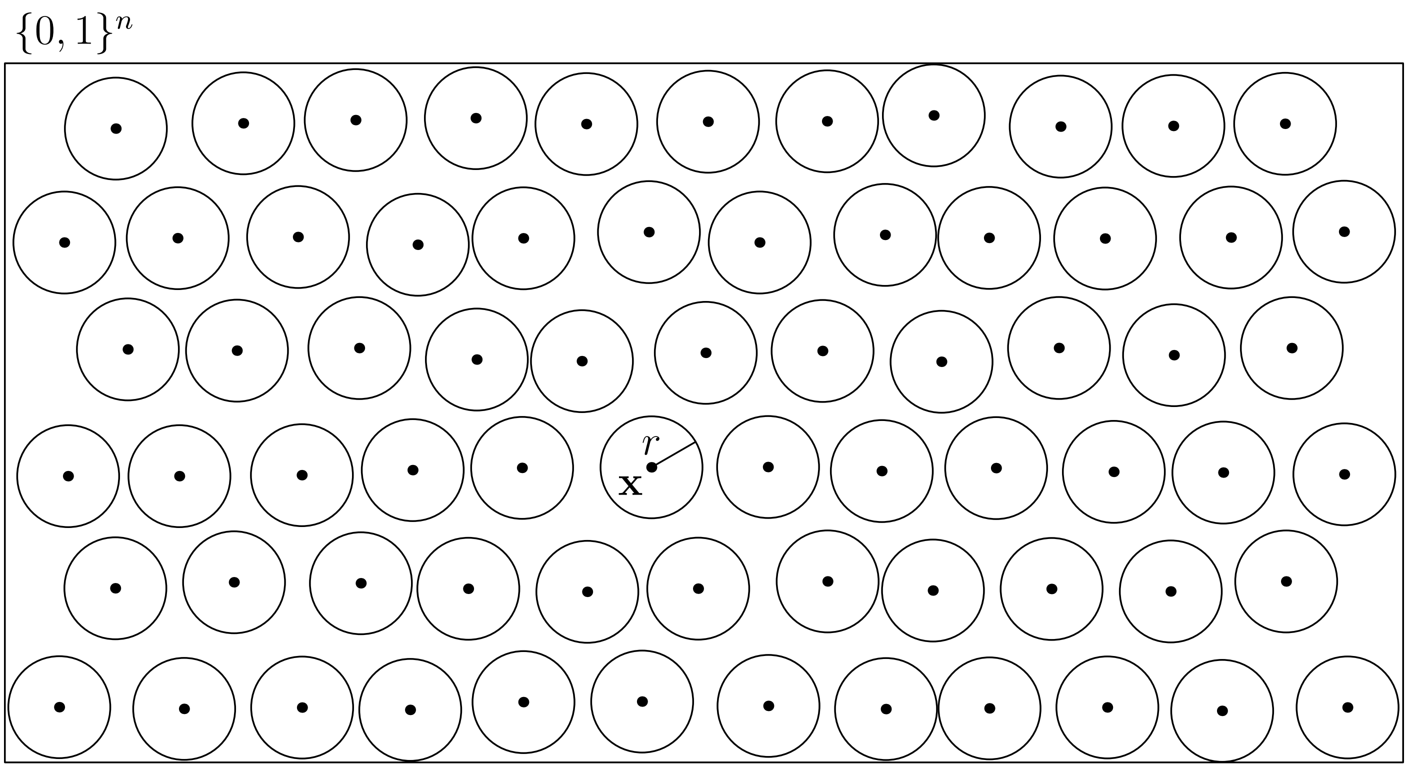
Advanced Machine Learning

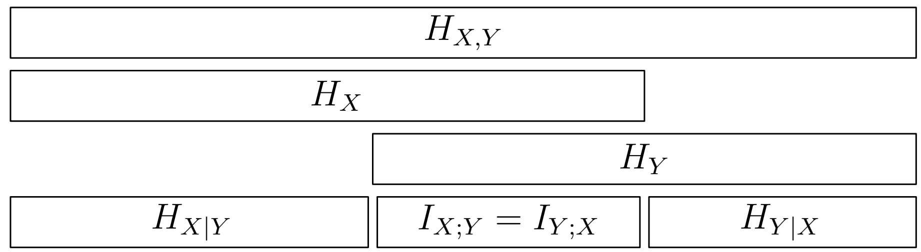
Information Theory



Information, KL-divergence, Minimum Description Length

Outline

1. **Information Theory**
2. KL-Divergence
3. Minimum Description Length
4. Variational Auto-Encoders



Communicating Via a Noisy Channel

- Information theory considers communicating down a (noisy) channel

$$X \sim \mathbb{P}(X) \xrightarrow{\text{noisy channel}} Y \sim \mathbb{P}(Y \mid X)$$

- We send a message X (with probability $\mathbb{P}(X)$) and receive a message Y with probability $\mathbb{P}(Y \mid X)$

- The uncertainty of the message sent, given we received a message y is

$$H_{X|Y=y} = -\sum_{x \in \mathcal{X}} \mathbb{P}(X = x \mid Y = y) \log(\mathbb{P}(X = x \mid Y = y))$$

- The expected uncertainty in the message sent is

$$H_{X|Y} = \sum_{y \in \mathcal{Y}} \mathbb{P}(Y = y)\, H_{X|Y=y} = -\sum_{x,y} \mathbb{P}(X = x, Y = y) \log(\mathbb{P}(X = x \mid Y = y))$$

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Joint Entropy

- We can define the **joint entropy**

$$H_{X,Y} = -\sum_{x,y} P_{X,Y}(x,y) \log(P_{X,Y}(x,y))$$

- If the message we receive is independent of the message that is sent then $H_{X,Y} = H_X + H_Y$ (we saw this in the last lecture)

- This is different when X and Y are correlated

- Since $\mathbb{P}(X,Y) = \mathbb{P}(Y|X)\,\mathbb{P}(X) = \mathbb{P}(X|Y)\,\mathbb{P}(Y)$ if follows

$$H_{X,Y} = H_X + H_{Y|X} = H_Y + H_{X|Y}$$

- Or $H_X - H_{X|Y} = H_Y - H_{Y|X}$

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Mutual Information

- The amount of uncertainty about the message being sent, X, before receiving the message is $H_X = -\mathbb{E}_X[\log \mathbb{P}(X)]$

