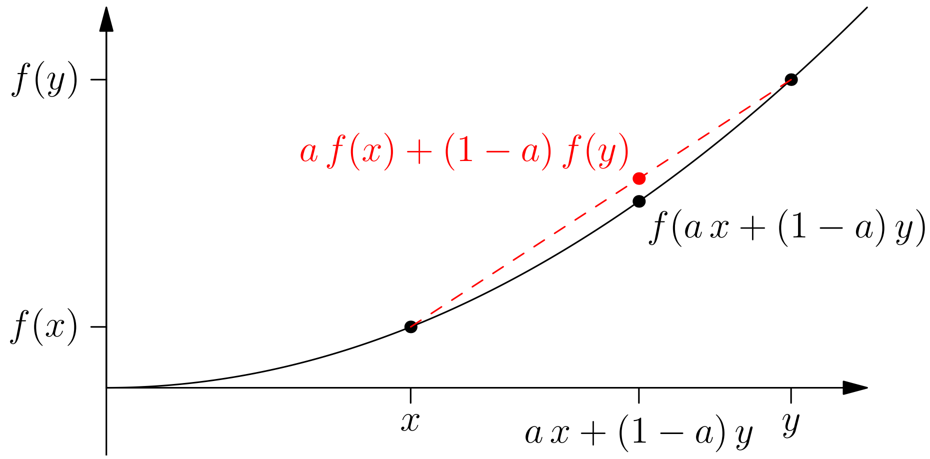
Advanced Machine Learning

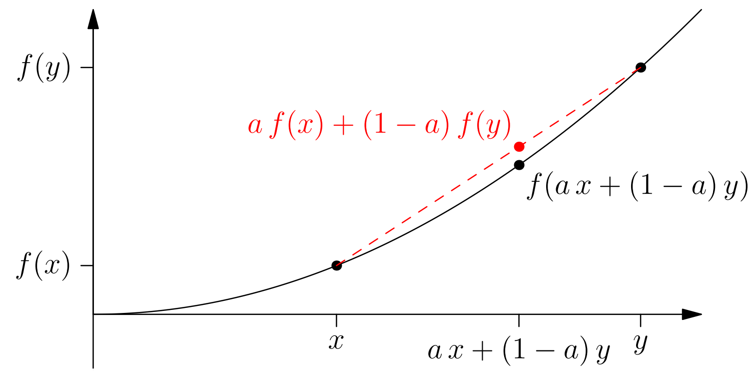
Convexity



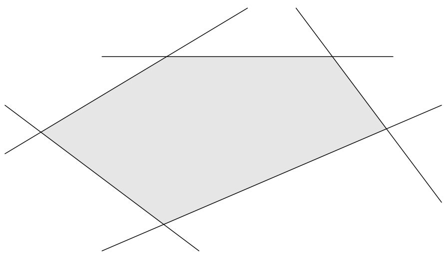
Convex sets, convex functions, Jensen's inequality

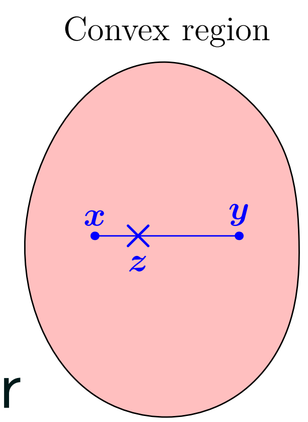
Outline

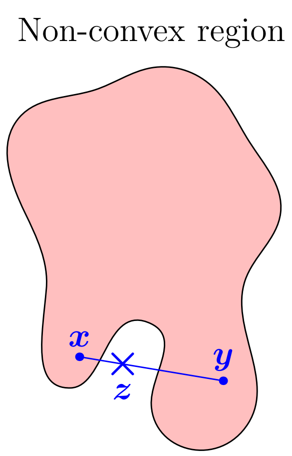
1. **Convex sets**
2. Convex functions
3. Jensen's inequality



Convex Regions







- Convex regions are familiar

- For any two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in a region $\mathcal{R}$ then for any $a \in [0,1]$ if

$$\boldsymbol{z} = a\,\boldsymbol{x} + (1-a)\,\boldsymbol{y} \in \mathcal{R}$$

- then $\mathcal{R}$ is a convex region

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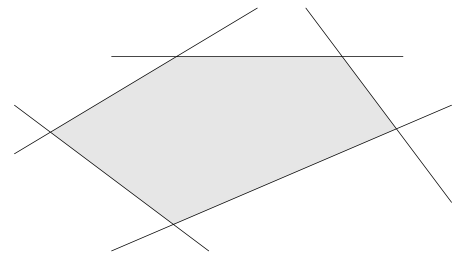
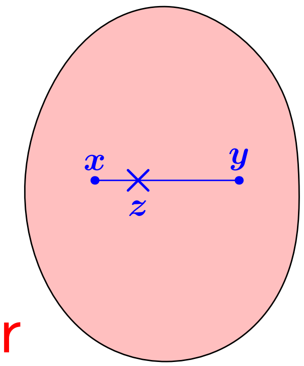


Convex region



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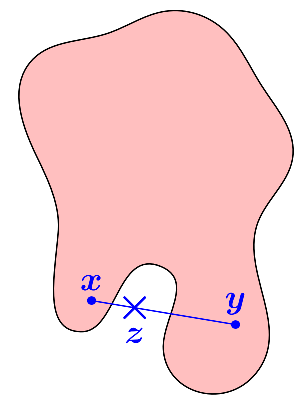
Non-convex region



Convex Sets

- For any set, $\mathcal{S}$, where addition and scalar multiplication is defined then:

  If for any two elements $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$ and any $a \in [0, 1]$

  $$\boldsymbol{z} = a\,\boldsymbol{x} + (1 - a)\,\boldsymbol{y} \in \mathcal{S}$$

  then $\mathcal{S}$ is said to be a convex set

Positive Semi-Definite Matrices

- Recall that a matrix $\mathbf{M}$ is positive semi-definite if for any vector $\boldsymbol{v}$

$$\boldsymbol{v}^{\mathsf{T}} \mathbf{M} \, \boldsymbol{v} \geq 0$$

- (We showed this also implies that all the eigenvalues are non-negative)
- We denote the fact that $\mathbf{M}$ is positive semi-definite by $\mathbf{M} \succeq 0$, and $\mathbf{M} \succ 0$ if it is positive definite
- The set of positive semi-definite (PSD) matrices (or kernels) form a convex set

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Proof

- Consider any two arbitrarily chosen PSD matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ and any $a \in [0, 1]$ then let

$$\mathbf{M}_3 = a\,\mathbf{M}_1 + (1-a)\,\mathbf{M}_2$$

- Then for any vector $\boldsymbol{v}$

$$\begin{aligned}
\boldsymbol{v}^{\mathsf{T}} \mathbf{M}_3 \boldsymbol{v} &= \boldsymbol{v}^{\mathsf{T}} \left(a\,\mathbf{M}_1 + (1-a)\,\mathbf{M}_2\right) \boldsymbol{v} \\
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  where $m_1 = \boldsymbol{v}^{\mathsf{T}} \mathbf{M}_1 \boldsymbol{v}$ and $m_2 = \boldsymbol{v}^{\mathsf{T}} \mathbf{M}_2 \boldsymbol{v}$

- But $m_1, m_2 \geq 0$ since $\mathbf{M}_1, \mathbf{M}_2 \succeq 0$. Thus $a\,m_1 + (1-a)\,m_2 \geq 0$ and so $\mathbf{M}_3 \succeq 0 \quad \square$

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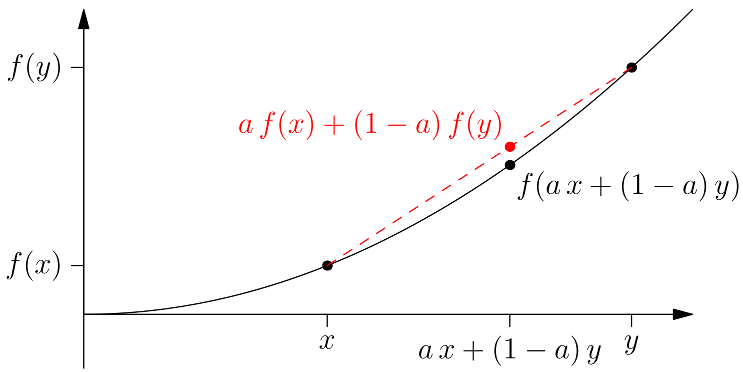

Convex Functions

- Any function $f(x)$ is said to be a **convex function** if for any two points x and y and any $a \in [0, 1]$

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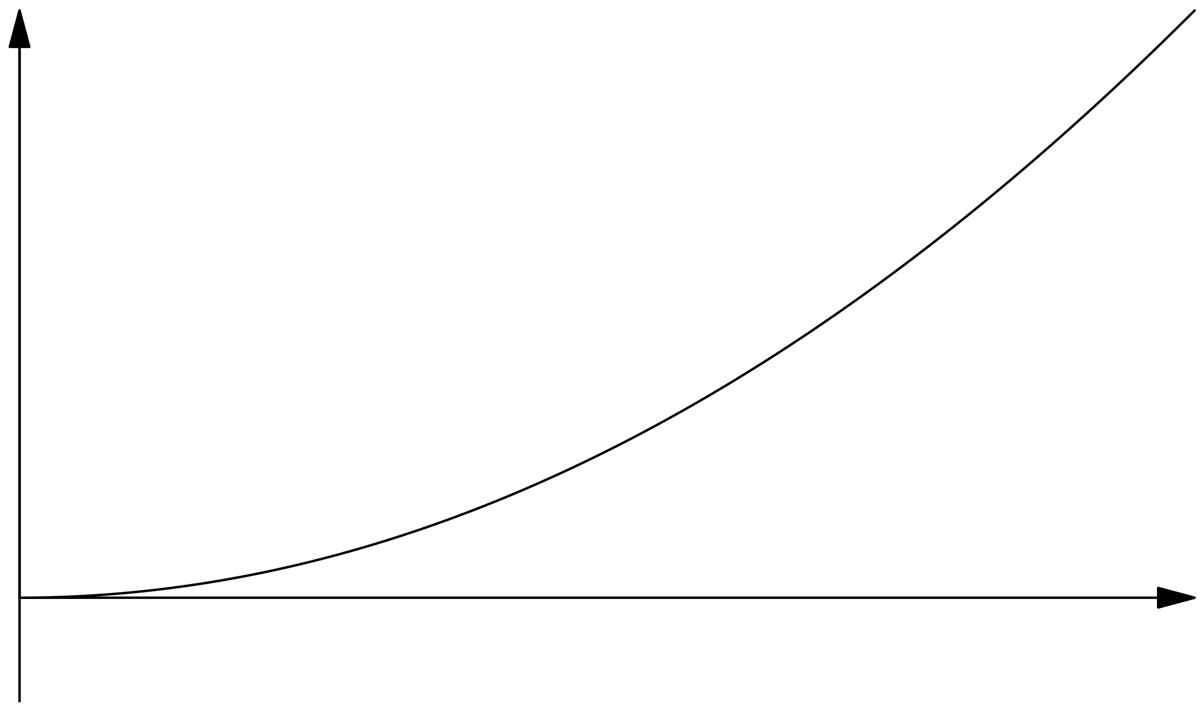

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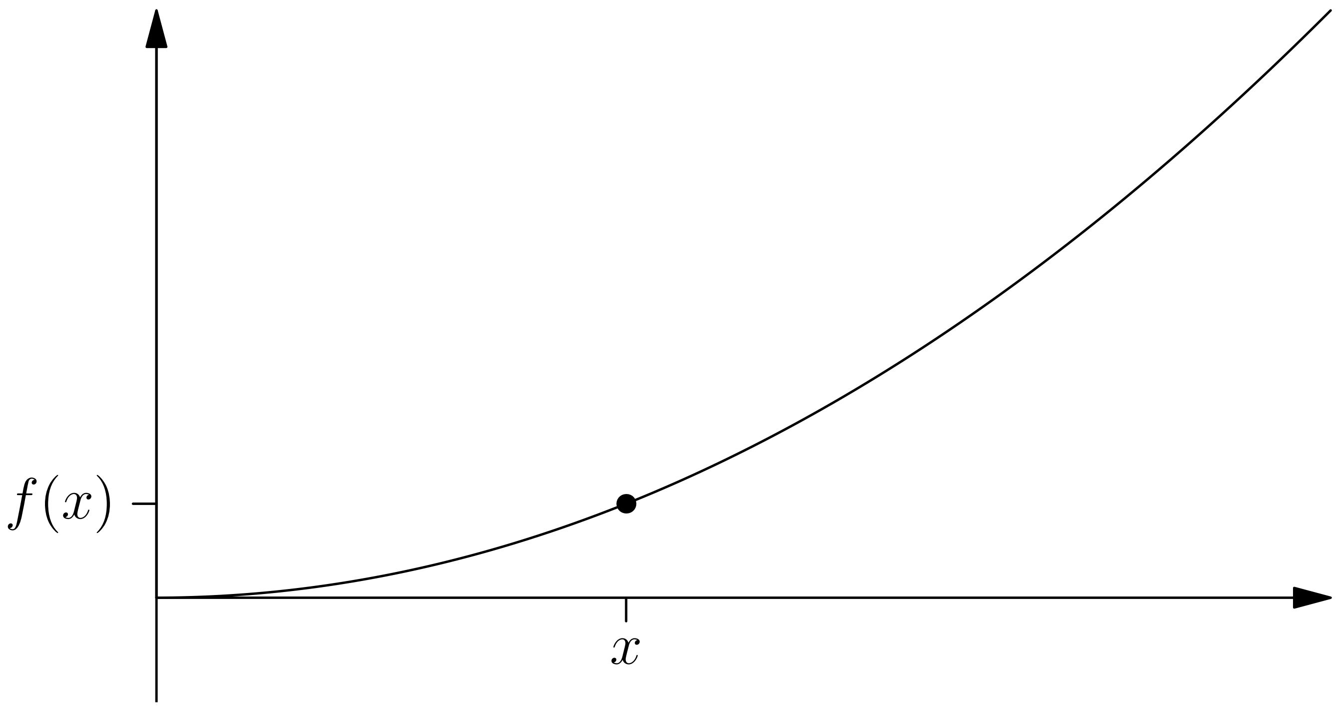

