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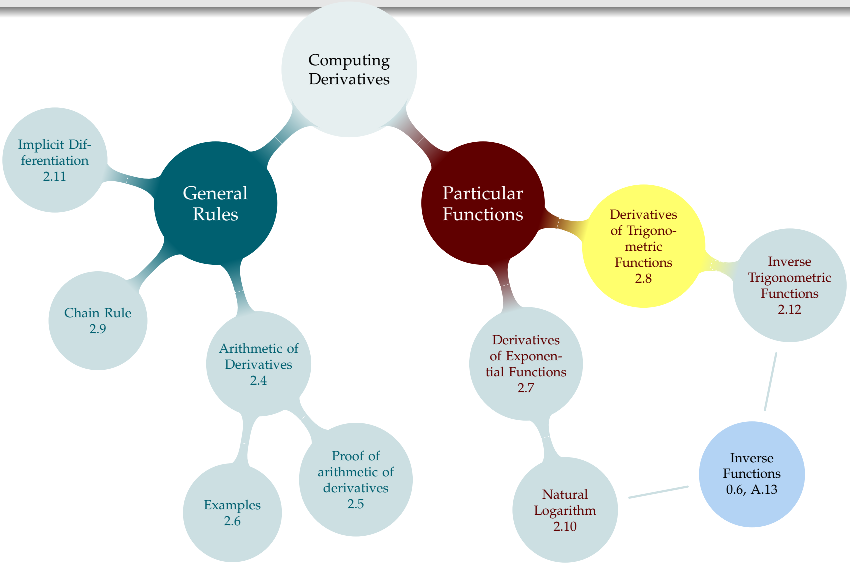
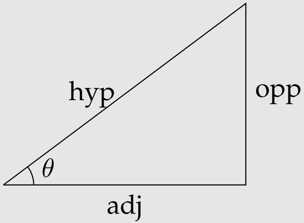


Basic Trig Functions



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$

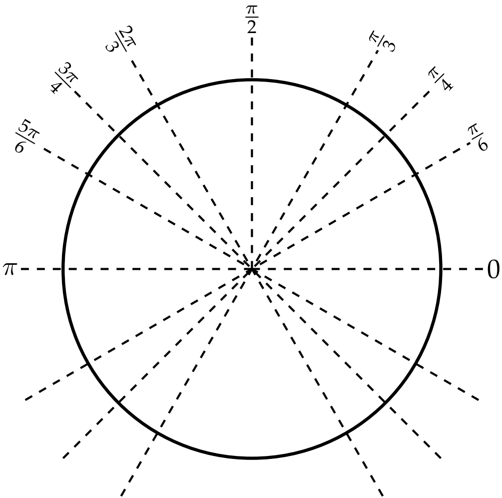
$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Commonly used facts

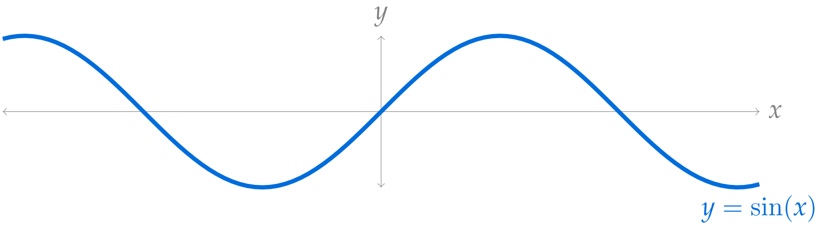
- Graphs of sine, cosine, tangent
- Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- How to use reference angles to find sine, cosine and tangent of other angles
- Identities: $\sin^2 x + \cos^2 x = 1$; $\quad \tan^2 x + 1 - \sec^2 x$;
  $\sin^2 x = \dfrac{1-\cos(2x)}{2}$; $\quad \cos^2 x = \dfrac{1+\cos 2x}{2}$
- Conversion between radians and degrees

CLP-1 has an appendix on high school trigonometry that you should be familiar with.

