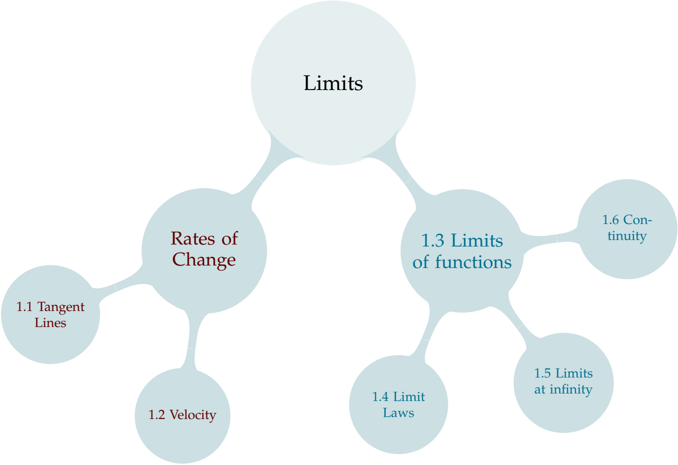
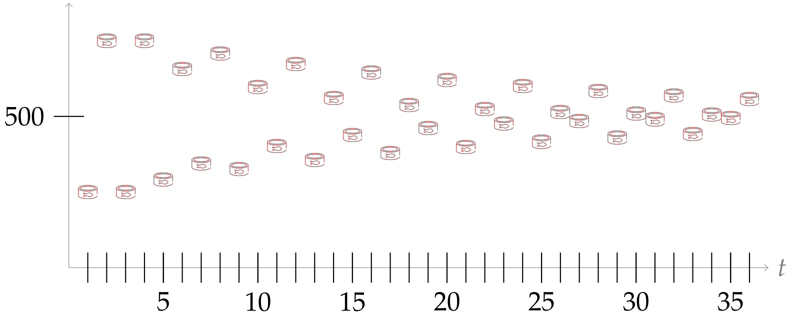
Table of Contents

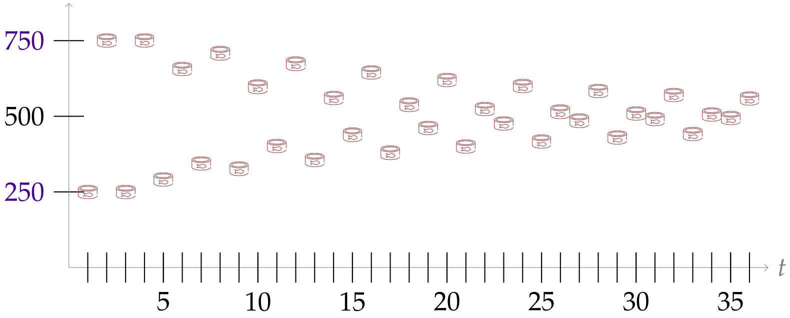
1.8 (Optional) Making Infinite Limits a Little More Formal



You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.

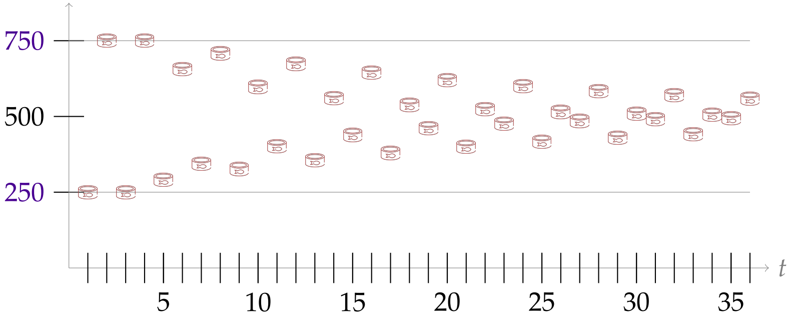


You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



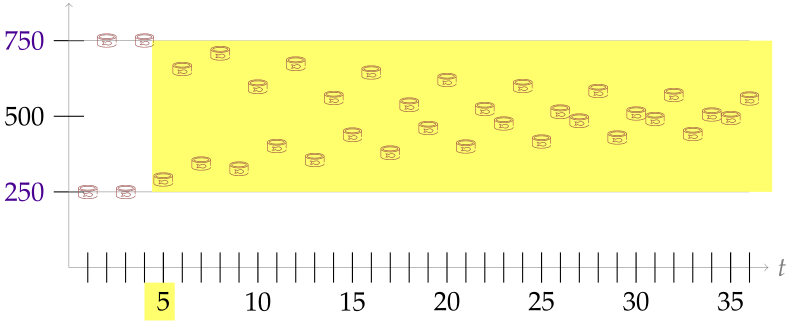
Was there a time after which your error always *less than* 250 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



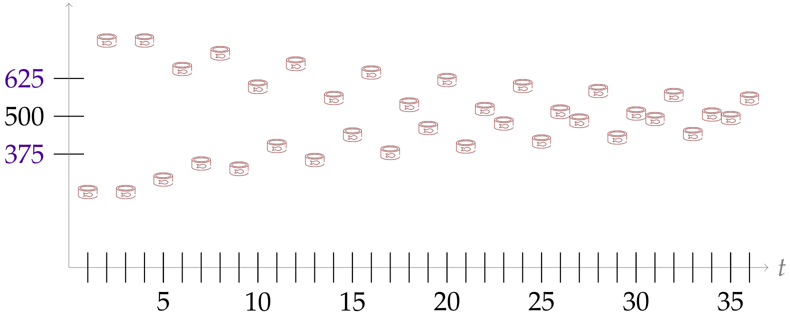
Was there a time after which your error always *less than* 250 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



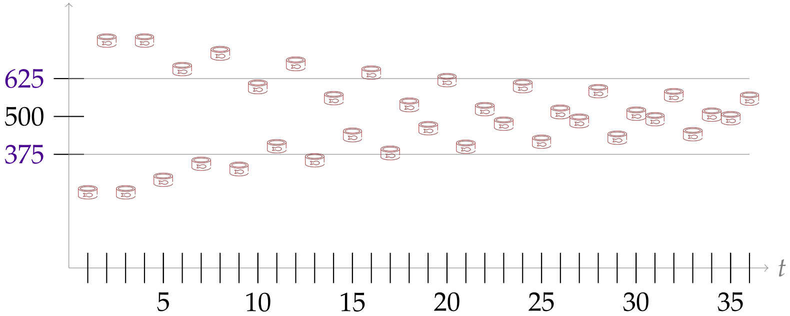
Was there a time after which your error always *less than* 250 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