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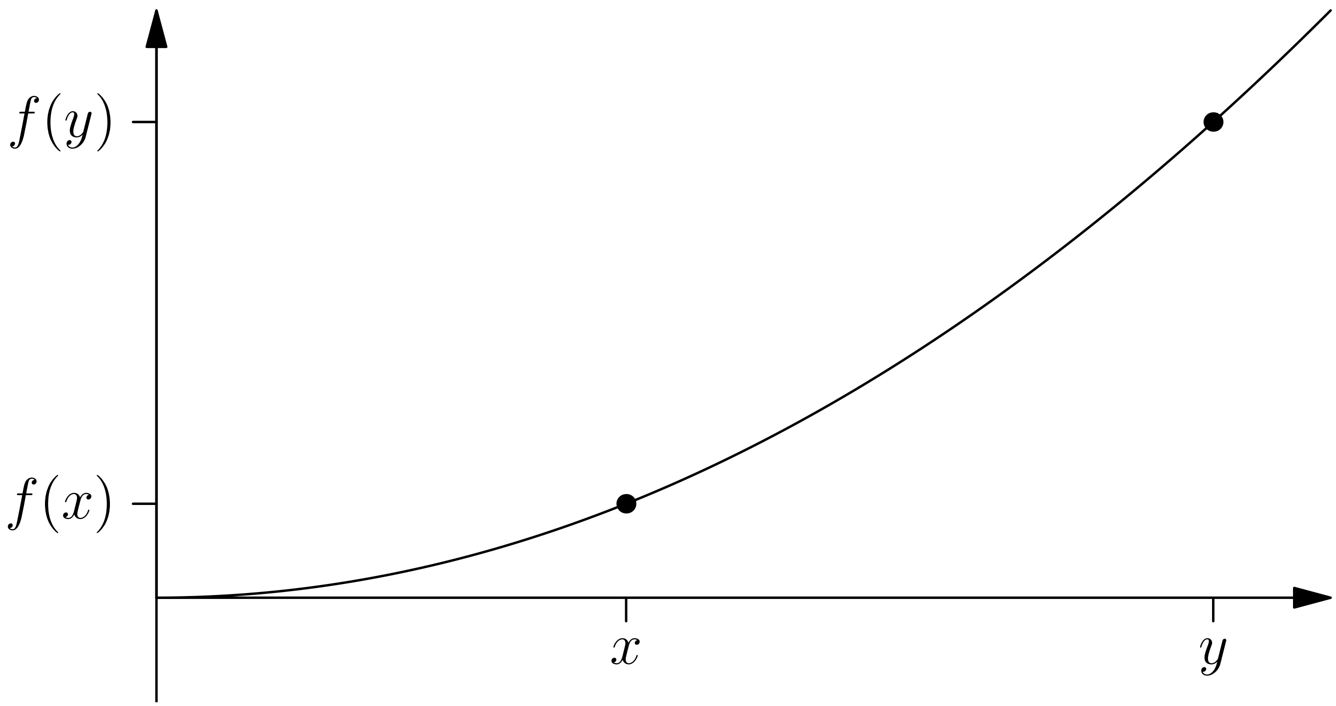

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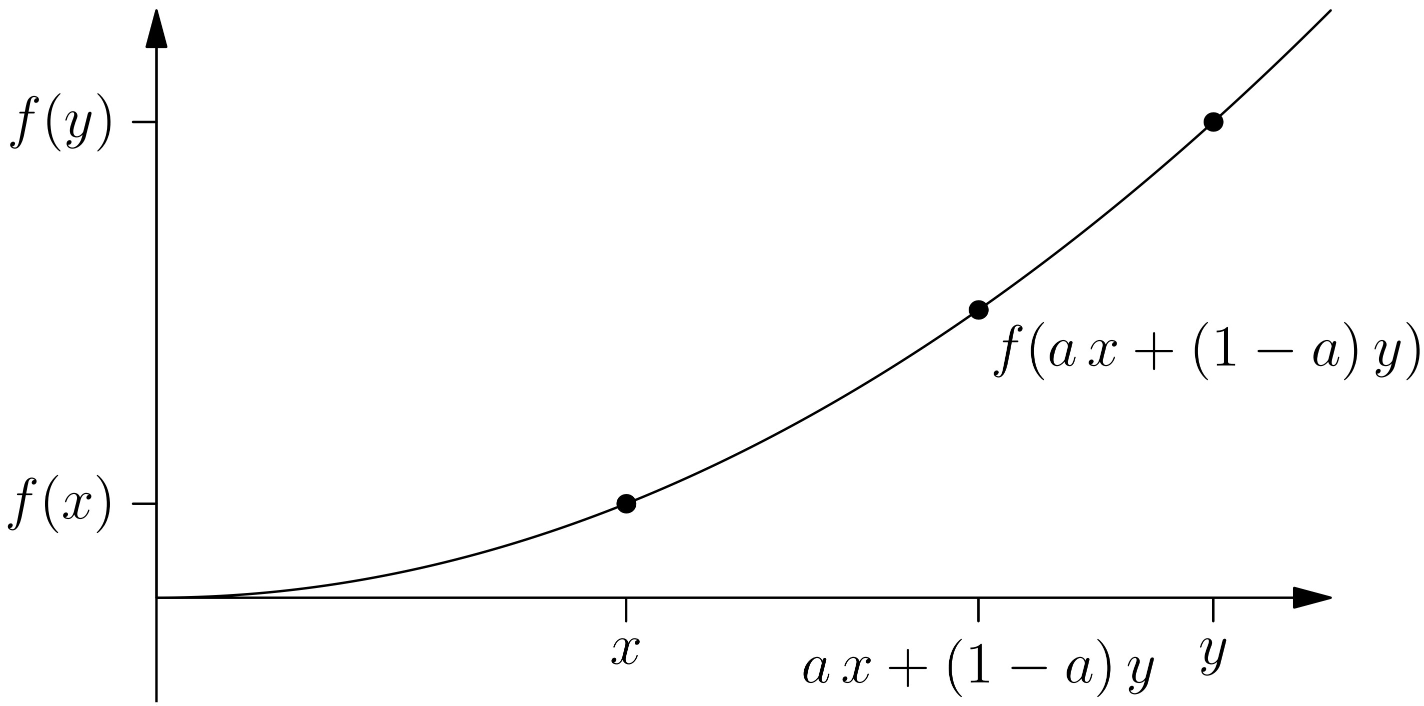

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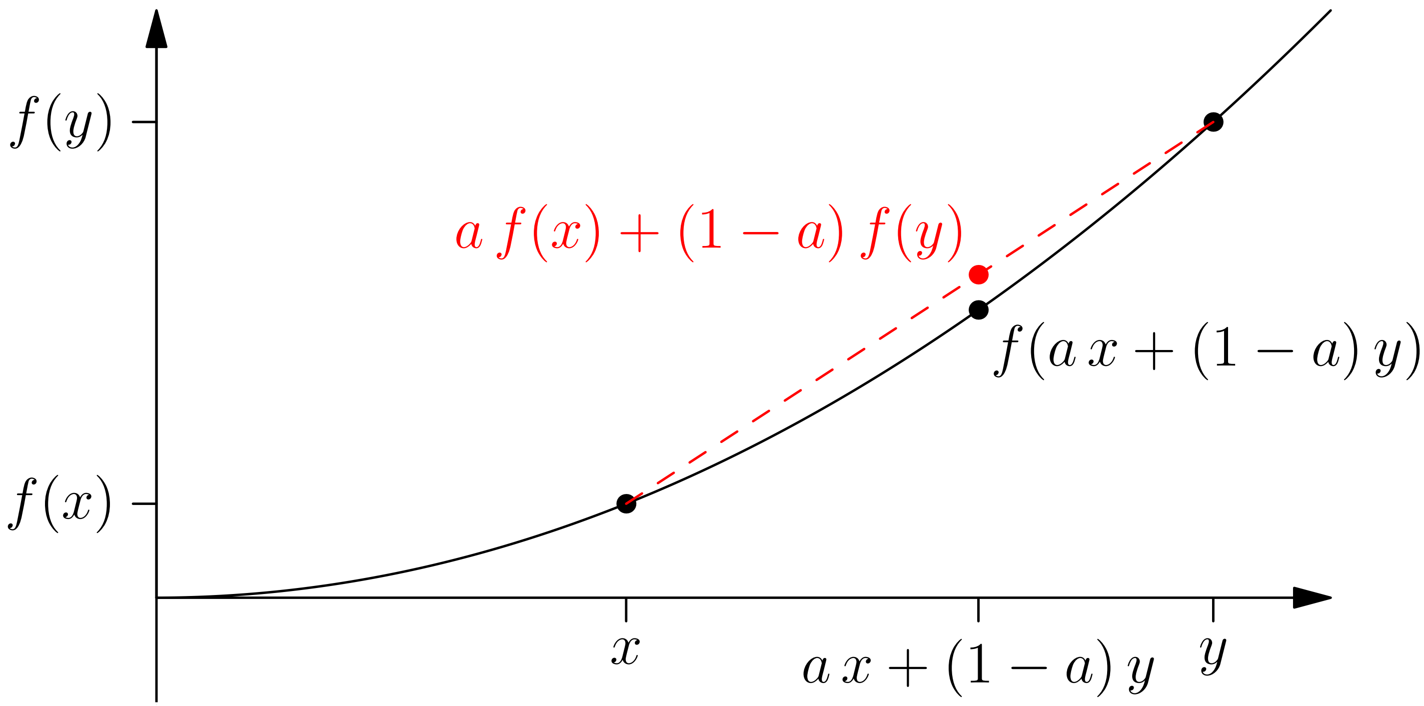
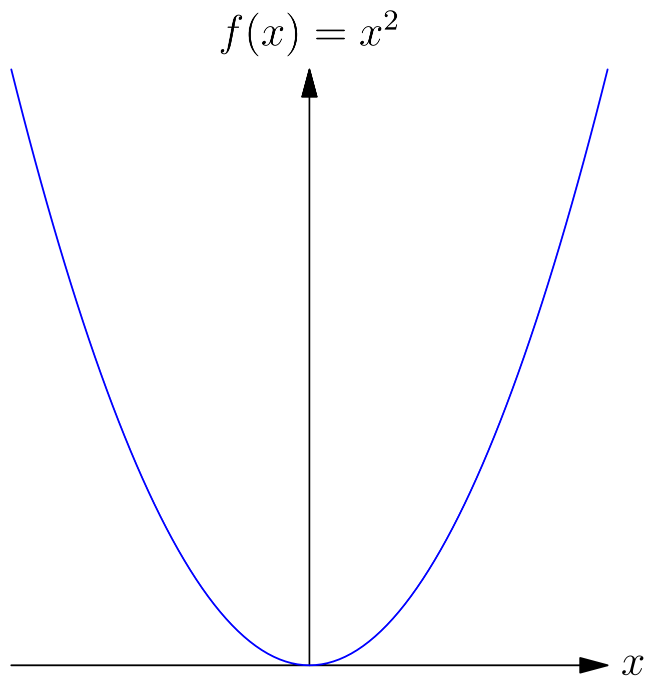
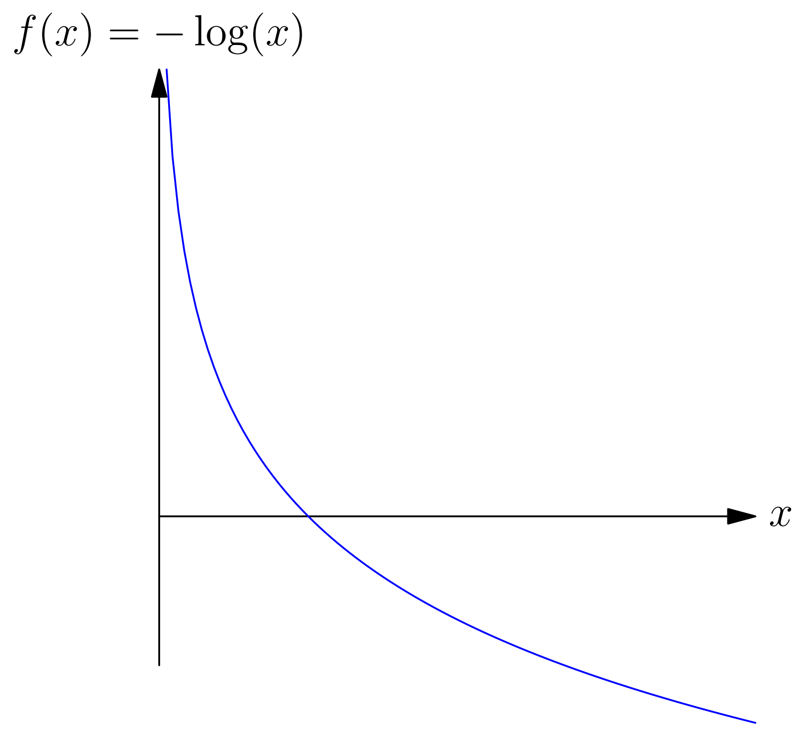


