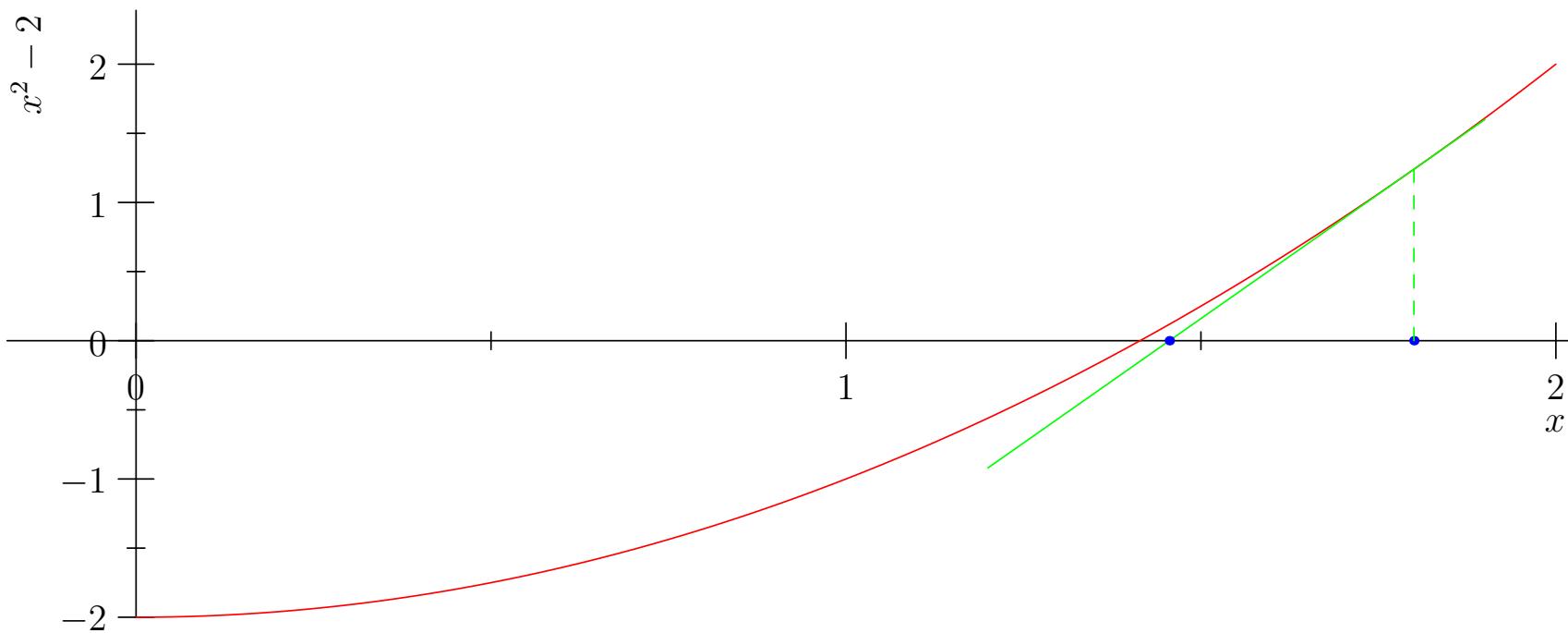


# Algorithms and Analysis

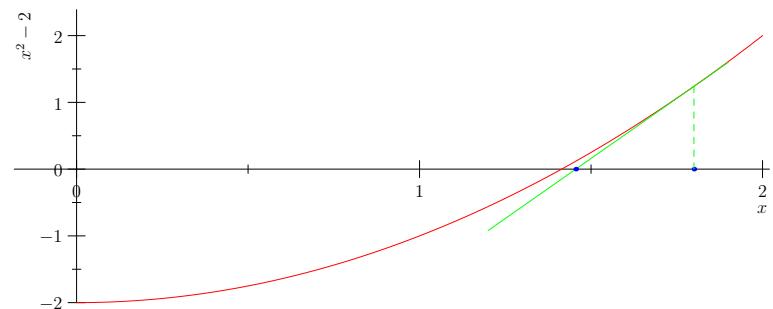
## Lesson 30: *Understand Numerics*



*Representing reals, rounding error, convergence, stability, conditioning*

# Outline

1. Numerical Approximations
2. Iterating to a Solution
3. Linear Algebra



# Numerical Analysis

- Numerical algorithms are usually taught separately from the “discrete algorithms” we have predominantly looked at
- The main difference stems from the fact that numerical algorithms model continuous variables
- Computers can only approximate continuous variables
- Numerical algorithms have to take into account this approximation

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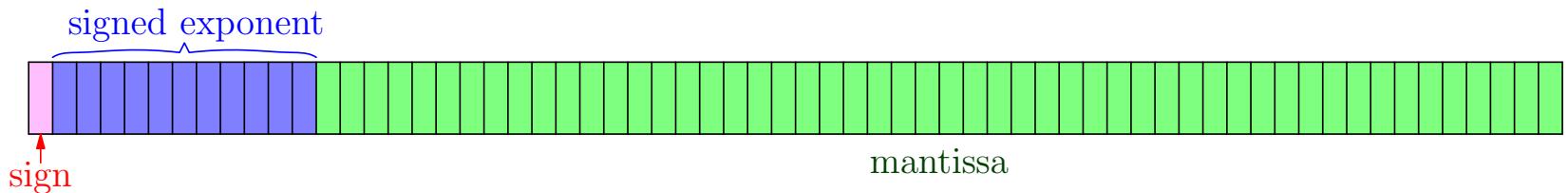
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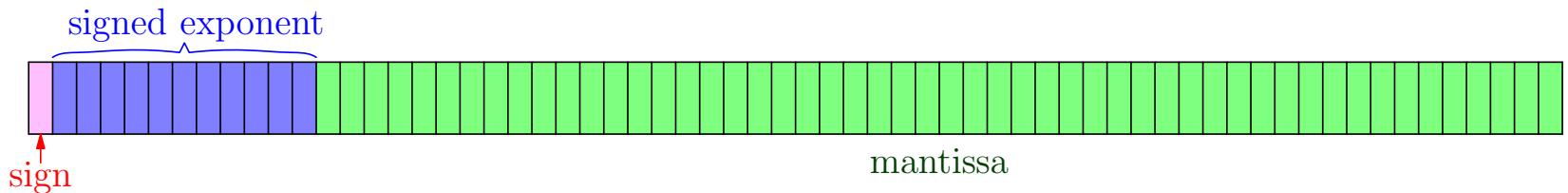
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- $x = m \times 2^{e-t}$
- $t$  is precision so that if  $e = t$ , then  $0.5 \leq x < 1$
- For IEEE double  $t = 1023$ ,  $\text{expon}_{\min} = -1021$ ,  $\text{expon}_{\max} = 1024$
- Typical rounding error is  $u = 1 \times 10^{-16}$

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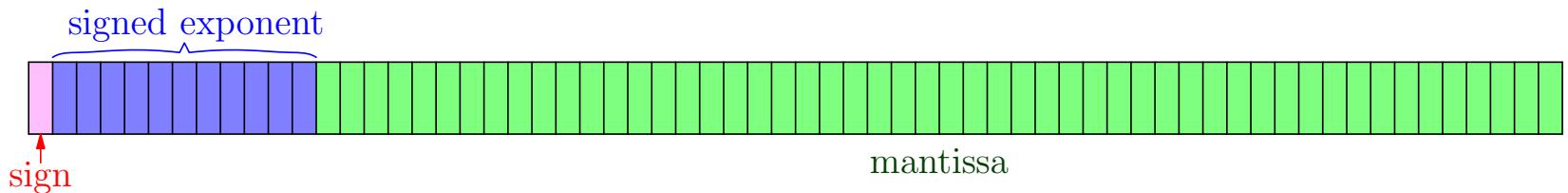
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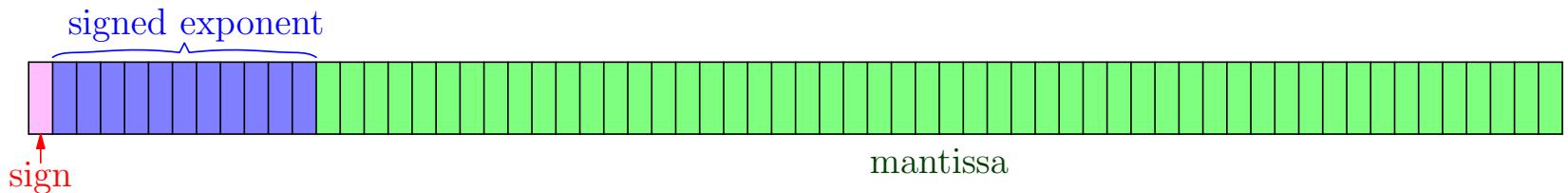
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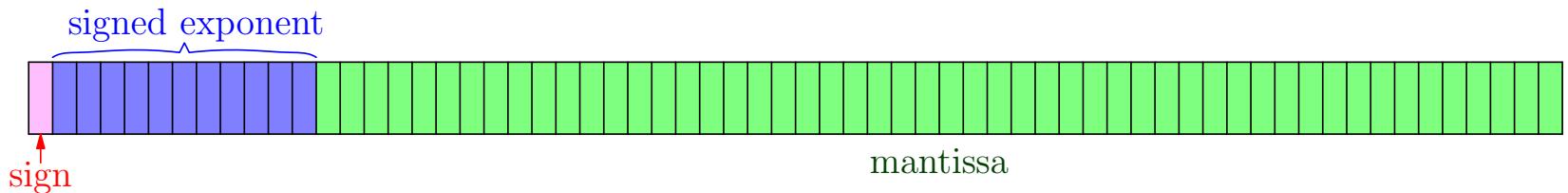
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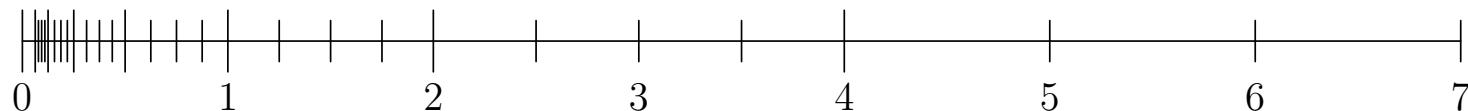
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# The Number Line

- We approximate the continuous number line by a set of discrete values
- Imagine using a mantissa of 3 bits and an exponent of 2 bits (and a sign)
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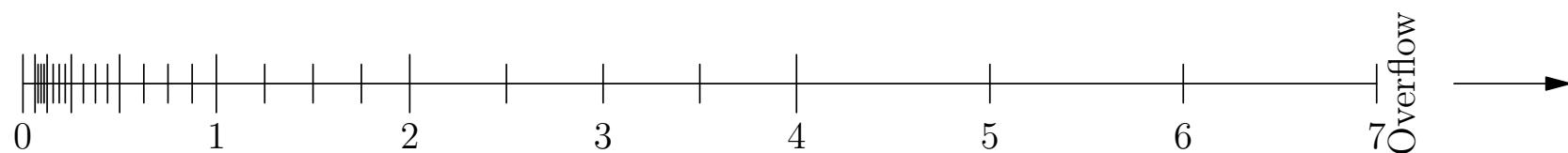
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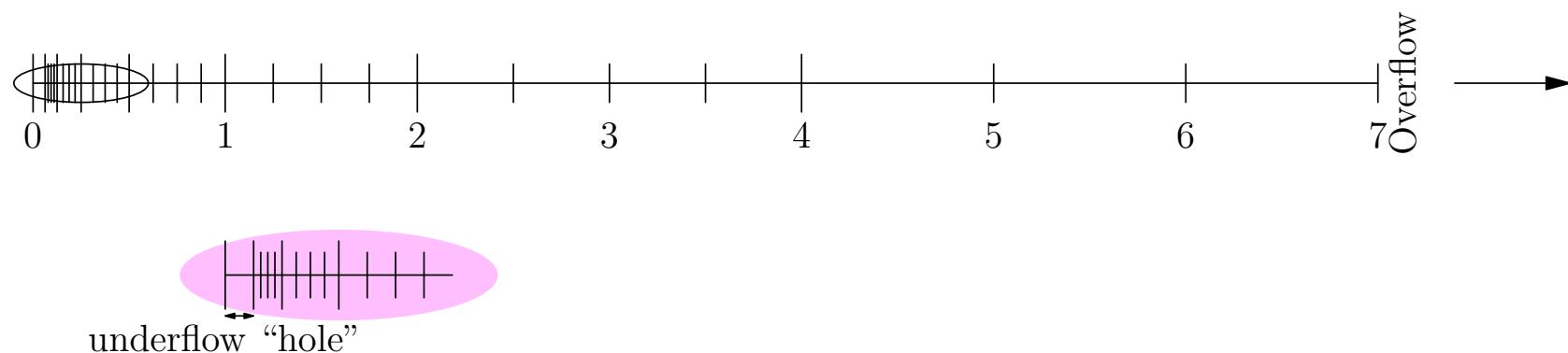
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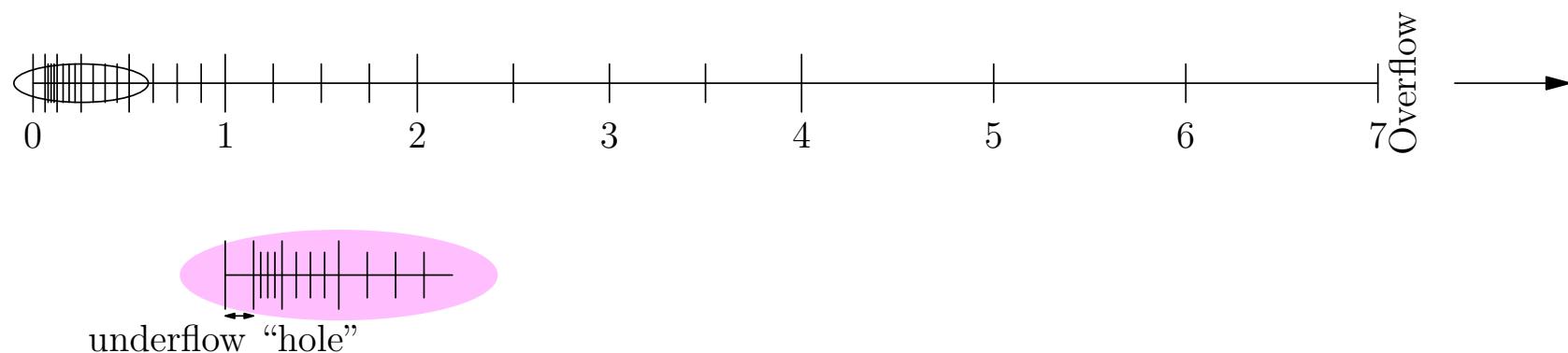
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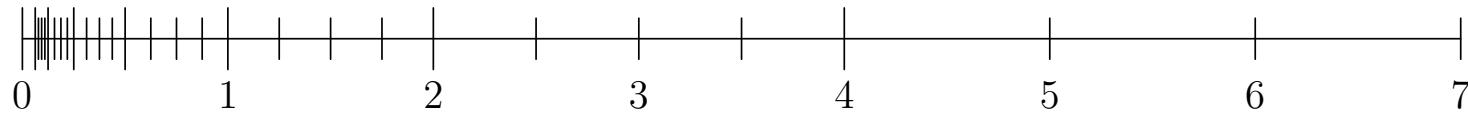
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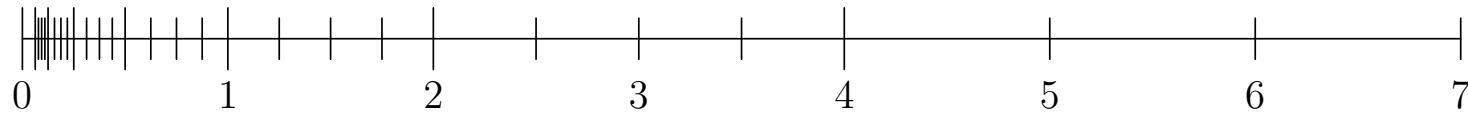


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$$\text{Relative error} = \left| \frac{\text{Approx} - \text{Exact}}{\text{Exact}} \right|$$

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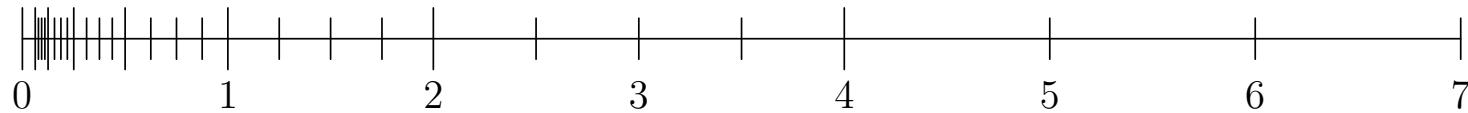


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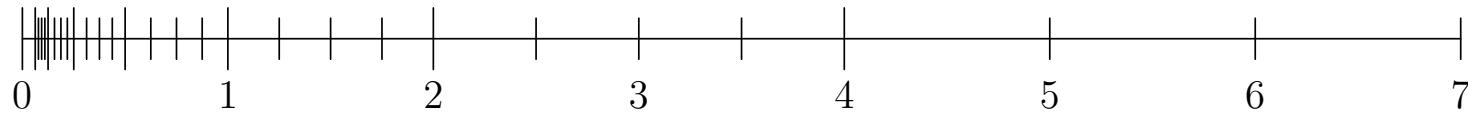


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# Losing Precision

- There seems to be plenty of precision, so what's the problem?
- One issue is that it's easy to lose precision
- Consider estimating derivatives by finite differencing

$$f'(x) \approx \frac{f(x + \epsilon) - f(x - \epsilon)}{2\epsilon}$$

- The problem is  $f(x + \epsilon)$  and  $f(x - \epsilon)$  are very close so in taking their difference we lose precision
- $f(x) = \sin(x)$ ,  $f'(x) = \cos(x)$  at  $x = 1.0$

$\epsilon$	$10^{-6}$	$10^{-8}$	$10^{-10}$	$10^{-12}$	$10^{-14}$
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# Solving Quadratic Equations

- A classic example where you can lose precision is in solving a quadratic equation  $a x^2 + b x + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a}$$

- If  $b^2 \gg |4 a x|$  then for one solution we end up subtracting numbers very close
- We rather use this equation to compute one solution

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- This is sometimes true, but we often use “for loops” where we might be losing accuracy all the time

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x = 1.6;  
for (i=0; i<50; i++)  
    x = sqrt(x);  
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- Gave the answer 1.2840 (if I run the for loop 60 times it gives the answer 1 for almost any input)

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- Nothing is exact so to check that  $x = y$  we use

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Math.abs(x-y) < 1.0e-10 // a small constant
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**long double**
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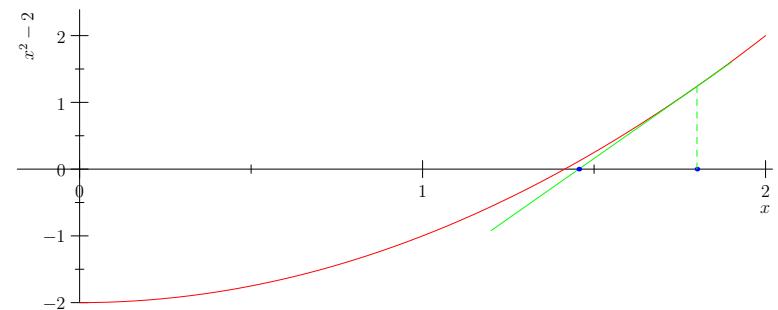
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# Iterative Algorithms

- We solve many numerical tasks by obtaining successively better solutions

$$x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, \dots$$

- We often stop when the change in solution is below some threshold, e.g.  $|x^{(i+1)} - x^{(i)}| \leq \epsilon \approx u$
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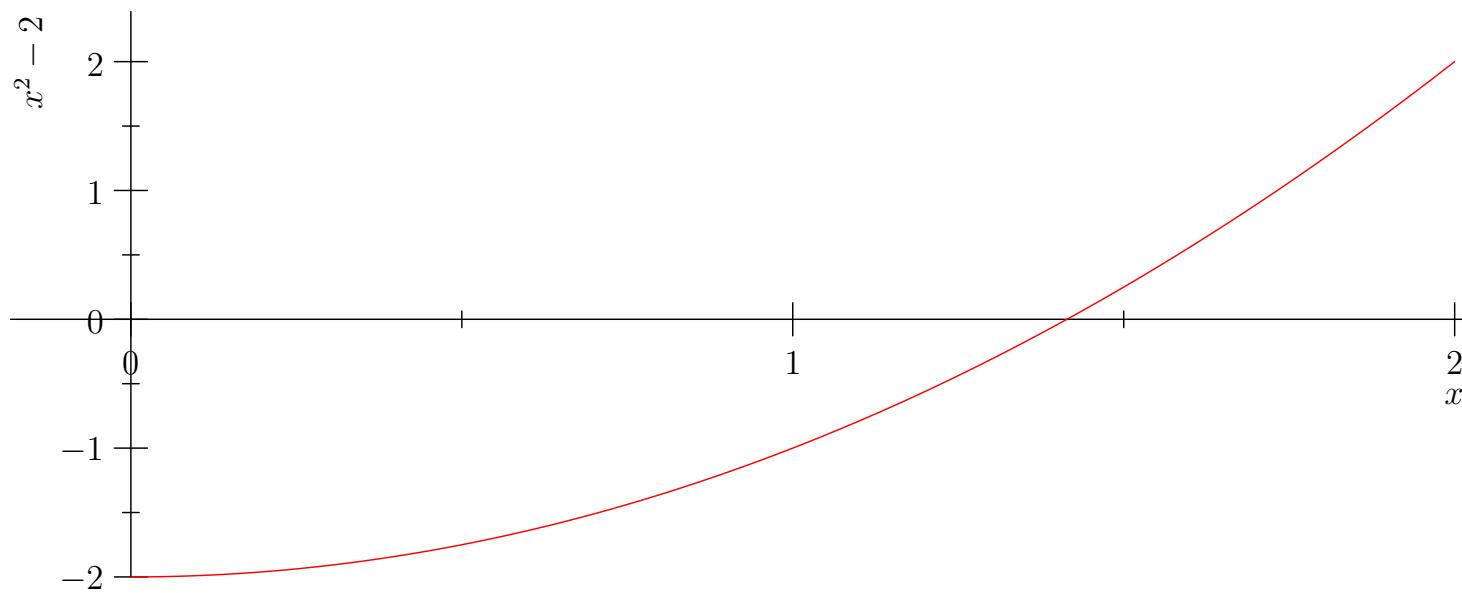
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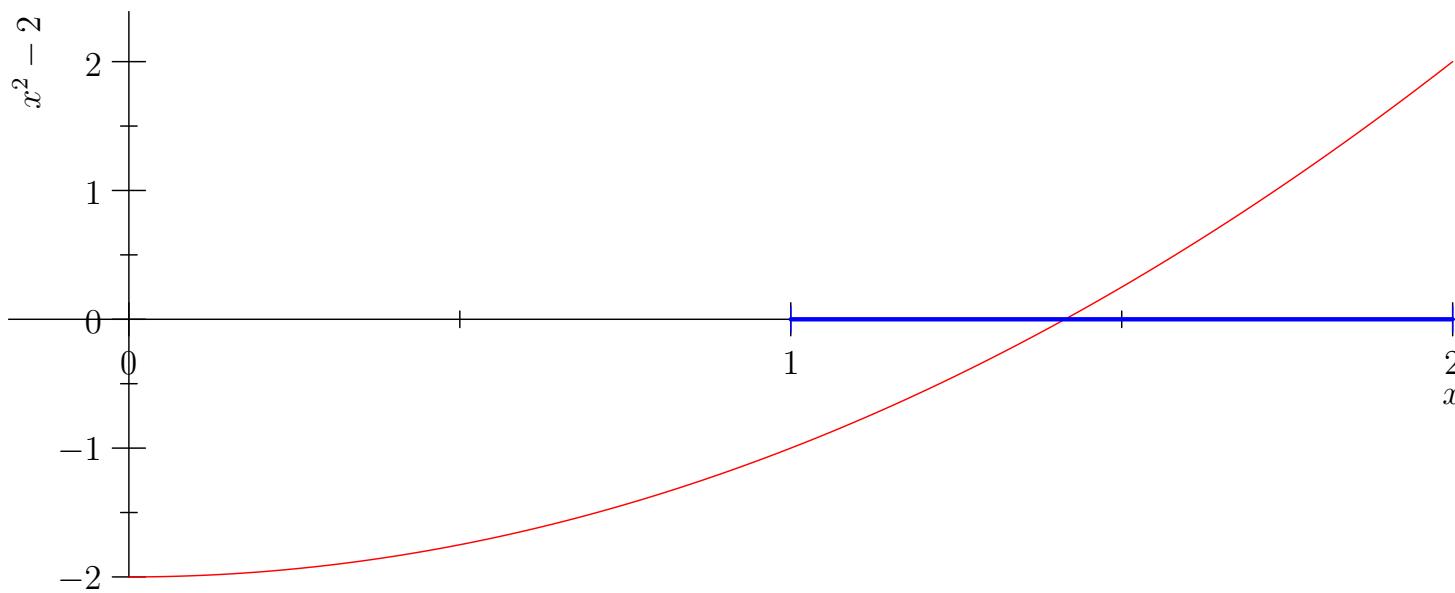


