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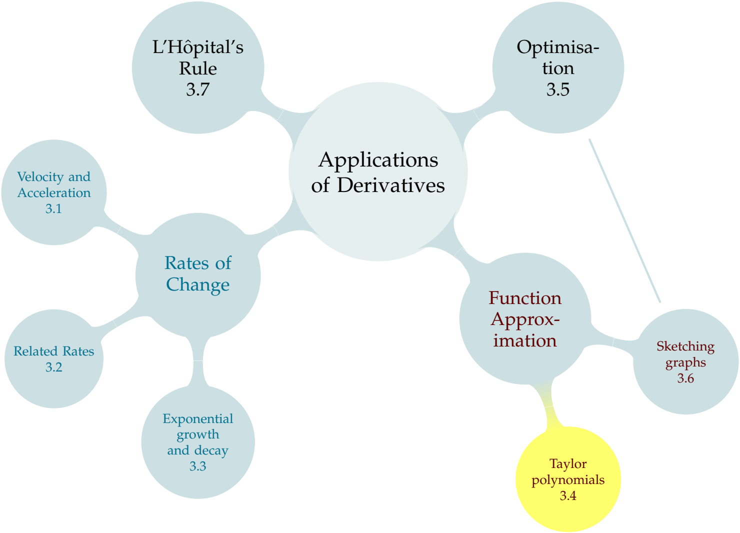
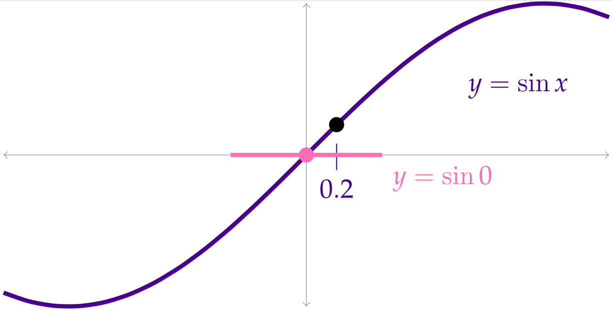


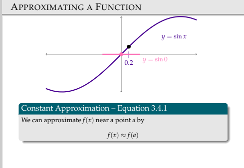
Approximating a Function



Constant Approximation – Equation 3.4.1

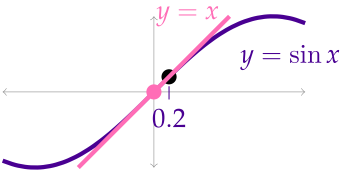
We can approximate $f(x)$ near a point a by

$$f(x) \approx f(a)$$



Mention: it's worth noting that even Google (or any other calculator) is also only providing an approximation, not an exact answer.

Approximating a Function



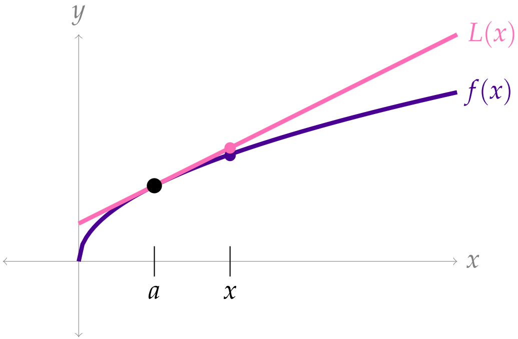
Linear Approximation (Linearization) – Equation 3.4.3

We can approximate $f(x)$ near a point a by the tangent line to $f(x)$ at a, namely

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

To find a linear approximation of $f(x)$ at a particular point x, pick a point a near to x, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$



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$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

Let $f(x) = \sqrt{x}$. Approximate $f(8.9)$.

CAN WE COMPUTE?

Suppose we want to approximate the value of $\cos(1.5)$. Which of the following linear approximations could we calculate by hand? (You can leave things in terms of π.)

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = 3/2$

C. both

D. neither

CAN WE COMPUTE?

