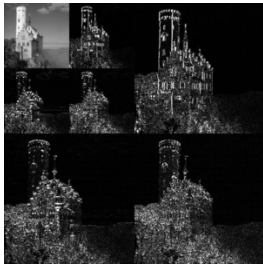


Lesson 24: Use Smart Encoding!



File compression, Huffman codes, wavelets

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File Compression

- File compression comes in two varieties
 - Exact compression (e.g. zip used on text files)
 - Lossy compression (e.g. jpeg used on pictures—jpeg can also be loss-less or exact)
- Good exact compression (also known as entropy encodings) can give a compression ratio around 25%
- Lossy compression can give a compression ratio from 10-1%
- Important for saving space, but lossy compression can also be used for noise reduction
- Even used for plagiarism detection!

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Huffman Coding

- Given a sequence of symbols and their probabilities of occurrence, Huffman code provides a way of coding up the information
- It is an example of a **greedy** strategy that happens to be optimal
- Like many greedy strategies it is easily implemented using a priority queue
- It is used in the UNIX compress program and in the exact part of JPEG
- The idea is to assign short codes to commonly used symbols

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Encoding

- We want to assign a code to each symbol
- To save space we want to assign short codes to frequently used symbols
- There is a problem: decoding
- If we assigned a code

$e \rightarrow 0$ $a \rightarrow 1$ $r \rightarrow 01$
 $o \rightarrow 10$ $i \rightarrow 11$ $t \rightarrow 000$

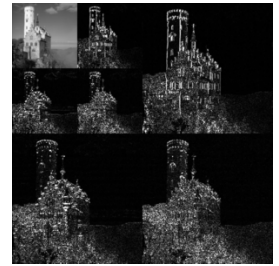
etc. we could compress a document very efficiently but we could never decode it uniquely

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- Huffman codes
- Wavelets



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Entropy Encoding

- Exact encodings use the principle of using short words for frequently occurring sequences (symbols) and longer words for sequences that occur less often
- Claude Shannon showed that for an alphabet of n symbols where the probability of symbol i occurring is p_i no code exists which can transmit information in less than

$$-\sum_{i=1}^n p_i \log_2(p_i) \text{ bits}$$

asymptotically this compression can be achieved

- Different encoding schemes differ in the way they identify symbols of the alphabet—this is rather specialist and we won't go into this

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Symbol Frequency

- We start from an alphabet describing the original document
 - This might be the set of characters
 - For an image it might be the set of pixel values
 - It might be pairs of pixel values
- We compute the number of occurrences of each symbol

Symbol	# Occurrences
a	145
b	67
⋮	⋮

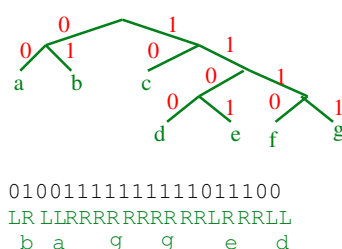
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Huffman Trees

- Once again tree come to the rescue!
- We assign each symbol to a leaf of a binary tree
- We use the position of the branch as an encoding of the symbol



$a \rightarrow 00$
 $b \rightarrow 01$
 $c \rightarrow 10$
 $d \rightarrow 1100$
 $e \rightarrow 1101$
 $f \rightarrow 1110$
 $g \rightarrow 1111$

01001111111111011100
 LR LRRRRRRRRRLRRLL
 b a g g e d

- The decoding is unique

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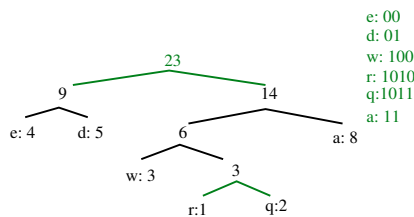
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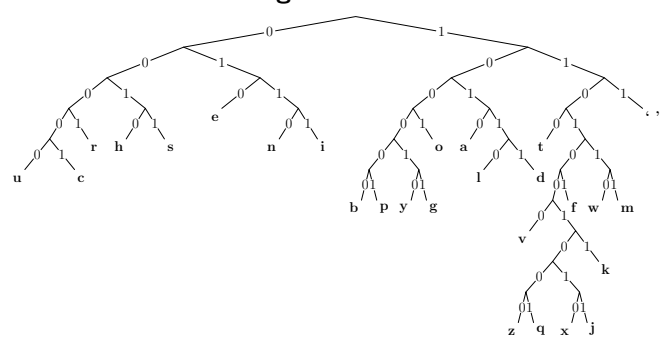
Generating the Huffman Tree

