Advanced Machine Learning

Probability

$$Y = g(X)$$

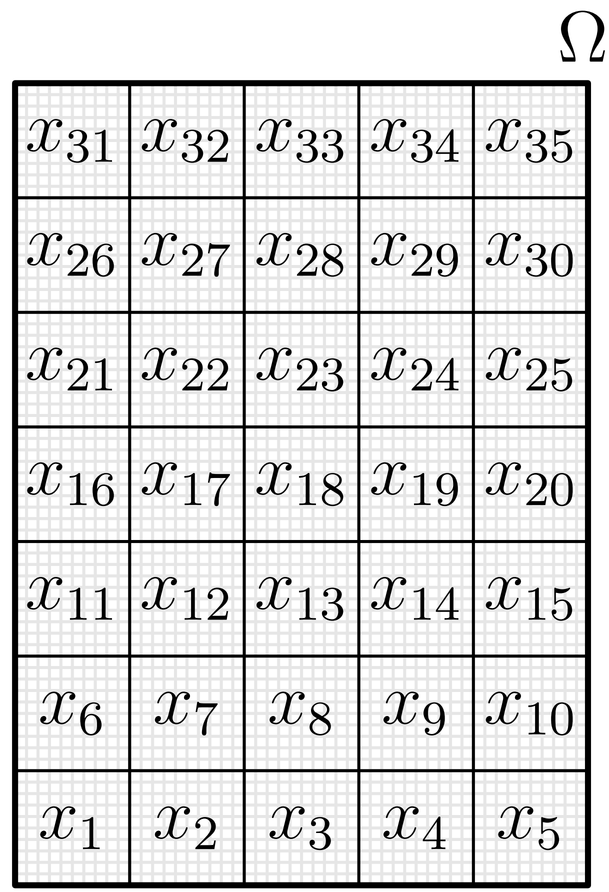
Ω

$y_{13} = g(x_{13})$	$y_{14} = g(x_{14})$	$y_{15} = g(x_{15})$	$y_{16} = g(x_{16})$
$y_9 = g(x_9)$	$y_{10} = g(x_{10})$	$y_{11} = g(x_{11})$	$y_{12} = g(x_{12})$
$y_5 = g(x_5)$	$y_6 = g(x_6)$	$y_7 = g(x_7)$	$y_8 = g(x_8)$
$y_1 = g(x_1)$	$y_2 = g(x_2)$	$y_3 = g(x_3)$	$y_4 = g(x_4)$

Probability, Random Variables, Expectations

Outline

1. **Random Variables**
2. Expectations
3. Calculus of Probabilities



Modelling Uncertainty

- To model a world with uncertainty we consider some set of **elementary events** or **outcomes** Ω
- For the outcome of rolling a dice $\Omega = \{1,2,3,4,5,6\}$
- The elementary events ω_i are **mutually exclusive** $\omega_i \cap \omega_j = \emptyset$ and **exhaustive** $\bigcup_i \omega_i = \Omega$
- We consider **events** $\mathcal{E} = \bigcup_{i \in \mathcal{I}} \omega_i$
- E.g. For a dice throw $\mathcal{E} = \{2,4,6\}$

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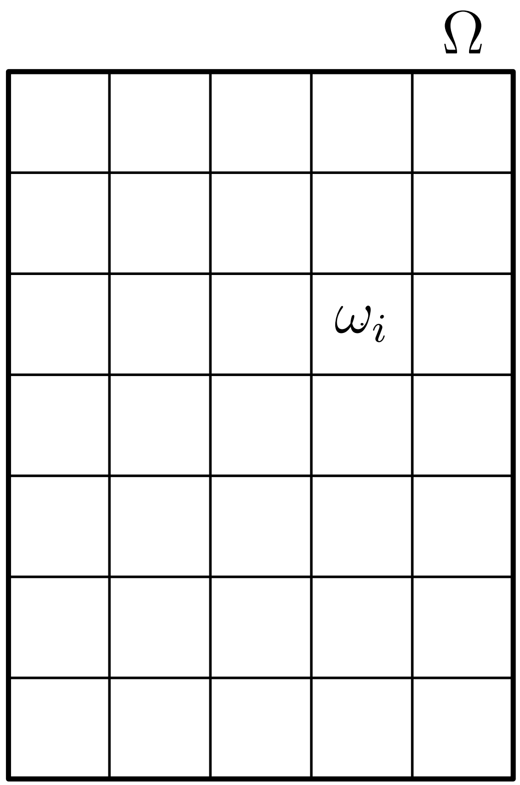

