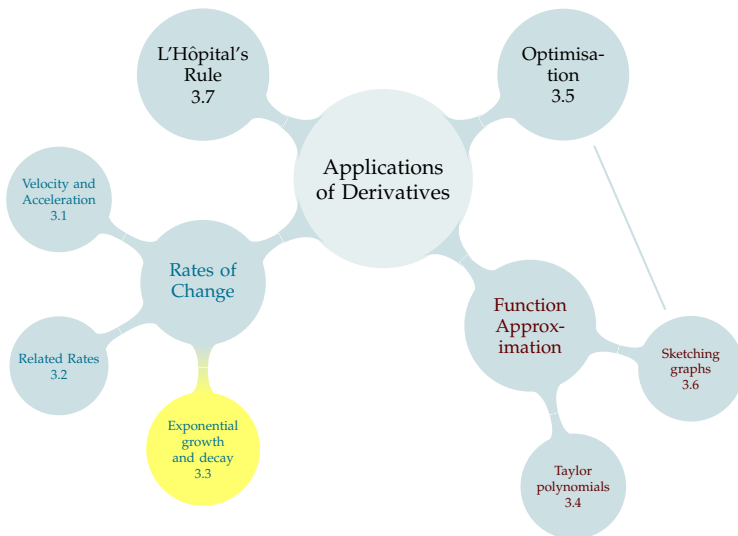


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# RADIOACTIVE DECAY

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

# RADIOACTIVE DECAY

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

# Differential Equation

Let  $Q = Q(t)$  be the amount of a radioactive substance at time  $t$ . Then for some positive constant  $k$ :

$$\frac{dQ}{dt} = -kQ$$













# RADIOACTIVE DECAY

## Quantity of a Radioactive Isotope

$$Q(t) = Ce^{-kt}$$

$Q(t)$ : quantity at time  $t$

What is the sign of  $Q(t)$ ?

- A. positive or zero
- B. negative or zero
- C. could be either
- D. I don't know

What is the sign of  $C$ ?

- A. positive or zero
- B. negative or zero
- C. could be either
- D. I don't know





Every 30 seconds, the size of the sample halves. What are  $C$  and  $k$ ?









$$Q'(t) = kQ(t)$$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.



$$Q'(t) = kQ(t)$$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

The rate of growth of a population in a given time interval is proportional to the number of individuals in the population, when the population has ample resources.



## Exponential Growth – Theorem 3.3.2

Let  $Q = Q(t)$  satisfy:

$$\frac{dQ}{dt} = kQ$$

for some constant  $k$ . Then for some constant  $C = Q(0)$ ,

$$Q(t) = Ce^{kt}$$



$$\frac{dy}{dt} + 3y = 0 \quad \text{and} \quad y(1) = 2.$$

What is  $y(t)$ ?



















$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$  is the temperature of the object,  $A$  is the ambient temperature,  $K$  is some constant.

What is true of  $K$ ?

- A.  $K \geq 0$   
B.  $K \leq 0$   
C.  $K = 0$   
D.  $K$  could be positive, negative, or zero, depending on the object  
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## Newton's Law of Cooling – Equation 3.3.7

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If  $T(10) < A$ , then:

- A.  $K > 0$
- B.  $T(0) > 0$
- C.  $T(0) > A$
- D.  $T(0) < A$

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Evaluate  $\lim_{t \rightarrow \infty} T(t)$ .

- A.  $A$
- B.  $0$
- C.  $\infty$
- D.  $T(0)$

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Evaluate  $\lim_{t \rightarrow \infty} T(t)$ .

- A.  $A$
- B.  $0$
- C.  $\infty$
- D.  $T(0)$



What assumptions are we making that might not square with the real world?

## Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt} = K[T(t) - A]$$

$T(t)$  is the temperature of the object,  $A$  is the ambient temperature, and  $K$  is some constant.

## Temperature of a Cooling Body – Corollary 3.3.8

$$T(t) = [T(0) - A]e^{Kt} + A$$

A farrier forms a horseshoe heated to  $400^\circ\text{C}$ , then dunks it in a river at room-temperature ( $25^\circ\text{C}$ ). The water boils for 30 seconds. The horseshoe is safe for the horse when it's  $40^\circ\text{C}$ . When can the farrier put on the horseshoe?



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$$T(t) = [T(0) - A]e^{Kt} + A$$



We know:  $T(0) = 400$ ,  $T(30) = 100$ , and  $A = 25$ . We want to find  $K$ .

$$100 = T(30) = [T(0) - A]e^{30K} + A = 375e^{30K} + 25$$

$$\Rightarrow 75 = 375e^{30K} \Rightarrow \frac{1}{5} = e^{30K} \Rightarrow K = \frac{-\log 5}{30}$$

Now, we set  $T(t) = 40$  and solve for  $t$ :

$$40 = T(t) = 375e^{\frac{-\log 5}{30}t} + 25$$

$$15 = 375e^{\frac{-\log 5}{30}t} = 375 \cdot 5^{-t/30}$$

$$\frac{1}{25} = 5^{-t/30}$$

$$25 = 5^{t/30}$$

$$2 = t/30$$

So the farrier can put the shoe on after 60 seconds in the water.



A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is  $40^\circ$ , and after 20 minutes, the tea is  $25^\circ$ . What is the temperature outside?





$$T(0) = 100, \text{ so}$$

$$T(10) = [100 - A]e^{10K} + A = 100e^{10K} + A(1 - e^{10K}) = 40$$

$$T(20) = [100 - A]e^{20K} + A = 100e^{20K} + A(1 - e^{20K}) = 25$$

$$\text{Solving both for } A, \text{ we get } A = \frac{40 - 100e^{10K}}{1 - e^{10K}} = \frac{25 - 100e^{20K}}{1 - e^{20K}}$$

Although this looks complicated, if we set  $x = e^{10k}$ , it simplifies to something we can easily solve.



$$A = \frac{40 - 100e^{10K}}{1 - e^{10K}} = \frac{25 - 100e^{20K}}{1 - e^{20K}}$$

$$A = \frac{40 - 100x}{1 - x} = \frac{25 - 100x^2}{1 - x^2}$$

$$(40 - 100x)(1 - x^2) = (25 - 100x^2)(1 - x)$$

$$(40 - 100x)(1 + x)(1 - x) = (25 - 100x^2)(1 - x)$$

$$(40 - 100x)(1 + x) = 25 - 100x^2$$

$$40 - 60x - 100x^2 = 25 - 100x^2$$

$$40 - 60x = 25$$

$$x = \frac{1}{4}$$

$$A = \frac{40 - 100x}{1 - x} = \frac{40 - \frac{100}{4}}{1 - \frac{1}{4}} = 20$$

It is 20 degrees outside.



