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graph LR
    GR((General Rules)) --- CD((Computing Derivatives))
    GR --- PF((Particular Functions))
    GR --- AD((Arithmetic of Derivatives))
    GR --- CR((Chain Rule))
    CD --- ID((Implicit Differentiation))
    CD --- CR2((Chain Rule))
    PF --- DTF((Derivatives of Trigonometric Functions))
    PF --- DEF((Derivatives of Exponential Functions))
    PF --- NL((Natural Logarithm))
    AD --- EX((Examples))
    AD --- PAD((Proof of arithmetic of derivatives))
    DTF --- ITF((Inverse Trigonometric Functions))
    DEF --- NL2((Natural Logarithm))
    NL --- IF((Inverse Functions))
  
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The mind map illustrates the structure of a calculus textbook, centered around 'General Rules'. The main branches are 'Computing Derivatives', 'Particular Functions', 'Arithmetic of Derivatives', and 'Chain Rule'. 'Computing Derivatives' further branches into 'Implicit Differentiation' and 'Chain Rule'. 'Particular Functions' branches into 'Derivatives of Trigonometric Functions', 'Derivatives of Exponential Functions', and 'Natural Logarithm'. 'Arithmetic of Derivatives' branches into 'Examples' and 'Proof of arithmetic of derivatives'. 'Derivatives of Trigonometric Functions' branches into 'Inverse Trigonometric Functions'. 'Derivatives of Exponential Functions' branches into 'Natural Logarithm'. 'Natural Logarithm' branches into 'Inverse Functions'.

- ▶ Content from Section 2.4 can be found in the following locations:
  - ▶ 22: derivative of sum and difference
  - ▶ 32: Theorem 2.4.3, the product rule
  - ▶ 40: Theorem 2.4.5, the quotient rule
- ▶ Content from Section 2.5 can be found in the following locations:
  - ▶ 18, 20: proof of the linearity of differentiation
  - ▶ 31: proof of the product rule
- ▶ Content from Section 2.6 can be found in the following locations:
  - ▶ 23, 25: linearity of differentiation
  - ▶ 35, 37: product rule
  - ▶ 38: Example 2.6.6, product of three functions
  - ▶ 41, 43: quotient rule
  - ▶ 53: Lemma 2.6.9, derivative of  $x^n$ .
  - ▶ 70, 74: power rule
  - ▶ 45, 49, 51, 72: various rules

