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graph TD; Derivatives((Derivatives)) --- Definition22((Definition 2.2)); Derivatives --- ComputingDerivatives((Computing Derivatives 2.4-2.12)); Derivatives --- Applications((Applications)); Definition22 --- RevisitingTangentLines((Revisiting tangent lines 2.1)); Definition22 --- HigherOrderDerivatives((Higher Order Derivatives 2.14)); Applications --- Interpretations23((Interpretations 2.3)); Applications --- MeanValueTheorem213((Mean Value Theorem 2.13));
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Derivatives

- Definition 2.2
 - Revisiting tangent lines 2.1
 - Higher Order Derivatives 2.14
- Computing Derivatives 2.4-2.12
- Applications
 - Interpretations 2.3
 - Mean Value Theorem 2.13

Definition 2.2.1

$$\text{So, } f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

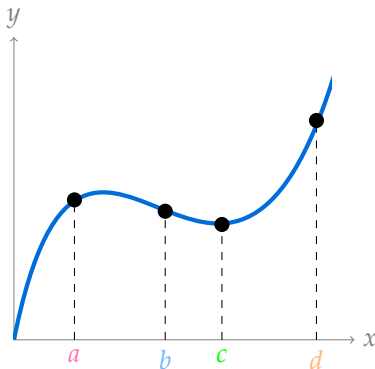
$f'(a)$ is also the **instantaneous rate of change** of f at a .

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) < 0$, then f is **decreasing** at a . Its graph “points down.”

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PRACTICE: INCREASING AND DECREASING



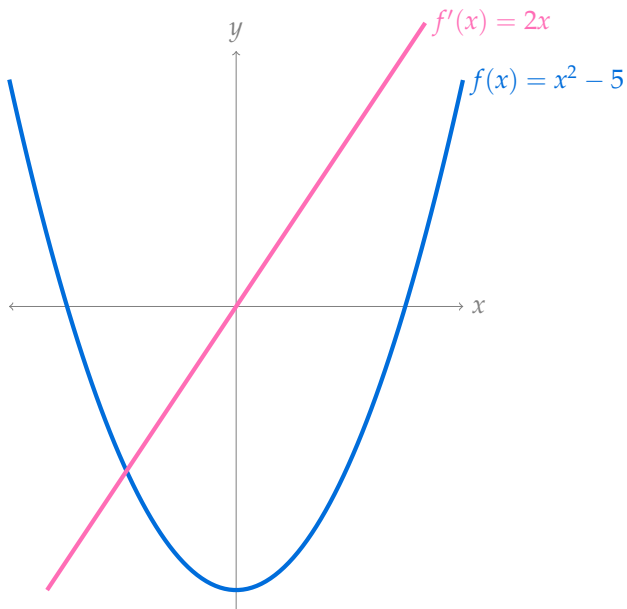
Where is $f'(x) < 0$?

Where is $f'(x) > 0$?

Where is $f'(x) \approx 0$?

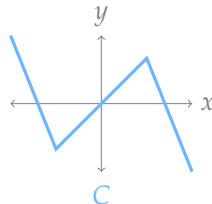
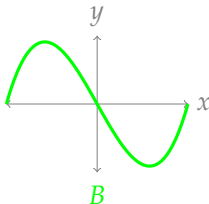
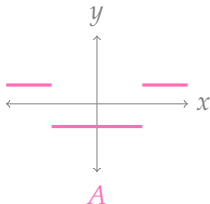
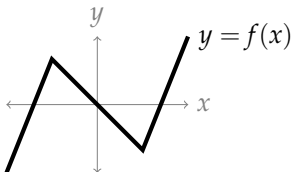
Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.

Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$.
We can also find its derivative at an arbitrary point x :



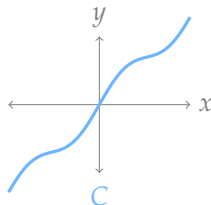
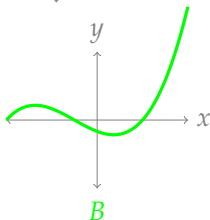
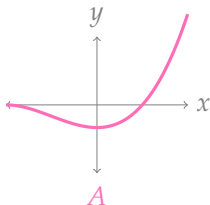
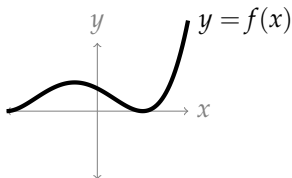
INCREASING AND DECREASING

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



INCREASING AND DECREASING

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.

The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that x will be a part of your final expression: this is a **function**.

If $f'(x)$ exists for all x in an interval (a, b) , we say that f is **differentiable on (a, b)** .