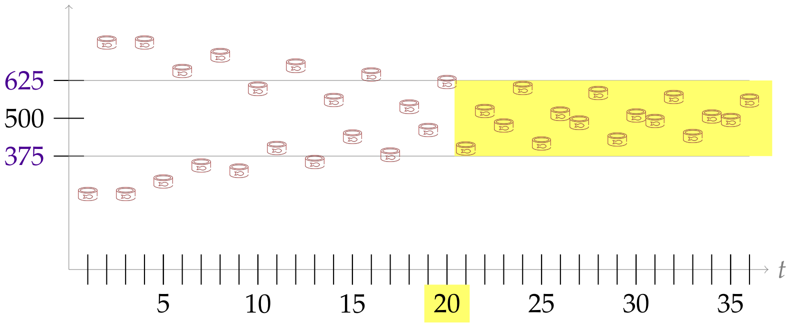
Was there a time after which your error always *less than* 125 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



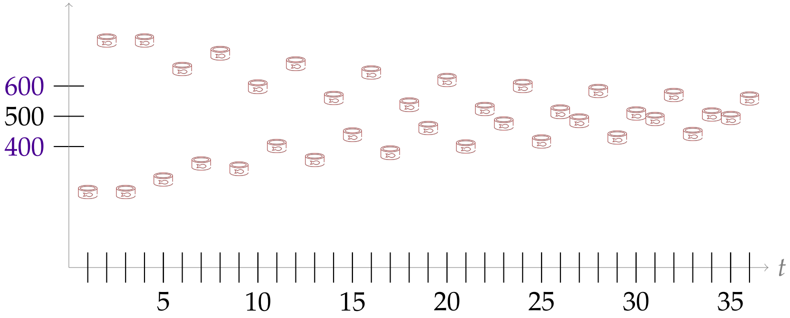
Was there a time after which your error always *less than* 125 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



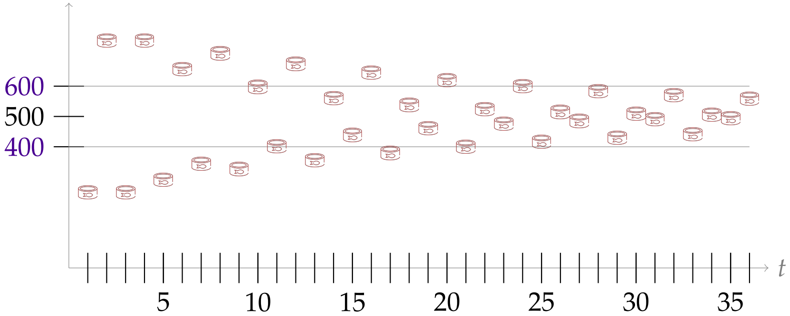
Was there a time after which your error always *less than* 125 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



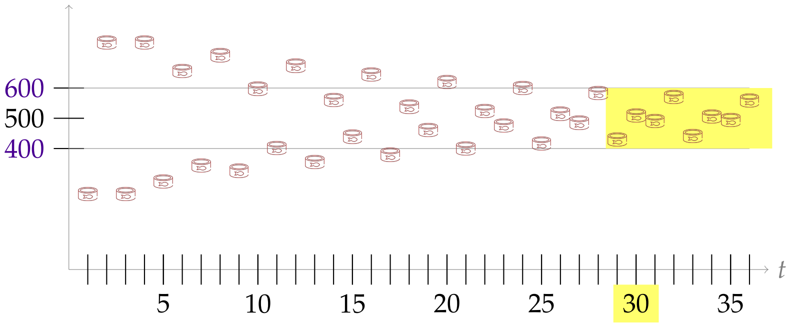
Was there a time after which your error always *less than* 100 g?

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



Was there a time after which your error always *less than* 100 g?

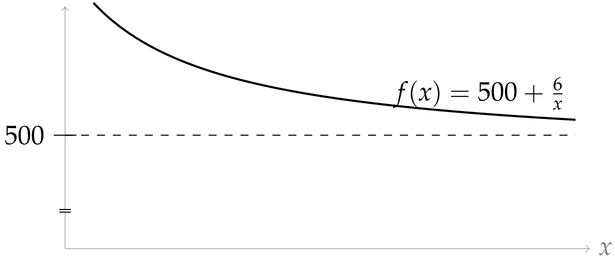
You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



Was there a time after which your error always *less than* 100 g?

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than 3 g. Find such a time.

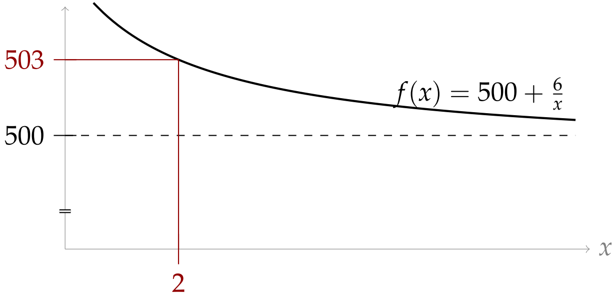


When $x >$ then $|f(x) - 500| < 3$.

ans

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

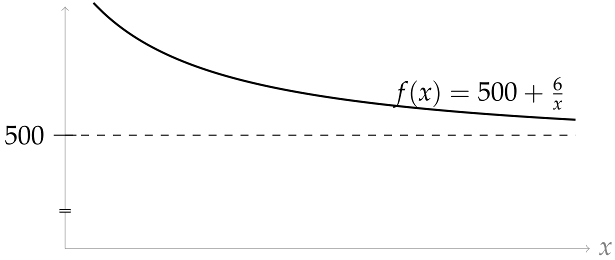
You need to reassure your boss that, after some time, your error is never more than 3 g. Find such a time.



When $x > 2$ then $|f(x) - 500| < 3$.

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than 2 g. Find such a time.

