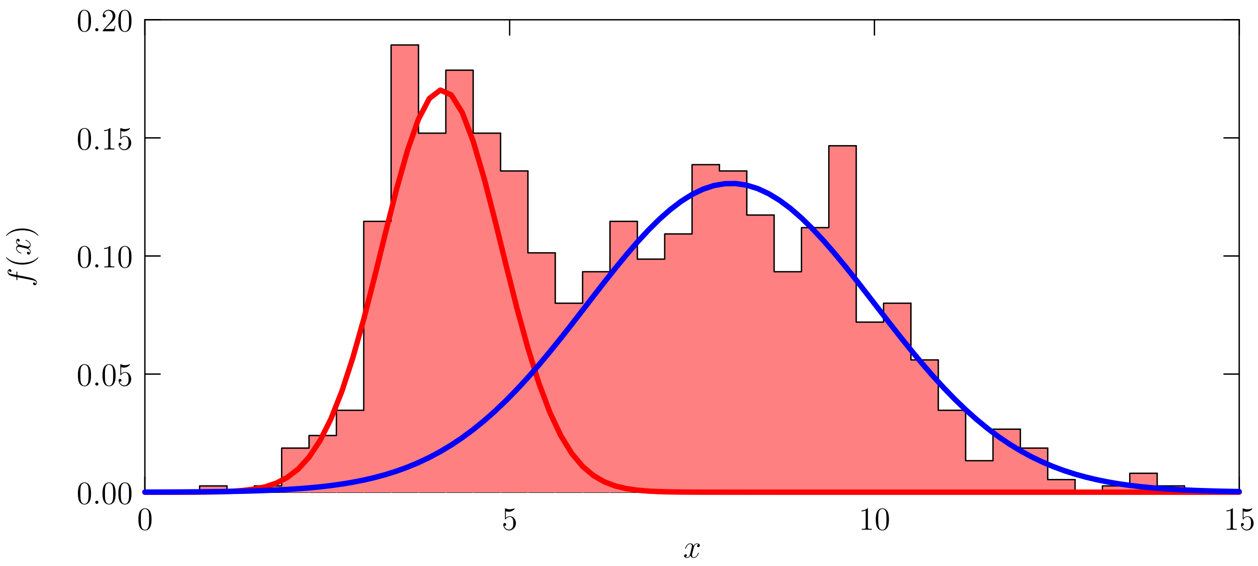
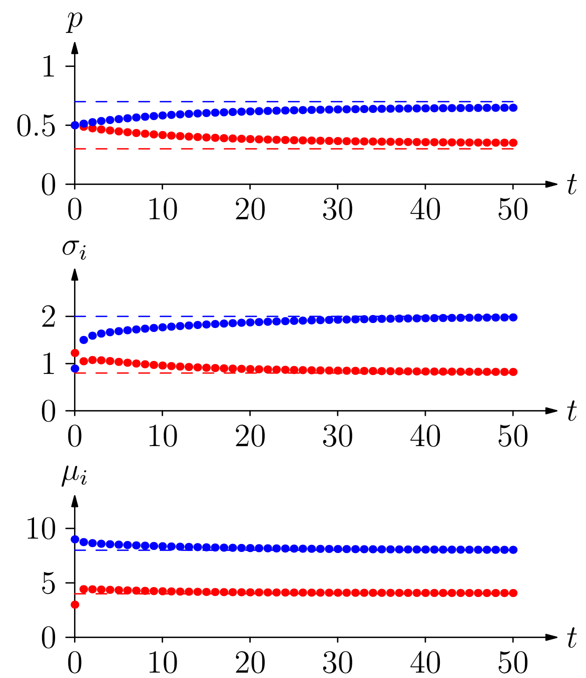
# Advanced Machine Learning

## Probabilistic Inference

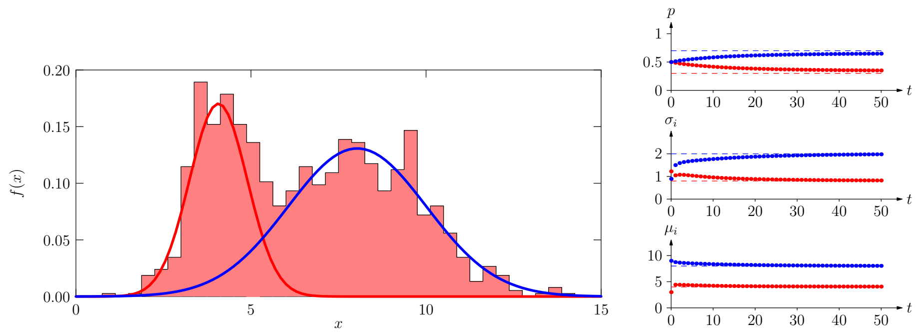



*Hierarchical Models, Mixture of Gaussians, Expectation Maximisation*

# Outline

1. **Building Probabilistic Models**
2. Mixture of Gaussians
3. Expectation Maximisation



# Building Probabilistic Models

- To describe a system with uncertainty we use random variables, $X$, $Y$, $Z$, etc.
- We use the convention of writing random variables in capitals (this is sometimes confusing as when you observe a random variables it is no longer random)
- The variables are described by probability mass function $\mathbb{P}(X, Y, Z)$ or if our variables are continuous, but probability densities $f_{X,Y,Z}(x, y, z)$
- A major rule of probability is

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# Conditional Probabilities

- When developing models it is often useful to consider conditional probabilities e.g. $\mathbb{P}(X, Y | Z)$ or $f_{X|Y,Z}(x|y, z)$

- A second major rule in probabilistic modelling is

$$\mathbb{P}(X, Y) = \mathbb{P}(X|Y)\,\mathbb{P}(Y) = \mathbb{P}(Y|X)\,\mathbb{P}(X)$$

- This is a mathematical identity that does not imply causality (it defines conditional probability)

- It is the origins of Bayes' rule: $\mathbb{P}(X|Y) = \dfrac{\mathbb{P}(Y|X)\,\mathbb{P}(X)}{\mathbb{P}(Y)}$

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# Discriminative Models

- We often think of our observations as given and the predictions as random variables
- For example we might be given some features $\boldsymbol{x}$ and we wish to predict a class $C \in \mathcal{C}$
- Our objective is then to find the probability $\mathbb{P}\left(C|\boldsymbol{x}\right)$
- This is known as a **discriminative model**
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# Generative Models

- Sometimes it is easy to think about the joint process of generating the features and outputs together
- This leads to a joint distribution $\mathbb{P}(\boldsymbol{X}, Y)$ where $\boldsymbol{X}$ are your features and $Y$ is your output you are trying to predict
- This is known as a **generative model**
- Generative models are often more natural to think about
- We can use them to do discrimination using

$$\mathbb{P}(Y|\boldsymbol{X}) = \frac{\mathbb{P}(\boldsymbol{X}, Y)}{\mathbb{P}(\boldsymbol{X})} = \frac{\mathbb{P}(\boldsymbol{X}, Y)}{\sum_{Y} \mathbb{P}(\boldsymbol{X}, Y)}$$

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# Latent Variables

- Sometimes we have models that involve random variables that we don't observe and we don't care about
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# Modelling Virus

- Suppose we want to estimate the number of hospitalisation from Corona virus in the next month
- Our observable is the number of reported cases
- In our model we might want to estimate the number of actual cases
- This would be a latent variable (it is not an observable or our final target, but it is very useful intermediate in our model)
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# Probabilistic Inference

- We can use Bayes' rules to learn a set of parameter $\boldsymbol{\Theta}$ that occur in our likelihood function

$$\mathbb{P}\left(\boldsymbol{\Theta}|\mathcal{D}\right) = \frac{\mathbb{P}\left(\mathcal{D}|\boldsymbol{\Theta}\right)\,\mathbb{P}\left(\boldsymbol{\Theta}\right)}{\mathbb{P}\left(\mathcal{D}\right)}$$

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# Problem with Bayes

- Bayes is problematic because it is often hard
- The posterior is often not expressible as a nice probability function
- We need to compute the *evidence* or *margin likelihood* we use

$$\mathbb{P}(\mathcal{D}) = \sum_{\boldsymbol{\Theta}} \mathbb{P}(\mathcal{D}|\boldsymbol{\Theta})\, \mathbb{P}(\boldsymbol{\Theta})$$

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- One work around is to compute the mode of the posterior

$$\boldsymbol{\Theta}_{\text{MAP}} = \underset{\boldsymbol{\Theta}}{\operatorname{argmax}}\, f(\mathcal{D}|\boldsymbol{\Theta})\, f(\boldsymbol{\Theta}) = \underset{\boldsymbol{\Theta}}{\operatorname{argmax}} \log(f(\mathcal{D}|\boldsymbol{\Theta})) + \log(f(\boldsymbol{\Theta}))$$

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# Maximum Likelihood

- When we assume a uniform prior then the MAP solution is just maximising the likelihood
- Weirdly this hack was accepted as part of mainstream statistics even when Bayesian statistics was considered unscientific
- Maximum likelihood is often sufficient for *government work*, but it isn't the best you can do
- Often in high-dimensional problems using a non-uniform prior can make a big difference
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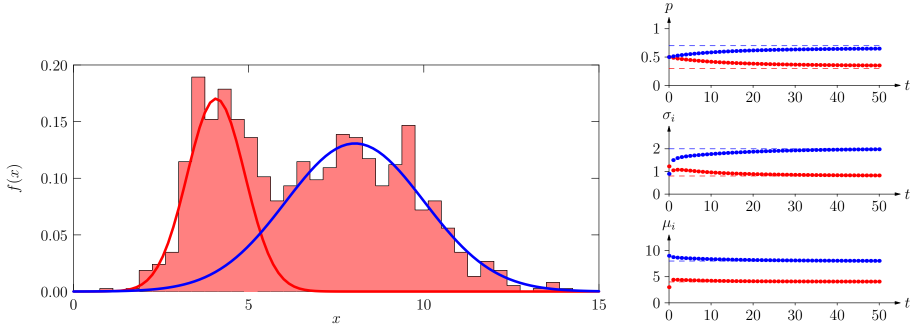

# Mixture of Gaussians

- Suppose we were observing the decays from two types of short-lived particle, $A$ or $B$
- We observe the half life, $X_i$, but not the particle type
- We assume $X_i$ is normally distributed with unknown means and variances: $\boldsymbol{\Theta} = \{\mu_A,\, \sigma_A^2,\, \mu_B,\, \sigma_B^2\}$
- Let $Z_i \in \{0, 1\}$ be an indicator that particle $i$ is of type $A$
- The probability of $X_i$ is given by

$$f(X_i|Z_i, \boldsymbol{\Theta}) = Z_i \, \mathcal{N}\big(X_i \big| \mu_A, \sigma_A^2\big) + (1 - Z_i) \, \mathcal{N}\big(X_i \big| \mu_B, \sigma_B^2\big)$$

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# Data

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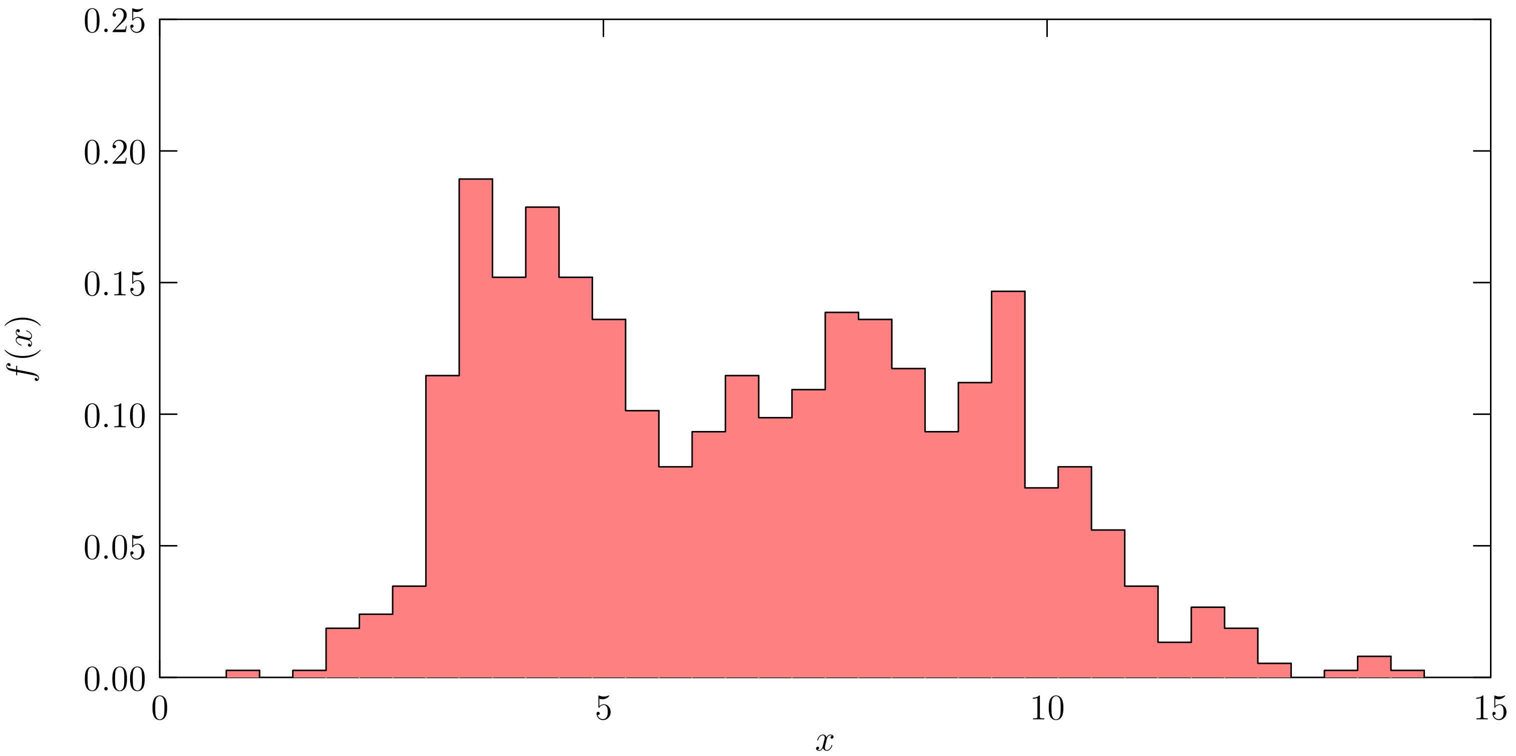

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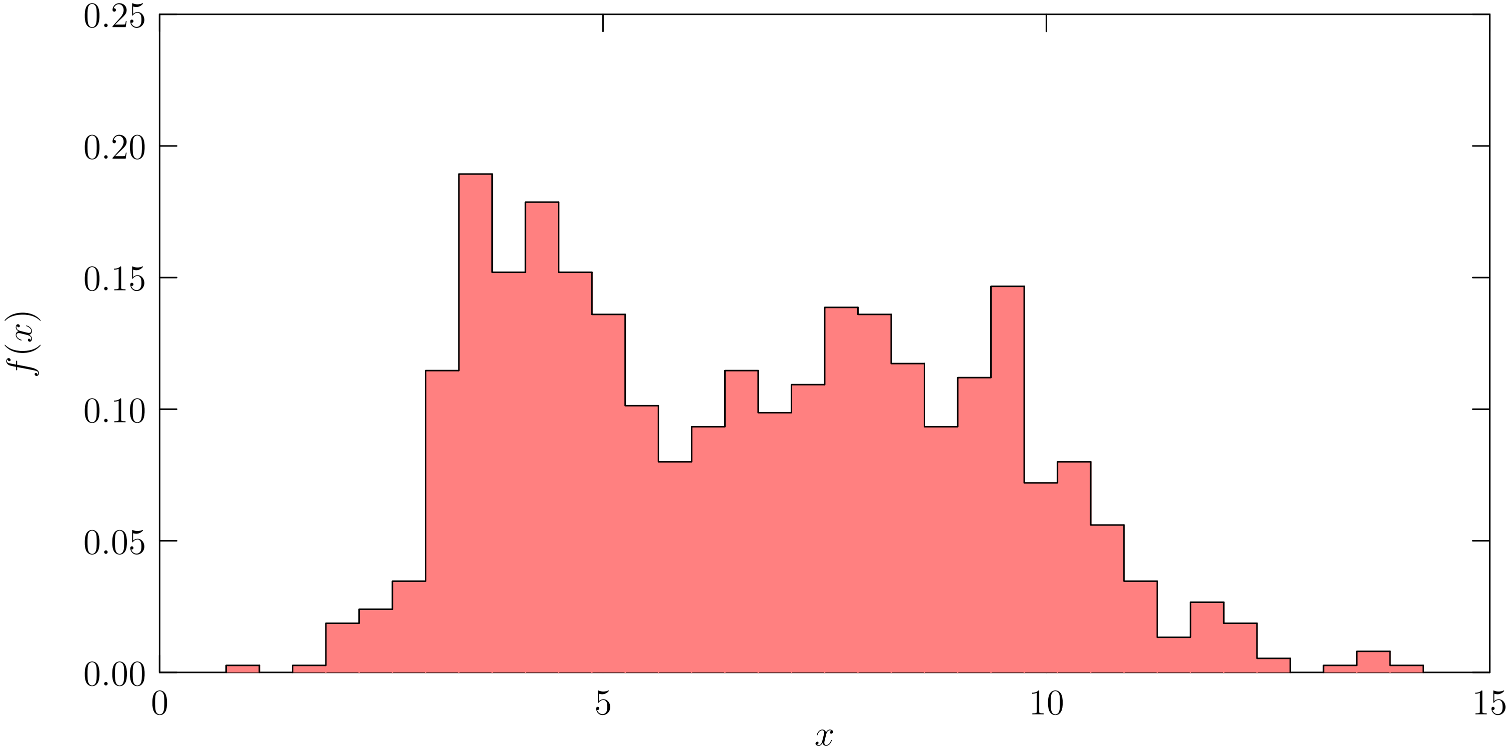

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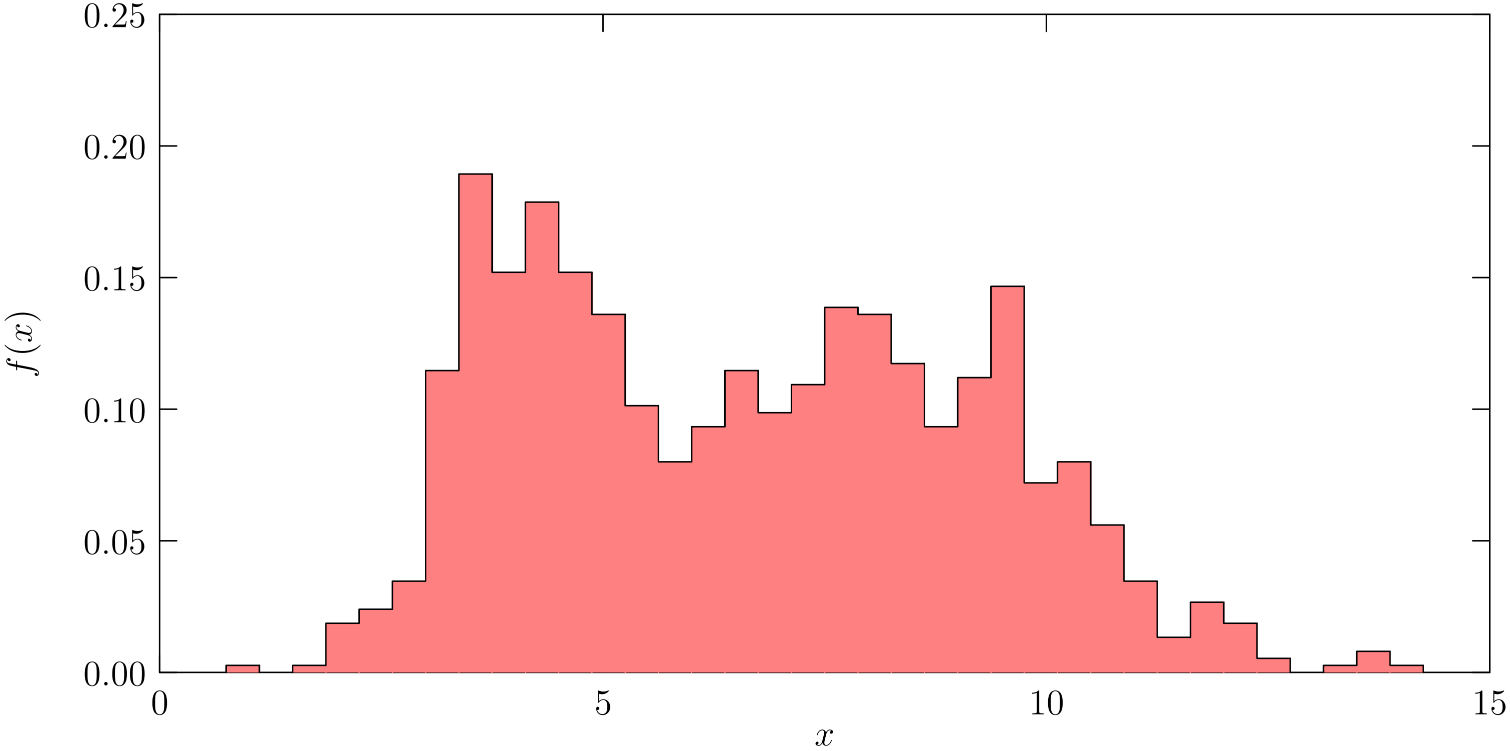

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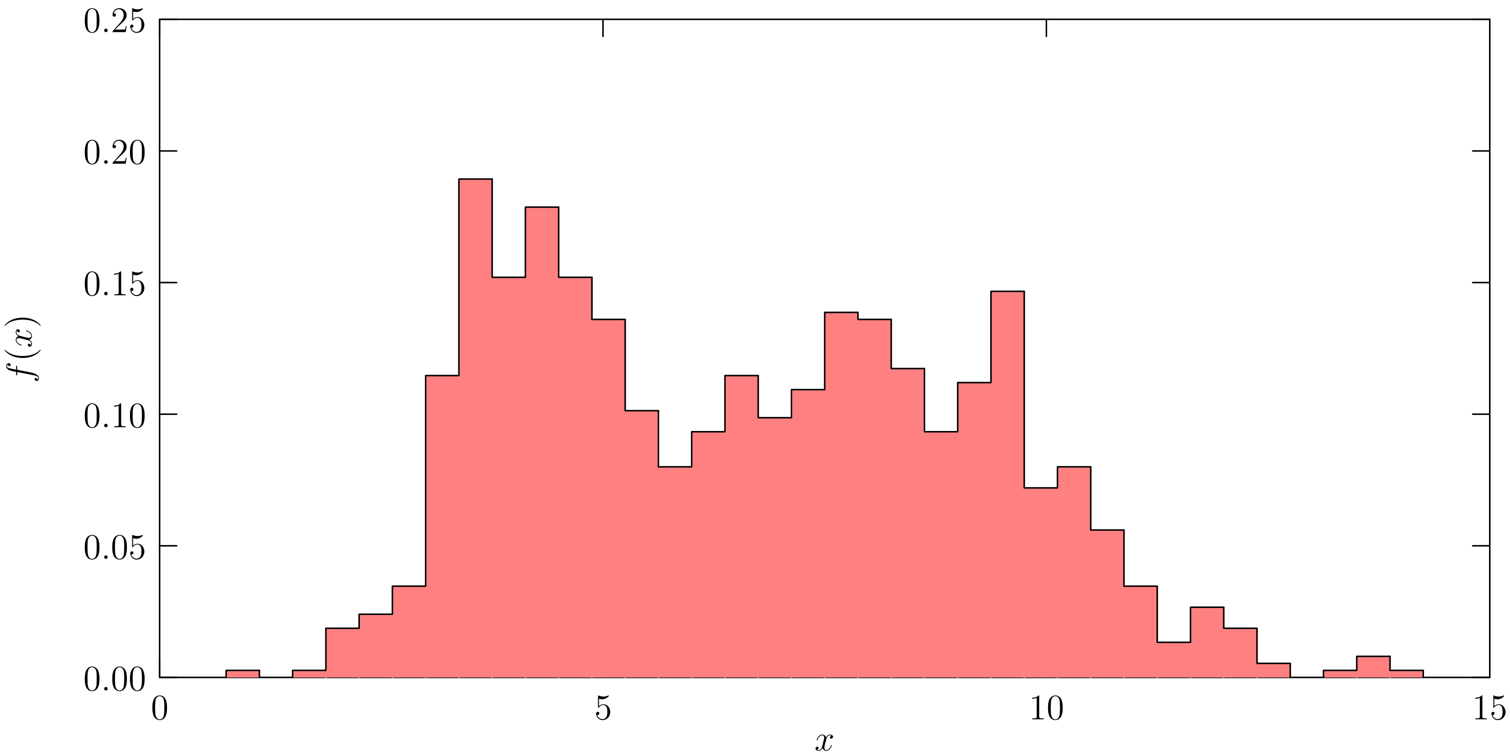

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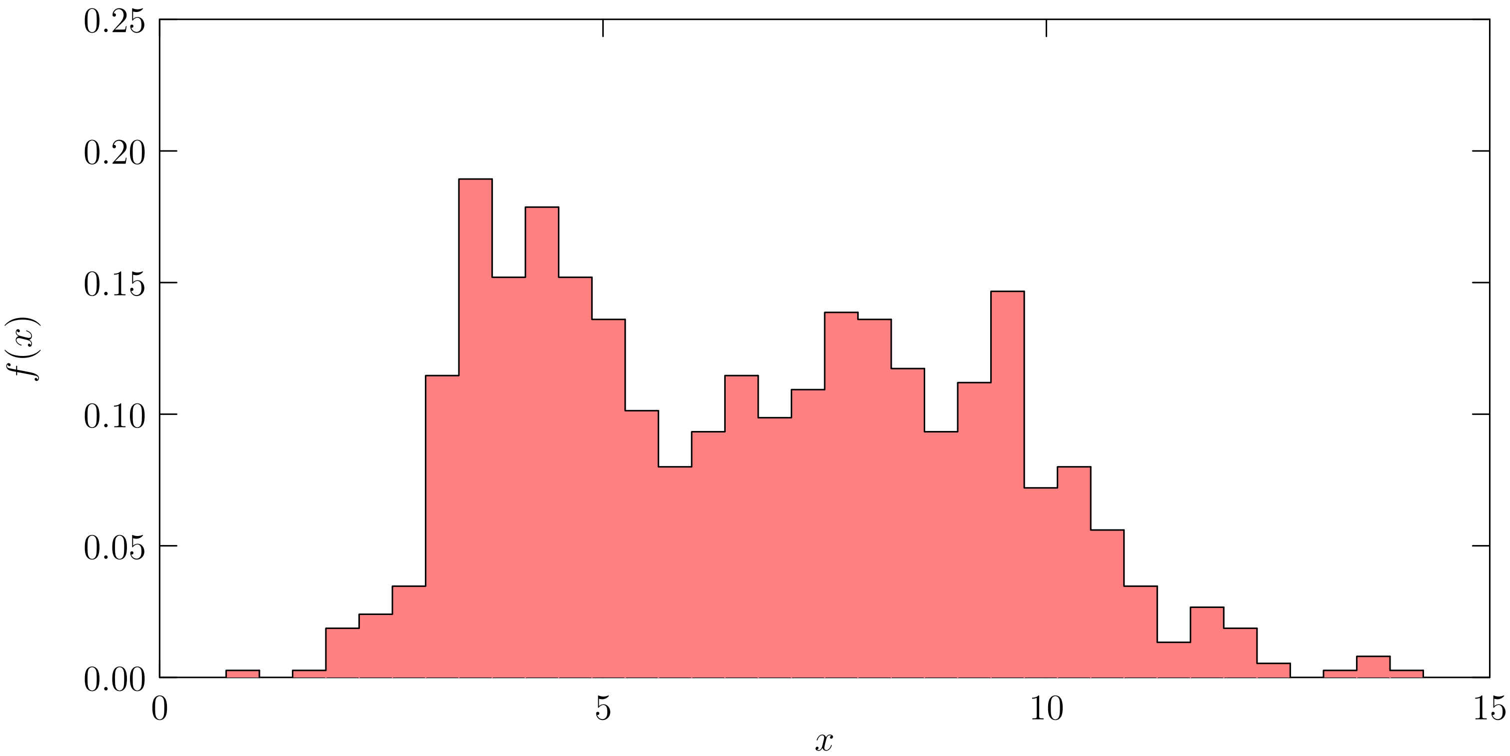

# Maximum Likelihood

- To solve the model as a Bayesian we would have to assign priors to our parameters $\boldsymbol{\Theta} = (\mu_A, \sigma_A, \mu_B, \sigma_B, p)$
- This is doable, but complicated (we would also end up with a distribution for our parameters)
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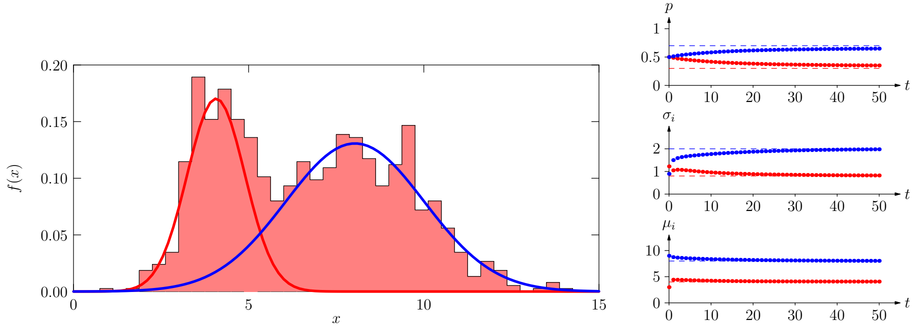
# Outline

