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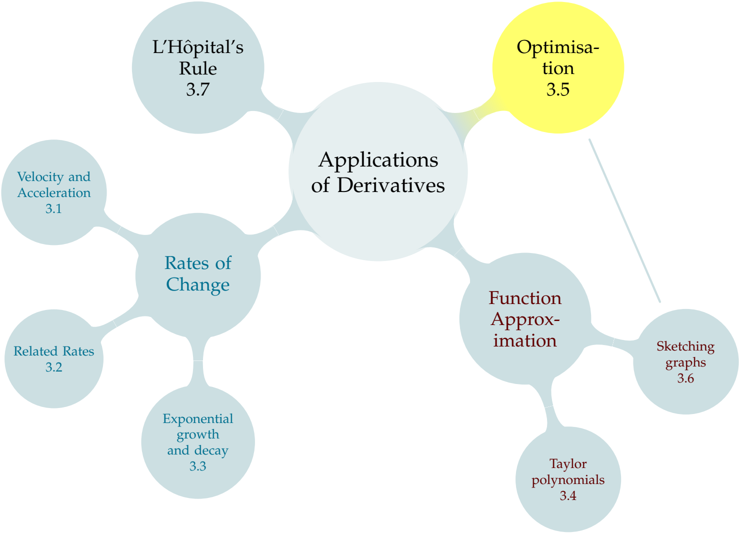

Optimisation:
finding the biggest/smallest/highest/lowest, etc.

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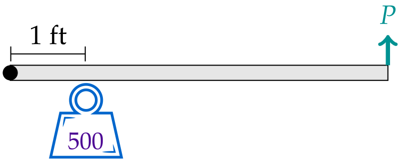
finding the biggest/smallest/highest/lowest, etc.

Lots of non-standard problems! Opportunities to work on your problem-solving skills.

ENGINEERING DESIGN EXAMPLE

A lever of density 3 lbs/ft is being used to lift a 500-pound weight, attached one foot from the fixed point.



For an L-foot-long lever, the force P required to lift the system satisfies

$$500(1) + 3L\left(\frac{L}{2}\right) - PL = 0$$

What length of lever will require the least amount of force to lift?

Source: Drexel (2006)

Medical Dosing Example

Let D be the size of a dose, α be the absorption rate, and β the elimination rate of a drug.
Caffeine is absorbed and eliminated by first-order kinetics. Its blood concentration over time is modelled as

$$c(t) = \frac{D}{1 - \beta/\alpha} \left(e^{-\beta t} - e^{-\alpha t}\right)$$

Will the blood concentration reach a toxic level?

Source (including links to a study): Vectornaut (2015)

Circuit Example

When a critically damped RLC circuit is connected to a voltage source, the current I in the circuit varies with time according to the equation

$$I(t) = \left(\frac{V}{L}\right) te^{-\frac{Rt}{2L}}$$

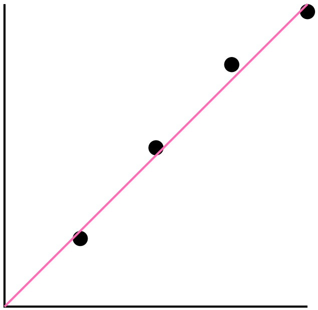
where V is the applied voltage, L is the inductance, and R is the resistance (all of which are constant).

We need to choose wires that will be able to safely carry the current at all times.

Source: Belk (2014)

LEAST SQUARES EXAMPLE

You have a lot of data that more-or-less resembles a line. Which line does it most resemble?



Extrema – Definition 3.5.3

Let I be an interval, and let the function $f(x)$ be defined for all $x \in I$. Now let $c \in I$.

- We say that $f(x)$ has a global (or absolute) minimum on the interval I at the point $x = c$ if $f(x) \geq f(c)$ for all $x \in I$.
- We say that $f(x)$ has a global (or absolute) maximum on I at $x = c$ if $f(x) \leq f(c)$ for all $x \in I$.
- We say that $f(x)$ has a local minimum at $x = c$ if $f(x) \geq f(c)$ for all $x \in I$ that are near c.
- We say that $f(x)$ has a local maximum at $x = c$ if $f(x) \leq f(c)$ for all $x \in I$ that are near c.

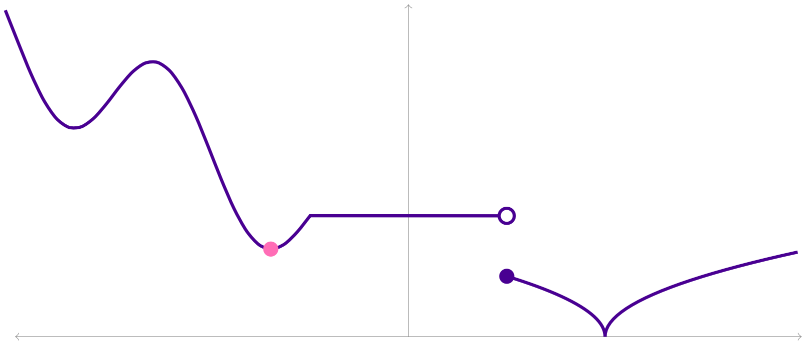
The maxima and minima of a function are called the extrema of that function.

Critical and Singular Points – Definition 3.5.6

Let $f(x)$ be a function and let c be a point in its domain. Then

- If $f'(c)$ exists and is zero we call $x = c$ a critical point of the function, and
- If $f'(c)$ does not exist then we call $x = c$ a singular point of the function.

Anatomy of a Function



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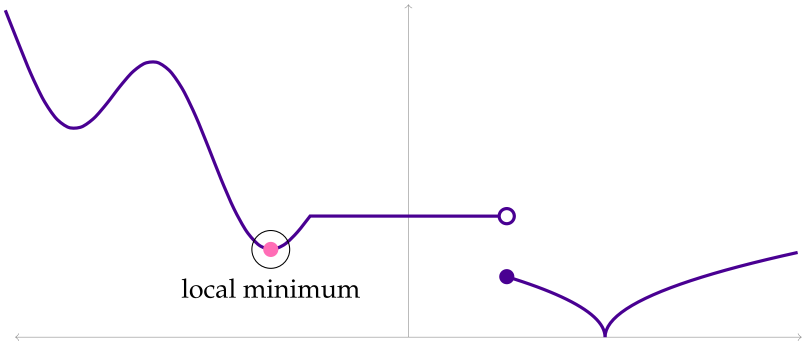

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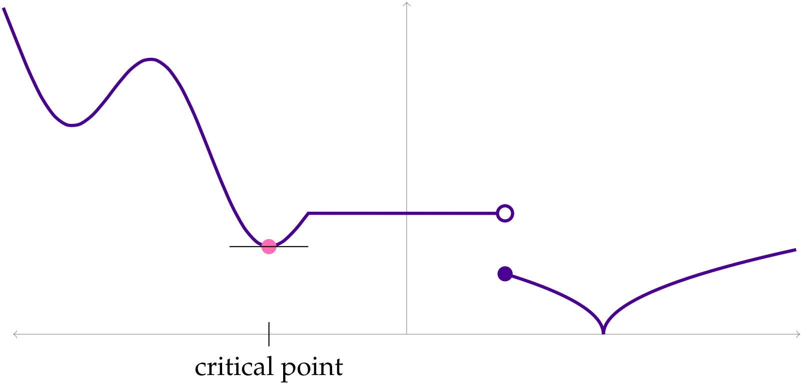

c is a critical point if $f'(c) = 0$.

