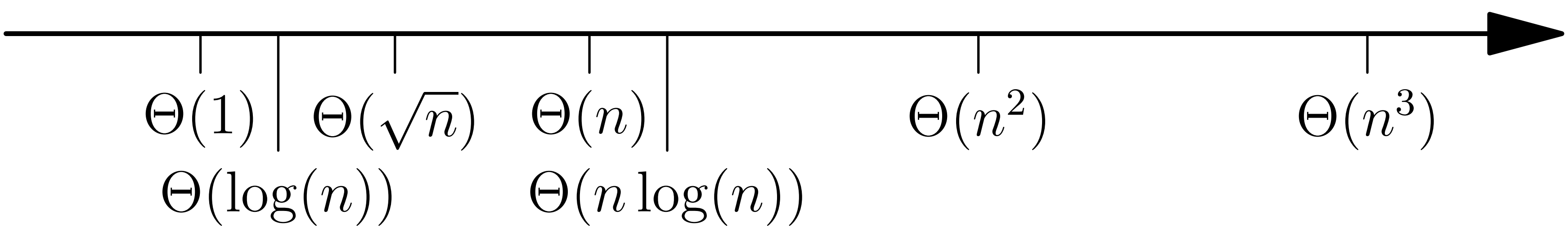
- If an algorithm is $O(g(n))$ then its time complexity is no more than $\Theta(g(n))$



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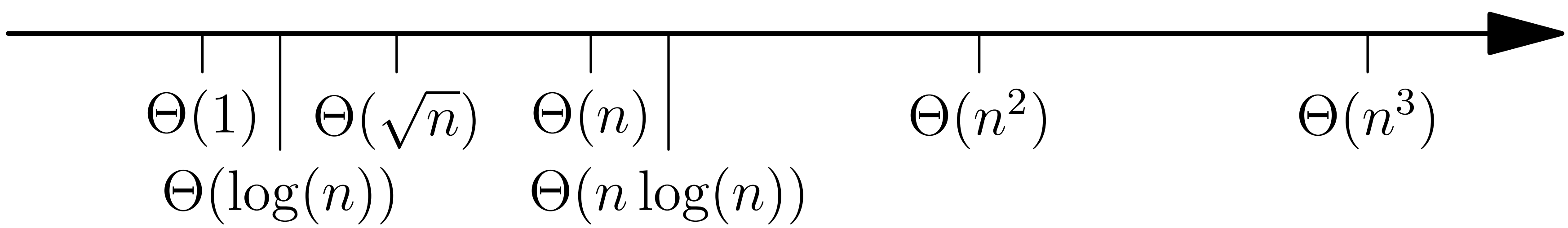

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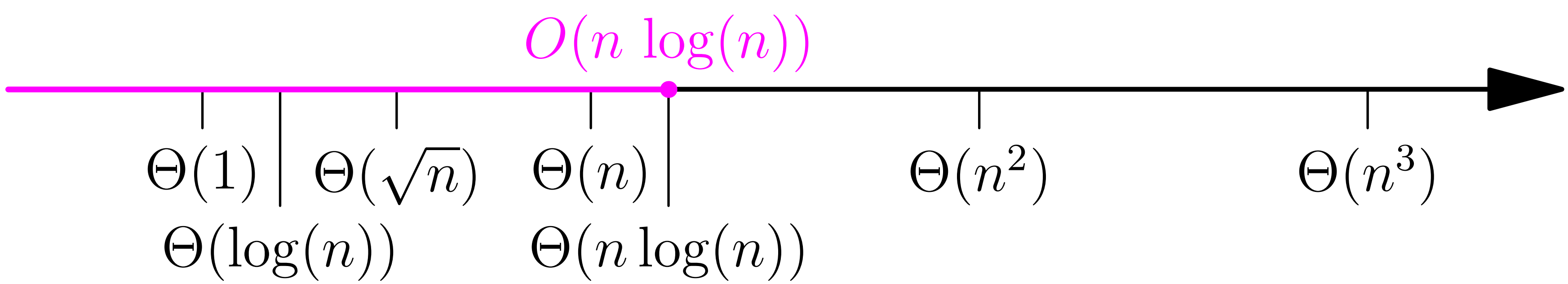

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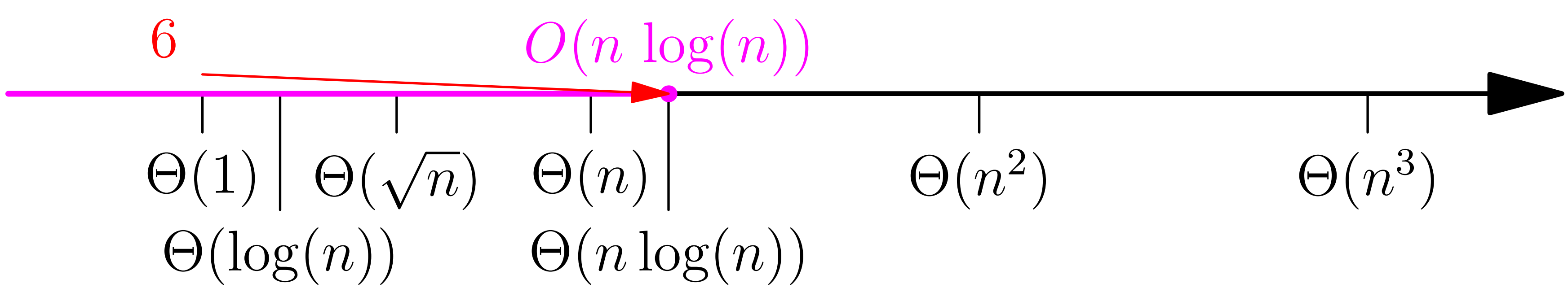

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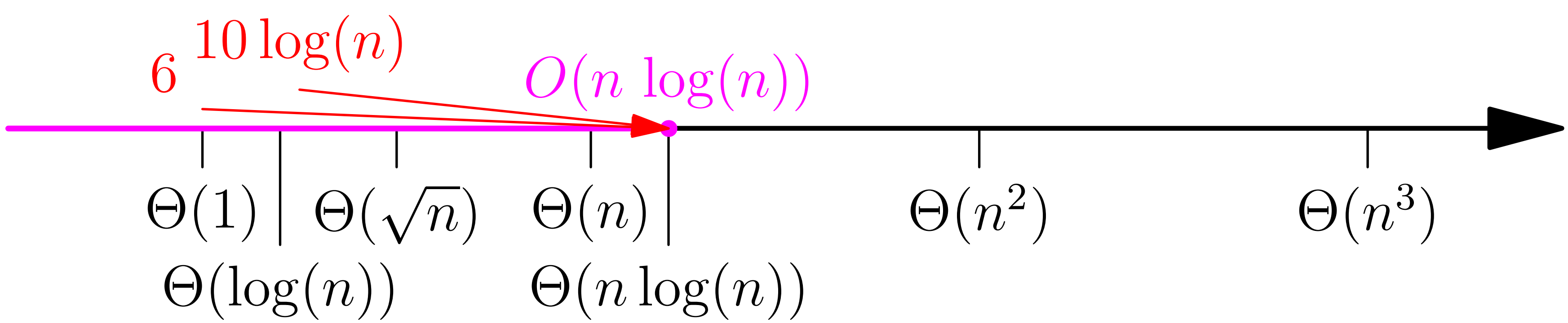