1. Building Probabilistic Models
2. Mixture of Gaussians
3. **Expectation Maximisation**



# EM Algorithm

- The maximum likelihood is a non-linear function of the parameters so cannot be immediately maximised

- We could use a standard optimiser but there is an alternative iterative scheme that goes by the name of **expectation maximisation** or the **EM algorithm**

- We proceed iteratively by maximising the expected log-likelihood with respect to the current set of parameters

$$\Theta^{(t+1)} = \underset{\boldsymbol{\Theta}}{\operatorname{argmax}} \sum_{\boldsymbol{Z}} \mathbb{P}\left(\boldsymbol{Z} \middle| \mathcal{D}, \boldsymbol{\Theta}^{(t)}\right) \log(f(\mathcal{D}|\boldsymbol{Z}, \boldsymbol{\Theta}))$$

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# Why EM Algorithm Works

- The argument around why this works is quite involved
- Note that at each step we maximise

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- We need to compute the distribution of latent variables conditioned on the data and current estimated parameters

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$$\mathbb{P}\left(\boldsymbol{Z}\middle|\mathcal{D}, \boldsymbol{\Theta}^{(t)}\right) = \prod_{i=1}^{m} \mathbb{P}\left(Z_i\middle|X_i, \boldsymbol{\Theta}^{(t)}\right)$$

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# EM for Mixture of Gaussians

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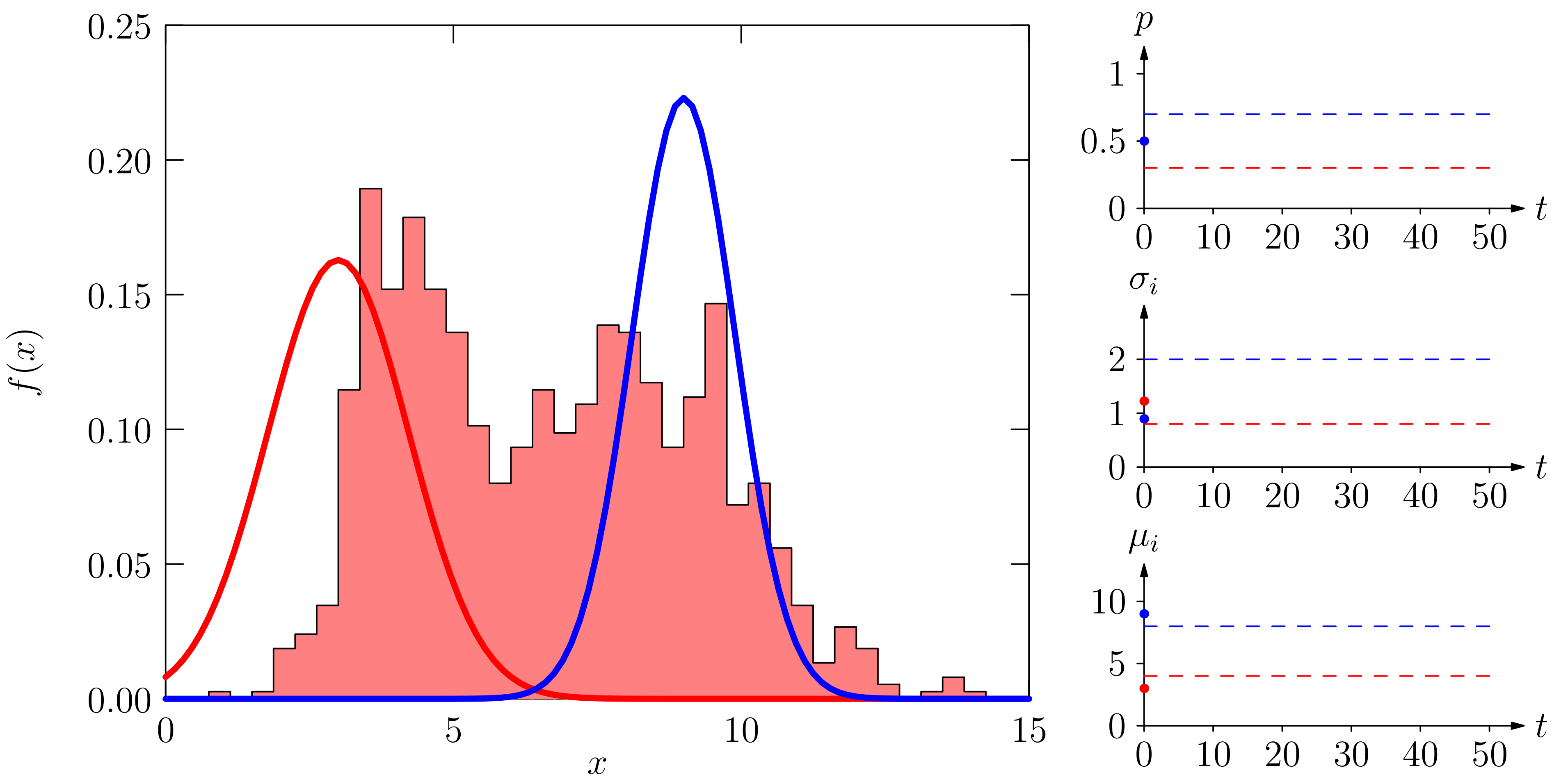

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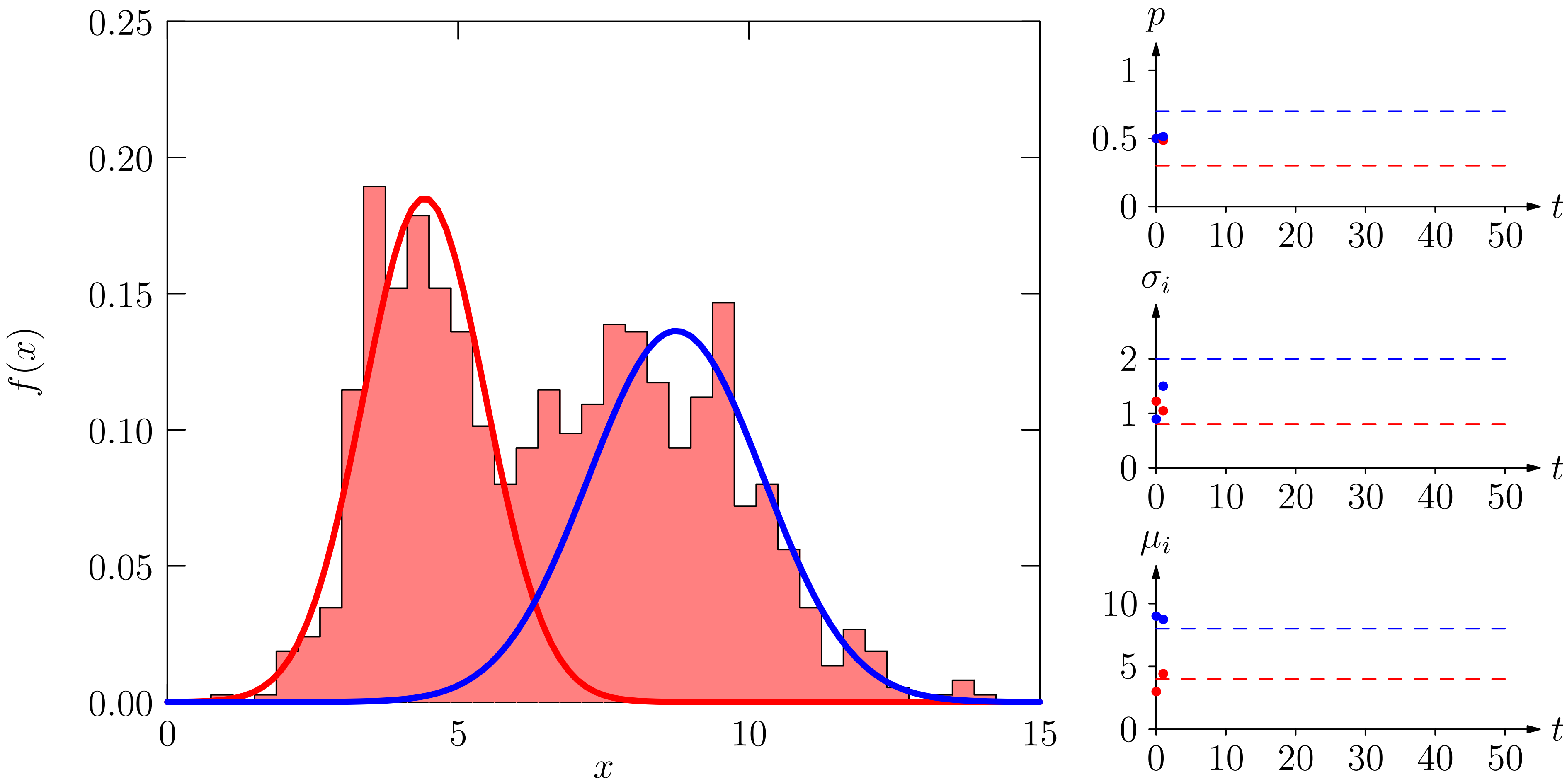

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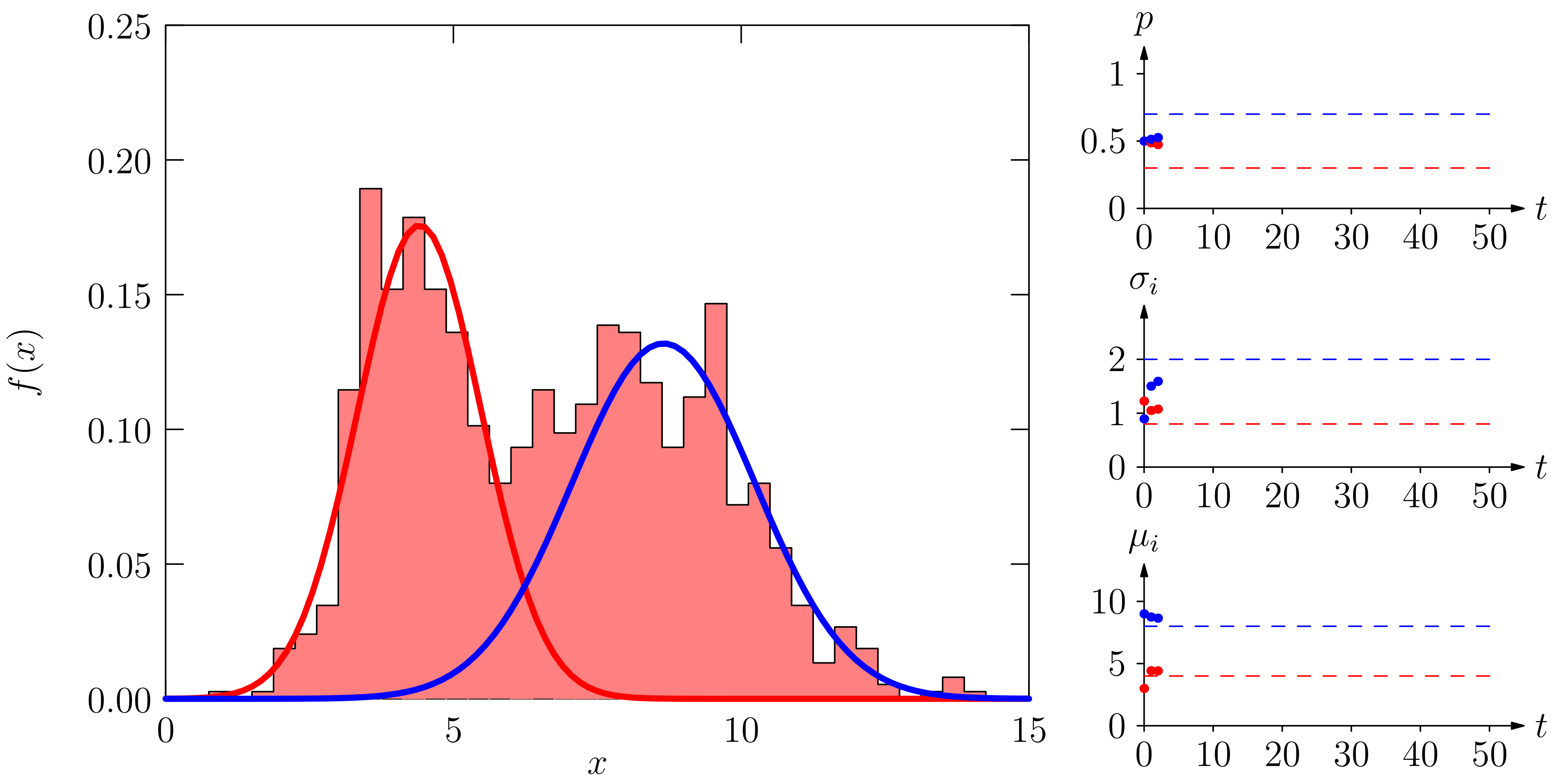

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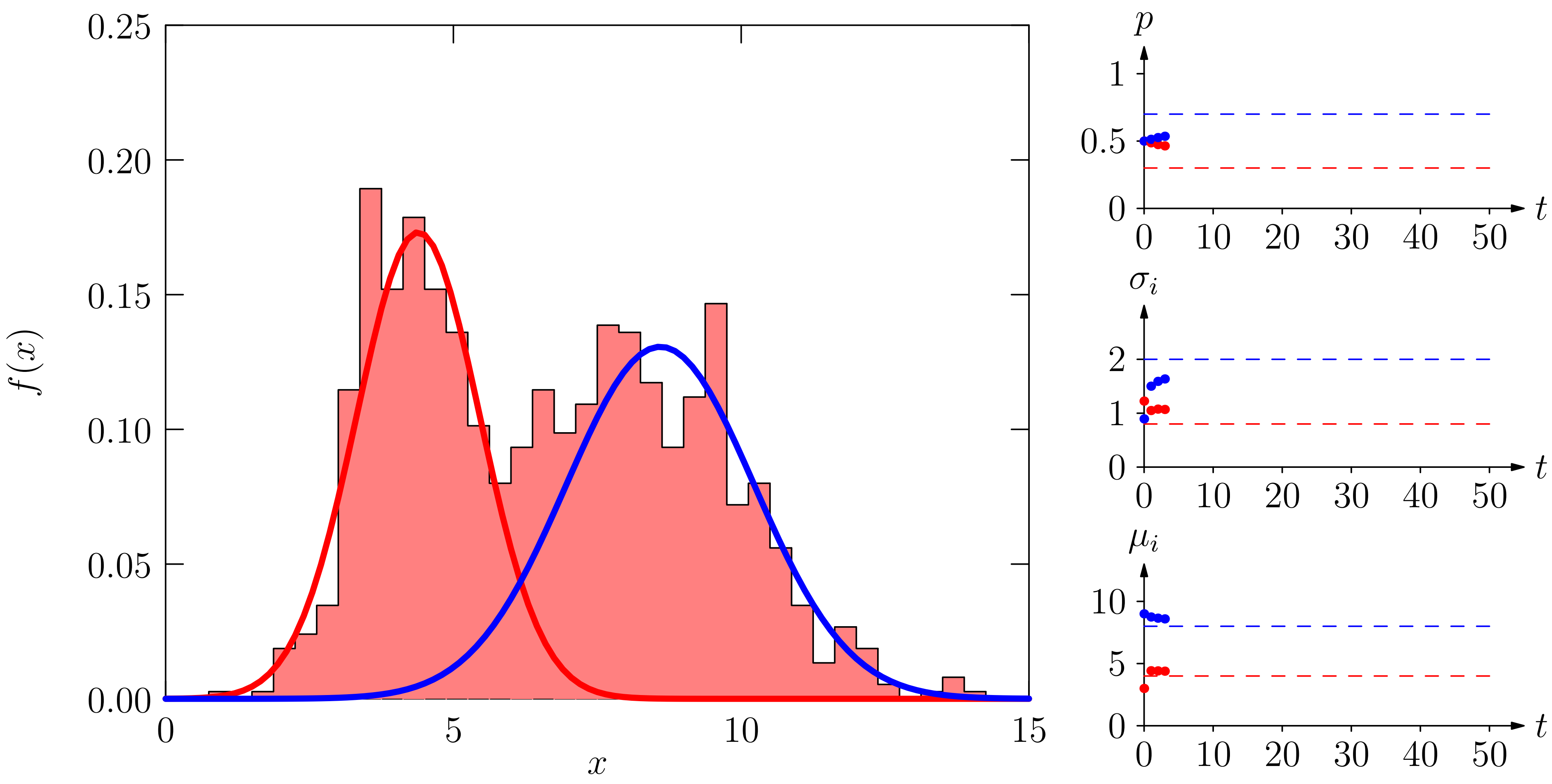
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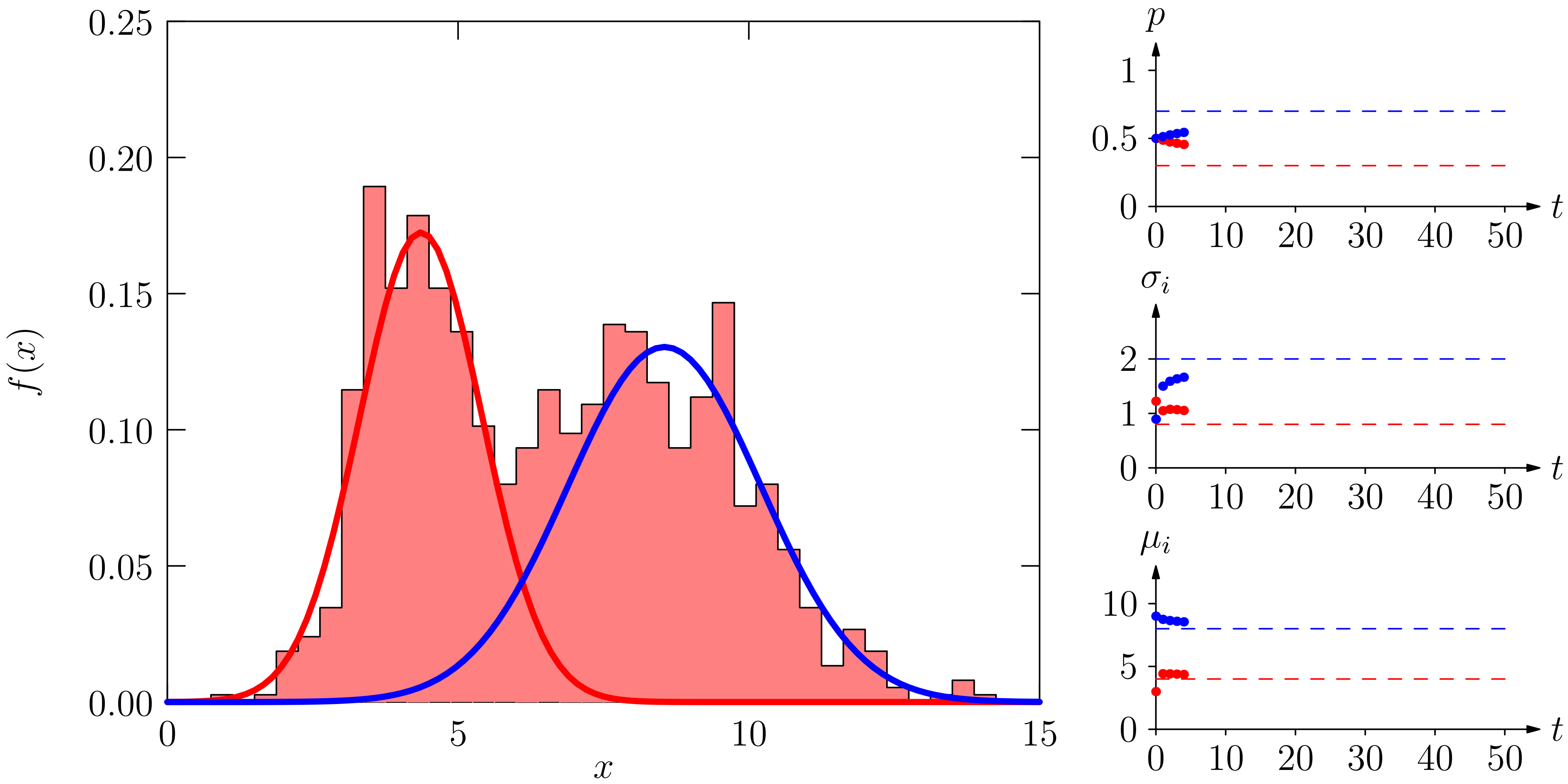
# Example



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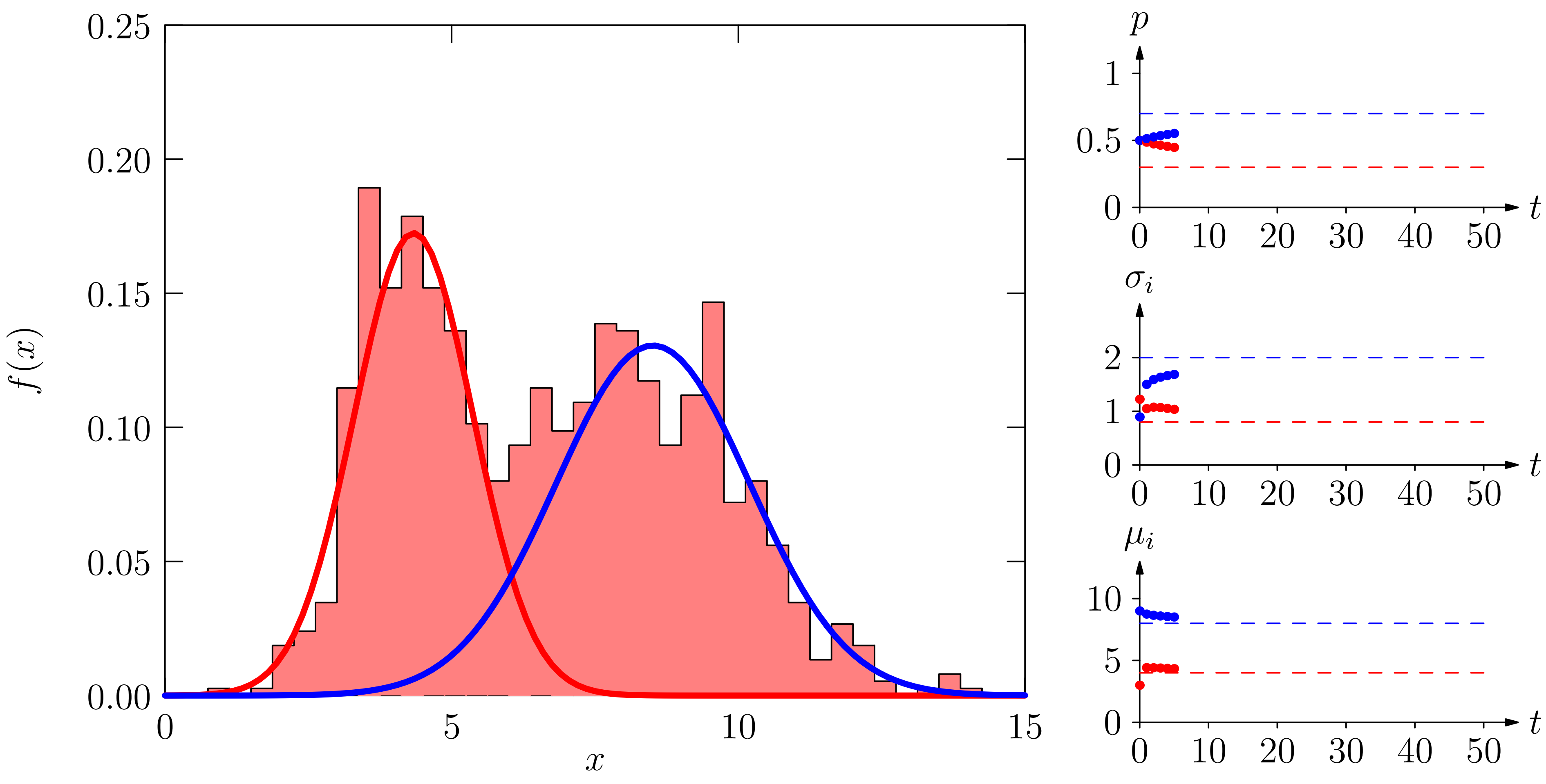
# Example



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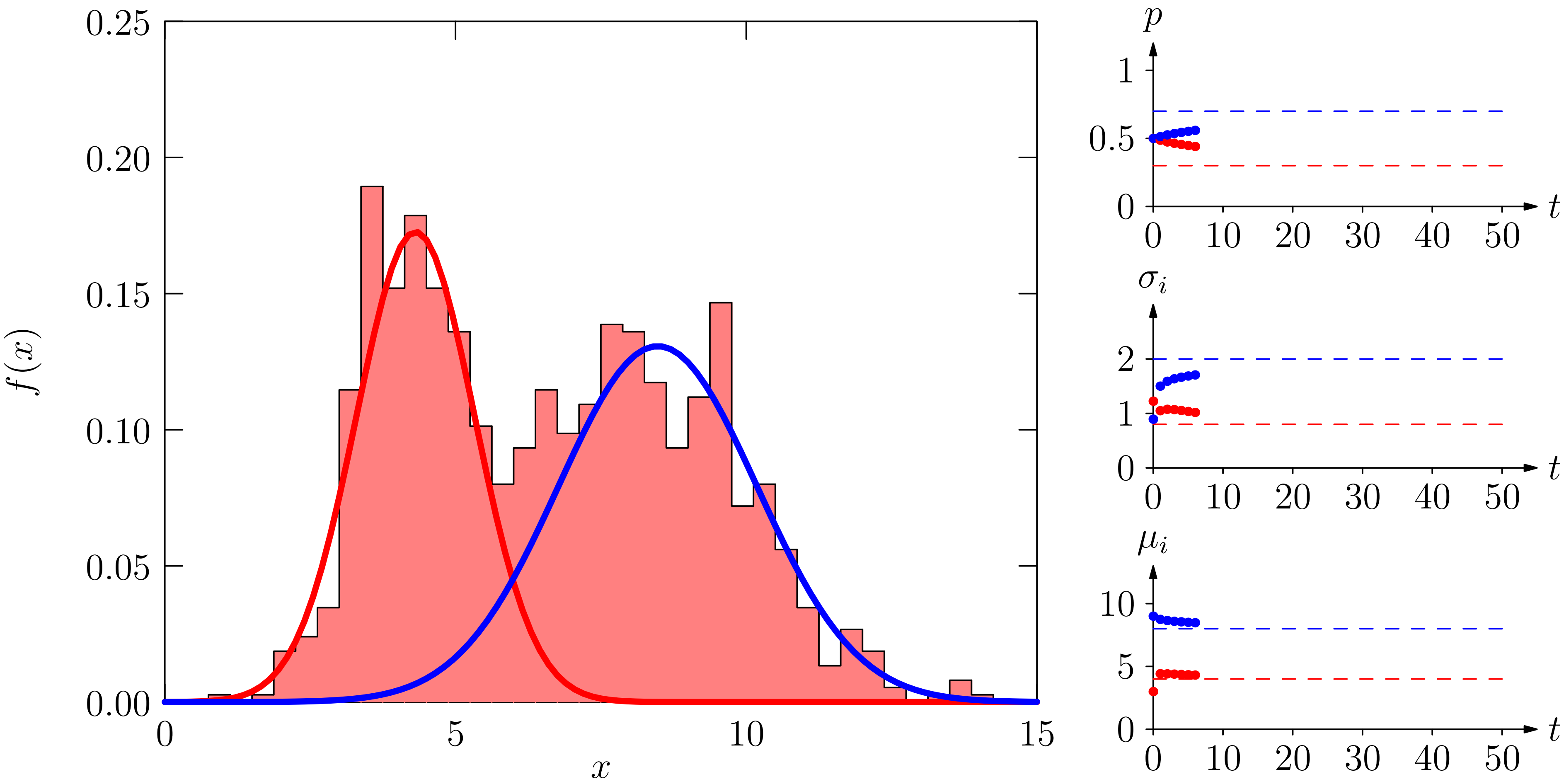
# Example