- We are left with the problem of constructing the Huffman tree such that frequently occurring letters have short codes
- A greedy approach is to iteratively build a tree by
 - combine the two most infrequent symbols into a subtree
 - Add their scores and treat them as a single symbol

aaaeedwqqadewwaaddreaad → 1111110000011001011101111...



English Letters



the quick brown fox jumps over the lazy dog
344 bits
11000010010111110100100100000011100001110100111111000000001100111011001101111101
0110011101001010111110100101100000110111100001001111100111010000100001111110000
100101111011010101101001000100010111101111001100011
211 bits

Implementing Huffman Encoding

- To implement Huffman encoding you need
 - A class to build Huffman trees by combining subtrees
 - A way to find the least frequently used symbols or symbol combinations
- Priority queues are ideal for this application
- They allow you to find the least frequently used symbols (`removeMin`) and to add new symbols (`add`)
- To decode you follow the Huffman tree

Greedy Strategy

- Huffman encoding is an example of a **Greedy solution pattern**
- That is we look for local optimality (i.e. we combine the two least frequently used symbols)
- In this case, we obtain global optimality (i.e. the Huffman tree obtained gives an optimal Huffman code)
- There are a number of important problems where greedy algorithms lead to global optimality (we saw this earlier)
- For these algorithms priority queues commonly are used for implementing the algorithm

Advanced Techniques

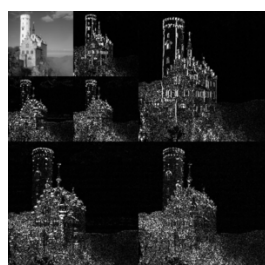
- Huffman code is optimal given the frequency of symbols
- However, there is considerable art in identifying which 'symbols' to use
- Advanced compression algorithms (LZ78, LZW Lempel-Ziv-Welch) build dictionaries of sequences seen in the files—they tend to be rather specialised
- Some recent algorithms (e.g. Burrows-Wheeler) transform the file in such a way that similar symbols are mapped to adjacent sites—depends on the generating mechanism of the language

File Compression and Plagiarism Detection

- One way of spotting plagiarism is to compare the compressed lengths of two files and the length of the compressed file when the two files are concatenated first
- If the files have the same structure the concatenated version can often be significantly reduced
- Also used in identifying closeness of species in constructing phylogenetic trees

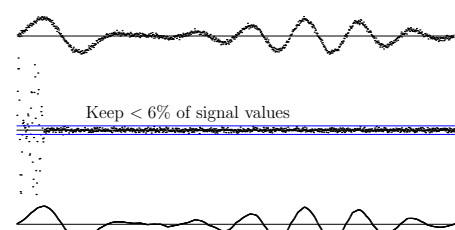
Outline

- Huffman codes
- Wavelets



Signals and Energies

- We consider compressing a signal $x = (x_0, x_1, \dots, x_{n-1})$



Wavelets

- We can define the “energy” as the squared deviations

$$E = \sum_{i=1}^n x_i^2$$

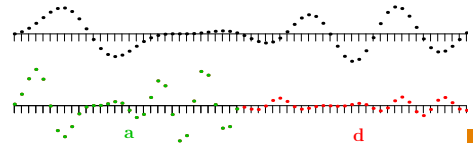
- Our strategy in lossy compression is to transmit as much “energy” in as few bits as possible
- There are different strategies to achieve good compress

- With wavelets we try to re-represent the signal so as to squeeze as much energy as possible into fewer bits