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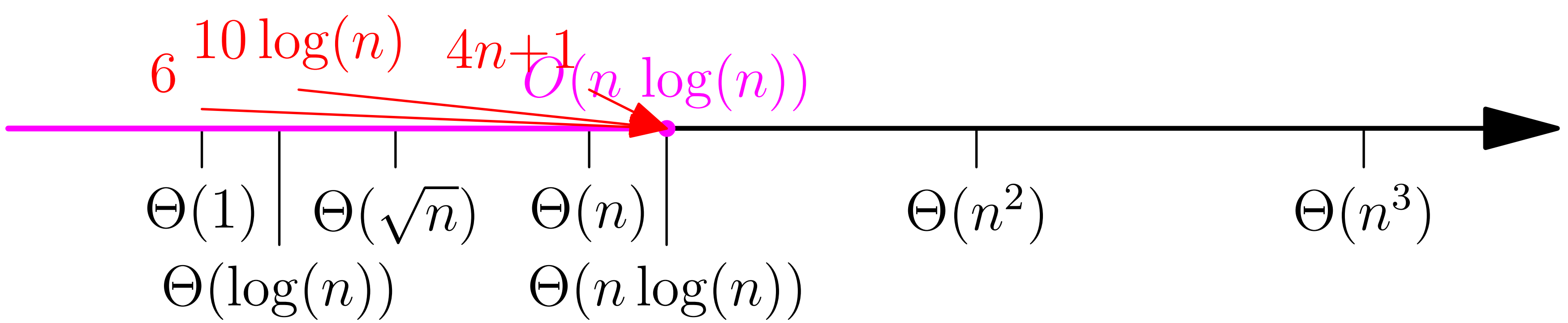

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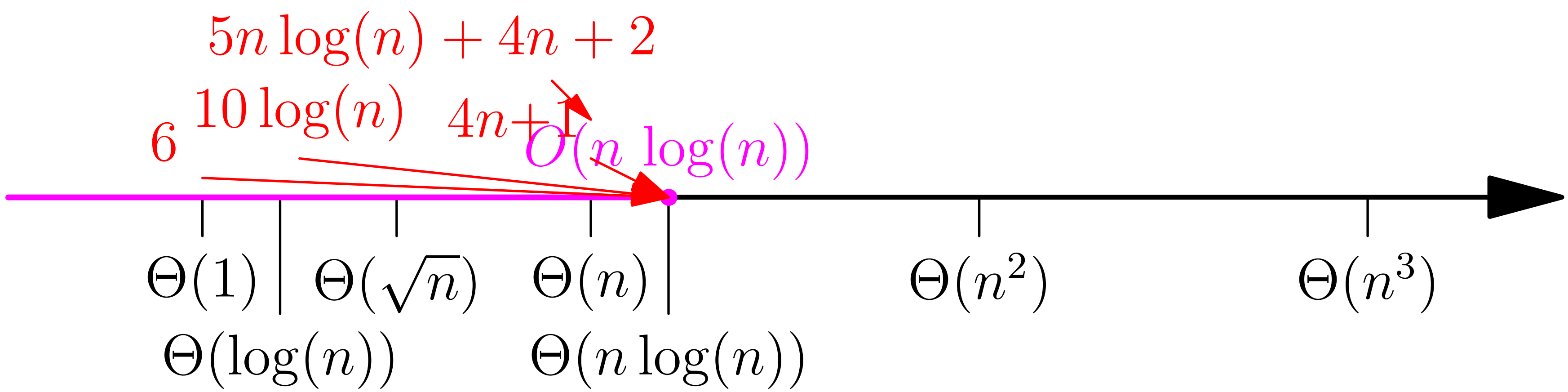

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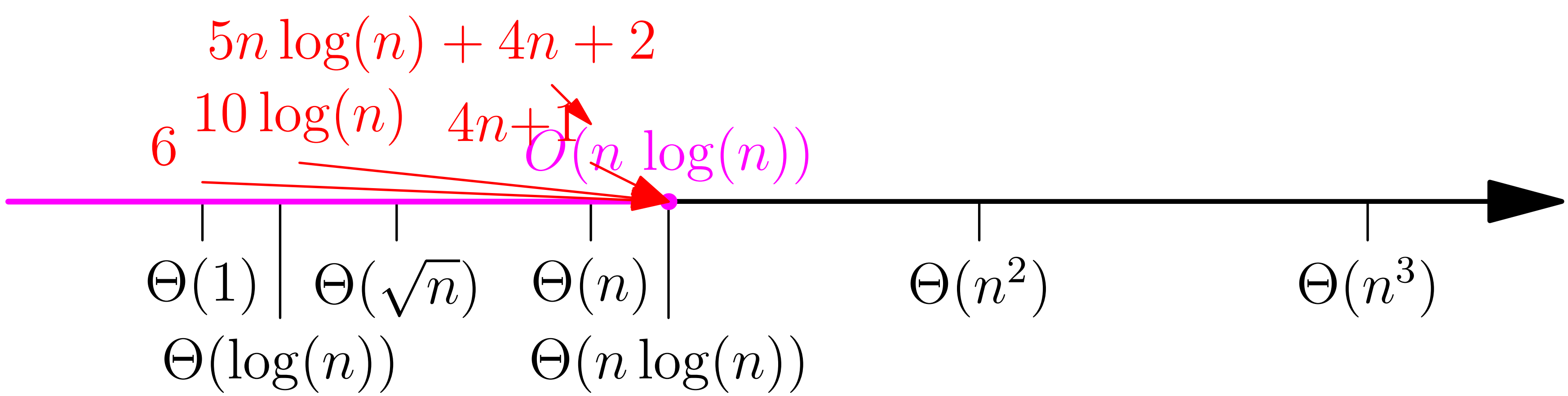

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Upper Bounding Time Complexity