Which of the following tangent lines is probably the most accurate in approximating $\cos(1.5)$?

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$
B. tangent line to $f(x) = \cos x$ when $x = \pi/4$
C. constant approximation: $\cos 1.5 \approx \cos(\pi/2) = 0$
D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

Linear Approximation

Approximate $\sin(3)$ using a linear approximation. You may leave your answer in terms of π.

Linear Approximation

Approximate $e^{1/10}$ using a linear approximation.
If $f(x) = e^x$ and $a = 0$:

Linear Approximation

Approximate $e^{1/10}$ using a linear approximation.
If $f(x) = e^x$ and $a = 0$:

The question often comes up here, what would we do on an assessment? I like to reassure students that questions will be worded in a way that makes expectations clear. For example, "your answer may depend on e," or "your answer should be rational."

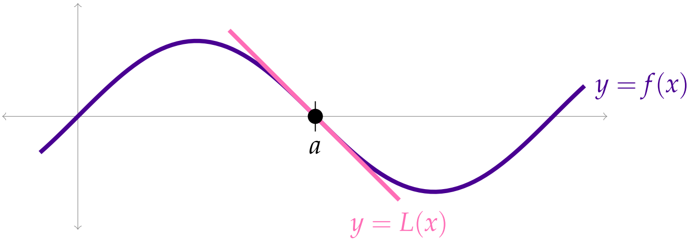
Linear Approximation Wrap-Up

Let $L(x) = f(a) + f'(a)(x-a)$, so $L(x)$ is the linear approximation (linearization) of $f(x)$ at a.

What is $L(a)$?

What is $L'(a)$?

What is $L''(a)$? (Recall $L''(x)$ is the derivative of $L'(x)$.)



Linear Approximation Wrap-Up

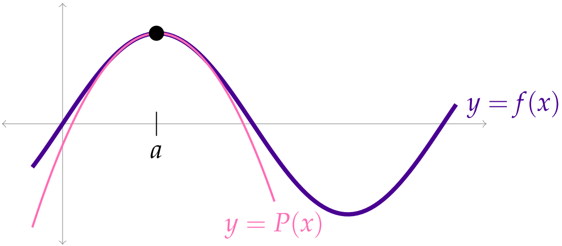
Let $L(x)$ be a linear approximation of $f(x)$.

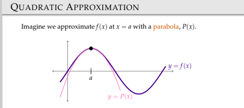
$f(a)$	$L(a)$	same
$f'(a)$	$L'(a)$	same
$f''(a)$	$L''(a)$	different[1]

[1] unless $f''(a) = 0$

Quadratic Approximation

Imagine we approximate $f(x)$ at $x = a$ with a parabola, $P(x)$.





"We want our approximation and our function to be as similar as possible (except our approximation should be easy to compute – like a polynomial). That's why we might think to match as many derivatives as possible at the point $x = a$.""You won't have to solve these equations, but this is where our quadratic approximation comes from"

	Constant	Linear	Quadratic
Function value matches at $x = a$	$\checkmark$	$\checkmark$	$\checkmark$
First derivative matches at $x = a$	$\times$	$\checkmark$	$\checkmark$
Second derivative matches at $x = a$	$\times$	$\times$	$\checkmark$

Constant:	$f(x) \approx f(a)$
Linear:	$f(x) \approx f(a) + f'(a)(x-a)$
Quadratic:	$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

3.4.3: Quadratic

Constant:	$f(x) \approx f(a)$
Linear:	$f(x) \approx f(a) + f'(a)(x-a)$
Quadratic:	$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

"We can see that each successive type adds a little accuracy by adding a higher-degree term"

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.

