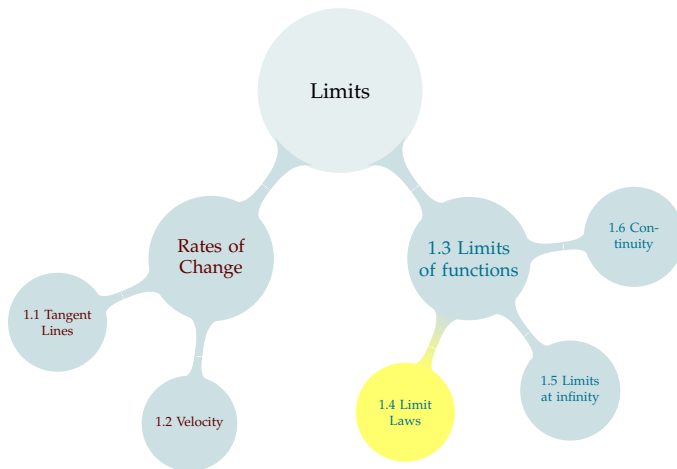


# TABLE OF CONTENTS



# CALCULATING LIMITS IN SIMPLE SITUATIONS

## Direct Substitution – Theorem 1.4.10

If  $f(x)$  is a polynomial or rational function, and  $a$  is in the domain of  $f$ , then:

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Calculate:  $\lim_{x \rightarrow 3} \left( \frac{x^2 - 9}{x + 3} \right)$

Calculate:  $\lim_{x \rightarrow 3} \left( \frac{x^2 - 9}{x - 3} \right)$

## Algebra with Limits: Theorem 1.4.2

Suppose  $\lim_{x \rightarrow a} f(x) = F$  and  $\lim_{x \rightarrow a} g(x) = G$ , where  $F$  and  $G$  are both real numbers. Then:

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$
- $\lim_{x \rightarrow a} (f(x)g(x)) = FG$
- $\lim_{x \rightarrow a} (f(x)/g(x)) = F/G$  provided  $G \neq 0$

Calculate:  $\lim_{x \rightarrow 1} \left[ \frac{2x+4}{x+2} + 13 \left( \frac{x+5}{3x} \right) \left( \frac{x^2}{2x-1} \right) \right]$

# LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A.  $4^{\frac{1}{2}}$

B.  $(-4)^{\frac{1}{2}}$

C.  $4^{-\frac{1}{2}}$

D.  $(-4)^{-\frac{1}{2}}$

E.  $8^{1/3}$

F.  $(-8)^{1/3}$

G.  $8^{-1/3}$

H.  $(-8)^{-1/3}$

## 1.4 Calculating Limits with Limit Laws

### Limits involving Powers and Roots

#### LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A.  $4^{\frac{1}{2}}$

B.  $(-4)^{\frac{1}{2}}$

C.  $4^{-\frac{1}{2}}$

D.  $(-4)^{-\frac{1}{2}}$

E.  $8^{1/3}$

F.  $(-8)^{1/3}$

G.  $8^{-1/3}$

H.  $(-8)^{-1/3}$

Raise your hand if you think A is real / not; Raise your hand if you think B is real/not; etc. Ask students to turn to their neighbours and describe a rule for when  $A^B$  is real, and when it is not.

## Powers of Limits – Theorem 1.4.8

If  $n$  is a positive integer, and  $\lim_{x \rightarrow a} f(x) = F$  (where  $F$  is a real number), then:

$$\lim_{x \rightarrow a} (f(x))^n = F^n.$$

Furthermore, **unless**  $n$  is even and  $F$  is negative,

$$\lim_{x \rightarrow a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x \rightarrow 4} (x + 5)^{1/2}$$

## 1.4 Calculating Limits with Limit Laws

### Powers of Limits - Theorem 1.4.8

If  $n$  is a positive integer, and  $\lim_{x \rightarrow a} f(x) = F$  (where  $F$  is a real number), then:

$$\lim_{x \rightarrow a} (f(x))^n = F^n.$$

Furthermore, unless  $n$  is even and  $F$  is negative,

$$\lim_{x \rightarrow a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x \rightarrow a} (x + 5)^{1/2}$$

Takeaway: when calculating limits, you can start by trying to "plug in." But, the MINUTE you divide by zero, or see 0/0, or if infinity shows up anywhere, YOU NEED TO DO SOMETHING ELSE

## CAUTIONARY TALES

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{(5+x)^2 - 25}{x}$$

$$\blacktriangleright \lim_{x \rightarrow 3} \left( \frac{x-6}{3} \right)^{1/8}$$

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{32}{x}$$

$$\blacktriangleright \lim_{x \rightarrow 5} (x^2 + 2)^{1/3}$$



Suppose you want to evaluate  $\lim_{x \rightarrow 1} f(x)$ , but  $f(1)$  doesn't exist. What does that tell you?

- A  $\lim_{x \rightarrow 1} f(x)$  may exist, and it may not exist.
- B We can find  $\lim_{x \rightarrow 1} f(x)$  by plugging in 1 to  $f(x)$ .
- C Since  $f(1)$  doesn't exist, it is not meaningful to talk about  $\lim_{x \rightarrow 1} f(x)$ .
- D Since  $f(1)$  doesn't exist, automatically we know  $\lim_{x \rightarrow 1} f(x)$  does not exist.
- E  $\lim_{x \rightarrow 1} f(x)$  does not exist if we are "dividing by zero," but may exist otherwise.

## 1.4 Calculating Limits with Limit Laws

Suppose you want to evaluate  $\lim_{x \rightarrow 1} f(x)$ , but  $f(1)$  doesn't exist. What does that tell you?

