

Further Mathematics and Algorithms

Lesson 11: *Keep Trees Balanced*



AVL trees, red-black trees, TreeSet, TreeMap

Outline

1. **Deletion**
2. Balancing Trees
 - Rotations
3. AVL
4. Red-Black Trees
 - TreeSet
 - TreeMap

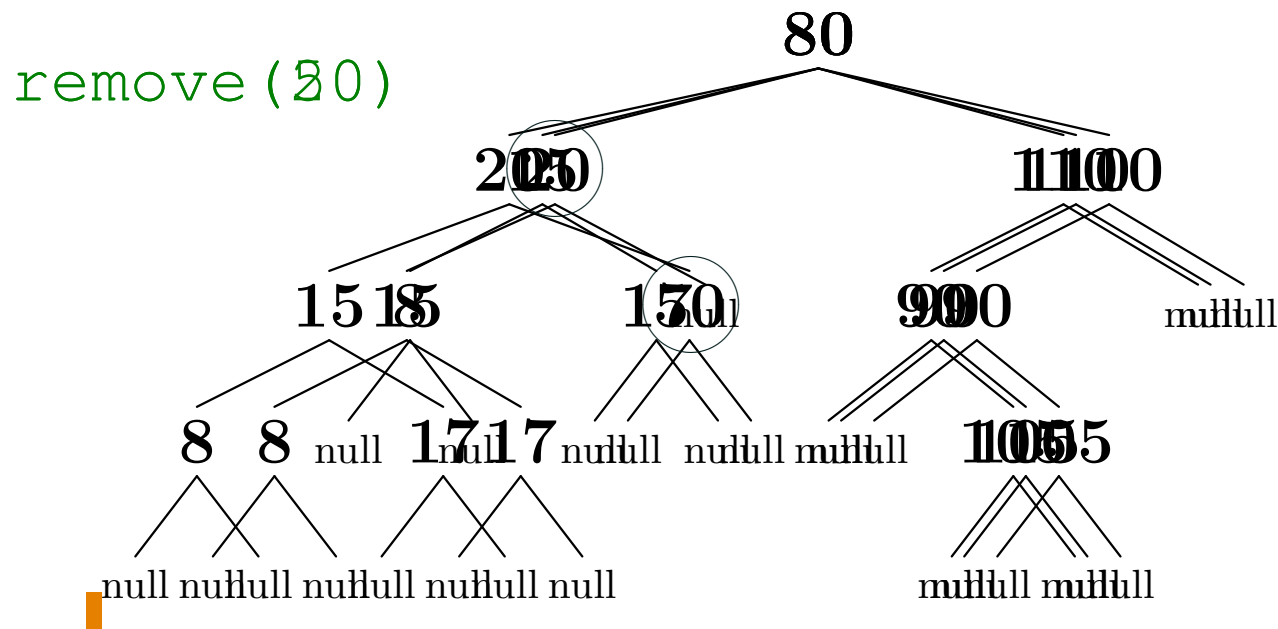


Recap

- Binary search trees are commonly used to store data because we need to only look down one branch to find any element■
- We saw how to implement many methods of the binary search tree
 - ★ `find`
 - ★ `insert`
 - ★ `successor` (in outline)■
- One method we missed was `remove`■

Deletion

- Suppose we want to delete some elements from a tree
- It is relatively easy if the element is a leaf node (e.g. 50)
- It is not so hard if the node has one child (e.g. 20)

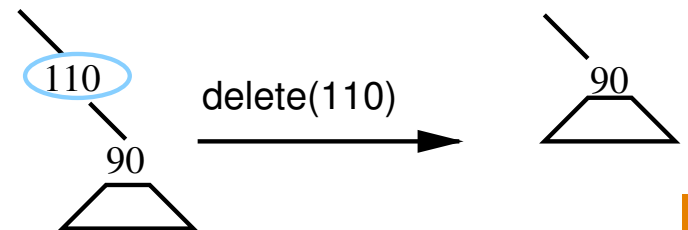
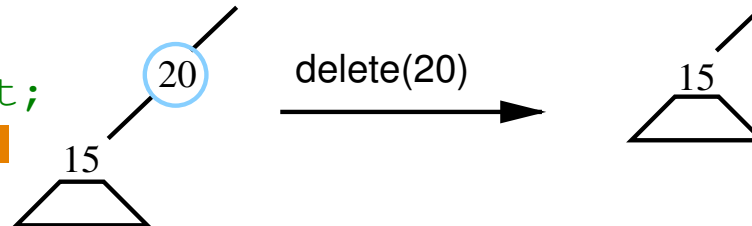
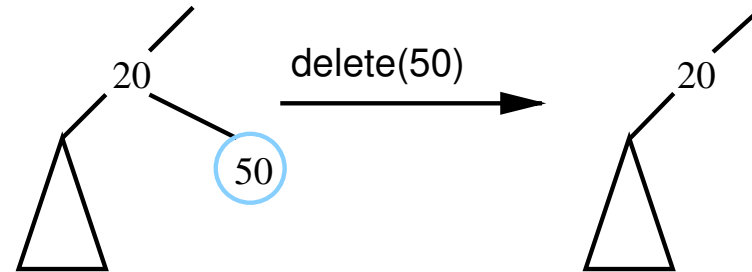


