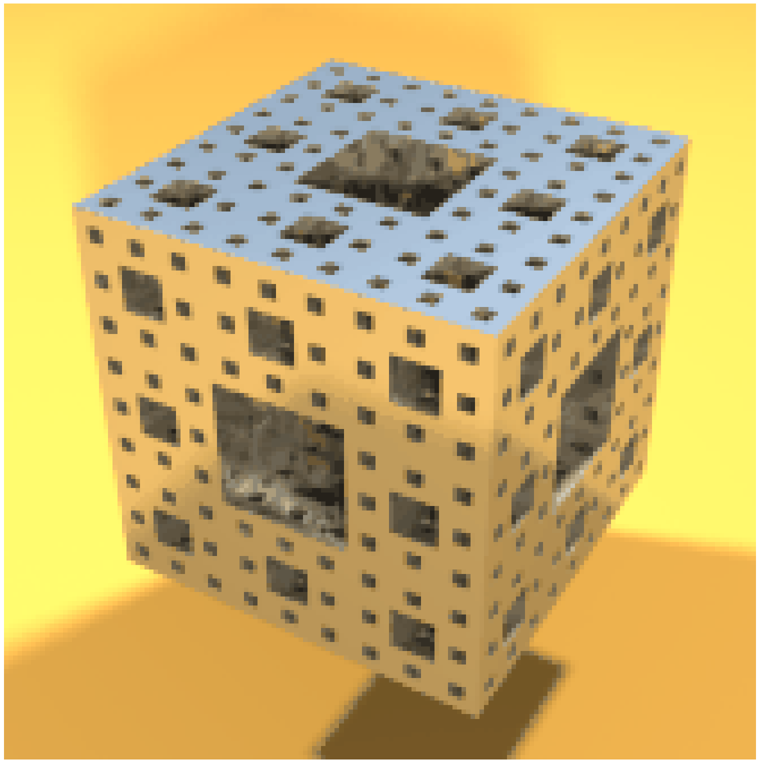
Algorithms and Analysis

Lesson 7: *Recurse!*



Induction, integer power, towers of hanoi, analysis

Outline

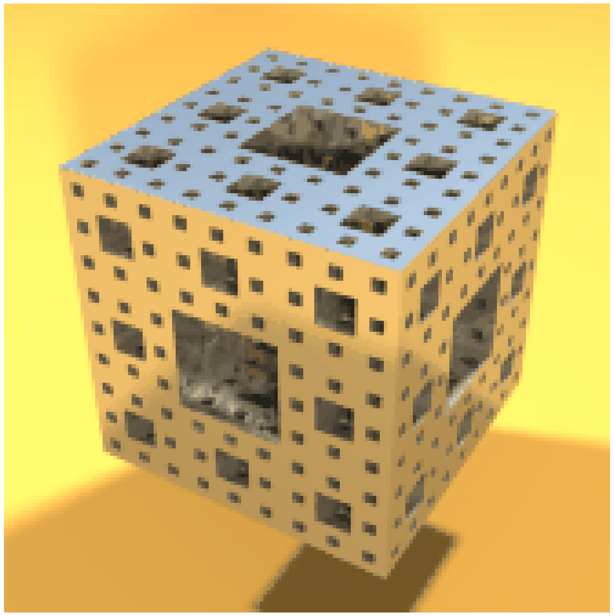
1. **Simple Recursion**

2. Programming Recursively

   - Simple Examples
   - Thinking about Recursion

3. Analysis of Recursion

   - Integer Powers
   - Towers of Hanoi



Recursion

- Recursion is a strategy whereby we reduce a problem to a smaller problem of the same type
- We repeat this until we reach a trivial case we can solve by some other means
- Recursion can also be used to describe situations in a succinct manner using references to itself. E.g.
  - ⋆ Definition of factorial:
    $n! = n \times (n-1)!$ with $0! = 1$
  - ⋆ Definition of ancestor: X is the ancestor of Y if X is the parent of Y or Z is the parent of Y and X is the ancestor of Z