In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?

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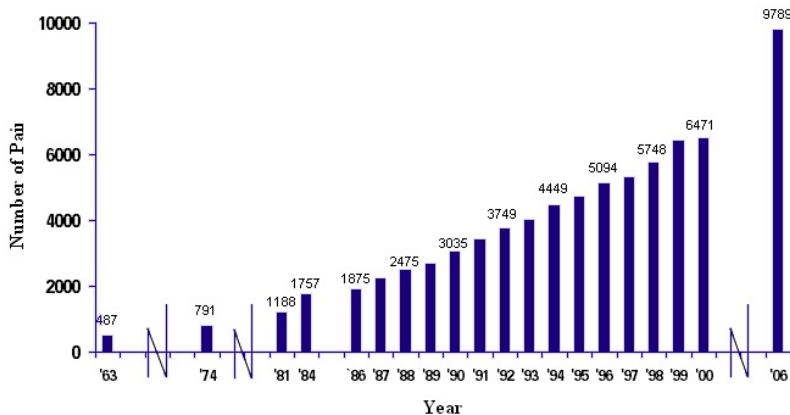
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Since we don't have a better model, let's assume the population  $P$  of nesting pairs follows:

$$P(t) = P(0)e^{Kt}$$

for some constant  $K$ .

To fit the data we have, let  $t = 0$  represent 1963, so  $P(0) = 487$ . Then

$$4015 = P(30) = 487e^{30K}$$

$$\text{so } e^K = \left(\frac{4015}{487}\right)^{1/30}.$$

Now we use this to predict  $P(43)$  (since 2006 is 43 years after 1963) and  $P(52)$  (since 2015 is 52 years after 1963).

$$P(43) = 487(e^K)^{43} = 487 \left(\frac{4015}{487}\right)^{43/30} \approx 10016$$

So we guess in 2016 there were about 10,016 breeding pairs in the lower 48.

$$P(52) = 487(e^K)^{52} = 487 \left(\frac{4015}{487}\right)^{52/30} \approx 18860$$

## link: Wood Bison Restoration in Alaska, Alaska Department of Fish and Game

### Excerpt:

*Based on experience with reintroduced populations elsewhere, wood bison would be expected to increase at a rate of 15%-25% annually after becoming established.... With an average annual growth rate of 20%, an initial precalving population of 50 bison would increase to 500 in approximately 13 years.*

NOW  
YOU



Are they using our same model?

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NOW  
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Are they using our same model?

Our model gives the same result.

# COMPOUND INTEREST

Suppose you invest \$10,000 in an account that accrues interest each month. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?



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Compound interest is calculated according to the formula  $Pe^{rt}$ , where  $r$  is the interest rate and  $t$  is time.

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Compound interest is calculated according to the formula  $Pe^{rt}$ , where  $r$  is the interest rate and  $t$  is time.

Measuring time in months,

$$10000e^{r \cdot 1} = 10100$$

$$e^r = \frac{10100}{10000} = 1.01$$

$$10000e^{12r} = 10000 \cdot (e^r)^{12} = 10000 \cdot 1.01^{12} \approx 11268.25$$



# CARRYING CAPACITY

For a population of size  $P$  with unrestricted access to resources, let  $\beta$  be the average number of offspring each breeding pair produces per generation, where a generation has length  $t_g$ . Then  $b = \frac{\beta-2}{2t_g}$  is the net birthrate (births minus deaths) per member per unit time. This yields  $\frac{dP}{dt}(t) = bP(t)$ , hence:



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$$P(t) = P(0)e^{bt}$$



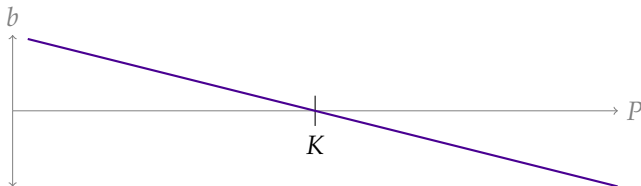
# CARRYING CAPACITY

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$$P(t) = P(0)e^{bt}$$

But as resources grow scarce,  $b$  might change.

$b$  is the net birthrate (births minus deaths) per member per unit time.  
 If  $K$  is the carrying capacity of an ecosystem, we can model  
 $b = b_0(1 - \frac{P}{K})$ .

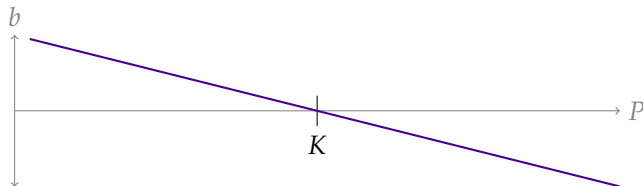


# CARRYING CAPACITY

$b$  is the net birthrate (births minus deaths) per member per unit time.

If  $K$  is the carrying capacity of an ecosystem, we can model

$$b = b_0 \left(1 - \frac{P}{K}\right).$$



NOW  
YOU



Describe to your neighbour what the following mean in

terms of the model:

- ▶  $b > 0, b = 0, b < 0$
- ▶  $P = 0, P > 0, P < 0$



# CARRYING CAPACITY

Then:

$$\frac{dP}{dt}(t) = b_0 \underbrace{\left(1 - \frac{P(t)}{K}\right)}_{\text{per capita birthrate}} P(t)$$



# CARRYING CAPACITY

Then:

$$\frac{dP}{dt}(t) = b_0 \underbrace{\left(1 - \frac{P(t)}{K}\right)}_{\text{per capita birthrate}} P(t)$$

This is an example of a differential equation that we don't have the tools to solve. (If you take more calculus, though, you'll learn how!) It's also an example of a way you might tweak a model so its assumptions better fit what you observe.

# RADIOCARBON DATING

Researchers at Charlie Lake in BC have found evidence<sup>1</sup> of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago.

