TOPIC 33
Isodynamic Point

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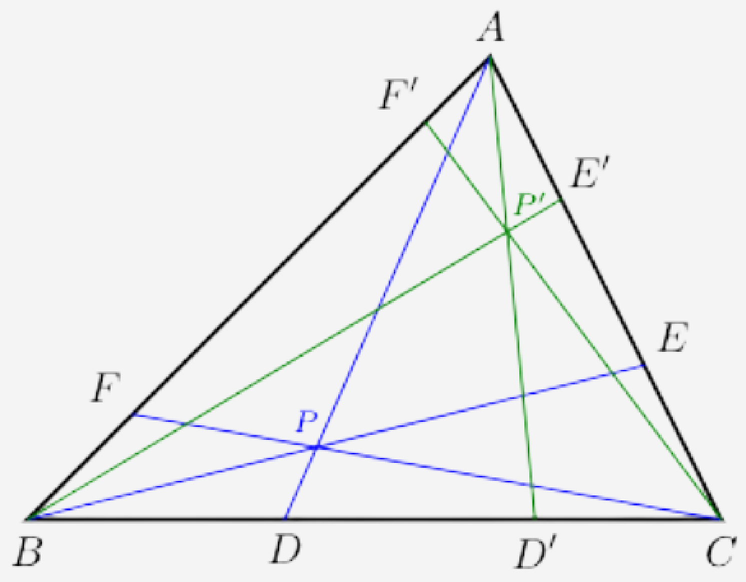
01 Isogonal Conjugate Point

Definition 1. (Isogonal Conjugate Points)

Let P be any point. Assume that AP intersects BC at D; BP intersects CA at E; and CP intersects AB at F. The line AD' is called the isogonal conjugate line of AD, if $\angle CAD' = \angle BAD$. Let BE' and CF' be the corresponding isogonal conjugate lines similarly defined. Then AD', BE', CF' are concurrent at a point P', which is called the isogonal conjugate point of P.

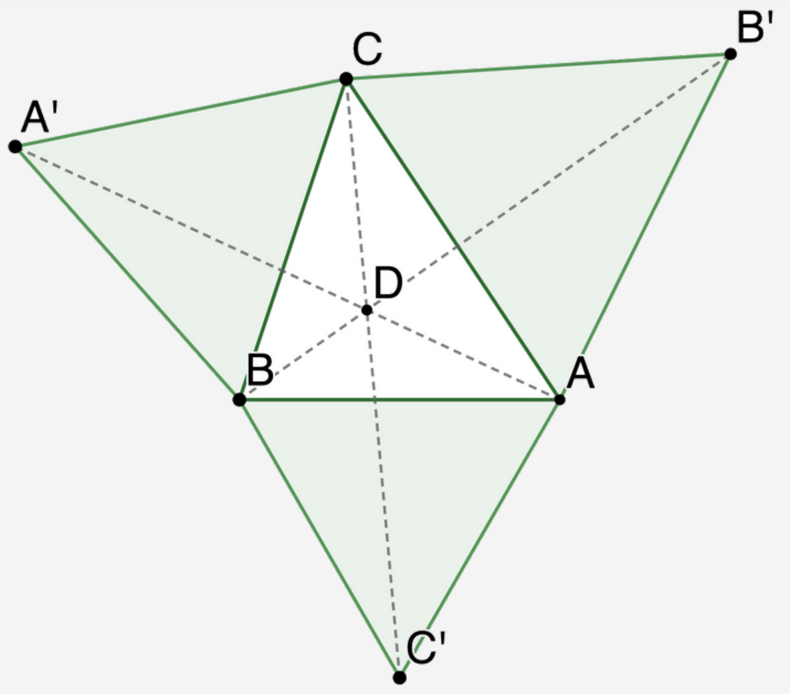
Isogonal points are reflexive, that is, if P' is the isogonal conjugate point of P, then P is the isogonal conjugate point of P'.



For more info, please refer to Topic 7

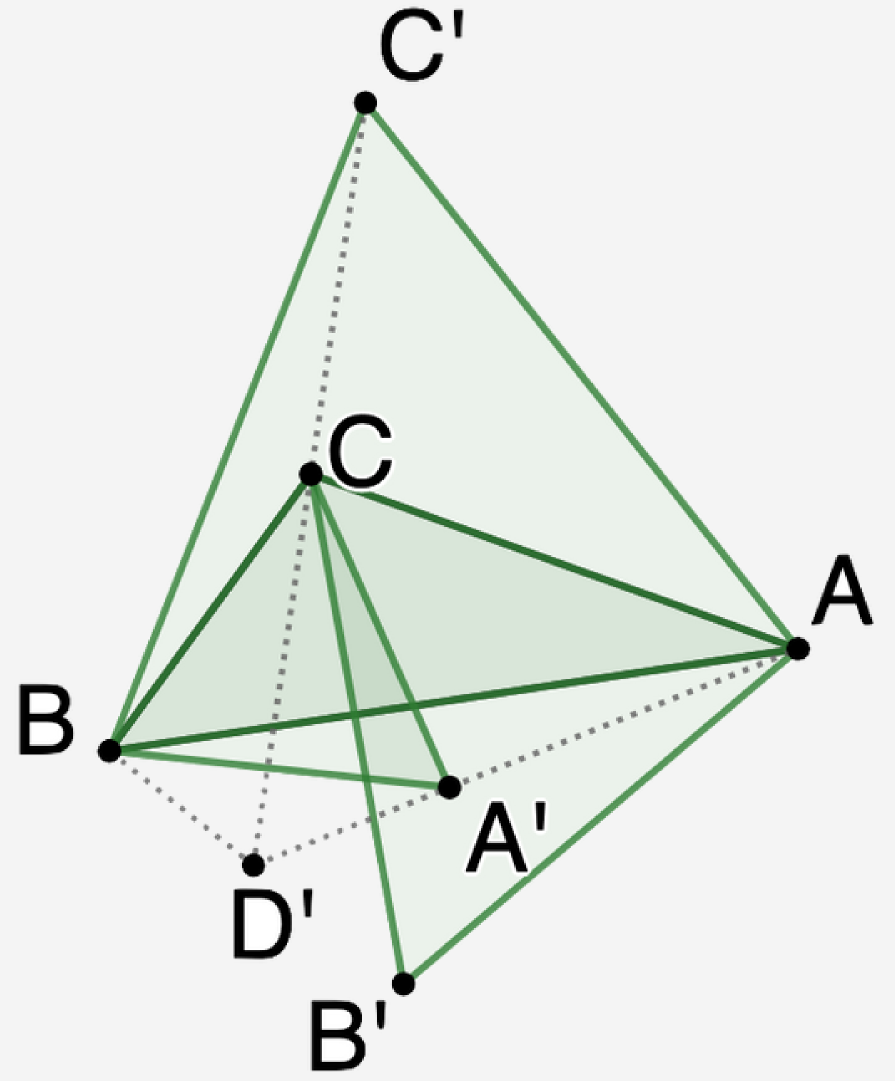
02 Isogonal Center

The positive isogonal center of a triangle can be drawn by making equilateral triangles of each edge outside and connect them with the other point of the original triangle:



02 Isogonal Center

If we draw equilateral triangles inside, we could get the negative isogonal center of a triangle.



