Algorithms and Analysis

Lesson 11: *Use Heaps!*

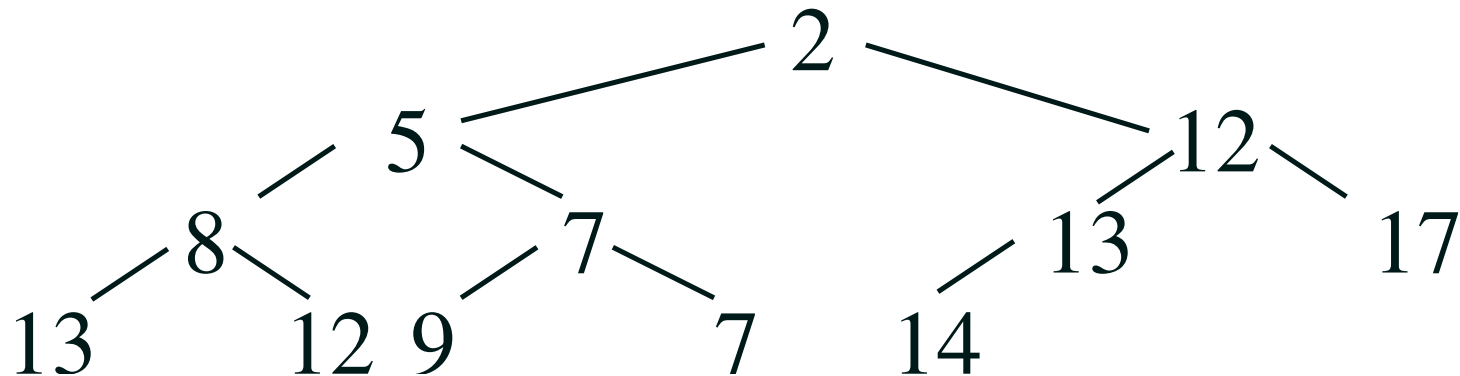*Heaps, Priority queues, Heap Sort, Other heaps*

Outline

1. **Heaps**

2. Priority Queues

   • Array Implementation

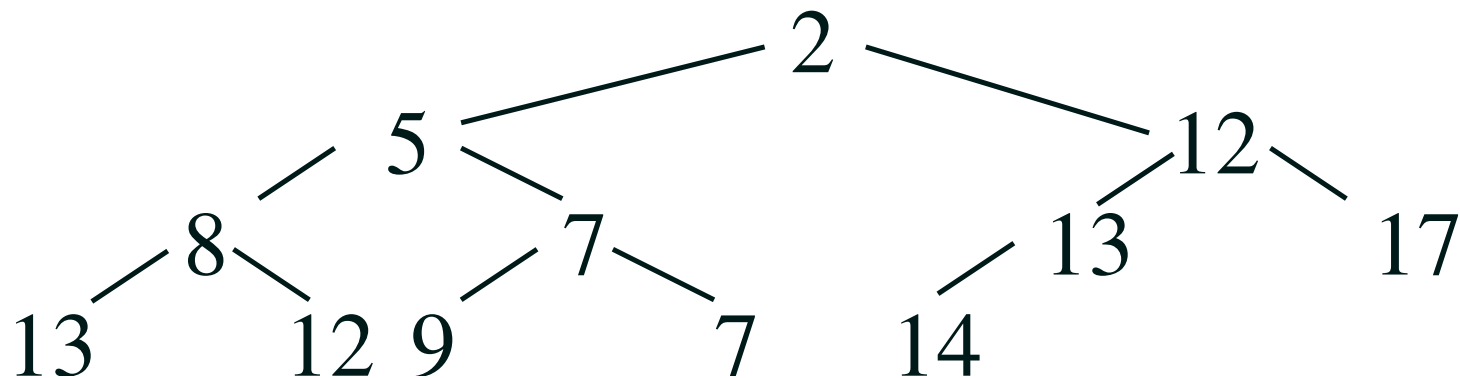3. Heap Sort

4. Other Heaps

Heaps

- A (min-)heap is (from one perspective) a binary tree

- It is a binary tree satisfying two constraints

  - ⋆ It is a **complete** tree
  - ⋆ Each child has a value 'greater than or equal to' its parent

```
                    2
            5               12
        8       7       13      17
      13  12  9   7   14
```

- **complete** means that every level is fully occupied above the lowest level and the nodes on the lowest level are all to the left

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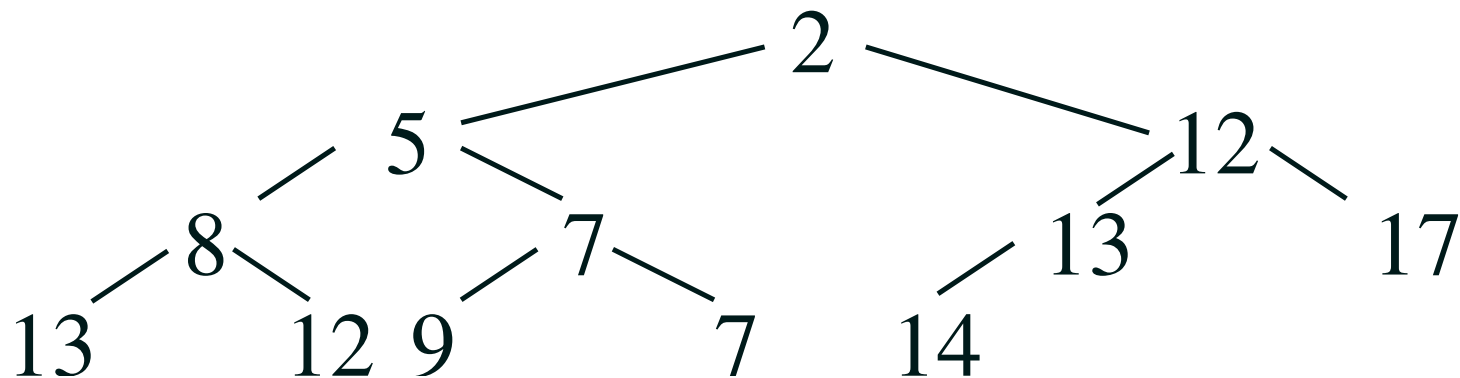
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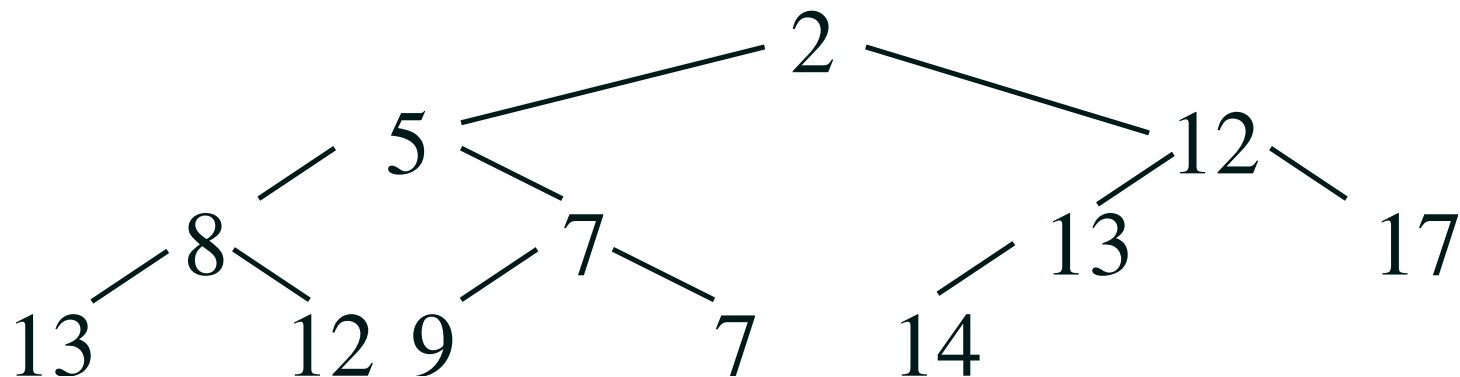
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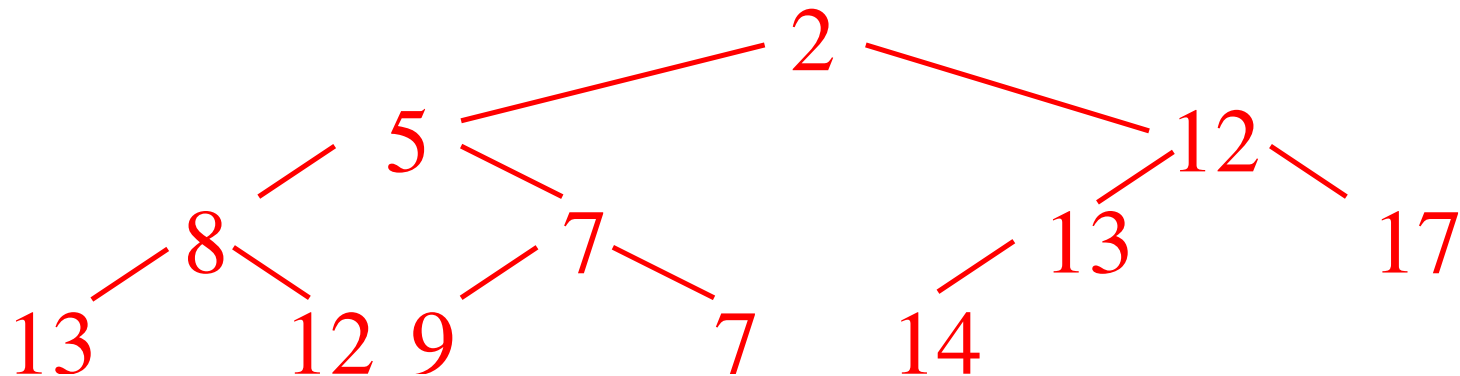
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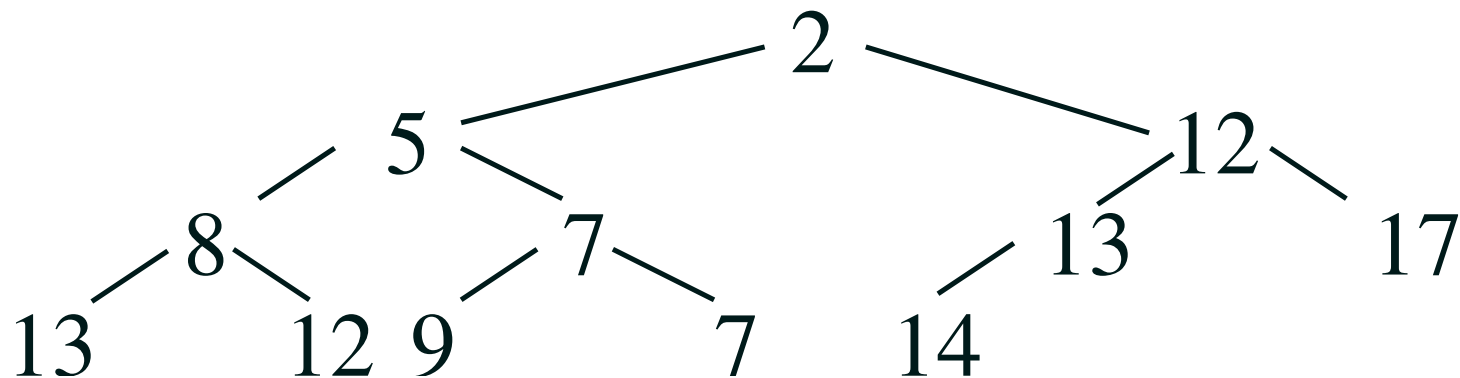
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Outline

1. Heaps

2. **Priority Queues**

   - Array Implementation

3. Heap Sort

4. Other Heaps

Priority Queues

- One of the prime uses of heaps is to implement a priority queue

- A Priority Queue is a queue with priorities

- That is, we assign a priority to each element we add

- The head of the queue is the element with highest priority (smallest number)

- Used, for example, in simulating real time events

- Used to implement "greedy algorithms"

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Priority Queue

- A simple Priority Queue might include

  ⋆ `unsigned size()` returning the the number of elements
  ⋆ `bool empty()` returns true if empty
  ⋆ `void push(T element, int priority)` adds an element
  ⋆ `T top()` returns head of queue
  ⋆ `void pop()` dequeues head of queue

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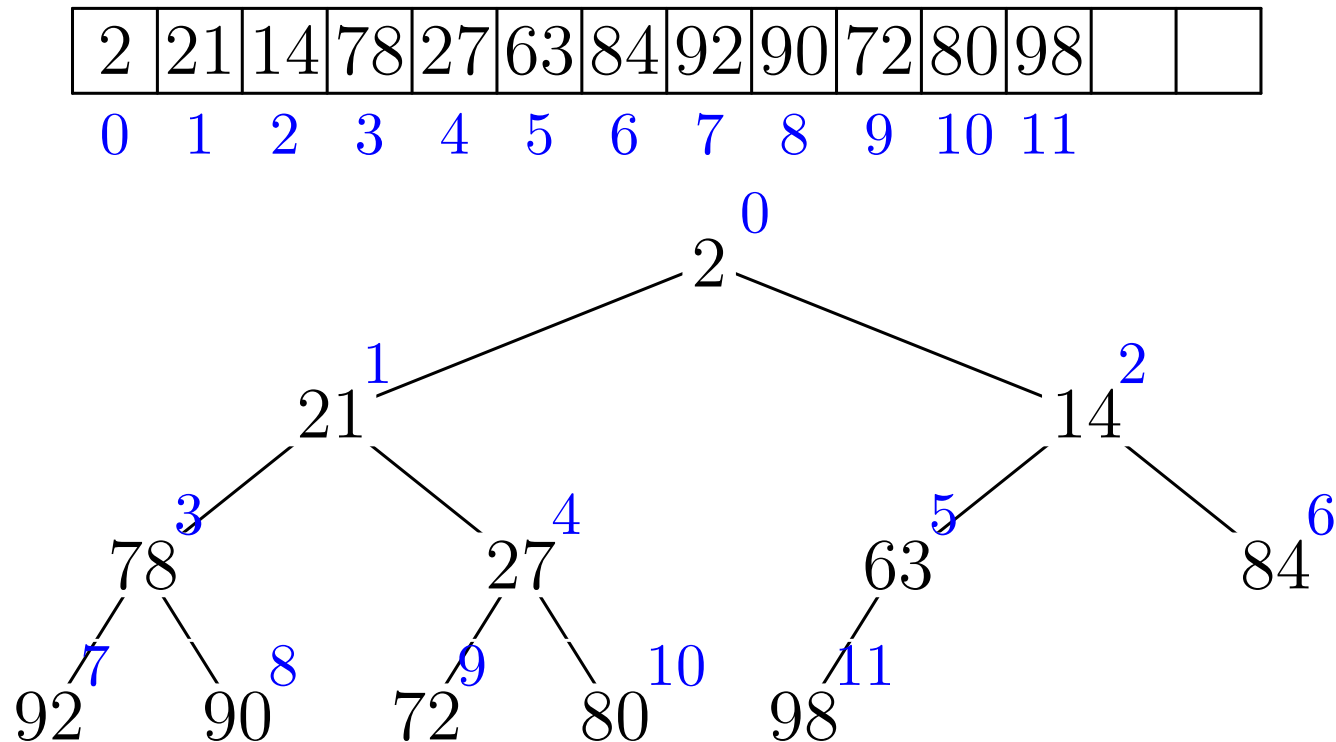

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Array Implementation of Heaps

