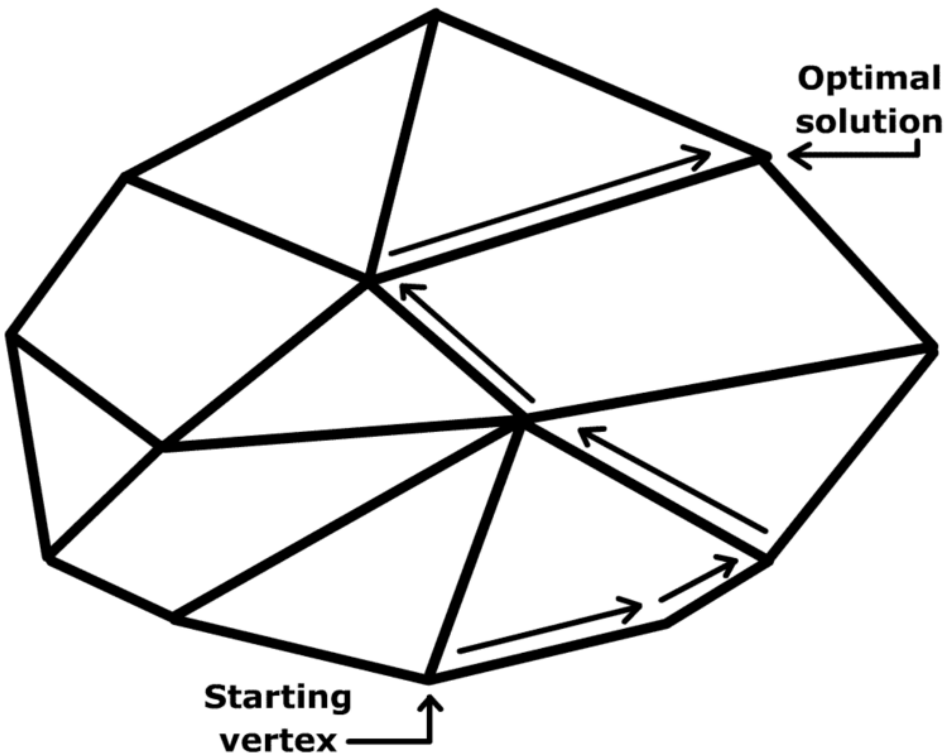
# Algorithms and Analysis

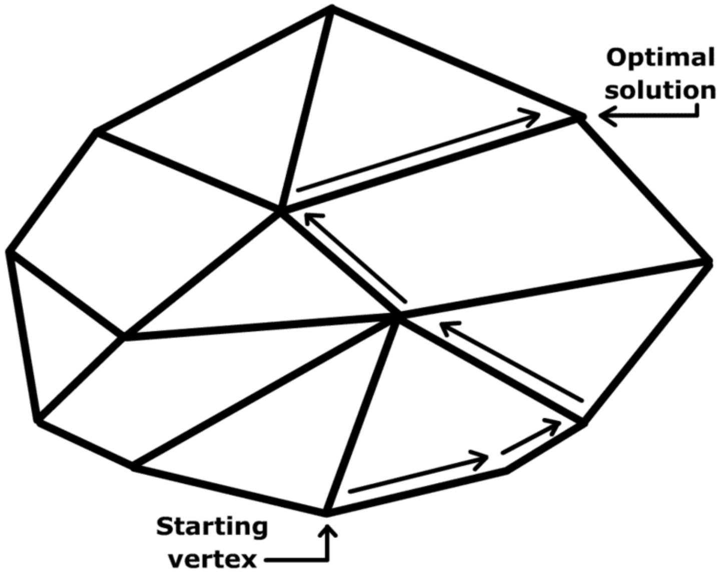
## Lesson 28: *Solving Linear Programs*




*linear programming, simplex methods, iterative search*

# Outline

1. **Recap**
2. Basic Feasible Solutions
3. Simplex Method
4. Classic LP Problems