Code to remove Node n

```

if (n->left==0 && n->right==0) {
    if (n == n->parent->left)
        n->parent->left = 0;
    else
        n->parent->right = 0;
} else if (n->right==0) {
    if (n == n->parent->left)
        n->parent->left = n->left;
    else
        n->parent->right = n->left;
    n->left->parent = n->parent;
} else if (n->left==0) {
    if (n == n->parent->left)
        n->parent->left = n->right;
    else
        n->parent->right = n->right;
    n->right->parent = n->parent;
}
delete n;

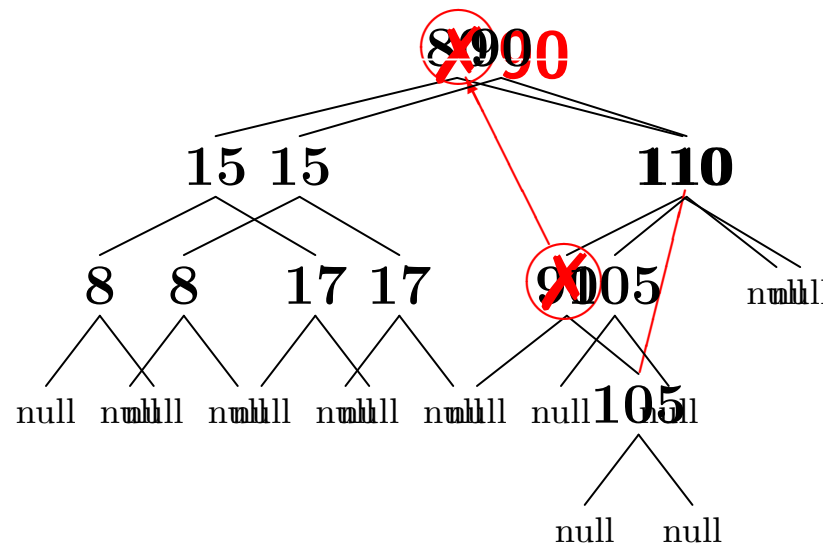
```



Removing Element with Two Children

- If an element has two children then
 - ★ replace that element by its successor
 - ★ and then remove the successor using the above procedure

remove (80)



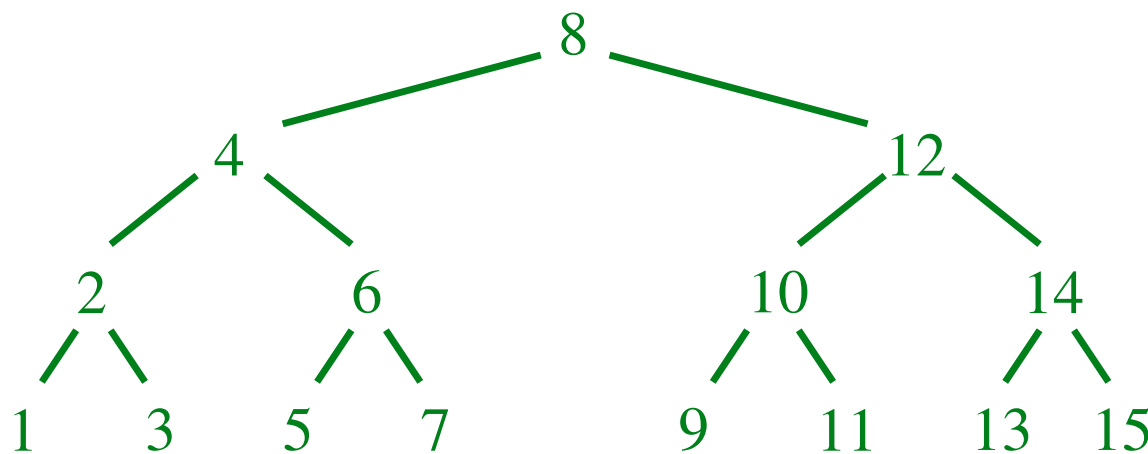
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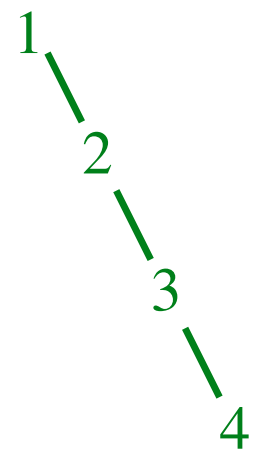