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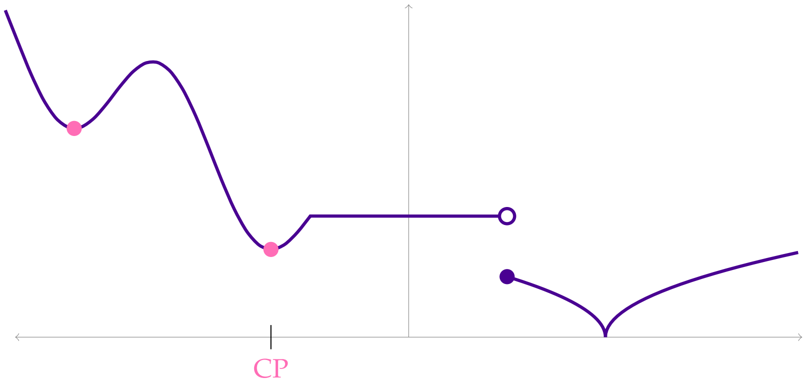

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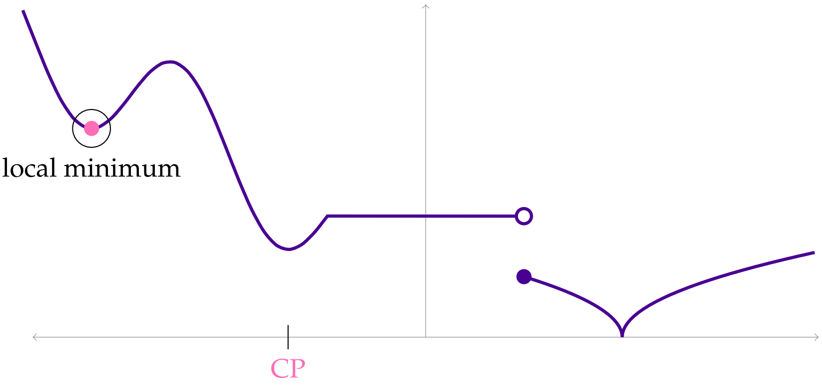

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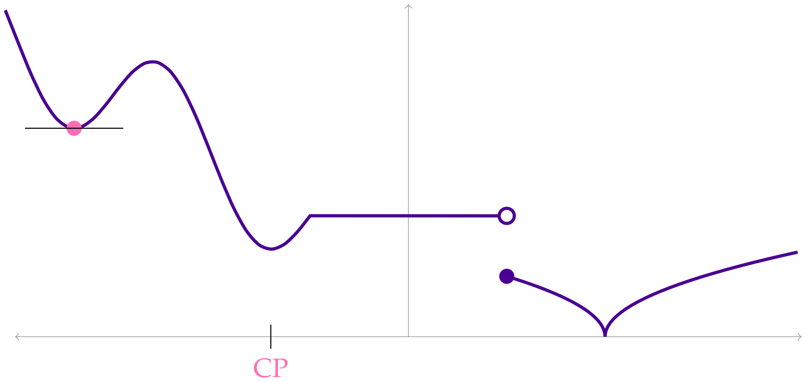

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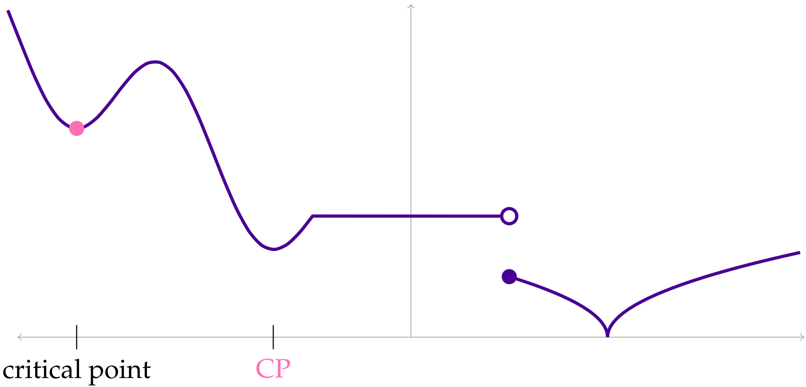

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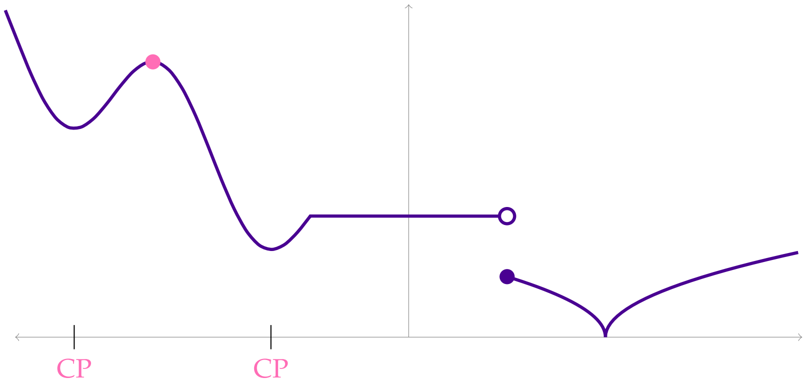

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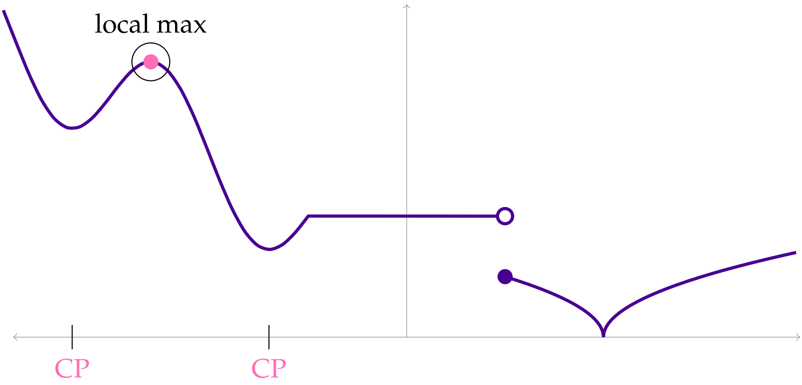

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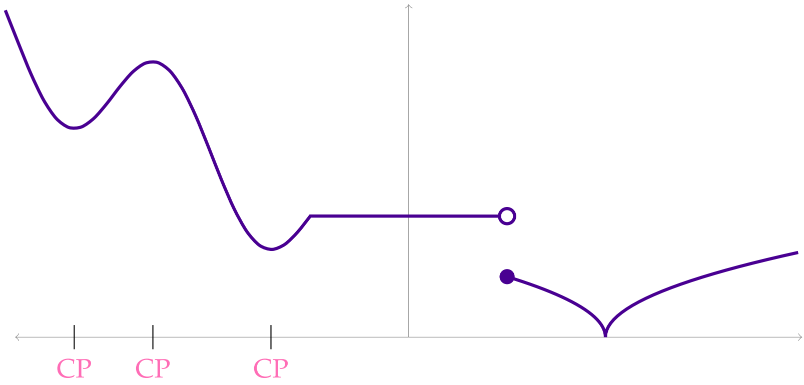

