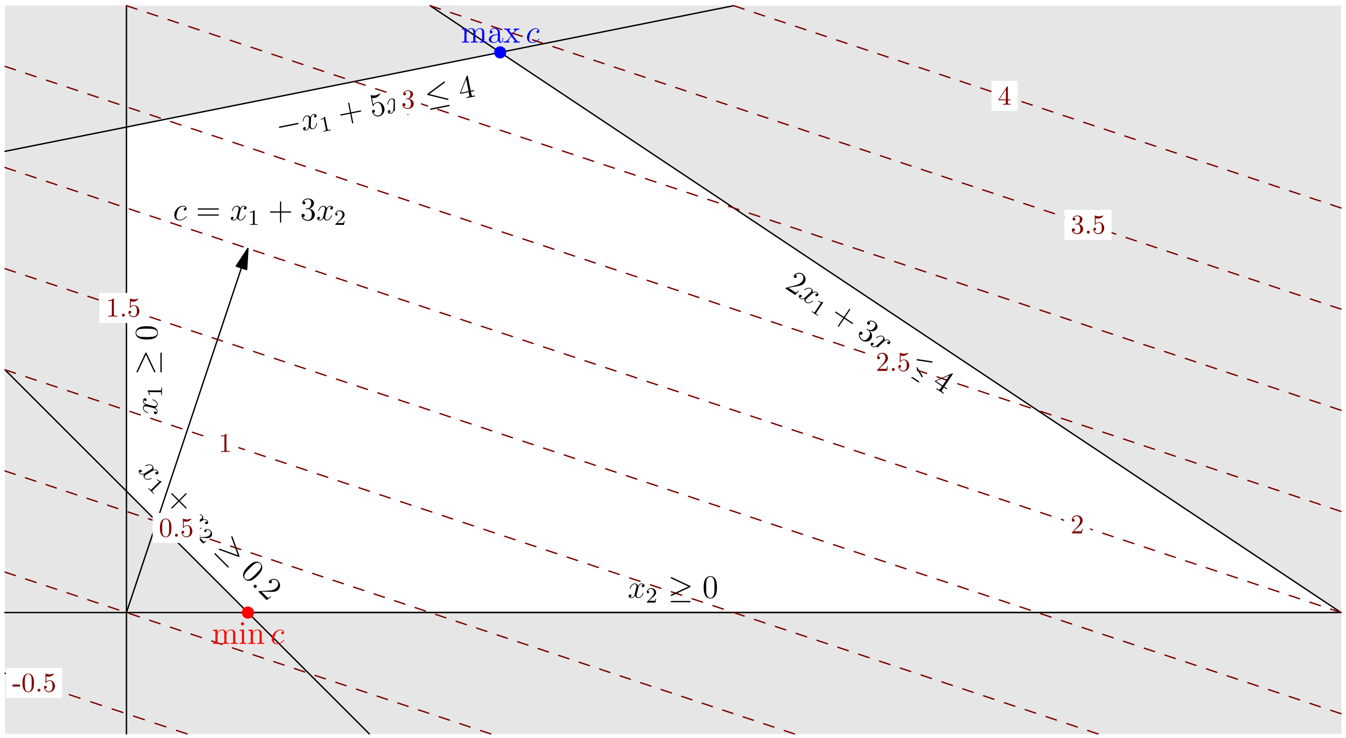
# Algorithms and Analysis

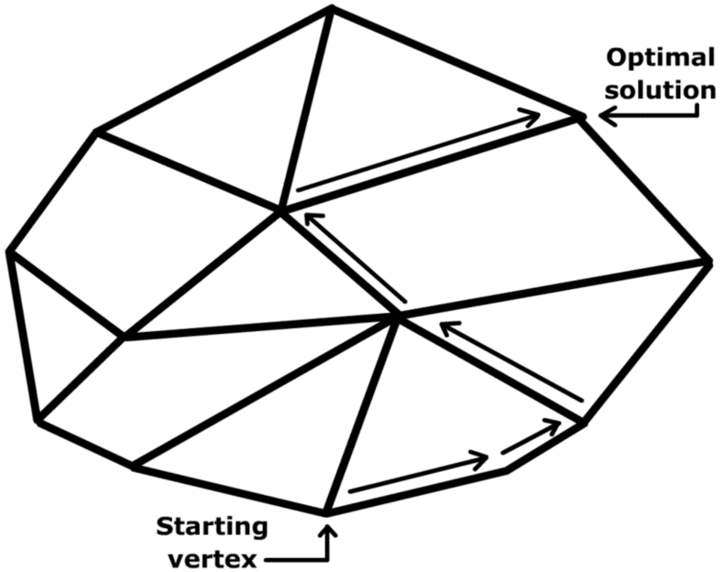
## Lesson 27: *Use Linear Programmings*




*linear programming, applications*

# Outline

1. **Examples**
2. Linear Programs
3. Properties of Solution
4. Normal Form



# Going Shopping

- Suppose we have a number of food stuffs which we label with indices $f \in \mathcal{F}$
- The price of food stuff $f$ per kilogram we denote $p_f$
- We are interested in buying a selection of foods $\boldsymbol{x} = (x_f | f \in \mathcal{F})$ where $x_f$ is the quantity (in kg) of food $f$
- We want to minimise the total price $\sum_f p_f \, x_f = \boldsymbol{p} \cdot \boldsymbol{x}$
- However we want to ensure that the food has enough vitamins

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# Nutrition

- We consider the set of vitamins $\mathcal{V}$
- Let $A_{v\,f}$ be the quantity of vitamin $v$ in food stuff $f$
- Let $b_v$ be the minimum daily requirement of vitamin $v$
- We therefore require

$$\forall v \in \mathcal{V} \qquad \sum_{f \in \mathcal{F}} A_{v\,f}\, x_f \geq b_v$$

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- We can write the food shopping problem as

$$\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \quad \text{subject to} \quad \mathbf{A}\boldsymbol{x} \geq \boldsymbol{b} \quad \text{and} \quad \boldsymbol{x} \geq \mathbf{0}$$

- Note that the inequalities involving vectors means that each component must be satisfied, i.e.

$$\begin{aligned} \mathbf{A}\boldsymbol{x} \geq \boldsymbol{b} \quad &\Rightarrow \quad \forall v \in \mathcal{V} \quad \sum_{f \in \mathcal{F}} A_{vf}\, x_f \geq b_v \\ \boldsymbol{x} \geq \mathbf{0} \quad &\Rightarrow \quad \forall f \in \mathcal{F} \quad x_f \geq 0 \end{aligned}$$

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# Transportation

- We consider a set of factories $\mathcal{F}$ producing a set of commodities $\mathcal{C}$
- The amount of commodity $c$ produced by factory $f$ we denote by $x_{cf}$
- The shipping cost of commodity $c$ from factory $f$ to the retailer of $c$ we denote by $p_{cf}$
- We want to choose $x_{cf}$ to minimise the transportation costs

$$\sum_{c \in \mathcal{C}, f \in \mathcal{F}} p_{cf}\, x_{cf}$$

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# Constraints

- Each factory can only produce a certain overall tonnage of commodities

$$\sum_{c \in \mathcal{C}} x_{cf} \leq b_f \qquad \forall f \in \mathcal{F}$$

where $b_f$ is the maximum production capacity of factory $f$

- The total demand for each commodity is $d_c$ so

$$\sum_{f \in \mathcal{F}} x_{cf} = d_c \qquad \forall c \in \mathcal{C}$$

- We can only produce positive amounts, i.e. $x_{cf} \geq 0$

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# Linear Program

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subject to

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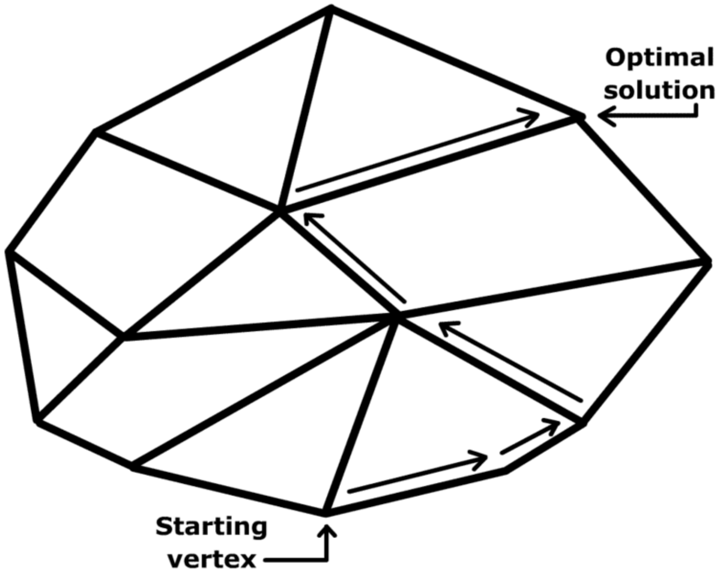

# General Linear Programs

- Linear programs are problems that can be formulated as follows

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subject to

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# Maximising

- We can also maximise rather than minimise
- Whether we want to maximise or minimise will depend on the application
- Note that

$$\max_{\boldsymbol{x}} \boldsymbol{c} \cdot \boldsymbol{x} \quad \equiv \quad \min_{\boldsymbol{x}} (-\boldsymbol{c}) \cdot \boldsymbol{x}$$

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# Linear Program Applications

- A huge number of problems can be mapped to linear programming problems
- Or modelled as linear (even when they're not, e.g. oil extraction)
- Realistic problems might have many more constraints and large number of variables
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# Key Features

- There are three key features of linear programs

  1. The cost (objective function) is linear in $x_i$ ($\boldsymbol{c} \cdot \boldsymbol{x}$)
  2. The constraints are linear in $x_i$ (e.g. $\mathbf{A}_1 \boldsymbol{x} \leq b_1$)
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# History

- Linear programming was “invented” by Leonid Kantorovich in 1939 to help Soviet Russia maximise its production
- It was kept secret during the war, but was finally made public in 1947 when George Dantzig published the **simplex method** which still today is a standard method for solving linear programs
- John von Neumann developed the idea of duality (you can turn a maximisation problem for a set of variables $\boldsymbol{x}$ into a minimisation problem for a dual set of variables $\boldsymbol{\lambda}$ associated with each constraint)
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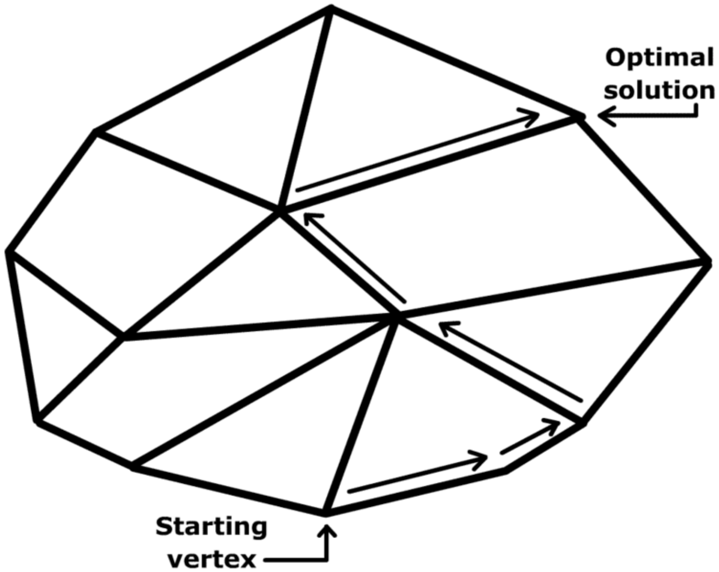

