Algorithms and Analysis

Lesson 12: *Use Heaps!*

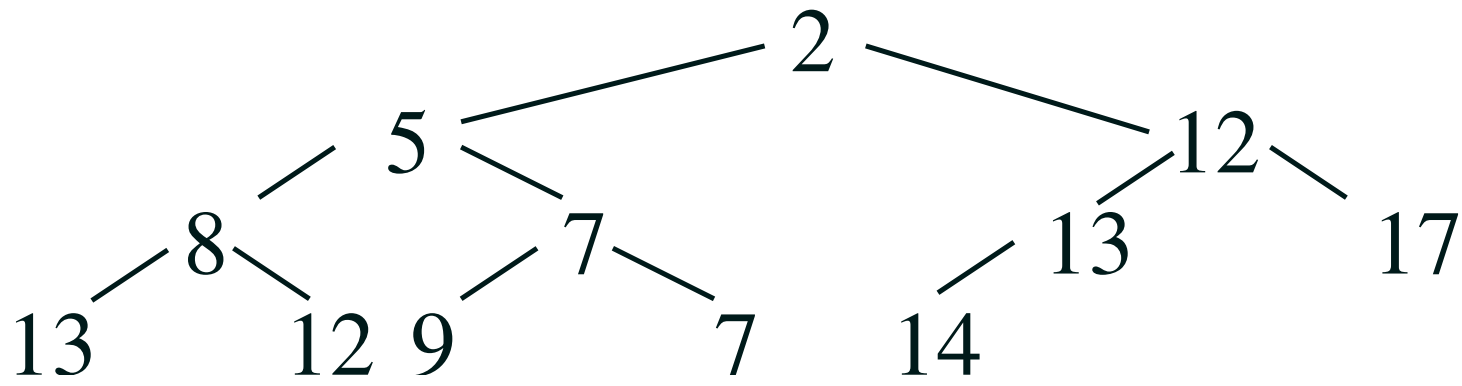
Heaps, Priority queues, Heap Sort, Other heaps

Outline

1. **Heaps**
2. Priority Queues
   - Array Implementation
3. Heap Sort
4. Other Heaps

Heaps

- A (min-)heap is (from one perspective) a binary tree
- It is a binary tree satisfying two constraints
  - ⋆ It is a **complete** tree
  - ⋆ Each child has a value 'greater than or equal to' its parent



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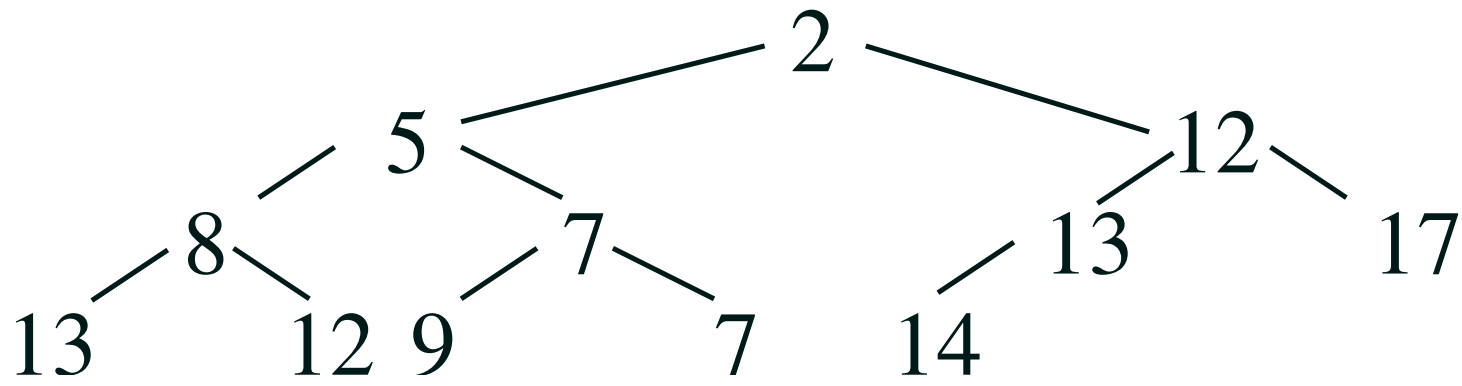

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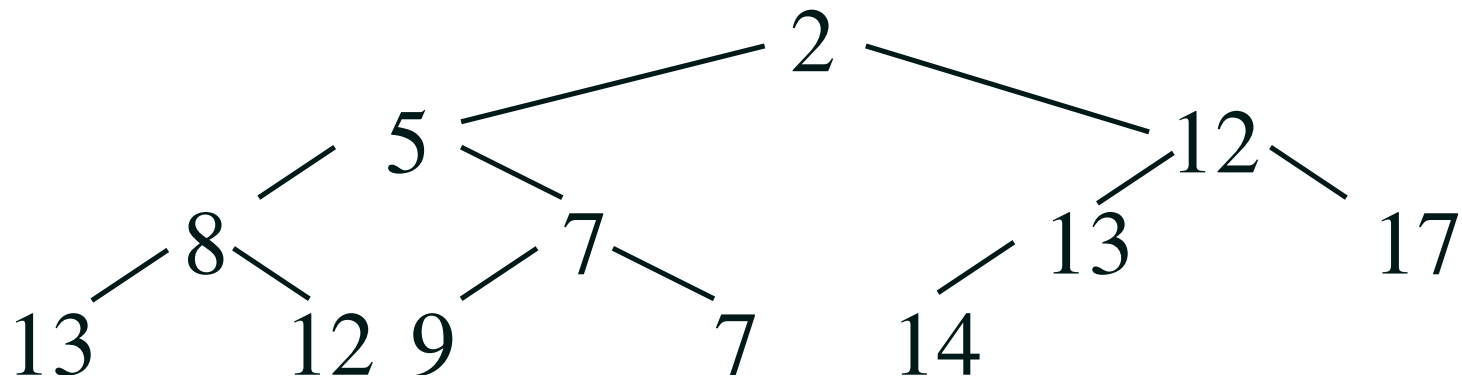

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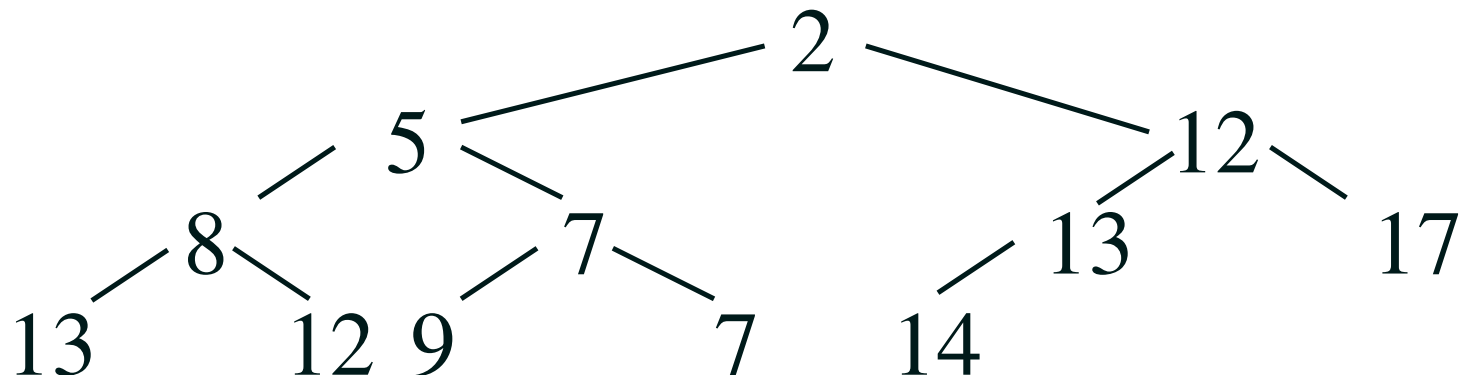

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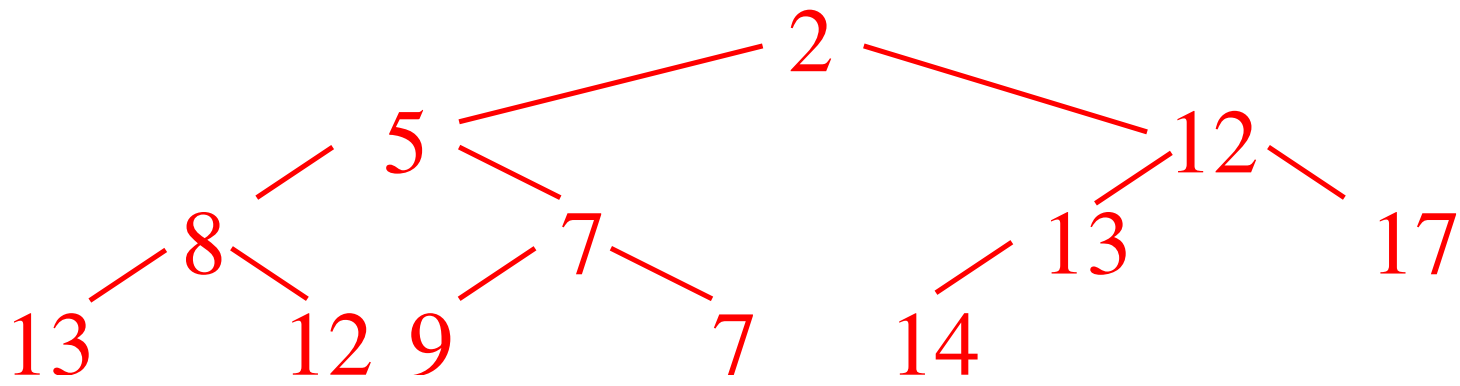

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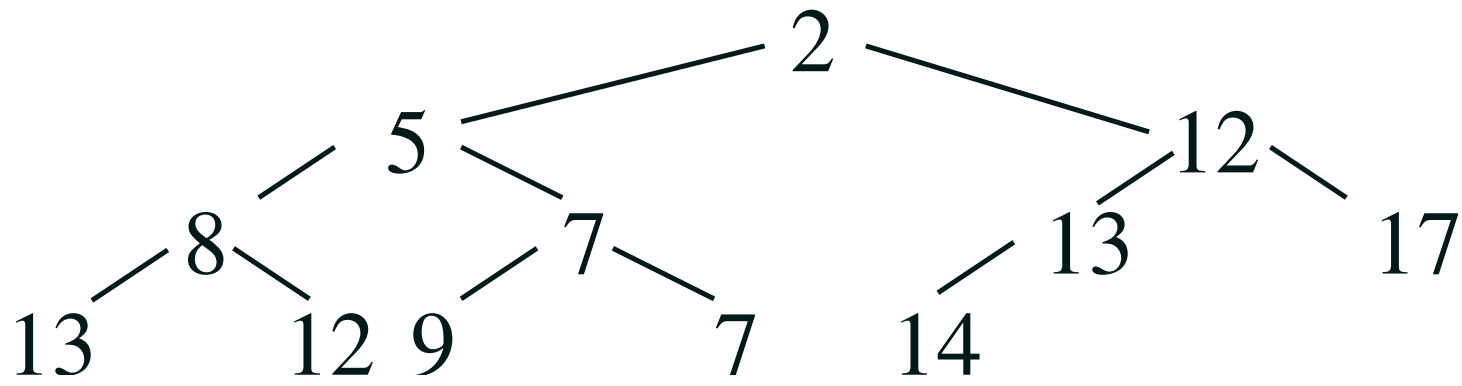

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Priority Queues

- One of the prime uses of heaps is to implement a priority queue
- A Priority Queue is a queue with priorities
- That is, we assign a priority to each element we add
- The head of the queue is the element with highest priority (smallest number)
- Used, for example, in simulating real time events
- Used to implement "greedy algorithms"

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Priority Queue

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  - ⋆ `unsigned size()` returning the the number of elements
  - ⋆ `bool empty()` returns true if empty
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  - ⋆ `T top()` returns head of queue
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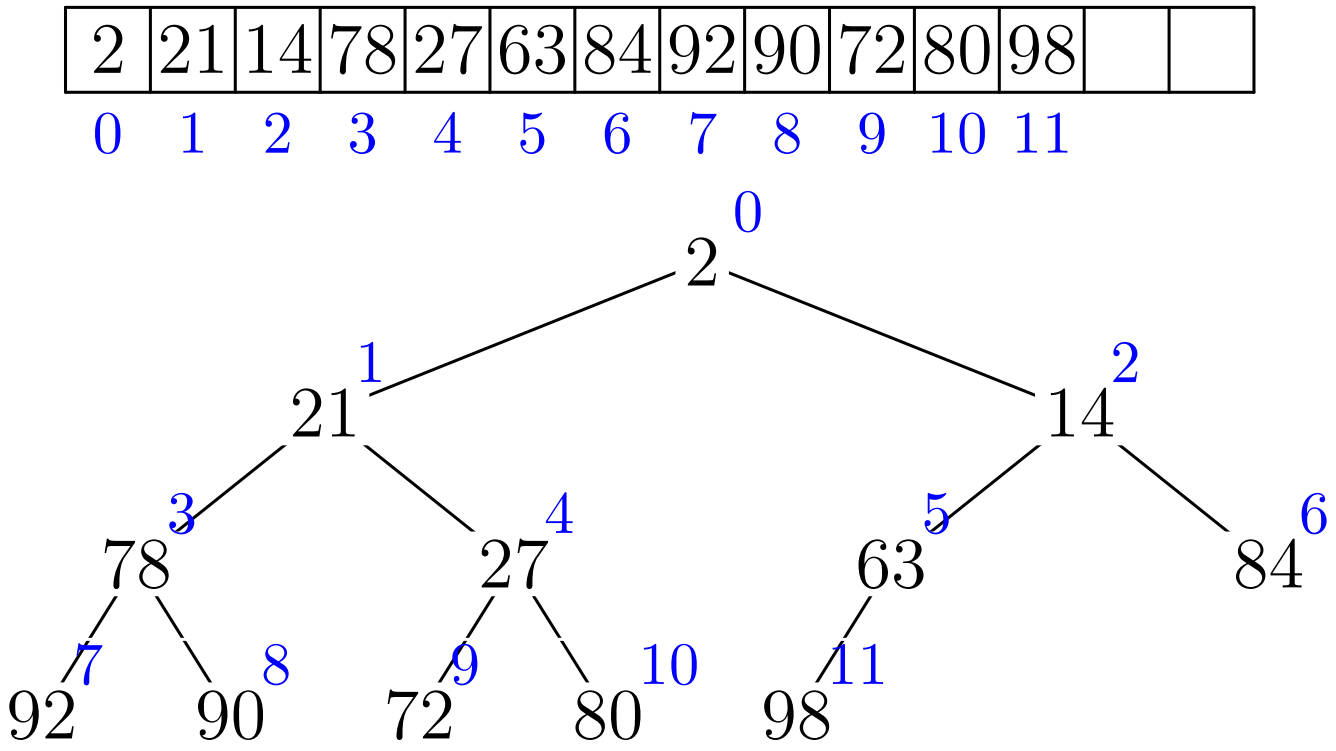

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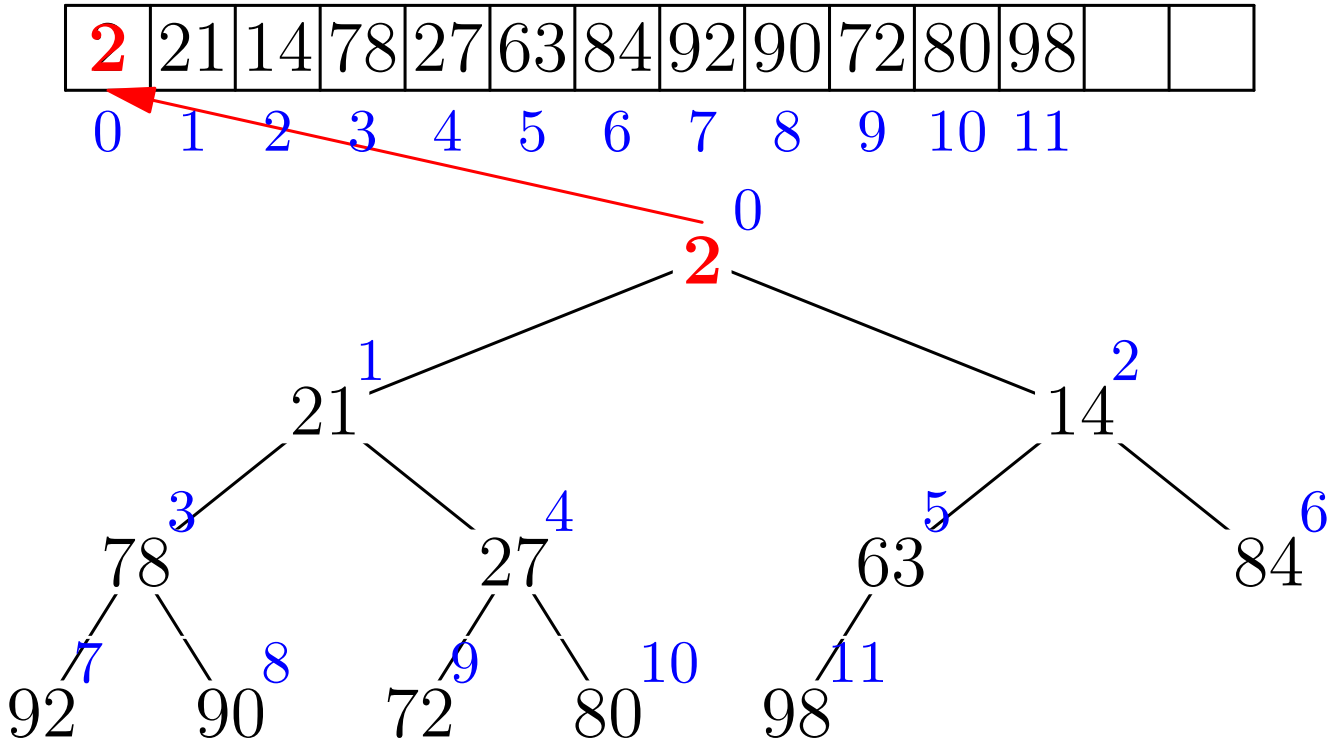

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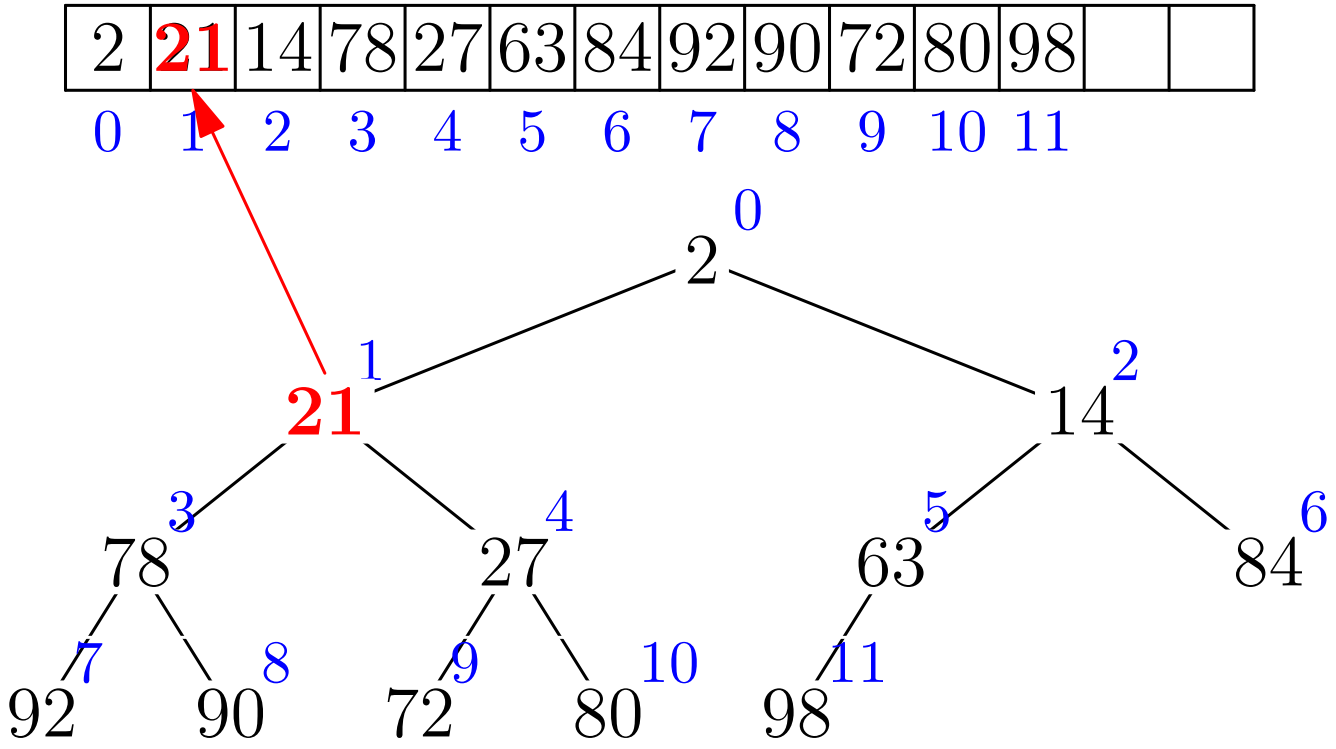

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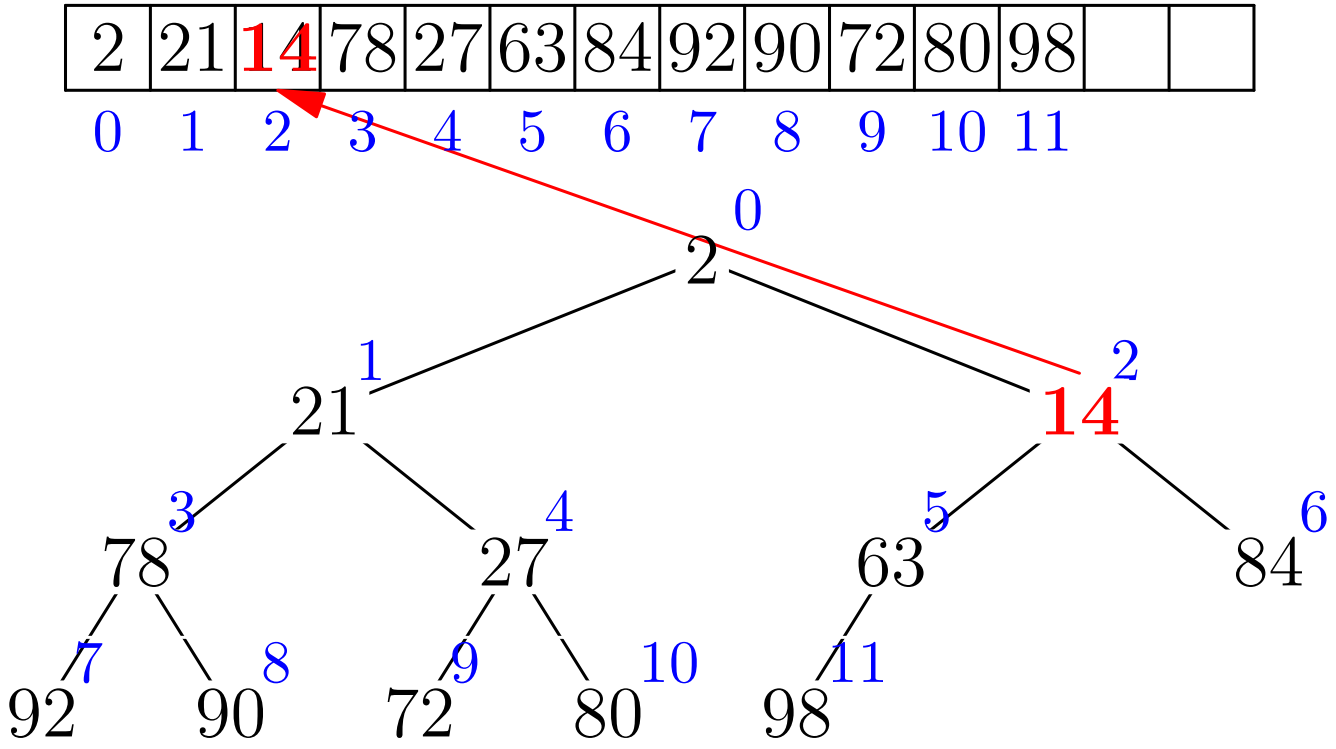

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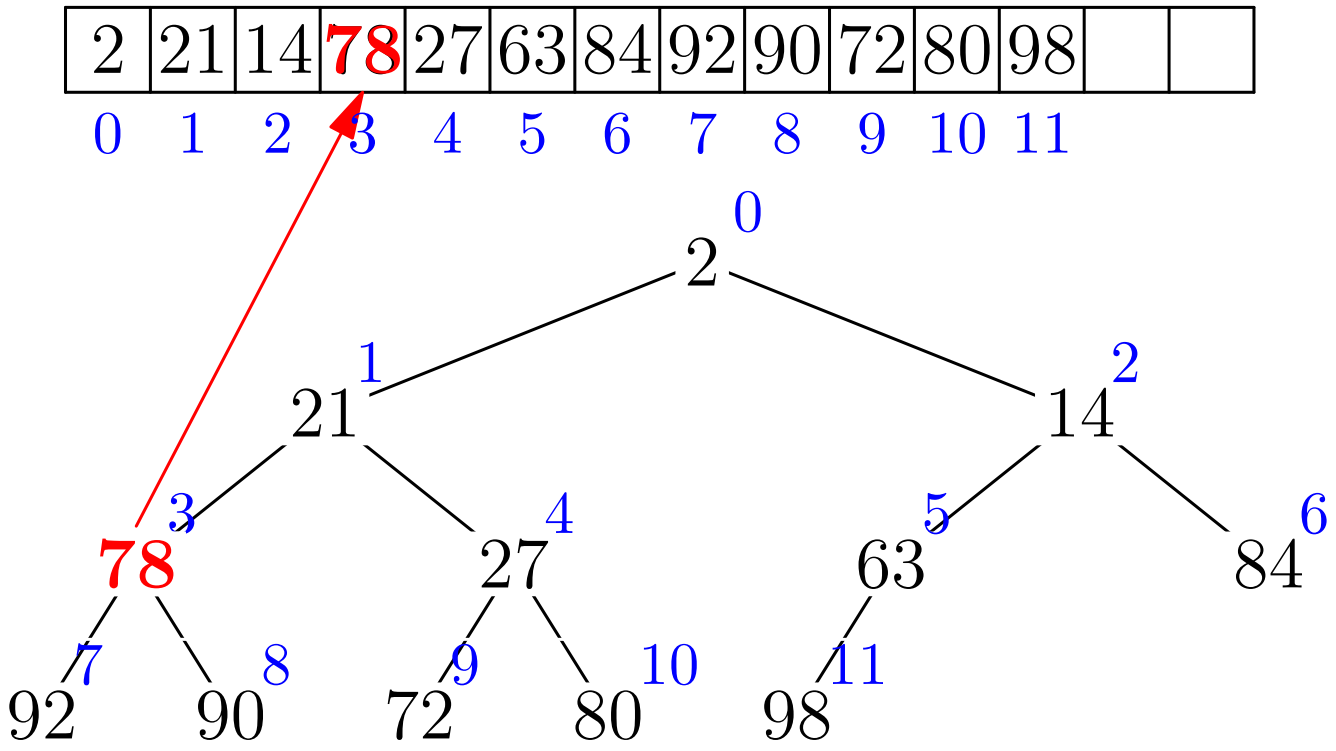

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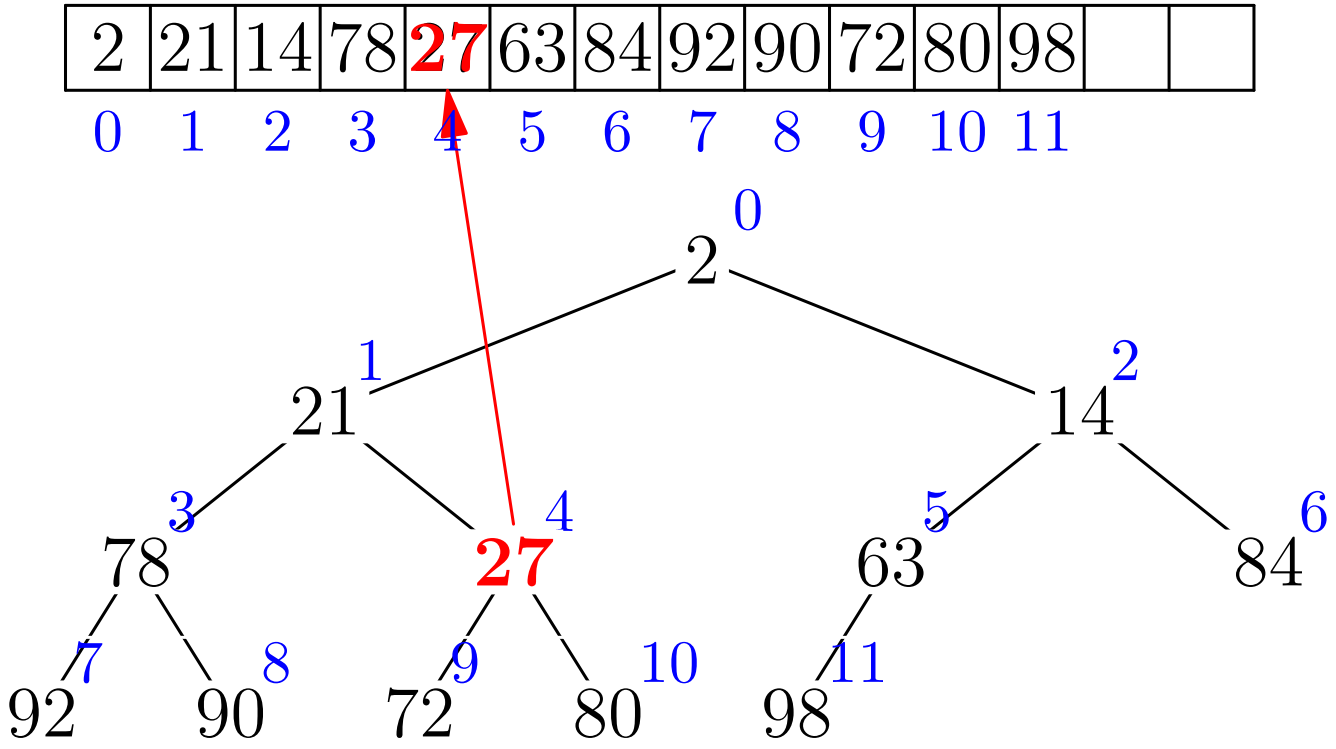

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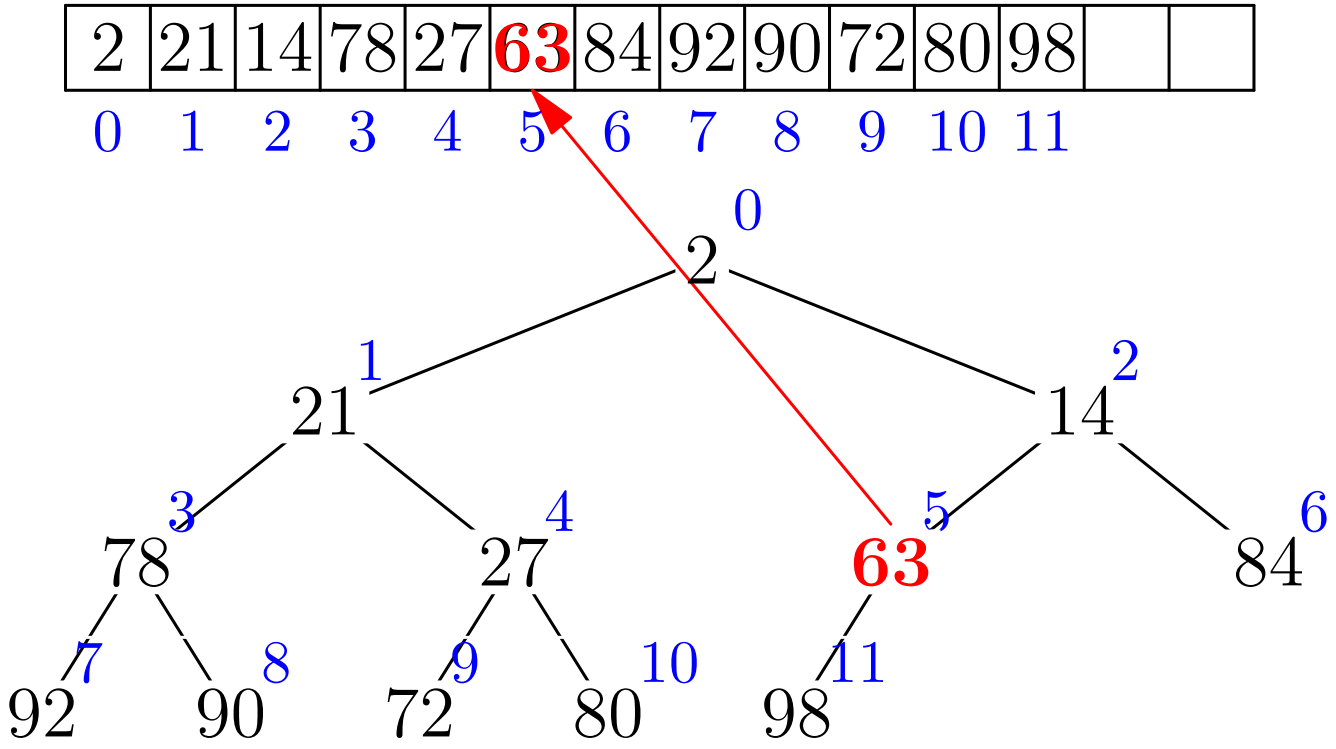

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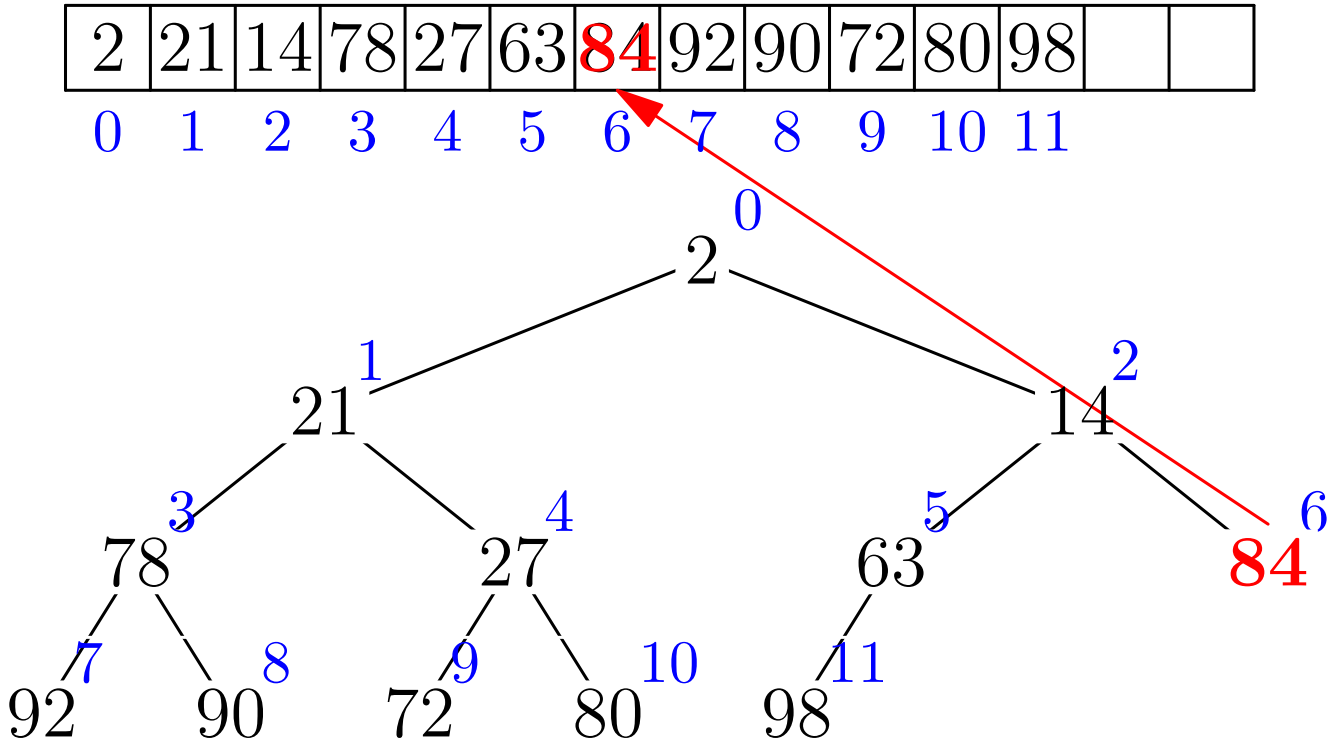

