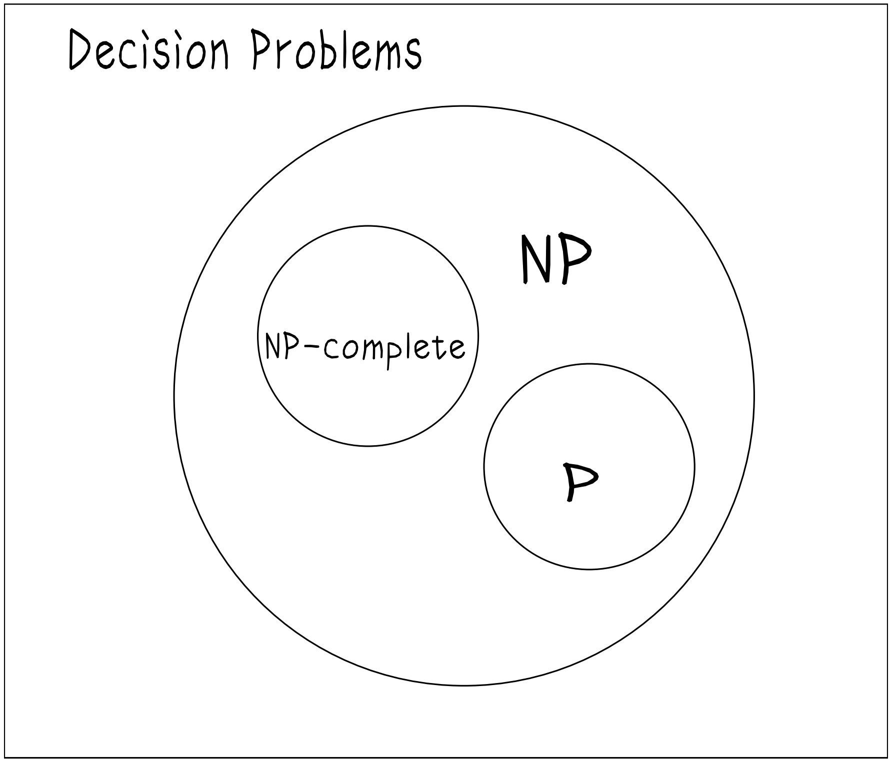
# Further Mathematics and Algorithms

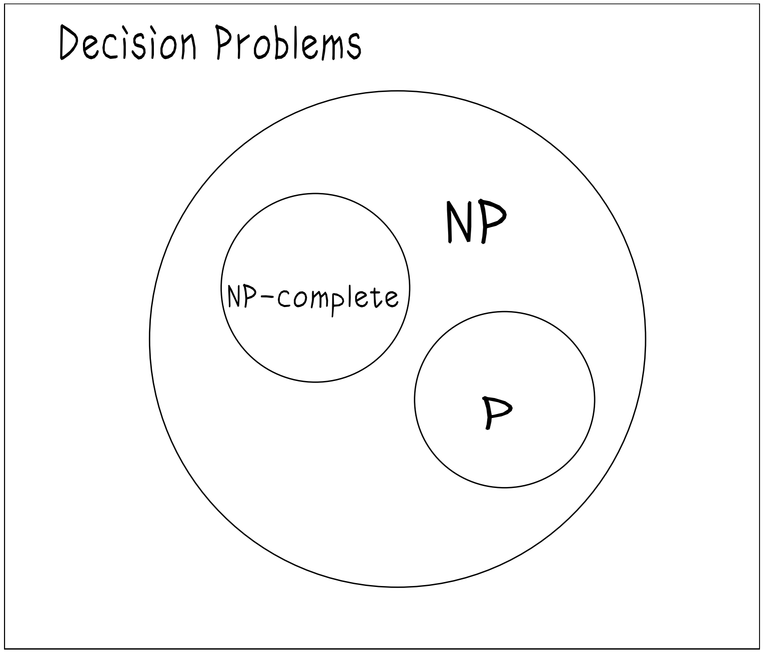
## Lesson 20: *Know What's Possible*




*Combinatorial optimisation, NP-completeness, polynomial reduction*

# Outline

1. **Motivation**
2. P, NP and NP-complete
3. Polynomial Reduction



# Exponentially Large Search Spaces

- We have seen a large number of decision problems and optimisation problems involving an exponentially large search space
- For some of these we have found efficient algorithms (greedy algorithms, divide and conquer, dynamic programming, . . . )
- For other problems we have found good algorithms (backtracking, branch and bound), but they are not necessarily polynomial
- Can we say anything general about how easy they are to solve

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# Types of Problems

- We concentrate here on two types of problems
  - ⋆ Decision Problems
  - ⋆ Combinatorial Optimisation Problems
- Decision problems are problems with a true/false answer, e.g. is it possible to cross all the bridges of Königsberg once?
- We showed earlier that backtracking can be used to find a solution which answers the decision problems, e.g. Hamiltonian circuit problem
- There are many other decision problems, but the most famous is satisfiability or SAT

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# SAT

- Given $n$ Boolean variables $X_i \in \{T, F\}$
- $m$ disjunctive (or's) clauses, e.g.

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\begin{aligned}
c_1 &= X_1 \vee \neg X_2 \vee X_3 \\
c_2 &= \neg X_2 \vee X_3 \vee X_5 \\
&\vdots \\
c_m &= X_2 \vee \neg X_4 \vee \neg X_5
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- Find an assignment, $\boldsymbol{X} \in \{T, F\}^n$ which satisfies all the clauses
- We can view this as finding an assignment that makes the formula $f(\boldsymbol{X})$ true where

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# Decisions and Optimisation Problems

- Often we can cast a decision problem as an optimisation problem
- E.g. the MAX-SAT problem is to find an assignment of variables that satisfies the most clauses
- If we can solve the MAX-SAT optimisation problem we can solve the decision problem
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- In the set of discrete optimisation problems an important class are those that involve *combinatorial objects* such as permutations, binary string, etc.

- Optimisation problems involving such objects are termed **combinatorial optimisation problems**

- Classical examples of such problems include
  - Travelling Salesperson Problem (TSP)
  - Graph colouring
  - Maximum Satisfiability (MAX-SAT)
  - Many scheduling problems
  - Bin-packing
  - Quadratic integer problems

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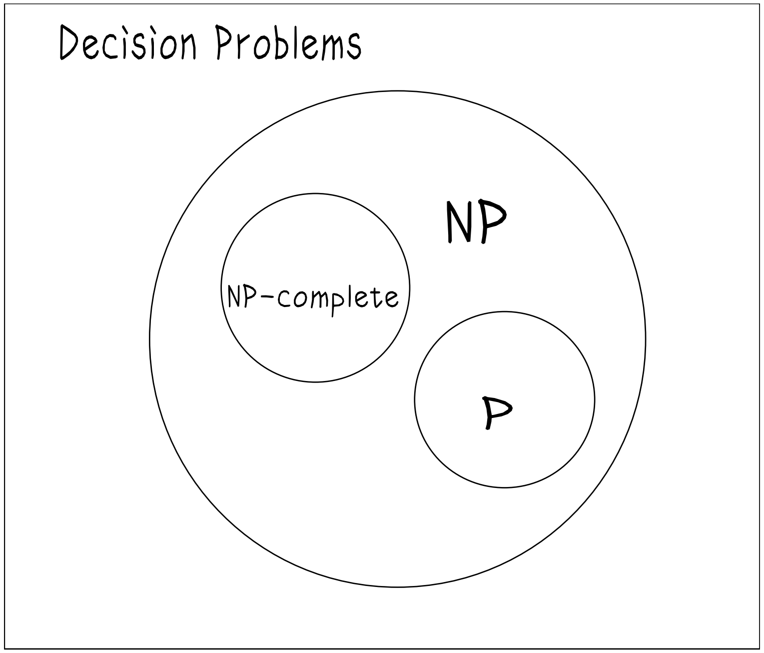

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# NP-Hard Problems

- Is it possible to solve the TSP in polynomial time?
- Answer:
- However, no one has discovered such an algorithm and if they do it will have huge implications
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# Decision Problems