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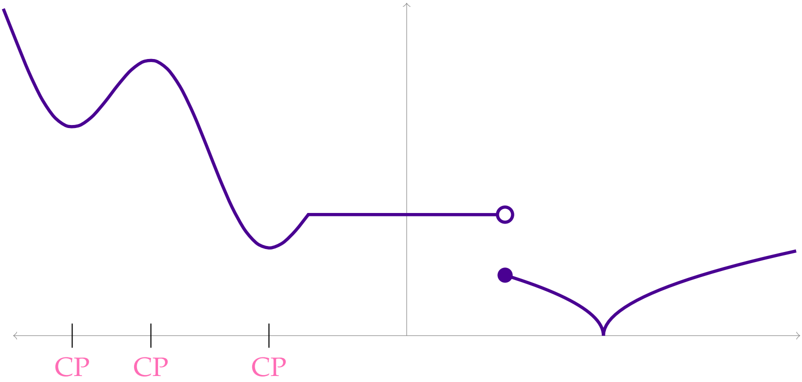
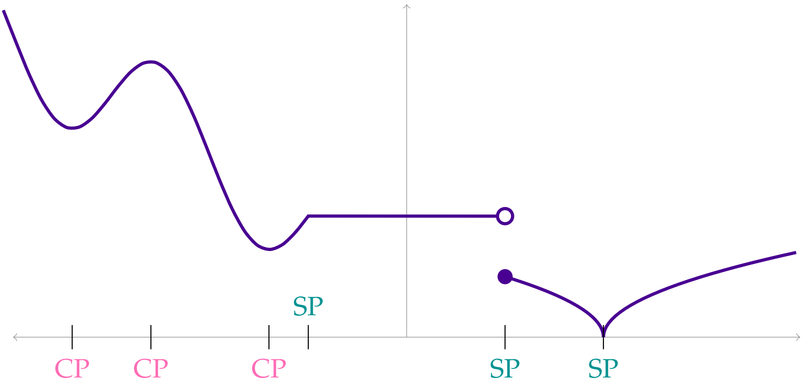


c is a singular point if $f'(c)$ does not exist.

Anatomy of a Function



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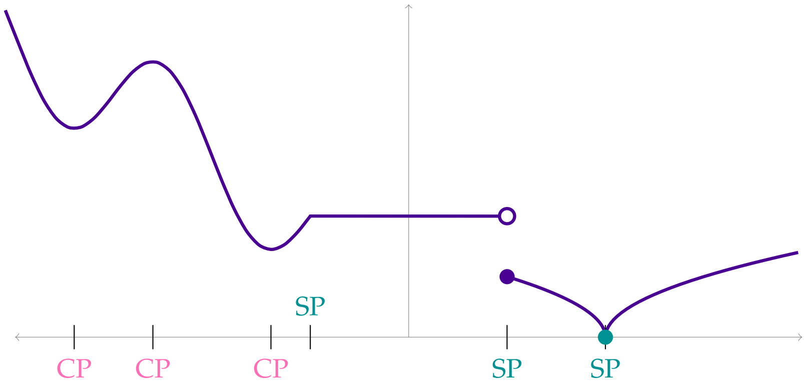

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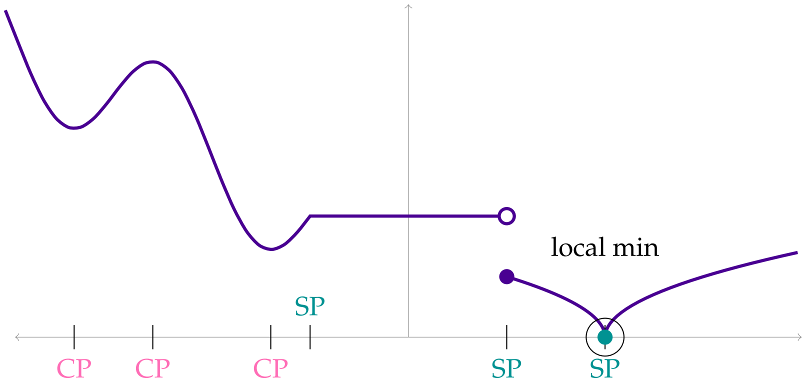

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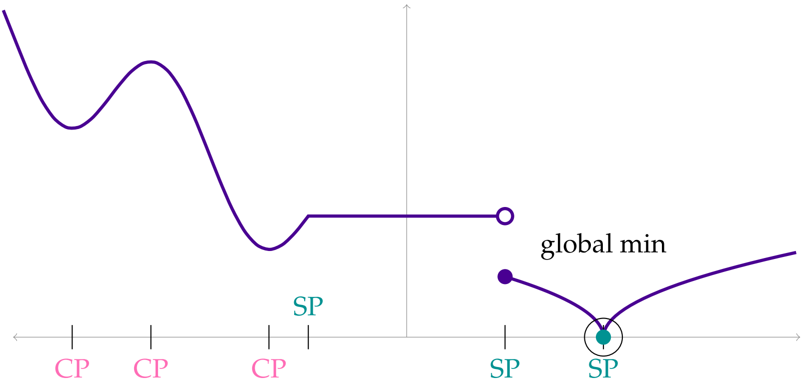

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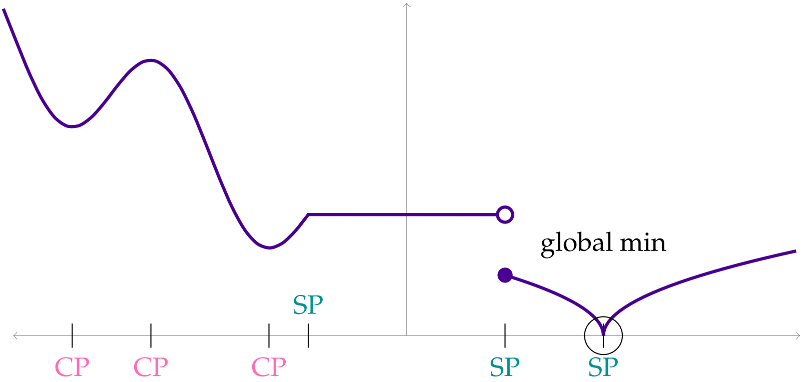
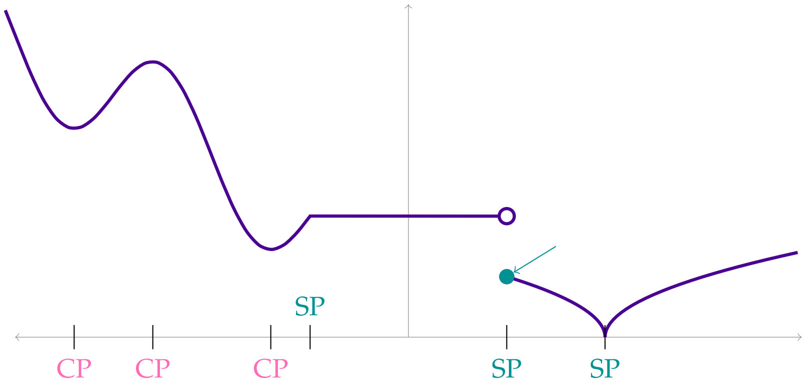


This function as shown has no global maximum (if it continues as shown).

