Algorithms and Analysis

Lesson 8: *Keep Trees Balanced*

AVL trees, red-black trees, TreeSet, TreeMap

Outline

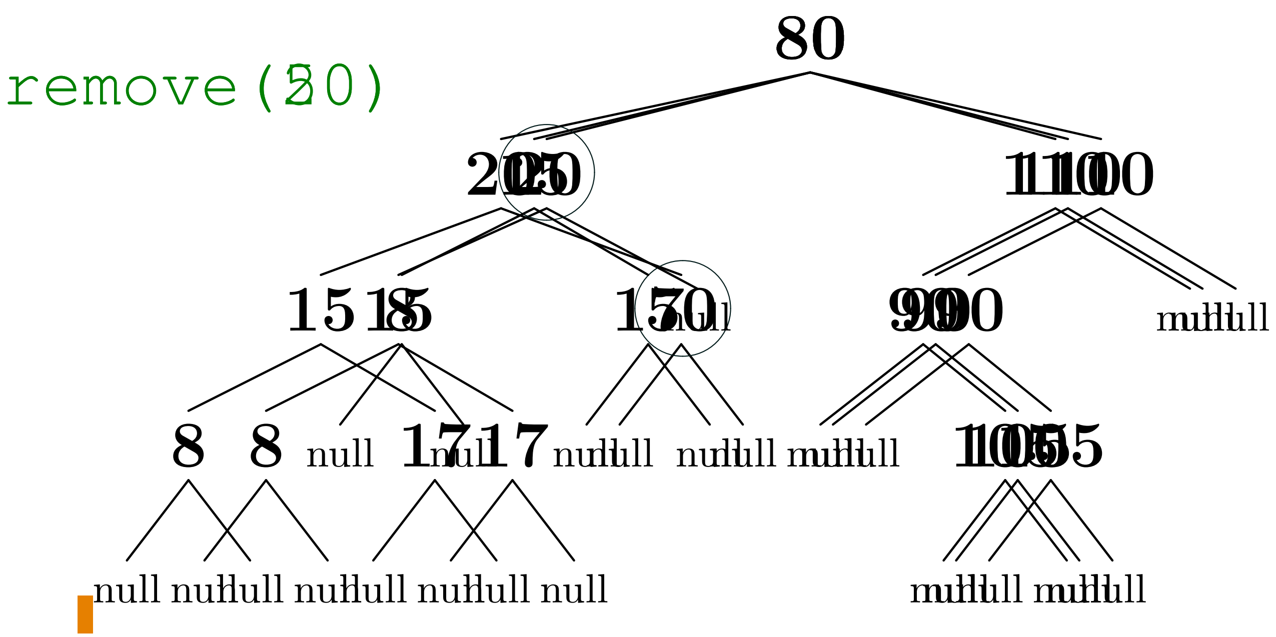
1. **Deletion**
2. Balancing Trees
   - Rotations
3. AVL
4. Red-Black Trees
   - TreeSet
   - TreeMap

Recap

- Binary search trees are commonly used to store data because we need to only look down one branch to find any element
- We saw how to implement many methods of the binary search tree
  - ⋆ `find`
  - ⋆ `insert`
  - ⋆ `successor` (in outline)
- One method we missed was `remove`

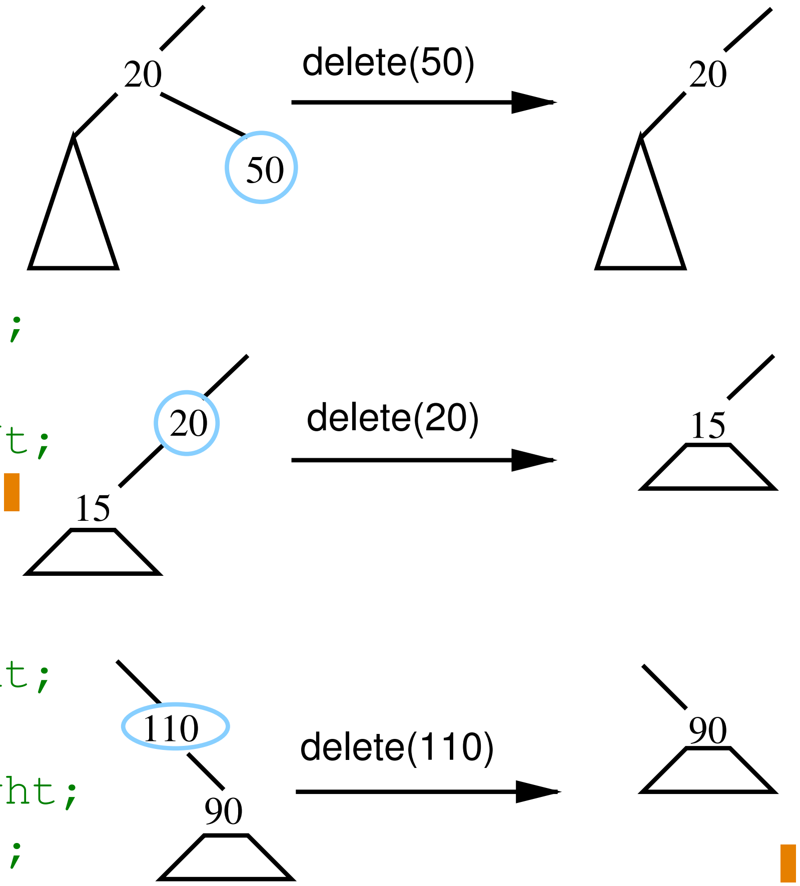
Deletion

- Suppose we want to delete some elements from a tree
- It is relatively easy if the element is a leaf node (e.g. 50)
- It is not so hard if the node has one child (e.g. 20)



