

# Further Mathematics and Algorithms

## Lesson 2: *Know How Long A Program Takes*



*TSP, Sorting, time complexity, Big-Theta, Big-O, Big-Omega*

# Outline

1. **TSP**
2. Sorting
3. Big O



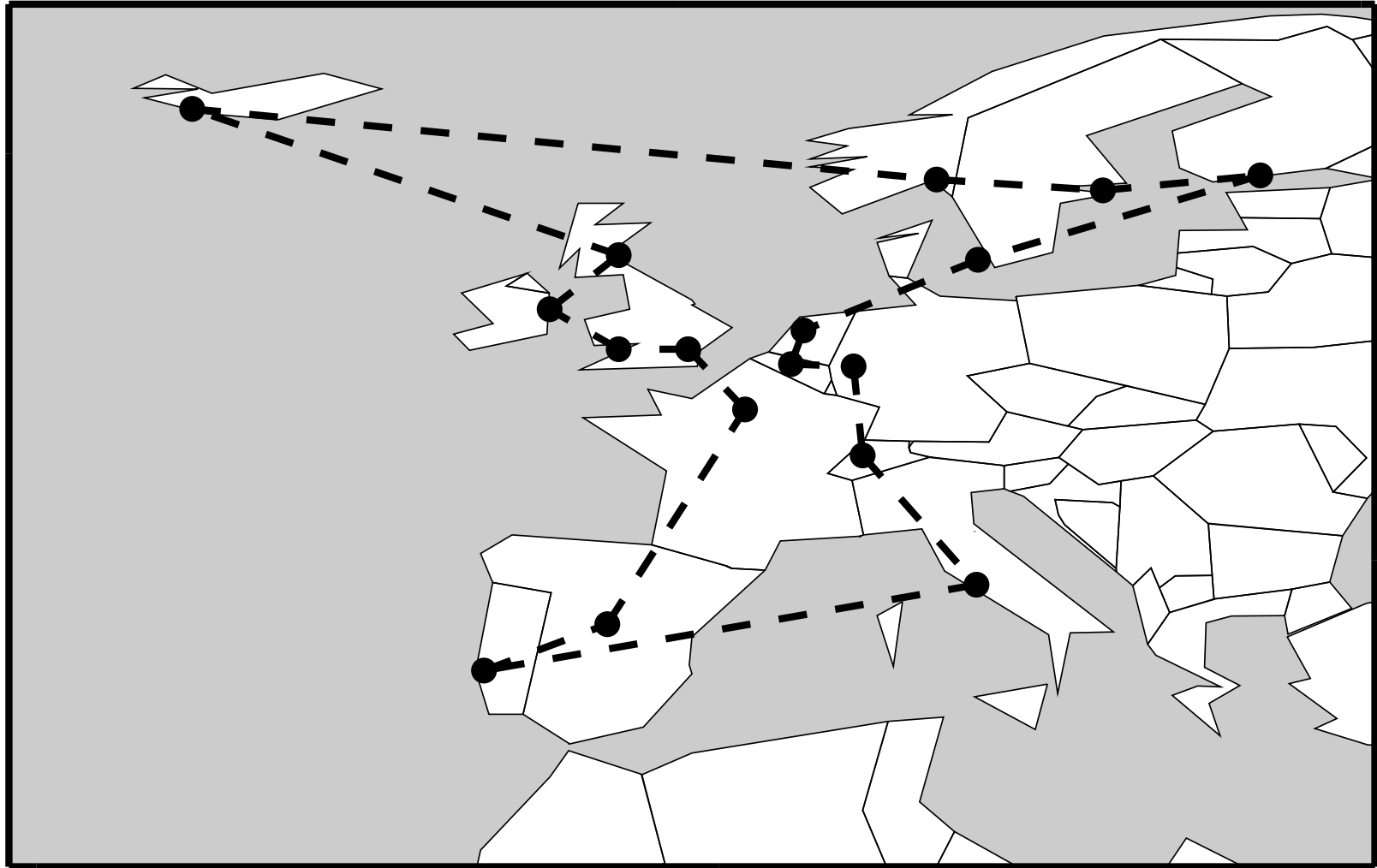
# Travelling Salesperson Problem

- Given a set of cities■
- A table of distances between cities■
- Find the shortest tour which goes through each city and returns to the start■

