Advanced Machine Learning Subsidary Notes

Lecture 23: Probabilistic Inference

Adam Prügel-Bennett

February 9, 2024

1 Keywords

- Hierarchical Models, Mixture of Gaussians, Expectation Maximisation

2 Main Points

2.1 Laws of Probability

- We quickly review probabilities (you should know all this)
- Probabilities and events
  - We typically associate probabilities with events
  - The probability of an event lie between 0 and 1
  - If the set of events $\mathcal{E}$ are exhaustive and mutually exclusive then

$$\sum_{A\in\mathcal{E}} \mathbb{P}[A] = 1$$

- Often we associate numbers to events
  - These are known as *random variables*
  - Conventionally random variables are denoted by capitals, while we use lower-case letters to represent the value the random variable takes
  - We associate probability distributions to random variables
  - For discrete random variables these are known as *probability mass functions* and are denoted $\mathbb{P}[X = x]$
  - When our events are continuous we often associate the outcome to a continuous random variable
  - In this case the probability of a continuous random variable taking a particular value is typically 0
  - We then look at probability densities

$$f_X(x) = \lim_{\delta x\to 0} \frac{\mathbb{P}[x \le X \le x + \delta x]}{\delta x}$$

    * Densities are not probabilities (they can be greater than 1 although

$$\mathbb{P}[a \le X \le b] = \int_a^b f_X(x)\,\mathrm{d}x$$

    is a probability and is always less than or equal to 1)

- Probabilities become interesting when we have multiple events
  - The *joint probability* of both event A and B occurring is denoted $\mathbb{P}[A, B]$
  - The probability of random variables X and Y taking values x and y is denoted by
    - $\mathbb{P}[X = x, Y = y]$ for discrete random variables or
    - $f_{X,Y}(x, y)$ for continuous random variables (where this is now a probability density)
    - sometimes we write $\mathbb{P}[X, Y]$ or $f(x, y)$ when the context is clear
  - The *conditional probability* of an event A happening given and event B has happened is denoted by $\mathbb{P}[A|B]$ for discrete random variables or $f_{X|Y}(x|y)$ for continuous random variables
    - sometimes we write $\mathbb{P}[X|Y]$ or $f(x|y)$ when the context is clear
    - conditional probability doesn't imply any causation
    - Note that $\mathbb{P}[X|Y]$ or $f(x|y)$ are probabilities or densities of X

$$\sum_x \mathbb{P}[X = x|Y = y] = 1 \qquad \int f(x|y)\,\mathrm{d}x = 1$$

    - But they are not probabilities or densities of Y
  - One of the most important rules in probabilities is
    - $\mathbb{P}[X, Y] = \mathbb{P}[X|Y]\ \mathbb{P}[Y] = \mathbb{P}[Y|X]\ \mathbb{P}[X]$
    - $f(x, y) = f(x|y)\, f(y) = f(y|x)\, f(x)$
    - Clearly this is where Bayes' rule comes from
  - A second rule that we use all the time is

$$\sum_y \mathbb{P}[X, Y = y] = \mathbb{P}[X] \qquad \int f(x, y)\,\mathrm{d}y = f(x)$$

  - All this generalises to more the two random variables
    - $\mathbb{P}[X = x, Y = y|Z = z]$ is the probability that both $X = x$ and $Y = y$ given that $Z = z$
    - $\mathbb{P}[X = x|Y = y, Z = z]$ is the probability that $X = x$ given that $Y = y$ and $Z = z$
  - Random variables are *independent* of each other if $\mathbb{P}[X, Y] = \mathbb{P}[X]\ \mathbb{P}[Y]$
  - Random variables X and Y are conditionally independent of each other given Z if

$$\mathbb{P}[X, Y|Z] = \mathbb{P}[X|Z]\ \mathbb{P}[Y|Z]$$

- We often consider *random vectors* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ where each component is a random variable
- I am expecting to you to know this material

2.2 Probabilistic Inference

- Most probabilistic inference involves constructing a model of the underlying data generation process and using Bayes' rule or maximum likelihood to learn unknown parameters of the model
- In modelling physical processes it is often easier to specify conditional probabilities where the there is some causal relationship
- **Discriminative Models**
  - Often in machine learning our goal is to learn the probability distribution $\mathbb{P}[Y|\boldsymbol{X}]$ where Y is our target and $\boldsymbol{X}$ is a data point

