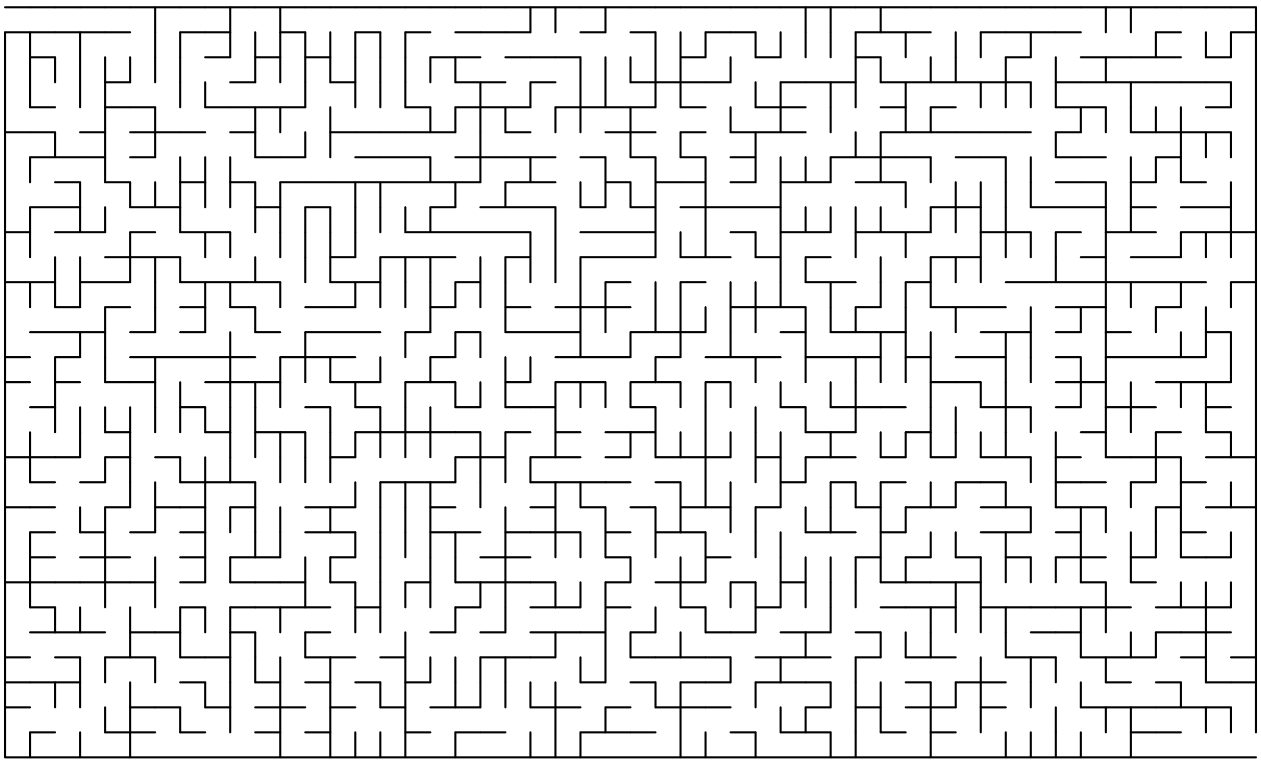
Lesson 15: *Use Arrays for Fast Set Algorithms*



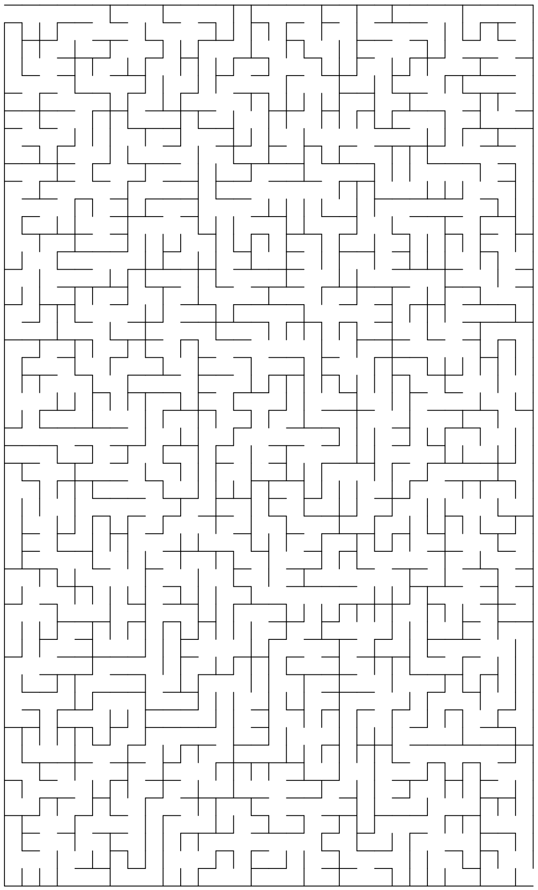
Equivalent classes, Disjoint Set, Fast Sets

Outline

1. **Equivalent Classes**

2. Disjoint Sets

3. Fast Sets



Equivalence Relations

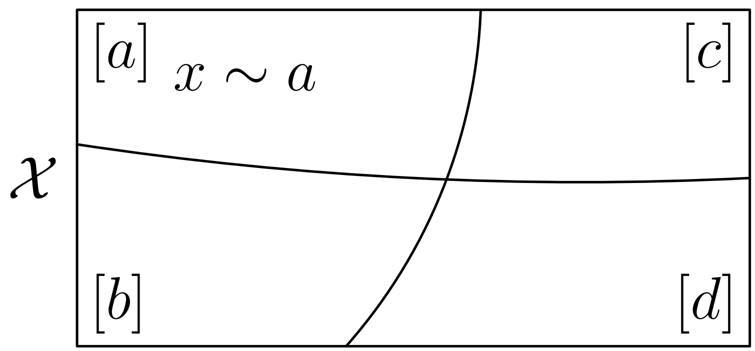
- Given a set of elements $\mathcal{X} = \{x_1, x_2, \ldots\}$ and a binary relationship $\sim$ with the following properties

  (Reflexivity) For every element $x \in \mathcal{X}$, $x \sim x$

  (Symmetry) For every two elements $x, y \in \mathcal{X}$ if $x \sim y$ then $y \sim x$

  (Transitivity) For every three elements $x, y, z \in \mathcal{X}$ if $x \sim y$ and $y \sim z$ then $x \sim z$

- Then $\sim$ defines a partitioning of the set into **equivalence classes**



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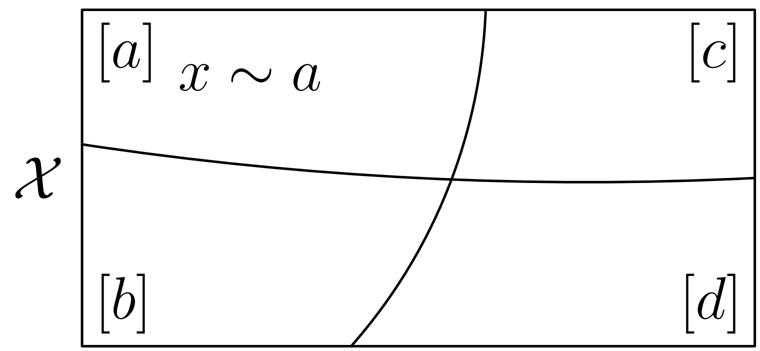

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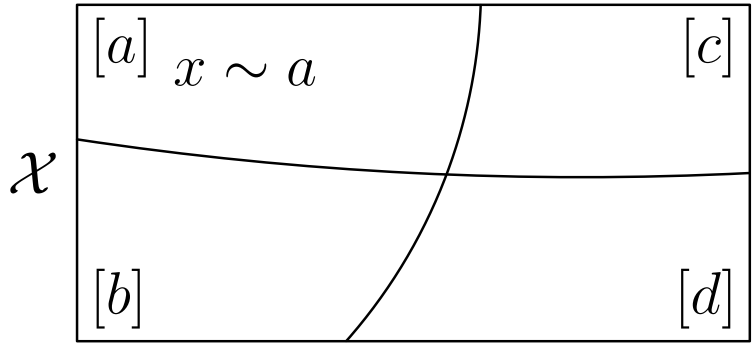

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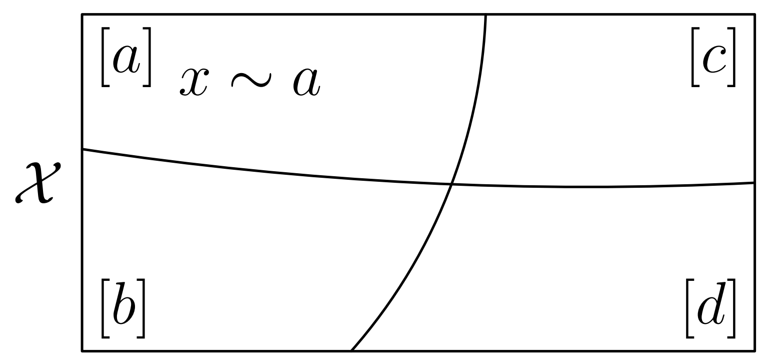

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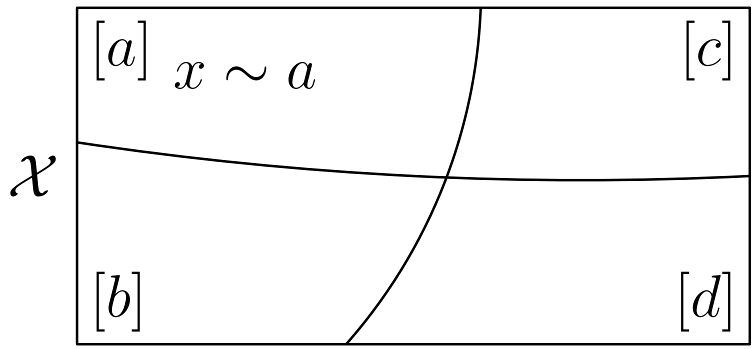