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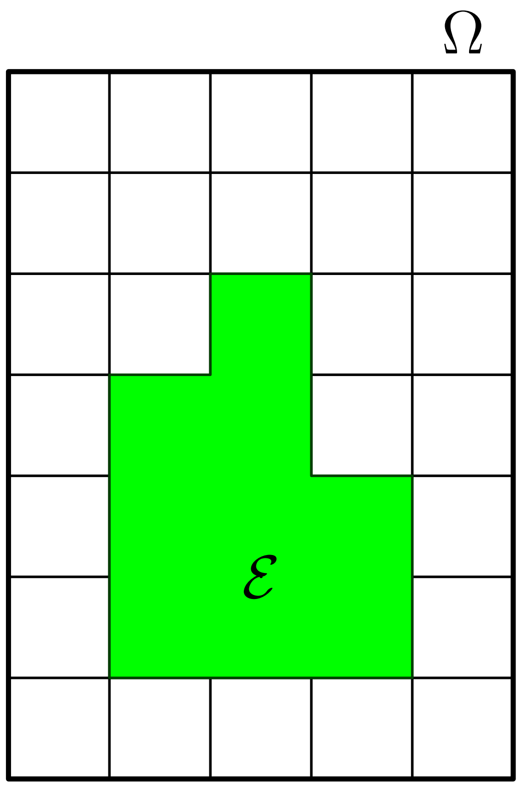

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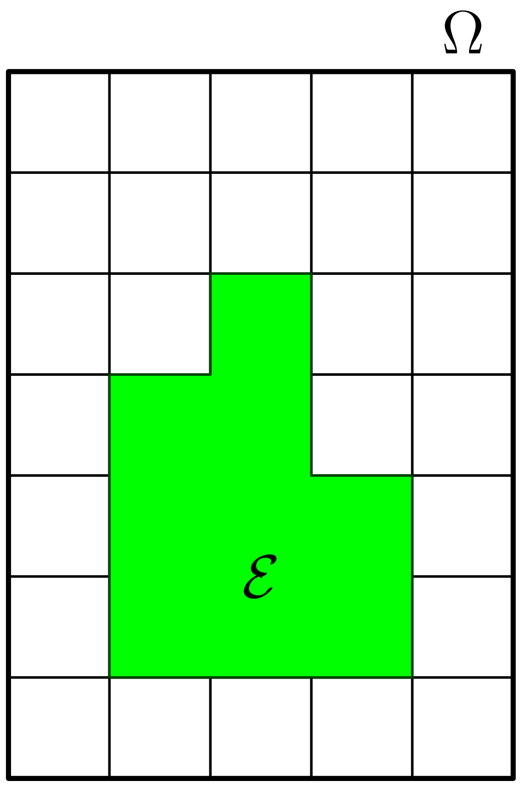
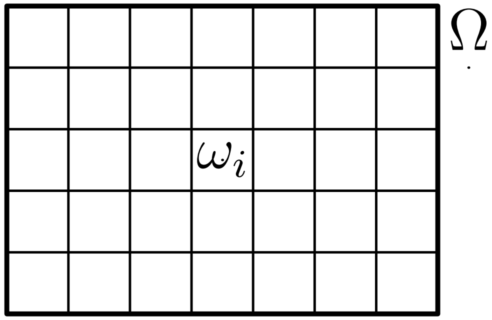