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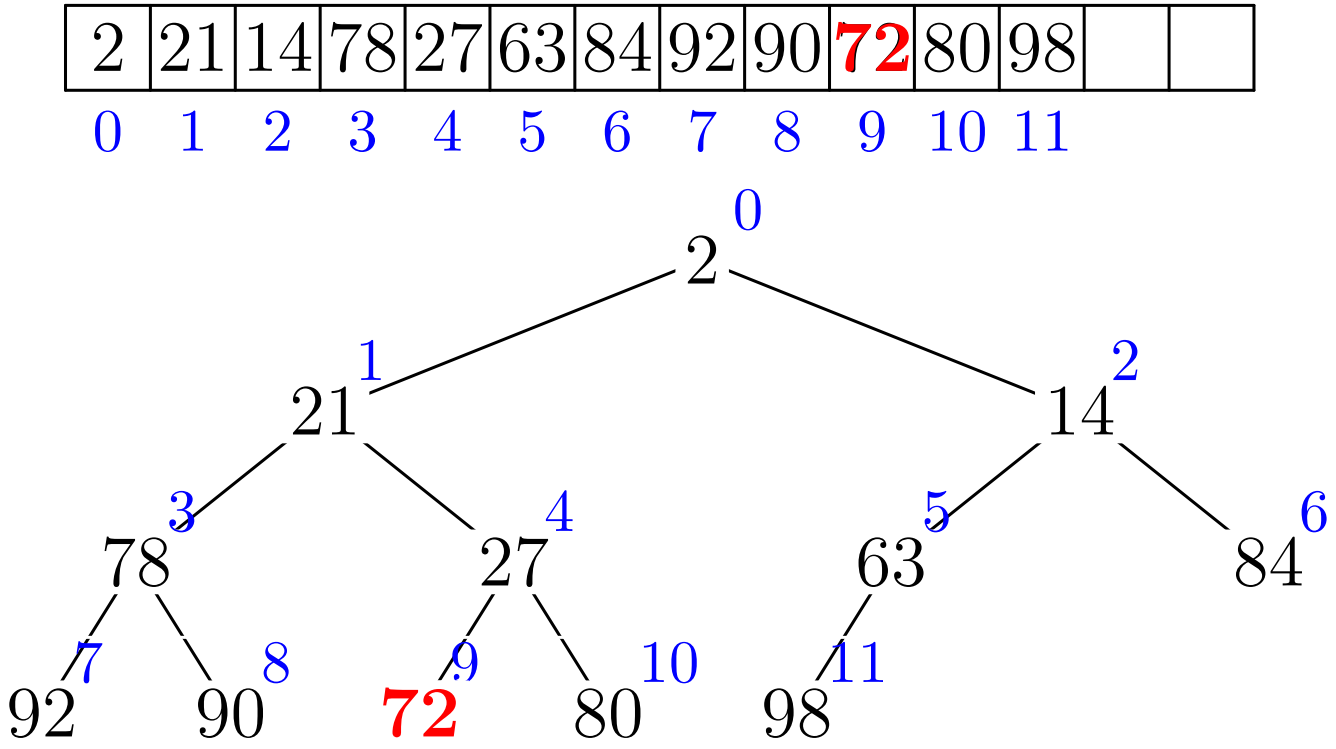

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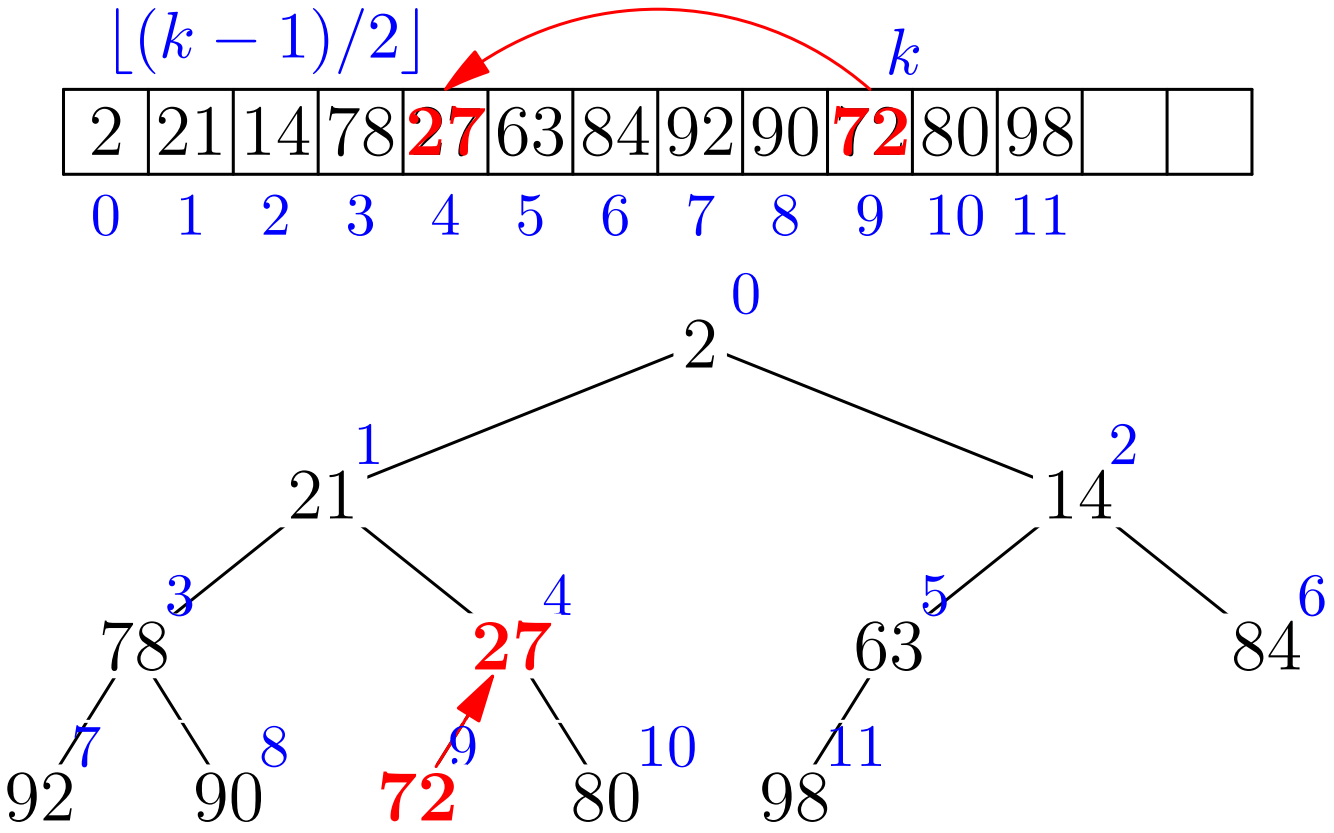

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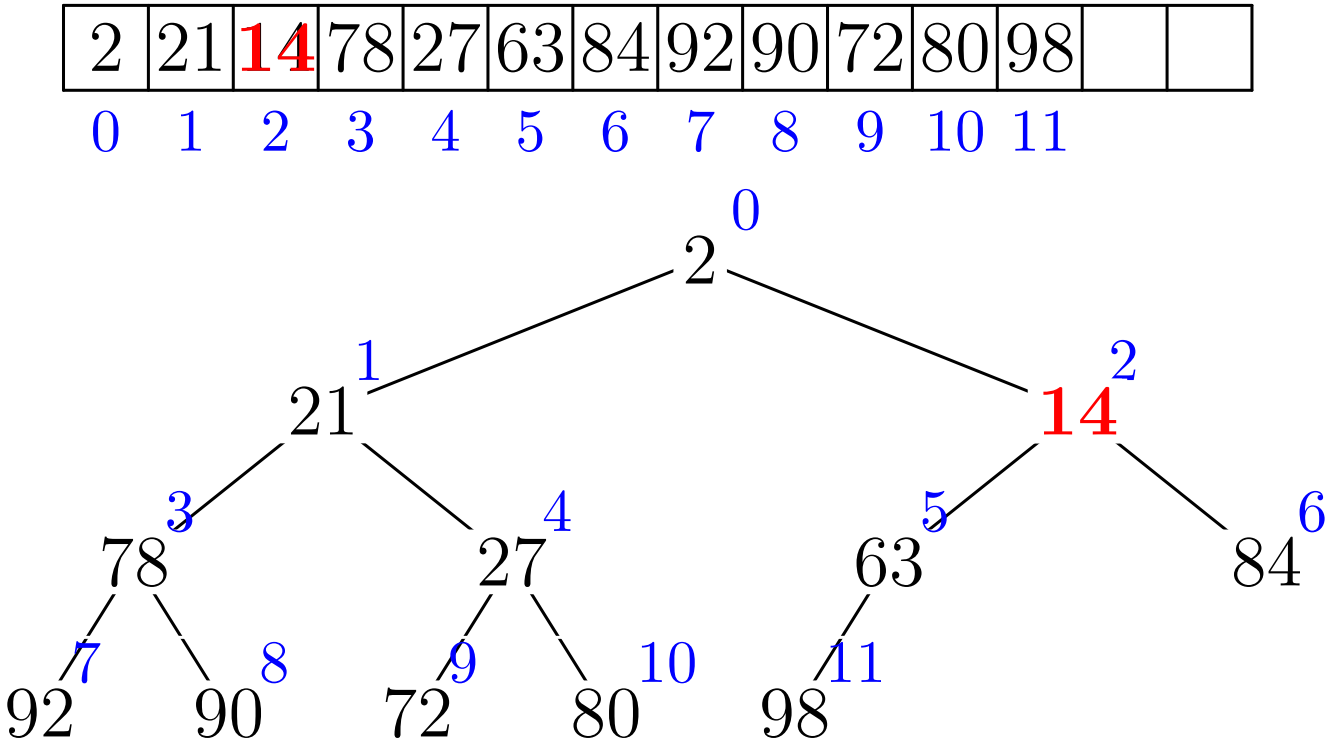

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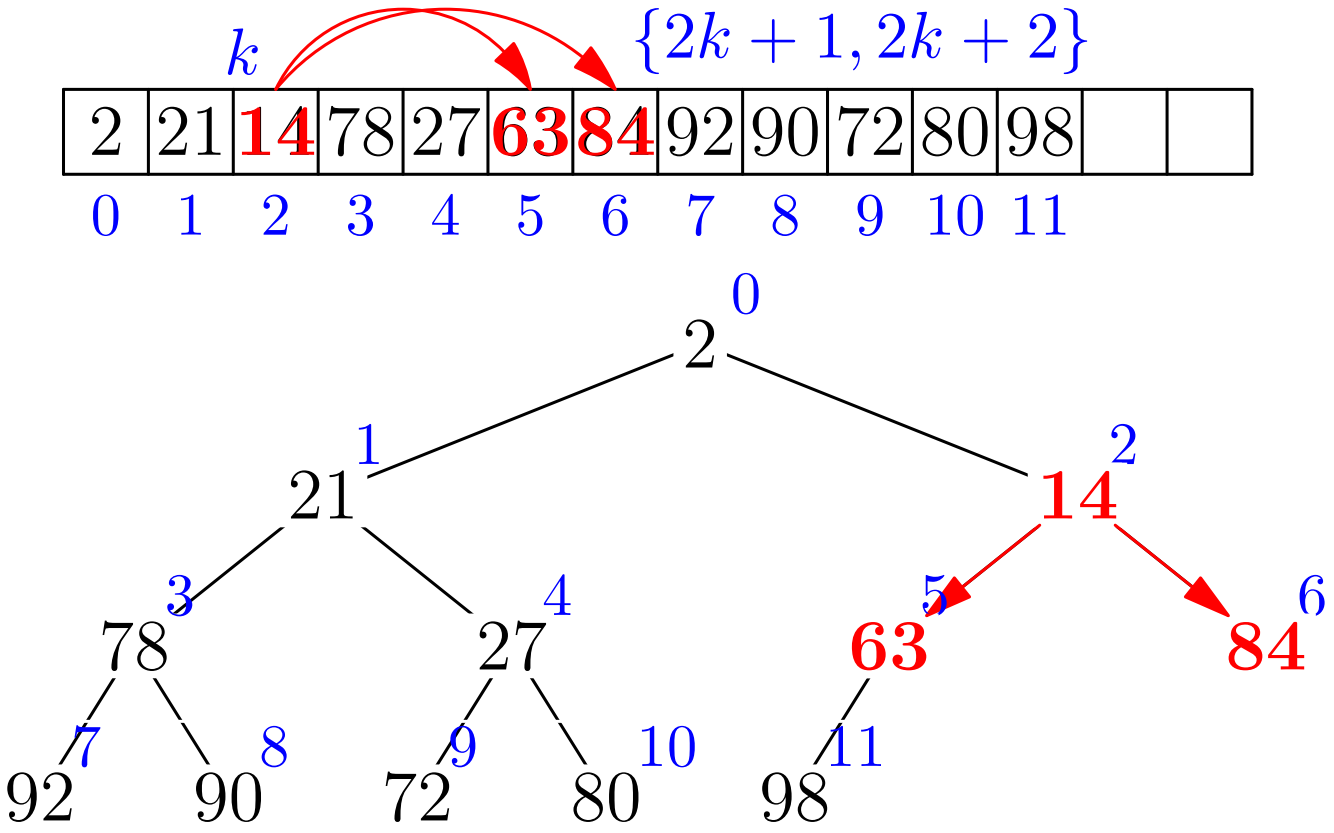

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  const T& top() {return array[0].first;}
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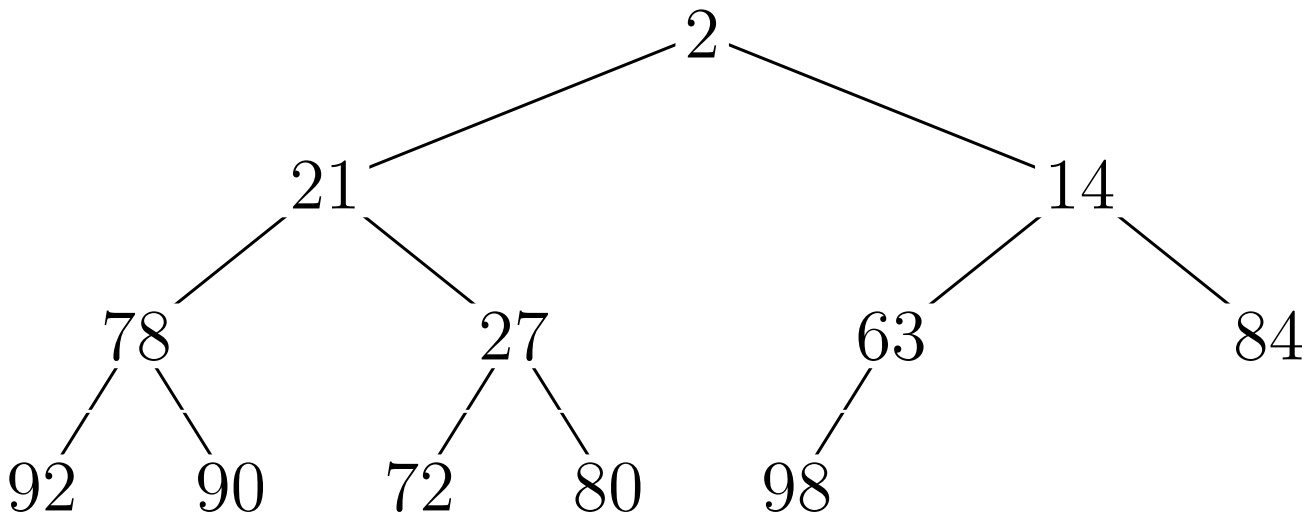
Adding an Element

2	21	14	78	27	63	84	92	90	72	80	98		



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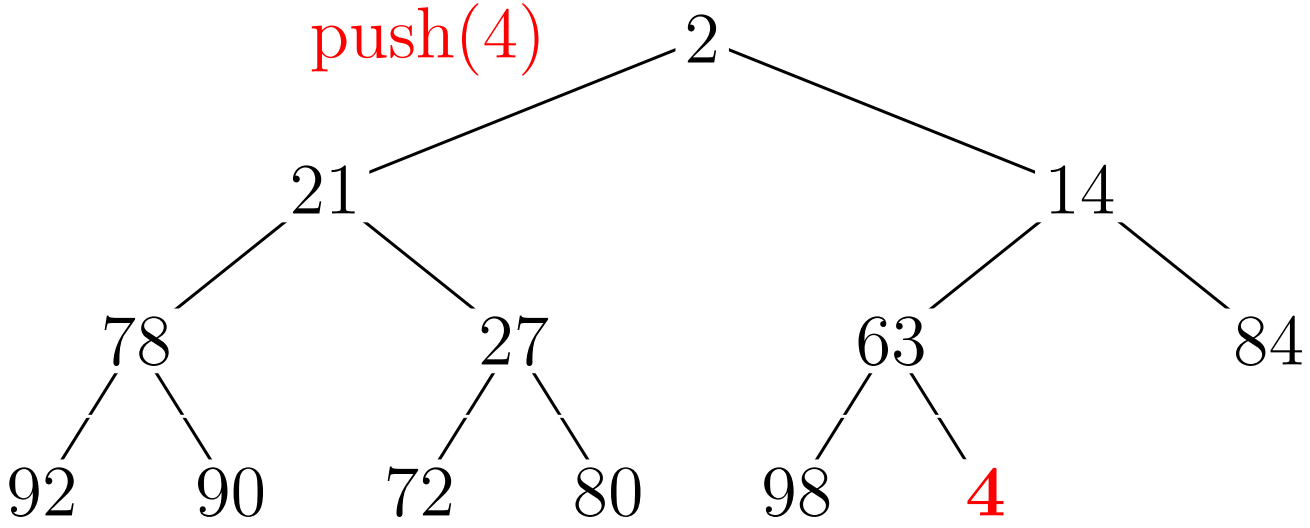

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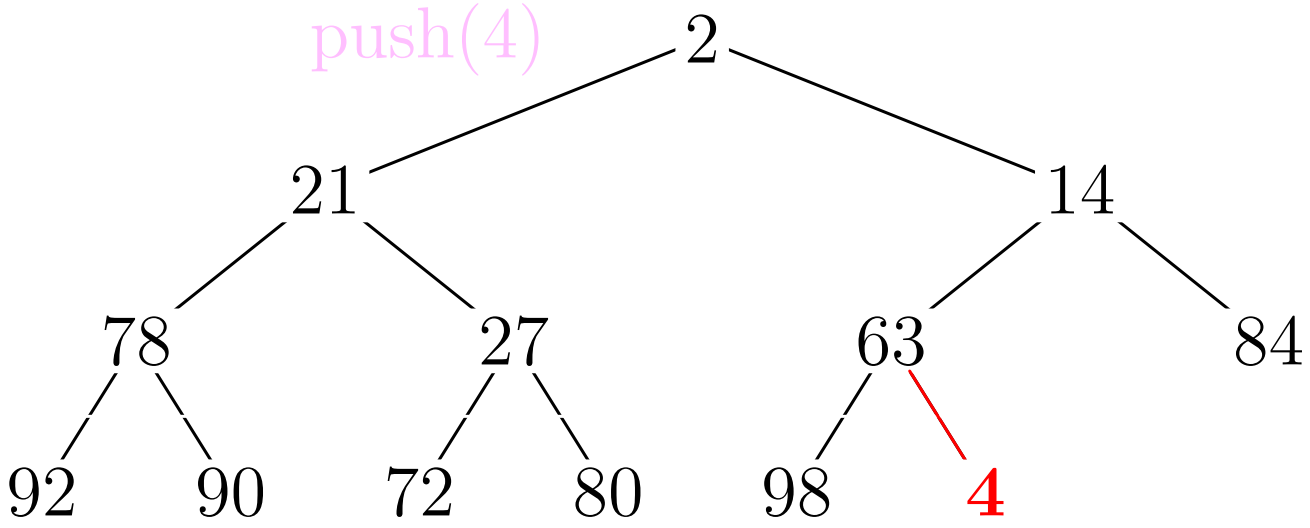

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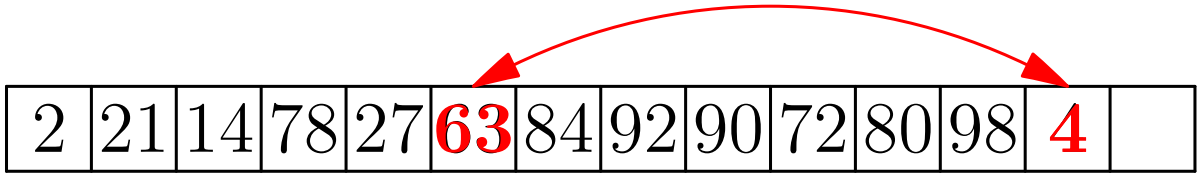

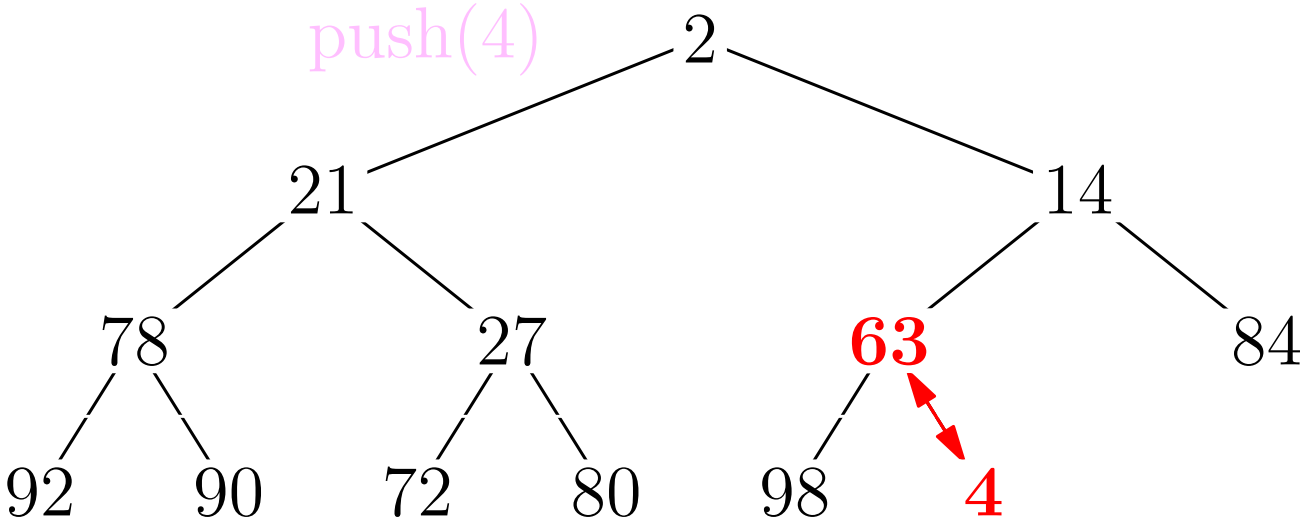

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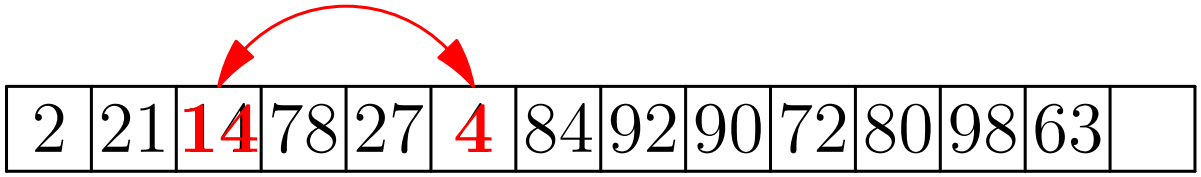

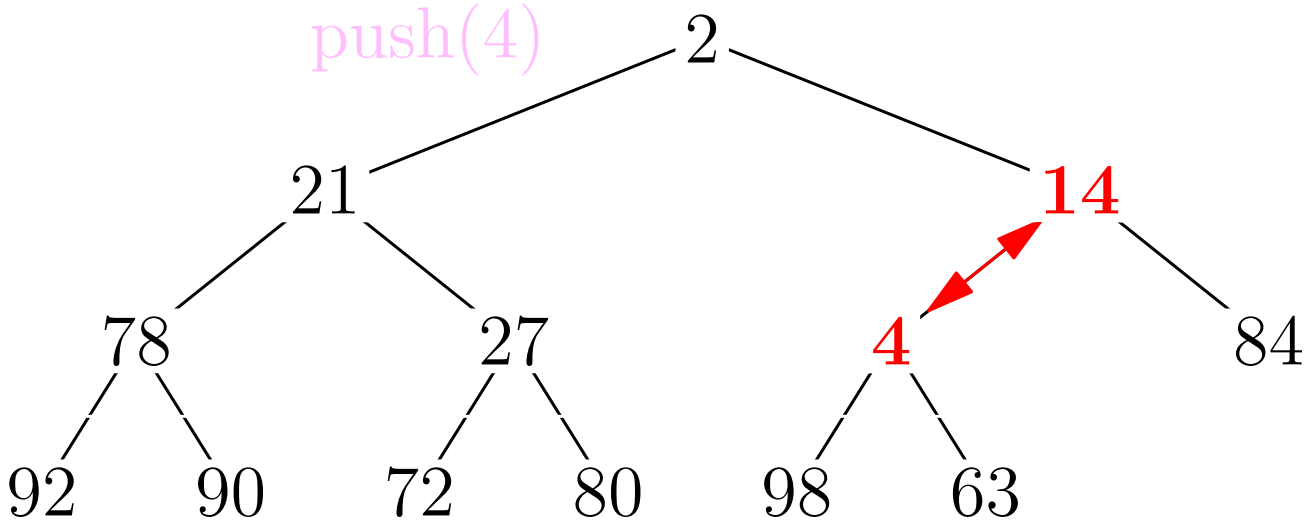

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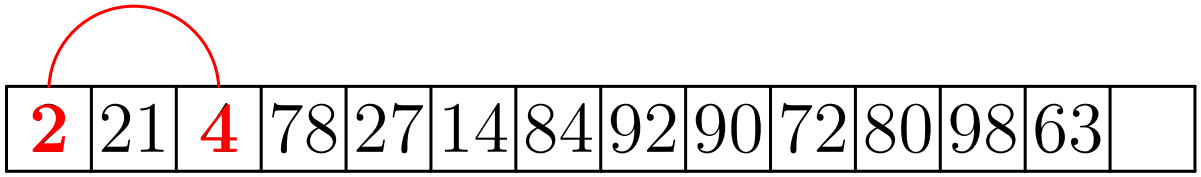

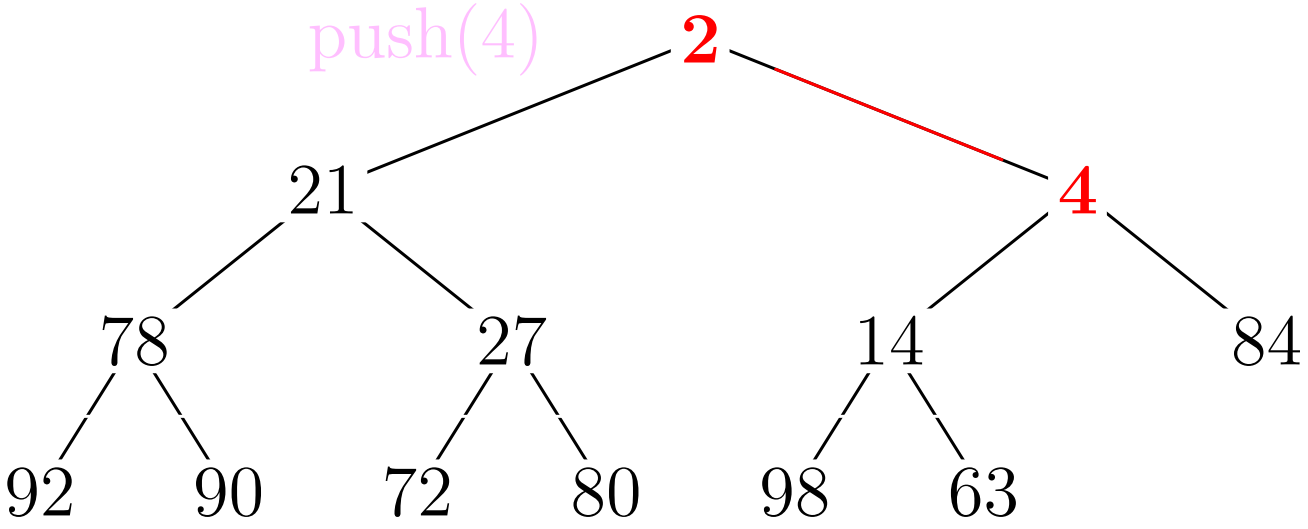

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  pair<T,P> tmp(value, priority);
  array.push_back(tmp);
  unsigned child = size() - 1;

  /* Percolate Up */
  while(child!=0) {
    unsigned parent = (child-1)>>1;  // floor((child-1)/2)
    if (array[parent].second < array[child].second)
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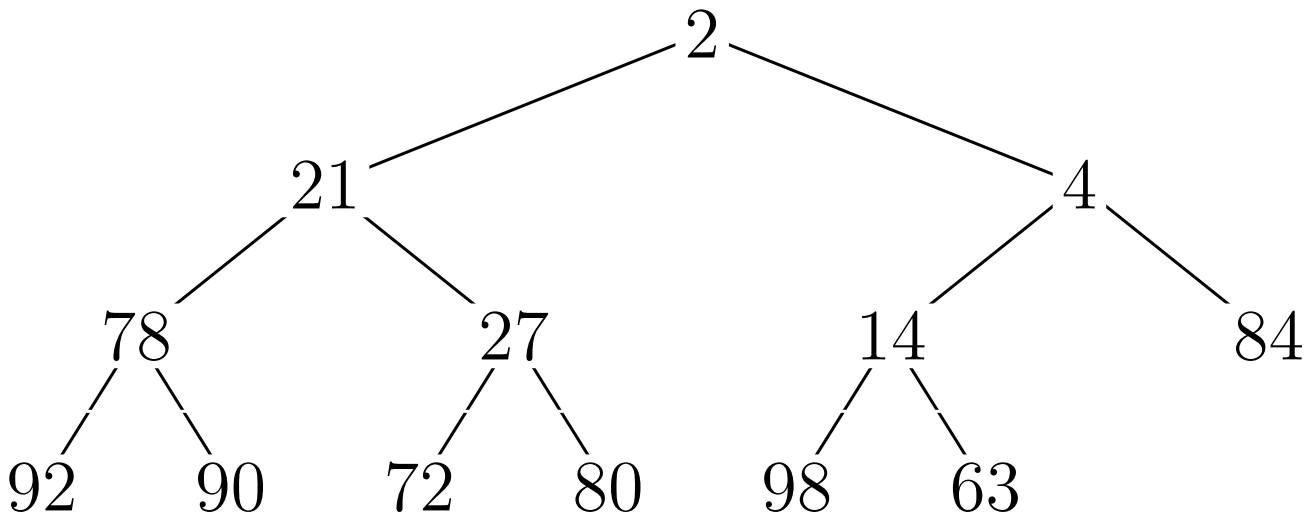

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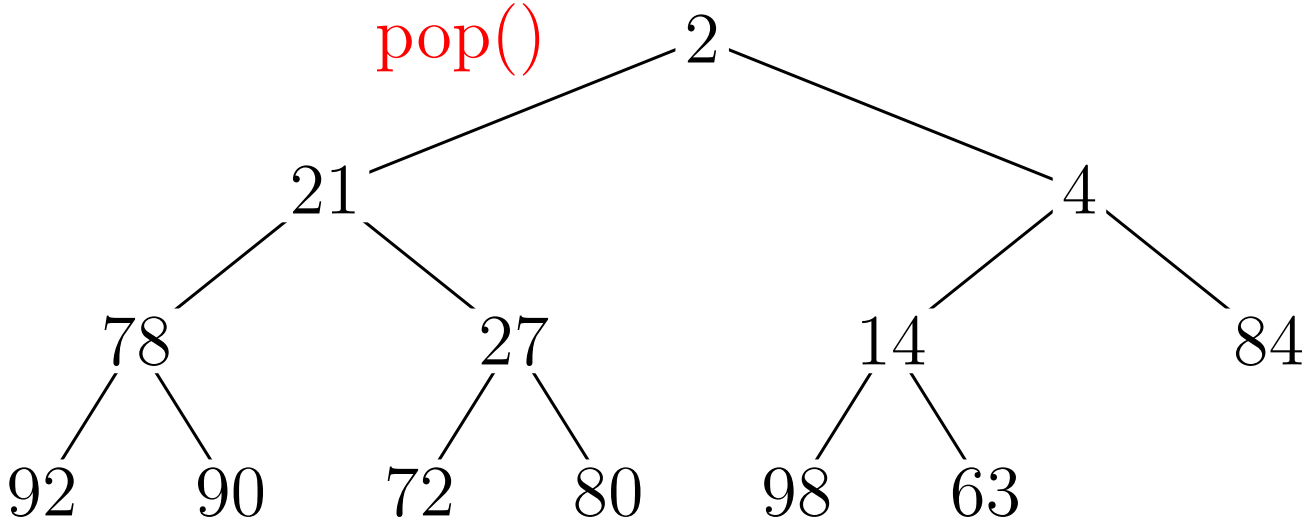

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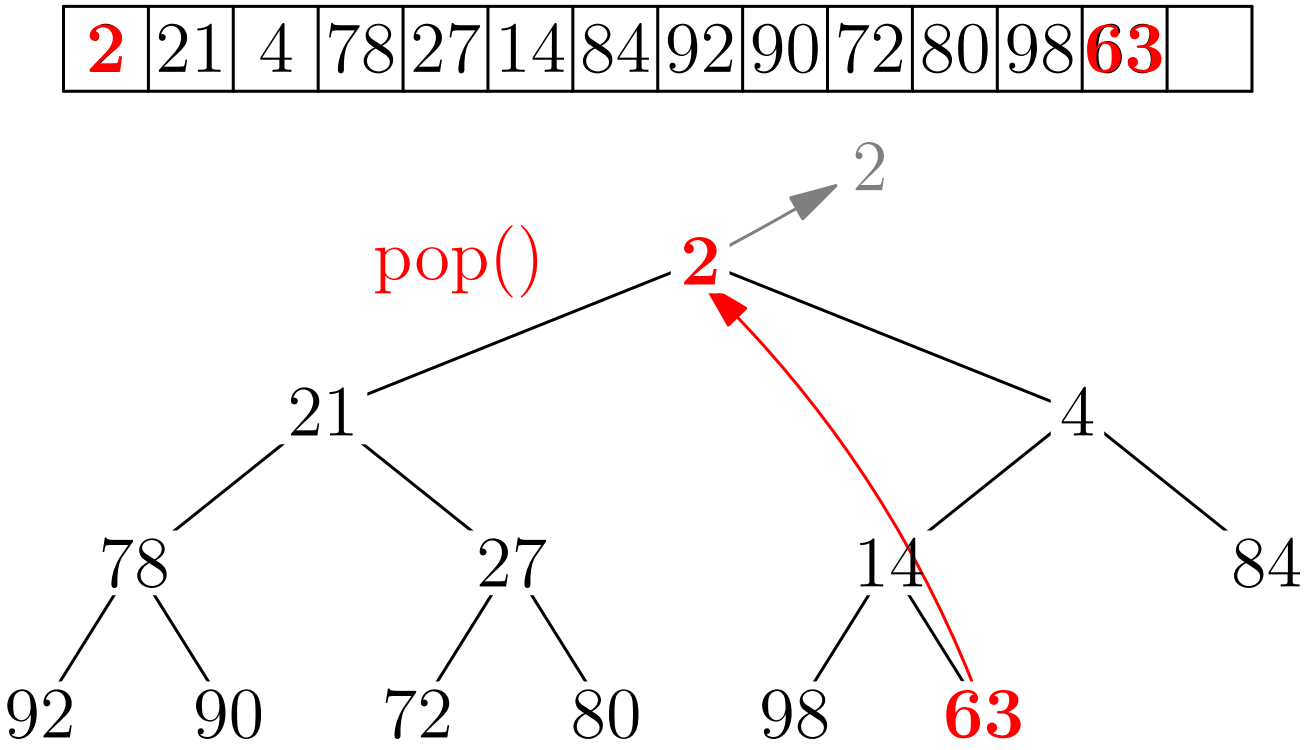

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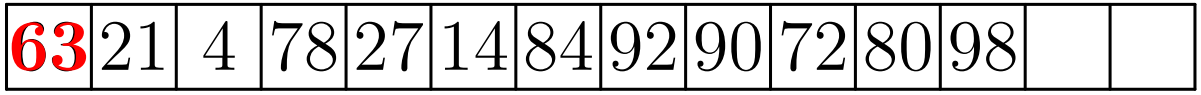

