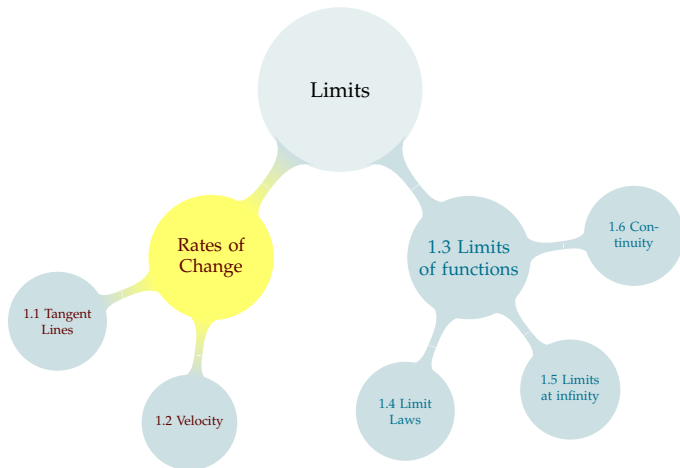


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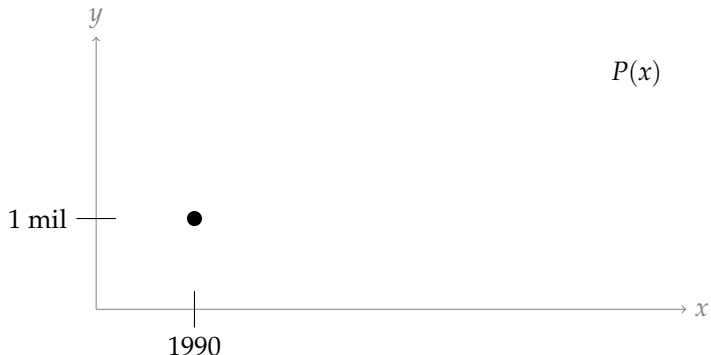


RATES OF CHANGE

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.