Probabilities

- We attribute a **probability**, $\mathbb{P}(\mathcal{E})$, to an event, $\mathcal{E}$, with the requirements
  - ⋆ $0 \leq \mathbb{P}(\mathcal{E}) \leq 1$
  - ⋆ $\mathbb{P}(\mathcal{E}) + \mathbb{P}(\neg\mathcal{E}) = 1$ where $\neg\mathcal{E} = \Omega \setminus \mathcal{E}$



- In some cases we can interpret $\mathbb{P}(\mathcal{E})$ as the expected frequency of occurrence of a repetitive trial
- But $\mathbb{P}(\text{Pass COMP6208 exam})$ is something you do once
- Can think of probability as an informed belief that something might happen
- When our knowledge changes the probability changes

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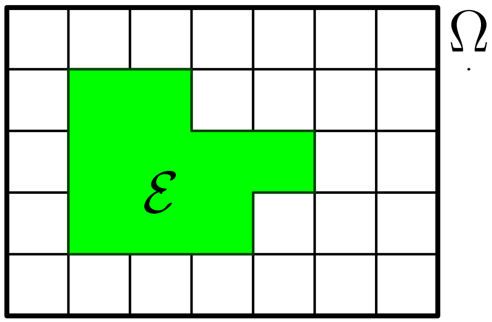

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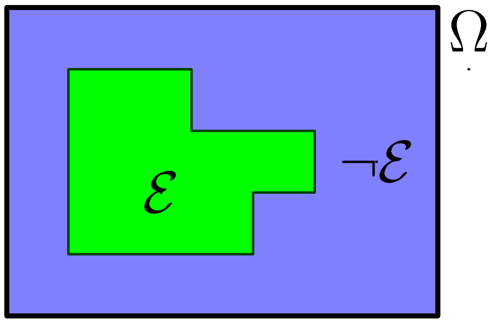

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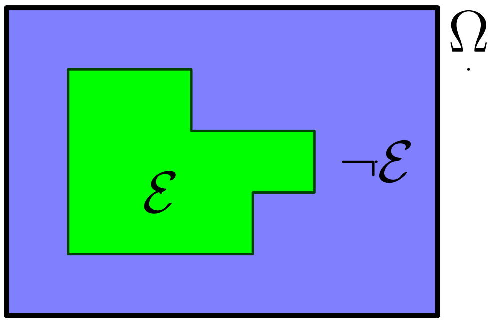

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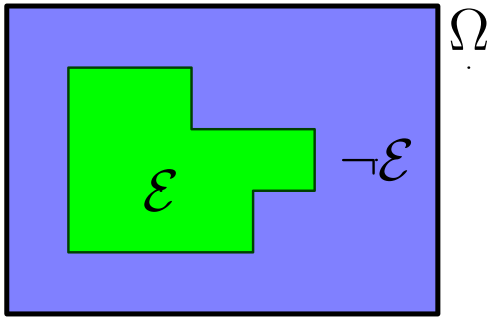

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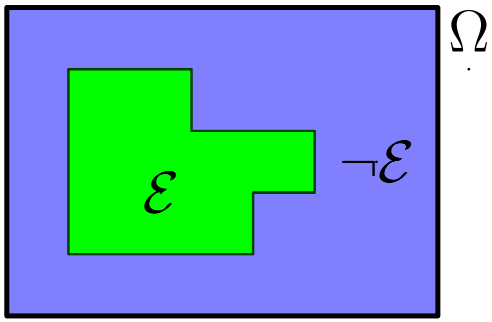

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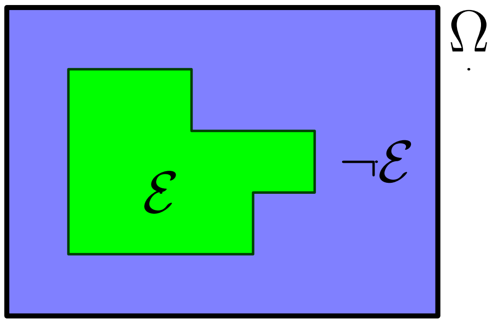

