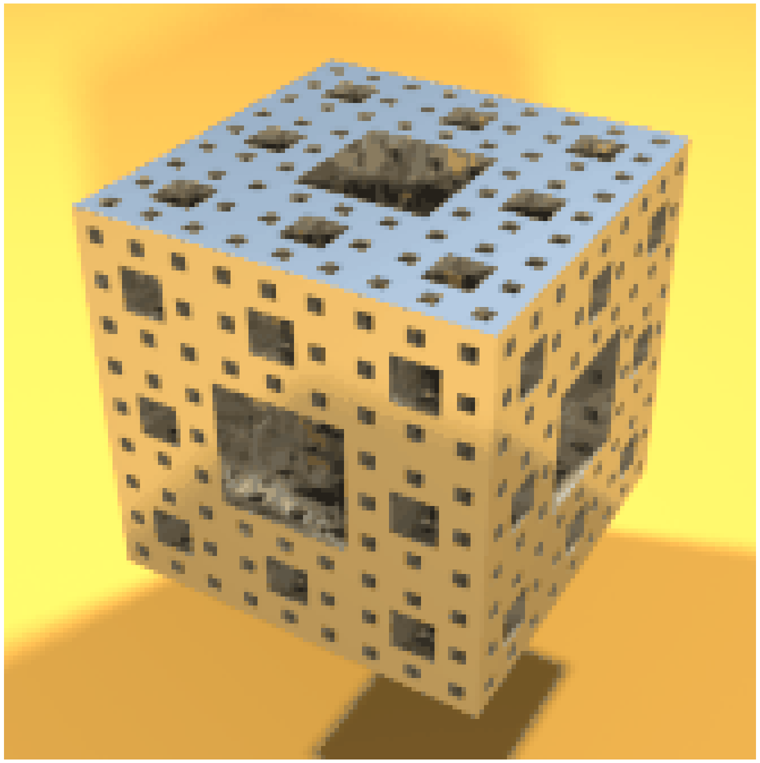
Algorithms and Analysis

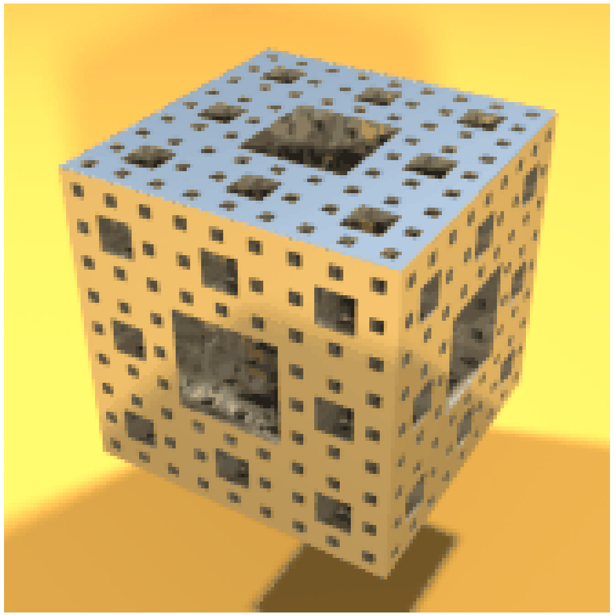
Lesson 6: *Recurse!*



Induction, integer power, towers of hanoi, analysis

Outline

1. **Simple Recursion**
2. Programming Recursively
   - Simple Examples
   - Thinking about Recursion
3. Analysis of Recursion
   - Integer Powers
   - Towers of Hanoi



Recursion

- Recursion is a strategy whereby we reduce a problem to a smaller problem of the same type
- We repeat this until we reach a trivial case we can solve by some other means
- Recursion can also be used to describe situations in a succinct manner using references to itself. E.g.
  - ⋆ Definition of factorial:
    $n! = n \times (n-1)!$ with $0! = 1$
  - ⋆ Definition of ancestor: X is the ancestor of Y if X is the parent of Y or Z is the parent of Y and X is the ancestor of Z

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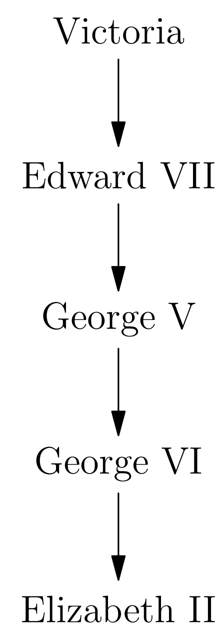

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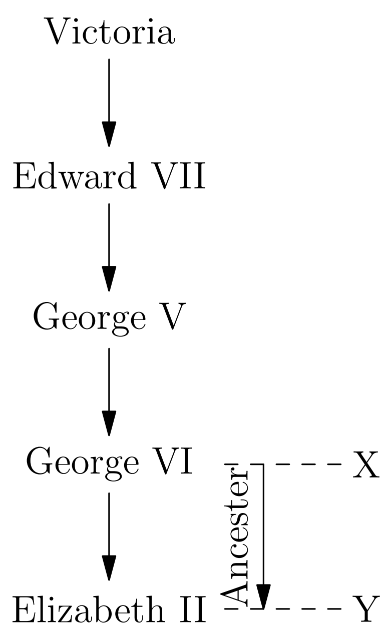

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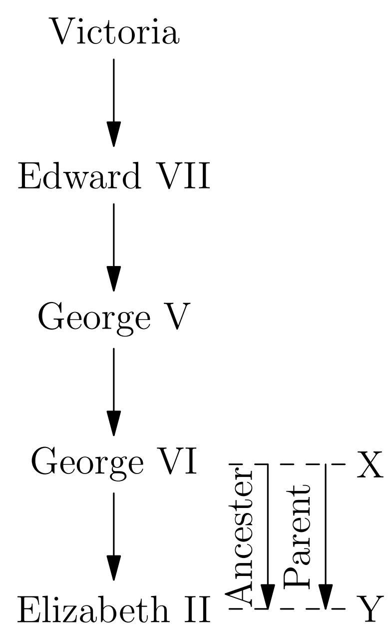

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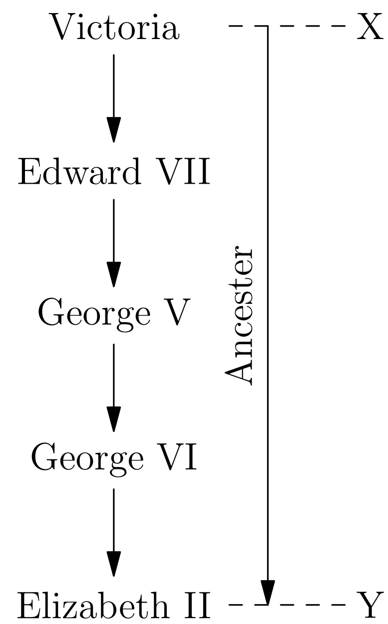

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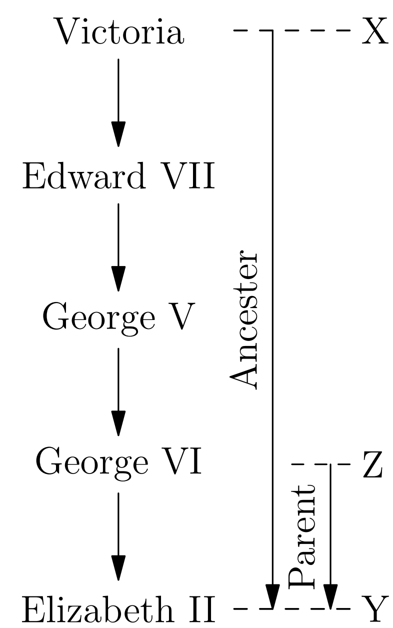

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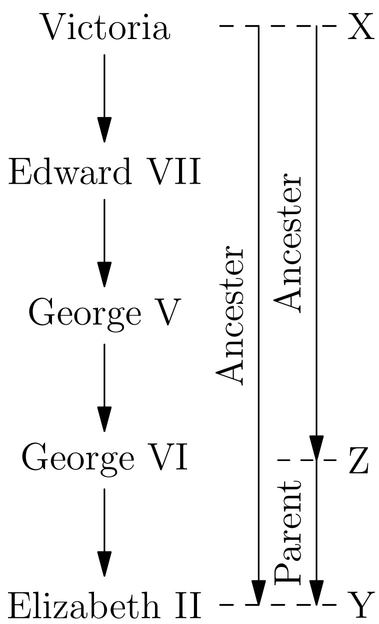

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- Notice that these are *self-referential* definitions

- A recursive definition consists of two elements
  - ⋆ **The Base Case:** or boundary cases where the problem is trivial
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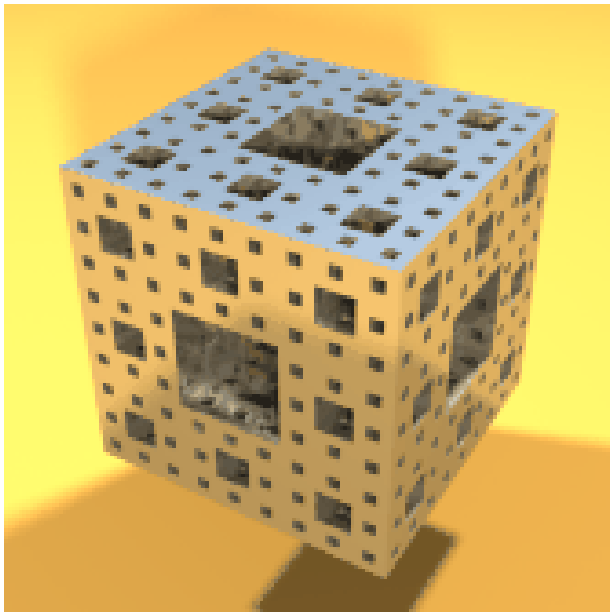

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- That is they allow functions/methods to be defined in terms of themselves

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$$0.95^{25} = 0.95 \times (0.95)^{24}$$

Integer Powers

- How do we compute 0.95^{25}?
- One way is to multiply together 0.95 twenty five times
- A more efficient way is to observe

$$x^{2n} = (x^n)^2$$
$$x^{2n+1} = x \times x^{2n}$$

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six multiplications rather than 24!

