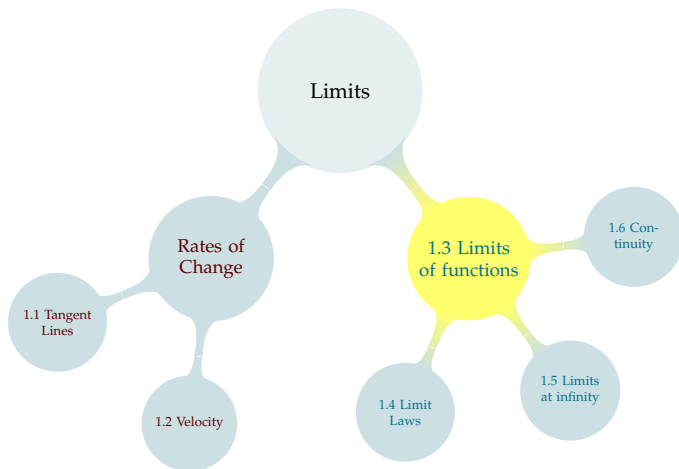


# TABLE OF CONTENTS



## Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \rightarrow a} f(x) = L$$

where  $a$  and  $L$  are real numbers

We read the above as “the limit as  $x$  goes to  $a$  of  $f(x)$  is  $L$ .”

Its meaning is: as  $x$  gets very close to (but not equal to)  $a$ ,  $f(x)$  gets very close to  $L$ .

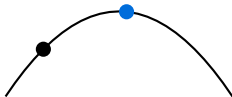
# FINDING SLOPES OF TANGENT LINES

We NEED limits to find slopes of tangent lines.



# FINDING SLOPES OF TANGENT LINES

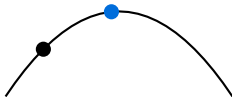
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Slope of secant line:  $\frac{\Delta y}{\Delta x}$ ,  $\Delta x \neq 0$ .

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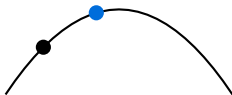
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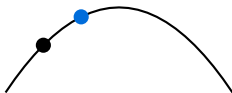
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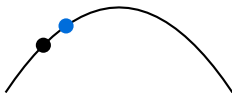
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If the position of an object at time  $t$  is given by  $s(t)$ , then its instantaneous velocity is given by

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Slope of tangent line: *can't do the same way*.

If the position of an object at time  $t$  is given by  $s(t)$ , then its instantaneous velocity is given by

$$\lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$$

# EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate  $\lim_{x \rightarrow 1} f(x)$ .

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What is  $f(1)$ ?

# EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate  $\lim_{x \rightarrow 1} f(x)$ .

What is  $f(1)$ ? **DNE** (can't divide by zero)

## EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate  $\lim_{x \rightarrow 1} f(x)$ .

Use the tables below to guess  $\lim_{x \rightarrow 1} f(x)$

$x$	$f(x)$	$x$	$f(x)$
0.9	3.61	1.1	4.41
0.99	3.9601	1.01	4.0401
0.999	3.99600	1.001	4.00400
0.9999	3.99960	1.0001	4.00040

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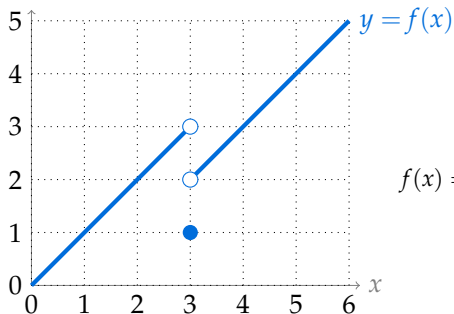
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$$\lim_{x \rightarrow 1} f(x) = 4$$





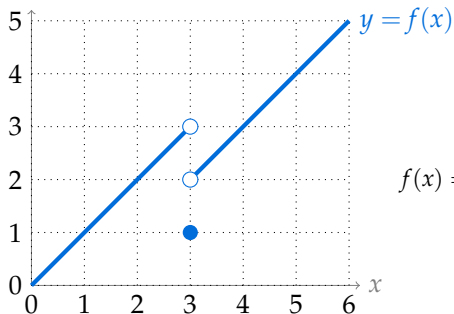
## ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think  $\lim_{x \rightarrow 3} f(x)$  should be?

## ONE-SIDED LIMITS

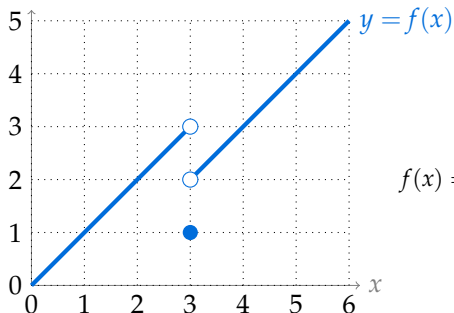


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DNE

## ONE-SIDED LIMITS

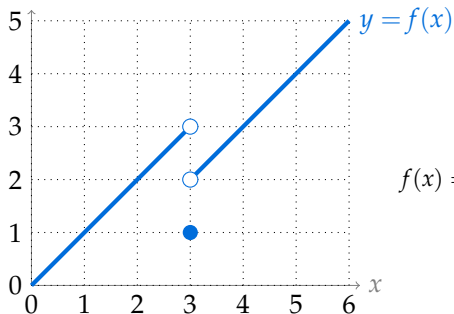


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Evaluate:  $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}}$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}}$

