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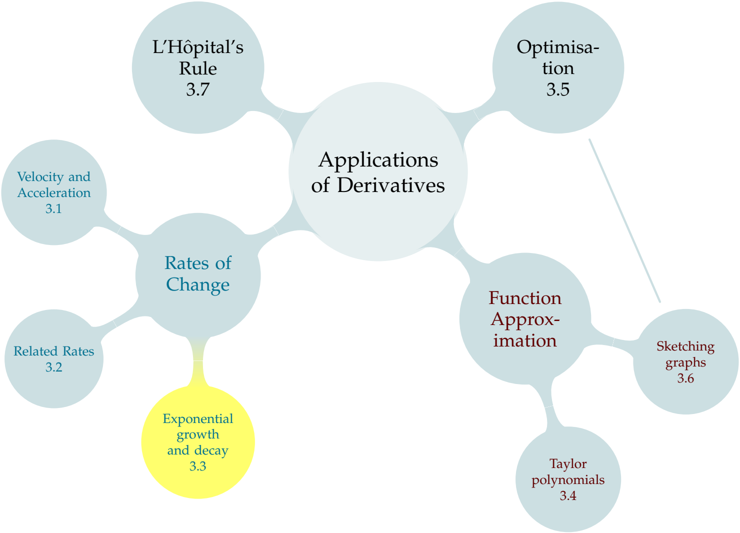

Radioactive Decay

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

Differential Equation

Let $Q = Q(t)$ be the amount of a radioactive substance at time t. Then for some positive constant k:

$$\frac{dQ}{dt} = -kQ$$

Solution – Theorem 3.3.2

Let $\boxed{Q(t) = Ce^{-kt}}$, where k and C are constants. Then:

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This is a first look at DEs. Take some time to point out how this equation looks different from what we're used to writing.emphasize we're looking for a function that makes the DE true, not just a number

RADIOACTIVE DECAY

Quantity of a Radioactive Isotope

$$Q(t) = Ce^{-kt}$$

$Q(t)$: quantity at time t

What is the sign of $Q(t)$?

A. positive or zero
B. negative or zero
C. could be either
D. I don't know

What is the sign of C?

A. positive or zero
B. negative or zero
C. could be either
D. I don't know

Seaborgium Decay

The amount of ^{266}Sg (Seaborgium-266) in a sample at time t (measured in seconds) is given by

$$Q(t) = Ce^{-kt}$$

Let's approximate the half life of ^{266}Sg as 30 seconds. That is, every 30 seconds, the size of the sample halves.

What are C and k?

A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope. In the year 2005, it is tested and found to contain only 2 units of that same isotope. How many units of the isotope were present in the year 2000?

$Q'(t) = kQ(t)$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

The rate of growth of a population in a given time interval is propotional to the number of individuals in the population, when the population has ample resources.

The amount of interest a bank account accrues in a given time interval is proportional to the balance in that bank account.

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Note the broader applicability of these equations

Exponential Growth – Theorem 3.3.2

Let $Q = Q(t)$ satisfy:

$$\frac{dQ}{dt} = kQ$$

for some constant k. Then for some constant $C = Q(0)$,

$$Q(t) = Ce^{kt}$$

Suppose $y(t)$ is a function with the properties that

$$\frac{dy}{dt} + 3y = 0 \quad \text{and} \quad y(1) = 2.$$

What is $y(t)$?

Population Growth

Suppose a petri dish starts with a culture of 100 bacteria cells and a limited amount of food and space. The population of the culture at different times is given in the table below. At approximately what time did the culture start to show signs of limited resources?

time	population
0	100
1	1000
3	100000
5	1000000

Flu Season

The CDC keeps records (link) on the number of flu cases in the US by week. At the start of the flu season, the 40th week of 2014, there are 100 cases of a particular strain. Five weeks later (at week 45), there are 506 cases. What do you think was the first week to have 5,000 cases? What about 10,000 cases?

Newton's Law of Cooling – Equation 3.3.7

The rate of change of temperature of an object is proportional to the difference in temperature between that object and its surroundings.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

where $T(t)$ is the temperature of the object at time t, A is the (constant) ambient temperature of the surroundings, and K is some constant depending on the object.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, K is some constant.

What is true of K?

A. $K \geq 0$

B. $K \leq 0$

C. $K = 0$

D. K could be positive, negative, or zero, depending on the object

E. I don't know

Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, and K is some constant.

$$T(t) = [T(0) - A]e^{Kt} + A$$

is the only function satisfying Newton's Law of Cooling

If $T(10) < A$, then:

A. $K > 0$
B. $T(0) > 0$
C. $T(0) > A$
D. $T(0) < A$

Evaluate $\lim\limits_{t \to \infty} T(t)$.

A. A
B. 0
C. ∞
D. $T(0)$

What assumptions are we making that might not square with the real world?

Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt} = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, and K is some constant.