# Structure of Linear Programs

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- The set of $x$ that satisfy all the constraints is known as the set of **feasible solutions**
- The set of feasible solutions may be empty in which case it is impossible to satisfy all the constraints
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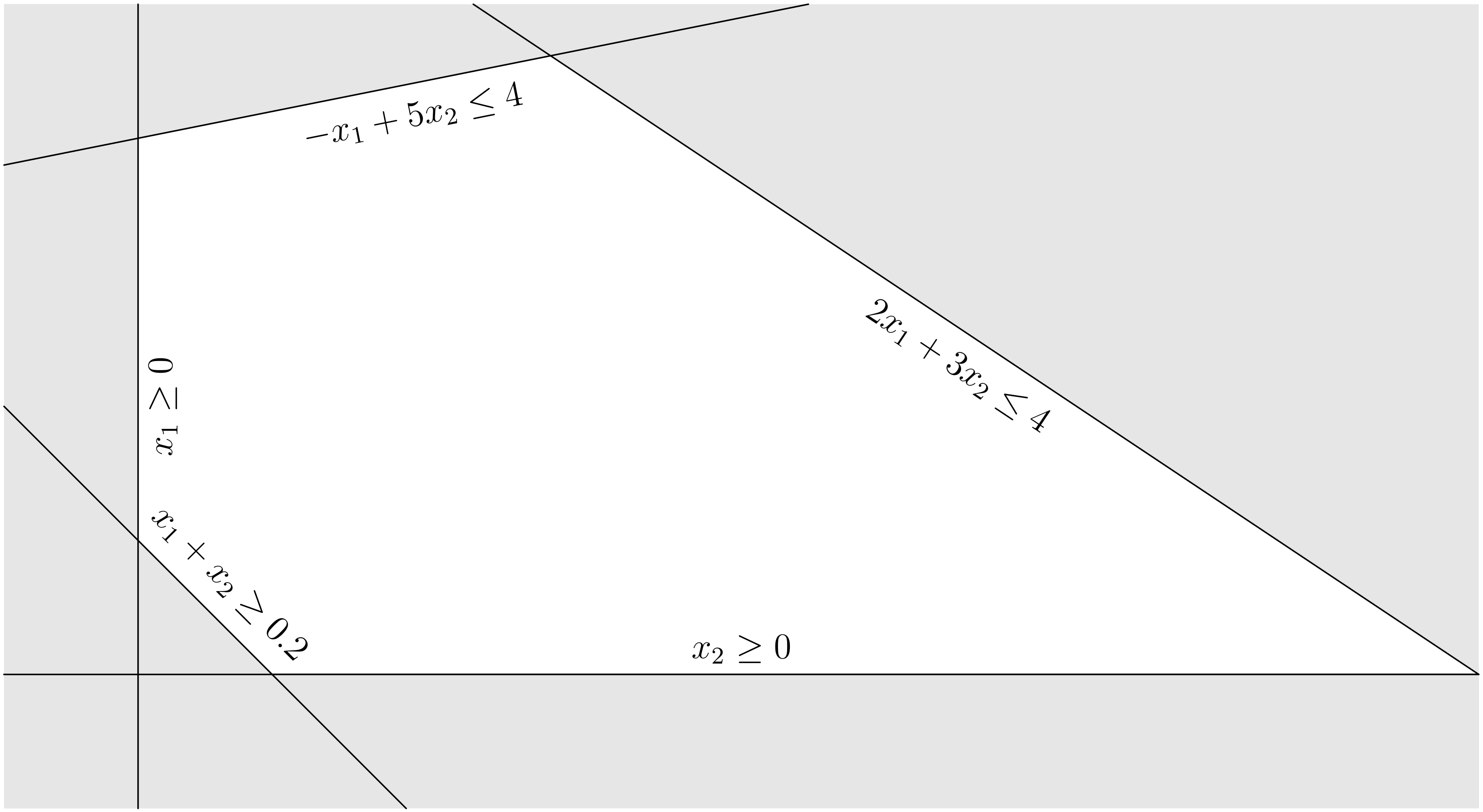

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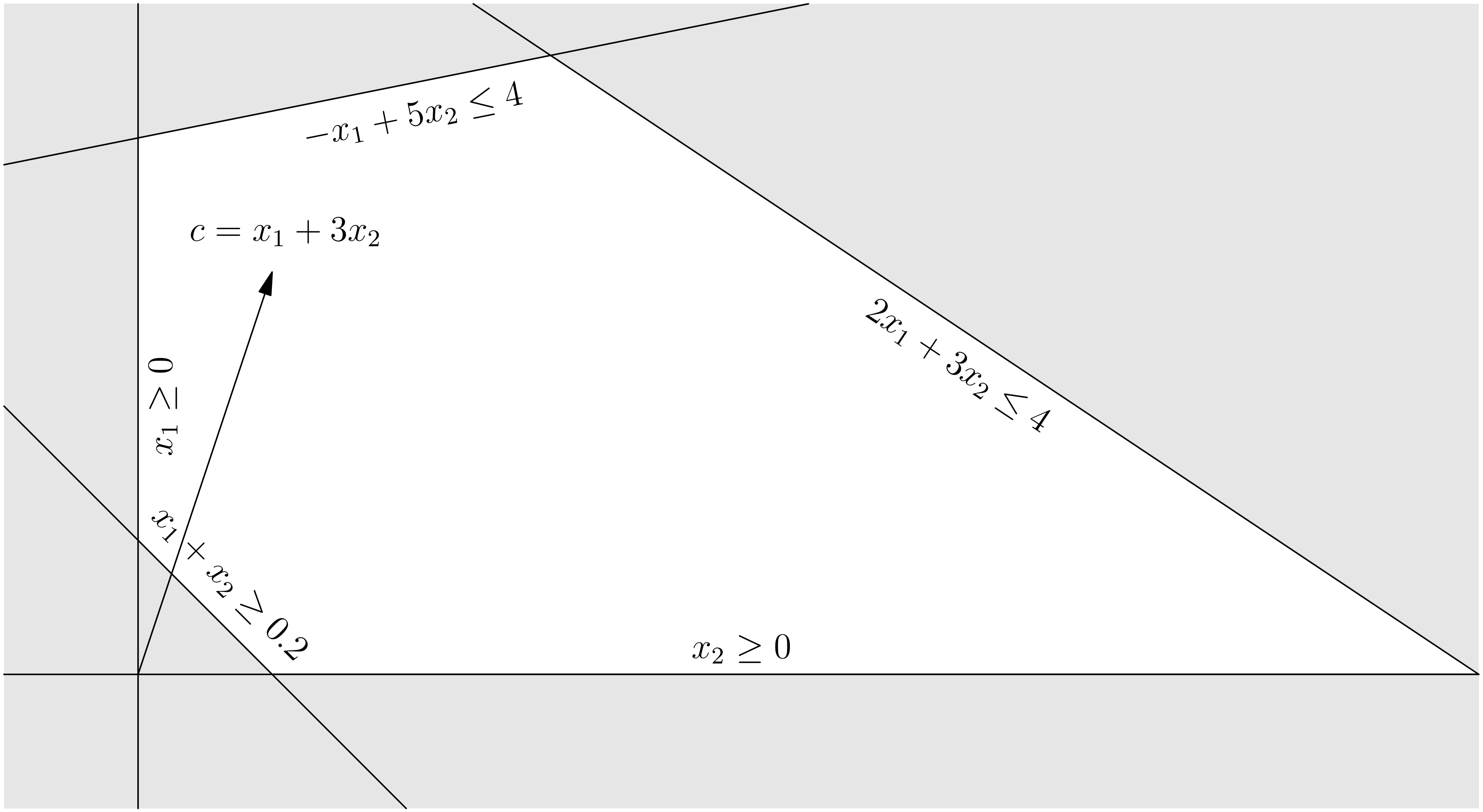

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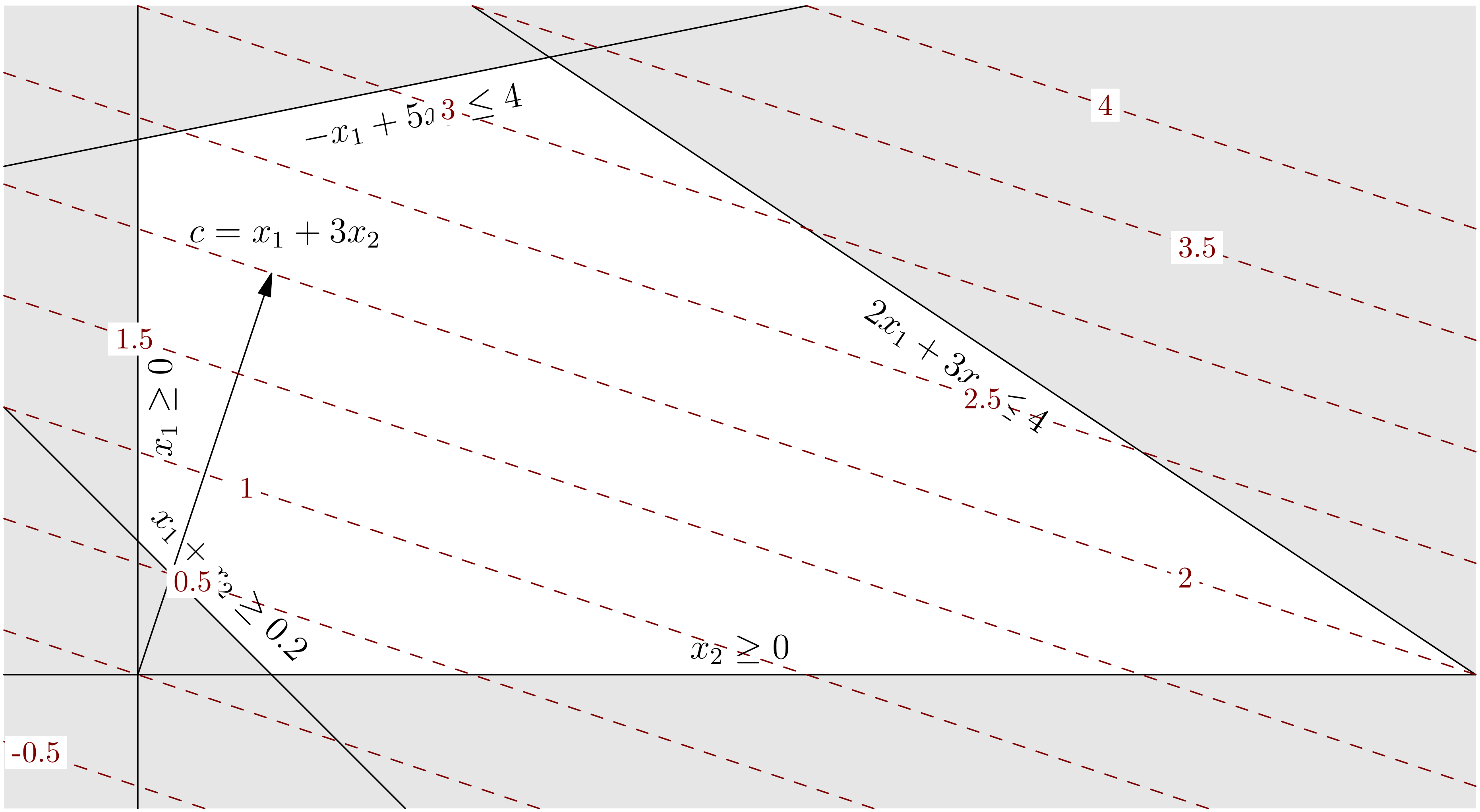

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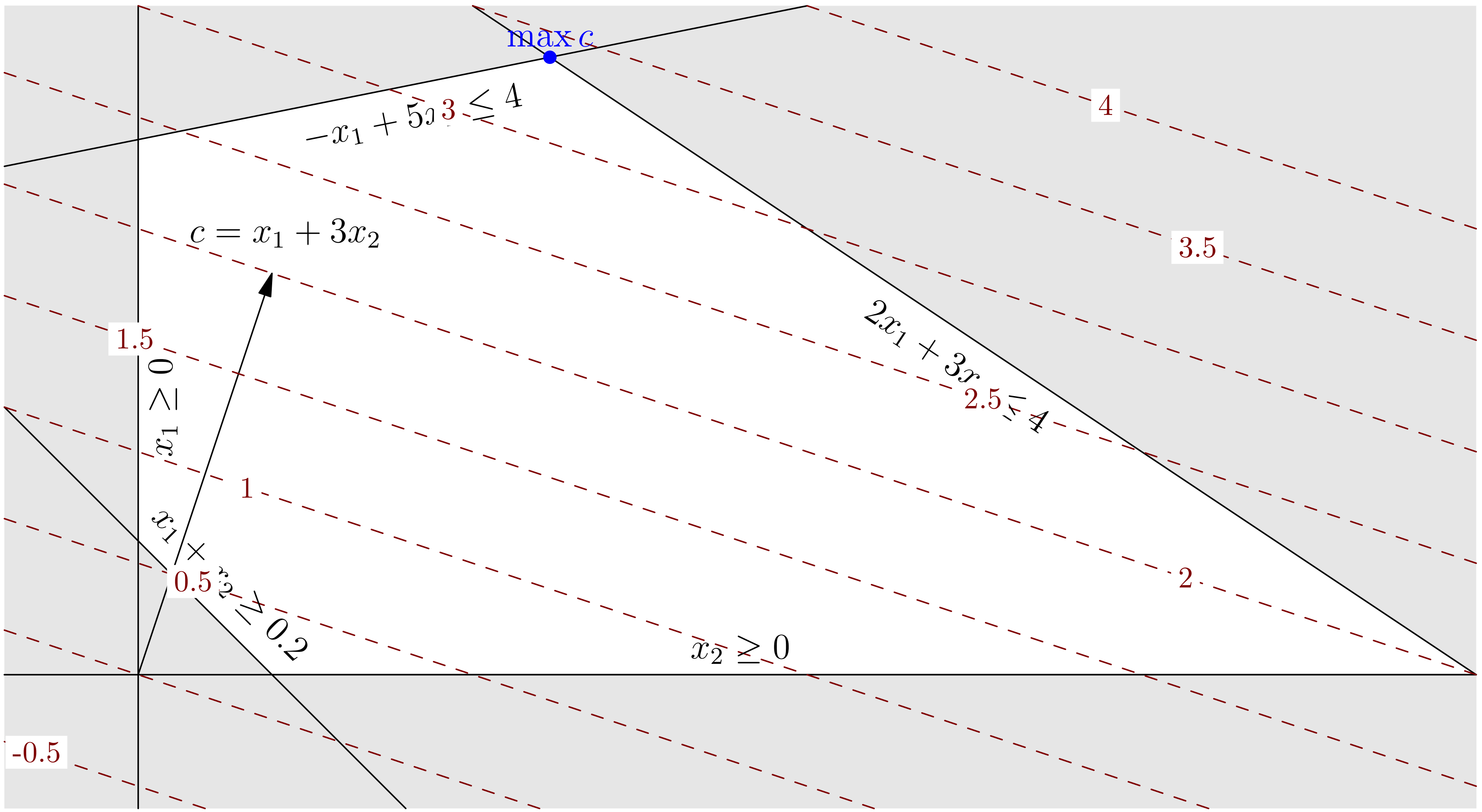