- Decision problems are problems with a true or false answer
- E.g. does a TSP have a tour less than $2\,000$ miles?
- Algorithmic complexity theory deals with classes of decision problems that have some characteristic size, $n$
- E.g. $n$ is the number of cities in a TSP
- Any decision problem that can be answered with an algorithm that runs in polynomial time ($n^a$) on a normal computer (Turing machine) is said to be in class P
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# Turing Machines

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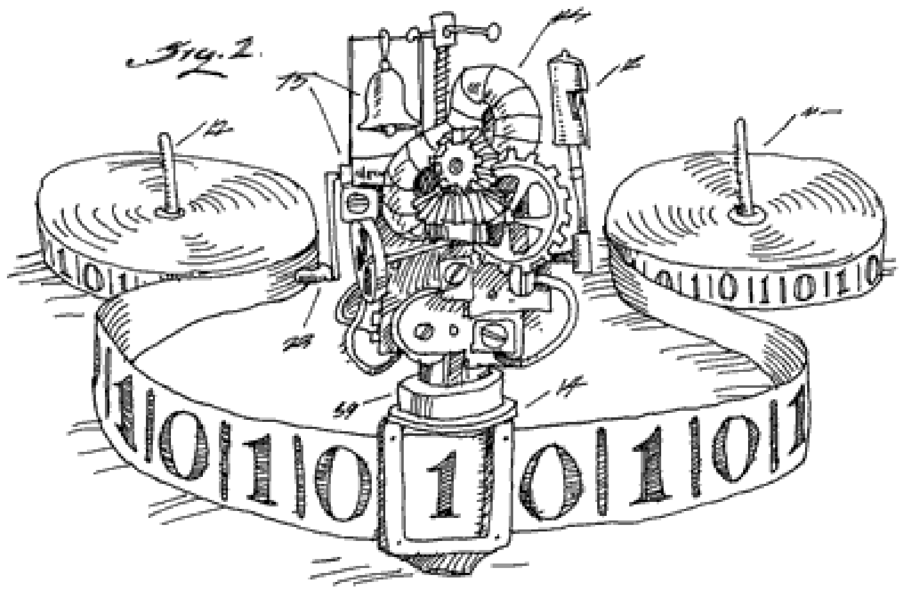

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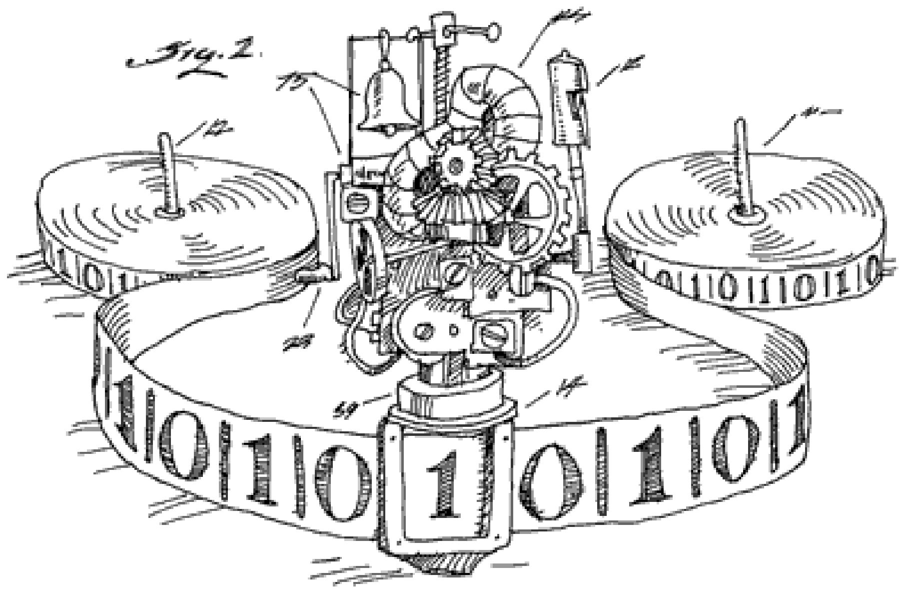

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- A whole lot of decision problems belong to class NP

- All decision problems with polynomial algorithms are also in class NP (P$\subset$NP)

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- For a problem to belong to class NP it must
  - ⋆ be a decision problem (true of false)
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  - ⋆ be verifiable if true in polynomial time on a normal Turing machine (e.g. a computer)
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- Thus if you could solve SAT in polynomial time you can use that to simulate a non-deterministic Turing machine in polynomial time
- SAT is therefore an example of one of the hardest problems in NP since if you can solve SAT in polynomial time you can solve all problems in NP in polynomial time
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- Cook showed that a non-deterministic Turing machine could be encoded as a big SAT formula
- The evolution of the state and tape was represented by a big tableau ($n^k \times n^k$-table where $n^k$ is the time it takes for the Turing machine to verify the answer)
- The structure of the clauses reflect the rules the Turing machine operates
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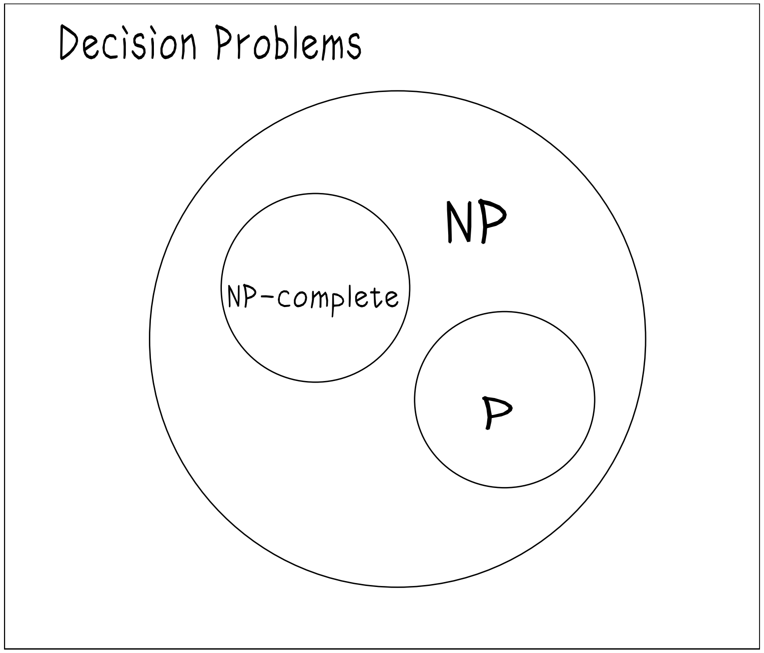
- The evolution of the state and tape was represented by a big tableau ($n^k \times n^k$-table where $n^k$ is the time it takes for the Turing machine to verify the answer)

- The structure of the clauses reflect the rules the Turing machine operates

- If the clauses are simultaneously satisfiable then there exists an input that satisfies the conditions

# Outline

1. Motivation
2. P, NP and NP-complete
3. **Polynomial Reduction**



# Polynomial Reductions

- Given two decision problems A and B we say there is a **polynomial reduction** from A to B if
  - ⋆ Every instance of A can be mapped to an instance of B:
  - ⋆ The truth of the instance A is the same as the corresponding instance B
- We can therefore use B to solve A
- So: $B \in P \rightarrow A \in P$
- The contrapositive of this statement is

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# SAT to 3-SAT

- We can reduce a clause with 4 variable to a clause with 3

$$X_1 \vee \neg X_3 \vee X_6 \vee \neg X_{10} \equiv (X_1 \vee \neg X_3 \vee Z) \wedge (\neg Z \vee X_6 \vee \neg X_{10})$$