Code to remove Node n

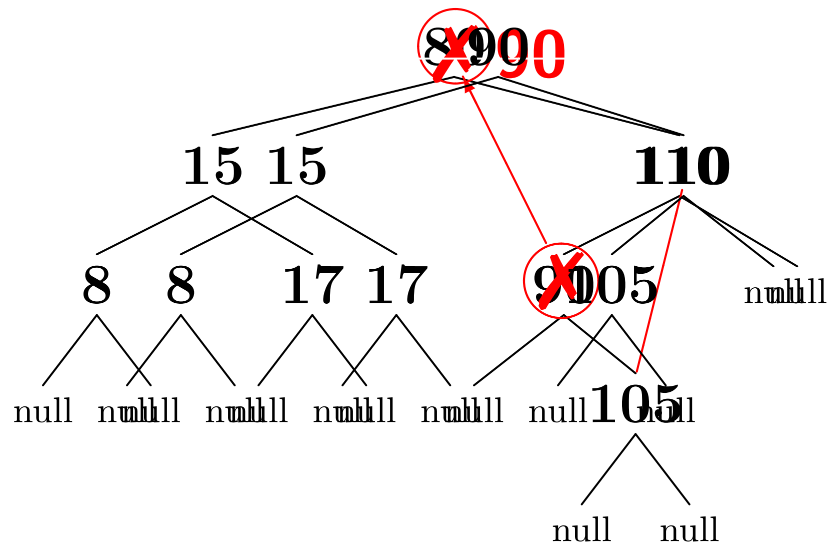
```
if (n->left==0 && n->right==0) {
   if (n == n->parent->left)
      n->parent->left = 0;
   else
      n->parent->right = 0;
} else if (n->right==0) {
   if (n == n->parent->left)
      n->parent->left = n->left;
   else
      n->parent->right = n->left;
   n->left->parent = n->parent;
} else if (n->left==0) {
   if (n == n->parent->left)
      n->parent->left = n->right;
   else
      n->parent->right = n->right;
   n->right->parent = n->parent;
}
delete n;
```



Removing Element with Two Children

- If an element has two children then
  - ⋆ replace that element by its successor
  - ⋆ and then remove the successor using the above procedure

`remove(80)`



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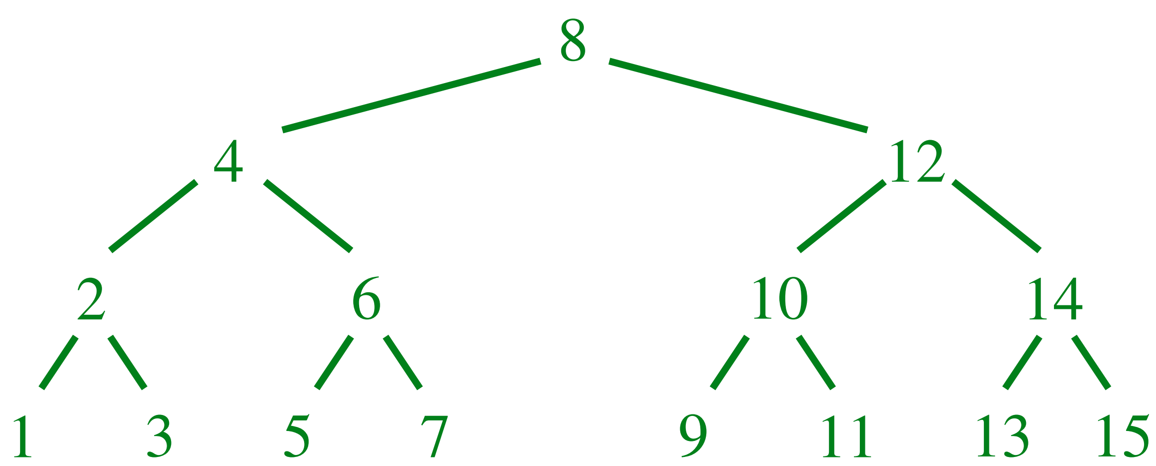
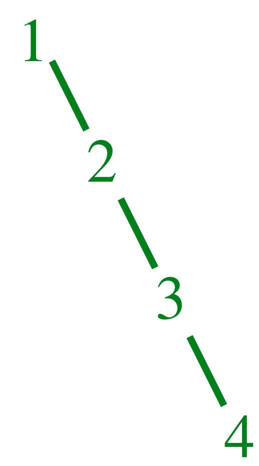
Why Balance Trees

- The number of comparisons to access an element depends on the depth of the node
- The average depth of the node depends on the shape of the tree



full tree



sparse tree

- The shape of the tree depends on the order the elements are added

Time Complexity

- In the best situation (a full tree) the number of elements in a tree is $n = \Theta(2^l)$ the depth is l so that the maximum depth is $\log_2(n)$

