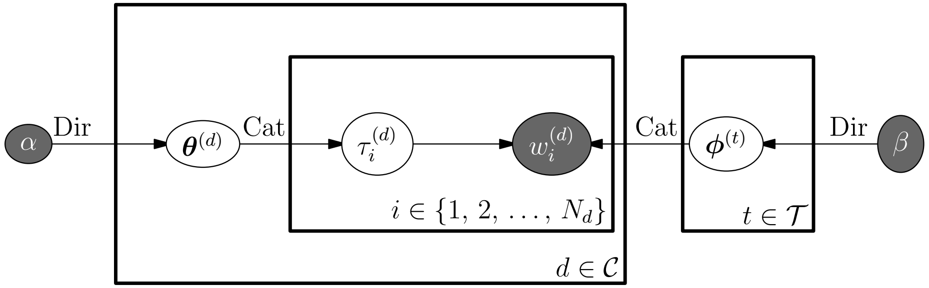
Advanced Machine Learning

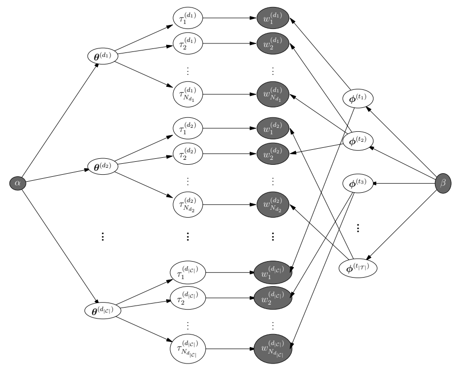
Graphical Models



Conditional Independence, Graphical models, LDA

Outline

1. **Graphical Models**
2. Cakes!
3. Latent Dirichlet Allocation



Graphical Models

- If we want to build large probabilistic inference systems
  - AI Doctor
  - Fault diagnostic system for a computer

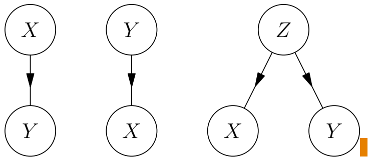
  we can describe this by introducing random variables, but it is helpful to graphically represent causal connections
- Graphical models allow us to do this
- It allows us to build a joint probability from which we can compute everything we want

Dependencies Between Variables

- In building a probabilistic model we want to know which random variables depend on each other directly and which don't
- Variables that don't will typically still be correlated
- If two random variables X and Y are correlated then
  - X could affect Y
  - Y could affect X
  - X and Y could not influence each other, but both be affected by another random variable Z

Graphical Models

- **Bayesian Belief Networks** are a type of graphical models where we use a directed graphs to show causal relationships between random variables
- We could represent the three conditions described above by



- We can use these graphical representations to work out how to efficiently average over latent variables

Statistical Independence

- Two random variables are statistically independent if

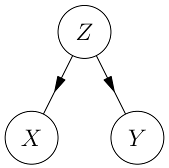
$$\mathbb{P}(X,Y) = \mathbb{P}(X)\,\mathbb{P}(Y)$$

- Equally this implies $\mathbb{P}(X|Y) = \mathbb{P}(X)$ and $\mathbb{P}(Y|X) = \mathbb{P}(Y)$
- Statistically independent variables are uncorrelated
- But statistical independence is often too powerful

Conditional Independence

- A weaker notion is conditional independence

$$\mathbb{P}(X,Y|Z) = \mathbb{P}(X|Z)\,\mathbb{P}(Y|Z)$$



- Conditional independence implies that there is no direct causation
- But it doesn't imply zero correlation
- Conditional independence reduces computational complexity, e.g.

$$\mathbb{E}[XY] = \sum_{X,Y,Z} X\,Y\,\mathbb{P}(X,Y,Z) = \sum_{Z} P(Z) \left(\sum_{X} X\,P(X|Z) \right) \left(\sum_{Y} Y\,P(Y|Z) \right)$$

