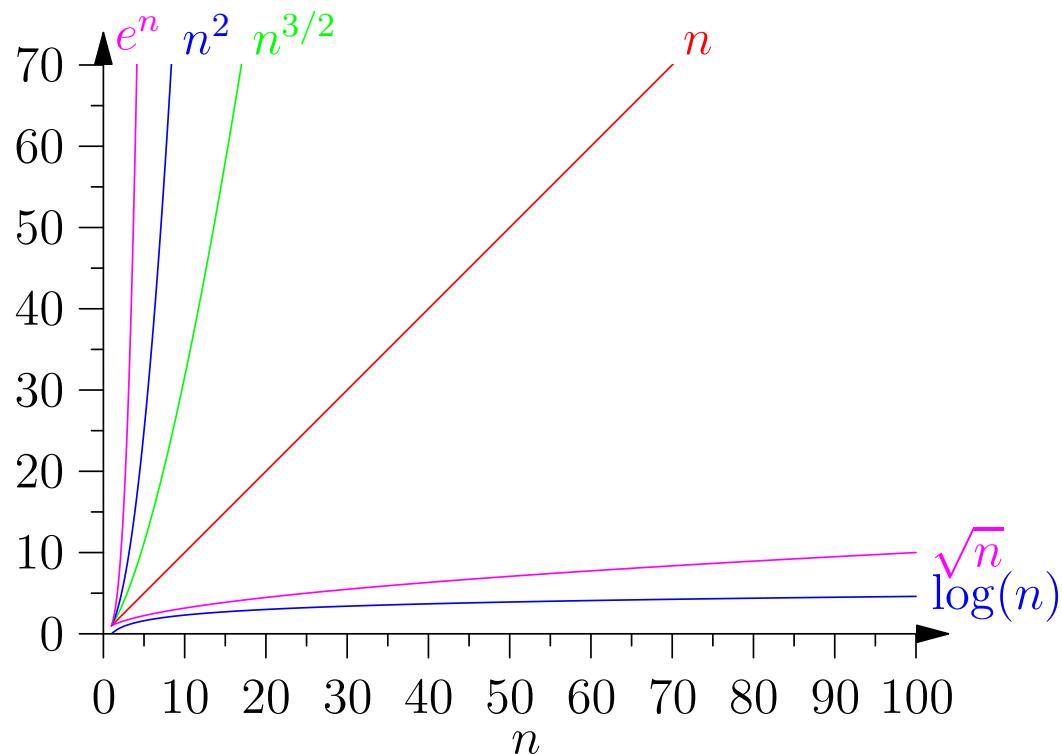


# Algorithms and Analysis

## Lesson 31: *Understand Time Complexity*



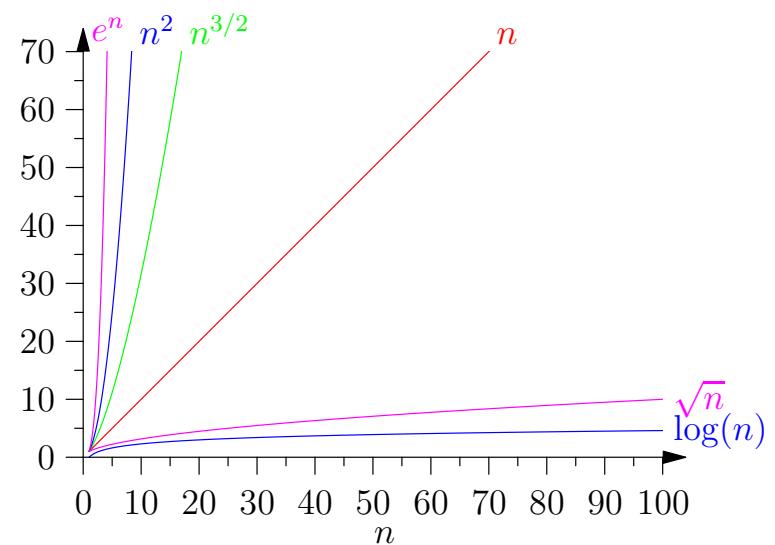
*Theta, Big-O, little-o, Big-Omega, little-omega*

# Outline

## 1. Time Complexity Classes

- Theta— $\Theta$
- Big O
- Little o
- Big Omega— $\Omega$
- Little omega— $\omega$

## 2. Computing Time Complexity



# Recap

- We have seen many algorithms taking times of order  $1$ ,  $\log(n)$ ,  $n$ ,  $n \log(n)$ ,  $n^2$ , etc
- Sometimes these are worst time, average time or best time results
- We have lots of different notations, e.g.  $O(1)$ ,  $\Theta(\log(n))$ ,  $\Omega(n^2)$ , etc.
- What does it all mean?

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# Complexity Class Sets

- The correct way to think about complexity classes is in terms of sets
- Suppose we have an algorithm which takes an input of size  $n$  and computes an output in  $f(n)$  operations
- E.g.  $f(n) = 4n^2 + 2n + 3$
- We can partition all run times into sets by considering only the leading order term and ignoring the constant term
- We denote these sets by  $\Theta(g(n))$ 
  - ★  $4n^2 + 2n + 3 \in \Theta(n^2)$
  - ★  $5n \log(n) + 3n + 2 \in \Theta(n \log(n))$

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# Defining $\Theta(g(n))$

- A function  $f(n) \in \Theta(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c \quad 0 < c < \infty$$

- E.g.

$$\lim_{n \rightarrow \infty} \frac{4n^2 + 2n + 3}{n^2} = 4$$

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- Nevertheless, ignoring the constant is often essential to make analysis of algorithms doable

# Ordering Complexity Classes

- We can define the relation  $\Theta(f(n)) < \Theta(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

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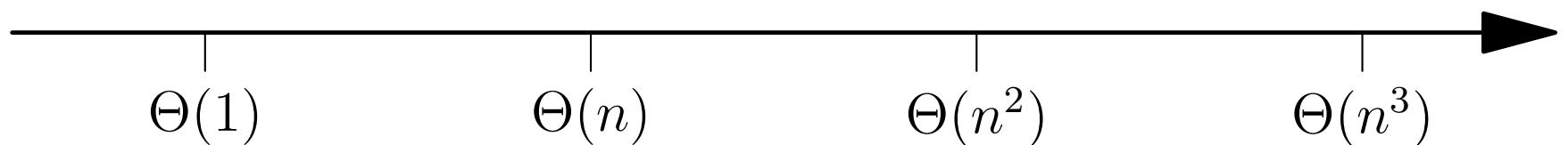
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# The Complexity Line

- We can order all complexity classes. E.g.

