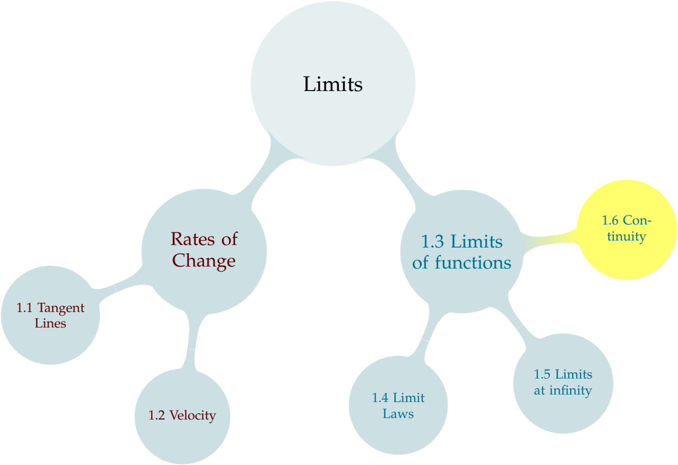
# Table of Contents



# Continuity

## Definition 1.6.1

A function $f(x)$ is continuous at a point $a$ if $\lim_{x \to a} f(x)$ exists AND is equal to $f(a)$.

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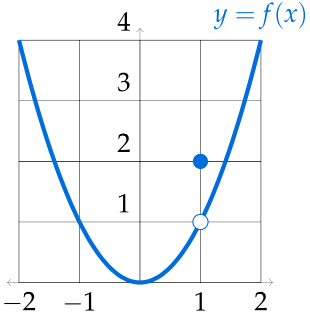


Does $f(x)$ exist at $x = 1$?
Is $f(x)$ continuous at $x = 1$?

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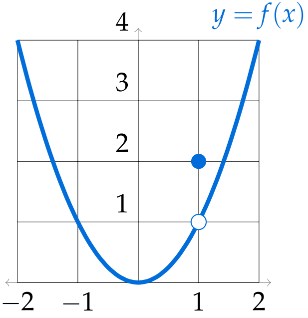


Does $f(x)$ exist at $x = 1$? Yes.
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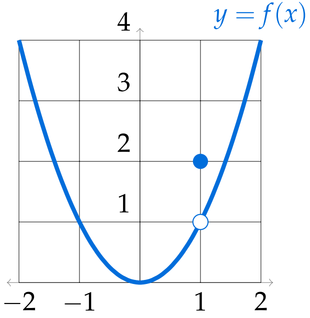


Does $f(x)$ exist at $x = 1$? Yes.
Is $f(x)$ continuous at $x = 1$? No.

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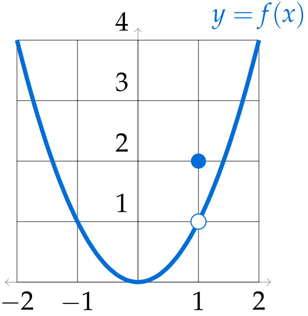
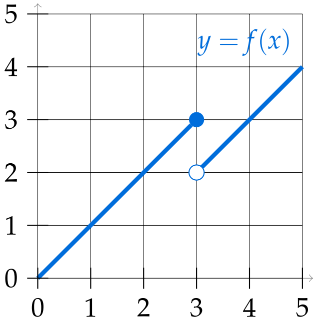



Does $f(x)$ exist at $x = 1$? Yes.
Is $f(x)$ continuous at $x = 1$? No.

This kind of discontinuity is called removable.

## Definitions 1.6.1 and 1.6.2

A function $f(x)$ is continuous at a point $a$ if $\lim_{x\to a} f(x)$ exists AND is equal to $f(a)$.




Is $f(x)$ continuous at $x = 3$?