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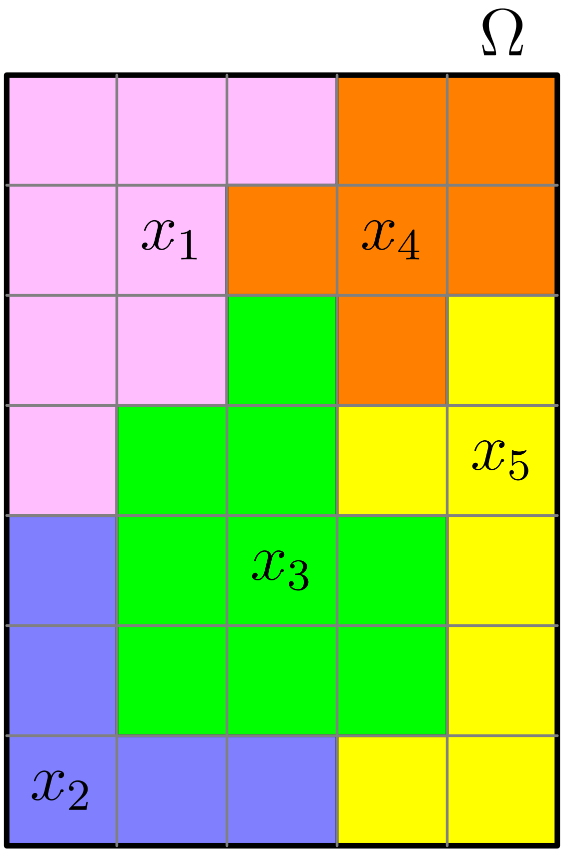
Random Variables

- We can define a **random variable**, X, by partition the set of outcomes Ω and assign a numbers to each partition

- E.g. for a dice

$$X = \begin{cases} 0 & \text{if } \omega \in \{1,3,5\} \\ 1 & \text{if } \omega \in \{2,4,6\} \end{cases}$$

- $\mathbb{P}(X = x_i) = \mathbb{P}(\mathcal{E}_i)$ where $\mathcal{E}_i$ is the event that corresponding to the partition with value x_i



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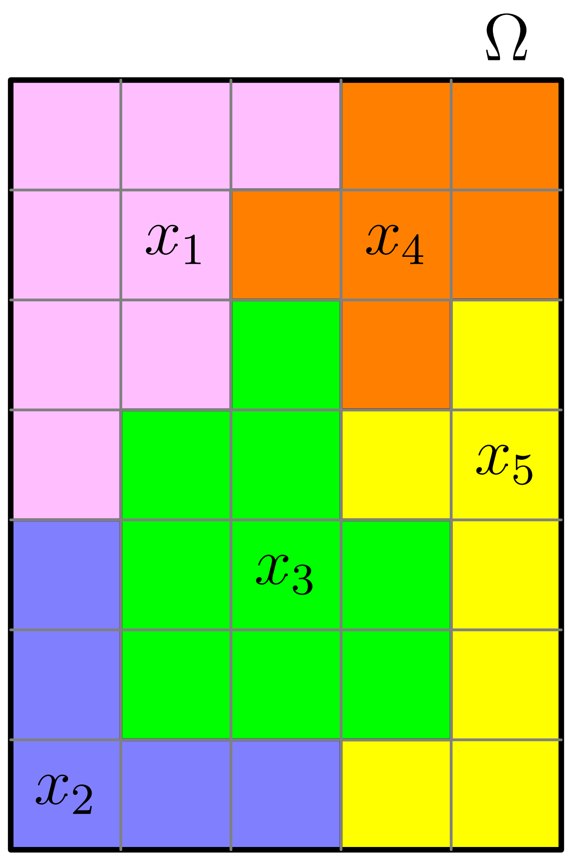

