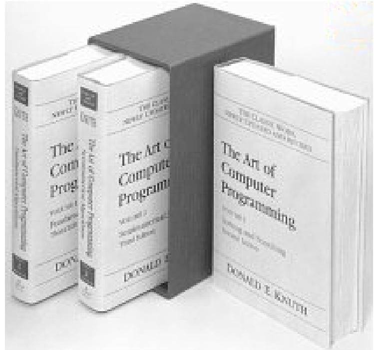
Algorithms and Analysis

Lesson 16: *Analyse!*

Pseudo code, binary search, insertion sort, selection sort, lower bound complexity

Outline

1. **Algorithm Analysis**
2. Search
3. Simple Sort
   - Insertion Sort
   - Selection Sort
4. Lower Bound



Algorithm Analysis

- We've covered most of the basic data structures
- The rest of the course is going to focus more on algorithms
- We will look predominantly at
  - Searching
  - Sorting
  - Graph Algorithms
- Emphasise general solution strategies

Code and Pseudo Code

- C++ code is often difficult to read—there are often programming details we don't care abour
- It contains details such as throwing exception which are repetitive and often depends on who you are writing the code for
- Algorithms are not language dependent (data structures are a bit more language dependent)
- To focus on what is important we will use a stylised programming language called **pseudo code**

Pseudo Code

- There is no standard for pseudo code
- The commands are not too dissimilar to C++
- The one strange convention is that assignments use an arrow $\leftarrow$
- Arrays are written in bold $\boldsymbol{a}$ with elements a_i
- In pseudo-code you are free to invent any operations that can be easily interpreted

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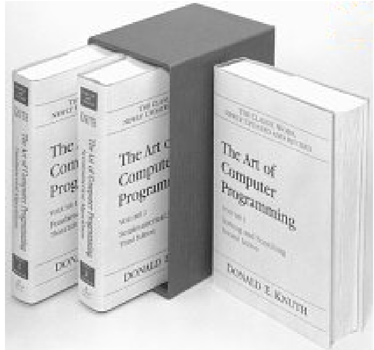


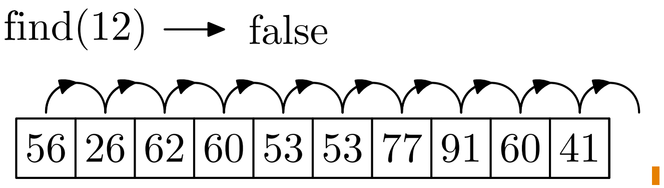
Dumb Search

```
DUMBSEARCH(a, x)
{
  /* search array a = (a_1, ... a_n) */
  /* for x return true */
  /* if successful else false */
  for i←1 to n
    if (a_i = x)
      return true
    endif
  endfor

  return false
}
```

```
bool search(T a[], T x)
{
  for (int i=0; i<n; i++) {
    if (a[i] == x)
      return true;
  }

  return false;
}
```



Time Complexity

- Worst case:
  - ⋆ The worst case for a successful search is when the element is in the last location in the array
  - ⋆ This takes n comparisons: worst case is $\Theta(n)$
- Best case:
  - ⋆ The best case is when the element is in the first location
  - ⋆ This takes 1 comparison: best case is $\Theta(1)$
- Average case:
  - ⋆ Assume every location is equally likely to hold the key

$$\frac{1+2+\ldots+n}{n} = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2}$$

- For an unsuccessful search n comparison are necessary

Binary Search

- If the array is ordered we can do better
- At each step we bisect the array

```
BinarySearch(a, x)
{
  low ←1
  high ←n
  while (low ≤ high)
    mid ←⌊(low + high)/2⌋
    if x > a_mid
      low ←mid + 1
    elseif x < a_mid
      high ←mid −1
    else
      return true
    endif
  endwhile
  return false
}
```

⋆ Based on a **divide-and-conquer** strategy

⋆ We check the middle of the array

$$\underbrace{a_1, a_2, \cdots, a_{m-1}}_{x<a_m}, \overbrace{a_m}^{x=a_m}, \underbrace{a_{m+1}, \cdots a_n}_{x>a_m}$$