Implementing Integer Power

- Integer power looks rather intimidating to code
- However, the recursive definition is easy
- We can easily code this function recursively

```
double power(double x, long n)   // (Overflow is possible)
{
    return  n <  0  ? 1 / power(x,-n)            // Negative power
        :  n == 0   ? 1                          // Special case
        :  n == 1   ? x                          // Base case
        :  n%2 == 0 ? (x = power(x, n/2)) * x    // Even power
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Helper Functions

- This is a slick implementation from the web, but not terribly efficient
- We only need to do the first two checks once
- A more efficient implementation would use a helper function

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Writing Recursive Programs

- You need to make sure that you catch the base case **before** you recurse
- The recursive case can call itself, possibly many times, provided the inductive argument is closer to the base case
- That is,
  - ⋆ Ensure that you use a 'smaller problem'
  - ⋆ Assume that you can solve the 'smaller problem'

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The Cost of Recursion

- Recursion acts just like any other function call
- The values of all local variables in scope are put on a stack
- The function is called and
  - ⋆ returns a value or
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Unrolling Recursion

- Recursion can frequently be replaced
- E.g. we can easily write a factorial function

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long factorial(long n)
{
    if (n<0)
        throw new IllegalArgumentException();
    long res = 1;
    for (int i=2; i<=n; i++)
        res *= i;
    return res;
}
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The Greatest Common Denominator

- One of the most famous algorithms is Euclid's algorithm for calculating the greatest common denominator
- The greatest common denominator of A and B is the largest integer, C, which exactly divides A and B
- E.g. the greatest common denominator of 70 and 25 is 5
- Euclid's algorithms uses the fact that
  - ⋆ $\gcd(A, B) = \gcd(B, A \mod B)$
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Implementation of GCD

- The implementation of gcd is trivial using recursion

```
long gcd(long a, long b)
{

    if (b==0) {
        return a;
    }
    return gcd(b, a%b);
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long gcd(long a, long b)
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    a = b;
    b = c;
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Implementation of GCD

- The implementation of gcd is trivial using recursion

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long gcd(long a, long b)
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  while(true) {
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        return a;
     }
     long c = a%b;
     a = b;
     b = c;
   }
}
```

- Example of tail recursion

When Definitely not to Recurse

- A classic recursively defined sequence is the Fibonacci series
  - ⋆ $f_n = f_{n-1} + f_{n-2}$
  - ⋆ $f_1 = f_2 = 1$

- We might be tempted to write a recursive function to define the series

```
long fibonacci(long n)
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  if (n<=2)
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Why shouldn't you want to do this?

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fib(6)
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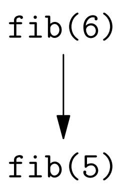

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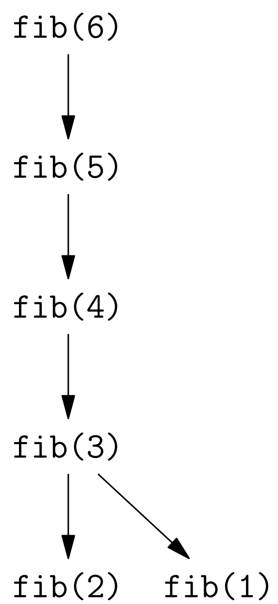

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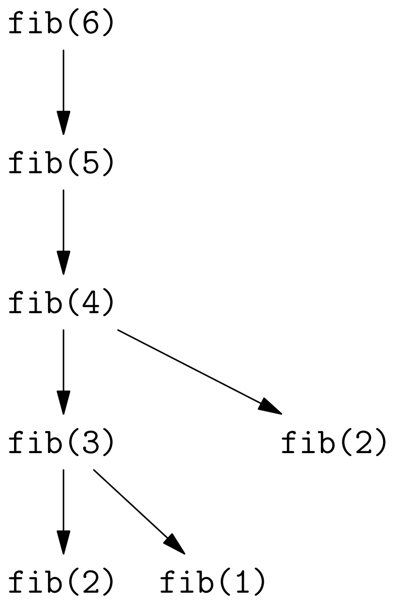

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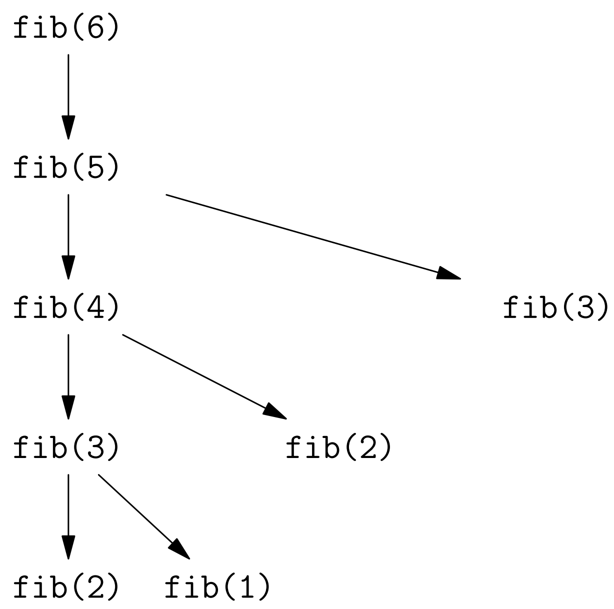

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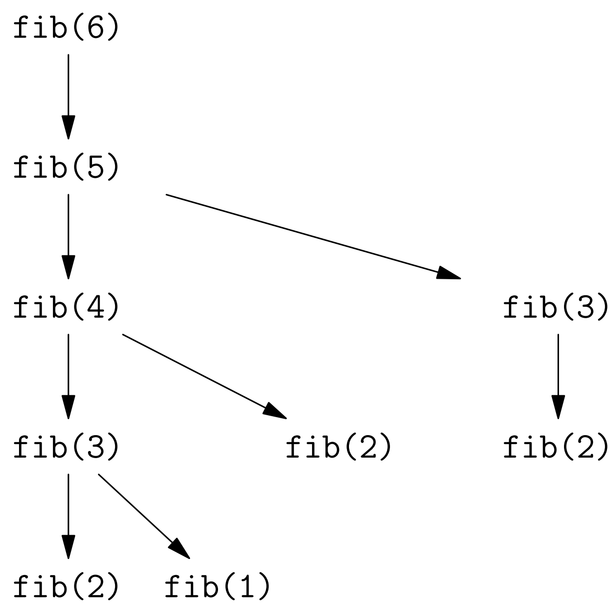

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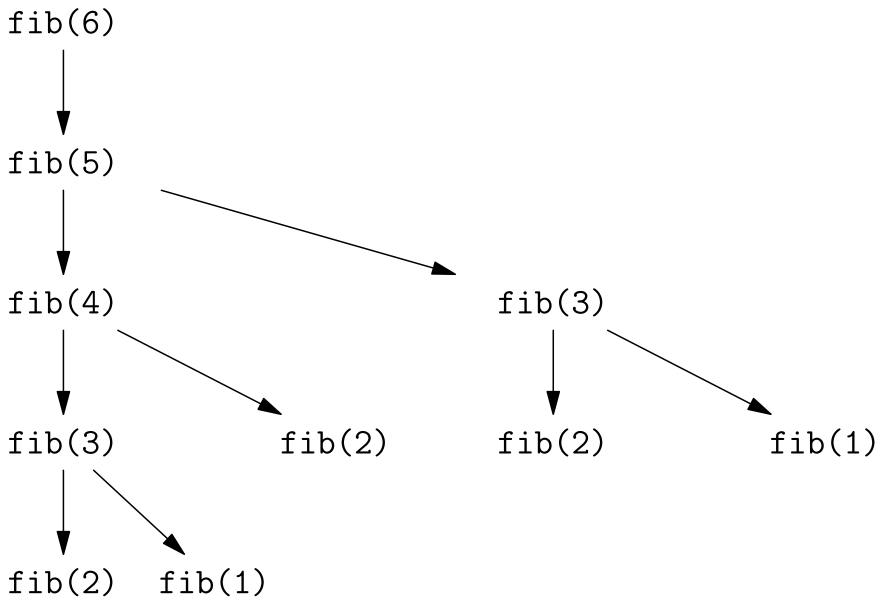