- We may parameterise this distribution with some parameters $\boldsymbol{\Theta}$ and our task would be to learn these parameters based on training data

- **Generative Models**
  - Surprisingly it is often easier to model the joint probability $\mathbb{P}[Y, \boldsymbol{X}]$
  - This means that we model the process of both generating the targets and the feature vectors together
  - These are known as *generative models* as they allow us to generate random examples
  - We don't necessary want to use them to generate random samples; it just makes the modelling process easier (although you need to get used to this as it feel counter-intuitive)
  - We can use generative models to do discrimination since $\mathbb{P}[Y|\boldsymbol{X}] = \mathbb{P}[Y, \boldsymbol{X}]\,/\mathbb{P}[Y]$ where $\mathbb{P}[Y] = \sum_{\boldsymbol{X}} \mathbb{P}[Y, \boldsymbol{X}]$
  - Examples of generative models include *Hidden Markov Models* and *Topic Models* (covered later)

- **Latent Variables**
  - In building probabilistic models we often model quite complicated processes
  - To do this we introduce intermediate processes described by random variables that we never observe
  - These are known as **latent variable**
  - Often our model will involve many different layers between the inputs $\boldsymbol{X}$ and targets Y: this construction is sometimes known as a *hierarchical model*

- **Difficulty of Bayes**
  - Bayesian inference is difficult because for most likelihoods there is no conjugate prior and the posterior is a mess
  - In this case it can be very difficult to compute the *evidence* or *marginal likelihood*

$$\mathbb{P}[\mathcal{D}] = \sum_{\boldsymbol{\Theta}} \mathbb{P}[\mathcal{D}|\boldsymbol{\Theta}]\ \mathbb{P}[\boldsymbol{\Theta}]$$

    or

$$f(\mathcal{D}) = \int f(\mathcal{D}|\boldsymbol{\theta})\, f(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta}$$

    - this is hard when $\boldsymbol{\Theta}$ takes on too many values (e.g. it might be a high dimensional vector or a continuous variable)
  - One solution to this is to obtain samples from the posterior distribution (this approach uses Monte Carlo methods which are very powerful, but can be slow)
  - Another approach is to seek the **maximum a-posterirori** or **MAP** solution

$$\boldsymbol{\theta}_{\text{MAP}} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, f(\mathcal{D}|\boldsymbol{\theta})\, f(\boldsymbol{\theta}) = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} \log(\mathcal{D}|\boldsymbol{\theta})) + \log(f(\boldsymbol{\theta}))$$

    - this is much easier than the full Bayesian approach as we don't need to compute the marginal likelihood $f(\mathcal{D})$
    - but it isn't really Bayesian (although some users will claim it is)
    - it throws away the posterior and replaces this with its mode
    - we have lost a measure of uncertainty
  - We can go one step further and assume a uniform prior
    - This leads to the **maximum likelihood estimate**
    - This was first proposed by Ronald Fisher in the time when Bayesian inference was considered taboo
    - Despite its strong connection to Bayesian inference it was accepted by the statistical community

2.3 Mixtures of Gaussians

- To illustrate latent variables and a simple hierarchical model we consider a classic probabilistic model known as *mixture of Gaussians*
- We consider a concrete scenario
- We suppose we are observing the decay of two types (A and B) of short-lived particles [1]
- We can measure their half lives, X_i, but we don't know the type of particle
- We have a measurement error of the half-life
- Let $Z_i \in \{0, 1\}$ equal 1 if particle i is of type A and 0 if it is of type B
- The probability distribution of the half-life measurement is therefore

$$f(X_i|Z_i, \boldsymbol{\Theta}) = Z_i \mathcal{N}\big(X_i\big|\mu_A, \sigma_A^2\big) + (1 - Z_i)\,\mathcal{N}\big(X_i\big|\mu_B, \sigma_B^2\big)$$

  - where μ_A and μ_B are the expected half-lives for particles of type A and B respectively
  - σ_A and σ_B are the standard deviations in the measurements
  - this says that if the i^{th} particle is of type A then the probability of X_i is $\mathcal{N}\big(X_i\big|\mu_A, \sigma_A^2\big)$, while if it of type B, then X_i is distributed according to $\mathcal{N}\big(X_i\big|\mu_B, \sigma_B^2\big)$
- We show some typical data from $m = 1\,000$ observations in Figure 1

