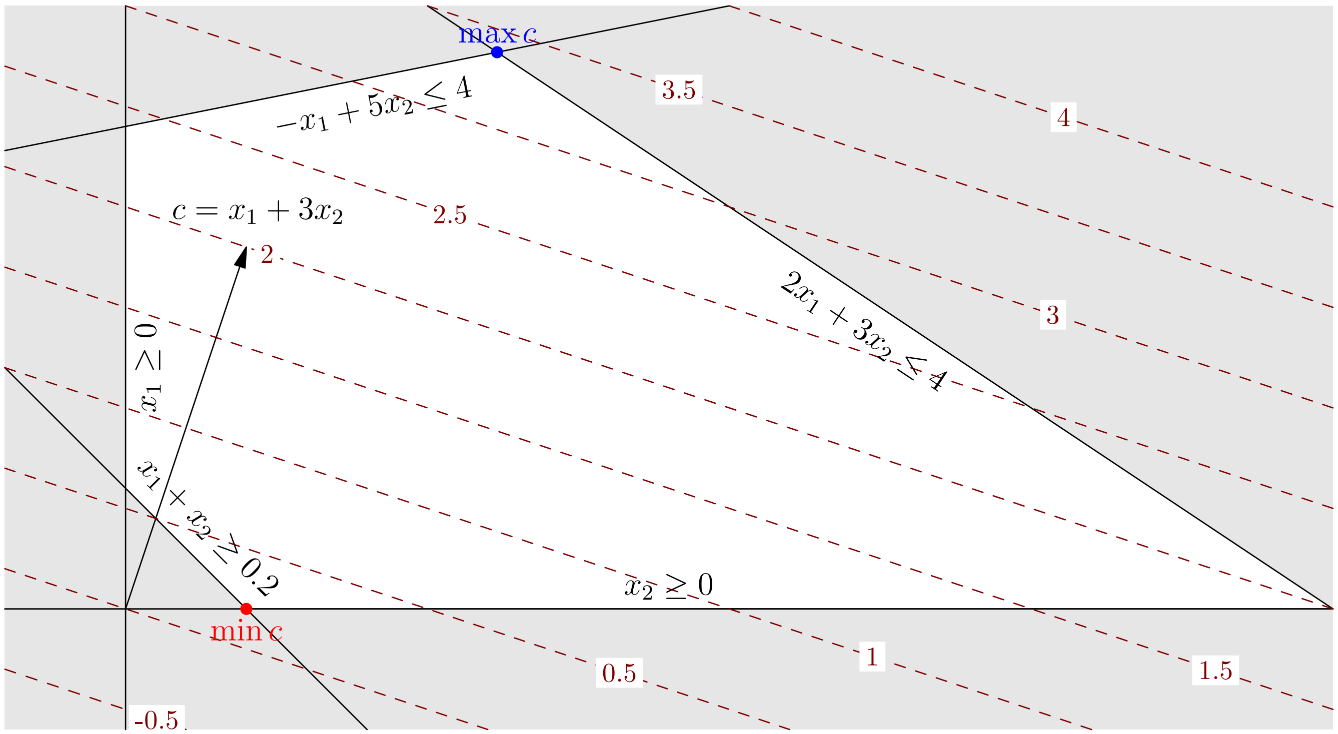
Algorithms and Analysis

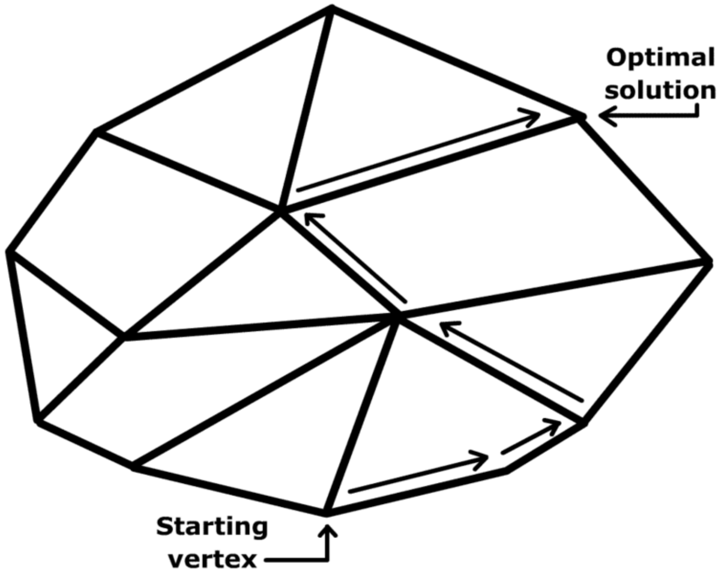
Lesson 26: *Use Linear Programmings*



linear programming, applications

Outline

1. **Examples**
2. Linear Programs
3. Properties of Solution
4. Normal Form



Going Shopping

- Suppose we have a number of food stuffs which we label with indices $f \in \mathcal{F}$
- The price of food stuff f per kilogram we denote p_f
- We are interested in buying a selection of foods $\boldsymbol{x} = (x_f | f \in \mathcal{F})$ where x_f is the quantity (in kg) of food f
- We want to minimise the total price $\sum_f p_f\, x_f = \boldsymbol{p} \cdot \boldsymbol{x}$
- However we want to ensure that the food has enough vitamins

Nutrition

- We consider the set of vitamins $\mathcal{V}$
- Let $A_{v\,f}$ be the quantity of vitamin v in food stuff f
- Let b_v be the minimum daily requirement of vitamin v
- We therefore require

$$\forall v \in \mathcal{V} \qquad \sum_{f \in \mathcal{F}} A_{v\,f}\, x_f \geq b_v$$

Optimisation Problem

- We can write the food shopping problem as

$$\min_{\boldsymbol{x}} \boldsymbol{p} \cdot \boldsymbol{x} \quad \text{subject to} \quad \mathbf{A}\boldsymbol{x} \geq \boldsymbol{b} \quad \text{and} \quad \boldsymbol{x} \geq \mathbf{0}$$

- Note that the inequalities involving vectors means that each component must be satisfied, i.e.

$$\mathbf{A}\boldsymbol{x} \geq \boldsymbol{b} \quad \Rightarrow \quad \forall v \in \mathcal{V} \quad \sum_{f \in \mathcal{F}} A_{v\,f}\, x_f \geq b_v$$