# Recap

- Linear programs are problems that can be formulated as follows

$$\min_{\boldsymbol{x}} \boldsymbol{c} \cdot \boldsymbol{x}$$

subject to

$$\mathbf{A}^{\leq}\boldsymbol{x} \leq \boldsymbol{b}^{\leq}, \quad \mathbf{A}^{\geq}\boldsymbol{x} \geq \boldsymbol{b}^{\geq}, \quad \mathbf{A}^{=}\boldsymbol{x} = \boldsymbol{b}^{=}, \quad \boldsymbol{x} \geq \mathbf{0}$$

- Where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$

- $\mathbf{A}^*$ are matrices and we interpret the inequalities to mean

$$\forall k \qquad \sum_{j=1}^{n} A^{\leq}_{kj} x_j \leq b^{\leq}_k$$

# Optima and Vertices

- Because the objective function is linear $(\boldsymbol{c} \cdot \boldsymbol{x})$ there is a direction where the objective is always improving
- Thus, the optima cannot lie in the interior of the search space
- When we meet a constraint that limits the direction we can move, but we can still move along the constraint
- We then meet another constraint which restricts the direction we can move by two degrees of freedom
- Eventually, we will reach $n$ constraints which defines a vertex of the feasible region and is optimal

# Transforming Linear Programs

- We can always transform an inequality constraint into an equality constraint by adding slack variables

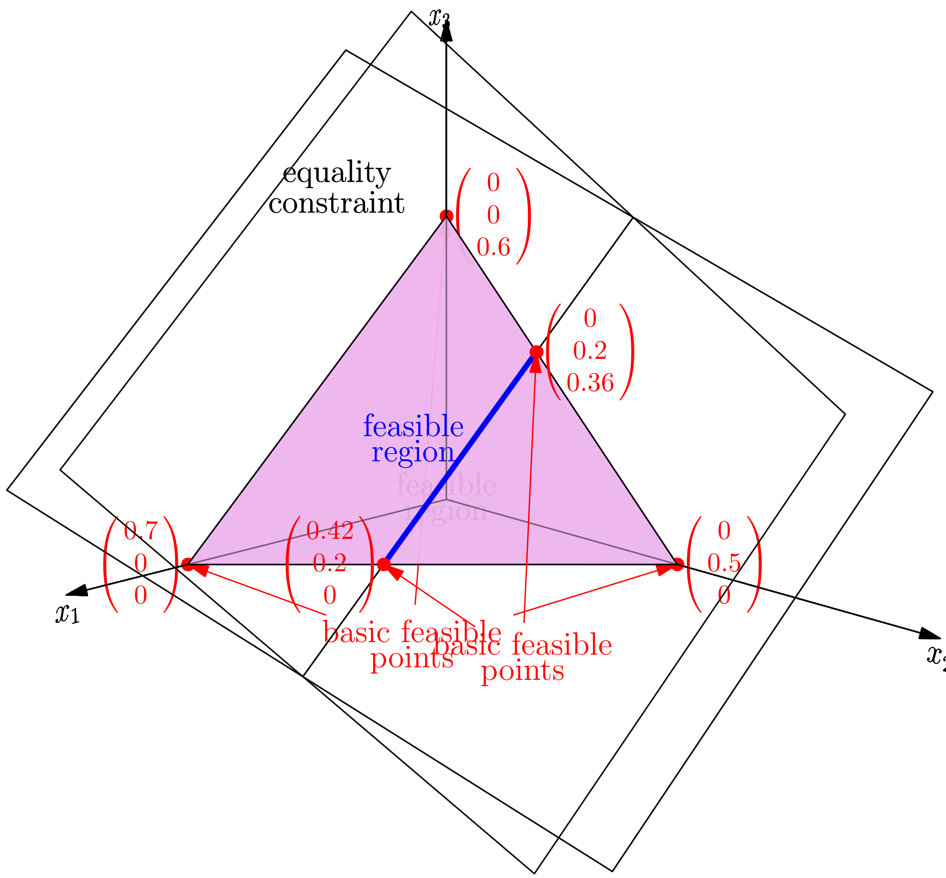
- E.g.

$$
\begin{array}{lcll}
\boldsymbol{a}_1 \cdot \boldsymbol{x} \geq 0 & \Rightarrow & \boldsymbol{a}_1 \cdot \boldsymbol{x} - z_1 = 0 & z_1 \geq 0 \\
\boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 0 & \Rightarrow & \boldsymbol{a}_2 \cdot \boldsymbol{x} + z_2 = 0 & z_2 \geq 0
\end{array}
$$