- In doing so we increase the number of variables and the number of clauses to satisfy
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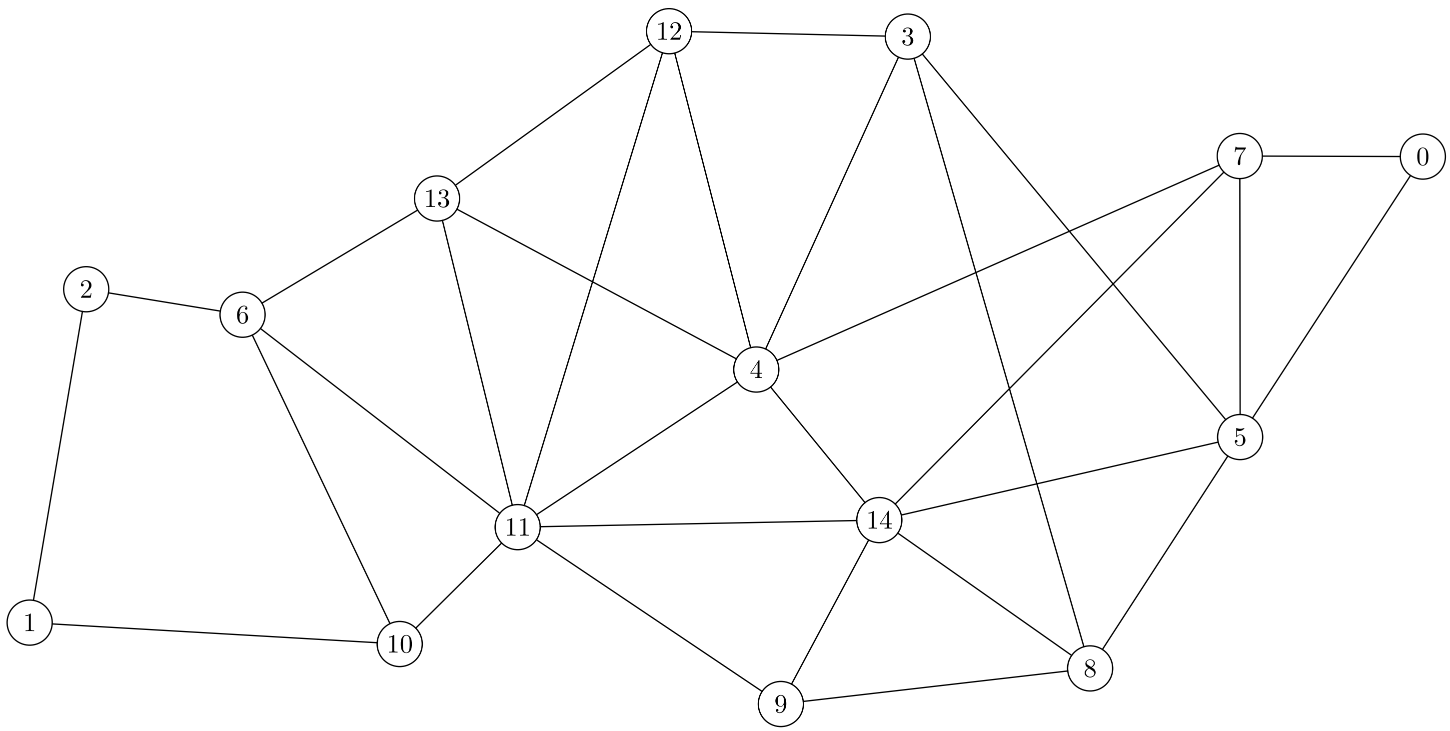
- The vertex cover problem is: "can we choose $K$ (9) vertices of a graph such that every edge is connected to a chosen vertex?"



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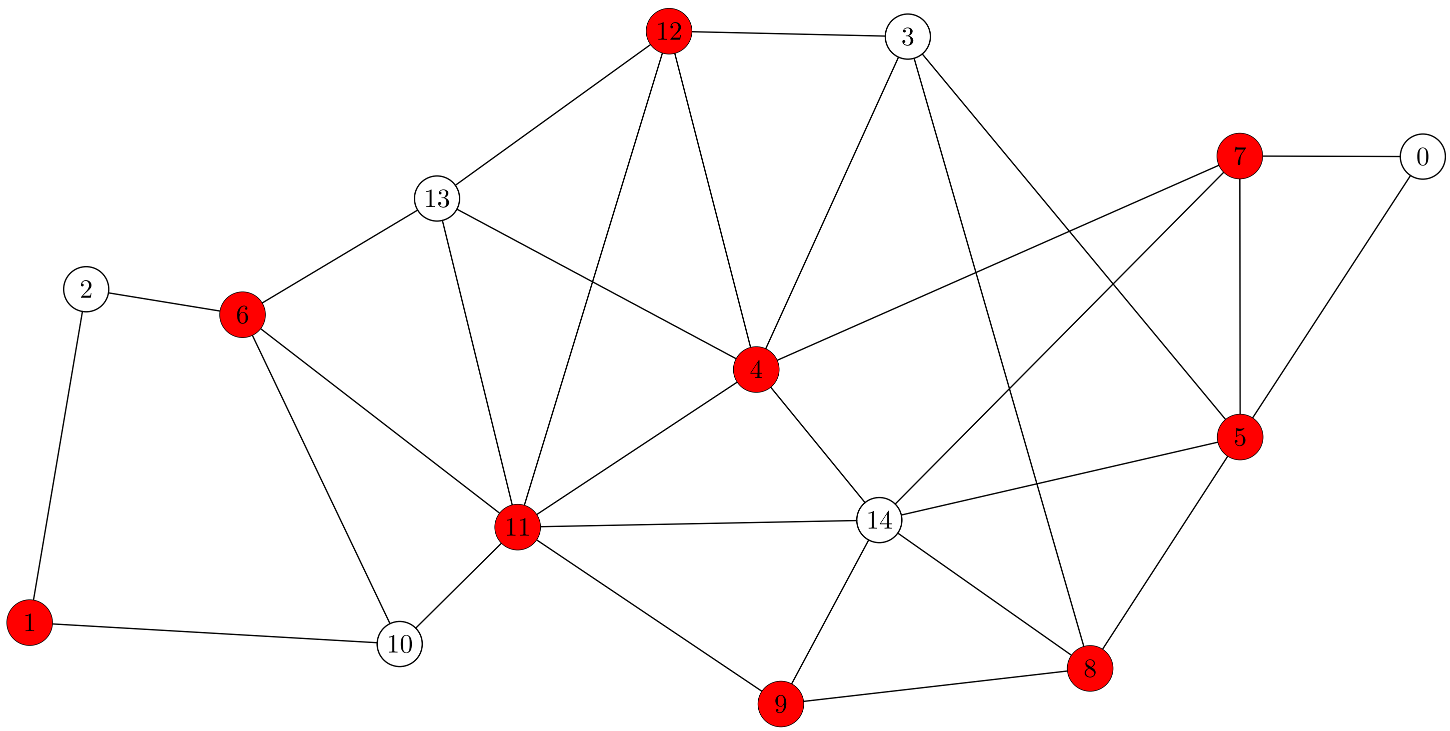

# Vertex Cover is NP-complete

- Vertex cover is obviously in NP as a set of $K$ vertices acts as a witness, i.e. it can be checked that it covers all edges
- To show vertex cover is NP-complete we show that every instance of 3-SAT is reducible to vertex cover
- The idea is to show that any 3-SAT problem with variables $\{X_1, X_2, \ldots, X_n\}$ and clauses $\{c_1, c_2, \ldots, c_m\}$ can be encoded as a vertex cover problem