For more info, please refer to *Topic 14*

03 Apollonian Circle

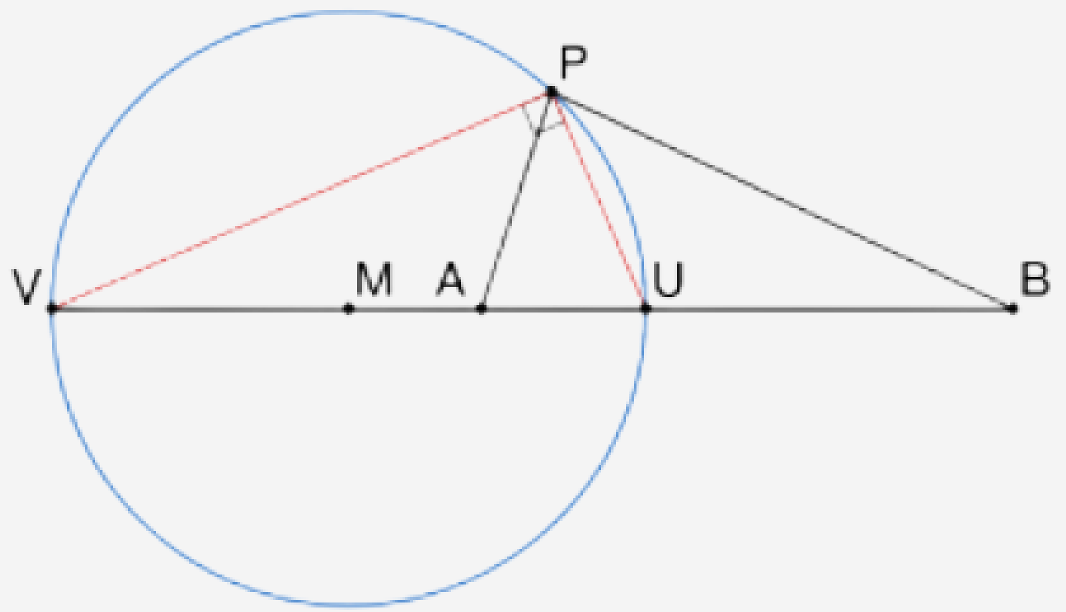


Figure 1

In figure 1, A,B are two fixed points. Find P such that $\frac{AP}{PB} = r$, where r is given and $r \neq 1$. Now we have:

$$AU : UB = AV : VB = AP : PB = r$$

By *Bisector Theorem*, we could see that PU and PV are internal and external angle bisectors of $\angle APB$. It is easy to show that $\angle VPU$ is a right angle. Build a circle around points P,V,U, such circle is called *Apollonian Circle* of $\triangle APB$.

Isodynamic Point

Isodynamic point was first researched by Joseph Neuberg, a Luxembourg mathematician who worked primarily in geometry.

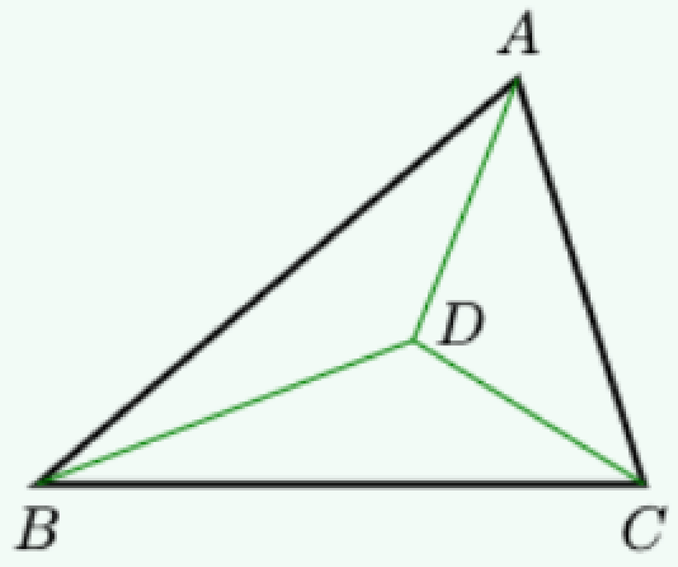
Isodynamic Point

Definition 3. (Isodynamic Point)

Let $\triangle ABC$ be a triangle, and let D be a point such that

$$AD \cdot BC = BD \cdot CA = CD \cdot AB.$$

Then we call D an isodynamic point.



Isodynamic Point

If $\triangle ABC$ is not equilateral, then there are exactly two isodynamic points, which could be found by creating *isogonal conjugate points* of its *positive and negative isogonal centers*.

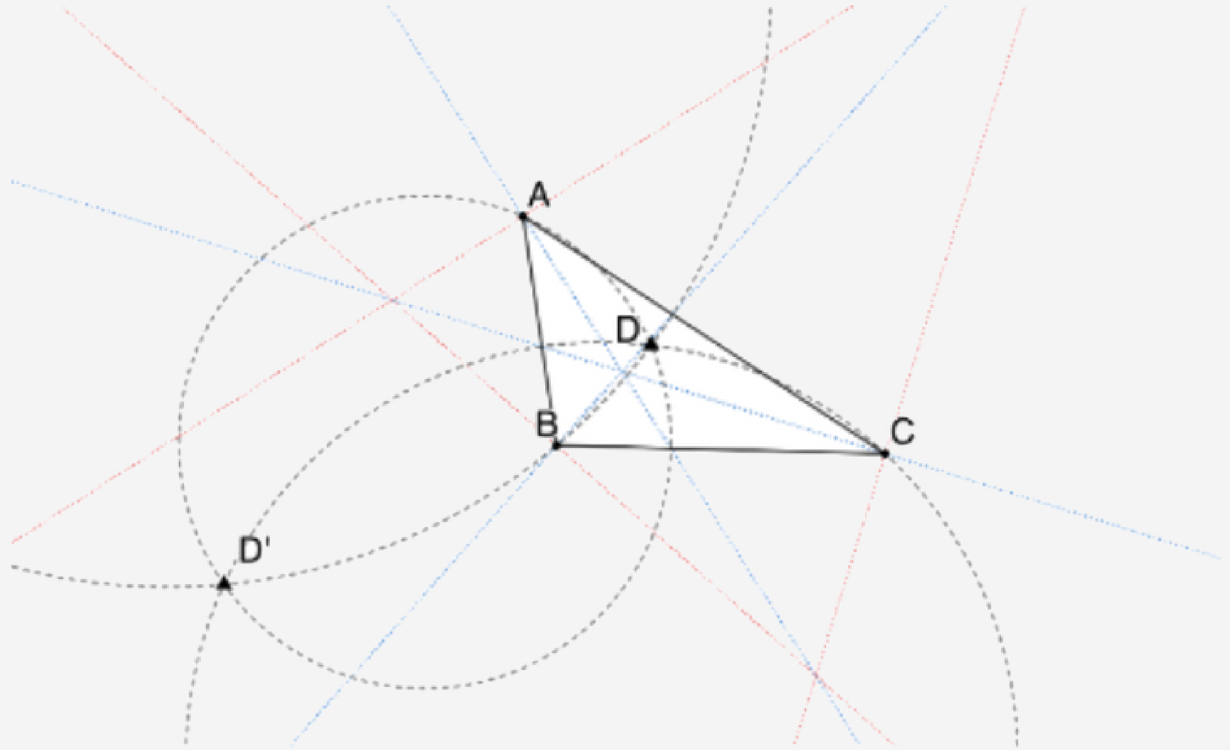


Figure 2: points D and D' are two isodynamic points

In Figure 2, all the red and blue lines are external and internal bisectors of $\angle A$, $\angle B$, and $\angle C$. They are used to construct three *Apollonian circles* .If $\triangle ABC$ is an equilateral triangle, then there is only one isodynamic point, which locates at the incenter of the triangle.

