Further Mathematics and Algorithms

Lesson 2: *Know How Long A Program Takes*

TSP, Sorting, time complexity, Big-Theta, Big-O, Big-Omega

Outline

1. **TSP**
2. Sorting
3. Big O

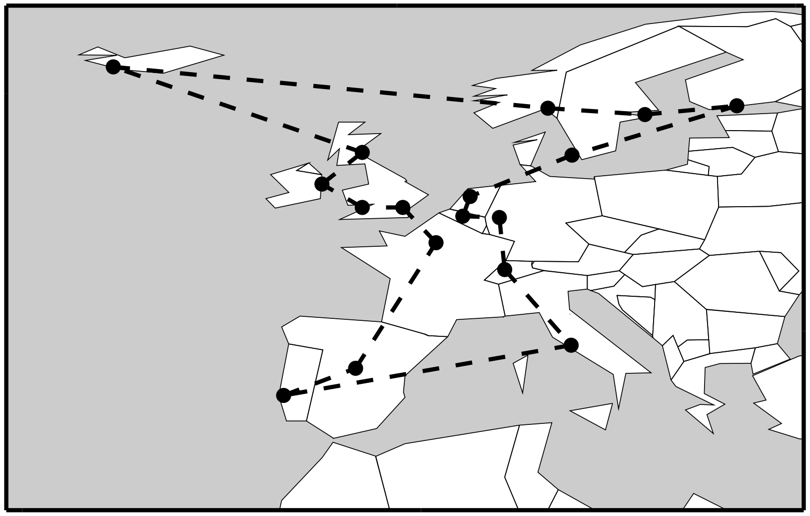
Travelling Salesperson Problem

- Given a set of cities
- A table of distances between cities
- Find the shortest tour which goes through each city and returns to the start

Example of Distance Table

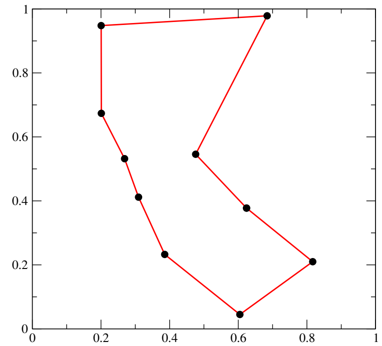
	Lon	Car	Dub	Edin	Reyk	Oslo	Sto	Hel	Cop	Amst	Bru	Bonn	Bern	Rome	Lisb	Madr	Par
London	0	223	470	538	1896	1151	1426	1816	950	349	312	503	743	1429	1587	1265	337
Cardiff	223	0	290	495	1777	1277	1589	1985	1139	564	533	725	927	1600	1492	1233	492
Dublin	470	290	0	350	1497	1267	1628	2026	1239	756	775	956	1207	1886	1638	1449	777
Edinburgh	538	495	350	0	1374	933	1314	1708	984	662	758	896	1243	1931	1964	1728	872
Reykjavik	1896	1777	1497	1374	0	1746	2134	2418	2104	2020	2130	2255	2617	3304	2949	2892	2232
Oslo	1151	1277	1267	933	1746	0	416	788	481	917	1088	1048	1459	2011	2739	2390	1343
Stockholm	1426	1589	1628	1314	2134	416	0	398	518	1126	1281	1181	1542	1978	2987	2593	1543
Helsinki	1816	1985	2026	1708	2418	788	398	0	881	1504	1650	1530	1856	2203	3360	2950	1910
Copenhagen	950	1139	1239	984	2104	481	518	881	0	625	769	662	1036	1538	2479	2076	1030
Amsterdam	349	564	756	662	2020	917	1126	1504	625	0	173	235	629	1296	1860	1480	428
Brussels	312	533	775	758	2130	1088	1281	1650	769	173	0	194	489	1174	1710	1315	262
Bonn	503	725	956	896	2255	1048	1181	1530	662	235	194	0	422	1067	1843	1420	400
Bern	743	927	1207	1243	2617	1459	1542	1856	1036	629	489	422	0	689	1630	1156	440
Rome	1429	1600	1886	1931	3304	2011	1978	2203	1538	1296	1174	1067	689	0	1862	1365	1109
Lisbon	1587	1492	1638	1964	2949	2739	2987	3360	2479	1860	1710	1843	1630	1862	0	500	1452
Madrid	1265	1233	1449	1728	2892	2390	2593	2950	2076	1480	1315	1420	1156	1365	500	0	1054
Paris	337	492	777	872	2232	1343	1543	1910	1030	428	262	400	440	1109	1452	1054	0