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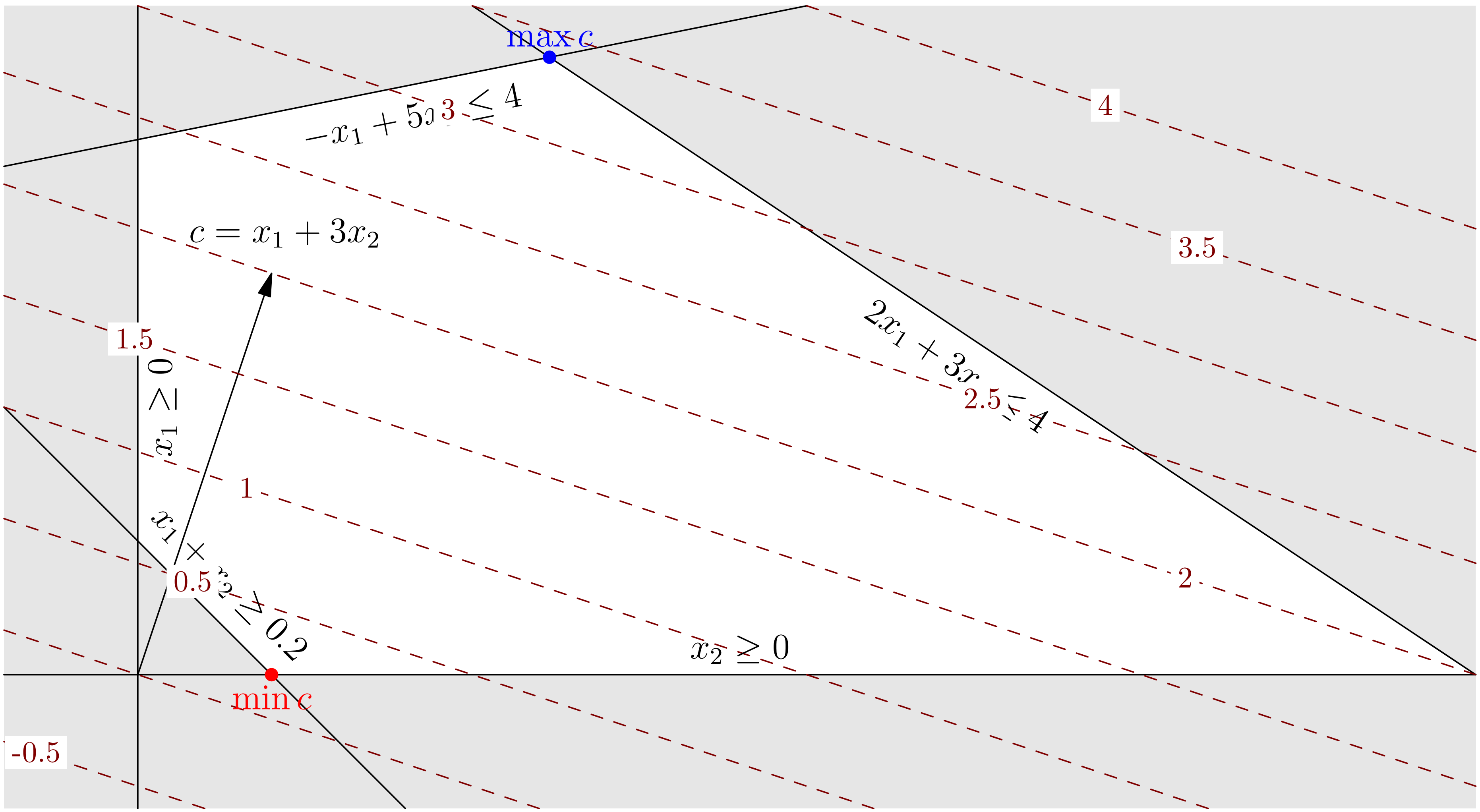

# Vertices of Polytope

- The space of feasible solutions is a polyhedra or polytope
- The maximum or minimum solution will always lie at a vertex of the polytope
- Our solution policy will be to start at a vertex and move to a neighbouring vertex that gives the best improvement in cost
- When this isn't possible then we are finished
- However, there is still a lot of work to realise this solution strategy

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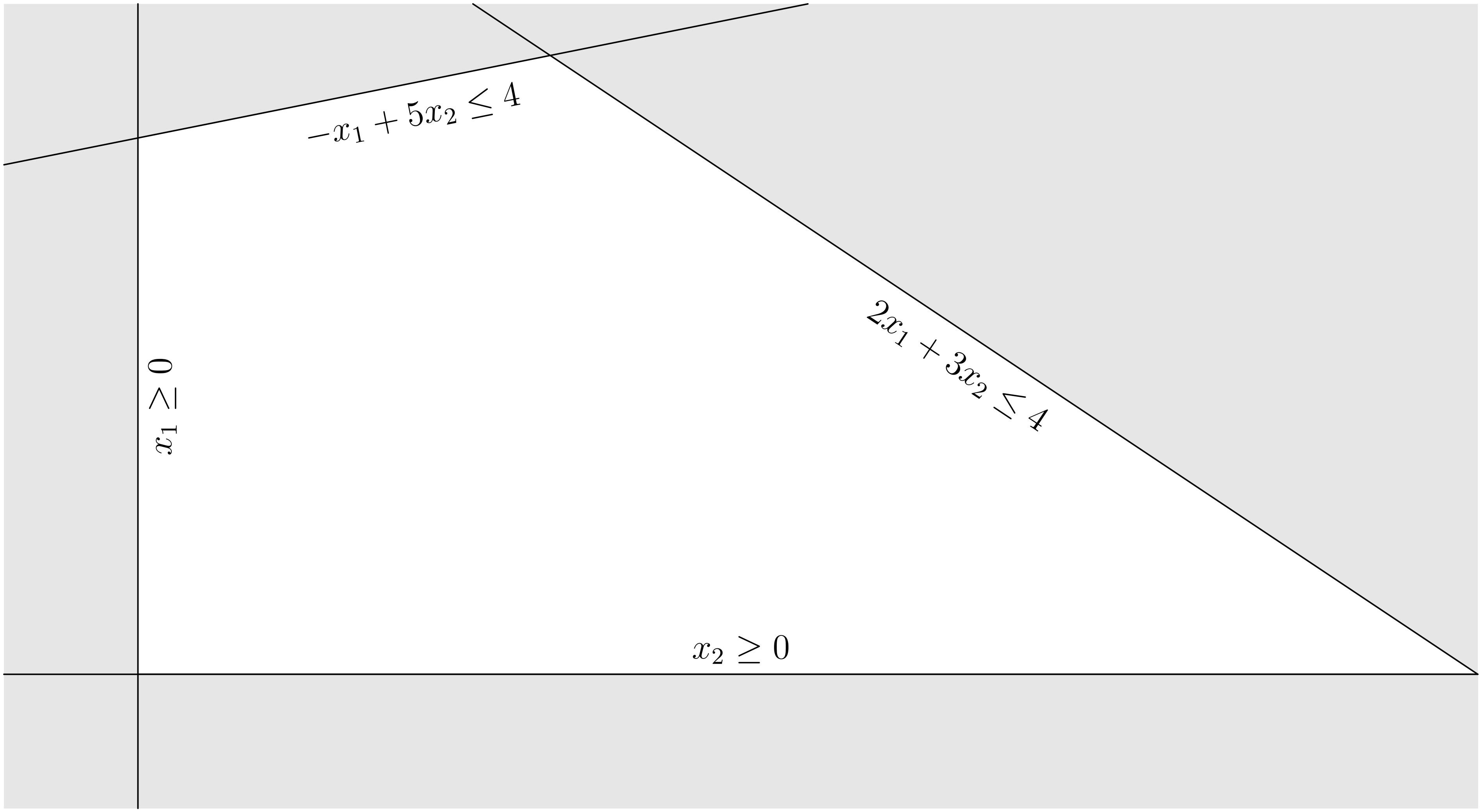

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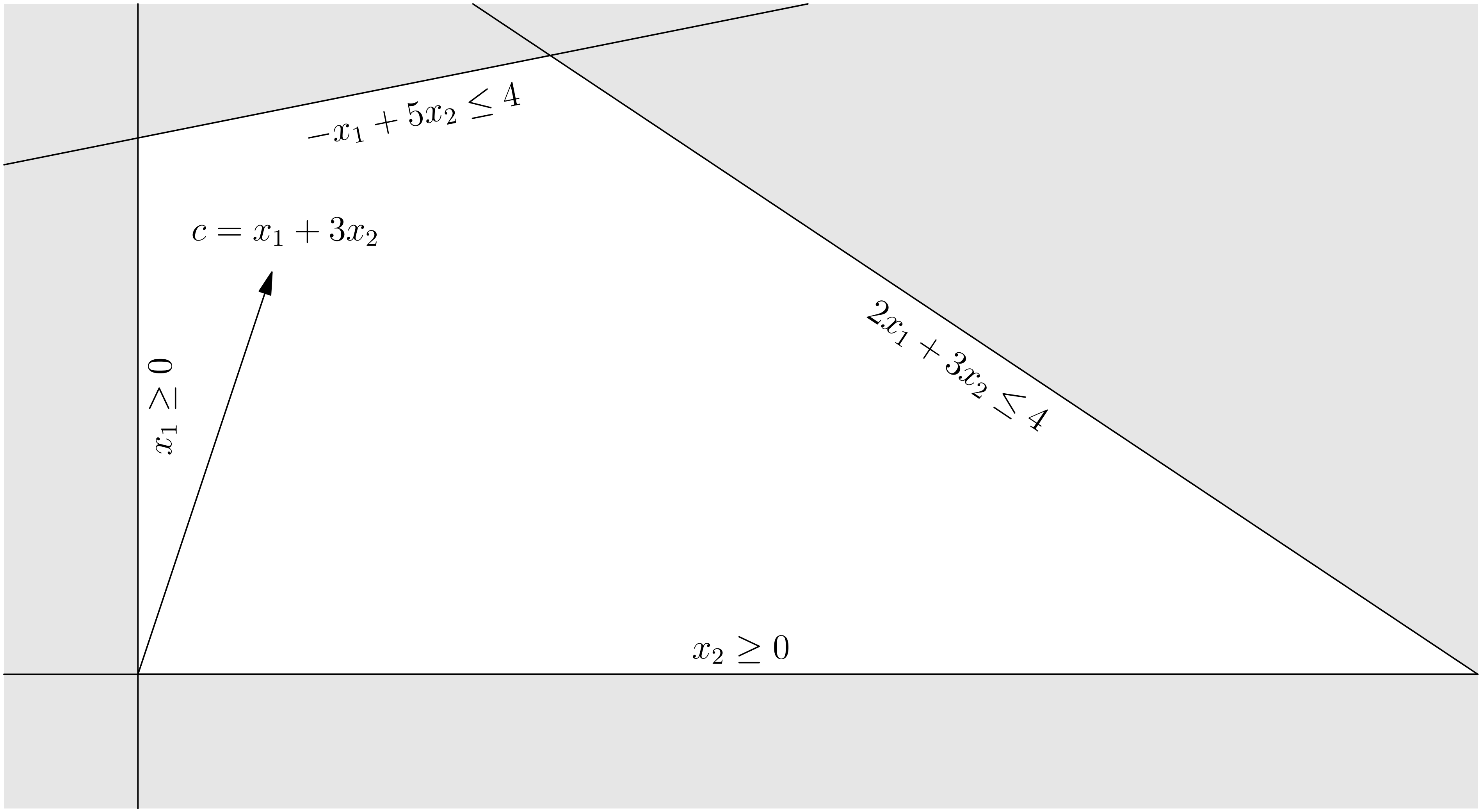