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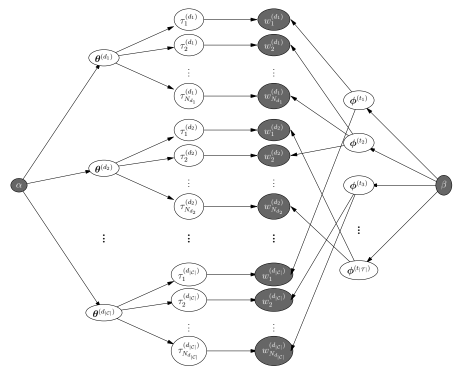


Let Them Eat Cakes

- I will go through a very simple example involving cakes
- It illustrates some simple principles
- In the subsidiary notes I present a very simple program for computing all the probabilities—I would encourage you to do this as it makes things much clearer

The Cake Scenario

- Abi and Ben both bake cakes and bring them into the coffee room
- Abi will bring in cakes 20% of the time: $\mathbb{P}(A = 1) = 0.2$
- Ben will bring in cakes 10% of the time: $\mathbb{P}(B = 1) = 0.1$
- 90% of the time if either Abi or Ben have put cakes in the coffee room there is some left when I enter $\mathbb{P}(C = 1|A = 1, B = 0) = \mathbb{P}(C = 1|A = 0, B = 1) = 0.9$
- If they both make cake then there is always cake left $\mathbb{P}(C = 1|A = 1, B = 1) = 1$
- If neither Abi or Ben has made cake there is still a 5% chance someone else has put cake in the coffee room $\mathbb{P}(C = 1|A = 0, B = 0) = 0.05$

Computing with Probabilities

- Other probabilities I can deduce, e.g. $\mathbb{P}(C = 0|A, B) = 1 - \mathbb{P}(C = 1|A, B)$
- I can depict the causal relationship as

$$A \longrightarrow C \longleftarrow B$$

- The quantity that I really want is the joint probability

$$\begin{aligned}\mathbb{P}(A, B, C) &= \mathbb{P}(C, B|A)\,\mathbb{P}(A)\\ &= \mathbb{P}(C|A, B)\,\mathbb{P}(B|A)\,\mathbb{P}(A) = \mathbb{P}(C|A, B)\,\mathbb{P}(B)\,\mathbb{P}(A)\end{aligned}$$

- Because $\mathbb{P}(B|A) = \mathbb{P}(B)$

Computing Expectations

- By using the joint probability and summing over all unknown quantities, we can compute expectations of anything we are interested in
- These sums are often sped up using knowledge of conditional independence
- To compute the probability of and event $\mathcal{E}$ we introduce an indicator function $[\![\mathcal{E}]\!]$ which is equal to 1 if the event happens and 0 otherwise

$$\mathbb{P}(\mathcal{E}) = \mathbb{E}[[\![\mathcal{E}]\!]]$$

- If E is a random variable equal to 1 if event $\mathcal{E}$ happens and 0 otherwise then $E = [\![\mathcal{E}]\!]$

Are There Any Cakes Left?

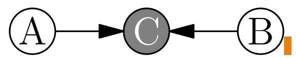
- We can use our model to compute the probabilities of there being cakes in the coffee room

$$\begin{aligned}\mathbb{P}(C=1) &= \sum_{A,B,C\in\{0,1\}} [\![C=1]\!]\,\mathbb{P}(A,B,C)\\ &= \sum_{A,B\in\{0,1\}} \mathbb{P}(C=1|A,B)\mathbb{P}(A)\mathbb{P}(B) = 0.29\end{aligned}$$

- The probability that Abi baked a cake is just 0.2 and for Ben its 0.1 (which is what we assume at the start)
- The probability of them both baking on a particular day is 0.02

Making Observation

- Making observations changes probabilities
- In graphical models observed random variables are shaded



- The probabilities conditioned on C is given by

$$\mathbb{P}(A,B|C) = \frac{\mathbb{P}(A,B,C)}{\mathbb{P}(C)}$$

where

$$\mathbb{P}(C) = \sum_{A,B\in\{0,1\}} \mathbb{P}(A,B,C)$$

Who Made Those Cakes?

