17-S3-Q5
Solution + Discussion

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Abstract

In this document we will go through the solution to the 17-S3-Q5 question and provide a discussion of the question at the end. There are also hints on the first page to aid you in finding a solution. There is no single method that results in an answer to a STEP question, there are a multitude of different paths that end up at the same solution. However, some methods are more straight forward and you are encouraged to take the path of least resistance.

Hints

First part: Using the fundamental polar coordinate equations to find $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ and $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ then use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Second part: What is the relationship between the gradients of the curves if they meet at a right angle?

Third part: Use the relation in the first part to form a differential equation for f(θ).

Solution

Using our fundamental polar coordinate equations

$$x = \mathrm{f}(\theta)\cos\theta \quad \text{and} \quad y = \mathrm{f}(\theta)\sin\theta$$

we can use the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x}$$

with

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \mathrm{f}'(\theta)\sin\theta + \mathrm{f}(\theta)\cos\theta\\ \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \mathrm{f}'(\theta)\cos\theta - \mathrm{f}(\theta)\sin\theta\end{aligned}$$

to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{f}'(\theta)\sin\theta + \mathrm{f}(\theta)\cos\theta}{\mathrm{f}'(\theta)\cos\theta - \mathrm{f}(\theta)\sin\theta} = \frac{\mathrm{f}'(\theta)\tan\theta + \mathrm{f}(\theta)}{\mathrm{f}'(\theta) - \mathrm{f}(\theta)\tan\theta}.$$

If we have two curves that meet at right angles at a point, then their gradients must also meet at right angles at that same point.

The gradient of the curves $r = \mathrm{f}(\theta)$

$$m_{\mathrm{f}} = \frac{\mathrm{f}'(\theta)\tan\theta + \mathrm{f}(\theta)}{\mathrm{f}'(\theta) - \mathrm{f}(\theta)\tan\theta}$$

and $r = \mathrm{g}(\theta)$

$$m_{\mathrm{g}} = \frac{\mathrm{g}'(\theta)\tan\theta + \mathrm{g}(\theta)}{\mathrm{g}'(\theta) - \mathrm{g}(\theta)\tan\theta}$$

must have a product of -1,

$$m_{\mathrm{f}}\ m_{\mathrm{g}} = -1 \ \Rightarrow \ m_{\mathrm{f}} = -\frac{1}{m_{\mathrm{g}}}.$$

We see that

$$\frac{\mathrm{f}'(\theta)\tan\theta + \mathrm{f}(\theta)}{\mathrm{f}'(\theta) - \mathrm{f}(\theta)\tan\theta} = \frac{\mathrm{g}(\theta)\tan\theta - \mathrm{g}'(\theta)}{\mathrm{g}'(\theta)\tan\theta + \mathrm{g}(\theta)}$$

gives,

$$\left(\tan^2\theta + 1\right)(\mathrm{fg} + \mathrm{f}'\mathrm{g}') = 0$$

after some simplification. Thus where they meet we have $\tan^2\theta + 1 \equiv \sec^2\theta = 0$ which would imply that $\cos\theta = 0$, or

$$\mathrm{f}(\theta)\mathrm{g}(\theta) + \mathrm{f}'(\theta)\mathrm{g}'(\theta) = 0.$$

Using the above relation with $\mathrm{g}(\theta) = a(1 + \sin\theta)$ gives

$$a\mathrm{f}(\theta)(1 + \sin\theta) + a\mathrm{f}'(\theta)\cos\theta = 0$$

for each value of a. Solving the differential equation

$$\frac{\mathrm{df}}{\mathrm{d}\theta} = -\frac{1+\sin\theta}{\cos\theta}\mathrm{f}$$

involves the integral

$$-\int \frac{1+\sin\theta}{\cos\theta}\,\mathrm{d}\theta.$$

Using the substitution $u = 1 + \sin\theta$,

$$-\int \frac{1+\sin\theta}{\cos\theta}\,\mathrm{d}\theta = -\int \frac{1}{2-u}\,\mathrm{d}u = \ln(1-\sin\theta) + c$$

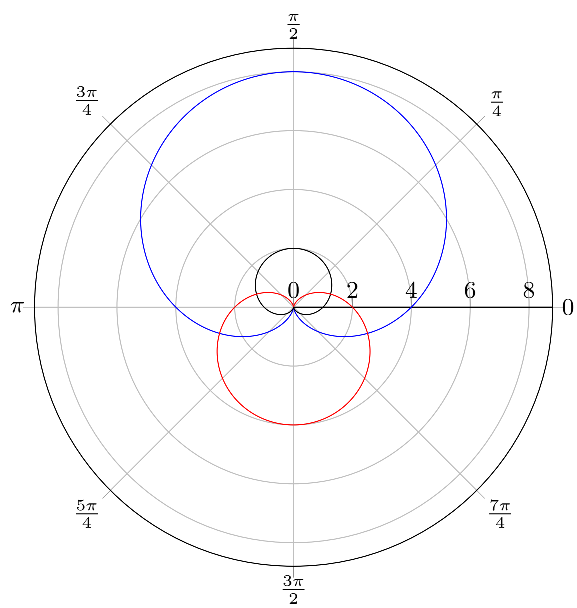
and thus we have that

$$\ln(\mathrm{f}) = \ln(1-\sin\theta) + c \Rightarrow \mathrm{f}(\theta) = A(1-\sin\theta)$$

for some A. With our initial condition $\mathrm{f}(-\frac{\pi}{2}) = 4$,

$$\mathrm{f}(\theta) = 2(1-\sin\theta).$$

Plotting $r = 1 + \sin\theta$, $r = 4(1+\sin\theta)$ and $\mathrm{f}(\theta)$,



with $r = 1 + \sin\theta$ in black, $r = 4(1+\sin\theta)$ in blue and $\mathrm{f}(\theta)$ in red.

Discussion

Familiarity with the equations of polar coordinates, sketching polar coordinates and Cartesian geometry are key to a understanding of this question.

Starting with finding an expression for the Cartesian derivative in terms of $r = \mathrm{f}(\theta)$ and θ can seem abstract. Writing down the fundamental equations

$$x = \mathrm{f}(\theta)\cos\theta \quad \text{and} \quad y = \mathrm{f}(\theta)\sin\theta$$

and the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x}$$

helps clarify what we know and what we need. The rest is differentiation and algebraic manipulation.

Showing the relation

$$\mathrm{f}(\theta)\mathrm{g}(\theta) + \mathrm{f}'(\theta)\mathrm{g}'(\theta) = 0$$

is conceptually the easiest if we consider the relation of the Cartesian form of the curves $r = \mathrm{f}(\theta)$ and $r = \mathrm{g}(\theta)$ as in this form we are comfortable with the fact that their gradients must have a product of -1.

Substituting our given curves leads to a standard separable differential equation which can be solved to find $\mathrm{f}(\theta)$.

To sketch a polar curve it is typically a good rule of thumb to recognise the standard curves (lines and circles in polar coordinates) in combination with finding the values of θ where $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$,$r = 0$ and values of r for certain $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \cdots$.

Overall, proficiency with polar coordinates and sketching polar curves will lead to a smooth experience in tackling this question.