08-S3-Q5
Solution + Discussion

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Abstract

In this document we will go through the solution to the 08-S3-Q5 question and provide a discussion of the question at the end. There are also hints on the first page to aid you in finding a solution. There is no single method that results in an answer to a STEP question, there are a multitude of different paths that end up at the same solution. However, some methods are more straight forward and you are encouraged to take the path of least resistance.

Hints

1st part: In the induction step substitute $\mathrm{T}_{k+2}(x)$ and $\mathrm{T}_k(x)$ out using the recurrence relation, expanding and simplifying.

2nd part: Use $(\mathrm{T}_n(x))^2 - \mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x) = 0$ to find $\mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_4$ in terms of $\mathrm{r}(x)$ and T_0 then use induction.

3rd part: Consider writing $\mathrm{T}_2(x) - 2x\mathrm{T}_1(x) + \mathrm{T}_0(x) = 0$ and $(\mathrm{T}_1(x))^2 - \mathrm{T}_0(x)\mathrm{T}_2(x) = 0$ in terms of T_0 and $\mathrm{r}(x)$.

Solution

Our base case is for $n = 1$,

$$\mathrm{T}_1^2(x) - \mathrm{T}_0(x)\mathrm{T}_2(x) = \mathrm{f}(x)$$

which is clearly true.

Our induction hypothesis is that

$$\mathrm{T}_k^2(x) - \mathrm{T}_{k-1}(x)\mathrm{T}_{k+1}(x) = \mathrm{f}(x)$$

is true for some $n = k$.

Considering the $n = k + 1$ case,

$$\begin{aligned}
&\mathrm{T}_{k+1}^2(x) - \mathrm{T}_k(x)\mathrm{T}_{k+2}(x) \\
&= \mathrm{T}_{k+1}^2(x) - \left[\frac{1}{2x}\mathrm{T}_{k+1}(x) + \frac{1}{2x}\mathrm{T}_{k-1}(x)\right]\left[2x\mathrm{T}_{k+1}(x) - \mathrm{T}_k(x)\right] \\
&= \mathrm{T}_{k+1}^2(x) - \left[\mathrm{T}_{k+1}^2(x) - \frac{1}{2x}\mathrm{T}_{k+1}(x)\mathrm{T}_k(x) + \mathrm{T}_{k+1}(x)\mathrm{T}_{k-1}(x) - \frac{1}{2x}\mathrm{T}_k(x)\mathrm{T}_{k-1}(x)\right] \\
&= \frac{\mathrm{T}_k(x)}{2x}\left[\mathrm{T}_{k+1}(x) + \mathrm{T}_{k-1}(x)\right] - \mathrm{T}_{k+1}(x)\mathrm{T}_{k-1}(x) \\
&= \mathrm{T}_k^2(x) - \mathrm{T}_{k-1}(x)\mathrm{T}_{k+1}(x) = \mathrm{f}(x)
\end{aligned}$$

by the induction hypothesis, hence true for $n = k + 1$.

As true for $n = 1$, $n = k + 1$ and we assumed true for $n = k$ then true for all $n \geq 1$ by induction.

Using the fact that $\mathrm{f}(x) \equiv 0$,

$$\mathrm{f}(x) = \mathrm{T}_n^2(x) - \mathrm{T}_{n-1}(x)\mathrm{T}_{n+1}(x)$$

we can find the terms $\mathrm{T}_2(x), \mathrm{T}_3(x), \mathrm{T}_4(x)$ in terms of $\mathrm{T}_0(x)$ and $\mathrm{r}(x)$:

$$\begin{aligned}
\mathrm{T}_2(x) &= \frac{(\mathrm{T}_1(x))^2}{\mathrm{T}_0(x)} = \mathrm{T}_0(x)\mathrm{r}^2(x) \\
\mathrm{T}_3(x) &= \frac{(\mathrm{T}_2(x))^2}{\mathrm{T}_1(x)} = \mathrm{T}_0(x)\mathrm{r}^3(x) \\
\mathrm{T}_4(x) &= \frac{(\mathrm{T}_3(x))^2}{\mathrm{T}_2(x)} = \mathrm{T}_0(x)\mathrm{r}^4(x)
\end{aligned}$$

