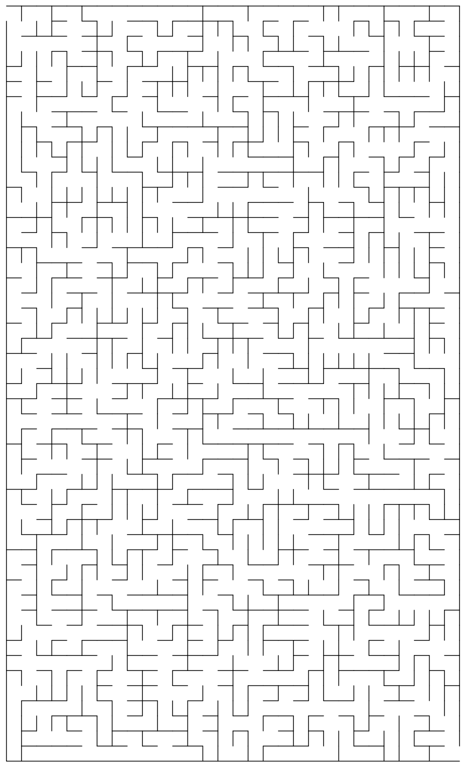
Algorithms and Analysis

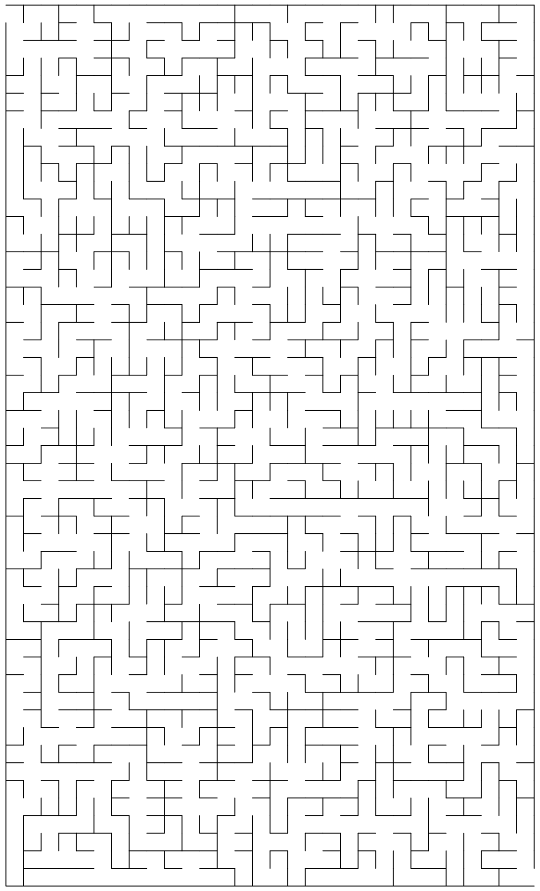
Lesson 14: *Use Arrays for Fast Set Algorithms*



Equivalent classes, Disjoint Set, Fast Sets

Outline

1. **Equivalent Classes**
2. Disjoint Sets
3. Fast Sets



Equivalence Relations

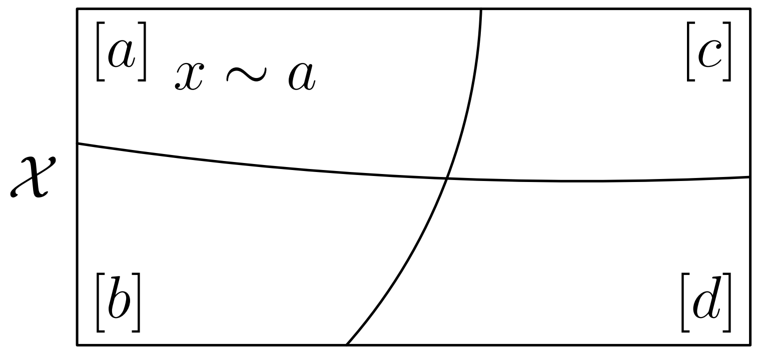
- Given a set of elements $\mathcal{X} = \{x_1, x_2, \ldots\}$ and a binary relationship $\sim$ with the following properties

  (Reflexivity) For every element $x \in \mathcal{X}$, $x \sim x$

  (Symmetry) For every two elements $x, y \in \mathcal{X}$ if $x \sim y$ then $y \sim x$

  (Transitivity) For every three elements $x, y, z \in \mathcal{X}$ if $x \sim y$ and $y \sim z$ then $x \sim z$

- Then $\sim$ defines a partitioning of the set into **equivalence classes**



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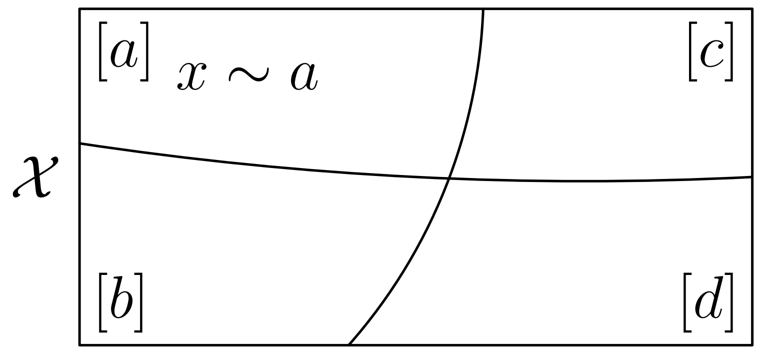

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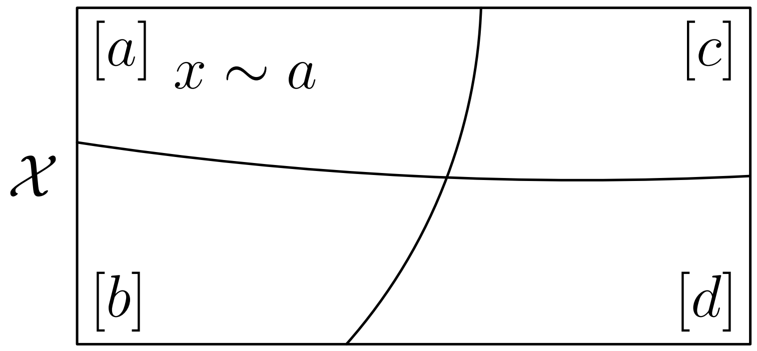

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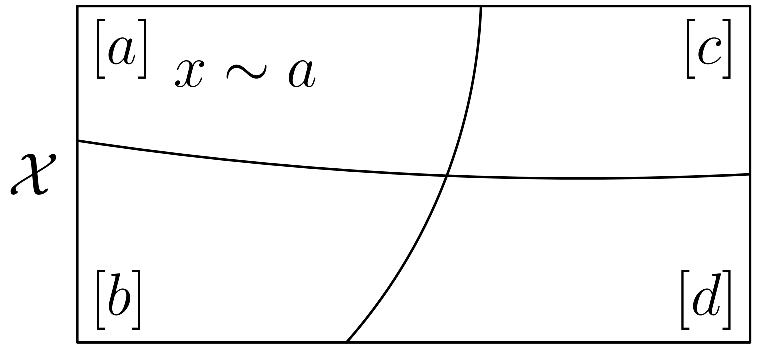

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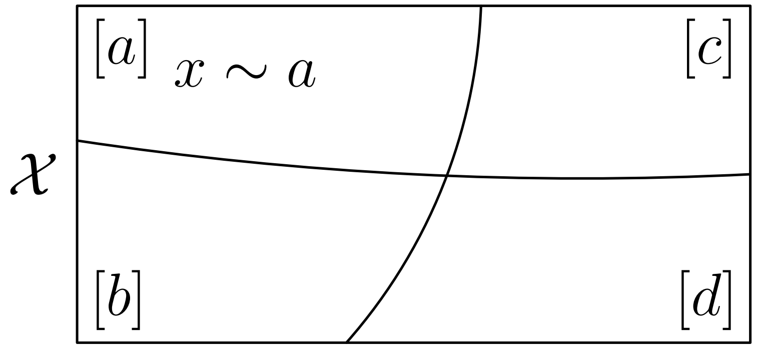

Example of Equivalence Classes

- Although, equivalent classes sound very mathematical they often provide a useful formalisation of the real world
- E.g. Pairs of web pages with a link in each direction between them
- Consider web pages in the same equivalence class if you can get from one to the other by clicking links
- Partitions the web into linked domains
- Friendship relations in social media

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Dynamic Equivalence Classes

- Finding equivalence classes is rather easy using graph traversal algorithms
- However, as our web example suggests, there are applications where equivalence classes change over time
- Adding a link could join two domains which were separate
- We will see this is a useful idea both for building mazes and (in a later lecture) for finding minimum spanning trees
- Building a data structure which finds equivalence classes where the equivalence relation changes over time is challenging

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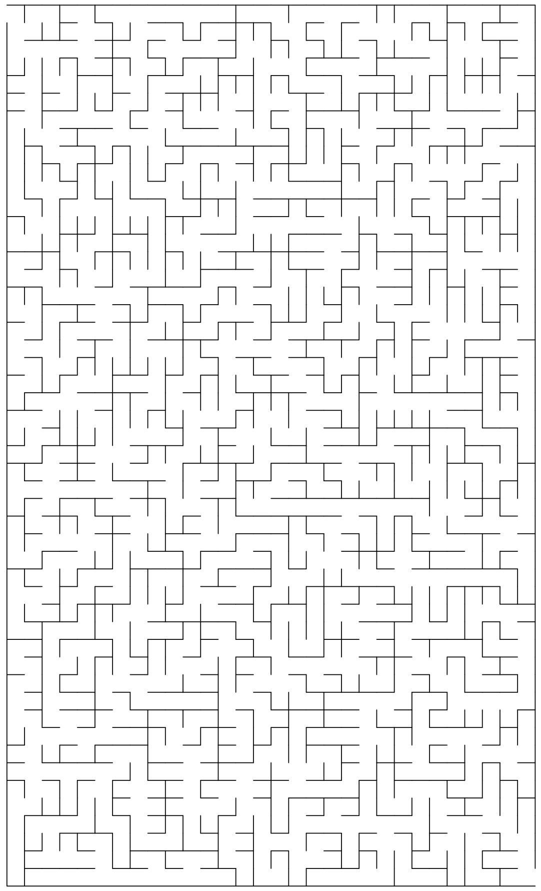