Suppose a comparable bone of a bison alive today contains  $1\mu\text{g}$  of  $^{14}\text{C}$ . If the half-life of  $^{14}\text{C}$  is about 5730 years, roughly how much  $^{14}\text{C}$  do you think the researchers found in the sample?

---

<sup>1</sup><http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf>

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- A. About  $\frac{1}{10,500} \mu\text{g}$
- B. About  $\frac{1}{4} \mu\text{g}$
- C. About  $\frac{1}{2} \mu\text{g}$

- D. About  $1 \mu\text{g}$
- E. I'm not sure how to estimate this



# RADIOCARBON DATING

First, an estimate; 10500 is not so far off from  $2(5730)$ , i.e. two half-lives, so we might guess that there is roughly a  $(\frac{1}{2})^2 = \frac{1}{4}$  of a microgram left.

We know  $Q(t) = Ce^{-kt} = e^{-kt} \mu\text{g}$ . We want to find  $Q(10500)$ , so we need to solve for  $k$ . Since we know the half-life: to do this, solve

$$\frac{1}{2} = e^{-k \cdot 5730} \quad \text{to get} \quad k = \frac{\log 2}{5730}$$

Now:

$$Q(10500) = e^{-\frac{\log 2}{5730} \cdot 10500} = 2^{-\frac{10500}{5730}} \approx 0.28 \mu\text{g}$$



Suppose a body is discovered at 3:45 pm, in a room held at  $20^\circ$ , and the body's temperature is  $27^\circ$ , not the normal  $37^\circ$ . At 5:45 pm, the temperature of the body has dropped to  $25.3^\circ$ . When did the inhabitant of the body die?

Set our time so that  $t = 0$  is 3:45pm and  $t = 2$  is 5:45pm. Then  $T(0) = 27$ ,  $T(2) = 25.4$ , and  $A = 20$ . Now:

$$T(t) = [27 - 20]e^{Kt} + 20 = 7e^{Kt} + 20$$

Using what we know about 5:45pm:

$$7e^{2K} + 20 = T(2) = 25.3$$

so

$$7e^{2K} = 5.3 \implies e^{2K} = \frac{5.3}{7} \implies e^K = \left(\frac{5.3}{7}\right)^{1/2}$$

Now:

$$T(t) = 7e^{Kt} + 20 = 7\left(\frac{5.3}{7}\right)^{t/2} + 20$$

So we set  $T(t) = 37$  and solve for  $t$ .



$$7 \left( \frac{5.3}{7} \right)^{t/2} + 20 = 37$$

$$7 \left( \frac{5.3}{7} \right)^{t/2} = 17$$

$$\left( \frac{5.3}{7} \right)^{t/2} = \frac{17}{7}$$

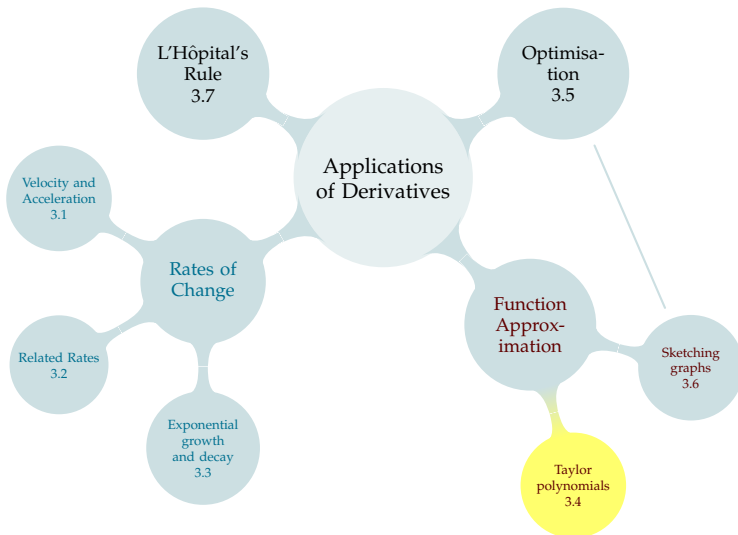
$$\frac{t}{2} = \frac{\log(17/7)}{\log(5.3/7)}$$

$$t = 2 \frac{\log(17/7)}{\log(5.3/7)} \approx -6.4$$

So the person died about 6.4 hours before 3:45pm. Now 0.4 hours is 24 minutes. So 6 hours and 24 minutes before 3:45 pm is 6 hours before 3:21pm, which is 9:21 am.

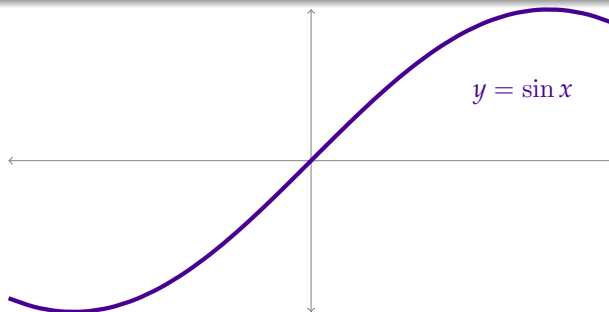


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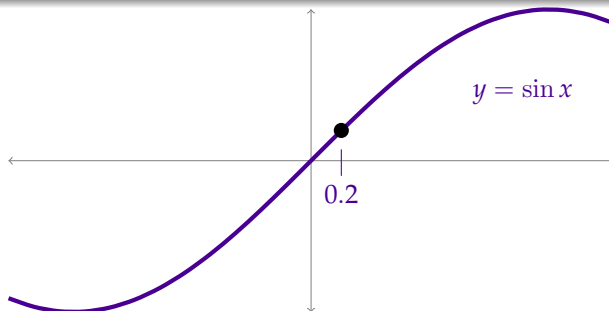




