

$$z = e^{-(x+1)^2 - (y-1)^2} + 0.6e^{-(x-1)^2 - 0.5(y+1)^2 + 0.1(x-3)(y-3)}$$

Gradient descent, quadratic minima, differing length scales

## ML = Optimisation

- Many learning machines can be thought of as functions of the form  

$$\hat{y} = f(\mathbf{x}|\mathbf{w})$$
 (or more generally  $\hat{y} = f(\mathbf{x}|\mathbf{w})$ )
- Given an input pattern (set of features)  $\mathbf{x}$  the learning machine makes a prediction  $\hat{y}$
- We try to choose the parameters  $\mathbf{w}$  so that the predictions are good
- In practice training a learning machine comes down to optimising some loss function

## Training

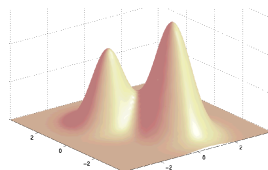
- Given a (labelled) training dataset  

$$\mathcal{D} = \{(\mathbf{x}_k, y_k) | k = 1, \dots, m\}$$
- We define an error or loss function that we want to minimise  

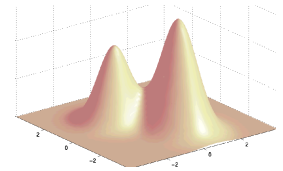
$$L(\mathbf{w}|\mathcal{D}) = \frac{1}{m} \sum_{k=1}^m (f(\mathbf{x}_k|\mathbf{w}) - y_k)^2$$
- We then use the machine with the weights  $\mathbf{w}^*$  which minimise  $L(\mathbf{w}|\mathcal{D})$

## Outline

- Motivation
- Gradient Descent
- Why Gradient Descent is Difficult

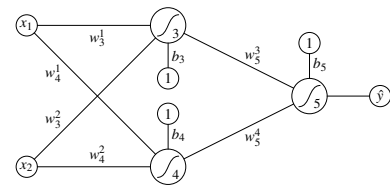


- Motivation
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## MLP

- We can depict a neural network such as an MLP by a diagram

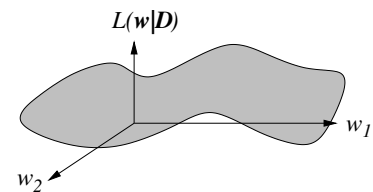


- Stands for the function  $(\hat{y} = f(\mathbf{x}|\mathbf{w}))$   

$$\hat{y} = g(w_5^3 g(w_3^1 x_1 + w_3^2 x_2 + b_3) + w_5^4 g(w_4^1 x_1 + w_4^2 x_2 + b_4) + b_5)$$
 where, for example,  $g(V) = \frac{1}{1+e^{-V}}$

## Computing Gradients

- $L(\mathbf{w}|\mathcal{D})$  is a complex function of the weights  $\mathbf{w}$

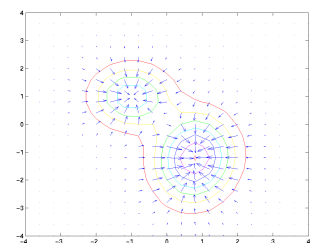
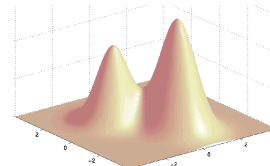


- To minimise we  $L(\mathbf{w}|\mathcal{D})$  we compute the gradient  $\nabla L(\mathbf{w}|\mathcal{D})$
- For neural networks an efficient algorithm for computing the gradient is known as back-propagation of errors (*back-prop*)

## Gradient Optimisation

- A maximum or minimum occurs when  $\nabla L(\mathbf{w}|\mathcal{D}) = \mathbf{0}$
- E.g.  