## ONE-SIDED LIMITS



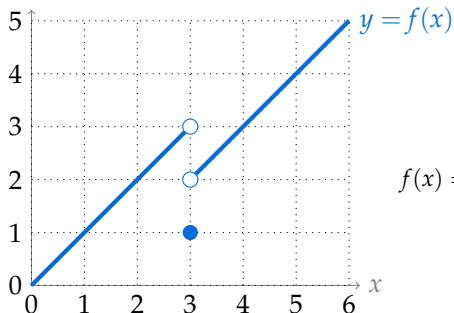
$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate:  $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}} = 3$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}}$



## ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate:  $\lim_{x \rightarrow 3^-} f(x) = 3$   
 from the left

$\lim_{x \rightarrow 3^+} f(x) = 2$   
 from the right



### Definition 1.3.7

The limit as  $x$  goes to  $a$  **from the left** of  $f(x)$  is written

$$\lim_{x \rightarrow a^-} f(x)$$

We only consider values of  $x$  that are **less than**  $a$ .

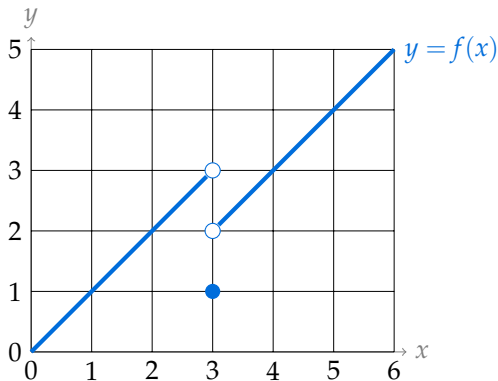
The limit as  $x$  goes to  $a$  **from the right** of  $f(x)$  is written

$$\lim_{x \rightarrow a^+} f(x)$$

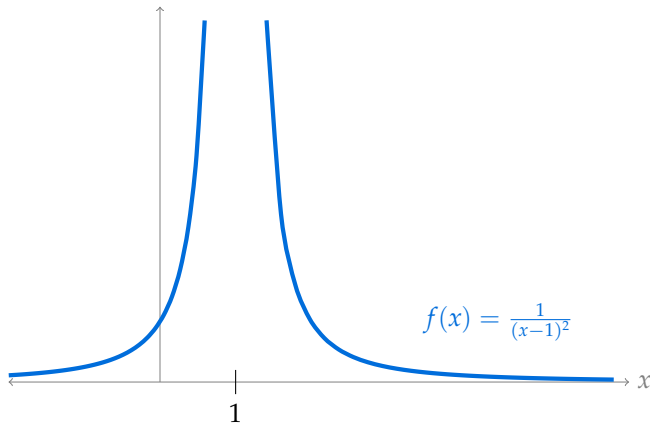
We only consider values of  $x$  **greater than**  $a$ .

## Theorem 1.3.8

In order for  $\lim_{x \rightarrow a} f(x)$  to exist, both one-sided limits must exist and be equal.



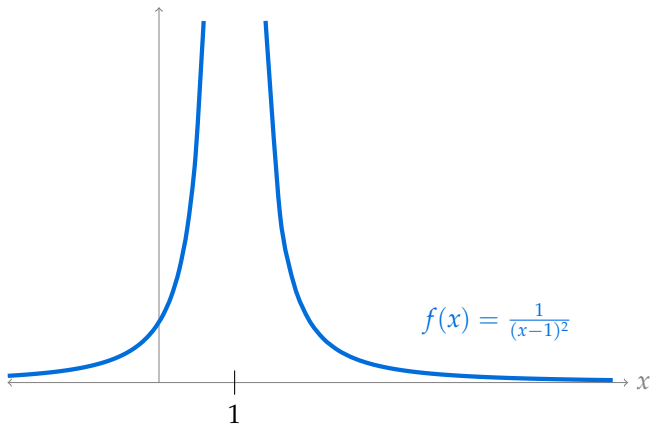
Consider the function  $f(x) = \frac{1}{(x-1)^2}$ . For what value(s) of  $x$  is  $f(x)$  **not** defined?





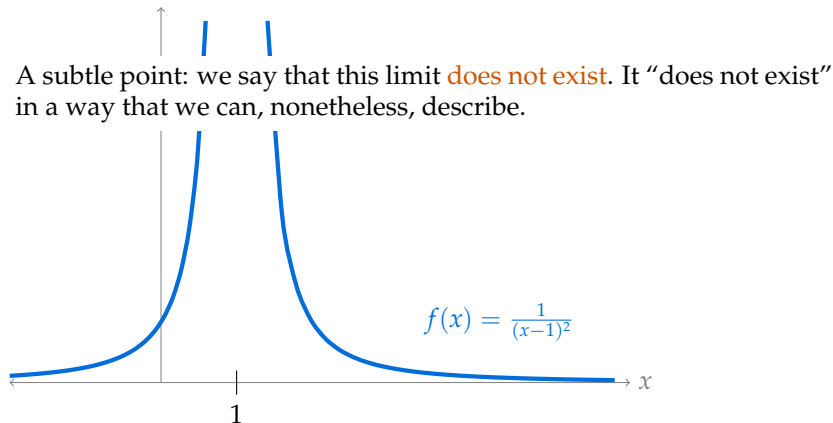
Based on the graph below, what would you like to write for:

$$\lim_{x \rightarrow 1} f(x) =$$

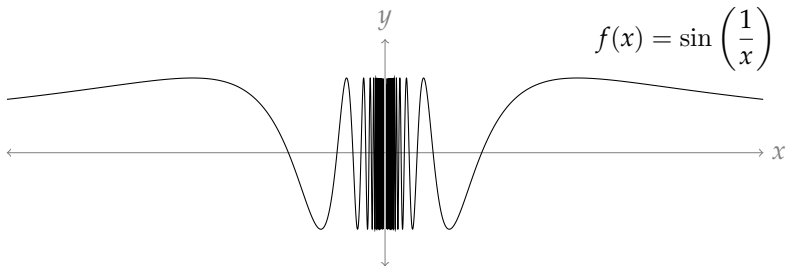


Based on the graph below, what would you like to write for:

$$\lim_{x \rightarrow 1} f(x) = \infty$$

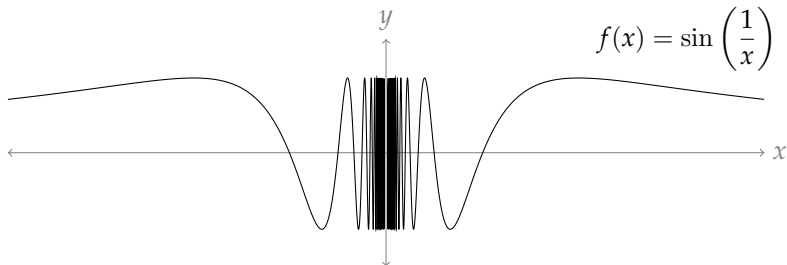


# A STRANGER LIMIT EXAMPLE



What is  $\lim_{x \rightarrow \infty} f(x)$ ?

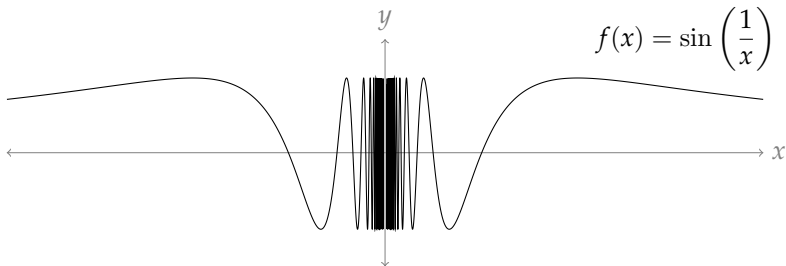
## A STRANGER LIMIT EXAMPLE



What is  $\lim_{x \rightarrow \infty} f(x)$ ?

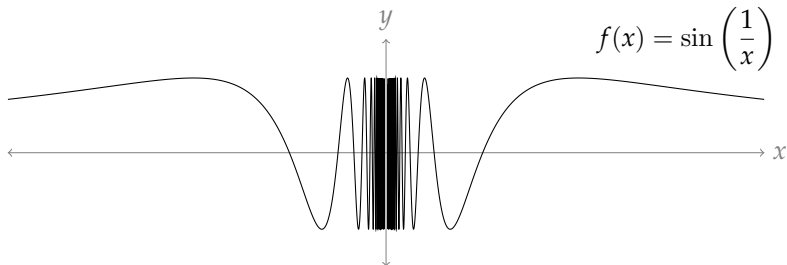
$$\lim_{x \rightarrow \infty} f(x) = 0$$

# A STRANGER LIMIT EXAMPLE



What is  $\lim_{x \rightarrow 0} f(x)$  ?

# A STRANGER LIMIT EXAMPLE

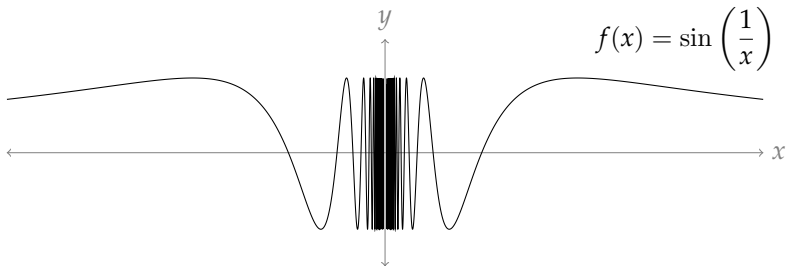


What is  $\lim_{x \rightarrow 0} f(x)$ ?

$\lim_{x \rightarrow 0} f(x)$  does not exist.