• Because the tree is complete we can implemented the heap efficiently using an array
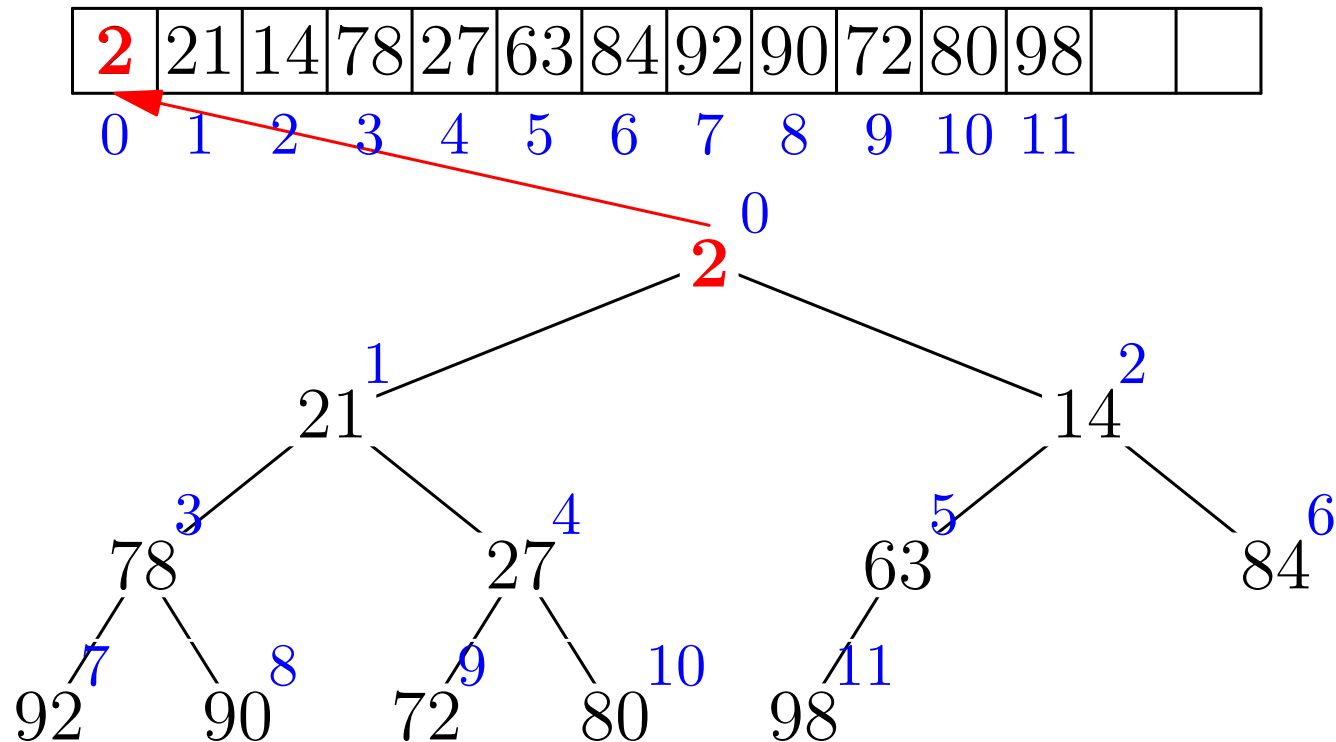
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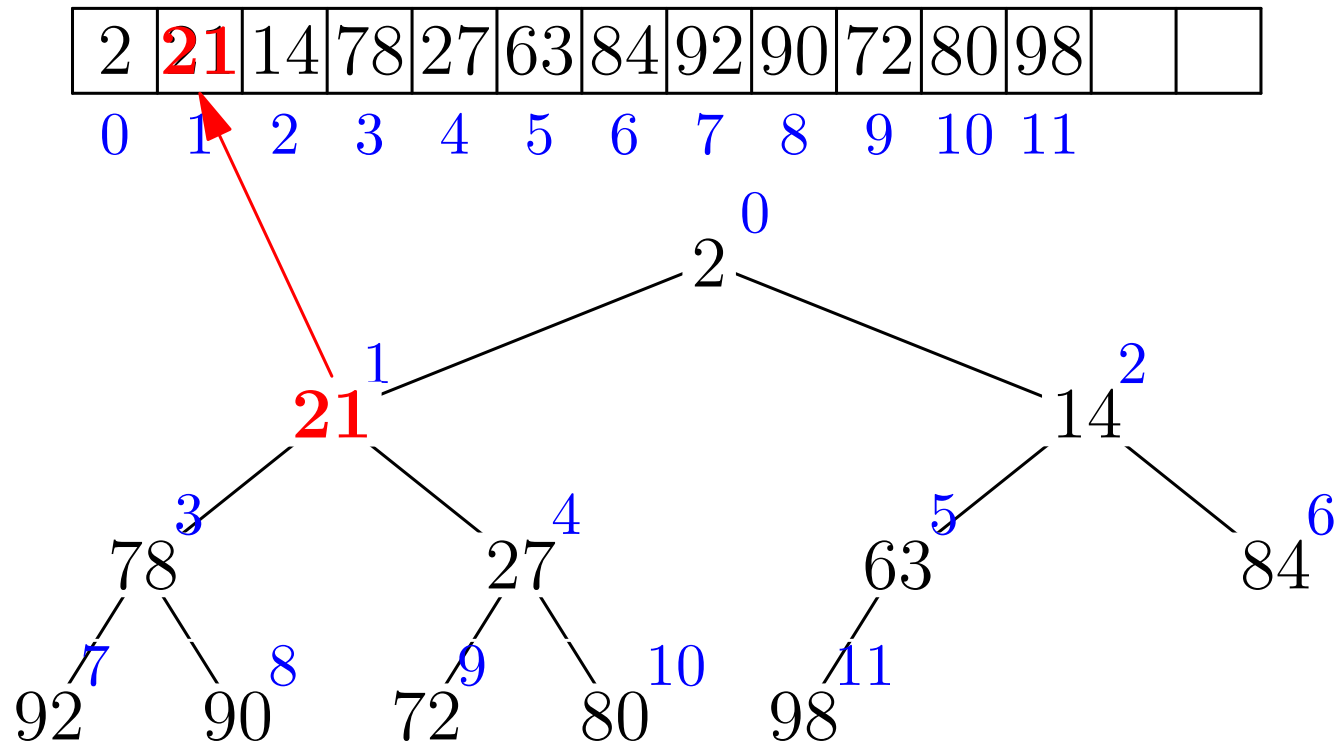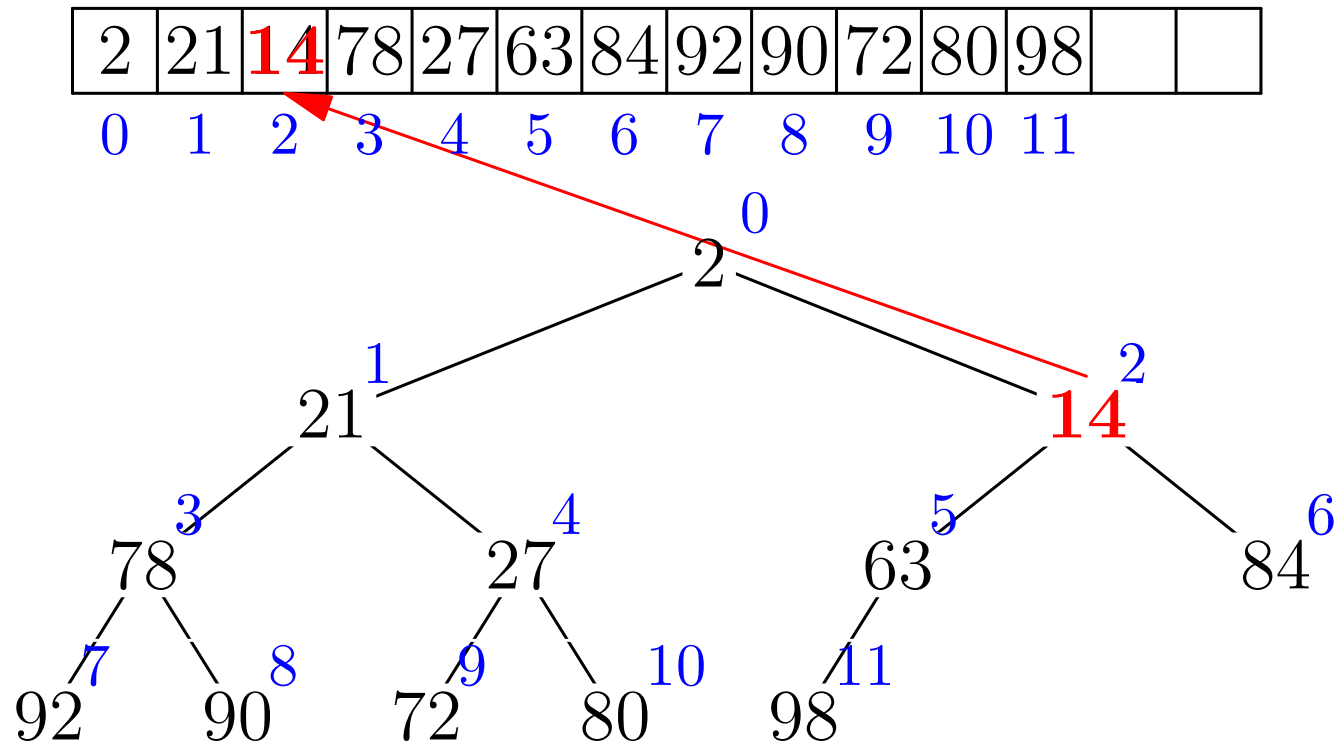
2	21	14	78	27	63	84	92	90	72	80	98		
0	1	2	3	4	5	6	7	8	9	10	11		

