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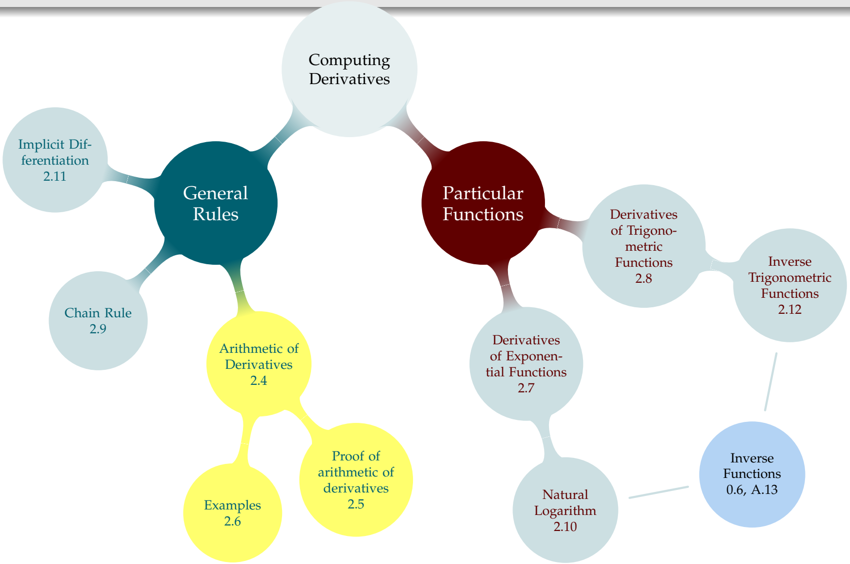

Derivatives of Lines

$$f(x) = 2x - 15$$

The equation of the tangent line to $f(x)$ at $x = 100$ is:

$f'(1) =$

A. 0 B. 1 C. 2 D. -15 E. -13

$f'(5) =$

$f'(-13) =$

$$g(x) = 13$$

$g'(1) =$

A. 0 B. 1 C. 2 D. 13

Adding a Constant

Adding or subtracting a constant to a function does not change its derivative.

We saw

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3-0.8t^2\right)\Big|_{t=1}=-1.6$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(10-0.8t^2\right)\Big|_{t=1}=$$

Differentiating Sums

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{f(x)+g(x)\right\} =$$

Differentiating Sums

$$\frac{d}{dx}\{f(x)+g(x)\} =$$

This is a really good intro to showing how the definition can be used to prove rules. Alternately put, how rules are just shortcuts for calculations starting with the definition.

Constant Multiple of a Function

Let a be a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{a \cdot f(x)\right\} =$$

Let a be a constant.

$$\frac{d}{dx}\{a \cdot f(x)\} =$$

Ask students to explain to their neighbours which step wouldn't have worked if a weren't constant

Rules – Lemma 2.4.1

Suppose $f(x)$ and $g(x)$ are differentiable, and let c be a constant number. Then:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+g(x)\} = f'(x)+g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)-g(x)\} = f'(x)-g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{cf(x)\} = cf'(x)$

For instance: let $f(x) = 10\left((2x-15)+13-\sqrt{x}\right)$. Then $f'(x) =$

NOW YOU

Suppose $f'(x) = 3x$, $g'(x) = -x^2$, and $h'(x) = 5$.

Calculate:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + 5g(x) - h(x) + 22\}$$

A. $3x - 5x^2$

B. $3x - 5x^2 - 5$

C. $3x - 5x^2 - 5 + 22$

D. none of the above

Derivatives of Products

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x\} = 1$$

True or False:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{2x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{x+x\}\\ &= [1]+[1]\\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{x\cdot x\}\\ &= [1]\cdot[1]\\ &= 1\end{aligned}$$

DERIVATIVES OF PRODUCTS

$$\frac{d}{dx}\{x\} = 1$$

True or False:

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True or False:

$$\begin{aligned}\frac{d}{dx}\{x^2\} &= \frac{d}{dx}\{x\cdot x\}\\ &= [1]\cdot[1]\\ &= 1\end{aligned}$$

Students have seen these rules before and often feel bored here. I like to remind them how weird the rules are. It wasn't obvious that the product rule would be what it is, but they've known it for so long, it *feels* obvious. So I like to show the contrasting patterns below.

What to do with Products?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x. What about $f(x)g(x)$?

What to do with Products?

What to do with Products?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x. What about $f(x)g(x)$?

