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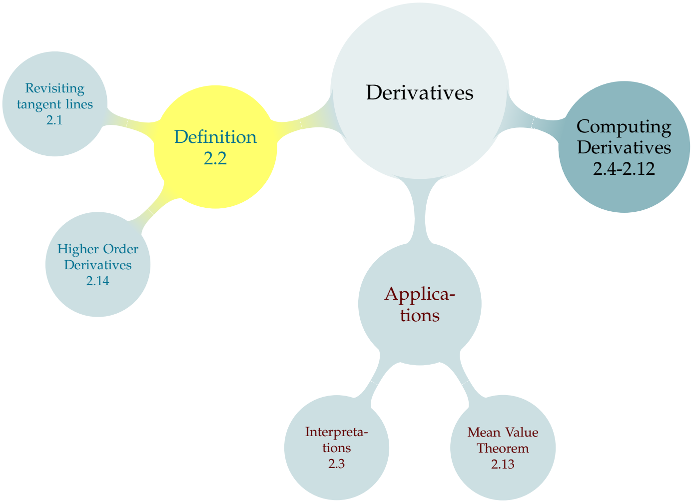

Derivative at a Point

Definition 2.2.1

Given a function $f(x)$ and a point a, the slope of the tangent line to $f(x)$ at a is the derivative of f at a, written $f'(a)$.

So, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$f'(a)$ is also the instantaneous rate of change of f at a.

Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then f is at a.

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Derivative

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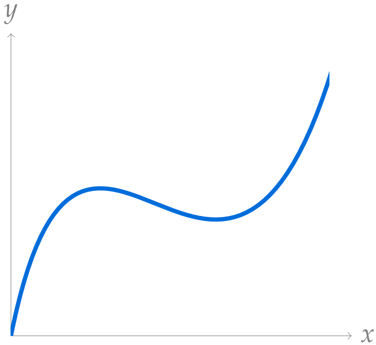
If $f'(a) > 0$, then f is increasing at a. Its graph "points up."

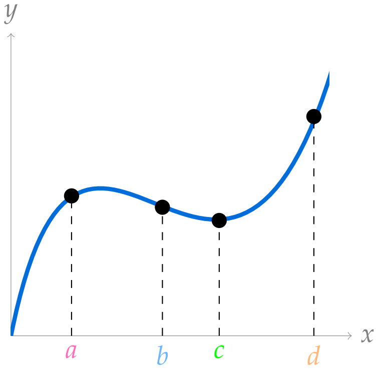
If $f'(a) < 0$, then f is decreasing at a. Its graph "points down."

If $f'(a) = 0$, then f looks constant or flat at a.

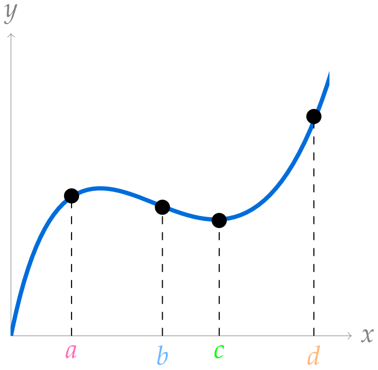
Practice: Increasing and Decreasing



Practice: Increasing and Decreasing



Practice: Increasing and Decreasing

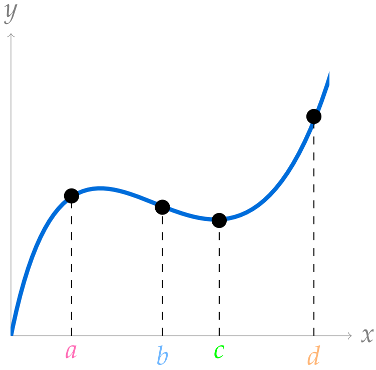


Where is $f'(x) < 0$?

Q

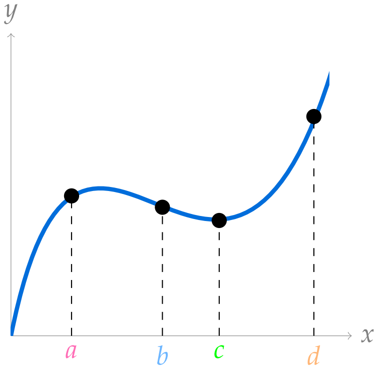
ans

Practice: Increasing and Decreasing



Where is $f'(x) < 0$? $f'(b) < 0$

Practice: Increasing and Decreasing

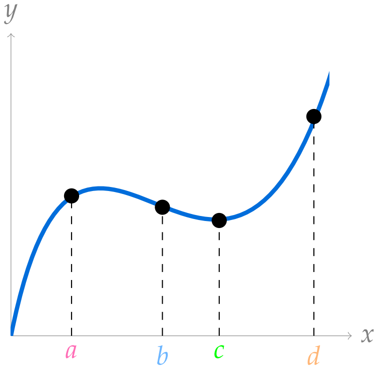


Where is $f'(x) > 0$?