- Shannon define the *mutual information* to be the expected loss in uncertainty when we receive a message

$$I_{X;Y} = H_X - H_{X|Y}$$

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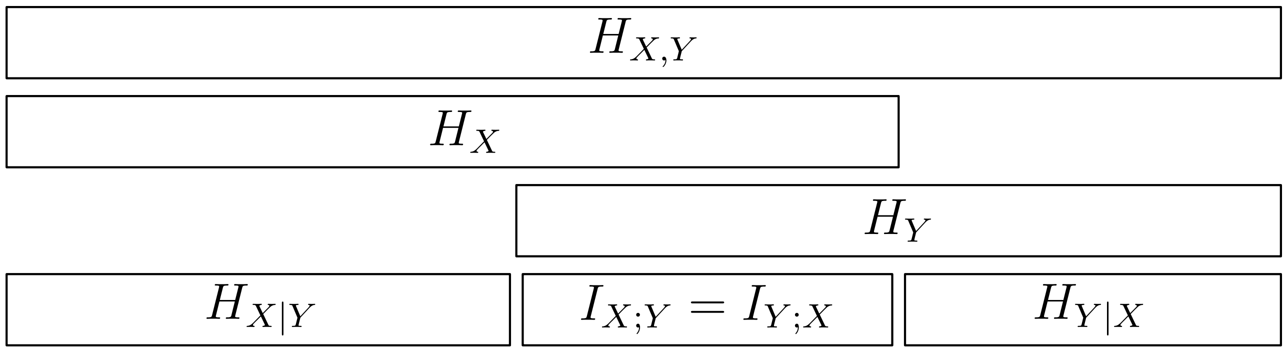
Channel Capacity

- We can summarise these relationships diagrammatically



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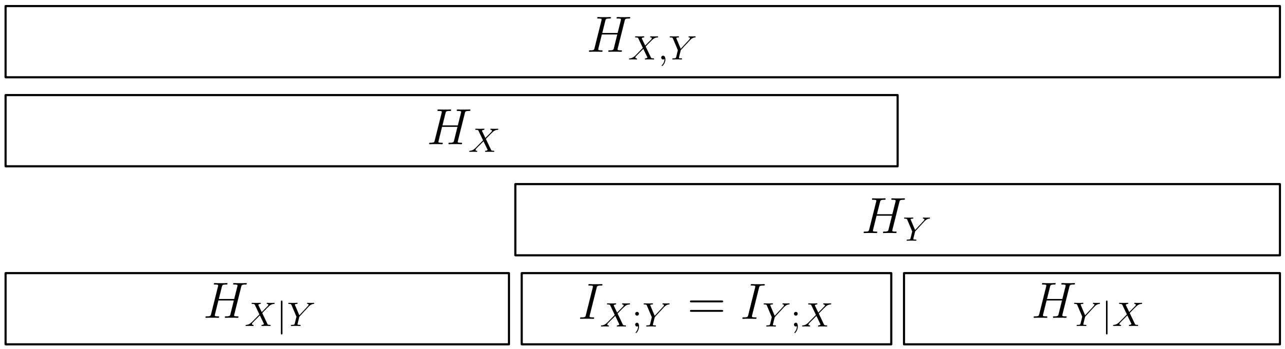

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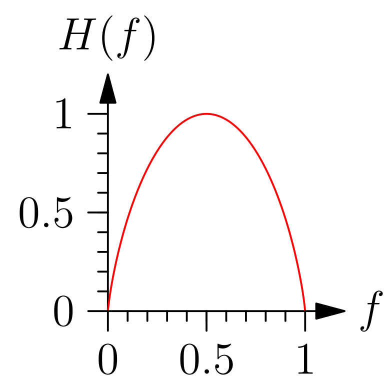
Independent Noise

- The simplest model of a noisy channel is a binary channel where each symbol is corrupted independently with a probability f

$$\mathbb{P}(X = 1|Y = 0) = \mathbb{P}(X = 0|Y = 1) = f$$



- An elementary calculations shows that

$$H_{X_i|Y_i} = -(1-f)\log(1-f) - f\log(f) = H(f)$$

- For a message of length n, $H_{X|Y} = nH(f)$

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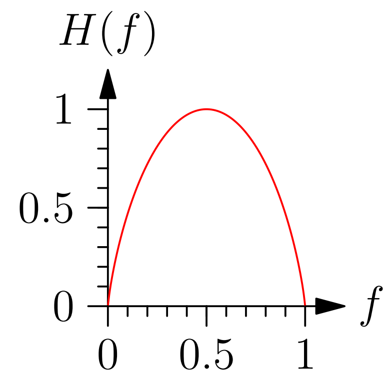

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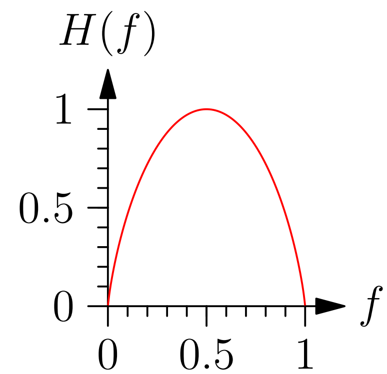

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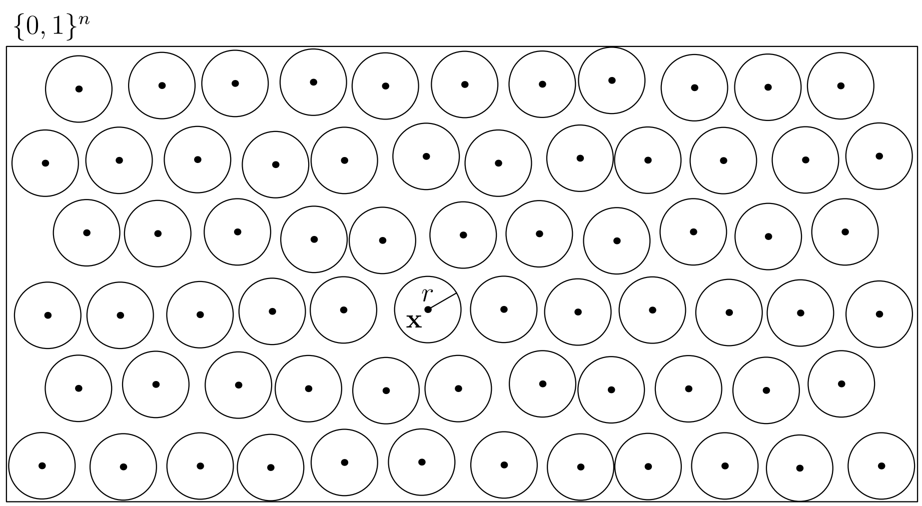
Error Correcting Codes

