Further Mathematics and Algorithms

Lesson 14: *Use Heaps!*

Heaps, Priority queues, Heap Sort, Other heaps

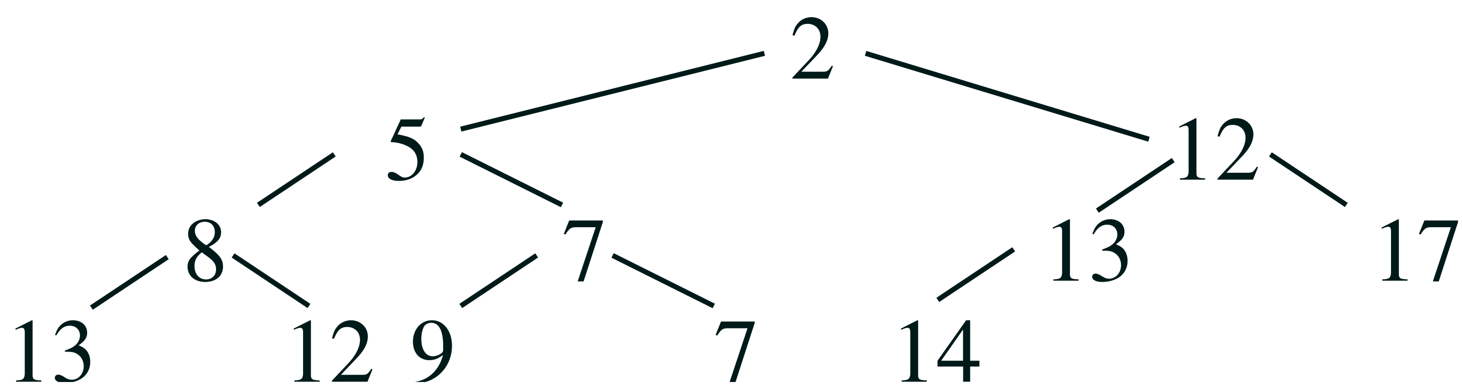
Outline

1. **Heaps**
2. Priority Queues
   - Array Implementation
3. Heap Sort
4. Other Heaps

Heaps

- A (min-)heap is (from one perspective) a binary tree

- It is a binary tree satisfying two constraints
  - ⋆ It is a **complete** tree
  - ⋆ Each child has a value 'greater than or equal to' its parent



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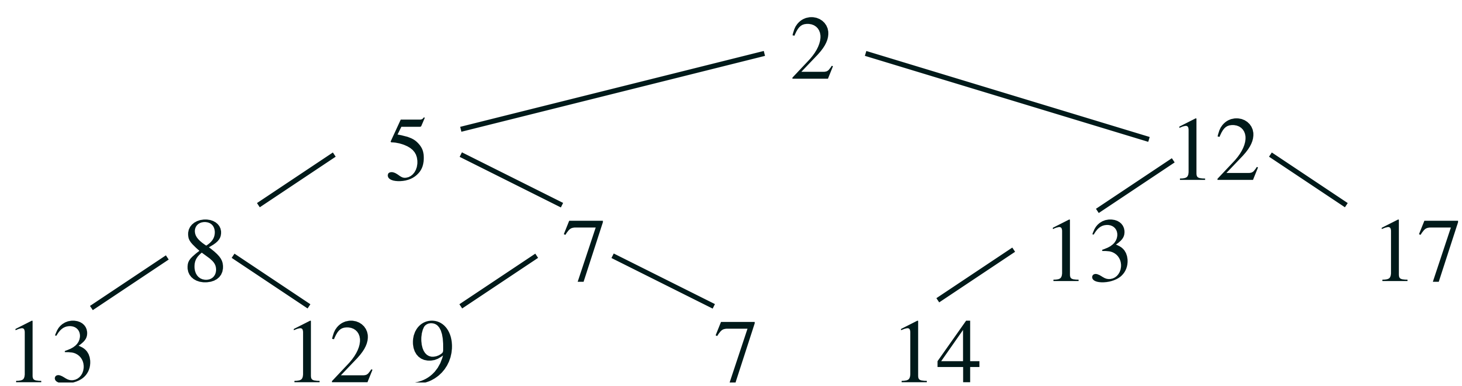

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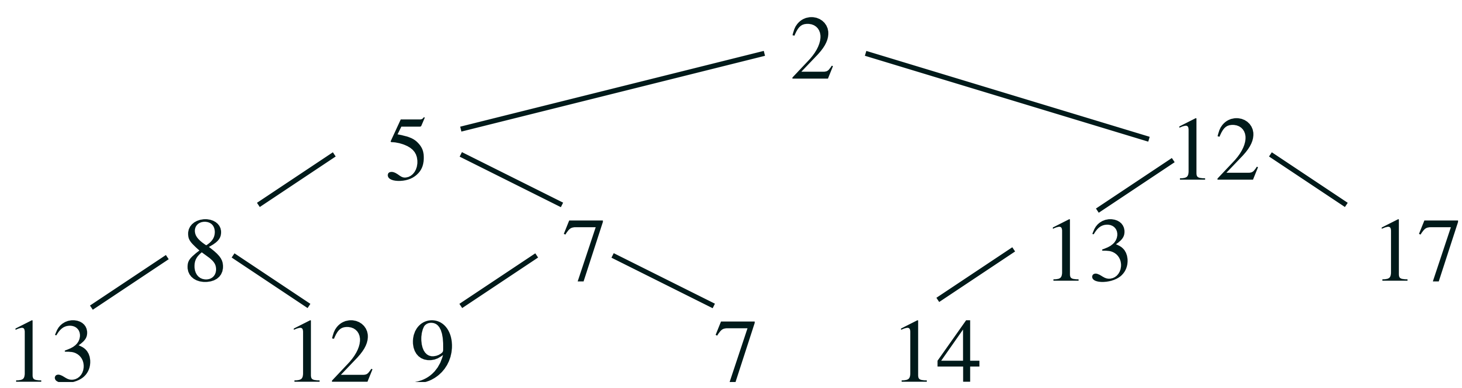

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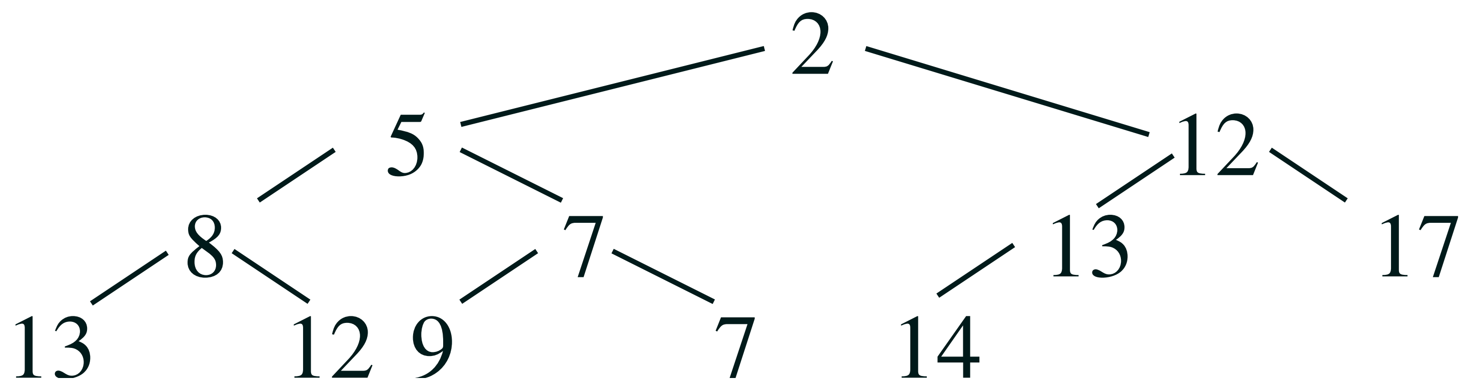

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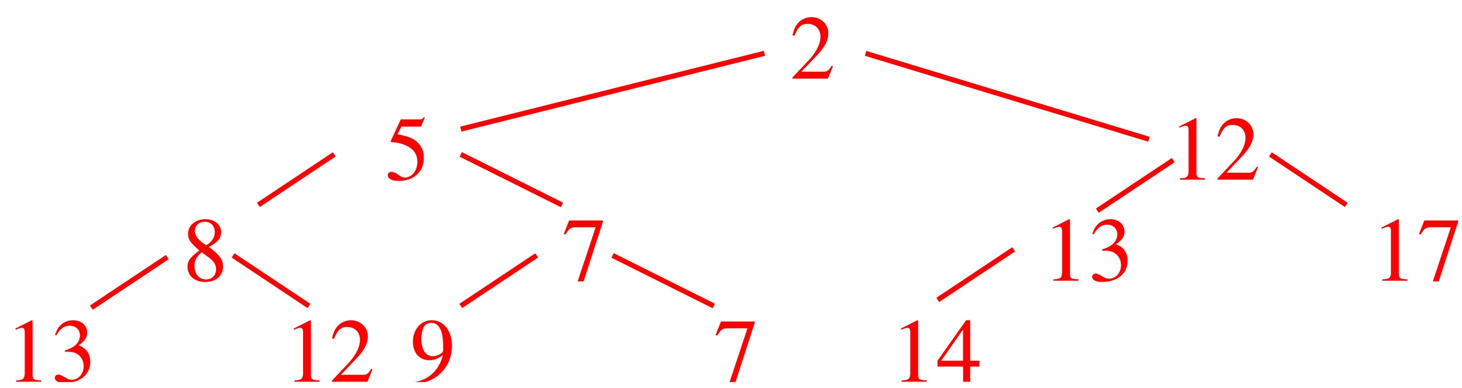

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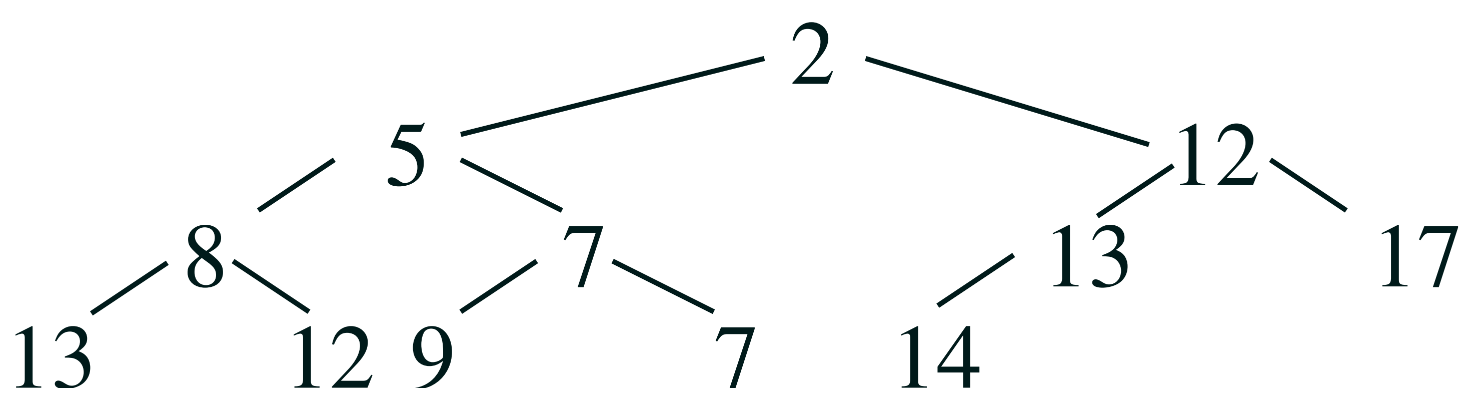

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Priority Queues

- One of the prime uses of heaps is to implement a priority queue
- A Priority Queue is a queue with priorities
- That is, we assign a priority to each element we add
- The head of the queue is the element with highest priority (smallest number)
- Used, for example, in simulating real time events
- Used to implement “greedy algorithms”

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Priority Queue

- A simple Priority Queue might include
  - ⋆ `unsigned size()` returning the the number of elements
  - ⋆ `bool empty()` returns true if empty
  - ⋆ `void push(T element, int priority)` adds an element
  - ⋆ `T top()` returns head of queue
  - ⋆ `void pop()` dequeues head of queue

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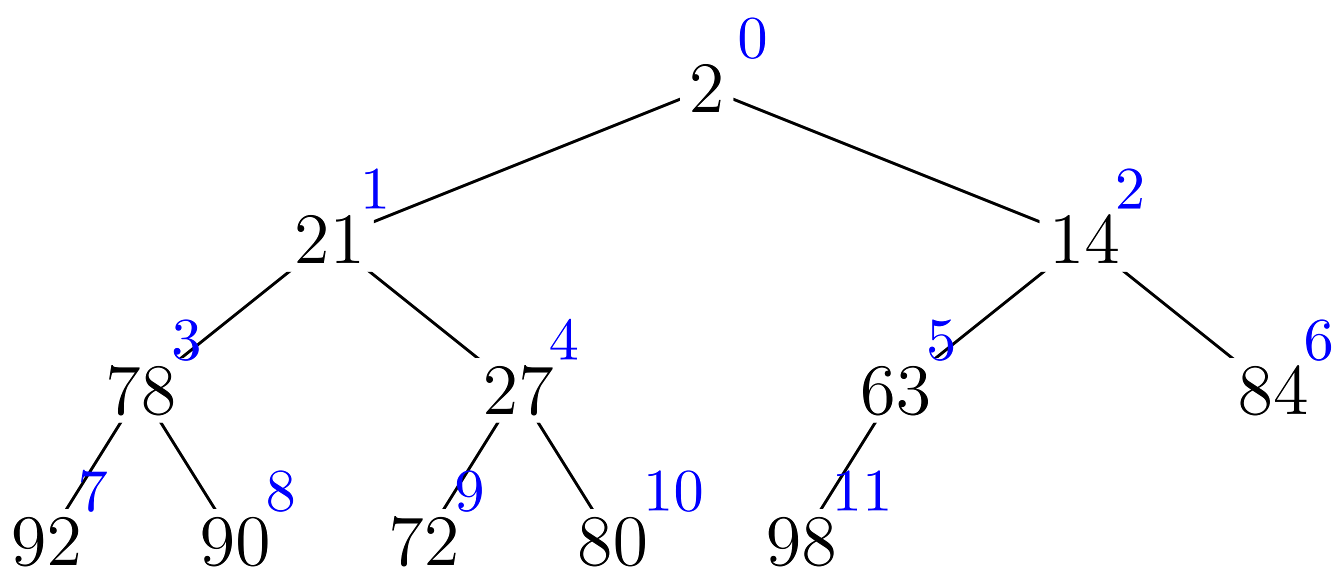
0	1	2	3	4	5	6	7	8	9	10	11		



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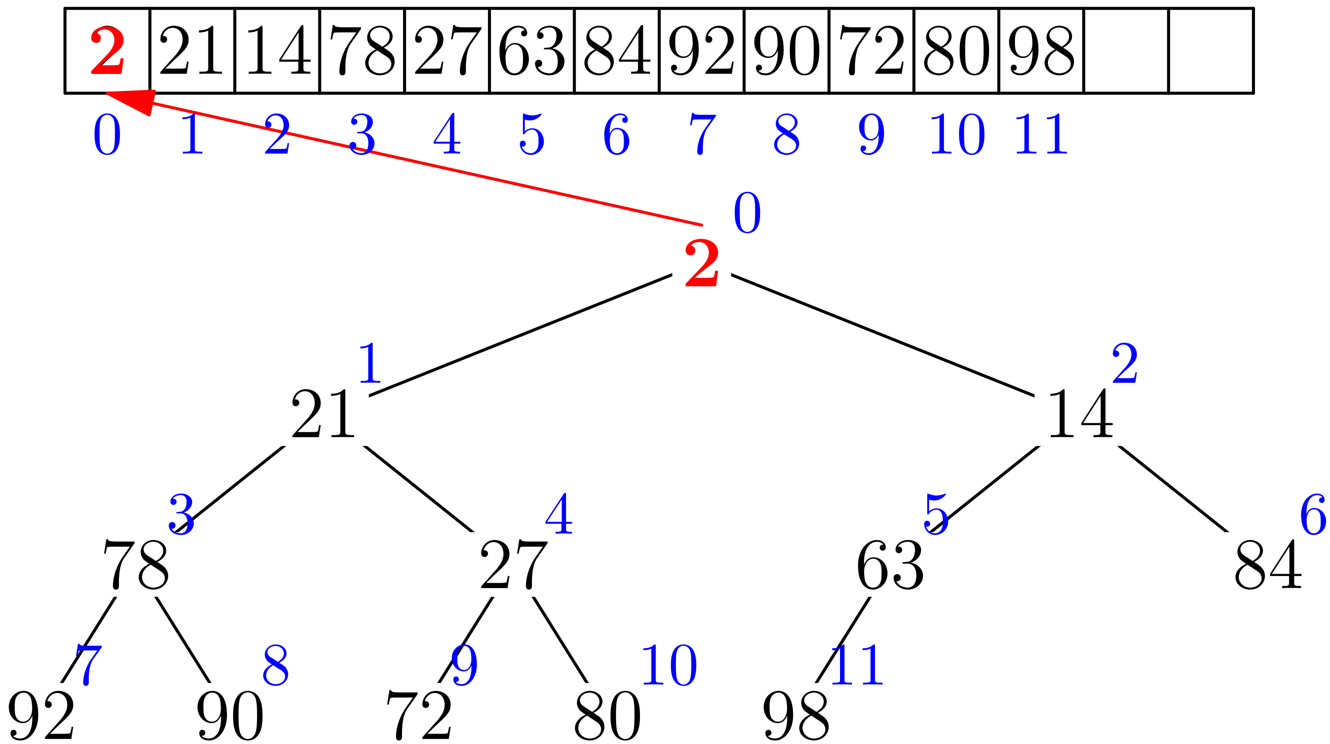

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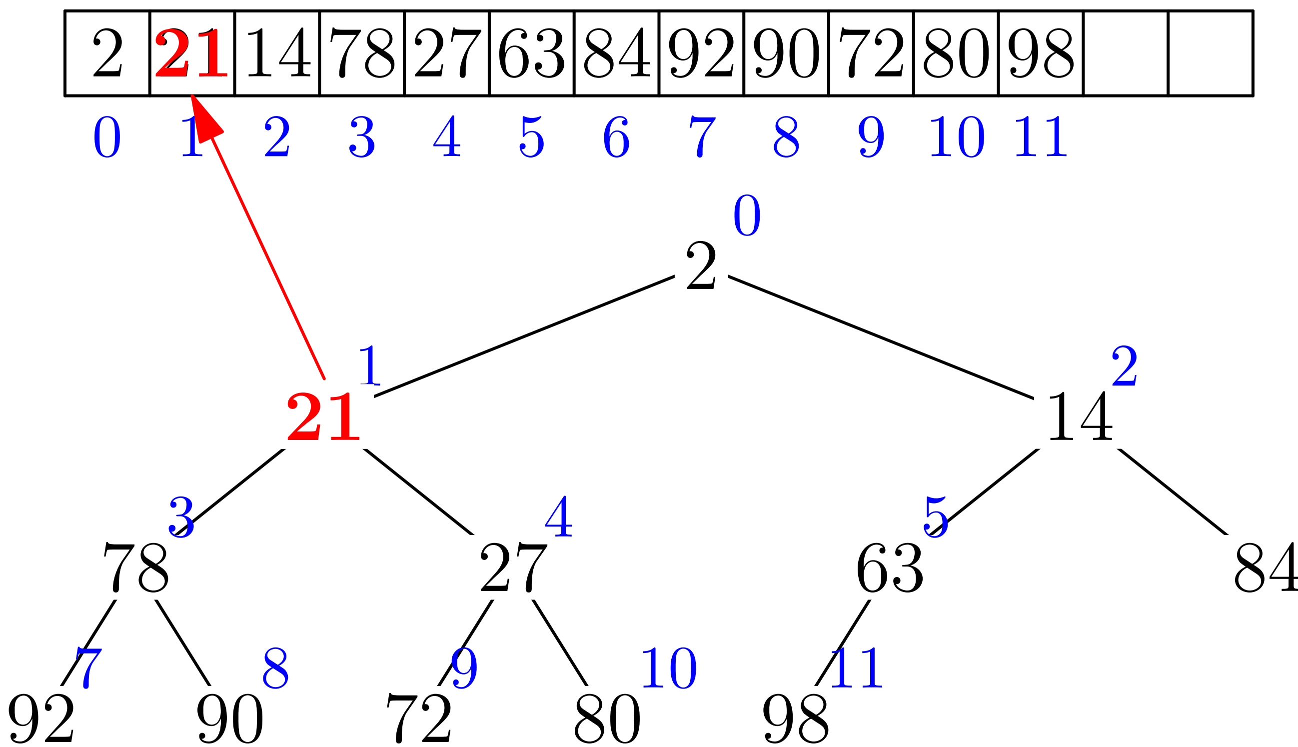

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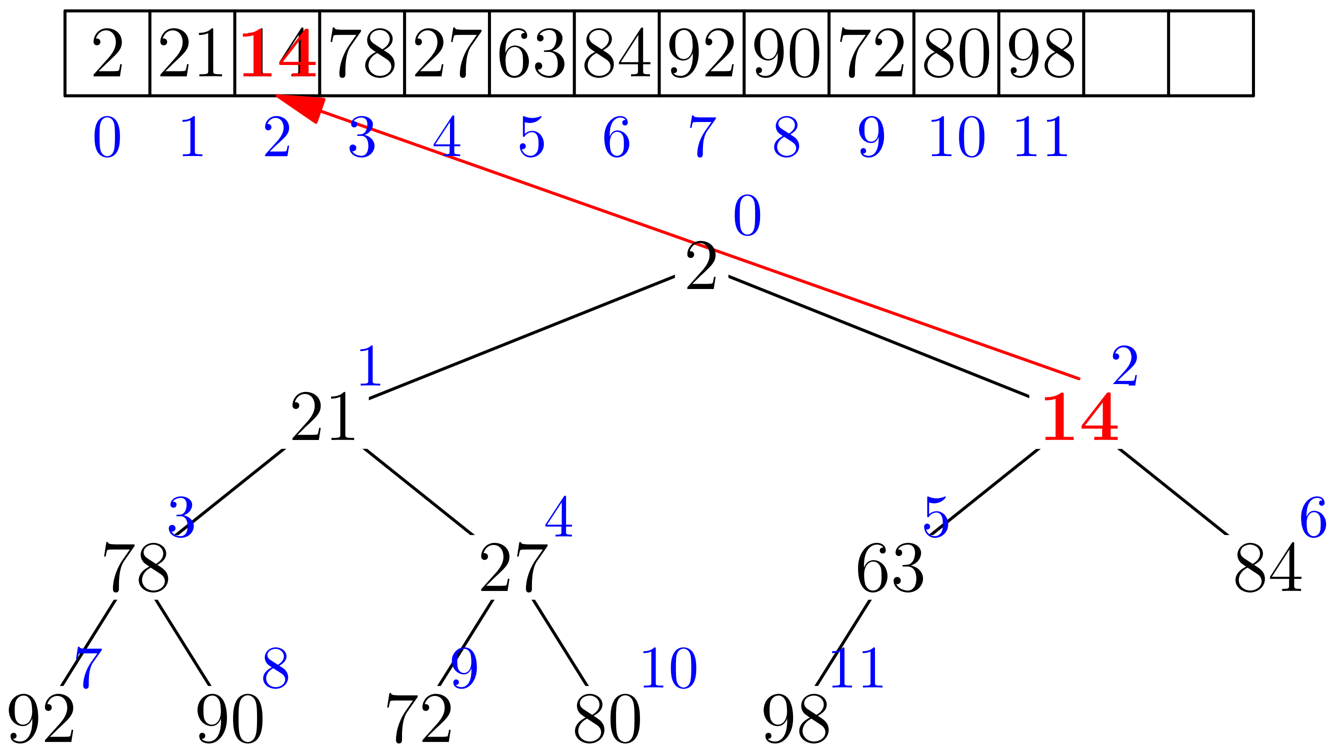

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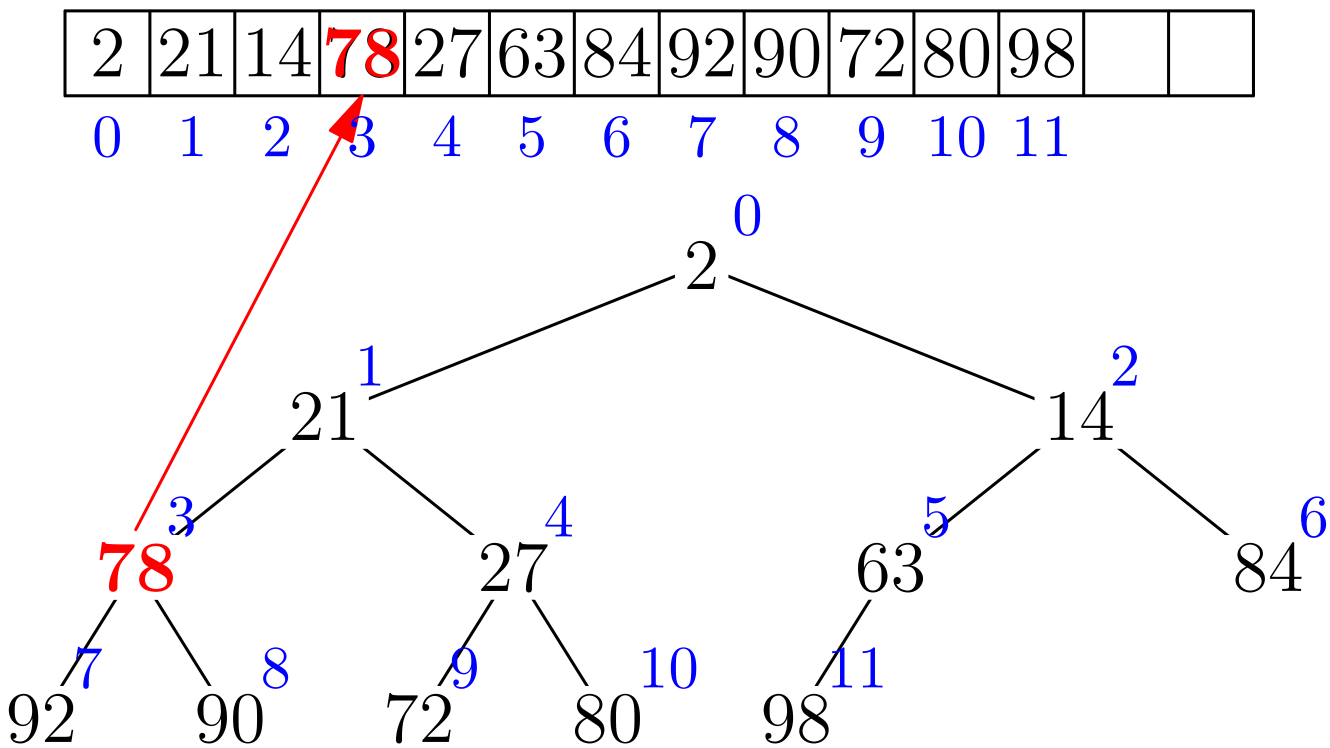

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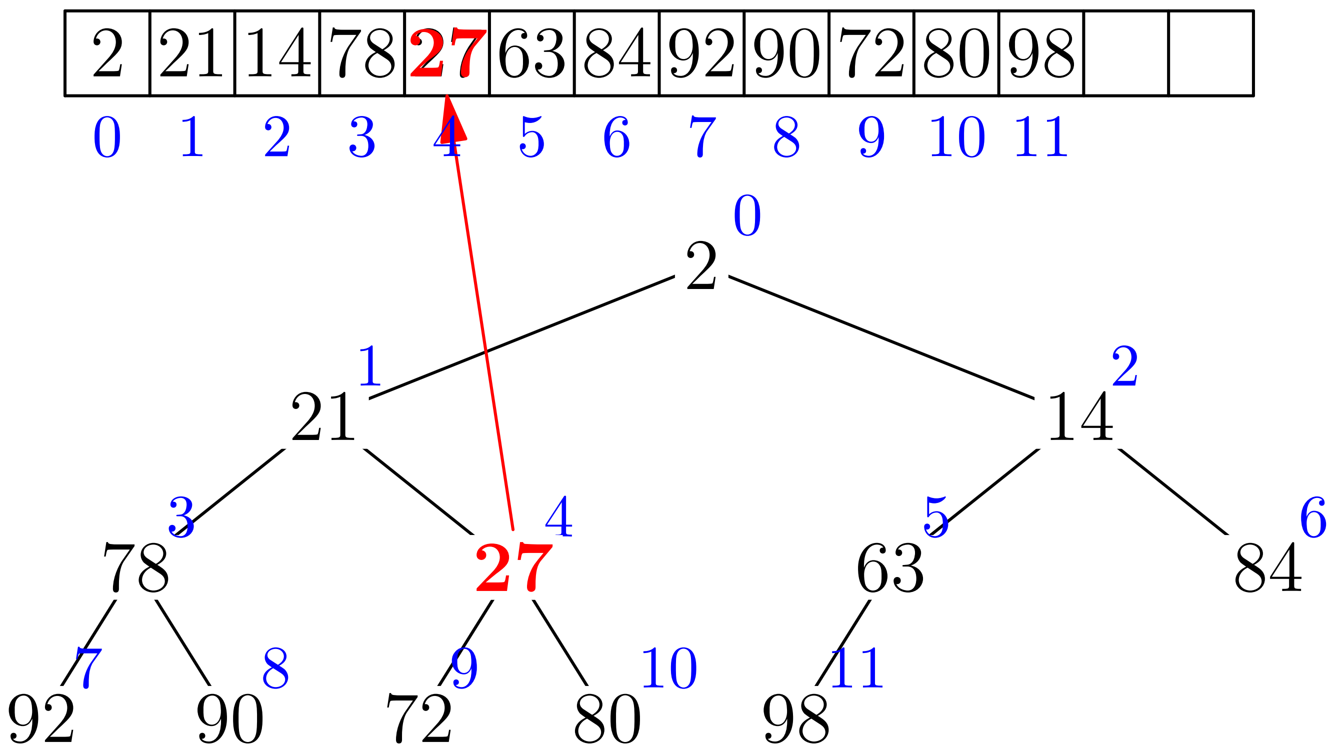

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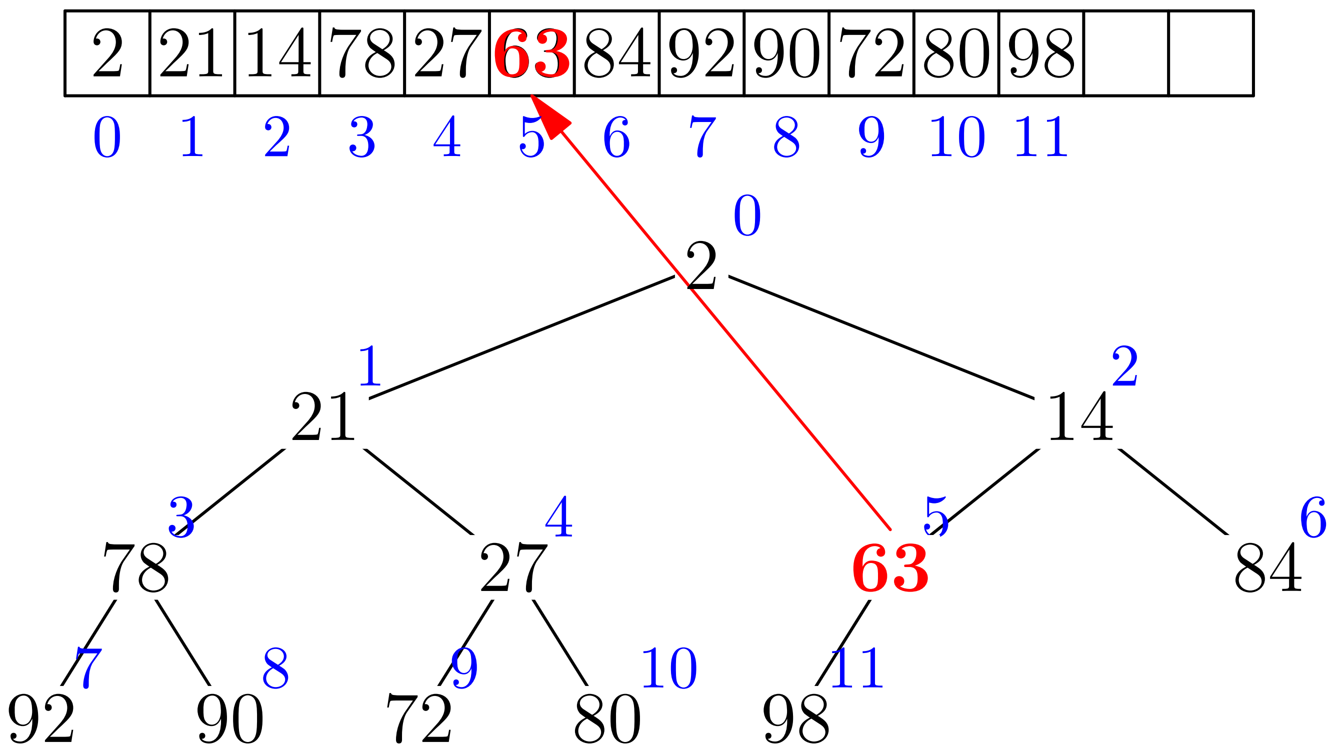

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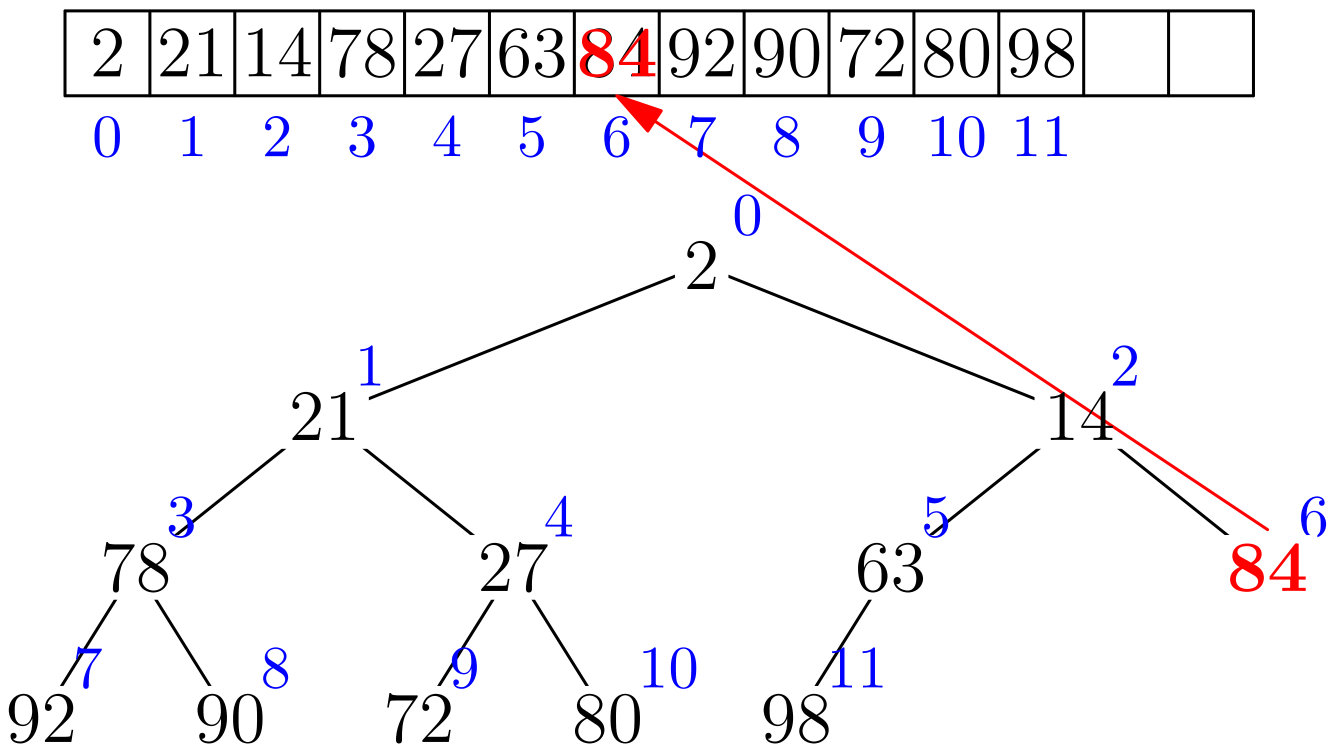

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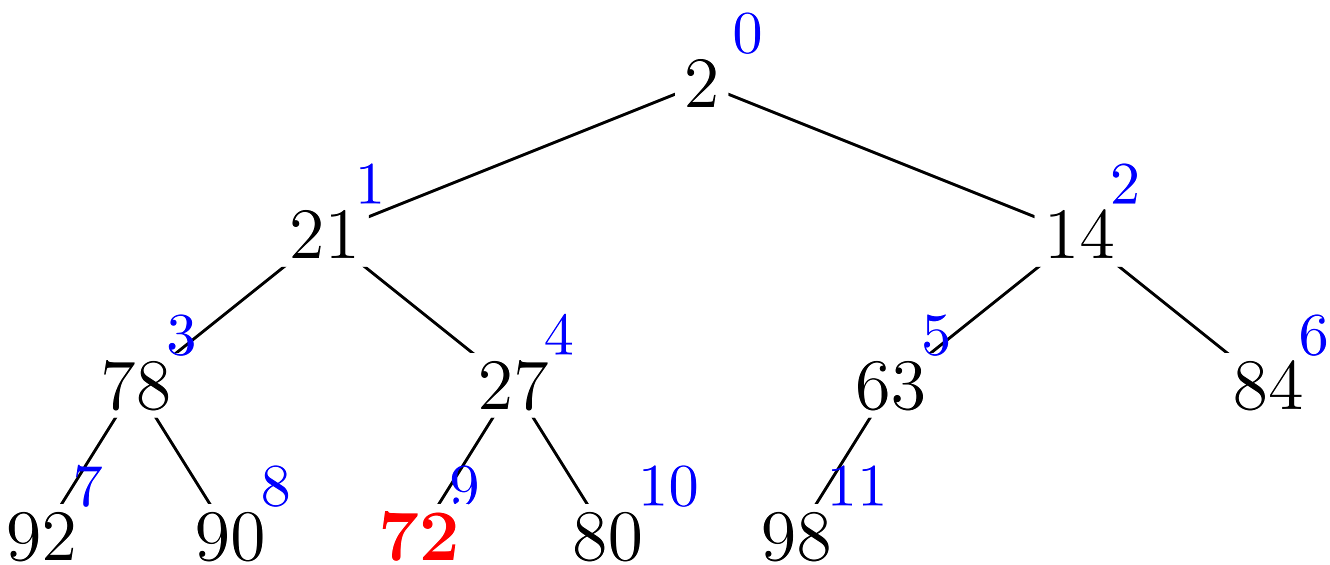