⋆ Based on a recursive idea

Binary Search in Action

BinarySearch(a, 97) not found

14	19	27	33	36	39	47	51	55	60	62	63	71	76	78	79	84	91	91	95

low high low mid high mid high mid low mid low mid high mid low mid high low

Analysis

- We count the number of comparisons (counting each `if/else if` statement as a single comparison)
- Let $C(n)$ be the number of comparisons needed to search in an array of size n
- After one comparison we are left (in the worst case) with having to search an array not larger than $\lfloor n/2 \rfloor$, thus

$$C(n) < C(\lfloor n/2 \rfloor) + 1$$

- We've seen this relation before (lesson on Recursion)
- Easy to show $C(n) < \lfloor \log_2(n) \rfloor + 1 = O(\log(n))$

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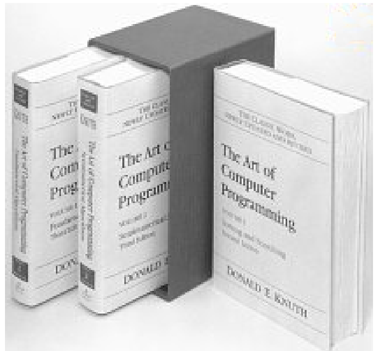


Sort Characteristics

- Sort is one of the best studied algorithms. We care about stability, space and time complexity
- A sort algorithm is said to be **stable** if it does not change the order of elements that have the same value
- Space Complexity. Sort is said to be
  - ⋆ **In-place** if the memory used is $O(1)$
- Time Complexity. In particular we are interested in
  - ⋆ Worst case
  - ⋆ Average case
  - ⋆ Best case

Insertion Sort

- In insertion sort we keep a subsequence of elements on the left in sorted order
- This subsequence is increased by *inserting* the next element into its correct position

```
InsertionSort(a)
{
  for i←2 to n
    v←a_i
    j←i-1
    while j ≥ 1 and a_j>v
      a_{j+1}←a_j
      j←j-1
    endwhile
    a_{j+1}←v
  endfor
}
```

23	37	39	50	66	69	69	74	84	90

sorted unsorted

Properties of Insertion Sort

- Insertion sort is **stable**. We only swap the ordering of two elements if one is strictly less than the other
- It is **in-place**
- Worst time complexity
  - ⋆ Occurs when the array is in inverse order
  - ⋆ Every element has to be moved to front of the array
  - ⋆ Number of comparisons for an array of size $C_w(n)$

$$C_w(n) = \sum_{i=2}^{n} (i-1) = 1 + 2 + \cdots n - 1 = \frac{n(n-1)}{2} \in \Theta(n^2)$$

Time Complexity

- Average Time Complexity
  - ⋆ On average we can expect that each new element being sorted moves half the way down sorted list
  - ⋆ This gives us an average time complexity, $C_a(n)$ of half the worst time

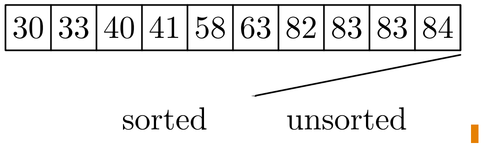
$$C_a(n) = \frac{n(n-1)}{4} \in \Theta(n^2)$$

- Best Time Complexity
  - ⋆ This occurs if the array is already sorted
  - ⋆ In this case we only need $C_b(n) = n - 1 \in \Theta(n)$ comparisons
- Insertion sort is a good sort for small arrays because it is stable, in-place and is efficient when the arrays are almost sorted

Selection Sort

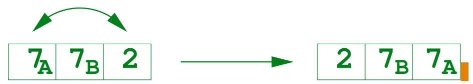
- A more direct **brute force** method is to find the least element iteratively
- We can make this an in-place method by swapping the least element with the first element, the second least element with the second element, etc.

```
SELECTIONSORT(a)
{
 for i←1 to n-1
   min ←i
   for j←i+1 to n
     if a_j<a_min
       min←j
     end if
   end for
   swap a_i  and a_min
 end for
}
```