Q

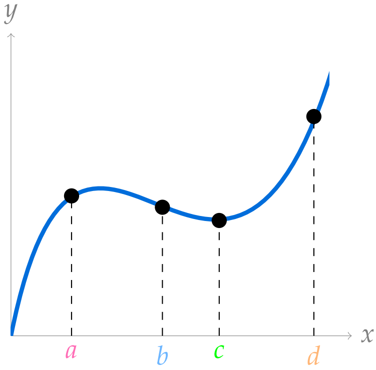
ans

Practice: Increasing and Decreasing



Where is $f'(x) > 0$? $f'(a) > 0$ and $f'(d) > 0$

Practice: Increasing and Decreasing

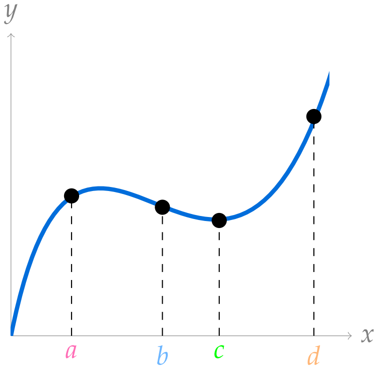


Where is $f'(x) \approx 0$?

Q

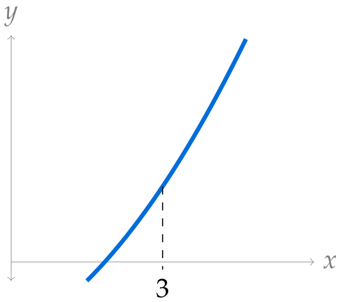
ans

Practice: Increasing and Decreasing

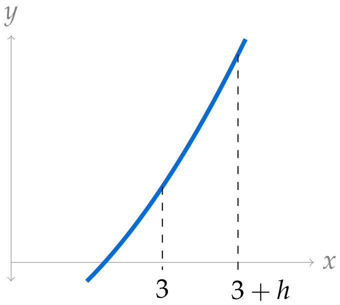


Where is $f'(x) \approx 0$? $f'(c) \approx 0$

Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.



Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.



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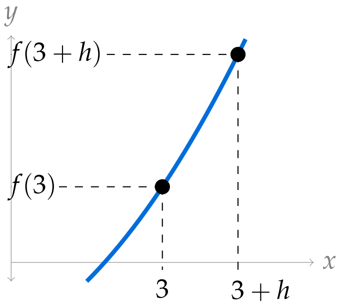
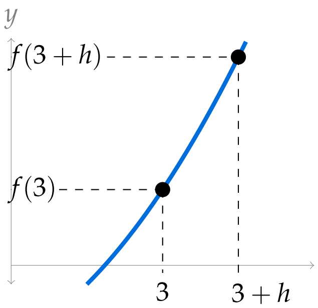


Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.



$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{\left((3+h)^2 - 5\right) - (3^2 - 5)}{h} \\ &= \lim_{h\to 0} \frac{\left(9 + 6h + h^2 - 5\right) - 4}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 6h}{h} \\ &= \lim_{h\to 0} h + 6 = 6 \end{aligned}$$

Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$. We can also find its derivative at an arbitrary point x:

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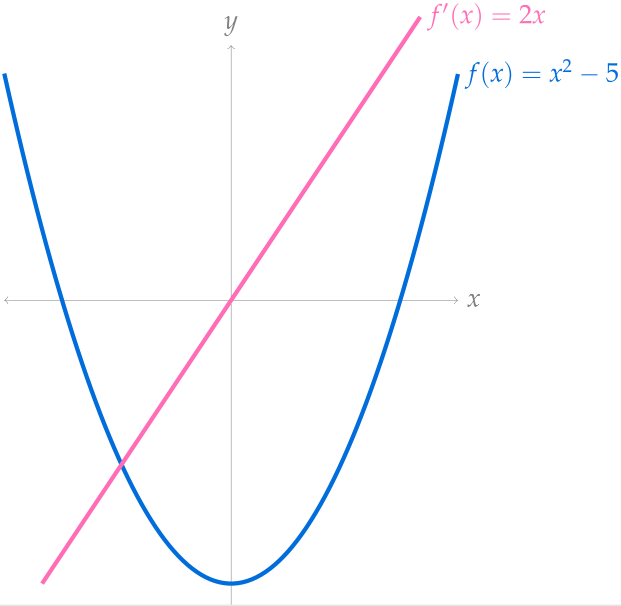
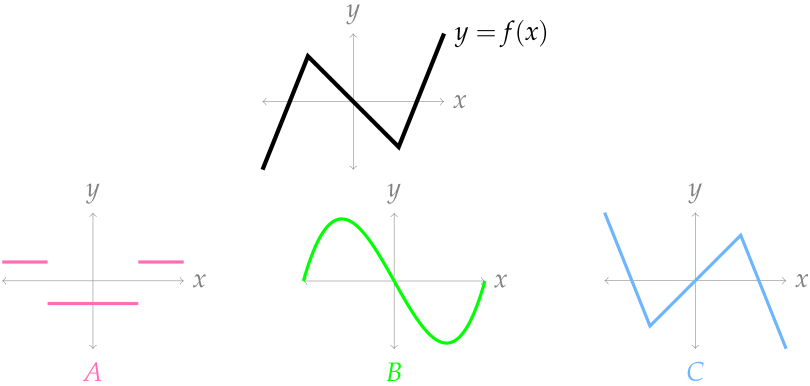
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 5 - (x^2 - 5)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h = 2x \qquad \text{(In particular, } f'(3) = 6.\text{)}
\end{aligned}$$



