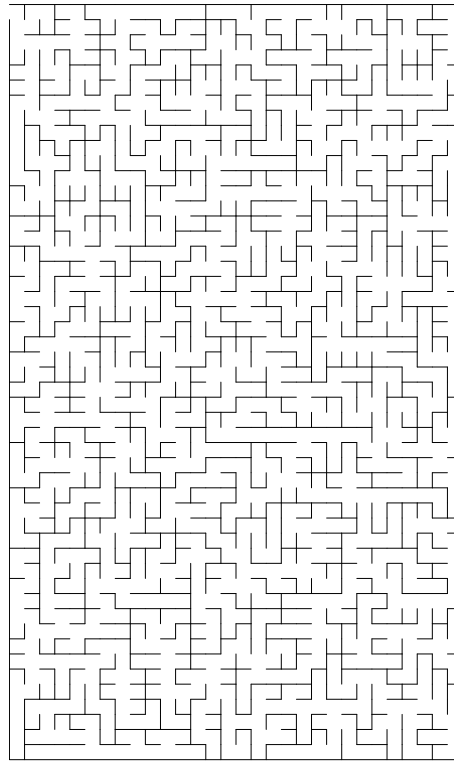


Algorithms and Analysis

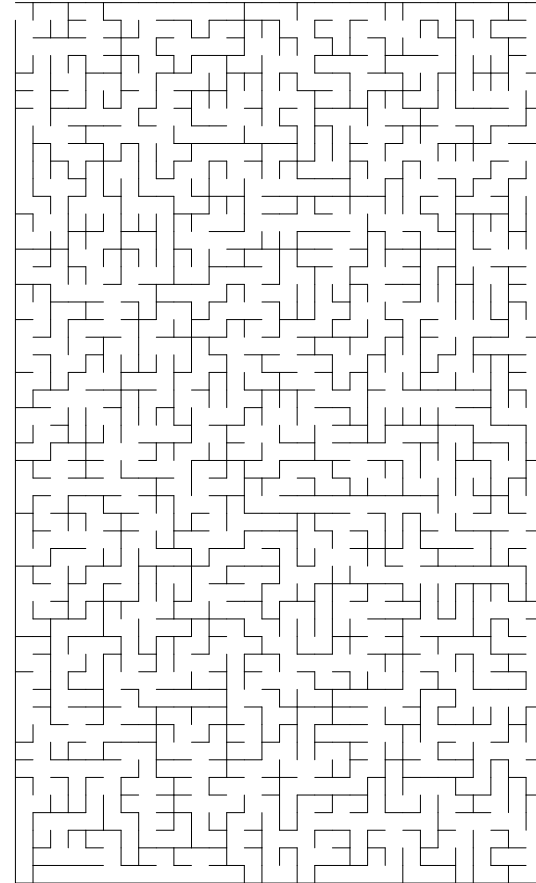
Lesson 14: *Use Arrays for Fast Set Algorithms*



Equivalent classes, Disjoint Set, Fast Sets

Outline

1. **Equivalent Classes**
2. Disjoint Sets
3. Fast Sets



Equivalence Relations

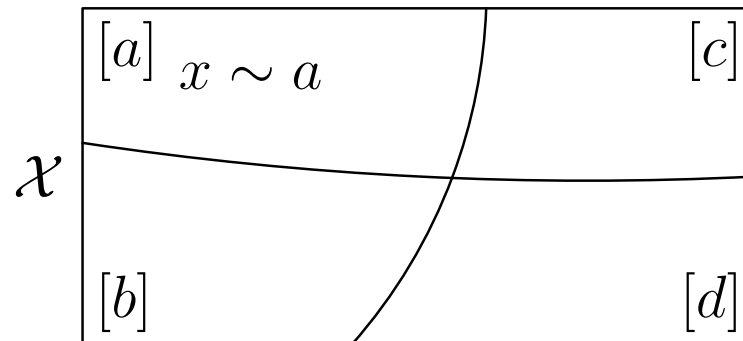
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(Reflexivity) For every element $x \in \mathcal{X}$, $x \sim x$

(Symmetry) For every two elements $x, y \in \mathcal{X}$ if $x \sim y$ then
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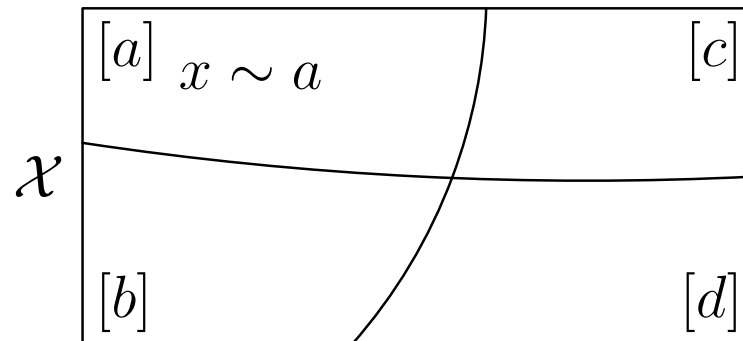
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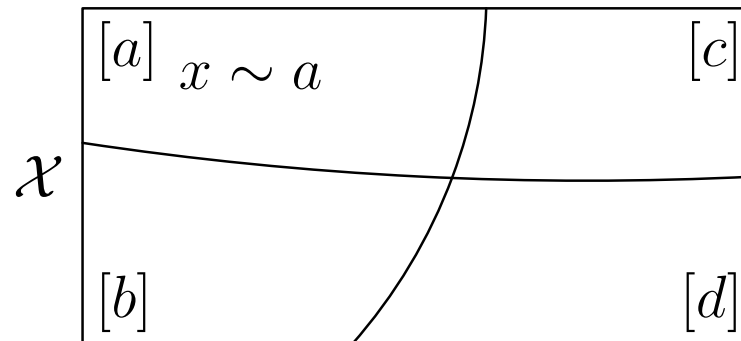
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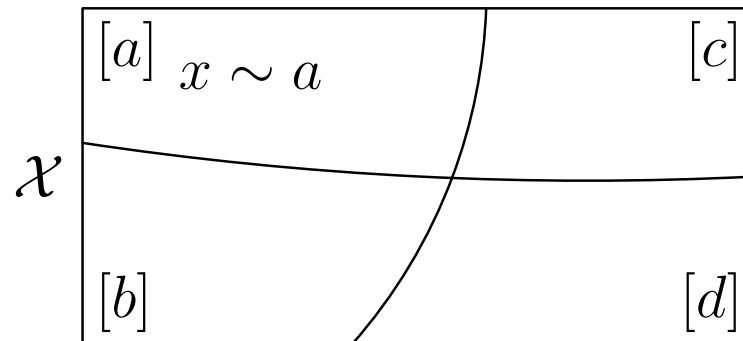
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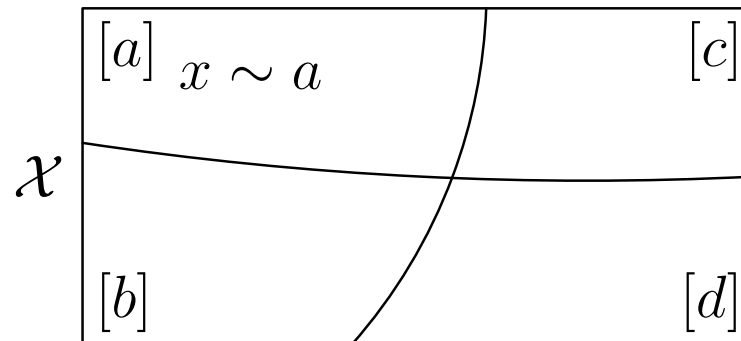
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Example of Equivalence Classes

- Although, equivalent classes sound very mathematical they often provide a useful formalisation of the real world
- E.g. Pairs of web pages with a link in each direction between them
- Consider web pages in the same equivalence class if you can get from one to the other by clicking links
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Dynamic Equivalence Classes

- Finding equivalence classes is rather easy using graph traversal algorithms
- However, as our web example suggests, there are applications where equivalence classes change over time
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- We will see this is a useful idea both for building mazes and (in a later lecture) for finding minimum spanning trees
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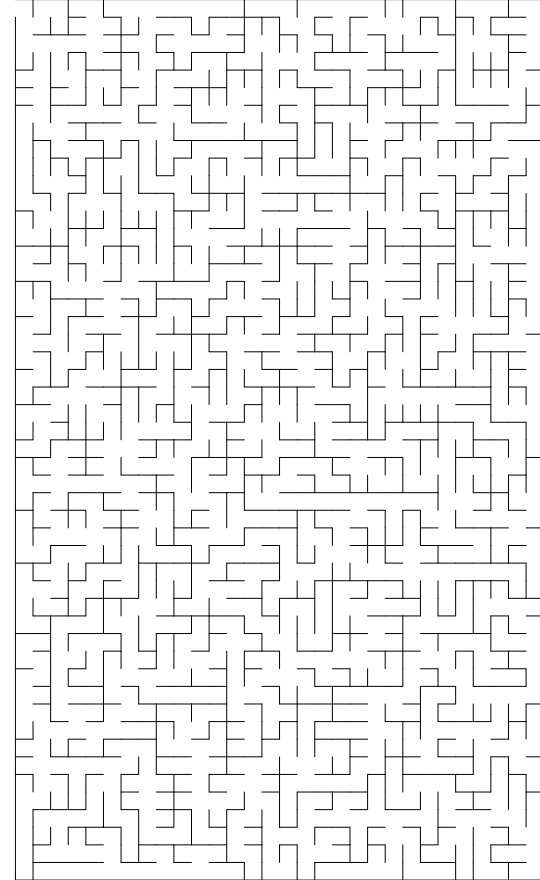
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Outline