- If we observe there are cakes

$$\mathbb{P}(A,B|C=1) = \mathbb{P}(A,B,C=1)/\mathbb{P}(C=1)$$

- A straightforward if tedious calculation shows

$$\begin{aligned}\mathbb{P}(A=1|C=1) = 0.628, \quad \mathbb{P}(B=1|C=1) = 0.317\\ \mathbb{P}(A=1,B=1|C=1) = 0.069\end{aligned}$$

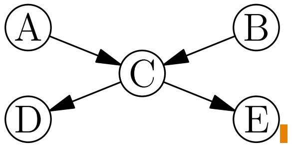
- Note $\mathbb{P}(A=1,B=1|C=1) \neq \mathbb{P}(A=1|C=1)\mathbb{P}(B=1|C=1)$
- When we observe C then A and B are no longer independent

Elaborate Cakes

- We can elaborate on our cake model
- We suppose that Dave likes cakes so if there is a cake in the coffee room there is a 80% chance that I will see him eating a cake: $\mathbb{P}(D=1|C=1) = 0.8$
- Even if there are no cakes in the coffee room there is a 10% chance that Dave has bought his own cake: $\mathbb{P}(D=1|C=0) = 0.1$
- Eli also likes cakes: there is a 60% chance that I will see her eating cakes if there are cakes in the coffee room: $\mathbb{P}(E=1|C=1) = 0.6$
- But she never buys herself cakes $\mathbb{P}(E=1|C=0) = 0$

Elaborate Graphical Model

- We can depict this situation as



- This allows us to break down the joint probability

$$\begin{aligned}\mathbb{P}(A,B,C,D,E) &= \mathbb{P}(C,D,E|A,B)\,\mathbb{P}(B)\,\mathbb{P}(A)\\ &= \mathbb{P}(D|C)\,\mathbb{P}(E|C)\,\mathbb{P}(C|A,B)\,\mathbb{P}(B)\,\mathbb{P}(A)\end{aligned}$$

- We use the conditional independence of D and E given C

Dependencies

- If we don't observe cakes then the probability of Dave and Eli eating cake are not independent

$$\mathbb{P}(D=1) = 0.303, \qquad \mathbb{P}(E=1) = 0.174$$
$$\mathbb{P}(D=1, E=1) = 0.1392$$

so $\mathbb{P}(D,E) \neq \mathbb{P}(D)\,\mathbb{P}(E)$

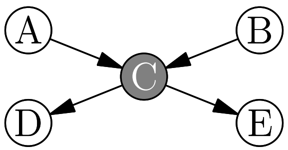
- This changes if we know there are cakes in the coffee room

$$\mathbb{P}(D=1|C=1) = 0.8 \qquad \mathbb{P}(E=1|C=1) = 0.6$$
$$\mathbb{P}(D=1, E=1|C=1) = 0.48$$

so $\mathbb{P}(D=1, E=1|C=1) = \mathbb{P}(D=1|C=1)\,\mathbb{P}(E=1|C=1)$

Observations and Independence

- Making observations changes the probabilities and in some case the dependencies of random variables on each other



$$\cancel{A \not\perp\!\!\!\perp B}$$

$$\cancel{D \not\perp\!\!\!\perp E}$$

- There are rules to deduce the conditional independence from a graphical model given which variables have been observed—but these are details that you can look up if needed

Graphical Model Frameworks

- There are sophisticated frameworks for computing probabilities in Bayesian Belief Networks efficiently
- If our graph is a tree then we can evaluate probabilities efficiently
- When there are loops (so that a random variable both influences and is influenced by another random variables) then exact evaluation of expectations requires exhaustive summing over variables (which is often not tractable)
- There are various message passing algorithms designed to obtain approximations of expectations

