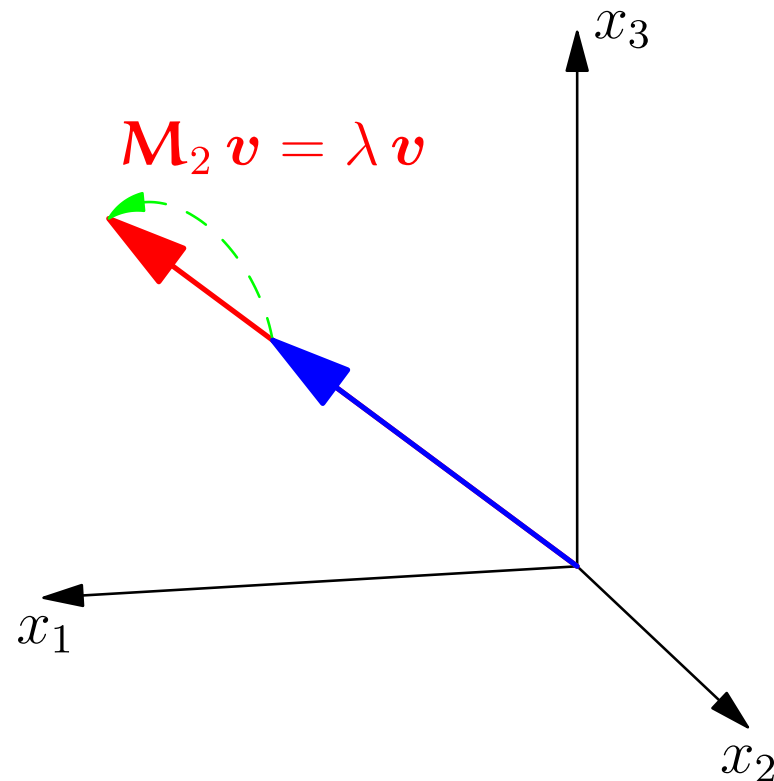


Advanced Machine Learning

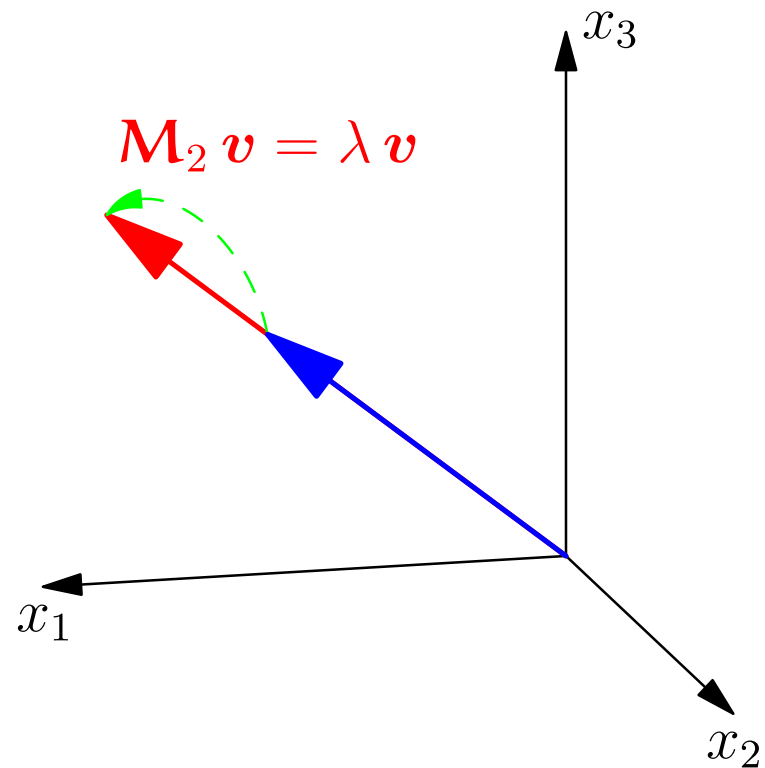
Eigensystems



Eigenvectors, Orthogonal Matrices, Eigenvector Decomposition, Rank

Outline

1. **Eigenvectors**
2. Orthogonal Matrices
3. Eigen Decomposition
4. Low Rank Approximation



Eigenvector equation

- Eigen-systems help us to understand mappings
- A vector v is said to be an **eigenvector** if

$$Mv = \lambda v$$

- M is square (i.e. $n \times n$)
- Where the number λ is the **eigenvalue**
- Eigenvalues play a fundamental role in understanding operators

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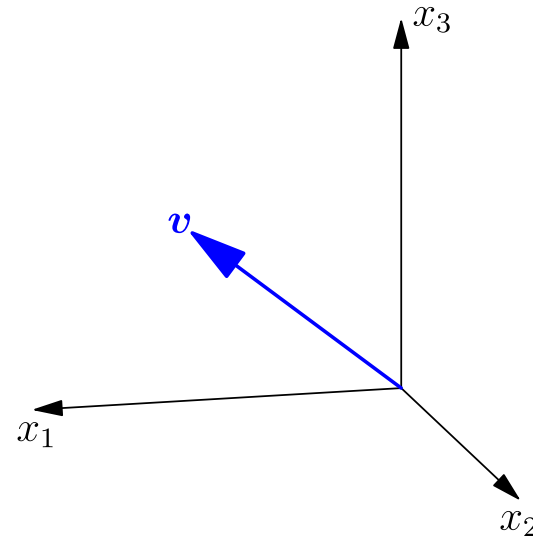
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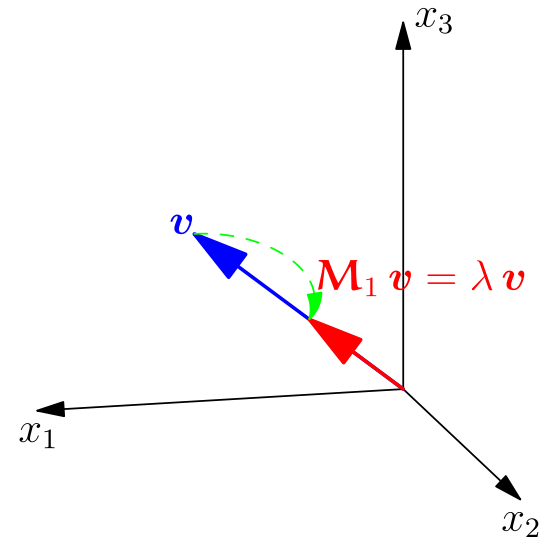


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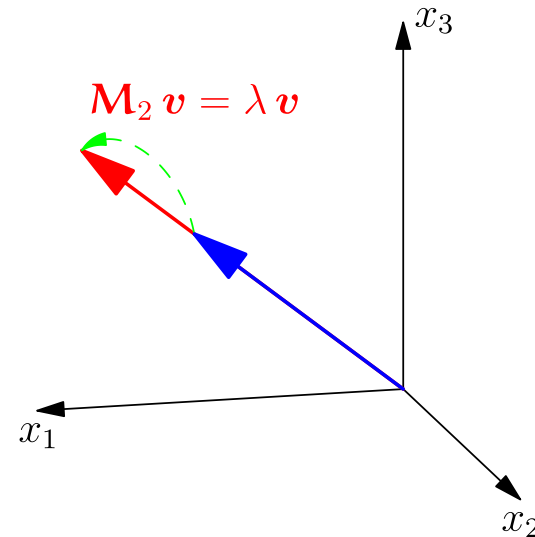


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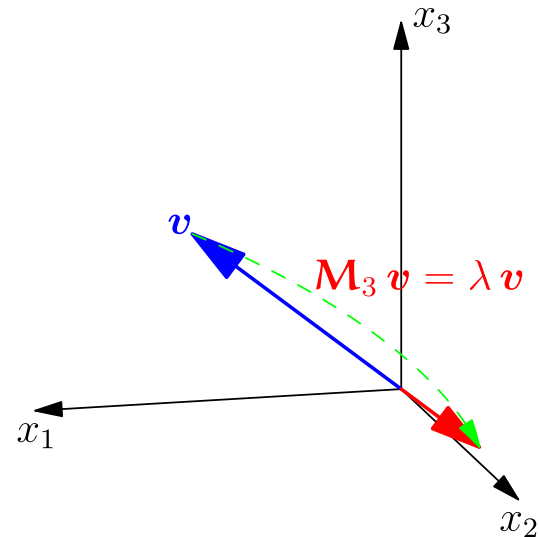