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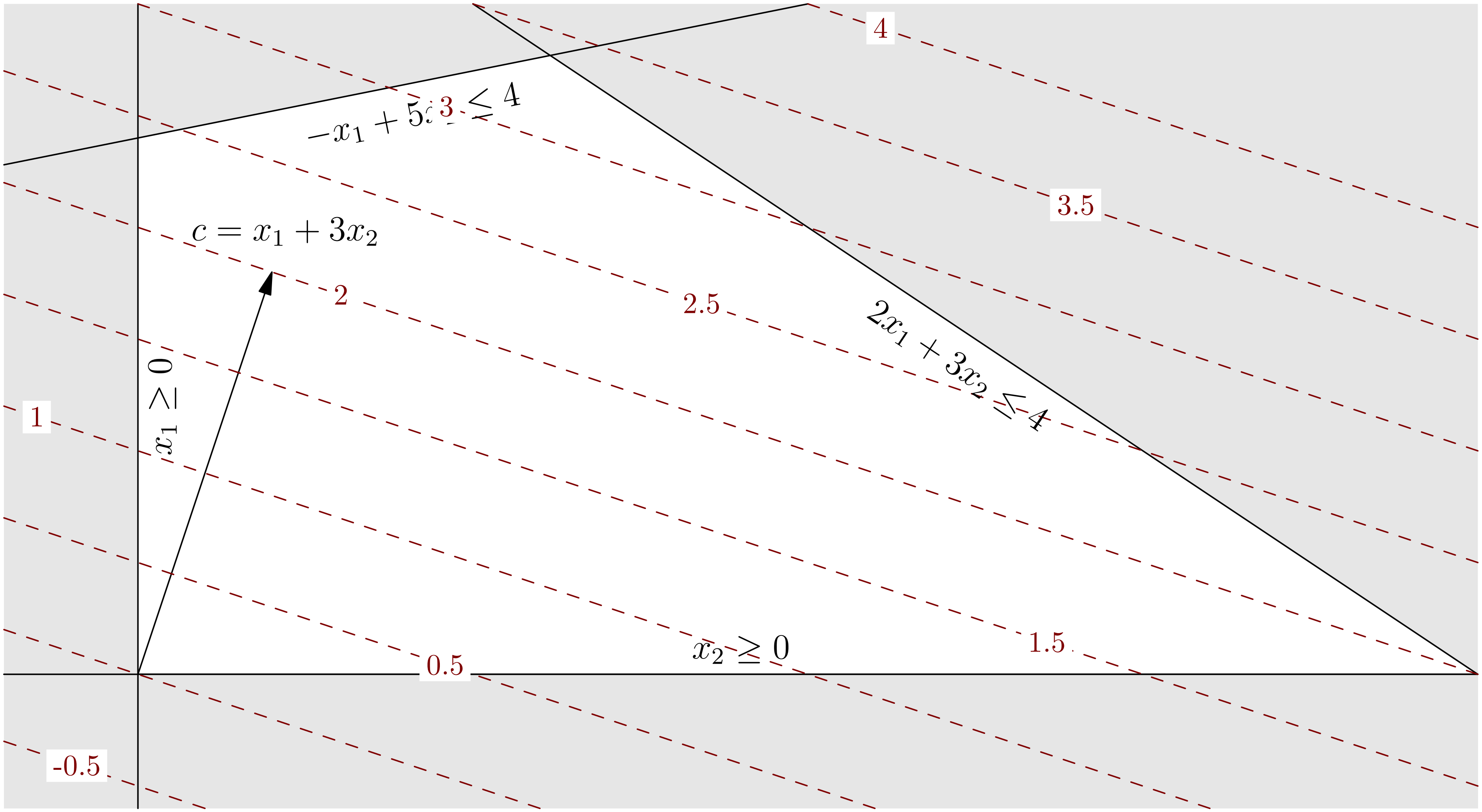

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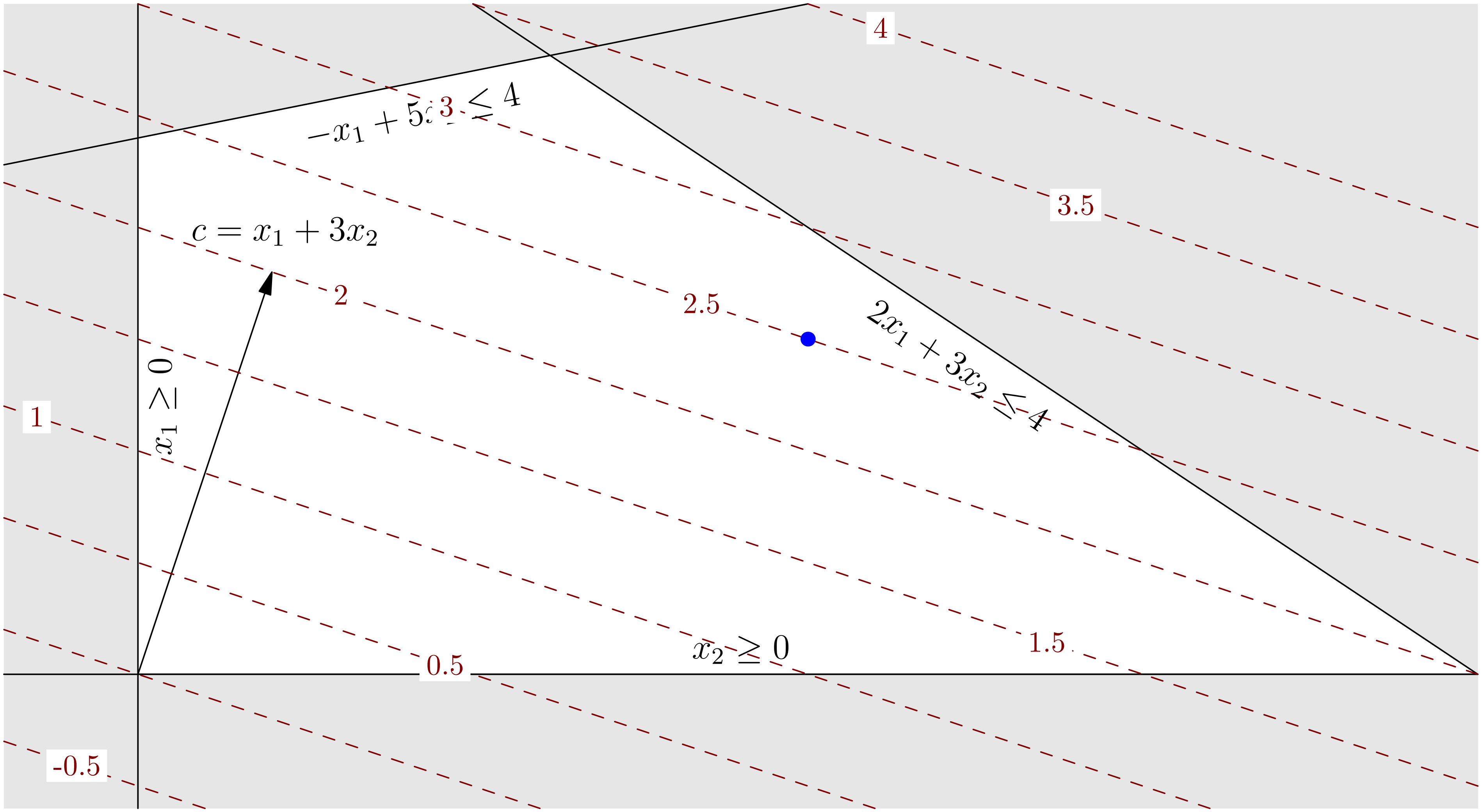

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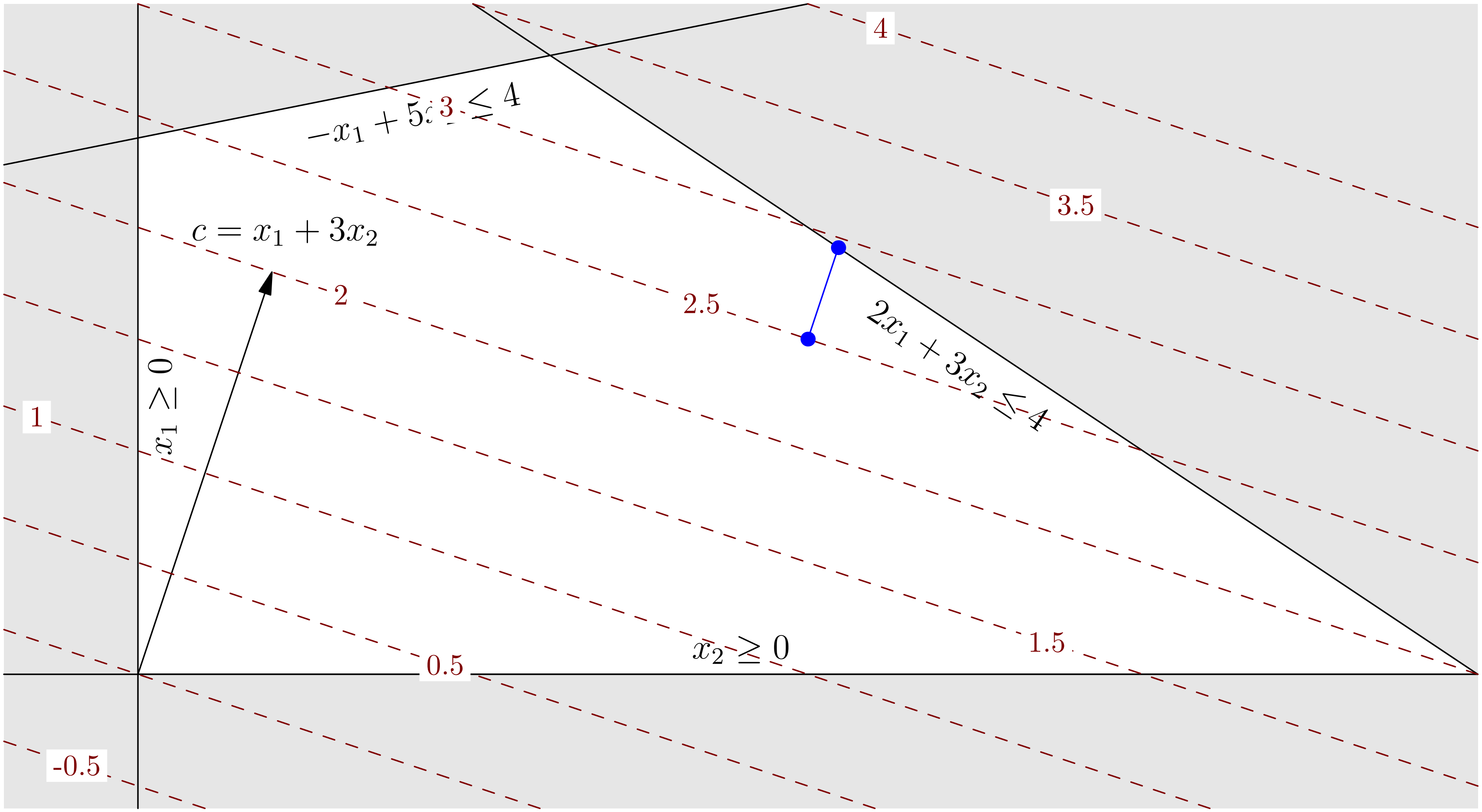