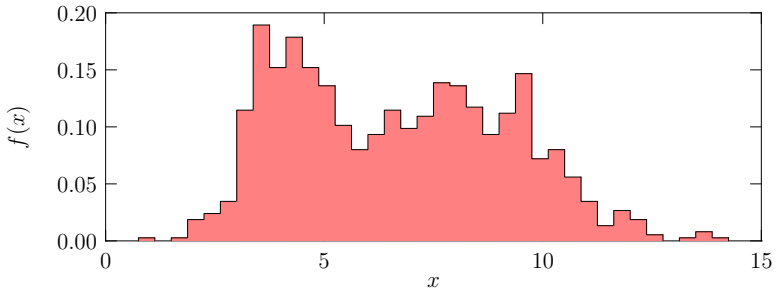

Figure 1: Example of data measuring the half-lives of two types of particles

- Our job is to infer the random variables $\boldsymbol{\Theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, p)$, where $p = \mathbb{P}[Z_i = 1]$ is the probability of the particle being type A
- We can do a full Bayesian calculation, but let us just use maximum likelihood
- The maximum likelihood of the data $\mathcal{D} = \{X_i | i = 1, 2, \ldots, m\}$ is

$$\begin{aligned} f(\mathcal{D}|\boldsymbol{\Theta}) &\overset{(1)}{=} \sum_{\boldsymbol{Z}\in\{0,1\}^m} f(\mathcal{D}, \boldsymbol{Z}|\boldsymbol{\Theta}) \\ &\overset{(2)}{=} \prod_{i=1}^{m} \sum_{Z_i\in\{0,1\}} f(X_i, Z_i|\boldsymbol{\Theta}) \overset{(3)}{=} \prod_{i=1}^{m} \sum_{Z_i\in\{0,1\}} f(X_i|Z_i, \boldsymbol{\Theta})\,\mathbb{P}[Z_i] \end{aligned}$$

  (1) where we marginalise out the latent variables $\boldsymbol{Z} = (Z_1, Z_2, \ldots Z_n)$

  (2) we assume the data is independent

  (3) we use the identity $f(X_i, Z_i|\boldsymbol{\Theta}) = f(X_i|Z_i, \boldsymbol{\Theta})\,\mathbb{P}[Z_i]$

[1] If you prefer, you can think of an autonomous vehicle using lidar where it detects reflections from two different, but close by, objects, A and B. We make multiple noisy measurements of the distance from the two objects.

- It is usually easier working with the log-likelihood

$$\begin{aligned}\log(f(\mathcal{D}|\boldsymbol{\Theta})) &= \sum_{i=1}^{m} \log\big(f(X_i|Z_i=1)\,\mathbb{P}[Z_i=1] + f(X_i|Z_i=0)\,\mathbb{P}[Z_i=0]\big) \\ &= \sum_{i=1}^{m} \log\big(p\,\mathcal{N}\big(X_i\big|\mu_A,\sigma_A\big) + (1-p)\,\mathcal{N}\big(X_i\big|\mu_B,\sigma_B\big)\big)\end{aligned}$$

- We could do gradient descent on this, but it is an ugly expression to work with

2.4 Expectation Maximisation

- Rather than maximise the likelihood directly we can iteratively maximise the expected log-likelihood starting form some initial guess $\boldsymbol{\Theta}^{(0)}$; we get an improved guess

$$\boldsymbol{\Theta}^{(t+1)} = \underset{\boldsymbol{\Theta}}{\operatorname{argmax}} \sum_{\boldsymbol{Z}\in\{0,1\}^m} \mathbb{P}\Big[\boldsymbol{Z}\Big|\mathcal{D},\boldsymbol{\Theta}^{(t)}\Big]\,\log(f(\mathcal{D},\boldsymbol{Z}|\boldsymbol{\Theta})) \tag{1}$$

- This is a general optimisation strategy that is regularly used when we have latent variables
- It is known as **expectation maximisation** or the **EM-algorithm**
- This looks very different to maximising the log-likelihood: it takes some effort to understand why this works
- To understand this we note

$$f(\mathcal{D},\boldsymbol{Z}|\boldsymbol{\Theta}) = f(\mathcal{D}|\boldsymbol{Z},\boldsymbol{\Theta})\,\mathbb{P}[\boldsymbol{Z}|\boldsymbol{\Theta}]$$