$$\boldsymbol{x} \geq \mathbf{0} \quad \Rightarrow \quad \forall f \in \mathcal{F} \quad x_f \geq 0$$

- This is an example of a "**linear program**"

Transportation

- We consider a set of factories $\mathcal{F}$ producing a set of commodities $\mathcal{C}$
- The amount of commodity c produced by factory f we denote by x_{cf}
- The shipping cost of commodity c from factory f to the retailer of c we denote by p_{cf}
- We want to choose x_{cf} to minimise the transportation costs

$$\sum_{c \in \mathcal{C}, f \in \mathcal{F}} p_{cf} \, x_{cf}$$

- However, we have constraints. . .

Constraints

- Each factory can only produce a certain overall tonnage of commodities

$$\sum_{c \in \mathcal{C}} x_{cf} \leq b_f \qquad \forall f \in \mathcal{F}$$

where b_f is the maximum production capacity of factory f

- The total demand for each commodity is d_c so

$$\sum_{f \in \mathcal{F}} x_{cf} = d_c \qquad \forall c \in \mathcal{C}$$

- We can only produce positive amounts, i.e. $x_{cf} \geq 0$

Linear Program

- We can write the full problem as

$$\min_{\boldsymbol{x}} \sum_{c\in\mathcal{C}, f\in\mathcal{F}} p_{cf}\, x_{cf}$$

subject to

$$\sum_{c\in\mathcal{C}} x_{cf} \le b_f \qquad \forall f \in \mathcal{F}$$

$$\sum_{f\in\mathcal{F}} x_{cf} = d_c \qquad \forall c \in \mathcal{C}$$

$$x_{cf} \ge 0 \qquad \forall c \in \mathcal{C}, \quad \forall f \in \mathcal{F}$$