- To reduce the chance of misinterpreting a message we need to build an error correcting code

- We can do this dividing the space of binary messages into a set of Hamming balls



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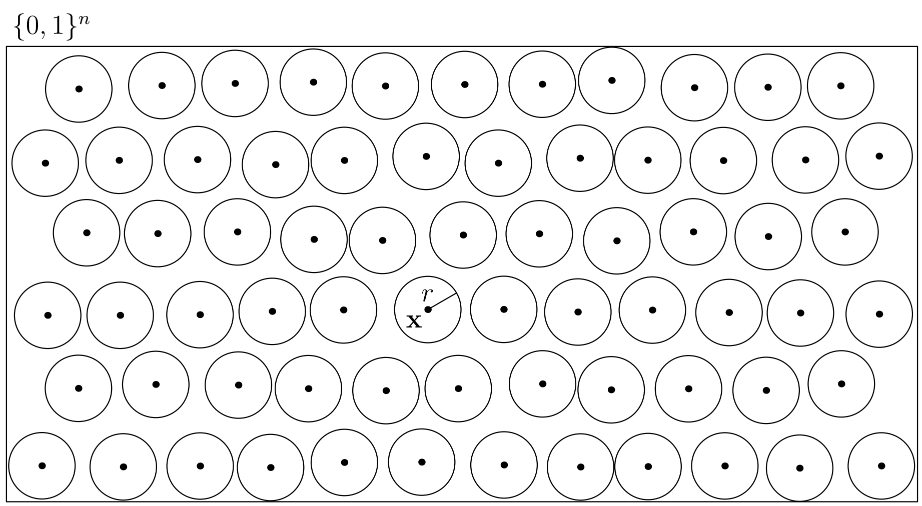

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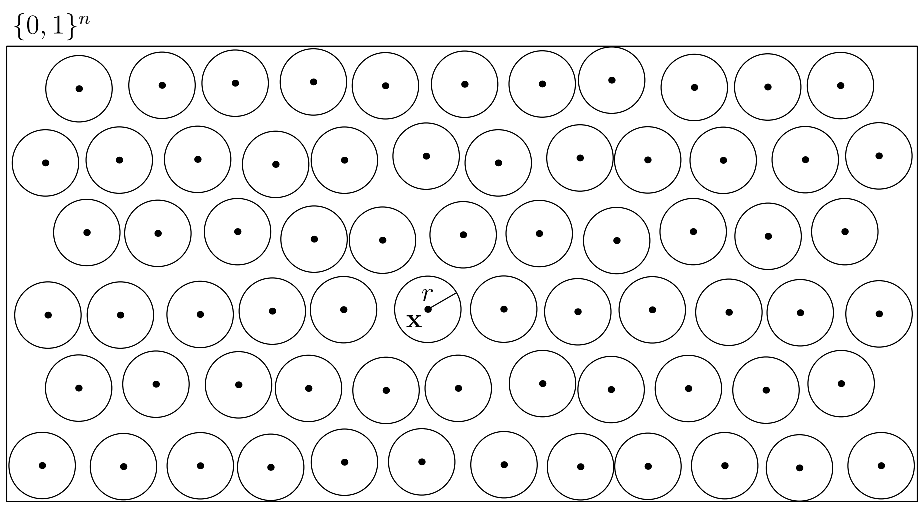

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Volume of Coding Space

- The expected number of errors in a string of length n given an error rate of f is $n\,f$
- For sufficiently large n we would expect all errors are smaller that $(f+\epsilon)n$ (for $\epsilon > 0$)
- If we make the radius of the Hamming ball $r = (f+\epsilon)n$ ($\epsilon > 0$) then we would expect no error for sufficiently large n
- An upper bound on the number of code words we can send in a string of length n is

$$\frac{2^n}{|B(\boldsymbol{x}_i, r)|} = c\sqrt{n}2^{I_{X;Y}}$$

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Lower Bounds

- Shannon also showed that choosing random strings as the centre of the Hamming balls we could fit $2^{I_{X;Y}}$ codes in a string of length n

- This means that we can send information at rate of $I_{X;Y}$

- The maximum rate is given by the channel capacity $\max I_{X;Y}$

- If $f = 0.1$ then $C = I_{X;Y} = 0.469\text{bits}$ so we need codes of just over twice as long to communicate accurately over a noisy channel with a 10% corruption rate

- Unfortunately, we can't efficiently decode random code positions, so although we know Shannon's bound is achievable we don't have practical codes that do this

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  - ⋆ Wikipedia mentions 14
- Suppose we want to align two sets of images through some non-linear transformations
- One way of doing this is to choose the non-linear transformations that maximise the mutual information (or normalised mutual information) between the two sets of images

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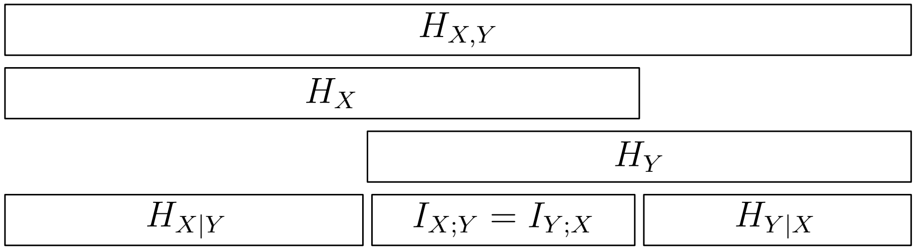

KL-Divergence

- We have met the Kullback-Leibler divergence

$$\begin{aligned}\mathrm{KL}\left(p\|q\right) &= \mathbb{E}_{X\sim p(X)}\left[\log\left(\frac{p(X)}{q(X)}\right)\right]\\ &= -\mathbb{E}_{X\sim p(X)}[\log(q(X))] - H_X\end{aligned}$$