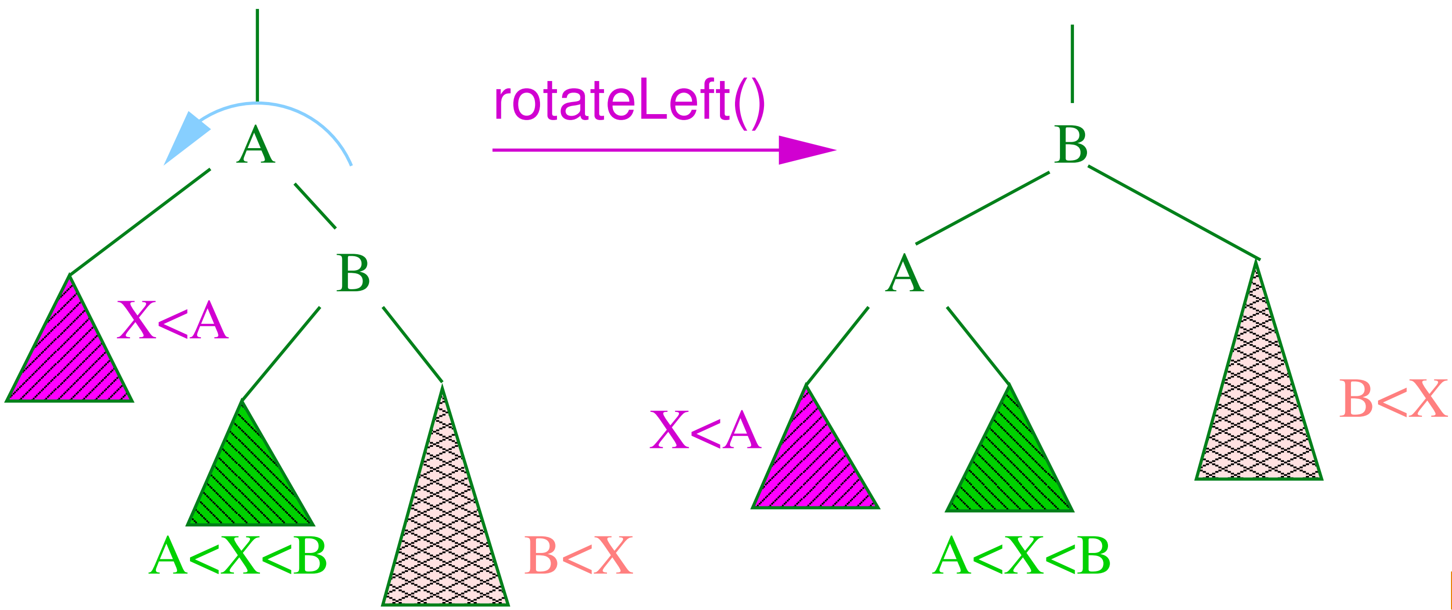
- It turns out for random sequences the average depth is $\Theta(\log(n))$

- In the worst case (when the tree is effectively a linked list), the average depth is $\Theta(n)$

- Unfortunately, the worst case happens when the elements are added $in\ order$ (not a rare event)