Example of Equivalence Classes

- Although, equivalent classes sound very mathematical they often provide a useful formalisation of the real world
- E.g. Pairs of web pages with a link in each direction between them
- Consider web pages in the same equivalence class if you can get from one to the other by clicking links
- Partitions the web into linked domains
- Friendship relations in social media

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Dynamic Equivalence Classes

- Finding equivalence classes is rather easy using graph traversal algorithms
- However, as our web example suggests, there are applications where equivalence classes change over time
- Adding a link could join two domains which were separate
- We will see this is a useful idea both for building mazes and (in a later lecture) for finding minimum spanning trees
- Building a data structure which finds equivalence classes where the equivalence relation changes over time is challenging

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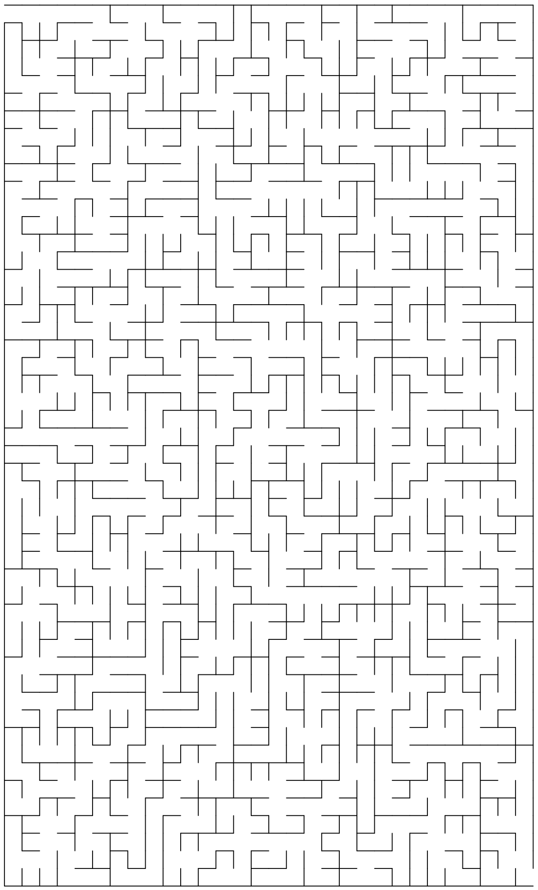
2. **Disjoint Sets**
3. Fast Sets



Union-Find

- In the union-find algorithm we have a set of objects $x \in \mathcal{S}$ which are to be grouped into subsets $\mathcal{S}_1$, $\mathcal{S}_2$, . . .
- Initially each object is in its individual subset (no relationships)
- We want to make the **union** of two subsets (add relationship between elements)
- We also want to **find** the subset given an element
- This is a common problem for which we will write a class `DisjointSets` to perform fast `union`s and `find`s

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DisjointSets

- We want to create a class

```
public class DisjointSets
{
    public DisjointSets(int numElements) {/* Constructor */}

    public int find(int x) {/* Find root */}

    public void union(int root1, int root2) {/* Union */}

    private int[] s;
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- Where `find(x)` returns a unique identifier for the subset which element `x` belongs to

- The array `s` contains labelling information to implement `find(x)`

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The Union-Find Dilemma

- A natural algorithm to perform finds is to maintain an array returning a subset label for each element—this makes `find` fast
- However, every time we combine two subset we have to change all the labels in this array (taking $O(n)$ operations)
- If we are unlucky the cost of performing n unions is $\Theta(n^2)$
- If we ensure that we relabel the smaller subset then the time complexity is $\Theta(n\log(n))$
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Fast Union

- To achieve fast unions we can represent our disjoint sets as a forest (many disjoint trees)

- Every time we perform a union we make one of the trees point to the head of the other tree

- The cost of `find` depends on the depth of the tree

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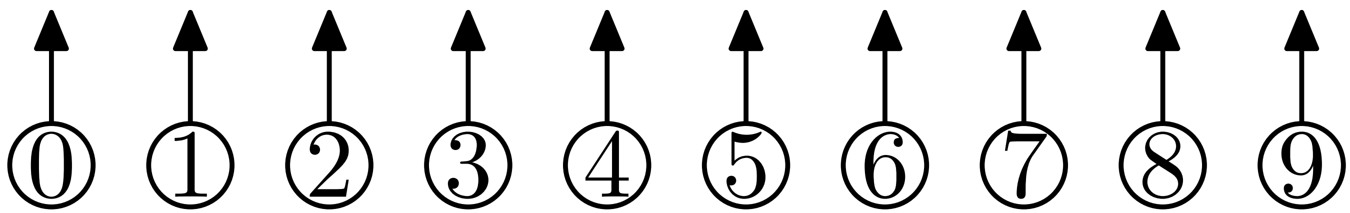
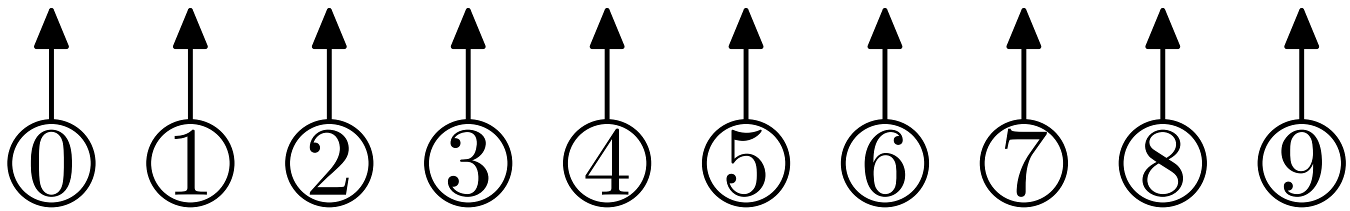
Putting it Together



-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(5),find(6))`



-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
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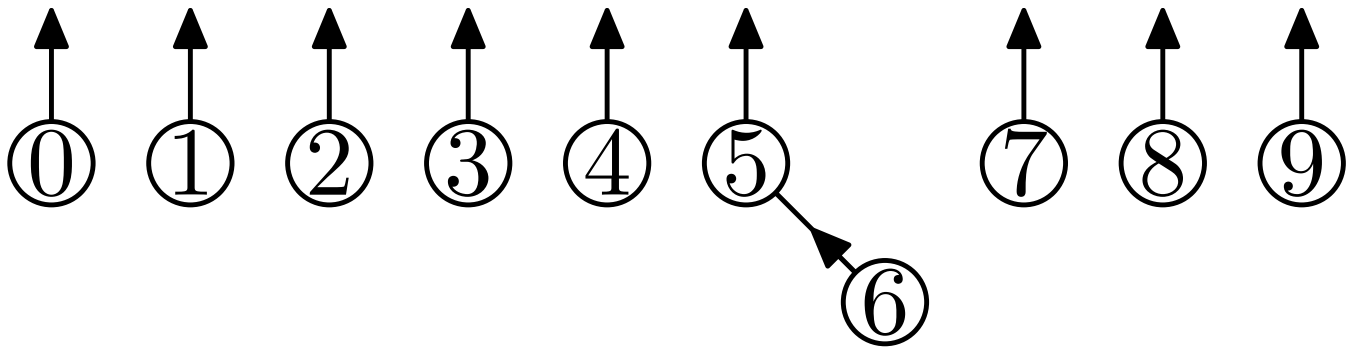
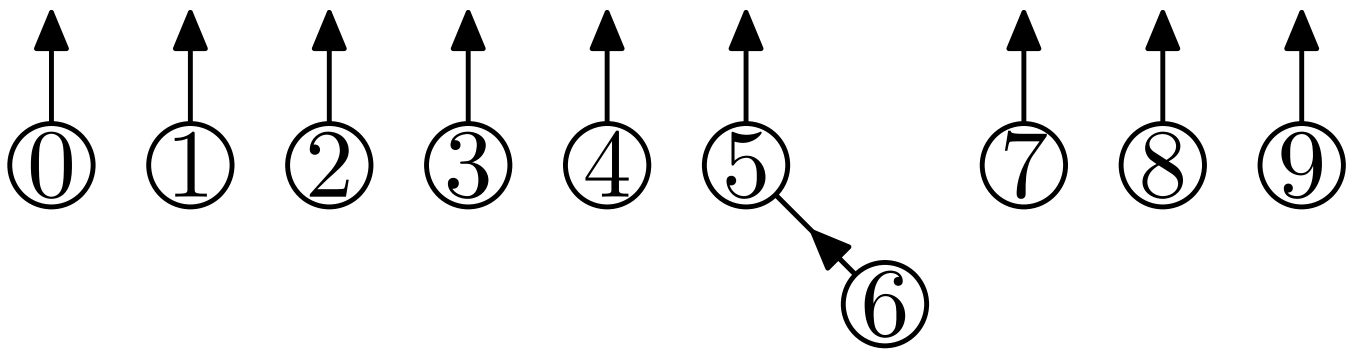


-1	-1	-1	-1	-1	-2	5	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(7),find(8))`



-1	-1	-1	-1	-1	-2	5	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