- Consider a program

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for(int i=0; i<n; i++)  {
  // do something
  if (/* some condition */) {
    for (int j=0; j<n; j++) {
      // do other stuff
    }
  }
}
// clean up
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- If the `if` statements is never true this is a $\Theta(n)$ algorithm if it is always true it is a $\Theta(n^2)$ algorithm

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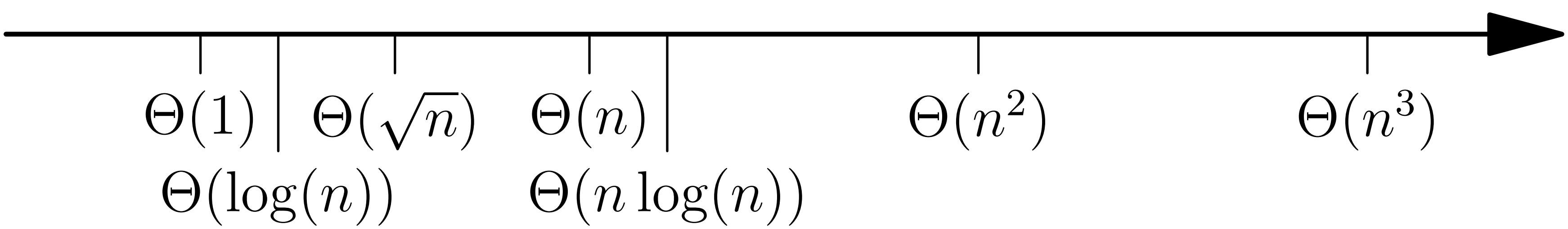
    }
  }
}
// clean up
```

- If the `if` statements is never true this is a $\Theta(n)$ algorithm if it is always true it is a $\Theta(n^2)$ algorithm

- If we don't know then we can at least say that the run time is in $O(n^2)$—we assume the worst, but the worst may never happen

Little-o

