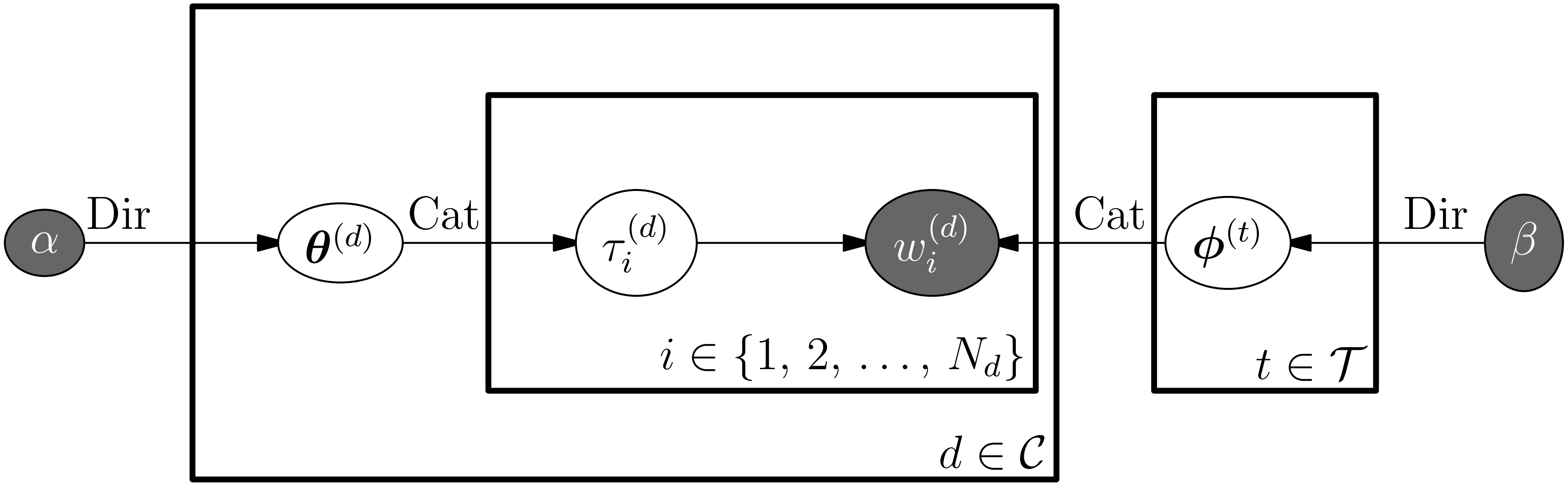
Advanced Machine Learning

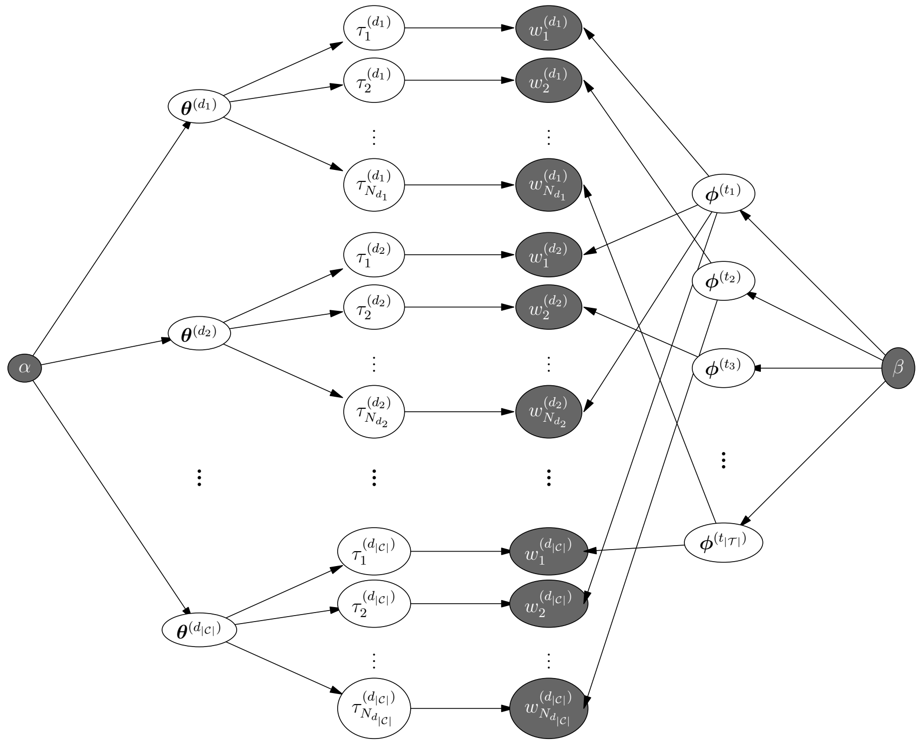
Generative Models



Generative models, graphical models, LDA

Outline

1. **Building Probabilistic Models**
2. Graphical Models
3. Latent Dirichlet Allocation



Building Probabilistic Models

- To describe a system with uncertainty we use random variables, X, Y, Z, etc.
- We use the convention of writing random variables in capitals (this is sometimes confusing as when you observe a random variables it is no longer random)
- The variables are described by probability mass function $\mathbb{P}(X, Y, Z)$ or if our variables are continuous, but probability densities $f_{X,Y,Z}(x, y, z)$
- We build in dependencies in this joint distribution

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Discriminative Models

- We often think of our observations as given and the predictions as random variables

- For example we might be given some features $\boldsymbol{x}$ and we wish to predict a class $C \in \mathcal{C}$

- Our objective is then to find the probability $\mathbb{P}\left(C|\boldsymbol{x}\right)$

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Generative Models

- Sometimes it is easy to think about the joint process of generating the features and outputs together
- This leads to a joint distribution $\mathbb{P}(\boldsymbol{X}, Y)$ where $\boldsymbol{X}$ are your features and Y is your output you are trying to predict
- This is known as a **generative model**
- Generative models are often more natural to think about
- We can use them to do discrimination using

$$\mathbb{P}(Y|\boldsymbol{X}) = \frac{\mathbb{P}(\boldsymbol{X}, Y)}{\mathbb{P}(\boldsymbol{X})} = \frac{\mathbb{P}(\boldsymbol{X}, Y)}{\sum_{Y} \mathbb{P}(\boldsymbol{X}, Y)}$$

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Latent Variables

- Sometimes we have models that involve random variables that we don't observe and we don't care about

- These are called **latent variables**

- If we have a latent variable Z and observed variable $\boldsymbol{X}$ and we are predicting a variable Y then we would **marginalise** over the latent variable

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Mixture of Gaussians

- Suppose we were observing the decays from two types of short-lived particle
- We observe the half life, X, but not the particle type
- We assume X is normally distributed with unknown means and variances: $\boldsymbol{\Theta} = \{\mu_1,\, \sigma_1^2,\, \mu_2,\, \sigma_2^2\}$
- Let $Z \in \{0,1\}$ be an indicator that it is particle 1
- The probability of X is given by

$$f(X|Z,\boldsymbol{\Theta}) = Z\,\mathcal{N}\big(X\big|\mu_1,\sigma_1^2\big) + (1-Z)\,\mathcal{N}\big(X\big|\mu_2,\sigma_2^2\big)$$

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- Note that

$$f(X|\boldsymbol{\Theta}) = \sum_{Z\in\{0,1\}} f(X,Z|\boldsymbol{\Theta}) = \sum_{Z\in\{0,1\}} f(X|Z,\boldsymbol{\Theta})\,\mathbb{P}(Z)$$

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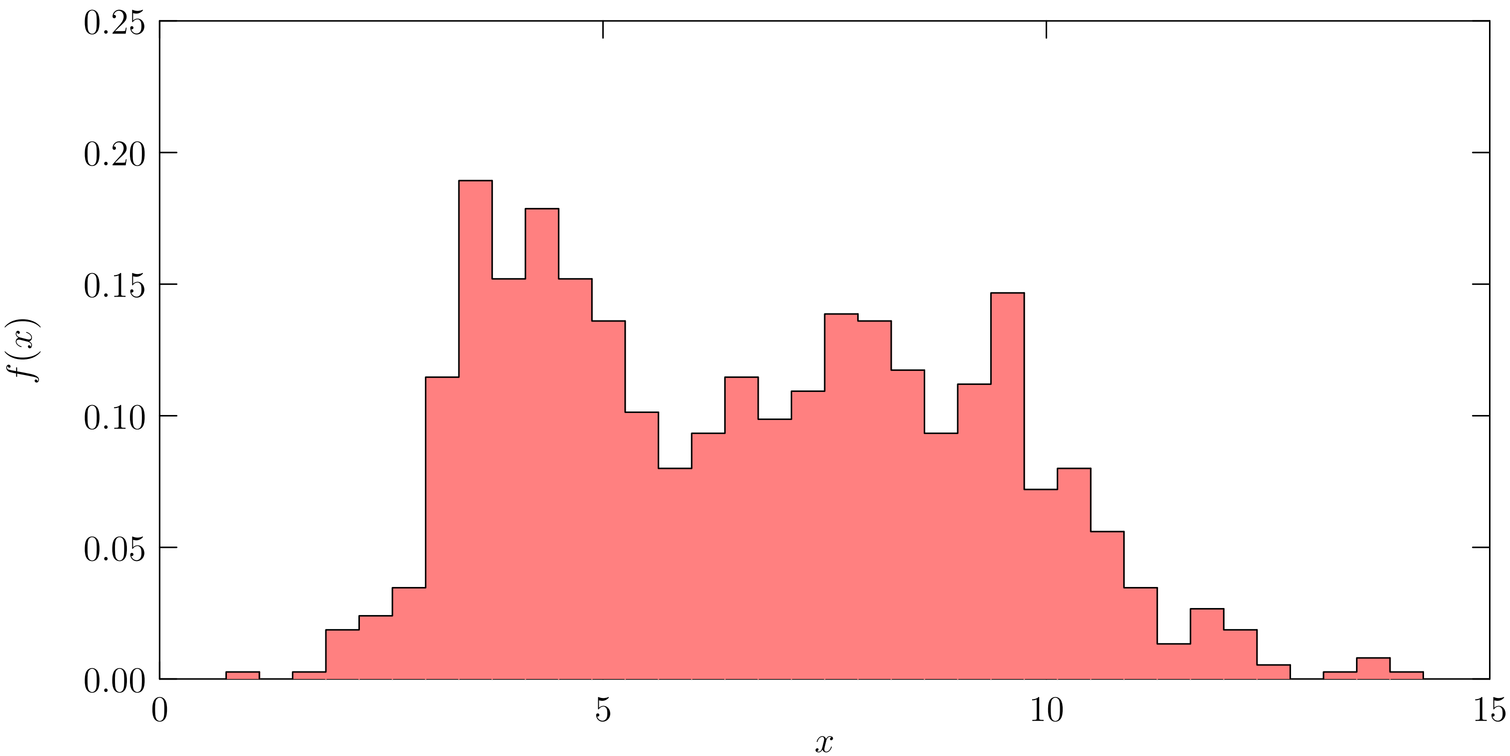

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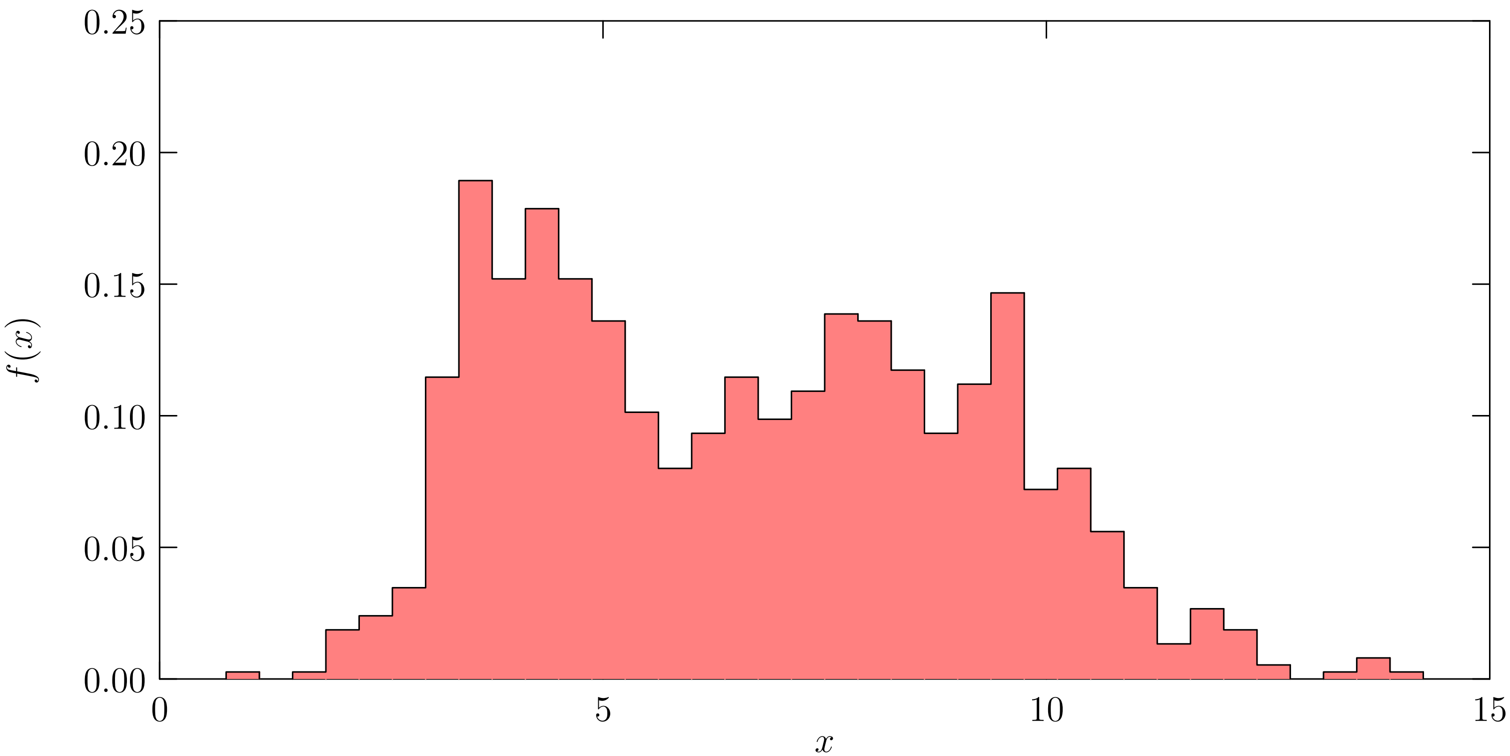

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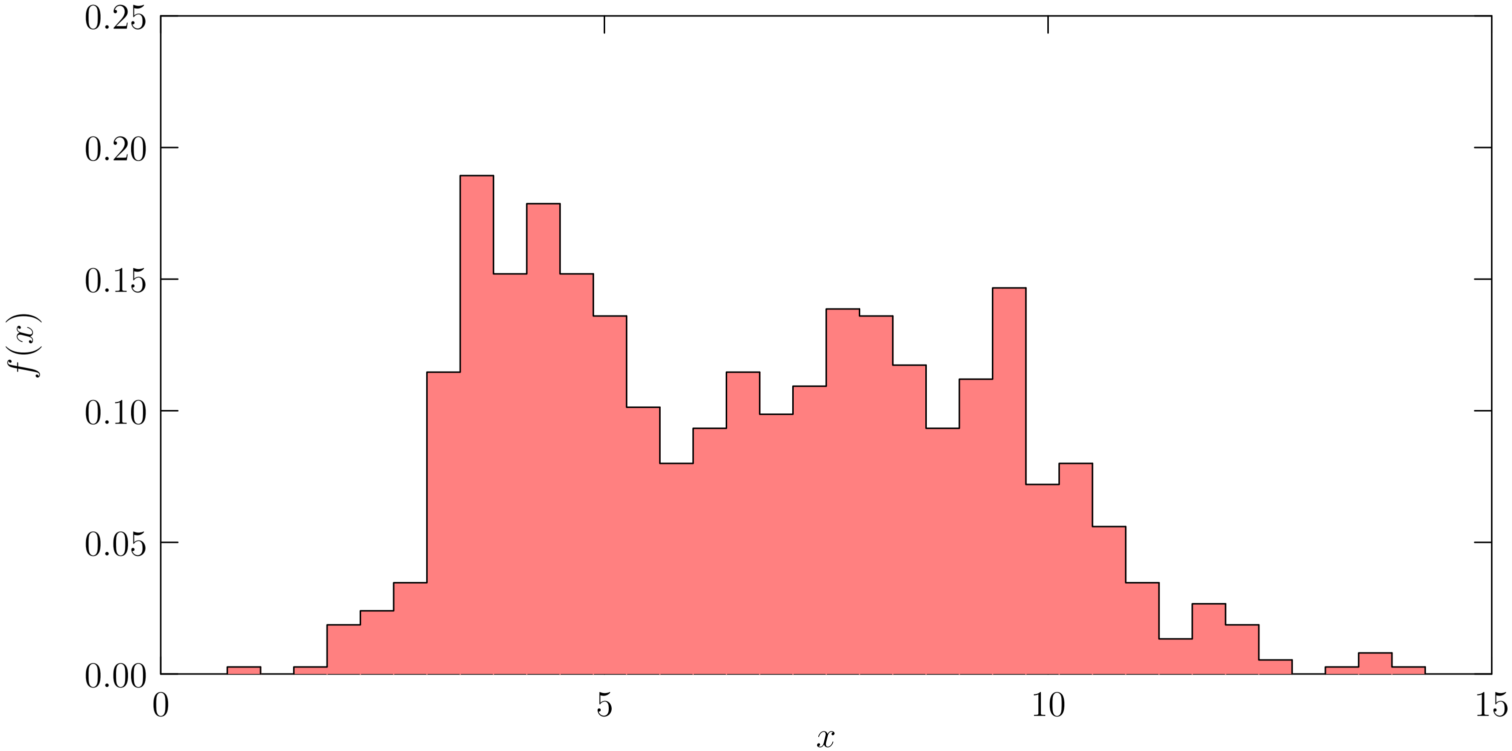

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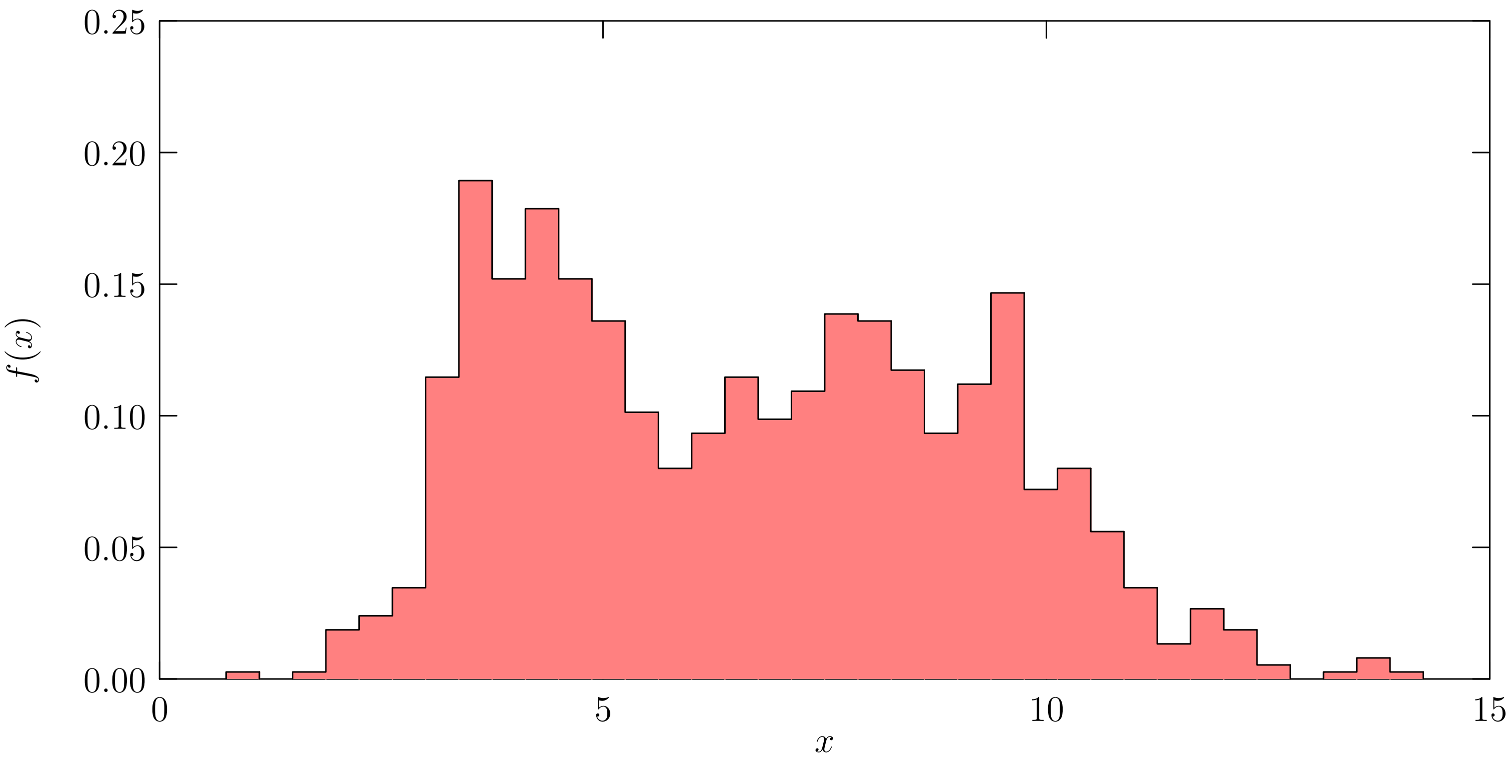

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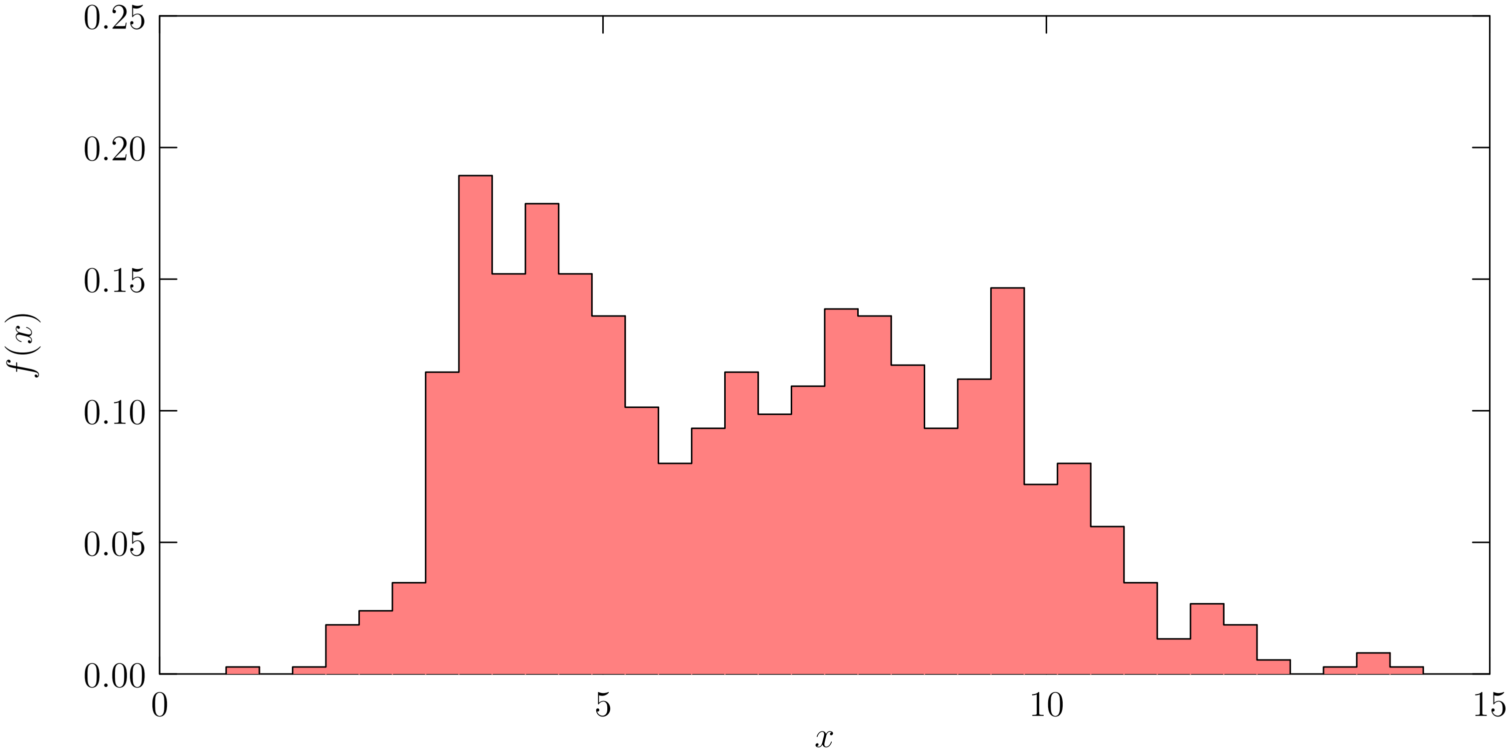

Maximum Likelihood

- To solve the model as a Bayesian we would have to assign priors to our parameters $\boldsymbol{\Theta} = (\mu_1, \sigma_1, \mu_2, \sigma_2, p)$
- This is doable, but complicated (we would also end up with a distribution for our parameters)
- Often we only want a reasonable estimate for some of our parameters (e.g. the half-lives μ_1 and μ_2)
- A reasonable approach is to seek those parameters that maximise the likelihood of our observed data

$$f(\mathcal{D}|\boldsymbol{\Theta}) = \prod_{X \in \mathcal{D}} f(X|\boldsymbol{\Theta})$$

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EM Algorithm

- The maximum likelihood is a non-linear function of the parameters so cannot be immediately maximised
- We have a difficulty in that our latent variable Z will depend on the parameter $\boldsymbol{\Theta}$
- And our likelihood will depend on the latent variable
- We therefore proceed iteratively by maximising the expected log-likelihood with respect to the current set of parameters

$$\Theta^{(t+1)} = \underset{\boldsymbol{\Theta}}{\operatorname{argmax}} \sum_{\boldsymbol{Z}} \mathbb{P}\left(\boldsymbol{Z}|\mathcal{D}, \boldsymbol{\Theta}^{(t)}\right) \log(f(\mathcal{D}|\boldsymbol{Z}, \boldsymbol{\Theta}))$$

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EM for Mixture of Gaussians

- Maximise with respect to parameters $\boldsymbol{\theta}$

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) = \sum_{\boldsymbol{Z}} \mathbb{P}\left(\boldsymbol{Z}|\mathcal{D},\boldsymbol{\Theta}^{(t)}\right) \log(f(\mathcal{D}|\boldsymbol{Z},\boldsymbol{\Theta}))$$

$$= \sum_{i=1}^{n} \sum_{Z_i \in \{1,2\}} \mathbb{P}\left(Z_i|X_i,\boldsymbol{\theta}_i\right) \Bigg(Z_i \log(p) + (1-Z_i)\log(1-p) + \frac{(X_i-\mu_{Z_i})^2}{2\,\sigma_{Z_i}^2} - \log\left(\sqrt{2\,\pi}\,\sigma_{Z_i}\right)\Bigg)$$

- Compute update equations

$$\frac{\partial\, Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})}{\partial\, \mu_k} = 0, \qquad \frac{\partial\, Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})}{\partial\, \sigma_k} = 0, \qquad \frac{\partial\, Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})}{\partial\, p} = 0$$

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&\qquad + \frac{(X_i - \mu_{Z_i})^2}{2\,\sigma_{Z_i}^2} - \log\left(\sqrt{2\,\pi}\,\sigma_{Z_i}\right)\bigg)
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Update Equations

- Means

$$\mu_{Z_i}^{(t+1)} = \frac{\sum_{i=1}^{n} \mathbb{P}\left(Z_i | X_i, \boldsymbol{\theta}^{(t)}\right) X_i}{\sum_{i=1}^{n} \mathbb{P}\left(Z_i | X_i, \boldsymbol{\theta}^{(t)}\right)},$$

- Variances

$$(\sigma_{Z_i}^{(t+1)})^2 = \frac{\sum_{i=1}^{n} \mathbb{P}\left(Z_i | X_i, \boldsymbol{\theta}^{(t)}\right) (X_i - \mu_{Z_i}^{(t+1)})^2}{\sum_{i=1}^{n} \mathbb{P}\left(Z_i | X_i, \boldsymbol{\theta}^{(t)}\right)}$$

- Probability of being type 1

$$p^{(t+1)} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}\left(Z_i | X_i, \boldsymbol{\theta}_i\right)$$

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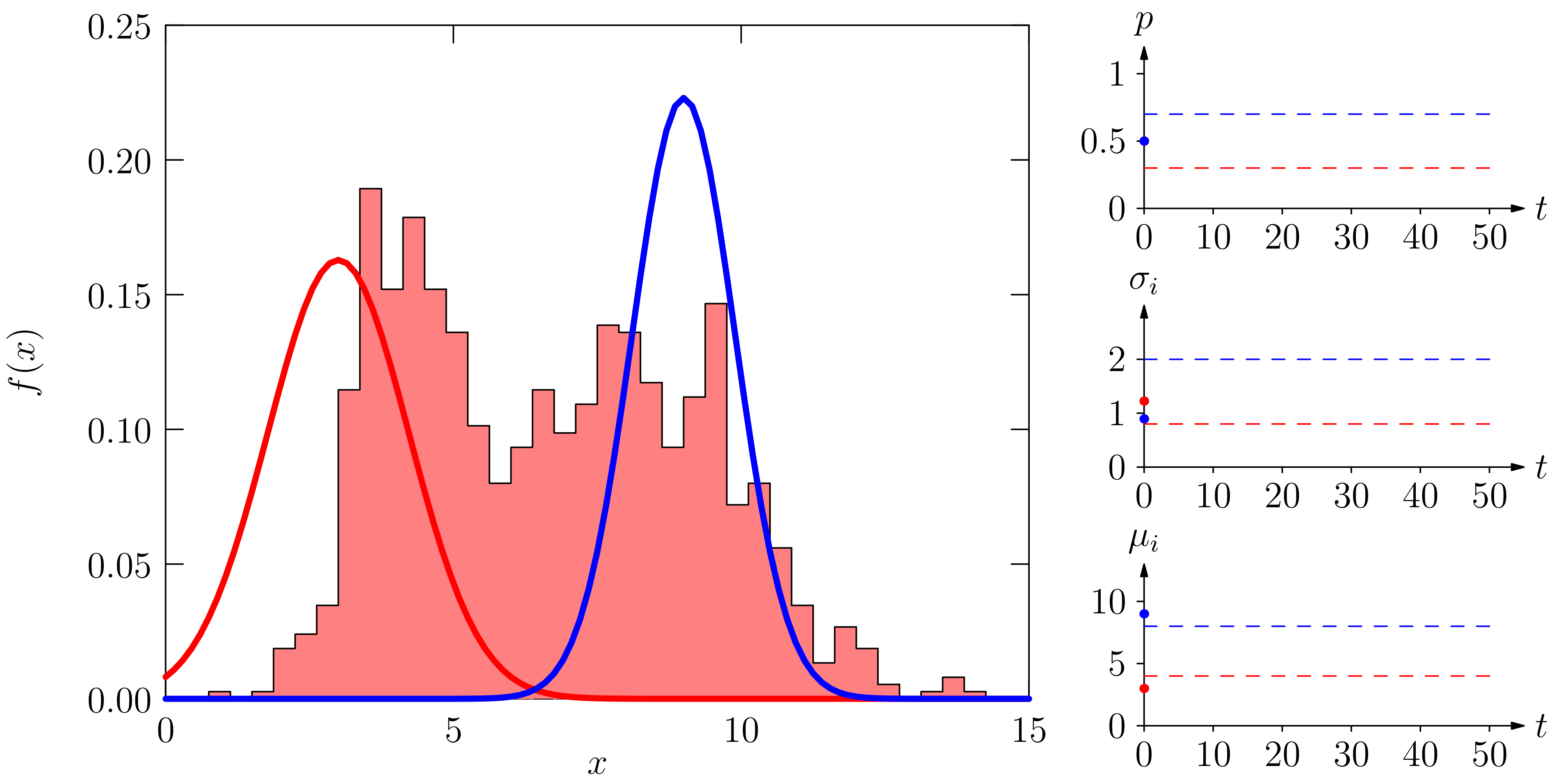