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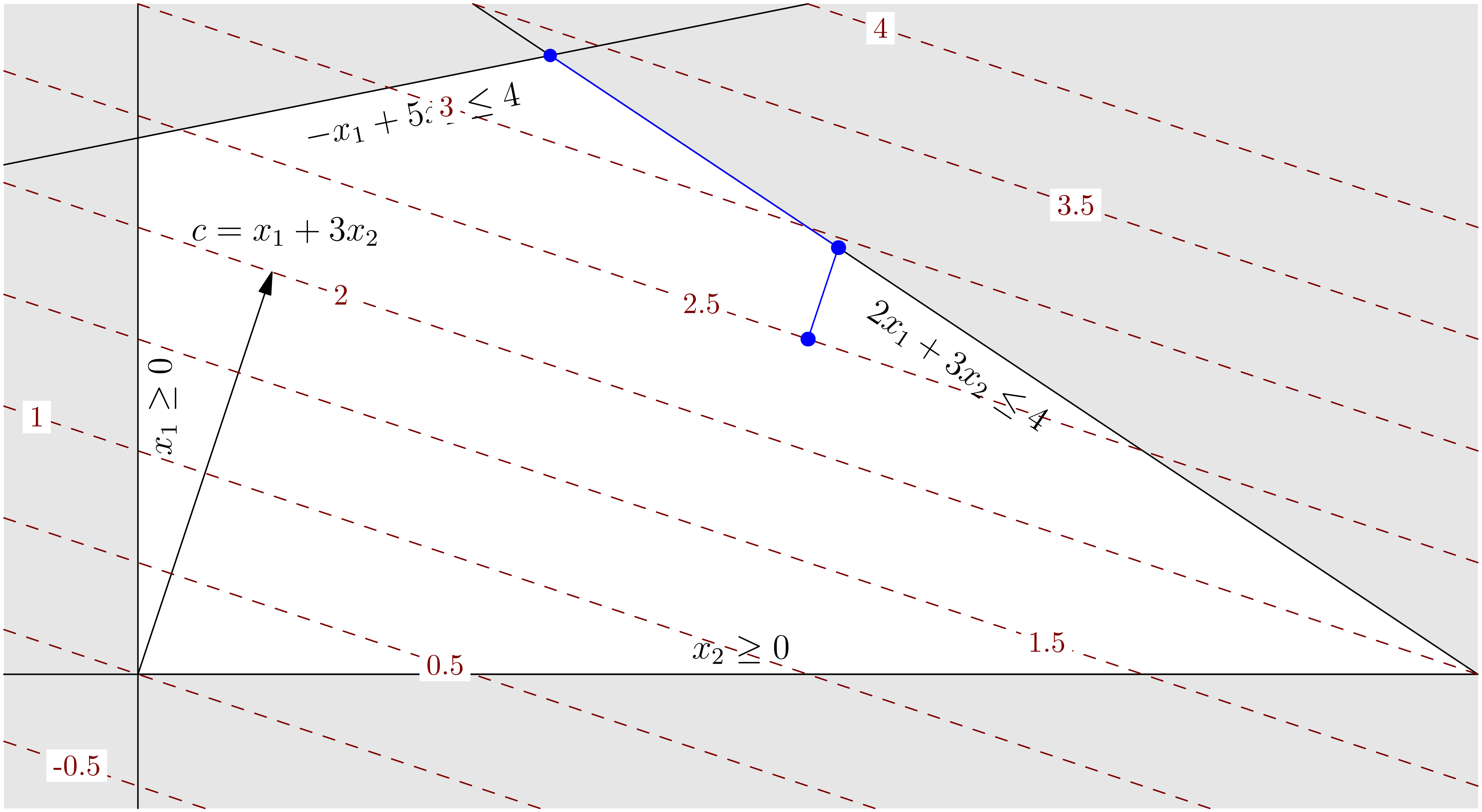

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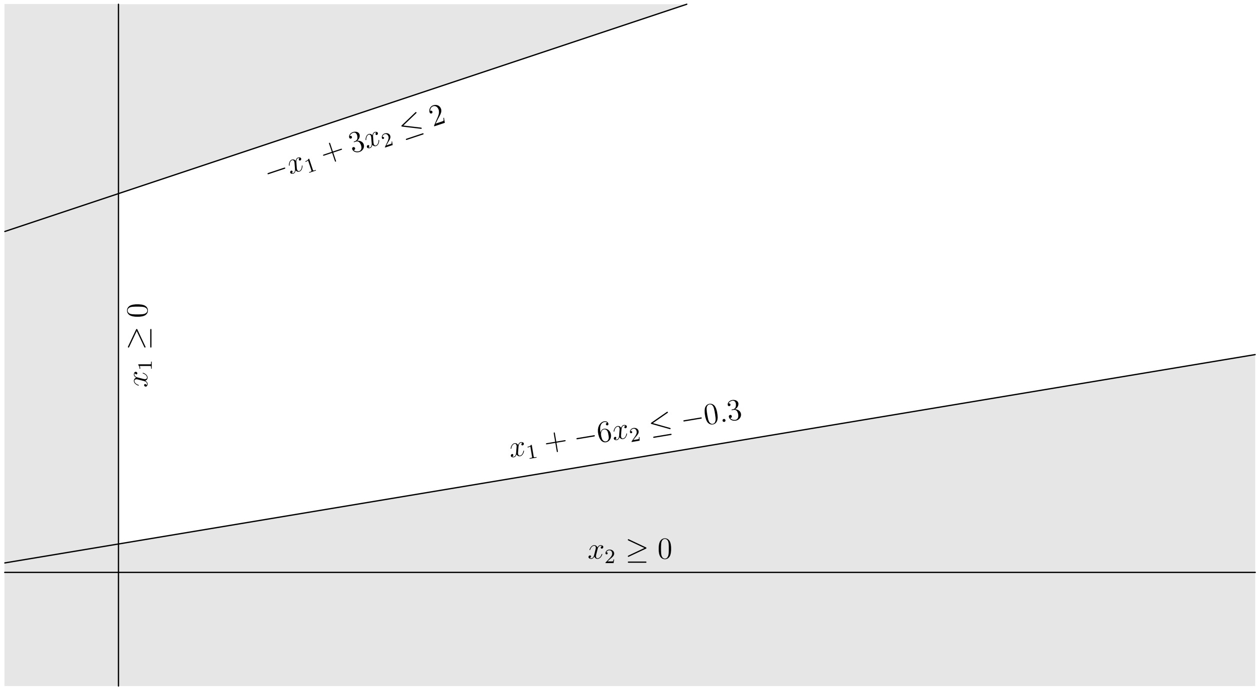

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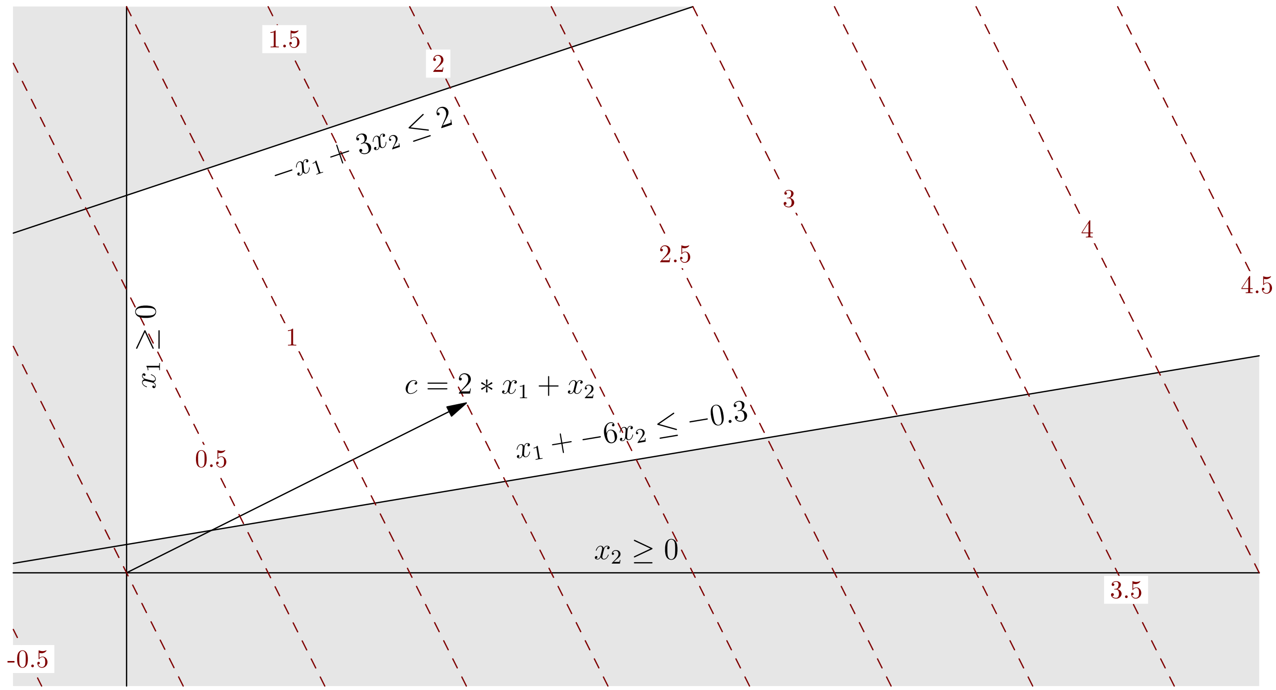

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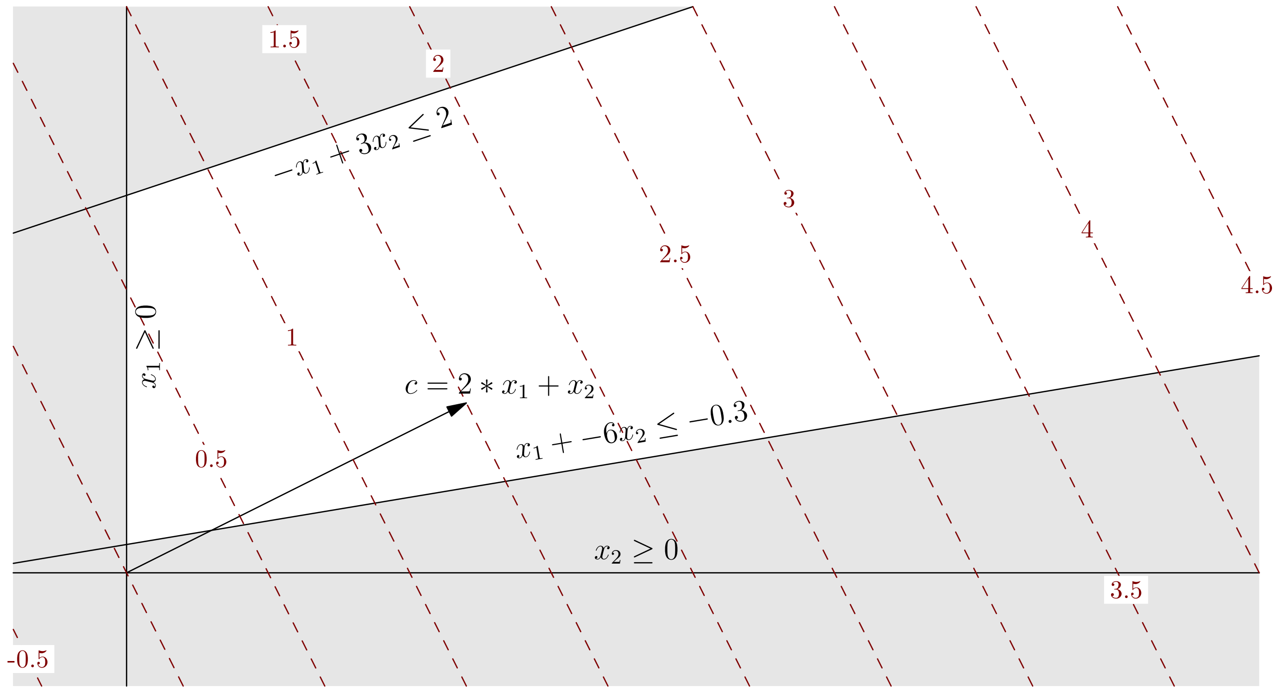
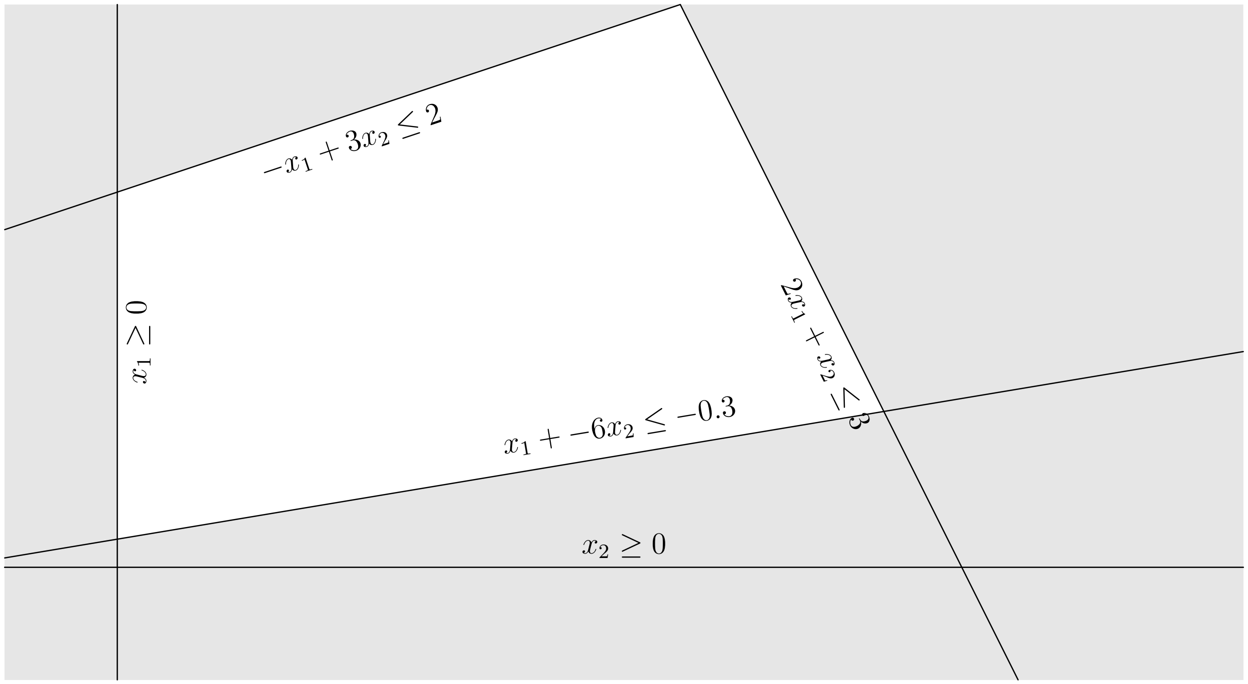