Union-Find

- In the union-find algorithm we have a set of objects $x \in \mathcal{S}$ which are to be grouped into equivalence classes $\mathcal{S}_1$, $\mathcal{S}_2$, . . .
- Initially each object is in its own equivalence class
- **union** combines two equivalence classes
- **find** returns a unique member of the equivalence class (e.g $[a]$)
- This is a common problem for which we will write a class `DisjointSets` to perform fast `union`s and `find`s

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DisjointSets

- We want to create a class

```
class DisjointSets
{
    DisjointSets(int numElements) {/* Constructor */}
    int find(int x) {/* Find root */}
    void union_(int root1, int root2) {/* Union */}

  private:
    int* s;
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- Where `find(x)` returns a unique identifier for the subset which element `x` belongs to

- The array `s` contains labelling information to implement `find(x)`

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The Union-Find Dilemma

- A natural algorithm to perform finds is to maintain an array returning a subset label for each element—this makes `find` fast
- However, every time we combine two subset we have to change all the labels in this array (taking $O(n)$ operations)
- If we are unlucky the cost of performing n unions is $\Theta(n^2)$
- If we ensure that we relabel the smaller subset then the time complexity is $\Theta(n \log(n))$
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Fast Union

- To achieve fast unions we can represent our disjoint sets as a forest (many disjoint trees)
- Every time we perform a union we make one of the trees point to the head of the other tree
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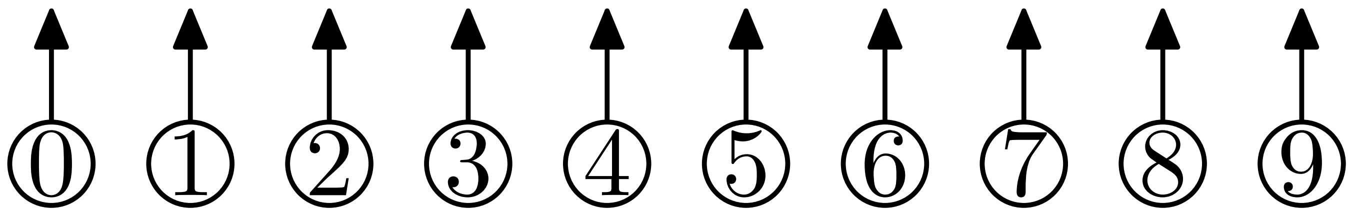
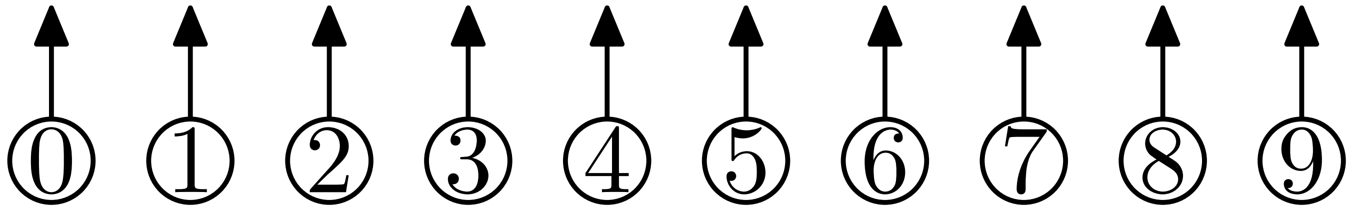
Putting it Together



-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(5),find(6))`



-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
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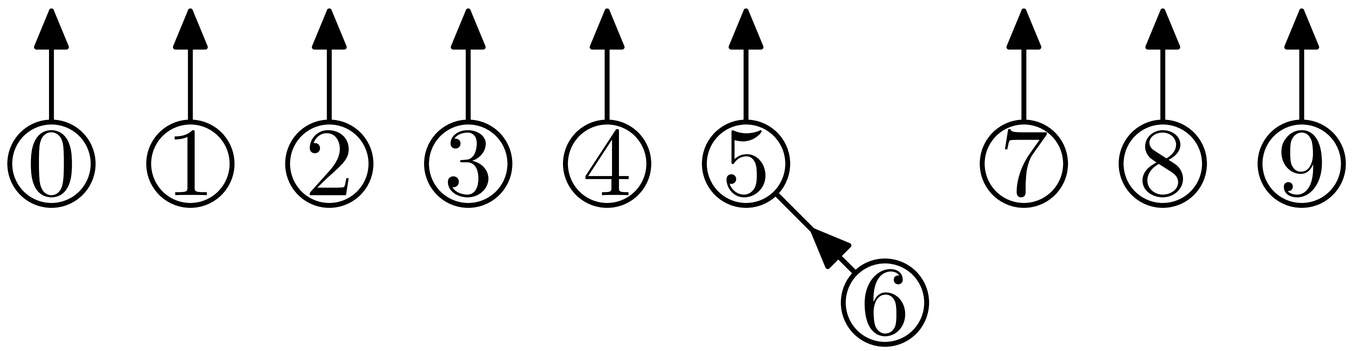
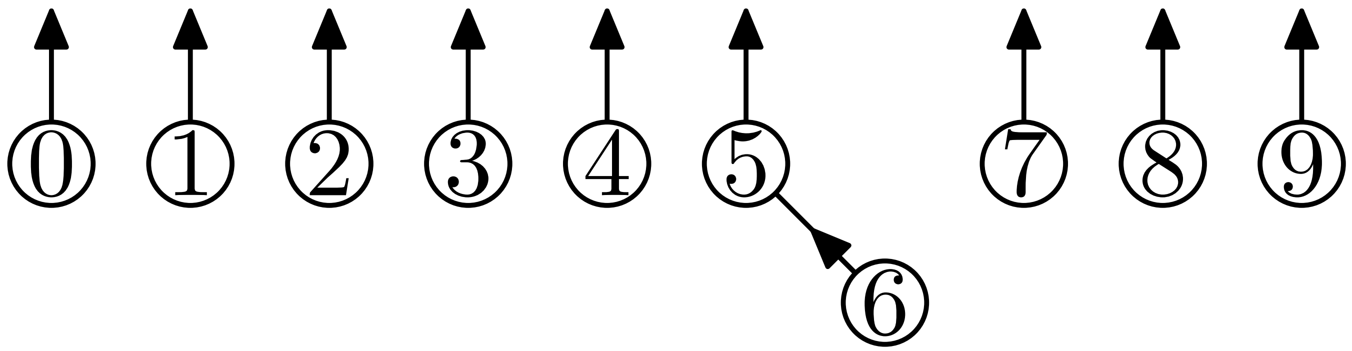


-1	-1	-1	-1	-1	-2	5	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(7),find(8))`



0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	-1	-2	5	-1	-1	-1

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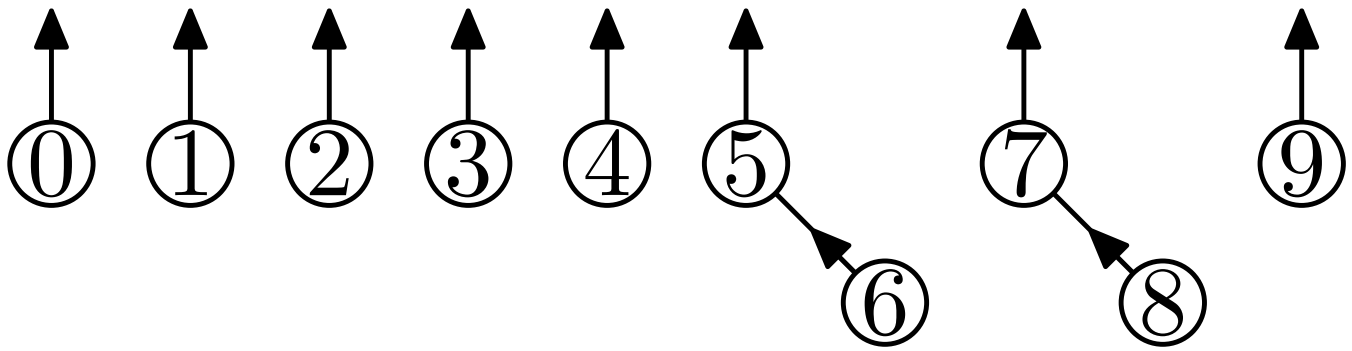
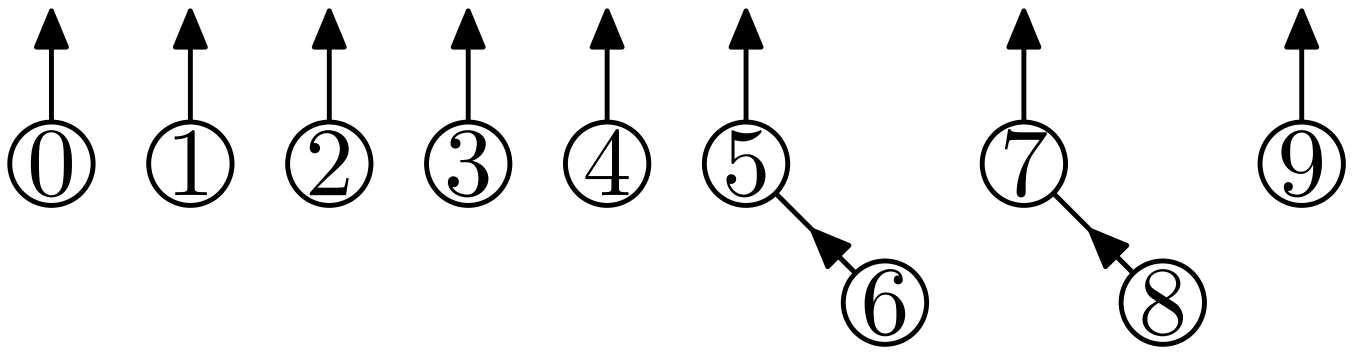


-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

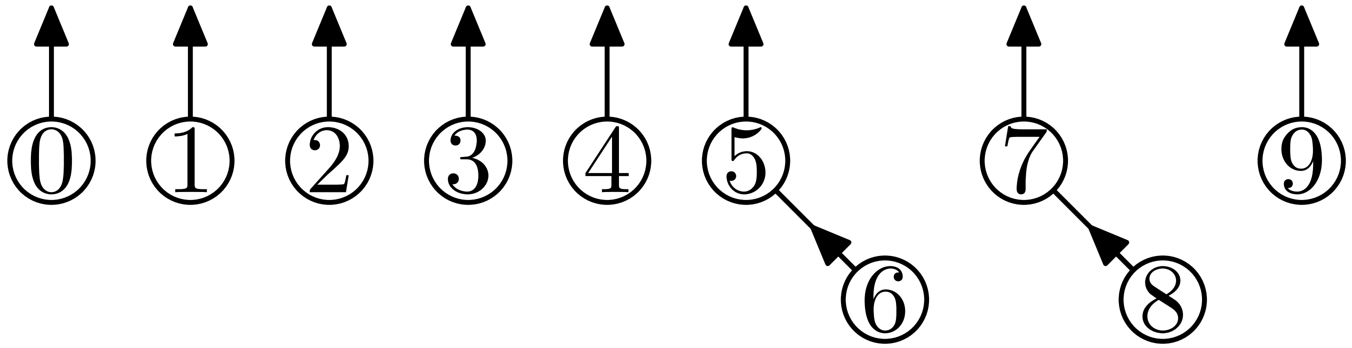