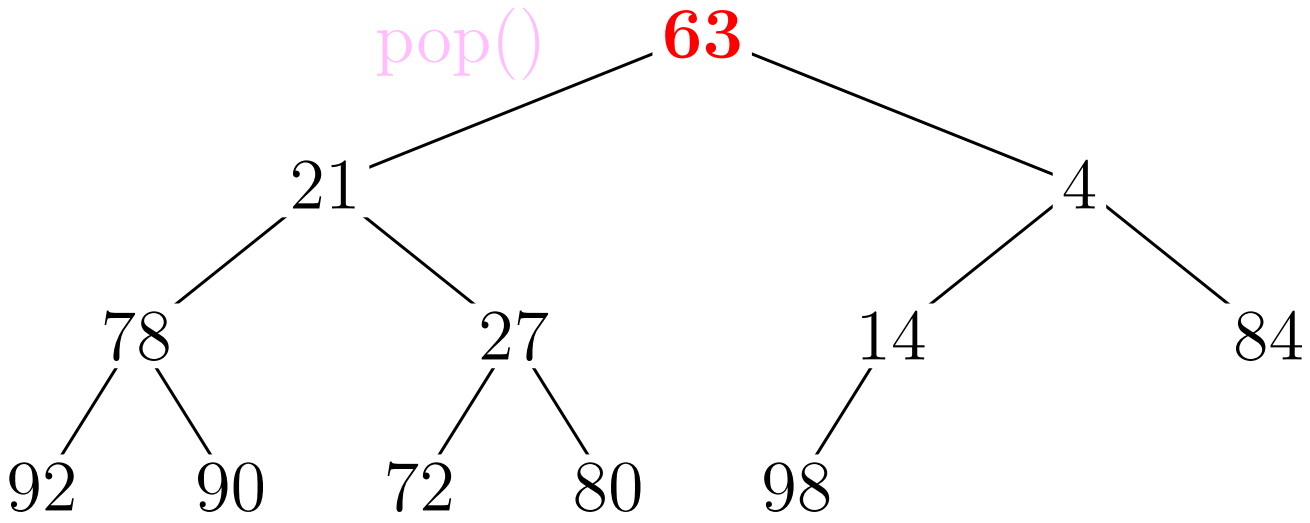

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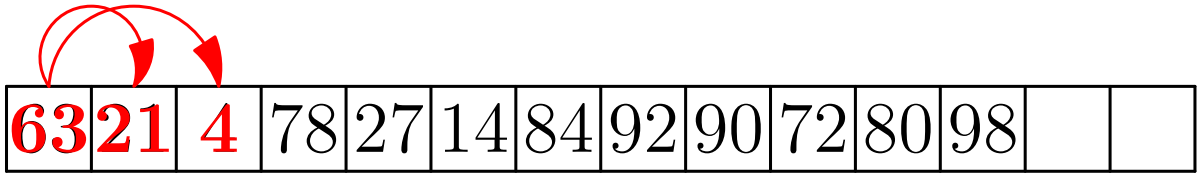

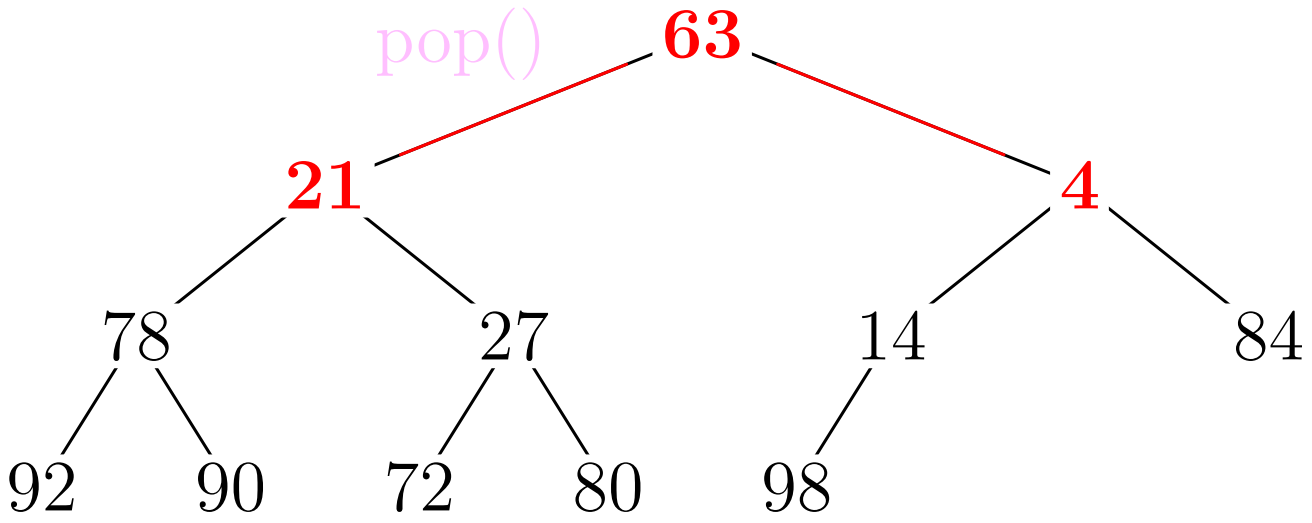

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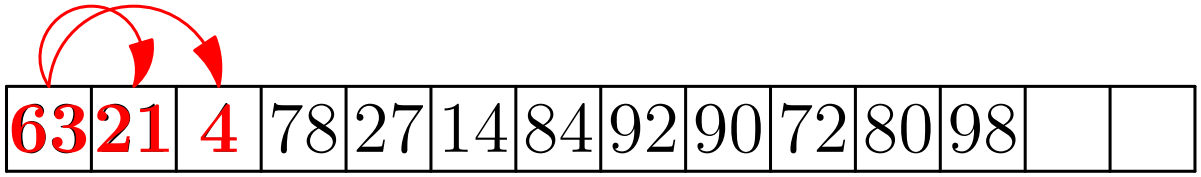

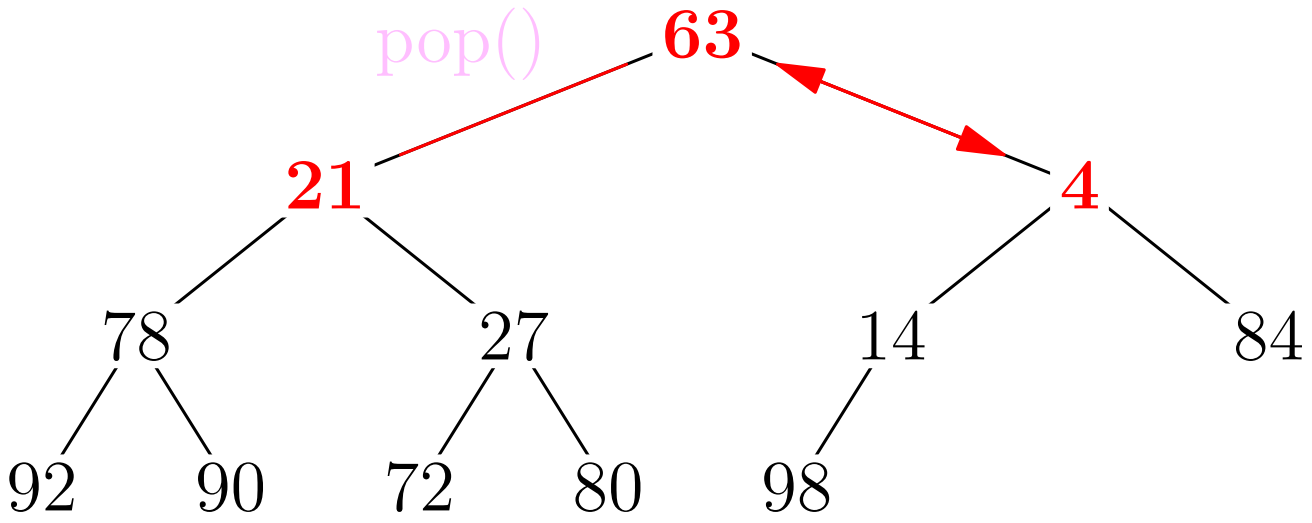

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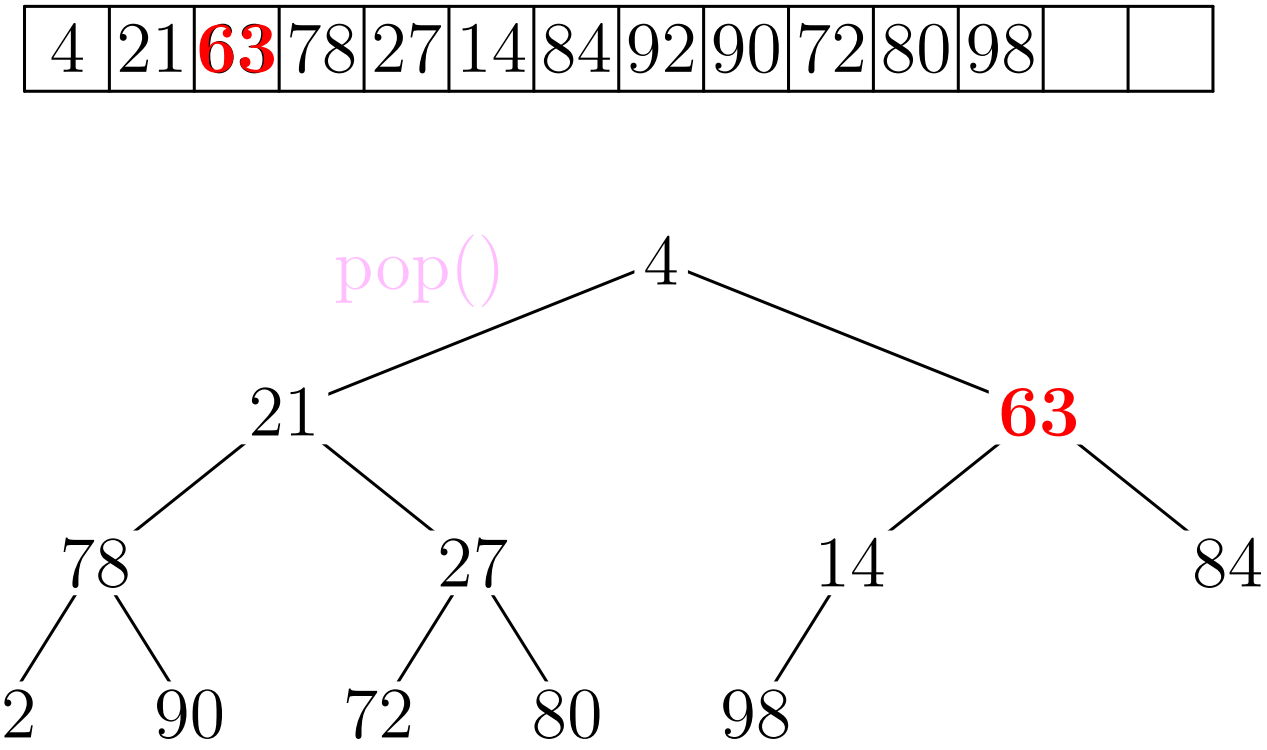

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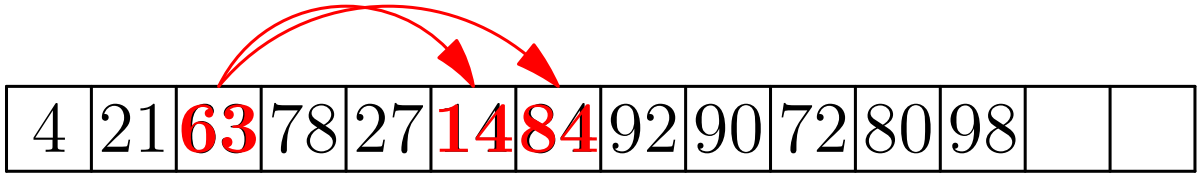

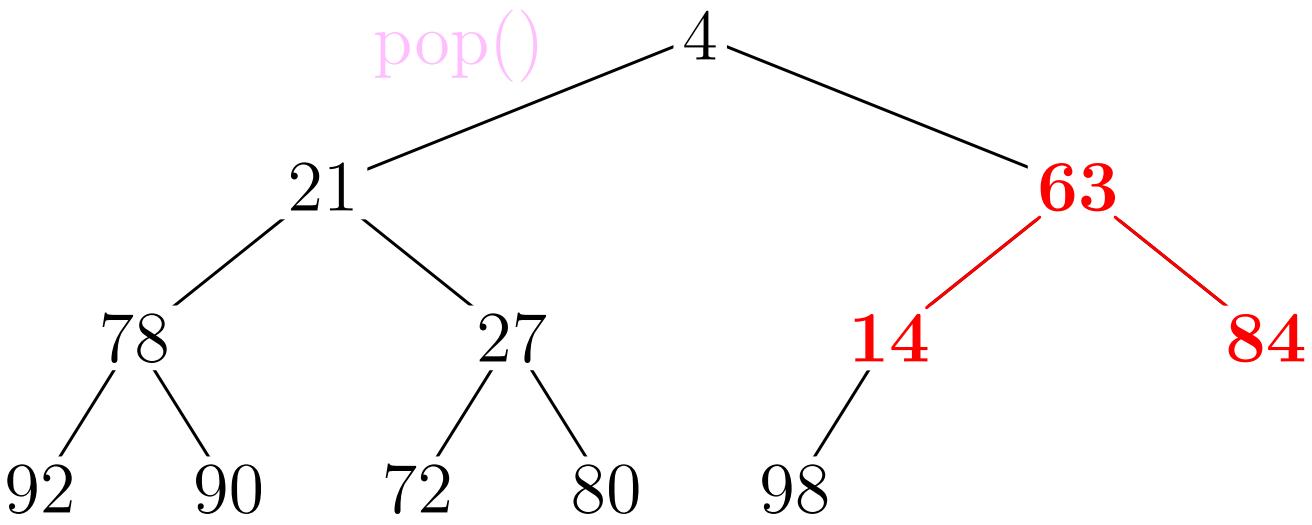

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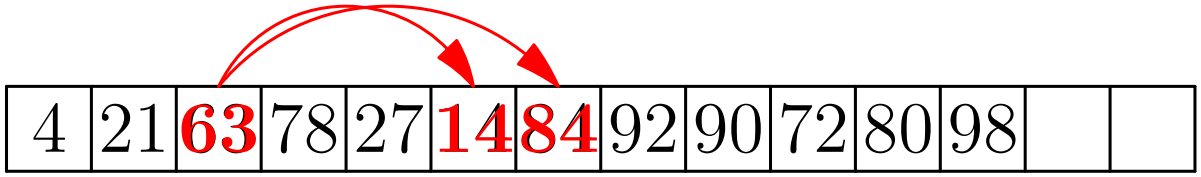

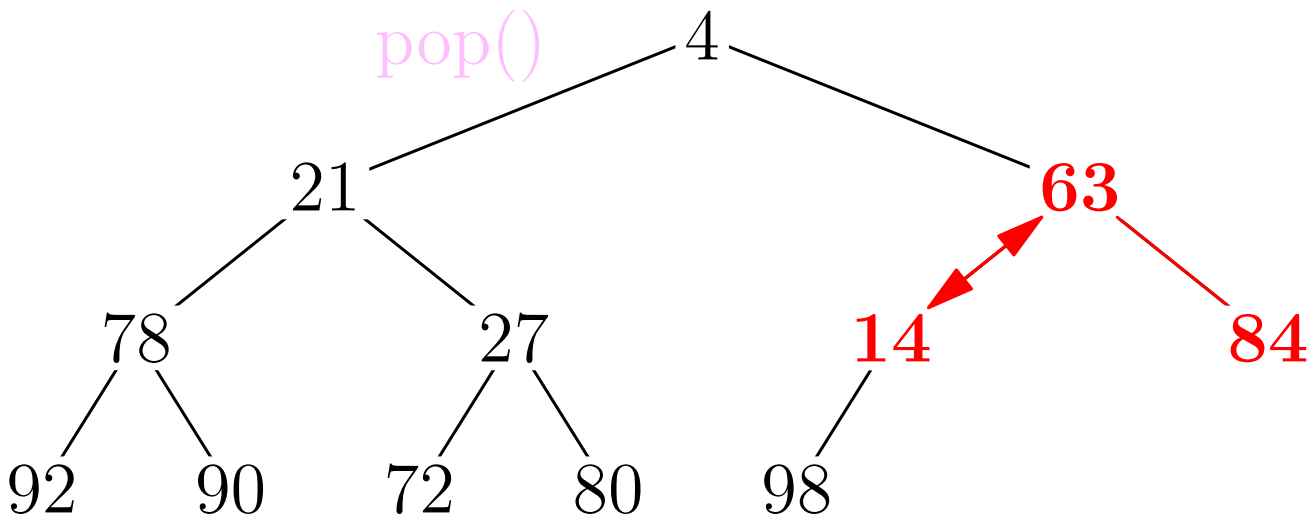

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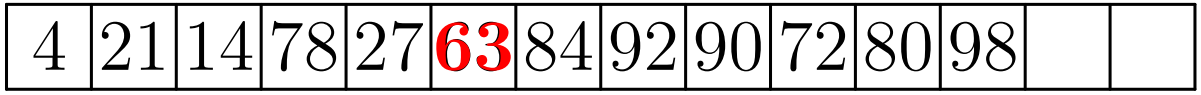

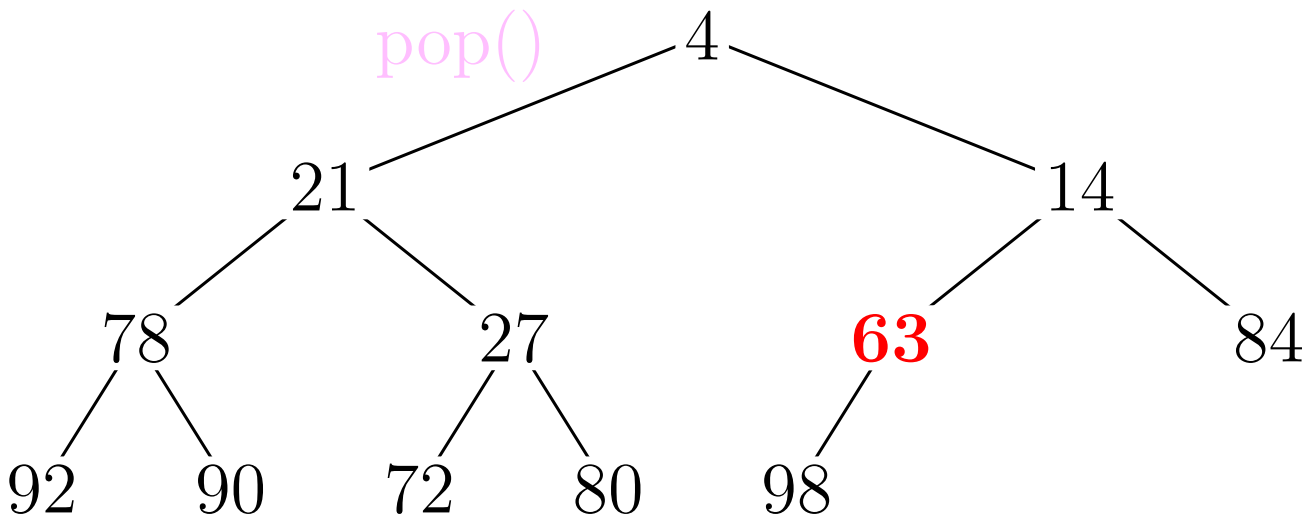

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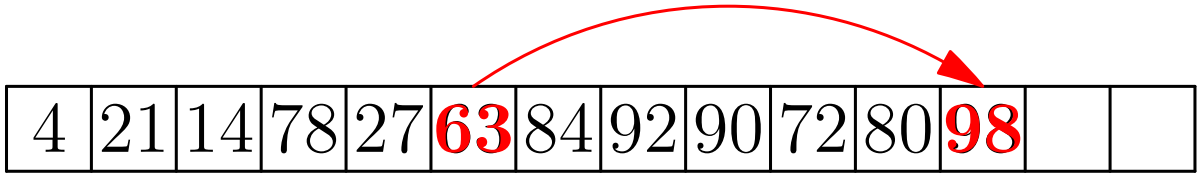
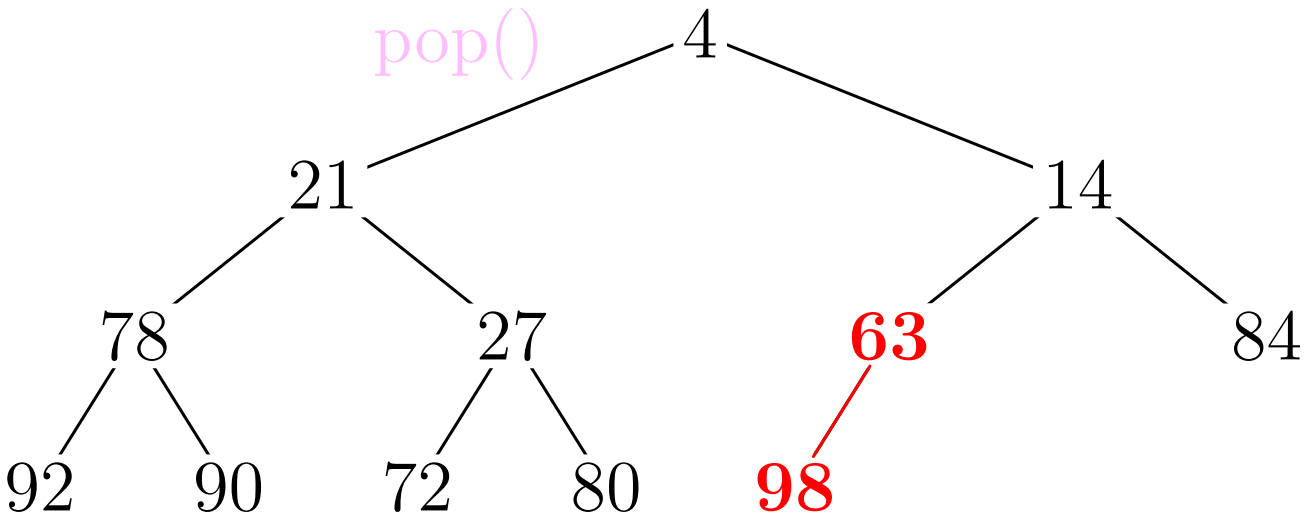




Popping the Top

```
void pop() {
unsigned parent = 0;
pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
```

Popping the Top

```
void pop() {
unsigned parent = 0;
pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
unsigned child = 1;

/* Percolate down */
while(child<size()) {
```

Popping the Top

```
void pop() {
unsigned parent = 0;
pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
unsigned child = 1;

/* Percolate down */
while(child<size()) {
  if (child+1<=size() && array[child+1].second < array[child].second)
    ++child;
```

Popping the Top

```
void pop() {
unsigned parent = 0;
pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
unsigned child = 1;

/* Percolate down */
while(child<size()) {
  if (child+1<=size() && array[child+1].second < array[child].second)
    ++child;
  if (array[child].second > array[parent].second)
    return;
```

Popping the Top

```
  void pop() {
  unsigned parent = 0;
  pair<T, P> tmp = array.back();
  array[0] = tmp;
  array.pop_back();
  unsigned child = 1;

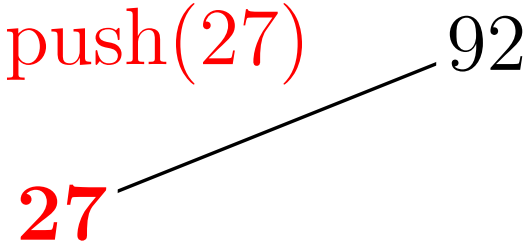
  /* Percolate down */
  while(child<size()) {
    if (child+1<=size() && array[child+1].second < array[child].second)
      ++child;
    if (array[child].second > array[parent].second)
      return;
    array[parent] = array[child];
    array[child] = tmp;
    parent = child;
    child = 2*parent + 1;
  }
}
```

Heaps in Action

Heaps in Action

push(92) **92**

Heaps in Action



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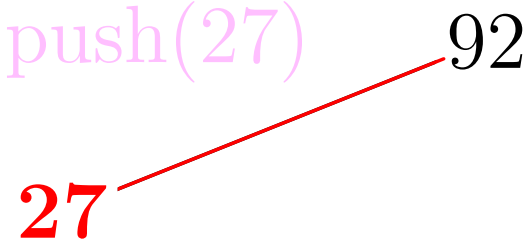

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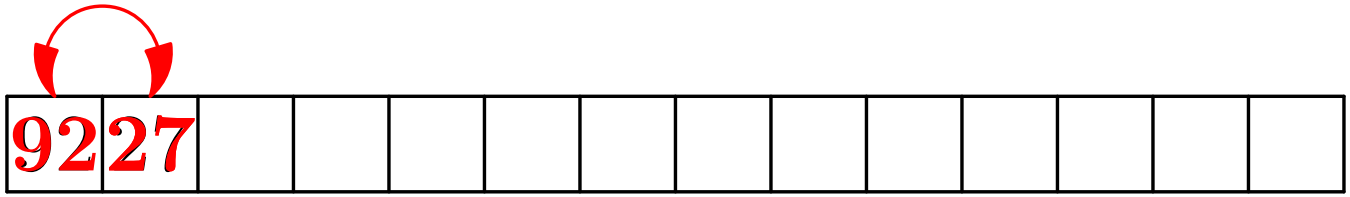

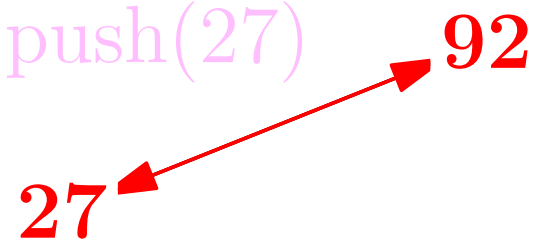

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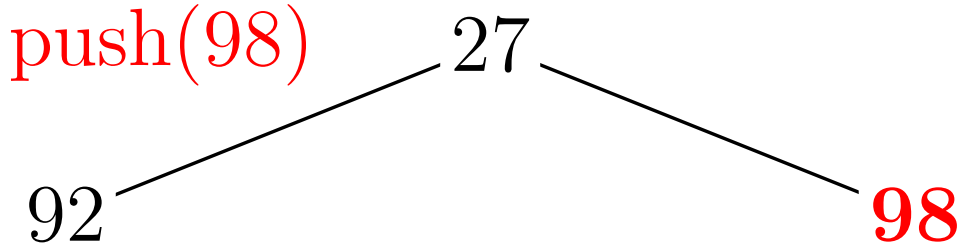

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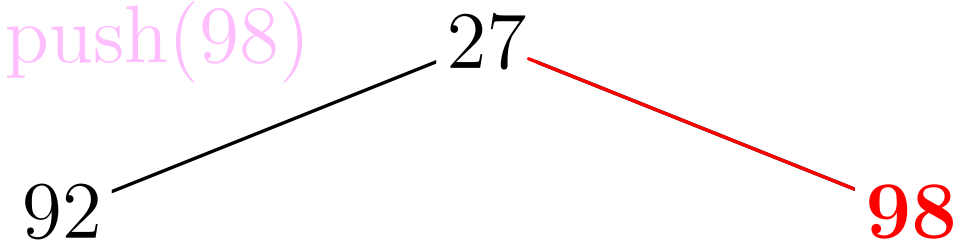

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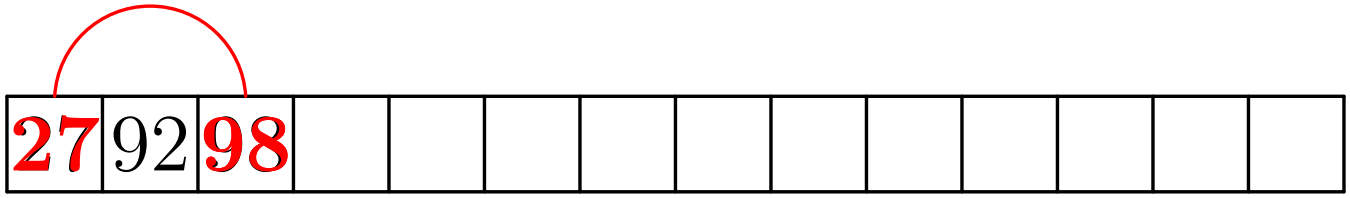

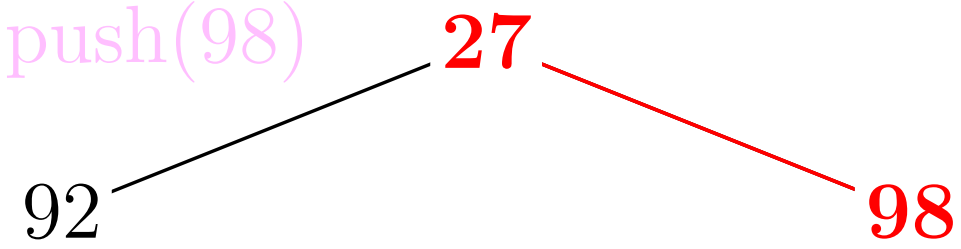

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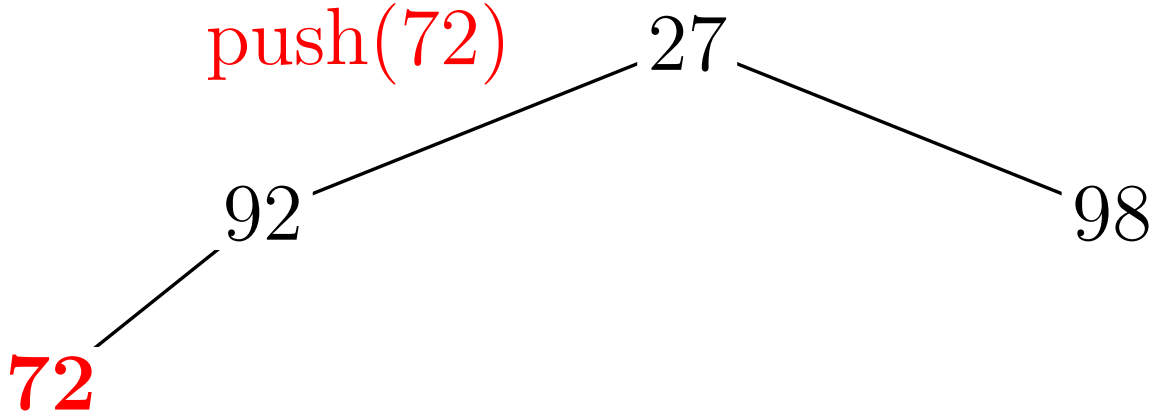

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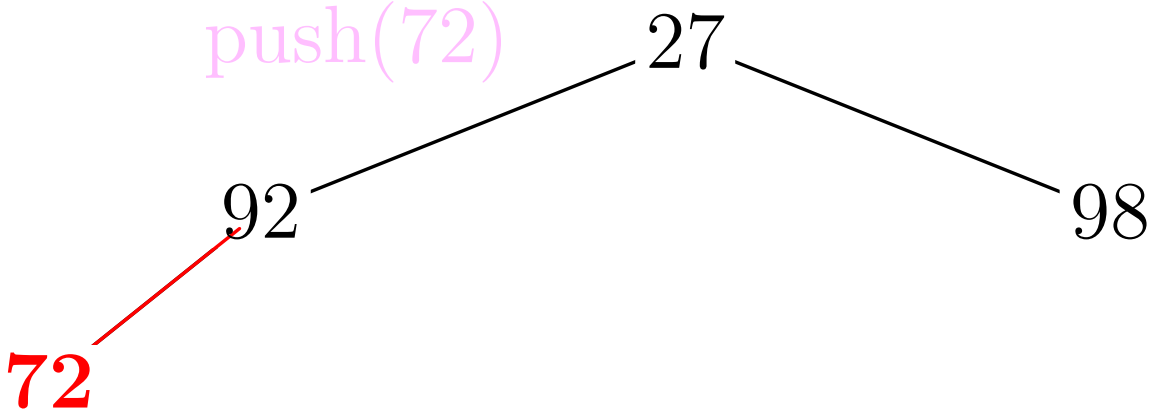

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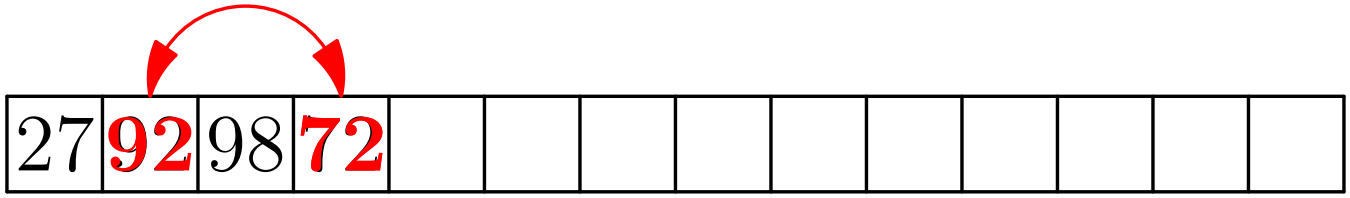

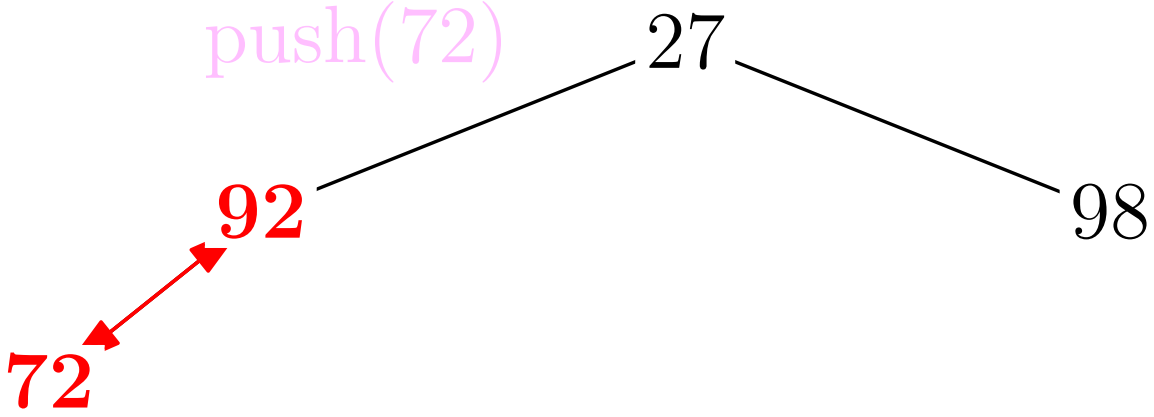

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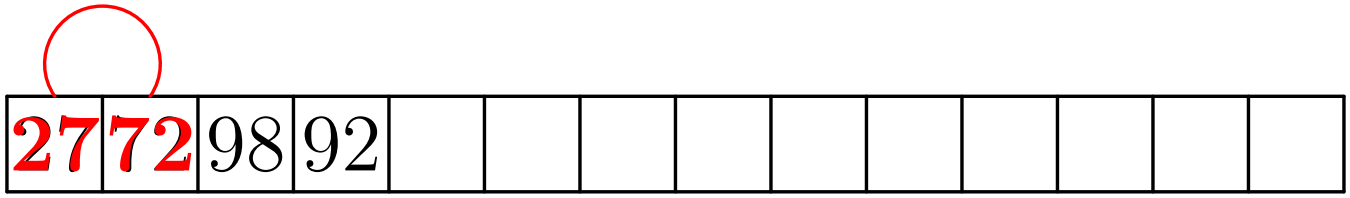

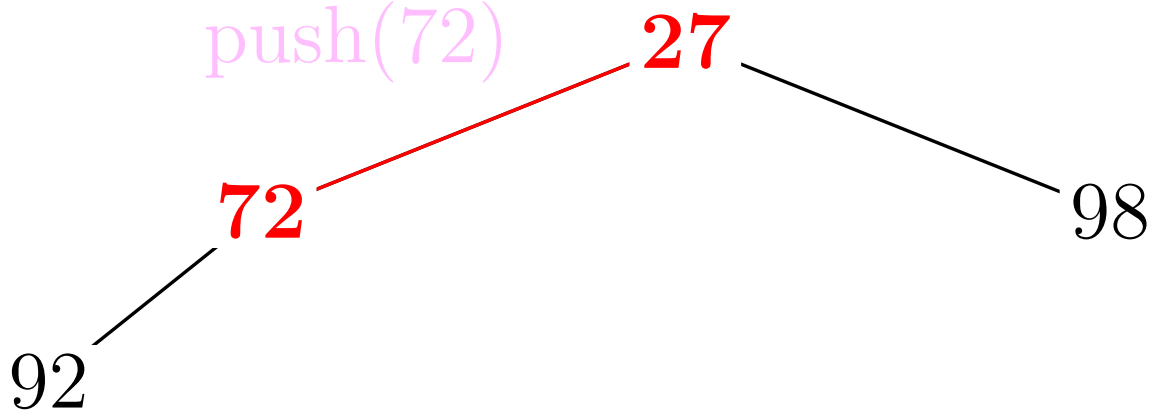

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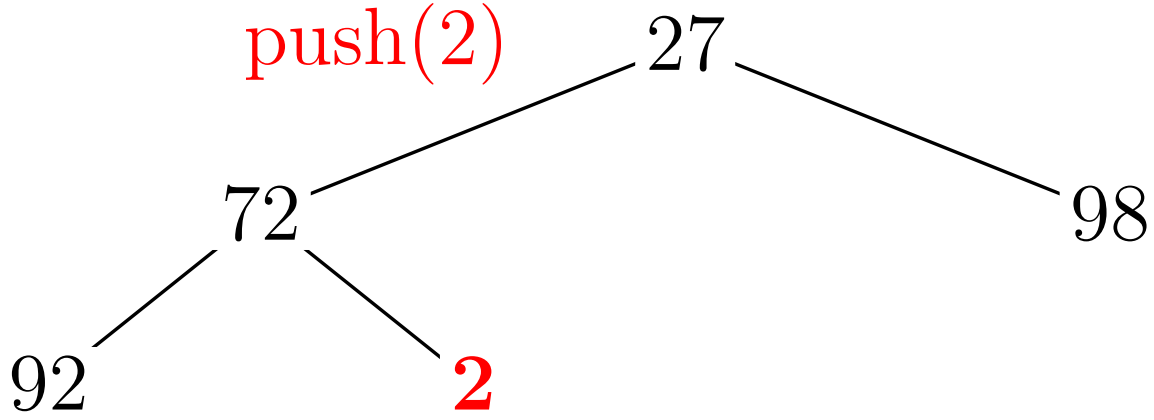

