Advanced Machine Learning Subsidary Notes

Lecture 24: Generative Models

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1 Keywords

- Conditional Independence, Graphical models, LDA

2 Main Points

2.1 Overview

- We have so far considered building rather simple probabilistic models
- But what if we want to do inference on a more complicated problem, e.g.
  - We might want to write a fault diagnosis system for a car
  - Or we want to create an AI doctor
  - The model of the spread of a virus
- Here we have a vast number of random variables with complicated relationships between them
- To help design our system we can build a *graphical model* showing the causal relationships between random variables

2.2 Conditional Independence

- **Independence**
  - Two random variable X and Y are independent if

$$\mathbb{P}[X,Y] = \mathbb{P}[X]\ \mathbb{P}[Y]$$

  - Independence can significantly speed up calculations, e.g.

$$\mathbb{E}\left[X^2\,Y\right] = \sum_{X,Y} X^2\,Y\,\mathbb{P}[X,Y] = \left(\sum_X X^2\,\mathbb{P}[X]\right)\ \left(\sum_Y Y\mathbb{P}[Y]\right)$$

    - If X and Y takes n and m values then without independence the double sum $\sum_{X,Y}$ would be over $n \times m$ possible values
    - With independence we can compute these sums independently so it just takes $m+n$ additions
  - When we have large systems with many independent variables then the time saving is often the difference between calculations being feasible or infeasible
  - Unfortunately in most complex systems there is likely to be some dependence between random variables

- **Conditional Independence**
  - A weaker notion than full independence is *conditional independence*
  - We say that X and Y are conditionally independent given Z if

$$\mathbb{P}[X, Y|Z] = \mathbb{P}[X|Z]\ \mathbb{P}[Y|Z]$$

  - Again this can lead to significant speed up in evaluating expectations, e.g.

$$\begin{aligned}\mathbb{E}\left[X^2\,Y\,Z^3\right] &= \sum_{X,Y,Z} \mathbb{P}[X,Y,Z]\,X^2\,Y\,Z^3 = \sum_{X,Y,Z} \mathbb{P}[X,Y|Z]\ \mathbb{P}[Z]\,X^2\,Y\,Z^3 \\ &= \sum_{Z} Z^3 \left(\sum_{X} X^2\,\mathbb{P}[X|Z]\right)\left(\sum_{Y} Y\,\mathbb{P}[Y|Z]\right)\end{aligned}$$

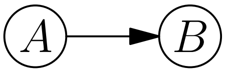
    - If X, Y and Z have l, m and n values respectively, then, ignoring conditional independence, this expectation would require $l \times m \times n$ additions; using conditional independence it only requires $n \times (l+m)$ additions
  - Although conditional dependence doesn't imply causality, if random variables X and Y are not directly causally related they will be conditionally independent
  - This is important prior information we can build into our model

2.3 Graphical Models

- In graphical models we represent each random variable as a node in a graph

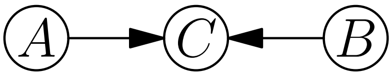
- There are two main classes of graphical models
  - Bayesian Belief Networks
    - use directed graphs
    - we will spend most of our time discussing these
  - Markov Fields
    - uses adirected graphs
    - these are used in graphics a lot
    - won't really discuss these

- Each causal connection we represent as a directed edge in a Bayesian Belief Network
  - if A directly influences B we represent this as



- **Cakes**
  - We consider the following example
  - Abi and Ben both bake cakes and like to bring them into the coffee room
  - They do this randomly without consulting with each other
  - Abi will bring in cakes 20% of the time: $\mathbb{P}[A=1] = 0.2$
  - Ben will bring in cakes 10% of the time: $\mathbb{P}[B=1] = 0.1$
  - 90% of the time if either Abi or Ben have put cakes in the coffee room there is some left when I enter $\mathbb{P}[C=1|A=1,B=0] = \mathbb{P}[C=1|A=0,B=1] = 0.9$
  - If they both make cake then there is always cake left $\mathbb{P}[C=1|A=1,B=1] = 1$
  - If neither Abi or Ben has made cake there is still a 5% chance someone else has put cake in the coffee room $\mathbb{P}[C=1|A=0,B=0] = 0.05$

- We note that $\mathbb{P}[C=0|A,B] = 1 - \mathbb{P}[C=1|A,B]$ as $\sum_{C\in\{0,1\}} \mathbb{P}[C|A,B] = 1$
- We can draw the causal relationships as