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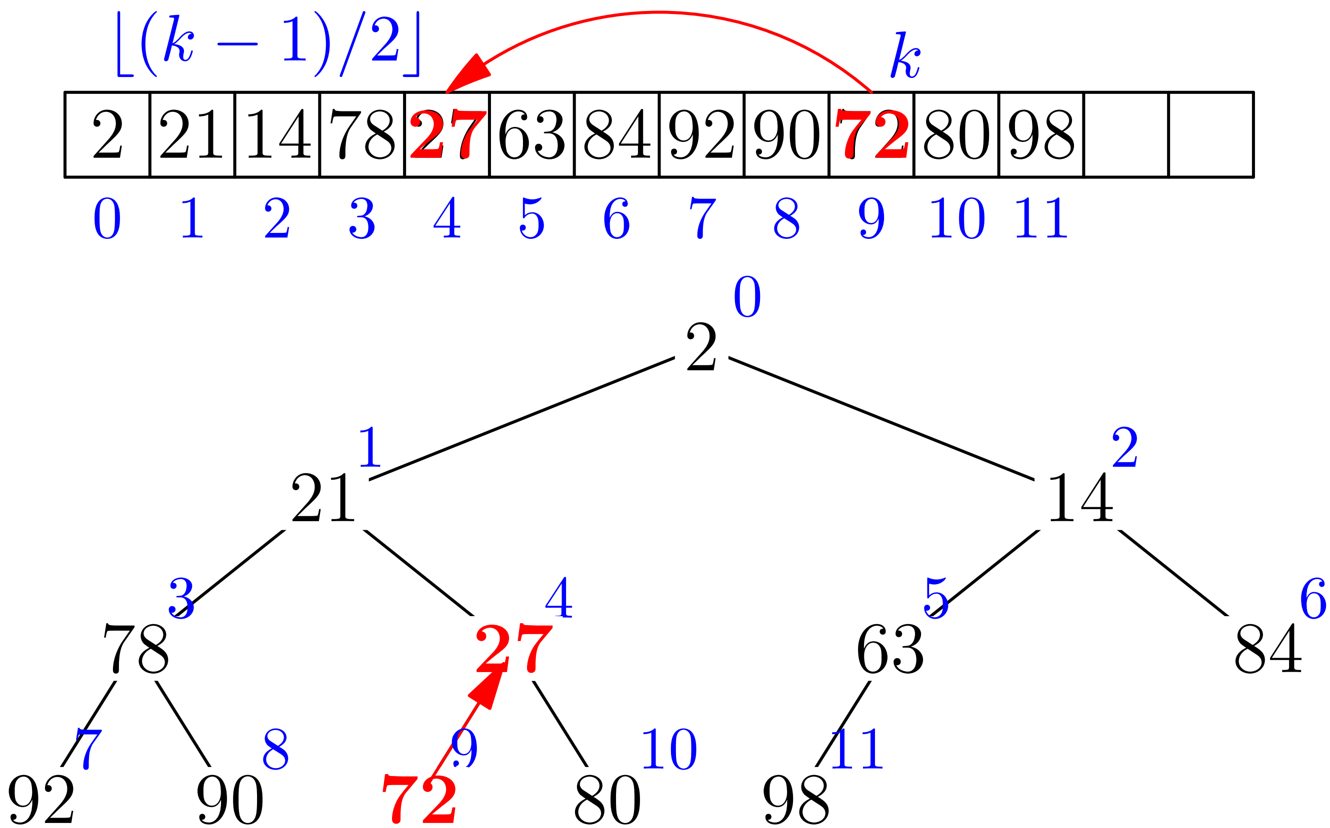

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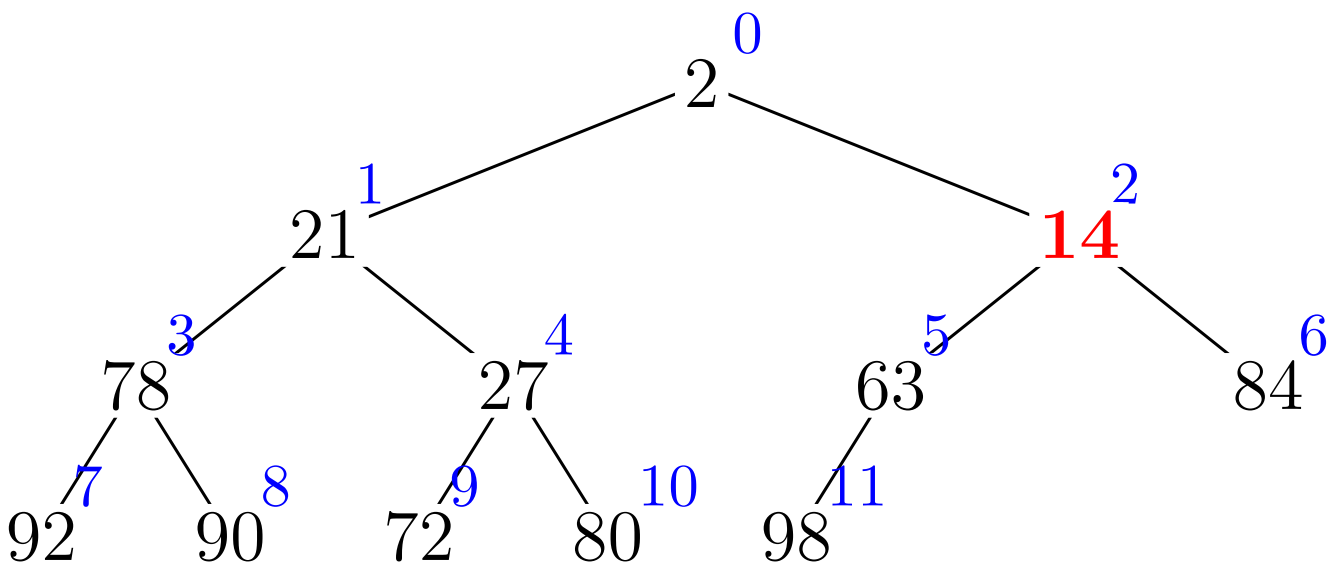

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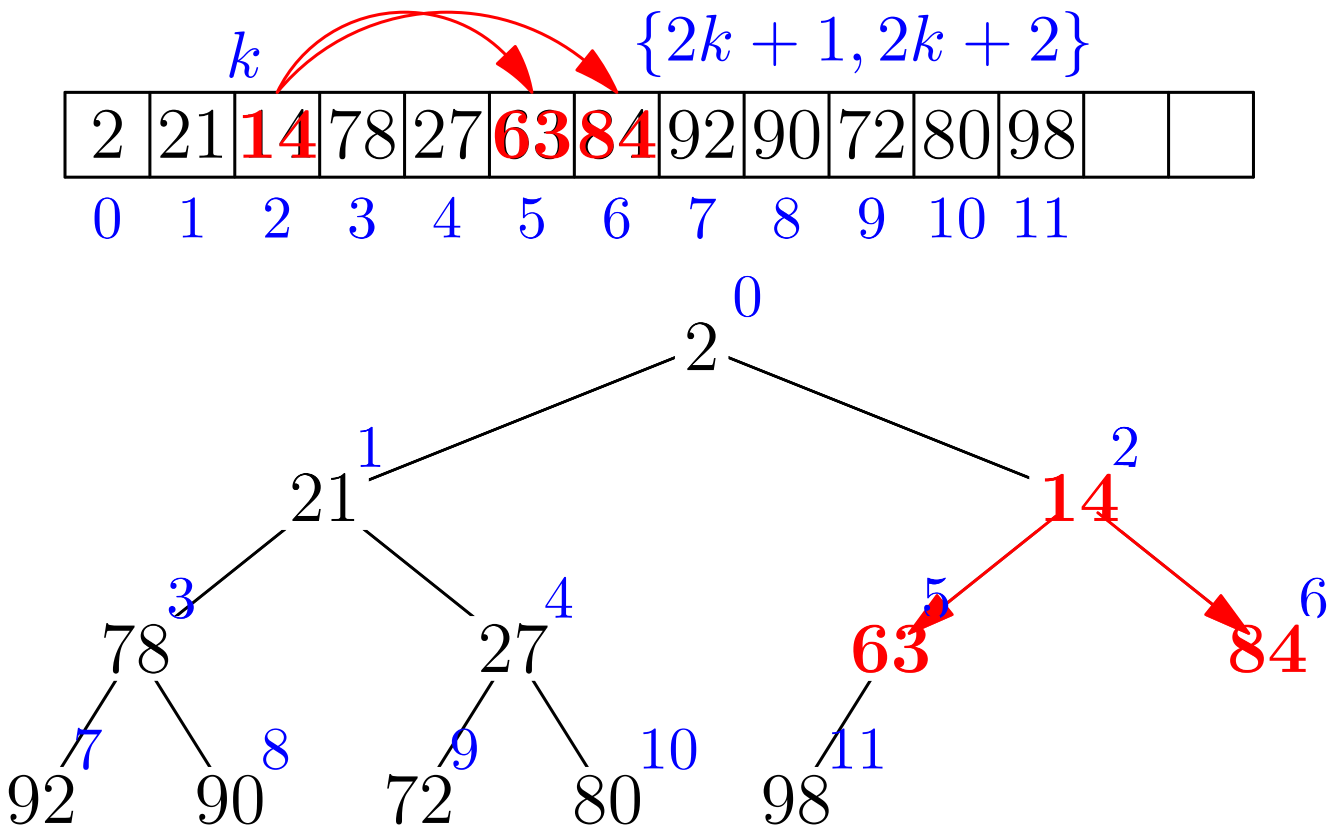

Code for a Priority Queue

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using namespace std;

template <typename T, typename P>
class heapPQ {
private:
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  const T& top() {return array[0].first;}
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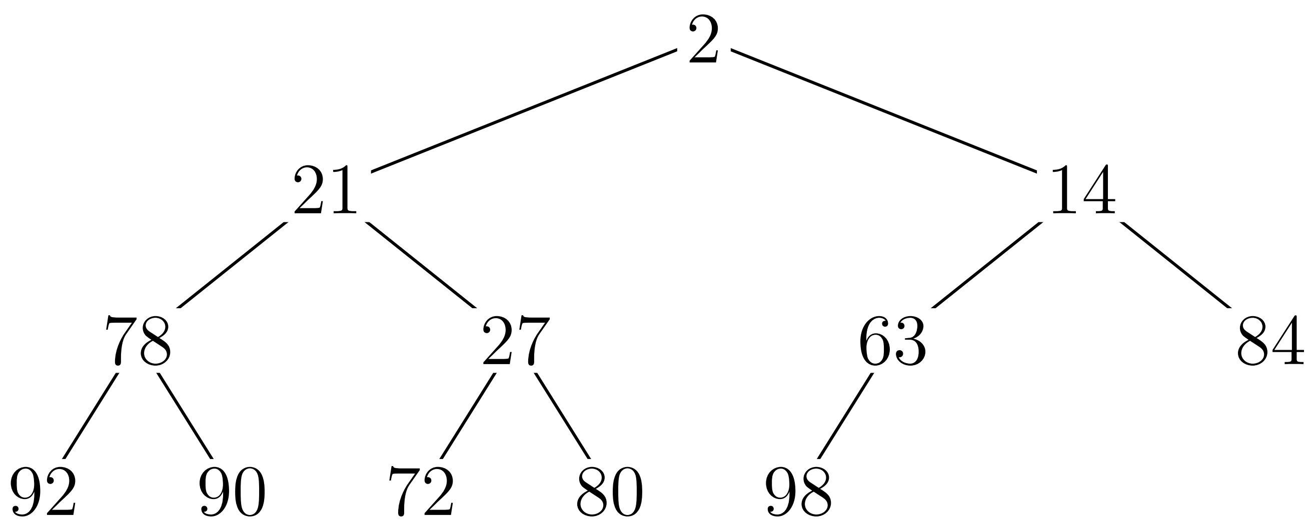
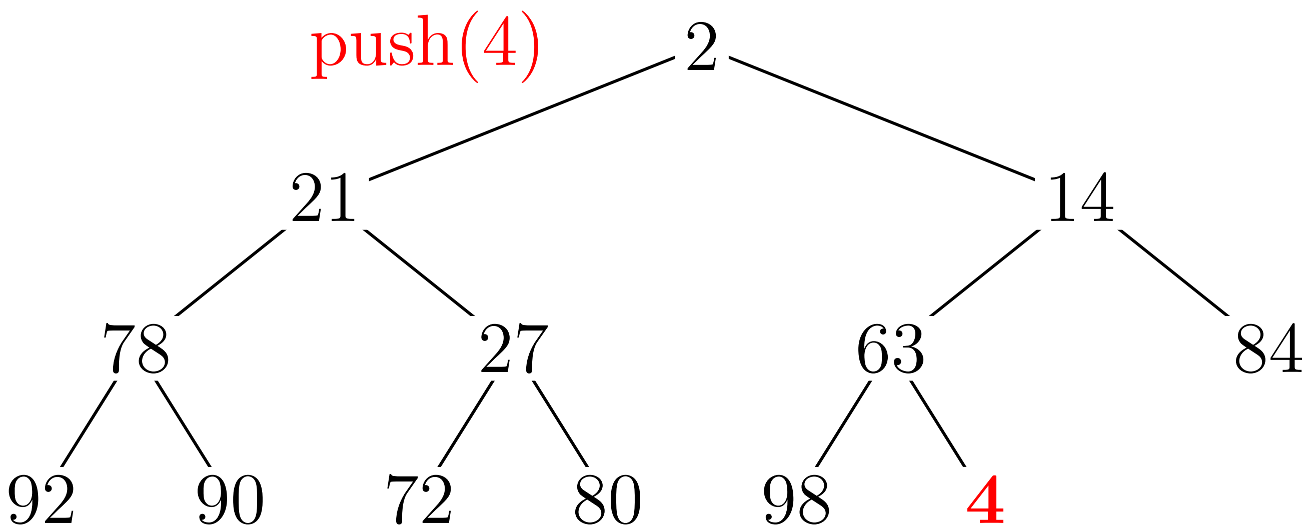
Adding an Element

2	21	14	78	27	63	84	92	90	72	80	98		



Adding an Element

2	21	14	78	27	63	84	92	90	72	80	98	**4**	



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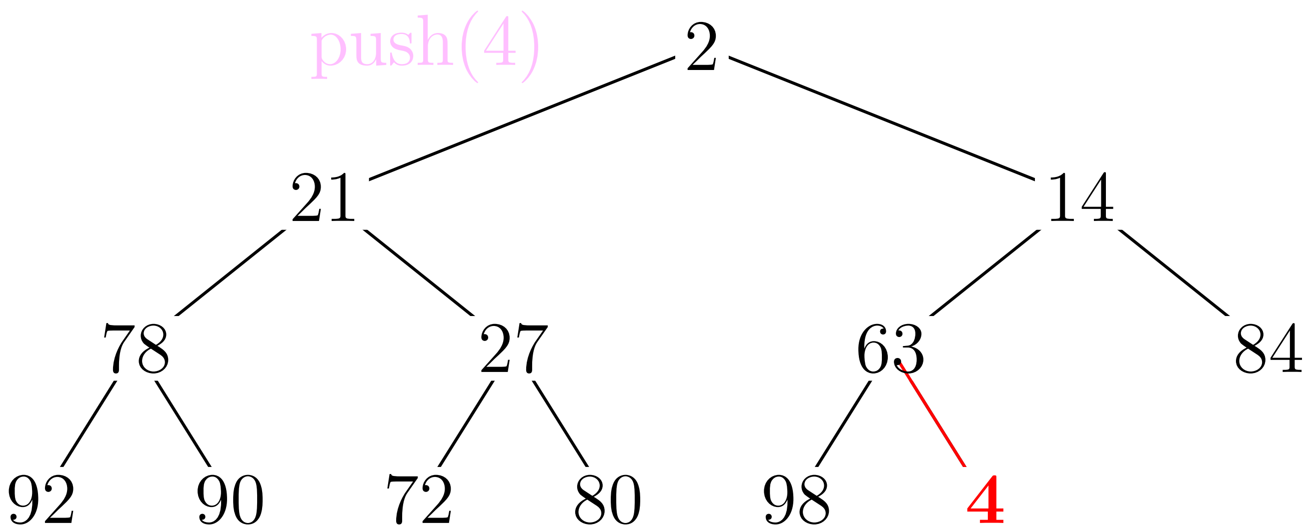

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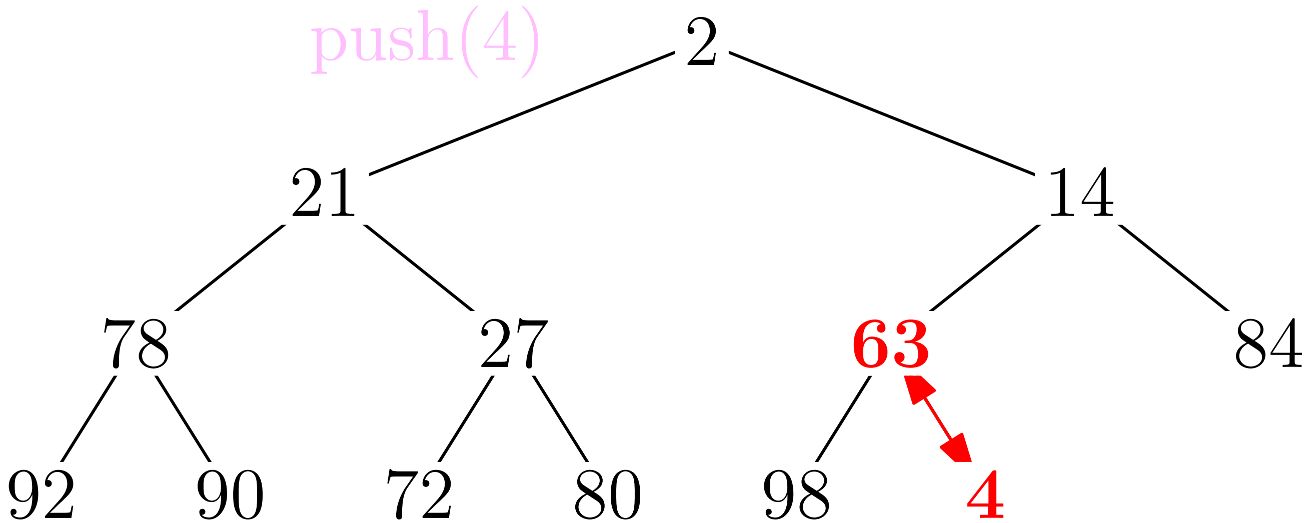

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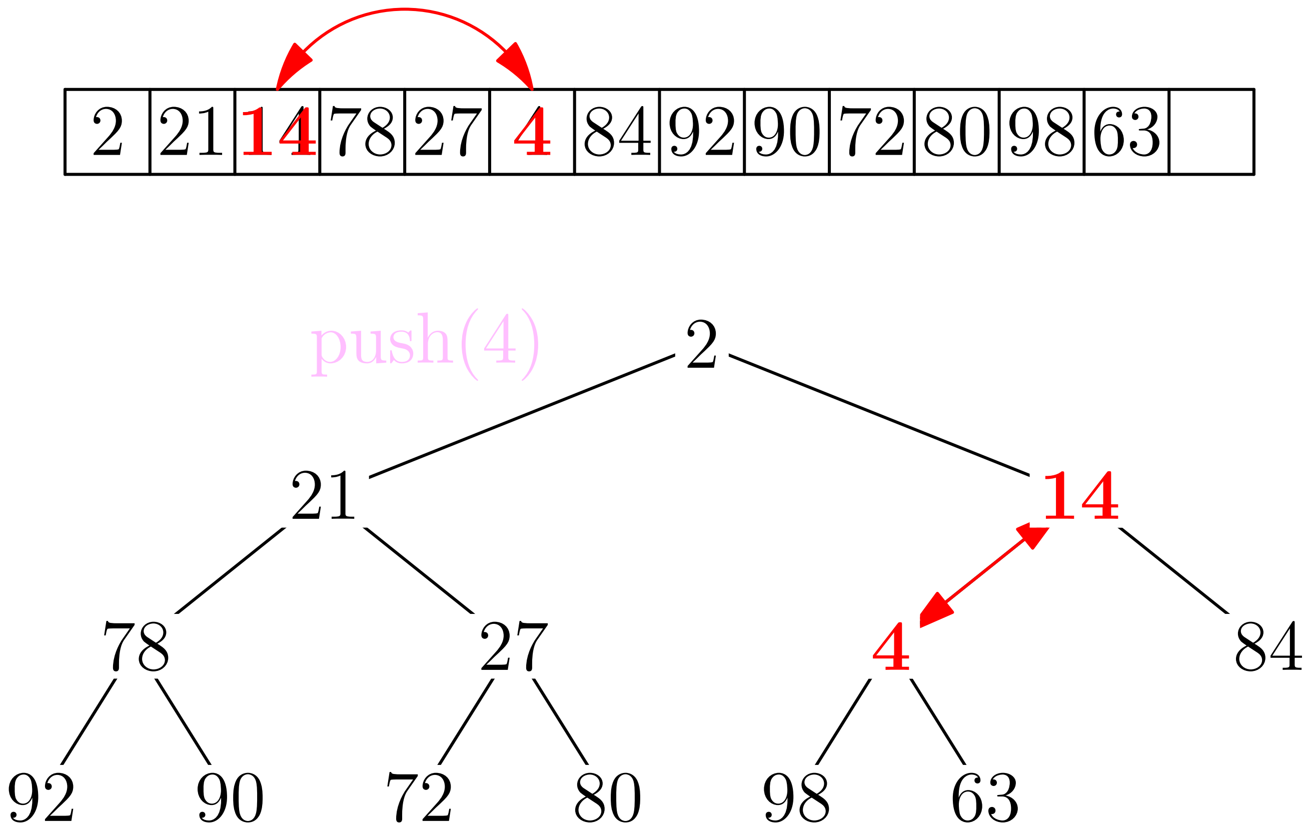

Adding an Element

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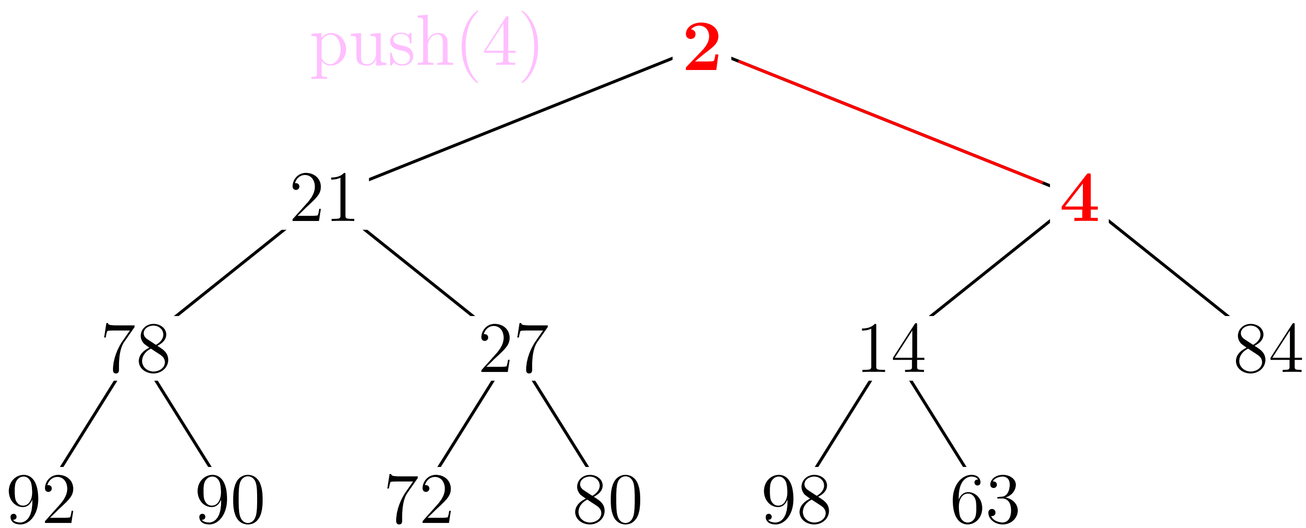

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void push(T value, P priority) {
  pair<T,P> tmp(value, priority);
  array.push_back(tmp);
  unsigned child = size() - 1;

  /* Percolate Up */
  while(child!=0) {
    unsigned parent = (child-1)>>1;  // floor((child-1)/2)
    if (array[parent].second < array[child].second)
      return;
    array[child] = array[parent];
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}
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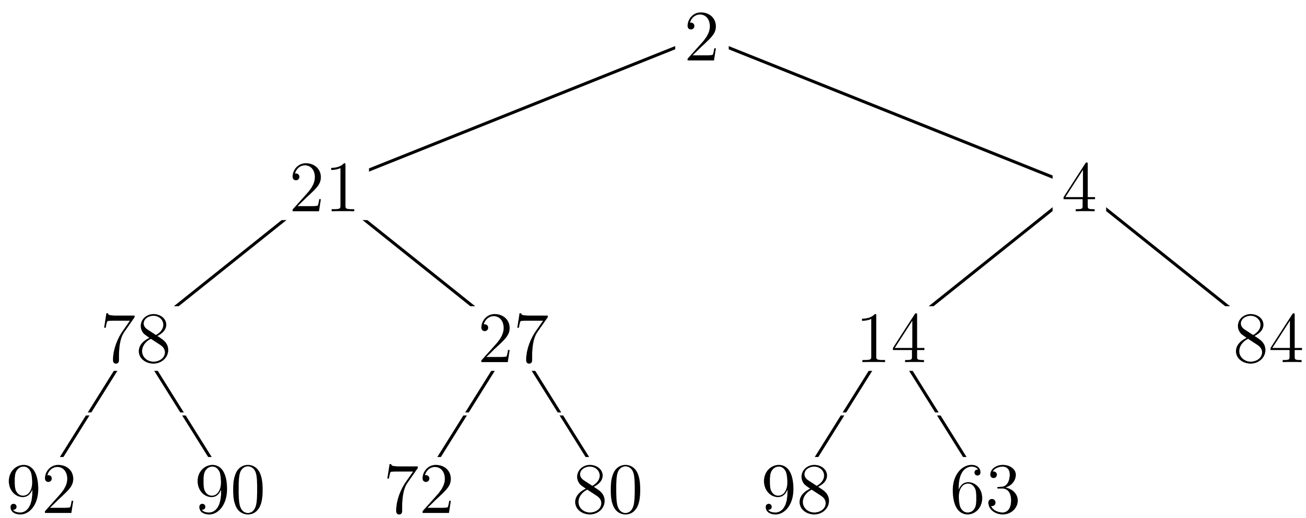
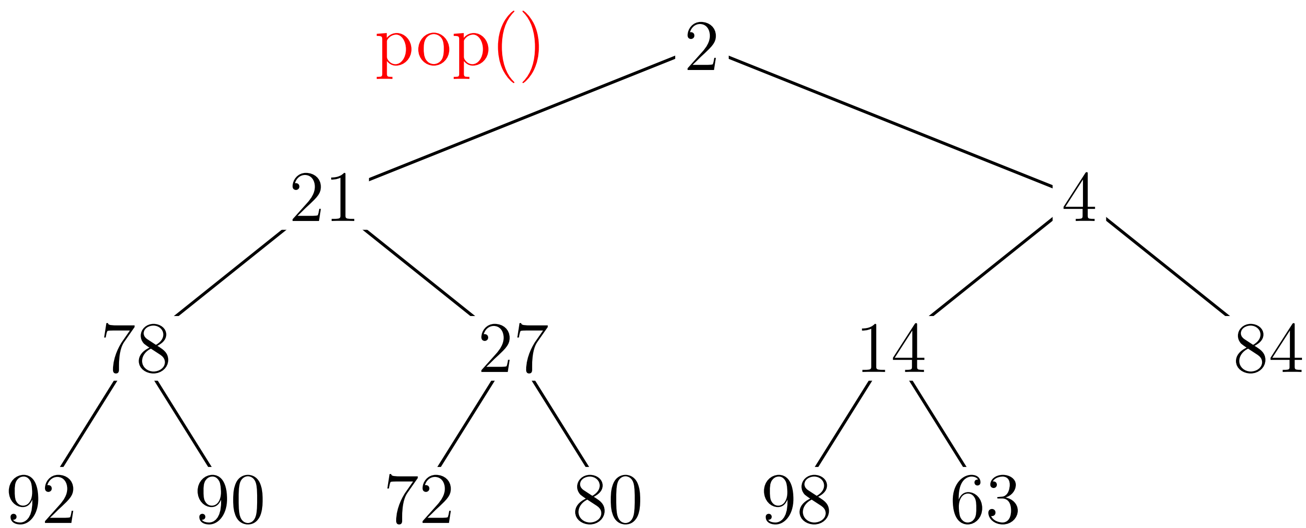
Popping the Top

2	21	4	78	27	14	84	92	90	72	80	98	63	



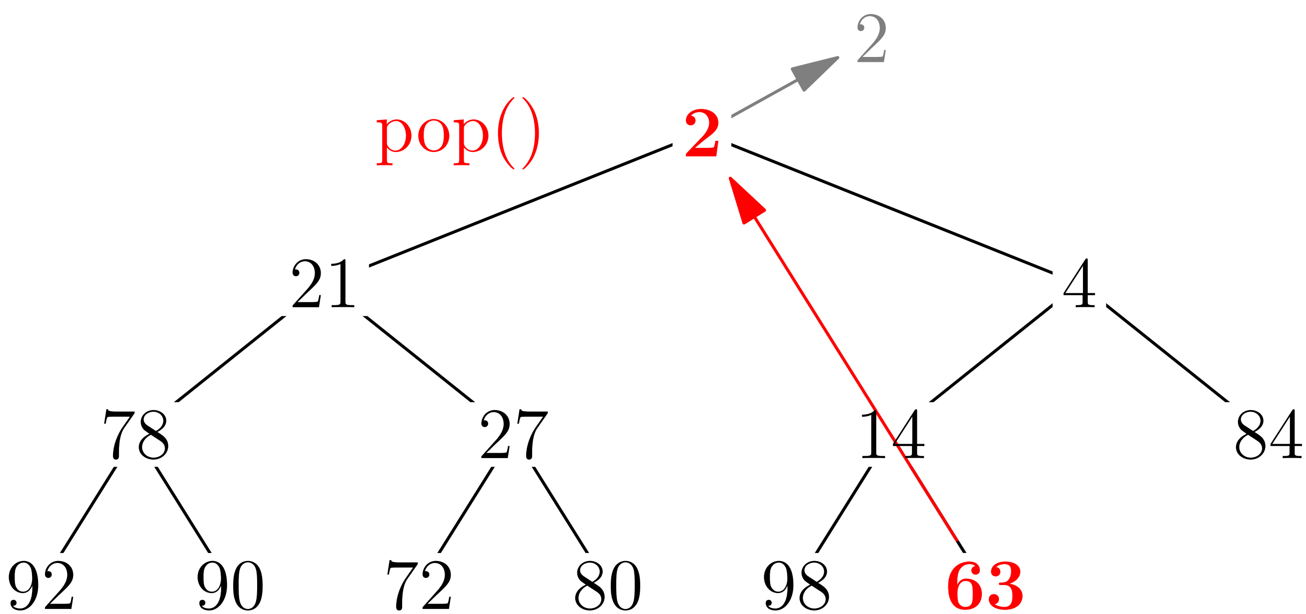
Popping the Top

2	21	4	78	27	14	84	92	90	72	80	98	63	



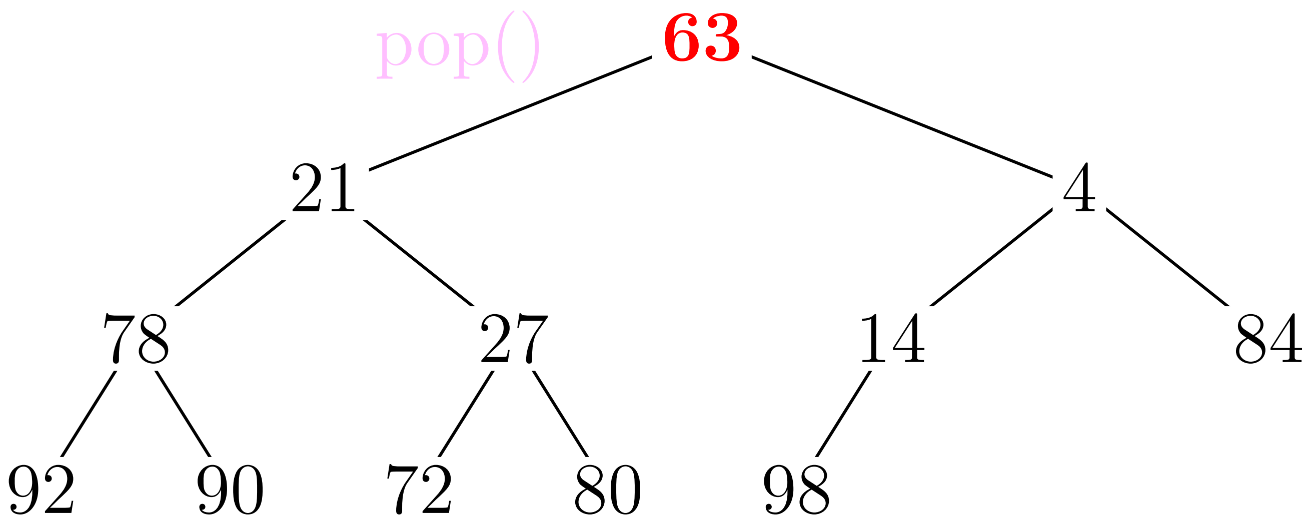
Popping the Top

2	21	4	78	27	14	84	92	90	72	80	98	**63**	



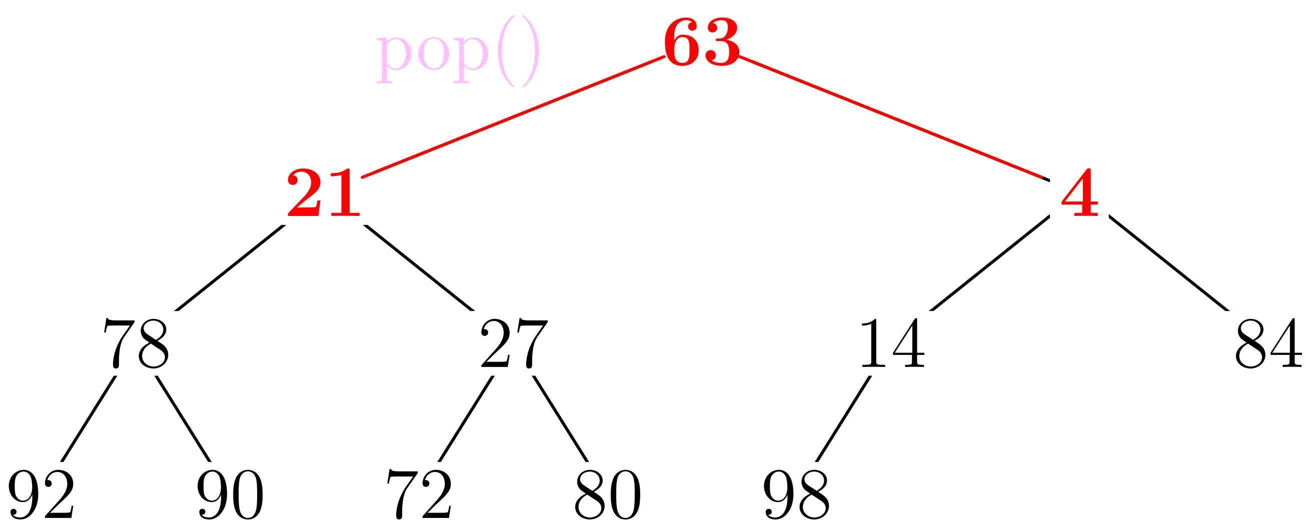
Popping the Top

63	21	4	78	27	14	84	92	90	72	80	98		



Popping the Top

63	**21**	**4**	78	27	14	84	92	90	72	80	98		



Popping the Top

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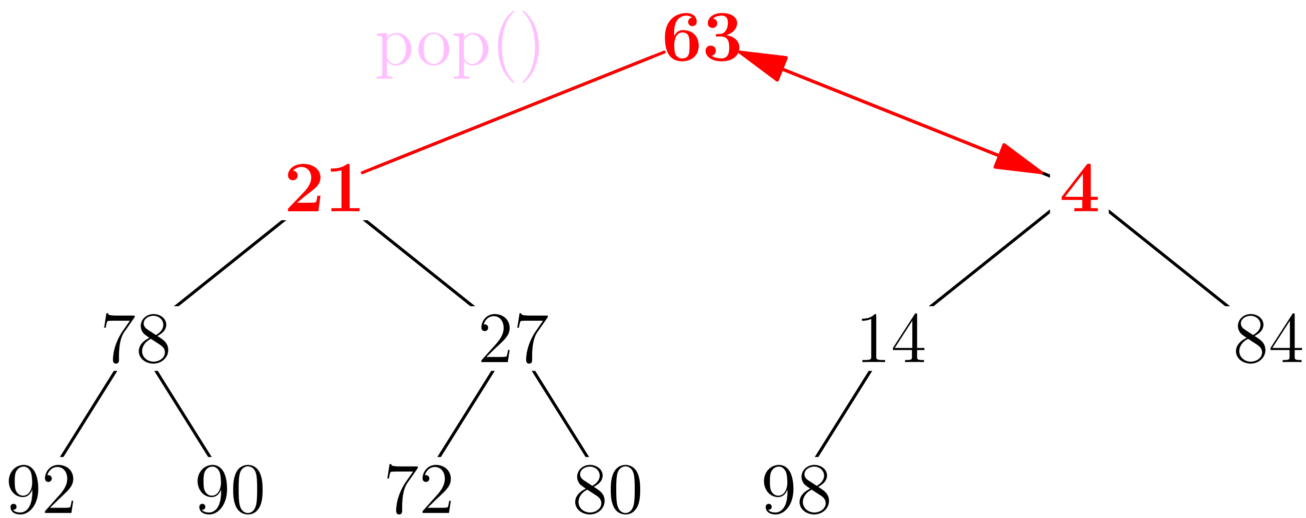

