Advanced Machine Learning

Vector Spaces

$$\mathbf{Mx=b}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}$$

$$\mathbf{b=M^{-1}x}$$

$$\nabla_X \mathrm{Tr}(X^{-1}A) = -X^{-1}AX^{-1}$$

$$\mathbf{Mv}_i = \lambda_i \mathbf{v}_i$$

Vectors, metric spaces, norms

Outline

1. **Vector Spaces**
2. Metrics (distances)
3. Norms

$\mathbf{Mx=b}$

$\mathbf{Mv}_i = \lambda_i \mathbf{v}_i$

$\mathbf{b=M}$

Matrices, Vectors and All That

- The language of machine learning is mathematics
- Sometimes we draw pretty pictures to explain the mathematics
- Much of the mathematics we will use involves vectors, matrices and functions
- You need to master the language of mathematics, otherwise you won't understand the algorithms
- I'm going to spend this lecture and the next revising the mathematics you need to know

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Scalars (Fields)

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- These are quantities we can add together $(a+b)$ and multiply together $(a \times b)$

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Vectors

- We often work with objects with many components (features)
- To help handle this we will use vector notation

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- We represent vectors by bold symbols
- All our vectors are column vectors by default
- We treat them as $n \times 1$ matrix

- We write row vectors as transposes of column vectors

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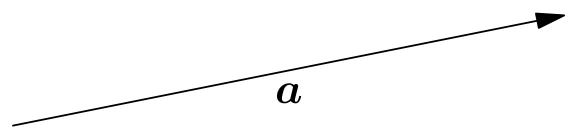
Basic Vector Operations

- The basic vector operations are adding



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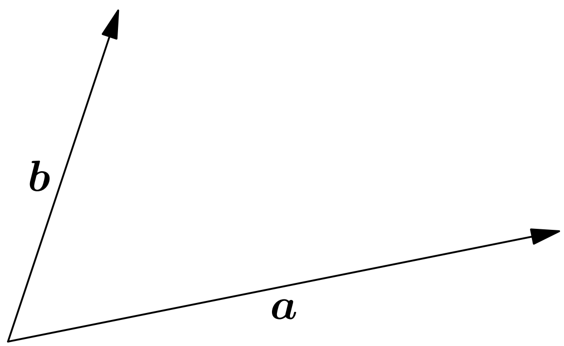

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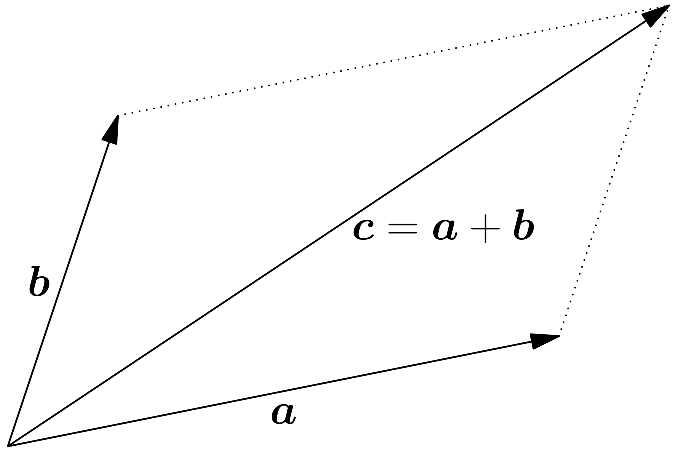

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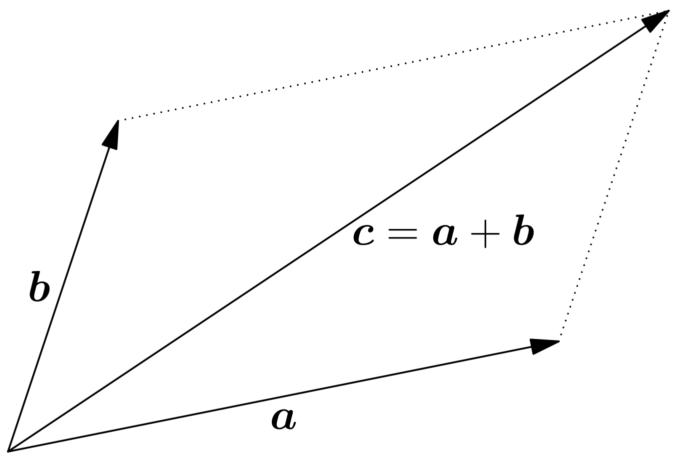