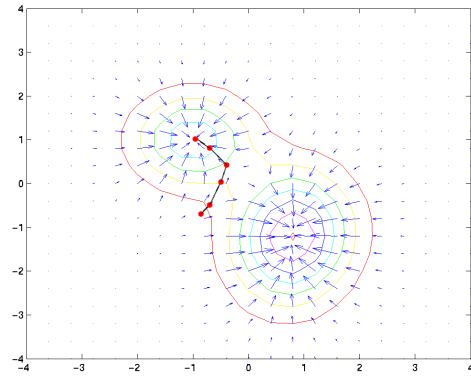
$$L = e^{-(x+1)^2 - (y-1)^2} + 0.6e^{-(x-1)^2 - 0.5(y+1)^2 + 0.1(x-3)(y-3)}$$



## Gradient Descent

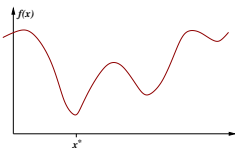
- For a simple function  $L(w|\mathcal{D})$  we can solve  $\nabla L(w|\mathcal{D}) = 0$  explicitly! E.g. the linear perceptron!
- For a non-linear functions we usually can't solve this set of simultaneous equations!
- We can find a maximum or minimum **iteratively**!
- If we know the gradient then we can follow the gradient!
  - Maximisation:  $w \rightarrow w' = w + r \nabla L(w|\mathcal{D})$
  - Minimisation:  $w \rightarrow w' = w - r \nabla L(w|\mathcal{D})$

## Hill-Climbing



## What Goes Right

- Almost all minima are quadratic! (Morse's theorem)



- Taylor expanding around a minimum  $x^*$ 

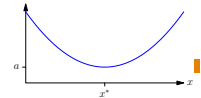
$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + \dots$$

$$= f(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + \frac{1}{3!}(x - x^*)^3 f'''(x^*) + \dots$$
- If  $x - x^*$  is sufficiently small the higher order terms are negligible!

## Newton's Method

- If we were in a quadratic minimum

$$f(x) = a + \frac{b}{2}(x - x^*)^2$$



- then

$$f'(x) = b(x - x^*), \quad f''(x) = b$$

- so

$$x - x^* = \frac{f'(x)}{b} = \frac{f'(x)}{f''(x)}$$

- or

$$x^* = x - \frac{f'(x)}{f''(x)}$$

## Newton's Method

- This is Newton's method!
- For non-quadratic functions Newton's method converges **quadratically** provided we are sufficiently close to a minimum!
- If we are at a distance  $x - x^* = \epsilon$  from the minima then after one cycle we will be at a distance  $\epsilon^2$ ! after two cycles we will be at a distance  $\epsilon^4$ , etc.!
- If we are too far from a minimum we might go anywhere!
- We should follow the gradient until we are near the minimum!

## Taylor's Expansion in High Dimensions

- We can generalise these results to many dimensions!
- The Taylor expansion of a function  $f(x)$  about  $x_0$

$$f(x) = f(x_0) + (x - x_0)^T \nabla f(x_0) + \frac{1}{2}(x - x_0)^T \mathbf{H}(x - x_0) + \dots$$

where  $\mathbf{H}$  is the **Hessian** matrix with elements

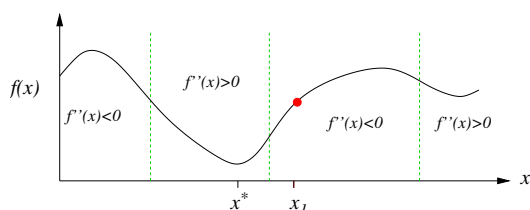
$$H_{ij} = \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j}$$

- Newton's method in high dimension is

$$x^* = x - \mathbf{H}^{-1} \nabla f(x)$$

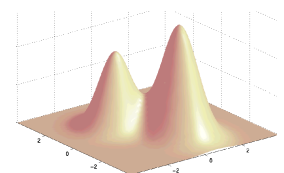
## Using the Second Derivative

- If we are optimising  $N$  parameters the Hessian is an  $N \times N$  matrix!
- It is time-consuming to compute! (and prone to errors when coding)!—for deep learning it is impossible even to store the Hessian!
- Away from minima they can be misleading