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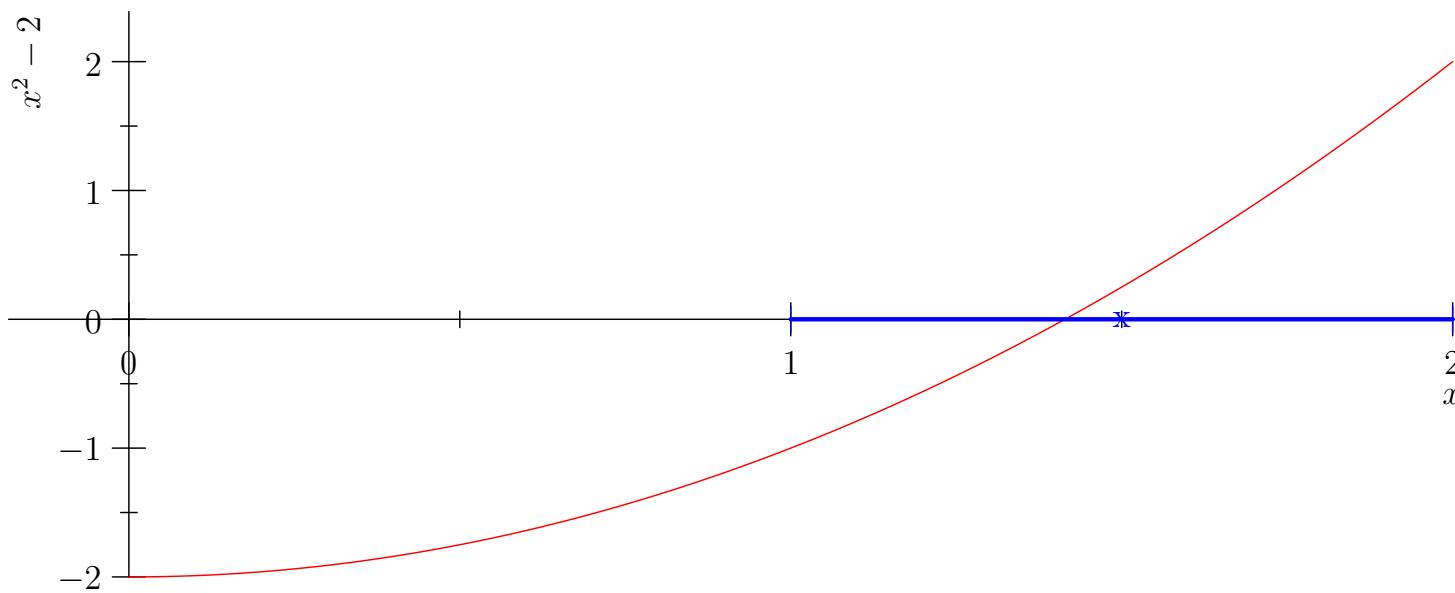


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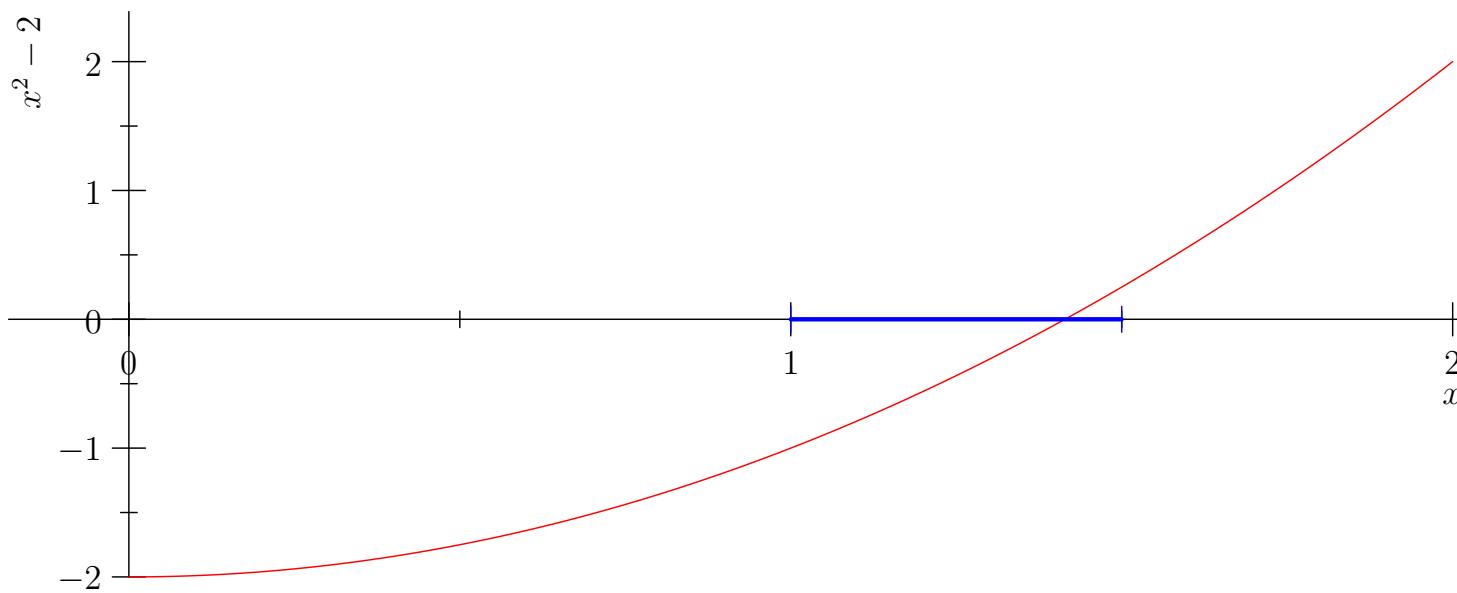


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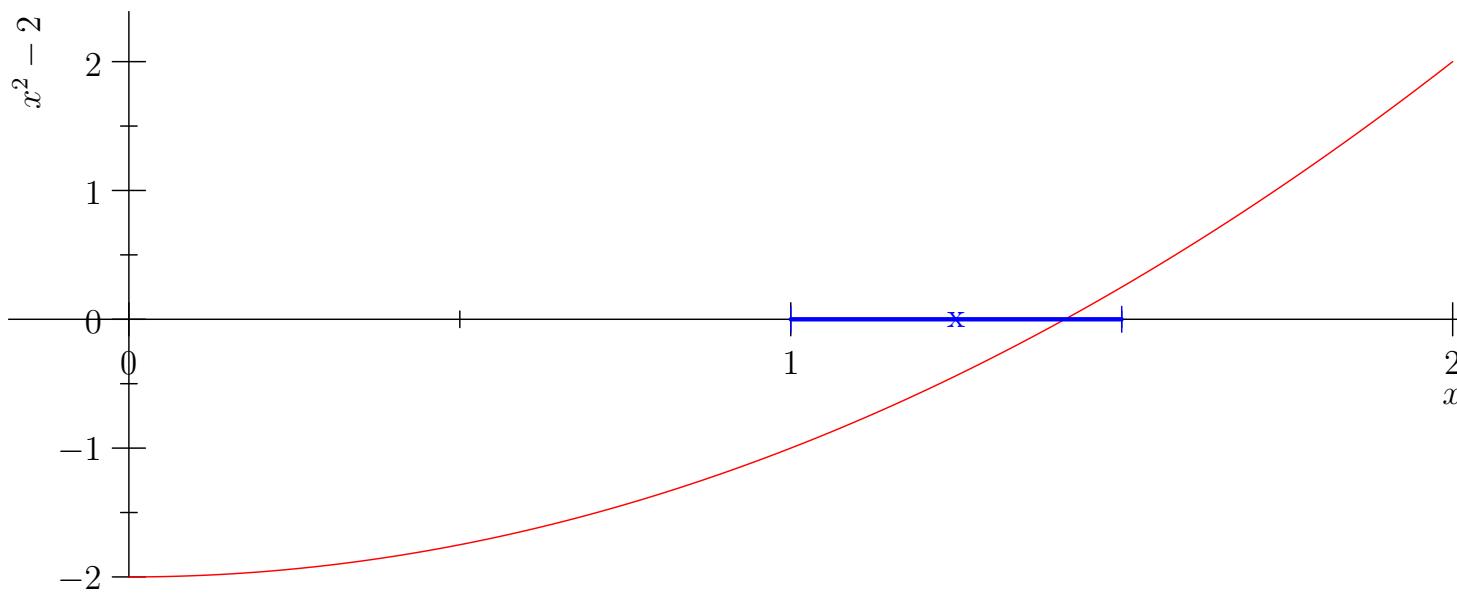


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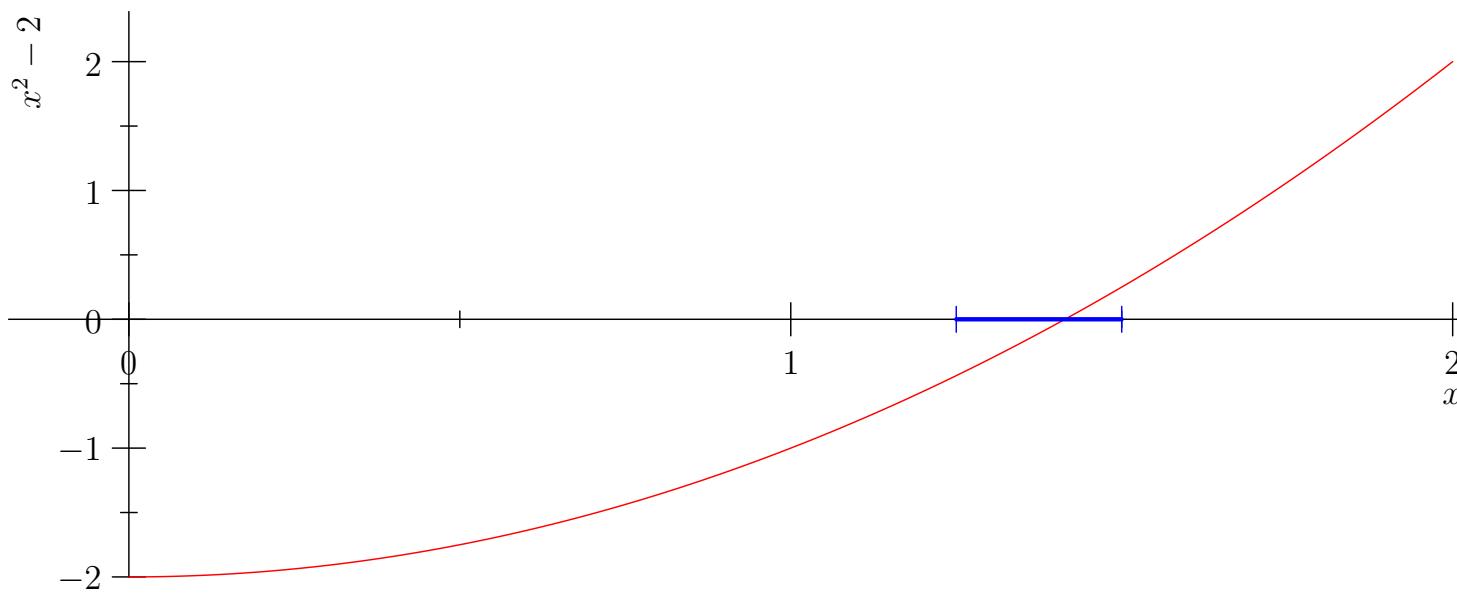


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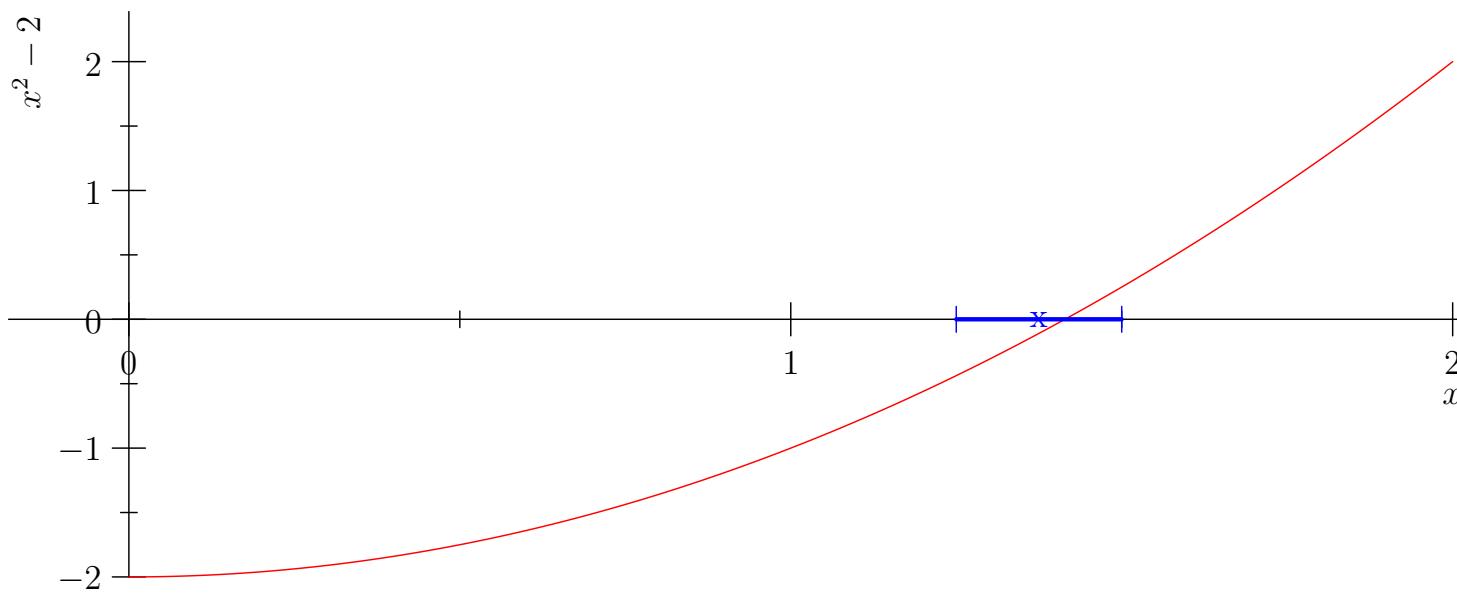


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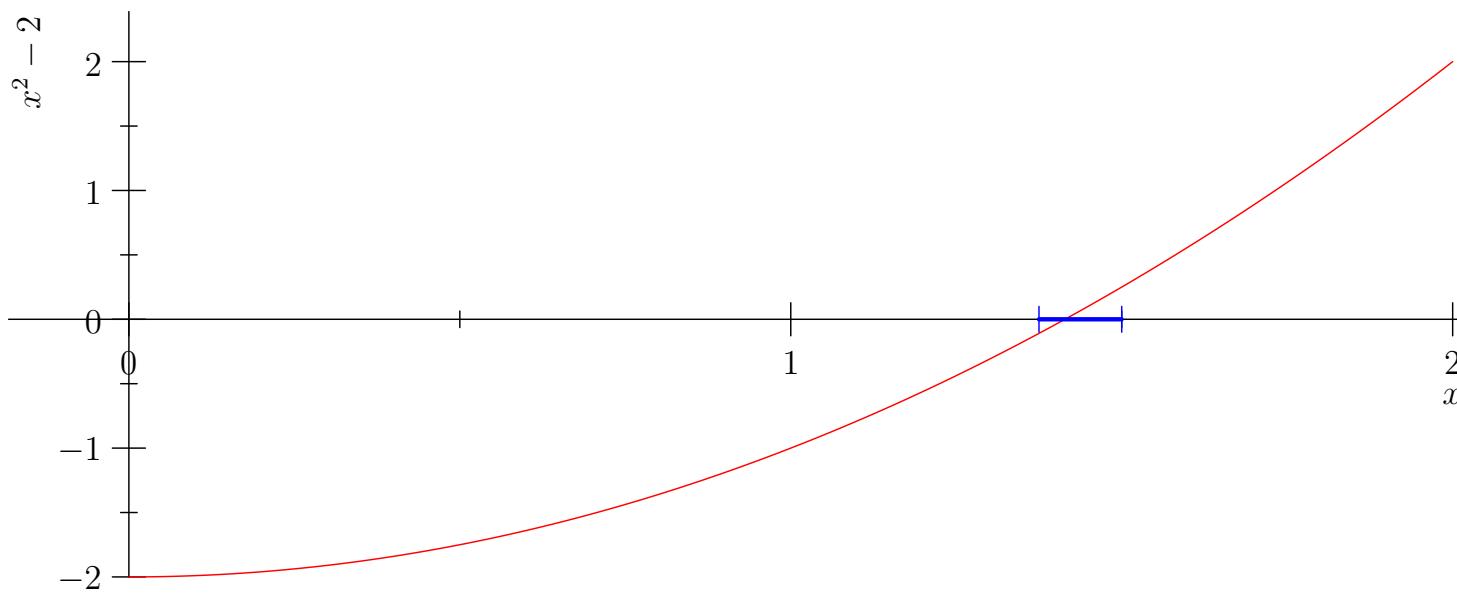


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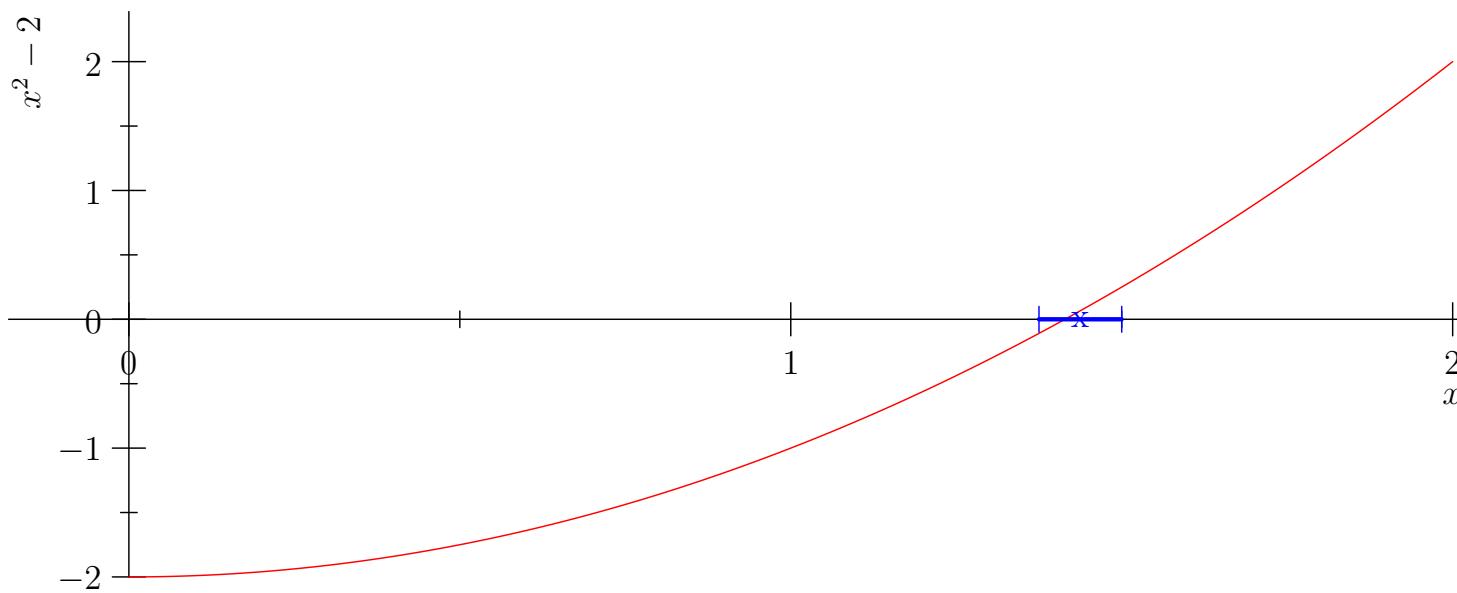