$$\Theta(1) < \Theta(\log(n)) < \Theta(\sqrt{n}) < \Theta(n) < \Theta(n^2)$$

- We can depict this as a complexity line



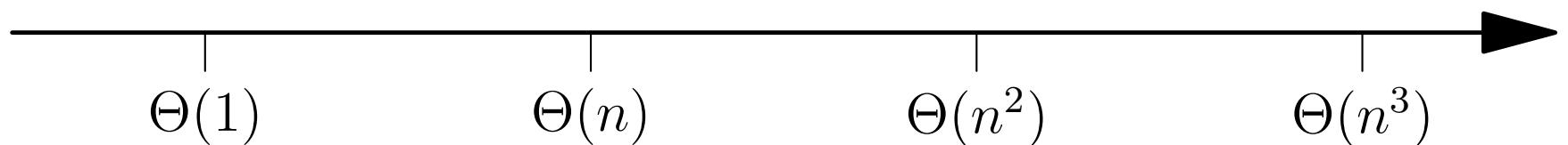
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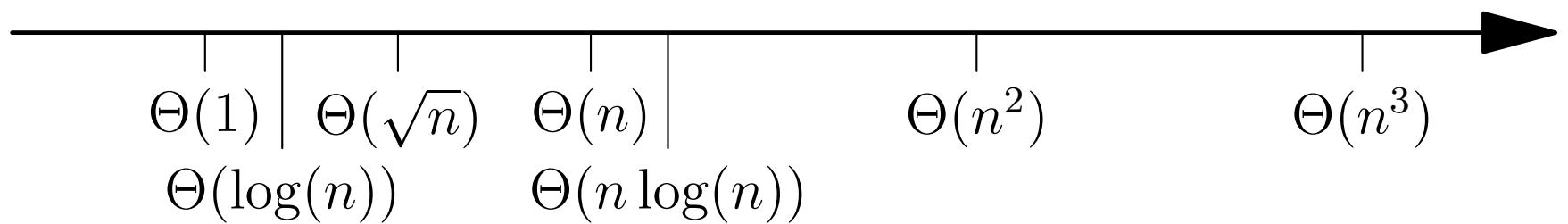
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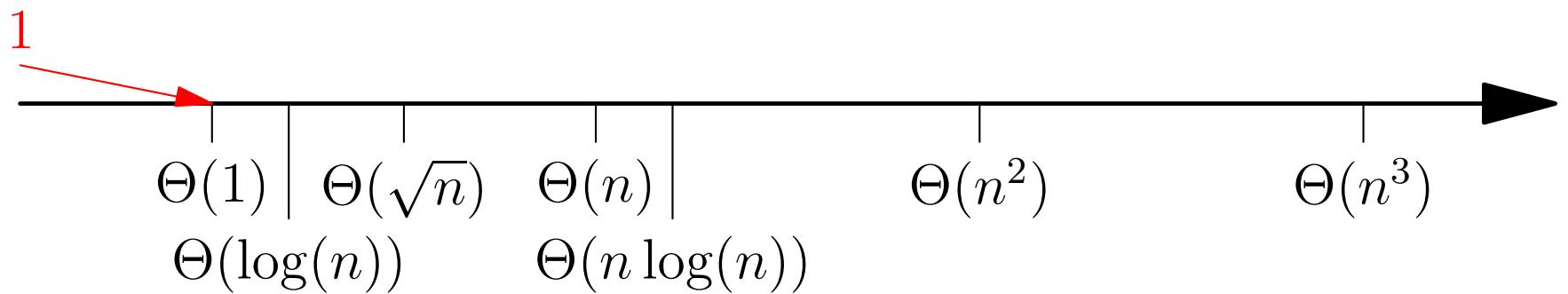
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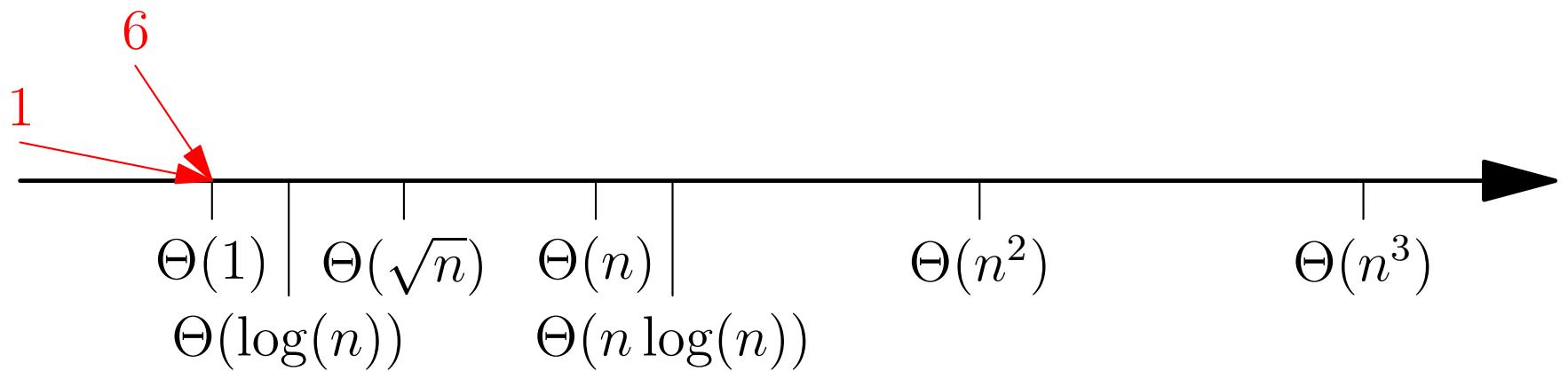
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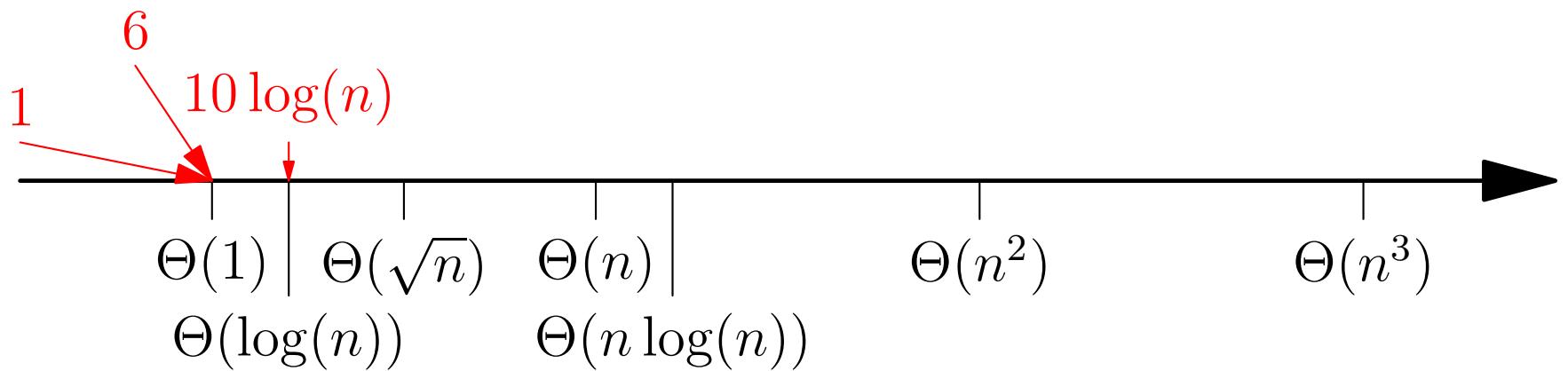
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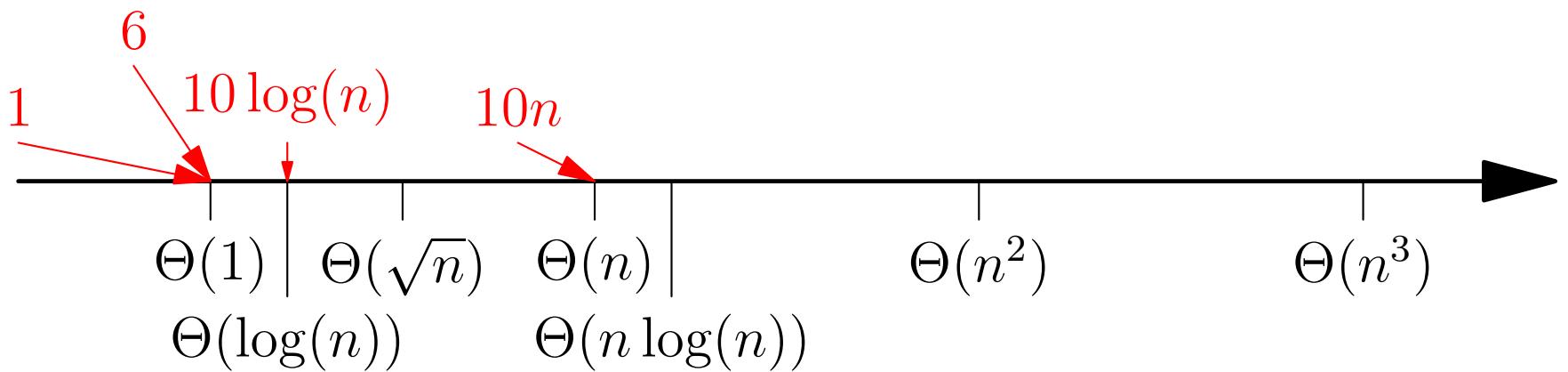
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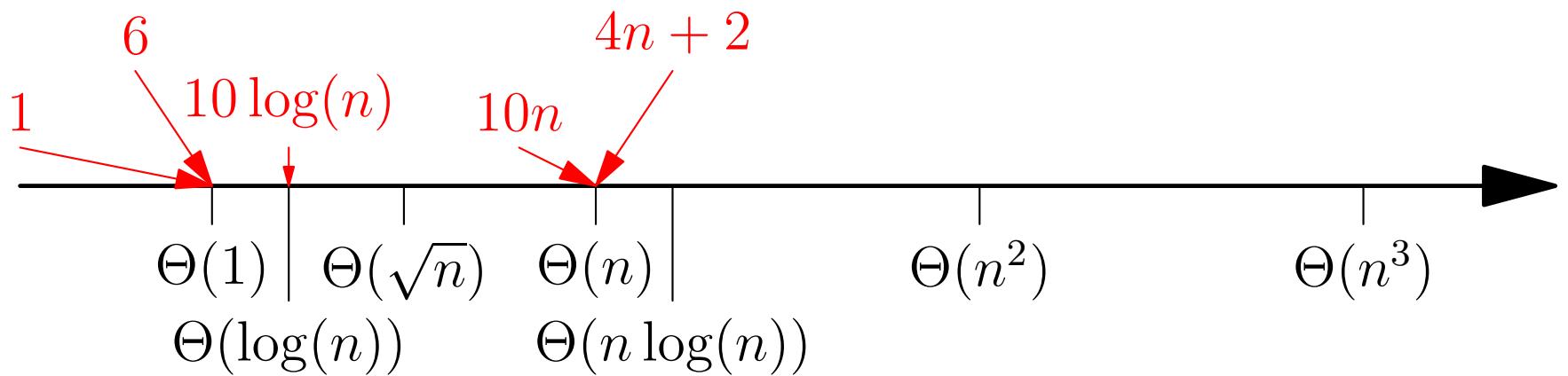
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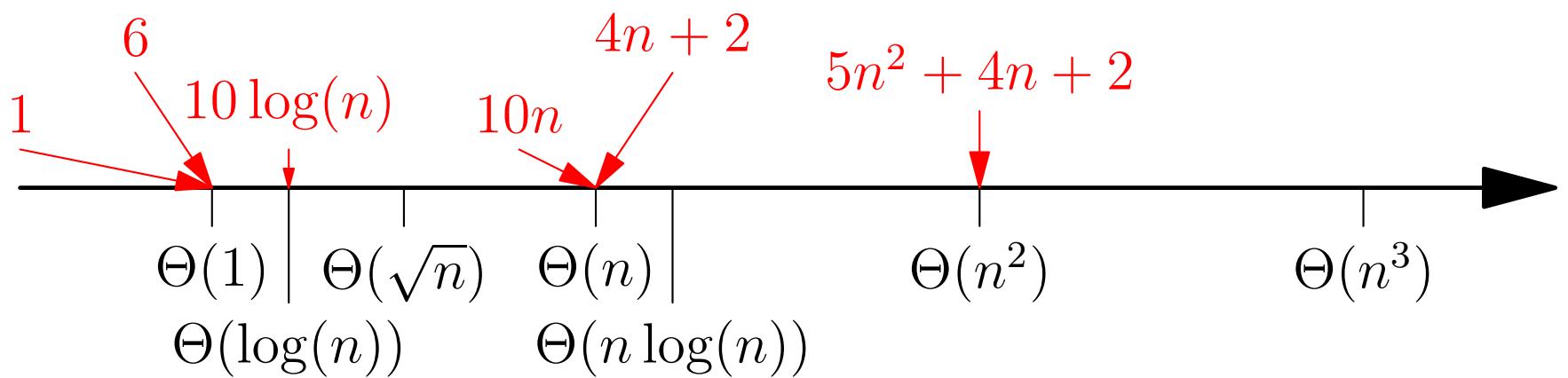
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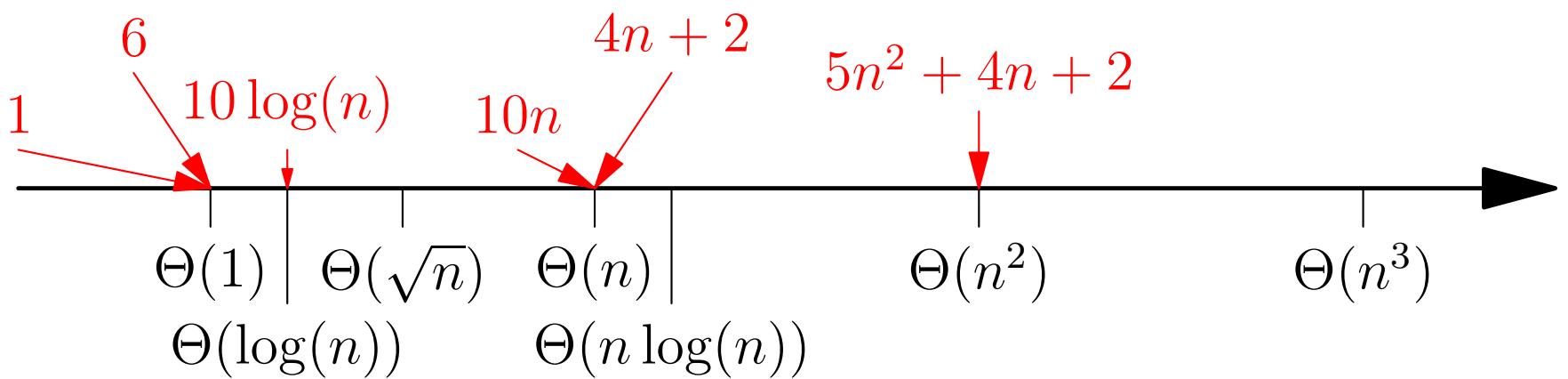
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# Complexity Dependent on Inputs