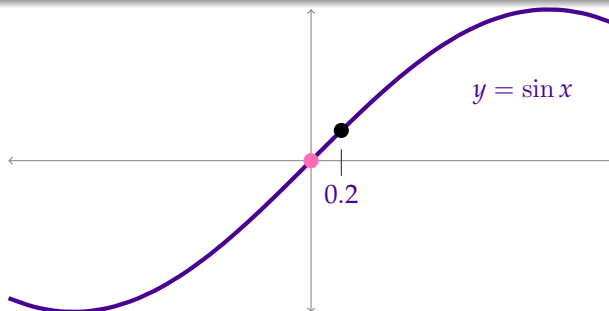
# APPROXIMATING A FUNCTION



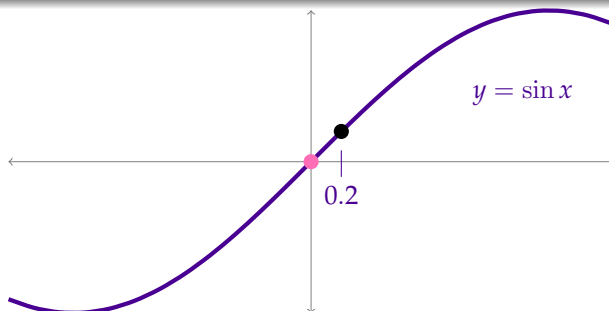
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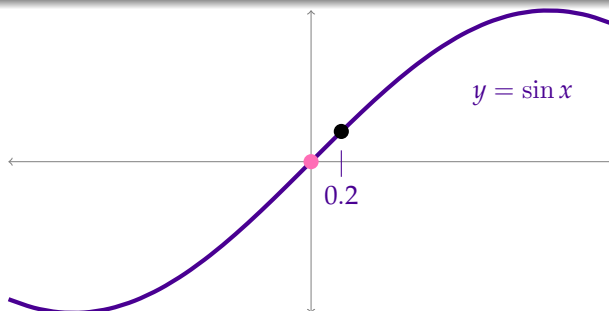


## Constant Approximation – Equation 3.4.1

We can approximate  $f(x)$  near a point  $a$  by

$$f(x) \approx f(a)$$

# APPROXIMATING A FUNCTION



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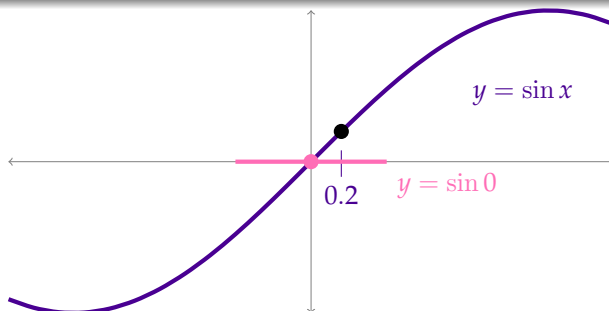
We can approximate  $f(x)$  near a point  $a$  by

$$f(x) \approx f(a)$$

Google:  $\sin(0.2) \approx 0.198669\dots$

Constant approx:  $\sin(0.2) \approx 0$

# APPROXIMATING A FUNCTION



## Constant Approximation – Equation 3.4.1

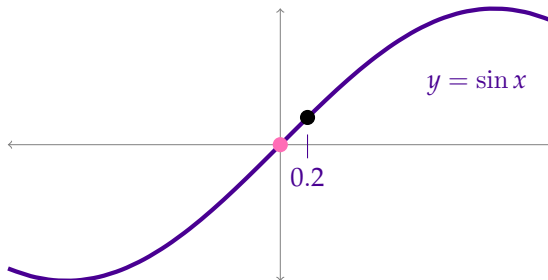
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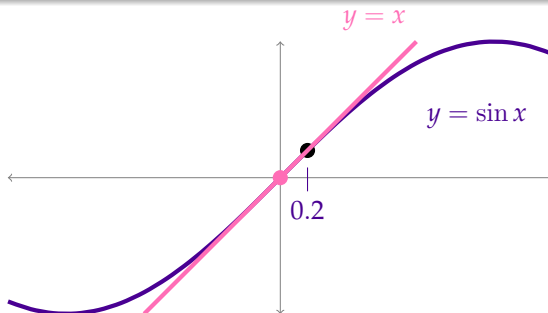
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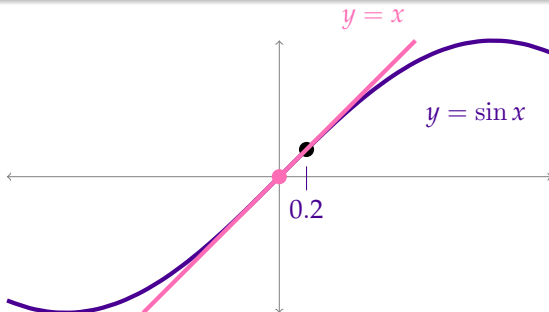


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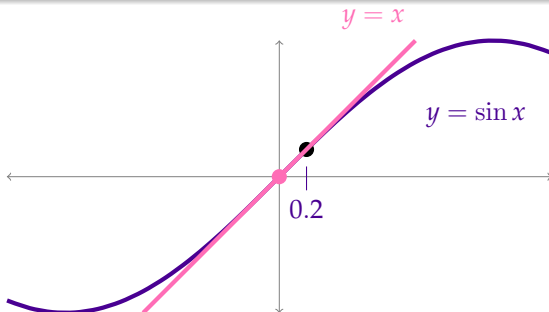


## Linear Approximation (Linearization) – Equation 3.4.3

We can approximate  $f(x)$  near a point  $a$  by the tangent line to  $f(x)$  at  $a$ , namely

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

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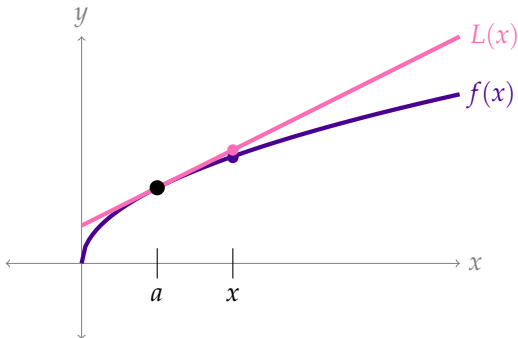
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Linear approx:

$$\sin(0.2) \approx 0 + 1(0.2 - 0) = 0.2$$

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$


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To find a linear approximation of  $f(x)$  at a particular point  $x$ , pick a point  $a$  **near to  $x$** , such that  $f(a)$  and  $f'(a)$  are **easy to calculate**.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Let  $f(x) = \sqrt{x}$ . Approximate  $f(8.9)$ .

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Let  $f(x) = \sqrt{x}$ . Approximate  $f(8.9)$ .

First we note that  $8.9 \approx 9$  and we can easily calculate  $f(9) = 3$ .