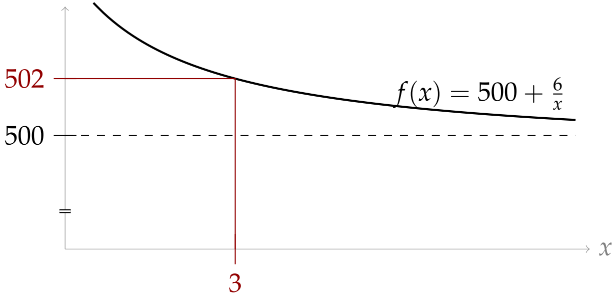


When $x >$ then $|f(x) - 500| < 2$.

ans

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

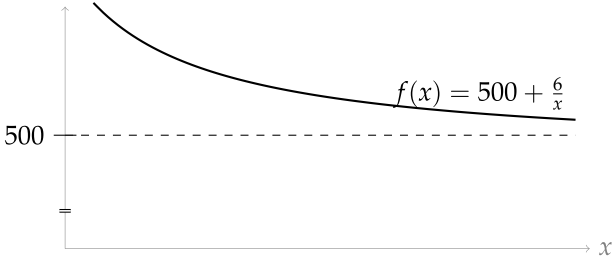
You need to reassure your boss that, after some time, your error is never more than 2 g. Find such a time.



When $x > 3$ then $|f(x) - 500| < 2$.

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than 1 g. Find such a time.



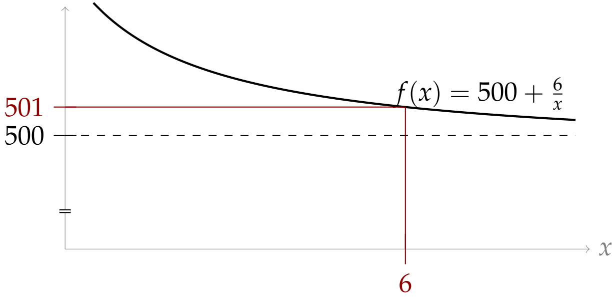
When $x >$ then $|f(x) - 500| < 1$.

Q

ans

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

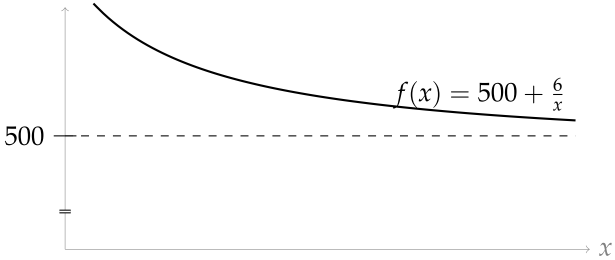
You need to reassure your boss that, after some time, your error is never more than 1 g. Find such a time.



When $x > 6$ then $|f(x) - 500| < 1$.

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than $\frac{1}{1000}$ g. Find such a time.



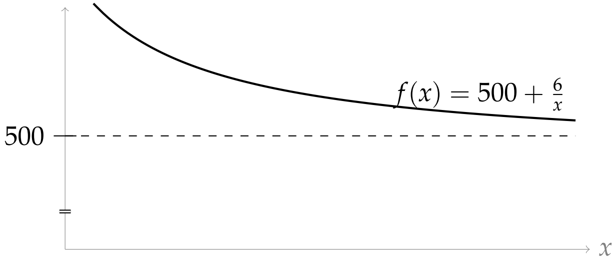
When $x >$ then $|f(x) - 500| < \frac{1}{1000}$.

Q

ans

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

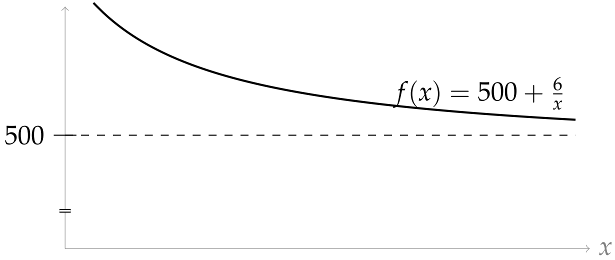
You need to reassure your boss that, after some time, your error is never more than $\frac{1}{1000}$ g. Find such a time.



When $x > 6000$ then $|f(x) - 500| < \frac{1}{1000}$.

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than ϵ g. Find such a time.



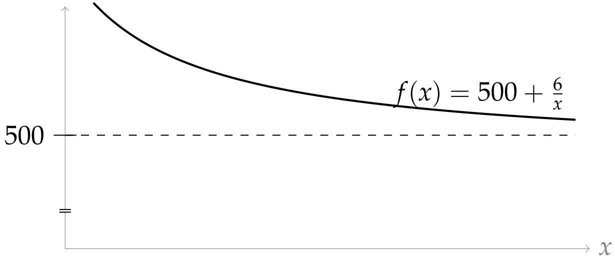
When $x >$ then $|f(x) - 500| < \epsilon$.

Q

ans

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than ϵ g. Find such a time.



When $x > \frac{6}{\epsilon}$ then $|f(x) - 500| < \epsilon$.
No matter how exacting your boss is, if they give you a non-zero error allowance, you can *always* schedule a time after which you will meet their standards.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that

the limit as x approaches ∞ of $f(x)$ is L

and write

$$\lim_{x\to\infty} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.
Similarly we write

$$\lim_{x\to-\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

Let f be a function defined on the whole real line.

$f(x)$: actual can weights

Let f be a function defined on the whole real line.

We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

$$\lim_{x \to \infty} f(x) = L$$

$f(x)$: actual can weights

L: weight on the label that you want to match

Let f be a function defined on the whole real line.

$f(x)$: actual can weights

We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

L: weight on the label that you want to match

$$\lim_{x \to \infty} f(x) = L$$

if and only if for every $\epsilon > 0$

ϵ: amount of allowable error

Let f be a function defined on the whole real line.

$f(x)$: actual can weights

We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

L: weight on the label that you want to match

$$\lim_{x\to\infty} f(x) = L$$

if and only if for every $\epsilon > 0$

ϵ: amount of allowable error

there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

M: time after which your weights are always off by less than ϵ

$|f(x)-L|$: error (difference between actual amount and label)

Let f be a function defined on the whole real line.