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- Thus $-\mathbb{E}_{X\sim p(X)}[\log(q(X))]$ is the expected length of message needed to code $X \sim p(X)$ using the optimal code for the distribution $q(X)$ that than $p(X)$

- $D_{KL}(p,q)$ is also known as the **relative entropy** and measures the expected extra length in coding $X \sim p(X)$ if we use the wrong distribution $q(X)$

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Use of KL-Divergence

- KL-Divergences are regularly used in machine learning
- Often we have a parameterised distribution $q(X|\boldsymbol{\theta})$ and we have some complex distribution $p(X)$
- We then use

$$\boldsymbol{\theta} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \, \mathrm{KL}\big(p \big\| q(\cdot|\boldsymbol{\theta})\big)$$

  to approximate $p(X)$ with a simpler distribution
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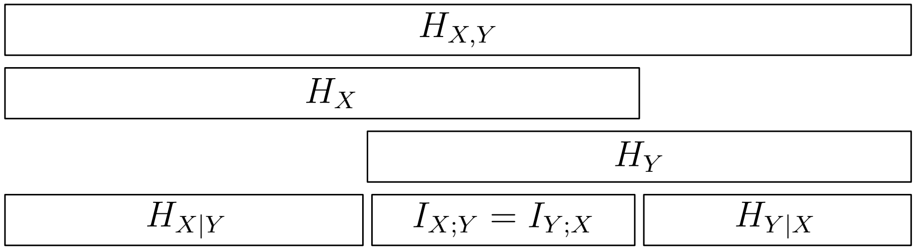

Compression and Model Selection

- Outside of Bayesian framework it is difficult to do model selection
- When is it better to accept a more complex model for a better fit and when are we just over-fitting?
- One principled approach is to use the model that allows us to maximally compress the data
- If we are compressing the data then we are capturing features of the data

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Alice and Bob

- Suppose Alice has data $\mathcal{D} = \{(\boldsymbol{x}_i, y_i) \mid i = 1, 2, \ldots, m\}$ while Bob has only the feature vectors $\{\boldsymbol{x}_i \mid i = 1, 2, \ldots, m\}$
- Alice wants to communicate y_i to Bob as efficiently as possible
- We suppose Alice & Bob have available a model $y = \hat{f}(\boldsymbol{x}|\boldsymbol{\theta})$
- Rather than sending the complete list $\{y_i \mid i = 1, 2, \ldots, m\}$ Alice can send bob the parameter $\boldsymbol{\theta}$ and the errors

$$\delta_i = y_i - \hat{f}(\boldsymbol{x}_i|\boldsymbol{\theta})$$

- Assuming the δ_i's have a distribution p_δ then the cost of communicating an error to accuracy Δ is $-\log(p_\delta(\delta_i) \times \Delta)$

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Description Length

- The **description length** for $\{y_i \mid i = 1{,}2{,}\ldots{,}m\}$ is then the cost of transmitting $\boldsymbol{\theta}$ plus the cost of transmitting the errors

$$L = \sum_{k=1}^{n} \ell(\theta_k) - \sum_{i=1}^{m} \log\Big(p_\delta\Big(y_i - \hat{f}(\boldsymbol{x}_i|\boldsymbol{\theta})\Big)\Big) - \log(\Delta)$$

  where $\ell(\theta_i)$ is the number of bits need to communicate θ_k (we get to choose the accuracy if is worth encoding the parameters)

- To select between models we choose the model with the **minimum description length**

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Minimum Description Length (MDL) Method

- The minimum description length method can be a powerful way of choosing between models
- Often it is the only principled method available
- It allows you to trade model accuracy against model complexity
- It can be fiddly as we need to determine the accuracy to which we should store the parameters of our model
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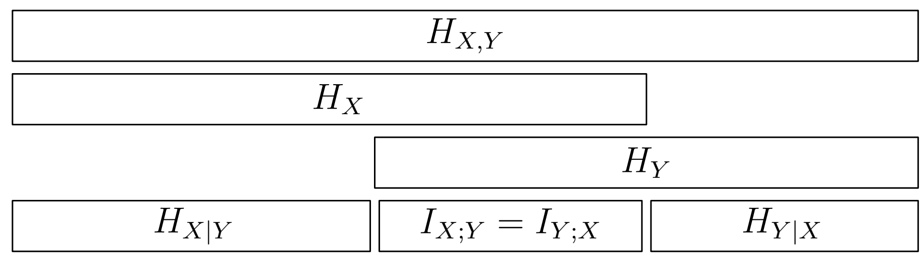
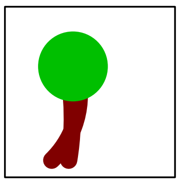
Outline

1. Information Theory
2. KL-Divergence
3. Minimum Description Length
4. **Variational Auto-Encoders**



Variational Auto-Encoders VAE

$\boldsymbol{x} \sim \mathcal{D}$



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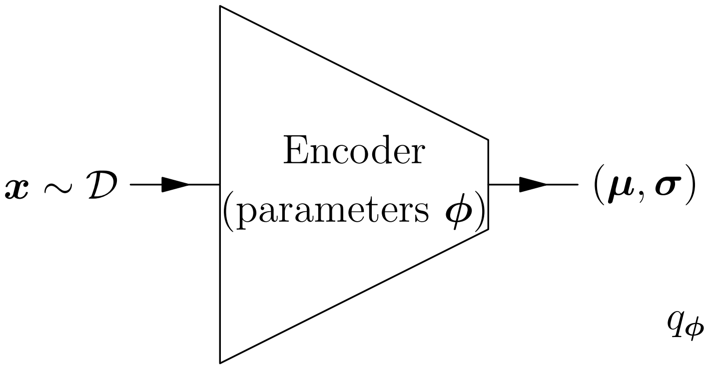
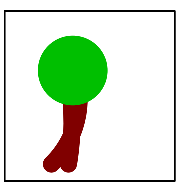
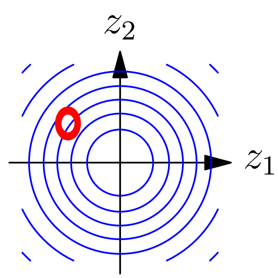


$$q_{\boldsymbol{\phi}}(\boldsymbol{z}|\boldsymbol{x}) = \mathcal{N}(\boldsymbol{z}|\boldsymbol{\mu}, \mathrm{diag}(\boldsymbol{\sigma}^2))$$