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Linear approximation: Using  $a = 9$ ,

$$f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$f(8.9) \approx f(9) + f'(9)(8.9 - 9) = 3 + \frac{1}{6}(-.1)$$

$$f(8.9) \approx 3 - \frac{1}{60} = 2.98\overline{33}$$

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Google:  $\sqrt{8.9} = 2.98328677804\dots$





Accurate

Possible to calculate: add, subtract, multiply, divide. Use integers or known constants

# CAN WE COMPUTE?

Suppose we want to approximate the value of  $\cos(1.5)$ . Which of the following linear approximations could we calculate by hand? (You can leave things in terms of  $\pi$ .)

- A. tangent line to  $f(x) = \cos x$  when  $x = \pi/2$   
 B. tangent line to  $f(x) = \cos x$  when  $x = 3/2$   
 C. both  
 D. neither







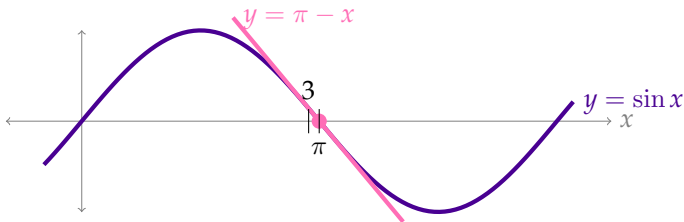
# LINEAR APPROXIMATION

Approximate  $\sin(3)$  using a linear approximation. You may leave your answer in terms of  $\pi$ .



# LINEAR APPROXIMATION

Approximate  $\sin(3)$  using a linear approximation. You may leave your answer in terms of  $\pi$ .



Let  $f(x) = \sin x$  and  $a = \pi$ . Then

$$f(3) \approx f(\pi) + f'(\pi)(3 - \pi) = \sin(\pi) + \cos(\pi)(3 - \pi) = \boxed{\pi - 3} \approx 0.14159$$

Google:  $\sin(3) = 0.14112000806\dots$



Approximate  $e^{1/10}$  using a linear approximation.  
If  $f(x) = e^x$  and  $a = 0$  :



## LINEAR APPROXIMATION

Approximate  $e^{1/10}$  using a linear approximation.

If  $f(x) = e^x$  and  $a = 0$  :

$$f'(x) = e^x$$

$$f(1/10) \approx f(0) + f'(0)(1/10 - 0) = e^0 + e^0(1/10 - 0) = 1 + 1/10 = 1.1$$

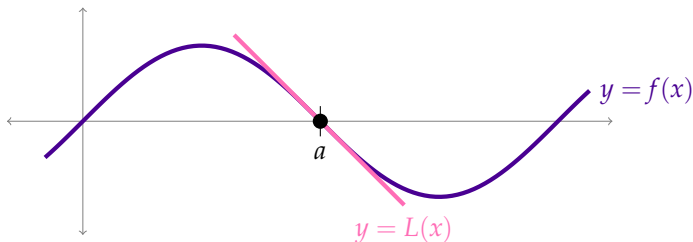
Google:  $e^{1/10} = 1.10517091808\dots$





# LINEAR APPROXIMATION WRAP-UP

Let  $L(x) = f(a) + f'(a)(x - a)$ , so  $L(x)$  is the linear approximation (linearization) of  $f(x)$  at  $a$ .

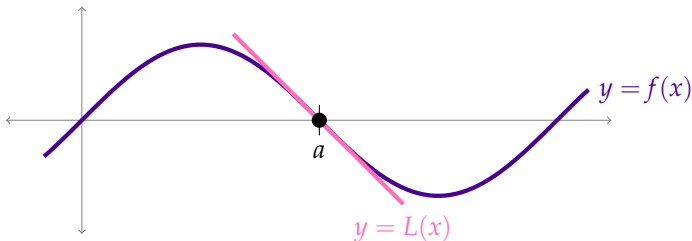




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What is  $L(a)$ ?

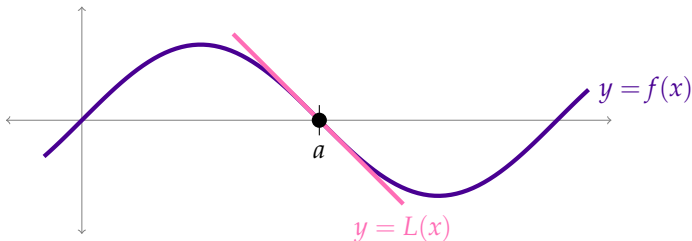


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What is  $L(a)$ ?

What is  $L'(a)$ ?









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What is  $L(a)$ ?

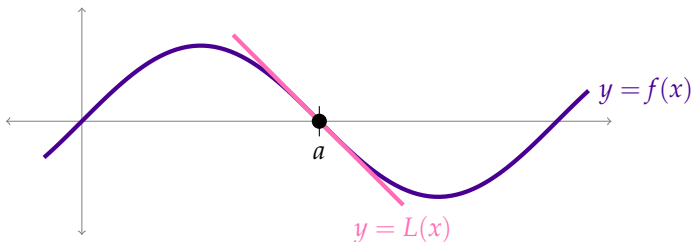
$$L(a) = f(a)$$

What is  $L'(a)$ ?

$$L'(a) = f'(a)$$

What is  $L''(a)$ ? (Recall  $L''(x)$  is the derivative of  $L'(x)$ .)

$$L''(a) = 0$$



# LINEAR APPROXIMATION WRAP-UP

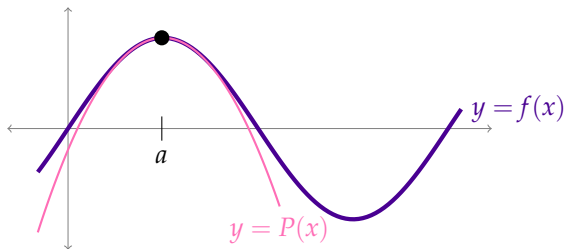
Let  $L(x)$  be a linear approximation of  $f(x)$ .

$f(a)$	$L(a)$	same
$f'(a)$	$L'(a)$	same
$f''(a)$	$L''(a)$	different <sup>1</sup>

<sup>1</sup>unless  $f''(a) = 0$

## QUADRATIC APPROXIMATION

Imagine we approximate  $f(x)$  at  $x = a$  with a **parabola**,  $P(x)$ .