Epigraph

- The **epigraph** of a function is the area that lies above the function

- The epigraph of a convex function is a convex region





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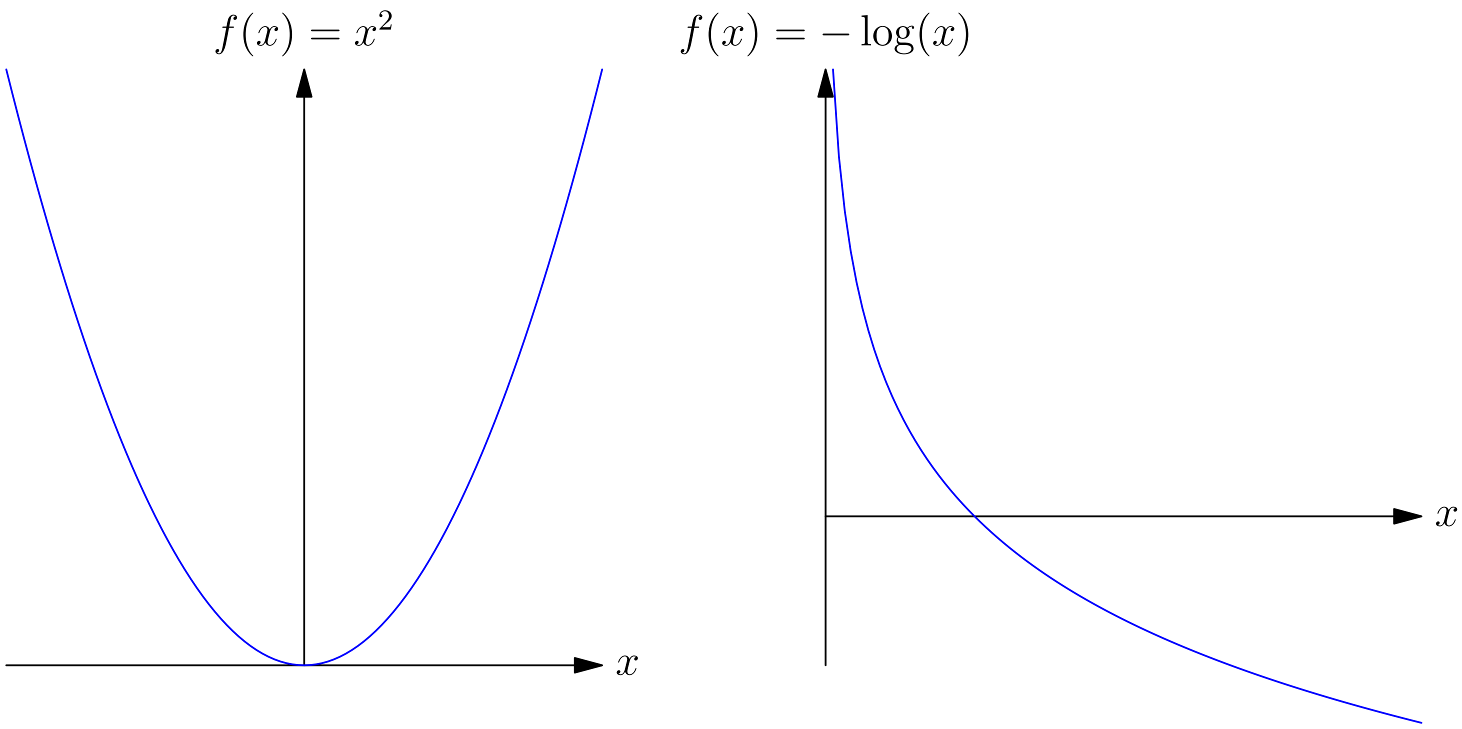

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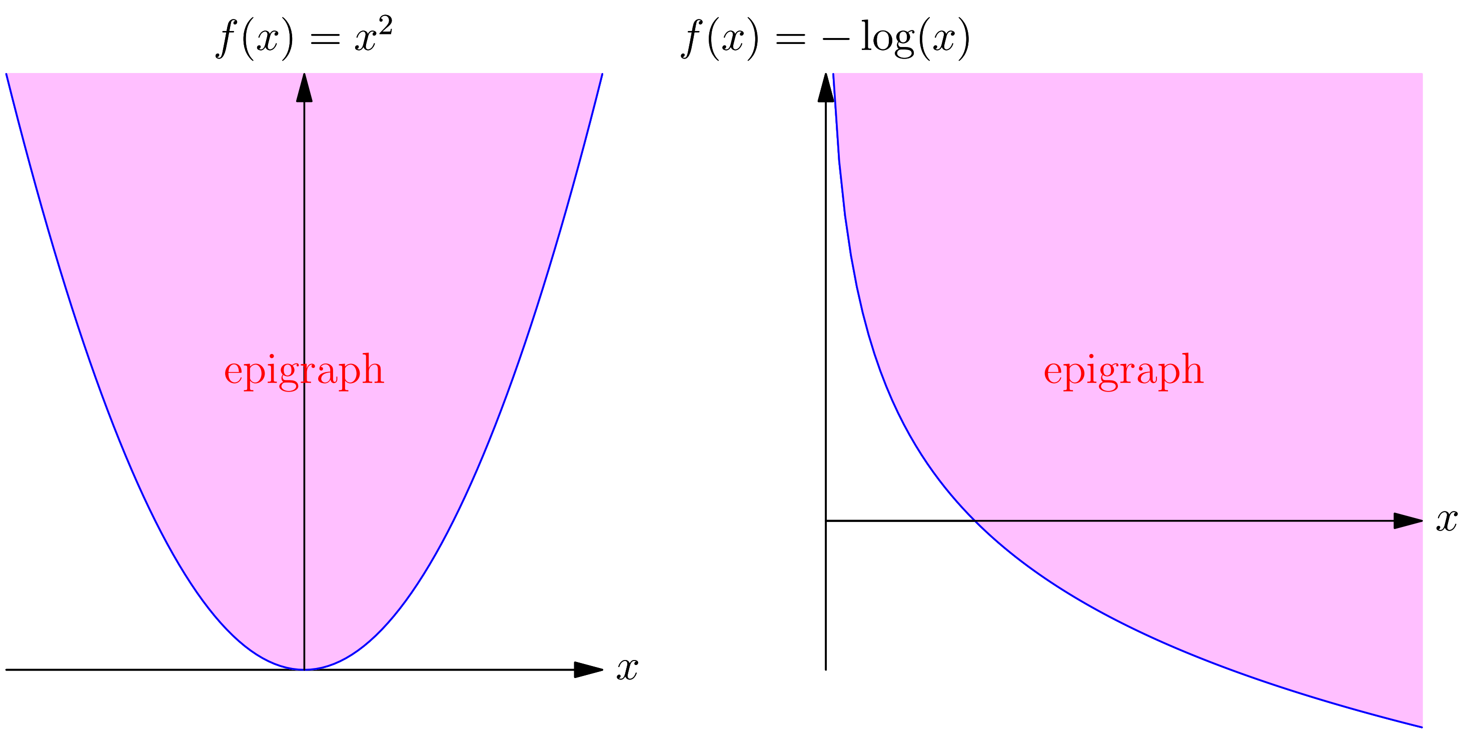

Convex-Down or Concave Functions

- Any function, $f(x)$, that satisfies the inverse inequality

$$f(a\,x + (1-a)\,y) \geq a f(x) + (1-a) f(y)$$

  for any points x and y and any $a \in [0,1]$ is said to be a **convex-down** or **concave** function

- Everything true for a convex(-up) function carries over to a convex-down function with a small modification

- If $f(x)$ is a convex-up function then $g(x) = -f(x)$ is a convex-down function

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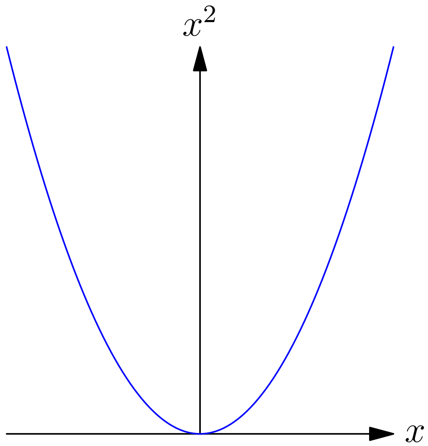
Examples

Convex-Up Functions



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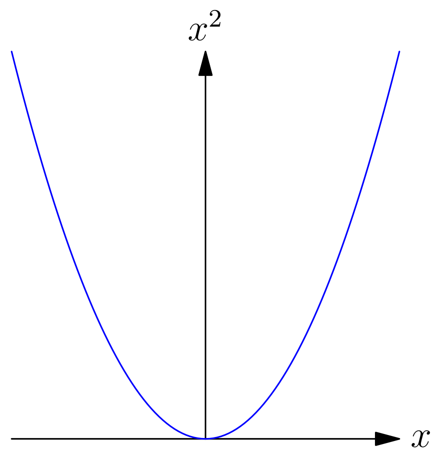

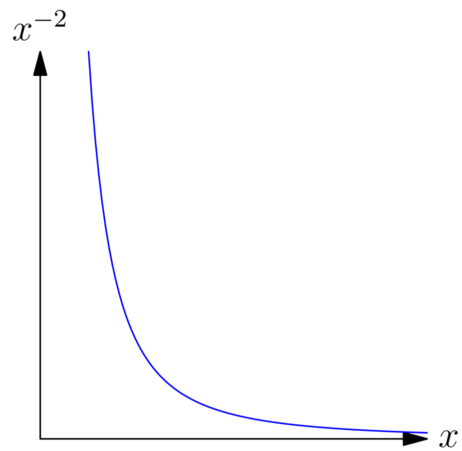

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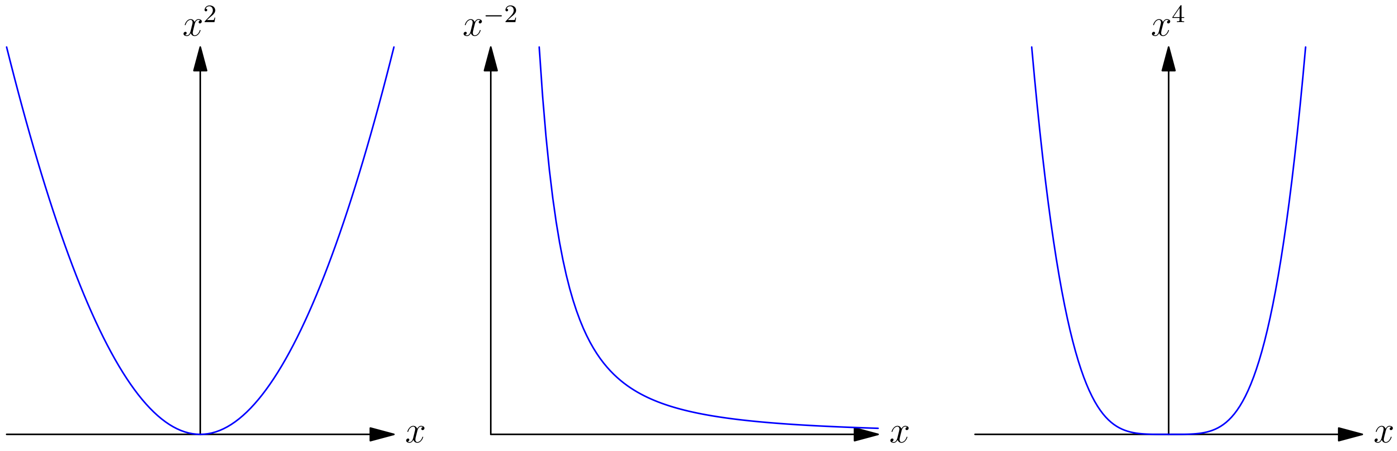