$f'(x) = 2x$
$f(x) = x^2 - 5$
y
x

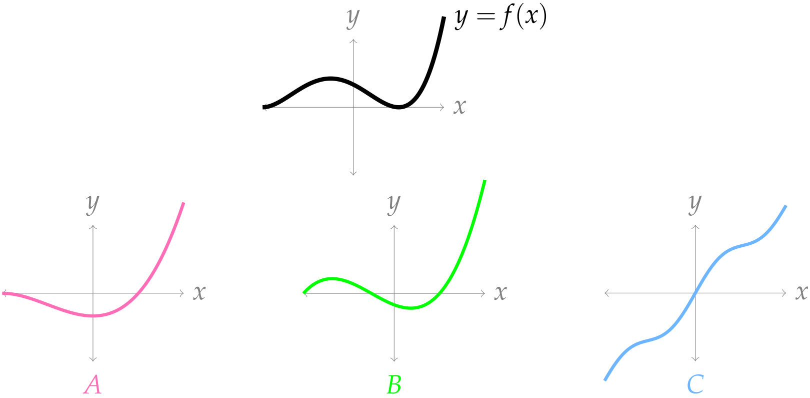
Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



INCREASING AND DECREASING

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



Q

ans

Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.
The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that x will be a part of your final expression: this is a function.

If $f'(x)$ exists for all x in an interval (a, b), we say that f is differentiable on (a, b).

Notation 2.2.8

The "prime" notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(a) \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) \qquad \left.\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right|_{x=a}$$

Notation 2.2.8

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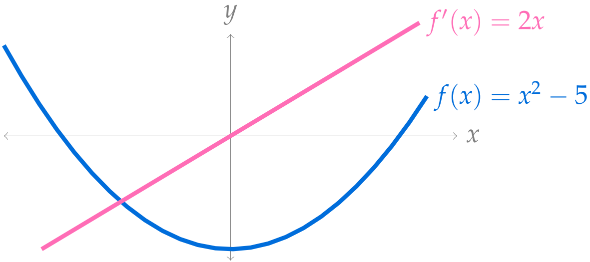
$\dfrac{\mathrm{d}f}{\mathrm{d}x}$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)\Big\vert_{x=a}$
function	number	function	number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibnitz Notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(3) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \qquad \left.\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right|_{x=3} =$$



ans

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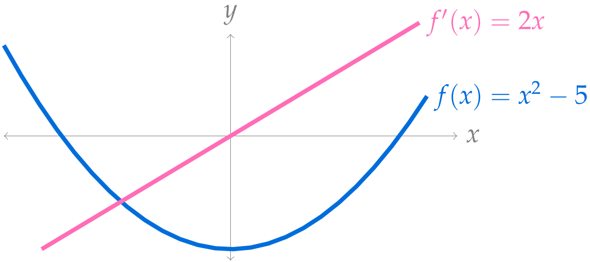

ans

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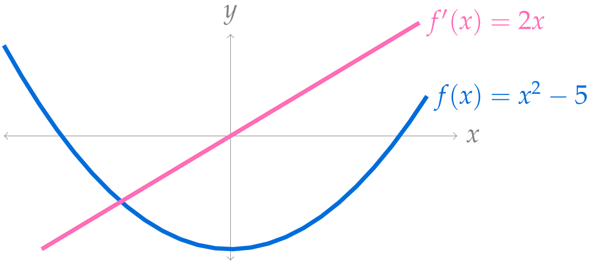