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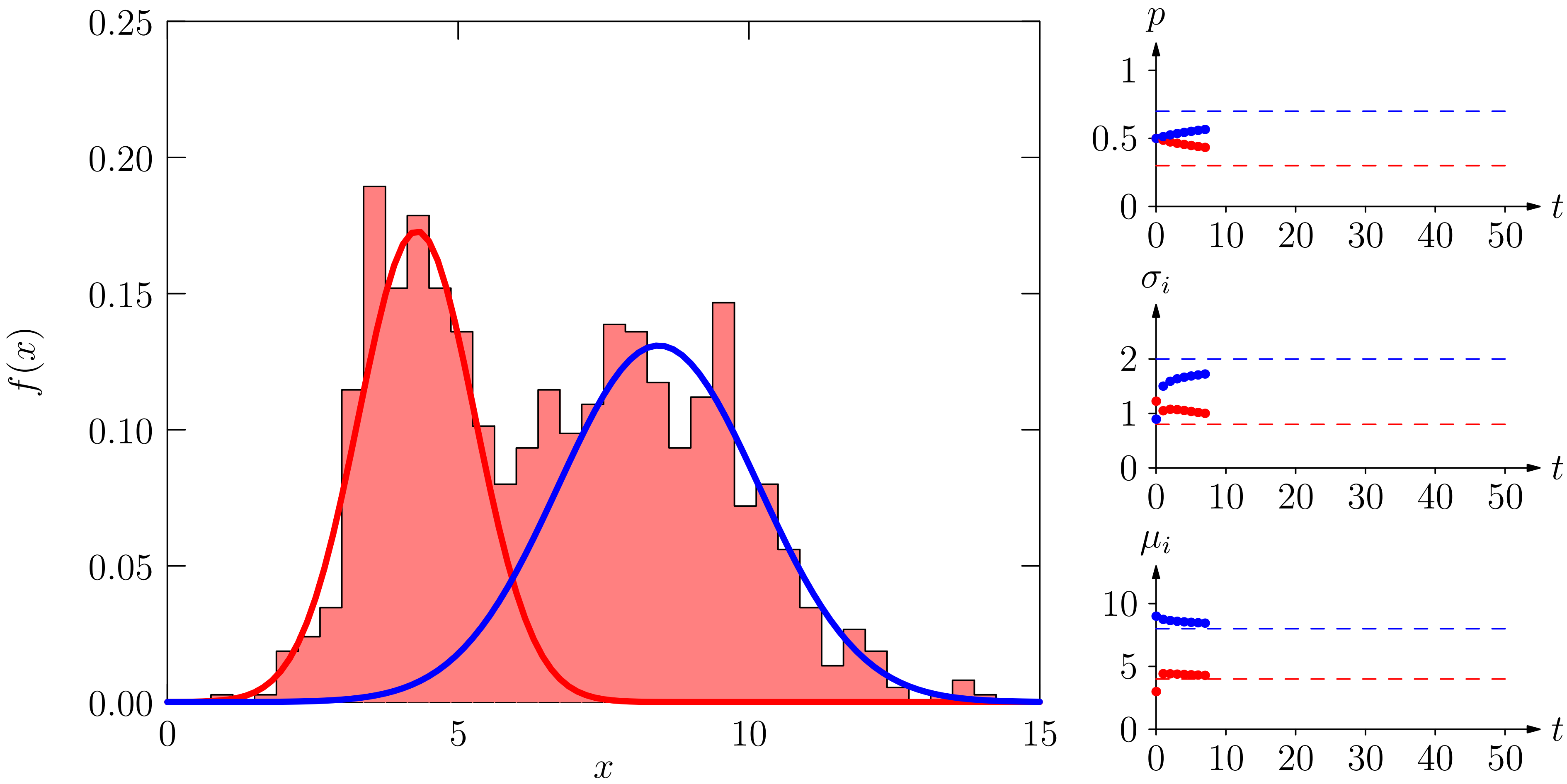
# Example



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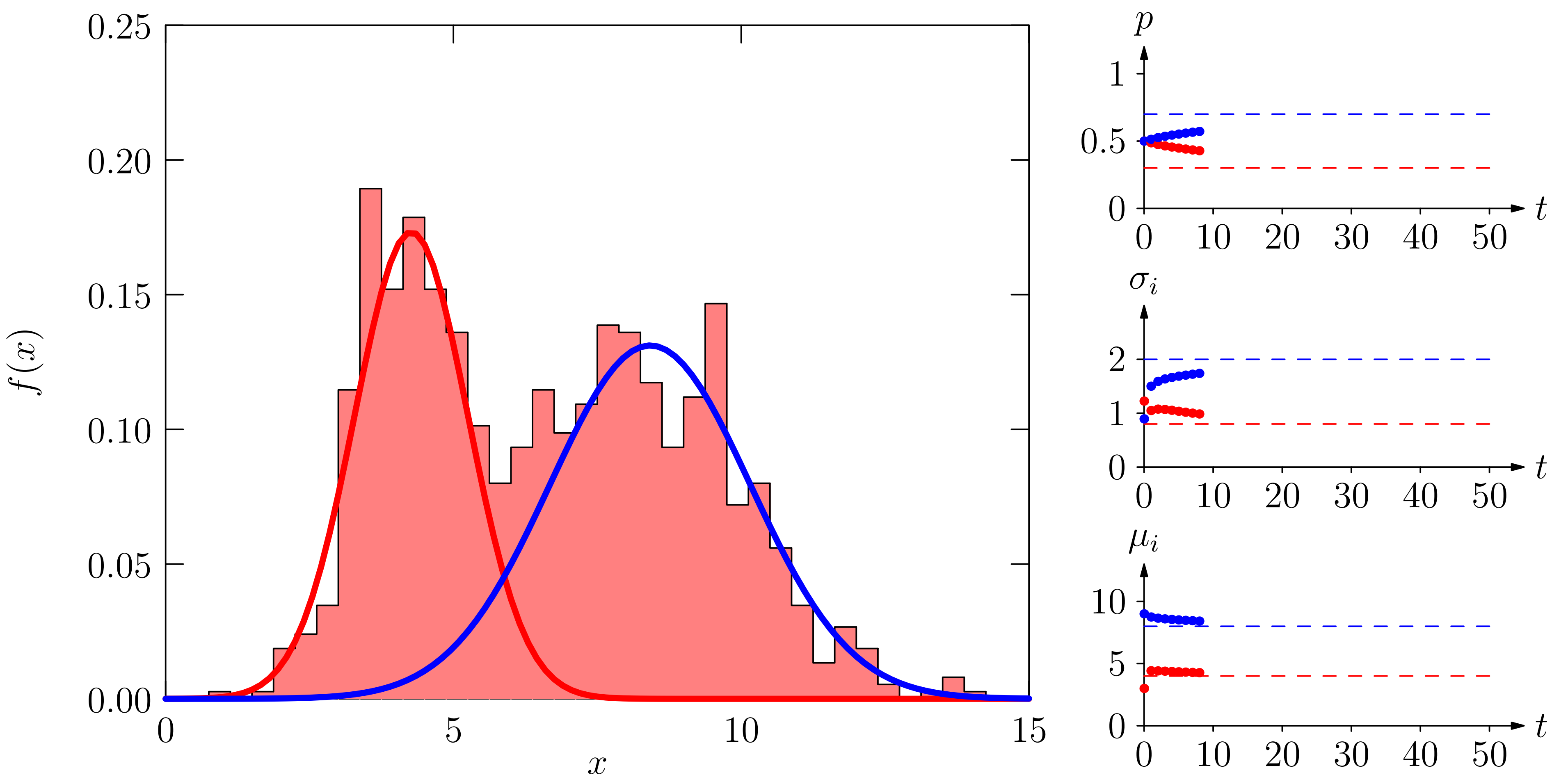
# Example



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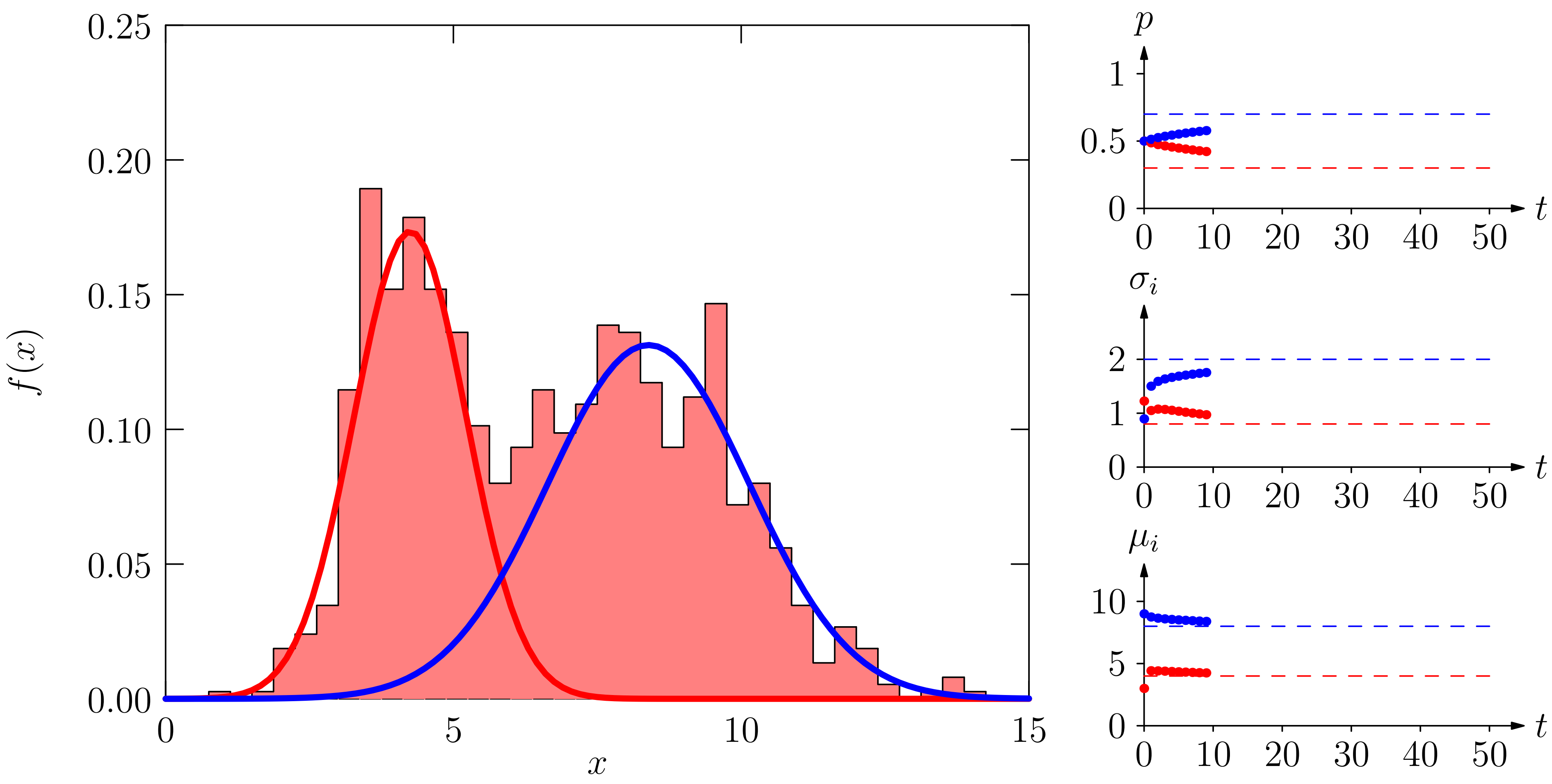
# Example



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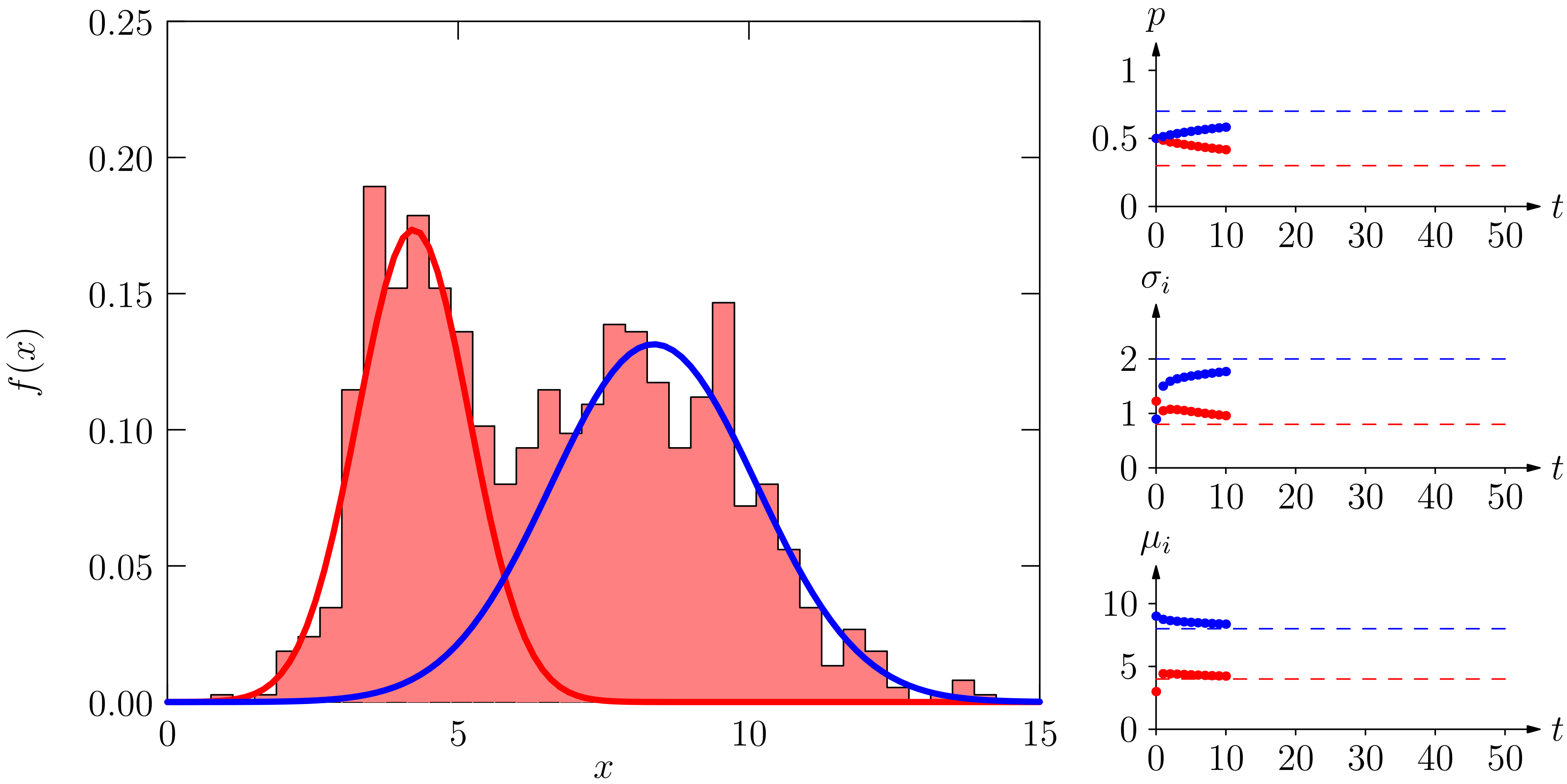

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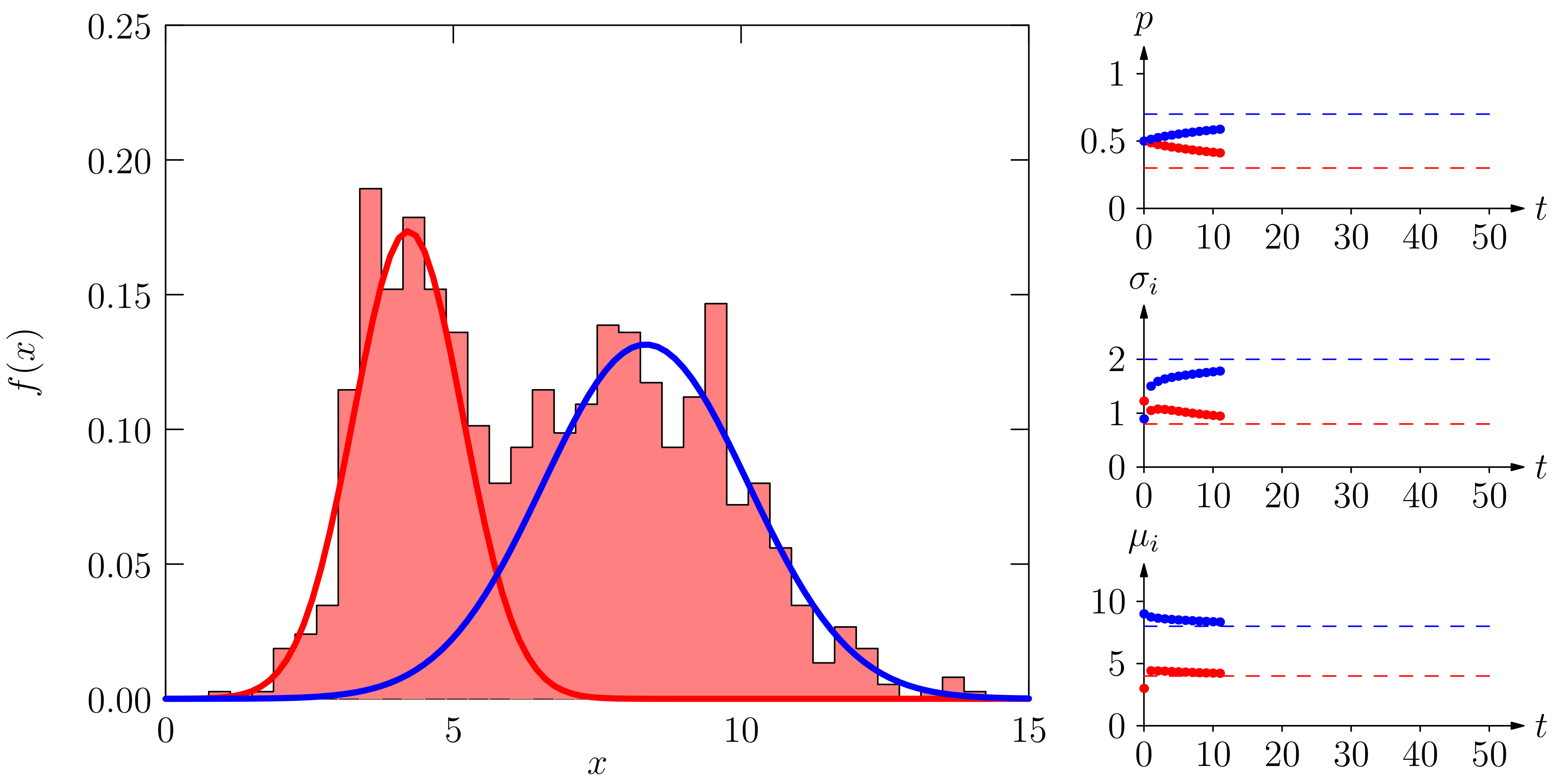

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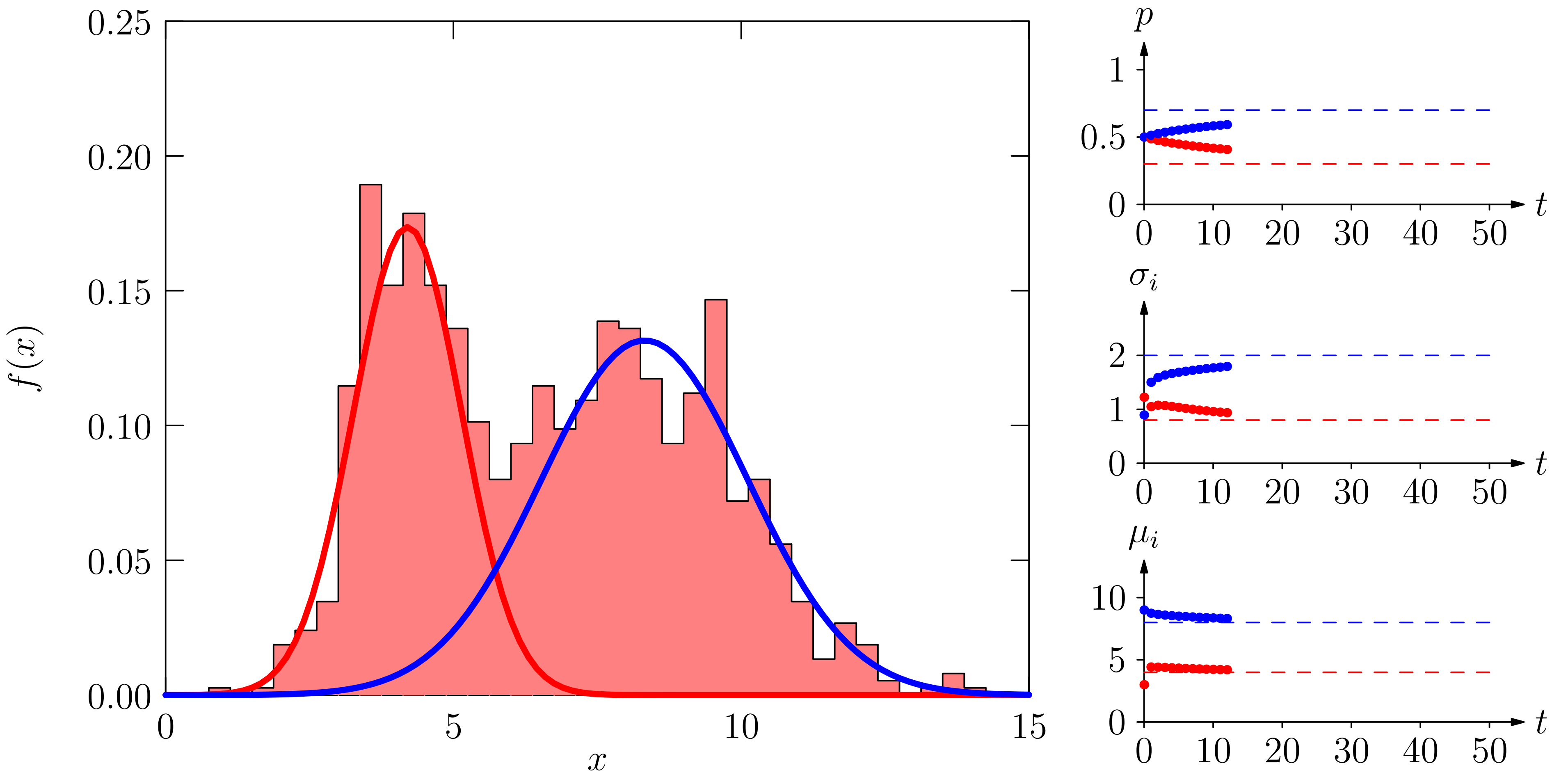

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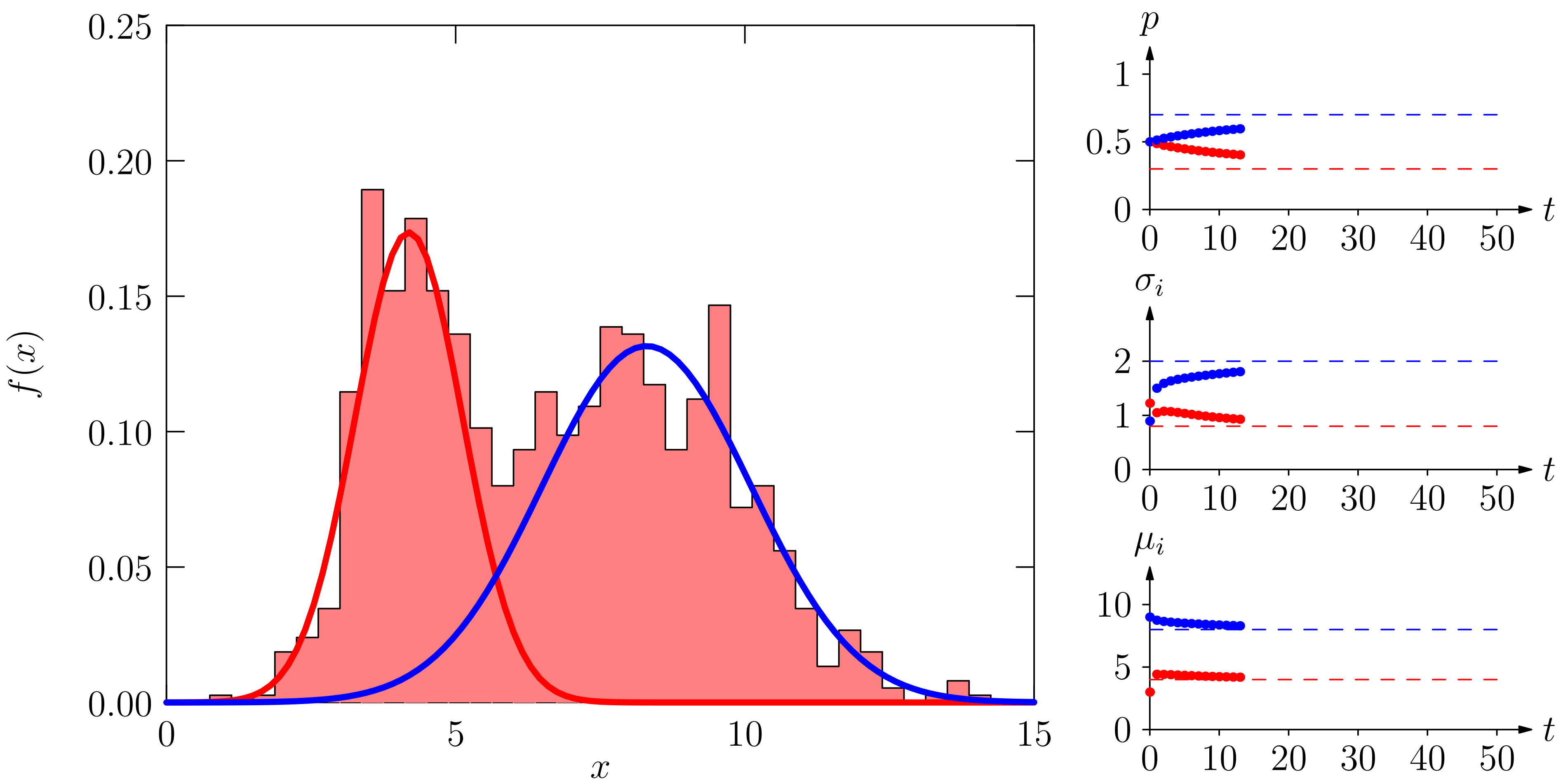
# Example



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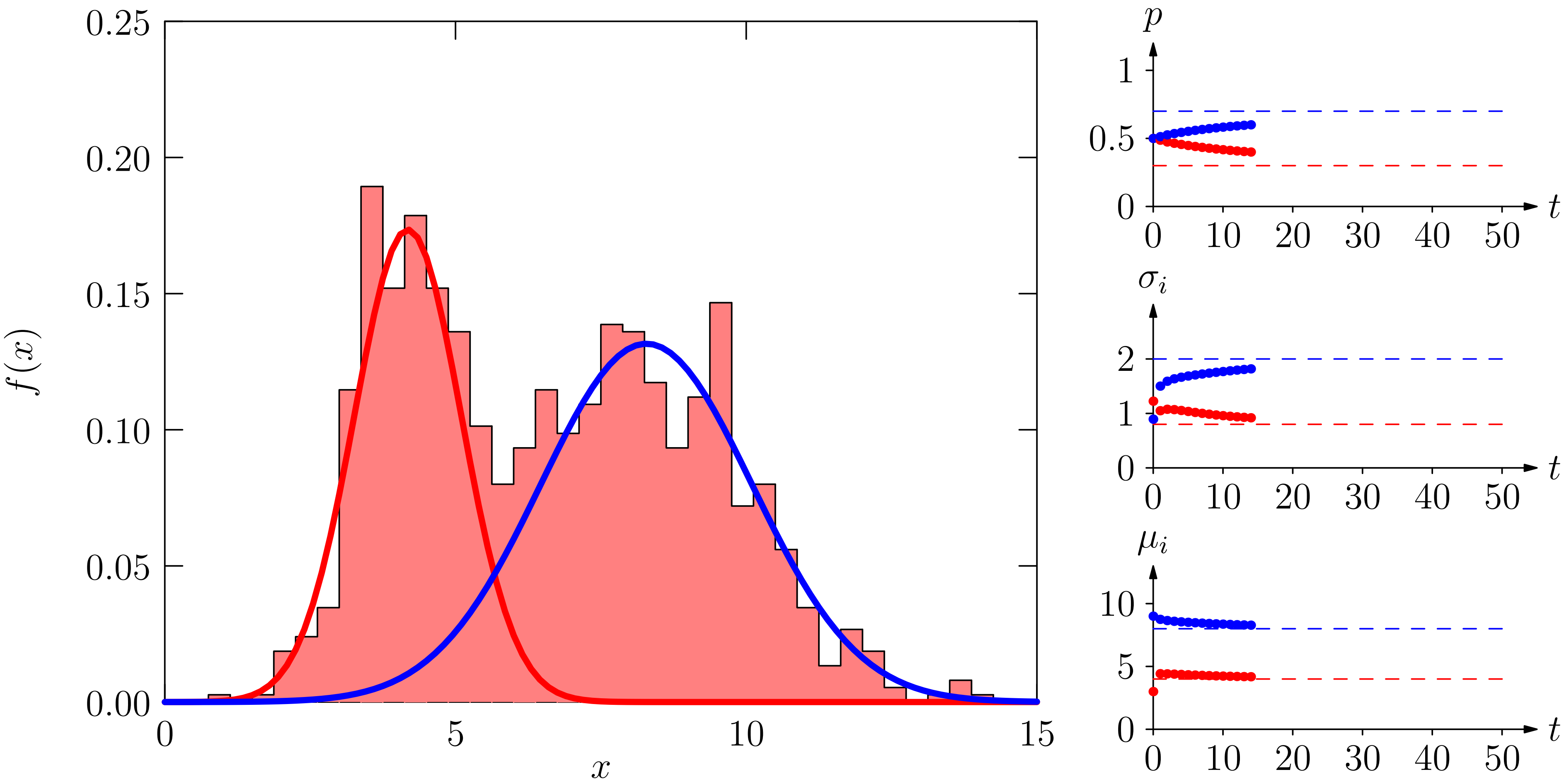