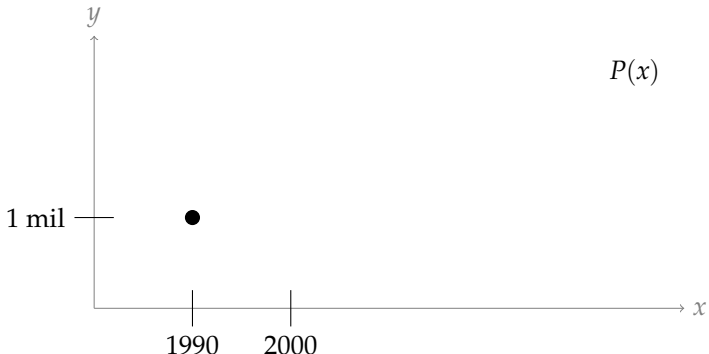
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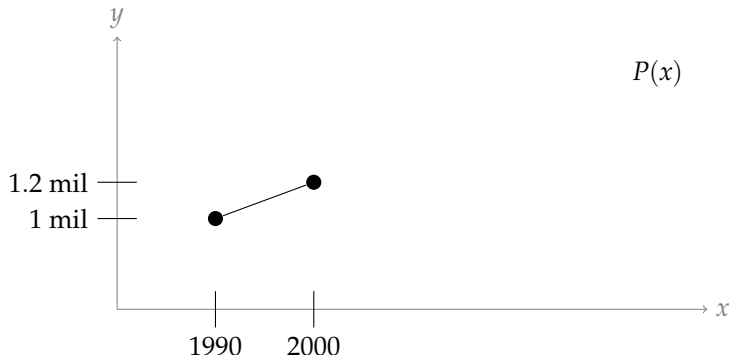
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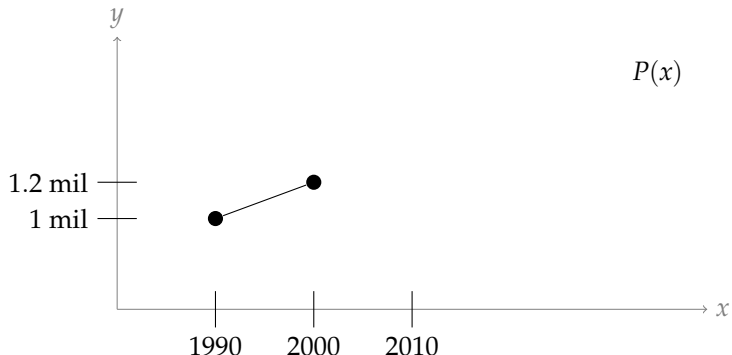
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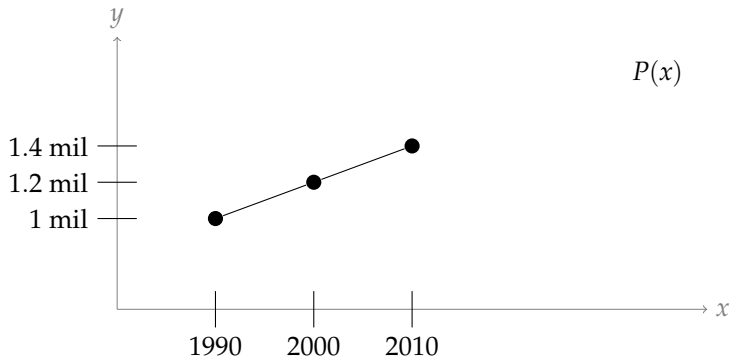
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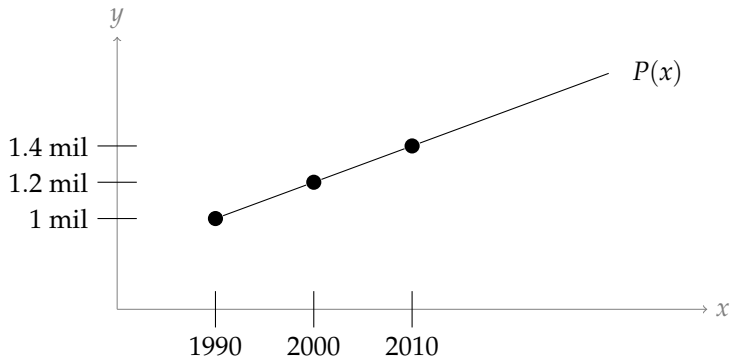
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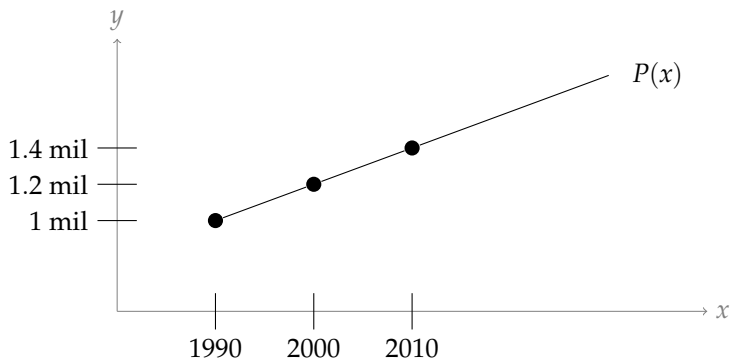
Definition

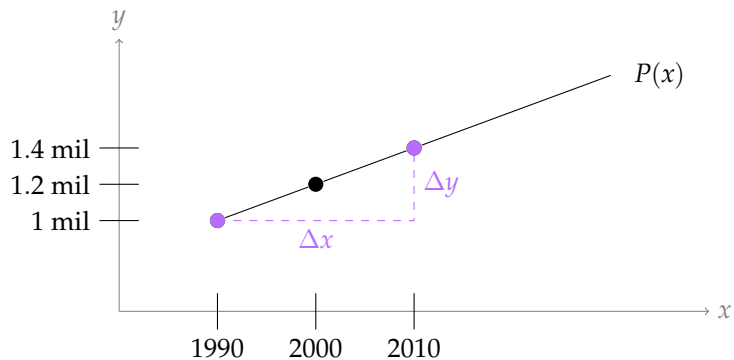
The **slope** of a line that passes through the points (x_1, y_1) and (x_2, y_2) is “rise over run”