```
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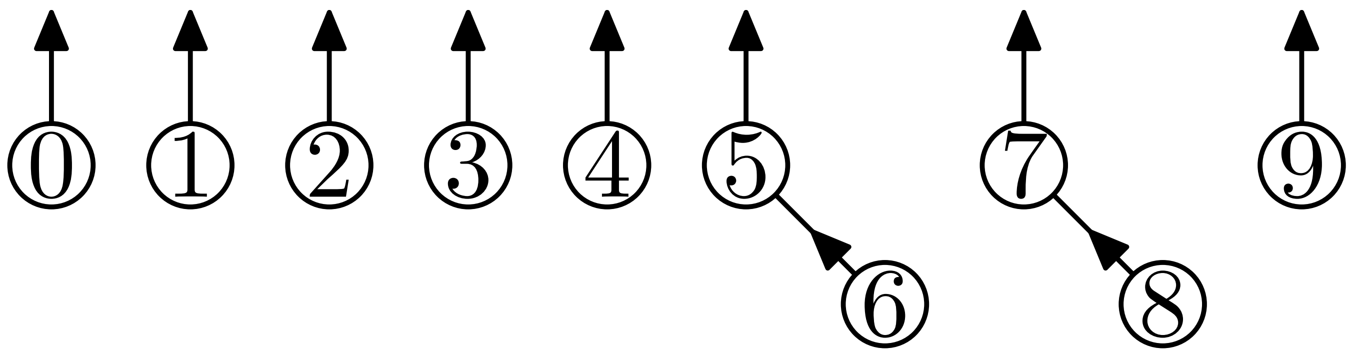
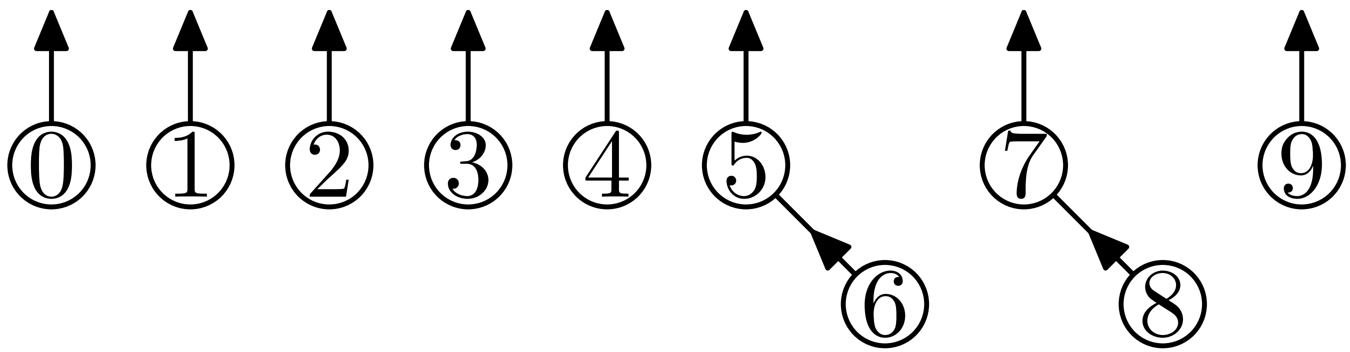


-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

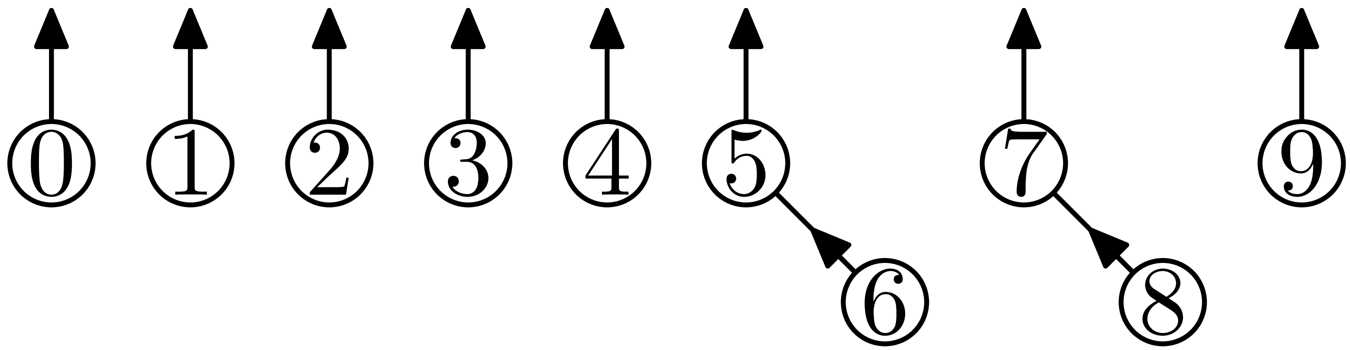
`find(6)`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

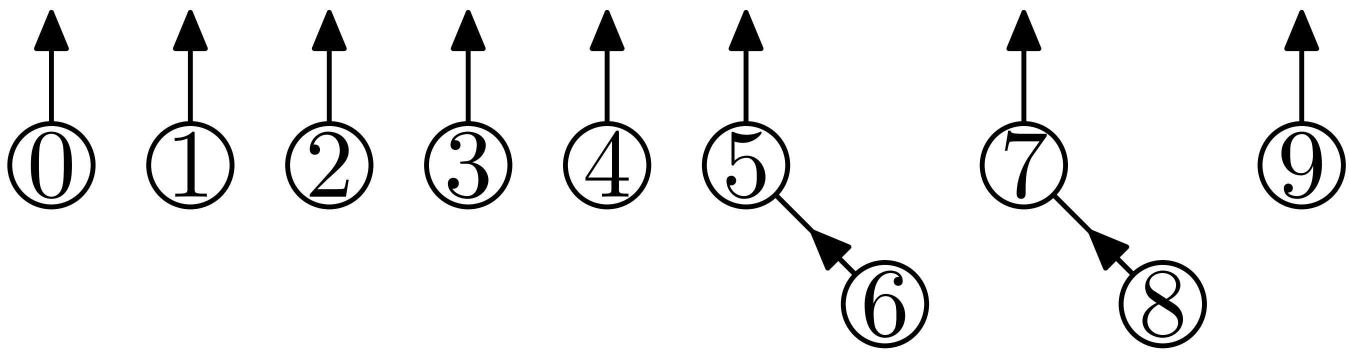
`find(6)=5`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(3),find(6))`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

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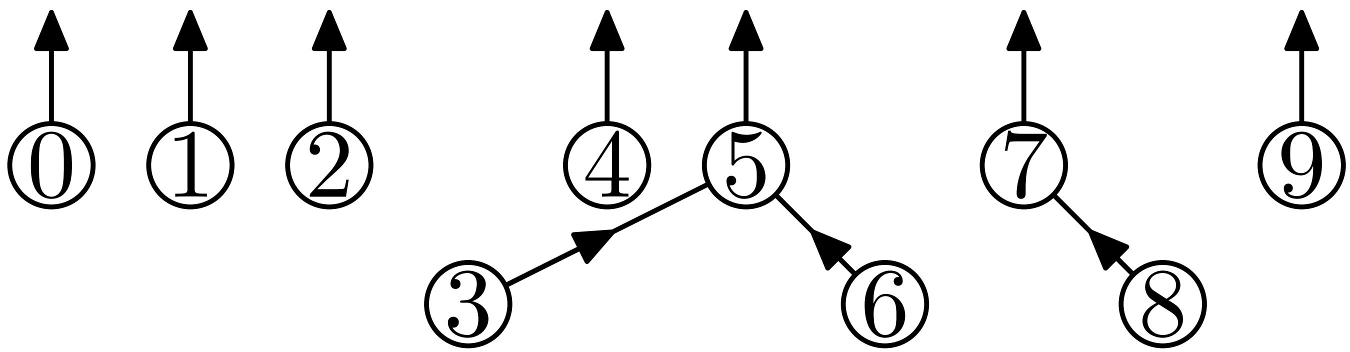
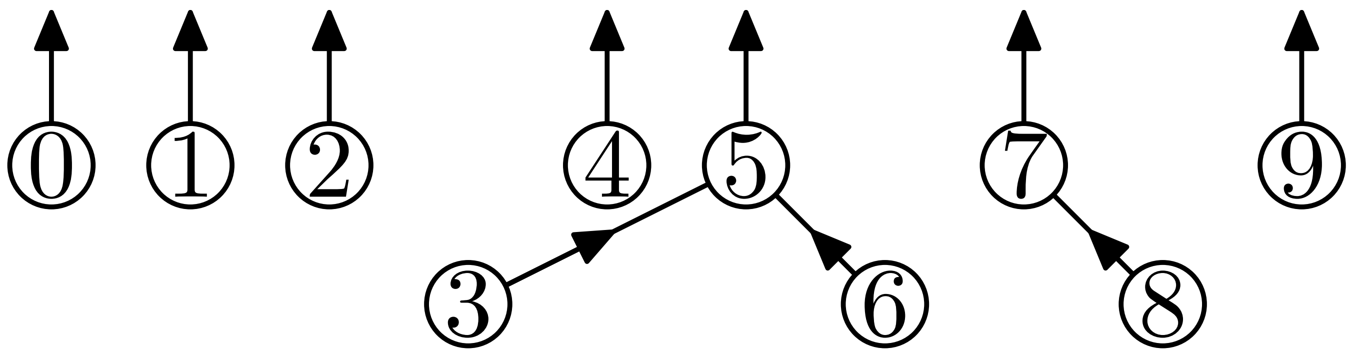