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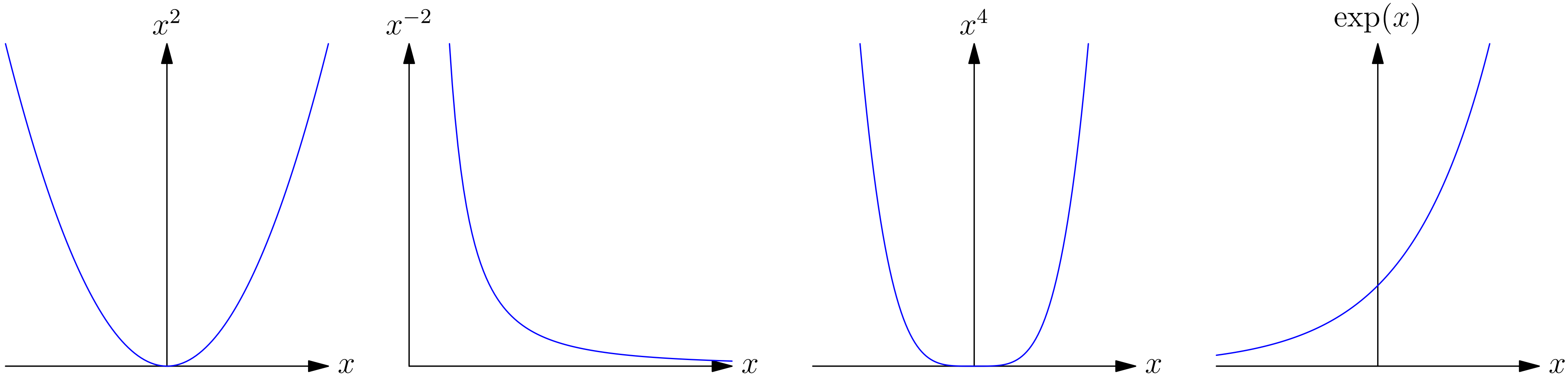

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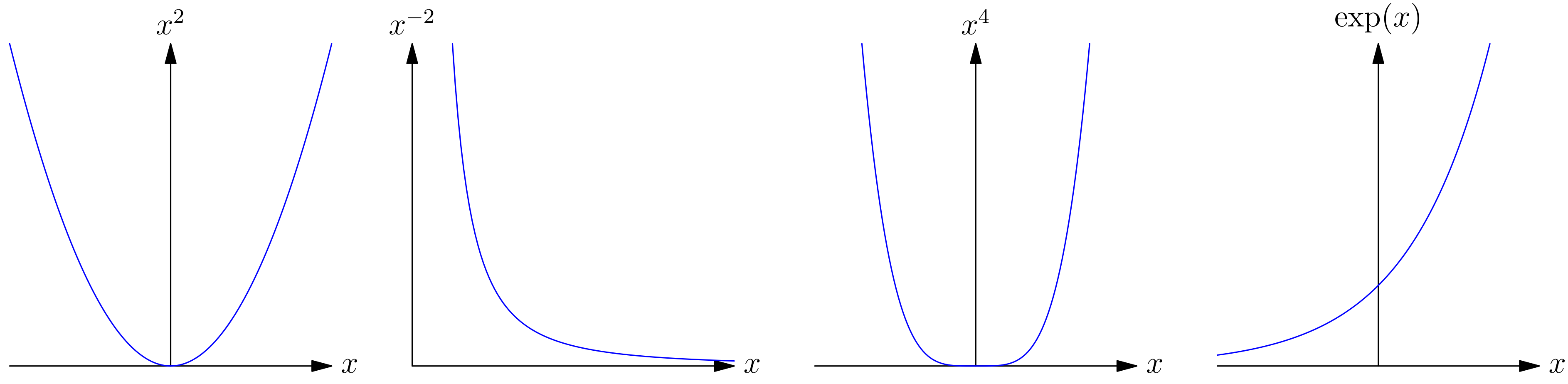
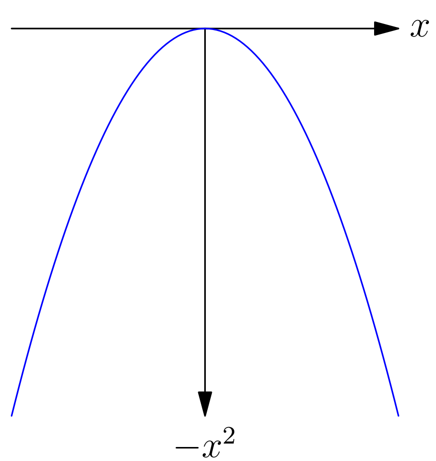


Convex-Down Functions



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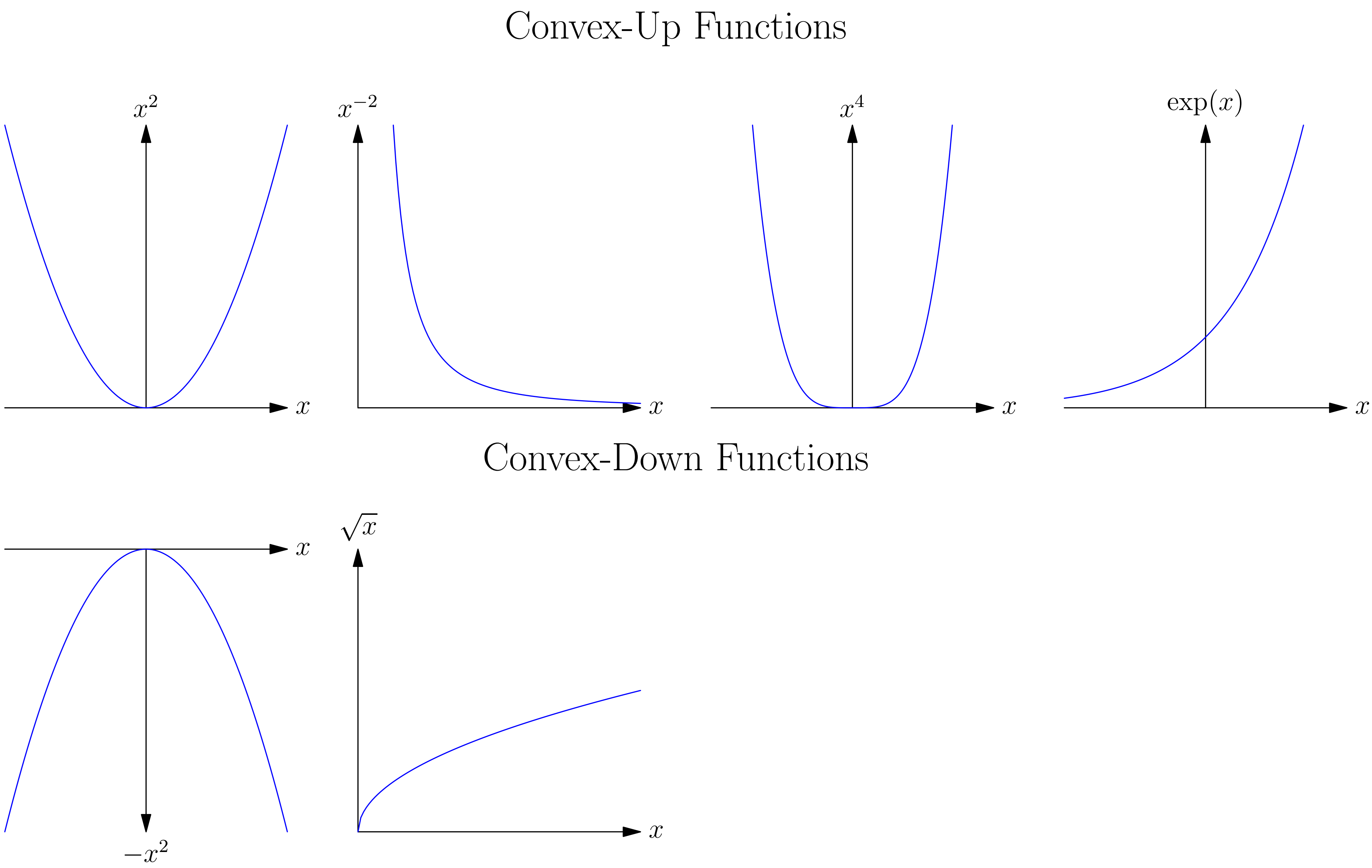
Convex-Up Functions
x^2
x
x^{-2}
x
x^4
x
$\exp(x)$
x
Convex-Down Functions
x
$-x^2$
$\sqrt{x}$
x

Examples

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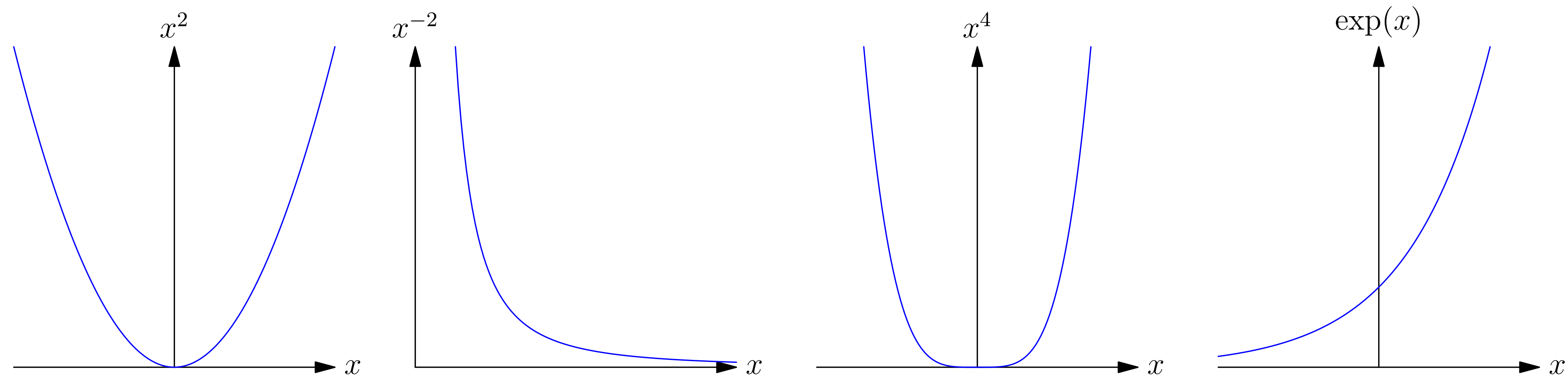

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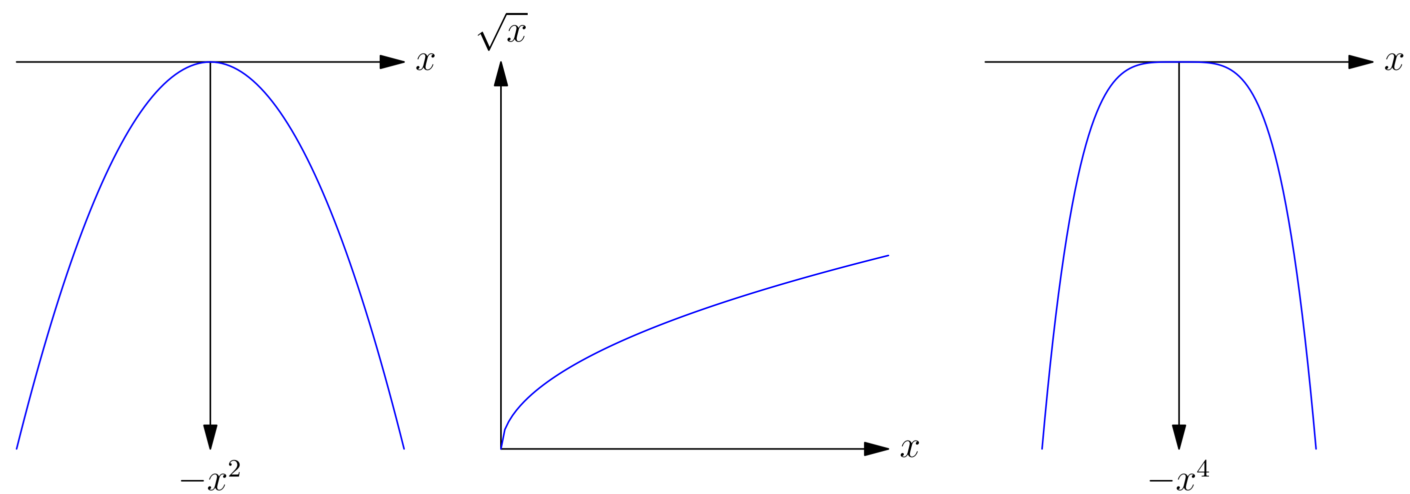