1. Equivalent Classes
2. **Disjoint Sets**
3. Fast Sets



Union-Find

- In the union-find algorithm we have a set of objects $x \in \mathcal{S}$ which are to be grouped into equivalence classes $\mathcal{S}_1, \mathcal{S}_2, \dots$
- Initially each object is in its own equivalence class
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DisjointSets

- We want to create a class

```
class DisjointSets
{
    DisjointSets(int numElements) { /* Constructor */}
    int find(int x) { /* Find root */}
    void union_(int root1, int root2) { /* Union */}

private:
    int* s;
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- Where `find(x)` returns a unique identifier for the subset which element `x` belongs to
- The array `s` contains labelling information to implement `find(x)`

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The Union-Find Dilemma

- A natural algorithm to perform finds is to maintain an array returning a subset label for each element—this makes `find` fast
- However, every time we combine two subset we have to change all the labels in this array (taking $O(n)$ operations)
- If we are unlucky the cost of performing n unions is $\Theta(n^2)$
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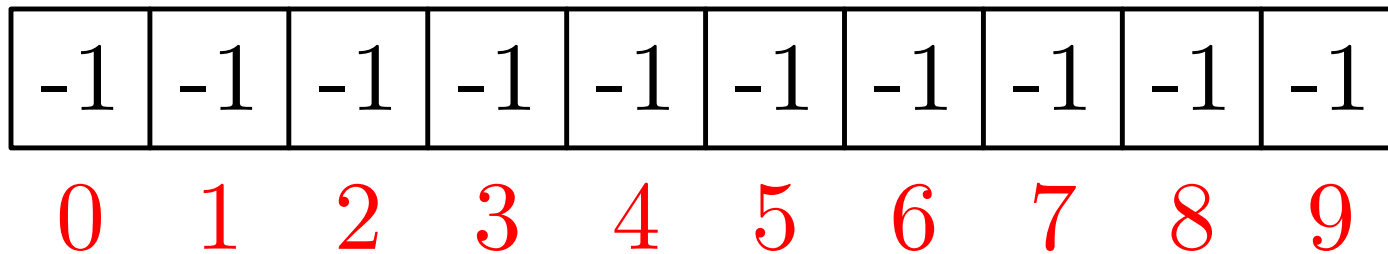
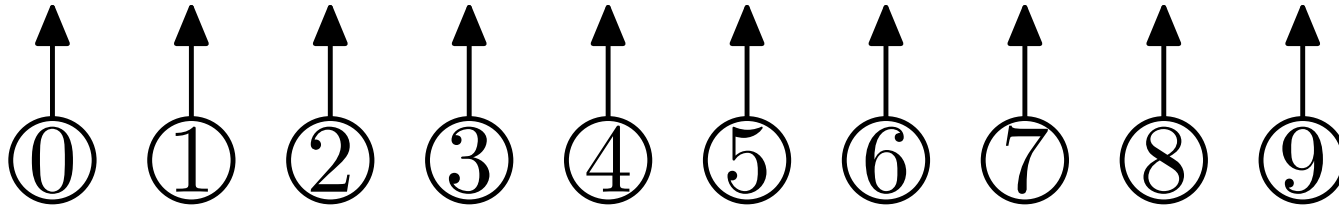
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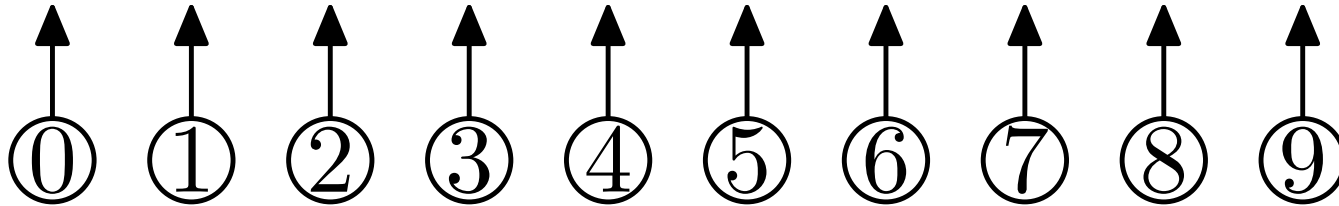
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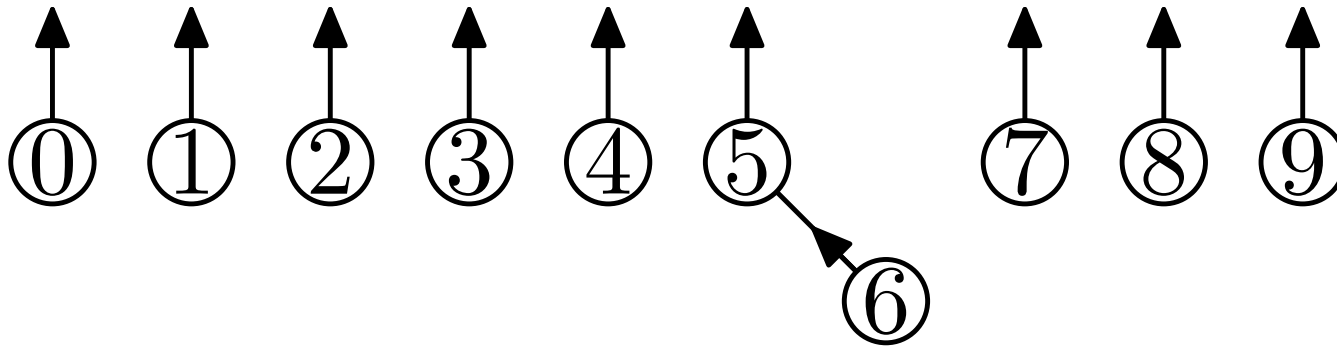
`union(find(5), find(6))`



-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

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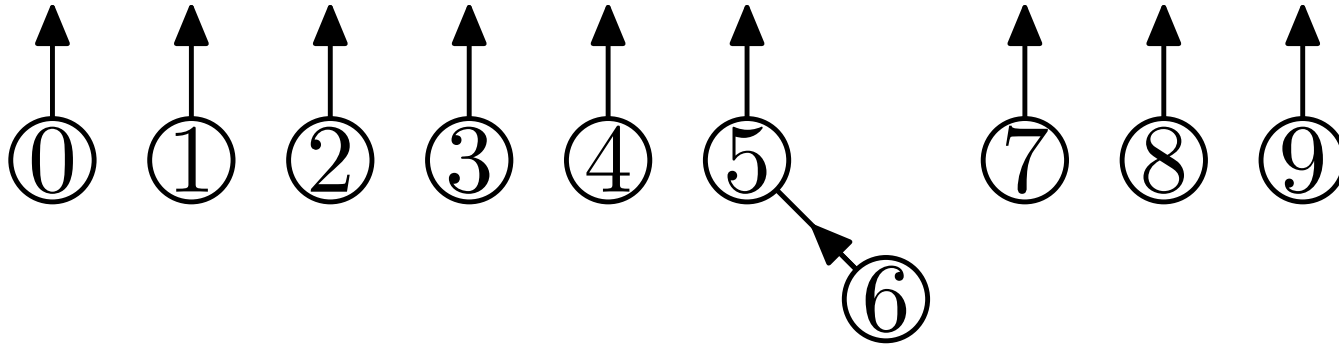
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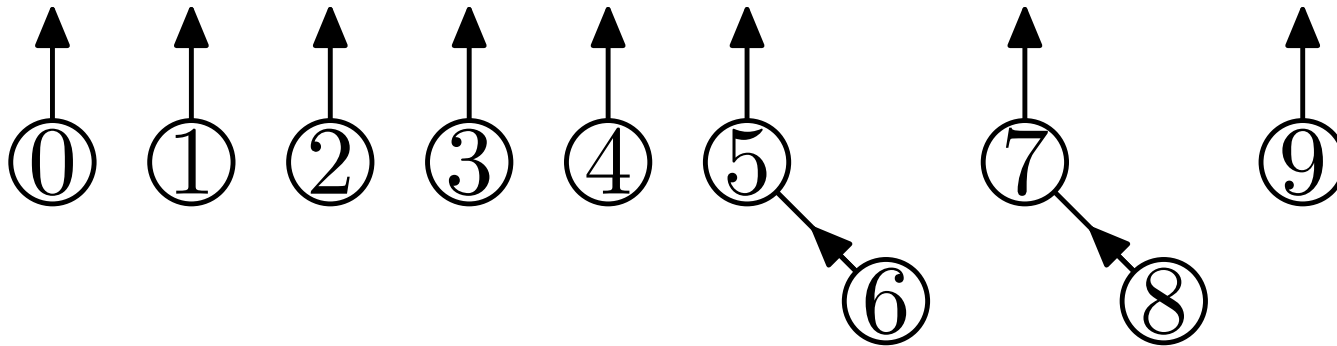
`union(find(7), find(8))`



-1	-1	-1	-1	-1	-2	5	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

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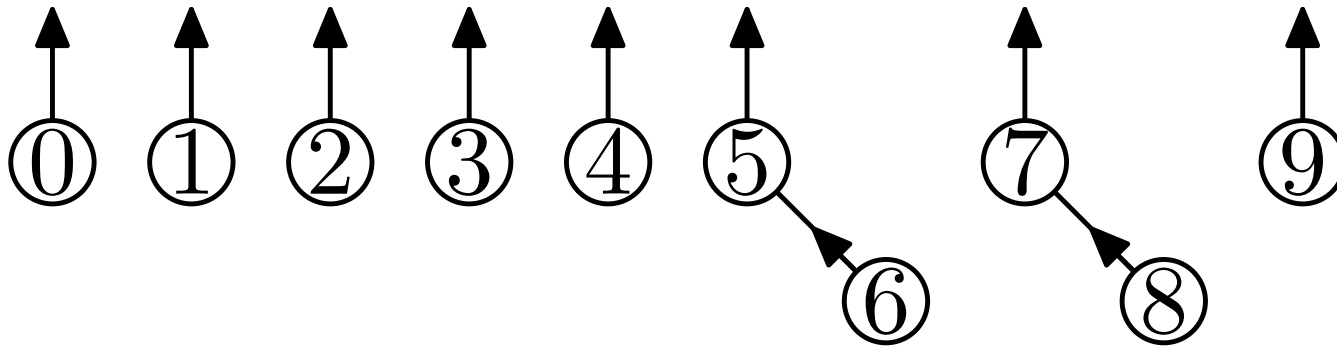
