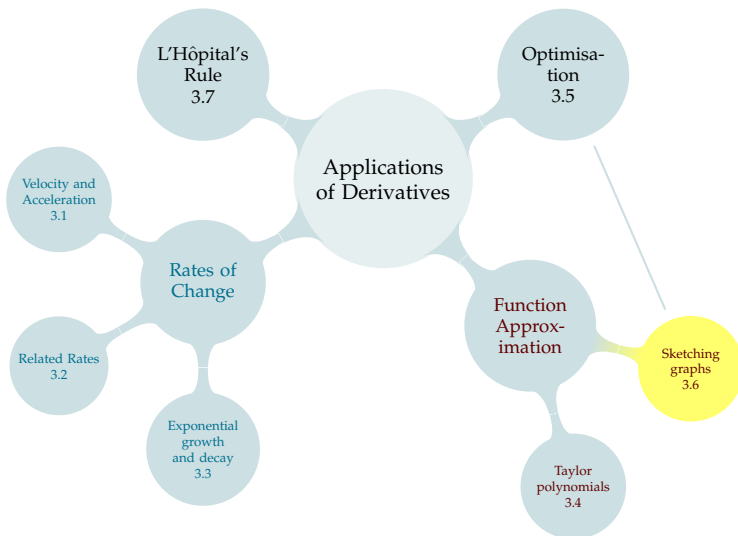


TABLE OF CONTENTS



CURVE SKETCHING

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Where might you expect $f(x)$ to have a vertical asymptote? What does the function look like nearby?

(Recall: a vertical asymptote occurs at $x = a$ if the function has an infinite discontinuity at a . That is, $\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$.)

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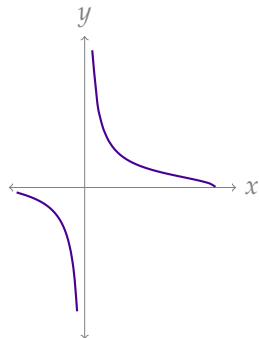
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Where is $f(x) = 0$?

What happens to $f(x)$ near its other endpoint, $x = -1$?





CURVE SKETCHING

Good things to check:

- Domain
- Vertical asymptotes: $\lim_{x \rightarrow a} f(x) = \pm\infty$
- Intercepts: $x = 0, f(x) = 0$
- Horizontal asymptotes and end behavior: $\lim_{x \rightarrow \pm\infty} f(x)$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x - 2}{(x + 3)^2}$$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

- ▶ Domain: $x \neq -3$
- ▶ Vertical asymptote: $x = -3$
- ▶ Intercepts: $(2, 0)$, $(0, -\frac{2}{9})$
- ▶ Horizontal asymptote: $y = 0$ in both directions

<https://www.desmos.com/calculator/hyzl5cyq7i>

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

CURVE SKETCHING

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$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

- ▶ Domain: $x \neq 0, 5$
- ▶ Vertical asymptotes: $x = 0, x = 5$
- ▶ Intercepts: $(-2, 0), (3, 0)$
- ▶ Horizontal asymptote: none

<https://www.desmos.com/calculator/ploa0q7bxn>

FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

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Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

Domain all real numbers

Intercepts Factor $f(x) = x^2(\frac{1}{2}x^2 - \frac{4}{3}x - 15)$. Intercept at the origin; use the quadratic formula to find x -intercepts at $x = \frac{4 \pm \sqrt{286}}{3}$, so $x \approx 7$ and $x \approx -4.3$.

End behaviour $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$

This is the information we've already talked about gathering. Now let's add the first derivative.

FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

$$\begin{aligned} f'(x) &= 2x^3 - 4x^2 - 30x \\ &= 2x(x^2 - 2x - 15) \\ &= 2x(x - 5)(x + 3) \end{aligned}$$

So, critical points are $x = 0$, $x = -3$, and $x = 5$. No singular points. At the critical points $f(0) = 0$, $f(-3) = -58.5$, $f(5) = -229.1\bar{6}$.

$x \approx -4.3$	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$0 < x < 5$	$x = 5$	$x > 5$	$x \approx 7$
$f(x) = 0$	$f' < 0$	CP	$f' > 0$	CP	$f' < 0$	CP	$f' > 0$	$f(x) = 0$
intercept	decr	loc min	incr	loc max	decr	loc min	incr	intercept

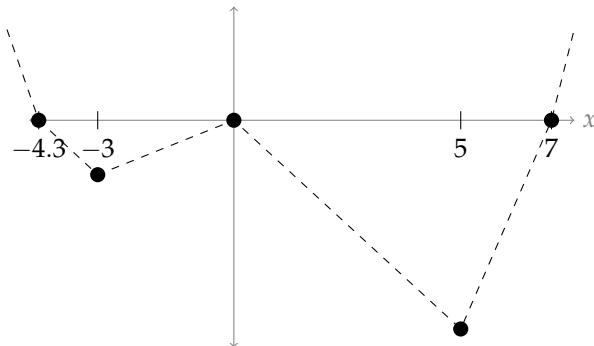
This gives us enough information to draw a skeleton.



FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



<https://www.desmos.com/calculator/lxdlgmhns1>

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

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$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

Domain all real numbers.

**End
behaviour** $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$

Intercepts

$$f(0) = 24$$

$$\begin{aligned} f(x) &= \frac{1}{3}x^2(x+6) + 4(x+6) \\ &= \left(\frac{1}{3}x^2 + 4\right)(x+6) \end{aligned}$$

Only two intercepts: $(0, 24)$ and $(-6, 0)$.

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

**Critical
points**

$$f'(x) = x^2 + 4x + 4 = (x + 2)^2 \quad \text{so } x = -2 \text{ is the location of the only critical point.}$$
$$f(-2) = 24 - \frac{8}{3} = 21 + \frac{1}{3}.$$

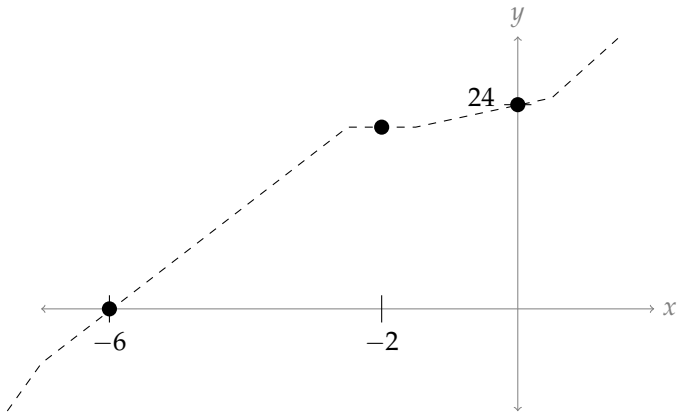
**Increasing,
decreasing**

$$f'(x) > 0 \text{ except at } x = -2, \text{ so apart from the critical point, } f(x) \text{ is increasing}$$

This is enough for us to draw a skeleton.

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$



<https://www.desmos.com/calculator/xum0mstmiv>

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$$f(x) = e^{\frac{x+1}{x-1}}$$

Domain $x \neq 1$

We need to consider what happens as x approaches 1 from the left and the right.

Vertical asymptotes

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty \implies \lim_{x \rightarrow 1^-} f(x) = \lim_{A \rightarrow -\infty} e^A = 0$$
$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \infty \implies \lim_{x \rightarrow 1^+} f(x) = \lim_{A \rightarrow +\infty} e^A = \infty$$

End behaviour

$$\lim_{x \rightarrow \pm\infty} \frac{x+1}{x-1} = 1, \text{ so } \lim_{x \rightarrow \pm\infty} f(x) = e$$

Intercepts $f(0) = \frac{1}{e}$; there are no roots



What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

**Critical
points**

$$f'(x) = e^{\frac{x+1}{x-1}} \left(\frac{-2}{(x-1)^2} \right) \quad \text{no critical points}$$

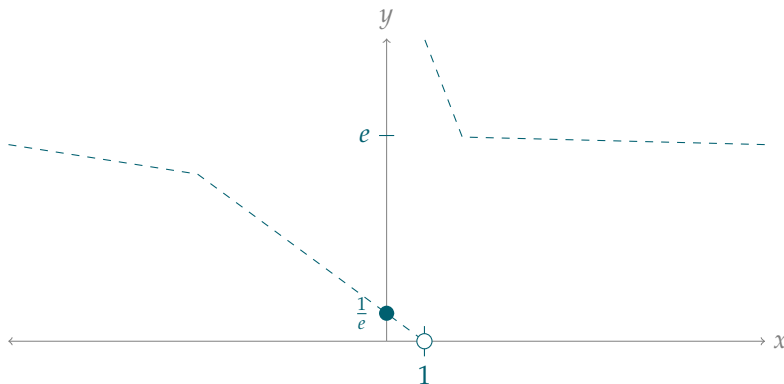
**Increasing,
decreasing**

$f(x)$ is decreasing everywhere it is defined

This information is enough to draw a skeleton.

What does the graph of the following function look like?