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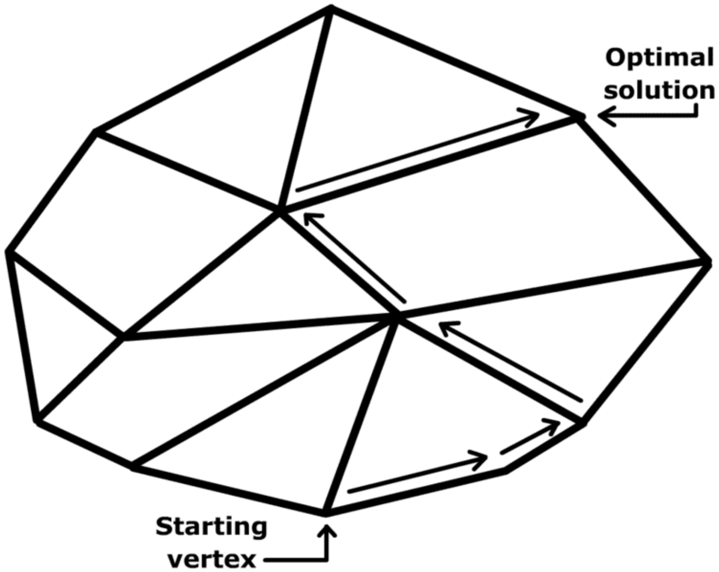

General Linear Programs

- Linear programs are problems that can be formulated as follows

$$\min_{\boldsymbol{x}} \boldsymbol{c} \cdot \boldsymbol{x}$$

subject to

$$\mathbf{A}^{\leq}\boldsymbol{x} \leq \boldsymbol{b}^{\leq}, \quad \mathbf{A}^{\geq}\boldsymbol{x} \geq \boldsymbol{b}^{\geq}, \quad \mathbf{A}^{=}\boldsymbol{x} = \boldsymbol{b}^{=}, \quad \boldsymbol{x} \geq \mathbf{0}$$

- Note in the previous example it was convenient to use two indices c and f to denote the components x_{cf}, however, it still has this structure

Maximising

- We can also maximise rather than minimise
- Whether we want to maximise or minimise will depend on the application
- Note that

$$\max_{\boldsymbol{x}} \boldsymbol{c} \cdot \boldsymbol{x} \quad \equiv \quad \min_{\boldsymbol{x}} (-\boldsymbol{c}) \cdot \boldsymbol{x}$$

- We can thus always reformulate a maximisation problem as a minimisation problem and vice versa

Linear Program Applications

- A huge number of problems can be mapped to linear programming problems
- Or modelled as linear (even when they're not, e.g. oil extraction)
- Realistic problems might have many more constraints and large number of variables
- State of the art solvers can deal with problems with hundreds of thousands or even millions of variables

Key Features

- There are three key features of linear programs
  1. The cost (objective function) is linear in x_i ($\boldsymbol{c} \cdot \boldsymbol{x}$)
  2. The constraints are linear in x_i (e.g. $\mathbf{A}_1 \boldsymbol{x} \leq b_1$)
  3. The component of $\boldsymbol{x}$ are non-negative (i.e. $x_i \geq 0$)
- These are very special features, very often they don't apply, but a surprising large number of problems can be formulated as linear programming problems

History

- Linear programming was “invented” by Leonid Kantorovich in 1939 to help Soviet Russia maximise its production
- It was kept secret during the war, but was finally made public in 1947 when George Dantzig published the **simplex method** which still today is a standard method for solving linear programs
- John von Neumann developed the idea of duality (you can turn a maximisation problem for a set of variables x into a minimisation problem for a dual set of variables $\boldsymbol{\lambda}$ associated with each constraint)
- von Neumann used this idea as the basis for “game theory”