- Sometimes we want to say that the time complexity for an algorithm $\Theta(f(n))$ is **strictly less** than a known time complexity $\Theta(g(n))$



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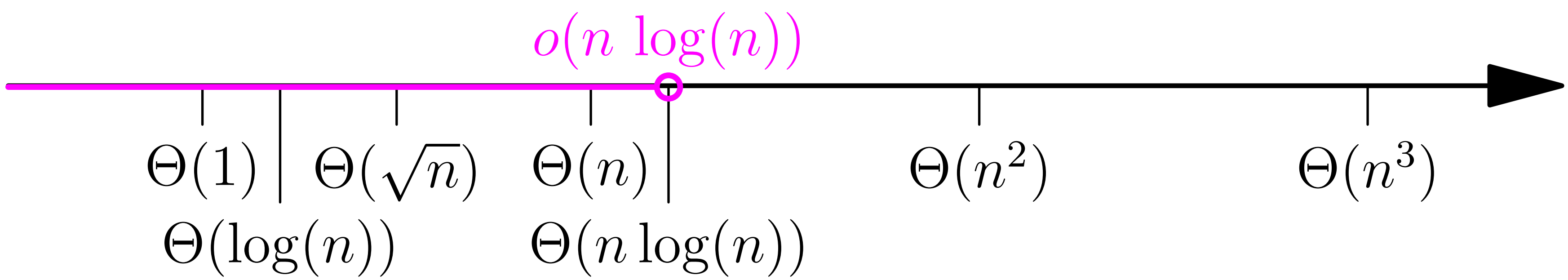

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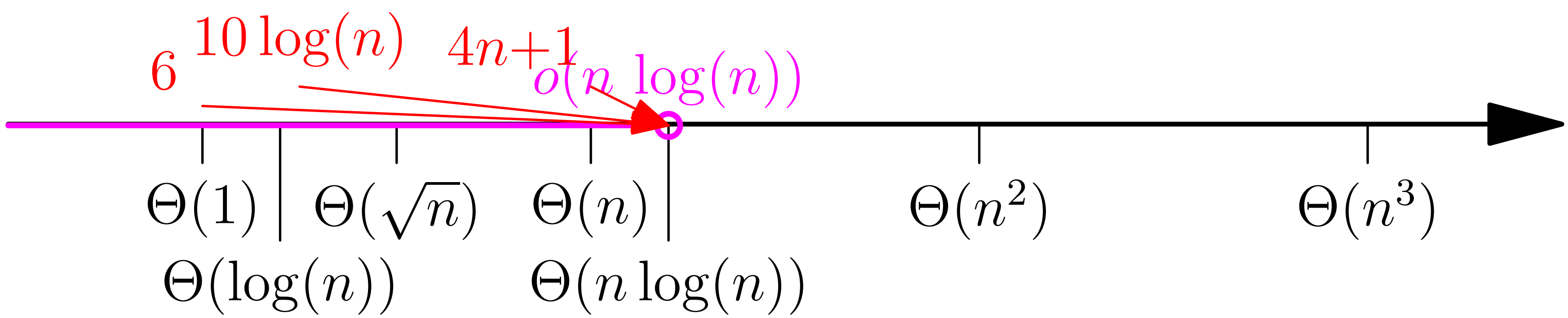

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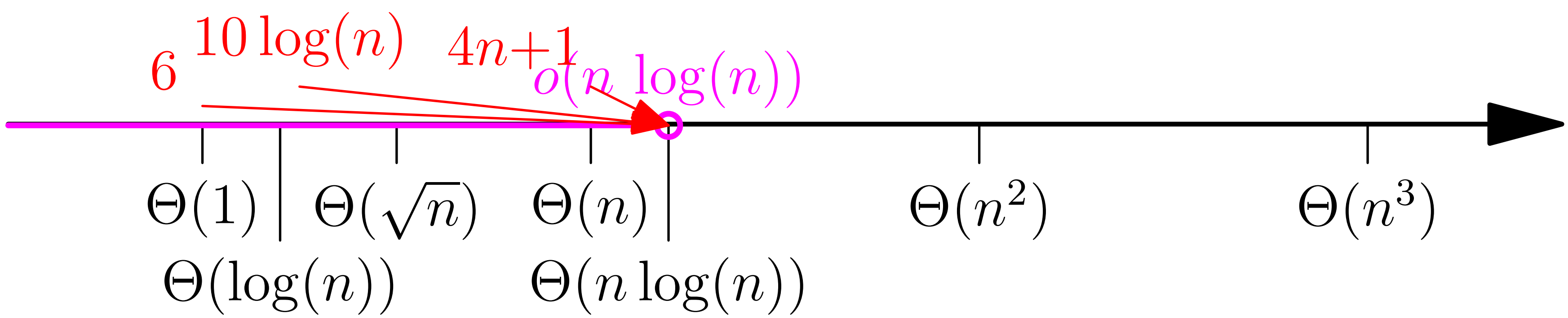

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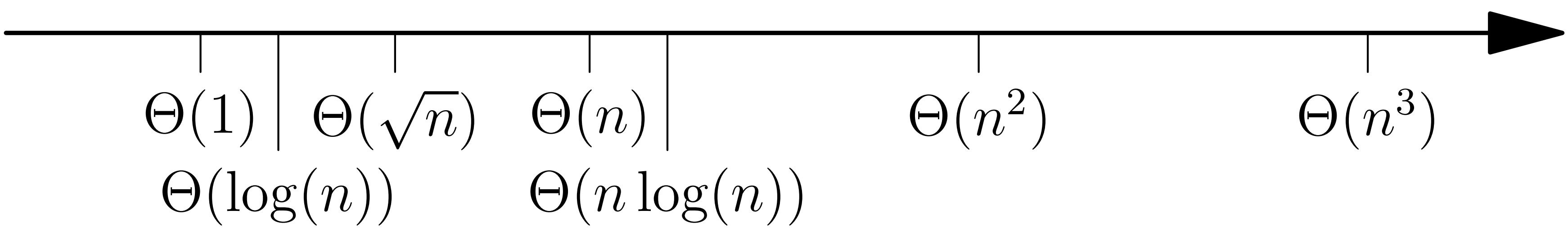

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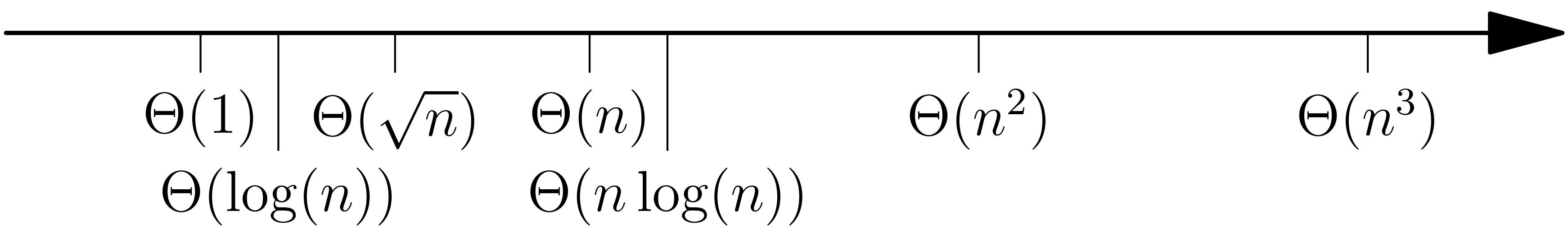

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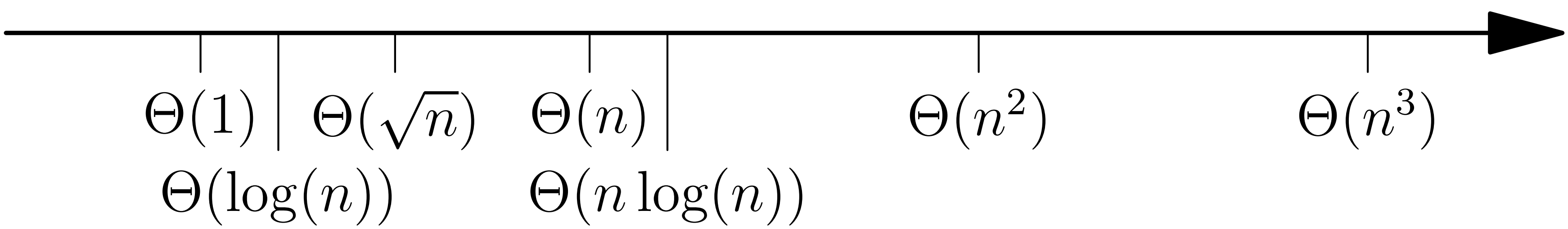

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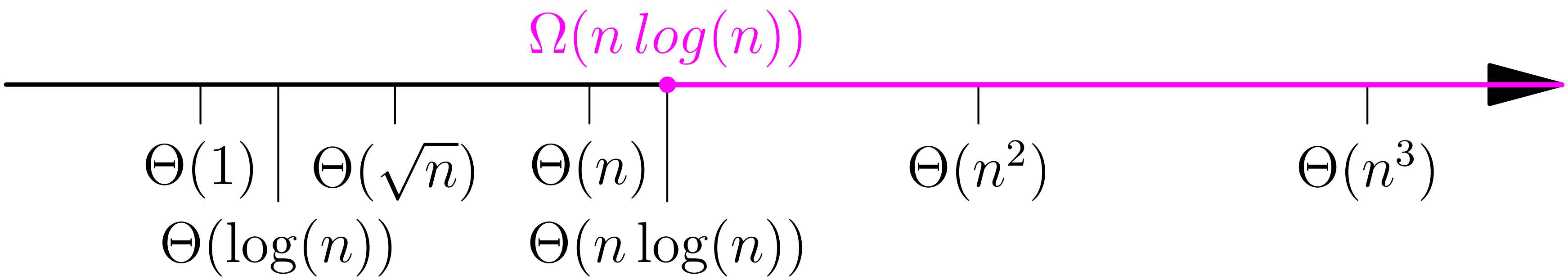