# DERIVATIVES OF LINES

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$$f'(5) =$$

$$f'(-13) =$$

$$g(x) = 13$$

$$g'(1) =$$

A. 0

B. 1

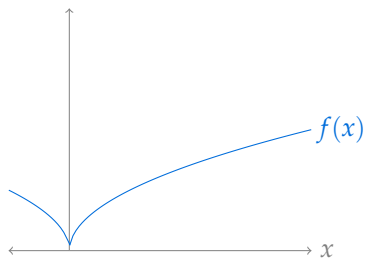
C. 2

D. 13

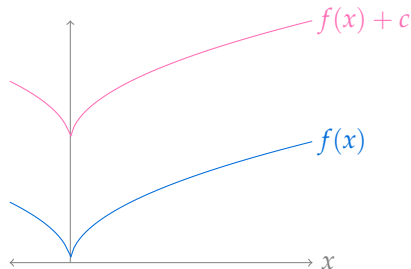




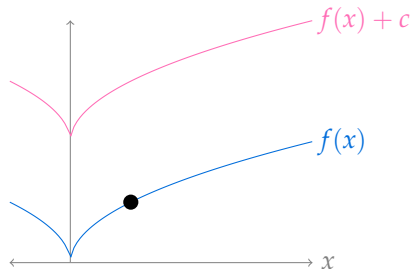
# ADDING A CONSTANT



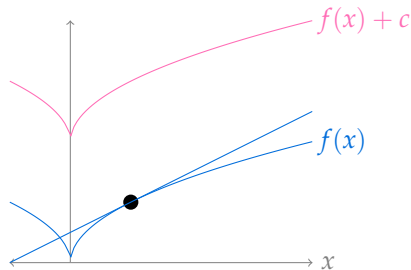
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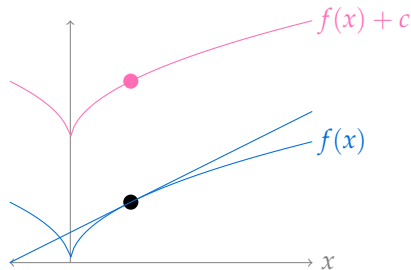
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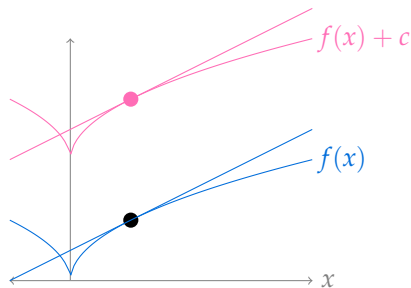
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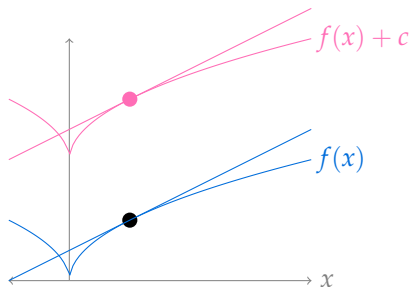
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We saw

$$\left. \frac{d}{dx} (3 - 0.8t^2) \right|_{t=1} = -1.6$$

So,

$$\left. \frac{d}{dx} (10 - 0.8t^2) \right|_{t=1} =$$



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# DIFFERENTIATING SUMS

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$$\begin{aligned}
 \frac{d}{dx} \{f(x) + g(x)\} &= \lim_{h \rightarrow 0} \left[ \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} \right] + \lim_{h \rightarrow 0} \left[ \frac{g(x+h) - g(x)}{h} \right] \\
 &= f'(x) + g'(x)
 \end{aligned}$$

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Let  $a$  be a constant.

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## Rules – Lemma 2.4.1

Suppose  $f(x)$  and  $g(x)$  are differentiable, and let  $c$  be a constant number. Then:

- ▶  $\frac{d}{dx} \{f(x) + g(x)\} = f'(x) + g'(x)$
- ▶  $\frac{d}{dx} \{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶  $\frac{d}{dx} \{cf(x)\} = cf'(x)$

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For instance: let  $f(x) = 10((2x - 15) + 13 - \sqrt{x})$ . Then  $f'(x) =$

## Rules – Lemma 2.4.1

Suppose  $f(x)$  and  $g(x)$  are differentiable, and let  $c$  be a constant number. Then:

- ▶  $\frac{d}{dx} \{f(x) + g(x)\} = f'(x) + g'(x) \leftarrow \text{Add a constant: no change}$
- ▶  $\frac{d}{dx} \{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶  $\frac{d}{dx} \{cf(x)\} = cf'(x) \leftarrow \text{Multiply by a constant: keep the constant}$

For instance: let  $f(x) = 10 \left( (2x - 15) + 13 - \sqrt{x} \right)$ . Then  $f'(x) = 10 \left( (2) + 0 - \frac{1}{2\sqrt{x}} \right)$ .



Now  
YouSuppose  $f'(x) = 3x$ ,  $g'(x) = -x^2$ , and  $h'(x) = 5$ .

Calculate:

$$\frac{d}{dx} \{f(x) + 5g(x) - h(x) + 22\}$$

- A.  $3x - 5x^2$
- B.  $3x - 5x^2 - 5$
- C.  $3x - 5x^2 - 5 + 22$
- D. none of the above

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True or False:

$$\begin{aligned}\frac{d}{dx}\{2x\} &= \frac{d}{dx}\{x + x\} \\ &= [1] + [1] \\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{d}{dx}\{x^2\} &= \frac{d}{dx}\{x \cdot x\} \\ &= [1] \cdot [1] \\ &= 1\end{aligned}$$

# WHAT TO DO WITH PRODUCTS?

Suppose  $f(x)$  and  $g(x)$  are differentiable functions of  $x$ . What about  $f(x)g(x)$ ?