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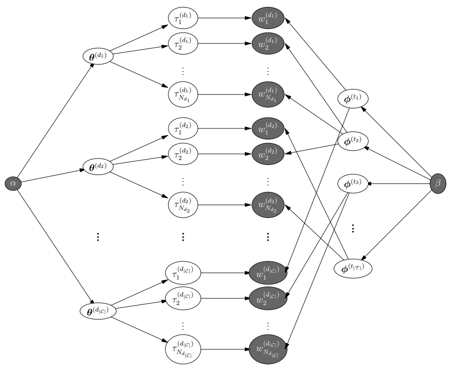
3. **Latent Dirichlet Allocation**



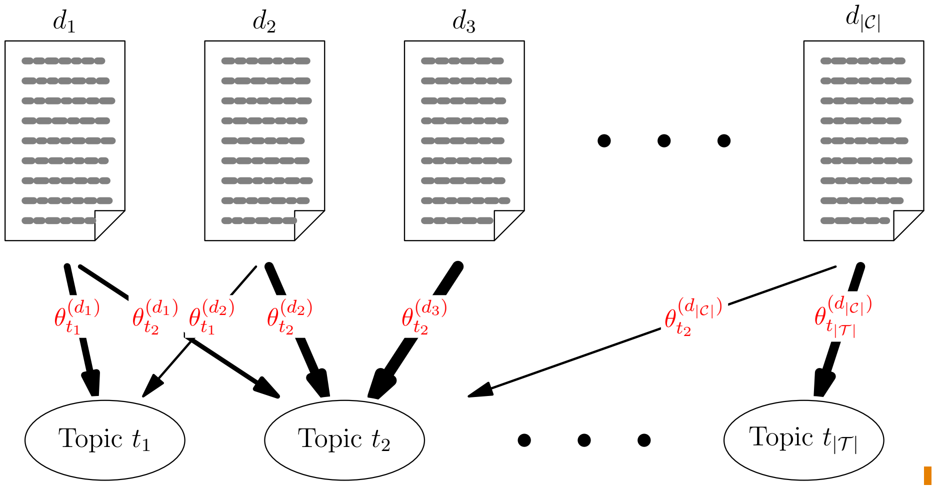
Model for Documents

- We consider a model for the words in a set of documents (we ignore word order)
- We consider a corpus $\mathcal{C} = \{d_i | i = 1, 2, \ldots |\mathcal{C}|\}$
- With documents consisting of words

$$d = \left(w_1^{(d)}, w_2^{(d)}, \ldots, w_{N_d}^{(d)}\right)$$

- We assume that there is a set of topics $\mathcal{T} = \{t_1, t_2, \ldots, t_{|\mathcal{T}|}\}$
- We associate a probability, $\theta_t^{(d)}$, that a word in document d relates to a topic t

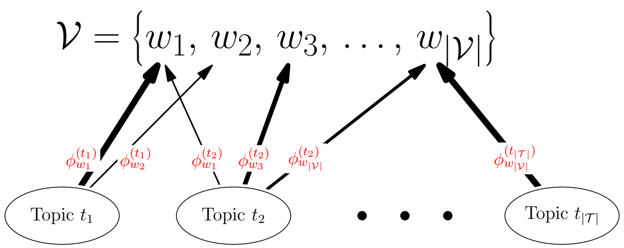
Documents and Topic



$$\boldsymbol{\theta}^{(d)} = (\theta_t^{(d)} | t \in \mathcal{T}) \qquad \theta_t^{(d)} \geq 0 \qquad \sum_{t=1}^{|\mathcal{T}|} \theta_t^{(d)} = 1$$

Words and Topic

- We associate a probability $\phi_w^{(t)}$ that a word, w, is related to a topic t



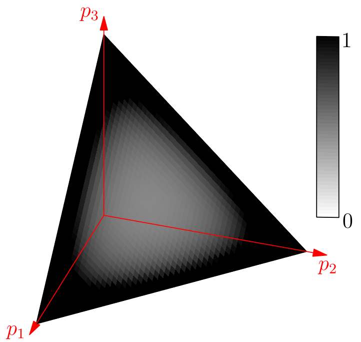
$$\boldsymbol{\phi}^{(t)} = (\phi_w^{(t)} | w \in \mathcal{V})$$