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Algorithms and Analysis

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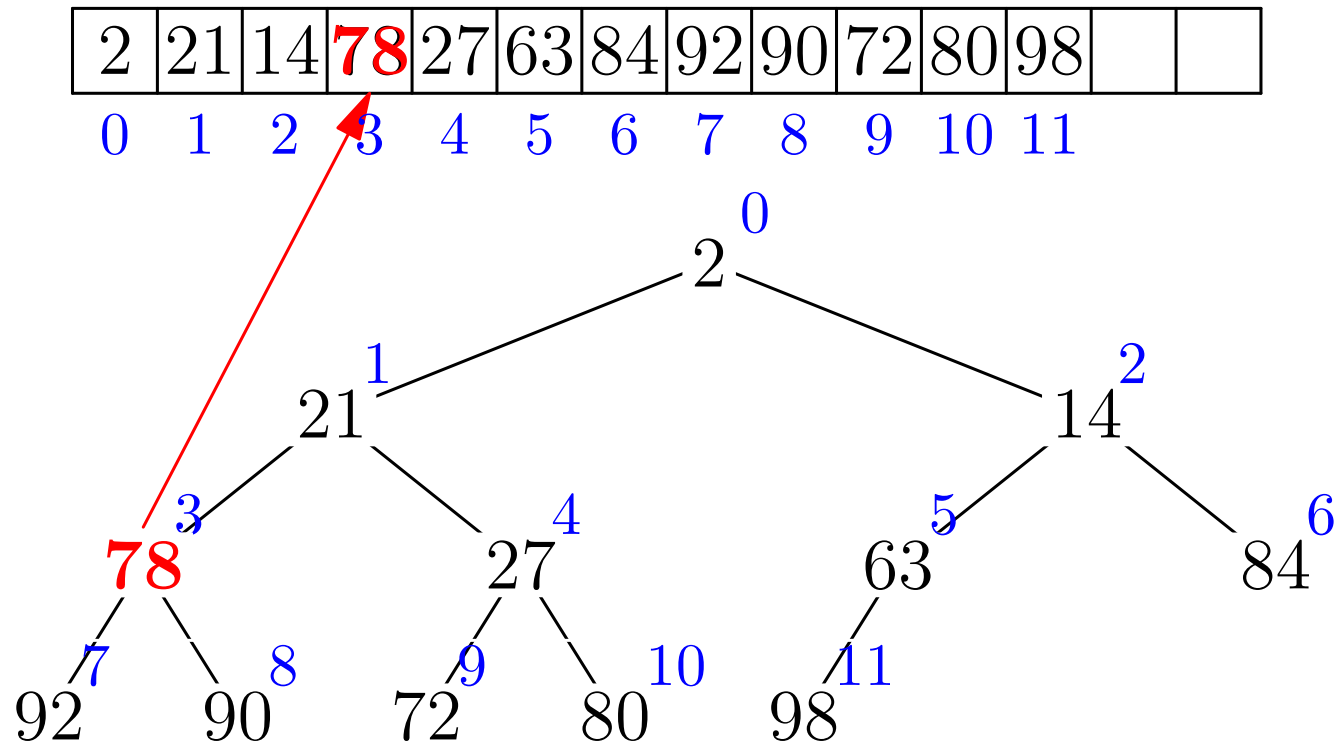
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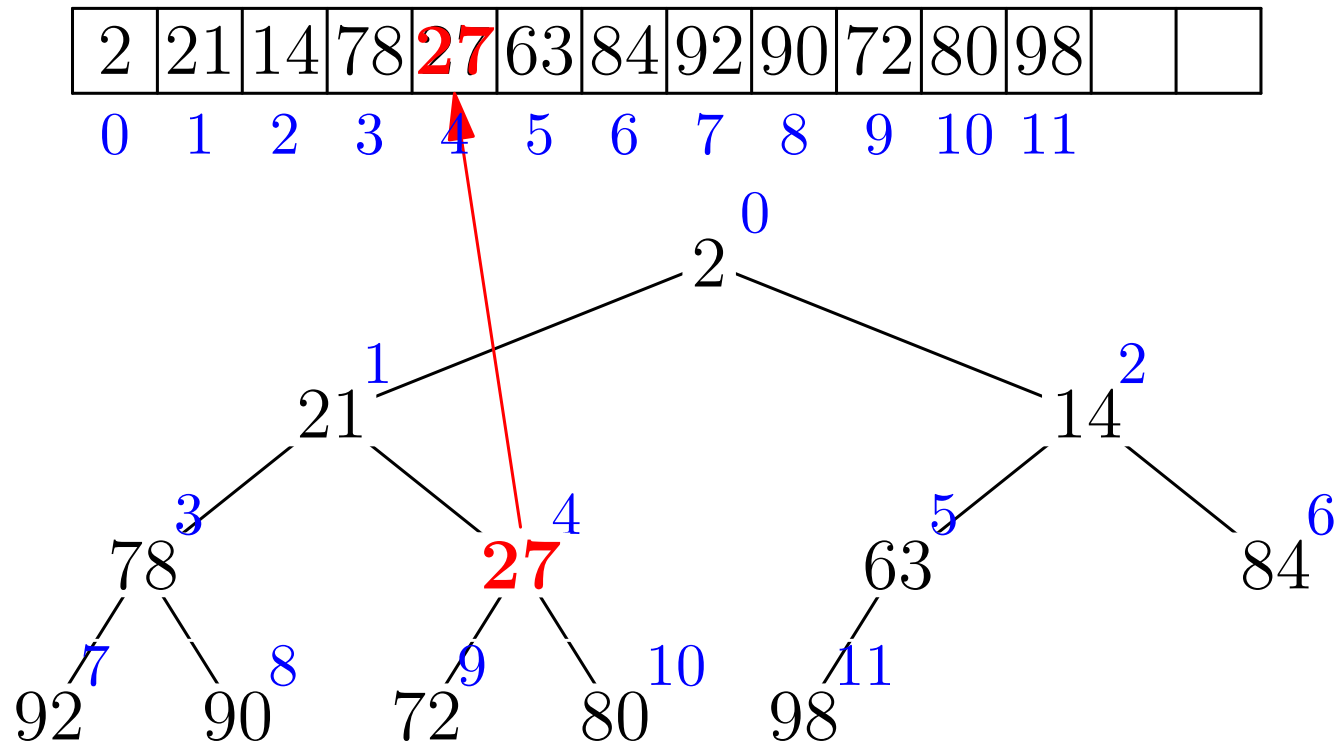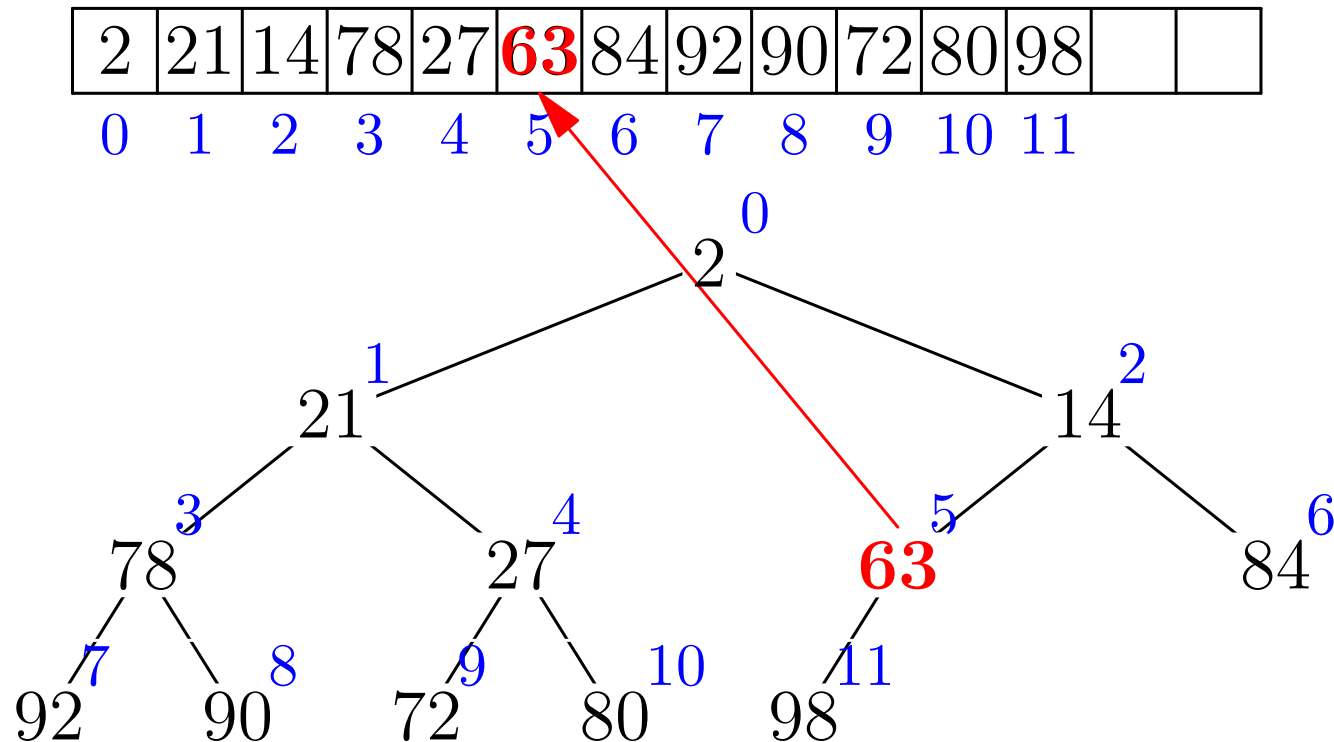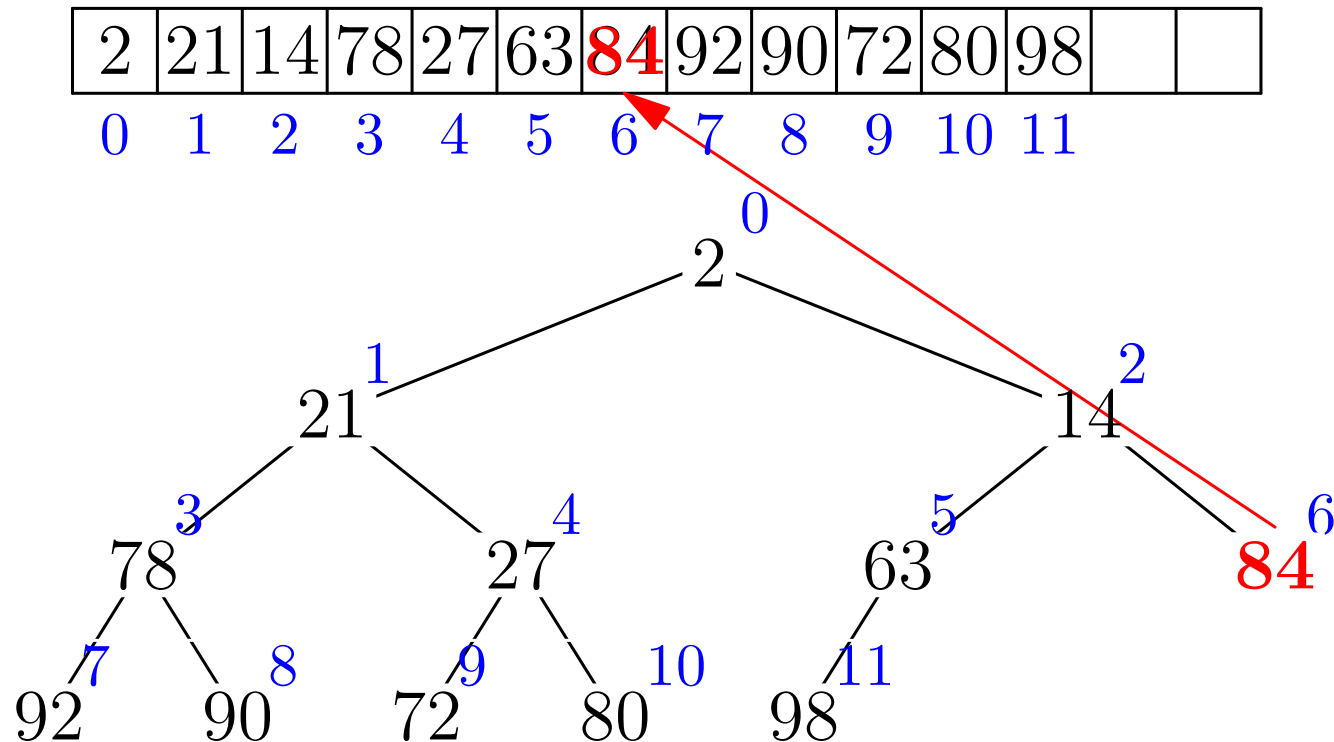
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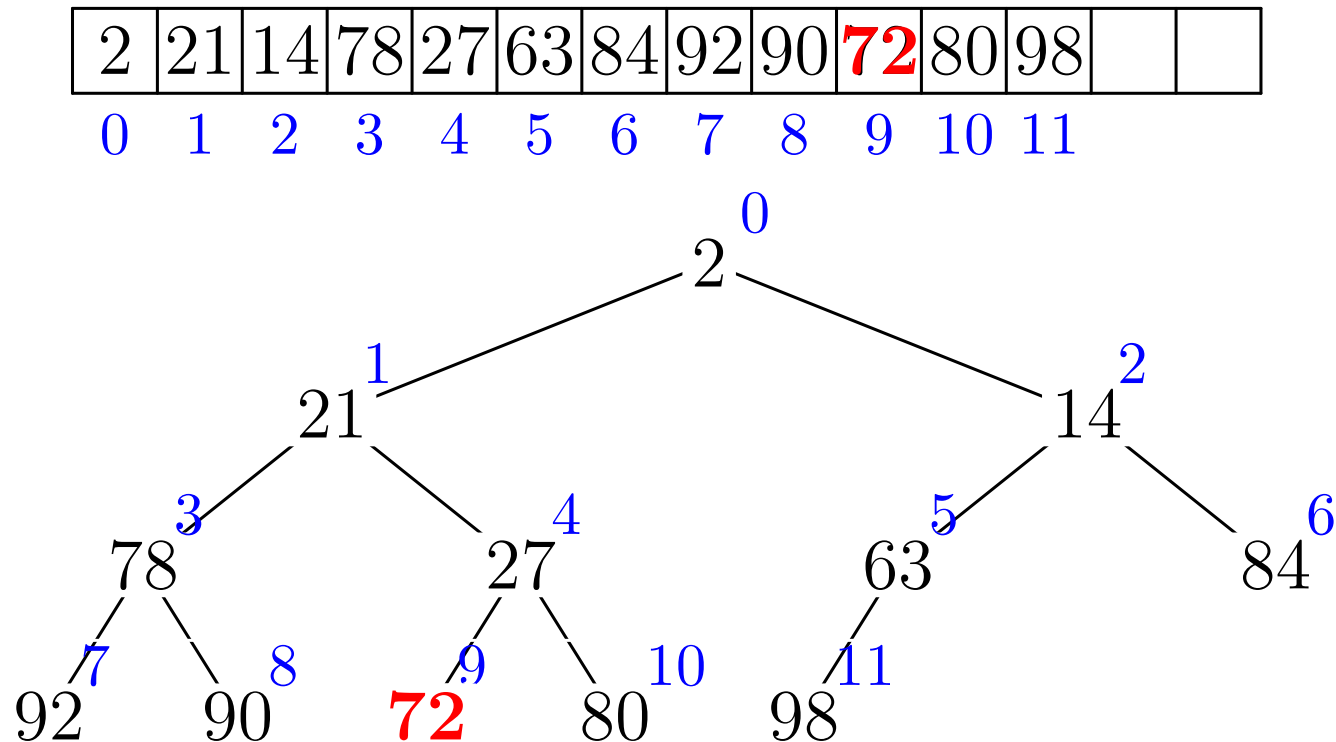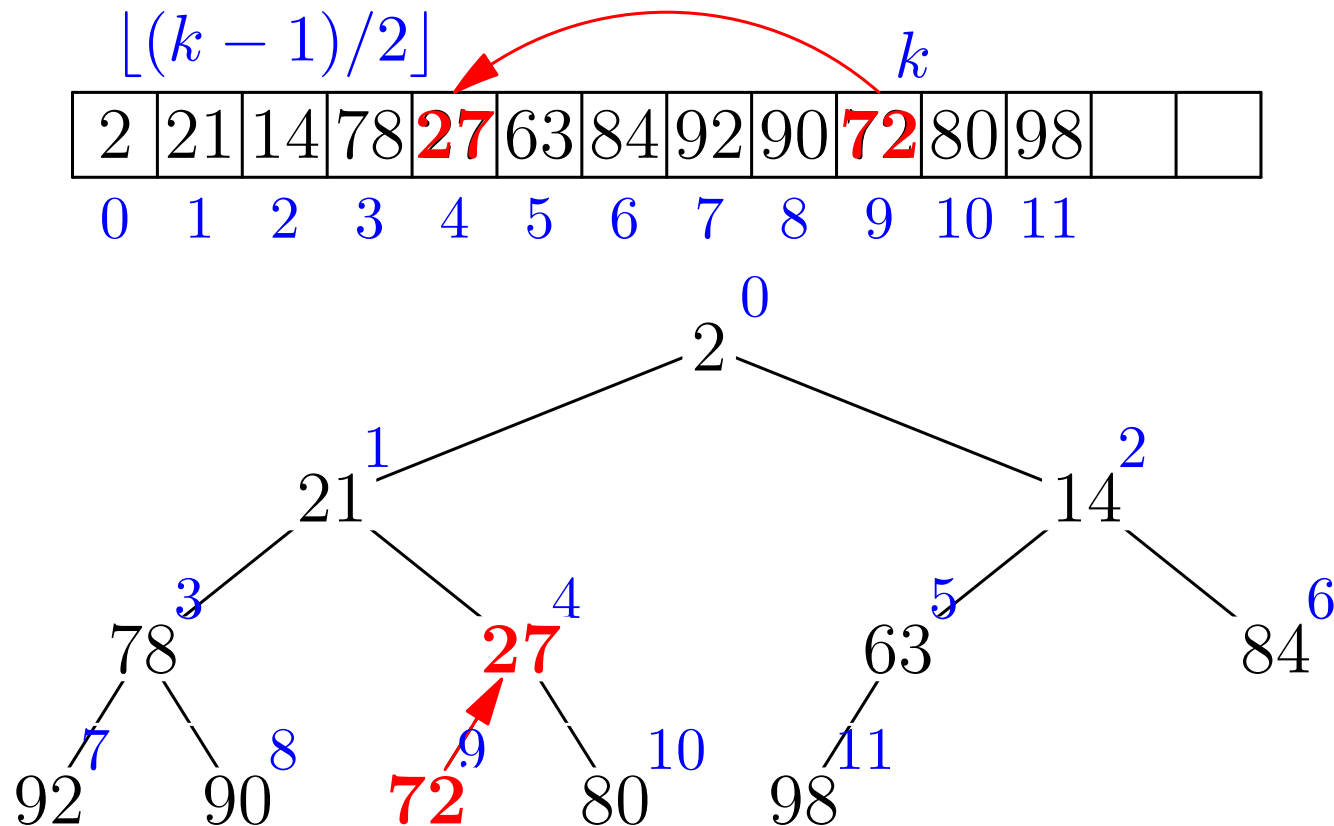- Because the tree is complete we can implemented the heap efficiently using an array
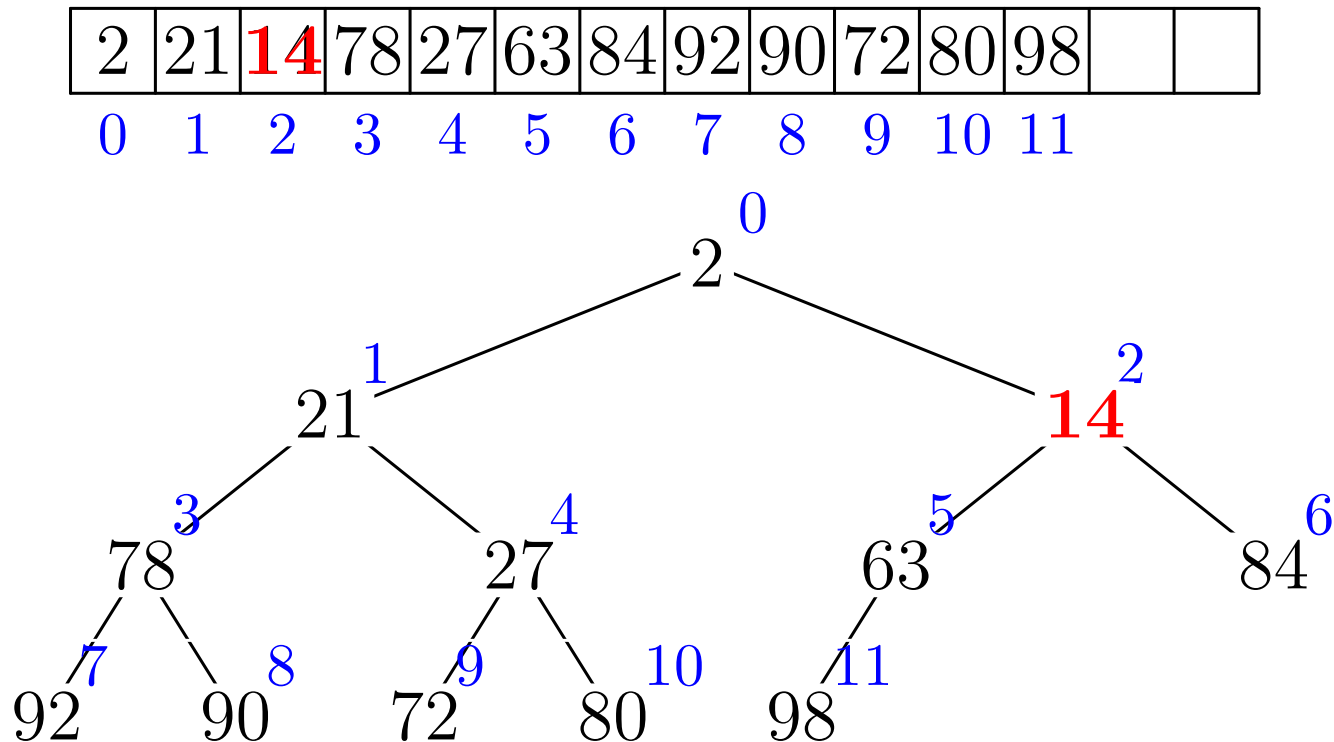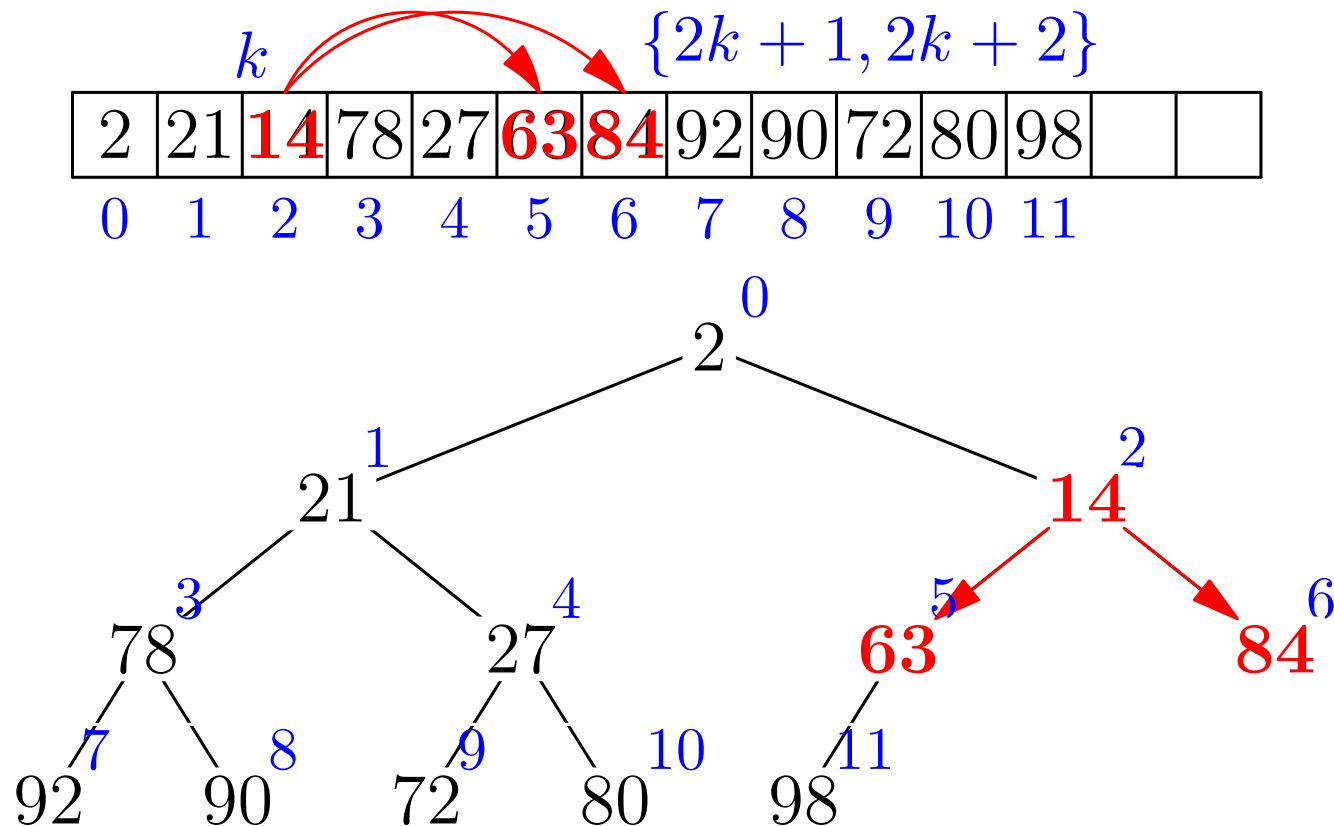
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$$\lfloor (k-1)/2 \rfloor \qquad\qquad k$$