- But usually this would not happen because of the problem definition

# Multiple Solutions

- You can also get multiple solutions if a constraint is orthogonal to the objective function



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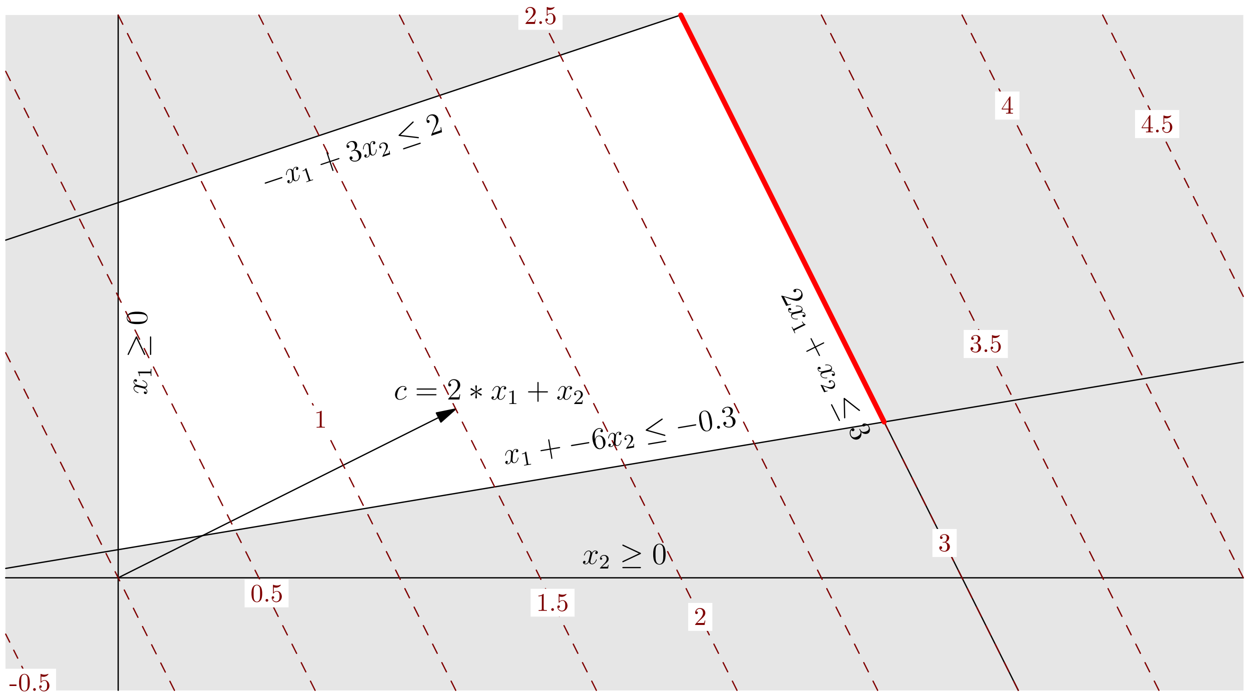

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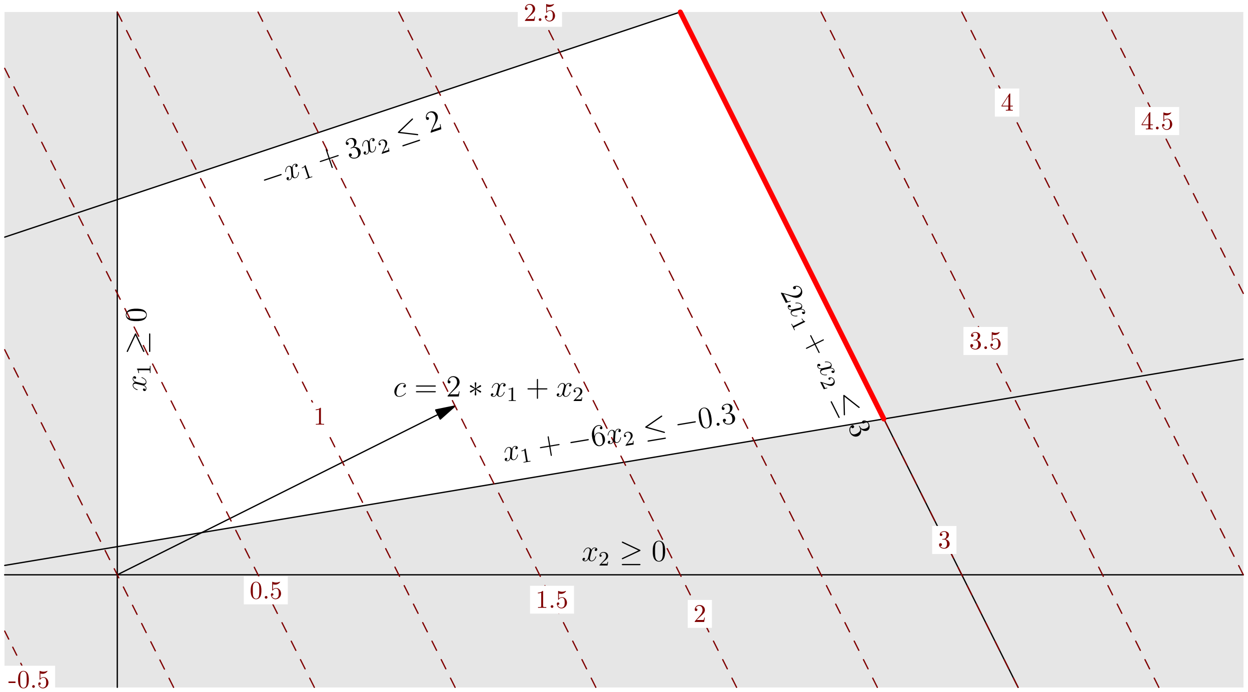
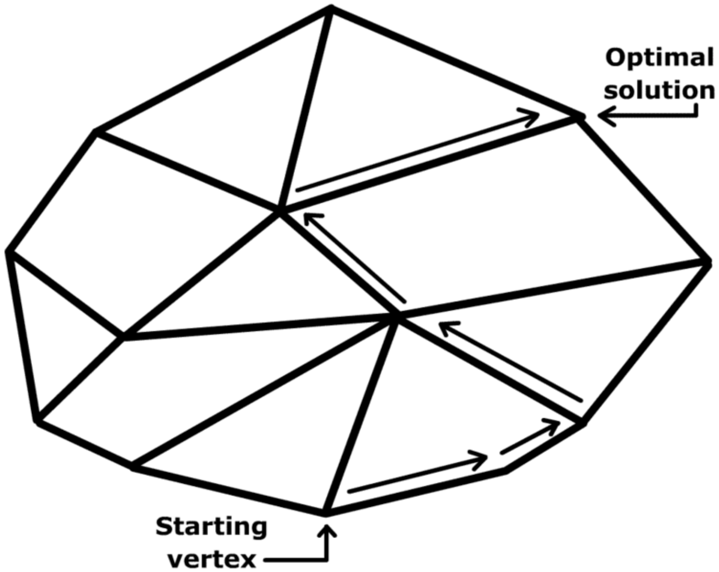



- Nevertheless the optimal will be at a vertex

# Outline

1. Examples
2. Linear Programs
3. Properties of Solution
4. **Normal Form**



# Converting Linear Programs

- Solving full linear programs is difficult
- However, it is much easier to solve linear programs in **normal form**
- This is basically a form where we get rid of all inequalities and rewriting the equalities
- Fortunately its rather easy to convert linear programs to normal form

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# Slack Variables

- We can change an inequality into an equality by introducing a new "**slack**" variable

- E.g.

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\begin{aligned}
\boldsymbol{a}_1 \cdot \boldsymbol{x} \geq 0 \qquad &\Rightarrow \qquad \boldsymbol{a}_1 \cdot \boldsymbol{x} - z_1 = 0 \quad z_1 \geq 0 \\
\boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 0 \qquad &\Rightarrow \qquad \boldsymbol{a}_2 \cdot \boldsymbol{x} + z_2 = 0 \quad z_2 \geq 0
\end{aligned}
$$

  $z_1$ (the excess) and $z_2$ (the deficit) are known as slack variables

- We eliminate inequalities at the expense of increasing the number of variables

- We can treat the slack variables on an equivalent footing to the normal variables (they just provide a different way of describing the original problem)

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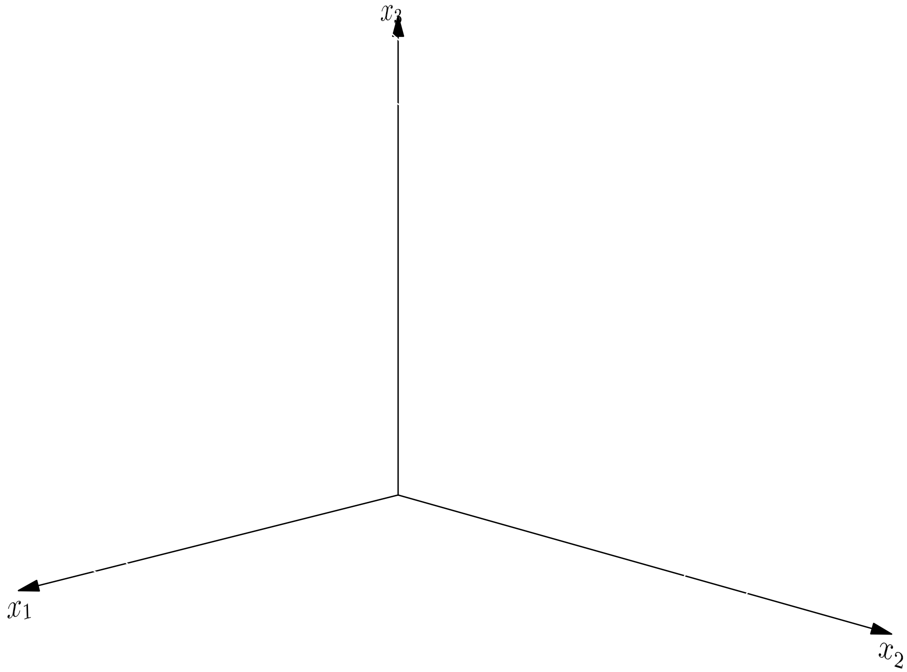
# Solving Linear Programming



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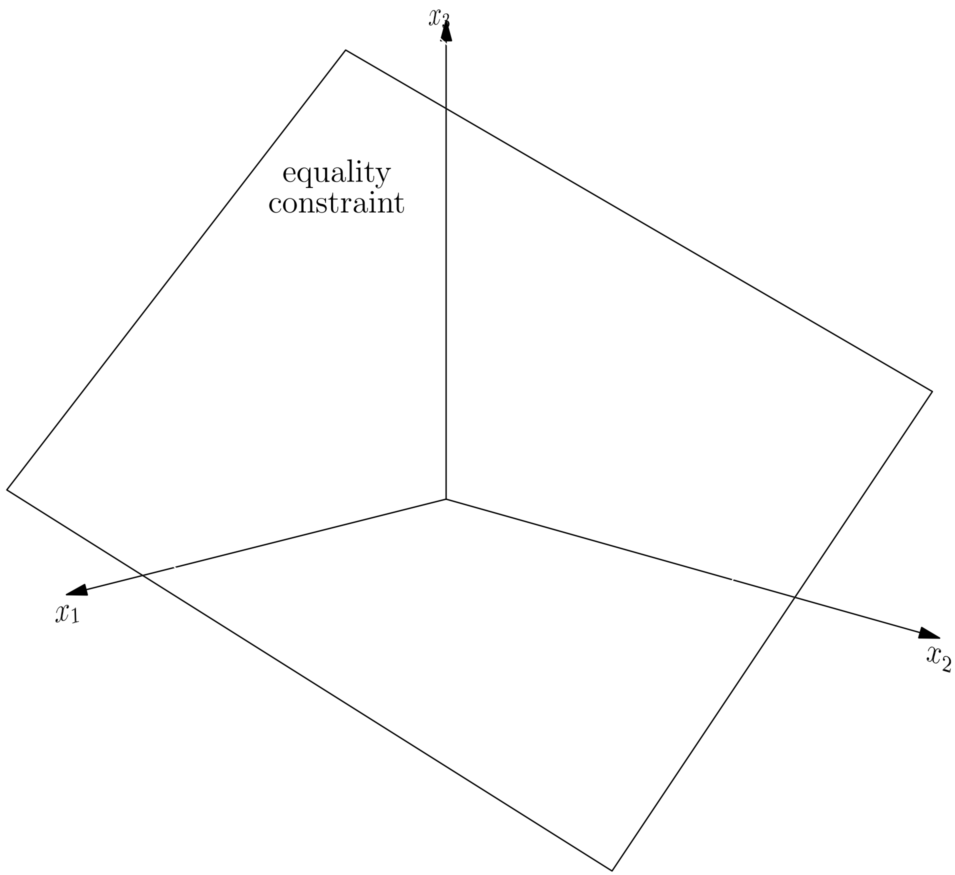