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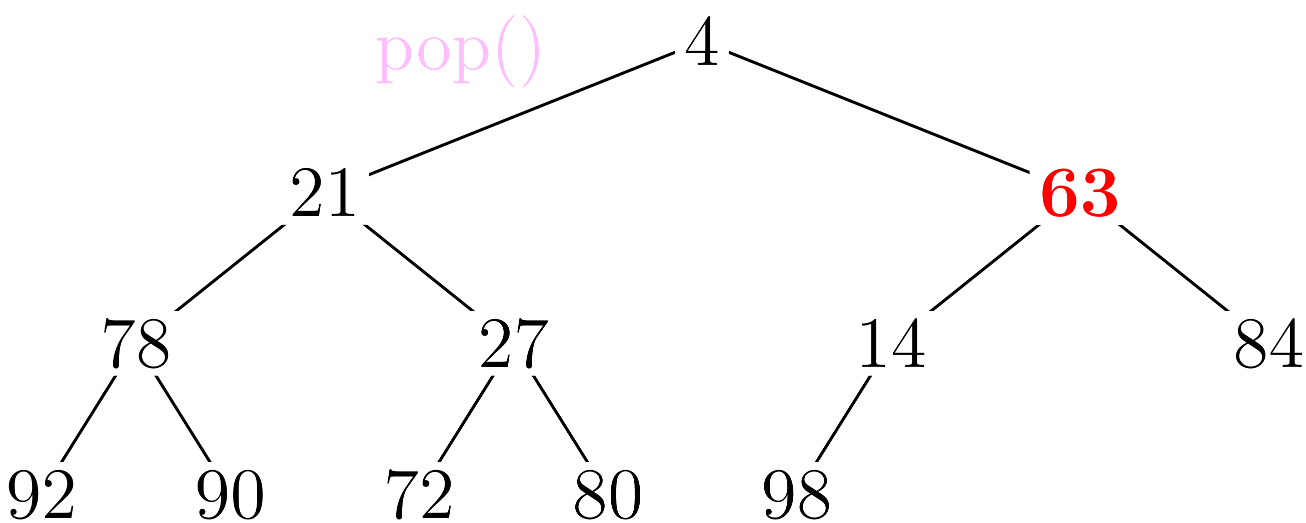
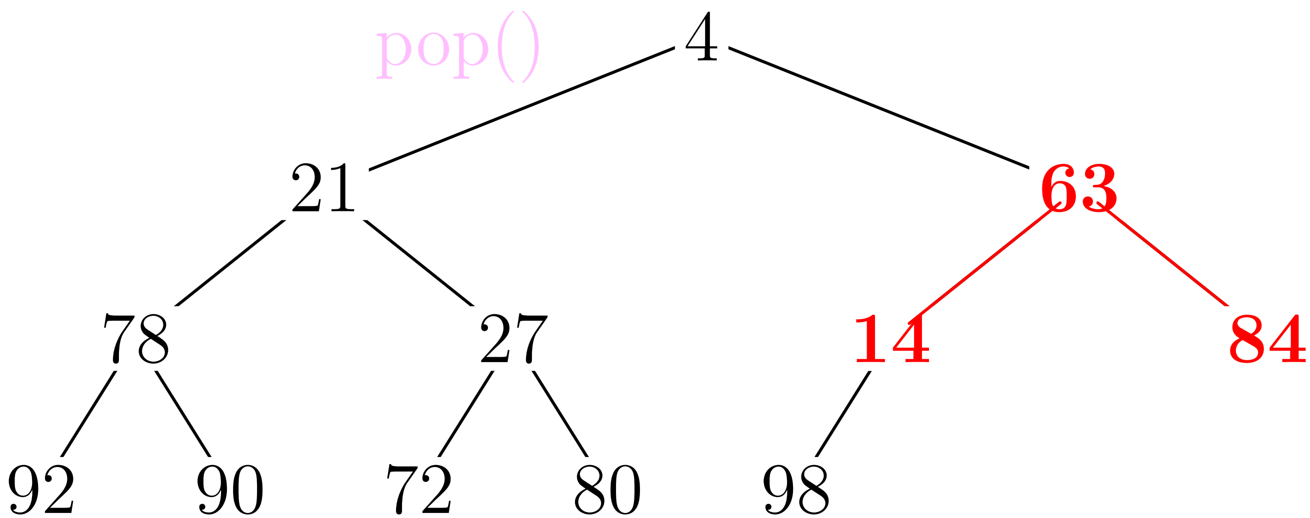


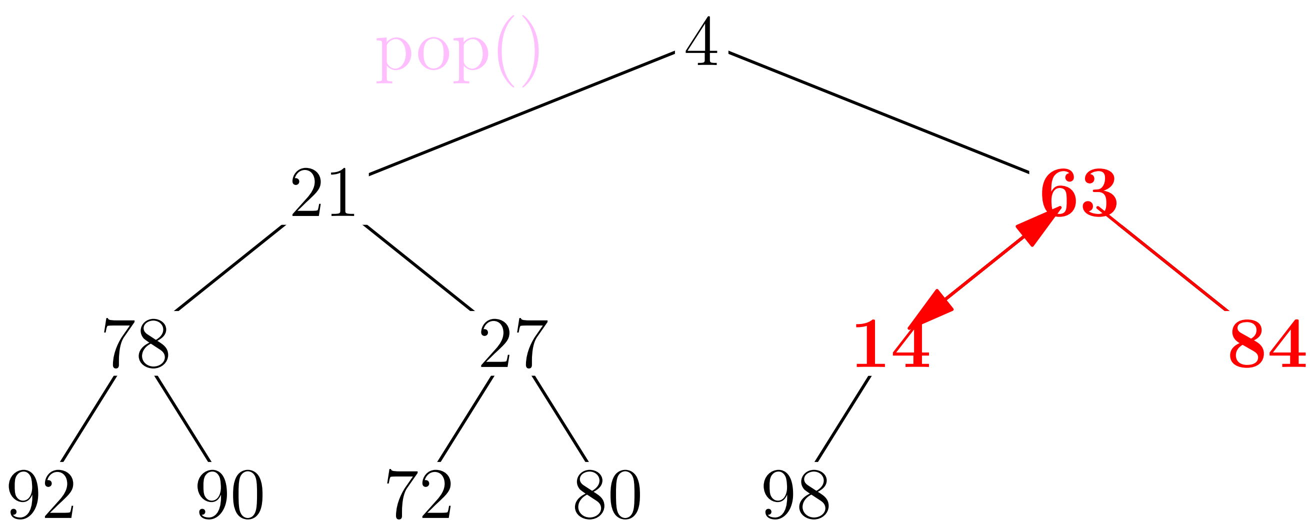
Popping the Top

4	21	**63**	78	27	**14**	**84**	92	90	72	80	98		



Popping the Top

4	21	**63**	78	27	**14**	**84**	92	90	72	80	98		



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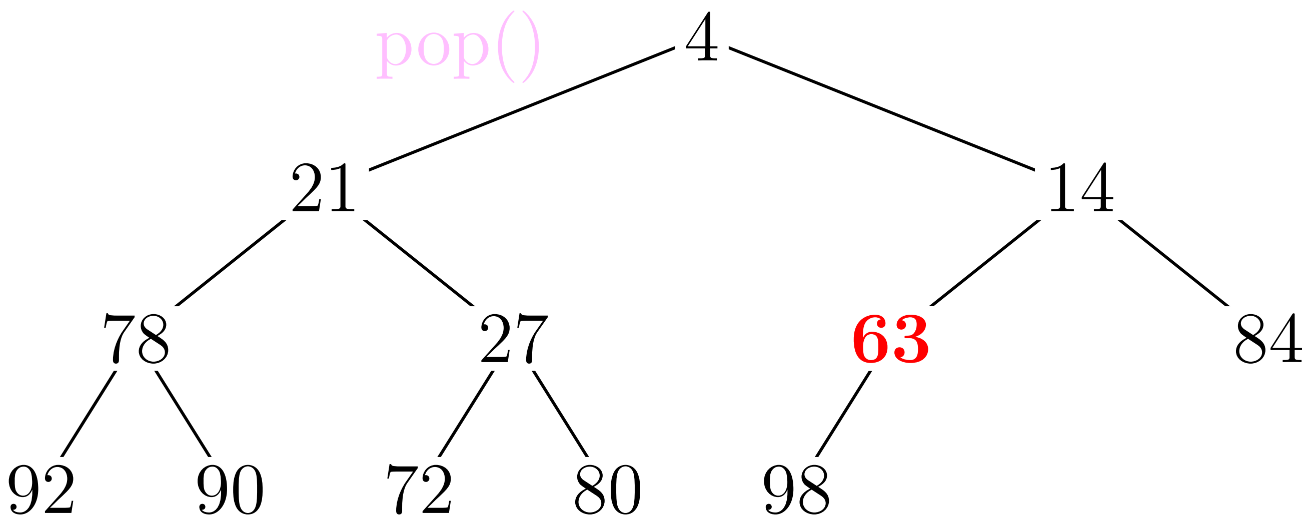
4	21	14	78	27	**63**	84	92	90	72	80	98		



Popping the Top

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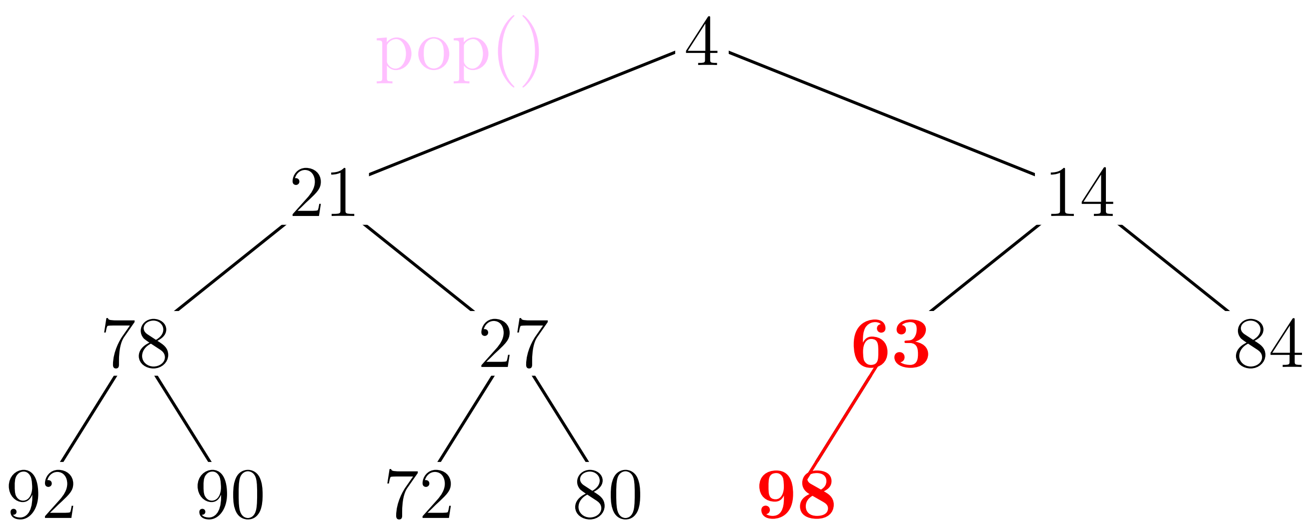