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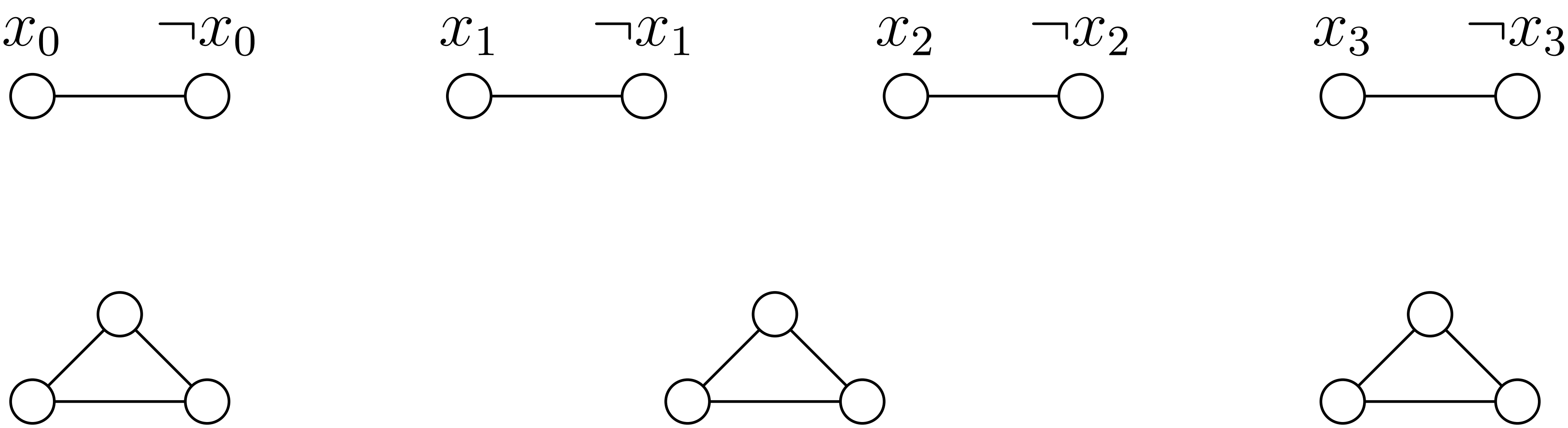

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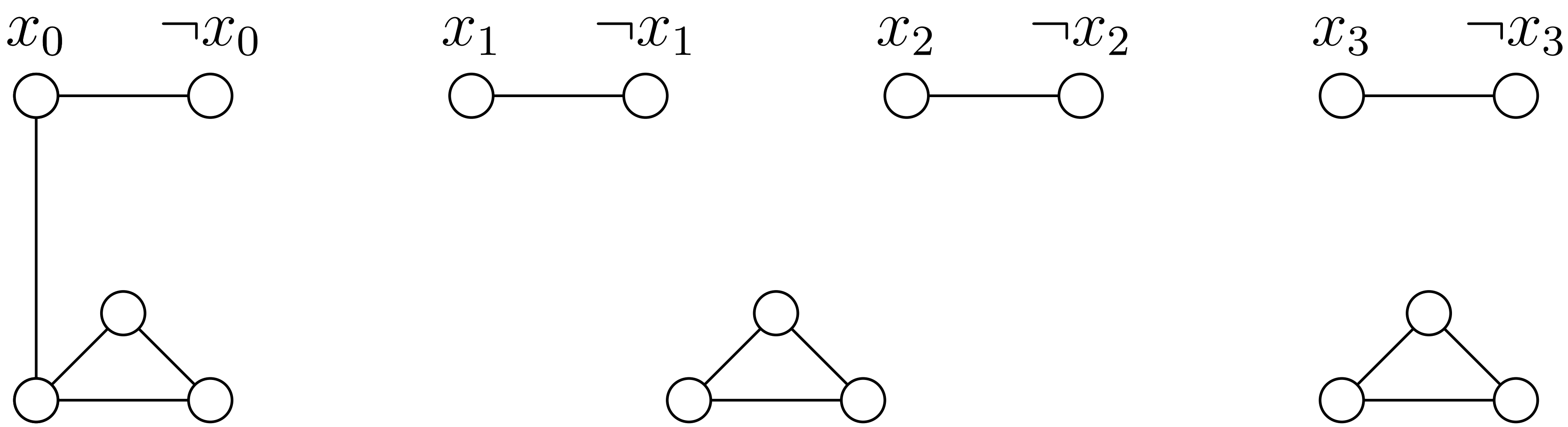

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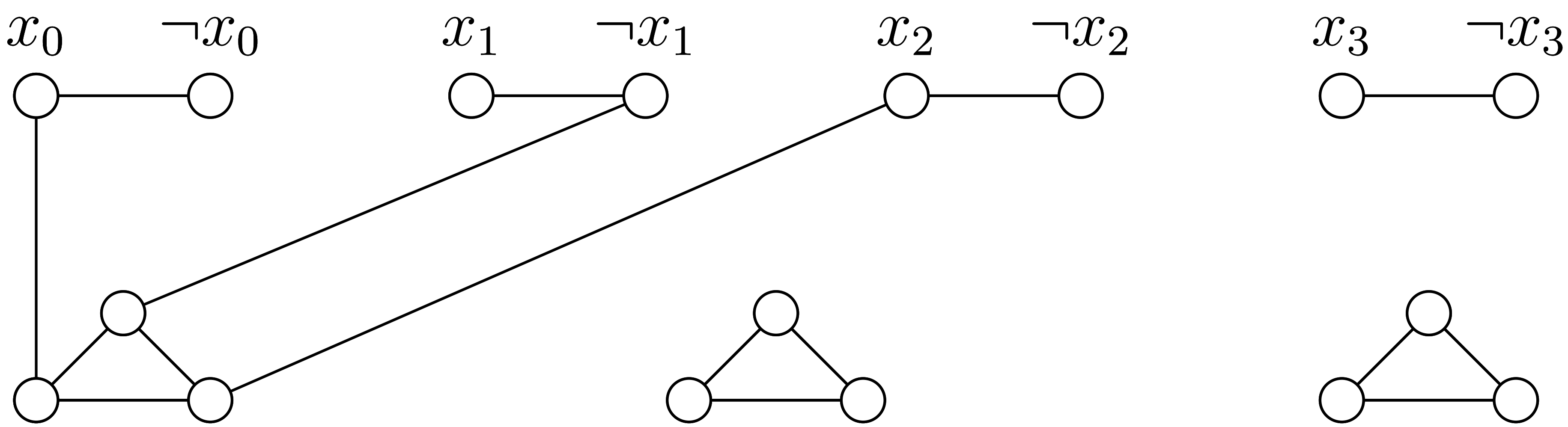

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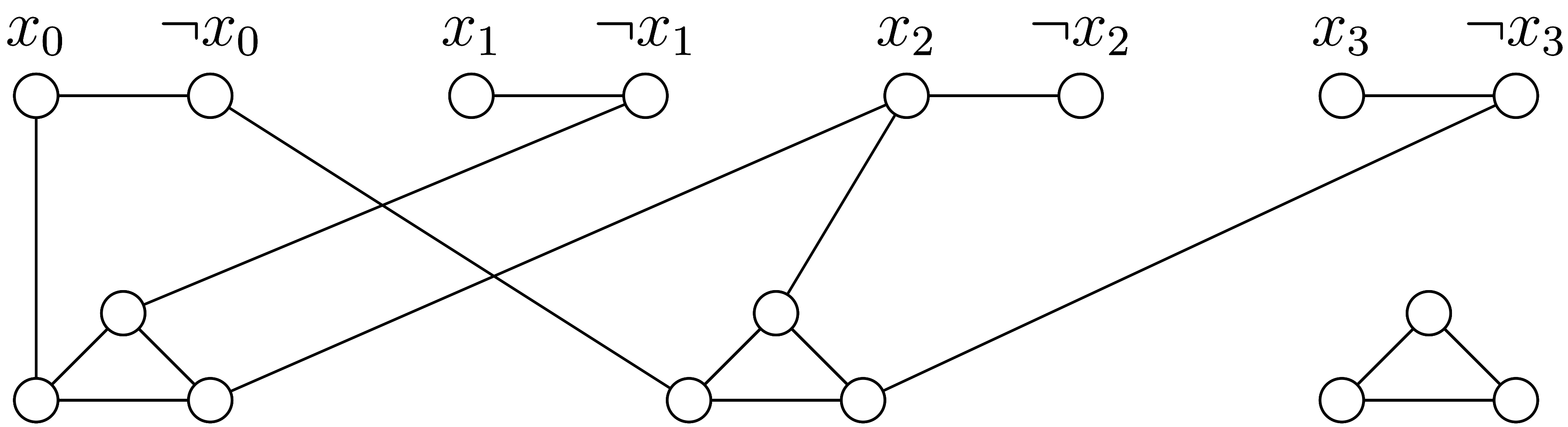

