

Information, KL-divergence, Minimum Description Length

Communicating Via a Noisy Channel

- Information theory considers communicating down a (noisy) channel

$$X \sim \mathbb{P}(X) \xrightarrow{\text{noisy channel}} Y \sim \mathbb{P}(Y | X)$$

- We send a message X (with probability $\mathbb{P}(X)$) and receive a message Y with probability $\mathbb{P}(Y | X)$
- The uncertainty of the message sent, given we received a message y is

$$H_{X|Y=y} = - \sum_{x \in \mathcal{X}} \mathbb{P}(X = x | Y = y) \log(\mathbb{P}(X = x | Y = y))$$

- The expected uncertainty in the message sent is

$$H_{X|Y} = \sum_{y \in \mathcal{Y}} \mathbb{P}(Y = y) H_{X|Y=y} = - \sum_{x,y} \mathbb{P}(X = x, Y = y) \log(\mathbb{P}(X = x | Y = y))$$

Mutual Information

- The amount of uncertainty about the message being sent, X , before receiving the message is $H_X = -\mathbb{E}_X[\log \mathbb{P}(X)]$
- Shannon define the *mutual information* to be the expected loss in uncertainty when we receive a message

$$I_{X;Y} = H_X - H_{X|Y}$$

- Since $H_X - H_{X|Y} = H_Y - H_{Y|X}$ it follows

$$I_{X;Y} = I_{Y;X}$$

Independent Noise

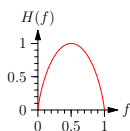
- The simplest model of a noisy channel is a binary channel where each symbol is corrupted independently with a probability f

$$\mathbb{P}(X = 1 | Y = 0) = \mathbb{P}(X = 0 | Y = 1) = f$$

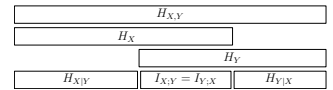
- An elementary calculations shows that

$$H_{X_i|Y_i} = -(1-f) \log(1-f) - f \log(f) = H(f)$$

- For a message of length n , $H_{X|Y} = nH(f)$



1. Information Theory
2. KL-Divergence
3. Minimum Description Length
4. Variational Auto-Encoders



Joint Entropy

- We can define the **joint entropy**

$$H_{X,Y} = - \sum_{x,y} P_{X,Y}(x,y) \log(P_{X,Y}(x,y))$$

- If the message we receive is independent of the message that is sent then $H_{X,Y} = H_X + H_Y$ (we saw this in the last lecture)

- $H_{X,Y} \neq H_X + H_Y$ if X and Y are correlated

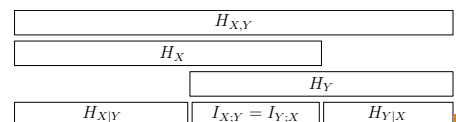
- Since $\mathbb{P}(X,Y) = \mathbb{P}(Y|X)\mathbb{P}(X) = \mathbb{P}(X|Y)\mathbb{P}(Y)$ it follows

$$H_{X,Y} = H_X + H_{Y|X} = H_Y + H_{X|Y}$$

- Or $H_X - H_{X|Y} = H_Y - H_{Y|X}$

Channel Capacity

- We can summarise these relationships diagrammatically



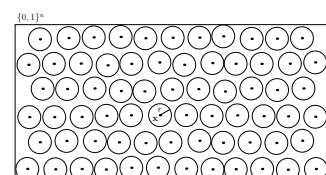
- Shannon defined the *capacity* of a noisy channel as

$$C = \max_{\mathbb{P}(X)} I_{X;Y}$$

- That is, you choose the probability distribution of the message to maximise the information gain

Error Correcting Codes

- To reduce the chance of misinterpreting a message we need to build an error correcting code
- We can do this dividing the space of binary messages into a set of Hamming balls



- A Hamming ball $B(x, r)$ is the set of strings that differ from n -dimensional binary string, x , by at most r digits

- The expected number of errors in a string of length n given an error rate of f is nf
- For sufficiently large n we would expect all errors are smaller than $(f + \epsilon)n$ (for $\epsilon > 0$)
- If we make the radius of the Hamming ball $r = (f + \epsilon)n$ ($\epsilon > 0$) then we would expect no error for sufficiently large n
- An upper bound on the number of code words we can send in a string of length n is

$$\frac{2^n}{|B(x_i, r)|} = c\sqrt{n}2^{I_{X;Y}}$$

Using Mutual Information

- Mutual information is used quite often in machine learning
 - Wikipedia mentions 14 applications
- Suppose we want to align two sets of images through some non-linear transformations
- One way of doing this is to choose the non-linear transformations that maximise the mutual information (or normalised mutual information) between the two sets of images