Example Tour

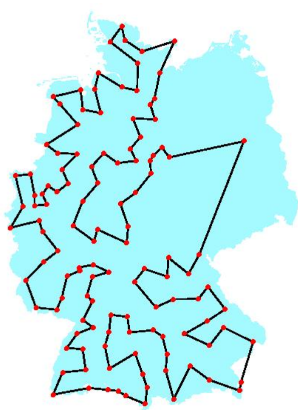


Brute Force

- I wrote a program to solve TSP by enumerating every path and finding the shortest
- I checked that it worked on some problems with 10 cities
- It takes just under half a second to solve this problem
- I set the program running on a 100 city problem—**How long will it take to finish?**

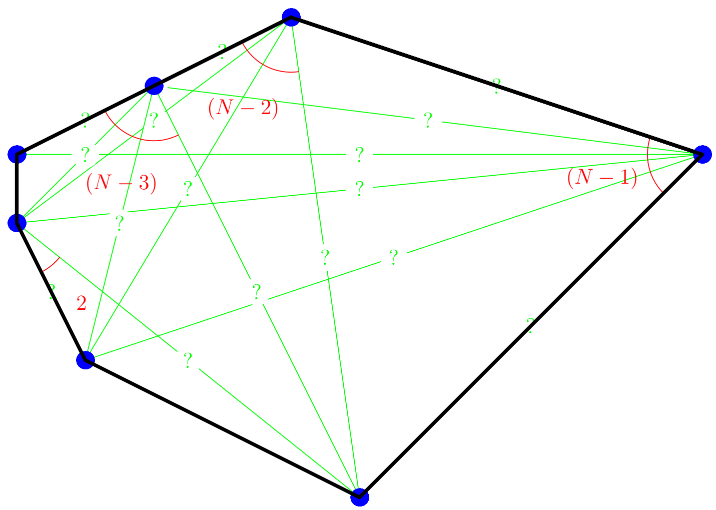


How Many Possible Tours Are There?



- For 100 cities how many possible tours are there?
- It doesn't matter where we start
- Starting from Berlin there are 99 cites we can try next

Counting Tours



$$\text{Number of tours} = (N-1) \times (N-2) \times (N-3) \times \cdots 2 \times 1 = (N-1)!$$

How Long Does It Take?

- The direction we go in is irrelevant
- Total number of tours is $99!/2$
- **Any more guesses how long it will take?**

How Big is 99 Factorial?

- $99! = 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1 = ?$
- Upper bound

$$\begin{aligned} 99! &= 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1 \\ 99! &< 99 \cdot 99 \cdot 99 \cdots 99 \cdot 99 = 99^{99} \end{aligned}$$

- Lower bound

$$\begin{aligned} 99! &= 99 \cdot 98 \cdot 97 \cdots 50 \cdot 49 \cdots 2 \cdot 1 \\ 99! &> 50 \cdot 50 \cdot 50 \cdots 50 \cdot 1 \cdots 1 \cdot 1 = 50^{50} \end{aligned}$$

How Long Does It Take?

- For $N > 1$

$$\left(\frac{N}{2}\right)^{N/2} < N! < N^N$$

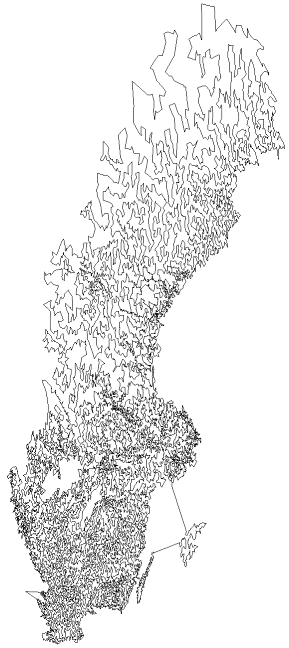
- $99!/2 = 4.666 \times 10^{155}$
- How long does it take to search all possible tours?
  - We computed about $200\,000$ tours in half a second
  - 3.15×10^7sec $= 1$ year
  - Age of Universe ≈ 15 billion years

Answer

- 2.72×10^{132} ages of the universe!
- Incidental

$$\begin{aligned} 99!/2 = {} & 4666310772197207634084961942813335 0 \\ & 24535798413219081073429648194760879 \\ & 99966149578044707319880782591431268 \\ & 48960413611879125592605458432000000 \\ & 0000000000000000 \end{aligned}$$

Record TSP Solved—15 112 and 24 978 Cities



In Case You're Curious

- Number of tours: $15111!/2 = 7.3 \times 10^{56592}$
- Current record 24 978 cities with 1.9×10^{98992} tours
- The algorithm for finding the optimum path does not look at every possible path
- If your interested look for the TSP homepage on the web `http://www.math.uwaterloo.ca/tsp/`

Lessons

- Even relatively small problems can take you an astronomical time to solve using simple algorithms
- As a professional programmer you need to have an estimate for how long an algorithm takes—otherwise you can look silly
- For the 100 city problem, if
  - I had 10^{87} cores, one for every particle in the Universe
  - I could compute a tour distance in 3×10^{-24} seconds, the time it takes light to cross a proton
  - It would still take $10^{39} \times$ the age of the universe
- Smart algorithms can make a much larger difference than fast computers!