  $z_1$ (the excess) and $z_2$ (the deficit) are known as slack variables

- A linear program with just equality constraints and non-negativity constraints is said to be in normal form

# Solving Linear Programming




- The basic feasible points for LP problems with $n$ variables and $m$ constraints have at least $n-m$ zero variables
- Typical number of basic feasible solutions is $\binom{n}{m} \geq \left(\frac{n}{m}\right)^m$
- Simplex algorithm organises iterative search for global solutions

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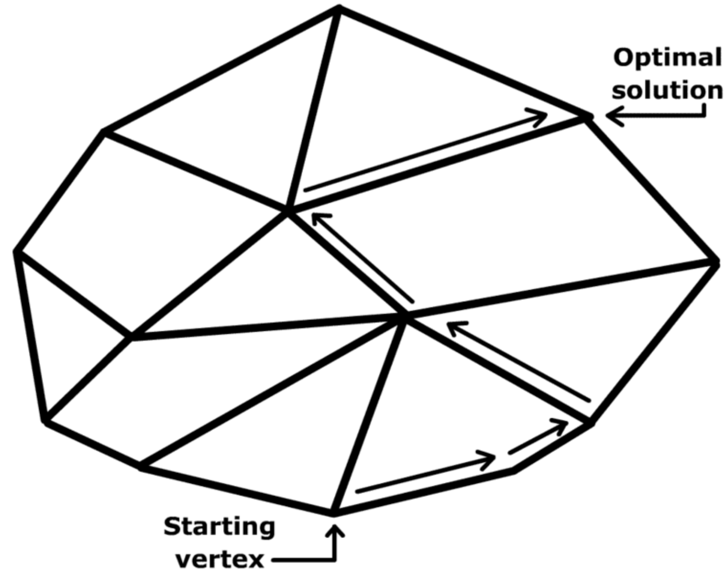

# Basic Feasible Solution

- A *basic feasible solution* or *basic feasible point* is a solution that lies at a vertex of the feasible space
- To solve a linear program we will start at a basic feasible point and move to the neighbour which best improves the objective function
- When we cannot find a better solution we are at the optimal solution
- This is an example of an iterative improvement algorithm which gives an optimal solution

# Constraints

- There are two types of constraints

  1. $n$ non-negativity constraints $x_i \geq 0$
  2. $m$ additional constraints, which we can take to be equalities $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

- Note that some of the variables might be slack variables

- We consider the case when there are more variables than additional constraints, i.e. $n > m$

- This is usually be the case, but. . .

- If this isn't true it turns out you can consider an equivalent problem (dual problem) where you have a variable for each constraint and a constraint for each variable

# Basic Variable

- In total we have $n$ equality and $m$ non-negativity constraints
- $n$ constraints must be satisfied to be at a vertex of feasible region
- So at least $n - m$ of the non-negativity constraints are satisfied (i.e. $x_i = 0$)
- The $n - m$ variables that are zero are said to be **non-basic variables**
- The other $m$ variables are said to be **basic variables**

# Initial Basic Feasible Solution

- One of the tricky bits of tackling a linear program is to find an initial feasible solution
- We do this in **phase one** of the simplex program
- To do this for each additional constraint we add a new **auxiliary variable** $\xi_k$, e.g.

$$\forall\, k \in \{1, 2, \ldots, m\} \qquad \xi_k + \sum_i A_{ki} x_i = b_k \geq 0$$

- We then can find a basic feasible solution by setting $x_i = 0$ so

$$\xi_k = b_k \quad \forall\, k \in \{1, 2, \ldots, m\}$$

# Eliminating Auxiliary Variables

- In phase one we run a simplex algorithm with an auxiliary cost function

$$\min f_{\text{aux}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{k=1}^{m} \xi_k$$

- This should find a solution where all the $\xi_k = 0$

- If no solution exists it means there is no feasible solution and we're finished

- If there is a solution then we can eliminate the auxiliary variables and we have a feasible solution