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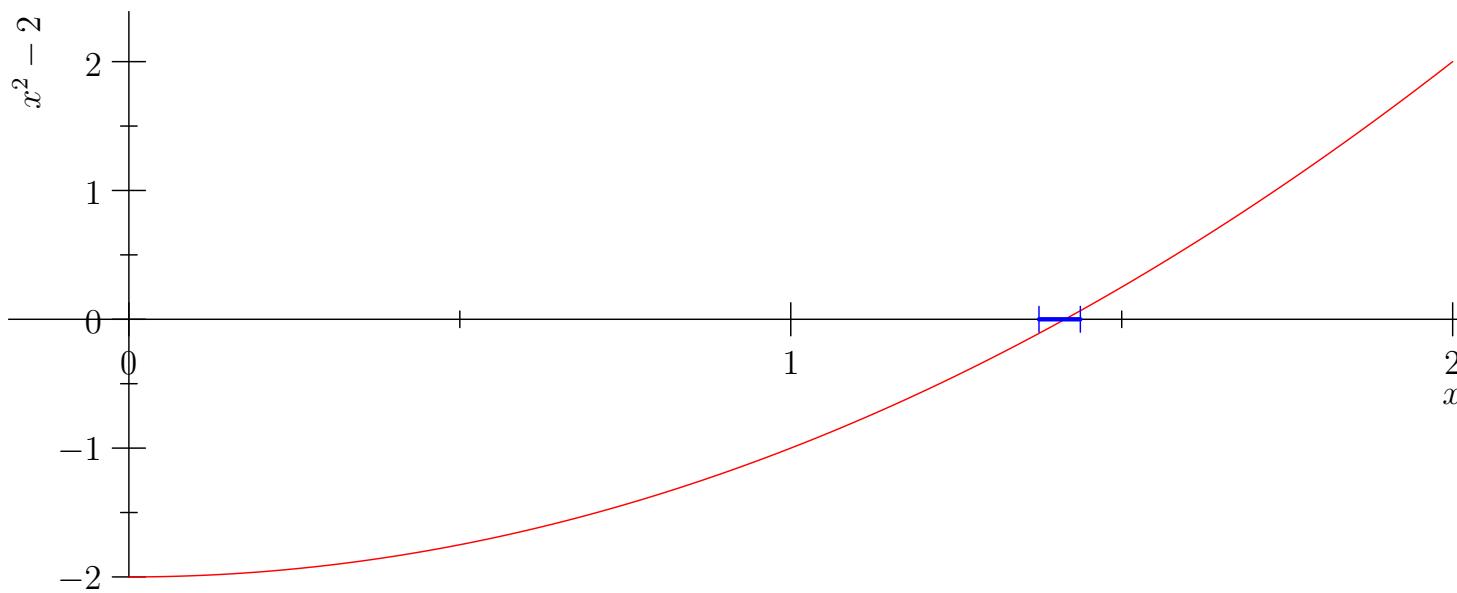


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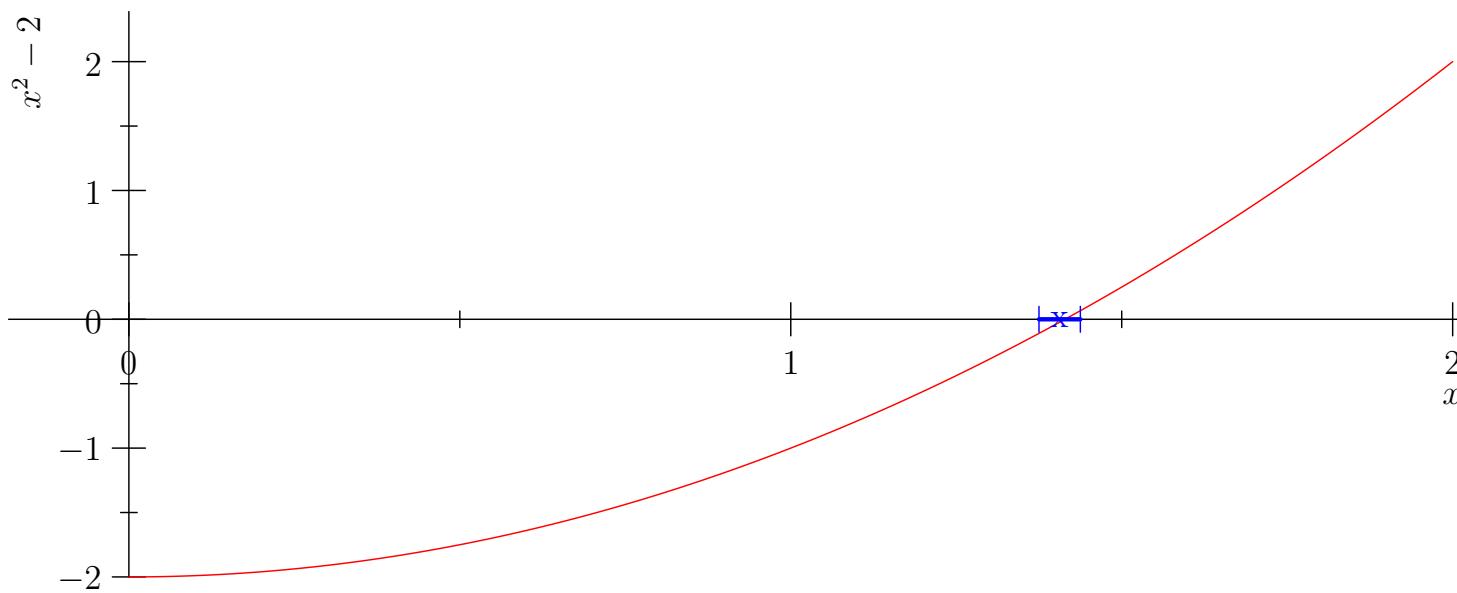


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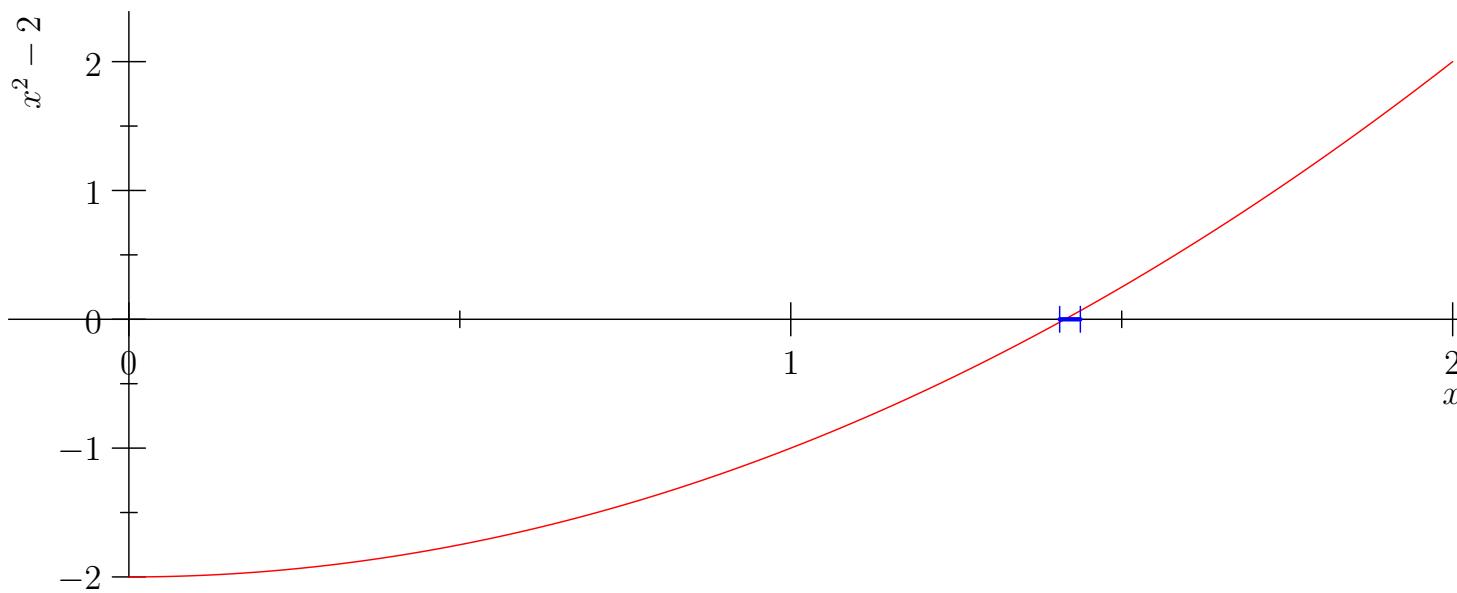


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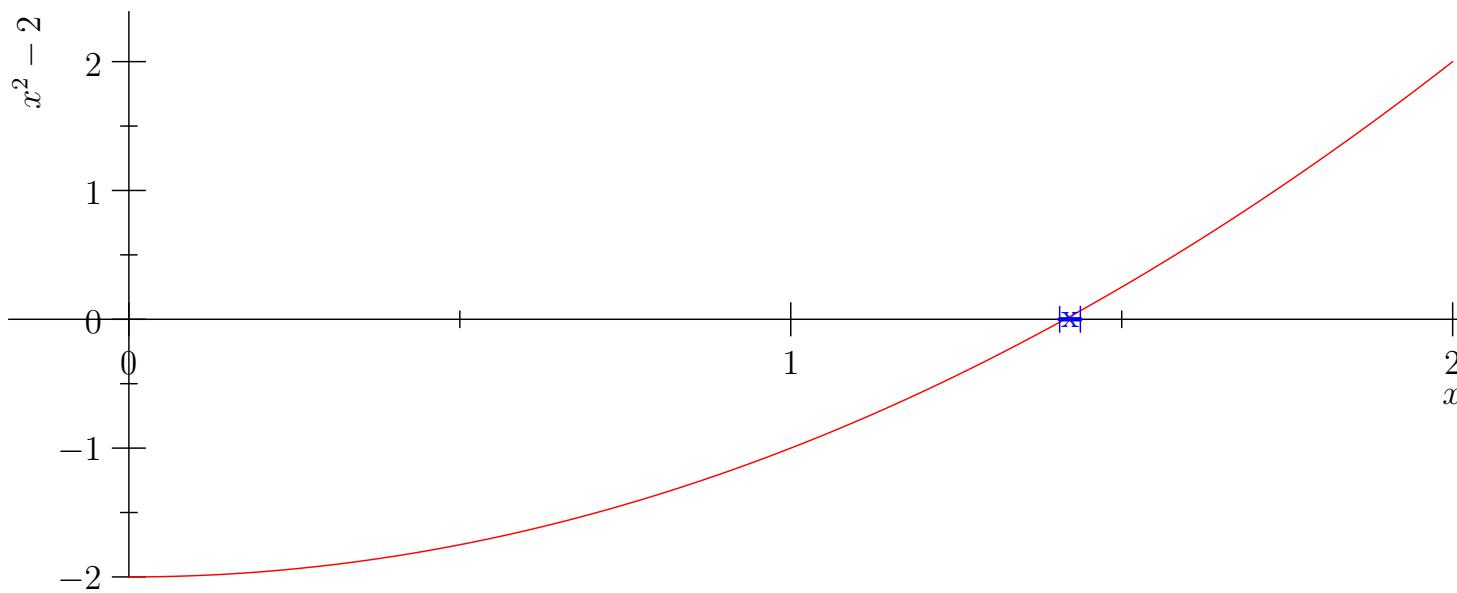


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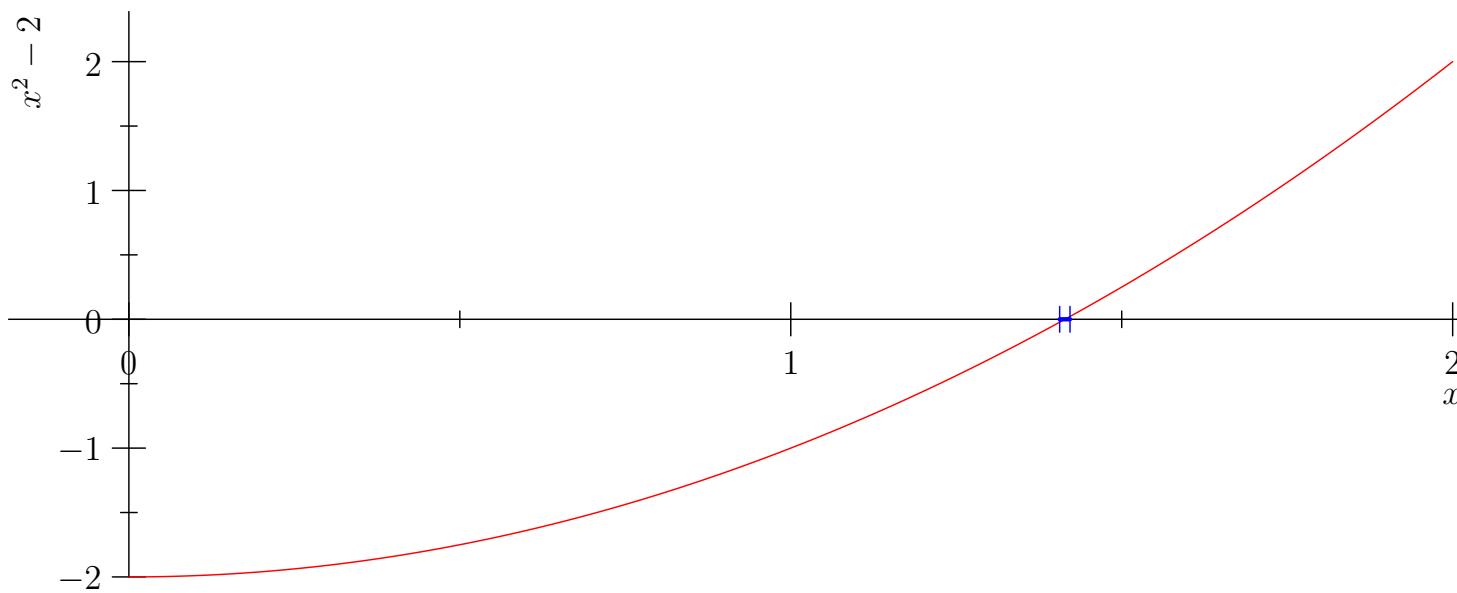


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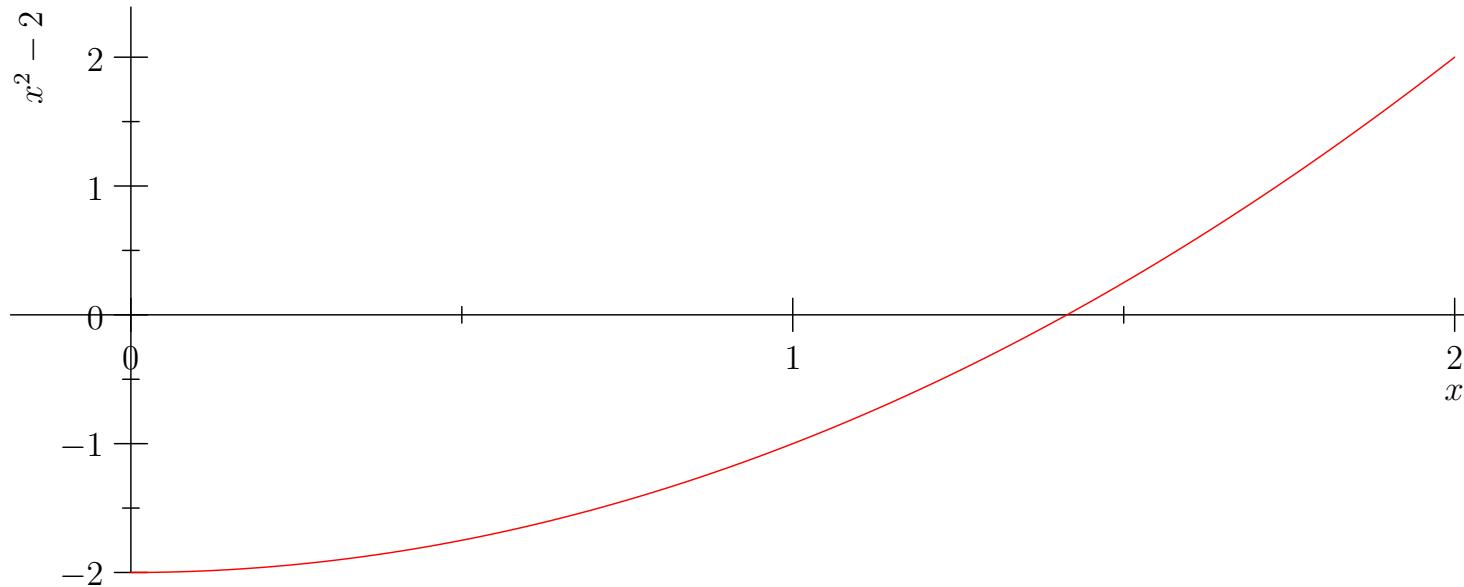


# Newton Raphson

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$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$$

- For  $f(x) = x^2 - 2$  so  $x^{(i+1)} = ((x^{(i)})^2 - 1)/(2 x^{(i)})$

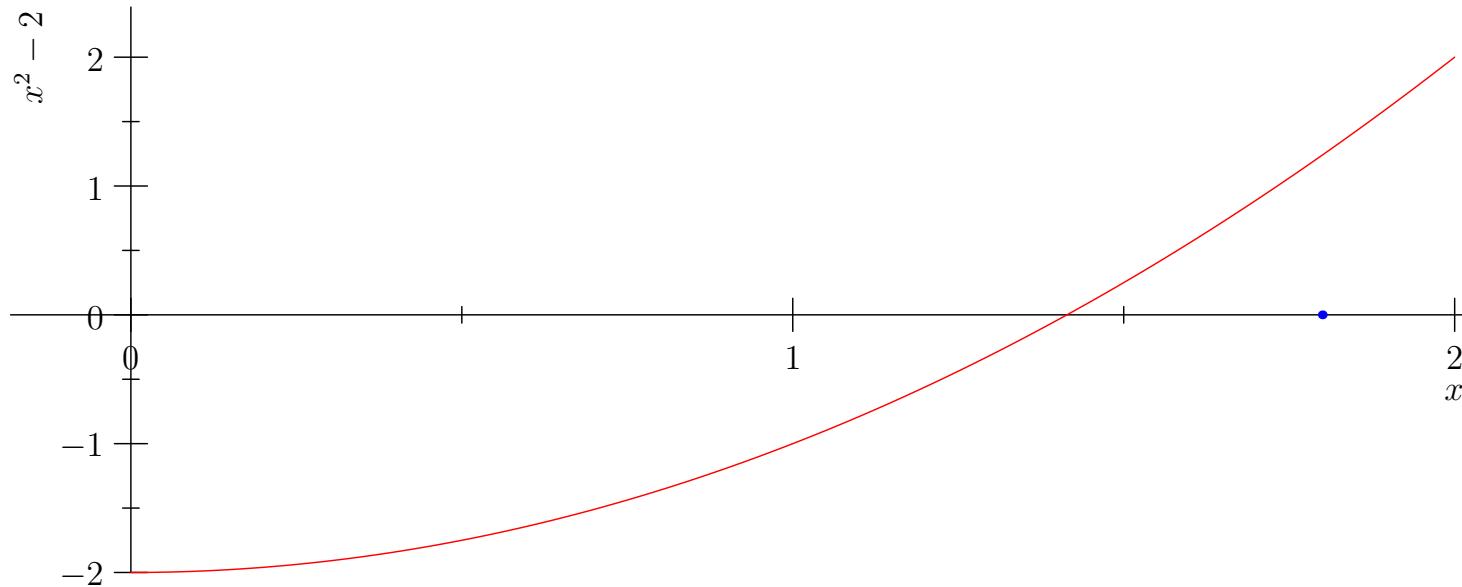


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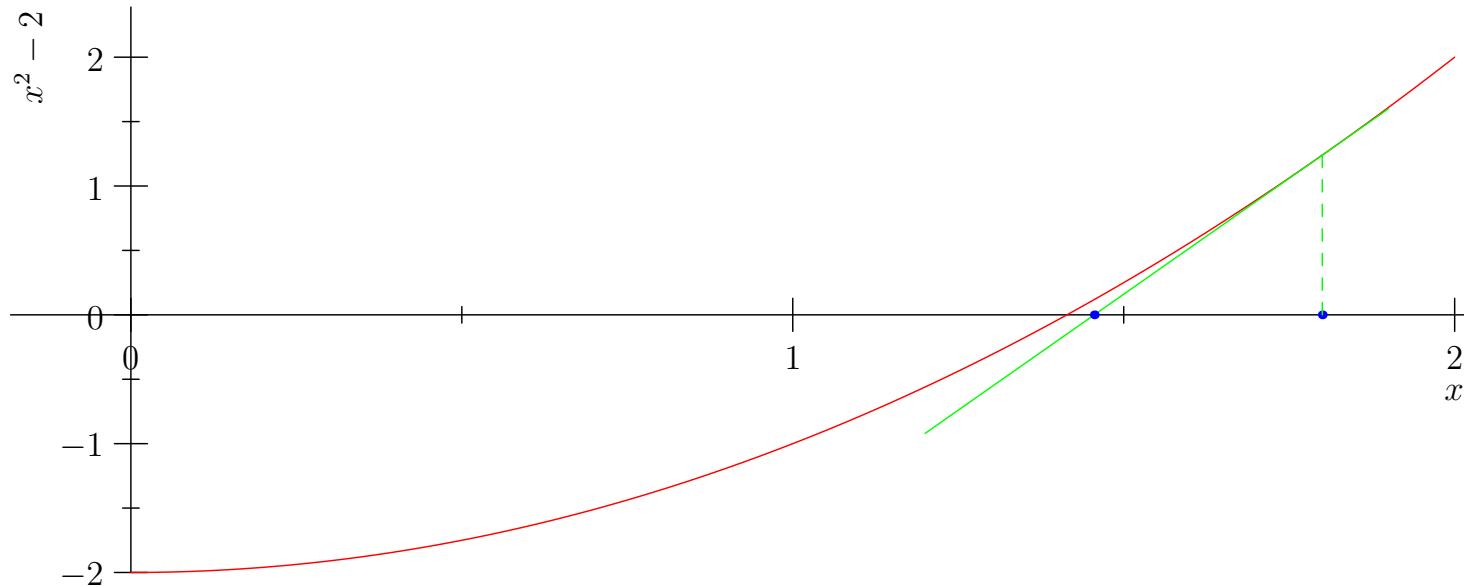


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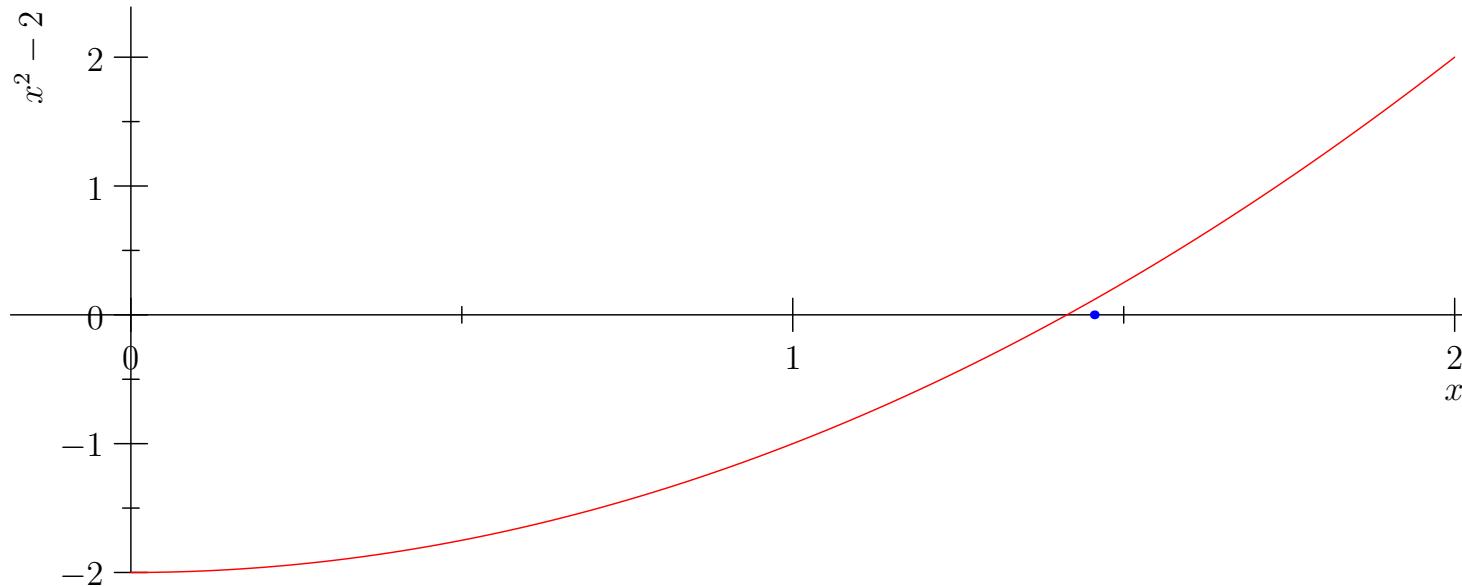


