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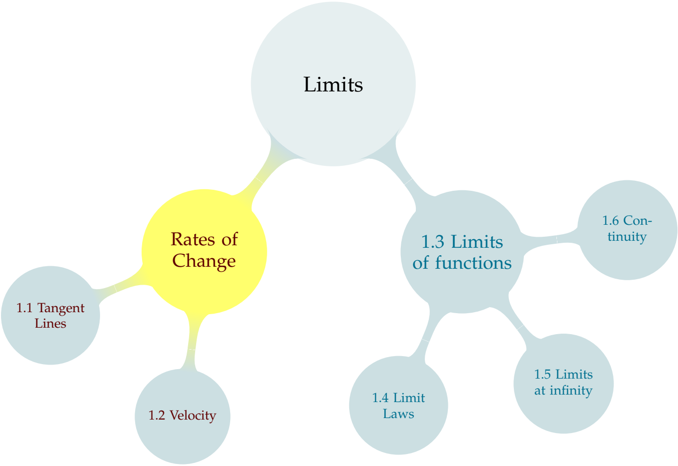
SKIP CHAPTER INTRO



Rates of Change

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.

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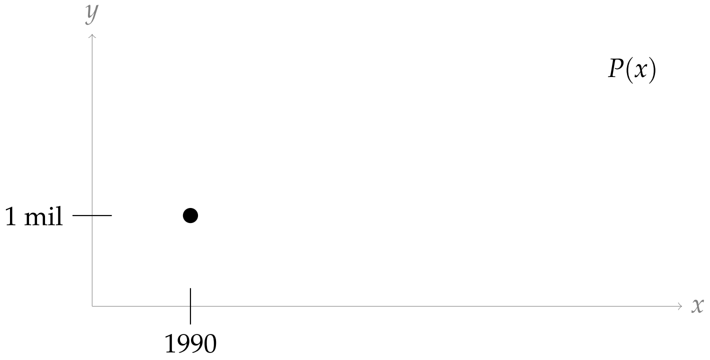

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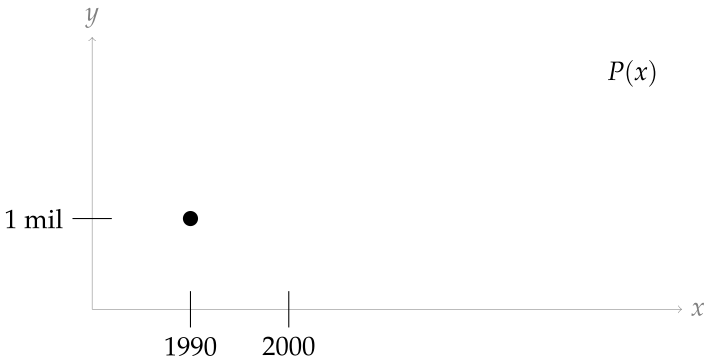

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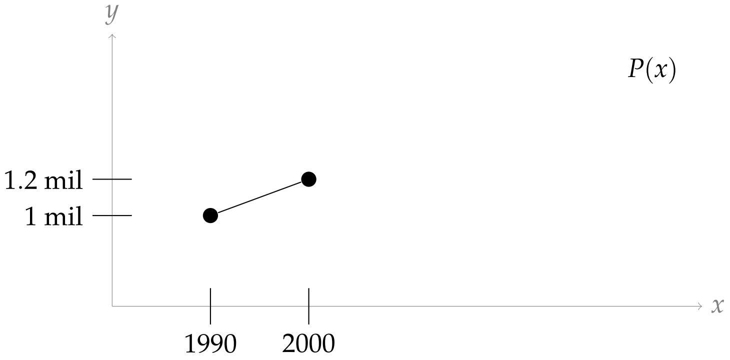

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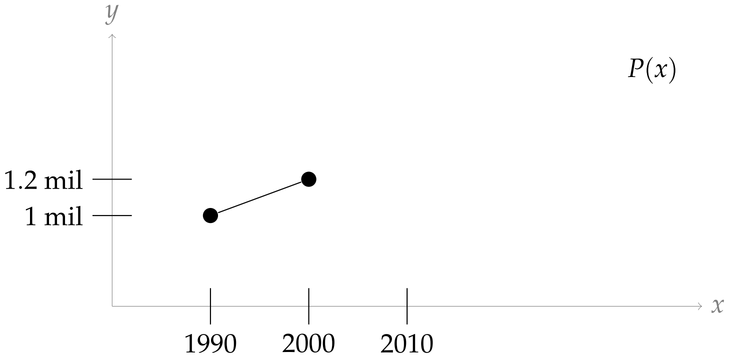

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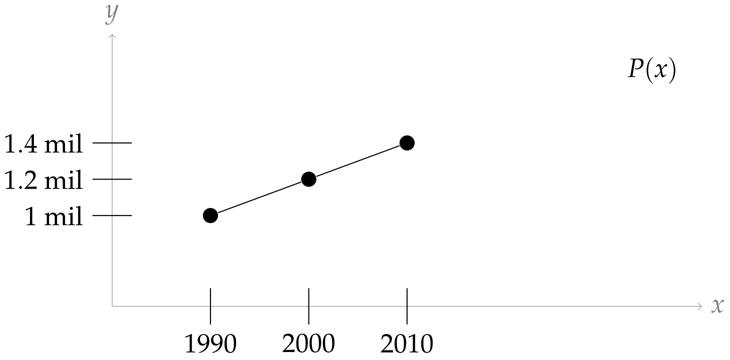

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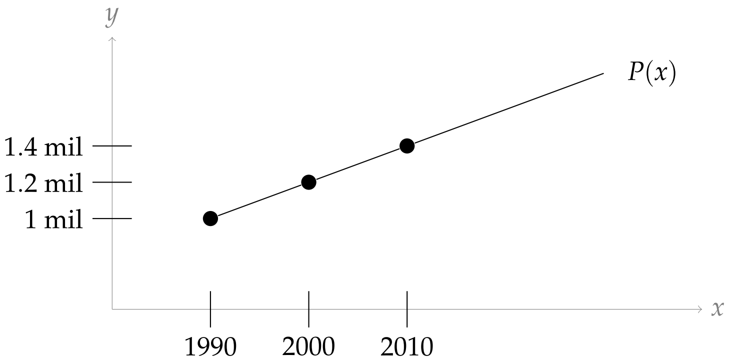
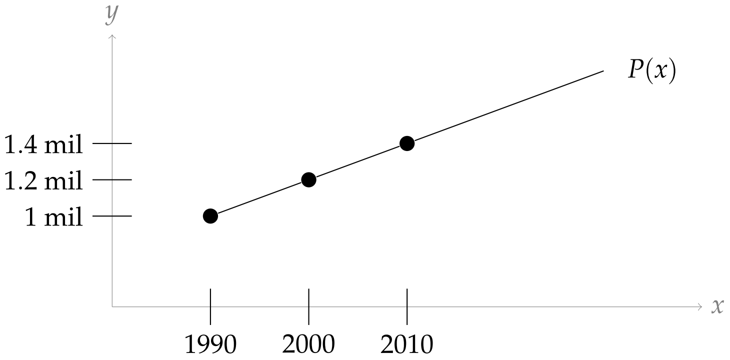


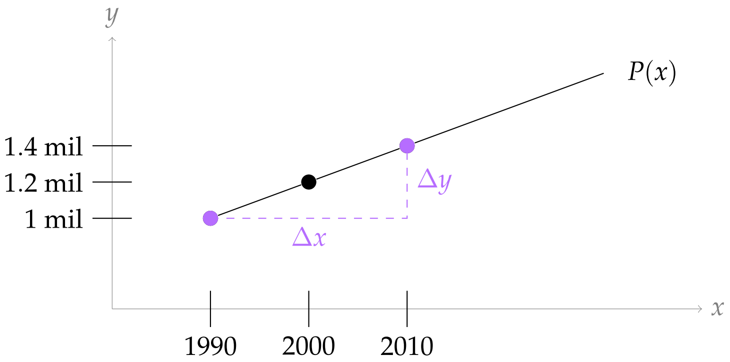
Definition

The **slope** of a line that passes through the points (x_1, y_1) and (x_2, y_2) is "rise over run"

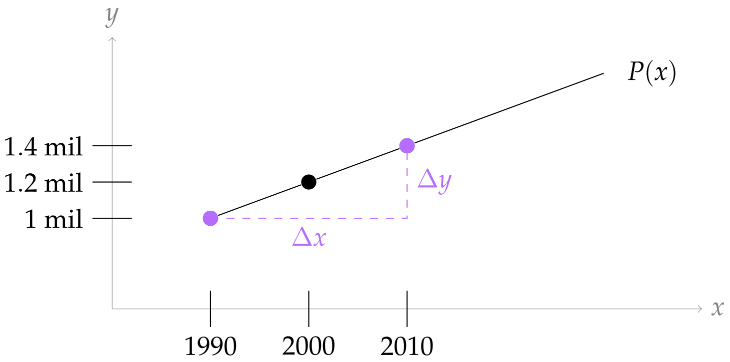
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is also called the **rate of change** of the function.
If a line has equation $y = mx + b$, its slope is m.

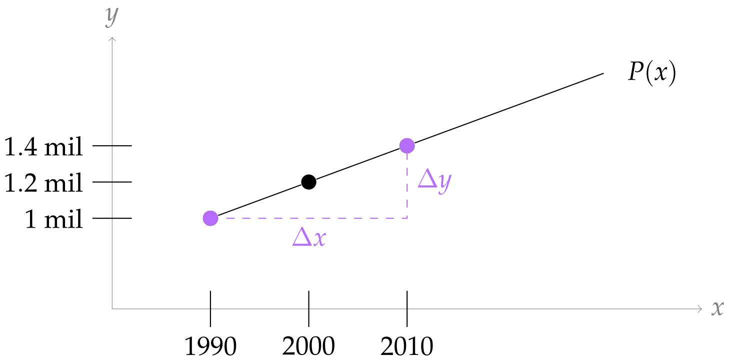

y
P(x)
1.4 mil
1.2 mil
1 mil
x
1990
2000
2010



y
P(x)
1.4 mil
1.2 mil
1 mil
Δy
Δx
x
1990
2000
2010



Rate of change: $\dfrac{400,000 \text{ people}}{20 \text{ years}} = 20,000\ \dfrac{\text{people}}{\text{year}}$



Rate of change: $\dfrac{400,000 \text{ people}}{20 \text{ years}} = 20,000 \ \dfrac{\text{people}}{\text{year}}$

(doesn't depend on the year)

Suppose the population of a small country is given in the chart below.

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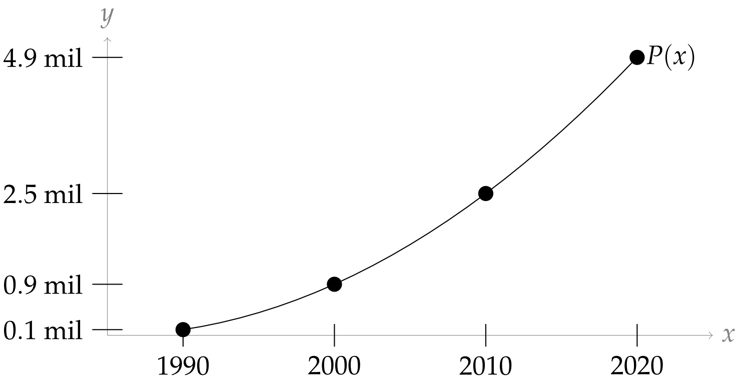

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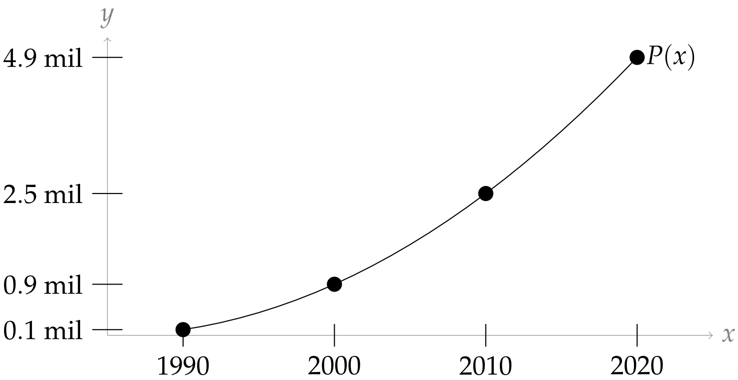

Rate of change $\dfrac{\Delta \text{ pop}}{\Delta \text{ time}}$

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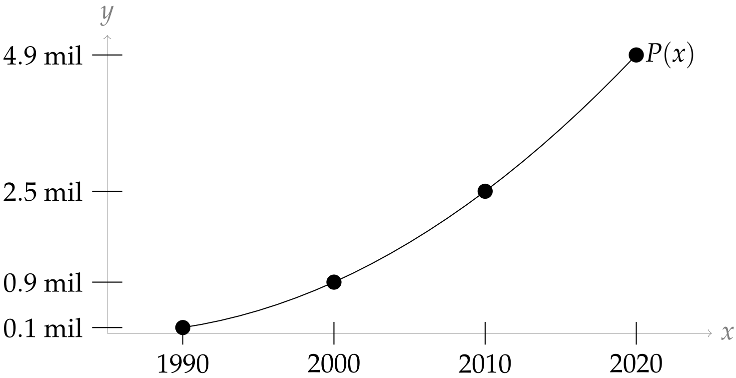
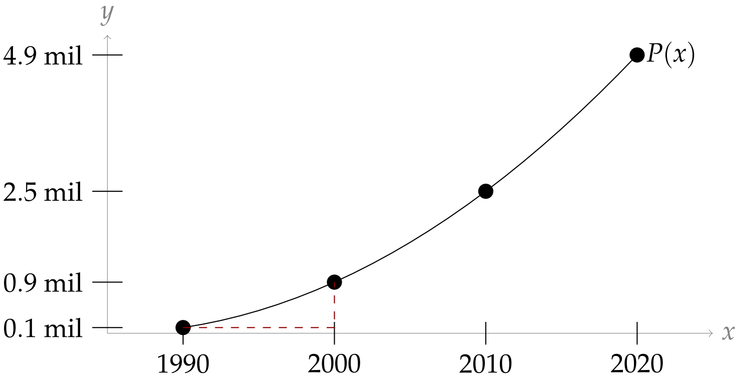


Rate of change $\dfrac{\Delta \text{ pop}}{\Delta \text{ time}}$ depends on time interval

Suppose the population of a small country is given in the chart below.