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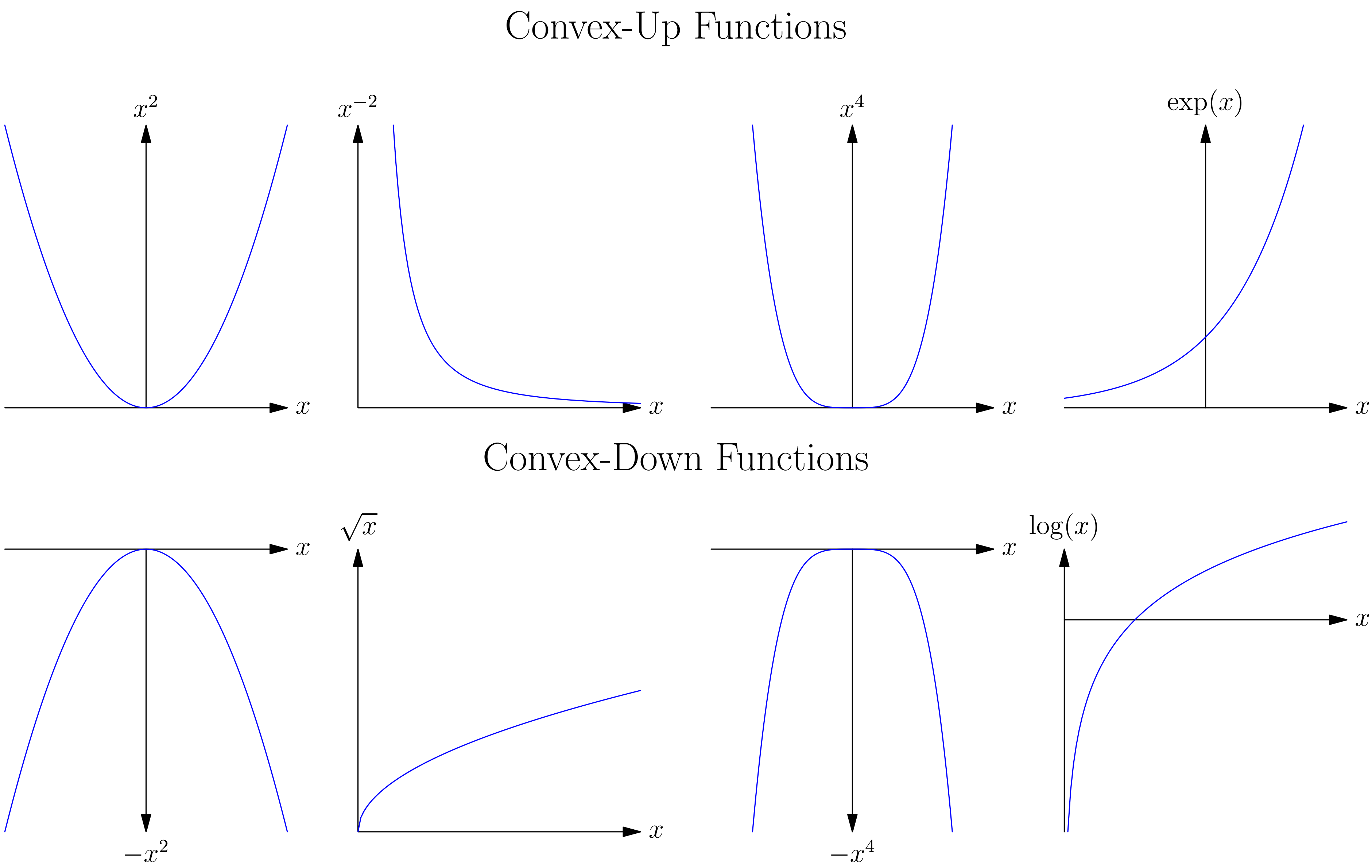


Convex-Up Functions
x^2
x
x^{-2}
x
x^4
x
$\exp(x)$
x
Convex-Down Functions
x
$-x^2$
$\sqrt{x}$
x
x
$-x^4$
$\log(x)$
x

Linear Functions

- Linear functions are given by

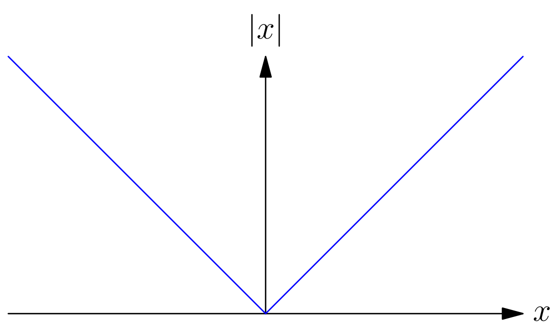
$$f(x) = m\,x + c$$

- They satisfy the **equality**

$$f(a\,x + (1-a)\,y) = a f(x) + (1-a) f(y)$$

- As such they are both convex(-up) and convex-down function

- $|x|$ is a convex-up function



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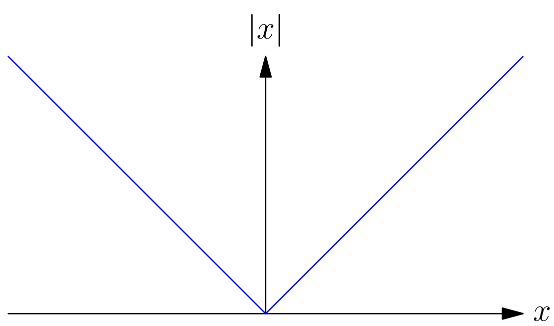

Linear Functions

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$$f(x) = m\,x + c$$

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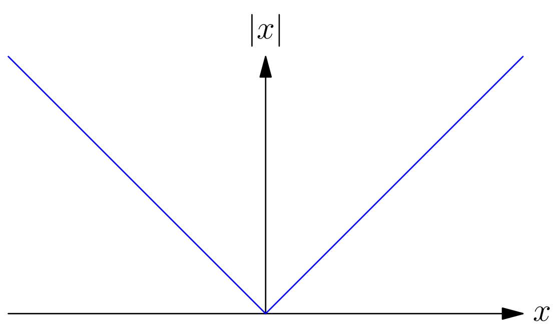

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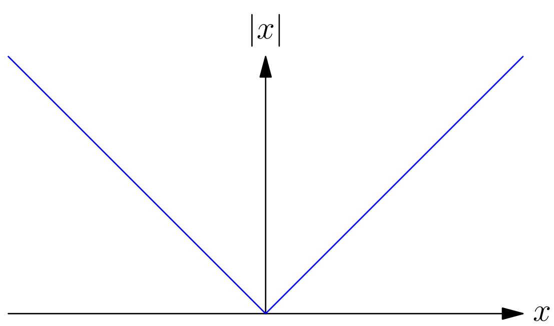

Strictly Convex Function

- Functions that satisfy the strict inequality (for $0 < a < 1$ and $x \neq y$)

$$f(a\,x + (1-a)\,y) < a f(x) + (1-a) f(y)$$

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Convexity in High Dimensions

- If $f : \mathbb{R}^n \to \mathbb{R}$ (i.e. $f(\boldsymbol{x})$ maps high dimensional point $\boldsymbol{x} \in \mathbb{R}^n$ to a real value) satisfies

$$f(a\,\boldsymbol{x} + (1-a)\,\boldsymbol{y}) \le a\,f(\boldsymbol{x}) + (1-a)\,f(\boldsymbol{y})$$

  for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $a \in [0,1]$ then $f(\boldsymbol{x})$ is a convex function

- $\|\boldsymbol{x}\|_2^2 = \sum_i x_i^2$ is a (strictly) convex function

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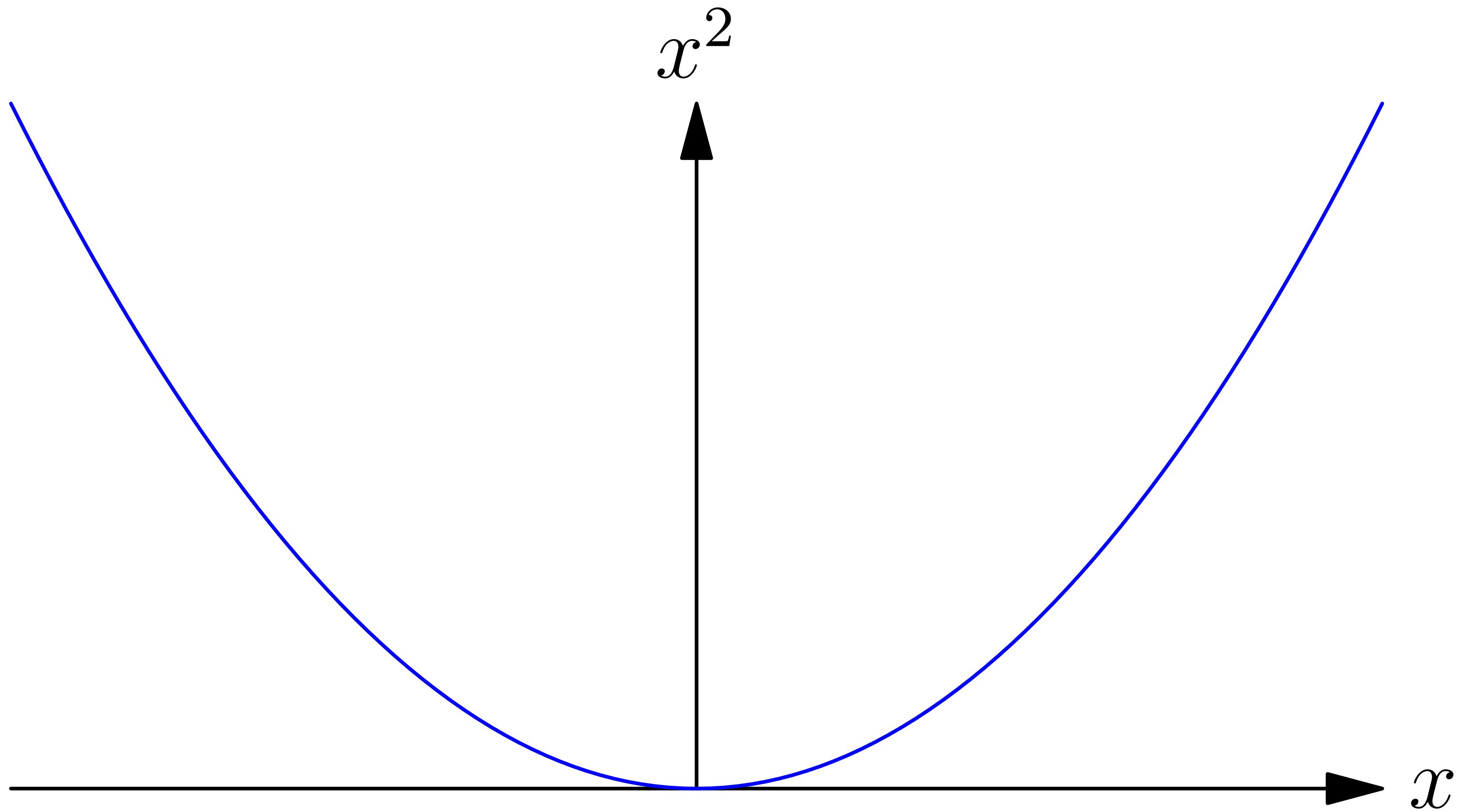

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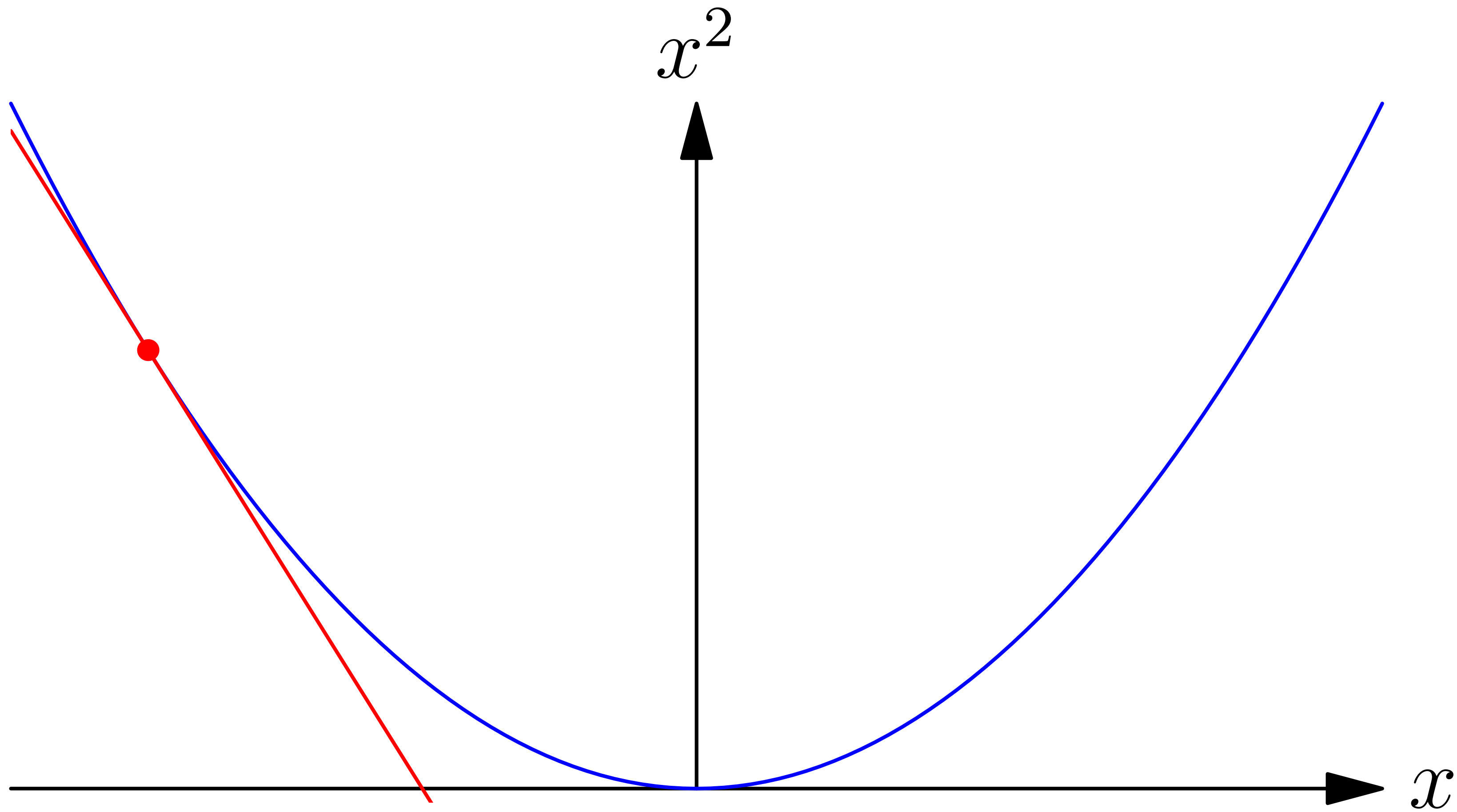