# Example of Distance Table

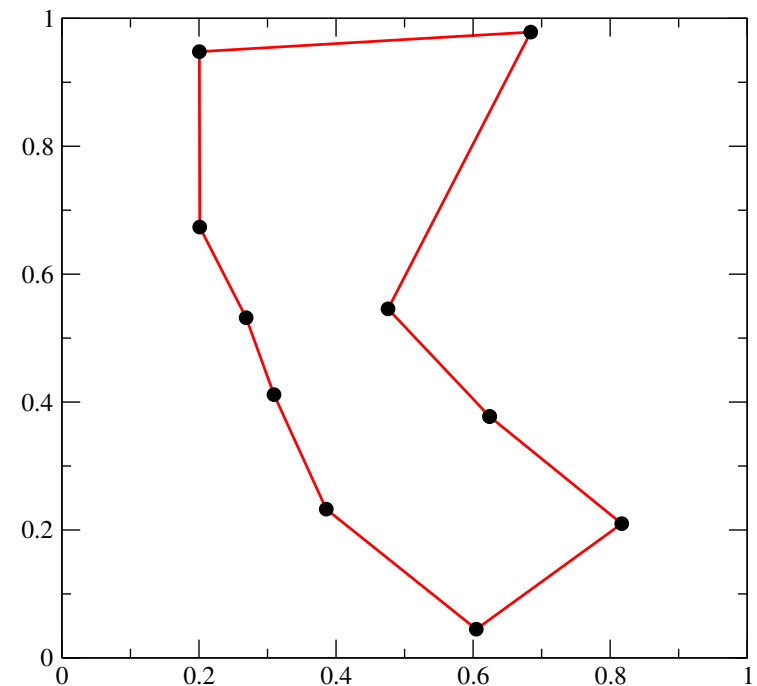
	Lon	Car	Dub	Edin	Reyk	Oslo	Sto	Hel	Cop	Amst	Bru	Bonn	Bern	Rome	Lisb	Madr	Par
London	0	223	470	538	1896	1151	1426	1816	950	349	312	503	743	1429	1587	1265	337
Cardiff	223	0	290	495	1777	1277	1589	1985	1139	564	533	725	927	1600	1492	1233	492
Dublin	470	290	0	350	1497	1267	1628	2026	1239	756	775	956	1207	1886	1638	1449	777
Edinburgh	538	495	350	0	1374	933	1314	1708	984	662	758	896	1243	1931	1964	1728	872
Reykjavik	1896	1777	1497	1374	0	1746	2134	2418	2104	2020	2130	2255	2617	3304	2949	2892	2232
Oslo	1151	1277	1267	933	1746	0	416	788	481	917	1088	1048	1459	2011	2739	2390	1343
Stockholm	1426	1589	1628	1314	2134	416	0	398	518	1126	1281	1181	1542	1978	2987	2593	1543
Helsinki	1816	1985	2026	1708	2418	788	398	0	881	1504	1650	1530	1856	2203	3360	2950	1910
Copenhagen	950	1139	1239	984	2104	481	518	881	0	625	769	662	1036	1538	2479	2076	1030
Amsterdam	349	564	756	662	2020	917	1126	1504	625	0	173	235	629	1296	1860	1480	428
Brussels	312	533	775	758	2130	1088	1281	1650	769	173	0	194	489	1174	1710	1315	262
Bonn	503	725	956	896	2255	1048	1181	1530	662	235	194	0	422	1067	1843	1420	400
Bern	743	927	1207	1243	2617	1459	1542	1856	1036	629	489	422	0	689	1630	1156	440
Rome	1429	1600	1886	1931	3304	2011	1978	2203	1538	1296	1174	1067	689	0	1862	1365	1109
Lisbon	1587	1492	1638	1964	2949	2739	2987	3360	2479	1860	1710	1843	1630	1862	0	500	1452
Madrid	1265	1233	1449	1728	2892	2390	2593	2950	2076	1480	1315	1420	1156	1365	500	0	1054
Paris	337	492	777	872	2232	1343	1543	1910	1030	428	262	400	440	1109	1452	1054	0