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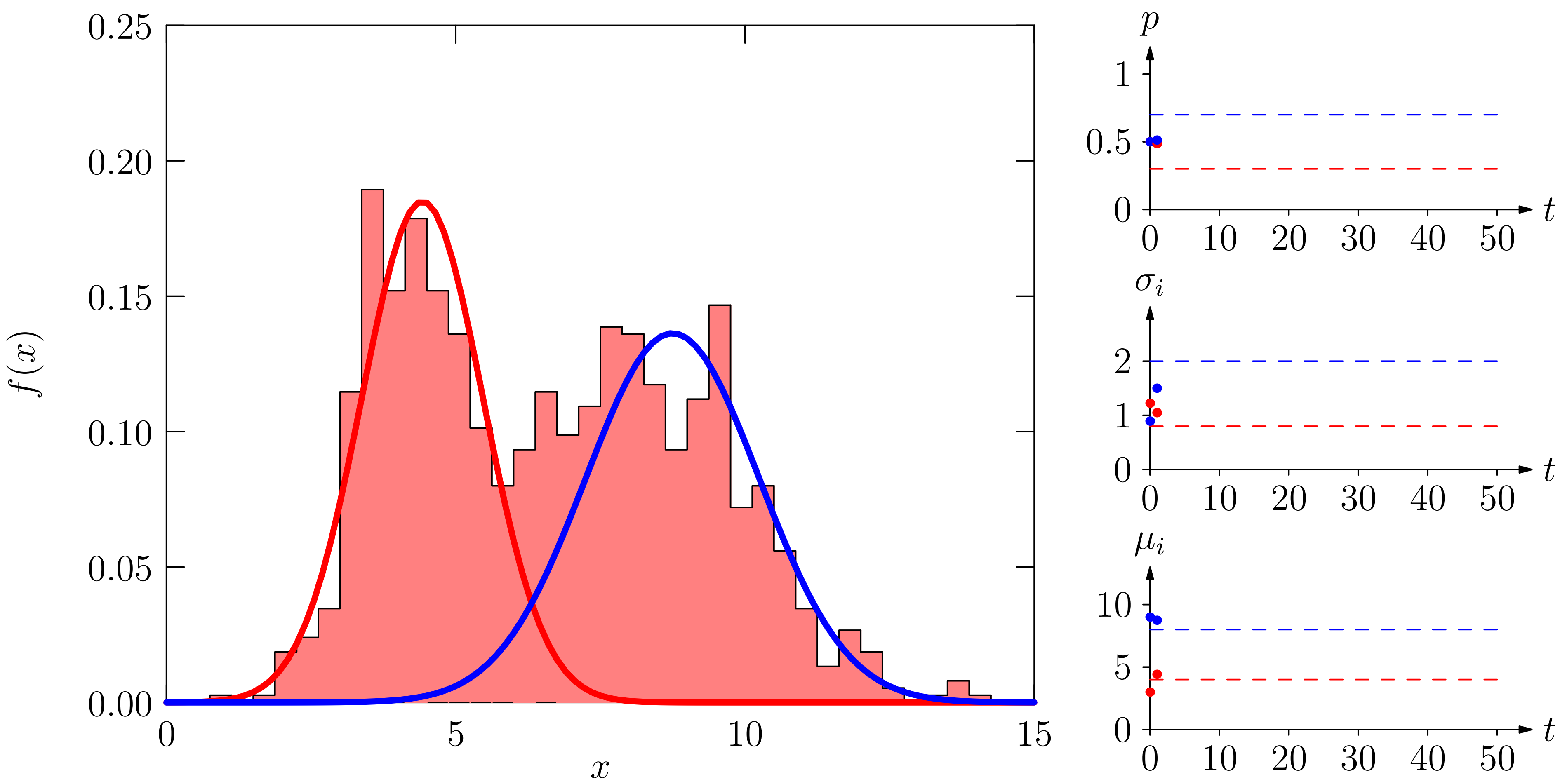

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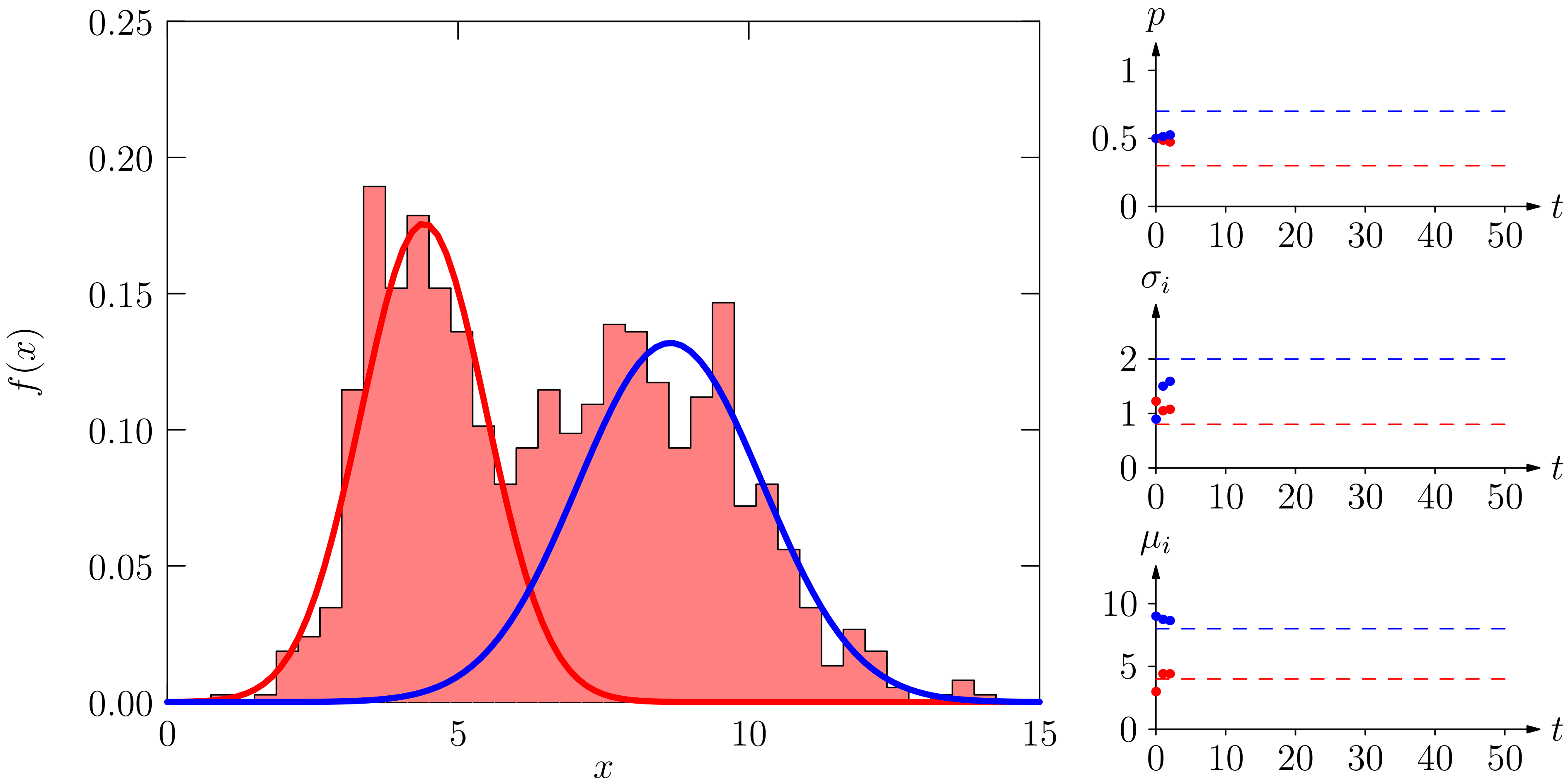
Example



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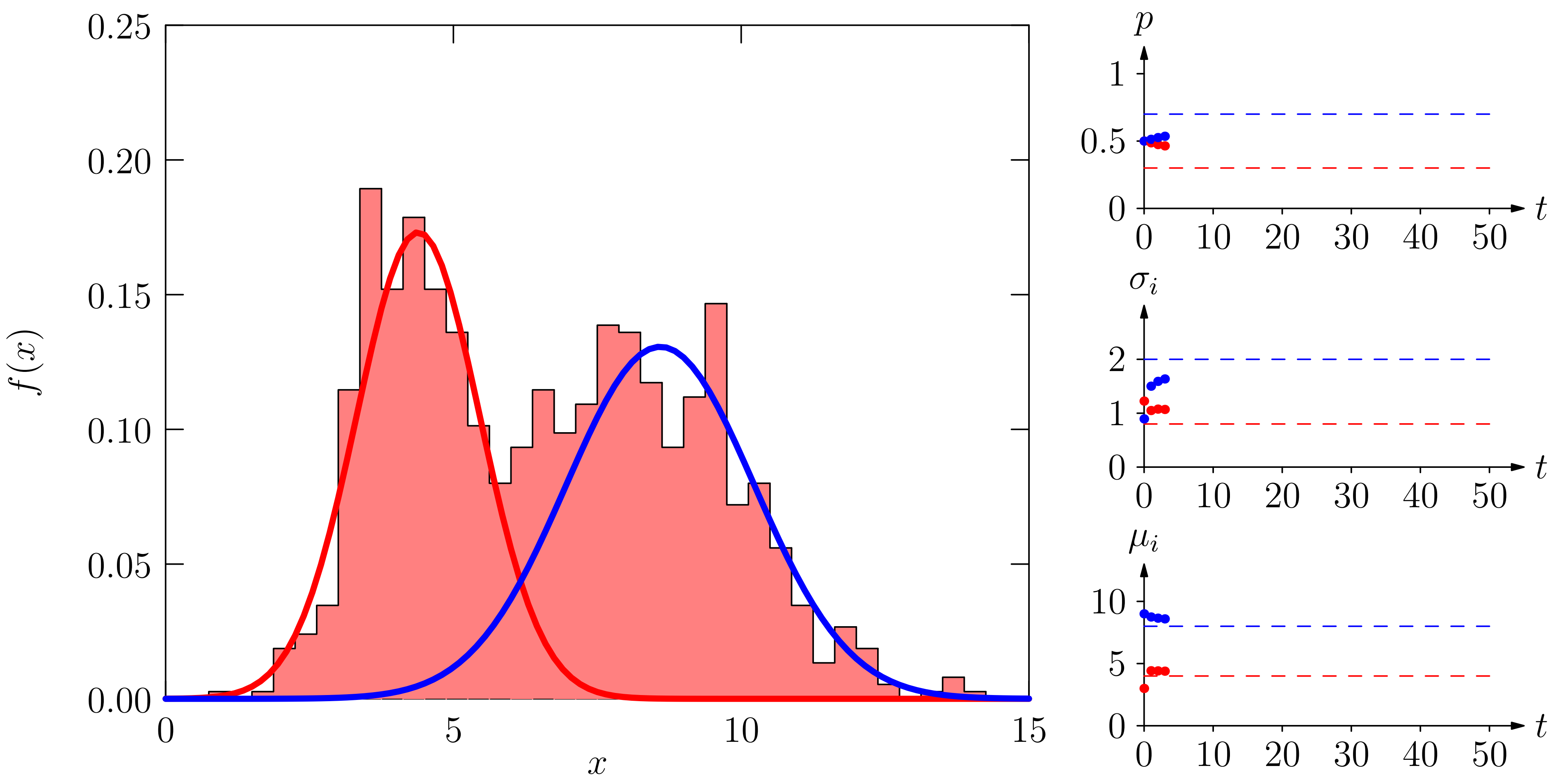
Example



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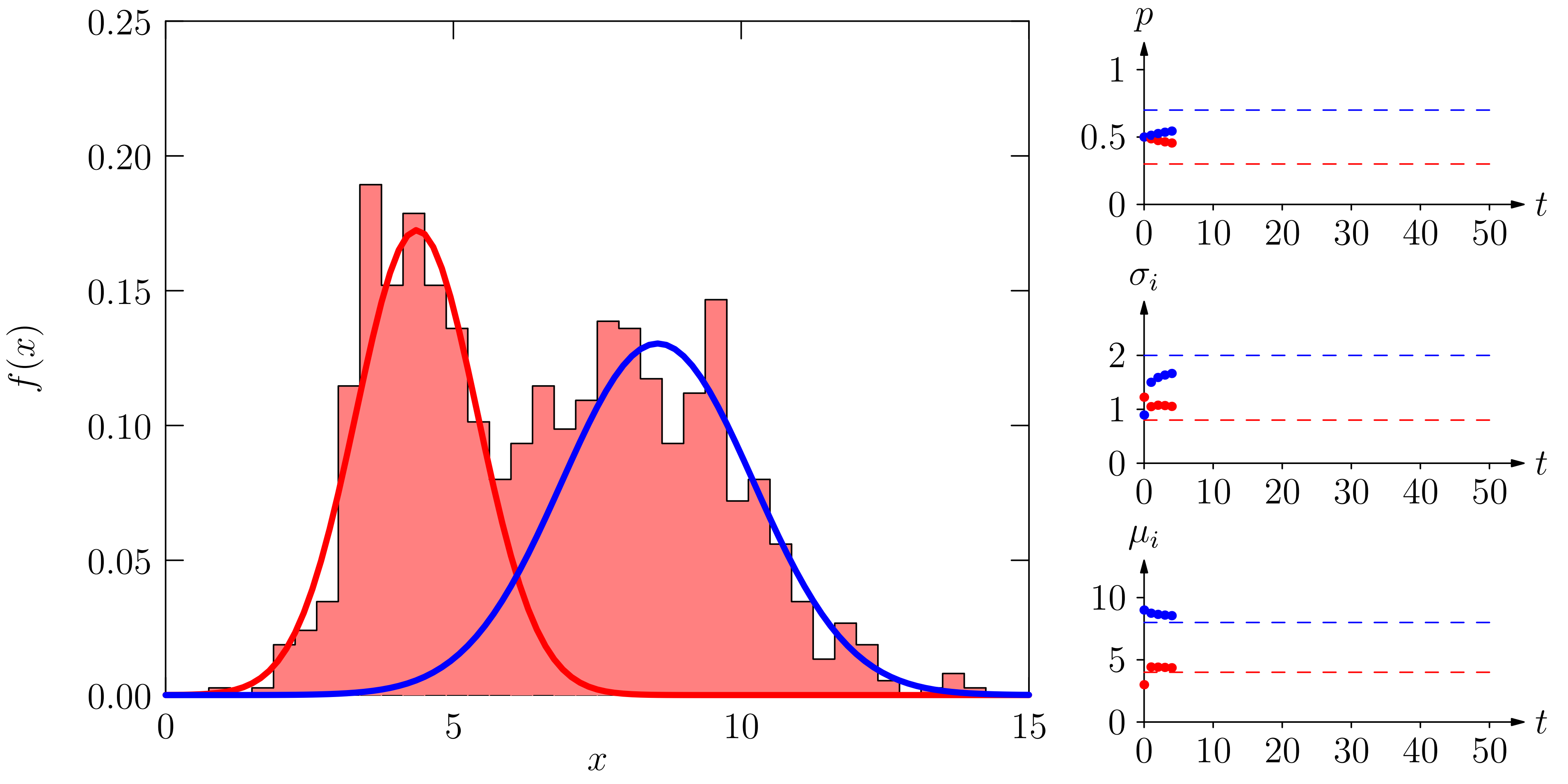
Example



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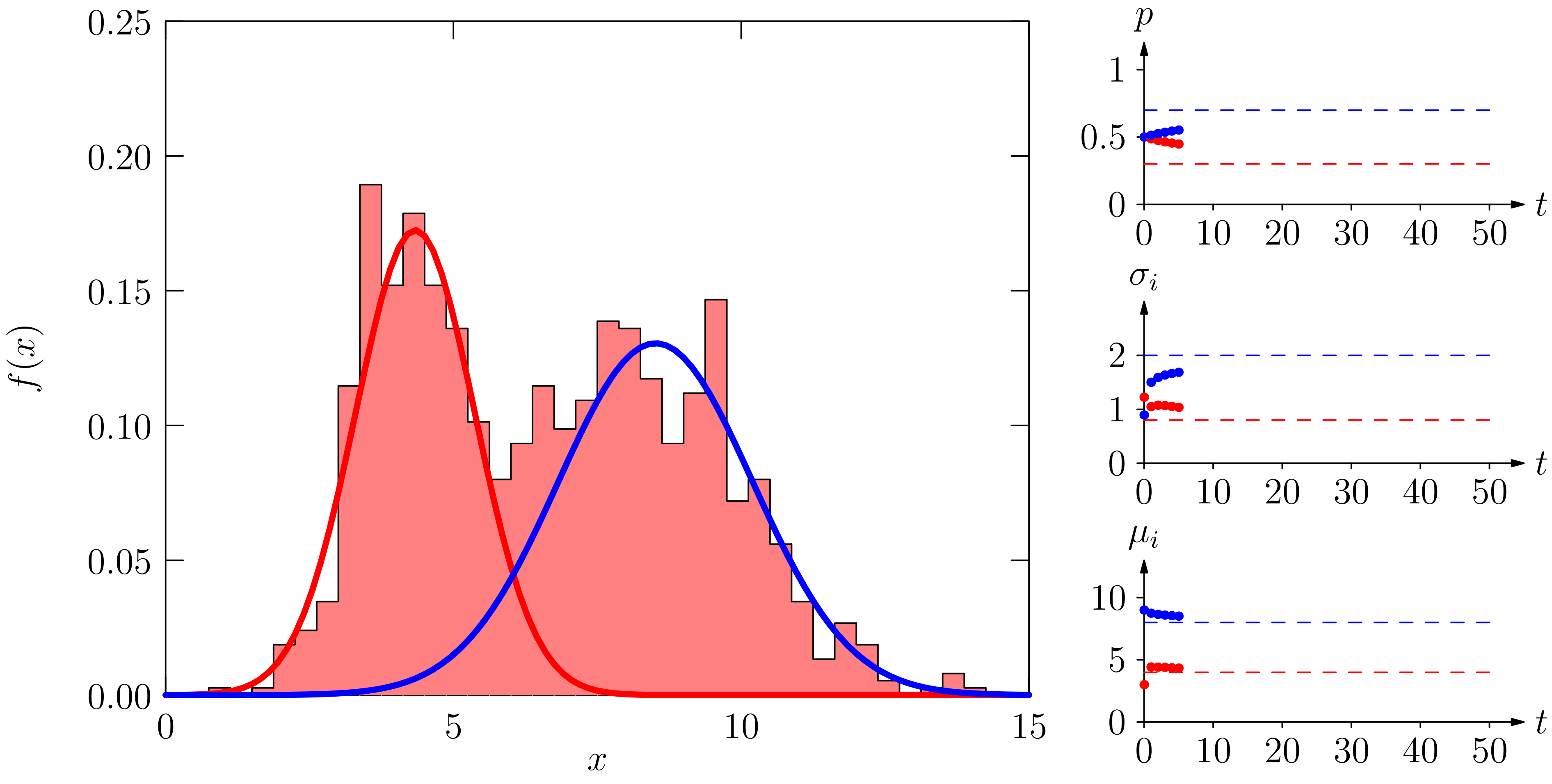

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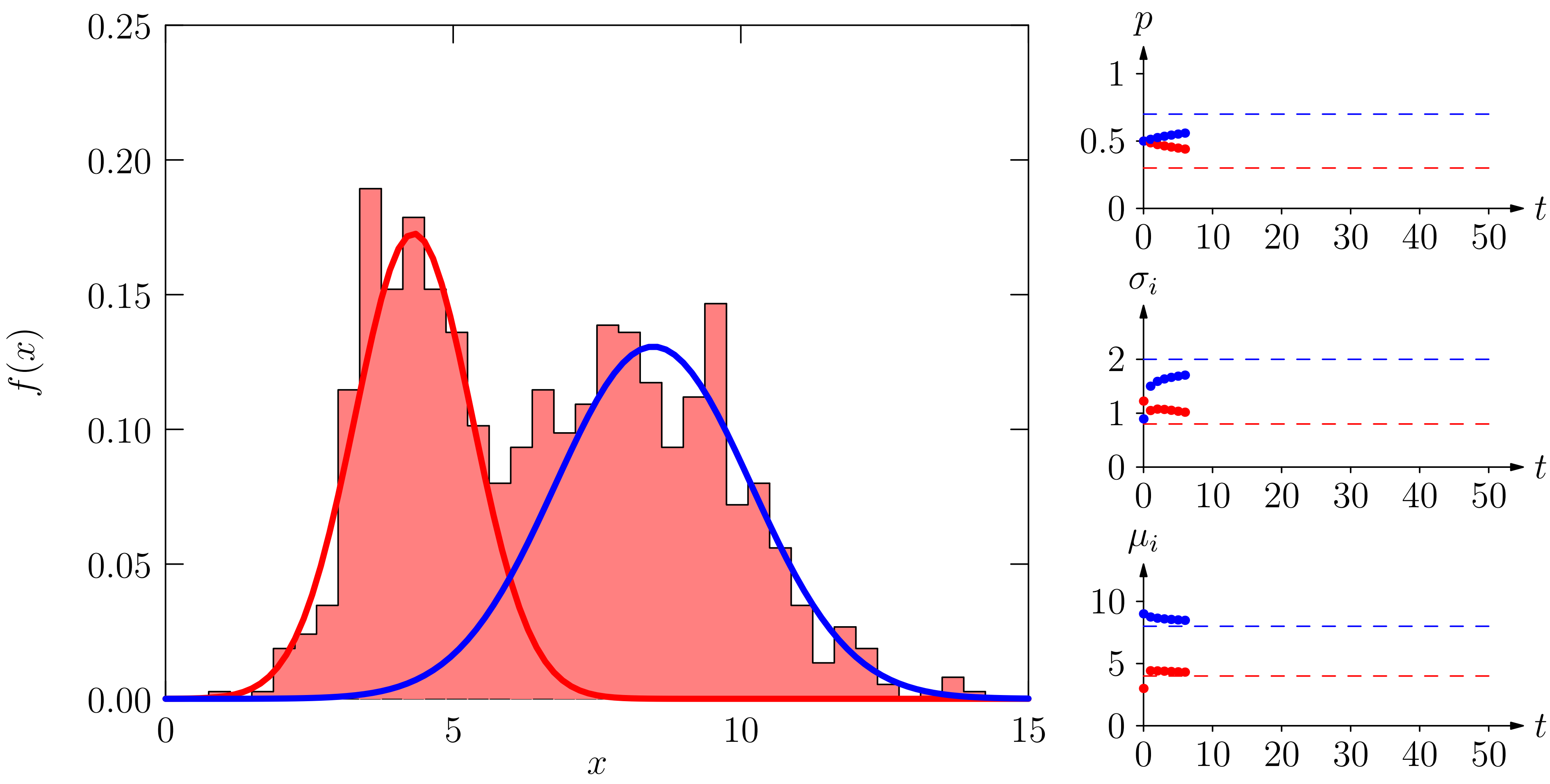
Example



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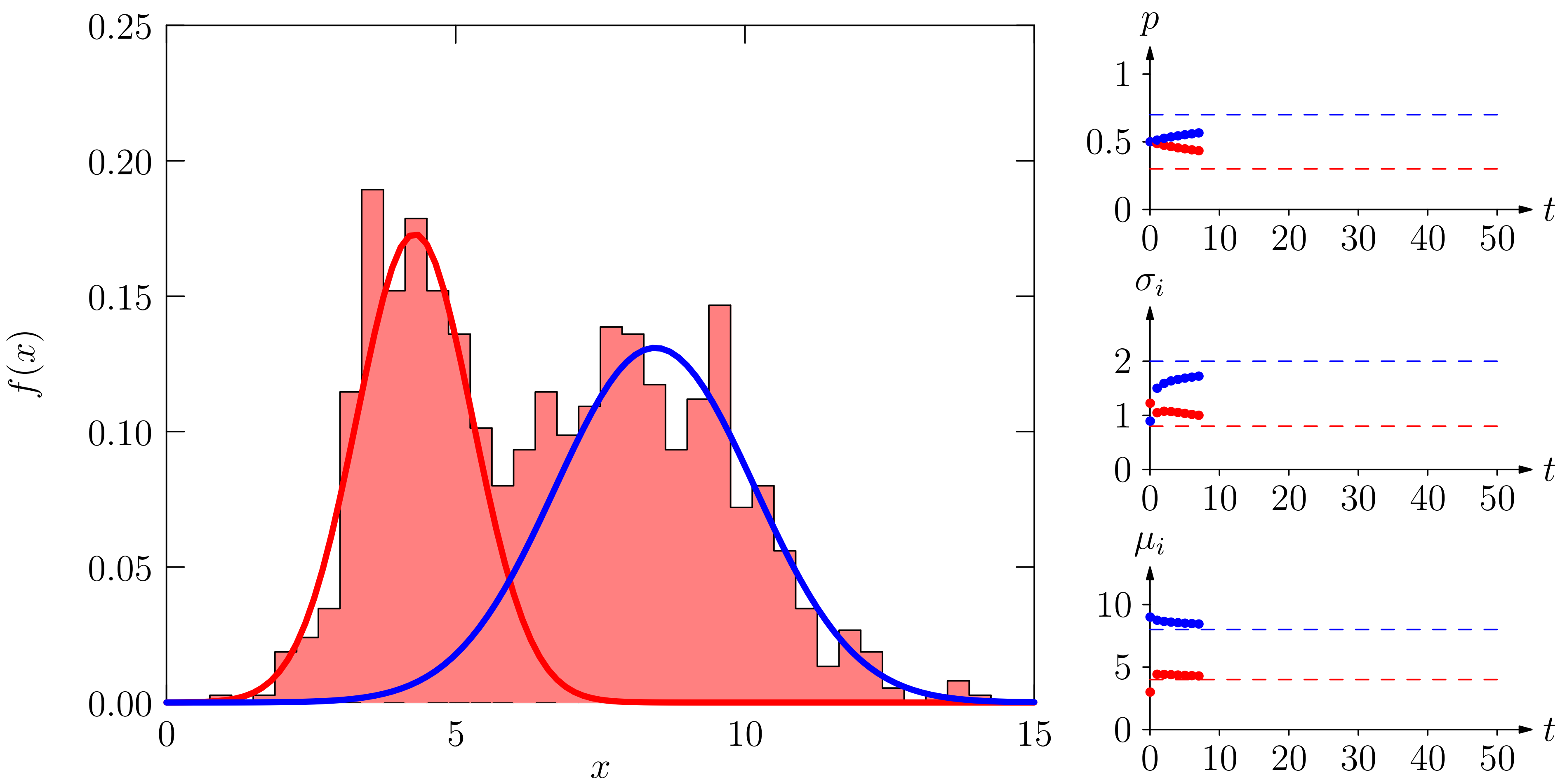
Example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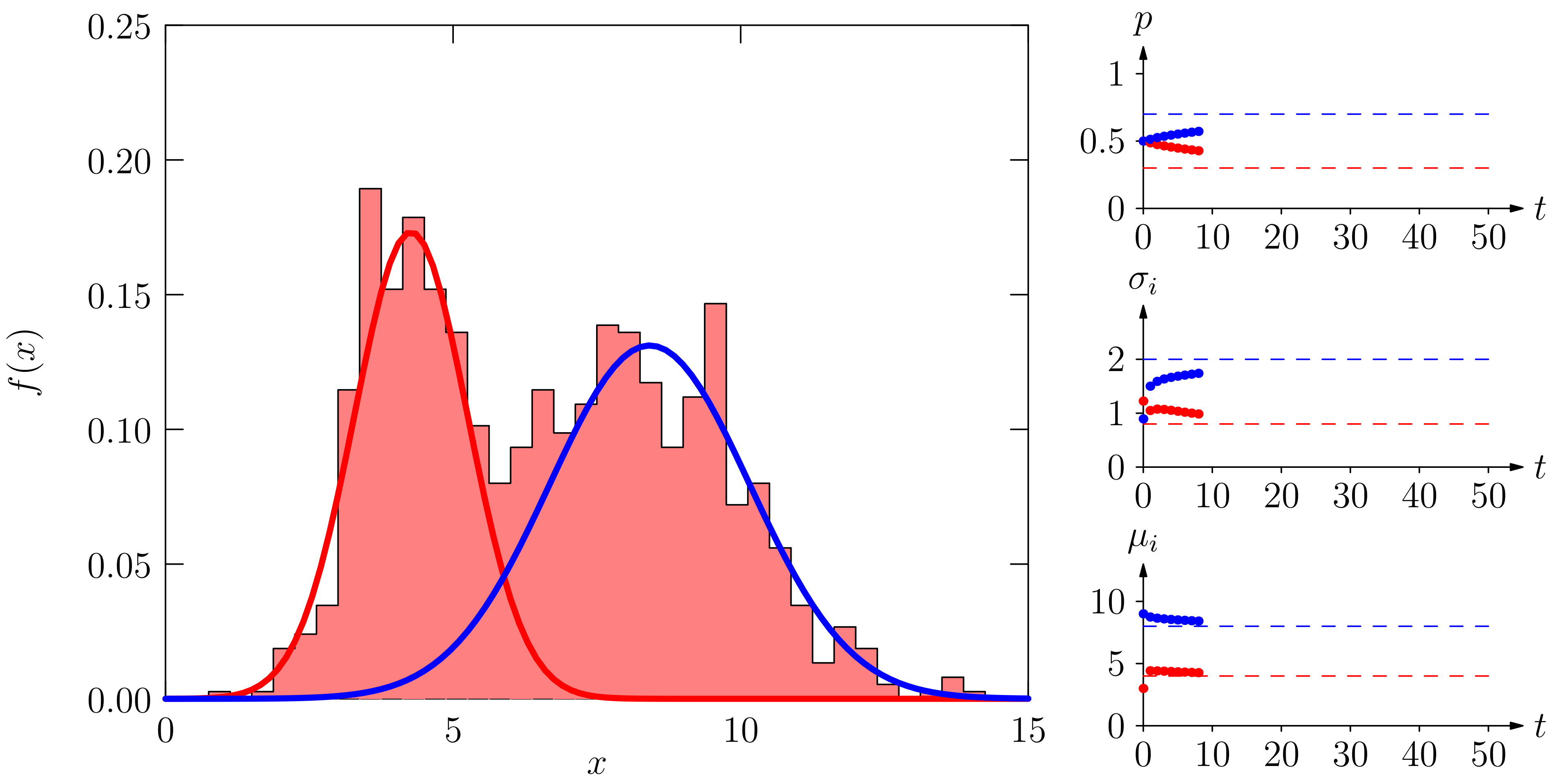
Example



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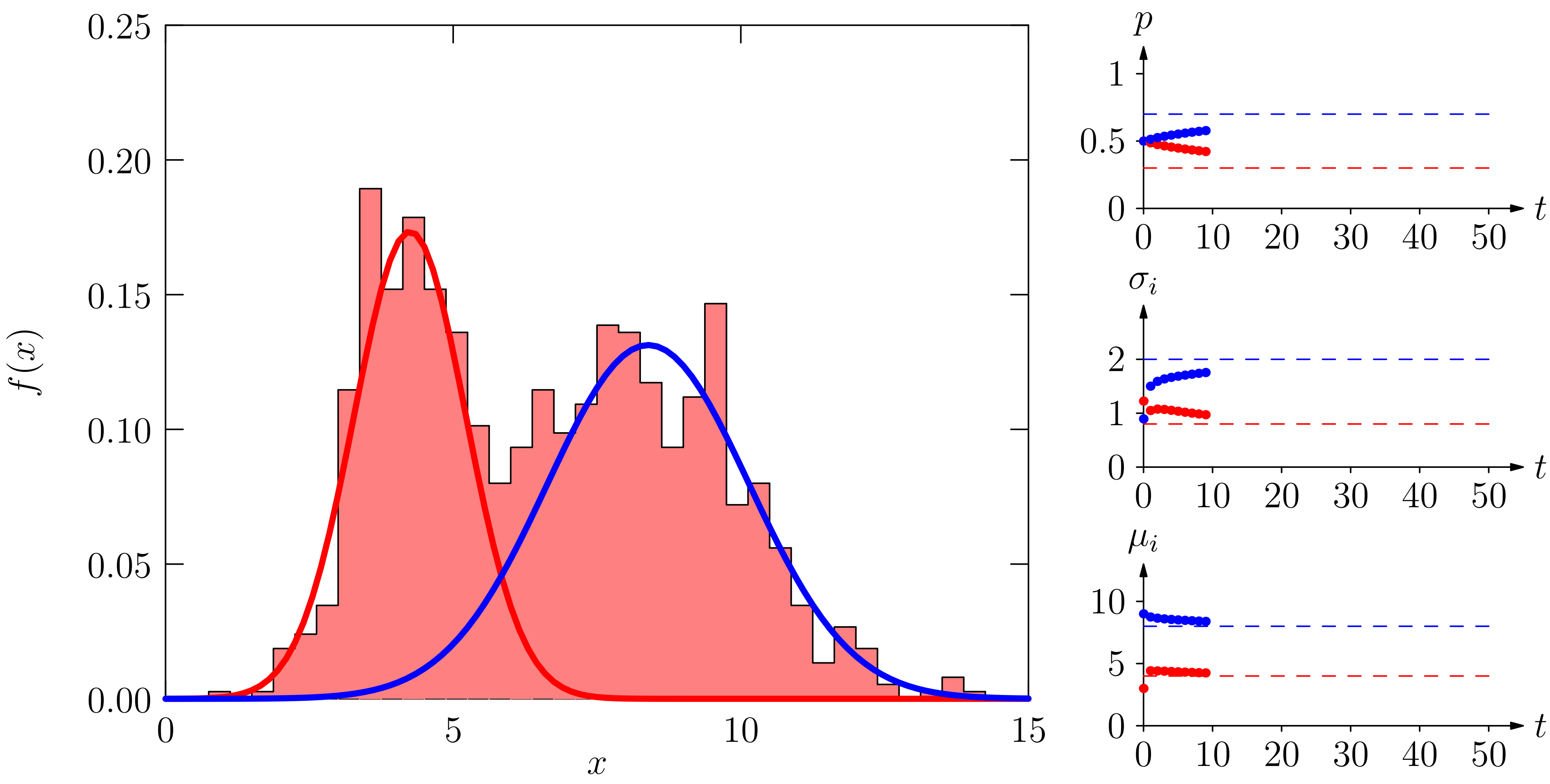