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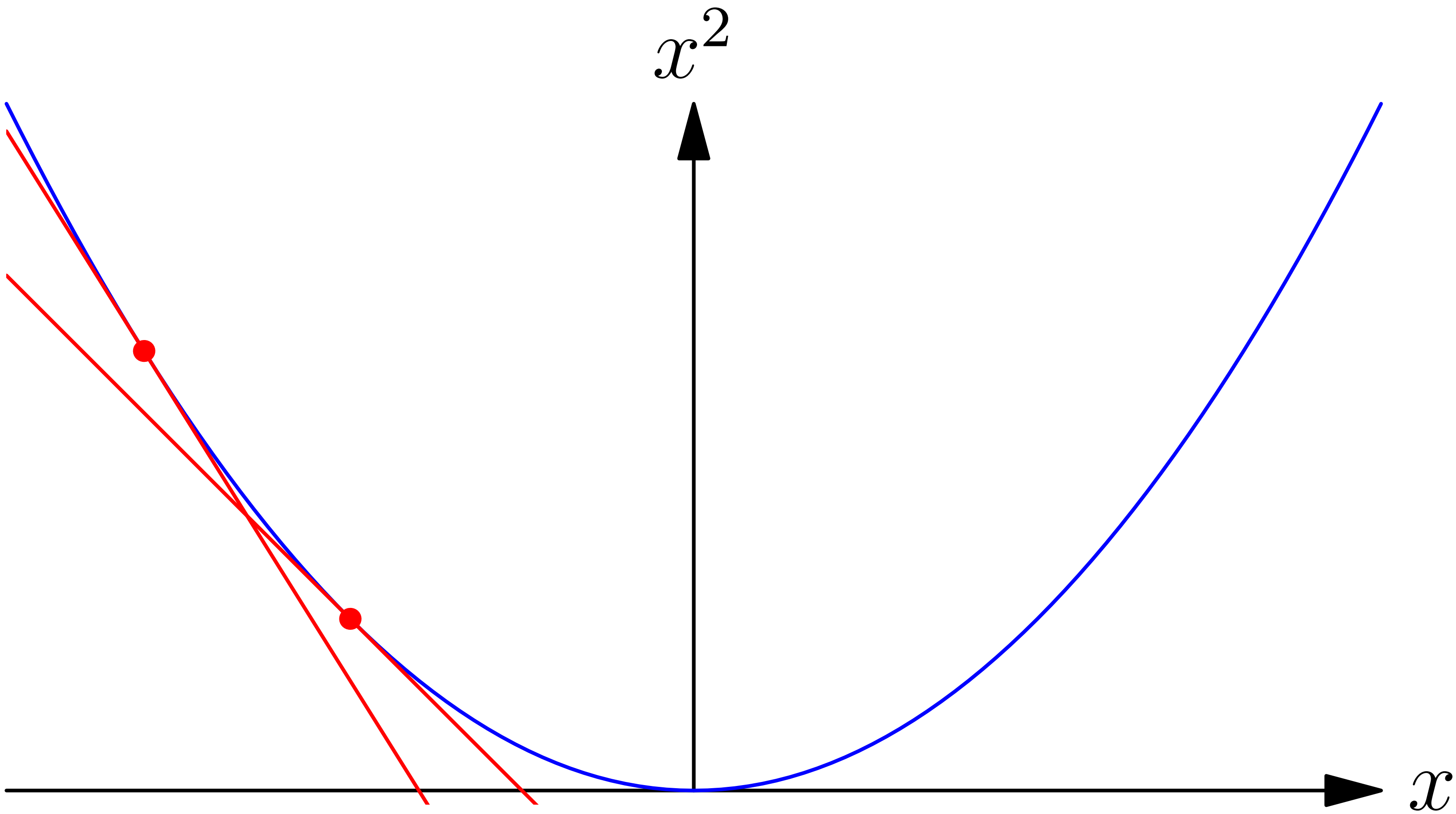

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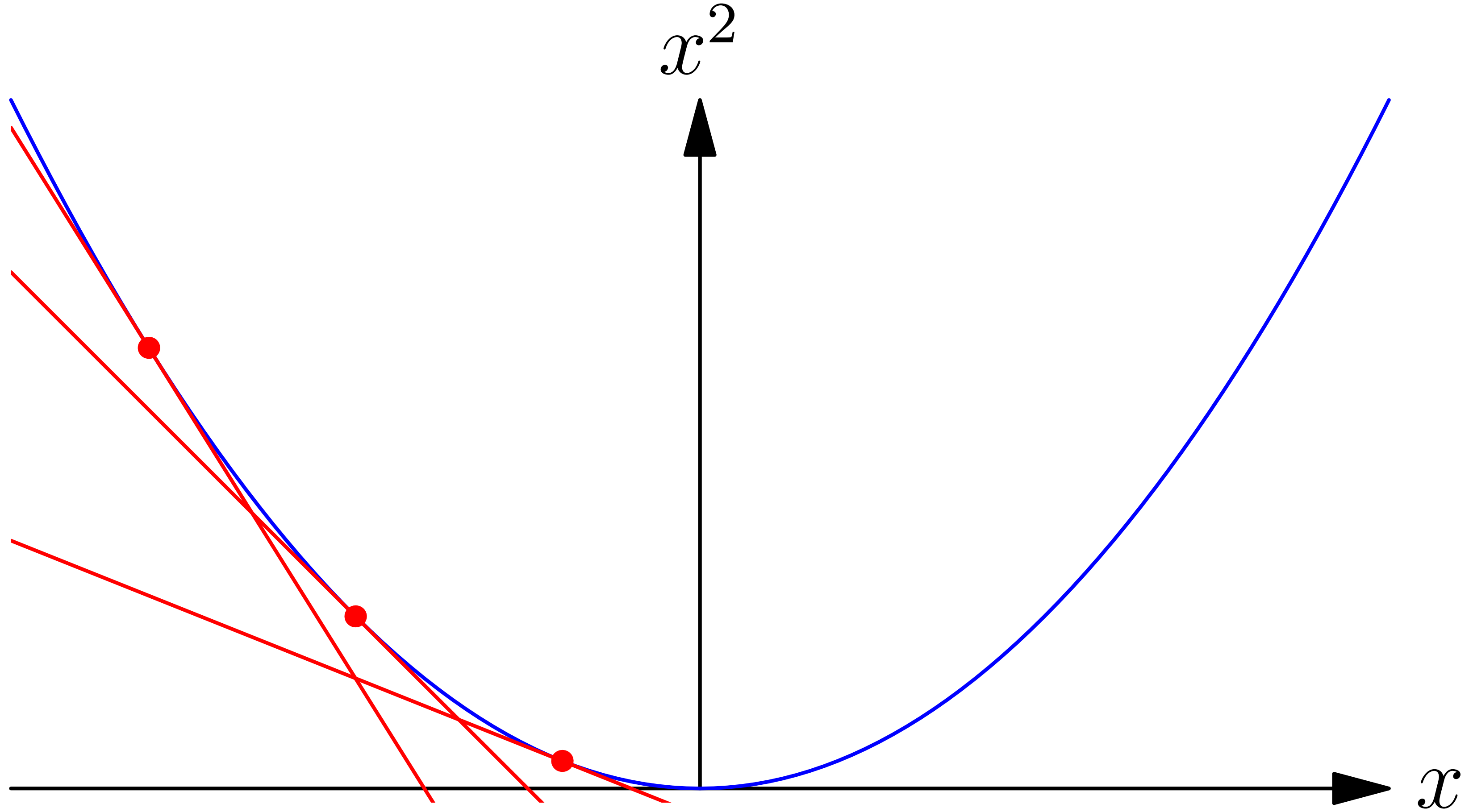

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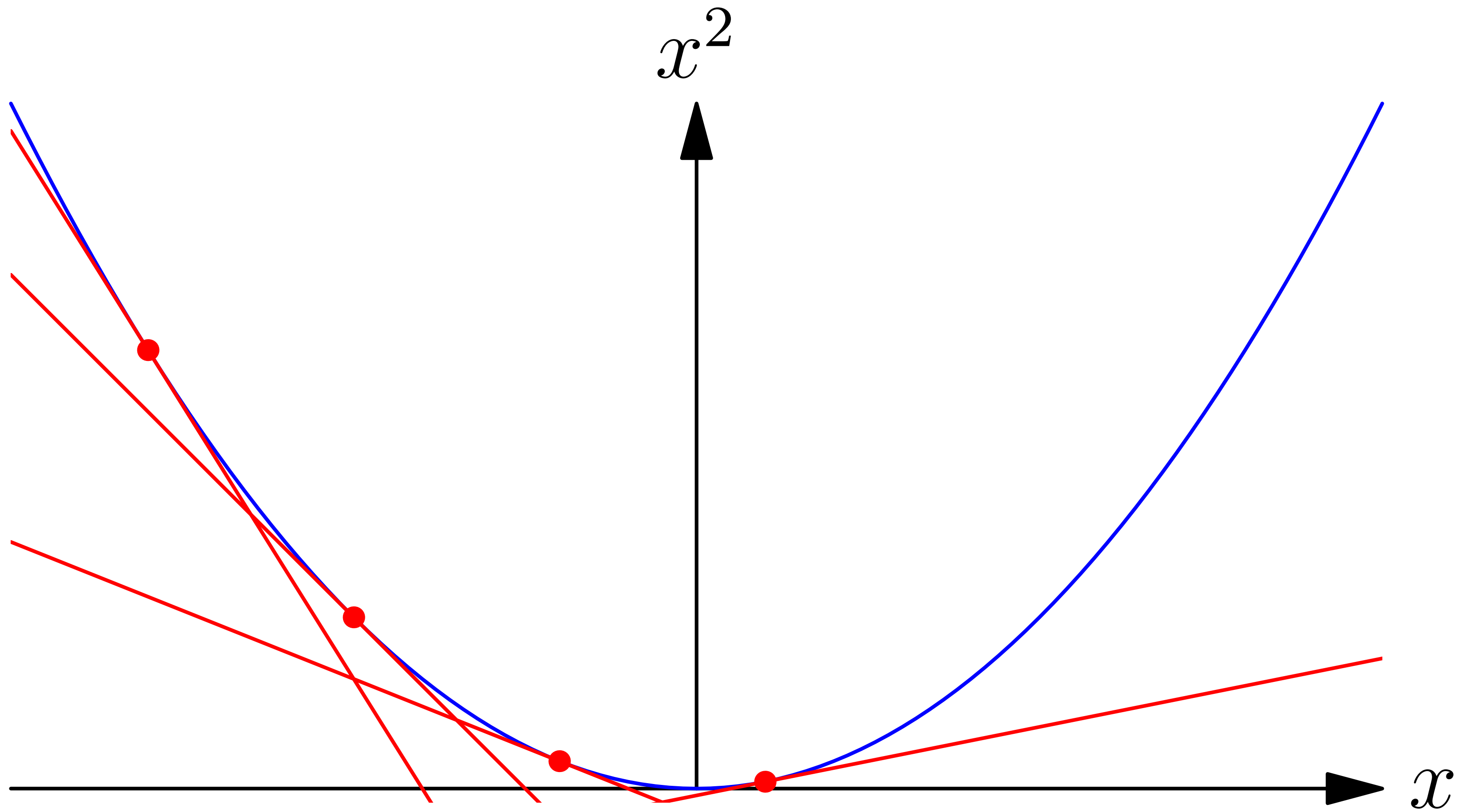