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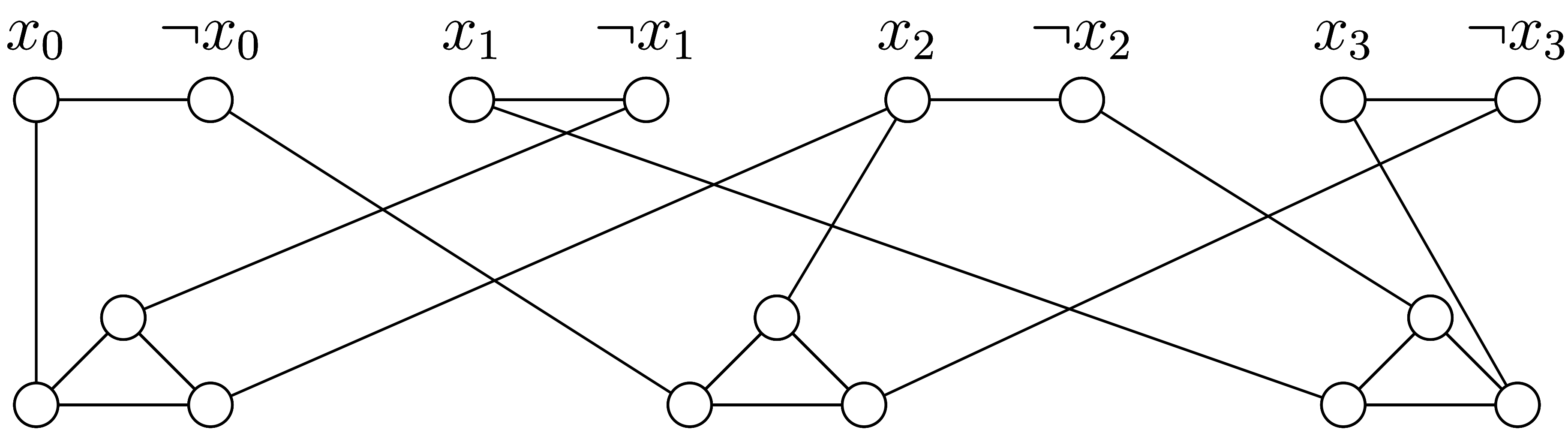

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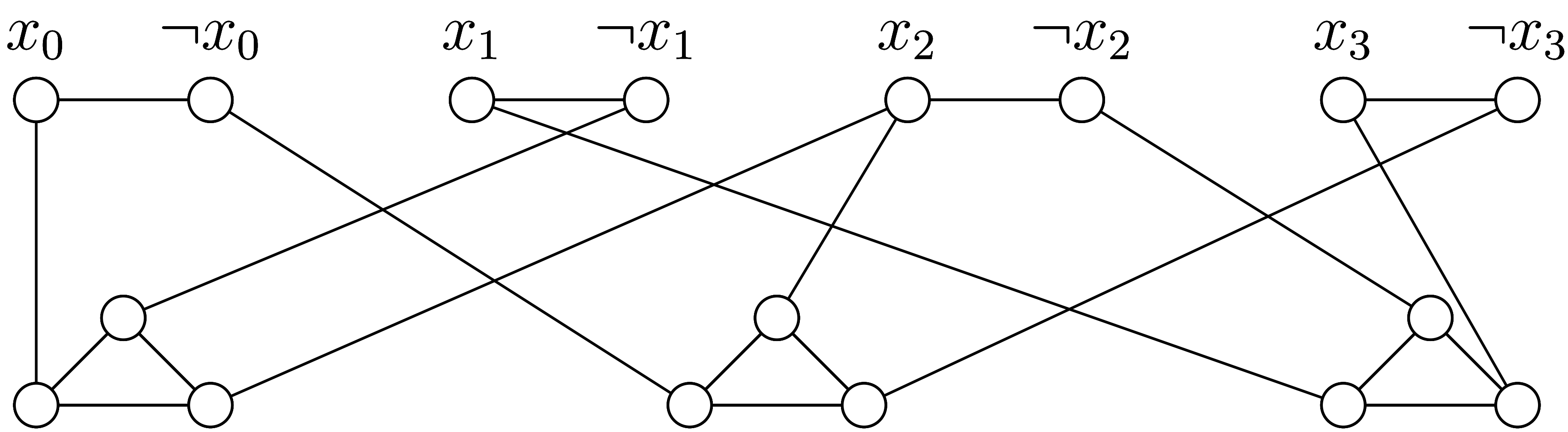


Cover all edges using $n + 2m = 4 + 2 \times 3 = 10$ vertices

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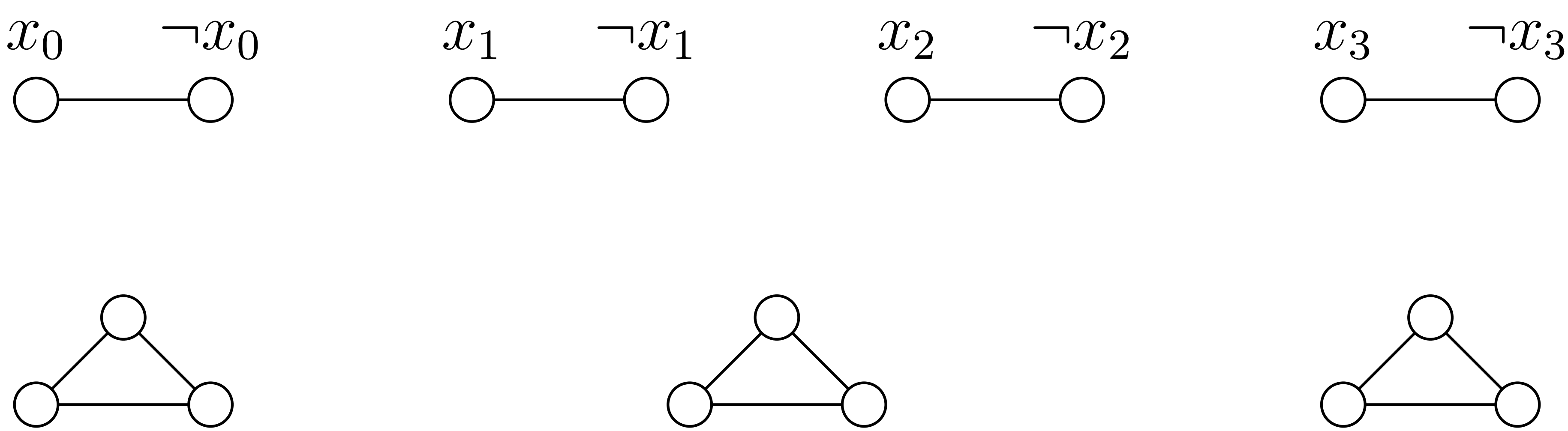


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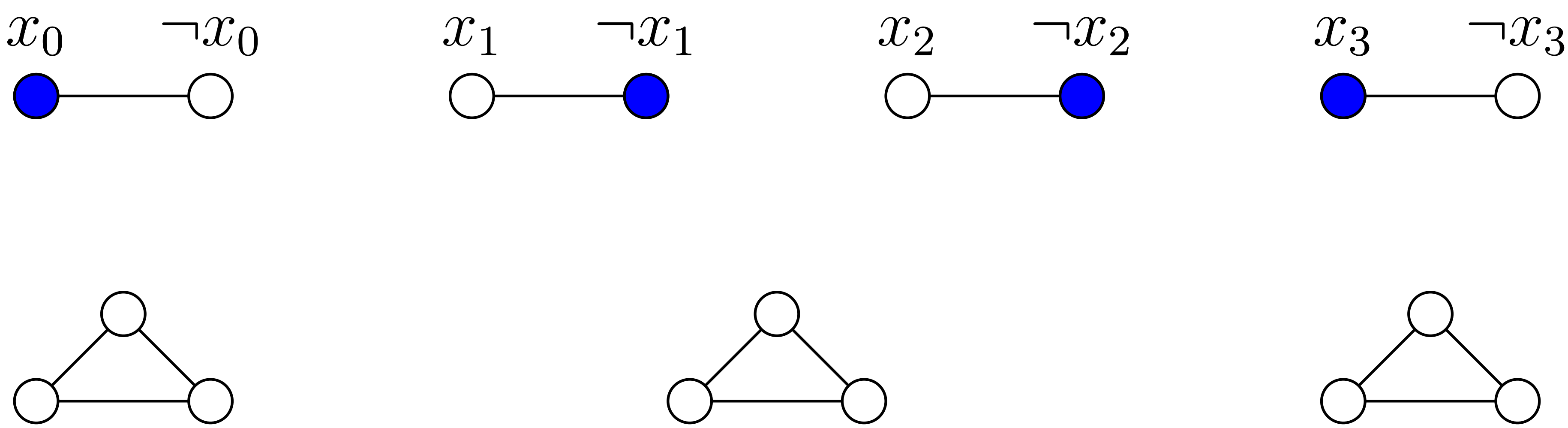