  From which we can deduce (taking logs and rearranging)

$$\log(f(\mathcal{D}|\boldsymbol{Z},\boldsymbol{\Theta})) = \log(f(\mathcal{D},\boldsymbol{Z}|\boldsymbol{\Theta})) - \log(\mathbb{P}[\boldsymbol{Z}|\boldsymbol{\Theta}])$$

- We now consider the probability distribution $\mathbb{P}\big[\boldsymbol{Z}\big|\mathcal{D},\boldsymbol{\Theta}^{(t)}\big]$, that tells us the probability that $Z_i = 1$ given X_i and the parameters $\boldsymbol{\Theta}^{(t)}$ (this is different to the prior distribution $\mathbb{P}\big[\boldsymbol{Z}|\boldsymbol{\Theta}^t\big] = p^{(t)}$)
- If we not take expectations of $\log(f(\mathcal{D}|\boldsymbol{\Theta}))$ give above with respect to this distribution then

$$\begin{aligned}\log(f(\mathcal{D}|\boldsymbol{\Theta})) &= \mathbb{E}_{\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}}[\log(f(\mathcal{D},\boldsymbol{Z}|\boldsymbol{\Theta}))] - \mathbb{E}_{\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}}[\log(\mathbb{P}[\boldsymbol{Z}|\boldsymbol{\Theta}])] \\ &= Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)}) + S(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)})\end{aligned}$$

  - Note that the left-hand side does not involve the latent variables so when we take the expectation we get itself
  - The first term on the right-hand side is

$$Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)}) = \mathbb{E}_{\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}}[\log(f(\mathcal{D},\boldsymbol{Z}|\boldsymbol{\Theta}))] = \sum_{\boldsymbol{Z}\in\{0,1\}^m} \mathbb{P}\Big[\boldsymbol{Z}\Big|\mathcal{D},\boldsymbol{\Theta}^{(t)}\Big]\,\log(f(\mathcal{D}|\boldsymbol{Z},\boldsymbol{\Theta}))$$

  - This is the term we are optimising in equation (1)
  - The second term is

$$S(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)}) = -\mathbb{E}_{\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}}[\log(\mathbb{P}[\boldsymbol{Z}|\boldsymbol{\Theta}])] = -\sum_{\boldsymbol{Z}\in\{0,1\}^m} \mathbb{P}\Big[\boldsymbol{Z}\Big|\mathcal{D},\boldsymbol{\Theta}^{(t)}\Big]\,\log(\mathbb{P}[\boldsymbol{Z}|\boldsymbol{\Theta}])$$

- Using the identity for the log-likelihood we can write the change in log-likelihood when we update our parameters

$$\begin{aligned}\Delta L &= \log\Big(f(\mathcal{D}|\boldsymbol{\Theta}^{(t+1)})\Big) - \log\Big(f(\mathcal{D}|\boldsymbol{\Theta}^{(t)})\Big)\\ &= Q(\boldsymbol{\Theta}^{(t+1)}|\boldsymbol{\Theta}^{(t)}) - Q(\boldsymbol{\Theta}^{(t)}|\boldsymbol{\Theta}^{(t)}) + S(\boldsymbol{\Theta}^{(t+1)}|\boldsymbol{\Theta}^{(t)}) - S(\boldsymbol{\Theta}^{(t)}|\boldsymbol{\Theta}^{(t)})\\ &= Q(\boldsymbol{\Theta}^{(t+1)}|\boldsymbol{\Theta}^{(t)}) - Q(\boldsymbol{\Theta}^{(t)}|\boldsymbol{\Theta}^{(t)}) + \mathrm{KL}\Big(\mathbb{P}\Big[\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}\Big]\Big\|\mathbb{P}\Big[\boldsymbol{Z}|\boldsymbol{\Theta}^{(t+1)}\Big]\Big)\end{aligned}$$