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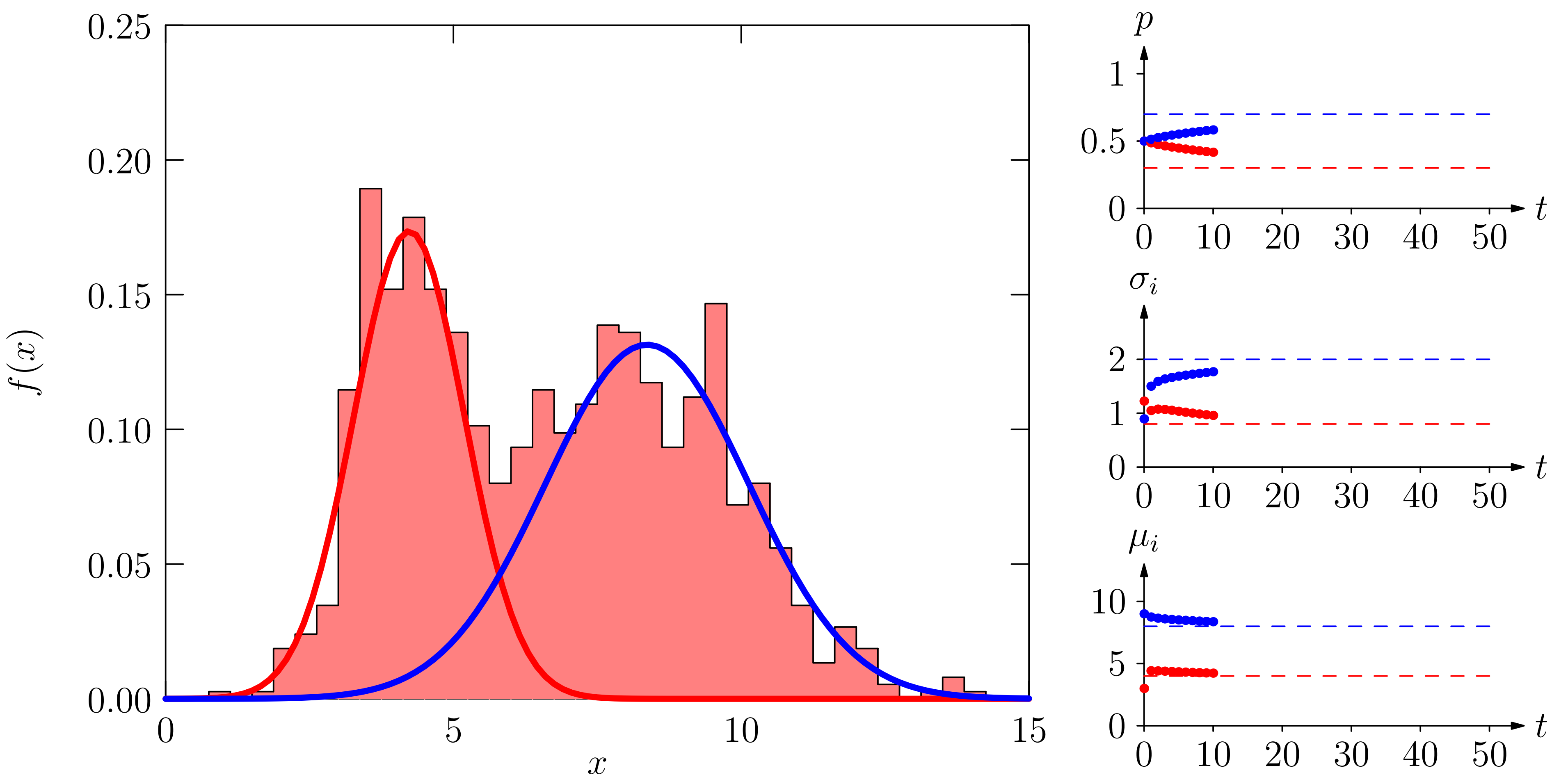

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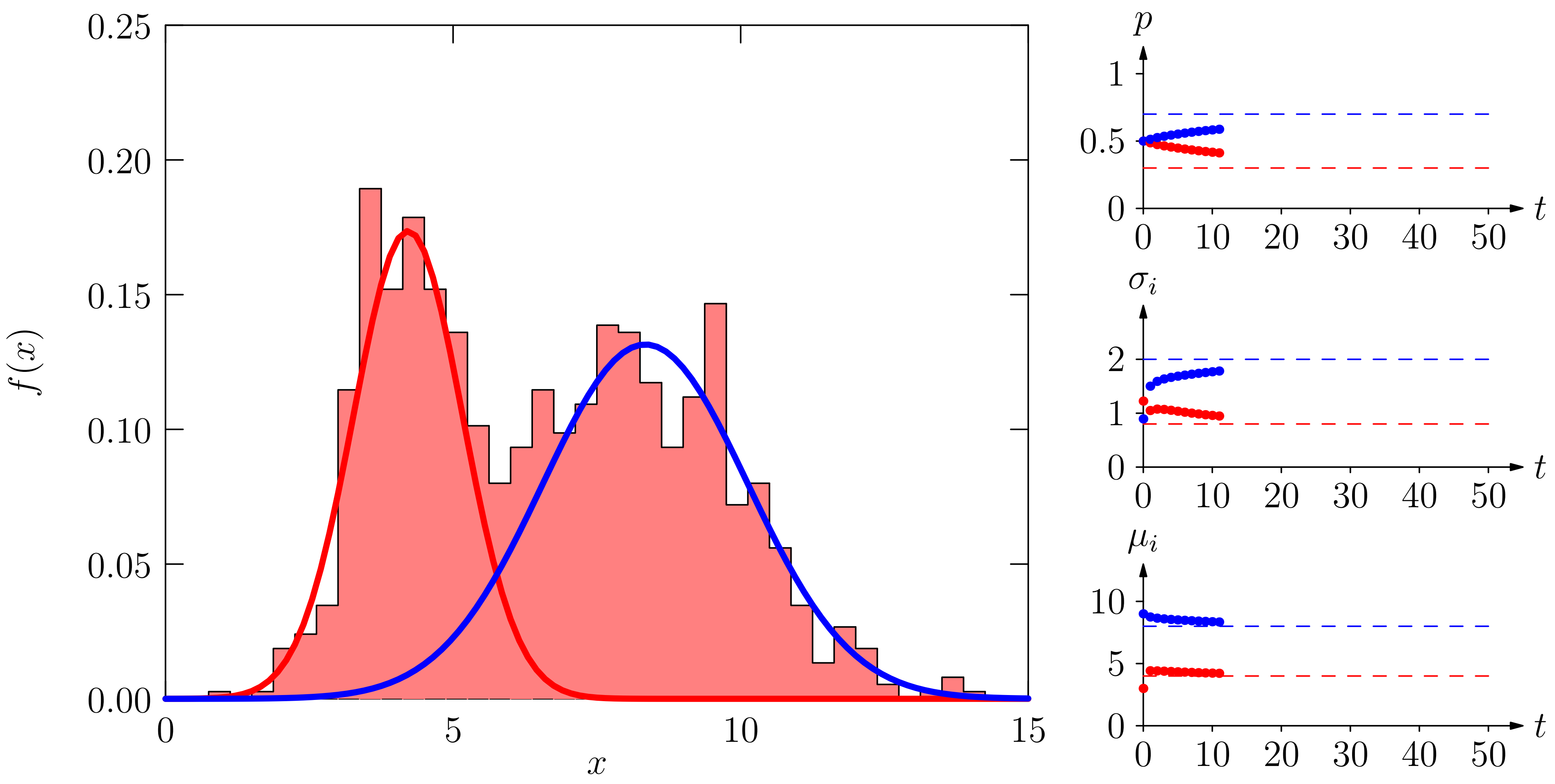

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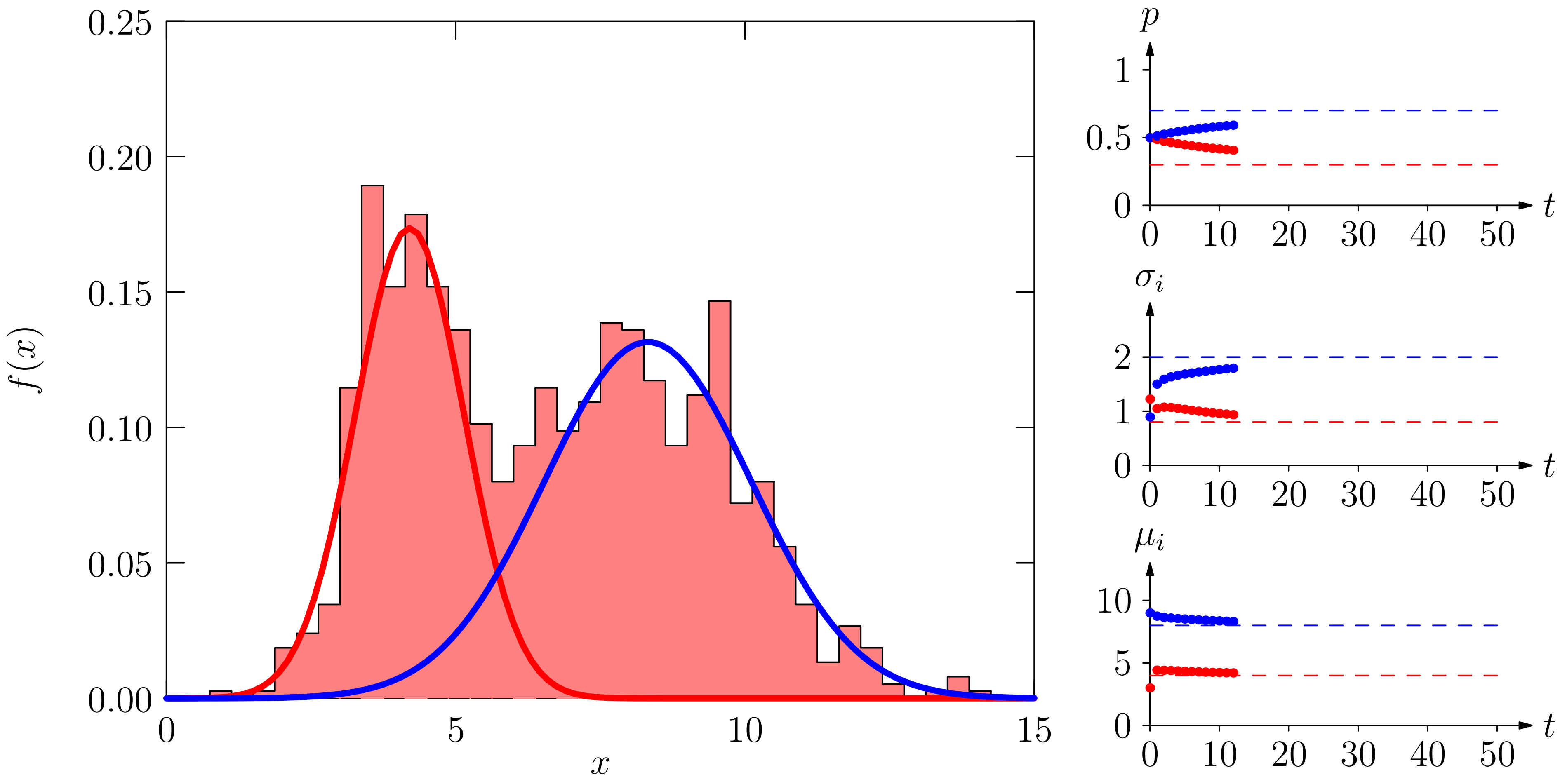

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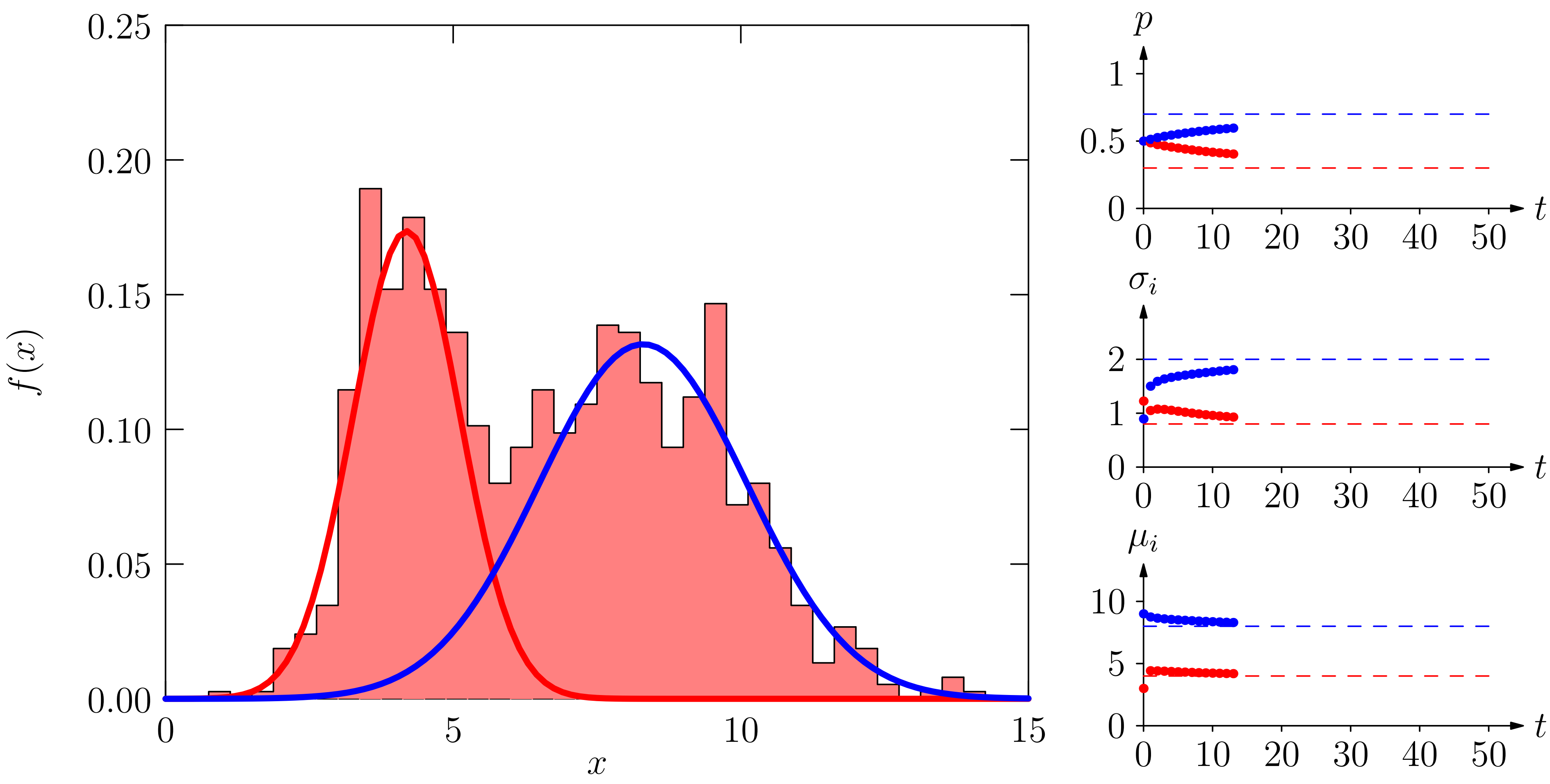

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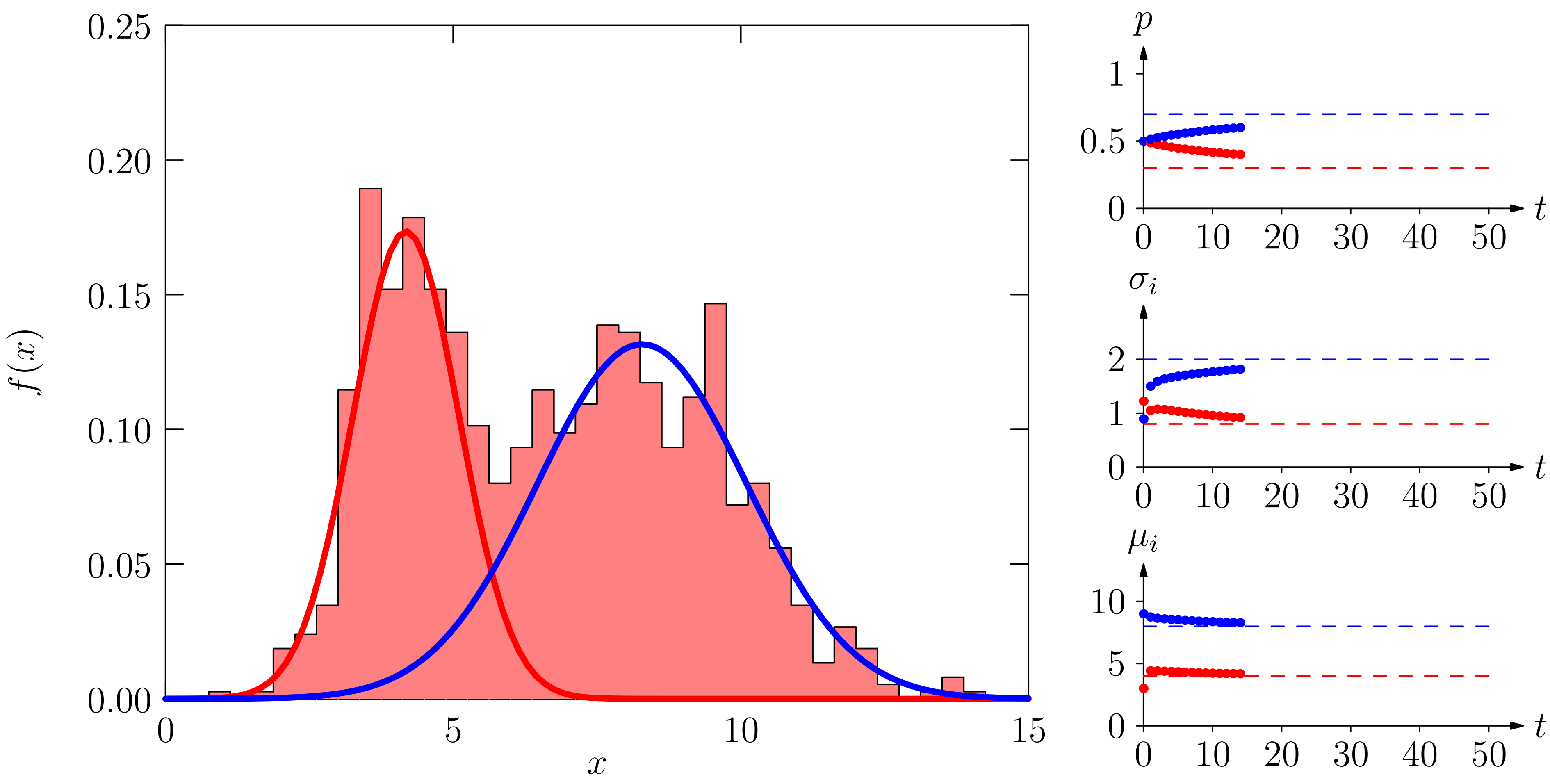

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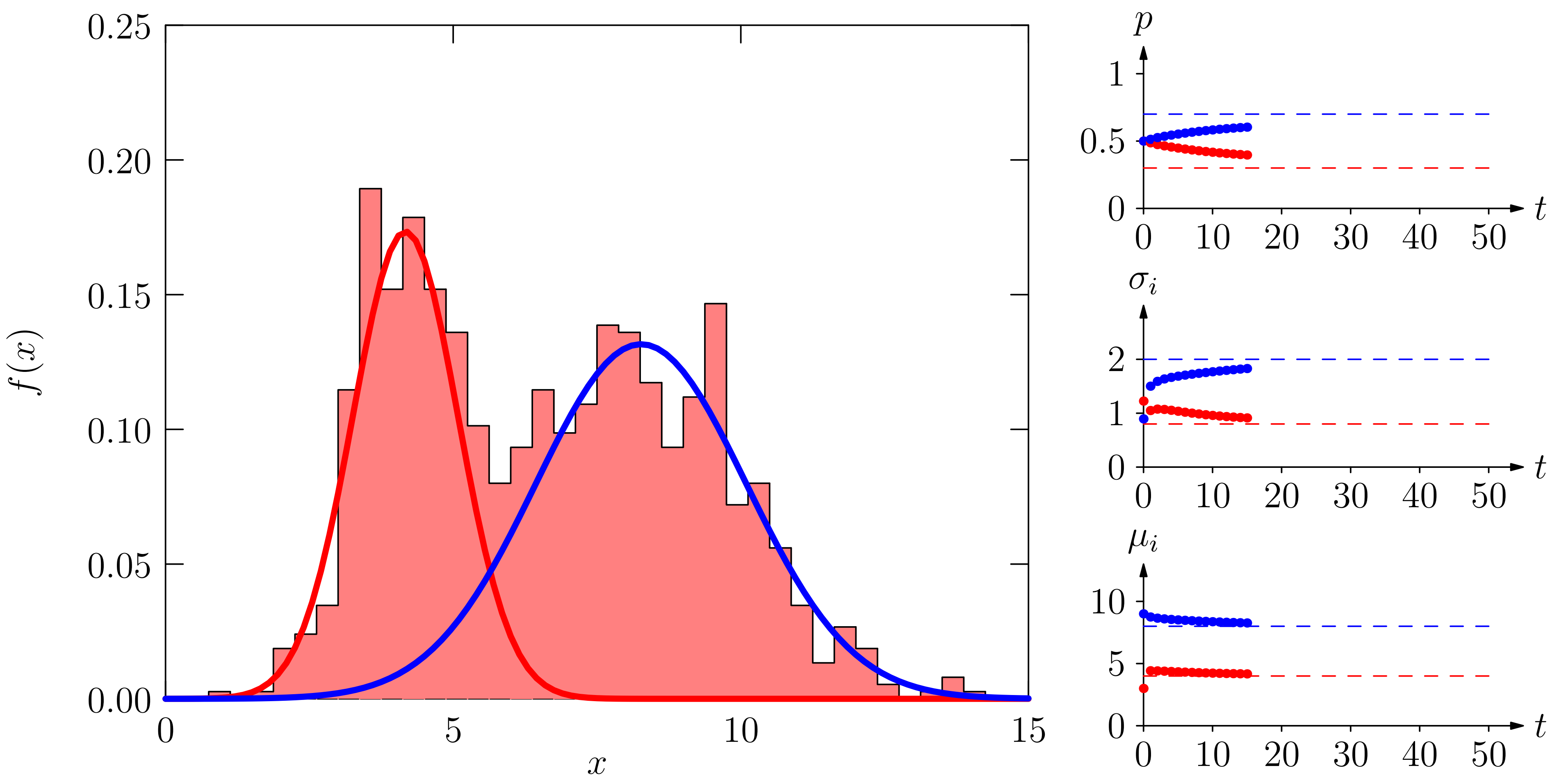

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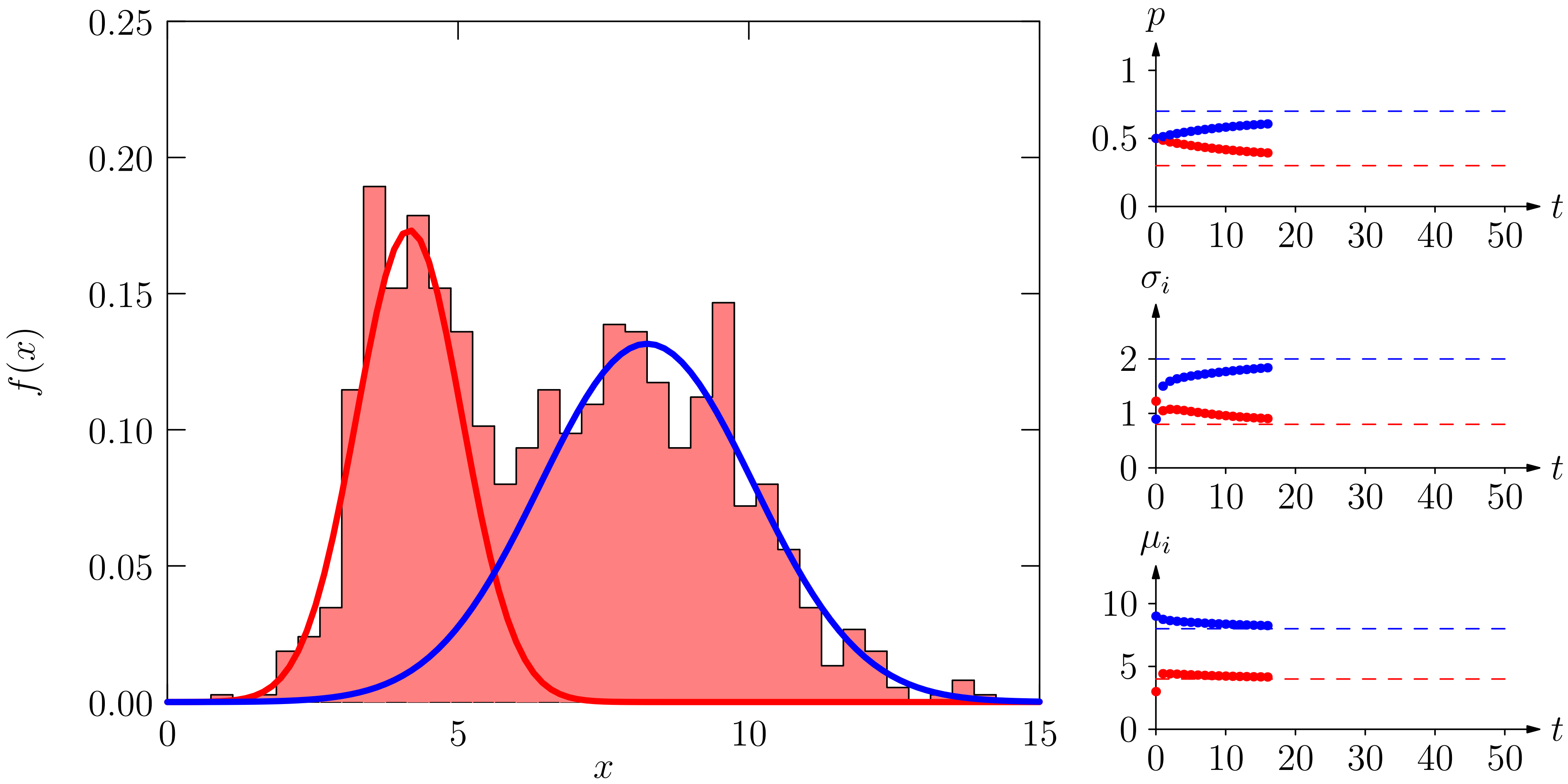

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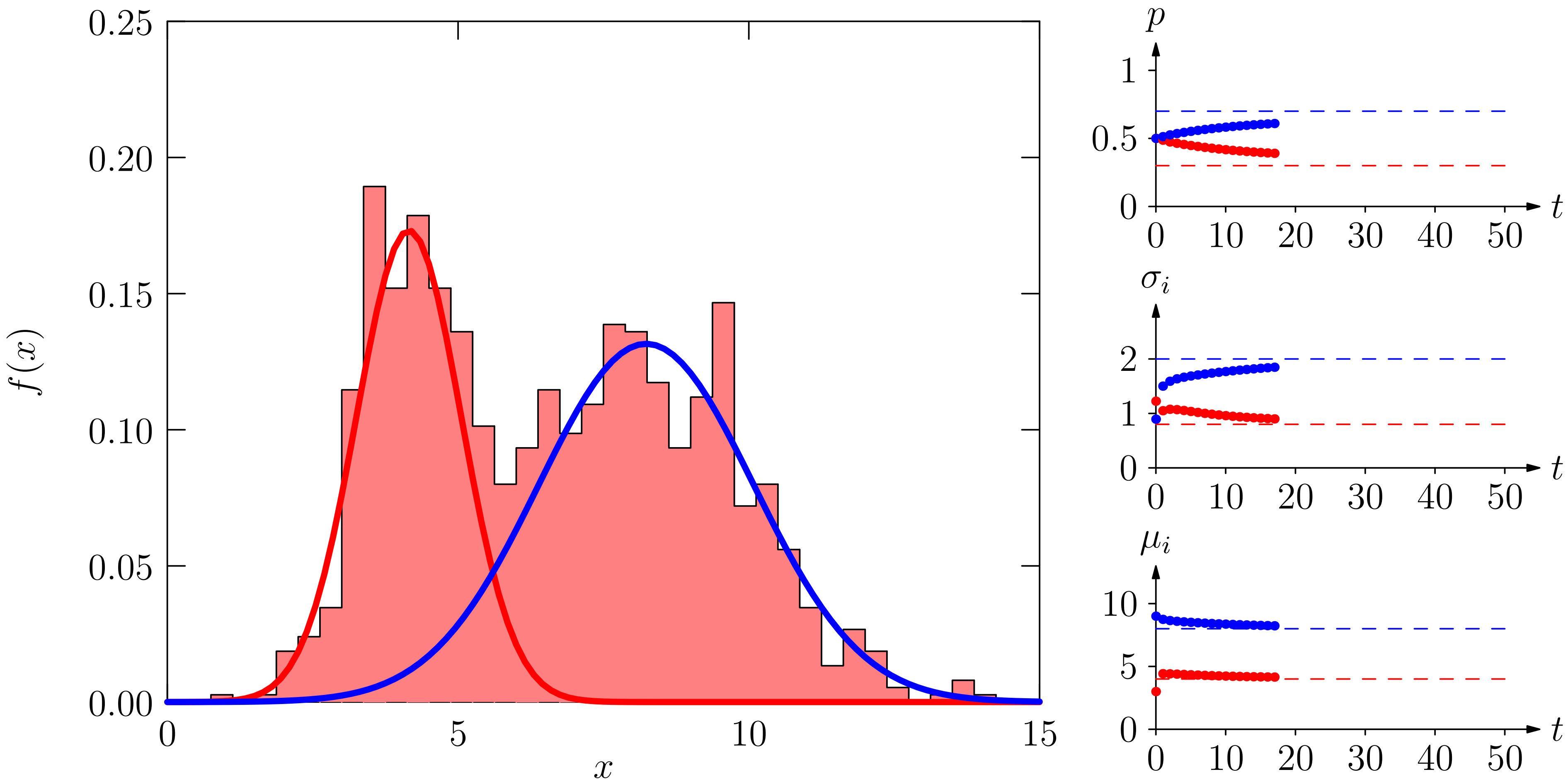
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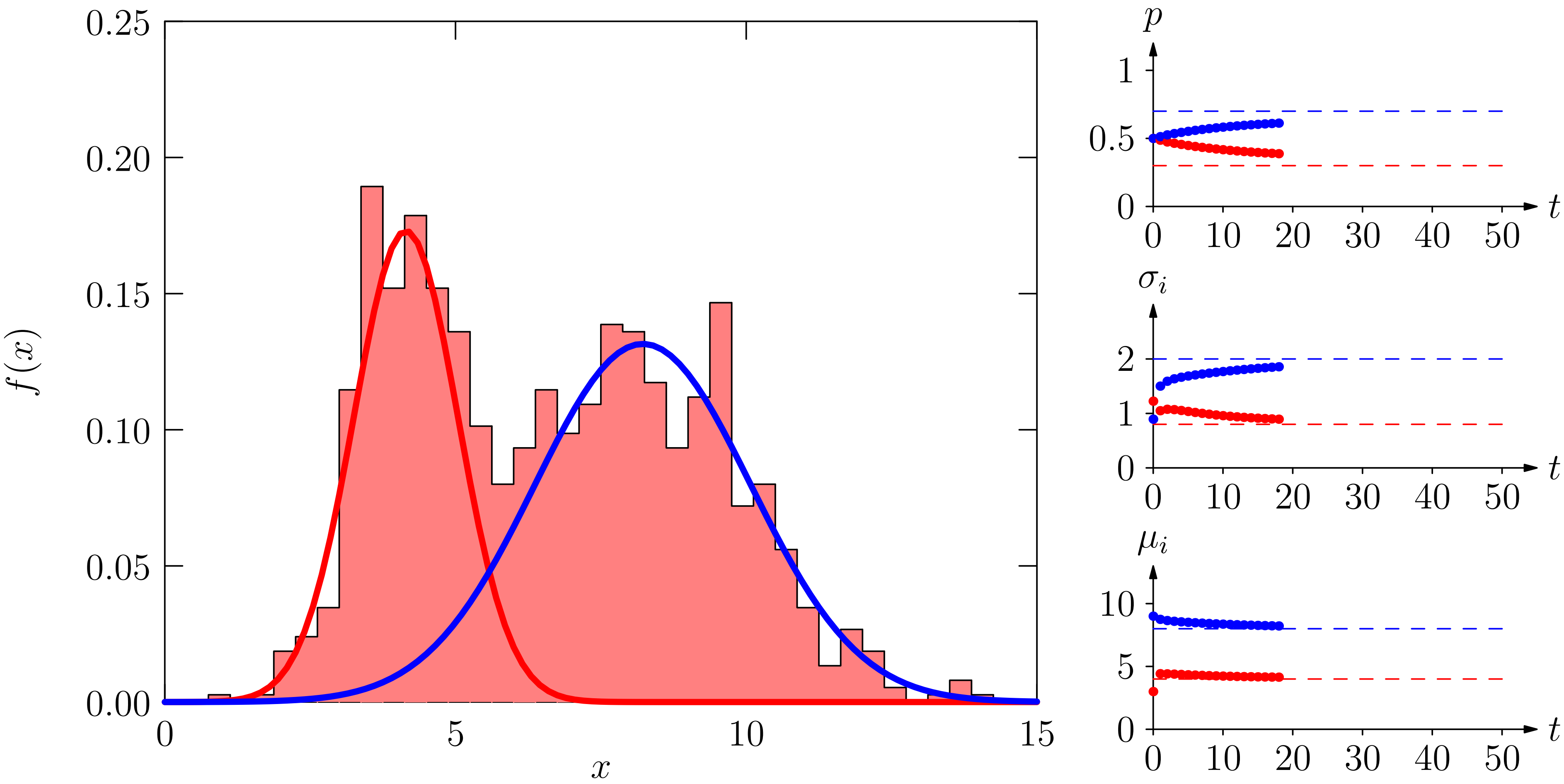
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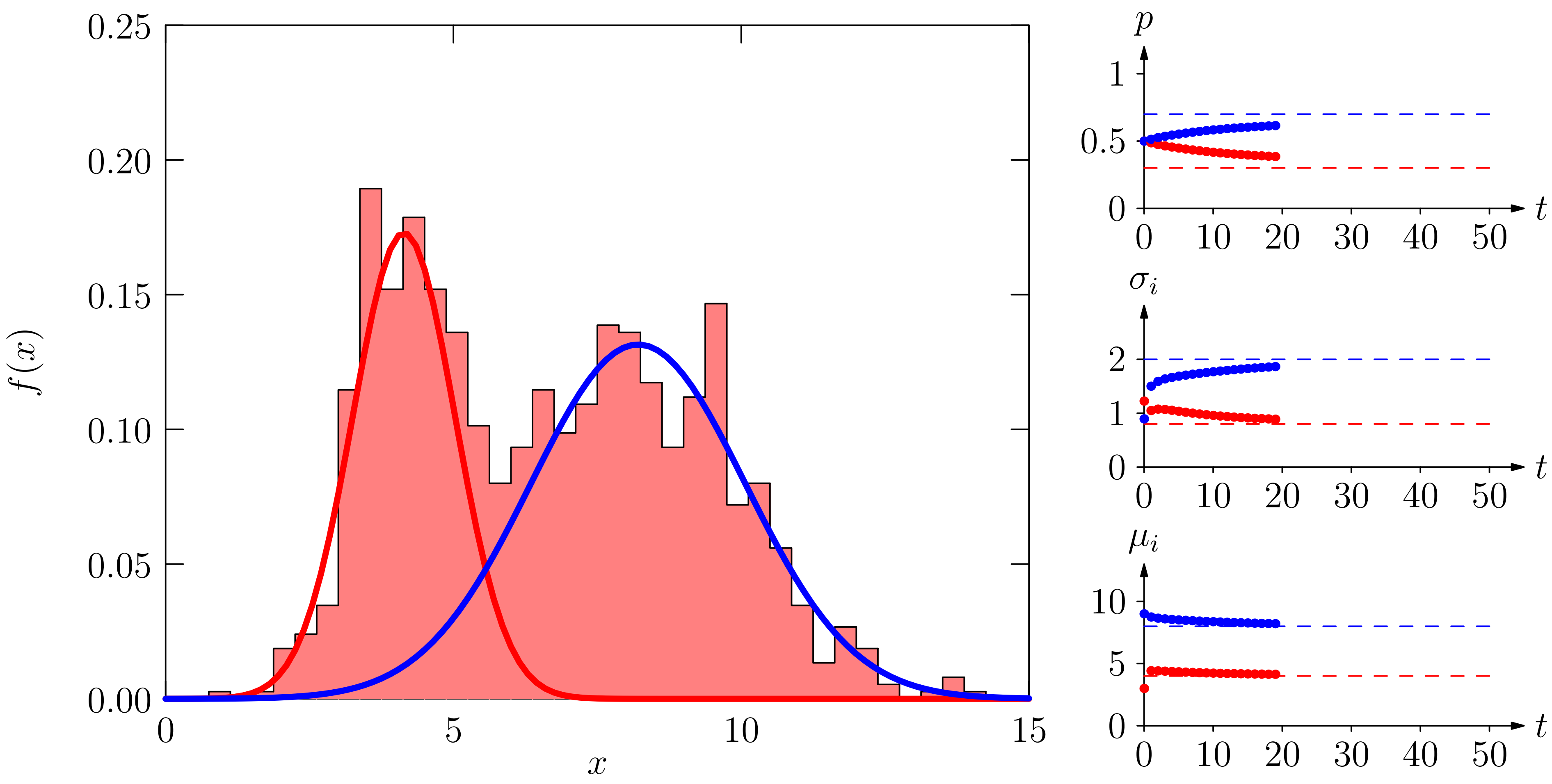