Why Balance Trees

- The number of comparisons to access an element depends on the depth of the node
- The average depth of the node depends on the shape of the tree



full tree



sparse tree

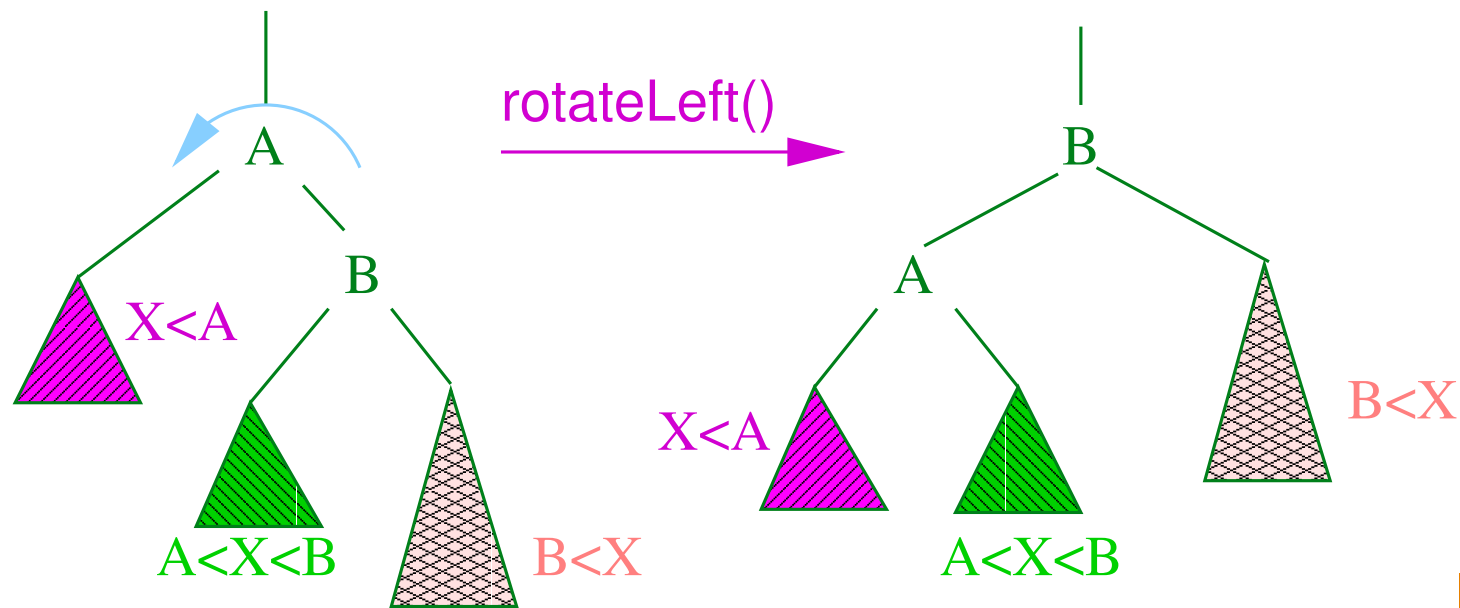
- The shape of the tree depends on the order the elements are added

Time Complexity

- In the best situation (a full tree) the number of elements in a tree is $n = \Theta(2^l)$ the depth is l so that the maximum depth is $\log_2(n)$ ■
- It turns out for random sequences the average depth is $\Theta(\log(n))$ ■
- In the worst case (when the tree is effectively a linked list), the average depth is $\Theta(n)$ ■
- Unfortunately, the worst case happens when the elements are added *in order* (not a rare event)■