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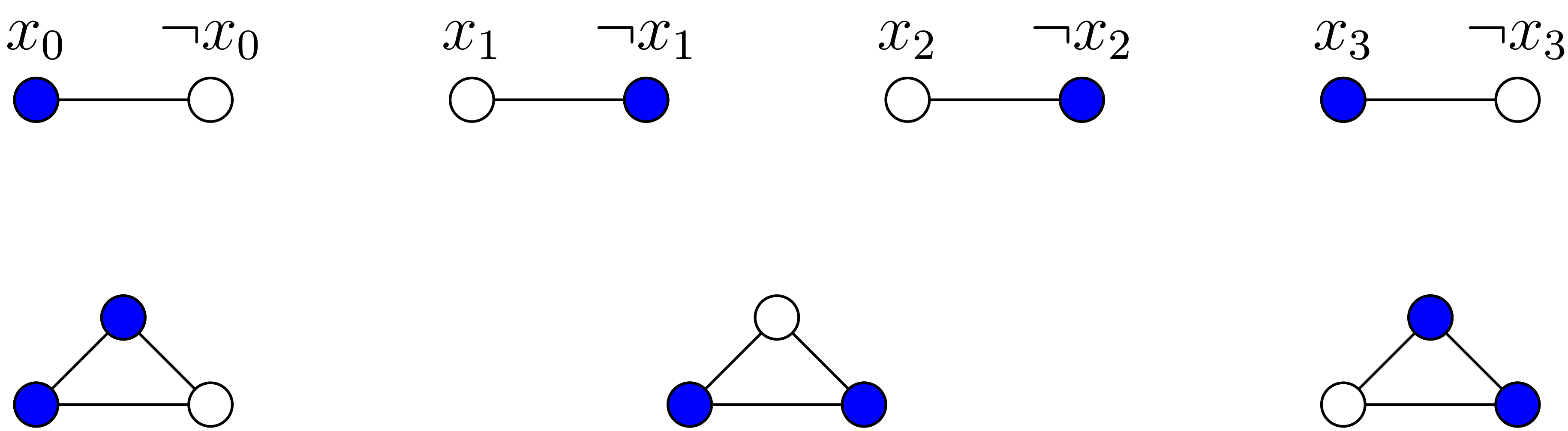


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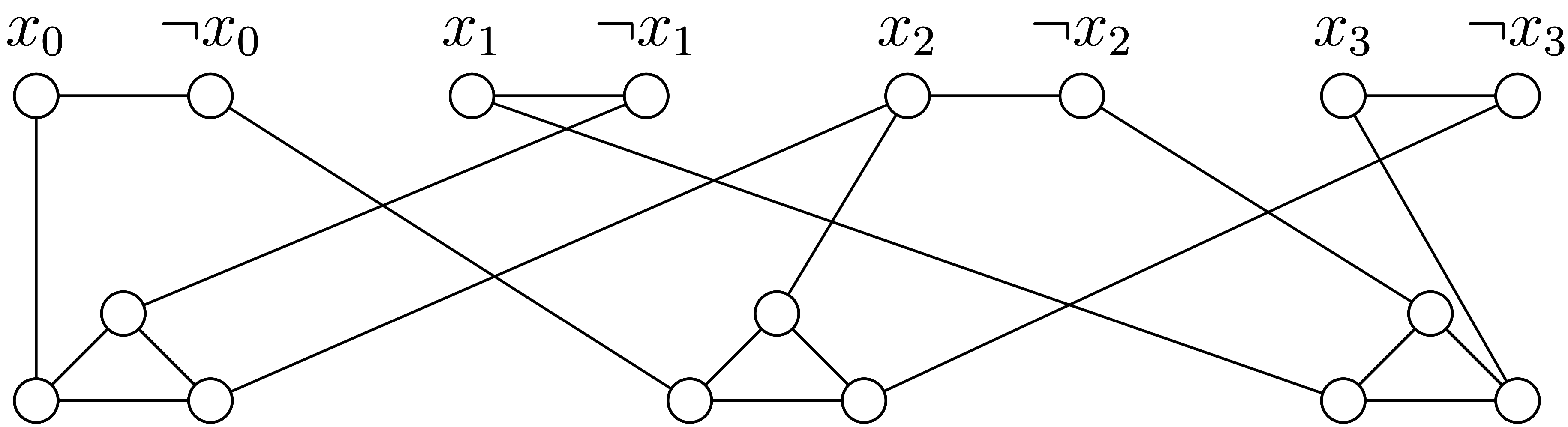


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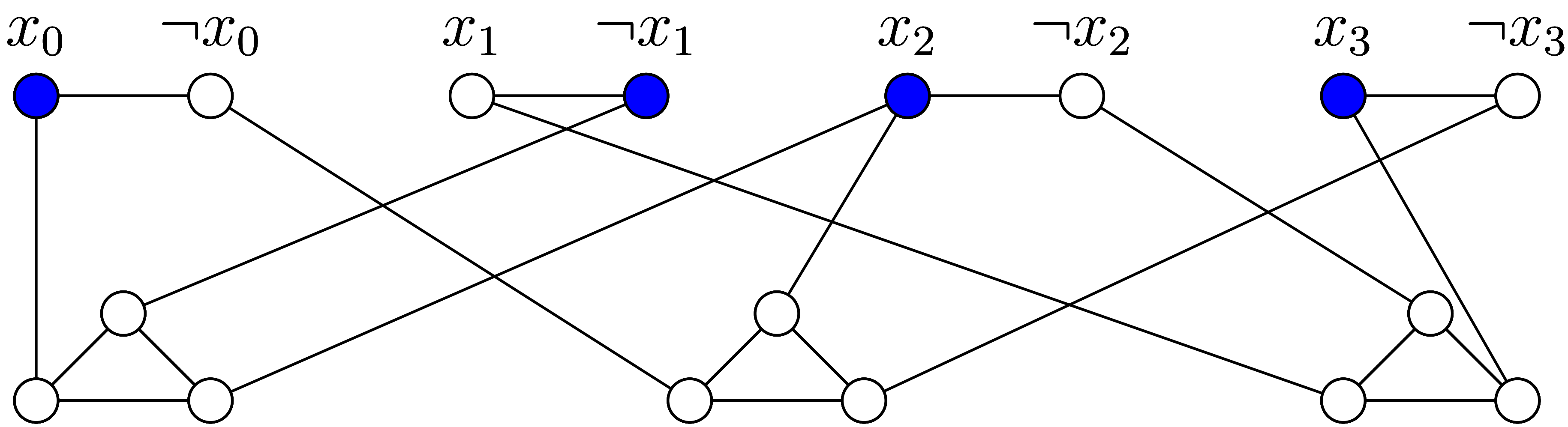