  - where

$$\begin{aligned}\mathrm{KL}\Big(\mathbb{P}\Big[\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}\Big]\Big\|\mathbb{P}\Big[\boldsymbol{Z}|\boldsymbol{\Theta}^{(t+1)}\Big]\Big) &= S(\boldsymbol{\Theta}^{(t+1)}|\boldsymbol{\Theta}^{(t)}) - S(\boldsymbol{\Theta}^{(t)}|\boldsymbol{\Theta}^{(t)})\\ &= -\sum_{\boldsymbol{Z}\in\{0,1\}^m} \mathbb{P}\Big[\boldsymbol{Z}\Big|\mathcal{D},\boldsymbol{\Theta}^{(t)}\Big]\,\log\left(\frac{\mathbb{P}\big[\boldsymbol{Z}|\boldsymbol{\Theta}^{(t+1)}\big]}{\mathbb{P}\big[\boldsymbol{Z}|\boldsymbol{\Theta}^{(t)}\big]}\right)\end{aligned}$$

  - We have shown in a previous lecture that KL-divergences are non-negative

- Now in expectation maximisation we choose

$$\boldsymbol{\Theta}^{(t+1)} = \operatorname*{argmax}_{\boldsymbol{\Theta}} Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)})$$

  which implies $Q(\boldsymbol{\Theta}^{(t+1)}|\boldsymbol{\Theta}^{(t)}) \geq Q(\boldsymbol{\Theta}^{(t)}|\boldsymbol{\Theta}^{(t)})$

- Thus $\Delta L \geq 0$, so in each step we increase the log-likelihood

- This gives us a relative simple procedure for maximising the likelihood (we can also use this to maximise the *a posteriori* solution); we choose $\boldsymbol{\Theta}$ to maximise

$$Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)}) = \sum_{\boldsymbol{Z}\in\{0,1\}^m} \mathbb{P}\Big[\boldsymbol{Z}\Big|\mathcal{D},\boldsymbol{\Theta}^{(t)}\Big]\,\log(f(\mathcal{D}|\boldsymbol{Z},\boldsymbol{\Theta}))$$

- Let us return to the problem of working out the half-life statistics of our two types of particles A and B

- Recall $f(\mathcal{D},\boldsymbol{Z}|\boldsymbol{\Theta}) = \prod_{i=1}^{m} f(X_i|Z_i,\boldsymbol{\Theta})\,\mathbb{P}[Z_i]$ where

$$f(X_i,Z_i|\boldsymbol{\Theta}) = p\,Z_i\,\mathcal{N}\big(X_i\big|\mu_A,\sigma_A^2\big) + (1-p)\,(1-Z_i)\,\mathcal{N}\big(X_i\big|\mu_B,\sigma_B^2\big)$$

- Let

$$\begin{aligned}p_i^{(t)} &= \mathbb{P}\Big[Z_i=1\Big|X_i,\boldsymbol{\Theta}^{(t)}\Big] = \frac{p^{(t)}\,\mathcal{N}\Big(X_i\big|\mu_A^{(t)},\sigma_A^{2(t)}\Big)}{p^{(t)}\,\mathcal{N}\Big(X_i\big|\mu_A^{(t)},\sigma_A^{2(t)}\Big) + (1-p^{(t)})\,\mathcal{N}\Big(X_i\big|\mu_B^{(t)},\sigma_B^{2(t)}\Big)}\\ q_i^{(t)} &= \mathbb{P}\Big[Z_i=0\Big|X_i,\boldsymbol{\Theta}^{(t)}\Big] = \frac{(1-p^{(t)})\,\mathcal{N}\Big(X_i\big|\mu_B^{(t)},\sigma_B^{2(t)}\Big)}{p^{(t)}\,\mathcal{N}\Big(X_i\big|\mu_A^{(t)},\sigma_A^{2(t)}\Big) + (1-p^{(t)})\,\mathcal{N}\Big(X_i\big|\mu_B^{(t)},\sigma_B^{2(t)}\Big)} = 1-p_i^{(t)}\end{aligned}$$

- Then

$$\begin{aligned}Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)}) &= \sum_{i=1}^{m} p_i^{(t)}\log\Big(p^{(t)}\,\mathcal{N}\big(X_i\big|\mu_A,\sigma_A^2\big)\Big) + q_i^{(t)}\log\Big((1-p^{(t)})\,\mathcal{N}\big(X_i\big|\mu_B,\sigma_B^2\big)\Big)\\ &= \sum_{i=1}^{m} p_i^{(t)}\left(\log(p) - \frac{(X_i-\mu_A)^2}{2\sigma_A^2} - \frac{1}{2}\log\big(2\,\pi\,\sigma_A^2\big)\right)\\ &\qquad + q_i^{(t)}\left(\log(1-p) - \frac{(X_i-\mu_B)^2}{2\sigma_B^2} - \frac{1}{2}\log\big(2\,\pi\,\sigma_B^2\big)\right)\end{aligned}$$

