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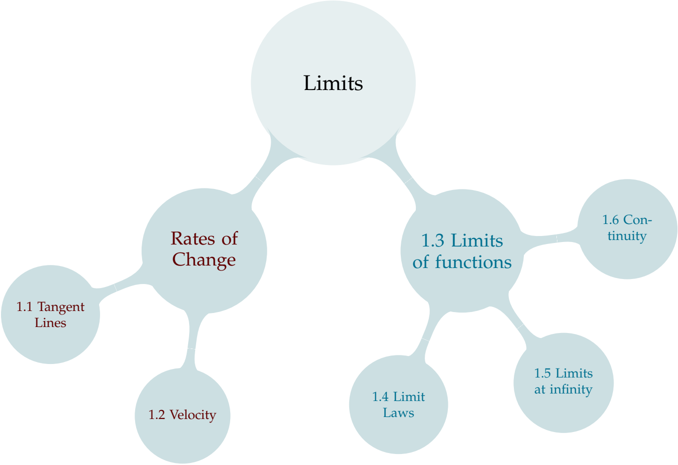
## 1.7 (Optional) Making the Informal a Little More Formal



Now that we've seen the limits of functions as $x$ goes to positive and negative infinity, let's look at limits as $x$ approaches a real number.

## └─1.7 (Optional) Making the Informal a Little More Formal

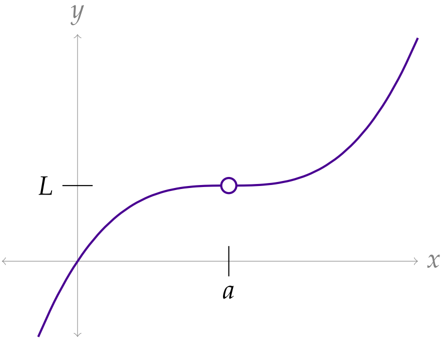
The actual computations for limits as $x$ goes to infinity are generally easier, so I like to teach 1.8 before 1.7.

A lot of the same language from the canning analogy can be re-used here: $\epsilon$ as error, for instance.

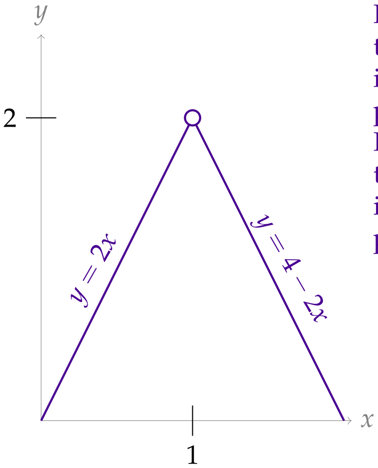
Now that we've seen the limits of functions as $x$ goes to positive and negative infinity, let's look at limits as $x$ approaches a real number.

$$\lim_{x \to a} f(x) = L$$

Informally: If $x$ is close enough (but not equal to) $a$, then $y$ is close enough to $L$.



Let $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$. Then $\lim_{x \to 1} |x| = 2$.




Find a positive number $\delta$ such that $|f(x) - 2| < \frac{1}{2}$ for all $x$ in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

Find a positive number $\delta$ such that $|f(x) - 2| < \frac{1}{4}$ for all $x$ in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

## Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of $a$, except possibly at $a$. We say that

*the limit as x approaches a of $f(x)$ is L*

and write

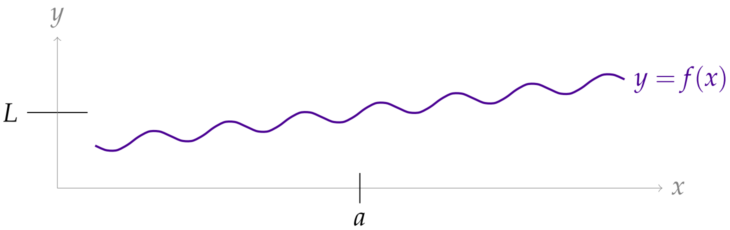
$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Note that an equivalent way of writing this very last statement is

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

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## Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of $a$, except possibly at $a$. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that $\lim_{x \to -1} |x + 1| = 0$.