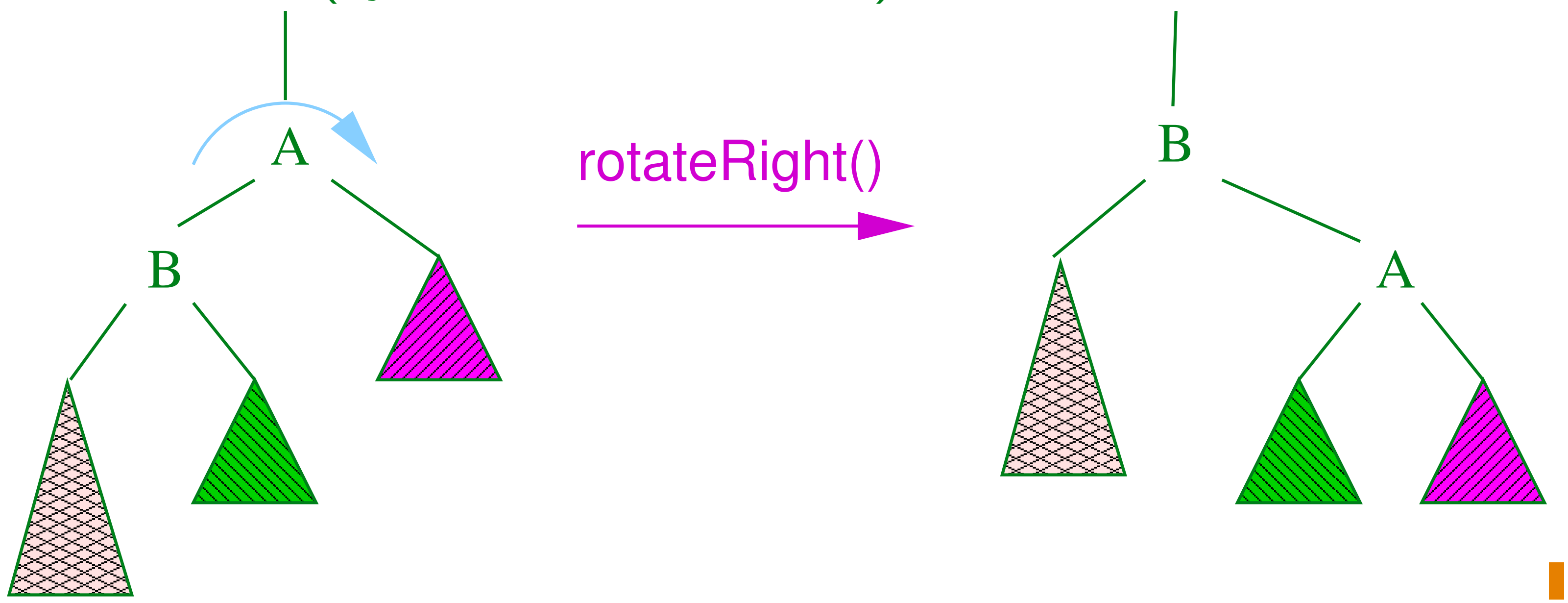
Rotations

- To avoid unbalanced trees we would like to modify the shape
- This is possible as the shape of the tree is not uniquely defined (e.g. we could make any node the root)
- We can change the shape of a tree using **rotations**
- E.g. left rotation



Types of Rotations

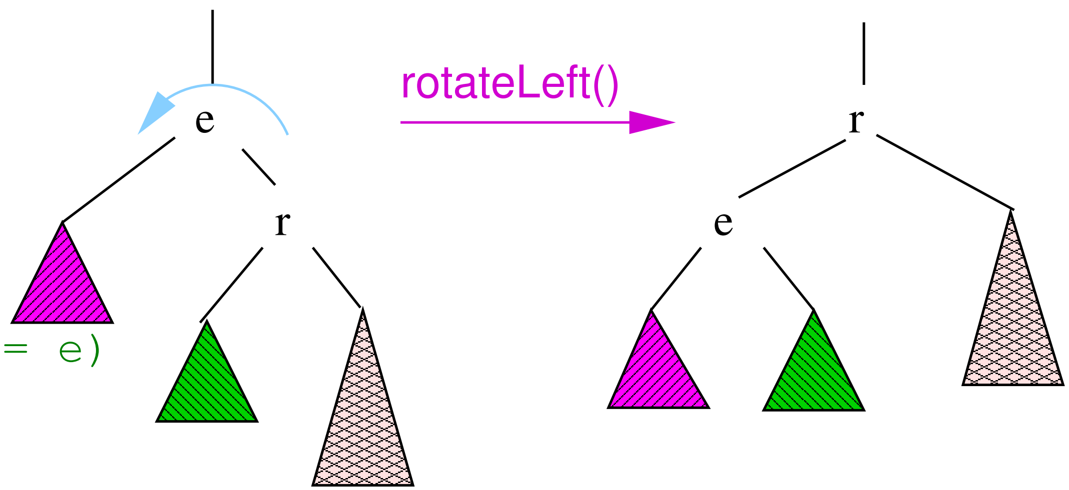
- We can get by with 4 types of rotations
  - ⋆ Left rotation (as above)
  - ⋆ Right rotation (symmetric to above)



  - ⋆ Left-right double rotation
  - ⋆ Right-left double rotation

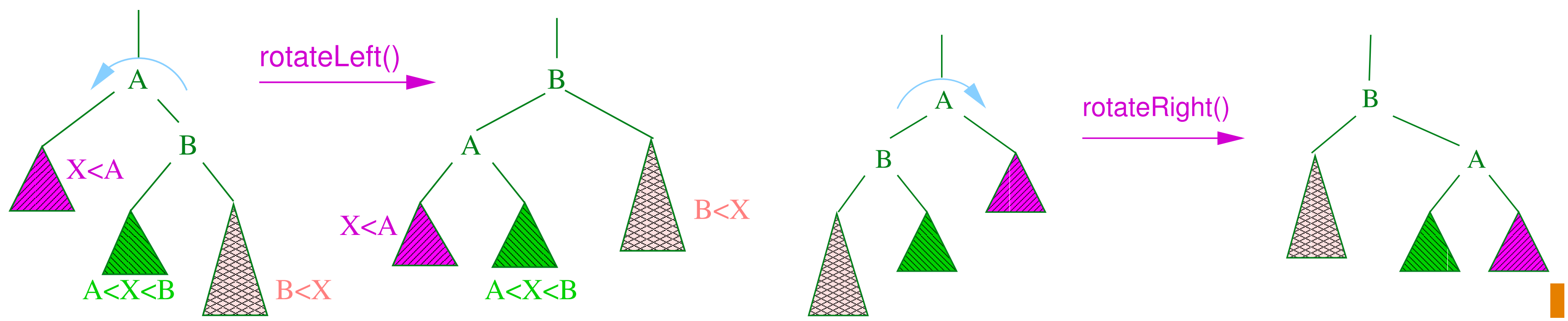
Coding Rotations

```
void rotateLeft(Node<T>* e)
{
  Node<T>* r = e->right;
  e->right = r->left;
  if (r->left != 0)
     r->left->parent = e;
  r->parent = e->parent;
  if (e->parent == 0)
     root = r;
  else if (e->parent->left == e)
     e->parent->left = r;
  else
     e->parent->right = r;
  r->left = e;
  e->parent = r;
}
```