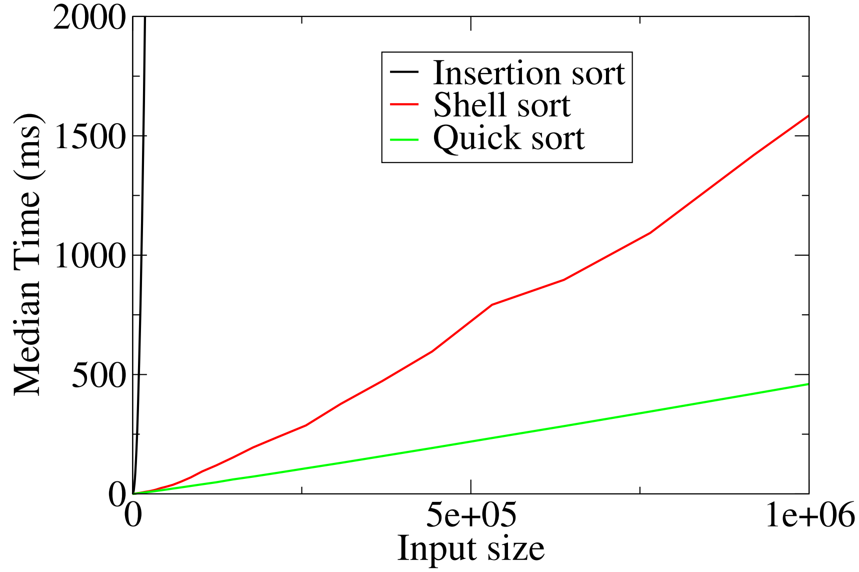
Outline

1. TSP
2. **Sorting**
3. Big O

Sort

- Comparison between common sort algorithms
  - Insertion sort—an easy algorithm to code
  - Shell sort—invented in 1959 by Donald Shell
  - Quick sort—invented in 1961 by Tony Hoare
- These take an array of numbers and returns a sorted array
- Sort is very commonly used algorithm so you care about how long it takes

Empirical Run Times



Lessons

- There is a right and wrong way to do easy problems
- You only really care when you are dealing with large inputs
- Good algorithms are difficult to come up with, but they exist
- We would like to quantify the performance of an algorithm—how much better is quick sort than insertion sort?

Outline

1. TSP
2. Sorting
3. **Big O**

Estimating Run Times

- We would like to estimate the run times of algorithms
- This depends on the hardware (how fast is your computer)
- We could count number of elementary operations, but
  - ⋆ different machines have different elementary operations
  - ⋆ many algorithms use complex functions such as `sqrt` (matrix inversion using Cholesky decomposition) or `sin` and `cos` (FFT)
  - ⋆ would need to count memory accesses which you shouldn't need to think about
  - ⋆ code after compiling can be very different from code before compiling

Engineering Solution

- Compute the **asymptotic leading functional behaviour**
- Lets take that statement to pieces
- Suppose we have an algorithm that takes $4n^2 + 12n + 199$ operations (clock cycles)
  - ⋆ **asymptotic**: what happens when n becomes very large
  - ⋆ **leading**: ignore the $12n + 199$ part as it is dominated by $4n^2$ (i.e. for large enough n we have $4n^2 \gg 12n + 199$)
  - ⋆ **functional behaviour**: ignore the constant 4
- We call this an order n^2, or quadratic time, algorithm
- We can write this in 'Big-Theta' notation as $\Theta(n^2)$
- This notion of 'run time' is known as **time complexity**

Advantages of Big-Theta Notation

- Doesn't depend on what computer we are running
- Don't need to know how many elementary operations are required for a non-elementary operation
- Can estimate run times by measuring run time on a small problem
  - ⋆ If I have a $\Theta(n^2)$ algorithm
  - ⋆ It takes x seconds on an input of 100
  - ⋆ It will take about $\frac{x \times n^2}{100^2}$ seconds on a problem of size n
    ($T(100) \approx c\,100^2 = x$ therefore $c = x/100^2$
    thus $T(n) = c\,n^2 = x\,n^2/100^2$)