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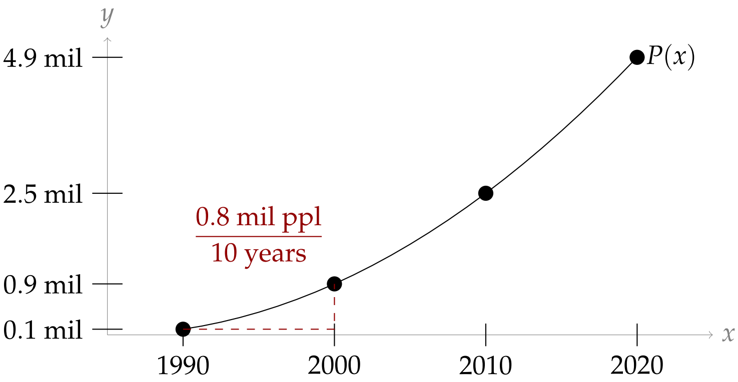

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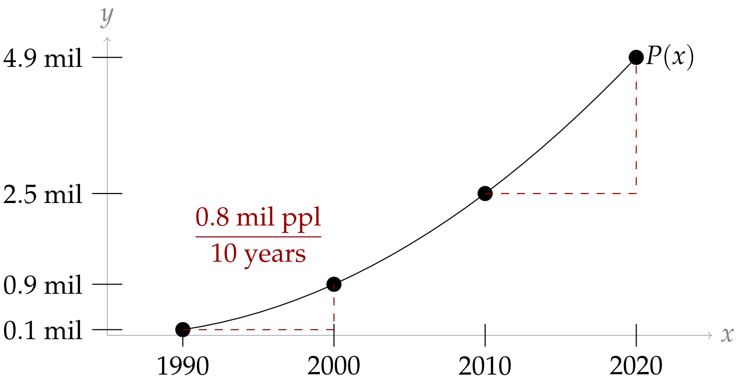

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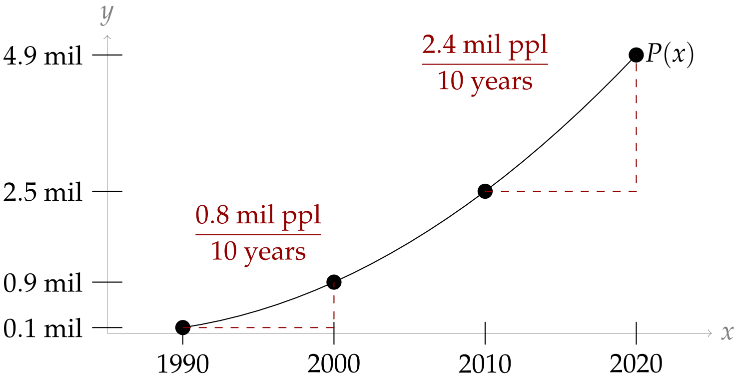
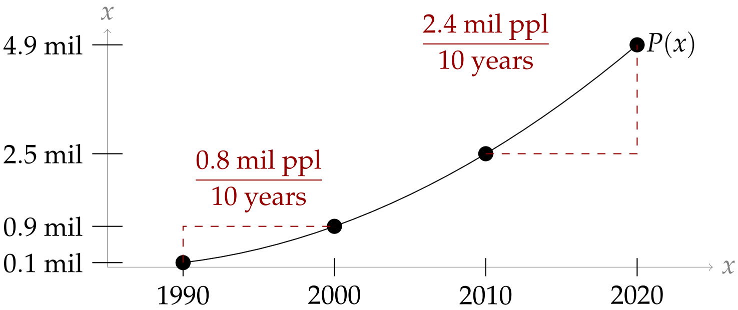


Rate of change $\dfrac{\Delta \text{ pop}}{\Delta \text{ time}}$ depends on time interval

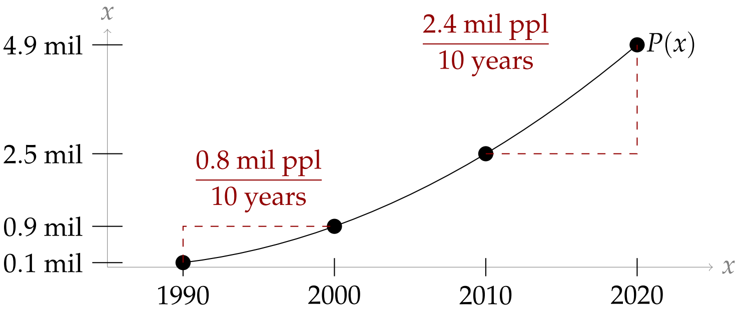
Definition

Let $y = f(x)$ be a curve that passes through (x_1, y_1) and (x_2, y_2). Then the **average rate of change** of $f(x)$ when $x_1 \leq x \leq x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

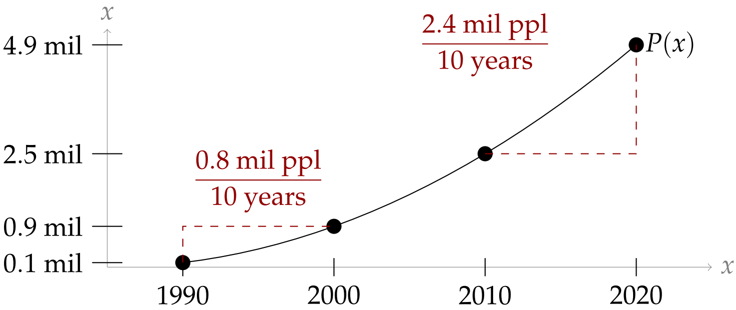


Average rate of change from 1990 to 2000:



Average rate of change from 1990 to 2000:
$80,000$ people per year.

Average rate of change from 2010 to 2020:



Average rate of change from 1990 to 2000:
$80,000$ people per year.

Average rate of change from 2010 to 2020:
$240,000$ people per year.

Average Rate of Change and Slope

The average rate of change of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is "change in output" divided by "change in input:"

$$\frac{f(b) - f(a)}{b - a}$$

Average Rate of Change and Slope

The average rate of change of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is "change in output" divided by "change in input:"

$$\frac{f(b) - f(a)}{b - a}$$

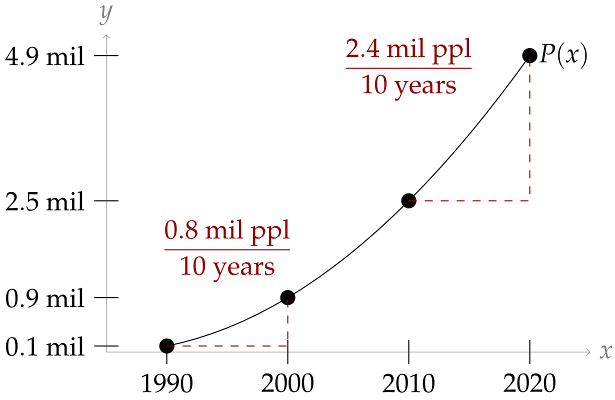
If the function $f(x)$ is a line, then the slope of the line is "rise over run,"

$$\frac{f(b) - f(a)}{b - a}$$

If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

How fast was this population growing in the year 2010? (What was its instantaneous rate of change?)



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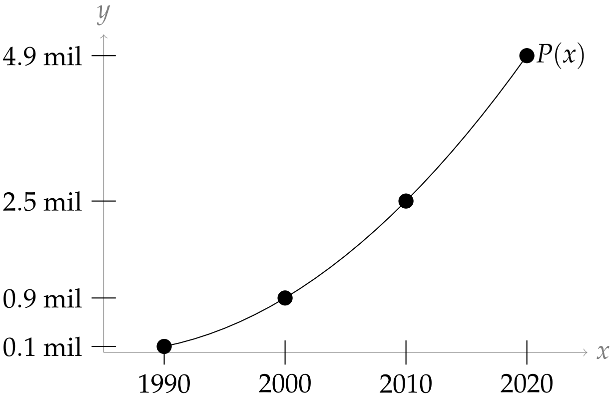

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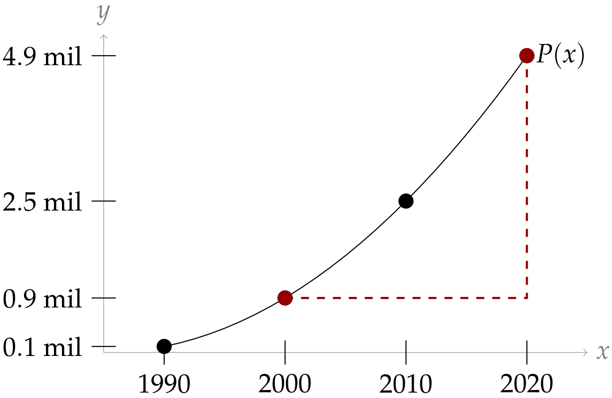

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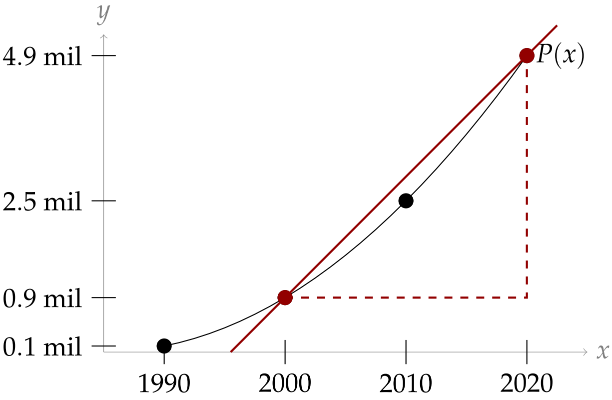

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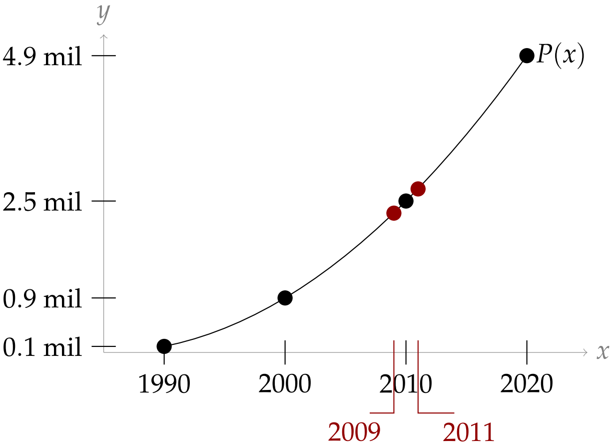

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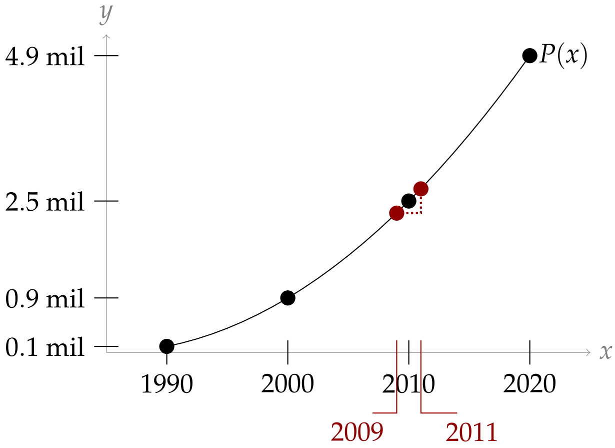