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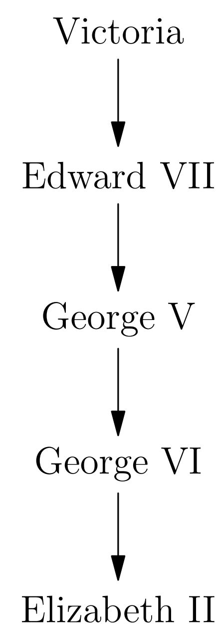

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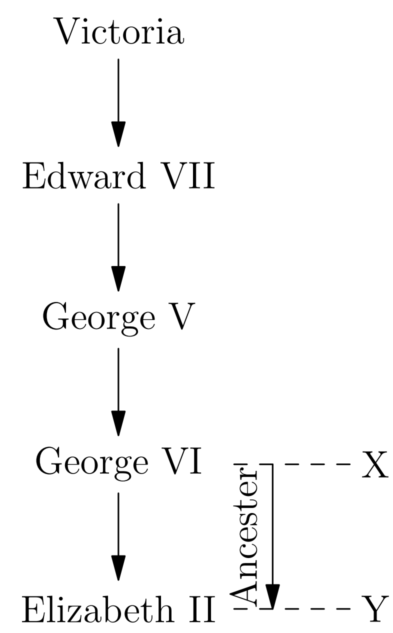

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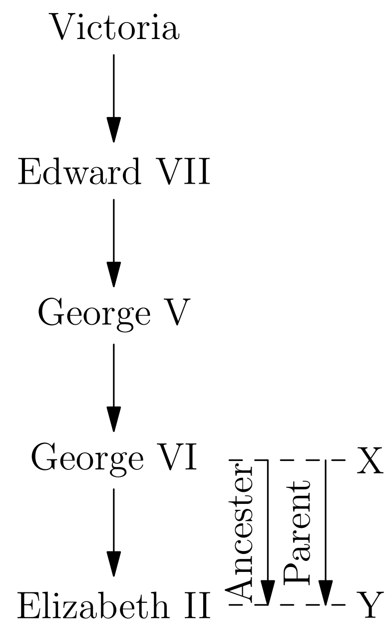

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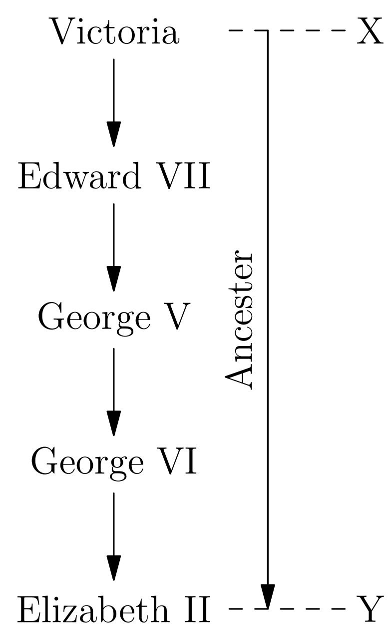

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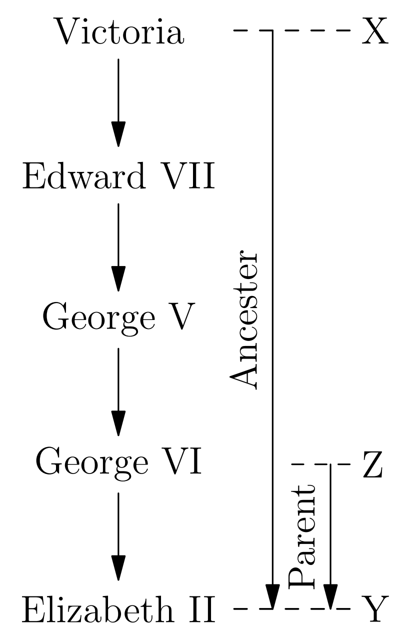

