Tandon CS Bridge: HW #5

Due on August 13, 2023

Ratan Dey Extended 24-week

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Problem 3

Solve the following questions from the Discrete Math ZyBook:

1. Excercise 4.1.3, sections b, c
   Which of the following functions are from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

   (a) $f(x) = 1/(x^2 - 4)$

   (b) $f(x) = \sqrt{x^2}$

   Solution

   (a) The range of this function is $\{x \in \mathbb{R} : \ x \neq -2 \ and \ x \neq 2\}$.

   (b) The range of this function is from $\mathbb{R}$ to $\mathbb{R}$.

2. Excercise 4.1.5, sections b, d, h, i, l
   Express the range of each function using roster notation.

   (a) Let $A = \{2, 3, 4, 5\}$.
   $f : A \to \mathbb{Z}$ such that $f(x) = x^2$

   (b) $f : \{0, 1\}^5 \to \mathbb{Z}$

   (c) Let $A = \{1, 2, 3\}$.
   $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$

   (d) Let $A = \{1, 2, 3\}$
   $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = f(x, y + 1)$

   (e) Let$A = \{1, 2, 3\}$.
   $f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$

   Solution

   (a) $\{4, 9, 16, 25\}$

   (b) $\{0, 1, 2, 3, 4, 5\}$

   (c) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

   (d) $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

   (e) $\{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Problem 4

1. Solve the following questions from the Discrete ZyBook:

   (a) Excercise 4.2.2, sections c, g, k
   For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

   i. $f : \mathbb{Z} \to \mathbb{Z}.h(x) = x^3$
   ii. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}.f(x, y) = (x + 1, 2y)$
   iii. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}.f(x, y) = (\lceil \frac{x}{5} \rceil, 5y - 2)$

   Solution

   i. This is a one-to-one and onto function.
   ii. This is a one-to-one but not onto function.
   iii. The function is onto but not one-to-one.

   (b) Excercise 4.2.4, sections b, c, d, g
   For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

   i. $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1.
   ii. $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.
   iii. $f : \{0, 1\}^3 \to \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string.
   iv. Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B = \{1\}$. $f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - B$.

   Solution

   i. This function is one-to-one but not onto.
   ii. This function is both one-to-one and onto.
   iii. This function is one-to-one but not onto.
   iv. This function is one-to-one but not onto.

2. Give an example of a function from the set of integers to the set of positive integers that is:

   (a) one-to-one, but not onto.
   (b) onto, but not one-to-one.
   (c) one-to-one and onto.
   (d) neither one-to-one nor onto.

   Solution

   (a)
   $$f(x) = \begin{cases} x, & x <= 0 \\ x + 2, & x > 0 \end{cases}$$

   (b)
   $$f(x) = \begin{cases} x, & x \textit{ is odd} \\ x/2, & x \textit{ is even} \end{cases}$$

   (c) $f(x) = x$
   (d) $f(x) = 1$

Problem 5

Solve the following questions from the Discrete Math ZyBook:

1. Excercise 4.3.2, sections c, d, g, i
   For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

   (a) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 2x + 3$

   (b) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
   $f : P(A) \to \{1, 2, 3, 4, 5, 6, 7, 8\}$.
   For $X \subseteq A$, $f(X) = |X|$

   (c) $f : \{0, 1\}^3 \to \{0, 1\}^3$, The output of f is obtained by taking the input string and reversing the bits.

   (d) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 5, y - 2)$

   Solution

   (a) The function has a well-defined inverse: $f^{-1} : \mathbb{R} \to \mathbb{R}$. $f^{-1}(y) = \frac{y-3}{2}$

   (b) The function does not have a well-defined inverse.

   (c) The function has a well-defined inverse: $f^{-1} : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.

   (d) The function has a well-defined inverse: $f^{-1} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) \to (x - 5, y + 2)$.

2. Excercise 4.4.8, sections c, d
   The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:

$$\begin{cases} f(x) = 2x + 3 \\ g(x) = 5x + 7 \\ h(x) = x^2 + 1 \end{cases}$$

   Give an explicit formula for each function given below.

   (a) f o h

   (b) h o f

   Solution

   (a) $f(h(x)) = 2x^2 + 5$

   (b) $h(f(x)) = 4x^2 + 6x + 10$

3. Excercise 4.4.2, section b-d
   Consider three functions f, g, and h, whose domain and target are $\mathbb{Z}$. Let:

$$\begin{cases} f(x) = x^2 \\ g(x) = 2^x \\ h(x) = \lceil \frac{x}{5} \rceil \end{cases}$$

   (a) Evaluate $(f \ o \ h)(52)$

   (b) Evaluate $(g \ o \ h \ o \ f)(4)$

   (c) Give a mathematical expression for h o f.

   (d) Give a mathematical expression for f o g.

Solution

(a) $(f\ o\ h) = (\lceil \frac{x}{5} \rceil)^2$
$(f\ o\ h)(52) = (\lceil \frac{52}{5} \rceil)^2 = 11^2 = 121$

(b) $(g\ o\ h\ o\ f) = 2^{\lceil \frac{x^2}{5} \rceil}$
$(g\ o\ h\ o\ f)(4) = 2^{\lceil \frac{4^2}{5} \rceil} = 2^4 = 16$

(c) $(h\ o\ f) = \lceil \frac{x^2}{5} \rceil$

(d) $(f\ o\ g) = (2^x)^2 = 2^{2x}$

4. Excercise 4.4.6, sections c-e
Define the following functions f, g, and h:

$$\begin{cases} f : \{0,1\}^3 \to \{0,1\}^3, \text{ The output of f is obtained by taking the input string and replacing the first bit by 1,} \\ \text{regardless of whether the first bit is a 0 or 1.} \\ g : \{0,1\}^3 \to \{0,1\}^3, \text{ The output of g is obtained by taking the input string and reversing the bits.} \\ h : \{0,1\}^3 \to \{0,1\}^3, \text{ The output of h is obtained by taking the input string x,} \\ \text{and replacing the last bit with a copy of the first bit.} \end{cases}$$

(a) What is (h o f)(010)

(b) What is the range of h o f?

(c) What is the range of g o f?

Solution

(a) (111)

(b) $\{101, 111\}$

(c) $\{000, 100, 010, 001, 101, 011, 110, 111\}$

5. Excercise 4.4.4, sections c, d
Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

(a) Is it possible that f is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

(b) Is it possible that g is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Solution

(a) Yes. Suppose $f(x) = 1$ and $g(x) = x$, then $g(f(x))$ should be a one-to-one function.

(b) No. If $g(x)$ id not a one-to-one function, then $g(f(x))$ is not a one-to-one fuction as well. Because for $g(x)$ there is always at least one element in the target that has more than one element in its domain.