- The easiest way to do this is with Haar wavelets

$$a_i = \frac{x_{2i} + x_{2i+1}}{\sqrt{2}} \quad d_i = \frac{x_{2i} - x_{2i+1}}{\sqrt{2}}$$

- Define new signal $(a_0, a_1, a_2, \dots, a_{n/2-1}, d_0, d_1, \dots, d_{n/2-1})$



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Carrier and Difference Signals

- The terms $a_i = (x_{2i} + x_{2i+1})/\sqrt{2}$ takes the “average” of the signal, but compresses it in half the space
- The terms $d_i = (x_{2i} - x_{2i+1})/\sqrt{2}$ takes the difference and is small if the signal does not change much
- The energy is conserved since

$$\begin{aligned} a_i^2 + d_i^2 &= \left(\frac{x_{2i} + x_{2i+1}}{\sqrt{2}} \right)^2 + \left(\frac{x_{2i} - x_{2i+1}}{\sqrt{2}} \right)^2 \\ &= \frac{x_{2i}^2 + 2x_{2i}x_{2i+1} + x_{2i+1}^2 + x_{2i}^2 - 2x_{2i}x_{2i+1} + x_{2i+1}^2}{2} = x_{2i}^2 + x_{2i+1}^2 \end{aligned}$$

- Attempt to push all the energy into the carrier signal, a_i

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Inverse Transform

- The wavelet transform can be easily reversed

$$\begin{aligned} a_i &= \frac{x_{2i} + x_{2i+1}}{\sqrt{2}} & d_i &= \frac{x_{2i} - x_{2i+1}}{\sqrt{2}} \\ x_{2i} &= \frac{a_i + d_i}{\sqrt{2}} & x_{2i+1} &= \frac{a_i - d_i}{\sqrt{2}} \end{aligned}$$

- Can compute transform using vectors (wavelets)

$$a_i = \mathbf{V}_i \cdot \mathbf{x} \quad d_i = \mathbf{W}_i \cdot \mathbf{x}$$

- These vectors are orthogonal to each other ($\mathbf{V}_i \cdot \mathbf{V}_j = 0$, $\mathbf{V}_i \cdot \mathbf{W}_j = 0$, etc.)

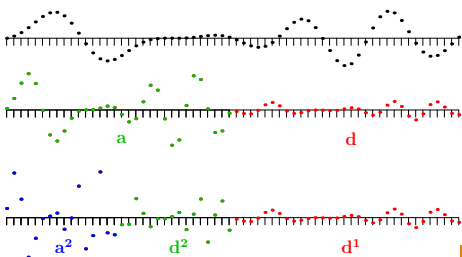
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And So On. . .

- We can repeat the process again to concentrate the energy further
- We apply the Haar transform just to the carry part $\mathbf{a} = (a_0, a_1, \dots, a_{n/2-1})$



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Properties of Daub4

- Similar to the Haar transform

$$c_0 + c_1 + c_2 + c_3 = \sqrt{2}, \quad c_3 - c_2 + c_1 - c_0 = 0$$

so the carrier signal (a_i) is approximately $\sqrt{2}$ times the original and the difference part (d_i) is equal to 0 for a flat signal, \mathbf{x}

- However in addition

$$0c_3 - 1c_2 + 2c_1 - 3c_0 = 0$$

so the difference part (d_i) is equal to 0 for any linear signal, \mathbf{x}

Daubechies Wavelets

- Ingrid Daubechies suggested a host of wavelets which do better than Haar for smooth signals
- The simplest is Daub4 defined by

$$\begin{aligned} a_i &= c_0x_{2i} + c_1x_{2i+1} + c_2x_{2i+2} + c_3x_{2i+3} \\ d_i &= c_3x_{2i} - c_2x_{2i+1} + c_1x_{2i+2} - c_0x_{2i+3} \end{aligned}$$

$$c_0 = \frac{1 + \sqrt{3}}{4\sqrt{2}} \quad c_1 = \frac{3 + \sqrt{3}}{4\sqrt{2}} \quad c_2 = \frac{3 - \sqrt{3}}{4\sqrt{2}} \quad c_3 = \frac{1 - \sqrt{3}}{4\sqrt{2}}$$