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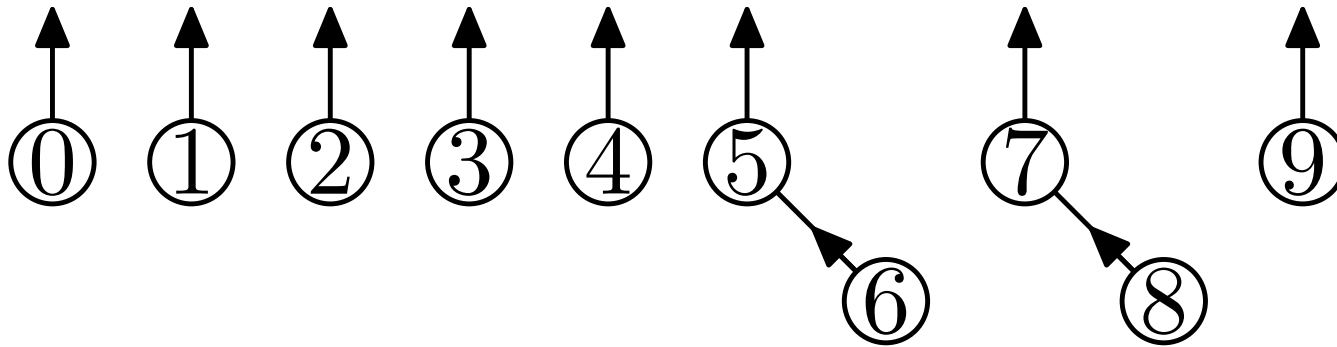
find(6)



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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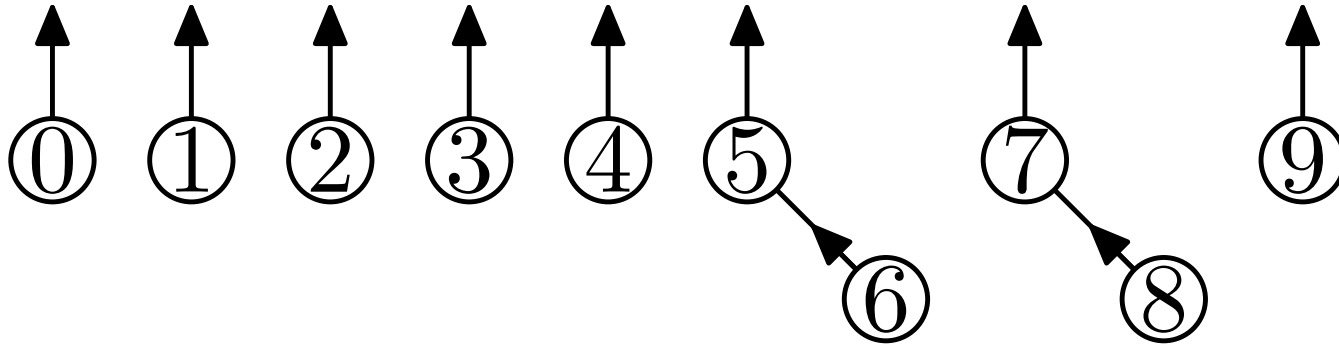
find(6)=5



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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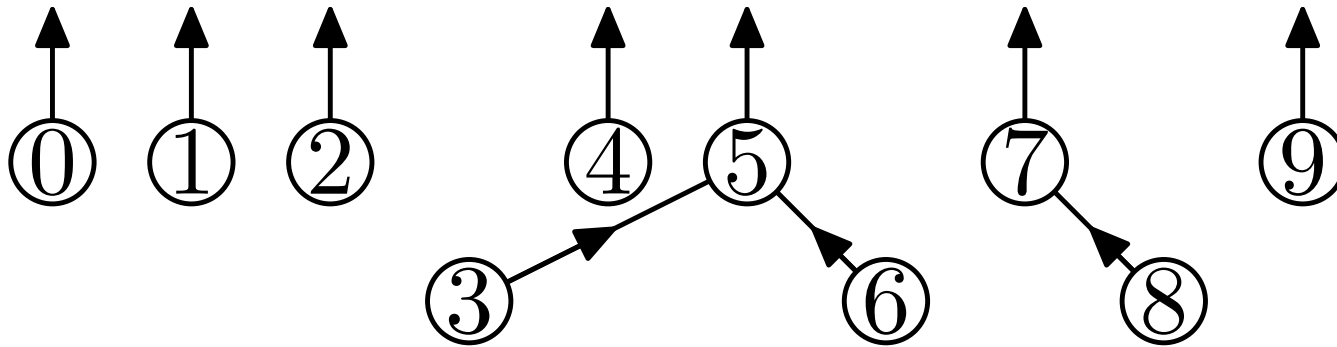
`union(find(3), find(6))`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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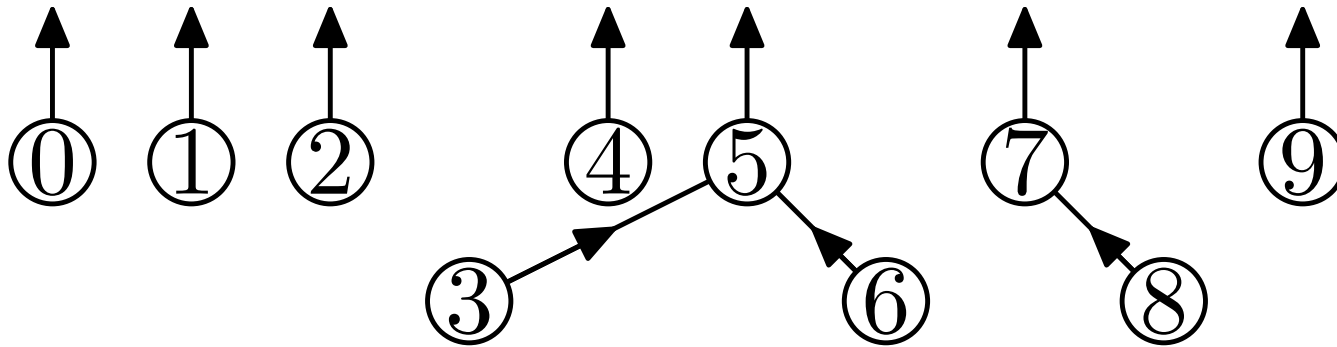
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-1	-1	-1	5	-1	-2	5	-2	7	-1
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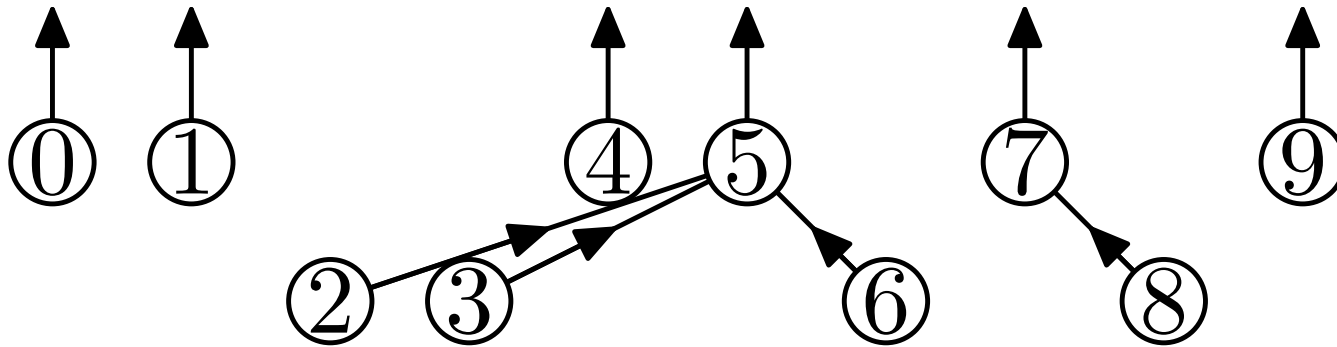
`union(find(2), find(6))`



-1	-1	-1	5	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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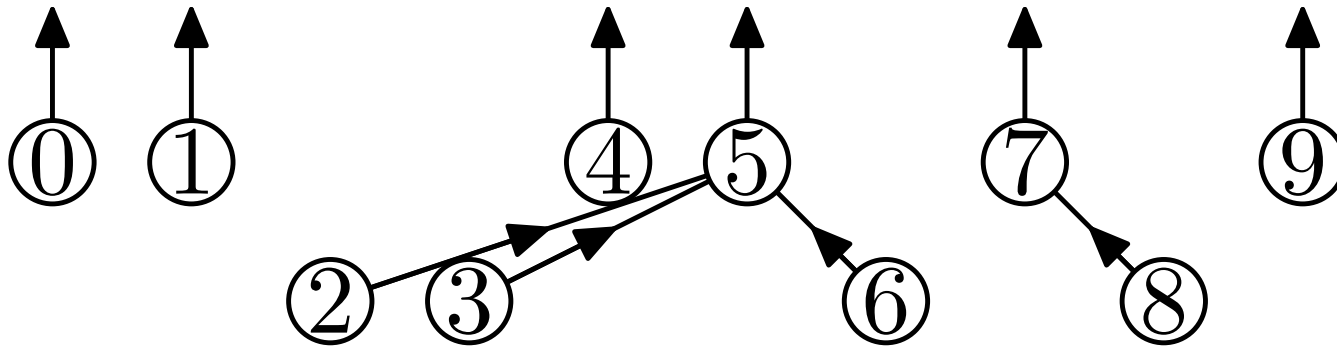