- This allows us to break down the joint probability as

$$\begin{aligned}\mathbb{P}[A,B,C] &\overset{(1)}{=} \mathbb{P}[C,B|A]\ \mathbb{P}[A] \\ &\overset{(2)}{=} \mathbb{P}[C|A,B]\ \mathbb{P}[B|A]\ \mathbb{P}[A] \overset{(3)}{=} \mathbb{P}[C|A,B]\ \mathbb{P}[B]\ \mathbb{P}[A]\end{aligned}$$

  (1) Using the definition of conditional probability (this is always true)

  (2) Using the definition of conditional probability again (this is always true)

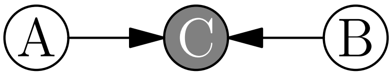
  (3) Using the fact that B and A are independent (there is no arrow between them in the graphical representation) so $\mathbb{P}[B|A] = \mathbb{P}[B]$

    * From the graphical representation we can immediately write down a simple form for this joint distribution

- We can use this decomposition to help us compute various probabilities
- (To compute probabilities we use the fact that the expectation of an indicator function $[\![\text{predicate}]\!]$ is equal to the probability $\mathbb{P}[\text{predicate}] = \mathbb{E}[\,[\![\text{predicate}]\!]]$
  * The indicator function $[\![\text{predicate}]\!]$ equals 1 if the predicate is true and 0 otherwise)
- Let's compute the probability there are cakes

$$\mathbb{P}[C=1] = \sum_{A,B,C\in\{0,1\}} [\![C=1]\!]\,\mathbb{P}[A,B,C] = \sum_{A,B\in\{0,1\}} \mathbb{P}[C=1|A,B]\,\mathbb{P}[A]\ \mathbb{P}[B] = 0.303$$

  * See Section 3.1 for details of the calculation (this is an exercise that should really help)
  * Here we exhaustively sum over all variables
- Let us consider what happens when we observe a random variable
  * In graphical models we often shade observed random variables



  * Let's compute quantities conditioned on an observation of C

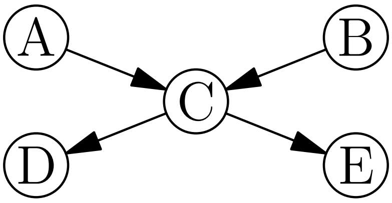
$$\mathbb{P}[A,B|C] = \frac{\mathbb{P}[A,B,C]}{\mathbb{P}[C]}$$

  * Thus $\mathbb{P}[A,B|C=1] = \mathbb{P}[A,B,C=1]\,/\mathbb{P}[C=1]$
  * Using this we can compute

$$\mathbb{P}[A=1,B=1|C=1] = 0.066, \quad \mathbb{P}[A=1|C=1] = 0.630, \quad \mathbb{P}[B=1|C=1] = 0.317$$

  * We note that $\mathbb{P}[A=1,B=1|C=1] \neq \mathbb{P}[A=1|C=1]\ \mathbb{P}[B=1|C=1]$
  * That is once we observe C then A and B are no longer independent
- We can extend our model further
  * We suppose that Dave likes cakes so if there is a cake in the coffee room there is a 80% chance that I will see him eating a cake: $\mathbb{P}[D=1|C=1] = 0.8$
  * Even if there are no cakes in the coffee room there is a 10% chance that Dave has bought his own cake: $\mathbb{P}[D=1|C=0] = 0.1$
  * Eli also likes cakes: there is a 60% chance that I will see her eating cakes if there are cakes in the coffee room: $\mathbb{P}[E=1|C=1] = 0.6$

- ∗ But she never buys herself cakes $\mathbb{P}[E = 1|C = 0] = 0$
- ∗ We can depict the dependencies of the this large model

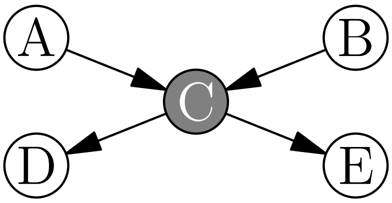


$$\begin{aligned}\mathbb{P}[A, B, C, D, E] &= \mathbb{P}[C, D, E|A, B]\ \mathbb{P}[B]\ \mathbb{P}[A] \\ &= \mathbb{P}[D|C]\ \mathbb{P}[E|C]\ \mathbb{P}[C|A, B]\ \mathbb{P}[B]\ \mathbb{P}[A]\end{aligned}$$