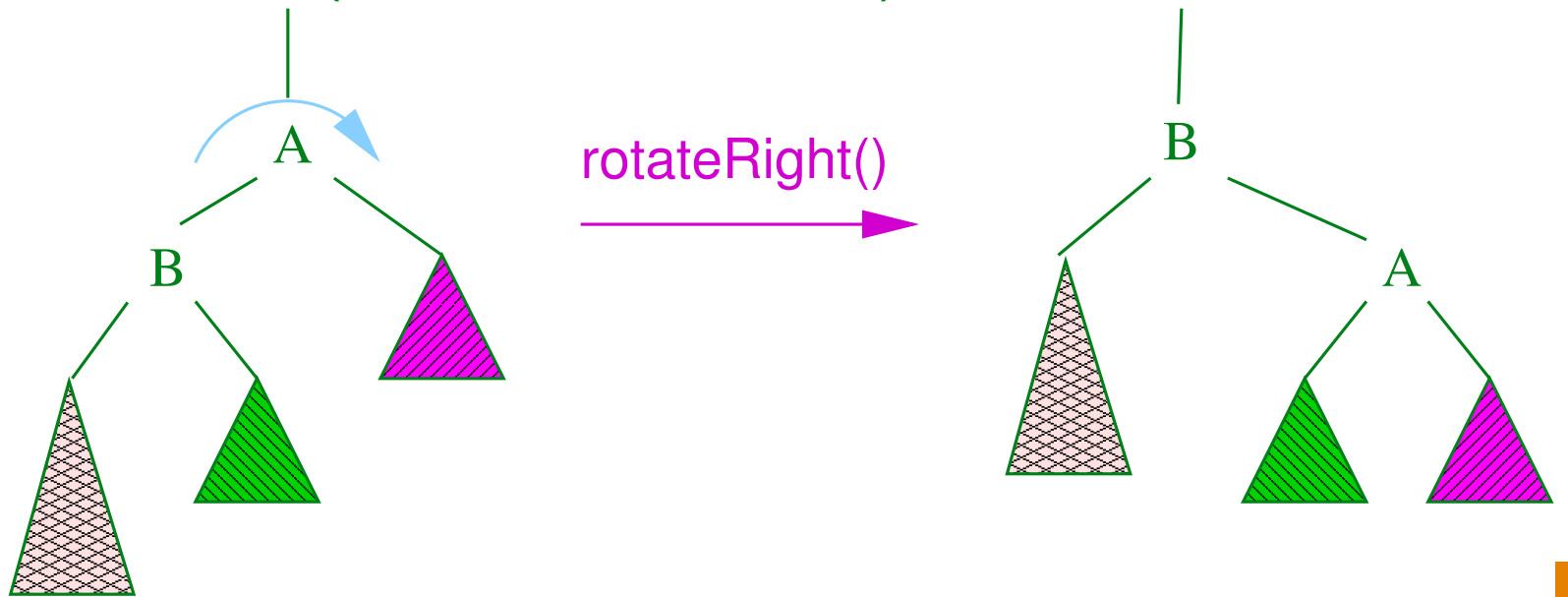
Rotations

- To avoid unbalanced trees we would like to modify the shape
- This is possible as the shape of the tree is not uniquely defined (e.g. we could make any node the root)
- We can change the shape of a tree using **rotations**
- E.g. left rotation



Types of Rotations

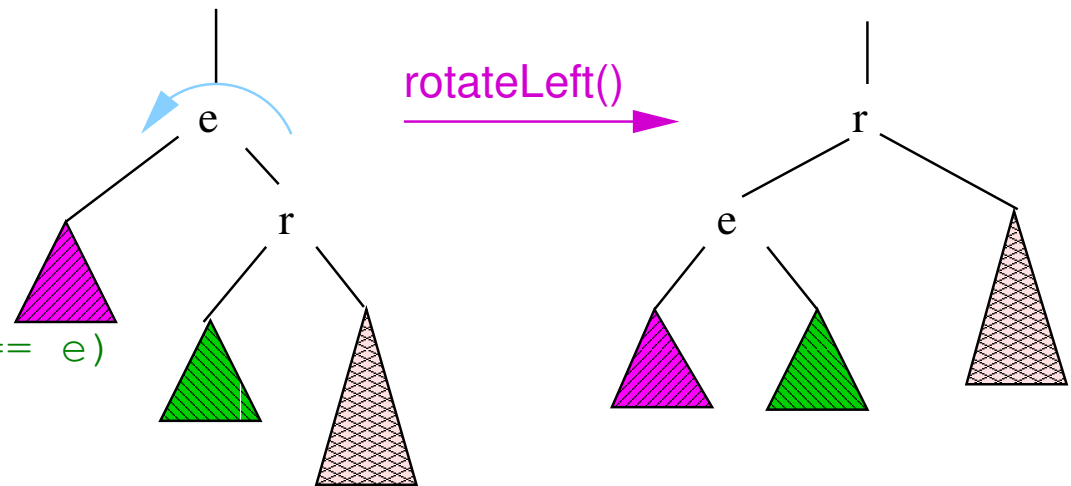
- We can get by with 4 types of rotations
- ★ Left rotation (as above)
- ★ Right rotation (symmetric to above)



- ★ Left-right double rotation
- ★ Right-left double rotation

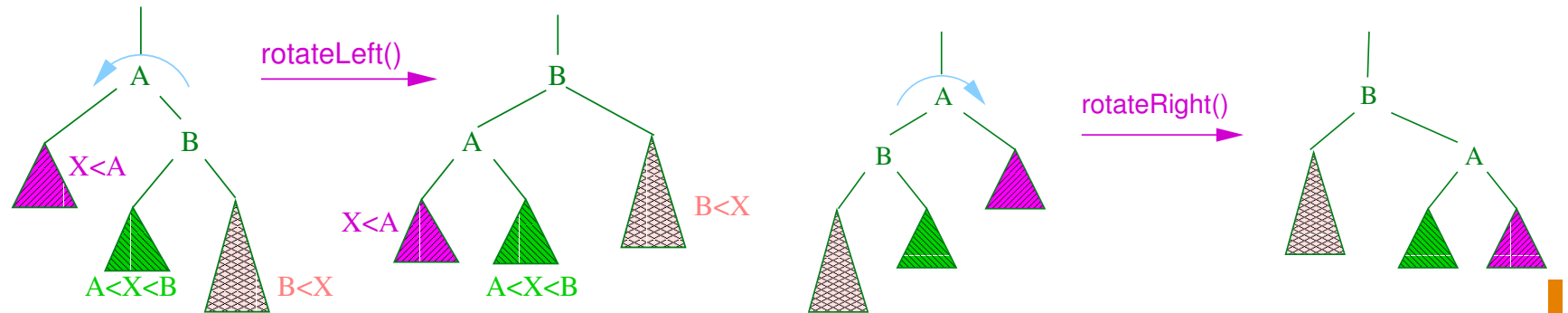
Coding Rotations

```
void rotateLeft(Node<T>* e)
{
    Node<T>* r = e->right;
    e->right = r->left;
    if (r->left != 0)
        r->left->parent = e;
    r->parent = e->parent;
    if (e->parent == 0)
        root = r;
    else if (e->parent->left == e)
        e->parent->left = r;
    else
        e->parent->right = r;
    r->left = e;
    e->parent = r;
}
```