$f(x)$: actual can weights

We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

L: weight on the label that you want to match

$$\lim_{x\to\infty} f(x) = L$$

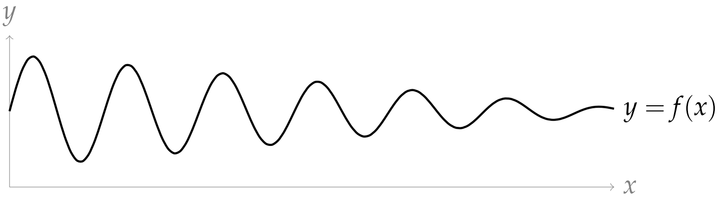
if and only if for every $\epsilon > 0$

ϵ: amount of allowable error

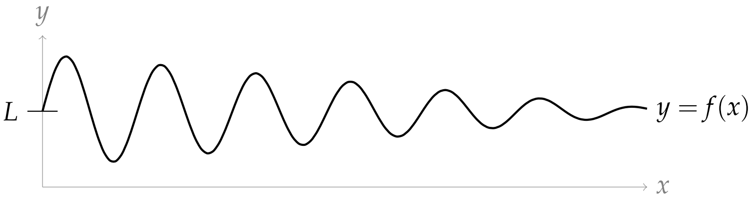
there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

M: time after which your weights are always off by less than ϵ

$|f(x) - L|$: error (difference between actual amount and label)

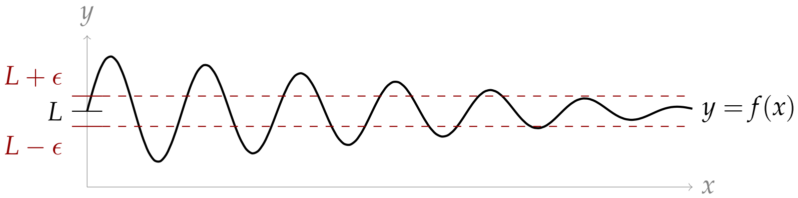


Let f be a function defined on the whole real line.



Let f be a function defined on the whole real line. We say that “the limit as x approaches ∞ of $f(x)$ is L” and write

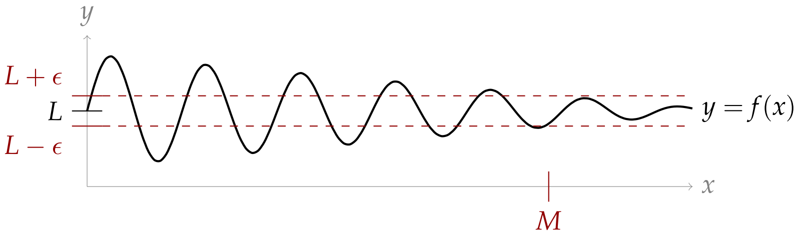
$$\lim_{x\to\infty} f(x) = L$$



Let f be a function defined on the whole real line. We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

$$\lim_{x \to \infty} f(x) = L$$

if and only if for every $\epsilon > 0$



Let f be a function defined on the whole real line. We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

$$\lim_{x\to\infty} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2}{x} + 1\right] = 1$.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty}\left[\frac{2}{x}+1\right] = 1$.

Let ϵ be any positive constant. We need to find M such that, for $x > M$:

$$\begin{aligned} |f(x) - L| &< \epsilon \\ \left|\left(\frac{2}{x}+1\right) - 1\right| &< \epsilon \\ \left|\frac{2}{x}\right| &< \epsilon \end{aligned}$$

It suffices to find a positive M. Then x is positive too.

$$\begin{aligned} \frac{2}{x} &< \epsilon \\ x &> \frac{2}{\epsilon} \end{aligned}$$

So, we choose $M = \frac{2}{\epsilon}$. Now we can go through the definition.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2}{x} + 1\right] = 1$.

Proof: Let $f(x) = \frac{2}{x} + 1$. For any $\epsilon > 0$, let $M = \frac{2}{\epsilon}$. Then for any $x > M$:

$$|f(x) - 1| = \left|\left(\frac{2}{x} + 1\right) - 1\right| = \left|\frac{2}{x}\right| = \frac{2}{x}$$

$$< \frac{2}{M} = \frac{2}{\frac{2}{\epsilon}} = \epsilon$$

Therefore $\lim_{x\to\infty} \left[\frac{2}{x} + 1\right] = 1$. $\square$

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim\limits_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim\limits_{x\to\infty} \left[5e^{-x}\right] = 0$

Q

ans

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim\limits_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim\limits_{x\to\infty} \left[5e^{-x}\right] = 0$

First, we need to find which M goes with ϵ.

$$|f(x) - L| = \left|5e^{-x} - 0\right| = 5e^{-x} = \frac{5}{e^x} < \epsilon$$

$$e^x > \frac{5}{\epsilon}$$

$$x > \log_e\left(\frac{5}{\epsilon}\right)$$

So, in our proof, we should use $M = \log_e\left(\frac{5}{\epsilon}\right)$

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[5e^{-x}\right] = 0$

Proof: Set $f(x) = 5e^{-x}$. For any $\epsilon > 0$, let $M = \log_e\left(\frac{5}{\epsilon}\right)$. Then for all x that are greater than M:

$$\begin{aligned} |f(x) - 0| &= \left|5e^{-x}\right| = \frac{5}{e^x} \\ &< \frac{5}{e^M} = \frac{5}{e^{\log_e\left(\frac{5}{\epsilon}\right)}} = \frac{5}{\frac{5}{\epsilon}} = \epsilon \end{aligned}$$

Therefore, $\lim_{x\to\infty} \left[5e^{-x}\right] = 0.$ $\square$

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Now You

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{\sin x}{x}\right] = 0$

Q

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim\limits_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim\limits_{x\to\infty} \left[\dfrac{\sin x}{x}\right] = 0$

First, we find M for an arbitrary ϵ. We want

$$|f(x) - 0| = \left|\frac{\sin x}{x}\right| < \epsilon$$

We need to find values of x that are large enough for this to be true, but we don't have to find the values of x that make equality hold. (Think about the first canning example where we didn't even have numbers.) So, we simplify things by noting $|\sin x| < 1$ for all x. For $x > 0$,

$$\left|\frac{\sin x}{x}\right| < \left|\frac{1}{x}\right| = \frac{1}{x} < \epsilon$$

$$x > \frac{1}{\epsilon}$$

So, we use $M = \frac{1}{\epsilon}$.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{\sin x}{x}\right] = 0$