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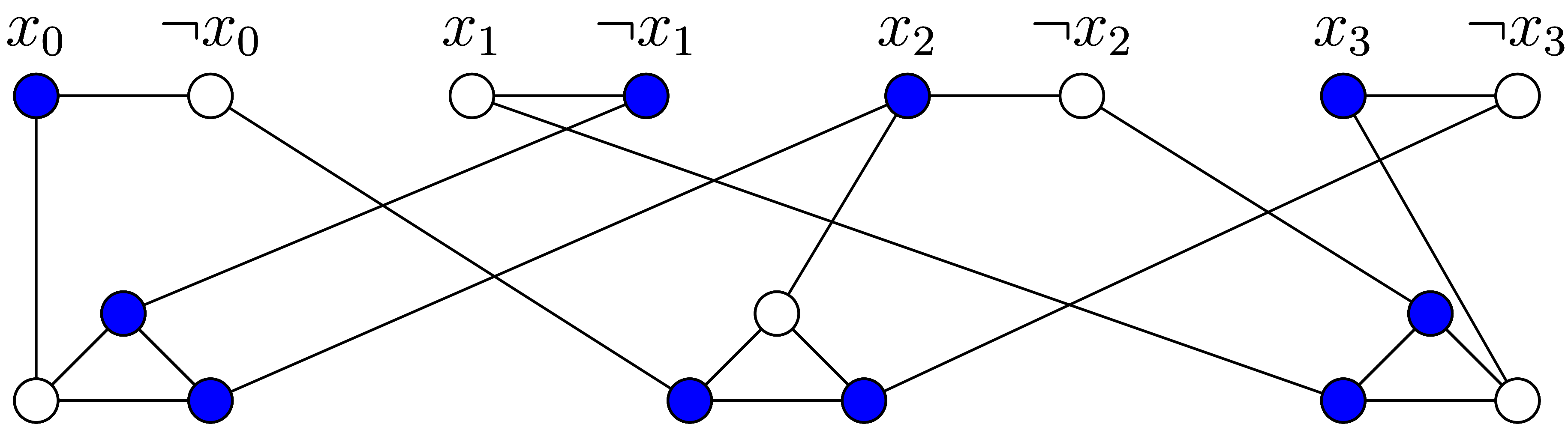


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# Other Examples of NP-complete

- As we can polynomial reduce any instance of SAT to vertex cover then vertex cover is also NP-complete

- Lots of problems have been shown to be in class NP-complete—some $10\,000$, or so, to date

- These include
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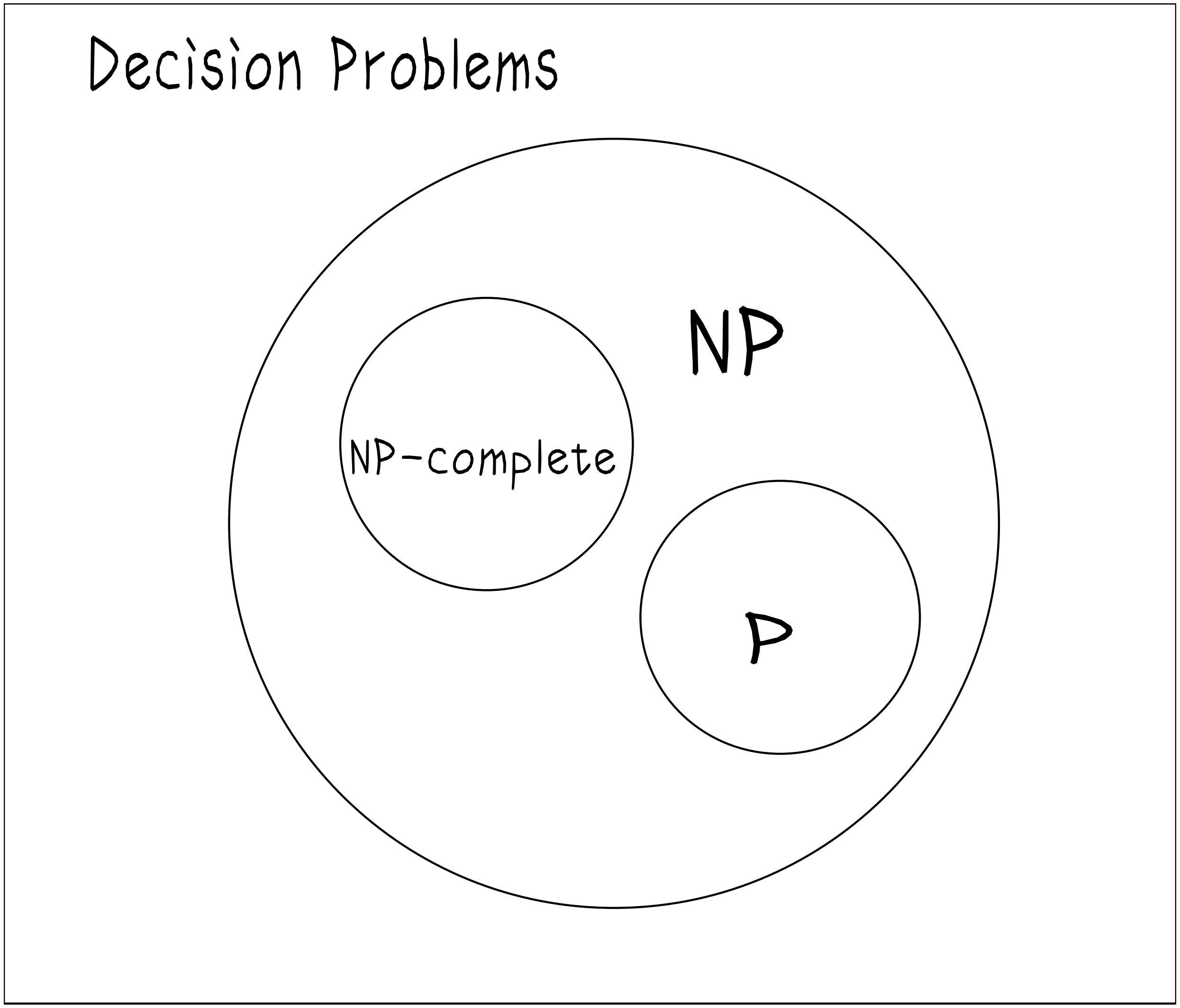
# Structure of Decision Problems



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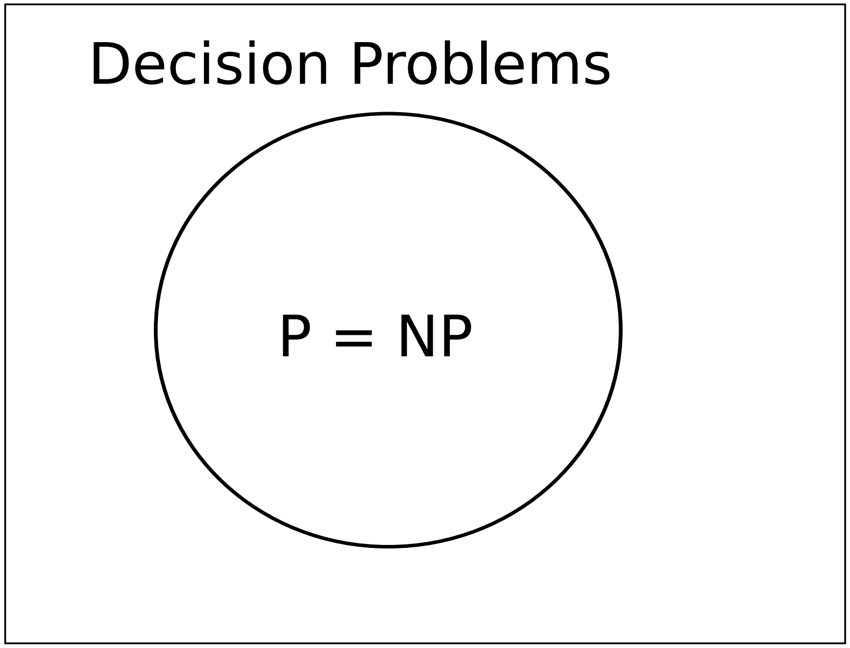