Dirichlet Allocation

- Most documents are predominantly about a few topics and most topic have a small number of words associated to them
- We can generate sparse vectors $\boldsymbol{\theta}^{(d)}$ and $\boldsymbol{\phi}^{(t)}$ from a Dirichlet distribution with small parameters $\boldsymbol{\alpha}$

$$\mathrm{Dir}(\boldsymbol{p}|\boldsymbol{\alpha}) = \Gamma\left(\sum_i \alpha_i\right) \prod_{i=1}^{n} \frac{p_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}$$

- $\sum_i p_i = 1$



$$\boldsymbol{\theta}^{(d)} \sim \mathrm{Dir}(\alpha \mathbf{1})$$
$$\boldsymbol{\phi}^{(t)} \sim \mathrm{Dir}(\beta \mathbf{1})$$

Generating Document

- To generate a document we choose a topic for each word and a word for each topic

$$\forall d \in \mathcal{C} \quad \boldsymbol{\theta}^{(d)} \sim \mathrm{Dir}(\alpha \mathbf{1})$$
$$\forall t \in \mathcal{T} \quad \boldsymbol{\phi}^{(t)} \sim \mathrm{Dir}(\beta \mathbf{1})$$
$$\forall d \in \mathcal{C} \;\wedge\; \forall i \in \{1, 2, \ldots, N_d\} \quad \tau_i^{(d)} \sim \mathrm{Cat}(\boldsymbol{\theta}^{(d)}), \, w_i^{(d)} \sim \mathrm{Cat}(\boldsymbol{\phi}^{(\tau_i^{(d)})})$$

- Where $\mathrm{Cat}(i|\boldsymbol{p}) = p_i$ is the categorical distribution (we choose one of a number of options)
- This model is known as **Latent Dirichlet Allocation**

LDA Graphical Model (version 1)

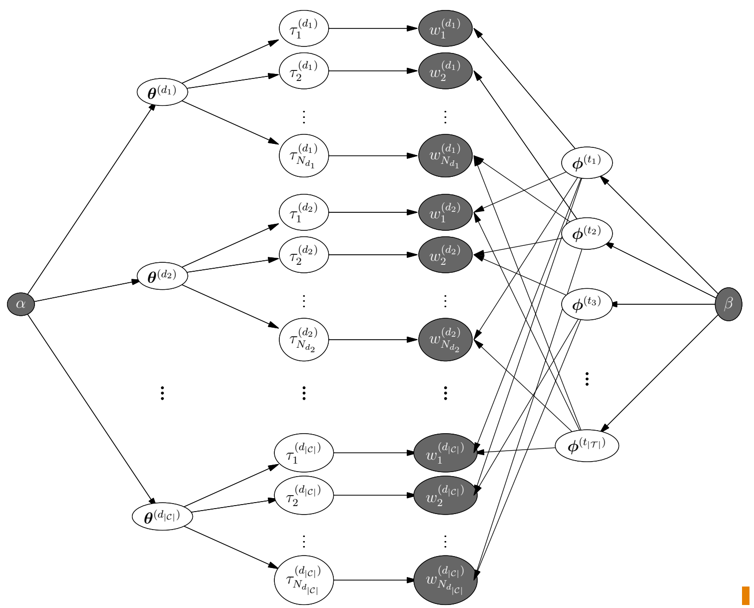
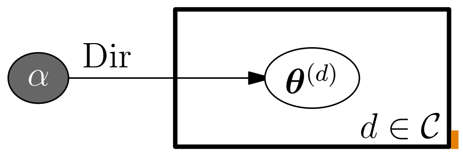