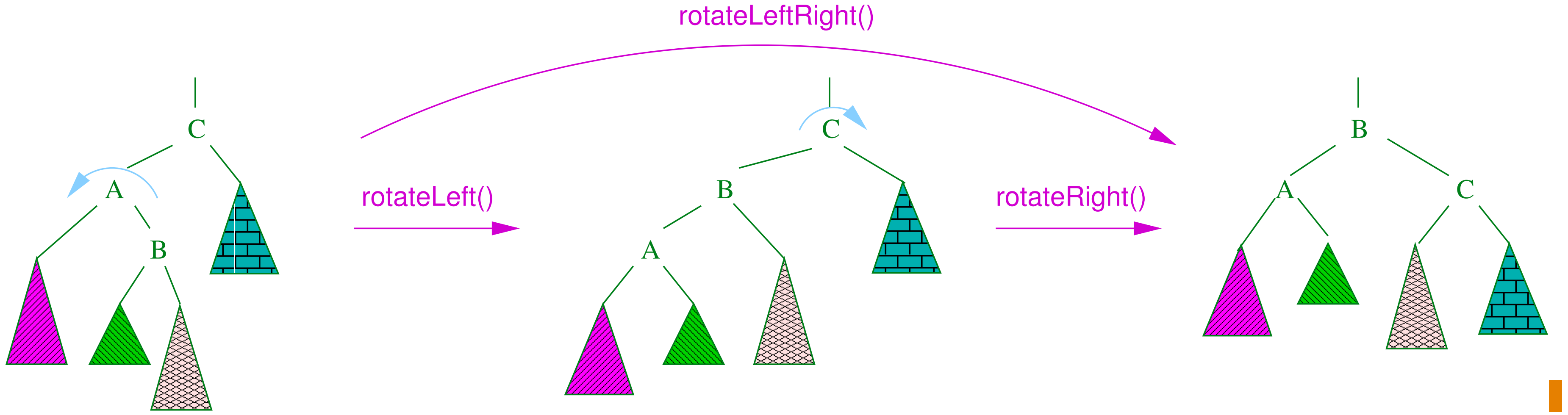
When Single Rotations Work

- Single rotations balance the tree when the unbalanced subtree is on the outside



Double Rotations

- If the unbalanced subtree is on the inside we need a double rotation




```
leftRotation(C.left);
rightRotation(C);
```

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Balancing Trees

- There are different strategies for using rotations for balancing trees
- The three most popular are
  - AVL-trees
  - Red-black trees
  - Splay trees
- They differ in the criteria they use for doing rotations

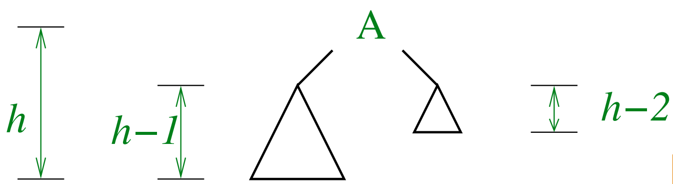
AVL Trees

- AVL-trees were invented in 1962 by two Russian mathematicians Adelson-Velski and Landis
- In AVL trees
  1. The heights of the left and right subtree differ by at most 1
  2. The left and right subtrees are AVL trees
- This guarantees that the worst case AVL tree has logarithmic depth

Minimum Number of Nodes

- Let $m(h)$ be the minimum number of nodes in a tree of height h
- This has to be made up of two subtrees: one of height $h-1$; and, in the worst case, one of height $h-2$
- Thus, the least number of nodes in a tree of height h is

$$m(h) = m(h-1) + m(h-2) + 1$$



- with $m(1) = 1$, $m(2) = 2$

Proof of Exponential Number of Nodes

- We have $m(h) = m(h-1) + m(h-2) + 1$ with $m(1) = 1$, $m(2) = 2$
- This gives us a sequence $1, 2, 4, 7, 12, \cdots$
- Compare this with Fibonacci $f(h) = f(h-1) + f(h-2)$, with $f(1) = f(2) = 1$
- This gives us a sequence $1, 1, 2, 3, 5, 8, 13, \cdots$
- It looks like $m(h) = f(h+2) - 1$
- Proof by substitution

Proof of Logarithmic Depth

- $m(h) = m(h-1) + m(h-2) + 1$ with $m(1) = 1$, $m(2) = 2$
- We can prove by inductions, $m(h) \geq (3/2)^{h-1}$
- $m(1) = 1 \geq (3/2)^0 = 1$, $m(2) = 2 \geq (3/2)^1 = 3/2$ ✓

$$m(h) \geq \left(\tfrac{3}{2}\right)^{h-3} \left(\tfrac{3}{2} + 1 + \left(\tfrac{3}{2}\right)^{3-h}\right) \geq \left(\tfrac{3}{2}\right)^{h-3} \tfrac{5}{2} = \left(\tfrac{3}{2}\right)^{h-3} \tfrac{10}{4} \geq \left(\tfrac{3}{2}\right)^{h-3} \tfrac{9}{4} = \left(\tfrac{3}{2}\right)^{h-1} \checkmark$$