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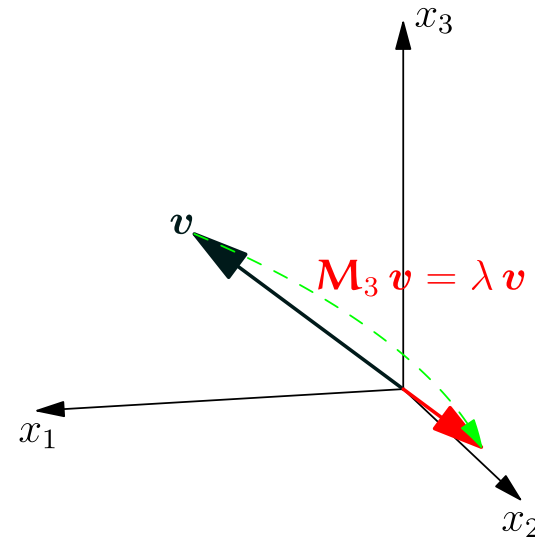


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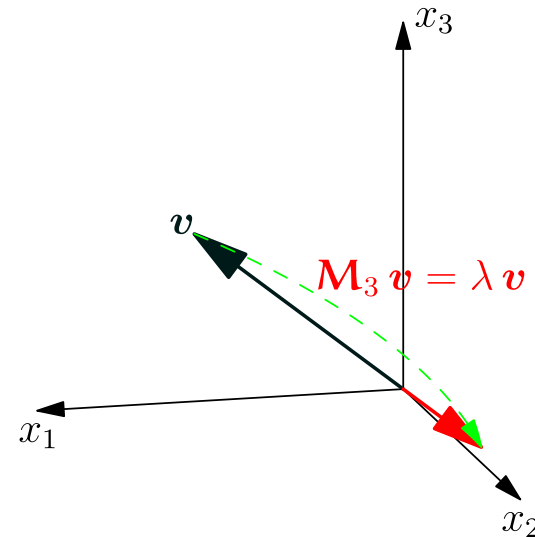


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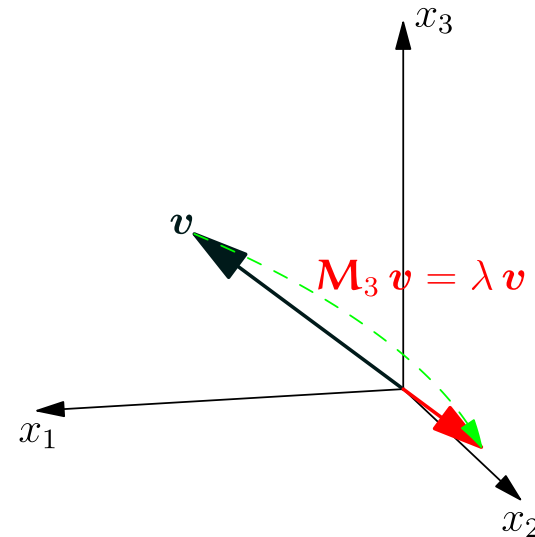


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Symmetric Matrices

- If \mathbf{M} is an $n \times n$ **symmetric** matrix then it has n real orthogonal eigenvectors with real eigenvalues
- We denote the i^{th} eigenvector by \mathbf{v}_i and the corresponding eigenvalue by λ_i so that

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

- Orthogonal means that if $i \neq j$ then

$$\mathbf{v}_i^T \mathbf{v}_j = 0$$

- (We can always normalise eigenvectors if we want)

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Proof of Orthogonality

- $(\mathbf{M}\mathbf{v}_i = \lambda_i\mathbf{v}_i)^\top$ implies $\mathbf{v}_i^\top \mathbf{M}^\top = \lambda_i\mathbf{v}_i^\top$
- When \mathbf{M} is symmetric then $\mathbf{M}\mathbf{v}_i = \lambda_i\mathbf{v}_i \Rightarrow \mathbf{v}_i^\top \mathbf{M} = \lambda_i\mathbf{v}_i^\top$
- Consider two eigenvectors \mathbf{v}_i and \mathbf{v}_j of \mathbf{M}

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- If $\lambda_i = \lambda_j$ then any linear combination of \mathbf{v}_i and \mathbf{v}_j is an eigenvector ($\mathbf{M}(a\mathbf{v}_i + b\mathbf{v}_j) = \lambda_i(a\mathbf{v}_i + b\mathbf{v}_j)$)

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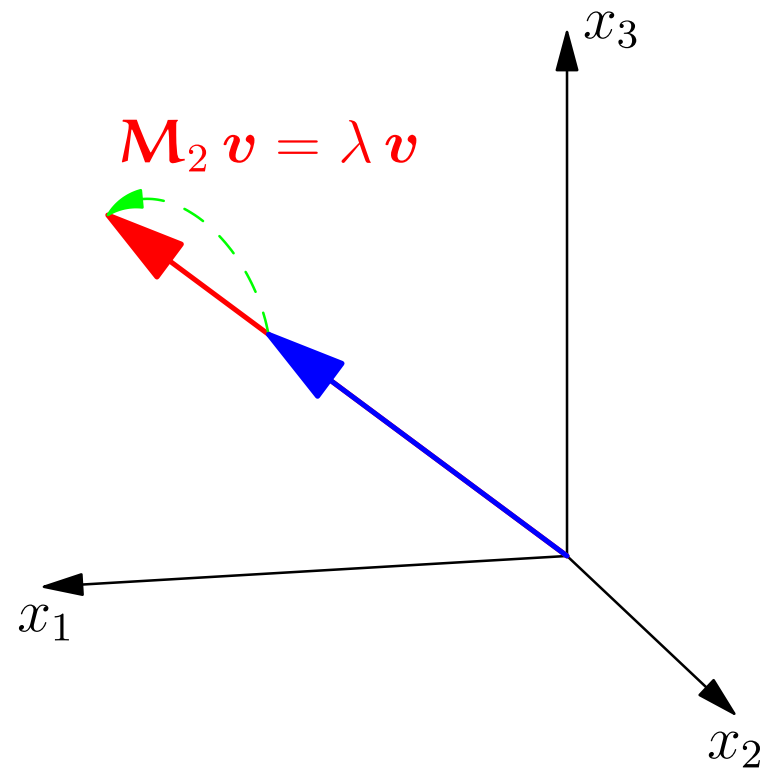
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Outline

1. Eigenvectors
2. **Orthogonal Matrices**
3. Eigen Decomposition
4. Low Rank Approximation



Orthogonal Matrices

- We can construct an **orthogonal** matrix **V** from the eigenvectors

$$\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$$

- Matrix **V** is an $n \times n$ matrix
- Because of the orthogonality of the vectors \mathbf{v}_i

$$\mathbf{V}^T \mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T \mathbf{v}_1 & \mathbf{v}_1^T \mathbf{v}_2 & \cdots & \mathbf{v}_1^T \mathbf{v}_n \\ \mathbf{v}_2^T \mathbf{v}_1 & \mathbf{v}_2^T \mathbf{v}_2 & \cdots & \mathbf{v}_2^T \mathbf{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_n^T \mathbf{v}_1 & \mathbf{v}_n^T \mathbf{v}_2 & \cdots & \mathbf{v}_n^T \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I}$$

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The Other Way Around

- We have shown that $\mathbf{V}^T \mathbf{V} = \mathbf{I}$
- Thus multiply both sides on the left by \mathbf{V}

$$\mathbf{V} \mathbf{V}^T \mathbf{V} = \mathbf{V}$$

- \mathbf{V} will have an inverse, \mathbf{V}^{-1} , such that $\mathbf{V} \mathbf{V}^{-1} = \mathbf{I}$
- Multiplying the equation on the right by \mathbf{V}^{-1}

$$(\mathbf{V} \mathbf{V}^T) \mathbf{V} \mathbf{V}^{-1} = \mathbf{V} \mathbf{V}^{-1}$$

- Note that, $\mathbf{V}^{-1} = \mathbf{V}^T$ (definition of orthogonal matrix)

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$$\begin{aligned} (\mathbf{V} \mathbf{V}^T) \mathbf{V} \mathbf{V}^{-1} &= \mathbf{V} \mathbf{V}^{-1} \\ \mathbf{V} \mathbf{V}^T &= \mathbf{I} \end{aligned}$$

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Invertible Matrices

- A matrix, \mathbf{M} , will be singular (uninvertible) if there exists a vector \mathbf{x} ($\neq \mathbf{0}$) such that

$$\mathbf{M}\mathbf{x} = \mathbf{0}$$

- Now if there exists such a vector such that $\mathbf{V}\mathbf{x} = \mathbf{0}$ then multiply by \mathbf{V}^T we get