Proof: Let $f(x) = \frac{\sin x}{x}$. For any $\epsilon > 0$, let $M = \frac{1}{\epsilon}$. Whenever $x > M$:

$$|f(x) - 0| = \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right| = \frac{1}{x}$$

$$< \frac{1}{M} = \epsilon$$

So $\lim_{x\to\infty} \frac{\sin x}{x} = 0.$ $\square$

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

NOW YOU Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2x^2}{x^2+1}\right] = 2$

Q

ans

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\dfrac{2x^2}{x^2+1}\right] = 2$

First, we find M based on ϵ.

$$\epsilon > |f(x) - 2| = \left|\frac{2x^2}{x^2+1} - 2\right| = \left|\frac{2x^2 - 2(x^2+1)}{x^2+1}\right| = \left|\frac{-2}{x^2+1}\right| = \frac{2}{x^2+1}$$

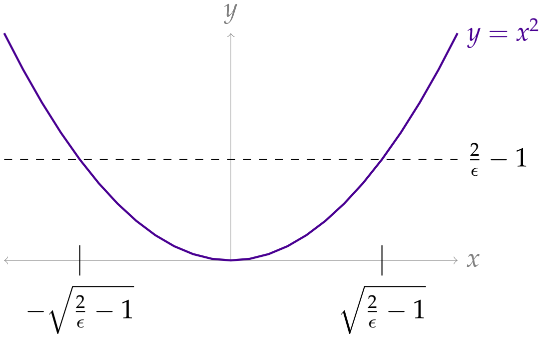
$$x^2 + 1 > \frac{2}{\epsilon} \implies x^2 > \frac{2}{\epsilon} - 1 \implies x > \sqrt{\frac{2}{\epsilon} - 1} = M$$

(If $\epsilon > 2$, we can take $M = 0$.) Note that the values of x that make $x^2 > \frac{2}{\epsilon} - 1$ true are $(-\infty, -a) \cup (a, \infty)$ for $a = \sqrt{\frac{2}{\epsilon} - 1}$. Since we only care about large x, we use the second interval. (See graph on next slide.)

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim\limits_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim\limits_{x\to\infty} \left[\dfrac{2x^2}{x^2+1}\right] = 2$



It **is true** that for all $x > \sqrt{\frac{2}{\epsilon} - 1}$, $x^2 > \frac{2}{\epsilon} - 1$.

It **is not true** that for all $x > -\sqrt{\frac{2}{\epsilon} - 1}$, $x^2 > \frac{2}{\epsilon} - 1$.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2x^2}{x^2+1}\right] = 2$

Proof: : Let $f(x) = \frac{2x^2}{x^2+1}$. For any $2 \geq \epsilon > 0$, set $M = \sqrt{\frac{2}{\epsilon} - 1}$. (For $\epsilon > 2$, set $M = 0$.) Then for any $x > M$:

$$|f(x) - 2| = \left|\frac{2x^2}{x^2+1} - 2\right| = \left|\frac{-2}{x^2+1}\right| = \frac{2}{x^2+1}$$

$$< \frac{2}{M^2+1} = \frac{2}{\left(\sqrt{\frac{2}{\epsilon}-1}\right)^2+1} = \frac{2}{\frac{2}{\epsilon} - 1 + 1} = \epsilon$$

So $\lim_{x\to\infty} \left[\frac{2x^2}{x^2+1}\right] = 2.$ $\square$

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} 5 = 5$

Q

ans

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} 5 = 5$

Proof: : Let $f(x) = 5$, and let $M = 1$. For any $\epsilon > 0$, and for any $x > M$:

$$|f(x) - 5| = 5 - 5 = 0 < \epsilon$$

So, $\lim_{x\to\infty} 5 = 5$. $\square$

Note: you actually could have chosen *any* value for M.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

True or False? $\lim_{x\to\infty} \sin x = 0$

Q

ans

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

True or False? $\lim_{x\to\infty} \sin x = 0$

False.

Let $f(x) = \sin x$ and consider $\epsilon = \frac{1}{2}$. Note that when $x = \frac{\pi}{2} + n\pi$ for any integer n, then $\sin x = \pm 1$, so

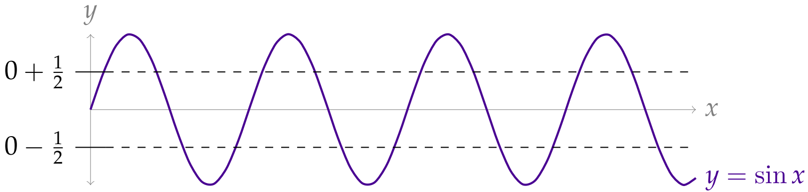
$$|f(x) - 0| = |\pm 1| = 1 > \frac{1}{2}$$

So there is *no* point after which $f(x)$ is always within $\frac{1}{2}$ of 0. Therefore $\lim_{x\to\infty} \sin x \neq 0$.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

True or False? $\lim_{x\to\infty} \sin x = 0$



USEFUL GENERAL PRINCIPLES

When we showed $\lim\limits_{x\to\infty}\left[\dfrac{\sin x}{x}\right] = 0$, we chose M using:

$$\left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right| = \frac{1}{x} < \epsilon$$

USEFUL GENERAL PRINCIPLES

When we showed $\lim_{x\to\infty} \left[\frac{\sin x}{x}\right] = 0$, we chose M using:

$$\left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right| = \frac{1}{x} < \epsilon$$

- $\left|\frac{1}{x}\right| = \frac{1}{x}$ only when x is positive. We want to show that an inequality holds for *large enough* values of x, so if it helps our cause, we can say "make sure x is larger than *blah*." Then we just choose M to be at least that number *blah*.

Useful General Principles

When we showed $\lim_{x\to\infty}\left[\frac{\sin x}{x}\right]=0$, we chose M using:

$$\left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right| = \frac{1}{x} < \epsilon$$

- $\left|\frac{1}{x}\right| = \frac{1}{x}$ only when x is positive. We want to show that an inequality holds for *large enough* values of x, so if it helps our cause, we can say "make sure x is larger than *blah*." Then we just choose M to be at least that number *blah*.
- If $a < b < c$, then $a < c$. So if you want to solve $a < c$, but it's too hard to find *exactly* when that's true, see whether you can replace a with a larger, easier expression b.
  That's what we did when we said $\left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$.