# Example Tour

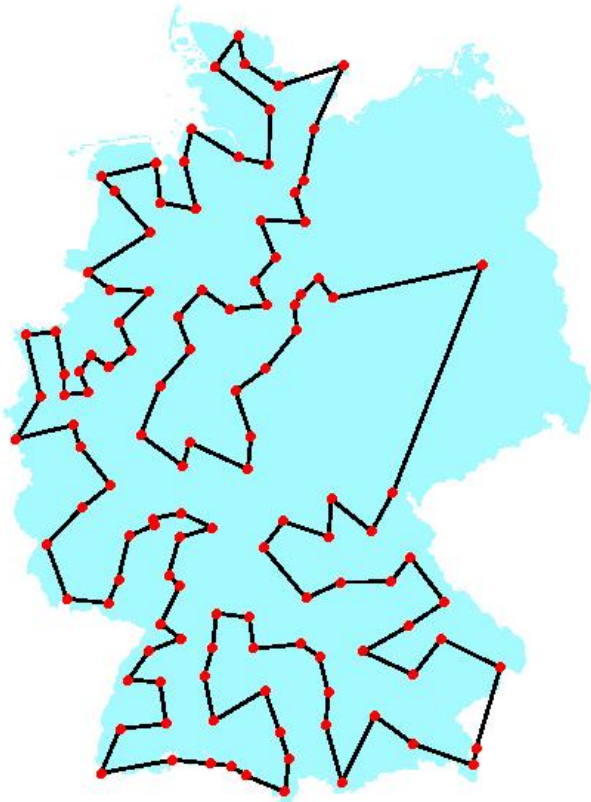


# Brute Force

- I wrote a program to solve TSP by enumerating every path and finding the shortest
- I checked that it worked on some problems with 10 cities
- It takes just under half a second to solve this problem
- I set the program running on a 100 city problem — **How long will it take to finish?**