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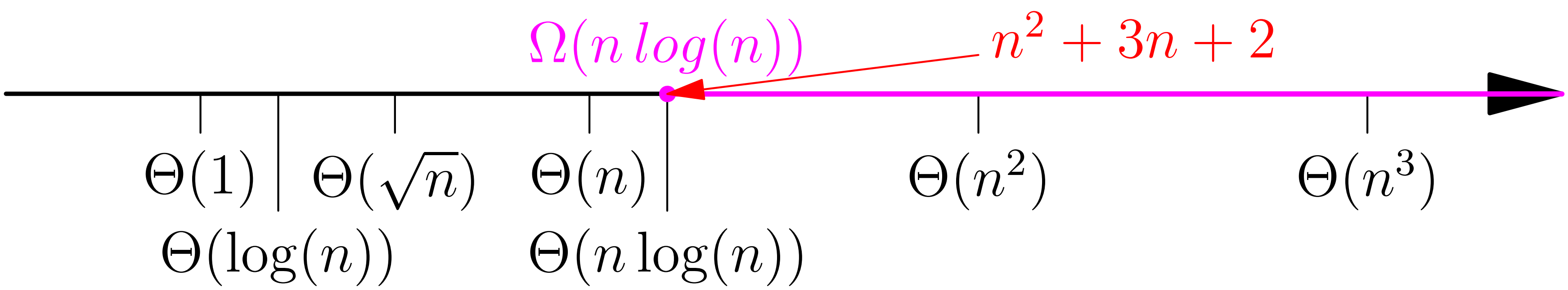

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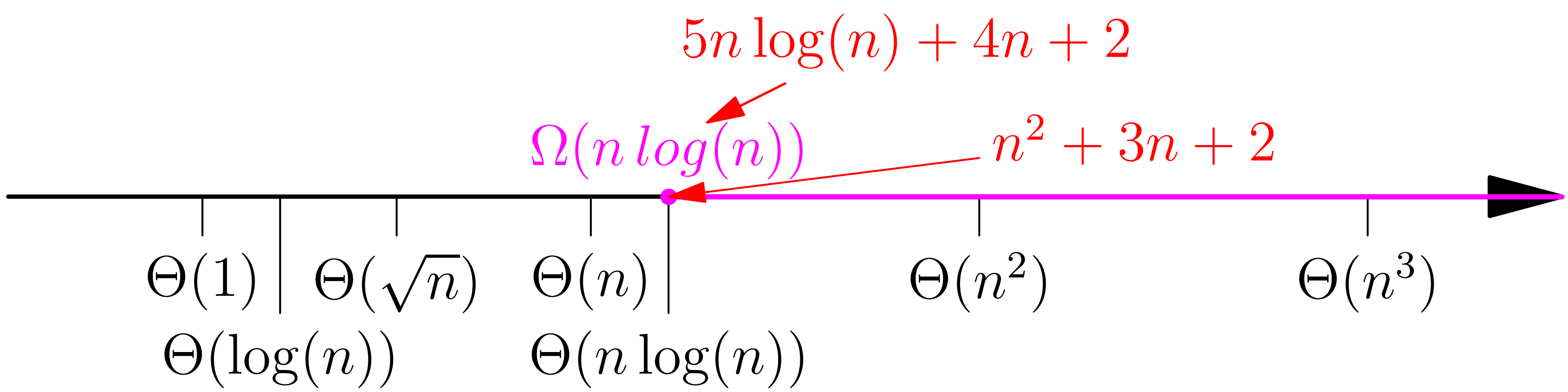

Lower Bounding Time Complexity

- Returning to the program

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// define stuff
for(int i=0; i<n; i++)  {
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- We might not know how frequently the `if` statement is true, but we know in all cases the first `for` loop iterates over n

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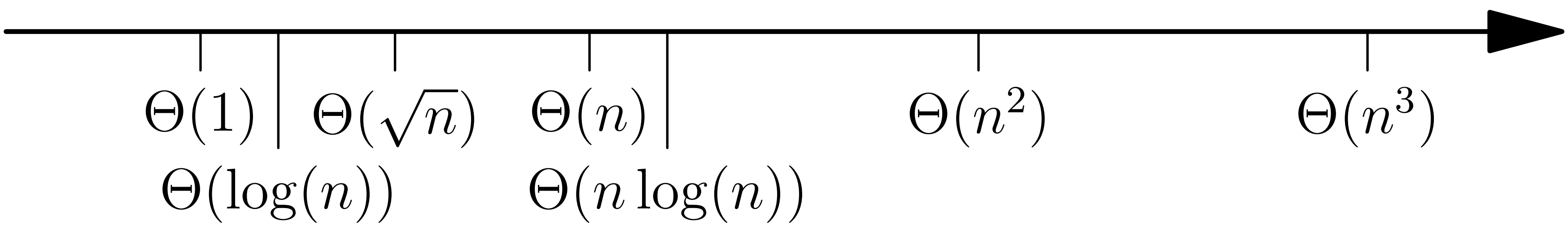

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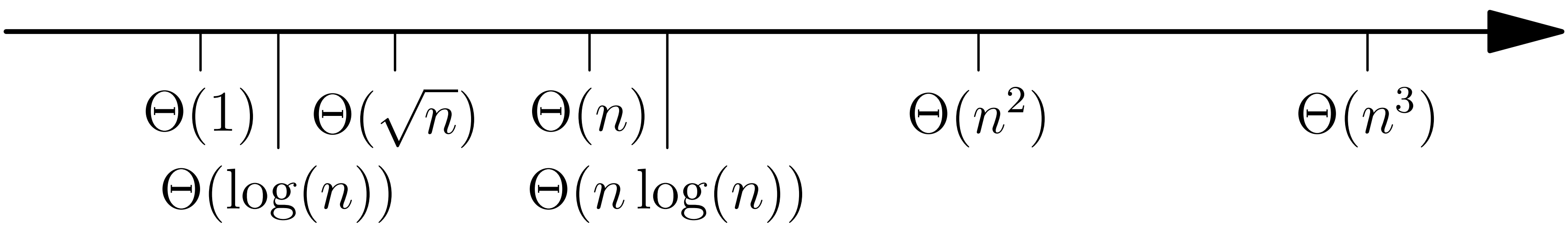

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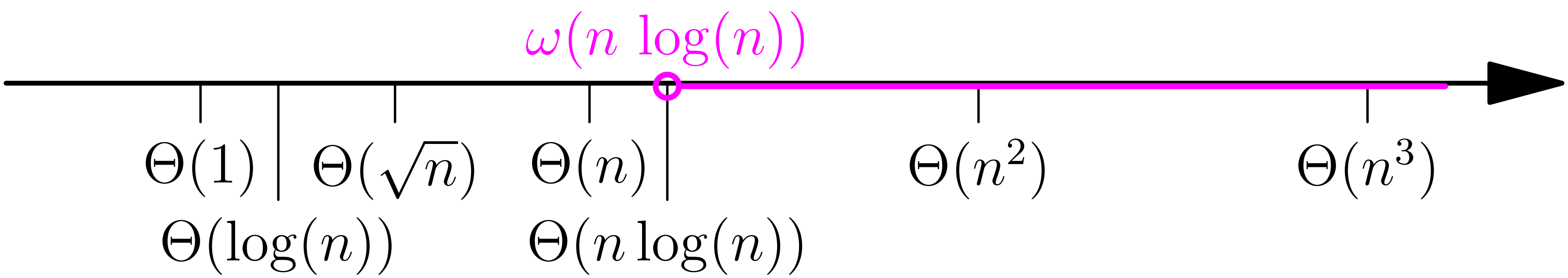