ANATOMY OF A FUNCTION



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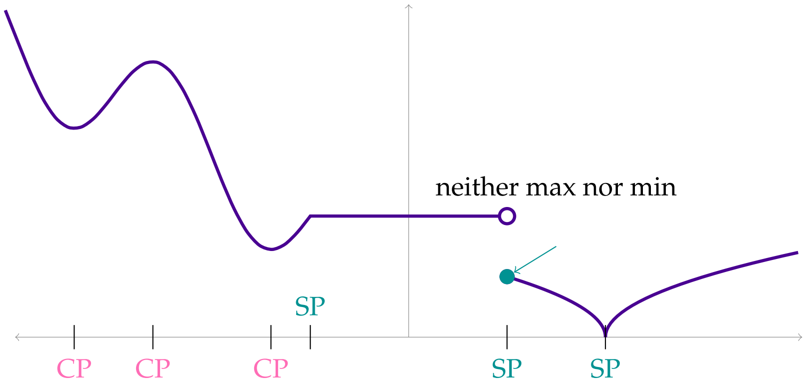

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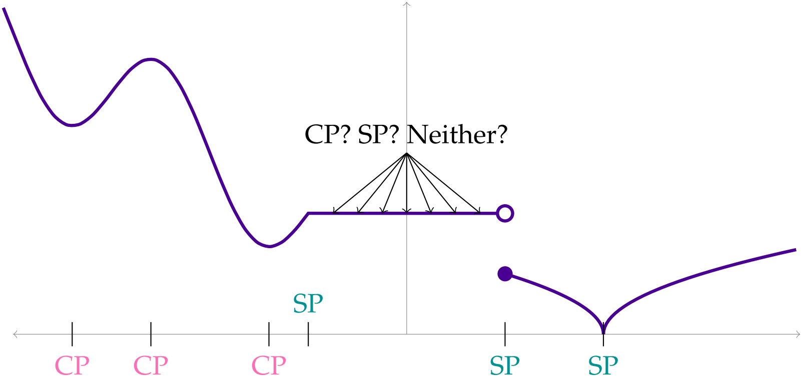

ans

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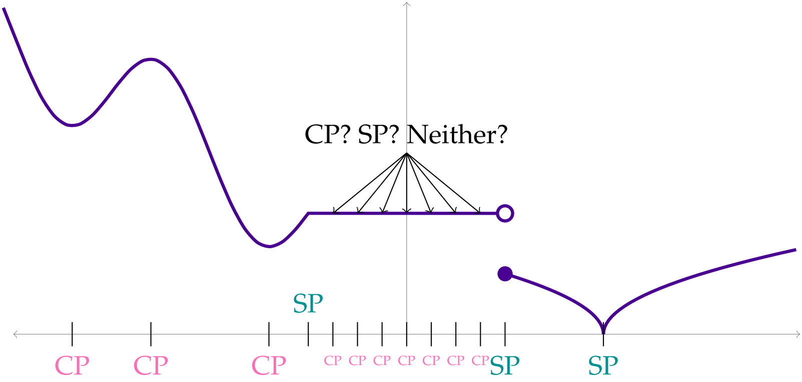

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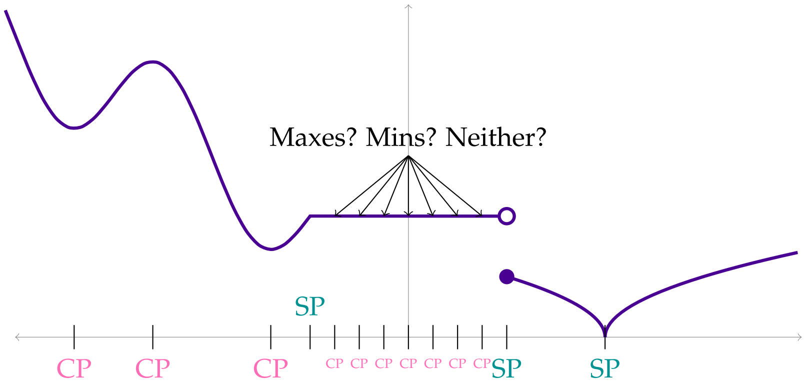

Theorem 3.5.4

If a function $f(x)$ has a local maximum or local minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

Multiple Choice

Suppose $f(x)$ has domain $(-\infty, \infty)$.
If $f'(5) = 0$, then:

A. $f'(5)$ DNE
B. f has a local maximum at 5
C. f has a local minimum at 5
D. f has a local extremum (maximum or minimum) at 5
E. f may or may not have a local extremum (max or min) at 5

Q

ans

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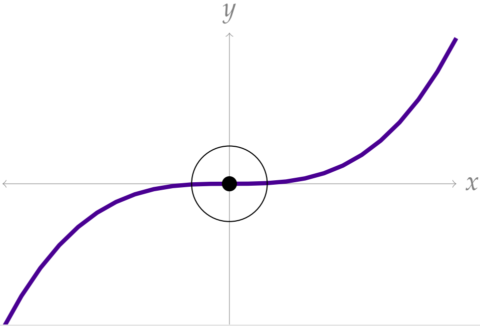

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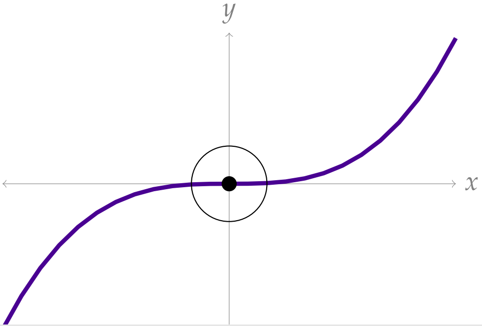


SKETCH

Draw a continuous function $f(x)$ with a local maximum at $x = 3$ and a local minimum at $x = -1$.

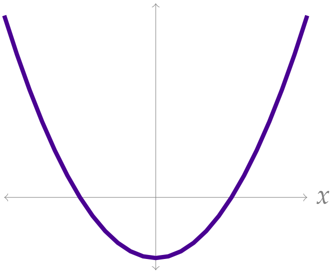
Draw a continuous function $f(x)$ with a local maximum at $x = 3$ and a local minimum at $x = -1$, but $f(3) < f(-1)$.

Draw a function $f(x)$ with a singular point at $x = 2$ that is NOT a local maximum, or a local minimum.

Q

ans

SECOND DERIVATIVES



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- Is second derivative positive, negative, or zero?
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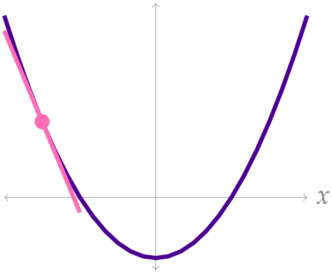

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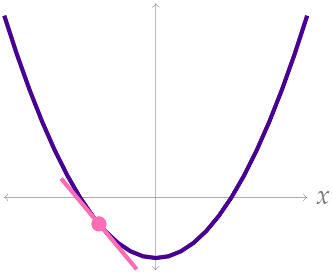

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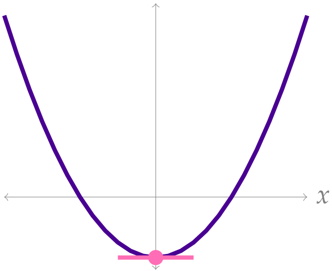

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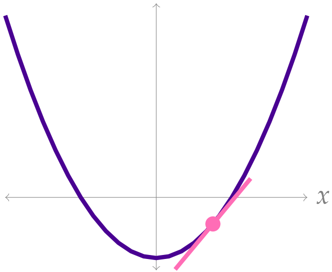