To avoid panic, emphasize that students are not responsible for reproducing this. The rule is surprising, we're just explaining why it's true. Also useful to emphasize that all the "tricks" are just *shortcuts for teh definition of the derivative.* Otherwise students often wonder why we bother with the definition at all.

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] =$$

Example: suppose $f(x) = 3x^2, f'(x) = 6x$, $g(x) = \sin(x), g'(x) = \cos(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[3x^2\sin(x)\right] =$$

Given $\frac{d}{dx}[2x+5] = 2,\quad \frac{d}{dx}\left[\sin(x^2)\right] = 2x\cos(x^2),\quad \frac{d}{dx}\left[x^2\right] = 2x$

Now You 

$$f(x) = (2x+5)\sin(x^2)$$

A. $f'(x) = (2)\left(2x\cos(x^2)\right)(2x)$

B. $f'(x) = (2)\left(2x\cos(x^2)\right)$

C. $f'(x) = (2x+5)(2) + \sin(x^2)\left(2x\cos(x^2)\right)$

D. $f'(x) = (2x+5)\left(2x\cos(x^2)\right) + (2)\sin(x^2)$

E. none of the above

Now You

$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is $f'(x)$?

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

Quotient Rule – Theorem 2.4.5

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Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x+5}{3x-6}\right\} =$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{5x}{\sqrt{x}-1}\right\} =$$

NOW YOU

Differentiate the following.

$$f(x) = 2x + 5$$

$$g(x) = (2x + 5)(3x - 7) + 25$$

$$h(x) = \frac{2x + 5}{8x - 2}$$

$$j(x) = \left(\frac{2x + 5}{8x - 2}\right)^2$$

Rules

Product: $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

Now You Differentiate the following.

$f(x) = 2x + 5$

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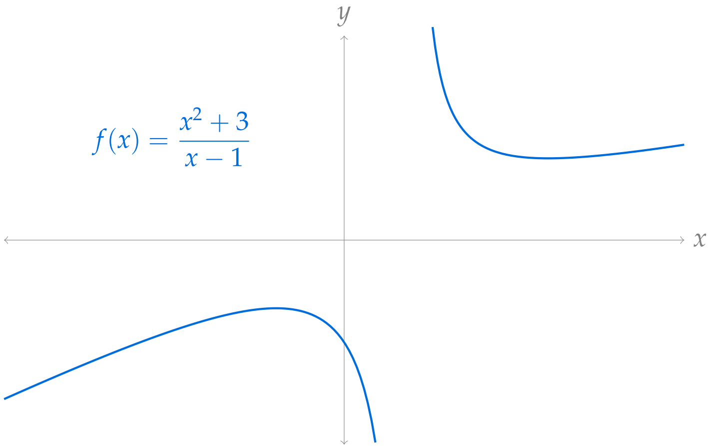
Rules

Product: $\frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

Students probably remember power and chain rules from high school, but these can be done without power and chain rules. The last example is there to give the faster workers something to work on, while the slower workers get quality time with the foundational examples.

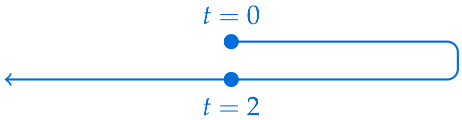


For which values of x is the tangent line to the curve horizontal?

The position of an object moving left and right at time t, $t \geq 0$, is given by

$$s(t) = -t^2(t-2)$$

where a positive position means it is to the right of its starting position, and a negative position means it is to the left. First it moves to the right, then it moves left forever.



What is the farthest point to the right that the object reaches?

More About the Product Rule

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{d}{dx}\{x^4\} = \frac{d}{dx}\{x \cdot x^3\}$$
$$= x(3x^2) + x^3(1) = 4x^3$$

Where are these functions defined?

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{30}	$30x^{29}$
x^n	nx^{n-1}

Cautionary Tale

With *functions* raised to a power, it's more complicated.

Differentiate $(2x+1)^2$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

$$\frac{\mathrm{d}}{\mathrm{d}x}\{3x^5 + 7x^2 - x + 15\} =$$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Differentiate $\dfrac{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{2x+5}$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Suppose a motorist is driving their car, and their position is given by $s(t) = 10t^3 - 90t^2 + 180t$ kilometres. At $t = 1$ (t measured in hours), a police officer notices they are driving erratically. The motorist claims to have simply suffered a lack of attention: they were in the act of pressing the brakes even as the officer noticed their speed.

At $t = 1$, how fast was the motorist going, and were they pressing the gas or the brake?

Challenge: What about $t = 2$?