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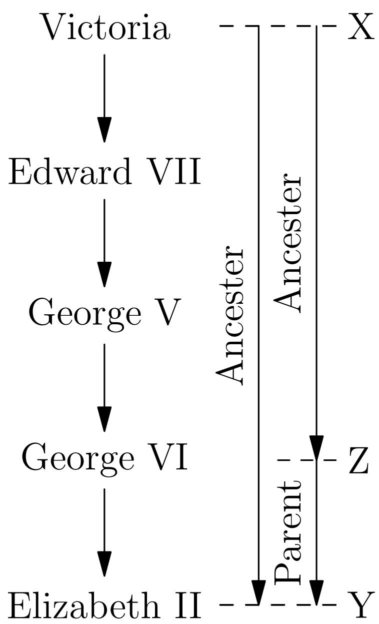

Structure of Recursion

- Notice that these are *self-referential* definitions
- A recursive definition consists of two elements
  - ⋆ **The Base Case:** or boundary cases where the problem is trivial
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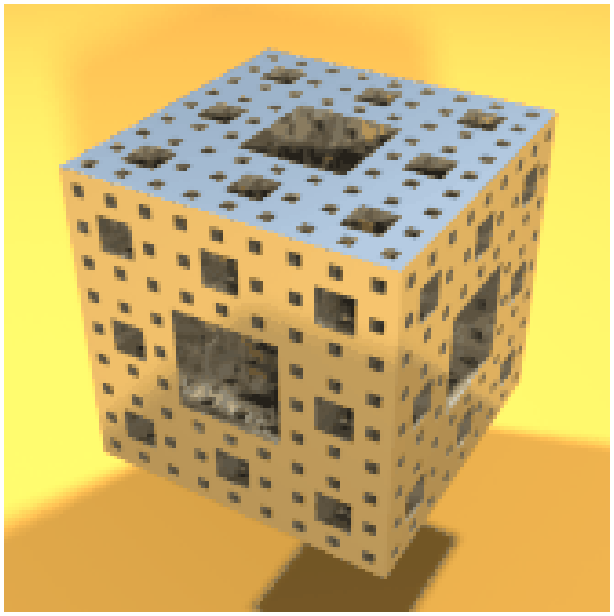
   - Thinking about Recursion
3. Analysis of Recursion
   - Integer Powers
   - Towers of Hanoi



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$$0.95^{25} = 0.95 \times (0.95)^{24}$$

Integer Powers

- How do we compute 0.95^{25}?
- One way is to multiply together 0.95 twenty five times
- A more efficient way is to observe

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six multiplications rather than 24!

Implementing Integer Power

- Integer power looks rather intimidating to code
- However, the recursive definition is easy
- We can easily code this function recursively

```
double power(double x, long n)    // (Overflow is possible)
{
    return  n <  0  ? 1 / power(x,-n)           // Negative power
          : n == 0  ? 1                          // Special case
          : n == 1  ? x                          // Base case
          : n%2 == 0 ? (x = power(x, n/2)) * x  // Even power
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Helper Functions

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Writing Recursive Programs

- You need to make sure that you catch the base case **before** you recurse

- The recursive case can call itself, possibly many times, provided the inductive argument is closer to the base case

- That is,

  - ⋆ Ensure that you use a 'smaller problem'
  - ⋆ Assume that you can solve the 'smaller problem'

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- Recursion acts just like any other function call
- The values of all local variables in scope are put on a stack
- The function is called and
  - ⋆ returns a value or
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Unrolling Recursion

- Recursion can frequently be replaced
- E.g. we can easily write a factorial function

```
long factorial(long n)
{
    if (n<0)
        throw new IllegalArgumentException();
    long res = 1;
    for (int i=2; i<=n; i++)
        res *= i;
    return res;
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with no function calls this will run much faster than the recursive version

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The Greatest Common Denominator

- One of the most famous algorithms is Euclid's algorithm for calculating the greatest common denominator

- The greatest common denominator of A and B is the largest integer, C, which exactly divides A and B

- E.g. the greatest common denominator of 70 and 25 is 5

- Euclid's algorithms uses the fact that
  - ⋆ $\gcd(A, B) = \gcd(B, A \mod B)$
  - ⋆ $\gcd(A, 0) = A$

