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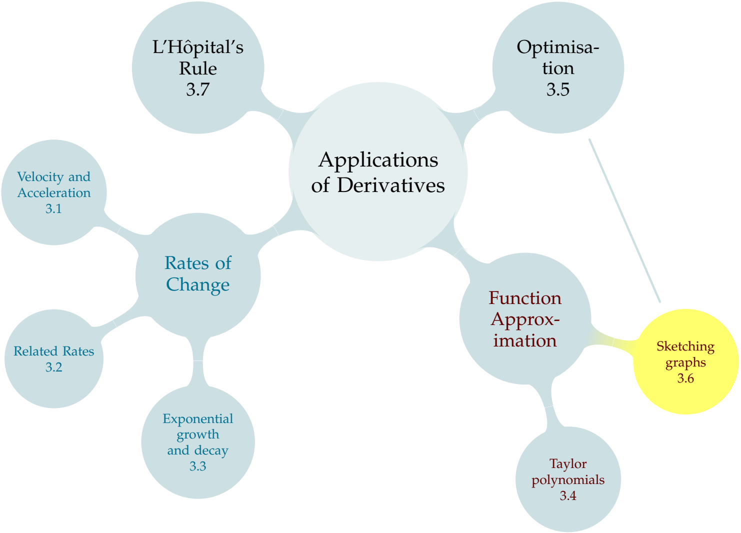
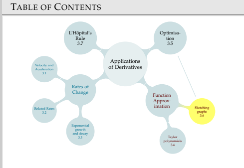




It can be very frustrating to learn how to solve a problem only partially, but this is what we do in this section. There are many pieces to curve sketching, and by focusing on one at a time, we teach students how to start sketching before we teach them how to finish sketching. It can be helpful in class to remind students that the remaining pieces are going to be taught soon.

CURVE SKETCHING

Review: find the domain of the following function.

$$f(x) = \frac{\sqrt{3-x^2}}{\log(x+1)}$$

Where might you expect $f(x)$ to have a vertical asymptote? What does the function look like nearby?
(Recall: a vertical asymptote occurs at $x = a$ if the function has an infinite discontinuity at a. That is, $\lim\limits_{x\to a^{\pm}} f(x) = \pm\infty$.)

Where is $f(x) = 0$?

What happens to $f(x)$ near its other endpoint, $x = -1$?

Curve Sketching

Good things to check:

- Domain
- Vertical asymptotes: $\lim_{x \to a} f(x) = \pm\infty$
- Intercepts: $x = 0, f(x) = 0$
- Horizontal asymptotes and end behavior: $\lim_{x \to \pm\infty} f(x)$

Curve Sketching

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

Curve Sketching

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

Graphing takes many steps, and it's often frustrating to see a final result without knowing how to get there. So when I show graphs that we haven't generated in class, I like to add information like: we will learn how to do this part later; look at how the intercepts are where we expect; look at the horizontal asymptotes that we predicted; etc.

Curve Sketching

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

First Derivative

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

Signs of Factored Functions

$$f(x) = \quad (x-1) \quad (x-2)^2 \quad (x-3)$$

$$\longleftrightarrow x$$

Signs of Factored Functions

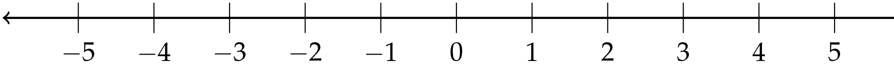
$$f(x) = (x-1)\quad(x-2)^2\quad(x-3)$$

Students often learned in high school that they should test points between consecutive roots, which is much slower.