`find(6)`



0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	-1	-2	5	-2	7	-1

Putting it Together

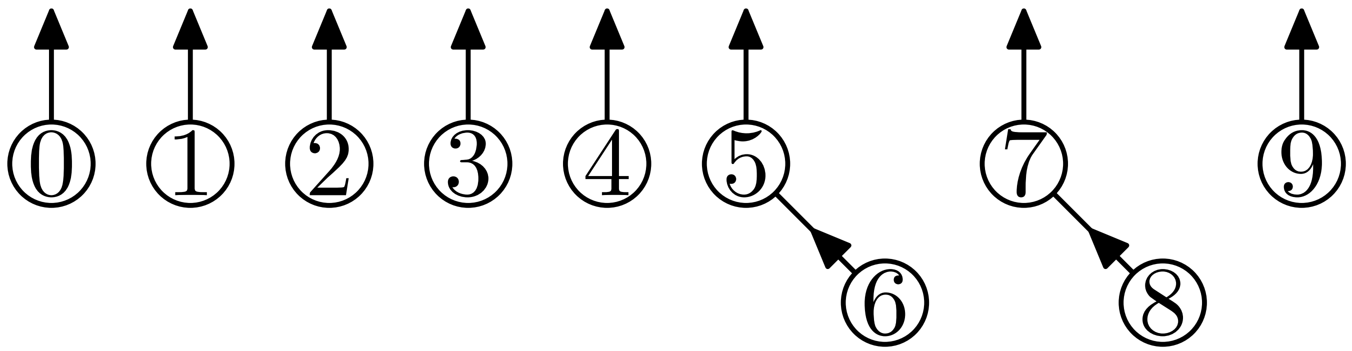
`find(6)=5`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(3),find(6))`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
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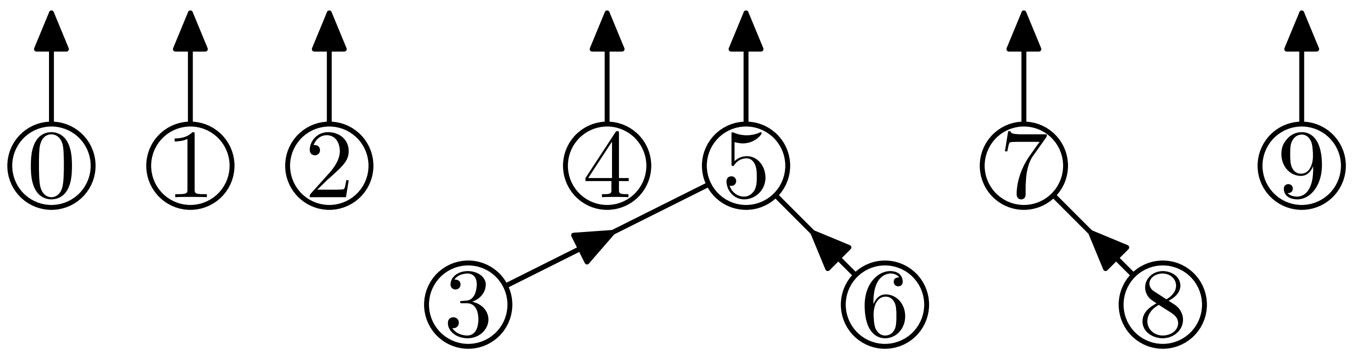
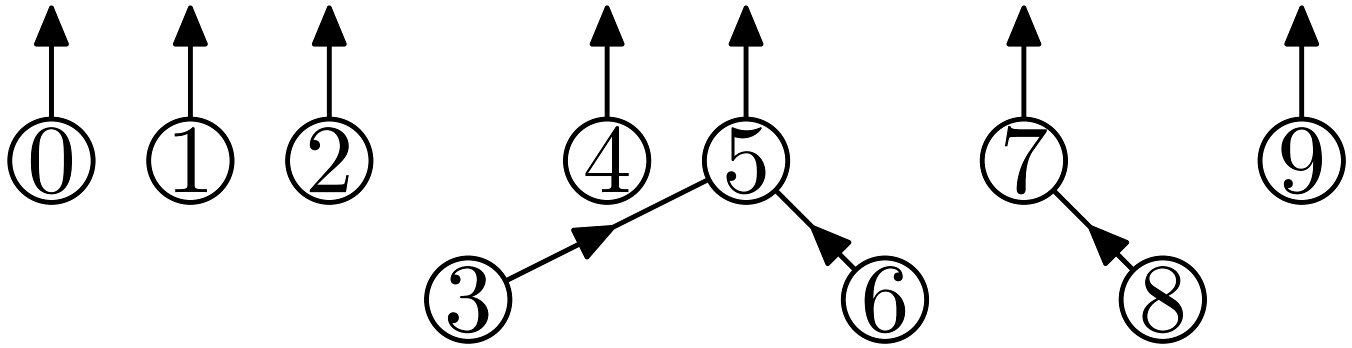


-1	-1	-1	5	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(2),find(6))`



0	1	2	3	4	5	6	7	8	9
-1	-1	-1	5	-1	-2	5	-2	7	-1

Putting it Together

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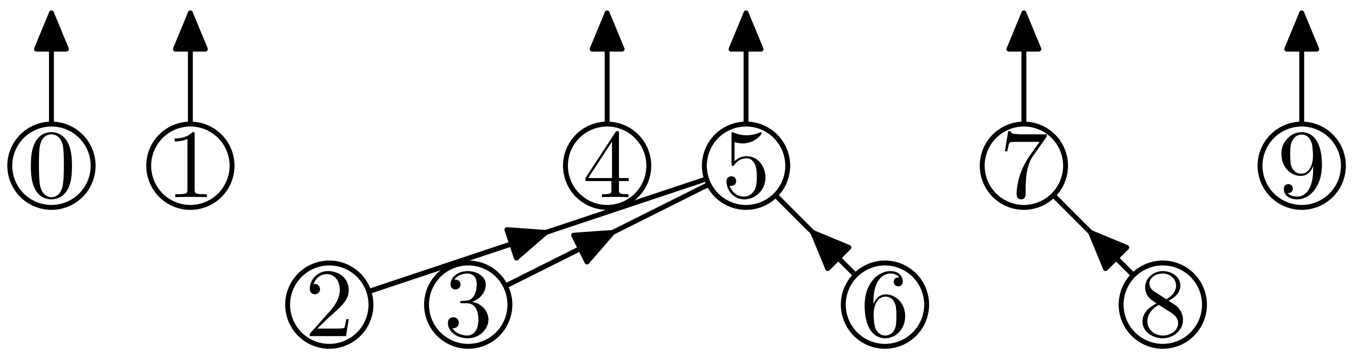

-1	-1	5	5	-1	-2	5	-2	7	-1
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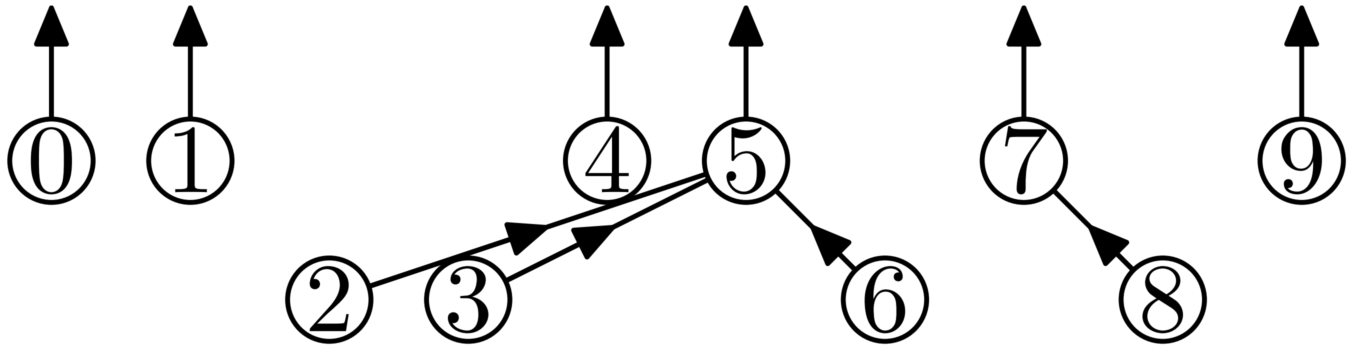
Putting it Together

`union(find(9),find(8))`



-1	-1	5	5	-1	-2	5	-2	7	-1
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Putting it Together

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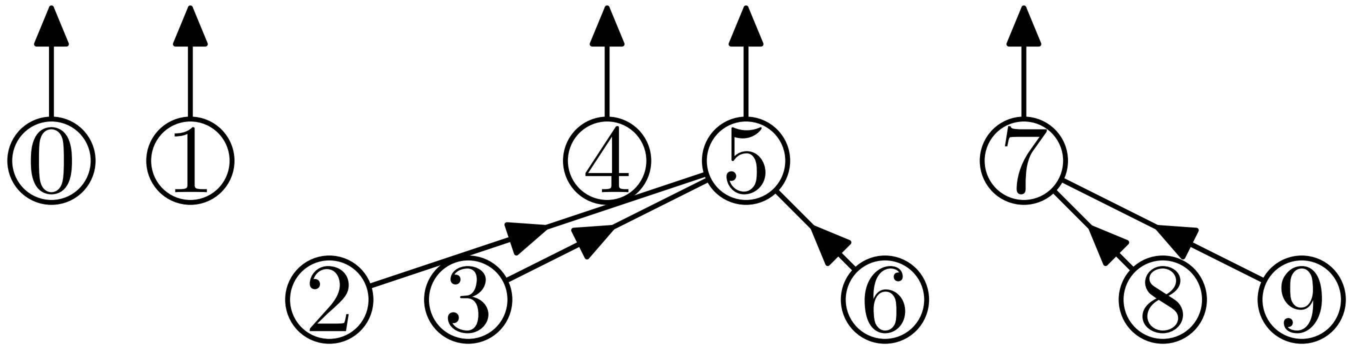
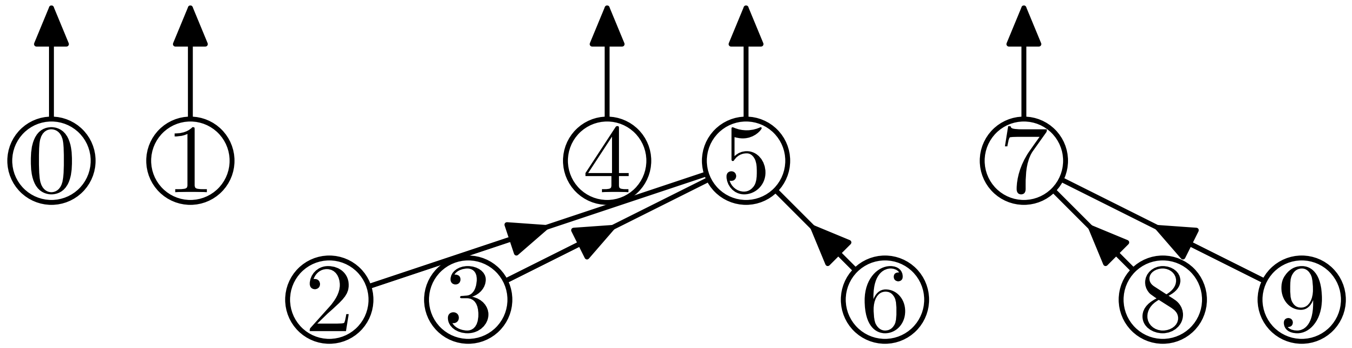


-1	-1	5	5	-1	-2	5	-2	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(9),find(3))`



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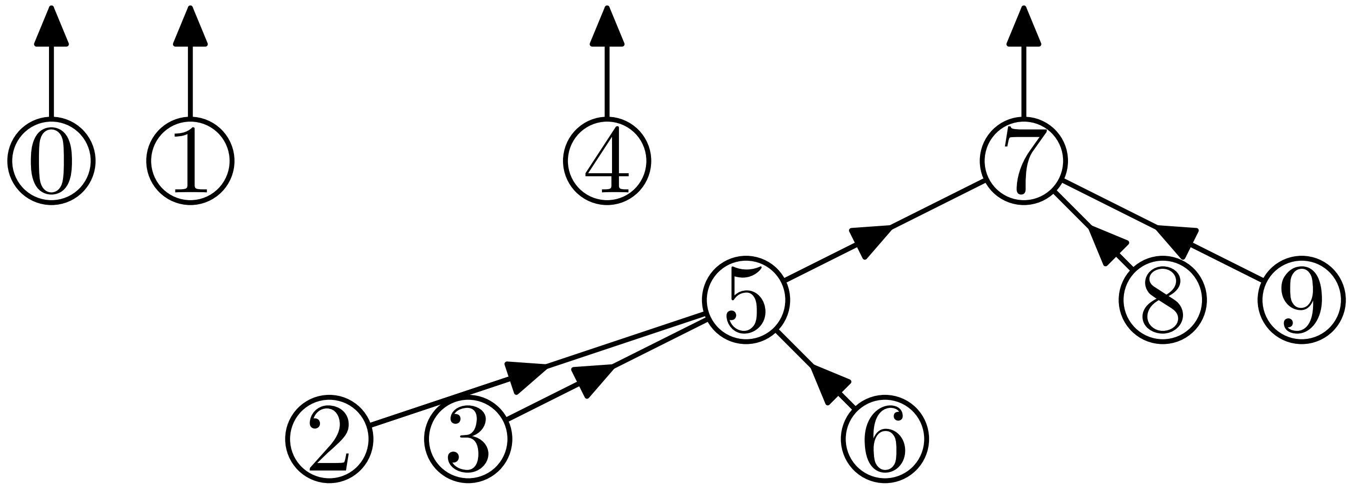

-1	-1	5	5	-1	7	5	-3	7	7
