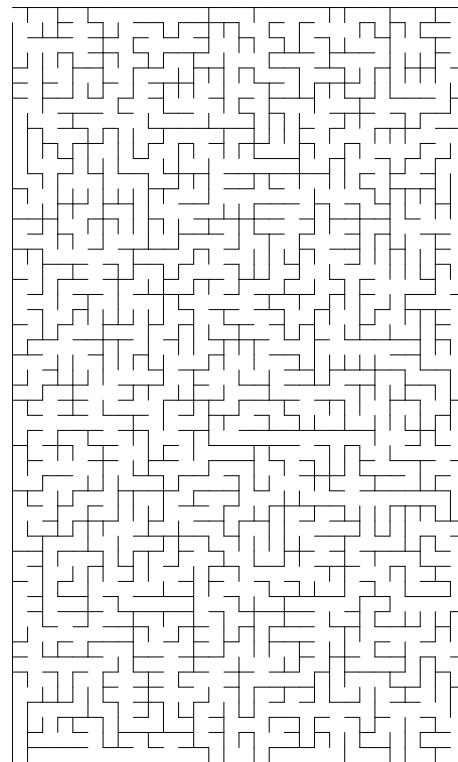


# Algorithms and Analysis

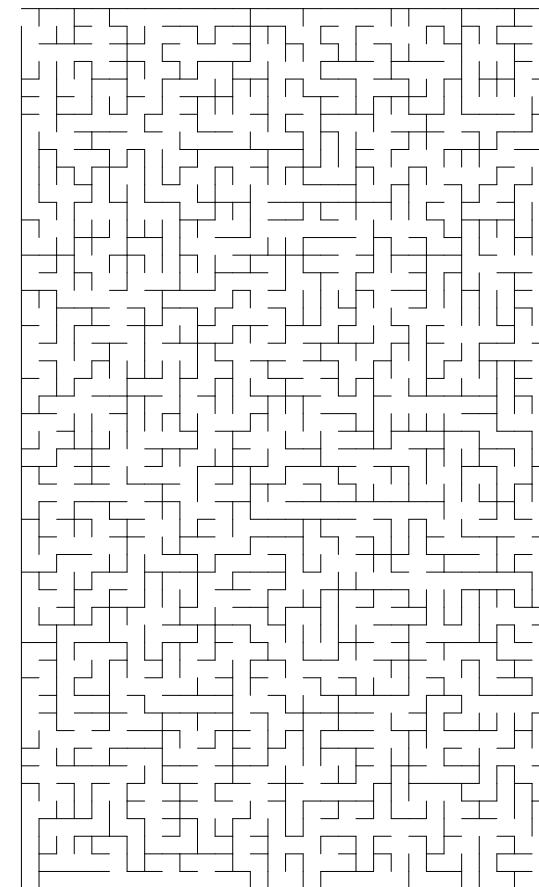
## Lesson 14: *Use Arrays for Fast Set Algorithms*



*Equivalent classes, Disjoint Set, Fast Sets*

# Outline

1. Equivalent Classes
2. Disjoint Sets
3. Fast Sets



# Equivalence Relations

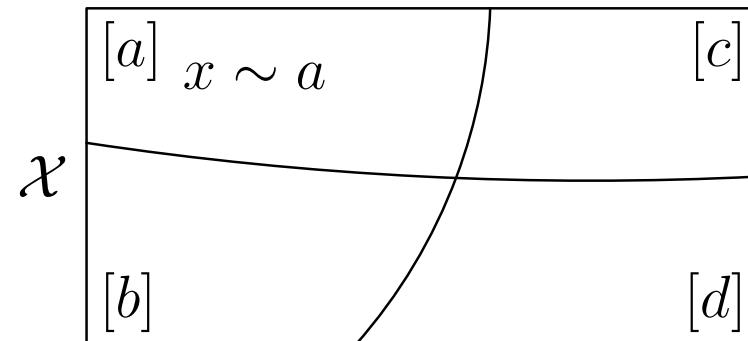
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**(Reflexivity)** For every element  $x \in \mathcal{X}$ ,  $x \sim x$

**(Symmetry)** For every two elements  $x, y \in \mathcal{X}$  if  $x \sim y$  then  $y \sim x$

**(Transitivity)** For every three elements  $x, y, z \in \mathcal{X}$  if  $x \sim y$  and  $y \sim z$  then  $x \sim z$

- Then  $\sim$  defines a partitioning of the set into **equivalence classes**



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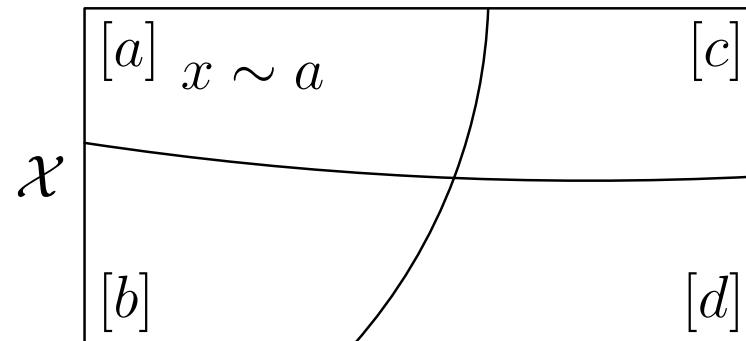
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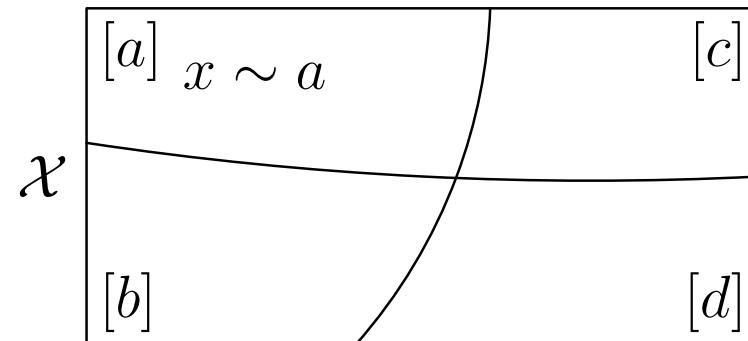
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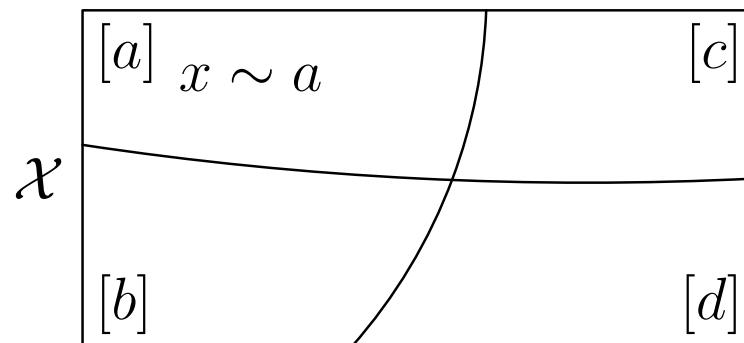
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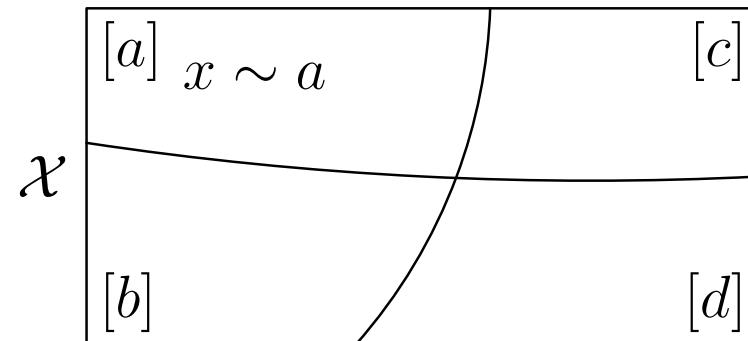
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# Example of Equivalence Classes

- Although, equivalent classes sound very mathematical they often provide a useful formalisation of the real world
- E.g. Pairs of web pages with a link in each direction between them
- Consider web pages in the same equivalence class if you can get from one to the other by clicking links
- Partitions the web into linked domains
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- Finding equivalence classes is rather easy using graph traversal algorithms
- However, as our web example suggests, there are applications where equivalence classes change over time
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- We will see this is a useful idea both for building mazes and (in a later lecture) for finding minimum spanning trees
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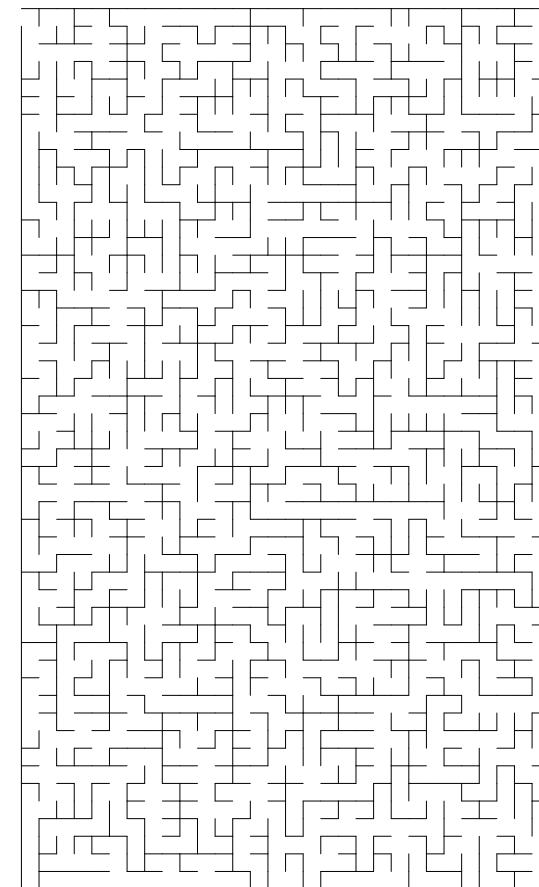
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# Outline