A  $\lim_{x \rightarrow 1} f(x)$  may exist, and it may not exist.

B We can find  $\lim_{x \rightarrow 1} f(x)$  by plugging in 1 to  $f(x)$ .

C Since  $f(1)$  doesn't exist, it is not meaningful to talk about  $\lim_{x \rightarrow 1} f(x)$ .

D Since  $f(1)$  doesn't exist, automatically we know  $\lim_{x \rightarrow 1} f(x)$  does not exist.

E  $\lim_{x \rightarrow 1} f(x)$  does not exist if we are "dividing by zero," but may exist otherwise.

We're identifying things that make limits harder to find. A limit being hard to find is not the same as the limit not existing, it just means you have to look harder. In a moment, we'll talk about what to do in these cases.

Which of the following statements is true about  $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x}$ ?

A  $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 0}{0^3 - 0^2 + 0} = \frac{0}{0}$

B Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of  $\frac{\sin x}{x^3 - x^2 + x}$  are both 0 when  $x = 0$ , the limit exists.

D Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not defined at 0, plugging in  $x = 0$  will not tell us the limit.

E Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about  $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x}$ ?

A  $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not rational, its limit at 1 does not exist.

C Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not defined at 1, plugging in  $x = 1$  will not tell us the limit.

D Since the numerator and denominator of  $\frac{\sin x}{x^3 - x^2 + x}$  are both 0 when  $x = 1$ , the limit exists.

## 1.4 Calculating Limits with Limit Laws

Which of the following statements is true about  $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x}$ ?

A  $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not rational, its limit at 1 does not exist.

C Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not defined at 1, plugging in  $x = 1$  will not tell us the limit.

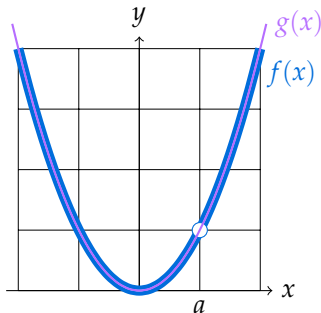
D Since the numerator and denominator of  $\frac{\sin x}{x^3 - x^2 + x}$  are both 0 when  $x = 1$ , the limit exists.

We're identifying things that make limits harder to find. A limit being hard to find is not the same as the limit not existing. We'll talk now about more things you can do to evaluate a limit in these trickier situations.

## Functions that Differ at a Single Point – Theorem 1.4.12

Suppose  $\lim_{x \rightarrow a} g(x)$  exists, and  $f(x) = g(x)$   
when  $x$  is close to  $a$  (but not necessarily equal to  $a$ ).

Then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ .



Evaluate  $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x - 1}$ .

Evaluate  $\lim_{x \rightarrow 5} \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$



# A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can **directly substitute** (plug in). If your function is made up of the **sum, difference, product, quotient, or power of polynomials**, you can do this **provided** the function exists where you're taking the limit.

$$\lim_{x \rightarrow 1} \left( \sqrt{35 + x^5} + \frac{x - 3}{x^2} \right)^3 =$$

## └ 1.4 Calculating Limits with Limit Laws

### └ A Few Strategies for Calculating Limits

#### A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can **directly substitute** (plug in). If your function is made up of the **sum, difference, product, quotient, or power of polynomials**, you can do this **provided** the function exists where you're taking the limit.

$$\lim_{x \rightarrow 1} \left( \sqrt{35 + x^2} + \frac{x-3}{x^2} \right)^5 =$$

Verbally: Separate piecewise functions from everything else. It's likely that the only times they'll see functions with discontinuities inside their domains, they will be written piecewise.

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to **simplify and cancel**.

$$\lim_{x \rightarrow 0} \frac{x + 7}{\frac{1}{x} - \frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

## DENOMINATORS APPROACHING ZERO

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}$$

## 1.4 Calculating Limits with Limit Laws

### └ Denominators Approaching Zero

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1} \frac{-1}{(x-1)^3}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}$$

Good to remind students about division at this point. A small number goes into any number lots of times: if I have one cake, and I cut it into tiny pieces, I get a lot of pieces. They often think these limits have some kind of unknowable magic to them, so it's good to bring them back to a place where things make intuitive sense.

# DENOMINATORS APPROACHING ZERO

NOW  
YOU



$$\lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4}$$

$$\lim_{x \rightarrow 2^-} \frac{x}{4 - x^2}$$

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$$

## Squeeze Theorem – Theorem 1.4.17

Suppose, when  $x$  is near (but not necessarily equal to)  $a$ , we have functions  $f(x)$ ,  $g(x)$ , and  $h(x)$  so that

$$f(x) \leq g(x) \leq h(x)$$

and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$ . Then  $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x)$ .

$$\lim_{x \rightarrow 0} x^2 \sin \left( \frac{1}{x} \right)$$

## 1.4 Calculating Limits with Limit Laws

### Squeeze Theorem – Theorem 1.4.17

Suppose, when  $x$  is near (but not necessarily equal to)  $a$ , we have functions  $f(x)$ ,  $g(x)$ , and  $h(x)$  so that

$$f(x) \leq g(x) \leq h(x)$$

and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$ . Then  $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x)$ .

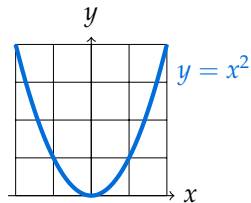
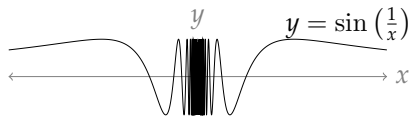
$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

Let's start by graphing the function



Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$



$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

## Included Work



'Brain' by [Eucalyp](#) is licensed under [CC BY 3.0](#) (accessed 8 June 2021), 22