

Tandon CS Bridge: HW #5

Due on August 13, 2023

Ratan Dey Extended 24-week

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Problem 3

Solve the following questions from the Discrete Math ZyBook:

1. Exercise 4.1.3, sections b, c

Which of the following functions are from \mathbb{R} to \mathbb{R} ? If f is a function, give its range.

(a) $f(x) = 1/(x^2 - 4)$

(b) $f(x) = \sqrt{x^2}$

Solution

(a) The range of this function is $\{x \in \mathbb{R} : x \neq -2 \text{ and } x \neq 2\}$.

(b) The range of this function is from \mathbb{R} to \mathbb{R} .

2. Exercise 4.1.5, sections b, d, h, i, l

Express the range of each function using roster notation.

(a) Let $A = \{2, 3, 4, 5\}$.

$$f : A \rightarrow \mathbb{Z} \text{ such that } f(x) = x^2$$

(b) $f : \{0, 1\}^5 \rightarrow \mathbb{Z}$

(c) Let $A = \{1, 2, 3\}$.

$$f : A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}, \text{ where } f(x, y) = (y, x)$$

(d) Let $A = \{1, 2, 3\}$

$$f : A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}, \text{ where } f(x, y) = f(x, y + 1)$$

(e) Let $A = \{1, 2, 3\}$.

$$f : P(A) \rightarrow P(A). \text{ For } X \subseteq A, f(X) = X - \{1\}$$

Solution

(a) $\{4, 9, 16, 25\}$

(b) $\{0, 1, 2, 3, 4, 5\}$

(c) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(d) $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

(e) $\{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Problem 4

1. Solve the following questions from the Discrete ZyBook:

(a) Exercise 4.2.2, sections c, g, k

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both.

If the function is not onto or not one-to-one, give an example showing why.

- i. $f : \mathbb{Z} \rightarrow \mathbb{Z}. h(x) = x^3$
- ii. $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}. f(x, y) = (x + 1, 2y)$
- iii. $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}. f(x, y) = (\lceil \frac{x}{5} \rceil, 5y - 2)$

Solution

- i. This is a one-to-one and onto function.
- ii. This is a one-to-one but not onto function.
- iii. The function is onto but not one-to-one.

(b) Exercise 4.2.4, sections b, c, d, g

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both.

If the function is not onto or not one-to-one, give an example showing why.

- i. $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1.
- ii. $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.
- iii. $f : \{0, 1\}^3 \rightarrow \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string.
- iv. Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B = \{1\}$. $f : P(A) \rightarrow P(A)$. For $X \subseteq A, f(X) = X - B$.

Solution

- i. This function is one-to-one but not onto.
- ii. This function is both one-to-one and onto.
- iii. This function is one-to-one but not onto.
- iv. This function is one-to-one but not onto.

2. Give an example of a function from the set of integers to the set of positive integers that is:

- (a) one-to-one, but not onto.
- (b) onto, but not one-to-one.
- (c) one-to-one and onto.
- (d) neither one-to-one nor onto.

Solution

(a)

$$f(x) = \begin{cases} x, & x \leq 0 \\ x + 2, & x > 0 \end{cases}$$

(b)

$$f(x) = \begin{cases} x, & x \text{ is odd} \\ x/2, & x \text{ is even} \end{cases}$$

(c) $f(x) = x$

(d) $f(x) = 1$

Problem 5

Solve the following questions from the Discrete Math ZyBook:

1. Exercise 4.3.2, sections c, d, g, i

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1} .

(a) $f : \mathbb{R} \rightarrow \mathbb{R}$. $f(x) = 2x + 3$

(b) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

$f : P(A) \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$.

For $X \subseteq A$, $f(X) = |X|$

(c) $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$, The output of f is obtained by taking the input string and reversing the bits.

(d) $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 5, y - 2)$

Solution

(a) The function has a well-defined inverse: $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$. $f^{-1}(y) = \frac{y-3}{2}$

(b) The function does not have a well-defined inverse.