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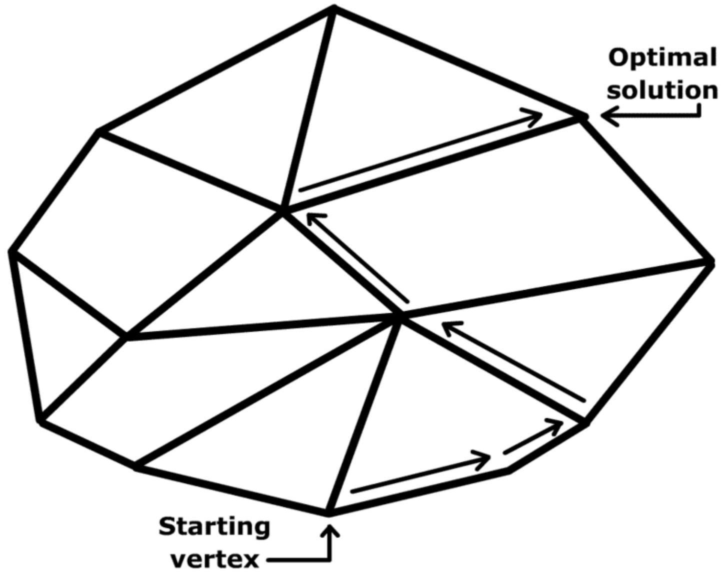

# Phase Two

- In phase two we now have an initial basic feasible solution (with $n - m$ zero variables)
- We then run the simplex algorithm on the original objective function $f(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x}$
- That is we move to a neighbouring vertex which gives the best increase in the objective function
- To help organise this search we write the objective function and constraints in a **restricted normal form** and then build a **tableau** showing the *basic variables* and the *non-basic variables*

# Restricted Normal Form

- To perform the moves between vertices it helps to represent the problem in a **restricted normal form**
- Starting from a basic feasible point we have a constraint for each basic (non-zero) variable
- We write the constraints as an equality between basic and non-basic (zero valued) variables
- Similarly we write the objective function in terms of non-basic variables
- This is always possible as we can use the constraints to eliminate the basic variables

# Tableau

$\max_{\mathbf{x}} \; f(\mathbf{x}) =$ [illegible]

where [illegible]

$x_1 = x_4 = x_5 = x_7 = x_8 = x_9 = 0$ [illegible]

[illegible]

# Awkward Problems

- If there are any column with all entries positive then this variable can be increase forever—this is a signal that the linear programming problem is unbounded
- You can also find that a basic variable becomes zero—this is known as a degenerate feasible vector
- It can by removed by exchanging variables on the left of the inequality with variables on the right
- This makes the algorithm a bit more complex to implement

# High Performance Solvers

- Although the tableau method is the "classic solver" it doesn't cut the mustard for large scale problems
- The simplex update can also be viewed as solving a linear set of equations which is facilitated by performing an LU-decomposition
- However, the constraints are often very sparse so good solvers try to take advantage of the sparsity
- Top end simplex algorithms are rather complex
- There is a second approach known as the interior point method which is competitive on large problems

# Time Complexity of Simplex

- The time complexity of the updates is $O(n^2)$
- The critical question is how many updates are necessary
- It turns out that typically this is $O(n)$ making the simplex algorithm $O(n^3)$
- However, it is possible to cook up problems where there is a "long path" from the initial solution to the optimum which is exponentially big
- Thus the worst case time is exponential, although this almost never happens in practice

# Interior Point Method

- An alternative to the simplex method is the interior point method which always remains in the feasible region, away from the constraints
- These method iterate towards the constraints and are provably polynomial
- For small linear programming problems they are out-performed in practice by the simplex method
- On large and very large problems they seem to perform as well if not better than the simplex method
- The high-end solvers will have a variety of interior point methods tailored to the particular problem