Power Rule – Corollary 2.6.17

$\frac{d}{dx}\{x^a\} = ax^{a-1}$ (where defined)

Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.
Suppose you are winding twine into a gigantic twine ball, filming the process, and trying to make a viral video. You can wrap one cubic meter of twine per hour. (In other words, when we have V cubic meters of twine, we're at time V hours.) How fast is the radius of your spherical twine ball increasing?

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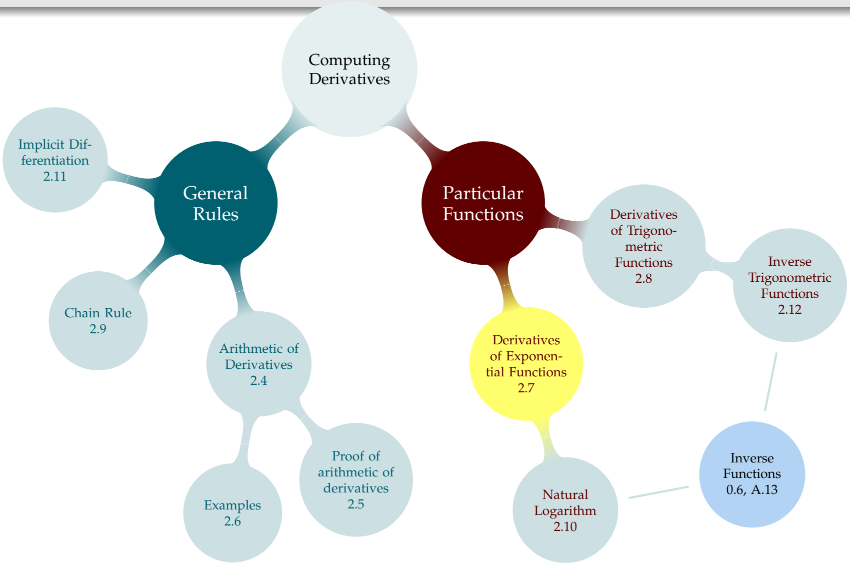
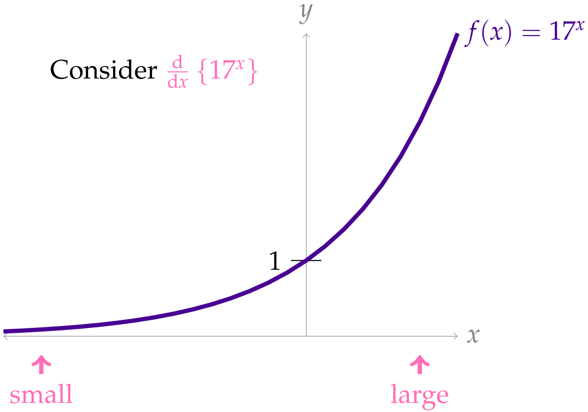


Exponential Functions

Consider $\frac{d}{dx}\{17^x\}$



$f(x)$ is always increasing, so $f'(x)$ is always positive.
$f'(x)$ might look similar to $f(x)$.

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0} \frac{(17^h - 1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\lim_{h\to 0} \frac{17^h - 1}{h} = 0$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be 0, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

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A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be 0, that wouldn't make sense.
C. I do not feel equipped to answer this question.

The second question is there as a second chance for students who missed the first one

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0} \frac{(17^h - 1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0} \frac{17^h - 1}{h} = \infty$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be ∞, that wouldn't make sense.
C. I do not feel equipped to answer this question.

2.7: Derivs of Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0}\frac{17^h-1}{h} = \infty$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be ∞, that wouldn't make sense.
C. I do not feel equipped to answer this question.

The second question is there as a second chance for students who missed the first one

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0} \frac{(17^h - 1)}{h}}_{\text{constant}}$$

h	$\dfrac{17^h - 1}{h}$
0.001	2.83723068608
0.00001	2.83325347992
0.0000001	2.83321374583
0.000000001	2.83321344163

2.7: Derivs of Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0} \frac{(17^h - 1)}{h}}_{\text{constant}}$$

h	$\frac{17^h - 1}{h}$
0.001	2.83723068608
0.00001	2.83325347992
0.0000001	2.83321374583
0.000000001	2.83321344163

"It's not clear what constant this is, but it is a constant."