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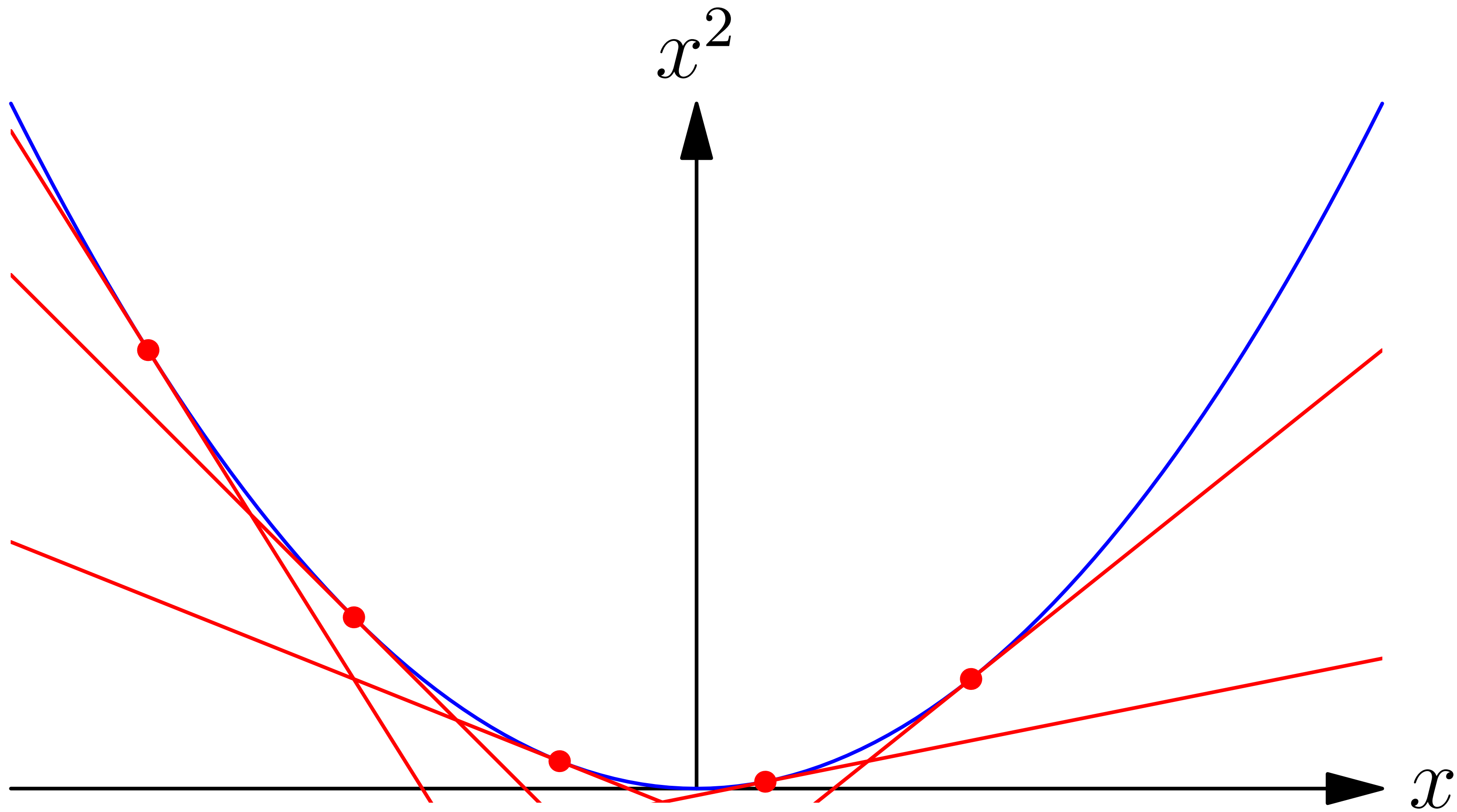

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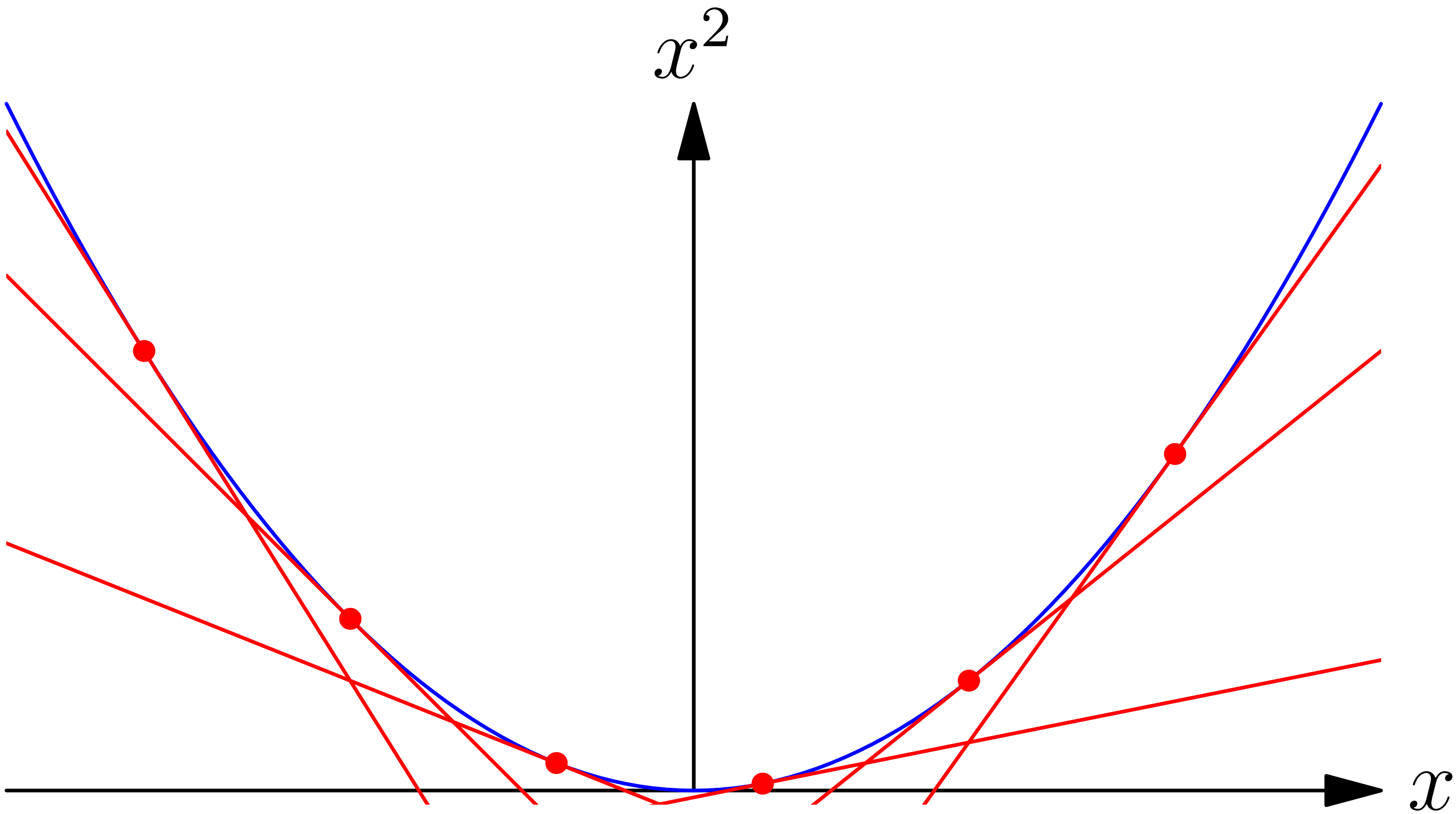

Second Derivatives

- As $f(x)$ lies on or above its tangent line then for any $\epsilon > 0$

$$f'(x+\epsilon) \geq f'(x)$$

therefore $f''(x) \geq 0$ at all points x

- In high dimensions a convex function lies above its tangent plane

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + (\boldsymbol{x} - \boldsymbol{x}^*)^\mathsf{T} \boldsymbol{\nabla} f(\boldsymbol{x}^*)$$

- The matrix of second derivatives (the Hessian) must be positive semi-definite at all points $\boldsymbol{x}$

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Proving Convexity

- $f(x) = x^2$ is strictly convex as $f''(x) = 2 > 0$
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- For any set of convex functions $f_1(x)$, $f_2(x)$, ... and any set of non-negative scalars a_1, a_2, ... then

$$g(x) = \sum_i a_i \, f_i(x)$$

  is convex

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$$g''(x) = \sum_i a_i \, f_i''(x)$$

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- All the properties we have discussed hold for functions defined on a convex set

- $\sin(x)$ is not generally neither convex up or down

- $\sin(x)$ for $x \in [0, \pi]$ is convex-down

- For a convex function defined on a non-convex set, $\mathcal{S}$, there exists points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$ such that for some $a \in [0, 1]$ there will be points $\boldsymbol{z} = a\,\boldsymbol{x} + (1-a)\boldsymbol{y} \notin \mathcal{S}$

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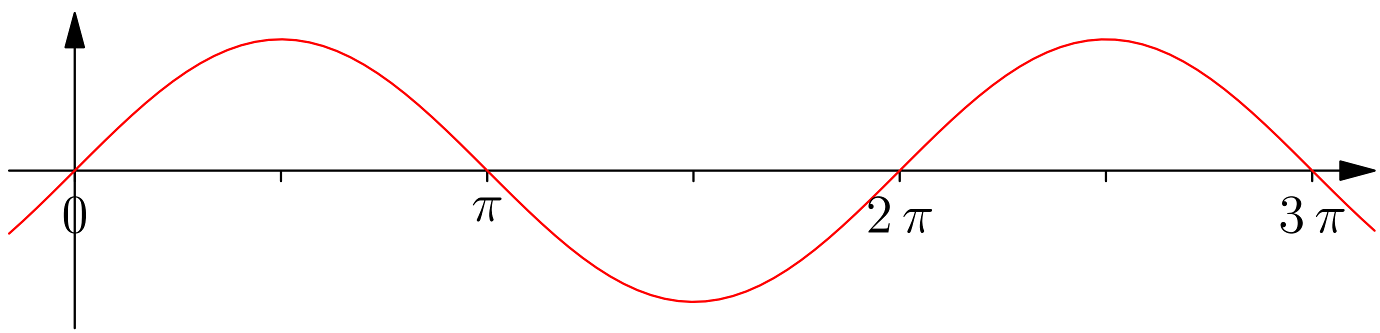

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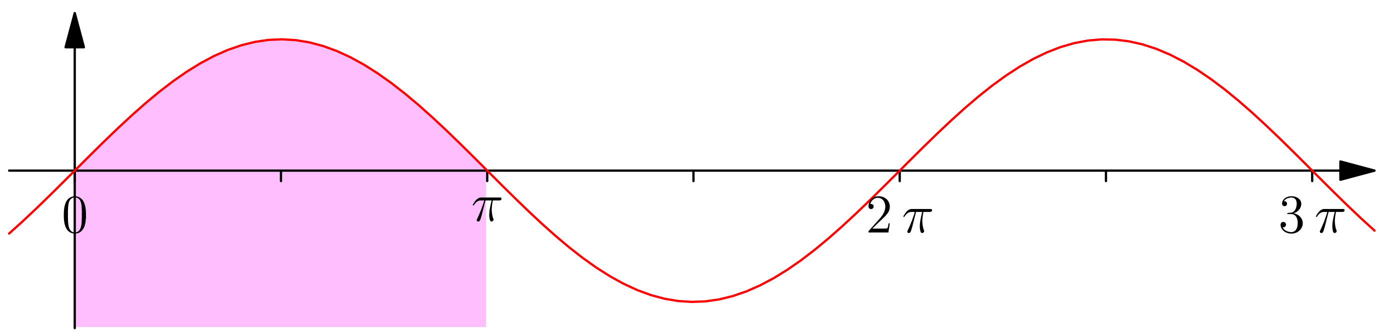

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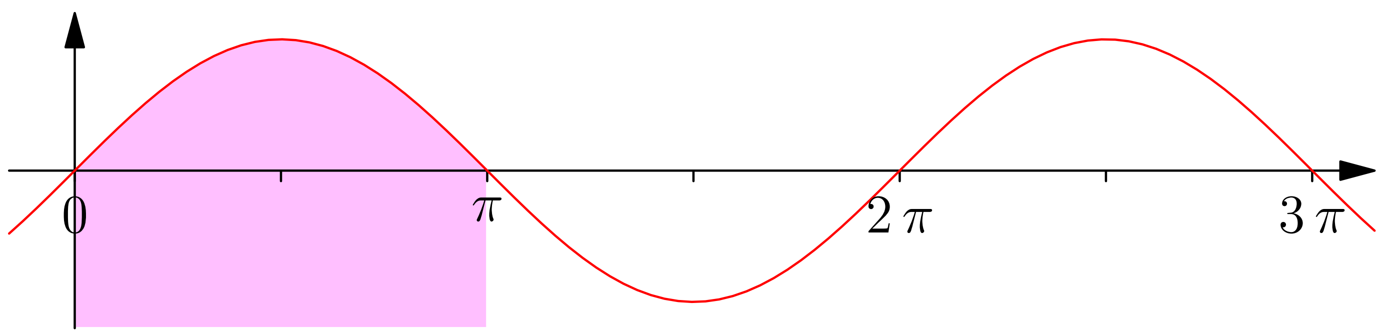

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Convex Functions Defined on Convex Sets

- All the properties we have discussed hold for functions defined on a convex set

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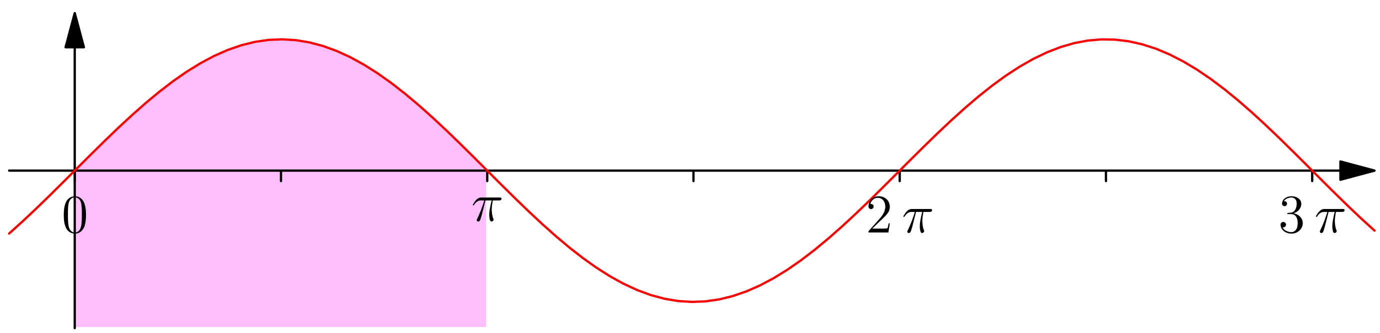

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Constraints

- Often we impose constraints on the set of points, e.g.

