

This file contains questions spanning CLP-1. It should not be taken as a complete review of the course, but rather as a jumping-off point. If you struggle with one question, go back to review its entire section. Sections are noted at the bottom of each page.

# SHORT ANSWER

S1. Find all solutions to  $x^3 - 3x^2 - x + 3 = 0$

[▶ solution S1](#)

S2. Compute the limit  $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

[▶ solution S2](#)

S3. Find all values of  $c$  such that the following function is continuous:

[▶ solution S3](#)

$$f(x) = \begin{cases} 8 - cx & \text{if } x \leq c \\ x^2 & \text{if } x > c \end{cases}$$

Use the definition of continuity to justify your answer.

S4. Compute

[▶ solution S4](#)

$$\lim_{x \rightarrow -\infty} \frac{3x+5}{\sqrt{x^2+5}-x}$$

## SHORT ANSWER

S5. Find the equation of the tangent line to the graph of  $y = \cos(x)$  at  $x = \frac{\pi}{4}$ .

[▶ solution S5](#)

S6. For what values of  $x$  does the derivative of  $\frac{\sin(x)}{x^2 + 6x + 5}$  exist?

[▶ solution S6](#)

S7. Find  $f'(x)$  if  $f(x) = (x^2 + 1)^{\sin(x)}$ .

[▶ solution S7](#)

S8. Consider a function of the form  $f(x) = Ae^{kx}$  where  $A$  and  $k$  are constants. If  $f(0) = 3$  and  $f(2) = 5$ , find the constants  $A$  and  $k$ .

[▶ solution S8](#)

## SHORT ANSWER

S9. Consider a function  $f(x)$  which has  $f'''(x) = \frac{x^3}{10 - x^2}$ . Show that when we approximate  $f(1)$  using its second Maclaurin polynomial, the absolute error is less than  $\frac{1}{50} = 0.02$ .

► solution S9

S10. Estimate  $\sqrt{35}$  using a linear approximation

► solution S10

S11. Let  $f(x) = x^2 - 2\pi x - \sin(x)$ . Show that there exists a real number  $c$  such that  $f'(c) = 0$ .

► solution S11

S12. Find the intervals where  $f(x) = \frac{\sqrt{x}}{x+6}$  is increasing.

► solution S12

# LONG ANSWER

L1. Compute the limit  $\lim_{x \rightarrow 1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$ .

[▶ solution L1](#)

L2. Show that there exists at least one real number  $c$  such that  $2 \tan(c) = c + 1$ .

[▶ solution L2](#)

L3. Determine whether the derivative of following function exists at  $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

[▶ solution L3](#)

# LONG ANSWER

L4. If  $x^2 \cos(y) + 2xe^y = 8$ , then find  $y'$  at the points where  $y = 0$ . You must justify your answer.

[▶ solution L4](#)

L5. Two particles move in the cartesian plane. Particle A travels on the  $x$ -axis starting at  $(10, 0)$  and moving towards the origin with a speed of 2 units per second. Particle B travels on the  $y$ -axis starting at  $(0, 12)$  and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point  $(4, 0)$ ?

[▶ solution L5](#)

L6. Find the global maximum and the global minimum for  $f(x) = x^3 - 6x^2 + 2$  on the interval  $[3, 5]$ .

[▶ solution L6](#)

# Solutions

---

[◀ back to questions](#)

S1.

Find all solutions to  $x^3 - 3x^2 - x + 3 = 0$

---



[◀ back to questions](#)**S1.**

Find all solutions to  $x^3 - 3x^2 - x + 3 = 0$

$$\begin{aligned}x^3 - 3x^2 - x + 3 &= x^2(x - 3) - (x - 3) \\&= (x^2 - 1)(x - 3) \\&= (x + 1)(x - 1)(x - 3)\end{aligned}$$

The solutions are  $x = 1$ ,  $x = 3$ , and  $x = -1$ .

---

[◀ back to questions](#)

S2.

Compute the limit  $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$

---

[◀ back to questions](#)

S2.

Compute the limit  $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

---

[◀ back to questions](#)

S3.

Find all values of  $c$  such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if } x \leq c \\ x^2 & \text{if } x > c \end{cases}$$

Use the definition of continuity to justify your answer.

---

[◀ back to questions](#)

S3.