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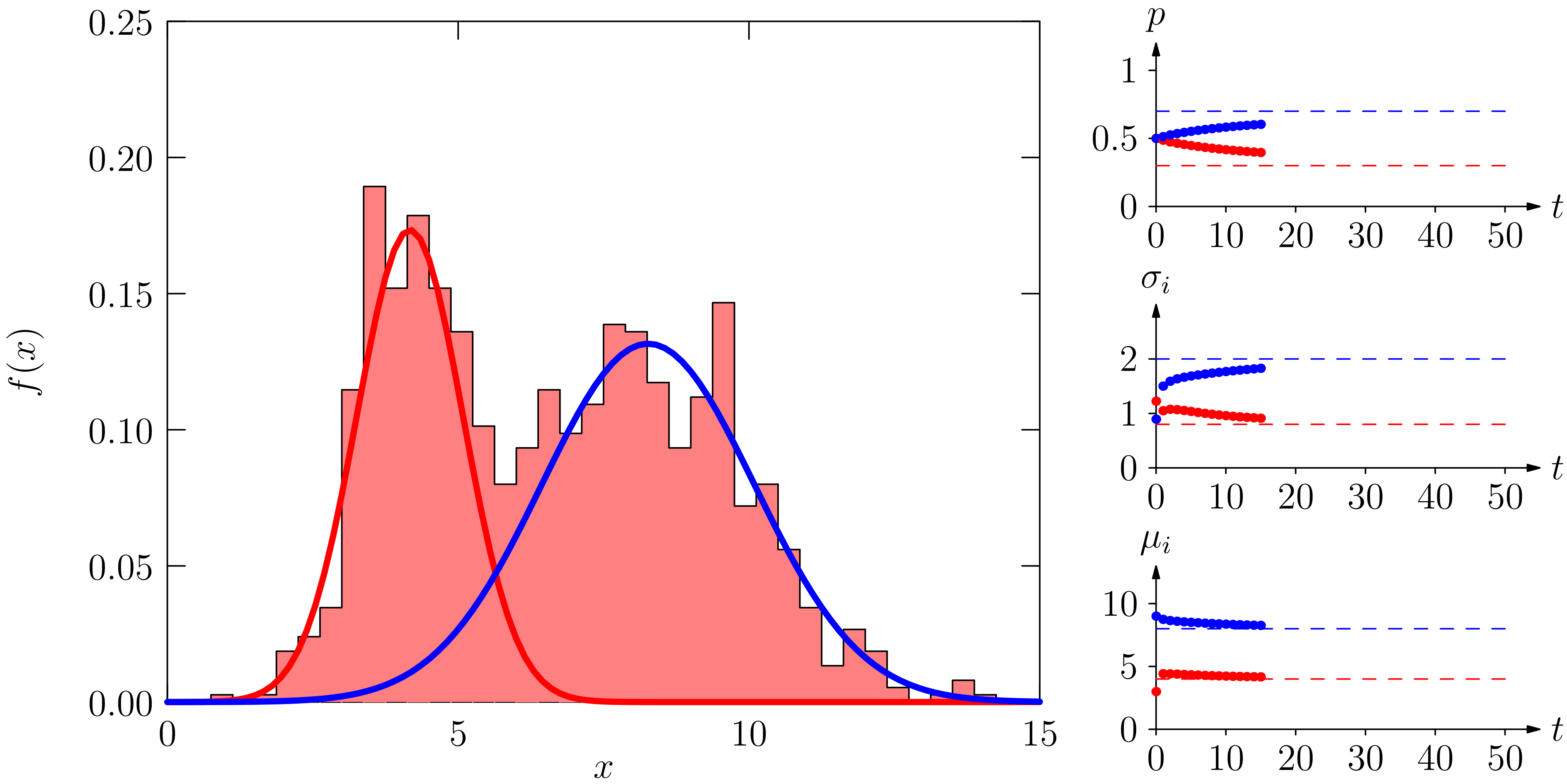
# Example



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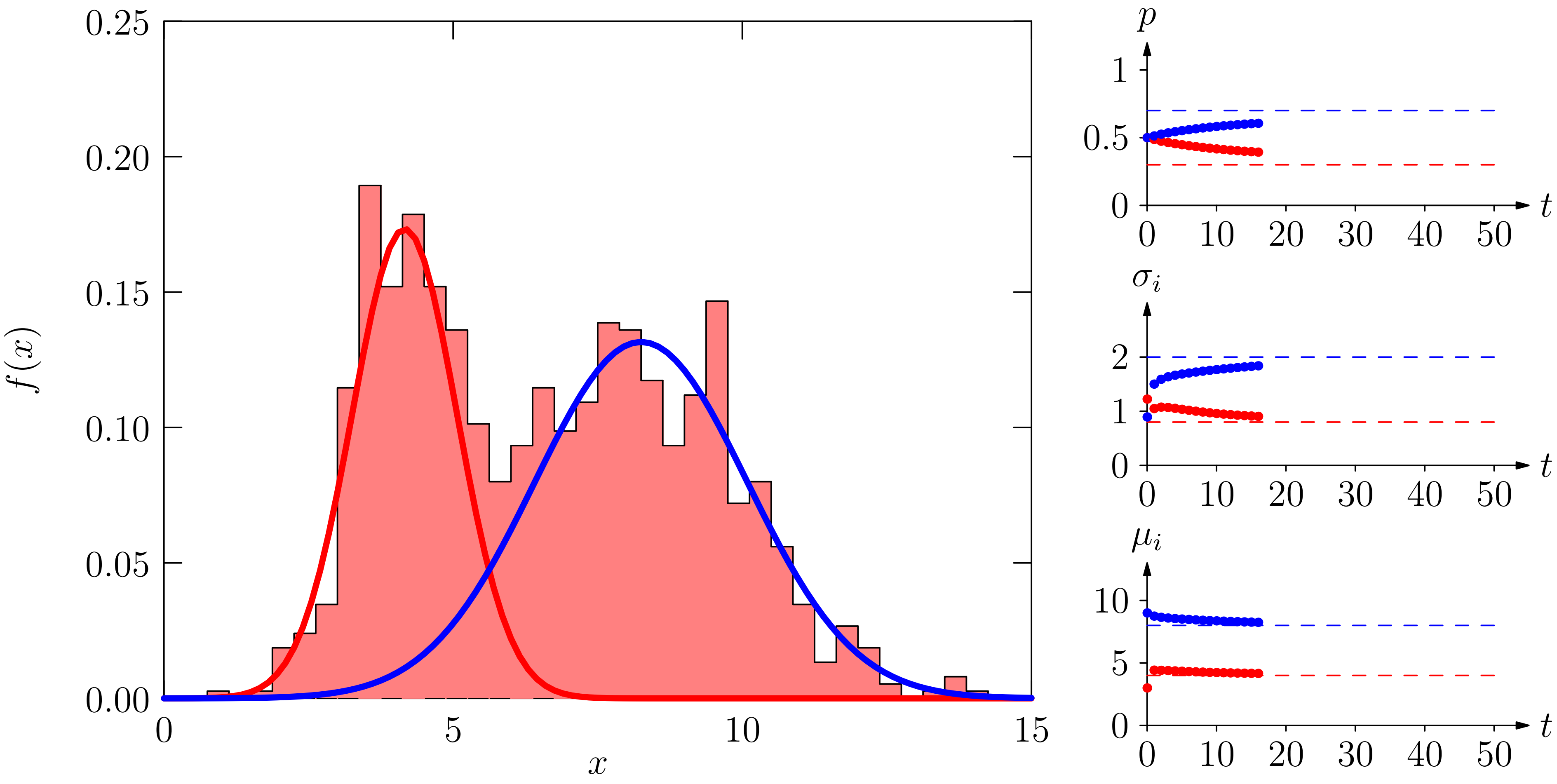

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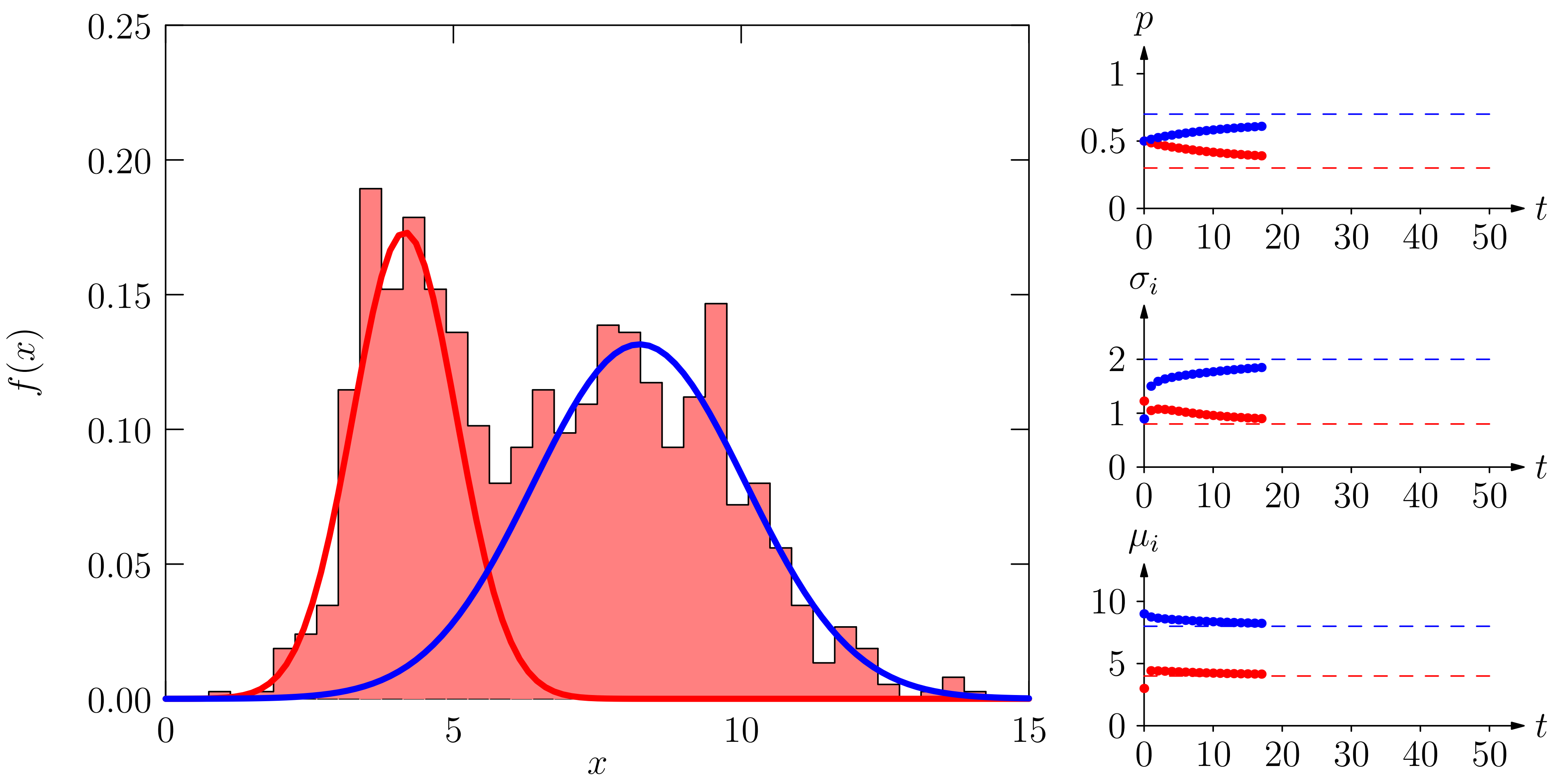

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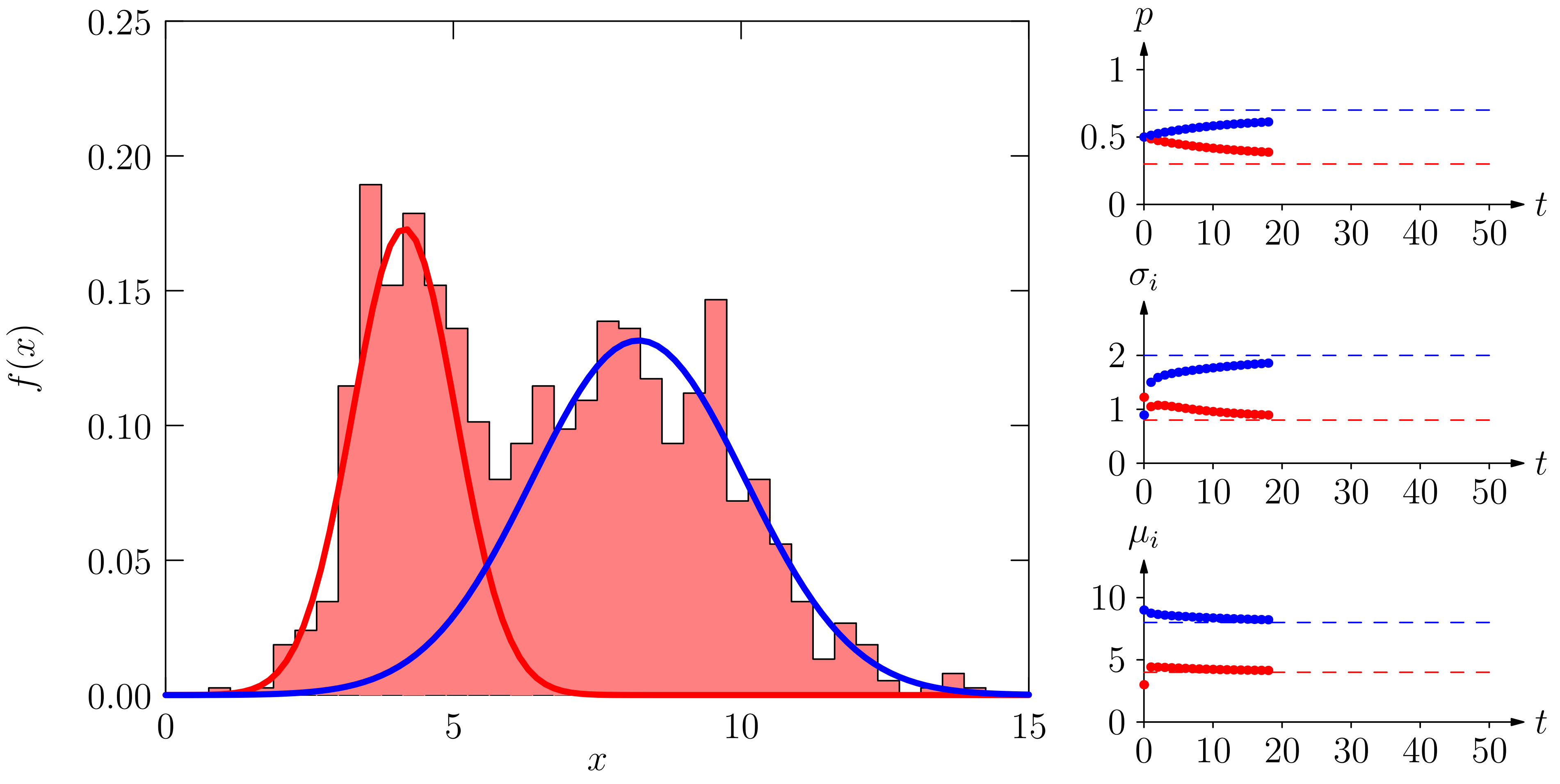

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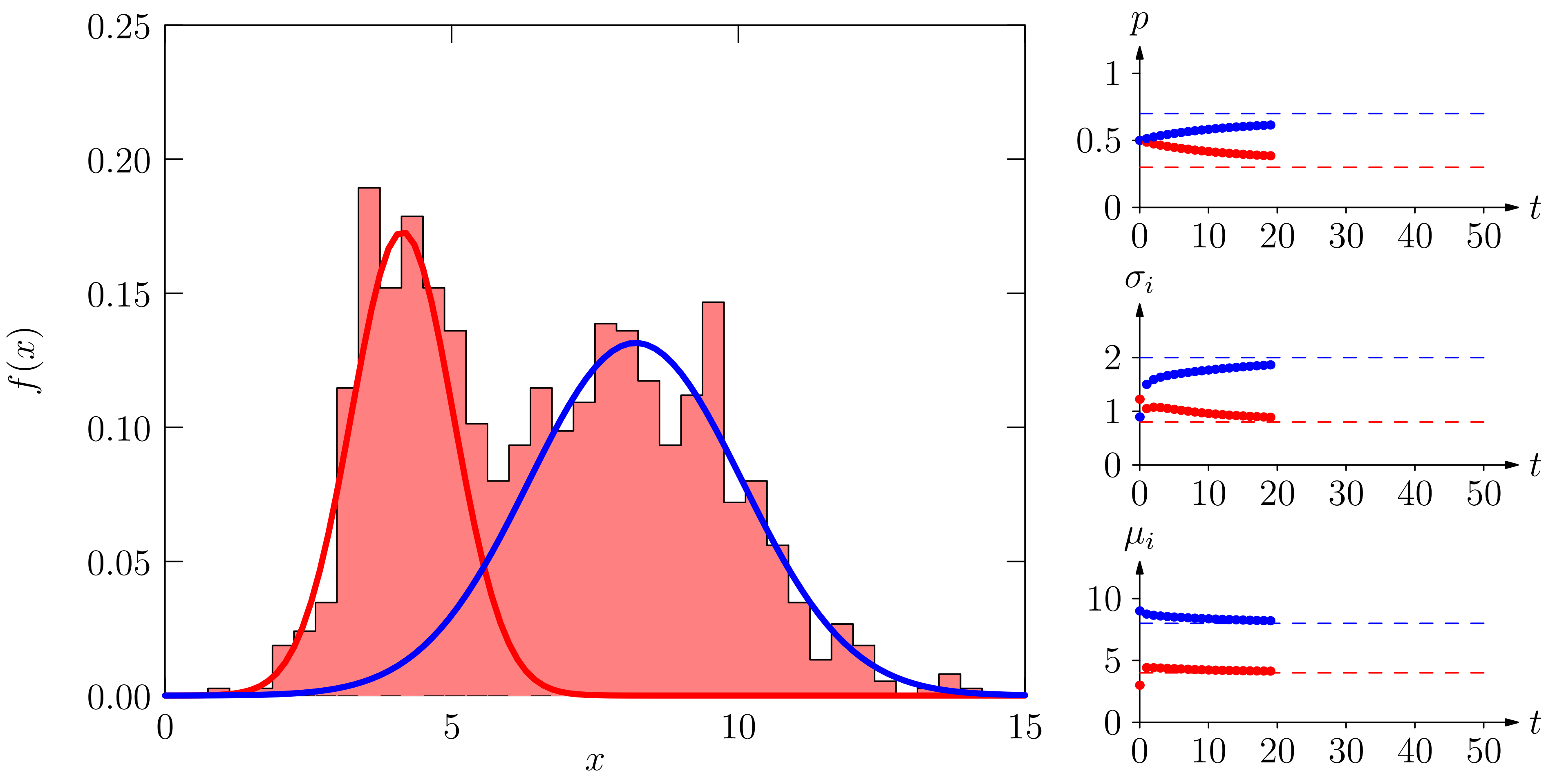

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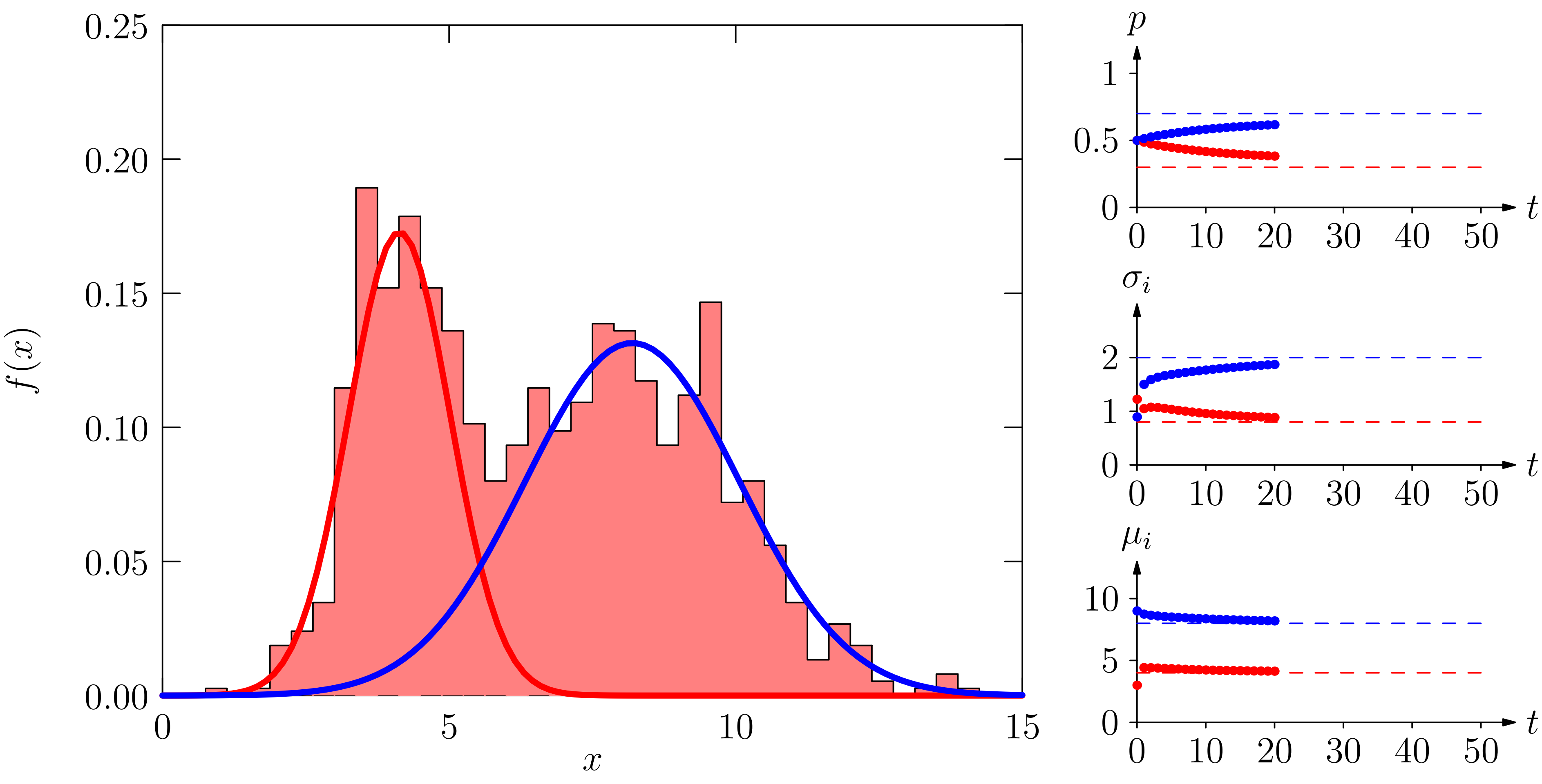

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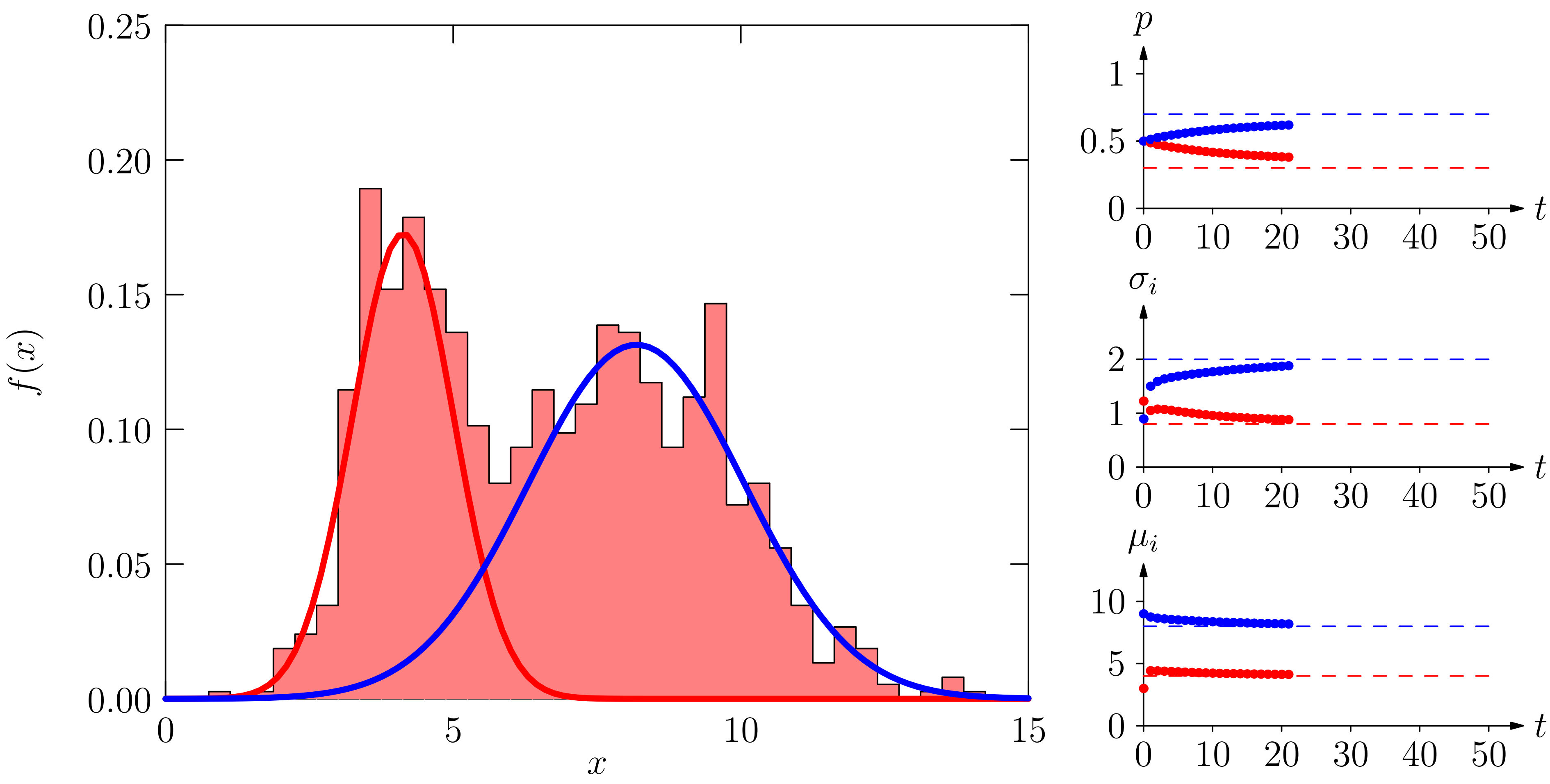

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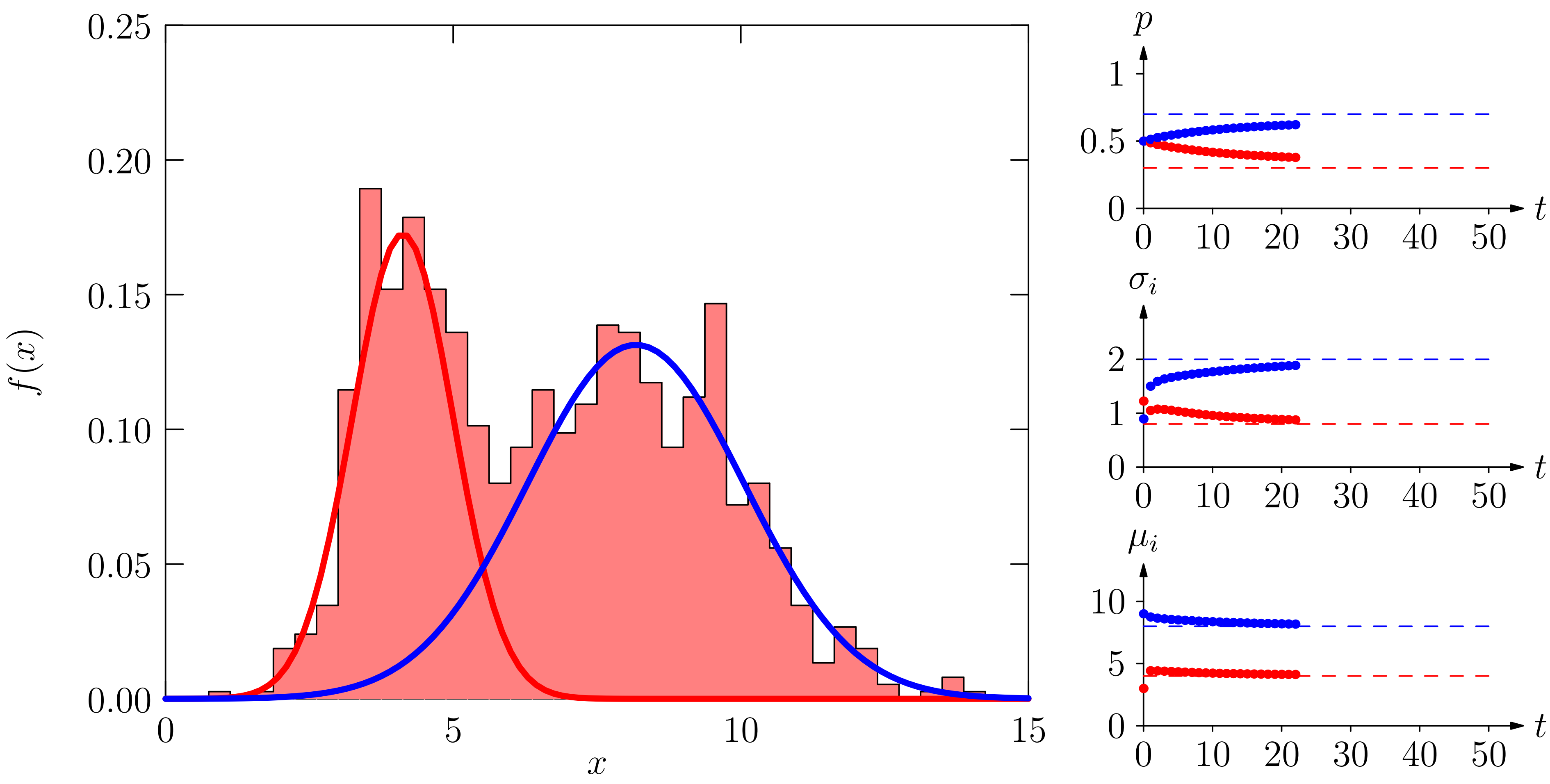
# Example



