Vector Calculus

David Puerta

Based on Lecture Notes by David Tong and Dexter Chua

Abstract

These are a set of notes I complied when teaching myself vector calculus. Mainly using David Tong's amazing notes supporting myself with Dexter Chua's notes.

Contents

1 Curves

1.1 Coordinates

We will look at maps of the form

$$f : \mathbb{R} \to \mathbb{R}^n$$

which are called *parametric curves* and are often denoted as the curve C.

As the image of the curve is $\mathbb{R}^n$ we will need an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^n$for $\mathbb{R}^n$. A vector $\mathbf{x} \in \mathbb{R}^n$ will be represented as

$$\mathbf{x} = (x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} x^i \mathbf{e}_i = x^i \mathbf{e}_i \tag{1}$$

where we have used the convention of implicitly summing over $i = 1, 2, \cdots, n$ and each component of $\mathbf{x}$ is x^i instead of x_i which you may have seen before.

1.2 Differentiating a Curve

The vector function $\mathbf{x}(t)$ is differentiable if as $\delta t \to 0$, we can write

$$\mathbf{x}(t + \delta t) - \mathbf{x}(t) = \dot{\mathbf{x}}(t)\delta t + \mathcal{O}(\delta t^2) \tag{2}$$

where $\mathcal{O}(\delta t^2)$ is the “big-O” notation meaning terms that scale δt^2 or smaller as $\delta t \to 0$. The term $\dot{\mathbf{x}}(t)$ is the *derivative* of $\mathbf{x}(t)$; a vector function that is differentiable everywhere is called *smooth*.

We can write

$$\delta\mathbf{x}(t) = \mathbf{x}(t + \delta t) - \mathbf{x}(t)$$

to have the derivative of the vector function $\mathbf{x}(t)$ as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{\mathbf{x}}(t) = \lim_{\delta t \to 0} \frac{\delta\mathbf{x}}{\delta t}$$

To differentiate a vector $\mathbf{x}(t)$ we can differentiate component wise

$$\mathbf{x}(t) = x^i \mathbf{e}_i \Rightarrow \dot{\mathbf{x}}(t) = \dot{x}^i \mathbf{e}_i$$

where our choice of basis $\{\mathbf{e}_i\}$ is independent of t. We will look later at the case when $\{\mathbf{e}_i\}$ are functions of t.

In addition, we have the following rules for differentiating a function $f(t)$ and vector functions $\mathbf{g}(t)$ and $\mathbf{h}(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\cdot\mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\cdot\mathbf{h} + \mathbf{g}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\times\mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\times\mathbf{h} + \mathbf{g}\times\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$

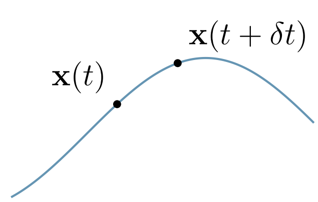
1.3 Tangent Vectors

The vector $\dot{\mathbf{x}}$ is referred to as the *tangent vector*.

The direction of the tangent vector is independent of the choice of parametrisation of the curve C. In contrast, the magnitude $|\dot{\mathbf{x}}(t)|$ of the tangent vector does dependent of the choice of parametrisation. A curve has a *regular* parametrisation if $\dot{\mathbf{x}}(t) \neq \mathbf{0}$ for all t.

1.4 Arc Length

Suppose that we have a parametrised curve with a parametrisation t.



The distance δs between two nearby points $\mathbf{x}(t)$ and $\mathbf{x}(t+\delta t)$ is

$$\delta s = |\delta \mathbf{x}| + \mathcal{O}(|\delta \mathbf{x}|^2) = |\dot{\mathbf{x}}(t)\delta t| + \mathcal{O}(\delta t^2)$$

where the $\mathcal{O}(|\delta \mathbf{x}|^2)$ comes from the fact that the distance between the two points is not exactly a straight line along the curve and contains higher order terms in $\delta \mathbf{x}^2$. We then have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \pm \left| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \right|$$

where we get the plus sign for increasing t and the minus sign for decreasing t.

Hence, if we choose a starting point t_0 and an end point t (with $t_0 < t$) then the distance along the curve from t_0 to t is

$$s = \int_{t_0}^{t} \mathrm{d}t' \, |\dot{\mathbf{x}}(t')| \tag{3}$$

This is called the *arc length* of the curve.