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Code for a Priority Queue

```cpp
#include <vector>
using namespace std;

template <typename T, typename P>
class heapPQ {
private:
  vector<pair<T, P> > array;
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  unsigned size() {return array.size();}

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  const T& top() {return array[0].first;}
```

Adding an Element
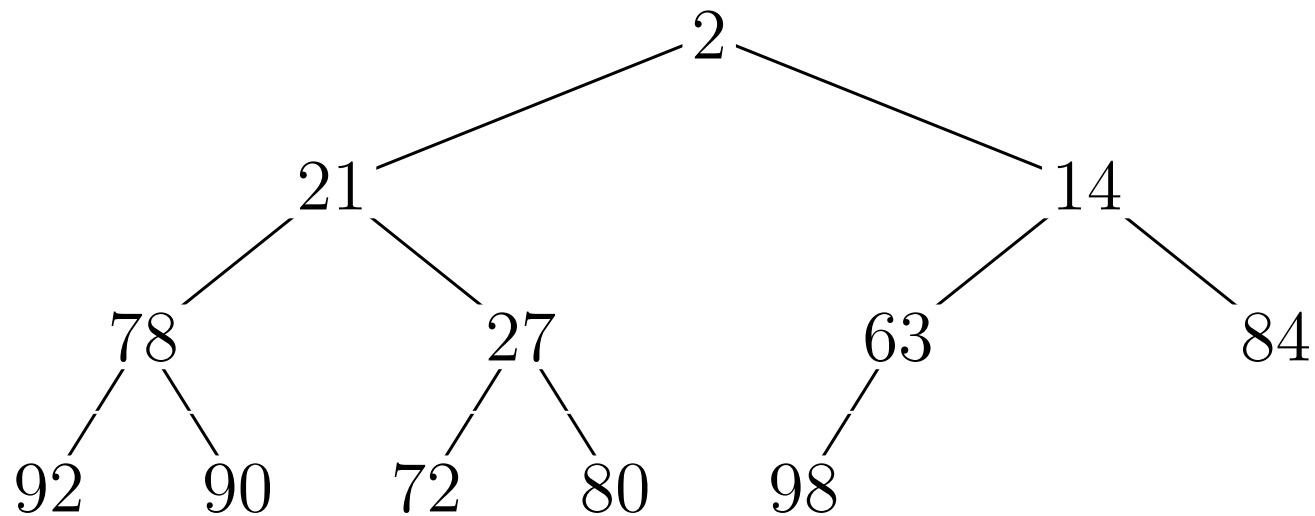
| 2 | 21 | 14 | 78 | 27 | 63 | 84 | 92 | 90 | 72 | 80 | 98 | | |

```
                          2
                 21               14
            78       27       63      84
          92  90   72  80   98
```

Adding an Element

2	21	14	78	27	63	84	92	90	72	80	98	**4**	

push(4)

```
                              2
                21                        14
          78          27            63          84
        92   90     72   80      98    4
```

Adding an Element

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```
                              2
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           78         27            63          84
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Adding an Element

| 2 | 21 | 14 | 78 | 27 | **63** | 84 | 92 | 90 | 72 | 80 | 98 | **4** | |

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```
                            2
                 21                      14
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Algorithms and Analysis
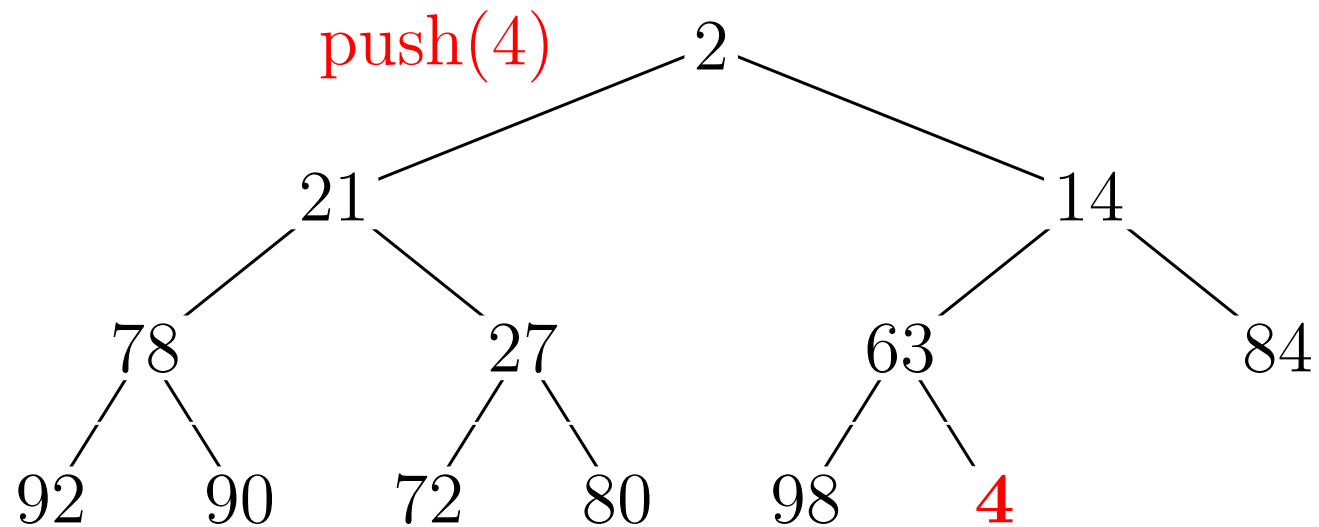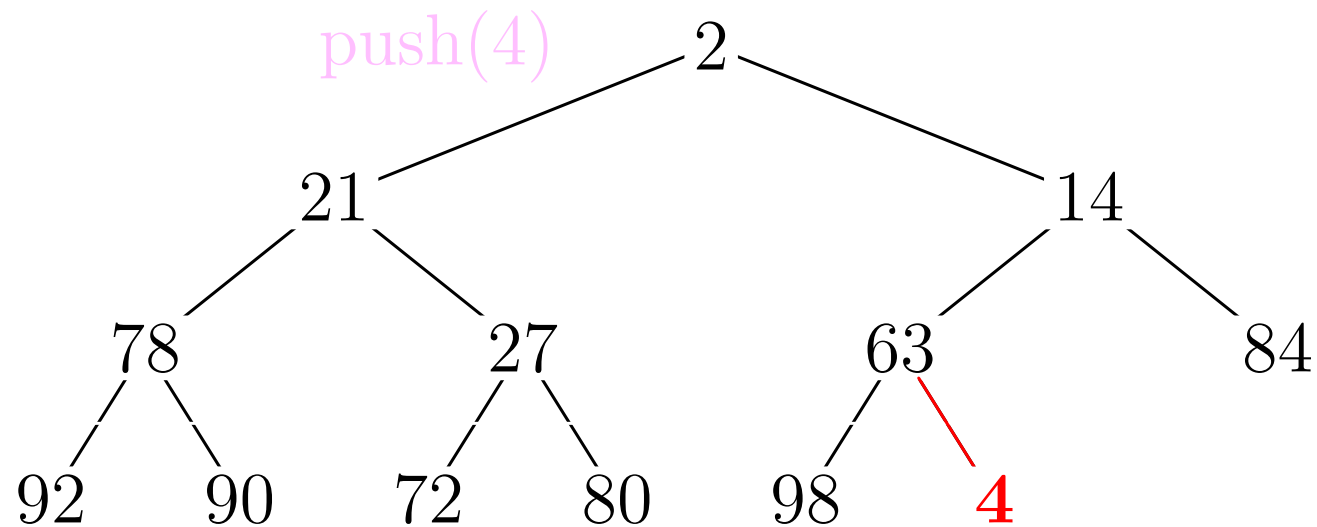
Adding an Element

Adding an Element

2	21	4	78	27	14	84	92	90	72	80	98	63	

push(4)

```
              2
           /     \
         21        4
        /  \      / \
      78    27  14    84
     / \    / \  / \
   92  90 72  80 98  63
```

Adding an Element

```cpp
void push(T value, P priority) {
  pair<T,P> tmp(value, priority);
  array.push_back(tmp);
  unsigned child = size() - 1;

  /* Percolate Up */
  while(child!=0) {
    unsigned parent = (child-1)>>1;   // floor((child-1)/2)
    if (array[parent].second < array[child].second)
      return;
    array[child] = array[parent];
    array[parent] = tmp;
    child = parent;
  }
}
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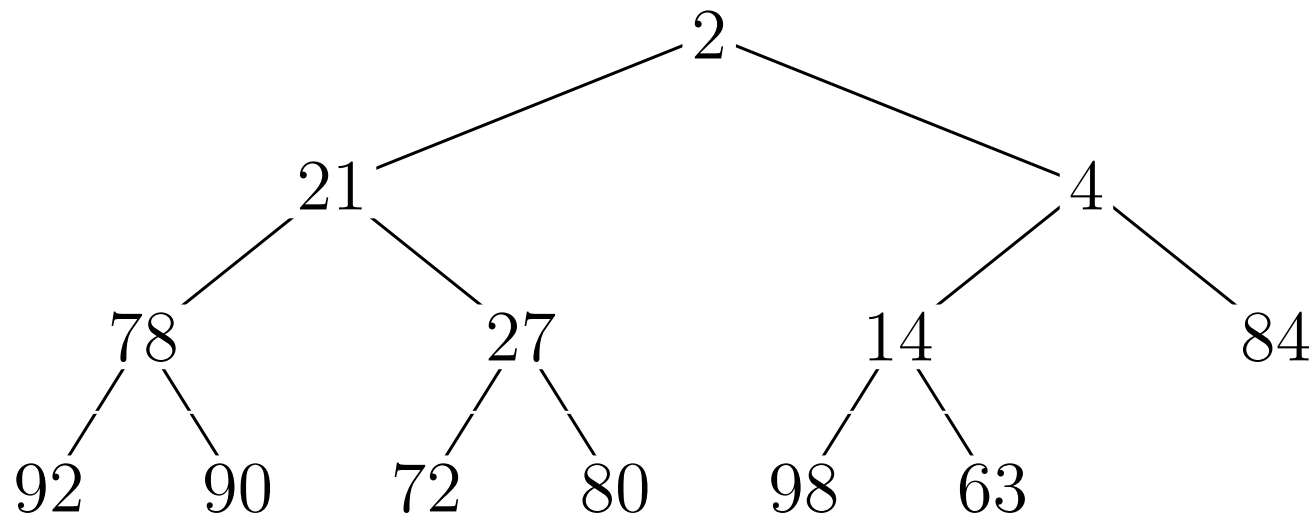
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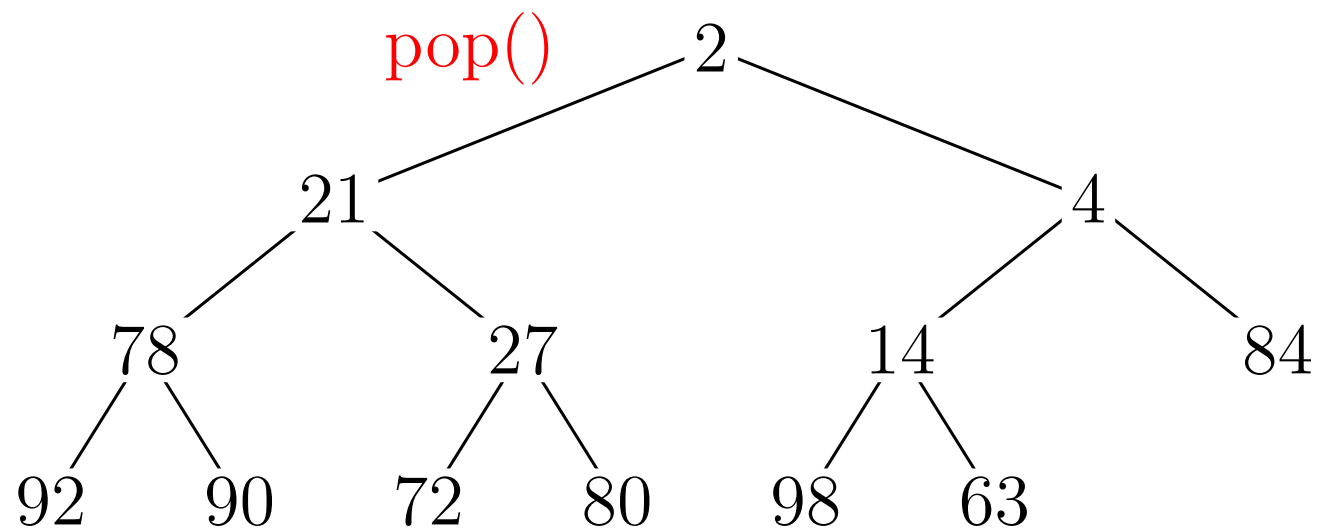
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}
```