Popping the Top

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void pop() {
unsigned parent = 0;
pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
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while(child<size()) {
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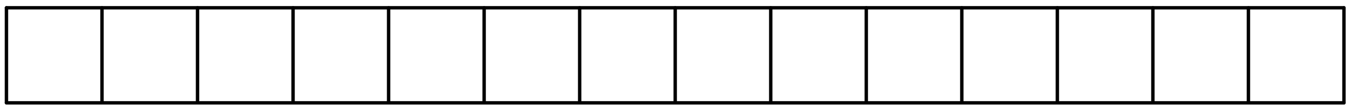
  parent = child;
  child = 2*parent + 1;
 }
}
```

Heaps in Action

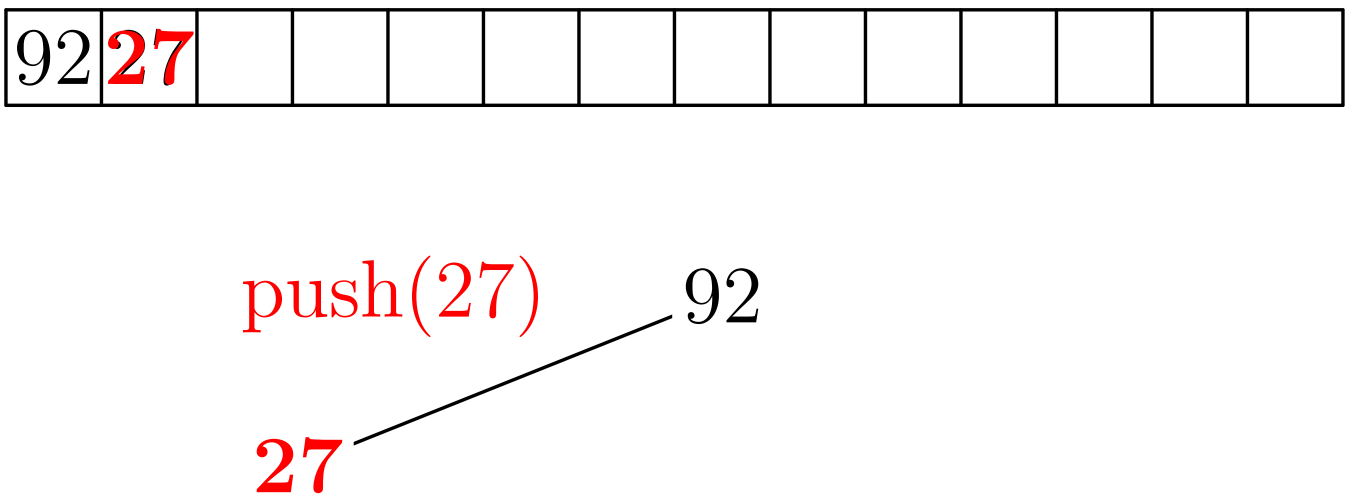


Heaps in Action

92													

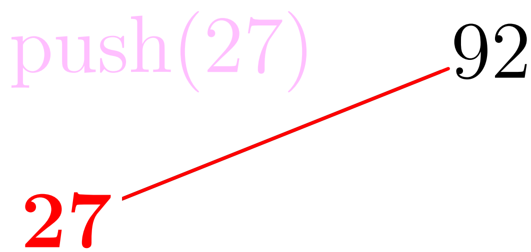
push(92) **92**

Heaps in Action



Heaps in Action

92	**27**												



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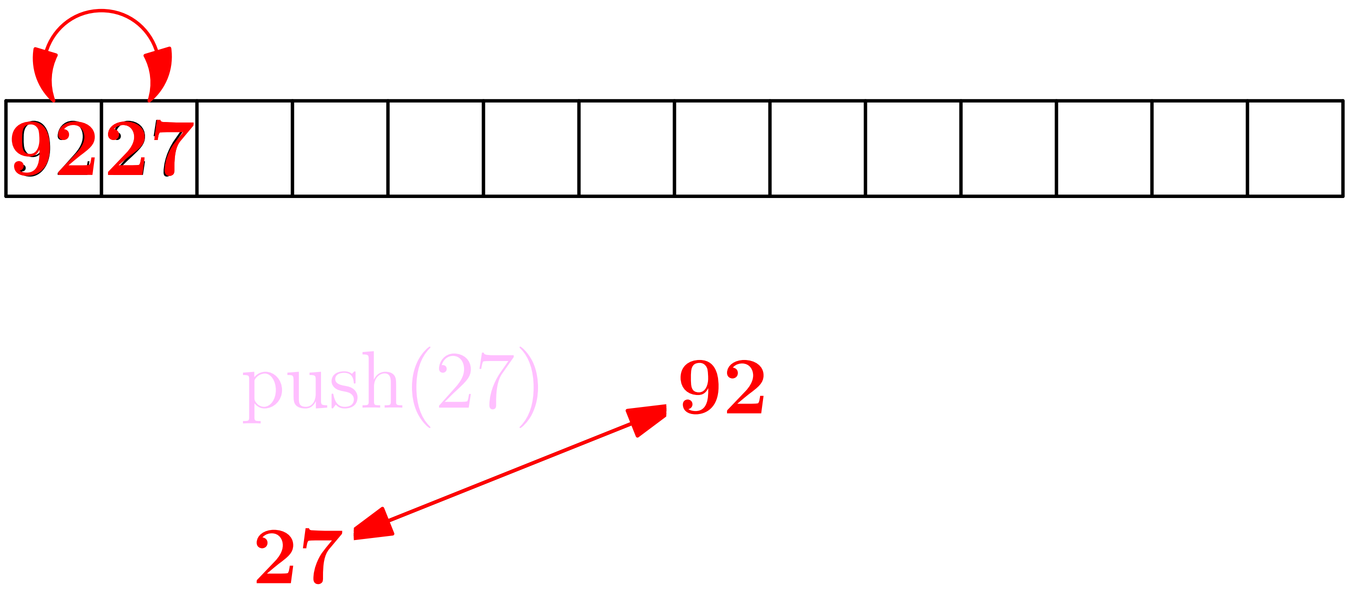

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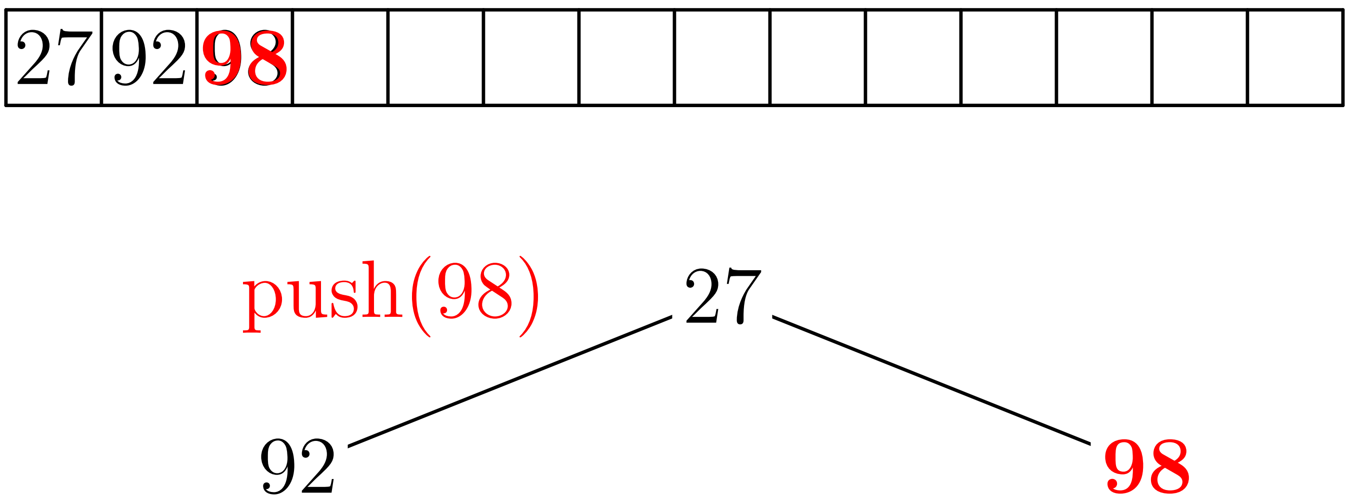

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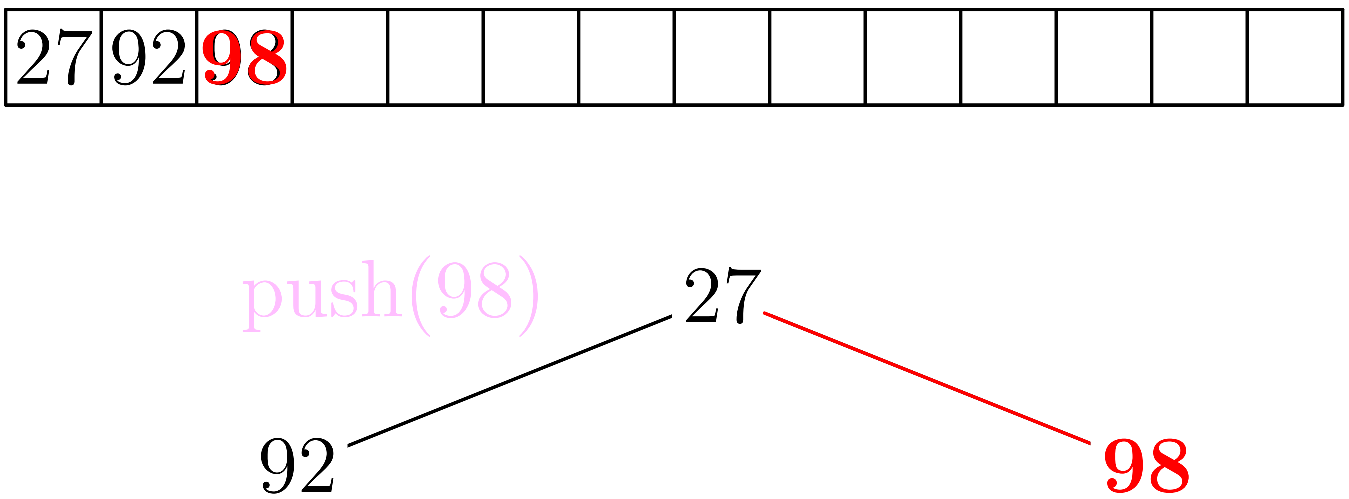

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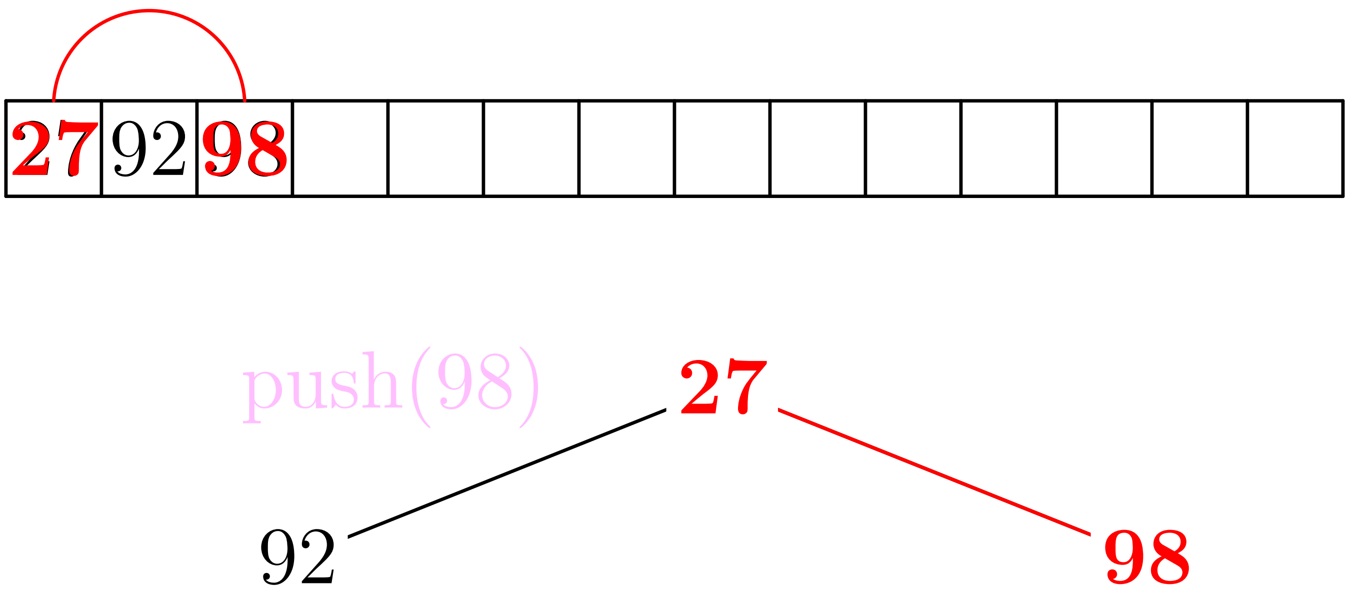
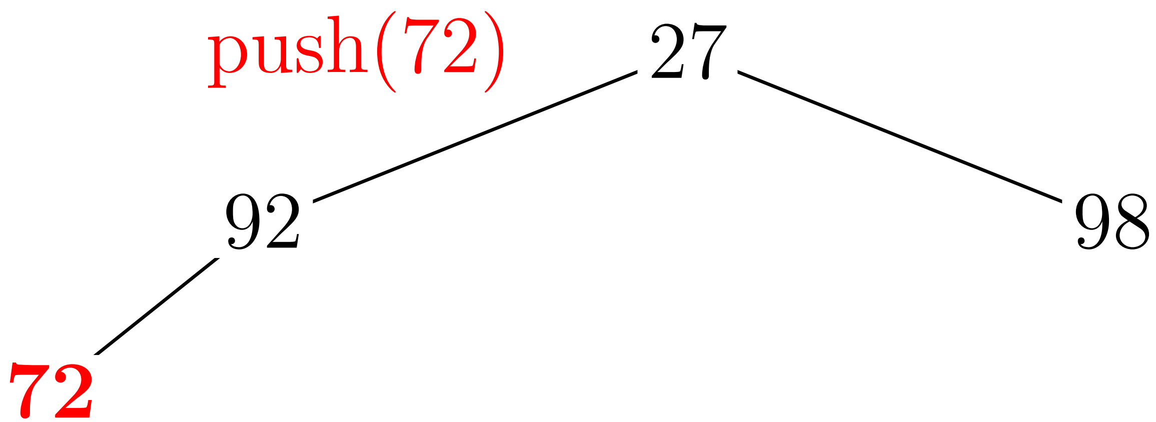


Heaps in Action

27	92	98	**72**										



Heaps in Action

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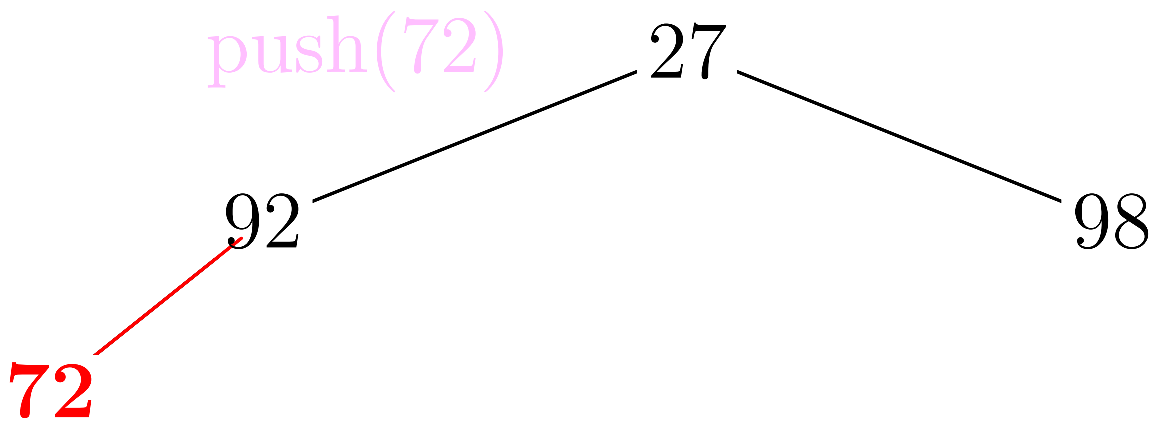

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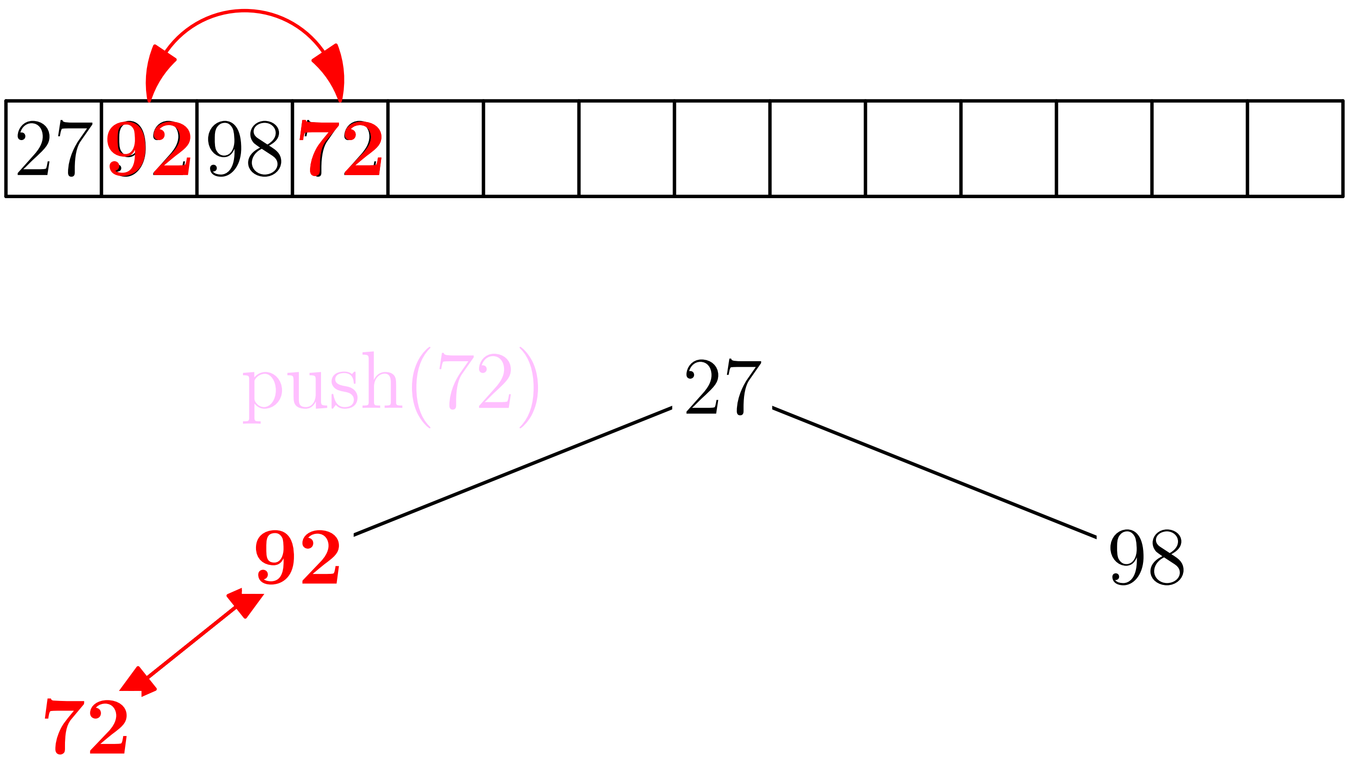

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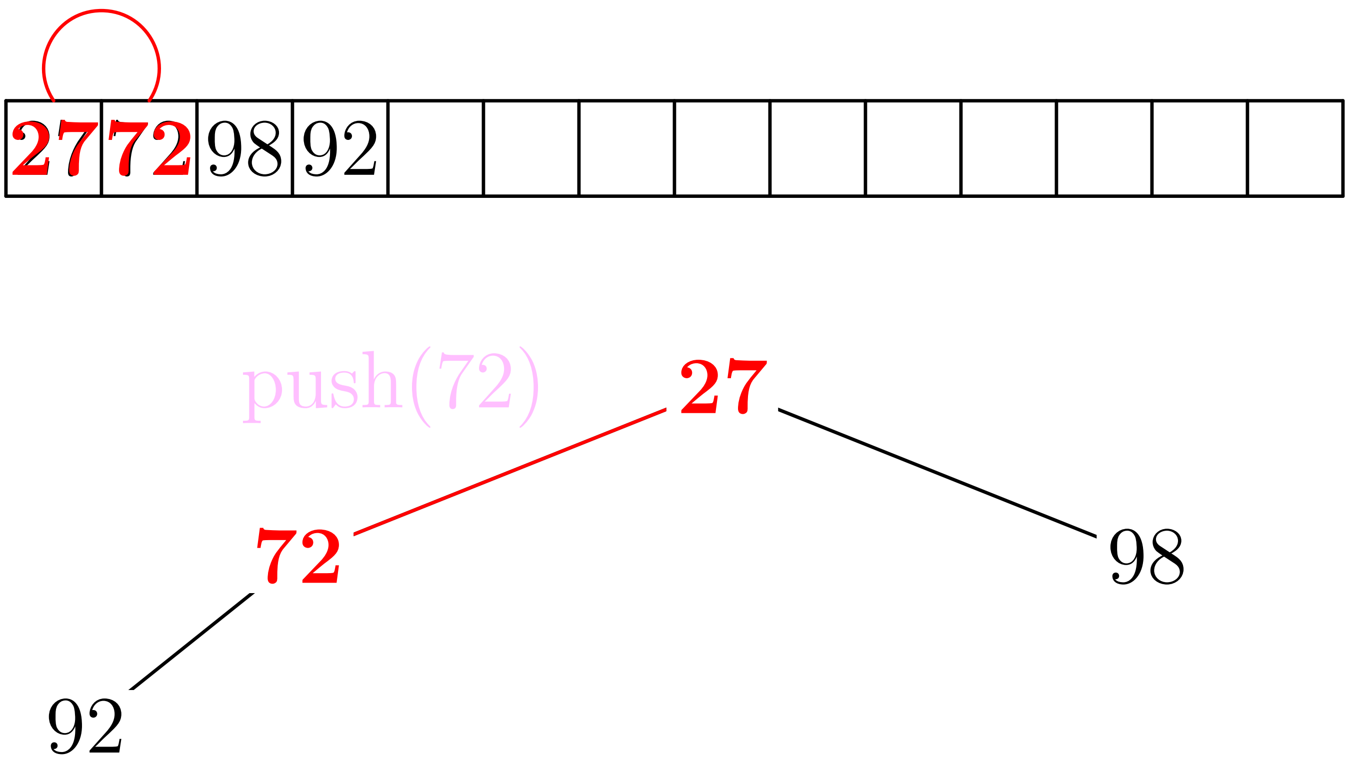
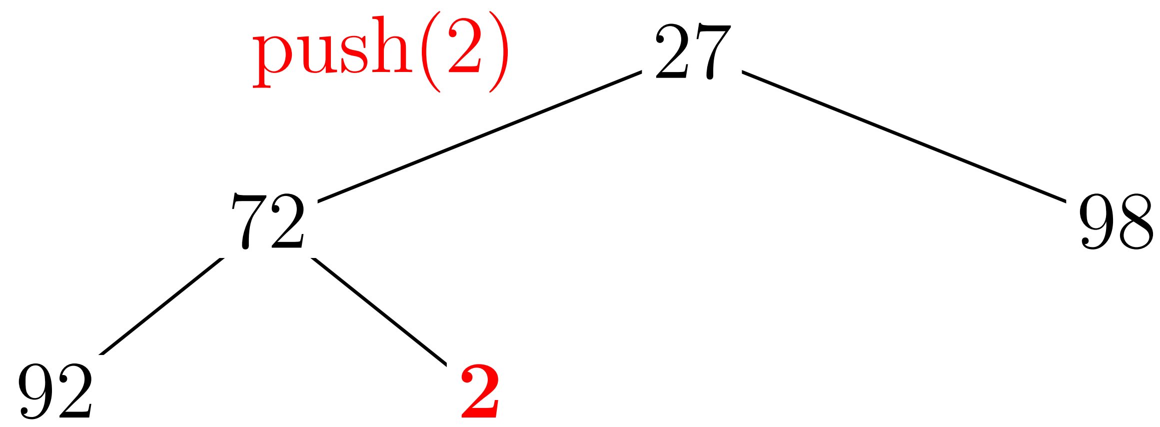


Heaps in Action

27	72	98	92	**2**									



Heaps in Action

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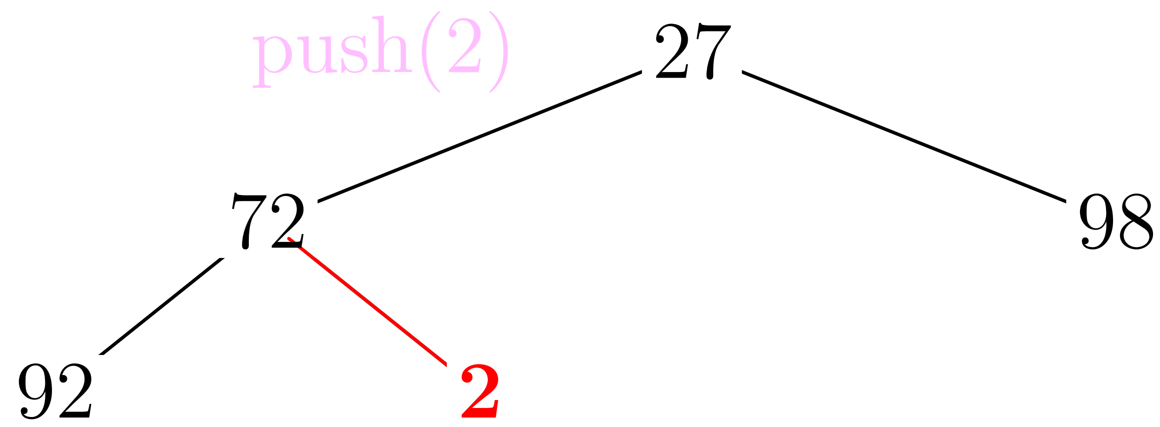

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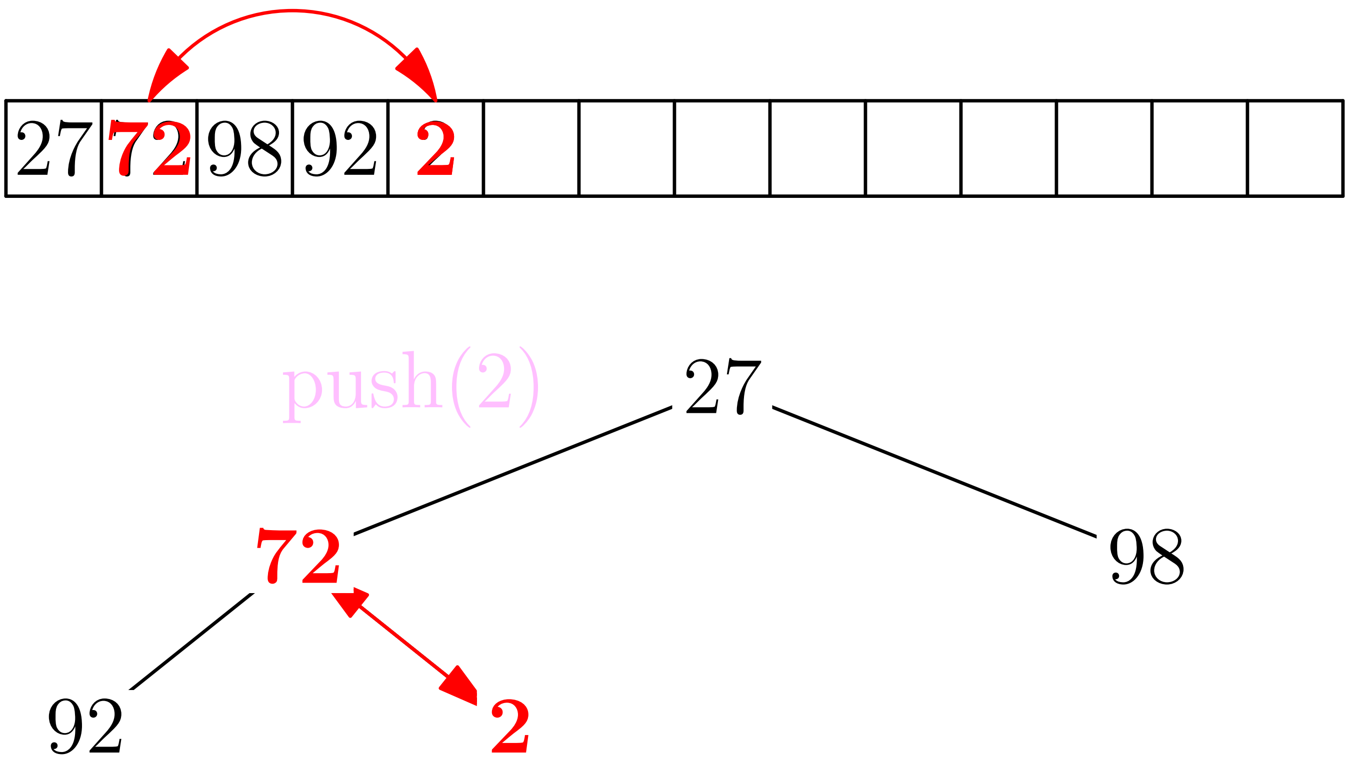

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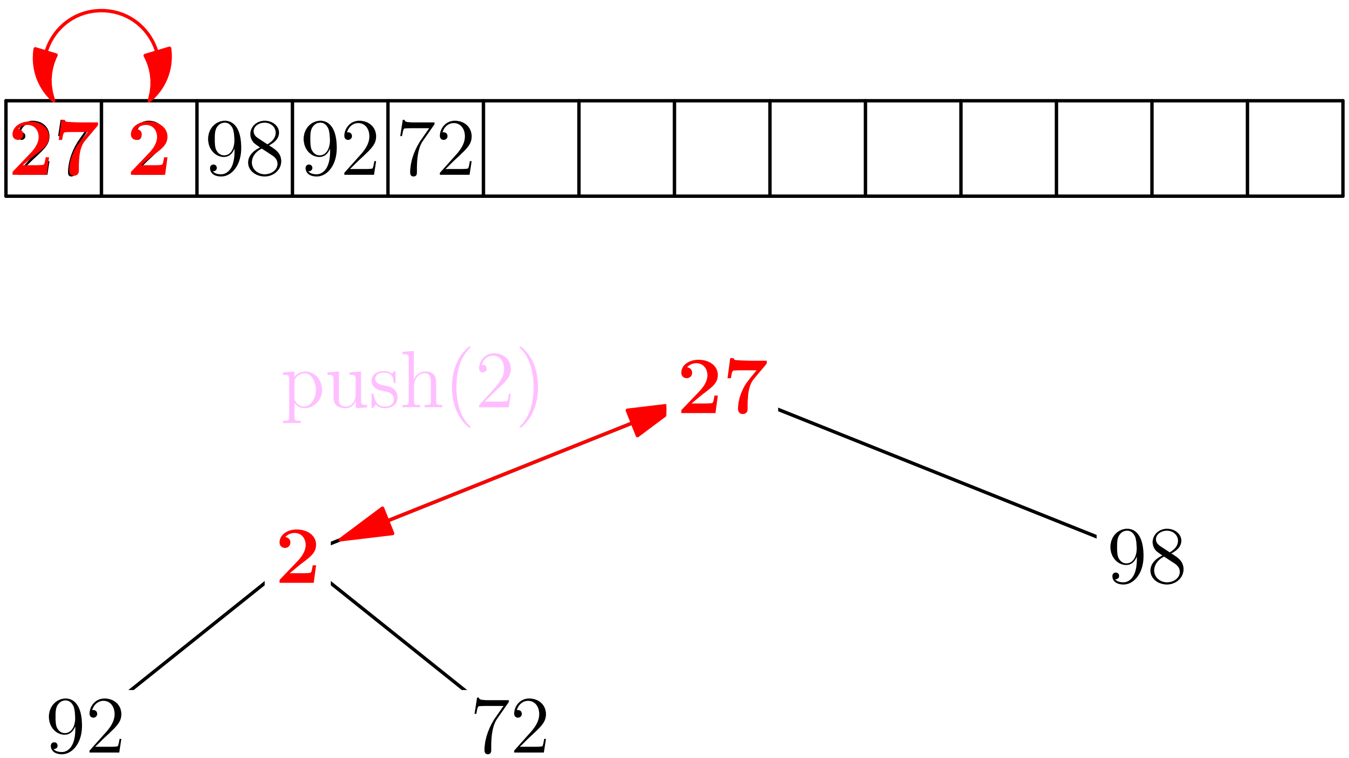
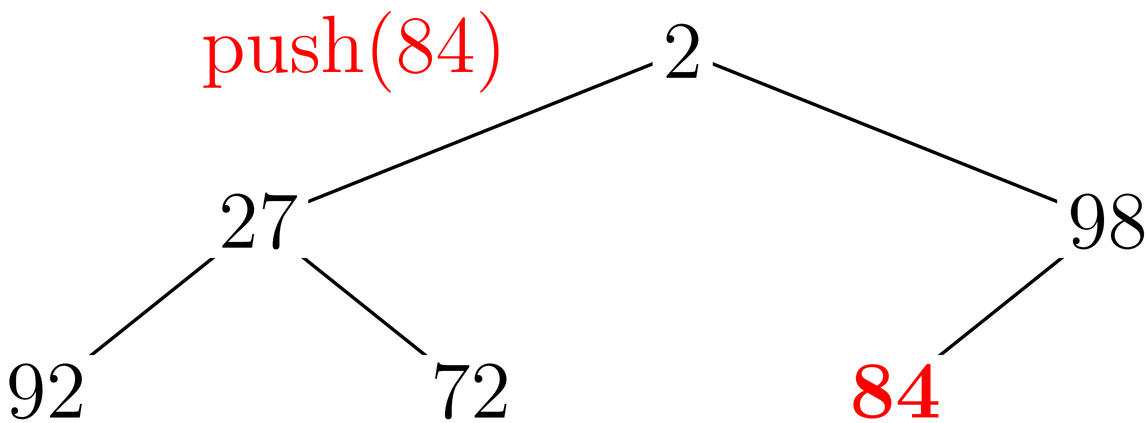


Heaps in Action

2	27	98	92	72	**84**								



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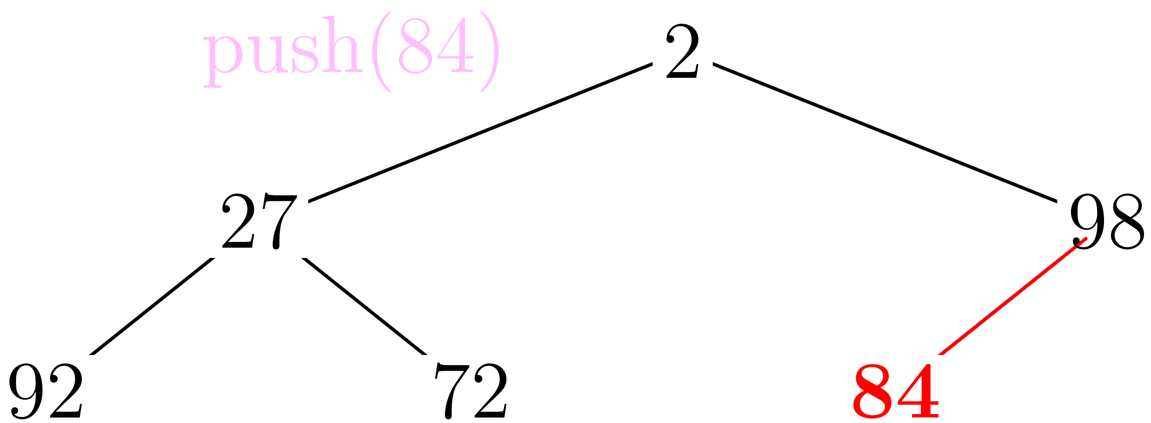

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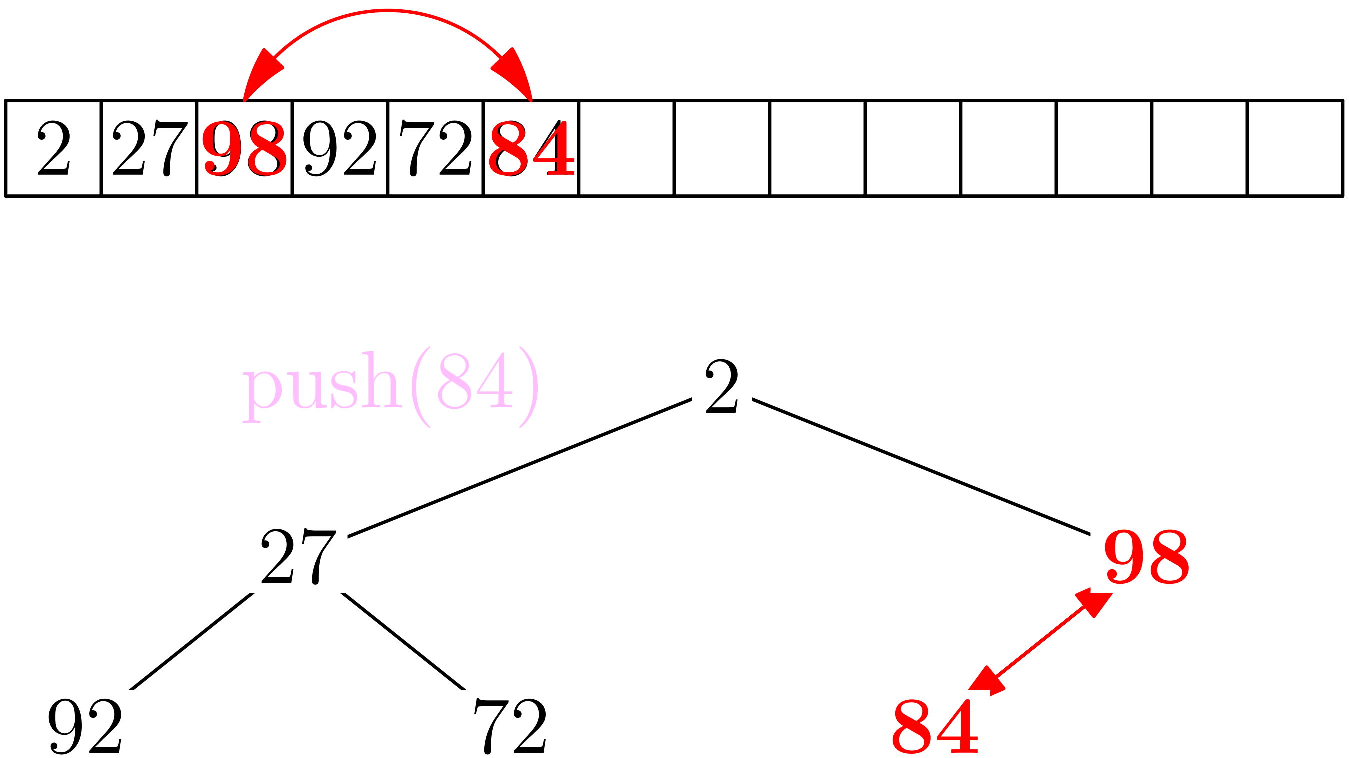
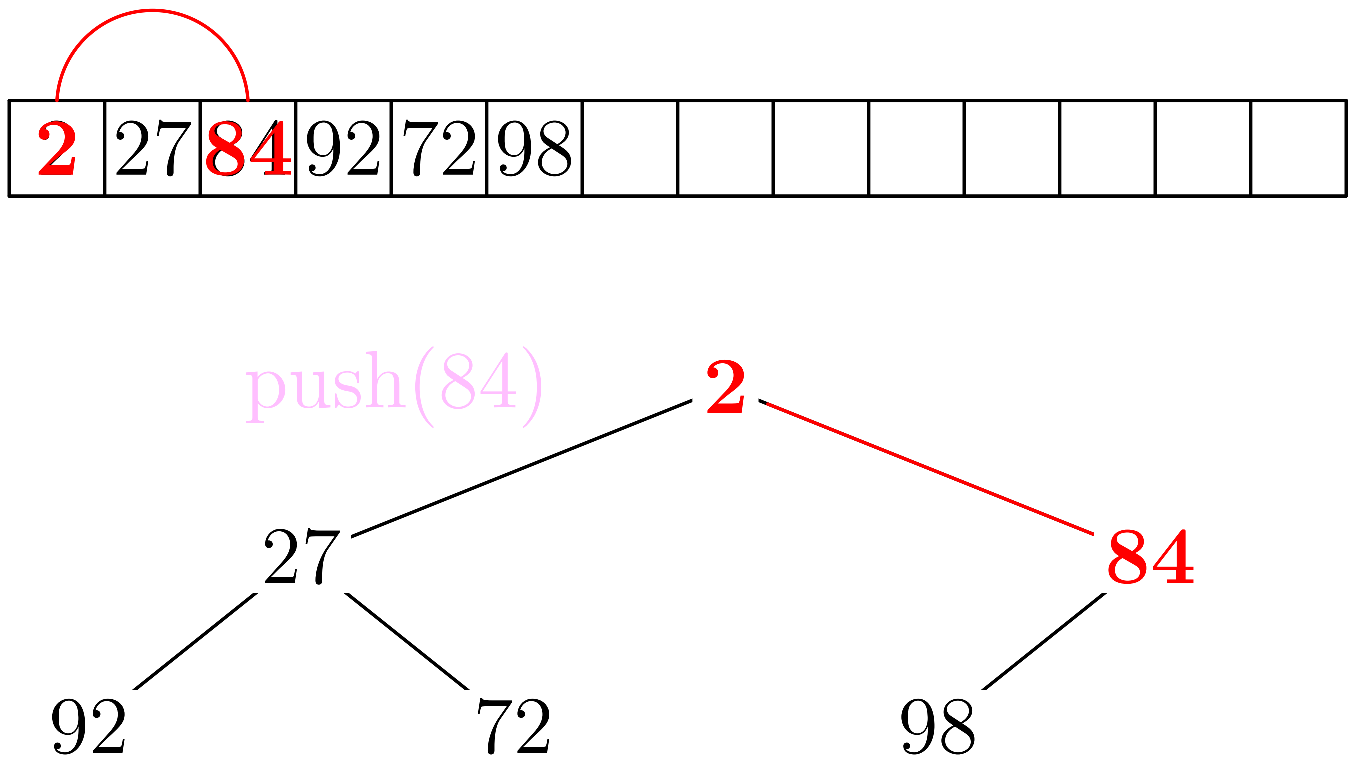
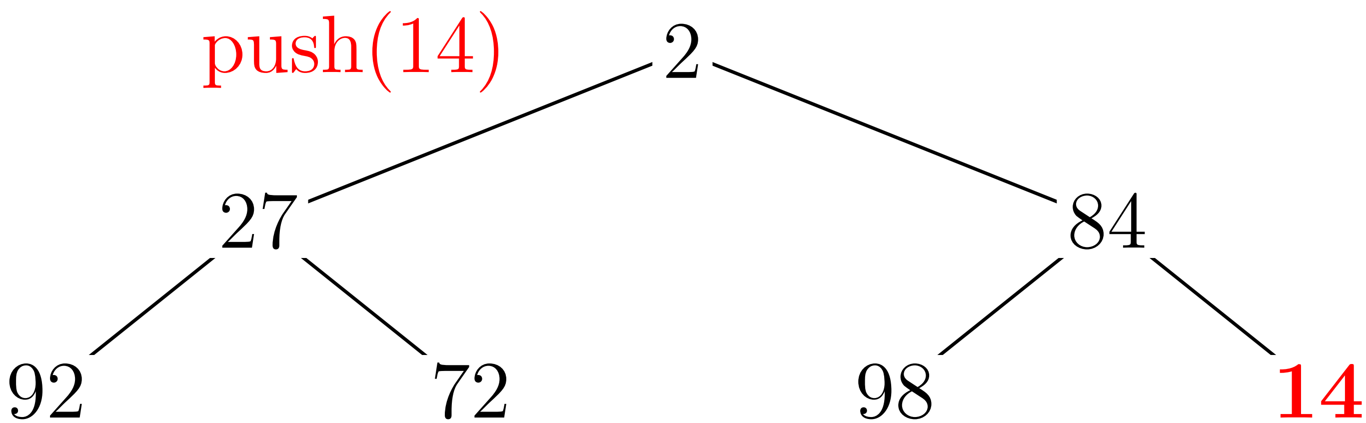


Heaps in Action



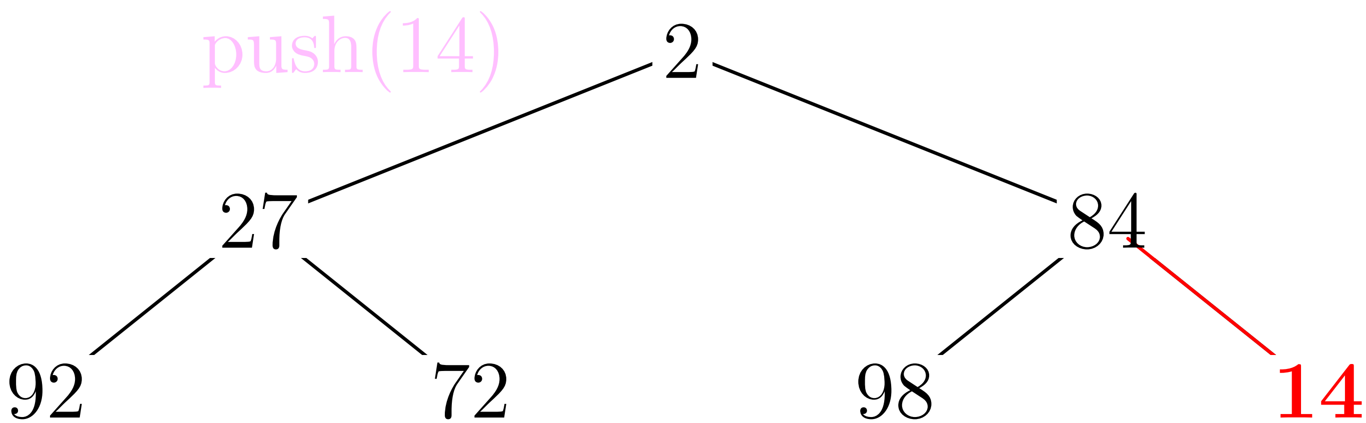
Heaps in Action

2	27	84	92	72	98	**14**							

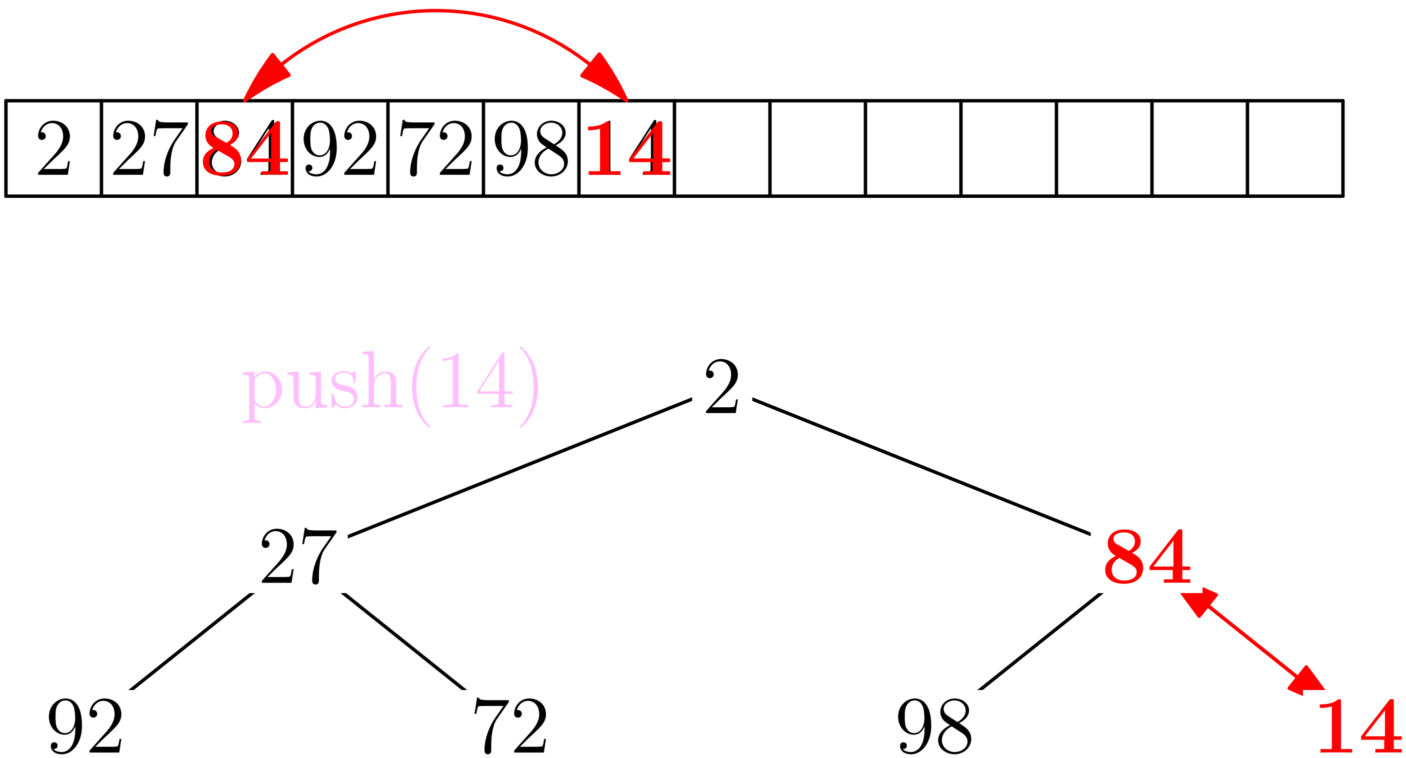


Heaps in Action

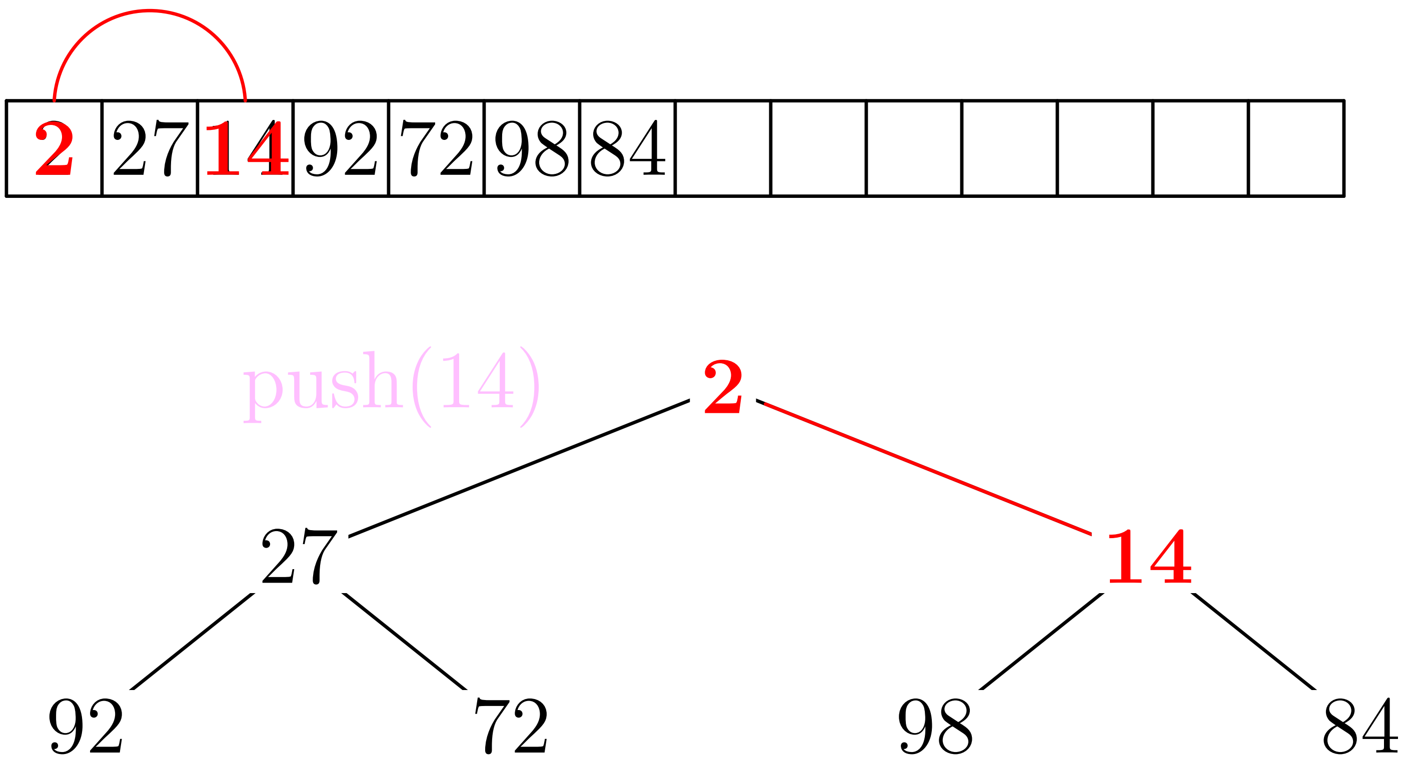
2	27	84	92	72	98	**14**							



Heaps in Action

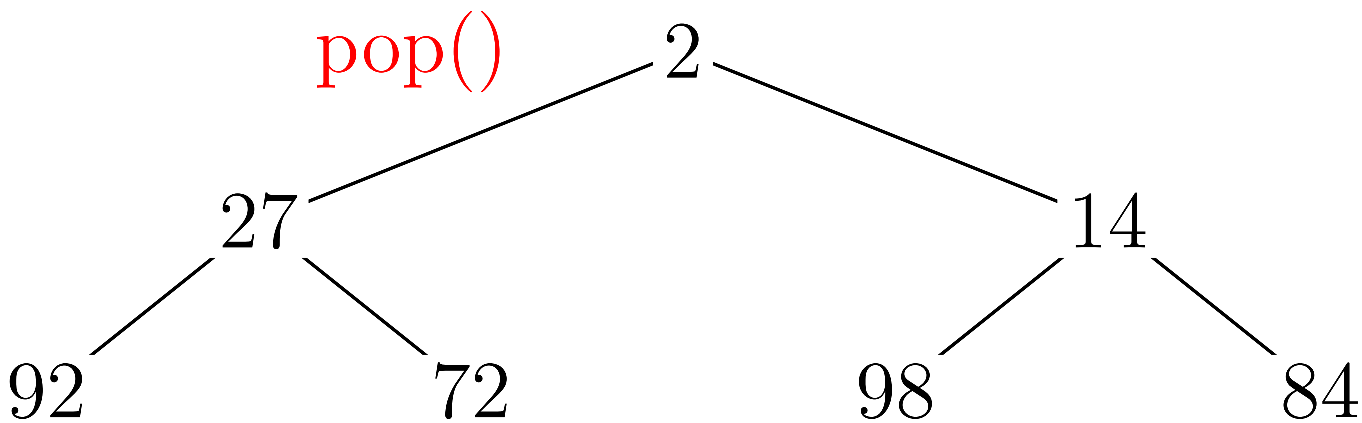


Heaps in Action

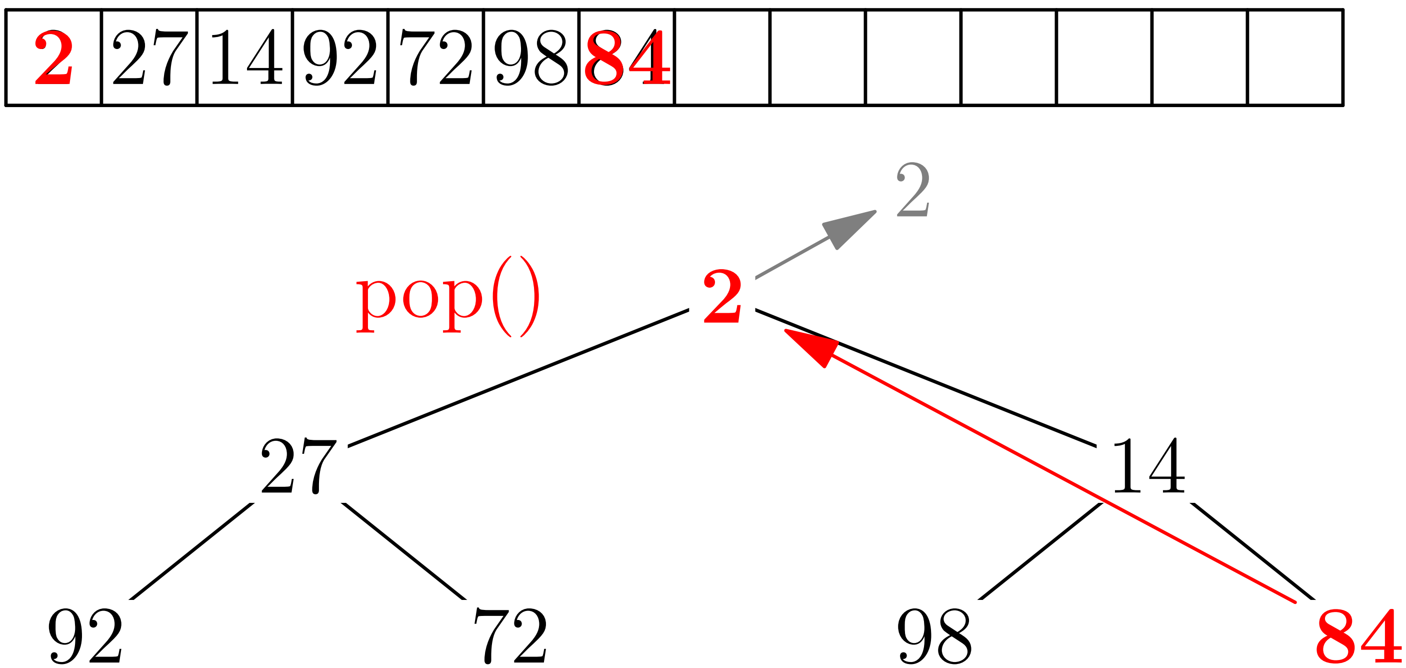


Heaps in Action

2	27	14	92	72	98	84							

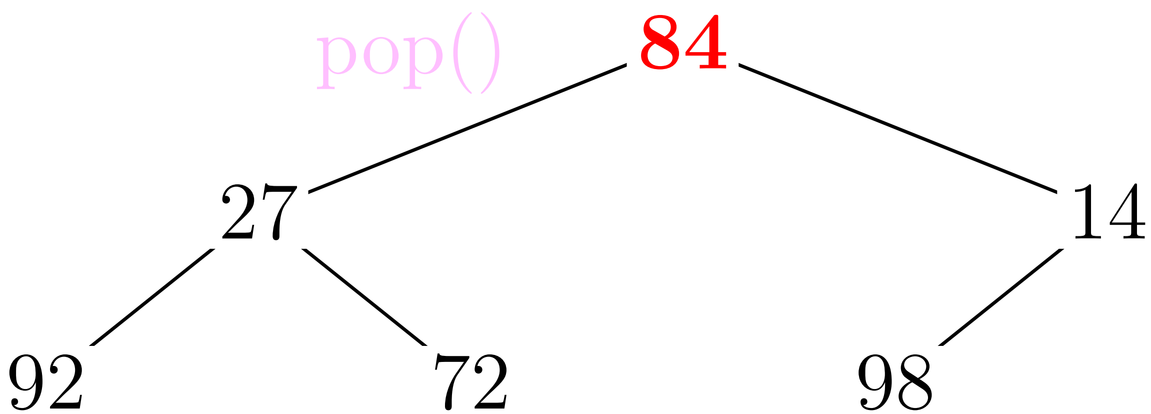


Heaps in Action

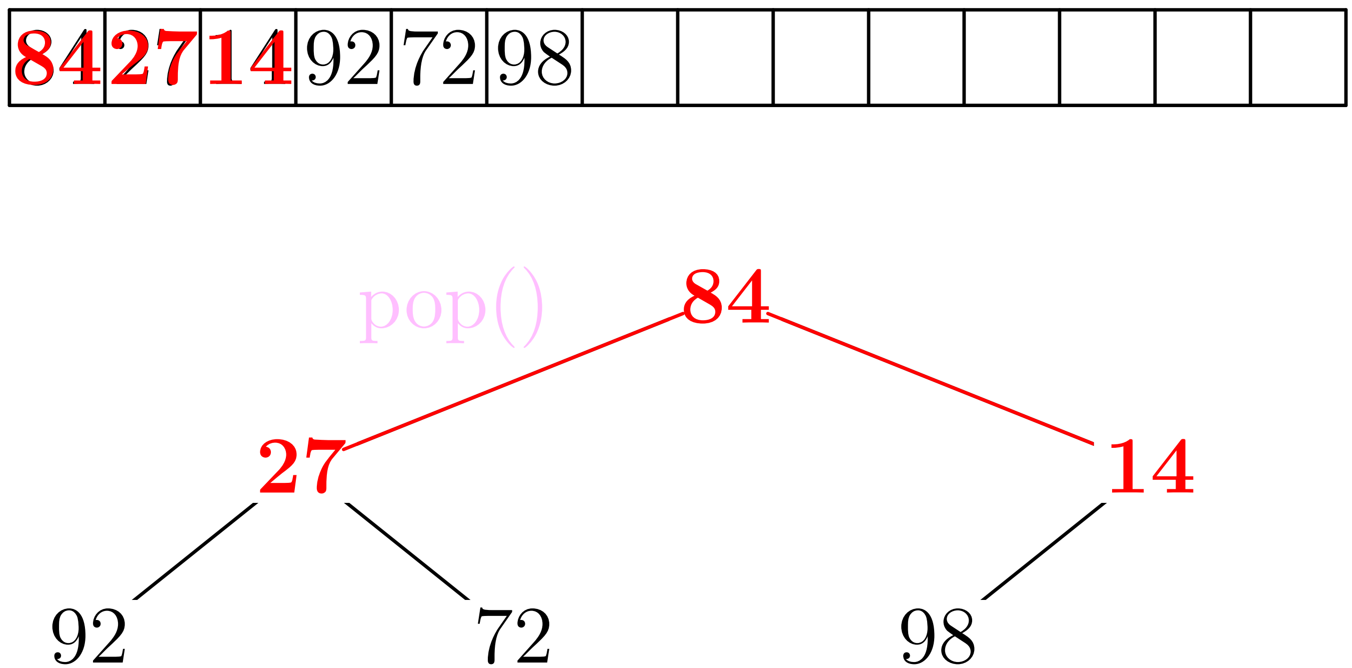