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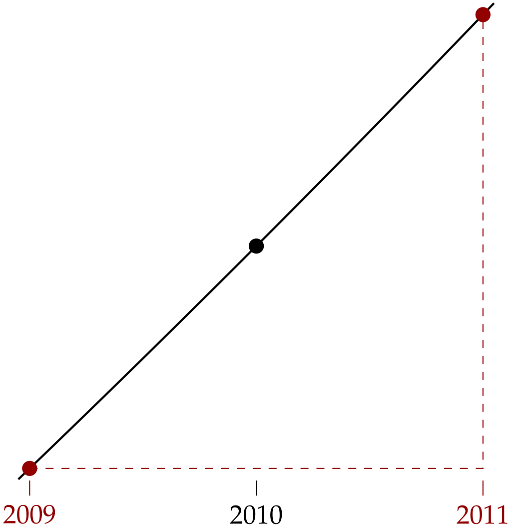

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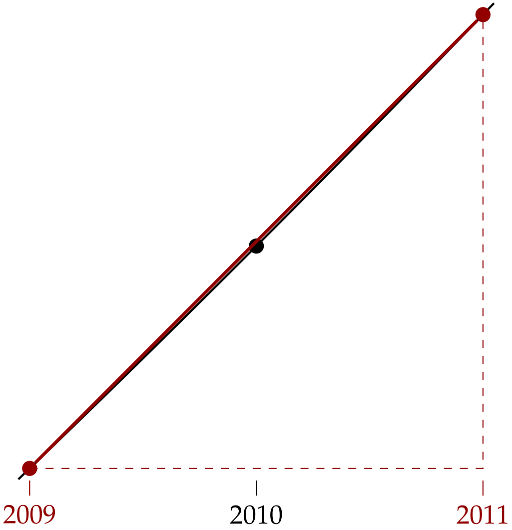

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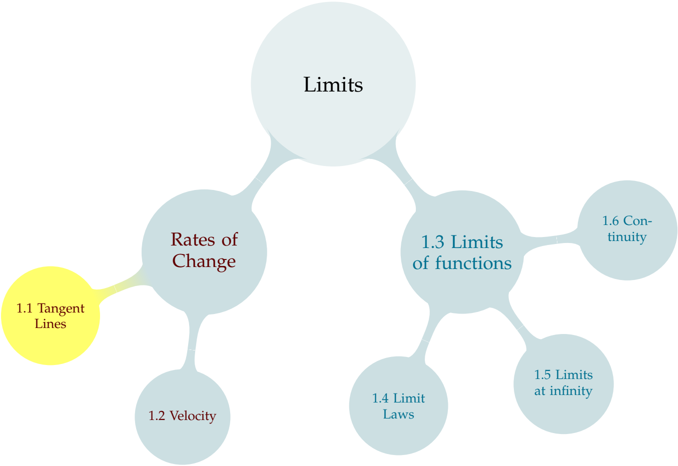
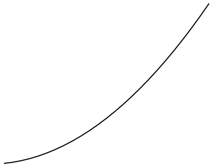


Definition

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q.



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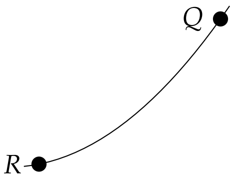

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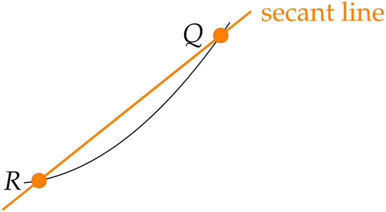
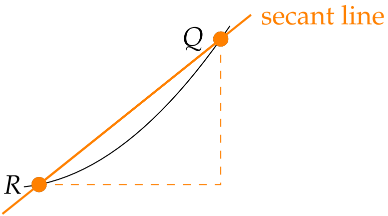


Definition

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q.

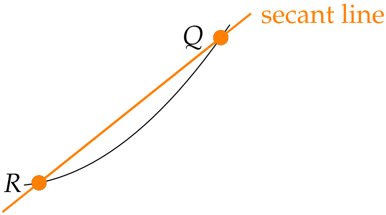
We call the slope of the secant line the **average rate of change of** $f(x)$ **from** R **to** Q**.**



Definition

The **tangent line** to the curve $y = f(x)$ at point P is a line that

- passes through P and
- has the same slope as $f(x)$ at P.



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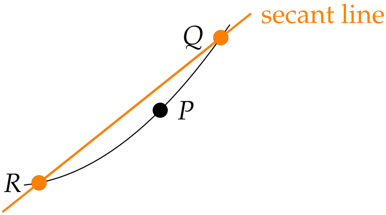

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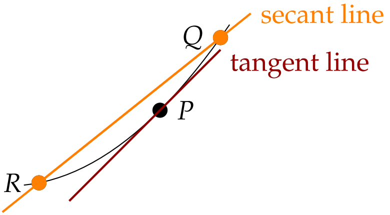
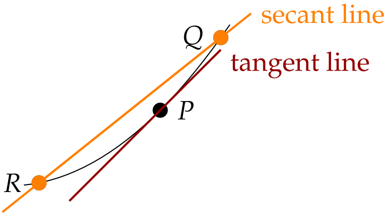


Definition

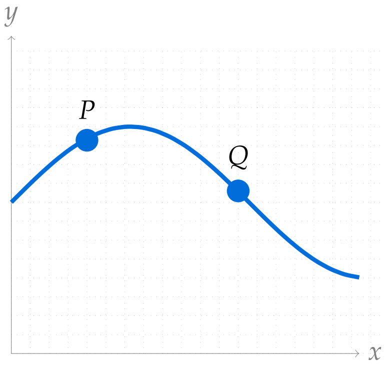
The **tangent line** to the curve $y = f(x)$ at point P is a line that

- passes through P and
- has the same slope as $f(x)$ at P.

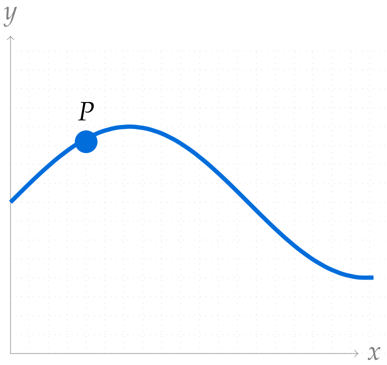
We call the slope of the tangent line the **instantaneous rate of change of $f(x)$ at P.**



On the graph below, draw the secant line to the curve through points P and Q.



On the graph below, draw the tangent line to the curve at point P.



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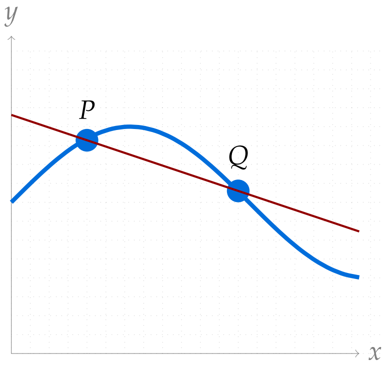

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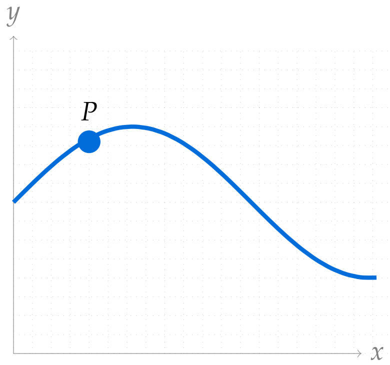

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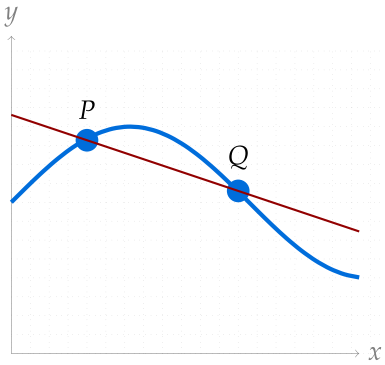

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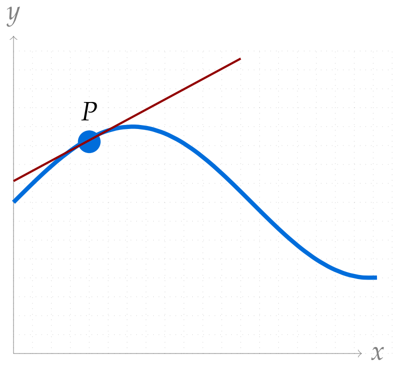

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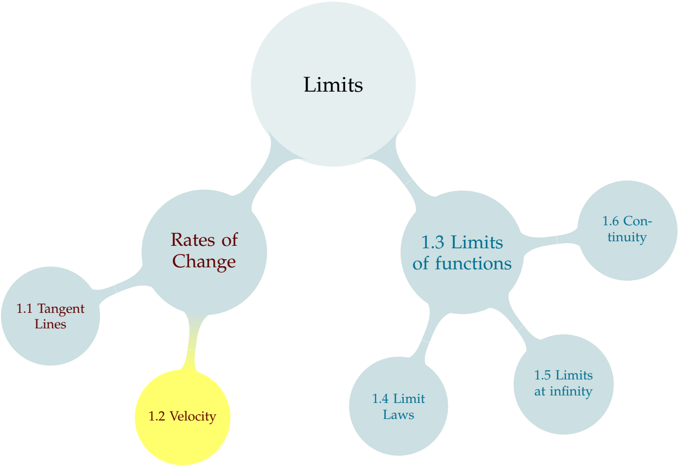
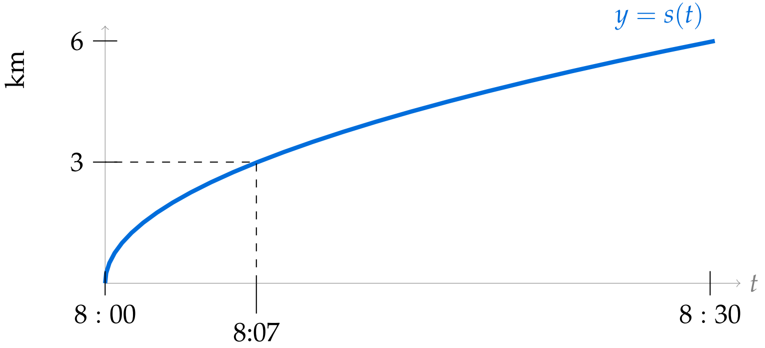


$y = s(t)$
km
6
3
t
8 : 00
8 : 30


$y = s(t)$
km
6
3
t
8 : 00
8:07
8 : 30

It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$

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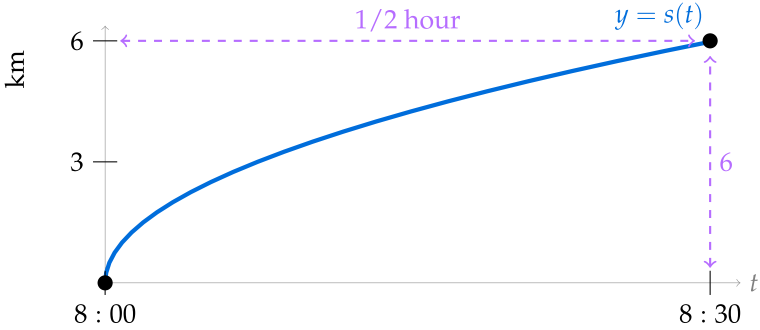

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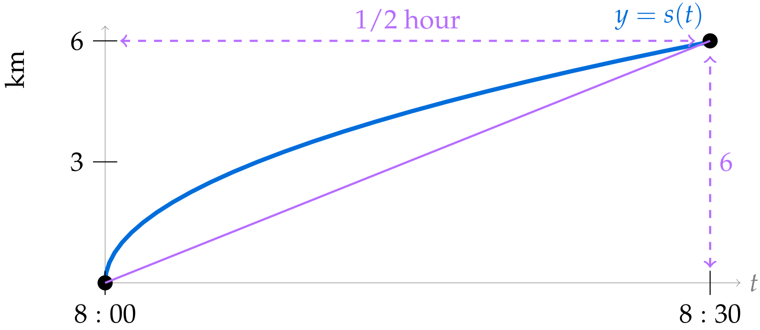
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$