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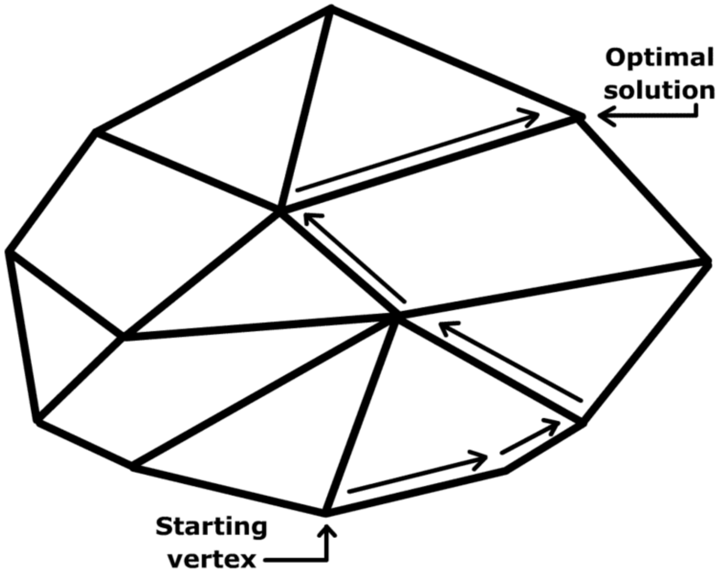


# LP Problems

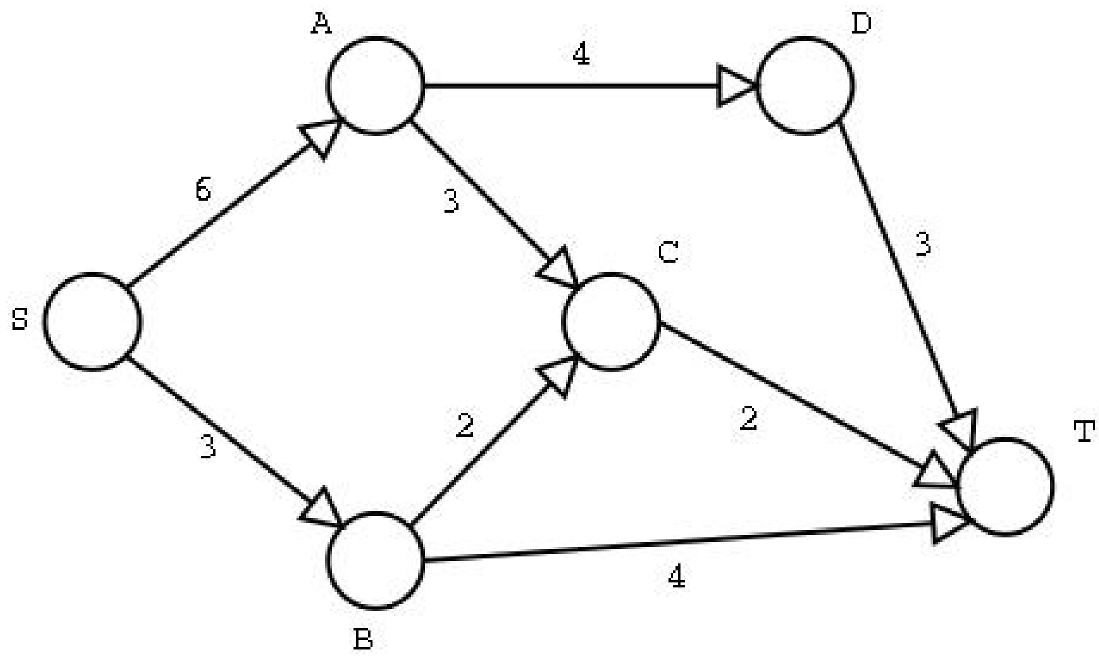
- Any problem that can be set up as a linear program can be solved in polynomial time
- One way is just to feed it to a LP-solver
- Sometimes the problems are important enough and have such a distinctive formulation that faster specialised algorithms have been developed
- We consider a couple of classic problems: *maximum flow* and *linear assignment*

# Maximum Flow

- In maximum flow we consider a directed graph representing a network of pipes
- We choose one vertex as the source and a second vertex as a sink
- Each edge has a flow capacity that cannot be exceeded
- The problem is to maximise the flow between source of sink
- This can be used to model the flow of a fluid, parts in an assembly line, current in an electrical circuit or packets through a communication network

# Example

- Consider a firm that has to ship haggis from Edinburgh to Southampton
- The shipping firm transports this in crates which it sends through intermediate cities
- The number of crates is limited by the size of the lorries it uses