## Outline

- Motivation
- Gradient Descent
- Why Gradient Descent is Difficult**



## Step Size

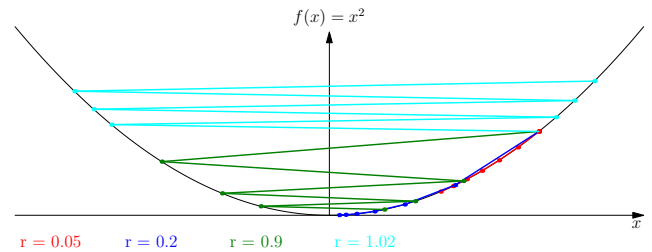
- Gradient descent

$$x' = x - r \nabla f(x)$$

- Need to choose the learning rate of step size,  $r$
- Too small steps takes lots of time
- Too large steps takes you away from a minimum

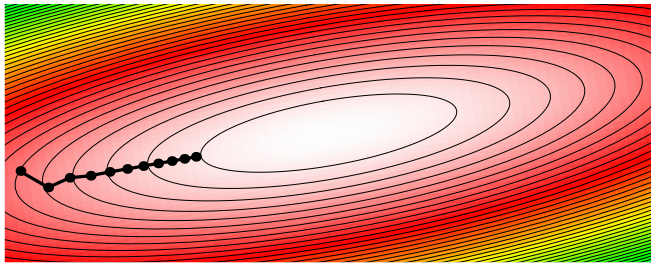
## Step Size

$$x \leftarrow x - r f'(x)$$



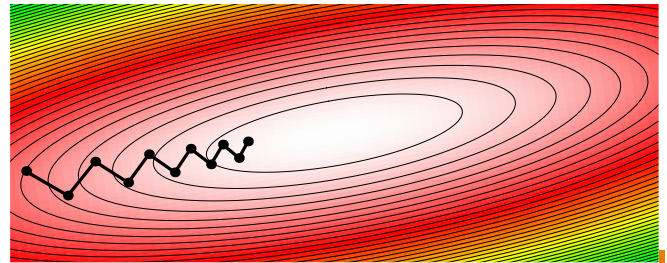
## Higher Dimensions

- In higher dimensions the problem is that there are some directions you need to move a long way



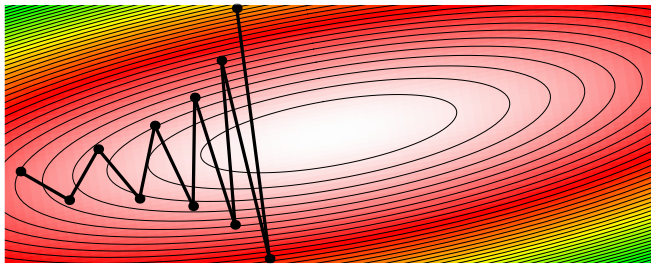
## Getting There Quicker

- Increasing the step size speeds up convergence, but the direction of steepest descent doesn't point to the minimum



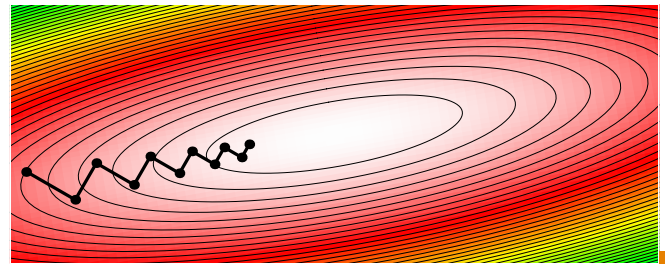
## More Haste Less Speed

- Increasing the step size, just a little further, increases the rate of converge in one direction, but ...



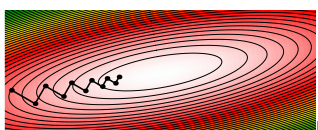
## Line Minimisation

- We can systematically seek the minimum along a line of the gradient



## Zig-Zag

- Note that in high dimensions gradient descent tends to zigzag



- If we computed the Hessian and used Newton's method we would jump straight to the minimum if we were in a quadratic potential
- However computing the Hessian is time consuming and misleading if we are not in a quadratic potential (i.e. far from the optimum)

## Better Optimisation Algorithms

- Good optimisation algorithms often compute an approximation of the Hessian
- E.g. Conjugate gradient
  - ★ Performs Line Minimisation
  - ★ Uses gradient, but does not go along it
  - ★ For a quadratic minimum in  $d$  dimensions it reaches the minimum in  $d$  steps
- E.g. Levenberg-Marquardt
  - ★ Used on least squares problem only
  - ★ Uses linear approximation of function to approximate Hessian
  - ★ Adapts from hill-climbing to Newton method
  - ★ Avoids line-minimisation