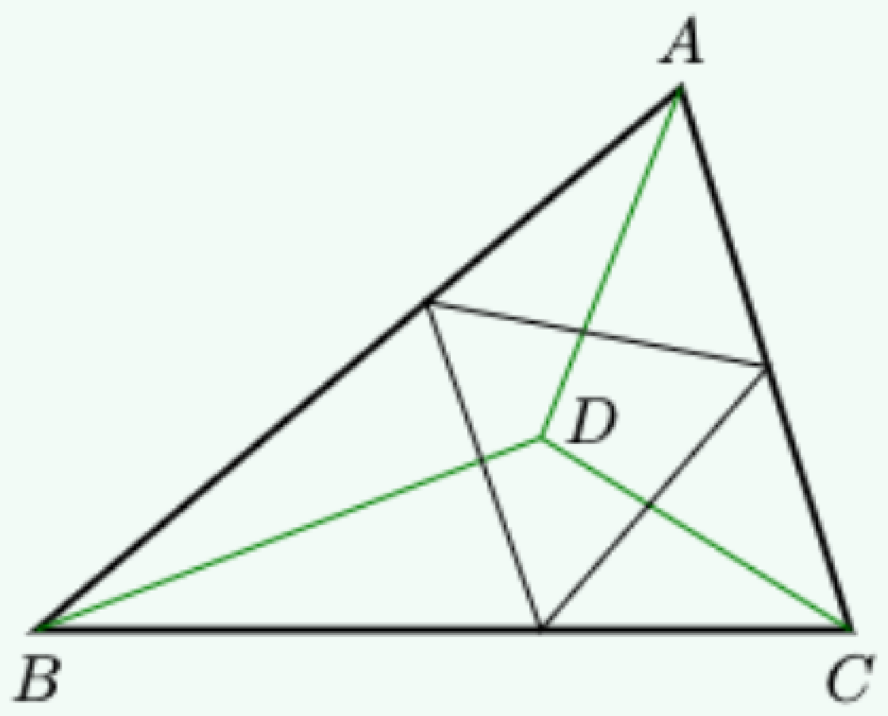
Isodynamic Point

The isodynamic point inside the triangle is called the **first isodynamic point**, and the other one is called the **second isodynamic point**. We assume all isodynamic points mentioned for future problems are **first isodynamic points** if there's no extra information.

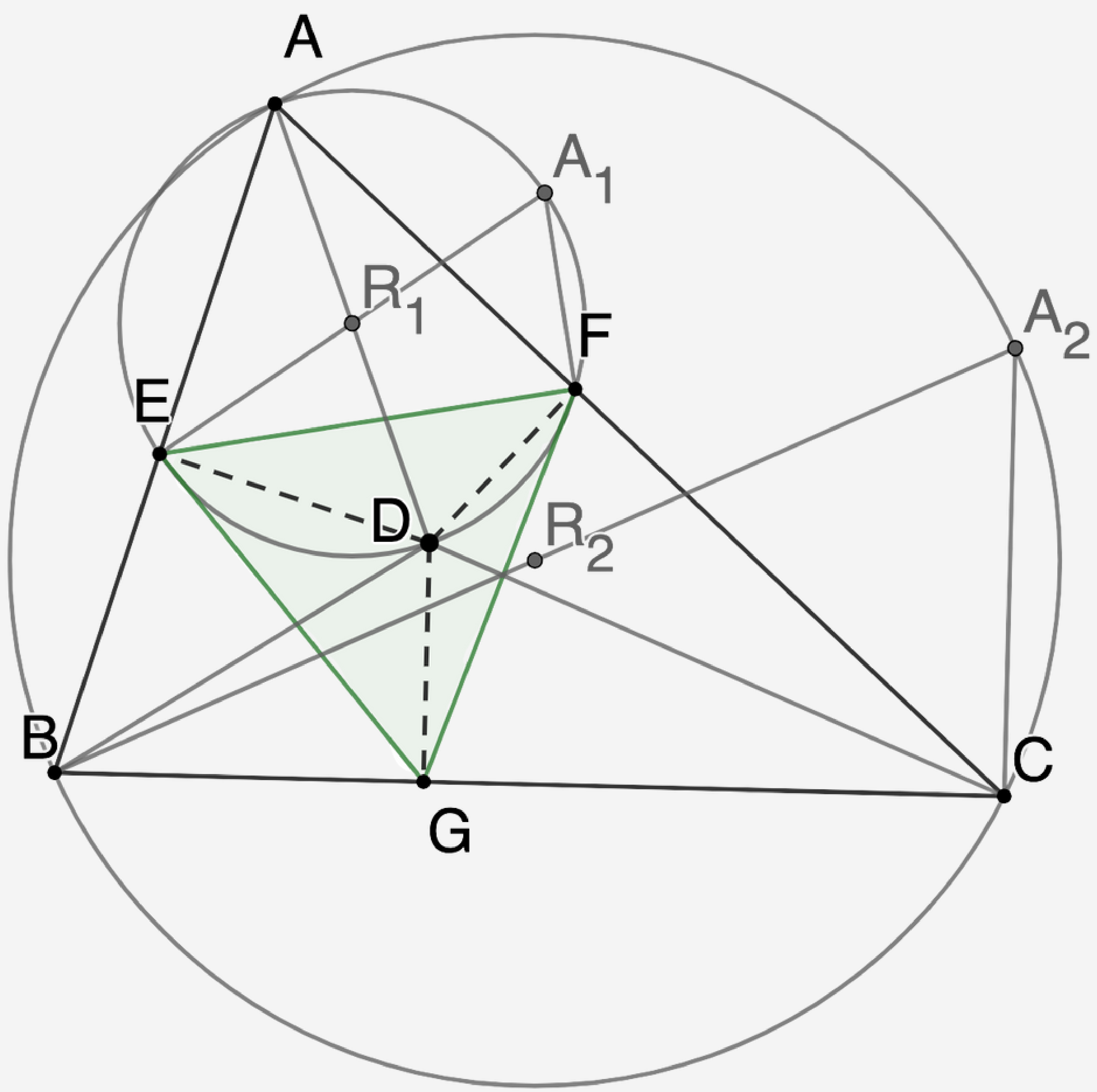
Isodynamic Point

Definition 4

The isodynamic point of a triangle could also be defined as the point whose pedal triangle *is an equilateral triangle.*



Proof:



Note:

- Quadrilateral AEDF is cyclic
- Draw two circles:
  - R1 passed AEFD
  - R2 passes ABC
- Transform Angle A twice

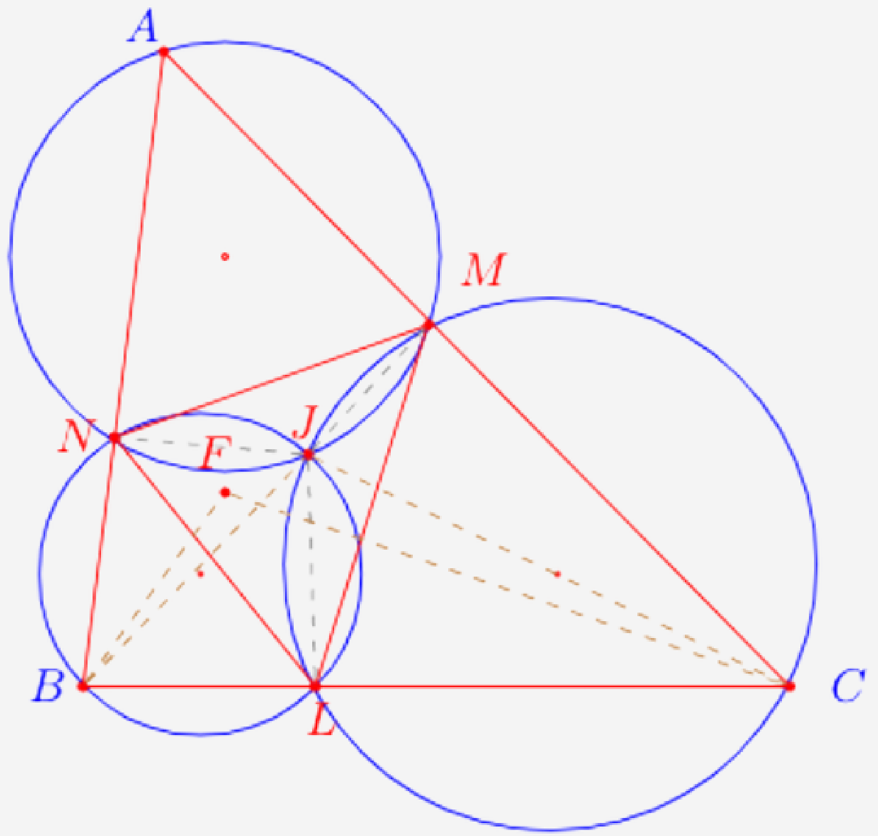
Properties of Isodynamic Point

Theorem 1

The isodynamic point and the Fermat point are isogonal conjugates.