- The run time of many algorithms depends on the input
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  - ★ Worst case time complexity (the longest time an algorithm will take)
  - ★ Average complexity (the expected time averaged over all possible inputs)
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- Algorithms are often rather complicated and knowing the exact time complexity (for either worst, average or best cases) might not be known
- In reality it will have some run time (e.g.  $f(n) = 3n^2 \log(n) + 2n^2 - n + 3$ ) and will belong to a  $\Theta$  time complexity set (e.g.  $\Theta(n^2 \log(n))$ ) but we might not be able to calculate it
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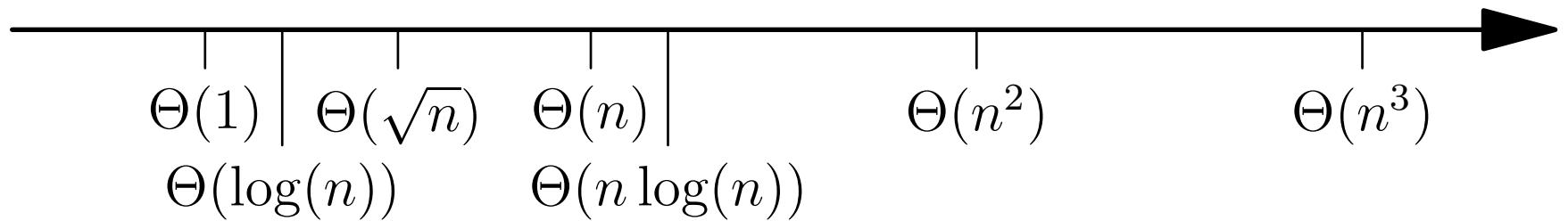
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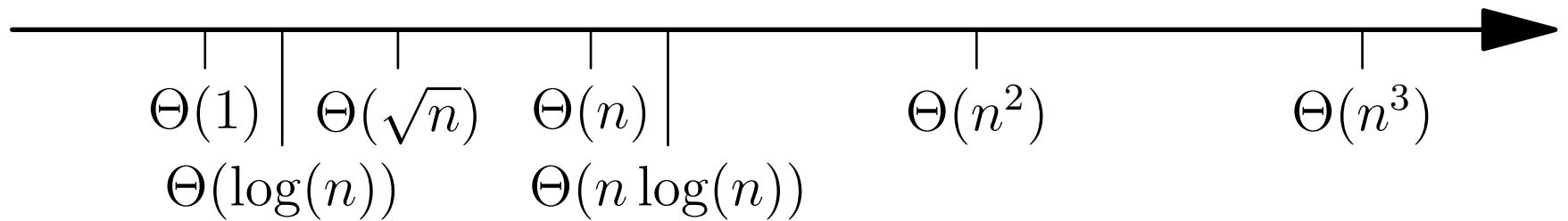
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- If an algorithm is  $O(g(n))$  then its time complexity is no more than  $\Theta(g(n))$