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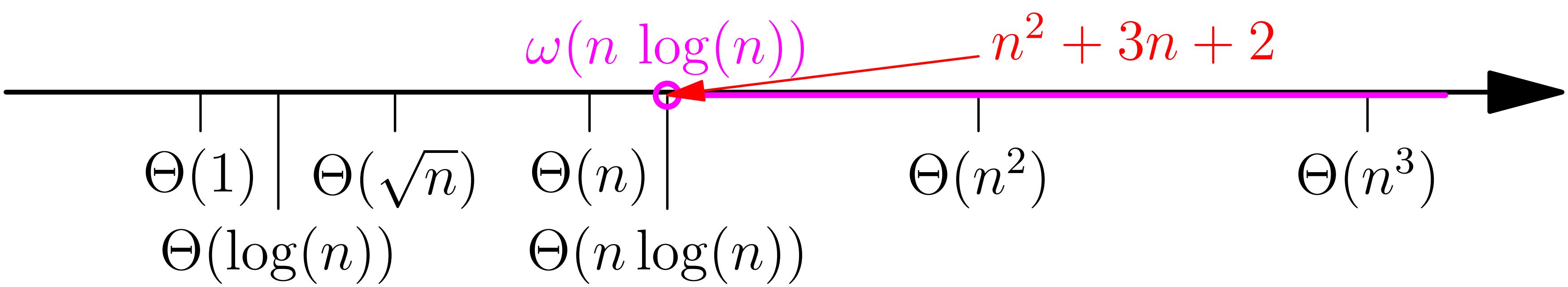

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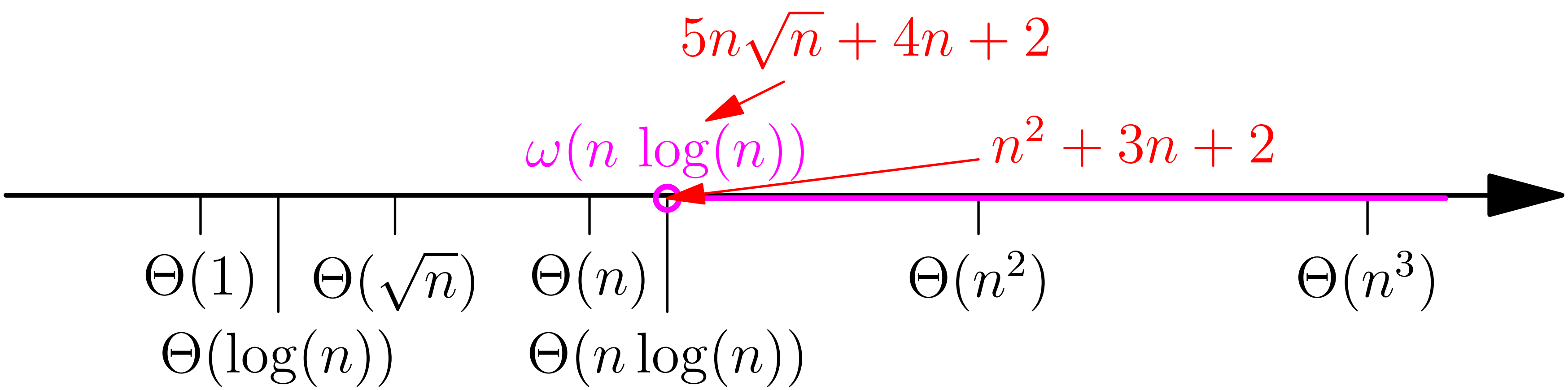

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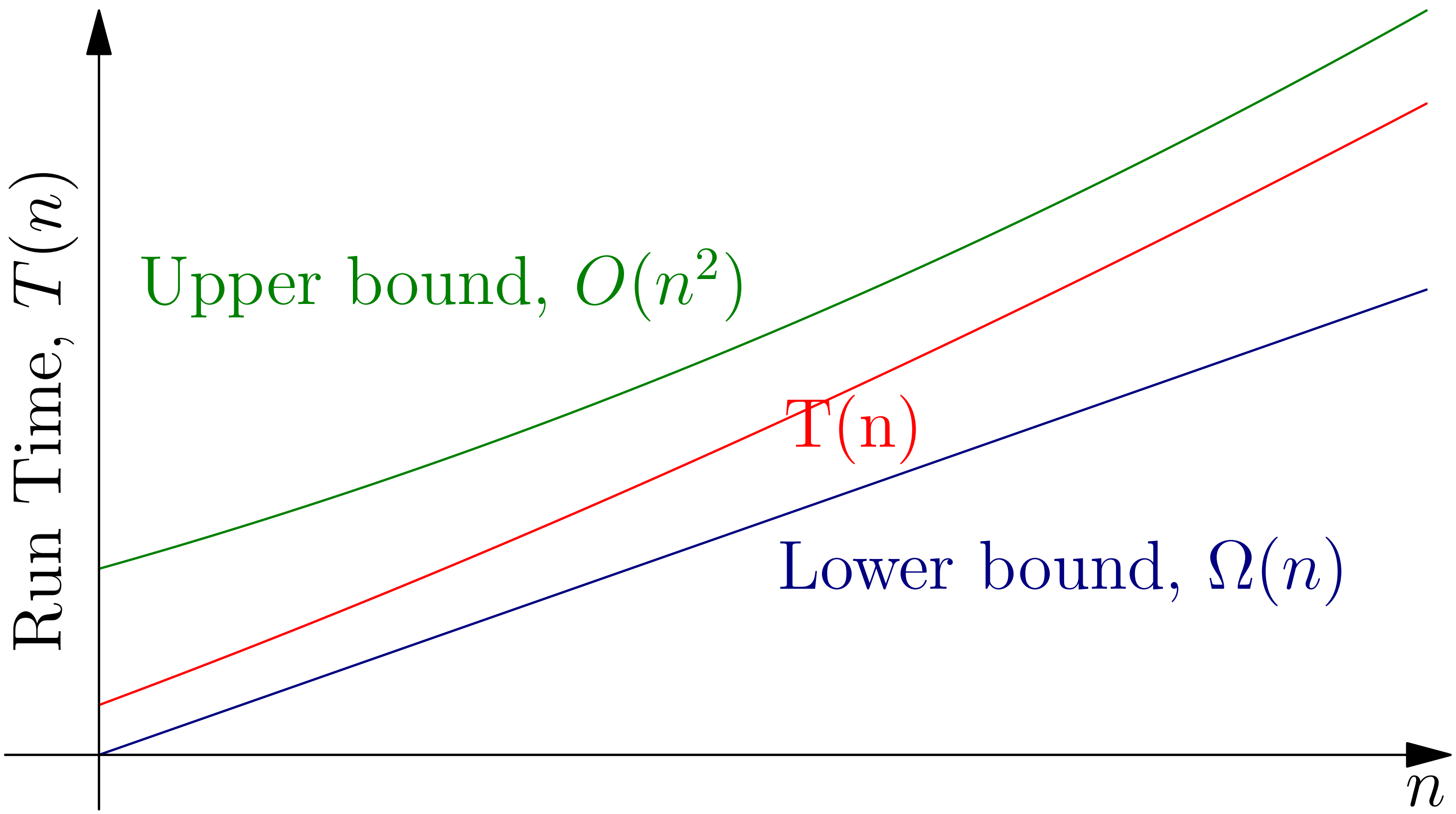

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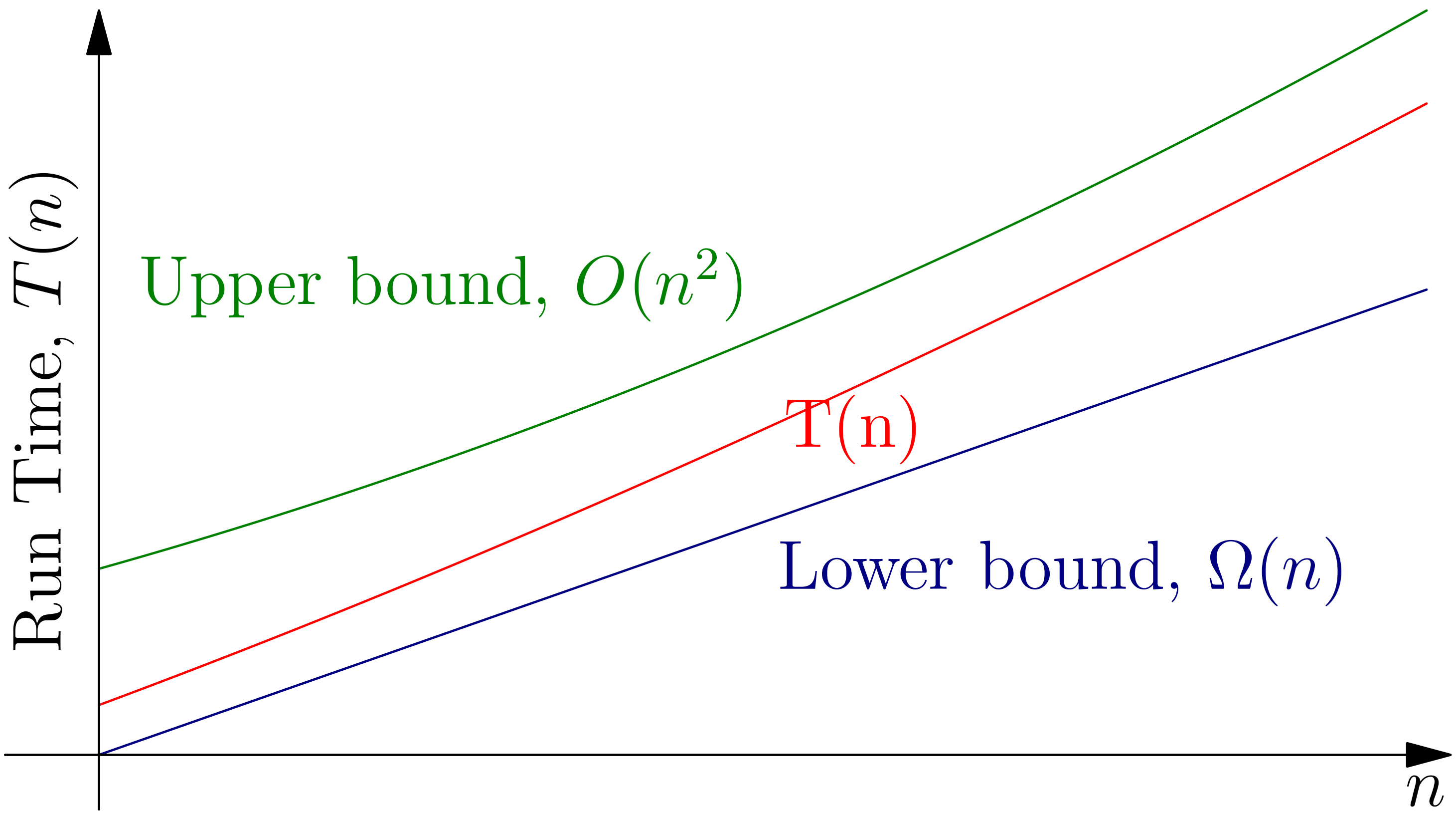

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- If we know an algorithm is
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Meaning of Time Complexity

- Insertion sort has time complexity $\Theta(n^2)$
- Because it consists of two `for` loops
- It takes 2 seconds to sort $100\,000$ items
- How long does it take to sort $1\,000\,000$ items?

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- Time taken is approximately 200 seconds or around 3.5 minutes