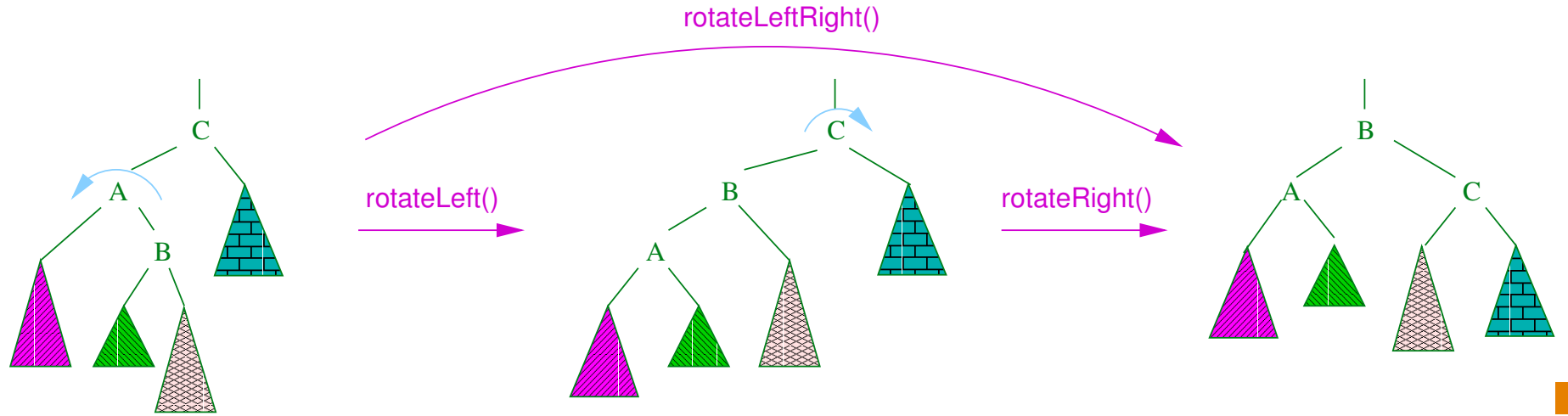
When Single Rotations Work

- Single rotations balance the tree when the unbalanced subtree is on the outside



Double Rotations

- If the unbalanced subtree is on the inside we need a double rotation



```
leftRotation(C.left);  
rightRotation(C);
```

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Balancing Trees

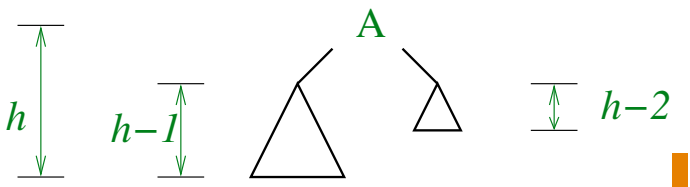
- There are different strategies for using rotations for balancing trees■
- The three most popular are
 - ★ AVL-trees
 - ★ Red-black trees
 - ★ Splay trees■
- They differ in the criteria they use for doing rotations■

AVL Trees

- AVL-trees were invented in 1962 by two Russian mathematicians Adelson-Velski and Landis■
- In AVL trees
 1. The heights of the left and right subtree differ by at most 1■
 2. The left and right subtrees are AVL trees■
- This guarantees that the worst case AVL tree has logarithmic depth■

Minimum Number of Nodes

- Let $m(h)$ be the minimum number of nodes in a tree of height h ■
- This has to be made up of two subtrees: one of height $h - 1$; and, in the worst case, one of height $h - 2$ ■
- Thus, the least number of nodes in a tree of height h is

$$m(h) = m(h - 1) + m(h - 2) + 1$$


- with $m(1) = 1$, $m(2) = 2$ ■

Proof of Exponential Number of Nodes

- We have $m(h) = m(h - 1) + m(h - 2) + 1$ with $m(1) = 1$, $m(2) = 2$ ■
- This gives us a sequence $1, 2, 4, 7, 12, \dots$ ■
- Compare this with Fibonacci $f(h) = f(h - 1) + f(h - 2)$, with $f(1) = f(2) = 1$ ■
- This gives us a sequence $1, 1, 2, 3, 5, 8, 13, \dots$ ■
- It looks like $m(h) = f(h + 2) - 1$ ■
- Proof by substitution ■

Proof of Logarithmic Depth

- $m(h) = m(h-1) + m(h-2) + 1$ with $m(1) = 1$, $m(2) = 2$ ■
- We can prove by inductions, $m(h) \geq (3/2)^{h-1}$ ■
- $m(1) = 1 \geq (3/2)^0 = 1$, $m(2) = 2 \geq (3/2)^1 = 3/2$ ■ ✓ ■

$$m(h) \geq \left(\frac{3}{2}\right)^{h-3} \left(\frac{3}{2} + 1 + \left(\frac{3}{2}\right)^{3-h}\right) \geq \left(\frac{3}{2}\right)^{h-3} \frac{5}{2} \geq \left(\frac{3}{2}\right)^{h-3} \frac{10}{4} \geq \left(\frac{3}{2}\right)^{h-3} \frac{9}{4} \geq \left(\frac{3}{2}\right)^{h-1} \quad \text{■} \checkmark \text{■}$$