1. Equivalent Classes
2. Disjoint Sets
3. Fast Sets



# Union-Find

- In the union-find algorithm we have a set of objects  $x \in \mathcal{S}$  which are to be grouped into equivalence classes  $\mathcal{S}_1, \mathcal{S}_2, \dots$
- Initially each object is in its own equivalence class
- **union** combines two equivalence classes
- **find** returns a unique member of the equivalence class (e.g  $[a]$ )
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# DisjointSets

- We want to create a class

```
class DisjointSets
{
    DisjointSets(int numElements) /* Constructor */
    int find(int x) /* Find root */
    void union_(int root1, int root2) /* Union */

private:
    int* s;
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- Where `find(x)` returns a unique identifier for the subset which element `x` belongs to
- The array `s` contains labelling information to implement `find(x)`

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- A natural algorithm to perform finds is to maintain an array returning a subset label for each element—this makes find fast
- However, every time we combine two subset we have to change all the labels in this array (taking  $O(n)$  operations)
- If we are unlucky the cost of performing  $n$  unions is  $\Theta(n^2)$
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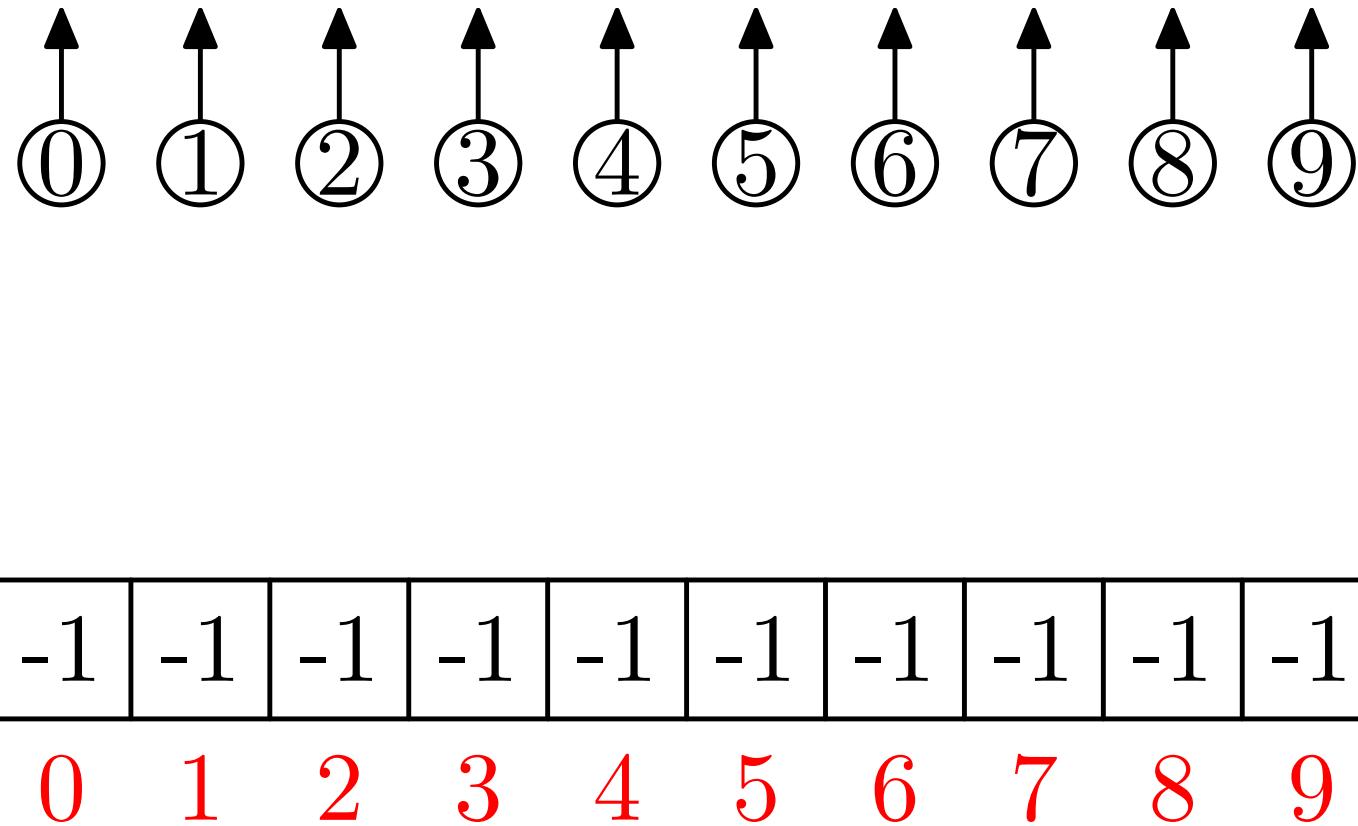
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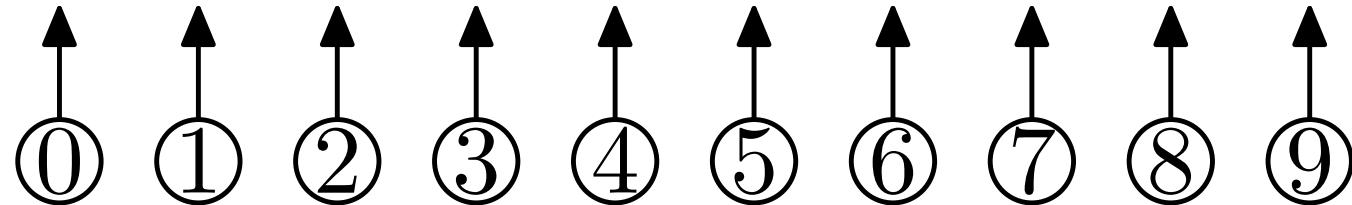
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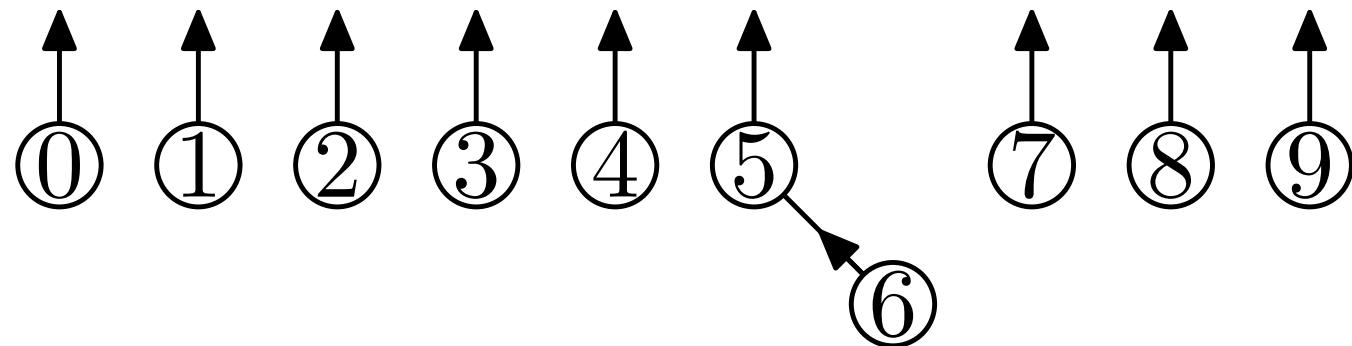
`union(find(5),find(6))`



-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

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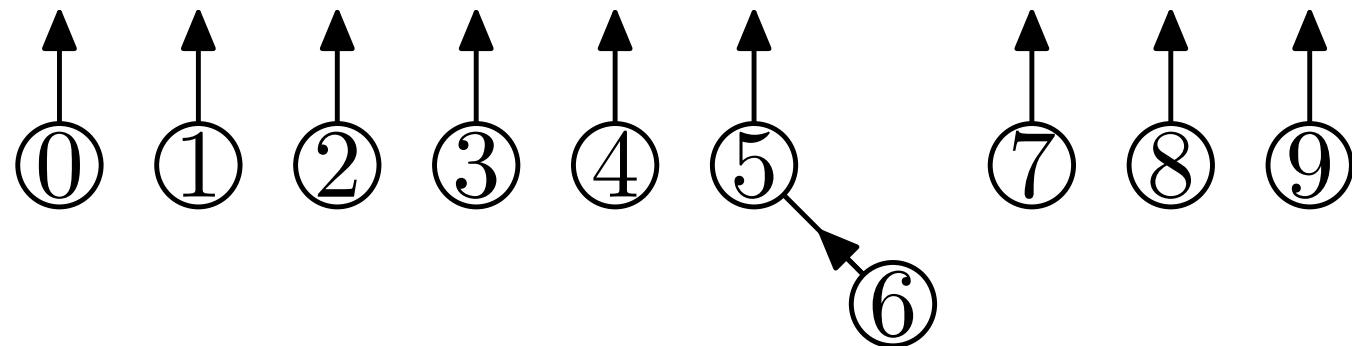
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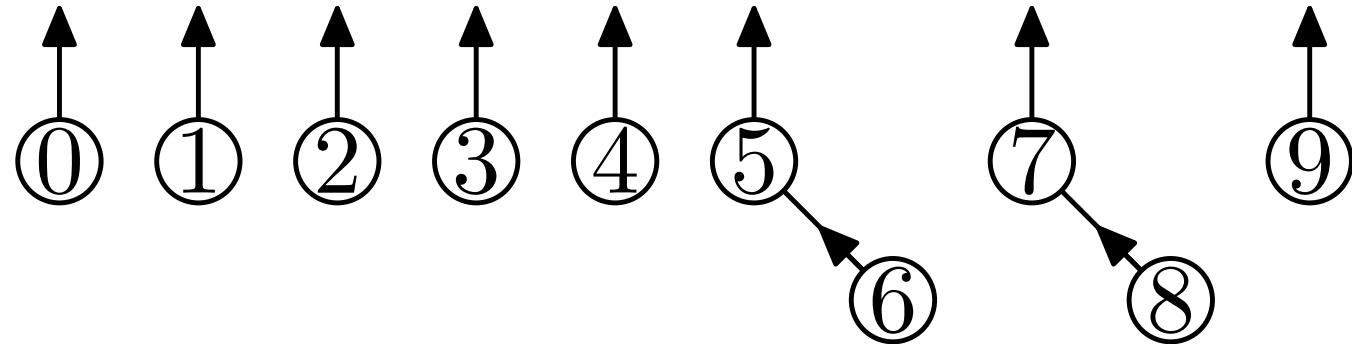
`union(find(7),find(8))`



-1	-1	-1	-1	-1	-2	5	-1	-1	-1
0	1	2	3	4	5	6	7	8	9

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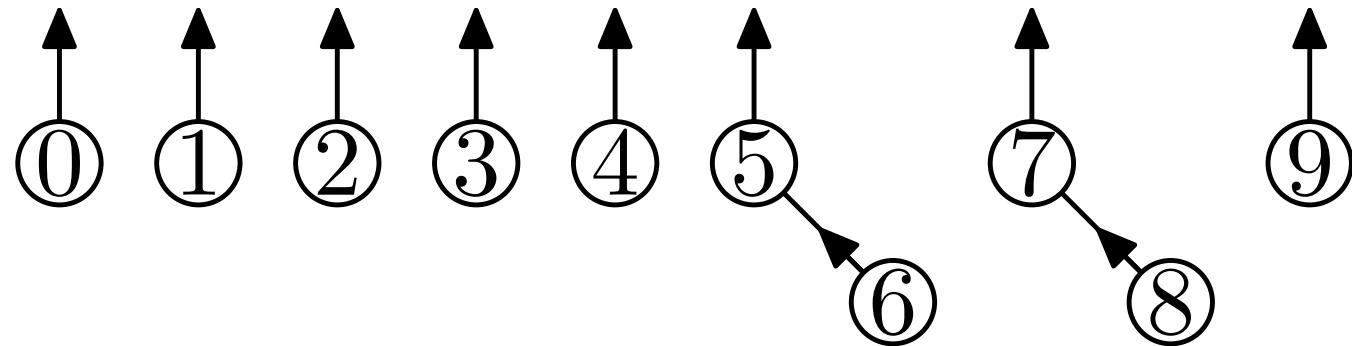
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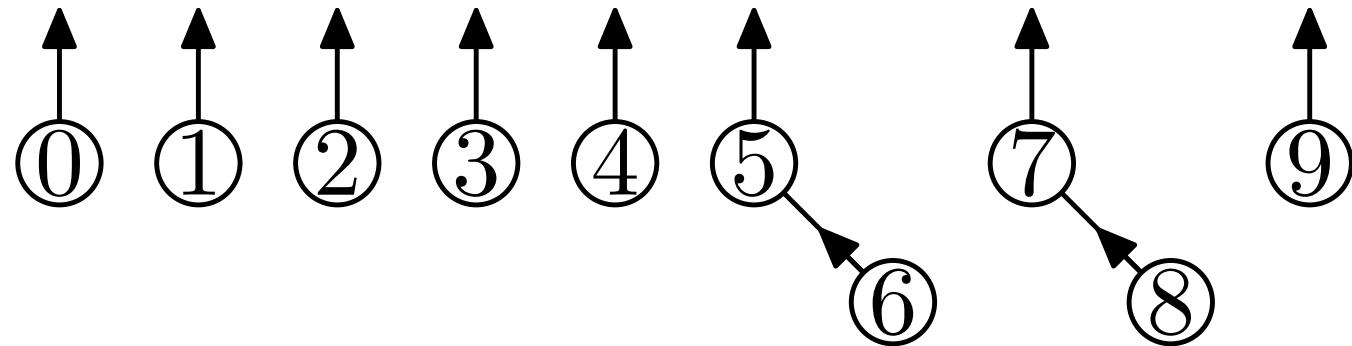
`find(6)`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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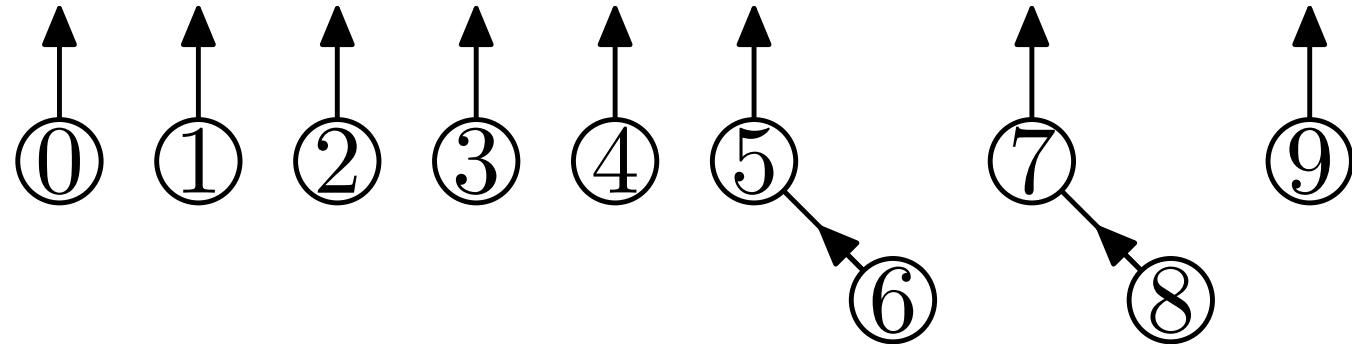
$\text{find}(6)=5$



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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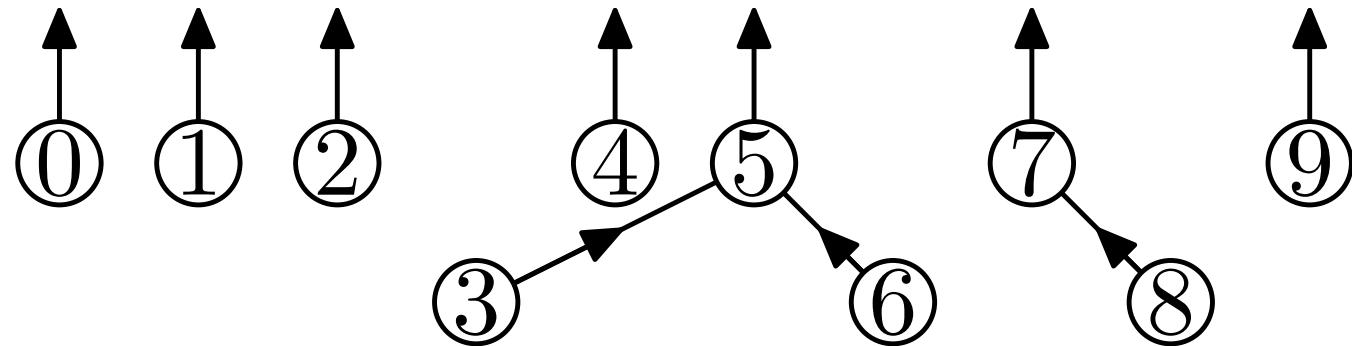
`union(find(3),find(6))`



-1	-1	-1	-1	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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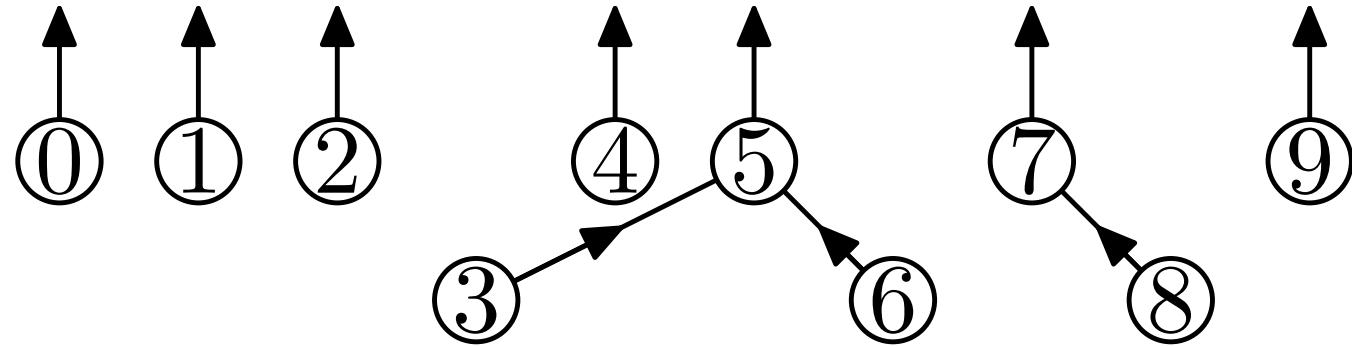