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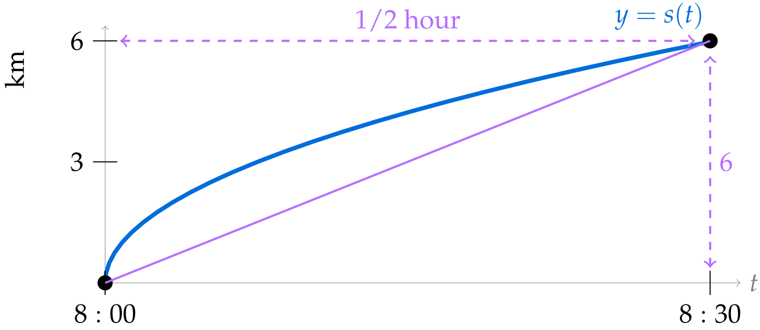
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$



It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$

At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

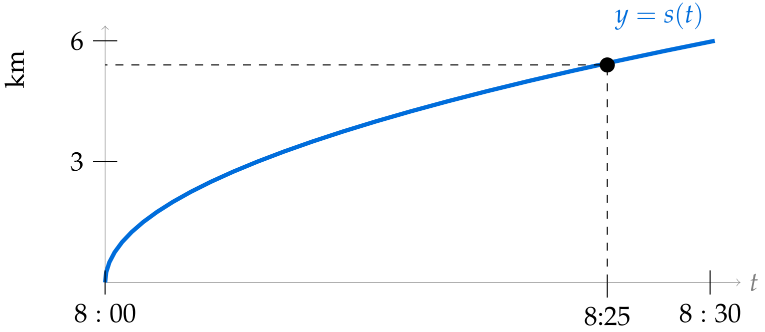
A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

C. tangent line to $y = s(t)$ at $t = 8:25$

D. slope of the tangent line to $y = s(t)$ at $t = 8:25$

ans



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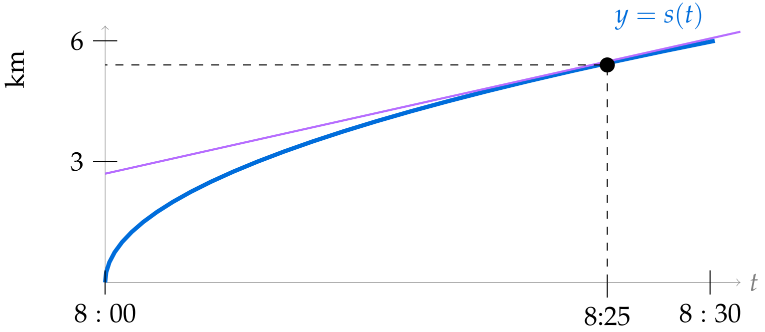
A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

C. tangent line to $y = s(t)$ at $t = 8:25$

D. slope of the tangent line to $y = s(t)$ at $t = 8:25$

ans



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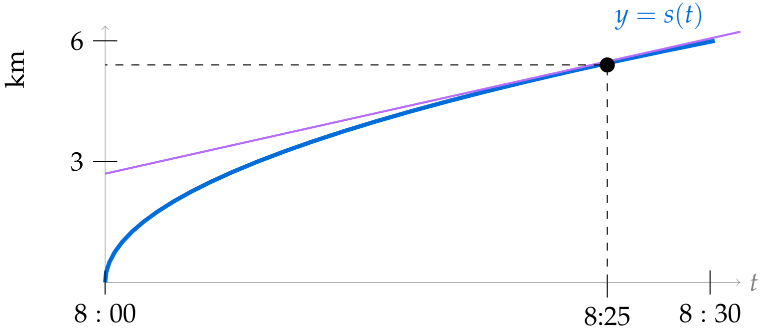

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Suppose the distance from the ground s (in meters) of a helium-filled balloon at time t over a 10-second interval is given by $s(t) = t^2$. Try to estimate how fast the balloon is rising when $t = 5$.

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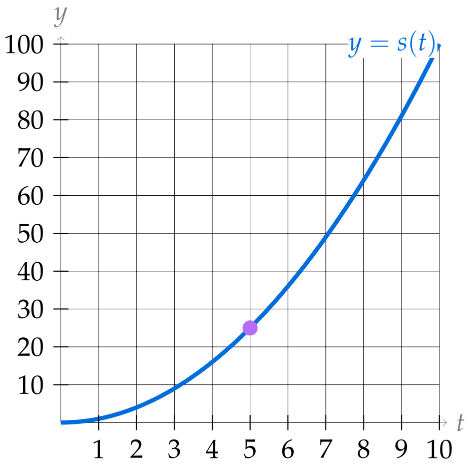

One way: Estimate the slope of the tangent line to the curve

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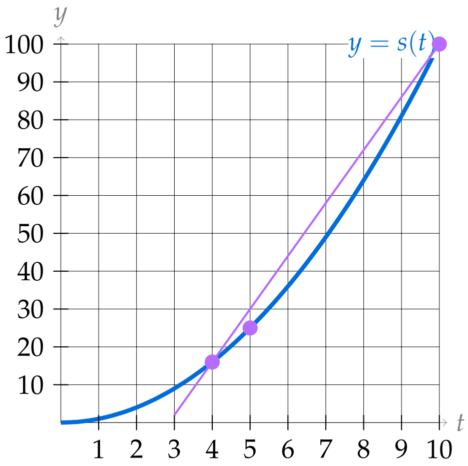

Another way: Calculate average rate of change for intervals around 5 that get smaller and smaller.

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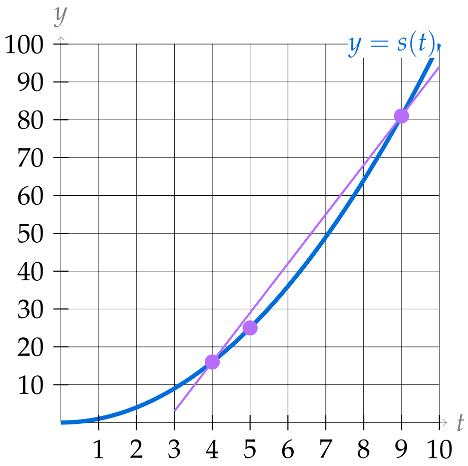

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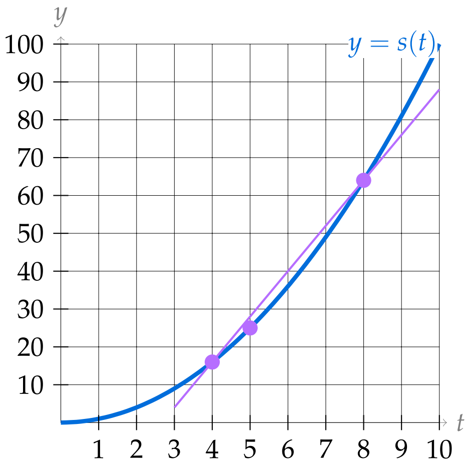

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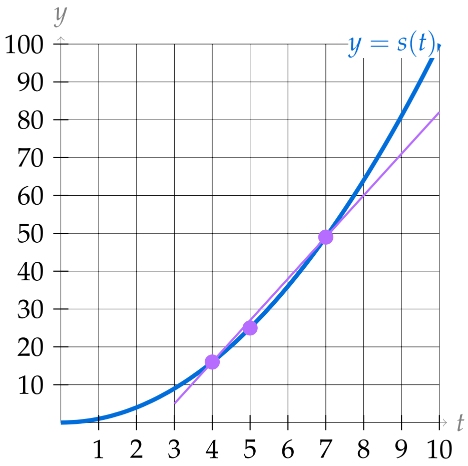

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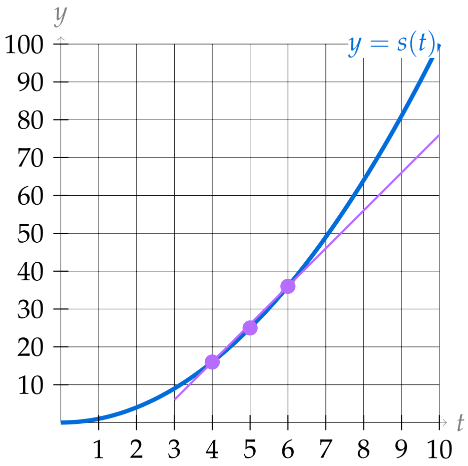

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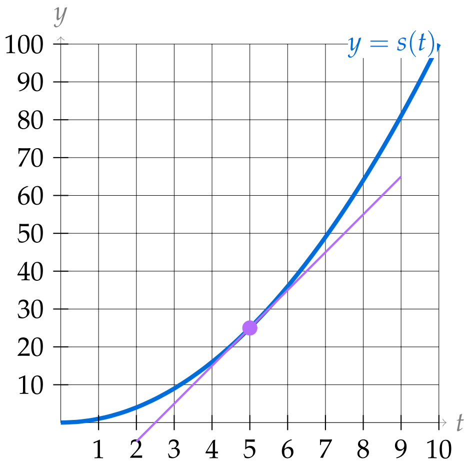
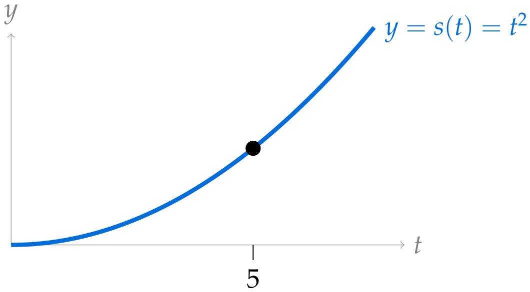
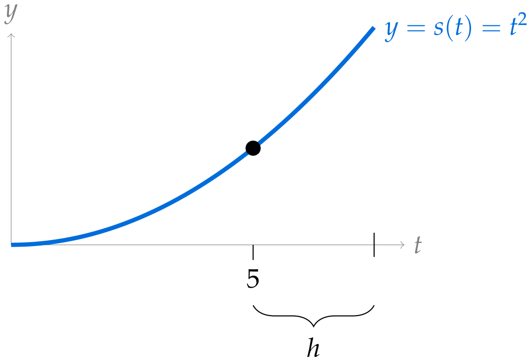


Another way: Calculate average rate of change for intervals around 5 that get smaller and smaller.

Let's look for an algebraic way of determining the velocity of the balloon when $t = 5$.



Suppose the interval $[5, \quad]$ has length h. What is the right endpoint of the interval?



Q

ans

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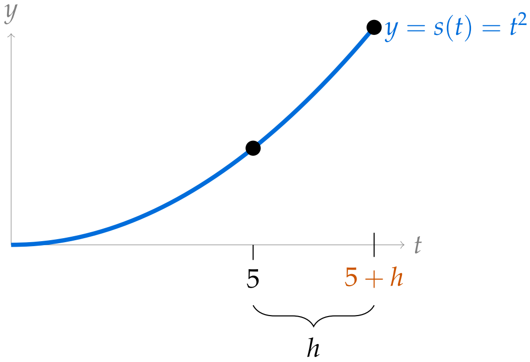
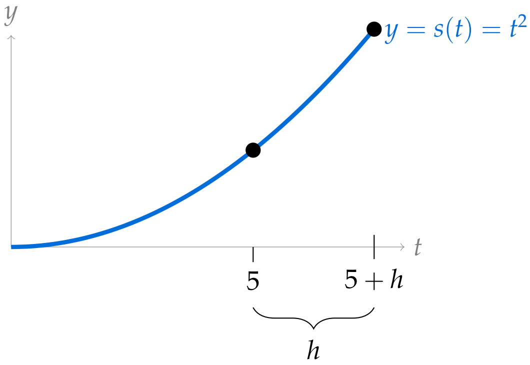


Write the equation for the average (vertical) velocity from $t = 5$ to $t = 5 + h$.



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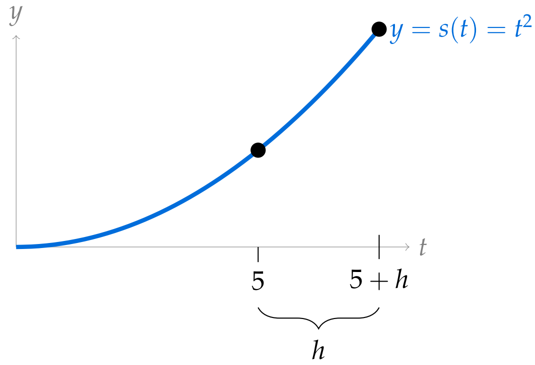
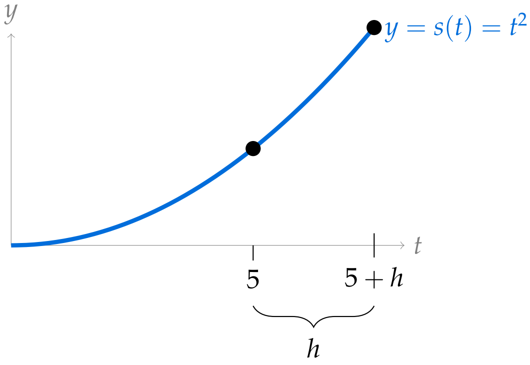


$$\text{vel} = \frac{\Delta\text{ height}}{\Delta\text{ time}} = \frac{s(5+h) - s(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

What happens to the velocity when h is very, very small?



