This file contains questions spanning CLP-1. It should not be taken as a complete review of the course, but rather as a jumping-off point. If you struggle with one question, go back to review its entire section. Sections are noted at the bottom of each page.

Short Answer

S1. Find all solutions to $x^3 - 3x^2 - x + 3 = 0$

solution S1

S2. Compute the limit $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-4}$

solution S2

S3. Find all values of c such that the following function is continuous:

solution S3

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \leq c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

S4. Compute

solution S4

$$\lim_{x \to -\infty} \frac{3x+5}{\sqrt{x^2+5} - x}$$

Short Answer

S5. Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \dfrac{\pi}{4}$.

▸ solution S5

S6. For what values of x does the derivative of $\dfrac{\sin(x)}{x^2 + 6x + 5}$ exist?

▸ solution S6

S7. Find $f'(x)$ if $f(x) = (x^2 + 1)^{\sin(x)}$.

▸ solution S7

S8. Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

▸ solution S8

Short Answer

S9. Consider a function $f(x)$ which has $f'''(x) = \dfrac{x^3}{10 - x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

solution S9

S10. Estimate $\sqrt{35}$ using a linear approximation

solution S10

S11. Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

solution S11

S12. Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

solution S12

Long Answer

L1. Compute the limit $\lim_{x\to 1} \dfrac{\sqrt{x+2}-\sqrt{4-x}}{x-1}$.

solution L1

L2. Show that there exists at least one real number c such that $2\tan(c) = c+1$.

solution L2

L3. Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \le 0 \\ x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

solution L3

Long Answer

L4. If $x^2 \cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$. You must justify your answer.

solution L4

L5. Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at $(0, 12)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

solution L5

L6. Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

solution L6

Solutions

◂ back to questions

S1.

Find all solutions to $x^3 - 3x^2 - x + 3 = 0$

back to questions

S1.
Find all solutions to $x^3 - 3x^2 - x + 3 = 0$

$$\begin{aligned} x^3 - 3x^2 - x + 3 &= x^2(x-3) - (x-3) \\ &= (x^2-1)(x-3) \\ &= (x+1)(x-1)(x-3) \end{aligned}$$

The solutions are $x = 1$, $x = 3$, and $x = -1$.

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S2.

Compute the limit $\displaystyle\lim_{x\to 2}\frac{x-2}{x^2-4}$

back to questions

S2.

Compute the limit $\lim_{x \to 2} \frac{x-2}{x^2-4}$

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$$

back to questions

S3.

Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \leq c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

back to questions

S3.

Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \leq c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

When $x \neq c$, $f(x)$ is continuous. The only difficult spot is when $x = c$.

- $f(c) = 8 - c^2$
- $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^-} (8 - cx) = 8 - c^2$
- $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^+} (x^2) = c^2$

Since $f(x)$ is continuous at c only if $f(c) = \lim\limits_{x \to c} f(x)$, we see the only values of c that make f continuous are those that satisfy $c^2 = 8 - c^2$. That is, $c = \pm 2$.

◂ back to questions

S4.

Compute

$$\lim_{x \to -\infty} \frac{3x+5}{\sqrt{x^2+5}-x}$$

back to questions

S4.

Compute

$$\lim_{x\to-\infty} \frac{3x+5}{\sqrt{x^2+5}-x}$$

We start by factoring out x from both the top and the bottom.

$$\lim_{x\to-\infty} \frac{3x+5}{\sqrt{x^2+5}-x}\left(\frac{1/x}{1/x}\right) = \lim_{x\to-\infty} \frac{3+\frac{5}{x}}{\frac{1}{x}\sqrt{x^2+5}-1}$$

Since x is approaching negative infinity, we can assume $x<0$. Then $x=-|x|=-\sqrt{x^2}$. We'll use this form to push the $\frac{1}{x}$ into the square root.

$$= \lim_{x\to-\infty} \frac{3+\frac{5}{x}}{-\frac{1}{\sqrt{x^2}}\sqrt{x^2+5}-1}$$

$$= \lim_{x\to-\infty} \frac{3+\frac{5}{x}}{-\sqrt{1+\frac{5}{x^2}}-1} = \frac{3+0}{-\sqrt{1}-1} = -\frac{3}{2}$$

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S5.

Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \dfrac{\pi}{4}$.

back to questions

S5.

Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \frac{\pi}{4}$.

$$f(x) = \cos x \qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Recall the equation of the tangent line to $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a)$

$$y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

back to questions

S6.

For what values of x does the derivative of $\dfrac{\sin(x)}{x^2+6x+5}$ exist?

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S6.

For what values of x does the derivative of $\dfrac{\sin(x)}{x^2+6x+5}$ exist?

First, note that $x^2 + 6x + 5 = (x+1)(x+5)$, so the function does not exist at either $x = -1$ or $x = -5$. For other values of x, using the quotient rule, we see

$$f'(x) = \frac{(x^2+6x+5)(\cos x) - \sin x(2x+6)}{(x^2+6x+5)^2}$$

which exists over the domain of $f(x)$.

All together, the derivative exists for all values of x except -1 and -5.