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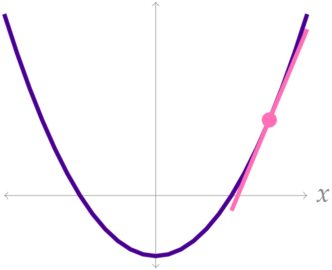

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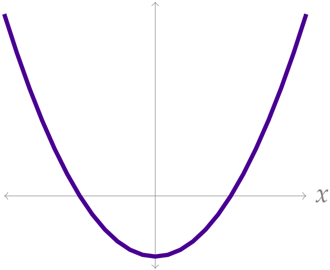

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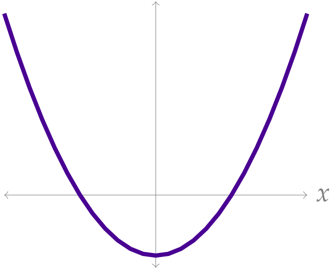

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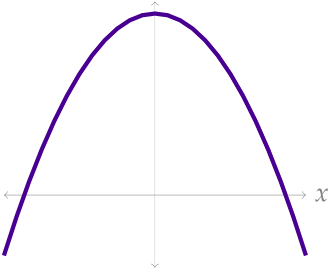

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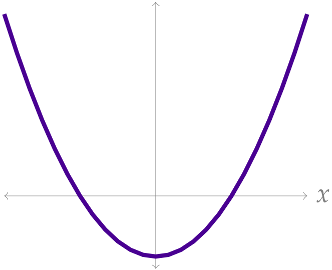

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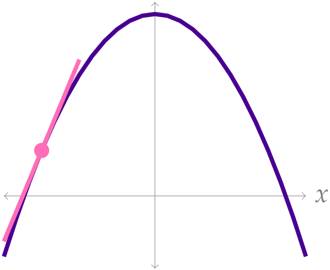

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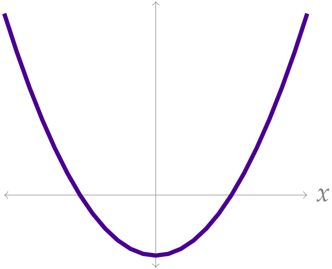

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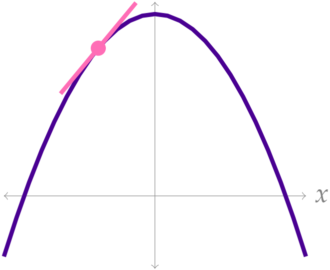

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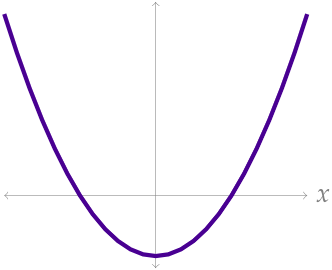

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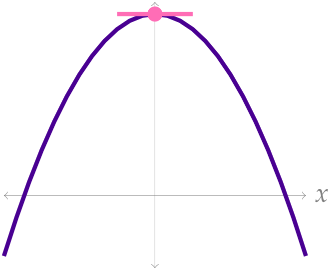

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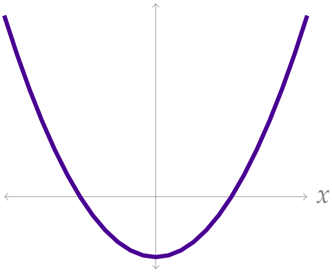

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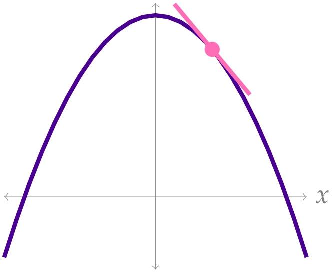

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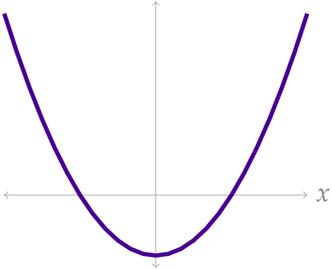

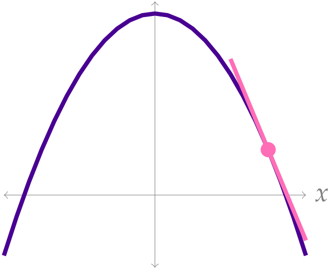

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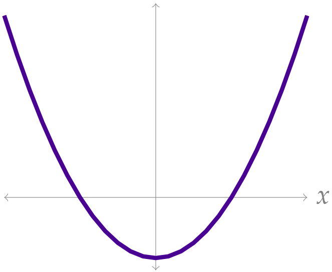

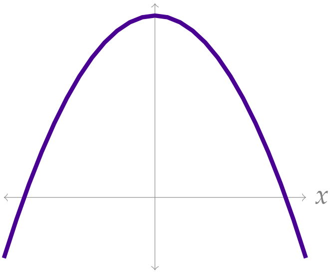

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Suppose $f'(x) = (x+5)^2(x-5)$. Then f has no singular points, and its critical points are ± 5. Identify whether the critical points are local maxima, local minima, or neither.

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Second Derivative Test:
Suppose $f'(a) = 0$ and $f''(a) > 0$. Then $x = a$ is a local

Suppose $f'(a) = 0$ and $f''(a) < 0$. Then $x = a$ is a local

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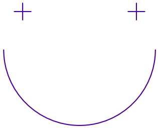

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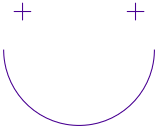
Suppose $f'(a) = 0$ and $f''(a) > 0$.
Then $x = a$ is a local **minimum**.



Suppose $f'(a) = 0$ and $f''(a) < 0$.
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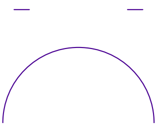
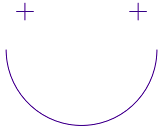
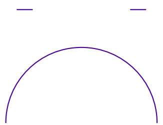


Suppose $f'(x) = (x+5)^2(x-5)$. Then f has no singular points, and its critical points are ± 5. Identify whether the critical points are local maxima, local minima, or neither.

Second Derivative Test:
Suppose $f'(a) = 0$ and $f''(a) > 0$.
Then $x = a$ is a local **minimum**.

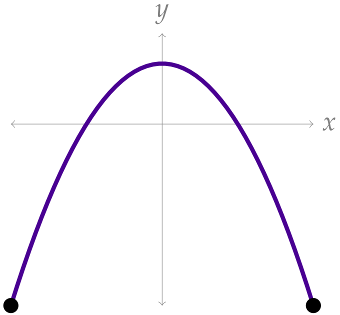


Suppose $f'(a) = 0$ and $f''(a) < 0$.
Then $x = a$ is a local **maximum**.



We see that, when we are close to -5, whether x is less than or greater than -5, still $f'(x)$ is negative. So, $f(x)$ is decreasing before $x = -5$ and also after it. So, -5 is not a local max or a local min.
Now consider $x = 5$. When x is a little less, $f'(x)$ is negative; when x is a little more than 5, $f'(x)$ is positive. So, f is decreasing till 5, then increasing after: so 5 is a local min.
Indeed, $x = 5$ is the site of a global min.