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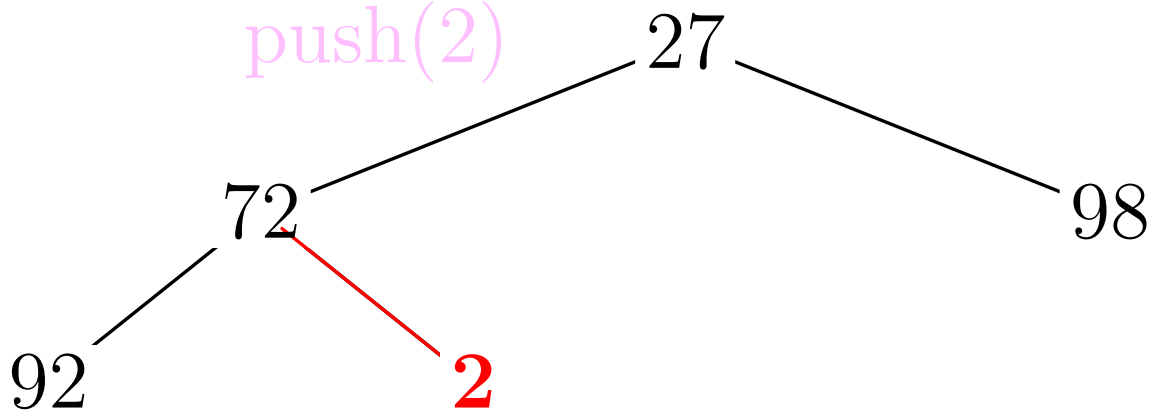

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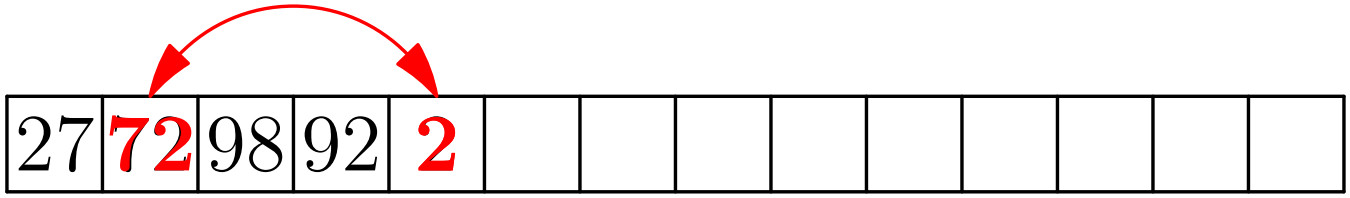

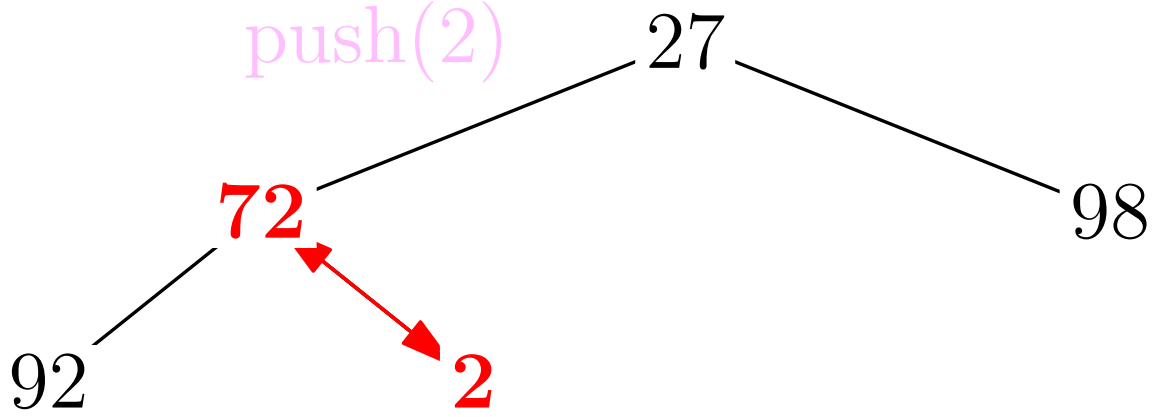

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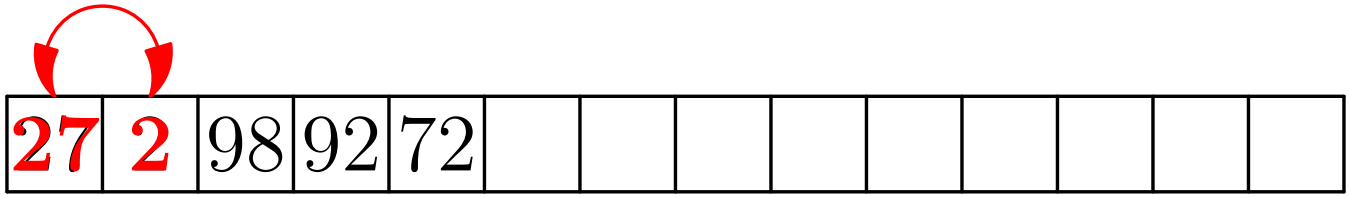

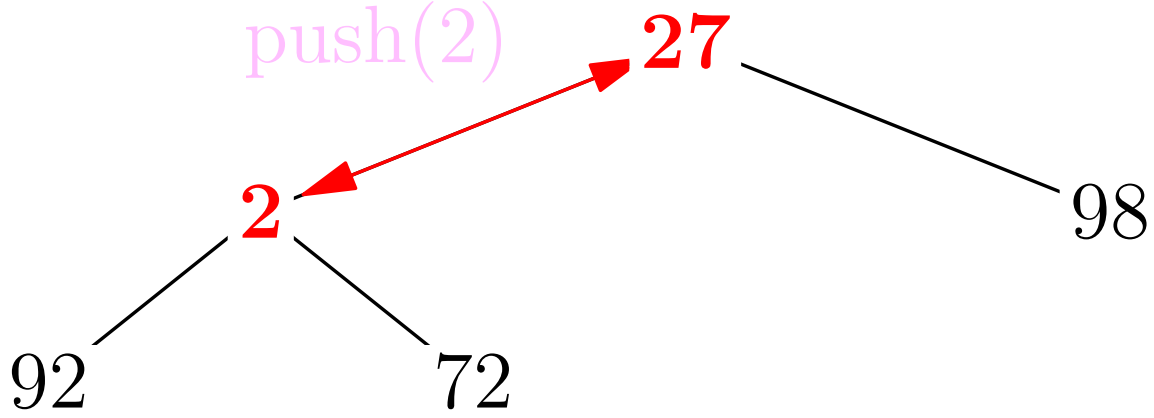

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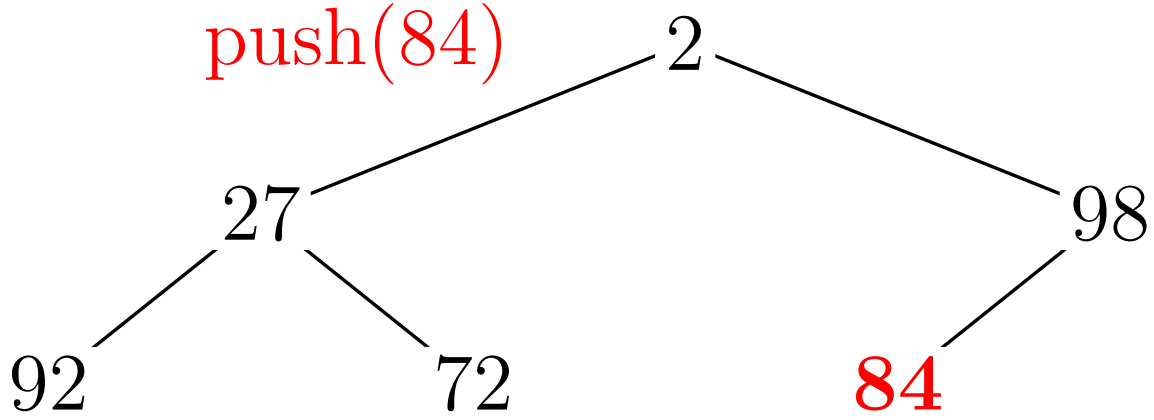

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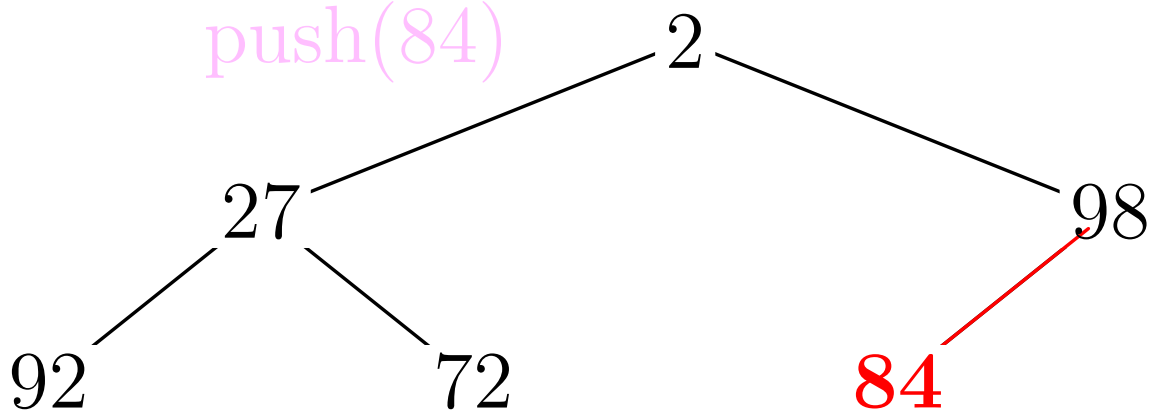

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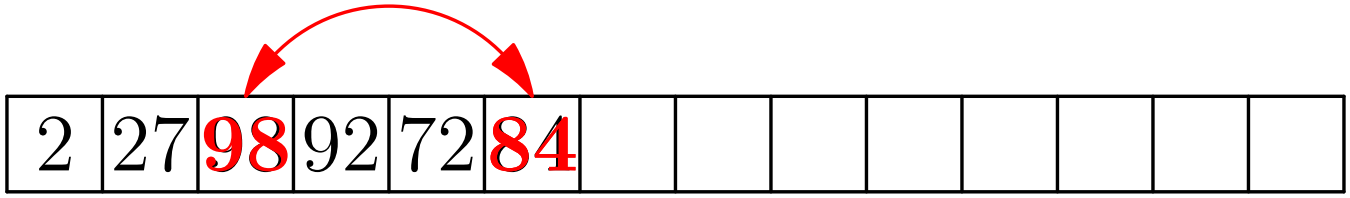

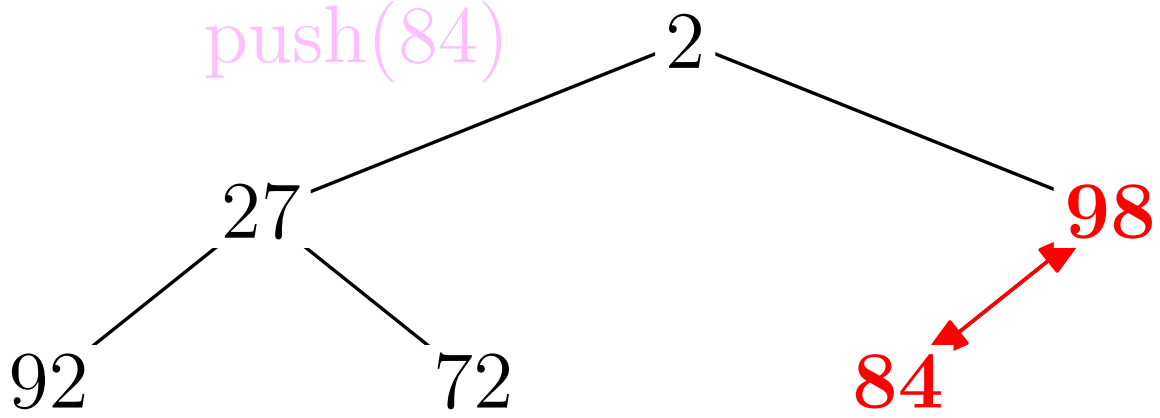

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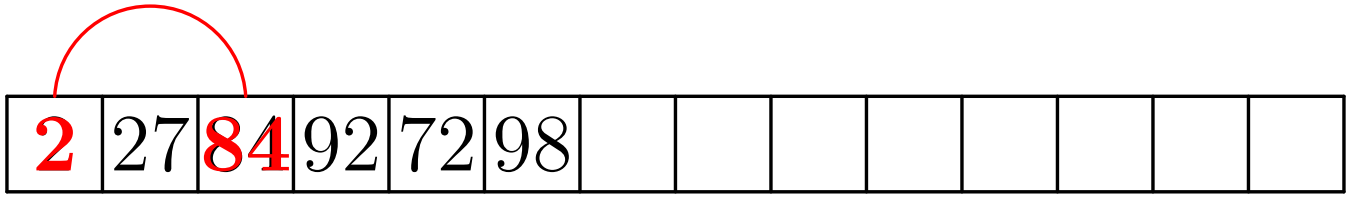

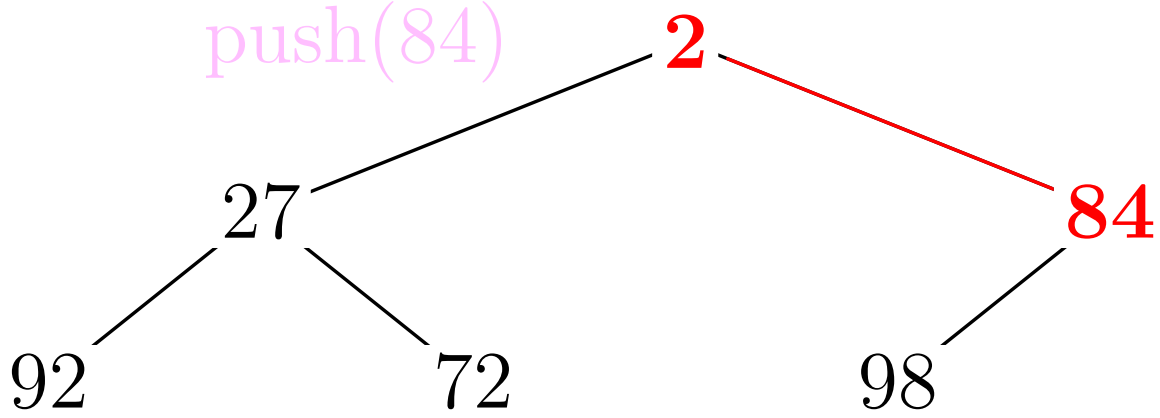

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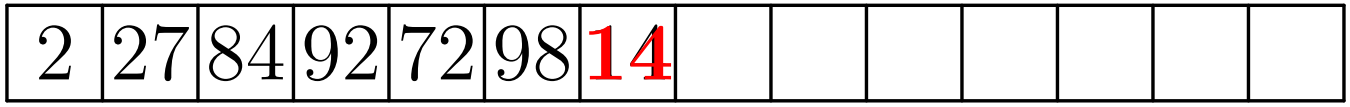

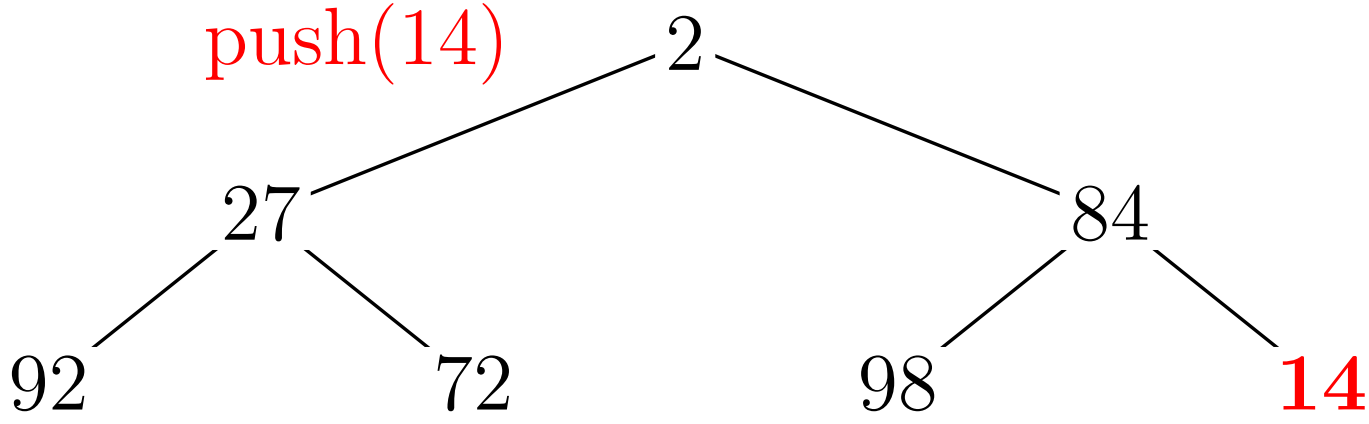

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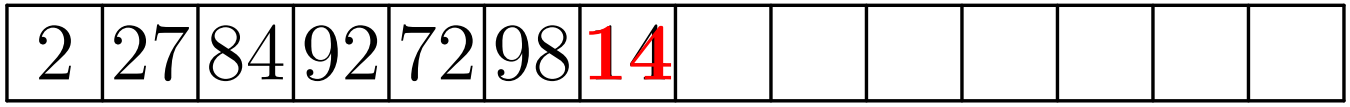

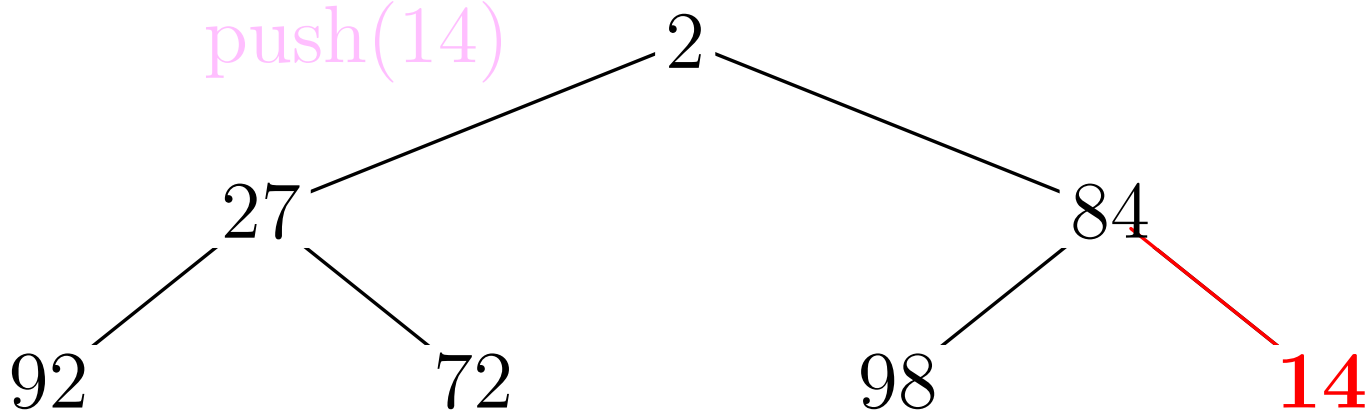

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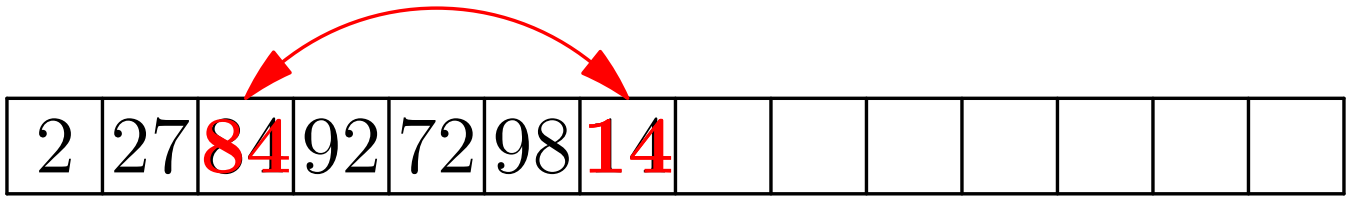

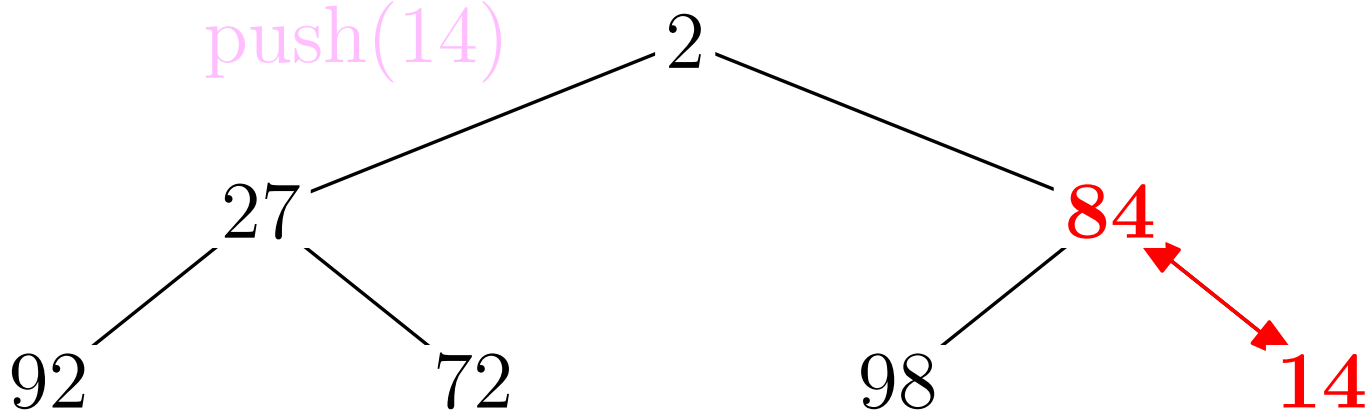

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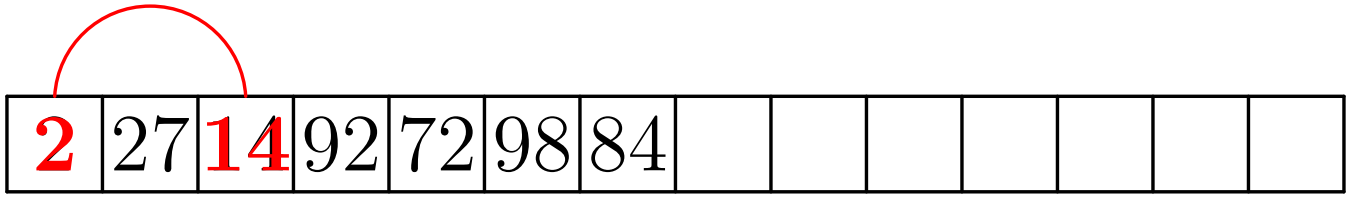

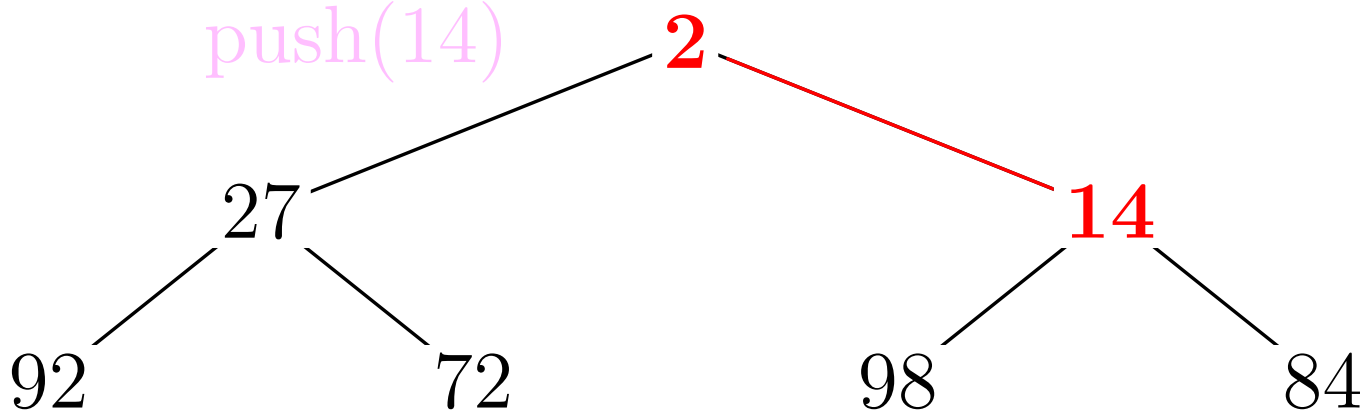

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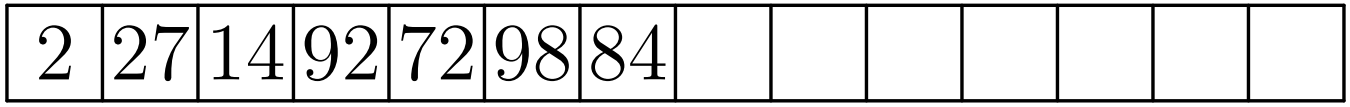

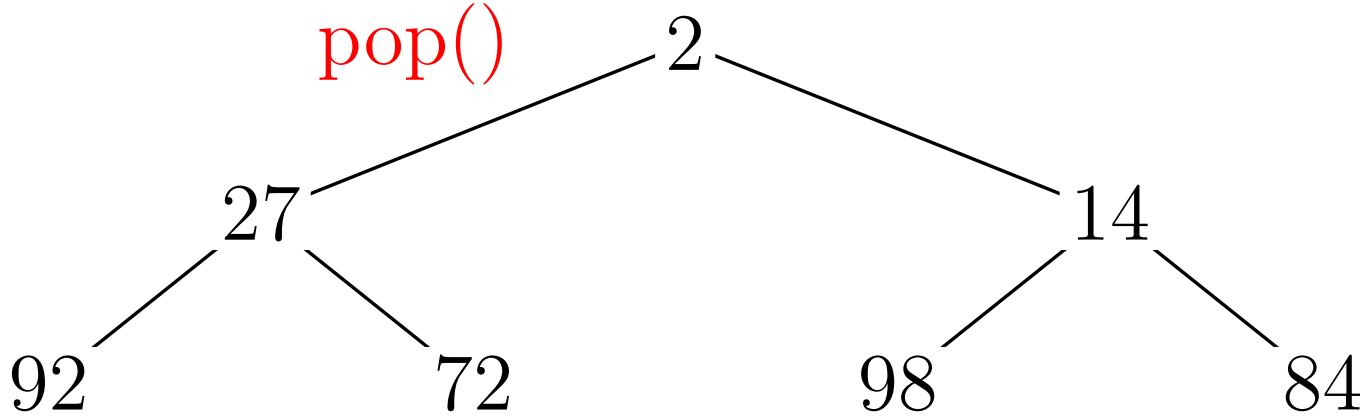

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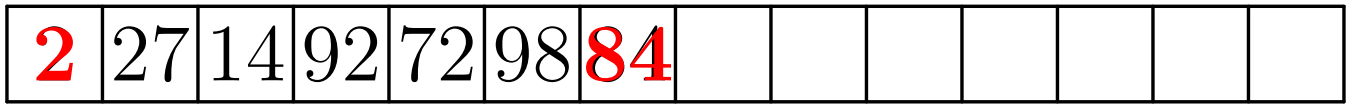

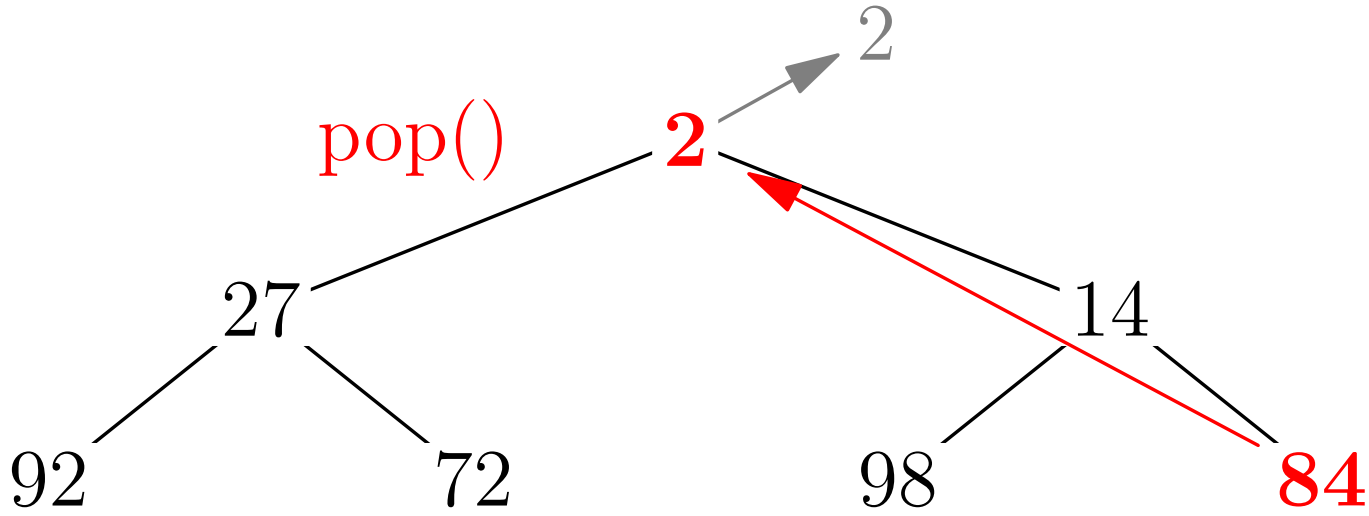

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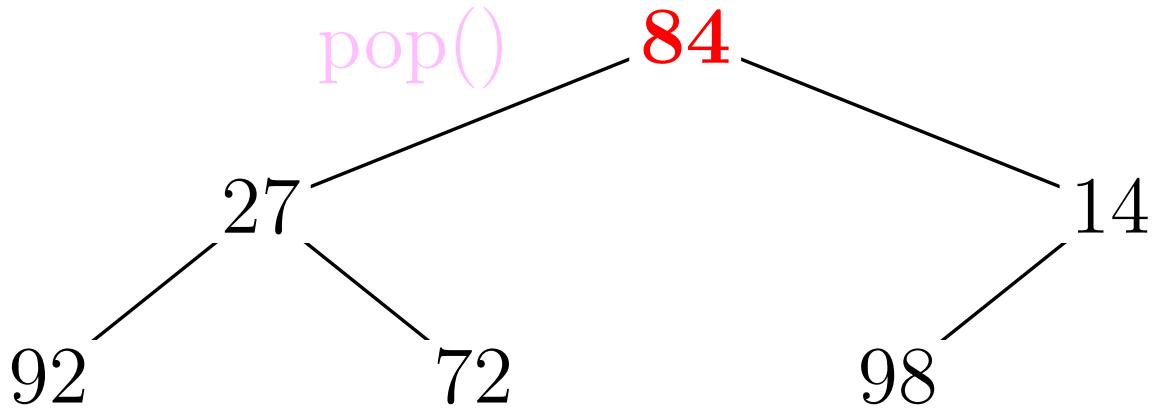

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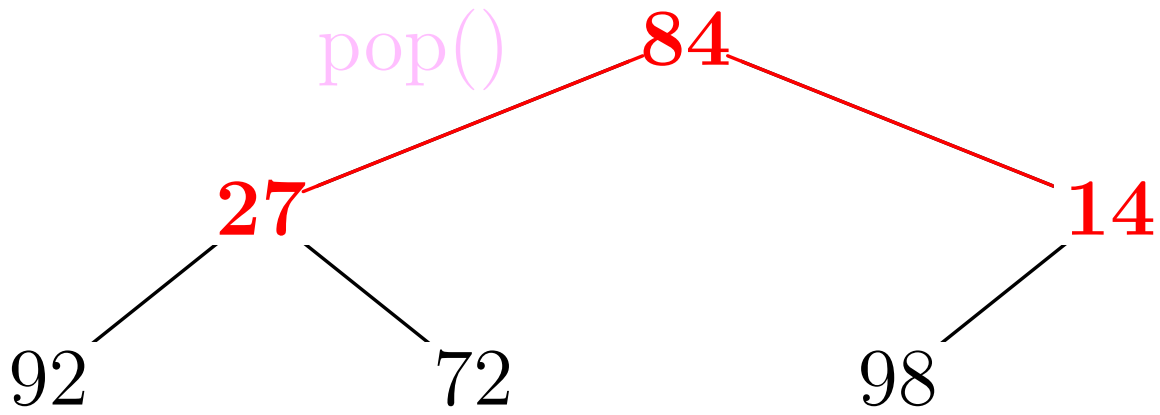

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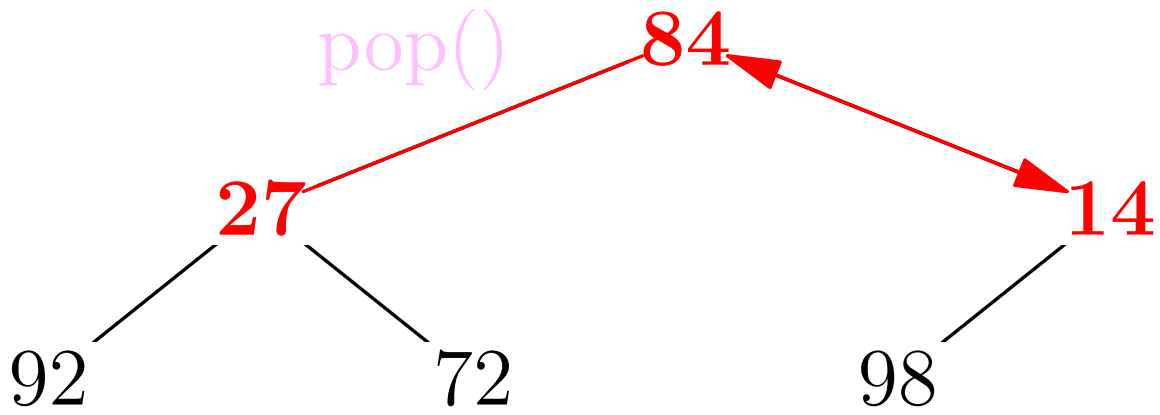

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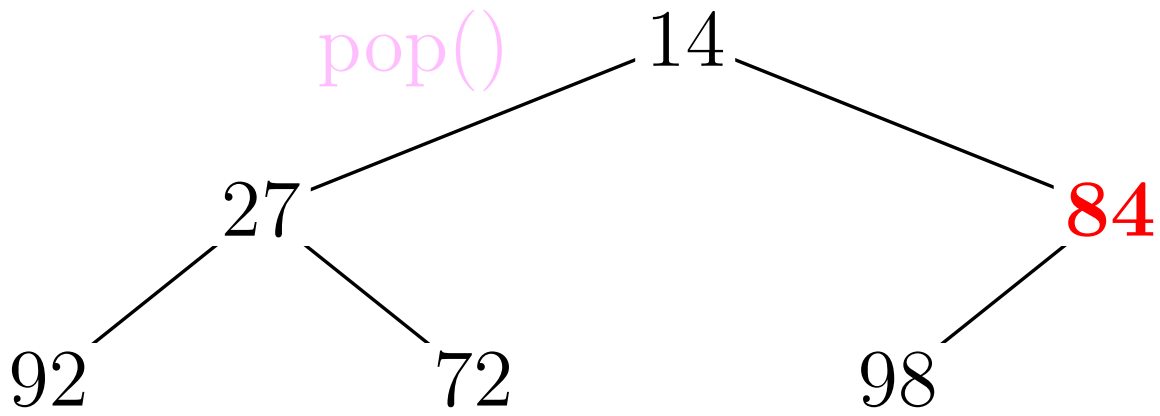