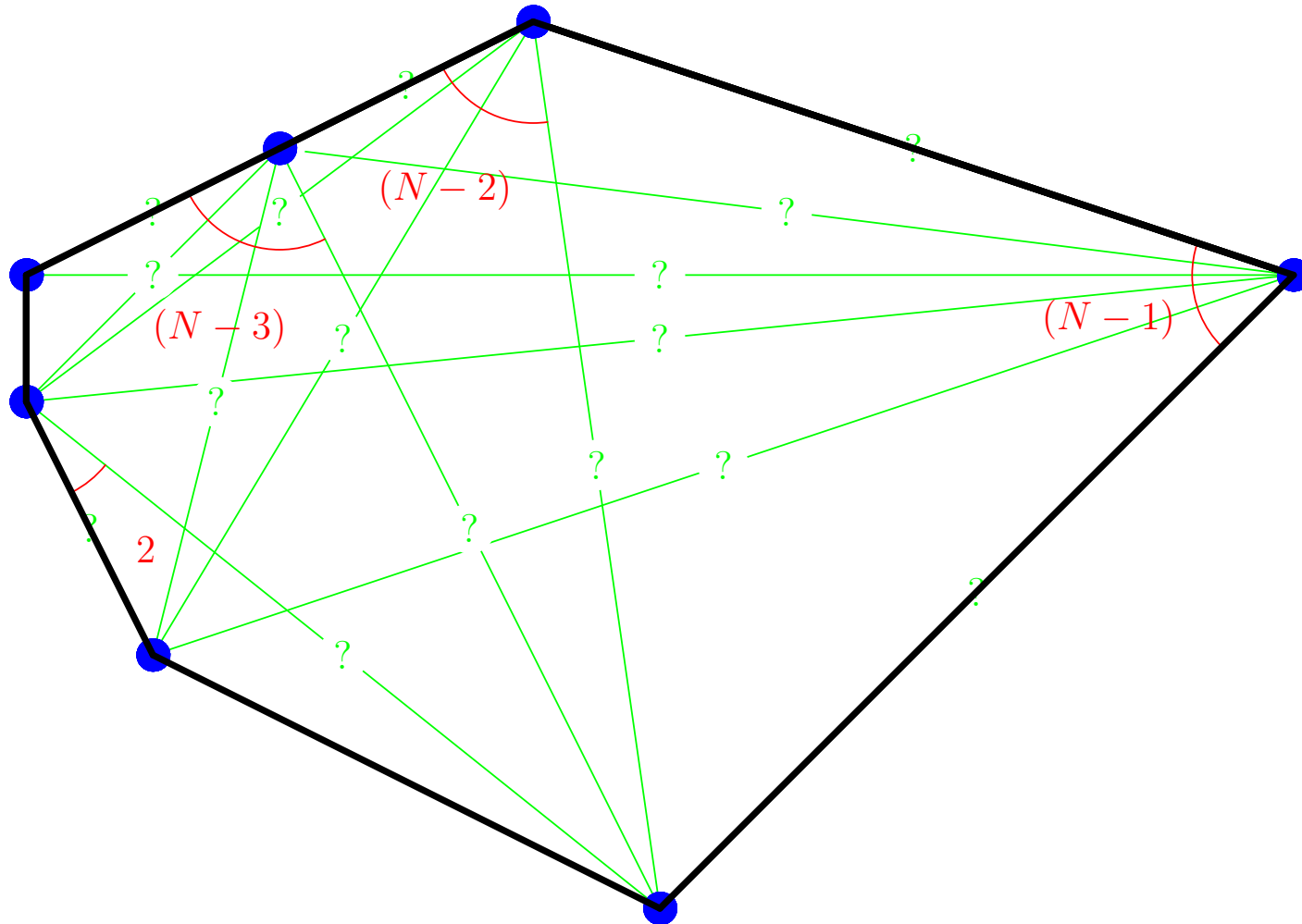


# How Many Possible Tours Are There?



- For 100 cities how many possible tours are there?■
- It doesn't matter where we start■
- Starting from Berlin there are 99 cities we can try next■

# Counting Tours



$$\text{Number of tours} = (N-1) \times (N-2) \times (N-3) \times \cdots 2 \times 1 = (N-1)!$$



# How Long Does It Take?

- The direction we go in is irrelevant■
- Total number of tours is  $99!/2$ ■
- **Any more guesses how long it will take?■**