## Definitions 1.6.1 and 1.6.2

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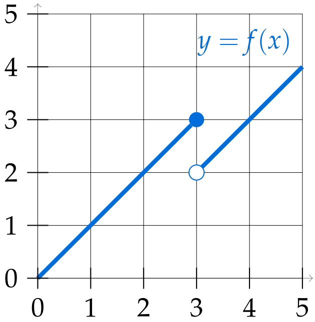
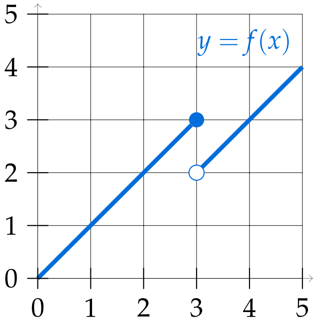



Is $f(x)$ continuous at $x = 3$? No.

This kind of discontinuity is called a jump.

## Definitions 1.6.1 and 1.6.2

A function $f(x)$ is continuous from the left at a point $a$ if $\lim\limits_{x \to a^-} f(x)$ exists AND is equal to $f(a)$.




Is $f(x)$ continuous at $x = 3$? No.

## Definitions 1.6.1 and 1.6.2

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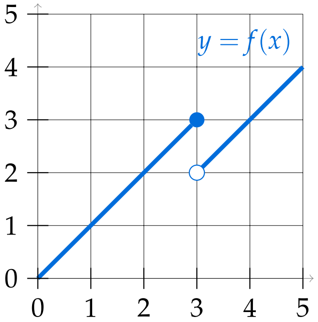


Is $f(x)$ continuous at $x = 3$? No.

Is $f(x)$ continuous from the left at $x = 3$?

Is $f(x)$ continuous from the right at $x = 3$?

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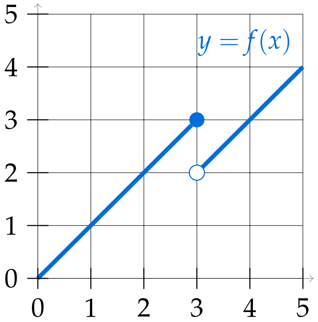


Is $f(x)$ continuous at $x = 3$? No.

Is $f(x)$ continuous from the left at $x = 3$? Yes.

Is $f(x)$ continuous from the right at $x = 3$?

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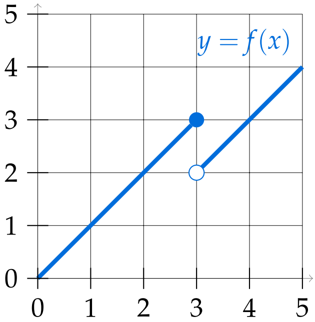


Is $f(x)$ continuous at $x = 3$? No.

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Is $f(x)$ continuous from the right at $x = 3$? No.

## Definition

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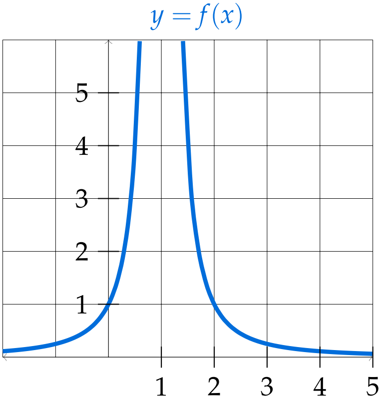

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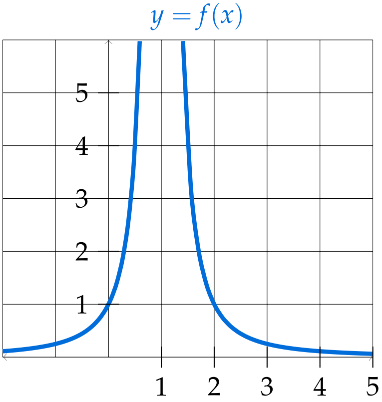


Since no one-sided limits exist at $x = 1$, there's no hope for continuity there – not even "from the left" or "from the right."

This is called an infinite discontinuity

## Definition

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$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ continuous at 0?

# Continuous Functions

Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point in their domain.

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$$f(x) = \frac{x^2}{2x-10} - \left( \frac{x^2+2x-1}{x-1} + \frac{\sqrt[5]{25-x} - \frac{1}{x}}{x+2} \right)^{1/3}$$

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$f(x)$ is continuous at every real number except $5, 1, 0$, and $-2$.