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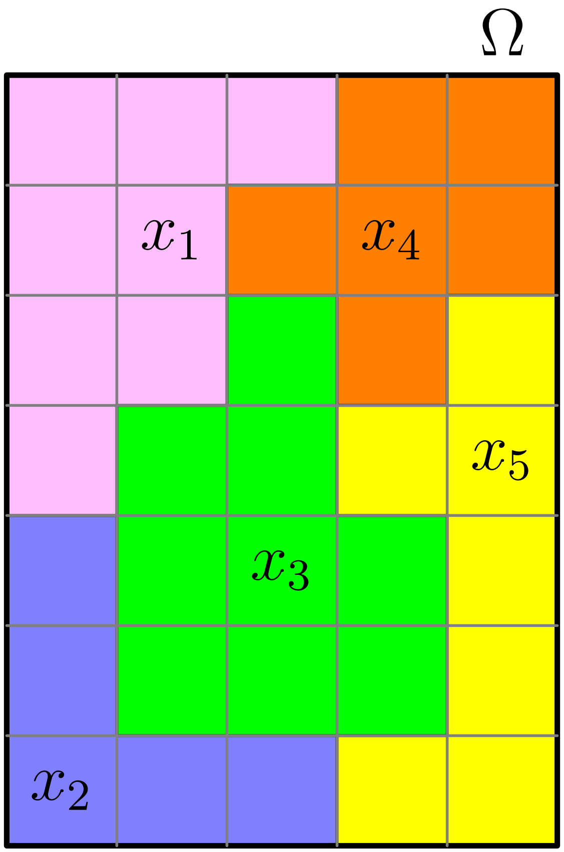

What's In A Name

- We denote random variables with capital letters, X, Y, Z, etc.
- The symbol denote an object that can take one of a number of different values, but which one is still to be decided by chance
- When we write $\mathbb{P}(X)$ we can view this as short-hand for

$$(\mathbb{P}(X = x) \mid x \in \mathcal{X}) = (\mathbb{P}(X = x_1), \mathbb{P}(X = x_2), \ldots \mathbb{P}(X = x_n))$$

  where $\mathcal{X}$ is the set of possible values that X can take

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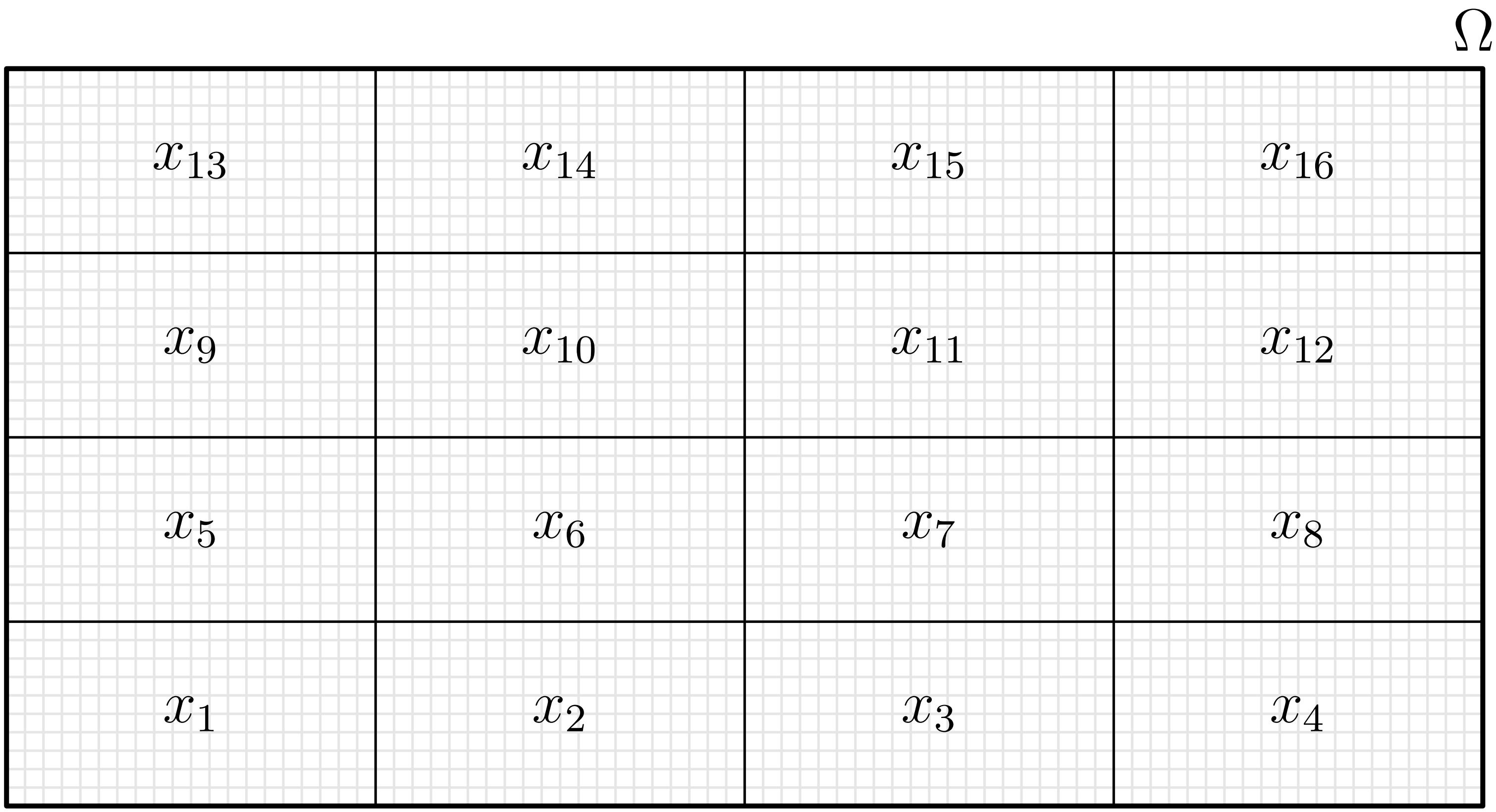