- To optimise this we just set the derivatives to 0
  - Optimising with respect to p

$$\frac{\partial Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)})}{\partial p} = \frac{1}{p}\sum_{i=1}^{m} p_i^{(t)} - \frac{1}{1-p}\sum_{i=1}^{m} q_i^{(t)} = 0$$

    solving for p

$$p^{(t+1)} = \frac{\sum_{i=1}^{m} p_i^{(t)}}{\sum_{i=1}^{m}(p_i^{(t)} + q_i^{(t)})} = \frac{1}{m}\sum_{i=1}^{m} p_i^{(t)}$$

  - Optimising with respect to μ_A

$$\frac{\partial Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)})}{\partial \mu_A} = -\sum_{i=1}^{m} p_i^{(t)} \frac{X_i - \mu_A}{\sigma_A^2}$$

    solving for μ_A (and performing a similar optimisation for μ_B)

$$\mu_A^{(t+1)} = \frac{\sum_{i=1}^{m} p_i^{(t)} X_i}{\sum_{i=1}^{m} p_i^{(t)}}, \qquad \mu_B^{(t+1)} = \frac{\sum_{i=1}^{m} q_i^{(t)} X_i}{\sum_{i=1}^{m} q_i^{(t)}}$$

  - Putting in the optimal value for $\mu_A^{(t)}$ and optimising with respect to σ_A^2

$$\frac{\partial Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(t)})}{\partial \sigma_A^2} = \frac{1}{2\,\sigma_A^4}\sum_{i=1}^{m} p_i^{(t)}(X_i - \mu_A^{(t)})^2 - \frac{1}{\sigma_A^2}\sum_{i=1}^{m} p_i^{(t)}$$

    Solving for σ_A^2 (and performing a similar optimisation for σ_B^2)

$$\sigma_A^2 = \frac{\sum_{i=1}^{m} p_i^{(t)}(X_i - \mu_A^{(t)})^2}{\sum_{i=1}^{m} p_i^{(t)}}, \qquad \sigma_B^2 = \frac{\sum_{i=1}^{m} q_i^{(t)}(X_i - \mu_B^{(t)})^2}{\sum_{i=1}^{m} q_i^{(t)}}$$

- These are very natural update equations
  - we make an estimate, $p_i^{(t)}$ of the probability that observation X_i is a particle of type A or B base on our current parameters
  - we then update all our parameters based on these estimates
- We are guaranteed that our EM-algorithm never decreases the likelihood (although it could reach a local rather than global optimum)
- For the data set we showed earlier (which was a random sample of size 1000 generated using $p = 0.3$, $\mu_A = 4$, $\sigma_A = 0.8$, $\mu_B = 8$ and $\sigma_B = 2$) we get the results shown in Figure 2
- The EM algorithm often leads to very natural update equations, but its convergence is often rather slow

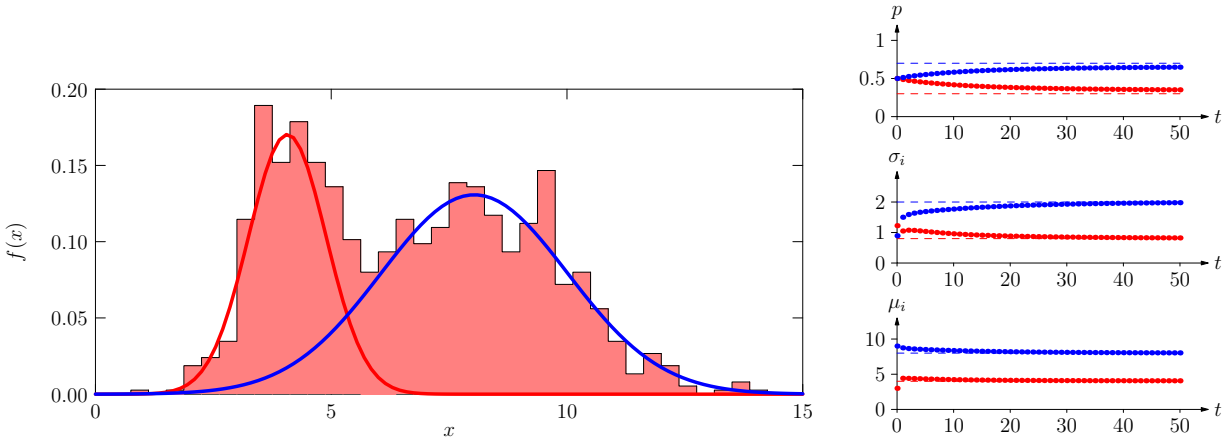

Figure 2: Example of EM algorithm to compute the statistics for the half-lives of our two particles