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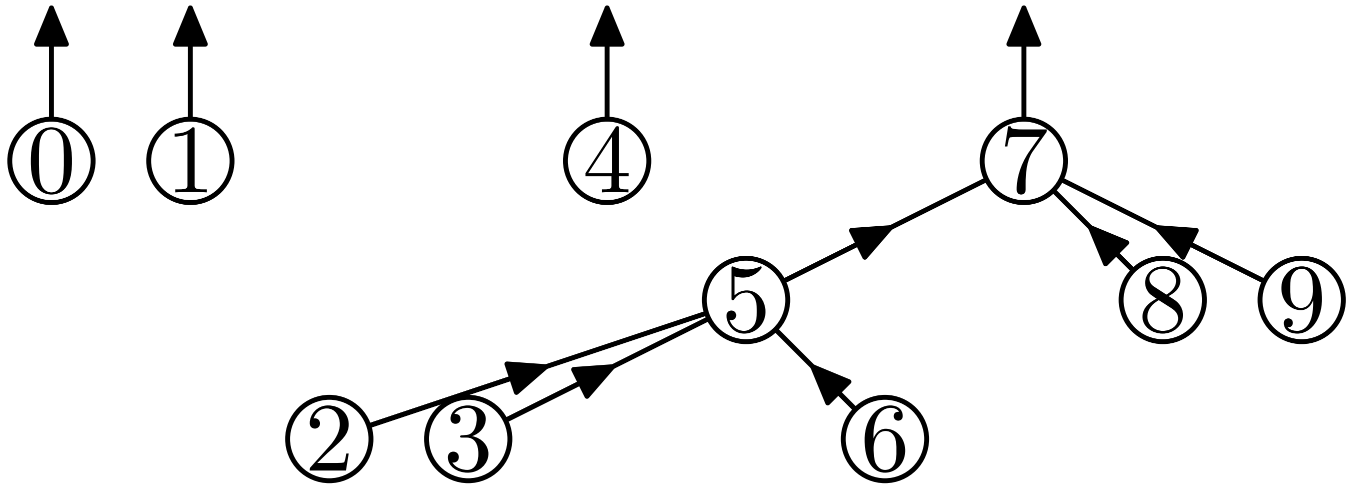
Putting it Together

`find(3)`



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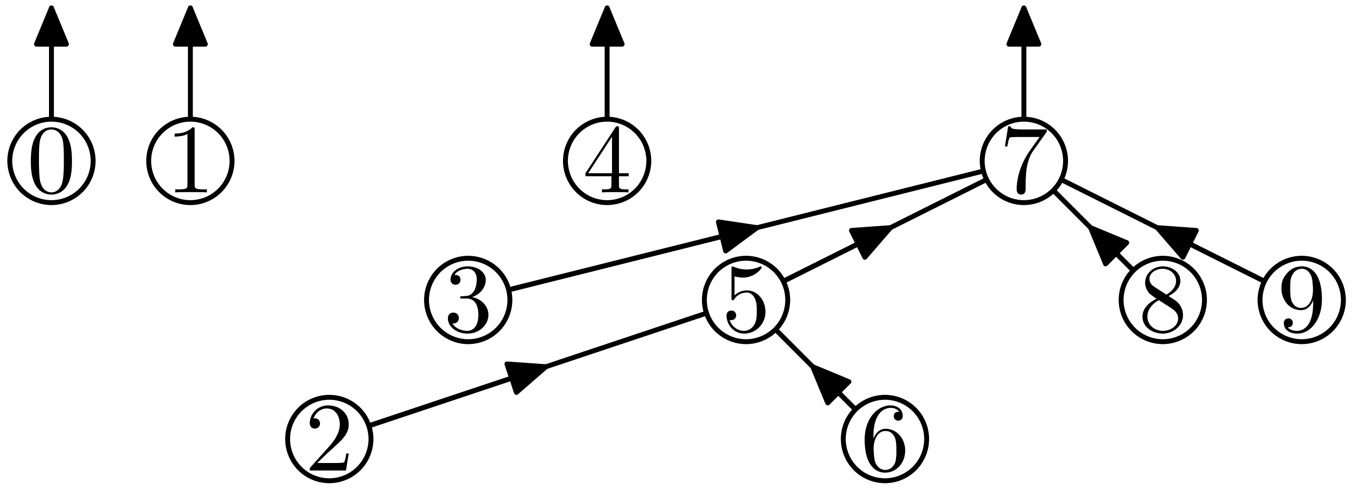
Putting it Together

`find(3)=7`



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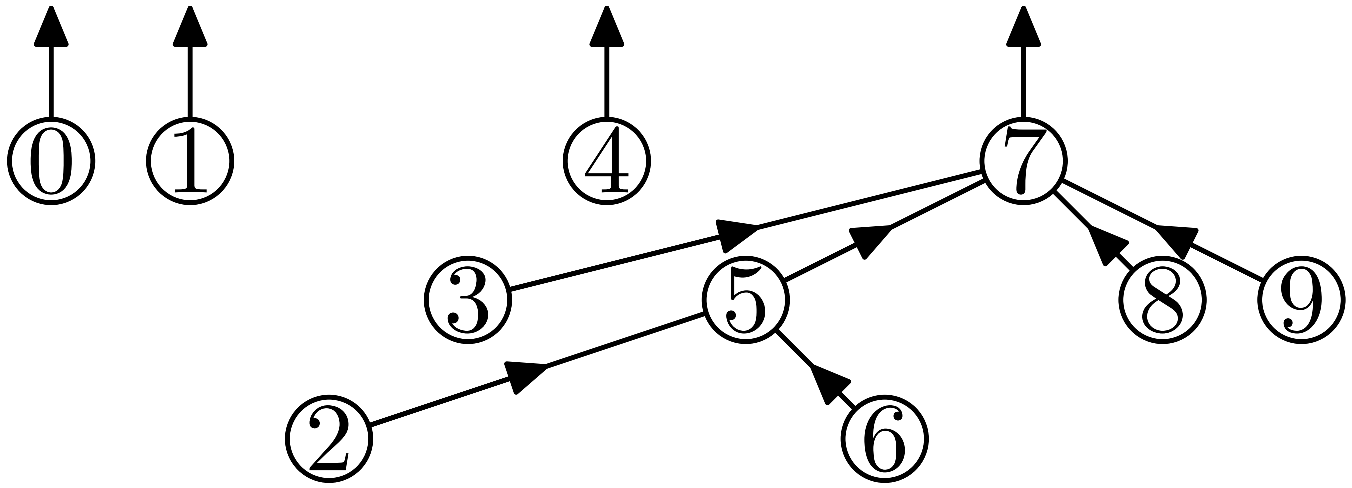
Putting it Together

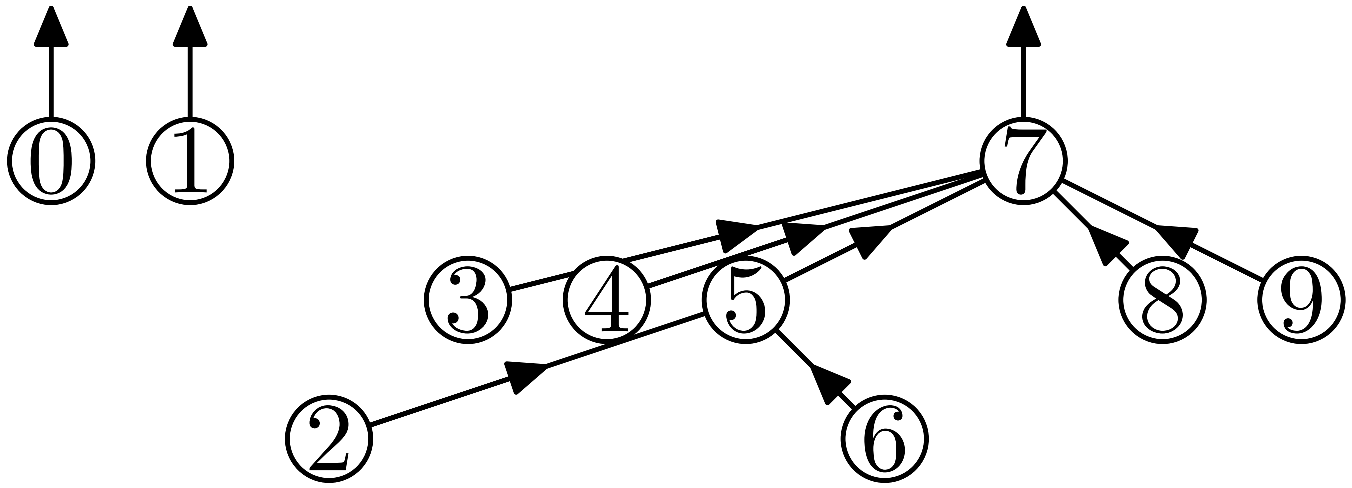
`union(find(3),find(4))`



-1	-1	5	7	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

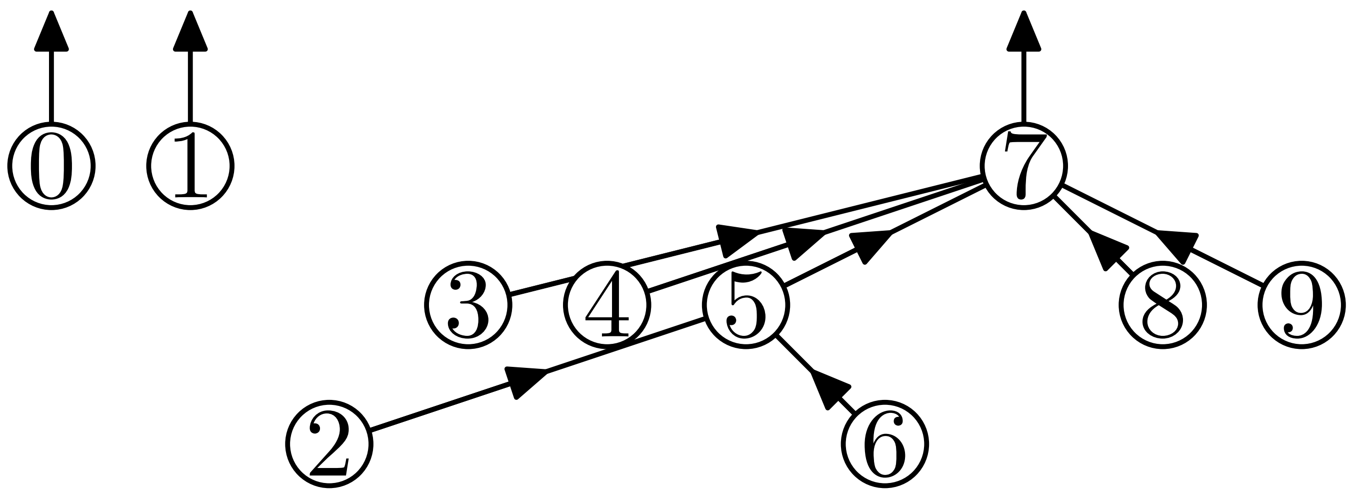
```
union(find(3),find(4))
```



-1	-1	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

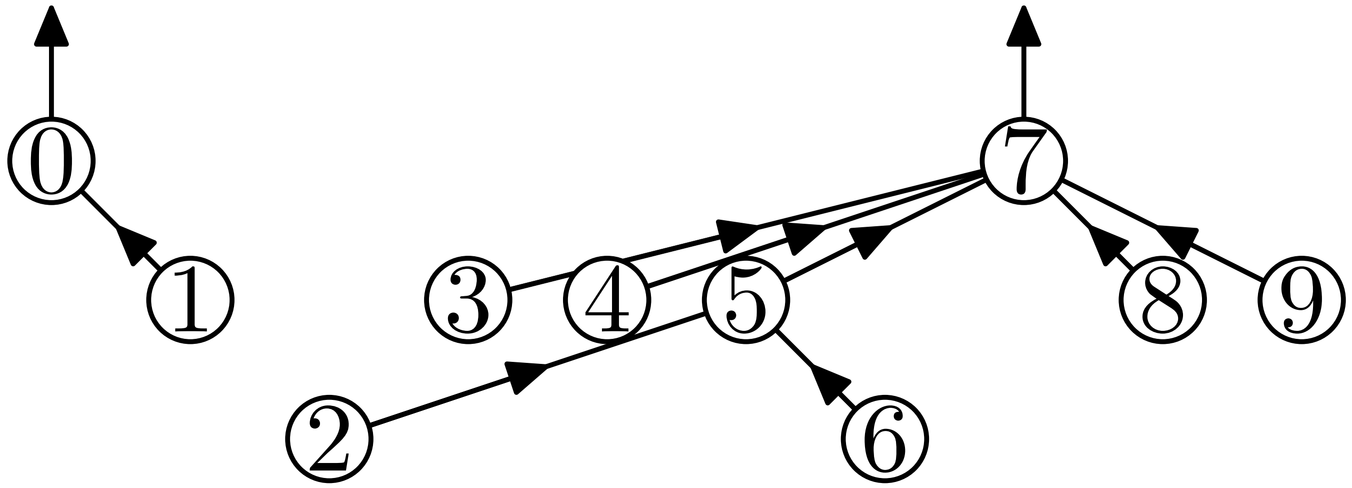
`union(find(0),find(1))`



-1	-1	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

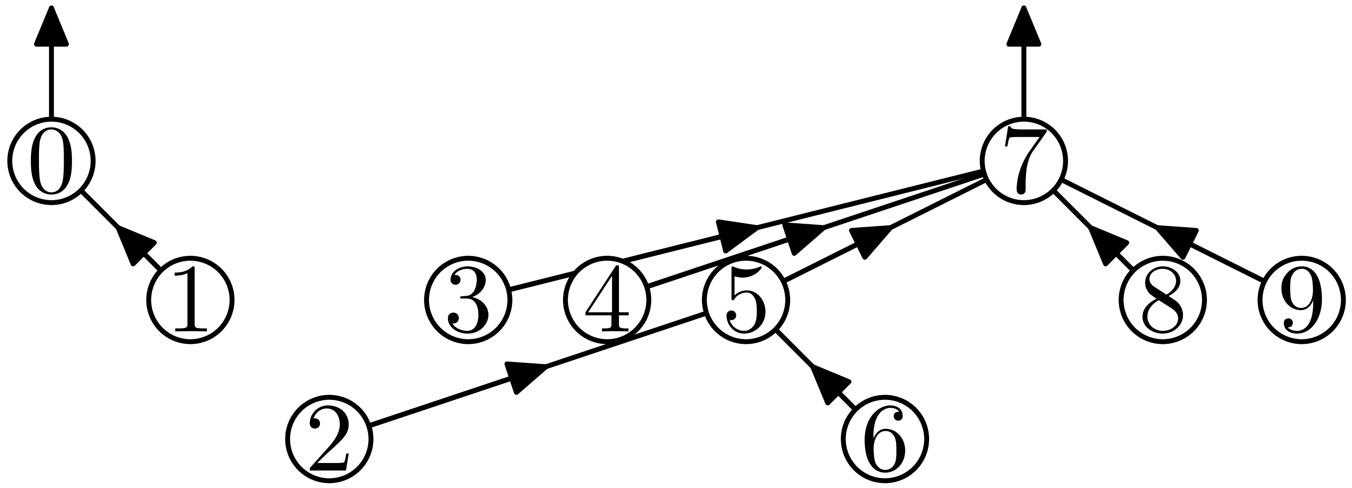
`union(find(0),find(1))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

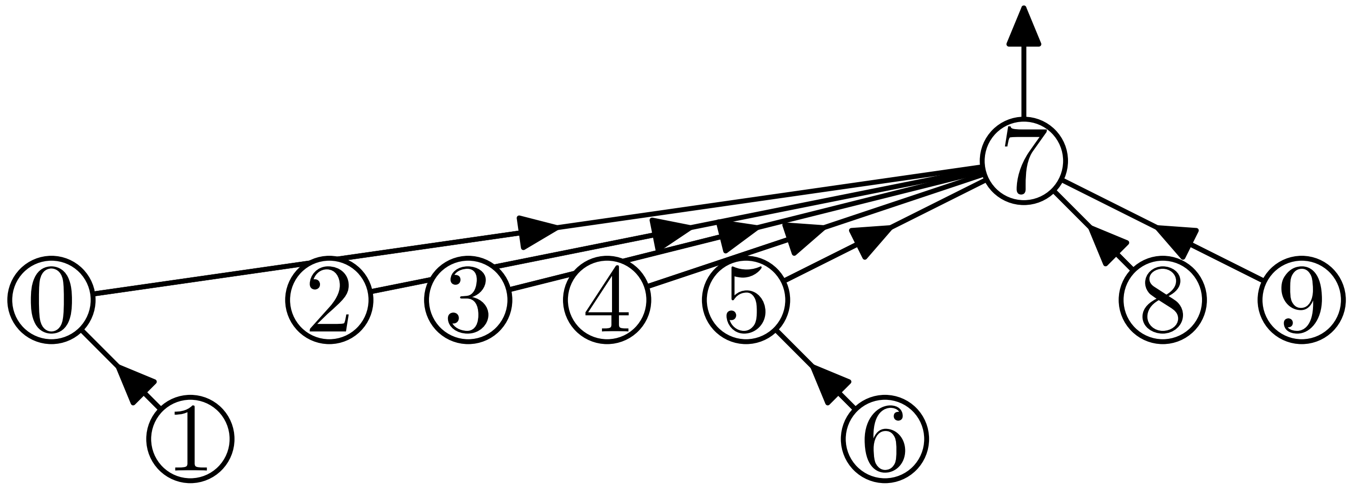
`union(find(1),find(2))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

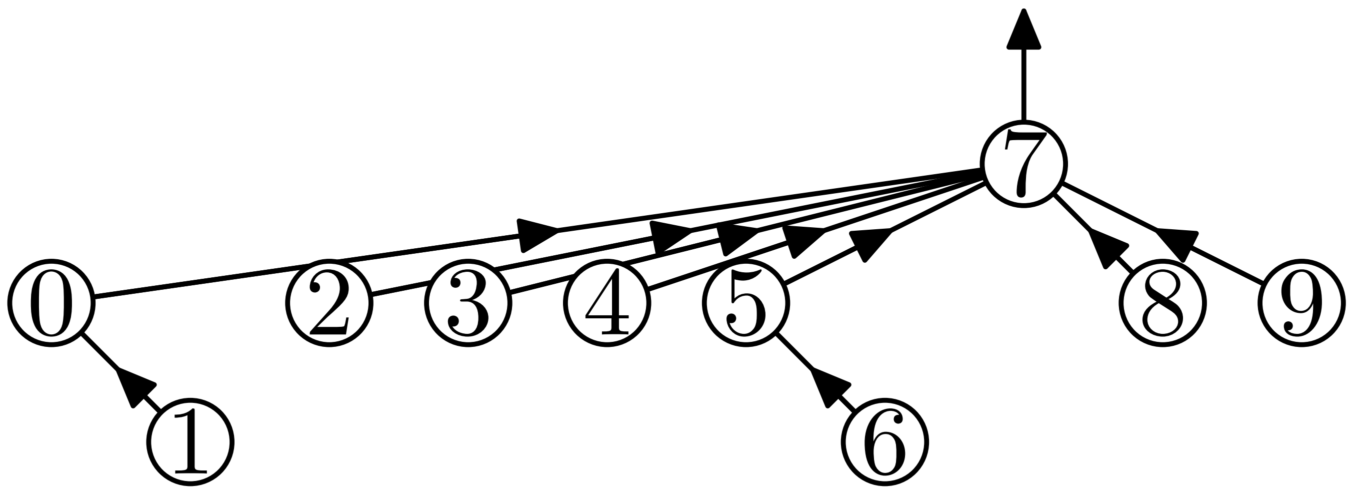
`union(find(1),find(2))`



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

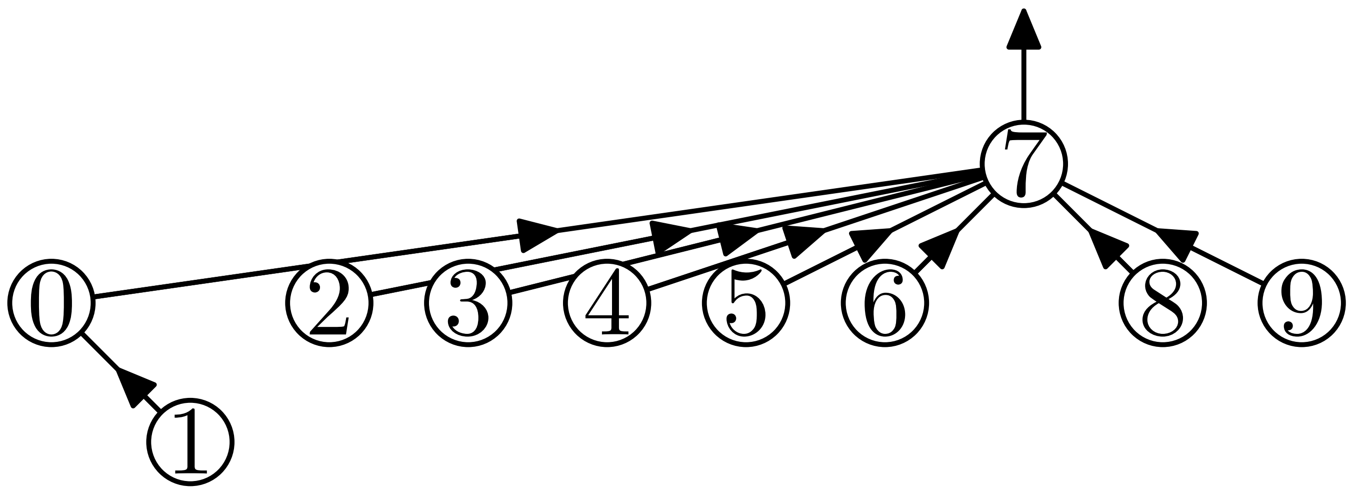
`find(6)`



0	1	2	3	4	5	6	7	8	9
7	0	7	7	7	7	5	-3	7	7

Putting it Together

