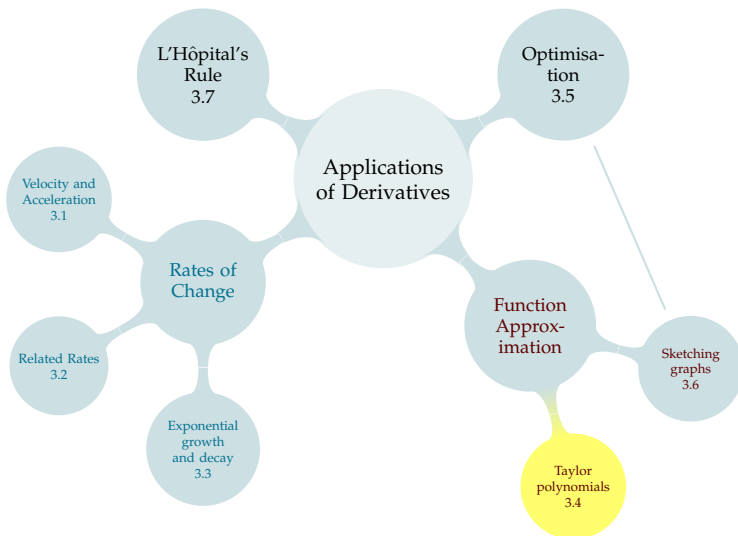
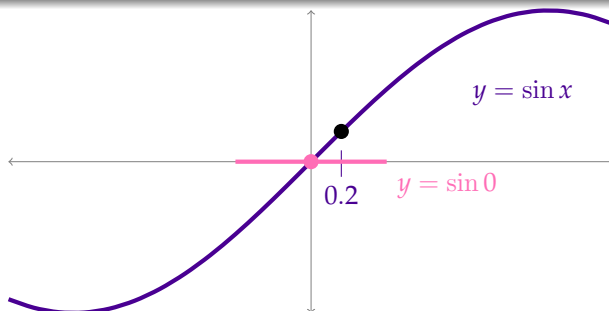


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# APPROXIMATING A FUNCTION

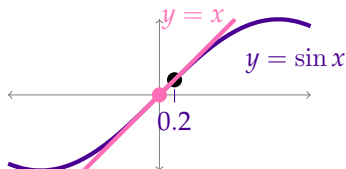


## Constant Approximation – Equation 3.4.1

We can approximate  $f(x)$  near a point  $a$  by

$$f(x) \approx f(a)$$

# APPROXIMATING A FUNCTION



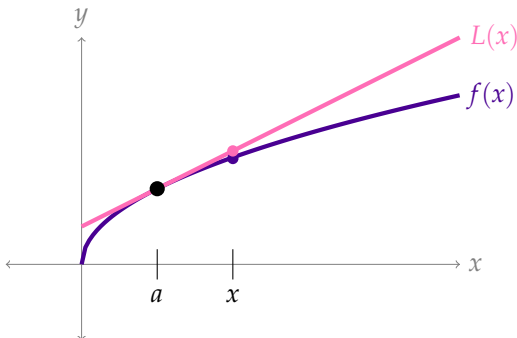
## Linear Approximation (Linearization) – Equation 3.4.3

We can approximate  $f(x)$  near a point  $a$  by the tangent line to  $f(x)$  at  $a$ , namely

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

To find a linear approximation of  $f(x)$  at a particular point  $x$ , pick a point  $a$  **near to  $x$** , such that  $f(a)$  and  $f'(a)$  are **easy to calculate**.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$



To find a linear approximation of  $f(x)$  at a particular point  $x$ , pick a point  $a$  **near to  $x$** , such that  $f(a)$  and  $f'(a)$  are **easy to calculate**.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Let  $f(x) = \sqrt{x}$ . Approximate  $f(8.9)$ .

# CAN WE COMPUTE?

Suppose we want to approximate the value of  $\cos(1.5)$ . Which of the following linear approximations could we calculate by hand? (You can leave things in terms of  $\pi$ .)

- A. tangent line to  $f(x) = \cos x$  when  $x = \pi/2$
- B. tangent line to  $f(x) = \cos x$  when  $x = 3/2$
- C. both
- D. neither

# CAN WE COMPUTE?

Which of the following tangent lines is probably the most accurate in approximating  $\cos(1.5)$ ?