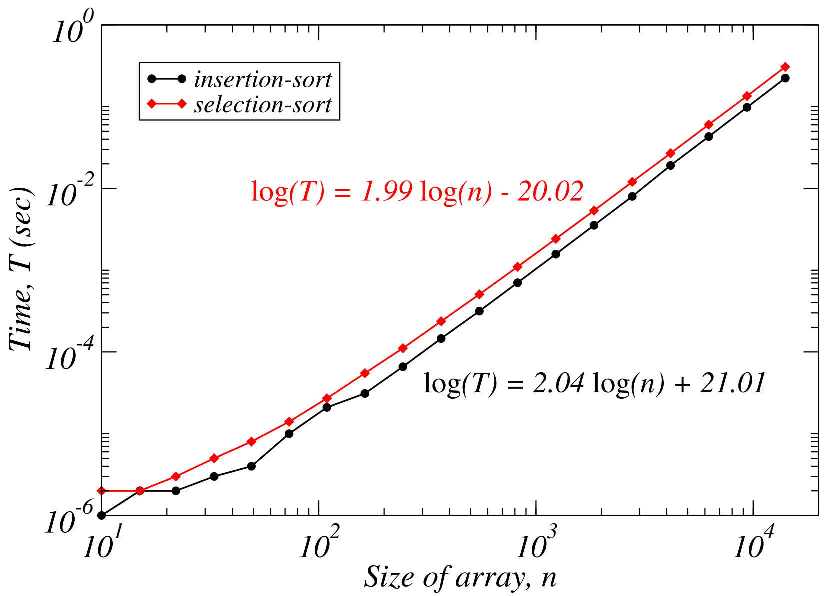
Analysis of Selection Sort

- Selection sort is in-place
- It isn't stable



- Selection sort always requires $n(n-1)/2$ comparisons so has the same worst case, but worse average case and best case complexity as insertion sort
- It only performs $n-1$ swaps—this makes it attractive (insertion sort moved more elements)

Insertion versus Selection Sort



Bubble Sort

- There are many other simple sort strategies
- One popular one is bubble sort—keep on swapping neighbours until the array is sorted
- It is stable and in-place
- This again has $O(n^2)$ complexity
- This isn't bad for a simple sort, but it does do more work than insertion sort and selection sort
- Apart from its name it just doesn't have anything going for it

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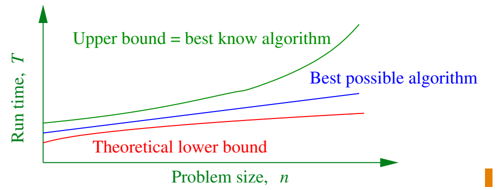
How Well Can You Do?

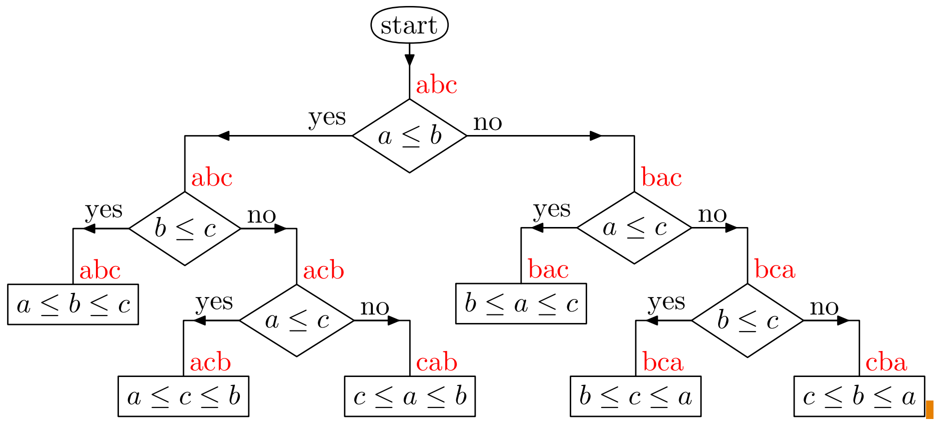
- Given a problem we would like to know what is the time complexity of the best possible program
- Usually there is no way of knowing this
- We can get an upper bound—if we know the time complexity of any algorithm that solves the problem we have an upper bound
- Lower bounds are far trickier
- A lower bound of $f(n)$ is a guarantee that we spend at least $f(n)$ operations to solve the problem