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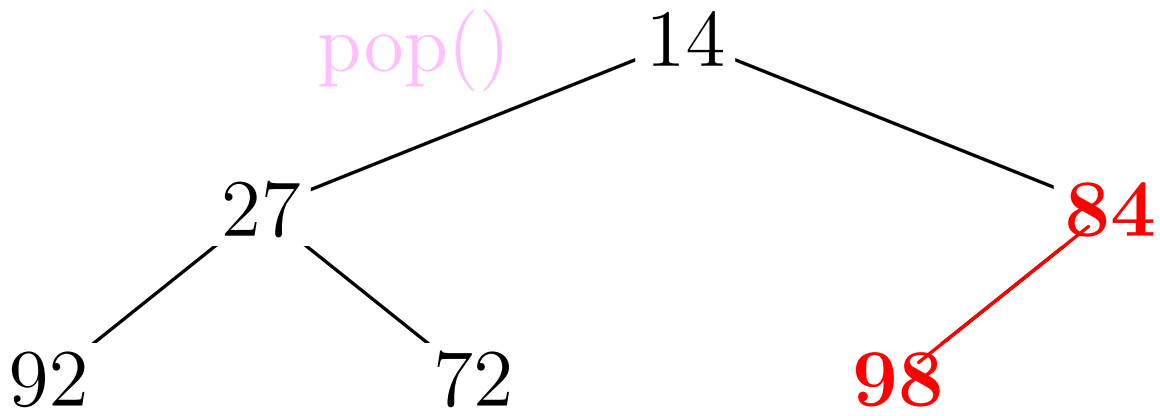

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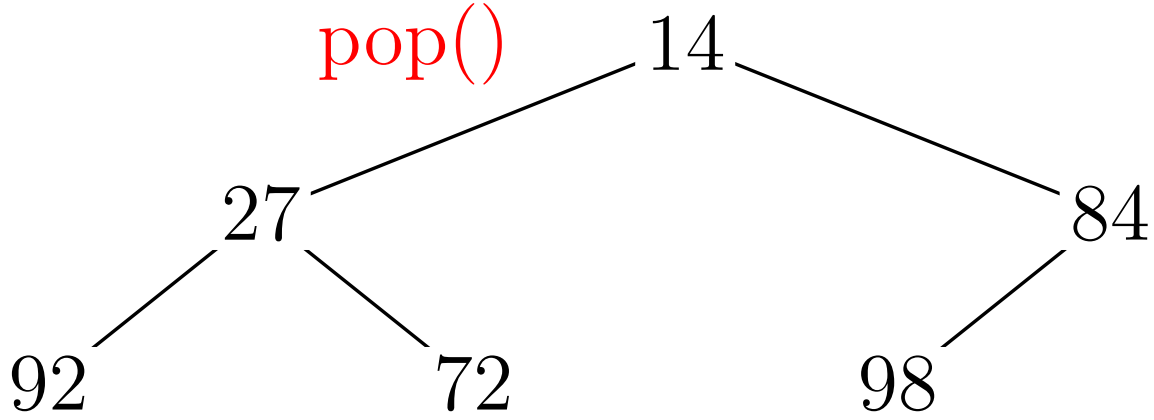

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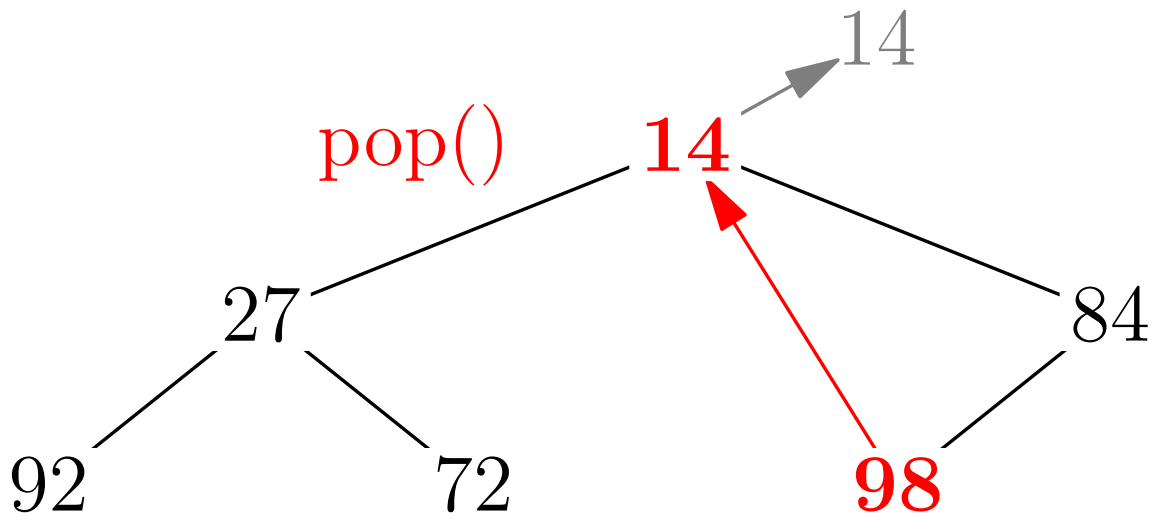
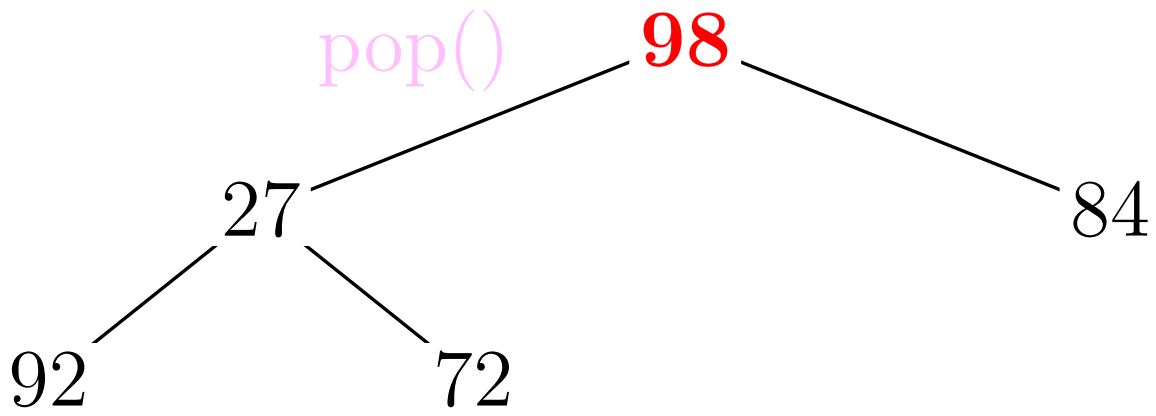


Heaps in Action



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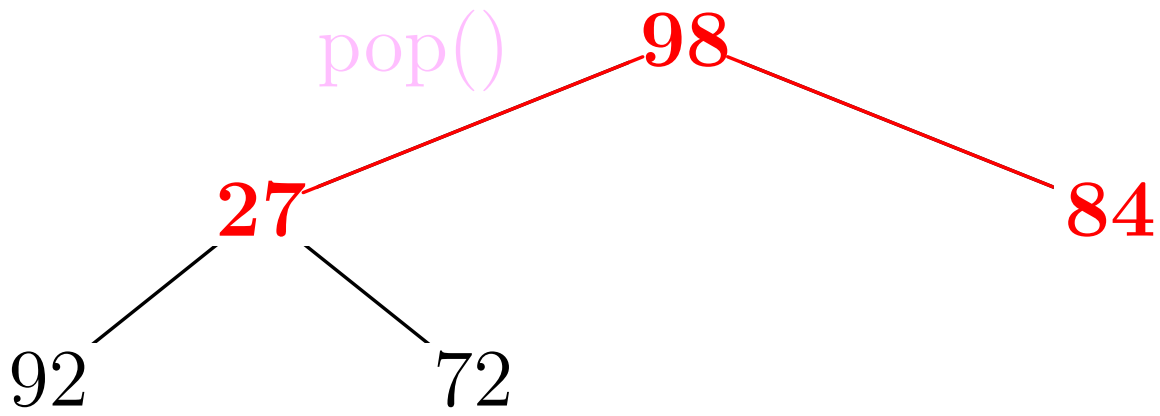

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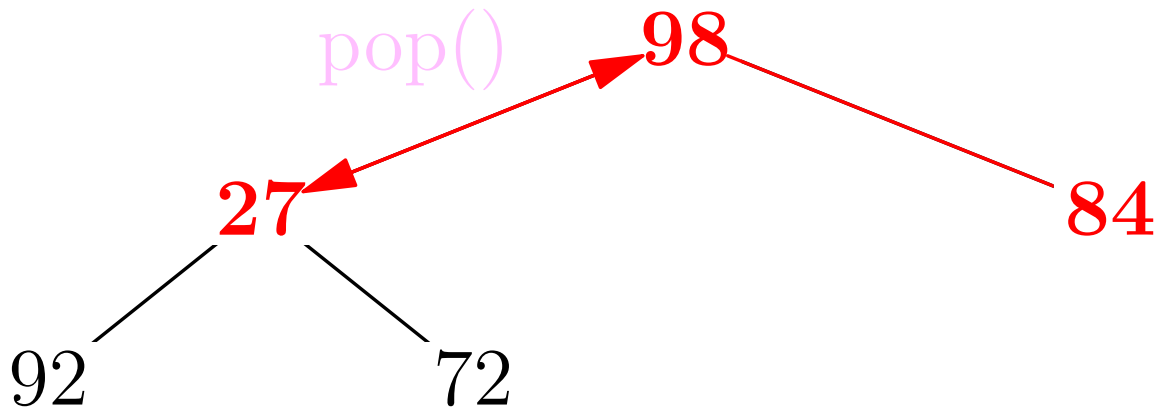

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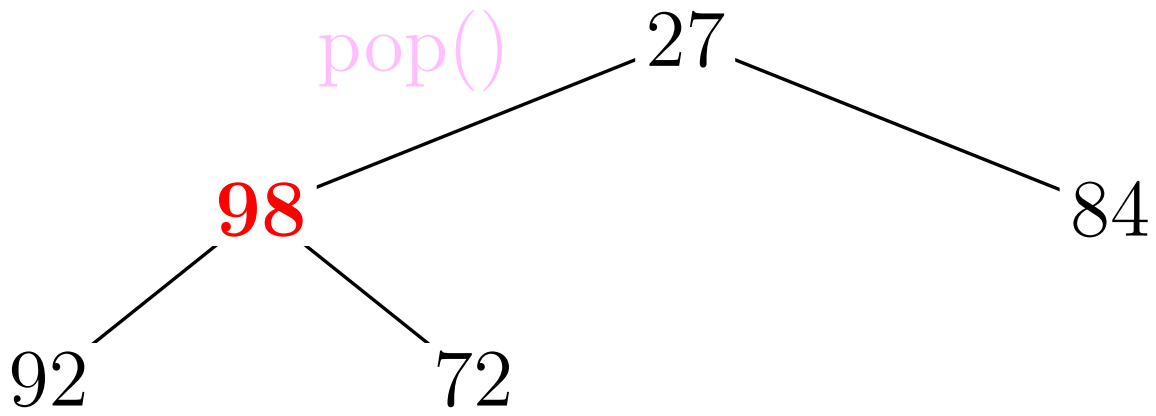

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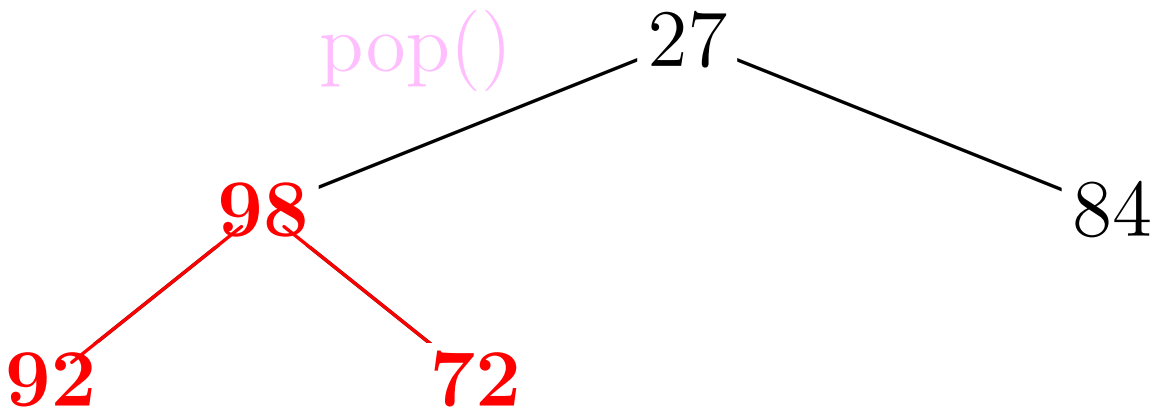

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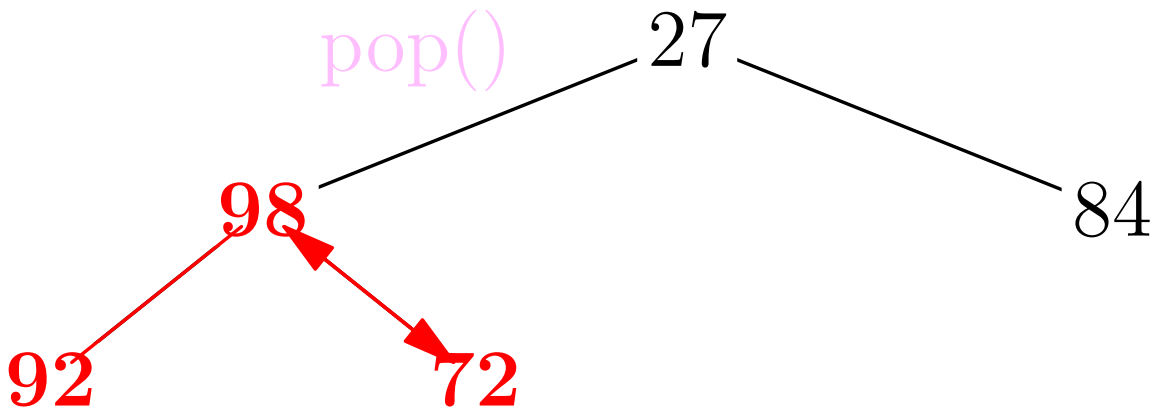

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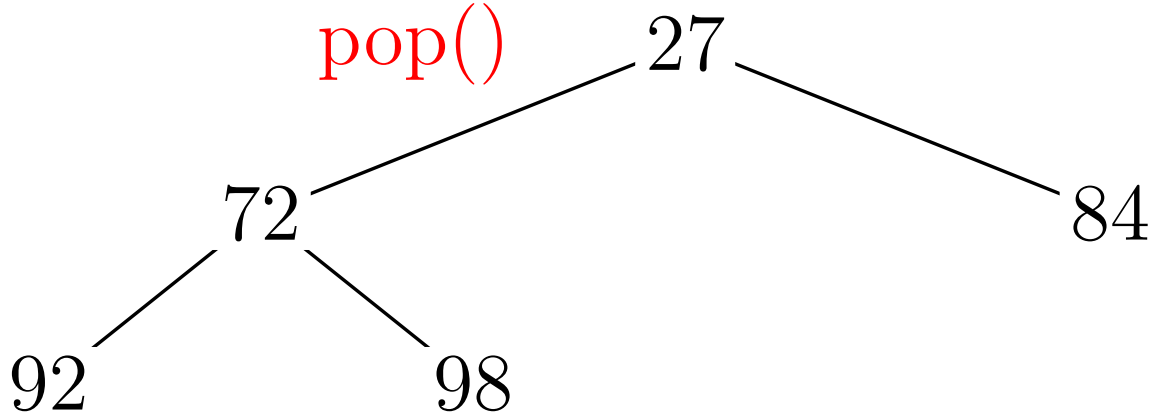

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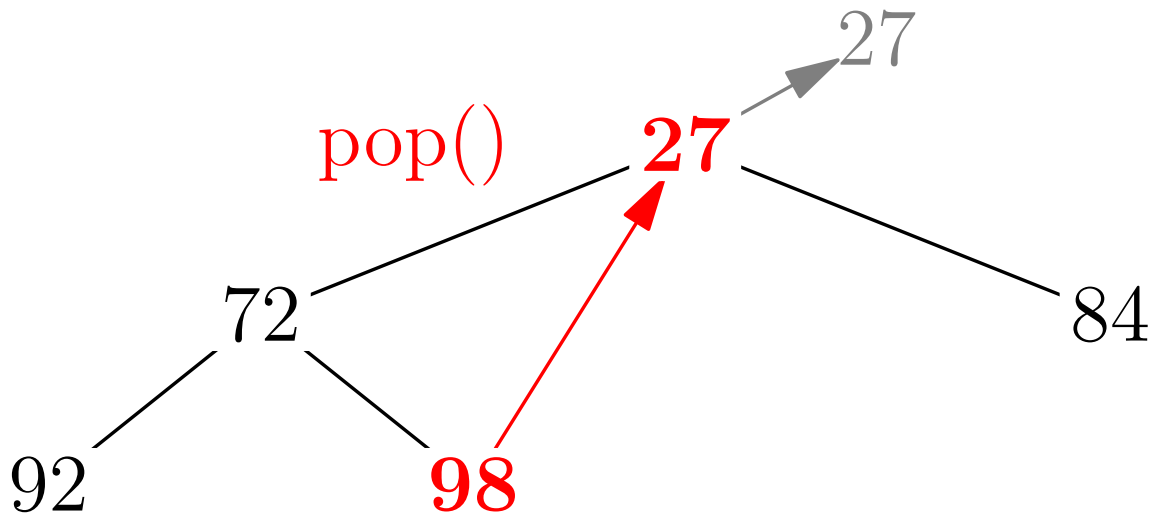

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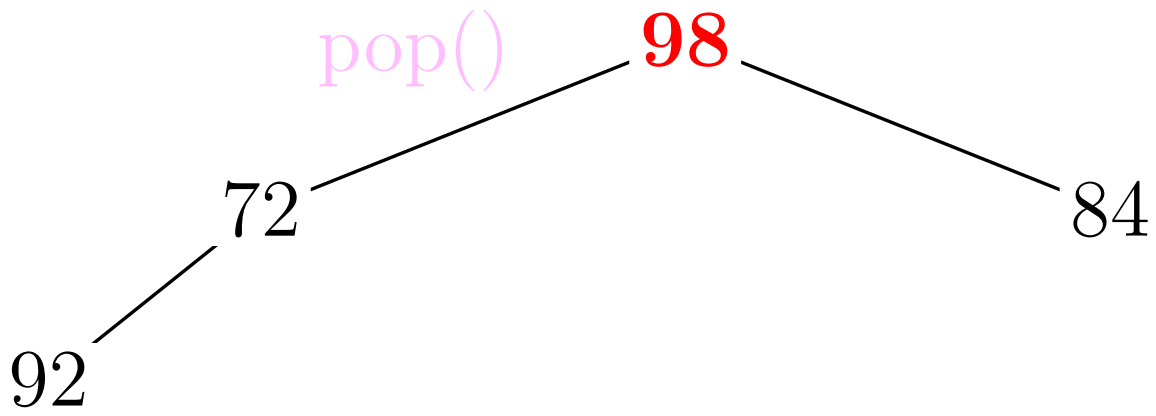

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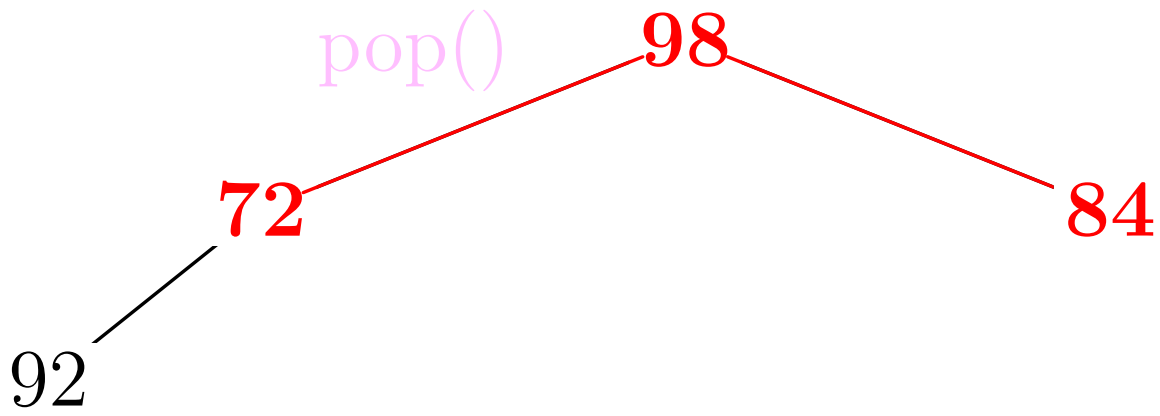

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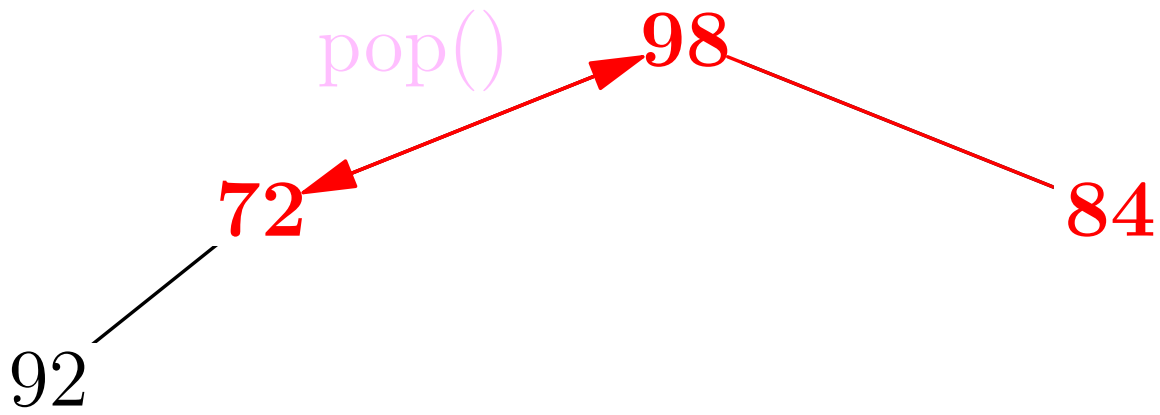

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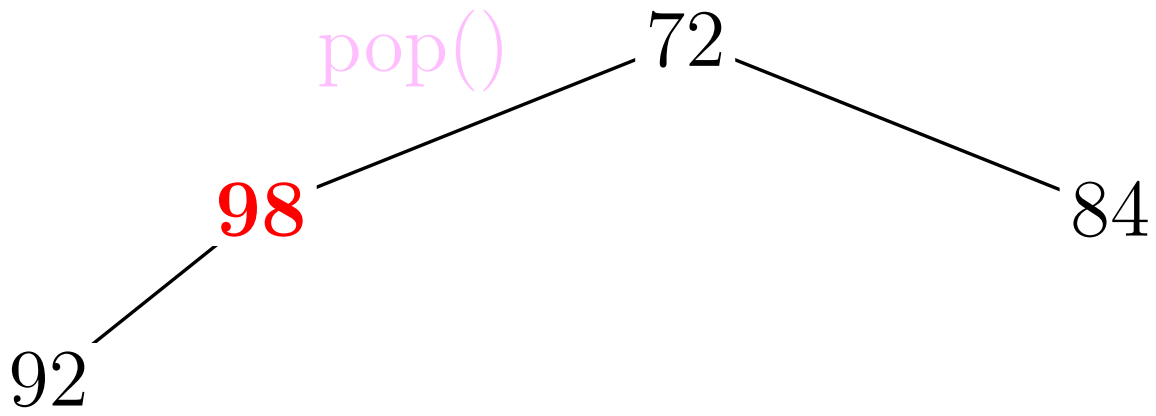

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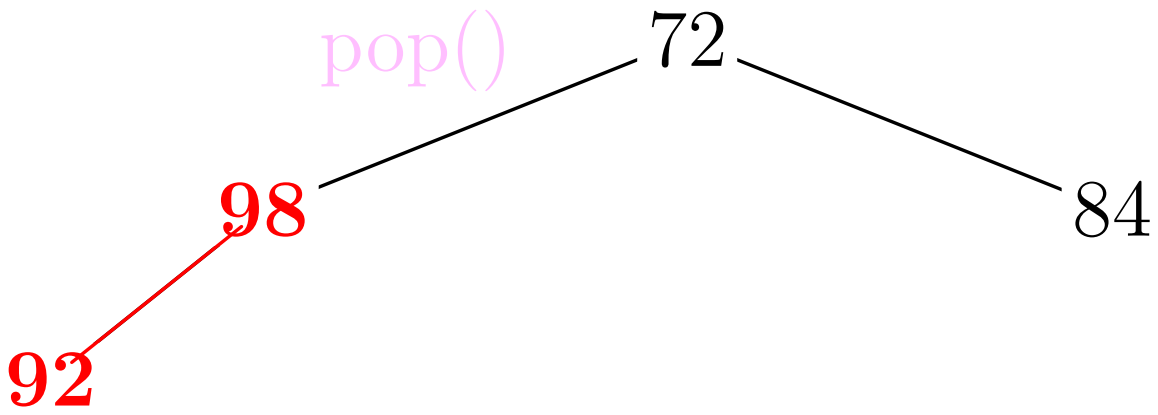

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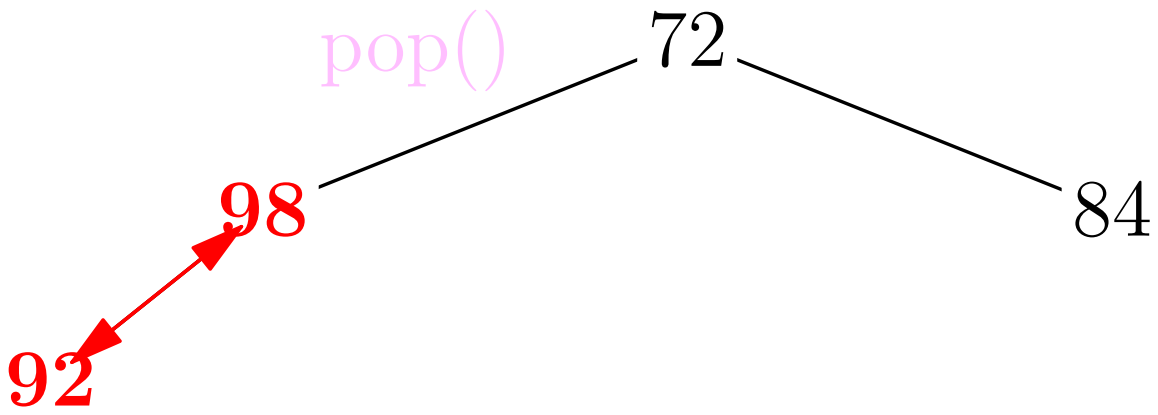

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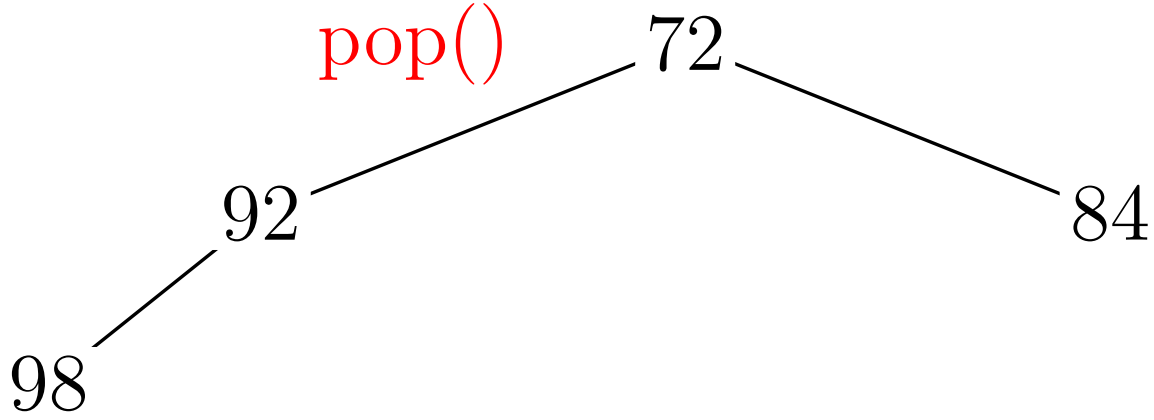

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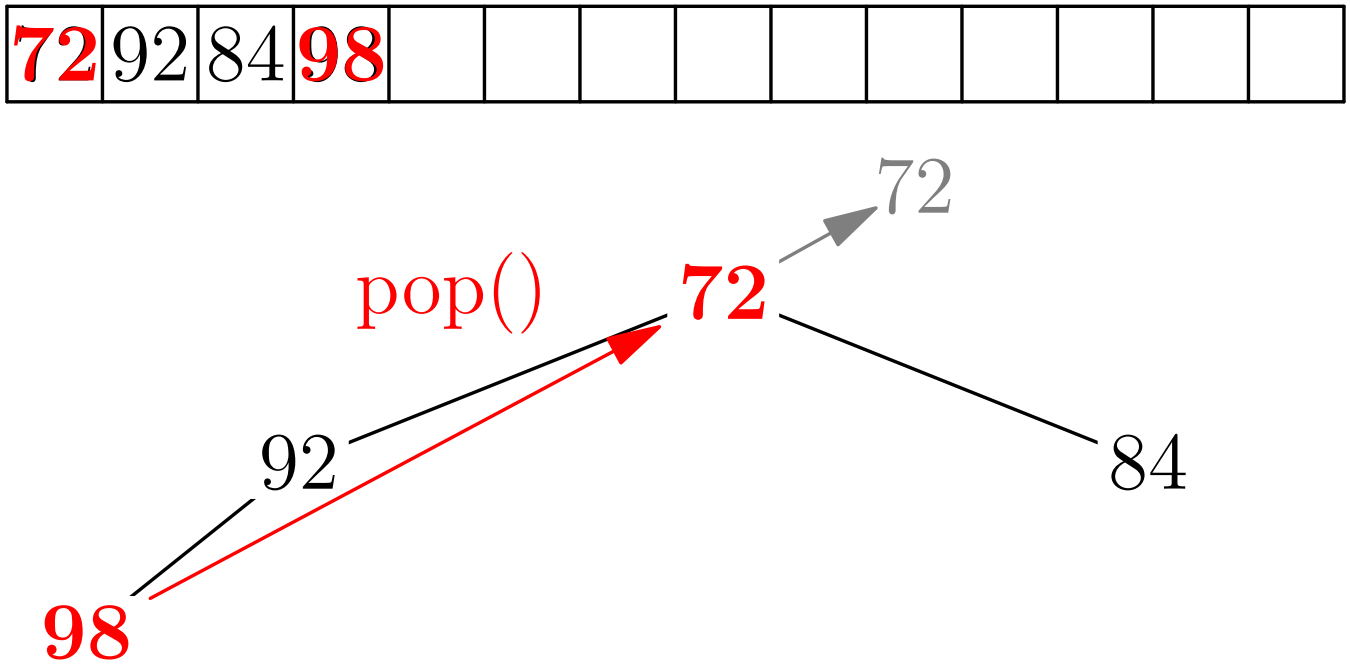

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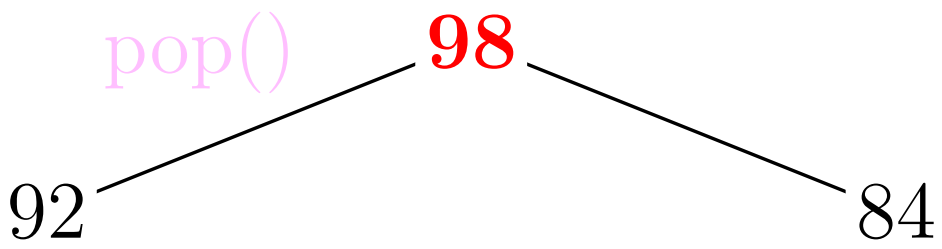

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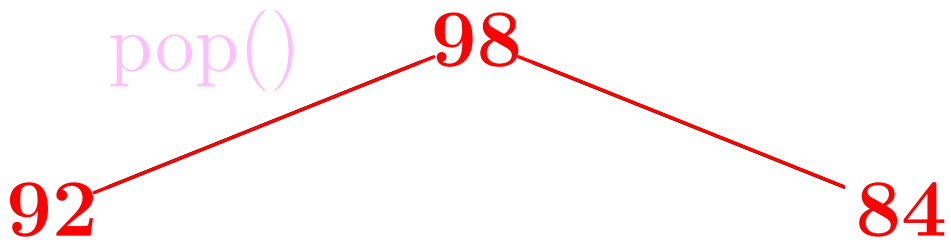

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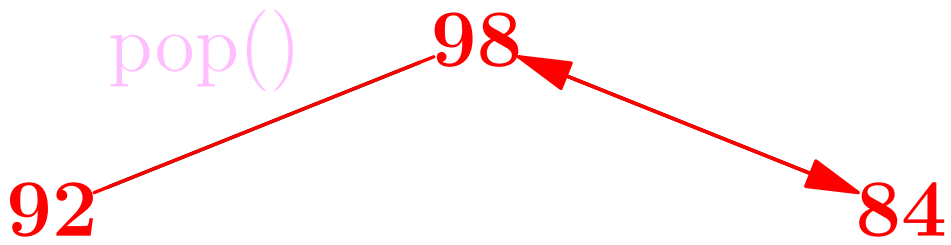

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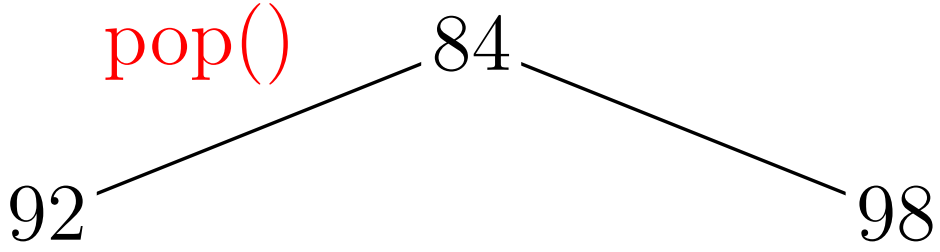

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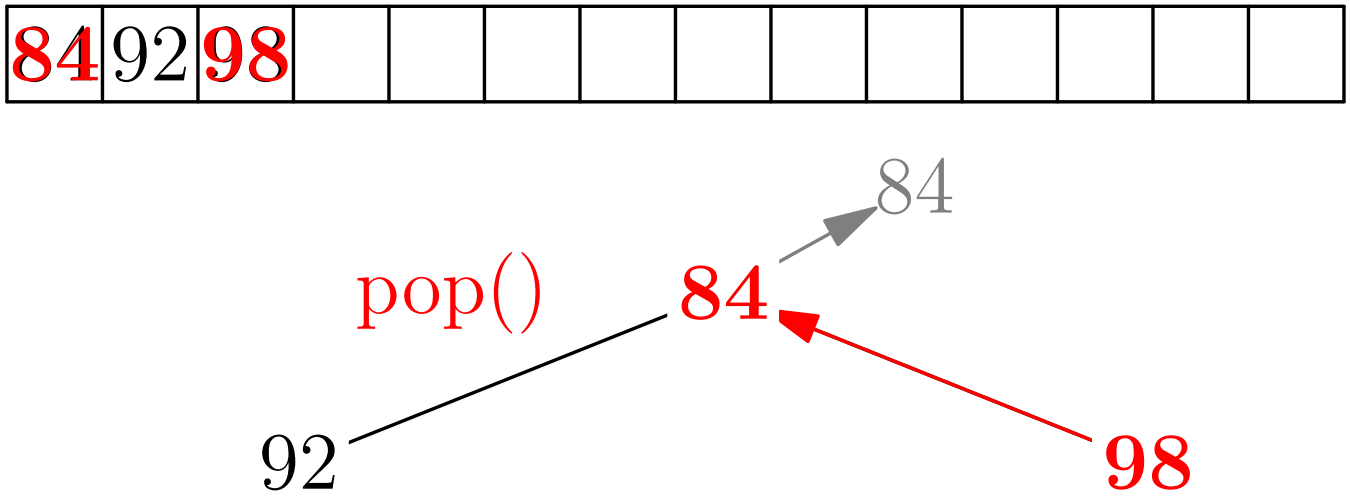

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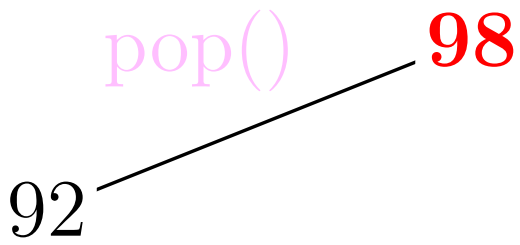

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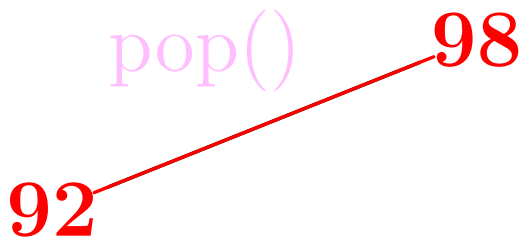