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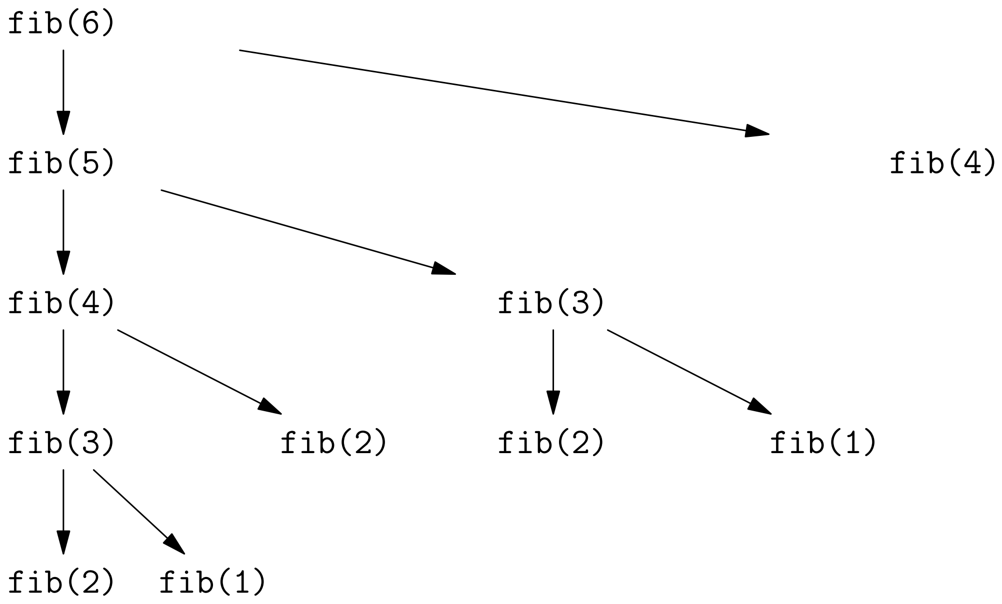

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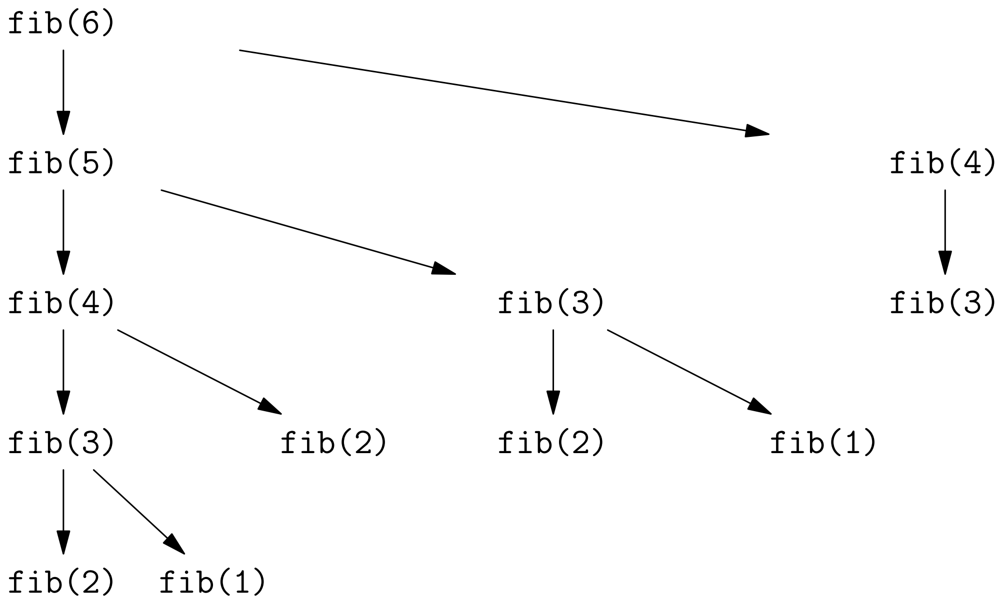

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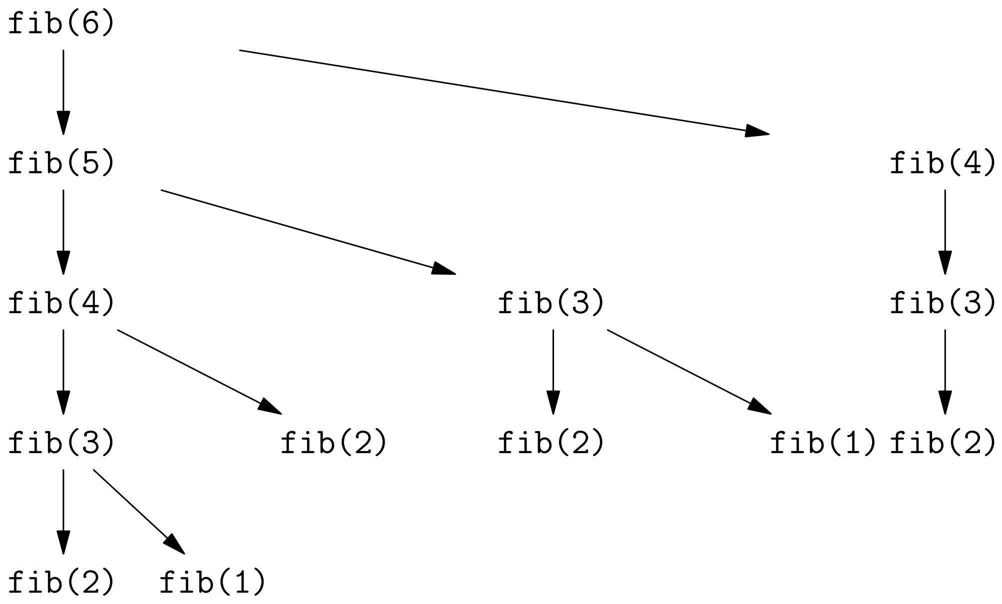

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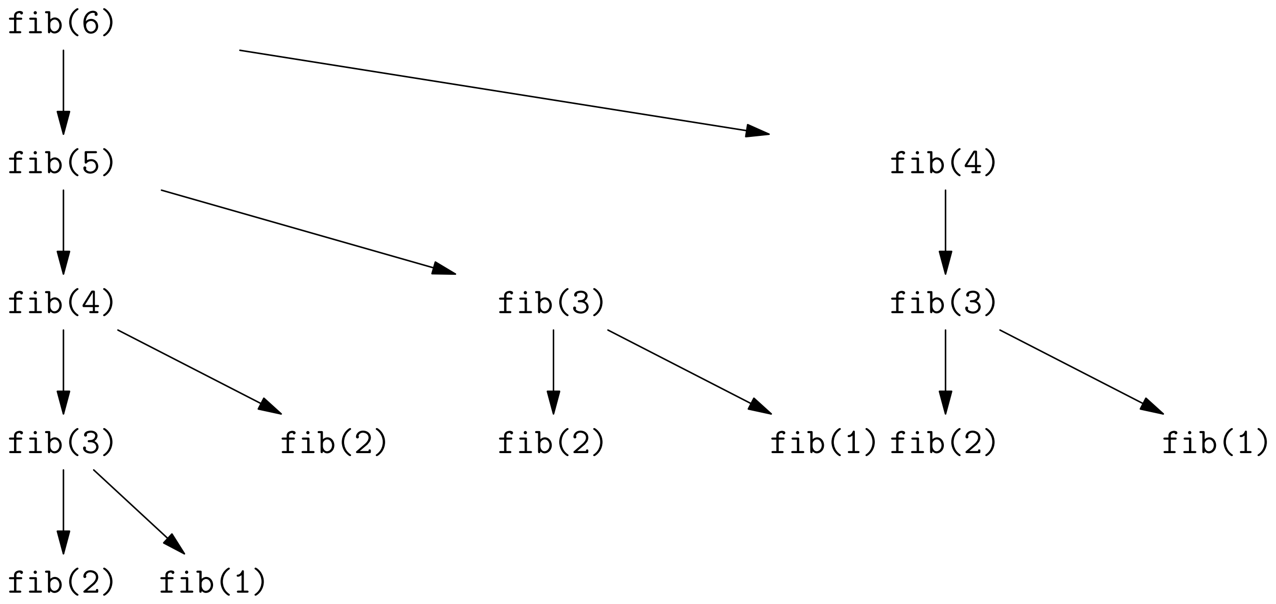
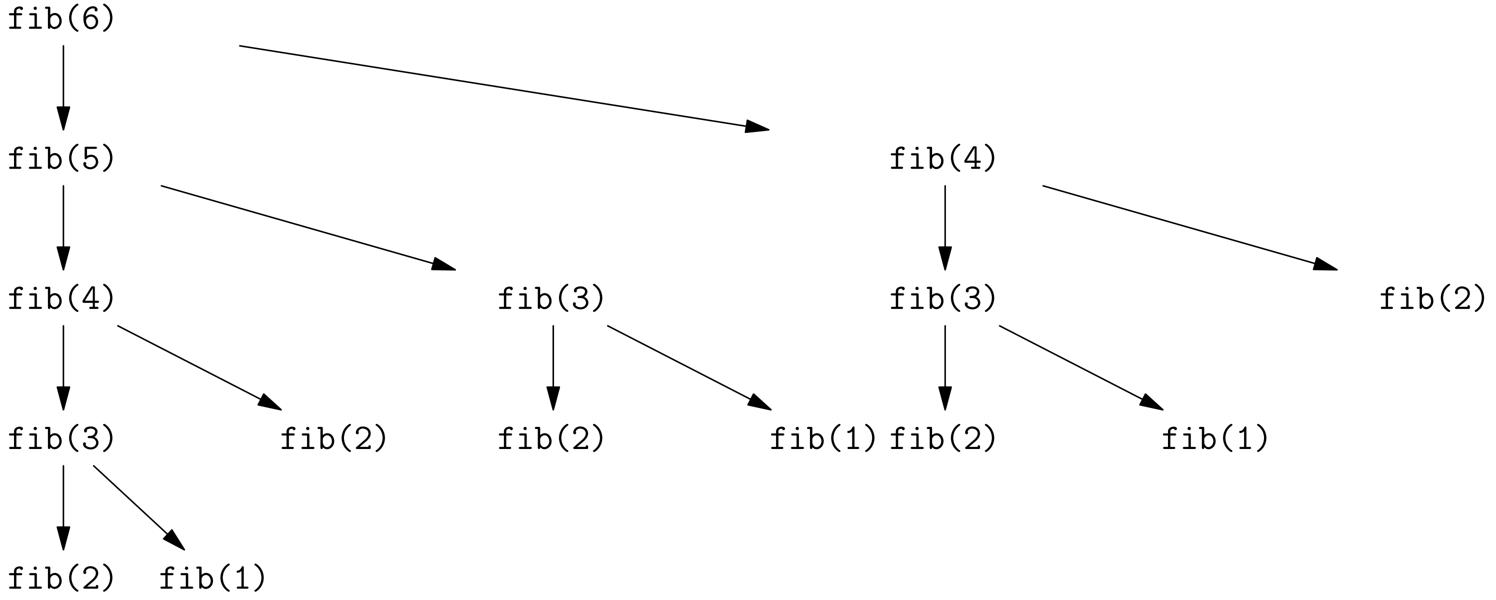