Heaps in Action

84	27	14	92	72	98								



Heaps in Action



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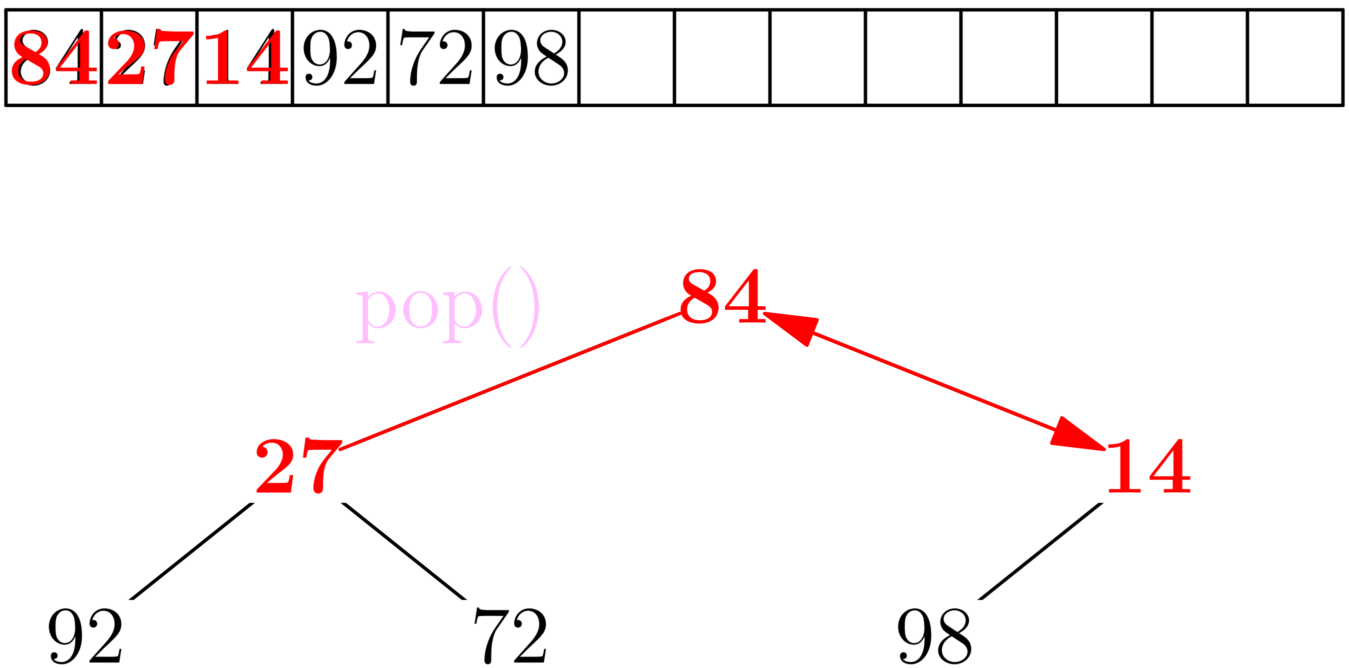
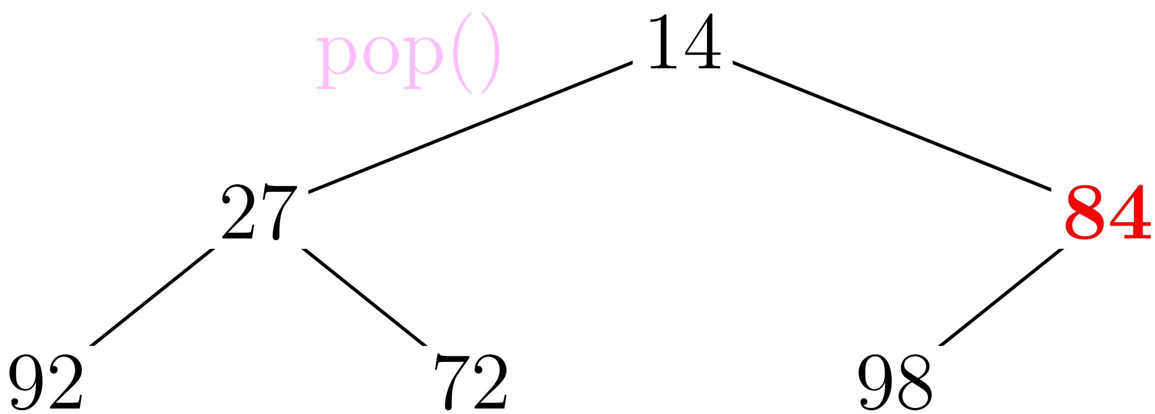


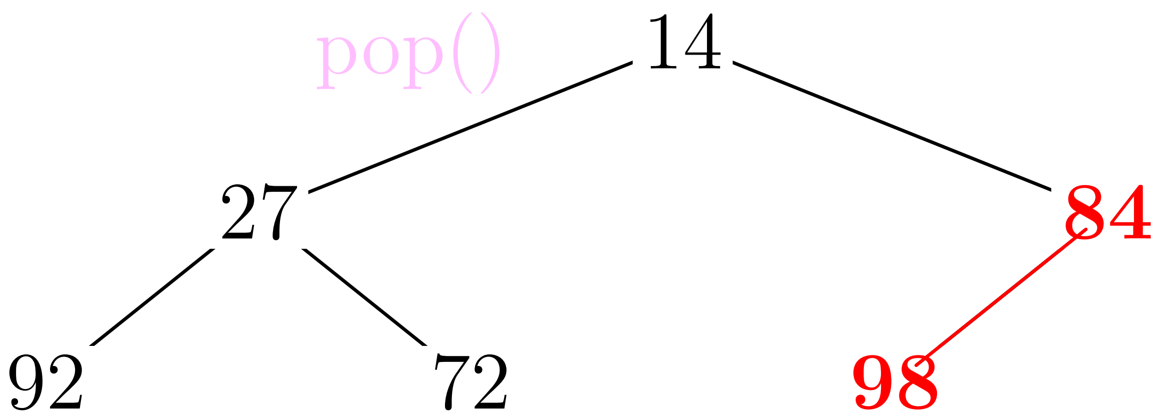
Heaps in Action

14	27	**84**	92	72	98								



Heaps in Action

14	27	**84**	92	72	**98**								



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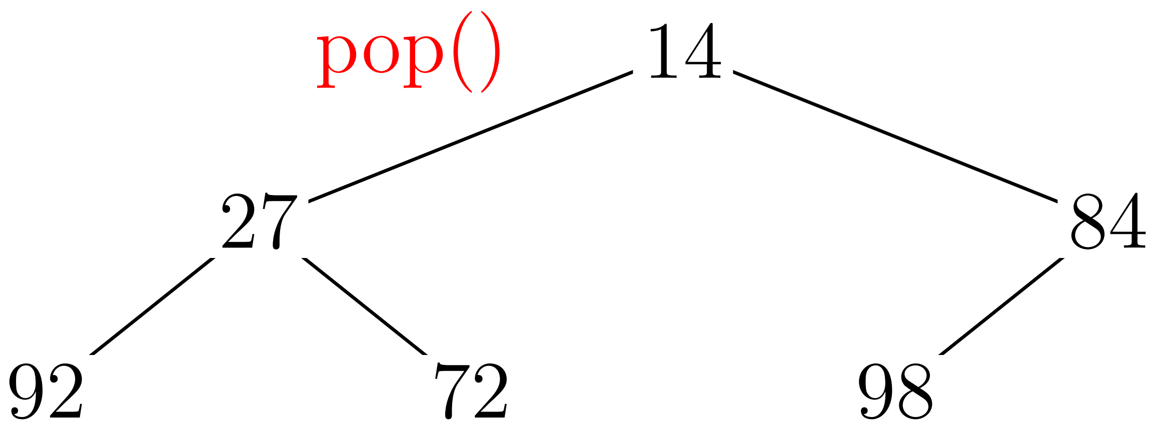
14	27	84	92	72	98								



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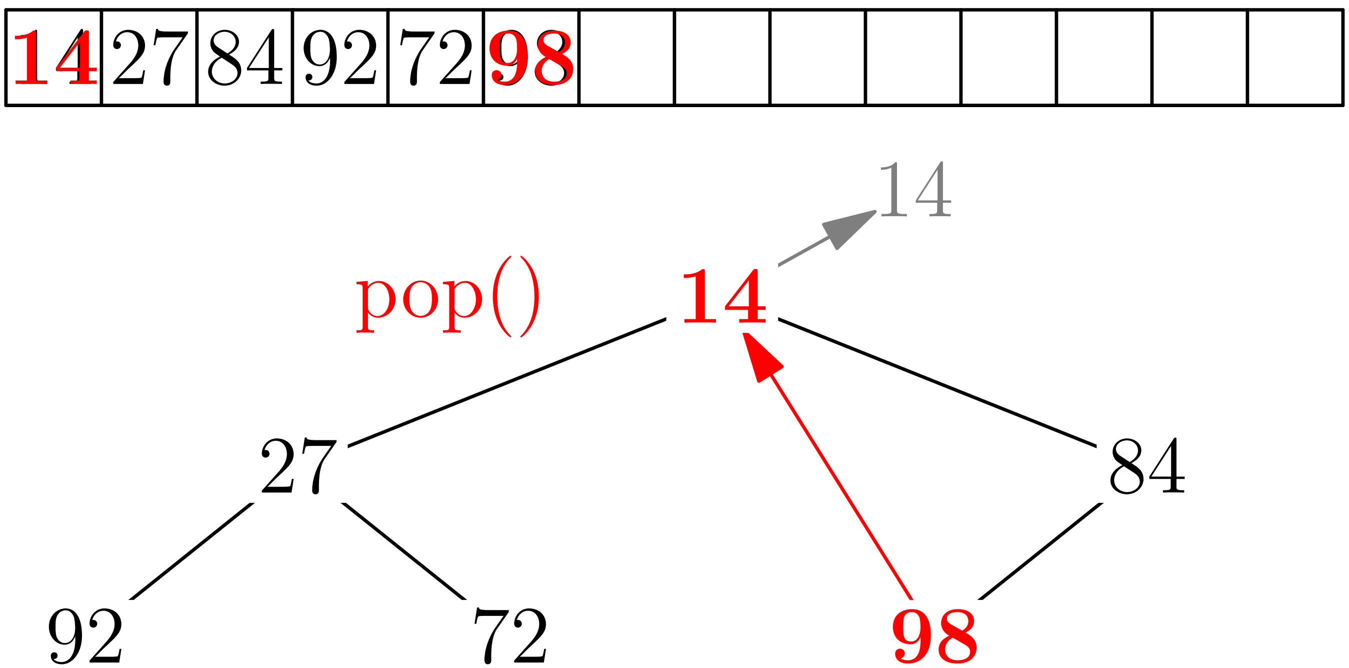

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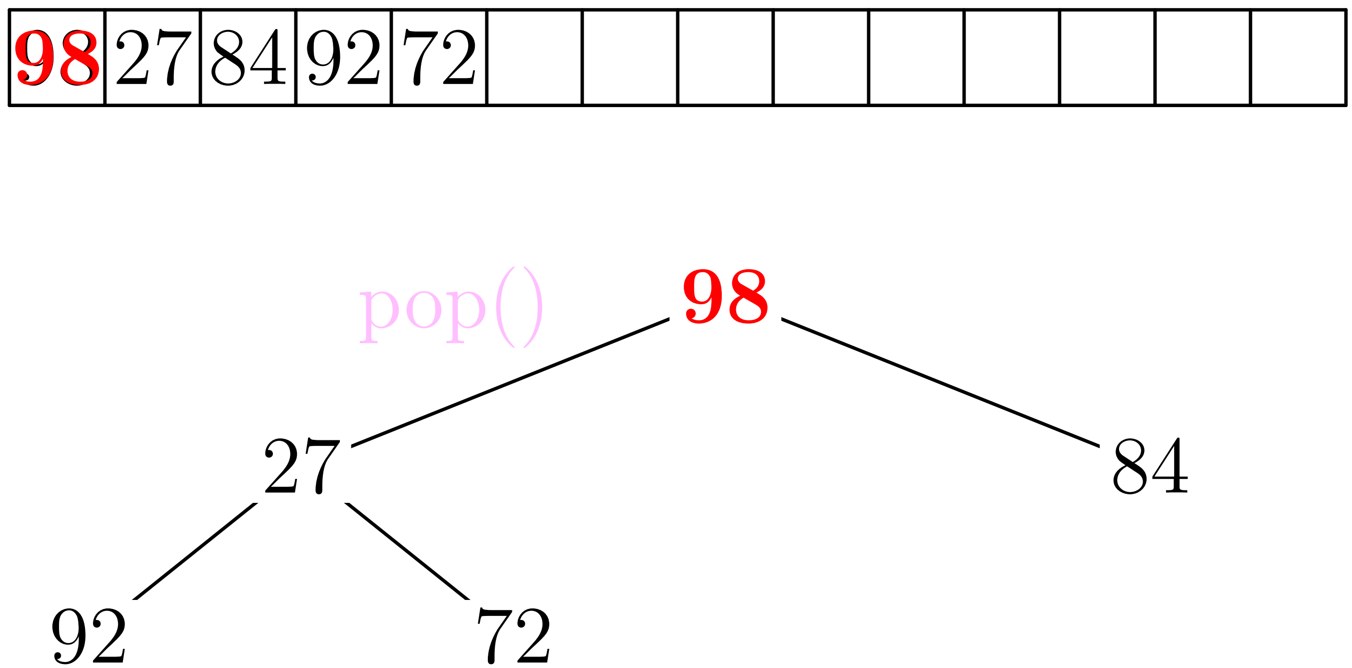

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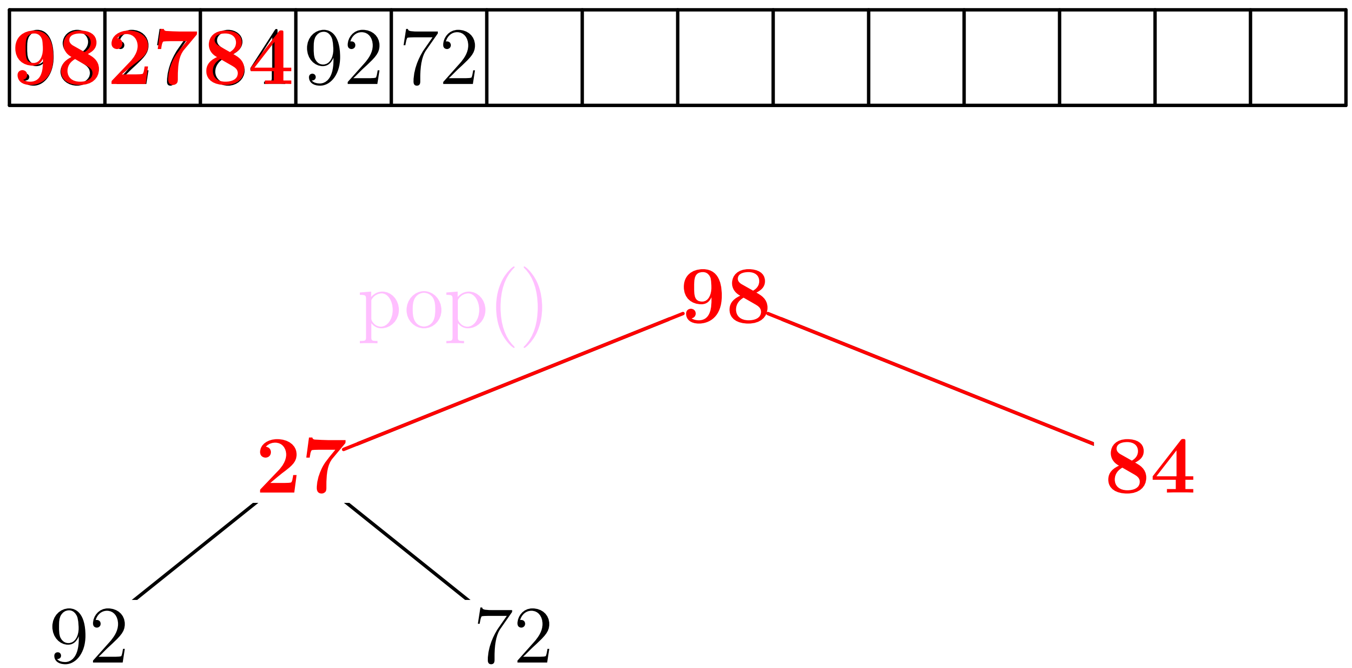

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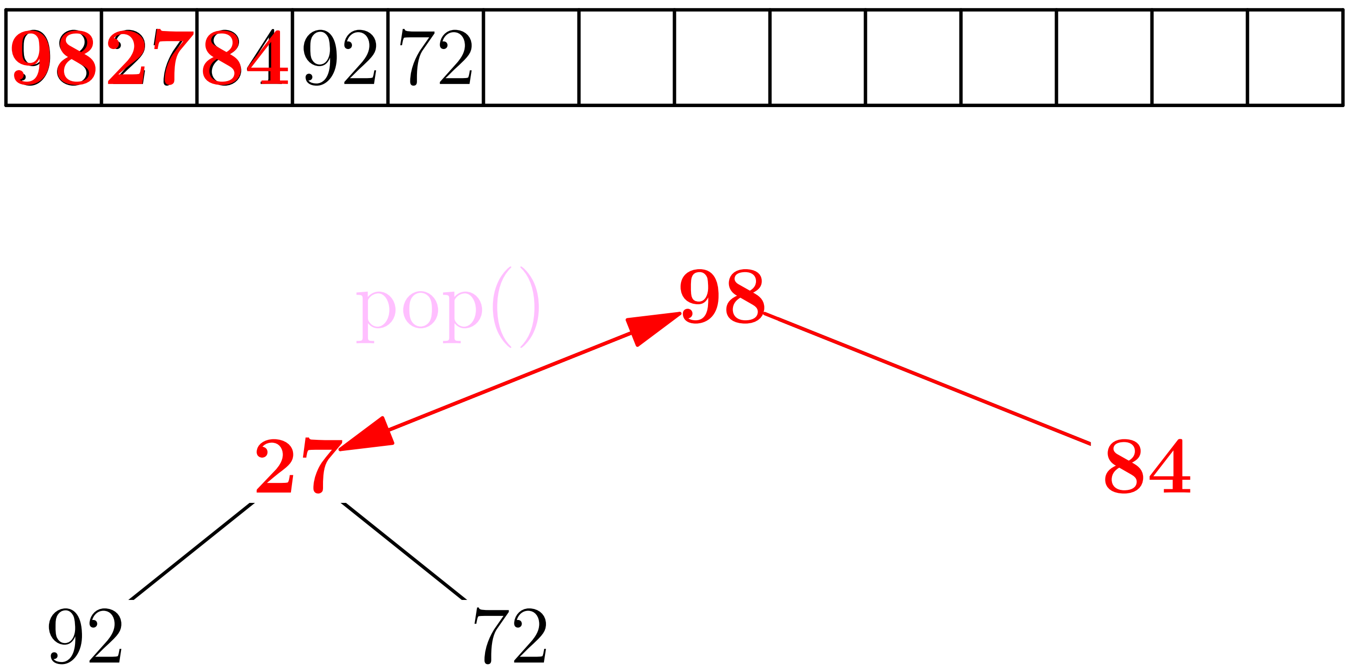

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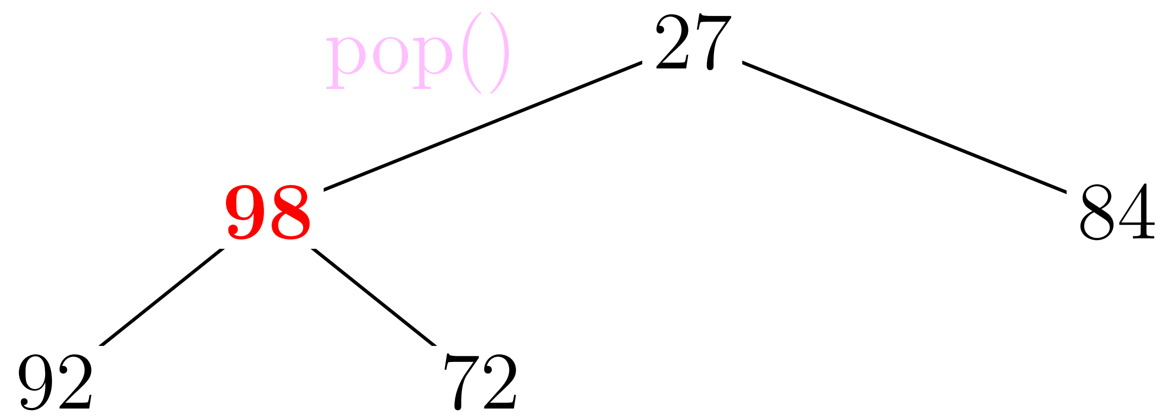

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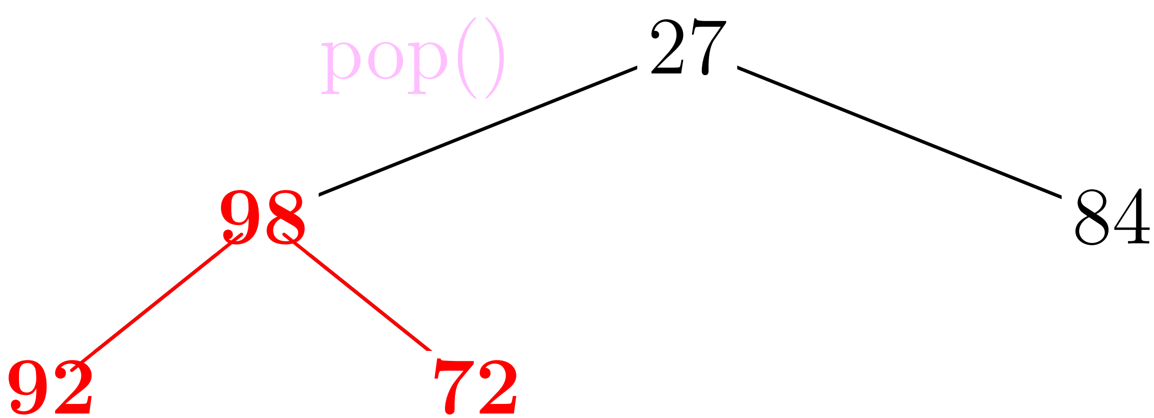

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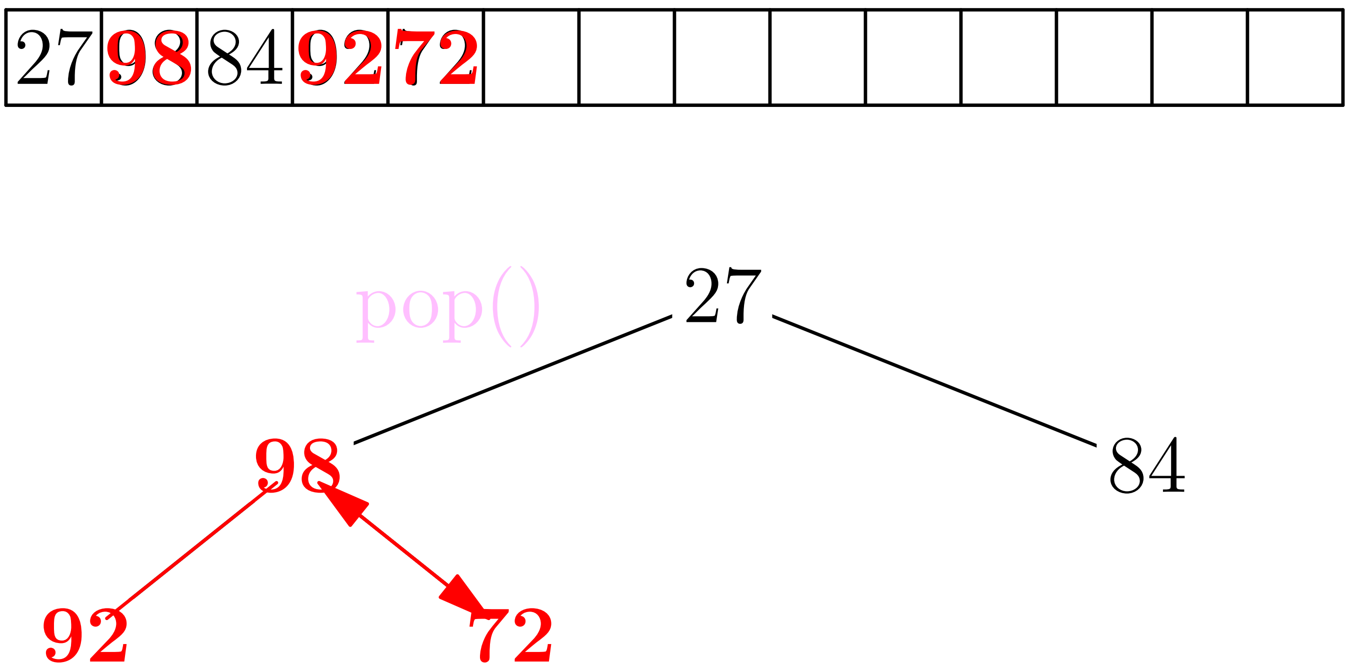

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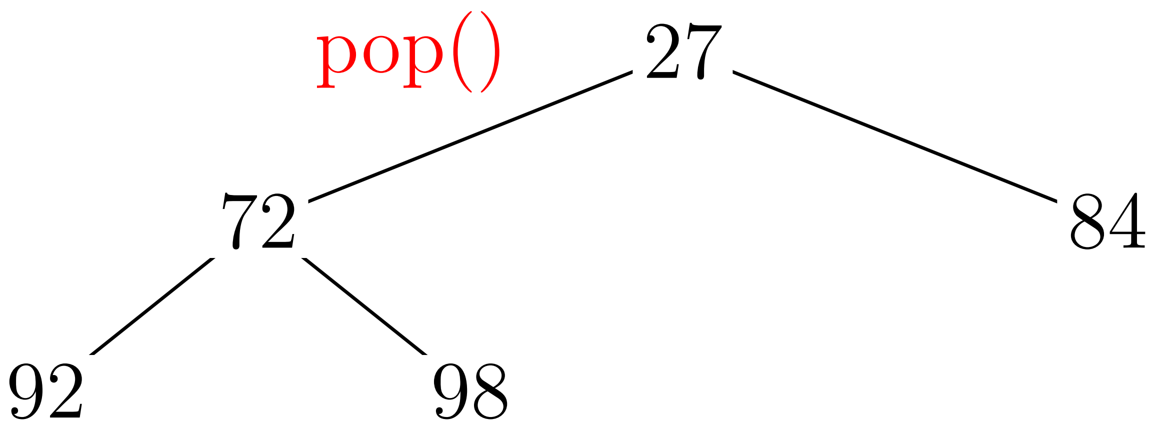

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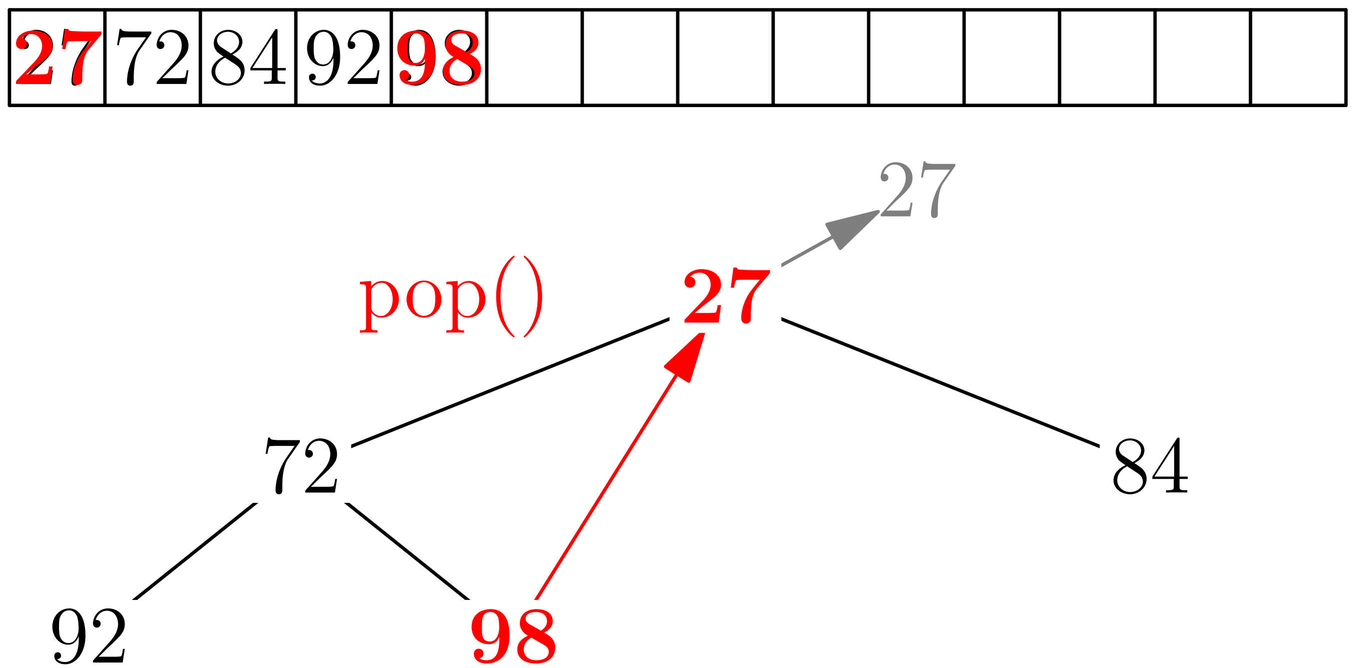

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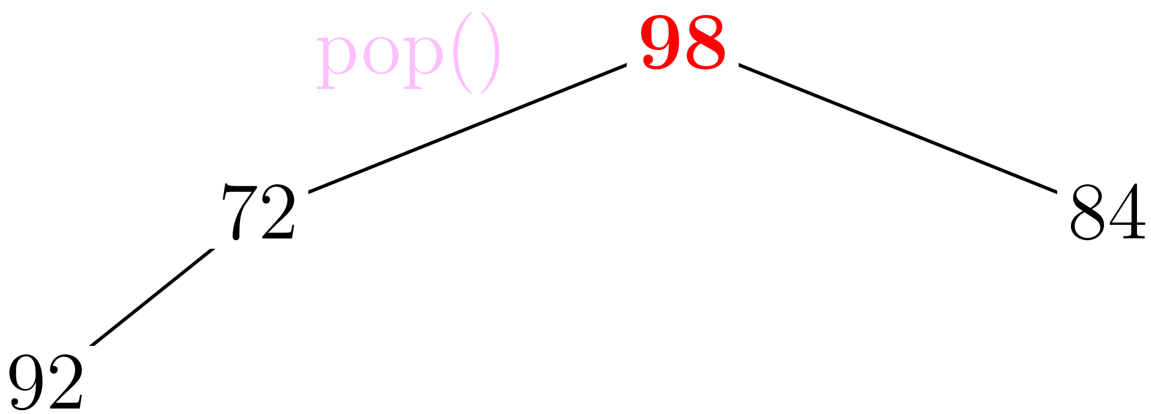

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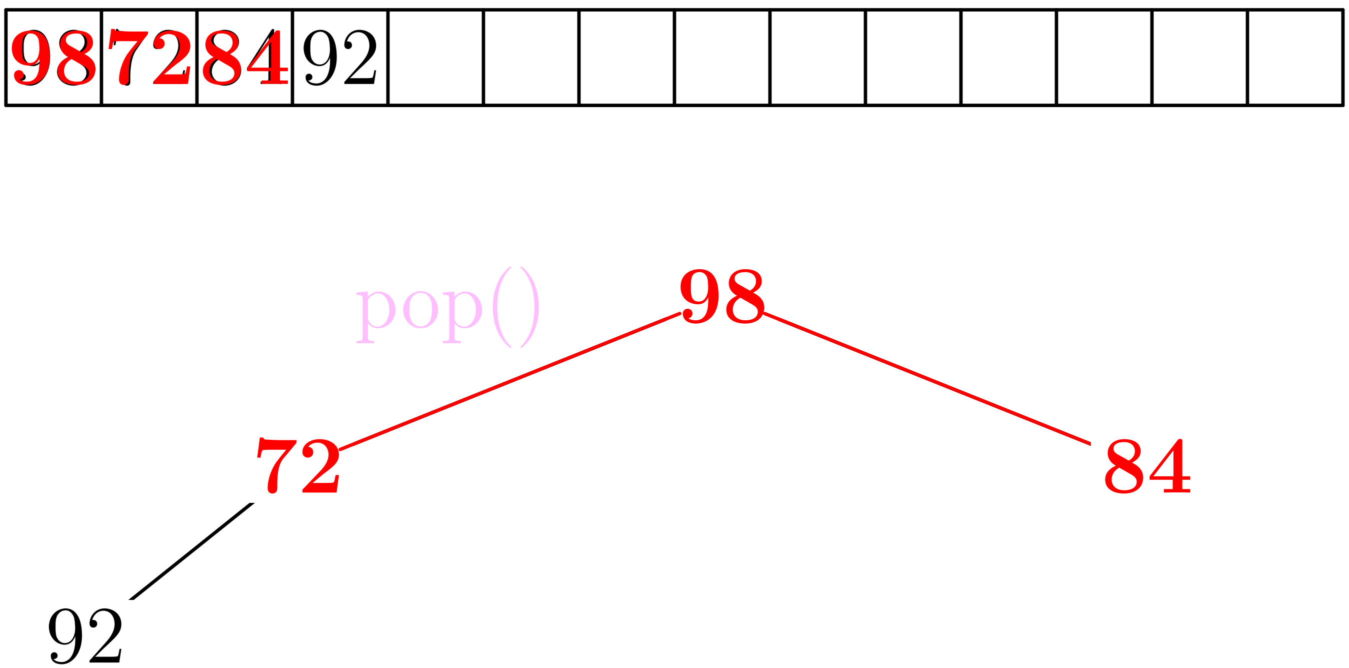

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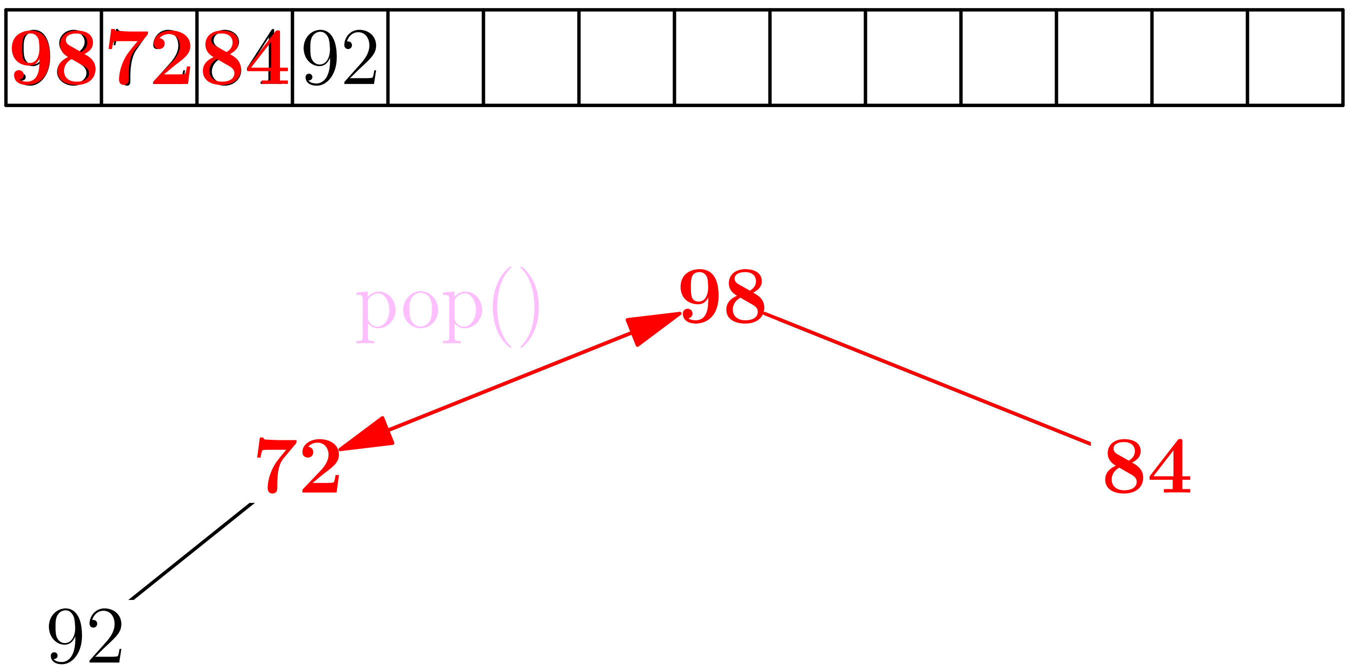

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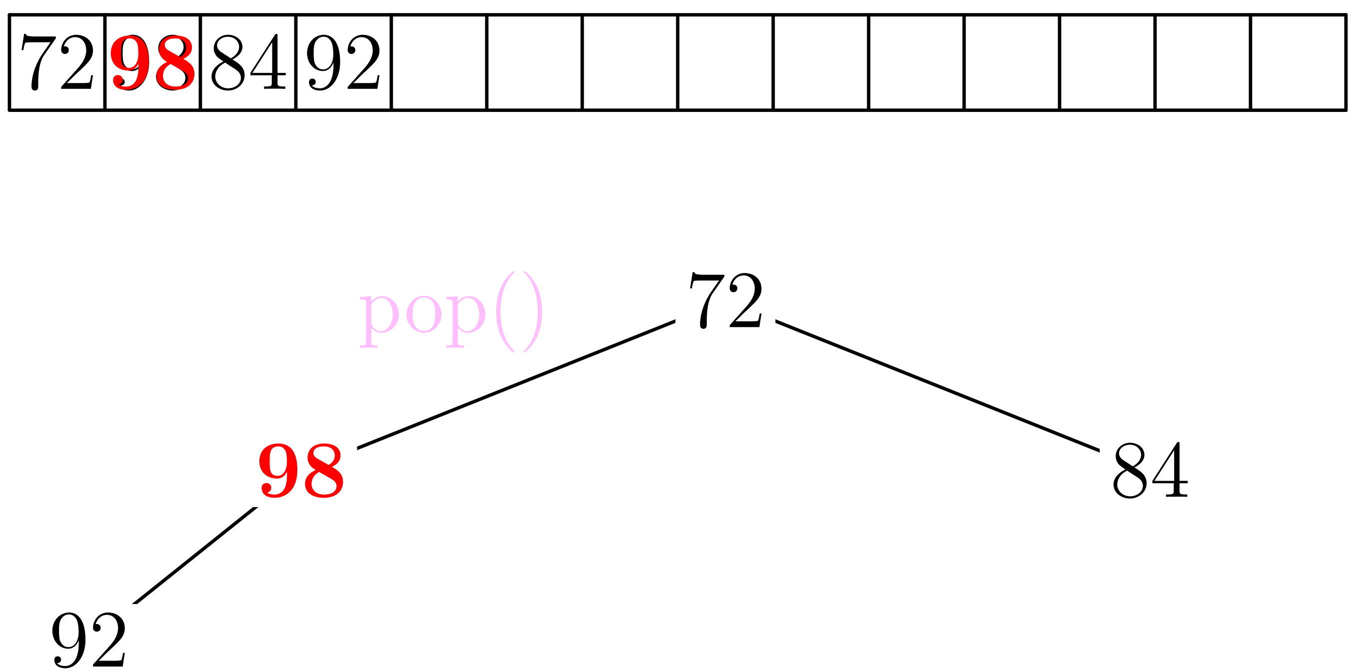

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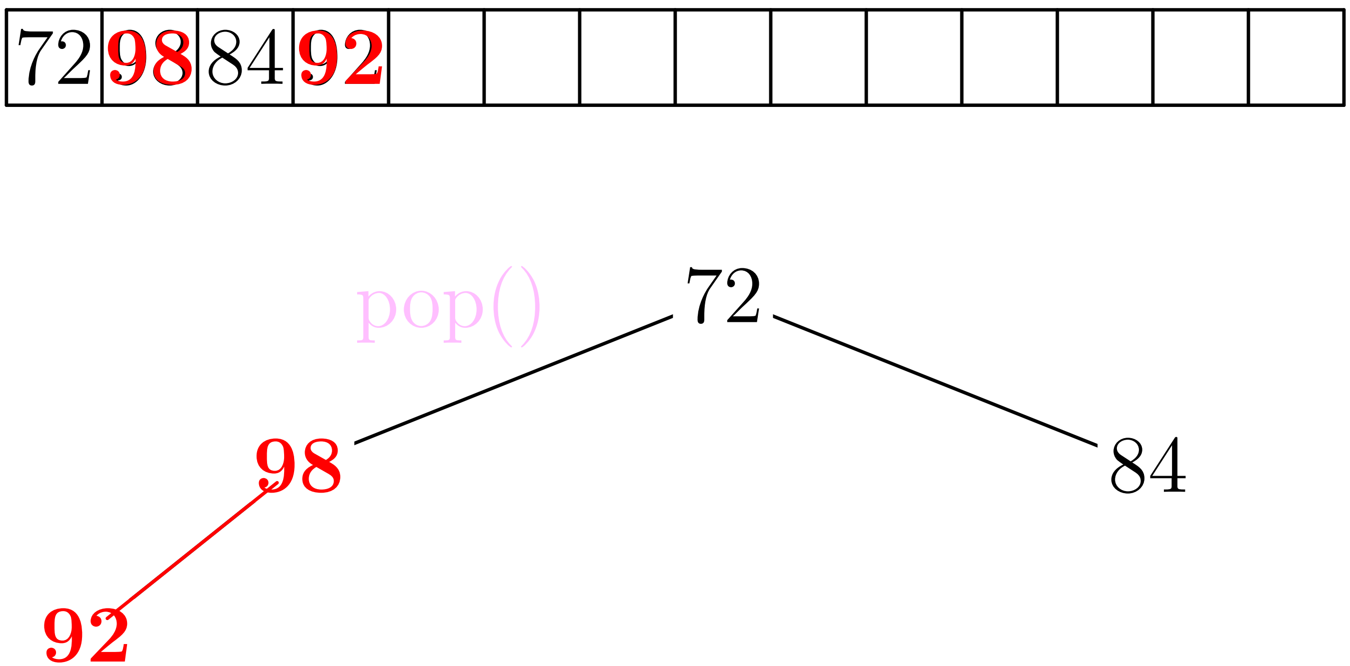

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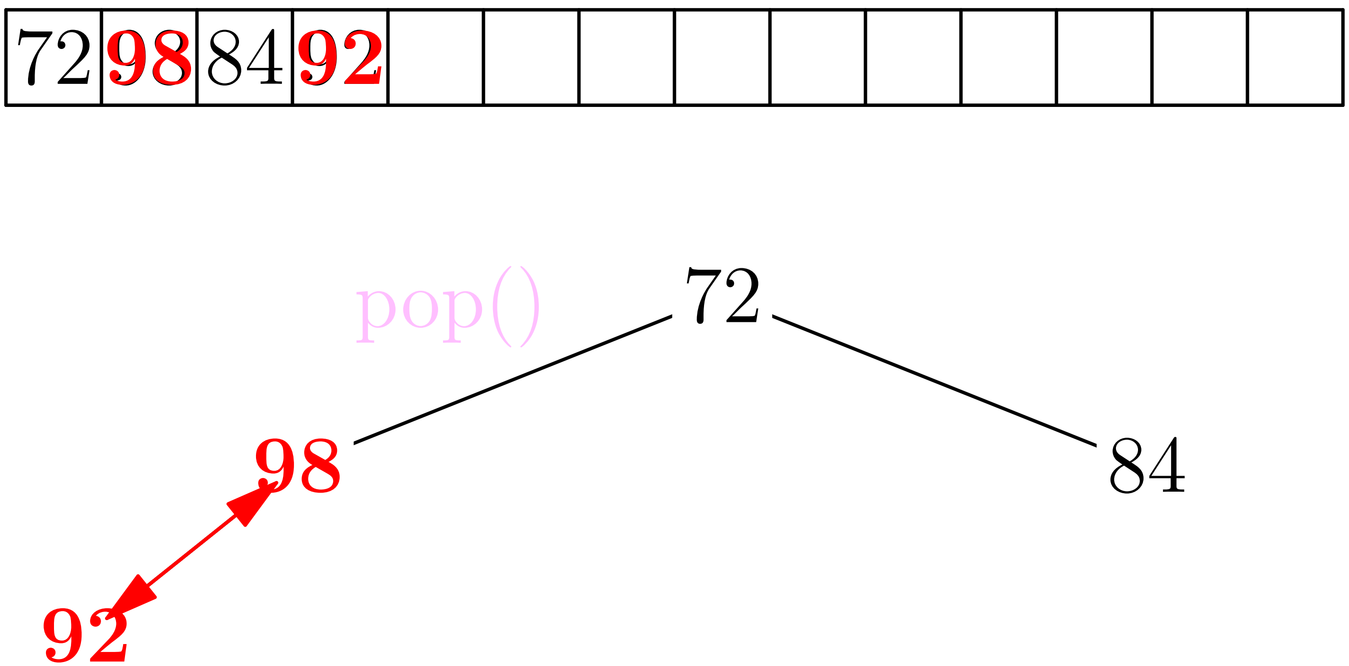

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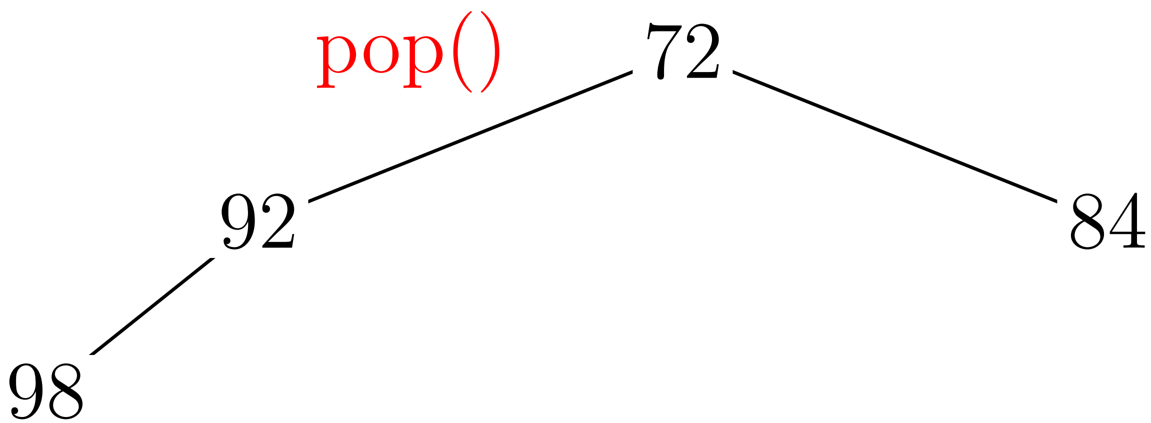

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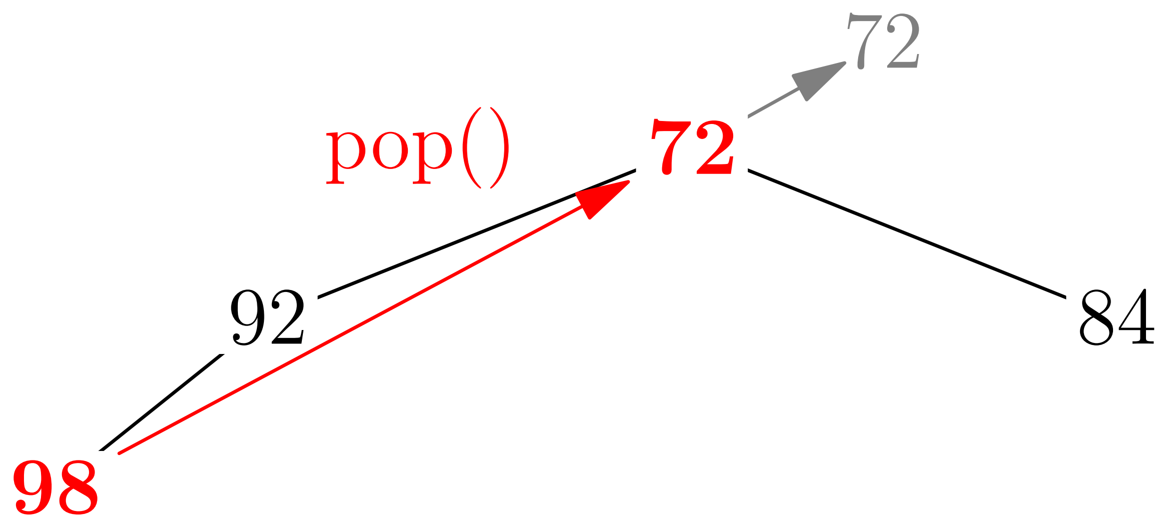

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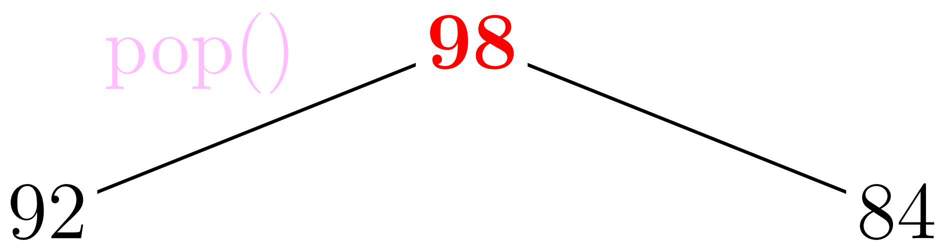

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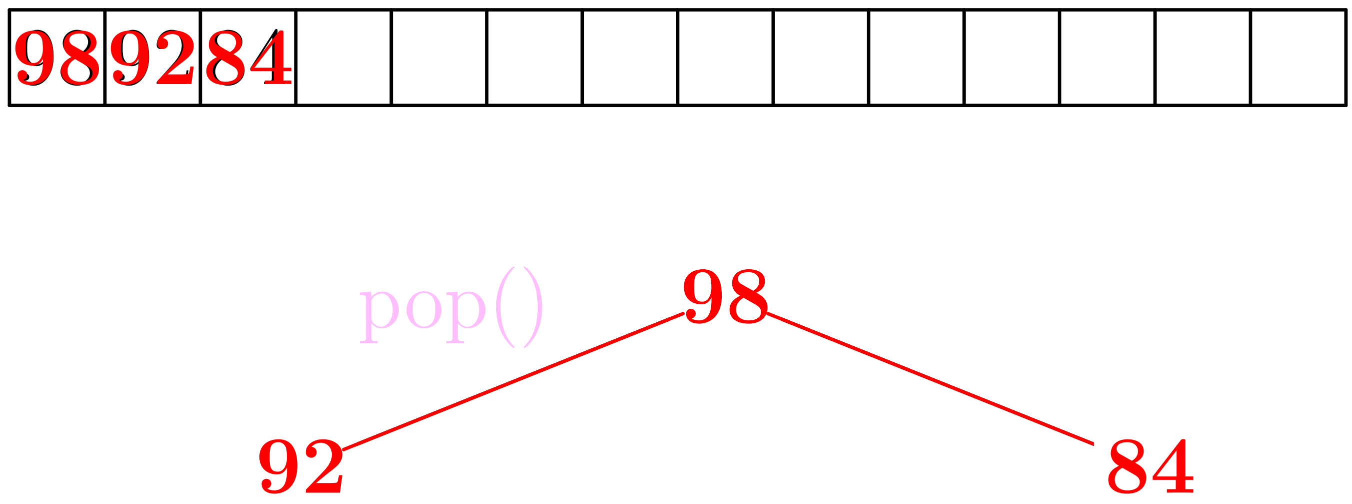

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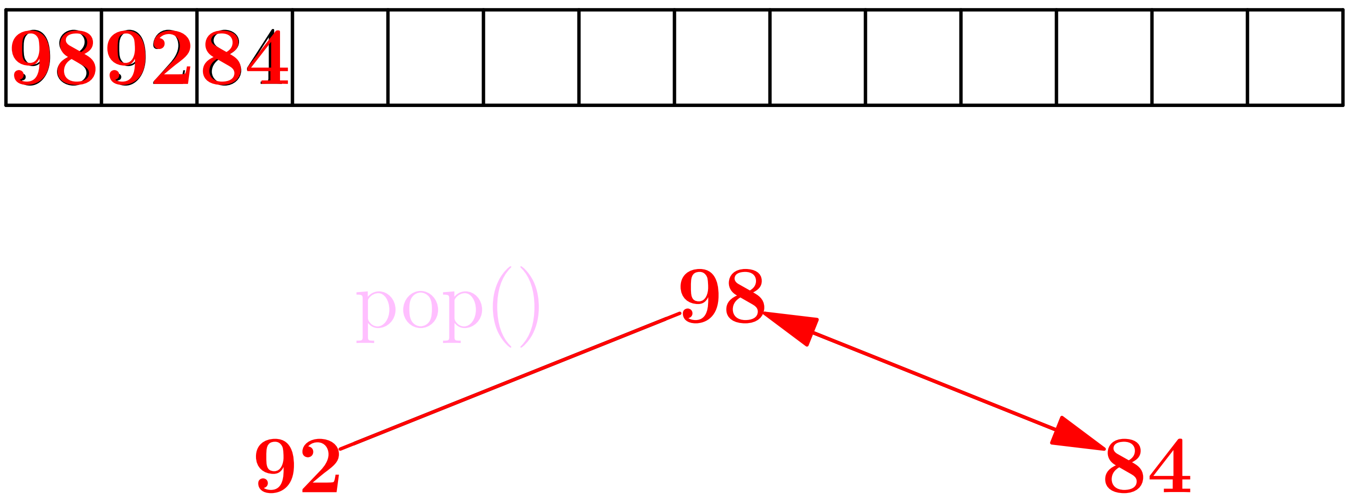

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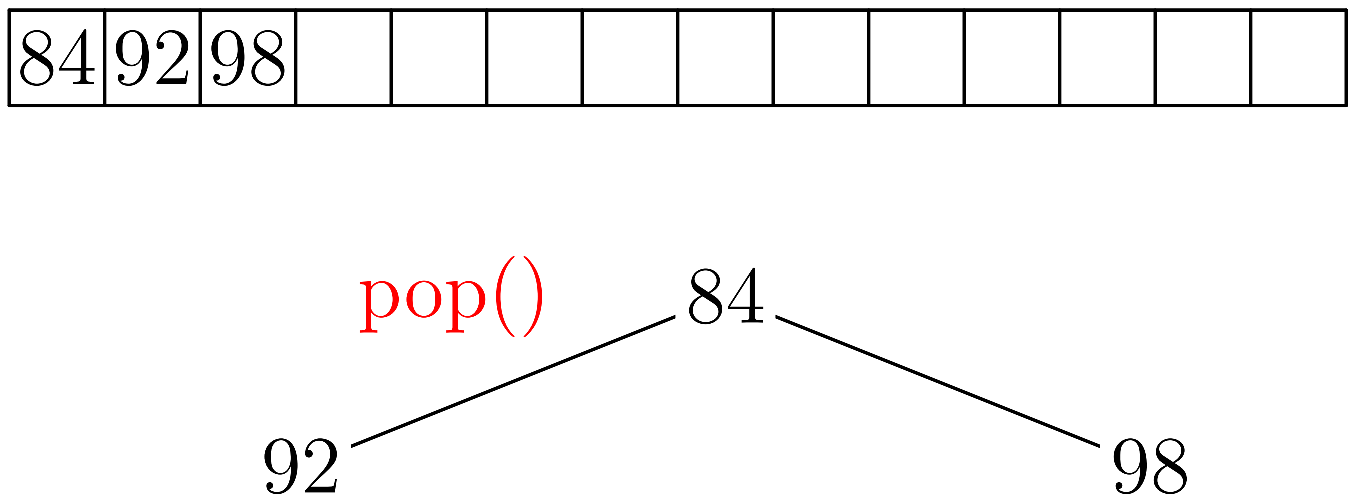

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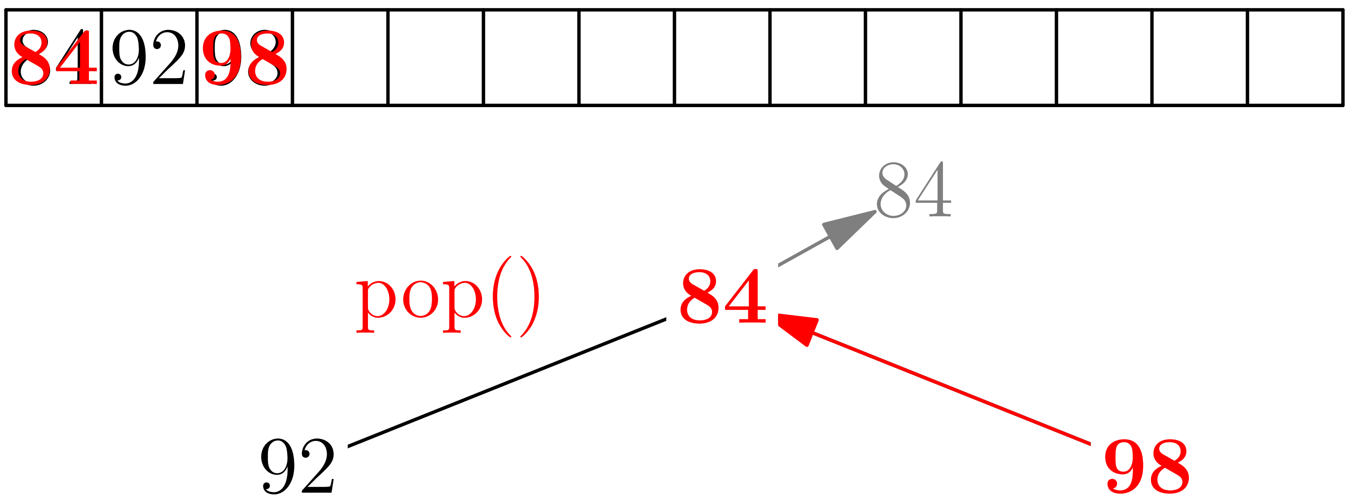