Reference Angles



Derivative of Sine



Consider the derivative of $f(x) = \sin(x)$.

$$\frac{d}{dx}\{\sin x\} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

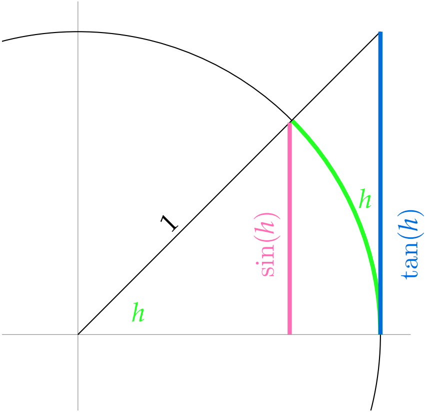
$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x)\lim_{h\to 0}\frac{\cos(0+h)-\cos(0)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\,\frac{d}{dx}\{\cos(x)\}\Big|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \qquad = \qquad \boxed{\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}}$$

since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.


1
h
$\sin(h)$
h
$\tan(h)$

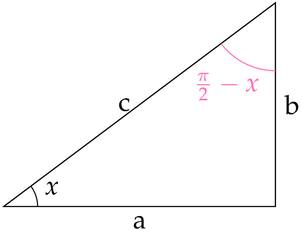
Derivatives of Sine and Cosine

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h} = \cos(x)$$

Derivative of Cosine

Now for the derivative of cos. We already know the derivative of sin, and it is easy to convert between sin and cos using trig identities.



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

When we use radians:

Derivatives of Trig Functions

$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} = -\sin(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} =$

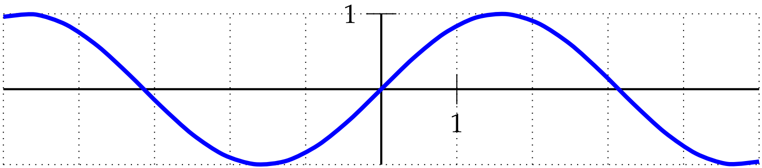
$\frac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} =$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} =$

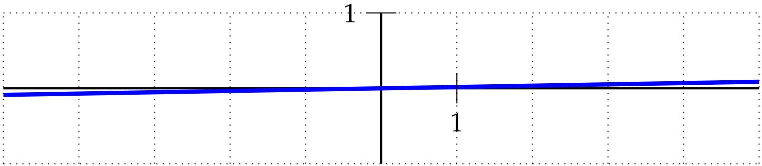
$\frac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} =$

Honorable Mention

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



$y = \sin x$, radians



$y = \sin x$, degrees

Other Trig Functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Other Trig Functions

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}
\frac{d}{dx}[\sec(x)] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] \\
&= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} \\
&= \sec(x)\tan(x)
\end{aligned}$$

Other Trig Functions

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\csc(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sin(x)}\right] \\
&= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin(x)}\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x)
\end{aligned}$$

Other Trig Functions

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\cot(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{\sin(x)}\right] \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}$$

Memorize

$$\frac{d}{dx}\{\sin(x)\} = \cos(x) \qquad \frac{d}{dx}\{\sec(x)\} = \sec(x)\tan(x)$$

$$\frac{d}{dx}\{\cos(x)\} = -\sin(x) \qquad \frac{d}{dx}\{\csc(x)\} = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\{\tan(x)\} = \sec^2(x) \qquad \frac{d}{dx}\{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.

Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Practice and Review

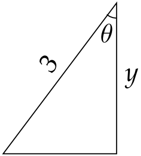
$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

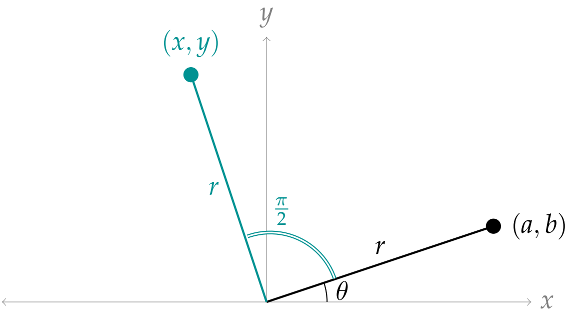
$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x-a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ?
When is the top of the ladder sinking the fastest? The slowest?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ $(0 \leq \theta \leq \frac{\pi}{2})$, is rotated $\pi/2$ radians. What is its new coordinate (x, y)? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ?