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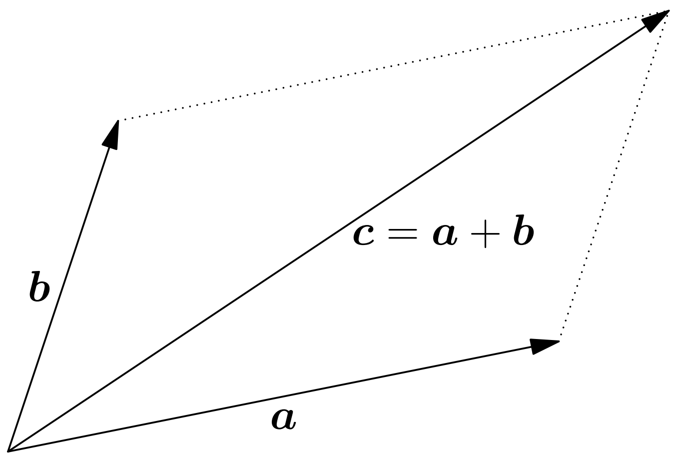

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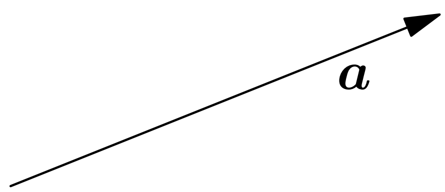

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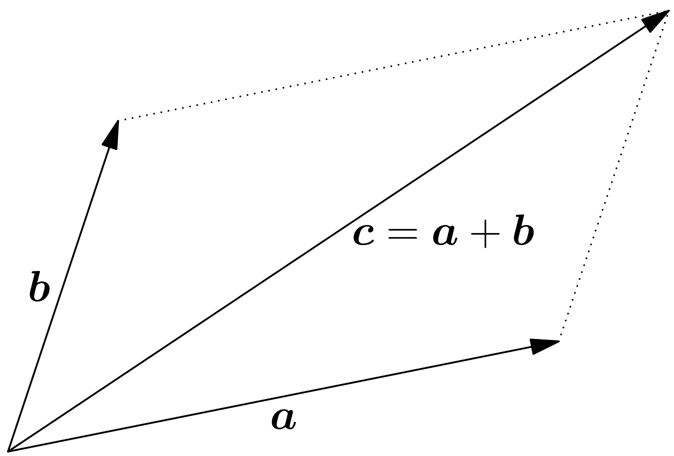

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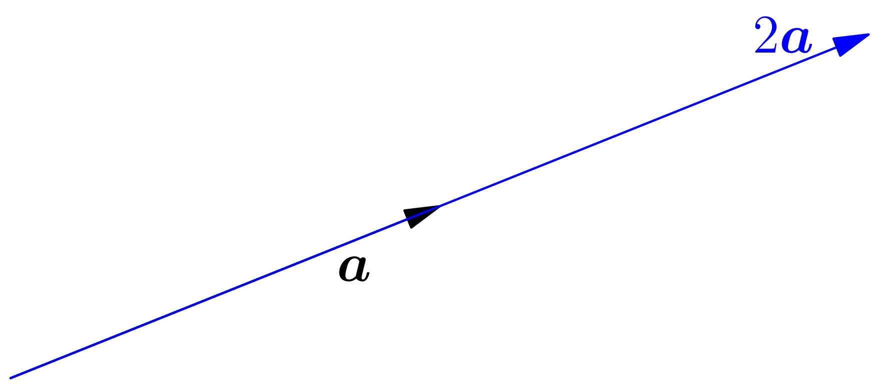

Vector Space

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  1. if $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ then $a\boldsymbol{v} \in \mathcal{V}$ and $\boldsymbol{v} + \boldsymbol{w} \in \mathcal{V}$ (closure)
  2. $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$ (commutativity of addition)
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- Just from these properties we can deduce other properties

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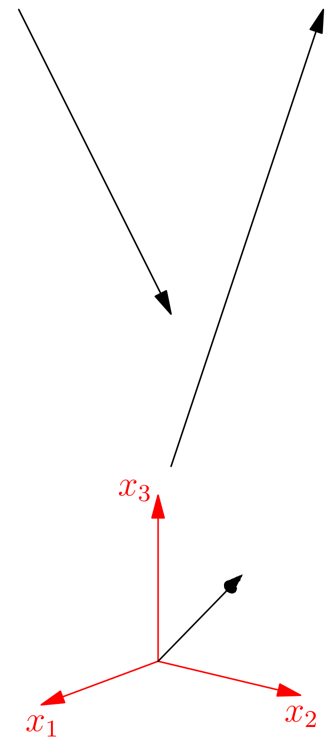
$\mathbb{R}^n$