- Taking logs: $\log(m(h)) \geq (h-1)\log(3/2)$ or

$$h \leq \frac{\log(m(h))}{\log(3/2)} + 1 = O\left(\log(m(h))\right)$$

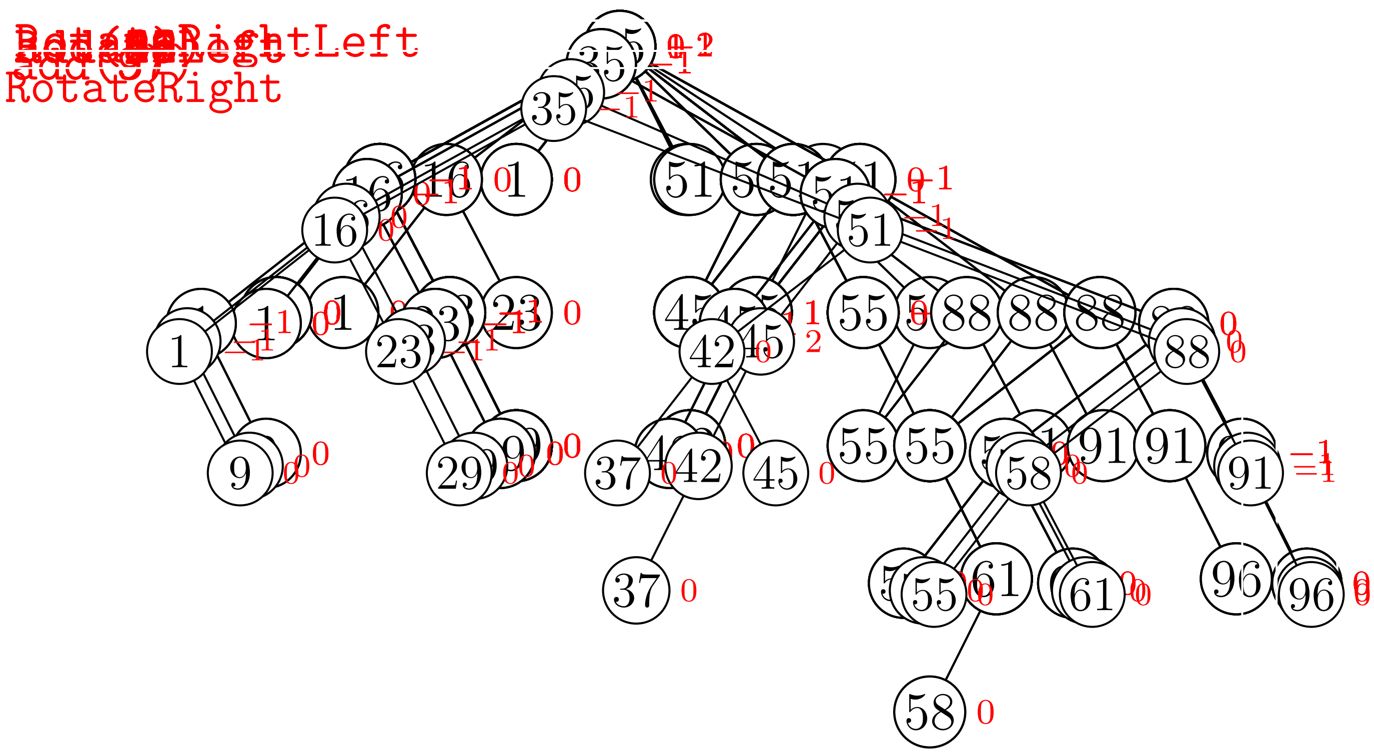
- The number of elements, n, we can store in an AVL tree is $n \geq m(h)$ thus

$$h \leq O(\log(n))$$

Implementing AVL Trees

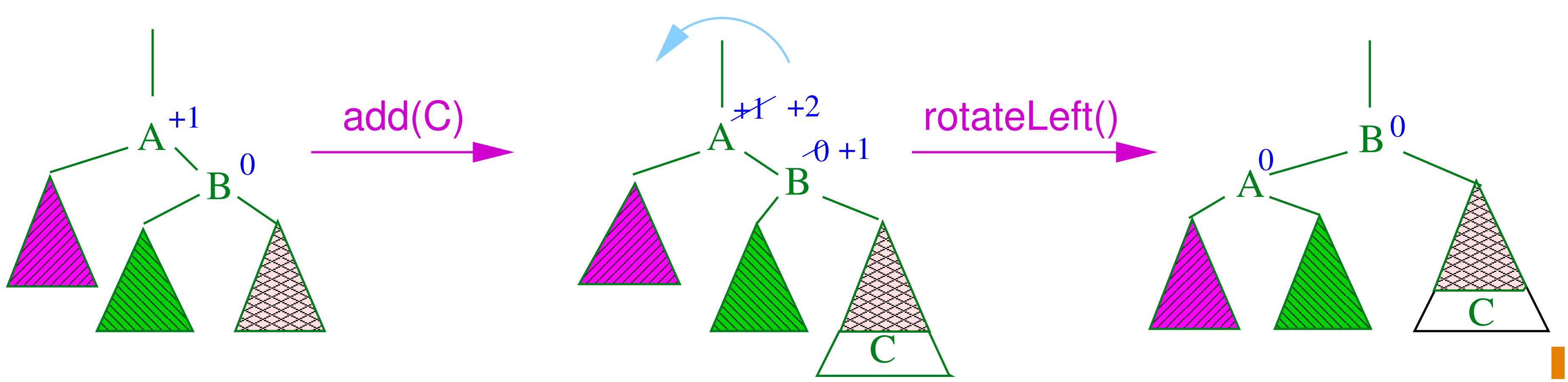
- In practice to implement an AVL tree we include additional information at each node indicating the balance of the subtrees