Popping the Top

2	21	4	78	27	14	84	92	90	72	80	98	63	

Popping the Top

| 2 | 21 | 4 | 78 | 27 | 14 | 84 | 92 | 90 | 72 | 80 | 98 | 63 | |

pop()

Popping the Top

| 2 | 21 | 4 | 78 | 27 | 14 | 84 | 92 | 90 | 72 | 80 | 98 | **63** | |

pop()   **2**   2

21   4

78   27   14   84

92   90   72   80   98   **63**

Popping the Top

63	21	4	78	27	14	84	92	90	72	80	98		

pop()   **63**

```
                        63
              21                    4
         78       27           14        84
       92   90   72   80      98
```

Popping the Top

| **63** | **21** | **4** | 78 | 27 | 14 | 84 | 92 | 90 | 72 | 80 | 98 | | |

pop()   **63**

21   **4**

78   27   14   84

92   90   72   80   98

Popping the Top

Popping the Top

| 4 | 21 | **63** | 78 | 27 | 14 | 84 | 92 | 90 | 72 | 80 | 98 | | |

pop()

```
                        4
           21                        63
      78        27              14        84
   92    90   72    80       98
```

Popping the Top

Popping the Top

| 4 | 21 | **63** | 78 | 27 | **14** | **84** | 92 | 90 | 72 | 80 | 98 | | |

pop()

4
21   63
78   27   14   84
92 90 72 80 98

Popping the Top

| 4 | 21 | 14 | 78 | 27 | **63** | 84 | 92 | 90 | 72 | 80 | 98 | | |

pop()

```
                          4
              21                      14
          78      27            63        84
        92  90  72  80        98
```

Popping the Top

| 4 | 21 | 14 | 78 | 27 | **63** | 84 | 92 | 90 | 72 | 80 | **98** | | |

pop()

4
├── 21
│   ├── 78
│   │   ├── 92
│   │   └── 90
│   └── 27
│   ├── 72
│   └── 80
└── 14
    ├── **63**
    │   └── **98**
    └── 84

Popping the Top

```
void pop() {
unsigned parent = 0;
pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
```

Popping the Top

```
void pop () {
unsigned parent = 0;
pair<T, P> tmp = array.back ();
array[0] = tmp;
array.pop_back ();
unsigned child = 1;

/* Percolate down */
while (child<size ()) {
```

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pair<T, P> tmp = array.back();
array[0] = tmp;
array.pop_back();
unsigned child = 1;

/* Percolate down */
while(child<size()) {
  if (child+1<=size() && array[child+1].second < array[child].second)
    ++child;
```

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array.pop_back();
unsigned child = 1;

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    ++child;
  if (array[child].second > array[parent].second)
    return;
```

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array.pop_back();
unsigned child = 1;

/* Percolate down */
while(child<size()) {
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    ++child;
  if (array[child].second > array[parent].second)
    return;
  array[parent] = array[child];
  array[child] = tmp;
  parent = child;
  child = 2*parent + 1;
}
}
```
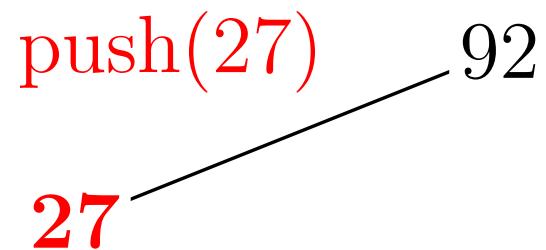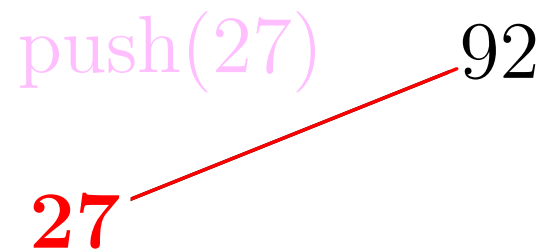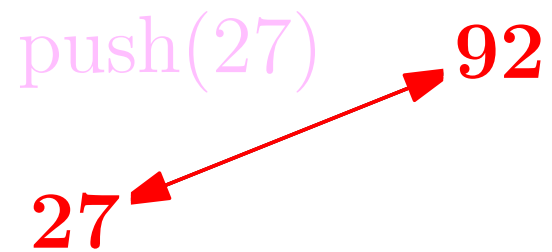
Heaps in Action

Algorithms and Analysis

Heaps in Action

92														

$$\text{push}(92) \qquad 92$$

Heaps in Action

92	**27**												

$$\text{push}(27) \quad 92$$

27

Heaps in Action

92	**27**												

push(27)   92

27

Heaps in Action

92 27

push(27)

92

27

Heaps in Action

27	92	**98**												

push(98)

```
        27
       /   \
     92     98
```

Heaps in Action

27	92	**98**												

push(98)

```
        27
       /  \
      92   98
```

Heaps in Action

27	92	**98**												

push(98)

27

92

98

Heaps in Action

27	92	98	**72**										

push(72)   27

         92   98

  72

Heaps in Action

27 | 92 | 98 | **72** | | | | | | | | | | |

push(72)

27
92   98
72

Heaps in Action

$$27 \quad \mathbf{92} \quad 98 \quad \mathbf{72}$$

push(72)

```
            27
           /    \
         92       98
        /
      72
```

Heaps in Action

27	**72**	98	92										

push(72)

27

72

98

92

Heaps in Action

27	72	98	92	**2**										

$\text{push}(2)$

```
           27
          /    \
        72      98
       /  \
     92    2
```

Heaps in Action

27	72	98	92	**2**										

push(2)   27
                  /   \
              72   98
             /   \
           92   **2**

Heaps in Action

| 27 | **72** | 98 | 92 | **2** | | | | | | | | | |

push(2)

27

72

98

92

2

Heaps in Action

Heaps in Action

2	27	98	92	72	**84**								

push(84)

2
27
98
92
72
84

Heaps in Action

2	27	98	92	72	**84**									

push(84)

```
              2
         27        98
      92    72   84
```

Heaps in Action

2	27	**98**	92	72	**84**										

push(84)

```
              2
             / \
           27   98
          /  \    \
        92    72   84
```

Heaps in Action

| 2 | 27 | 84 | 92 | 72 | 98 | | | | | | | | | |

push(84)

```
          2
        /    \
      27       84
     /  \        \
   92    72       98
```

Heaps in Action

2	27	84	92	72	98	**14**							

push(14)

```
                           2
                27                    84
          92         72         98         14
```

Heaps in Action

2	27	84	92	72	98	**14**								

push(14)

Heaps in Action

Heaps in Action

Heaps in Action

2	27	14	92	72	98	84								

pop()
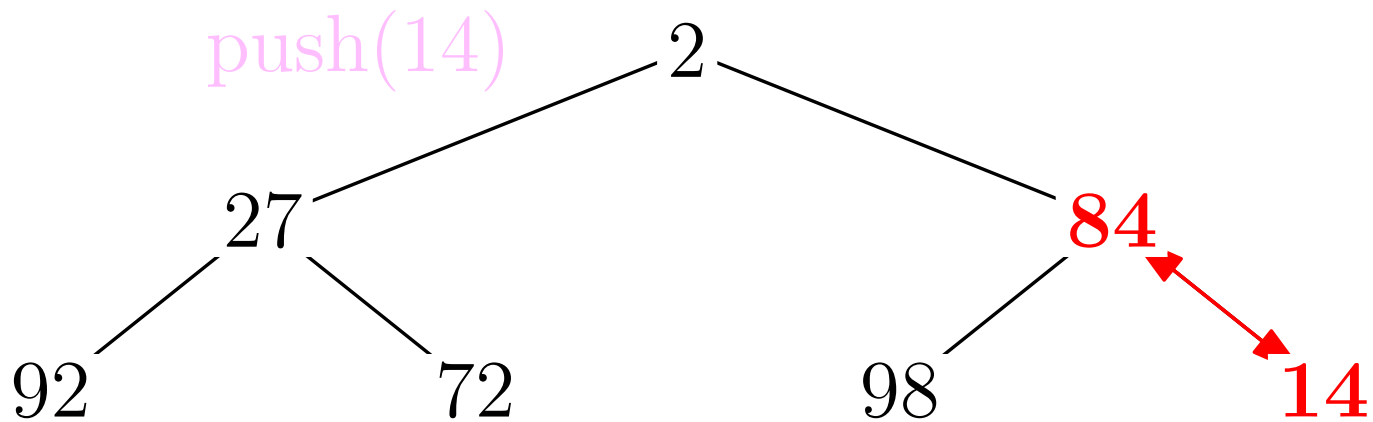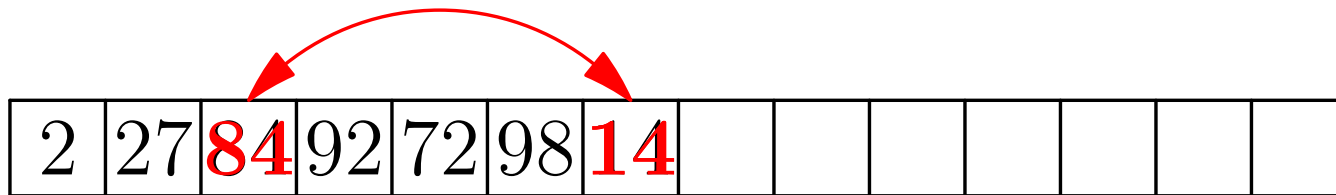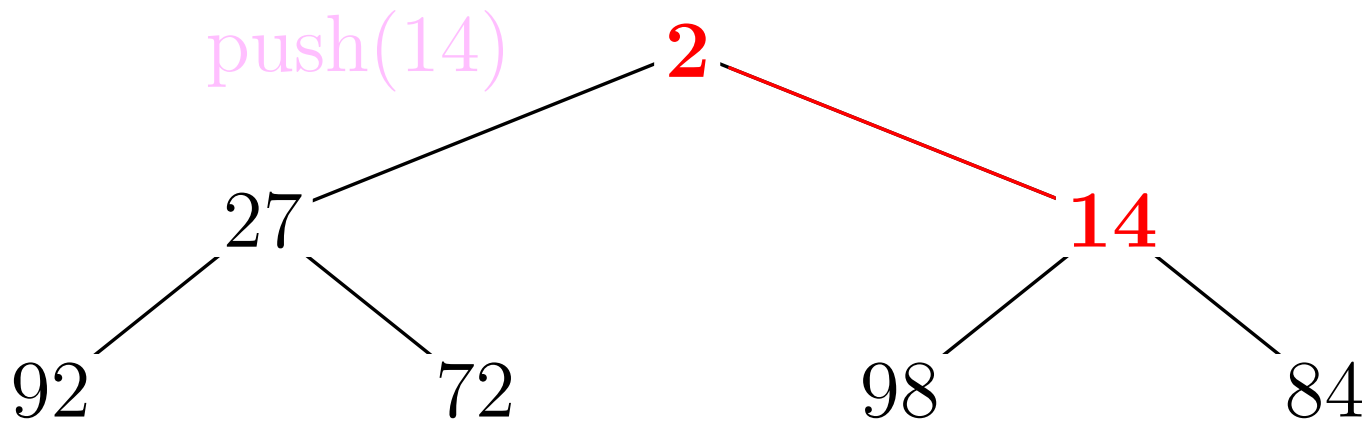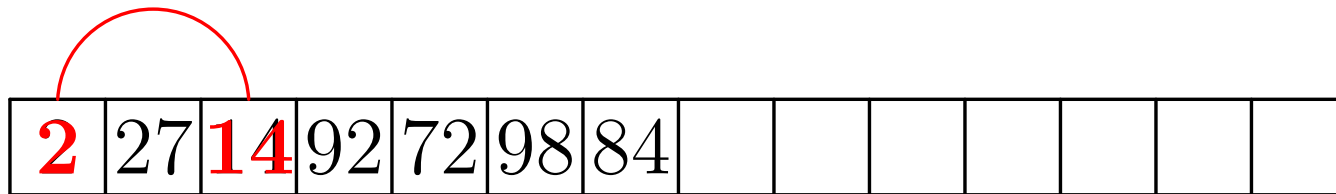
```
              2
        27         14
      92   72    98   84
```

Heaps in Action

2	27	14	92	72	98	**84**								

pop()

Heaps in Action

| 84 | 27 | 14 | 92 | 72 | 98 | | | | | | | | | |

pop()   **84**

```
              84
          /        \
        27          14
       /  \           \
     92    72          98
```

Heaps in Action

84	27	14	92	72	98									

pop()

84
 / \
27   14
/ \   \
92 72 98

Heaps in Action

| 84 | 27 | 14 | 92 | 72 | 98 | | | | | | | | | | |

pop()

84

27   14

92   72   98

Heaps in Action

14	27	**84**	92	72	98									

pop()

```
              14
         27        84
      92    72   98
```

Heaps in Action

14	27	**84**	92	72	**98**										

pop() 14

27   **84**

92   72   **98**

Heaps in Action

14	27	84	92	72	98										

pop()

```
                    14
          27                    84
      92      72            98
```

Heaps in Action

14	27	84	92	72	**98**									

14

pop() **14**

27   84

92   72   **98**

Heaps in Action

98	27	84	92	72										

pop()

98

27

84

92

72

Heaps in Action

98	**27**	**84**	92	72										

pop()   **98**
                    /   \
              27   **84**
             /   \
           92   72

Heaps in Action

98	27	84	92	72											

pop()

```
        98
      /    \
    27      84
   /  \
  92   72
```

Heaps in Action

27	**98**	84	92	72										

pop()

```
            27
           /  \
        98      84
       /  \
     92    72
```

Heaps in Action

| 27 | **98** | 84 | **92** | **72** | | | | | | | | | |

pop()   27

98   84

92   **72**

Heaps in Action

27	**98**	84	**92**	**72**										

pop()   27

98   84

92   **72**

Heaps in Action

27	72	84	92	98										

Heaps in Action

27	72	84	92	**98**										

pop()

27

27

72

84

92

98

Heaps in Action

98	72	84	92										

pop() **98**

72

84

92

Heaps in Action

98	72	84	92										

pop() 98

72

92

84

Heaps in Action

98	**72**	**84**	92										

pop()

98

72

84

92

Heaps in Action

72	**98**	84	92											

pop()   72

98   84

92

Heaps in Action

| 72 | **98** | 84 | **92** | | | | | | | | | | | |

pop()

72

98

84

92

Heaps in Action

72	**98**	84	**92**										

pop()   72

98

92

84

Heaps in Action

72	92	84	98										

pop()