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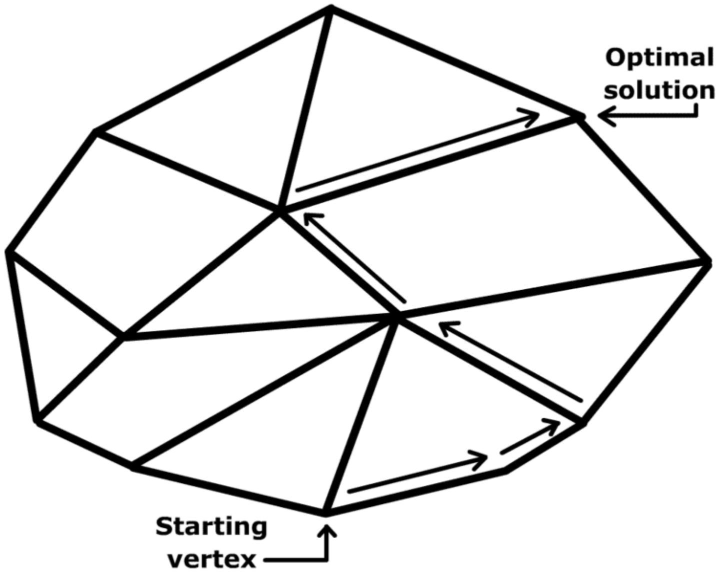
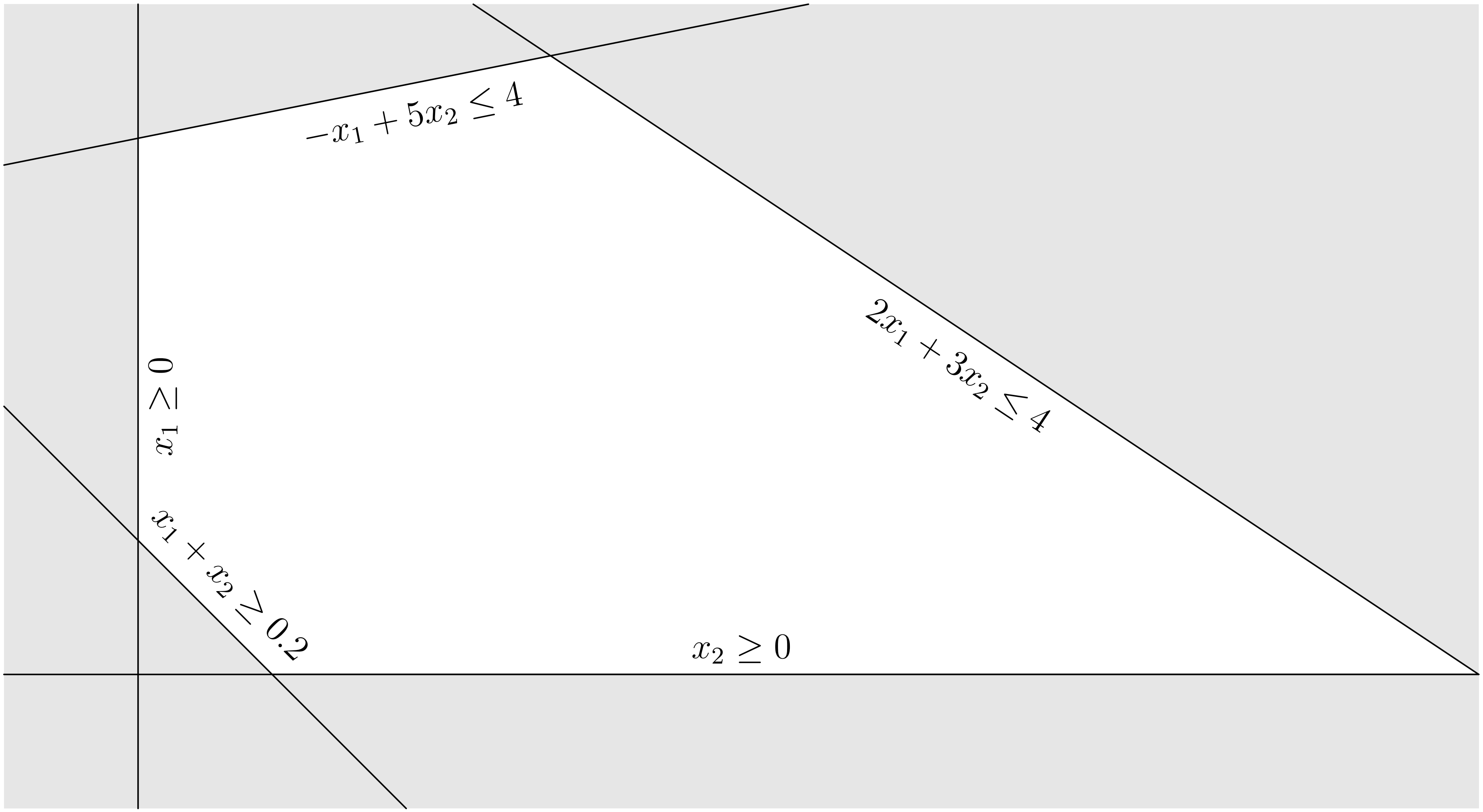