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A continuous function is continuous for every point in $\mathbb{R}$.

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We say $f(x)$ is continuous over $(a, b)$ if it is continuous at every point in $(a, b)$. So, $f(x)$ is continuous over its domain, $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, 5) \cup (5, \infty)$.

## Common Functions – Theorem 1.6.8

Functions of the following types are continuous over their domains:

- polynomials and rationals
- roots and powers
- trig functions and their inverses
- exponential and logarithm
- The products, sums, differences, quotients, powers, and compositions of continuous functions

Where is the following function continuous?

$$f(x) = \left( \frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

Where is the following function continuous?

$$f(x) = \left( \frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

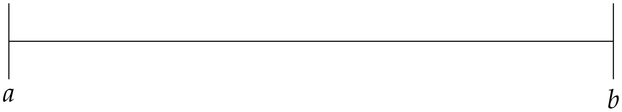
Over its domain: $[0, 2) \cup (2, \infty)$.

# A Technical Point

## Definition 1.6.3

A function $f(x)$ is continuous on the closed interval $[a, b]$ if:

- $f(x)$ is continuous over $(a, b)$, and
- $f(x)$ is continuous from the left at , and
- $f(x)$ is continuous from the right at



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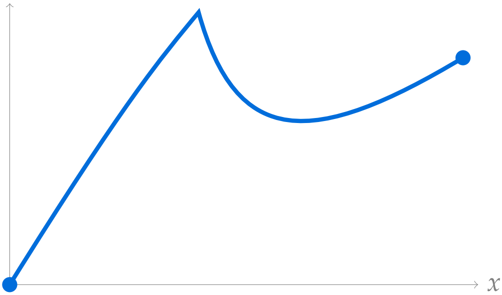
## Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If $y$ is any number between $f(a)$ and $f(b)$, then there exists $c$ in $(a, b)$ such that $f(c) = y$.



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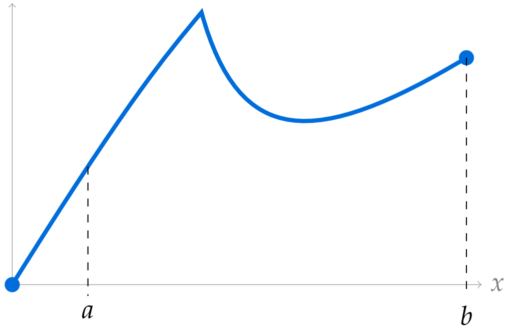

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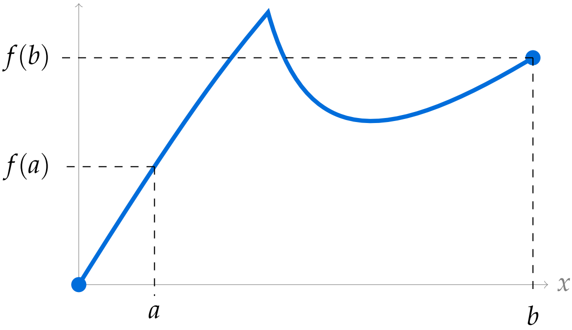

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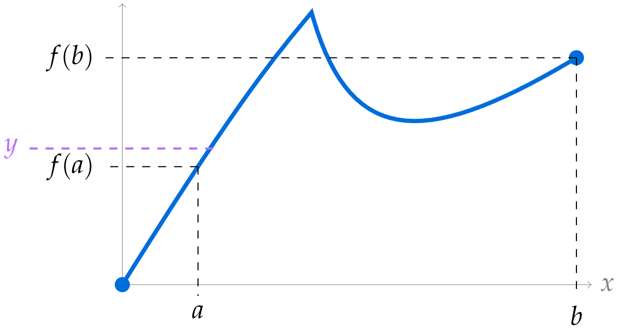