```
              72
          /        \
        92          84
       /
     98
```

Heaps in Action

| **72** | 92 | 84 | **98** | | | | | | | | | | | |

pop()

72

72

92

84

98

Heaps in Action

98	92	84												

pop()   **98**

92   84

Heaps in Action

98	92	84											

pop()

```
        98
       /  \
     92    84
```

Heaps in Action

| 98 | 92 | 84 | | | | | | | | | | | |

pop()

98

92

84

Heaps in Action

84	92	98												

pop()

```
          84
         /  \
       92    98
```

Heaps in Action

Heaps in Action

98	92												

pop()   98

92

Heaps in Action

98	92												

pop()   98

92

Heaps in Action

98	92												

pop() 98

92

Heaps in Action

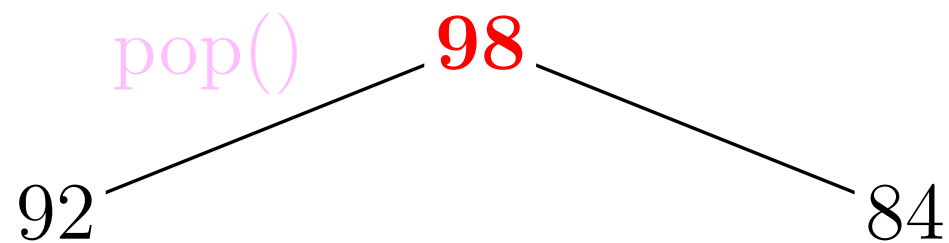| 92	98													

pop() 92

98

Heaps in Action

Heaps in Action

98													

pop()   98

Heaps in Action

98

pop()   **98**   98

Time Complexity of Heaps

- The two important operation are `add` and `removeMin`

- These both either percolating an element up the tree or percolating an element down the tree

- The number of operations depends on the depth of the tree which is $\Theta(\log(n))$

- Thus `add` and `removeMin` are $O(\log(n))$

- Except `add` could also require resizing the array, but the amortised cost of this is low

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Algorithms and Analysis

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Real Time Simulation

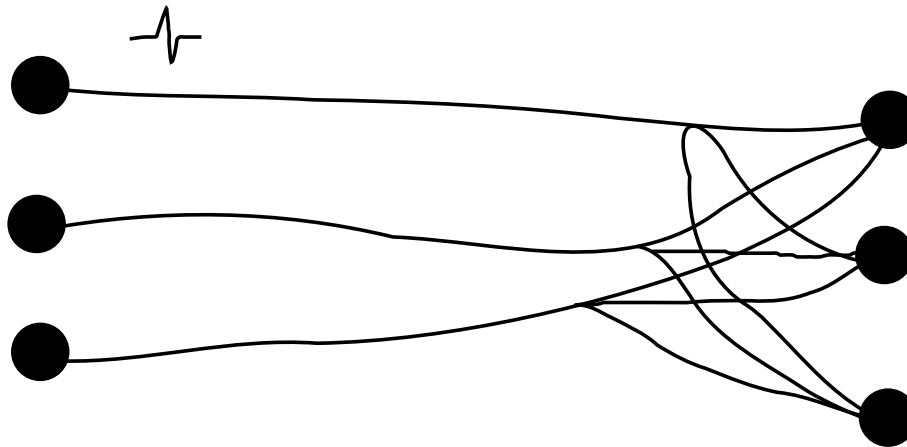- A nice application of priority queues is to perform real time simulations

- I was once modelling a neural network where neuron fired an impulse which would then be received by other neurons

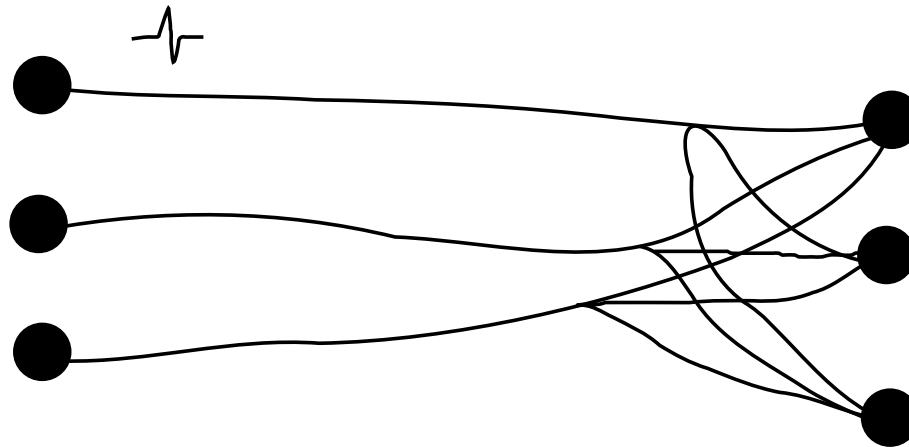---

Algorithms and Analysis

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Synchronised Firing

- We wanted to show that if a group of neurons fired together they could make another group of neurons fire in synchrony despite the fact that it would take different times for the receiving neurons to feel the pulse (due to the different lengths of the axons)

- A famous Israeli group had "proved" this couldn't happen

- Using a priority queue we modelled the neurons

  ⋆ When a neuron fired the receiving neurons would be put on a priority queue according to when they received the pulse
  ⋆ If the receiving neurons received enough pulse in a short enough time they would then fire

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Efficient Modelling

• Using a priority queue meant we knew when the next event would happen

• We did not have to run a clock where most of the time nothing happened

• This allowed us to perform a very large simulation efficiently

• The simulation showed that the pulse of neurons synchronised despite the "proof" that this wouldn't happen

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Outline

1. Heaps

2. Priority Queues

   • Array Implementation

3. **Heap Sort**

4. Other Heaps

Heap Sort

- A priority queue suggests a very simple way of performing sort

- We simply add elements to a heap and then take them off again

```
template <typename T>
void sort(vector<T> aList)
{
    HeapPQ<T> aHeap = new HeapPQ<T>(aList.size());
    for (T element: aList)
      aHeap.push(element, element);

    aList.clear();
    while(aHeap.size() > 0) {
      aList.push_back(aHeap.top());
      aHeap.pop();
    }
}
```

- Note that this is not an in-place sort algorithm (i.e. it uses lots of memory)

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Example of Heap Sort

92	27	98	72	2	84	14	90	78	21	80	63

Example of Heap Sort

92	27	98	72	2	84	14	90	78	21	80	63

92

Example of Heap Sort

| 92 | 27 | 98 | 72 | 2 | 84 | 14 | 90 | 78 | 21 | 80 | 63 |

```
        27
   92
```

Example of Heap Sort

92	27	98	72	2	84	14	90	78	21	80	63

```
            27
          /    \
       92        98
```

Example of Heap Sort

| 92 | 27 | 98 | 72 | 2 | 84 | 14 | 90 | 78 | 21 | 80 | 63 |

```
              27
            /    \
          72      98
         /
       92
```

Example of Heap Sort

| 92 | 27 | 98 | 72 | 2 | 84 | 14 | 90 | 78 | 21 | 80 | 63 |

Example of Heap Sort

| 92 | 27 | 98 | 72 | 2 | 84 | 14 | 90 | 78 | 21 | 80 | 63 |

Example of Heap Sort

| 92 | 27 | 98 | 72 | 2 | 84 | 14 | 90 | 78 | 21 | 80 | 63 |

```
                    2
          27                 14
      92      72          98      84
```

Example of Heap Sort

$$\boxed{92}\boxed{27}\boxed{98}\boxed{72}\boxed{2}\boxed{84}\boxed{14}\boxed{\textcolor{red}{90}}\boxed{78}\boxed{21}\boxed{80}\boxed{63}$$

Example of Heap Sort

92	27	98	72	2	84	14	90	78	21	80	63

Example of Heap Sort

| 92 | 27 | 98 | 72 | 2 | 84 | 14 | 90 | 78 | 21 | 80 | 63 |

```
                    2
              ┌─────┴─────────────┐
             21                   14
          ┌───┴───┐           ┌────┴────┐
         78       27         98         84
        ┌─┴─┐    ┌┘
       92  90   72
```

Example of Heap Sort

92	27	98	72	2	84	14	90	78	21	80	63

Example of Heap Sort

92	27	98	72	2	84	14	90	78	21	80	63

```
                           2
                 21                  14
           78        27        63         84
         92  90    72  80    98
```

Example of Heap Sort

Example of Heap Sort

Example of Heap Sort

| 2 | | | | | | | | | | | | |

```
                      14
             21                63
        78        27      98        84
      92  90    72  80
```

Example of Heap Sort

2												

Example of Heap Sort
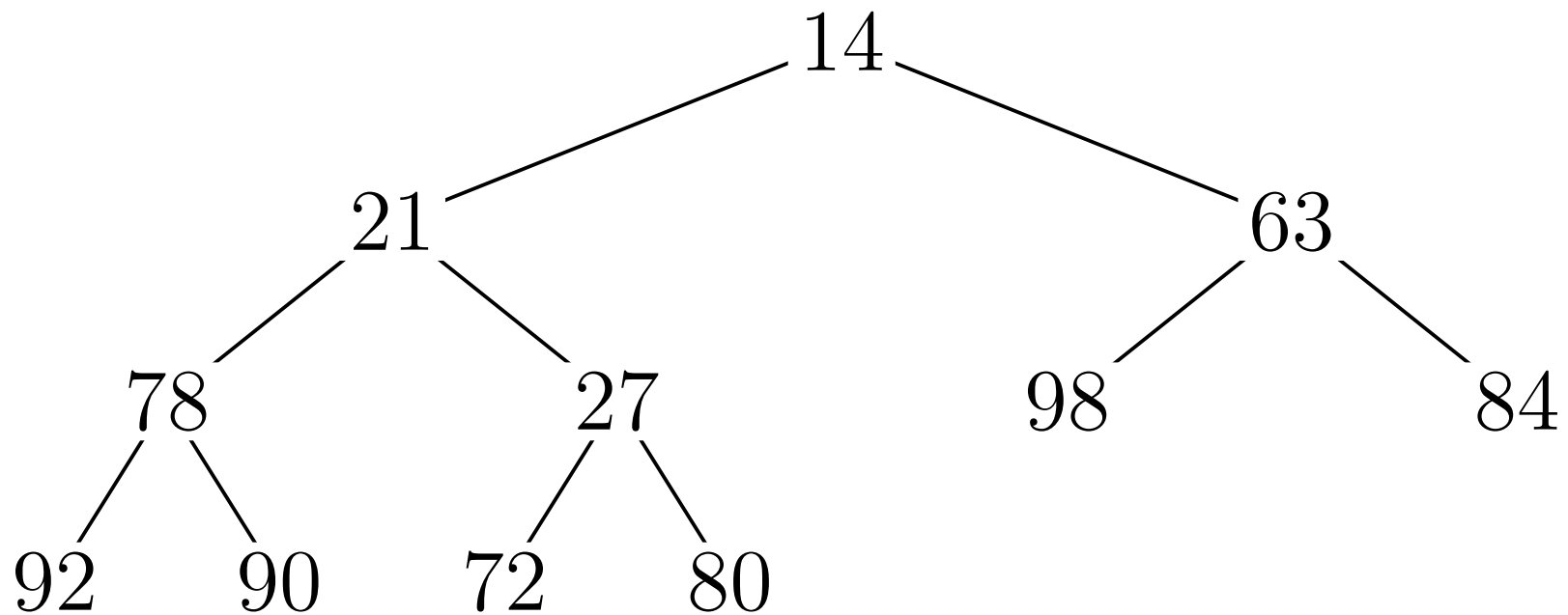
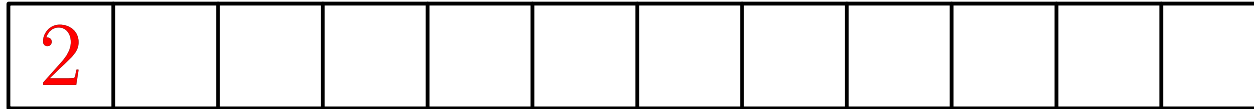| 2	14											

Example of Heap Sort

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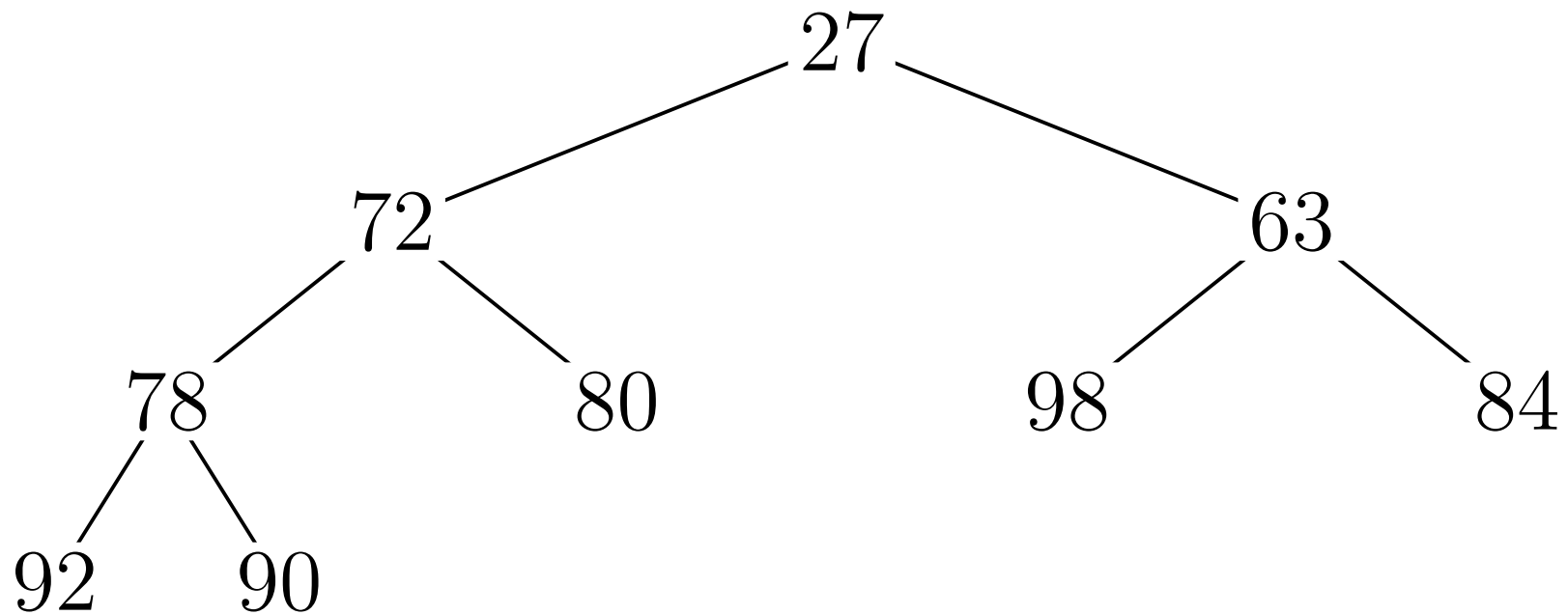
| 2 | 14 | 21 | | | | | | | | | | |

Example of Heap Sort
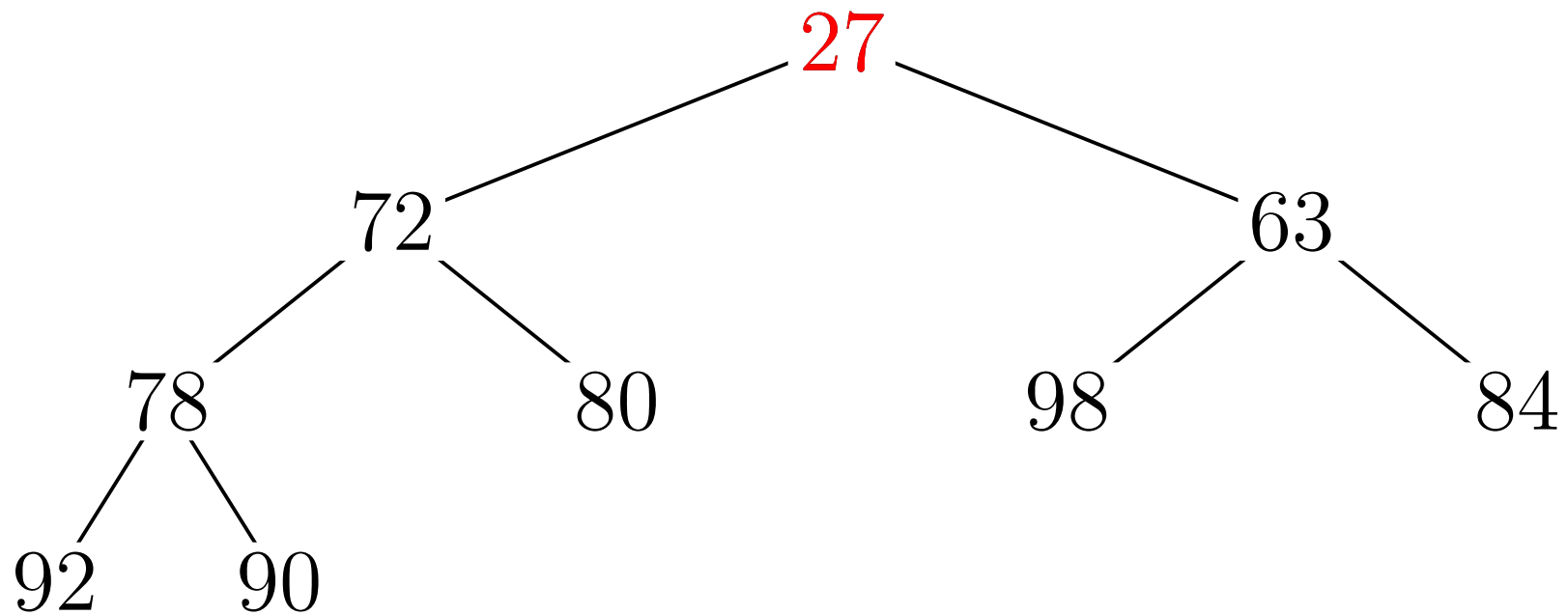
2	14	21										

Example of Heap Sort

| 2 | 14 | 21 | 27 | | | | | | | | |

Example of Heap Sort
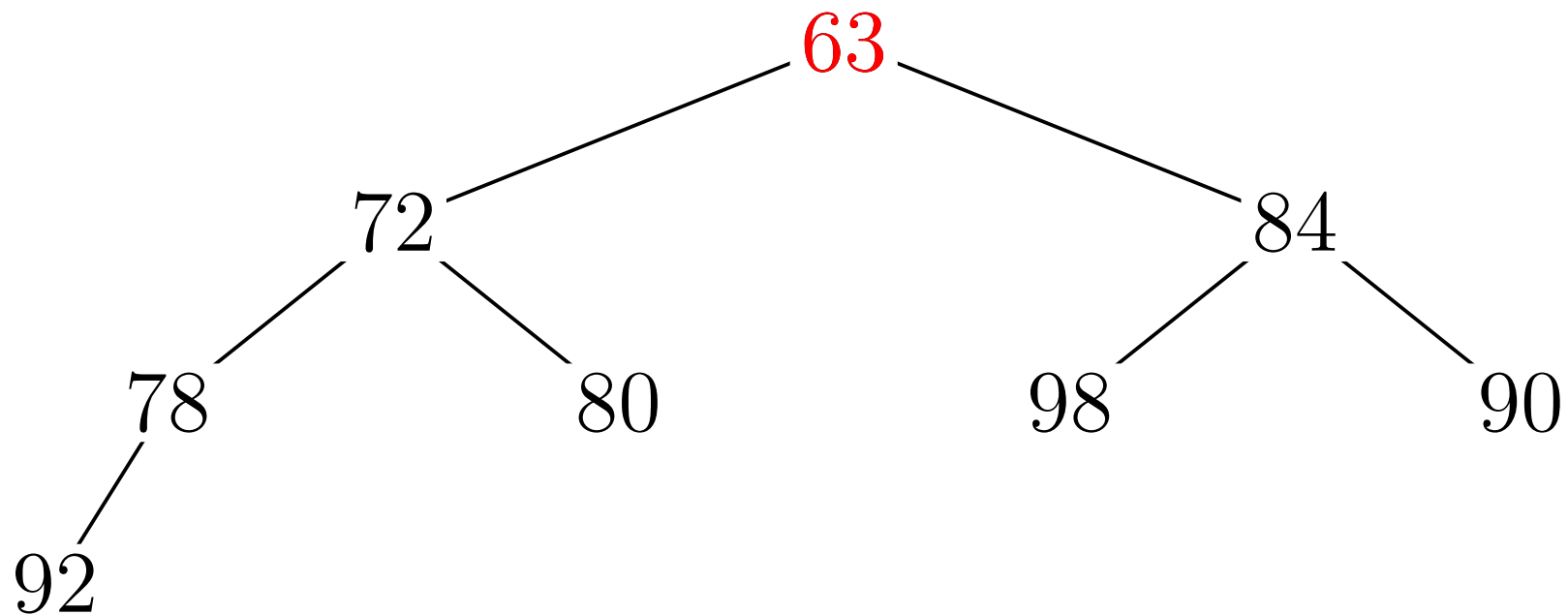
| 2 | 14 | 21 | 27 | | | | | | | | | |