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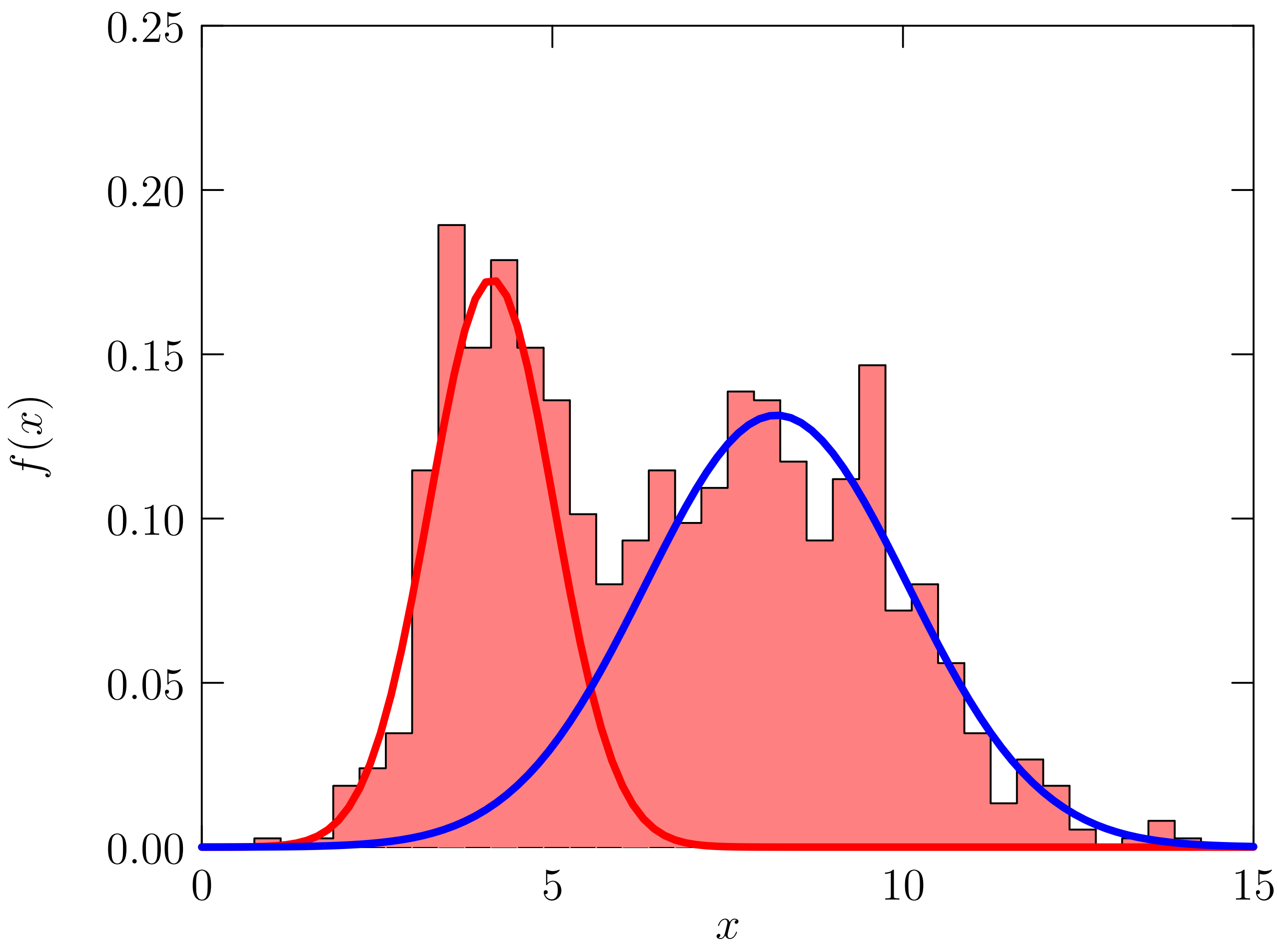

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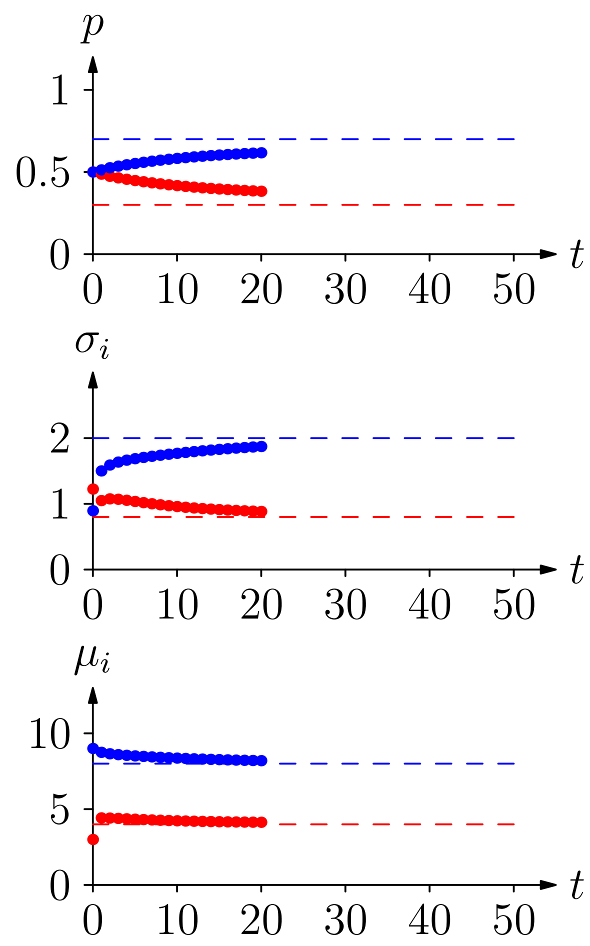

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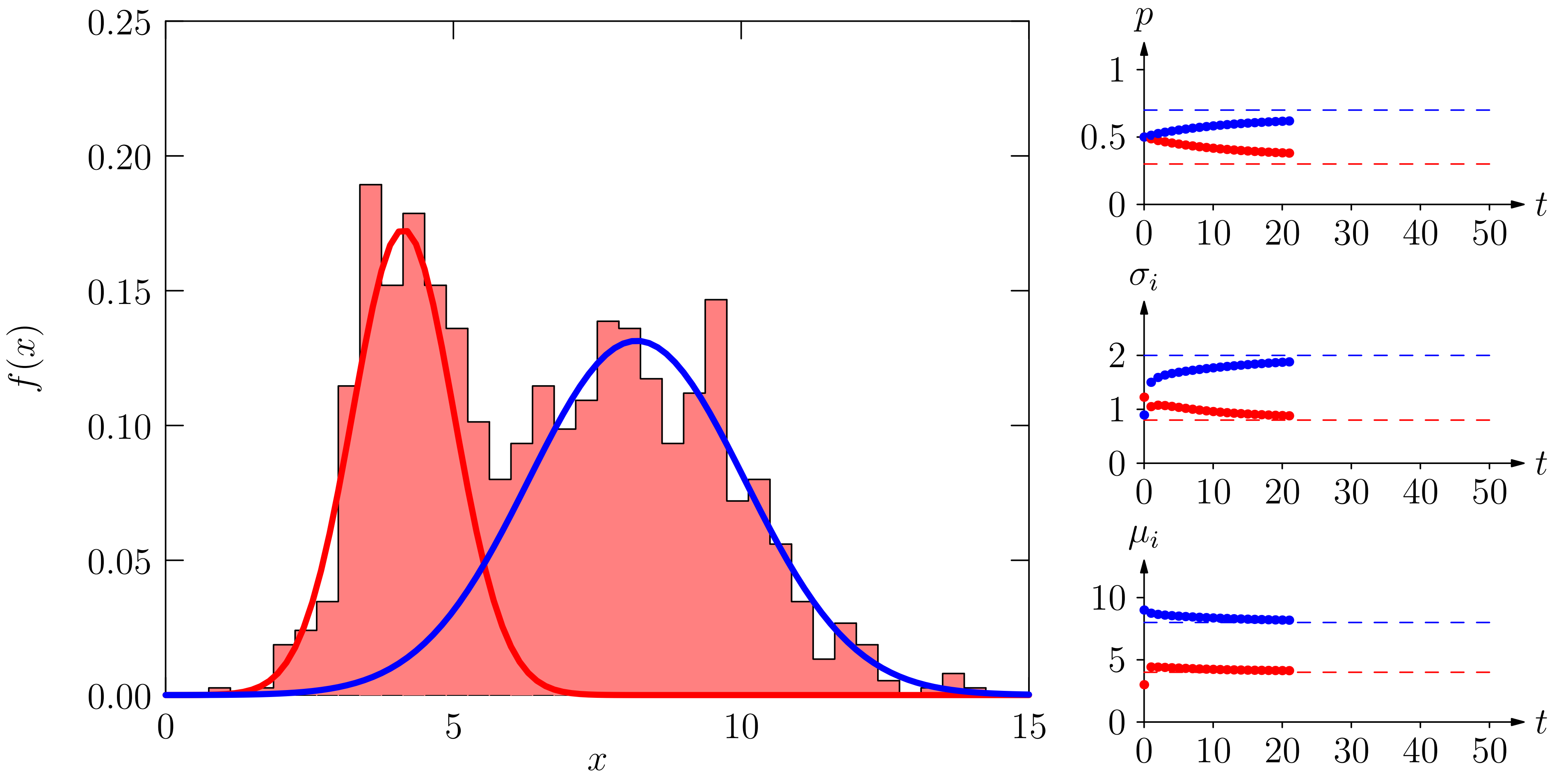
Example



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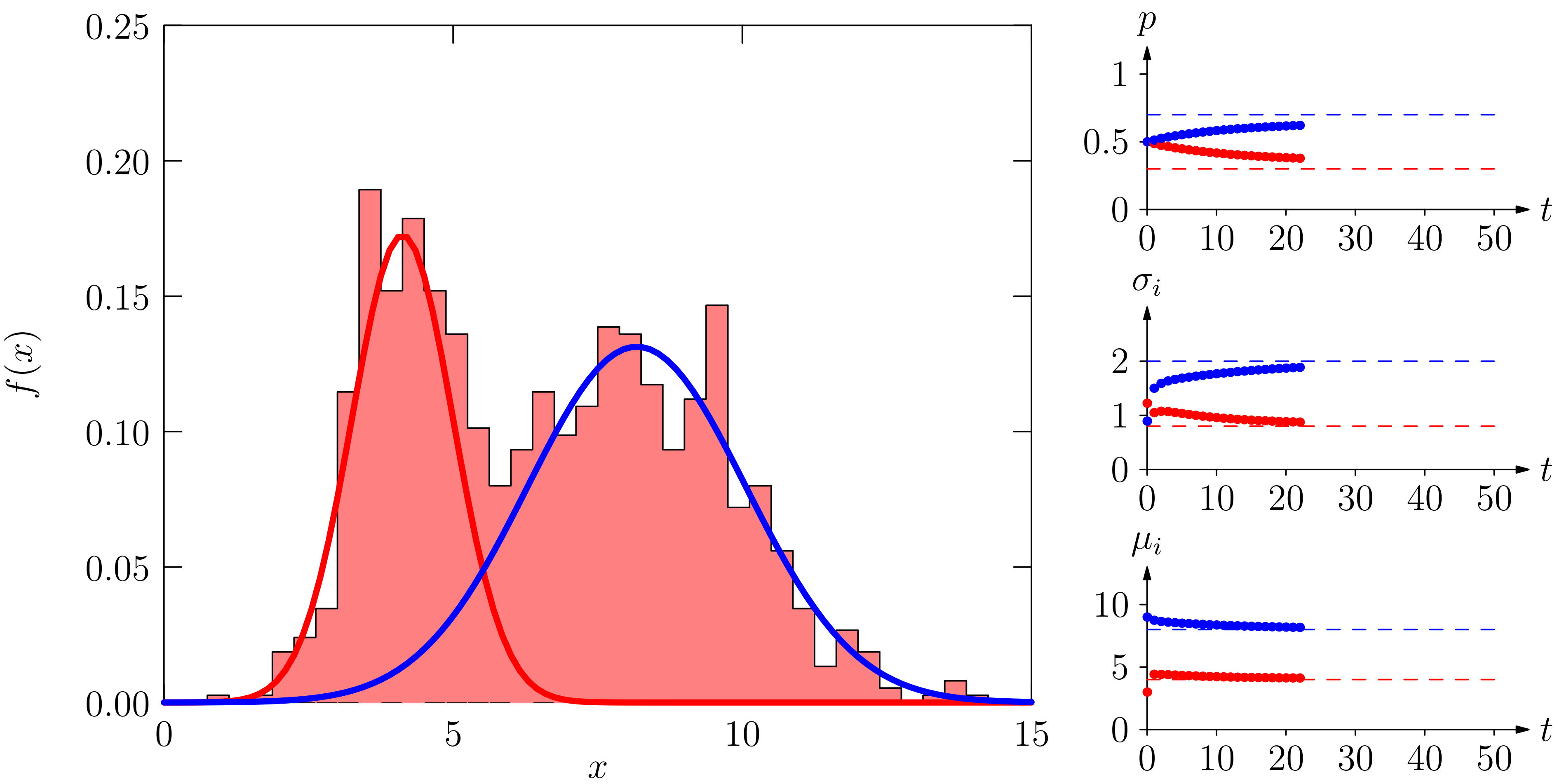
Example



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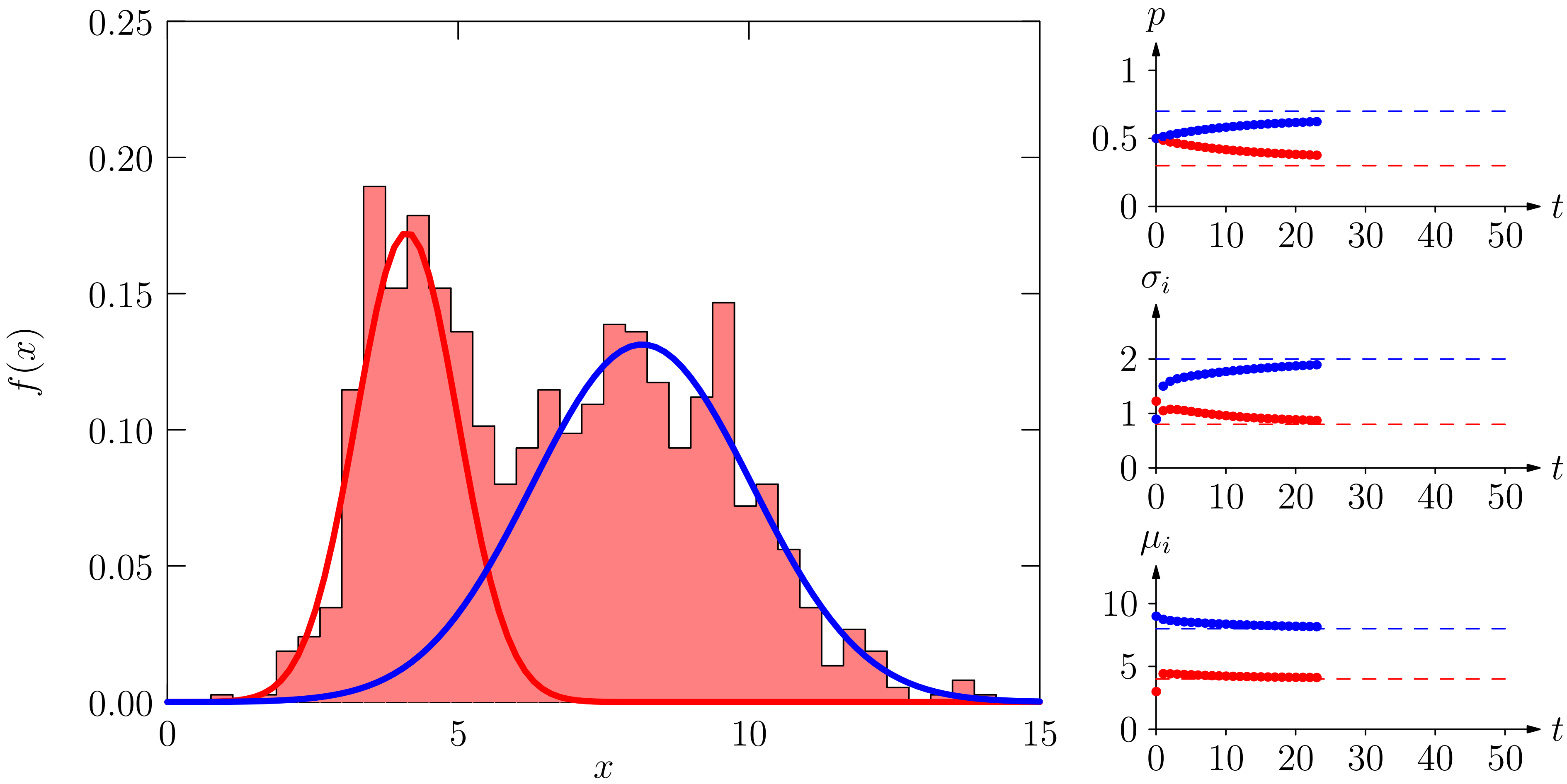
Example



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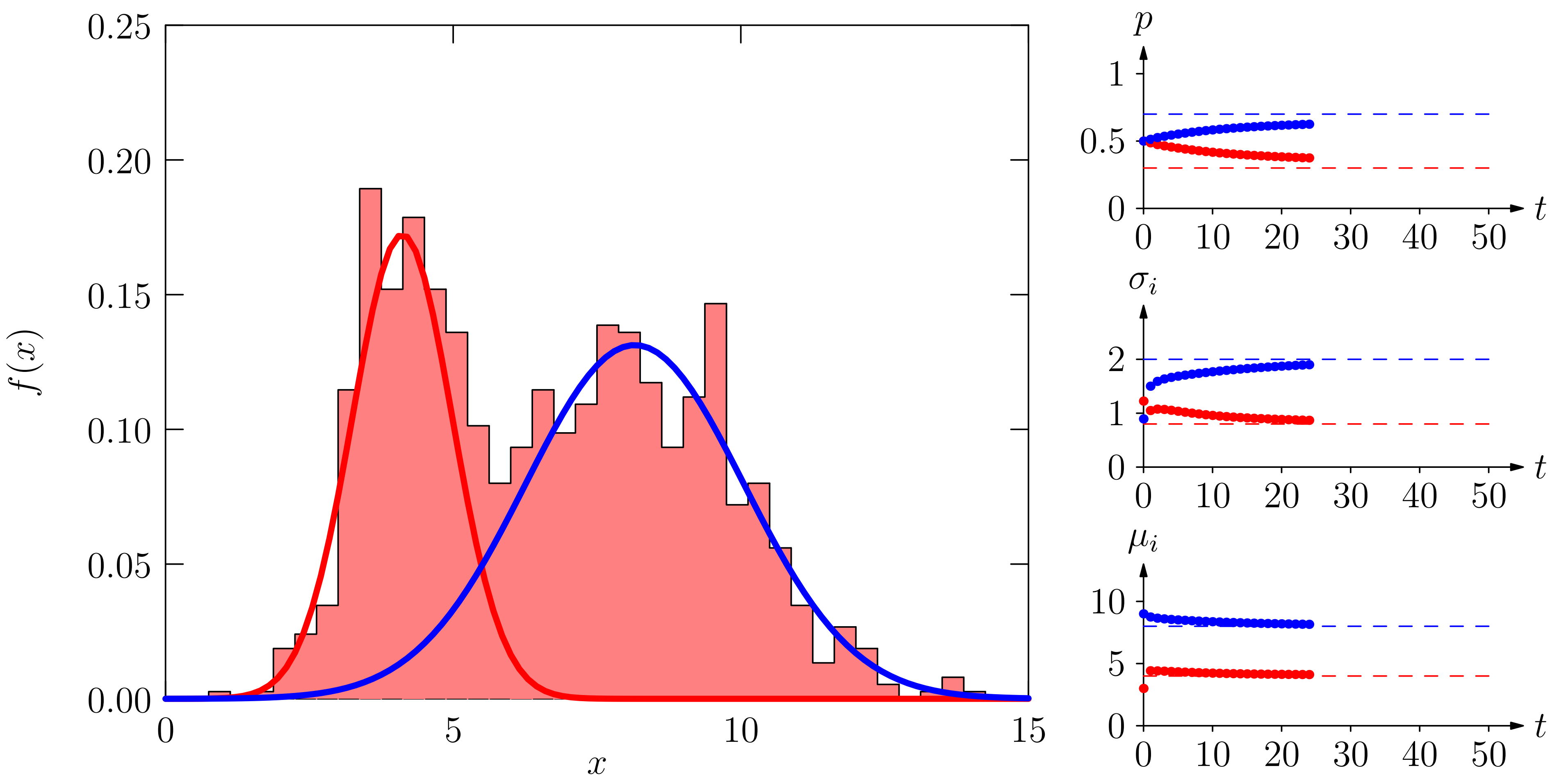
Example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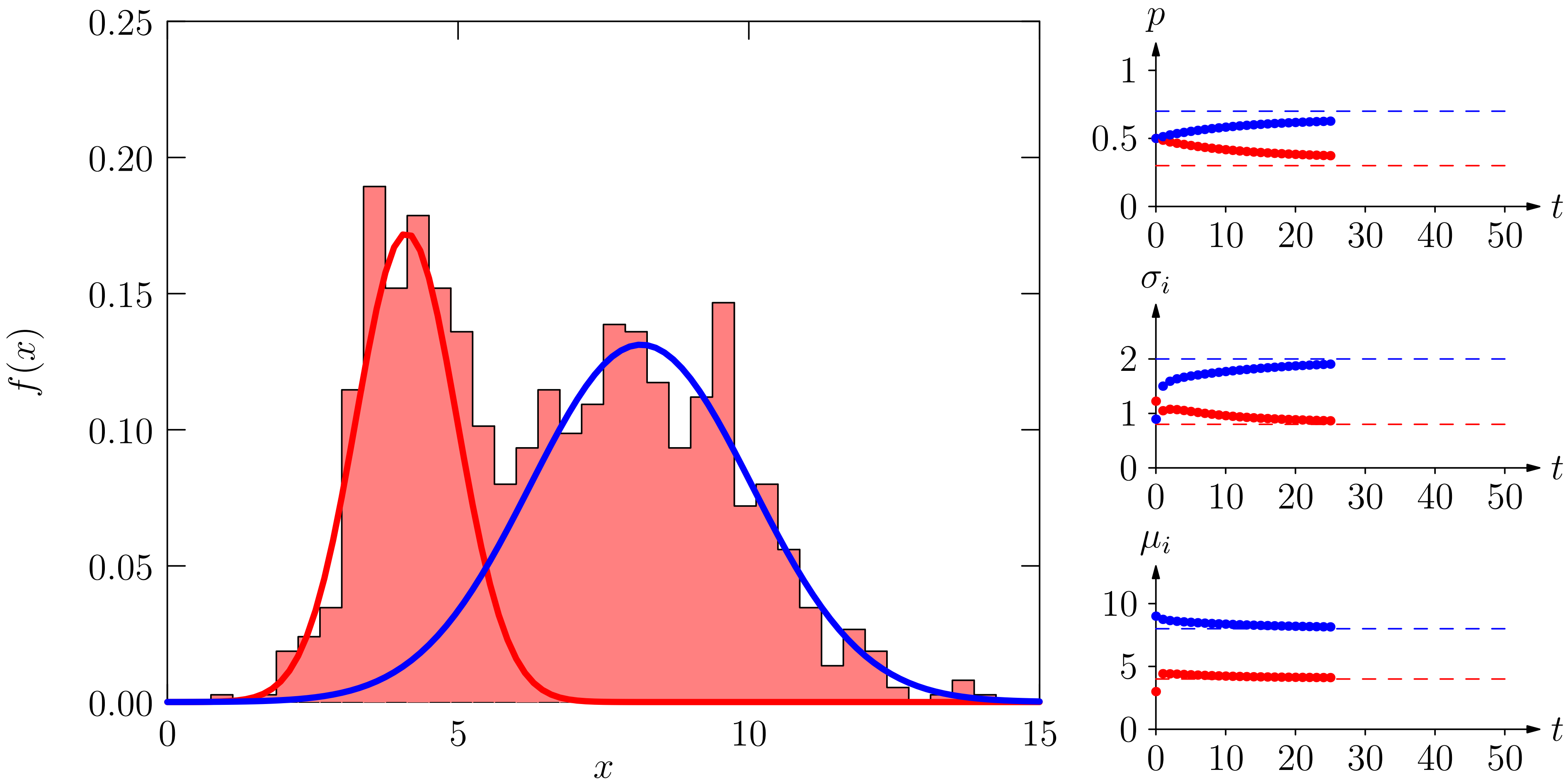
Example



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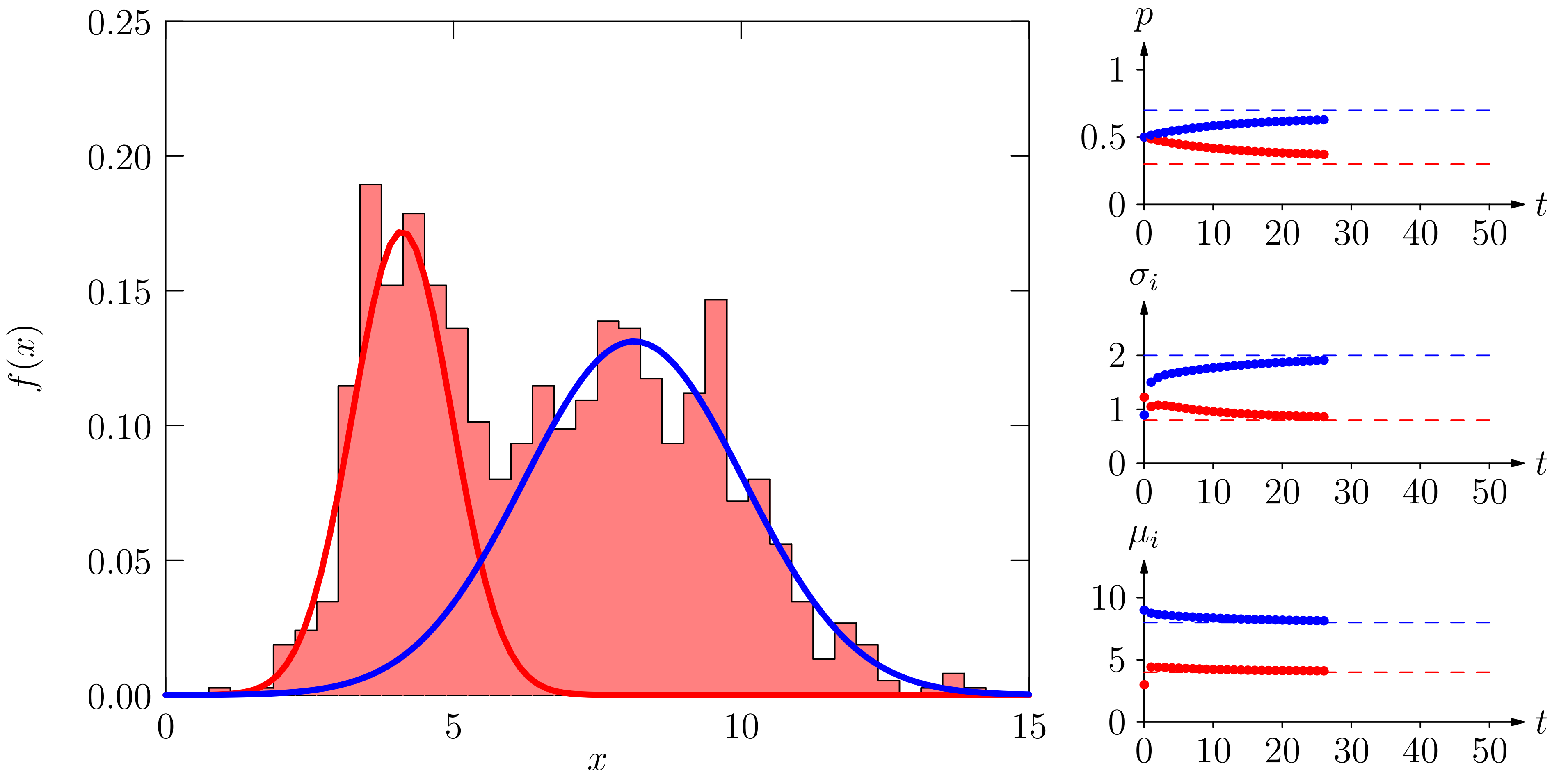
Example



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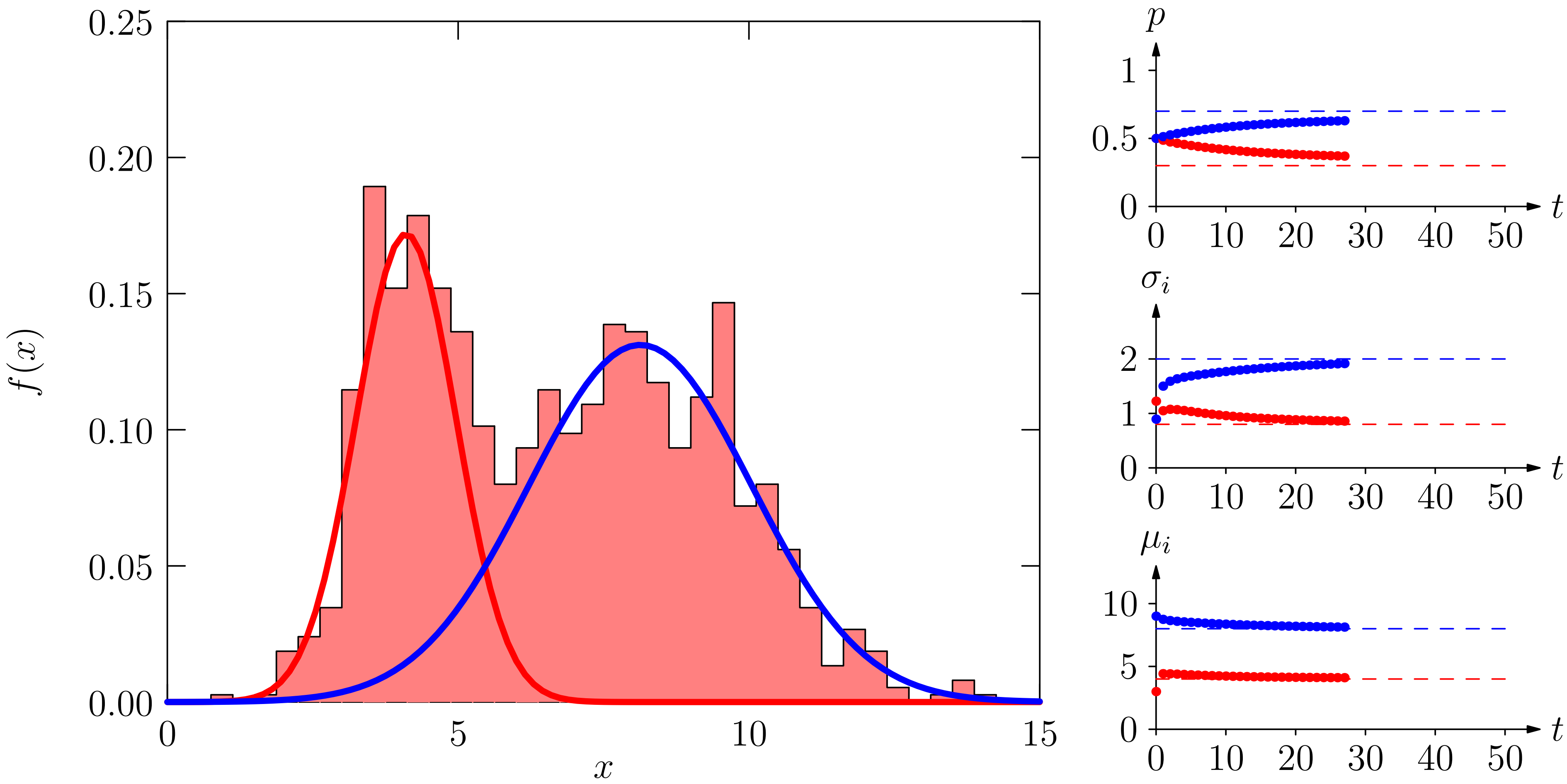