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Ω

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Continuous Spaces

- If the space of elementary events is continuous (e.g. for darts $\boldsymbol{x} = (x, y)$) then $\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) = 0$

- But if we consider a region, $\mathcal{R}$, then we can assign a probability to landing in the region $\mathbb{P}(\boldsymbol{X} \in \mathcal{R})$

- It is useful to work with **probability densities function** (PDF)

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{\mathbb{P}(\boldsymbol{X} \in \mathcal{B}(\boldsymbol{x}, \epsilon))}{|\mathcal{B}(\boldsymbol{x}, \epsilon)|}$$

  where $\mathcal{B}(\boldsymbol{x}, \epsilon)$ is a ball of radius ϵ around the point $\boldsymbol{x}$ and $|\mathcal{B}(\boldsymbol{x}, \epsilon)|$ is the volume of the ball

- If we make a change of variable the volume $|\mathcal{B}(\boldsymbol{x}, \epsilon)|$ might change so $f_{\boldsymbol{X}}(\boldsymbol{x})$ will change

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Change of Variables

- Consider a region $\mathcal{R}$—we can describe this using different coordinate systems x or $y = g(x)$

- But

$$\mathbb{P}(X \in \mathcal{R}) = \int_{\mathcal{R}} f_X(x)\mathrm{d}x = \mathbb{P}(Y \in \mathcal{R}) = \int_{\mathcal{R}} f_Y(y)\mathrm{d}y$$

- As this is true for any region $\mathcal{R}$: $f_X(x)|\mathrm{d}x| = f_Y(y)|\mathrm{d}y|$

- Or

$$f_X(x) = f_Y(y)\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| = f_Y(g(x))|g'(x)|$$

- The probability density measured in units of probability per cm is different to that measured in units of probability per inch

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Jacobian

- In high dimension if we make a change of variables $\boldsymbol{x} \to \boldsymbol{y}(\boldsymbol{x})$ (which can be seen as a change of random variables $\boldsymbol{X} \to \boldsymbol{Y}(\boldsymbol{X})$)
- Then

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{Y}}(\boldsymbol{y})|\det(\mathbf{J})|$$

where $\mathbf{J}$ is the Jacobian matrix

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$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{Y}}(\boldsymbol{y})|\det(\mathbf{J})|$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