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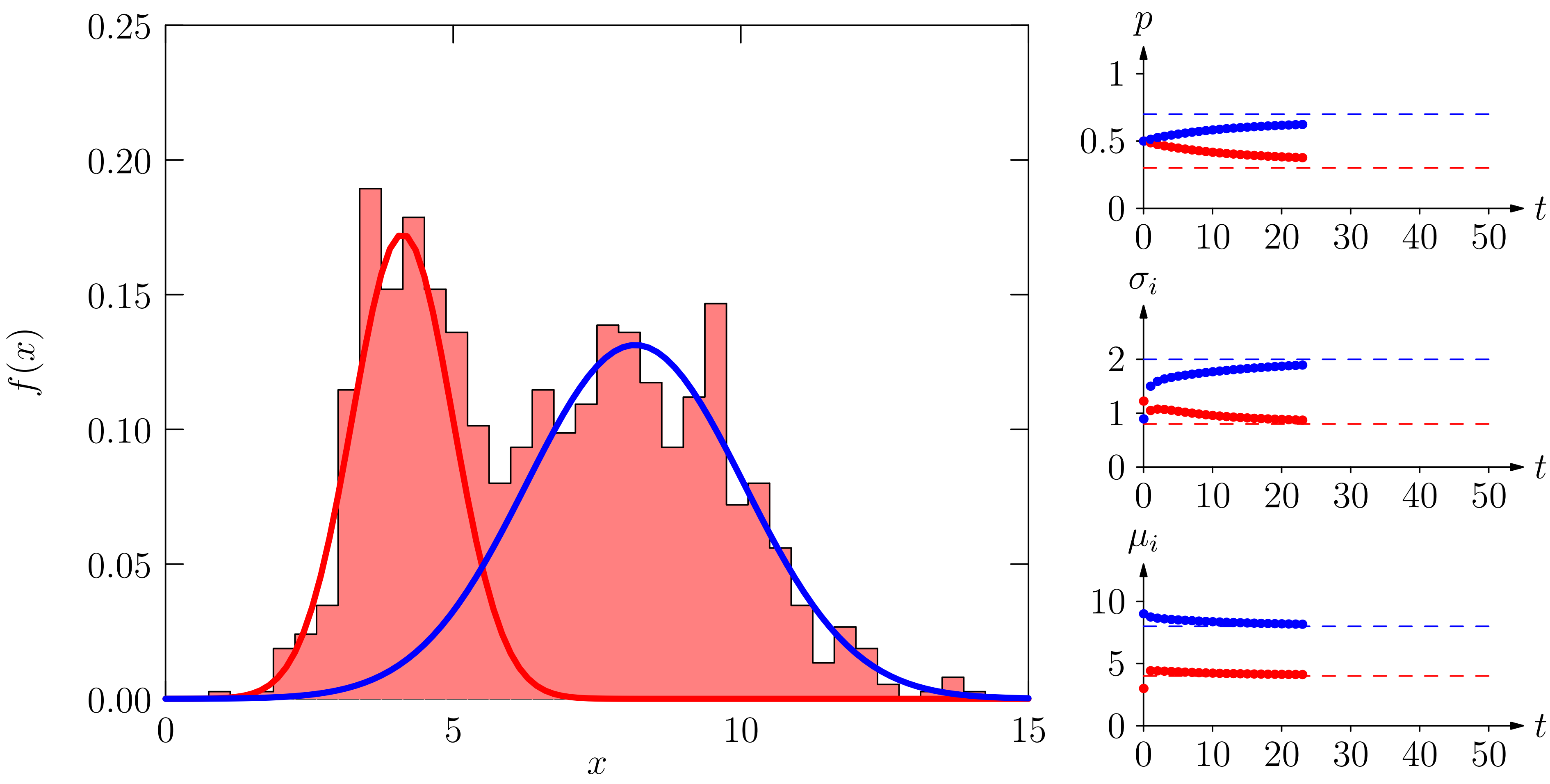

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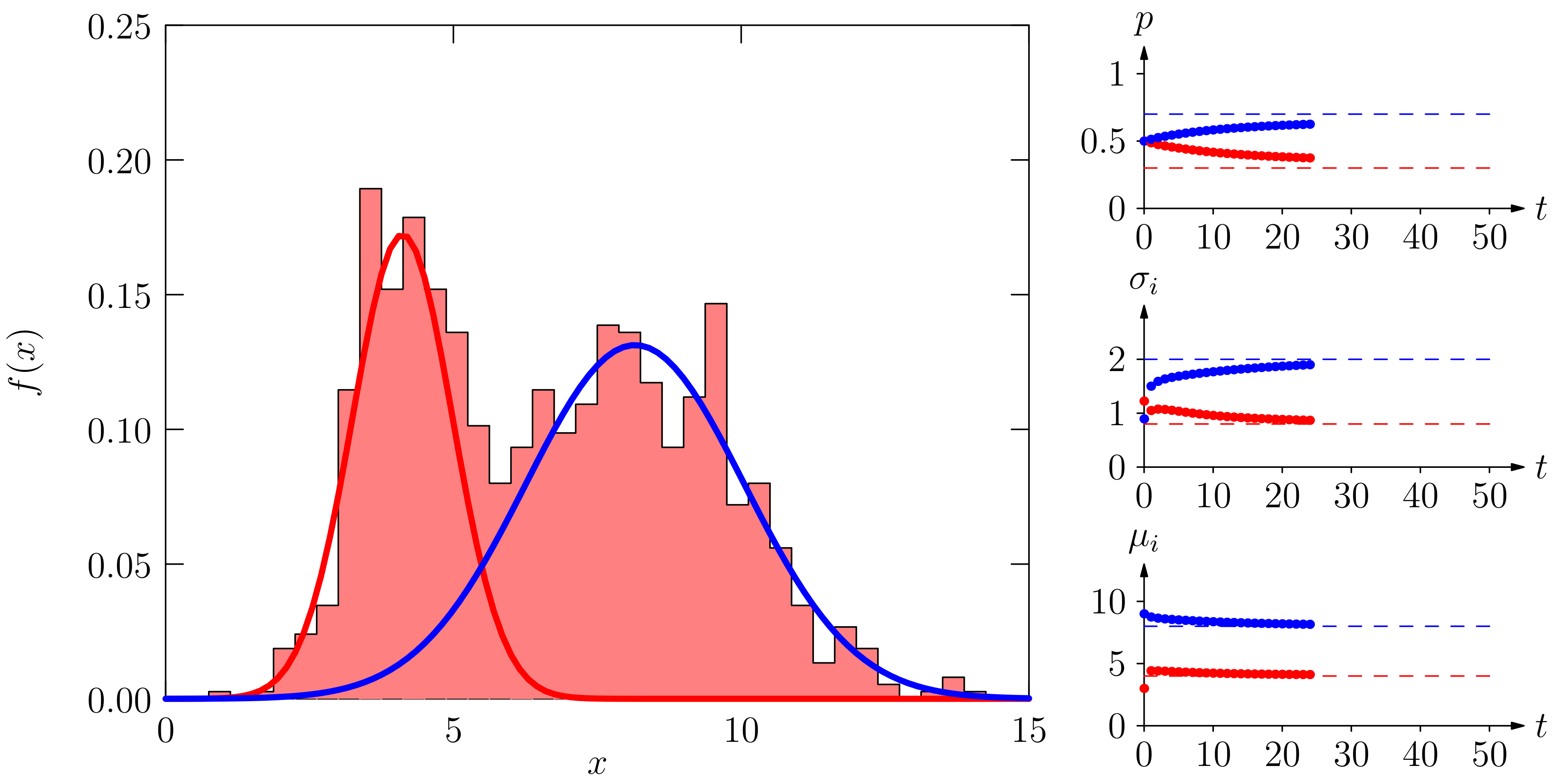
# Example



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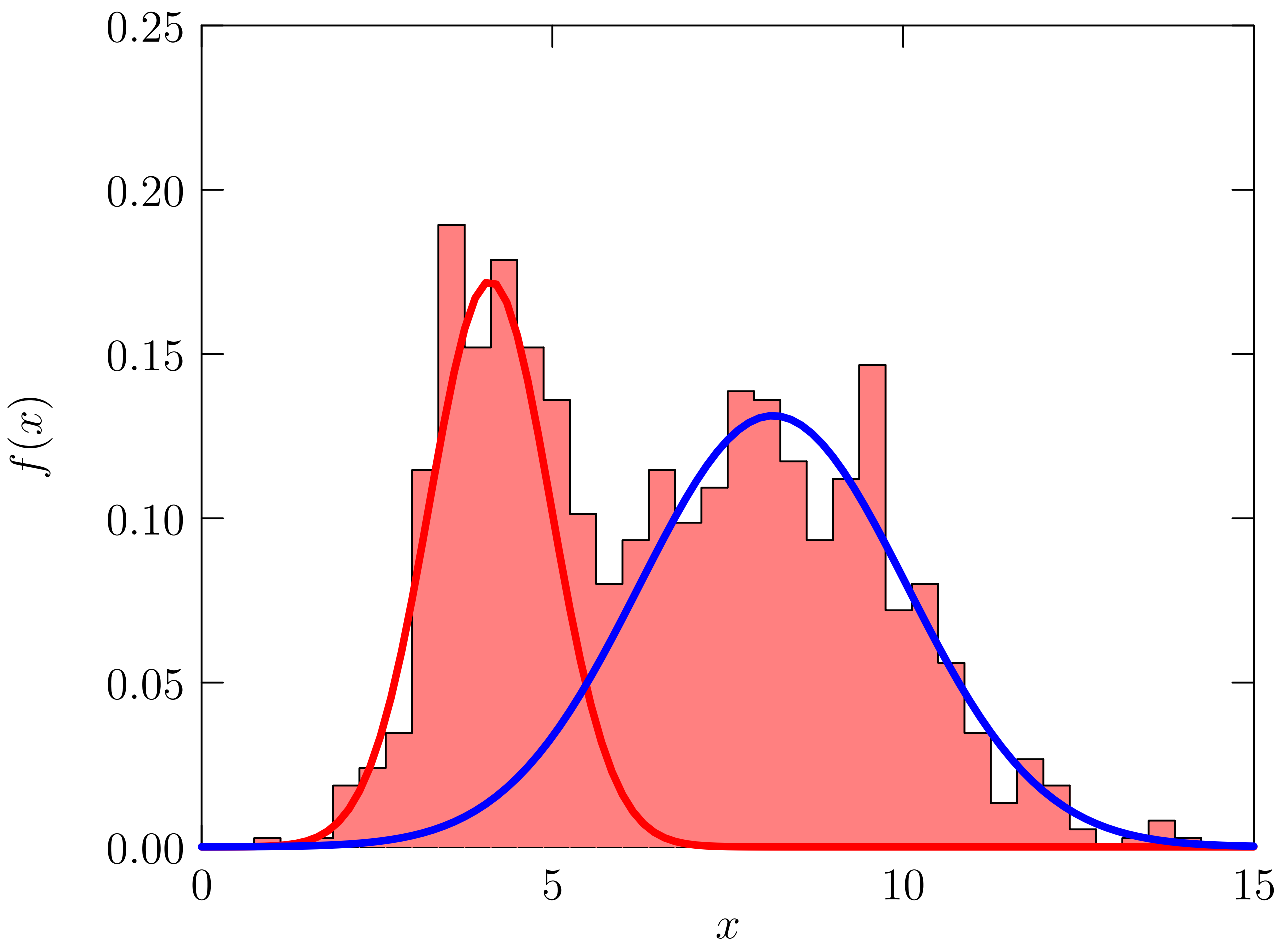
# Example



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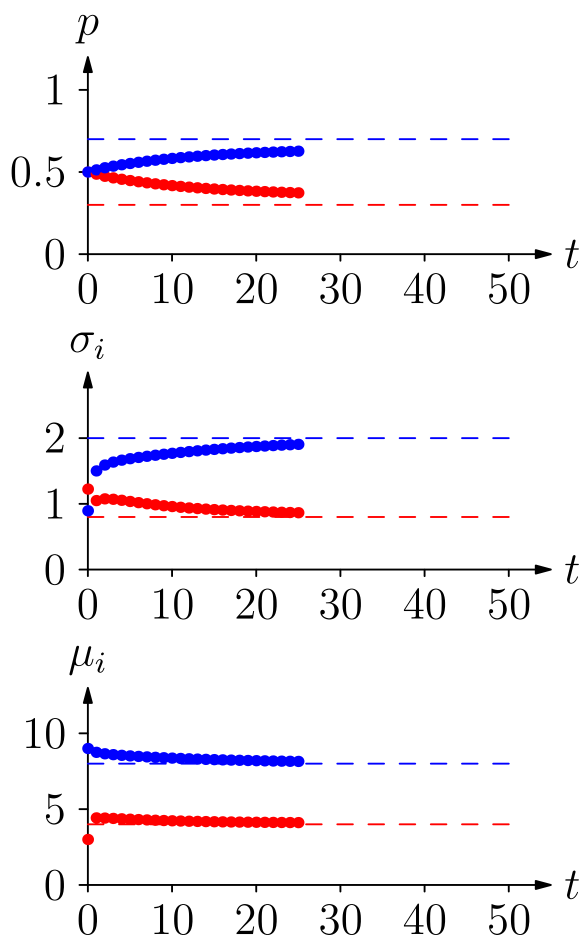

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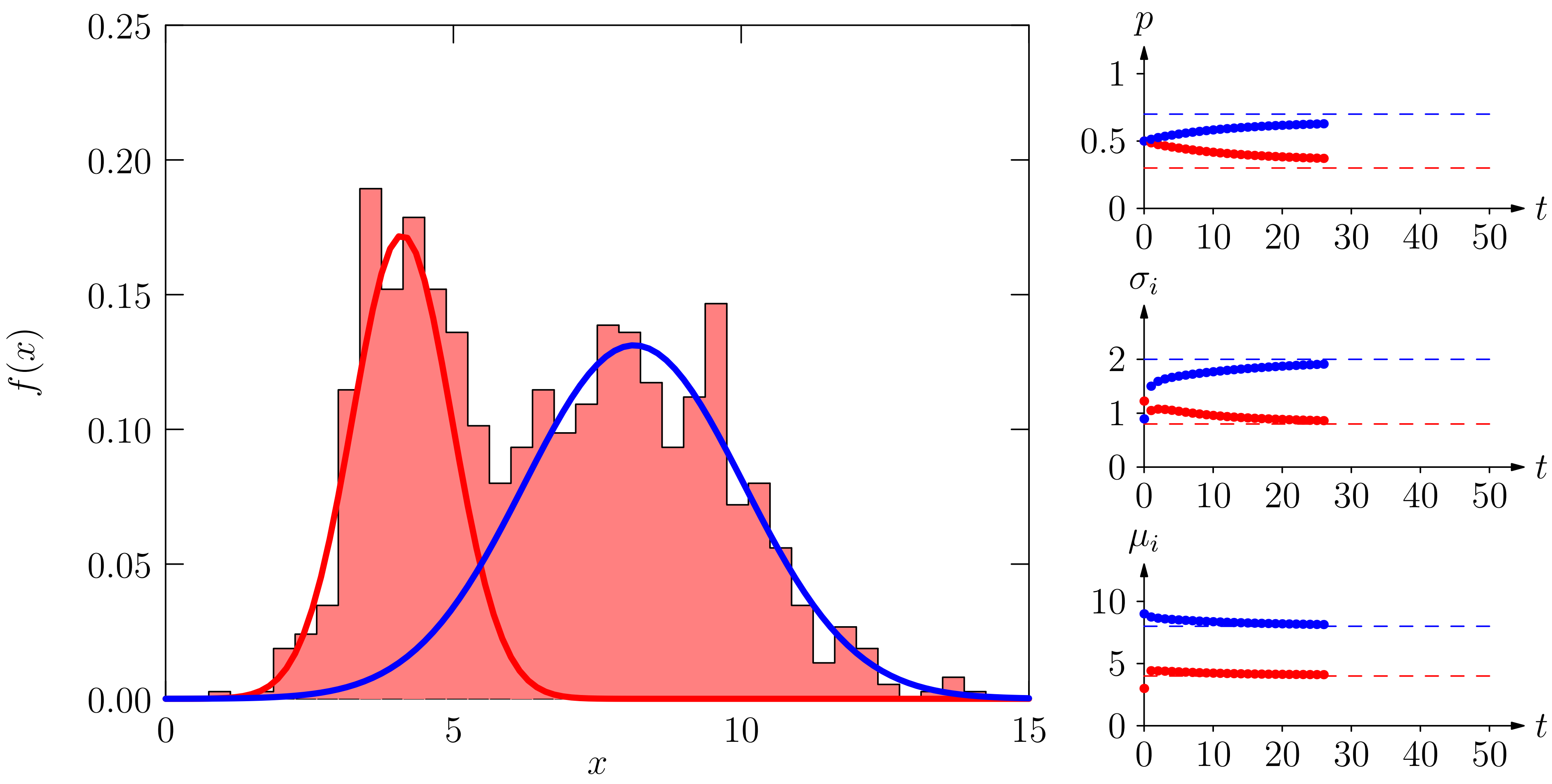

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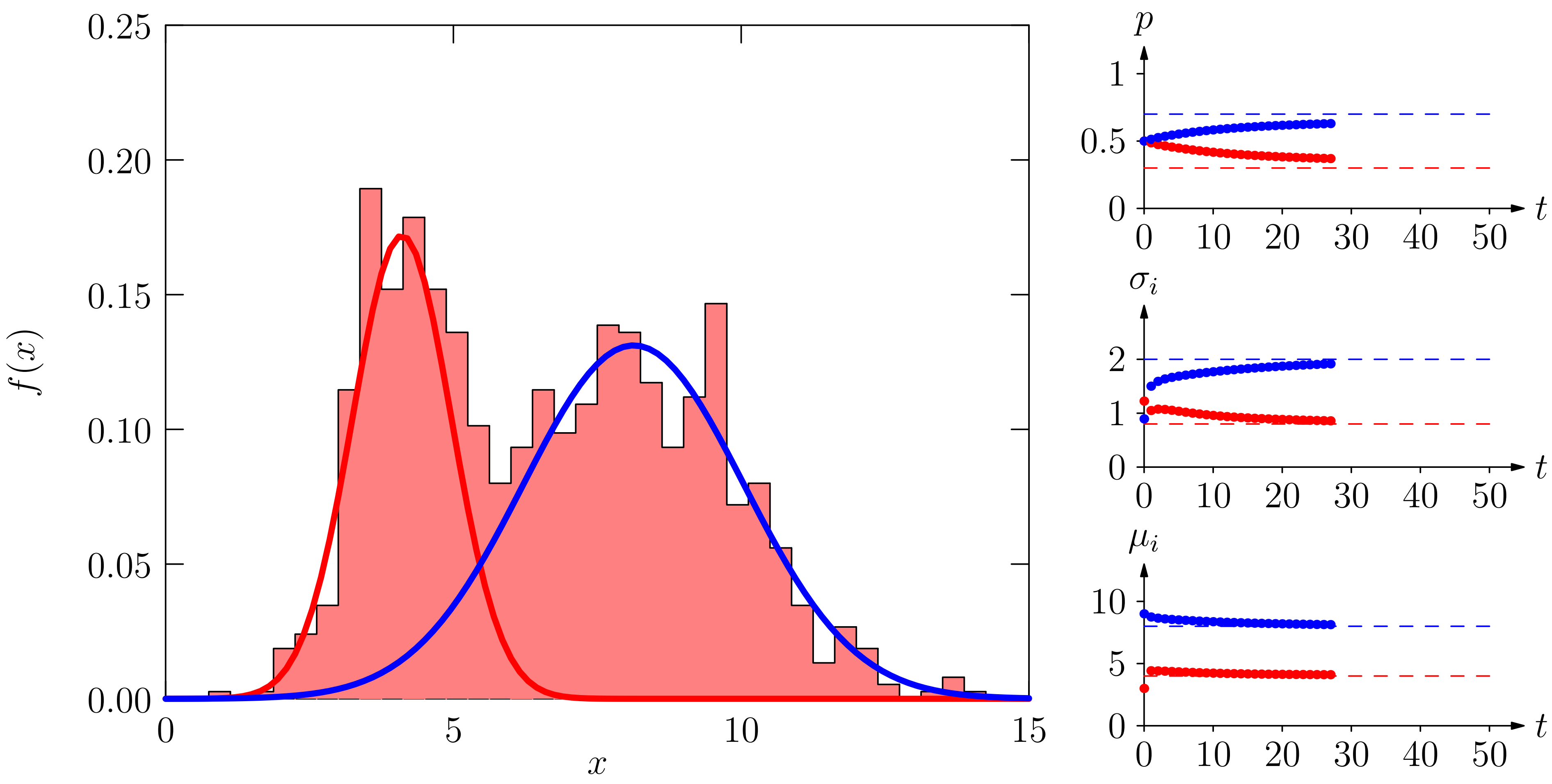
# Example



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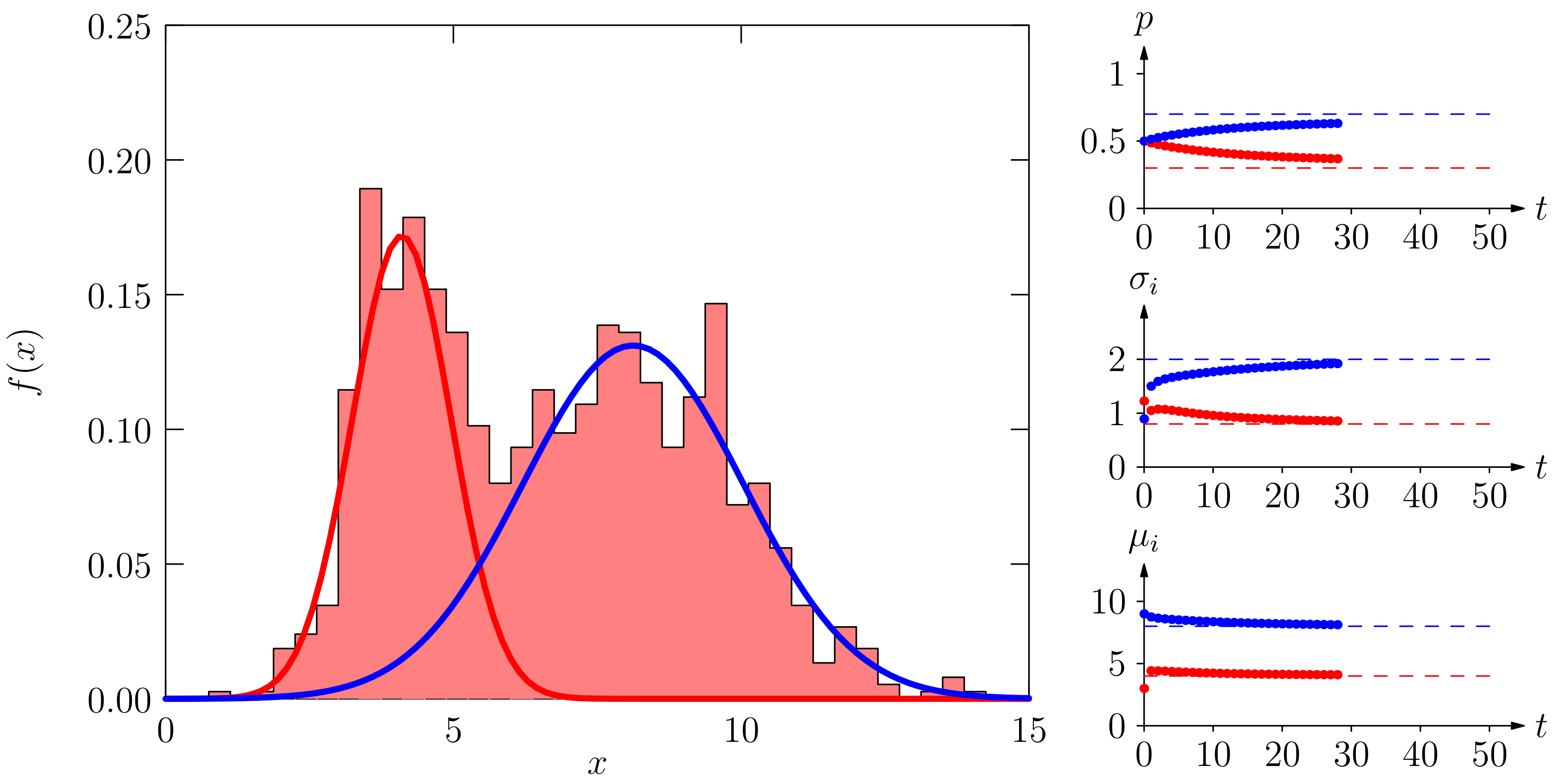
# Example



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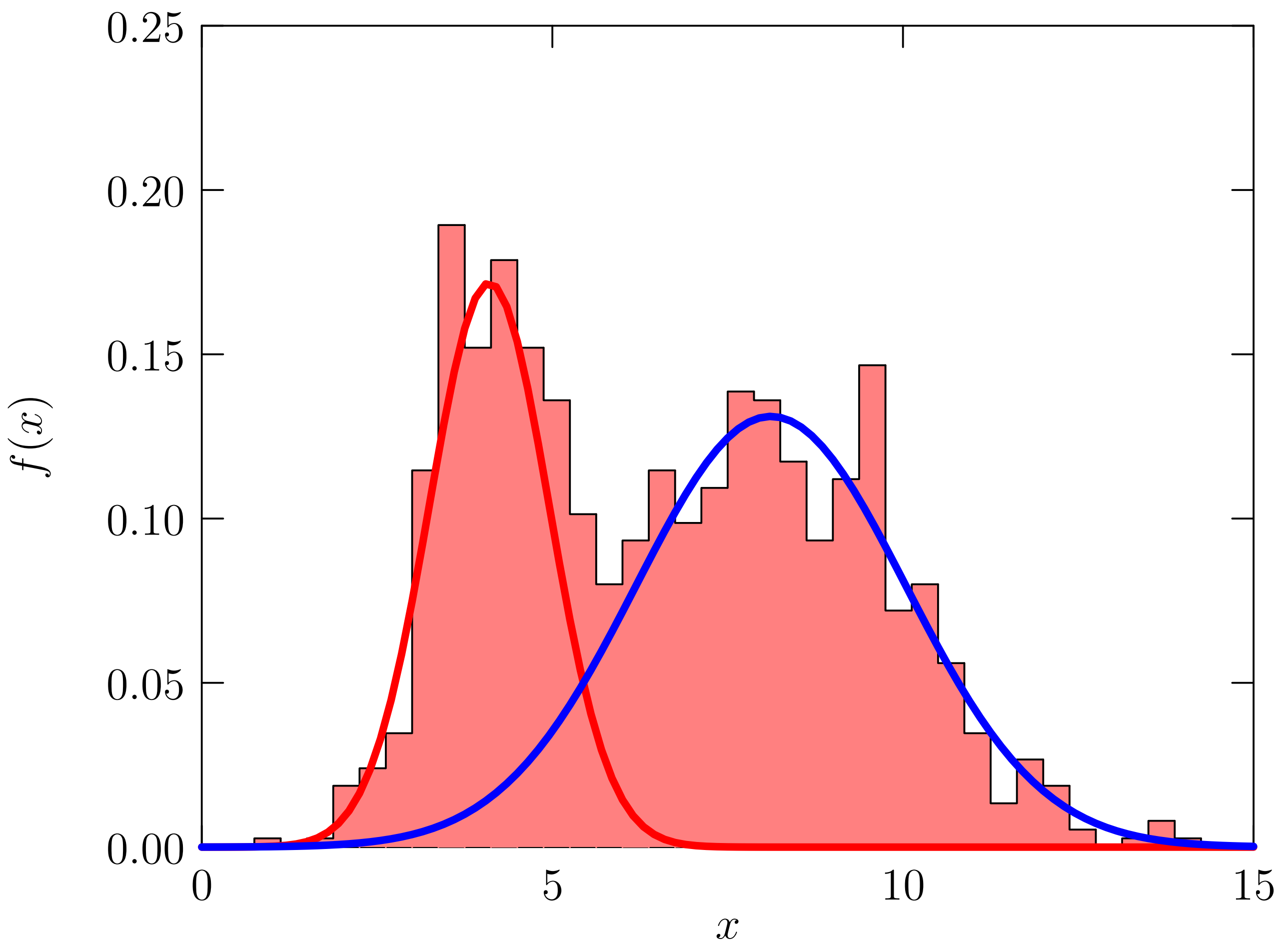
# Example