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-1	-1	5	5	-1	-2	5	-2	7	-1
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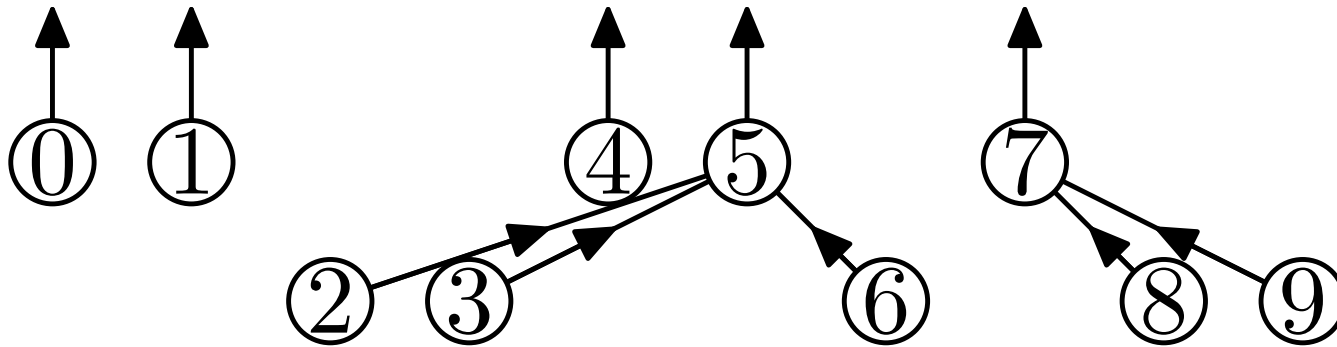
`union(find(9), find(8))`



-1	-1	5	5	-1	-2	5	-2	7	-1
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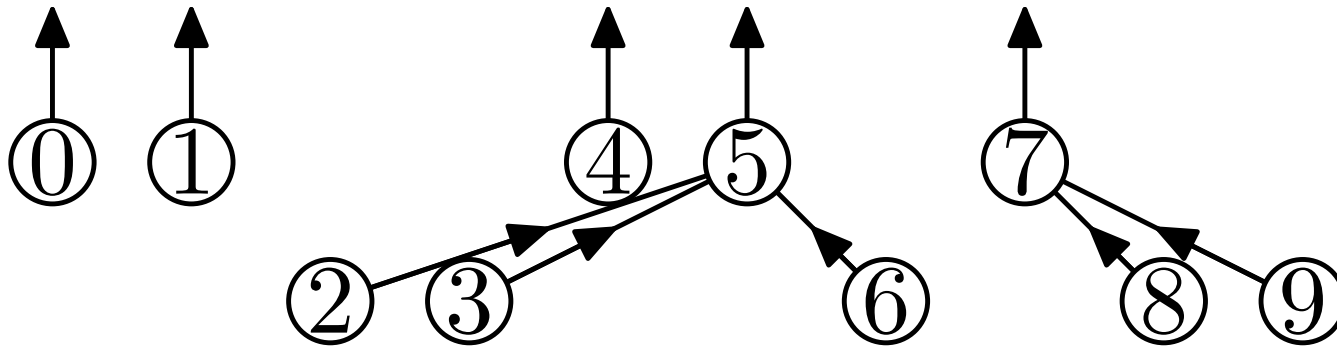
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-1	-1	5	5	-1	-2	5	-2	7	7
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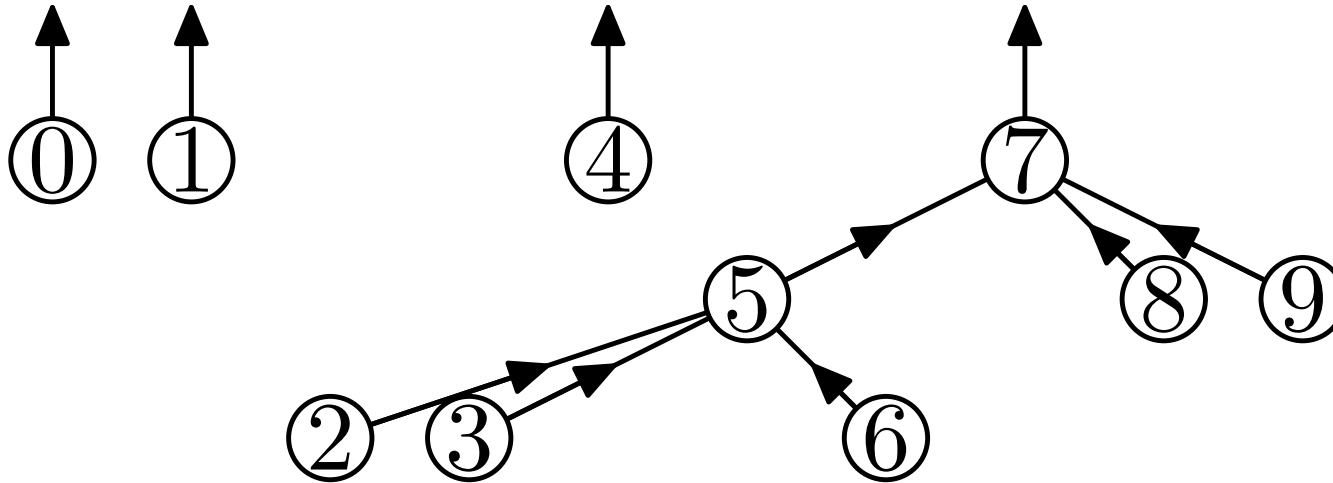
`union(find(9), find(3))`



-1	-1	5	5	-1	-2	5	-2	7	7
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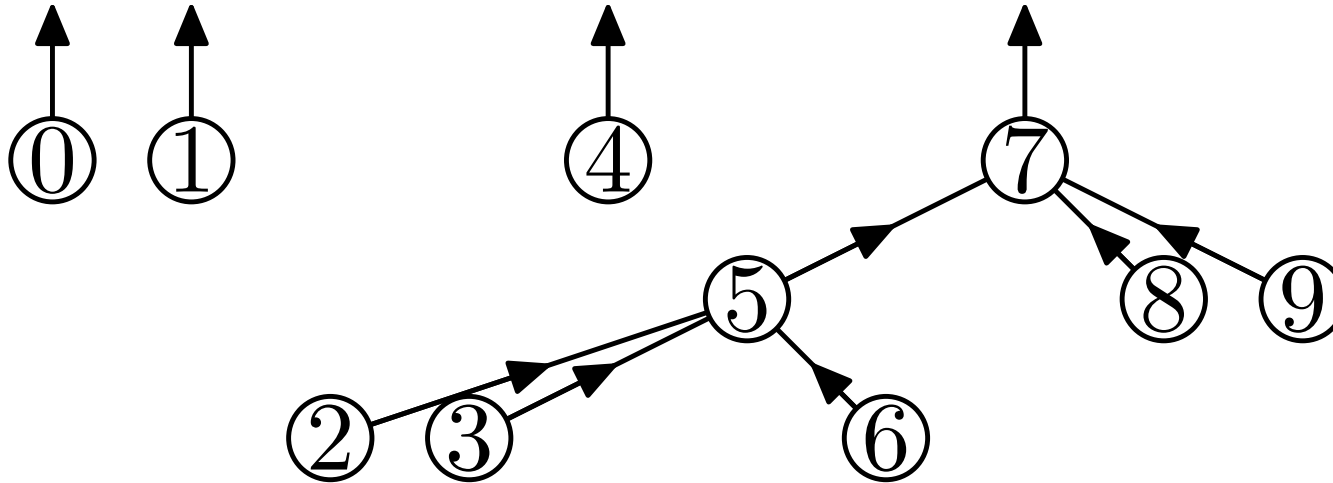
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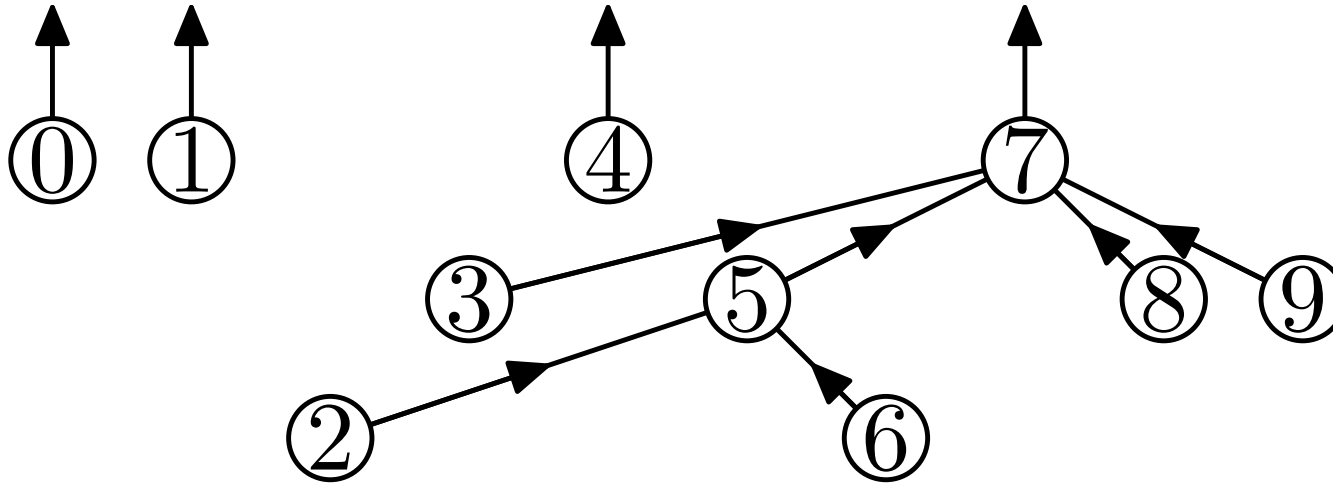
find(3)



-1	-1	5	5	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

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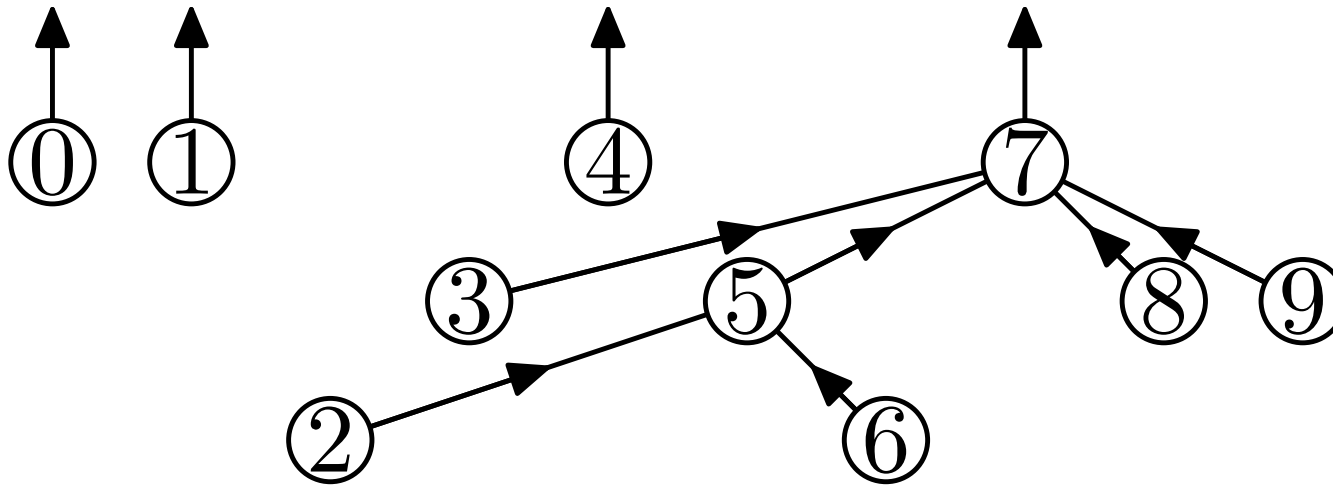
find(3)=7



-1	-1	5	7	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