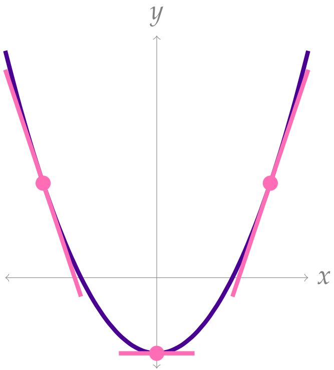
Signs of Factored Functions

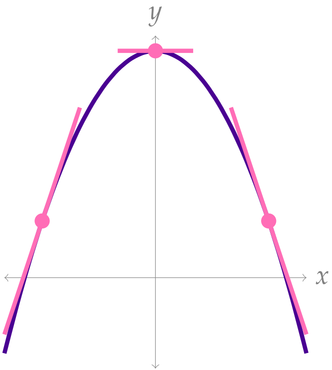
$$f(x) = (x-3)(x-1)^2 x(x+2)^3(x+5)^4$$

Where is $f(x)$ positive? Where is it negative?



Concavity



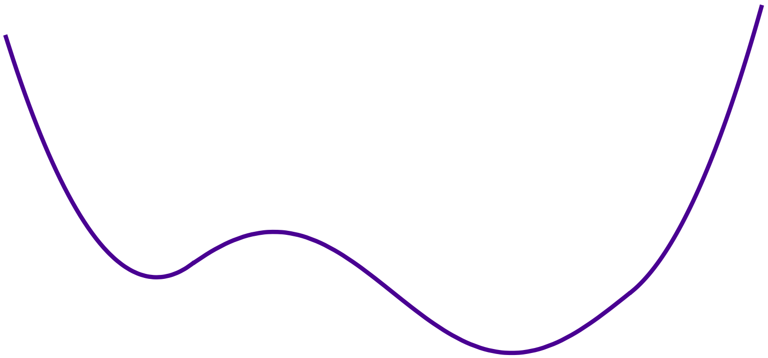


- Slopes are increasing
- $f''(x) > 0$
- "concave up"
- tangent line below curve

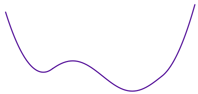
- Slopes are decreasing
- $f''(x) < 0$
- "concave down"
- tangent line above curve

MNEMONIC

CONCAVITY

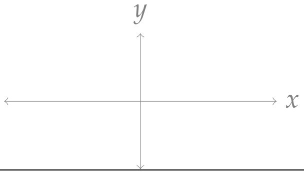
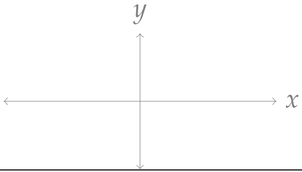
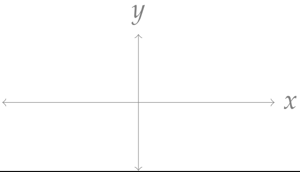
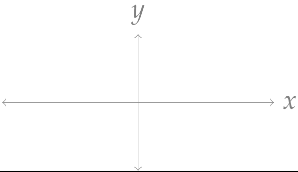


Concavity



"Just by staring at it, decide where this function is concave up, and where it is concave down."

Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

It is a very common misconception that e.g. concave up functions are also increasing. Be wary of conflating first and second derivatives.

Poll Questions

Describe the concavity of the function $f(x) = e^x$.

A. concave up
B. concave down
C. concave up for $x < 0$; concave down for $x > 0$
D. concave down for $x < 0$; concave up for $x > 0$
E. I'm not sure

Is it possible to be concave up and decreasing?

A. Yes B. No C. I'm not sure

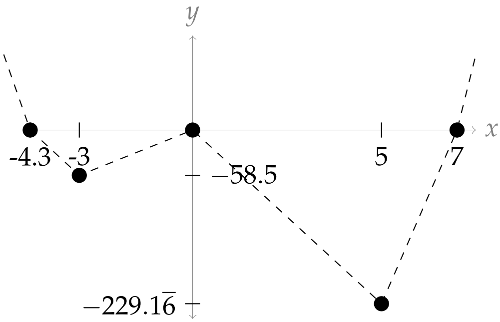
Suppose a function $f(x)$ is defined for all real numbers, and is concave up on the interval $[0, 1]$. Which of the following must be true?