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-1	-1	-1	5	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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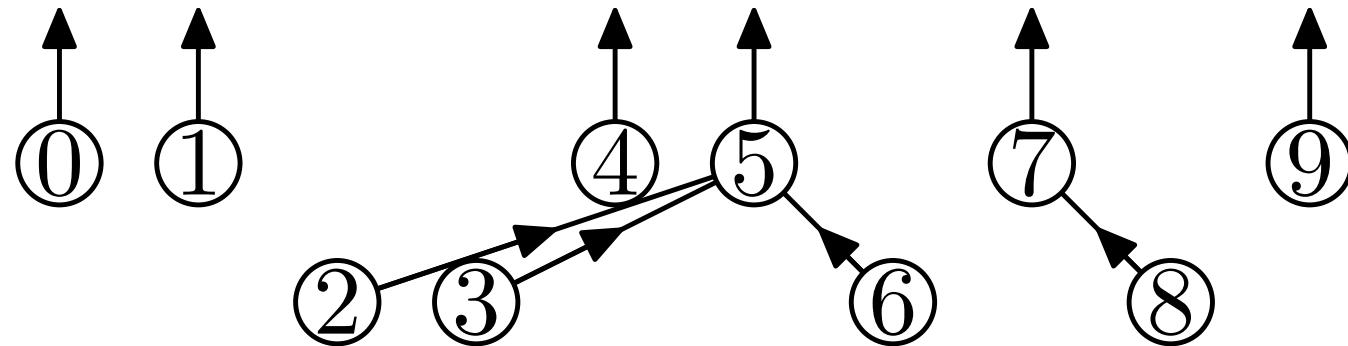
`union(find(2),find(6))`



-1	-1	-1	5	-1	-2	5	-2	7	-1
0	1	2	3	4	5	6	7	8	9

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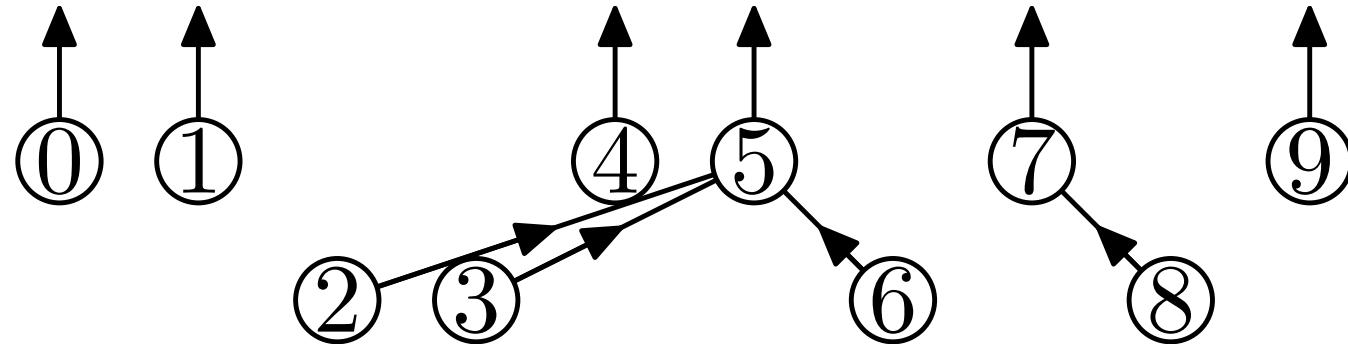
`union(find(2),find(6))`



-1	-1	5	5	-1	-2	5	-2	7	-1
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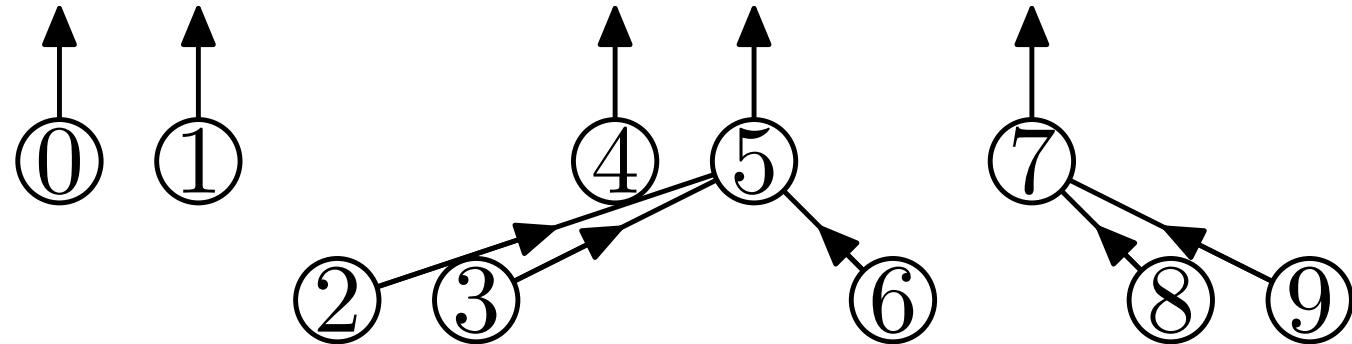
`union(find(9),find(8))`



-1	-1	5	5	-1	-2	5	-2	7	-1
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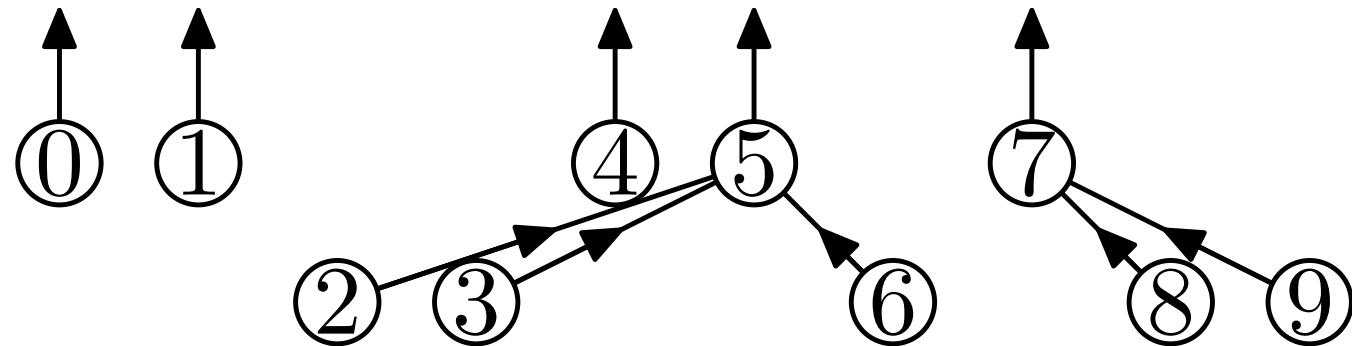
`union(find(9),find(8))`



-1	-1	5	5	-1	-2	5	-2	7	7
0	1	2	3	4	5	6	7	8	9

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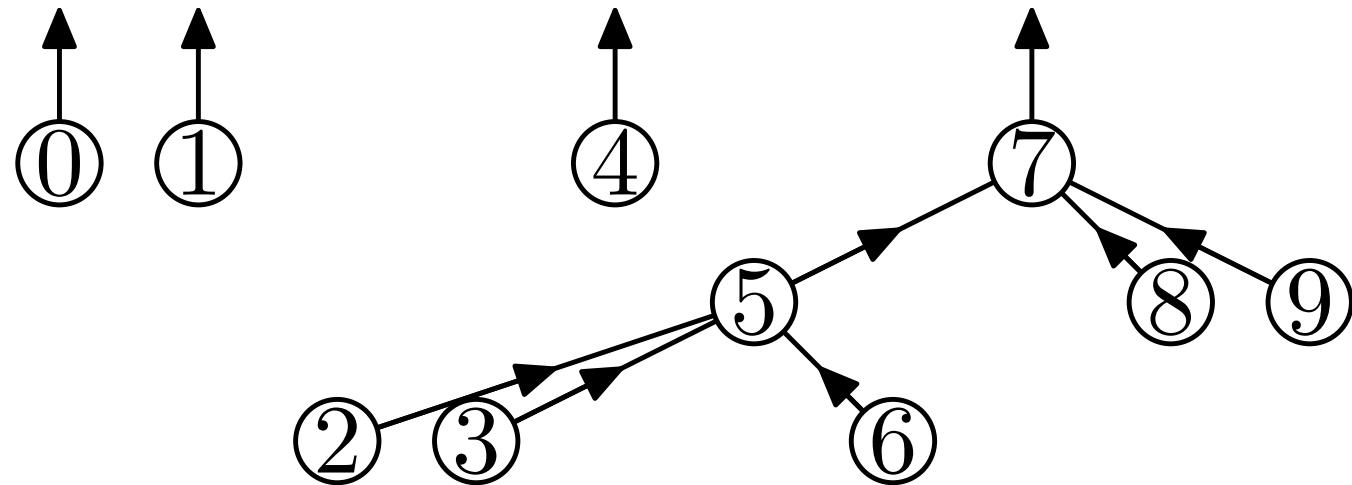
`union(find(9),find(3))`



-1	-1	5	5	-1	-2	5	-2	7	7
0	1	2	3	4	5	6	7	8	9

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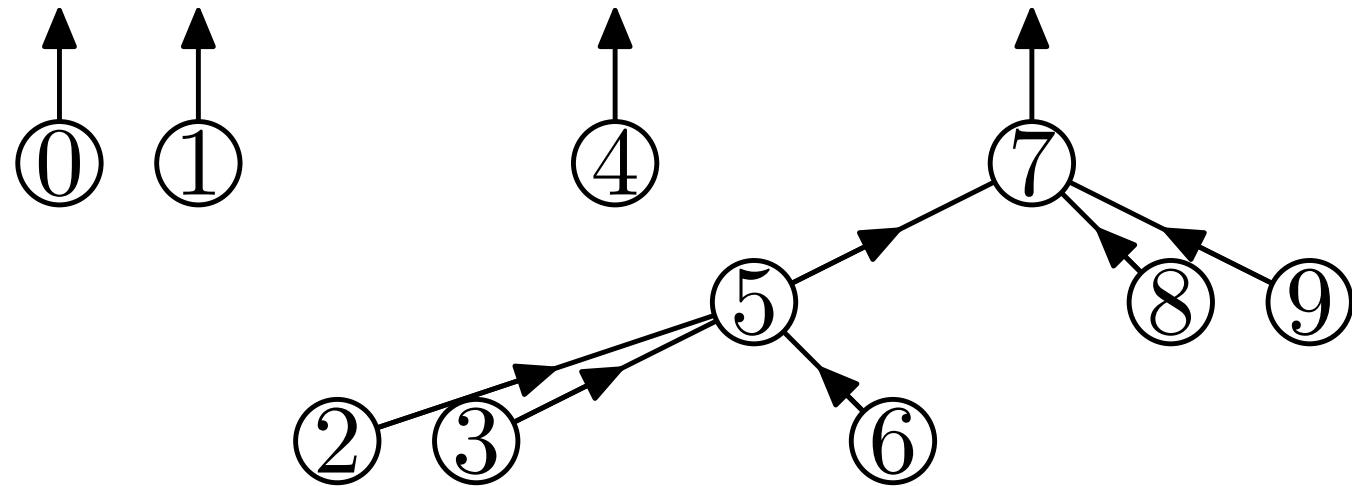
`union(find(9),find(3))`



-1	-1	5	5	-1	7	5	-3	7	7
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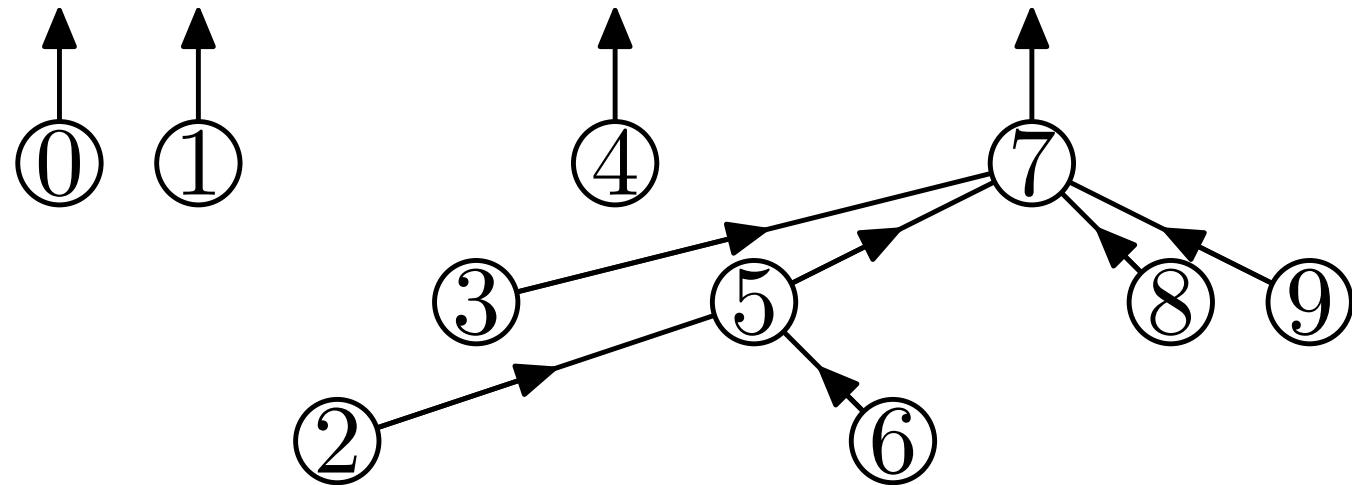
`find(3)`



-1	-1	5	5	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

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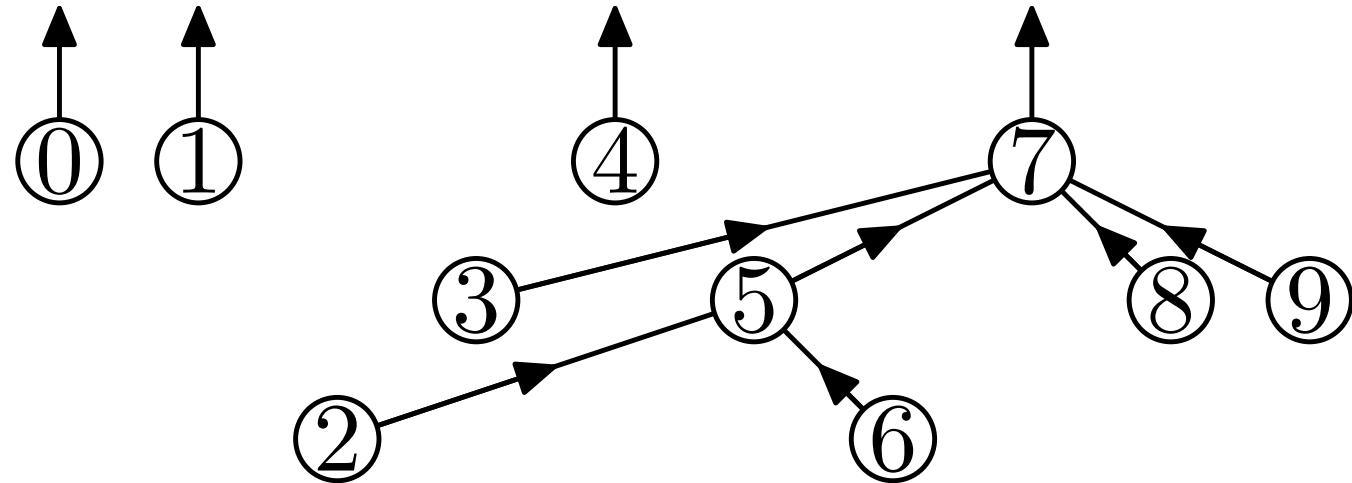