- · where we have used the conditional independence
- · note that D and E are conditionally independent of A and B given C
- · That is, these probabilities will depend on events A and B, but once I know there are cakes in the coffee room it doesn't matter who put them there
- · We can compute probabilities for this larger system

$$\mathbb{P}[D = 1] = 0.3121, \qquad \mathbb{P}[E = 1] = 0.1818, \qquad \mathbb{P}[D = 1, E = 1] = 0.14544$$

  so $\mathbb{P}[D, E] \neq \mathbb{P}[D]\ \mathbb{P}[E]$
- · D and E are not independent variables as they coupled through C
- · However when we observe C



  then $\mathbb{P}[D, E|C] \overset{(1)}{=} \mathbb{P}[D|C]\ \mathbb{P}[E|C]$
- · E.g.

$$\mathbb{P}[D = 1|C = 1] = 0.8 \quad \mathbb{P}[E = 1|C = 1] = 0.6 \quad \mathbb{P}[D = 1, E = 1|C = 1] = 0.48$$

2.4 Latent Dirichlet Allocation

- Most probabilistic models can be represented as a graphical model
- There are times when this isn't particularly useful
- But it can just help us to understand what is going on
- We consider an example of this called **Latent Dirichlet Allocation**
  - This is sometimes known as LDA, but should not be confused with *linear discriminant analysis*
- LDA is used to model topics in a set of documents (or *corpus*)
- We want to identify a set of topics
- The topics are associated with particular words
- The documents will be associated with a small number of topics
- To model this we build a *generative model*

  - This is natural to build

  - Although it seems the wrong way around—we don't want to build a corpus of documents

  - But Bayes's rule allows us to invert this

- Let us start with some definitions

  - We consider generating a corpus of documents

$$\mathcal{C} = \{d_i | i = 1, 2, \dots |\mathcal{C}|\}$$

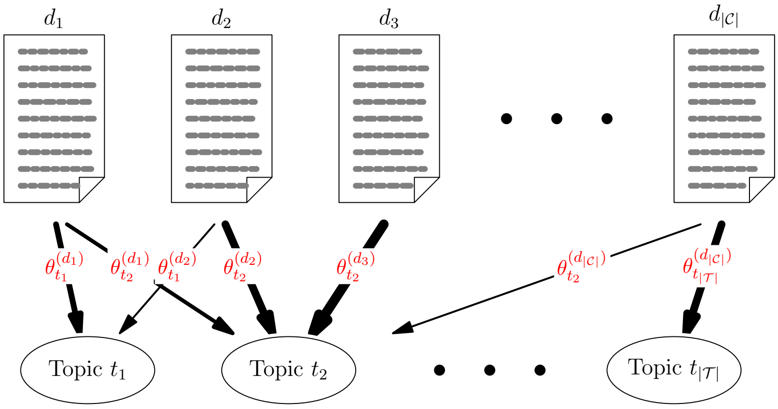
  - Each document consists of a set of words

$$d = \left(w_1^{(d)}, w_2^{(d)}, \dots, w_{N_d}^{(d)}\right)$$

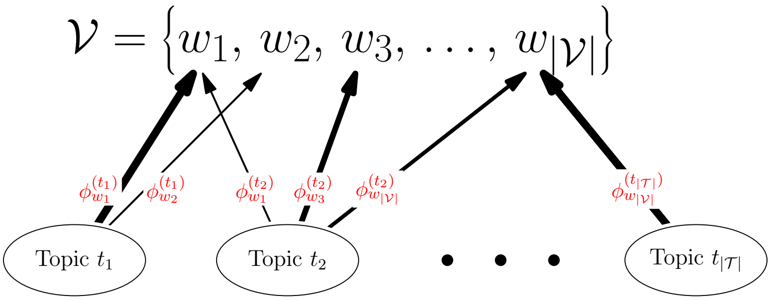
  - We assume there is a set of topics

$$\mathcal{T} = \{t_1, t_2, \dots, t_{|\mathcal{T}|}\}$$

  - We associate a probability, $\theta_t^{(d)}$, that a word in document d relates to a topic t



  - We associate a probability $\phi_w^{(t)}$ that a word, w, is related to a topic t