Note: Fermat Point is a point such that the sum of the three distances from each of the three vertices of the triangle to the point is the smallest possible. When a triangle's largest angle is smaller than 120°, its fermat point is the same as its positive isogonal center. For more information, please refer to Topic 14.

Proof:



let F be the Fermat Point of $\triangle ABC$. J is the isodynamic point of $\triangle ABC$. Two lemmas from *Fermat point* and *Isogonal conjugate points* will be used in proving this theorem. To prove these lemmas, please refer to their sections on links above.

Lemma 1

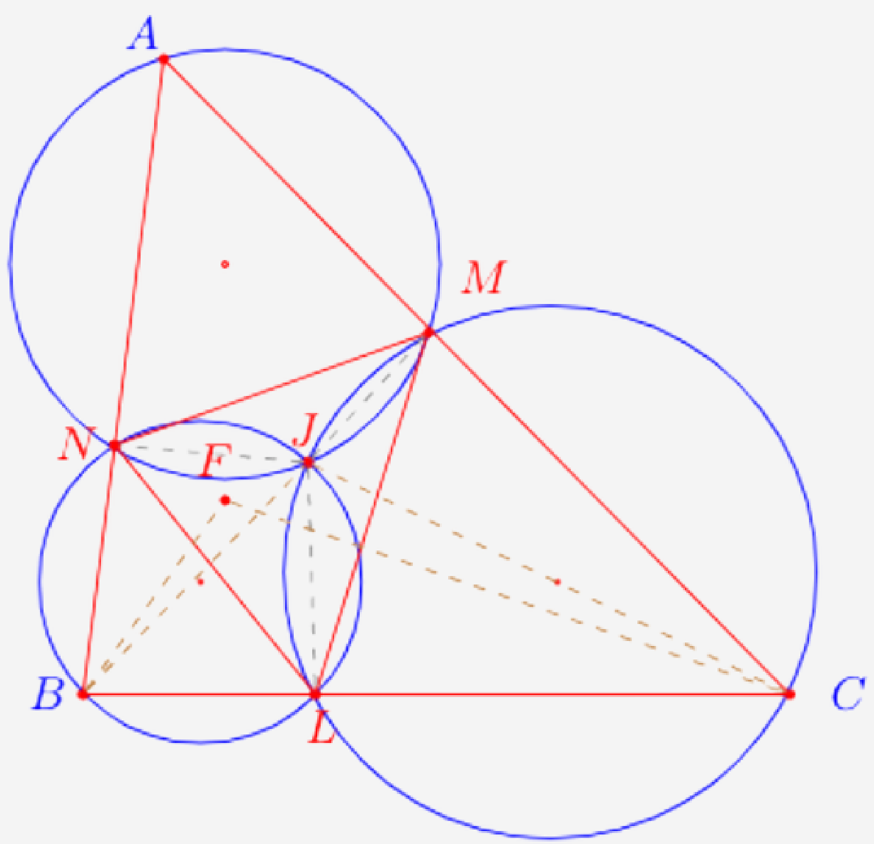
1. *In $\triangle ABC$, the Fermat point F is the only point in the triangle that satisfies:*

$$\angle AFB = \angle BFC = \angle CFA = 120^\circ$$

2. *Two points F and J are isogonal conjugate points if and only if:*

$$\angle BFC + \angle BJC = 180^\circ + \angle A.$$

Proof:



So it will be sufficient to show that points F and J satisfies both lemma. Let LMN be the pedal triangle of J. Then from the cyclic quadrilaterals $JMAN$, $JNBL$, and $JLCM$ we have

$$\begin{aligned}\angle BJC &= \angle JBA + \angle A + \angle JCB = \angle JLN + \angle A + \angle MLJ \\ &= 60^\circ + \angle A \\ \iff \angle BFC &= 180^\circ + \angle A - (60^\circ + \angle A) = 120^\circ.\end{aligned}$$

Similarly we can show that

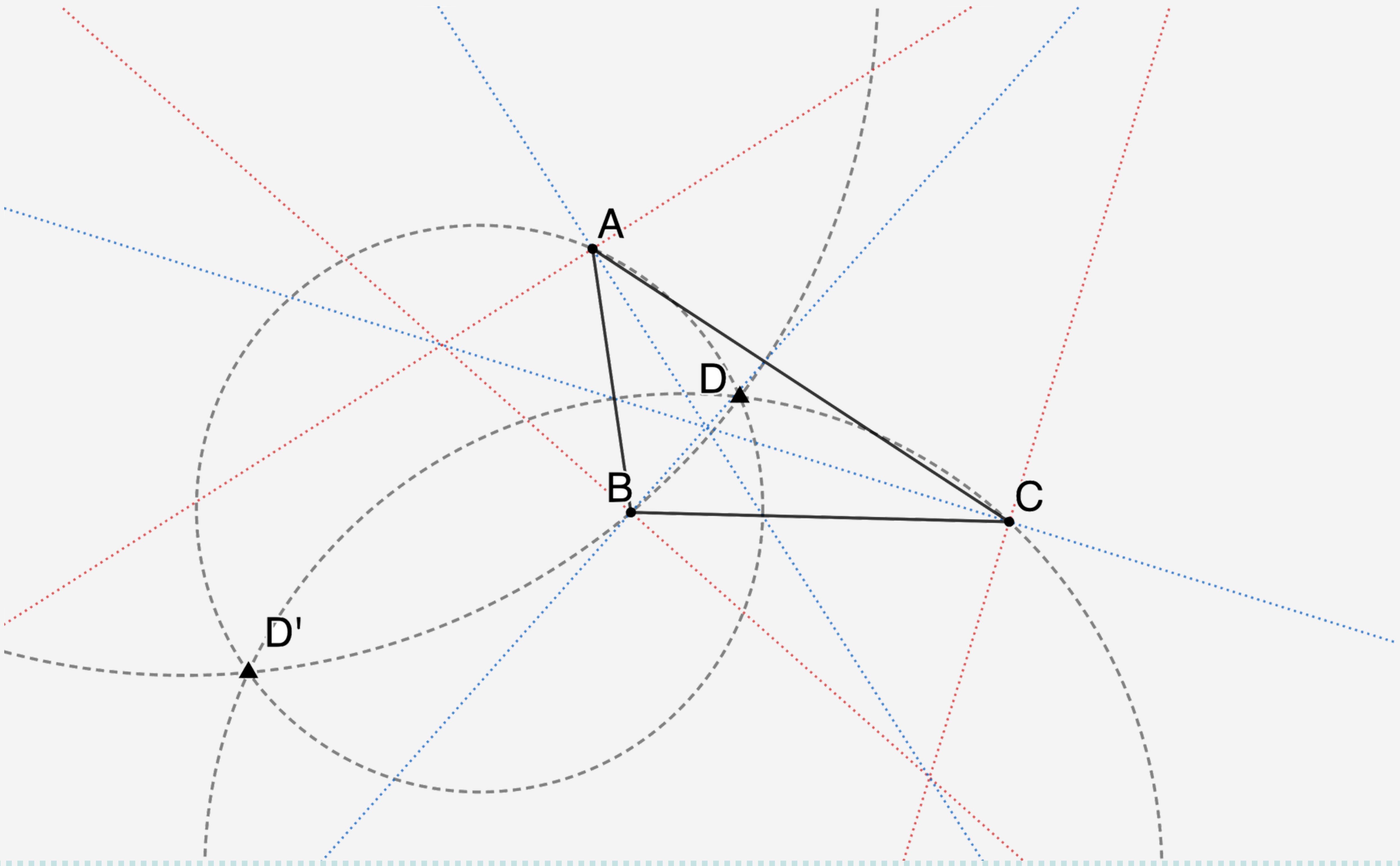
$$\angle AFB = \angle CFA = 120^\circ.$$

So F is the isogonal conjugate of J.