$\text{find}(3)=7$



-1	-1	5	7	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

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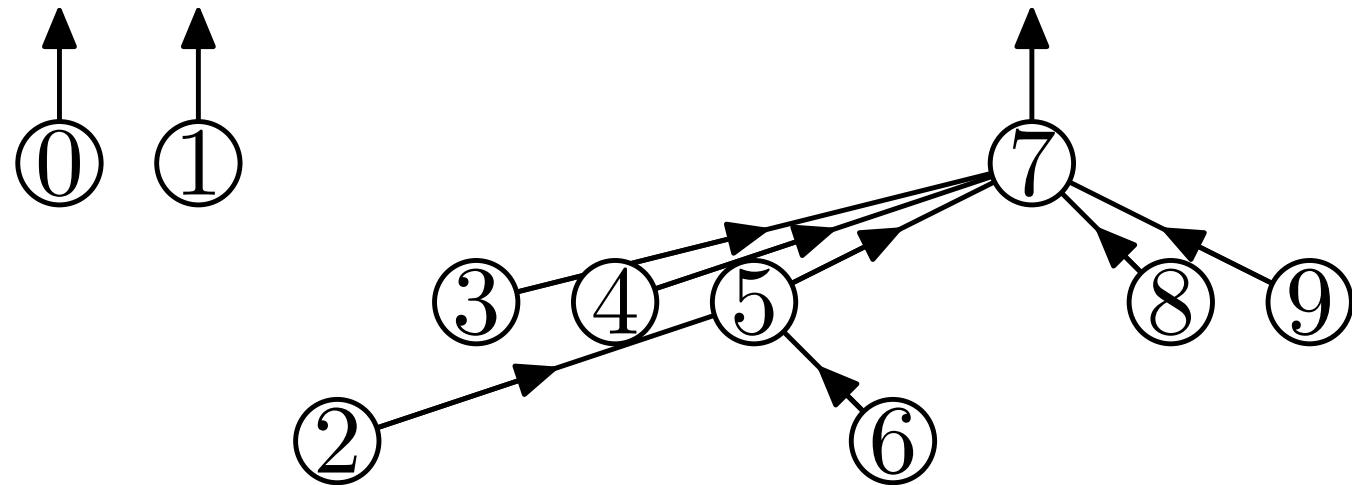
`union(find(3),find(4))`



-1	-1	5	7	-1	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

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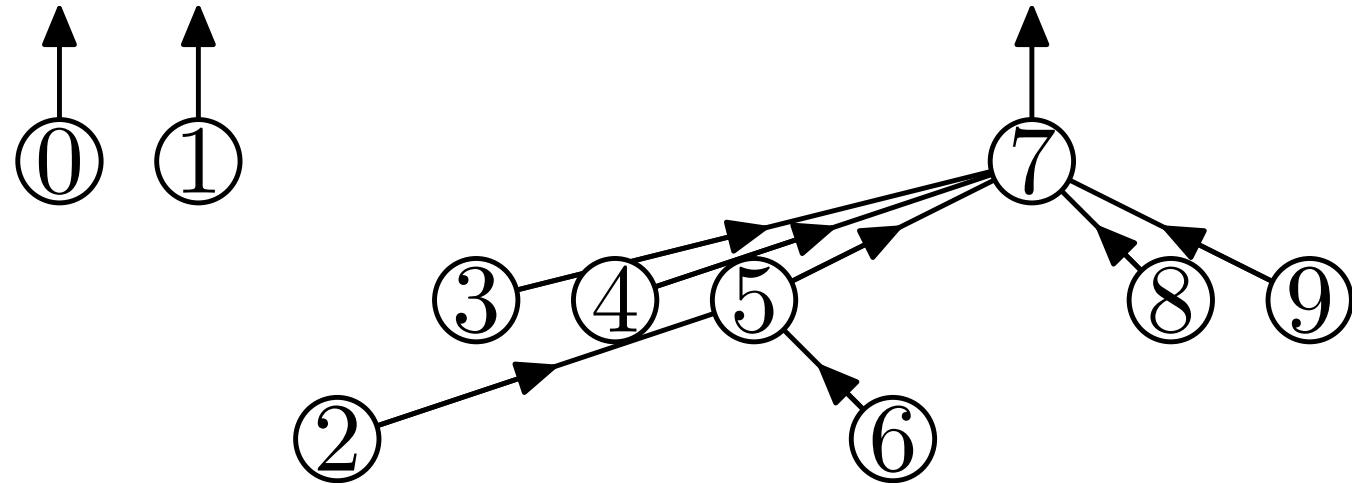
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0	1	2	3	4	5	6	7	8	9

# Putting it Together

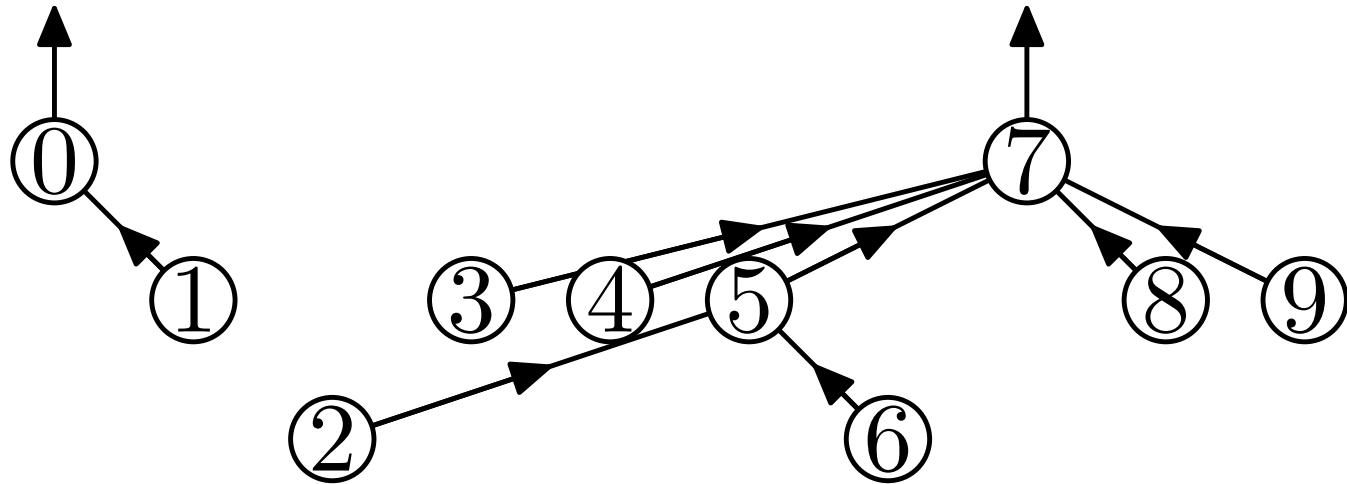
`union(find(0),find(1))`



-1	-1	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

# Putting it Together

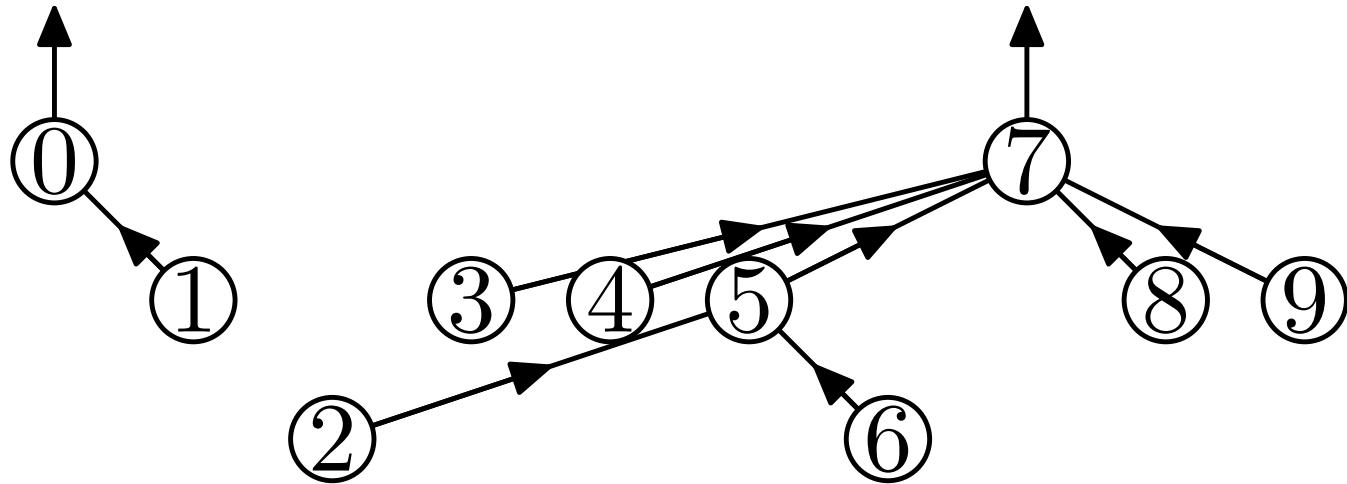
`union(find(0),find(1))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

# Putting it Together

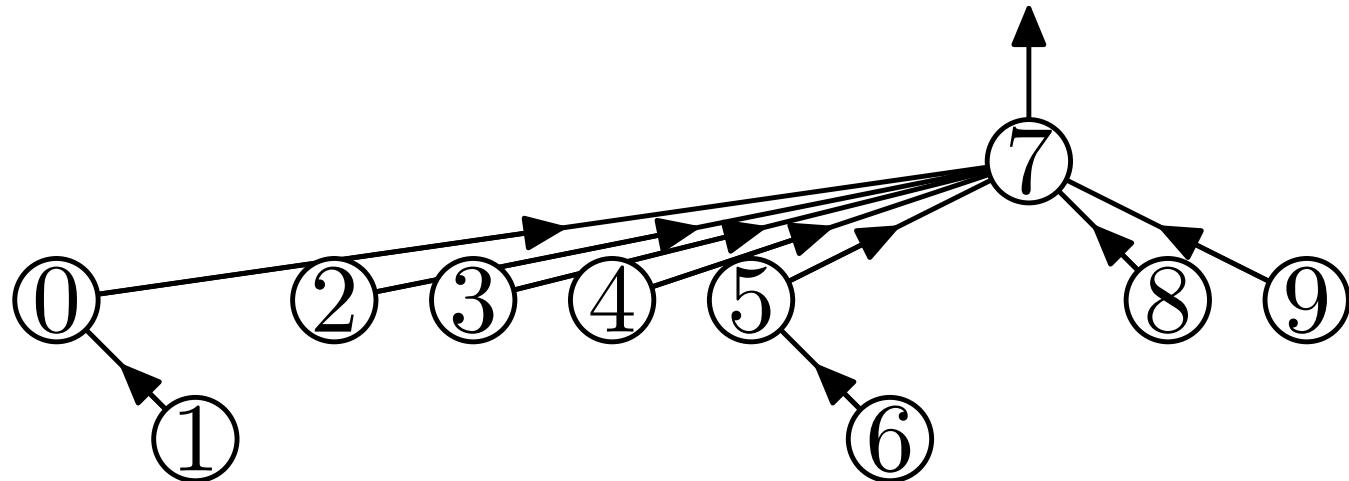
`union(find(1),find(2))`



-2	0	5	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

# Putting it Together

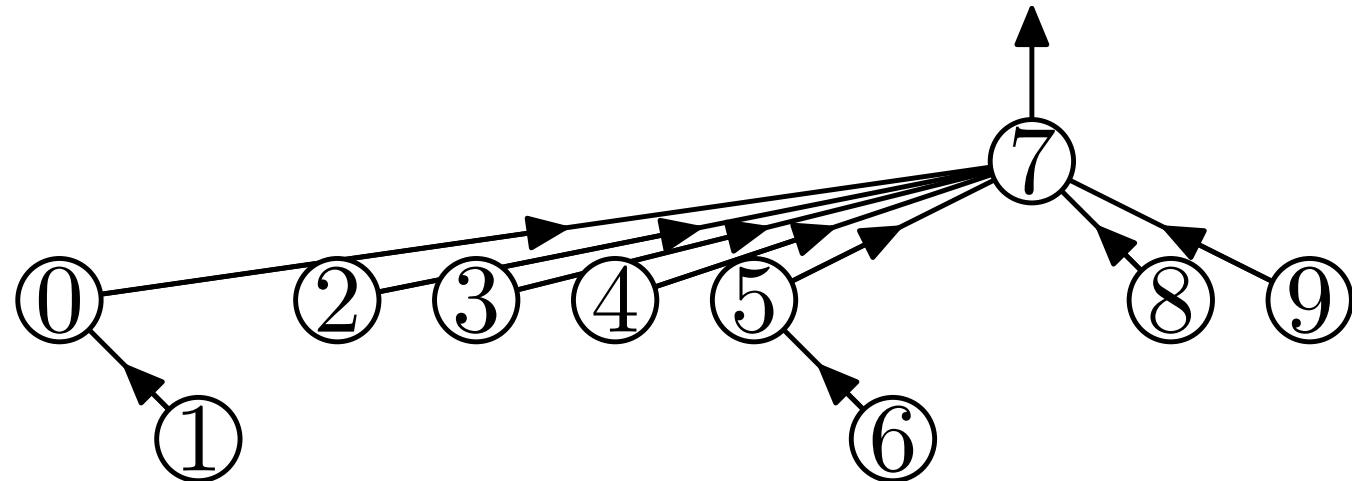
`union(find(1),find(2))`



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

# Putting it Together

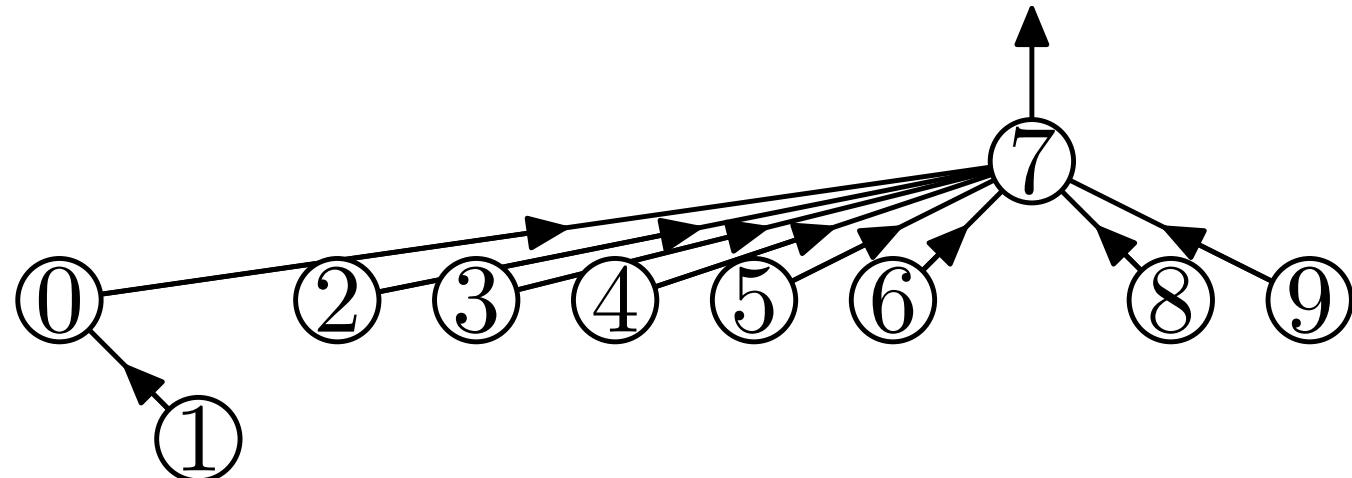
find(6)



7	0	7	7	7	7	5	-3	7	7
0	1	2	3	4	5	6	7	8	9

# Putting it Together

$\text{find}(6)=7$

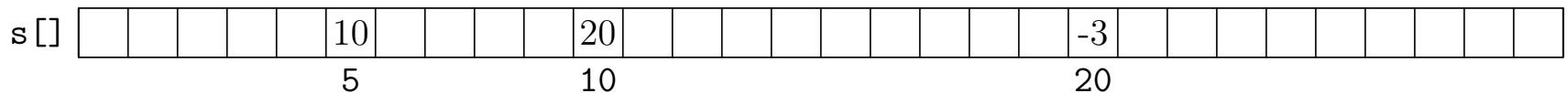


7	0	7	7	7	7	7	-3	7	7
0	1	2	3	4	5	6	7	8	9

# Smart Union