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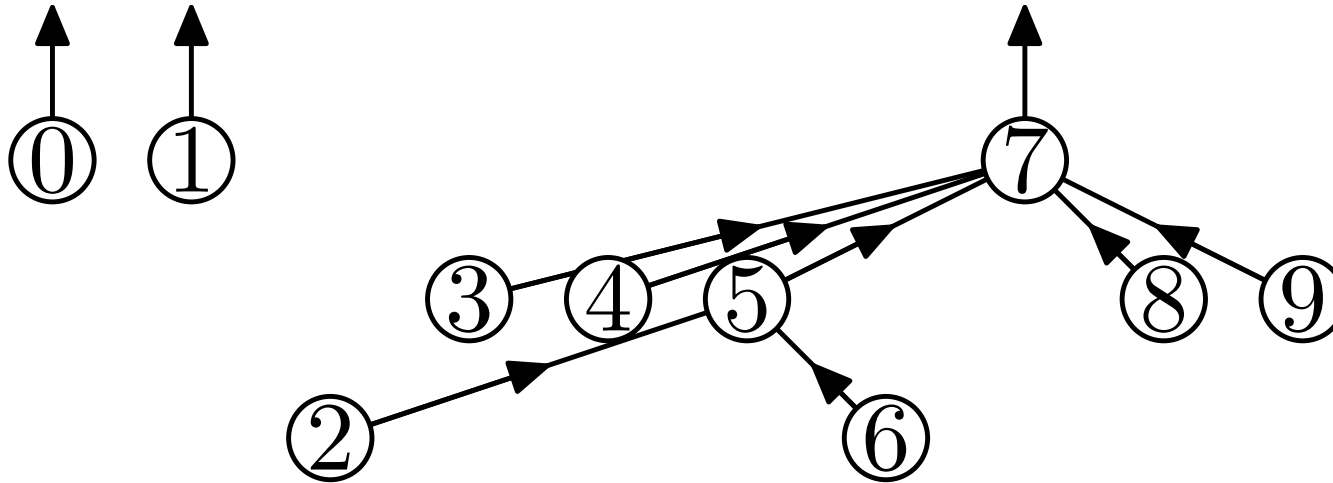
`union(find(3), find(4))`



-1	-1	5	7	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

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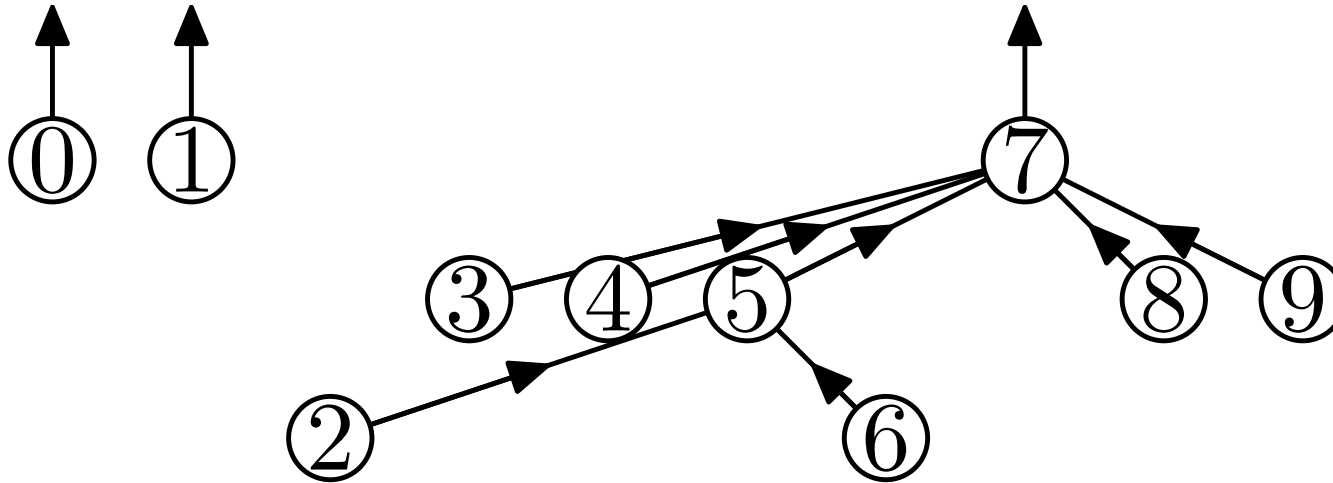
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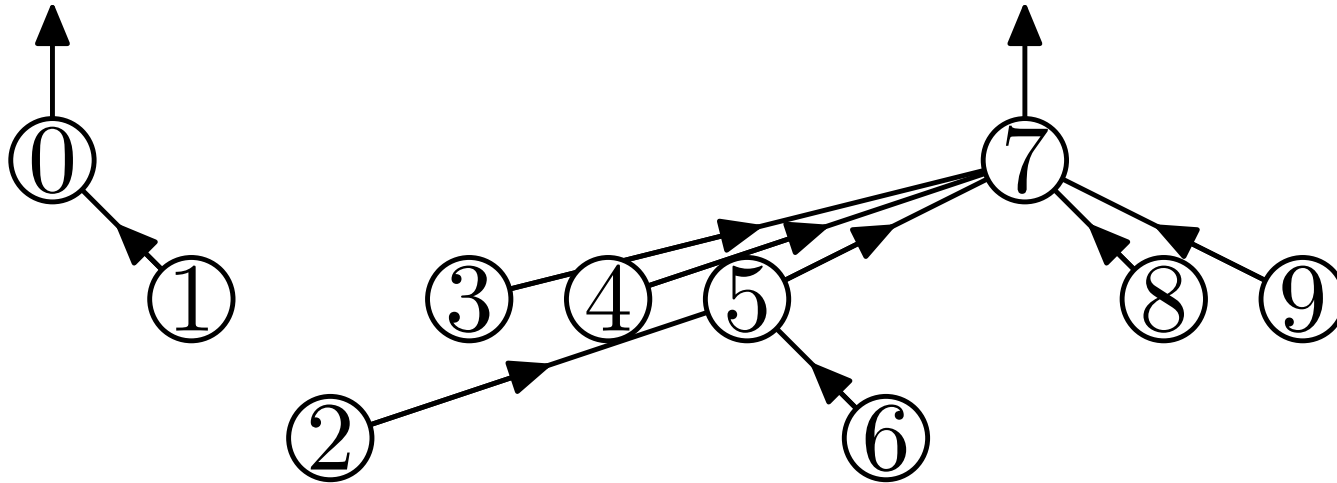
`union(find(0), find(1))`



-1	-1	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

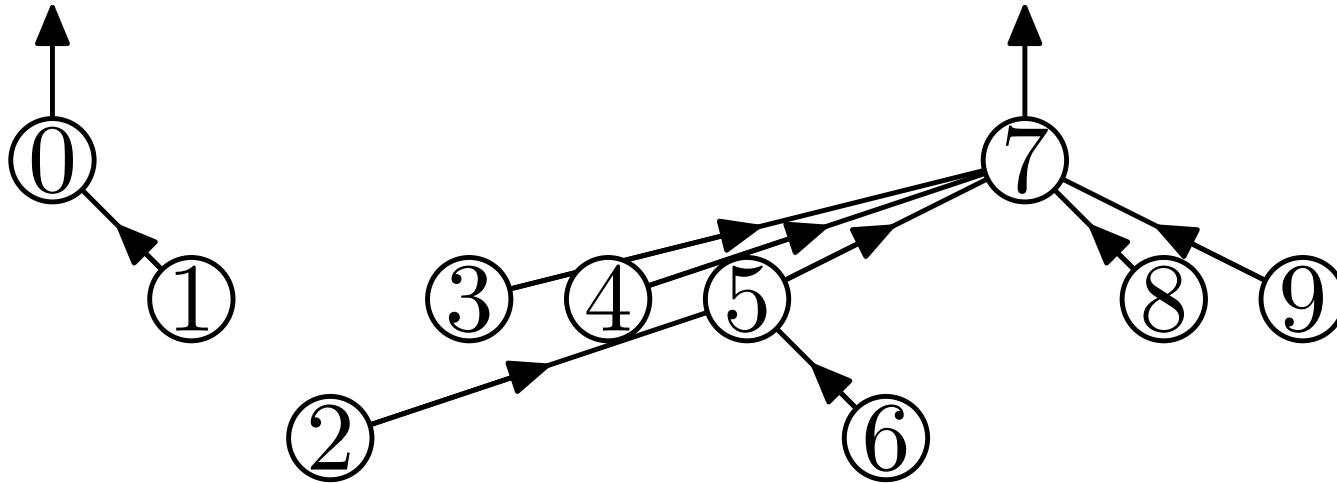
`union(find(0), find(1))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

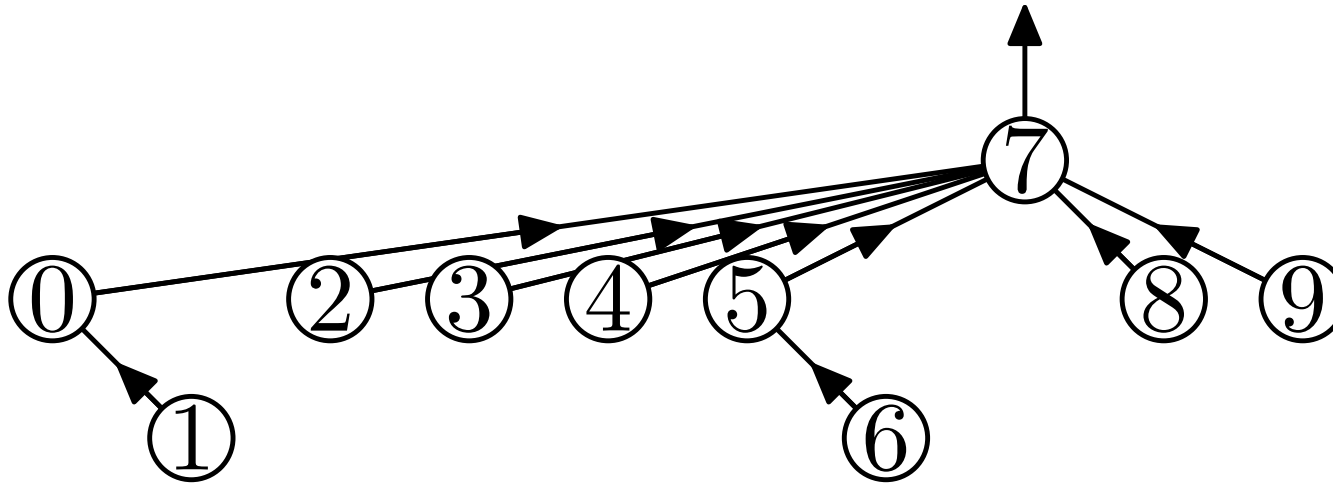
`union(find(1), find(2))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

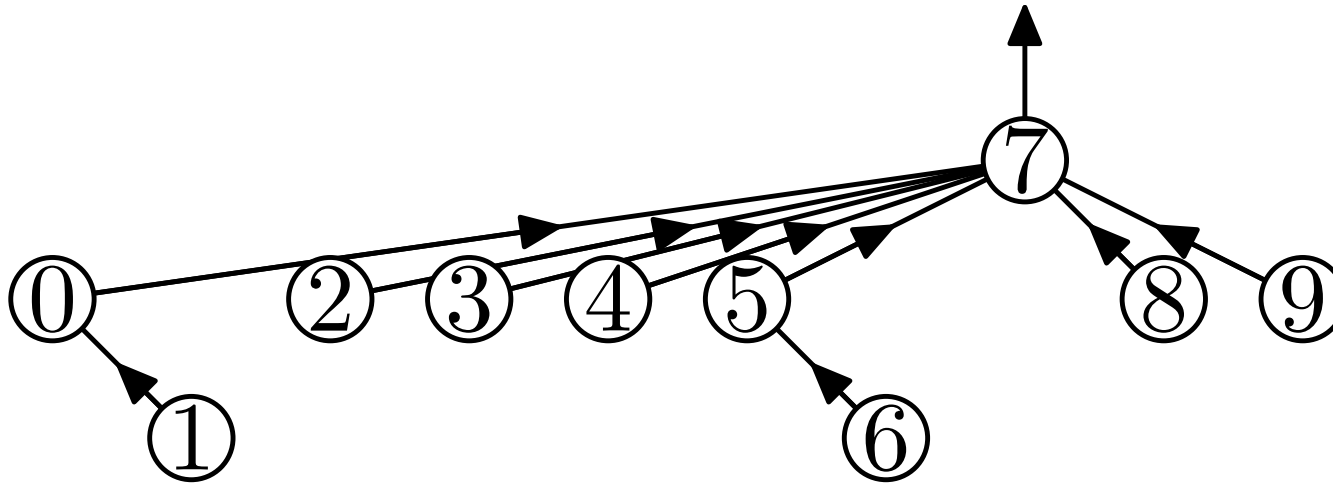
`union(find(1), find(2))`



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

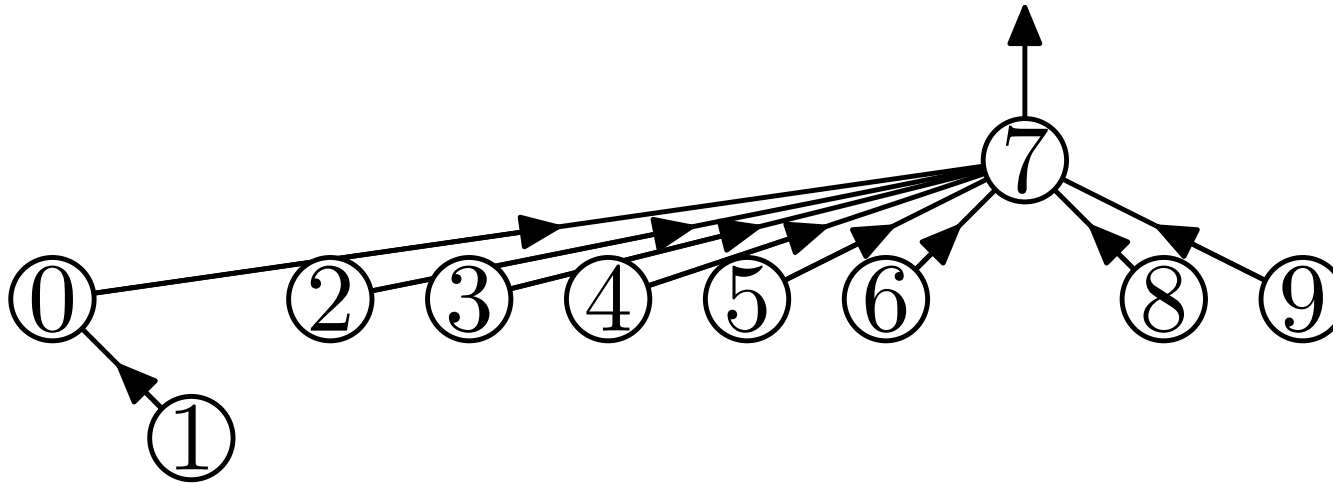
find(6)



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

$\text{find}(6)=7$



7	0	7	7	7	7	7	-3	7	7
0	1	2	3	4	5	6	7	8	9

Smart Union