- Ensures integrals over volumes are the same

Meaning of Probability Densities

- Probability densities are not probabilities
- They are positive, but don't need to be less than 1
- Note that

$$f_X(x) = \lim_{\delta x \to 0} \frac{\mathbb{P}(x \leq X < x + \delta x)}{\delta x}$$

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Cumulative Distribution Functions

- We can define the **cumulative distribution function** (CDF)

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} \sum_{i:x_i \leq x} \mathbb{P}(X = x_i) \\ \int_{-\infty}^{x} f_X(y)\,\mathrm{d}y \end{cases}$$

- This is a function that goes from 0 to 1 as x goes from $-\infty$ to ∞

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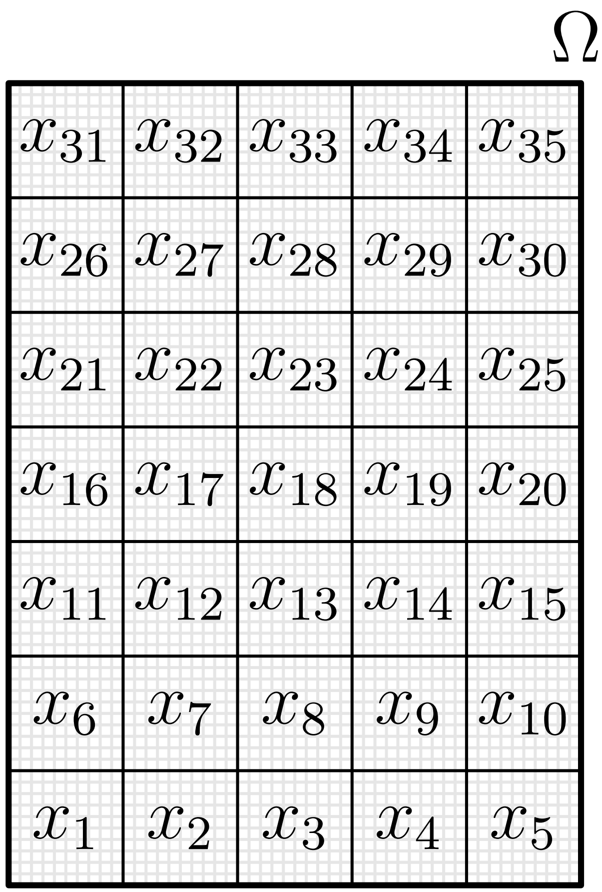
1. Random Variables
2. **Expectations**
3. Calculus of Probabilities



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- We can define the expectation of $\boldsymbol{Y} = g(\boldsymbol{X})$ as

$$\mathbb{E}_{\boldsymbol{X}}[g(\boldsymbol{X})] = \begin{cases} \displaystyle\sum_{\boldsymbol{x}\in\mathcal{X}} g(\boldsymbol{x})\,\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) \\ \displaystyle\int g(\boldsymbol{x})\, f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{cases}$$

- The expectation of a constant c is

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- Because sums and integrals are linear operators

$$\sum_i (a x_i + b y_i) = a \left(\sum_i x_i \right) + b \left(\sum_i y_i \right)$$

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- An indicator function has the property

$$[\![predicate]\!] = \begin{cases} 1 & \text{if } predicate \text{ is True} \\ 0 & \text{if } predicate \text{ is False} \end{cases}$$

(sometimes written $\boldsymbol{I}_A(x)$ where $A(x)$ is the predicate)