Counting Instructions

- Big-Theta run times are often easy to calculate
- a $\Theta(n)$ algorithm

```
// define stuff
for(int i=0; i<n; i++)  {
  // do something
}
// clean up
```

- a $\Theta(n^2)$ algorithm

```
// define stuff
for(int i=0; i<n; i++)  {
  // do something
  for (int j=0; j<n; j++) {
    // do other stuff
  }
}
// clean up
```

Disadvantage with Big-Theta notation

- Can't compare algorithms with the same Big-Theta time complexity
- For small inputs Big-Theta time complexity can be misleading. E.g.
  - algorithm A takes $n^3 + 2n^2 + 5$ operations
  - algorithm B takes $20n^2 + 100$ operations
  - algorithm A is $\Theta(n^3)$ and algorithm B is $\Theta(n^2)$
  - algorithm A is faster than algorithm B for $n < 18$

  but who cares?
- In some cases Big-Theta time complexity is hard to compute

Not So Sure

- Some algorithms are harder to compute

```
// define stuff
for(int i=0; i<n; i++)  {
  // do something
  if (/* some condition */) {
    for (int j=0; j<n; j++) {
      // do other stuff
    }
  }
}
// clean up
```

- Time complexity now depends on the `if` statement
- If the condition is often satisfied we have a $\Theta(n^2)$ algorithm
- If the condition is true only rarely then we have a $\Theta(n)$ algorithm

Bounds

- To avoid having to think really hard we define upper and lower bounds
- The upper bound we write using **big-O** notation
  - The above algorithm is an $O(n^2)$ algorithm
  - I.e. it runs in no more than order n^2 operations
- The lower bound we write using **big-Omega** notation
  - The above algorithm is a $\Omega(n)$ algorithm
  - I.e. it runs in no less than order n operations

Precise Definitions of $O(n)$

- An algorithm that runs in $f(n)$ operations is $O(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \qquad \text{where } c \text{ is a constant (could be zero)}$$

- E.g.. $f(n) = 3\,n^2 + 2\,n + 12$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{3\,n^2 + 2\,n + 12}{n^2} = 3 \quad \Rightarrow 3\,n^2 + 2\,n + 12 = O(n^2)$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{3\,n^2 + 2\,n + 12}{n^3} = 0 \quad \Rightarrow 3\,n^2 + 2\,n + 12 = O(n^3)$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{3\,n^2 + 2\,n + 12}{n} = \infty \quad \Rightarrow 3\,n^2 + 2\,n + 12 \neq O(n)$$

Lower Bound Definition

- An algorithm that runs in $f(n)$ operations is $\Omega(g(n))$ if

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = c \qquad \text{where } c \text{ is a constant (could be zero)}$$

- E.g. $f(n) = 3\,n^2 + 2\,n + 12$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n^2}{3\,n^2 + 2\,n + 12} = \frac{1}{3} \Rightarrow 3\,n^2 + 2\,n + 12 = \Omega(n^2)$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n^3}{3\,n^2 + 2\,n + 12} = \infty \Rightarrow 3\,n^2 + 2\,n + 12 \neq \Omega(n^3)$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n}{3\,n^2 + 2\,n + 12} = 0 \quad \Rightarrow 3\,n^2 + 2\,n + 12 = \Omega(n)$$

Big-Theta

- An algorithm that runs in $f(n)$ operations is $\Theta(g(n))$ if

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = c \qquad \text{where } c \text{ is a non-zero constant}$$

- That is, $f(n) = \Theta(g(n))$ if

$$f(n) = O(g(n)) \quad \text{and} \quad f(n) = \Omega(g(n))$$

- I.e. the lower bound is identical to the upper bound
- Often the most straightforward way of obtaining big-Theta is to show the upper and lower bounds are the same

Use and Misuse

- Note: big-O notation is most commonly used
- often people say they have a $O(n^2)$ when in fact they mean they have a $\Theta(n^2)$ algorithm (a much stronger result)
- Note that an $O(n^2)$ algorithm is also a $O(n^3)$ algorithm
- Strictly a $O(n^2)$ algorithm **may not** be faster than a $O(n^3)$ algorithm when n becomes larger
- A $\Theta(n^2)$ algorithm **will** be faster than a $\Theta(n^3)$ algorithm when n becomes larger

Lessons to Learn

- Run times (computational time complexity) matters
- Choosing an algorithm with the best time complexity is important
- Understand the meaning of big-Theta, big-O and big-Omega
- Know how to estimate time complexity for simple algorithms (loop counting)