- Thus $\gcd(70, 25) = \gcd(25, 20) = \gcd(20, 5) = \gcd(5, 0) = 5$

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Implementation of GCD

- The implementation of gcd is trivial using recursion

```
long gcd(long a, long b)
{

    if (b==0) {
        return a;
    }
    return gcd(b, a%b);
}
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Implementation of GCD

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long gcd(long a, long b)
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     a = b;
     b = c;
     return gcd(a, b);
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Implementation of GCD

- The implementation of gcd is trivial using recursion

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long gcd(long a, long b)
{
  while(true) {
     if (b==0) {
        return a;
     }
     long c = a%b;
     a = b;
     b = c;
   }
}
```

- Example of tail recursion

When Definitely not to Recurse

- A classic recursively defined sequence is the Fibonacci series
  - ⋆ $f_n = f_{n-1} + f_{n-2}$
  - ⋆ $f_1 = f_2 = 1$

- We might be tempted to write a recursive function to define the series

```
long fibonacci(long n)
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Why shouldn't you want to do this?

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```
fib(6)
```

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}
```

$$\begin{tikzcd} \texttt{fib(6)} \arrow[d] \\ \texttt{fib(5)} \end{tikzcd}$$

Why Use Recursion At All?

- Both `factorial` and `gcd` could be written without using recursion
- The programs would probably run faster
- The `gcd` program would be less clear
- The cost of the additional function calls is often insignificant
- It would considerably harder to write many programs such as `power` non-recursively