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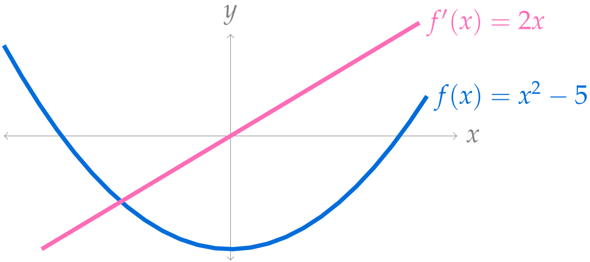

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Newtonian Notation:

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Leibniz Notation:

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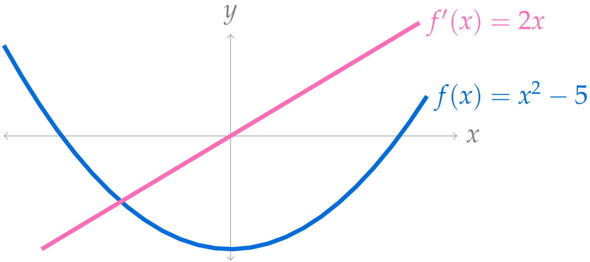


Alternate Definition – Definition 2.2.1

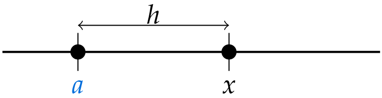
Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.



Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.

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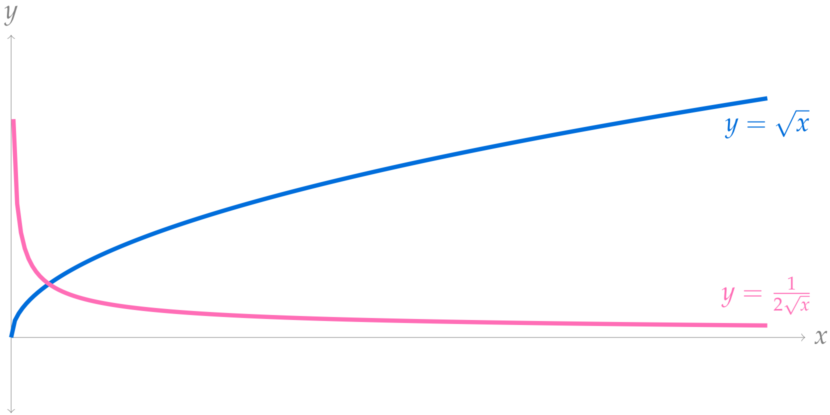
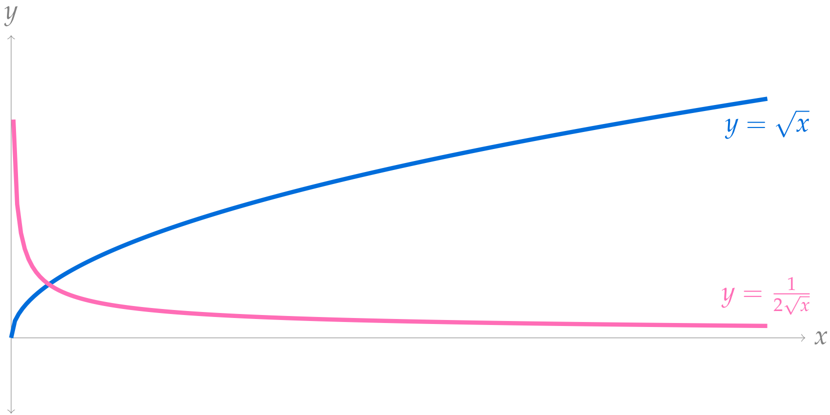
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h\to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}
$$