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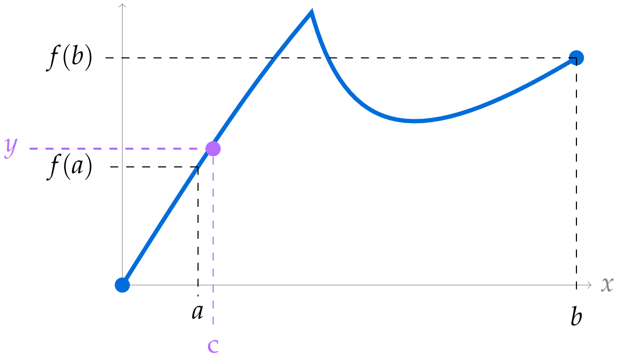

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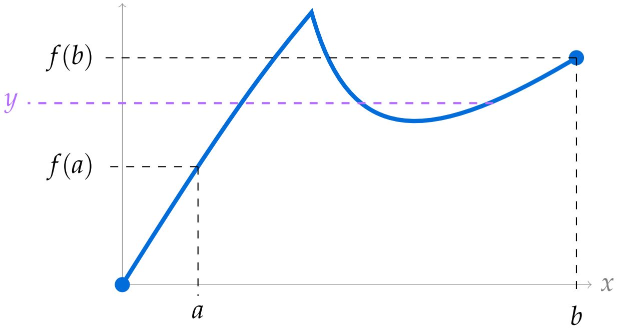

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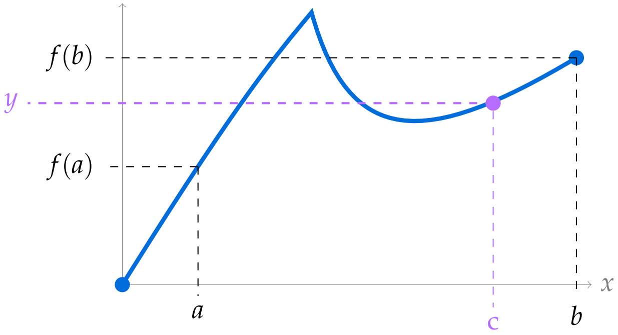

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Suppose your favourite number is 45.54. At noon, your car is parked, and at 1pm you're driving 100kph. By the Intermediate Value Theorem, at some point between noon and 1pm you were going exactly 45.54 kph.

# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$.

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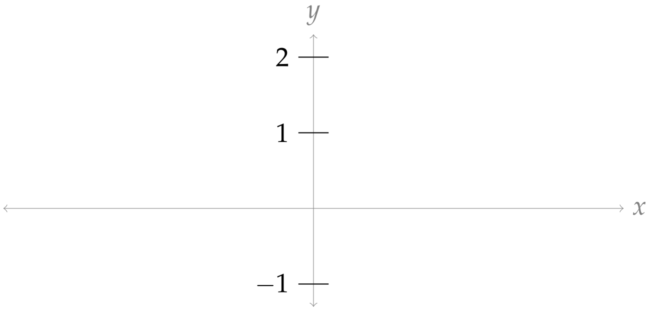
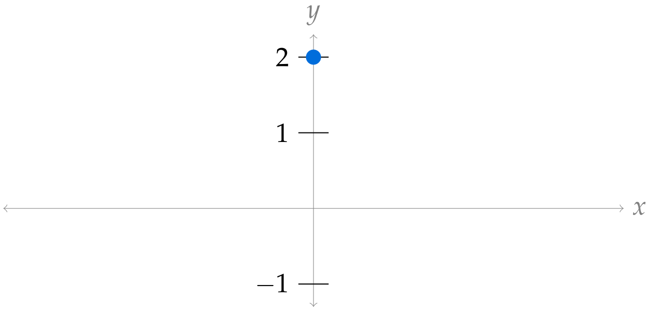


# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$. Let's find some points:

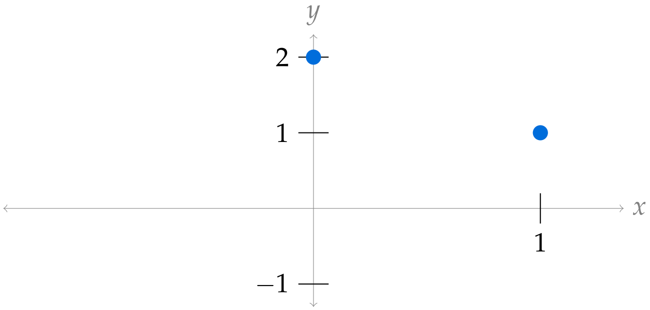
$f(0) = 2$



# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$. Let's find some points:

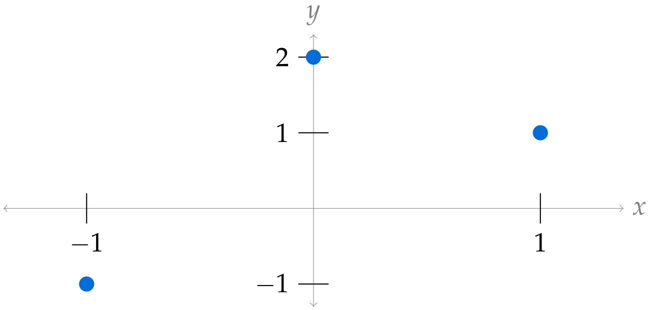
$f(0) = 2$ $\qquad$ $f(1) = 1$



# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$. Let's find some points:

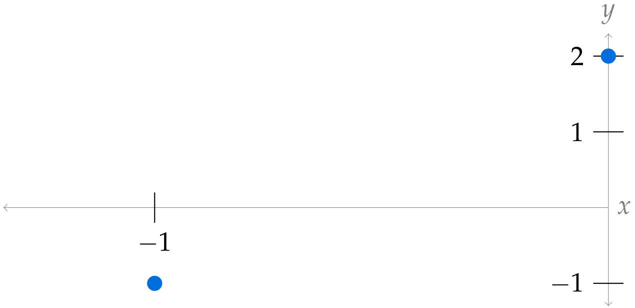
$f(0) = 2$ $f(1) = 1$ $f(-1) = -1$



# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$.

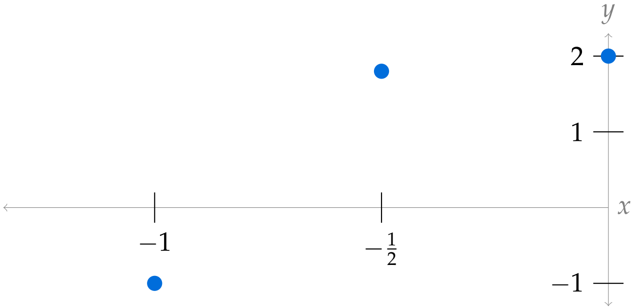
$f(0) = 2, f(-1) = -1$



# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$.

$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84$



# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$.

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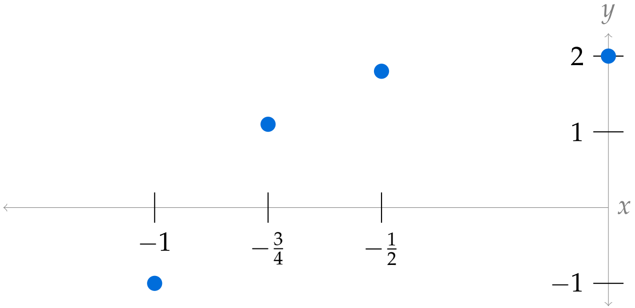
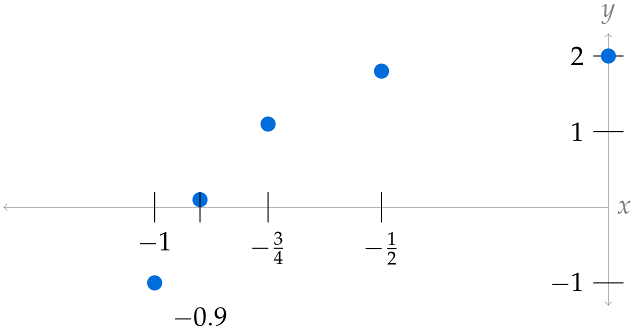


# Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value $x$ for which $f(x) = 0$.