Find all values of  $c$  such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if } x \leq c \\ x^2 & \text{if } x > c \end{cases}$$

Use the definition of continuity to justify your answer.

When  $x \neq c$ ,  $f(x)$  is continuous. The only difficult spot is when  $x = c$ .

- ▶  $f(c) = 8 - c^2$
- ▶  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^-} (8 - cx) = 8 - c^2$
- ▶  $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^+} (x^2) = c^2$

Since  $f(x)$  is continuous at  $c$  only if  $f(c) = \lim_{x \rightarrow c} f(x)$ , we see the only values of  $c$  that make  $f$  continuous are those that satisfy  $c^2 = 8 - c^2$ . That is,  $c = \pm 2$ .

[◀ back to questions](#)

S4.

Compute

$$\lim_{x \rightarrow -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x}$$

[◀ back to questions](#)

S4.

Compute

$$\lim_{x \rightarrow -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x}$$

We start by factoring out  $x$  from both the top and the bottom.

$$\lim_{x \rightarrow -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x} \left( \frac{1/x}{1/x} \right) = \lim_{x \rightarrow -\infty} \frac{3 + \frac{5}{x}}{\frac{1}{x} \sqrt{x^2 + 5} - 1}$$

Since  $x$  is approaching negative infinity, we can assume  $x < 0$ . Then  $x = -|x| = -\sqrt{x^2}$ . We'll use this form to push the  $\frac{1}{x}$  into the square root.

$$\begin{aligned} &= \lim_{x \rightarrow -\infty} \frac{3 + \frac{5}{x}}{-\frac{1}{\sqrt{x^2}} \sqrt{x^2 + 5} - 1} \\ &= \lim_{x \rightarrow -\infty} \frac{3 + \frac{5}{x}}{-\sqrt{1 + \frac{5}{x^2}} - 1} = \frac{3 + 0}{-\sqrt{1} - 1} = -\frac{3}{2} \end{aligned}$$

---

[◀ back to questions](#)

S5.

Find the equation of the tangent line to the graph of  $y = \cos(x)$  at  $x = \frac{\pi}{4}$ .

---



[◀ back to questions](#)

S5.

Find the equation of the tangent line to the graph of  $y = \cos(x)$  at  $x = \frac{\pi}{4}$ .

$$\begin{aligned}f(x) &= \cos x & f\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} \\f'(x) &= -\sin x & f'\left(\frac{\pi}{4}\right) &= -\frac{1}{\sqrt{2}}\end{aligned}$$

Recall the equation of the tangent line to  $y = f(x)$  at  $x = a$  is  
 $y = f(a) + f'(a)(x - a)$

$$y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \left(x - \frac{\pi}{4}\right)$$

---

[◀ back to questions](#)

S6.

For what values of  $x$  does the derivative of  $\frac{\sin(x)}{x^2 + 6x + 5}$  exist?

---

[◀ back to questions](#)

S6.

For what values of  $x$  does the derivative of  $\frac{\sin(x)}{x^2 + 6x + 5}$  exist?

First, note that  $x^2 + 6x + 5 = (x + 1)(x + 5)$ , so the function does not exist at either  $x = -1$  or  $x = -5$ . For other values of  $x$ , using the quotient rule, we see

$$f'(x) = \frac{(x^2 + 6x + 5)(\cos x) - \sin x(2x + 6)}{(x^2 + 6x + 5)^2}$$

which exists over the domain of  $f(x)$ .

All together, the derivative exists for all values of  $x$  except  $-1$  and  $-5$ .

---

[◀ back to questions](#)

S7.

Find  $f'(x)$  if  $f(x) = (x^2 + 1)^{\sin(x)}$ .

---

[◀ back to questions](#)

S7.

Find  $f'(x)$  if  $f(x) = (x^2 + 1)^{\sin(x)}$ .

