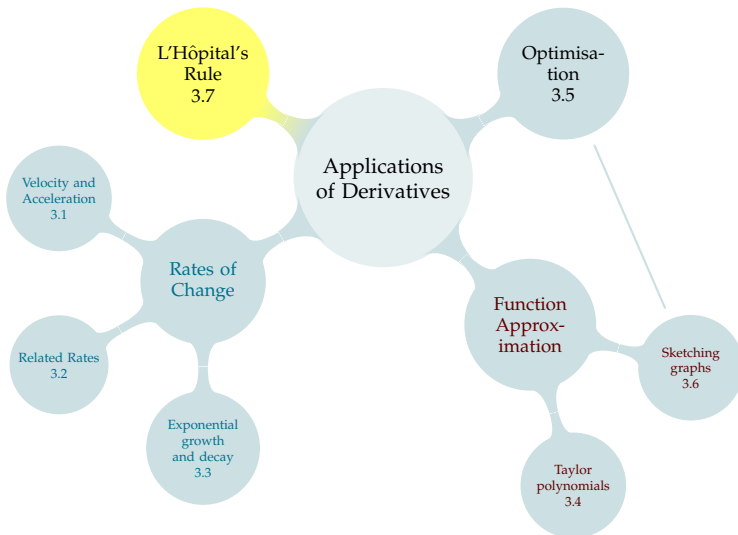


# TABLE OF CONTENTS



# BACK TO LIMITS!

$$\lim_{x \rightarrow \infty} \frac{x^2}{5}$$

$$\lim_{x \rightarrow \infty} \frac{5}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{x^2}{5}$$

$$\lim_{x \rightarrow 0} \frac{5}{x^2}$$

## Indeterminate Forms – Definition 3.7.1

Suppose  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ . Then the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

is an **indeterminate form** of the type  $\frac{0}{0}$ .

Suppose  $\lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} G(x) = \infty$  (or  $-\infty$ ). Then the limit

$$\lim_{x \rightarrow a} \frac{F(x)}{G(x)}$$

is an **indeterminate form** of the type  $\frac{\infty}{\infty}$ .

**When you see an indeterminate form, you need to do more work.**

### 3.7: L'Hôpital's Rule and Indeterminate Forms

Back to Limits!

#### BACK TO LIMITS!

$$\lim_{x \rightarrow 0} \frac{x^2}{e^x} \quad \lim_{x \rightarrow \infty} \frac{5}{x^2} \quad \lim_{x \rightarrow 0} \frac{x^2}{e^x} \quad \lim_{x \rightarrow \infty} \frac{5}{x^2}$$

#### Indeterminate Forms - Definition 3.7.1

Suppose  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ . Then the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

is an *indeterminate form* of the type  $\frac{0}{0}$ .

Suppose  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$  (or  $-\infty$ ). Then the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

is an *indeterminate form* of the type  $\frac{\infty}{\infty}$ .

When you see an indeterminate form, you need to do more work.

Often students misinterpret the phrase “indeterminate” as “unknowable,” so I like to emphasize the “more work needed” part. This is one of two common mistakes, the other being the assumption that  $n/n = 1$  even when  $n = 0$  or  $n$  is not a number.

# INDETERMINATE FORMS

$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$$

indeterminate form of the type  $\frac{0}{0}$

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 4x + 2}{8x^2 - 5}$$

indeterminate form of the type  $\frac{\infty}{\infty}$

# INDETERMINATE FORMS AND THE DERIVATIVE

$$\lim_{x \rightarrow 0} \frac{3 \sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type  $\frac{0}{0}$

## L'Hôpital's Rule: First Part – Theorem 3.7.2

Let  $f$  and  $g$  be functions such that  $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$ .

If  $f'(a)$  and  $g'(a)$  exist and  $g'(a) \neq 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ .

If  $f$  and  $g$  are differentiable on an open interval containing  $a$ , and if  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  exists, then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .

This works even for  $a = \pm\infty$ .

**Extremely Important Note:**  
L'Hôpital's Rule only works on indeterminate forms.

## L'Hôpital's Rule: Second Part – Theorem 3.7.2

Let  $f$  and  $g$  be functions such that  $\lim_{x \rightarrow a} f(x) = \infty = \lim_{x \rightarrow a} g(x)$ .

If  $f'(a)$  and  $g'(a)$  exist and  $g'(a) \neq 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ .

If  $f$  and  $g$  are differentiable on an open interval containing  $a$ , and if  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  exists, then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .

This works even for  $a = \pm\infty$ .

**Extremely Important Note:**  
L'Hôpital's Rule only works on indeterminate forms.

Evaluate:

$$\lim_{x \rightarrow 2} \frac{3x \tan(x - 2)}{x - 2}$$



## LITTLE HARDER

$$\lim_{x \rightarrow 0} \frac{x^4}{e^x - \cos x - x}$$

indeterminate form of the type  $\frac{0}{0}$

Evaluate:

$$\lim_{x \rightarrow \infty} \frac{\log x}{\sqrt{x}}$$

# OTHER INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} e^{-x} \log x$$

form  $0 \cdot \infty$

## VOTE VOTE VOTE

Which of the following can you immediately apply L'Hôpital's rule to?

A.  $\frac{e^x}{2e^x + 1}$

B.  $\lim_{x \rightarrow 0} \frac{e^x}{2e^x + 1}$

C.  $\lim_{x \rightarrow \infty} \frac{e^x}{2e^x + 1}$

D.  $\lim_{x \rightarrow \infty} e^{-x}(2e^x + 1)$

E.  $\lim_{x \rightarrow 0} \frac{e^x}{x^2}$

## VOTEY McVOTEFACE

Suppose you want to use L'Hôpital's rule to evaluate  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ , which has the form  $\frac{0}{0}$ . How does the quotient rule fit into this problem?

- A. You should use the quotient rule because the function you are differentiating is a quotient.
- B. You will not use the quotient rule because you differentiate the numerator and the denominator separately
- C. You may use the quotient rule because perhaps  $f(x)$  or  $g(x)$  is itself in the form of a quotient
- D. You will not use L'Hôpital's rule because  $\frac{0}{0}$  is not an appropriate indeterminate form
- E. You will not use L'Hôpital's rule because, since the top has limit zero, the whole function has limit 0

# MORE QUESTIONS

Which of the following is NOT an indeterminate form?

- A.  $\frac{\infty}{\infty}$  for example,  $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$
- B.  $\frac{0}{0}$  for example,  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$
- C.  $\frac{0}{\infty}$  for example,  $\lim_{x \rightarrow 0^+} \frac{x}{\log x}$
- D.  $0 \cdot \infty$  for example,  $\lim_{x \rightarrow \infty} x(\arctan(x) - \pi/2)$
- E. all of the above are indeterminate forms

# I HAVE SO MANY QUESTIONS

Which of the following is NOT an indeterminate form?

A.  $1^\infty$  for example,  $\lim_{x \rightarrow \infty} \left( \frac{x+1}{x} \right)^x$

B.  $0^\infty$  for example,  $\lim_{x \rightarrow \infty} \left( \frac{1}{x} \right)^x$

C.  $\infty^0$  for example,  $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$

D.  $0^0$  for example,  $\lim_{x \rightarrow 0^+} x^x$

E. all of the above are indeterminate forms

F. none of the above are indeterminate forms

### 3.7: L'Hôpital's Rule and Indeterminate Forms

I have so many questions

$1^\infty$  definitely takes some explaining

#### I HAVE SO MANY QUESTIONS

Which of the following is NOT an indeterminate form?

- A.  $1^\infty$  for example,  $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x}\right)^x$
- B.  $0^\infty$  for example,  $\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)^x$
- C.  $\infty^0$  for example,  $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$
- D.  $0^0$  for example,  $\lim_{x \rightarrow 0^+} x^x$
- E. all of the above are indeterminate forms
- F. none of the above are indeterminate forms



# EXPONENTIAL INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} x^{1/x}$$

# EXPONENTIAL INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{3x}$$

Evaluate:

$$\lim_{x \rightarrow \infty} \frac{\log x}{\log \sqrt{x}}$$

$$\lim_{x \rightarrow \infty} (\log x)^{\sqrt{x}}$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x}$$

# MORE EXAMPLES

$$\lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} - \sqrt{x^2 + x}$$

$$\lim_{x \rightarrow 0} \sqrt[x^2]{\sin^2 x}$$

$$\lim_{x \rightarrow 0} \sqrt[x^2]{\cos x}$$

Sketch the graph of  $f(x) = x \log x$ .

Note: when you want to know  $\lim_{x \rightarrow 0} f(x)$ , you'll need to use L'Hôpital.

Evaluate  $\lim_{x \rightarrow 0^+} (\csc x)^x$