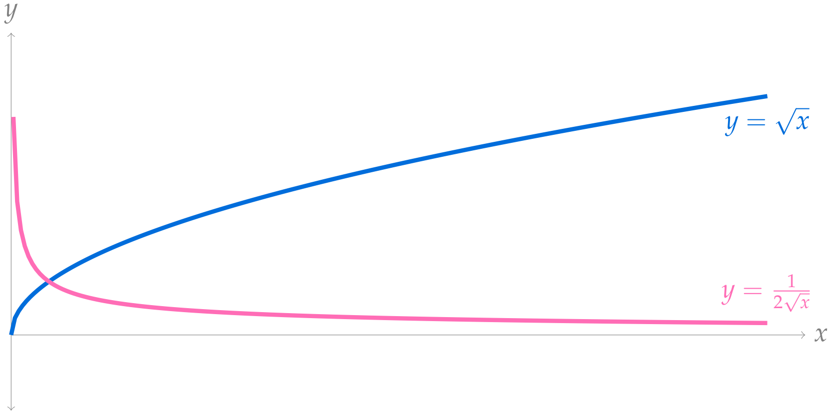

y
$y = \sqrt{x}$
$y = \frac{1}{2\sqrt{x}}$
x



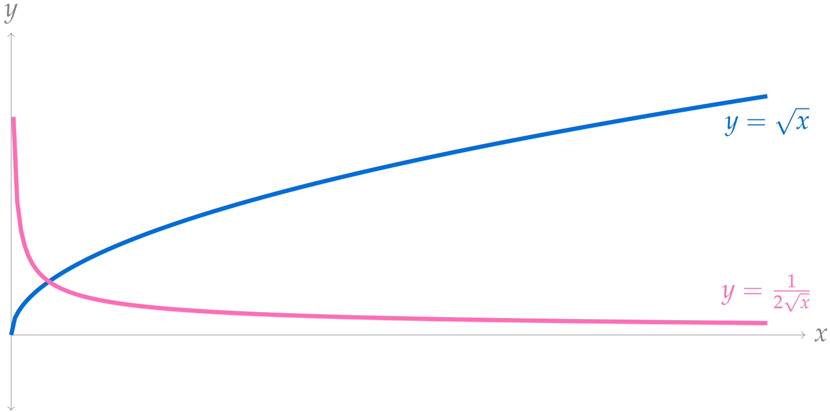
Review: $\lim_{x\to\infty} \sqrt{x} =$ $\lim_{x\to\infty} \frac{1}{2\sqrt{x}} =$

Q

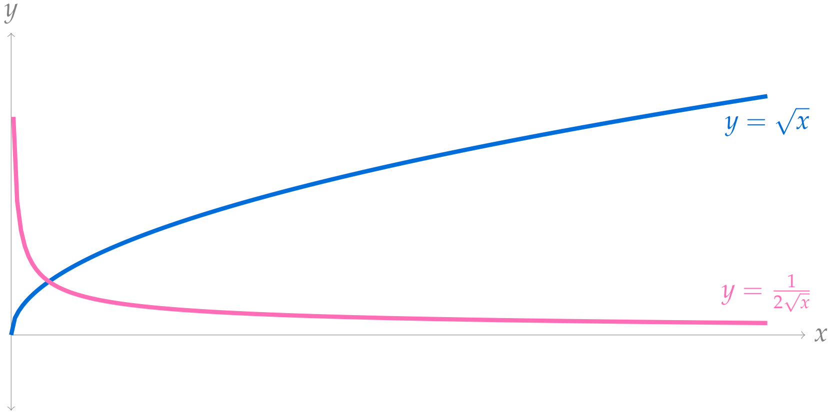
ans



$\boxed{\text{Review:}}$ $\lim_{x\to\infty} \sqrt{x} = \infty$ $\lim_{x\to\infty} \frac{1}{2\sqrt{x}} =$



$\boxed{\text{Review:}}$ $\lim_{x\to\infty} \sqrt{x} = \infty$ $\lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$

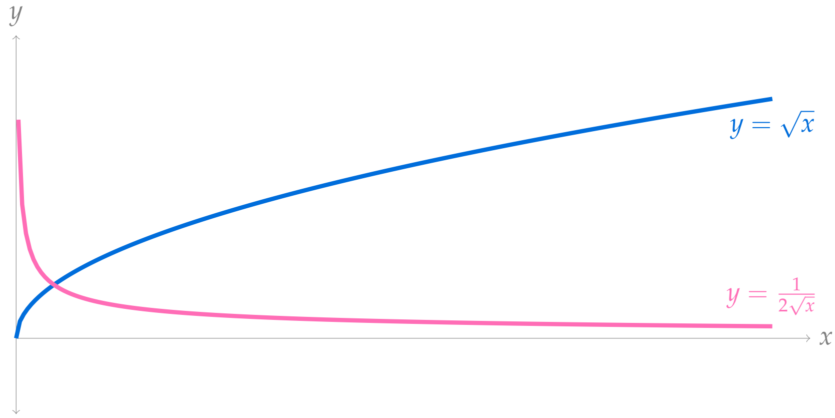


$\boxed{\text{Review:}}$ $\lim_{x\to\infty} \sqrt{x} = \infty$ $\lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$

$\lim_{x\to 0^+} \sqrt{x} =$ $\lim_{x\to 0^+} \frac{1}{2\sqrt{x}} =$

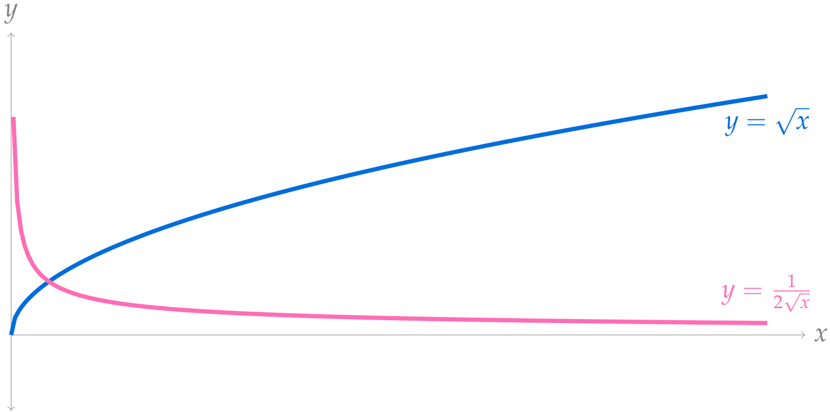
Q

ans



$\boxed{\text{Review:}}$ $\lim_{x \to \infty} \sqrt{x} = \infty$ $\lim_{x \to \infty} \frac{1}{2\sqrt{x}} = 0$