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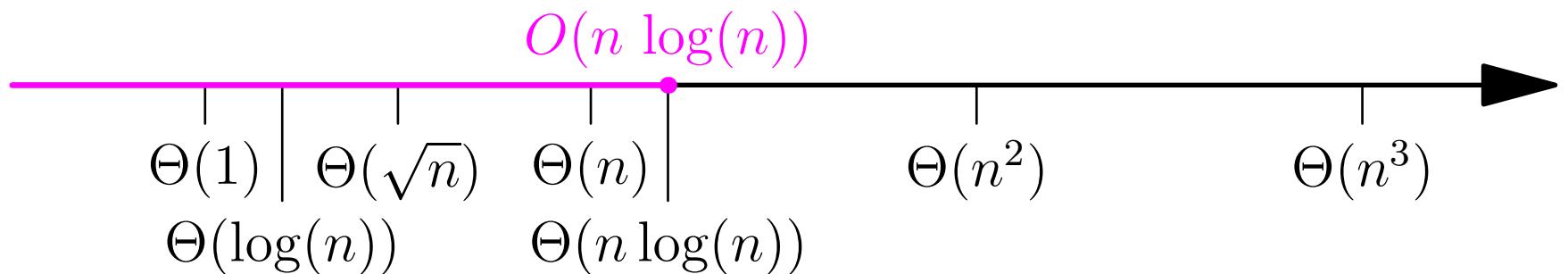
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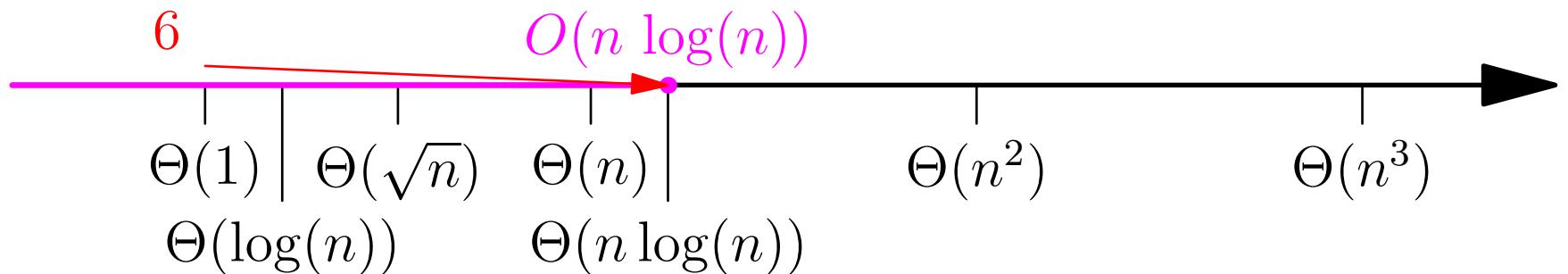
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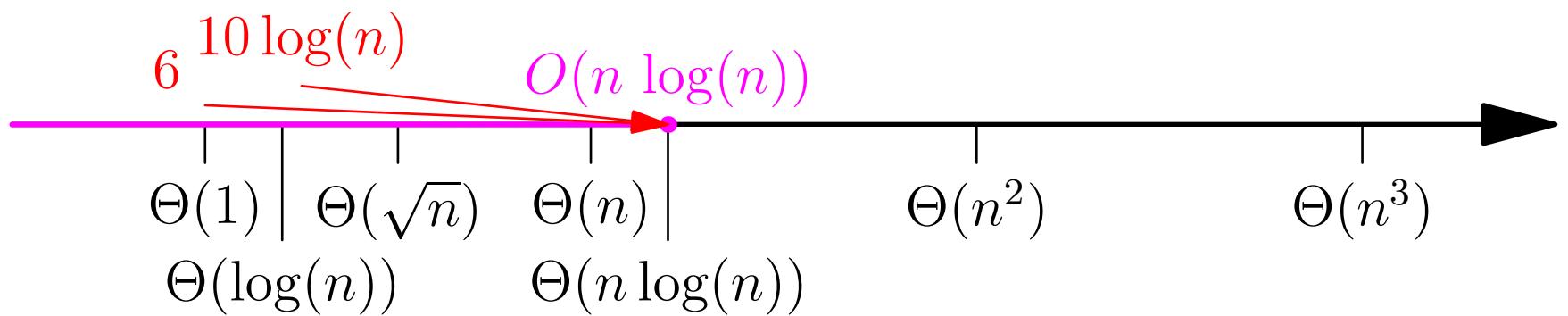
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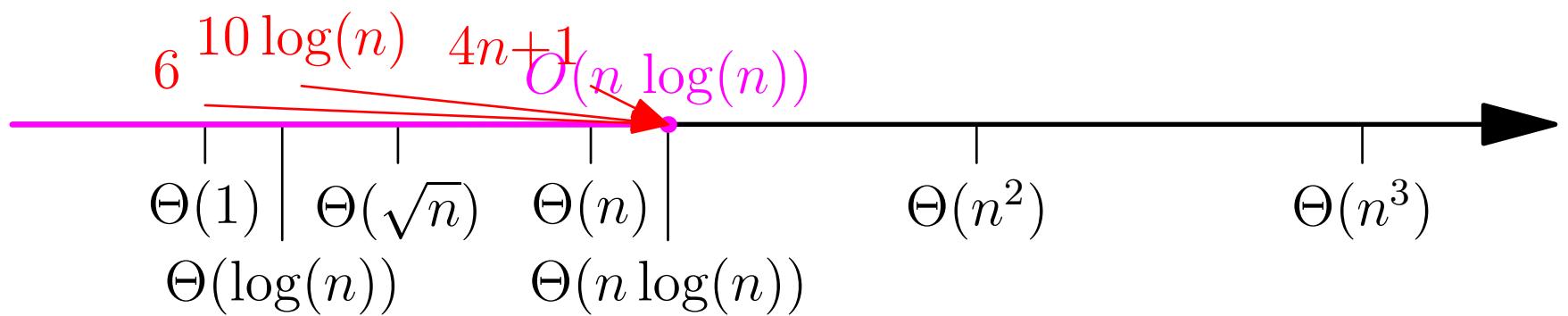
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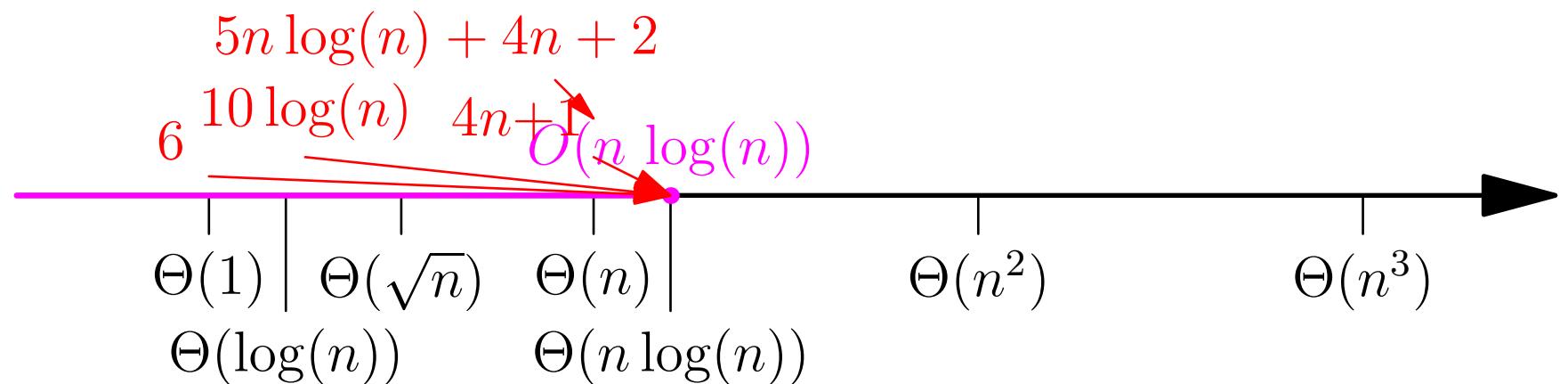
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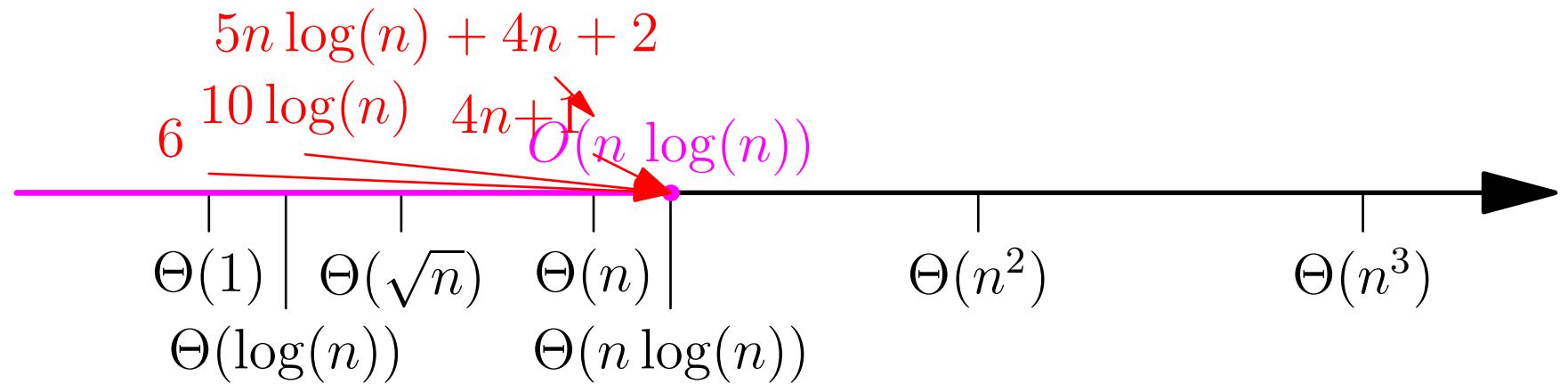
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# Upper Bounding Time Complexity

- Consider a program

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// define stuff
for(int i=0; i<n; i++)  {
    // do something
    if /* some condition */ {
        for (int j=0; j<n; j++) {
            // do other stuff
        }
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}
// clean up
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- If the **if** statements is never true this is a  $\Theta(n)$  algorithm if it is always true it is a  $\Theta(n^2)$  algorithm
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- If the **if** statements is never true this is a  $\Theta(n)$  algorithm if it is always true it is a  $\Theta(n^2)$  algorithm
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# Upper Bounding Time Complexity