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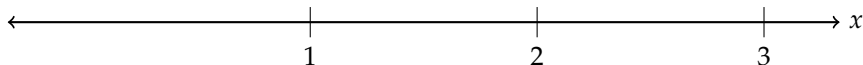
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SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

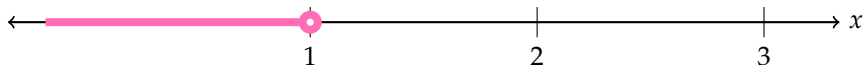
$$f(x) = (x - 1)(x - 2)(x - 3)$$



SIGNS OF FACTORED FUNCTIONS

▶ SKIP SIGN CHANGES

$$f(x) = \underset{-}{(x-1)} (x-2) (x-3)$$



SIGNS OF FACTORED FUNCTIONS

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+



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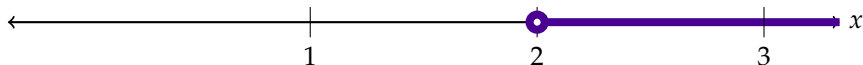


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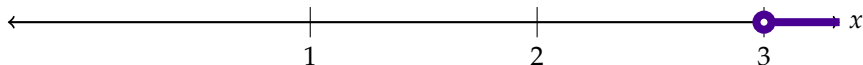
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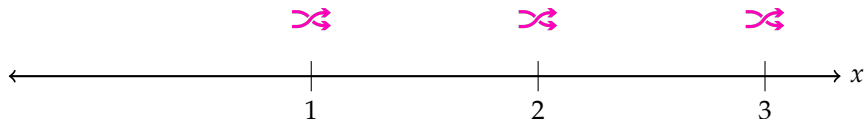
$$f(x) = (x - 1)(x - 2)(x - 3) +$$



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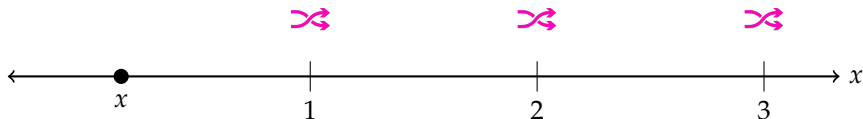


Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

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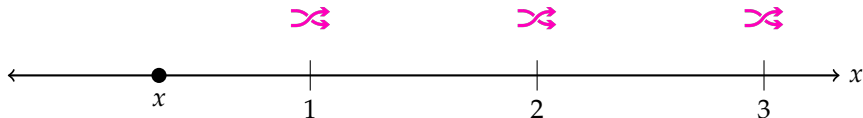
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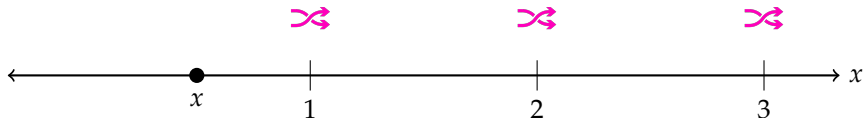
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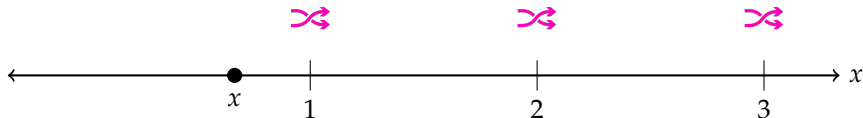
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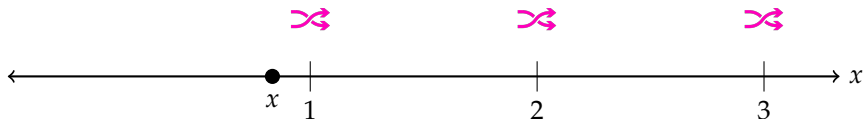
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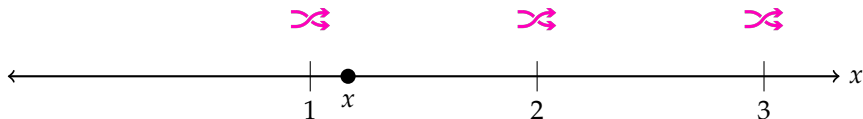
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$$f(x) = \begin{array}{ccc} (x-1) & (x-2) & (x-3) \\ + & - & - \end{array}$$



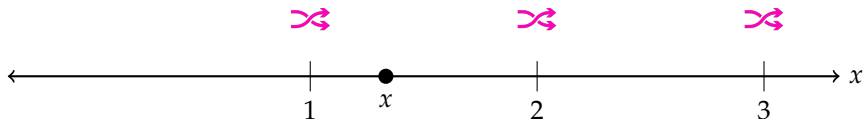
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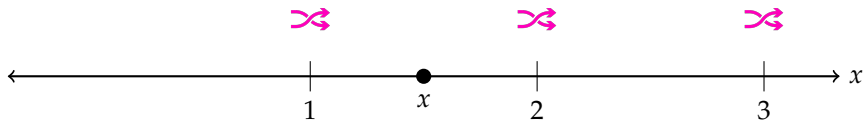
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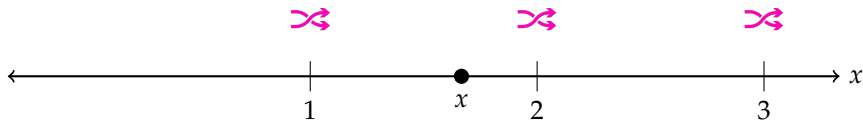
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$$f(x) = \underset{+}{(x-1)} \underset{-}{(x-2)} \underset{-}{(x-3)}$$



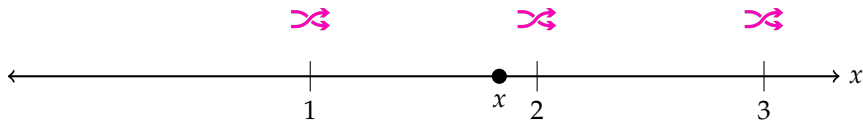
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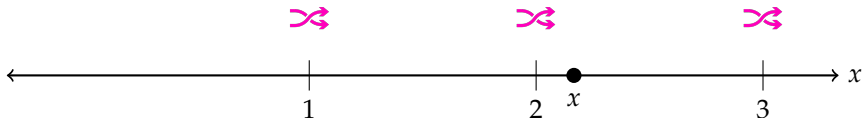


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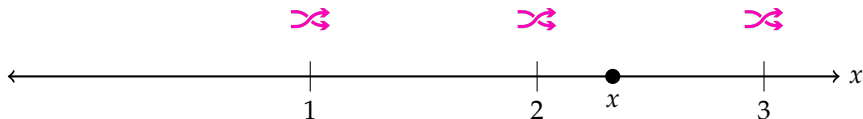
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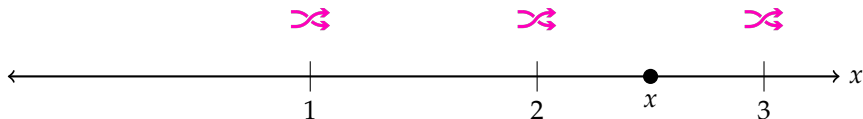
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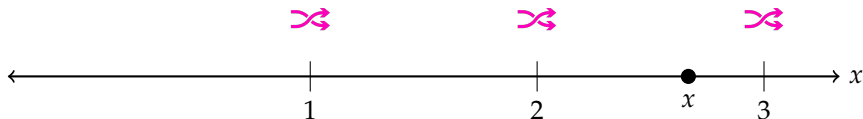
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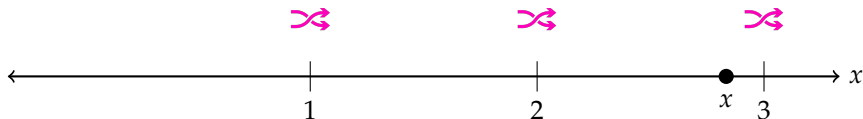
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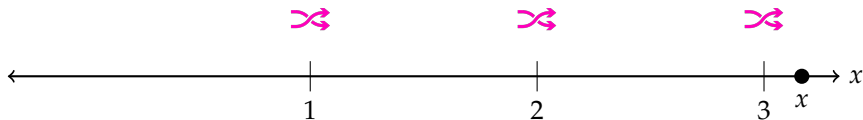
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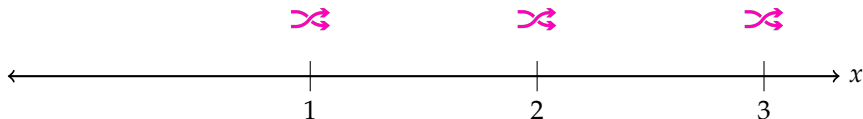
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SIGNS OF FACTORED FUNCTIONS

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$$f(x) = (x - 1)(x - 2)(x - 3)$$

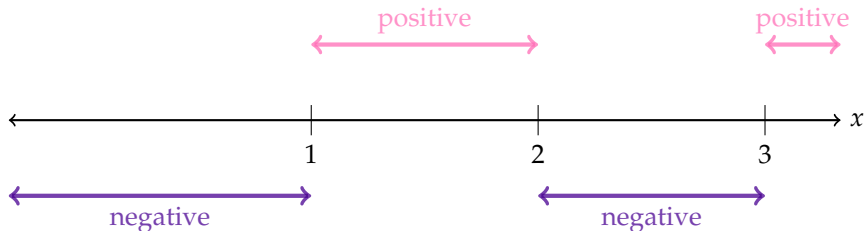


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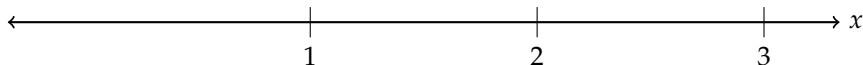


Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 1) (x - 2)^2 (x - 3)$$



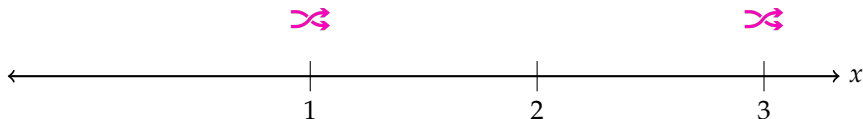
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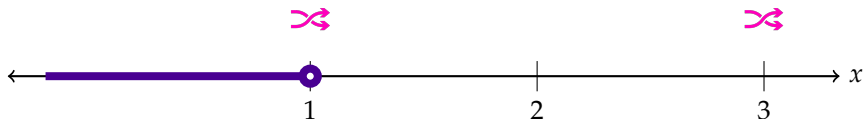


Sign of entire function:

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$$f(x) = \underset{-}{(x-1)} \underset{+}{(x-2)^2} \underset{-}{(x-3)}$$



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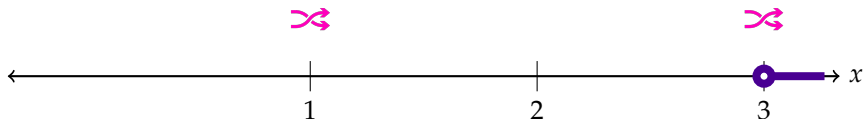


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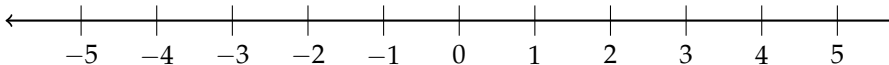
Sign of entire function:

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$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

Where is $f(x)$ positive? Where is it negative?