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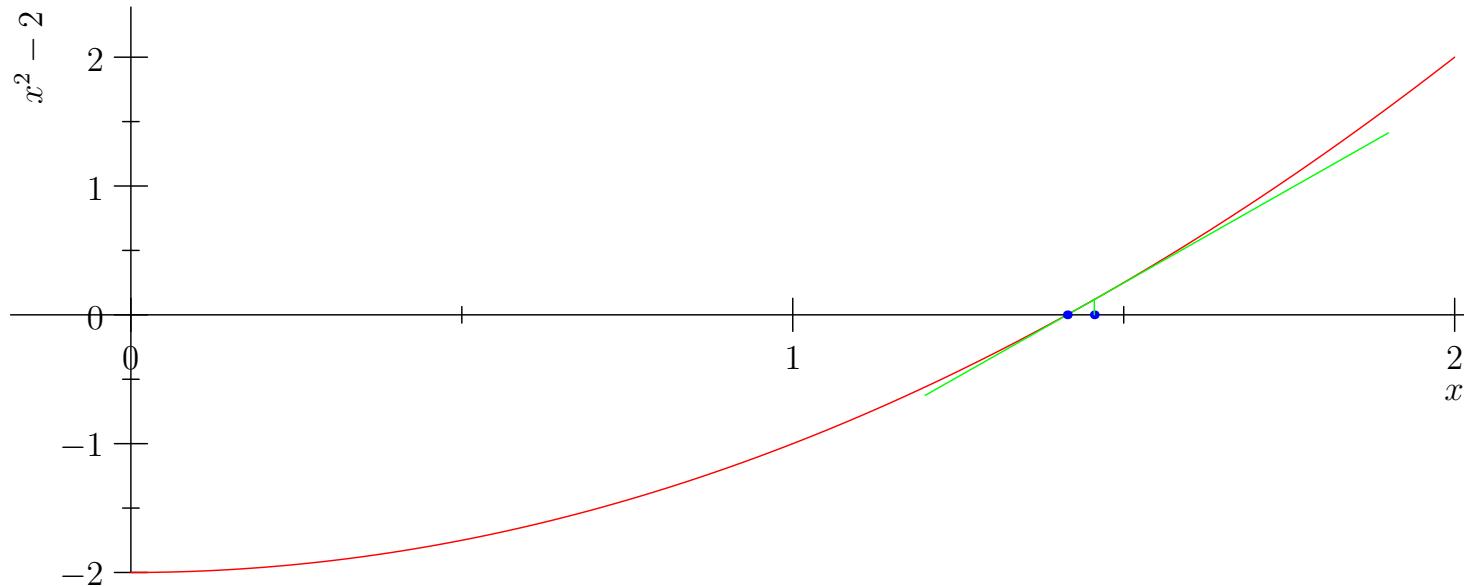


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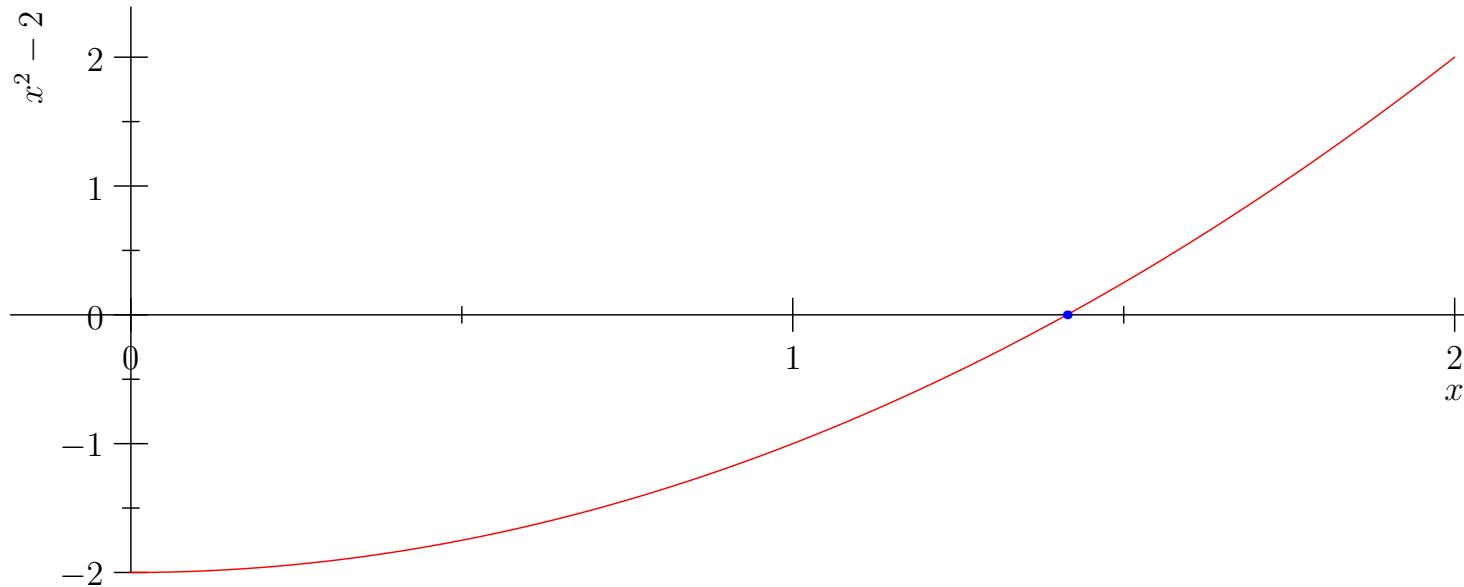


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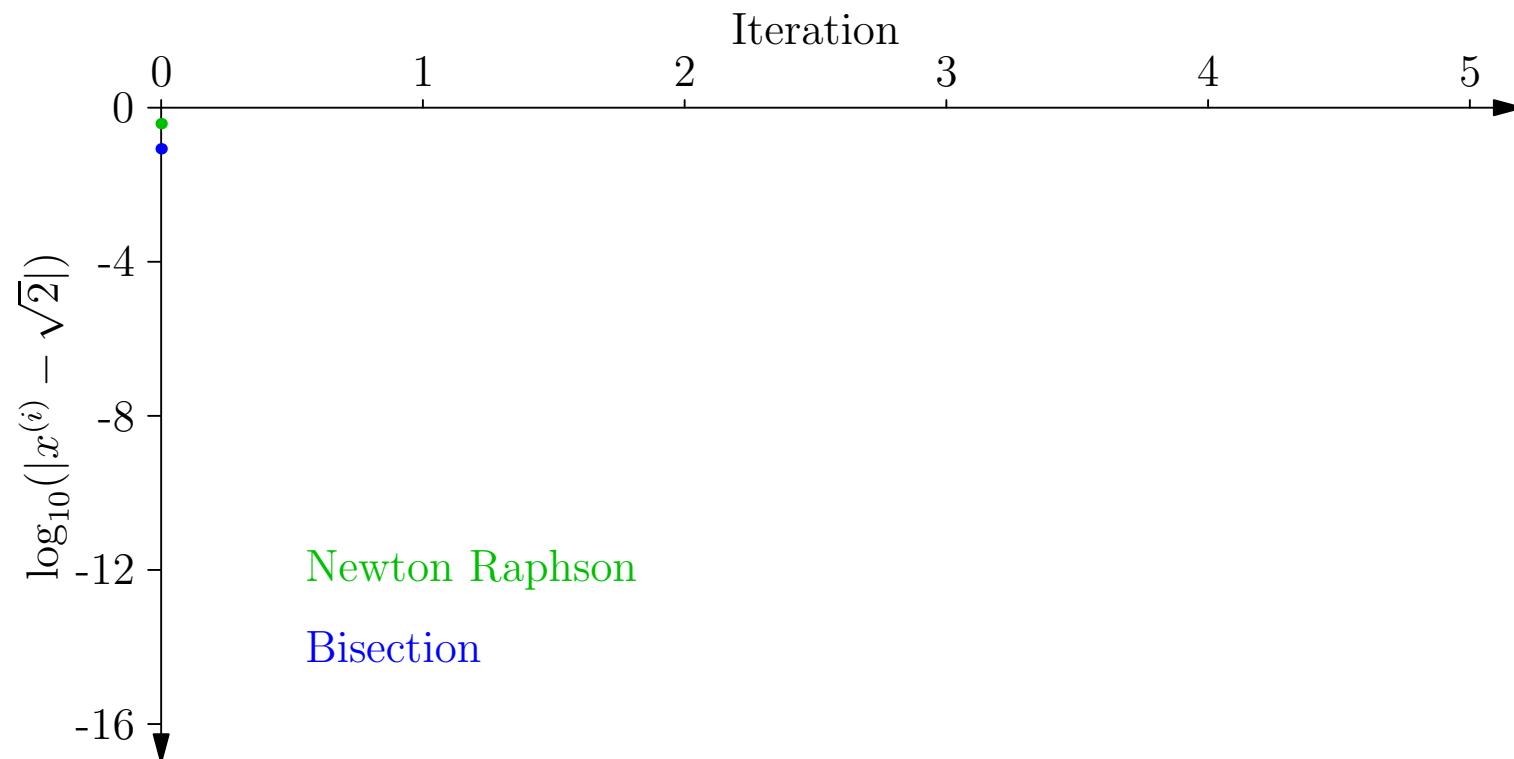
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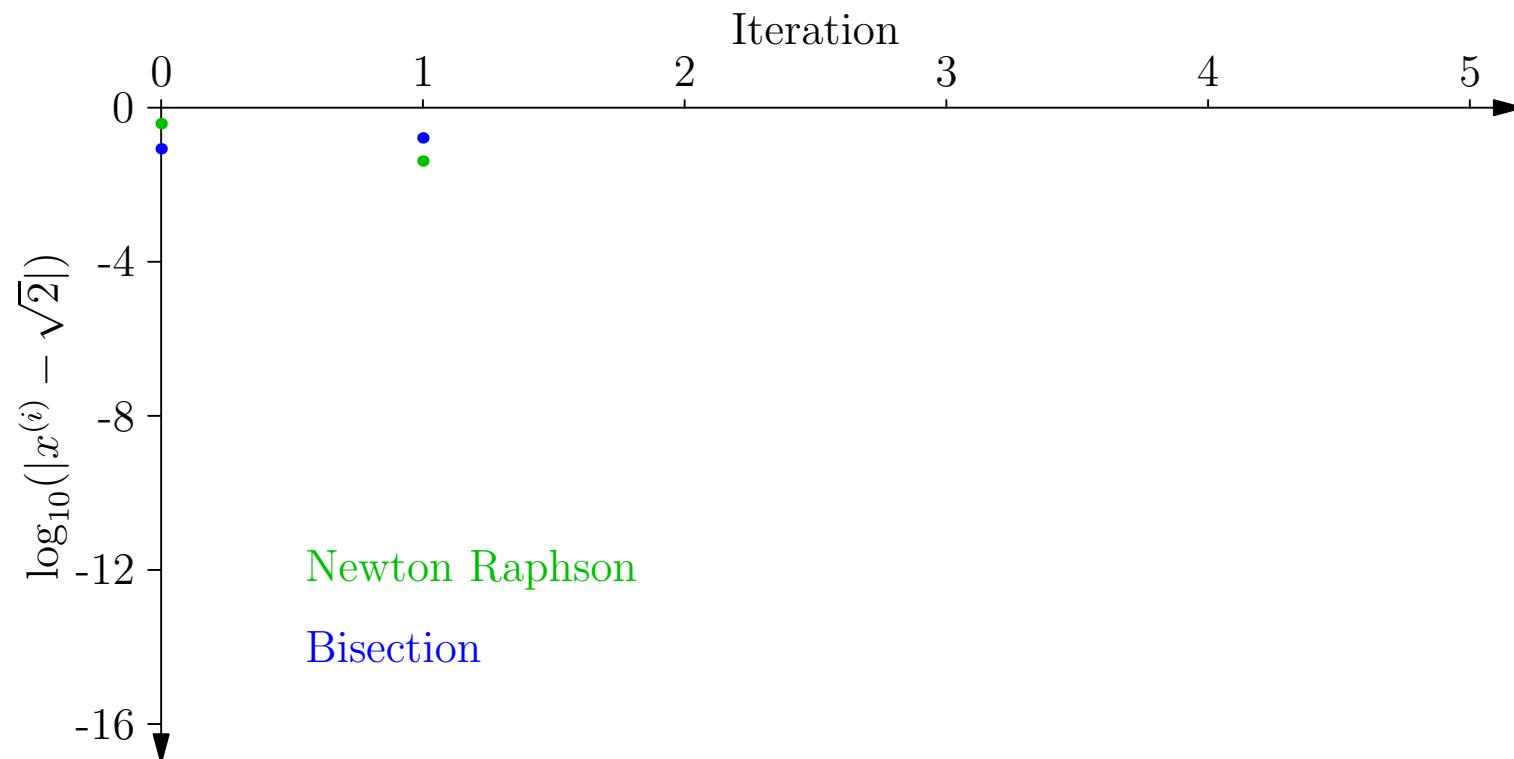
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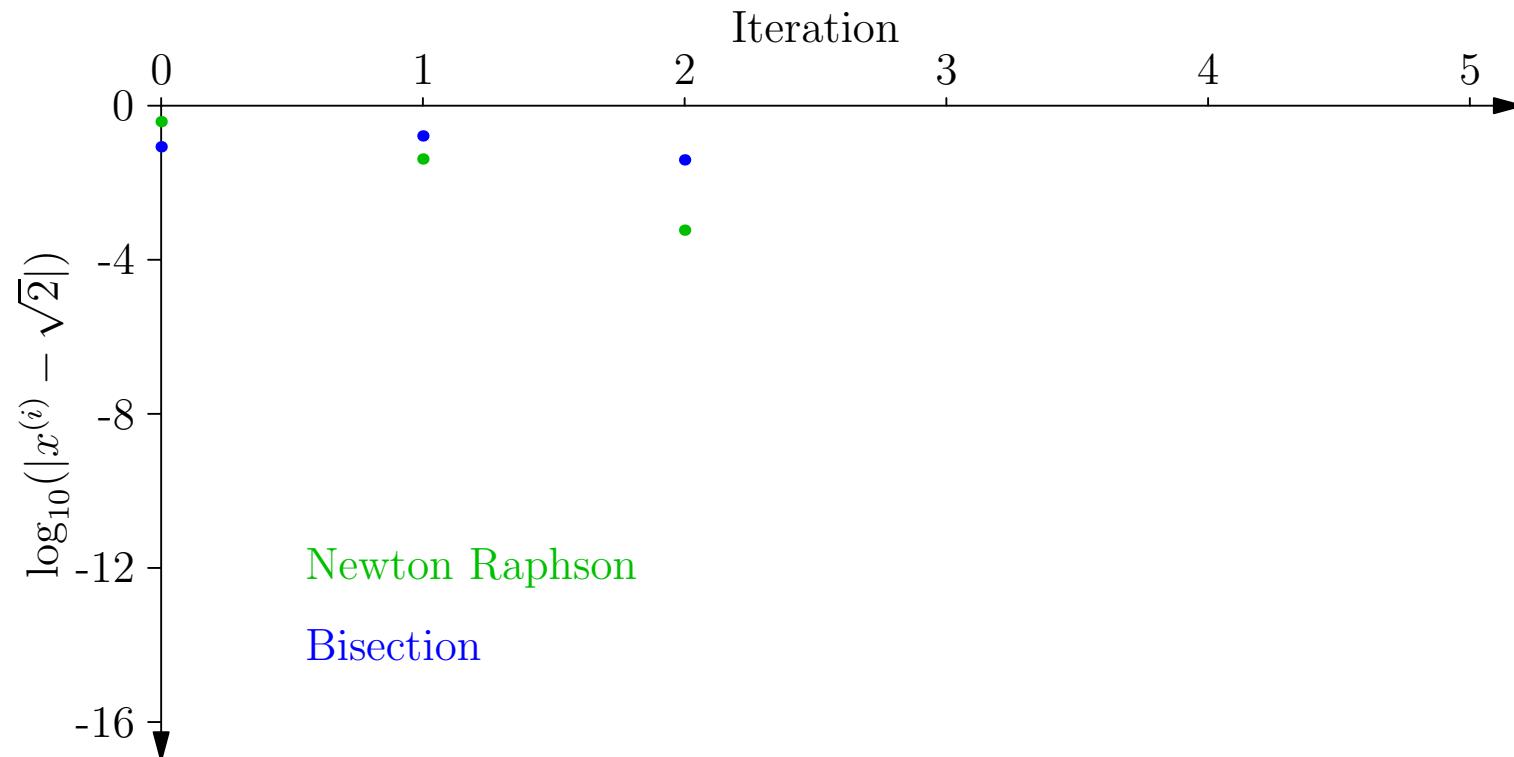
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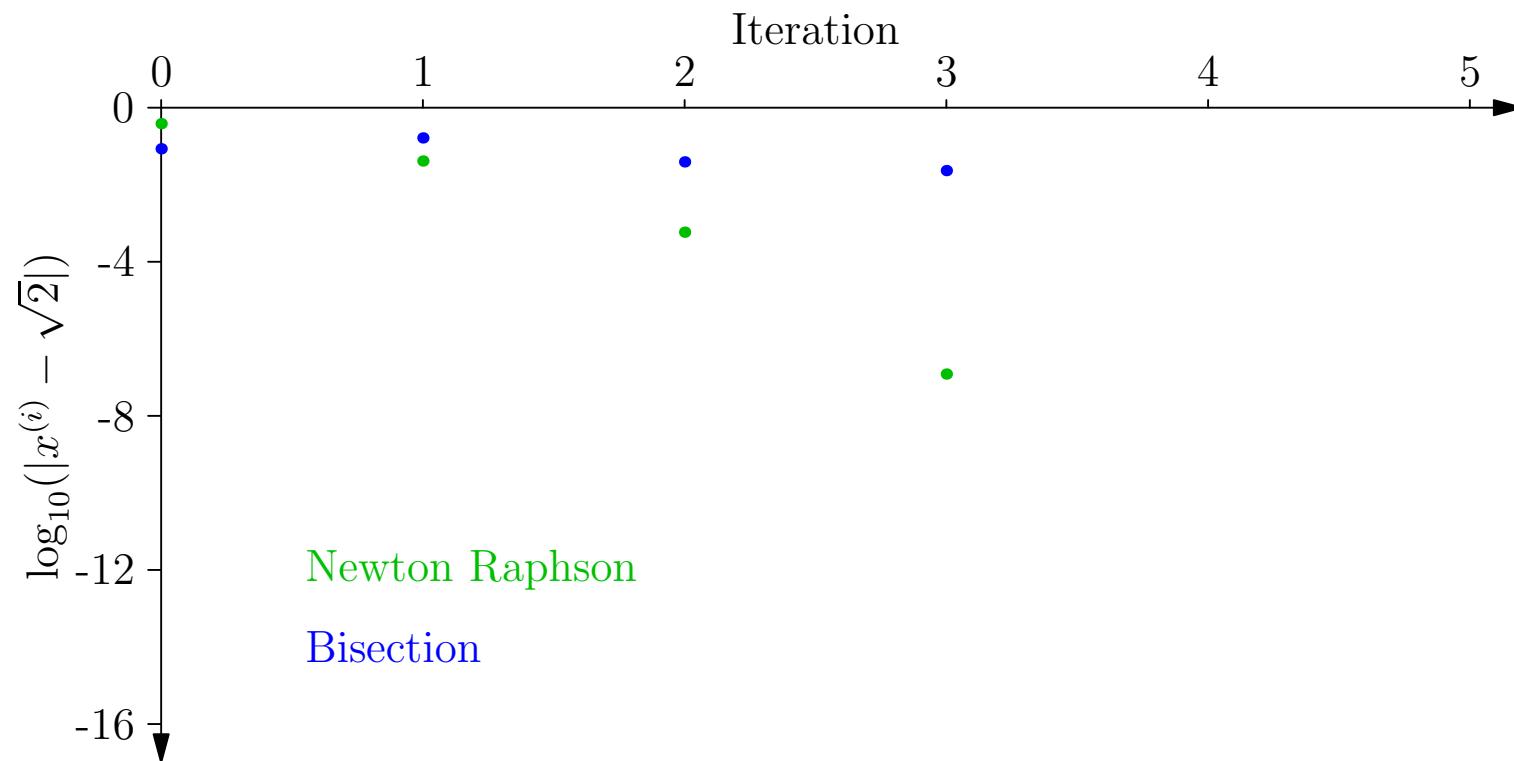
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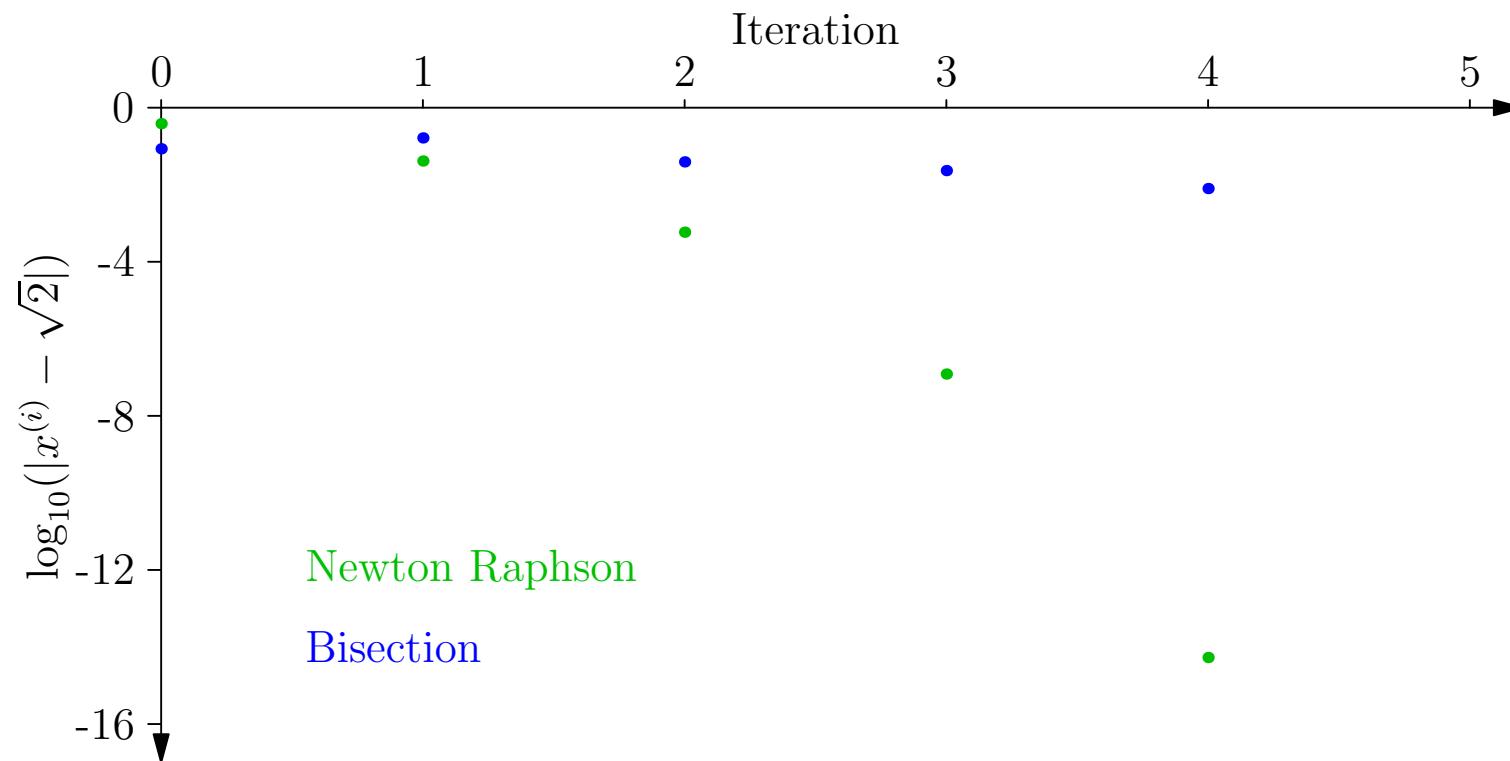
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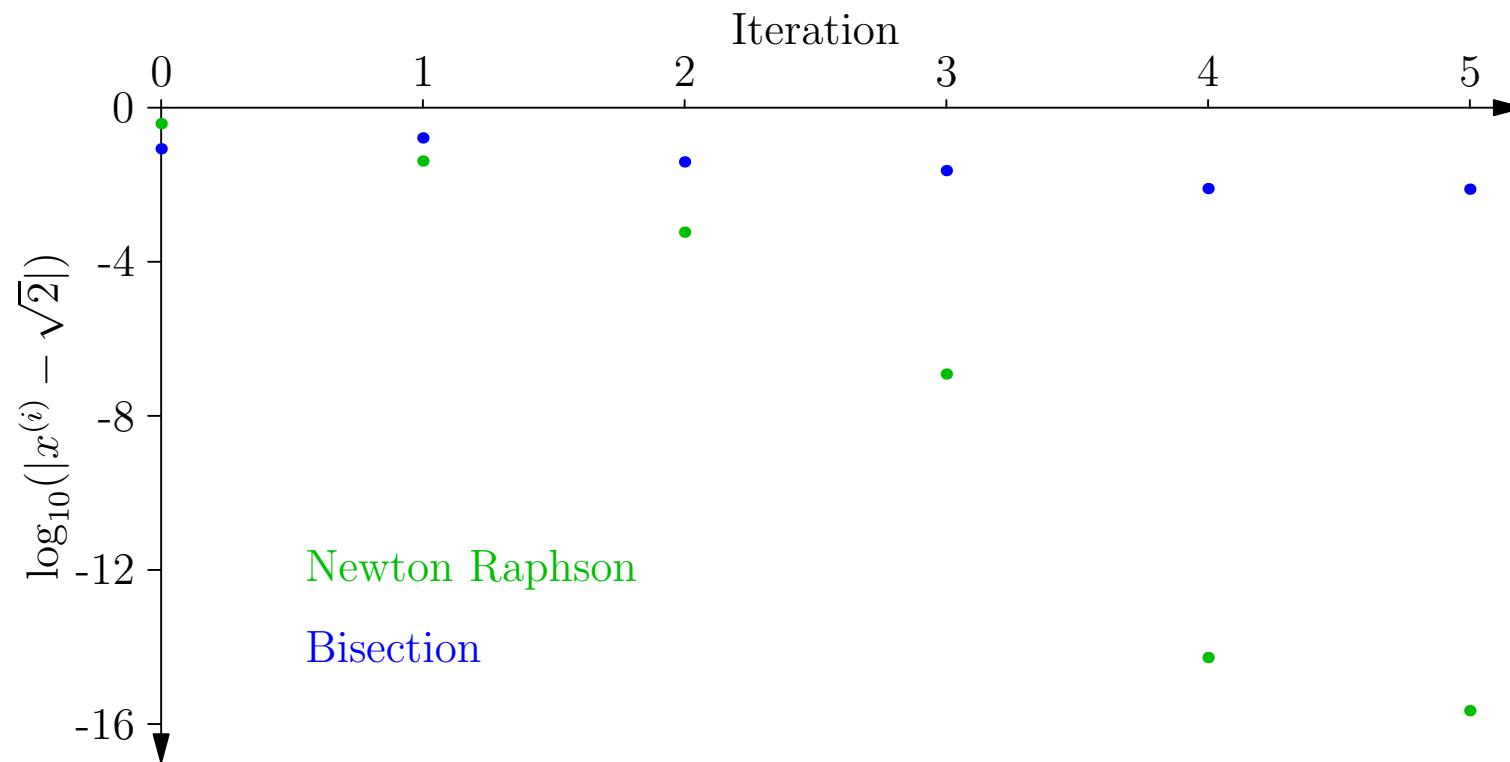
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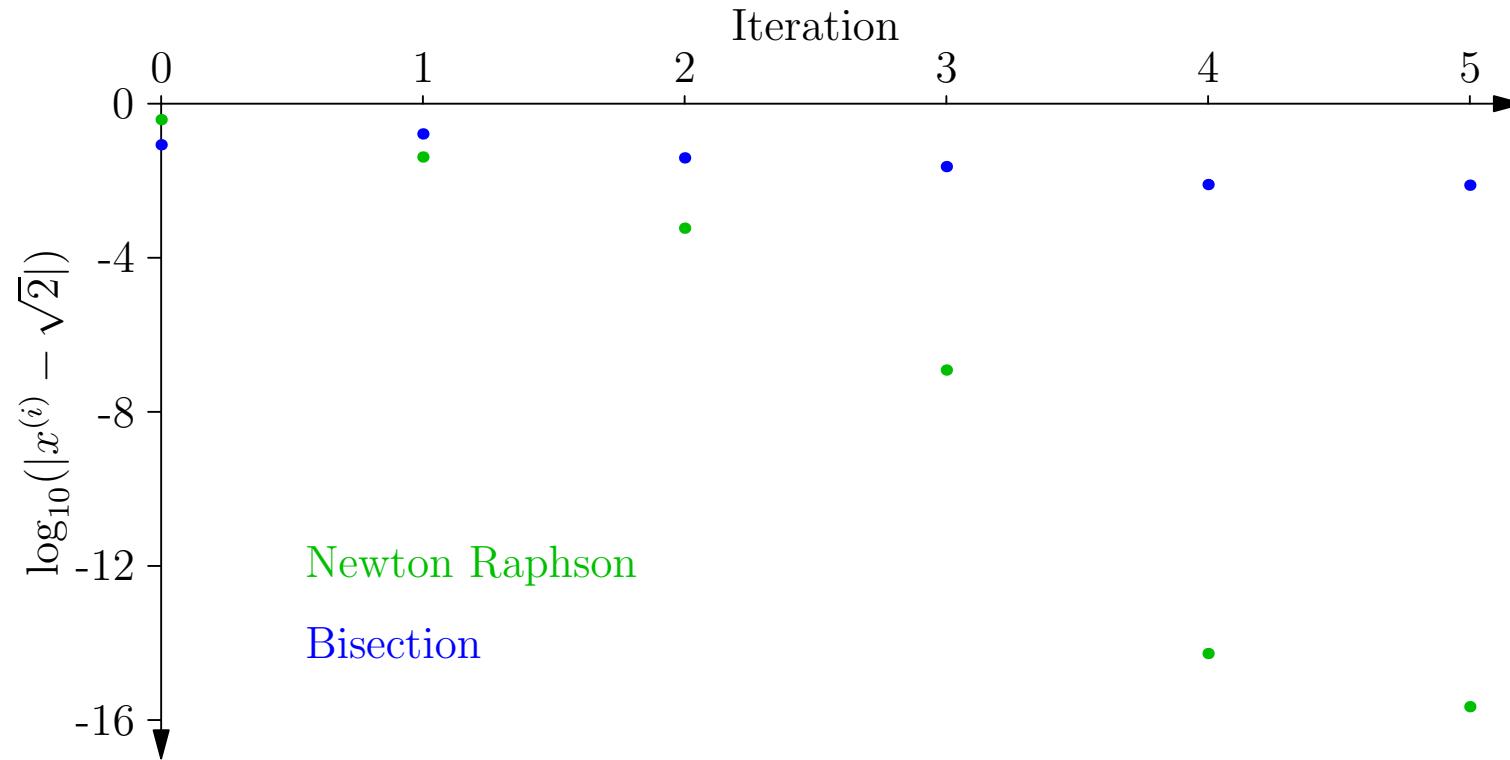
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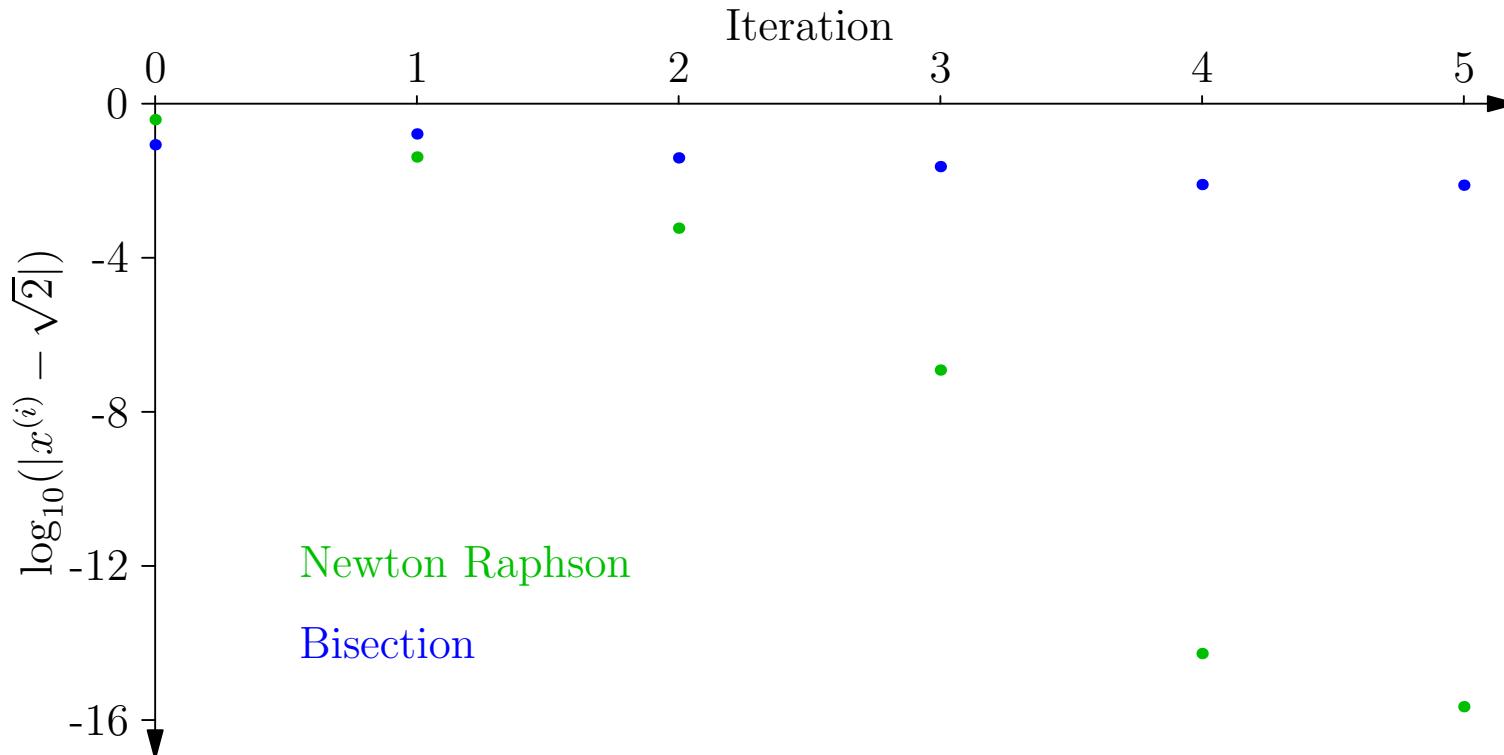