$\lim_{x \to 0^+} \sqrt{x} = 0$ $\lim_{x \to 0^+} \frac{1}{2\sqrt{x}} =$



Review:

$$\lim_{x\to\infty} \sqrt{x} = \infty \qquad \lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$$

$$\lim_{x\to 0^+} \sqrt{x} = 0 \qquad \lim_{x\to 0^+} \frac{1}{2\sqrt{x}} = \infty$$

Now You

Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\right\}.$$

Q

ans

NOW YOU

Using the definition of the derivative, calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\right\}$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\right] &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\
&= \lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} \\
&= \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}
\end{aligned}$$

Now You

Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x}{x+1}\right\}.$$

Now You

Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x^2+x}}\right\}.$$

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{2x}{x+1}\right\}$.

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{2x}{x+1}\right\}$.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{2(x+h)}{x+h+1}-\frac{2x}{x+1}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{2(x+h)(x+1)}{(x+h+1)(x+1)}-\frac{2x(x+h+1)}{(x+1)(x+h+1)}\right)\\
&= \lim_{h\to 0}\frac{2}{h}\left(\frac{(x^2+x+xh+h)-(x^2+xh+x)}{(x+h+1)(x+1)}\right)\\
&= \lim_{h\to 0}\frac{2}{h}\left(\frac{h}{(x+h+1)(x+1)}\right)\\
&= \lim_{h\to 0}\frac{2}{(x+h+1)(x+1)} = \frac{2}{(x+1)^2}
\end{aligned}$$

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{\sqrt{x^2+x}}\right\}$.

Q

ans

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{\sqrt{x^2+x}}\right\}$.

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{\sqrt{(x+h)^2+x+h}}-\frac{1}{\sqrt{x^2+x}}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{x^2+x}}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}}-\frac{\sqrt{(x+h)^2+x+h}}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{x^2+x}-\sqrt{(x+h)^2+x+h}}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}}\right)\left(\frac{\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}}{\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{(x^2+x)-[(x+h)^2+x+h]}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}\left[\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}\right]}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{-(2xh+h^2+h)}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}\left[\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}\right]}\right)\\
&= \lim_{h\to 0}\frac{-(2x+h+1)}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}\left[\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}\right]} = \frac{-(2x+1)}{2(x^2+x)^{3/2}}
\end{aligned}
$$

A

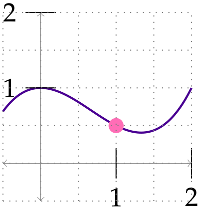
Memorize

The derivative of a function f at a point a is given by the following limit, if it exists:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Zooming In

For a smooth function, if we zoom in at a point, we see a line:



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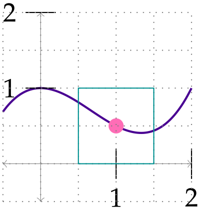

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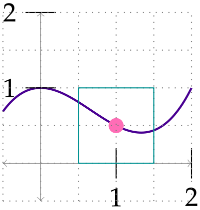

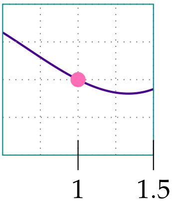

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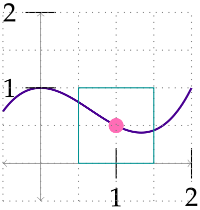

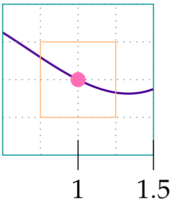

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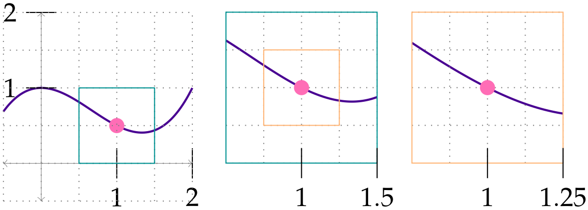