Structure of Linear Programs

- Before we go into the details of solving linear programs its useful to consider the structure of the solutions
- The set of x that satisfy all the constraints is known as the set of **feasible solutions**
- The set of feasible solutions may be empty in which case it is impossible to satisfy all the constraints
- This is rather disappointing, but usually doesn't happen if we have formulated a sensible problem

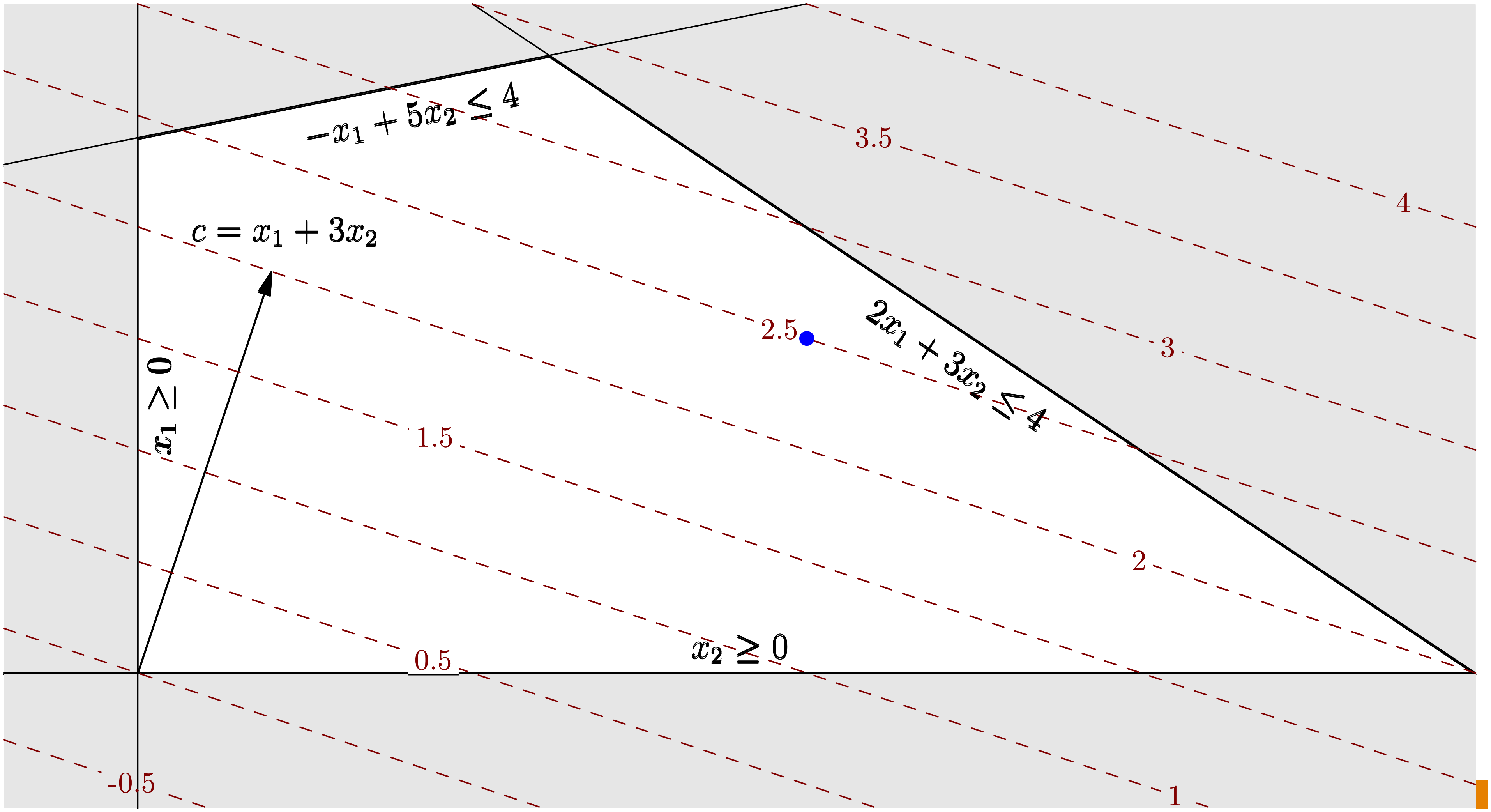