# Convergence



- Bisection shows linear convergence (exponential increase in accuracy)

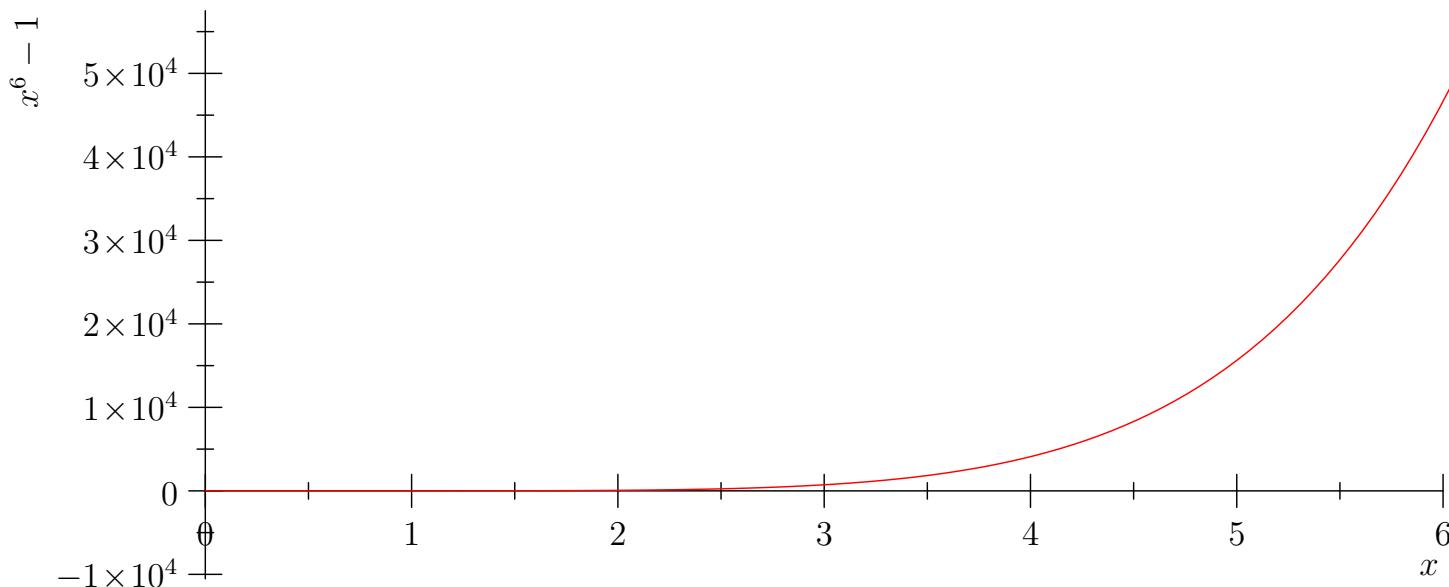
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- Bisection shows linear convergence (exponential increase in accuracy)
- Newton Raphson shows quadratic convergence