# How Big is 99 Factorial?

- $99! = 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1 = ?$

- Upper bound

$$99! = 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1$$

$$99! < 99 \cdot 99 \cdot 99 \cdots 99 \cdot 99 = 99^{99}$$

- Lower bound

$$99! = 99 \cdot 98 \cdot 97 \cdots 50 \cdot 49 \cdots 2 \cdot 1$$

$$99! > 50 \cdot 50 \cdot 50 \cdots 50 \cdot 1 \cdots 1 \cdot 1 = 50^{50}$$

# How Long Does It Take?

- For  $N > 1$

$$\left(\frac{N}{2}\right)^{N/2} < N! < N^N$$

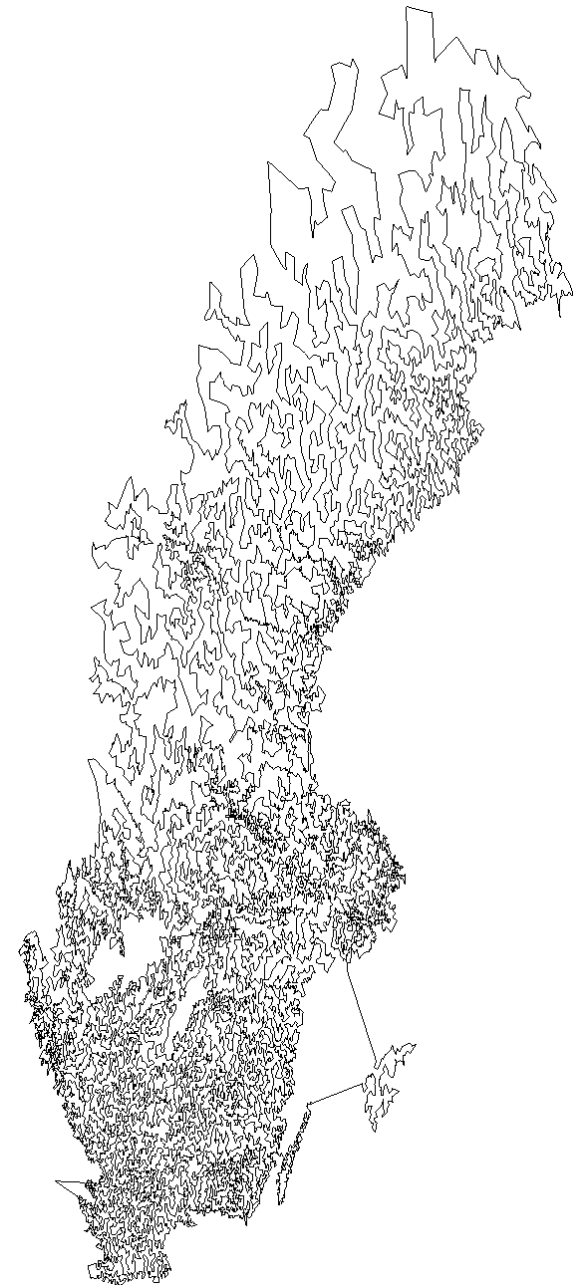
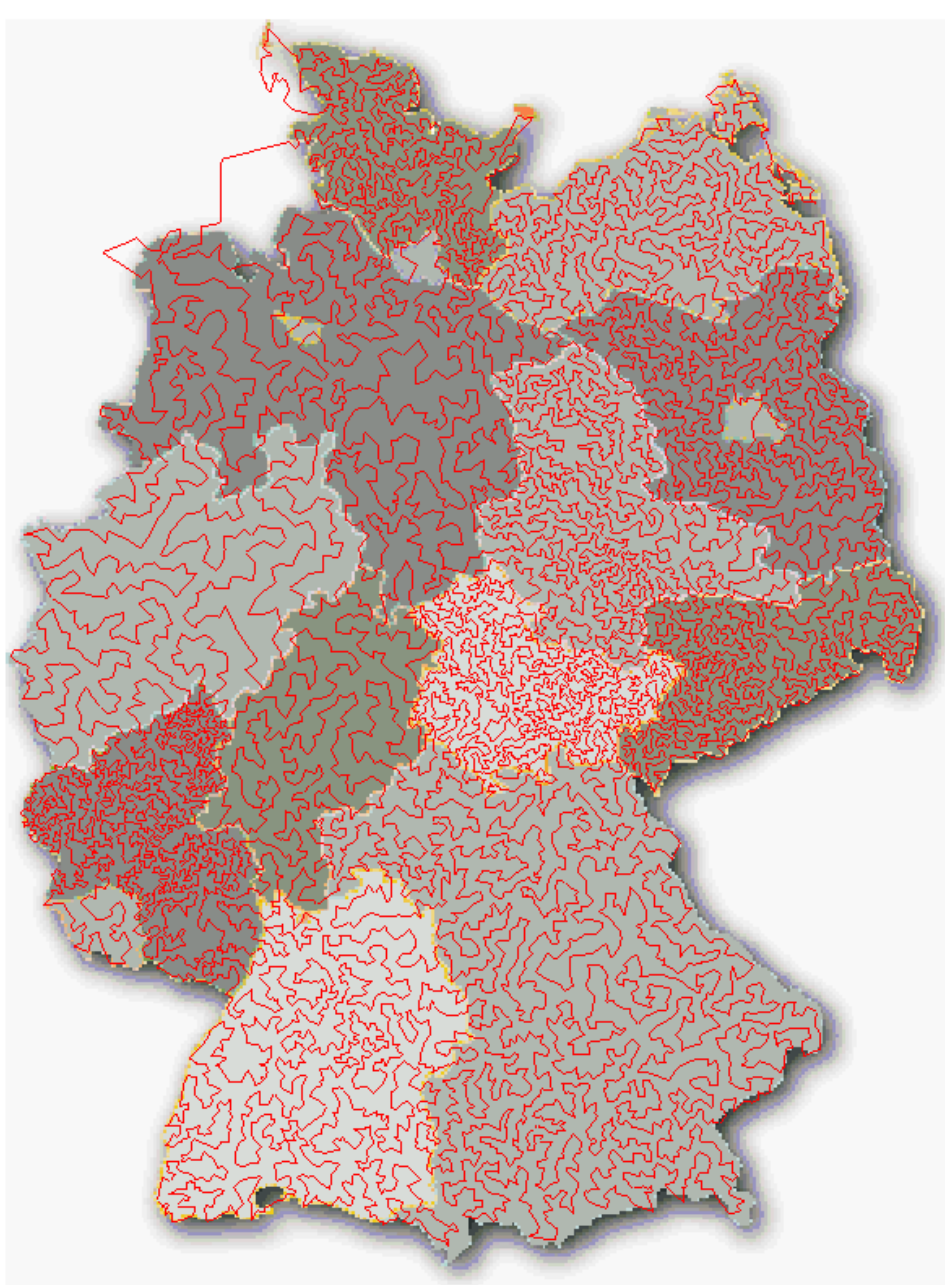
- $99!/2 = 4.666 \times 10^{155}$
- How long does it take to search all possible tours?
  - ★ We computed about 200 000 tours in half a second
  - ★  $3.15 \times 10^7 \text{sec} = 1 \text{ year}$
  - ★ Age of Universe  $\approx 15 \text{ billion years}$