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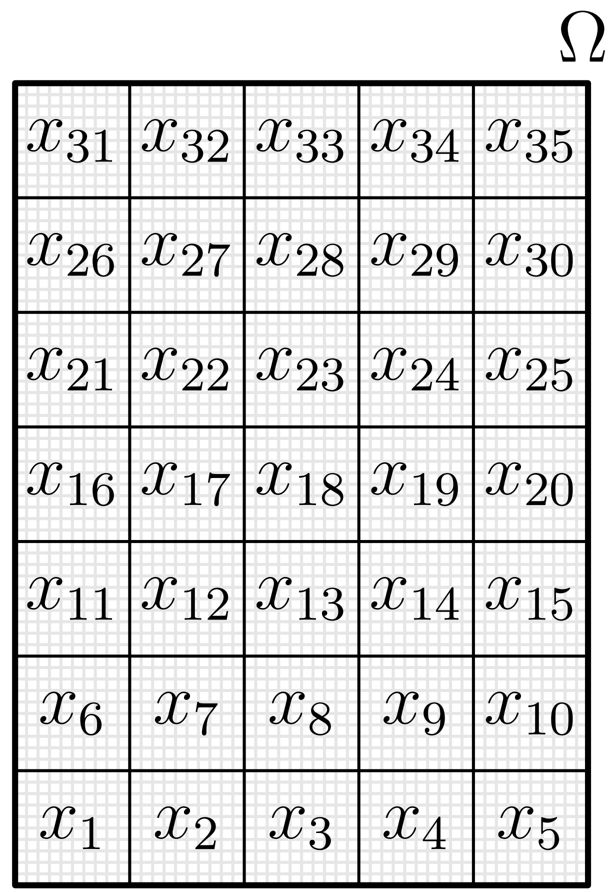

Joint Probabilities

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- But if we know $\mathbb{P}(\text{study}) = 0.8$ then we can compute

$$\mathbb{P}(\text{pass}, \text{study}) = \mathbb{P}(\text{pass} \mid \text{study})\, \mathbb{P}(\text{study}) = 0.9 \times 0.8 = 0.72$$

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Independence

- Random variables X and Y are said to be **independent** if

$$\mathbb{P}(X,Y) = \mathbb{P}(X)\,\mathbb{P}(Y)$$

- Because $\mathbb{P}(X,Y) = \mathbb{P}(X|Y)\,\mathbb{P}(Y)$ and $\mathbb{P}(X,Y) = \mathbb{P}(Y|X)\,\mathbb{P}(X)$ independence implies

$$\mathbb{P}(X|Y) = \mathbb{P}(X) \qquad\qquad \mathbb{P}(Y|X) = \mathbb{P}(Y)$$

- Probabilistic independence implies a mathematical co-incident not necessarily causal independence
- However causal independence implies probabilistic independence
- If $X \in \{0,1\}$ represents the outcome of tossing a coin and $Y \in \{1,2,3,4,5,6\}$ the outcome of rolling a dice then X and Y are independent

Independence

- Random variables X and Y are said to be **independent** if

$$\mathbb{P}(X,Y) = \mathbb{P}(X)\,\mathbb{P}(Y)$$

- Because $\mathbb{P}(X,Y) = \mathbb{P}(X|Y)\,\mathbb{P}(Y)$ and $\mathbb{P}(X,Y) = \mathbb{P}(Y|X)\,\mathbb{P}(X)$ independence implies

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Well Conducted Experiments

- In well conducted experiments we expect the results we obtain are independent
- Let $\mathcal{D} = (X_1, X_2, \ldots, X_m)$ represents possible outcomes from a set of m well conducted experiments then

$$\mathbb{P}(\mathcal{D}) = \prod_{i=1}^{m} \mathbb{P}(X_i)$$

- Denoting a possible sentence I might say by $\mathcal{S} = (W_1, W_2, \ldots, W_m)$ then

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otherwise it's time I retired

Conditional Independence

- Let $K(d)$ be a random variable measuring the amount you know about ML on day d of your revision

- From you revision schedule you can write down your belief

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- But a very reasonable model is

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  what you are going to know today will just depend on what you knew yesterday

- We say that $K(d)$ is **conditionally independent** on $K(d-2)$, $K(d-3)$, etc. given $K(d-1)$

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Conclusion

- To work with probabilities you need to know
  - ⋆ How to go back and forward between joint probabilities and conditional probabilities
  - ⋆ How to marginalise out variables
- You need to understand that for continuous outcomes, it makes sense to talk about the probability density
- You need to know that expectations are linear operators and the expectation of a constant is the constant
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