- A. tangent line to  $f(x) = \cos x$  when  $x = \pi/2$
- B. tangent line to  $f(x) = \cos x$  when  $x = \pi/4$
- C. constant approximation:  $\cos 1.5 \approx \cos(\pi/2) = 0$
- D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

# LINEAR APPROXIMATION

Approximate  $\sin(3)$  using a linear approximation. You may leave your answer in terms of  $\pi$ .



# LINEAR APPROXIMATION

Approximate  $e^{1/10}$  using a linear approximation.

If  $f(x) = e^x$  and  $a = 0$  :

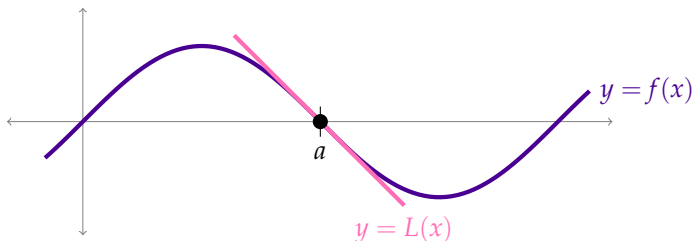
# LINEAR APPROXIMATION WRAP-UP

Let  $L(x) = f(a) + f'(a)(x - a)$ , so  $L(x)$  is the linear approximation (linearization) of  $f(x)$  at  $a$ .

What is  $L(a)$ ?

What is  $L'(a)$ ?

What is  $L''(a)$ ? (Recall  $L''(x)$  is the derivative of  $L'(x)$ .)



# LINEAR APPROXIMATION WRAP-UP

Let  $L(x)$  be a linear approximation of  $f(x)$ .

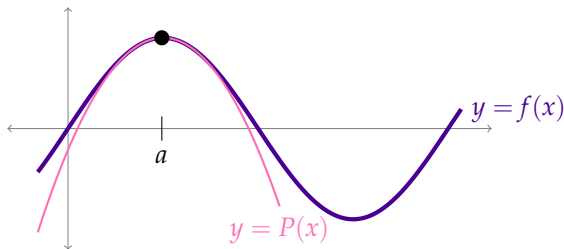
$f(a)$	$L(a)$	same
$f'(a)$	$L'(a)$	same
$f''(a)$	$L''(a)$	different <sup>1</sup>

---

<sup>1</sup>unless  $f''(a) = 0$

# QUADRATIC APPROXIMATION

Imagine we approximate  $f(x)$  at  $x = a$  with a parabola,  $P(x)$ .



	Constant	Linear	Quadratic
Function value matches at $x = a$	✓	✓	✓
First derivative matches at $x = a$	✗	✓	✓
Second derivative matches at $x = a$	✗	✗	✓

Constant:

$$f(x) \approx f(a)$$

Linear:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Quadratic:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

# QUADRATIC APPROXIMATION

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

Approximate  $\log(1.1)$  using a quadratic approximation.

# QUADRATIC APPROXIMATION

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

Approximate  $\sqrt[3]{28}$  using a quadratic approximation.

*You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.*



Determine what  $f(x)$  and  $a$  should be so that you can approximate the following using a quadratic approximation.

$$\log(.9)$$

$$e^{-1/30}$$

$$\sqrt[5]{30}$$

$$(2.01)^6$$

	Constant	Linear	Quadratic	degree $n$
match $f(a)$	✓	✓	✓	✓
match $f'(a)$	×	✓	✓	✓
match $f''(a)$	×	×	✓	✓
...				
match $f^{(n)}(a)$	×	×	×	✓
match $f^{(n+1)}(a)$	×	×	×	×

Constant:

$$f(x) \approx f(a)$$

Linear:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Quadratic:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Degree- $n$ :

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \dots?$$

## BRIEF DETOUR: SIGMA (SUMMATION) NOTATION

$$\sum_{i=a}^b f(i)$$

- ▶  $a, b$  (integers) “bounds”
- ▶  $i$  “index”: runs over integers from  $a$  to  $b$
- ▶  $f(i)$  “summand”: compute for every  $i$ , add

# SIGMA NOTATION

$$\sum_{i=2}^4 (2i + 5)$$

# SIGMA NOTATION

$$\sum_{i=1}^4 (i + (i - 1)^2)$$

Write the following expressions in sigma notation:

1.  $3 + 4 + 5 + 6 + 7$

2.  $8 + 8 + 8 + 8 + 8$

3.  $1 + (-2) + 4 + (-8) + 16$

## Factorial – Definition 3.4.9

We read “ $n!$ ” as “ $n$  factorial.”