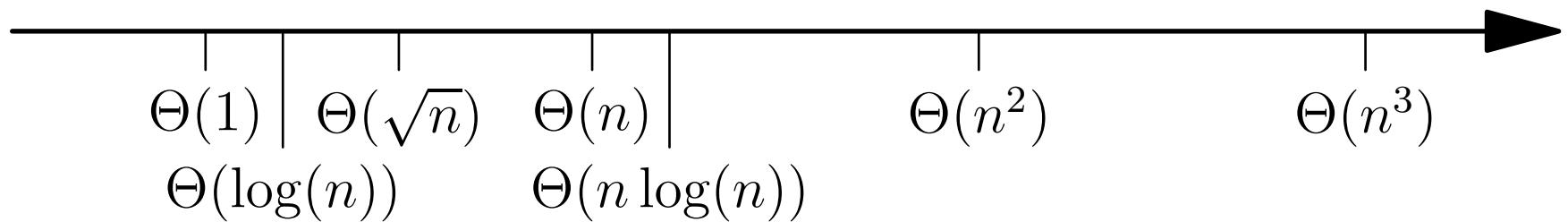
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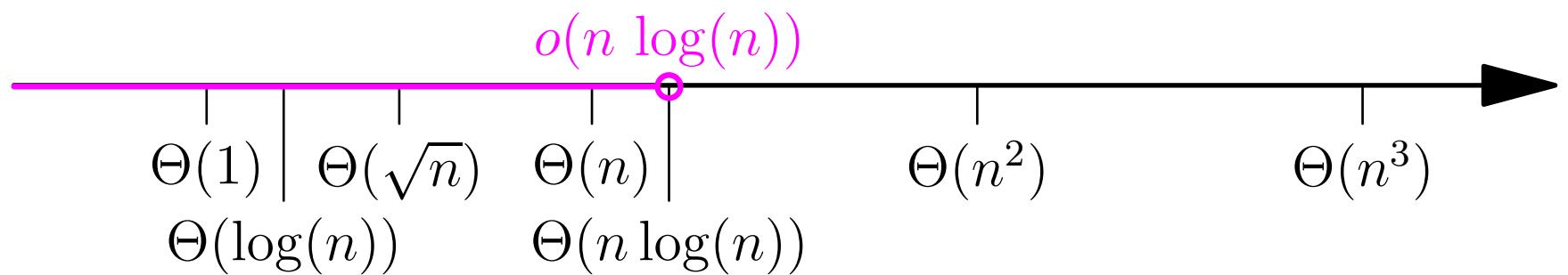
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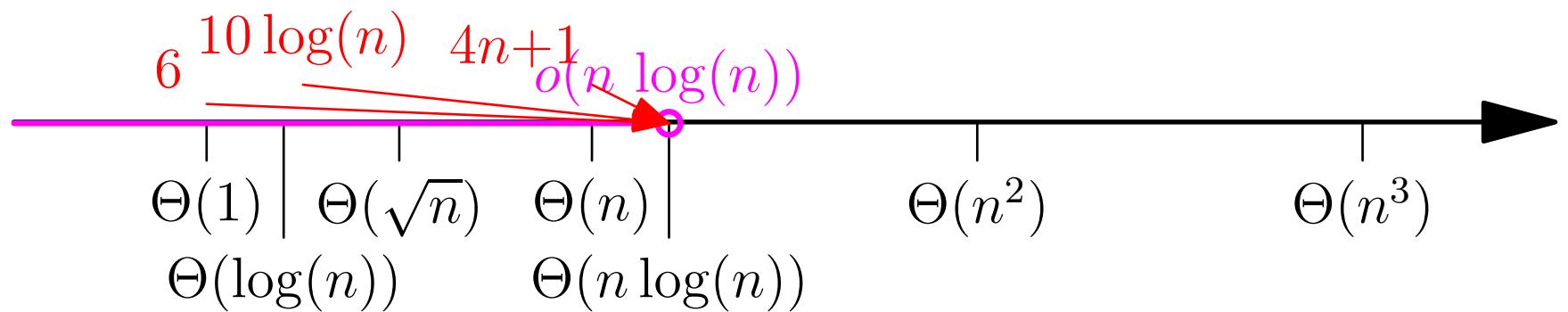
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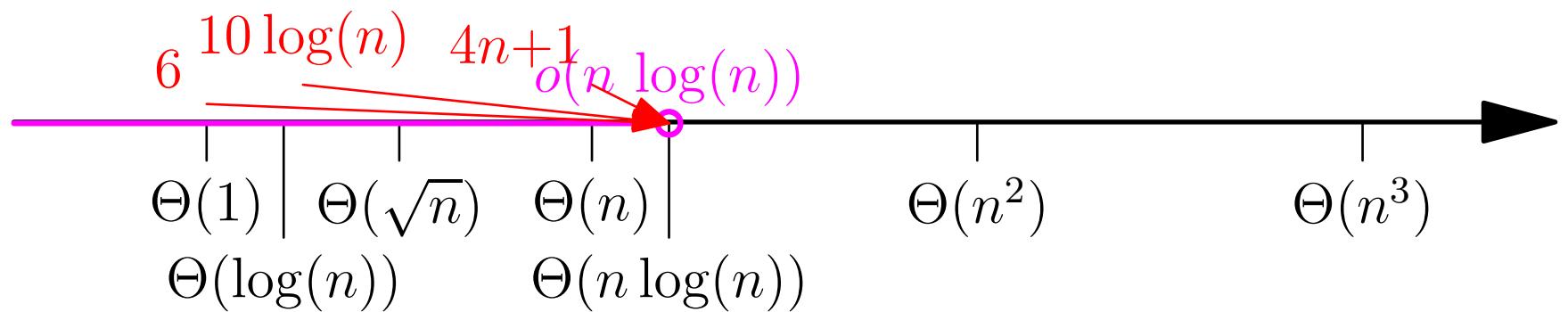
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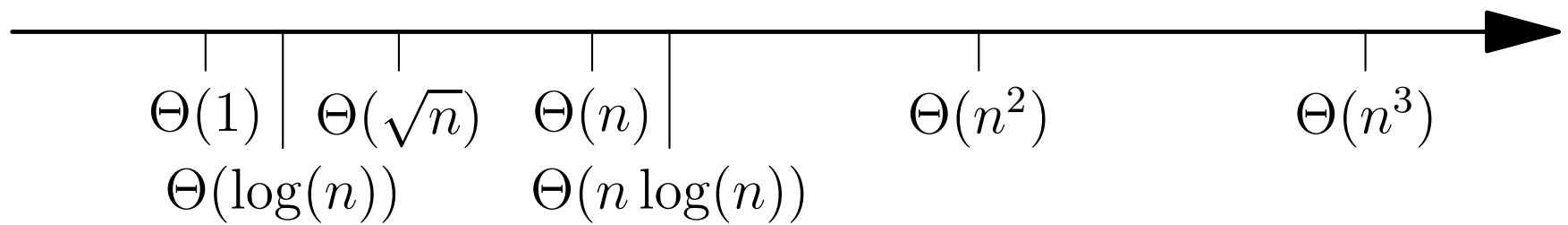
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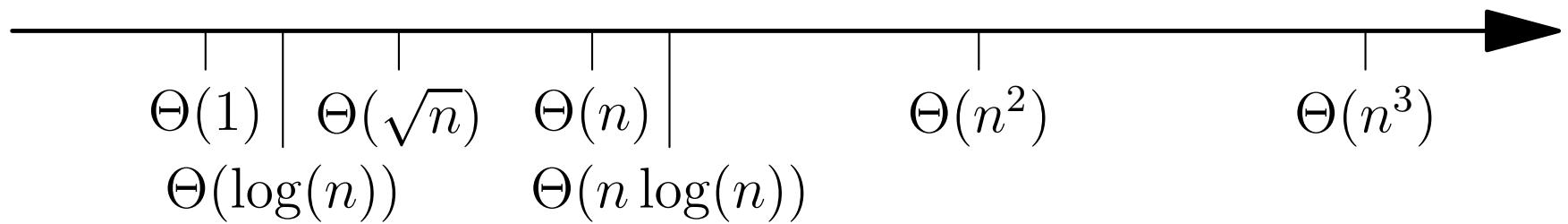
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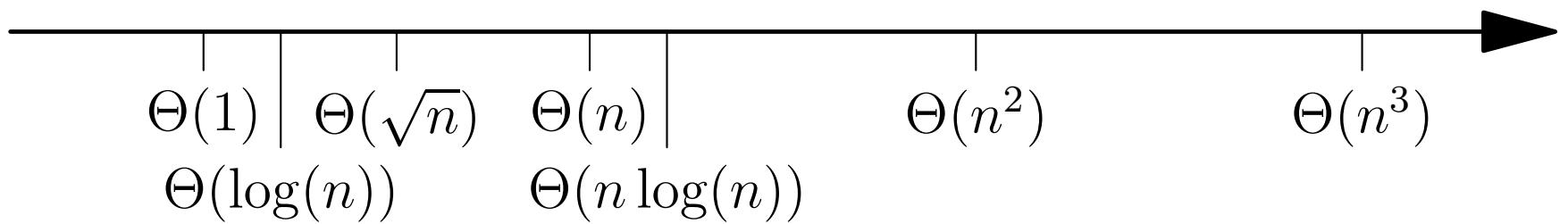
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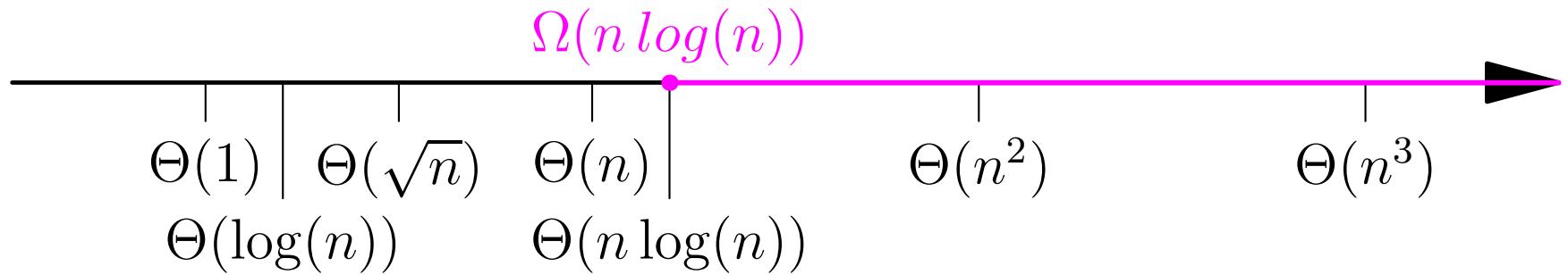
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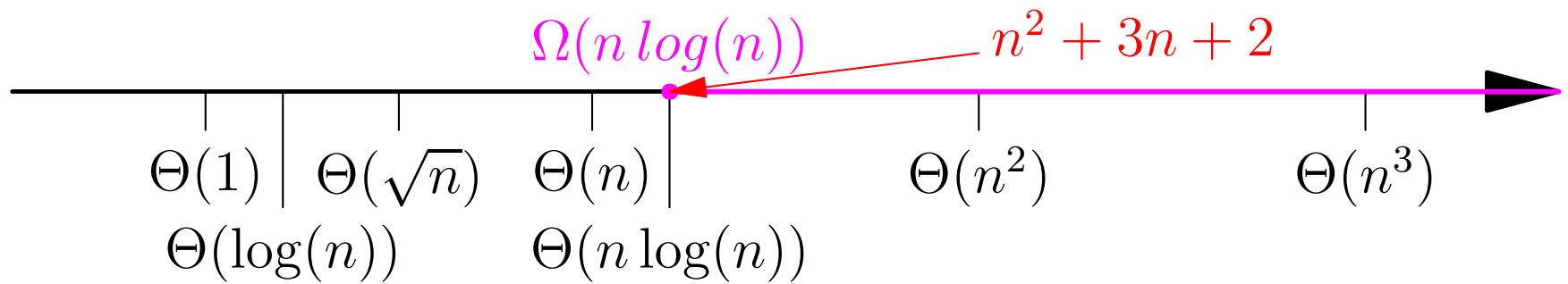