$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84, f\left(-\frac{3}{4}\right) \approx 1.13, f(-.9) = 0.097$



Use the Intermediate Value Theorem to show that there exists some solution to the equation $\ln x \cdot e^x = 4$, and give a reasonable interval where that solution might occur.

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- The function $f(x) = \ln x \cdot e^x$ is continuous over its domain, which is $(0, \infty)$. In particular, then, it is continuous over the interval $(1, e)$.
- $f(1) = \ln(1)e = 0 \cdot e = 0$ and $f(e) = \ln(e) \cdot e^e = e^e$. Since $e > 2$, we know $f(e) = e^e > 2^2 = 4$.
- Then 4 is between $f(1)$ and $f(e)$.
- By the Intermediate Value Theorem, $f(c) = 4$ for some $c$ in $(1, e)$.

Now You Use the Intermediate Value Theorem to give a reasonable interval where the following is true: $e^x = \sin(x)$. (Don't use a calculator – use numbers you can easily evaluate.)

**Now You** Use the Intermediate Value Theorem to give a reasonable interval where the following is true: $e^x = \sin(x)$. (Don't use a calculator – use numbers you can easily evaluate.)

We can rearrange this: let $f(x) = e^x - \sin(x)$, and note $f(x)$ has roots exactly when $e^x = \sin(x)$.

- The function $f(x) = e^x - \sin x$ is continuous over its domain, which is all real numbers. In particular, then, it is continuous over the interval $\left(-\frac{3\pi}{2}, e\right)$.
- $f(0) = e^0 - \sin 0 = 1 - 0 = 1 > 0$ and $f\left(-\frac{3\pi}{2}\right) = e^{-\frac{3\pi}{2}} - \sin\left(\frac{-3\pi}{2}\right) = e^{-\frac{3\pi}{2}} - 1 < e^0 - 1 = 1 - 1 = 0.$
- Then 0 is between $f(0)$ and $f\left(-\frac{3\pi}{2}\right)$.
- By the Intermediate Value Theorem, $f(c) = 0$ for some $c$ in $\left(-\frac{3\pi}{2}, 0\right)$.
- Therefore, $e^c = \sin c$ for some $c$ in $\left(-\frac{3\pi}{2}, 0\right)$.

Now You

Is there any value of $x$ so that $\sin x = \cos(2x) + \frac{1}{4}$?

Q



ans

Now You

Is there any value of $x$ so that $\sin x = \cos(2x) + \frac{1}{4}$?

Yes, somewhere between 0 and $\frac{\pi}{2}$.

Now You

Is the following reasoning correct?

- $f(x) = \tan x$ is continuous over its domain, because it is a trigonometric function.
- In particular, $f(x)$ is continuous over the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.
- $f\left(\frac{\pi}{4}\right) = 1$, and $f\left(\frac{3\pi}{4}\right) = -1$.
- Since $f\left(\frac{3\pi}{4}\right) < 0 < f\left(\frac{\pi}{4}\right)$, by the Intermediate Value Theorem, there exists some number $c$ in the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ such that $f(c) = 0$.

Now You

Is the following reasoning correct?

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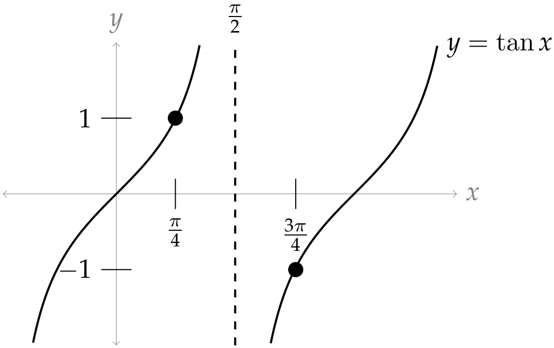

$y$
$\frac{\pi}{2}$
$y = \tan x$
$1$
$-1$
$x$
$\frac{\pi}{4}$
$\frac{3\pi}{4}$

# Continuity

Section 1.6 Review

Suppose $f(x)$ is continuous at $x = 1$. Does $f(x)$ have to be defined at $x = 1$?