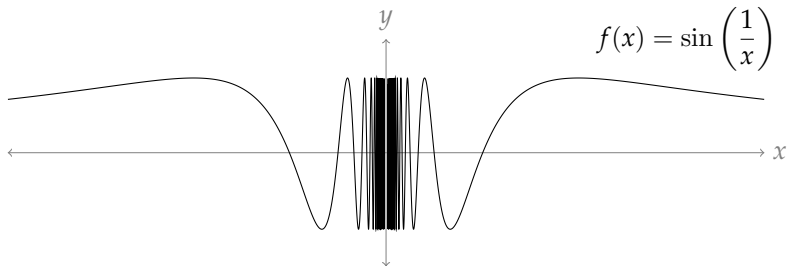
We can call this behaviour “infinite wiggling.”

## A STRANGER LIMIT EXAMPLE



What is  $\lim_{x \rightarrow \pi} f(x)$  ?

## A STRANGER LIMIT EXAMPLE



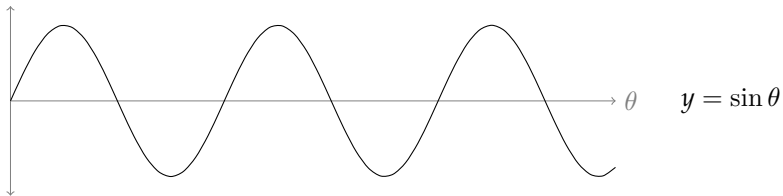
What is  $\lim_{x \rightarrow \pi} f(x)$ ?

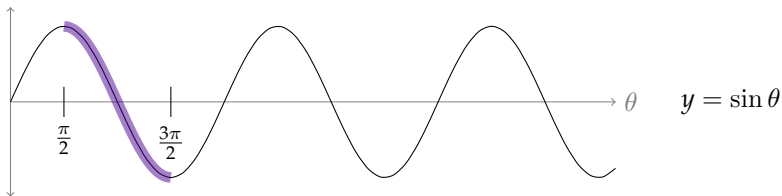
$$\lim_{x \rightarrow \pi} f(x) = \sin\left(\frac{1}{\pi}\right)$$



OPTIONAL: SKETCHING  $f(x) = \sin\left(\frac{1}{x}\right)$ 

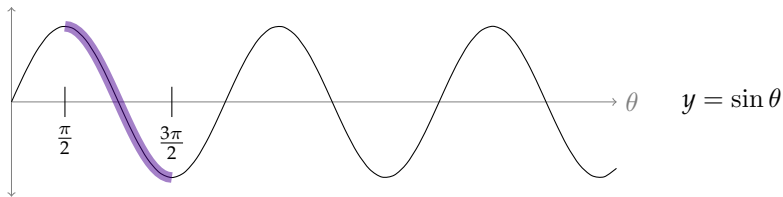
▶ SKIP SKETCHING



OPTIONAL: SKETCHING  $f(x) = \sin\left(\frac{1}{x}\right)$ [▶ SKIP SKETCHING](#)

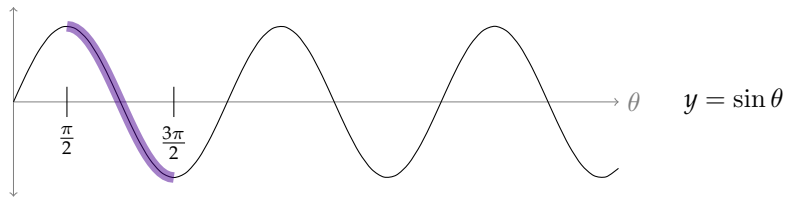
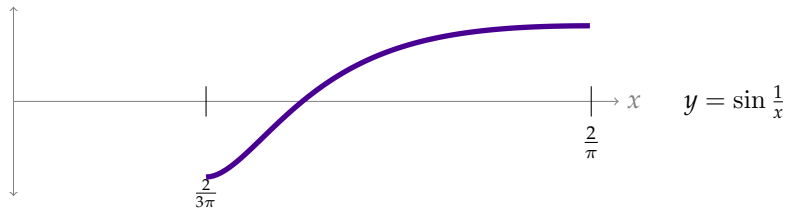
OPTIONAL: SKETCHING  $f(x) = \sin\left(\frac{1}{x}\right)$ 

▶ SKIP SKETCHING



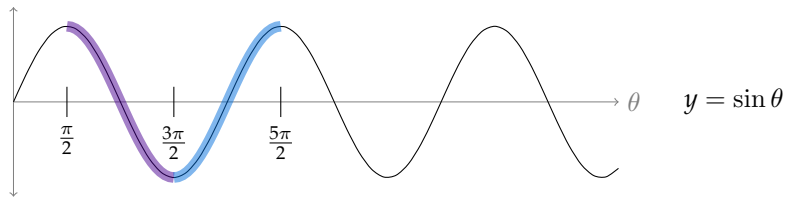
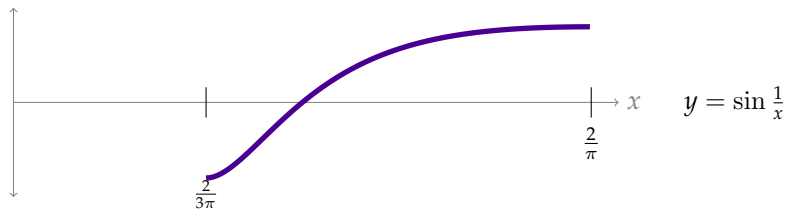
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▶ SKIP SKETCHING