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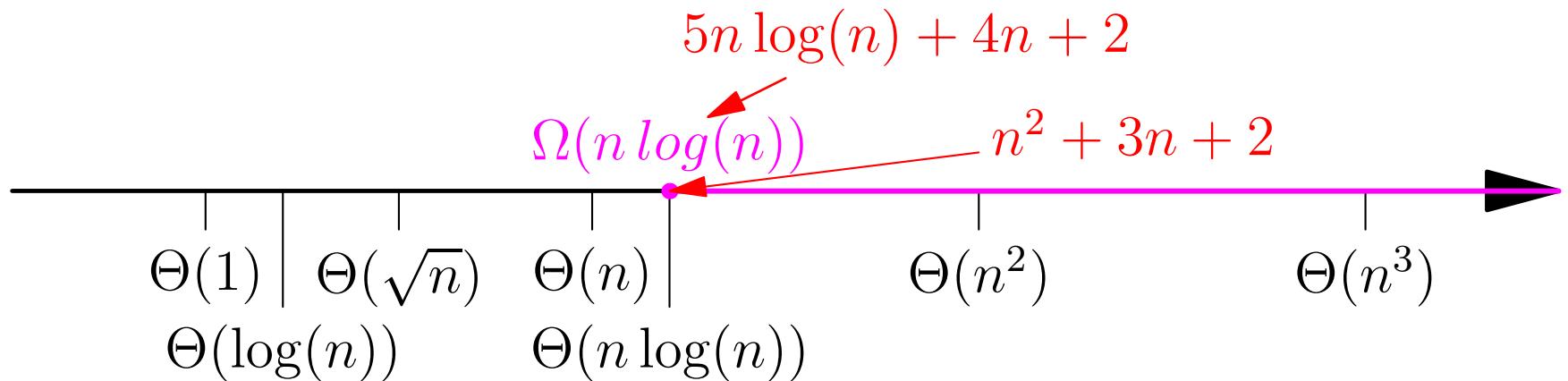
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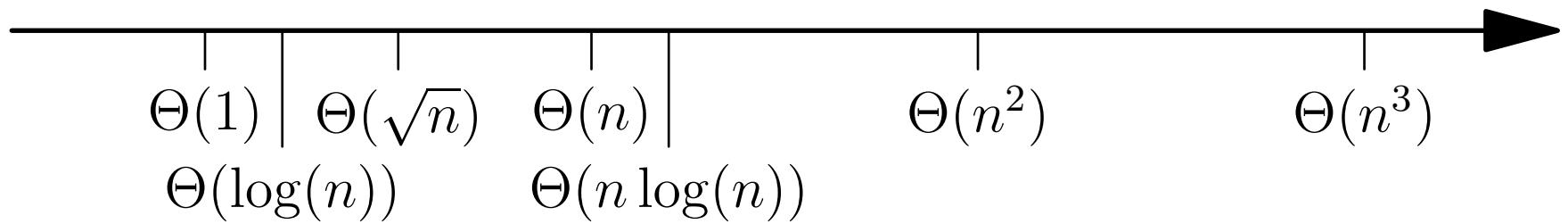
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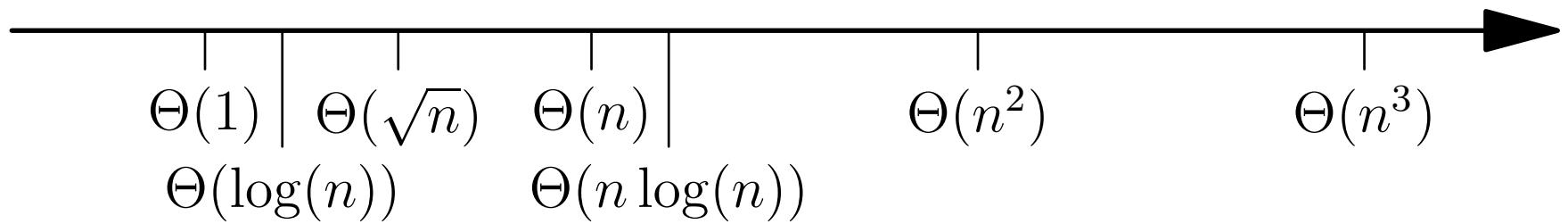
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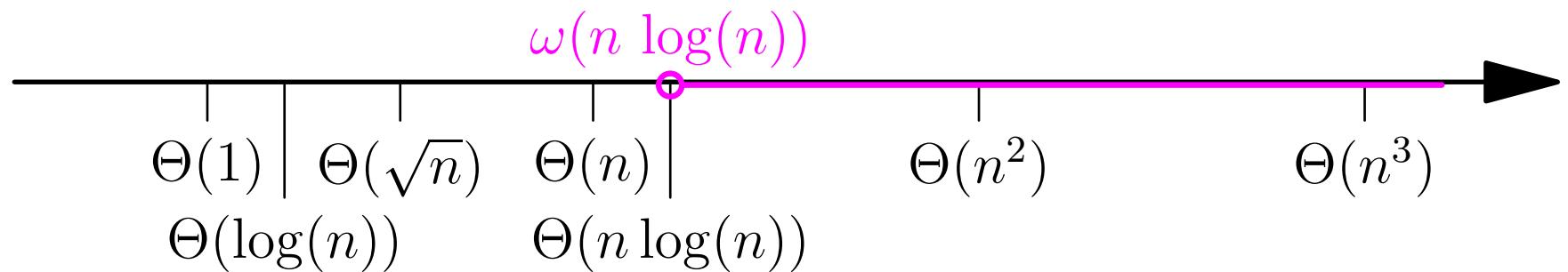
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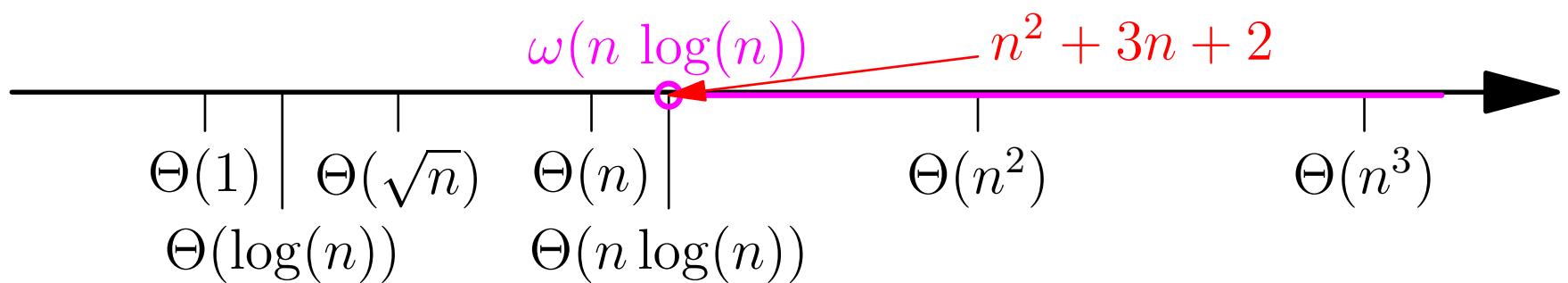
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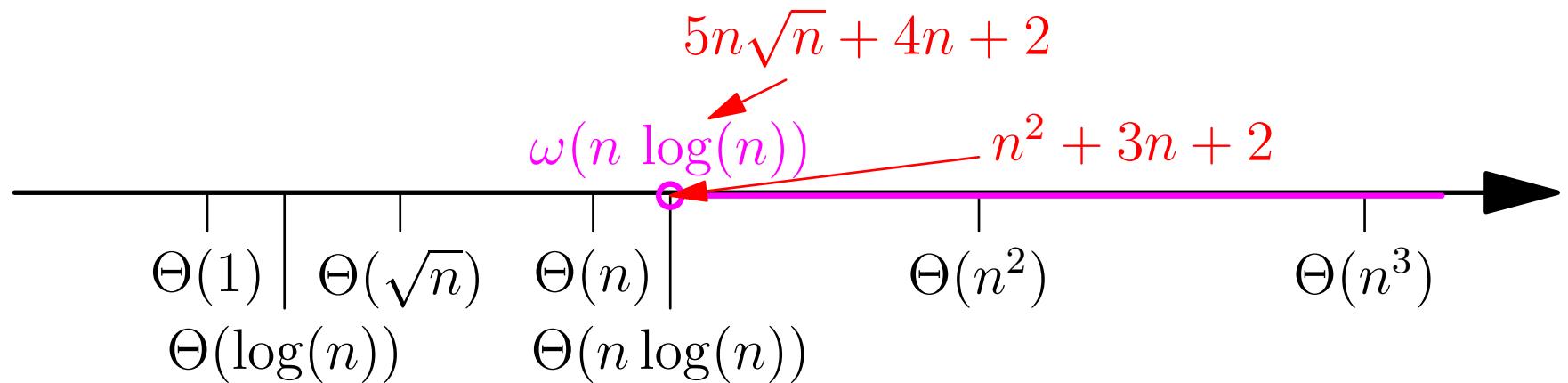
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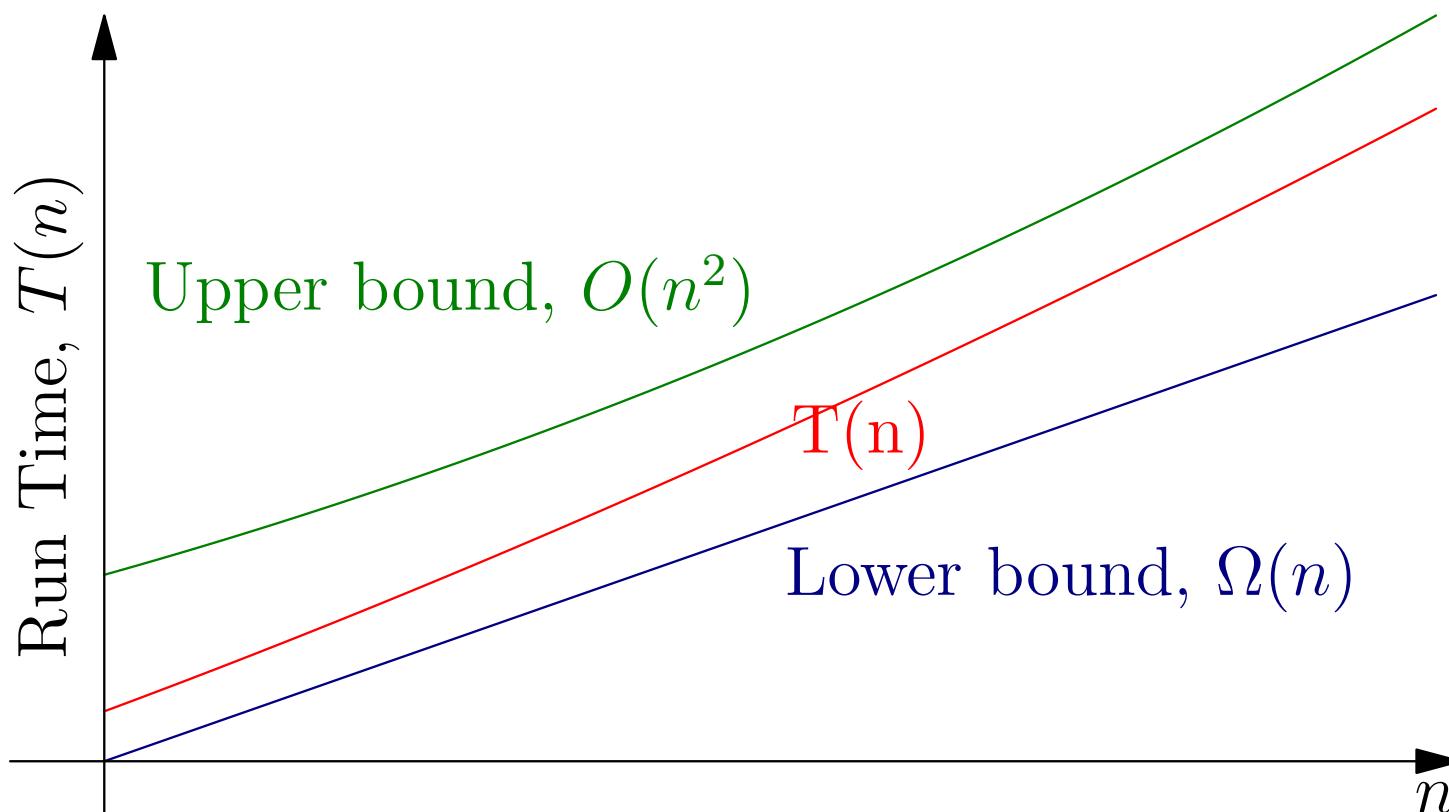
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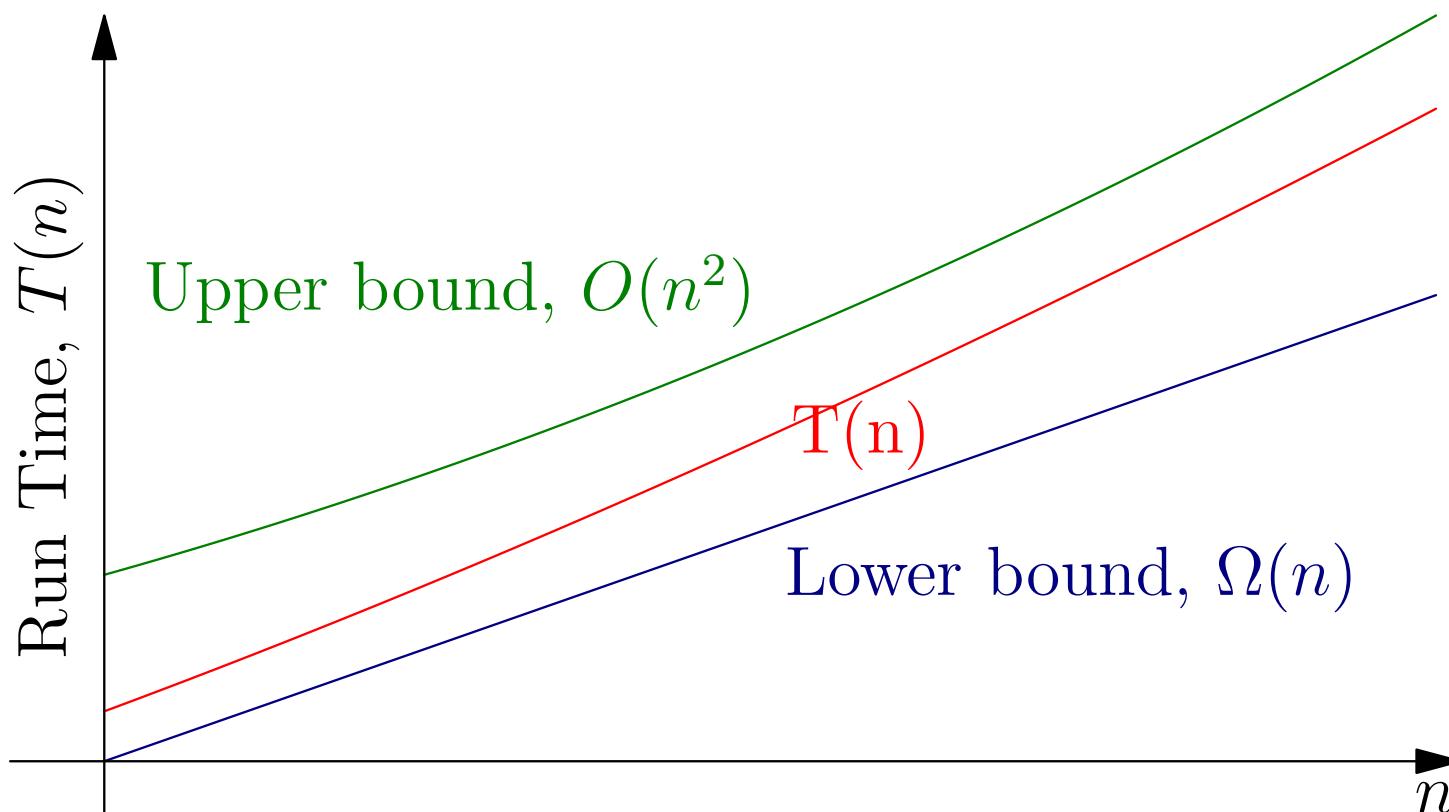
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- Time taken is approximately 200 seconds or around 3.5 minutes

