Tandon CS Bridge: HW #11

Due on September 22, 2023

Ratan Dey Extended 24-week

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Problem 5

1. Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$.
   Solution
   1. **Base case**: When $n = 1$, $n^3 + 2n = 3$, which can be divided by 3.
   2. **Inductive step**: Assume that when $n = k$, where k is some integer, 3 divides $n^3 + 2n = k^3 + 2k$. When $n = k+1$, $n^3 + 2n = (k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = k^3 + 2k + 3(k^2 + k + 3)$. Because 3 divides $3(k^2 + k + 3)$ and $k^3 + 2k$ by the induction hypothesis. So, 3 divides $n^3 + 2n$ when $n = k + 1$.
   Therefore, for any positive integer n, 3 divides $n^3 + 2n$.

2. Use strong induction to prove that any positive integer $n(n \geq 2)$ can be written as a product of primes.
   Solution
   1. **Base case**: When $n = 2$, it is a product of the prime number 2 of itself.
   2. **Inductive step**: Assume that for $k \geq 2$, any integer $j \in [2, k]$ can be expressed as a product of prime numbers. If $k + 1$ is prime, then it is already a product of the prime number of itself.
   If $k + 1$ is not prime, it can be expressed as the product of two integer a and b, which are both greater than 2.
   $k + 1 = a \times b \leftrightarrow a = (k+1)/b$. Because b is greater than 2, $a = (k+1)/b < k + 1$. Then $a \leq k$.
   Symmetrically we get $b \leq k$. Thus, a and b can be expressed as $a = p_1 \cdot p_2 \cdots p_m$ and $b = q_1 \cdot q_2 \cdots q_n$, respectively.
   Therefore, $k + 1$ can be expressed as a product of primes $k + 1 = a \cdot b = (p_1 \cdot p_2 \cdots p_m) \cdot q_1 \cdot q_2 \cdots q_n$

Problem 6

Solve the following questions from the Discrete Math ZyBook:

1. Define $P(n)$ to be the assertion that: $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. **Solution**

   (a) $P(3) = \sum_{j=1}^{3} j^2 = 14 = \frac{3(3+1)(2\cdot3+1)}{6}$

   (b) $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

   (c) $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

   (d) In the base case, we need to prove that when $n = 1$, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

   (e) In the inductive step, we need to prove that when $n = k+1$, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

   (f) Inductive hypothesis: when $n = k$, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

   (g) $\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

   $= \frac{(k+1)[k(2k+1)+(k+1)]}{6} = \frac{(k+1)(k+2)(k+3)}{6}$ Therefore, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

2. Prove that for $n \geq 1, \sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$ **Solution**

   Proof. 1. **Base case**: When $n = 1$, $\sum_{j=1}^{n} \frac{1}{j^2} = 1 = 2 - \frac{1}{n}$
   2. **Inductive step**: Assume that for $n = k, \sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n} = 2 - \frac{1}{k}$
   When $n = k+1, \sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$
   $\because \frac{(\frac{1}{k} - \frac{1}{(k+1)^2})}{\frac{1}{(k+1)}} = 1 + \frac{k+1}{k} - \frac{1}{k+1} = 1 + \frac{k^2+k+1}{k^2+k} > 1$
   $\therefore \frac{1}{k} - \frac{1}{(k+1)^2} > \frac{1}{k+1}$
   $\therefore 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$
   $\therefore \sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$
   Therefore, for $n \geq 1, \sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$ □

3. Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

   Proof. 1. **Base case**: When $n = 1$, 4 evenly divides $3^{2\cdot1} - 1 = 8$.
   2. **Inductive step**: Assume that for $n = k$, 4 divides $3^{2\cdot k} - 1$
   When $n = k+1, 3^{2\cdot(k+1)} - 1 = 3^{2k+2} - 1 = 3^2 \cdot 3^{2k} - 3^2 + 3^2 - 1 = 3^2(3^{2k} - 1) + 8$
   $\because$ 4 evenly divides $3^{2\cdot k} - 1$ and 8
   $\therefore$ 4 evenly divides $3^{2\cdot(k+1)} - 1$
   Therefore, for any positive integer n, 4 evenly divides $3^{2n} - 1$. □