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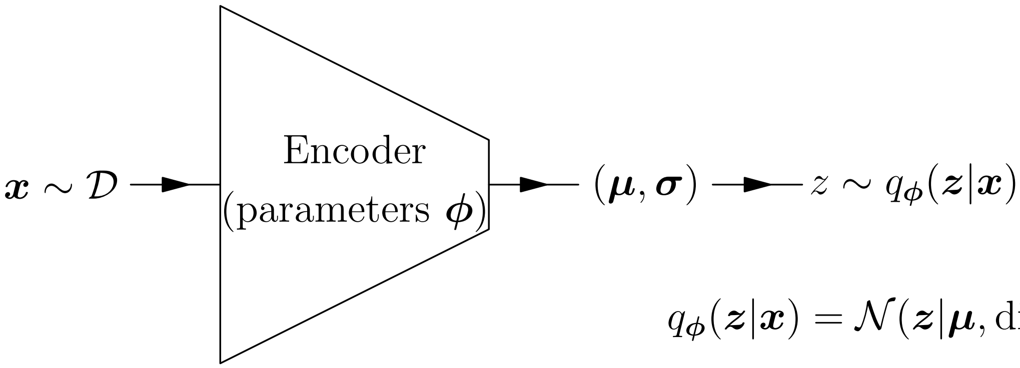

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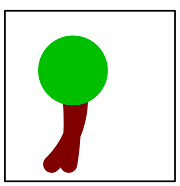

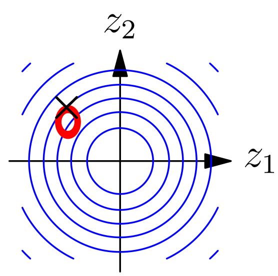

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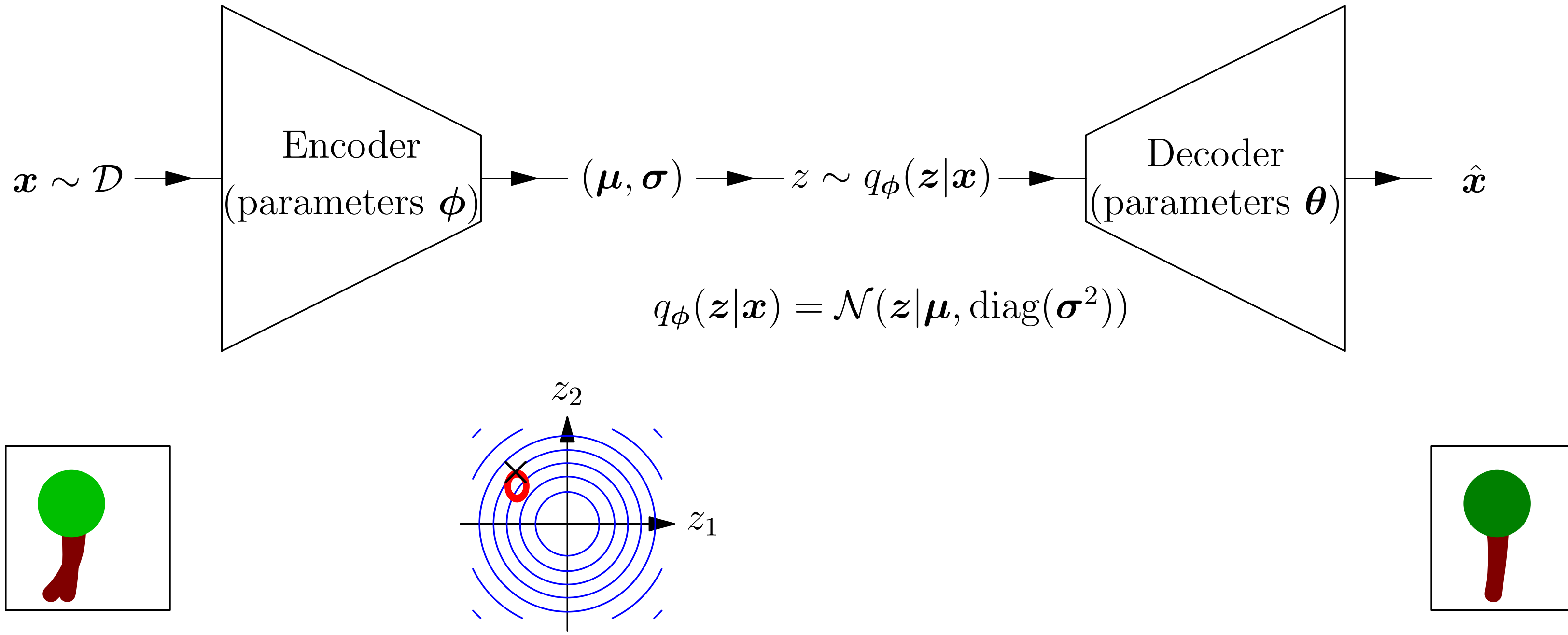

