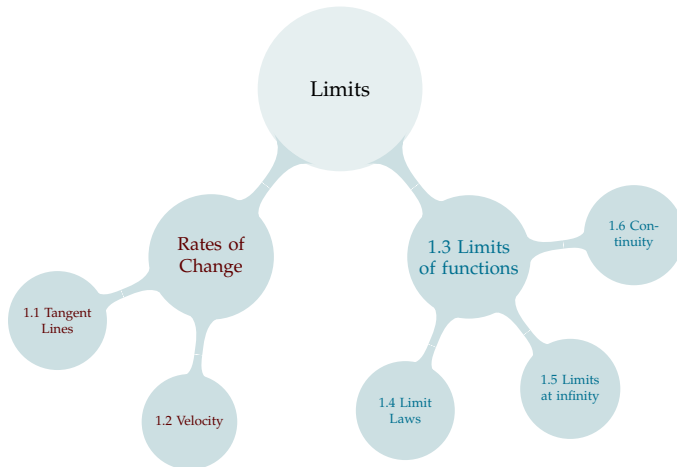


TABLE OF CONTENTS

1.7 (Optional) Making the Informal a Little More Formal



Now that we've seen the limits of functions as x goes to positive and negative infinity, let's look at limits as x approaches a real number.

1.7 (Optional) Making the Informal a Little More Formal

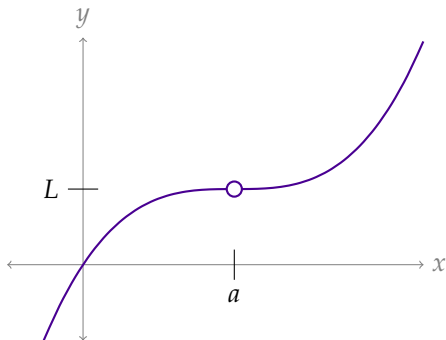
Now that we've seen the limits of functions as x goes to positive and negative infinity, let's look at limits as x approaches a real number.

The actual computations for limits as x goes to infinity are generally easier, so I like to teach 1.8 before 1.7.

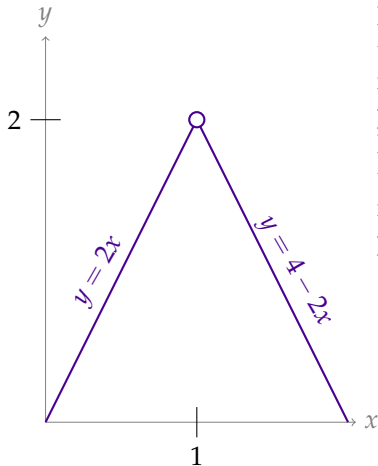
A lot of the same language from the canning analogy can be re-used here: ϵ as error, for instance.

$$\lim_{x \rightarrow a} f(x) = L$$

Informally: If x is close enough (but not equal to) a , then y is close enough to L .



Let $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$. Then $\lim_{x \rightarrow 1} |f(x)| = 2$.



Find a positive number δ such that $|f(x) - 2| < \frac{1}{2}$ for all x in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

Find a positive number δ such that $|f(x) - 2| < \frac{1}{4}$ for all x in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a , except possibly at a . We say that

the limit as x approaches a of $f(x)$ is L

and write

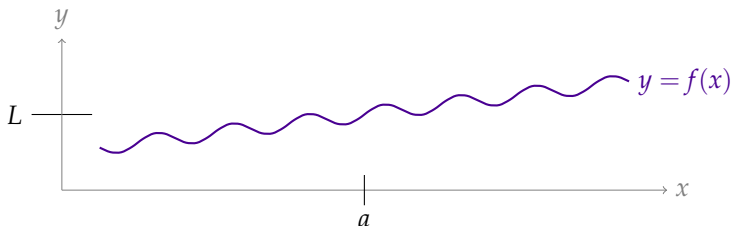
$$\lim_{x \rightarrow a} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Note that an equivalent way of writing this very last statement is

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$



Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a , except possibly at a .

We write

$$\lim_{x \rightarrow a} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

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Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a , except possibly at a . We say that $\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that $\lim_{x \rightarrow -1} |x + 1| = 0$.

Definition 1.7.1

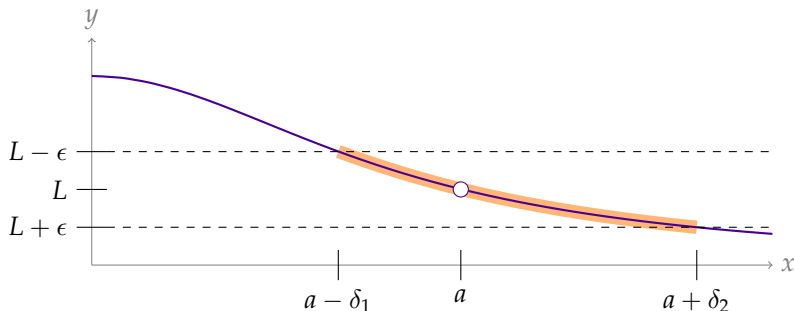
Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a , except possibly at a . We say that $\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

$$\text{Let } f(x) = \begin{cases} x + 1 & x < 0 \\ 1 - x^2 & x > 0 \end{cases}.$$

Using Definition 1.7.1, prove that $\lim_{x \rightarrow 0} f(x) = 1$.

