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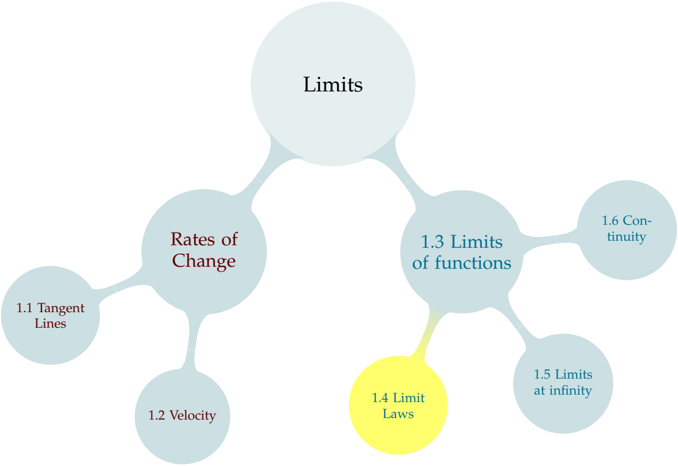

Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

$$\lim_{x\to a} f(x) = f(a).$$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x+3}\right)$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x\to a}(f(x) + g(x)) = F + G$
- $\lim_{x\to a}(f(x) - g(x)) = F - G$
- $\lim_{x\to a}(f(x)g(x)) = FG$
- $\lim_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

Calculate: $\lim_{x\to 1}\left[\frac{2x+4}{x+2} + 13\left(\frac{x+5}{3x}\right)\left(\frac{x^2}{2x-1}\right)\right]$

LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

Limits involving Powers and Roots

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Raise your hand if you think A is real / not; Raise your hand if you think B is real/not; etc.Ask students to turn to their neighbours and describe a rule for when A^B is real, and when it is not.

Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x\to a} f(x) = F$ (where F is a real number), then:

$$\lim_{x\to a} (f(x))^n = F^n.$$

Furthermore, unless n is even and F is negative,

$$\lim_{x\to a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x\to 4}(x+5)^{1/2}$$

Powers of Limits – Theorem 1.4.8

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Takeaway: when calculating limits, you can start by trying to "plug in." But, the MINUTE you divide by zero, or see 0/0, or if infinity shows up anywhere, YOU NEED TO DO SOMETHING ELSE

Cautionary Tales

- $\lim_{x \to 0} \dfrac{(5+x)^2 - 25}{x}$
- $\lim_{x \to 3} \left(\dfrac{x-6}{3}\right)^{1/8}$
- $\lim_{x \to 0} \dfrac{32}{x}$
- $\lim_{x \to 5} \left(x^2+2\right)^{1/3}$

Suppose you want to evaluate $\lim_{x\to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x\to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x\to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x\to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x\to 1} f(x)$ does not exist.

E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

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We're identifying things that make limits harder to find. A limit being hard to find is not the same as the limit not existing, it just means you have to look harder. In a moment, we'll talk about what to do in these cases.

Which of the following statements is true about $\lim_{x \to 0} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \to 0} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 0}{0^3 - 0^2 + 0} = \dfrac{0}{0}$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about $\lim_{x\to 1} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x\to 1} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

C Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

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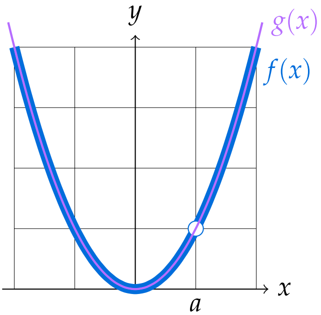
We're identifying things that make limits harder to find. A limit being hard to find is not the same as the limit not existing. We'll talk now about more things you can do to evaluate a limit in these trickier situations.

Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim_{x \to a} g(x)$ exists, and $f(x) = g(x)$ when x is close to a (but not necessarily equal to a).

Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.



Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 + x^2 - x - 1}{x - 1}$.

Evaluate $\lim\limits_{x \to 5} \dfrac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

Verbally: Separate piecewise functions from everything else. It's likely that the only times they'll see functions with discontinuities inside their domains, they will be written piecewise.

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x \to 0} \frac{x+7}{\frac{1}{x} - \frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

Denominators Approaching Zero

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

$$\lim_{x\to 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{1}{x-1}$$

$$\lim_{x\to 1^+} \frac{1}{x-1}$$

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$\lim_{x \to 1^+} \frac{1}{x-1}$

Good to remind students about division at this point. A small number goes into any number lots of times: if I have one cake, and I cut it into tiny pieces, I get a lot of pieces. They often think these limits have some kind of unknowable magic to them, so it's good to bring them back to a place where things make intuitive sense.

Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4}$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2}$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \le g(x) \le h(x)$$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. Then $\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

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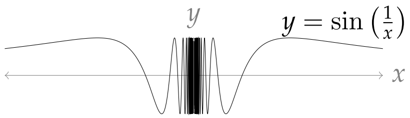
and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

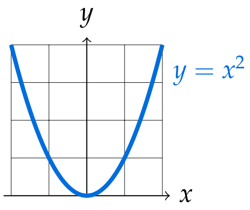
$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Let's start by graphing the function

Evaluate:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$





$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \quad \leq \quad \sin\left(\frac{1}{x}\right) \quad \leq \quad 1$$

Included Work

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