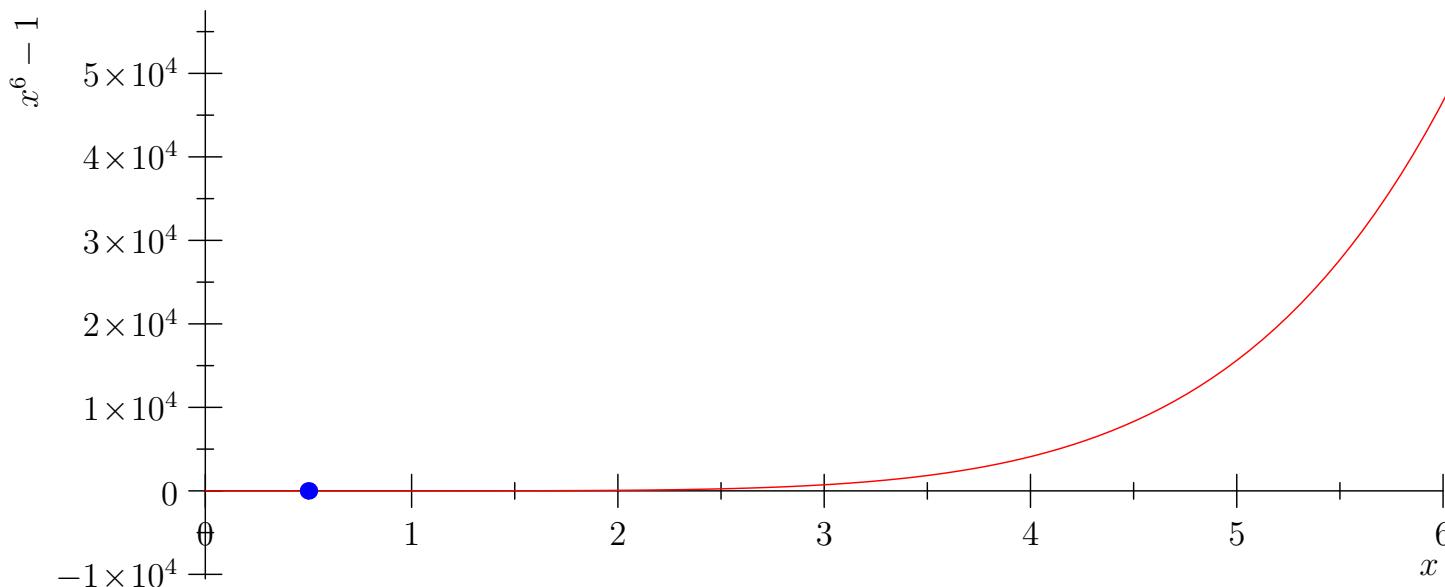
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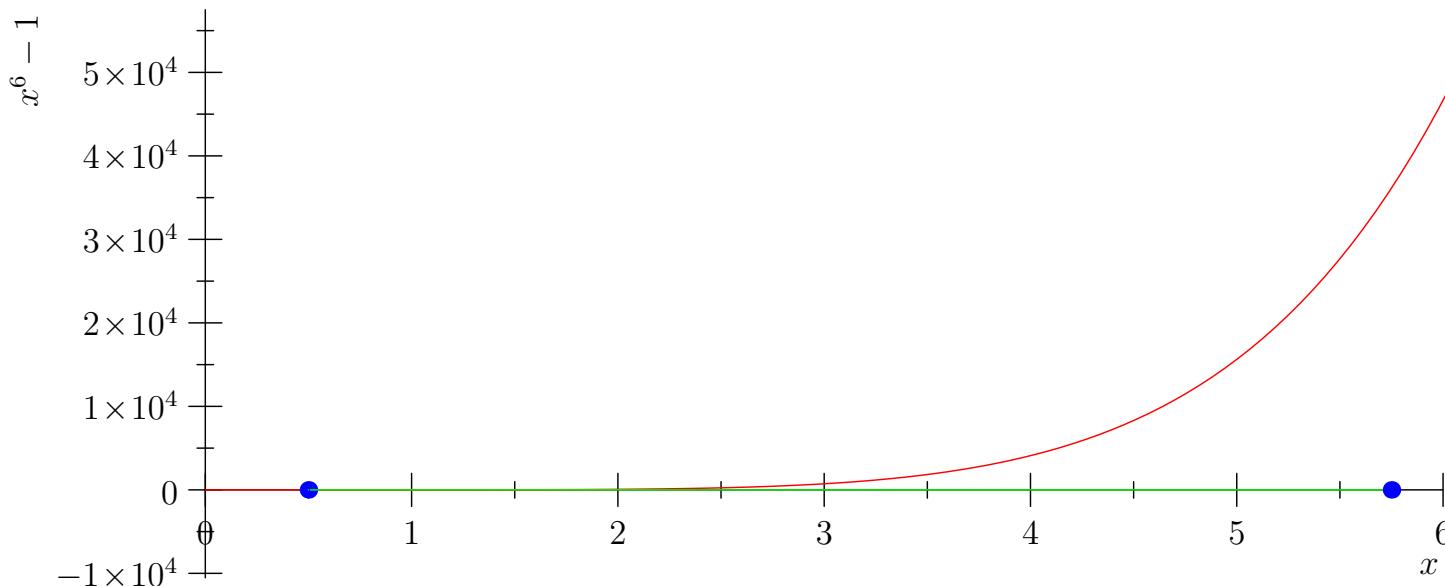
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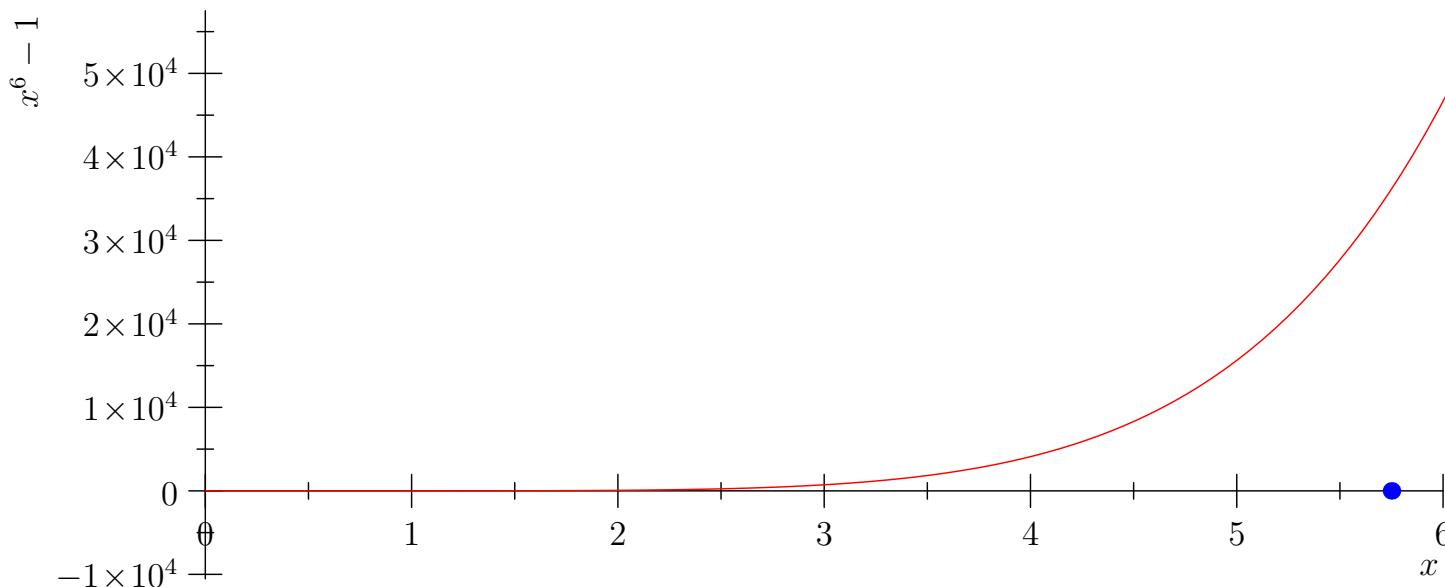
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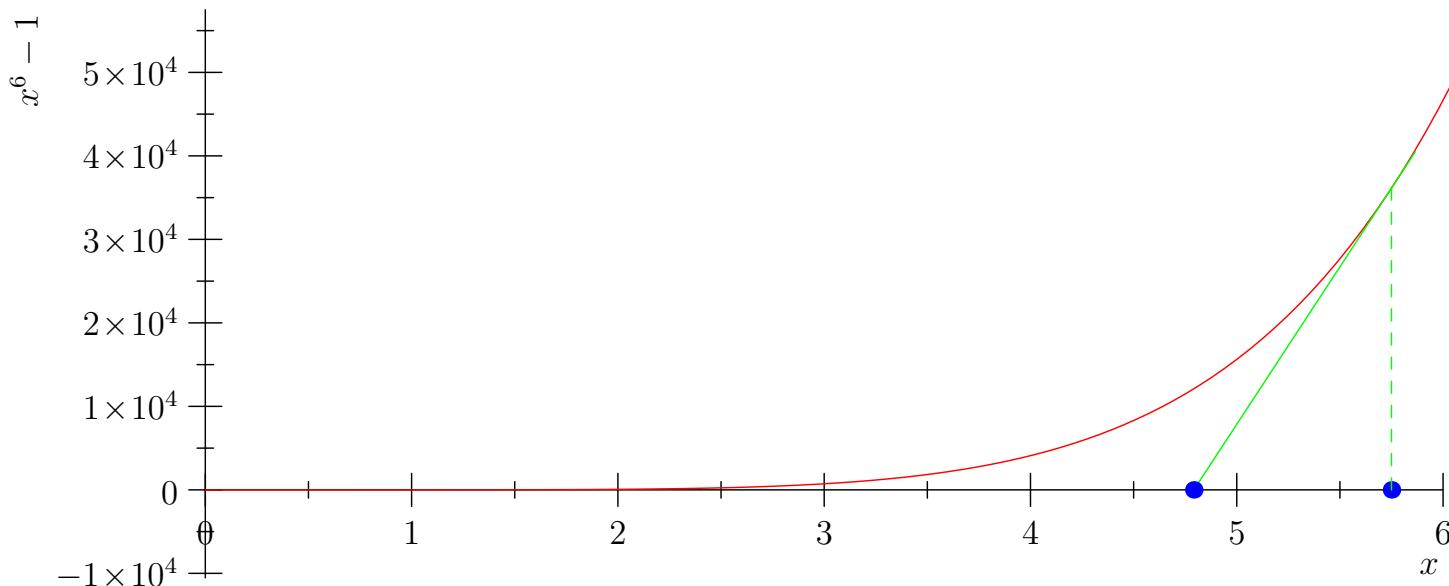
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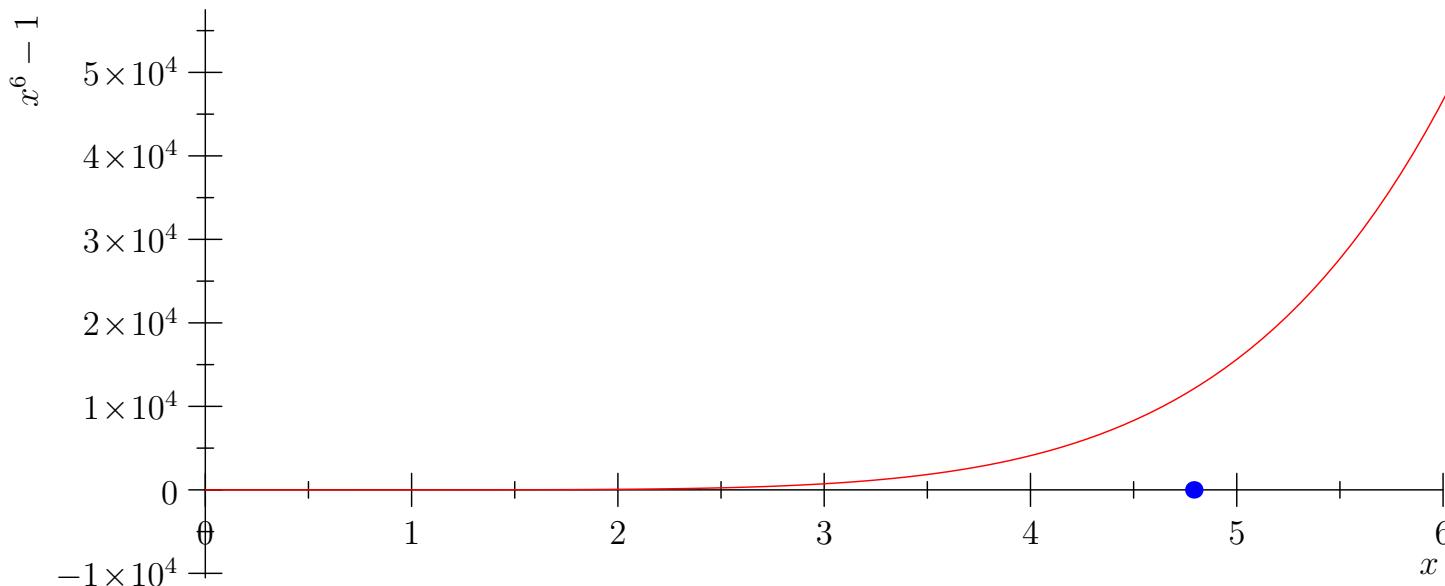
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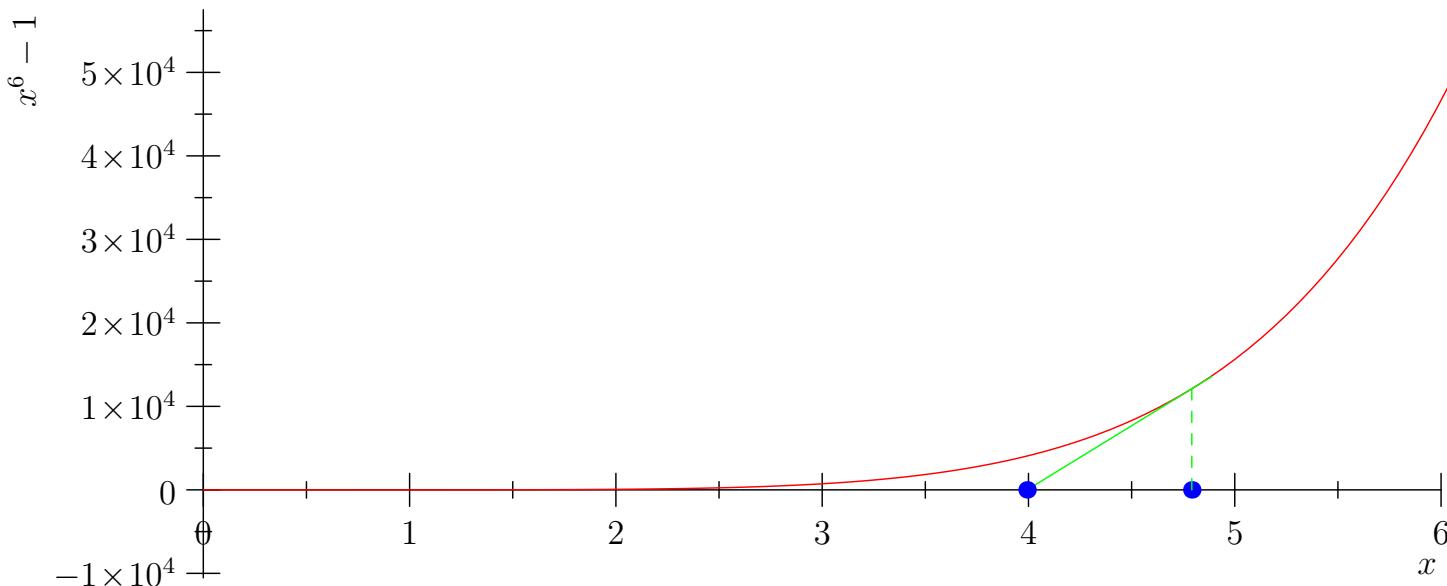
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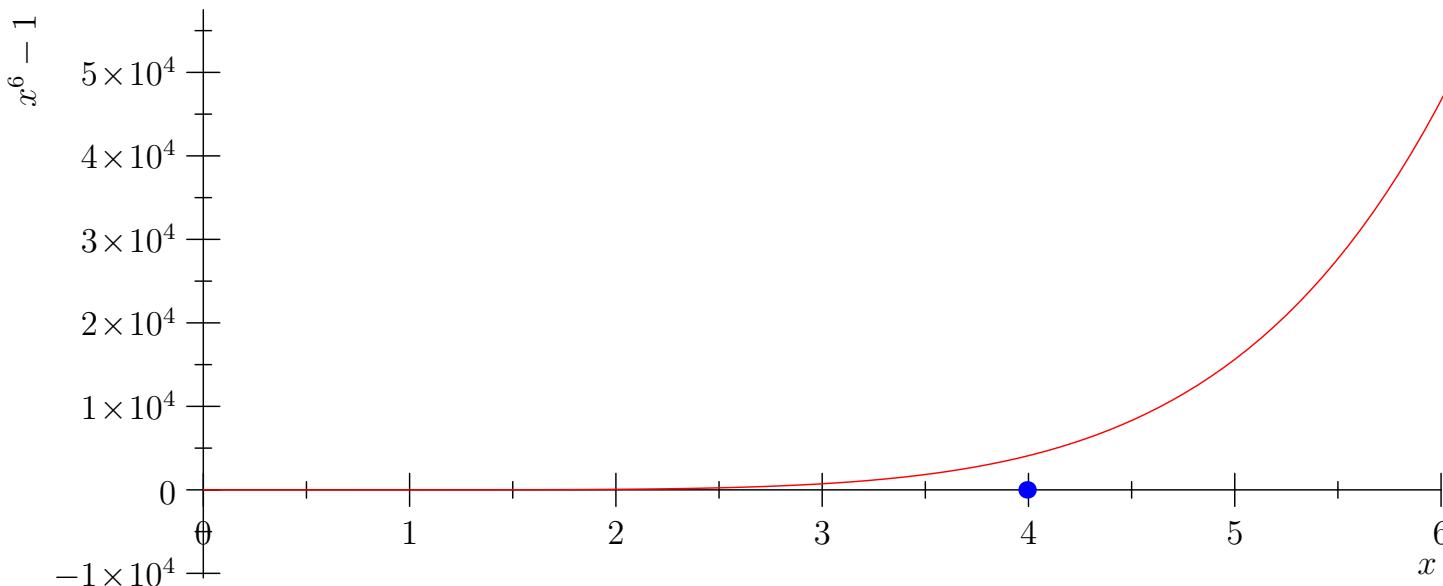
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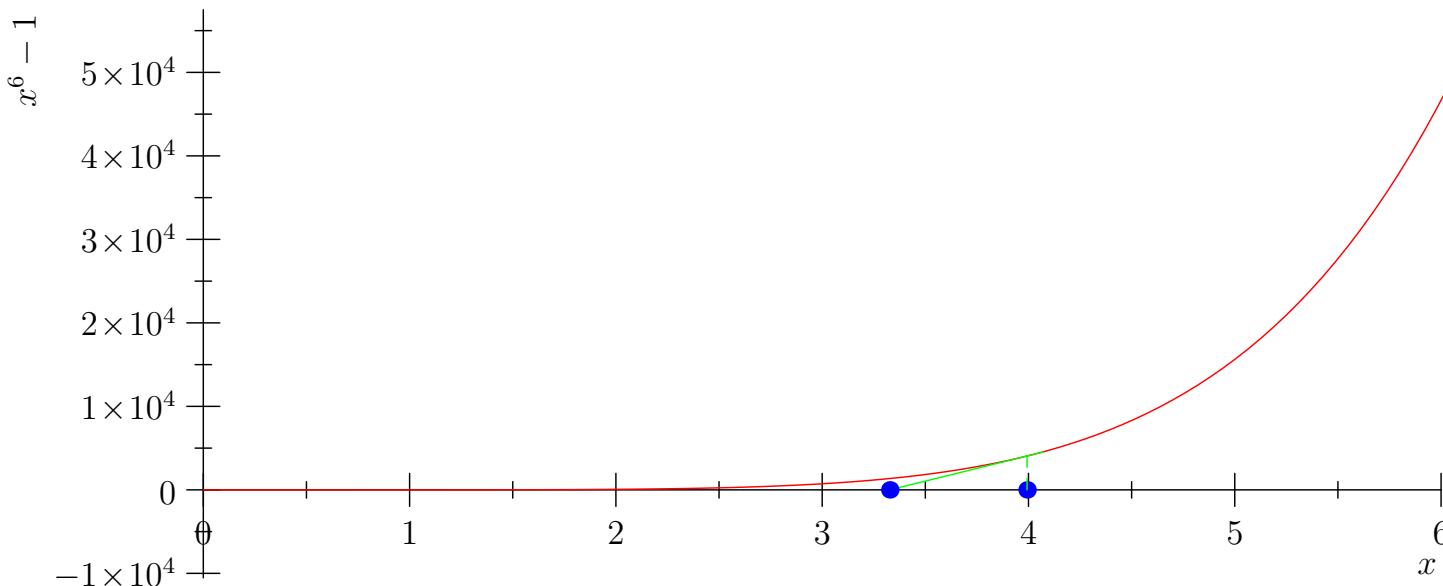
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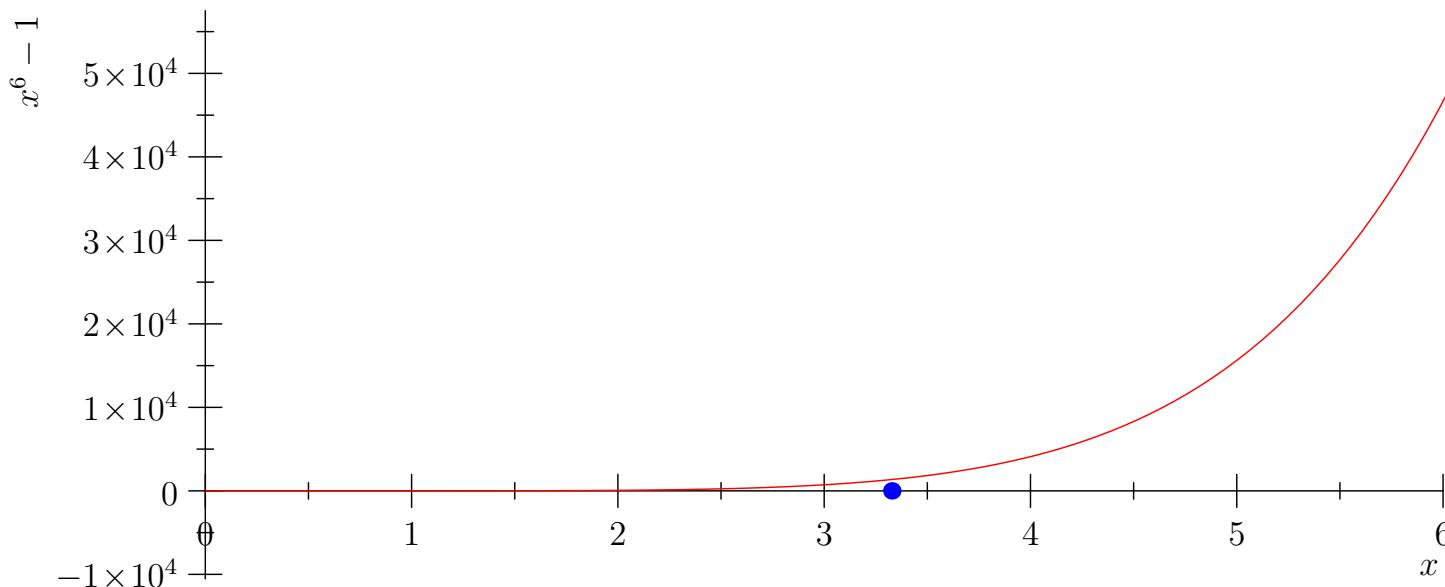
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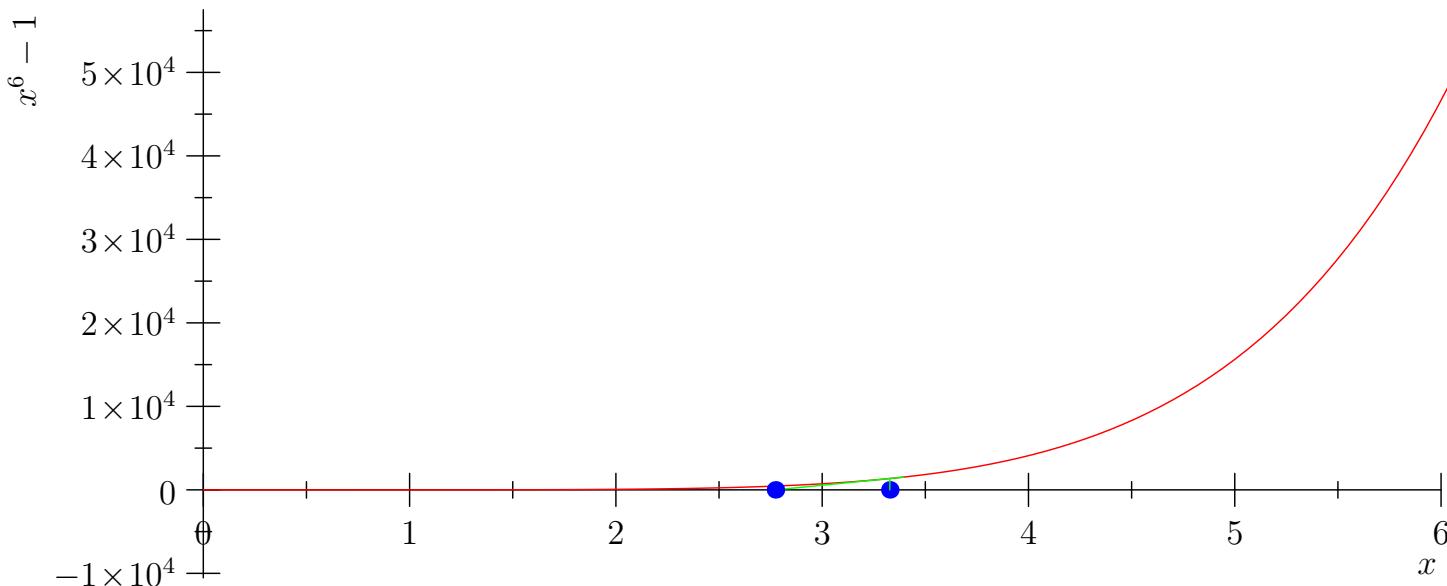
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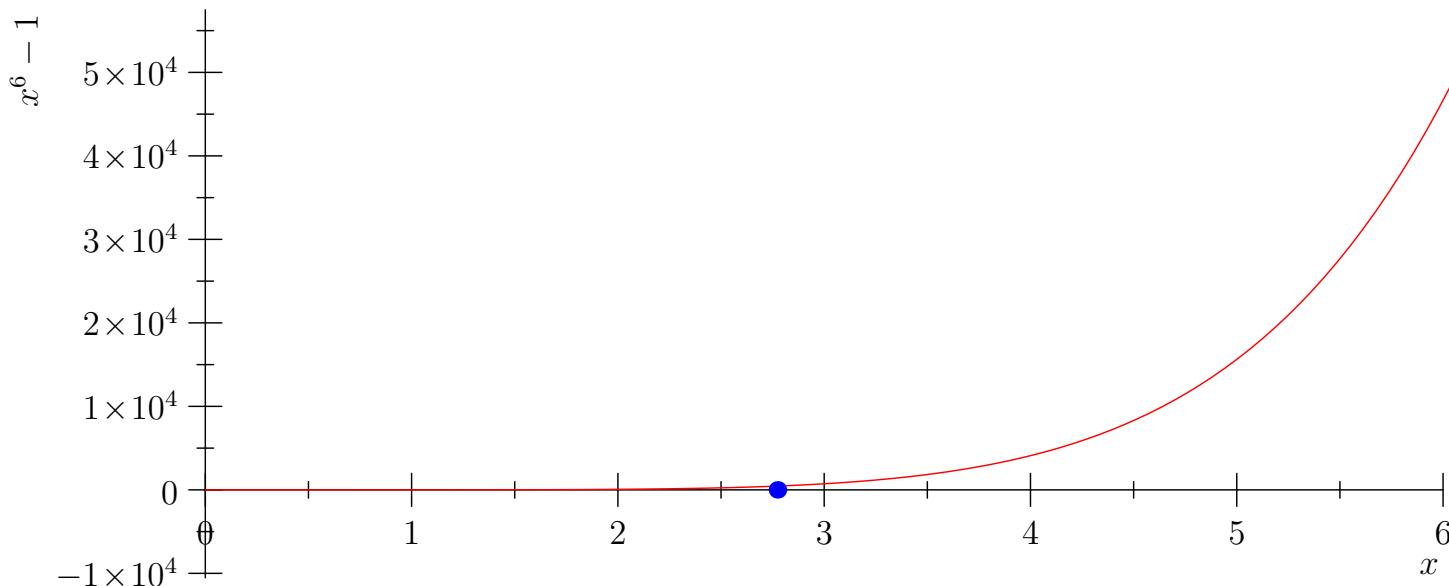
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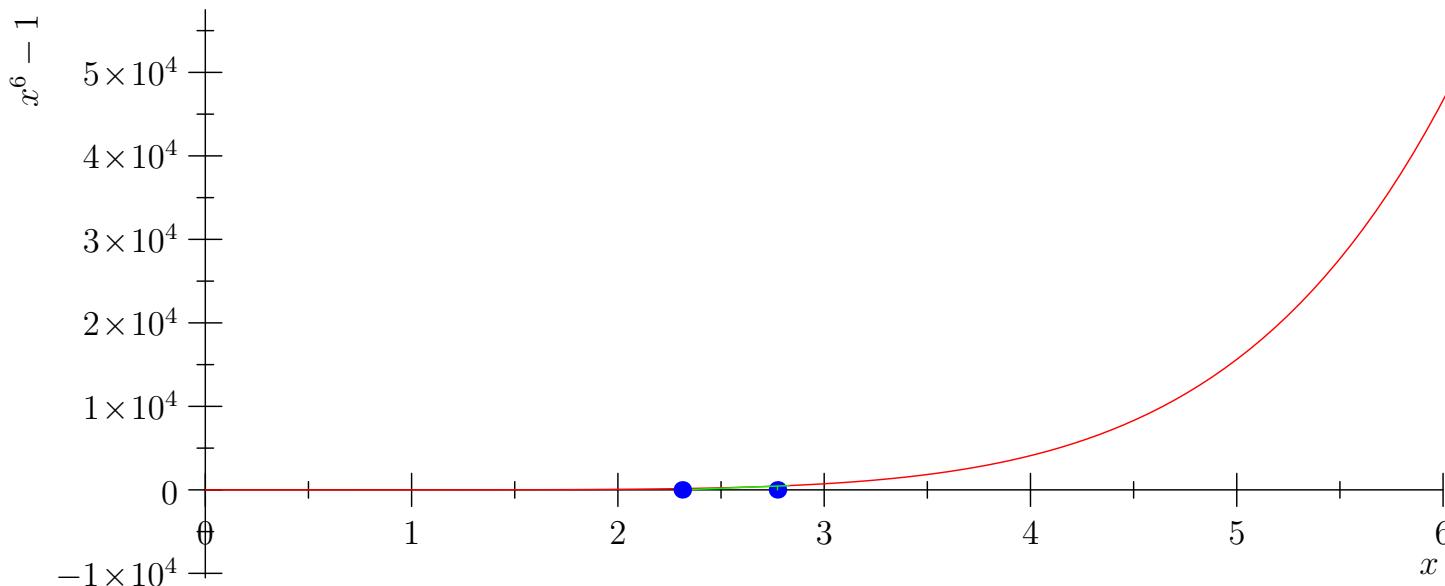
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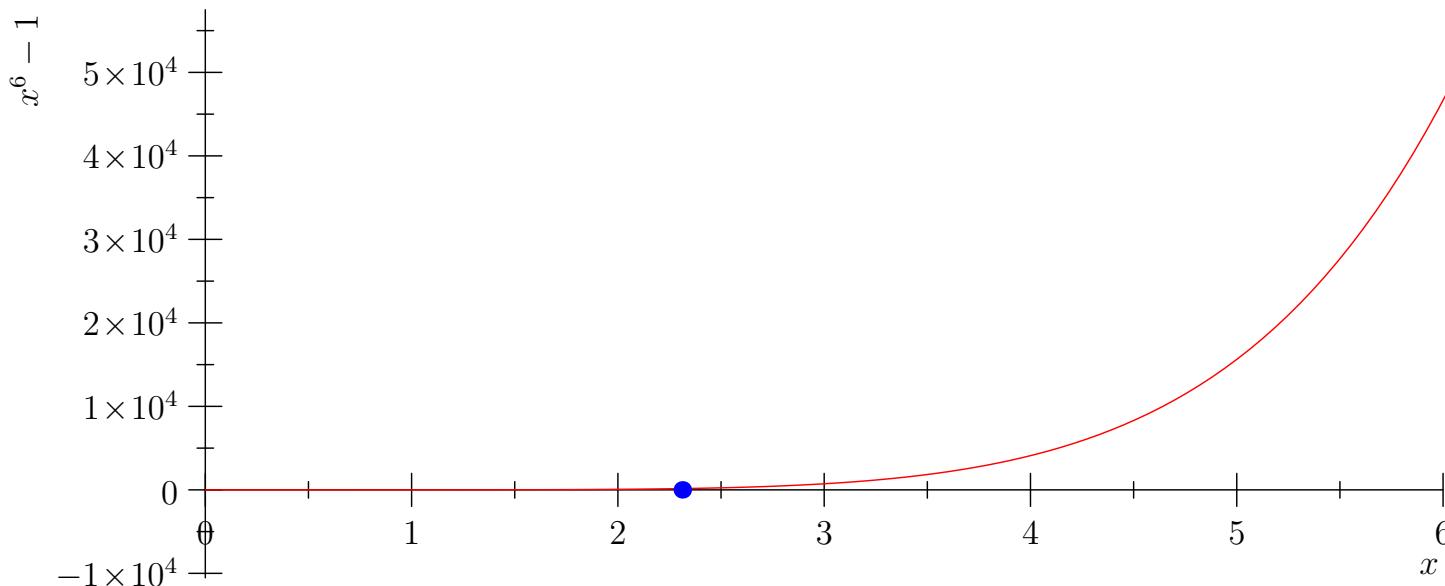
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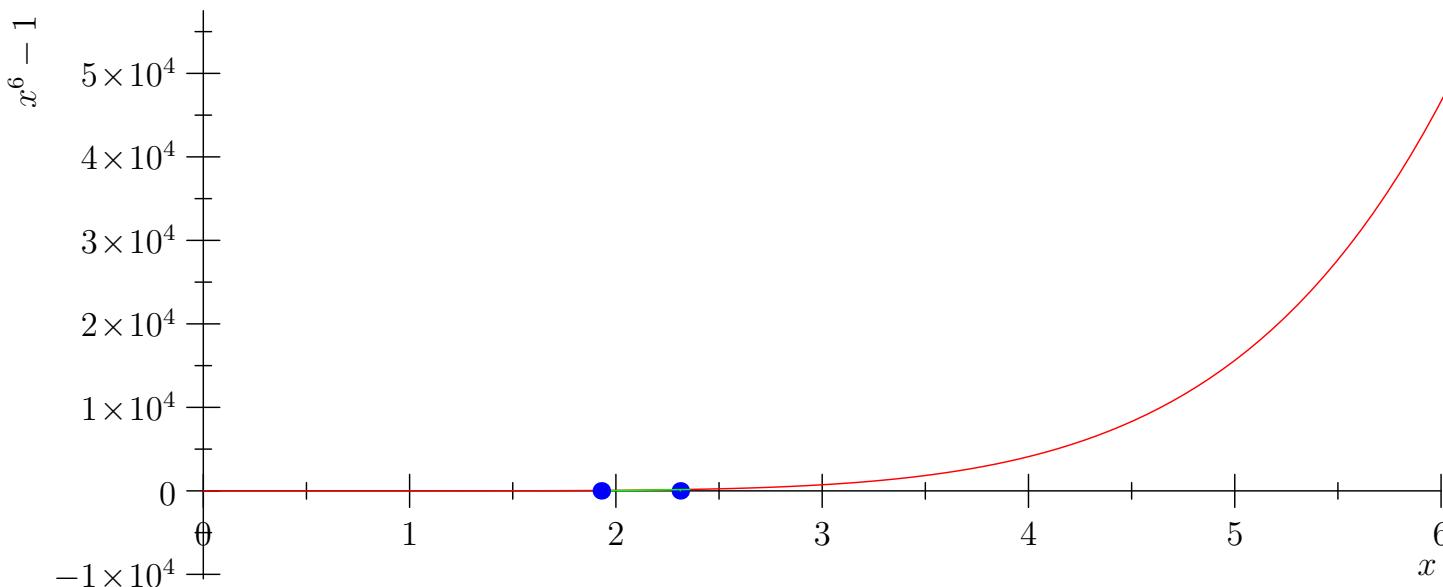
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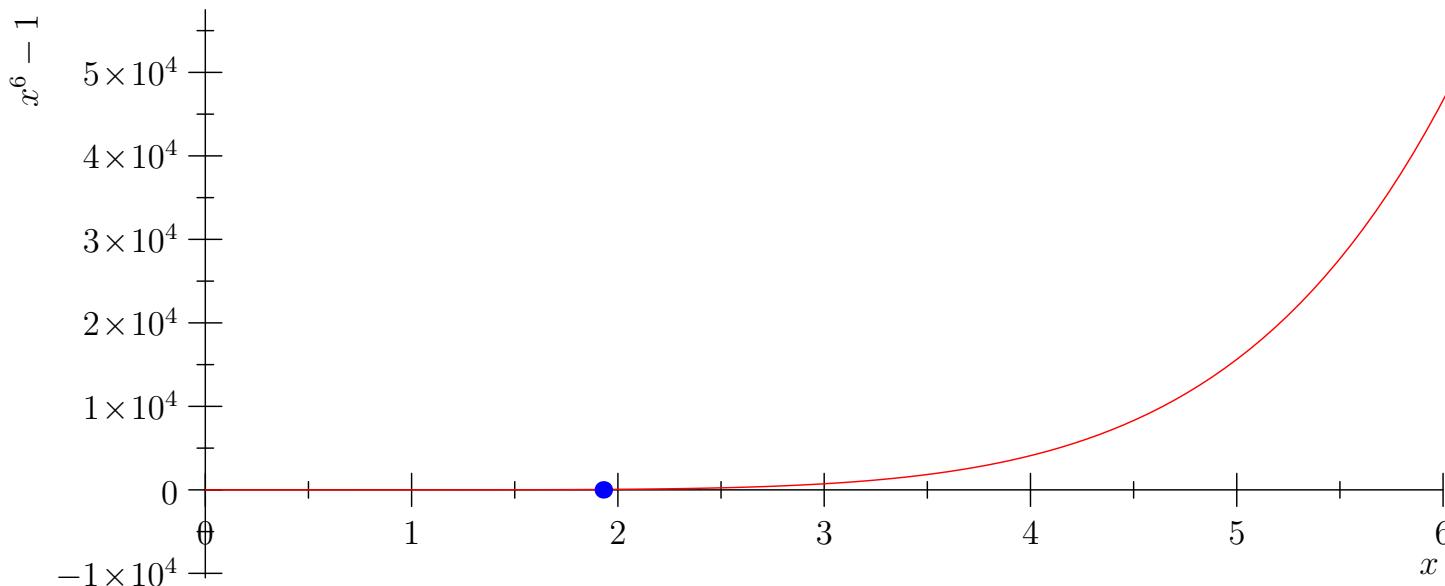
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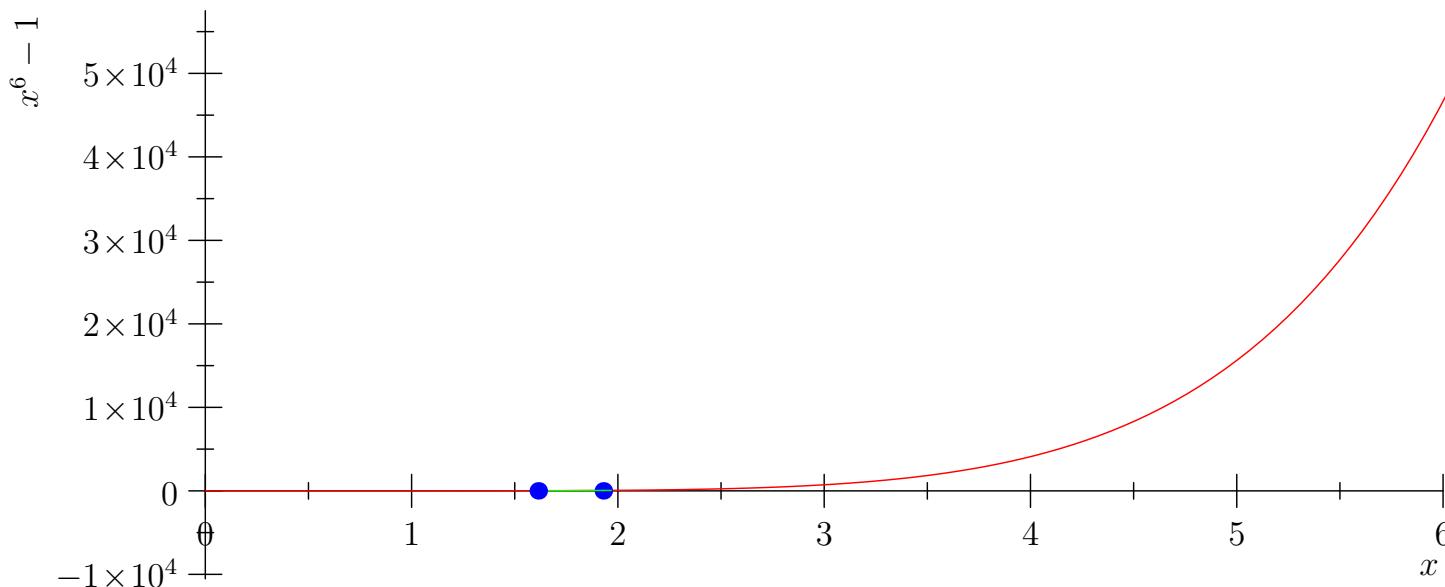
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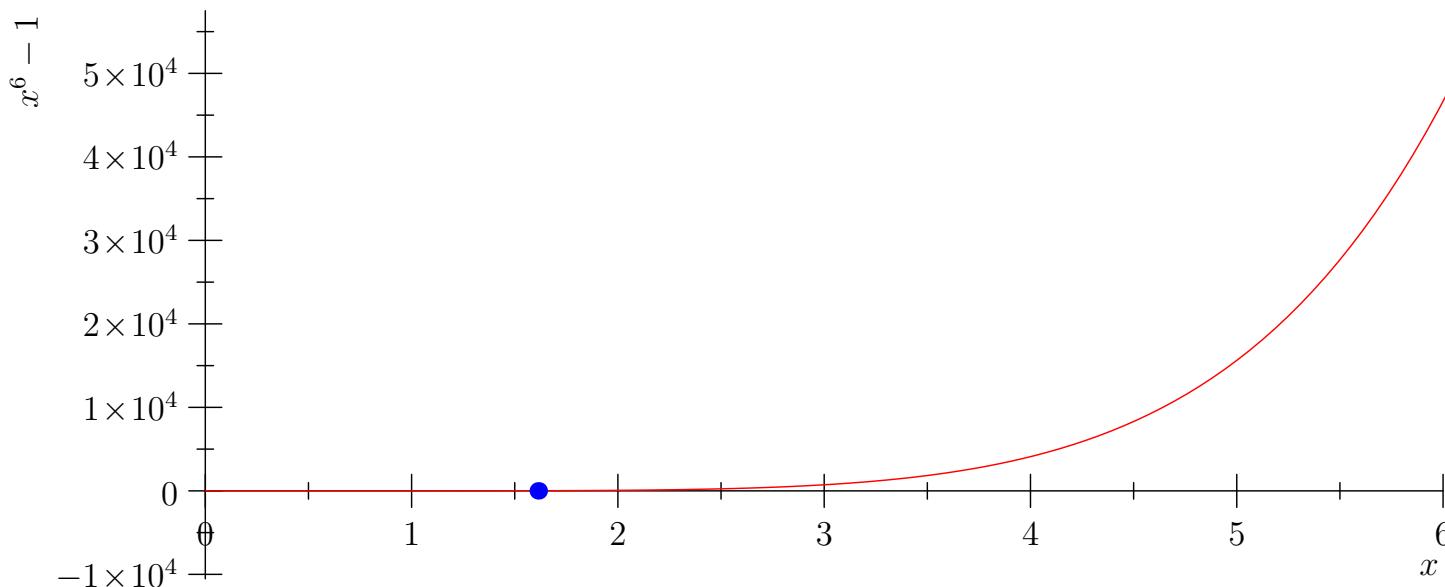
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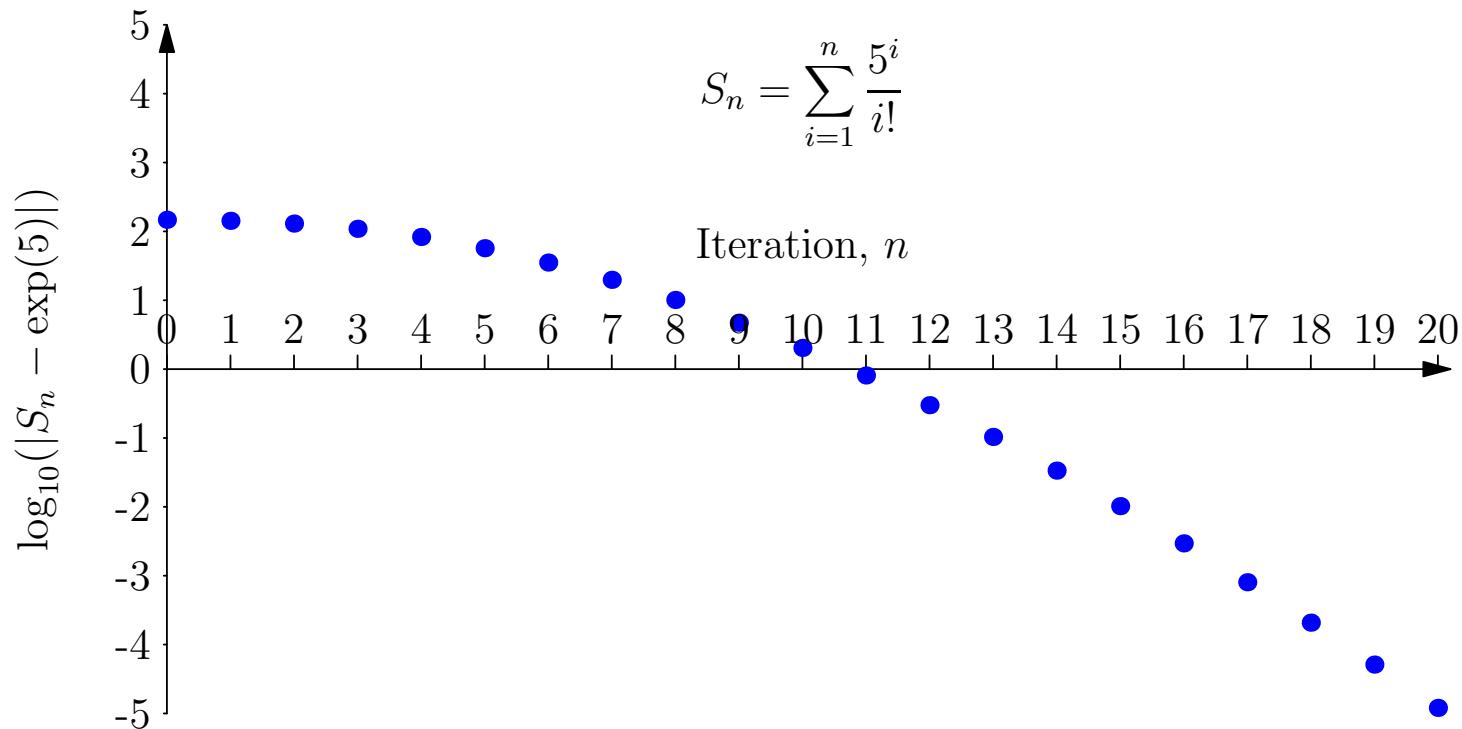
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# Evaluating Functions