The Space of Feasible Solutions



Vertices of Polytope

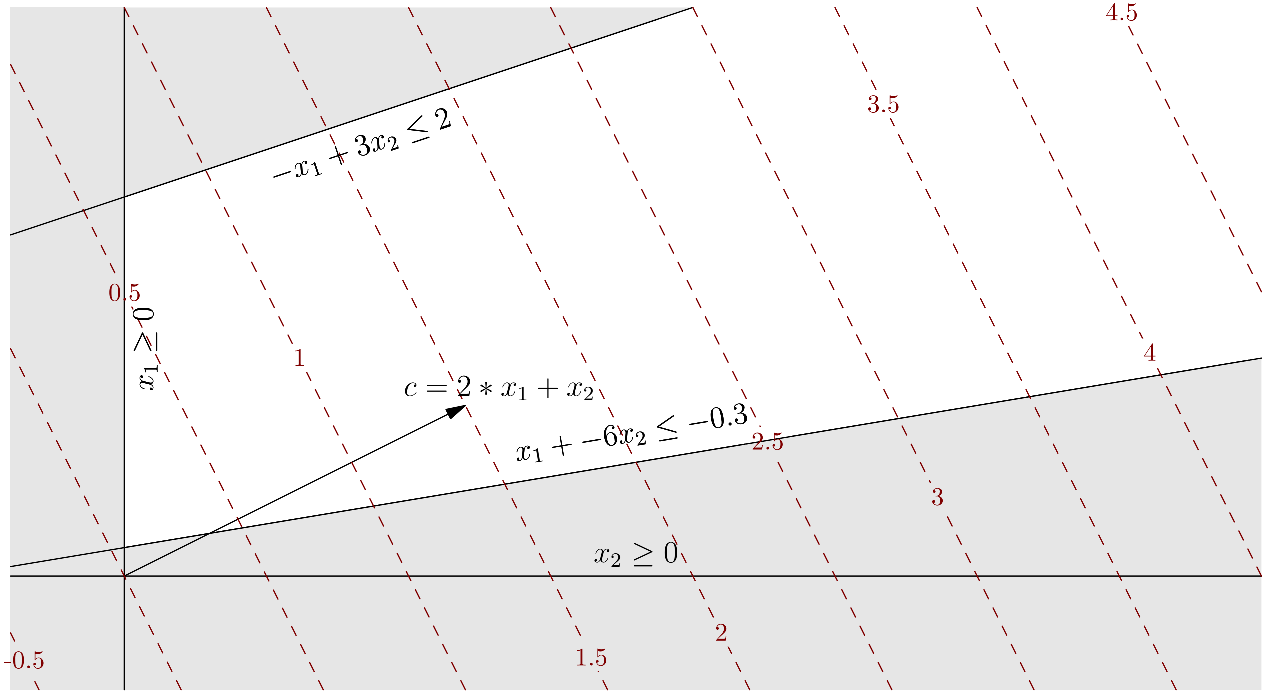
- The space of feasible solutions is a polyhedra or polytope
- The maximum or minimum solution will always lie at a vertex of the polytope
- Our solution policy will be to start at a vertex and move to a neighbouring vertex that gives the best improvement in cost
- When this isn't possible then we are finished
- However, there is still a lot of work to realise this solution strategy