Limit as x goes to negative infinity

Definition 1.8.1 (a)

We write

$$\lim_{x \to -\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

LIMIT AS x GOES TO NEGATIVE INFINITY

Definition 1.8.1 (a)

We write

$$\lim_{x\to-\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

Use Definition 1.8.1 to prove $\displaystyle\lim_{x\to-\infty} \frac{x^3}{x^3+1} = 1$

Q

ans

Use Definition 1.8.1 to prove $\lim_{x \to -\infty} \dfrac{x^3}{x^3+1} = 1$

Q

ans

Use Definition 1.8.1 to prove $\lim\limits_{x\to-\infty} \dfrac{x^3}{x^3+1} = 1$

Let $f(x) = \frac{x^3}{x^3+1}$ and $\epsilon > 0$.

Since we are now concerned with highly *negative* values of x, we can assume that x is a large negative number.
To find N, we solve the inequality:

$$\epsilon > |f(x) - 1| = \left|\frac{x^3}{x^3+1} - 1\right|$$

$$= \left|\frac{x^3}{x^3+1} - \frac{x^3+1}{x^3+1}\right|$$

$$= \left|\frac{-1}{x^3+1}\right|$$

Since we will be choosing highly negative values of x, the denominator x^3+1 is a negative number. Then $\frac{-1}{x^3+1}$ is a positive number.

$$= \frac{-1}{x^3+1}$$

So we have

$$\epsilon > \frac{-1}{x^3+1}$$

Use Definition 1.8.1 to prove $\displaystyle\lim_{x\to-\infty} \frac{x^3}{x^3+1} = 1$

$$\epsilon > \frac{-1}{x^3+1}$$

The denominator is negative, so when we multiply both sides by it, we flip the inequality

$$\begin{aligned}
\epsilon(x^3+1) &< -1 \\
\epsilon x^3 + \epsilon &< -1 \\
\epsilon x^3 &< -1-\epsilon \\
x^3 &< \frac{-1-\epsilon}{\epsilon} = -\left(1+\frac{1}{\epsilon}\right) \\
x &< -\left(1+\frac{1}{\epsilon}\right)^{1/3}
\end{aligned}$$

Choose $N = -\left(1+\frac{1}{\epsilon}\right)^{1/3}$.

Use Definition 1.8.1 to prove $\displaystyle\lim_{x\to-\infty}\frac{x^3}{x^3+1}=1$

Proof: : Set $f(x)=\frac{x^3}{x^3+1}$. For any $\epsilon>0$, let $N=-\left(1+\frac{1}{\epsilon}\right)^{1/3}$. Then for any $x<N$:

$$\begin{aligned}
|f(x)-1| &= \left|\frac{x^3}{x^3+1}-1\right| = \left|\frac{-1}{x^3+1}\right| = \frac{-1}{x^3+1}\\
&< \frac{-1}{N^3+1} = \frac{-1}{\left(-\left(1+\frac{1}{\epsilon}\right)^{1/3}\right)^3+1} = \frac{-1}{-(1+\frac{1}{\epsilon})+1}\\
&= \frac{-1}{-\frac{1}{\epsilon}} = \epsilon
\end{aligned}$$

So, $\displaystyle\lim_{x\to-\infty}\frac{x^3}{x^3+1}=1.$ $\square$

LIMIT AS x GOES TO NEGATIVE INFINITY

Definition 1.8.1 (a)

We write

$$\lim_{x\to-\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

Use Definition 1.8.1 to prove $\displaystyle\lim_{x\to-\infty} \frac{\cos x}{\sqrt{x^4+x^2+1}} = 0$

Q

ans

Use Definition 1.8.1 to prove $\displaystyle\lim_{x \to -\infty} \frac{\cos x}{\sqrt{x^4 + x^2 + 1}} = 0$

Q

ans

Use Definition 1.8.1 to prove $\lim_{x\to-\infty} \frac{\cos x}{\sqrt{x^4+x^2+1}} = 0$

Let's start by getting a handle on the inequality we know we'll be solving:

$$\epsilon > |f(x) - 0| = \left|\frac{\cos x}{\sqrt{x^4+x^2+1}}\right| = \frac{|\cos x|}{\sqrt{x^4+x^2+1}}$$

We've seen something similar with sine. Since $|\cos x| \le 1$, we can solve instead the right-most inequality below:

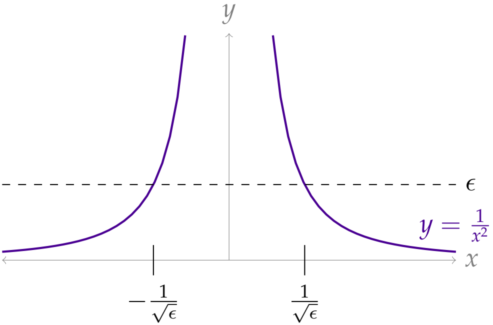
$$\frac{|\cos x|}{\sqrt{x^4+x^2+1}} \le \frac{1}{\sqrt{x^4+x^2+1}} < \epsilon$$

But why stop there? Let's solve the right-most inequality below:

$$\frac{|\cos x|}{\sqrt{x^4+x^2+1}} \le \frac{1}{\sqrt{x^4+x^2+1}} < \frac{1}{\sqrt{x^4}} = \frac{1}{x^2} < \epsilon$$

So, we set $N = -\frac{1}{\sqrt{\epsilon}}$. (Note $\frac{1}{x^2}$ is *not* always less than ϵ if x only has to be less than $\frac{1}{\sqrt{\epsilon}}$. A graph can help explain this – see next slide.)

Use Definition 1.8.1 to prove $\lim_{x\to-\infty} \dfrac{\cos x}{\sqrt{x^4+x^2+1}} = 0$



It **is true** that for all $x < -\frac{1}{\sqrt{\epsilon}}$, $\frac{1}{x^2} < \epsilon$.

It **is not true** that for all $x < \frac{1}{\sqrt{\epsilon}}$, $\frac{1}{x^2} < \epsilon$.