-1	-1	-1	5	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

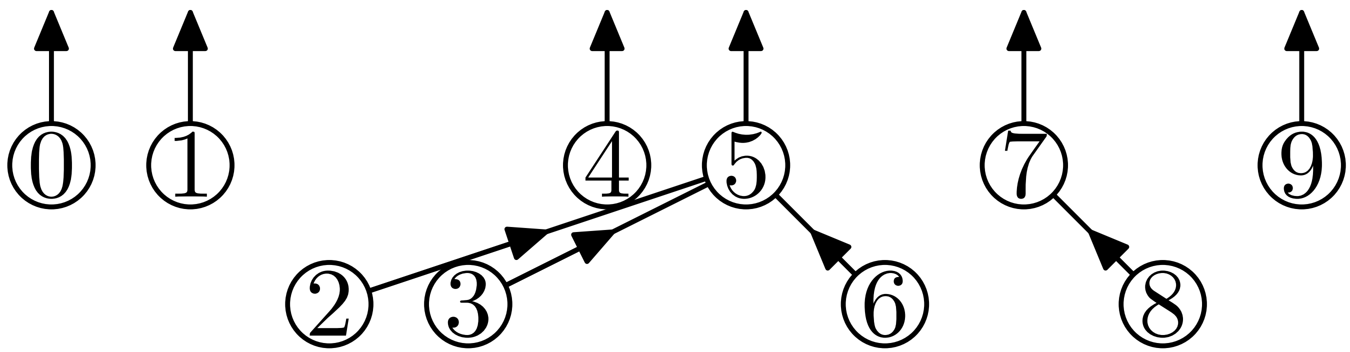
`union(find(2),find(6))`



0	1	2	3	4	5	6	7	8	9
-1	-1	-1	5	-1	-2	5	-2	7	-1

Putting it Together

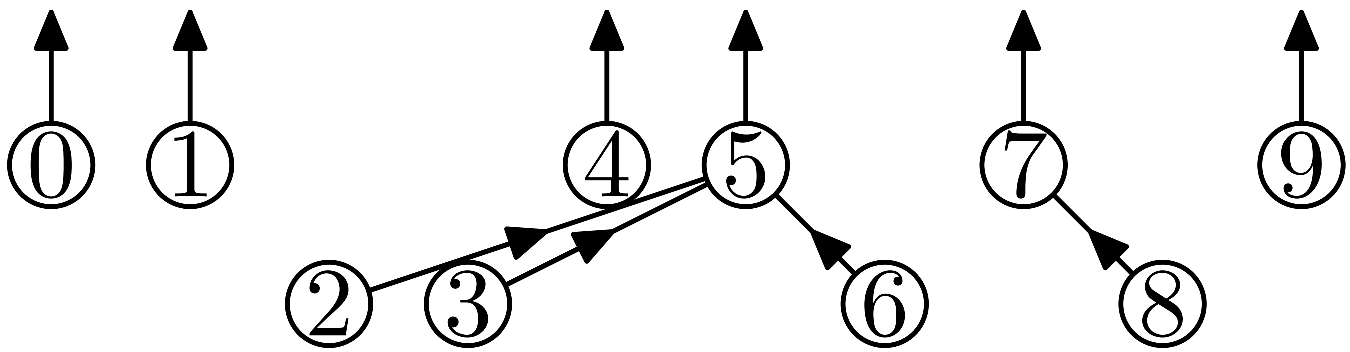
```
union(find(2),find(6))
```



-1	-1	5	5	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

Putting it Together

`union(find(9),find(8))`



-1	-1	5	5	-1	-2	5	-2	7	-1
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Putting it Together

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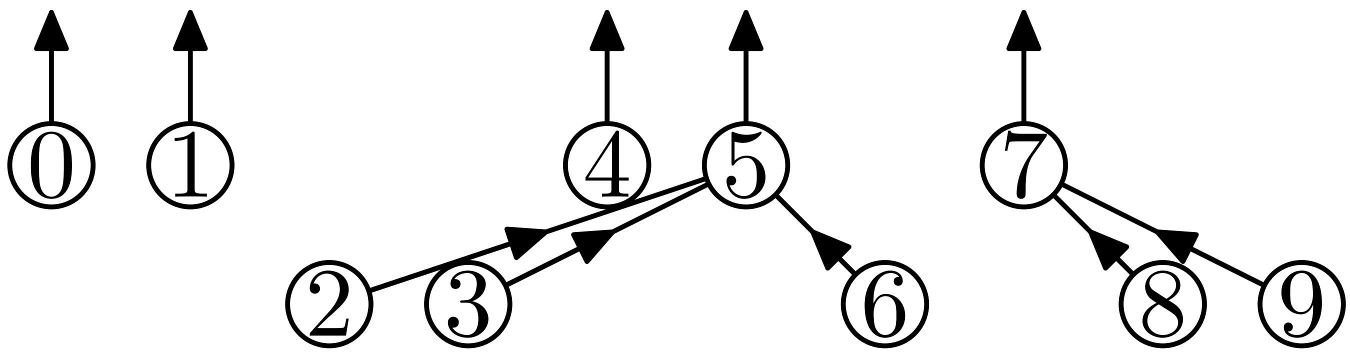
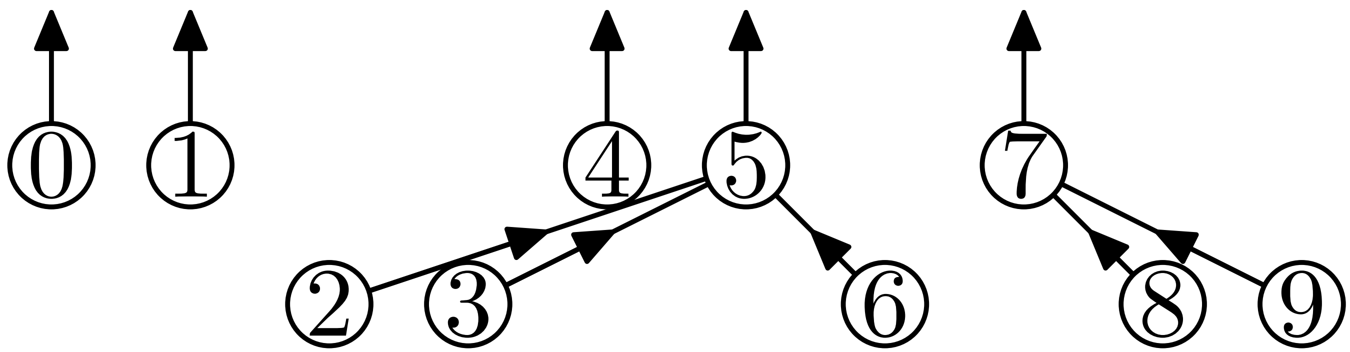


-1	-1	5	5	-1	-2	5	-2	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

```
union(find(9),find(3))
```



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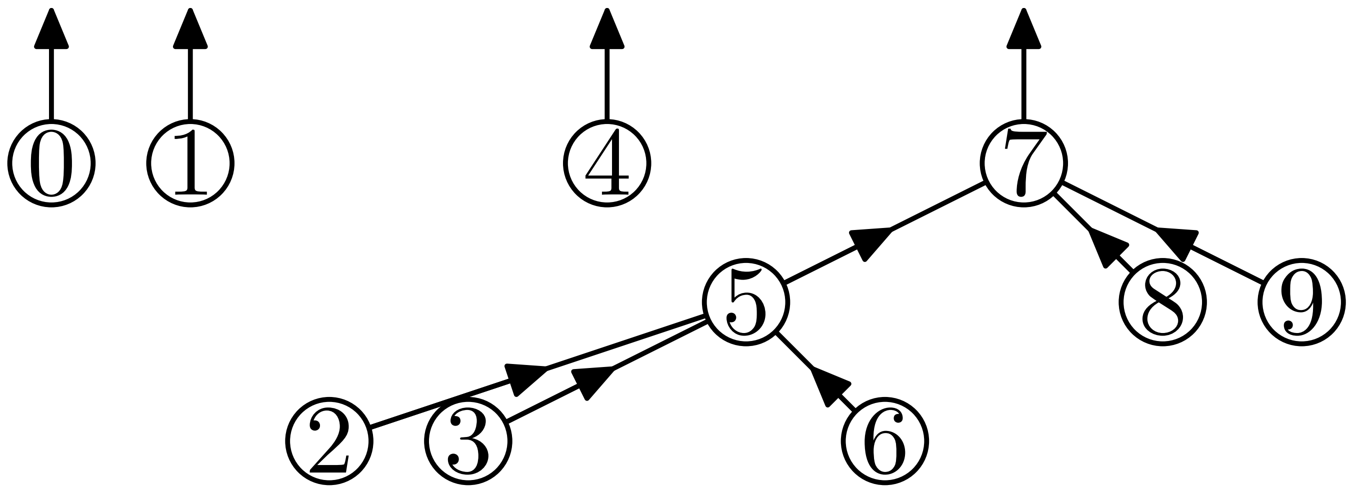
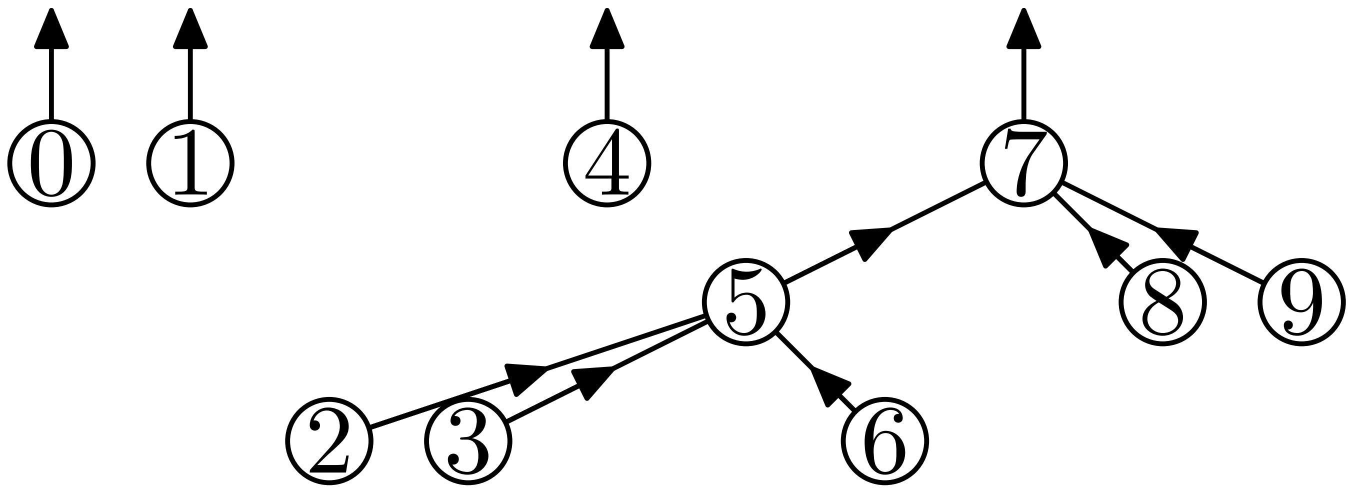