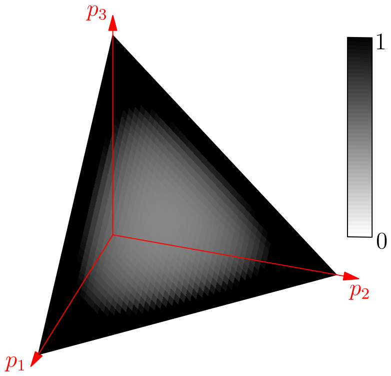

$\mathcal{V} = \left\{w_1, w_2, w_3, \dots, w_{|\mathcal{V}|}\right\}$
$\phi_{w_1}^{(t_1)}$
$\phi_{w_2}^{(t_1)}$
$\phi_{w_1}^{(t_2)}$
$\phi_{w_3}^{(t_2)}$
$\phi_{w_{|\mathcal{V}|}}^{(t_2)}$
$\phi_{w_{|\mathcal{V}|}}^{(t_{|\mathcal{T}|})}$
Topic t_1
Topic t_2
Topic $t_{|\mathcal{T}|}$

  - Most documents are predominantly about a few topics and most topic have a small number of words associated to them

  - We can generate probability vectors $\boldsymbol{\theta}^{(d)}$ and $\boldsymbol{\phi}^{(t)}$ from a Dirichlet distribution

$$\mathrm{Dir}(\boldsymbol{p}|\boldsymbol{\alpha}) = \Gamma\left(\sum_i \alpha_i\right) \prod_{i=1}^{n} \frac{p_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}$$

- $\boldsymbol{\theta}^{(d)} \sim \mathrm{Dir}(\alpha\,\mathbf{1})$ and $\boldsymbol{\phi}^{(t)} \sim \mathrm{Dir}(\beta\,\mathbf{1})$
- By choosing a Dirichlet distribution with a small components, α_i, we ensure that have most of its probability density lies around the edges



- By drawing $\boldsymbol{\theta}^{(d)}$ and $\boldsymbol{\phi}^{(t)}$ from a Dirichlet distribution with small parameters α and β we ensure that most components are very small with a few large components
- To generate a document we choose a topic for each word and a word for each topic
- We use the categorical distribution
  * if $\boldsymbol{p}$ is a vector of non-negative values that sum to 1 then $\mathrm{Cat}(i|\boldsymbol{p}) = p_i$
  * That is if $I \sim \mathrm{Cat}(\boldsymbol{p})$ then I will be an integer, i with probability p_i
- Thus for word i of document d we first choose a topic $\tau_i^{(d)} \sim \mathrm{Cat}(\boldsymbol{\theta}^d)$ and then we choose a word $w_i^{(d)} \sim \mathrm{Cat}(\boldsymbol{\phi}^{\tau_i^{(d)}})$
  * It is a slightly crazy model in that words are randomly chosen from the topics of the document with no ordering
- We could represent this by a rather ugly graphical model (see Figure 1)
- To make graphical models more manageable people have invented a graphical means of showing repeats
- For example, to illustrate that we have a probability vector $\boldsymbol{\theta}^{(d)}$ drawn from a Dirichlet distribution with parameter α for each document d in our corpus $\mathcal{C}$ we can use a **plate diagram**

../lectures/figures/GMplateSimple.pdf

- Using a plate diagram we can represent the LDA as shown in Figure 2
- It takes a bit of time to decode this

../lectures/figures/LDAGM.pdf

Figure 1: Graphical Model for Latent Dirichlet Allocation

../lectures/figures/GMplateLDA.pdf

Figure 2: Graphical Model for Latent Dirichlet Allocation

    - We have a probability vector $\boldsymbol{\theta}^{(d)}$ for every document in our corpus
        - this tells us the distribution of topics in the document
        - $\boldsymbol{\theta}^{(d)}$ is drawn from a Dirichlet distribution with parameters $\boldsymbol{\alpha} = (\alpha, \alpha, \alpha, \ldots, \alpha)$
    - We have a probability $\boldsymbol{\phi}^{(\tau)}$ for every topic
        - this tells us the distribution of words associated with a topic
        - $\boldsymbol{\phi}^{(\tau)}$ is drawn from a Dirichlet distribution with parameters $\boldsymbol{\beta} = (\beta, \beta, \beta, \ldots, \beta)$
    - For each document, d, and each word, $w_i^{(d)}$ in the document we have
        - a topic $\tau_i^{(d)}$ drawn from $\boldsymbol{\theta}^{(d)}$
        - the words $w_i^{(d)}$ are drawn from $\boldsymbol{\phi}^{(\tau_i^{(d)})}$
        - that is it depends both on the topic $\tau_i^{(d)}$ and on the distributions of words associated with that topic
    - In practice I am usually given the documents with words (the words are observed)
    - I have shaded what is usually taken to be observed (for α and β we usually just choose these from the start—we could learn then so they would not be observed)