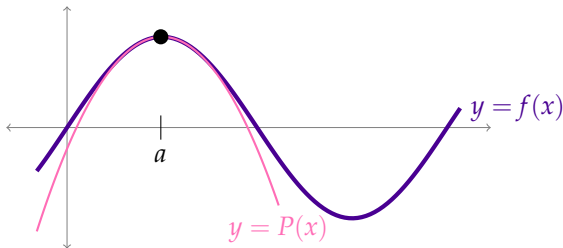






## QUADRATIC APPROXIMATION

Imagine we approximate  $f(x)$  at  $x = a$  with a **parabola**,  $P(x)$ .



$P(x) = A + Bx + Cx^2$	$P(a) = A + Ba + Ca^2$	$f(a)$
$P'(x) = B + 2Cx$	$P'(a) = B + 2Ca$	$f'(a)$
$P''(x) = 2C$	$P''(a) = 2C$	$f''(a)$



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Constant:

$$f(x) \approx f(a)$$

Linear:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Quadratic:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$





## QUADRATIC APPROXIMATION

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

Approximate  $\sqrt[3]{28}$  using a quadratic approximation.

*You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.*





$$(2.01)^6$$

Determine what  $f(x)$  and  $a$  should be so that you can approximate the following using a quadratic approximation.

$$\log(.9) \quad f(x) = \log(x), a = 1$$

$$e^{-1/30} \quad f(x) = e^x, a = 0$$

$$\sqrt[5]{30} \quad f(x) = \sqrt[5]{x}, a = 32 = 2^5$$

$$(2.01)^6 \quad f(x) = x^6, a = 2$$

It is possible to compute the last one without an approximation, but an approximation might save time while being sufficiently accurate for your purposes.

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## BRIEF DETOUR: SIGMA (SUMMATION) NOTATION

$$\sum_{i=a}^b f(i)$$

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$$\sum_{i=a}^b f(i)$$

- ▶  $a, b$  (integers) “bounds”
- ▶  $i$  “index”: runs over integers from  $a$  to  $b$
- ▶  $f(i)$  “summand”: compute for every  $i$ , add

## SIGMA NOTATION

$$\sum_{i=2}^4 (2i + 5)$$

## SIGMA NOTATION

$$\sum_{i=2}^4 (2i + 5)$$

$$\begin{aligned}\sum_{i=2}^4 (2i+5) &= \underbrace{(2 \cdot 2 + 5)}_{i=2} + \underbrace{(2 \cdot 3 + 5)}_{i=3} + \underbrace{(2 \cdot 4 + 5)}_{i=4} \\ &= 9 + 11 + 13 = 33\end{aligned}$$

## SIGMA NOTATION

$$\sum_{i=1}^4 (i + (i-1)^2)$$

## SIGMA NOTATION

$$\sum_{i=1}^4 (i + (i-1)^2)$$

$$= \underbrace{(1+0^2)}_{i=1} + \underbrace{(2+1^2)}_{i=2} + \underbrace{(3+2^2)}_{i=3} + \underbrace{(4+3^2)}_{i=4}$$

$$= 1 + 3 + 7 + 13 = 24$$

1.  $3 + 4 + 5 + 6 + 7$
2.  $8 + 8 + 8 + 8 + 8$
3.  $1 + (-2) + 4 + (-8) + 16$



### Factorial – Definition 3.4.9

We read “ $n!$ ” as “ $n$  factorial.”

For a natural number  $n$ ,  $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ .

By convention,  $0! = 1$ .





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By convention,  $0! = 1$ .

We write  $f^{(n)}(x)$  to mean the  $n^{\text{th}}$  derivative of  $f(x)$ . By convention,  $f^{(0)}(x) = f(x)$ .

## Taylor Polynomial – Definition 3.4.11

Given a function  $f(x)$  that is differentiable  $n$  times at a point  $a$ , the  $n$ -th degree **Taylor polynomial** for  $f(x)$  about  $a$  is

$$T_n(a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

If  $a = 0$ , we also call it a **Maclaurin polynomial**.

$$T_n(a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$\begin{aligned} T_n(a) &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \\ &= \underbrace{f(a)}_{k=0} + \underbrace{f'(a)(x-a)}_{k=1} + \underbrace{\frac{1}{2!}f''(a)(x-a)^2}_{k=2} + \\ &\quad \underbrace{\frac{1}{3!}f'''(a)(x-a)^3}_{k=3} + \underbrace{\frac{1}{4!}f^{(4)}(a)(x-a)^4}_{k=4} + \\ &\quad \cdots + \underbrace{\frac{1}{n!}f^{(n)}(a)(x-a)^n}_{k=n} \end{aligned}$$

$$= f(a)$$









$$T_n(a) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x - a)^n$$

Find the 7th degree Maclaurin<sup>2</sup> polynomial for  $e^x$ .

<sup>2</sup>A Maclaurin polynomial is a Taylor polynomial with  $a = 0$ .

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Maclaurin<sup>2</sup> polynomial for  $e^x$ .

Let  $f(x) = e^x$ . Then every derivative of  $e^x$  is just  $e^x$ , and  $e^0 = 1$ . So:

$$\begin{aligned} T_7(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{7!}f^{(7)}(0)(x-0)^7 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\ &= \sum_{k=0}^7 \frac{x^k}{k!} \end{aligned}$$

[e<sup>x</sup> approximations - link](#)

<sup>2</sup>A Maclaurin polynomial is a Taylor polynomial with  $a = 0$ .

Find the 8th degree Maclaurin polynomial for  $f(x) = \sin x$ .







### Notation 3.4.18

Let  $x, y$  be variables related such that  $y = f(x)$ . Then we denote a small change in the variable  $x$  by  $\Delta x$  (read as “delta  $x$ ”). The corresponding small change in the variable  $y$  is denoted  $\Delta y$  (read as “delta  $y$ ”).

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let  $y = f(x)$  be the amount of water needed to produce  $x$  apples in an orchard.

A farmer wants to know how a much water is needed to increase their crop yield.





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- Consider changing the number of apples grown from  $a$  to  $a + \Delta x$

A farmer wants to know how much water is needed to increase their crop yield.  $\Delta x$  is shorthand for some change in the number of apples, and  $\Delta y$  is shorthand for some change in the amount of water.