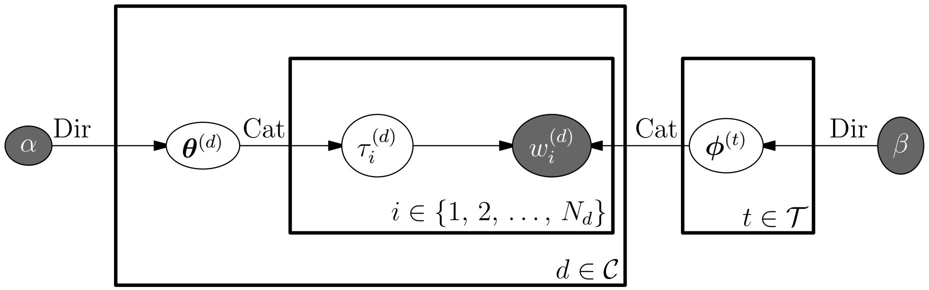
Plate Diagrams

- Drawing every random variable is tedious (and not really possible)
- A short-hand is to draw a box (plate) meaning repeat



- That is we generate vectors $\boldsymbol{\theta}^d$ from a Dirchelet distribution $\mathrm{Dir}(\boldsymbol{\theta}|\alpha \mathbf{1})$ for all documents in corpus $\mathcal{C}$

LDA Graphical Model (version 2)



- This is a lot more compact
- Personally, I find it hard to read, but you get used to it

Probabilistic Model

- The graphical Model is shorthand for the variables

$$\boldsymbol{W} = (\boldsymbol{w}^{(d)} | d \in \mathcal{C}) \quad \text{with} \quad \boldsymbol{w}^{(d)} = (w_1^{(d)}, w_2^{(d)}, \ldots, w_{N_d}^{(d)}), \quad \text{and} \quad w_i^{(d)} \in \mathcal{V}$$

$$\boldsymbol{T} = (\tau_i^{(d)} | d \in \mathcal{C} \wedge i \in \{1, 2, \ldots, N_d\}) \quad \text{with} \quad \tau_i^{(d)} \in \mathcal{T}$$

$$\boldsymbol{\Theta} = (\boldsymbol{\theta}^{(d)} | d \in \mathcal{C}) \quad \text{with} \quad \boldsymbol{\theta}^{(d)} = (\theta_t^{(d)} | t \in \mathcal{T}) \in \Lambda^{|\mathcal{T}|}$$

$$\boldsymbol{\Phi} = (\boldsymbol{\phi}^{(t)} | t \in \mathcal{T}) \quad \text{with} \quad \boldsymbol{\phi}^{(t)} = (\phi_w^{(t)} | w \in \mathcal{V}) \in \Lambda^{|\mathcal{V}|}$$

- Distributed according to

$$\mathbb{P}(\boldsymbol{W}, \boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{\Phi} | \alpha, \beta) = \left(\prod_{t \in \mathcal{T}} \mathrm{Dir}\left(\boldsymbol{\phi}^{(t)} | \beta \mathbf{1}\right) \right) \left(\prod_{d \in \mathcal{C}} \mathrm{Dir}\left(\boldsymbol{\theta}^{(d)} | \alpha \mathbf{1}\right) \prod_{i=1}^{N_d} \mathrm{Cat}\left(\tau_i^{(d)} | \boldsymbol{\theta}^{(d)}\right) \mathrm{Cat}\left(w_i^{(d)} | \boldsymbol{\phi}^{(\tau_i^{(d)})}\right) \right)$$

Finding Topics

- We are given the set of words $\boldsymbol{W}$ and don't really care about τ_i^d the topic associated with word i in document d
- But we are interested in the words associated with each topic $\boldsymbol{\phi}^{(t_i)}$
- And the topics associated with each document $\boldsymbol{\theta}^{(d)}$
- To compute them we need to sample the probability distribution
- One way to do this is using Monte Carlo methods (see next lecture)

Summary

- Building probabilistic models is an intricate process
- Graphical models provide a representation showing the causal relationship between random variables
- This allows us to break down the joint probability of all the variables into conditional probabilities
- This is useful for building the model, but also can speed up evaluating expectations
- Making observations changes the probabilities of random variables
- It is possible to generate very rich models such as Latent Dirichlet Allocation (LDA)