For a natural number  $n$ ,  $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ .

By convention,  $0! = 1$ .

We write  $f^{(n)}(x)$  to mean the  $n^{\text{th}}$  derivative of  $f(x)$ . By convention,  $f^{(0)}(x) = f(x)$ .

## Taylor Polynomial – Definition 3.4.11

Given a function  $f(x)$  that is differentiable  $n$  times at a point  $a$ , the  $n$ -th degree **Taylor polynomial** for  $f(x)$  about  $a$  is

$$T_n(a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$$

If  $a = 0$ , we also call it a **Maclaurin polynomial**.



$$T_n(a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$$

$$=$$

$$T_n(a) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

Find the 7th degree Maclaurin<sup>2</sup> polynomial for  $e^x$ .

---

<sup>2</sup>A Maclaurin polynomial is a Taylor polynomial with  $a = 0$ .

$$T_n(a) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

Find the 8th degree Maclaurin polynomial for  $f(x) = \sin x$ .

$$T_n(a) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$



Find the 7th degree Taylor polynomial for  $f(x) = \log x$ , centered at  $a = 1$ .

» skip  $\Delta x$  notation

## Notation 3.4.18

Let  $x, y$  be variables related such that  $y = f(x)$ . Then we denote a small change in the variable  $x$  by  $\Delta x$  (read as “delta  $x$ ”). The corresponding small change in the variable  $y$  is denoted  $\Delta y$  (read as “delta  $y$ ”).

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let  $y = f(x)$  be the amount of water needed to produce  $x$  apples in an orchard.

A farmer wants to know how much water is needed to increase their crop yield.  $\Delta x$  is shorthand for some change in the number of apples, and  $\Delta y$  is shorthand for some change in the amount of water.



- Consider changing the number of apples grown from  $a$  to  $a + \Delta x$
- Then the change in water requirements goes from  $y = f(a)$  to  $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

# LINEAR APPROXIMATION OF $\Delta y$

- Using a linear approximation, setting  $x = a + \Delta x$ :

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{linear approximation}$$

$$f(a + \Delta x) \approx f(a) + f'(a)(\Delta x) \quad \text{set } x = a + \Delta x$$

$$\Delta y = f(a + \Delta x) - f(a) \approx f'(a)\Delta x \quad \text{subtract } f(a) \text{ both sides}$$

## Linear Approximation of $\Delta y$ (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

If we set  $\Delta x = 1$ , then  $\Delta y \approx f'(a)$ . So, if we want to produce  $a + 1$  apples instead of  $a$  apples, the extra water needed for that one extra apple is about  $f'(a)$ . We call this the *marginal* water cost of the apple.

# QUADRATIC APPROXIMATION OF $\Delta y$

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

## Quadratic Approximation of $\Delta y$ (Equation 3.4.21)

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$



[▶ skip further examples](#)

Approximate  $\tan(65^\circ)$  three ways: using constant, linear, and quadratic approximation.

Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and  $\pi$ .

You measure an angle  $x \approx \frac{\pi}{2}$ , and use it to calculate  $y = \sin x \approx 1$ . However, you suspect the angle was not *exactly* equal to  $\frac{\pi}{2}$ , which means the actual value  $y$  is slightly *less than* 1. In order for your value of  $y$  to have an error of no more than  $\frac{1}{200}$ , how accurate does your measurement of  $\theta$  have to be?

## Definition 3.4.25

Let  $Q_0$  be the exact value of a quantity and let  $Q_0 + \Delta Q$  be the measured value. We call