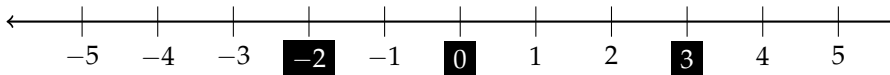


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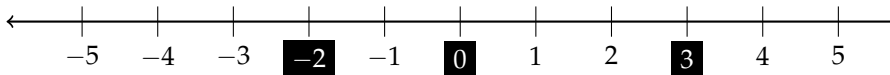


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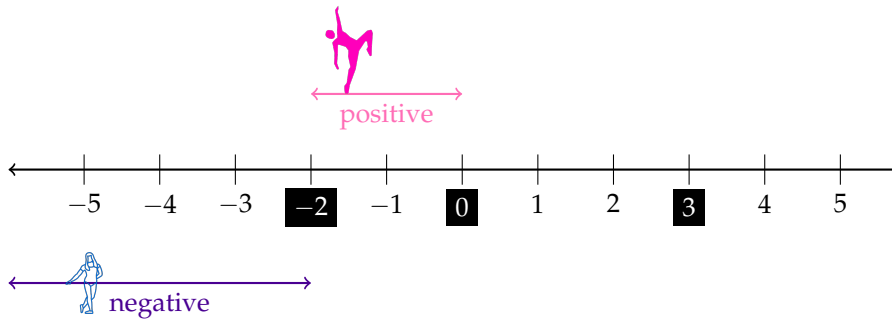


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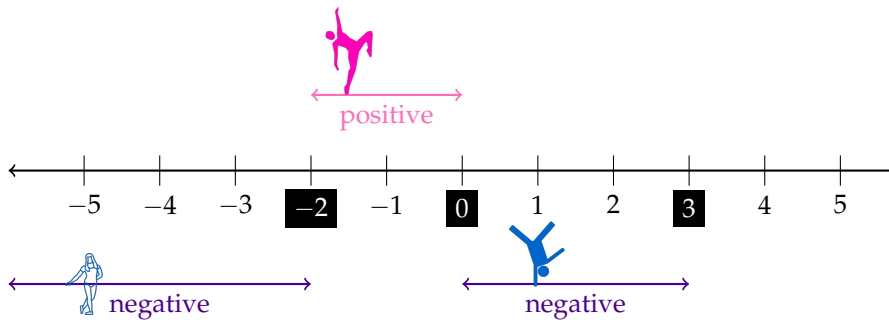


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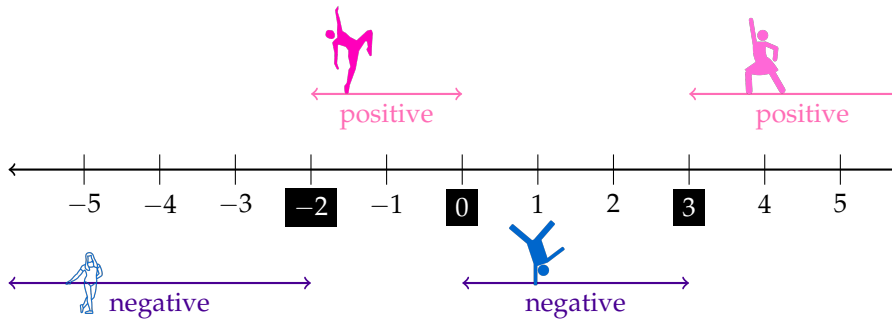


SIGNS OF FACTORED FUNCTIONS

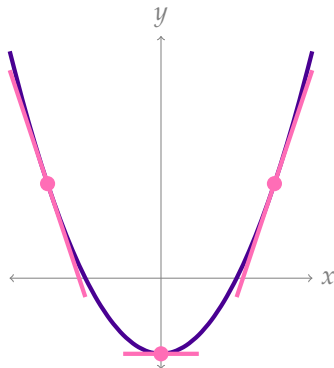
[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

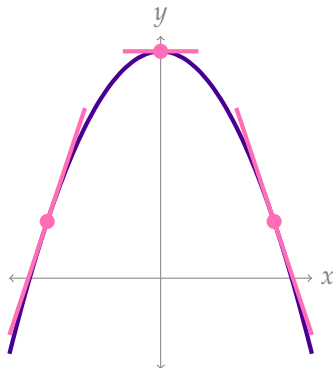
Where is $f(x)$ positive? Where is it negative?



CONCAVITY



- ▶ Slopes are increasing
- ▶ $f''(x) > 0$
- ▶ “concave up”
- ▶ tangent line below curve



- ▶ Slopes are decreasing
- ▶ $f''(x) < 0$
- ▶ “concave down”
- ▶ tangent line above curve

MNEMONIC

+

+

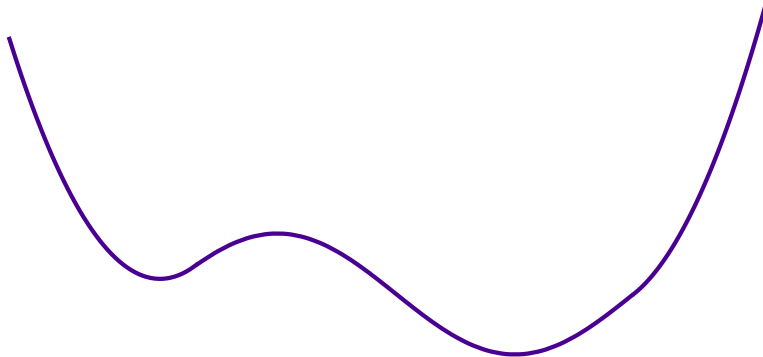


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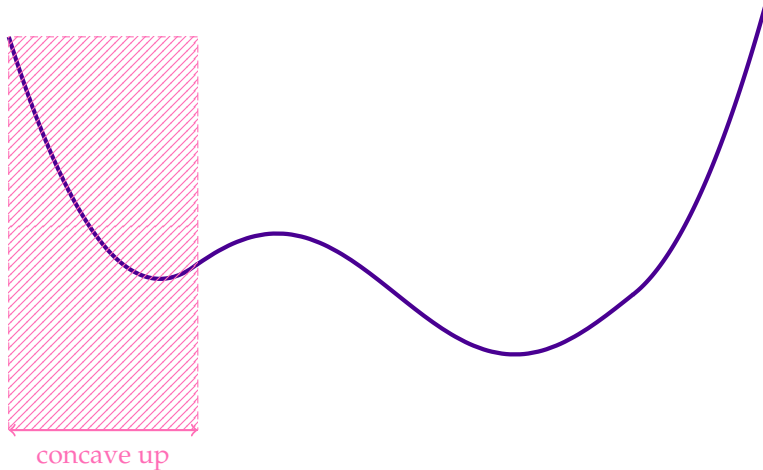
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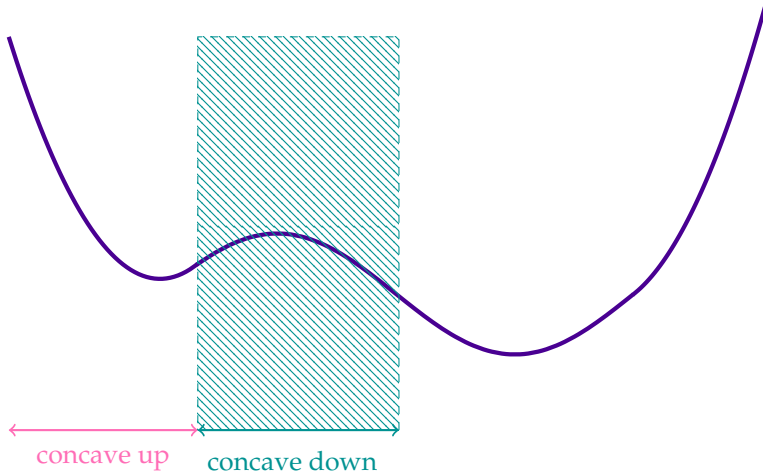
CONCAVITY