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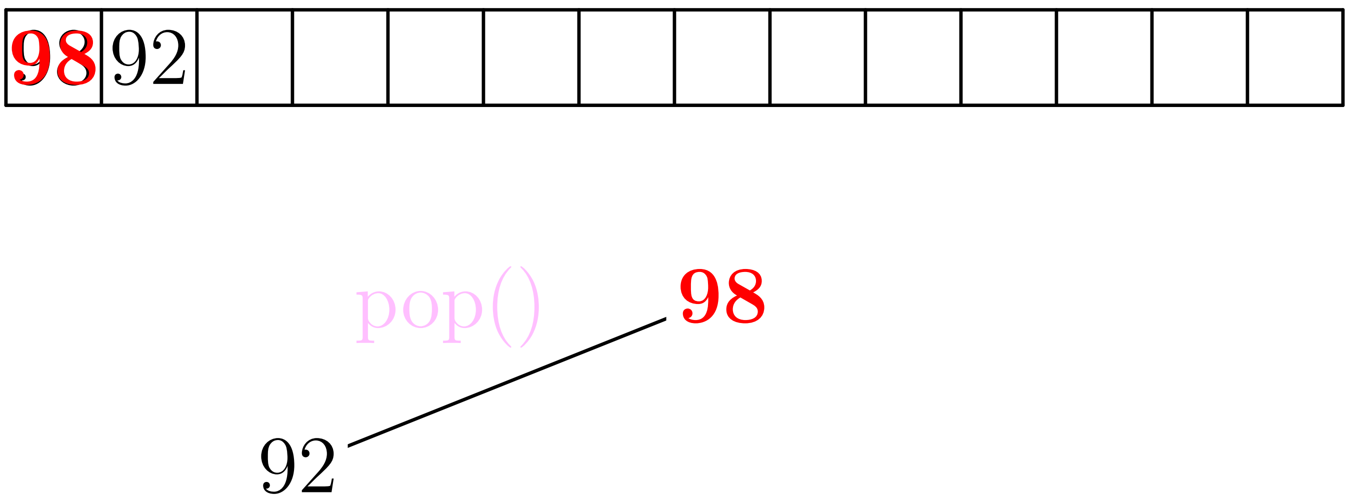

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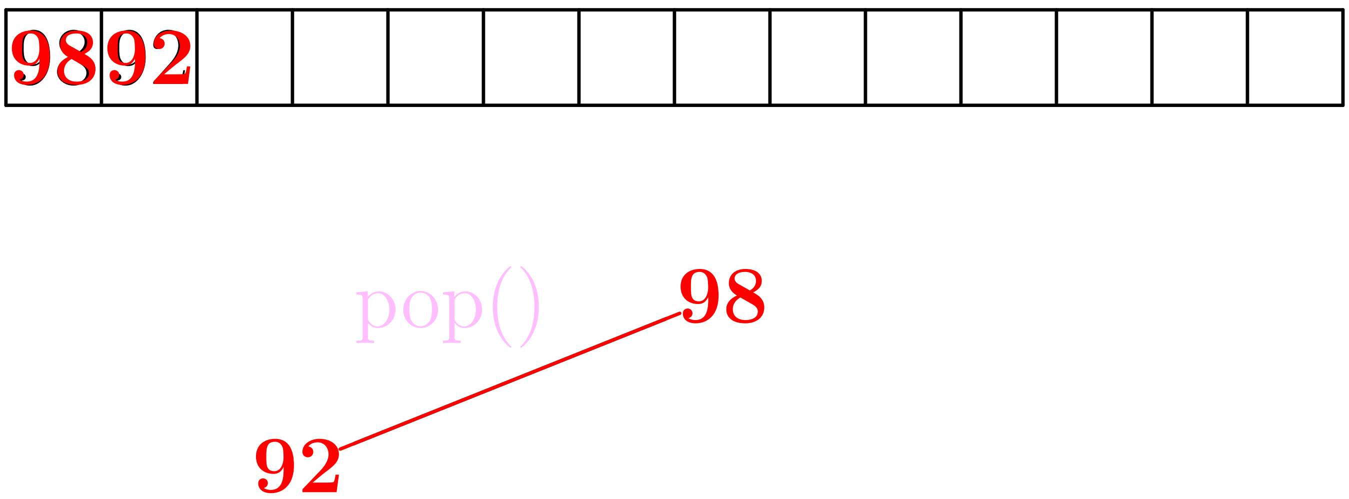

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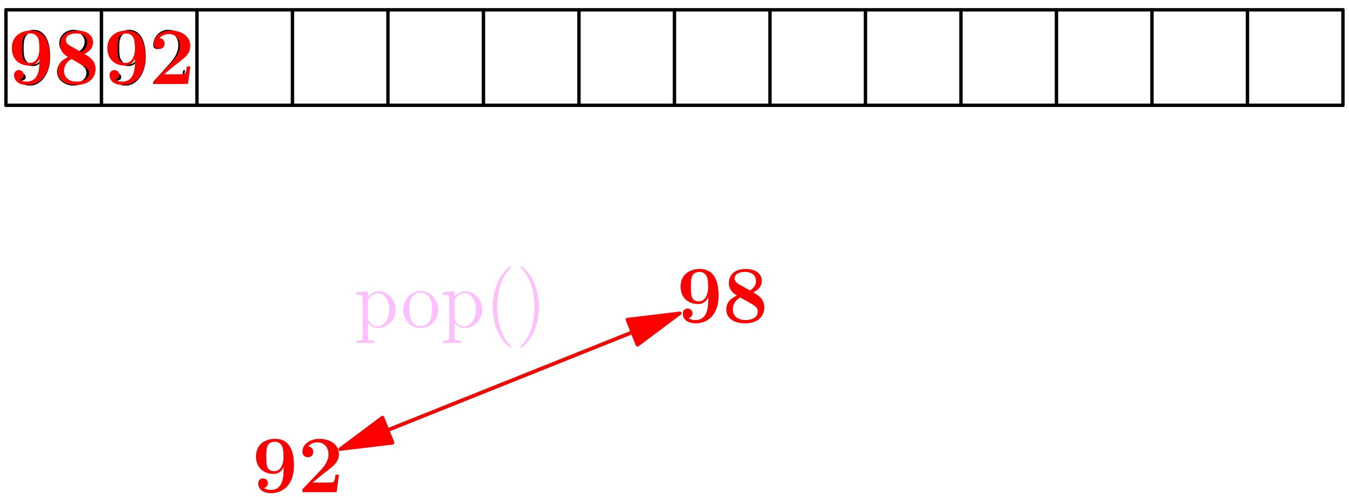

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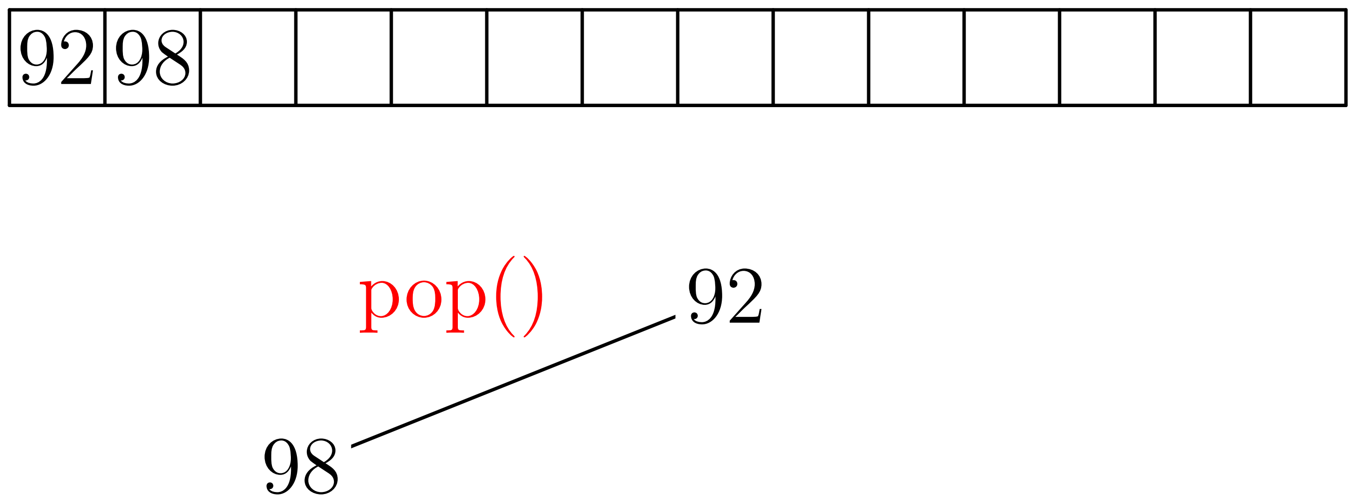

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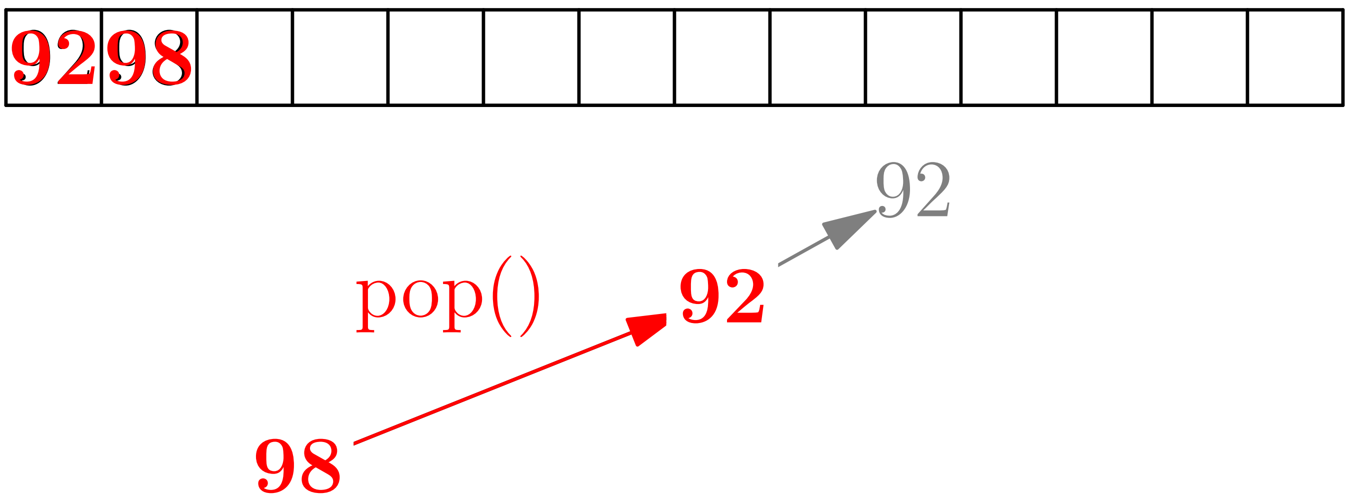

Heaps in Action

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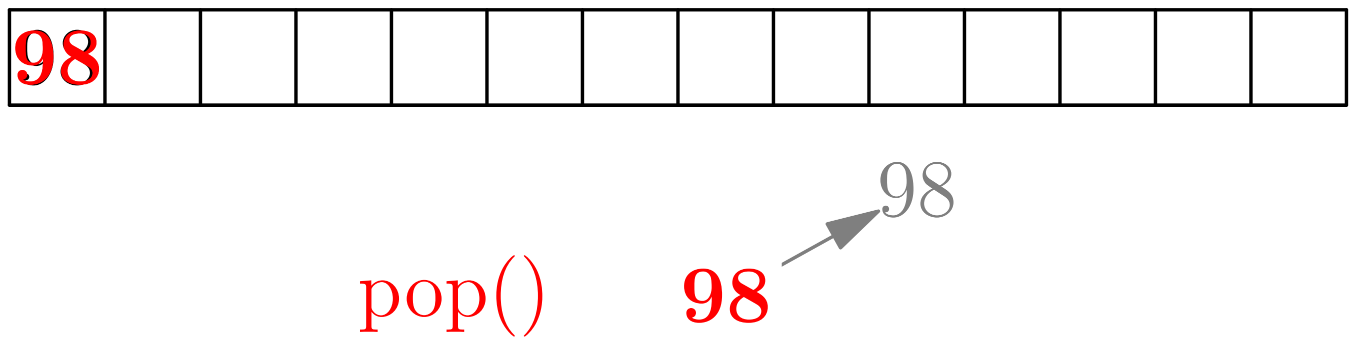
Heaps in Action



Time Complexity of Heaps

- The two important operation are `add` and `removeMin`
- These both either percolating an element up the tree or percolating an element down the tree
- The number of operations depends on the depth of the tree which is $\Theta(\log(n))$
- Thus `add` and `removeMin` are $O(\log(n))$
- Except `add` could also require resizing the array, but the amortised cost of this is low

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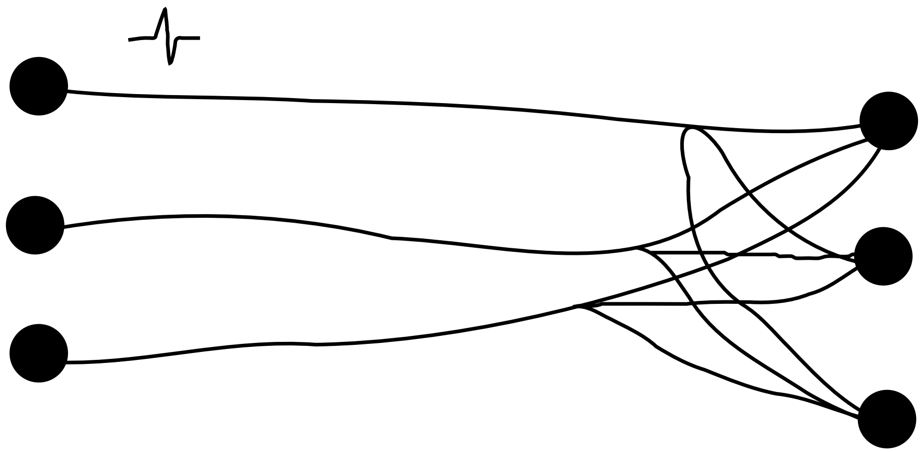
Real Time Simulation

- A nice application of priority queues is to perform real time simulations
- I was once modelling a neural network where neuron fired an impulse which would then be received by other neurons



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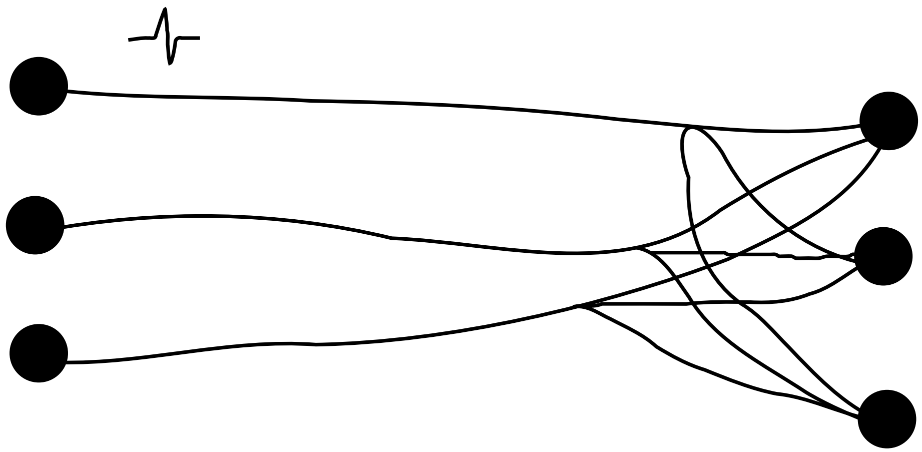

Synchronised Firing

- We wanted to show that if a group of neurons fired together they could make another group of neurons fire in synchrony despite the fact that it would take different times for the receiving neurons to feel the pulse (due to the different lengths of the axons)

- A famous Israeli group had "proved" this couldn't happen

- Using a priority queue we modelled the neurons
  - ⋆ When a neuron fired the receiving neurons would be put on a priority queue according to when they received the pulse
  - ⋆ If the receiving neurons received enough pulse in a short enough time they would then fire

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Outline

1. Heaps
2. Priority Queues
   - Array Implementation
3. **Heap Sort**
4. Other Heaps

Heap Sort

- A priority queue suggests a very simple way of performing sort

- We simply add elements to a heap and then take them off again

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template <typename T>
void sort(vector<T> aList)
{
    HeapPQ<T> aHeap = new HeapPQ<T>(aList.size());
    for (T element: aList)
      aHeap.push(element, element);

    aList.clear();
    while(aHeap.size() > 0) {
      aList.push_back(aHeap.top());