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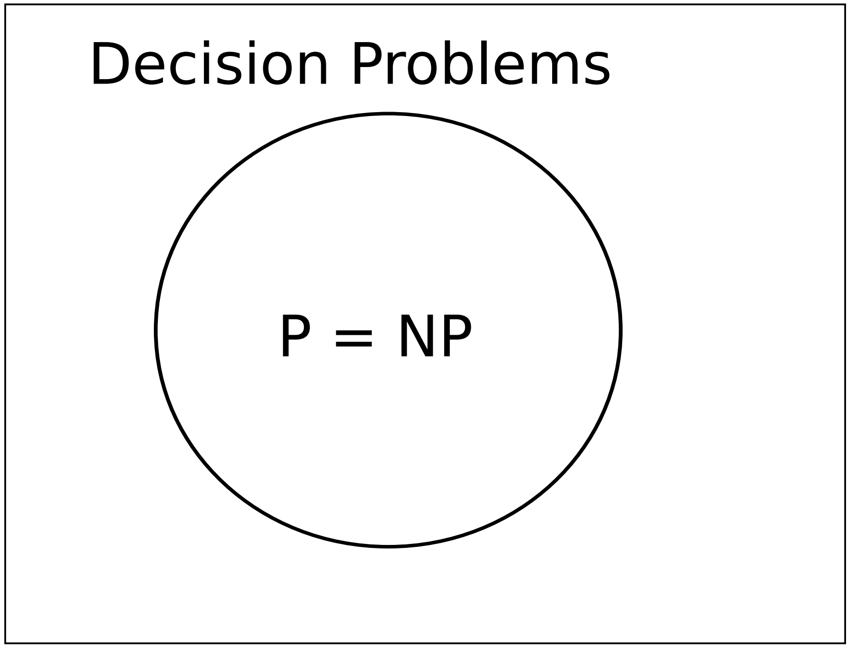

# NP-Hard

- TSP is not a decision problem—although we can make it into one—Is there a tour shorter that $L$?
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- Thus finding the shortest tour is at least as hard as solving the decision problems
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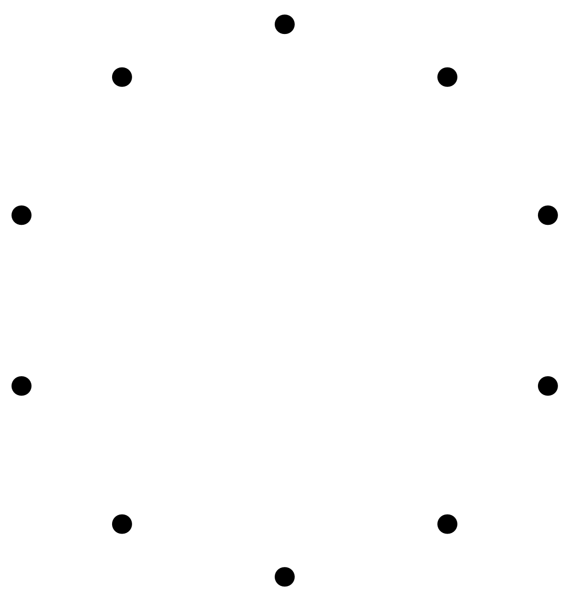

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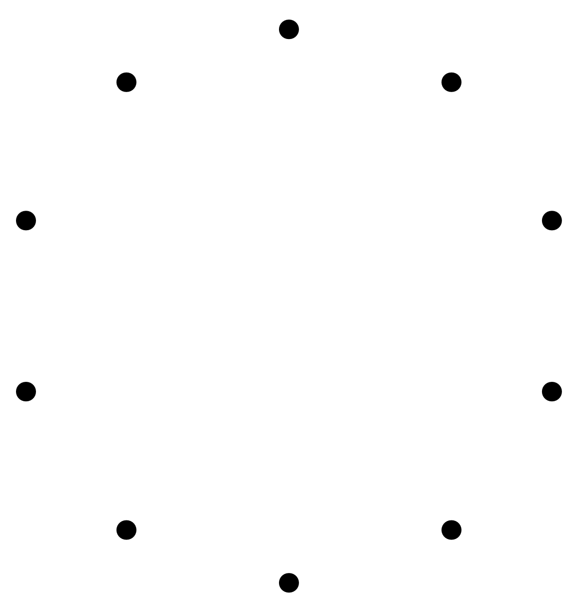

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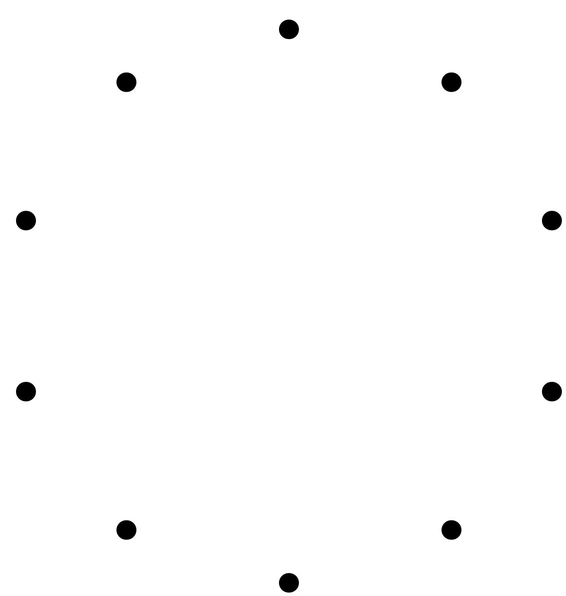

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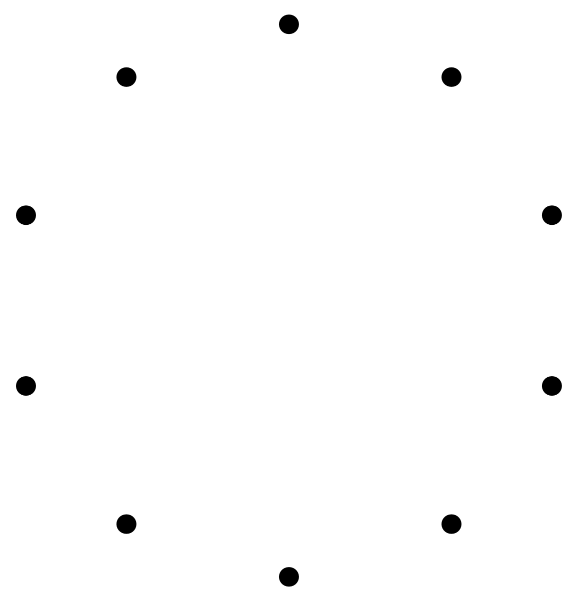
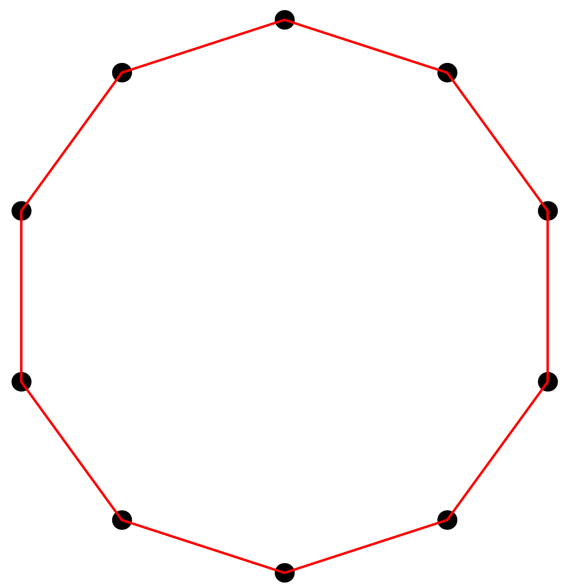


# Not All NP-Hard Problem Instances are Hard

- NP-hardness is a worst case analysis
- It means there exist some instance of the problem that we don't know how to solve in polynomial time
- Many instance of the problem might be rather easy to solve
- What is the optimal TSP tour for the problem below?



# Not All NP-Hard Problems are Hard

- For some problems almost all instances appear easy

- E.g. The subset-sum problem
  - ⋆ Given a set of numbers find a subset whose sums is as close as possible to some constant
  - ⋆ Subset-sum is in NP-hard but there exist a "pseudo-polynomial time" algorithm which solves almost every instance in polynomial time

- Many problems including subset-sum are known to be easy to approximate

- For other problems even finding a good approximation is known to be NP-hard

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# Lessons

- There exist efficient algorithms for many problems. . .
- . . . but probably not for all
- There are no known polynomial algorithm for any NP-complete problem
- These include many famous problems: TSP, graph-colouring, scheduling, . . .
- If you could find a polynomial algorithm for any of these problems then you could use it to solve all problems in NP in polynomial time

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