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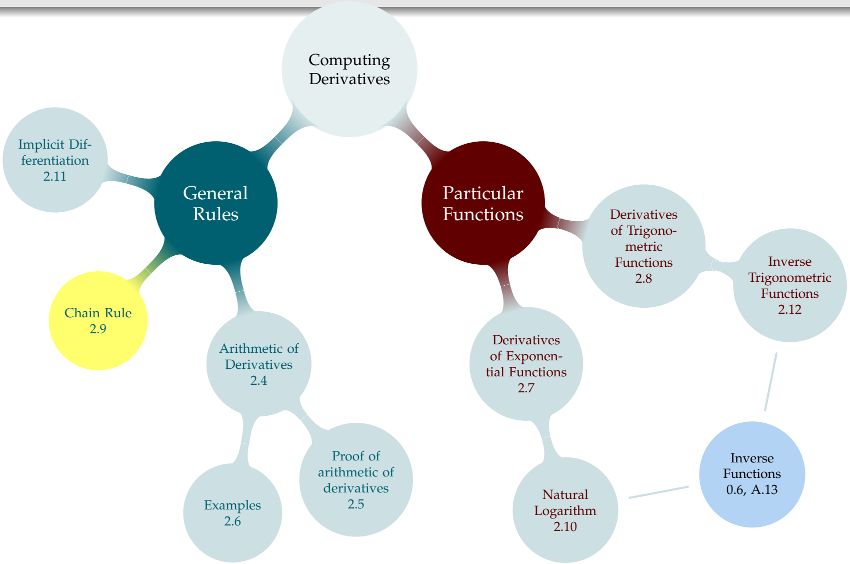
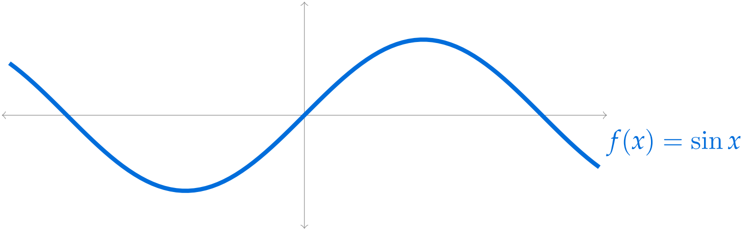
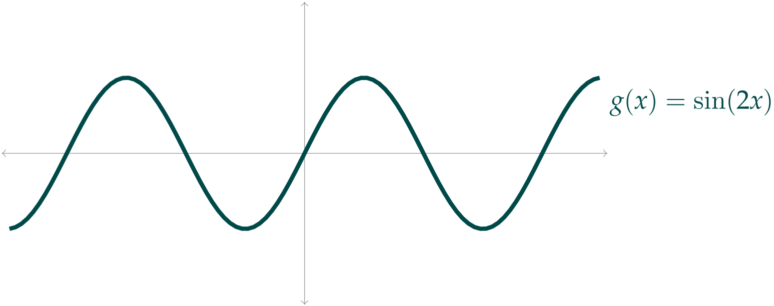


INTUITION: $\sin x$ VERSUS $\sin(2x)$





Compound Functions

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*. Available from https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp

Kelp Population

k	kelp population
u	urchin population
o	otter population
p	public policy

$k(u)$ $\qquad$ $k(u(o))$ $\qquad$ $k(u(o(p)))$

These are examples of compound functions.

Should $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o))$ be positive or negative?
A. positive $\qquad$ B. negative $\qquad$ C. I'm not sure

Should $k'(u)$ be positive or negative?
A. positive $\qquad$ B. negative $\qquad$ C. I'm not sure

Differentiating Compound Functions

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\left(\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right)\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f\left(\boxed{g(x+h)}\right)-f\left(\boxed{g(x)}\right)}{\boxed{g(x+h)}-\boxed{g(x)}}\cdot g'(x)
\end{aligned}
$$

Set $H = g(x+h) - g(x)$. As $h \to 0$, we also have $H \to 0$. So

$$
\begin{aligned}
&= \lim_{H\to 0}\frac{f(g(x)+H)-f(g(x))}{H}\cdot g'(x)\\
&= f'(g(x))\cdot g'(x)
\end{aligned}
$$

Chain Rule

Chain Rule – Theorem 2.9.3

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))\, g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

In the case of kelp, $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o)) = \frac{\mathrm{d}k}{\mathrm{d}u}(u(o))\,\frac{\mathrm{d}u}{\mathrm{d}o}(o)$

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f\left(g(x)\right)\} = f'\left(g(x)\right)g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}\left(g(x)\right)\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

$$F(v) = \left(\frac{v}{v^3 + 1} \right)^6$$

NOW YOU

Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

NOW YOU Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t}u\big(o(t)\big)$.

Note: your answer should depend only on t: not o.

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\frac{1}{x}\right)\right\}$

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

Included Work

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