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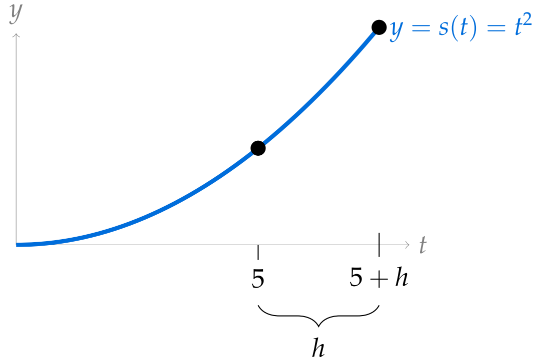
When h is very small,

$$\approx 10$$

What do you think is the slope of the tangent line to the graph when $t = 5$?



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Limit Notation

We write:

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We say: "The limit as h goes to 0 of $(10 + h)$ is 10."

It means: As h gets extremely close to 0, $(10 + h)$ gets extremely close to 10.

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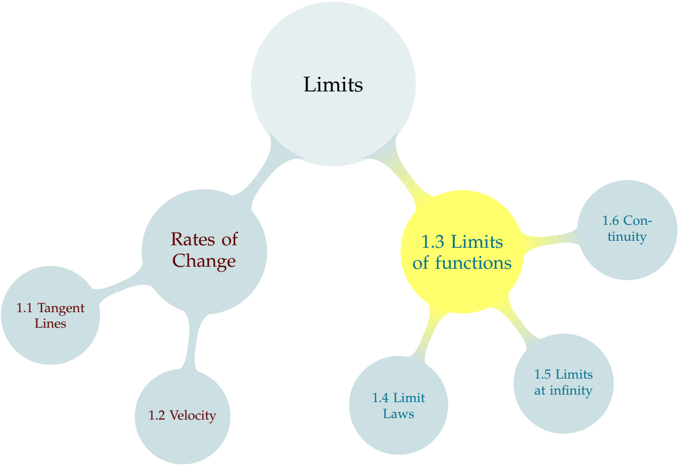


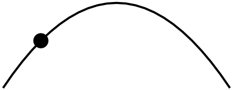
Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \to a} f(x) = L$$

where a and L are real numbers
We read the above as "the limit as x goes to a of $f(x)$ is L."
Its meaning is: as x gets very close to (but not equal to) a, $f(x)$ gets very close to L.

Finding Slopes of Tangent Lines

We NEED limits to find slopes of tangent lines.



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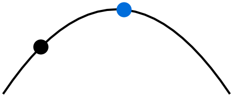

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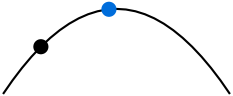

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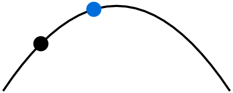

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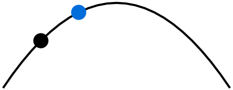

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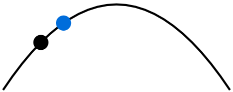

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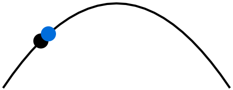

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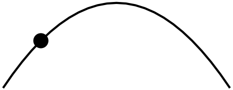

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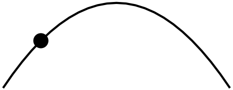

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$$\lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$$

EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \to 1} f(x)$.

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What is $f(1)$?

Q

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Evaluating Limits

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \to 1} f(x)$.

Use the tables below to guess $\lim_{x \to 1} f(x)$

x	$f(x)$
0.9	3.61
0.99	3.9601
0.999	3.99600
0.9999	3.99960

x	$f(x)$
1.1	4.41
1.01	4.0401
1.001	4.00400
1.0001	4.00040

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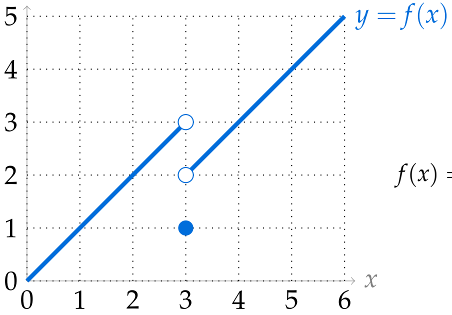
$$\boxed{\lim_{x \to 1} f(x) = 4}$$

A

One-Sided Limits



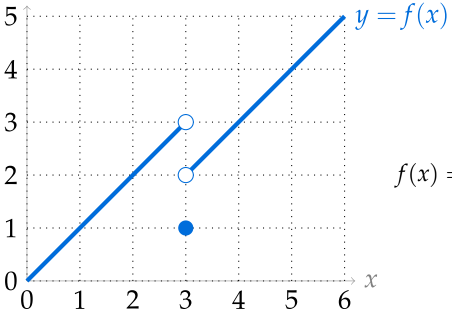
$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x-1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim\limits_{x \to 3} f(x)$ should be?

Q

ans

One-Sided Limits



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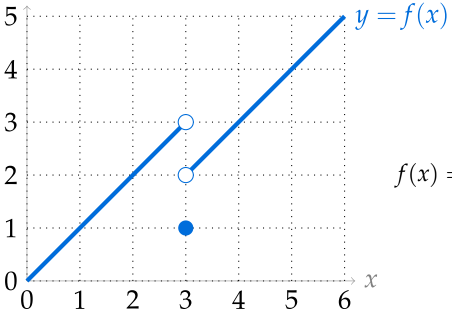


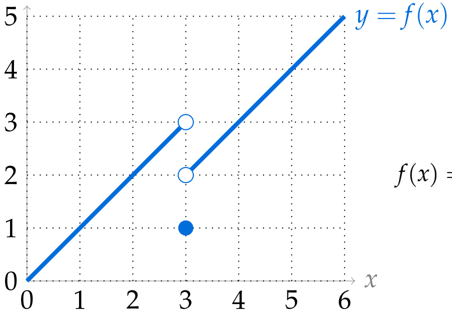
$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate: $\underbrace{\lim_{x \to 3^-} f(x)}_{\text{from the left}}$ $\underbrace{\lim_{x \to 3^+} f(x)}_{\text{from the right}}$

Q

ans

One-Sided Limits



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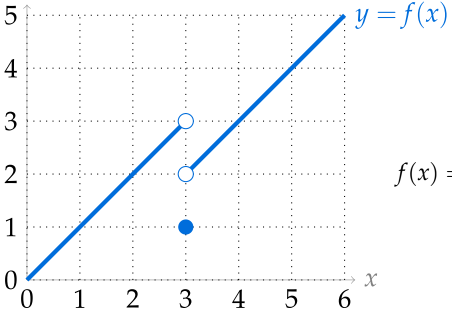
Evaluate: $\underbrace{\lim_{x \to 3^-} f(x)}_{\text{from the left}} = 3$ $\qquad$ $\underbrace{\lim_{x \to 3^+} f(x)}_{\text{from the right}}$

A

ans

One-Sided Limits



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate: $\underbrace{\lim_{x\to 3^-} f(x)}_{\text{from the left}} = 3$ $\qquad \underbrace{\lim_{x\to 3^+} f(x)}_{\text{from the right}} = 2$

Definition 1.3.7

The limit as x goes to a **from the left** of $f(x)$ is written

$$\lim_{x \to a^-} f(x)$$

We only consider values of x that are less than a.

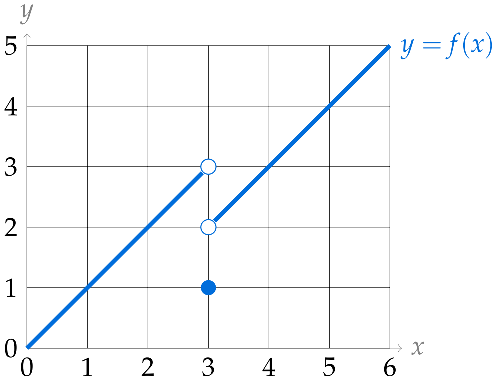
The limit as x goes to a **from the right** of $f(x)$ is written

$$\lim_{x \to a^+} f(x)$$

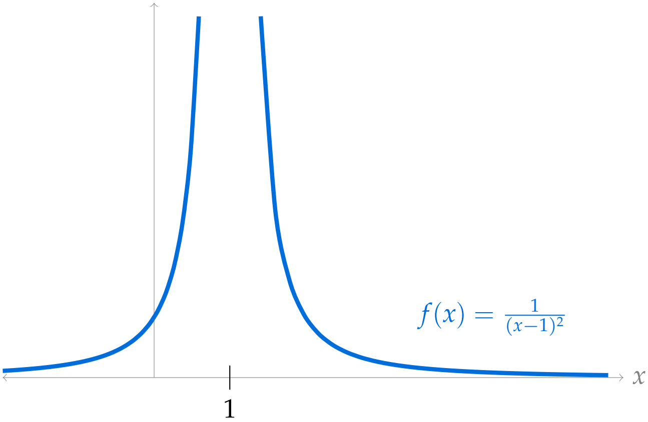
We only consider values of x greater than a.

Theorem 1.3.8

In order for $\lim_{x \to a} f(x)$ to exist, both one-sided limits must exist and be equal.



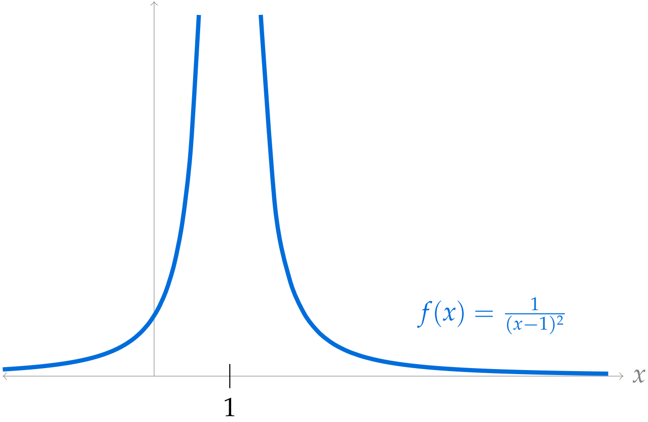
Consider the function $f(x) = \dfrac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ not defined?



ans

Based on the graph below, what would you like to write for:

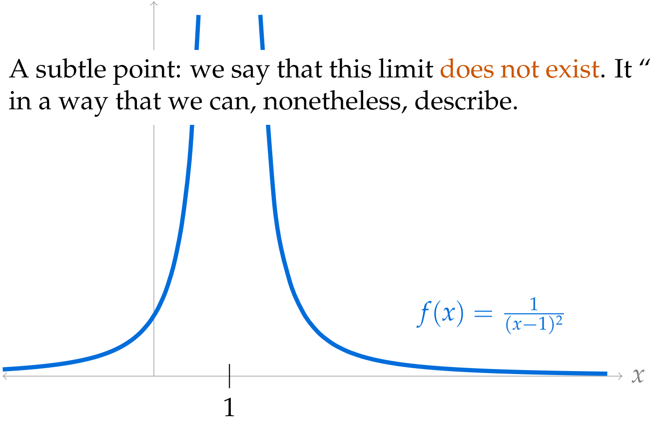
$$\lim_{x\to 1} f(x) =$$



ans

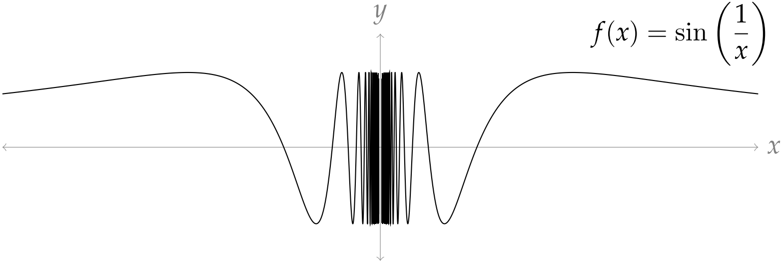
Based on the graph below, what would you like to write for:

$$\lim_{x\to 1} f(x) = \infty$$



A subtle point: we say that this limit does not exist. It "does not exist" in a way that we can, nonetheless, describe.

A Stranger Limit Example



What is $\lim_{x\to\infty} f(x)$?