- We can evaluate many functions using a series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

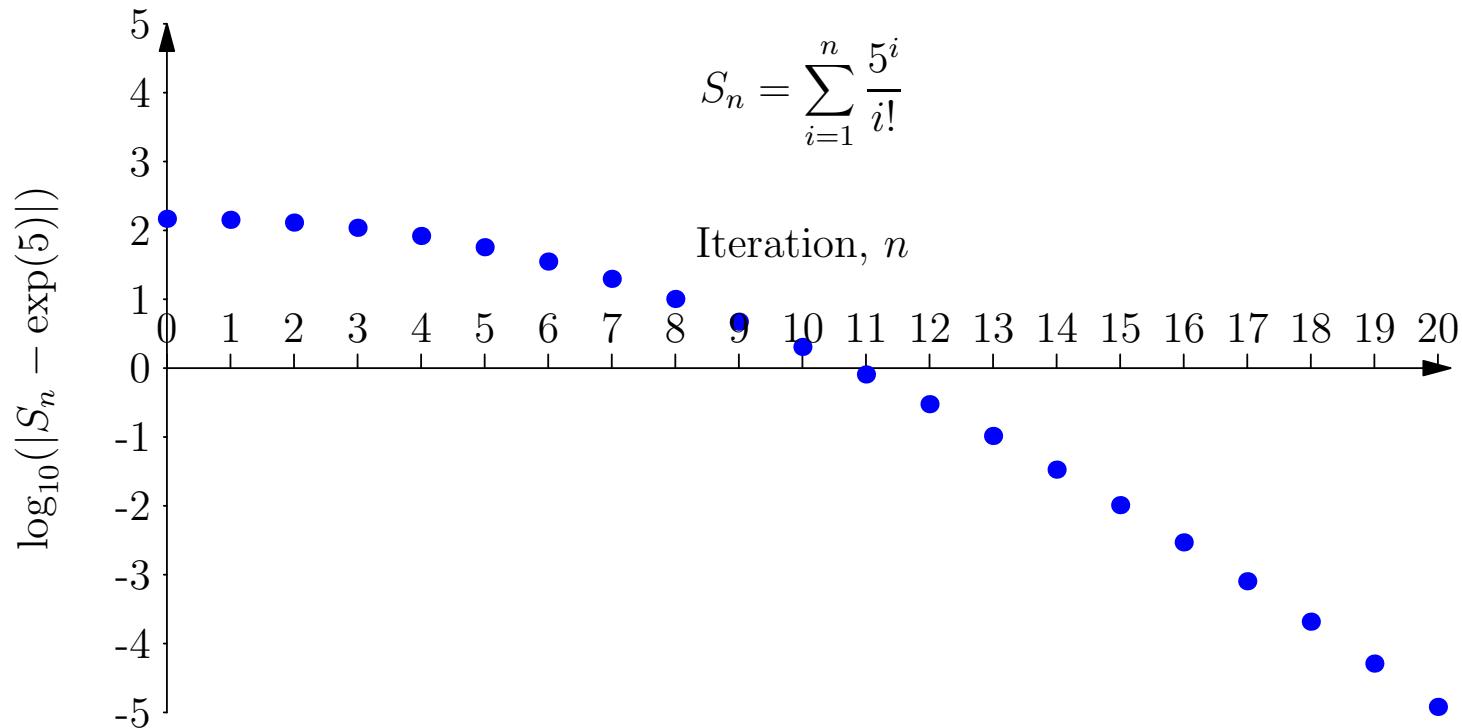


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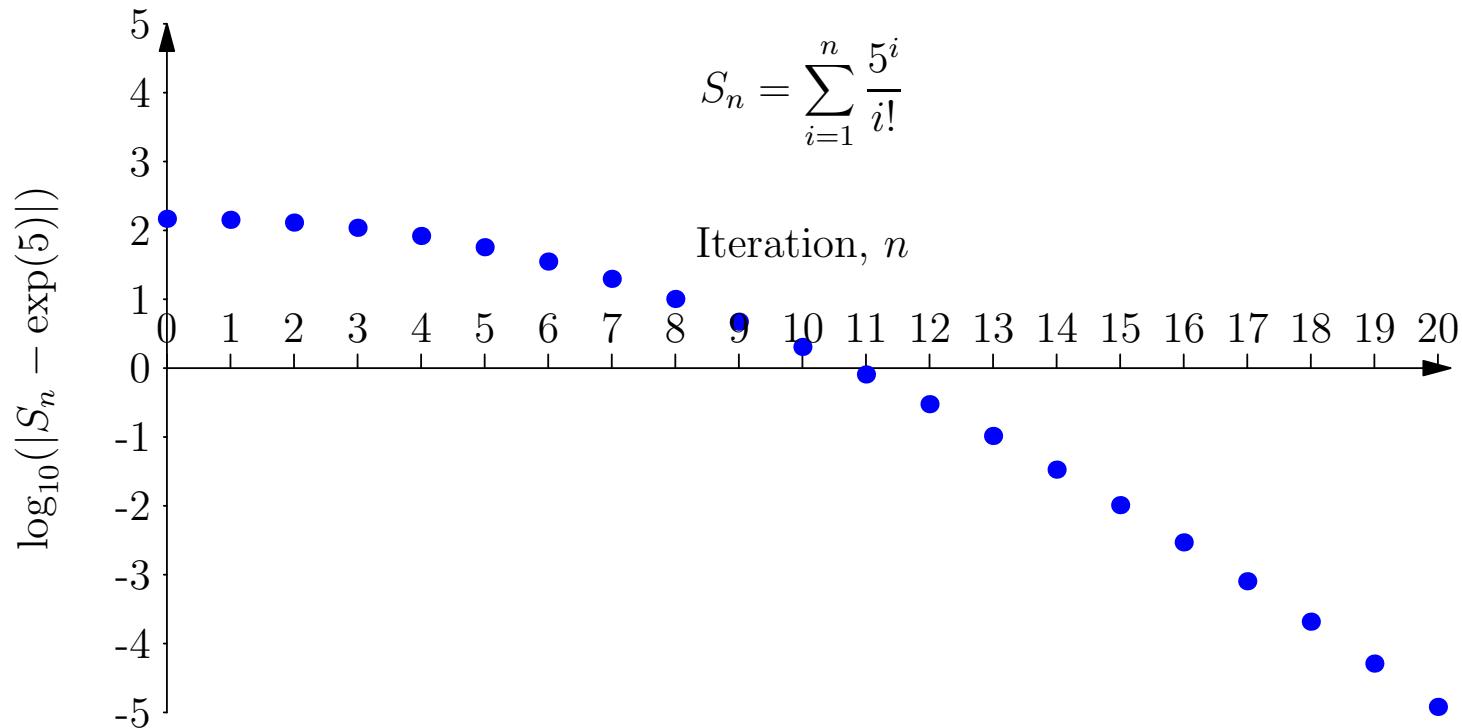