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Invertible Matrices

- A matrix, \mathbf{M} , will be singular (uninvertible) if there exists a vector \mathbf{x} ($\neq \mathbf{0}$) such that

$$\mathbf{M}\mathbf{x} = \mathbf{0}$$

- Now if there exists such a vector such that $\mathbf{V}\mathbf{x} = \mathbf{0}$ then multiply by \mathbf{V}^T we get

$$\mathbf{V}^T\mathbf{V}\mathbf{x} = \mathbf{V}^T\mathbf{0}$$

$$\mathbf{x} = \mathbf{0}$$

since $\mathbf{V}^T\mathbf{V} = \mathbf{I}$

- Thus \mathbf{V} is invertible

Rotations

- Orthogonal matrices satisfy $\mathbf{V}^T \mathbf{V} = \mathbf{V} \mathbf{V}^T = \mathbf{I}$
- As a consequent they define rotations (and possibly a reflection)
- Consider a vector \mathbf{x} and $\mathbf{x}' = \mathbf{V}\mathbf{x}$, now

$$\|\mathbf{x}'\|_2^2 = \mathbf{x}'^T \mathbf{x}' = (\mathbf{V}\mathbf{x})^T (\mathbf{V}\mathbf{x}) = \mathbf{x}^T \mathbf{V}^T \mathbf{V} \mathbf{x} = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|_2^2$$

- Similarly if additionally $\mathbf{y}' = \mathbf{V}\mathbf{y}$ then

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- Rotations and reflections preserve lengths and angles

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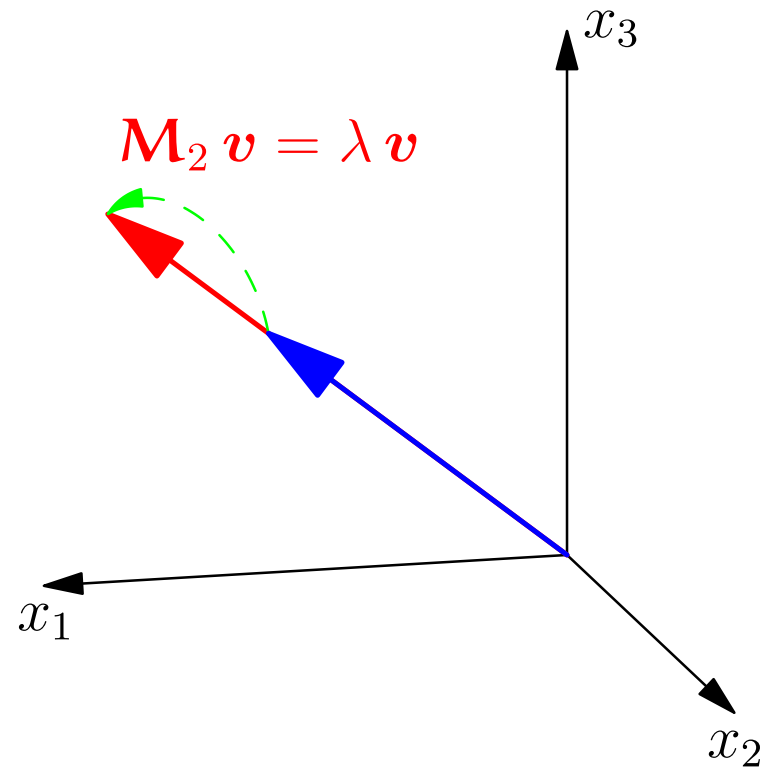
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Outline

1. Eigenvectors
2. Orthogonal Matrices
3. **Eigen Decomposition**
4. Low Rank Approximation



Matrix Decomposition

- Taking the matrix of eigenvectors, V , then

$$MV = M(v_1, v_2, \dots, v_n) = (\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n) = V\Lambda$$

- where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$

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$$M = MVV^T = V\Lambda V^T$$

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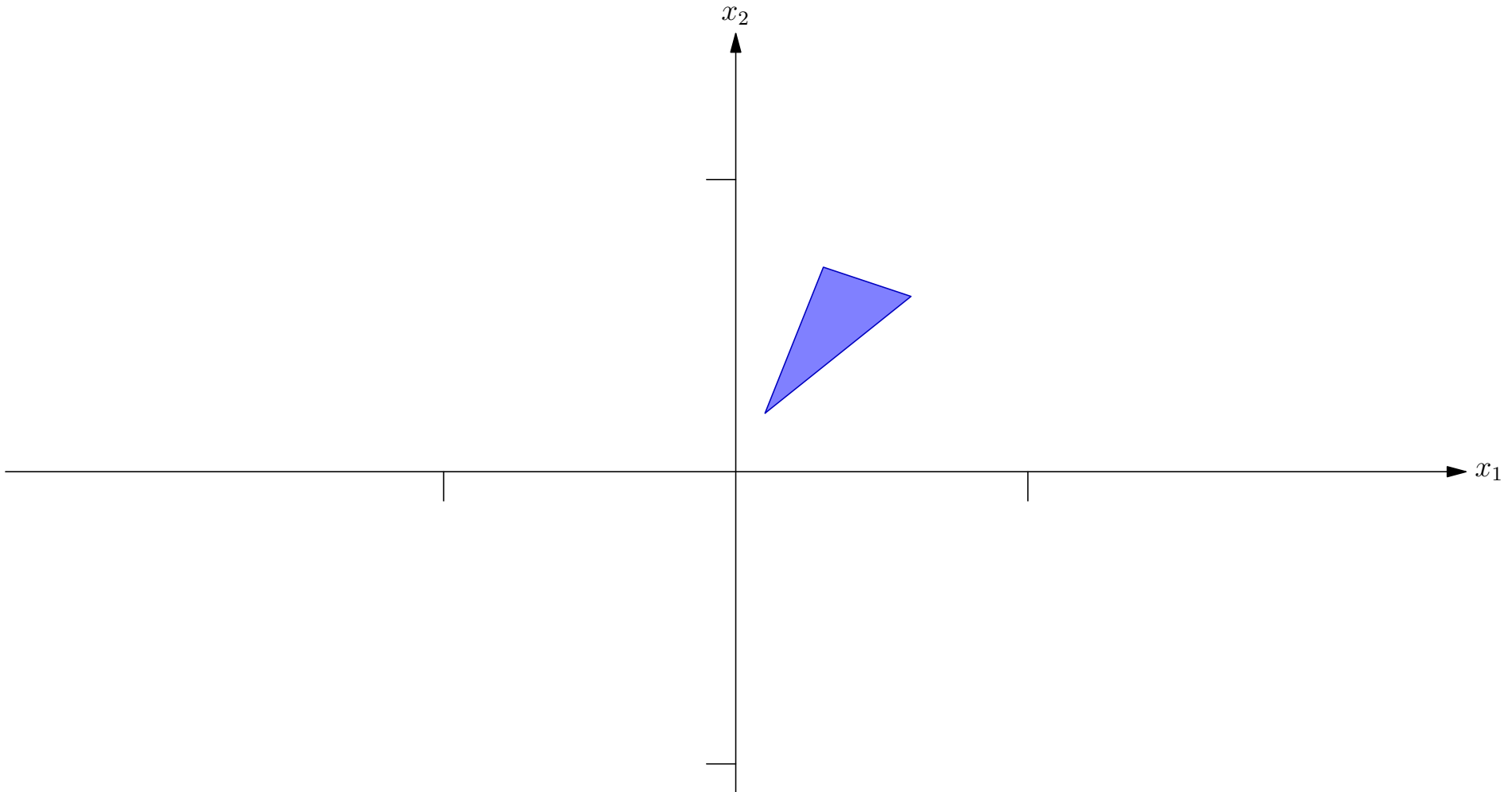
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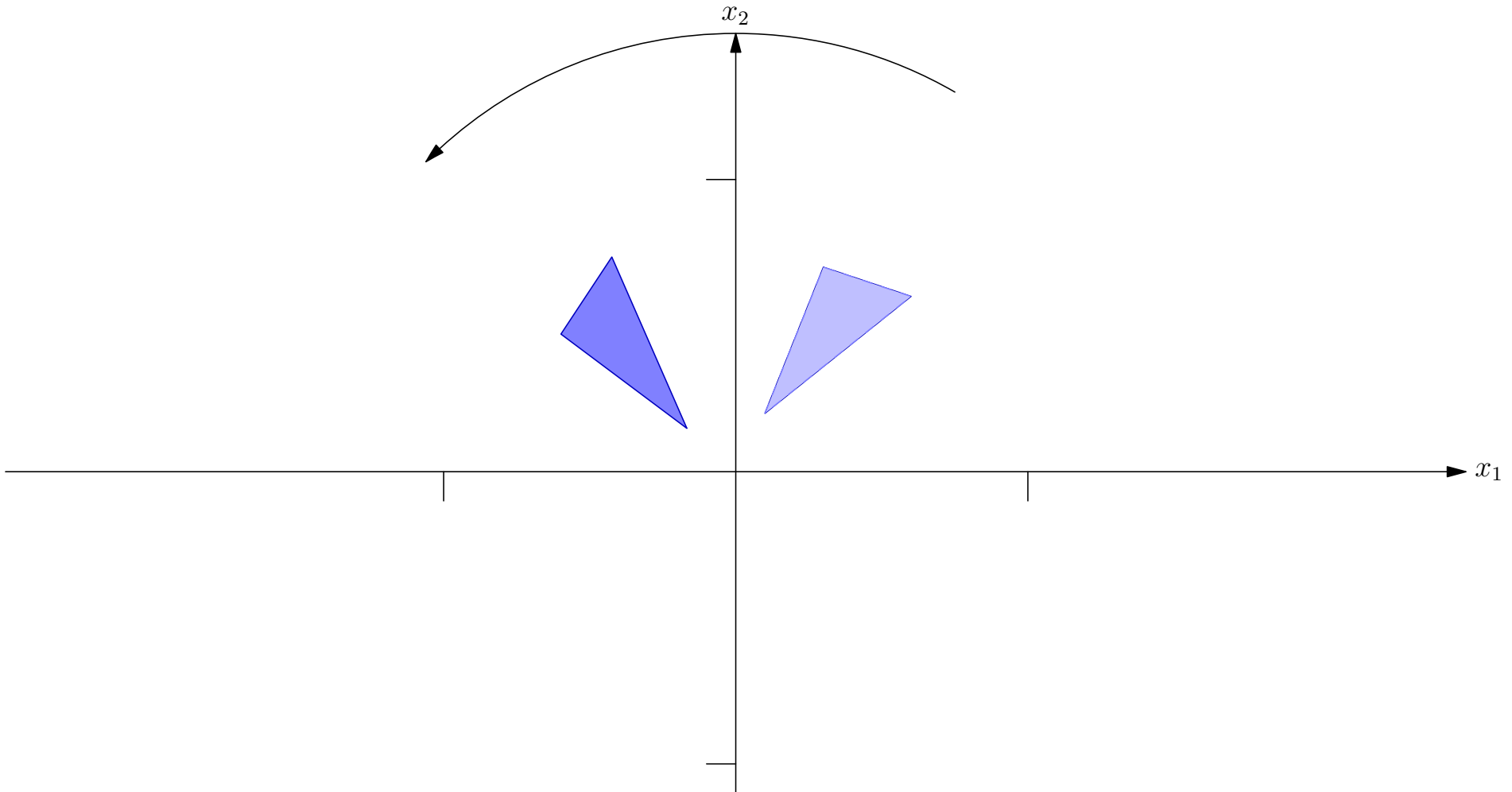
Mappings by Symmetric Matrices

