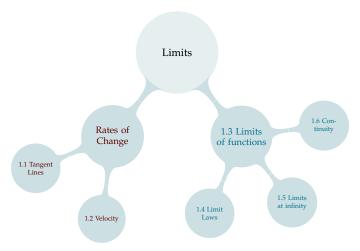
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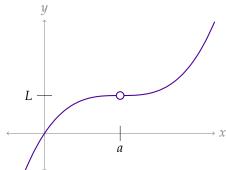
1.7 (Optional) Making the Informal a Little More Formal



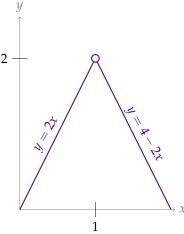
Now that we've seen the limits of functions as *x* goes to positive and negative infinity, let's look at limits as *x* approaches a real number.

$$\lim_{x \to a} f(x) = L$$

Informally: If x is close enough (but not equal to) a, then y is close enough to L.



Let
$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$$
. Then $\lim_{x \to 1} |x| = 2$.



Find a positive number δ such that $|f(x) - 2| < \frac{1}{2}$ for all x in the interval $(1 - \delta, 1 + \delta)$, except

possibly x = 1. Find a positive number δ such that $|f(x) - 2| < \frac{1}{4}$ for all x in the interval $(1 - \delta, 1 + \delta)$, except possibly x = 1.

Let $a \in \mathbb{R}$ and let f(x) be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that

the limit as x approaches a of f(x) is L

and write

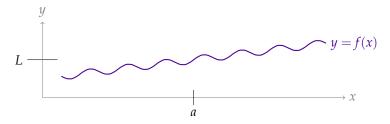
$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Note that an equivalent way of writing this very last statement is

if
$$0 < |x - a| < \delta$$
 then $|f(x) - L| < \epsilon$.



Let $a \in \mathbb{R}$ and let f(x) be a function defined everywhere in a neighbourhood of a, except possibly at a. We write

$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - L| < \epsilon$$
 whenever $0 < |x - a| < \delta$

Let $a \in \mathbb{R}$ and let f(x) be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that $\lim_{x\to -1} |x+1| = 0$.

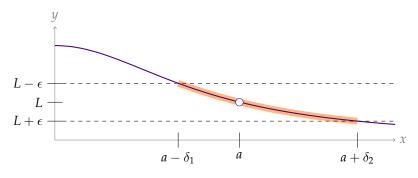
Let $a \in \mathbb{R}$ and let f(x) be a function defined everywhere in a neighbourhood of *a*, except possibly at *a*. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Let
$$f(x) = \begin{cases} x+1 & x < 0 \\ 1-x^2 & x > 0 \end{cases}$$

Using Definition 1.7.1, prove that $\lim_{x\to 0} f(x) = 1$.

GENERAL PRINCIPLES

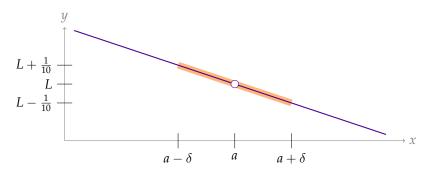
Suppose $|f(x) - L| < \epsilon$ whenever $a - \delta_1 < x < a$ and whenever $a < x < a + \delta_2$.



Consider values of x such that $0 < |x - a| < \min\{\delta_1, \delta_2\}$.

GENERAL PRINCIPLES

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.



Can you give values of *x* where $|f(x) - L| < \frac{1}{5}$?

 \Rightarrow skip ϵ small

GENERAL PRINCIPLES

Definition 1.7.1

Let $a \in \mathbb{R}$ and let f(x) be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

It is enough to show that for every ϵ such that $0 < \epsilon < c$ (where c is some constant) there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

That means it doesn't hurt your proof if you say something like "we assume $\epsilon < 1$ ".

In a previous example, we chose

$$\delta = \min\{\epsilon, \sqrt{\epsilon}\}\$$

It would be OK to say "we can assume $\epsilon < 1$; set $\delta = \epsilon$."

Let $a \in \mathbb{R}$ and let f(x) be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that
$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$$
.

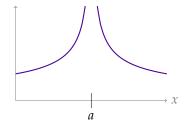
INFINITE LIMITS

Definition 1.8.1 (b)

Let *a* be a real number and f(x) be a function defined for all $x \neq a$. We write

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every P > 0 there exists $\delta > 0$ so that f(x) > P whenever $0 < |x - a| < \delta$.

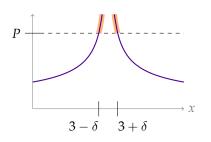


Definition 1.8.1 (b)

Let *a* be a real number and f(x) be a function defined for all $x \neq a$. We write

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every P > 0 there exists $\delta > 0$ so that f(x) > P whenever $0 < |x - a| < \delta$.



Let $f(x) = \frac{1}{(x-3)^2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \to 3} f(x) = \infty$$

Definition 1.8.1 (b)

Let *a* be a real number and f(x) be a function defined for all $x \neq a$. We write

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every P > 0 there exists $\delta > 0$ so that f(x) > P whenever $0 < |x - a| < \delta$.

Let $f(x) = \frac{1}{x-2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x\to 2} f(x) = \infty$$