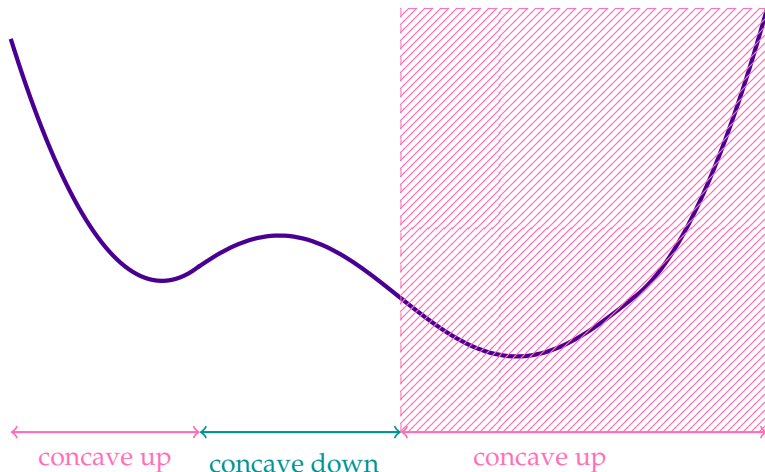
CONCAVITY



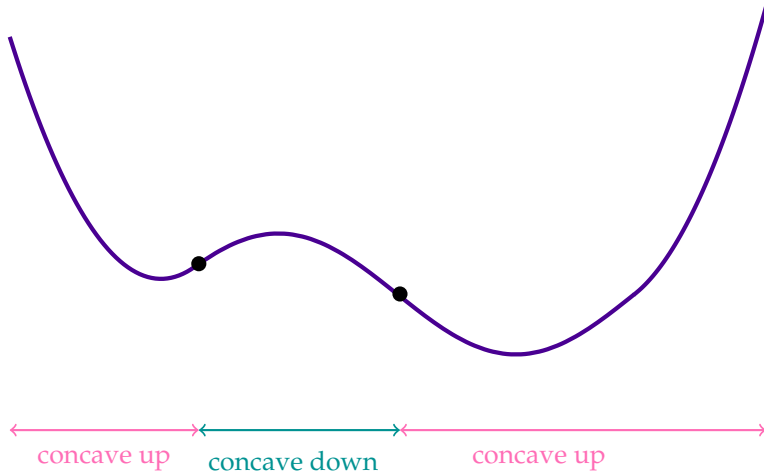
CONCAVITY



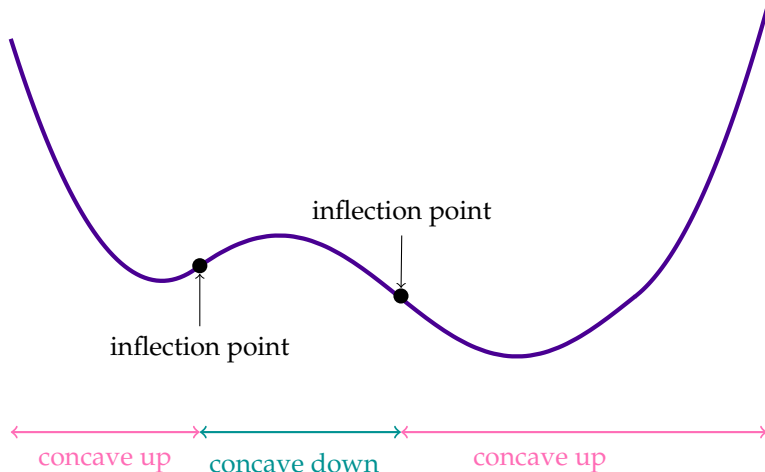
CONCAVITY



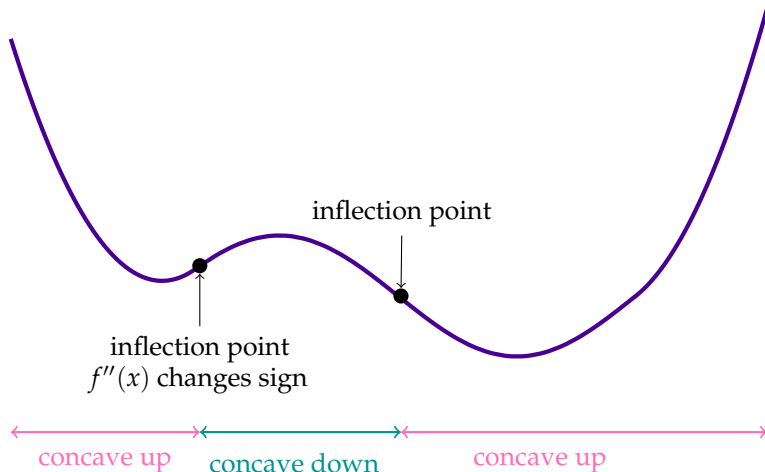
CONCAVITY



CONCAVITY



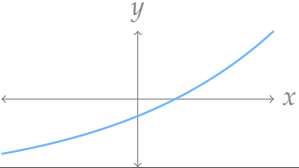
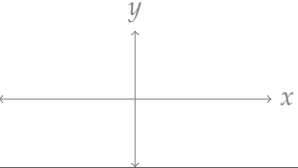
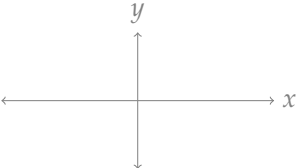
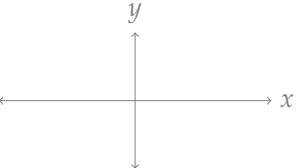
CONCAVITY



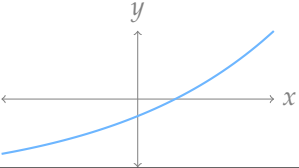
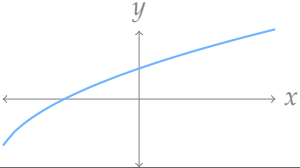
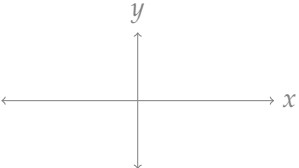
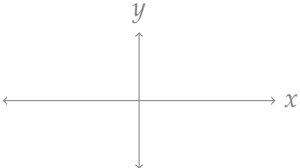
Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

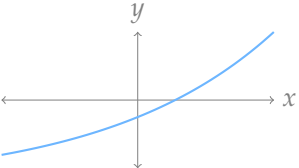
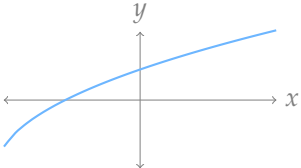
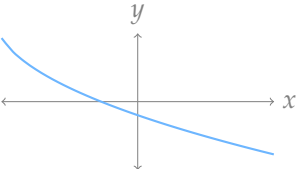
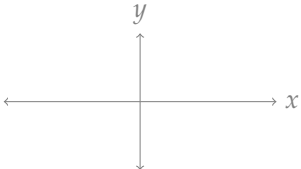
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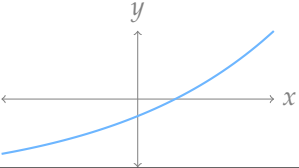
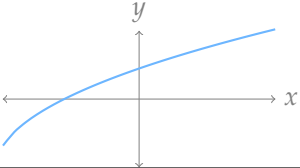
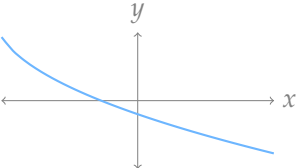
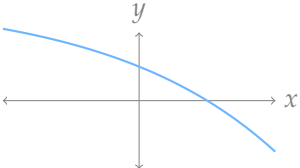
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decreasing		

POLL QUESTIONS

Describe the concavity of the function $f(x) = e^x$.

- A. concave up
- B. concave down
- C. concave up for $x < 0$; concave down for $x > 0$
- D. concave down for $x < 0$; concave up for $x > 0$
- E. I'm not sure

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Is it possible to be concave up and decreasing?

A. Yes

B. No

C. I'm not sure

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Suppose a function $f(x)$ is defined for all real numbers, and is concave up on the interval $[0, 1]$. Which of the following must be true?

- A. $f'(0) < f'(1)$
- B. $f'(0) > f'(1)$
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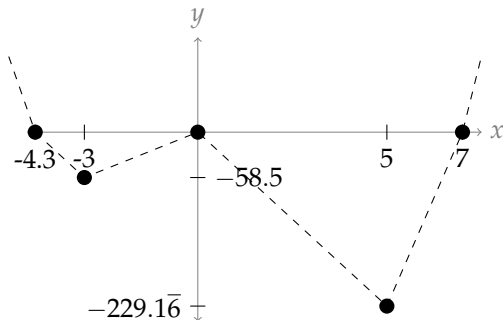
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REVISITING A PREVIOUS EXAMPLE

◀ original example

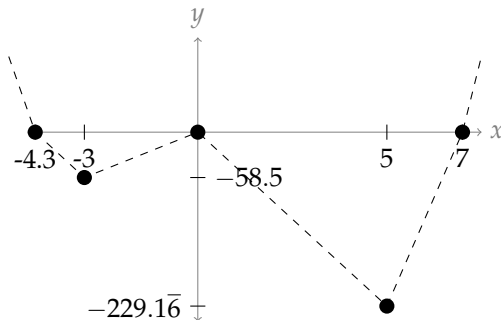
$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



REVISITING A PREVIOUS EXAMPLE

[◀ original example](#)

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

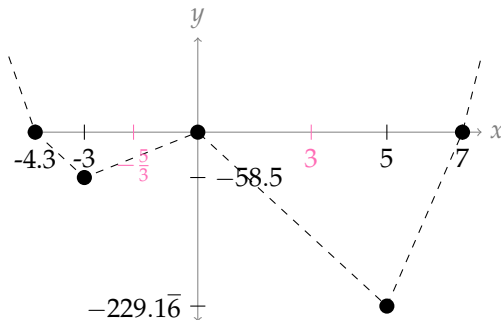


$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

REVISITING A PREVIOUS EXAMPLE

[◀ original example](#)

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

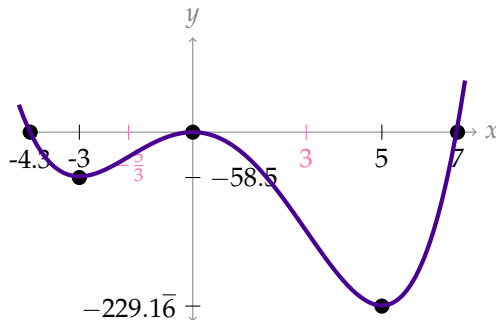


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REVISITING A PREVIOUS EXAMPLE

[◀ original example](#)