A. $f'(0) < f'(1)$
B. $f'(0) > f'(1)$
C. $f'(0)$ is positive
D. $f'(0)$ is negative
E. I'm not sure

Revisiting a previous example

original example

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

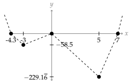


$$f''(x) = 6x^2 - 8x - 30 = 2(x-3)(3x+5)$$

Revisiting a previous example

original example

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



$f''(x) = 6x^2 - 8x - 30 = 2(x-3)(3x+5)$

Rather than start from scratch, here's an example we already have a skeleton for. The second derivative factors nicely, so we can quickly see how to add concavity to our sketches.

Sketch:

$$f(x) = x^5 - 15x^3$$

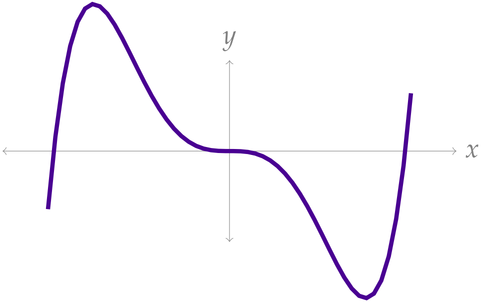
3.6.3: Concavity

Sketch:

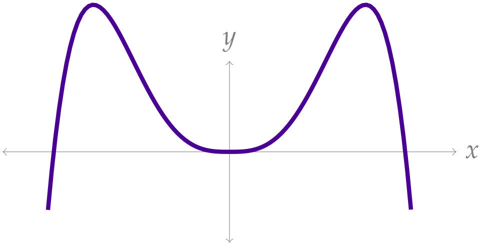
$$f(x) = x^5 - 15x^3$$

Mention symmetry, to motivate next subsection.

Even and Odd Functions



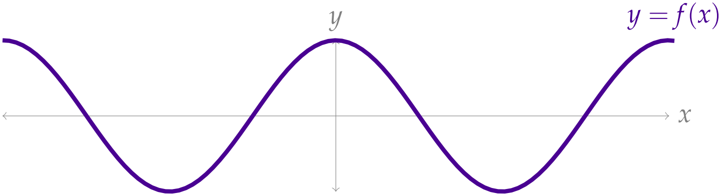
Even and Odd Functions



Even Function – Definition 3.6.5

A function $f(x)$ is even if, for all x in its domain,

$$f(-x) = f(x)$$



EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is even if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:

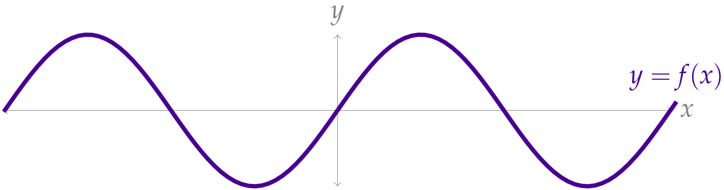
$f(x) = x^2$

$f(x) = x^4$

$f(x) = \cos(x)$

$$f(x) = \frac{x^4 + \cos(x)}{x^{16} + 7}$$

Odd Functions



Suppose $f(1) = 2$. Then $f(-1) =$
Suppose $f(3) = -2$. Then $f(-3) =$

Odd Function – Definition 3.6.6

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Odd Functions

Odd Function – Definition 3.6.6

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$f(x) = x$

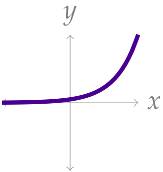
$f(x) = x^3$

$f(x) = \sin(x)$

$f(x) = \dfrac{x(1+x^2)}{x^2+5}$

Poll Tiiime

Pick out the odd function.

A:


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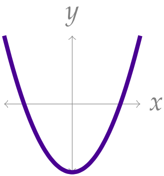
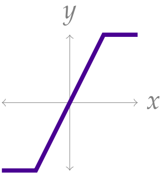


C:


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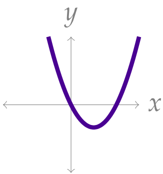
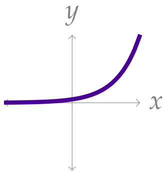
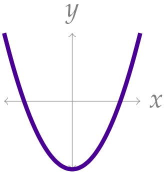
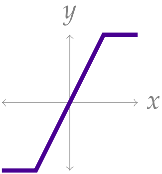


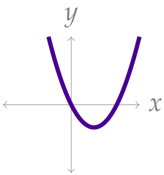
POLL TIIIME

Pick out the even function.