-1	-1	5	5	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

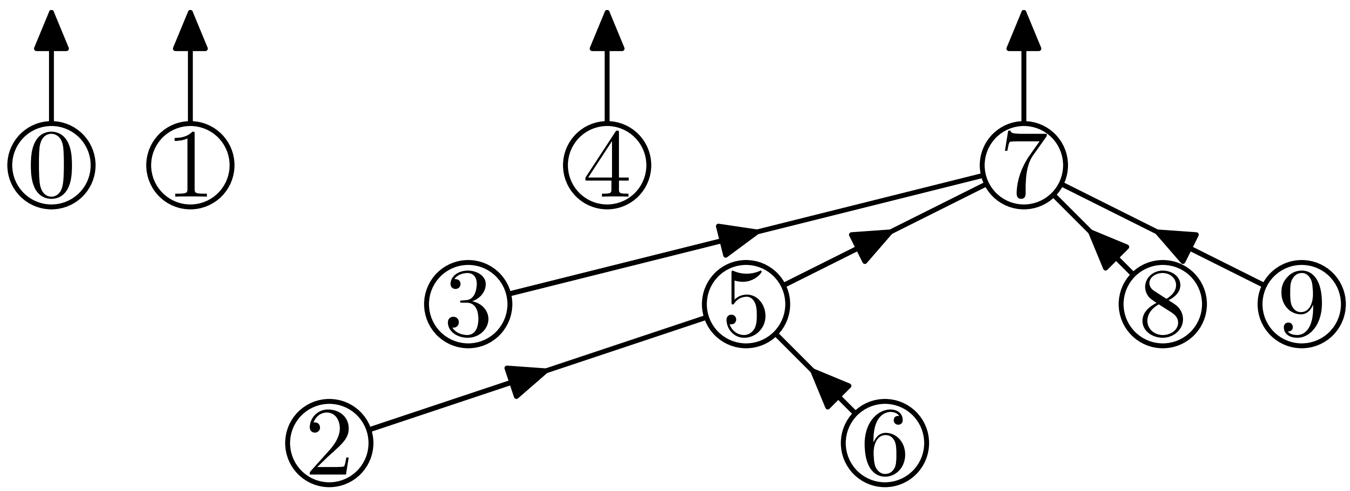
`find(3)`



-1	-1	5	5	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

`find(3)=7`



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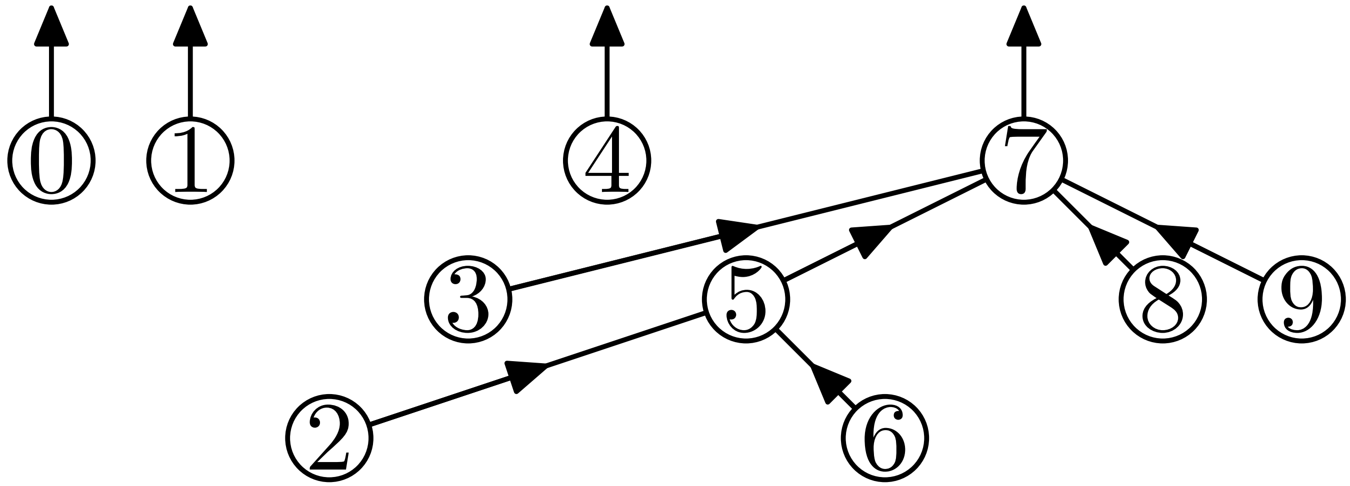
0	1	2	3	4	5	6	7	8	9

