# 08-S3-Q6
# Solution + Discussion

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**Abstract**

In this document we will go through the solution to the 08-S3-Q6 question and provide a discussion of the question at the end. There are also hints on the first page to aid you in finding a solution. There is no single method that results in an answer to a STEP question, there are a multitude of different paths that end up at the same solution. However, some methods are more straight forward and you are encouraged to take the path of least resistance.

### Hints

**Part (i)**: Differentiate $p$ implicitly just as you would differentiate $y$.

**Part (ii)**: Differentiate with respect to $x$.

**Solution**

**(i)** Differentiating with respect to $x$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y = p^2 + 2xp\right]$$

which leaves us with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(p^2 + 2xp).$$

As $p$ is a function of $x$ we can implicitly differentiate the right hand side of the above equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2p\frac{\mathrm{d}p}{\mathrm{d}x} + 2x\frac{\mathrm{d}p}{\mathrm{d}x} + 2p$$

and noting that $p = \frac{\mathrm{d}y}{\mathrm{d}x}$ we can rearrange to arrive at

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -2 - \frac{2x}{p}.$$

Notice that we can rearrange the equation to

$$\frac{\mathrm{d}x}{\mathrm{d}p} + \frac{2x}{p} = -2$$

which is a first order differential equation which can be solved with the integrating factor

$$I = \exp\left(\int \frac{2}{p}\,\mathrm{d}p\right) = \exp(2\ln(p)) = p^2.$$

Multiplying the differential equation by $I$,

$$\frac{\mathrm{d}}{\mathrm{d}p}[xp^2] = -2p^2$$

and after integrating both sides we arrive at our desired

$$x = -\frac{2}{3}p + Ap^{-2}$$

where $A$ is an arbitrary constant.

Using our initial conditions ($p = -3$ when $x = 2$) we find that

$$2 = -\frac{2}{3}(-3) + \frac{A}{9} \Rightarrow A = 0$$

and thus

$$x = -\frac{2}{3}p.$$

Replacing $p$ with $\frac{\mathrm{d}y}{\mathrm{d}x}$,

$$x = -\frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x}$$

which can be solved to get

$$y(x) = -\frac{3}{4}x^2 + C$$

where $C$ is an arbitrary constant.

Using the initial conditions in the original equation $y = p^2 + 2xp$ we get that $y = -3$ when $x = 2$. Hence we have

$$-3 = -\frac{3}{4}(4) + C \Rightarrow C = 0$$

and thus

$$y(x) = -\frac{3}{4}x^2.$$

**(ii)** Using the same idea as part (i), differentiating with respect to $x$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(2xp + p\ln(p)),$$

using the chain and product rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2p + 2x\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\mathrm{d}p}{\mathrm{d}x}\ln(p) + p\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}x}$$

and noting that $p = \frac{\mathrm{d}y}{\mathrm{d}x}$ we can rearrange the equation to

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -\frac{1}{p} - \frac{2x}{p} - \frac{\ln(p)}{p}$$

which can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}p} + \frac{2x}{p} = -\frac{1}{p} - \frac{\ln(p)}{p}.$$

This is again a first order differential equation for $x$ in terms of $p$ that can be solved with the same integrating factor ($I = p^2$) as before. Multiplying by $I$,

$$\frac{\mathrm{d}}{\mathrm{d}p}[xp^2] = -p - p\ln(p).$$

Integrating both sides,

$$xp^2 = \int -p - p\ln(p)\,\mathrm{d}p = -\int p\,\mathrm{d}p - \int p\ln(p)\,\mathrm{d}p$$
$$\Rightarrow xp^2 = -\frac{p^2}{2} - \left[\frac{p^2}{2}\ln(p) - \int \frac{1}{p}\cdot\frac{p^2}{2}\,\mathrm{d}p\right] = -\frac{p^2}{2}\left(\frac{1}{2} + \ln(p)\right) + B$$

where $B$ is an arbitrary constant. Hence we arrive at

$$x = -\frac{1}{2}\left[\frac{1}{2} + \ln(p)\right] + Bp^{-2}.$$

Using our initial conditions ($p = 1$ when $x = -\frac{1}{4}$) we see that

$$-\frac{1}{4} = -\frac{1}{4} + B \Rightarrow B = 0$$

thus

$$x = -\frac{1}{2}\ln(p) - \frac{1}{4}.$$

Rearranging for $p$ in terms of $x$ and using $p = \frac{\mathrm{d}y}{\mathrm{d}x}$,

$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = e^{-2x-\frac{1}{2}}$$

and integrating both sides,

$$y = -\frac{1}{2}e^{-2x-\frac{1}{2}} + C.$$

From $y = 2xp + p\ln(p)$ and our initial conditions ($p = 1$ when $x = -\frac{1}{4}$) we have that $y = -\frac{1}{2}$ when $x = -\frac{1}{4}$. Solving for $C$,

$$-\frac{1}{2} = -\frac{1}{2}e^{0} + C \Rightarrow C = 0$$

and hence we have

$$y = -\frac{1}{2}e^{-2x-\frac{1}{2}}.$$

### Discussion

As far as STEP question goes this one is pretty straight forward. If you ignore the initial discomfort of having $p = \mathrm{d}y/\mathrm{d}x$ and the having $\mathrm{d}x/\mathrm{d}p$ (differentiating $x$ as a function of $\mathrm{d}y/\mathrm{d}x$!) then the actual computation is clear-cut. To understand why viewing the differential equations

$$y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad y = 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\ln\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

using the substitution $p = \frac{\mathrm{d}y}{\mathrm{d}x}$ works in the first place is to see when it made the question easier.

Starting with

$$y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x}$$

we see that in our solution we arrive at

$$x = -\frac{2}{3}p + Ap^{-2}$$

with $A$ an arbitrary constant. If we remember what we are after ($y$ as a function of $x$), we see that we would like to find a solution for $p$ in terms of $x$ and integrate the result to find $y$. Yet this is only (reasonably) possible if we allow for initial conditions such that $A = 0$ - which is what we had in the question. If instead we require that $A \neq 0$ we arrive at a standstill as there is no reasonable way to solve for $p$.[1] Similarly with

$$y = 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\ln\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

we see that in our solution we arrive at

$$x = -\frac{1}{4} - \frac{1}{2}\ln(p) + Bp^{-2}$$

with $B$ an arbitrary constant. We see that if $B \neq 0$ we are in the same situation as part (i) - arguably worse as we have $\ln(p)$.

Overall, this question is accessible to all those who can follow instructions, comfortable with integrating factors and smile as everything falls into place.

[1] It should be noted that the equation can be written as a cubic in $p$ yet this leads to some ugly integration, try it yourself!