$$\texttt{balanceFactor} = \begin{cases} -1 & \text{right subtree deeper than left subtree} \\ 0 & \text{left and right subtrees equal} \\ +1 & \text{left subtree deeper than right subtree} \end{cases}$$



Balancing AVL Trees

- When `adding` an element to an AVL tree
  - ⋆ Find the location where it is to be inserted
  - ⋆ Iterate up through the parents re-adjusting the `balanceFactor`
  - ⋆ **If** the balance factor exceeds ± 1 `then` re-balance the tree and stop
  - ⋆ `else if` the balance factor goes to zero **then** stop



AVL Deletions

- AVL deletions are similar to AVL insertions
- One difference is that after performing a rotation the tree may still not satisfy the AVL criteria so higher levels need to be examined
- In the worst case $\Theta(\log(n))$ rotations may be necessary
- This may be relatively slow—but in many applications deletions are rare

AVL Tree Performance

- Insertion, deletion and search in AVL trees are, at worst, $\Theta(\log(n))$
- The height of an average AVL tree is $1.44 \log_2(n)$
- The height of an average binary search tree is $2.1 \log_2(n)$
- Despite being more compact insertion is slightly slower in AVL trees than binary search trees without balancing (for random input sequences)
- Search is, of course, quicker

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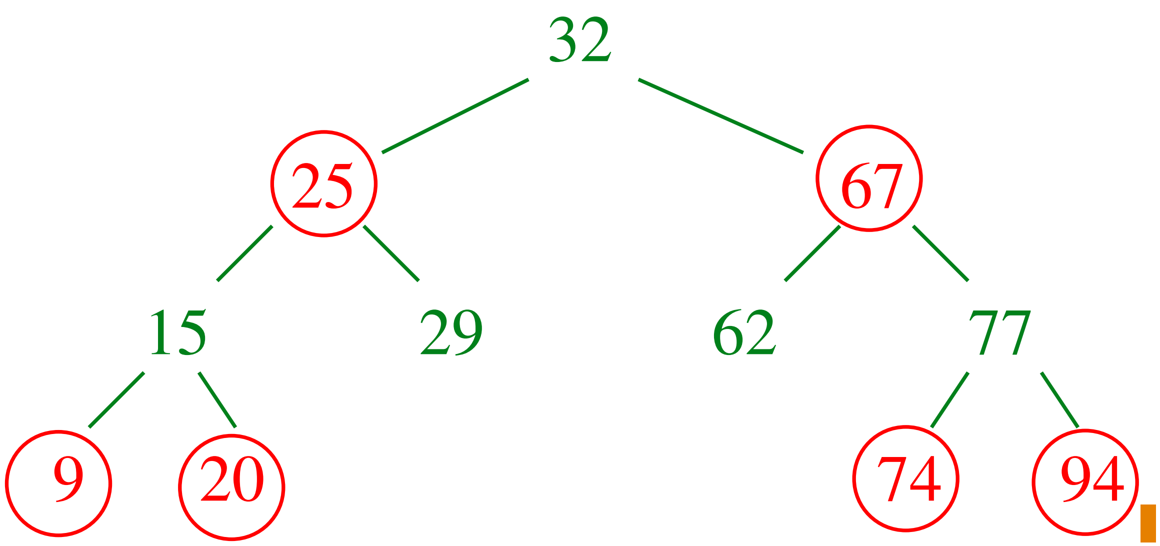
Red-Black Trees

- Red-black trees are another strategy for balancing trees
- Nodes are either *red* or *black*
- Two rules are imposed