Putting it Together

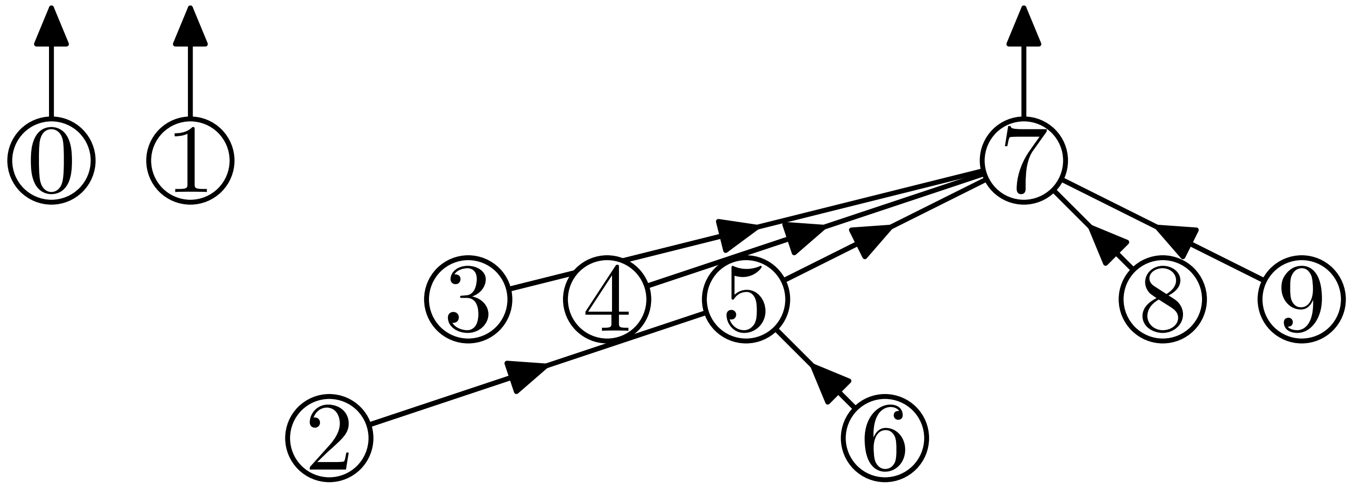
`union(find(3),find(4))`



-1	-1	5	7	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

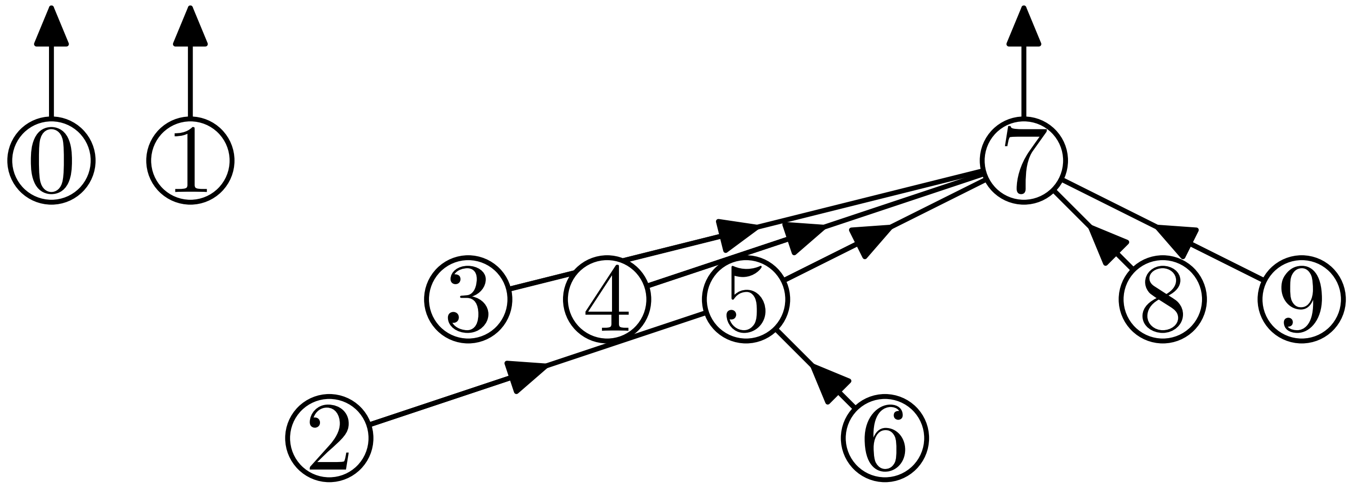
`union(find(3),find(4))`



-1	-1	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

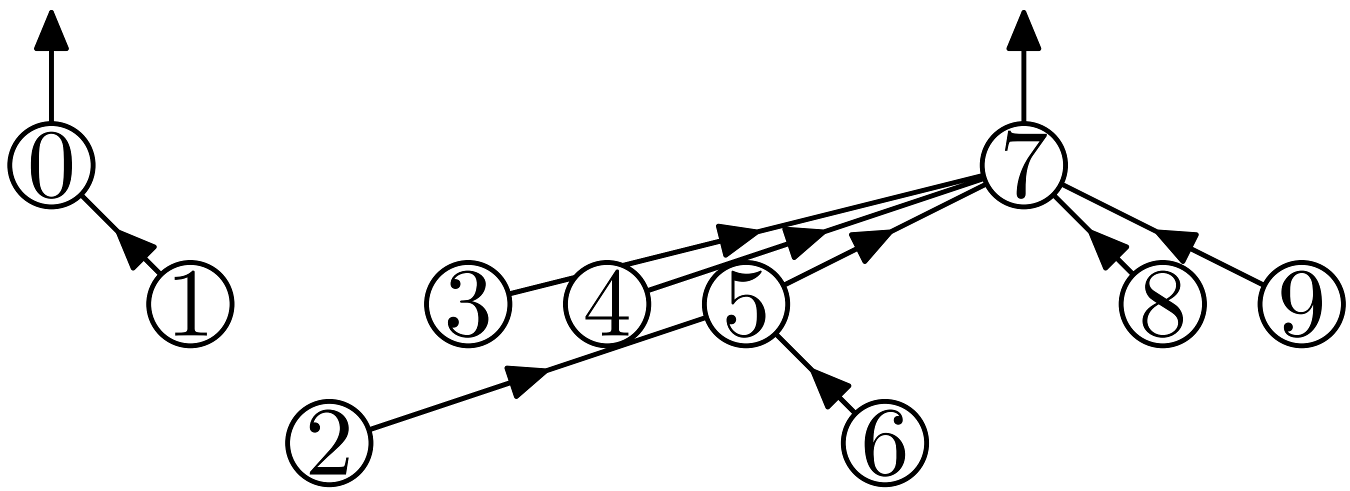
`union(find(0),find(1))`



-1	-1	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

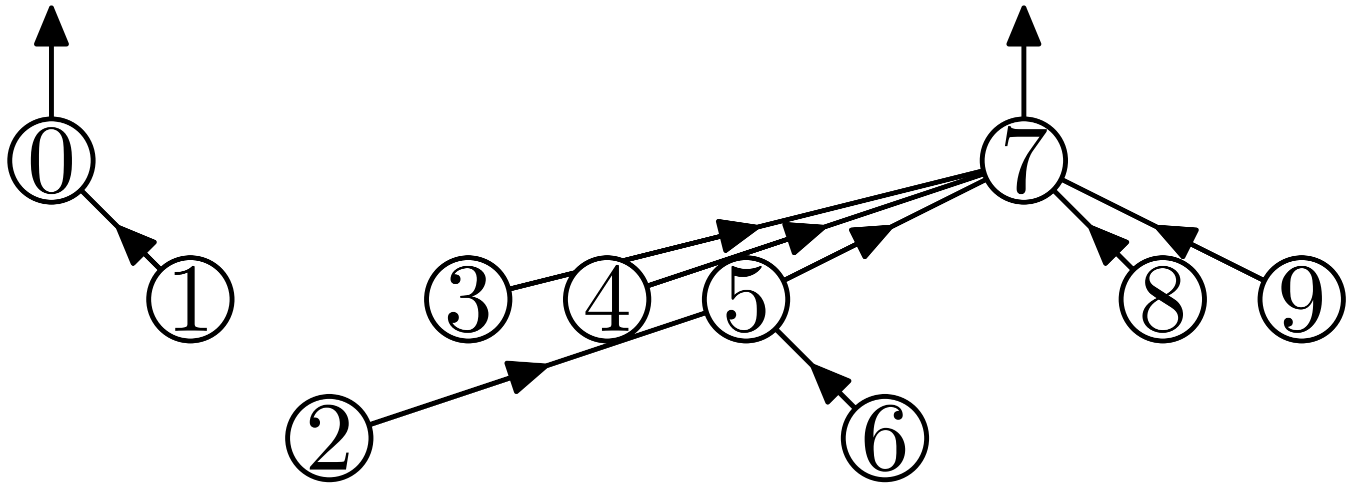
`union(find(0),find(1))`



0	1	2	3	4	5	6	7	8	9
-2	0	5	7	7	7	5	-3	7	7

Putting it Together

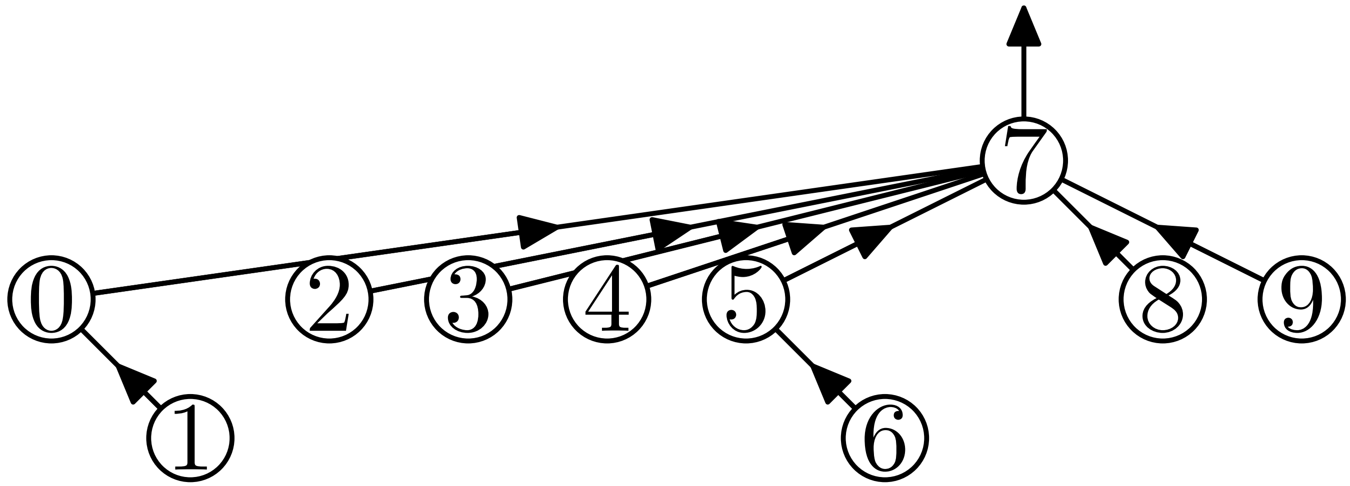
`union(find(1),find(2))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