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

Sketch:

$$f(x) = x^5 - 15x^3$$

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$$f(x) = x^5 - 15x^3$$

Domain	Defined and differentiable for all real numbers.
Intercepts	$f(x) = x^3(x^2 - 15)$: Roots are at $x = 0$ and $x = \pm\sqrt{15} \approx \pm 4$
End behaviour	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$
Critical points	$f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9)$. So the critical points are $x = 0, x = \pm 3$.
Increasing, decreasing	Increasing on $(-\infty, -3)$, decreasing on $(-3, 0)$ and $(0, 3)$, increasing on $(3, \infty)$
Local extrema	From intervals of increase and decrease: local max at $x = -3$ and local min at $x = 3$

Sketch:

$$f(x) = x^5 - 15x^3$$

Concavity

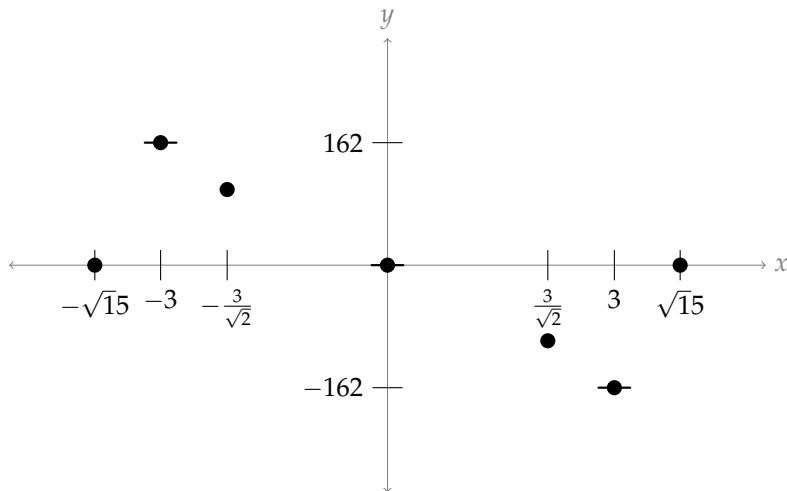
$f''(x) = 20x^3 - 90x = 10x(2x^2 - 9) = 0$ for $x = 0$ and $x = \pm \frac{3}{\sqrt{2}} \approx \pm 2.12$. All of these are inflection points; concave down $(-\infty, -\frac{3}{\sqrt{2}})$, concave up $(-\frac{3}{\sqrt{2}}, 0)$, concave down $(0, \frac{3}{\sqrt{2}})$, and concave up $(\frac{3}{\sqrt{2}}, \infty)$.

y-values of notable points

$f(3) = -162, f(-3) = 162, f(-3/\sqrt{2}) \approx 100, f(3/\sqrt{2}) \approx -100$

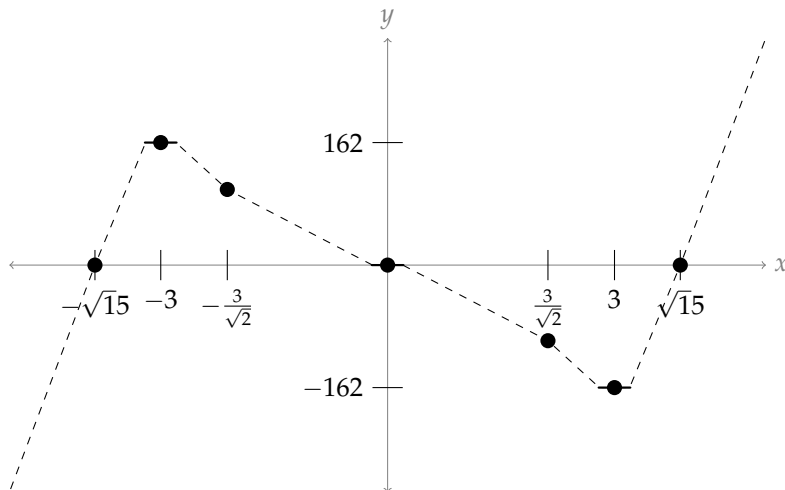
Sketch:

$$f(x) = x^5 - 15x^3$$



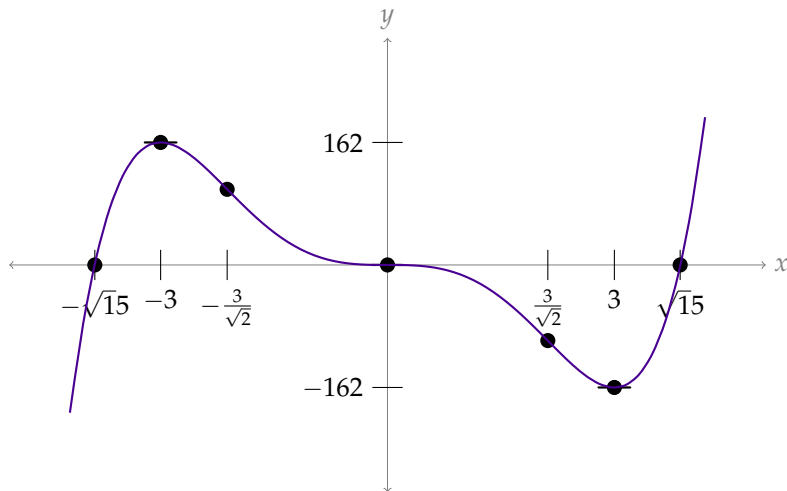
Sketch:

$$f(x) = x^5 - 15x^3$$



Sketch:

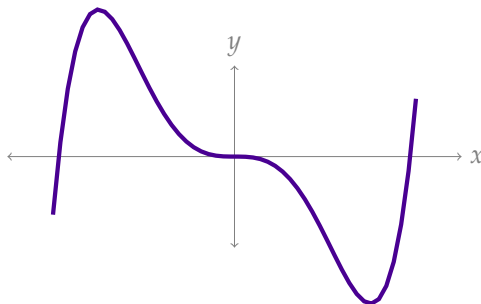
$$f(x) = x^5 - 15x^3$$



<https://www.desmos.com/calculator/uoi6nmgr8>

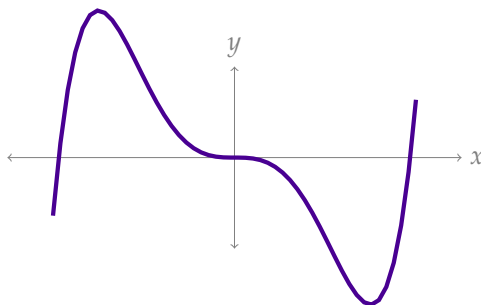


EVEN AND ODD FUNCTIONS



$$f(x) = x^5 - 15x^3$$

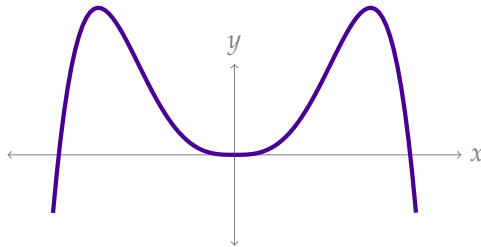
EVEN AND ODD FUNCTIONS



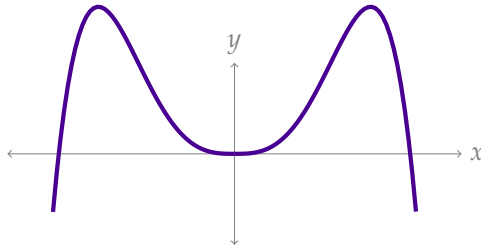
$$f(x) = x^5 - 15x^3$$

odd function

EVEN AND ODD FUNCTIONS



EVEN AND ODD FUNCTIONS

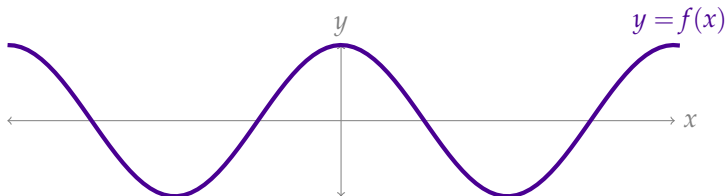


even function

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

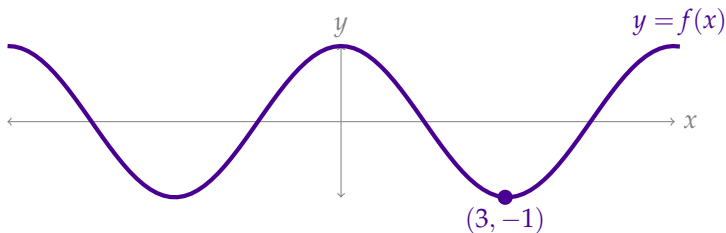
$$f(-x) = f(x)$$



Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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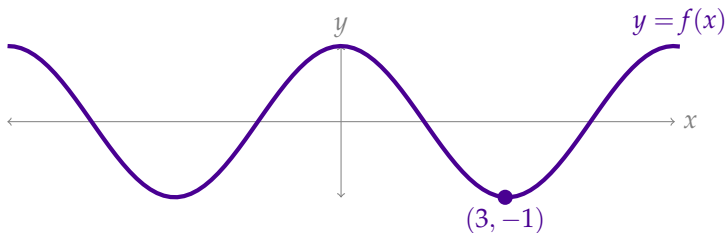


Suppose $f(3) = -1$.