and we can conjecture that for $n \geq 1$,

$$\mathrm{T}_n(x) = \mathrm{T}_0(x)\mathrm{r}^n(x).$$

We will prove this by strong induction. Our base case $n = 1$,

$$\mathrm{T}_{(1)}(x) = \mathrm{T}_0(x)\mathrm{r}(x)$$

is clearly true.

Our induction hypothesis is that

$$\mathrm{T}_k(x) = \mathrm{T}_0(x)\mathrm{r}^k(x) \quad \text{and} \quad \mathrm{T}_{k-1}(x) = \mathrm{T}_0(x)\mathrm{r}^{k-1}(x)$$

is true for some $n = k$ and $n = k - 1$ respectively.

Considering the $n = k + 1$ case,

$$\mathrm{T}_{k+1}(x) = \frac{\mathrm{T}_k^2(x)}{\mathrm{T}_{k-1}(x)} = \frac{(\mathrm{T}_0(x)\mathrm{r}^k(x))^2}{\mathrm{T}_0(x)\mathrm{r}^{k-1}(x)} = \mathrm{T}_0(x)\mathrm{r}^{k+1}(x)$$

by the induction hypothesis, hence true for $n = k + 1$.

As true for $n = 1$, $n = k + 1$ and assumed true for $n = k$ and $n = k - 1$ then true for all $n \geq 1$ by induction.

To find the possible values of $\mathrm{r}(x)$ we will look at the pair of equations

$$\mathrm{T}_2(x) - 2x\mathrm{T}_1(x) + \mathrm{T}_0(x) = 0 \quad \text{and} \quad (\mathrm{T}_1(x))^2 - \mathrm{T}_0(x)\mathrm{T}_2(x) = 0$$

which arrive from the recurrence relation and the fact that $\mathrm{f}(x) \equiv 0$. Rearranging the first equation to get

$$\mathrm{T}_2(x) = 2x\mathrm{T}_1(x) - \mathrm{T}_0(x)$$

and substituting into the second equation we arrive at

$$\mathrm{T}_1^2(x) - \mathrm{T}_0(x)\Big[2x\mathrm{T}_1(x) - \mathrm{T}_0(x)\Big] = \mathrm{T}_1^2(x) - 2x\mathrm{T}_0(x)\mathrm{T}_1(x) + \mathrm{T}_0^2(x) = 0.$$

Using $\mathrm{r}(x) = \mathrm{T}_1(x)/\mathrm{T}_0(x)$ we can write the equation above as

$$\mathrm{r}^2(x) - 2x\mathrm{r}(x) + 1 = 0,$$

solving the quadratic in $\mathrm{r}(x)$ we find that

$$\mathrm{r}(x) = x \pm \sqrt{x^2 - 1}.$$

Discussion

The hardest part of this question is the great possibility of cluttered working, mislabelling and general ambiguity. Success in this question relies on a clear line of working with the end goal constantly in mind. Comprising of two proofs and lots of manipulation, the goal (either the end result of the induction step or a solution of $\mathrm{r}(x)$ in terms of x) should always be in the back of your head when you are answering the question; this is particularly true for the induction step in the first part of the question.