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$$\log(T(n)) \in \Theta(n)$$

- This is true if
  - $T(n) = 2^n$
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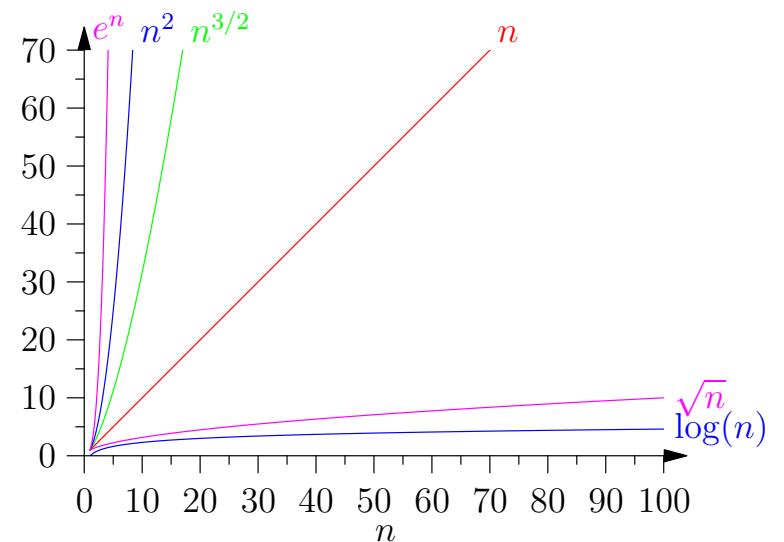
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# Outline