Determine what $f(x)$ and a should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$

$e^{-1/30}$

$\sqrt[5]{30}$

$(2.01)^6$

	Constant	Linear	Quadratic	degree n
match $f(a)$	✓	✓	✓	✓
match $f'(a)$	×	✓	✓	✓
match $f''(a)$	×	×	✓	✓
...				
match $f^{(n)}(a)$	×	×	×	✓
match $f^{(n+1)}(a)$	×	×	×	×

Constant:

$$f(x) \approx f(a)$$

Linear:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Quadratic:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Degree-n:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$$

3.4.4-5: Taylor Polynomial

Constant:
$f(x) \approx f(a)$

Linear:
$f(x) \approx f(a) + f'(a)(x-a)$

Quadratic:
$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Degree-n:
$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$

"We can see that each successive type adds a little accuracy by adding a higher-degree term"

Constant:
$f(x) \approx f(a)$

Linear:
$f(x) \approx f(a) + f'(a)(x-a)$

Quadratic:
$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Degree-n:
$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$

We'll use sum notation for Taylor Polynomials, so useful to review it. Students should have seen it in high school.

Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

- a, b (integers) "bounds"
- i "index": runs over integers from a to b
- $f(i)$ "summand": compute for every i, add

Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

Sigma Notation

$$\sum_{i=1}^{4}(i+(i-1)^2)$$

Write the following expressions in sigma notation:

1. $3+4+5+6+7$
2. $8+8+8+8+8$
3. $1+(-2)+4+(-8)+16$

Factorial – Definition 3.4.9

We read "n!" as "n factorial."
For a natural number n, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.
By convention, $0! = 1$.

We write $f^{(n)}(x)$ to mean the n^{th} derivative of $f(x)$. By convention, $f^{(0)}(x) = f(x)$.

Taylor Polynomial – Definition 3.4.11

Given a function $f(x)$ that is differentiable n times at a point a, the n-th degree **Taylor polynomial** for $f(x)$ about a is

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, we also call it a **Maclaurin polynomial**.

3.4.4-5: Taylor Polynomial

Factorial – Definition 3.4.9

We read "$n!$" as "n factorial."
For a natural number n, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.
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Taylor Polynomial – Definition 3.4.11

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If $a = 0$, we also call it a **Maclaurin polynomial**.

Students usually ask about $0! = 1$. Two explanations. First, it's a convention because people found it convenient. Second, $n!$ is the number of ways of ordering n distinct objects.

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=$$

3.4.4-5: Taylor Polynomial

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$=$$

Some students will undoubtedly struggle with interpreting the sigma notation, so I like to go through the indices until at least $k = 4$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Maclaurin[2] polynomial for e^x.

[2] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

Now You

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

skip Δx notation

Notation 3.4.18

Let x, y be variables related such that $y = f(x)$. Then we denote a small change in the variable x by Δx (read as "delta x"). The corresponding small change in the variable y is denoted Δy (read as "delta y").

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let $y = f(x)$ be the amount of water needed to produce x apples in an orchard.
A farmer wants to know how a much water is needed to increase their crop yield. Δx is shorthand for some change in the number of apples, and Δy is shorthand for some change in the amount of water.

- Consider changing the number of apples grown from a to $a + \Delta x$
- Then the change in water requirements goes from $y = f(a)$ to $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximation of Δy

- Using a linear approximation, setting $x = a + \Delta x$:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) && \text{linear approximation} \\ f(a+\Delta x) &\approx f(a) + f'(a)(\Delta x) && \text{set } x = a + \Delta x \\ \Delta y = f(a+\Delta x) - f(a) &\approx f'(a)\Delta x && \text{subtract } f(a) \text{ both sides} \end{aligned}$$

Linear Approximation of Δy (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

If we set $\Delta x = 1$, then $\Delta y \approx f'(a)$. So, if we want to produce $a+1$ apples instead of a apples, the extra water needed for that one extra apple is about $f'(a)$. We call this the *marginal* water cost of the apple.

Quadratic Approximation of Δy

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

Quadratic Approximation of Δy (Equation 3.4.21)

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and π.