- Later we will see algorithms like `quick sort` which rely on recursion

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Outline

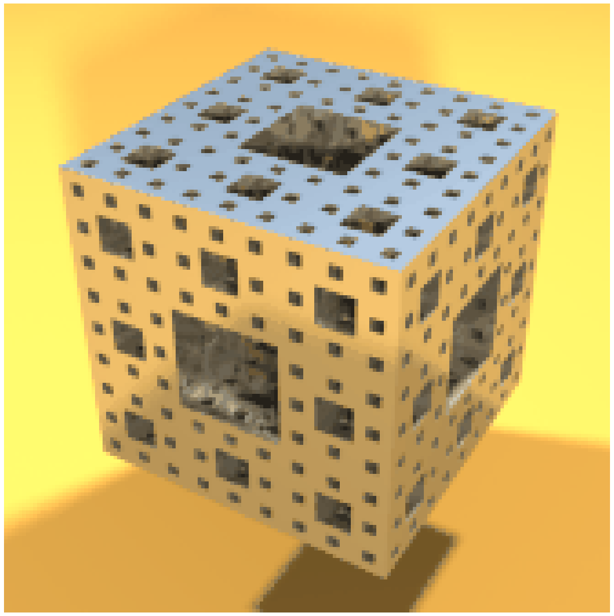
1. Simple Recursion

2. Programming Recursively

   - Simple Examples
   - Thinking about Recursion

3. **Analysis of Recursion**

   - Integer Powers
   - Towers of Hanoi



Analysis

- We can use recursion to compute the time complexity of a recursive program!
- To do this we denote the time taken to solve a problem of size n by $T(n)$
- To compute the time complexity of factorial, we note that to compute $n!$ we have to multiply n by $(n-1)!$
- That is, the number of multiplications we need to compute is

$$T(n) = T(n-1) + 1$$

- Now $T(0) = 0$ so

$$T(n) = T(n-1) + 1 = T(n-2) + 2 = \cdots = T(0) + n = n$$

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- That is, the number of multiplications we need to compute is

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Analysis

- We can use recursion to compute the time complexity of a recursive program!
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Time to Compute Power

- How long does it take to compute x^n?
- Remember

$$x^{2n} = (x^n)^2$$
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- Thus

$$T(n) = \begin{cases} T(n/2) + 1 & \text{if } n \text{ is even} \\ T((n-1)/2) + 2 & \text{if } n \text{ is odd} \end{cases}$$
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How many times?

- We want to solve $T(n) \leq T(\lfloor n/2 \rfloor) + 2$ with $T(1) = 0$
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- Denoting n by a binary number $n = b_m b_{m-1} \cdots b_2 b_1$
  - ⋆ $b_i \in \{0, 1\}$
  - ⋆ $b_m = 1$
  - ⋆ m is the number of digits in the binary representation of n