If we choose a different parametrisation $\tau(t)$ for the curve (which we take to be invertible and smooth with $\dot{\tau} > 0$ so that they measure "increasing time" in the same direction) then

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\tau}{\mathrm{d}t}$$

We can then compute the arc length with the parametrisation τ,

$$s = \int_{t_0}^{t} \mathrm{d}t' \, |\dot{\mathbf{x}}(t')| = \int_{\tau_0}^{\tau} \mathrm{d}\tau' \left| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau'} \cdot \frac{\mathrm{d}\tau'}{\mathrm{d}t'} \right| \frac{\mathrm{d}t'}{\mathrm{d}\tau'} = \int_{\tau_0}^{\tau} \mathrm{d}\tau' \left| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau'} \right|$$

This means that the arc length is independent of the choice of parametrisation t.

As the arc length of a curve is independent of the choice of parametrisation, then we can parametrise the curve by its arc length. We can think of $\mathbf{x}(s)$ with its tangent vector $\mathrm{d}\mathbf{x}/\mathrm{d}s$ which we also denote as $\mathbf{x}'(s)$.

In addition, as we are viewing $\mathbf{x}(s)$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \pm \left| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \right| \Rightarrow \left| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \right| = 1$$

This means that our tangent vector is a unit vector when working in terms of the arc length.

Example

Consider the curve $\mathbf{x} = (\cos t, \sin t, t)$ for $t \in [0, 4\pi]$.

We then have

$$\dot{\mathbf{x}}(t) = (-\sin t, \cos t, 1)$$

Calculating the arc length from $t = 0$ to t

$$s = \int_0^t \mathrm{d}t' \sqrt{2} = t\sqrt{2}$$

We then parametrise our curve by s

$$\mathbf{x}(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

With our tangent vector

$$\mathbf{x}'(s) = \left(-\frac{1}{\sqrt{2}} \sin \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Which has magnitude

$$|\mathbf{x}'(s)| = \frac{1}{2}\left(\sin^2 \frac{s}{\sqrt{2}} + \cos^2 \frac{s}{\sqrt{2}} + 1\right) = 1$$

as expected.

1.5 Curvature & Torsion

Given a curve C which is parametrised by its arc length s, we have shown that the tangent vector

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$$

has unit length. We further define the *curvature* as the magnitude of the derivative of the tangent vector with respect to its arc length

$$\kappa(s) = \left|\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2}\right| \tag{4}$$

If we normalise the derivative of the tangent vector with respect to its arc length (provided that $\kappa \neq 0$) we arrive at the *principle normal*

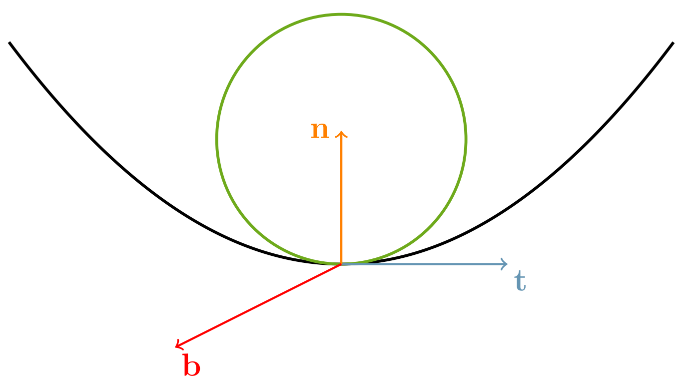
$$\mathbf{n} = \frac{1}{\kappa}\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2} = \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \tag{5}$$

Which has a unit length, $|\mathbf{n}| = 1$. Using the fact that $\mathbf{t}$ is a unit vector with $\mathbf{t}\cdot\mathbf{t} = 1$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{t}\cdot\mathbf{t}) = 2\kappa\mathbf{n}\cdot\mathbf{t} = 0$$

As $\kappa \neq 0$ then $\mathbf{n}\cdot\mathbf{t} = 0$ and the principle normal is orthogonal to the tangent vector. Hence, the vectors $\{\mathbf{t}, \mathbf{n}\}$ define a plane called the *osculating plane*.