Suppose $f(x)$ is continuous at $x = 1$. Does $f(x)$ have to be defined at $x = 1$?

Yes. Since $f(x)$ is continuous at $x = 1$, $\lim_{x \to 1} f(x) = f(1)$, so $f(1)$ must exist.

Suppose $f(x)$ is continuous at $x = 1$ and $\lim_{x\to 1^-} f(x) = 30$.

True or false: $\lim_{x\to 1^+} f(x) = 30$.

Q

ans

Suppose $f(x)$ is continuous at $x = 1$ and $\lim_{x \to 1^-} f(x) = 30$.

True or false: $\lim_{x \to 1^+} f(x) = 30$.

True. Since $f(x)$ is continuous at $x = 1$, $\lim_{x \to 1} f(x) = f(1)$, so $\lim_{x \to 1} f(x)$ must exist. That means both one-sided limits exist, and are equal to each other.

Suppose $f(x)$ is continuous at $x = 1$ and $f(1) = 22$. What is $\lim_{x \to 1} f(x)$?

Q



ans

Suppose $f(x)$ is continuous at $x = 1$ and $f(1) = 22$. What is $\lim_{x \to 1} f(x)$?

$22 = f(1) = \lim_{x \to 1} f(x)$.

Suppose $\lim_{x\to 1} f(x) = 2$. Must it be true that $f(1) = 2$?

Q



ans

Suppose $\lim_{x\to 1} f(x) = 2$. Must it be true that $f(1) = 2$?

No. In order to determine the limit as $x$ goes to 1, we ignore $f(1)$. (Perhaps $f(x)$ is not even defined at 1.)

$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of $a$ is $f(x)$ continuous?

$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of $a$ is $f(x)$ continuous?

We need $ax^2 = 3x$ when $x = 1$, so $a = 3$.

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of $a$ is $f(x)$ continuous at $x = -\sqrt{3}$?

Q



ans

By the definition of continuity, if $f(x)$ is continuous at $x = -\sqrt{3}$, then $f(-\sqrt{3}) = \lim_{x \to -\sqrt{3}} f(x)$. Note $f(-\sqrt{3}) = a$, and when $x$ is close to (but not equal to) $-\sqrt{3}$, then $f(x) = \frac{\sqrt{3}x+3}{x^2-3}$.

$$
\begin{aligned}
f(-\sqrt{3}) &= \lim_{x \to -\sqrt{3}} f(x) \\
a &= \lim_{x \to -\sqrt{3}} \frac{\sqrt{3}x + 3}{x^2 - 3} = \lim_{x \to -\sqrt{3}} \frac{\sqrt{3}(x + \sqrt{3})}{(x + \sqrt{3})(x - \sqrt{3})} \\
&= \lim_{x \to -\sqrt{3}} \frac{\sqrt{3}}{x - \sqrt{3}} = \frac{\sqrt{3}}{-\sqrt{3} - \sqrt{3}} = -\frac{1}{2}
\end{aligned}
$$

So we can use $a = -\frac{1}{2}$ to make $f(x)$ continuous at $x = -\sqrt{3}$.

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of $a$ is $f(x)$ continuous at $x = \sqrt{3}$?

Q



ans

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of $a$ is $f(x)$ continuous at $x = \sqrt{3}$?

By the definition of continuity, if $f(x)$ is continuous at $x = \sqrt{3}$, then $f(\sqrt{3}) = \lim_{x\to\sqrt{3}} f(x)$. When $x$ is close to (but not equal to) $\sqrt{3}$, then $f(x) = \frac{\sqrt{3}x+3}{x^2-3}$. However, as $x$ approaches $\sqrt{3}$, the denominator of this expression gets closer and closer to zero, while the top gets closer and closer to 6. So, this limit does not exist. Therefore, no value of $a$ will make $f(x)$ continuous at $x = \sqrt{3}$.

A

## Included Work

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