Endpoints



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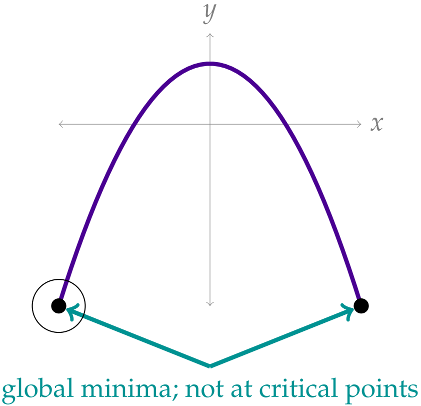
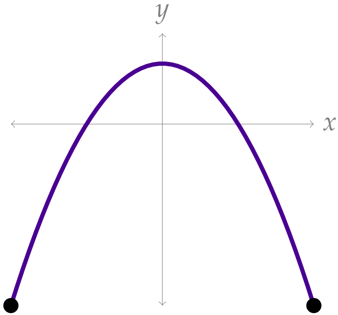


ENDPOINTS



Theorems 3.5.11 and 3.5.12

A function that is continuous on the interval $[a, b]$ (where a and b are real numbers–not infinite) has a global max and min, and they occur at endpoints, critical points, or singular points.

Determining Extrema

To find local extrema:

- Could be at
- Could be at
- Could be at
- At these points, check whether there is some interval around x where $f(x)$ is no larger than the other numbers, or no smaller. (A sketch helps. The signs of the derivatives on either side of x are also a clue.)

To find global extrema:

- Could be at
- Could be at
- Could be at
- Check the value of the function at all of these, and compare.

DETERMINING EXTREMA

To find local extrema:

- Could be at critical points ($f'(x) = 0$)
- Could be at singular points ($f'(x)$ DNE)
- Could be at endpoints
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To find global extrema:

- Could be at critical points ($f'(x) = 0$)
- Could be at singular points ($f'(x)$ DNE)
- Could be at endpoints;
  also check the limit as the function goes to $\pm\infty$.
- Check the value of the function at all of these, and compare.

Find All Extrema[1]:

$$f(x) = x^3 - 3x$$

[1] Extrema: local and global maxima and minima

Q

ans

Find All Extrema[1]:

$$f(x) = x^3 - 3x$$

Since there are no endpoints, we only need to find critical points and singular points. $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$. So there are no singular points, and the critical points are ± 1.

We know that cubic functions grow hugely positive in one direction, and hugely negative in the other. So, there's no global max or min. We need only decide whether $x = 1$ and $x = -1$ are local extrema.

We can easily graph $f'(x)$, and we see it is an upwards-pointing parabola. It is positive to the left of $x = -1$ and negative just to its right, so f is increasing up till $x = -1$, then decreasing after; so $x = 1$ is a local max.
Likewise, $f'(x)$ is negative just to the left of $x = 1$ and positive to the right of it; so it's decreasing till $x = 1$ and increasing after. Thus $x = 1$ is a local min.

[1] Extrema: local and global maxima and minima

Find All Extrema

$$f(x) = \sqrt[3]{x^2 - 64}, \quad x \text{ in } [-1, 10]$$

Q

ans

Find All Extrema

$$f(x) = \sqrt[3]{x^2 - 64}, \quad x \text{ in } [-1, 10]$$

The endpoints are -1 and 10. We differentiate to identify critical points and singular points: $f'(x) = \frac{1}{3}(x^2 - 64)^{-2/3}(2x) = \frac{2}{3}x(x^2 - 64)^{-2/3}$. So the critical point is $x = 0$ and the singular points are $x = \pm 8$; but since $x = -8$ is not our domain, we don't have to worry about it.

The global extrema are found by simply comparing the value of the function at the various interesting points.
$f(0) = \sqrt[3]{-64} = -4$; $f(8) = 0$; $f(-1) = -\sqrt[3]{63}$; $f(10) = \sqrt[3]{100 - 64} = \sqrt[3]{36}$
Of these, -4 is the smallest and $\sqrt[3]{36}$ is the largest, so the global max is $\sqrt[3]{36}$ at $x = 10$, and the global min is -4 at $x = 0$.

Then it's pretty clear that $x = 0$ is a local min. Regarding the endpoints, since $f'(-1) < 0$, $x = -1$ is the location of a local maximum. Since $f'(10) > 0$ $x = 10$ is the location of a local maximum as well. So, what of $x = 8$? When x is slightly smaller than 8, or slightly larger than 8, $f'(x)$ is positive; so $f(x)$ is increasing to the left of 8 and also to the right of 8. Then 8 is neither a local max nor a local min.

Find the largest and smallest value of $f(x) = x^4 - 18x^2$.

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There are no endpoints given, so we take the domain to be the domain of the function, which is all real numbers. As x goes to infinity or negative infinity, $f(x)$ goes to infinity, so there is no global max, hence no largest value.

To find the global min, we differentiate: $f'(x) = 4x^3 - 36x = 4x(x^2 - 9)$. So the critical points are 0 and ± 3, and there are no singular points.

$f(0) = 0$, and $f(3) = f(-3) = -81$, so the smallest value (and global min) is -81, and it occurs twice (which is fine): at 3 and -3.

Find the largest and smallest values of $f(x) = \sin^2 x - \cos x$.

Q

ans

Find the largest and smallest values of $f(x) = \sin^2 x - \cos x$.

Since this function is periodic of period 2π, we can restrict our search to x values in $[0, 2\pi)$.

$$f'(x) = 2\sin x \cos x + \sin x = \sin x(2\cos x + 1)$$

So our critical points occur when $\sin x = 0$ and when $\cos x = -1/2$. That is, when x is $0, \pi, 2\pi/3$, or $4\pi/3$.

We plug these in to find

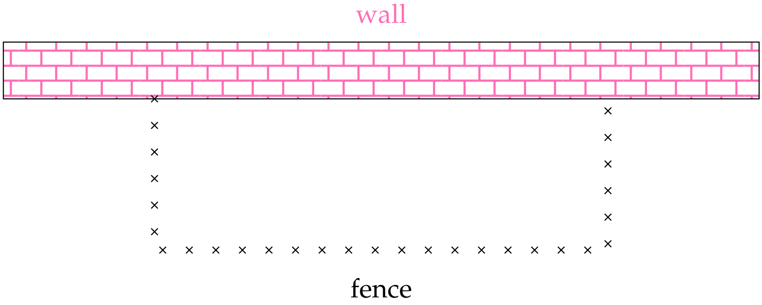
$$f(0) = -1, \qquad f(\pi) = 1, \qquad f(2\pi/3) = f(4\pi/3) = \frac{5}{4}$$

So the biggest this function gets is 1.25, and this occurs at $x = \frac{2\pi}{3} + 2n\pi$ and $\frac{4\pi}{3} + 2n\pi$ for any integer n. The smallest $f(x)$ gets is -1, and this occurs at $x = 2n\pi$, for any integer n.

A

MAX/MIN WORD PROBLEMS

A rancher wants to build a rectangular pen, using an existing wall for one side of the pen, and using 100m of fencing for the other three sides. What are the dimensions of the pen built this way that has the largest area?



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MAX/MIN WORD PROBLEMS

A rancher wants to build a rectangular pen, using an existing wall for one side of the pen, and using 100m of fencing for the other three sides. What are the dimensions of the pen built this way that has the largest area?