$$\mathbf{M} = \begin{pmatrix} 0.83 & -0.31 \\ -0.31 & 1.9 \end{pmatrix} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T = \begin{pmatrix} \cos(-75) & \sin(-75) \\ -\sin(-75) & \cos(-75) \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0.75 \end{pmatrix} \begin{pmatrix} \cos(75) & \sin(75) \\ -\sin(75) & \cos(75) \end{pmatrix}$$



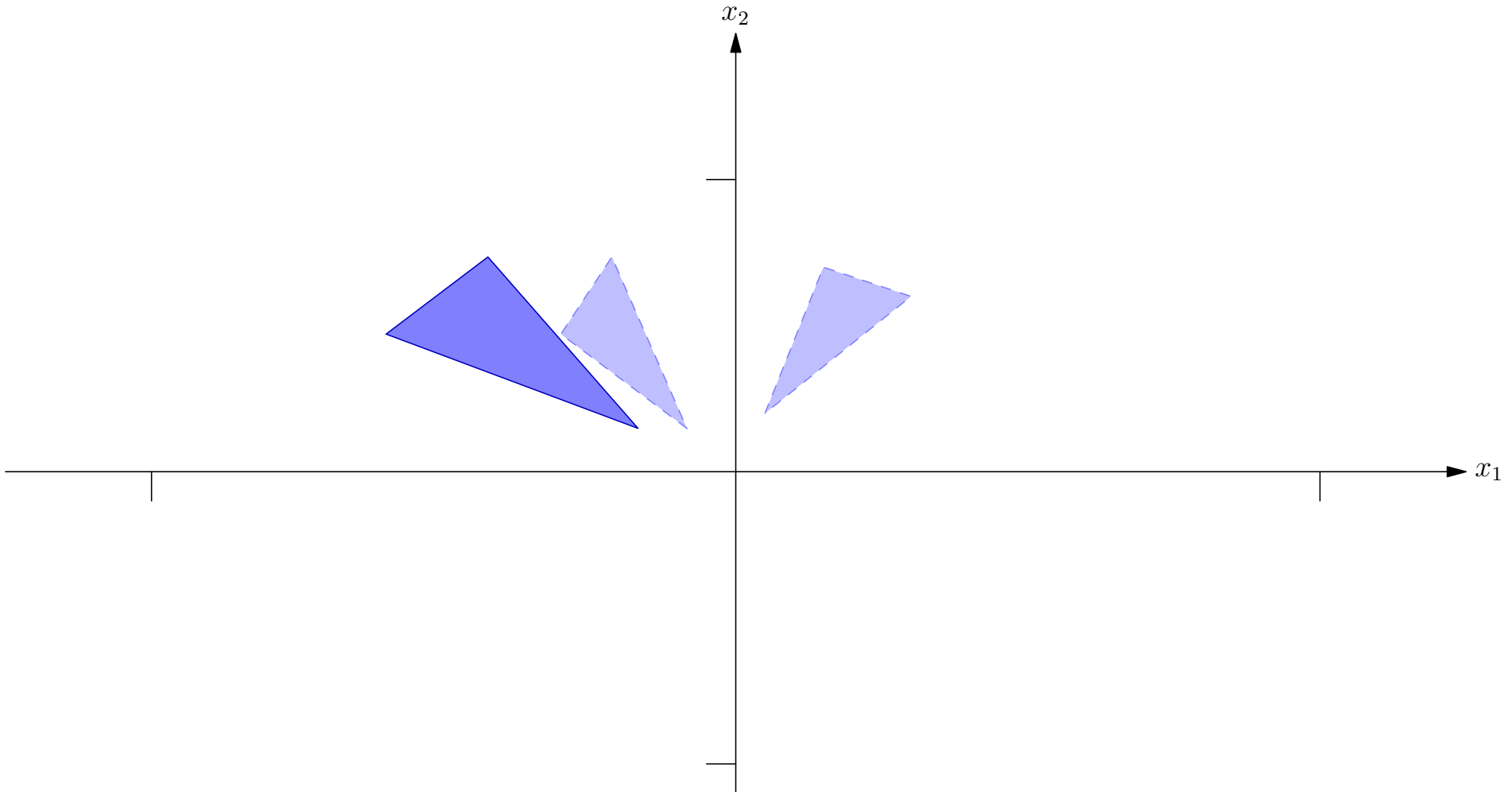
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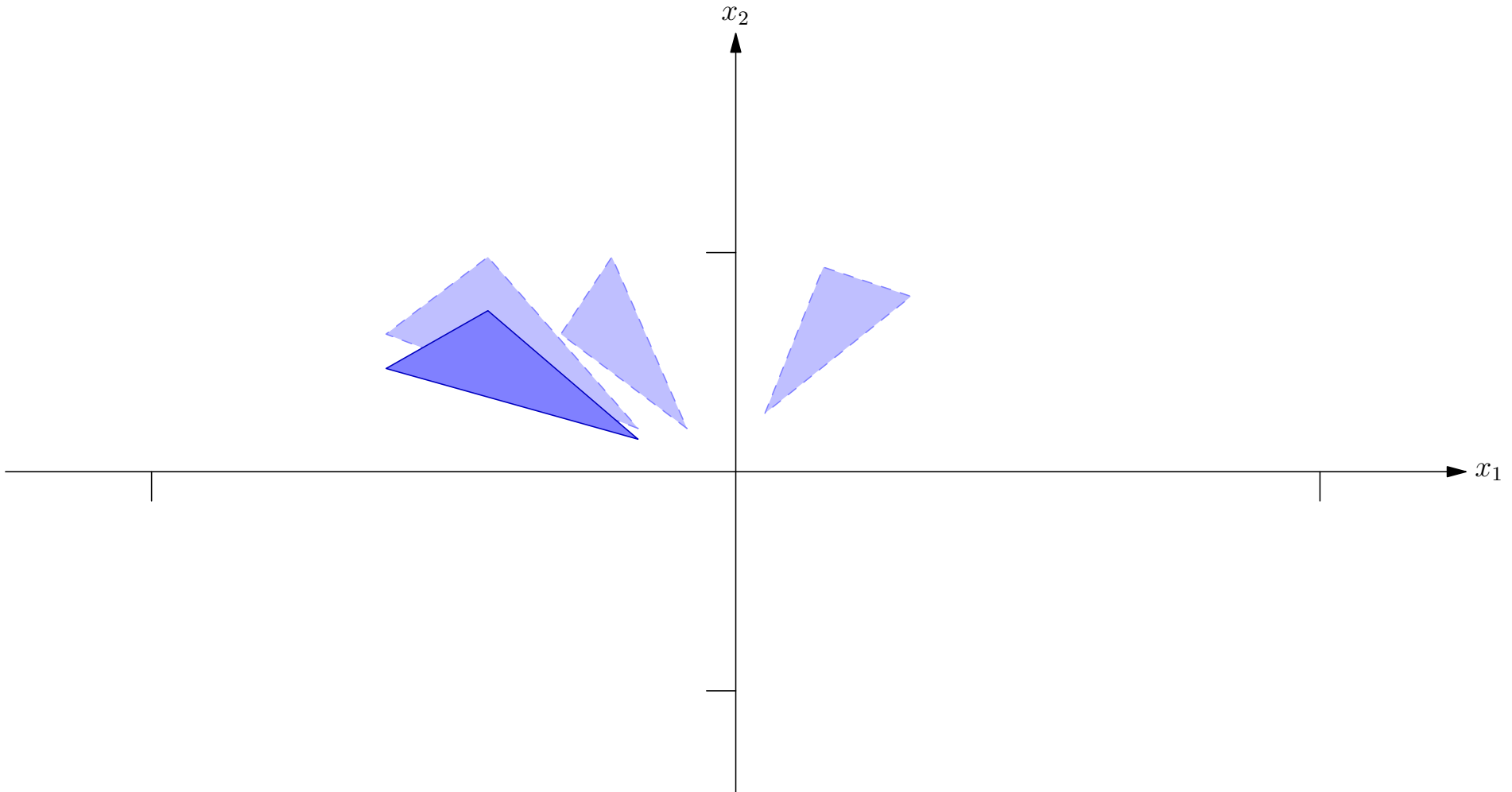
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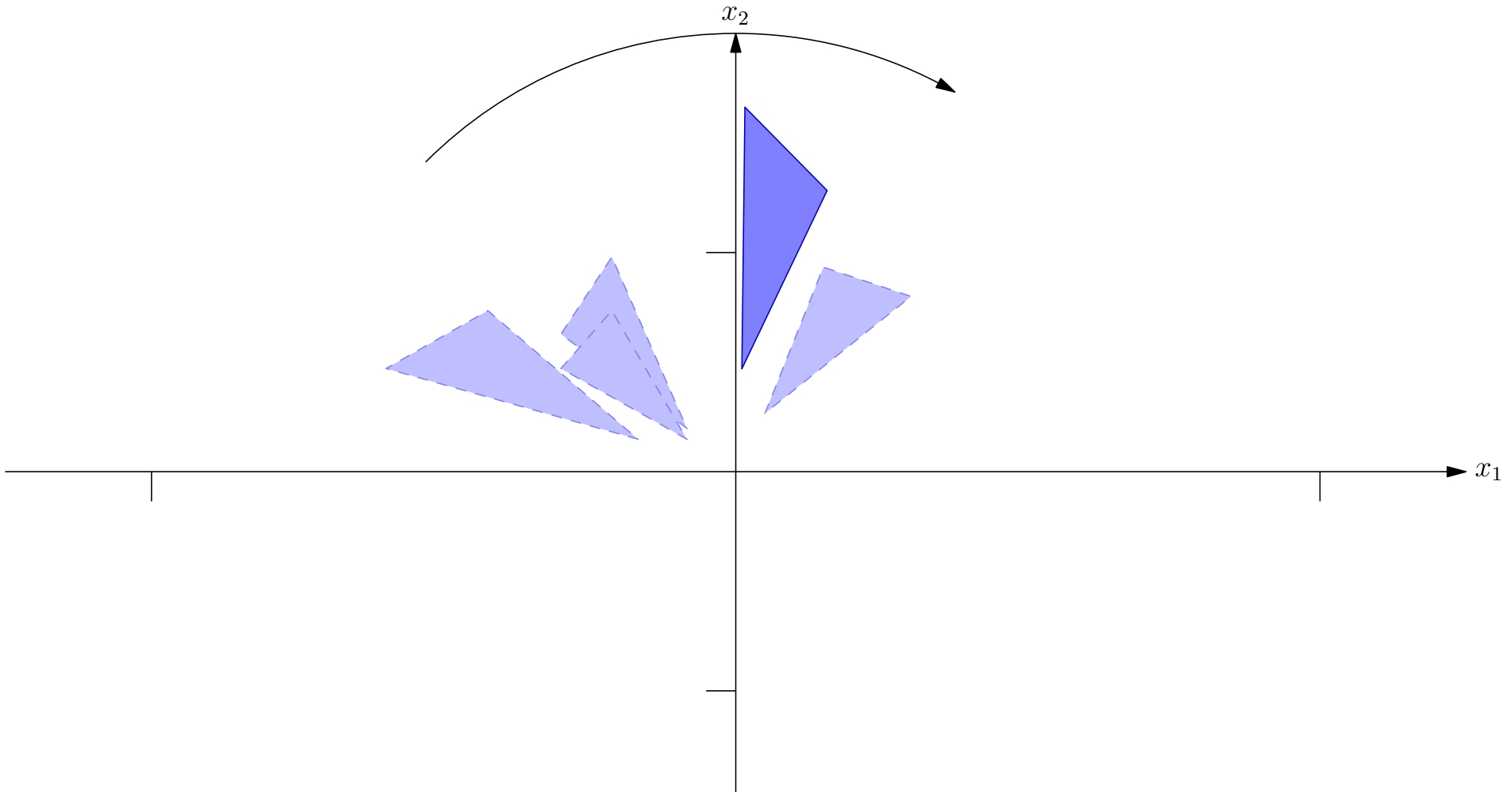
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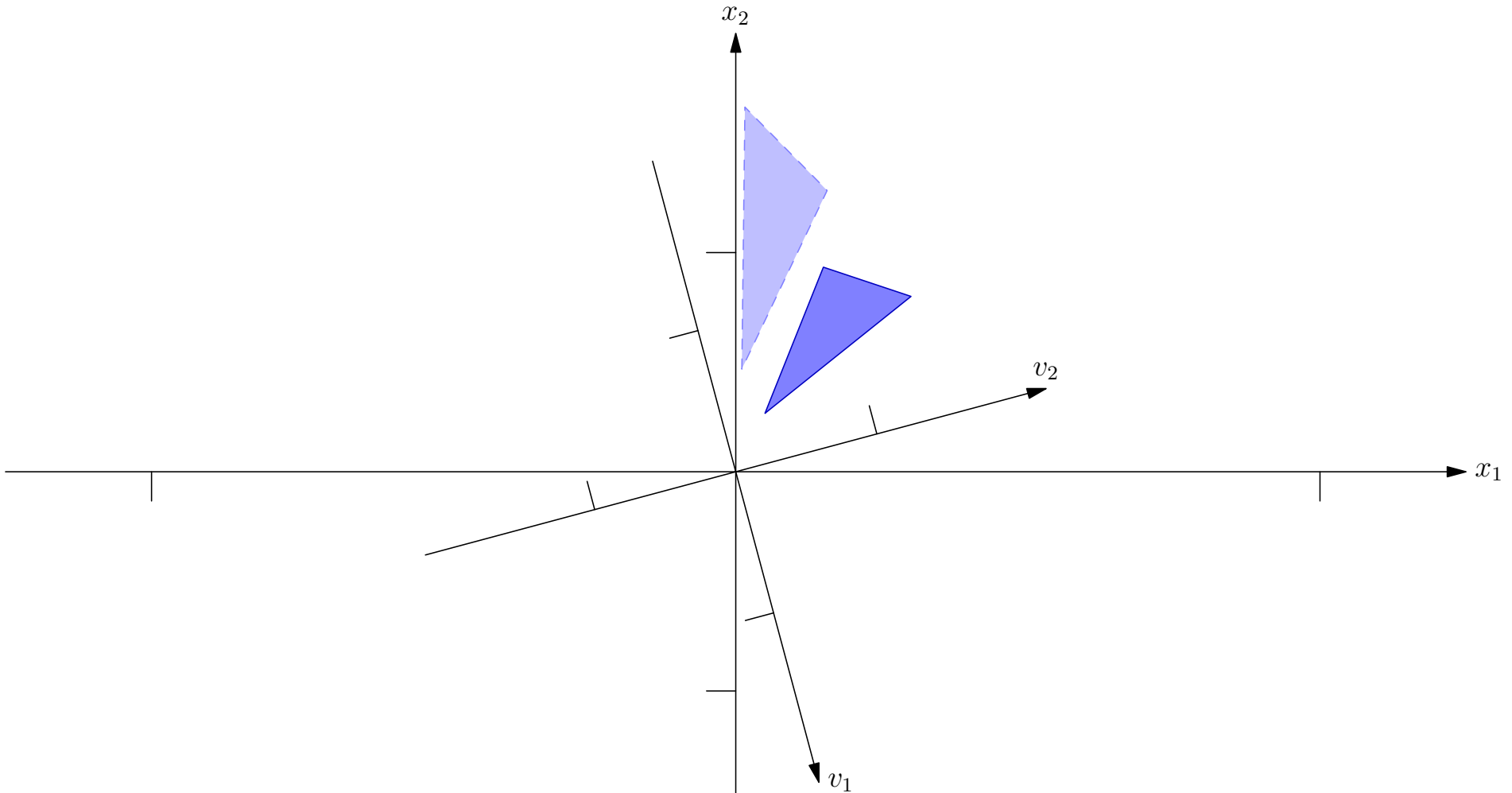