## Levenberg-Marquardt

- Want to minimise  $\|\epsilon(\mathbf{w})\|^2$  where  $\epsilon_i(\mathbf{w}) = f(\mathbf{x}_i|\mathbf{w}) - y_i$
- Use linear approximation

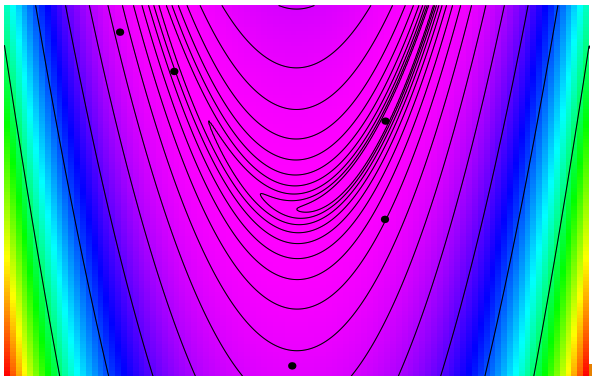
$$\epsilon_i(\mathbf{w}) \approx \epsilon_i(\mathbf{w}^{(k)}) + (\mathbf{w} - \mathbf{w}^{(k)})^\top \nabla \epsilon_i(\mathbf{w}^{(k)})$$

$$\text{with } \nabla \epsilon_i(\mathbf{w}^{(k)}) = \nabla f(\mathbf{x}_i|\mathbf{w}^{(k)})$$

- Solve quadratic minimisation of approximate error  $\arg\min_{\mathbf{w}} L_{approx}(\mathbf{w})$  with  $\mathbf{J} = \nabla \epsilon(\mathbf{w}^{(k)})$

$$\begin{aligned} L_{approx}(\mathbf{w}) &= \|\epsilon(\mathbf{w}^{(k)}) + \mathbf{J}(\mathbf{w} - \mathbf{w}^{(k)})\|^2 \\ &= \epsilon(\mathbf{w}^{(k)})^\top \epsilon(\mathbf{w}^{(k)}) + 2(\mathbf{w} - \mathbf{w}^{(k)})^\top \mathbf{J}^\top \epsilon(\mathbf{w}^{(k)}) \\ &\quad + (\mathbf{w} - \mathbf{w}^{(k)})^\top \mathbf{J}^\top \mathbf{J} (\mathbf{w} - \mathbf{w}^{(k)}) \end{aligned}$$

$$\epsilon_1 = 10(x_2 - x_1^2) \text{ and } \epsilon_2 = 1 - x_1$$



## Trust Region

- Solution given by  $\nabla_{\mathbf{w}} L_{approx}(\mathbf{w}) = 0$  gives

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - (\mathbf{J}^\top \mathbf{J})^{-1} \mathbf{J}^\top \epsilon(\mathbf{w}^{(k)})$$

- Can lead us in the wrong direction
- Instead use  $\mathbf{w}^{(k+1)} = \arg\min_{\mathbf{w}} L_{approx}(\mathbf{w}) + \nu \|\mathbf{w} - \mathbf{w}^{(k)}\|^2$

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - (\mathbf{J}^\top \mathbf{J} + \nu \mathbf{I})^{-1} \mathbf{J}^\top \epsilon(\mathbf{w}^{(k)})$$

- $\nu$  limits the step size
- If predicted reduction in error is accurate then reduce  $\nu$ , else if predicted reduction in error is very poor increase  $\nu$

## Summary

- There are some **non-gradient methods** (Nelder Mead, evolutionary strategies, Powell's method), but in very high dimensions these are not very competitive
- There are **gradient methods** (first order methods) that suffer from the problem of having to choose a single step size with conflicting requirements in different directions
- **Newton's method** (a second order method) requires computing the Hessian matrix, gives very fast convergent, but can take you in the wrong direction if you are not sufficiently close to a minimum
- There exist a number of **pseudo-Newton methods** (conjugate gradient, Levenberg-Marquardt, etc.) that approximates Newton's method often without explicitly computing the Hessian