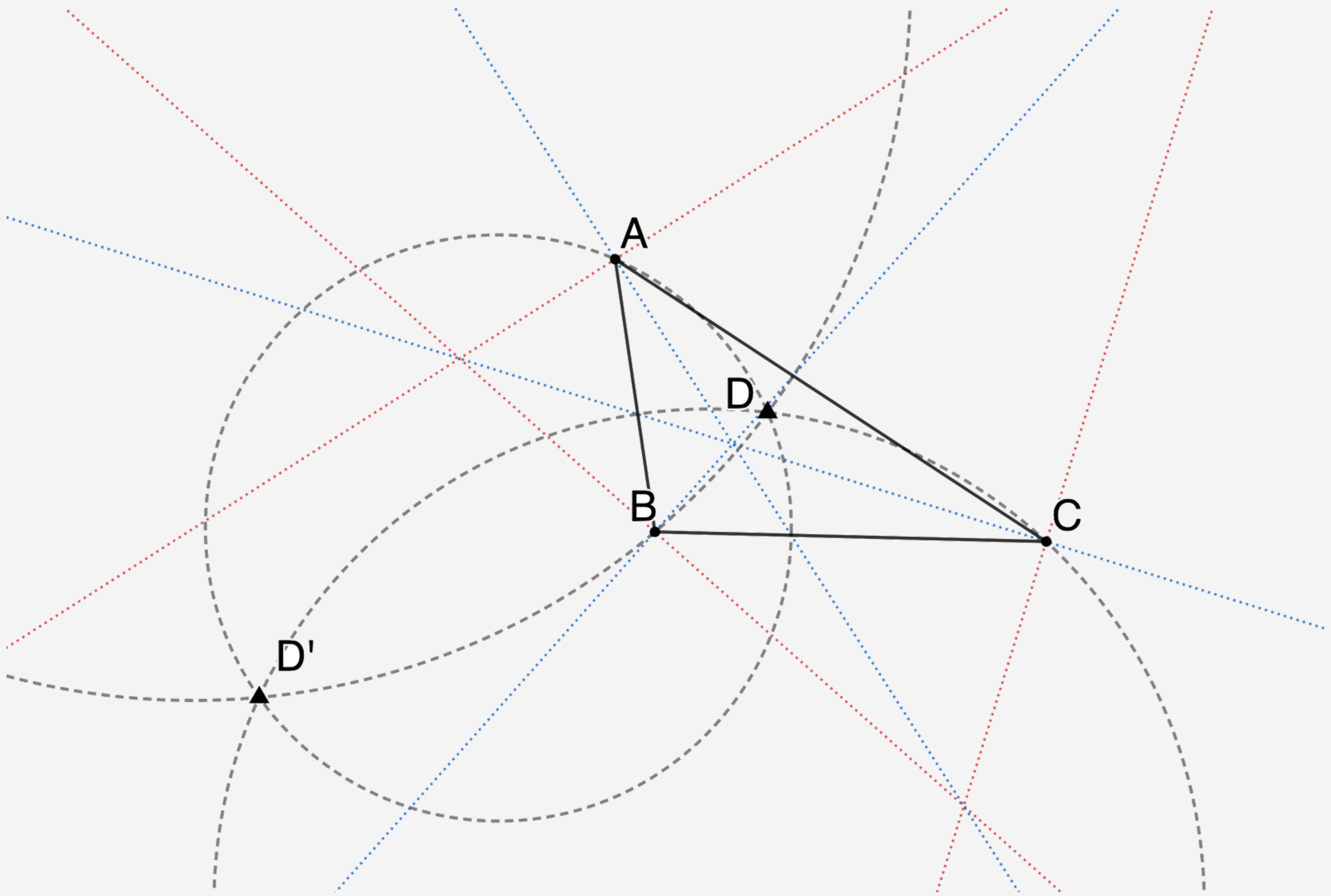
Properties of Isodynamic Point

Theorem 2

Three Apollonian circles of a non-equilateral triangle have exactly two isodynamic points as intersections.


A
B
C
D
D'

Proof:



Proof: Let D, D' be the intersections of three Apollonian circles of $\triangle ABC$. From the definition of Apollonian circles, we could have:

$$DB : DC = BA : CA$$

and

$$DC : DA = BC : BA$$

Thus

$$DB \cdot CA = BA \cdot DC$$

Similarly, we could get

$$DB \cdot CA = BA \cdot DC = DA \cdot BC$$

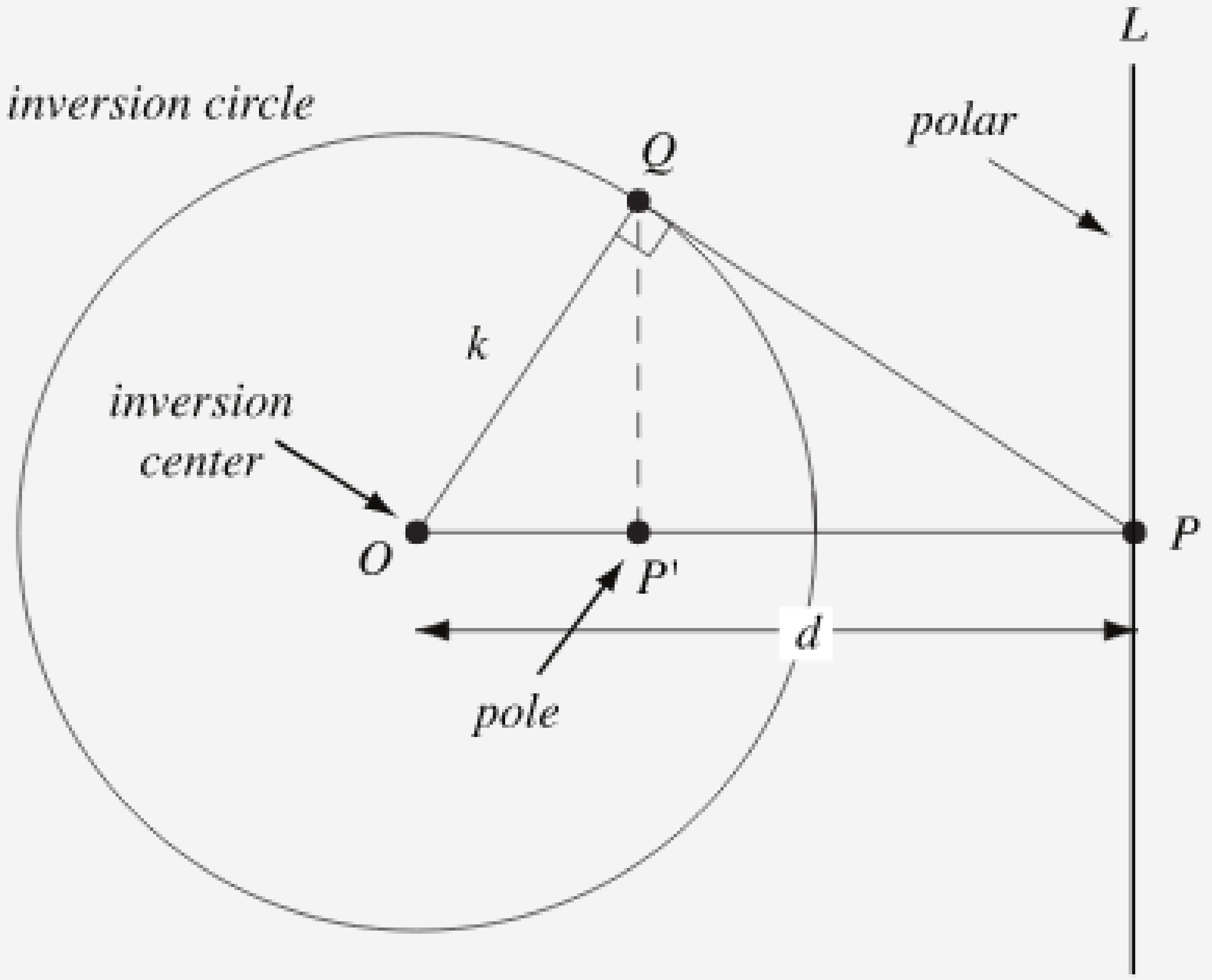
Therefore D is the isodynamic point of $\triangle ABC$.