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S7.

Find $f'(x)$ if $f(x) = (x^2 + 1)^{\sin(x)}$.

back to questions

S7.
Find $f'(x)$ if $f(x) = (x^2+1)^{\sin(x)}$.

$f(x)$ is neither an exponential function (with a constant base) nor a power function (with a constant power). When we see a function raised to a function, we differentiate using logarithmic differentiation.

$$\begin{aligned}
f(x) &= (x^2+1)^{\sin(x)} \\
\log f(x) &= \log\left[(x^2+1)^{\sin(x)}\right] = \sin x \cdot \log(x^2+1) \\
\frac{f'(x)}{f(x)} &= \sin x \cdot \frac{2x}{x^2+1} + \cos x \cdot \log(x^2+1) \\
f'(x) &= f(x)\left[\sin x \cdot \frac{2x}{x^2+1} + \cos x \cdot \log(x^2+1)\right] \\
&= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + \cos x \cdot \log(x^2+1)\right]
\end{aligned}$$

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S8.

Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

This is a review of high school material. This type of calculation comes up in Section 3.3: Exponential Growth and Decay

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S8.

Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

$$3 = f(0) = Ae^0 = A$$

$$5 = f(2) = 3e^{2k} \implies \frac{5}{3} = e^{2k}$$

$$\log_e\left(\frac{5}{3}\right) = 2k$$

$$k = \frac{\log_e(5/3)}{2}$$

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S9.

Consider a function $f(x)$ which has $f'''(x) = \dfrac{x^3}{10 - x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

back to questions

S9.

Consider a function $f(x)$ which has $f'''(x) = \dfrac{x^3}{10 - x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

For some c between 0 and 1:

$$\Big|\underbrace{f(1) - T_2(1)}_{\text{error}}\Big| = \left|\frac{f'''(c)}{3!}(1-0)^3\right| = \frac{1}{6}\left|\frac{c^3}{10-c^2}\right|$$

Since c is between 0 and 1, we note $0 < c^3 < 1$ and $9 < 10 - c^2 < 10$, so:

$$|f(1) - T_2(1)| < \frac{1}{6}\left|\frac{1}{9}\right| = \frac{1}{54} < \frac{1}{50}$$

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S10.

Estimate $\sqrt{35}$ using a linear approximation

back to questions

S10.

Estimate $\sqrt{35}$ using a linear approximation

The general form of a linear approximation is

$$L(x) = f(a) + f'(a)(x - a)$$

If $f(x) = \sqrt{x}$ and $a = 36$, then $f(a) = 6$ and $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{12}$. So,

$$L(x) = 6 + \frac{1}{12}(x - 36)$$

Then: $\sqrt{35} = f(35) \approx L(35) = 6 + \frac{1}{12}(35 - 36) = 6 - \frac{1}{12} = \frac{71}{12}$

back to questions

S11.

Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

back to questions

S11.

Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

We note that $f(x)$ is continuous and differentiable over all real numbers. Since $f(0) = f(2\pi) = 0$, by Rolle's Theorem (also by the Mean Value Theorem) there exists some c between 0 and 2π such that $f'(c) = 0$.

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S12.

Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

back to questions

S12.
Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

We find where the first derivative is positive.

$$\begin{aligned} 0 < f'(x) &= \frac{(x+6)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+6)^2} && \text{multiply by } (x+6)^2 \\ 0 &< (x+6)\frac{1}{2\sqrt{x}} - \sqrt{x} && \text{multiply by } 2\sqrt{x} \\ 0 &< (x+6) - 2x \\ x &< 6 \end{aligned}$$

Note, however, that the function's derivative *does not exist* when $x \leq 0$. So the interval is $(0, 6)$.

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L1.

Compute the limit $\lim_{x \to 1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$.

back to questions

L1.

Compute the limit $\lim_{x\to 1} \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1}$.

If we try to do the limit naively we get 0/0, so we simplify.

$$\begin{aligned}\frac{\sqrt{x+2}-\sqrt{4-x}}{x-1} &= \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1}\cdot\frac{\sqrt{x+2}+\sqrt{4-x}}{\sqrt{x+2}+\sqrt{4-x}}\\ &= \frac{(x+2)-(4-x)}{(x-1)(\sqrt{x+2}+\sqrt{4-x})}\\ &= \frac{2x-2}{(x-1)(\sqrt{x+2}+\sqrt{4-x})}\\ &= \frac{2}{\sqrt{x+2}+\sqrt{4-x}}\end{aligned}$$

$$\lim_{x\to 1}\frac{\sqrt{x+2}-\sqrt{4-x}}{x-1} = \lim_{x\to 1}\frac{2}{\sqrt{x+2}+\sqrt{4-x}} = \frac{2}{\sqrt{3}+\sqrt{3}} = \frac{1}{\sqrt{3}}$$

back to questions

L2.

Show that there exists at least one real number c such that $2\tan(c) = c + 1$.

back to questions

L2.
Show that there exists at least one real number c such that $2\tan(c) = c + 1$.