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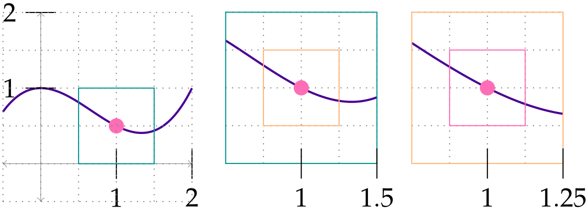

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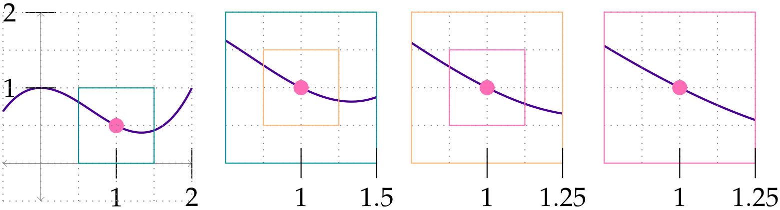

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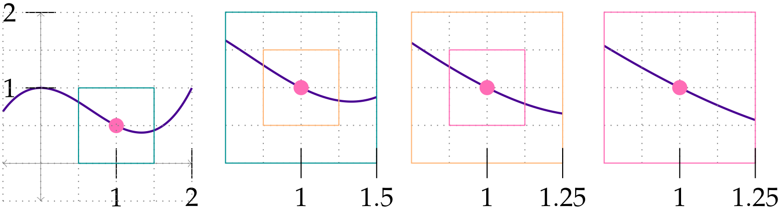


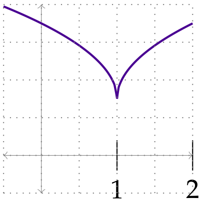
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



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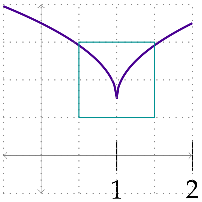

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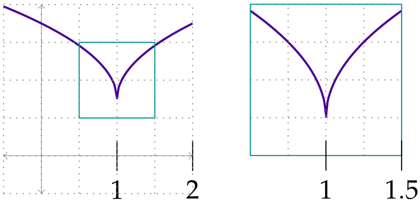

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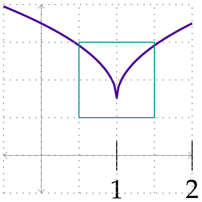

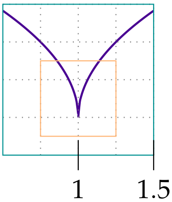

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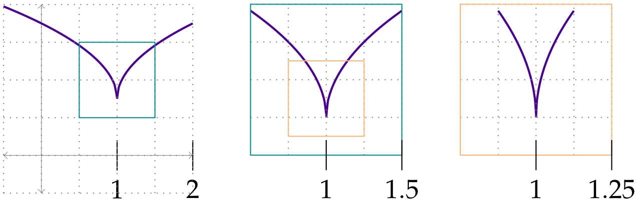

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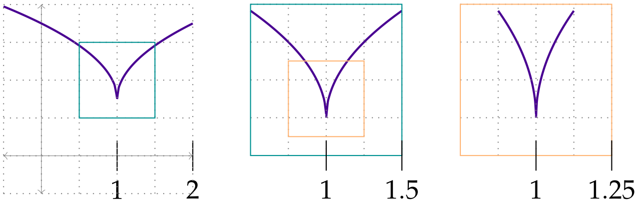
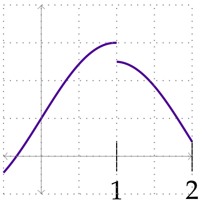


Discontinuity:



ZOOMING IN ON FUNCTIONS THAT AREN'T SMOOTH

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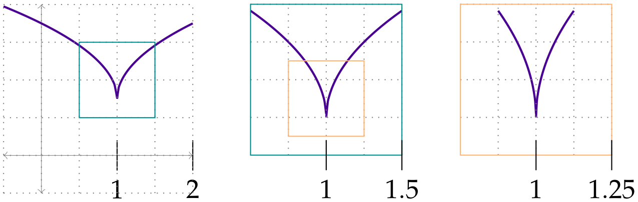
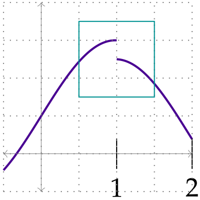


Discontinuity:



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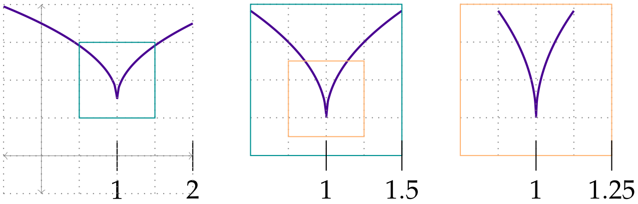