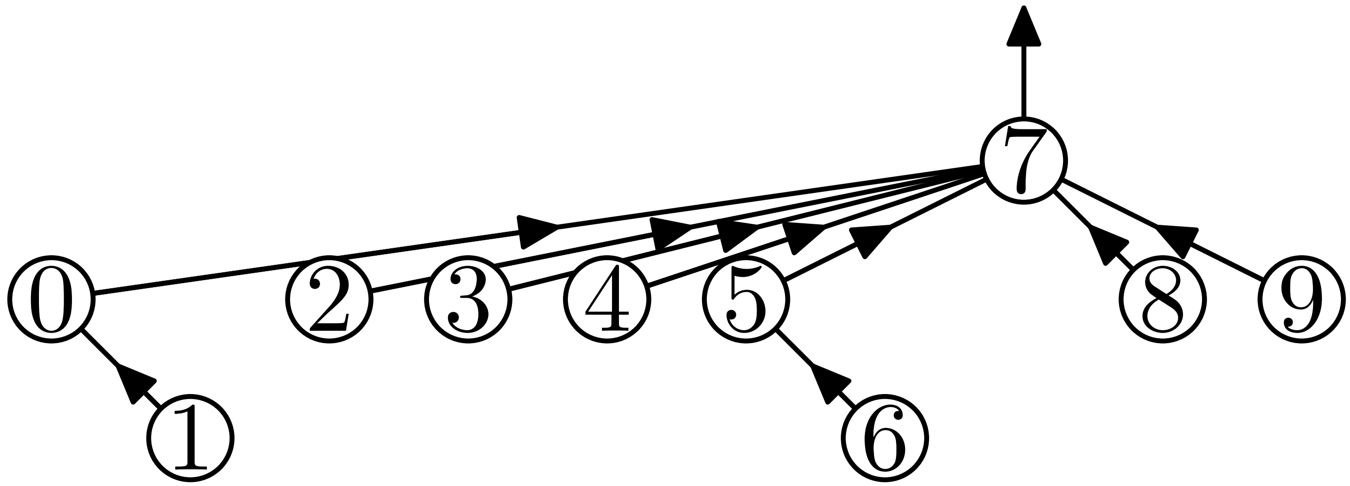
`union(find(1),find(2))`



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

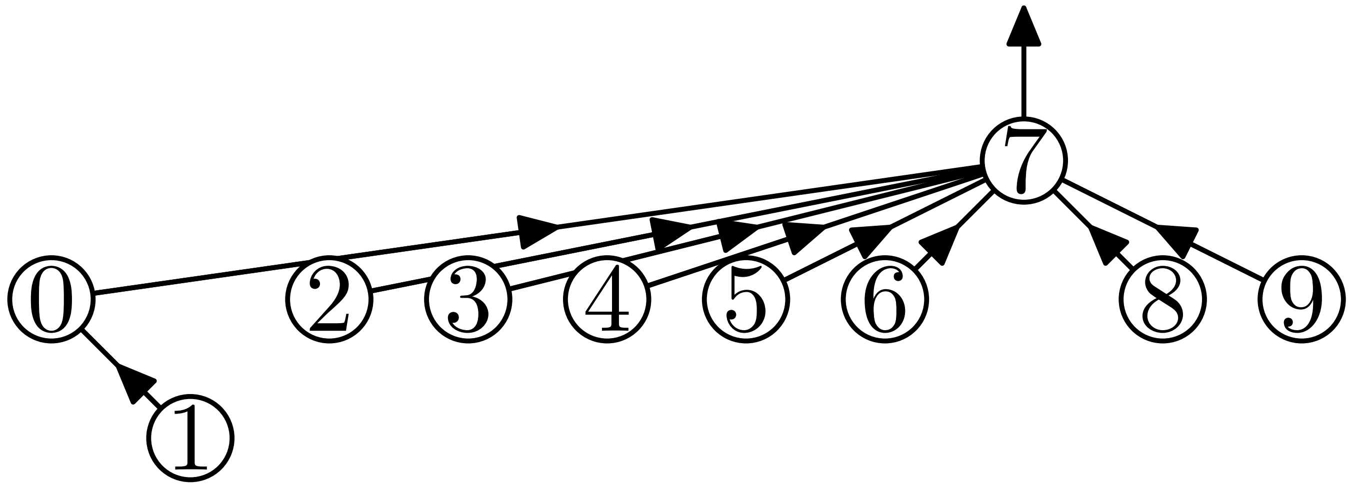
`find(6)`



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

Putting it Together

`find(6)=7`

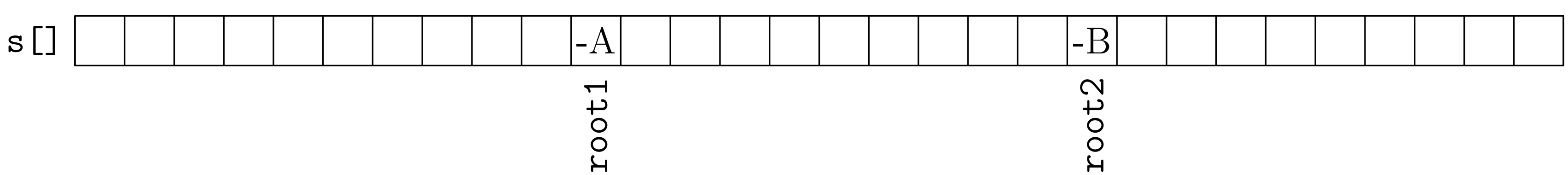


7	0	7	7	7	7	7	-3	7	7
0	1	2	3	4	5	6	7	8	9

Smart Union

```
public DisjointSets(int numElements)
{
    s = new int[numElements];
    for(int i=0; i<s.length; i++)
        s[i] = -1;                          // roots are negative number
}

public void union(int root1, int root2)
{
    if (s[root2]<s[root1]) {                // root2 is deeper
        s[root1] = root2;                   // make root2 the root
    } else {
        if (s[root1]==s[root2])
            s[root1]--;                     // update height if same
        s[root2] = root1;                   // make root1 new root
    }
}
```



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            s[root1]--;                 // update height if same
        s[root2] = root1;               // make root1 new root
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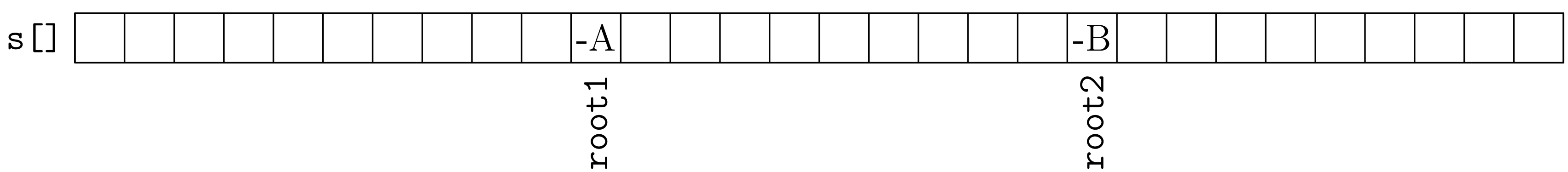
```



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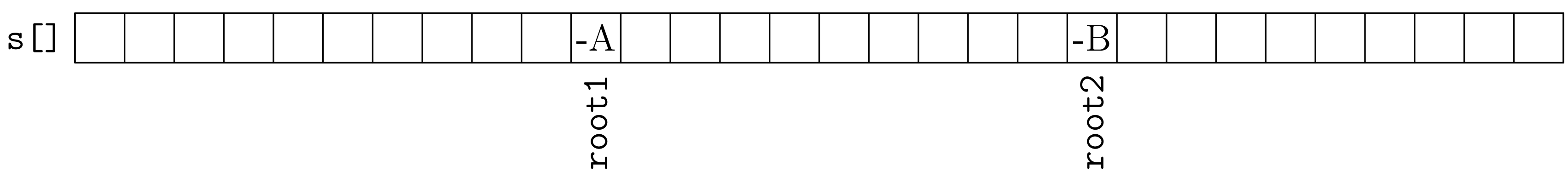

Path Compression

- To speed up `find` we relabel all nodes we visit during `find` by the root label