```
DisjointSets::DisjointSets(int numElements)
{
    s = new int[numElements];
    for(int i=0; i<numElements; i++)
        s[i] = -1;                                // roots are negative number
}

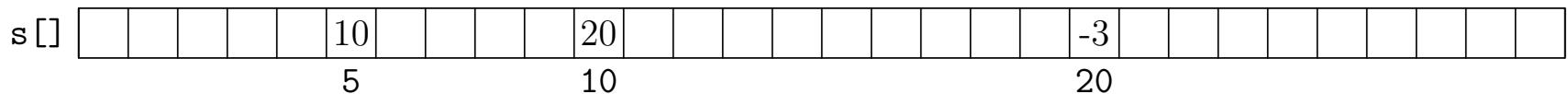
void DisjointSets::union_(int root1, int root2)
{
    if (s[root2]<s[root1]) {                  // root2 is deeper
        s[root1] = root2;                      // make root2 the root
    } else {
        if (s[root1]==s[root2])
            s[root1]--;                         // update height if same
        s[root2] = root1;                      // make root1 new root
    }
}
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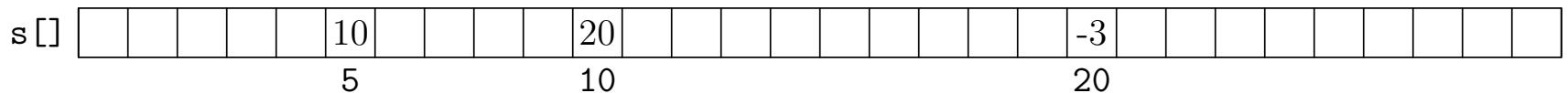
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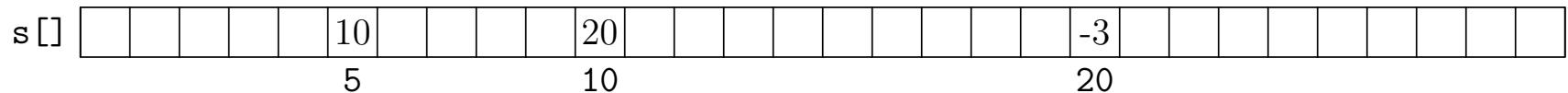
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```