- Consider changing the number of apples grown from  $a$  to  $a + \Delta x$
- Then the change in water requirements goes from  $y = f(a)$  to  $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

## LINEAR APPROXIMATION OF $\Delta y$

- Using a linear approximation, setting  $x = a + \Delta x$ :







# QUADRATIC APPROXIMATION OF $\Delta y$

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.







Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and  $\pi$ .



$$f''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{(1/2)^3} = 8\sqrt{3}$$

Constant:  $f(x) \approx f(a)$

$$f\left(\frac{13\pi}{36}\right) \approx f\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Linear:  $f(x) \approx f(a) + f'(a)(x - a)$

$$\begin{aligned} f\left(\frac{13\pi}{36}\right) &\approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right) \\ &= \sqrt{3} + 4\left(\frac{\pi}{36}\right) \end{aligned}$$

Quadratic:  $f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$

$$\begin{aligned} f\left(\frac{13\pi}{36}\right) &\approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right) + \frac{1}{2}f''\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)^2 \\ &= \sqrt{3} + 4\left(\frac{\pi}{36}\right) + \frac{1}{2}(8\sqrt{3})\left(\frac{\pi}{36}\right)^2 \\ &= \sqrt{3} + \frac{\pi}{9} + \frac{4\sqrt{3}\pi^2}{6^4} \end{aligned}$$

type	approx	decimal
constant	$\sqrt{3}$	1.732...
linear	$\sqrt{3} + \frac{\pi}{9}$	2.081...
quadratic	$\sqrt{3} + \frac{\pi}{9} + \frac{4\sqrt{3}\pi^2}{6^4}$	2.134...
actual	—	2.145...













## Definition 3.4.25

Let  $Q_0$  be the exact value of a quantity and let  $Q_0 + \Delta Q$  be the measured value. We call

$$|\Delta Q|$$

the **absolute error** of the measurement, and

$$100 \frac{|\Delta Q|}{Q_0}$$

the **percentage error** of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

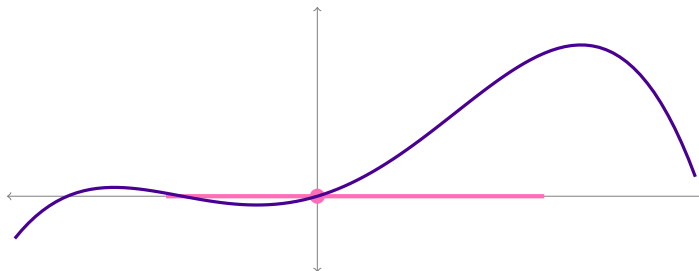


Once again, you find yourself in the position of measuring an angle  $x$ , which you use to compute  $y = \sin x$ . Let's say both  $x$  and  $y$  are positive. If your percentage error in measuring  $x$  is at most 1%, what is the corresponding maximum percentage error in  $y$ ? Use a linear approximation.

Using the linear approximation  $y(x_0 + \Delta x) \approx y(x_0) + y'(x_0)(\Delta x)$ :

$$\Rightarrow 100 \frac{|\Delta y|}{y_0} \approx 100 \frac{\frac{|x_0 \cos x_0|}{100}}{y_0} = \frac{x_0 |\cos x_0|}{y_0} = \frac{x_0 |\cos x_0|}{\sin x_0}$$

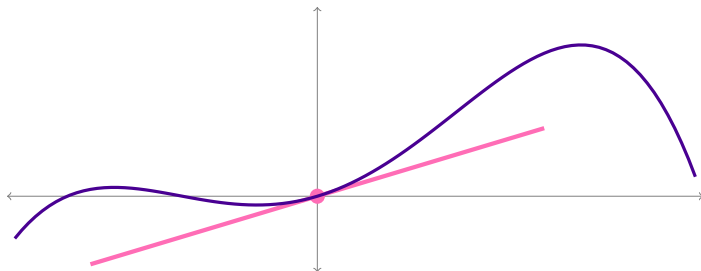
## ERROR: WHAT “CAUSES” ERROR IN AN ESTIMATION?



**Constant approximation:** We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

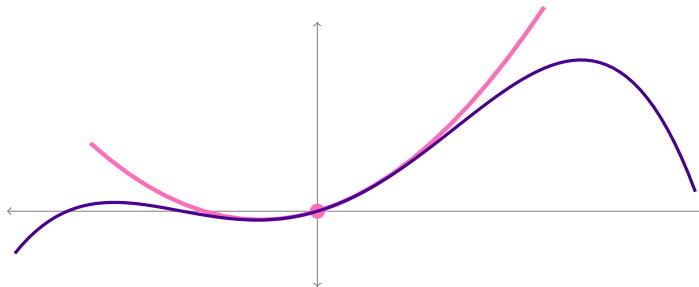


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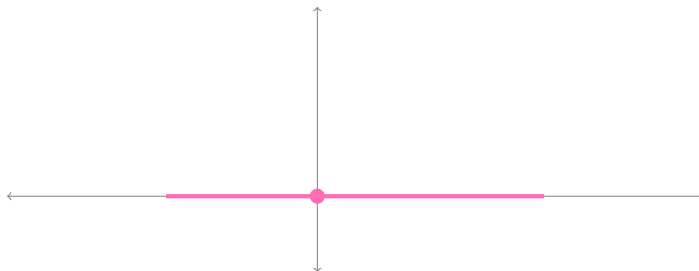
**Linear approximation:** We assume the function changes at a constant rate, but in fact the function changes at different rates (its second derivative is not always zero).

## ERROR: WHAT “CAUSES” ERROR IN AN ESTIMATION?



**Quadratic approximation:** We assume the function's derivative changes at a constant rate, but in fact the function's derivative changes at different rates (its third derivative is not always zero).