```
find(6)=7
```

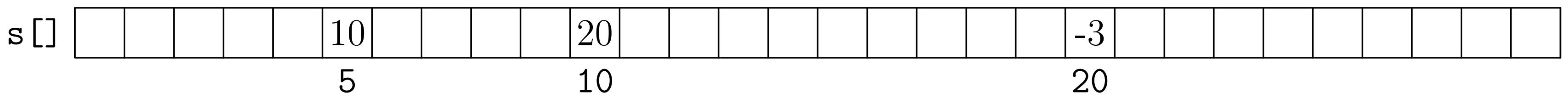


7	0	7	7	7	7	7	-3	7	7
0	1	2	3	4	5	6	7	8	9

Smart Union

```
DisjointSets::DisjointSets(int numElements)
{
    s = new int[numElements];
    for(int i=0; i<numElements; i++)
        s[i] = -1;                              // roots are negative number
}

void DisjointSets::union_(int root1, int root2)
{
    if (s[root2]<s[root1]) {                    // root2 is deeper
        s[root1] = root2;                       // make root2 the root
    } else {
        if (s[root1]==s[root2])
            s[root1]--;                         // update height if same
        s[root2] = root1;                       // make root1 new root
    }
}
```



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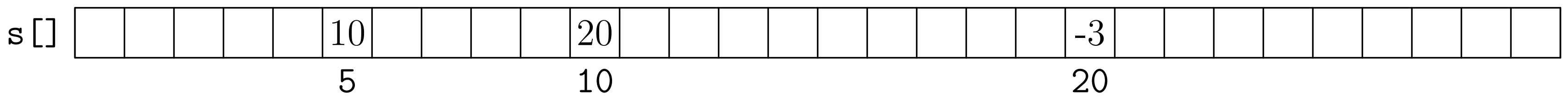

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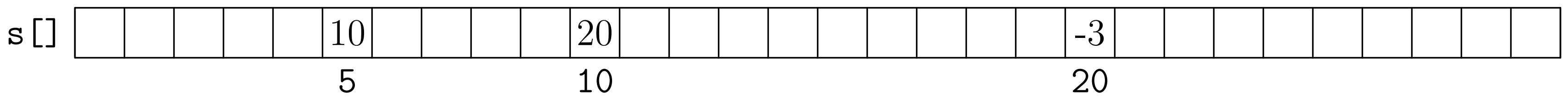
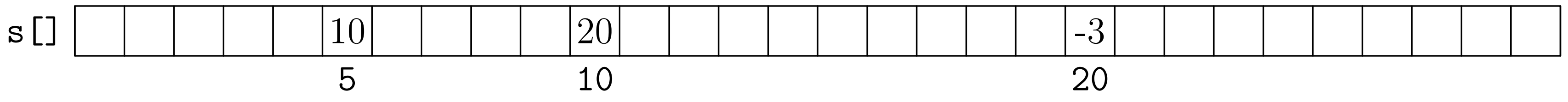
            s[root1]--;                     // update height if same
        s[root2] = root1;                   // make root1 new root
    }
}
```



Path Compression

- To speed up `find` we relabel all nodes we visit during `find` by the root label

```
int DisjointSets::find(int index)
{
    if (s[index]<0)
        return index;
    else
        return s[index] = find(s[index]);
}
```



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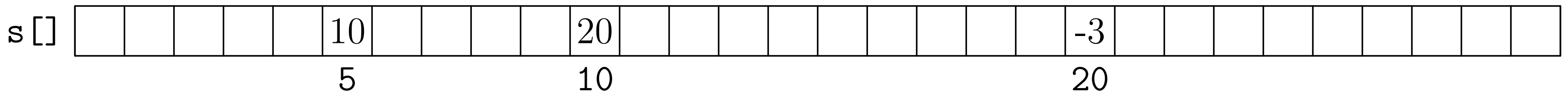
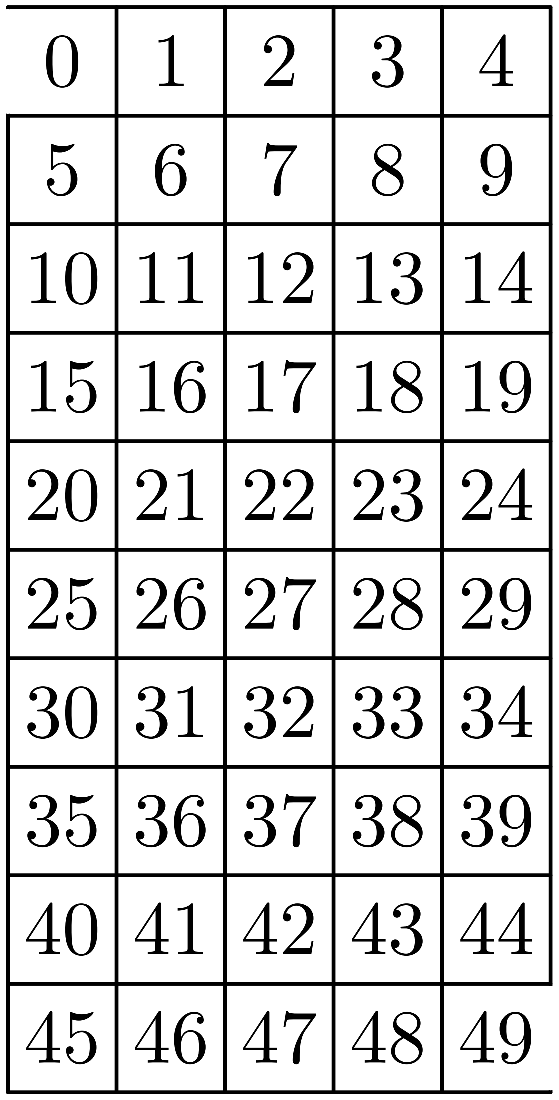