## 1. Time Complexity Classes

- Theta— $\Theta$
- Big O
- Little o
- Big Omega— $\Omega$
- Little omega— $\omega$

## 2. Computing Time Complexity



# Counting For Loops

- How long does the following code take?

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- Answer  $\Theta(n^2)$

# Recursion

- Determining time complexity is harder when we use recursion
- Consider Euclid's algorithm for determining the greatest common divisor

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long gcd(long m, long n)
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    while(n!=0) {
        long rem = m%n;
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- Example of Euclid's algorithm  $\text{gcd}(1989, 1590)$
- Sequence of remainders is 399, 393, 6, 3, 0
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- How long does it take compute  $\text{gcd}(n, m)$  with  $n > m$
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- An observation which makes the analysis relatively simple is that the remainder is reduced by at least 2 after two iterations
- To prove
  - ★ Using the recursion (assuming  $m, n < 0$ )

$$\gcd(m, n) = \gcd(n, \text{rem}(m, n)) = \gcd(\text{rem}(m, n), \text{rem}(n, \text{rem}(m, n)))$$

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- ★ Choose  $t = \lceil \log_2(n) \rceil$
  - ★ then  $2^{-t}n = 2^{-\lceil \log_2(n) \rceil}n \leq 2^{-\log_2(n)}n = \frac{n}{n} = 1$
  - ★ Thus  $T(2^{-t}n) < T(1) = 1$
  - ★  $T(n) < 1 + 2t = 1 + 2\lceil \log_2(n) \rceil \in O(\log(n))$
- A huge calculation shows the the average number of iterations is about  $(12 \log(2) \log(n))/\pi^2 + 1.47$

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# Probability of Relative Primes

- Consider the following program to compute the probability of relative primes for all numbers up to  $n$

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double probRelPrime(n)
{
    int rel=0, tot=0;
    for(int i=1; i<=n; i++)
        for(int j=i+1; j<=n; j++) {
            tot++;
            if (gcd(i, j)==1)
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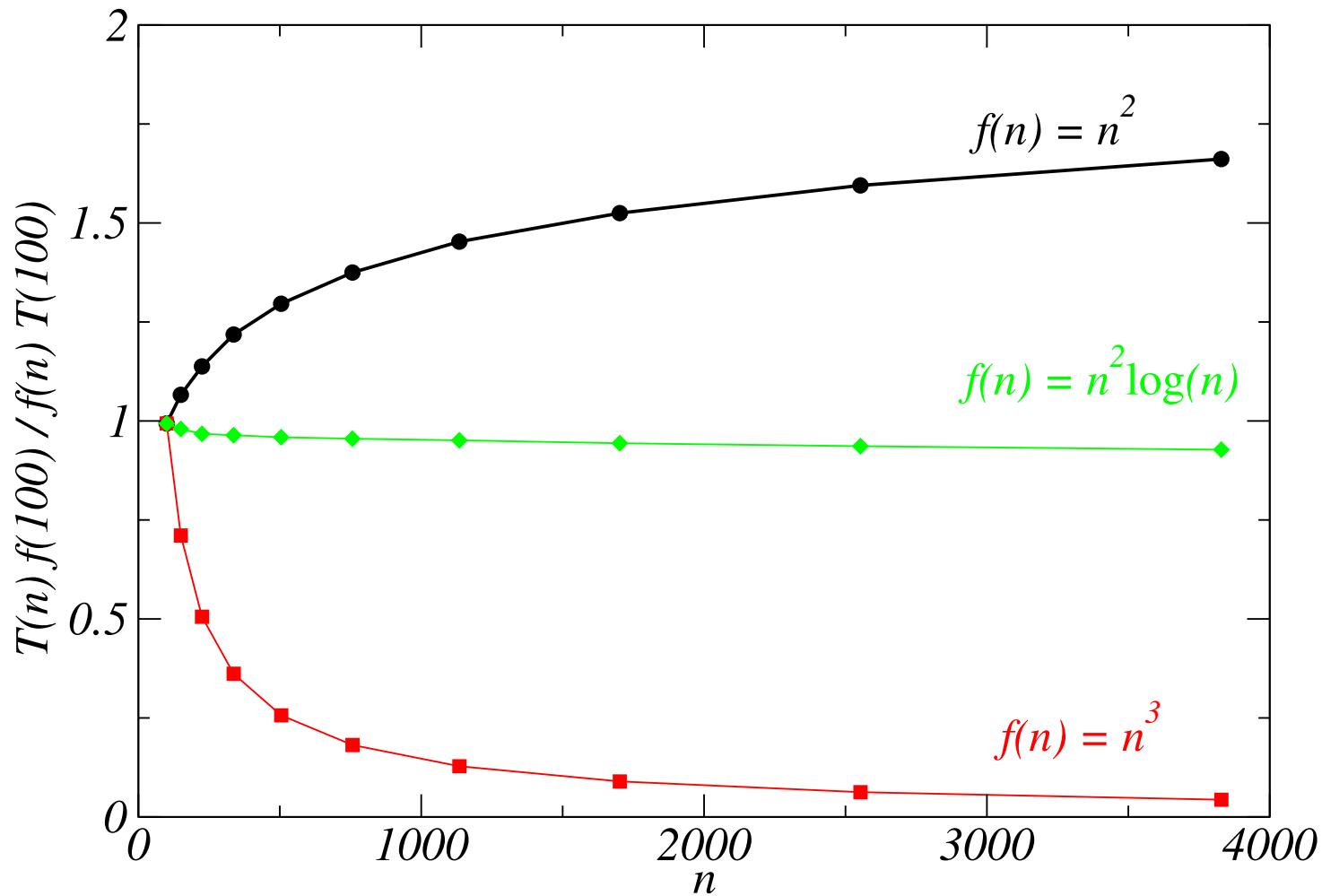
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# Testing Hypothesis

- We can test our hypothesis by scaling the run time by the complexity



# Conclusions

- You should understand the difference between  $\Theta$ ,  $O$ ,  $o$ ,  $\Omega$  and  $\omega$
- You need to be able to compute time complexity by loop counting
- To compute time complexity for recursive functions you need to be able to obtain recurrence equations
- You should be able to solve simple recurrence equations and sum up simple series
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- Thank you for attending the course