A: 

B: 

C: 

D: 

Even more Poll tiiiiime

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

Even more and more Poll tiiiiime

Suppose $f(x)$ is an even function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

OK OK... LAST ONE

Suppose $f(x)$ is an even function, differentiable for all real numbers. What can we say about $f'(x)$?

A. $f'(x)$ is also even

B. $f'(x)$ is odd

C. $f'(x)$ is constant

D. all of the above are true

E. none of the above are necessarily true

Periodicity

Periodic – Definition 3.6.9

A function is periodic with period $P > 0$ if

$$f(x) = f(x+P)$$

whenever x and $x+P$ are in the domain of f, and P is the smallest such (positive) number

Examples: $\sin(x)$, $\cos(x)$ both have period 2π; $\tan(x)$ has period π.

Ignoring concavity, sketch $f(x) = \sin(\sin x)$.

Challenge: ignoring exact locations of extrema, sketch $g(x) = \sin(2\pi \sin x)$.

3.6.4 : Symmetries

Ignoring concavity, sketch $f(x) = \sin(\sin x)$.

Challenge: ignoring exact locations of extrema, sketch $g(x) = \sin(2\pi \sin x)$.

The second one really is quite tough; you can let the faster students work on it while the rest of the class is working on $f(x)$. Then the solution is more-or-less in the slides if they want to look, with no need to do it in class.

Let's Graph

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2-64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2 + 64)}{3(x^2-64)^{5/3}}$$

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2-64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2+64)}{3(x^2-64)^{5/3}}$$

Takes too much time in class to differentiate, so derivatives given

Let's Graph

$$f(x) = \frac{x^2 + x}{(x+1)(x^2+1)^2}$$

Note: for $x \neq -1, f(x) = \dfrac{x(x+1)}{(x+1)(x^2+1)^2} = \dfrac{x}{(x^2+1)^2}$

$$g(x) := \frac{x}{(x^2+1)^2}$$

$$g'(x) = \frac{1-3x^2}{(x^2+1)^3}$$

$$g''(x) = \frac{12x(x^2-1)}{(x^2+1)^4}$$

LET'S GRAPH

$$f(x) = x(x-1)^{2/3}$$

- $f'(x) = \dfrac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \dfrac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

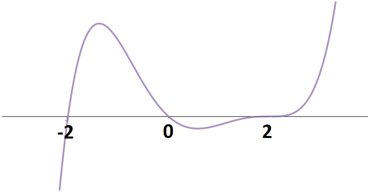
- $f(3/5) \approx 0.3$
- $f(6/5) \approx 0.4$

Ch 3.6 Review: matching

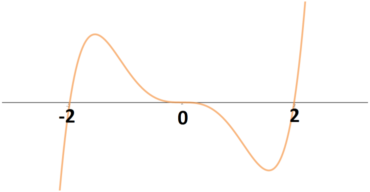
Matching (as opposed to sketching) is a nice way to review specific ideas, like when a function changes signs.

MATCH THE FUNCTION TO ITS GRAPH

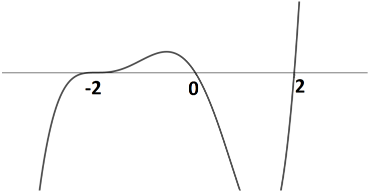
A. $f(x) = x^3(x+2)(x-2) = x^5 - 4x^3$
B. $f(x) = x(x+2)^3(x-2) = x^5 + 4x^4 - 16x^2 - 16x$
C. $f(x) = x(x+2)(x-2)^3 = x^5 - 4x^4 + 16x^2 - 16x$



I



III



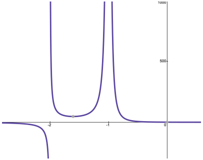
II

A. $f(x) = \dfrac{x-1}{(x+1)(x+2)}$

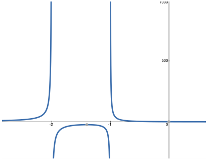
B. $f(x) = \dfrac{(x-1)^2}{(x+1)(x+2)}$

C. $f(x) = \dfrac{x-1}{(x+1)^2(x+2)}$

D. $f(x) = \dfrac{(x-1)^2}{(x+1)^2(x+2)}$



I.