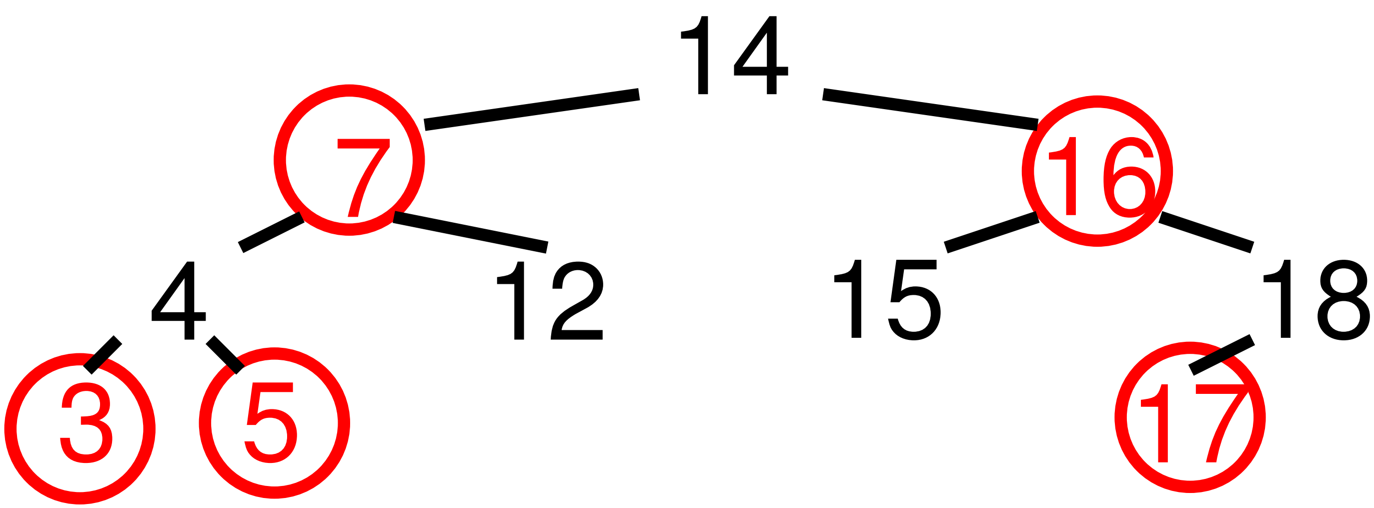
  Red Rule: the children of a red node must be black

  Black Rule: the number of black elements must be the same in all paths from the root to elements with no children or with one child



Restructuring

- When inserting a new element we first find its position
- If it is the root we colour it black otherwise we colour it red
- If its parent is red we must either relabel or restructure the tree



Performance of Red-Black Trees

- Red-black trees are slightly more complicated to code than AVL trees
- Red-black trees tend to be slightly less compact than AVL trees
- However, insertion and deletion run slightly quicker
- Both Java Collection classes and C++ STL use red-black trees

Set

- The standard template library (STL) has a class `std::set<T>`
- It also has a `std::underordered_set<T>` class (which uses a hash table covered later)
- As well as `std::multiset<T>` that implements a multiset (i.e. a set, but with repetitions)
- Using sets you can also implement **maps**

Maps

- One major abstract data type (ADT) we have not encountered is the map class
- The map class `std:map<Key,V>` contain key-value pairs `pair<Key,V>`
  - ⋆ The first element of type `Key` is the **key**
  - ⋆ The second element of type `V` is the **value**
- Maps work as content addressable arrays

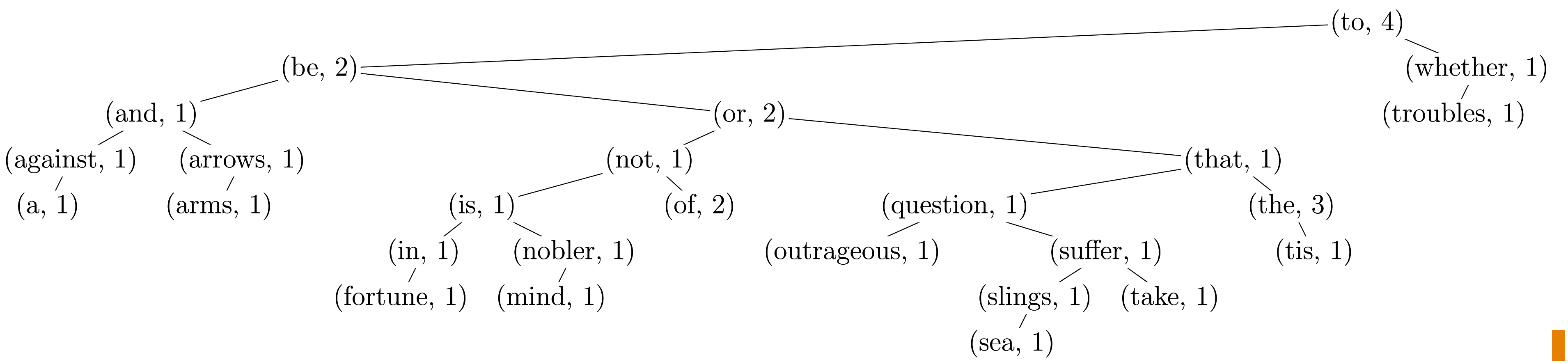
```
map<string, int> students;
student["John Smith"] = 89;
student["Terry Jones"] = 98;
cout << students["John Smith"];
```

Implementing a Map

- Maps can be implemented using a set by making each node hold a `pair<K,V>` objects

```
class pair<K,V>
{
  public:
  K first;
  V second;
}
```

- We can count words using the key for words and value to count



Lessons

- Binary search trees are very efficient (order $\log(n)$ insertion, deletion and search) provided they are balanced
- Balanced trees are achieved by performing rotations
- There are different strategies for deciding when to rotate including
  - ⋆ AVL trees
  - ⋆ Red-black trees
- Binary trees are used for implementing **sets** and **maps**