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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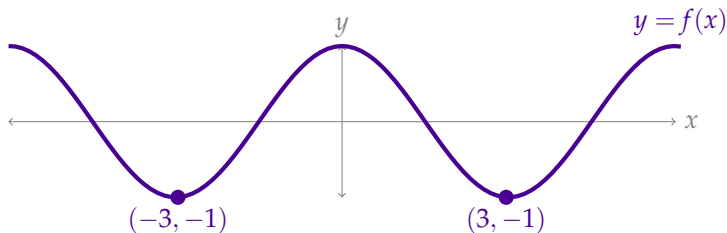


Suppose $f(3) = -1$. Then $f(-3) =$

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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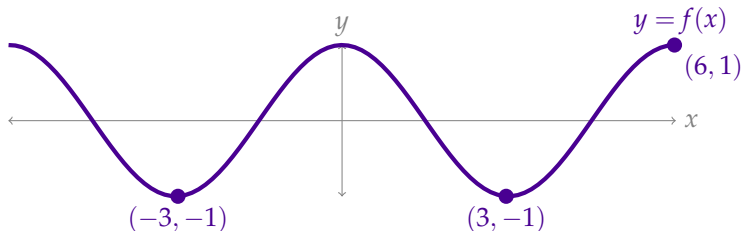


Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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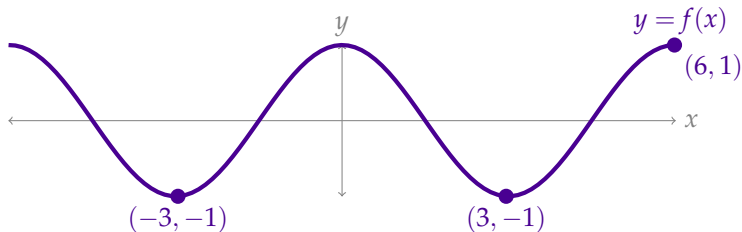
Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Suppose $f(6) = 1$.

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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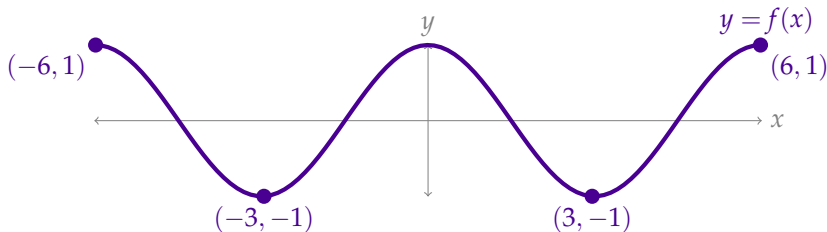
Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

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Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Suppose $f(6) = 1$. Then $f(-6) = 1$ also.

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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Examples:

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:

$$f(x) = x^2$$

EVEN FUNCTIONS

Even Function – Definition 3.6.5

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Examples:

$$f(x) = x^2$$

$$f(x) = x^4$$

EVEN FUNCTIONS

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A function $f(x)$ is **even** if, for all x in its domain,

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Examples:

$$f(x) = x^2$$

$$f(x) = x^4$$

$$f(x) = \cos(x)$$

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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Examples:

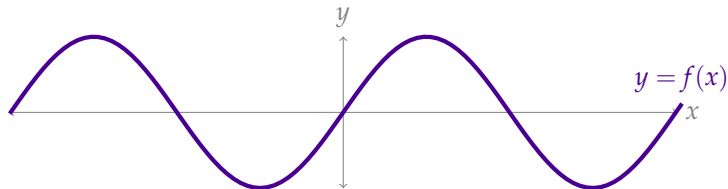
$$f(x) = x^2$$

$$f(x) = x^4$$

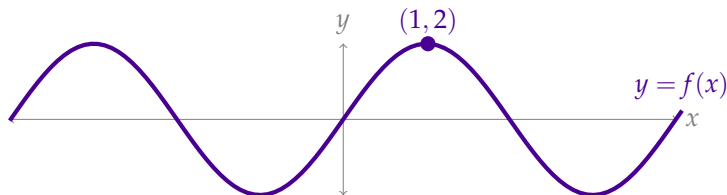
$$f(x) = \cos(x)$$

$$f(x) = \frac{x^4 + \cos(x)}{x^{16} + 7}$$

ODD FUNCTIONS

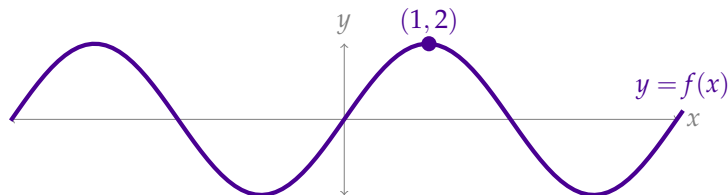


ODD FUNCTIONS



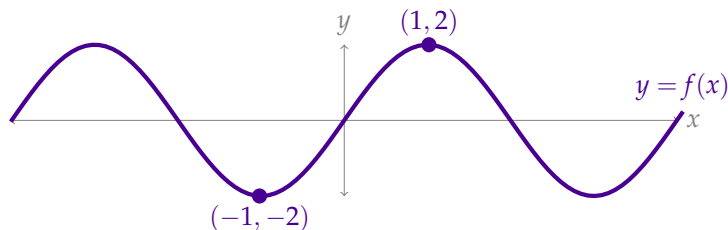
Suppose $f(1) = 2$.

ODD FUNCTIONS



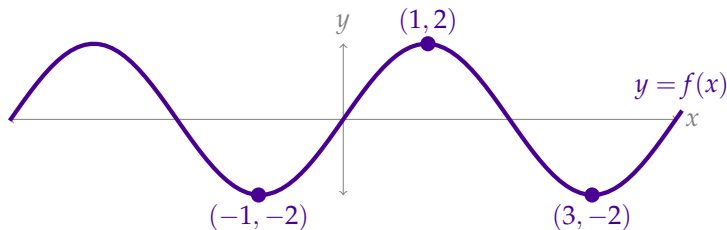
Suppose $f(1) = 2$. Then $f(-1) =$

ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

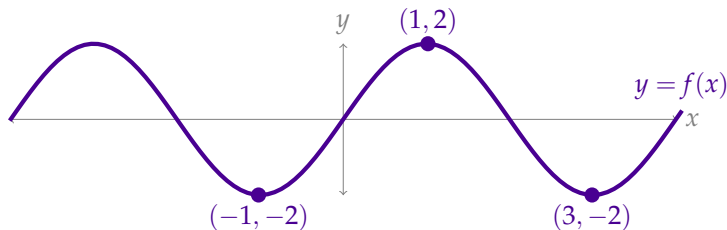
ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$.

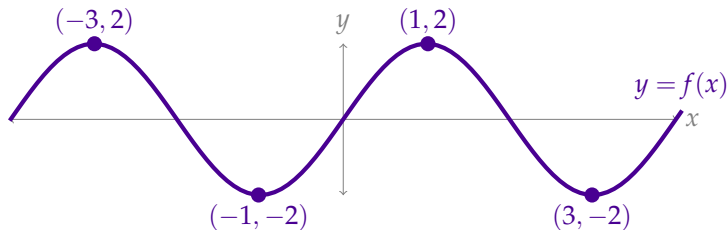
ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) =$

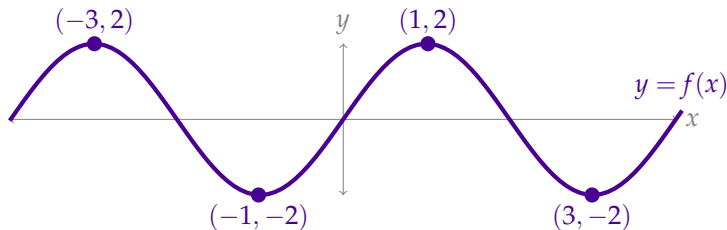
ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

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ODD FUNCTIONS



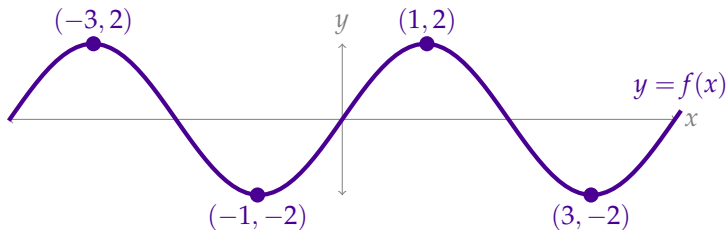
Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) = 2$.

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) = 2$.

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

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Examples:

ODD FUNCTIONS

Odd Function – Definition 3.6.6

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$$f(-x) = -f(x)$$

Examples:

$$f(x) = x$$

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$$f(x) = x$$

$$f(x) = x^3$$

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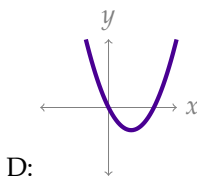
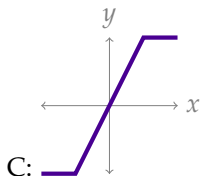
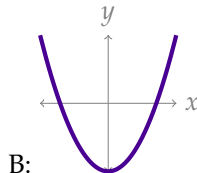
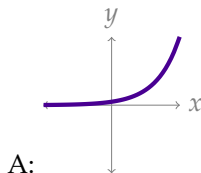
$$f(x) = x^3$$

$$f(x) = \sin(x)$$

$$f(x) = \frac{x(1 + x^2)}{x^2 + 5}$$

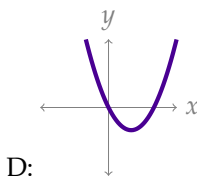
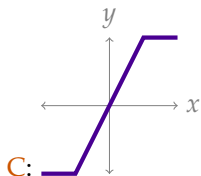
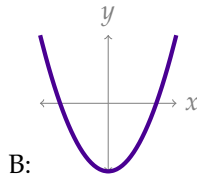
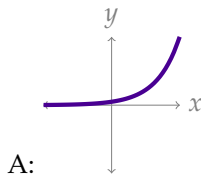
POLL TIME

Pick out the **odd** function.