# Path Compression

- To speed up `find` we relabel all nodes we visit during `find` by the root label

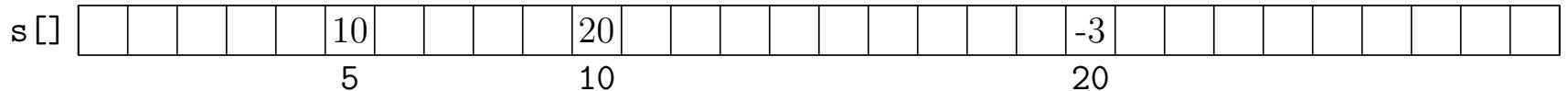
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int DisjointSets::find(int index)
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# Mazes

- Union-Find is a data structure which can occur in very different applications
- One application is building a maze
- Start from a complete lattice
- Remove a randomly chosen edge if it connects two unconnected regions
- Stop when the start and end cell are connected
- Or better after all cells are connected

0	1	2	3	4
5	6	7	8	9
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20	21	22	23	24
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# Time Complexity of Union-Find

- If we perform  $M$  finds and  $N$  unions then the time complexity is  $O(M \log_2^*(N))$
- Where  $\log_2^*(N)$  is the number of times you need to apply the logarithm function before you get a number less than 1
- In practice  $\log_2^*(N) \leq 5$  for all conceivable  $N$

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$$\log_2(10^{80}) = 265.75$$