- Taking logs: $\log(m(h)) \geq (h-1) \log(3/2)$ or

$$h \leq \frac{\log(m(h))}{\log(3/2)} + 1 = O(\log(m(h))) \quad \text{■}$$

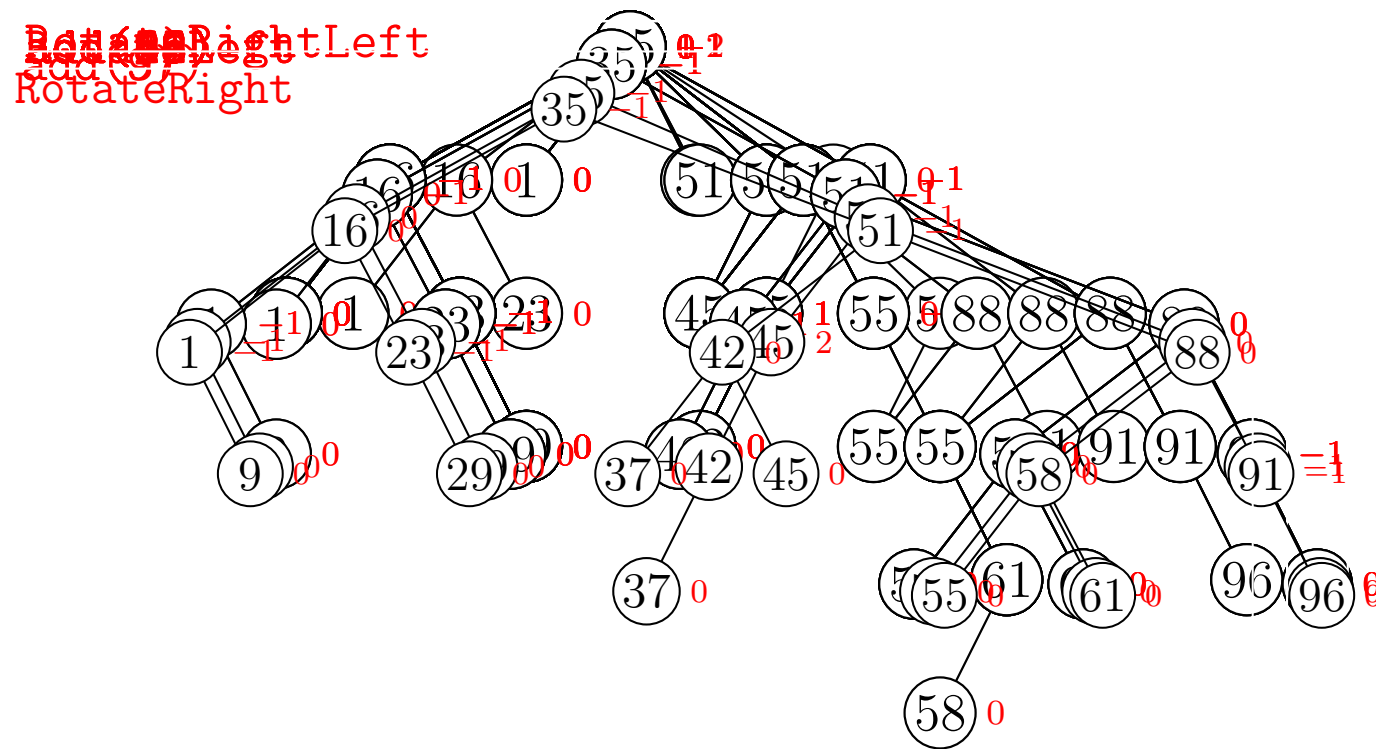
- The number of elements, n , we can store in an AVL tree is $n \geq m(h)$ thus

$$h \leq O(\log(n)) \quad \text{■}$$

Implementing AVL Trees

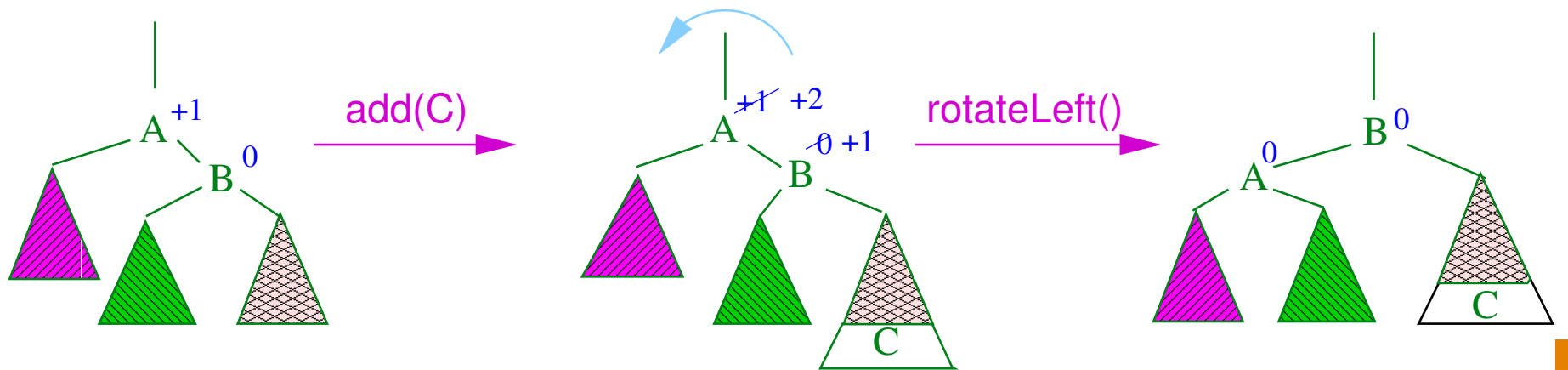
- In practice to implement an AVL tree we include additional information at each node indicating the balance of the subtrees