To get the desired result in a proof by induction it is often a good idea to remember the maxim: "What do I have and what do I need?". The $n = k + 1$ step leads to the "What do I have" portion being

$$(\mathrm{T}_{k+1}(x))^2 - \mathrm{T}_k(x)\mathrm{T}_{k+2}(x)$$

and the "What do I need" portion being

$$(\mathrm{T}_k(x))^2 - \mathrm{T}_{k-1}(x)\mathrm{T}_{k+1}(x)$$

as we would like to use the induction hypothesis. A useful observation is that what we have is $\mathrm{T}_{k+2}(x), \mathrm{T}_{k+1}(x), \mathrm{T}_k(x)$ terms while we need the $\mathrm{T}_{k+1}(x), \mathrm{T}_k(x)$ and $\mathrm{T}_{k-1}(x)$ terms. Using the recurrence relation to get $\mathrm{T}_{k+2}(x)$ in terms of $\mathrm{T}_{k+1}(x)$ and $\mathrm{T}_k(x)$ is a clear way forward as we do not have $\mathrm{T}_{k+2}(x)$ in our goal. However, the second step is not all too obvious as we are left with

$$(\mathrm{T}_{k+1}(x))^2 - \mathrm{T}_k(x)\Big[2x\mathrm{T}_{k+1}(x) - \mathrm{T}_{k-1}(x)\Big].$$

It is a good rule of thumb to do as little algebra if possible in a question as it has the highest possibility of introducing an error due to an algebraic misstep. We could try to use the recurrence relation to get $\mathrm{T}_{k+1}(x)$ in terms of $\mathrm{T}_k(x)$ and $\mathrm{T}_{k-1}(x)$ but this would lead to expanding a binomial, we could instead try and use the recurrence relation to get $\mathrm{T}_k(x)$ in terms of $\mathrm{T}_{k+1}(x)$ and $\mathrm{T}_{k-1}(x)$ which would be less work.

In general, when doing a proof by induction take the $n = k+1$ step studiously as when written up the proof is normally quite short yet may require a decent amount of exploring and perseverance to arrive at. Take it carefully, trying not to get caught up in a well of algebra.

Once we arrive at our two possible functions for $\mathrm{r}(x)$

$$\mathrm{r}_1(x) = x + \sqrt{x^2 - 1} \quad \text{and} \quad \mathrm{r}_2(x) = x - \sqrt{x^2 - 1}$$

we see that a linear combination of these will result in a closed form for $\mathrm{T}_n(x)$, namely

$$\mathrm{T}_n(x) = \mathrm{T}_0(x)\Big[\Big(x + \sqrt{x^2 - 1}\Big)^n + \Big(x - \sqrt{x^2 - 1}\Big)^n\Big] \quad (n \geq 1)$$

An interesting property can be found for these $\mathrm{T}_n(x)$'s by considering their derivatives when $\mathrm{T}_0(x) = 1$:

$$\begin{aligned}
\mathrm{T}'_n(x) &= \frac{n}{\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^n - \left(x-\sqrt{x^2-1}\right)^n\right] \\
\mathrm{T}''_n(x) &= \frac{n}{\sqrt{x^2-1}}\left[\frac{x}{x^2-1}\left(\left(x+\sqrt{x^2-1}\right)^n - \left(x-\sqrt{x^2-1}\right)^n\right)\right] \\
&+ \frac{n}{\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]
\end{aligned}$$

which we can write in a more attractive way as

$$\begin{aligned}
\mathrm{T}'_n(x) &= \frac{n}{\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^n - \left(x-\sqrt{x^2-1}\right)^n\right] \\
\mathrm{T}''_n(x) &= \frac{x}{x^2-1}\mathrm{T}'_n(x) + \frac{n}{\sqrt{x^2-1}}\mathrm{T}_n(x)
\end{aligned}$$

leaving us with

$$(x^2-1)\mathrm{T}''_n(x) = x\mathrm{T}'_n(x) + n\sqrt{x^2-1}\mathrm{T}_n(x).$$

This is telling us that our functions $\mathrm{T}_n(x)$ are in fact solutions to the differential equation

$$(x^2-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} - n\sqrt{x^2-1}y = 0$$

for $n \in N$.

These polynomials are in fact *Chebyshev Polynomials of the first kind*.