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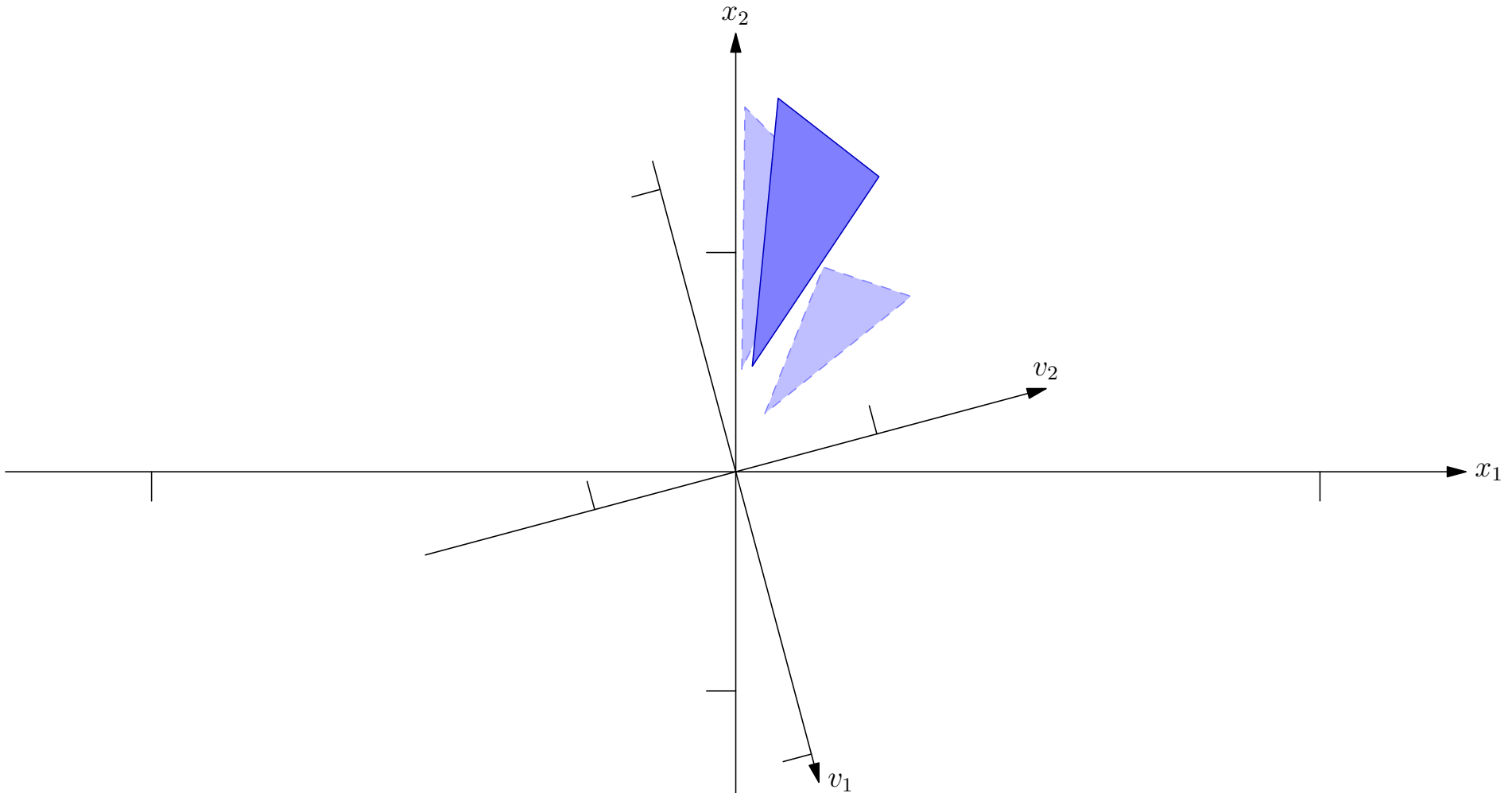
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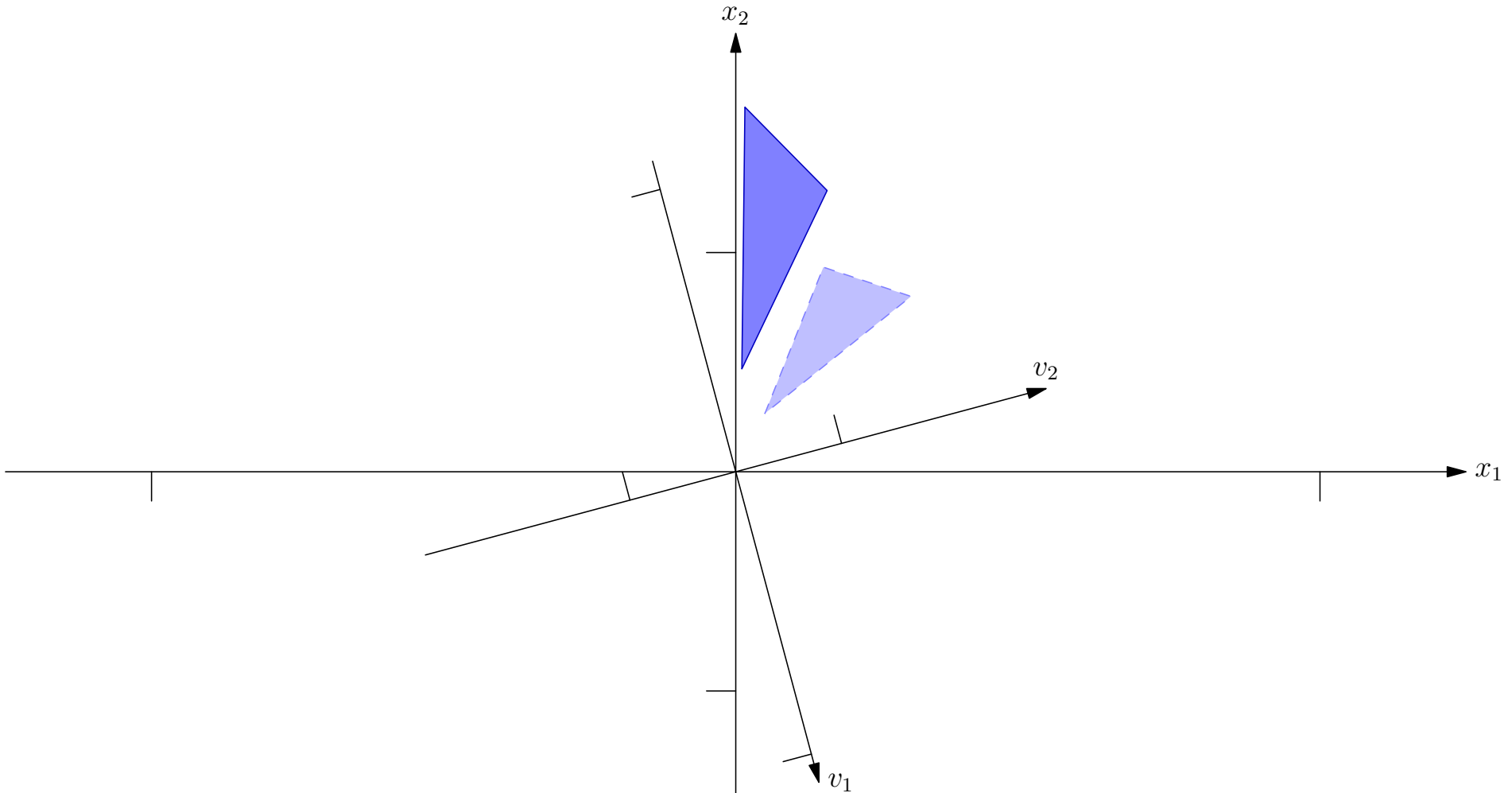
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$$\mathbf{M} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