Let the side of the fence parallel to the barn be w metres long, and the two sides of the fence touching the barn be l metres long. Then $2l + w = 100$, and $A = lw$, where A is the area of the pen.
We want to find where A is maximum, given $2l + w = 100$. So, we can turn A into a function of just one variable by substituting $w = 100 - 2l$. Then

$$A(l) = l(100 - 2l) = 100l - 2l^2$$

so A is a parabola pointing down; its maximum occurs at its only critical point. Now to find its critical point, we differentiate:

$$A'(l) = 100 - 4l$$

so $l = 25$ and $w = 50$ give the pen with the biggest area.

General Idea

We know how to find the global extrema of a function over an interval.

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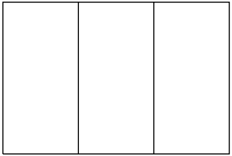
Problems often involve multiple variables, but we can only deal with functions of one variable.

Find all the variables in terms of ONE variable, so we can find extrema.

You want to build a pen, as shown below, in the shape of a rectangle with two interior divisions. If you have 1000m of fencing, what is the greatest area you can enclose?



Q

ans

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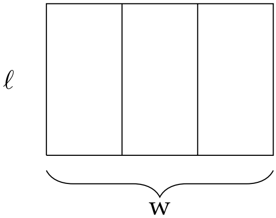

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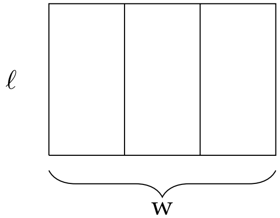

The area is $A = \ell w$, and the amount of fencing gives us $4\ell + 2w = 1000$. Then $w = 500 - 2\ell$, so

$$A(\ell) = (500 - 2\ell)\ell = 500\ell - 2\ell^2 \qquad A'(\ell) = 500 - 4\ell$$

We differentiate to find CPs, and the maximum of A over the domain $0 < \ell < 250$ is at $\ell = 125$, which gives $w = 250$ and $A = (125)(250)$ sq metres.

Suppose you want to make a rectangle with perimeter 400. What dimensions give you the maximum area?

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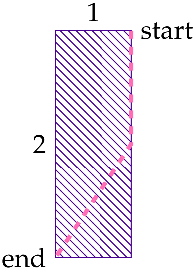
A 100-by-100 square.

You are standing on the bank of a river that is 1km wide, and you want to reach the opposite side, two km down the river. You can paddle 3 kilometres per hour, and walk 6 kph while carrying your boat. What route takes you to your desired destination in the least amount of time?



ans

Let $2-x$ be the distance you travel carrying the boat, and then $\sqrt{x^2+1}$ is the distance you row. The time it takes is given by

$$T(x)=\frac{1}{6}(2-x)+\frac{1}{3}\sqrt{x^2+1}$$

By differentiating,

$$T'(x)=-\frac{1}{6}+\frac{1}{3}\frac{x}{\sqrt{x^2+1}}$$

we see the only critical point for positive x is $x=\frac{1}{\sqrt{3}}$. This is the global minimum, so the fastest route is to portage (carry your boat) for $2-\frac{1}{\sqrt{3}}$ kilometres, then row the rest of the way.

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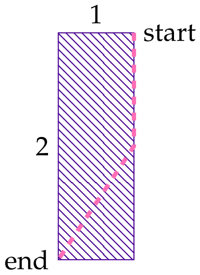

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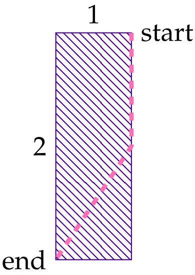
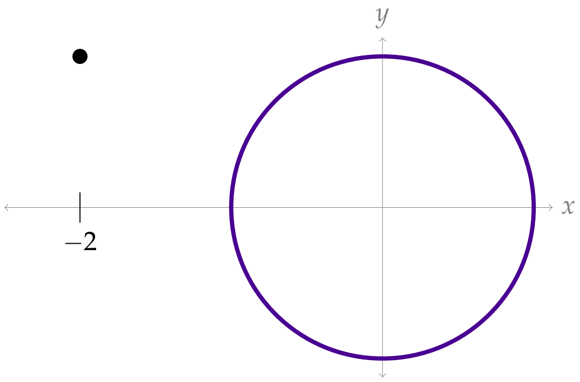


Paddle the whole way: it's faster, and more direct.

Let C be the circle given by $x^2 + y^2 = 1$. What is the closest point on C to the point $(-2, 1)$?



Q

ans

We can reasonably see that the y-coordinate will be positive, and the x-coordinate negative. For any point (x, y) on C, the distance D to $(-2, 1)$ is determined by:

$$\begin{aligned} D^2 &= \left(x-(-2)\right)^2 + (y-1)^2 \\ &= x^2 + 4x + 4 + \left(\sqrt{1-x^2}-1\right)^2 \\ &= x^2 + 4x + 4 + (1-x^2) - 2\sqrt{1-x^2} + 1 \\ &= 4x + 6 - 2\sqrt{1-x^2} \end{aligned}$$

where this only makes sense for x between -1 and 1. Minimizing D is the same as minimizing D^2; so we take the derivative of the above function:

$$\frac{\mathrm{d}}{\mathrm{d}x} D(x)^2 = 4 - 2\frac{-2x}{2\sqrt{1-x^2}} = 4 + \frac{2x}{\sqrt{1-x^2}}$$

There are singular points at the endpoints $x = \pm 1$, and a critical point at $x = -\sqrt{4/5}$. Since $D(-1)^2 = 2$, $D(1)^2 = 10$, and $D(-\sqrt{4/5})^2 = 6 - \frac{10}{\sqrt{5}} \approx 1.53$, the closest point is $\boxed{(-\sqrt{4/5}, \sqrt{1/5})}$.

Suppose you want to manufacture a closed cylindrical can on the cheap. If the can should have a volume of one litre (1000 cm^3), what is the smallest surface area it can have?

Q

Suppose you want to manufacture a closed cylindrical can on the cheap. If the can should have a volume of one litre ($1000\ \text{cm}^3$), what is the smallest surface area it can have?

Let the can have radius $r > 0$ and height $h > 0$. Then its volume is $1000 = V = \pi r^2 h$, and its surface area is $2(\pi r^2) + (2\pi r)h$.

Using the volume equation, we know $h = \frac{1000}{\pi r^2}$, so for surface area, this gives us

$$S(r) = 2\pi r^2 + 2\pi r\frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

$$S'(r) = 4\pi r - \frac{2000}{r^2} = 4\pi r\left(1 - \frac{500}{\pi r^3}\right)$$

The singular point $r = 0$ is not in the domain of S; the only critical point is $r = \sqrt[3]{\frac{500}{\pi}}$. We see that, for r smaller than the CP, $S'(r) < 0$ so that $S(r)$ is decreasing, and that, for r larger than the CP, $S'(r) > 0$ so that $S(r)$ is increasing. So the minimum of $S(r)$ occurs at $r = \sqrt[3]{\frac{500}{\pi}}$ and

$$\text{minimum surface area } = S\left(\sqrt[3]{\frac{500}{\pi}}\right) = 2\pi\left(\tfrac{500}{\pi}\right)^{2/3} + 2000\left(\tfrac{\pi}{500}\right)^{1/3}$$

A cylindrical can is to hold 20π cubic metres. The material for the top and bottom costs \$10 per square metre, and material for the side costs \$8 per square metre. Find the radius r and height h of the most economical can.