We want to find N that guarantees that $\frac{1}{x^2} < \epsilon$ whenever $x < N$.
That's why we choose $N = -\frac{1}{\sqrt{\epsilon}}$.
Now we're ready to start our proof.

Use Definition 1.8.1 to prove $\lim_{x\to-\infty} \dfrac{\cos x}{\sqrt{x^4+x^2+1}} = 0$

Proof: : Let $f(x) = \frac{\cos x}{\sqrt{x^4+x^2+1}}$. For any $\epsilon > 0$, set $N = -\frac{1}{\sqrt{\epsilon}}$. Then for any $x < N$:

$$|f(x) - 0| = \frac{|\cos x|}{\sqrt{x^4+x^2+1}} < \frac{1}{x^2}$$

Note that $x < N < 0$, so $|x| > |N|$ and $x^2 > N^2$. Then:

$$|f(x) - 0| < \frac{1}{x^2} < \frac{1}{N^2} = \frac{1}{\left(-\frac{1}{\sqrt{\epsilon}}\right)^2} = \epsilon$$

So, $\lim_{x\to-\infty} \dfrac{\cos x}{\sqrt{x^4+x^2+1}} = 0.$ $\square$

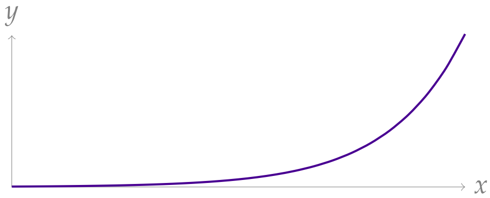
Infinite Limits

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write

$$\lim_{x \to \infty} f(x) = \infty$$

if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



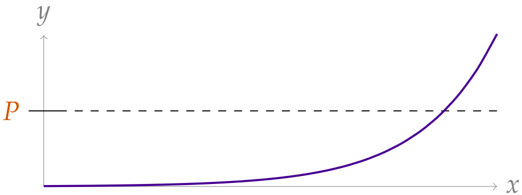
Infinite Limits

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write

$$\lim_{x\to\infty} f(x) = \infty$$

if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



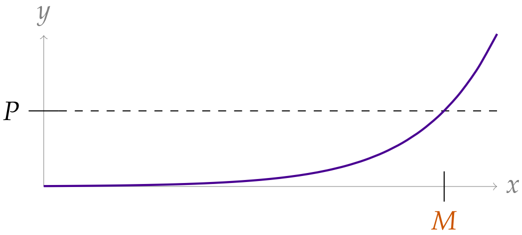
Infinite Limits

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write

$$\lim_{x\to\infty} f(x) = \infty$$

if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



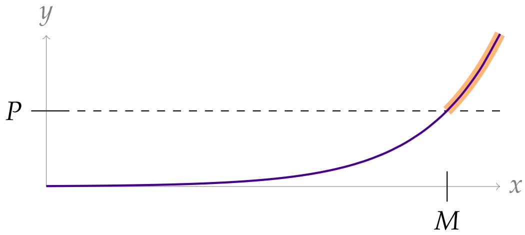
Infinite Limits

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write

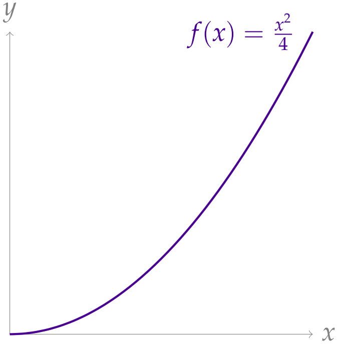
$$\lim_{x\to\infty} f(x) = \infty$$

if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



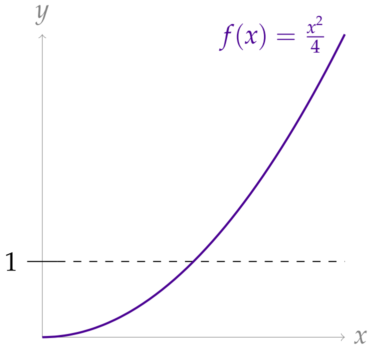
Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



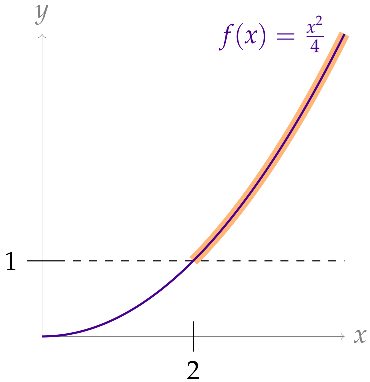
Let $P = 1$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

Q

ans

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

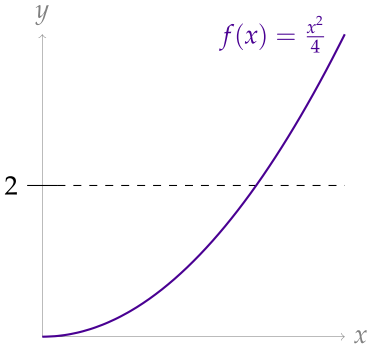


Let $P = 1$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

$$1 < \frac{x^2}{4}$$
$$4 < x^2$$
$$2 < x$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



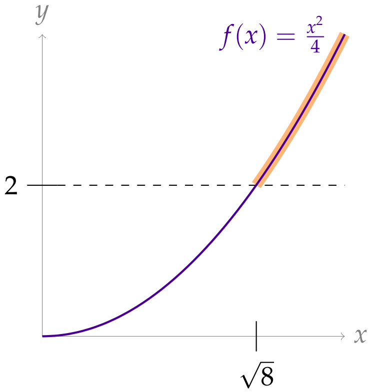
Let $P = 2$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

Q

ans

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

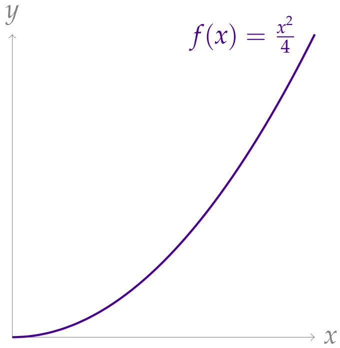


Let $P = 2$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

$$2 < \frac{x^2}{4}$$
$$8 < x^2$$
$$\sqrt{8} < x$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