Fibonacci

```
long fib(long n)
{
  if (n<=2)
    return 1;
  return fib(n-1) + fib(n-2);
}
```



Why Use Recursion At All?

- Both `factorial` and `gcd` could be written without using recursion
- The programs would probably run faster
- The `gcd` program would be less clear
- The cost of the additional function calls is often insignificant
- It would considerably harder to write many programs such as `power` non-recursively
- Later we will see algorithms like `quick sort` which rely on recursion

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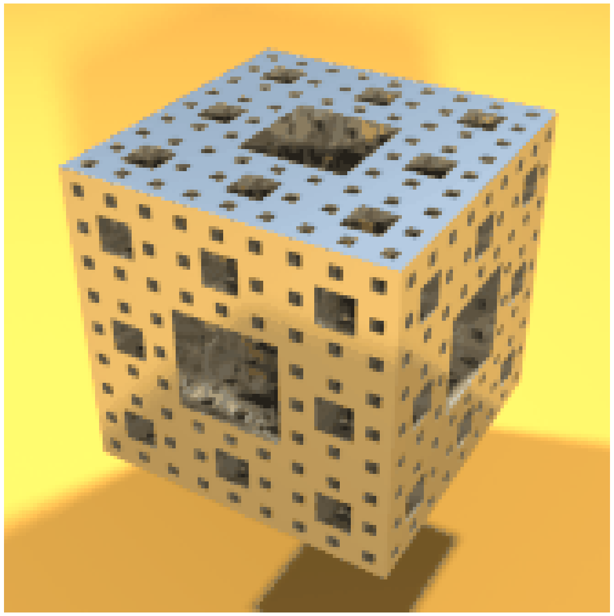
Outline

1. Simple Recursion
2. Programming Recursively
   - Simple Examples
   - Thinking about Recursion
3. **Analysis of Recursion**
   - Integer Powers
   - Towers of Hanoi



Analysis

- We can use recursion to compute the time complexity of a recursive program!
- To do this we denote the time taken to solve a problem of size n by $T(n)$
- To compute the time complexity of factorial, we note that to compute $n!$ we have to multiply n by $(n-1)!$
- That is, the number of multiplications we need to compute is

$$T(n) = T(n-1) + 1$$

- Now $T(0) = 0$ so

$$T(n) = T(n-1) + 1 = T(n-2) + 2 = \cdots = T(0) + n = n$$

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Time to Compute Power

- How long does it take to compute x^n?
- Remember

$$x^{2n} = (x^n)^2$$
$$x^{2n+1} = x \times x^{2n}$$

- Thus

$$T(n) = \begin{cases} T(n/2) + 1 & \text{if } n \text{ is even} \\ T((n-1)/2) + 2 & \text{if } n \text{ is odd} \end{cases}$$
$$\leq T(\lfloor n/2 \rfloor) + 2$$

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How many times?

- We want to solve $T(n) \leq T(\lfloor n/2 \rfloor) + 2$ with $T(1) = 0$
- How many times do we divide n by two until we reach 1?
- Denoting n by a binary number $n = b_m b_{m-1} \cdots b_2 b_1$
  - ⋆ $b_i \in \{0, 1\}$
  - ⋆ $b_m = 1$
  - ⋆ m is the number of digits in the binary representation of n
  - ⋆ $\lfloor n/2 \rfloor = b_m b_{m-1} \cdots b_2$
  - ⋆ After $m - 1$ 'divides' we reach 1
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- How many binary digits do you need to represent an integer n
- Note that an m digit number can represent a number from 2^m to $2^{m+1} - 1$
- Thus

$$2^m \leq n < 2^{m+1}$$
$$m \leq \log_2(n) < m + 1$$

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Towers of Hanoi

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Using the intermediate location, the monks began to move disks back and forth from the original pile to the pile at the new location, always keeping the piles in order (largest on the bottom, smallest on the top). According to the legend, before the monks could make the final move to complete the new pile in the new location, the temple would turn to dust and the world would end.

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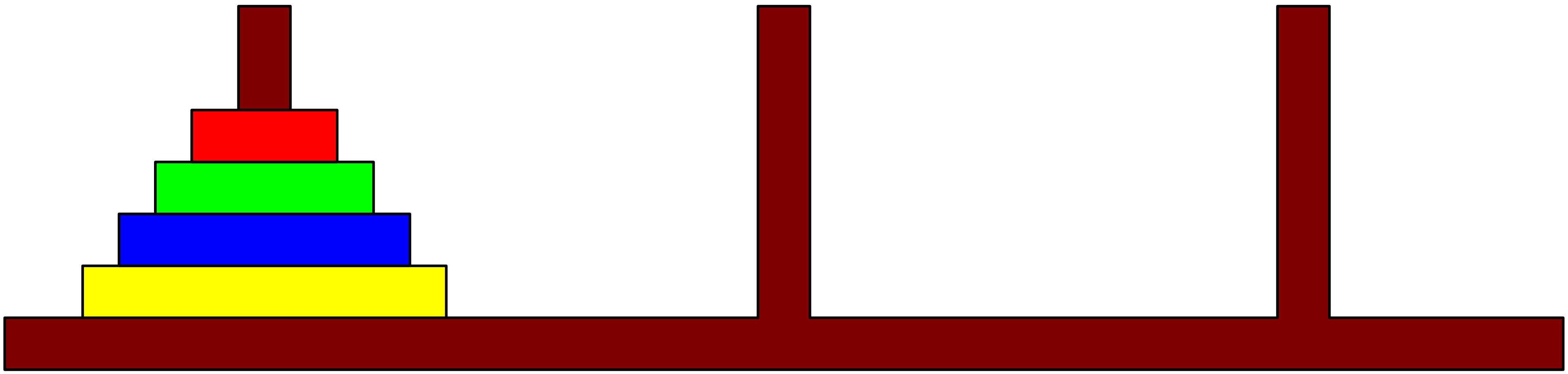

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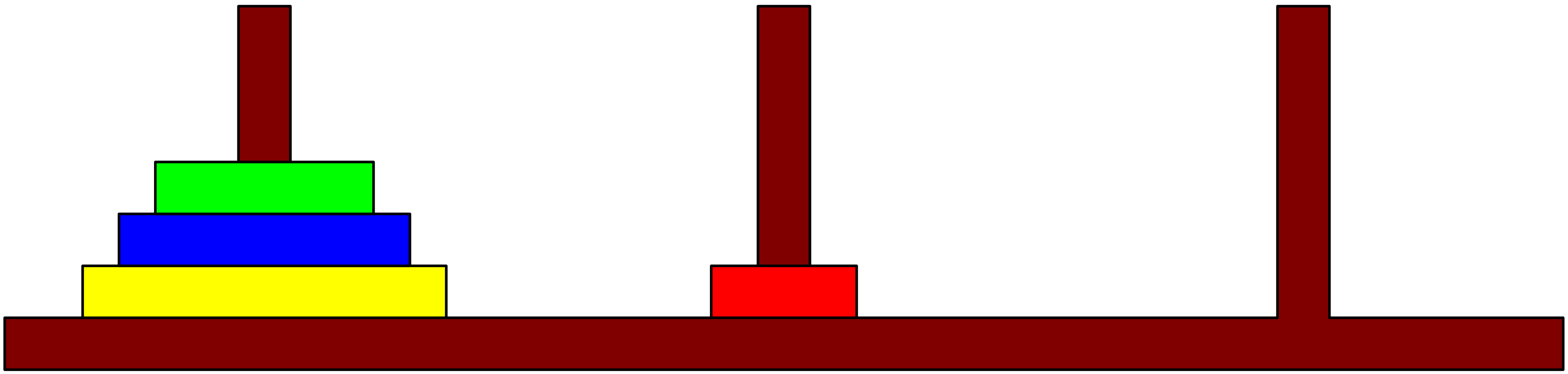