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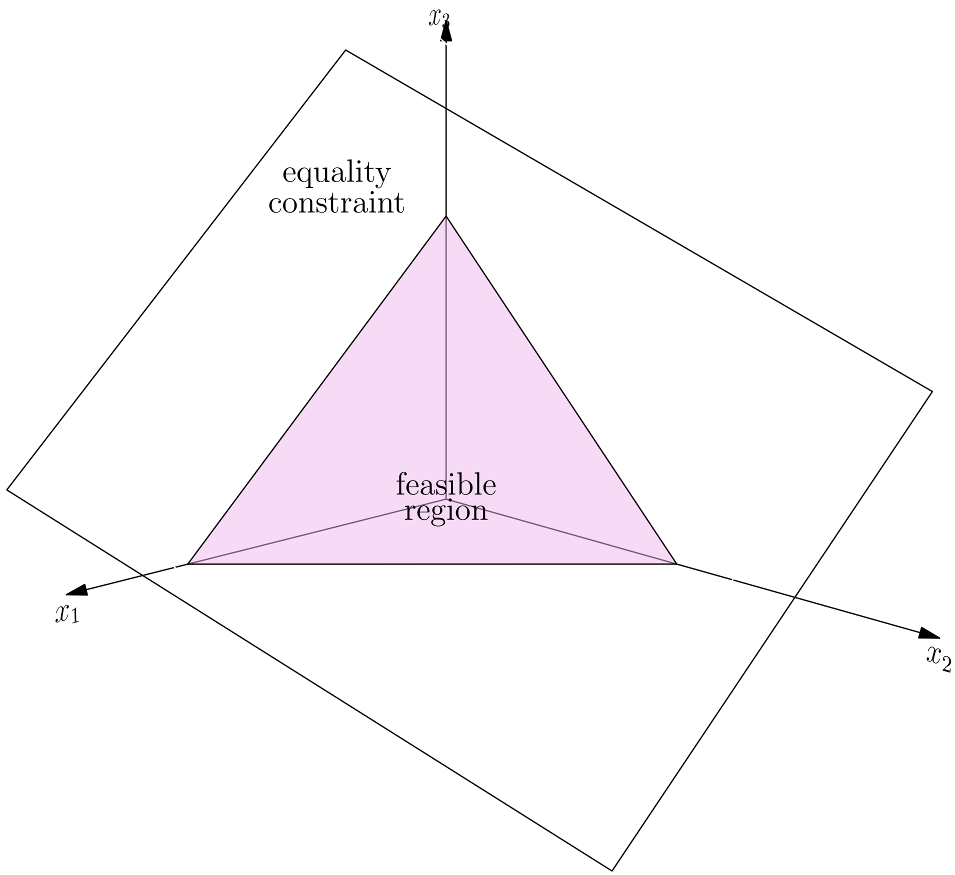

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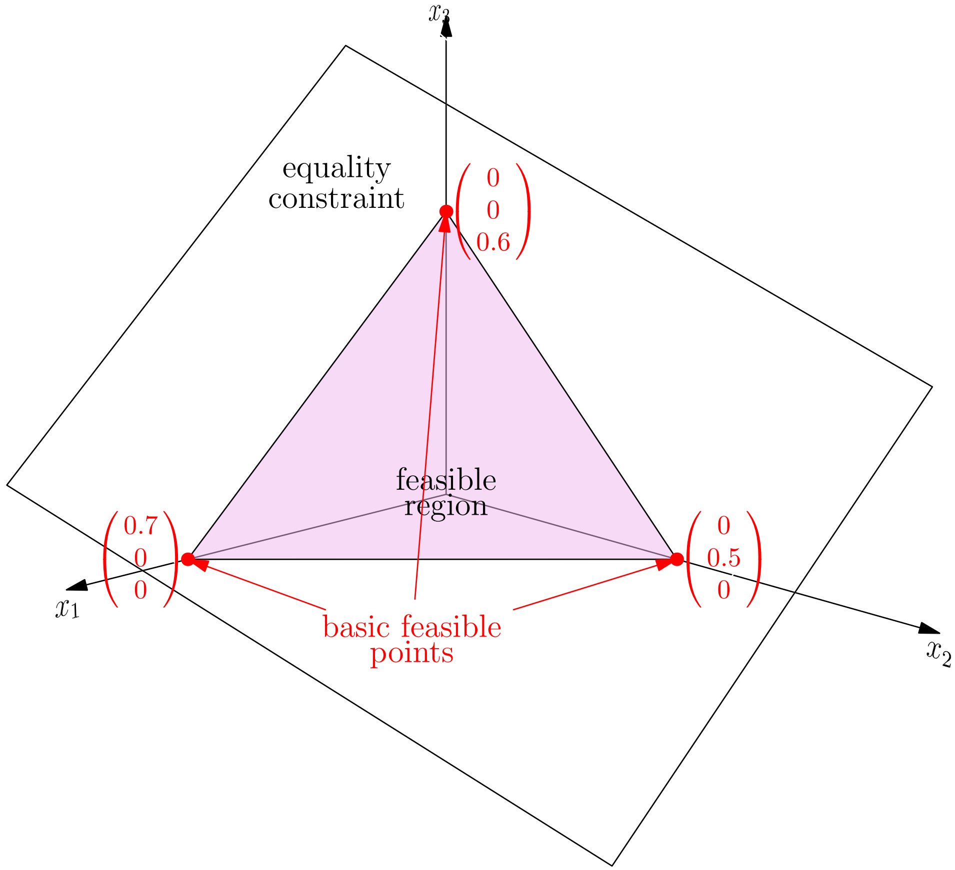

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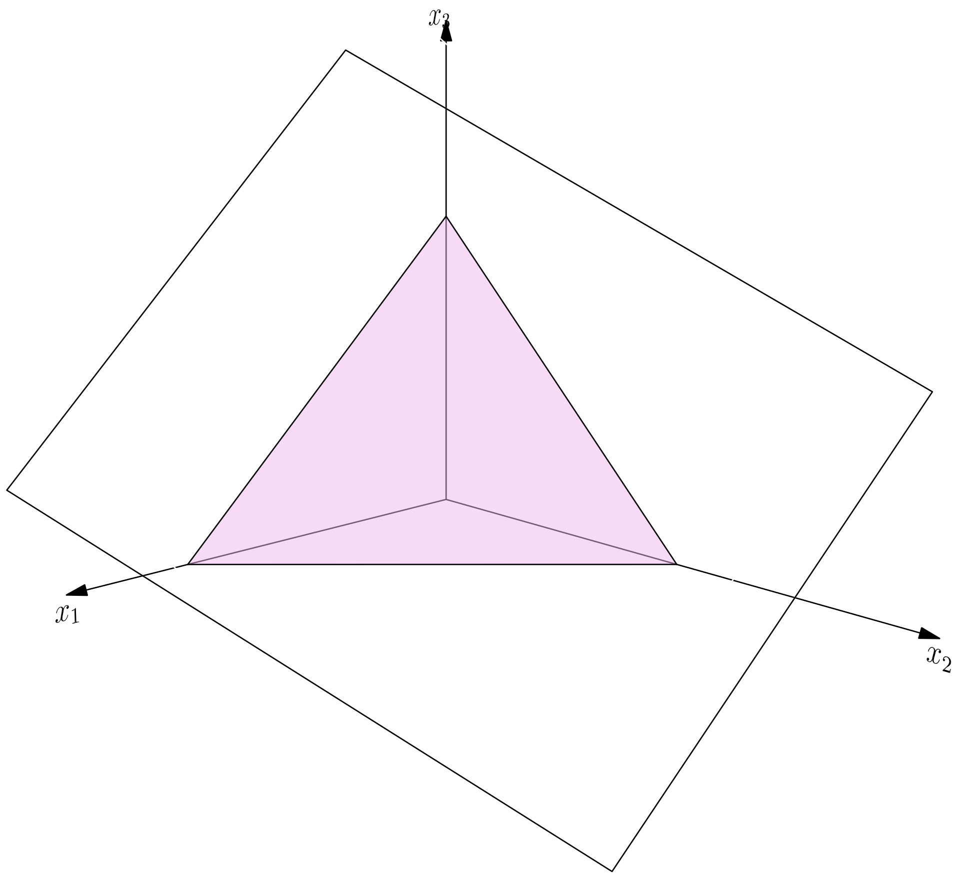

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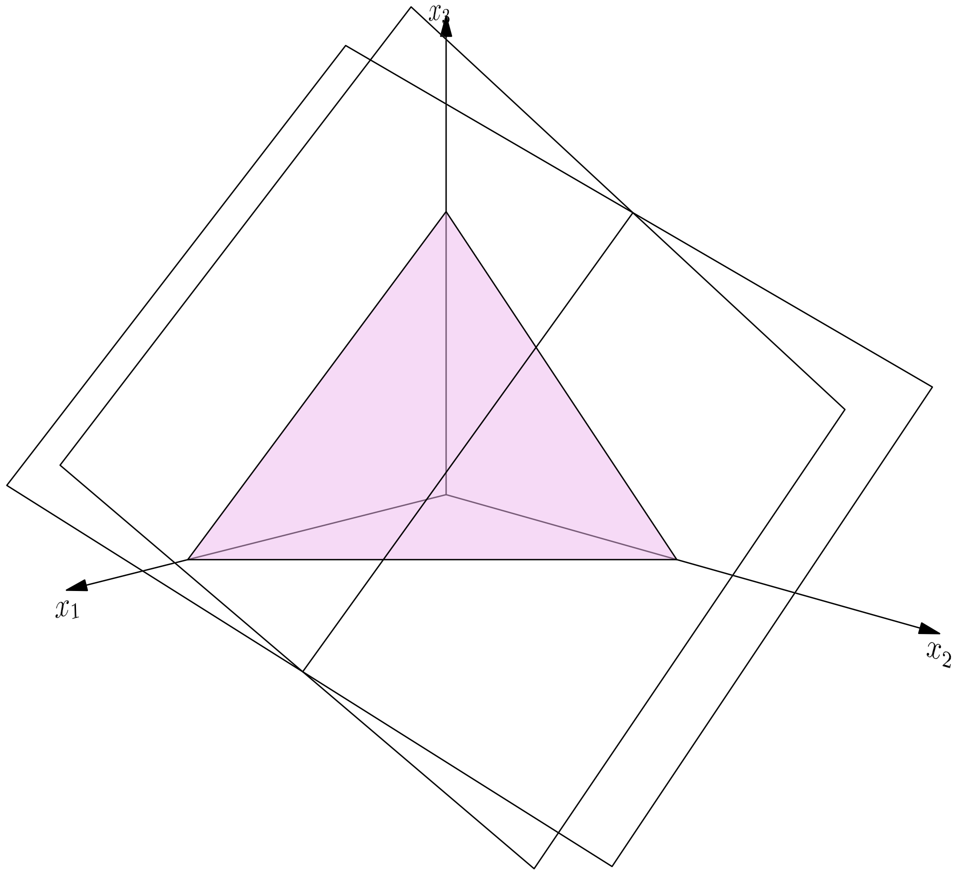