- The graphical model helps us write down the joint distribution

- We define matrices to denote all the variables

$$\begin{aligned}
\boldsymbol{W} &= (\boldsymbol{w}^{(d)} | d \in \mathcal{C}) \quad \text{with} \quad \boldsymbol{w}^{(d)} = (w_1^{(d)}, w_2^{(d)}, \ldots, w_{N_d}^{(d)}), \quad \text{and} \quad w_i^{(d)} \in \mathcal{V} \\
\boldsymbol{T} &= (\tau_i^{(d)} | d \in \mathcal{C} \ \wedge \ i \in \{1, 2, \ldots, N_d\}) \quad \text{with} \quad \tau_i^{(d)} \in \mathcal{T} \\
\boldsymbol{\Theta} &= (\boldsymbol{\theta}^{(d)} | d \in \mathcal{C}) \quad \text{with} \quad \boldsymbol{\theta}^{(d)} = (\theta_t^{(d)} | t \in \mathcal{T}) \in \Lambda^{|\mathcal{T}|} \\
\boldsymbol{\Phi} &= (\boldsymbol{\phi}^{(t)} | t \in \mathcal{T}) \quad \text{with} \quad \boldsymbol{\phi}^{(t)} = (\phi_w^{(t)} | w \in \mathcal{V}) \in \Lambda^{|\mathcal{V}|}
\end{aligned}$$

- Then the joint distribution is given by

$$\mathbb{P}\left[\boldsymbol{W}, \boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{\Phi} \middle| \alpha, \beta\right] = \left(\prod_{t \in \mathcal{T}} \mathrm{Dir}\left(\boldsymbol{\phi}^{(t)} \middle| \beta \mathbf{1}\right)\right) \left(\prod_{d \in \mathcal{C}} \mathrm{Dir}\left(\boldsymbol{\theta}^{(d)} \middle| \alpha \mathbf{1}\right) \prod_{i=1}^{N_d} \mathrm{Cat}\left(\tau_i^{(d)} \middle| \boldsymbol{\theta}^{(d)}\right) \mathrm{Cat}\left(w_i^{(d)} \middle| \boldsymbol{\phi}^{(\tau_i^{(d)})}\right)\right)$$

- It is now a technical exercise to compute the quantities of interest

- For example $f(\boldsymbol{\Theta}, \boldsymbol{\Phi} | \boldsymbol{W}, \alpha, \beta)$ will tell us about the words associated with the topics that are present in the corpus and the topics associated with each document

- Note that we would marginalise out $\boldsymbol{T}$

- There are different techniques for computing these probabilities, e.g. using MCMC or variational approximations

3 Exercise

3.1 Cakes

- Write a program to compute the probability of various events concerning cakes

- To compute all the probabilities (sometimes inefficiently) we can sum over all values our variables can take