3 Exercises

3.1 Mysterious Disease

- We assume that we are tracking some disease
- Let $Z(t)$ be the number of people that catch the disease on day t, but this is unknown (a latent variable)
- We assume the rate of growth of the disease is

$$\mathbb{P}[Z(t+1)] = \text{Poi}\Big(Z(t+1)\Big|\frac{r_0}{3}\left(Z(t)+Z(t-1)+Z(t-2)\right)\Big)$$

  - We are assuming that someone is virulent for the first three days after catching the disease
  - We assume $Z(1)=1$ and $Z(0)=Z(-1)=0$
  - In expectation everyone with the disease will infect r_0 new people
- We observe $X(t)$ which is some proportion of new patients such that

$$\mathbb{P}[X(t)=k] = \text{Binom}(k|Z(t),p) = \binom{Z(t)}{k} p^k (1-p)^{Z(t)-k}$$

  - We assume that each patient will be tested with a probability p
  - We are given a time series $(X(1), X(2), \ldots, X(T))$
- Build a probabilistic model to estimate p and k_0
- See answers

3.2 Experiments

3.3 Mysterious Disease

- Build a simulator of your models (assume you know p and r_0)
- Choose any language you are comfortable with
- If you are feeling very adventurous you could try to solve your model to predict p and r_0, but be warned this is hard (you probably need to use MCMC, but you could try an EM algorithm)

```
r0 = 2
r = r0/3
p = 0.2
T = 20
Z(1) = 1;
Z(2) = poissrnd(r*Z(1));
Z(3) = poissrnd(r*(Z(1)+Z(2)));
for t = 4:T
   Z(t) = poissrnd(r*(Z(t-1)+Z(t-2)+Z(t-3)));
endfor

for t= 1:T
  X(t) = binornd(Z(t),p);
endfor
```

4 Answers

4.1 Mysterious Disease

- We want to compute $f(p, k_0|\mathcal{D})$ where $\mathcal{D} = (X(1), X(2), \ldots, X(T))$
- We have a likelihood of
$$\mathbb{P}[X(t)|Z(t), p] = \text{Binom}(X(t)|Z(t), p)$$
where
$$\mathbb{P}[Z(t+1)|r_0, Z(t), Z(t-1), Z(t-2)] = \text{Poi}\Big(Z(t+1)\Big|\frac{r_0}{3}\left(Z(t) + Z(t-1) + Z(t-2)\right)\Big)$$
- To perform a Bayesian calculation we would have to put priors, $f(p)$ and $f(r_0)$, on p and r_0
  - A reasonable prior to use for p is $f(p) = \text{Beta}(p|a, b)$—you could use $a = b = 0$ as an uninformative priors, but you might have some prior knowledge, e.g. $a = b = 2$ say which says $f(p) = 6\,p\,(1-p)$
  - A reasonable prior for r_0 would be $\text{Gamma}(r_0|a, b)$—you could use $a = b = 0$ as an uninformative prior but again you might have some prior belief, e.g. $a = 2$, $b = 1$
- Bayes rule tells us
$$f(p, r, \boldsymbol{Z}|\boldsymbol{X}) = \frac{\mathbb{P}[X(t)|Z(t), p]\ \prod_{t=1}^{T} \mathbb{P}[Z(t)|Z(t-1), Z(t-2), Z(t-3), r_0]\ f(r_0)\, f(p)}{\mathbb{P}[\boldsymbol{X}]}$$
- To get estimates of p and r_0 we have to marginalise out $\boldsymbol{Z}$
- Now the problem is this is rather horrible to compute (your priors are not conjugate priors in this problem and the posterior is very complicated)
- Probably the best way to do this is to use Markov Chain Monte Carlo (MCMC), but you will have to wait before I get there