For each point s on the curve there is an associated osculating plane, draw a circle in each plane that touches the curve at the point s whose curvature is $\kappa(s)$; this is called the *osculating circle*.



Working in $\mathbb{R}^3$, we define the *binormal* as the normal to the osculating plane

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{6}$$

As both $\mathbf{t}$ and $\mathbf{n}$ are unit vectors, then $\mathbf{b}$ is a unit vector. Using the same argument as before, as $\mathbf{b}$ is a unit vector with $\mathbf{b}\cdot\mathbf{b} = 1$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{b}\cdot\mathbf{b}) = 2\mathbf{b}\cdot\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0 \Rightarrow \mathbf{b}\cdot\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$$

We also have that $\mathbf{t} \cdot \mathbf{b} = 0$ and $\mathbf{n} \cdot \mathbf{b} = 0$ from the definition of the cross product, which gives

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{t} \cdot \mathbf{b}) = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \cdot \mathbf{b} + \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \kappa(\mathbf{n} \cdot \mathbf{b}) + \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}$$
$$\Rightarrow \mathbf{t} \cdot \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$$

Which means that $\mathrm{d}\mathbf{b}/\mathrm{d}s$ is orthogonal to both $\mathbf{t}$ and $\mathbf{b}$ and thus must lie parallel to $\mathbf{n}$. We then define the *torsion* as how much the binormal changes

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\tau(s)\mathbf{n} \tag{7}$$

Intuitively, the curvature tells us how much the curve fails to be a straight line and the torsion how much the curve fails to be a plane.

Observe that using $\mathbf{b} = \mathbf{t} \times \mathbf{n}$,

$$\mathbf{b} \times \mathbf{t} = (\mathbf{t} \times \mathbf{n}) \times \mathbf{t} = (\mathbf{t} \cdot \mathbf{t})\mathbf{n} - (\mathbf{n} \cdot \mathbf{t})\mathbf{t} = \mathbf{n}$$

Which we can then use to get

$$\mathbf{n}' = (\mathbf{b} \times \mathbf{t})' = \mathbf{b}' \times \mathbf{t} + \mathbf{b} \times \mathbf{t}' = (-\tau\mathbf{n}) \times \mathbf{t} + \mathbf{b} \times (\kappa\mathbf{n}) = \tau\mathbf{b} - \kappa\mathbf{t}$$

using our previous relationships. We then have three sets of equations

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa\mathbf{n} \tag{8}$$
$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\tau\mathbf{n} \tag{9}$$
$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \tau\mathbf{b} - \kappa\mathbf{t} \tag{10}$$

which are referred to as the *Frent-Serret* equations. If we are given $\kappa(s)$ and $\tau(s)$ with initial conditions on $\mathbf{b}$ and $\mathbf{t}$, we can then solve for $\mathbf{b}$, $\mathbf{t}$ and the curve $\mathbf{x}(s)$.

1.6 Line Integral

Scalar Fields

We will first look at the map

$$\begin{aligned} \phi : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{x} &\to \phi(\mathbf{x}) \end{aligned}$$

We could define the function $\phi(\mathbf{x}(t))$ for a parametrised curve C in $\mathbb{R}^n$ and then integrate over t in the usual way. Yet this will depend on the parametrisation.

Instead, we will work with the arc length s. Integrating from the point $\mathbf{x}(s_a) = \mathbf{a}$ to $\mathbf{x}(s_b) = \mathbf{b}$ on a curve C, we define the line integral as

$$\int_C \phi \,\mathrm{d}s = \int_{s_a}^{s_b} \phi(\mathbf{x}(s)) \,\mathrm{d}s$$

If instead we are given a curve C with a different parametrisation $\mathbf{x}(t)$ with $\mathbf{x}(t_a) = \mathbf{a}$ and $\mathbf{x}(t_b) = \mathbf{b}$, then

$$\int_C \phi \,\mathrm{d}s = \int_{t_a}^{t_b} \phi(\mathbf{x}(t)) \frac{\mathrm{d}s}{\mathrm{d}t} \,\mathrm{d}t \quad \text{with} \quad \frac{\mathrm{d}s}{\mathrm{d}t} = \pm|\dot{\mathbf{x}}(t)|$$

where the sign depend if $t_a < t_b$ or $t_b < t_a$.