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- Where $\mathbf{\Lambda}^{-1} = \text{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}) = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda_n} \end{pmatrix}$

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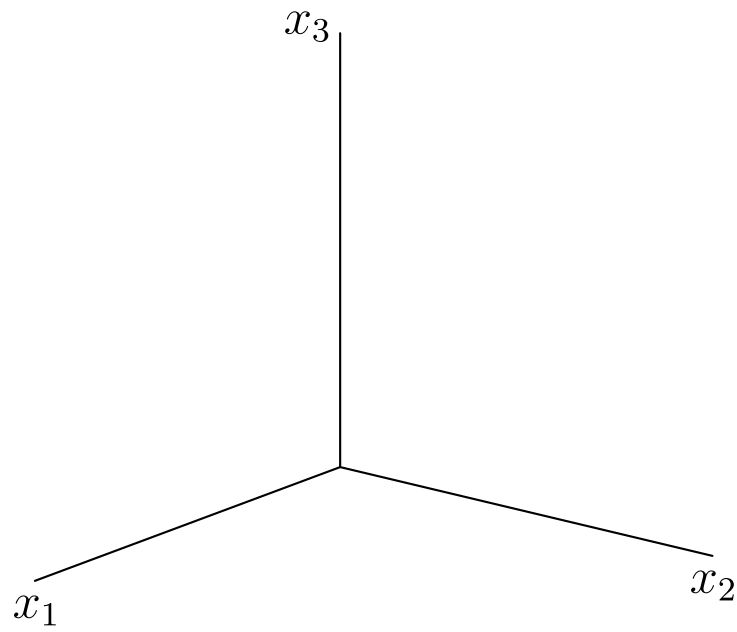
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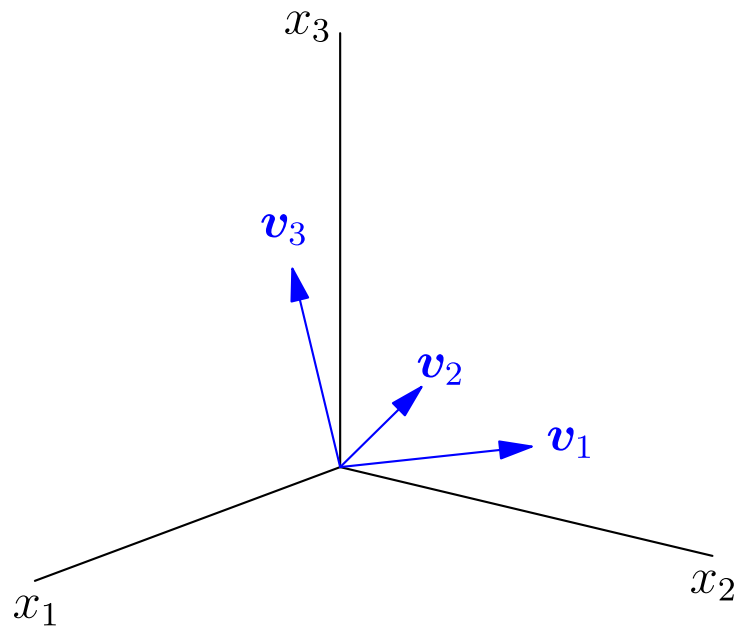
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III-Conditioning Again



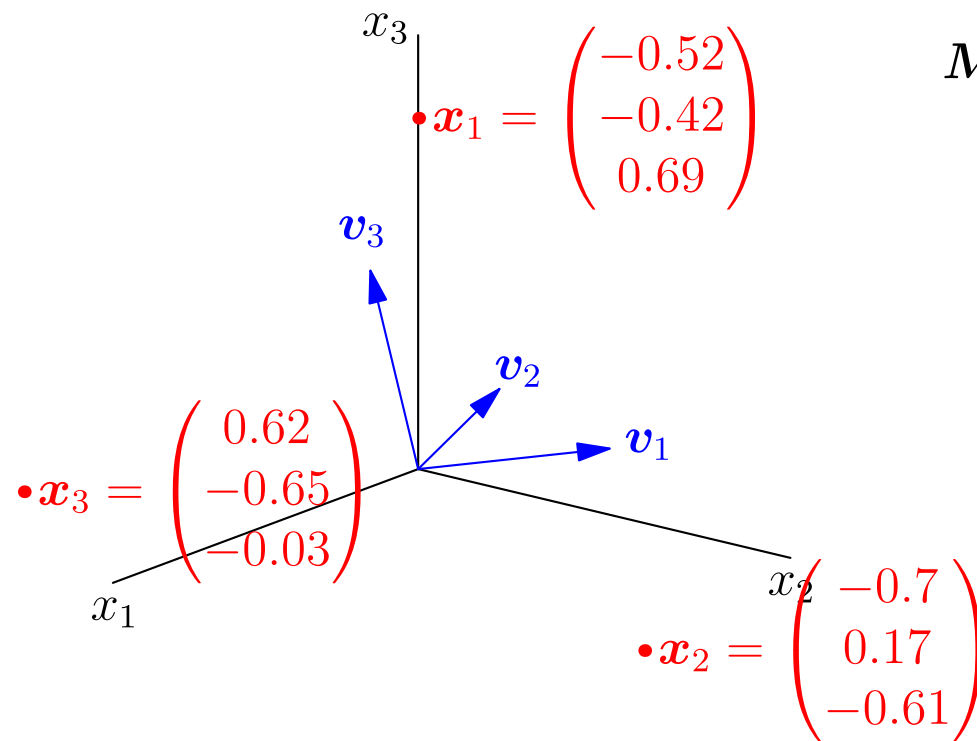
$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 0.80 & 0.01 & 0.25 \\ 0.01 & 0.85 & 0.48 \\ 0.25 & 0.48 & 0.5 \end{pmatrix} \\ &= \mathbf{V} \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \mathbf{V}^T \end{aligned}$$

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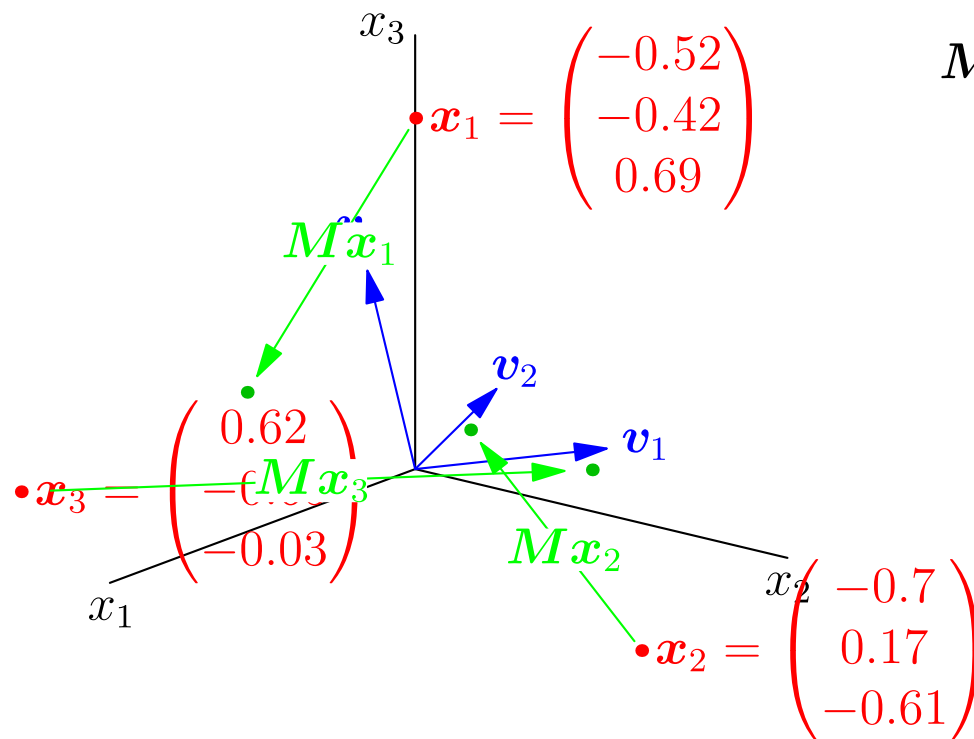
$$\begin{aligned} M &= \begin{pmatrix} 0.80 & 0.01 & 0.25 \\ 0.01 & 0.85 & 0.48 \\ 0.25 & 0.48 & 0.5 \end{pmatrix} \\ &= V \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} V^T \end{aligned}$$