Temperature of a Cooling Body – Corollary 3.3.8

$$T(t) = [T(0) - A]e^{Kt} + A$$

A farrier forms a horseshoe heated to 400° C, then dunks it in a river at room-temperature (25° C). The water boils for 30 seconds. The horseshoe is safe for the horse when it's 40° C. When can the farrier put on the horseshoe?

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A farrier is a person who puts horseshoes on horses. They often double as blacksmiths, heating the shoes up in a forge and hitting them on an anvil to shape them. Then to cool the shoes down, they may dunk them in a bucket of water, like the photo.

We're making the very gross assumptions that the water stops boiling when the shoe hits 100 degrees, and the temperature of the river water is constant.

A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°, and after 20 minutes, the tea is 25°. What is the temperature outside?

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This is a perennial frustration, but a good exercise. It anticipates a question in WeBWorK.

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Now a long list of questions, some of which you may have time for in class.

In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?

link: Wood Bison Restoration in Alaska, Alaska Department of Fish and Game

Excerpt:

> *Based on experience with reintroduced populations elsewhere, wood bison would be expected to increase at a rate of 15%-25% annually after becoming established.... With an average annual growth rate of 20%, an initial precalving population of 50 bison would increase to 500 in approximately 13 years.*

Now You

Are they using our same model?

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Now You: Are they using our same model?

Wood bison were thought to be extinct, except for populations that had interbred with plains bison. A pure-blooded population was discovered in Canada and bred in captivity to increase their numbers. Later, Canada sent some of these descendants to Alaska, where the Alaska Wildlife Conservation Center cared for them and increased their numbers, eventually releasing some. After being extinct in Alaska for around a century, there are once again wood bison in the wild.It can often seem like scientists have unknowable algorithms behind their predictions. When we understand where a model comes from, we can better understand its limitations. For example, the model we (and ADF&G) are using is simply that bison will reproduce in proportion to their population. It's useful to think about when that assumption might not hold.

Compound Interest

Suppose you invest \$10,000 in an account that accrues interest each month. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?

Compound interest is calculated according to the formula Pe^{rt}, where r is the interest rate and t is time.

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Compound interest is calculated according to the formula Pe^{rt}, where r is the interest rate and t is time.

Probably students have seen a derivation of the compound interest formula that didn't use DE. Since the balance grows proportionally to how much is in it, it system satisfies the same DE, so it'll have the same solution.

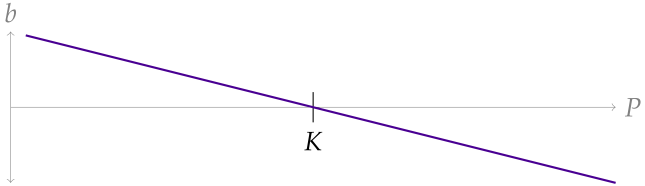
Carrying Capacity

For a population of size P with unrestricted access to resources, let β be the average number of offspring each breeding pair produces per generation, where a generation has length t_g. Then $b = \frac{\beta - 2}{2t_g}$ is the net birthrate (births minus deaths) per member per unit time. This yields $\frac{dP}{dt}(t) = bP(t)$, hence:

But as resources grow scarce, b might change.

Carrying Capacity

b is the net birthrate (births minus deaths) per member per unit time. If K is the carrying capacity of an ecosystem, we can model $b = b_0(1 - \frac{P}{K})$.



Now You

Describe to your neighbour what the following mean in terms of the model:

- $b > 0$, $b = 0$, $b < 0$
- $P = 0$, $P > 0$, $P < 0$

CARRYING CAPACITY

Then:

$$\frac{dP}{dt}(t) = \underbrace{b_0 \left(1 - \frac{P(t)}{K}\right)}_{\text{per capita birthrate}} P(t)$$

This is an example of a differential equation that we don't have the tools to solve. (If you take more calculus, though, you'll learn how!) It's also an example of a way you might tweak a model so its assumptions better fit what you observe.

Radiocarbon Dating

Researchers at Charlie Lake in BC have found evidence[1] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago.

Suppose a comparable bone of a bison alive today contains 1μg of ^{14}C. If the half-life of ^{14}C is about 5730 years, roughly how much ^{14}C do you think the researchers found in the sample?

A. About $\frac{1}{10{,}500}$ μg
B. About $\frac{1}{4}$ μg
C. About $\frac{1}{2}$ μg
D. About 1 μg
E. I'm not sure how to estimate this

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

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Read the question first. Note that we can do this two ways: there's an easy approximation in your head, and a pen-and-paper calculation. First, do the rough approximation.

Suppose a body is discovered at 3:45 pm, in a room held at 20°, and the body's temperature is 27°, not the normal 37°. At 5:45 pm, the temperature of the body has dropped to 25.3°. When did the inhabitant of the body die?

Included Work

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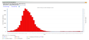 U.S. WHO/NREVSS Collaborating Laboratories and ILNet. ‘Stacked Column Chart WHO//NREVSS’ Centers for Disease Control and Prevention. No longer available from http://gis.cdc.gov/grasp/fluview/fluportaldashboard.html (accessed 20 October 2015), 11

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