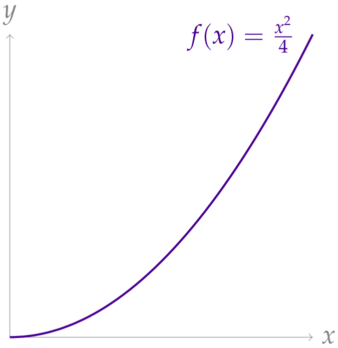
Let $P = 1\,000\,000$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

Q

ans

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

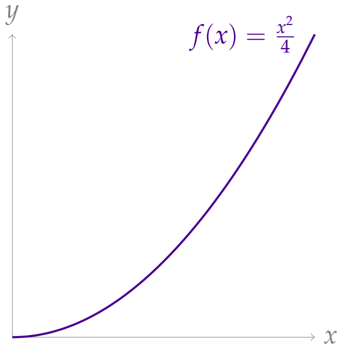


Let $P = 1\,000\,000$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

$$10^6 < \frac{x^2}{4}$$
$$4 \times 10^6 < x^2$$
$$2 \times 10^3 < x$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



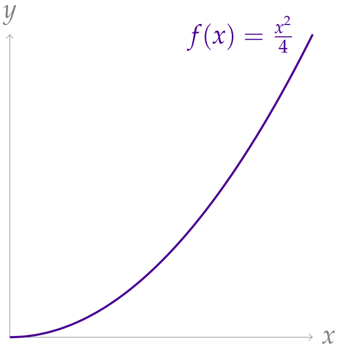
Let $P > 0$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

Q

ans

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



Let $P > 0$. Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

$$P < \frac{x^2}{4}$$
$$4P < x^2$$
$$2\sqrt{P} < x$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x \to \infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x \to \infty} \sqrt[3]{x} = \infty$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} \sqrt[3]{x} = \infty$$

Let $P > 0$ and $f(x) = \sqrt[3]{x}$. We should find a value of M so that $f(x) > P$ whenever $x > M$.

$$\begin{aligned} P < f(x) &= x^{1/3} \\ P^3 &< x \end{aligned}$$

So, we choose $M = P^3$.
Proof: For any $P > 0$, let $M = P^3$. Then whenever $x > M$, $\sqrt[3]{x} > \sqrt[3]{M} = \sqrt[3]{P^3} = P$. So, $\lim_{x\to\infty} \sqrt[3]{x} = \infty$.

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} x(\sin x + 1) = \infty$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x \to \infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

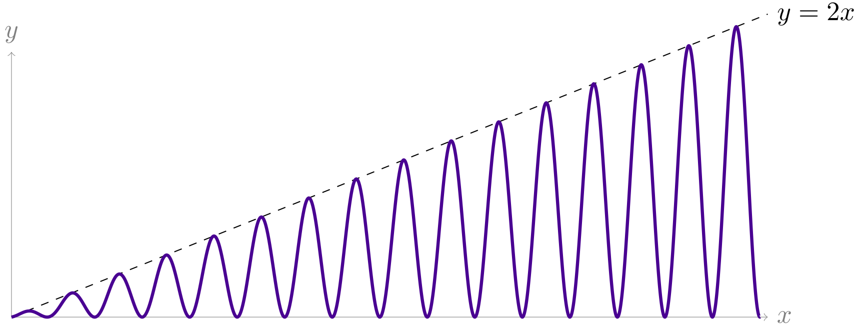
Using definition 1.8.1, prove or disprove the following:

$$\lim_{x \to \infty} x(\sin x + 1) = \infty$$

Let $f(x) = x(\sin x + 1)$. Note that when $x = (2n + 1.5)\pi$ for any integer n (e.g. $x = \frac{3}{2}\pi$, $x = \frac{7}{2}\pi$, $x = \frac{11}{2}\pi$), then $f(x) = x(-1 + 1) = 0$. So even if we narrow our focus to very large values of x, there will always be a value of x where $f(x) = 0$. That tells us that the statement is not true, and gives us the examples we need to disprove it.

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} x(\sin x + 1) = \infty$$

The statement is false.

Proof: Let $P = 1$, and let M be any positive number. There exists an integer n such that $(2n + 1.5)\pi > M$. For $x = (2n + 1.5)\pi$, we have both $x > M$ and $x(\sin x + 1) = 0 \not> P$.
That is: there exists some $P > 0$ such that there is *no* $M > 0$ with the property that $x(\sin x + 1) > P$ whenever $x > M$. So, $\lim_{x\to\infty} x(\sin x + 1) \neq \infty$.

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} x(\sin x + 2) = \infty$$

Q

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x \to \infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x \to \infty} x(\sin x + 2) = \infty$$

Note $x(\sin x + 2) \geq x(-1 + 2) = x$ for all values of $x > 0$. So if $x > P$, then $x(\sin x + 1) \geq x > P$.

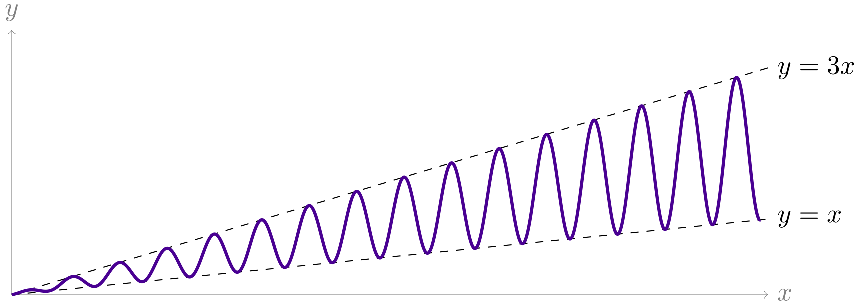
Proof: For any $P > 0$, let $M = P$. Whenever $x > M$, then

$$x(\sin x + 2) \geq x(-1 + 2) = x > M = P$$

So $\lim_{x \to \infty} x(\sin x + 2) = \infty$.

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



Included Work

 'Brain' by Eucalyp is licensed under CC BY 3.0 (accessed 8 June 2021), 38, 41

 tuna can by throwaway icons is licensed under CC BY 3.0, 2–11

 'Notebook' by Iconic is licensed under CC BY 3.0 (accessed 9 June 2021), 54, 59