```
DisjointSets::DisjointSets(int numElements)
{
    s = new int[numElements];
    for(int i=0; i<numElements; i++)
        s[i] = -1;           // roots are negative number
}
```

```
void DisjointSets::union_(int root1, int root2)
{
    if (s[root2]<s[root1]) {           // root2 is deeper
        s[root1] = root2;             // make root2 the root
    } else {
        if (s[root1]==s[root2])
            s[root1]--;               // update height if same
        s[root2] = root1;             // make root1 new root
    }
}
```

s[]

					10					20									-3									
					5					10									20									

Smart Union

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    }
}
```

Diagram illustrating an array `s[]` with 25 cells. The array is divided into three segments: the first 5 cells contain the value 10, the next 5 cells contain the value 20, and the last 5 cells contain the value -3. The indices 5, 10, and 20 are marked below the array.

Smart Union

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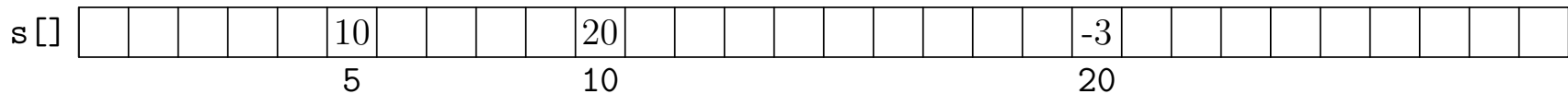
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    } else {
        if (s[root1]==s[root2])        // update height if same
            s[root1]--;
        s[root2] = root1;              // make root1 new root
    }
}
```

s []						10					20									-3								
						5					10									20								

Path Compression

- To speed up `find` we relabel all nodes we visit during `find` by the root label