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- Note that this is not an in-place sort algorithm (i.e. it uses lots of memory)

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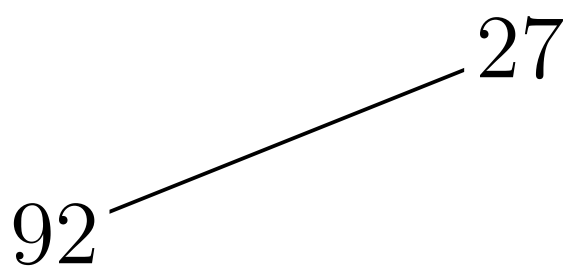

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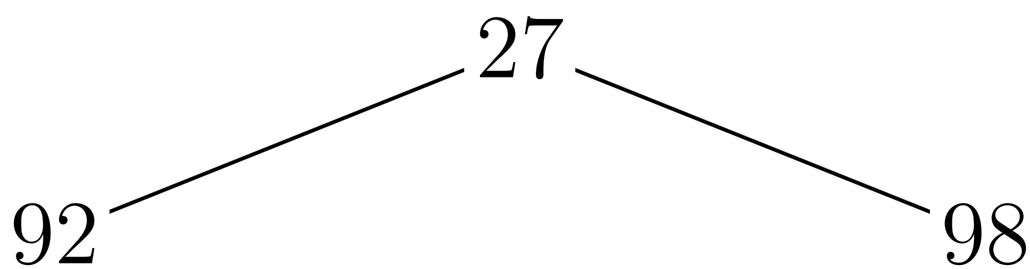

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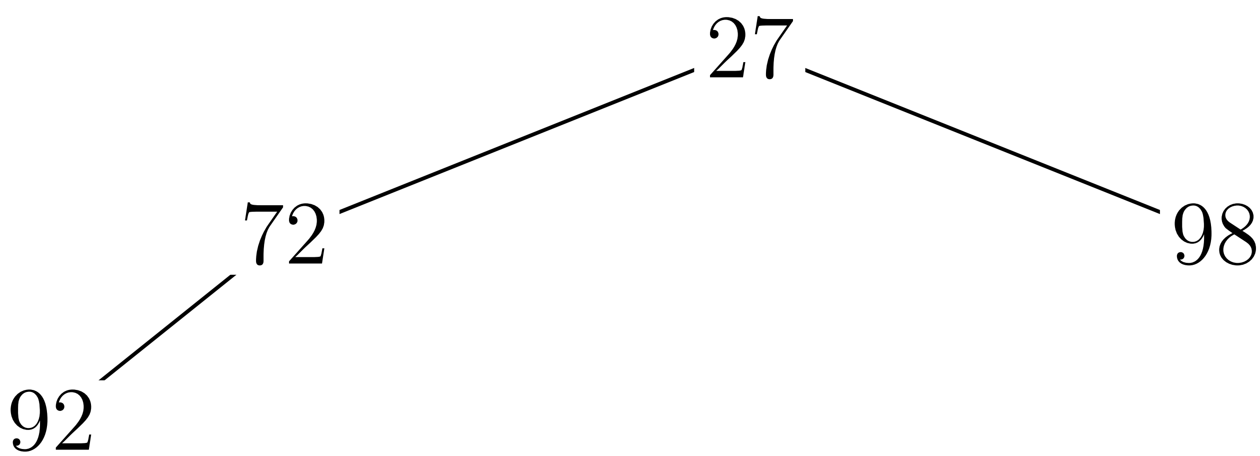

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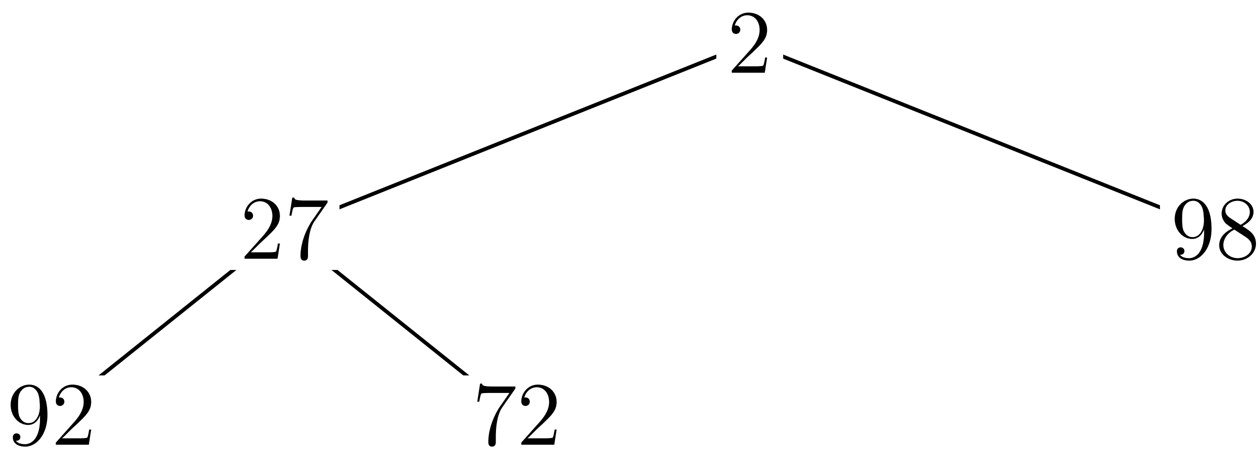

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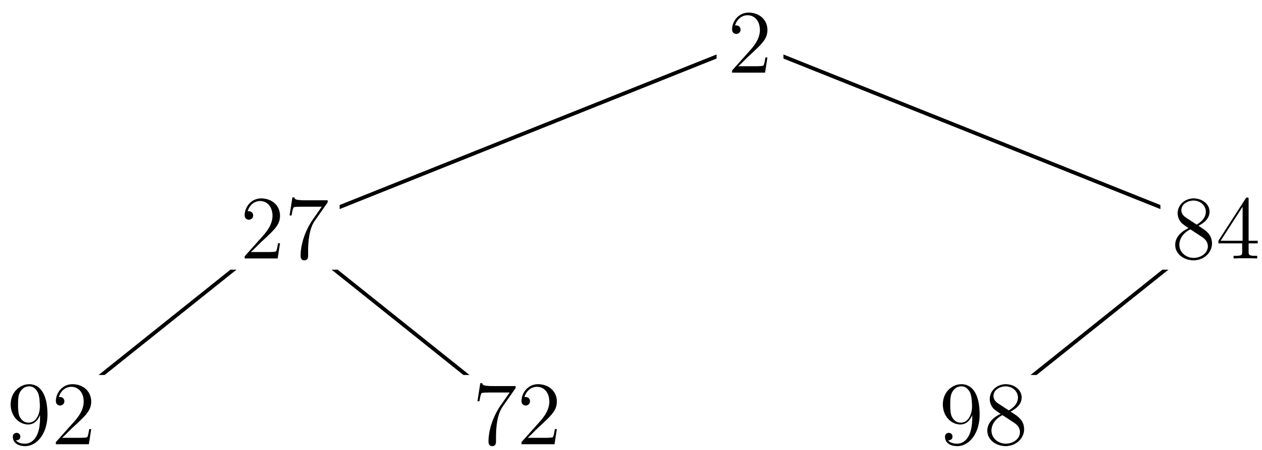

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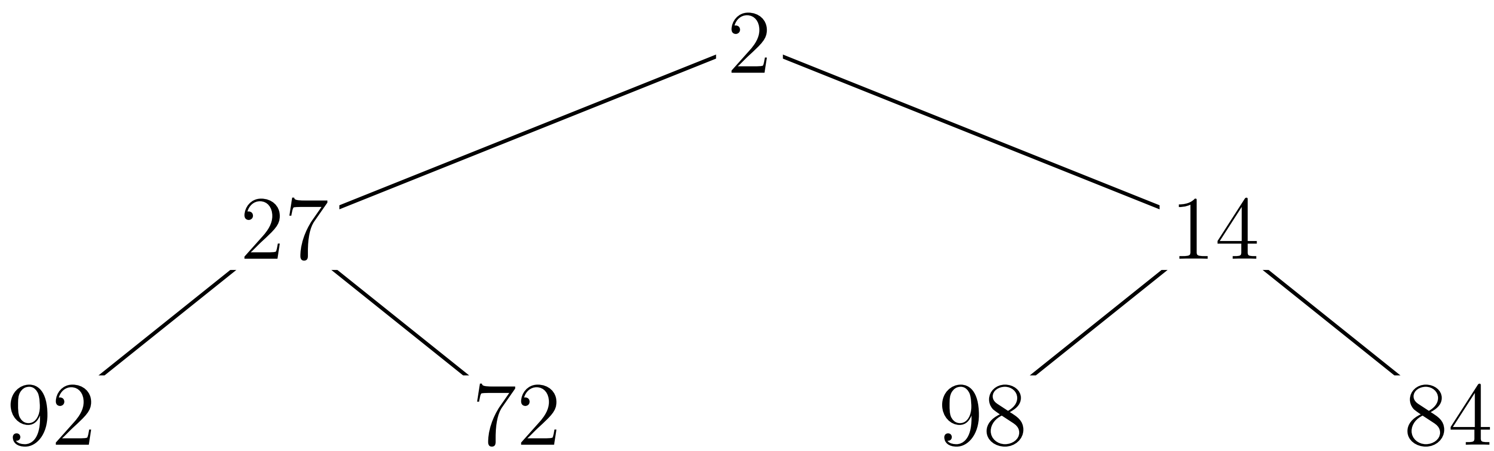

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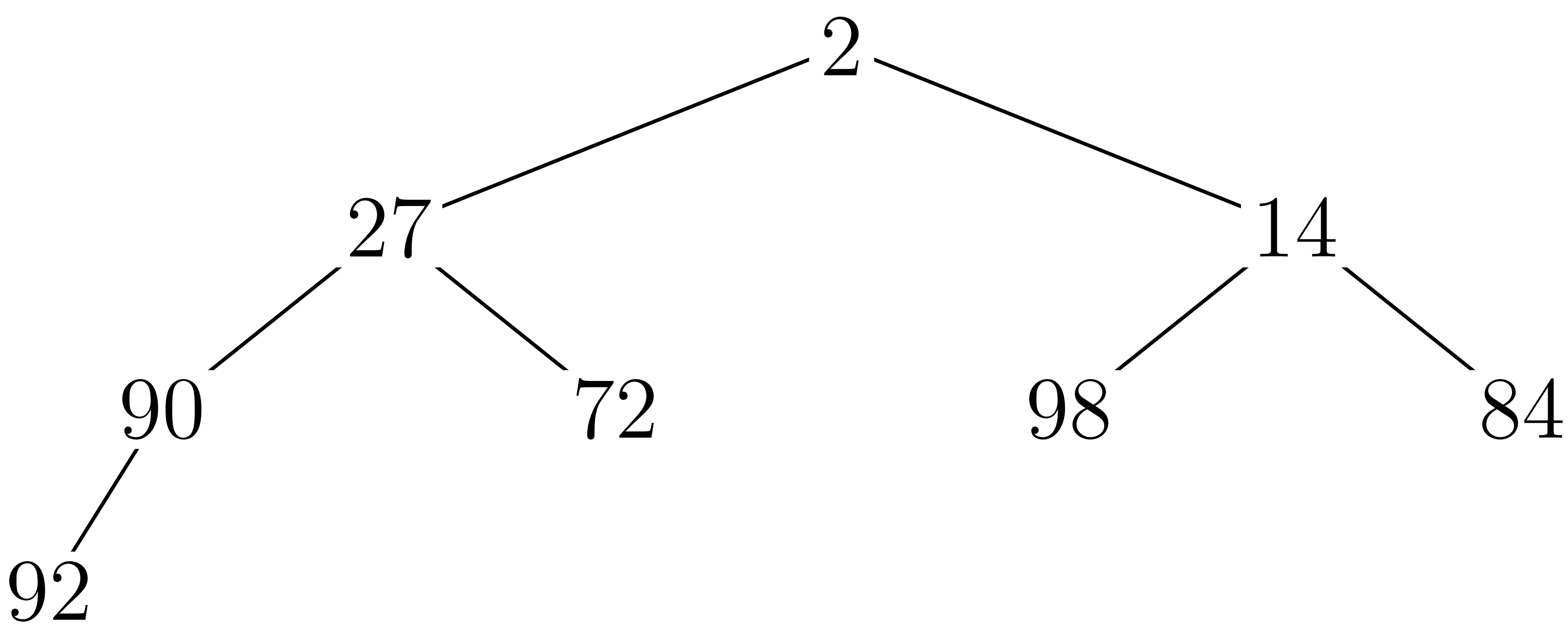

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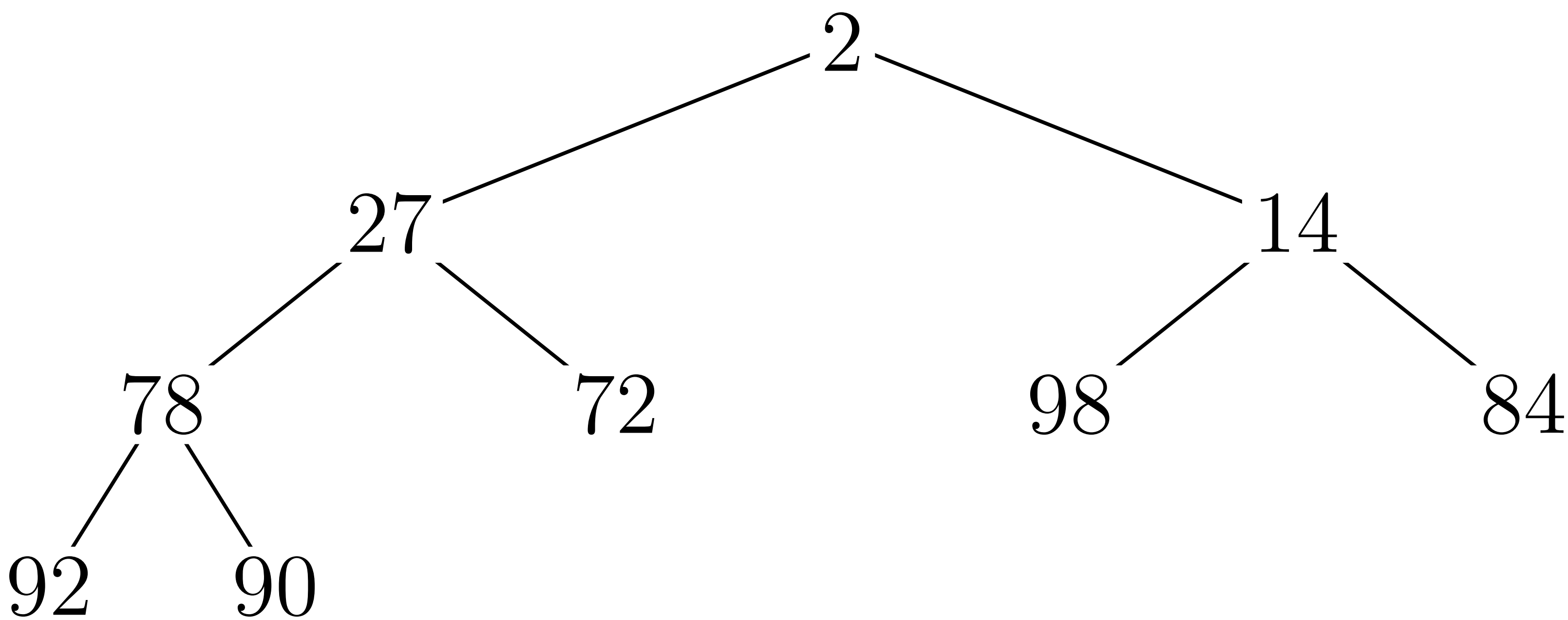

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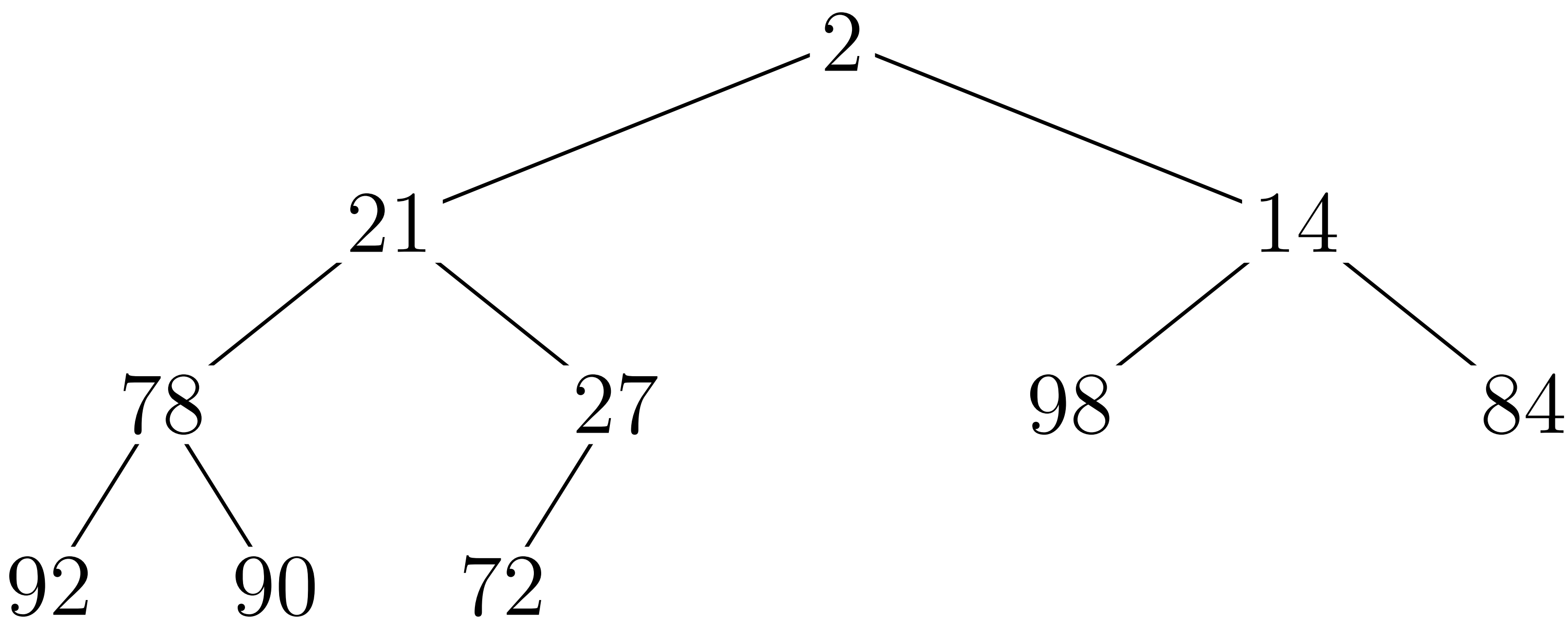

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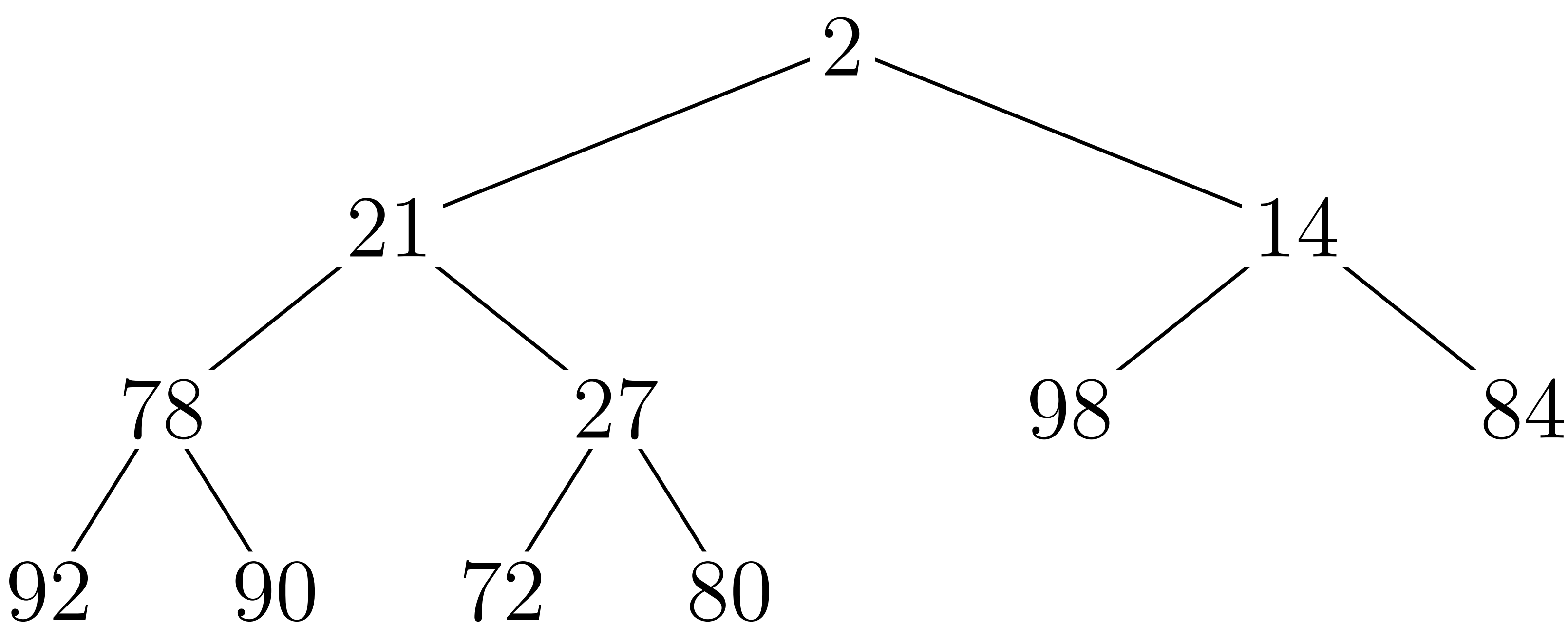

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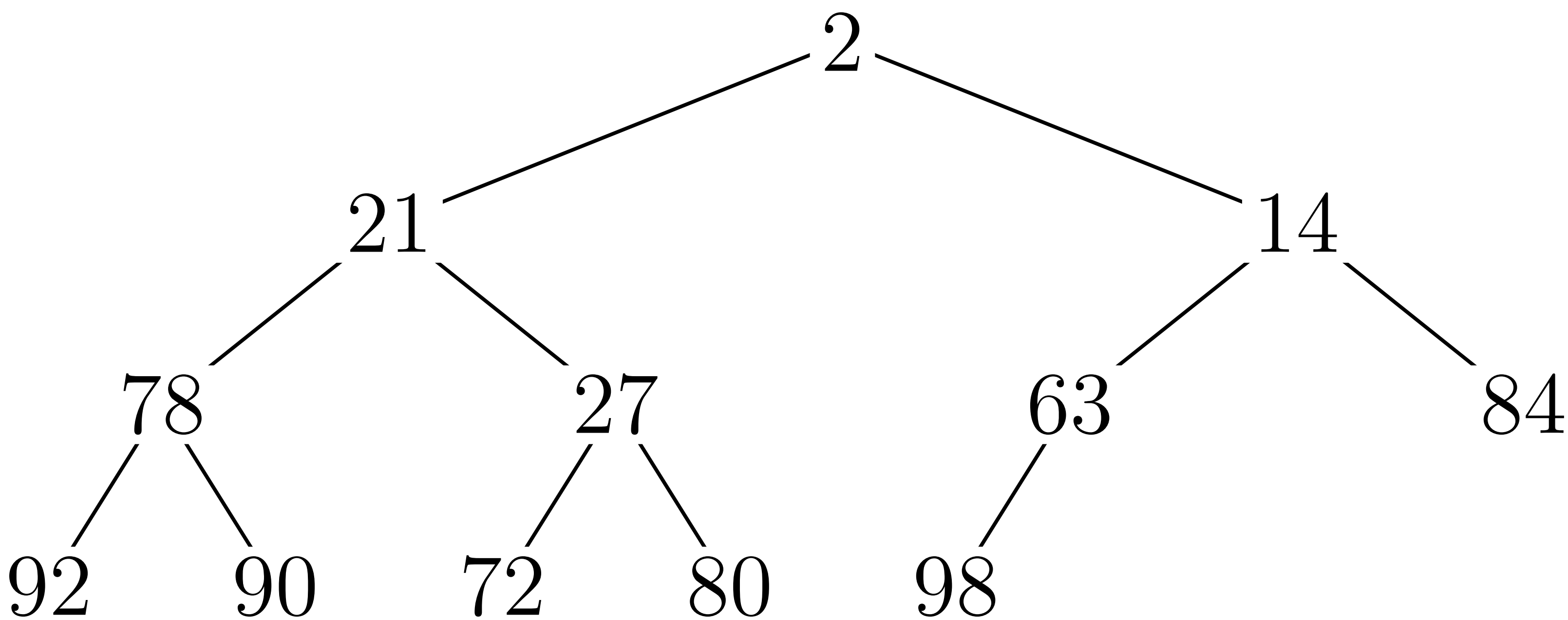

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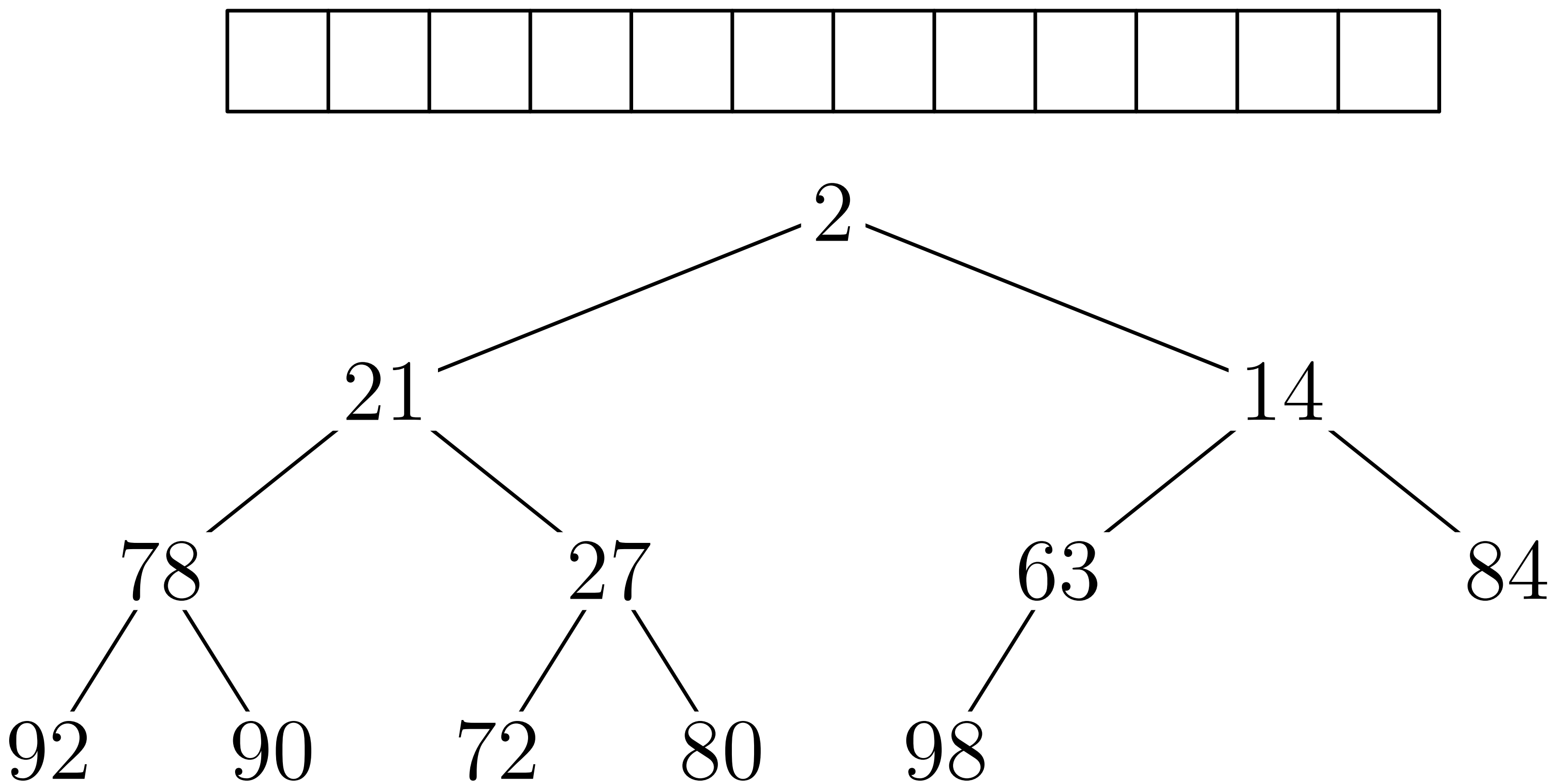

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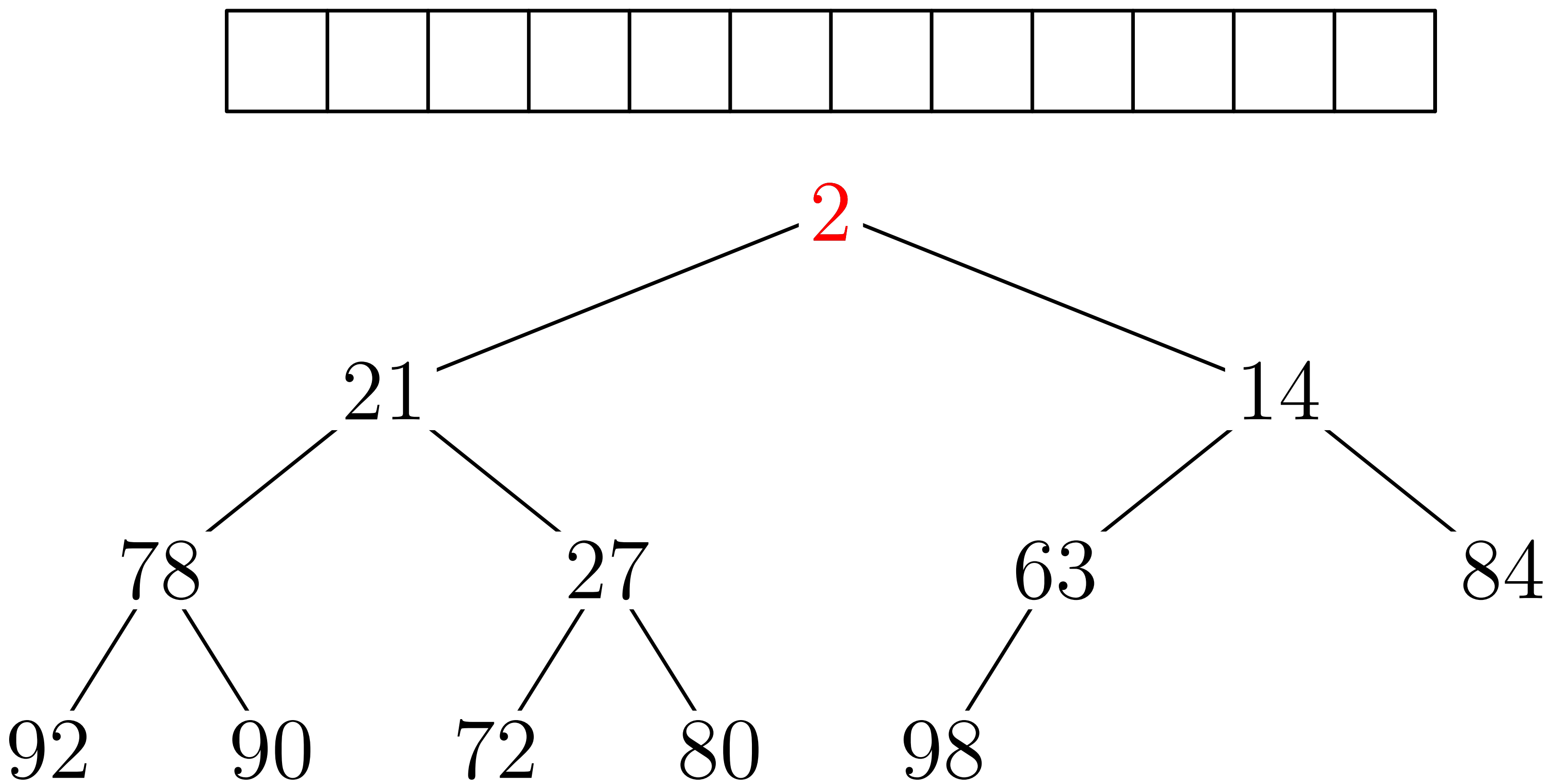

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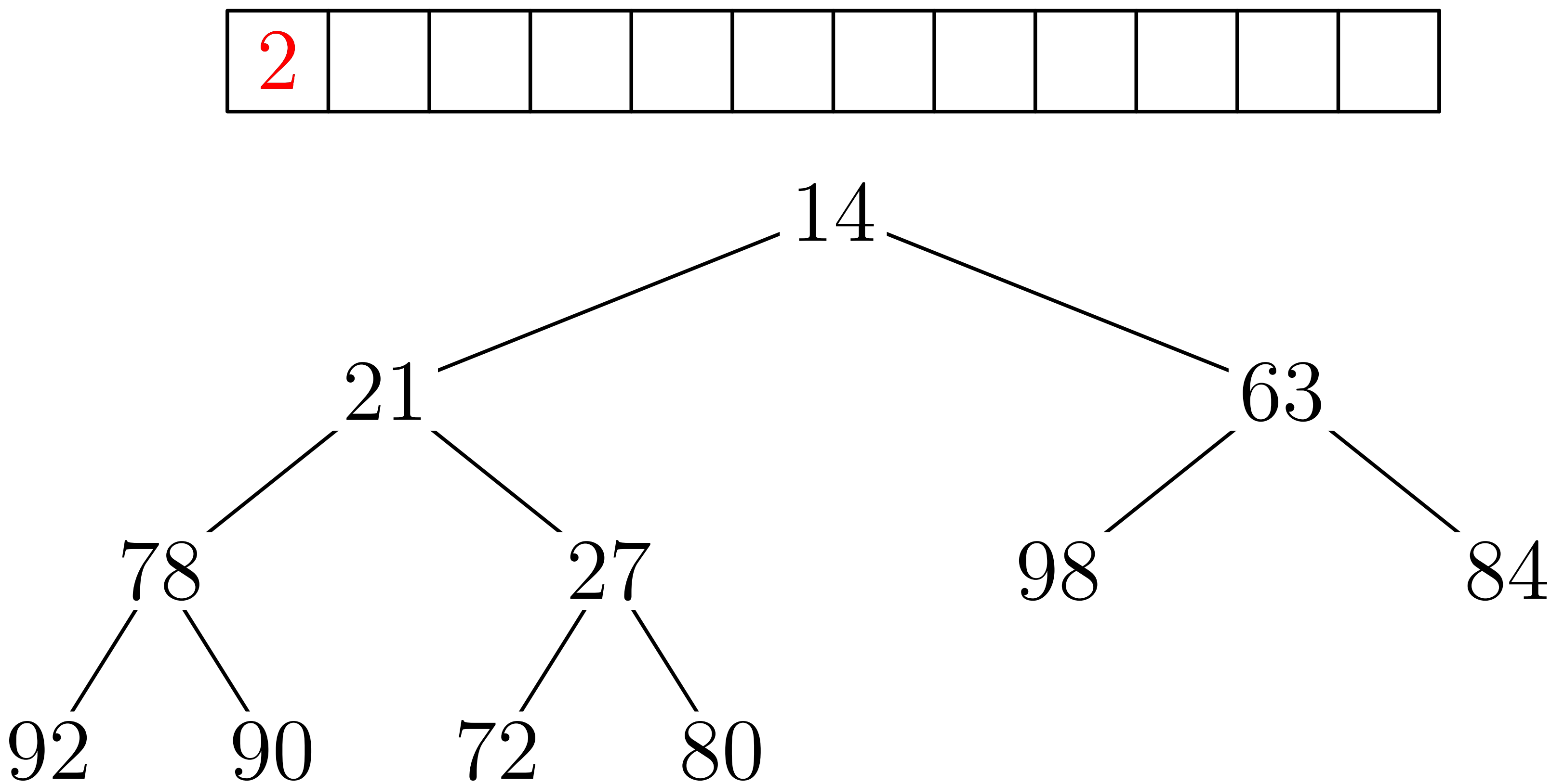

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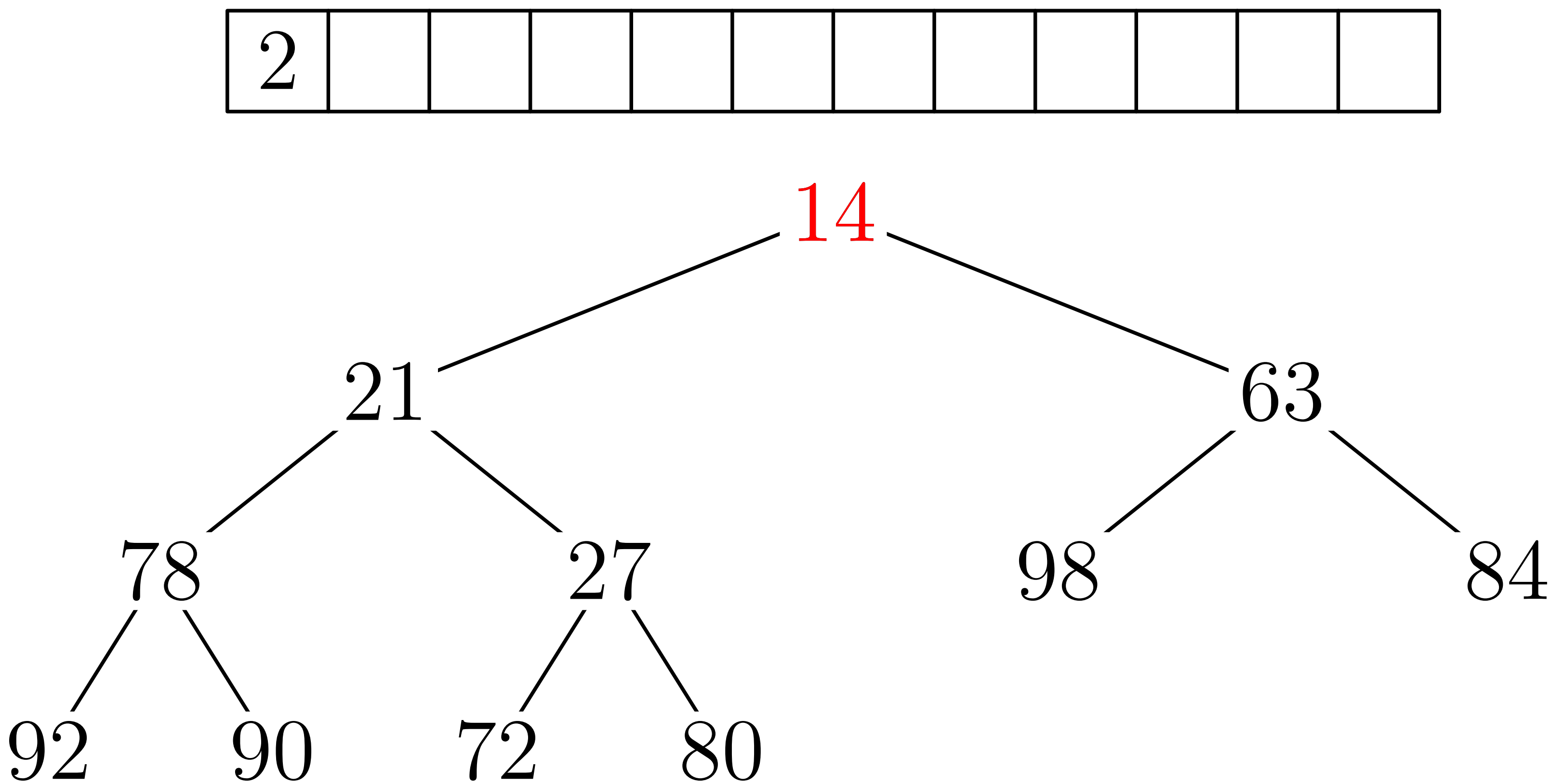

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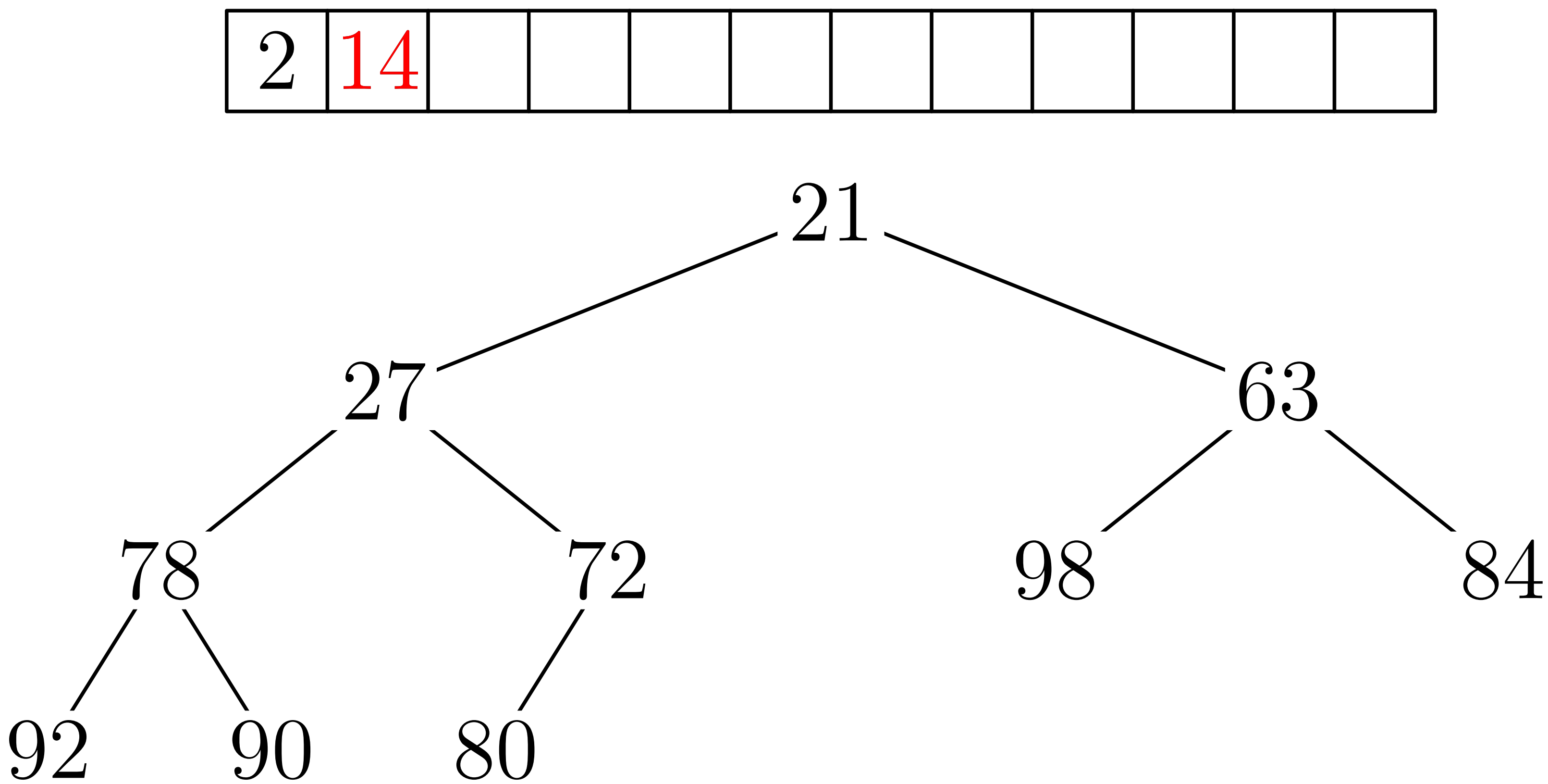

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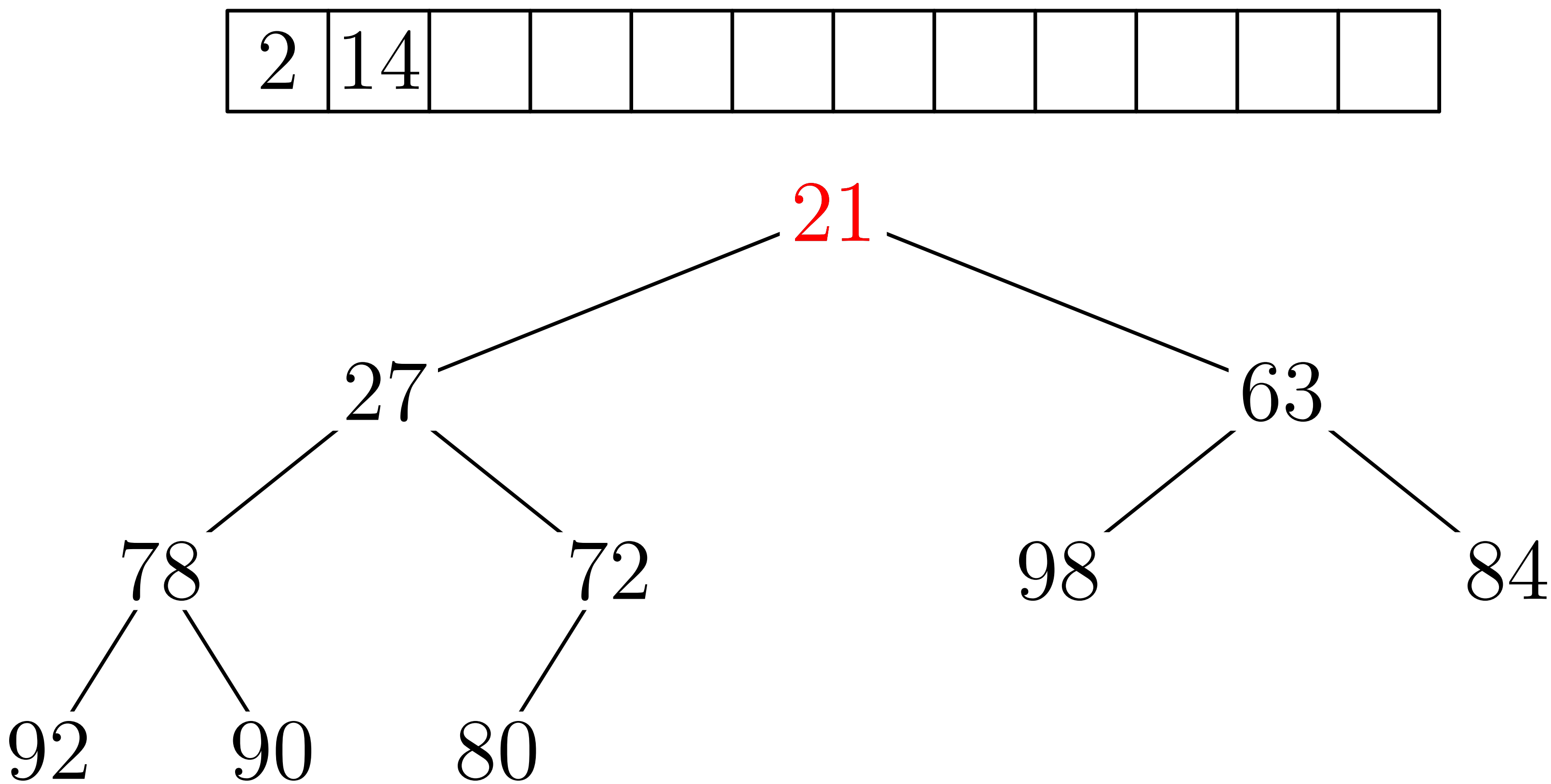

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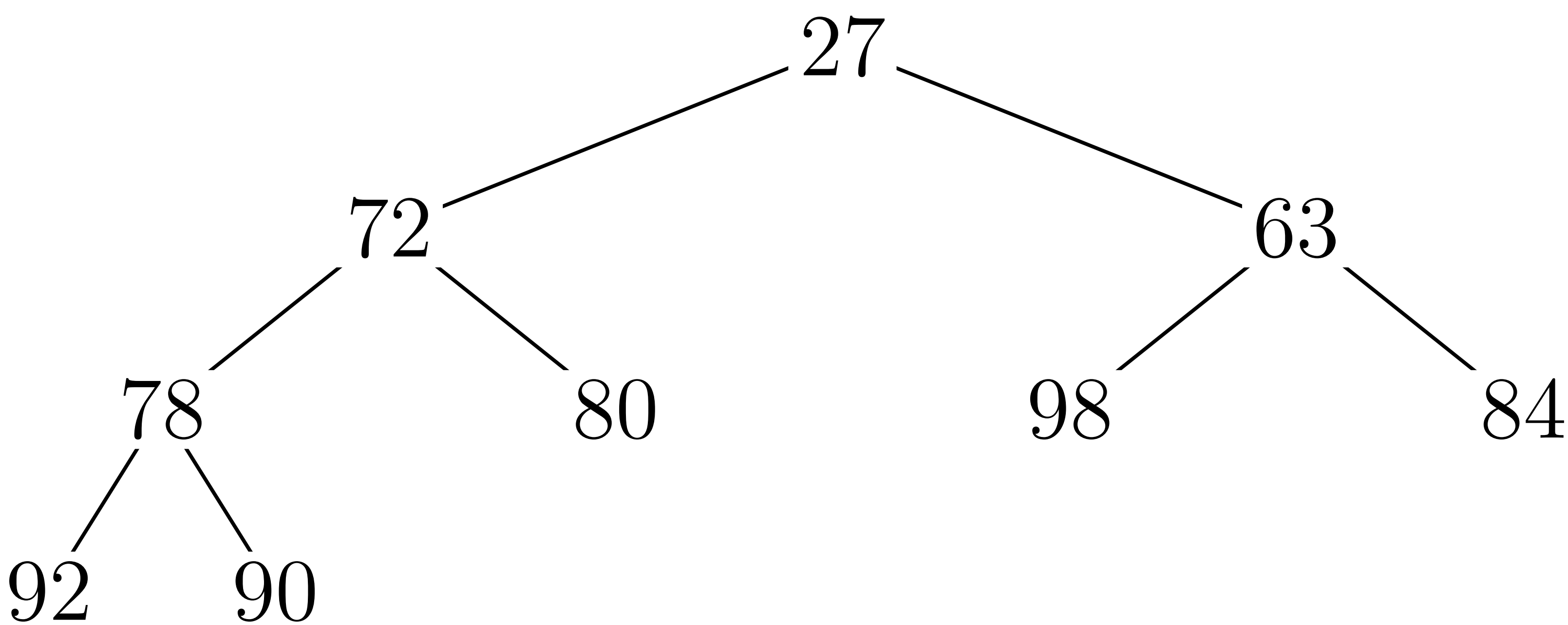
2	14	21									



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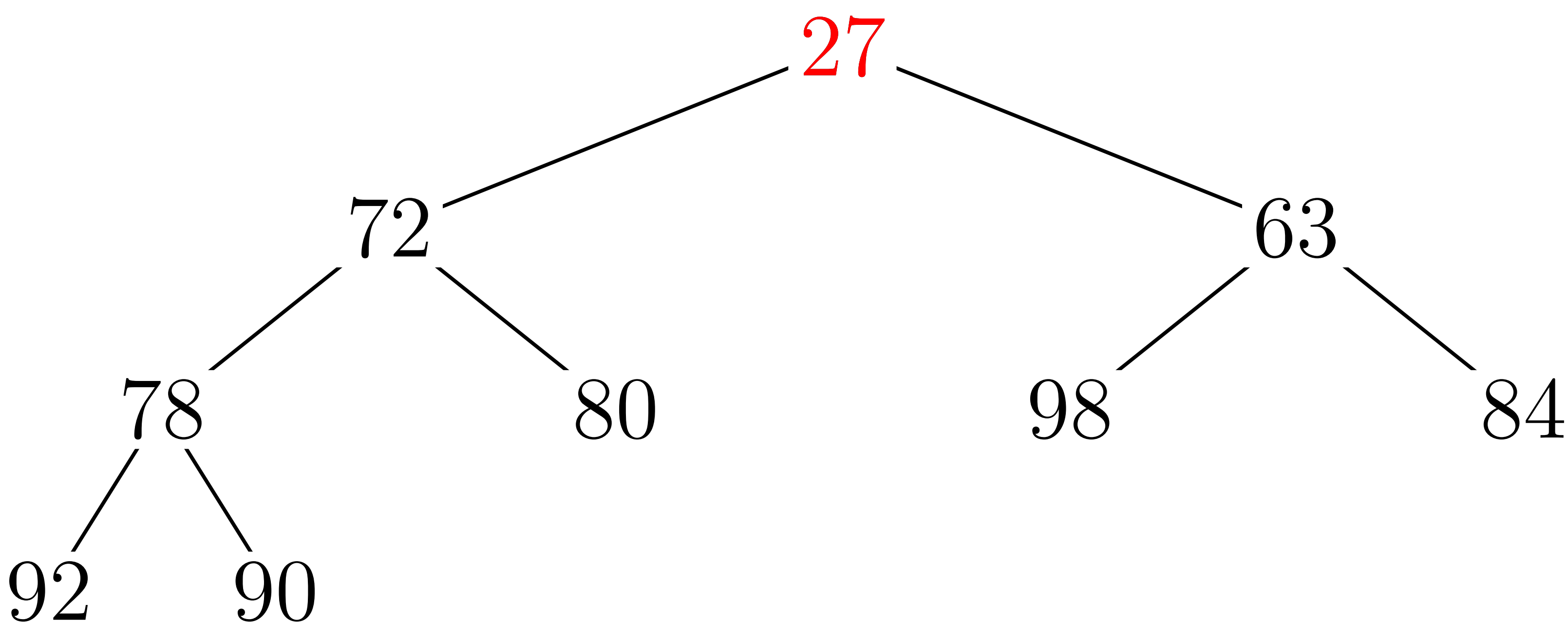

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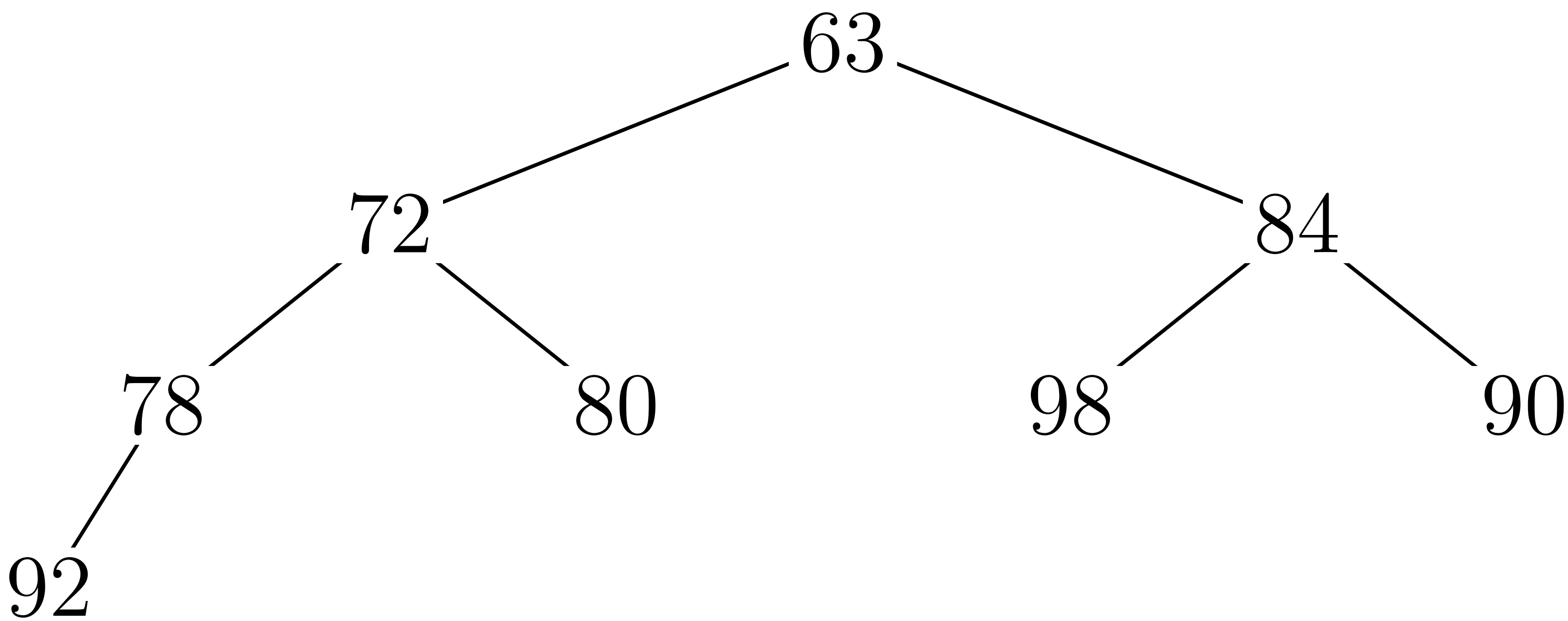

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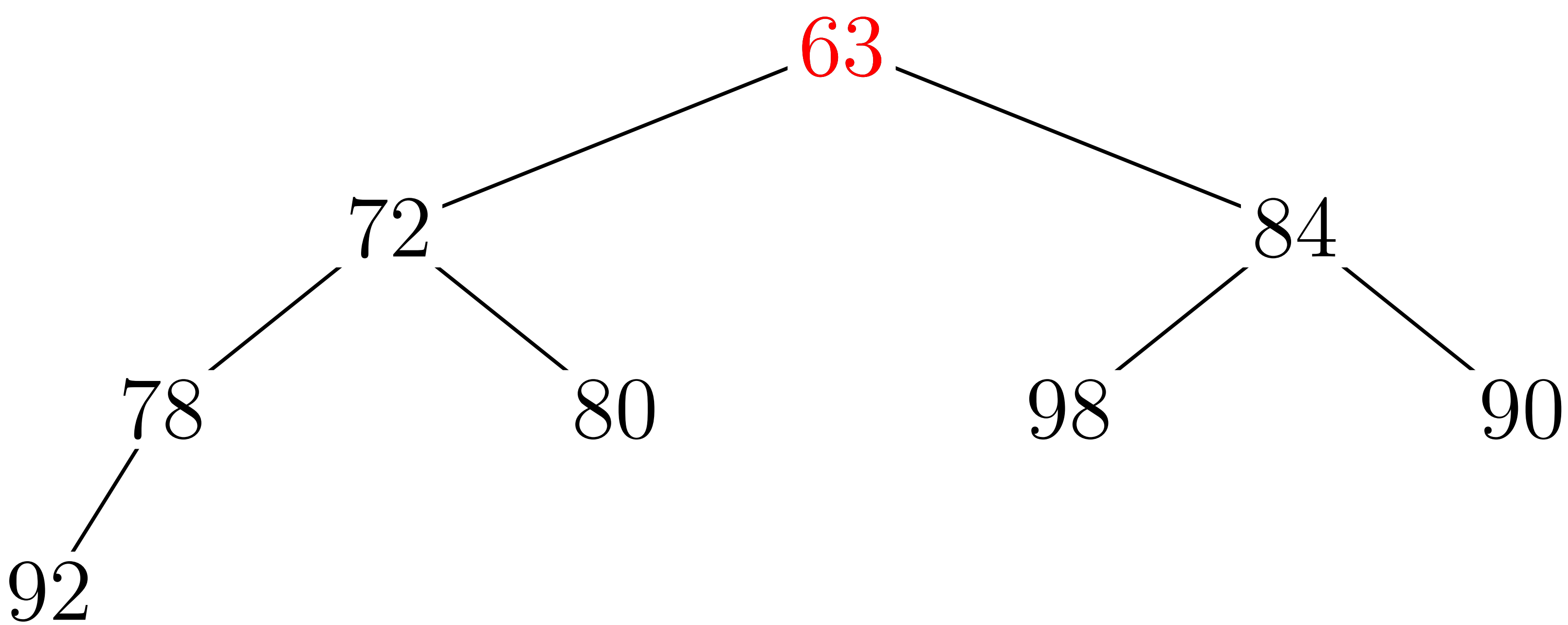

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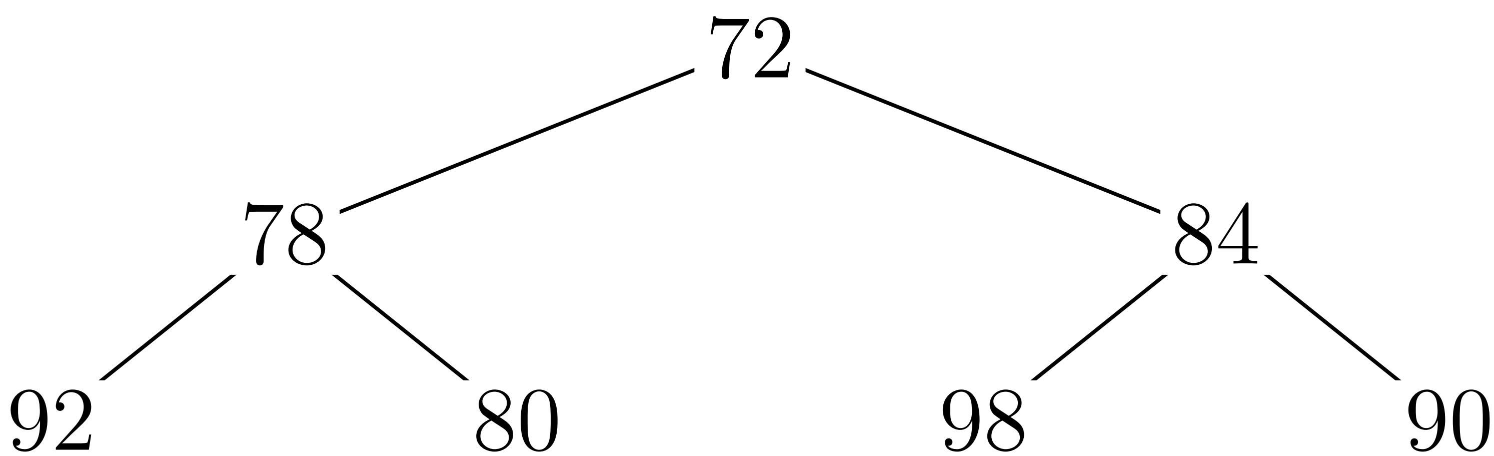

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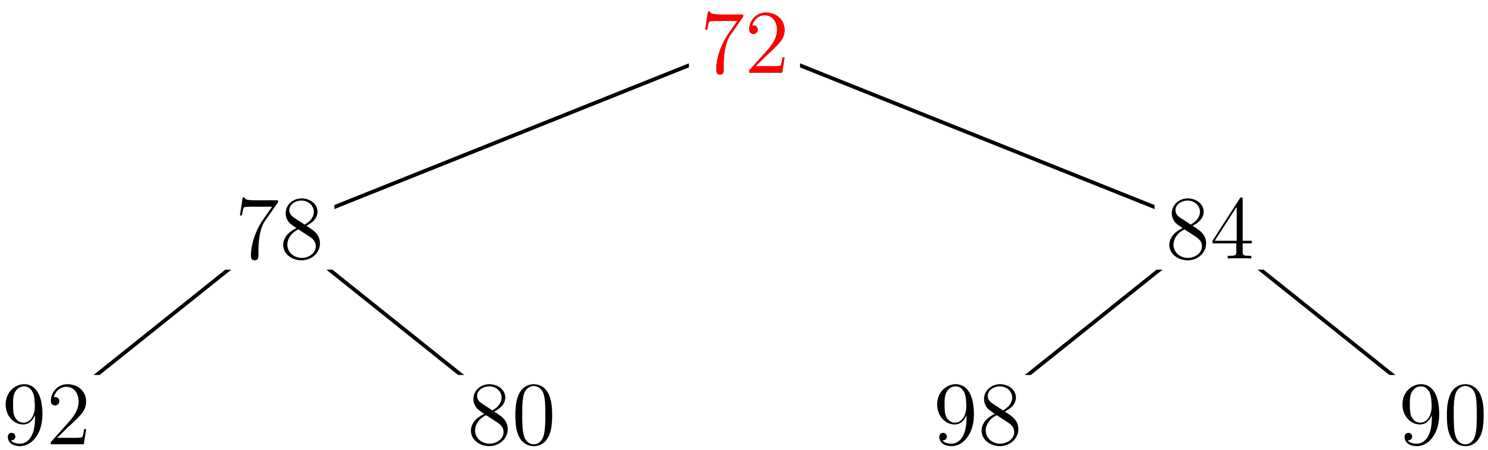

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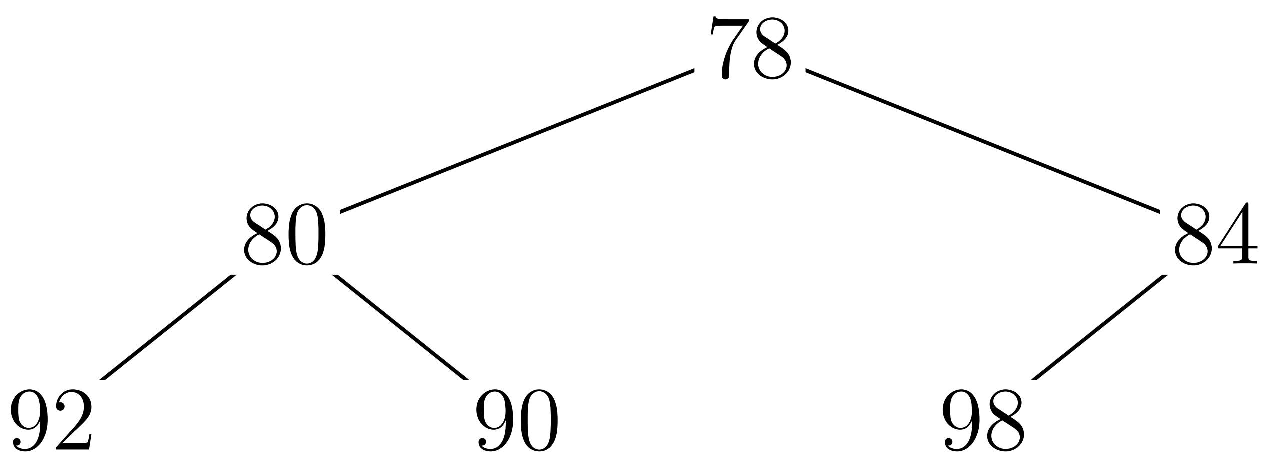

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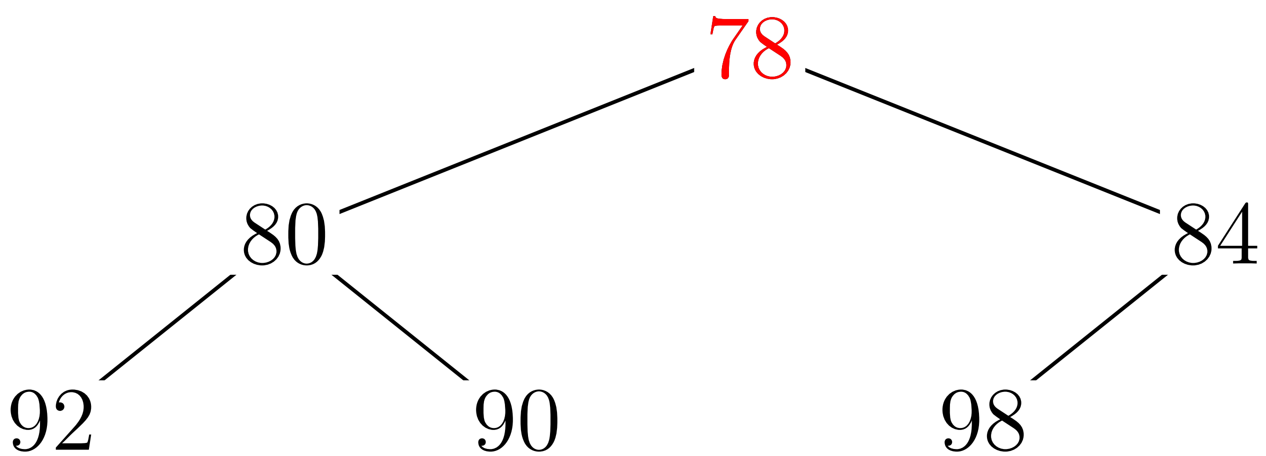

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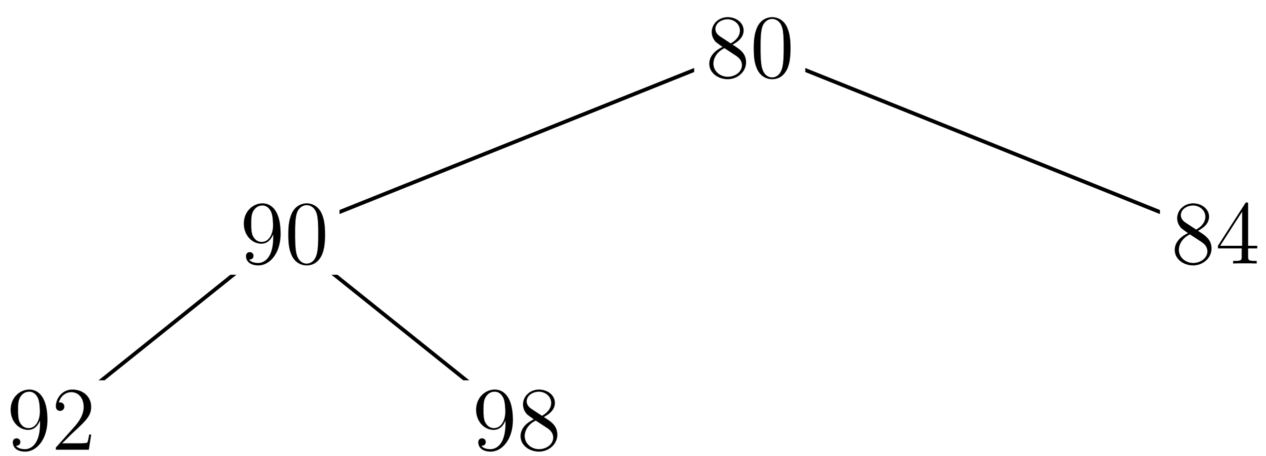