# Answer

- $2.72 \times 10^{132}$  ages of the universe!
- Incidental

$$\begin{aligned} 99!/2 = & 46663107721972076340849619428133350 \\ & 24535798413219081073429648194760879 \\ & 99966149578044707319880782591431268 \\ & 48960413611879125592605458432000000 \\ & 000000000000000000 \end{aligned}$$

# Record TSP Solved—15 112 and 24 978 Cities



# In Case You're Curious

- Number of tours:  $15111!/2 = 7.3 \times 10^{56592}$
- Current record 24 978 cities with  $1.9 \times 10^{98992}$  tours
- The algorithm for finding the optimum path does not look at every possible path
- If your interested look for the TSP homepage on the web  
<http://www.math.uwaterloo.ca/tsp/>

# Lessons

- Even relatively small problems can take you an astronomical time to solve using simple algorithms■
- As a professional programmer you need to have an estimate for how long an algorithm takes■—otherwise you can look silly■
- For the 100 city problem, if
  - ★ I had  $10^{87}$  cores■, one for every particle in the Universe■
  - ★ I could compute a tour distance in  $3 \times 10^{-24}$  seconds■, the time it takes light to cross a proton■
  - ★ It would still take  $10^{39} \times$  the age of the universe■
- Smart algorithms can make a much larger difference than fast computers!■