- Again conserves energy

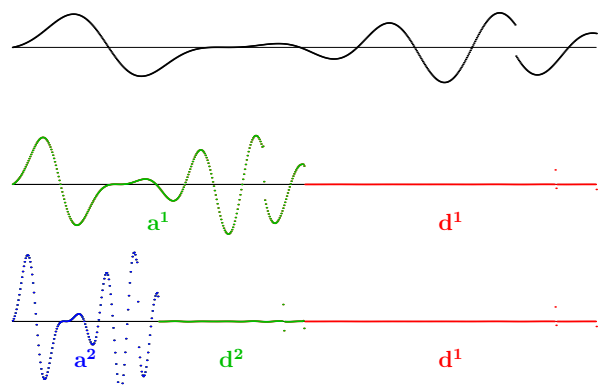
$$\sum_{i=1}^{n/2} a_i^2 + b_i^2 = \sum_{i=1}^n x_i^2$$

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Daub4



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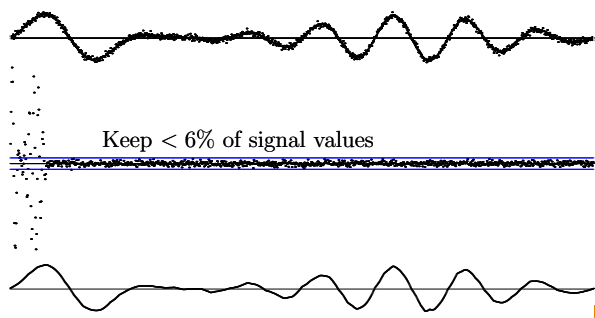
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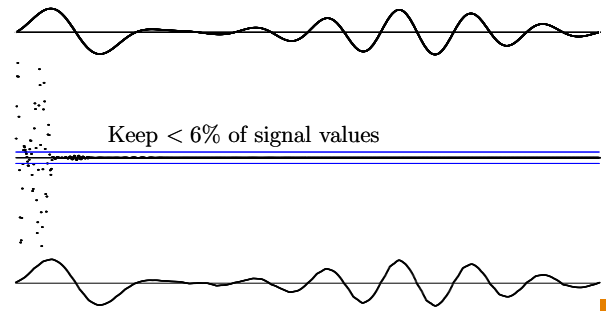
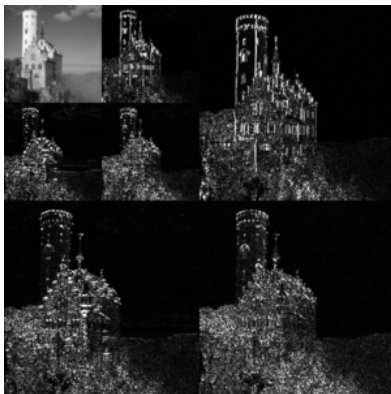
- To compress the signal we can set all components of the transformed signal whose magnitude lies below a threshold to 0
- We transmit the non-zero magnitude together with a binary mask showing the position of the non-zero magnitude
- We can reduce the accuracy (number of decimal places) of the non-zero magnitudes (quantisation)—this is repaired on inverting transform
- We can compress the binary mask using Huffman encoding or other scheme

Noise Reduction

- Can also be used in noise reduction



2-D Wavelets



Other Wavelets

- Can use high-order wavelets which captures more energy in the carrier signal, e.g. Daub10 or Daub20
- Many other wavelets capture other properties (e.g. Coiflets capture properties of a continuous signal sampled at discrete points)
- Efficiency of wavelets depend on how well the capture underlying properties of signals
- Can also construct 2-d wavelets for image compression (jpeg-2000)

Summary

- File compression is an important task in its own right
- Files may either be compressed losslessly or lossily
- Lossy compression is typically much more efficient (e.g. an order of magnitude smaller)
- Huffman encoding often lies at the lowest level in many compression algorithms
- Wavelets illustrate a strategy of changing the representation to concentrate the energy of a signal