```
int DisjointSets::find(int index)
{
    if (s[index]<0)
        return index;
    else
        return s[index] = find(s[index]);
}
```



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s[]					10					20								-3										
					5					10								20										

Mazes

- Union-Find is a data structure which can occur in very different applications
- One application is building a maze
- Start from a complete lattice
- Remove a randomly chosen edge if it connects two unconnected regions
- Stop when the start and end cell are connected
- Or better after all cells are connected

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29
30	31	32	33	34
35	36	37	38	39
40	41	42	43	44
45	46	47	48	49

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Time Complexity of Union-Find

- If we perform M finds and N unions then the time complexity is $O(M \log_2^*(N))$
- Where $\log_2^*(N)$ is the number of times you need to apply the logarithm function before you get a number less than 1
- In practice $\log_2^*(N) \leq 5$ for all conceivable N

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$$\log_2(10^{80}) = 265.75$$

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$$\log_2(\log_2(10^{80})) = 8.0539$$

Time Complexity of Union-Find

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$$\log_2(\log_2(\log_2(10^{80}))) = 3.0097$$

Time Complexity of Union-Find

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$$\log_2(\log_2(\log_2(\log_2(10^{80})))) = 1.5896$$

Time Complexity of Union-Find

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$$\log_2(\log_2(\log_2(\log_2(\log_2(10^{80})))) = 0.66868$$

Time Complexity of Union-Find

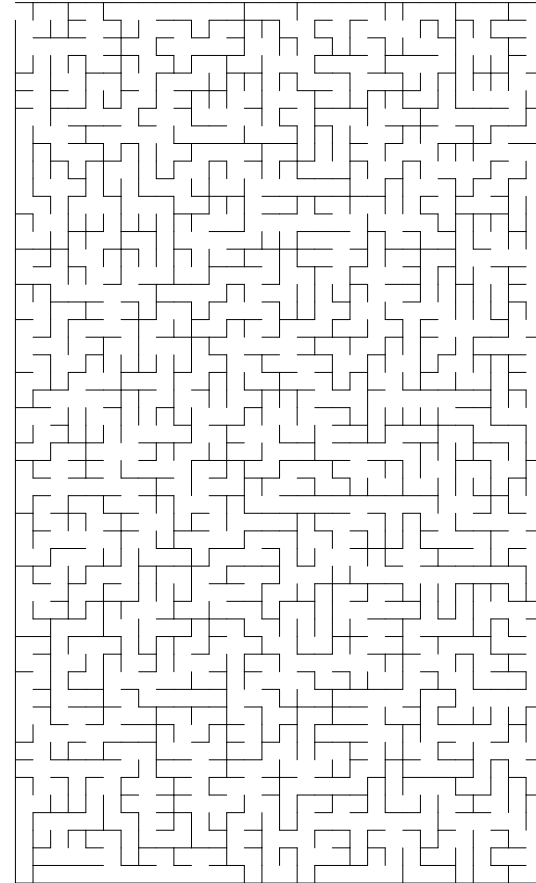
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$$\log_2(\log_2(\log_2(\log_2(\log_2(10^{80})))) = 0.66868$$

- The proof of this time complexity is rather involved

Outline

1. Equivalent Classes
2. Disjoint Sets
3. **Fast Sets**



Comparison of Sets

- Binary Search Trees: $O(\log_2(n))$, general purpose
- Hash tables: $O(1)$, but need to compute hash, slow iterator when sparse, general purpose
- B-trees: $O((k - 1) \log_k(n))$ very complicated, used for large amounts of data
- Tries: $O(\text{len}(\text{seq})) = O(\log_k(n))$ for large k expensive in memory, complicated to code efficiently

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What Set to Use?

- A PhD student and I were working on writing a fast solver for a combinatorial optimisation problem
- We had to choose one variable to change out of a small number of possible variables
- Each time we changed a variable then we had to update the list of possible variables (remove some variables add others)
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- We wanted a data structure which had quick add and remove and where we could choose a variable at random—what should we use?

Bounded Set

- One special feature is that we knew we only wanted the set to contain integers between 0 and n (where n might be 100 000)
- This allowed us to use an array to represent whether an integer belong to that set
- But how do we find a random element of the set quickly?

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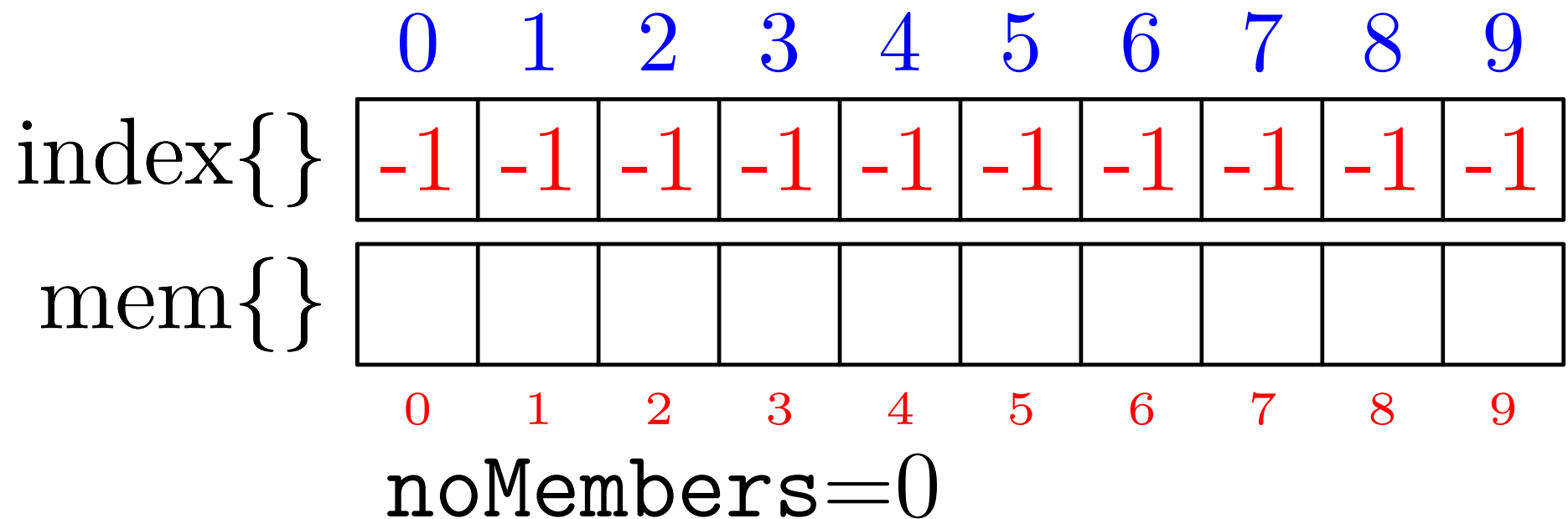
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Bounded Set

- One special feature is that we knew we only wanted the set to contain integers between 0 and n (where n might be 100 000)
- This allowed us to use an array to represent whether an integer belong to that set
- But how do we find a random element of the set quickly?
- Use another array of course!

FastSet

FastSet



FastSet

add(4)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
----	----	----	----	----	----	----	----	----	----

mem{}

--	--	--	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=0

FastSet

	true									
	0	1	2	3	4	5	6	7	8	9
index{}	-1	-1	-1	-1	0	-1	-1	-1	-1	-1
mem{}	4									
	0	1	2	3	4	5	6	7	8	9
noMembers=1										

FastSet

add(9)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	-1	-1	-1
----	----	----	----	---	----	----	----	----	----

mem{}

4									
---	--	--	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=1

FastSet

	true									
	0	1	2	3	4	5	6	7	8	9
index{}	-1	-1	-1	-1	0	-1	-1	-1	-1	1
mem{}	4	9								
	0	1	2	3	4	5	6	7	8	9
noMembers=2										

FastSet

add(7)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	-1	-1	1
----	----	----	----	---	----	----	----	----	---

mem{}

4	9								
---	---	--	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=2

FastSet

true										
0 1 2 3 4 5 6 7 8 9										
index{}	-1	-1	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7							
0 1 2 3 4 5 6 7 8 9										
noMembers=3										

FastSet

add(4)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	2	-1	1
----	----	----	----	---	----	----	---	----	---

mem{}

4	9	7							
---	---	---	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=3

FastSet

	false									
	0	1	2	3	4	5	6	7	8	9
index{}	-1	-1	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7							
	0	1	2	3	4	5	6	7	8	9
noMembers=3										

FastSet

add(1)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	2	-1	1
----	----	----	----	---	----	----	---	----	---

mem{}

4	9	7							
---	---	---	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=3

FastSet

	true									
	0	1	2	3	4	5	6	7	8	9
index{}	-1	3	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7	1						
	0	1	2	3	4	5	6	7	8	9
noMembers=4										

FastSet

contains(9)

0 1 2 3 4 5 6 7 8 9

index{}

-1	3	-1	-1	0	-1	-1	2	-1	1
----	---	----	----	---	----	----	---	----	---

mem{}

4	9	7	1						
---	---	---	---	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=4

FastSet

true										
	0	1	2	3	4	5	6	7	8	9
index{}	-1	3	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7	1						
	0	1	2	3	4	5	6	7	8	9
noMembers=4										

FastSet

contains(5)

0 1 2 3 4 5 6 7 8 9

index{}

-1	3	-1	-1	0	-1	-1	2	-1	1
----	---	----	----	---	----	----	---	----	---

mem{}

4	9	7	1						
---	---	---	---	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=4

FastSet

	false									
	0	1	2	3	4	5	6	7	8	9
index{}	-1	3	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7	1						
	0	1	2	3	4	5	6	7	8	9
noMembers=4										

FastSet

remove(9)

0 1 2 3 4 5 6 7 8 9

index{}

-1	3	-1	-1	0	-1	-1	2	-1	1
----	---	----	----	---	----	----	---	----	---

mem{}

4	9	7	1						
---	---	---	---	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=4

FastSet

true										
0 1 2 3 4 5 6 7 8 9										
index{}	-1	1	-1	-1	0	-1	-1	2	-1	-1
mem{}	4	1	7							
0 1 2 3 4 5 6 7 8 9										
noMembers=3										

Implementation

```
class FastSet {  
    private:  
        int* indexArray;  
        int* memberArray;  
        int noMembers;  
  
    public:  
        FastSet(int n) {  
            indexArray = new int[n];  
            memberArray = new int[n];  
            for(int i=0; i<n; i++) {  
                indexArray[i] = -1;  
            }  
            noMembers = 0;  
        }  
  
        ~FastSet() {  
            delete[] indexArray;  
            delete[] memberArray;  
        }  
  
        int size() { return noMembers; }  
}
```

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            delete[] indexArray;  
            delete[] memberArray;  
        }  
  
        int size() { return noMembers; }  
}
```

Add and Remove

```
bool add(int i) {  
    if (indexArray[i]>-1)  
        return false;  
    memberArray[noMembers] = i;  
    indexArray[i] = noMembers;  
    ++noMembers;  
    return true;  
}
```

```
bool remove(int i) {  
    if (indexArray[i]==-1)  
        return false;  
    --noMembers;  
    memberArray[indexArray[i]] = memberArray[noMembers];  
    indexArray[memberArray[noMembers]] = indexArray[i];  
    indexArray[i] = -1;  
    return true;  
}
```

Add and Remove

```
bool add(int i) {  
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    memberArray[noMembers] = i;  
    indexArray[i] = noMembers;  
    ++noMembers;  
    return true;  
}
```

```
bool remove(int i) {  
    if (indexArray[i]==-1)  
        return false;  
    --noMembers;  
    memberArray[indexArray[i]] = memberArray[noMembers];  
    indexArray[memberArray[noMembers]] = indexArray[i];  
    indexArray[i] = -1;  
    return true;  
}
```

Collection Methods

```
void clear() {  
    for(int i=0; i<noMembers; i++) {  
        indexArray[memberArray[i]] = -1;  
    }  
    noMembers = 0;  
}  
  
bool isEmpty() {  
    return noMembers==0;  
}  
  
int* begin() {return &memberArray[0];}  
int* end() {return &memberArray[noMembers];}  
  
}
```

Collection Methods

```
void clear() {  
    for(int i=0; i<noMembers; i++) {  
        indexArray[memberArray[i]] = -1;  
    }  
    noMembers = 0;  
}  
  
bool isEmpty() {  
    return noMembers==0;  
}  
  
int* begin() {return &memberArray[0];}  
int* end() {return &memberArray[noMembers];}  
  
}
```

Collection Methods

```
void clear() {  
    for(int i=0; i<noMembers; i++) {  
        indexArray[memberArray[i]] = -1;  
    }  
    noMembers = 0;  
}  
  
bool isEmpty() {  
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}  
  
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And Random?

- We can add additional methods taking advantage of the classes strength

private:

```
random_device rd; // Seed for the random number engine  
mt19937 gen(rd()); // Mersenne Twister RNG
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public:

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int getRandomElement() {  
    return memberArray[uniform_int_distribution<int>(0, noMembers)];  
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- Need to use FastSet signature to use this

```
FastSet fastSet(n);  
:  
int r = fastSet.getRandomElement();
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- We compared our algorithm to a very highly regarded “state-of-the-art” algorithm
- For large problems we were over 10 times faster because of this data structure
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- Why? The array solution isn't in the books

Lessons

- If you have a bounded set then using an array is usually going to be very fast $O(1)$ (or $O(\log^*(n))$)
- These data structures are not general purpose for solving every day problems (c.f. `vector<T>`, `set<T>` and `map<T>`)
- They are “back pocket” data structures that solve problems that come up often enough that they are worth knowing about
- Sometimes good algorithms are not documented, but it doesn't mean they don't exist

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