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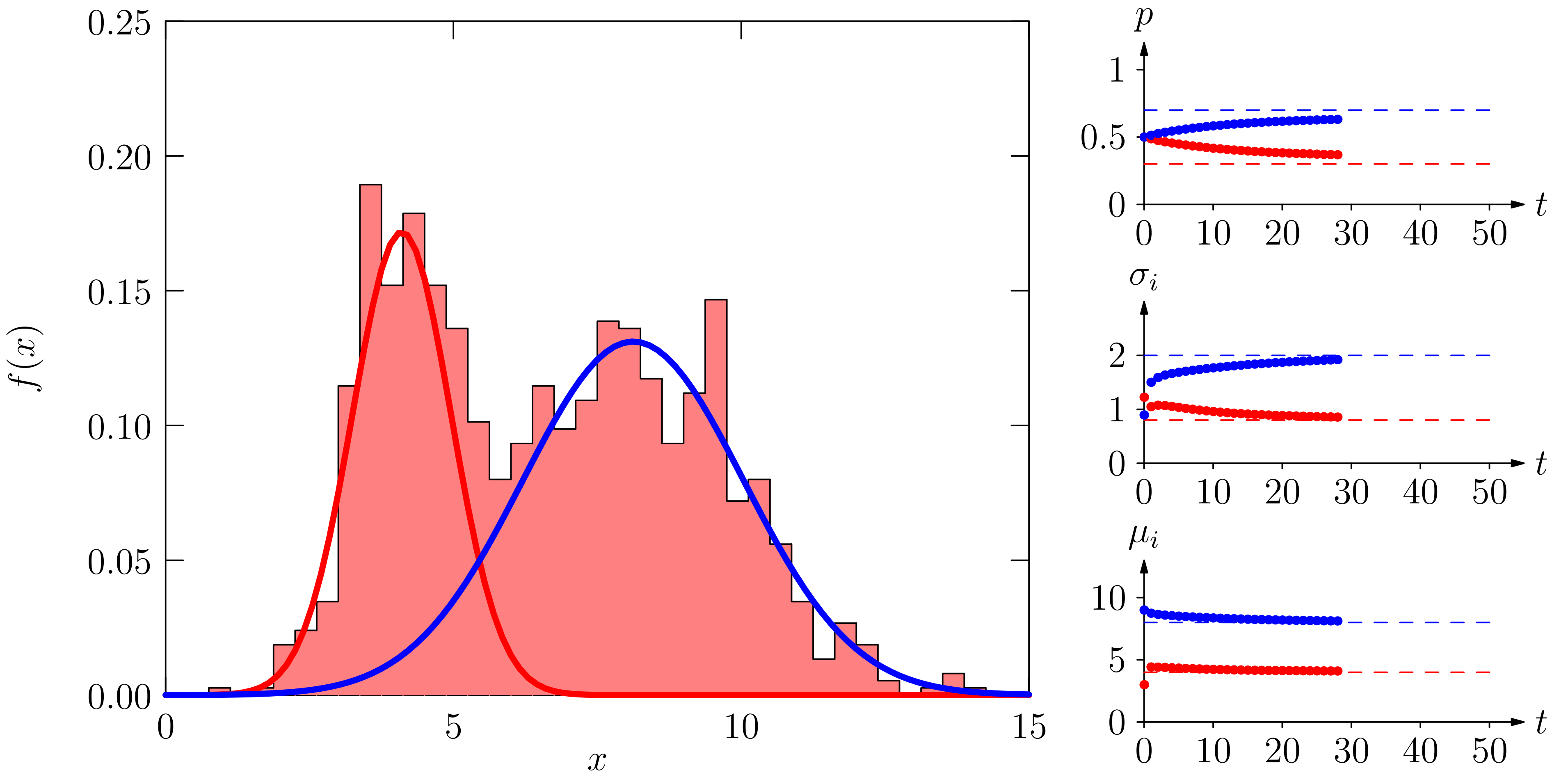
Example



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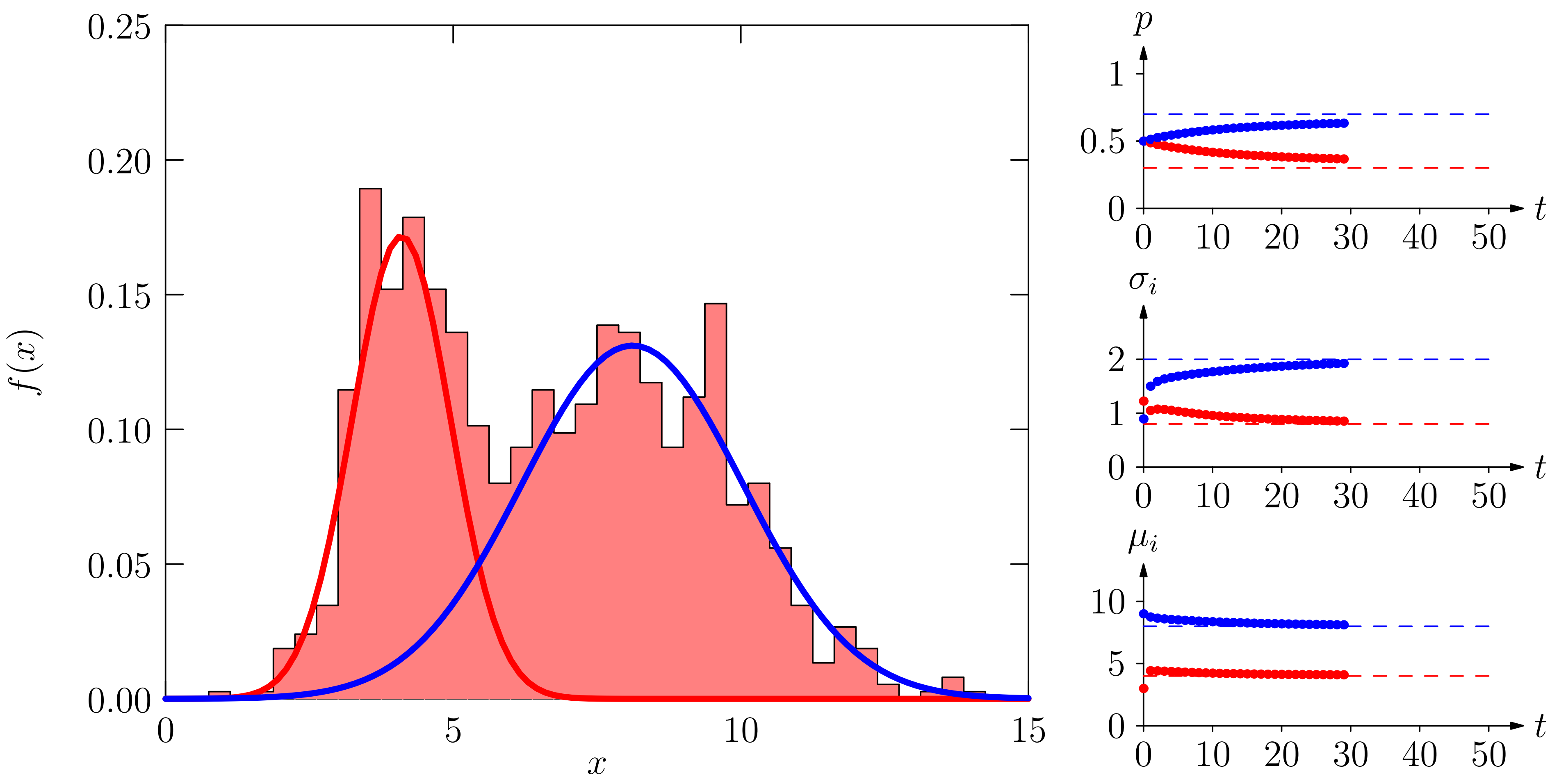

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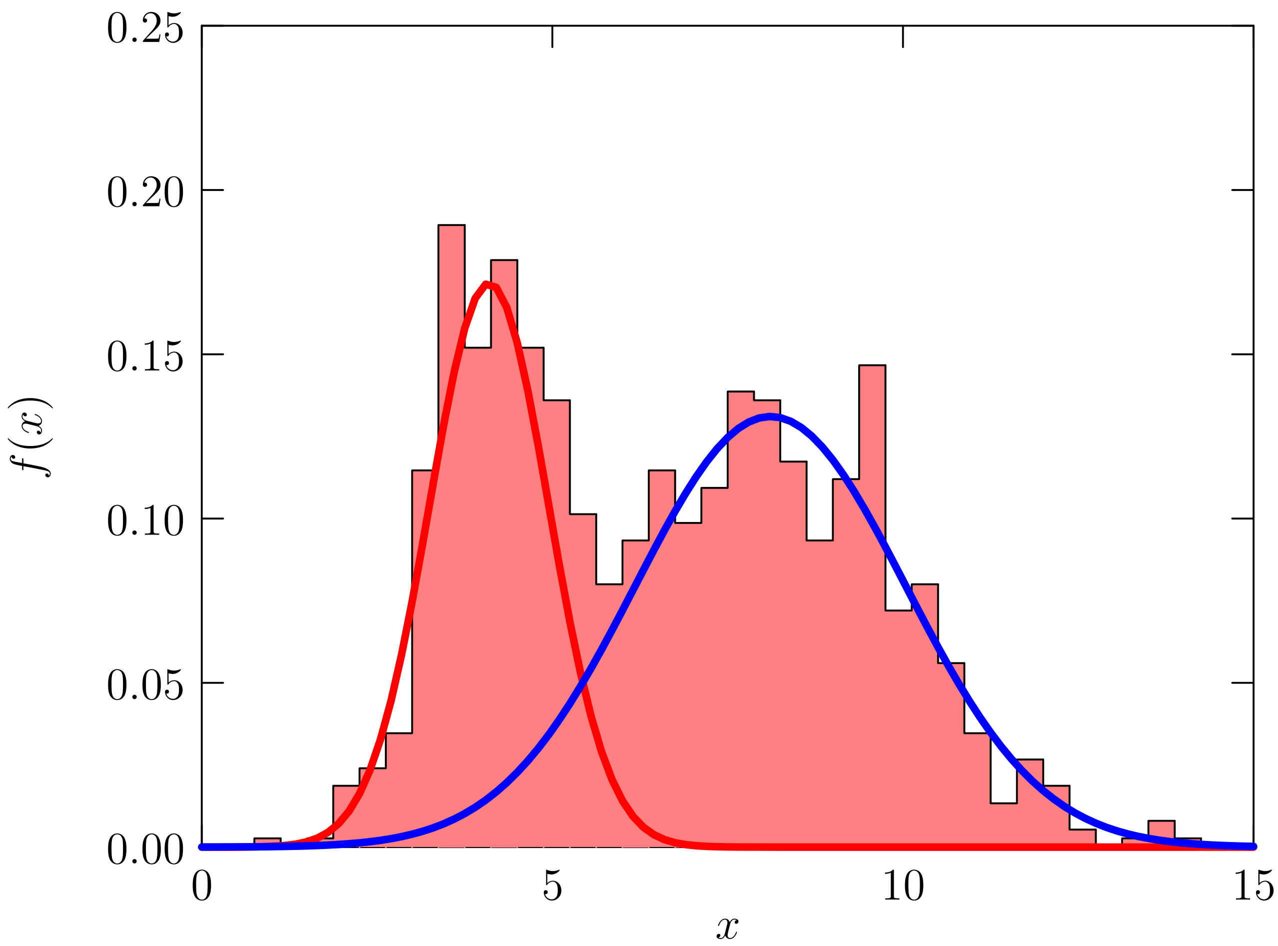

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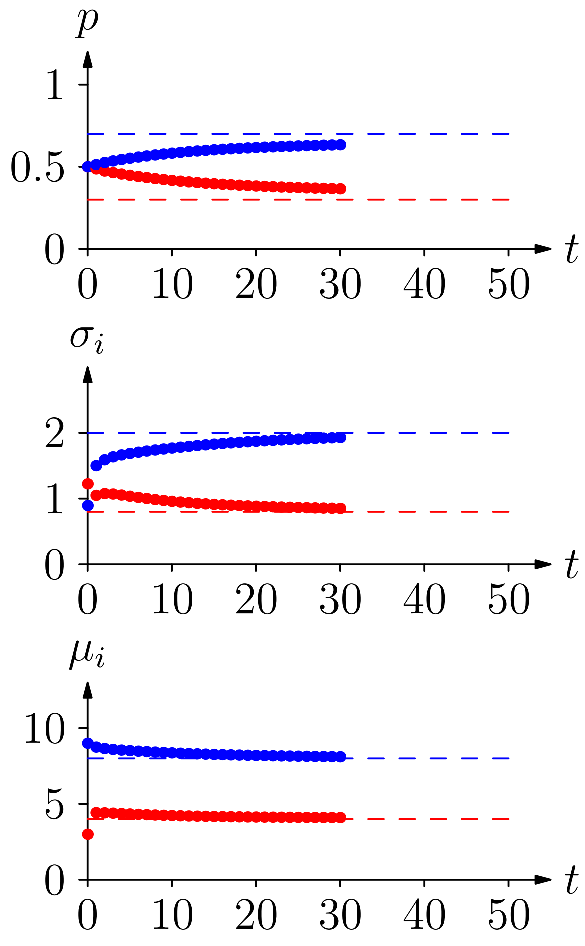

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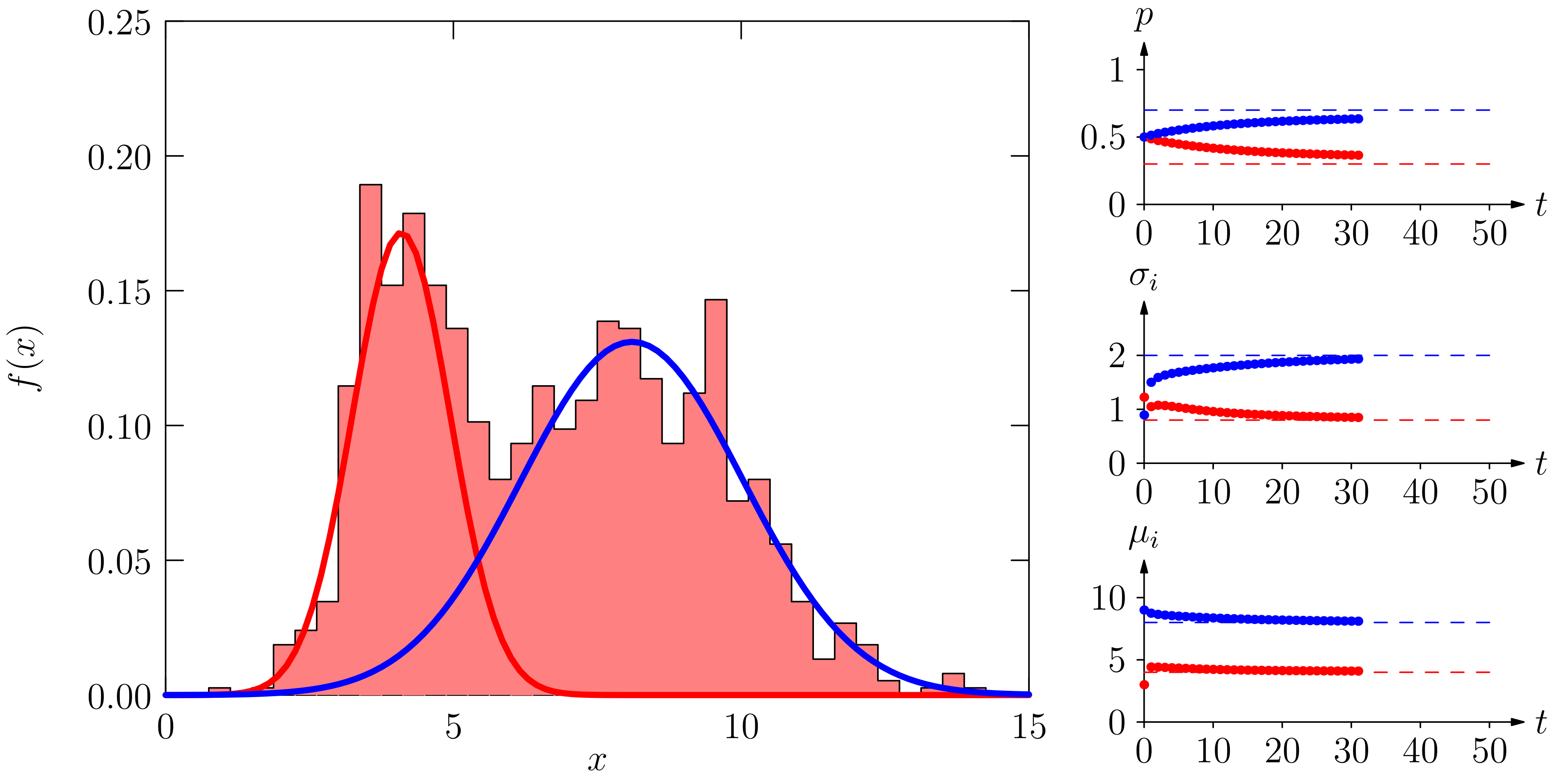

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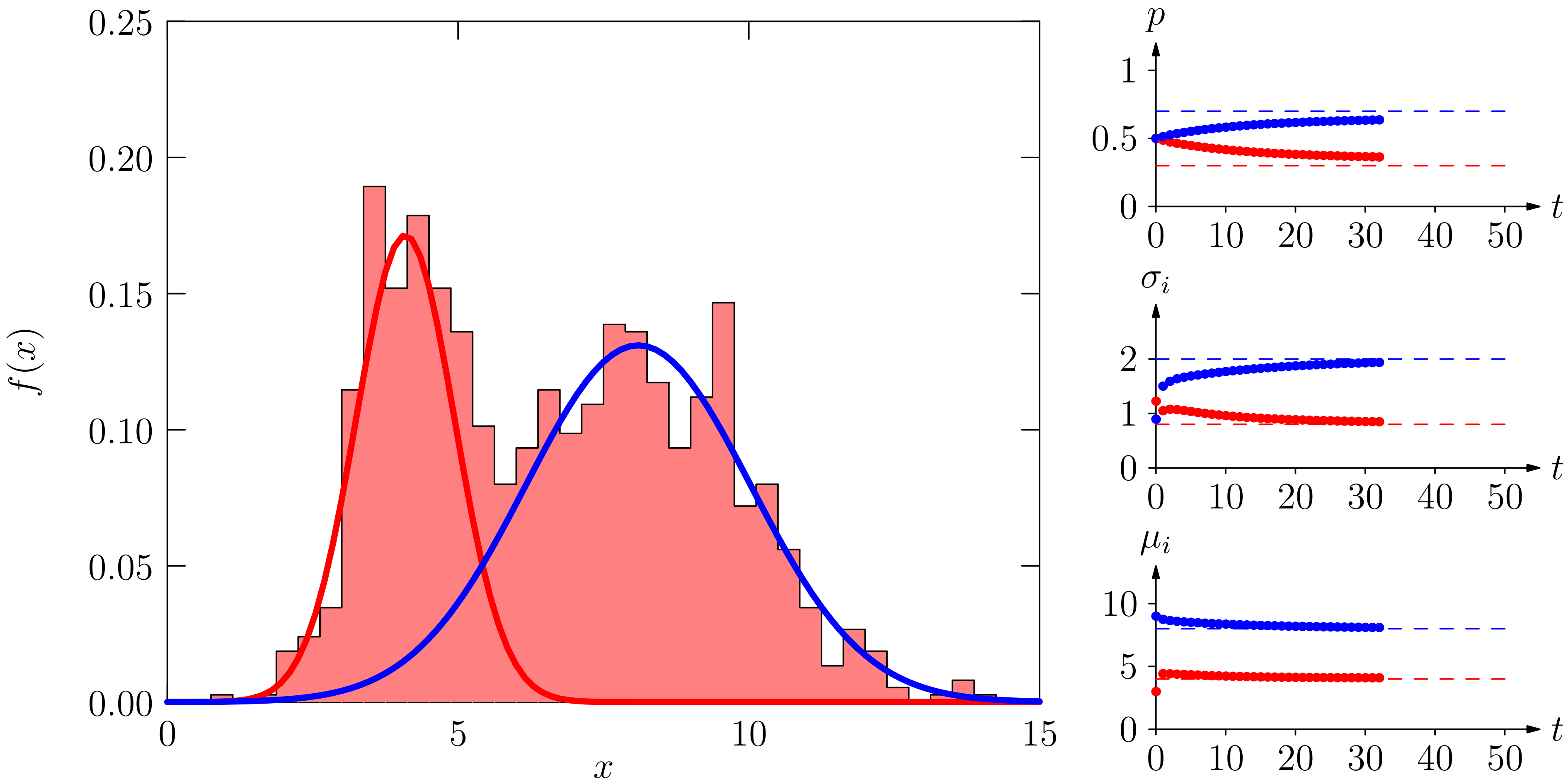

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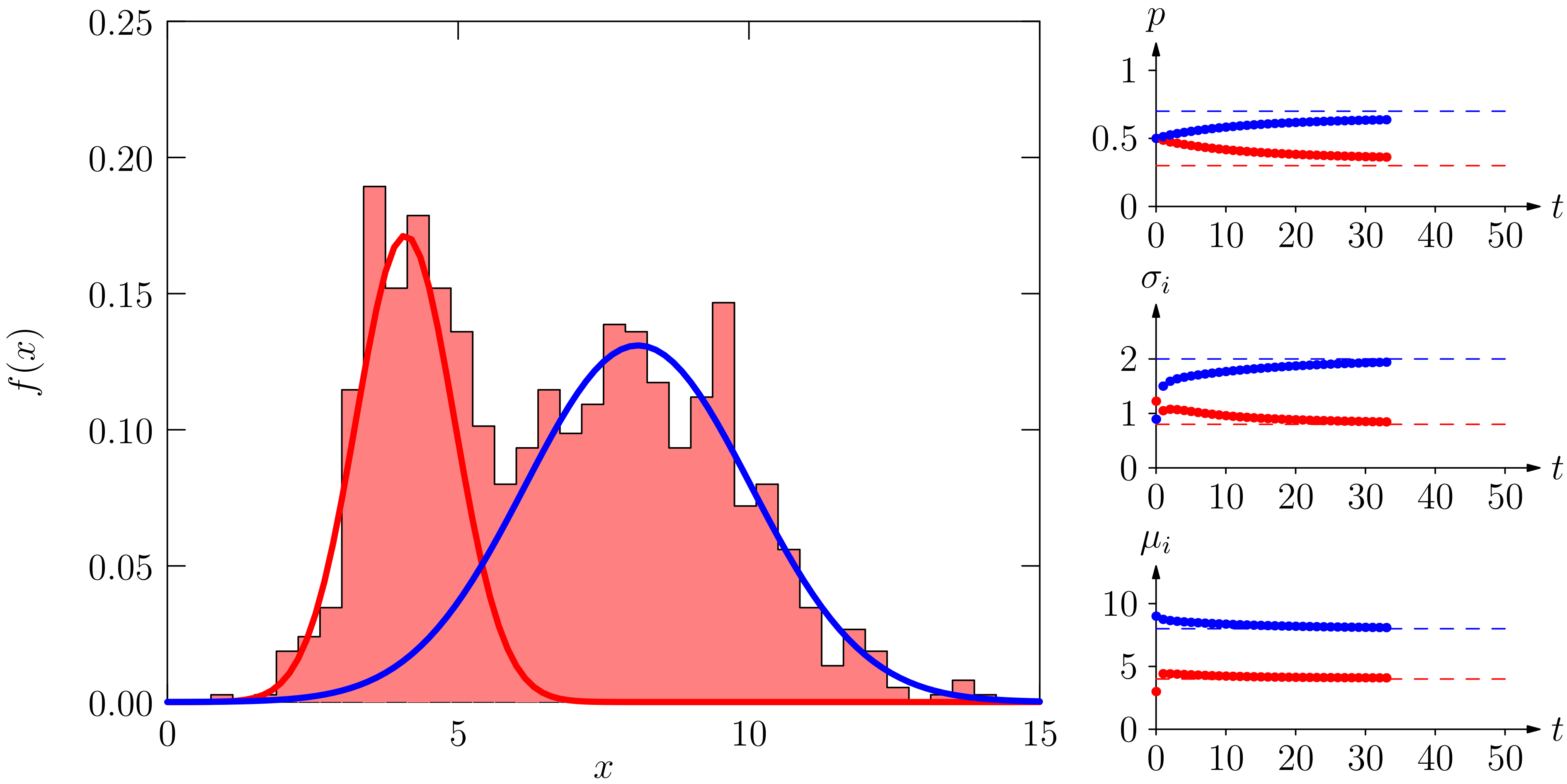

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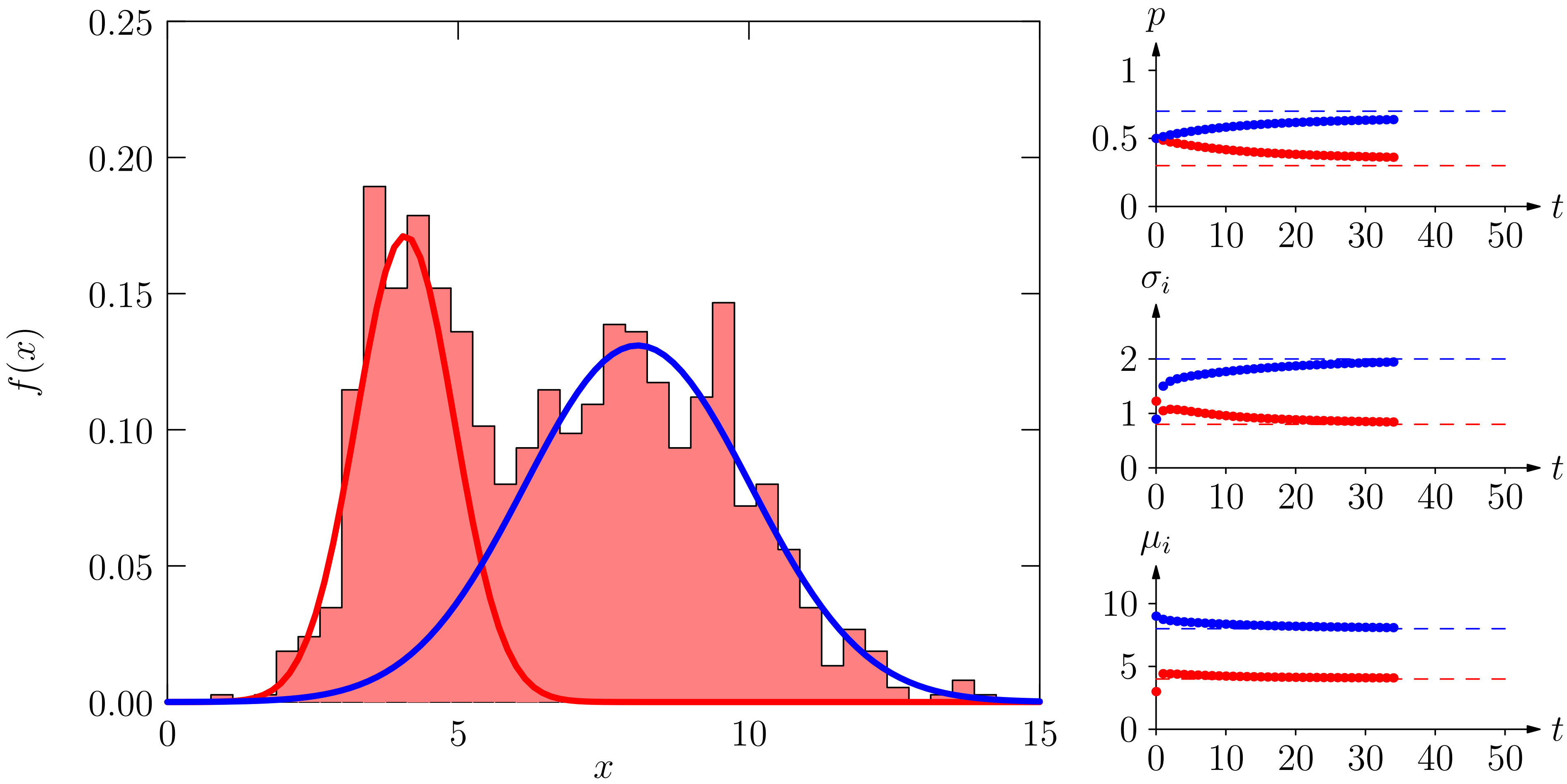
Example



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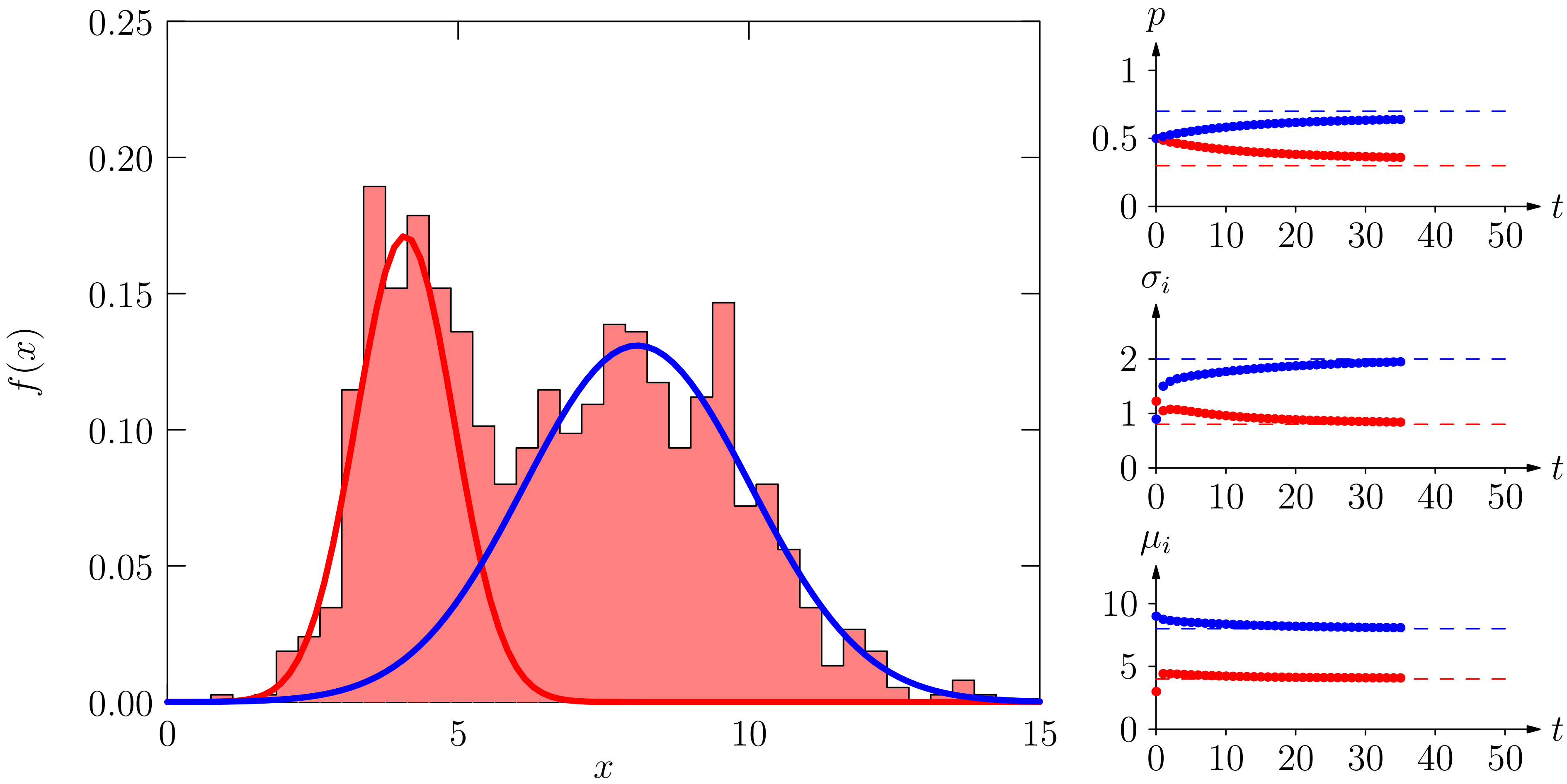