$$|\Delta Q|$$

the **absolute error** of the measurement, and

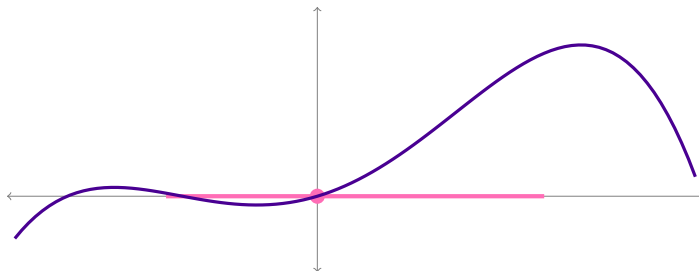
$$100 \frac{|\Delta Q|}{Q_0}$$

the **percentage error** of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

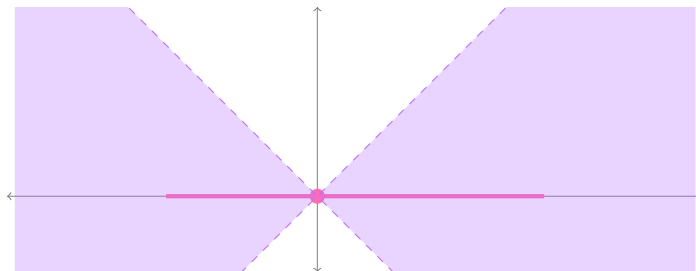
Once again, you find yourself in the position of measuring an angle  $x$ , which you use to compute  $y = \sin x$ . Let's say both  $x$  and  $y$  are positive. If your percentage error in measuring  $x$  is at most 1%, what is the corresponding maximum percentage error in  $y$ ? Use a linear approximation.

# ERROR: WHAT “CAUSES” ERROR IN AN ESTIMATION?



**Constant approximation:** We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

# CONTROLLING THE “CAUSE” OF THE ERROR



**Constant approximation:** We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).  
**BUT:** suppose we know the max and min values of the function's slope.

## Error

The error in an estimation  $f(x) \approx T_n(x)$  is  $f(x) - T_n(x)$ . We often use  $|f(x) - T_n(x)|$  if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

## Taylor's Theorem – Equation 3.4.33

For some  $c$  strictly between  $x$  and  $a$ ,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

The trick is bounding  $f^{(n+1)}(c)$ . It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error:  $|f(x) - T_n(x)|$ .

Third degree Maclaurin polynomial for  $f(x) = e^x$ :

$$\begin{aligned}T_3(x) &= f(0) + f'(0)(x - 0) + \frac{1}{2!}f''(0)(x - 0)^2 + \frac{1}{3!}f'''(0)(x - 0)^3 \\&= e^0 + e^0x + \frac{1}{2!}e^0x^2 + \frac{1}{3!}e^0x^3 \\&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\end{aligned}$$

Bound the error associated with using  $T_3(x)$  to approximate  $e^{1/10}$ .



## Taylor's Theorem – Equation 3.4.33

For some  $c$  strictly between  $x$  and  $a$ ,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Bound the error associated with using  $T_3(x)$  to approximate  $e^{1/10}$ .

## Taylor's Theorem – Equation 3.4.33

For some  $c$  strictly between  $x$  and  $a$ ,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at  $a = \pi/2$  to approximate  $f(x) = \cos x$ . What could the magnitude of the error be if we approximate  $\cos(2)$ ?

## Taylor's Theorem – Equation 3.4.33

For some  $c$  strictly between  $x$  and  $a$ ,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) (x-a)^{n+1}$$

Suppose we use a third degree Taylor polynomial centred at 4 to approximate  $f(x) = \sqrt{x}$ . If we use this Taylor polynomial to approximate  $\sqrt{4.1}$ , give a bound for our error.

## Taylor's Theorem – Equation 3.4.33

For some  $c$  strictly between  $x$  and  $a$ ,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) (x-a)^{n+1}$$

Suppose you want to approximate the value of  $e$ , knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for  $f(x) = e^x$  to approximate  $f(1) = e^1 = e$ . Bound your error.

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?

# WHICH DEGREE?

Suppose you want to approximate  $\sin 3$  using a Taylor polynomial of  $f(x) = \sin x$  centered at  $a = \pi$ . If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

# WHICH DEGREE?

Suppose you want to approximate  $e^5$  using a Maclaurin polynomial of  $f(x) = e^x$ . If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

# WHICH DEGREE?

Suppose you want to approximate  $\log \frac{4}{3}$  using a Taylor polynomial of  $f(x) = \log x$  centred at  $a = 1$ . If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?



# WHICH DEGREE?

Let  $f(x) = \sqrt[4]{x}$ . Suppose you use a second-degree Taylor polynomial of  $f(x)$  centered at  $a = 81$  to approximate  $\sqrt[4]{81.2}$ . Bound your error, and tell whether  $T_2(10)$  is an overestimate or underestimate.

## Included Work



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