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2. **Sorting**
3. Big O

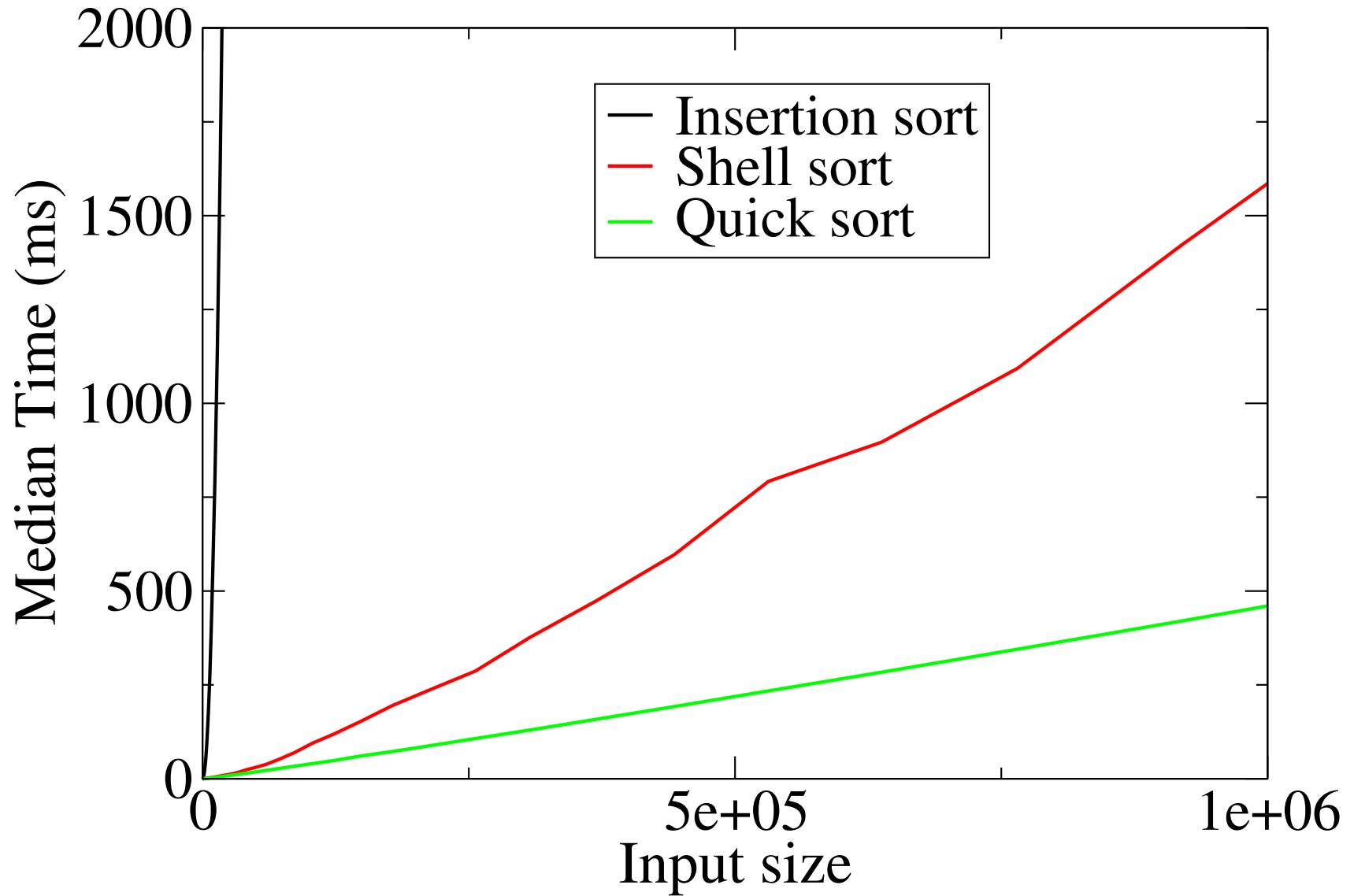




# Sort

- Comparison between common sort algorithms
  - ★ Insertion sort—an easy algorithm to code
  - ★ Shell sort—invented in 1959 by Donald Shell
  - ★ Quick sort—invented in 1961 by Tony Hoare
- 
- These take an array of numbers and returns a sorted array■
- Sort is very commonly used algorithm■so you care about how long it takes■

# Empirical Run Times



# Lessons

- There is a right and wrong way to do easy problems■
- You only really care when you are dealing with large inputs■
- Good algorithms are difficult to come up with, but they exist■
- We would like to quantify the performance of an algorithm■—how much better is quick sort than insertion sort?■

# Outline

1. TSP
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# Estimating Run Times

- We would like to estimate the run times of algorithms■
- This depends on the hardware (how fast is your computer)■
- We could count number of elementary operations■ but
  - ★ different machines have different elementary operations■
  - ★ many algorithms use complex functions such as `sqrt` (matrix inversion using Cholesky decomposition) or `sin` and `cos` (FFT)■
  - ★ would need to count memory accesses which you shouldn't need to think about■
  - ★ code after compiling can be very different from code before compiling ■

# Engineering Solution

- Compute the **asymptotic leading functional behaviour**■
- Lets take that statement to pieces■
- Suppose we have an algorithm that takes  $4n^2 + 12n + 199$  operations (clock cycles)
  - ★ **asymptotic**: what happens when  $n$  becomes very large■
  - ★ **leading**: ignore the  $12n + 199$  part as it is dominated by  $4n^2$  (i.e. for large enough  $n$  we have  $4n^2 \gg 12n + 199$ )■
  - ★ **functional behaviour**: ignore the constant 4■
- We call this an order  $n^2$ , or quadratic time, algorithm■
- We can write this in 'Big-Theta' notation as  $\Theta(n^2)$ ■
- This notion of 'run time' is known as **time complexity** ■

# Advantages of Big-Theta Notation

- Doesn't depend on what computer we are running■
- Don't need to know how many elementary operations are required for a non-elementary operation■
- Can estimate run times by measuring run time on a small problem
  - ★ If I have a  $\Theta(n^2)$  algorithm
  - ★ It takes  $x$  seconds on an input of 100
  - ★ It will take about  $\frac{x \times n^2}{100^2}$  seconds on a problem of size  $n$ ■  
 $(T(100) \approx c 100^2 = x \text{ therefore } c = x/100^2$   
thus  $T(n) = c n^2 = x n^2 / 100^2)$

# Counting Instructions

- Big-Theta run times are often easy to calculate

- a  $\Theta(n)$  algorithm

```
// define stuff  
for(int i=0; i<n; i++) {  
    // do something  
}  
// clean up
```

- a  $\Theta(n^2)$  algorithm

```
// define stuff  
for(int i=0; i<n; i++) {  
    // do something  
    for (int j=0; j<n; j++) {  
        // do other stuff  
    }  
}  
// clean up
```