Optimal Solution



Unbounded Solutions

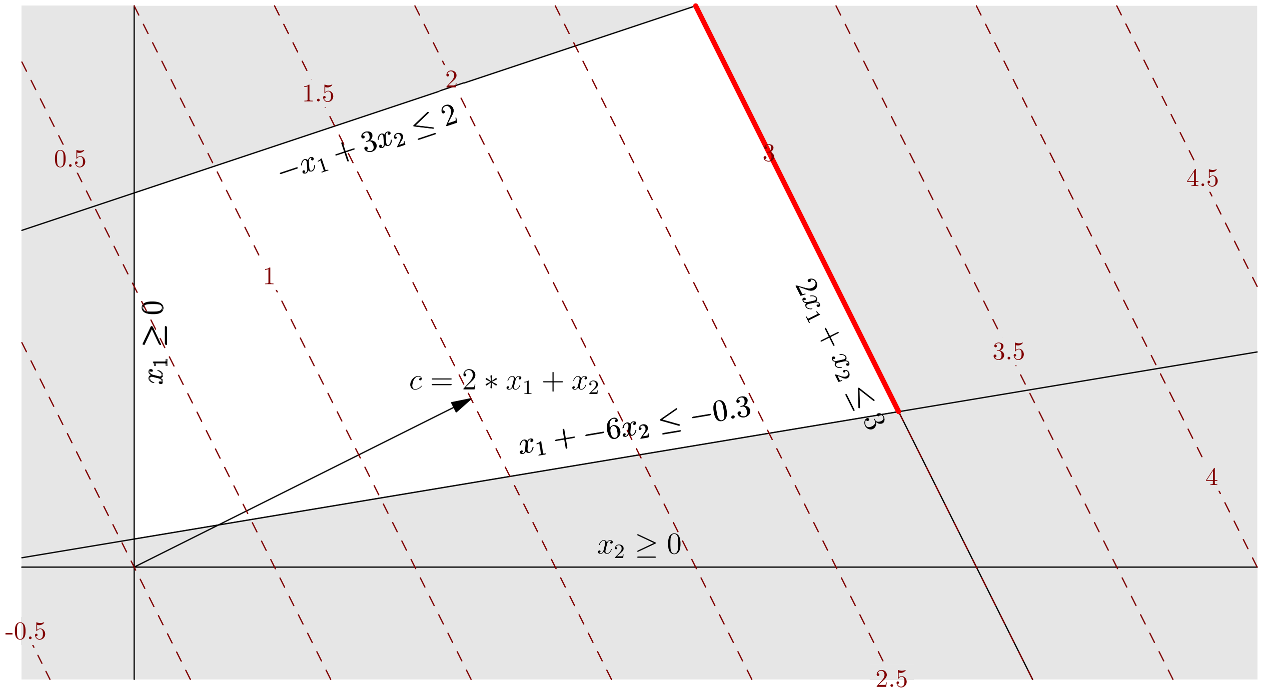
- If you are unlucky you might not have a bounded solution



- But usually this would not happen because of the problem definition

Multiple Solutions

- You can also get multiple solutions if a constraint is orthogonal to the objective function



- Nevertheless the optimal will be at a vertex

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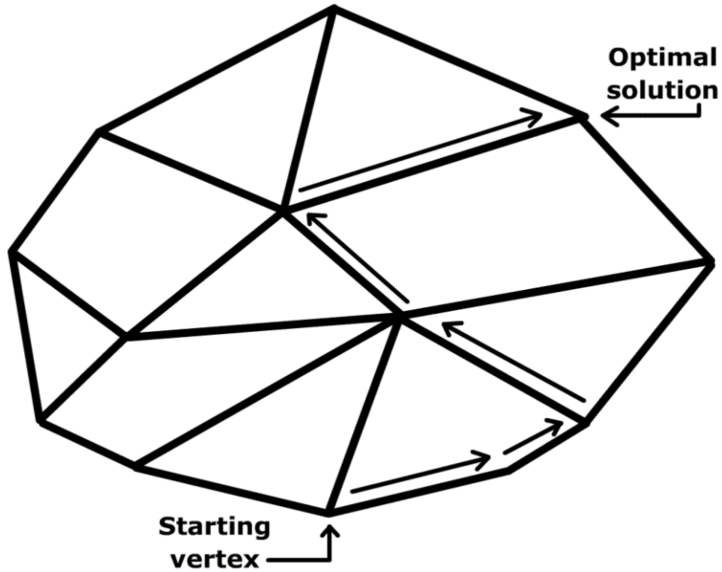


Converting Linear Programs

- Solving full linear programs is difficult
- However, it is much easier to solve linear programs in **normal form**
- This is basically a form where we get rid of all inequalities and rewriting the equalities
- Fortunately its rather easy to convert linear programs to normal form

Slack Variables

- We can change an inequality into an equality by introducing a new "**slack**" variable

- E.g.