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- How many binary digits do you need to represent an integer n
- Note that an m digit number can represent a number from 2^m to $2^{m+1} - 1$
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$$2^m \le n < 2^{m+1}$$

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Towers of Hanoi

In an ancient city, so the legend goes, monks in a temple had to move a pile of 64 sacred disks from one location to another. The disks were fragile; only one could be carried at a time. A disk could not be placed on top of a smaller, less valuable disk. In addition, there was only one other location in the temple (besides the original and destination locations) sacred enough for a pile of disks to be placed there.

Using the intermediate location, the monks began to move disks back and forth from the original pile to the pile at the new location, always keeping the piles in order (largest on the bottom, smallest on the top). According to the legend, before the monks could make the final move to complete the new pile in the new location, the temple would turn to dust and the world would end.

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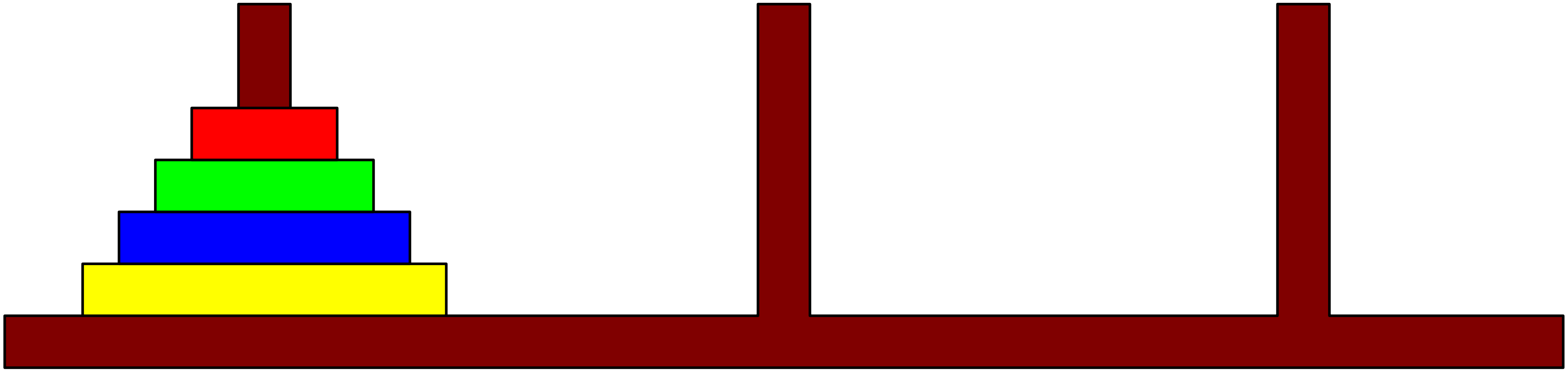

Algorithms in the Real World

- We require an algorithm to solve the towers of Hanoi
- Algorithms don't just apply to computers!
- If you try to solve the problem by hand you will discover that its quite fiddly
- There is a simple recursive solution which turns out to be optimal
- Let `move(X,Y)` denote the procedure of moving the top disk from peg `X` to peg `Y`
- Let `hanoi(n,X,Y,Z)` denote the procedure of moving the top `n` disks from peg `X` to peg `Z` using peg `Y`

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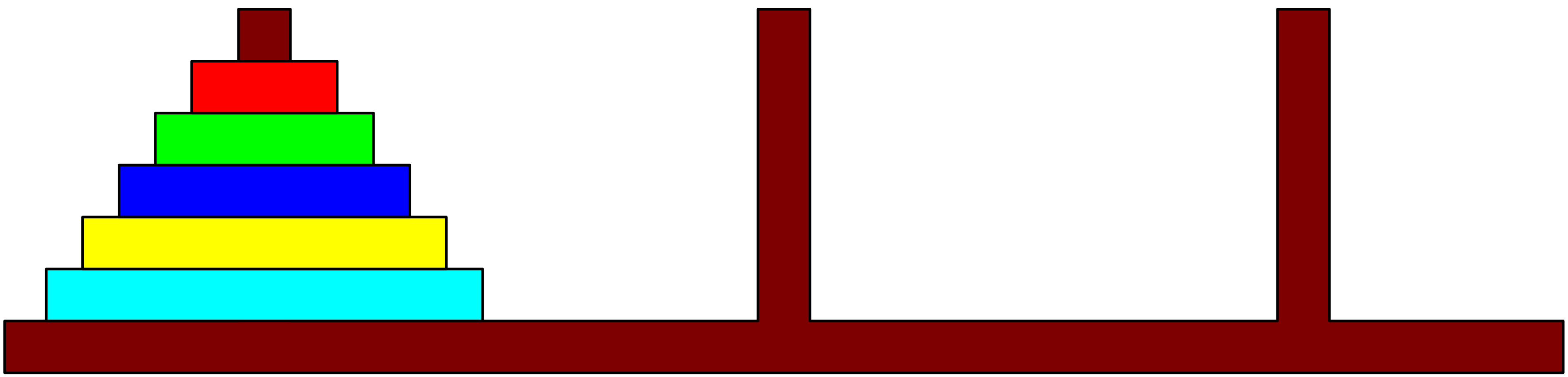