# Disadvantage with Big-Theta notation

- Can't compare algorithms with the same Big-Theta time complexity■
- For small inputs Big-Theta time complexity can be misleading.■  
E.g.
  - ★ algorithm A takes  $n^3 + 2n^2 + 5$  operations
  - ★ algorithm B takes  $20n^2 + 100$  operations
  - ★ algorithm A is  $\Theta(n^3)$  and algorithm B is  $\Theta(n^2)$
  - ★ algorithm A is faster than algorithm B for  $n < 18$ ■but who cares?■
- In some cases Big-Theta time complexity is hard to compute■

# Not So Sure

- Some algorithms are harder to compute

```
// define stuff
for(int i=0; i<n; i++) {
    // do something
    if (/* some condition */) {
        for (int j=0; j<n; j++) {
            // do other stuff
        }
    }
}
// clean up ■
```

- Time complexity now depends on the `if` statement ■
- If the condition is often satisfied we have a  $\Theta(n^2)$  algorithm ■
- If the condition is true only rarely then we have a  $\Theta(n)$  algorithm ■

# Bounds

- To avoid having to think really hard we define upper and lower bounds■
- The upper bound we write using **big-O** notation
  - ★ The above algorithm is an  $O(n^2)$  algorithm
  - ★ I.e. it runs in no more than order  $n^2$  operations■
- The lower bound we write using **big-Omega** notation
  - ★ The above algorithm is a  $\Omega(n)$  algorithm
  - ★ I.e. it runs in no less than order  $n$  operations■

# Precise Definitions of $O(n)$

- An algorithm that runs in  $f(n)$  operations is  $O(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c \quad \text{where } c \text{ is a constant (could be zero)} \blacksquare$$

- E.g..  $f(n) = 3n^2 + 2n + 12$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 12}{n^2} = 3 \blacksquare \Rightarrow 3n^2 + 2n + 12 = O(n^2) \blacksquare$$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 12}{n^3} = 0 \blacksquare \Rightarrow 3n^2 + 2n + 12 = O(n^3) \blacksquare$$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 12}{n} = \infty \blacksquare \Rightarrow 3n^2 + 2n + 12 \neq O(n) \blacksquare$$

# Lower Bound Definition

- An algorithm that runs in  $f(n)$  operations is  $\Omega(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = c \quad \text{where } c \text{ is a constant (could be zero)} \blacksquare$$

- E.g.  $f(n) = 3n^2 + 2n + 12$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \lim_{n \rightarrow \infty} \frac{n^2}{3n^2 + 2n + 12} = \frac{1}{3} \blacksquare \Rightarrow 3n^2 + 2n + 12 = \Omega(n^2) \blacksquare$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \lim_{n \rightarrow \infty} \frac{n^3}{3n^2 + 2n + 12} = \infty \blacksquare \Rightarrow 3n^2 + 2n + 12 \neq \Omega(n^3) \blacksquare$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \lim_{n \rightarrow \infty} \frac{n}{3n^2 + 2n + 12} = 0 \blacksquare \Rightarrow 3n^2 + 2n + 12 = \Omega(n) \blacksquare$$

# Big-Theta

- An algorithm that runs in  $f(n)$  operations is  $\Theta(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = c \quad \text{where } c \text{ is a non-zero constant} \blacksquare$$

- That is,  $f(n) = \Theta(g(n))$  if

$$f(n) = O(g(n)) \quad \text{and} \quad f(n) = \Omega(g(n)) \blacksquare$$

- I.e. the lower bound is identical to the upper bound  $\blacksquare$
- Often the most straightforward way of obtaining big-Theta is to show the upper and lower bounds are the same  $\blacksquare$

# Use and Misuse

- Note: big-O notation is most commonly used■
- often people say they have a  $O(n^2)$  when in fact they mean they have a  $\Theta(n^2)$  algorithm (a much stronger result)■
- Note that an  $O(n^2)$  algorithm is also a  $O(n^3)$  algorithm■
- Strictly a  $O(n^2)$  algorithm **may not** be faster than a  $O(n^3)$  algorithm when  $n$  becomes larger■
- A  $\Theta(n^2)$  algorithm **will** be faster than a  $\Theta(n^3)$  algorithm when  $n$  becomes larger■

# Lessons to Learn

- Run times (computational time complexity) matters■
- Choosing an algorithm with the best time complexity is important■
- Understand the meaning of big-Theta, big-O and big-Omega■
- Know how to estimate time complexity for simple algorithms (loop counting)■