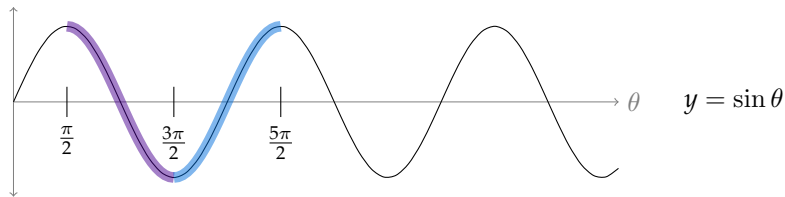
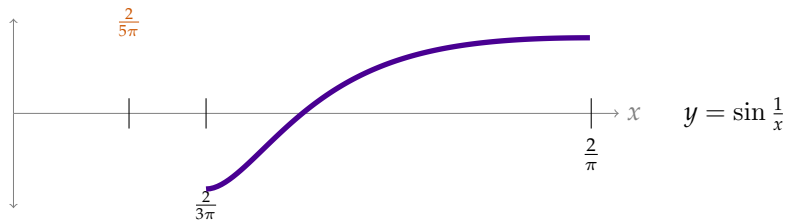
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▶ SKIP SKETCHING



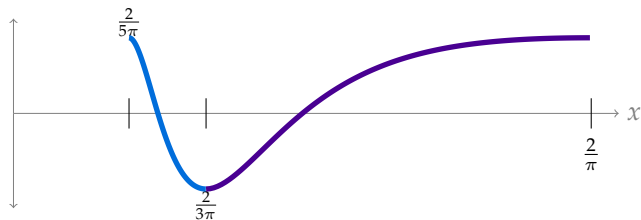
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▶ SKIP SKETCHING

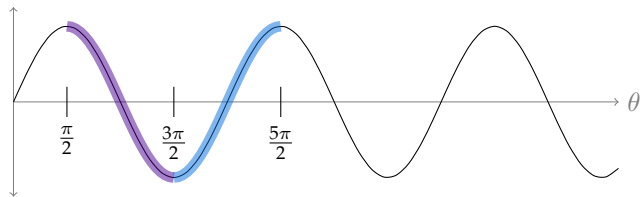


OPTIONAL: SKETCHING  $f(x) = \sin\left(\frac{1}{x}\right)$ 

▶ SKIP SKETCHING



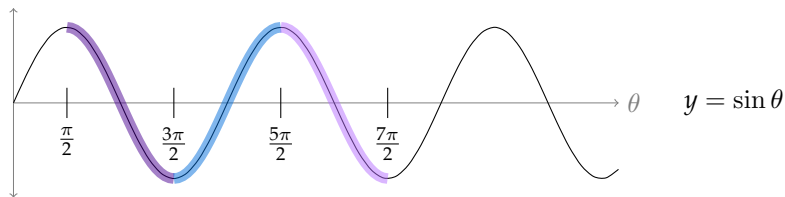
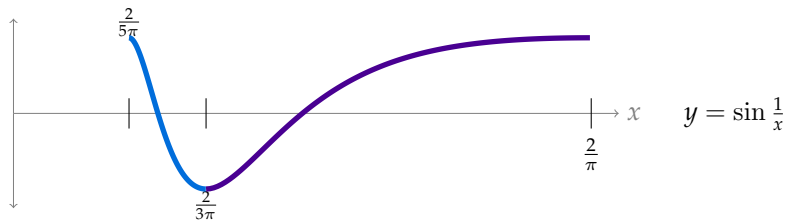
$$y = \sin \frac{1}{x}$$



$$y = \sin \theta$$

OPTIONAL: SKETCHING  $f(x) = \sin\left(\frac{1}{x}\right)$ 

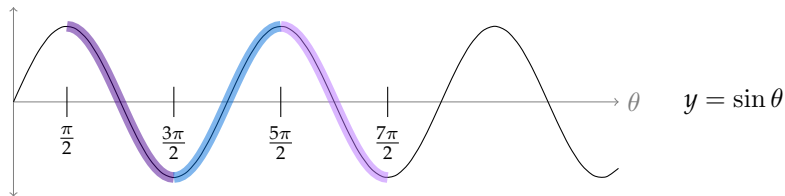
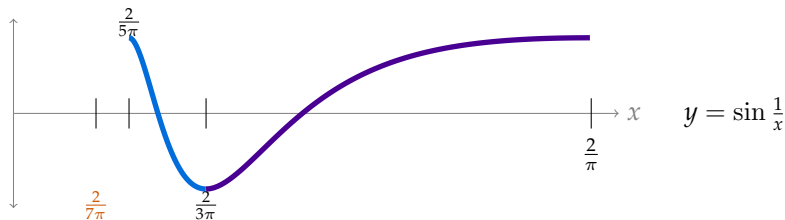
▶ SKIP SKETCHING