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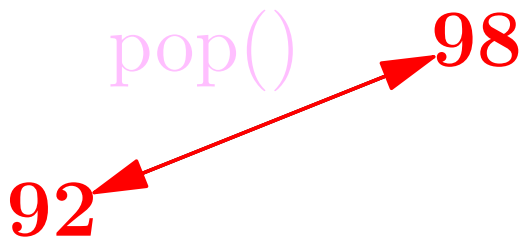

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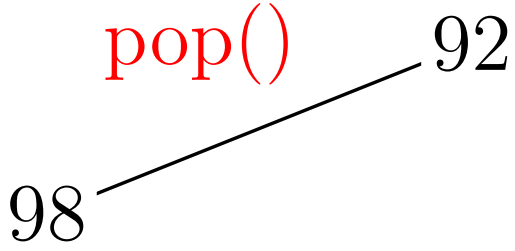

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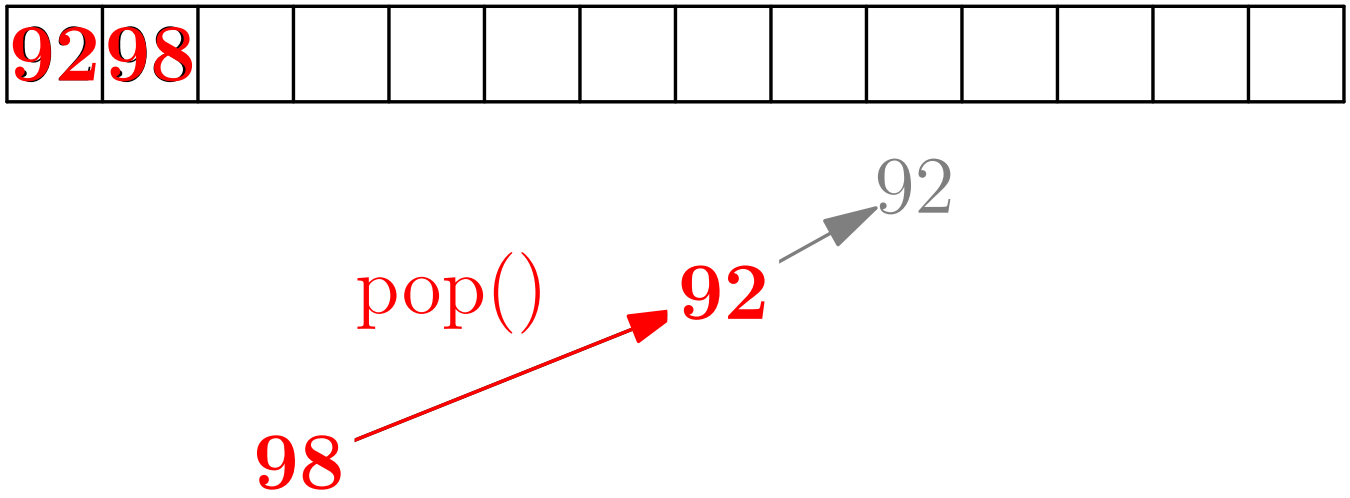

Heaps in Action

pop() 98

Heaps in Action

Time Complexity of Heaps

- The two important operation are `add` and `removeMin`
- These both either percolating an element up the tree or percolating an element down the tree
- The number of operations depends on the depth of the tree which is $\Theta(\log(n))$
- Thus `add` and `removeMin` are $O(\log(n))$
- Except `add` could also require resizing the array, but the amortised cost of this is low

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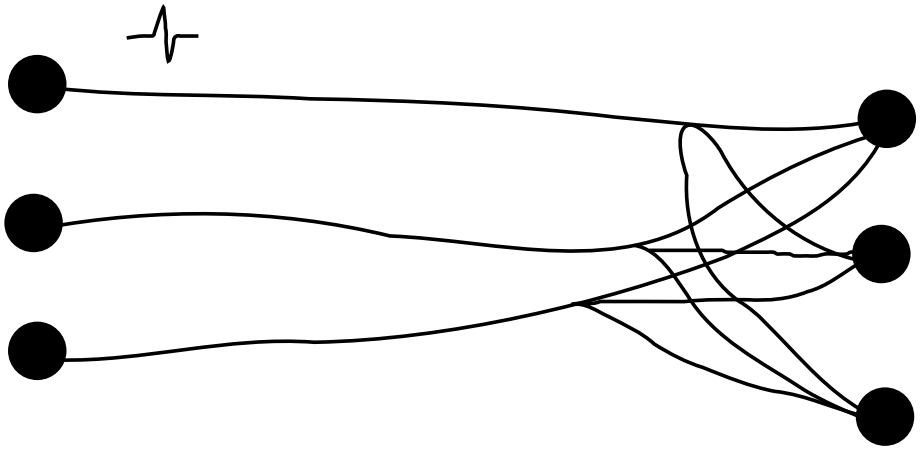
Real Time Simulation

- A nice application of priority queues is to perform real time simulations
- I was once modelling a neural network where neuron fired an impulse which would then be received by other neurons



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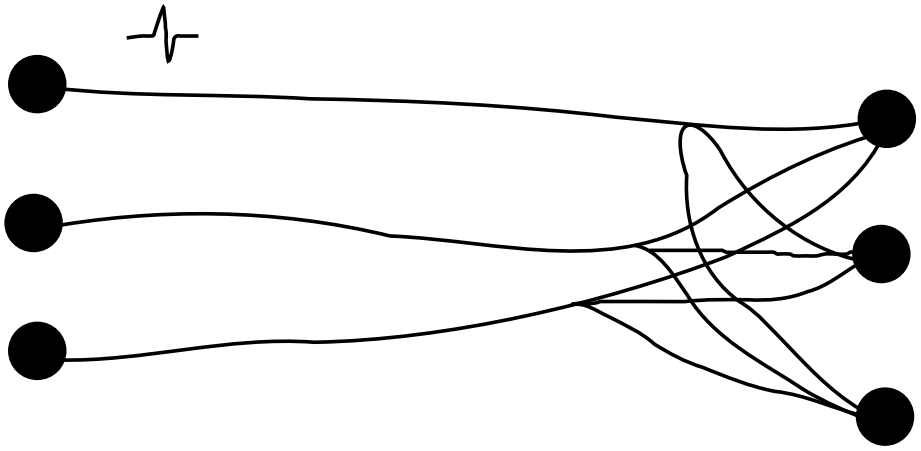

Synchronised Firing

- We wanted to show that if a group of neurons fired together they could make another group of neurons fire in synchrony despite the fact that it would take different times for the receiving neurons to feel the pulse (due to the different lengths of the axons)
- A famous Israeli group had "proved" this couldn't happen
- Using a priority queue we modelled the neurons
  - ⋆ When a neuron fired the receiving neurons would be put on a priority queue according to when they received the pulse
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Efficient Modelling

- Using a priority queue meant we knew when the next event would happen
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Outline