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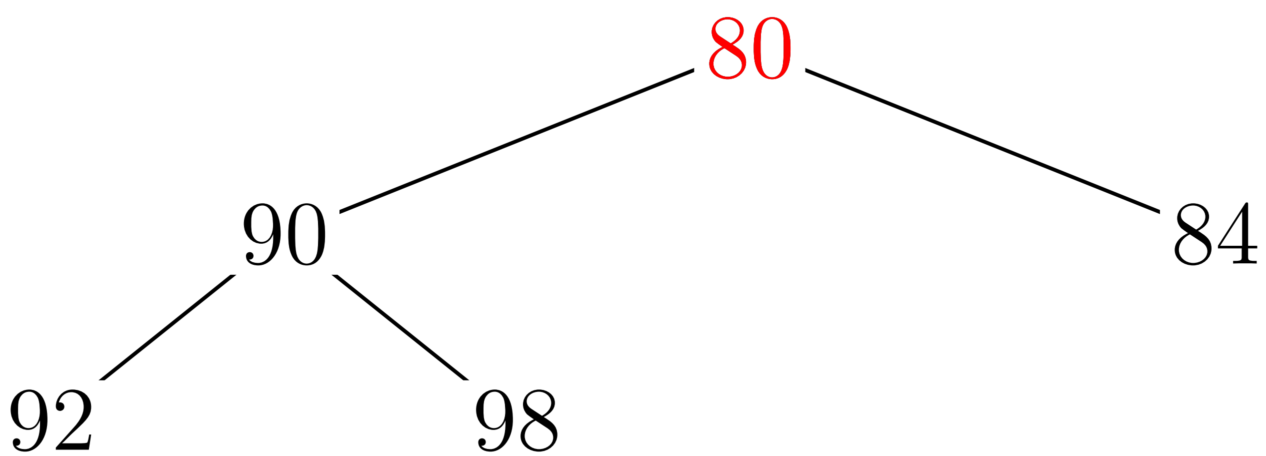

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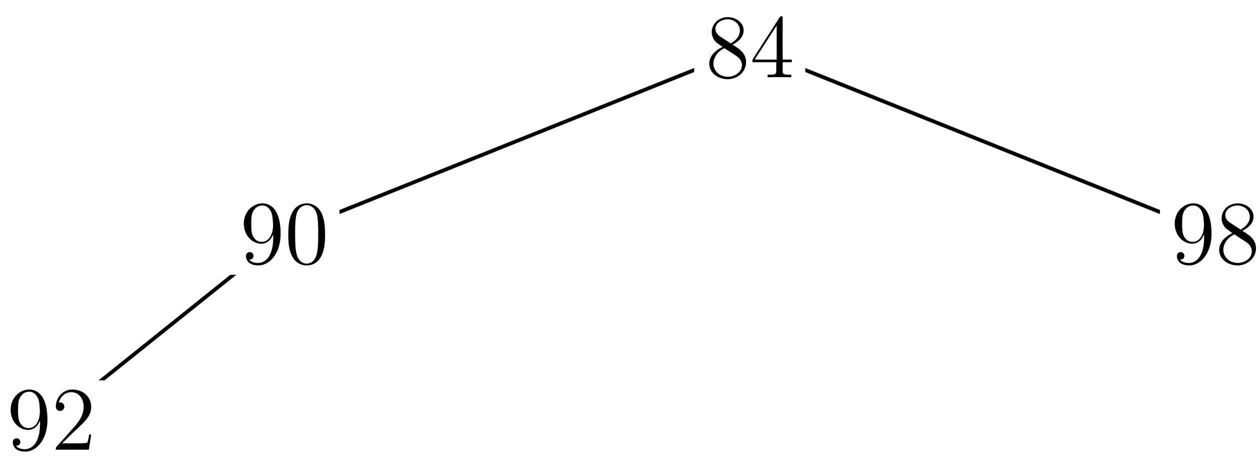
2	14	21	27	63	72	78	80				



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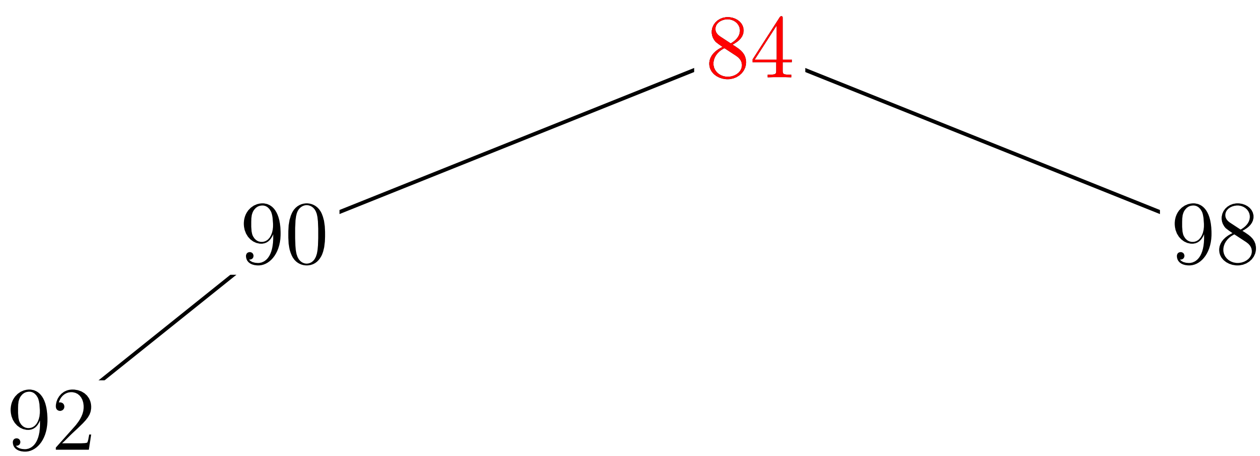

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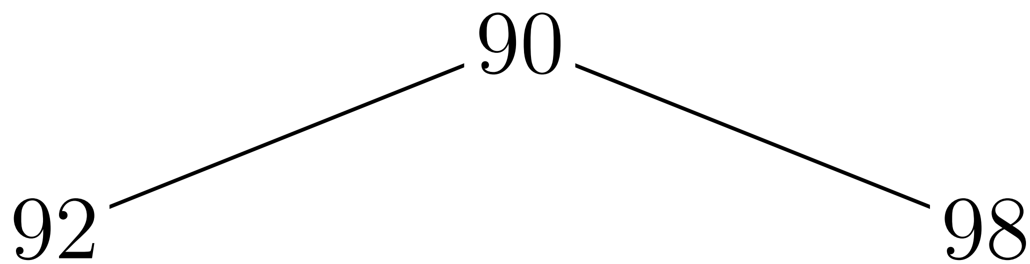

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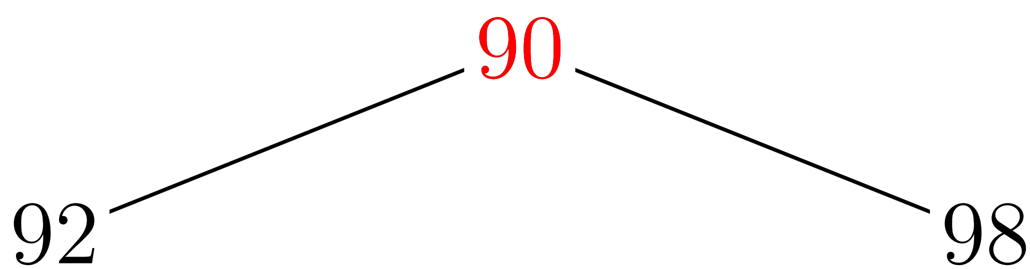

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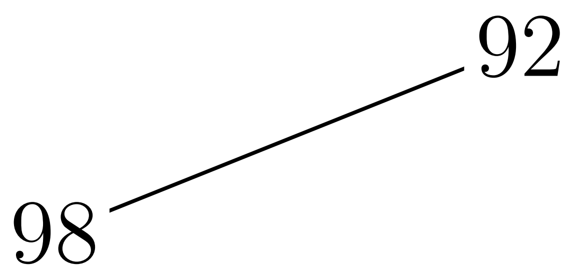

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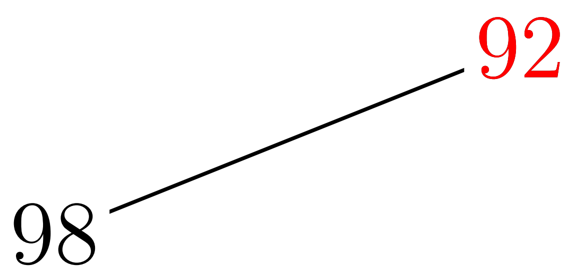

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98

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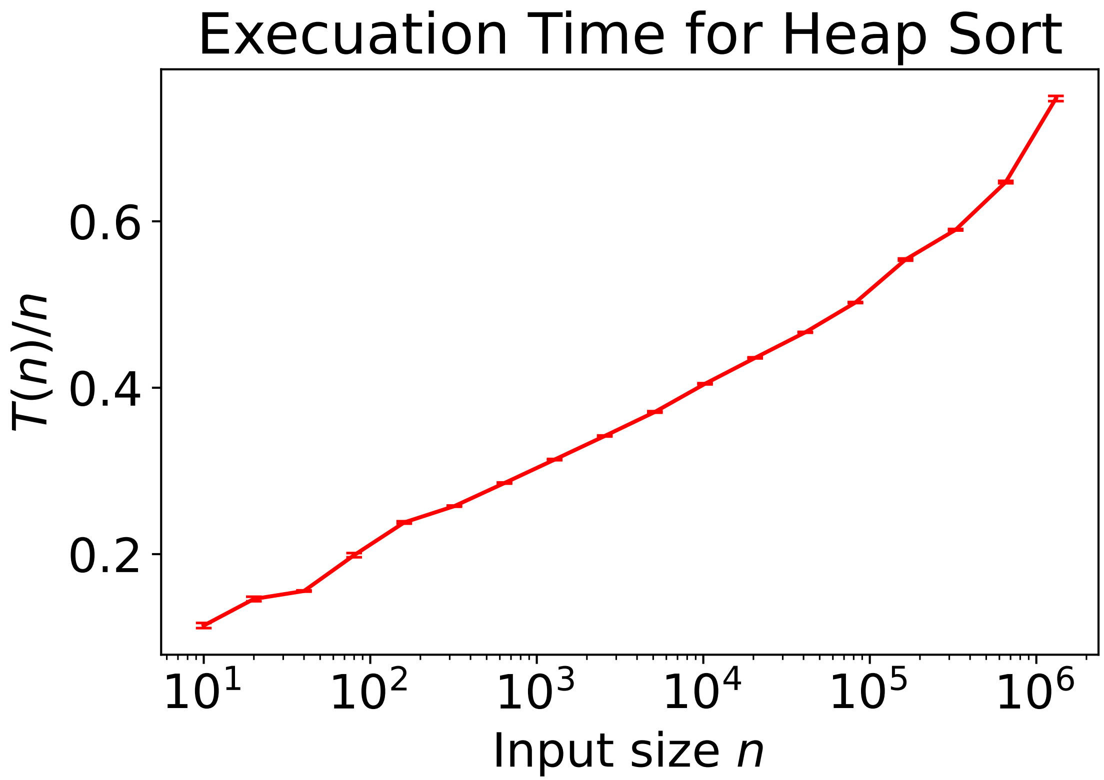
Complexity of Heap Sort

- As we have to add n elements and then remove n elements the time complexity is log-linear, i.e. $O(n \log(n))$



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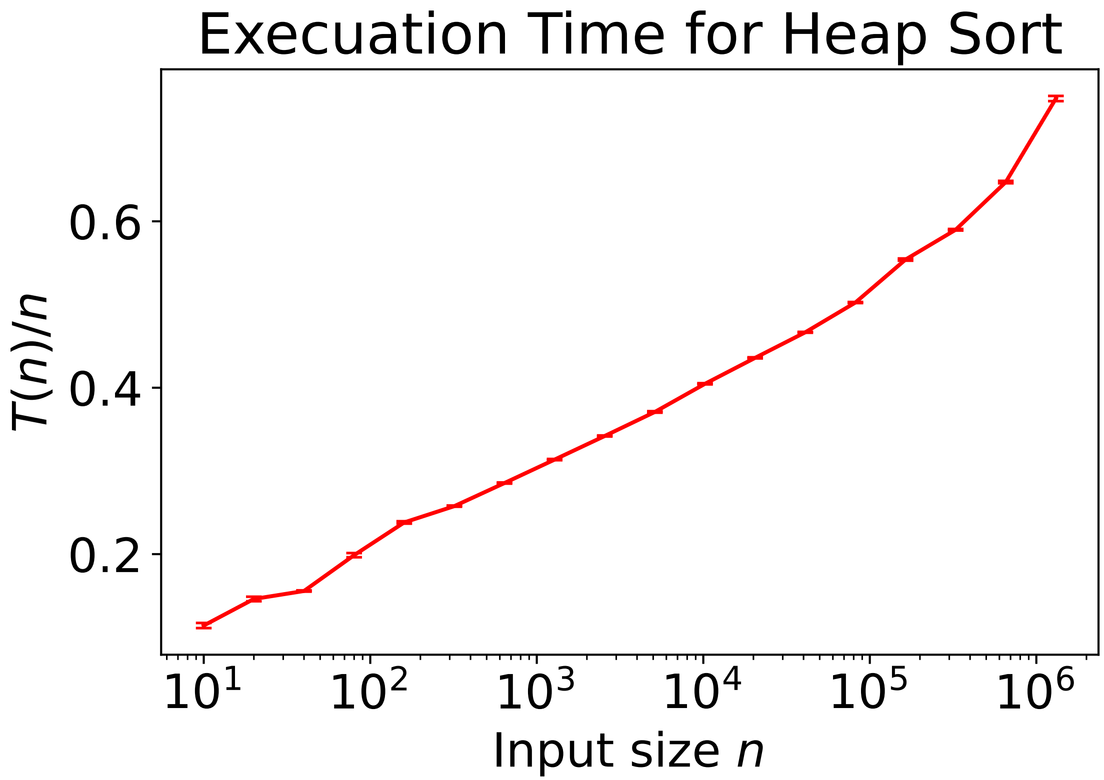

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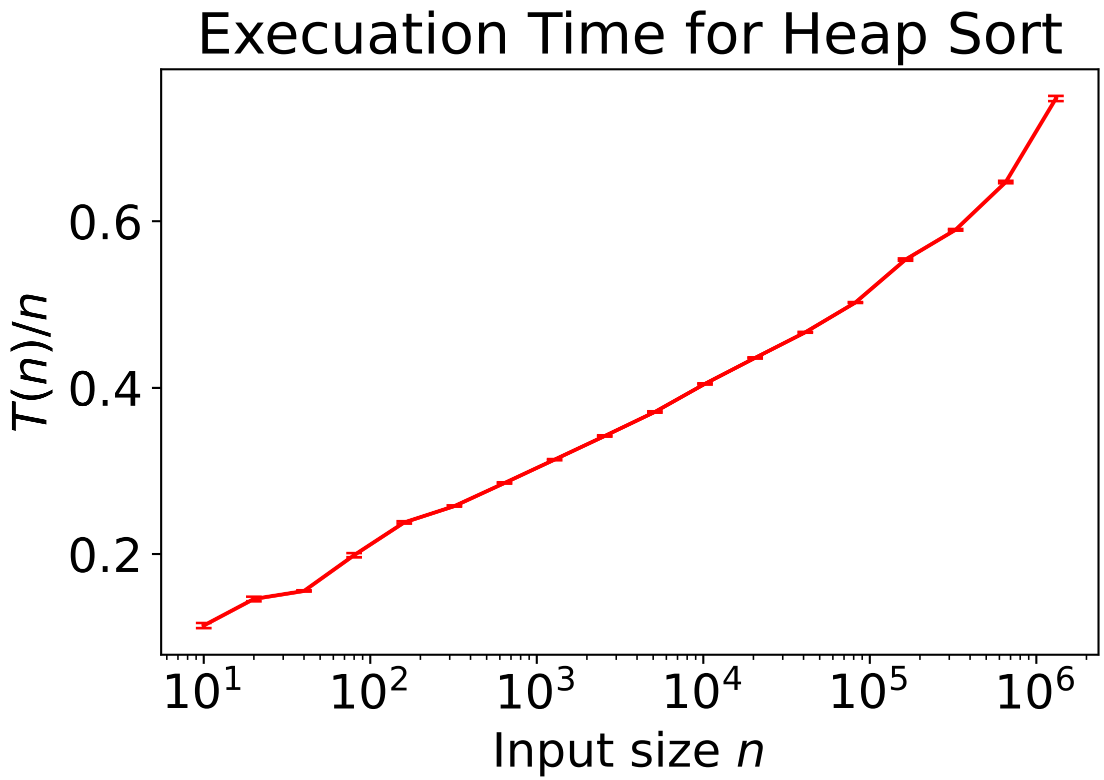

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Other Heaps

- Binary Heaps are so useful that other types of heaps have been developed
- The simplest enhancement is to combine a binary heap with a map which maintains a pointer to each element
- The map has to be updated every time elements are moved in the heap (fortunately only $O(\log(n))$ elements are move each time the heap is updated)
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Merging Heaps

- One common demand on a heaps is to merge two heaps
- Unfortunately binary heaps are not efficiently merged
- There are a variety of different heaps (leftist heaps, skew heaps, binomial queues, . . .) designed to be merged
- All these heaps are real binary trees (i.e. represented by pointers rather than put in an array)
- They are slower than a binary heap because indexing is slightly slower as is creating node objects
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Lessons

- Heaps are a powerful data structure—they are particularly useful for implementing priority queues

- Heaps are binary trees that can be implemented as arrays

- Priority queue have many (often surprising uses)
  - ⋆ They are used when you need a queue with priorities, e.g. in operating systems
  - ⋆ They can be used to perform pretty efficient sort
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  - ⋆ One important application is in real time simulations

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