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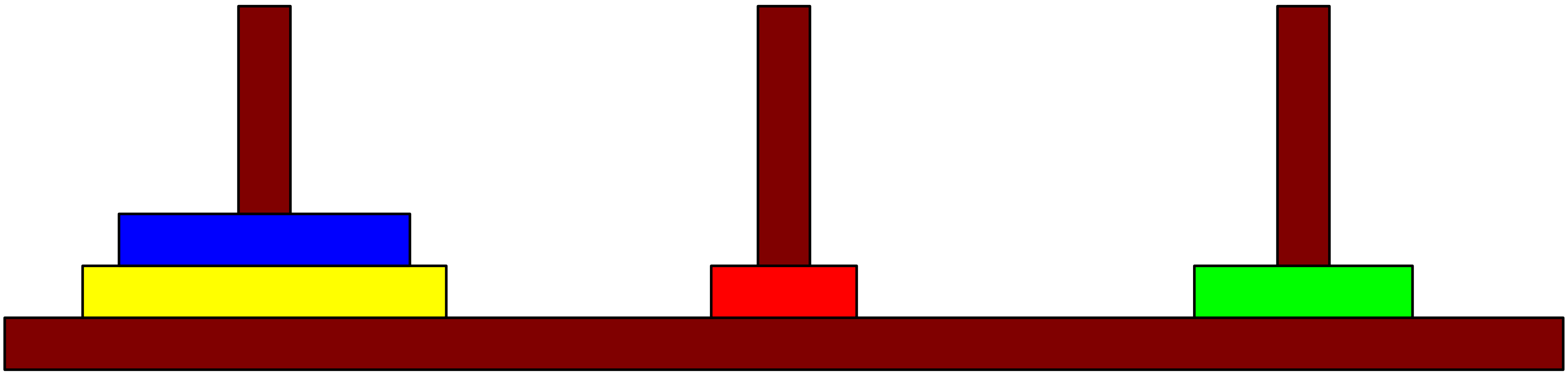

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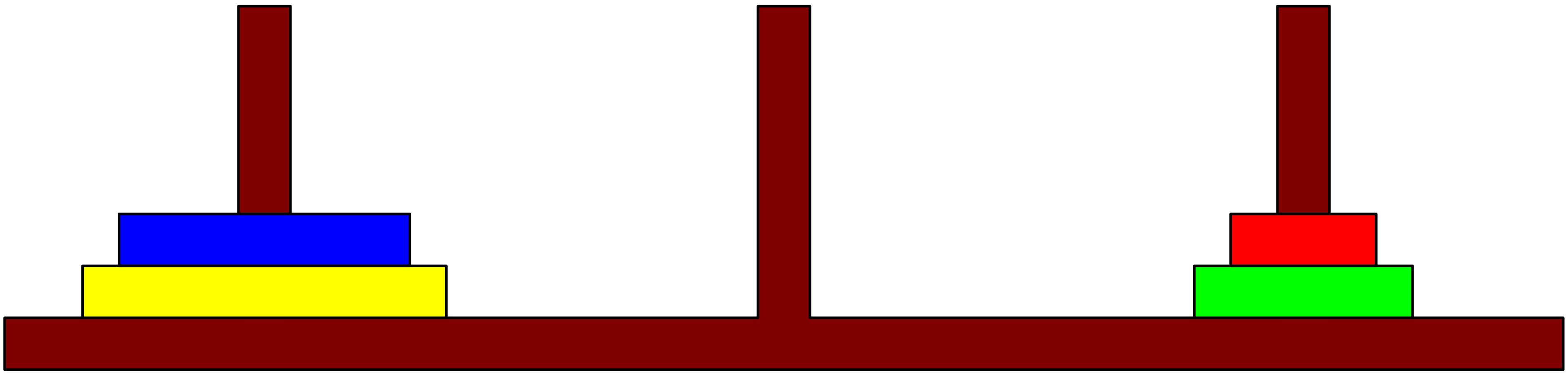

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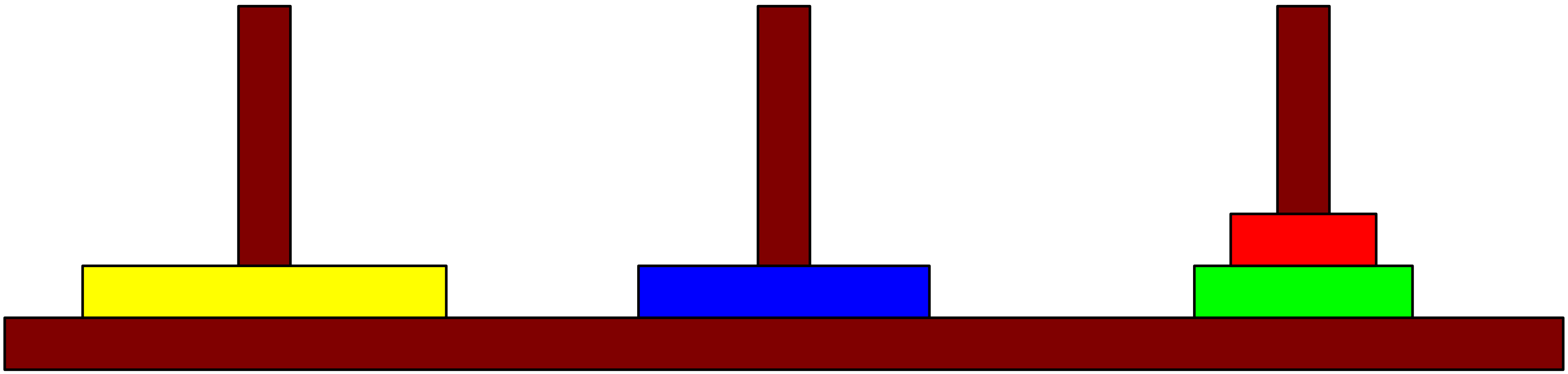

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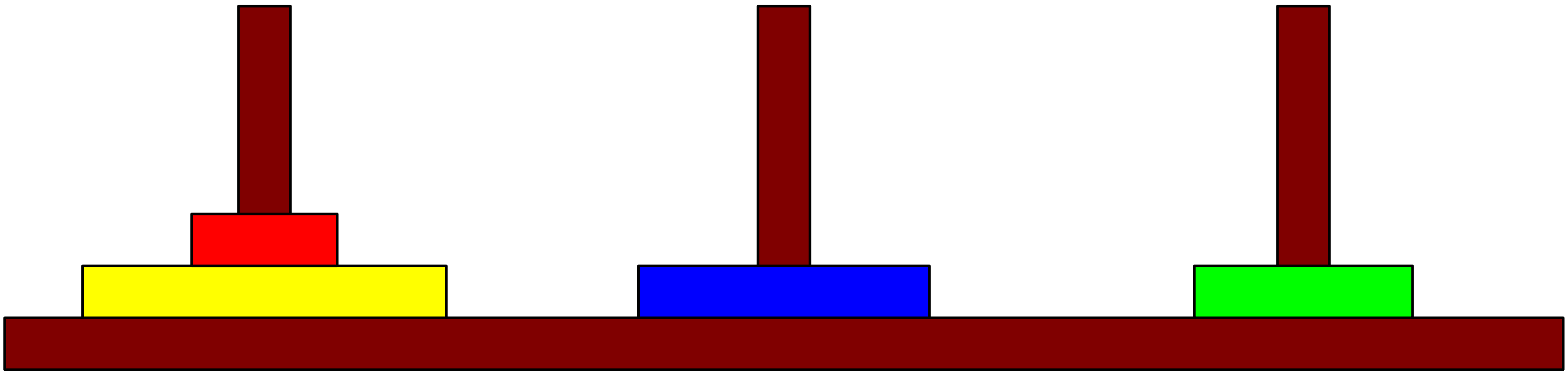