(c) The function has a well-defined inverse: $f^{-1} : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.

(d) The function has a well-defined inverse: $f^{-1} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$, $f(x, y) \rightarrow (x - 5, y + 2)$.

2. Exercise 4.4.8, sections c, d

The domain and target set of functions f , g , and h are \mathbb{Z} . The functions are defined as:

$$\begin{cases} f(x) = 2x + 3 \\ g(x) = 5x + 7 \\ h(x) = x^2 + 1 \end{cases}$$

Give an explicit formula for each function given below.

(a) $f \circ h$

(b) $h \circ f$

Solution

(a) $f(h(x)) = 2x^2 + 5$

(b) $h(f(x)) = 4x^2 + 6x + 10$

3. Exercise 4.4.2, section b-d

Consider three functions f , g , and h , whose domain and target are \mathbb{Z} . Let:

$$\begin{cases} f(x) = x^2 \\ g(x) = 2^x \\ h(x) = \lceil \frac{x}{5} \rceil \end{cases}$$

(a) Evaluate $(f \circ h)(52)$

(b) Evaluate $(g \circ h \circ f)(4)$

(c) Give a mathematical expression for $h \circ f$.

(d) Give a mathematical expression for $f \circ g$.

Solution

$$(a) \quad (f \circ h) = \left(\lceil \frac{x}{5} \rceil\right)^2$$

$$(f \circ h)(52) = \left(\lceil \frac{52}{5} \rceil\right)^2 = 11^2 = 121$$

$$(b) \quad (g \circ h \circ f) = 2^{\lceil \frac{x^2}{5} \rceil}$$

$$(g \circ h \circ f)(4) = 2^{\lceil \frac{4^2}{5} \rceil} = 2^4 = 16$$

$$(c) \quad (h \circ f) = \lceil \frac{x^2}{5} \rceil$$

$$(d) \quad (f \circ g) = (2^x)^2 = 2^{2x}$$

4. Exercise 4.4.6, sections c-e

Define the following functions f , g , and h :

$$\left\{ \begin{array}{l} f : \{0, 1\}^3 \rightarrow \{0, 1\}^3, \text{ The output of } f \text{ is obtained by taking the input string and replacing the first bit by 1,} \\ \text{regardless of whether the first bit is a 0 or 1.} \\ g : \{0, 1\}^3 \rightarrow \{0, 1\}^3, \text{ The output of } g \text{ is obtained by taking the input string and reversing the bits.} \\ h : \{0, 1\}^3 \rightarrow \{0, 1\}^3, \text{ The output of } h \text{ is obtained by taking the input string } x, \\ \text{and replacing the last bit with a copy of the first bit.} \end{array} \right.$$

$$(a) \quad \text{What is } (h \circ f)(010)$$

$$(b) \quad \text{What is the range of } h \circ f?$$

$$(c) \quad \text{What is the range of } g \circ f?$$

Solution

$$(a) \quad (111)$$

$$(b) \quad \{101, 111\}$$

$$(c) \quad \{000, 100, 010, 001, 101, 011, 110, 111\}$$

5. Exercise 4.4.4, sections c, d

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions.

$$(a) \quad \text{Is it possible that } f \text{ is not one-to-one and } g \circ f \text{ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for } f \text{ and } g.$$

$$(b) \quad \text{Is it possible that } g \text{ is not one-to-one and } g \circ f \text{ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for } f \text{ and } g.$$

Solution

$$(a) \quad \text{Yes. Suppose } f(x) = 1 \text{ and } g(x) = x, \text{ then } g(f(x)) \text{ should be a one-to-one function.}$$

$$(b) \quad \text{No. If } g(x) \text{ is not a one-to-one function, then } g(f(x)) \text{ is not a one-to-one function as well. Because for } g(x) \text{ there is always at least one element in the target that has more than one element in its domain.}$$