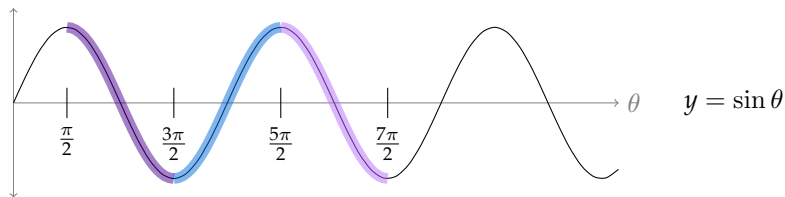
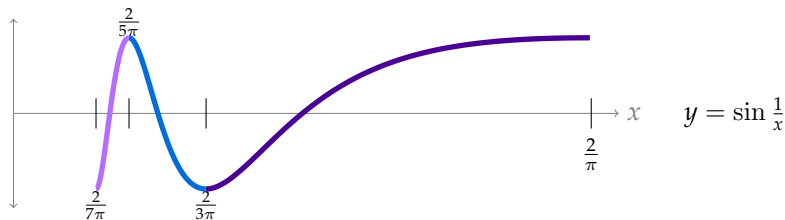
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▶ SKIP SKETCHING



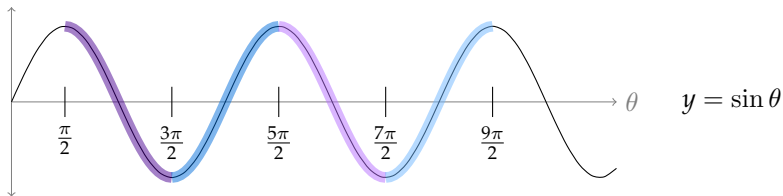
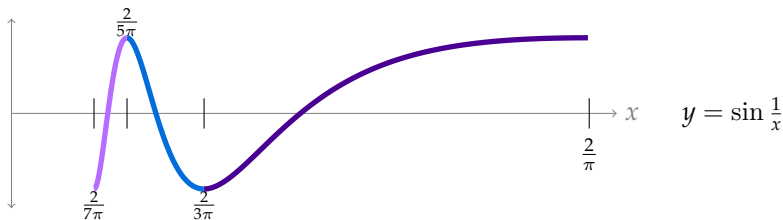
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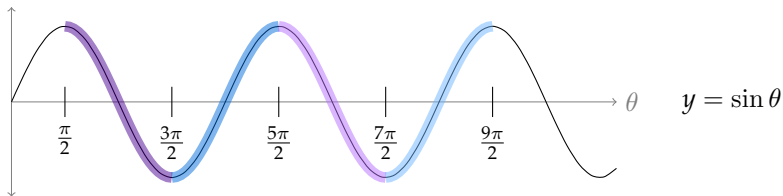
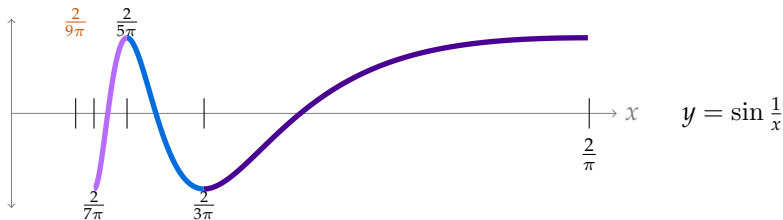
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▶ SKIP SKETCHING



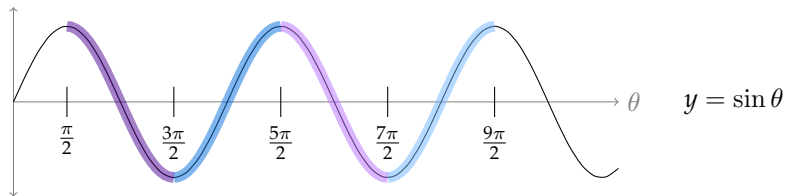
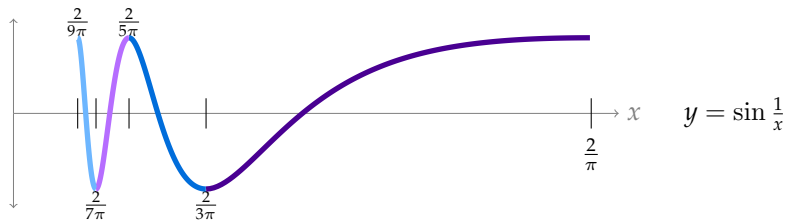
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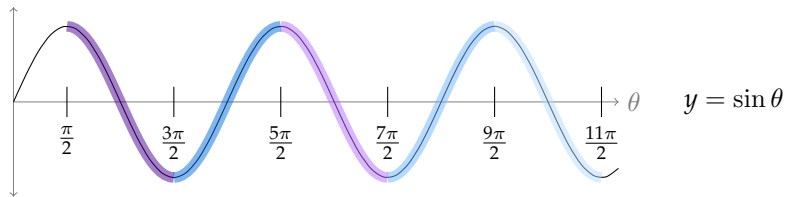
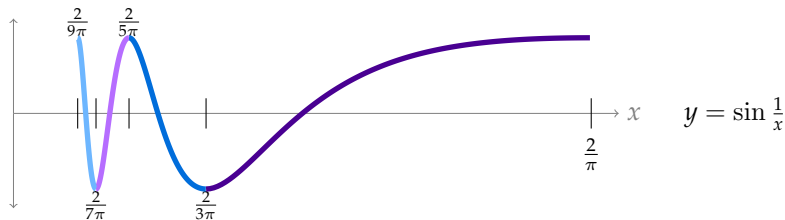
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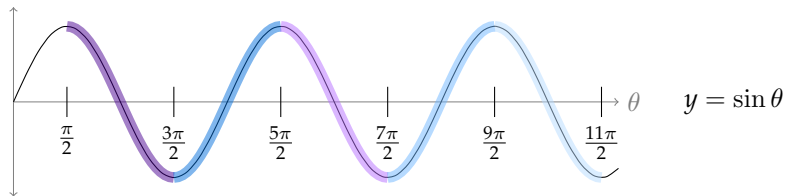
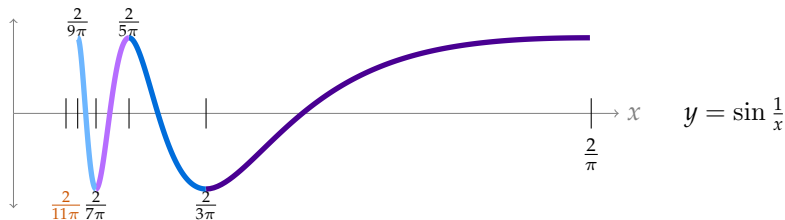
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▶ SKIP SKETCHING