```
              63
         /          \
       72            84
      /  \          /  \
    78    80      98    90
   /
  92
```

Algorithms and Analysis

Example of Heap Sort

2	14	21	27	63									

```
            72
          /    \
        78      84
       /  \     /  \
     92    80  98    90
```

Example of Heap Sort

2	14	21	27	63								

72

78   84

92   80   98   90

Example of Heap Sort

2	14	21	27	63	72						

```
            78
         /      \
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      /  \        /
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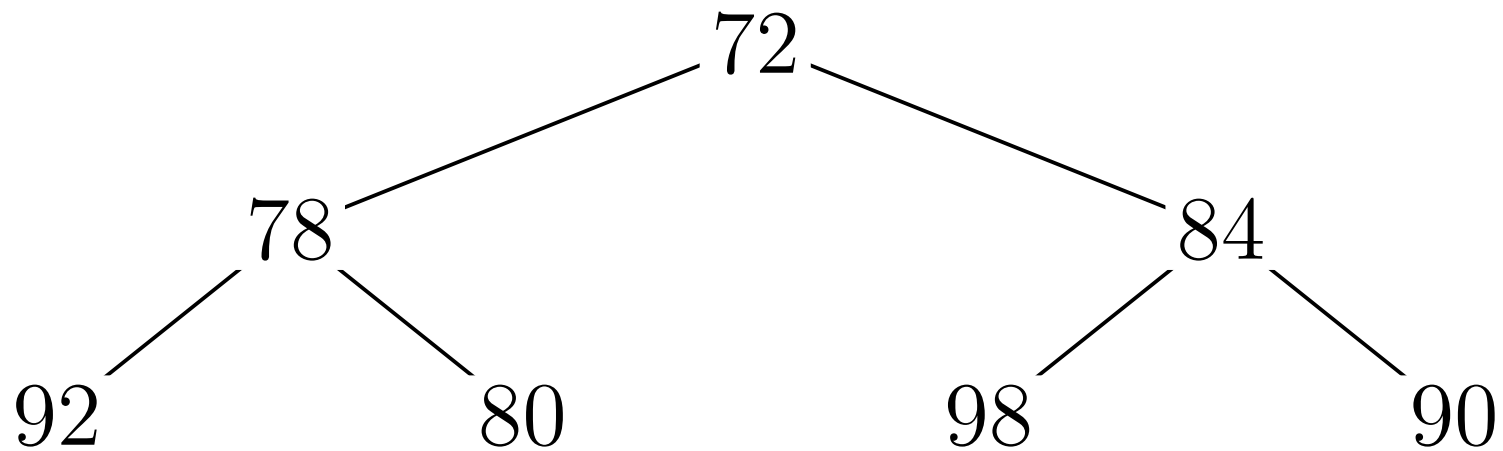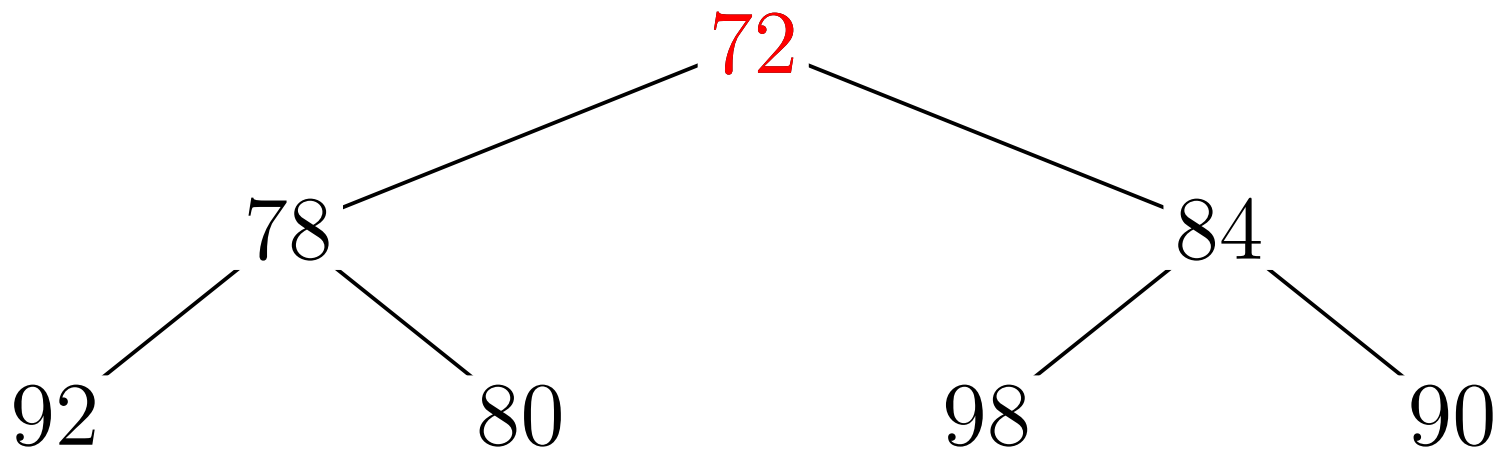
Example of Heap Sort