Vector Fields

Consider the map

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$$

We could integrate component wise, treating each component as a scalar field to give a vector as an answer.

On the other hand, we can integrate the component of the vector field that lies tangent to the curve to give a scalar as an answer.

Suppose our curve C has a parametrisation $\mathbf{x}(t)$ and we wish to integrate from $\mathbf{x}(t_a) = \mathbf{a}$ to $\mathbf{x}(t_b) = \mathbf{b}$. We define the line integral as

$$\int_C \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_{t_a}^{t_b} \mathbf{F}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)\, \mathrm{d}t$$

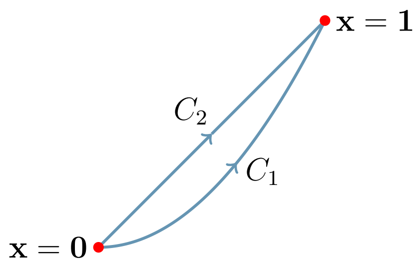
The orientation is in the direction along the curve, or the direction of the tangent vector $\dot{\mathbf{x}}$. The orientation is from $\mathbf{a}$ to $\mathbf{b}$ regardless if $t_a < t_b$ or $t_b < t_a$.

Example

Let $\phi(\mathbf{x}) = x + y + z$ and $\mathbf{F}(\mathbf{x}) = (xe^y, z^2, xy)$. We define the curves

$$C_1 : \mathbf{x}(t) = (t, t^2, t^3)$$
$$C_2 : \mathbf{x}(t) = (t, t, t)$$

and wish to integrate each of the fields along curves C_1 and C_2 from $\mathbf{x} = \mathbf{0}$ to $\mathbf{x} = \mathbf{1}$.



Starting with $\phi(\mathbf{x})$, we see that

$$\int_{C_1} \phi\, \mathrm{d}s = \int_0^1 \mathrm{d}t\, (t + t^2 + t^3) \cdot |(1, 2t, 3t^2)| = \int_0^1 \mathrm{d}t\, (t + t^2 + t^3)\sqrt{1 + 4t^2 + 9t^4} \approx 2.715$$

We can also re-parametrise C_2 in terms of its arc length

$$\mathbf{x}(s) = \left(\frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right) \Rightarrow \int_{C_2} \phi\, \mathrm{d}s = \int_0^{\sqrt{3}} \frac{3s}{\sqrt{3}}\, \mathrm{d}s = \frac{3\sqrt{3}}{2}$$

Secondly, integrating $\mathbf{F}$ along C_1

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \mathrm{d}t\, (te^{t^2}, t^6, t^3) \cdot (1, 2t, 3t^2) = \int_0^1 \mathrm{d}t\, (te^{t^2} + 2t^7 + 3t^5) = \frac{1}{4}(1 + 2e)$$

Finally, along C_2

$$\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{x} = \int_0^1 \mathrm{d}t\,(te^t, t^2, t^2)\cdot(1,1,1) = \int_0^1 \mathrm{d}t\,(te^t + 2t^2) = \frac{5}{3}$$

If we wish to integrate along a closed curve C who has the same start and end point, then we denote it as

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{x}$$

In addition, if we have a closed curve C such that $C = \sum_i C_i$ with each C_i smooth and joined at the end points, then

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{x} = \sum_i \int_{C_i} \mathbf{F}\cdot \mathrm{d}\mathbf{x}$$

We will also define the curve $-C$ for a given curve C which is in the opposite direction,

$$\int_{-C} \mathbf{F}\cdot \mathrm{d}\mathbf{x} = -\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{x}$$

Example

Define the curve $C = C_1 - C_2$ with $\mathbf{F} = (xe^y, z^2, xy)$, C_1 and C_2 from the previous example. As C is a closed curve then

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{x} = \int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{x} - \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{x} = \frac{1}{4}(1+2e) - \frac{5}{3}$$

1.7 The Gradient

Take a scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$, we have a partial derivative of $\phi(x^1, x^2, \cdots, x^n)$ as

$$\frac{\partial \phi}{\partial x^i} = \lim_{h\to 0} \frac{\phi(x^1, \cdots, x^i + h, \cdots, x^n) - \phi(x^1, \cdots, x^n)}{h}$$

for $i = 1, \cdots, n$. This is a derivative in the x^i direction, treating all other variables as constant.