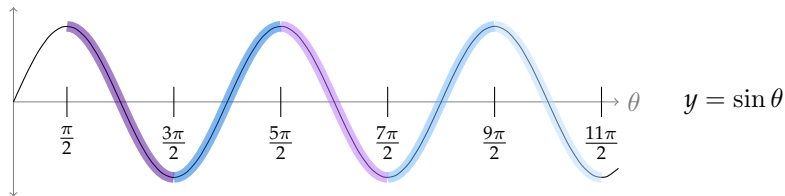
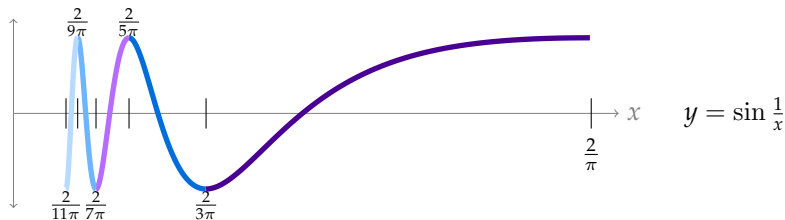
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▶ SKIP SKETCHING



OPTIONAL: SKETCHING  $f(x) = \sin\left(\frac{1}{x}\right)$ 

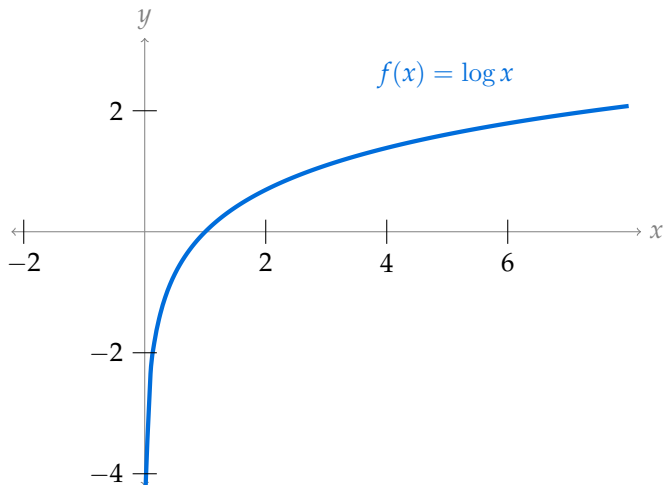
▶ SKIP SKETCHING





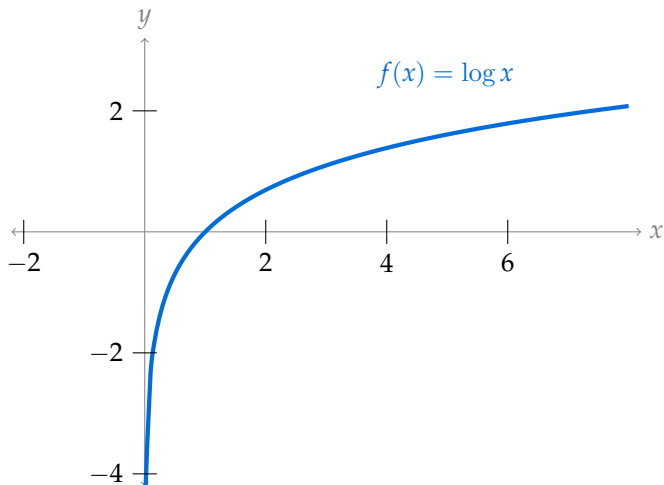
# LIMITS AND THE NATURAL LOGARITHM

Where is  $f(x)$  defined, and where is it not defined?