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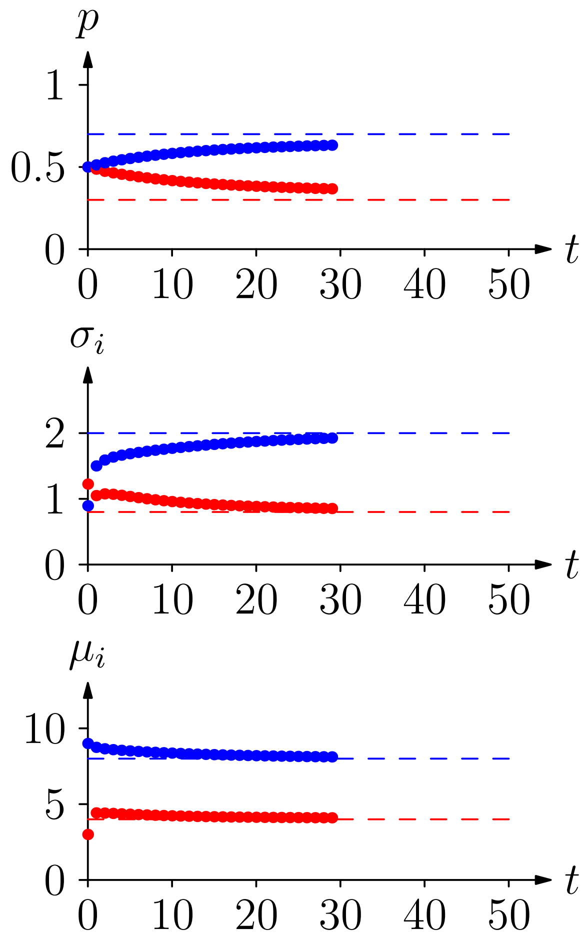

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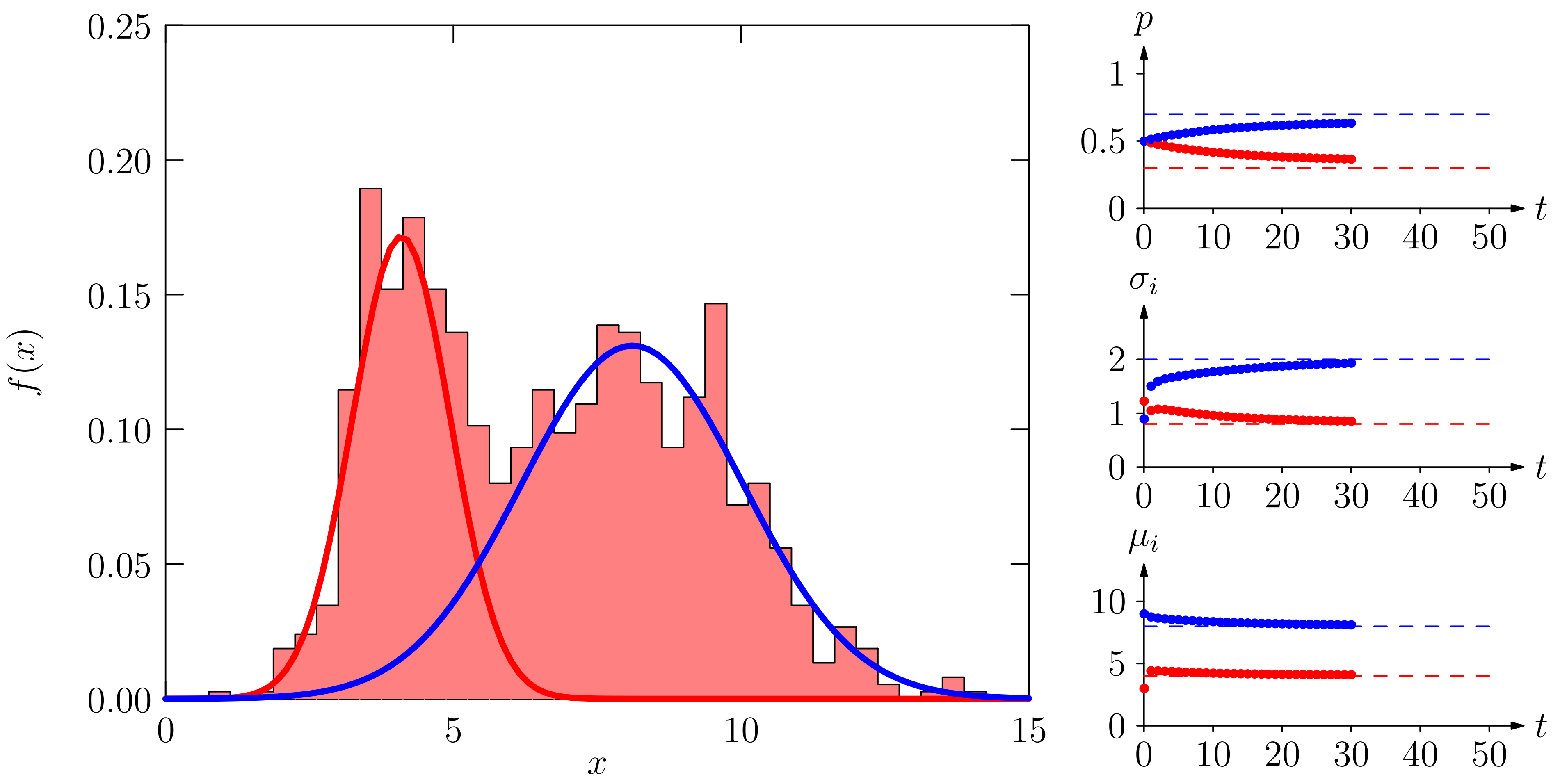

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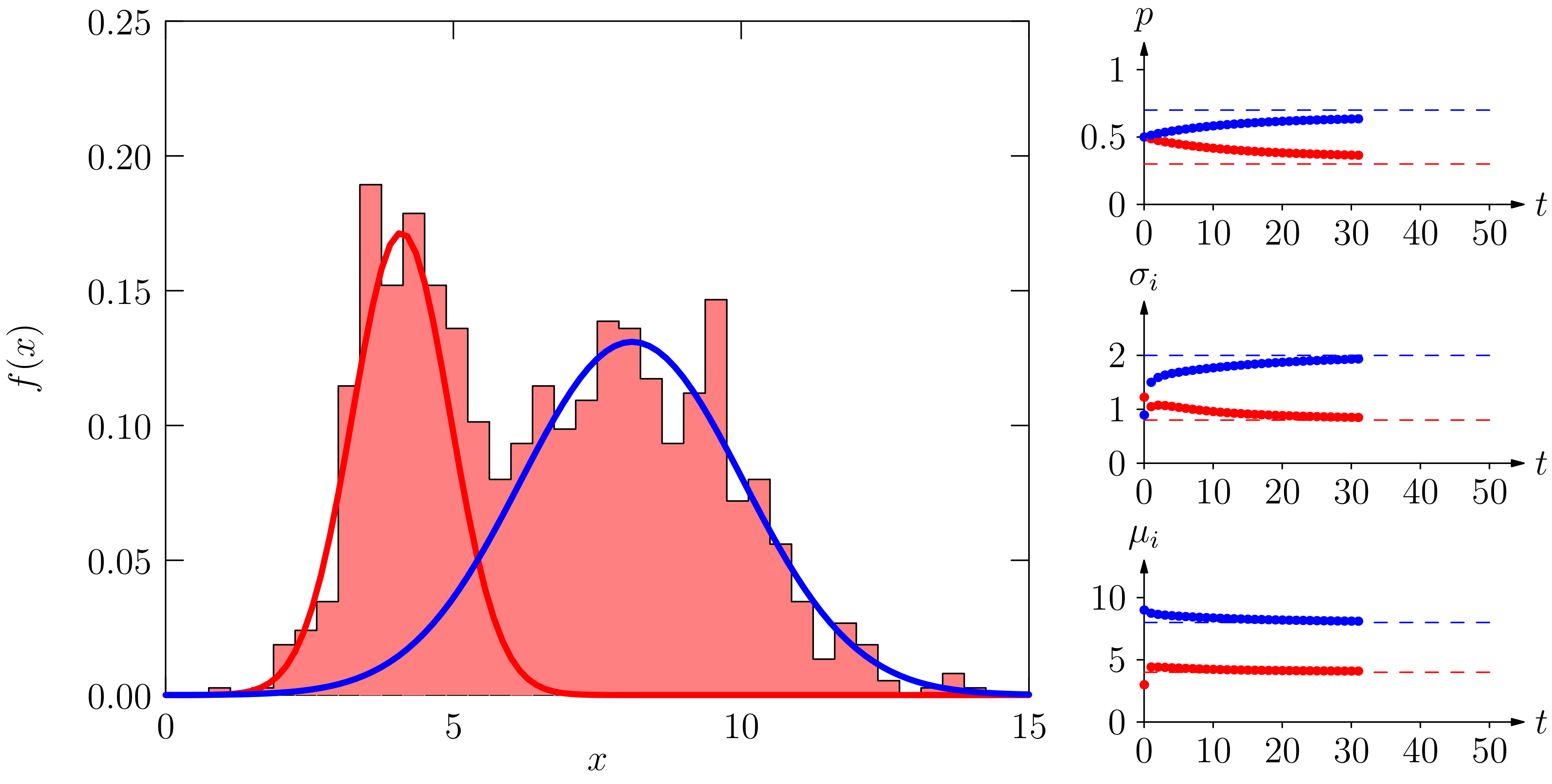

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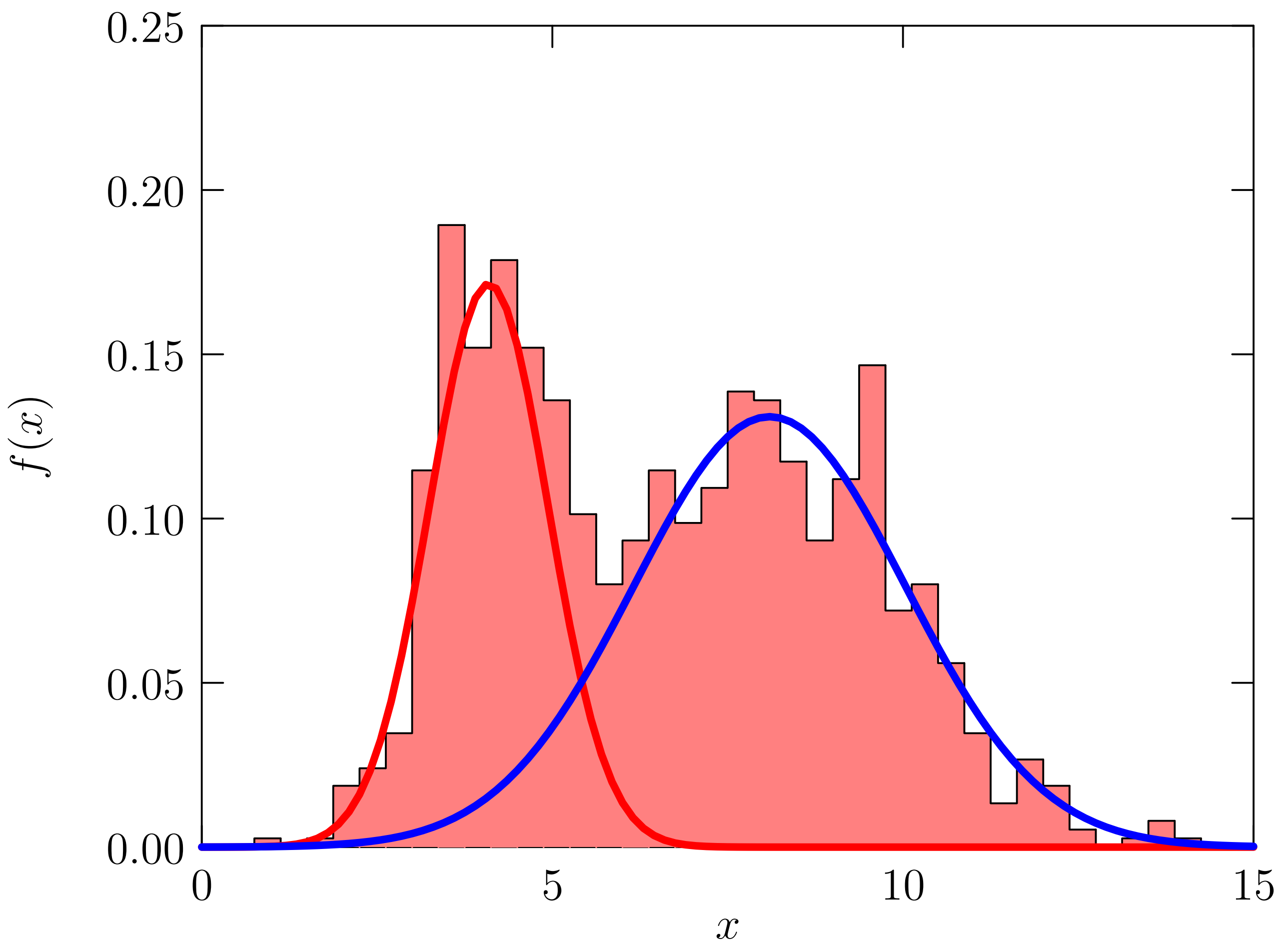
# Example



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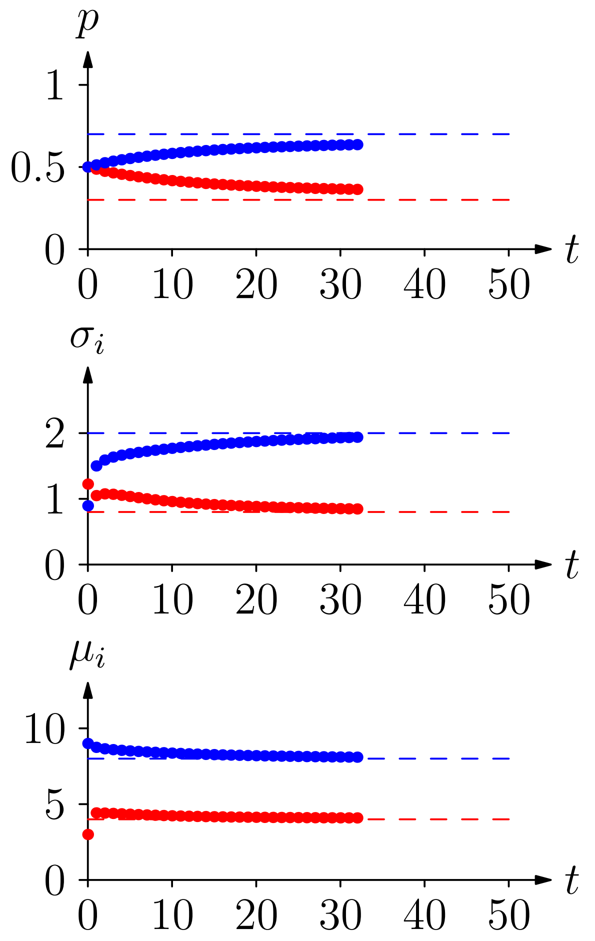

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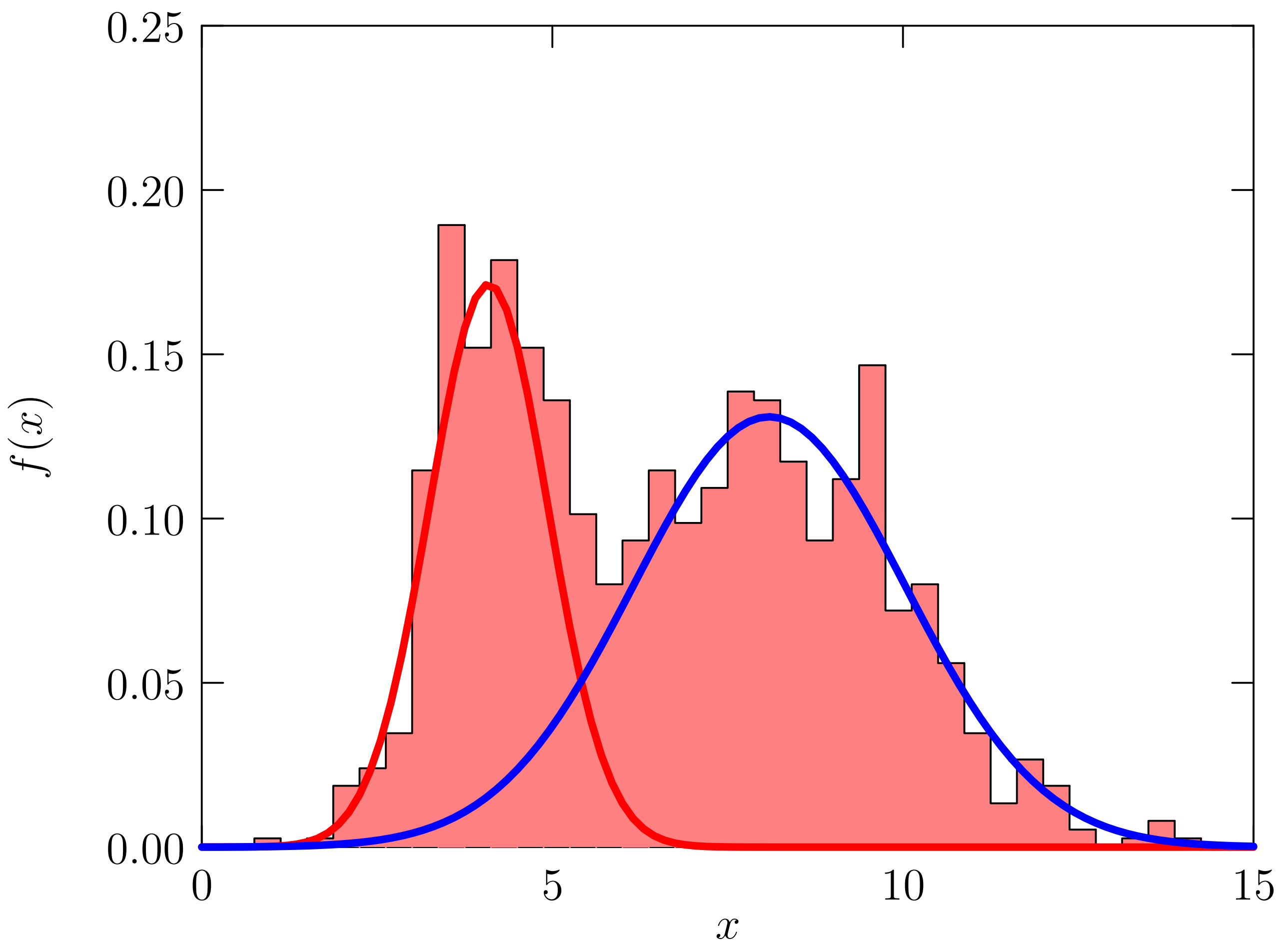

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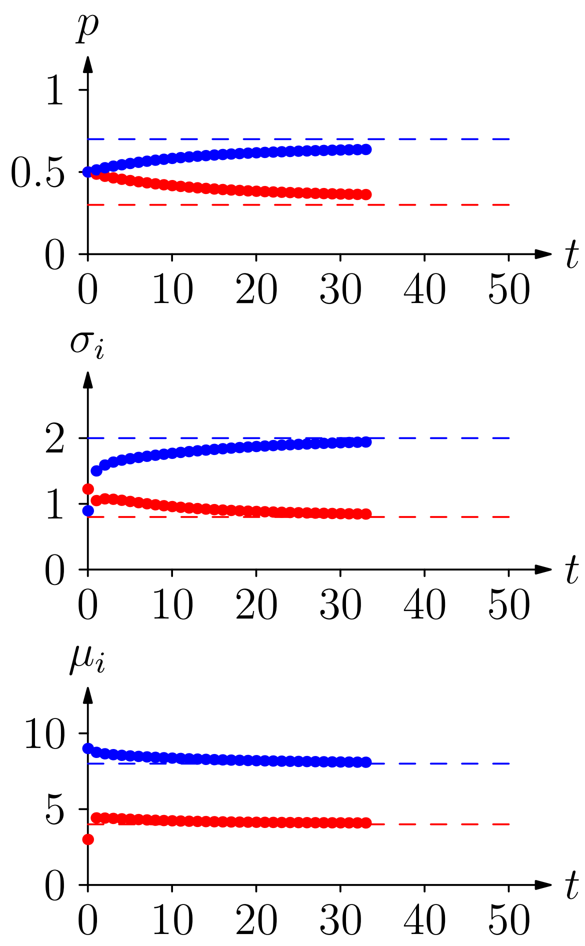

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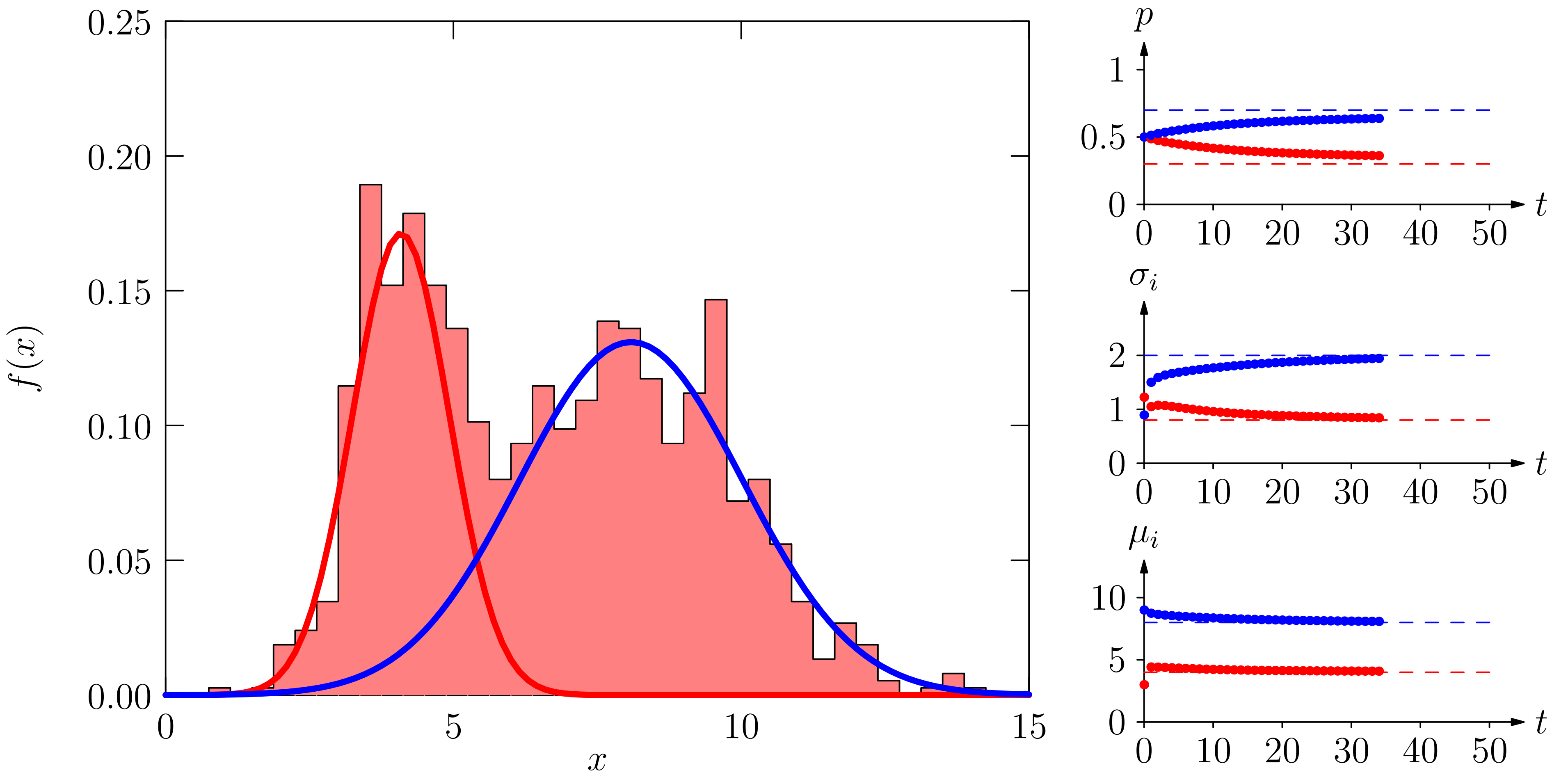

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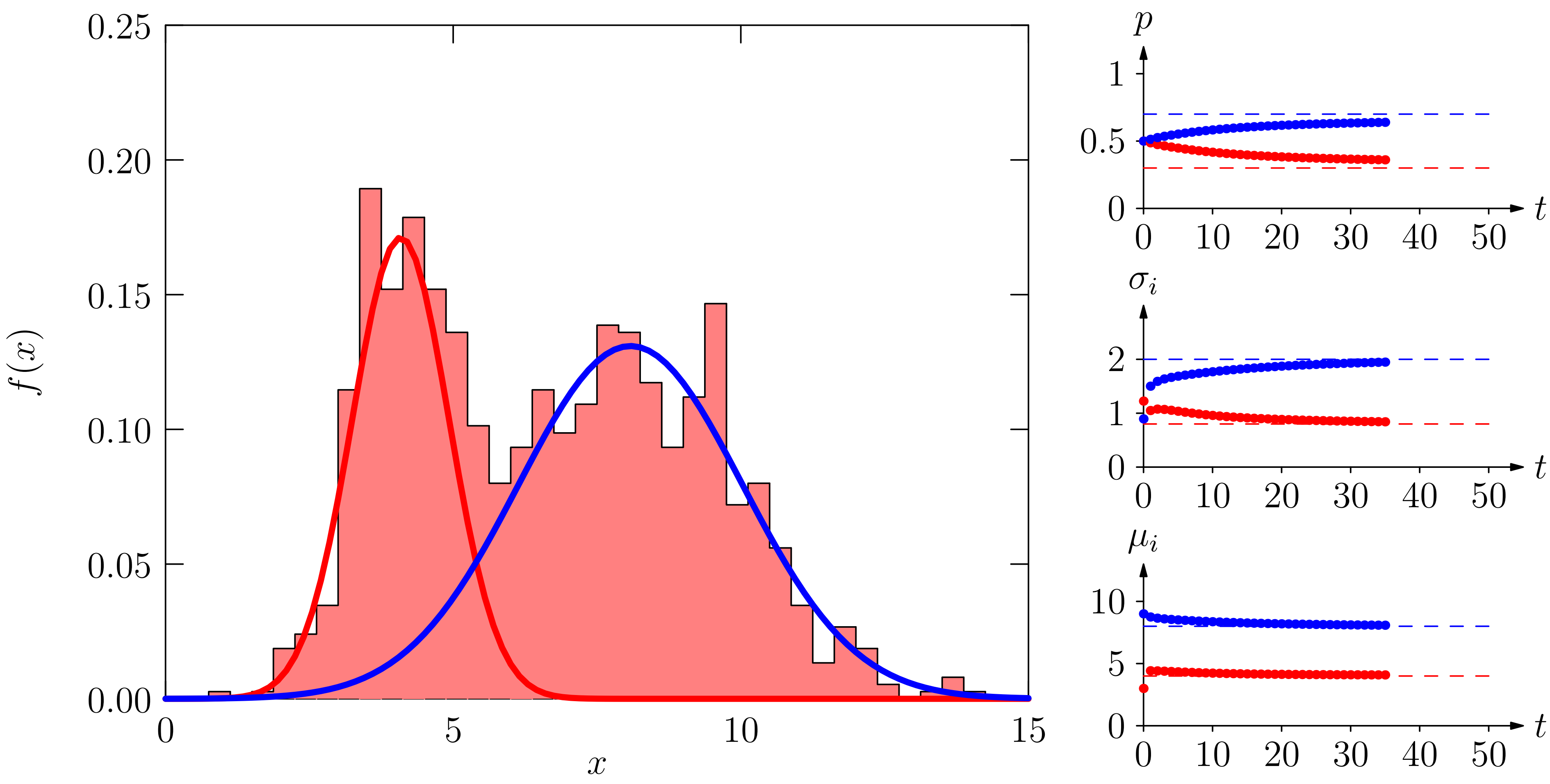
# Example