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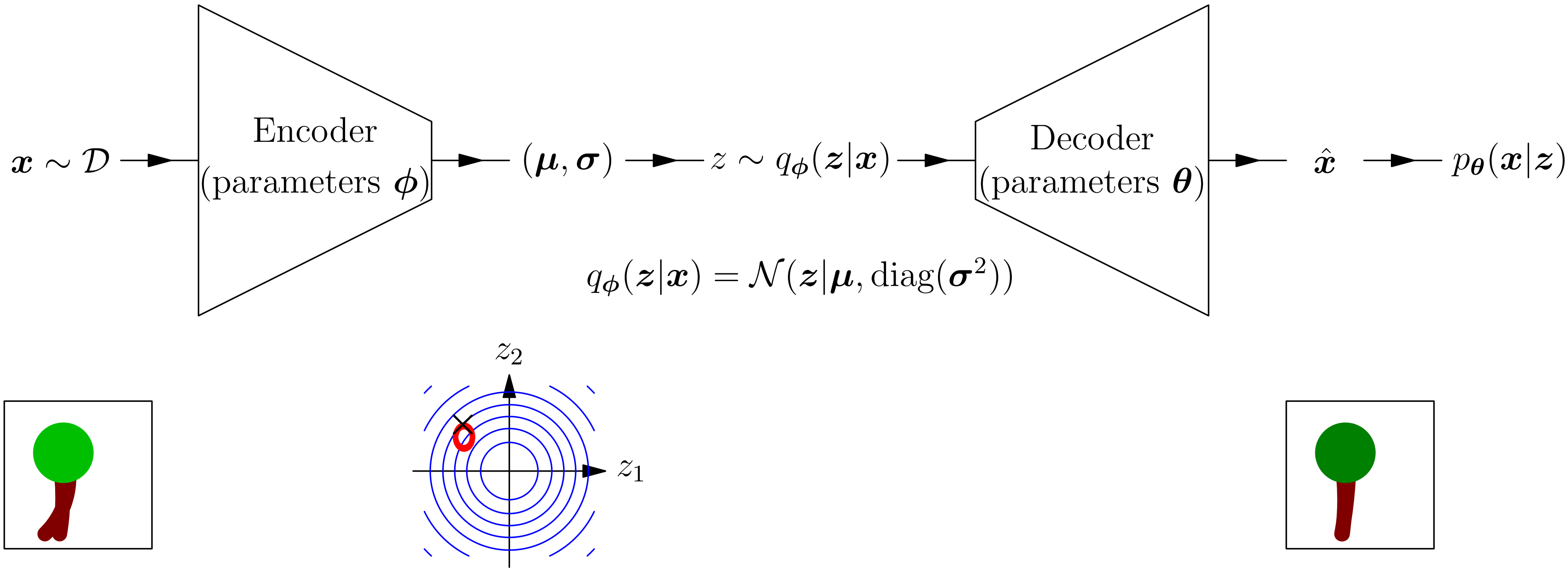

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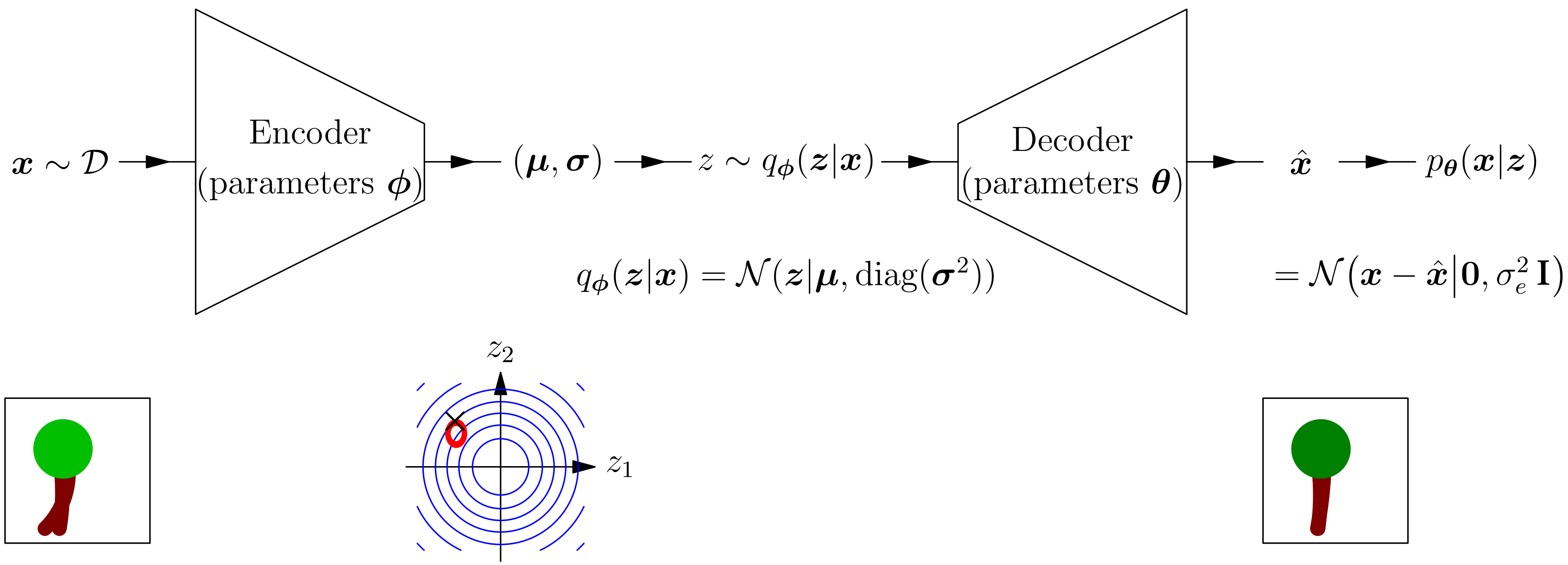

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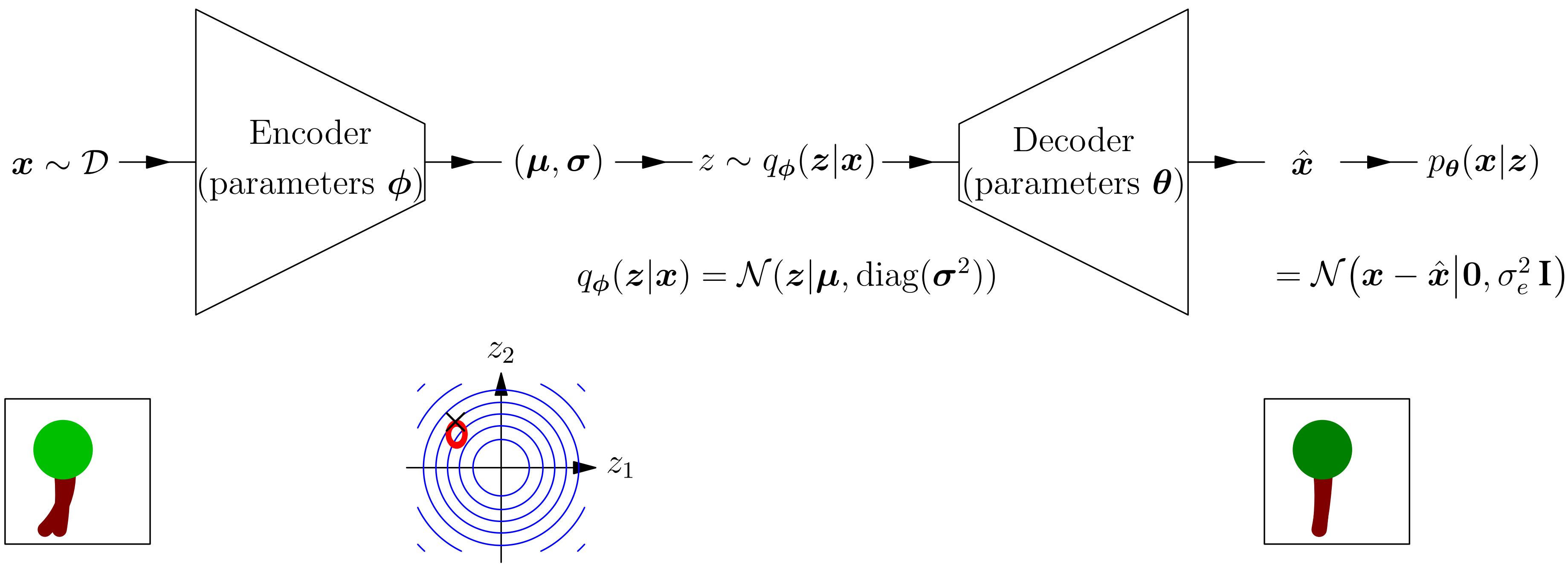