Mazes

- Union-Find is a data structure which can occur in very different applications
- One application is building a maze
- Start from a complete lattice
- Remove a randomly chosen edge if it connects two unconnected regions
- Stop when the start and end cell are connected
- Or better after all cells are connected



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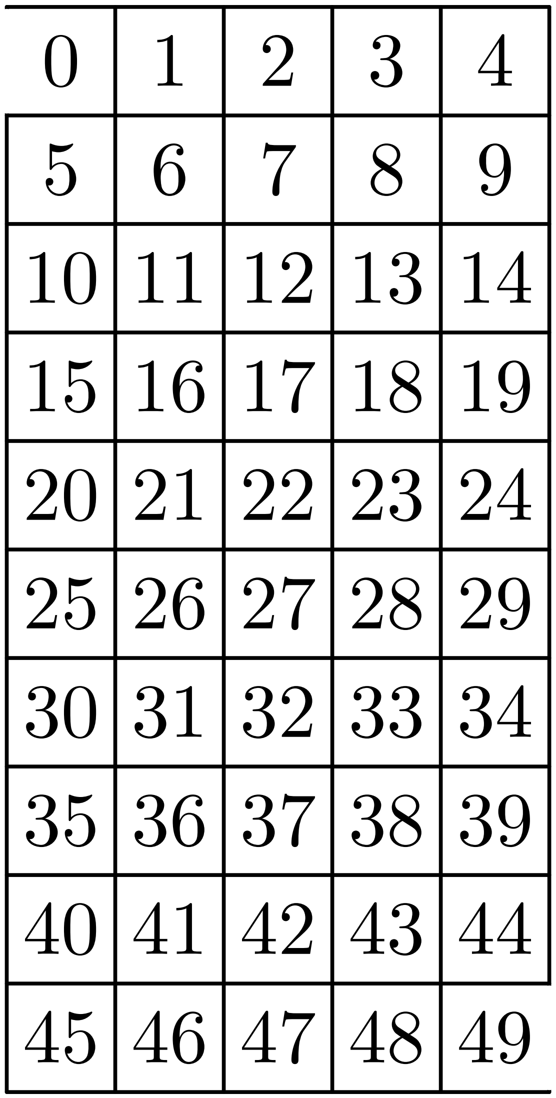

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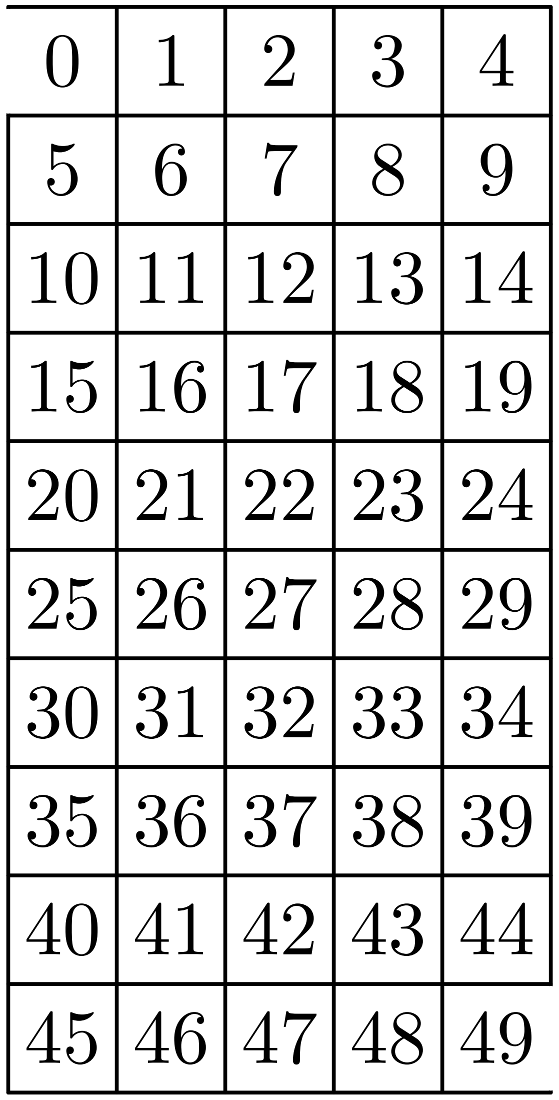

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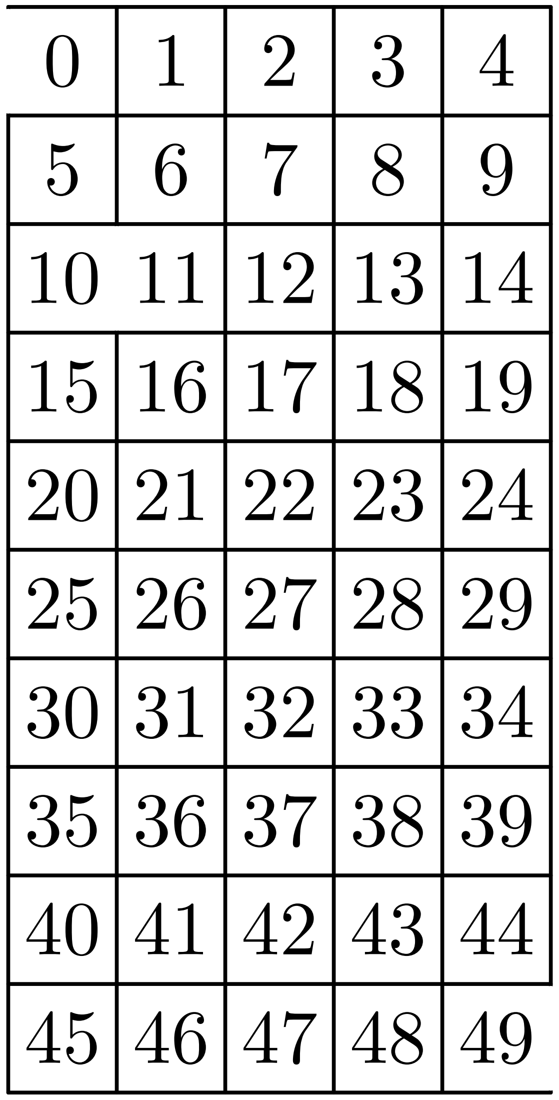

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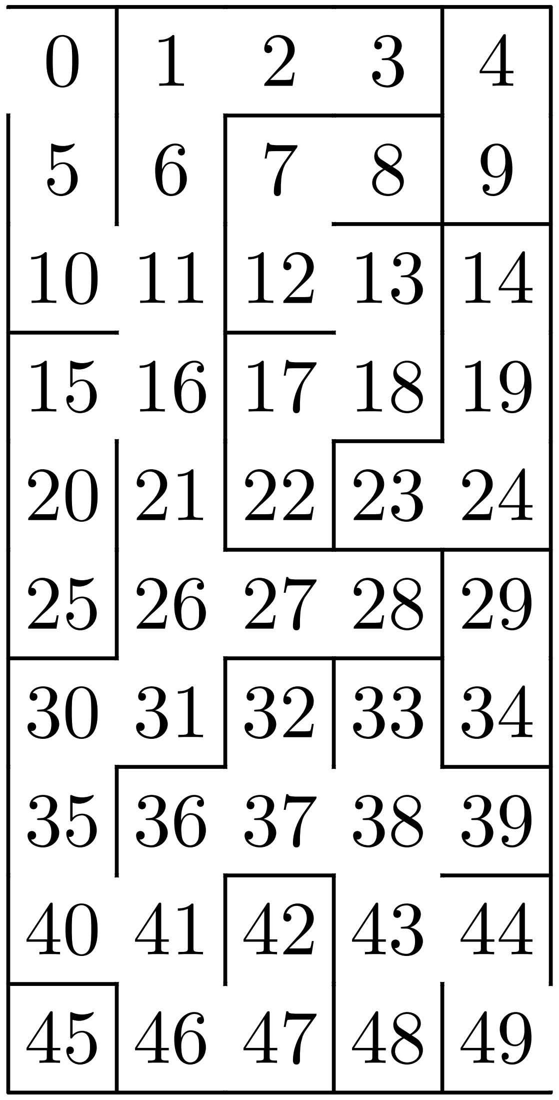

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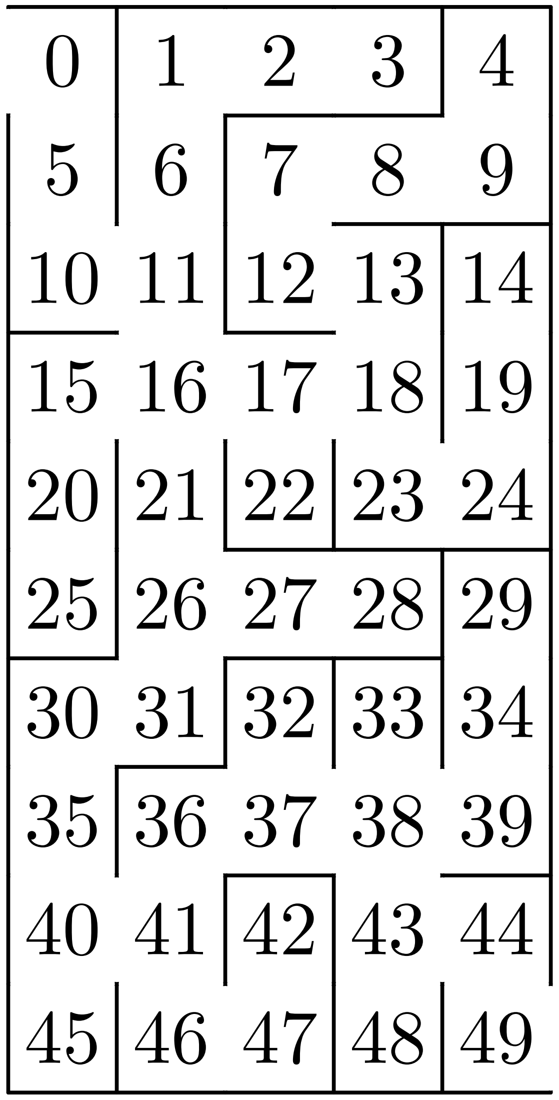