- I have done this somewhat inefficiently in the answers

4 Answers

4.1 Cakes

- I am using asymptote which I usually use for drawing diagrams, but its a language with C syntax
- Port this to a language of your choice

```
real pcGab(int a, int b, int c) { // P(C|A,B)
  real p;
  if (a==1 && b==1)
    p = 1;
  else if (a==1 || b==1)
    p = 0.95;
  else
    p = 0.05;
  if (c==1)
    return p;
  else
    return 1-p;
}

real pa(int a) { // P(A)
  return (a==1)? 0.2:0.8;
}

real pb(int b) { // P(B)
  return (b==1)? 0.1:0.9;
}

real pd(int d, int c) { // P(D|C)
  real p = (c==1)? 0.8:0.1;
  return (d==1)? p:1-p;
}

real pe(int e, int c) {// P(E|C)
  real p = (c==1)? 0.6:0;
  return (e==1)? p:1-p;
}



typedef real func(int, int, int, int, int); // define signature of general function

real expect(func f) { // compute expectations exhaustively
  real sum = 0;
  for (int a=0; a<=1; ++a) {
    for (int b=0; b<=1; ++b) {
      for (int c=0; c<=1; ++c) {
        for (int d=0; d<=1; ++d) {
          for (int e=0; e<=1; ++e) {
            sum += f(a,b,c,d,e)*pcGab(a,b,c)*pa(a)*pb(b)*pd(d,c)*pe(e,c);
          }
        }
      }
    }
  }
  return sum;
}

/* Define functions to find expectations */
/* These are all indicator funtions so I end up with probabilities */

real f(int a, int b, int c, int d, int e) {return 1;}
real fa(int a, int b, int c, int d, int e) {return a;}
real fb(int a, int b, int c, int d, int e) {return b;}
real fab(int a, int b, int c, int d, int e) {return a*b;}
real fc(int a, int b, int c, int d, int e) {return c;}
real fac(int a, int b, int c, int d, int e) {return a*c;}
real fbc(int a, int b, int c, int d, int e) {return b*c;}
real fabc(int a, int b, int c, int d, int e) {return a*b*c;}
real fd(int a, int b, int c, int d, int e) {return d;}
real fe(int a, int b, int c, int d, int e) {return e;}
real fde(int a, int b, int c, int d, int e) {return d*e;}
real fcd(int a, int b, int c, int d, int e) {return c*d;}
real fce(int a, int b, int c, int d, int e) {return c*e;}

real fcde(int a, int b, int c, int d, int e) {return c*d*e;}


write("Check joint probability is normalised: ", expect(f));
write("P(A=1) = ", expect(fa));
write("P(B=1) = ", expect(fb));
write("P(A=1)*P(B=1) = ", expect(fa)*expect(fb));
write("P(A=1,B=1) = ", expect(fab));
write("Note P(A=1,B=1) = P(A=1)*P(B=1)");
write("-");

real Pc = expect(fc);
write("P(C=1) = ", Pc);
real PaGc = expect(fac)/Pc;
real PbGc = expect(fbc)/Pc;
real PabGc = expect(fabc)/Pc;
write("P(A=1|C=1) = ", PaGc);
write("P(B=1|C=1) = ", PbGc);
write("P(A=1|C=1)*P(B=1|C=1) = ", PaGc*PbGc);
write("P(A=1,B=1|C=1) = ", PabGc);
write("Note: P(A=1,B=1|C=1) != P(A=1|C=1)*P(B=1|C=1)");
write("-");

write("P(D=1) = ", expect(fd));
write("P(E=1) = ", expect(fe));
write("P(D=1)*P(E=1) = ", expect(fd)*expect(fe));
write("P(D=1,E=1) = ", expect(fde));
write("Note: P(D=1,E=1) != P(D=1)*P(E=1)");
write("-");
write("P(D=1|C=1) = ", expect(fcd)/Pc);
write("P(E=|C=11) = ", expect(fce)/Pc);
write("P(D=1|C=1)∗P(E=1|C=1) = ", expect(fcd)/Pc∗expect(fce)/Pc);
write("P(D=1,E=1|C=1) = ", expect(fcde)/Pc);
write("Note: P(D=1,E=1|C=1) = P(D=1|C=1)∗P(E=1|C=1)");
```

4.2 Result from program

```
Check joint probability is normalised: 1
P(A=1) = 0.2
P(B=1) = 0.1
P(A=1)∗P(B=1) = 0.02
P(A=1,B=1) = 0.02
Note P(A=1,B=1) = P(A=1)∗P(B=1)
-
P(C=1) = 0.303
P(A=1|C=1) = 0.63036303630363
P(B=1|C=1) = 0.316831683168317
P(A=1|C=1)∗P(B=1|C=1) = 0.19971898179917
P(A=1,B=1|C=1) = 0.066006600660066
Note: P(A=1,B=1|C=1) != P(A=1|C=1)∗P(B=1|C=1)
-
P(D=1) = 0.3121
P(E=1) = 0.1818
P(D=1)∗P(E=1) = 0.05673978
P(D=1,E=1) = 0.14544
Note: P(D=1,E=1) != P(D=1)∗P(E=1)
-
P(D=1|C=1) = 0.8
P(E=|C=11) = 0.6
P(D=1|C=1)∗P(E=1|C=1) = 0.48
P(D=1,E=1|C=1) = 0.48
Note: P(D=1,E=1|C=1) = P(D=1|C=1)∗P(E=1|C=1)
```