- When we first learn about vectors we think of them arrows in 3-D space
- If we centre them all at the origin then there is a one-to-one correspondence between vectors and points in space
- We call this vector space $\mathbb{R}^3$
- Any set of quantities $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\mathsf{T}$ which satisfy the axioms above form a vector space $\mathbb{R}^n$
- Of course, we can't so easily draw pictures of high-dimensional vectors



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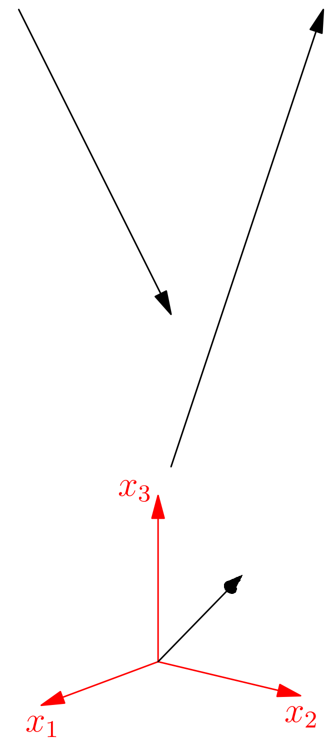

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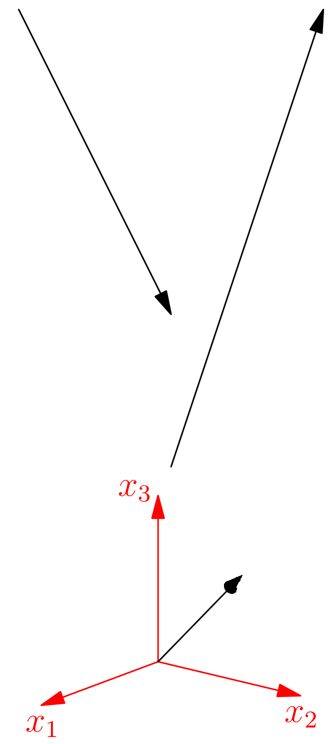

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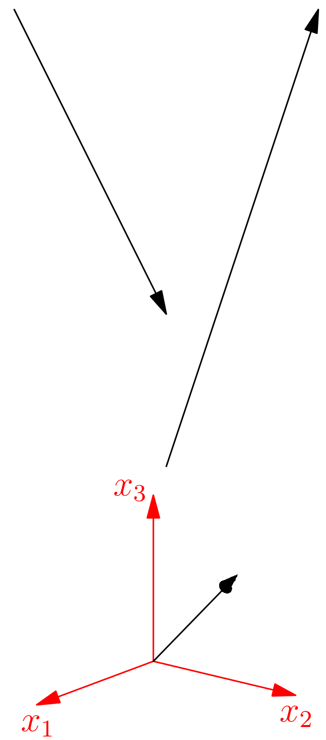

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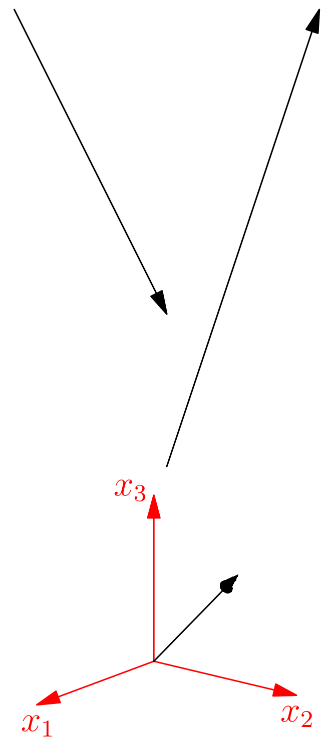

Other Vector Spaces

- Any set of object that satisfies the axioms of a vector spacer are *vectors*

- Matrices satisfy all the conditions of a vector space

- Infinite sequences form a vector space

- Functions form a vector space
  - ⋆ Let $C(a,b)$ be the set of functions defined on the interval $[a,b]$
  - ⋆ Note that if $f(x), g(x) \in C(a,b)$ then $a\, f(x) \in C(a,b)$ and $f(x) + g(x) \in C(a,b)$

- Bounded vectors in $\mathbb{R}^n$ **don't** form a vector space

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  - ⋆ Note that if $f(x),g(x) \in C(a,b)$ then $a\,f(x) \in C(a,b)$ and $f(x) + g(x) \in C(a,b)$
- Bounded vectors in $\mathbb{R}^n$ **don't** form a vector space

Other Vector Spaces

- Any set of object that satisfies the axioms of a vector spacer are $vectors$—not just $\boldsymbol{v} \in \mathbb{R}^n$
- Matrices satisfy all the conditions of a vector space
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Outline