# Flow

- We are given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each edge has a capacity $c(i,j)$
- We define the flow from $i$ to $j$ as $f(i,j)$ with $0 \le f(i,j) \le c(i,j)$
- For all vertices except the source ($s$) and sink ($t$) we assume

$$\forall i \in \mathcal{V}/\{s,t\} \qquad \sum_{j \in \mathcal{V} | (i,j) \in \mathcal{E}} f(i,j) = \sum_{j \in \mathcal{V} | (i,j) \in \mathcal{E}} f(j,i)$$

(i.e. no flow is lost from source to sink)

- We want to maximise the flow from the source

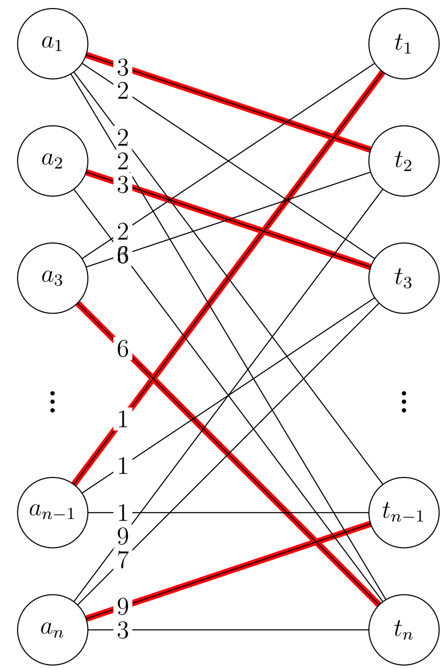
$$\sum_{i \in \mathcal{V} | (s,i) \in \mathcal{E}} f(s,i)$$

# Solving Maximum Flow

- As set up we have a linear objective function with linear constraints
- We can therefore solve this problem with a LP-solver
- (Note the solution will typically involve a fraction flow)
- However, this is such a classic problem with a distinctive structure that we can solve it more quickly with other algorithms
- The classic algorithm is the Ford-Fulkerson method with run time $O(|\mathcal{E}| \times f_{\max})$ where $f_{\max}$ is the maximum flow, although we won't cover this in the course

# Linear Assignment

- We are given a set of $n$ agents, $\mathcal{A}$, and $n$ tasks, $\mathcal{T}$
- Each agent has a cost associated with performing a task $c(a,t)$
- We want to assign an agent to one task so as to minimise the total cost
- Consider a taxi firm with taxi's at 5 different locations and 5 requests to fulfil. The cost is the distance to the clients. Which taxi should go to which client?



# LA as LP

- The linear assignment problem can be set as a linear programming problem

$$\min_{\boldsymbol{x}} \sum_{a\in\mathcal{A},t\in\mathcal{T}} c(a,t)x_{a,t}$$

subject to

$$\forall a \in \mathcal{A} \qquad \sum_{t\in\mathcal{T}} x_{a,t} = 1$$

$$\forall t \in \mathcal{T} \qquad \sum_{a\in\mathcal{A}} x_{a,t} = 1$$

$$\forall (a,t) \in (\mathcal{A},\mathcal{T}) \qquad x_{a,t} \geq 0$$

# Hungarian Algorithm

- Linear assignment is another classic problem that is commonly encountered
- Although it can be solved using a generic LP-solver this is not the most efficient algorithm
- The most efficient algorithm is the Hungarian algorithms
- This is rather complex (having once implemented it I can tell you from bitter experience it ain't easy)
- Its worst case time is $O(n^3)$ although it frequently takes $\Theta(n^2)$

# Quadratic Programming

- If we have linear constraints and a quadratic objective function then we have a quadratic programming problem
- Again this can be solved in polynomial time
- Many of the ideas used are the same as for linear programming
- This also has important applications in science and engineering

# Lessons

- Linear programming is a classic problem
- We know a huge number of problems are solvable in polynomial time because they can be formulated as linear programs
- Linear programs occur sufficiently often that they are hugely important
- They aren't easy to solve, although standard simplex is not massively complex
- For particular LP problems with distinctive structure there are sometimes better algorithms than generic LP-solvers