01 Transformation of Isodynamic Point

Theorem 3

The inversion of the triangle $\triangle ABC$ with respect to an isodynamic point transforms the original triangle into an equilateral triangle.

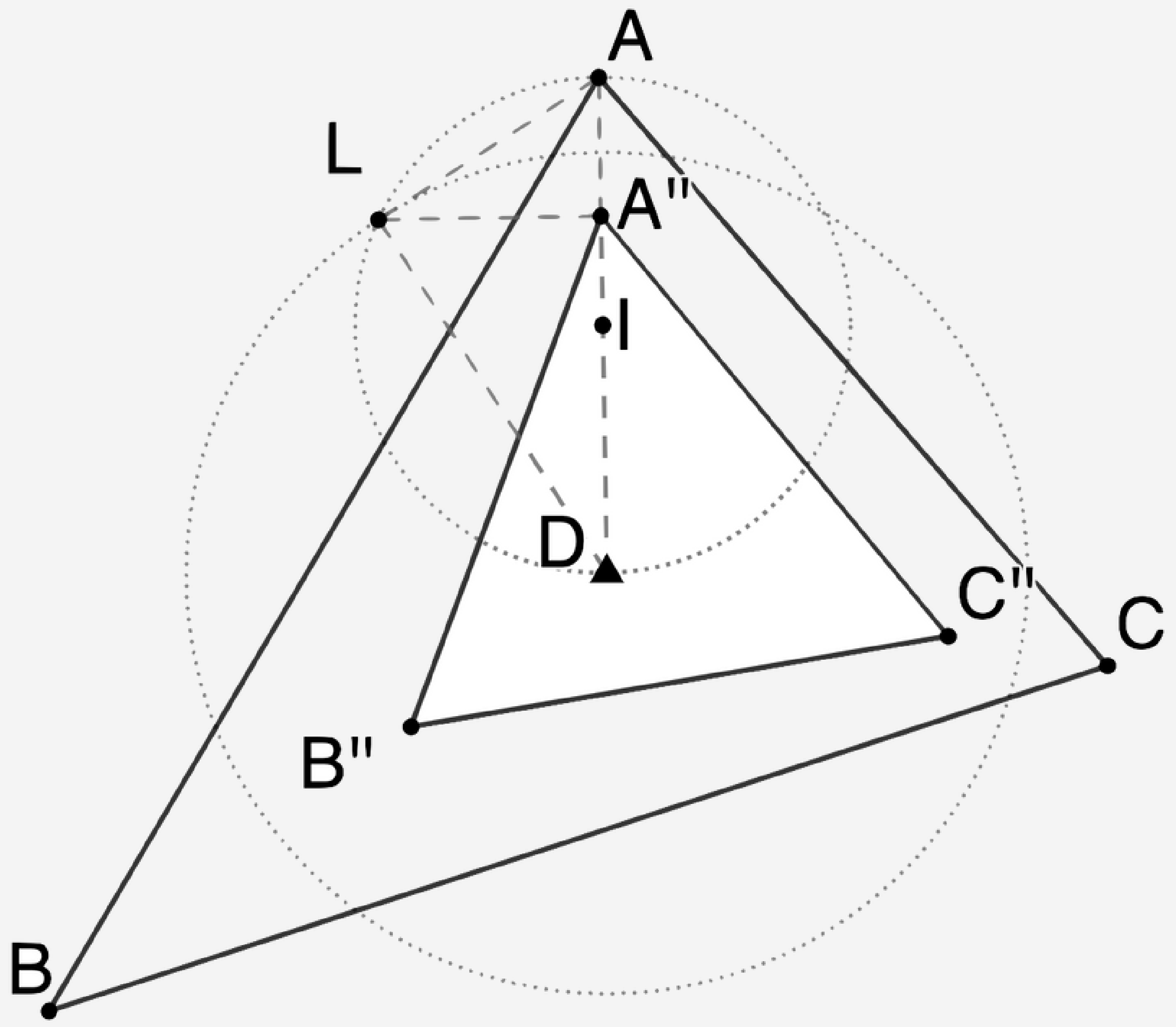
Inversion



It is easy to show the main properties:

k^2=OP • OP'

Inversion



Proof: By property of inversion,

$$k^2 = DC'' \cdot DC = DB'' \cdot DB = DA'' \cdot DA$$

Thus

$$\frac{AD}{DC''} = \frac{CD}{AD''}$$

Since $\angle ADC$ is a common angle, $\triangle DA''C''$ and $\triangle DCA$ are similar.Thus:

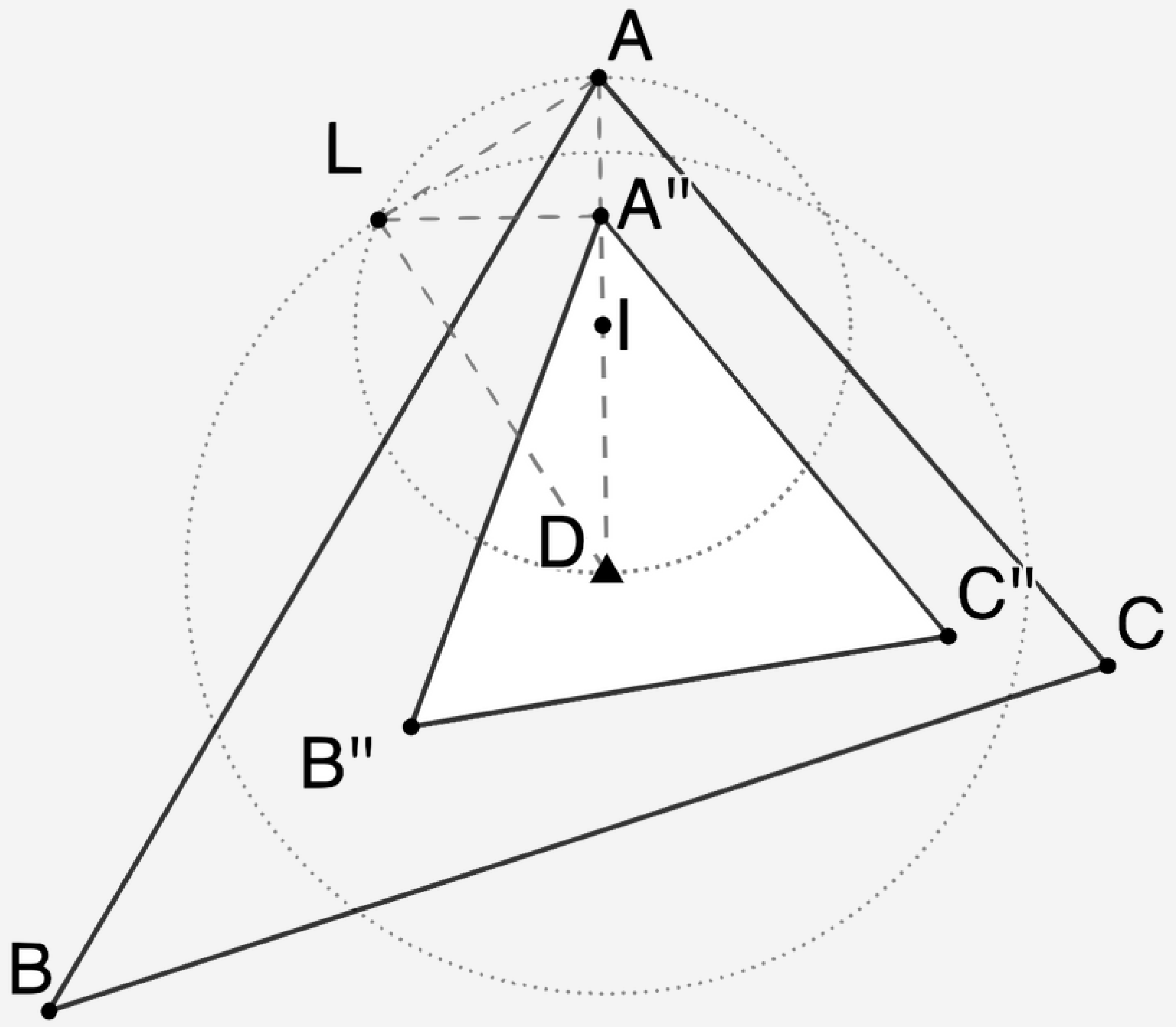
$$\frac{A''C''}{AC} = \frac{A''D}{CD}$$

$$A''C'' = \frac{A''D \cdot AC}{CD} \tag{1}$$

Similarly, we could get

$$A''B'' = \frac{A''D \cdot AB}{BD} \tag{2}$$

Proof:



By definition of isodynamic point,

$$AC \cdot BD = AB \cdot CD$$

Thus

$$\frac{AC}{CD} = \frac{AB}{BD}$$

Multiply both sides by $A''D$,

$$\frac{A''D \cdot AC}{CD} = \frac{A''D \cdot AB}{BD} \tag{3}$$

Combine equations (1), (2), (3), We could get

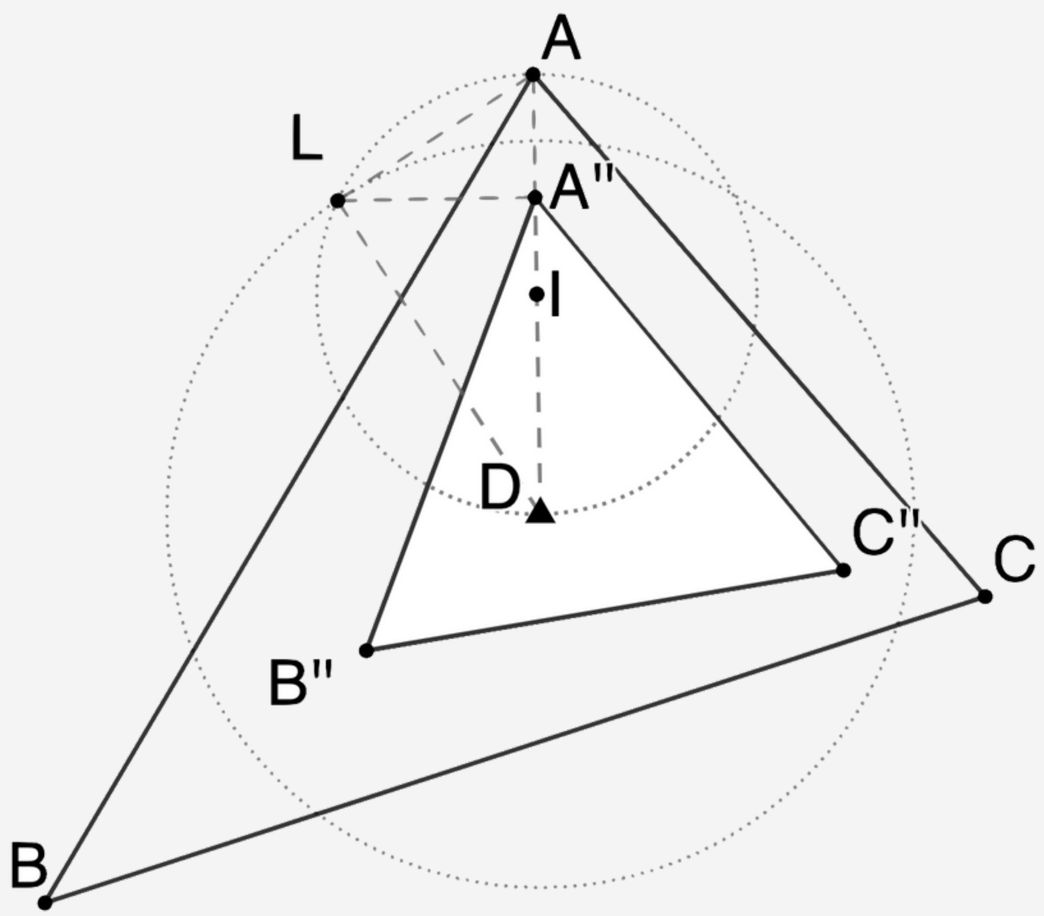
$$A''B'' = A''C''$$

Similarly, we could derive

$$A''B'' = A''C'' = B''C''$$

Hence $\triangle A''B''C''$ is equilateral. ∎

Transformation of Isodynamic Point



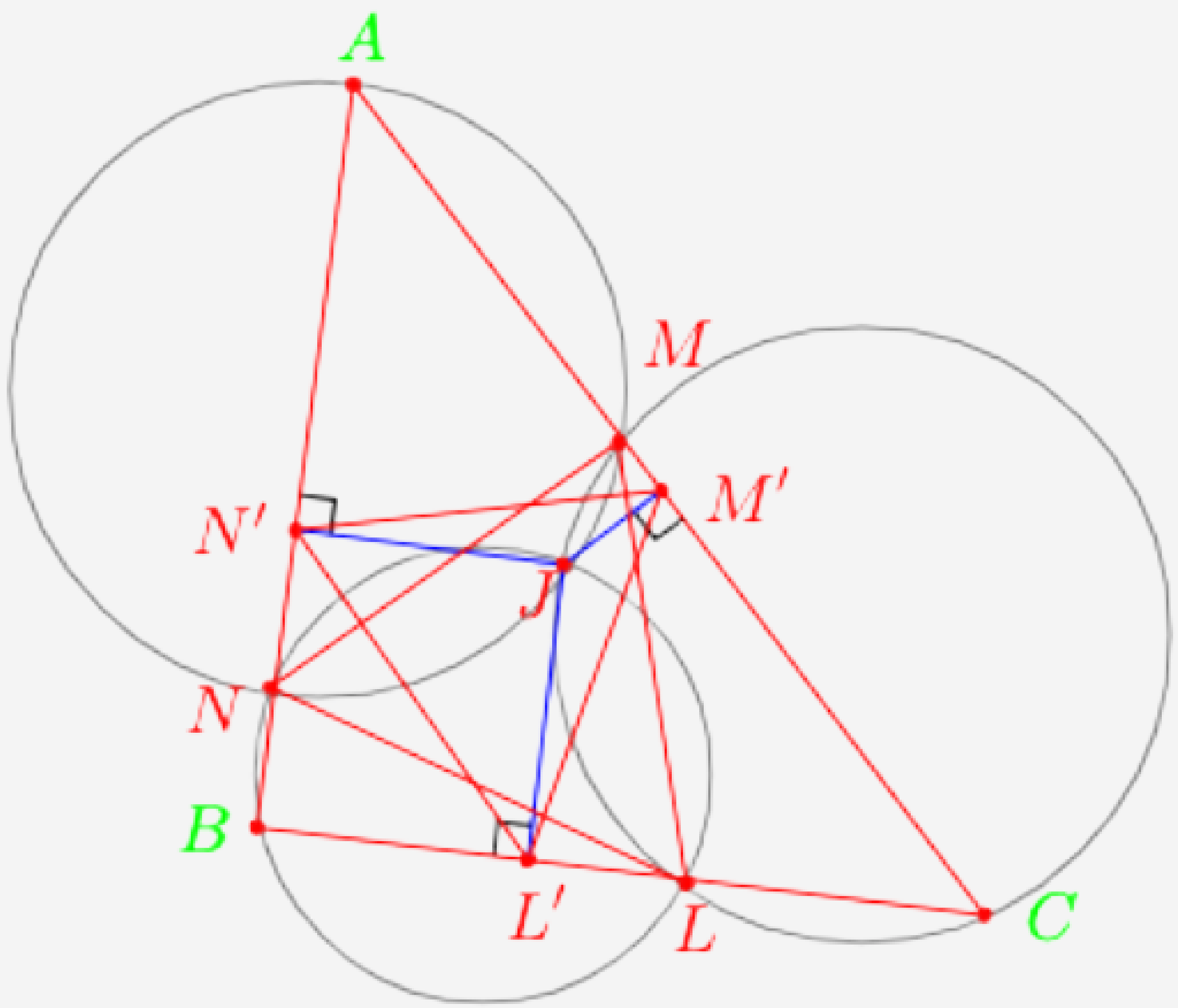
There are also many other types of transformations that could be connected with isodynamic point. For example, The individual isodynamic points are fixed by *Möbius* Transformations that map the interior of the circumcircle of $\triangle ABC$ to the interior of the circumcircle of the transformed triangle, and swapped by transformations that exchange the interior and exterior of the circumcircle.

02 Area of Pedal Triangle

Theorem 4

Among all equilateral triangles having vertices on the sides of a triangle, the pedal triangle of J, the first isodynamic point, has the minimum area.

Proof:

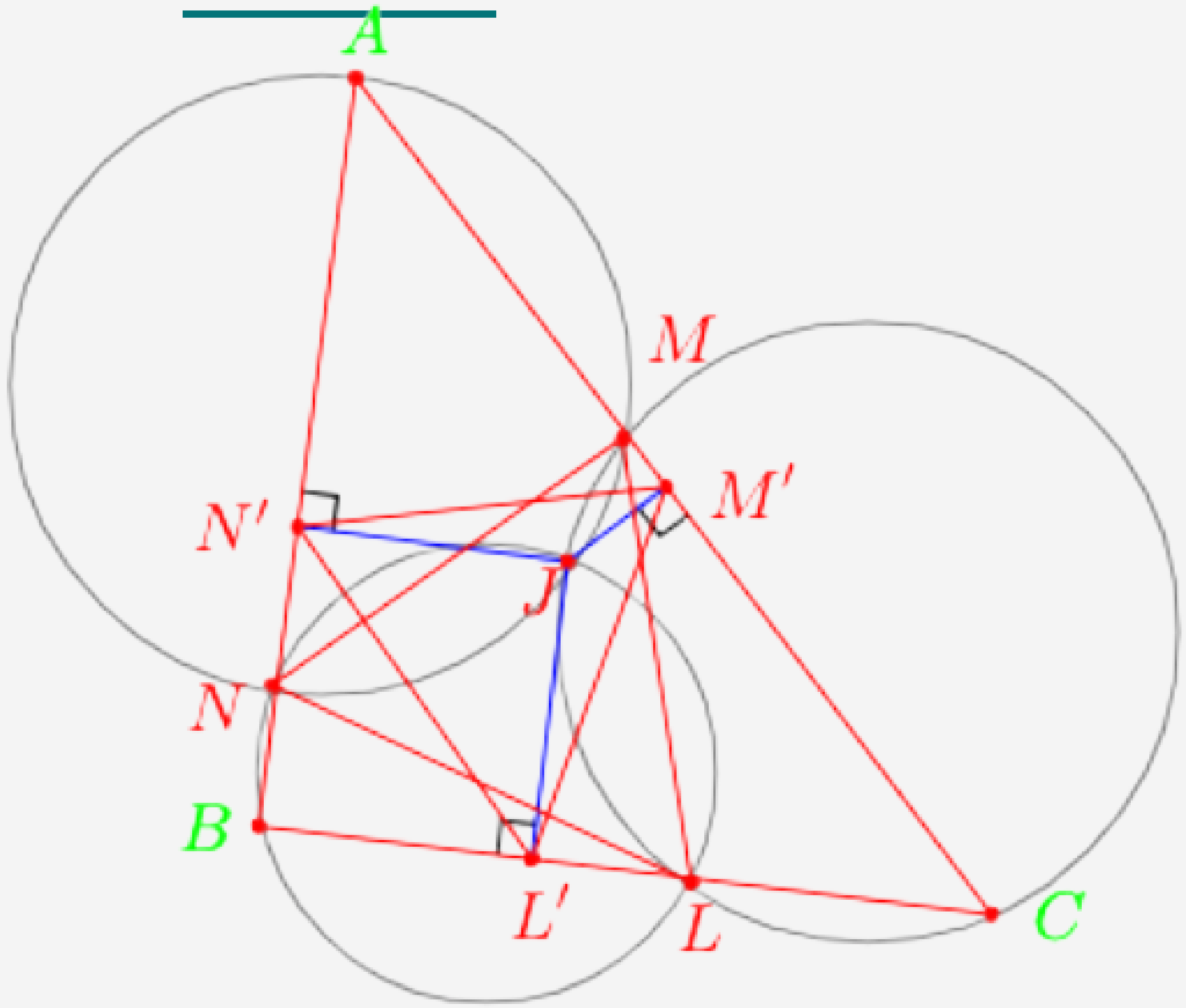


The following lemma is used to prove this theorem:

Lemma 2. Miquel's Theorem

The lines from the Miquel point to the marked points make equal angles with the respective sides. (Miquel point is the intersection of three circles drawn from one point of a triangle and two random points on nearby edges).

Proof:



Let LMN be an equilateral triangle which has vertices on the sides of $\triangle ABC$. If we draw the circumcircles of the triangles LCM, MAN, NBL, they will concur in a point J, by *Miquel's Theorem*. Now we draw the pedal triangle $L'M'N'$ of the point J. From the cyclic quadrilaterals we have

$$\angle JLM = \angle JCM = \angle JL'M'$$

$$\angle JLN = \angle JBN = \angle JL'N'.$$

Adding these two we get, $\angle MLN = \angle M'L'N' = 60°$. So a spiral similarity with center J, ratio $r = \frac{JL'}{JL} \leq 1$, and angle $\alpha = \angle LJL'$ maps $\triangle LMN \to \triangle L'M'N'$. From the definition, we deduce that J is the first isodynamic point of $\triangle ABC$. Hence the conclusion follows. $\blacksquare$

Thanks for Watching!

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