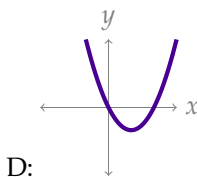
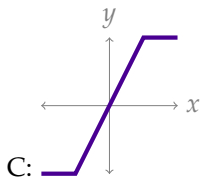
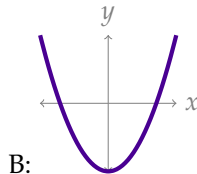
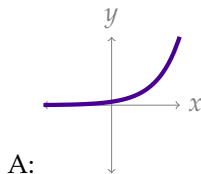
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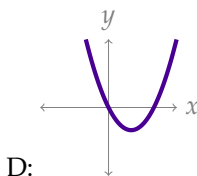
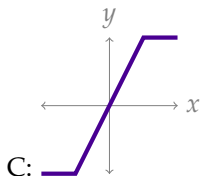
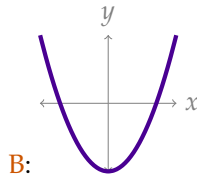
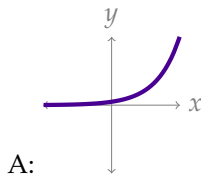
POLL TIME

Pick out the **even** function.



POLL TIME

Pick out the **even** function.



EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0)$
- B. $f(0) = -f(0)$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true



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EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions
- B. $f(0) = -f(0)$
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- D. all of the above are true
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EVEN MORE POLL TIIIIIME

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- A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions
- B. $f(0) = -f(0) \leftarrow$ only possible for $f(0) = 0$
- C. $f(0) = 0$
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EVEN MORE POLL TIIIIIME

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- A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions
- B. $f(0) = -f(0) \leftarrow$ only possible for $f(0) = 0$
- C. $f(0) = 0 \leftarrow$ this is equivalent to the choice above
- D. all of the above are true
- E. none of the above are necessarily true

EVEN MORE AND MORE POLL TIIIIIME

Suppose $f(x)$ is an **even** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0)$
- B. $f(0) = -f(0)$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true

EVEN MORE AND MORE POLL TIIIIIME

Suppose $f(x)$ is an **even** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0)$
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- E. none of the above are necessarily true



OK OK... LAST ONE

Suppose $f(x)$ is an **even** function, differentiable for all real numbers.
What can we say about $f'(x)$?

- A. $f'(x)$ is also even
- B. $f'(x)$ is odd
- C. $f'(x)$ is constant
- D. all of the above are true
- E. none of the above are necessarily true



OK OK... LAST ONE

Suppose $f(x)$ is an **even** function, differentiable for all real numbers.
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PERIODICITY

Periodic – Definition 3.6.9

A function is **periodic** with period $P > 0$ if

$$f(x) = f(x + P)$$

whenever x and $x + P$ are in the domain of f , and P is the smallest such (positive) number

Examples: $\sin(x)$, $\cos(x)$ both have period 2π ; $\tan(x)$ has period π .

Ignoring concavity, sketch $f(x) = \sin(\sin x)$.

Challenge: ignoring exact locations of extrema, sketch $g(x) = \sin(2\pi \sin x)$.

$$f(x) = \sin(\sin x)$$

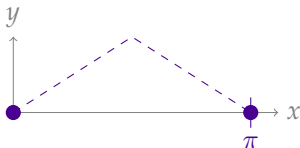
$$f(x) = \sin(\sin x)$$

Sin is periodic; since $\sin x = \sin(2\pi + x)$, then $\sin(\sin x) = \sin(\sin(2\pi + x))$, so $f(x)$ is also periodic. It suffices to sketch $f(x)$ for an interval of length 2π , because any such segment will repeat.

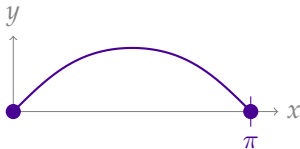
Since the function is also odd, if we sketch it on the interval $[0, \pi]$, then we can extrapolate to the interval $[-\pi, 0]$. So we consider the interval $[0, \pi]$.

- ▶ Intercepts: $(0, 0), (0, \pi)$
- ▶ First derivative: $f'(x) = \cos(\sin x) \cdot \cos(x)$
 - ▶ For $0 \leq x \leq \pi$, we have $0 \leq \sin x \leq 1 < \frac{\pi}{2}$ and hence $0 < \cos(\sin x) \leq 1$.
 - ▶ CP: $x = \frac{\pi}{2}$
 - ▶ increasing: $(0, \pi/2)$
 - ▶ decreasing: $(\pi/2, 0)$

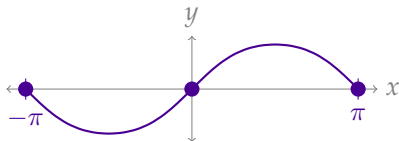
1. Interval $(0, \pi)$ skeleton, based on above work:



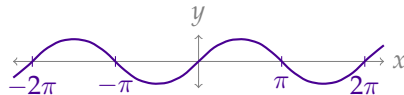
2. Make into a smooth curve:



3. Use odd symmetry to get interval $[-\pi, \pi]$



4. Use periodicity



» skip $g(x)$

$$g(x) = \sin(2\pi \sin x)$$

» skip $g(x)$

$$g(x) = \sin(2\pi \sin x)$$

Sin is periodic; since $2\pi \sin x = 2\pi \sin(2\pi + x)$, then

$$\sin(2\pi \sin x) = \sin(2\pi \sin(2\pi + x))$$

so $g(x)$ is also periodic. It suffices to sketch $g(x)$ for an interval of length 2π , because any such segment will repeat.

Note

$$\begin{aligned} g(-x) &= \sin(2\pi \sin(-x)) = \sin((2\pi)(-\sin x)) \\ &= \sin(-2\pi \sin x) = -\sin(2\pi \sin x) = -g(x) \end{aligned}$$

so $g(x)$ is odd. If we sketch it on the interval $[0, \pi]$, then we can extrapolate to the interval $[-\pi, 0]$. So we consider the interval $[0, \pi]$.

» skip $g(x)$ Intercepts in $[0, \pi]$:

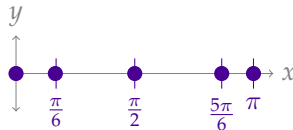
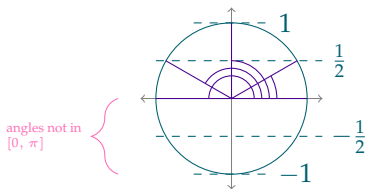
$$g(0) = 0$$

$$0 = g(0) = \sin(2\pi \sin x)$$

$$\Rightarrow 2\pi \sin x \in \{0, \pm\pi, \pm2\pi, \pm3\pi, \dots\}$$

$$\Rightarrow \sin x \in \left\{0, \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \dots\right\}$$

$$\Rightarrow x \in \left\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi\right\}$$

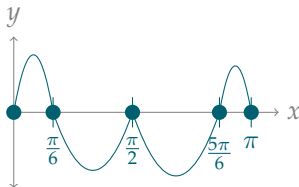


» skip $g(x)$

Now let's consider the sign of $g(x)$ between the intercepts. Since $g(x)$ isn't given as a factored product, our old shortcut isn't so useful.

interval	$(0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{5\pi}{6})$	$(\frac{5\pi}{6}, \pi)$
range of $\sin x$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, 1)$	$(0, \frac{1}{2})$
range of $2\pi \sin x$	$(0, \pi)$	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(0, \pi)$
sign of $\sin(2\pi \sin x)$	+	−	−	+

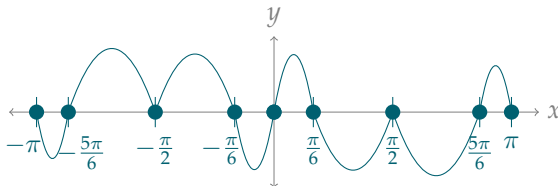
So, a rough sketch on the interval $[0, \pi]$ is:



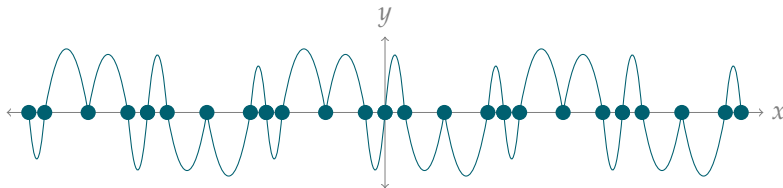
Yes, this is a rough sketch. The curve should be smooth at $\frac{\pi}{2}$.

» skip $g(x)$

Use odd symmetry, we sketch the interval $[-\pi, \pi]$:



Using periodicity:



LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2 - 64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2 + 64)}{3(x^2 - 64)^{5/3}}$$

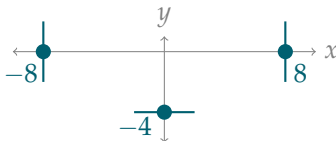
Domain all real numbers

End behaviour $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$

Intercepts $(0, -4), (\pm 8, 0)$

Critical point $(0, -4)$

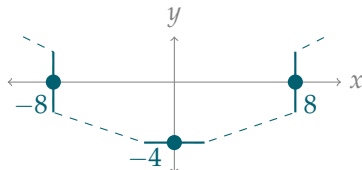
Singular points $(-8, 0), (8, 0)$
Near the singular points, $f'(x)$ gets very large, so $f(x)$ looks vertical.



**Increasing,
decreasing**

Numerator of $f'(x)$ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$. Denominator is positive where it exists. So $f(x)$ is decreasing on $(-\infty, 0)$ (where it is differentiable) and increasing on $(0, \infty)$ (where it is differentiable).

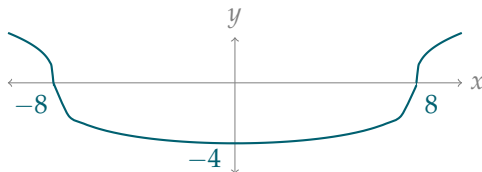
Skeleton



Concavity

Numerator of second derivative is negative everywhere. Denominator is positive (so $f''(x)$ is negative) on $(-\infty, -8) \cup (8, \infty)$ and denominator is negative (so $f''(x)$ is positive) on $(-8, 8)$.