Exponential Time Complexity

- When we talk about exponential time complexity we usually mean that

$$\log(T(n)) \in \Theta(n)$$

- This is true if
  - ⋆ $T(n) = 2^n$
  - ⋆ $T(n) = 6.1\,\mathrm{e}^{0.003n}$
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- Note that none of these are in complexity class $\Theta(\mathrm{e}^n)$

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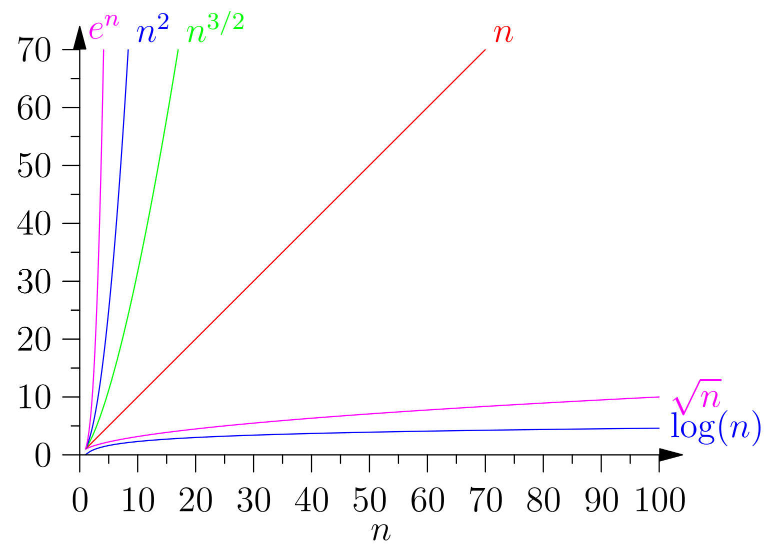
Outline

1. Time Complexity Classes
   - Theta—Θ
   - Big O
   - Little o
   - Big Omega—Ω
   - Little omega—ω
2. **Computing Time Complexity**



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- Answer $\Theta(n^2)$

Recursion

- Determining time complexity is harder when we use recursion
- Consider Euclid's algorithm for determining the greatest common divisor