GENERAL PRINCIPLES

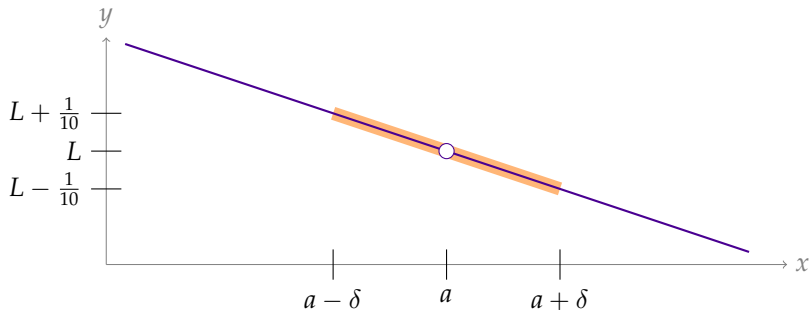
Suppose $|f(x) - L| < \epsilon$ whenever $a - \delta_1 < x < a$ and whenever $a < x < a + \delta_2$.



Consider values of x such that $0 < |x - a| < \min\{\delta_1, \delta_2\}$.

GENERAL PRINCIPLES

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.



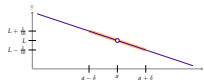
Can you give values of x where $|f(x) - L| < \frac{1}{5}$?

1.7 (Optional) Making the Informal a Little More Formal

General Principles

GENERAL PRINCIPLES

Suppose $|f(x) - L| < \frac{\epsilon}{2}$ for all x such that $0 < |x - a| < \delta$.



Can you give values of x where $|f(x) - L| < \frac{\epsilon}{2}$?

SKIP

WLOG prove only for small epsilon. This doesn't really come up so often at this level, which is why there's a skip button.

GENERAL PRINCIPLES

Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a , except possibly at a . We say that $\lim_{x \rightarrow a} f(x) = L$ if and only if **for every** $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

It is enough to show that **for every** ϵ such that $0 < \epsilon < c$ (where c is some constant) there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

That means it doesn't hurt your proof if you say something like "we assume $\epsilon < 1$ ".

In a previous example, we chose

$$\delta = \min\{\epsilon, \sqrt{\epsilon}\}$$

It would be OK to say "we can assume $\epsilon < 1$; set $\delta = \epsilon$."

Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a , except possibly at a . We say that $\lim_{x \rightarrow a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \frac{1}{4}$.

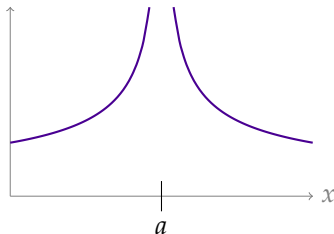
INFINITE LIMITS

Definition 1.8.1 (b)

Let a be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

$$\lim_{x \rightarrow a} f(x) = \infty$$

if and only if for every $P > 0$ there exists $\delta > 0$ so that $f(x) > P$ whenever $0 < |x - a| < \delta$.

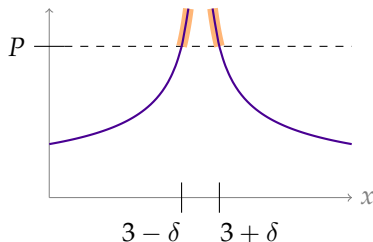


Definition 1.8.1 (b)

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Let $f(x) = \frac{1}{(x-3)^2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow 3} f(x) = \infty$$

1.7 (Optional) Making the Informal a Little More Formal

Definition 1.8.1 (b)

Let a be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

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if and only if for every $P > 0$ there exists $\delta > 0$ so that $f(x) > P$ whenever $0 < |x - a| < \delta$.



Let $f(x) = \frac{1}{x-a}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow a} f(x) = \infty$$

The generic picture is kept on the left, but it's nice to mention that it is, indeed, generic. In particular, the picture won't fit for $f(x) = \frac{1}{x-2}$.

Definition 1.8.1 (b)

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Let $f(x) = \frac{1}{x-2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow 2} f(x) = \infty$$

1.7 (Optional) Making the Informal a Little More Formal

Definition 1.8.1 (b)

Let a be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

$$\lim_{x \rightarrow a} f(x) = \infty$$

if and only if for every $P > 0$ there exists $\delta > 0$ so that $f(x) > P$ whenever $0 < |x - a| < \delta$.

Let $f(x) = \frac{1}{x-2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow 2} f(x) = \infty$$

The generic picture is kept on the left, but it's nice to mention that it is, indeed, generic. In particular, the picture won't fit for $f(x) = \frac{1}{x-2}$.