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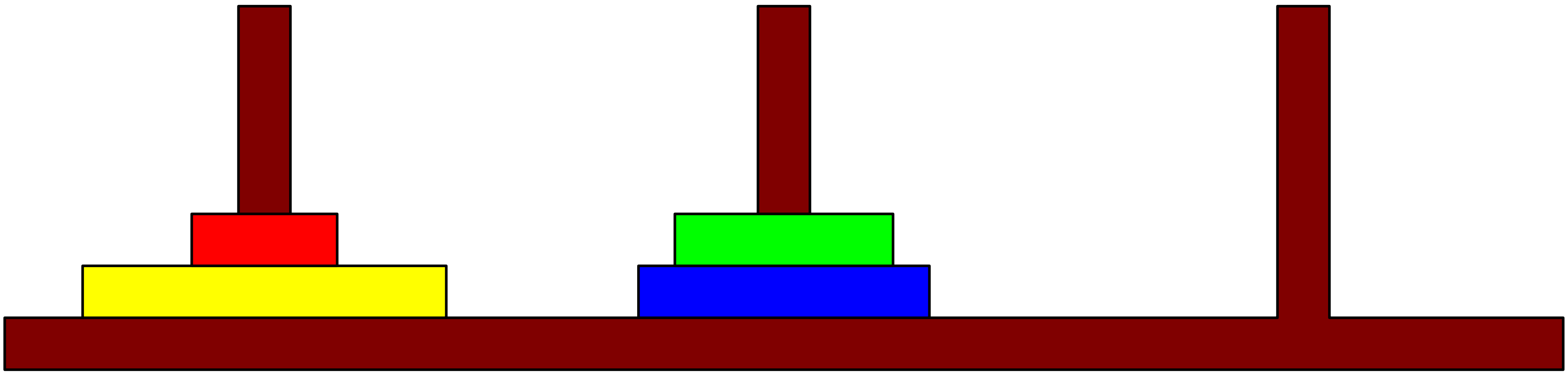

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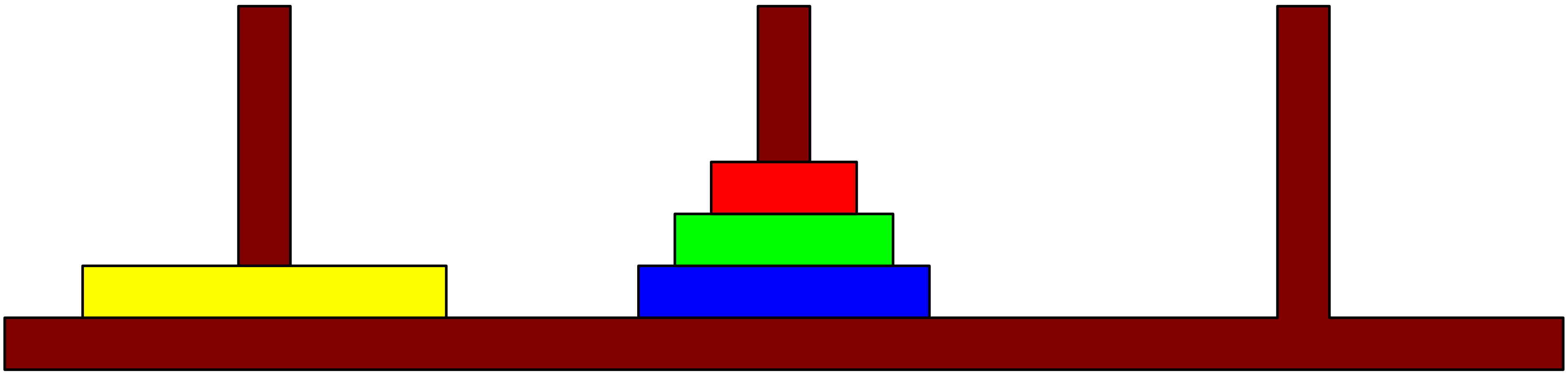

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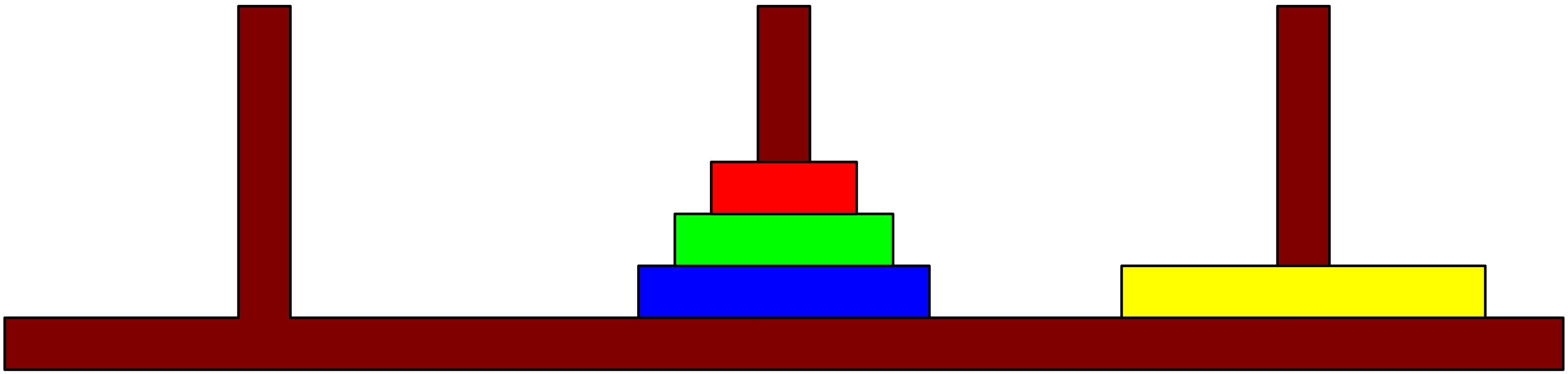

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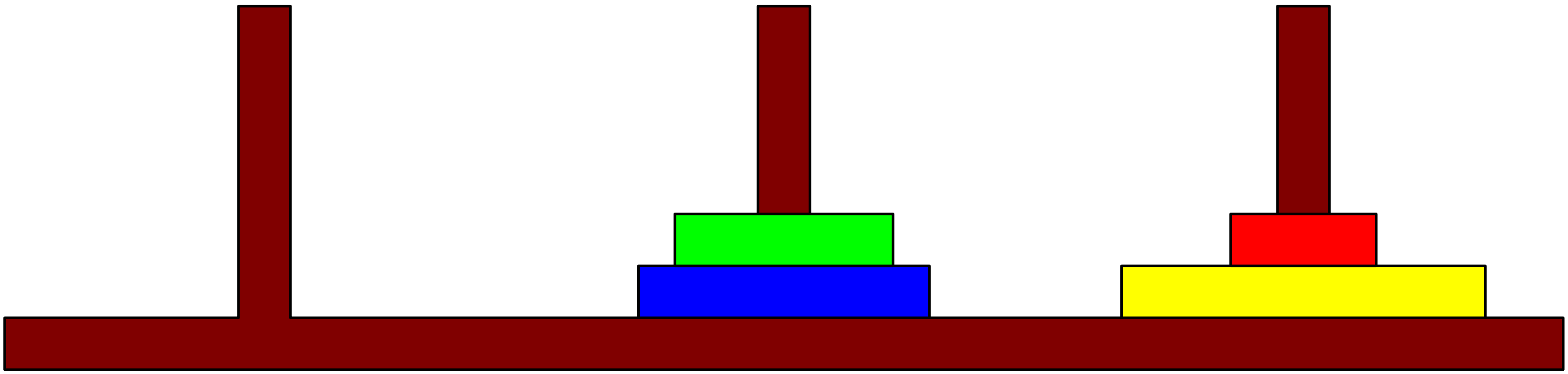

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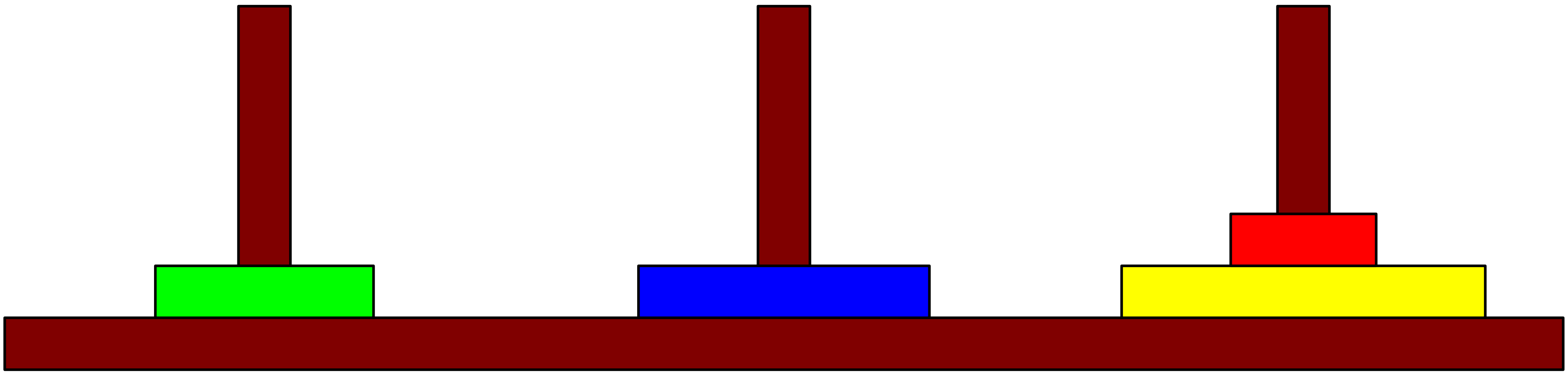