$f(x)$  is neither an exponential function (with a constant base) nor a power function (with a constant power). When we see a function raised to a function, we differentiate using logarithmic differentiation.

$$f(x) = (x^2 + 1)^{\sin(x)}$$

$$\log f(x) = \log \left[ (x^2 + 1)^{\sin(x)} \right] = \sin x \cdot \log(x^2 + 1)$$

$$\frac{f'(x)}{f(x)} = \sin x \cdot \frac{2x}{x^2 + 1} + \cos x \cdot \log(x^2 + 1)$$

$$\begin{aligned} f'(x) &= f(x) \left[ \sin x \cdot \frac{2x}{x^2 + 1} + \cos x \cdot \log(x^2 + 1) \right] \\ &= (x^2 + 1)^{\sin x} \left[ \frac{2x \sin x}{x^2 + 1} + \cos x \cdot \log(x^2 + 1) \right] \end{aligned}$$

---

[◀ back to questions](#)**S8.**

Consider a function of the form  $f(x) = Ae^{kx}$  where  $A$  and  $k$  are constants. If  $f(0) = 3$  and  $f(2) = 5$ , find the constants  $A$  and  $k$ .

---

---

[◀ back to questions](#)**S8.**

Consider a function of the form  $f(x) = Ae^{kx}$  where  $A$  and  $k$  are constants. If  $f(0) = 3$  and  $f(2) = 5$ , find the constants  $A$  and  $k$ .

---

$$3 = f(0) = Ae^0 = A$$

$$5 = f(2) = 3e^{2k} \implies \frac{5}{3} = e^{2k}$$

$$\log_e \left( \frac{5}{3} \right) = 2k$$

$$k = \frac{\log_e(5/3)}{2}$$

---

[◀ back to questions](#)

S9.

Consider a function  $f(x)$  which has  $f'''(x) = \frac{x^3}{10 - x^2}$ . Show that when we approximate  $f(1)$  using its second Maclaurin polynomial, the absolute error is less than  $\frac{1}{50} = 0.02$ .

---



[◀ back to questions](#)

S9.

Consider a function  $f(x)$  which has  $f'''(x) = \frac{x^3}{10 - x^2}$ . Show that when we approximate  $f(1)$  using its second Maclaurin polynomial, the absolute error is less than  $\frac{1}{50} = 0.02$ .

For some  $c$  between 0 and 1:

$$\underbrace{|f(1) - T_2(1)|}_{\text{error}} = \left| \frac{f'''(c)}{3!} (1 - 0)^3 \right| = \frac{1}{6} \left| \frac{c^3}{10 - c^2} \right|$$

Since  $c$  is between 0 and 1, we note  $0 < c^3 < 1$  and  $9 < 10 - c^2 < 10$ , so:

$$|f(1) - T_2(1)| < \frac{1}{6} \left| \frac{1}{9} \right| = \frac{1}{54} < \frac{1}{50}$$

---

[◀ back to questions](#)

S10.

Estimate  $\sqrt{35}$  using a linear approximation

---

[◀ back to questions](#)

S10.

Estimate  $\sqrt{35}$  using a linear approximation

The general form of a linear approximation is

$$L(x) = f(a) + f'(a)(x - a)$$

If  $f(x) = \sqrt{x}$  and  $a = 36$ , then  $f(a) = 6$  and  $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{12}$ . So,

$$L(x) = 6 + \frac{1}{12}(x - 36)$$

Then:  $\sqrt{35} = f(35) \approx L(35) = 6 + \frac{1}{12}(35 - 36) = 6 - \frac{1}{12} = \frac{71}{12}$

---

[◀ back to questions](#)**S11.**

Let  $f(x) = x^2 - 2\pi x - \sin(x)$ . Show that there exists a real number  $c$  such that  $f'(c) = 0$ .

---

---

[◀ back to questions](#)**S11.**

Let  $f(x) = x^2 - 2\pi x - \sin(x)$ . Show that there exists a real number  $c$  such that  $f'(c) = 0$ .

---

We note that  $f(x)$  is continuous and differentiable over all real numbers. Since  $f(0) = f(2\pi) = 0$ , by Rolle's Theorem (also by the Mean Value Theorem) there exists some  $c$  between 0 and  $2\pi$  such that  $f'(c) = 0$ .

---

[◀ back to questions](#)**S12.**

Find the intervals where  $f(x) = \frac{\sqrt{x}}{x+6}$  is increasing.

---

[◀ back to questions](#)

S12.

Find the intervals where  $f(x) = \frac{\sqrt{x}}{x+6}$  is increasing.

We find where the first derivative is positive.

$$0 < f'(x) = \frac{(x+6)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+6)^2} \quad \text{multiply by } (x+6)^2$$

$$0 < (x+6)\frac{1}{2\sqrt{x}} - \sqrt{x} \quad \text{multiply by } 2\sqrt{x}$$

$$0 < (x+6) - 2x$$

$$x < 6$$

Note, however, that the function's derivative *does not exist* when  $x \leq 0$ . So the interval is  $(0, 6)$ .