Time Complexity of Union-Find

- If we perform M finds and N unions then the time complexity is $O\big(M \log_2^*(N)\big)$
- Where $\log_2^*(N)$ is the number of times you need to apply the logarithm function before you get a number less than 1
- In practice $\log_2^*(N) \leq 5$ for all conceivable N

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$$\log_2(10^{80}) = 265.75$$

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$$\log_2(\log_2(10^{80})) = 8.0539$$

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$$\log_2(\log_2(\log_2(10^{80}))) = 3.0097$$

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$$\log_2(\log_2(\log_2(\log_2(10^{80})))) = 1.5896$$

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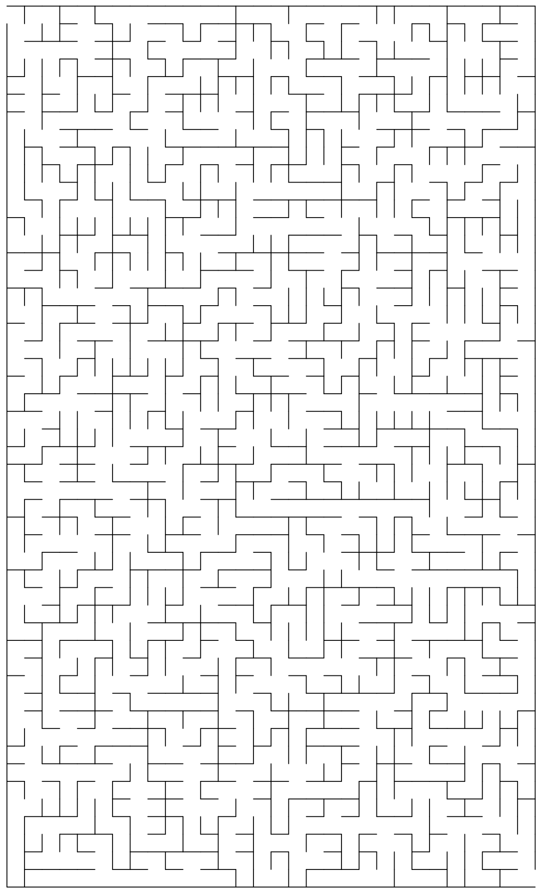
$$\log_2(\log_2(\log_2(\log_2(\log_2(10^{80}))))) = 0.66868$$

- The proof of this time complexity is rather involved

Outline

1. Equivalent Classes
2. Disjoint Sets
3. **Fast Sets**



Comparison of Sets

- Binary Search Trees: $O(\log_2(n))$, general purpose
- Hash tables: $O(1)$, but need to compute hash, slow iterator when sparse, general purpose
- B-trees: $O((k-1)\log_k(n))$ very complicated, used for large amounts of data
- Tries: $O(len(seq)) = O(\log_k(n))$ for large k expensive in memory, complicated to code efficiently

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What Set to Use?

- A PhD student and I were working on writing a fast solver for a combinatorial optimisation problem
- We had to choose one variable to change out of a small number of possible variables
- Each time we changed a variable then we had to update the list of possible variables (remove some variables add others)
- We wanted a data structure which had quick add and remove and where we could choose a variable at random

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- We wanted a data structure which had quick add and remove and where we could choose a variable at random—what should we use?

Bounded Set

- One special feature is that we knew we only wanted the set to contain integers between 0 and n (where n might be $100\,000$)
- This allowed us to use an array to represent whether an integer belong to that set
- But how do we find a random element of the set quickly?

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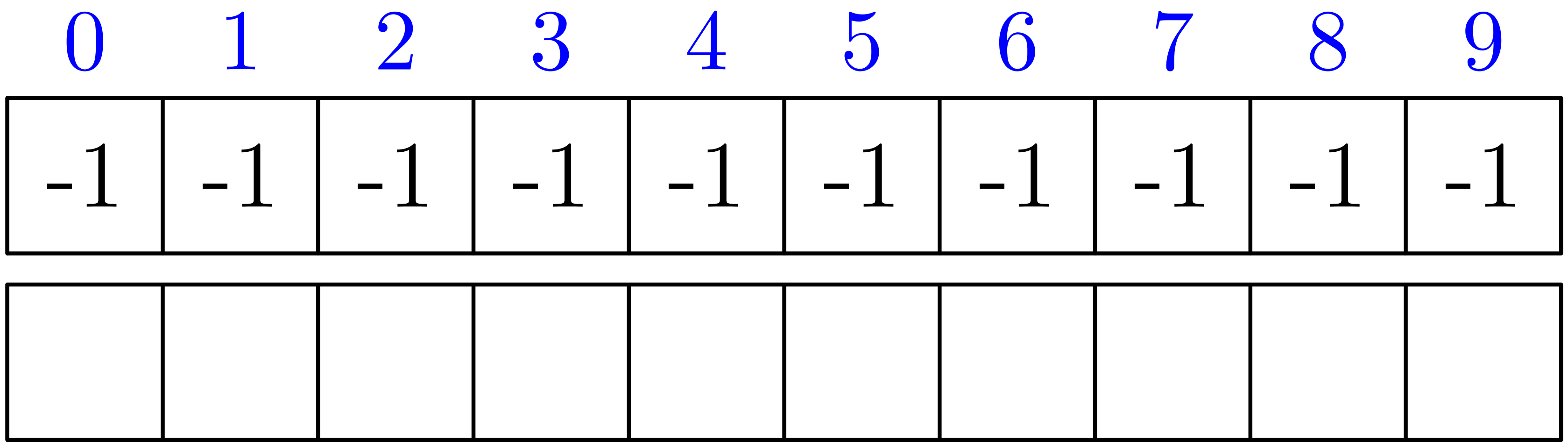
- But how do we find a random element of the set quickly?
- Use another array of course!

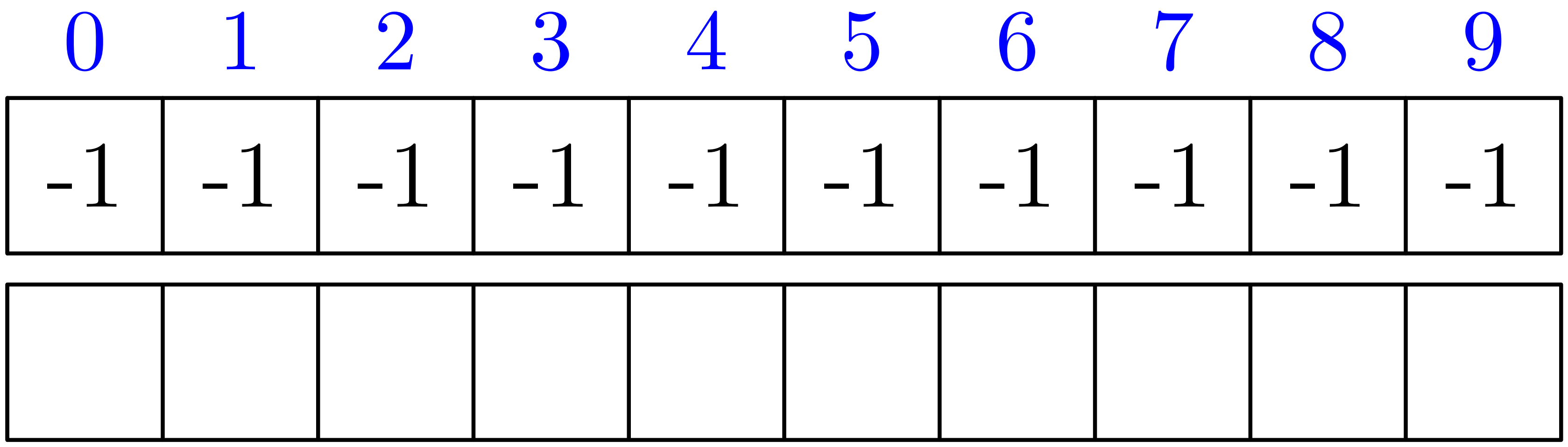
FastSet

FastSet



FastSet

`add(4)`



FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	-1
4									

FastSet

`add(9)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	-1

4									

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	1

4	9								

FastSet

`add(7)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	1

4	9								

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1
4	9	7							

FastSet

`add(4)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1

4	9	7							

FastSet

```
false
```

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1

4	9	7							

FastSet

`add(1)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1
4	9	7							

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1
4	9	7	1						

FastSet

`contains(9)`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1
4	9	7	1						

FastSet

`contains(5)`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

`false`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

```
remove(9)
```

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1
4	9	7	1						

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	1	-1	-1	0	-1	-1	2	-1	-1

4	1	7							

Implementation

```
class FastSet {
  private:
    int* indexArray;
    int* memberArray;
    int noMembers;

  public:
    FastSet(int n) {
      indexArray = new int[n];
      memberArray = new int[n];
      for(int i=0; i<n; i++) {
           indexArray [i] = -1;
      }
      noMembers = 0;
    }

    ~FastSet() {
      delete[] indexArray;
      delete[] memberArray;
    }

    int size() { return noMembers; }
```

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    }

    int size() { return noMembers; }
```

Add and Remove

```
bool add(int i) {
   if (indexArray[i]>-1)
      return false;
   memberArray[noMembers] = i;
   indexArray[i] = noMembers;
   ++noMembers;
   return true;
}

bool remove(int i) {
   if (indexArray[i]==-1)
         return false;
   --noMembers;
   memberArray[indexArray[i]] = memberArray[noMembers];
   indexArray[memberArray[noMembers]] = indexArray[i];
   indexArray[i] = -1;
   return true;
}
```

Add and Remove

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   --noMembers;
   memberArray[indexArray[i]] = memberArray[noMembers];
   indexArray[memberArray[noMembers]] = indexArray[i];
   indexArray[i] = -1;
   return true;
}
```

Collection Methods

```
void clear() {
   for(int i=0; i<noMembers; i++) {
      indexArray[memberArray[i]] = -1;
   }
   noMembers = 0;
}

bool isEmpty() {
   return noMembers==0;
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int* begin() {return &memberArray[0];}
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Collection Methods

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void clear() {
   for(int i=0; i<noMembers; i++) {
      indexArray[memberArray[i]] = -1;
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And Random?

- We can add additional methods taking advantage of the classes strength

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private:
  random_device rd;  // Seed for the random number engine
  mt19937 gen(rd()); // Mersenne Twister RNG

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- For large problems we were over 10 times faster because of this data structure
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Lessons

- If you have a bounded set then using an array is usually going to be very fast $O(1)$ (or $O(\log^*(n))$)
- These data structures are not general purpose for solving every day problems (c.f. `vector<T>`, `set<T>` and `map<T>`)
- They are “back pocket” data structures that solve problems that come up often enough that they are worth knowing about
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