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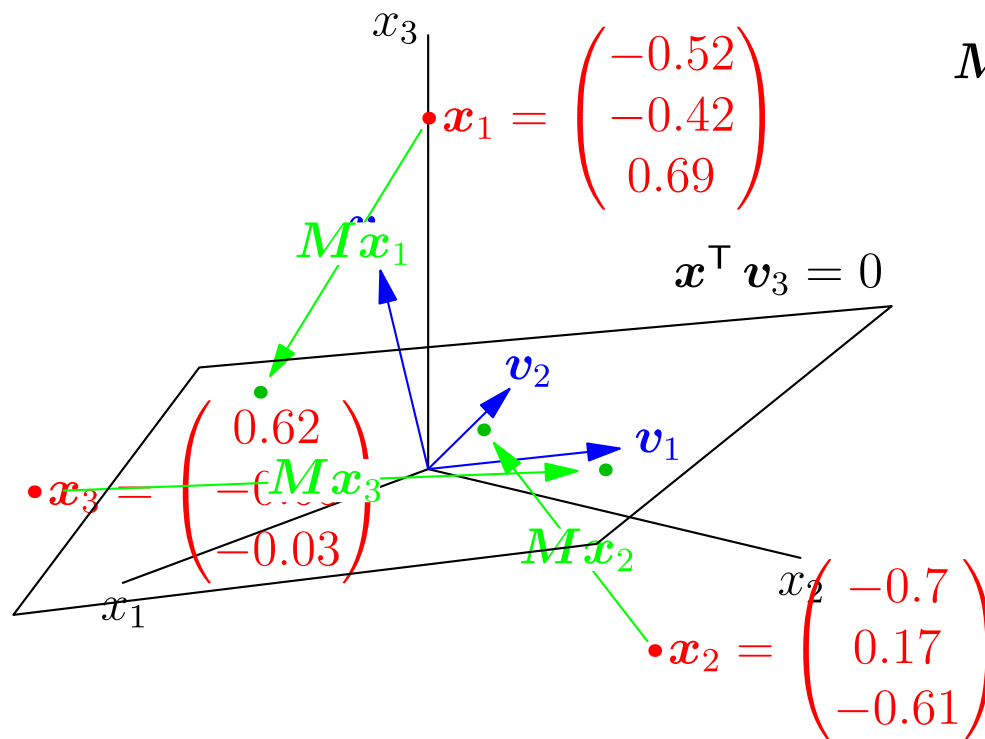
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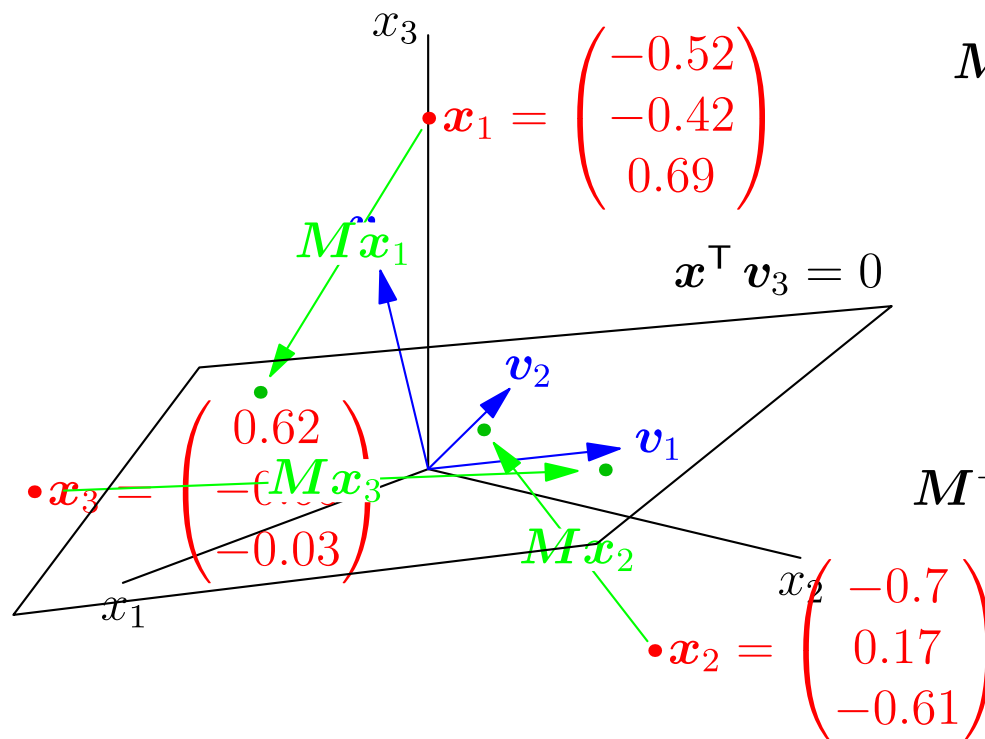
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$$M^{-1} = \begin{pmatrix} 1.9 & 1.17 & -2.1 \\ 1.17 & 3.34 & -3.8 \\ -2.1 & -3.8 & 6.8 \end{pmatrix}$$

$$= V \begin{pmatrix} 1.25 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 10 \end{pmatrix} V^T$$

Condition Number

- Taking matrix inverses can be inherently unstable
- Any small error can be amplified by taking the inverse
- The stability of the inverse depends on the ratio of smallest eigenvalue to the largest eigenvalue (i.e. the biggest possible amplification compared to the smallest)
- Note that the Hilbert-norm of a matrix is the absolute value of the largest eigenvalue
- The condition number is given by

$$\|\mathbf{M}\|_H \times \|\mathbf{M}^{-1}\|_H = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}$$

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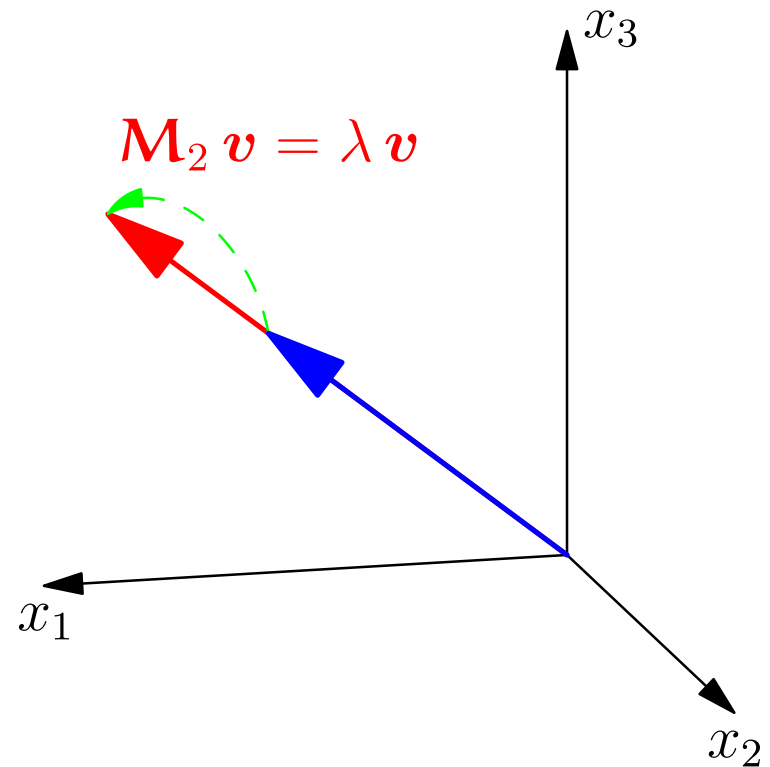
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Outline

1. Eigenvectors
2. Orthogonal Matrices
3. Eigen Decomposition
4. **Low Rank Approximation**



Rank of a Matrix

- The rank of a matrix, \mathbf{M} , is the number of non-zero eigenvalues
- The space spanned by the eigenvectors \mathbf{v}_a , \mathbf{v}_b , etc. with zero eigenvalue forms a **null space**
- Any vector in the null space will get projected to the zero vector

$$\mathbf{M}(a\mathbf{v}_a + b\mathbf{v}_b + \dots) = \mathbf{0}$$

- A square matrix is said to be **rank deficient** if it has any eigenvectors with eigenvalue equal to 0
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“Inverting” Rank Deficient Matrices

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- Although we don't know x we can find a vector, x , that satisfies $\mathbf{M}x = b$
- Given a symmetric $n \times n$ matrix with k non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ we can construct a “pseudo inverse” \mathbf{M}^+ as $\mathbf{V}\mathbf{\Lambda}^+\mathbf{V}^T$ where $\mathbf{\Lambda}^+ = \text{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_k}, 0, \dots, 0)$
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- Recall that matrices with large and small eigenvalues are ill-conditions so the inverse has the potential to greatly amplify any measurement error
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Summary

- Linear mappings are commonly used in machine learning algorithms such as regression
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- Any symmetric matrix can be decomposed as $\mathbf{M} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$
 - ★ where \mathbf{V} are orthogonal matrices whose rows are the eigenvector
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