Notation 2.2.8

The “prime” notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{df}{dx}$$

function

$$\frac{df}{dx}(a)$$

number

$$\frac{d}{dx}f(x)$$

function

$$\frac{d}{dx}f(x)\Big|_{x=a}$$

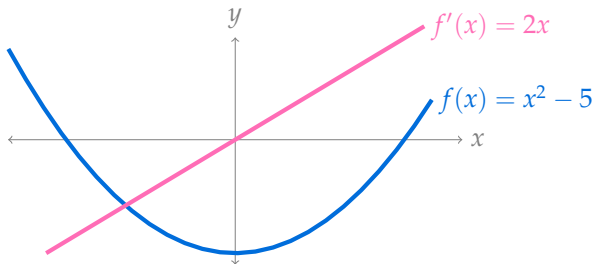
number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibnitz Notation:

$$\frac{df}{dx} = \qquad \frac{df}{dx}(3) = \qquad \frac{d}{dx}f(x) = \qquad \frac{d}{dx}f(x)\bigg|_{x=3} =$$



Alternate Definition – Definition 2.2.1

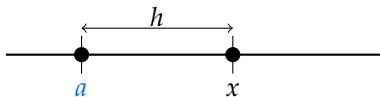
Calculating

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

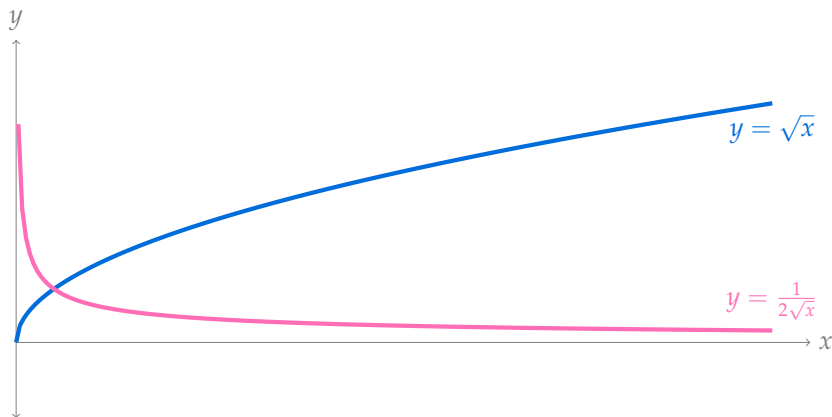
is the same as calculating

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.



Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.



Review:

$$\lim_{x \rightarrow 0^+} \sqrt{x} = \quad \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} =$$

$$\lim_{x \rightarrow 0^+} \frac{1}{2\sqrt{x}} =$$



NOW
YOU

Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{1}{x} \right\}.$$

Using the definition of the derivative, calculate $\frac{d}{dx} \left\{ \frac{2x}{x+1} \right\}$.

Using the definition of the derivative, calculate $\frac{d}{dx} \left\{ \frac{1}{\sqrt{x^2 + x}} \right\}$.

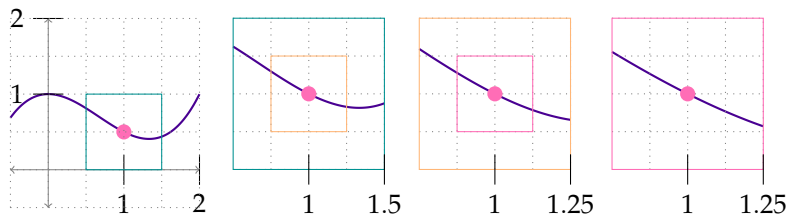
Memorize

The derivative of a function f at a point a is given by the following limit, if it exists:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

ZOOMING IN

For a smooth function, if we zoom in at a point, we see a line:



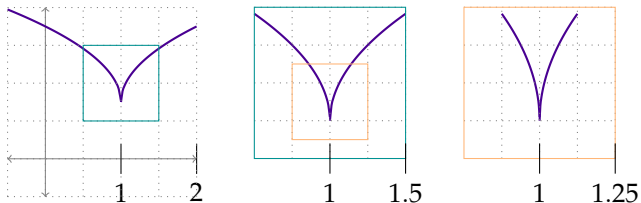
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

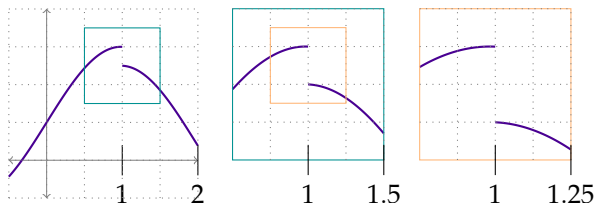
ZOOMING IN ON FUNCTIONS THAT AREN'T SMOOTH

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



Discontinuity:



Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ **does not exist** at $x = a$ if

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

does not exist.

Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

WHEN DERIVATIVES DON'T EXIST

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

└ 2.2 Definition of the Derivative

└ When Derivatives Don't Exist

As usual, it's nice to reassure students that we did not need to know the graph of this function to answer the question. Otherwise, they might unduly worry.

Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Let $f(x)$ be a function and let a be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$
$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$

Included Work



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