$$\begin{aligned} \boldsymbol{a}_1 \cdot \boldsymbol{x} \geq 0 \quad &\Rightarrow \quad \boldsymbol{a}_1 \cdot \boldsymbol{x} - z_1 = 0 \quad z_1 \geq 0 \\ \boldsymbol{a}_2 \cdot \boldsymbol{x} \leq 0 \quad &\Rightarrow \quad \boldsymbol{a}_2 \cdot \boldsymbol{x} + z_2 = 0 \quad z_2 \geq 0 \end{aligned}$$

  z_1 (the excess) and z_2 (the deficit) are known as slack variables

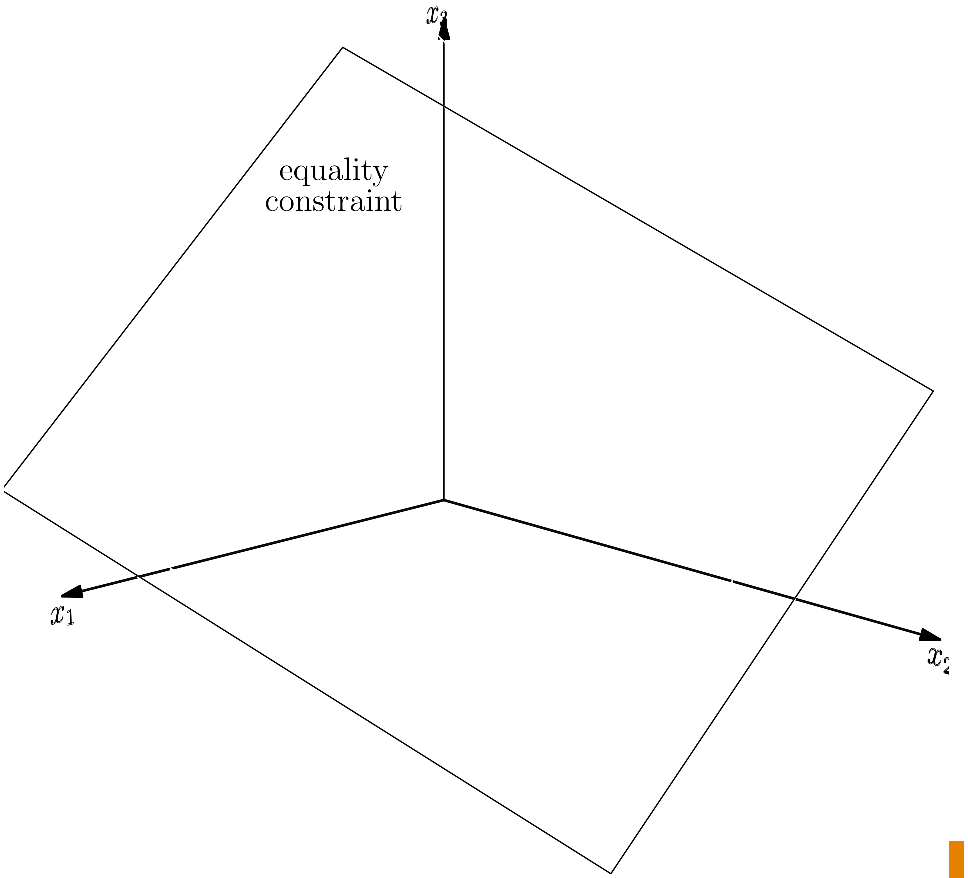
- We eliminate inequalities at the expense of increasing the number of variables

- We can treat the slack variables on an equivalent footing to the normal variables (they just provide a different way of describing the original problem)

Normal Form

- A linear program with only equality constraints is said to be in **normal form**
- We will find in the next lecture that this is a convenient form for solving linear programs
- An equality constraint restricts the solutions to a subspace (some lower dimensional space)

Solving Linear Programming



- The basic feasible points for LP problems with n variables and m constraints have at least $n-m$ zero variables
- Typical number of basic feasible solutions is $\binom{n}{m} \geq \left(\frac{n}{m}\right)^m$
- Simplex algorithm organises iterative search for global solutions

Lessons

- There are a huge number of problems that can be set up as linear programs
- They are particularly useful in resource allocation where the resources are all positive
- The solution to linear programming problems is at the vertex of the feasible space (intersection of constraints)
- We can search for solutions by moving from vertex to vertex
- We can transform inequality constraints to equality constraints using slack variables