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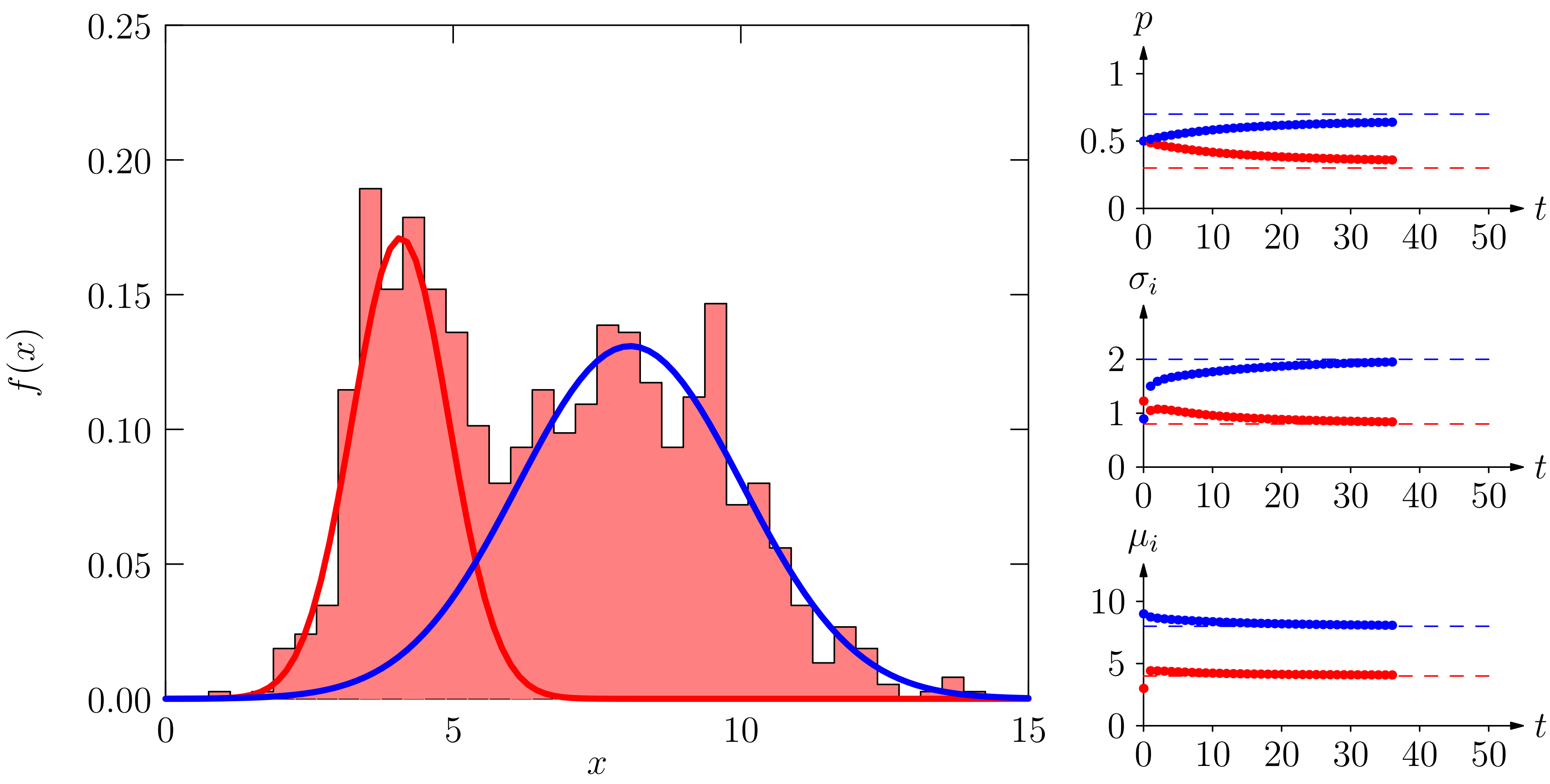

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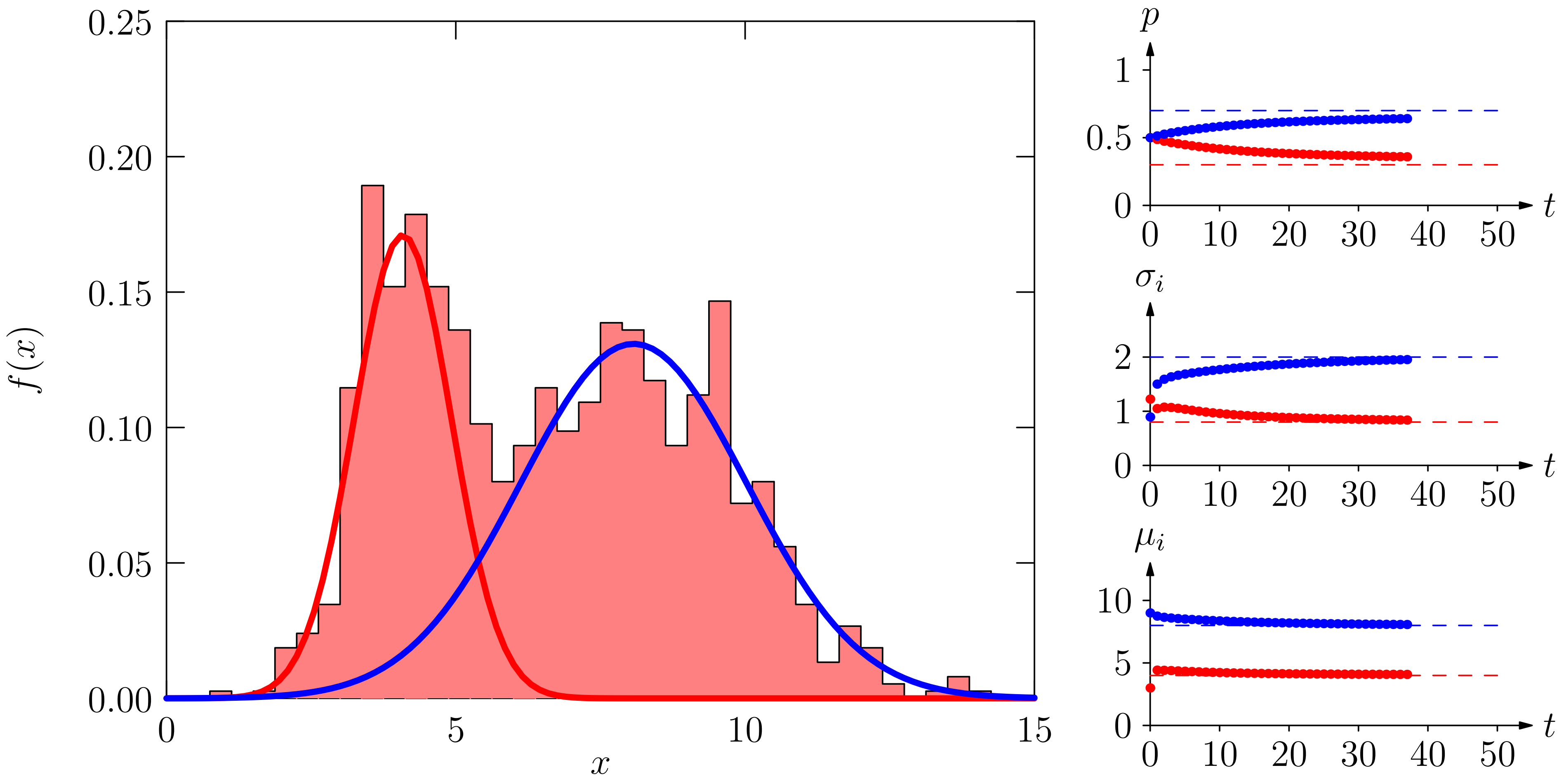

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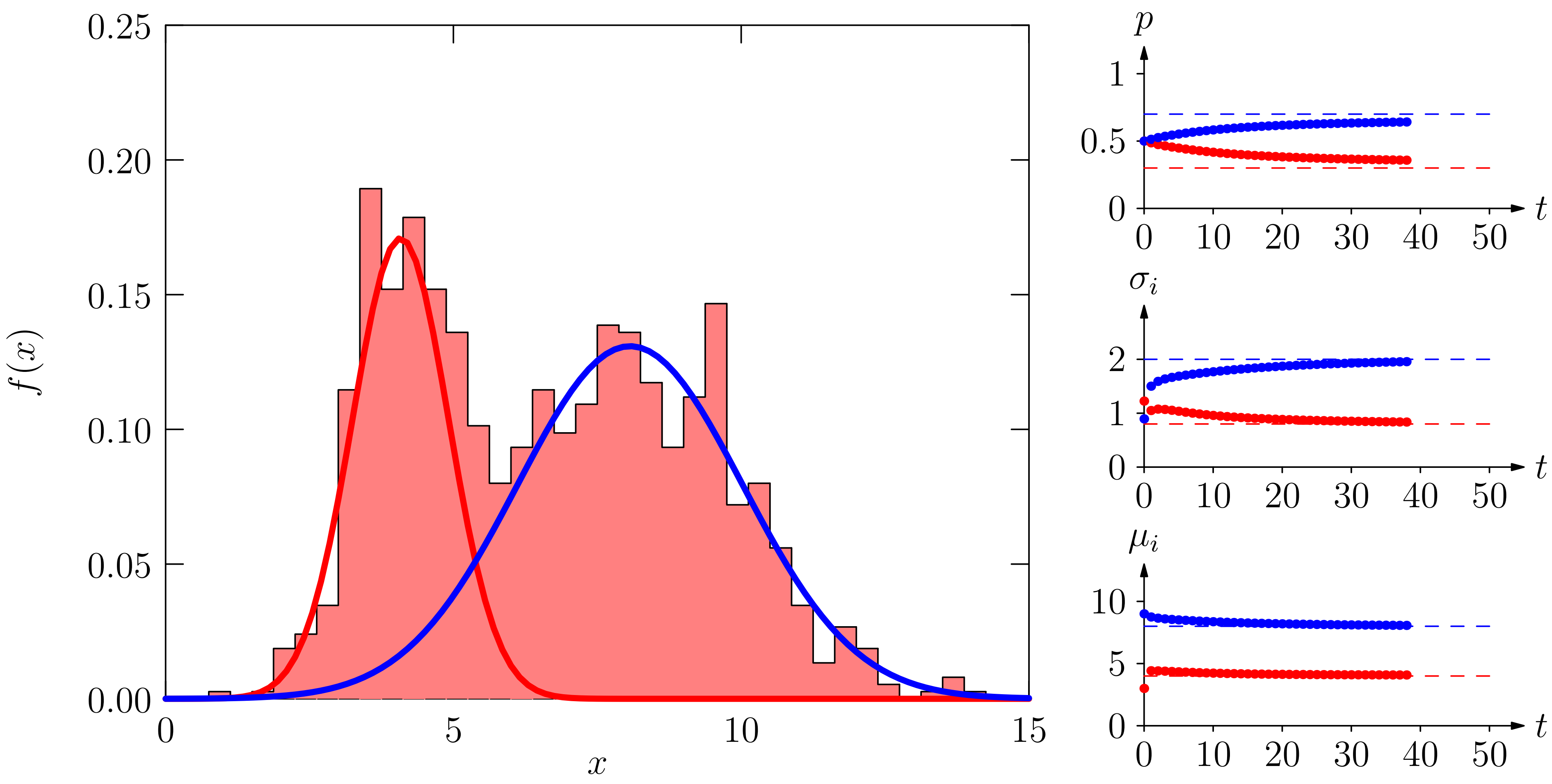

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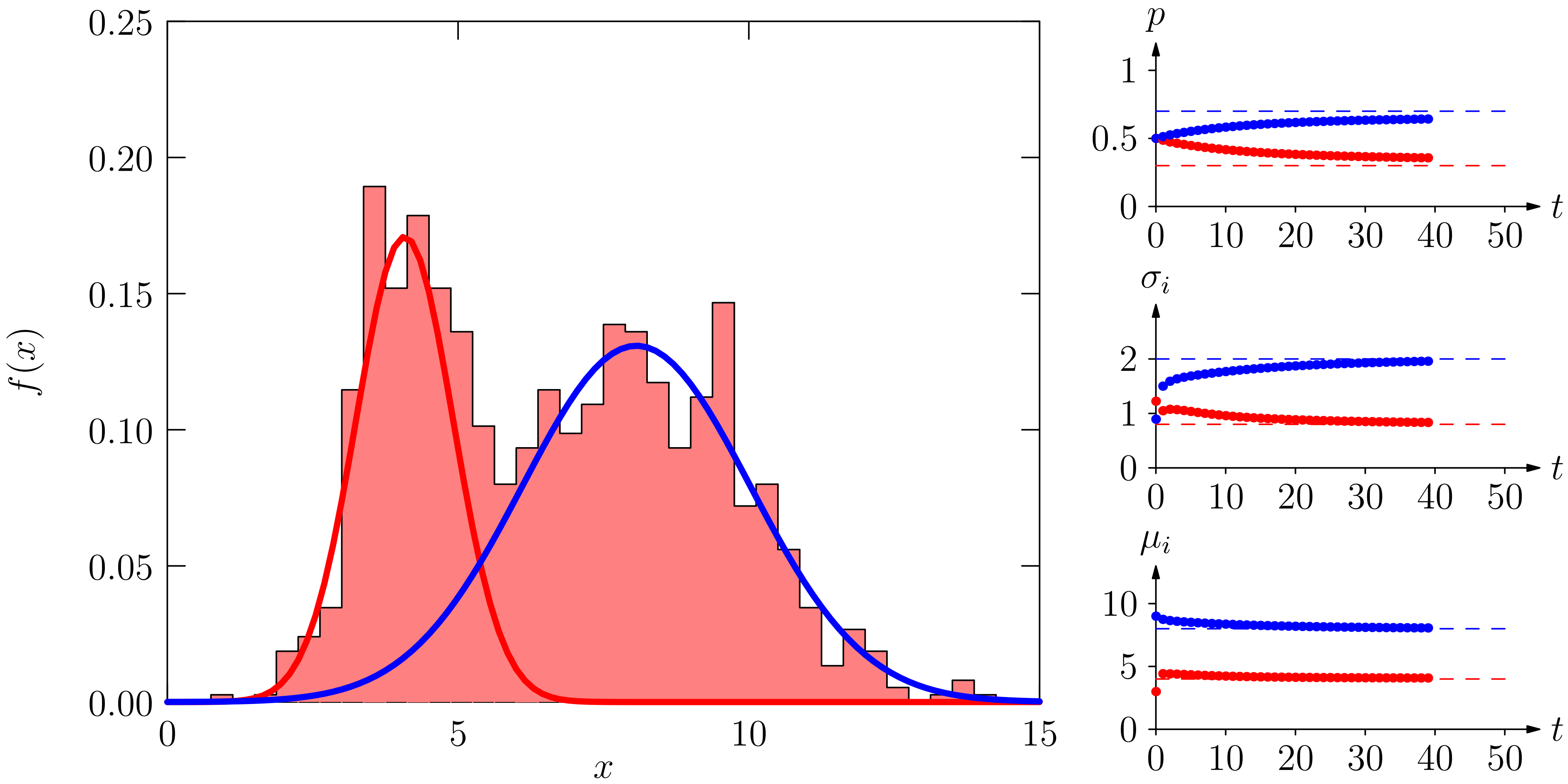
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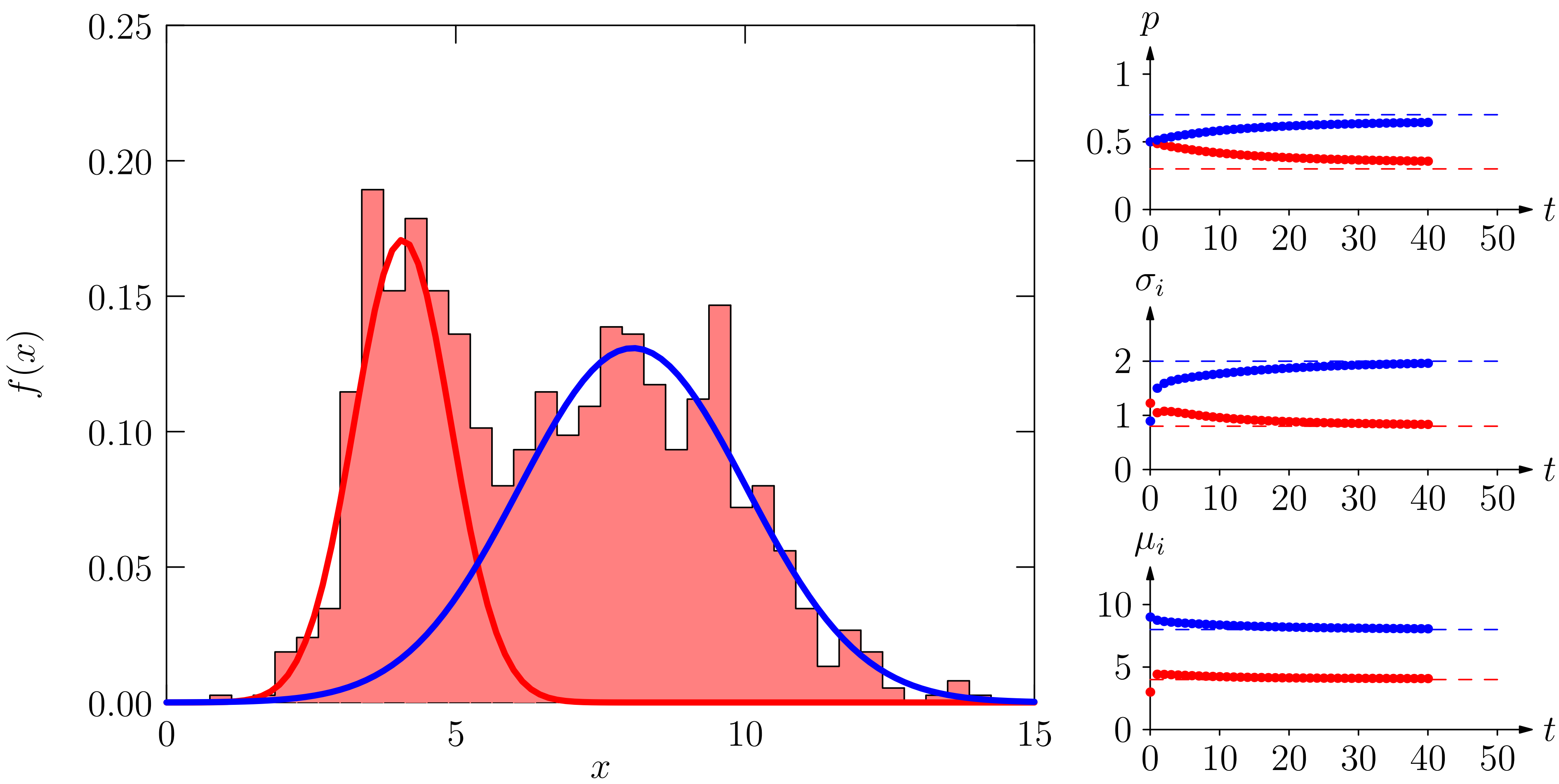

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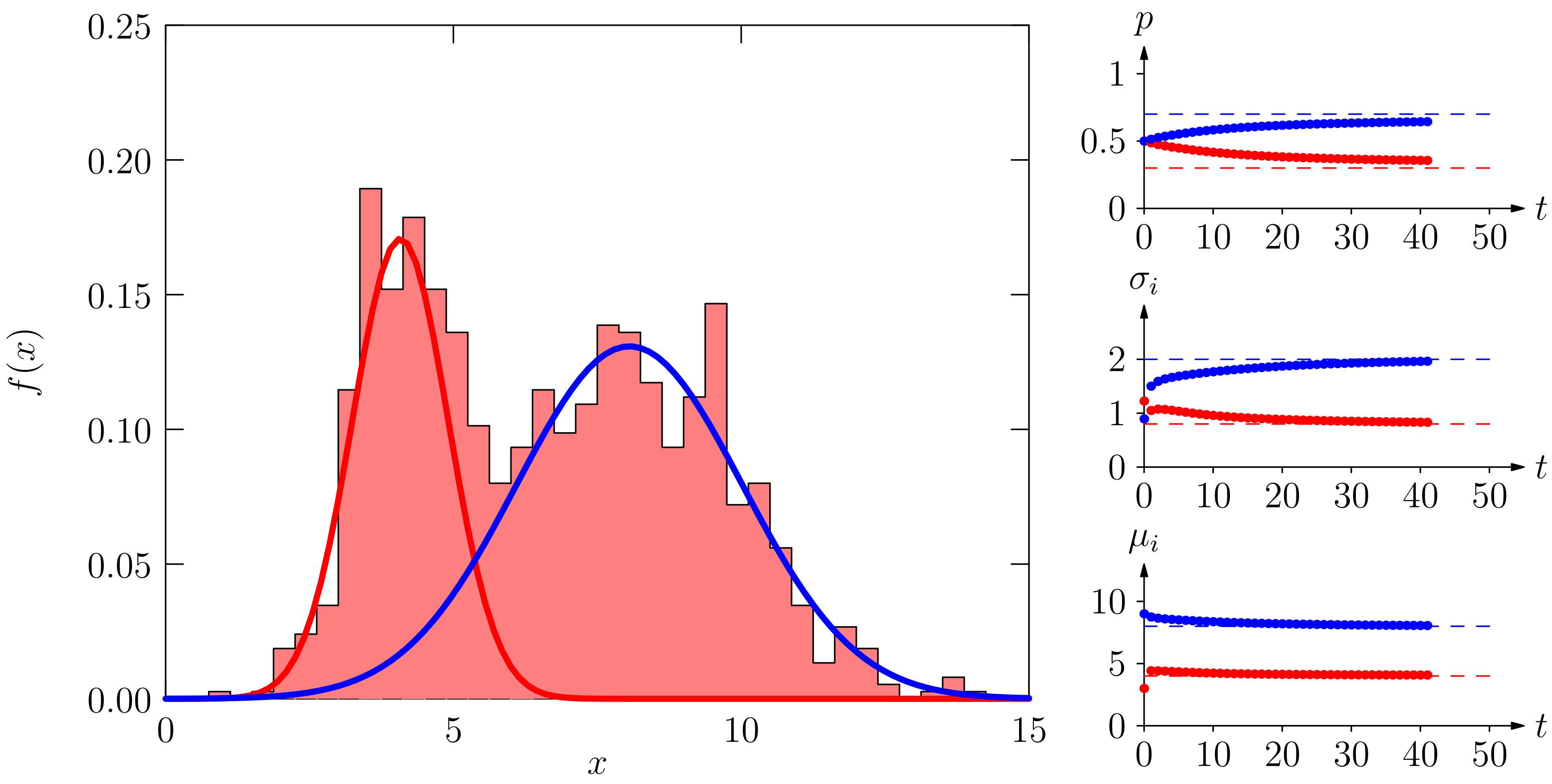

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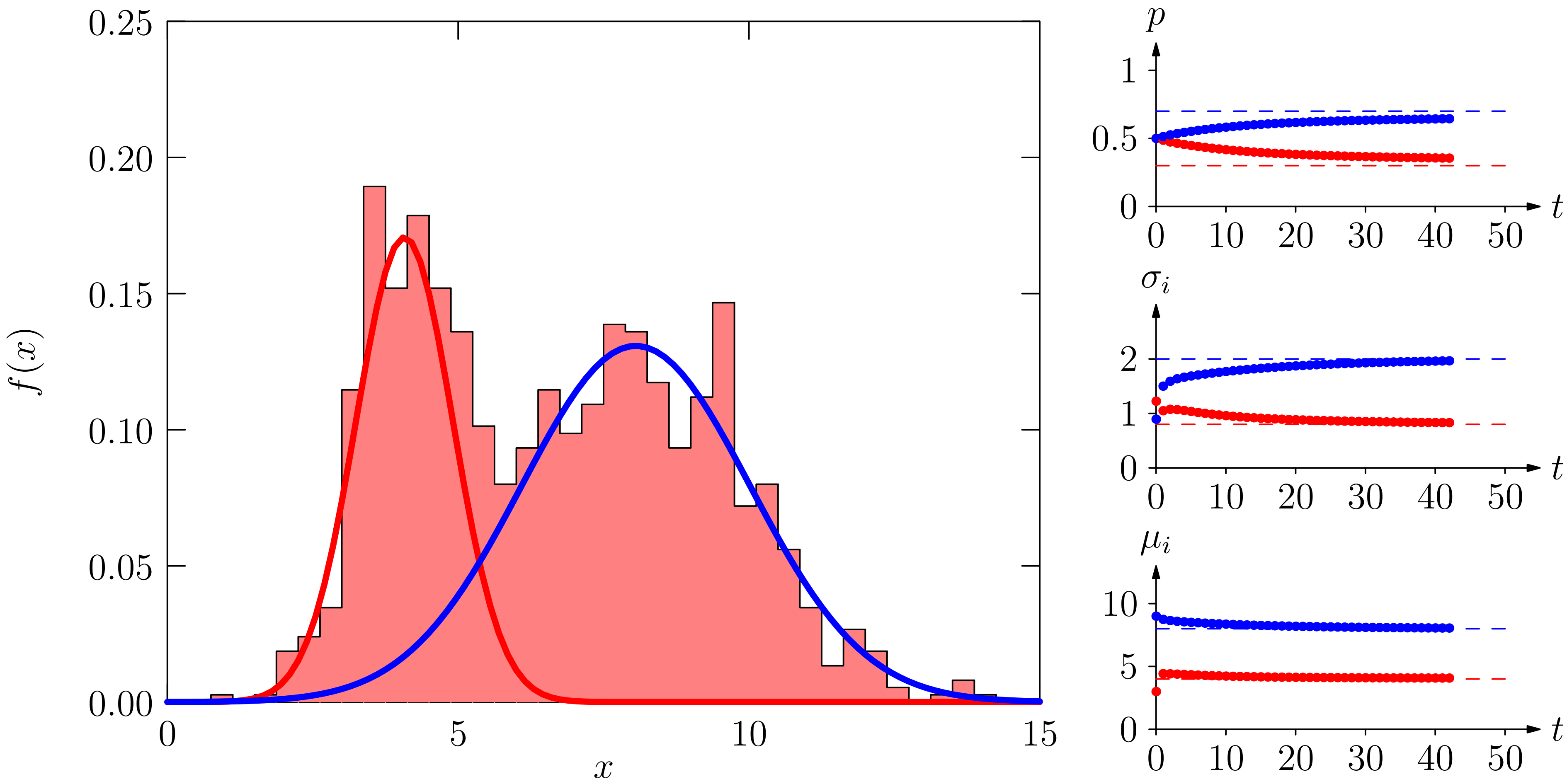

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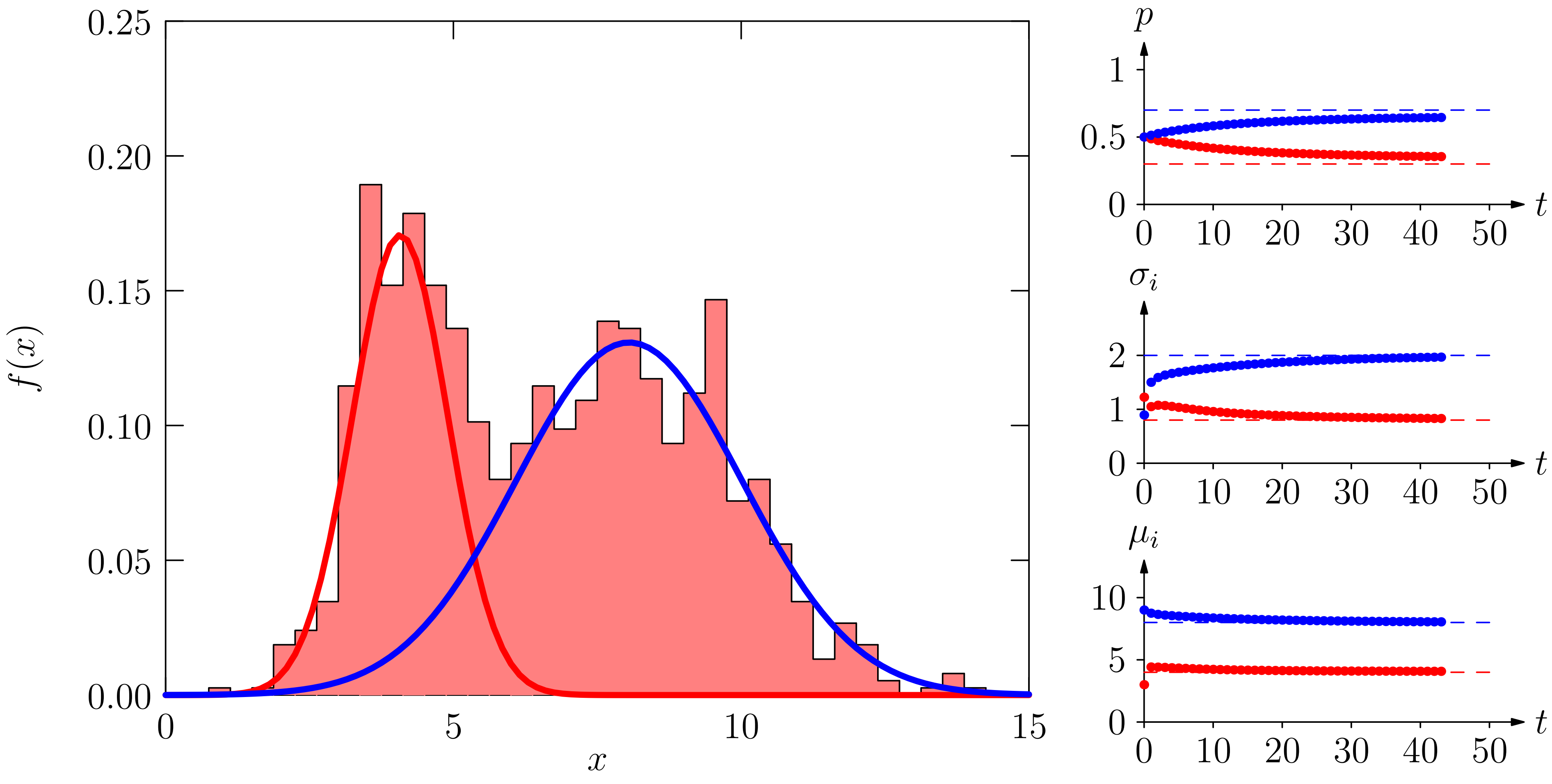

