89-S1-Q6
Solution + Discussion

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Abstract

In this document we will go through the solution to the 89-S1-Q6 question and provide a discussion of the question at the end. There are also hints on the first page to aid you in finding a solution. There is no single method that results in an answer to a STEP question, there are a multitude of different paths that end up at the same solution. However, some methods are more straight forward and you are encouraged to take the path of least resistance.

Hints

First part: Instead of using the equation of a line as $y - y_0 = m(x - x_0)$ use $z - y_0 = m(t - x_0)$.

Second part: Find the length of PQ.

Third part: What must be true in order for $(0, 2)$ to be the minimum point of the curve $y = \mathrm{f}(x)$? Can you then use this to eliminate one of the possible values of $\mathrm{f}'(x)$?

Solution

Differentiating the curve $y = \mathrm{f}(x)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$$

gives us the expression for the gradient of the normal at $P : (x, \mathrm{f}(x))$ as $m_p = -1/\mathrm{f}'(x)$. Using the equation of a straight line,

$$\text{Normal at } P \ : z - \mathrm{f}(x) = -\frac{1}{\mathrm{f}'(x)}(t - x)$$

where instead of the equation of the line being $y - y_0 = m(x - x_0)$, we have used $z - y_0 = m(t - x_0)$ as we have already used the variable x in our point P. This leaves us with,

$$\text{Normal at } P \ : z = -\frac{1}{\mathrm{f}'(x)}t + \frac{x}{\mathrm{f}'(x)} + \mathrm{f}(x)$$

giving

$$Q : \left(0, \mathrm{f}(x) + \frac{x}{\mathrm{f}'(x)}\right).$$

Using pythagoras' theorem to find the length PQ,

$$PQ^2 = (x - 0)^2 + \left(\frac{x}{\mathrm{f}'(x)} + \mathrm{f}(x) - \mathrm{f}(x)\right)^2 = x^2 + \left(\frac{x}{\mathrm{f}'(x)}\right)^2$$

and equating it with the given,

$$x^2 + \left(\frac{x}{\mathrm{f}'(x)}\right)^2 = x^2 + e^{x^2} \Rightarrow (\mathrm{f}'(x))^2 = x^2 e^{-x^2}.$$

We can solve for $\mathrm{f}'(x)$,

$$\mathrm{f}'(x) = \pm x e^{-x^2/2}$$

and find $\mathrm{f}''(x)$,

$$\mathrm{f}''(x) = \pm e^{-x^2/2} \mp x^2 e^{-x^2/2}$$

which evaluated at $(0, 2)$,

$$\mathrm{f}''(0) = \pm 1$$

which has to be greater than zero as we are given that $(0, 2)$ is a minimum.

Hence,

$$\mathrm{f}'(x) = x e^{-x^2/2}$$

and integrating,

$$\mathrm{f}(x) = C - e^{-x^2/2}$$

where C is an arbitrary constant. Using the point $(0, 2)$ to find C leaves us with

$$\mathrm{f}(x) = -1 - e^{-x^2/2}.$$

Discussion

As far as STEP questions go, this one is quite straight forward. Arguably the hardest part is in the initial equation of the normal through P and finding out which $\mathrm{f}'(x)$ to use. The phrasing of the question, writing $P : (x, \mathrm{f}(x))$, provides discomfort when attempting to use the traditional equation of the line $y - y_0 = m(x - x_0)$ as the variable x is already in use, yet a simple change of notation solves this problem. When attempting to form the differential equation, we arrive at

$$\left(\frac{\mathrm{df}}{\mathrm{d}x}\right)^2 = x^2 e^{-x^2}$$

and we have to deal with which root to pick, the positive or the negative? Using the given minimum leads to the use of

$$\mathrm{f}'(x) = xe^{-x^2/2}.$$

Overall, this question is quite straightforward and should be accessible to all those who don't mind some initial discomfort.