KL-Divergence

- We have met the Kullback-Leibler divergence

$$\text{KL}(p||q) = \mathbb{E}_{X \sim p(X)} \left[\log \left(\frac{p(X)}{q(X)} \right) \right]$$

$$= -\mathbb{E}_{X \sim p(X)} [\log(q(X))] - H_X$$
- Recall $-\log(q(X=x))$ is the length of code need to send a message x with a probability $q(X=x)$
- Thus $-\mathbb{E}_{X \sim p(X)} [\log(q(X))]$ is the expected length of message needed to code $X \sim p(X)$ using the optimal code for the distribution $q(X)$ that than $p(X)$
- $\text{KL}(p||q)$ is also known as the **relative entropy** and measures the expected extra length in coding $X \sim p(X)$ if we use the wrong distribution $q(X)$

Evidence Lower Bound (ELBO)

- We can re-write the variational free energy as

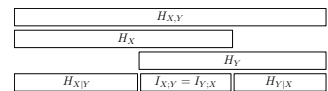
$$\begin{aligned} \Phi(\phi) &= - \int g(\theta | \phi) \log \left(\frac{g(\theta | \phi)}{(f(\theta, \mathcal{D})/f(\mathcal{D})) f(\mathcal{D})} \right) d\theta \\ &= - \int g(\theta | \phi) \left(\log \left(\frac{g(\theta | \phi)}{f(\theta | \mathcal{D})} \right) - \log(f(\mathcal{D})) \right) d\theta \\ &= -\text{KL}(g(\theta | \phi) || f(\theta | \mathcal{D})) + \log(f(\mathcal{D})) \end{aligned}$$

- If we maximise $\Phi(\phi)$, we end up minimising the KL divergence between g and f so that $g \approx f$ and $\Phi(\phi) \approx \log(f(\mathcal{D}))$
- That is, we choose the parameters of our simple factorised distribution so that it is close to the true posterior

- Shannon also showed that choosing $2^{I_{X;Y}}$ random strings of length n the Hamming distance between balls would be at least f with high probability
- This means that we can send information at rate of $I_{X;Y}$
- The maximum rate is given by the channel capacity $\max I_{X;Y}$
- If $f = 0.1$ then $C = I_{X;Y} = 0.469$ bits so we need codes of just over twice as long to communicate accurately over a noisy channel with a 10% corruption rate
- Unfortunately, we can't efficiently decode random code positions, so although we know Shannon's bound is achievable we don't have practical codes that do this

Outline

- Information Theory
- KL-Divergence
- Minimum Description Length
- Variational Auto-Encoders



Variational Approximation

- Recall we use MCMC in Bayesian inference because the posterior distribution is too complicated to write down in closed form
- In the variational approximation we approximate the posterior distribution by a simpler (typically factored distribution), e.g.

$$f(\theta | \mathcal{D}) \approx g(\theta | \phi) = \prod_i g(\theta_i | \phi_i)$$

- The standard method for solving this is to maximise the **variational free energy**

$$\Phi(\phi) = - \int g(\theta | \phi) \log \left(\frac{g(\theta | \phi)}{f(\theta, \mathcal{D})} \right) d\theta$$

Put Another Way

- We can rewrite the variational free energy as $\Phi(\phi) = L_q(\phi) + H_q(\phi)$ where

$$L_q(\phi) = \int g(\theta | \phi) (\log(f(\mathcal{D}|\theta)) + \log(f(\theta))) d\phi$$

acts like an expected posterior term that is maximised when the data is well modelled (we put the probability density, $g(\theta | \phi)$ where the $f(\theta, \mathcal{D})$ is large)

- The second term is an entropy

$$H_q(\phi) = - \int g(\theta | \phi) \log(g(\theta | \phi)) d\phi$$

That is, we maximise the uncertainty of the distribution $g(\theta | \phi)$

- Variational methods can be much faster than MCMC (although they tend to involve some iterations to minimise the variation free energy)■
- They can produce very good approximations, although this is not guaranteed (depends on the problem)■
- They can be extended (e.g. by minimising $\text{KL}(g\|f)$ rather than $\text{KL}(f\|g)$ —this is known as *belief propagation*)■
- MCMC is less elegant, but is a controlled approximation (we get better results by increasing the number of iterations)■
- MCMC is slower, but on modern computers this isn't usually a problem■

Compression and Model Selection

- Outside of the Bayesian framework it is difficult to do model selection—most of ML isn't Bayesian■
- When is it better to accept a more complex model for a better fit and when are we just over-fitting?■
- Usually we answer this using a validation set, but this is not always possible■
- One principled approach is to use the model that allows us to maximally compress the data■
- If we are compressing the data then we are capturing features of the data■

Description Length

- The **description length** for $\{y_i \mid i = 1, 2, \dots, m\}$ is then the cost of transmitting θ plus the cost of transmitting the errors