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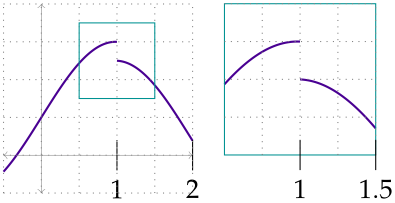

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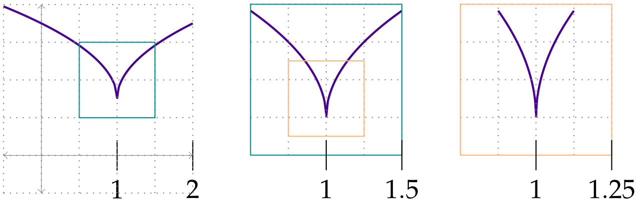

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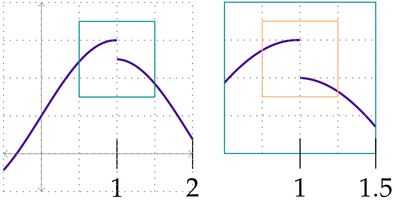

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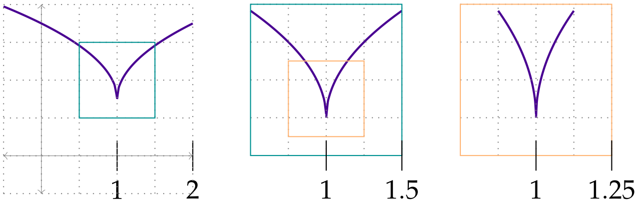

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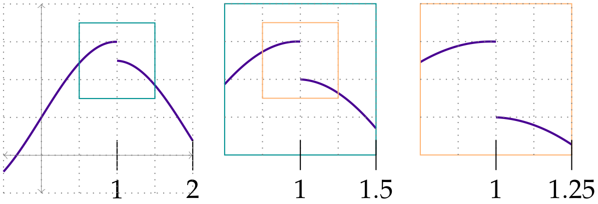

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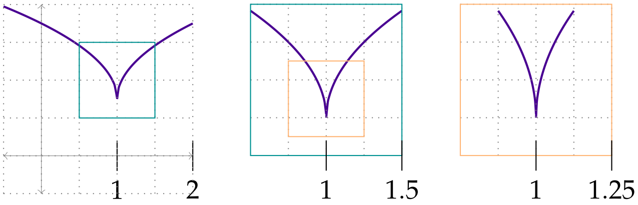

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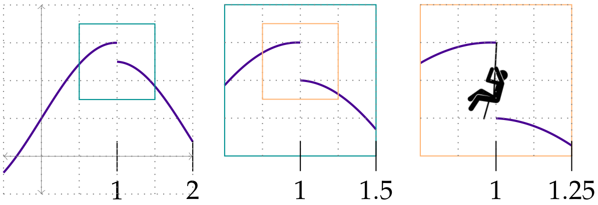


Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ does not exist at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

does not exist.
Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

When Derivatives Don't Exist

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

When Derivatives Don't Exist

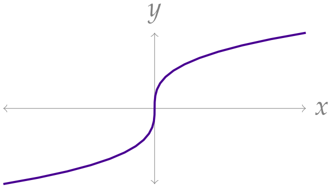
What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty \end{aligned}$$

Since the limit does not exist, we conclude $f'(x)$ is not defined at $x = 0$.

We can go a little farther: since the limit goes to infinity, the graph $y = f(x)$ looks vertical at $x = 0$.



Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof: If $f'(a)$ exists, that means:

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{exists}$$

$$\implies \lim_{h\to 0} \left[h \cdot \frac{f(a+h) - f(a)}{h} \right] = \left[\lim_{h\to 0} h \right] \cdot \left[\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \right]$$

$$\implies \lim_{h\to 0} \left[h \cdot \frac{f(a+h) - f(a)}{h} \right] = 0$$

$$\implies \lim_{h\to 0} [f(a+h) - f(a)] = 0$$

$$\implies \lim_{h\to 0} f(a+h) = f(a)$$

and that is the definition of $f(x)$ being continuous at $x = a$.

Let $f(x)$ be a function and let a be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ not differentiable at $x = a$
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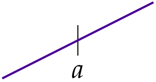
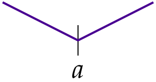
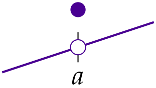
$f(x)$ continuous at $x=a$ $f(x)$ differentiable at $x=a$ 	$f(x)$ continuous at $x=a$ $f(x)$ not differentiable at $x=a$ 
$f(x)$ not continuous at $x=a$ $f(x)$ differentiable at $x=a$ impossible	$f(x)$ not continuous at $x=a$ $f(x)$ not differentiable at $x=a$ 

Included Work

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