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Suppose  $f(x)$  and  $g(x)$  are differentiable functions of  $x$ . What about  $f(x)g(x)$ ?

$$\begin{aligned}
 \frac{d}{dx} \{f(x)g(x)\} &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - \textcolor{red}{f(x+h)g(x)} + \textcolor{red}{f(x+h)g(x)} - f(x)g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) [g(x+h) - \textcolor{red}{g(x)}] + g(x) [\textcolor{red}{f(x+h)} - f(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \left[ f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ f(x+h) \frac{\textcolor{red}{g(x+h)} - \textcolor{red}{g(x)}}{h} + \textcolor{blue}{g(x)} \frac{f(x+h) - f(x)}{h} \right] \\
 &= f(x) \textcolor{red}{g'(x)} + g(x) f'(x)
 \end{aligned}$$

## Product Rule – Theorem 2.4.3

For differentiable functions  $f(x)$  and  $g(x)$ :

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$



## Product Rule – Theorem 2.4.3

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Example:

$$\frac{d}{dx} [x^2] =$$

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Example: suppose  $f(x) = 3x^2$ ,  $f'(x) = 6x$ ,  $g(x) = \sin(x)$ ,  $g'(x) = \cos(x)$ .

$$\frac{d}{dx} [3x^2 \sin(x)] =$$

## Product Rule – Theorem 2.4.3

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$$\frac{d}{dx} [3x^2 \sin(x)] = 3x^2 \cdot \cos(x) + \sin(x) \cdot 6x$$

Given  $\frac{d}{dx} [2x + 5] = 2$ ,  $\frac{d}{dx} [\sin(x^2)] = 2x \cos(x^2)$ ,  $\frac{d}{dx} [x^2] = 2x$

NOW  
YOU



$$f(x) = (2x + 5) \sin(x^2)$$

- A.  $f'(x) = (2) (2x \cos(x^2)) (2x)$
- B.  $f'(x) = (2) (2x \cos(x^2))$
- C.  $f'(x) = (2x + 5)(2) + \sin(x^2) (2x \cos(x^2))$
- D.  $f'(x) = (2x + 5) (2x \cos(x^2)) + (2) \sin(x^2)$
- E. none of the above

NOW  
YOU



$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is  $f'(x)$ ?

NOW  
YOU



$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is  $f'(x)$ ?

$$f(x) = [a(x)b(x)] c(x)$$

$$\begin{aligned} f'(x) &= [a(x)b(x)] c'(x) + c(x) \frac{d}{dx} \{a(x)b(x)\} \\ &= a(x)b(x)c'(x) + c(x) [a(x)b'(x) + a'(x)b(x)] \\ &= a'(x)b(x)c(x) + a(x)b'(x)c(x) + a(x)b(x)c'(x) \end{aligned}$$

## Quotient Rule – Theorem 2.4.5

Let  $f(x)$  and  $g(x)$  be differentiable and  $g(x) \neq 0$ . Then:

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.



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$$\frac{d}{dx} \left\{ \frac{2x + 5}{3x - 6} \right\} =$$

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Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{d}{dx} \left\{ \frac{2x+5}{3x-6} \right\} = \frac{(3x-6)(2) - (2x+5)(3)}{(3x-6)^2}$$

## Quotient Rule – Theorem 2.4.5

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$$\frac{d}{dx} \left\{ \frac{5x}{\sqrt{x} - 1} \right\} =$$

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Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{d}{dx} \left\{ \frac{5x}{\sqrt{x}-1} \right\} = \frac{(\sqrt{x}-1)(5) - (5x)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}-1)^2} = \frac{\frac{5}{2}\sqrt{x} - 5}{(\sqrt{x}-1)^2}$$

NOW  
YOU



Differentiate the following.

$$f(x) = 2x + 5$$

$$g(x) = (2x + 5)(3x - 7) + 25$$

$$h(x) = \frac{2x + 5}{8x - 2}$$

$$j(x) = \left( \frac{2x + 5}{8x - 2} \right)^2$$

## Rules

Product:  $\frac{d}{dx} \{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient:  $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