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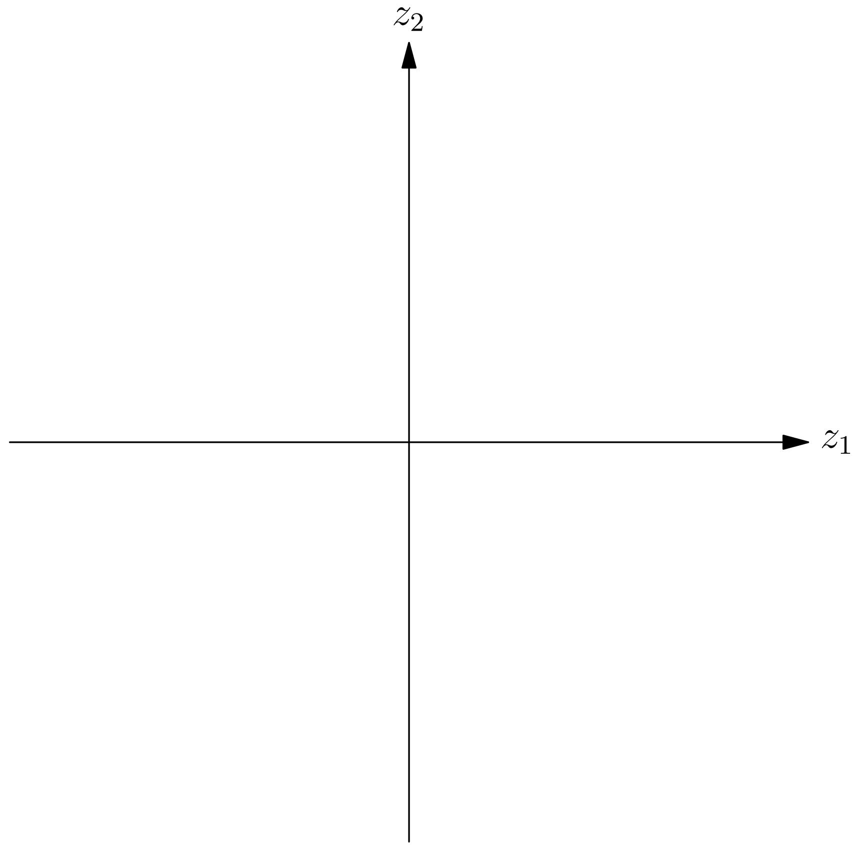
Latent Space



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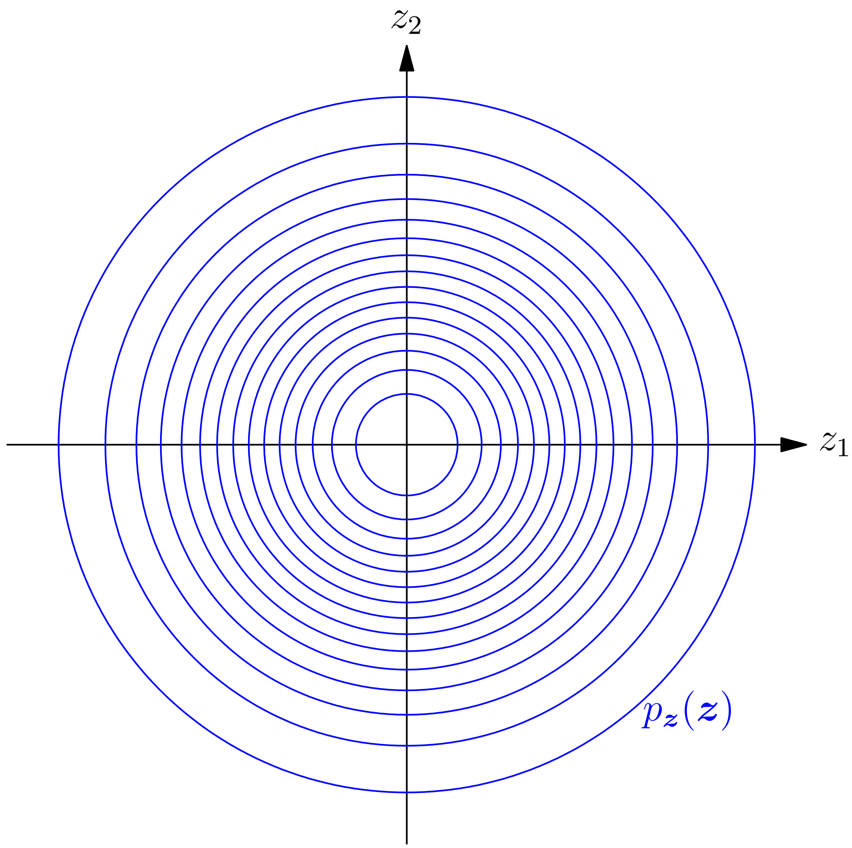