```
public int find(int index)
{
    if (s[index]<0)
        return index;
    else
        return s[index] = find(s[index]);
}
```

s[]						10					20										-3									
						5					10										20									

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Mazes

- Union-Find is a data structure which can occur in very different applications
- One application is building a maze
- Start from a complete lattice
- Remove a randomly chosen edge if it connects two unconnected regions
- Stop when the start and end cell are connected
- Or better after all cells are connected

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29
30	31	32	33	34
35	36	37	38	39
40	41	42	43	44
45	46	47	48	49

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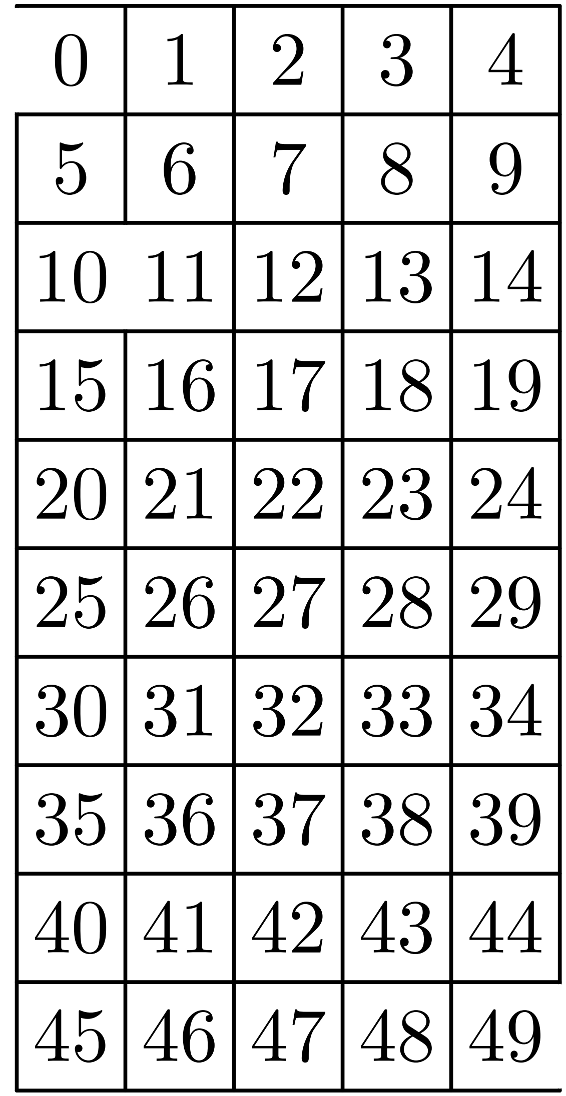

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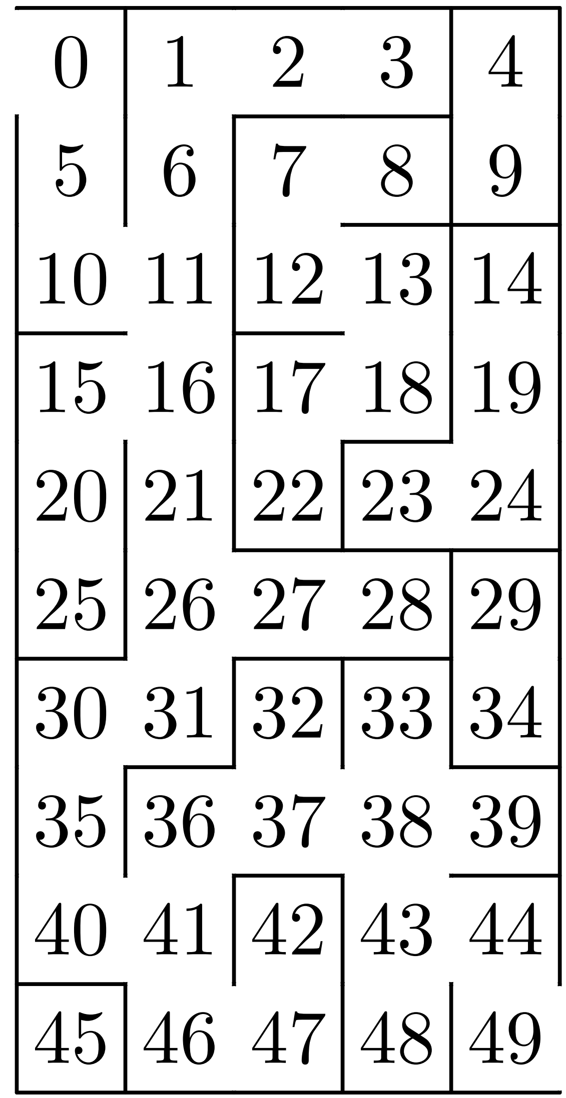

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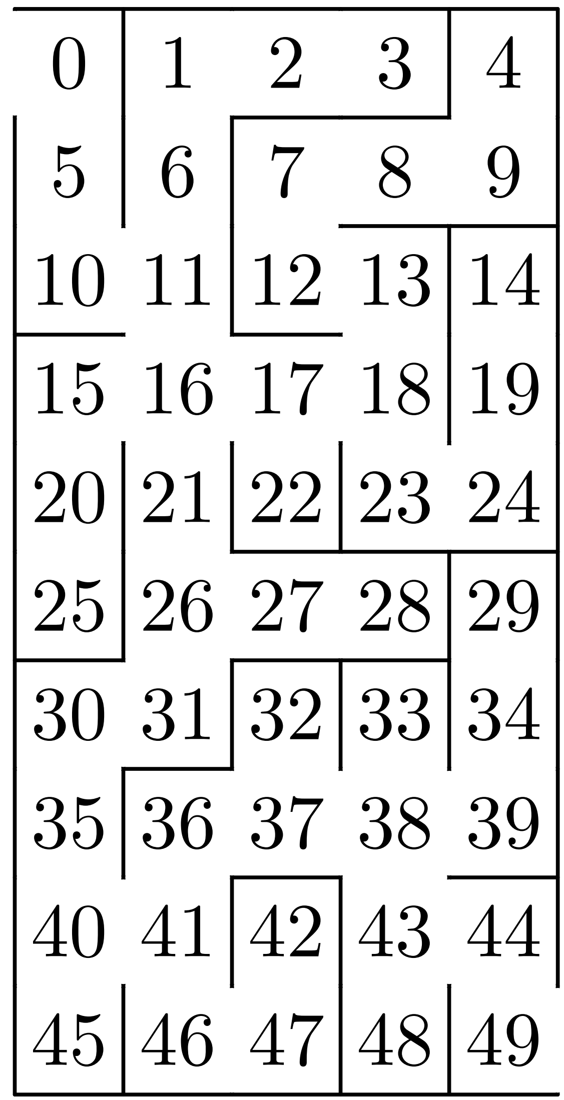

Time Complexity of Union-Find

- If we perform M finds and N unions then the time complexity is $O\big(M \log_2^*(N)\big)$
- Where $\log_2^*(N)$ is the number of times you need to apply the logarithm function before you get a number less than 1
- In practice $\log_2^*(N) \leq 5$ for all conceivable N

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$$\log_2(10^{80}) = 265.75$$

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$$\log_2(\log_2(10^{80})) = 8.0539$$

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$$\log_2(\log_2(\log_2(10^{80}))) = 3.0097$$

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$$\log_2(\log_2(\log_2(\log_2(10^{80})))) = 1.5896$$

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$$\log_2(\log_2(\log_2(\log_2(\log_2(10^{80}))))) = 0.66868$$

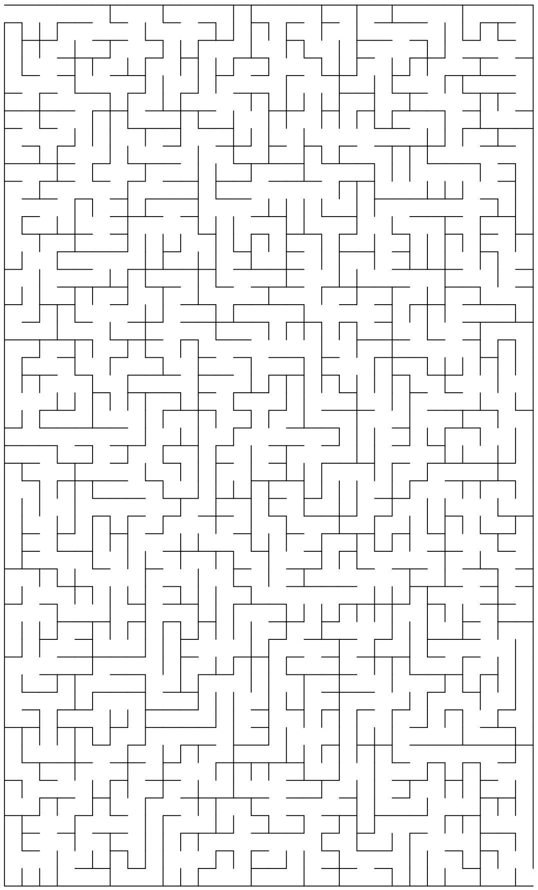
- The proof of this time complexity is rather involved

Outline

1. Equivalent Classes

2. Disjoint Sets

3. **Fast Sets**



Comparison of Sets

- Binary Search Trees: $O(\log_2(n))$, general purpose
- Hash tables: $O(1)$, but need to compute hash, slow iterator when sparse, general purpose
- B-trees: $O((k-1)\log_k(n))$ very complicated, used for large amounts of data
- Tries: $O(\log_k(n))$ for large k expensive in memory, complicated to code efficiently

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What Set to Use?

- A PhD student and I were working on writing a fast solver for a combinatorial optimisation problem
- We had to choose one variable to change out of a small number of possible variables
- Each time we changed a variable then we had to update the list of possible variables (remove some variables add others)
- We wanted a data structure which had quick add and remove and where we could choose a variable at random

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- We wanted a data structure which had quick add and remove and where we could choose a variable at random—what should we use?

Bounded Set

- One special feature is that we knew we only wanted the set to contain integers between 0 and n (where n might be $100\,000$)
- This allowed us to use an array to represent whether an integer belong to that set
- But how do we find a random element of the set quickly?

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Bounded Set

- One special feature is that we knew we only wanted the set to contain integers between 0 and n (where n might be 100 000)
- This allowed us to use an array to represent whether an integer belong to that set
- But how do we find a random element of the set quickly?
- Use another array of course!

FastSet

FastSet

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

FastSet

`add(4)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	-1

4									

FastSet

`add(9)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	-1

4									

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	1
4	9								

FastSet

`add(7)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	-1	-1	1

4	9								

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1
4	9	7							

FastSet

`add(4)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1

4	9	7							

FastSet

`false`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1

4	9	7							

FastSet

`add(1)`

0	1	2	3	4	5	6	7	8	9
-1	-1	-1	-1	0	-1	-1	2	-1	1

4	9	7							

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1
4	9	7	1						

FastSet

`contains(9)`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1
4	9	7	1						

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

`contains(5)`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

`false`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

`remove(9)`

0	1	2	3	4	5	6	7	8	9
-1	3	-1	-1	0	-1	-1	2	-1	1

4	9	7	1						

FastSet

`true`

0	1	2	3	4	5	6	7	8	9
-1	1	-1	-1	0	-1	-1	2	-1	-1
4	1	7							

Implementation

```
public class FastSet extends AbstractSet<Integer> {
    private int[] indexArray;
    private int[] memberArray;
    private int noMembers;

    public FastSet(int n) {
        indexArray = new int[n];
        memberArray = new int[n];
        for(int i=0; i<n; i++) {
            indexArray [i] = -1;
        }
        noMembers = 0;
    }

    public int size() {
       return noMembers;
    }
```

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```

Add and Remove

```
public boolean add(int i) {
   if (indexArray[i]>-1)
      return false;
   memberArray[noMembers] = i;
   indexArray[i] = noMembers;
   ++noMembers;
   return true;
}

public boolean remove(int i) {
   if (indexArray[i]==-1)
         return false;
   --noMembers;
   memberArray[indexArray[i]] = memberArray[noMembers];
   indexArray[memberArray[noMembers]] = indexArray[i];
   indexArray[i] = -1;
   return true;
}
```

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   indexArray[memberArray[noMembers]] = indexArray[i];
   indexArray[i] = -1;
   return true;
}
```

Collection Methods

```
public void clear() {
    for(int i=0; i<noMembers; i++) {
        indexArray[memberArray[i]] = -1;
    }
    noMembers = 0;
}

public boolean isEmpty() {
    return noMembers==0;
}

public Iterator<Integer> iterator() {
    return new FastSetIterator();
}
```

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Iterator

```java
private class FastSetIterator implements Iterator<Integer> {
   int current = 0;

   public boolean hasNext() {
      return current < noMembers;
   }

   public Integer next() throws NoSuchElementException {
      if (current>=noMembers) throw new NoSuchElementException();
      current++;
      return memberArray[current-1];
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   public void remove() throws IllegalStateException {
      if (current==0) throw new IllegalStateException();
      indexArray[memberArray[current-1]] = -1;
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And Random?

- So far we have just implemented a new `Set<Integer>` as part of the java Collection class
- We can add additional methods taking advantage of the classes strength

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private static Random rand = new Random();

public int getRandomElement() {
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- Need to use FastSet signature to use this

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- For large problems we were over 10 times faster because of this data structure
- The competitor algorithm used a complex tree structure instead of the simple array
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Lessons

- If you have a bounded set then using an array is usually going to be very fast $O(1)$ (or $O(\log^*(n))$)
- These data structures are not general purpose for solving every day problems (c.f. `List<T>`, `Set<T>` and `Map<T>`)
- They are "back pocket" data structures that solve problems that come up often enough that they are worth knowing about
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