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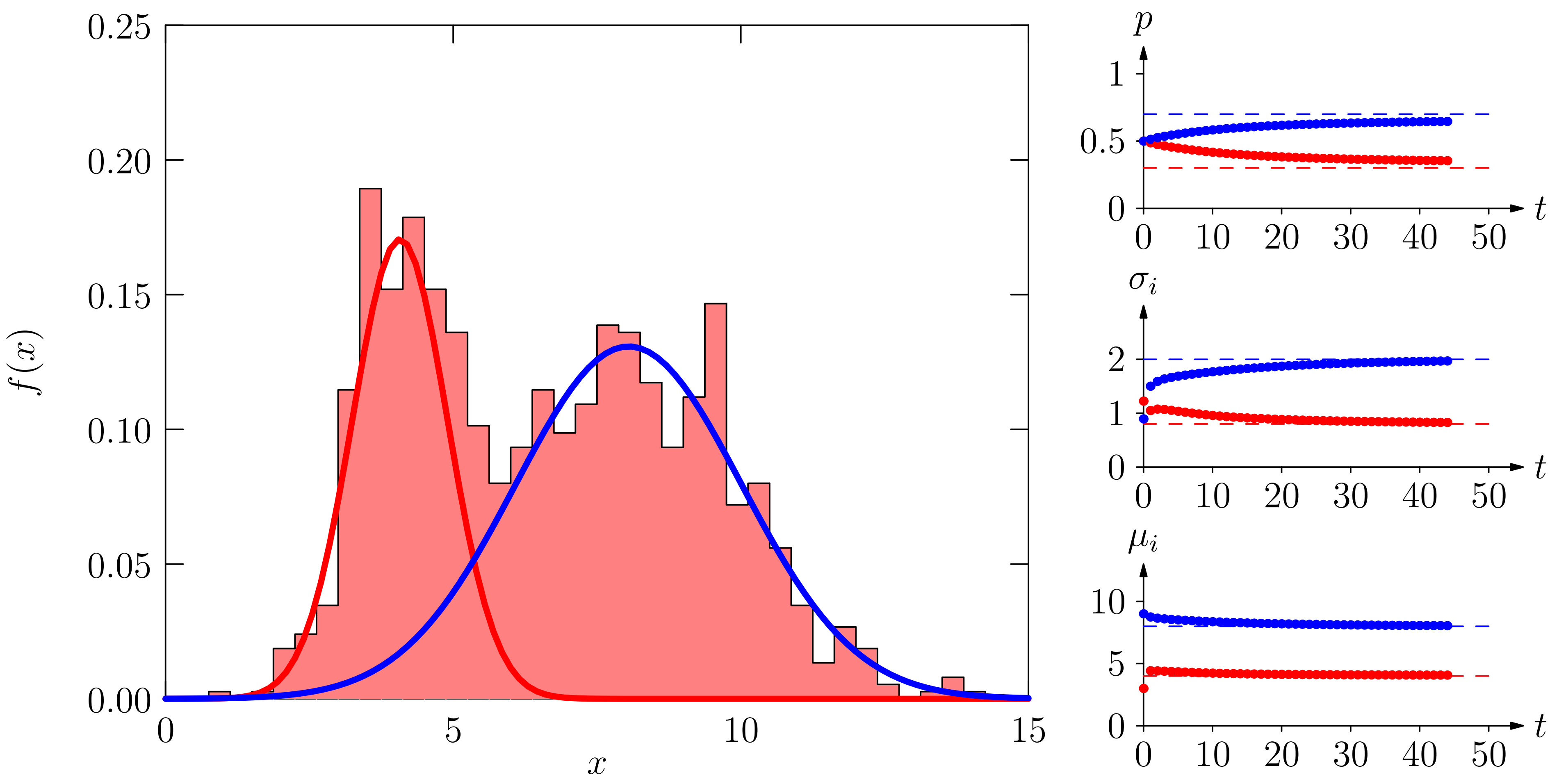
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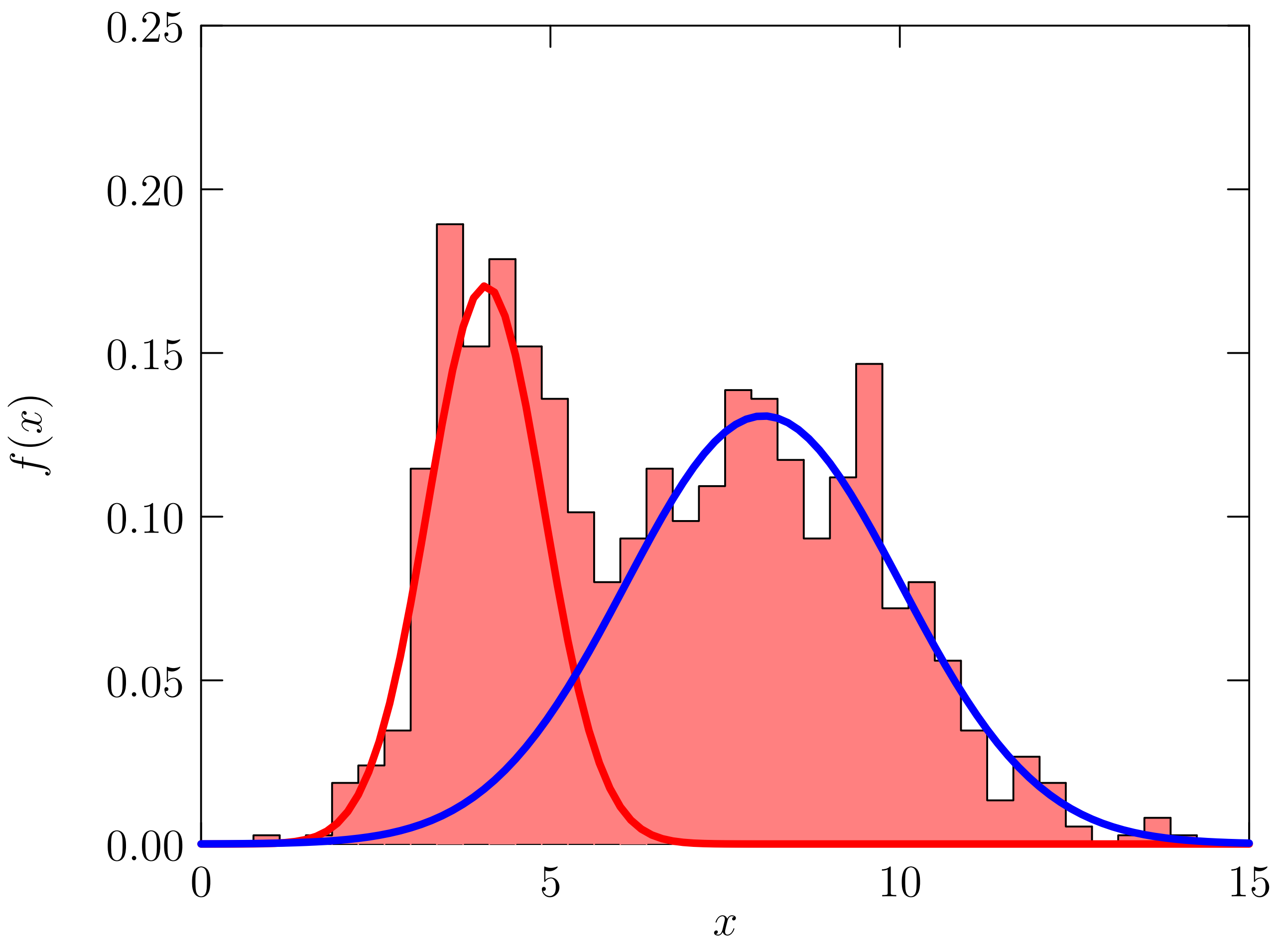

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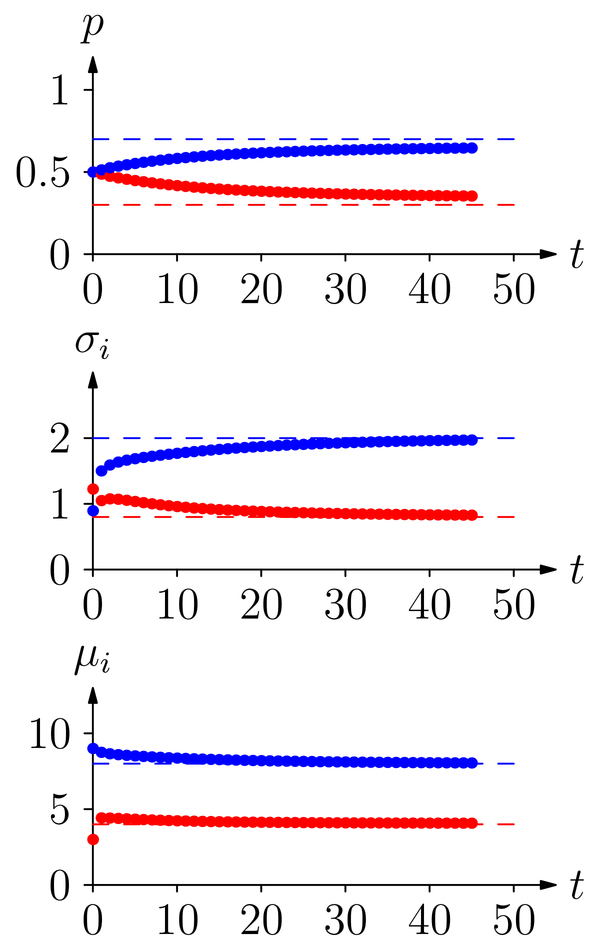

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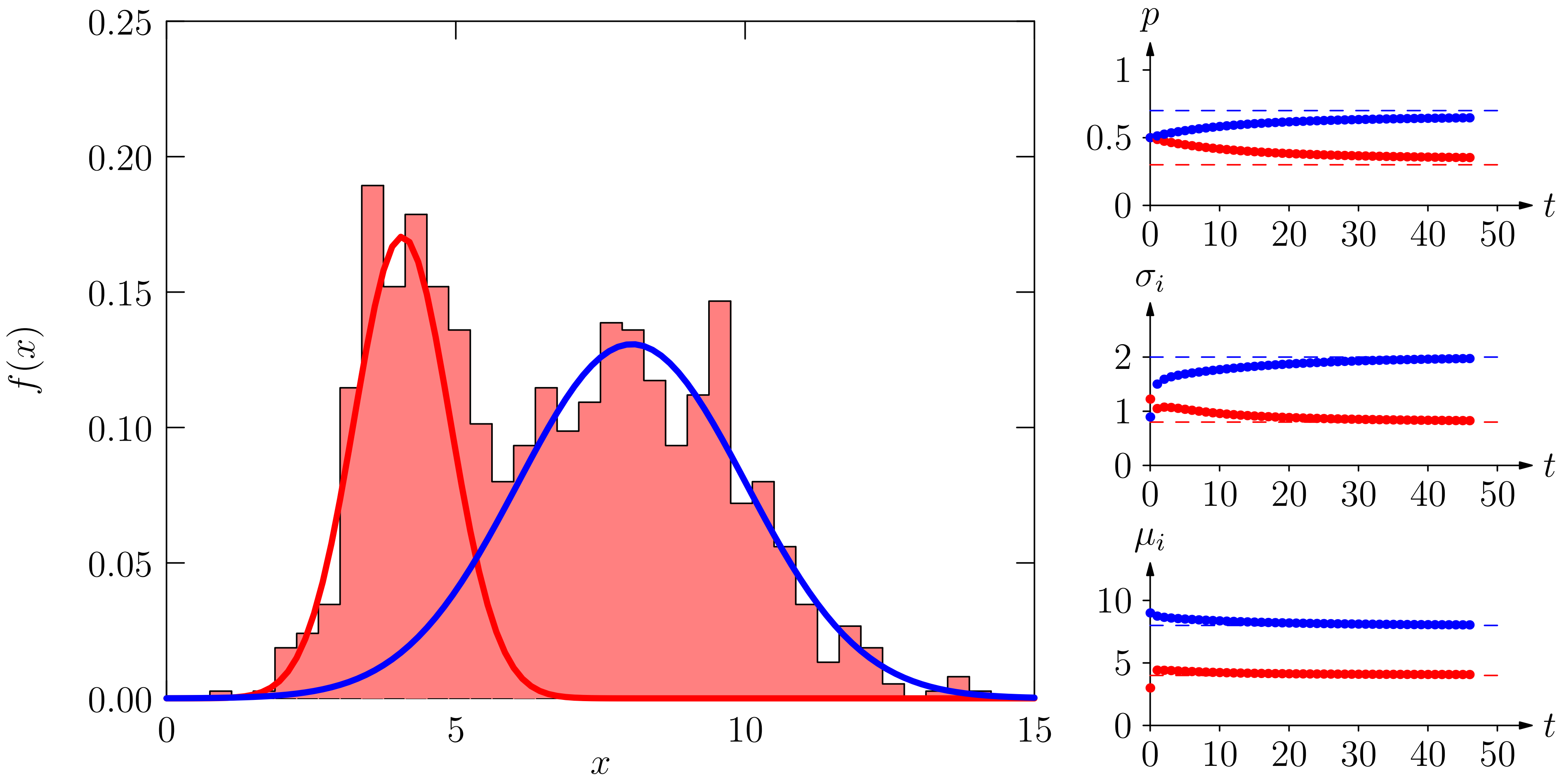

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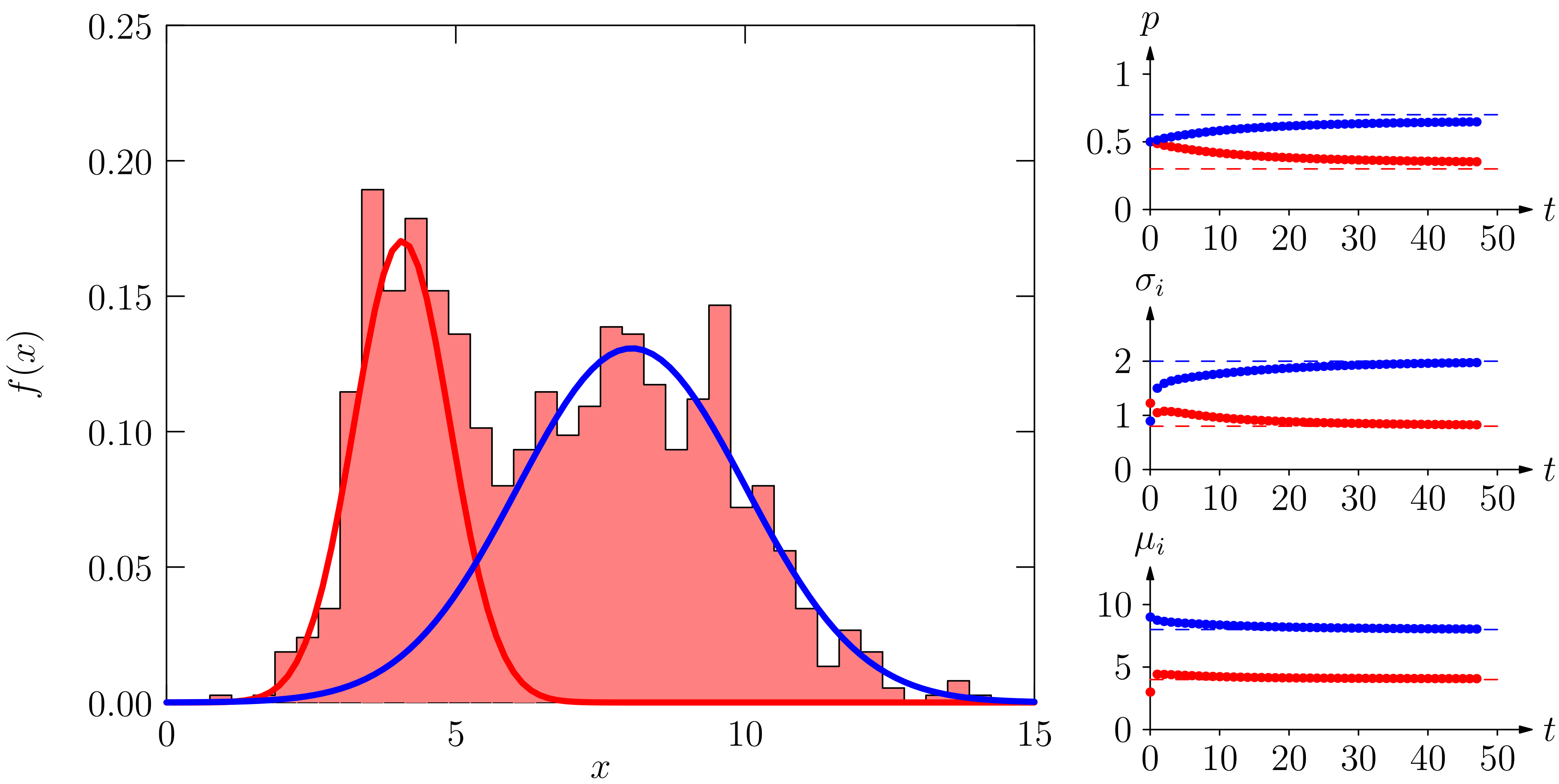
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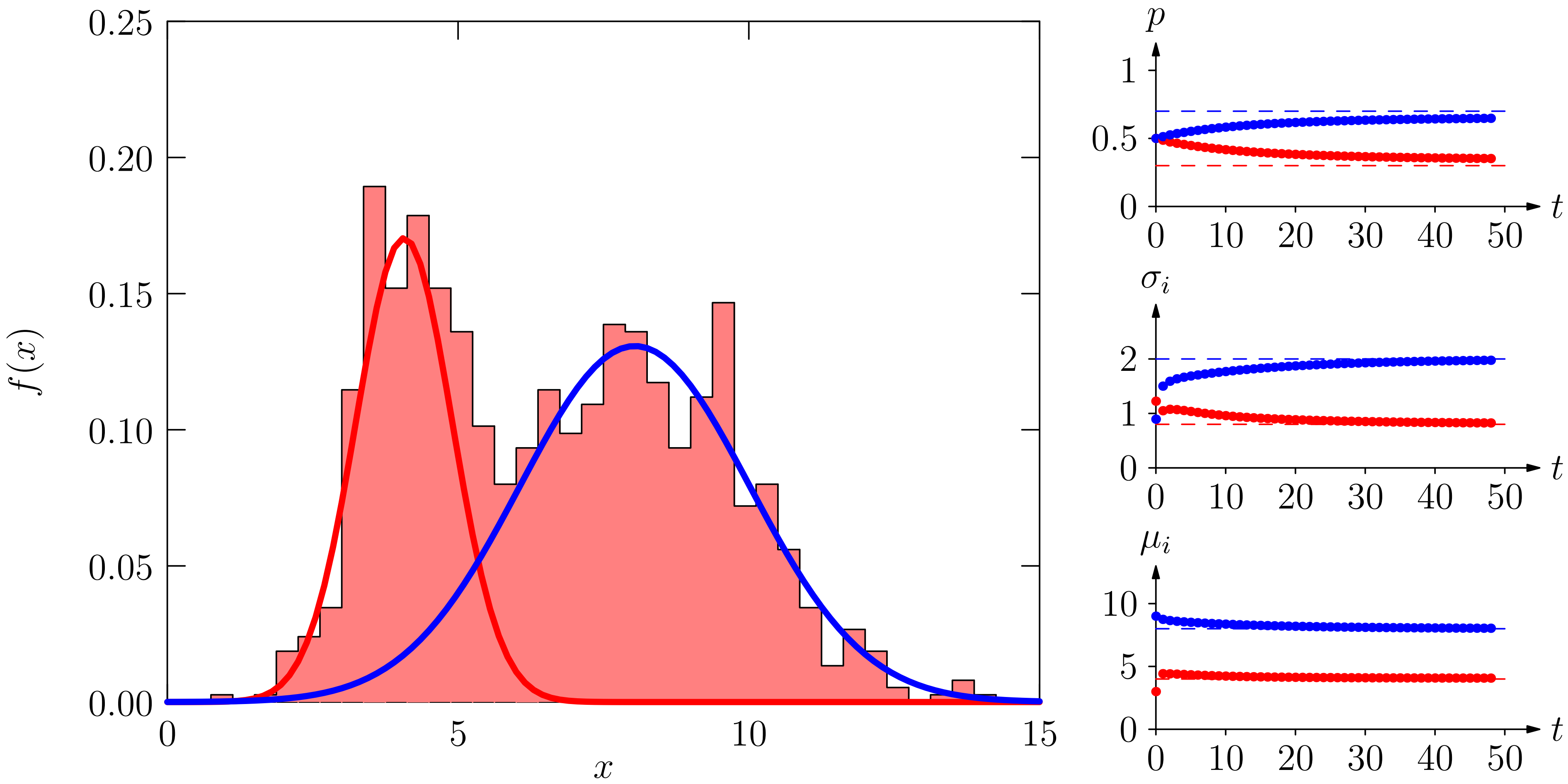

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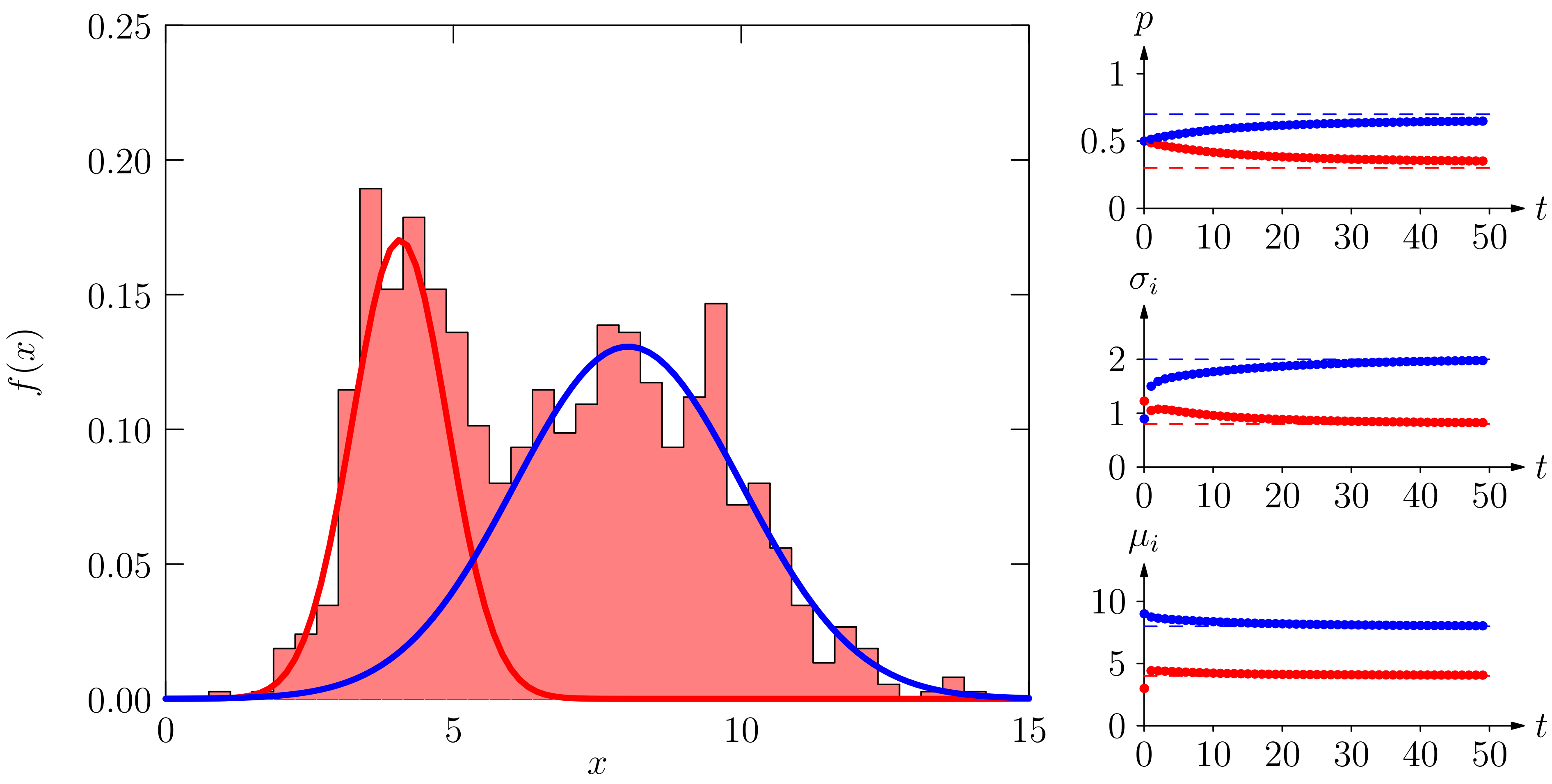
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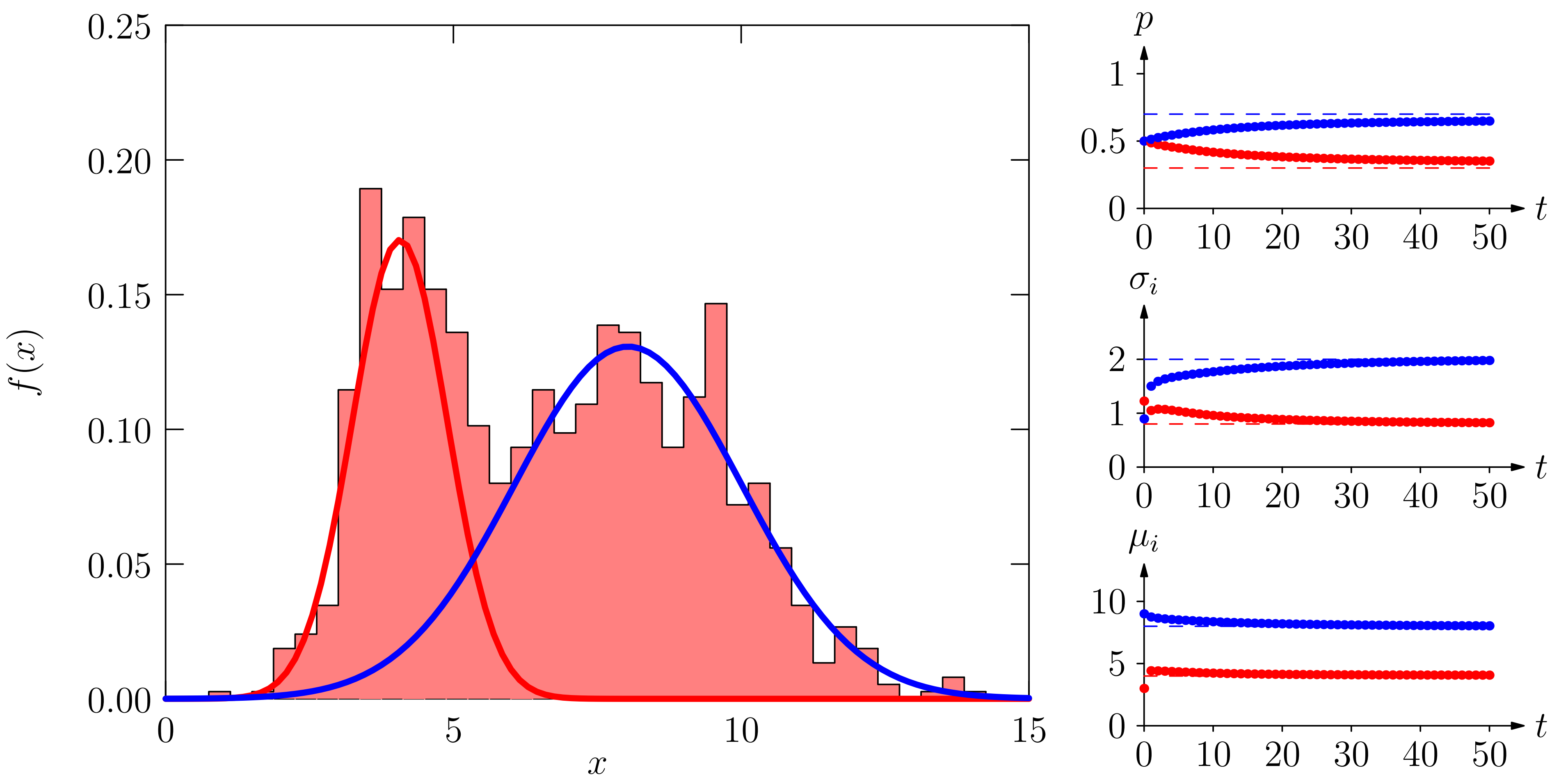

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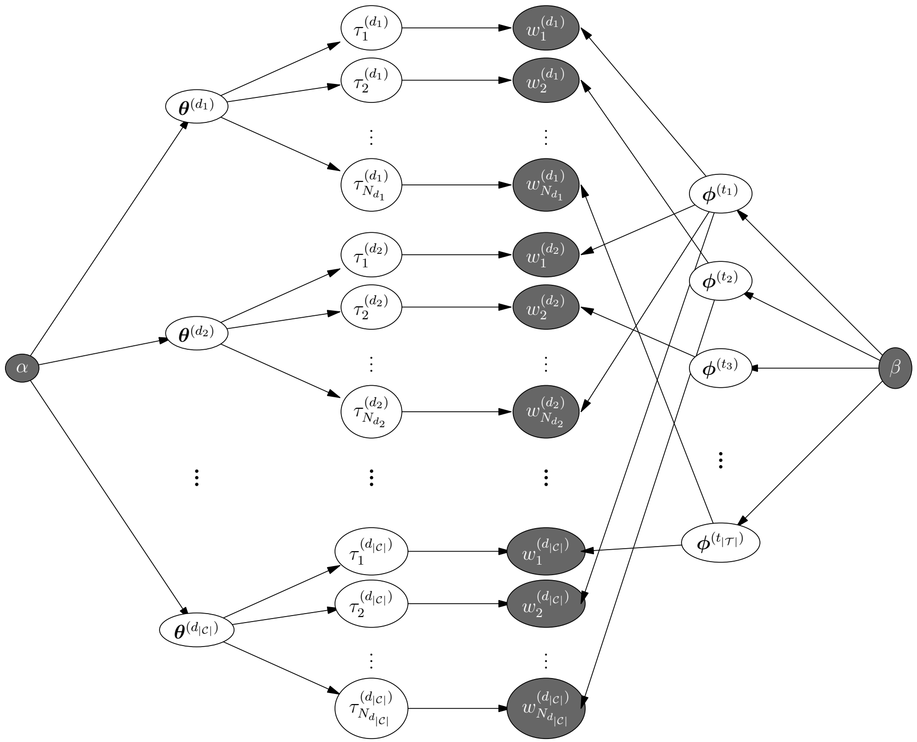
Outline

1. Building Probabilistic Models
2. **Graphical Models**
3. Latent Dirichlet Allocation



Dependencies Between Variables

- In building a probabilistic model we want to know which random variables depend on each other directly and which don't
- Variables that don't will typically still be correlated
- If two random variables X and Y are correlated then
  - ⋆ X could affect Y
  - ⋆ Y could affect X
  - ⋆ X and Y could not influence each other, but both be affected by another random variable Z

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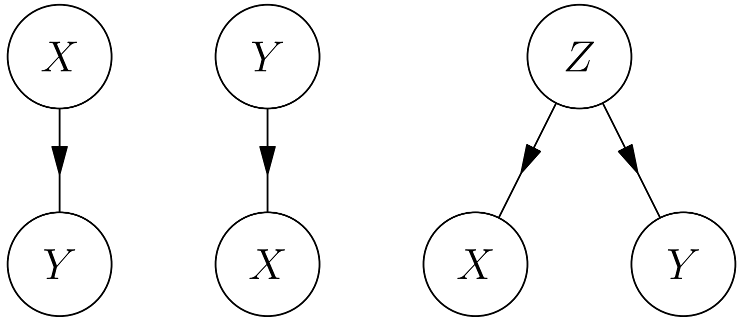
Graphical Models

- Graphical models are directed graphs that show causal relationships between random variables
- We could represent the three conditions described above by



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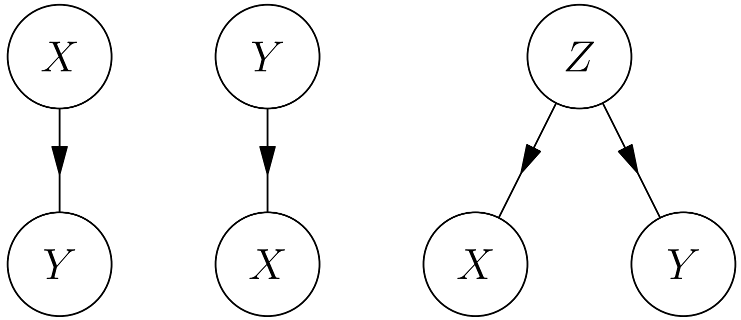

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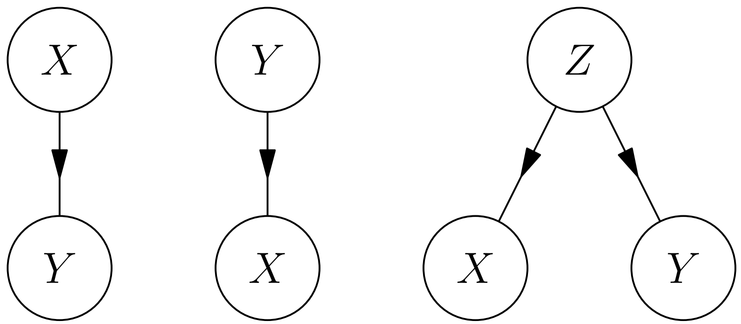

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Statistical Independence

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- Equally this implies $\mathbb{P}(X|Y) = \mathbb{P}(X)$ and $\mathbb{P}(Y|X) = \mathbb{P}(Y)$
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- A weaker notion is conditional independence

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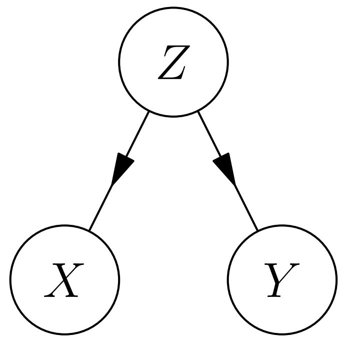

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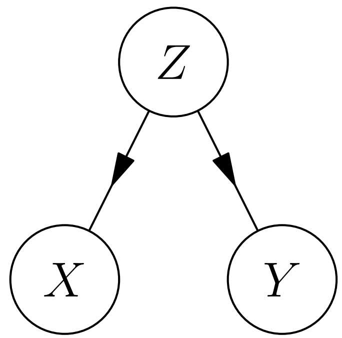

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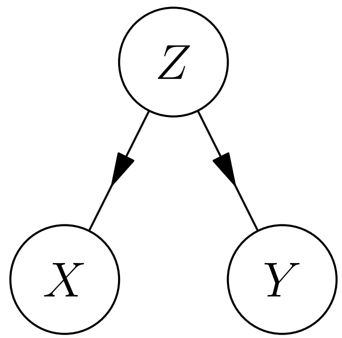

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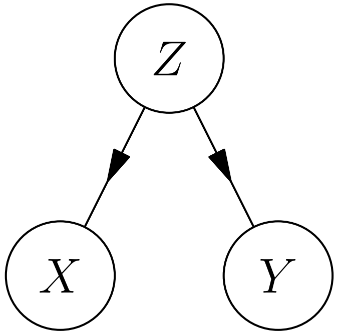

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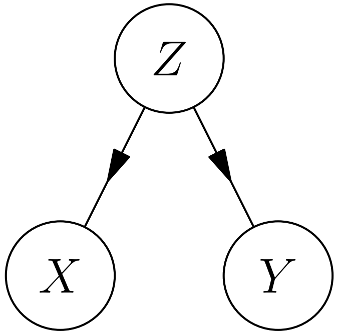

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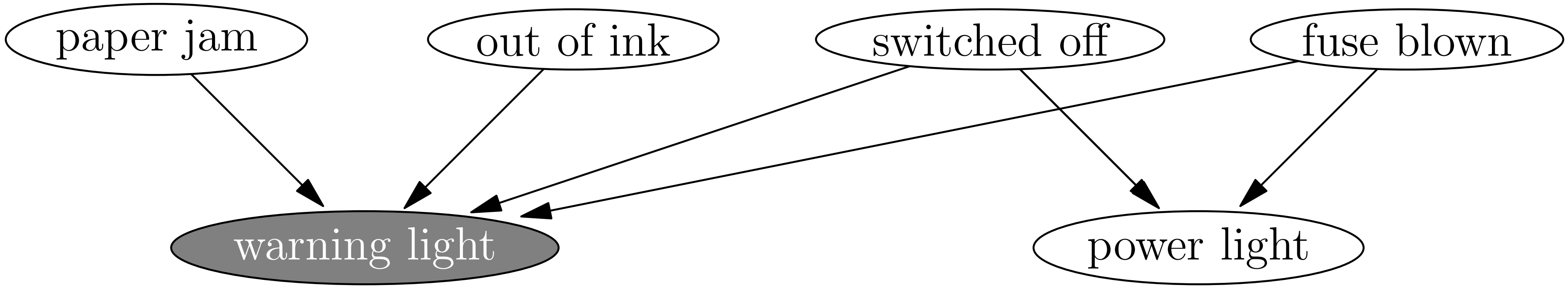
Graphical Models

- Graphical models often provide a quick way to represent the world



- In graphical models we shade nodes that we observe

- Note that the top events are conditionally independent if we make no observation, but are dependent if we observe a warning light!