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$$\log_2(\log_2(10^{80})) = 8.0539$$

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$$\log_2(\log_2(\log_2(10^{80}))) = 3.0097$$

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$$\log_2(\log_2(\log_2(\log_2(10^{80})))) = 1.5896$$

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$$\log_2(\log_2(\log_2(\log_2(\log_2(10^{80})))))) = 0.66868$$

# Time Complexity of Union-Find

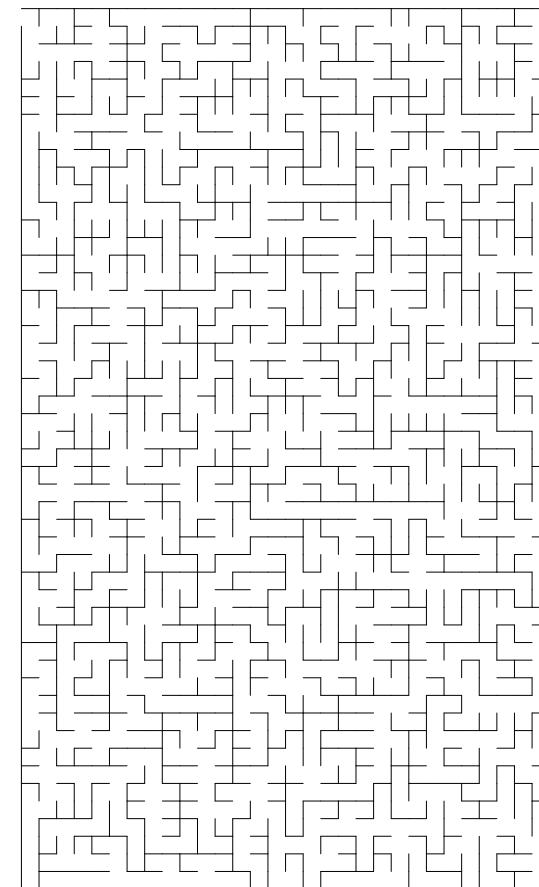
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- The proof of this time complexity is rather involved

# Outline

1. Equivalent Classes
2. Disjoint Sets
3. **Fast Sets**



# Comparison of Sets

- Binary Search Trees:  $O(\log_2(n))$ , general purpose
- Hash tables:  $O(1)$ , but need to compute hash, slow iterator when sparse, general purpose
- B-trees:  $O((k - 1) \log_k(n))$  very complicated, used for large amounts of data
- Tries:  $O(\text{len}(\text{seq})) = O(\log_k(n))$  for large  $k$  expensive in memory, complicated to code efficiently

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# What Set to Use?

- A PhD student and I were working on writing a fast solver for a combinatorial optimisation problem
- We had to choose one variable to change out of a small number of possible variables
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- We wanted a data structure which had quick add and remove and where we could choose a variable at random—**what should we use?**

# Bounded Set

- One special feature is that we knew we only wanted the set to contain integers between 0 and  $n$  (where  $n$  might be 100 000)
- This allowed us to use an array to represent whether an integer belongs to that set
- But how do we find a random element of the set quickly?

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- This allowed us to use an array to represent whether an integer belongs to that set
- But how do we find a random element of the set quickly?
- Use another array of course!

# FastSet

# FastSet

	0	1	2	3	4	5	6	7	8	9
index{}	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
mem{}										
	0	1	2	3	4	5	6	7	8	9

noMembers=0

# FastSet

add(4)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
----	----	----	----	----	----	----	----	----	----

mem{}

--	--	--	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=0

# FastSet

index{}	0	1	2	3	4	5	6	7	8	9
index{}	-1	-1	-1	-1	0	-1	-1	-1	-1	-1
mem{}	4									

noMembers=1

# FastSet

add(9)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	-1	-1	-1
----	----	----	----	---	----	----	----	----	----

mem{}

4									
---	--	--	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=1

# FastSet

index{}	0	1	2	3	4	5	6	7	8	9
mem{}	-1	-1	-1	-1	0	-1	-1	-1	-1	1
noMembers=2	4	9								

# FastSet

add(7)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	-1	-1	1
----	----	----	----	---	----	----	----	----	---

mem{}

4	9								
---	---	--	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=2

# FastSet

index{}	0	1	2	3	4	5	6	7	8	9
mem{}	-1	-1	-1	-1	0	-1	-1	2	-1	1
noMembers=3	4	9	7							

# FastSet

add(4)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	2	-1	1
----	----	----	----	---	----	----	---	----	---

mem{}

4	9	7							
---	---	---	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=3

# FastSet

false

	0	1	2	3	4	5	6	7	8	9
index{}	-1	-1	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7							

noMembers=3

# FastSet

add(1)

0 1 2 3 4 5 6 7 8 9

index{}

-1	-1	-1	-1	0	-1	-1	2	-1	1
----	----	----	----	---	----	----	---	----	---

mem{}

4	9	7							
---	---	---	--	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=3

# FastSet

index{}	0	1	2	3	4	5	6	7	8	9
index{}	-1	3	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7	1						

noMembers=4

# FastSet

contains(9)

0 1 2 3 4 5 6 7 8 9

index{}

-1	3	-1	-1	0	-1	-1	2	-1	1
----	---	----	----	---	----	----	---	----	---

mem{}

4	9	7	1						
---	---	---	---	--	--	--	--	--	--

noMembers=4

# FastSet

index{}	0	1	2	3	4	5	6	7	8	9
index{}	-1	3	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7	1						

noMembers=4

# FastSet

contains(5)

0 1 2 3 4 5 6 7 8 9

index{}

-1	3	-1	-1	0	-1	-1	2	-1	1
----	---	----	----	---	----	----	---	----	---

mem{}

4	9	7	1						
---	---	---	---	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=4

# FastSet

false

	0	1	2	3	4	5	6	7	8	9
index{}	-1	3	-1	-1	0	-1	-1	2	-1	1
mem{}	4	9	7	1						

noMembers=4

# FastSet

remove(9)

0 1 2 3 4 5 6 7 8 9

index{}

-1	3	-1	-1	0	-1	-1	2	-1	1
----	---	----	----	---	----	----	---	----	---

mem{}

4	9	7	1						
---	---	---	---	--	--	--	--	--	--

0 1 2 3 4 5 6 7 8 9

noMembers=4

# FastSet

true

0 1 2 3 4 5 6 7 8 9

index{}	-1	1	-1	-1	0	-1	-1	2	-1	-1
mem{}	4	1	7							

0 1 2 3 4 5 6 7 8 9

noMembers=3

# Implementation

```
class FastSet {  
    private:  
        int* indexArray;  
        int* memberArray;  
        int noMembers;  
  
    public:  
        FastSet (int n) {  
            indexArray = new int[n];  
            memberArray = new int[n];  
            for(int i=0; i<n; i++) {  
                indexArray [i] = -1;  
            }  
            noMembers = 0;  
        }  
  
        ~FastSet () {  
            delete[] indexArray;  
            delete[] memberArray;  
        }  
  
        int size () { return noMembers; }  
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```

# Add and Remove

```
bool add(int i) {
    if (indexArray[i]>-1)
        return false;
    memberArray [noMembers] = i;
    indexArray [i] = noMembers;
    ++noMembers;
    return true;
}

bool remove(int i) {
    if (indexArray[i]==-1)
        return false;
    --noMembers;
    memberArray [indexArray[i]] = memberArray [noMembers];
    indexArray [memberArray [noMembers]] = indexArray [i];
    indexArray [i] = -1;
    return true;
}
```

# Add and Remove

```
bool add(int i) {
    if (indexArray[i]>-1)
        return false;
    memberArray [noMembers] = i;
    indexArray[i] = noMembers;
    ++noMembers;
    return true;
}

bool remove(int i) {
    if (indexArray[i]==-1)
        return false;
    --noMembers;
    memberArray [indexArray[i]] = memberArray [noMembers];
    indexArray [memberArray [noMembers]] = indexArray[i];
    indexArray[i] = -1;
    return true;
}
```

# Collection Methods

```
void clear() {  
    for(int i=0; i<noMembers; i++) {  
        indexArray[memberArray[i]] = -1;  
    }  
    noMembers = 0;  
}  
  
bool isEmpty() {  
    return noMembers==0;  
}  
  
int* begin() {return &memberArray[0];}  
int* end() {return &memberArray[noMembers];}  
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# And Random?

- We can add additional methods taking advantage of the classes strength

**private:**

```
random_device rd; // Seed for the random number engine  
mt19937 gen(rd()); // Mersenne Twister RNG
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**public:**

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int getRandomElement () {  
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- Need to use FastSet signature to use this

```
FastSet fastSet(n);  
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int r = fastSet.getRandomElement();
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- We compared our algorithm to a very highly regarded “state-of-the-art” algorithm
- For large problems we were over 10 times faster because of this data structure
- The competitor algorithm used a complex tree structure instead of the simple array
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- Why? **The array solution isn't in the books**

# Lessons

- If you have a bounded set then using an array is usually going to be very fast  $O(1)$  (or  $O(\log^*(n))$ )
- These data structures are not general purpose for solving every day problems (c.f. `vector<T>`, `set<T>` and `map<T>`)
- They are “back pocket” data structures that solve problems that come up often enough that they are worth knowing about
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