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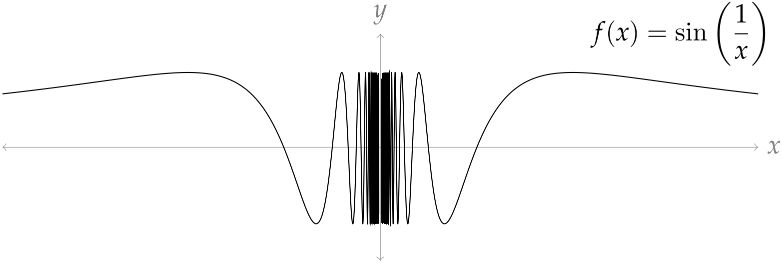
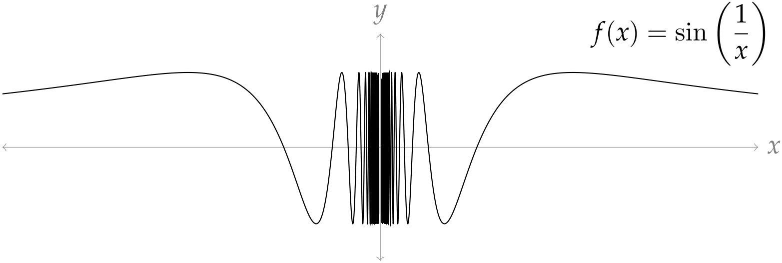


What is $\lim_{x\to\infty} f(x)$?

$$\lim_{x\to\infty} f(x) = 0$$

A Stranger Limit Example

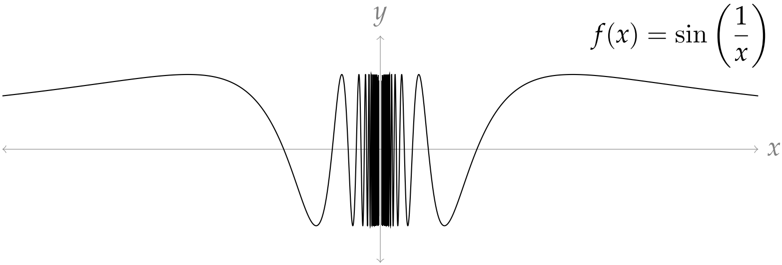


What is $\lim_{x\to 0} f(x)$?

Q

ans

A Stranger Limit Example

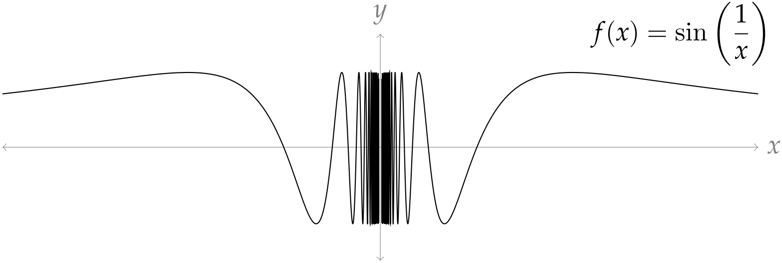


What is $\lim_{x \to 0} f(x)$?

$\lim_{x \to 0} f(x)$ does not exist.

We can call this behaviour "infinite wiggling."

A Stranger Limit Example

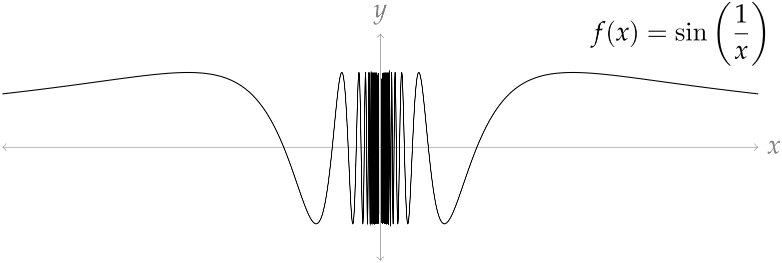


What is $\lim_{x \to \pi} f(x)$?

Q

ans

A Stranger Limit Example

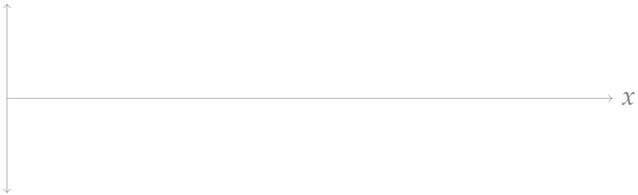


What is $\lim_{x\to\pi} f(x)$?

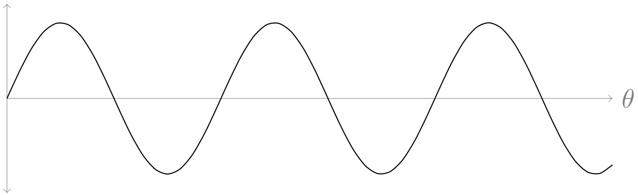
$\lim_{x\to\pi} f(x) = \sin\left(\frac{1}{\pi}\right)$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING



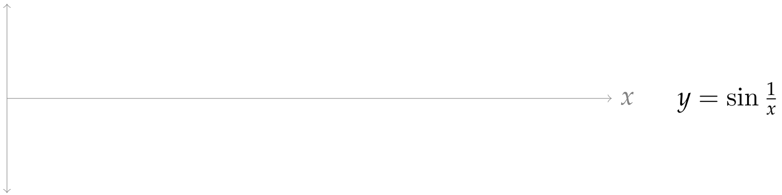
$y = \sin\frac{1}{x}$

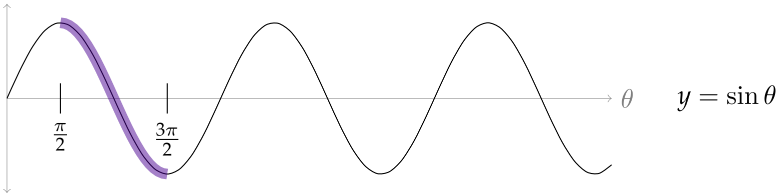


$y = \sin\theta$

Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING





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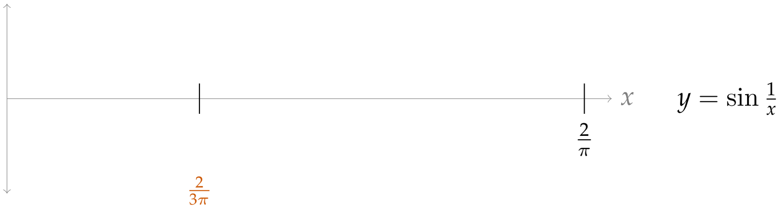
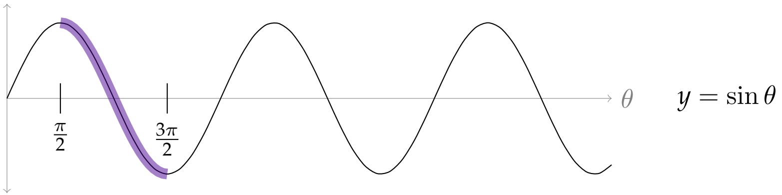
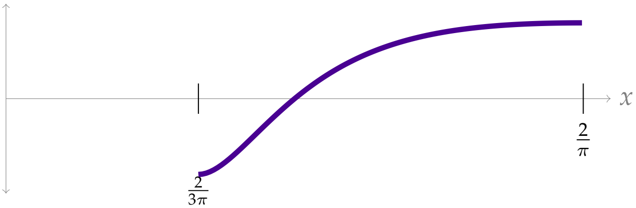
SKIP SKETCHING



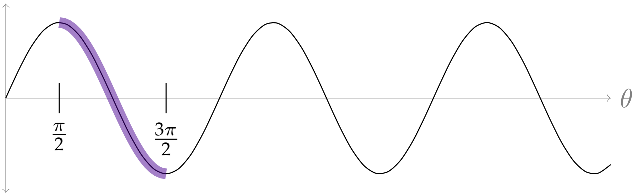


Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$

Skip sketching



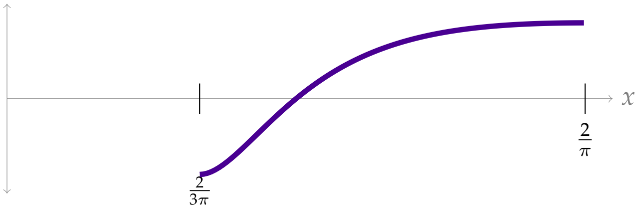
$y = \sin\frac{1}{x}$



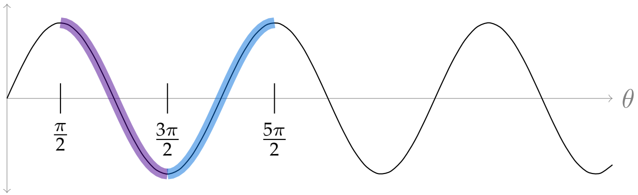
$y = \sin\theta$

Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING



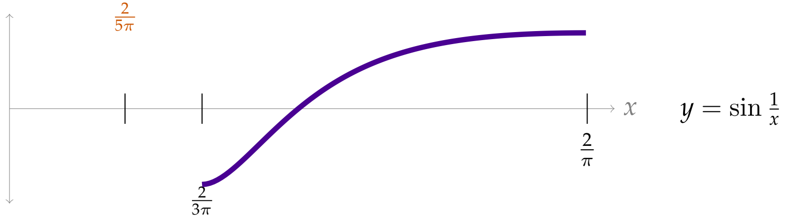
$y = \sin \frac{1}{x}$

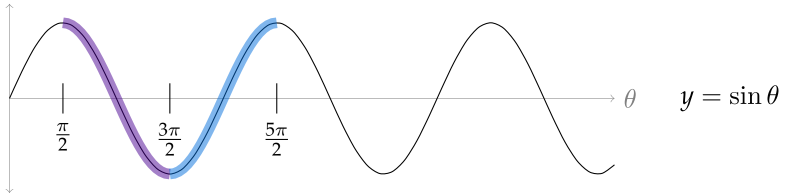


$y = \sin\theta$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

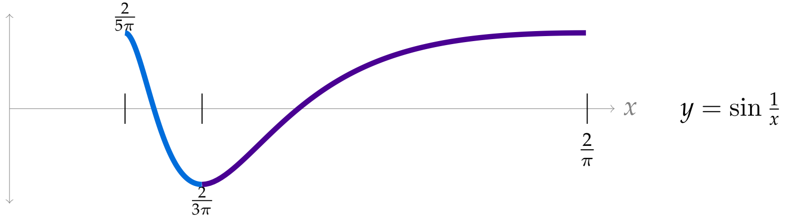
SKIP SKETCHING

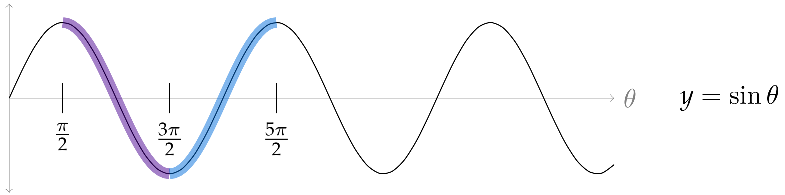




Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING





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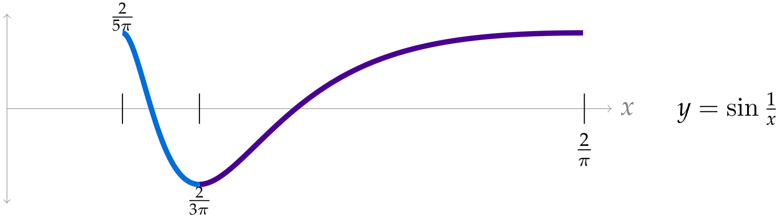
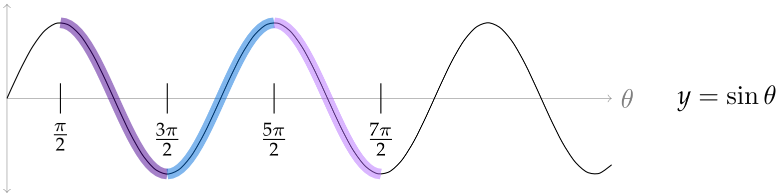
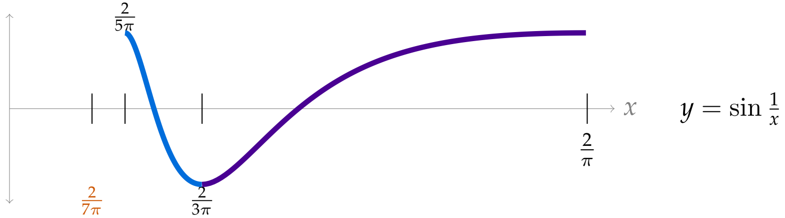
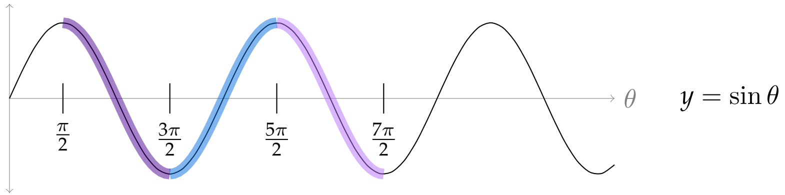




Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$

▶▶ SKIP SKETCHING





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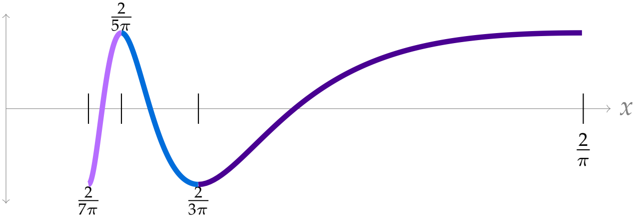
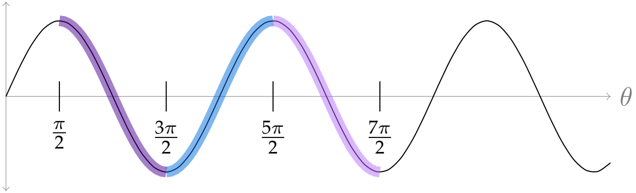
SKIP SKETCHING



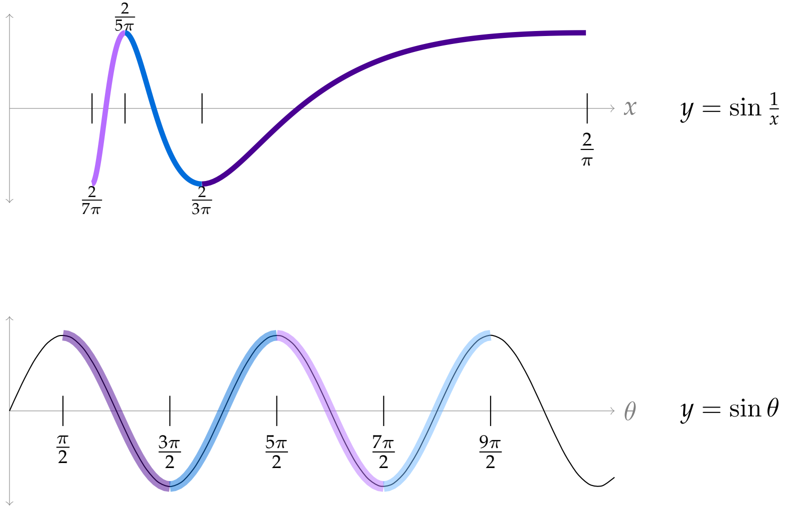
$y = \sin\frac{1}{x}$



$y = \sin\theta$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING



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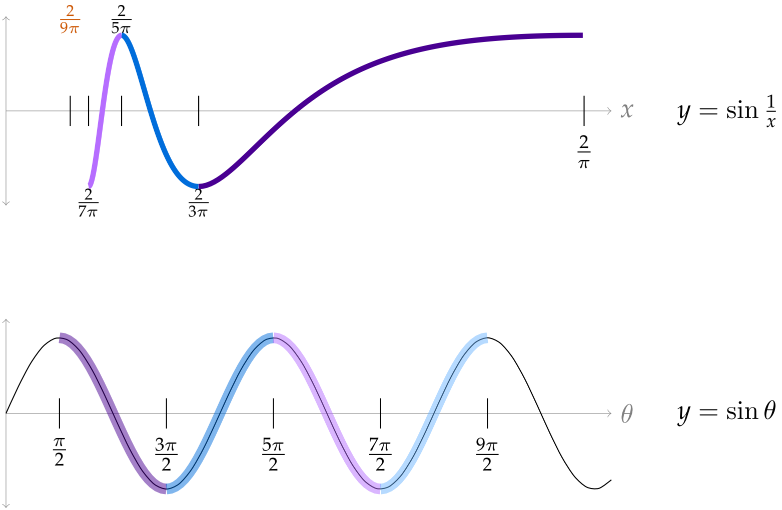
▶▶ SKIP SKETCHING



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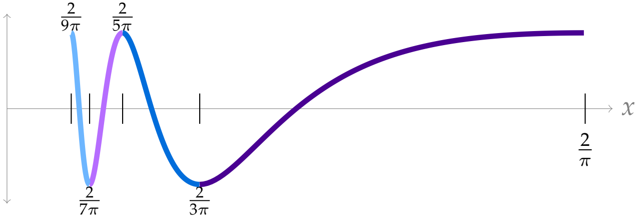
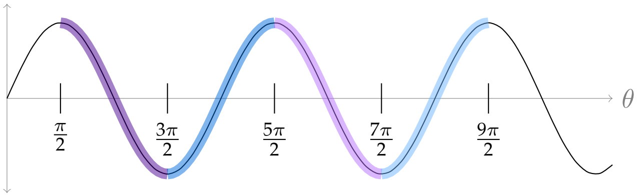
SKIP SKETCHING



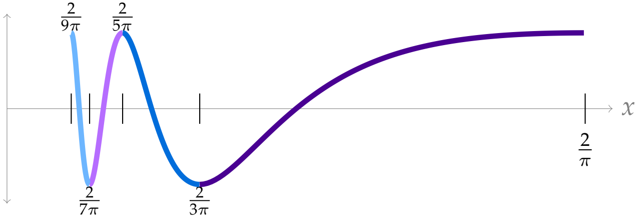
$y = \sin\frac{1}{x}$



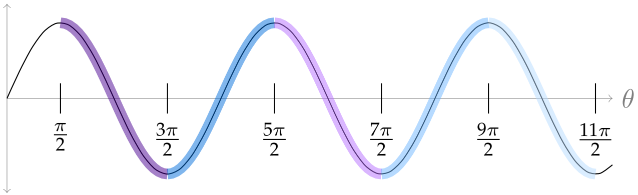
$y = \sin\theta$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING



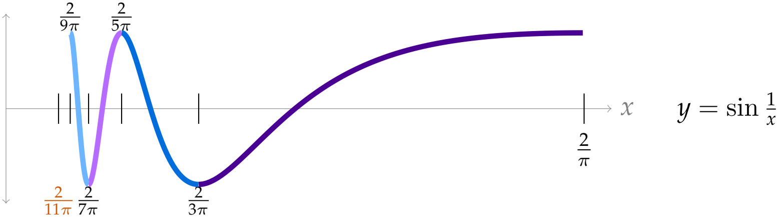
$y = \sin\frac{1}{x}$

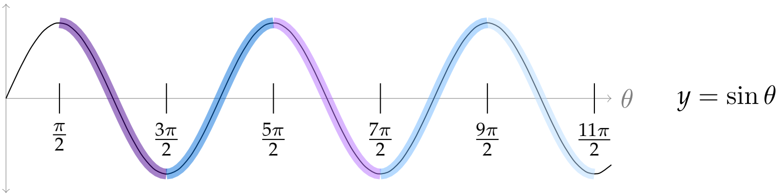


$y = \sin\theta$

Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$

SKIP SKETCHING





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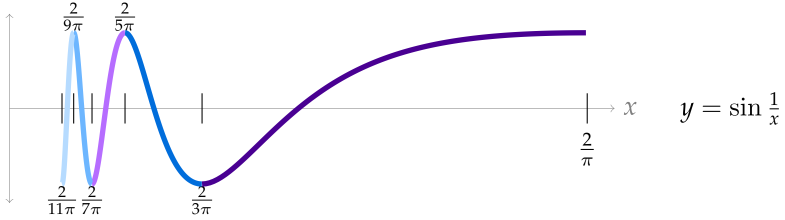
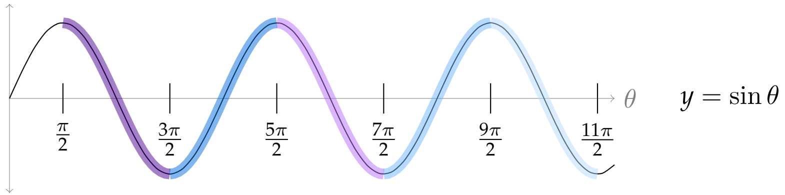
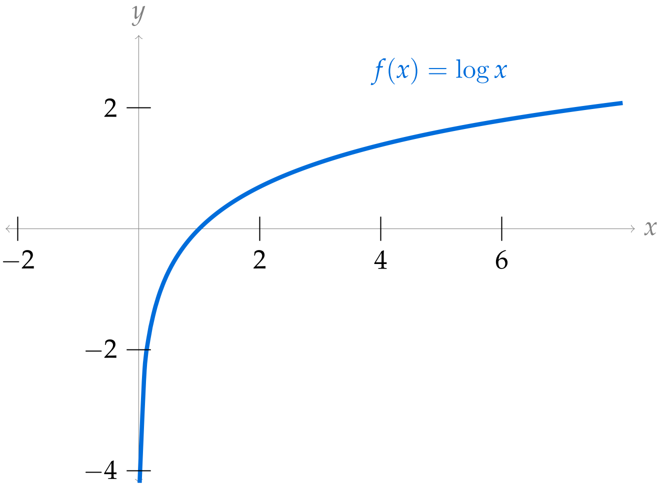
▶▶ SKIP SKETCHING





Limits and the Natural Logarithm

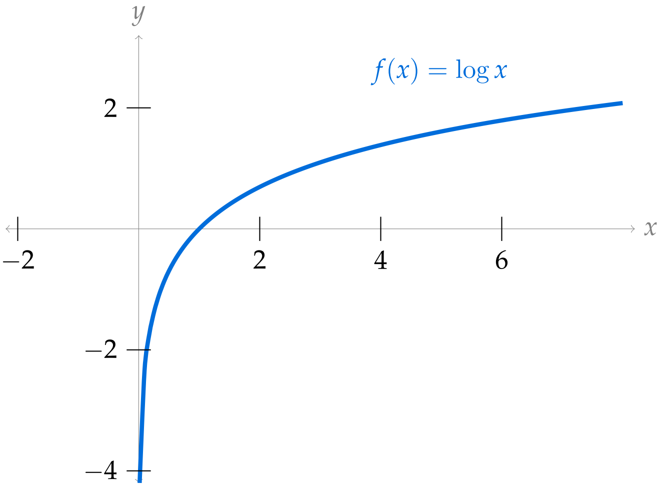
Where is $f(x)$ defined, and where is it not defined?



ans

Limits and the Natural Logarithm

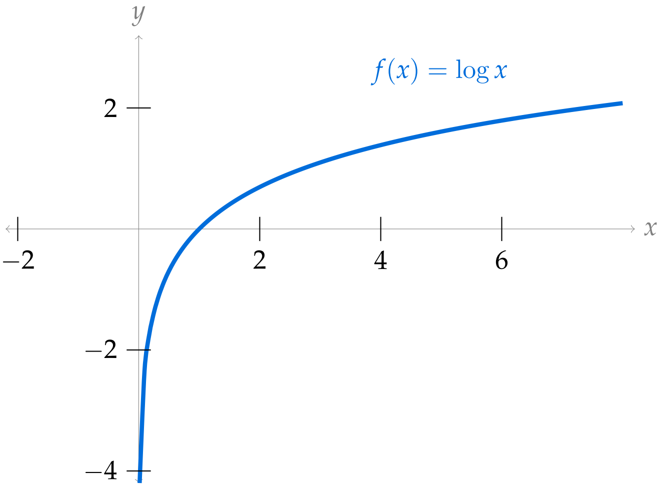
What can you say about the limit of $f(x)$ near 0?



ans

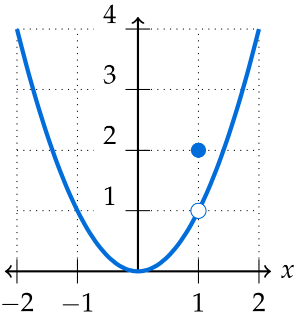
Limits and the Natural Logarithm

What can you say about the limit of $f(x)$ near 0? $\lim_{x \to 0^+} \log(x) = -\infty$



Section 1.3 Review

$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x \to 1} f(x)$?

A. $\lim_{x \to 1} f(x) = 2$

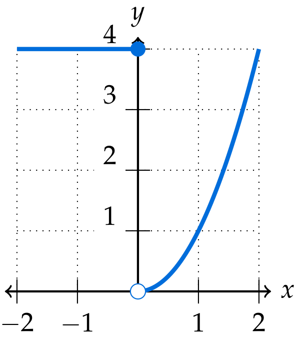
B. $\lim_{x \to 1} f(x) = 1$

C. $\lim_{x \to 1} f(x)$ DNE

D. none of the above

ans

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim\limits_{x \to 0} f(x)$?