We can write each of these partial derivatives as

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\mathbf{e}_i$$

where $\{\mathbf{e}_i\}$ is an orthonormal basis for $\mathbb{R}^n$; this is called the *gradient* of ϕ. This $\nabla\phi$ is a vector field yet it is convention not to write $\boldsymbol{\nabla}\phi$ and understand that $\nabla\phi$ is a vector.

Take a normal unit vector $\hat{\mathbf{n}}$, the rate of change of ϕ in the direction of $\hat{\mathbf{n}}$ is

$$\mathrm{D}_{\hat{\mathbf{n}}}\phi = \hat{\mathbf{n}} \cdot \nabla\phi$$

which is called the *directional derivative*. We see that this is maximised when $\hat{\mathbf{n}}$ is parallel to the gradient $\nabla\phi$, which tells us that $\nabla\phi$ points towards where $\phi(\mathbf{x})$ changes the quickest.

1.8 Conservative Fields

For a vector field $\mathbf{F}$, we say it is *conservative* if

$$\mathbf{F} = \nabla\phi$$

for some scalar field ϕ; this scalar field is called the *potential*.

Claim: The line integral around any closed curve vanishes if and only if the vector field $\mathbf{F}$ is conservative.

Proof: Suppose that $\mathbf{F} = \nabla\phi$. Integrating along a closed curve C from $\mathbf{x}(t_a) = \mathbf{a}$ to $\mathbf{x}(t_b) = \mathbf{b}$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{t_a}^{t_b} \mathrm{d}t\, \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \int_{t_a}^{t_b} \mathrm{d}t\, \frac{\partial\phi(\mathbf{x}(t))}{\partial x^i} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \int_{t_a}^{t_b} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{x}(t))$$

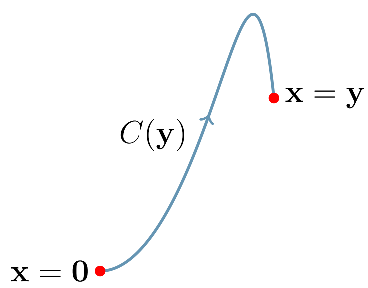
by the chain rule. Now we have the integral of a total derivative, which gives

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \phi(\mathbf{b}) - \phi(\mathbf{a}) = 0$$

Suppose that $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$ for any curve C. Define a scalar curve $\phi : \mathbb{R}^n \to \mathbb{R}$ with $\phi(\mathbf{0}) = 0$. At any point $\mathbf{x} = \mathbf{y}$, we define

$$\phi(\mathbf{y}) = \int_{C(\mathbf{y})} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

where $C(\mathbf{y})$ is the curve that starts at $\mathbf{x} = \mathbf{0}$ and ends at $\mathbf{x} = \mathbf{y}$.



By our assumption,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

for any closed curve C, so it does not matter which closed curve C we take, they all give the same answer.

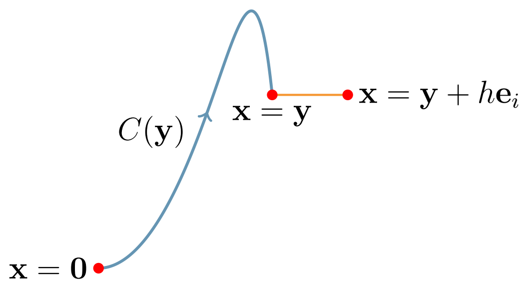
Consider,

$$\frac{\partial \phi}{\partial x^i}(\mathbf{y}) = \lim_{h \to 0} \frac{1}{h} \left\{ \int_{C(\mathbf{y}+h\mathbf{e}_i)} \mathbf{F} \cdot \mathrm{d}\mathbf{x} - \int_{C(\mathbf{y})} \mathbf{F} \cdot \mathrm{d}\mathbf{x} \right\}$$

The first integral goes along the line $C(\mathbf{y})$ and then along the red line in the $\mathbf{e}_i$ direction as shown in the figure below. The second integral goes back along the curve $C(\mathbf{y})$.