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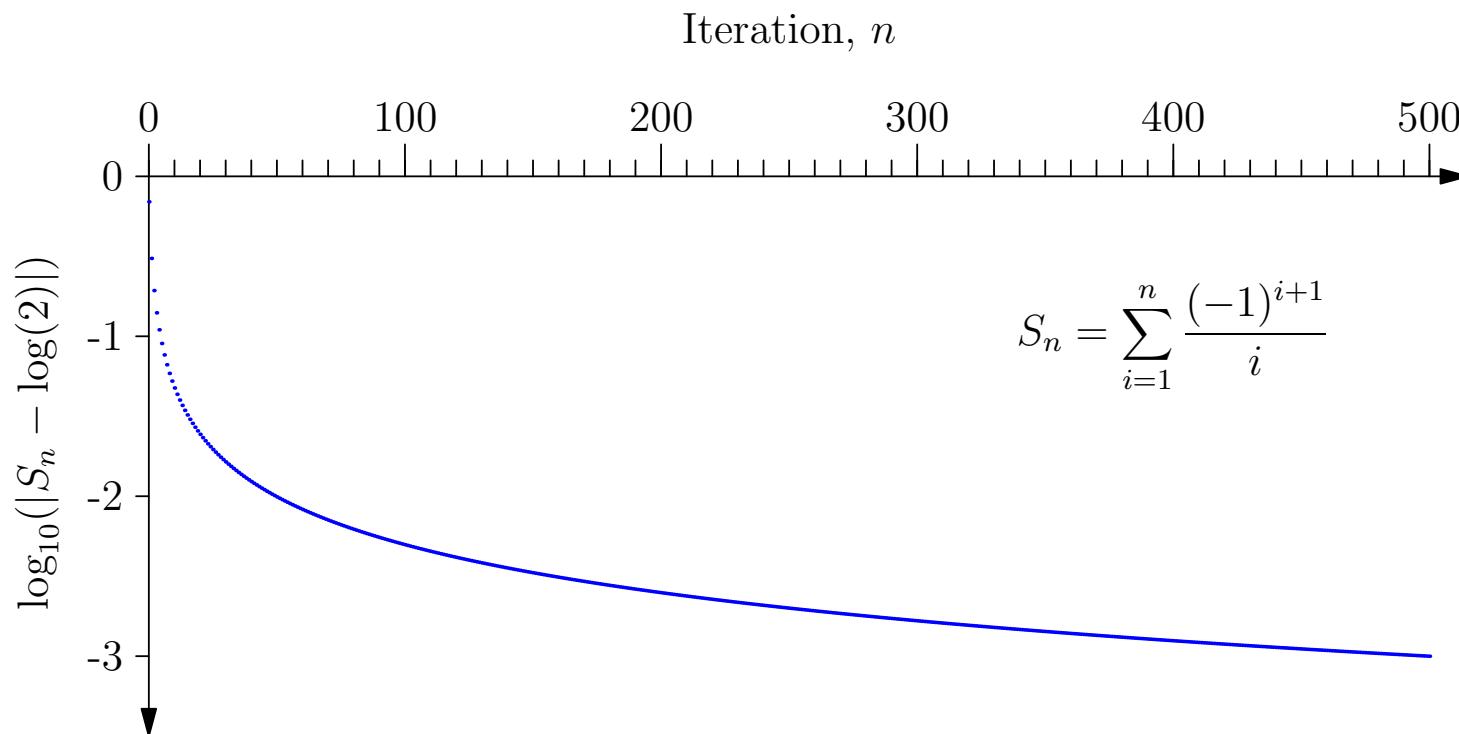
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- Converges for  $-1 < x \leq 1$ , but converges slowly for  $x = 1$

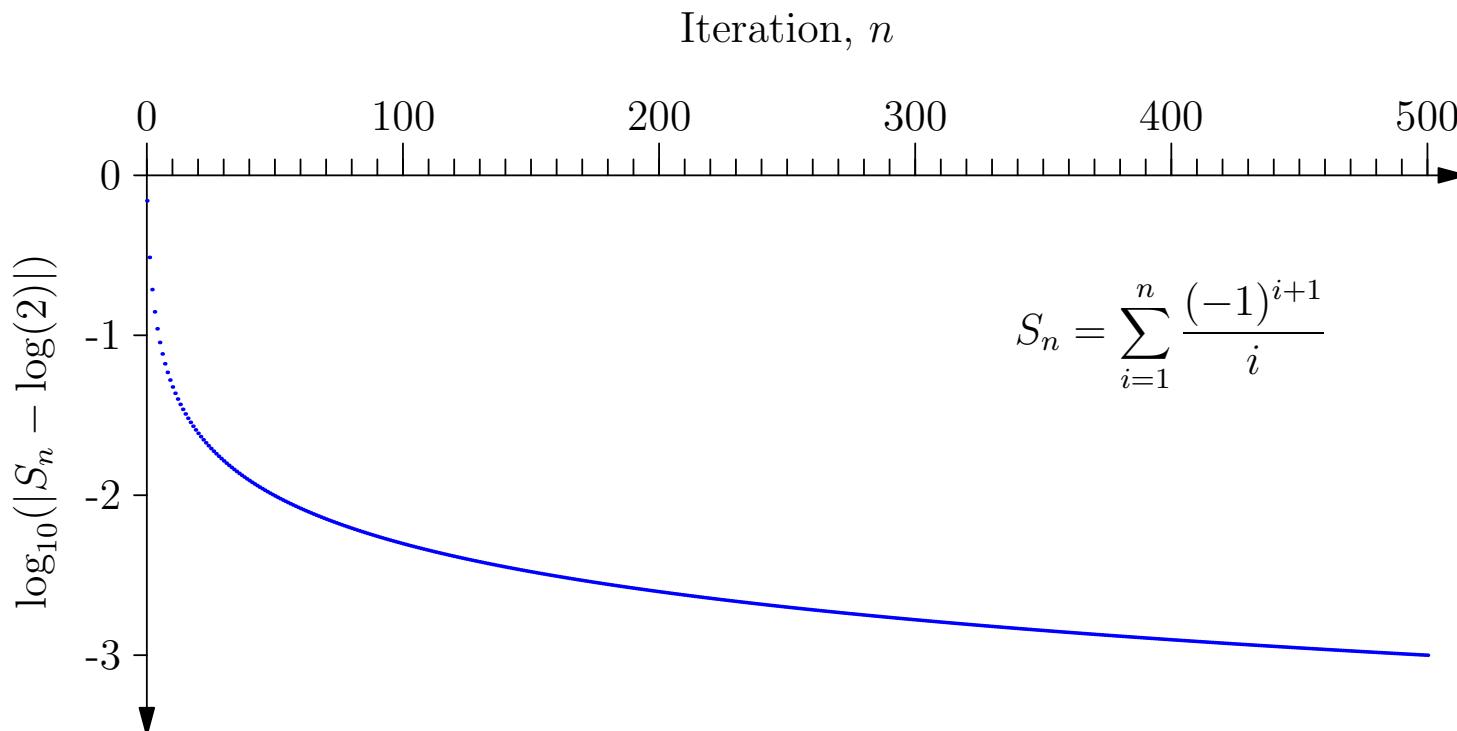


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- Many functions can be approximated by a sum
- We get a truncation error by taking only a finite number of elements
- We want the truncation error to be around machine accuracy
- For quick evaluation we need a strongly convergent series
- This often depend on the value of the argument we give to the function
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- Differential equations are used in many applications, for example in modelling the motion of object
- A typical equation of motion might be

$$\frac{d^2x(t)}{dt^2} = 2 \frac{dx(t)}{dt} + 3x(t)$$

- Which has a general solution  $x(t) = c_1 e^{-t} + c_2 e^{3t}$
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# Euler's Method

- To solve a differential equation we use an approximate update equation

$$x(t + \epsilon) \approx x(t) + \epsilon \dot{x}(t)$$

$$\dot{x}(t + \epsilon) \approx \dot{x}(t) + \epsilon \ddot{x}(t)$$

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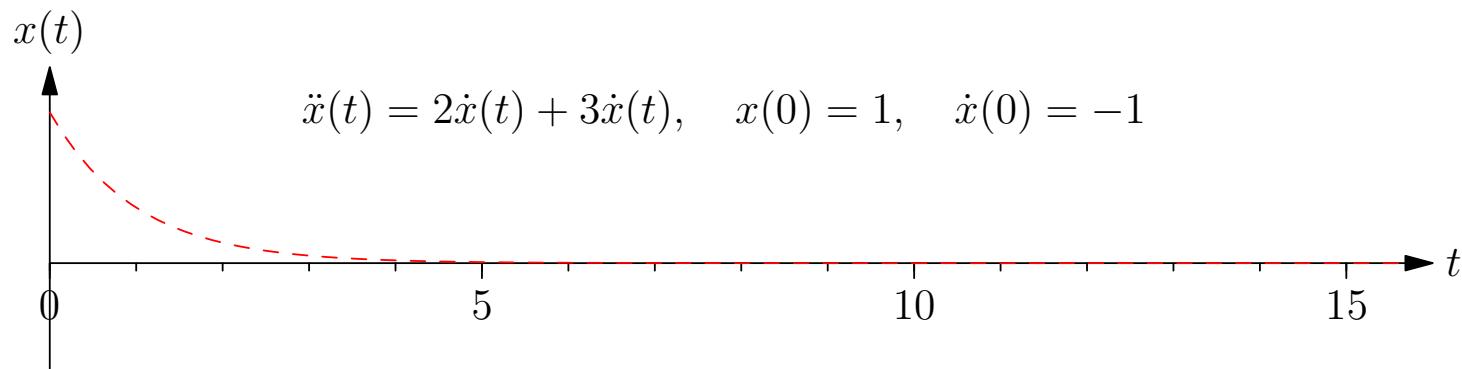
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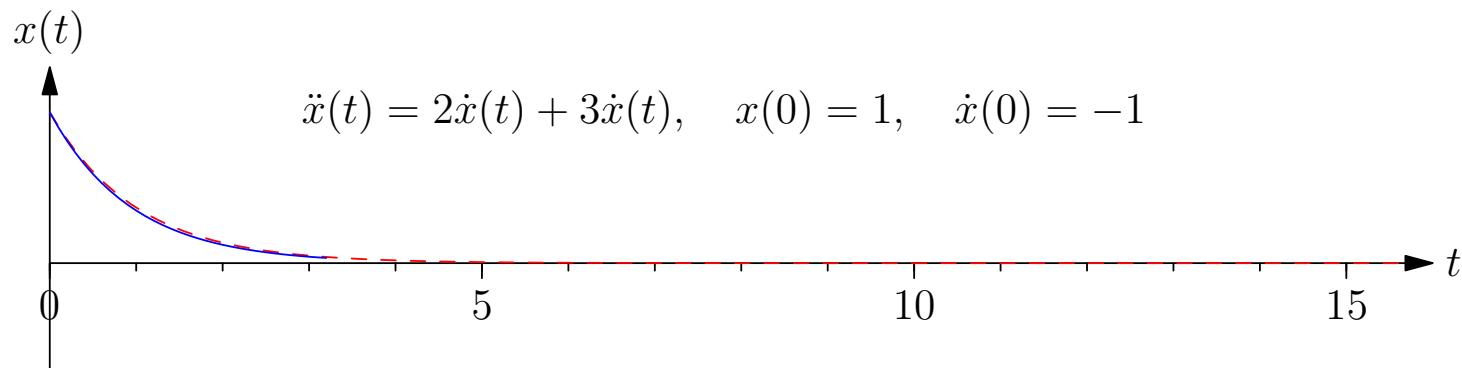
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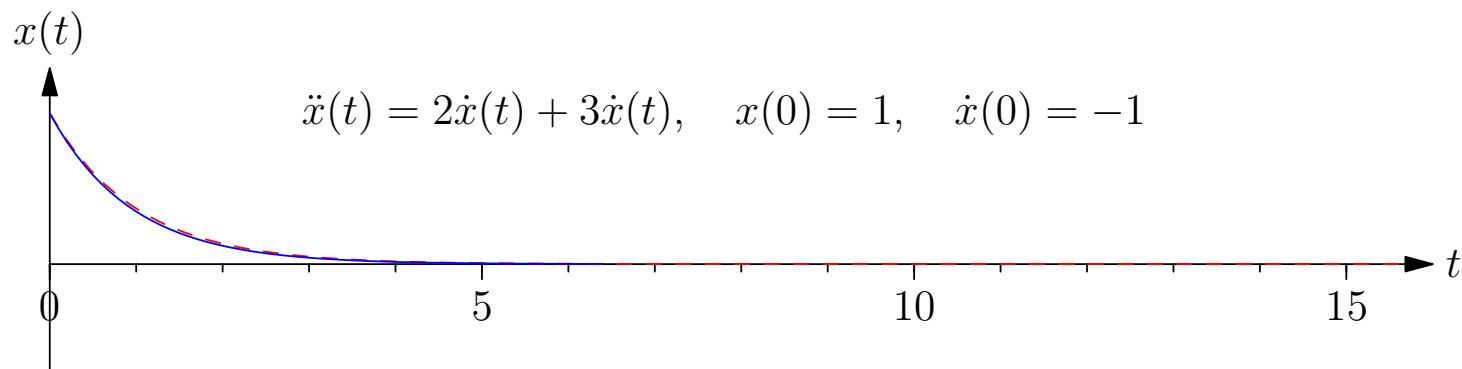
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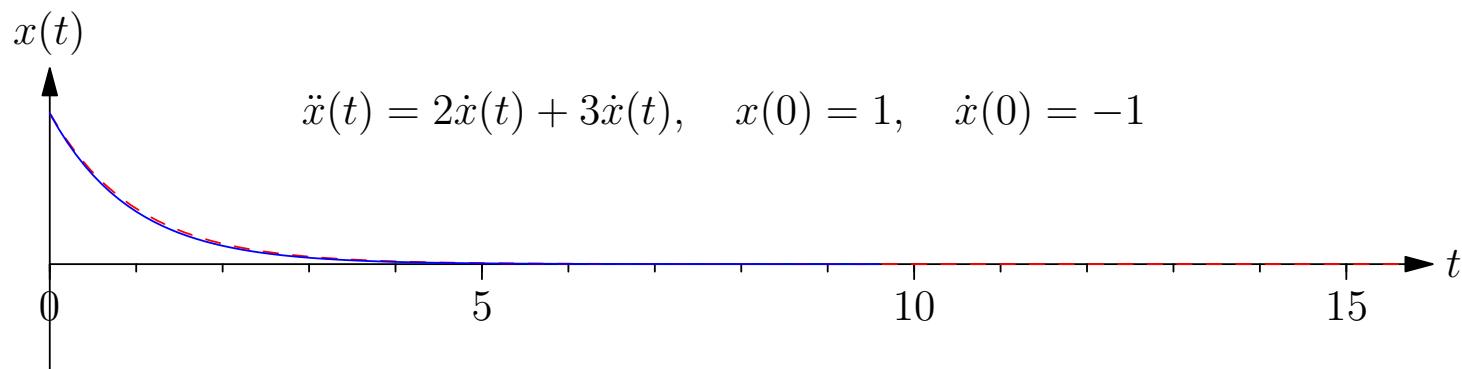
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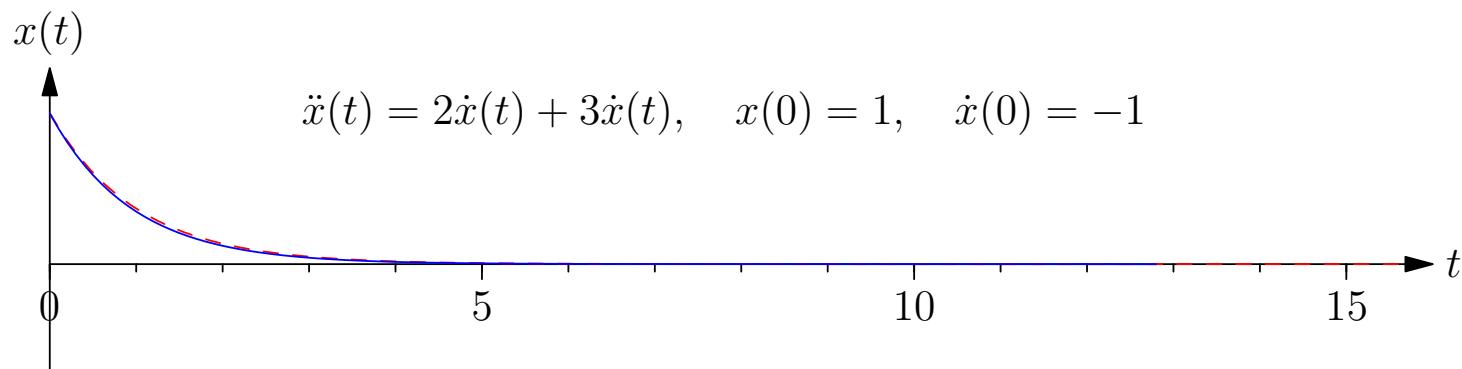
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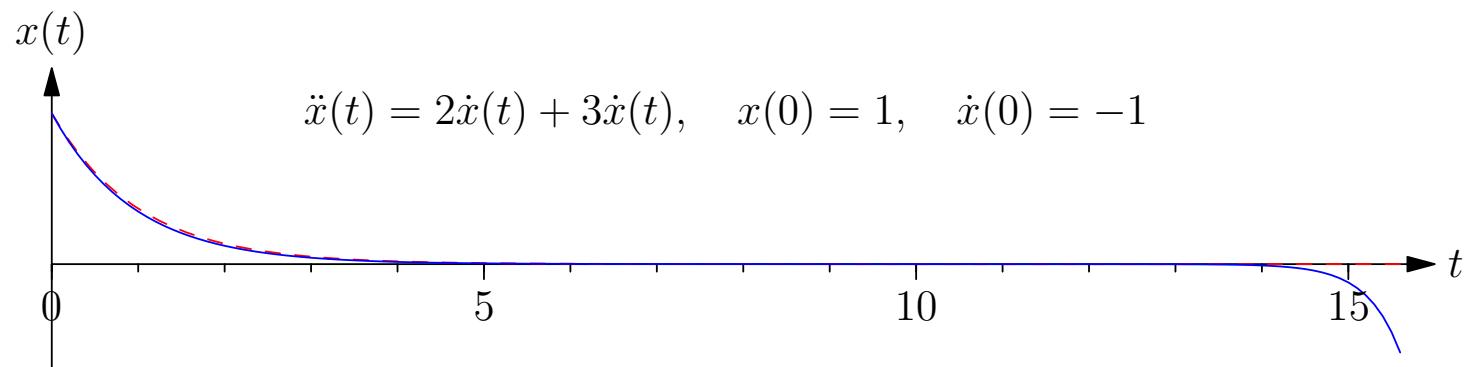
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# Stability

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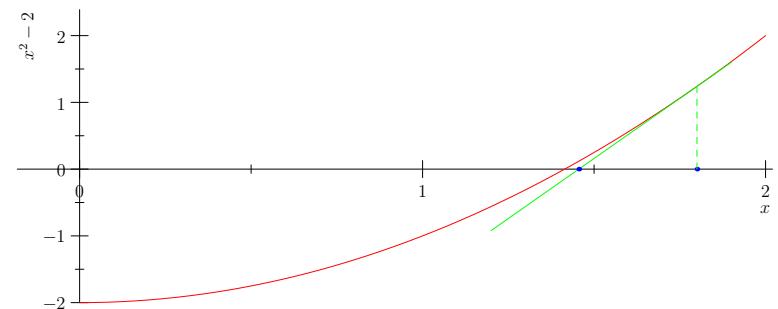
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# Outline

1. Numerical Approximations
2. Iterating to a Solution
3. Linear Algebra



# Solving Simultaneous Equations

- When problems involve many variables it is convenient to use matrices and vectors to store the numbers

$$\begin{aligned}3x + 2y &= 5 \\7x - 8y &= -11\end{aligned}\quad \begin{pmatrix} 3 & 2 \\ 7 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \end{pmatrix}$$

- Or  $\mathbf{Ax} = \mathbf{b}$  with solution  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- Linear algebra is an abstraction allowing mathematicians, scientists and engineers to write solutions at a higher level
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- There are a large number of problems with matrices that people care about
- The solution often depends on the problem
- These include
  - ★ Multiply matrices together
  - ★ Solving linear equations  $\mathbf{A}\mathbf{x} = \mathbf{b}$
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$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 6 \\ 3 & 5 & 9 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.75 & 1 & 0 \\ 0.25 & 0.428 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 6 \\ 0 & 3.5 & 4.5 \\ 0 & 0 & -4.28 \end{pmatrix} = \mathbf{LU}$$

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# LU-Decomposition

- LU-decomposition is achieved by Gaussian-elimination
- This is a straightforward procedure, but if done carelessly can lead to large rounding errors
- The standard solution is to permute the rows of the matrix (aka pivoting) to prevent loss of accuracy
- In addition we can “polish” solutions

$$\mathbf{A}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{b} = \boldsymbol{\epsilon}$$

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# Norms

- With some work we can get a good approximation to  $x$  such that  $\mathbf{A}x = b$
- But what if we have some error in  $b$ , this induces an error  $\delta x = \mathbf{A}^{-1} \delta b$
- How big is  $\delta x$ ?
- To measure the size of a vector we use a norm  $\|\delta x\|$ , which is a number encoding the size of  $\delta x$
- There are a number of different norms, e.g.

$$\|\delta x\|_2 = \sqrt{\delta x_1^2 + \cdots + \delta x_n^2}, \quad \|\delta x\|_1 = |\delta x_1| + \cdots + |\delta x_n|$$

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- The size of the error in  $x$ :  $\mathbf{A}x = b$  when  $b$  has error  $\delta b$  is

$$\|\delta x\| = \|\mathbf{A}^{-1}\delta b\| \leq \|\mathbf{A}^{-1}\| \|\delta b\|$$

- Where  $\|\mathbf{A}^{-1}\|$  provides a measure of the size of the error in the worst case
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- Care has to be taken to ensure that needless operations (such as inverting a matrix) are not done
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- The idea of poor conditioning (errors being amplified) is useful in understanding many numerical tasks

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# Lessons

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- Don't avoid numerical algorithms, they are hugely important with vast areas of applications
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