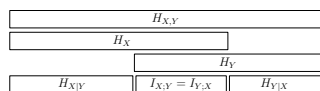
$$L = \sum_{k=1}^n \ell(\theta_k) - \sum_{i=1}^m \left(\log(p_\delta(y_i - \hat{f}(x_i|\theta))) + \log(\Delta) \right)$$

where $\ell(\theta_k)$ is the number of bits need to communicate θ_k (we get to choose the accuracy if it is worth encoding the parameters)■

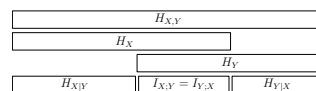
- To select between models we choose the model with the **minimum description length**■
- Note that the accuracy Δ will lead to the same cost, $-m \log(\Delta)$, for all models so doesn't affect which model is selected■

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Alice and Bob

- Suppose Alice has data $\mathcal{D} = \{(x_i, y_i) \mid i = 1, 2, \dots, m\}$ while Bob has only the feature vectors $\{x_i \mid i = 1, 2, \dots, m\}$ ■
- Alice wants to communicate y_i to Bob as efficiently as possible■
- We suppose Alice & Bob have available a model $\hat{f}(x|\theta)$ ■
- Rather than sending the complete list $\{y_i \mid i = 1, 2, \dots, m\}$ Alice can send Bob the parameter θ and the errors

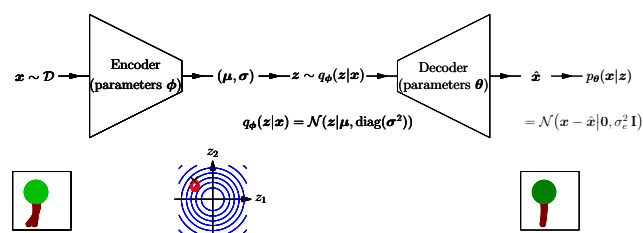
$$\delta_i = y_i - \hat{f}(x_i|\theta)$$

- Assuming the δ_i 's have a distribution p_δ then the cost of communicating an error to accuracy Δ is $-\log(p_\delta(\delta_i) \times \Delta)$ ■

Minimum Description Length (MDL) Method

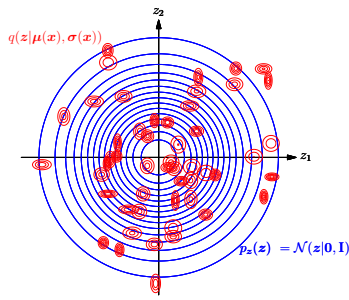
- The minimum description length method can be a powerful way of choosing between models■
- Often it is the only principled method available■
- It allows you to trade model accuracy against model complexity■
- It can be fiddly as we need to determine the accuracy to which we should store the parameters of our model■
- This isn't something we usually think about, but often we can get very good models even when we truncate the parameters to low precision■

Variational Auto-Encoders VAE



$$\mathcal{L} = \mathbb{E}_{x \sim \mathcal{D}} [\text{KL}(q_\theta(z|x) \parallel \mathcal{N}(\mathbf{0}, \mathbf{I})) - \log(p_\theta(x|z(x)))]$$

Latent Space



$$\text{KL}(q_{\theta}(z|x) \parallel \mathcal{N}(\mathbf{0}, \mathbf{I}))$$

Understanding the Loss Function

- The original paper derived the loss function as a variational approximation to maximising some posterior
- This is difficult to understand (at least, for me)
- It has a very natural explanation in terms of minimum description length
- Alice wants to communicate the images to Bob
- Alice uses the encoder to derive a (latent) code $q(z|x)$ which she communicates to Bob
- She also communicates the errors $\delta = x - \bar{x}$
- Bob uses the decoder to decode $q(z|x)$ and δ to repair the images

Description Length

- The loss

$$\mathcal{L} = \mathbb{E}_{x \sim \mathcal{D}} [\text{KL}(q_{\theta}(z|x) \parallel \mathcal{N}(\mathbf{0}, \mathbf{I})) - \log(p_{\theta}(x|z(x)))]$$

can be interpreted as

- ★ The cost of communicating the code $\text{KL}(q_{\theta}(z|x) \parallel \mathcal{N}(\mathbf{0}, \mathbf{I}))$
- ★ Plus the cost to send the repair $\log(p_{\theta}(x|z(x)))$
- We minimise the loss function equivalent to MDL
- What is really clever is that we can choose the accuracy of the code we send $q_{\theta}(z|x)$ to minimise the over-all cost

Conclusions

- Information theory has regularly been used in machine learning
- It requires some understanding and care to do it properly
- The KL-divergence (or relative entropy) is often used to make two probability distribution more alike
- The minimum description length is a powerful principle for model selection
- Variational Auto-Encoders have a very natural interpretation in terms of minimising a description length