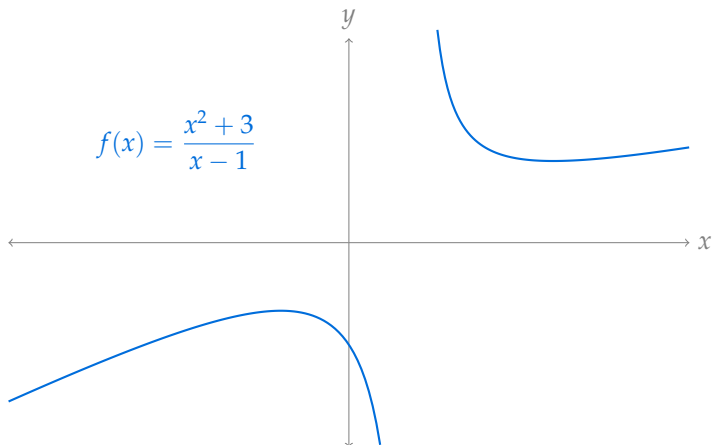
$$f(x) = 2x + 5$$

$$g(x) = (2x + 5)(3x - 7) + 25$$

$$h(x) = \frac{2x + 5}{8x - 2}$$

$$j(x) = \left( \frac{2x+5}{8x-2} \right)^2$$





For which values of  $x$  is the tangent line to the curve horizontal?

A horizontal line has slope 0, and the slope of the tangent line is the function's derivative. So, we should find where the function's derivative is 0.

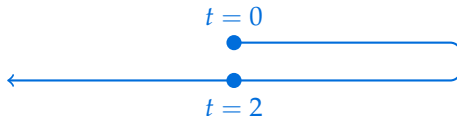
$$f'(x) = \frac{(x-1)(2x) - (x^2+3)(1)}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}$$

$$x = -1, x = 3$$

The position of an object moving left and right at time  $t$ ,  $t \geq 0$ , is given by

$$s(t) = -t^2(t - 2)$$

where a positive position means it is to the right of its starting position, and a negative position means it is to the left. First it moves to the right, then it moves left forever.



What is the farthest point to the right that the object reaches?

When the object turns to come back around,  $s'(t) = 0$ . If we can find the value of  $t$  that makes this true, then we plug it in to  $s(t)$  to find the farthest to the right reached by the object.

$$s'(t) = [-t^2](1) + (-2t)(t - 2) = -3t^2 + 4t = t(4 - 3t)$$

So, the object turns around when  $t = \frac{4}{3}$ .

Its position at that time is  $s\left(\frac{4}{3}\right) = \frac{32}{27}$  units to the right of its starting position.

# MORE ABOUT THE PRODUCT RULE

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\}$$

function	derivative
$x$	1

# MORE ABOUT THE PRODUCT RULE

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$

function	derivative
$x$	1

# MORE ABOUT THE PRODUCT RULE

$$\begin{aligned}\frac{d}{dx}\{x^2\} &= \frac{d}{dx}\{x \cdot x\} = x(1) + x(1) \\ &= 2x\end{aligned}$$

function	derivative
$x$	$1$
$x^2$	$2x$

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$$= 2x$$

$$\frac{d}{dx}\{x^3\}$$

function	derivative
$x$	$1$
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$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$

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$x$	$1$
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$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\} \\ = (x)(2x) + (x^2)(1)$$

function	derivative
$x$	$1$
$x^2$	$2x$

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$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\} \\ = (x)(2x) + (x^2)(1) = 3x^2$$

function	derivative
$x$	$1$
$x^2$	$2x$
$x^3$	$3x^2$

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$$\frac{d}{dx}\{x^4\} = \frac{d}{dx}\{x \cdot x^3\}$$

function	derivative
$x$	$1$
$x^2$	$2x$
$x^3$	$3x^2$

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function	derivative
$x$	$1$
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$x^3$	$3x^2$

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$$\begin{aligned}\frac{d}{dx}\{x^4\} &= \frac{d}{dx}\{x \cdot x^3\} \\ &= x(3x^2) + x^3(1) = 4x^3\end{aligned}$$

function	derivative
$x$	$1$
$x^2$	$2x$
$x^3$	$3x^2$
$x^4$	$4x^3$

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$x^3$	$3x^2$
$x^4$	$4x^3$
$x^{30}$	

# MORE ABOUT THE PRODUCT RULE

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Where are these functions defined?

# CAUTIONARY TALE

WITH *functions* RAISED TO A POWER, IT'S MORE COMPLICATED.