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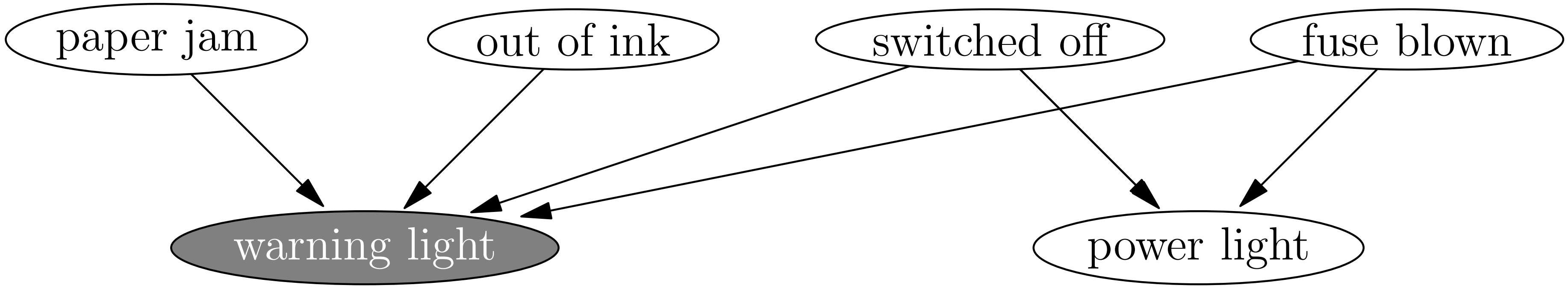

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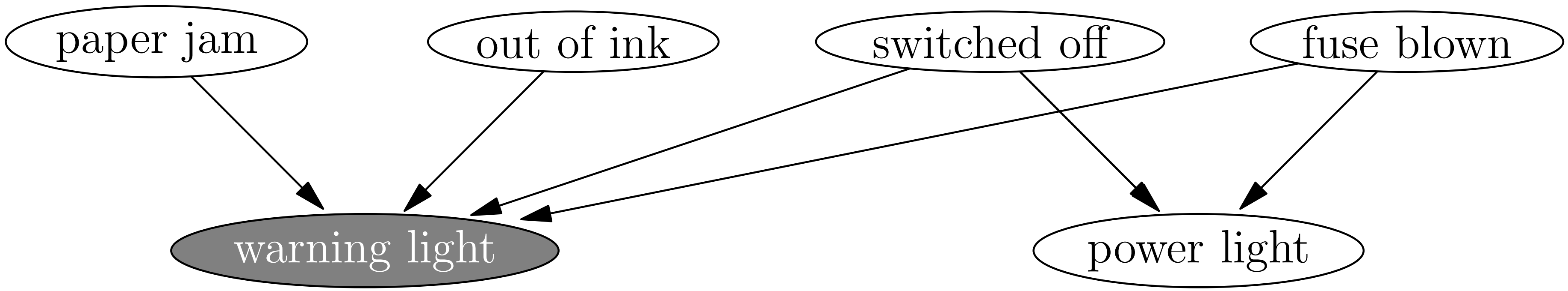

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Outline

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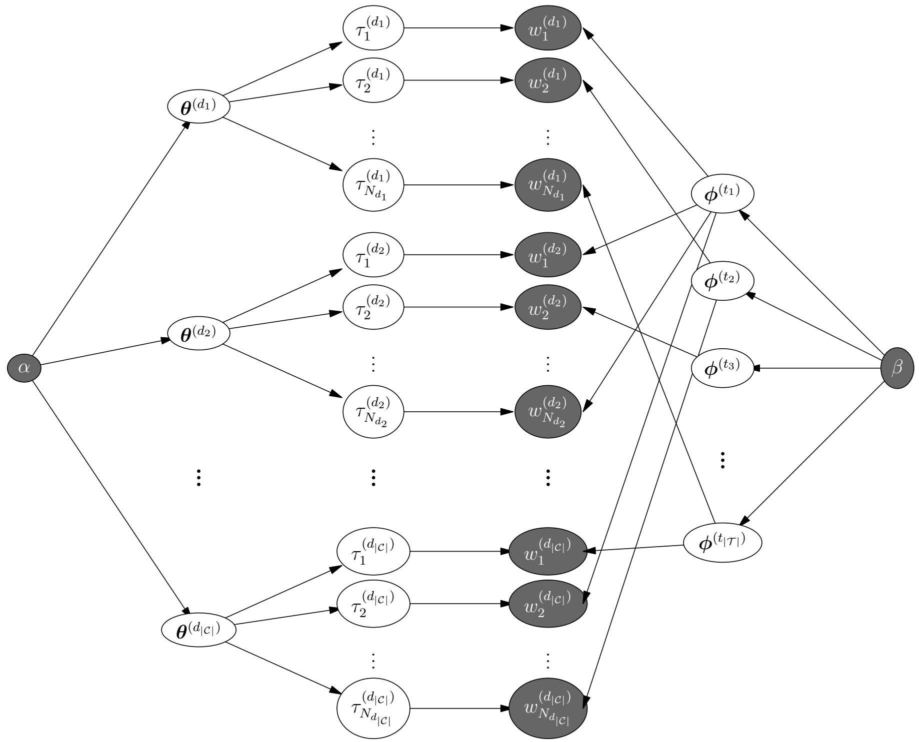
2. Graphical Models
3. **Latent Dirichlet Allocation**



Model for Documents

- We consider a model for the words in a set of documents (we ignore word order)
- We consider a corpus $\mathcal{C} = \{d_i | i = 1, 2, \ldots |\mathcal{C}|\}$
- With documents consisting of words

$$d = \left(w_1^{(d)}, w_2^{(d)}, \ldots, w_{N_d}^{(d)}\right)$$

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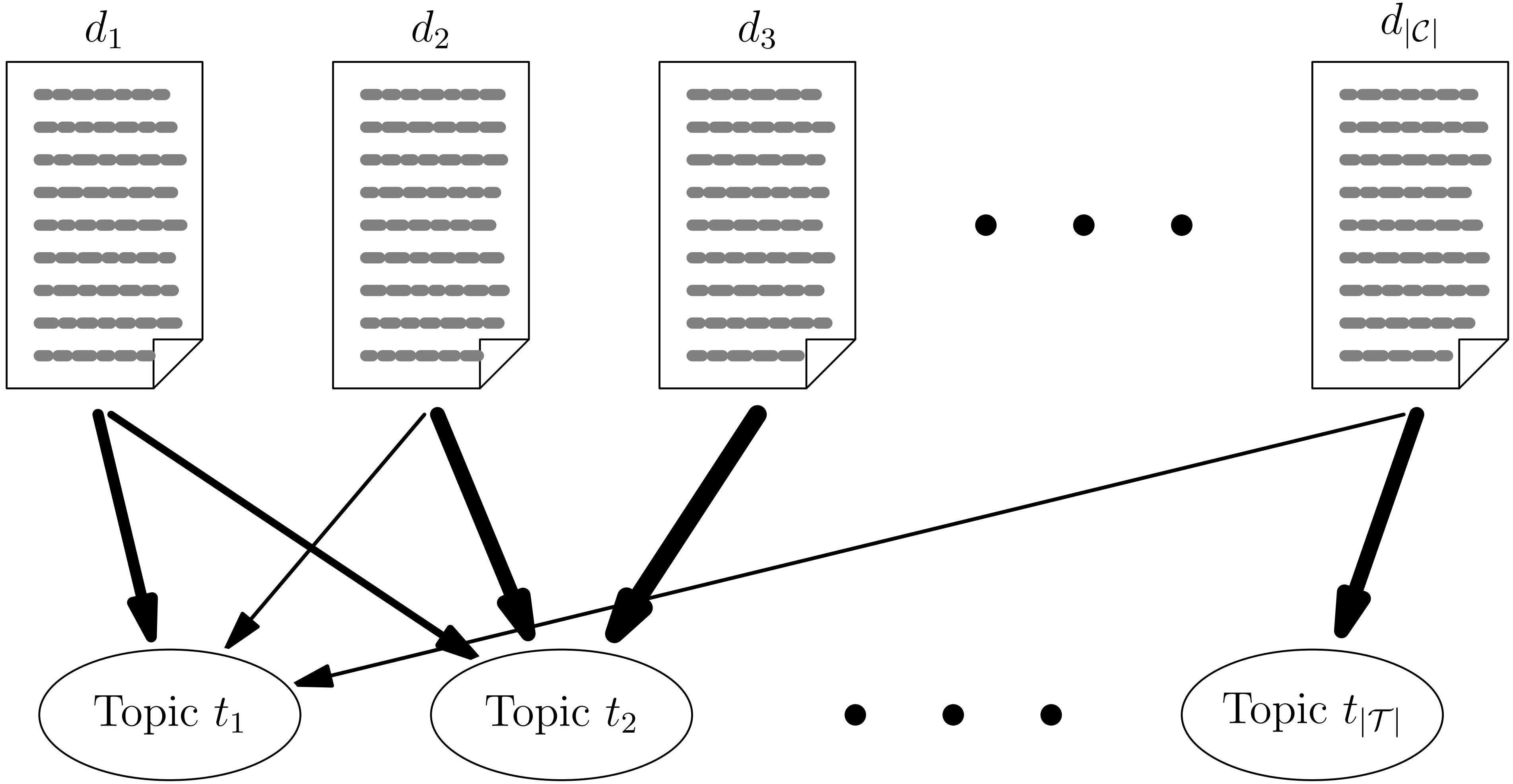
Documents and Topic



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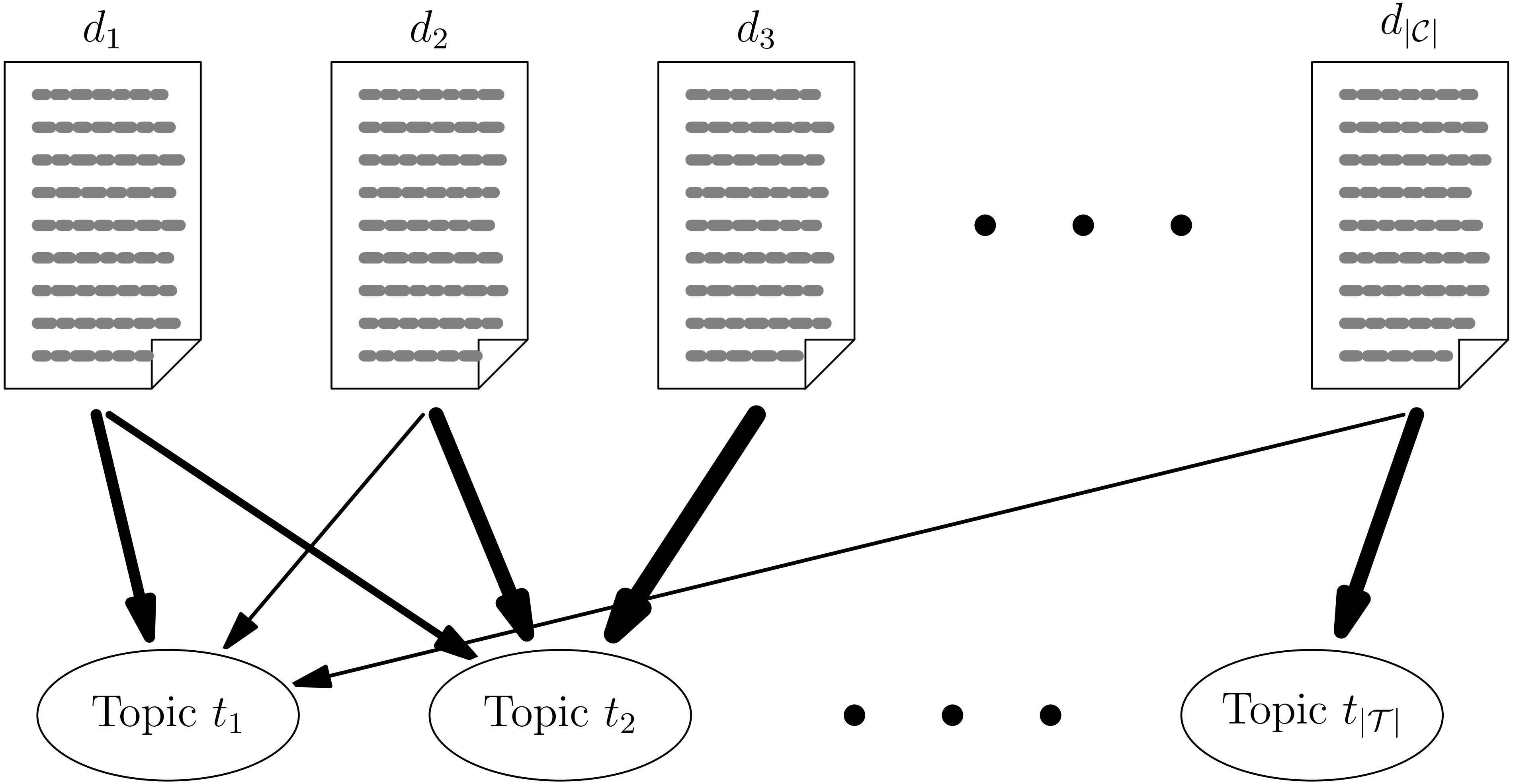

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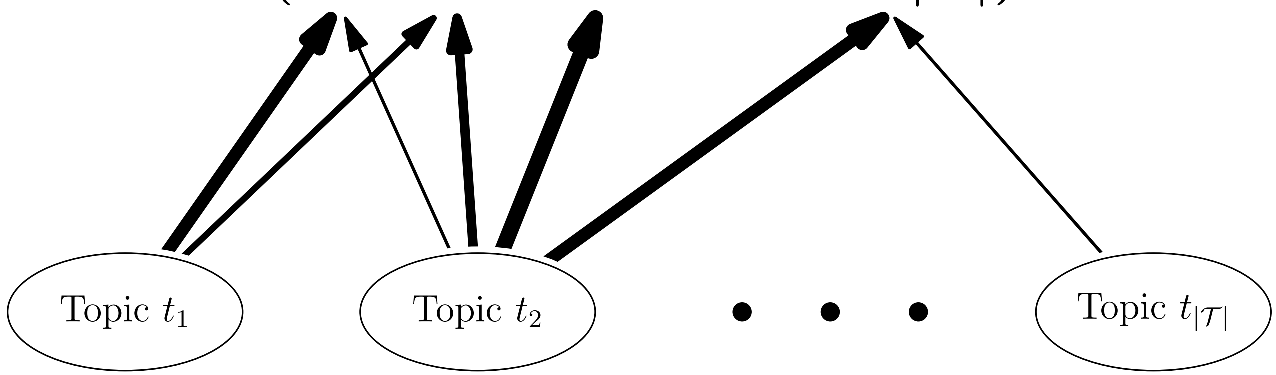

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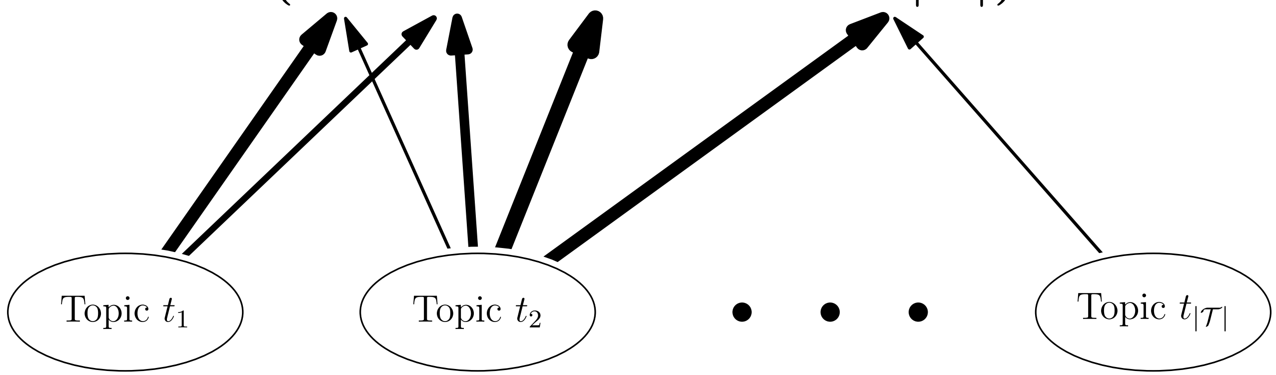

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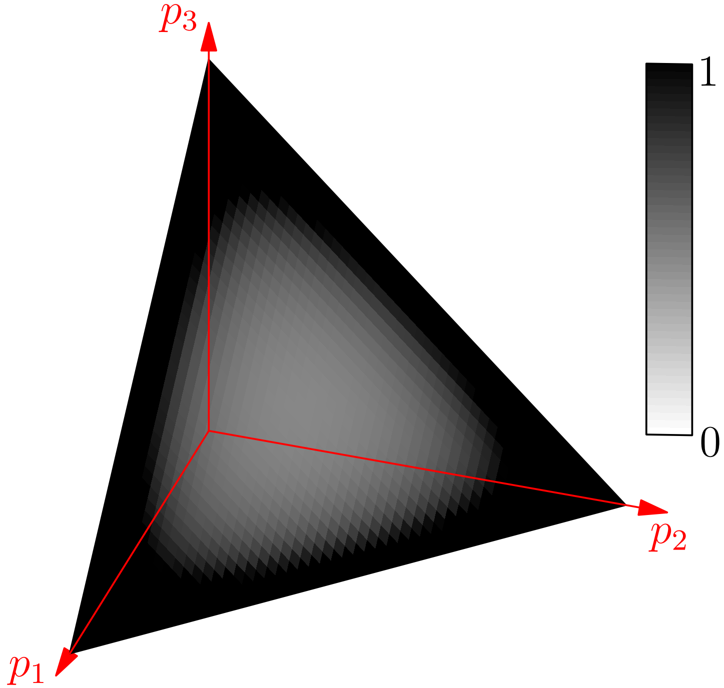
Dirichlet Allocation

- Most documents are predominantly about a few topics and most topic have a small number of words associated to them

- We can generate sparse vectors $\boldsymbol{\theta}^{(d)}$ and $\boldsymbol{\phi}^{(t)}$ from a Dirichlet distribution with small parameters $\boldsymbol{\alpha}$



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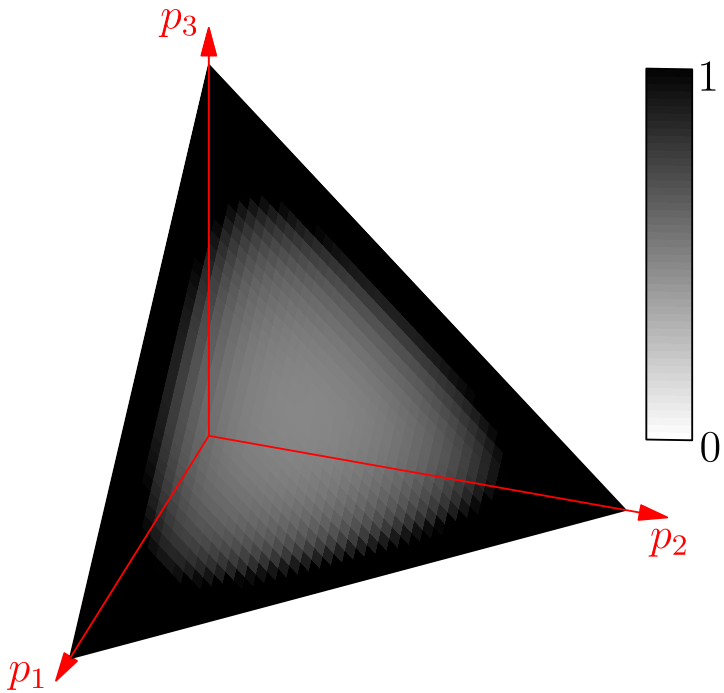

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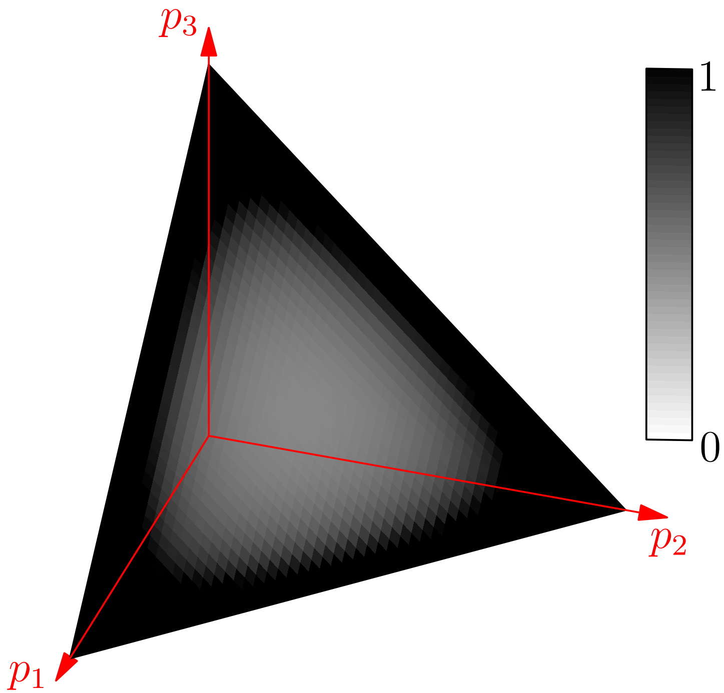

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LDA Graphical Model (version 1)

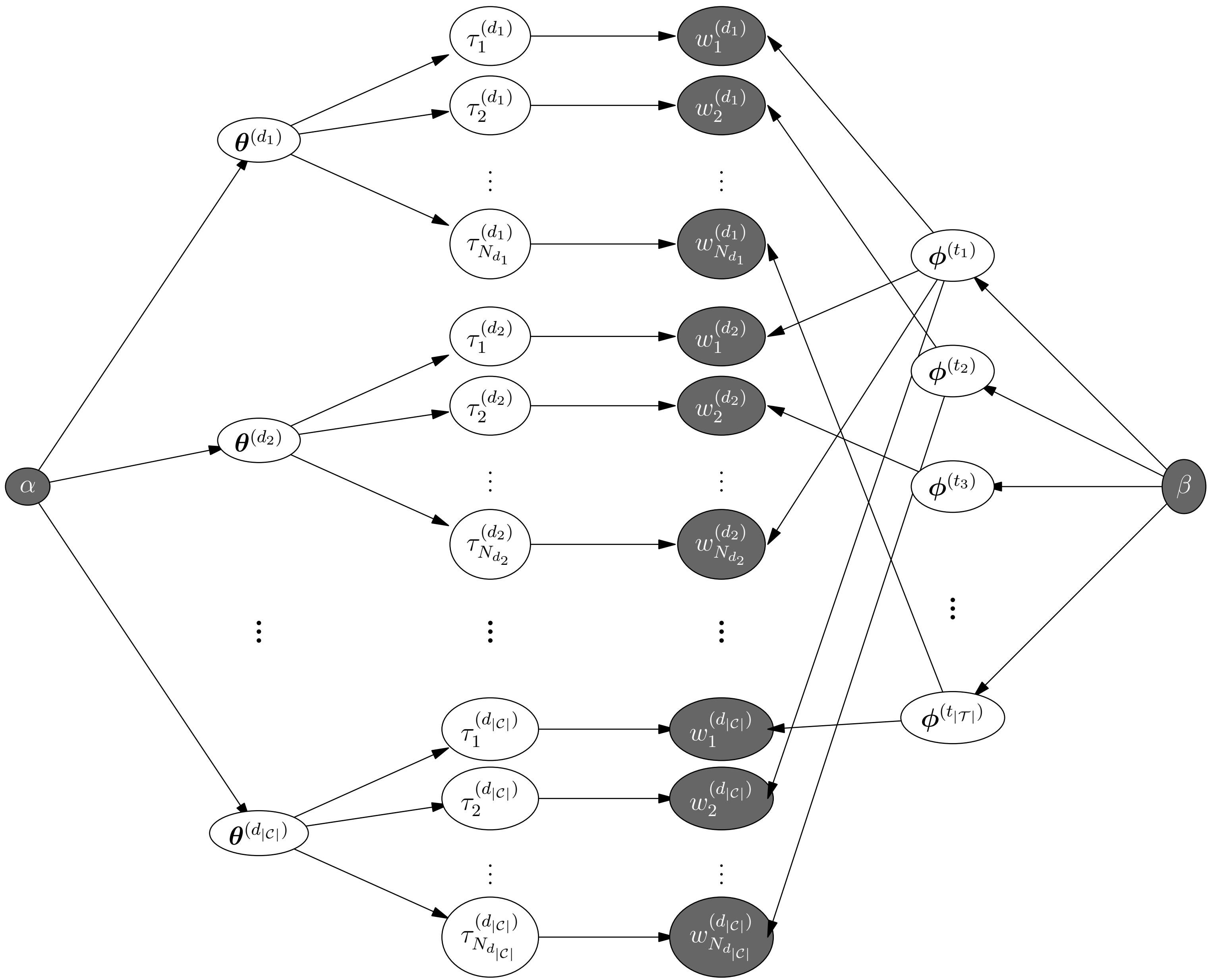
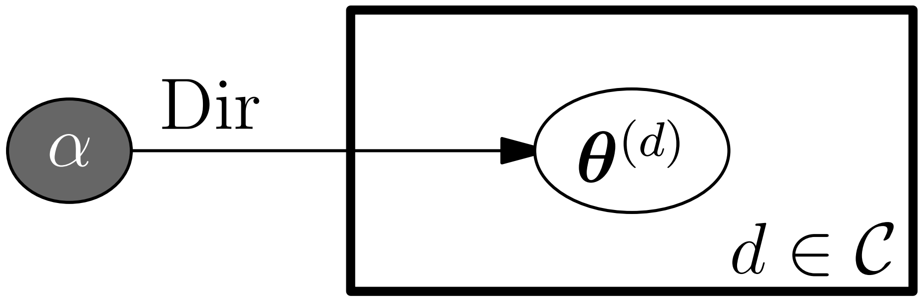


Plate Diagrams

- Drawing every random variable is tedious (and not really possible)
- A short-hand is to draw a box (plate) meaning repeat



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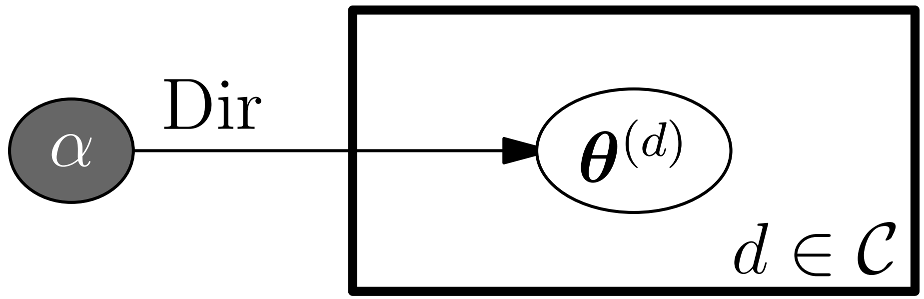

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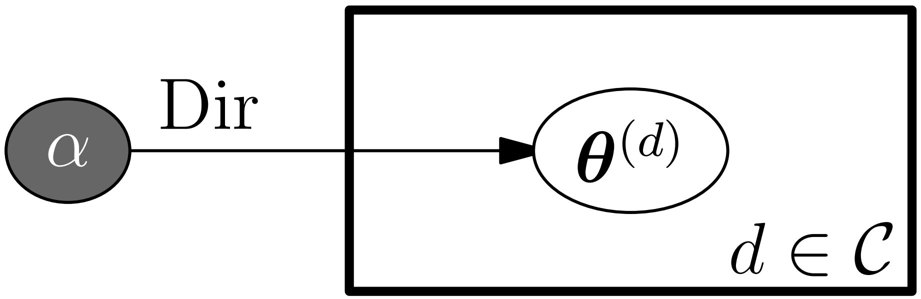

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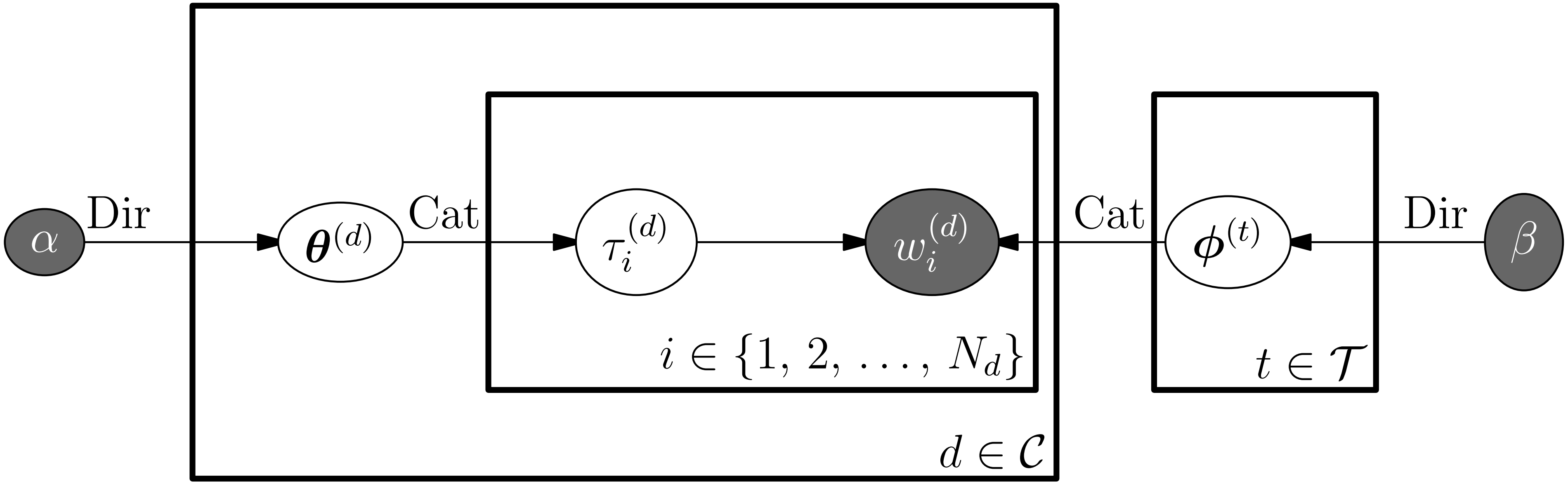

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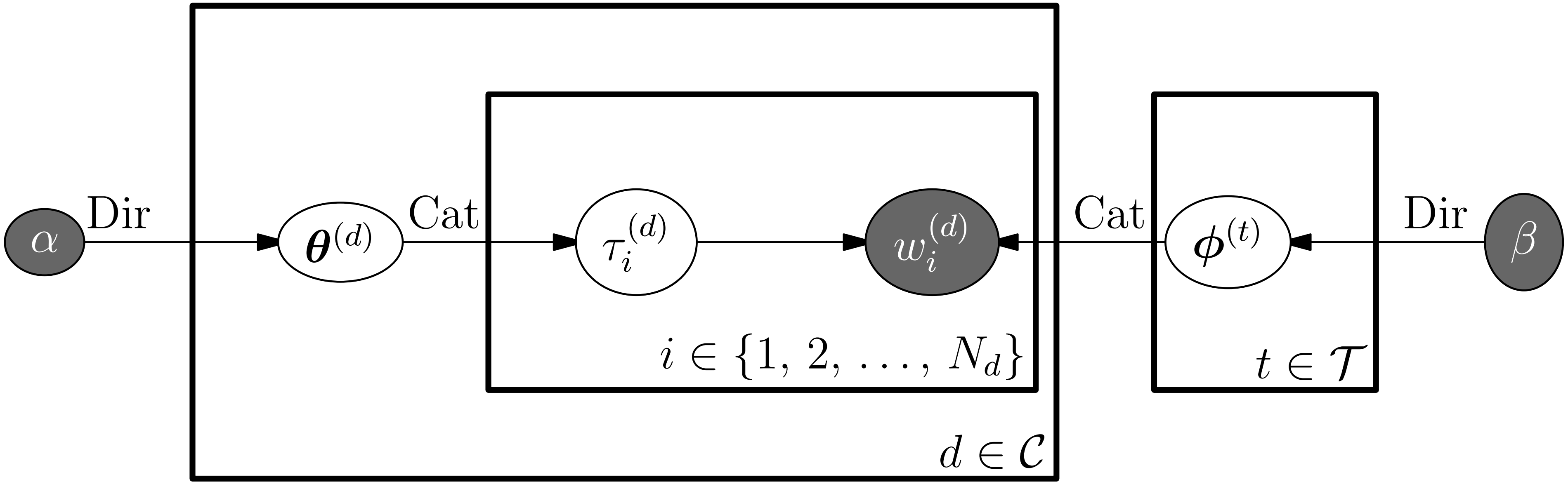

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Probabilistic Model

- The graphical Model is shorthand for the variables

$$\boldsymbol{W} = (\boldsymbol{w}^{(d)} | d \in \mathcal{C}) \quad \text{with} \quad \boldsymbol{w}^{(d)} = (w_1^{(d)}, w_2^{(d)}, \ldots, w_{N_d}^{(d)}), \quad \text{and} \quad w_i^{(d)} \in \mathcal{V}$$

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$$\mathbb{P}\left(\boldsymbol{W}, \boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{\Phi} \middle| \alpha, \beta\right) = \left(\prod_{t \in \mathcal{T}} \mathrm{Dir}\left(\boldsymbol{\phi}^{(t)} \middle| \beta \mathbf{1}\right)\right)$$
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- To compute them we need to sample the probability distribution
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