$$x_i > 0 \qquad\qquad \boldsymbol{a}^{\mathsf{T}}\boldsymbol{x} = b \qquad\qquad \boldsymbol{x}^{\mathsf{T}}\mathbf{M}\,\boldsymbol{x} \leq 1$$

- Linear constraints (e.g. $x_i > 0$ or $\boldsymbol{a}^{\mathsf{T}}\boldsymbol{x} = b$ or $\boldsymbol{a}^{\mathsf{T}}\boldsymbol{x} \leq b$) always define a convex region

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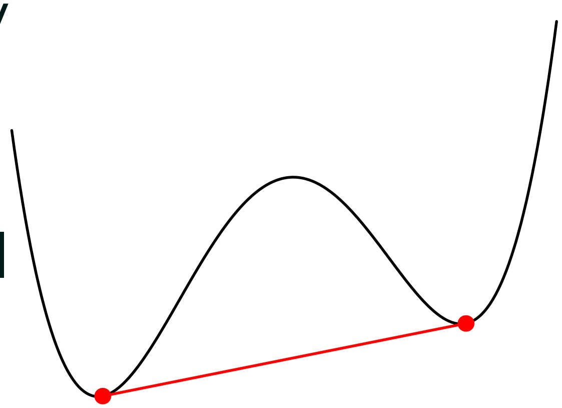
Unique Minimum

- Strictly convex function have a unique minimum
- The existence of a local minimum would break convexity
  - ⋆ The line connecting a local minimum to a global minimum would be strictly decreasing
  - ⋆ Thus there are points next to the local minimum with lower values
  - ⋆ This is a contradiction



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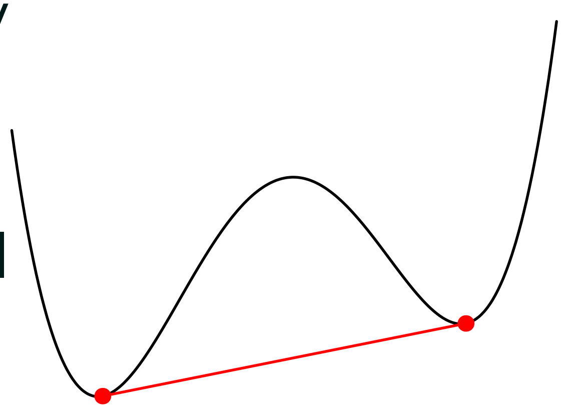

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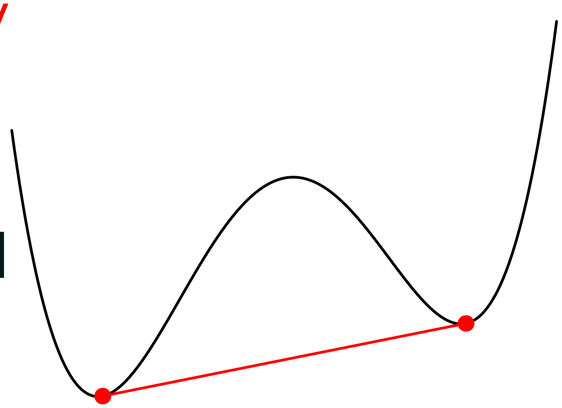

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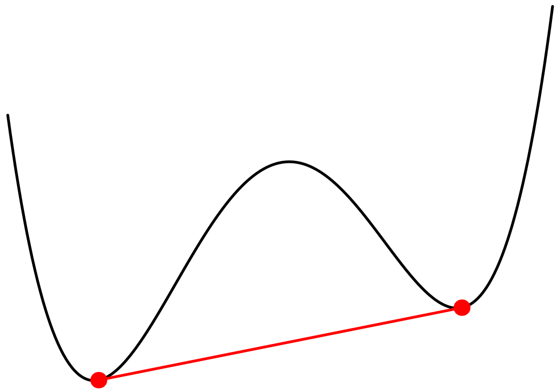

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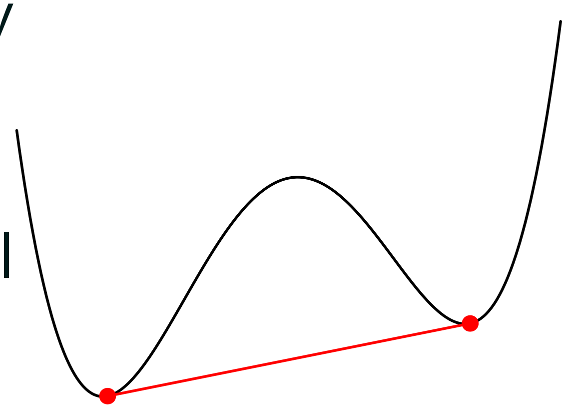

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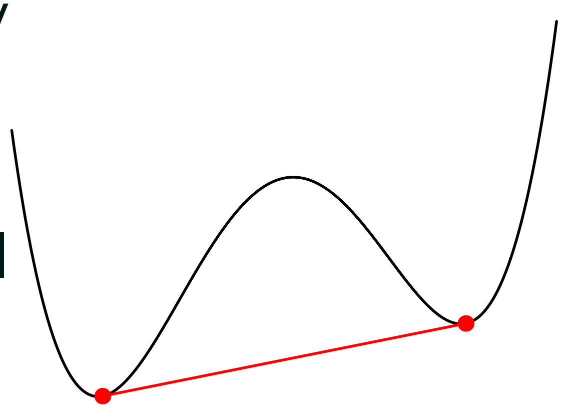

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Convex Set of Minima

- If $f(\boldsymbol{x})$ is **convex** but not **strictly convex** then there might exist a convex set $\mathcal{M} \subset \mathcal{X}$ of minima such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}$ and any $\boldsymbol{z} \in \mathcal{X}$ we have $f(\boldsymbol{x}) = f(\boldsymbol{y}) \leq f(\boldsymbol{z})$

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$$f''(x) + g''(x) > 0$$

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Linear Regression

- For linear regression the error function

$$E(\boldsymbol{w}) = \|\mathbf{X}\,\boldsymbol{w} - \boldsymbol{y}\|^2 = \boldsymbol{w}^{\mathsf{T}}\mathbf{X}^{\mathsf{T}}\mathbf{X}\boldsymbol{w} - 2\boldsymbol{w}^{\mathsf{T}}\mathbf{X}^{\mathsf{T}}\,\boldsymbol{y} + \boldsymbol{y}^{\mathsf{T}}\boldsymbol{y}$$

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- If $\mathbf{H} \succ 0$ there will be a unique minima, while if $\mathbf{H}$ has some zero eigenvalues there will be a family of solutions

Regularised Linear Regression

- In ridge regression we minimise a loss

$$L(\boldsymbol{w}) = \|\mathbf{X}\,\boldsymbol{w} - \boldsymbol{y}\|^2 + \eta\,\|\boldsymbol{w}\|^2 = \boldsymbol{w}^{\mathsf{T}}\left(\mathbf{X}^{\mathsf{T}}\mathbf{X} + \eta\mathbf{I}\right)\boldsymbol{w} - 2\boldsymbol{w}^{\mathsf{T}}\mathbf{X}^{\mathsf{T}}\boldsymbol{y} + \boldsymbol{y}^{\mathsf{T}}\boldsymbol{y}$$

- Because $\|\boldsymbol{w}\|^2$ is strictly convex the loss function is strictly convex and so will have a unique solution

- Using an L_1 regulariser (Lasso)

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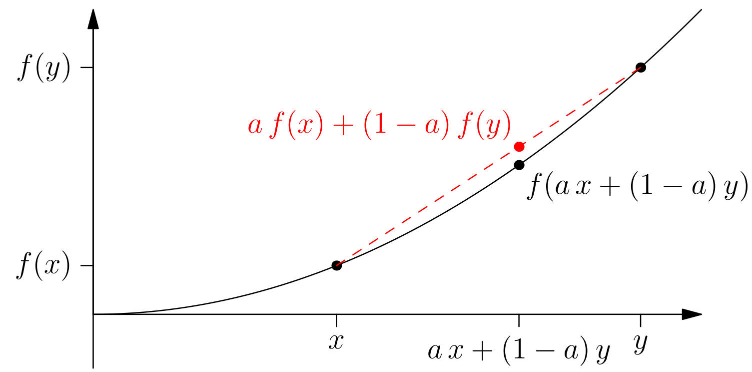
Outline

1. Convex sets
2. Convex functions
3. **Jensen's inequality**



Jensen's Inequality

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Proof

- We said before that a convex function must lie on or above its tangent plane at any point $\boldsymbol{x}^*$

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + (\boldsymbol{x} - \boldsymbol{x}^*)^{\mathsf{T}} \boldsymbol{\nabla} f(\boldsymbol{x}^*)$$

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Simple Proofs with Jensen's Inequality

- Since $f(x) = x^2$ is convex by Jensen's inequality

$$\mathbb{E}\left[X^2\right] \geq \mathbb{E}[X]^2 \quad \text{or} \quad \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \geq 0$$

(i.e. variance are non-negative)

- The KL-divergence $\mathrm{KL}(f\|g)$ between two categorical probability distributions $(f_1, f_2, \ldots)$ and $(g_1, g_2, \ldots)$ is define as

$$\mathrm{KL}(f\|g) = -\sum_i f_i \log\left(\frac{g_i}{f_i}\right)$$

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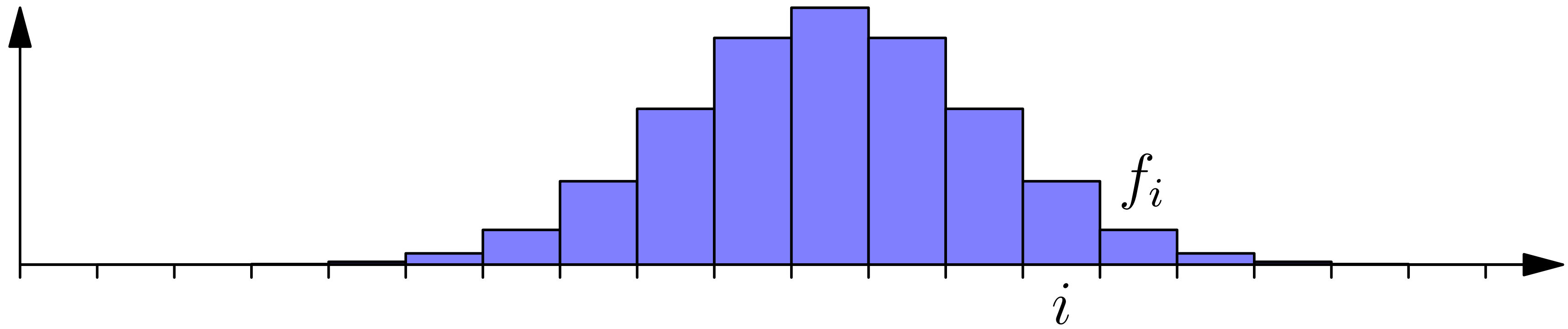
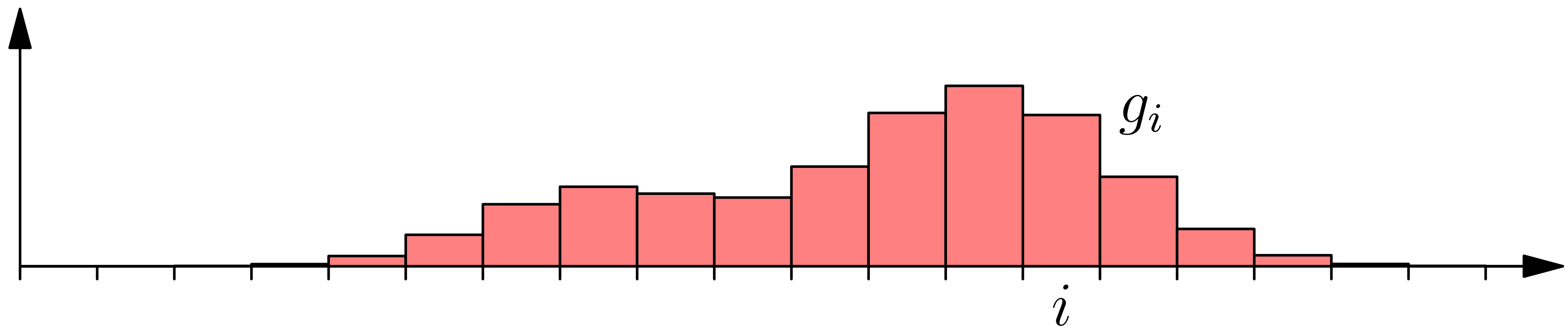
Kullback-Leibler Divergence





$$\mathrm{KL}(\boldsymbol{f}\|\boldsymbol{g}) = -\sum_{i=1}^{n} f_i \log\left(\frac{g_i}{f_i}\right) = 2.21$$

Proof of Gibbs' Inequality

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$$\mathrm{KL}(f\|g) = -\sum_i f_i \log\left(\frac{g_i}{f_i}\right) = -\mathbb{E}_f\left[\log\left(\frac{g_i}{f_i}\right)\right]$$

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- We will meet KL-divergences later on

Lessons

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- A lot of ML algorithms involve convex functions
- As such they will have a unique minimum (or a convex set of minima)
- Convexity is an elegant idea which is relatively easy to prove theorems about
- One of the most useful tools is Jensen's inequality

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