2	14	21	27	63	72						

Example of Heap Sort

| 2 | 14 | 21 | 27 | 63 | 72 | 78 | | | | | |

```
            80
          /    \
        90      84
       /  \
     92    98
```
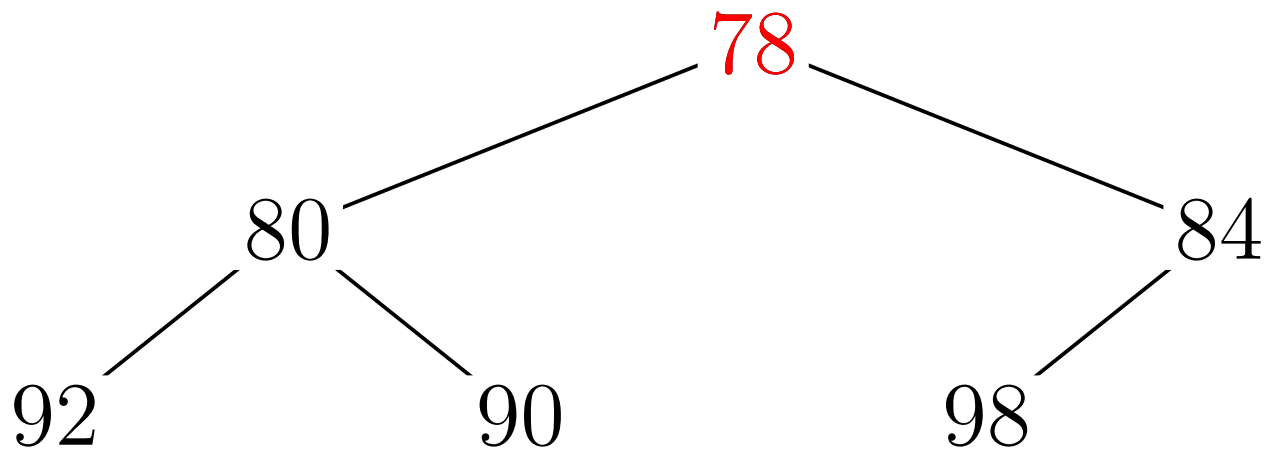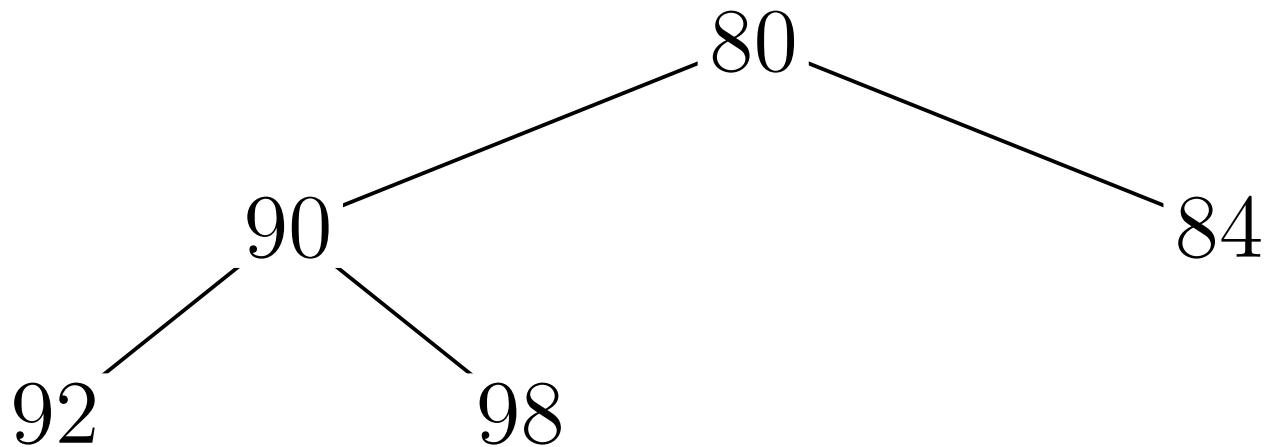
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```
              84
       90              98
    92
```
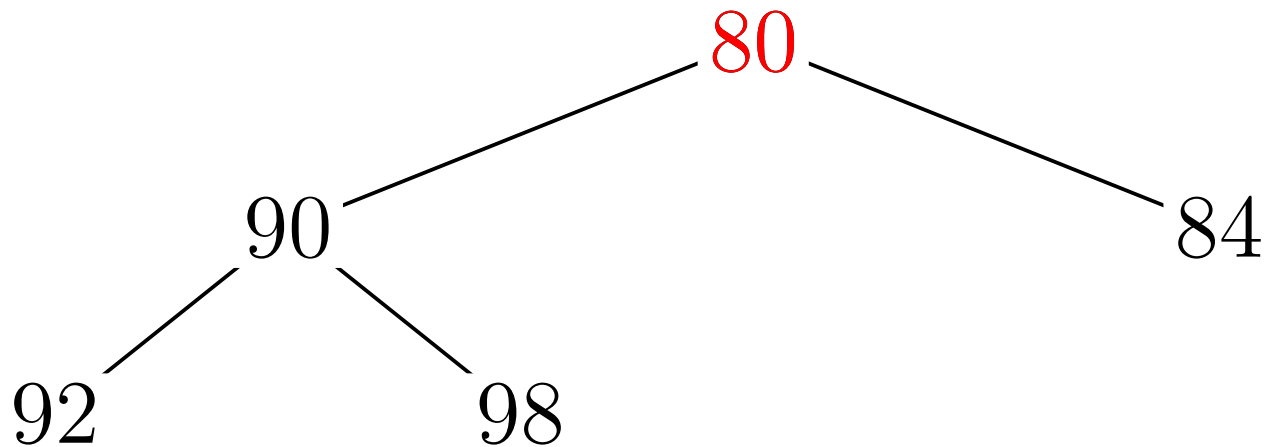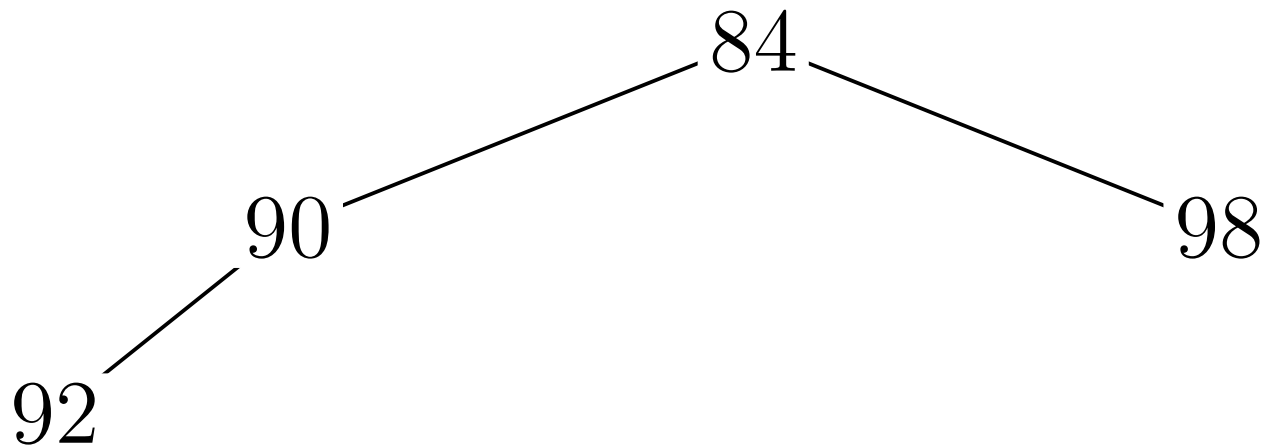
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```
            90
          /      \
       92          98
```
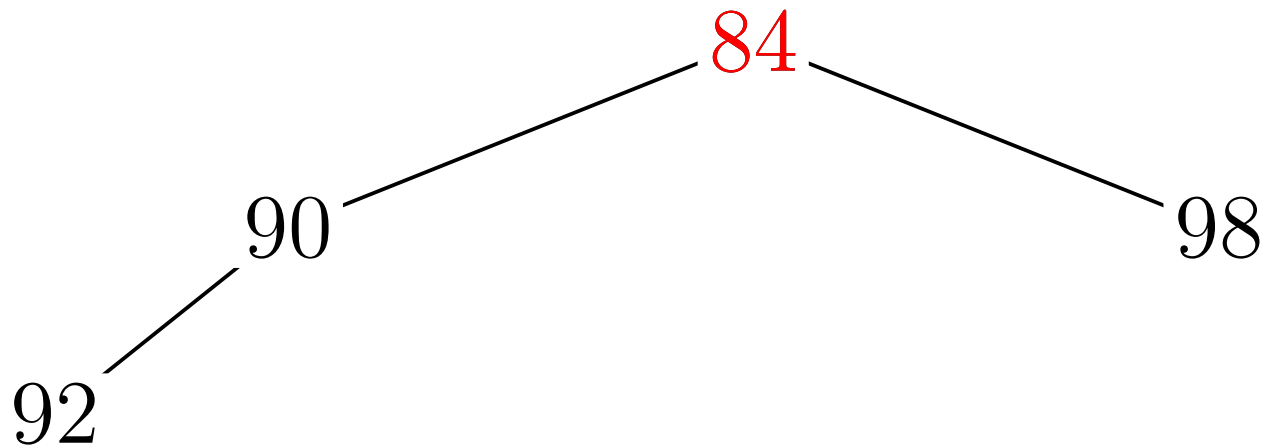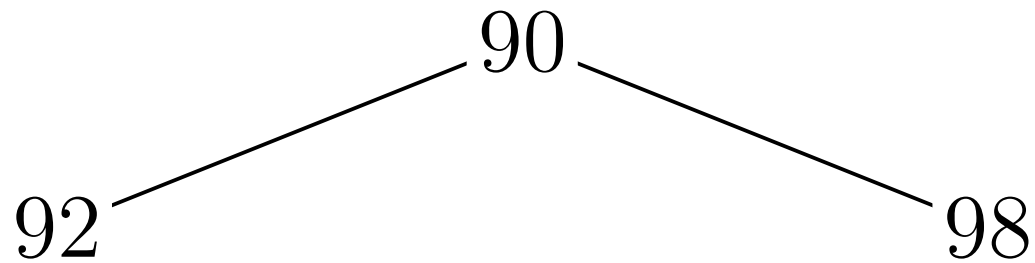
Example of Heap Sort

| 2 | 14 | 21 | 27 | 63 | 72 | 78 | 80 | 84 | | | |

Example of Heap Sort

| 2 | 14 | 21 | 27 | 63 | 72 | 78 | 80 | 84 | 90 | | |

92

98

Example of Heap Sort

| 2 | 14 | 21 | 27 | 63 | 72 | 78 | 80 | 84 | 90 | | |

92

98

Example of Heap Sort

2	14	21	27	63	72	78	80	84	90	92	

98

Example of Heap Sort

2	14	21	27	63	72	78	80	84	90	92	

98

Example of Heap Sort

2	14	21	27	63	72	78	80	84	90	92	98

Complexity of Heap Sort

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Execuation Time for Heap Sort

- This is actually a very efficient algorithm

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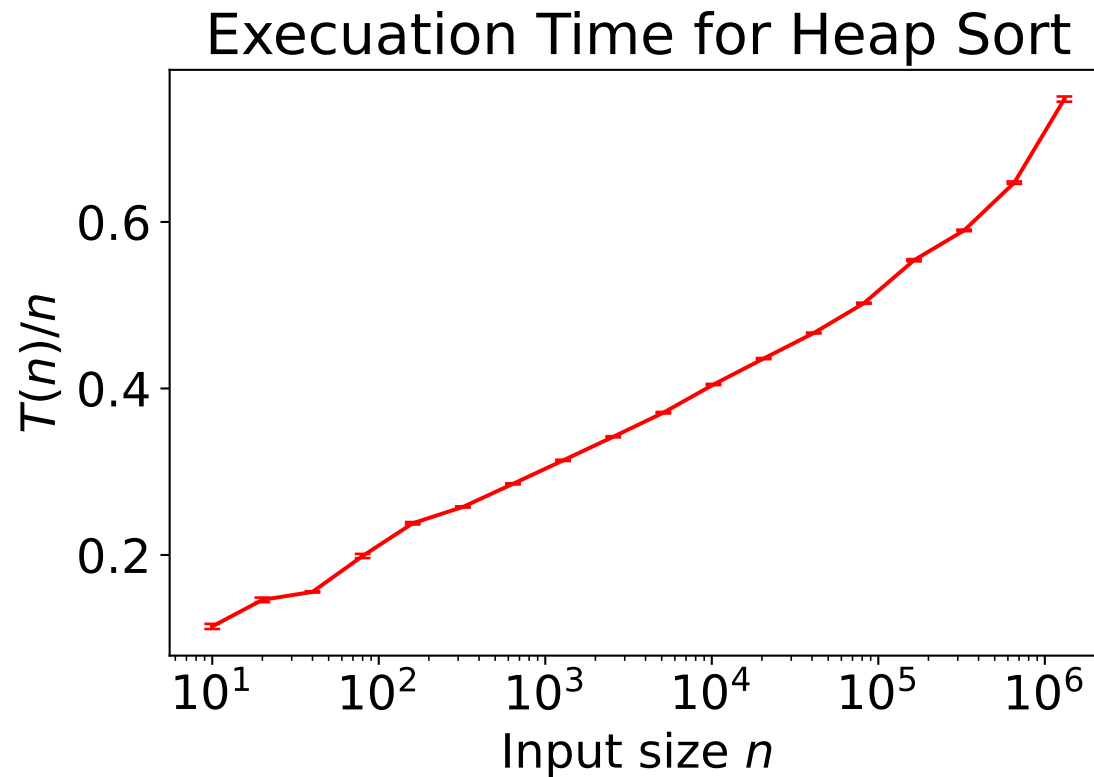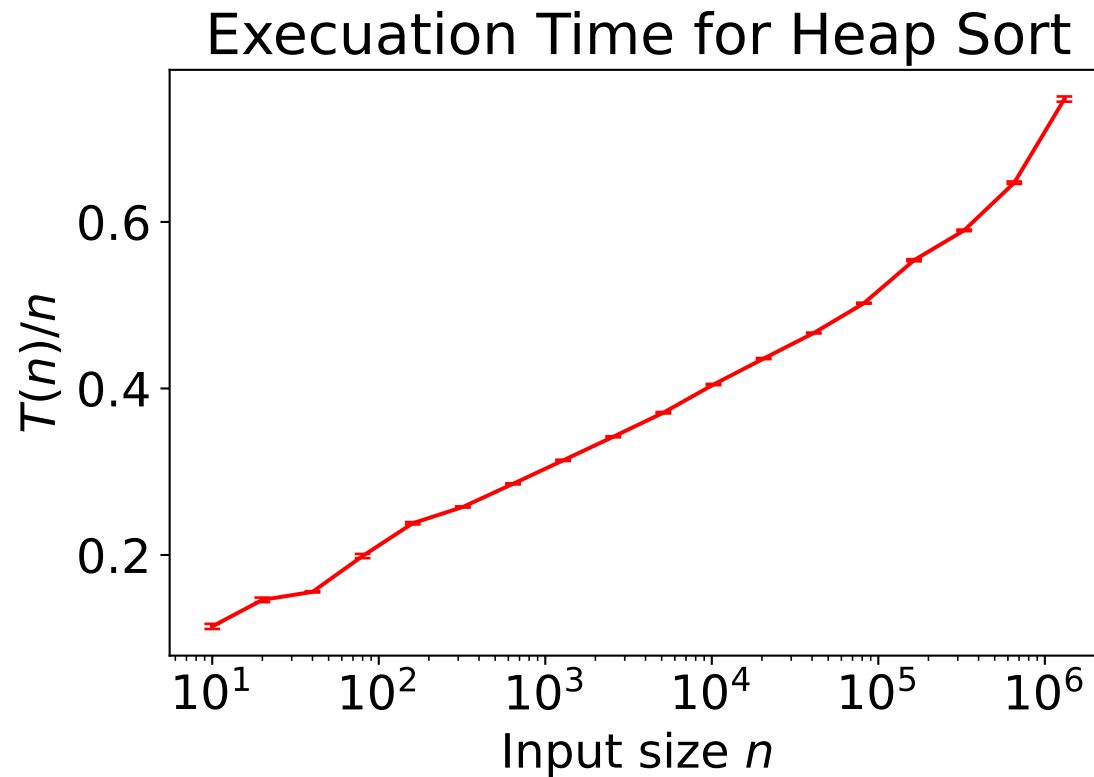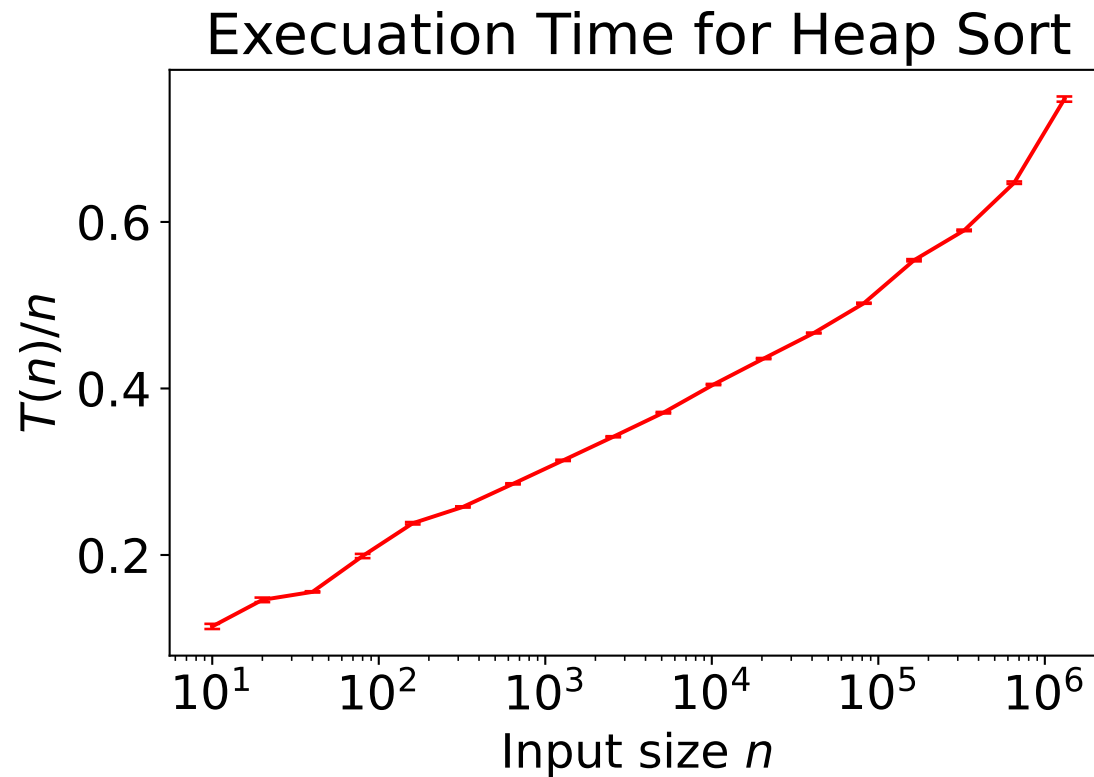
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Outline

1. Heaps

2. Priority Queues

   • Array Implementation

3. Heap Sort

4. **Other Heaps**

Other Heaps

- Binary Heaps are so useful that other types of heaps have been developed

- The simplest enhancement is to combine a binary heap with a map which maintains a pointer to each element

- The map has to be updated every time elements are moved in the heap (fortunately only $O(\log(n))$ elements are move each time the heap is updated)

- The advantage of this heap is that the priorities of elements can be changed (involving percolating elements up or down the tree)

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Merging Heaps

- One common demand on a heaps is to merge two heaps

- Unfortunately binary heaps are not efficiently merged

- There are a variety of different heaps (leftist heaps, skew heaps, binomial queues, . . .) designed to be merged

- All these heaps are real binary trees (i.e. represented by pointers rather than put in an array)

- They are slower than a binary heap because indexing is slightly slower as is creating node objects

- However, they allow merging

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- All other operations are achieved by merging

  - ⋆ Adding an element is achieved by merging the current heap with a heap of one element
  - ⋆ Removing the minimum element is achieved by removing the root and merging the left and right tree

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Lessons

- Heaps are a powerful data structure—they are particularly useful for implementing priority queues

- Heaps are binary trees that can be implemented as arrays

- Priority queue have many (often surprising uses)

    ⋆ They are used when you need a queue with priorities, e.g. in operating systems
    ⋆ They can be used to perform pretty efficient sort
    ⋆ They are often used for implementing greedy type algorithms
    ⋆ One important application is in real time simulations

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