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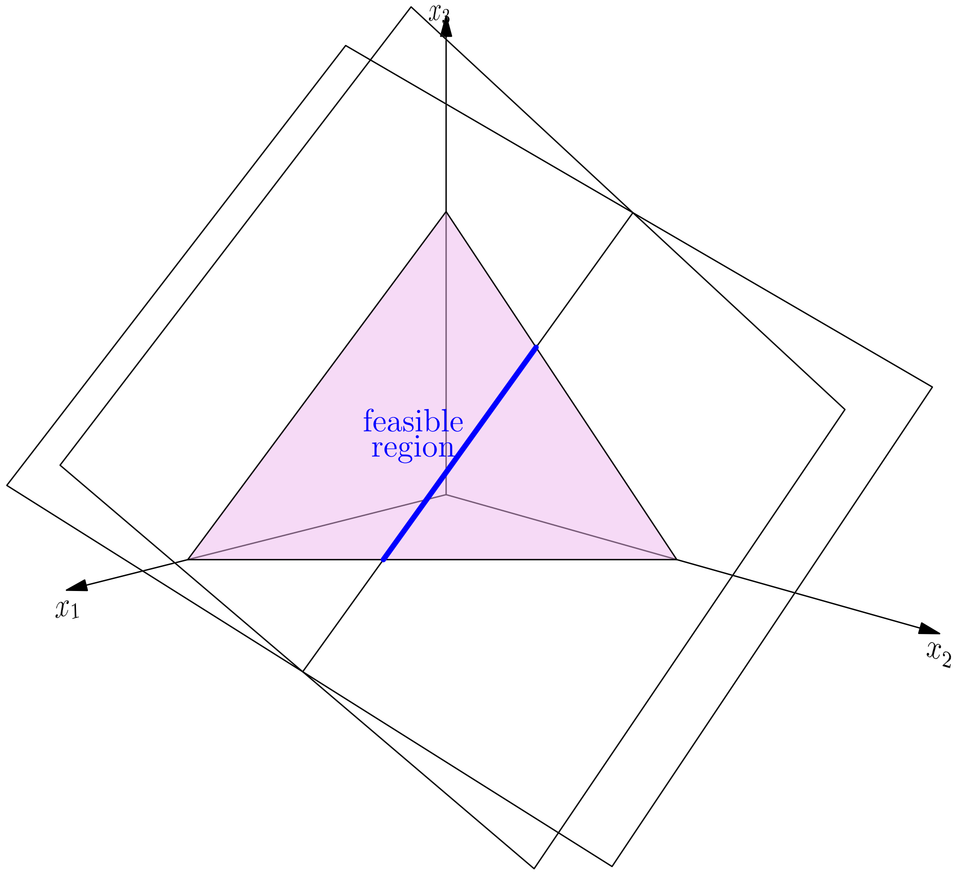

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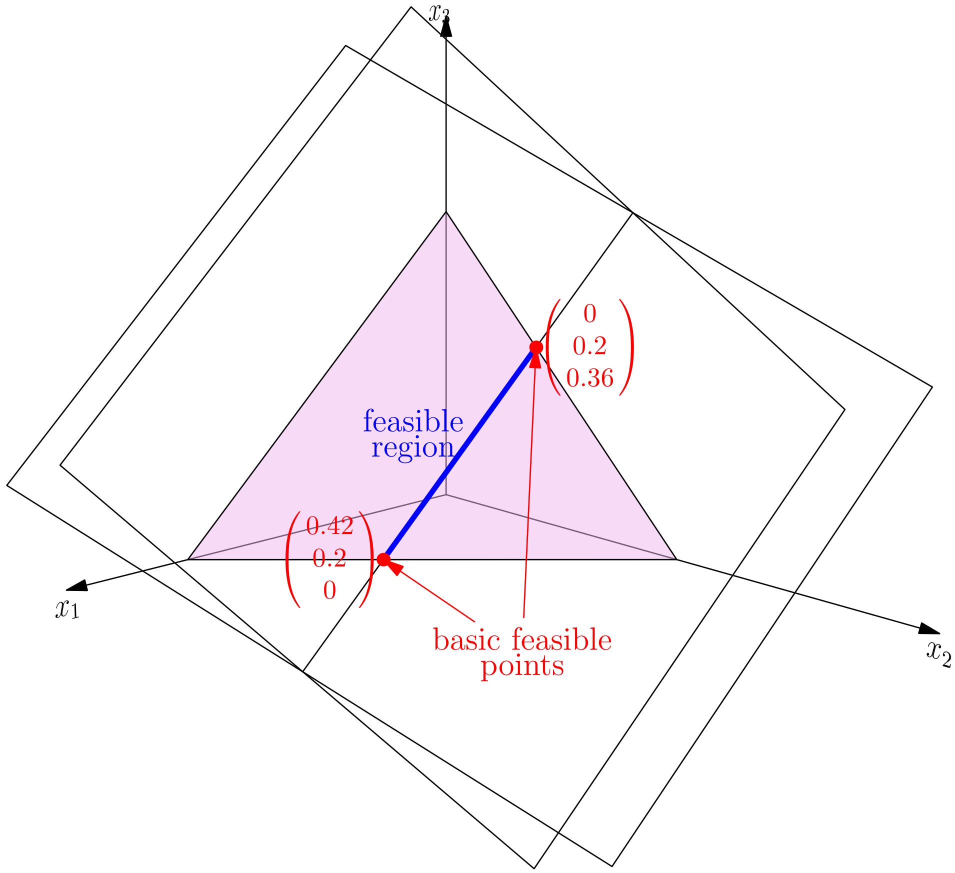

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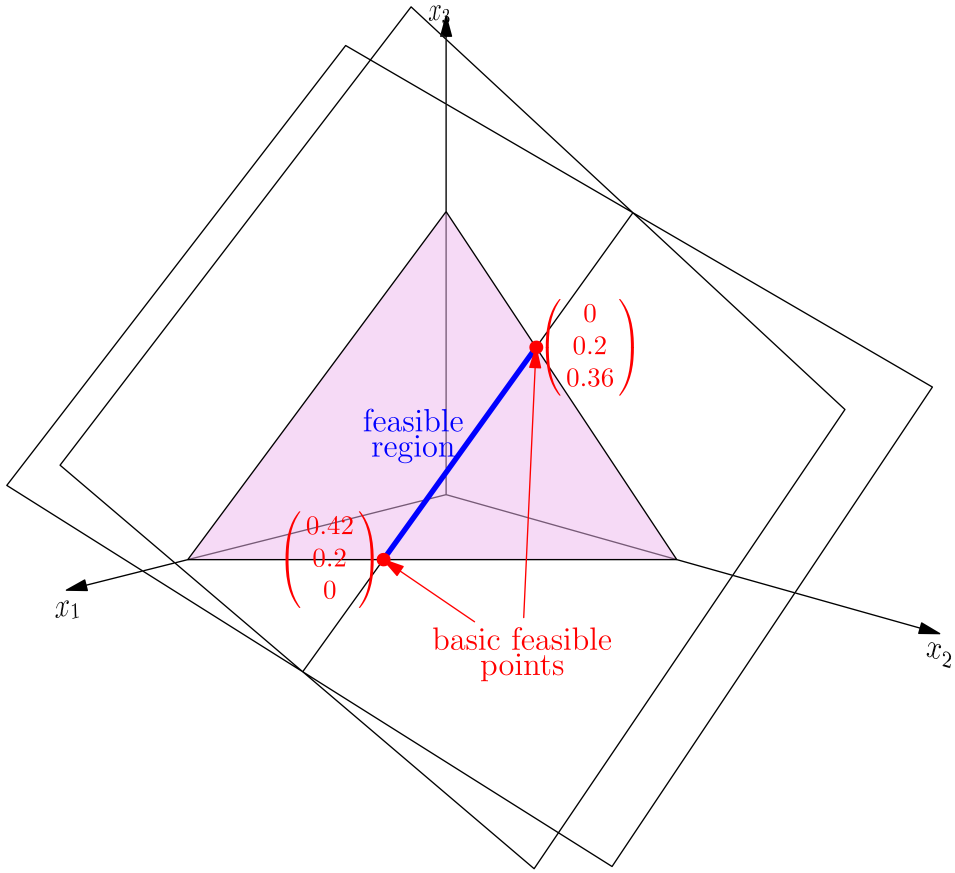


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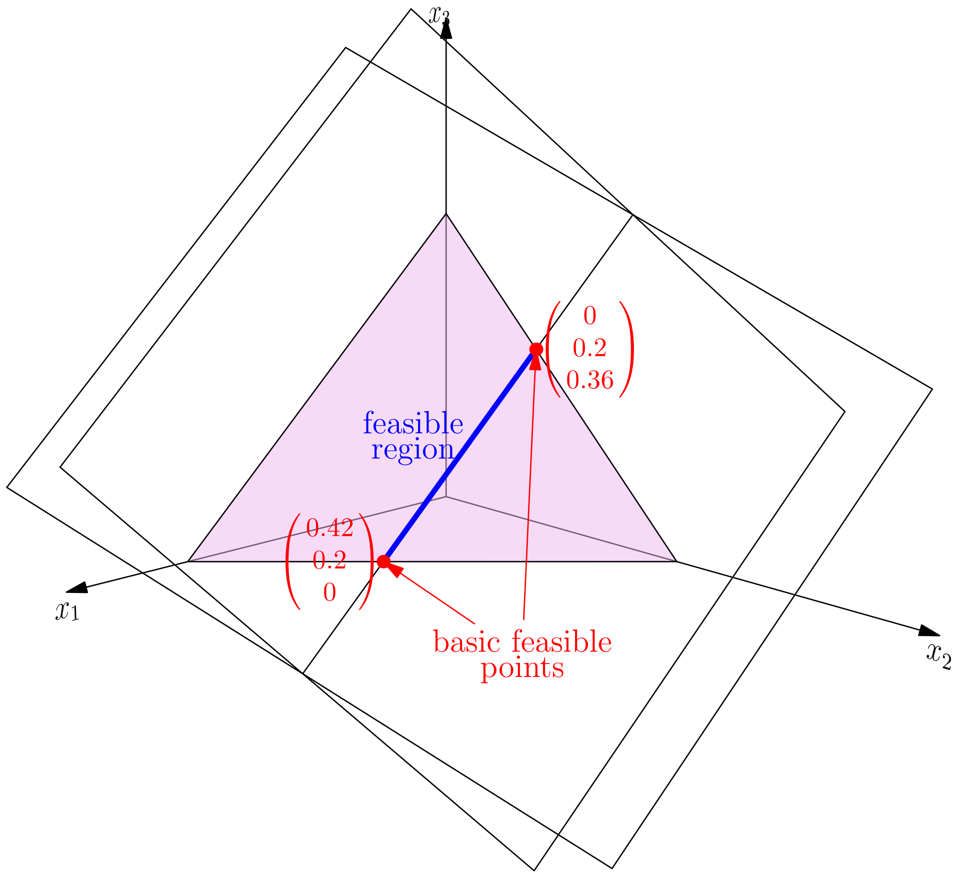


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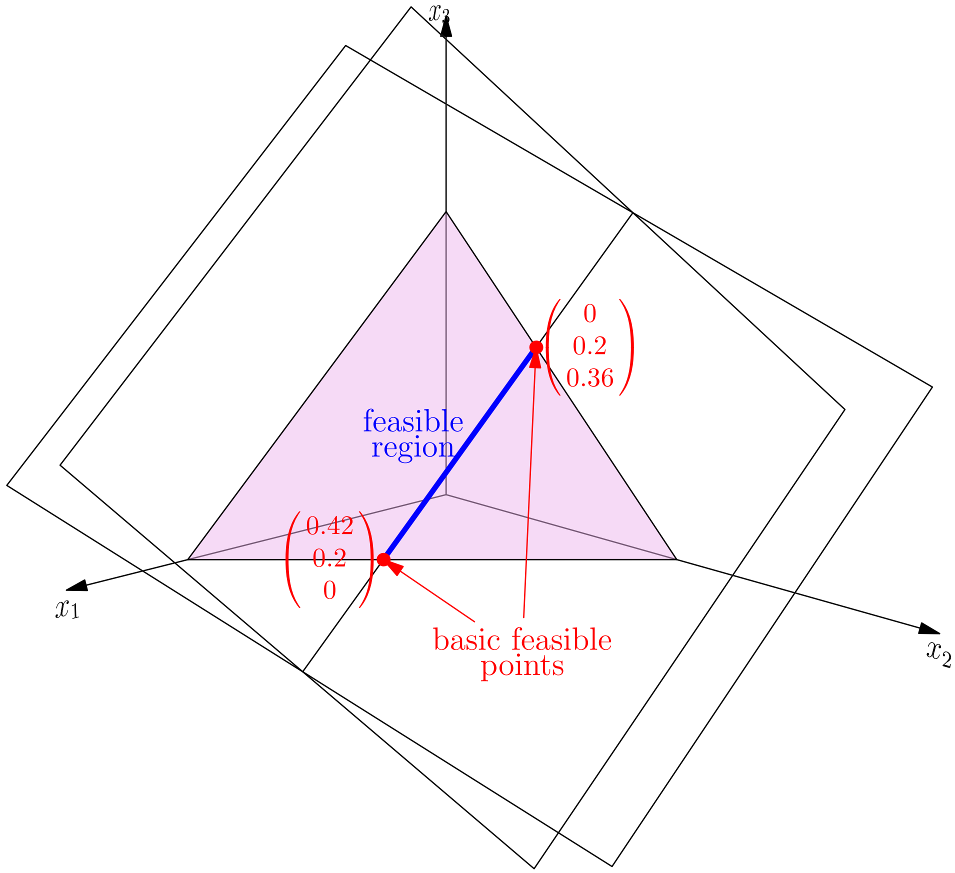


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- Typical number of basic feasible solutions is $\binom{n}{m} \geq \left(\frac{n}{m}\right)^m$
- Simplex algorithm organises iterative search for global solutions

# Lessons

- There are a huge number of problems that can be set up as linear programs
- They are particularly useful in resource allocation where the resources are all positive
- The solution to linear programming problems is at the vertex of the feasible space (intersection of constraints)
- We can search for solutions by moving from vertex to vertex
- We can transform inequality constraints to equality constraints using slack variables

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