```
long gcd(long m, long n)
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  while(n!=0) {
    long rem = m%n;
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    return m;
  else
    return gcd(n, m%n);
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Example of gcd

- Example of Euclid's algorithm `gcd(1989,1590)`
- Sequence of remainders is 399, 393, 6, 3, 0
- The greatest common divisor is 3
- How long does it take compute `gcd(n,m)` with $n > m$
- This is subtle as could depend in a complex way on the pair n and m

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- An observation which makes the analysis relatively simple is that the remainder is reduced by at least 2 after two iterations
- To prove
  - ⋆ Using the recursion (assuming $m, n < 0$)

$$\mathrm{gcd}(m, n) = \mathrm{gcd}(n, \mathrm{rem}(m, n)) = \mathrm{gcd}(\mathrm{rem}(m, n), \mathrm{rem}(n, \mathrm{rem}(m, n)))$$

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  - $\star$ Note that $T(1) = 1$

$$T(n) < T(2^{-1}n) + 2 < T(2^{-2}n) + 4 < \cdots < T(2^{-t}n) + 2t$$

  - $\star$ Choose $t = \lceil \log_2(n) \rceil$
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  - $\star$ Thus $T(2^{-t}n) < T(1) = 1$
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- To show that $T(n) \in O(\log(n))$ we observe
  - ⋆ Note that $T(1) = 1$

$$T(n) < T(2^{-1}n) + 2 < T(2^{-2}n) + 4 < \cdots < T(2^{-t}n) + 2t$$

  - ⋆ Choose $t = \lceil \log_2(n) \rceil$
  - ⋆ then $2^{-t}n = 2^{-\lceil \log_2(n) \rceil}n \leq 2^{-\log_2(n)}n = \frac{n}{n} = 1$
  - ⋆ Thus $T(2^{-t}n) < T(1) = 1$
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Probability of Relative Primes

- Consider the following program to compute the probability of relative primes for all numbers up to n

```
public static double probRelPrime(n)
{
  int rel=0, tot=0;
  for(int i=1; i<=n; i++)
     for(int j=i+1; j<=n; j++) {
       tot++;
       if (gcd(i,j)==1)
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  return (double) rel / tot;
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- What is the time complexity?

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- Program involves two nested loops of size $O(n)$
- Then we need to calculate `gcd(i, j)` at each iteration
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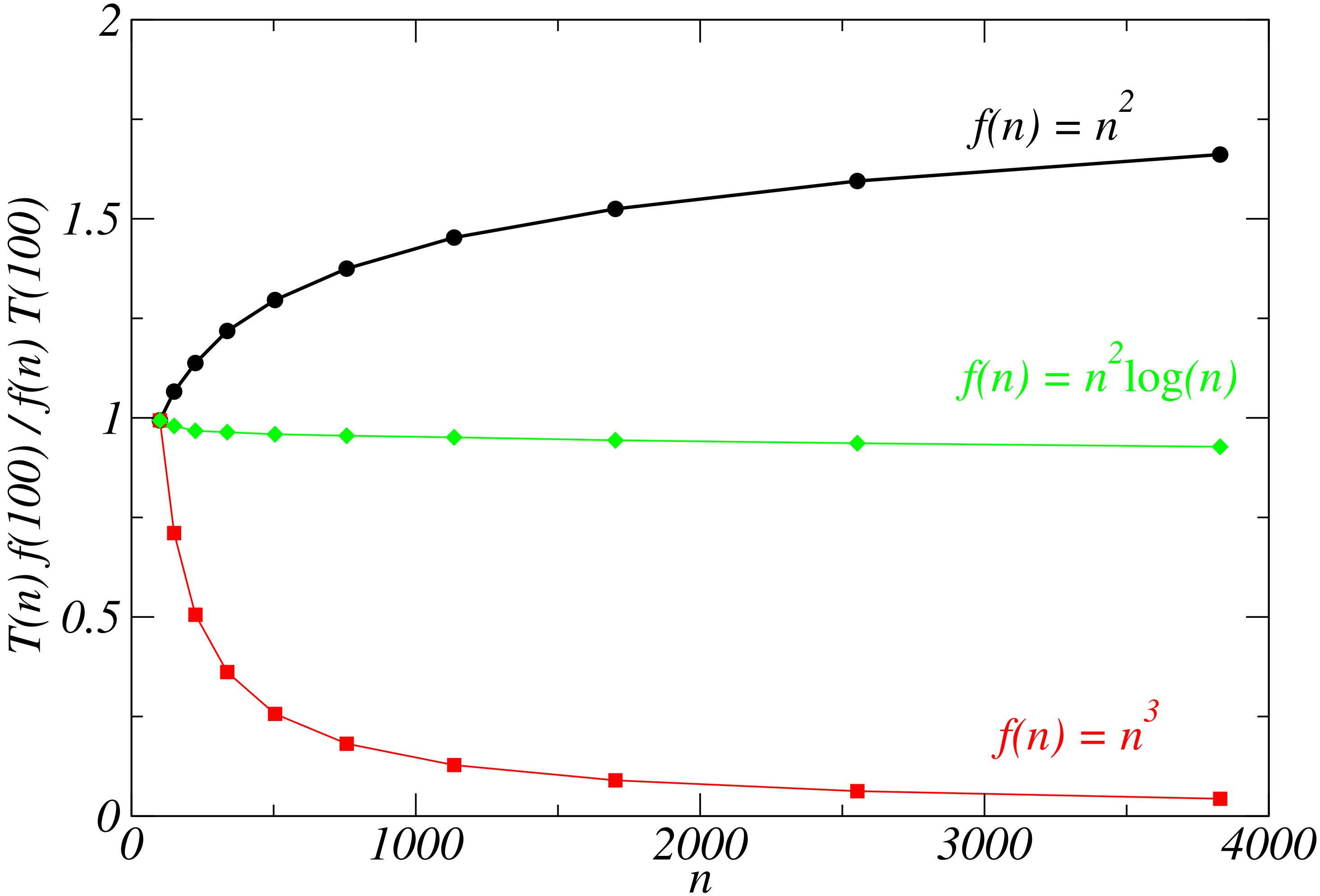
Testing Hypothesis

- We can test our hypothesis by scaling the run time by the complexity



Conclusions

- You should understand the difference between Θ, O,o, Ω and ω
- You need to be able to compute time complexity by loop counting
- To compute time complexity for recursive functions you need to be able to obtain recurrence equations
- You should be able to solve simple recurrence equations and sum up simple series
- You should be able to prove more complicated results using proof by induction

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- Thank you for attending the course