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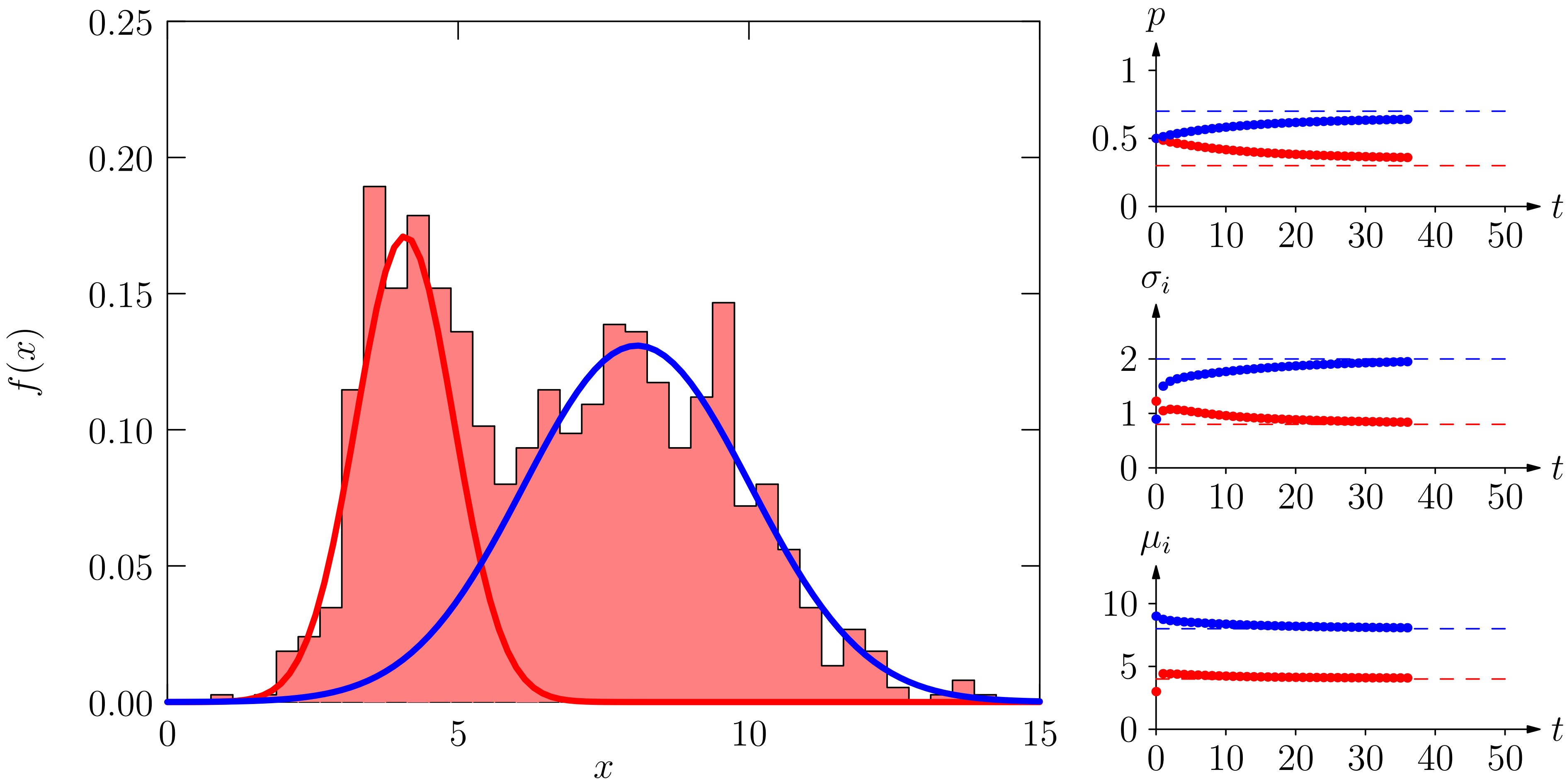

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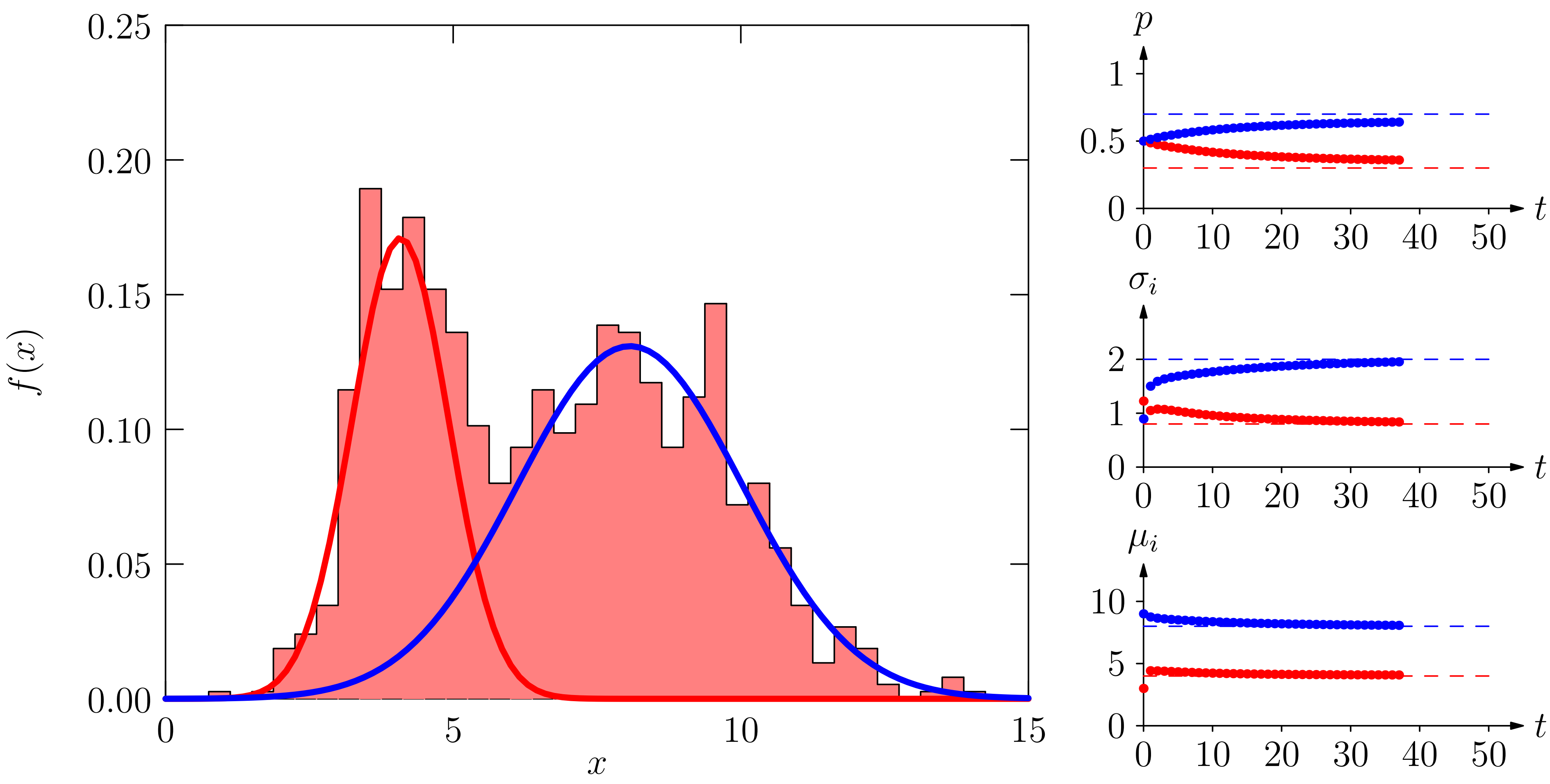

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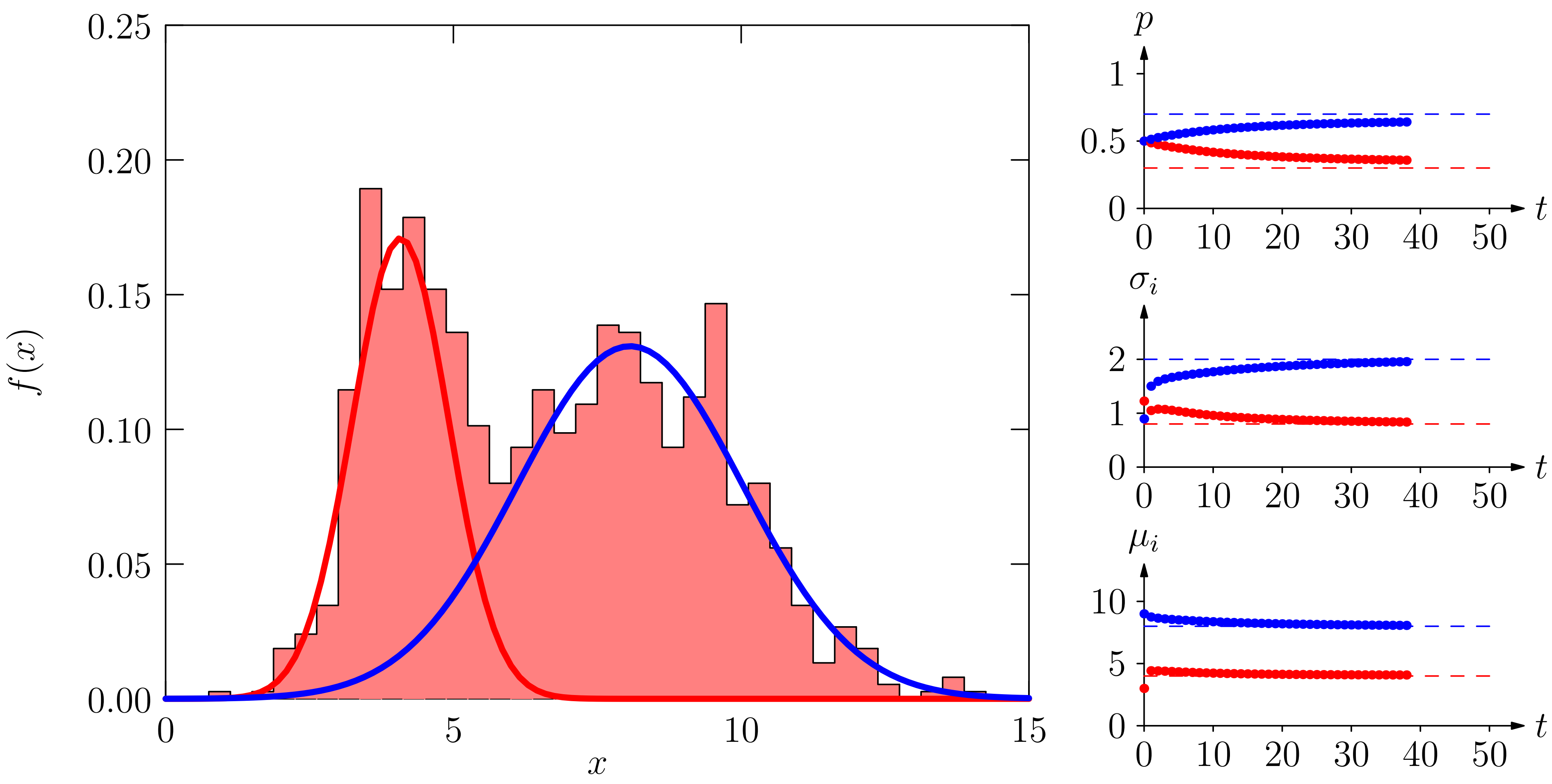

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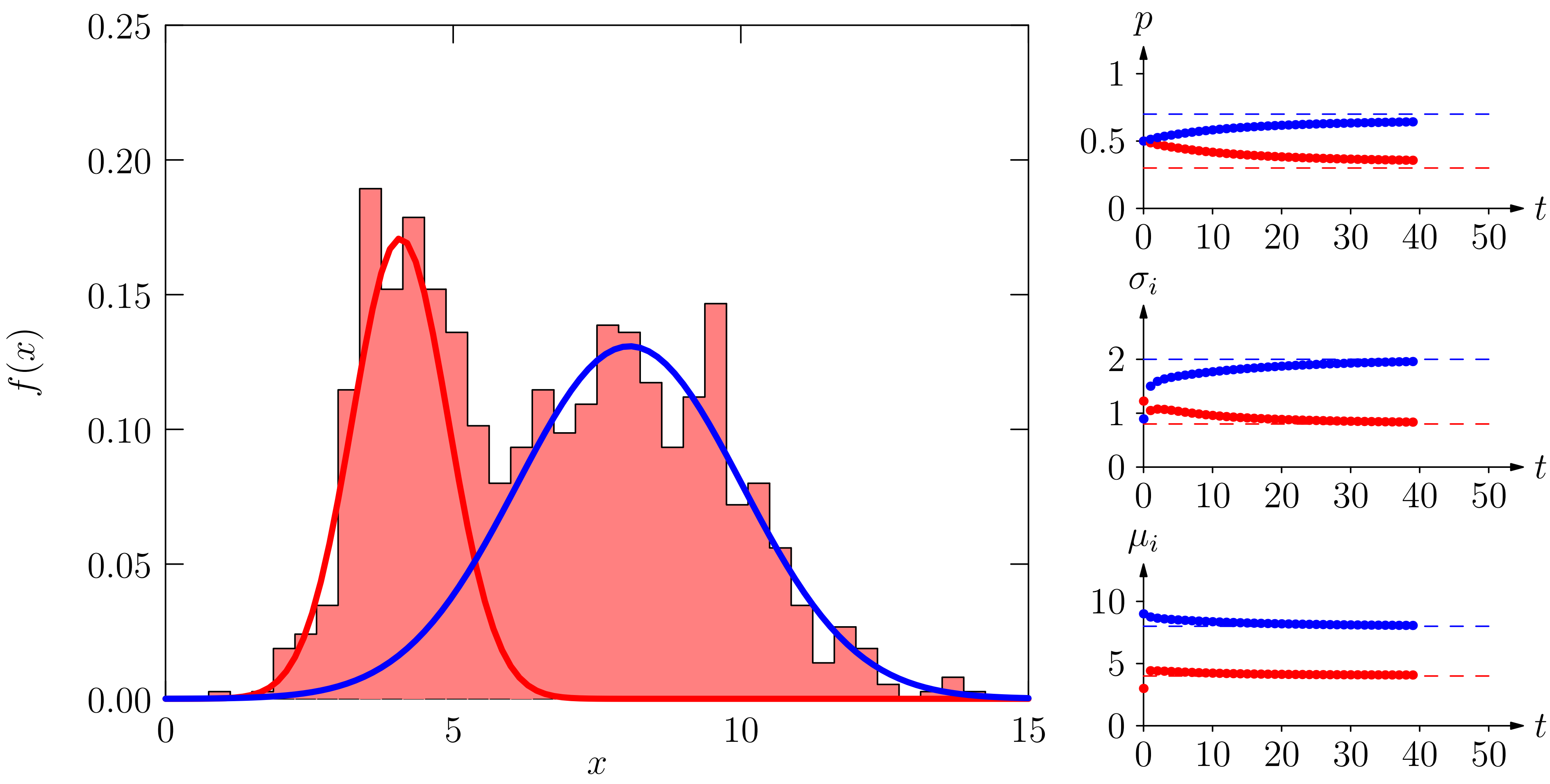

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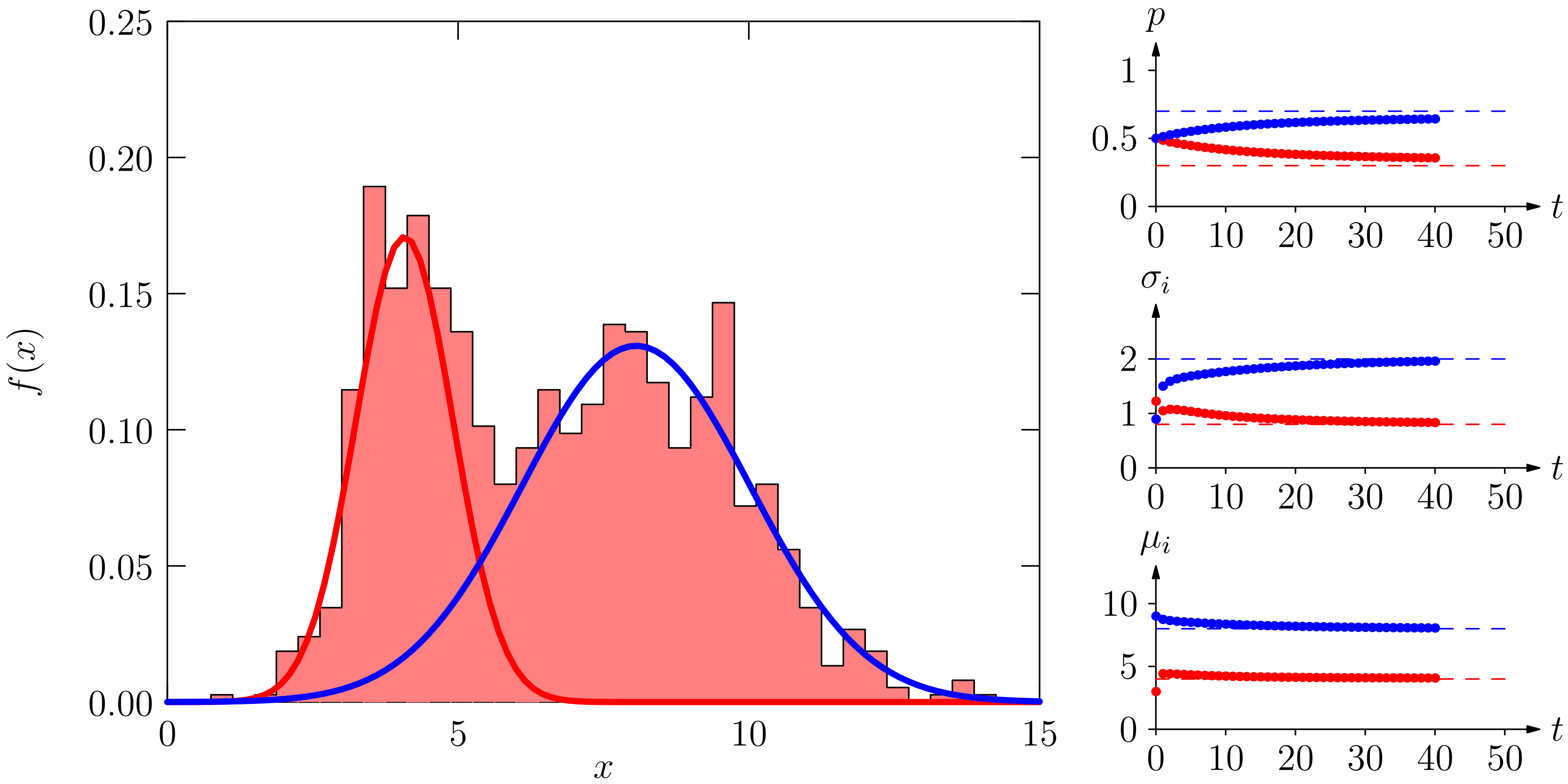

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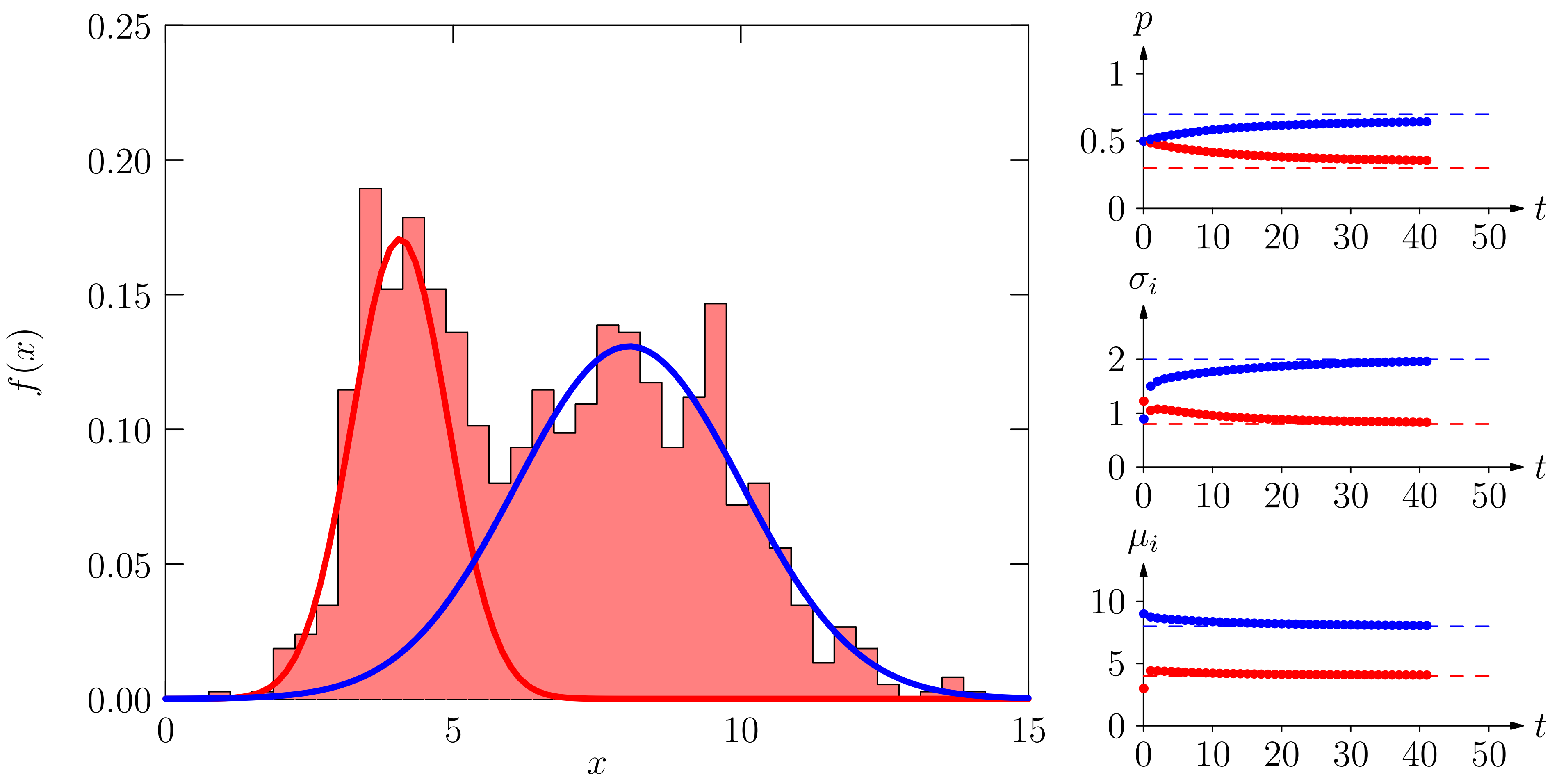

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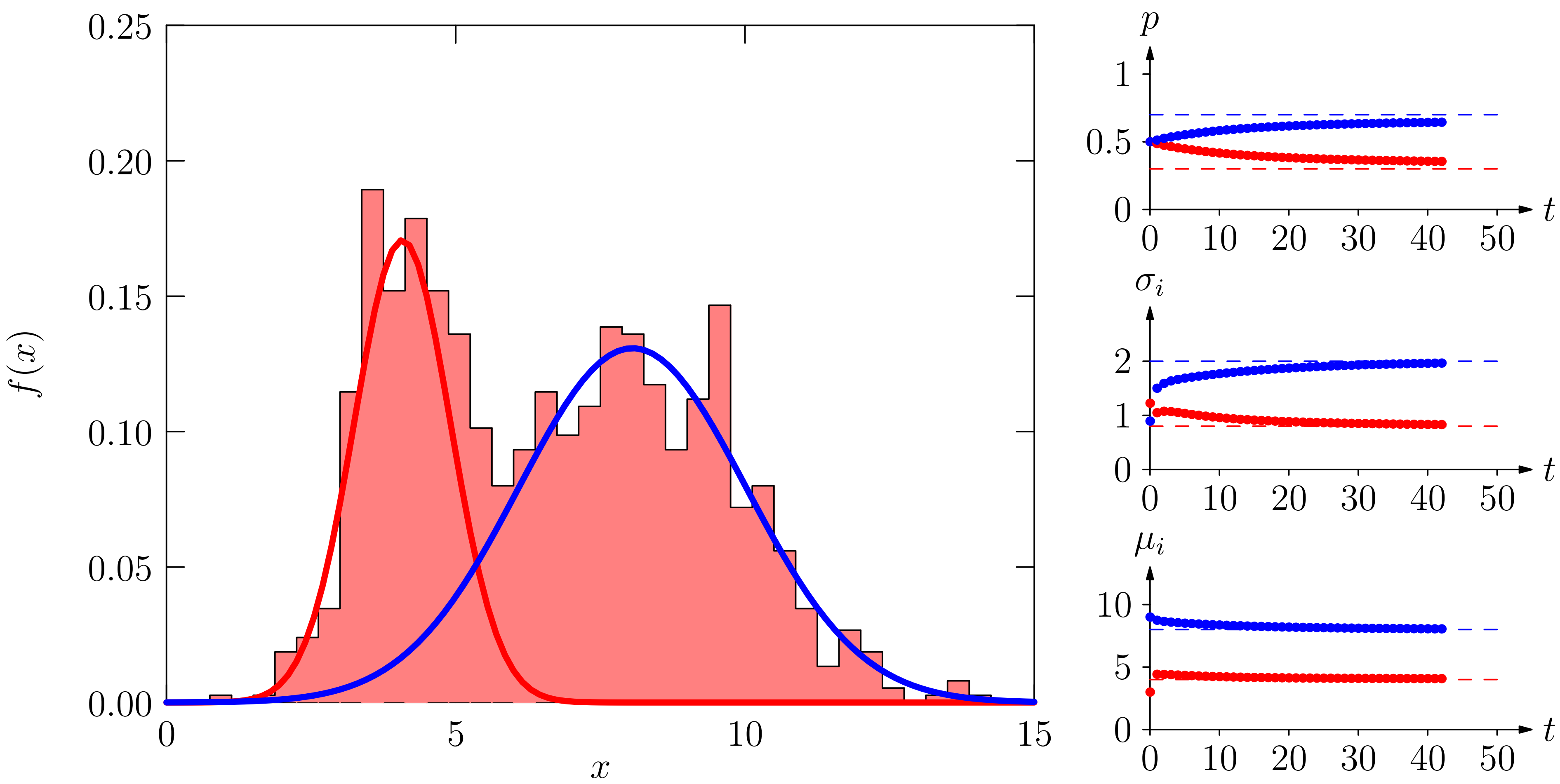

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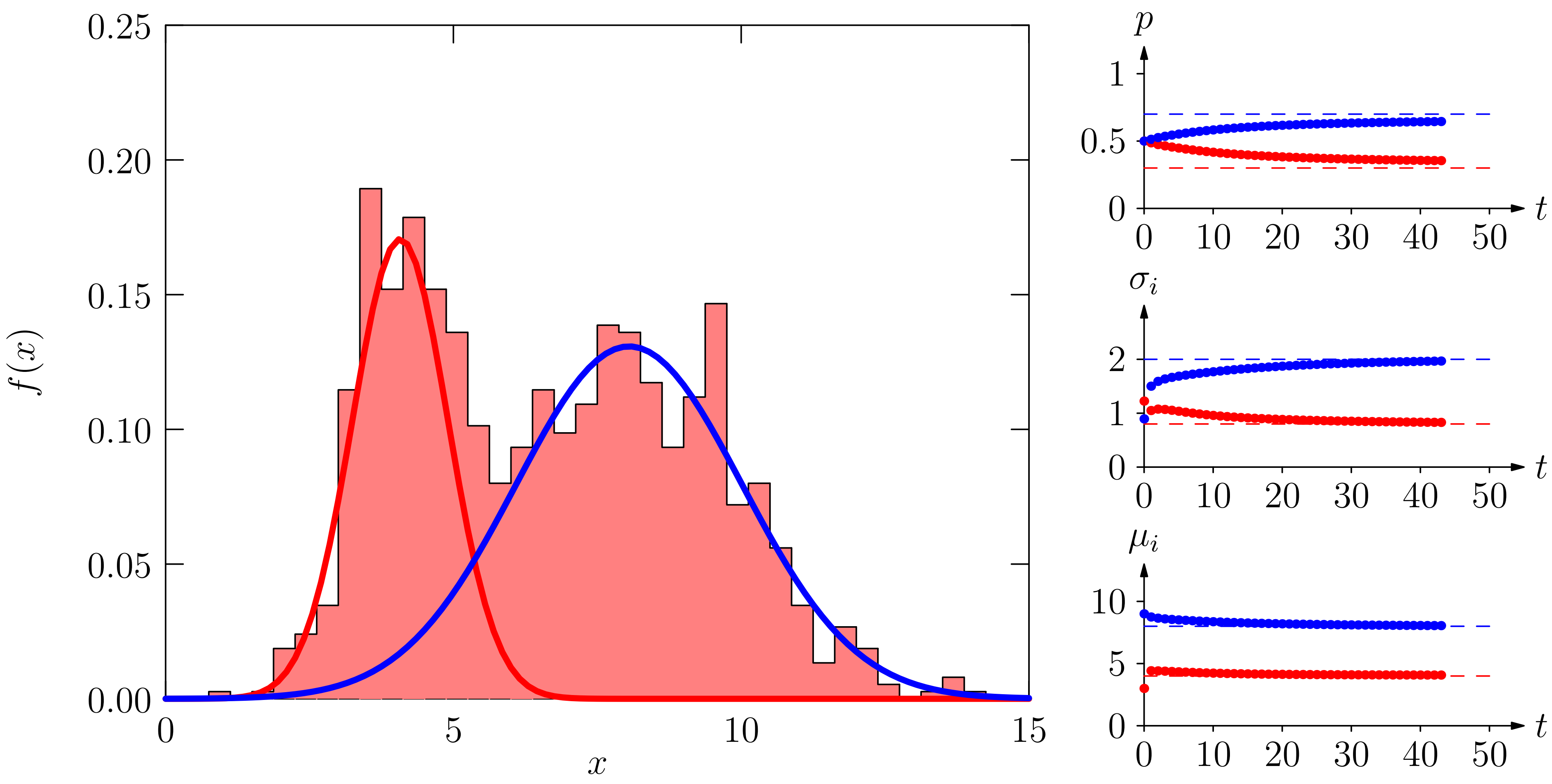