## Definition 1.7.1

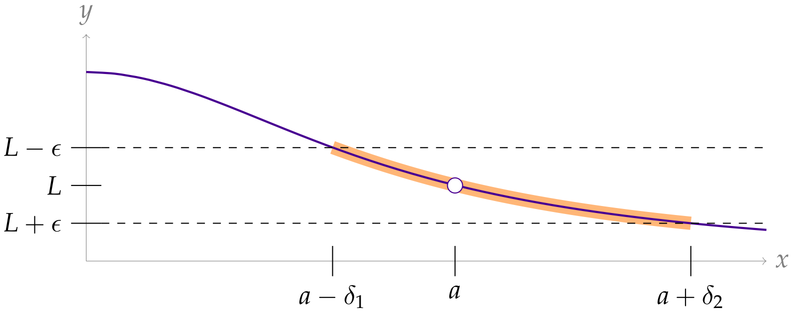
Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of $a$, except possibly at $a$. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Let $f(x) = \begin{cases} x + 1 & x < 0 \\ 1 - x^2 & x > 0 \end{cases}$.

Using Definition 1.7.1, prove that $\lim_{x \to 0} f(x) = 1$.

# General Principles

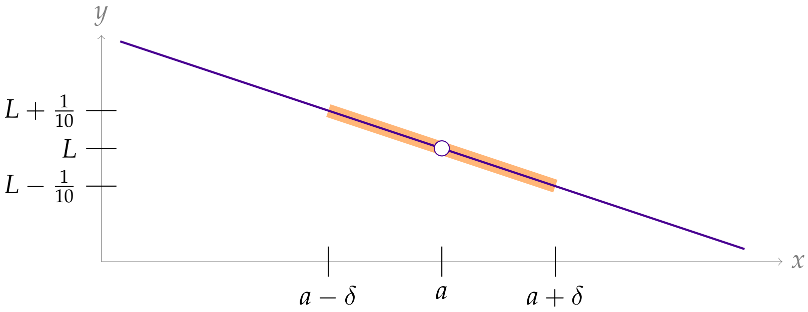
Suppose $|f(x) - L| < \epsilon$ whenever $a - \delta_1 < x < a$ and whenever $a < x < a + \delta_2$.




Consider values of $x$ such that $0 < |x - a| < \min\{\delta_1, \delta_2\}$.

# General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all $x$ such that $0 < |x - a| < \delta$.

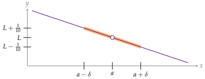



Can you give values of $x$ where $|f(x) - L| < \frac{1}{5}$?

skip $\epsilon$ small

GENERAL PRINCIPLES

Suppose $|f(x) - L| < \frac{1}{10}$ for all $x$ such that $0 < |x - a| < \delta$.




Can you give values of $x$ where $|f(x) - L| < \frac{1}{5}$?

skip ε small

WLOG prove only for small epsilon. This doesn't really come up so often at this level, which is why there's a skip button.

# General Principles

**Definition 1.7.1**

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of $a$, except possibly at $a$. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

It is enough to show that for every $\epsilon$ such that $0 < \epsilon < c$ (where $c$ is some constant) there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

That means it doesn't hurt your proof if you say something like "we assume $\epsilon < 1$".

In a previous example, we chose

$$\delta = \min\{\epsilon, \sqrt{\epsilon}\}$$

It would be OK to say "we can assume $\epsilon < 1$; set $\delta = \epsilon$."

## Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of $a$, except possibly at $a$. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that $\lim_{x \to 2} \dfrac{x-2}{x^2-4} = \dfrac{1}{4}$.

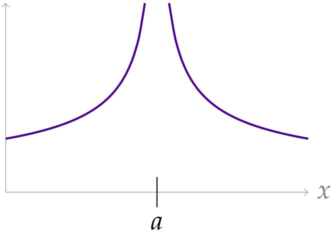
# Infinite Limits

## Definition 1.8.1 (b)

Let $a$ be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every $P > 0$ there exists $\delta > 0$ so that $f(x) > P$ whenever $0 < |x - a| < \delta$.



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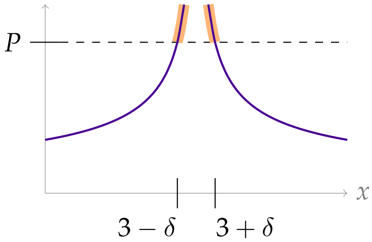
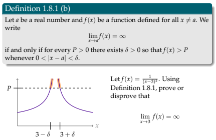



Let $f(x) = \frac{1}{(x-3)^2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \to 3} f(x) = \infty$$




The generic picture is kept on the left, but it's nice to mention that it is, indeed, generic. In particular, the picture won't fit for $f(x) = \frac{1}{x-2}$.

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Let $a$ be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

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Let $f(x) = \frac{1}{x-2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \to 2} f(x) = \infty$$

**Definition 1.8.1 (b)**

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Let $f(x) = \frac{1}{x-2}$. Using Definition 1.8.1, prove or disprove that

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The generic picture is kept on the left, but it's nice to mention that it is, indeed, generic. In particular, the picture won't fit for $f(x) = \frac{1}{x-2}$.