- $\tan x$ is continuous on the interval $(-\pi/2, \pi/2)$
- $x + 1$ is a polynomial and therefore continuous for all real numbers
- So, $f(x) = 2\tan(x) - x - 1$ is a continuous function on the interval $(-\pi/2, \pi/2)$.
- Set $a = 0$. Then a is in the interval $(-\pi/2, \pi/2)$ and
$$f(a) = 2\tan(0) - 0 - 1 = 0 - 1 = -1 < 0.$$
- Set $b = \frac{\pi}{4}$. Then b is in the interval $(-\pi/2, \pi/2)$ and
$$f(b) = 2\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - 1 = 2 - \frac{\pi}{4} - 1 = 1 - \frac{\pi}{4} = \frac{4-\pi}{4} > 0.$$
- All together: $f(x)$ is continuous on $[0, \pi/4]$, and $f(0) < 0$ while $f(\pi/4) > 0$. Then the Intermediate Value Theorem guarantees the existence of a real number $c \in (0, \pi/4)$ such that $f(c) = 0$.

◂ back to questions

L3.

Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

back to questions

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Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

The function is differentiable at $x = 0$ if the following limit exists:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 0}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

Note that we used the fact that $f(0) = 0$ following the definition of the first branch, which includes the point $x = 0$.

We compute left and right limits of $\frac{f(x)}{x}$ as x goes to 0.

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{2x^3 - x^2}{x} = \lim_{x\to 0^-} 2x^2 - x = 0$$
$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^+} x \cdot \sin\left(\frac{1}{x}\right)$$

Next use the squeeze theorem. Note, first, that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so that $-x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq x$. Note, second, that $\lim\limits_{x\to 0} x = \lim\limits_{x\to 0} -x = 0$. So, by the squeeze theorem,

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

Since the left and right limits match (they're both equal to 0), we conclude that indeed $f(x)$ is differentiable at $x = 0$ (and its derivative at $x = 0$ is actually equal to 0).

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L4.

If $x^2 \cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$. You must justify your answer.

back to questions

L4.

If $x^2 \cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$. You must justify your answer.

- First we find the x-coordinates where $y = 0$.

$$x^2 \cos(0) + 2xe^0 = 8$$
$$x^2 + 2x - 8 = 0$$
$$(x + 4)(x - 2) = 0$$

  So $x = 2, -4$.

- Now we use implicit differentiation to get y' in terms of x, y:

$$x^2 \cos(y) + 2xe^y = 8 \quad \text{differentiate both sides}$$
$$x^2 \cdot (-\sin y) \cdot y' + 2x \cos y + 2xe^y \cdot y' + 2e^y = 0$$

- Now set $y = 0$ to get

$$\begin{aligned} x^2 \cdot (-\sin 0) \cdot y' + 2x\cos 0 + 2xe^0 \cdot y' + 2e^0 &= 0 \\ 0 + 2x + 2xy' + 2 &= 0 \\ 2xy' &= -(2x+2) \\ y' &= -\frac{1+x}{x} \end{aligned}$$

- So at $(x, y) = (2, 0)$ we have $y' = -\frac{3}{2}$,
- and at $(x, y) = (-4, 0)$ we have $y' = -\frac{3}{4}$.

back to questions

L5.

Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at $(0, 12)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

back to questions

L5.

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The position of particle A along the x axis starts at $(10, 0)$, and moves toward the origin at 2 units per second, so its position is given by $\big(x(t), 0\big)$ with $x(t) = 10 - 2t$, where t is measured in seconds. Similarly, the position of B along the y axis is given by $\big(0, y(t)\big)$ with $y(t) = 12 - 3t$. The distance $z(t)$ between the two particles satisfies $z(t)^2 = x(t)^2 + y(t)^2$.

When $x(t) = 4$, we solve $4 = 10 - 2t$ for t and find $t = 3$, so $y(3) = 12 - 3(3) = 3$. Then $z = 5$ when $t = 3$.
Differentiating implicitly, $z(t)^2 = x(t)^2 + y(t)^2$ tells us

$$2z(t)\frac{\mathrm{d}z}{\mathrm{d}t}(t) = 2x(t)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + 2y(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t)$$

so, when $t = 3$,

$$2(5)\frac{\mathrm{d}z}{\mathrm{d}t}(3) = 2(4)(-2) + 2(3)(-3) = -34$$

Then the distance between the two particles is changing at $-\frac{17}{5}$ units per second.

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L6.

Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

back to questions

L6.

Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

We compute $f'(x) = 3x^2 - 12x$. So $f(x)$ has no singular points (i.e. it is differentiable for all x), but has two critical points obtained by solving

$$f'(x) = 3x(x-4) = 0$$

which yields the two critical points $x = 0$ and $x = 4$. Only the critical point $x = 4$ is in the allowed interval $[3, 5]$.
In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[3, 5]$, we compute the value of f at the allowed critical point and at the end points of the allowed interval.

$$f(3) = -25, \quad f(4) = -30 \quad \text{and} \quad f(5) = -23.$$

So, the global max is -23 while the global min is -30.