[◀ back to questions](#)

L1.

Compute the limit  $\lim_{x \rightarrow 1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$ .



[◀ back to questions](#)

L1.

Compute the limit  $\lim_{x \rightarrow 1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$ .

If we try to do the limit naively we get  $0/0$ , so we simplify.

$$\begin{aligned}\frac{\sqrt{x+2} - \sqrt{4-x}}{x-1} &= \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1} \cdot \frac{\sqrt{x+2} + \sqrt{4-x}}{\sqrt{x+2} + \sqrt{4-x}} \\&= \frac{(x+2) - (4-x)}{(x-1)(\sqrt{x+2} + \sqrt{4-x})} \\&= \frac{2x-2}{(x-1)(\sqrt{x+2} + \sqrt{4-x})} \\&= \frac{2}{\sqrt{x+2} + \sqrt{4-x}} \\ \lim_{x \rightarrow 1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1} &= \lim_{x \rightarrow 1} \frac{2}{\sqrt{x+2} + \sqrt{4-x}} = \frac{2}{\sqrt{3} + \sqrt{3}} = \frac{1}{\sqrt{3}}\end{aligned}$$

---

[◀ back to questions](#)

L2.

Show that there exists at least one real number  $c$  such that  $2 \tan(c) = c + 1$ .

---

[◀ back to questions](#)

L2.

Show that there exists at least one real number  $c$  such that  $2 \tan(c) = c + 1$ .

- ▶  $\tan x$  is continuous on the interval  $(-\pi/2, \pi/2)$
- ▶  $x + 1$  is a polynomial and therefore continuous for all real numbers
- ▶ So,  $f(x) = 2 \tan(x) - x - 1$  is a continuous function on the interval  $(-\pi/2, \pi/2)$ .
- ▶ Set  $a = 0$ . Then  $a$  is in the interval  $(-\pi/2, \pi/2)$  and

$$f(a) = 2 \tan(0) - 0 - 1 = 0 - 1 = -1 < 0.$$

- ▶ Set  $b = \frac{\pi}{4}$ . Then  $b$  is in the interval  $(-\pi/2, \pi/2)$  and

$$f(b) = 2 \tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - 1 = 2 - \frac{\pi}{4} - 1 = 1 - \frac{\pi}{4} = \frac{4 - \pi}{4} > 0.$$

- ▶ All together:  $f(x)$  is continuous on  $[0, \pi/4]$ , and  $f(0) < 0$  while  $f(\pi/4) > 0$ . Then the Intermediate Value Theorem guarantees the existence of a real number  $c \in (0, \pi/4)$  such that  $f(c) = 0$ .

---

[◀ back to questions](#)**L3.**

Determine whether the derivative of following function exists at  $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

---

[◀ back to questions](#)**L3.**

Determine whether the derivative of following function exists at  $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

The function is differentiable at  $x = 0$  if the following limit exists:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x) - 0}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x}$$

Note that we used the fact that  $f(0) = 0$  following the definition of the first branch, which includes the point  $x = 0$ .

We compute left and right limits of  $\frac{f(x)}{x}$  as  $x$  goes to 0.

$$\lim_{x \rightarrow 0^-} \frac{f(x)}{x} = \lim_{x \rightarrow 0^-} \frac{2x^3 - x^2}{x} = \lim_{x \rightarrow 0^-} 2x^2 - x = 0$$

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = \lim_{x \rightarrow 0^+} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \rightarrow 0^+} x \cdot \sin\left(\frac{1}{x}\right)$$

Next use the squeeze theorem. Note, first, that  $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ , so that  $-x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq x$ . Note, second, that  $\lim_{x \rightarrow 0} x = \lim_{x \rightarrow 0} -x = 0$ . So, by the squeeze theorem,

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = \lim_{x \rightarrow 0^+} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

Since the left and right limits match (they're both equal to 0), we conclude that indeed  $f(x)$  is differentiable at  $x = 0$  (and its derivative at  $x = 0$  is actually equal to 0).