1. Heaps
2. Priority Queues
   - Array Implementation
3. **Heap Sort**
4. Other Heaps

Heap Sort

- A priority queue suggests a very simple way of performing sort
- We simply add elements to a heap and then take them off again

```
template <typename T>
void sort(vector<T> aList)
{
    HeapPQ<T> aHeap = new HeapPQ<T>(aList.size());
    for (T element: aList)
      aHeap.push(element, element);

    aList.clear();
    while(aHeap.size() > 0) {
      aList.push_back(aHeap.top());
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- Note that this is not an in-place sort algorithm (i.e. it uses lots of memory)

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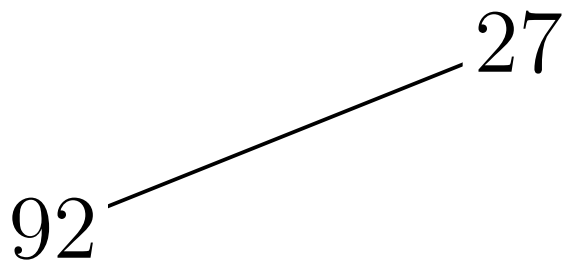

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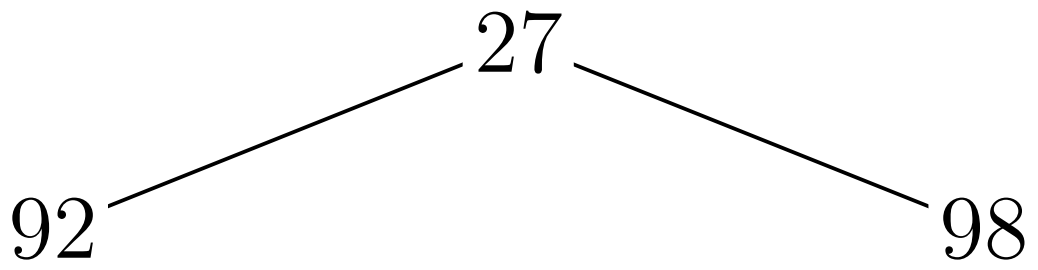

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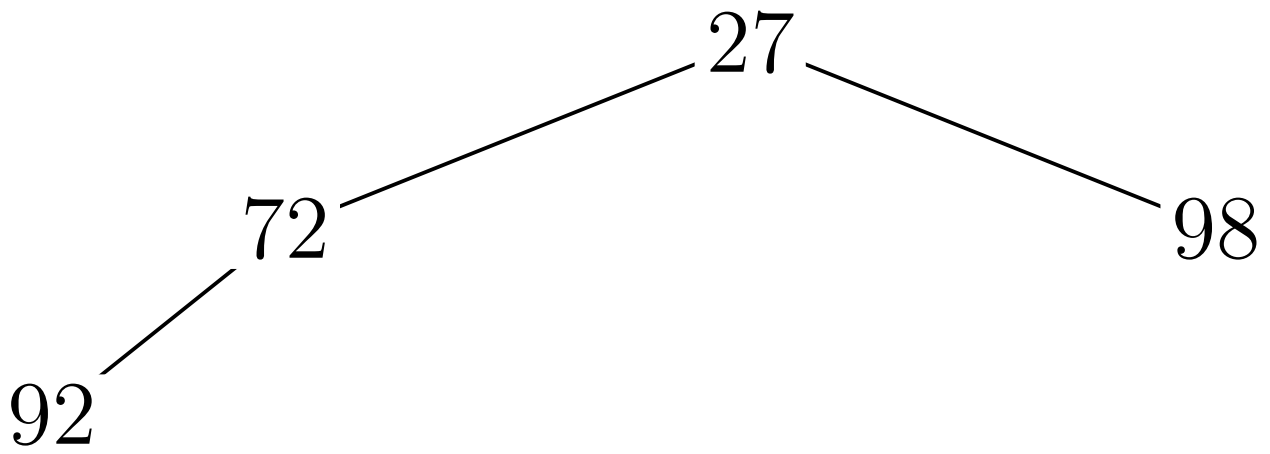

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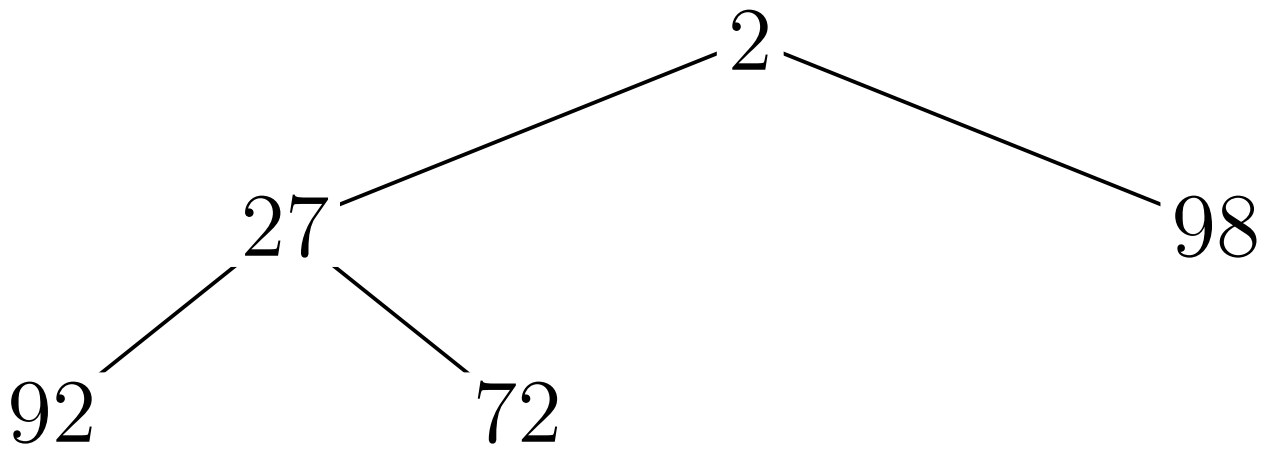

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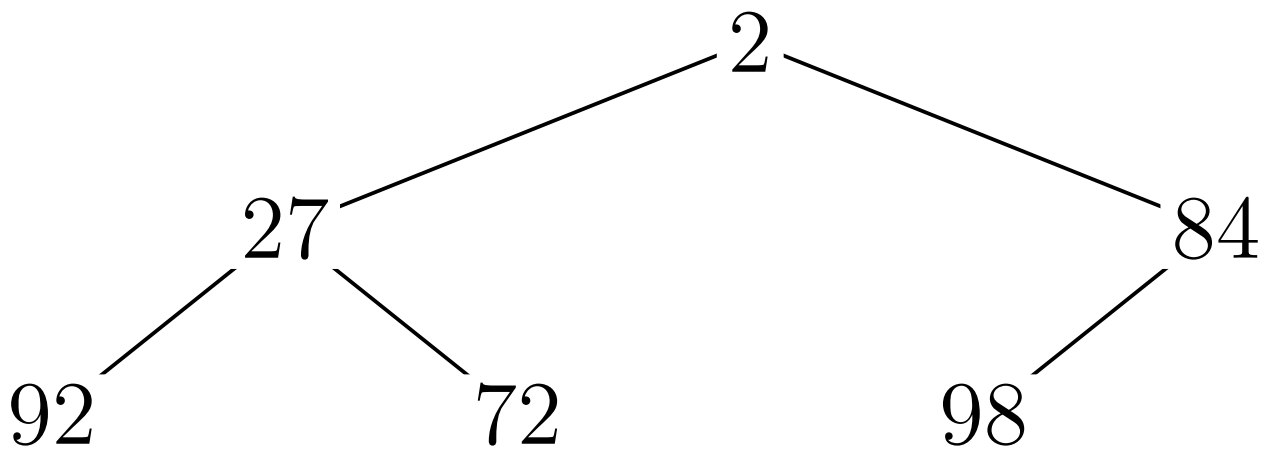

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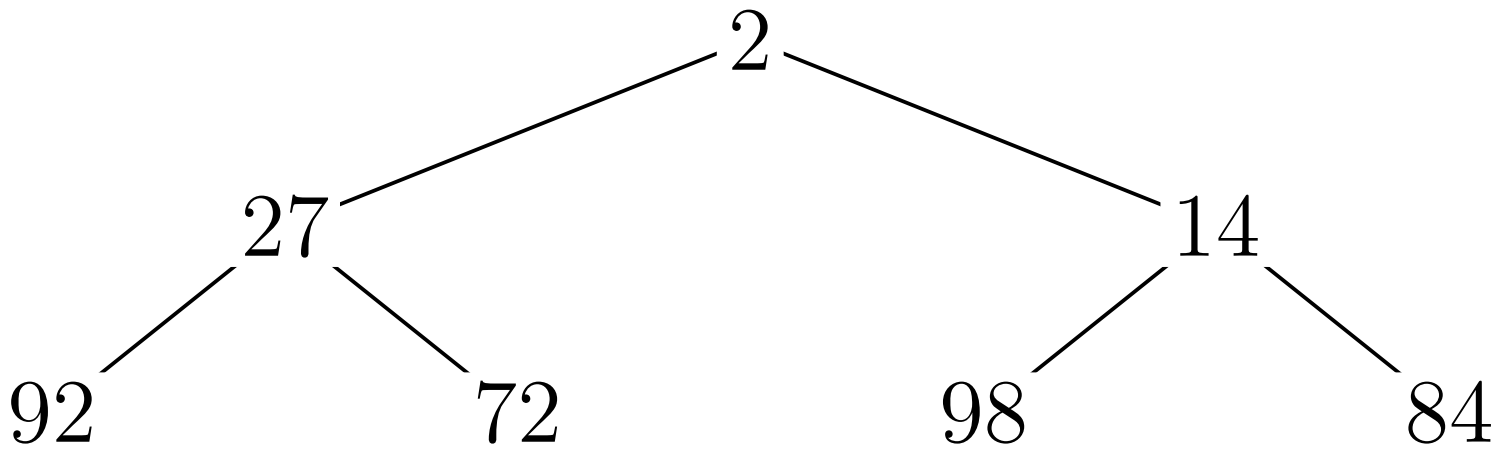

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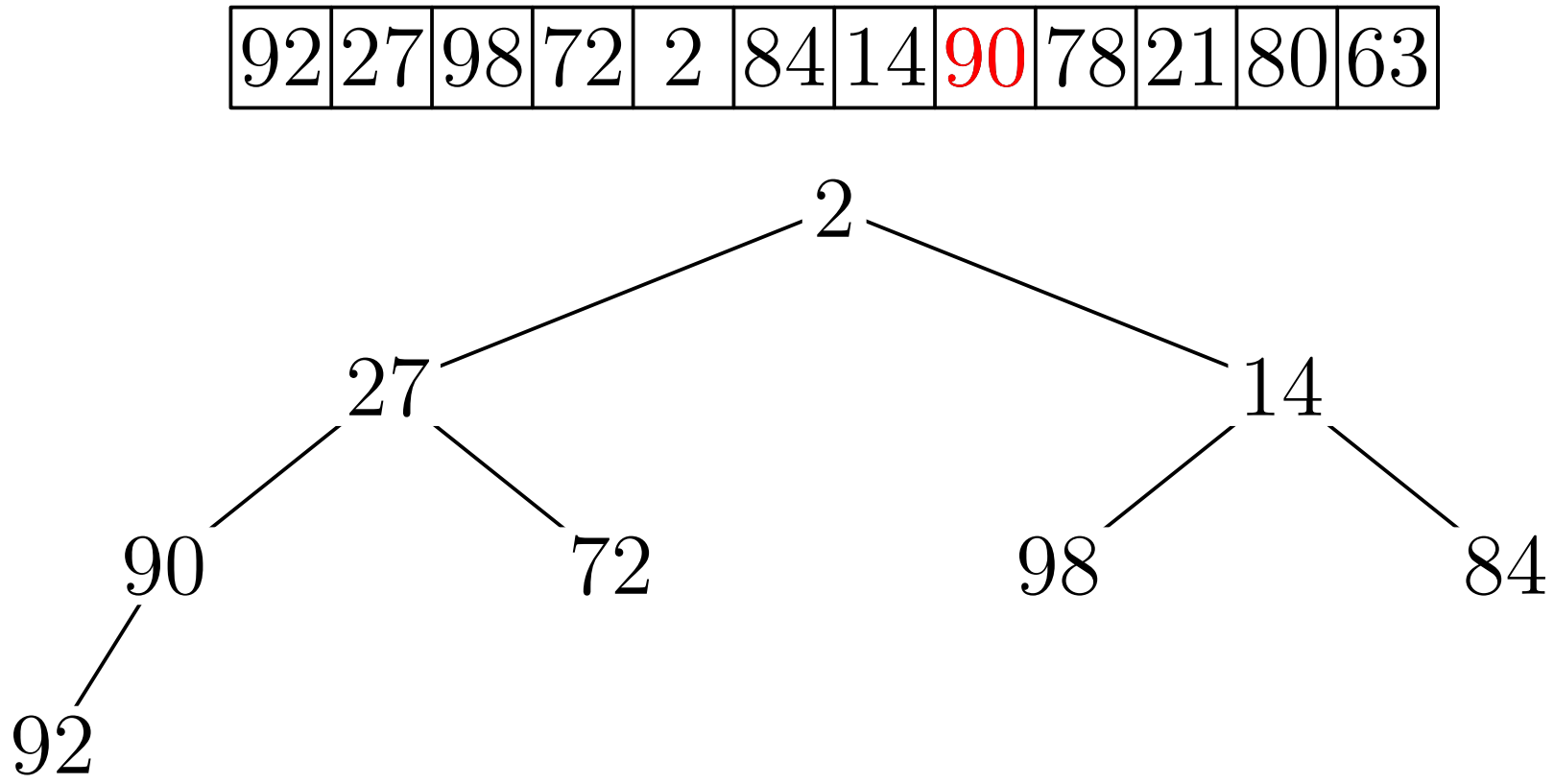

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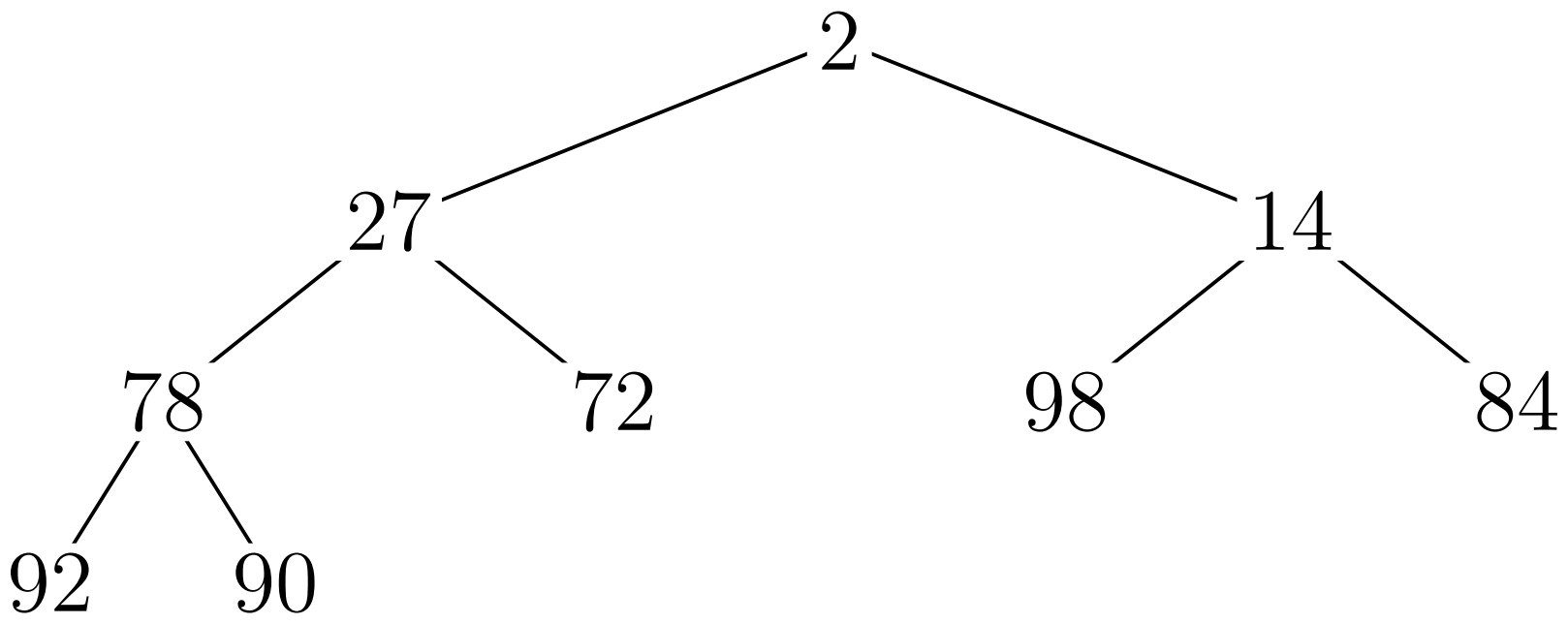

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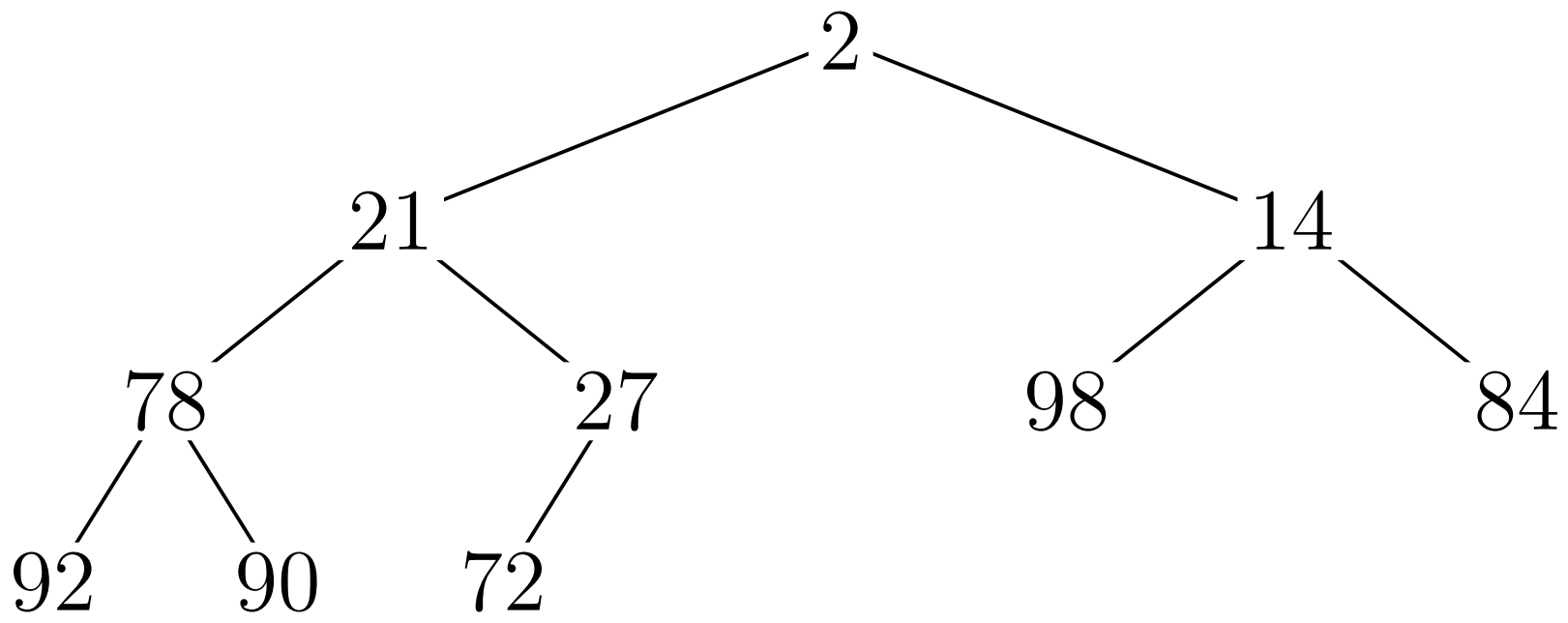
92	27	98	72	2	84	14	90	78	21	80	63



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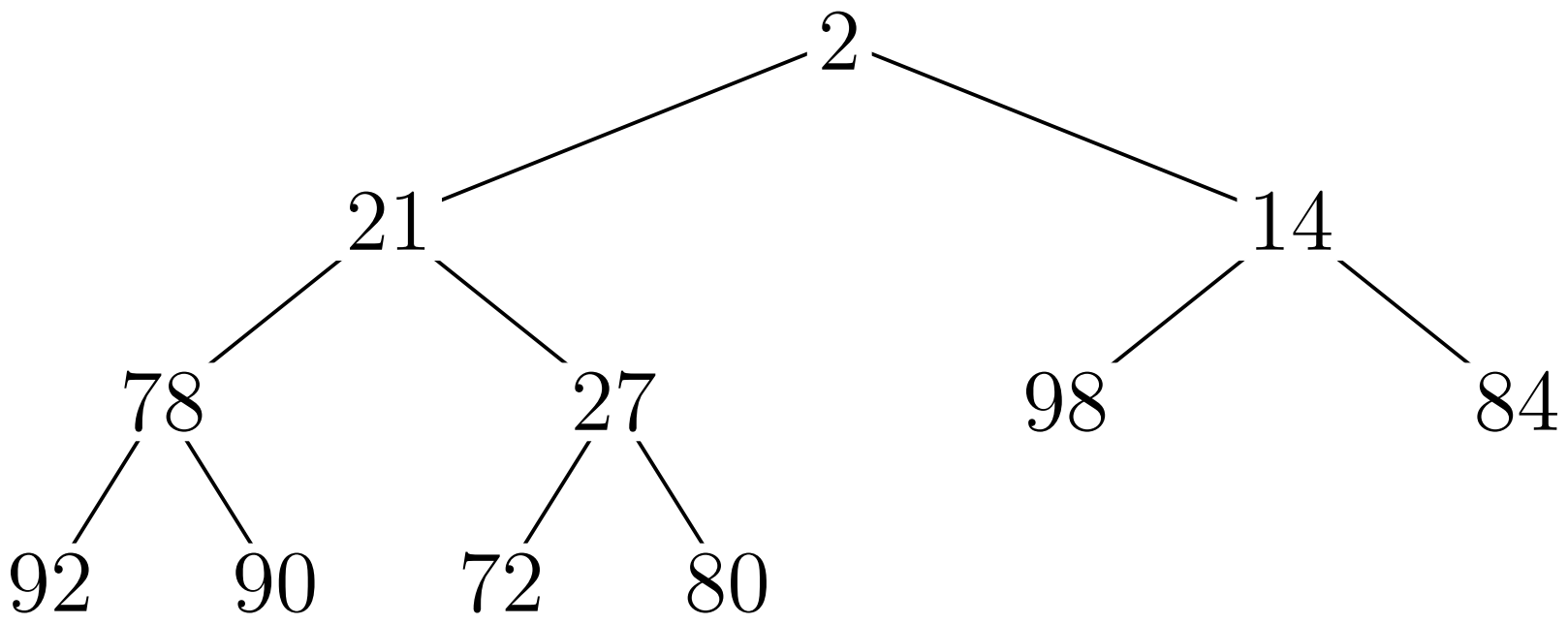

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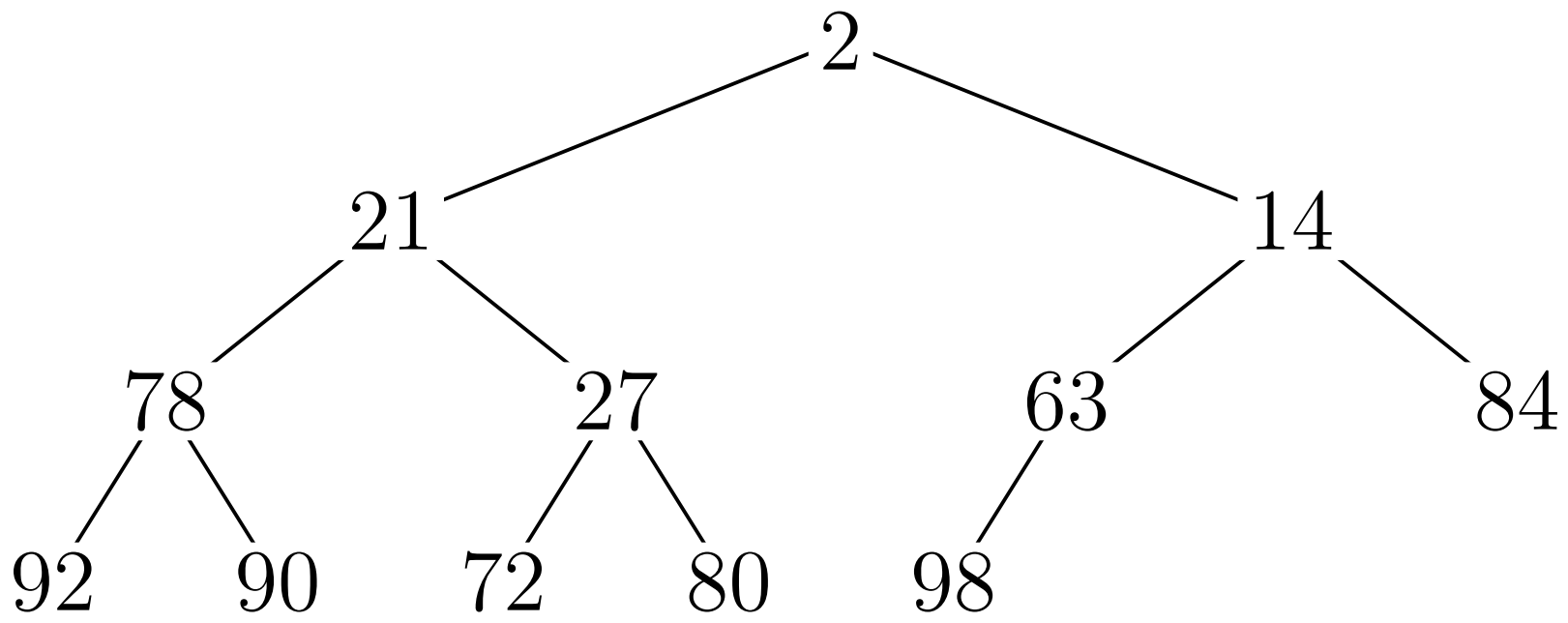

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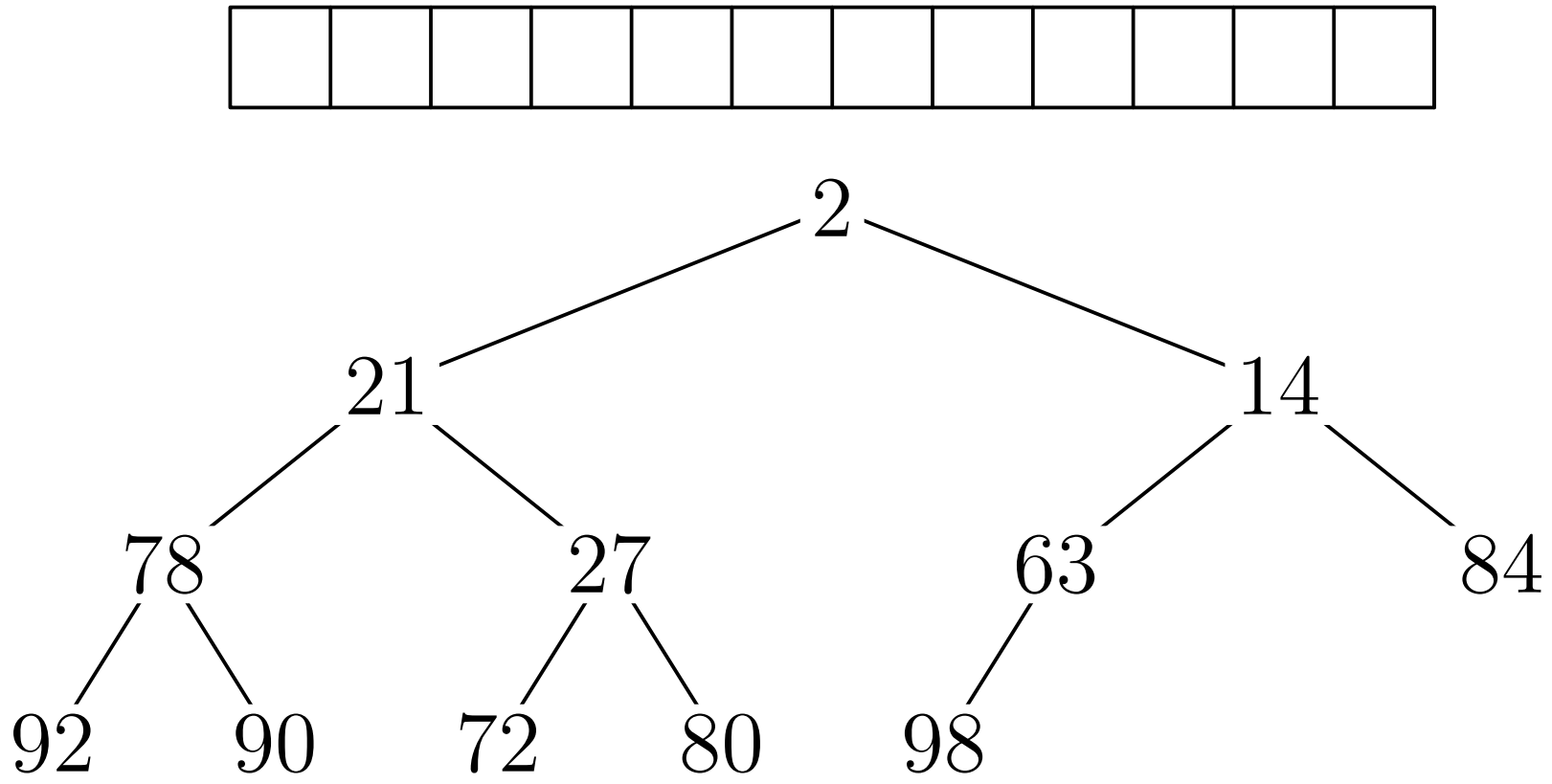

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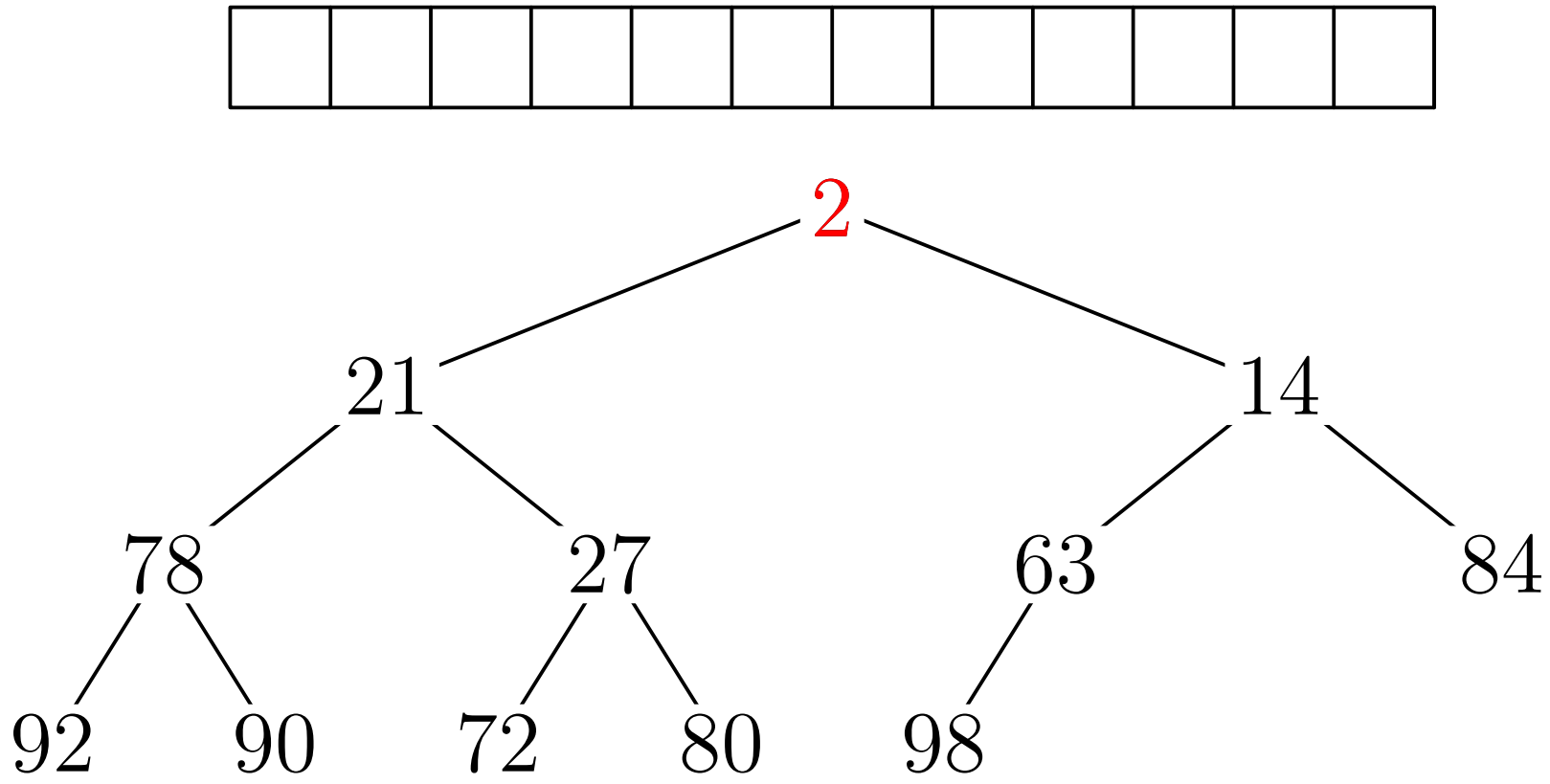

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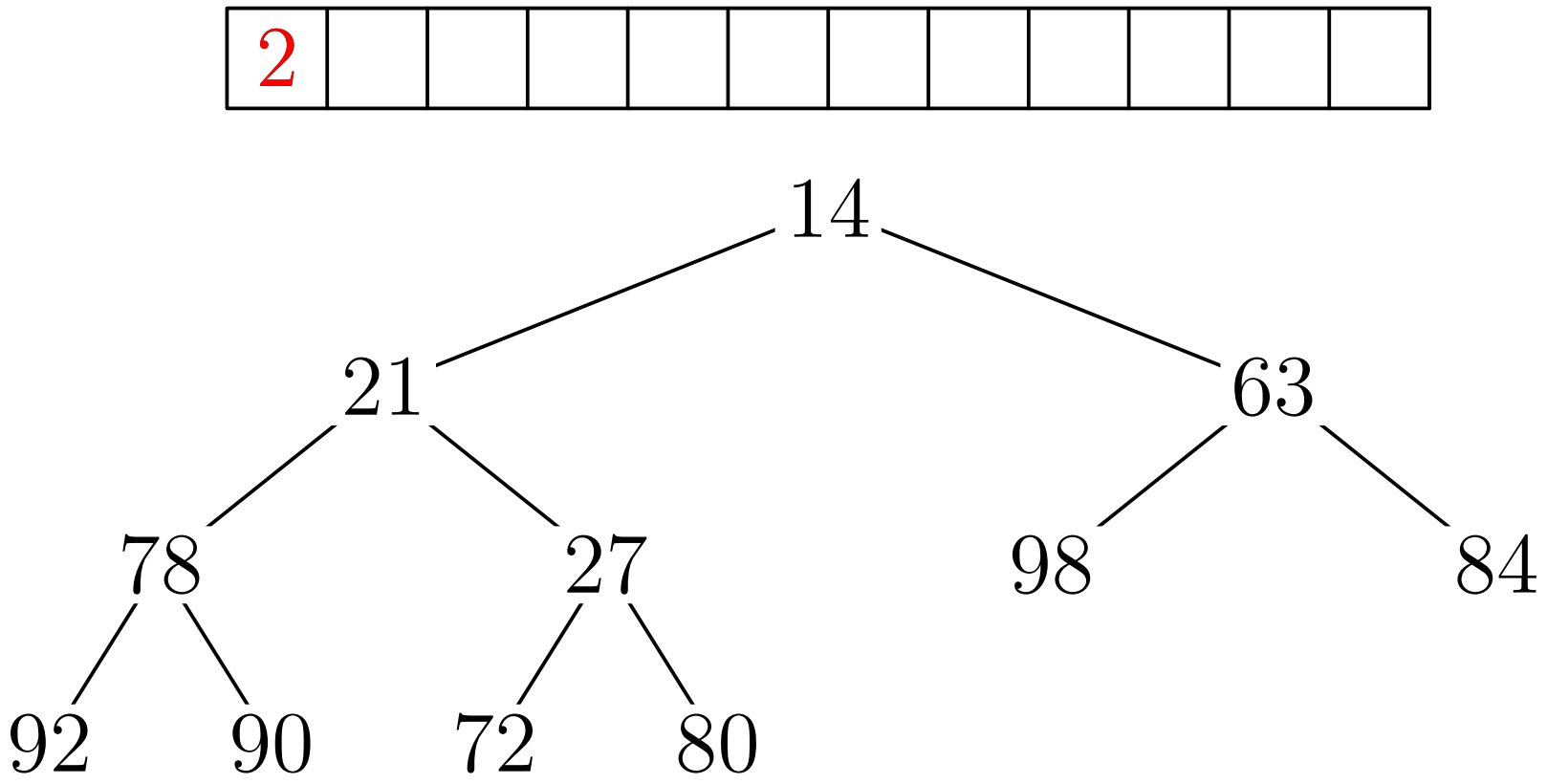

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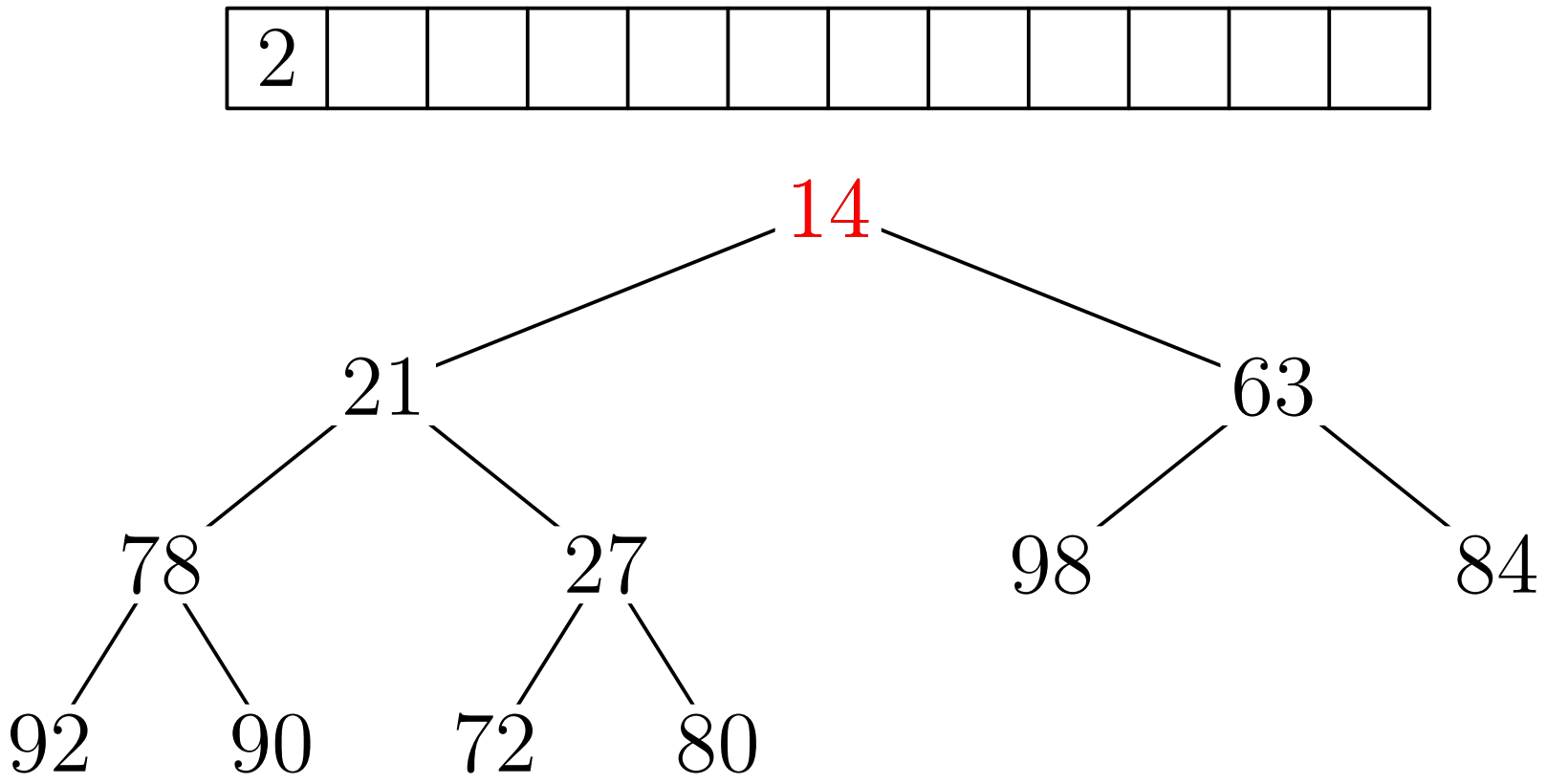

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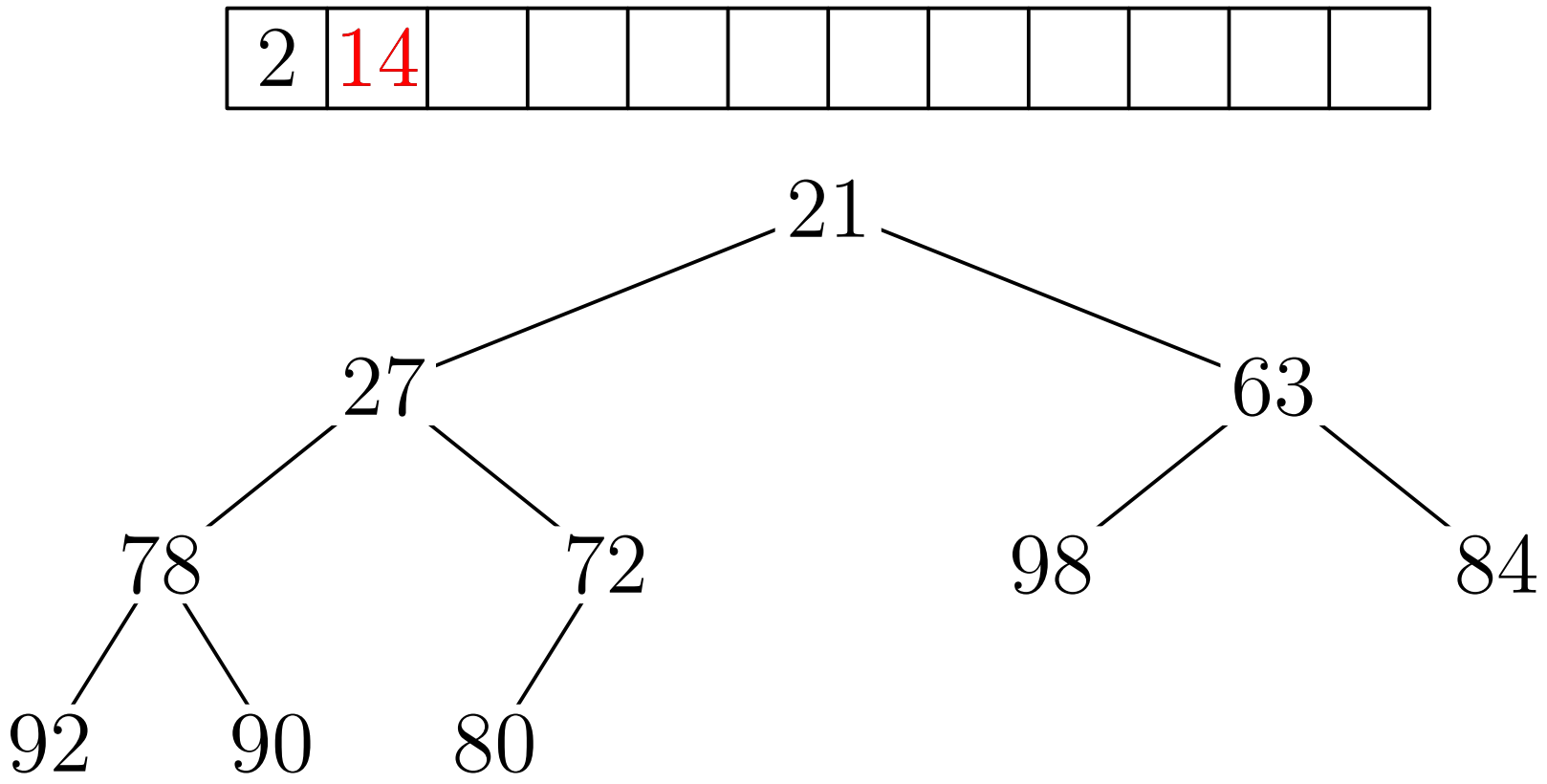

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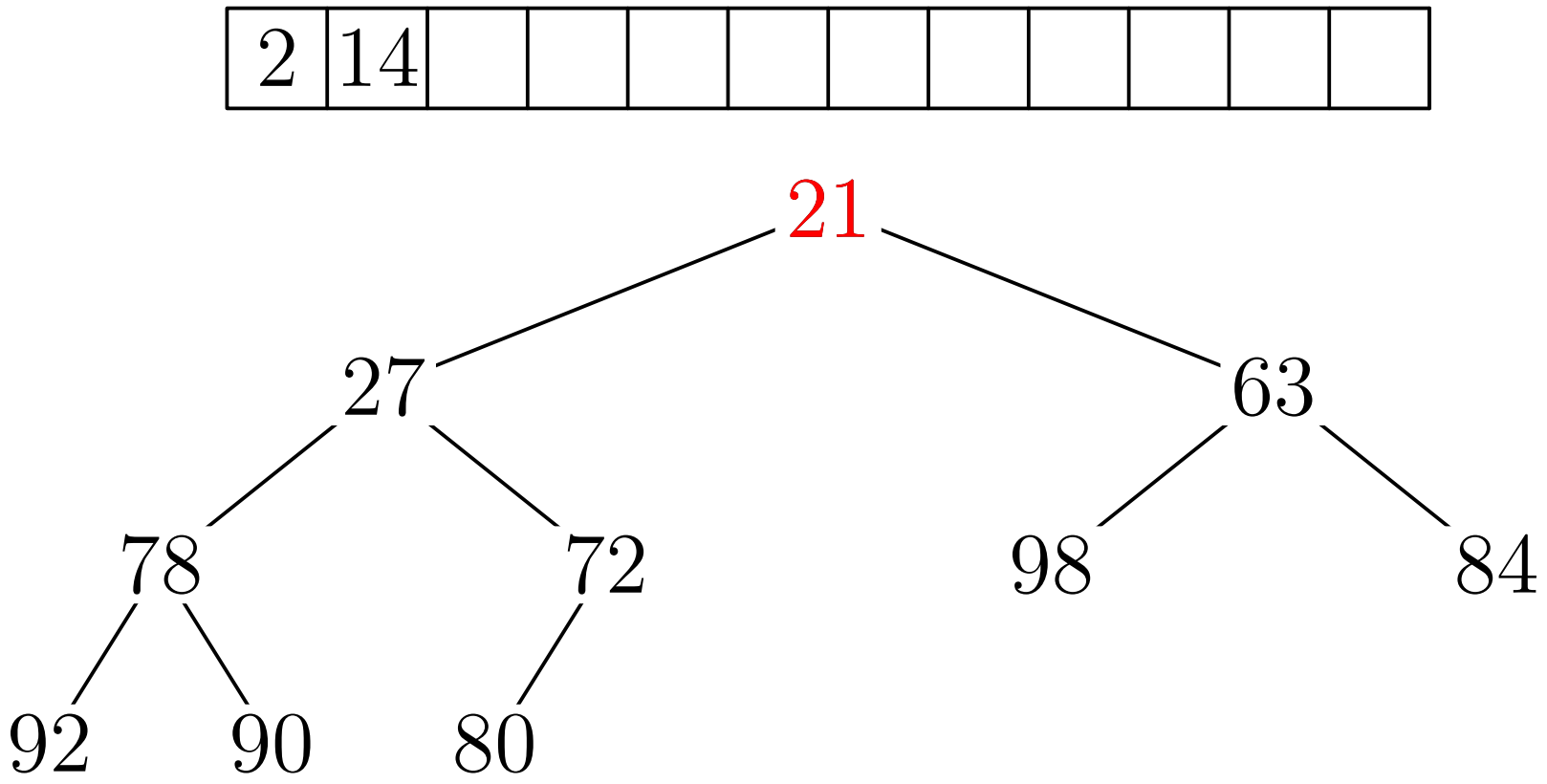

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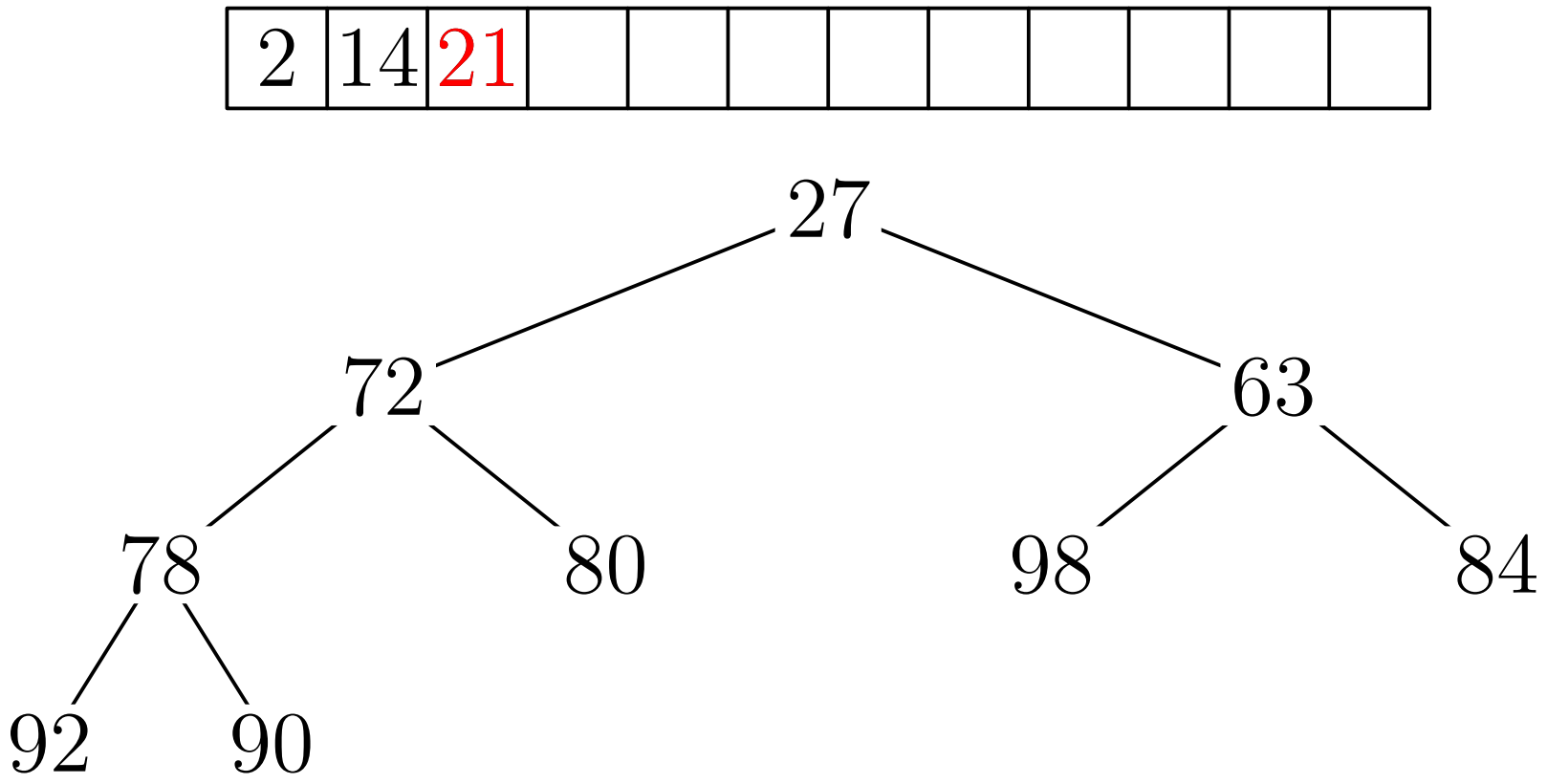

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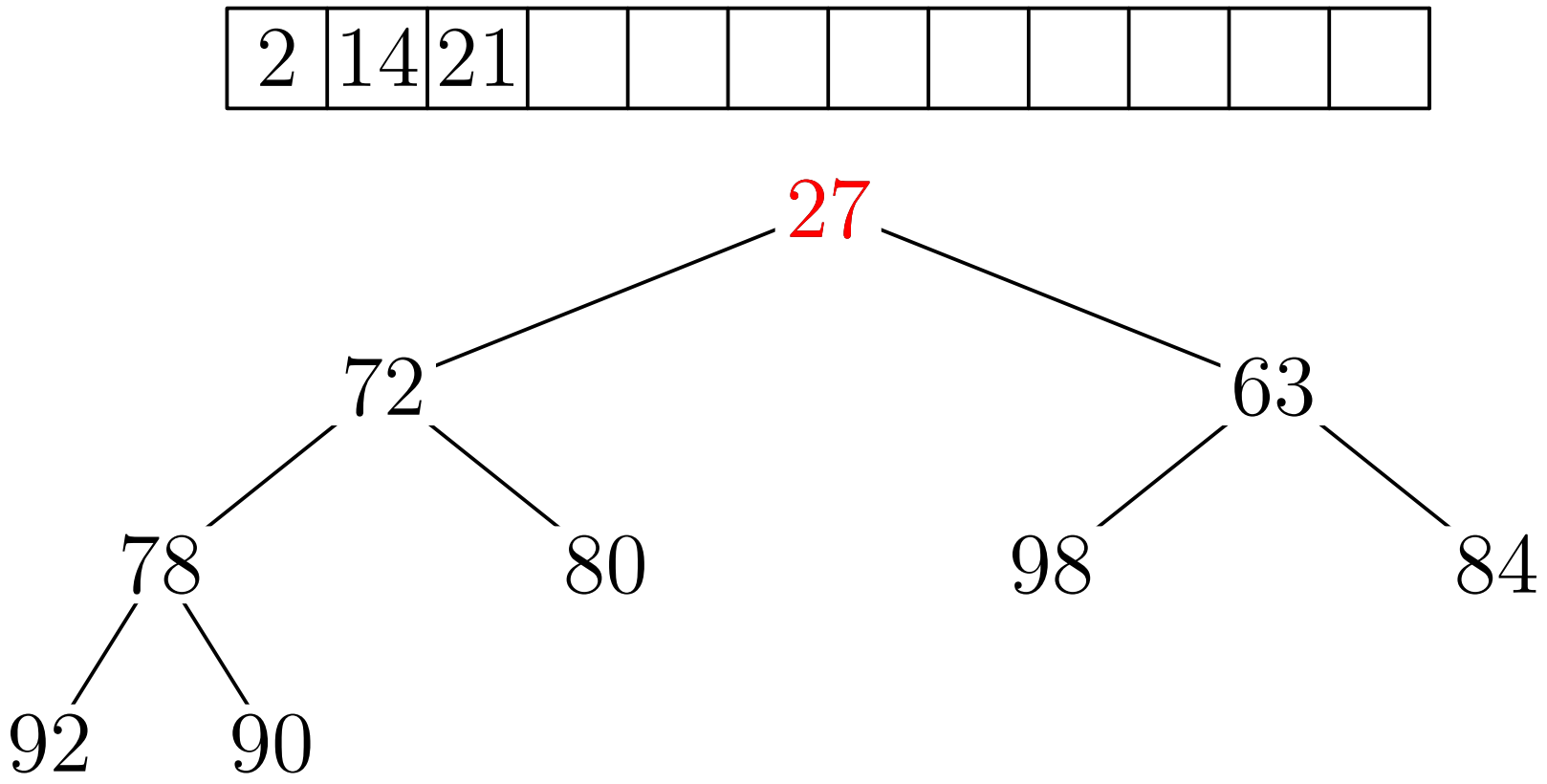

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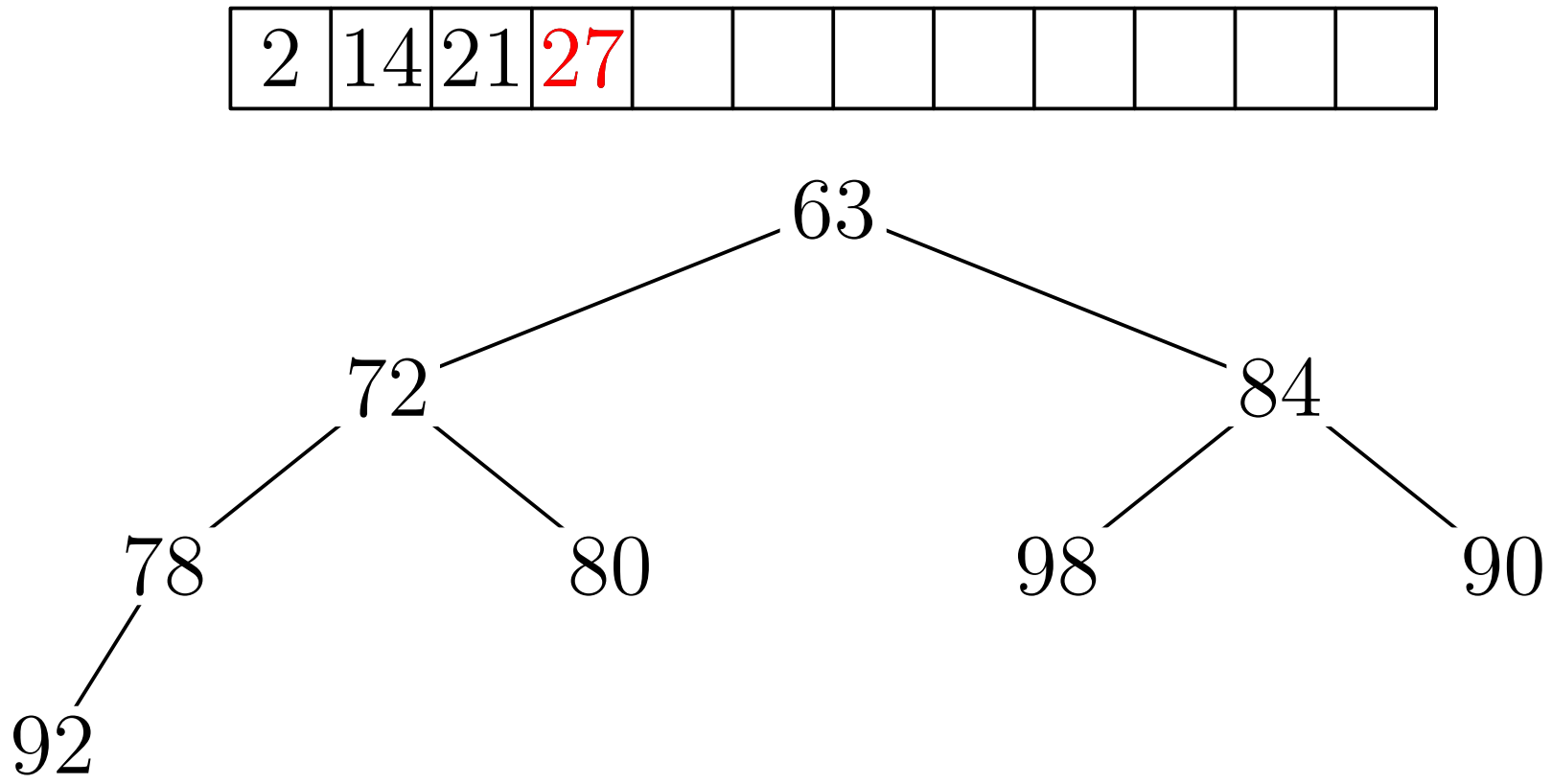

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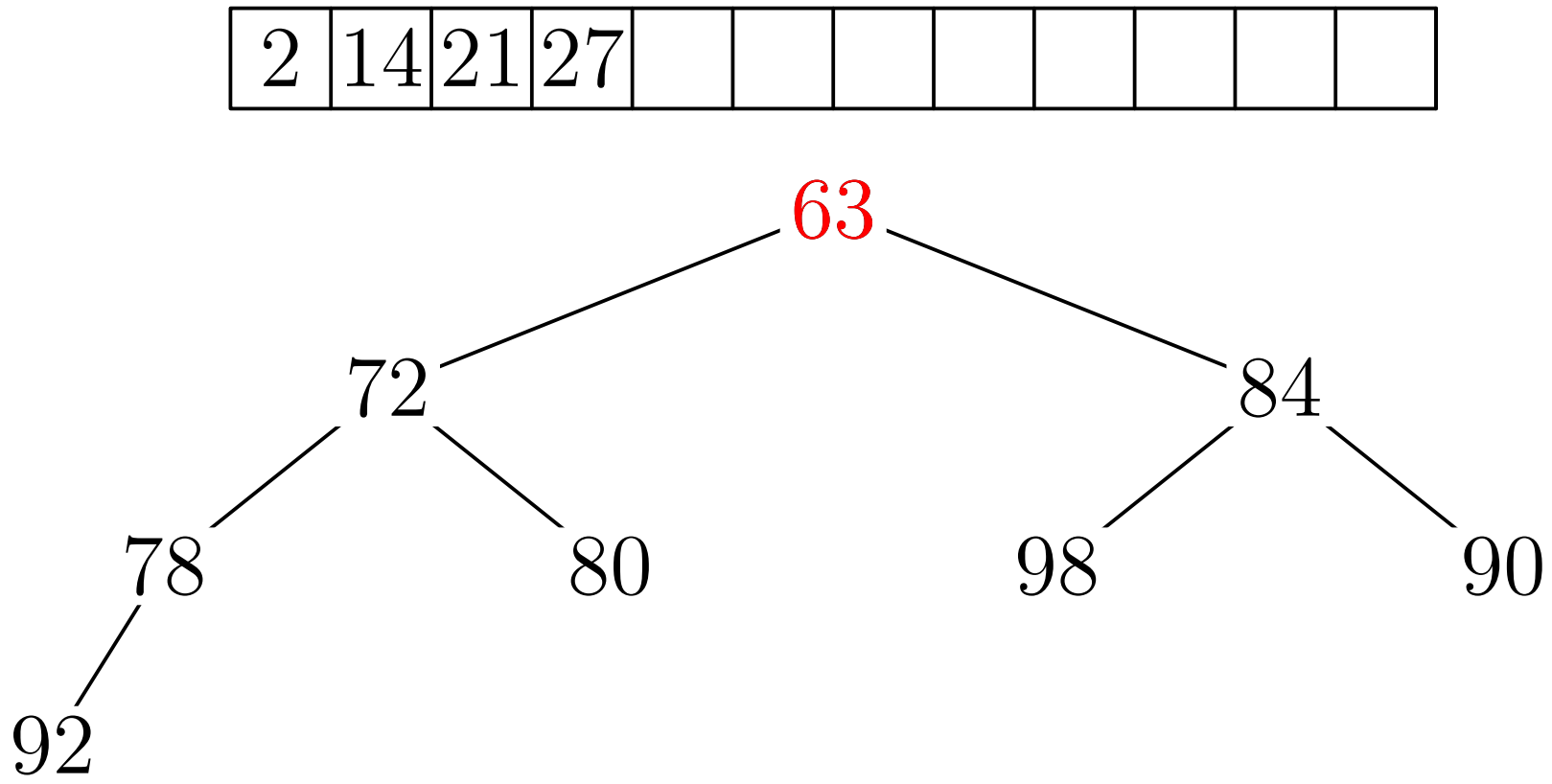

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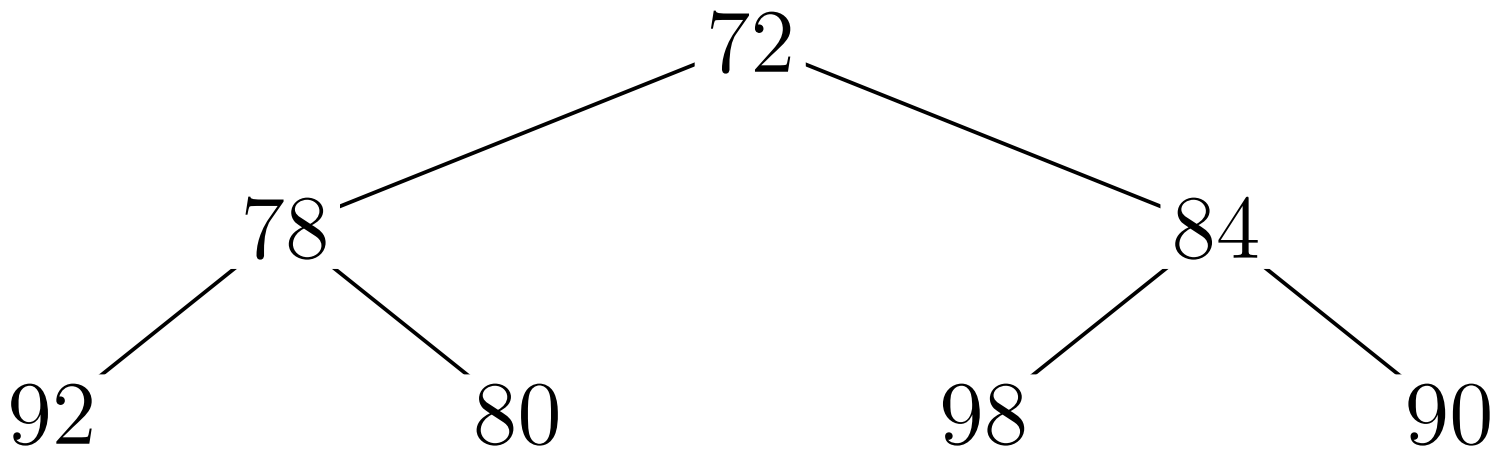

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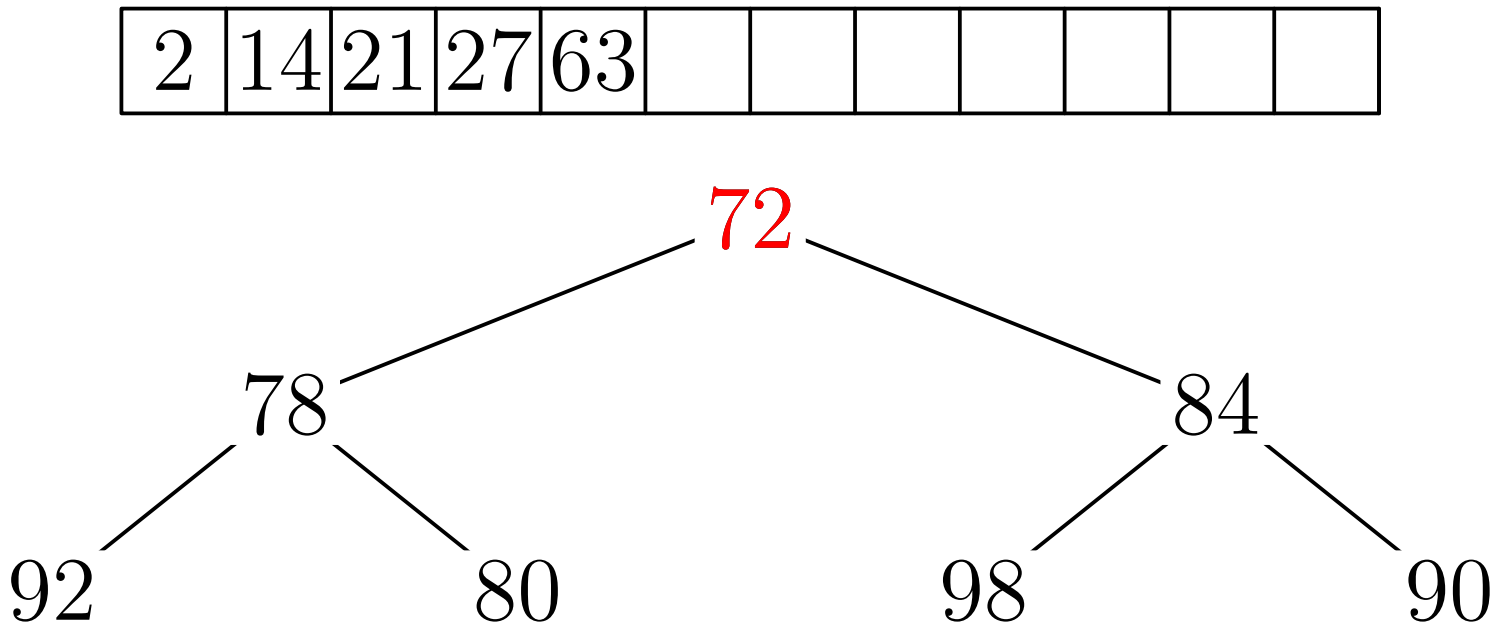

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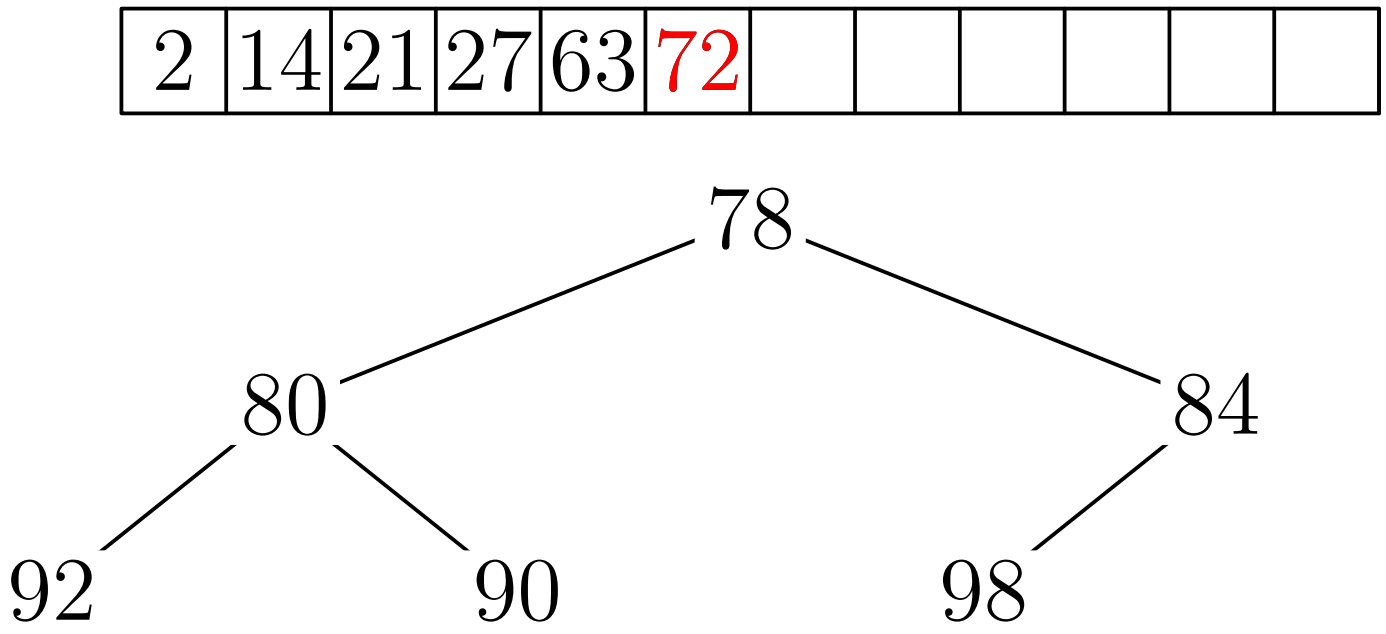

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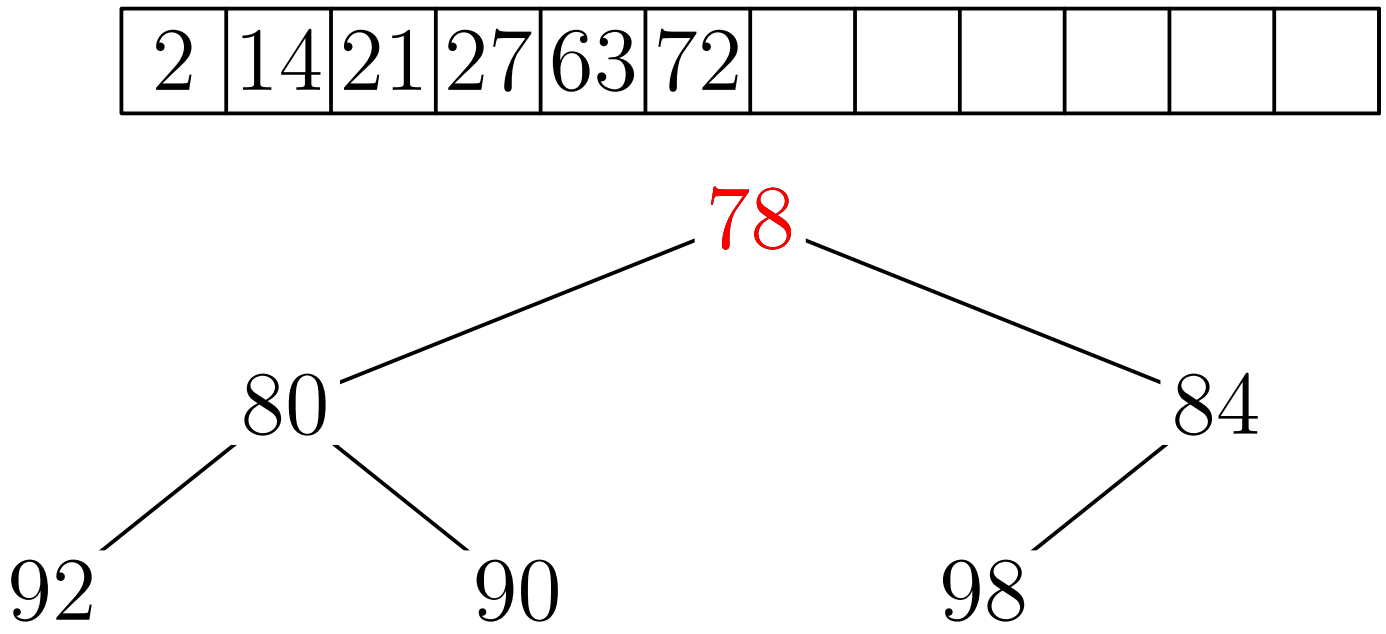

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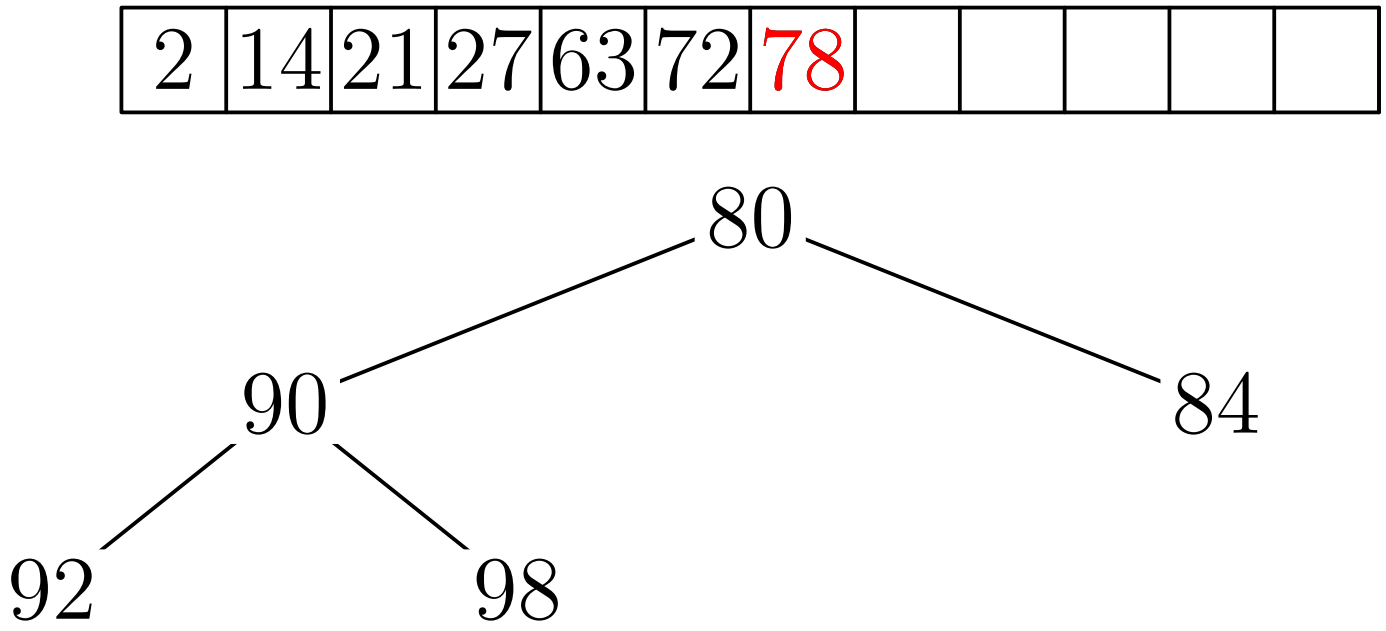