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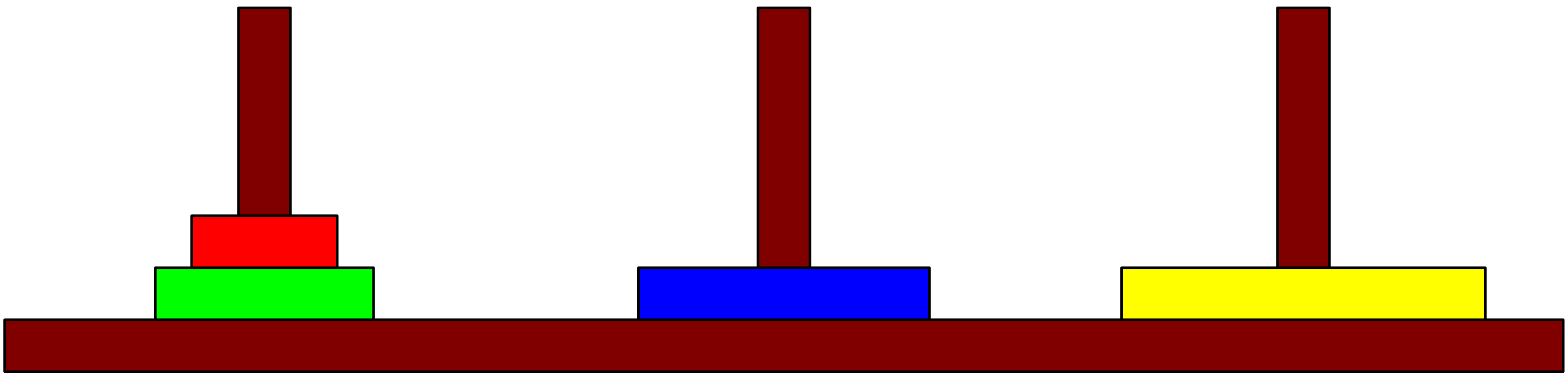

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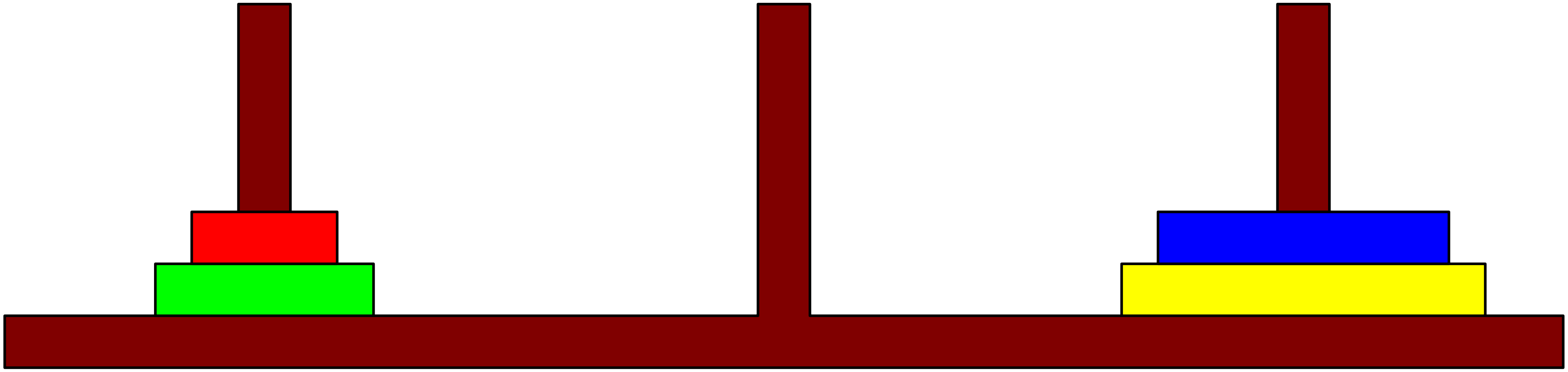

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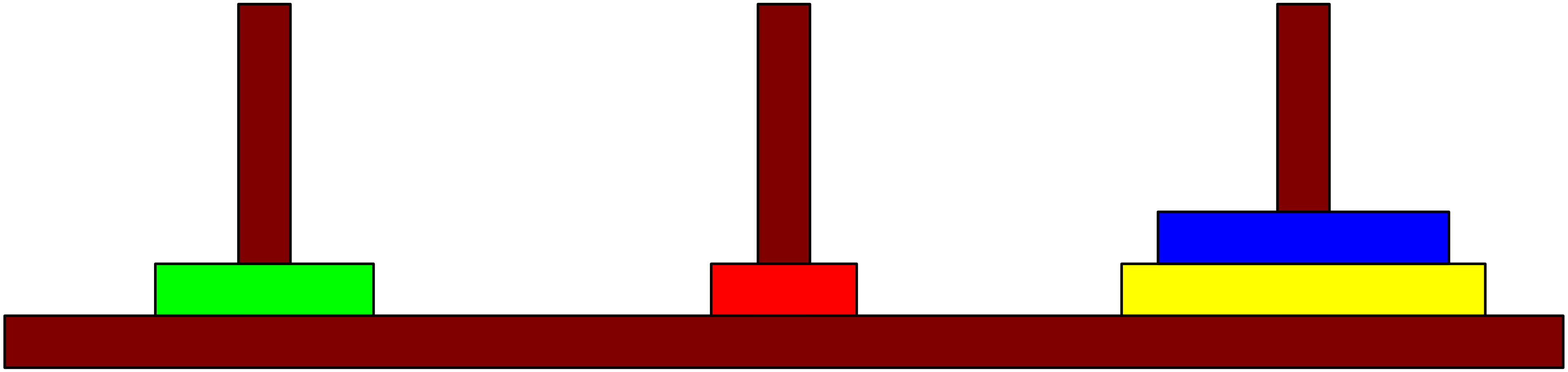

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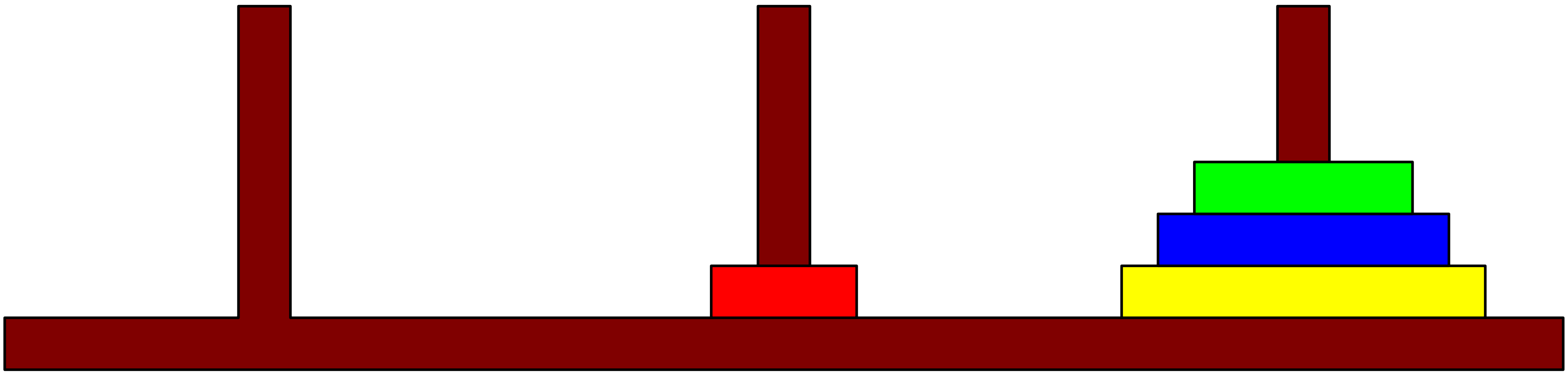

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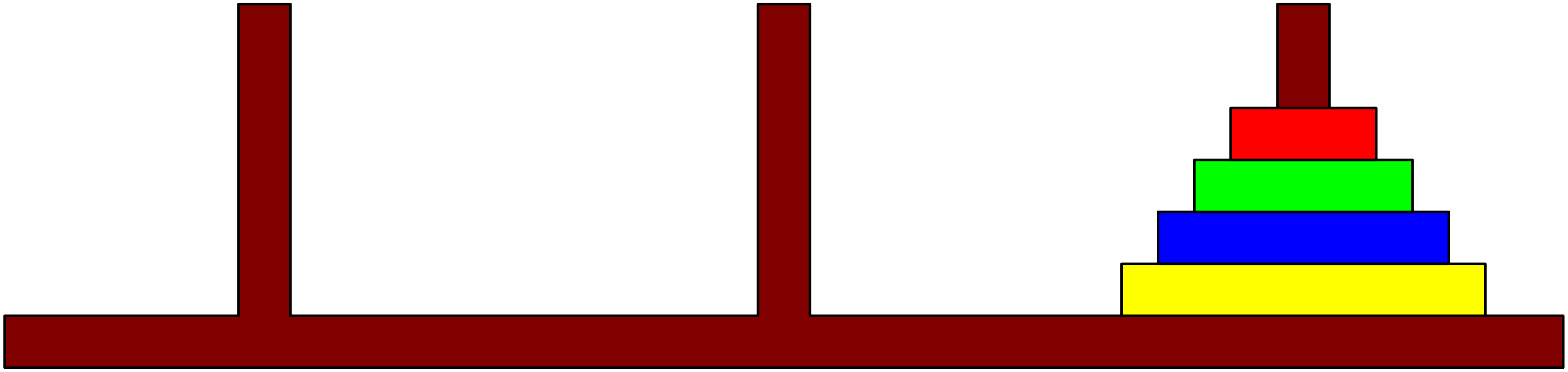