---

[◀ back to questions](#)**L4.**

If  $x^2 \cos(y) + 2xe^y = 8$ , then find  $y'$  at the points where  $y = 0$ .  
You must justify your answer.

---

[◀ back to questions](#)**L4.**

If  $x^2 \cos(y) + 2xe^y = 8$ , then find  $y'$  at the points where  $y = 0$ .  
You must justify your answer.

- First we find the  $x$ -coordinates where  $y = 0$ .

$$x^2 \cos(0) + 2xe^0 = 8$$

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

So  $x = 2, -4$ .

- Now we use implicit differentiation to get  $y'$  in terms of  $x, y$ :

$$x^2 \cos(y) + 2xe^y = 8 \quad \text{differentiate both sides}$$

$$x^2 \cdot (-\sin y) \cdot y' + 2x \cos y + 2xe^y \cdot y' + 2e^y = 0$$



- Now set  $y = 0$  to get

$$x^2 \cdot (-\sin 0) \cdot y' + 2x \cos 0 + 2xe^0 \cdot y' + 2e^0 = 0$$

$$0 + 2x + 2xy' + 2 = 0$$

$$2xy' = -(2x + 2)$$

$$y' = -\frac{1+x}{x}$$

- So at  $(x, y) = (2, 0)$  we have  $y' = -\frac{3}{2}$ ,  
► and at  $(x, y) = (-4, 0)$  we have  $y' = -\frac{3}{4}$ .

---

[◀ back to questions](#)**L5.**

Two particles move in the cartesian plane. Particle A travels on the  $x$ -axis starting at  $(10, 0)$  and moving towards the origin with a speed of 2 units per second. Particle B travels on the  $y$ -axis starting at  $(0, 12)$  and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point  $(4, 0)$ ?

---

[◀ back to questions](#)

L5.

Two particles move in the cartesian plane. Particle A travels on the  $x$ -axis starting at  $(10, 0)$  and moving towards the origin with a speed of 2 units per second. Particle B travels on the  $y$ -axis starting at  $(0, 12)$  and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point  $(4, 0)$ ?

The position of particle A along the  $x$  axis starts at  $(10, 0)$ , and moves toward the origin at 2 units per second, so its position is given by  $(x(t), 0)$  with  $x(t) = 10 - 2t$ , where  $t$  is measured in seconds.

Similarly, the position of B along the  $y$  axis is given by  $(0, y(t))$  with  $y(t) = 12 - 3t$ . The distance  $z(t)$  between the two particles satisfies  $z(t)^2 = x(t)^2 + y(t)^2$ .

When  $x(t) = 4$ , we solve  $4 = 10 - 2t$  for  $t$  and find  $t = 3$ , so  $y(3) = 12 - 3(3) = 3$ . Then  $z = 5$  when  $t = 3$ .

Differentiating implicitly,  $z(t)^2 = x(t)^2 + y(t)^2$  tells us

$$2z(t) \frac{dz}{dt}(t) = 2x(t) \frac{dx}{dt}(t) + 2y(t) \frac{dy}{dt}(t)$$

so, when  $t = 3$ ,

$$2(5) \frac{dz}{dt}(3) = 2(4)(-2) + 2(3)(-3) = -34$$

Then the distance between the two particles is changing at  $-\frac{17}{5}$  units per second.

---

[◀ back to questions](#)**L6.**

Find the global maximum and the global minimum for  $f(x) = x^3 - 6x^2 + 2$  on the interval  $[3, 5]$ .

---

[◀ back to questions](#)**L6.**

Find the global maximum and the global minimum for  $f(x) = x^3 - 6x^2 + 2$  on the interval  $[3, 5]$ .

We compute  $f'(x) = 3x^2 - 12x$ . So  $f(x)$  has no singular points (i.e. it is differentiable for all  $x$ ), but has two critical points obtained by solving

$$f'(x) = 3x(x - 4) = 0$$

which yields the two critical points  $x = 0$  and  $x = 4$ . Only the critical point  $x = 4$  is in the allowed interval  $[3, 5]$ .

In order to compute the global maximum and the global minimum for  $f(x)$  on the interval  $[3, 5]$ , we compute the value of  $f$  at the allowed critical point and at the end points of the allowed interval.

$$f(3) = -25, \quad f(4) = -30 \quad \text{and} \quad f(5) = -23.$$

So, the global max is  $-23$  while the global min is  $-30$ .