You measure an angle $x \approx \frac{\pi}{2}$, and use it to calculate $y = \sin x \approx 1$. However, you suspect the angle was not *exactly* equal to $\frac{\pi}{2}$, which means the actual value y is slightly *less than* 1. In order for your value of y to have an error of no more than $\frac{1}{200}$, how accurate does your measurement of θ have to be?

Definition 3.4.25

Let Q_0 be the exact value of a quantity and let $Q_0 + \Delta Q$ be the measured value. We call

$$|\Delta Q|$$

the absolute error of the measurement, and

$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

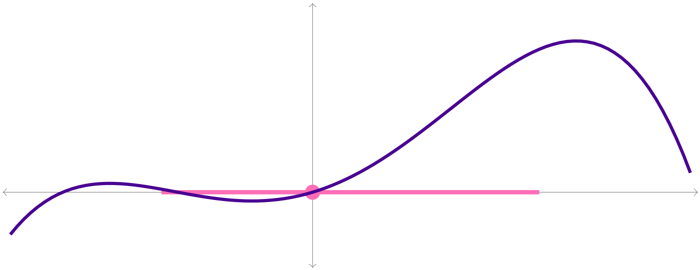
Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

3.4.8: Error in Taylor

Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

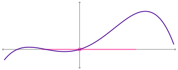
What we're going to do now is introduce an equation that can help us understand the error in our approximations when we use Taylor polynomials. We won't show you exactly where the equation comes from, but I want to give you a little intuition. So the following is another TED talk: it's background to help you understand what we'll be doing later, but you won't be assessed on it.

ERROR: WHAT "CAUSES" ERROR IN AN ESTIMATION?



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

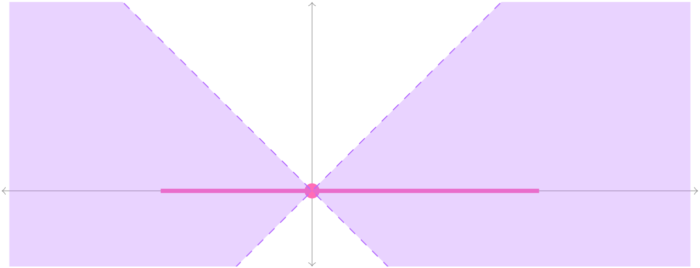
Error: what “causes” error in an estimation?



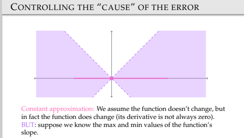
Constant approximation: We assume the function doesn’t change, but in fact the function does change (its derivative is not always zero).

“After linear, the explanations lose some intuitiveness, but it’s the same idea.”

Controlling the "cause" of the error



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero). BUT: suppose we know the max and min values of the function's slope.



“The absolute biggest the function could be is here, because it can’t grow any faster than this line; the absolute smallest...”TED talk over

Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

3.4.8: Error in Taylor

Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

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The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

The "sloppiness" allowed in choosing c can be very stressful for students. So it's nice to explain, as you're working problems, what would be reasonable and what would not.

Third degree Maclaurin polynomial for $f(x) = e^x$:

$$\begin{aligned}
T_3(x) &= f(0) + f'(0)(x-0) + \tfrac{1}{2!}f''(0)(x-0)^2 + \tfrac{1}{3!}f'''(0)(x-0)^3 \\
&= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}
\end{aligned}$$

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could the magnitude of the error be if we approximate $\cos(2)$?

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use a third degree Taylor polynomial centred at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for our error.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose you want to approximate the value of e, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound your error.

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?

Which Degree?

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Which Degree?

Suppose you want to approximate e^5 using a Maclaurin polynomial of $f(x) = e^x$. If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

WHICH DEGREE?

Suppose you want to approximate $\log \frac{4}{3}$ using a Taylor polynomial of $f(x) = \log x$ centred at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Which Degree?

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

Included Work

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 'old tree' by FayraLovers is licensed under CC BY 3.0 (accessed 21 July 2021), 38