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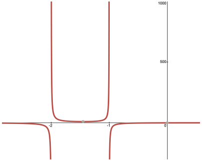
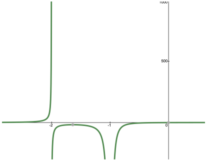


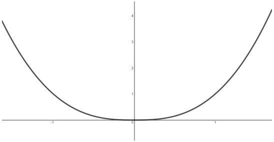
II.



IV.

Match the Function to its Graph

A. $f(x) = |x|^e$ B. $f(x) = e^{|x|}$ C. $f(x) = e^{x^2}$ D. $f(x) = e^{x^4 - x}$



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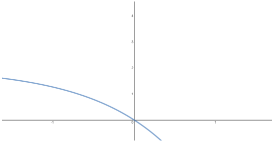
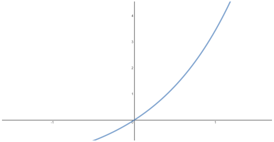
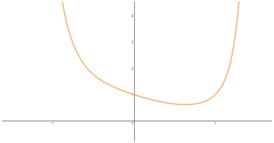


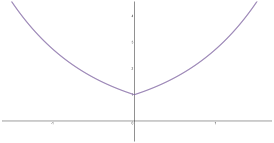
II



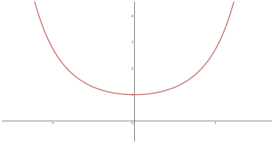
III



IV

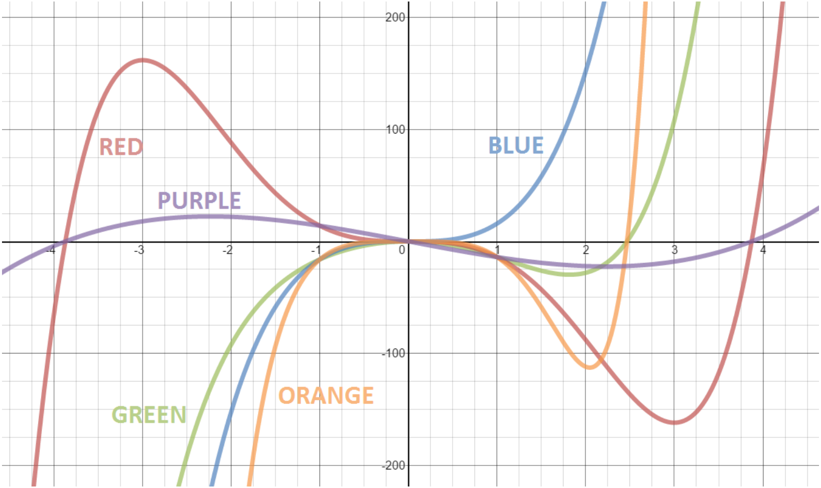


V

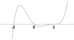
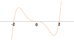
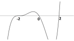


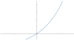
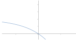
VI

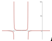
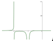
A. $f(x) = x^5 + 15x^3$ B. $f(x) = x^5 - 15x^3$ C. $f(x) = x^5 - 15x^2$
D. $f(x) = x^3 - 15x$ E. $f(x) = x^7 - 15x^4$

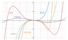


Included Work

, ,  screenshots of graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator (accessed 13 November 2015), 45

, , , , ,  screenshots of graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator, (accessed 13 November 2015), 47

, , ,  screenshots from graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator (accessed 16 July 2021), 46

 screenshot from graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator, with text added (accessed 13 November 2015), 48