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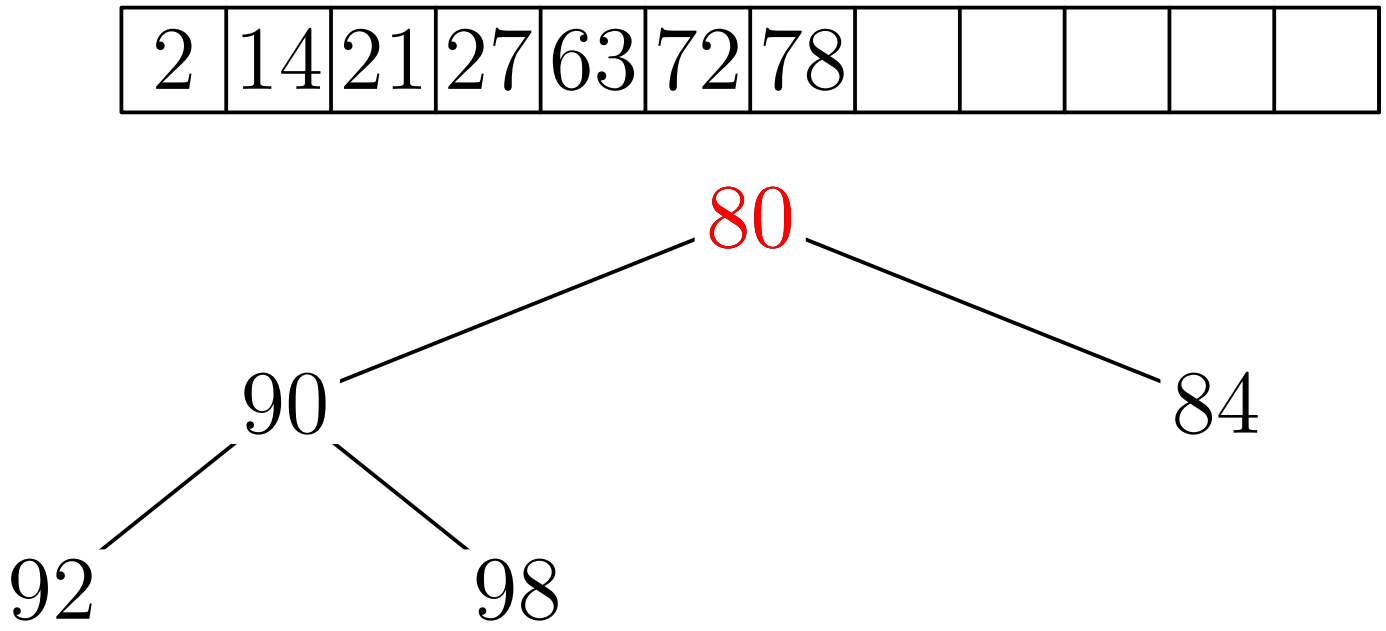

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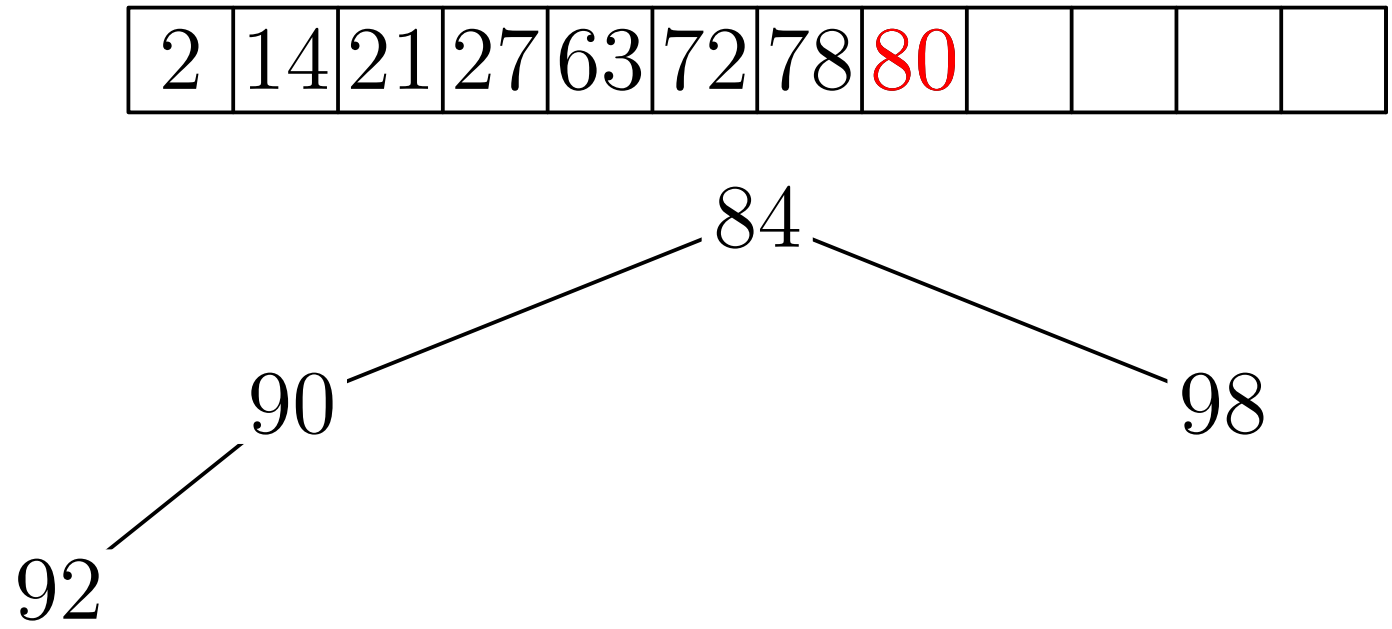

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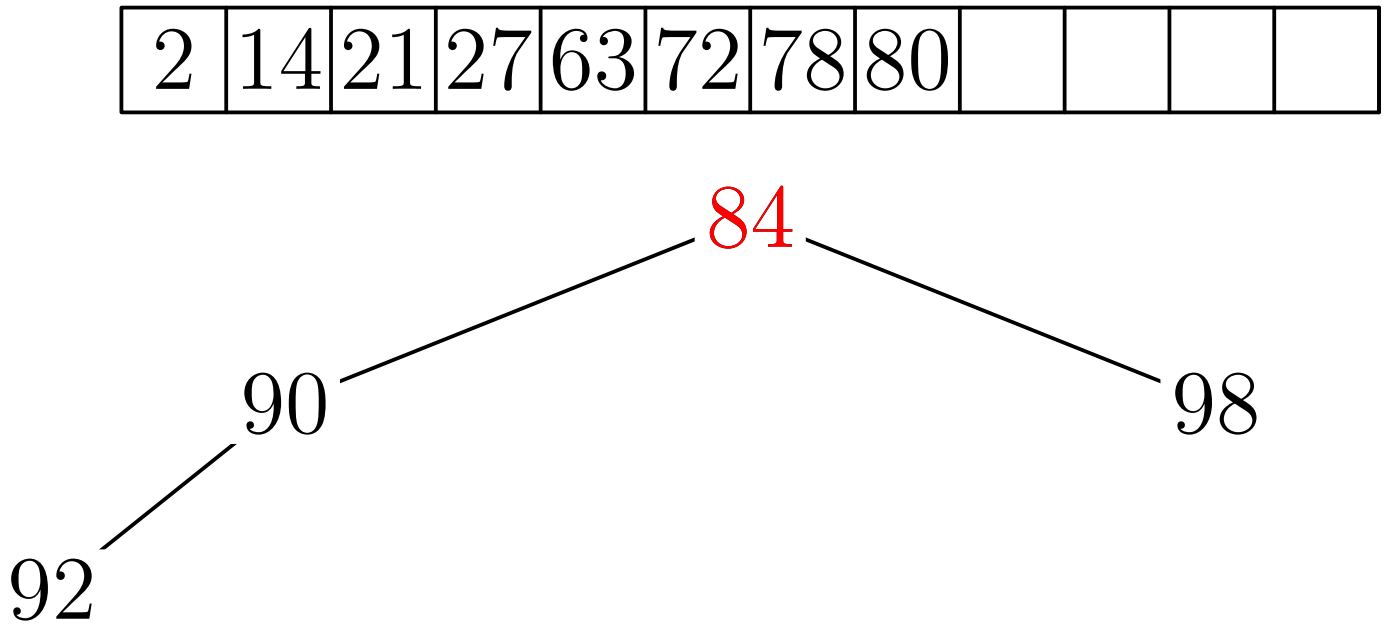

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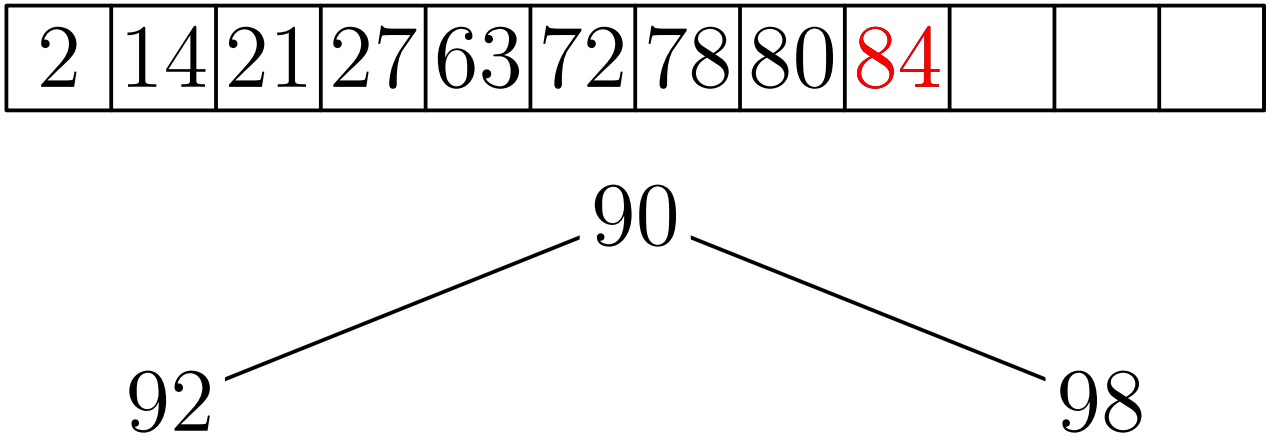

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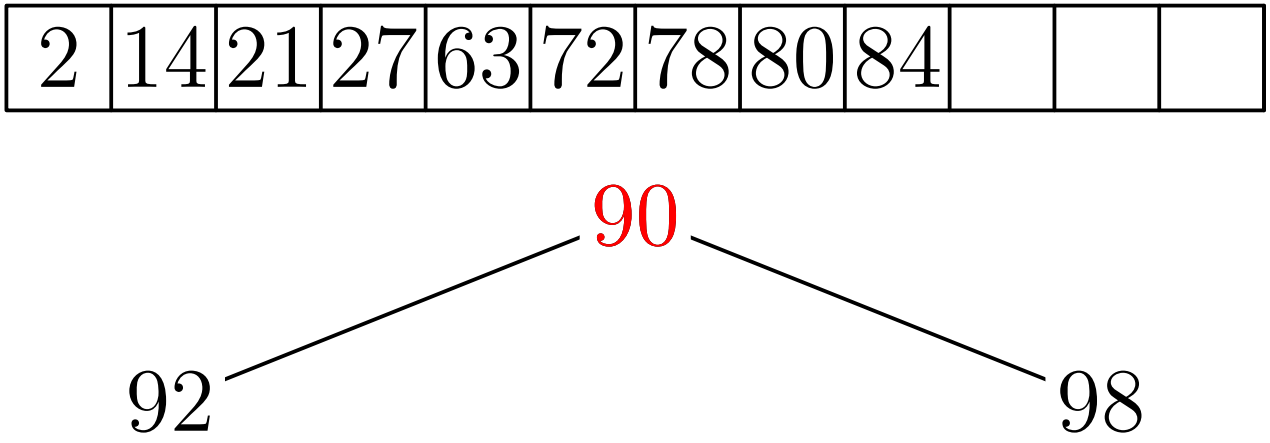

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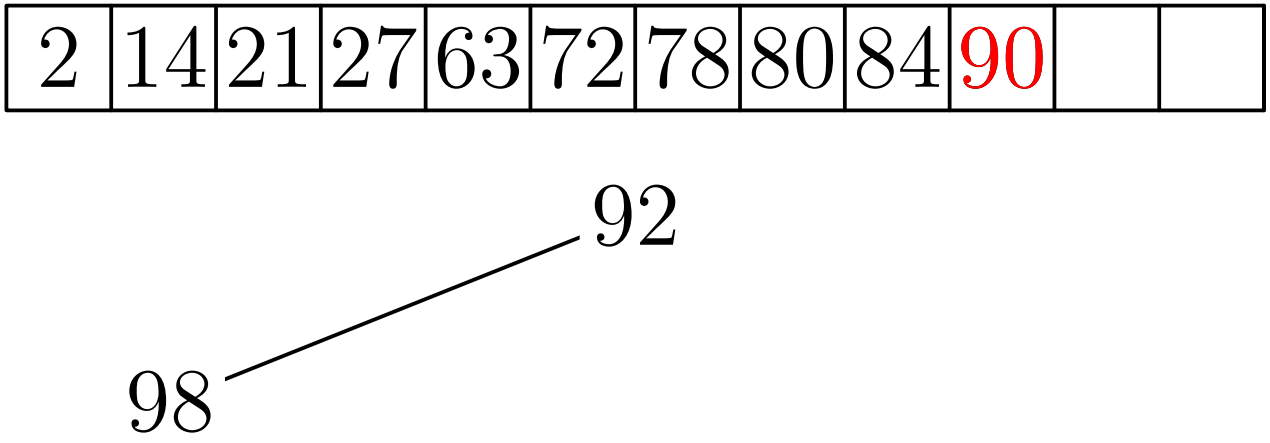

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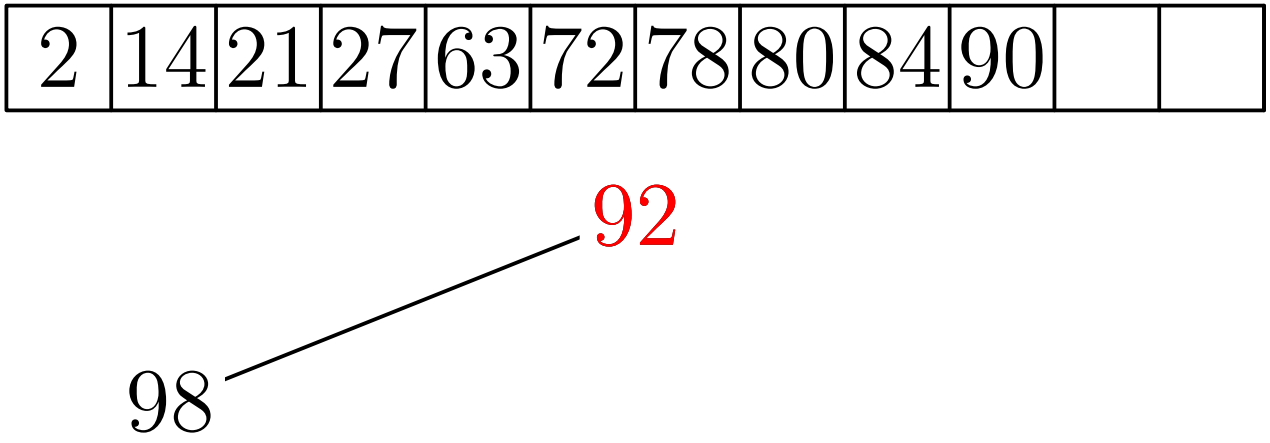

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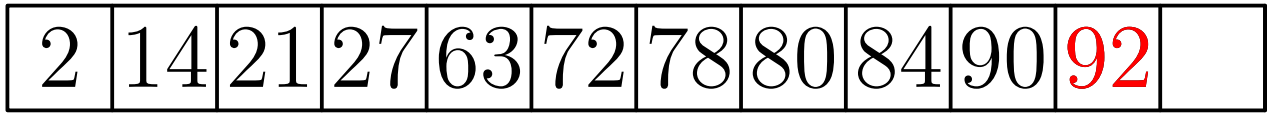

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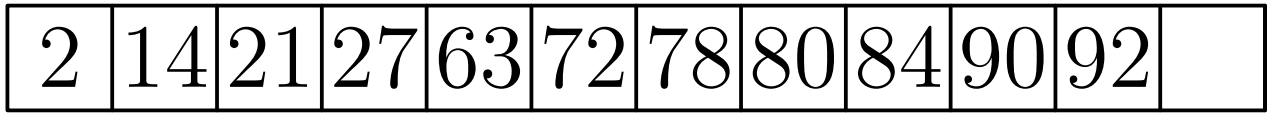

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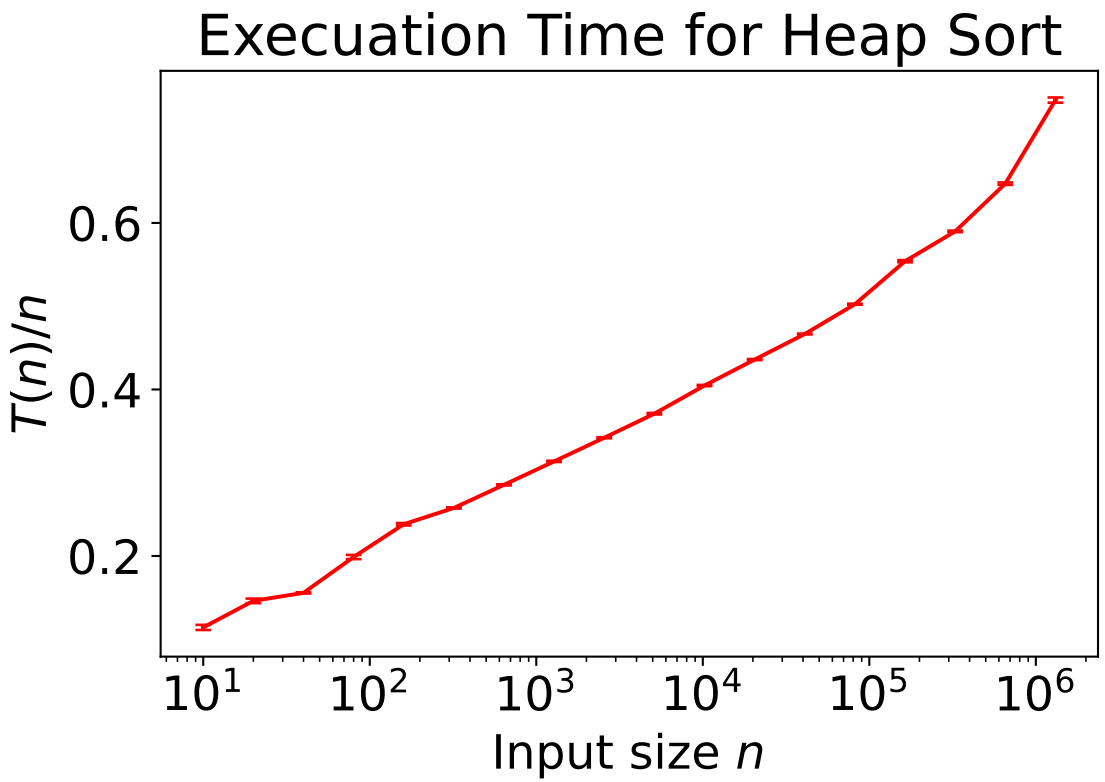
2	14	21	27	63	72	78	80	84	90	92	98

Complexity of Heap Sort

- As we have to add n elements and then remove n elements the time complexity is log-linear, i.e. $O(n \log(n))$



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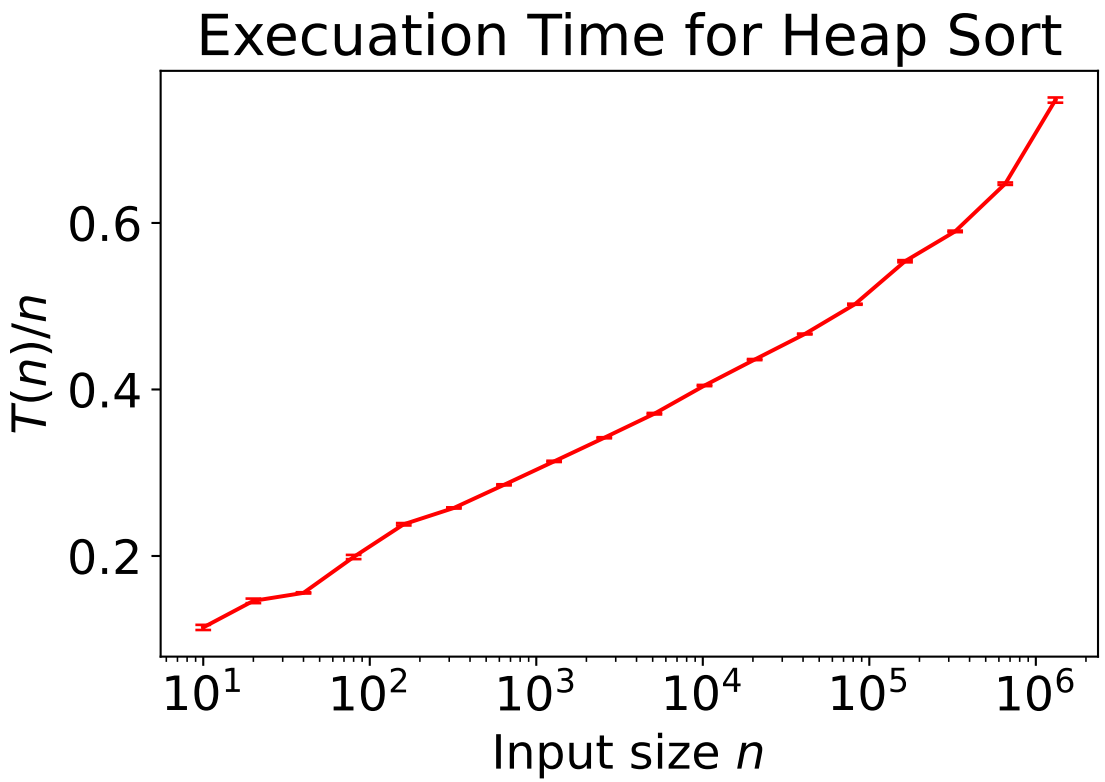

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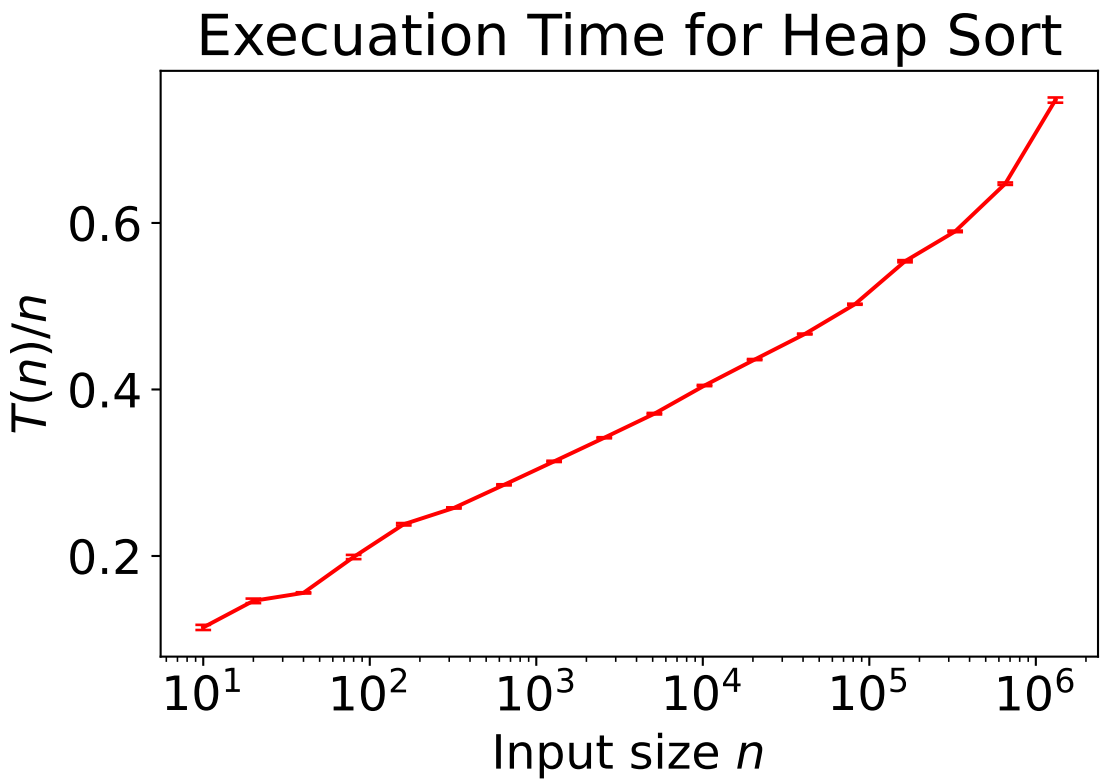

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Outline

1. Heaps
2. Priority Queues
   - Array Implementation
3. Heap Sort
4. **Other Heaps**

Other Heaps

- Binary Heaps are so useful that other types of heaps have been developed
- The simplest enhancement is to combine a binary heap with a map which maintains a pointer to each element
- The map has to be updated every time elements are moved in the heap (fortunately only $O(\log(n))$ elements are move each time the heap is updated)
- The advantage of this heap is that the priorities of elements can be changed (involving percolating elements up or down the tree)

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Merging Heaps

- One common demand on a heaps is to merge two heaps
- Unfortunately binary heaps are not efficiently merged
- There are a variety of different heaps (leftist heaps, skew heaps, binomial queues, . . .) designed to be merged
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Other Operations

- All other operations are achieved by merging
  - ⋆ Adding an element is achieved by merging the current heap with a heap of one element
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Lessons

- Heaps are a powerful data structure—they are particularly useful for implementing priority queues
- Heaps are binary trees that can be implemented as arrays
- Priority queue have many (often surprising uses)
  - ⋆ They are used when you need a queue with priorities, e.g. in operating systems
  - ⋆ They can be used to perform pretty efficient sort
  - ⋆ They are often used for implementing greedy type algorithms
  - ⋆ One important application is in real time simulations
- There exists many extensions of heaps

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