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The cost is $C = 8(2\pi rh) + 2 \cdot 10 \cdot \pi r^2$. Since $20\pi = V = \pi r^2 h$, we see $rh = \frac{20}{r}$, so

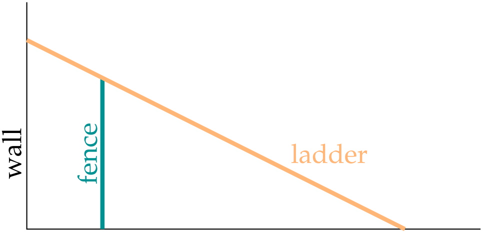
$$C(r) = 16\pi\frac{20}{r} + 20\pi r^2$$

As

$$C'(r) = 20\pi\left(-\frac{16}{r^2} + 2r\right)$$

$C(r)$ is minimized at $r = 2$, and hence $h = 5$.

Suppose a 2-metre high fence stands 1 metre away from a high wall. What is the shortest ladder that will reach over the fence to the wall?



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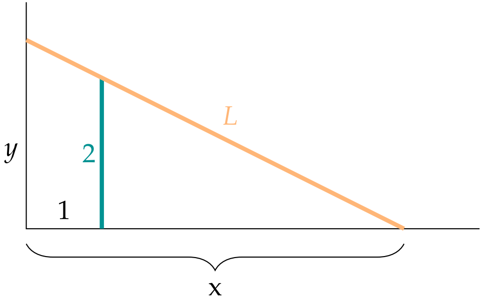


Q

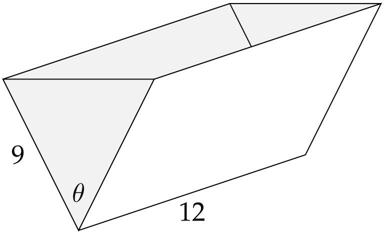
ans

To get y in terms of x, we notice similar triangles tell us $\frac{y}{x} = \frac{2}{x-1}$, hence $y = \frac{2x}{x-1}$. The values of x that make sense are $(1, \infty)$. Noting $L^2 = x^2 + y^2 = x^2 + \frac{4x^2}{(x-1)^2}$, we see L^2 gets very large as x gets close to 1 or as x gets very large. As

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} L(x)^2 &= 2x + \frac{8x}{(x-1)^2} - \frac{8x^2}{(x-1)^3} \\
&= 2x(x-1)^3 \left\{ (x-1)^3 + 4(x-1) - 4x \right\} \\
&= 2x(x-1)^3 \left\{ (x-1)^3 - 4 \right\}
\end{aligned}$$

the only critical point is at $x = \sqrt[3]{4} + 1$, and this gives us our minimum $L = \sqrt{(\sqrt[3]{4}+1)^2 + \frac{4(\sqrt[3]{4}+1)^2}{\sqrt[3]{4}^2}}$.

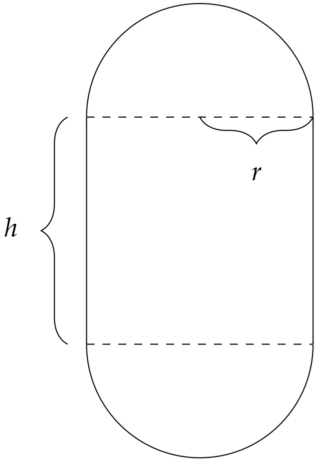
Suppose a file folder is 12 inches long and 9 inches wide. You want to make a box by opening the folder and capping the ends. What angle should you open the folder to, to make the box with the greatest volume?



Q

The volume of a shape like this is the area of the triangular base, times the height, which is 12. So we maximize the volume by maximizing the area of the triangle with two sides of length 9, and angle between them θ.
This triangle has height $h = 9\cos\frac{\theta}{2}$ and base $b = 2 \times 9\sin\frac{\theta}{2}$ and so has area $\frac{1}{2}bh = 9^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{9^2}{2}\sin\theta$.
The maximum occurs at $\pi/2$. So, you want the folder to make a right angle.

We want to bend a piece of wire into the perimeter of the shape shown below: a rectangle of height h and width $2r$, with a half circle of radius r on the top and bottom.



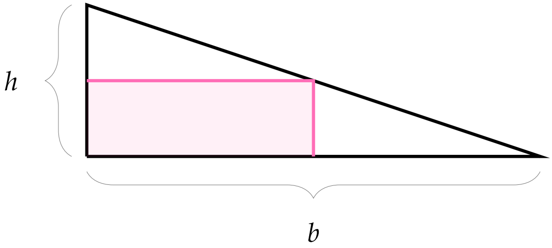
If you only have 100cm of wire, what values of r and h give the largest enclosed area?

The perimeter is $100 = 2\pi r + 2h$, so $h = 50 - \pi r$.
The area is $\pi r^2 + 2rh = \pi r^2 + 2r(50 - \pi r) = 100r - \pi r^2 = A(r)$.

As $A'(r) = 100 - 2\pi r$, the maximum of $A(r)$ occurs at $r = \frac{50}{\pi}$. Then $h = 0$. That is: there's no rectangular part, only a circle.

Circles are somehow more efficient at storing space than rectangles. More precisely, we mean that the ratio of area to perimeter is higher in circles than in rectangles. So the most efficient way to make the shape is to have the rectangular part have height 0.

Suppose we take a right triangle, with height h and base b. We inscribe a rectangle in it that shares a right angle, as shown below. What are the dimensions of the rectangle with the biggest area?



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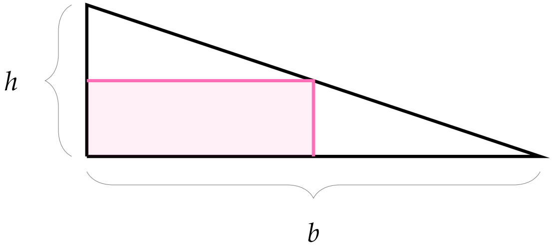
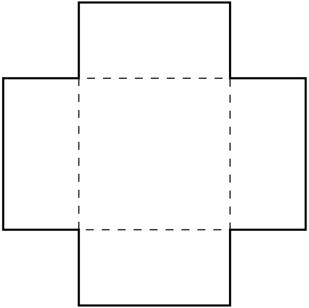


$b/2$ by $h/2$.

ACTIVITY

By cutting out squares from the corners, turn a piece of paper into an open-topped box that holds a lot of beans.



Q

ans

Included Work

Belk, J. (13 April 2014). *Bad Optimization Problems. I thought that Jack M made an interesting comment about this question*. [Comment on the online forum post *Optimization problems that today's students might actually encounter?*]. Stackexchange. https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter (accessed October 2019 or earlier), 6

 'Weight' by Bakunetsu Kaito is licensed under CC BY 3.0 (accessed 16 July 2021) , 4

Author unknown. Optimization Problems. (2006). *Drexel University Department of Mathematics, Calculus I Home Page Spring 2006, Calc 1 Spring lecture 6.* https://www.math.drexel.edu/%7Ejwd25/CALC1_SPRING_06/lectures/lecture9.html (accessed October 2019 or earlier), 4

Vectornaut. (10 May 2015). *When someone swallows a dose of a drug, it doesn't go into their bloodstream all at once.* [Comment on the online forum post *Optimization problems that today's students might actually encounter?*]. Stackexchange. https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter (accessed October 2019 or earlier), 5