As a result the difference in the integrals is simply the integral along the orange line

$$\frac{\partial \phi}{\partial x^i}(\mathbf{y}) = \lim_{h \to 0} \frac{1}{h} \int_{\text{orange line}} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$



As we are integrating along a straight line in the $\mathbf{e}_i$ direction, we are projecting the F_i component of $\mathbf{F}$. As h is small we get

$$\int_{\text{orange line}} \mathbf{F} \cdot \mathrm{d}\mathbf{x} \approx F_i h$$

and after taking the limit as $h \to 0$,

$$\frac{\partial \phi}{\partial x^i}(\mathbf{y}) = F_i(\mathbf{y})) \Rightarrow \mathbf{F} = \nabla \phi \qquad \square$$

In order to check if there exists a scalar field ϕ such that $\mathbf{F} = \nabla \phi$ we have

$$F_i = \frac{\partial \phi}{\partial x^i}$$

by definition, differentiating again gives

$$\frac{\partial F_i}{\partial x^j} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} = \frac{\partial F_j}{\partial x^i}$$

as for "nice" $\mathbf{F}$ we have $\partial^2 F/\partial x \partial y = \partial^2 F/\partial y \partial x$. Hence, for $\mathbf{F}$ to be conservative we need $\partial_i F_j = \partial_j F_i$ for all i, j.

Exact Differentials

Given a scalar field $\phi(\mathbf{x})$ on $\mathbb{R}^n$, we define the *differential* as

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x^i}\,\mathrm{d}x^i = \nabla \phi \cdot \mathrm{d}\mathbf{x}$$

This is a function of $\mathbf{x}$ and captures how much the function ϕ changes as we move in any direction around $\mathbf{x}$.

Consider a field $\mathbf{F}(\mathbf{x})$ on $\mathbb{R}^n$. Taking an inner product with an infinitesimal vector to get

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

is a *differential form*. If a differential form can be written as

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}\phi = \nabla\phi \cdot \mathrm{d}\mathbf{x}$$

for some ϕ it is said to be an *exact* differential form. This is a re-formalisation of our idea: a differential is exact if and only if the vector field is conservative.

An Application: Work & Potential Energy

Consider a particle with trajectory $\mathbf{x}(t)$. By Newton's second law

$$\mathbf{F}(\mathbf{x}) = m\ddot{\mathbf{x}}$$

where $\mathbf{F}(\mathbf{x})$ is the force field. Recall that a particle has kinetic energy given by $\mathrm{K} = \frac{1}{2}m\dot{\mathbf{x}}^2$; this kinetic energy changes in time as

$$\mathrm{K}(t_2) - \mathrm{K}(t_1) = \int_{t_1}^{t_2} \mathrm{d}t\, \frac{\mathrm{dK}}{\mathrm{d}t} = \int_{t_1}^{t_2} \mathrm{d}t\, (m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}) = \int_{t_1}^{t_2} \mathrm{d}t\, \dot{\mathbf{x}} \cdot \mathbf{F} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

where C is the trajectory of the curve. The line integral of the force $\mathbf{F}$ along the trajectory C of the particle is known as the *work done*. If our force field $\mathbf{F}$ is conservative, then

$$\mathbf{F} = -\nabla \mathrm{V}$$

for some V. From the above, for a conservative force the work done only depends on the end points of our trajectory

$$\mathrm{K}(t_2) - \mathrm{K}(t_1) = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = -\mathrm{V}(t_2) + \mathrm{V}(t_1)$$

$$\Rightarrow \mathrm{K}(t) + \mathrm{V}(t) = \text{constant}$$

A Subtlety

Consider the vector field given by

$$\mathbf{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

This is a conservative field with

$$\mathbf{F} = \nabla \phi \quad \text{and} \quad \phi = \arctan\left(\frac{y}{x}\right)$$

Integrating $\mathbf{F}$ along a closed curve C given by a circle of radius R about the origin $\mathbf{x}(t) = (R\cos t, R\sin t)$:

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^{2\pi} \mathrm{d}t\, \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \int_0^{2\pi} \mathrm{d}t\, \frac{1}{R^2} \left(R^2 \sin^2 t + R^2 \cos^2 t \right) = 2\pi$$

Yet we have $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \neq 0$ when $\mathbf{F}$ is a conservative field!? This does not vanish as our function ϕ is not continuos along the y-axis. In fact, we have implicitly assumed that our function ϕ is continuos. We should have $\mathbf{F} = \nabla \phi$ with ϕ continuos to guaranty that $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$.