Algorithms in the Real World

- We require an algorithm to solve the towers of Hanoi
- Algorithms don't just apply to computers!
- If you try to solve the problem by hand you will discover that its quite fiddly
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- Let `move(X,Y)` denote the procedure of moving the top disk from peg `X` to peg `Y`
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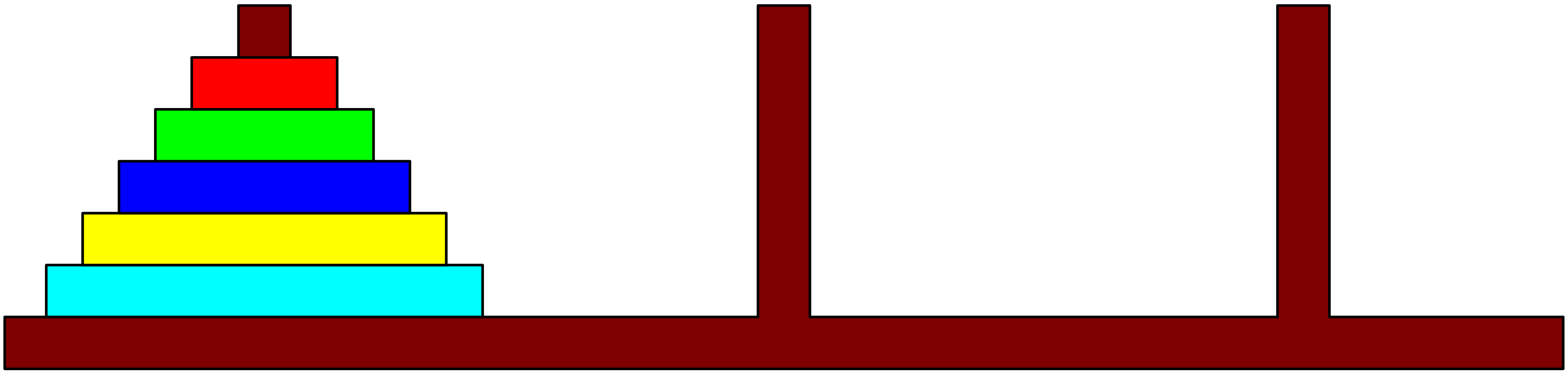

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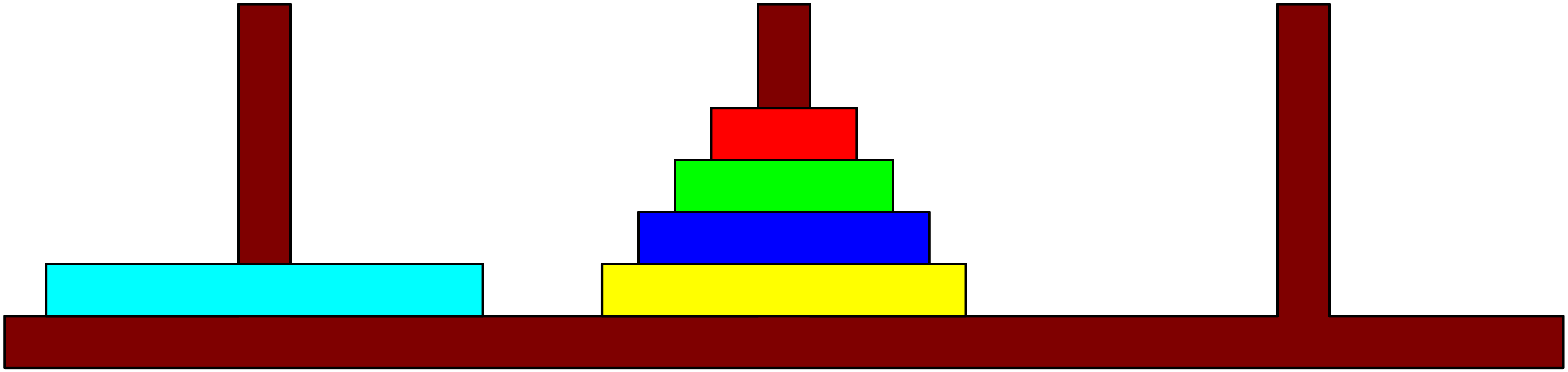

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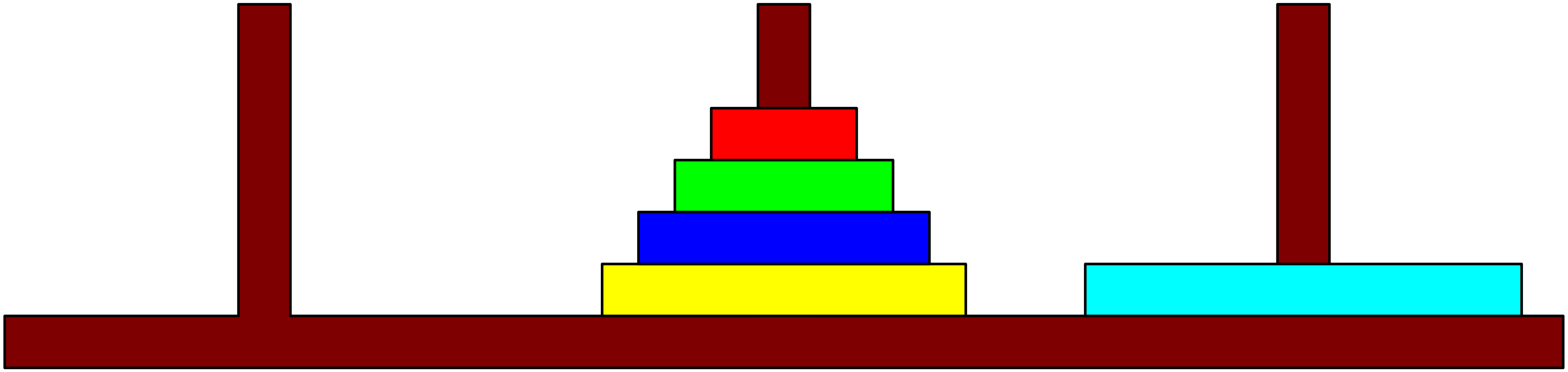

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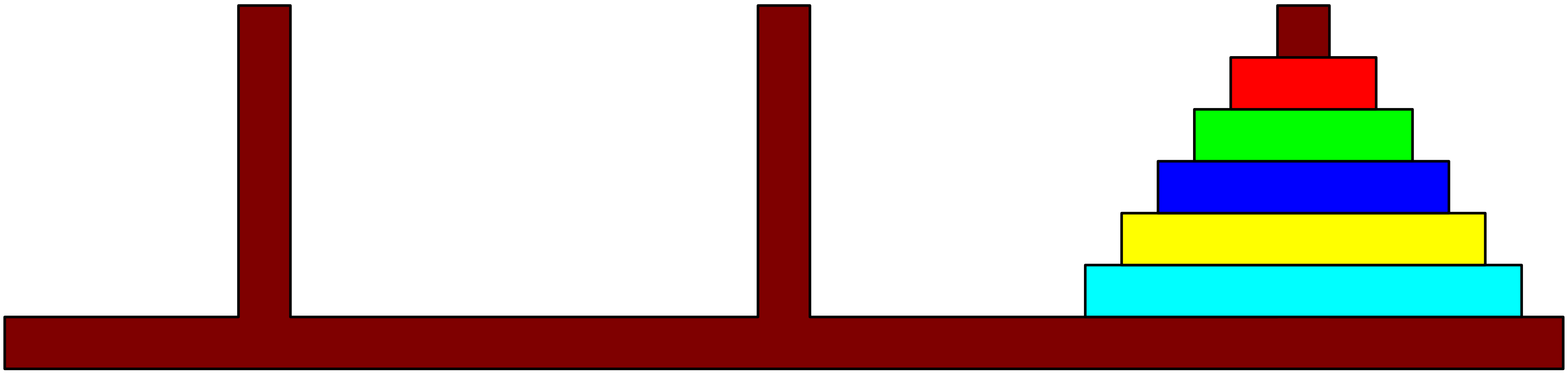

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Time Complexity of Recursion

- The time complexity for recursion can be tricky to calculate
- The procedure is to calculate the time, $T(n)$, taken for a problem of size n in terms of the time taken for a smaller problem
- The difficulty is to solve the recursion
- Recursive programs can be very quick (e.g. $O(\log n)$ for computing integer powers)
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- In case your interested, if it takes 1 second to move a disk it will take almost $585\,000\,000\,000$ years to move 64 disks

Lessons

- Recursion is a powerful tool for writing algorithms
- It often provides simple algorithms to otherwise complex problems
- Recursion comes at a cost (extra function calls)
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