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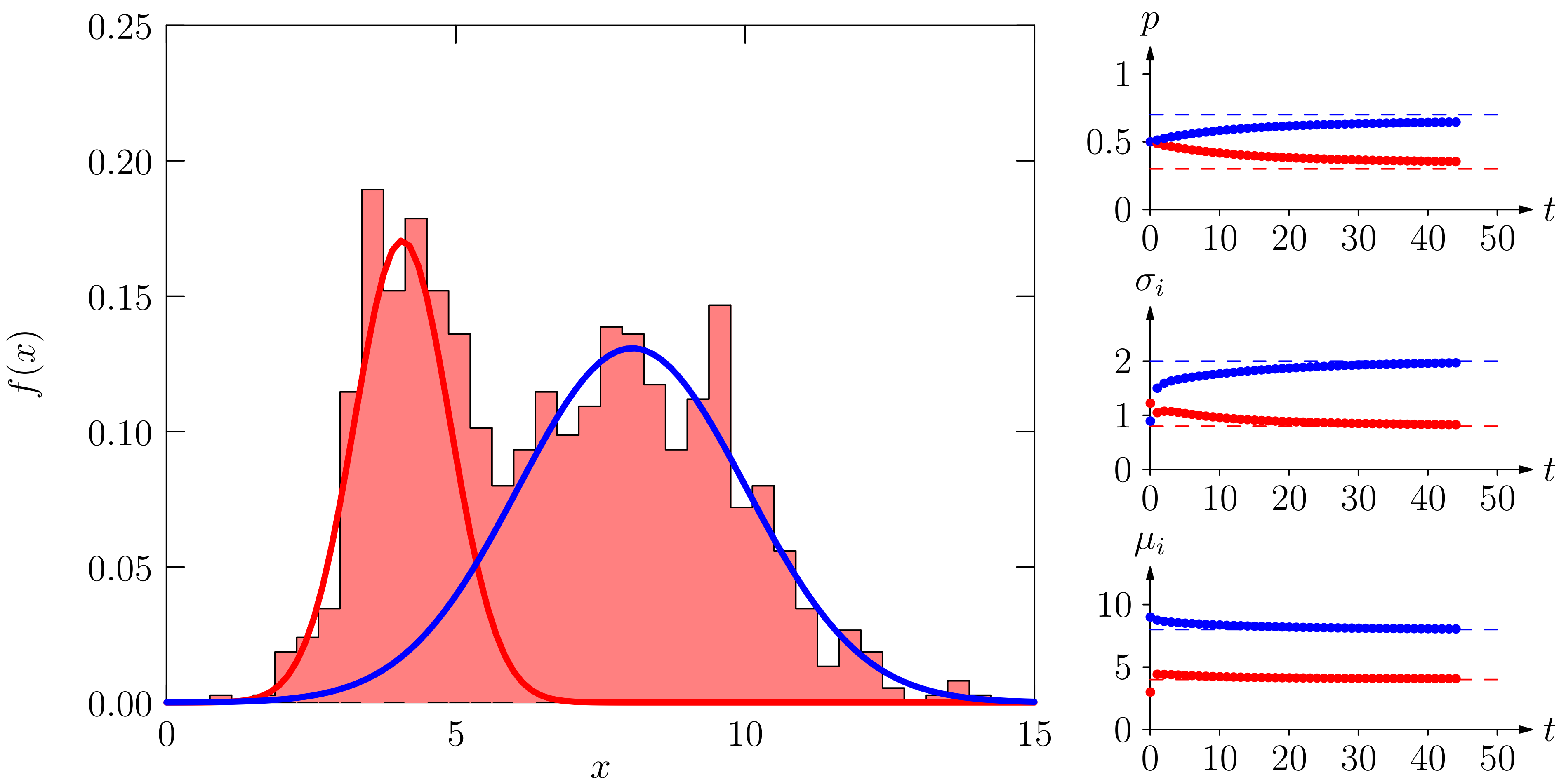

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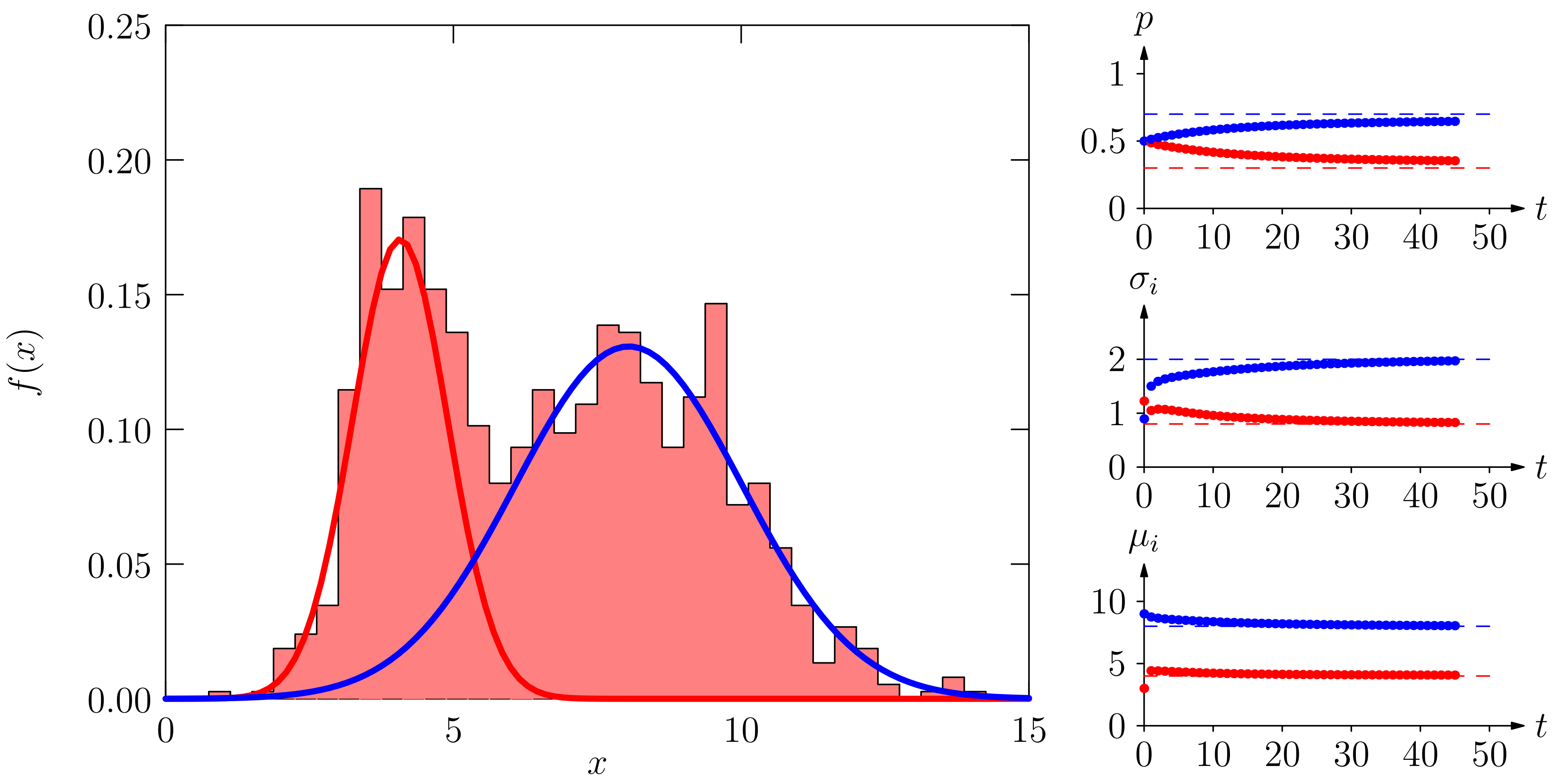

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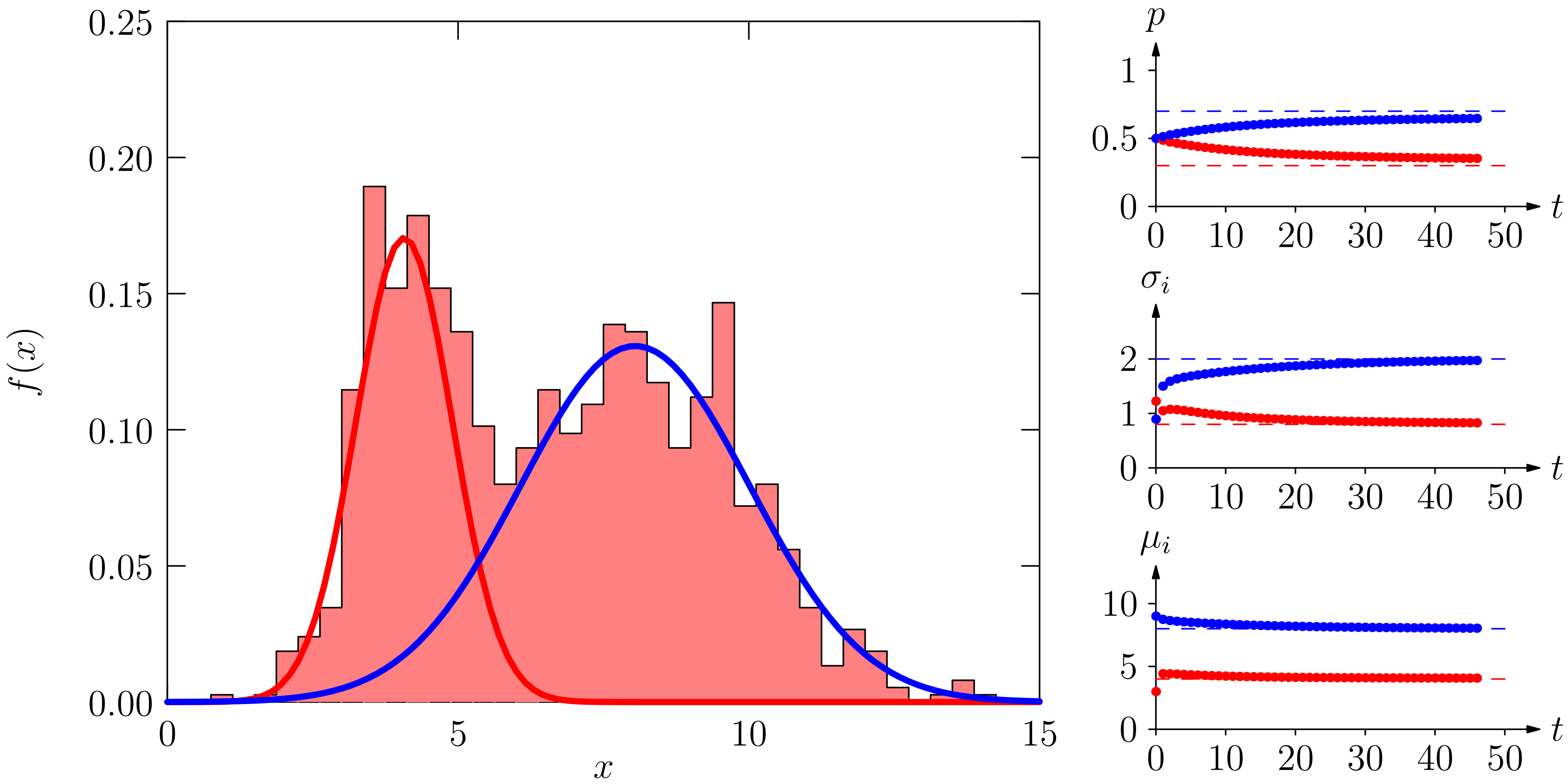

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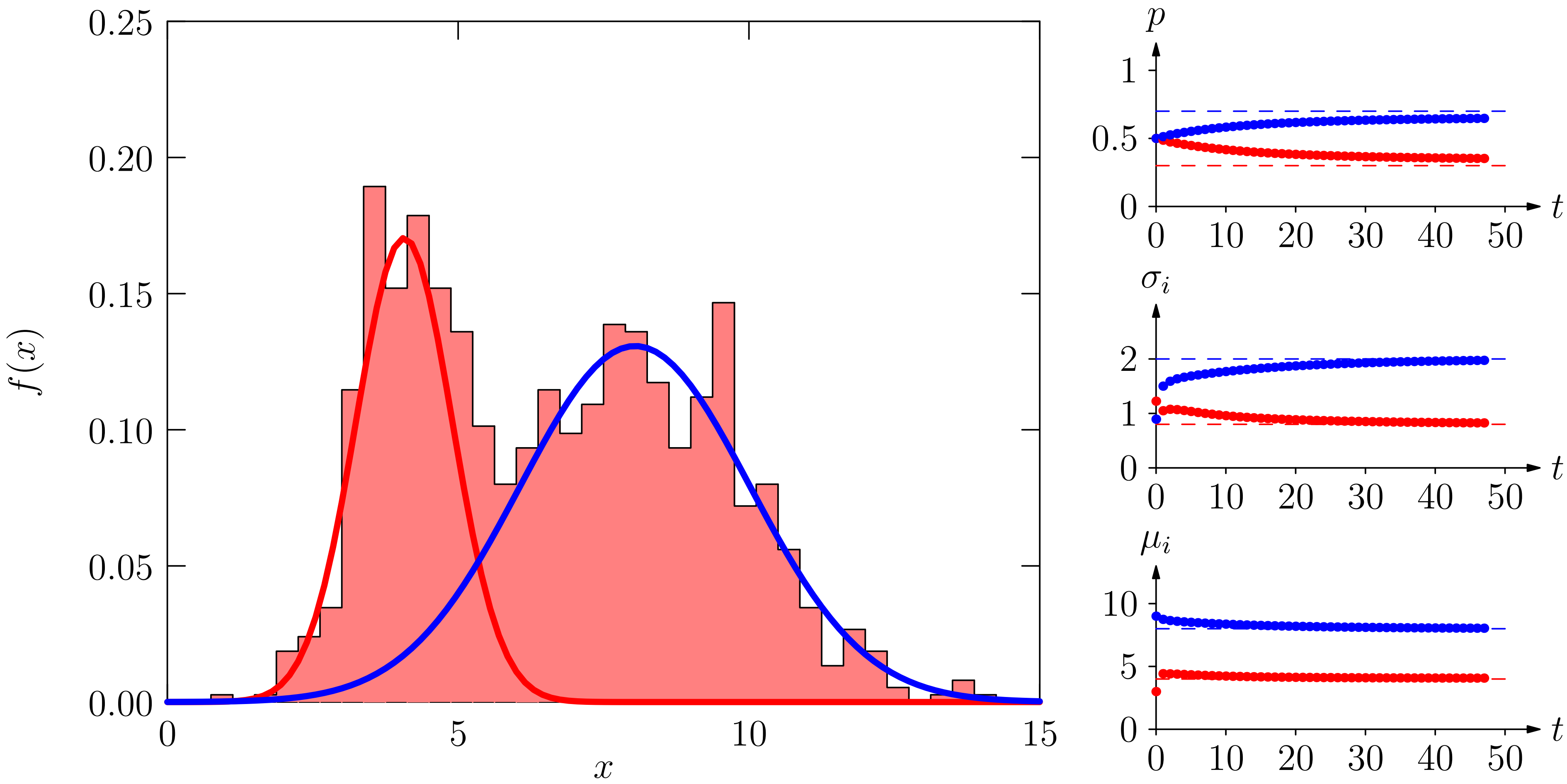
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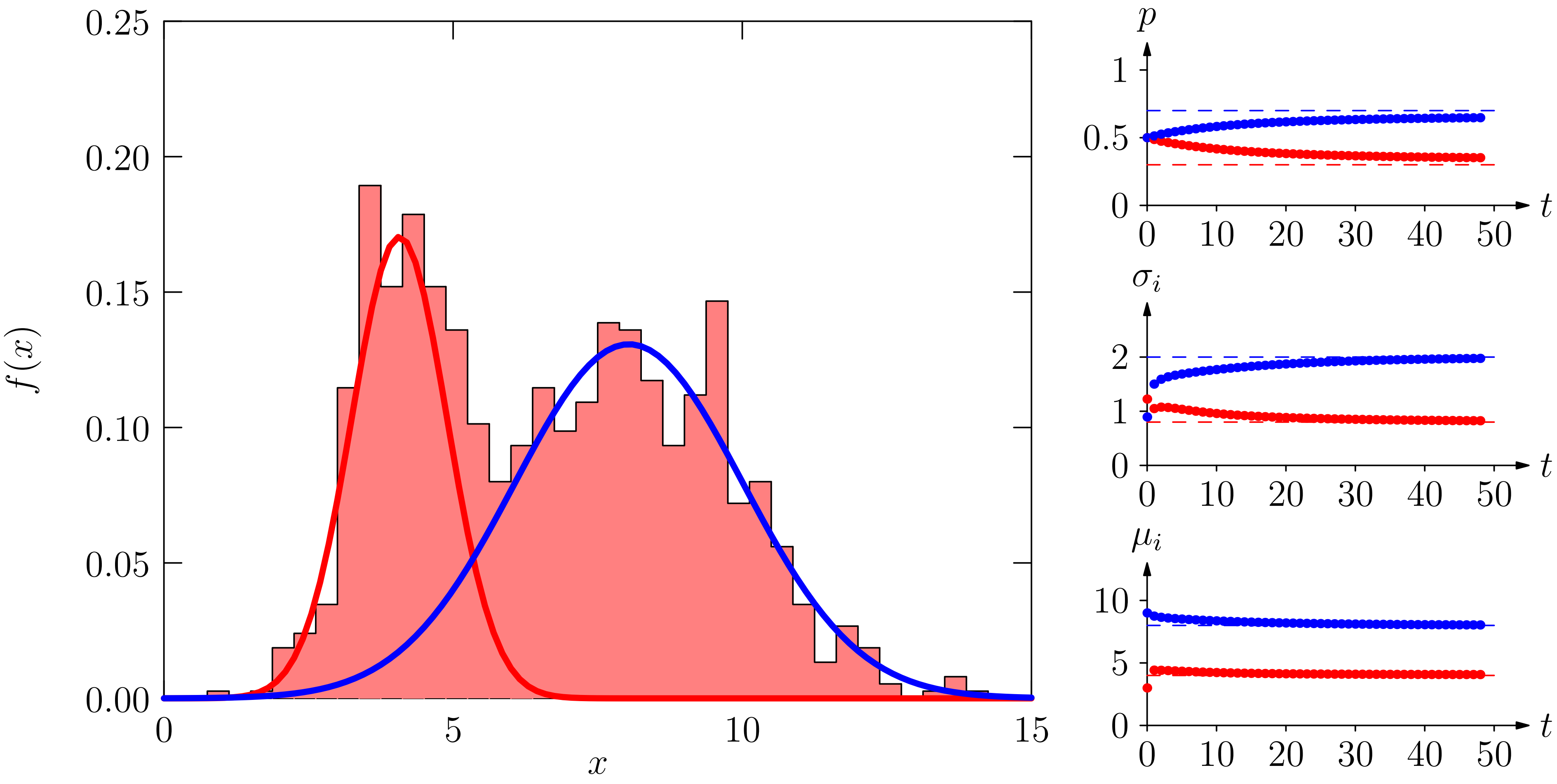

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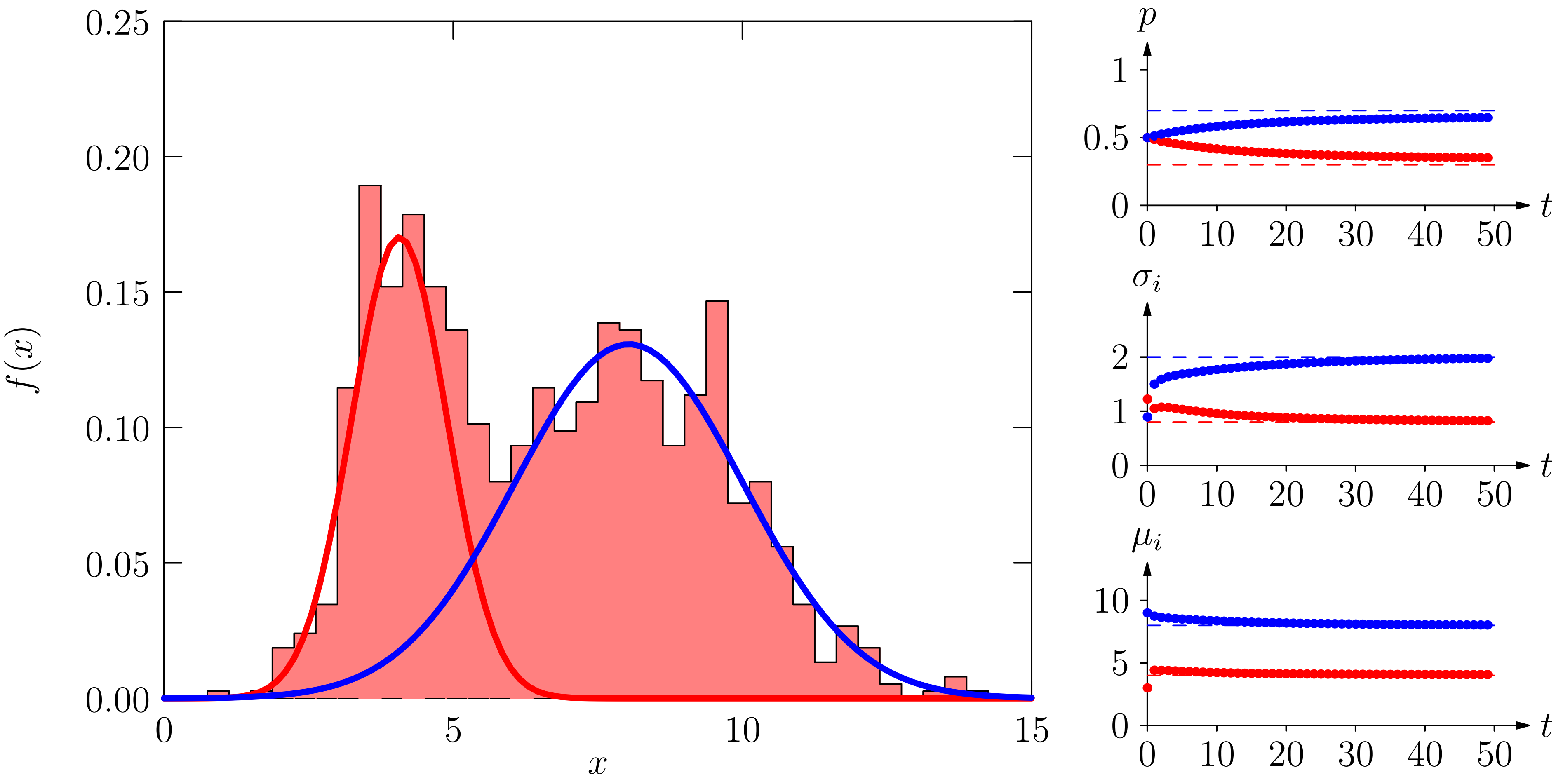

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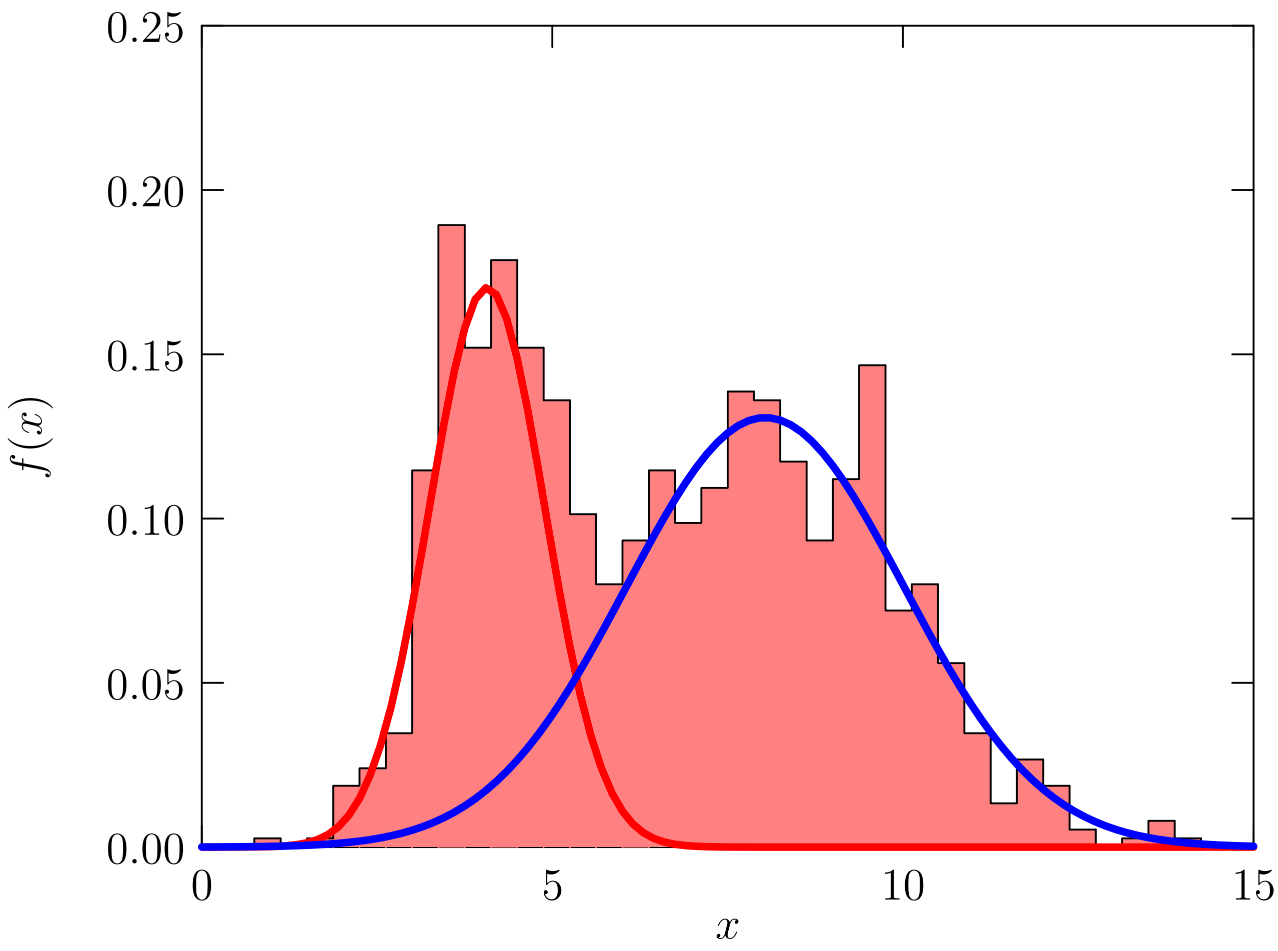

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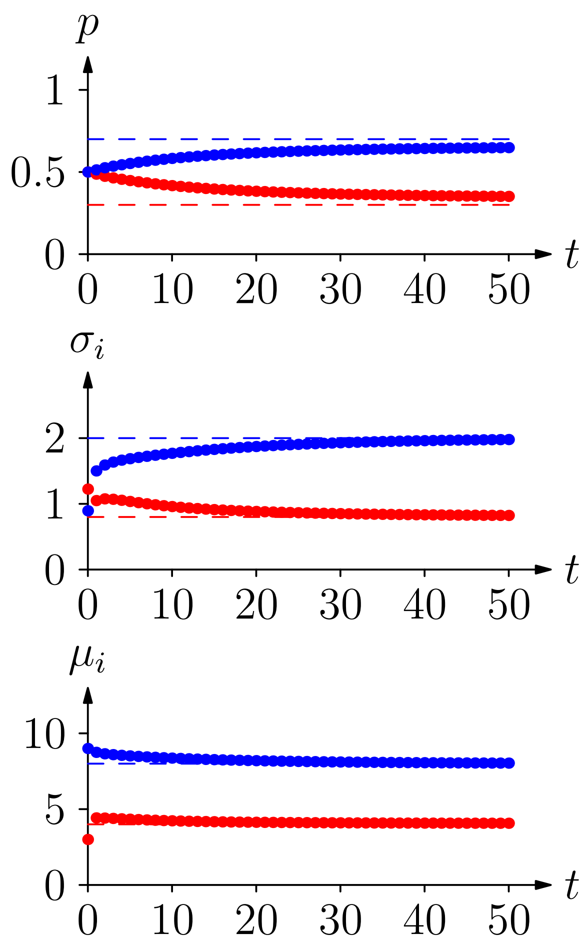

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# Summary

- Building probabilistic models is an intricate process
- Identifying random variables that describe the system is the first step
- Often we need to introduce variables that we don't observe and need to be marginalised out
- The EM algorithm provide one approach to maximising likelihoods or MAP solutions when we have latent variables
- It often gives nice update equations, but convergence can be slow

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