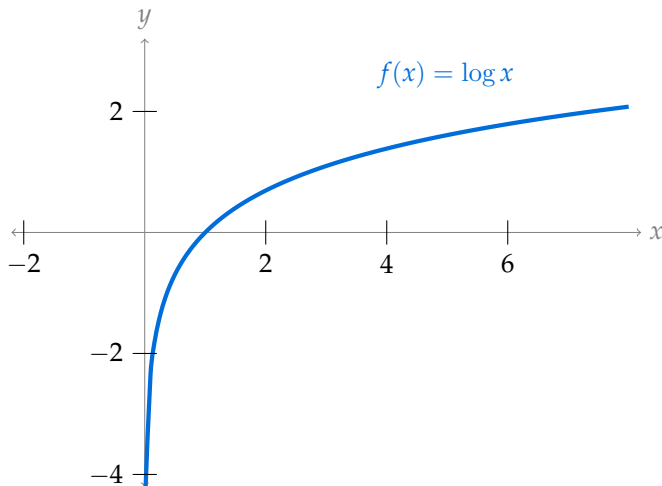
# LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of  $f(x)$  near 0?



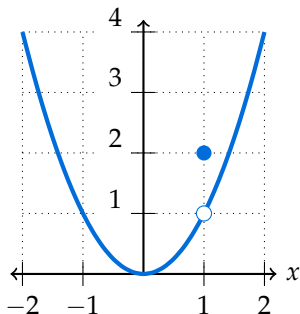
# LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of  $f(x)$  near 0?  $\lim_{x \rightarrow 0^+} \log(x) = -\infty$



## Section 1.3 Review

$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is  $\lim_{x \rightarrow 1} f(x)$ ?

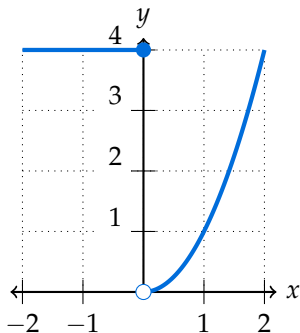
A.  $\lim_{x \rightarrow 1} f(x) = 2$

B.  $\lim_{x \rightarrow 1} f(x) = 1$

C.  $\lim_{x \rightarrow 1} f(x)$  DNE

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is  $\lim_{x \rightarrow 0} f(x)$ ?

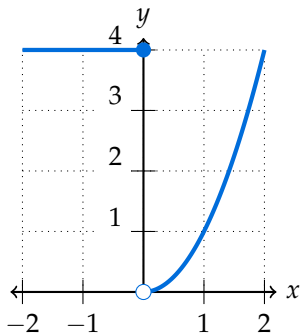
A.  $\lim_{x \rightarrow 0} f(x) = 4$

B.  $\lim_{x \rightarrow 0} f(x) = 0$

C.  $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is  $\lim_{x \rightarrow 0} f(x)$ ?

A.  $\lim_{x \rightarrow 0} f(x) = 4$

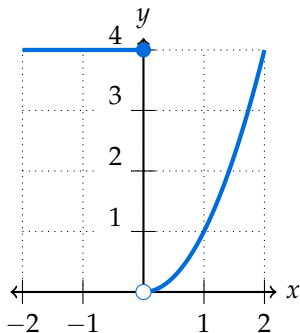
B.  $\lim_{x \rightarrow 0} f(x) = 0$

C.  $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$\lim_{x \rightarrow 0} f(x)$  DNE

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is  $\lim_{x \rightarrow 0^+} f(x)$ ?

A.  $\lim_{x \rightarrow 0} f(x) = 4$

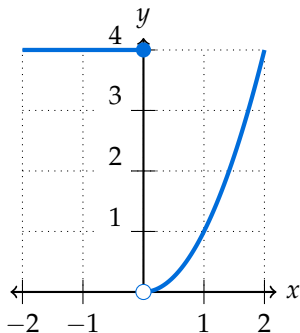
B.  $\lim_{x \rightarrow 0} f(x) = 0$

C.  $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above



$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is  $\lim_{x \rightarrow 0^+} f(x)$ ?

A.  $\lim_{x \rightarrow 0^+} f(x) = 4$

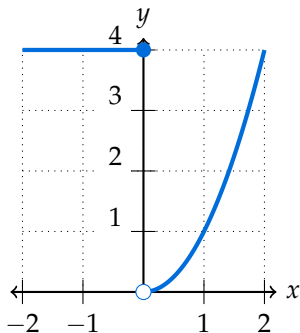
B.  $\lim_{x \rightarrow 0^+} f(x) = 0$

C.  $\lim_{x \rightarrow 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

What is  $f(0)$ ?



Suppose  $\lim_{x \rightarrow 3^-} f(x) = 1$  and  $\lim_{x \rightarrow 3^+} f(x) = 1.5$ .

Does  $\lim_{x \rightarrow 3} f(x)$  exist?

- A. Yes, certainly, because the limits from both sides exist.
- B. No, never, because the limit from the left is not the same as the limit from the right.
- C. Can't tell. For some functions it might exist, for others not.

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Suppose  $\lim_{x \rightarrow 3^-} f(x) = 22 = \lim_{x \rightarrow 3^+} f(x)$ .

Does  $\lim_{x \rightarrow 3} f(x)$  exist?

- A. Yes, certainly, because the limits from both sides exist and are equal to each other.
- B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.
- C. Can't tell. We need to know the value of the function at  $x = 3$ .

Suppose  $\lim_{x \rightarrow 3^-} f(x) = 22 = \lim_{x \rightarrow 3^+} f(x)$ .

Does  $\lim_{x \rightarrow 3} f(x)$  exist?

- A. Yes, certainly, because the limits from both sides exist and are equal to each other.
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## Included Work



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