1. Vector Spaces
2. **Metrics (distances)**
3. Norms

$\mathbf{Mx=b}$

$\mathbf{Mv}_i = \lambda_i \mathbf{v}_i$

$\mathbf{b=M}$

Metrics

- Vector spaces become more interesting if we have a notion of distance

- We say $d(\boldsymbol{x},\boldsymbol{y})$ is a **proper distance** or **metric** if

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ (non-negativity)
2. $d(\boldsymbol{x},\boldsymbol{y}) = 0$ iff $\boldsymbol{x} = \boldsymbol{y}$ (identity of indiscernibles)
3. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$ (symmetry)
4. $d(\boldsymbol{x},\boldsymbol{y}) \leq d(\boldsymbol{x},\boldsymbol{z}) + d(\boldsymbol{z},\boldsymbol{y})$ (triangular inequality)

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- We are often interested in functions that behave nicely

- E.g. They are continuous

Lipschitz Function

- One way to characterise well behaved function, $f(x)$ is if there exists a number $K < \infty$ such that for all x and y

$$d(f(x), f(y)) \leq K\, d(x, y)$$

- This is known as a **Lipschitz condition** and the function is said to be K-Lipschitz
- Note that such functions cannot have any jumps (i.e. they are continuous)
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Contractive Mappings

- An interesting class of function are those for which $K < 1$
- These are said to be contractive mappings
- A famous theorem that applies to contractive mappings is the Banach fixed-point theorem which says there exists a unique fixed point such that $f(x) = x$
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3. **Norms**

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Norms

- Vector spaces are even more interesting with a notion of length
- **Norms** provide some measure of the size of a vector
- To formalise this we define the **norm** of an object $\boldsymbol{v}$ as $\|\boldsymbol{v}\|$ satisfying

  1. $\|\boldsymbol{v}\| > 0$ if $\boldsymbol{v} \neq \mathbf{0}$ (non-negativity)
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Vector Norms

- The familiar vector norm is the (Euclidean) two norm

$$\|\boldsymbol{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- Other norms exist, such as the p-norm $(p \geq 1)$

$$\|\boldsymbol{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

- Special cases include the 1-norm and the infinite norm

$$\|\boldsymbol{v}\|_1 = \sum_{i=1}^{n} |v_i| \qquad \|\boldsymbol{v}\|_\infty = \max_i |v_i|$$

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Matrix Norms

- We can define norms for other objects
- The norm of a matrix encodes how large the mapping is
- The Frobenius norm is defined by

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|A_{ij}|^2}$$

- Many other norms exist including 1-norm, max-norm, etc.
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Compatible Norms

- A vector and matrix norm are said to be compatible if

$$\|\mathbf{M}\boldsymbol{v}\|_b \leq \|\mathbf{M}\|_a \times \|\boldsymbol{v}\|_b$$

  (Spectral and Euclidean norms are compatible)

- Norms provide quick ways to bound the maximum growth of a vector under a mapping induced by the matrix

- We will see that a measure of the sensitivity of a mapping is in terms of the ratio of its maximum effect to its minimum effect on a vector

- This is known as the **conditioning**, given by $\|\mathbf{M}\| \times \|\mathbf{M}^{-1}\|$

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Why Should You Care?

- Deep learning involves multiply the input (which we can think of as a vector $\boldsymbol{x}$) by many layers
- In CNNs we have convolutional layers and dense layers
- The effect of applying these layers can be represented by a matrix multiplication $\boldsymbol{x}_n = \mathbf{L}_n \boldsymbol{x}_{n-1}$
- We also do other things like applying ReLU's or pooling that changes the magnitude, $\boldsymbol{x}_n$, of our representation
- If you are developing new architectures you want $\|\boldsymbol{x}_n\|$ neither to blow up or vanish
- This can be controlled by carefully choosing $\|\mathbf{L}_n\|$

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- Deep learning involves multiply the input (which we can think of as a vector $\boldsymbol{x}$) by many layers
- In CNNs we have convolutional layers and dense layers
- The effect of applying these layers can be represented by a matrix multiplication $\boldsymbol{x}_n = \mathbf{L}_n \boldsymbol{x}_{n-1}$
- We also do other things like applying ReLU's or pooling that changes the magnitude, $\boldsymbol{x}_n$, of our representation
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Function Norms

- Functions can also have norms, for example, if $f(x)$ is defined in some interval $\mathcal{I}$

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- Vector spaces with a distance (metric spaces) and vector spaces with a norm (normed vector spaces) are interesting objects
- They allow you to define a topology (open/closed sets, etc.)
- You can build up ideas about connectedness, continuity, contractive maps, fixed-point theorems, . . .
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