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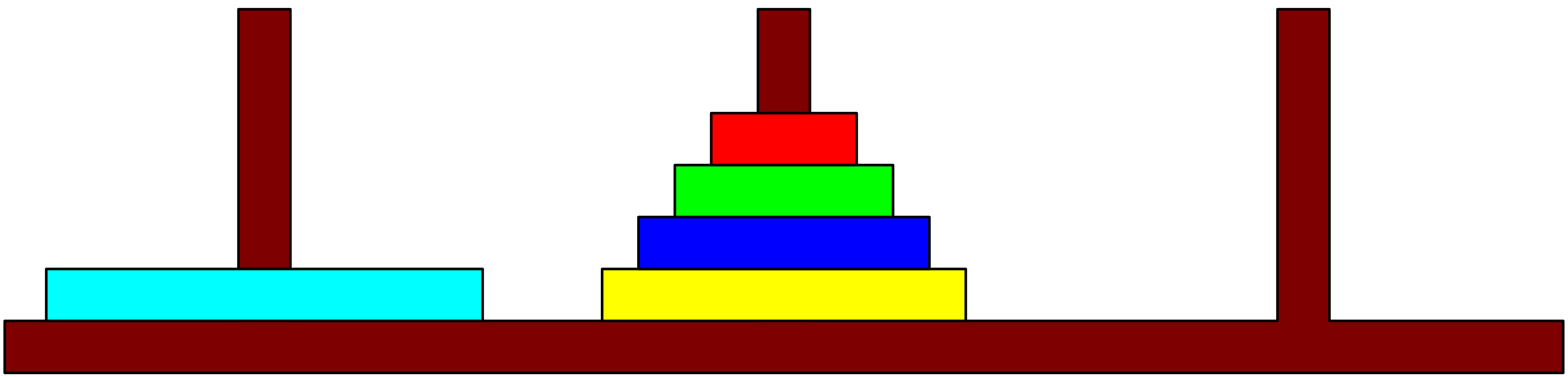

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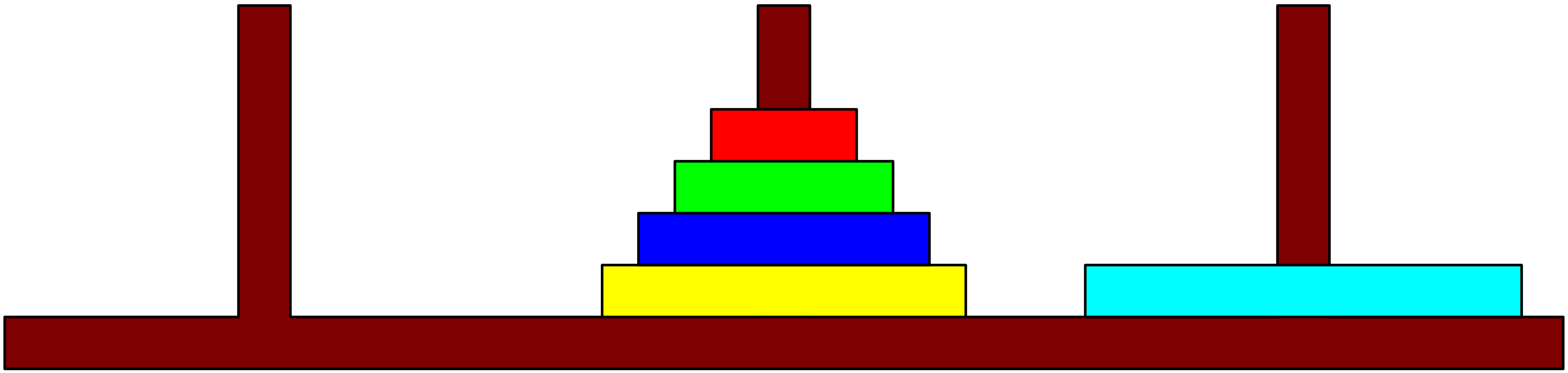

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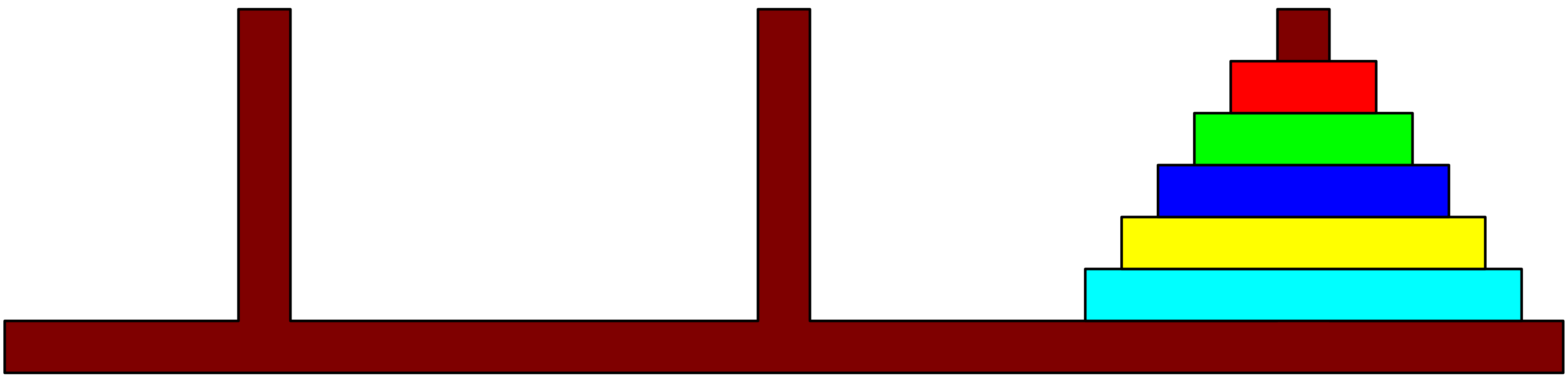

Optimality of Solution

- This is optimal because
  - ⋆ You have to move the largest disk from peg A to peg C
  - ⋆ We do this only once
  - ⋆ To make this move all the other disks must be on peg B
  - ⋆ Assuming that we solve the $n-1$ disk problem optimally then we solve the n disk problem optimally
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- Recursive case: Assume $T(n-1) = 2^{n-1} - 1$ then

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\begin{aligned}
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Time Complexity of Recursion

- The time complexity for recursion can be tricky to calculate
- The procedure is to calculate the time, $T(n)$, taken for a problem of size n in terms of the time taken for a smaller problem
- The difficulty is to solve the recursion
- Recursive programs can be very quick (e.g. $O(\log n)$ for computing integer powers)
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- In case your interested, if it takes 1 second to move a disk it will take almost $585\,000\,000\,000$ years to move 64 disks

Lessons

- Recursion is a powerful tool for writing algorithms
- It often provides simple algorithms to otherwise complex problems
- Recursion comes at a cost (extra function calls)
- There are times when you should avoid recursion (computing Fibonacci numbers)
- You need to be able to analyse the time complexity of recursion
- Used appropriately, recursion is fantastic!

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