Decision Trees

- Decision trees are a way to visualise (at least, in principle) many algorithms
- They will eventually give us a lower bound on the time complexity of sort using binary decisions
- A decision tree shows the series of decisions made during an algorithm
- For sort based on binary comparisons the decision tree shows what the algorithm does after every comparison

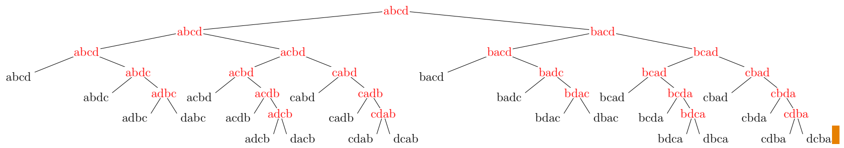
Decision Tree for Insertion Sort



- Note there is one leaf for every possible way of sorting the list

Decision Trees and Time Complexity

- The time taken to complete the task is the depth of the tree at which we finish (i.e. the leaf nodes)
- We can thus read of the time complexity
  - ⋆ worst case time: depth of the deepest of leaf
  - ⋆ best case time: depth of the shallowest of leaf
  - ⋆ average case time: average depth of leaves
- Different sort strategies will have different decision trees
- Decision trees are usually far too large to write out ☹



Requirements of Correct Sort

- Any sort based on binary comparisons must have a leaf of the tree for every possible way of sorting the list
- The array $[a, b, c]$ must be arranged differently for all combinations

$$[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]$$

- That is they must go through a different path of the decision tree
- If not sort won't work

Minimum Number of Leaves

- There must be, at least, one leaf node of the decision tree for each possible permutation of the list
- How many permutations are there of a list of size n?
- Start with a sequence $(a_1, a_2, \ldots, a_n)$
- To create a new permutation we can choose any member of the list as the first element
- We can choose any of the remaining $n-1$ elements of the list as the second element of the permutation
- The total number of permutation is $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$

Lower Bound Time Complexity for Sorting

- Any sort algorithm using binary comparisons must have a decision tree with at least $n!$ leaf nodes
- This will be a binary tree with some depth d
- The number of leaves at depth d is 2^d
- Thus the smallest depth tree must have a depth d such that $2^d \geq n!$
- That is, the depth of the decision tree satisfies $d \geq \log_2(n!)$
- But this is the number of comparisons needed in our sort
- We are left with a lower bound on the time complexity of $\log_2(n!)$

How Big is $\log_2(n!)$

- We showed in the second lecture that

$$\left(\frac{n}{2}\right)^{n/2} < n! < n^n$$

- It is not too difficult to show that asymptotically (i.e. as $n \to \infty$) that $n!$ approaches $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$—this is known as **Stirling's approximation**
- Thus

$$\begin{aligned}\log_2(n!) &\approx n\log_2(n) - n\log_2(e) + \frac{\log_2(n)}{2} + \frac{\log_2(2\pi)}{2} \\ &= \Theta(n\log_2(n))\end{aligned}$$

Complexity of Sorting

- We therefore have a lower bound on the time complexity of $\Omega(n\log(n))$
- This is true for any sort using binary comparisons
- We will see in the next lecture there exists algorithms with time complexity $O(n\log(n))$
- This means our lower bound is tight—i.e. it is the true cost of the best algorithm
- Having a lower bound we know we are not going to obtain a substantially faster algorithm

Lessons

- Analysis of algorithms is hard
- Analysis is important: without it we don't know if we have a good algorithm or whether we should try to find a more efficient one
- Lower bounds are particularly important