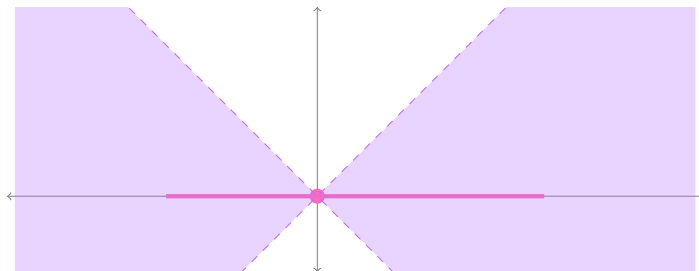
## CONTROLLING THE “CAUSE” OF THE ERROR



**Constant approximation:** We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

**BUT:** suppose we know the max and min values of the function's slope.

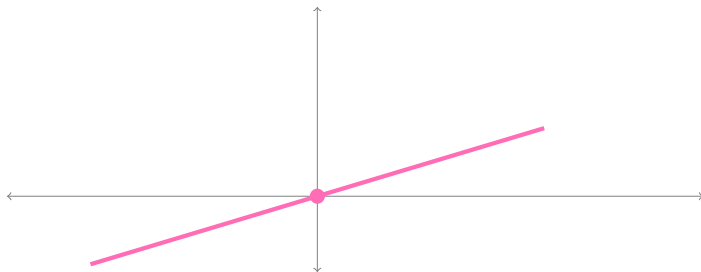
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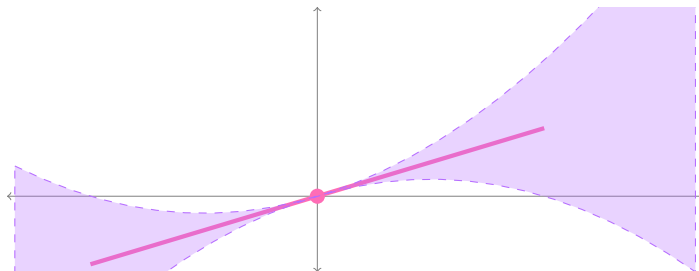
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## CONTROLLING THE “CAUSE” OF THE ERROR



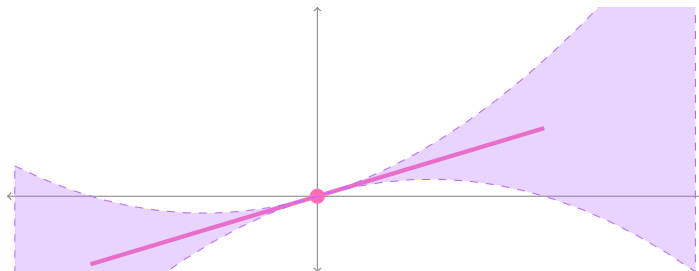
**Linear approximation:** We assume the function changes at a constant rate, but in fact the function changes at different rates (its first derivative is not always zero). **BUT:** suppose we know the max and min values of the function's second derivative.

## CONTROLLING THE “CAUSE” OF THE ERROR



**Linear approximation:** We assume the function changes at a constant rate, but in fact the function changes at different rates (its first derivative is not always zero). **BUT:** suppose we know the max and min values of the function's second derivative.

## CONTROLLING THE “CAUSE” OF THE ERROR



In general, if the “thing that causes the error” is big, then our error is big. We find the largest and smallest possible errors.







Third degree Maclaurin polynomial for  $f(x) = e^x$ :

Third degree Maclaurin polynomial for  $f(x) = e^x$ :

$$\begin{aligned} T_3(x) &= f(0) + f'(0)(x-0) + \frac{1}{2!}f''(0)(x-0)^2 + \frac{1}{3!}f'''(0)(x-0)^3 \\ &= e^0 + e^0x + \frac{1}{2!}e^0x^2 + \frac{1}{3!}e^0x^3 \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned}$$

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$$\begin{aligned} T_3(x) &= f(0) + f'(0)(x-0) + \frac{1}{2!}f''(0)(x-0)^2 + \frac{1}{3!}f'''(0)(x-0)^3 \\ &= e^0 + e^0x + \frac{1}{2!}e^0x^2 + \frac{1}{3!}e^0x^3 \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned}$$

Bound the error associated with using  $T_3(x)$  to approximate  $e^{1/10}$ .



For some  $c$  in  $(0, 0.1)$  :

$$\underbrace{f(0.1) - T_3(0.1)}_{\text{error}} = \frac{1}{4!} f^{(4)}(c) (.1 - 0)^4$$

$$= \frac{1}{4!} (0.0001) e^c$$

For  $c$  in  $(0, 0.1)$ ,  $1 \leq e^c < e^1 < 3$ , so

$$\begin{aligned} \frac{1}{4!}(0.0001) &\leq \underbrace{f(.1) - T_3(.1)}_{\text{error}} \\ &\leq \frac{1}{4!}(0.0001) \\ 4.2 \times 10^{-6} &\leq \underbrace{f(0.1) - T_3(0.1)}_{\text{error}} \leq 1.3 \times 10^{-5} \end{aligned}$$

## Taylor's Theorem – Equation 3.4.33

For some  $c$  strictly between  $x$  and  $a$ ,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at  $a = \pi/2$  to approximate  $f(x) = \cos x$ . What could the magnitude of the error be if we approximate  $\cos(2)$ ?

For some  $c$  in  $(\pi/2, 2)$ :

$$\underbrace{f(2) - T_5(2)}_{\text{error}} = \frac{1}{6!} f^{(6)}(c) (2 - \pi/2)^6$$

Note  $f^{(6)}(x)$  is going to be plus or minus sine or cosine, so  $-1 \leq f^{(6)}(c) \leq 1$ . Also,  $0 < 2 - \pi/2 < 1$ . Now:

$$\frac{-1}{6!} = \frac{1}{6!}(-1)(1)^6 \leq f(2) - T_5(2) \leq \frac{1}{6!}(1)(1)^6 = \frac{1}{6!}$$

And  $\frac{1}{6!} \approx 0.0014$ . Be very careful with positives and negatives here :)

We don't actually have to compute  $T_5(x)$ , but if you want to as an exercise, click [here](#) to see the result.









For some  $c$  in  $(0, 1)$ :



A

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?





## WHICH DEGREE?

Suppose you want to approximate  $e^5$  using a Maclaurin polynomial of  $f(x) = e^x$ . If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?





Suppose you want to approximate  $\log \frac{4}{3}$  using a Taylor polynomial of  $f(x) = \log x$  centred at  $a = 1$ . If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?





Let  $f(x) = \sqrt[4]{x}$ . Suppose you use a second-degree Taylor polynomial of  $f(x)$  centered at  $a = 81$  to approximate  $\sqrt[4]{81.2}$ . Bound your error, and tell whether  $T_2(10)$  is an overestimate or underestimate.



## Included Work



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