$$\text{balanceFactor} = \begin{cases} -1 & \text{right subtree deeper than left subtree} \\ 0 & \text{left and right subtrees equal} \\ +1 & \text{left subtree deeper than right subtree} \end{cases}$$



Balancing AVL Trees

- When adding an element to an AVL tree
 - ★ Find the location where it is to be inserted
 - ★ Iterate up through the parents re-adjusting the balanceFactor
 - ★ **If** the balance factor exceeds ± 1 then re-balance the tree and stop
 - ★ else if the balance factor goes to zero **then** stop



AVL Deletions

- AVL deletions are similar to AVL insertions
- One difference is that after performing a rotation the tree may still not satisfy the AVL criteria so higher levels need to be examined
- In the worst case $\Theta(\log(n))$ rotations may be necessary
- This may be relatively slow—but in many applications deletions are rare

AVL Tree Performance

- Insertion, deletion and search in AVL trees are, at worst, $\Theta(\log(n))$ ■
- The height of an average AVL tree is $1.44 \log_2(n)$ ■
- The height of an average binary search tree is $2.1 \log_2(n)$ ■
- Despite being more compact insertion is slightly slower in AVL trees than binary search trees without balancing (for random input sequences)■
- Search is, of course, quicker■

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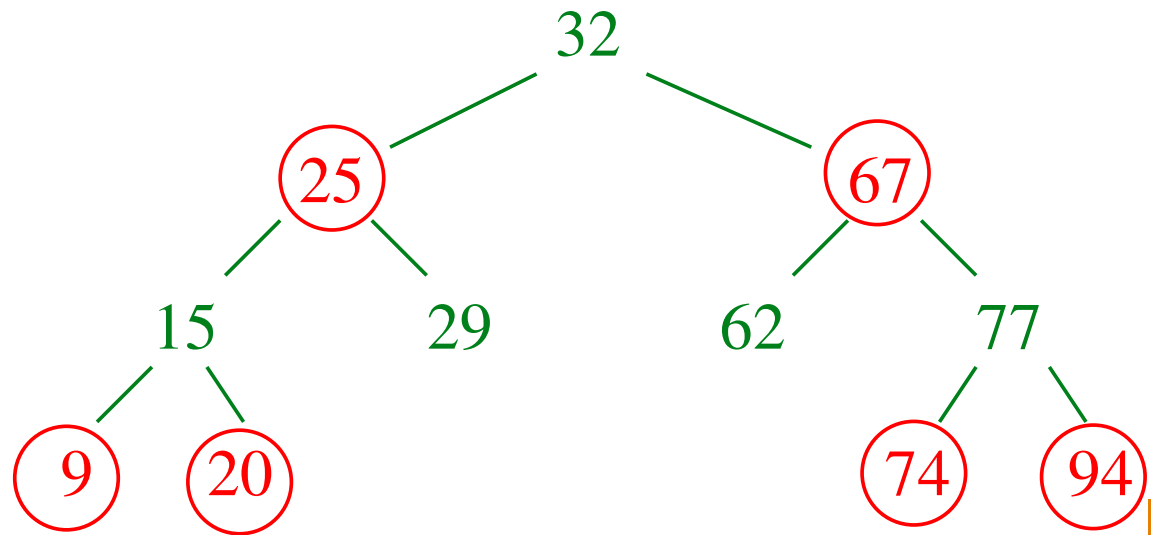


Red-Black Trees

- Red-black trees are another strategy for balancing trees
- Nodes are either *red* or *black*
- Two rules are imposed