Differentiate  $(2x + 1)^2$

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Differentiate  $(2x + 1)^2$

$$\begin{aligned}\frac{d}{dx} \{(2x + 1)^2\} &= \frac{d}{dx} \{(2x + 1)(2x + 1)\} \\ &= (2x + 1)(2) + (2x + 1)(2) \\ &= 4(2x + 1)\end{aligned}$$

## Power Rule – Corollary 2.6.17

$$\frac{d}{dx}\{x^a\} = ax^{a-1} \text{ (where defined)}$$

$$\frac{d}{dx}\{3x^5 + 7x^2 - x + 15\} =$$

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$$\frac{d}{dx} \{x^a\} = ax^{a-1} \text{ (where defined)}$$

$$\begin{aligned} \frac{d}{dx} \{3x^5 + 7x^2 - x + 15\} \\ &= 3 \cdot 5x^4 + 7 \cdot 2x - 1 \\ &= 15x^4 + 14x - 1 \end{aligned}$$

## Power Rule – Corollary 2.6.17

$$\frac{d}{dx} \{x^a\} = ax^{a-1} \text{ (where defined)}$$

Differentiate  $\frac{(x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x})}{2x + 5}$



## Power Rule – Corollary 2.6.17

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Differentiate  $\frac{(x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x})}{2x + 5}$

$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{(x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x})}{2x + 5} \right\} \\ &= \frac{(2x + 5) \cdot \frac{d}{dx} \{ (x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x}) \} - (x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x})(2)}{(2x + 5)^2} \\ &= \frac{(2x + 5) \left[ (x^4 + 1) \left( \frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4} \right) + 4x^3(\sqrt[3]{x} + \sqrt[4]{x}) \right]}{(2x + 5)^2} \\ &\quad - \frac{2(x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x})}{(2x + 5)^2} \end{aligned}$$

Suppose a motorist is driving their car, and their position is given by  $s(t) = 10t^3 - 90t^2 + 180t$  kilometres. At  $t = 1$  ( $t$  measured in hours), a police officer notices they are driving erratically. The motorist claims to have simply suffered a lack of attention: they were in the act of pressing the brakes even as the officer noticed their speed.

At  $t = 1$ , how fast was the motorist going, and were they pressing the gas or the brake?

Challenge: What about  $t = 2$ ?

Velocity is the rate of change of position, so the velocity of the car is given by:

$$v(t) = s'(t) = 30t^2 - 180t + 180$$

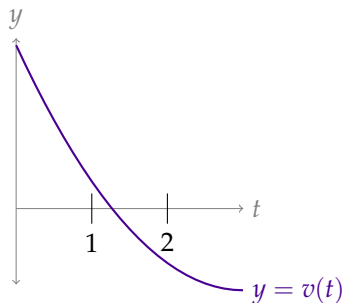
When  $t = 1$ ,  $v(1) = 30$ , so the motorist was going 30 kph.

$$v'(t) = 60t - 180$$

When  $t = 1$ , the velocity of the car was changing by  $v'(t) = -120$  kph per hour. Since the velocity was positive, but its rate of change is negative, the car was slowing down, i.e. decelerating, when  $t = 1$ .

$v(2) = -60$ , so the motorist is driving backwards at 60 kph when  $t = 2$ . Also  $v'(2) = -60$  so the motorist's velocity is becoming increasingly more negative. That is, the motorist is going backwards faster and faster, and so is accelerating.

The contrast between  $t = 1$  and  $t = 2$  can be a subtle point. To help illustrate it, consider the graph below.



The vertical axis corresponds to velocity, so the car is stopped when the curve crosses the  $t$ -axis. At  $t = 1$ , the curve is “heading towards” its  $t$ -intercept. So the velocity is approaching 0. At  $t = 2$ , it is “heading away” from its  $t$ -intercept. The velocity is moving away from 0.

Recall that a sphere of radius  $r$  has volume  $V = \frac{4}{3}\pi r^3$ .

Suppose you are winding twine into a gigantic twine ball, filming the process, and trying to make a viral video. You can wrap one cubic meter of twine per hour. (In other words, when we have  $V$  cubic meters of twine, we're at time  $V$  hours.) How fast is the radius of your spherical twine ball increasing?

## Included Work



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