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} &= \lim_{h\to 0}\frac{17^{x+h}-17^x}{h} \\
&= \lim_{h\to 0}\frac{17^x 17^h-17^x}{h} \\
&= \lim_{h\to 0}\frac{17^x(17^h-1)}{h} \\
&= 17^x \lim_{h\to 0}\frac{(17^h-1)}{h}
\end{aligned}$$

In general, for any positive number a,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

2.7: Derivs of Exponential Functions

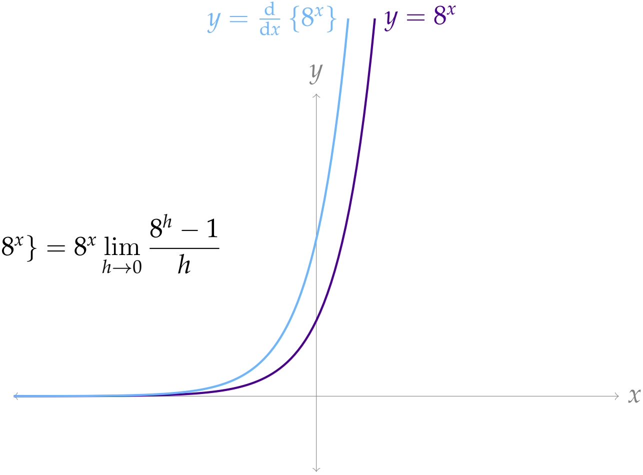
$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = \lim_{h\to 0}\frac{17^{x+h}-17^x}{h}$$
$$= \lim_{h\to 0}\frac{17^x 17^h - 17^x}{h}$$
$$= \lim_{h\to 0}\frac{17^x(17^h-1)}{h}$$
$$= 17^x \lim_{h\to 0}\frac{(17^h-1)}{h}$$

In general, for any positive number a,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

Point out that 17 could have been replaced with any other number. Go through and cross out 17, write in a.

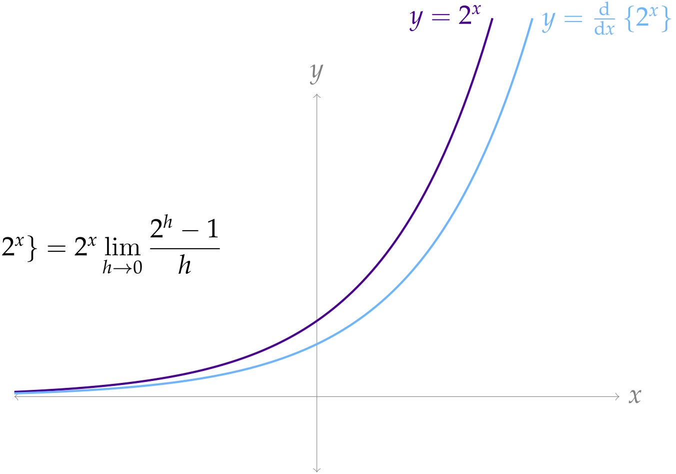
Exponential Functions



$$\frac{\mathrm{d}}{\mathrm{d}x}\{8^x\} = 8^x \lim_{h\to 0} \frac{8^h - 1}{h}$$

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{2^x\right\} = 2^x \lim_{h\to 0} \frac{2^h - 1}{h}$$



In general, for any positive number a, $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$

Euler's Number – Theorem 2.7.4

We define e to be the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$e \approx 2.7182818284590452353602874713526624\ldots$ (Wikipedia)

Theorem 2.7.4 and Corollary 2.10.6

Using this definition of e,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} = e^x \underbrace{\lim_{h\to 0}\frac{e^h-1}{h}}_{1} = e^x$$

In general, $\displaystyle\lim_{h\to 0}\frac{a^h-1}{h} = \log_e(a)$, so $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x\log_e(a)$

That $\displaystyle\lim_{h\to 0}\frac{a^h-1}{h} = \log_e(a)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x\log_e(a)$ are consequences of

$$a^x = \left(e^{\log_e(a)}\right)^x = e^{x\log_e(a)}$$

For the details, see the end of Section 2.7.

Things to Have Memorized

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^x\right\} = e^x$$

When a is any constant,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{a^x\right\} = a^x \log_e(a)$$

Let $f(x) = \dfrac{e^x}{3x^5}$. When is the tangent line to $f(x)$ horizontal?

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{3x}\right\}$

Evaluate $\frac{d}{dx}\left\{e^{3x}\right\}$

2 ways: product rule with $e^x \cdot e^x \cdot e^x$, or previous rule with $\left(e^3\right)^x$

Suppose the deficit, in millions, of a fictitious country is given by

$$f(x) = e^x(4x^3 - 12x^2 + 14x - 4)$$

where x is the number of years since the current leader took office. Suppose the leader has been in power for exactly two years.

1. Is the deficit increasing or decreasing?

2. Is the rate at which the deficit is growing increasing or decreasing?

Included Work

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