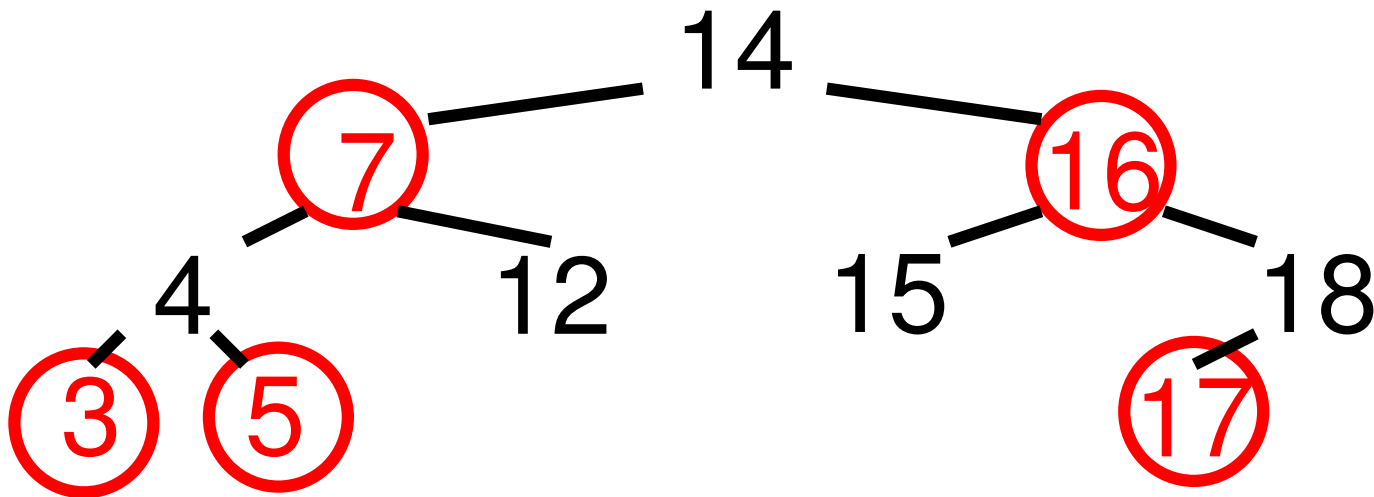
Red Rule: the children of a red node must be black

Black Rule: the number of black elements must be the same in all paths from the root to elements with no children or with one child



Restructuring

- When inserting a new element we first find its position■
- If it is the root we colour it black otherwise we colour it red■
- If its parent is red we must either relabel or restructure the tree■



Performance of Red-Black Trees

- Red-black trees are slightly more complicated to code than AVL trees■
- Red-black trees tend to be slightly less compact than AVL trees■
- However, insertion and deletion run slightly quicker■
- Both Java Collection classes and C++ STL use red-black trees■

Set

- The standard template library (STL) has a class `std::set<T>`■
- It also has a `std::unordered_set<T>` class (which uses a hash table covered later)■
- As well as `std::multiset<T>` that implements a multiset (i.e. a set, but with repetitions)■
- Using sets you can also implement **maps**■

Maps

- One major abstract data type (ADT) we have not encountered is the map class■
- The map class `std::map<Key, V>` contain key-value pairs `pair<Key, V>`■
 - ★ The first element of type `Key` is the **key**■
 - ★ The second element of type `V` is the **value**■
- Maps work as content addressable arrays■

```
map<string, int> students;  
student["John_Smith"] = 89;  
student["Terry_Jones"] = 98;  
cout << students["John_Smith"];
```

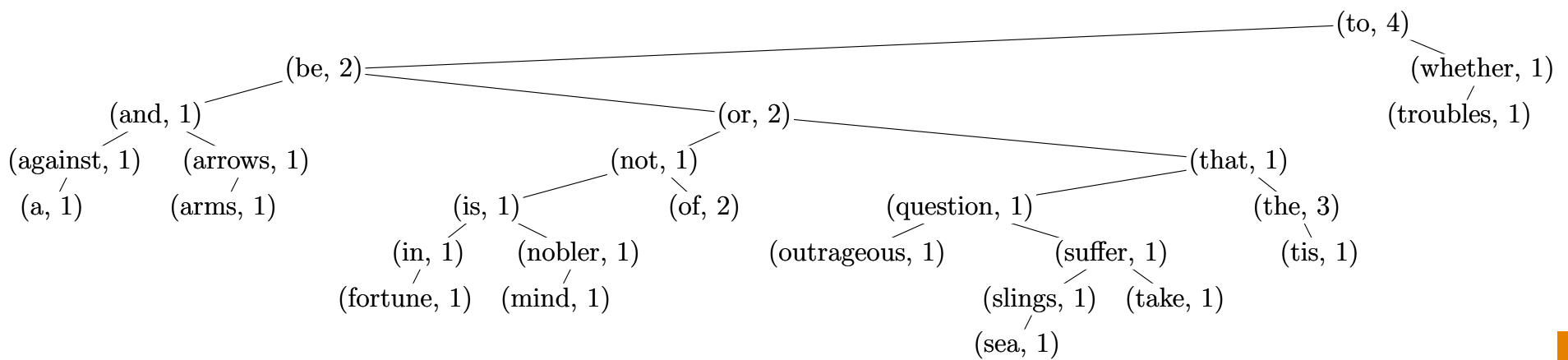


Implementing a Map

- Maps can be implemented using a set by making each node hold a `pair<K, V>` objects

```
class pair<K, V>
{
    public:
    K first;
    V second;
}
```

- We can count words using the key for words and value to count



Lessons

- Binary search trees are very efficient (order $\log(n)$ insertion, deletion and search) provided they are balanced■
- Balanced trees are achieved by performing rotations■
- There are different strategies for deciding when to rotate including
 - ★ AVL trees
 - ★ Red-black trees■
- Binary trees are used for implementing **sets** and **maps**■