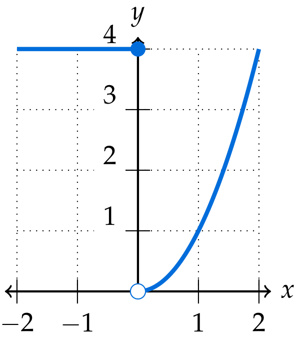
A. $\lim\limits_{x \to 0} f(x) = 4$

B. $\lim\limits_{x \to 0} f(x) = 0$

C. $\lim\limits_{x \to 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim\limits_{x\to 0} f(x)$?

A. $\lim\limits_{x\to 0} f(x) = 4$

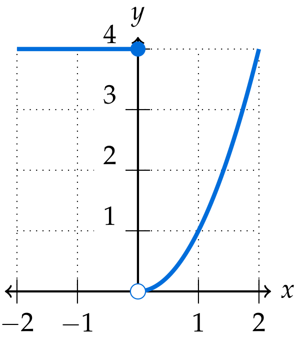
B. $\lim\limits_{x\to 0} f(x) = 0$

C. $\lim\limits_{x\to 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$\lim\limits_{x\to 0} f(x)$ DNE

$$f(x) = \begin{cases} 4 & x \le 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \to 0^+} f(x)$?

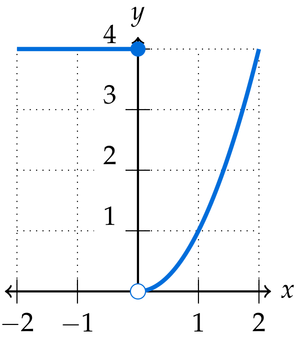
A. $\lim_{x \to 0} f(x) = 4$

B. $\lim_{x \to 0} f(x) = 0$

C. $\lim_{x \to 0} f(x) = \begin{cases} 4 & x \le 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x\to 0^+} f(x)$?

A. $\lim_{x\to 0^+} f(x) = 4$

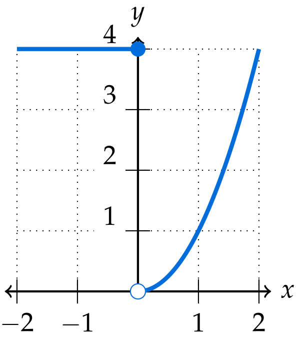
B. $\lim_{x\to 0^+} f(x) = 0$

C. $\lim_{x\to 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

What is $f(0)$?



Suppose $\lim_{x \to 3^-} f(x) = 1$ and $\lim_{x \to 3^+} f(x) = 1.5$.

Does $\lim_{x \to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist.

B. No, never, because the limit from the left is not the same as the limit from the right.

C. Can't tell. For some functions is might exist, for others not.

Suppose $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 1.5$.

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Suppose $\lim_{x\to 3^-} f(x) = 22 = \lim_{x\to 3^+} f(x)$.

Does $\lim_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist and are equal to each other.

B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.

C. Can't tell. We need to know the value of the function at $x = 3$.

Q

ans

Suppose $\lim_{x\to 3^-} f(x) = 22 = \lim_{x\to 3^+} f(x)$.

Does $\lim_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist and are equal to each other.

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A

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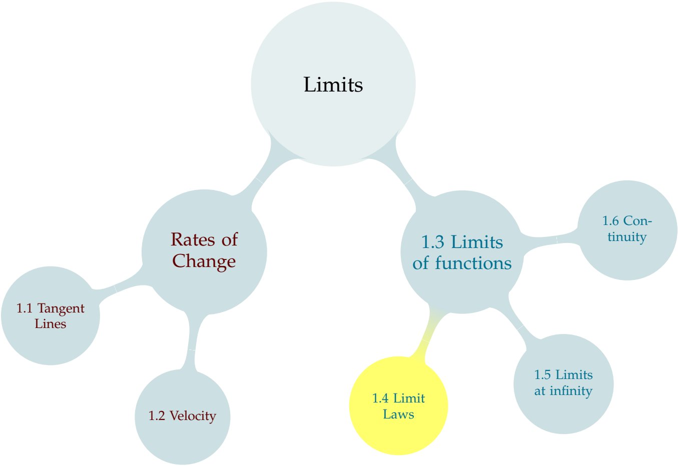

Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

$$\lim_{x\to a} f(x) = f(a).$$

Calculating Limits in Simple Situations

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Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x+3}\right)$

ans

Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

$$\lim_{x \to a} f(x) = f(a).$$

Calculate: $\displaystyle\lim_{x \to 3} \left(\frac{x^2 - 9}{x + 3} \right) = \left(\frac{3^2 - 9}{3 + 3} \right) = \frac{0}{6} = 0$

Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

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Calculate: $\lim_{x\to 3}\left(\frac{x^2-9}{x+3}\right) = \left(\frac{3^2-9}{3+3}\right) = \frac{0}{6} = 0$

Calculate: $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

Calculating Limits in Simple Situations

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If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

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Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x+3}\right) = \left(\frac{3^2-9}{3+3}\right) = \frac{0}{6} = 0$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

Can't find in the same way: 3 not in domain

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x\to a}(f(x) + g(x)) = F + G$
- $\lim_{x\to a}(f(x) - g(x)) = F - G$
- $\lim_{x\to a}(f(x)g(x)) = FG$
- $\lim_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

Algebra with Limits: Theorem 1.4.2

Suppose $\lim\limits_{x\to a} f(x) = F$ and $\lim\limits_{x\to a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim\limits_{x\to a}(f(x)+g(x)) = F+G$
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- $\lim\limits_{x\to a}(f(x)g(x)) = FG$
- $\lim\limits_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

Calculate: $\lim\limits_{x\to 1}\left[\dfrac{2x+4}{x+2} + 13\left(\dfrac{x+5}{3x}\right)\left(\dfrac{x^2}{2x-1}\right)\right]$

Calculate: $\lim_{x \to 1} \left[\frac{2x+4}{x+2} + 13 \left(\frac{x+5}{3x} \right) \left(\frac{x^2}{2x-1} \right) \right]$

Calculate: $\lim\limits_{x\to 1}\left[\dfrac{2x+4}{x+2}+13\left(\dfrac{x+5}{3x}\right)\left(\dfrac{x^2}{2x-1}\right)\right]$

$$=\lim_{x\to 1}\left(\frac{2x+4}{x+2}\right)+\left(\lim_{x\to 1}13\right)\left(\lim_{x\to 1}\frac{x+5}{3x}\right)\left(\lim_{x\to 1}\frac{x^2}{2x-1}\right)$$

$$=\left(\frac{2(1)+4}{1+2}\right)+(13)\left(\frac{(1)+5}{3(1)}\right)\left(\frac{1^2}{2(1)-1}\right)$$

$$=(2)+13(2)(1)$$

$$=28$$

Limits involving Powers and Roots

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

ans

Limits involving Powers and Roots

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

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Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x \to a} f(x) = F$ (where F is a real number), then:

$$\lim_{x \to a} \left(f(x)\right)^n = F^n.$$

Furthermore, unless n is even and F is negative,

$$\lim_{x \to a} \left(f(x)\right)^{1/n} = F^{1/n}$$

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$$\lim_{x\to 4} (x+5)^{1/2}$$

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$$\lim_{x\to 4}(x+5)^{1/2} = \left[\lim_{x\to 4}(x+5)\right]^{1/2} = 9^{1/2} = 3$$

Cautionary Tales

- $\lim_{x \to 0} \dfrac{(5+x)^2 - 25}{x}$

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- $\lim\limits_{x\to 0} \dfrac{32}{x}$

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- $\lim_{x\to 5} \left(x^2+2\right)^{1/3}$

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- $\lim\limits_{x\to 0} \dfrac{32}{x} \to \dfrac{32}{0}$; this expression is meaningless
- $\lim\limits_{x\to 5} \left(x^2+2\right)^{1/3} = (5^2+2)^{1/3} = \sqrt[3]{27} = 3$

Suppose you want to evaluate $\lim_{x \to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x \to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x \to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x \to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x \to 1} f(x)$ does not exist.

E $\lim_{x \to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

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A

Which of the following statements is true about $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 0}{0^3 - 0^2 + 0} = \dfrac{0}{0}$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

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A $\lim\limits_{x\to 1} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

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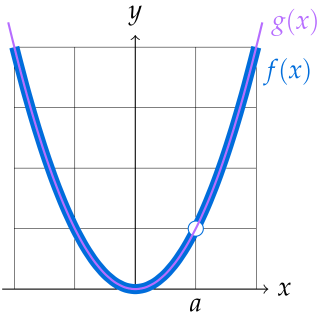
A

Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim\limits_{x \to a} g(x)$ exists, and $f(x) = g(x)$ when x is close to a (but not necessarily equal to a).

Then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$.



Evaluate $\lim_{x \to 1} \dfrac{x^3 + x^2 - x - 1}{x - 1}$.

ans

Evaluate $\displaystyle\lim_{x\to1}\frac{x^3+x^2-x-1}{x-1}$.

$$\begin{aligned}\frac{x^3+x^2-x-1}{x-1} &= \frac{(x+1)^2(x-1)}{x-1}\\ &= (x+1)^2 \text{ whenever } x\neq 1\end{aligned}$$

So, $\displaystyle\lim_{x\to1}\frac{x^3+x^2-x-1}{x-1} = \lim_{x\to1}(x+1)^2 = 4$

Evaluate $\displaystyle\lim_{x\to 5} \frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}$

ans

Evaluate $\displaystyle\lim_{x\to 5} \frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}$

$$\begin{aligned}
\frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5} &= \frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}\left(\frac{\sqrt{x+20}+\sqrt{4x+5}}{\sqrt{x+20}+\sqrt{4x+5}}\right)\\
&= \frac{(x+20)-(4x+5)}{(x-5)(\sqrt{x+20}+\sqrt{4x+5})}\\
&= \frac{-3x+15}{(x-5)(\sqrt{x+20}+\sqrt{4x+5})}\\
&= \frac{-3}{\sqrt{x+20}+\sqrt{4x+5}}
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to 5}\frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5} &= \lim_{x\to 5}\frac{-3}{\sqrt{x+20}+\sqrt{4x+5}}\\
&= \frac{-3}{\sqrt{5+20}+\sqrt{4(5)+5}} = \frac{-3}{10}
\end{aligned}$$

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

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A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

$$\left(\sqrt{35+1^5}+\frac{1-3}{1^2}\right)^3 = 64$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0}\frac{x+7}{\frac{1}{x}-\frac{1}{2x}}$$

ans

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0} \frac{x+7}{\frac{1}{x}-\frac{1}{2x}} = \lim_{x\to 0} \frac{x+7}{\frac{2}{2x}-\frac{1}{2x}}$$

$$= \lim_{x\to 0} \frac{x+7}{\frac{1}{2x}} = \lim_{x\to 0} 2x(x+7) = 0$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

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Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

Denominators Approaching Zero

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

$$\lim_{x\to 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{1}{x-1}$$

$$\lim_{x\to 1^+} \frac{1}{x-1}$$

Denominators Approaching Zero

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$$

$$\lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty$$

$$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty$$

Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4}$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2}$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4} = \infty$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2} = \infty$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \le g(x) \le h(x)$$

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

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$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Evaluate:

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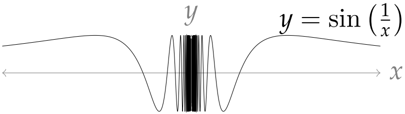

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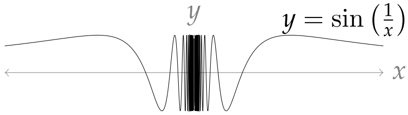
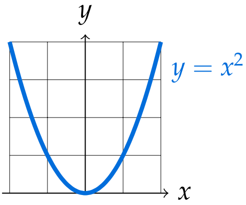
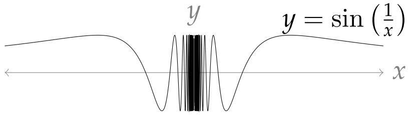
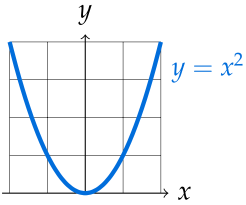


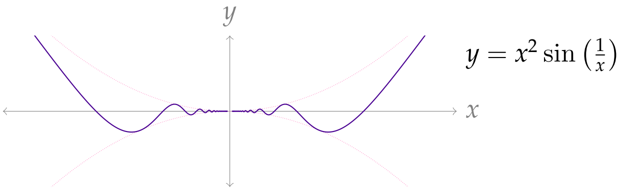


Evaluate:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$







$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\begin{aligned}
-1 &\leq \sin\left(\frac{1}{x}\right) \leq 1 \\
\text{so} \quad -x^2 &\leq x^2\sin\left(\frac{1}{x}\right) \leq x^2 \\
\text{and also} \quad \lim_{x\to 0} -x^2 &= 0 = \lim_{x\to 0} x^2
\end{aligned}$$

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\end{aligned}$$

Therefore, by the Squeeze Theorem, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Included Work

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