So our function is concave up on $(-8, 8)$ and concave down on $(-\infty, -8) \cup (8, \infty)$.



LET'S GRAPH

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

LET'S GRAPH

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

Note: for $x \neq -1$, $f(x) = \frac{x(x + 1)}{(x + 1)(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^2}$

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$$g(x) := \frac{x}{(x^2 + 1)^2}$$

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Note: for $x \neq -1$, $f(x) = \frac{x(x + 1)}{(x + 1)(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^2}$

$$g(x) := \frac{x}{(x^2 + 1)^2}$$

$$g'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3}$$

$$g''(x) = \frac{12x(x^2 - 1)}{(x^2 + 1)^4}$$

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

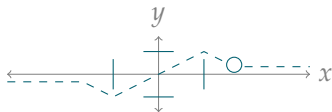
$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

When $x \neq 1$, $f(x) = g(x)$. So, $f(x)$ looks like $g(x)$ except it has a removable discontinuity (hole) at $x = 1$. Let's graph

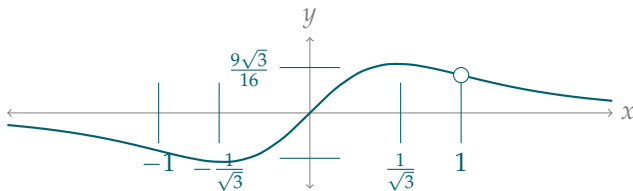
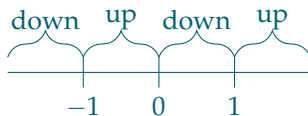
$$g(x) = \frac{x}{(x^2 + 1)^2}$$

- ▶ Domain: all real numbers
- ▶ HA: $y = 0$ on both sides
- ▶ VA: none
- ▶ Intercepts: $(0, 0)$
- ▶ Odd symmetry
- ▶ CP: $x = \pm \frac{1}{\sqrt{3}}$; associated points: $\left(-\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{16}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{16}\right)$

- Increasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$



- g'' is given (nearly) factored, so we can see its sign changes at $x = -1, 0, 1$. Concavity:



LET'S GRAPH

$$f(x) = x(x - 1)^{2/3}$$

- $f'(x) = \frac{5x - 3}{3\sqrt[3]{x - 1}}$
- $f''(x) = \frac{2(5x - 6)}{9(\sqrt[3]{x - 1})^4}$

LET'S GRAPH

$$f(x) = x(x-1)^{2/3}$$

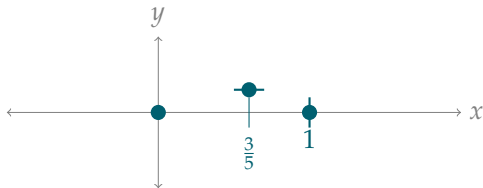
- $f'(x) = \frac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \frac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

► $f(3/5) \approx 0.3$

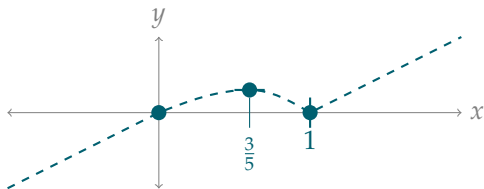
► $f(6/5) \approx 0.4$



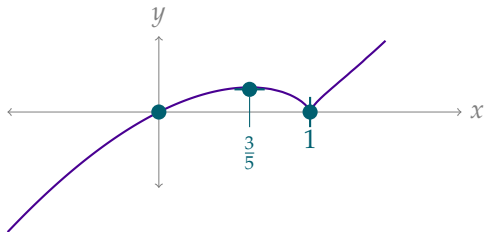
- ▶ Domain: all reals
- ▶ VA: none
- ▶ HA: none
- ▶ Intercepts: $(0, 0)$, $(1, 0)$
- ▶ Symmetry: not even, not odd, not periodic
- ▶ CP: $x = \frac{3}{5}$ ($y \approx 0.3$); SP $x = 1$ ($y = 0$).
Near the SP, first deriv is very large, so function looks vertical.



- First derivative changes sign at $x = \frac{3}{5}$ and $x = 1$. Function is increasing on $(-\infty, \frac{3}{5}) \cup (1, \infty)$ and decreasing on $(\frac{3}{5}, 1)$.



- Second derivative changes sign at $x = \frac{5}{6}$. (Note the denominator has an even power). Function is concave down on $(-\infty, \frac{5}{6})$ and concave up on $(\frac{5}{6}, \infty)$.



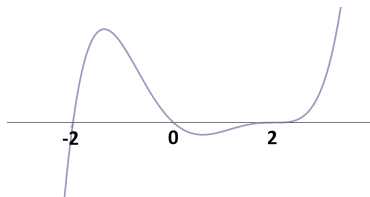
Ch 3.6 Review: matching

MATCH THE FUNCTION TO ITS GRAPH

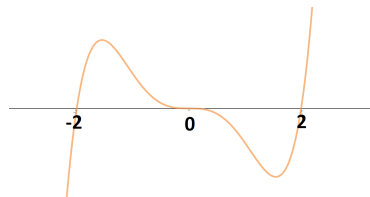
A. $f(x) = x^3(x+2)(x-2) = x^5 - 4x^3$

B. $f(x) = x(x+2)^3(x-2) = x^5 + 4x^4 - 16x^2 - 16x$

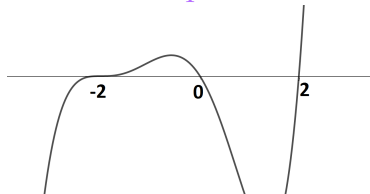
C. $f(x) = x(x+2)(x-2)^3 = x^5 - 4x^4 + 16x^2 - 16x$



I



III



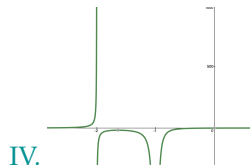
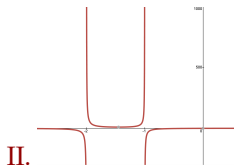
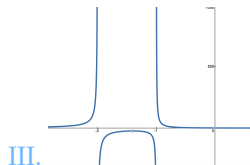
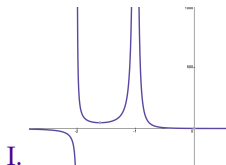
II

A. $f(x) = \frac{x-1}{(x+1)(x+2)}$

B. $f(x) = \frac{(x-1)^2}{(x+1)(x+2)}$

C. $f(x) = \frac{x-1}{(x+1)^2(x+2)}$

D. $f(x) = \frac{(x-1)^2}{(x+1)^2(x+2)}$



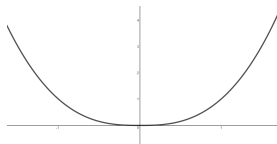
MATCH THE FUNCTION TO ITS GRAPH

A. $f(x) = |x|^e$

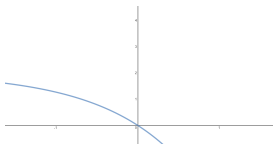
B. $f(x) = e^{|x|}$

C. $f(x) = e^{x^2}$

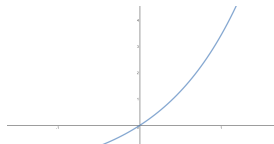
D. $f(x) = e^{x^4-x}$



I



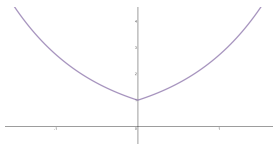
II



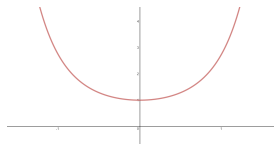
III



IV



V



VI

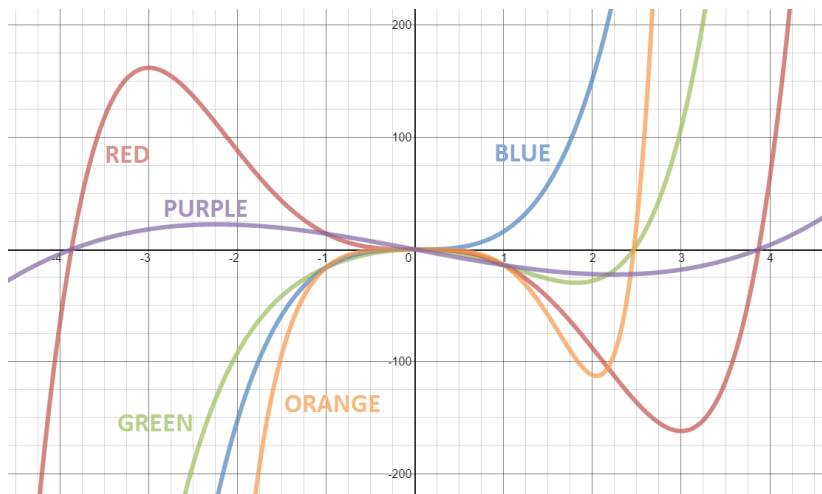
A. $f(x) = x^5 + 15x^3$

B. $f(x) = x^5 - 15x^3$

C. $f(x) = x^5 - 15x^2$

D. $f(x) = x^3 - 15x$

E. $f(x) = x^7 - 15x^4$



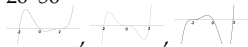
Included Work



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screenshots of graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator> (accessed 13 November 2015), 163



screenshots of graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator>, (accessed 13 November 2015), 165



screenshots from graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator> (accessed 16 July 2021), 164



screenshot from graphs generated using Desmos Graphing Calculator
<https://www.desmos.com/calculator>, with text added (accessed 13 November 2015), 166



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151–154