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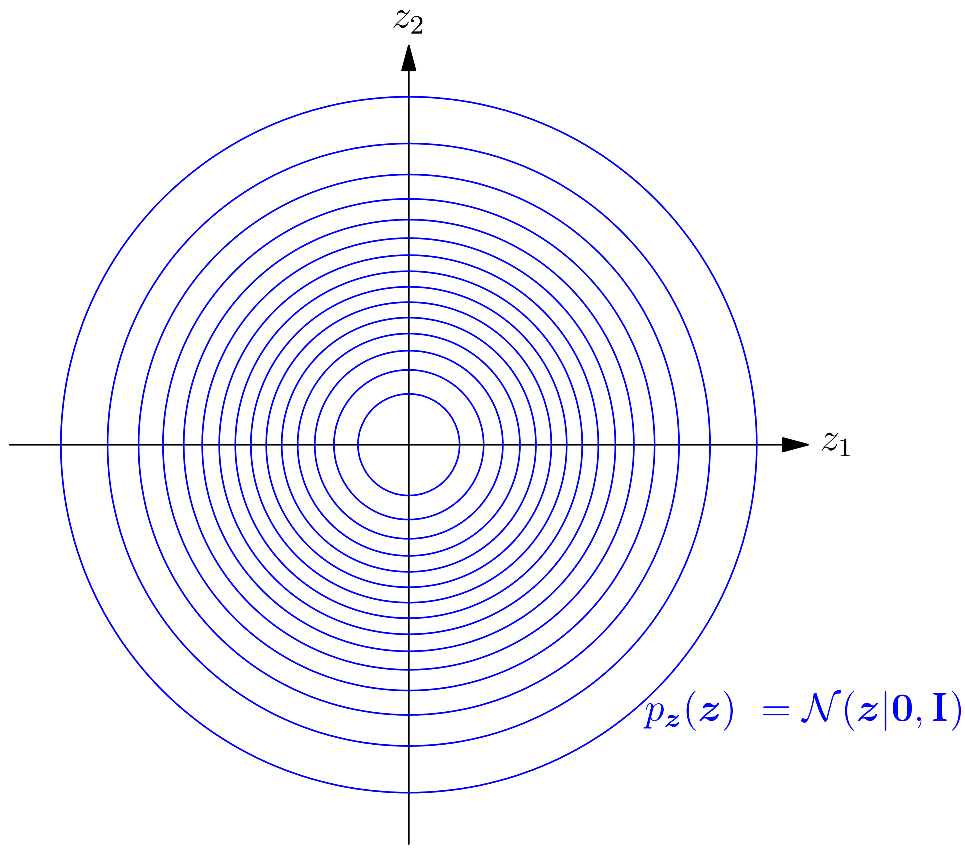

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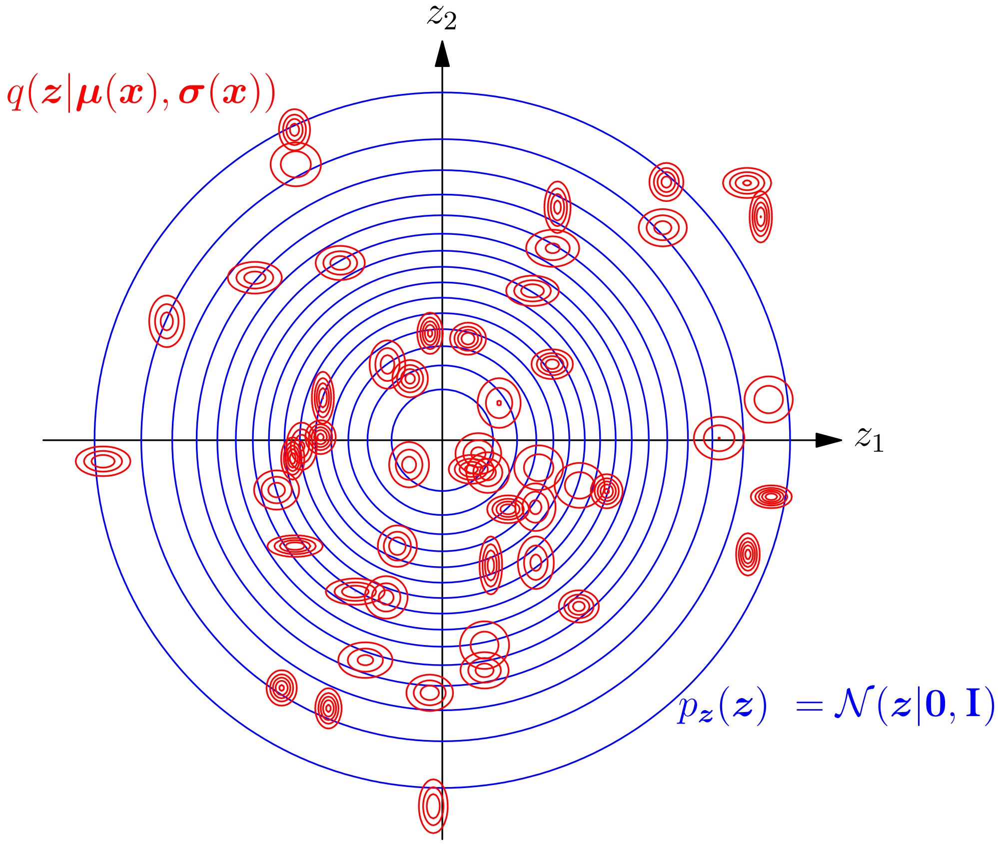

Understanding the Loss Function

- The original paper derived the loss function as a variational approximation to maximising some posterior
- This is difficult to understand (at least, for me)
- It has a very natural explanation in terms of minimum description length
- Alice wants to communicate the images to Bob
- Alice uses the encoder to derive a (latent) code $q(\boldsymbol{z}|\boldsymbol{x})$ which she communicates to Bob
- She also communicates the errors $\boldsymbol{\delta} = \boldsymbol{x} - \bar{\boldsymbol{x}}$
- Bob uses the decoder to decode $q(\boldsymbol{z}|\boldsymbol{x})$ and $\boldsymbol{\delta}$ to repair the images

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  can be interpreted as

  ⋆ The cost of communicating the code $\mathrm{KL}\Big(q_{\boldsymbol{\theta}}(\boldsymbol{z}|\boldsymbol{x})\Big\|\mathcal{N}(\mathbf{0},\mathbf{I})\Big)$
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- It requires some understanding and care to do it properly
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- The minimum description length is a powerful principle for model selection
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