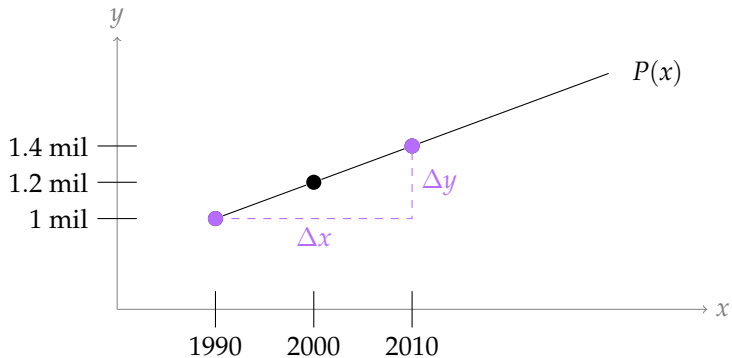
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is also called the **rate of change** of the function.

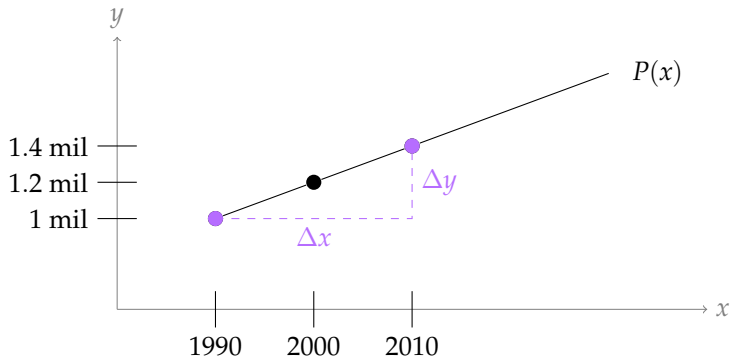
If a line has equation $y = mx + b$, its slope is m .







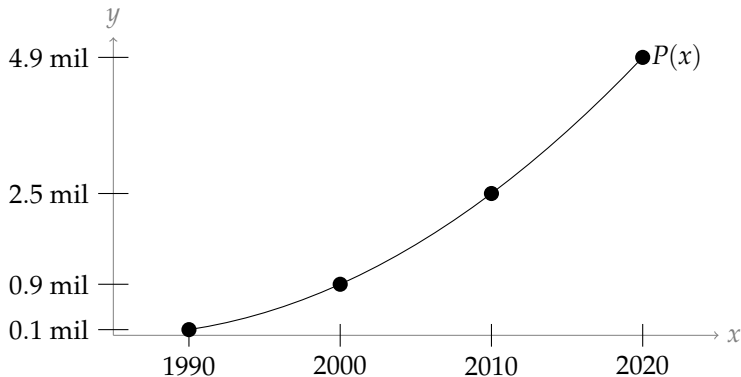
$$\text{Rate of change: } \frac{400,000 \text{ people}}{20 \text{ years}} = 20,000 \frac{\text{people}}{\text{year}}$$



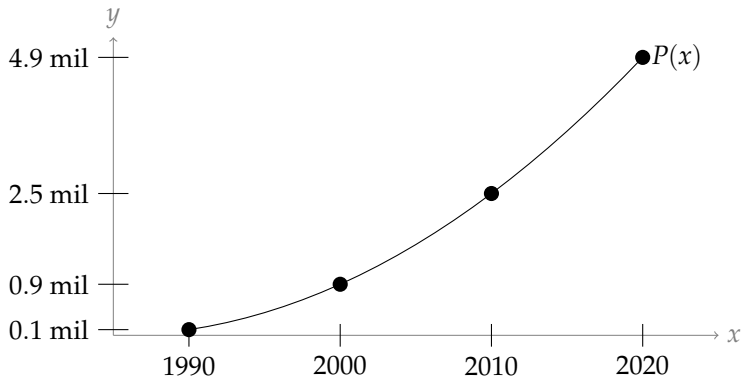
Rate of change: $\frac{400,000 \text{ people}}{20 \text{ years}} = 20,000 \frac{\text{people}}{\text{year}}$
(doesn't depend on the year)

Suppose the population of a small country is given in the chart below.

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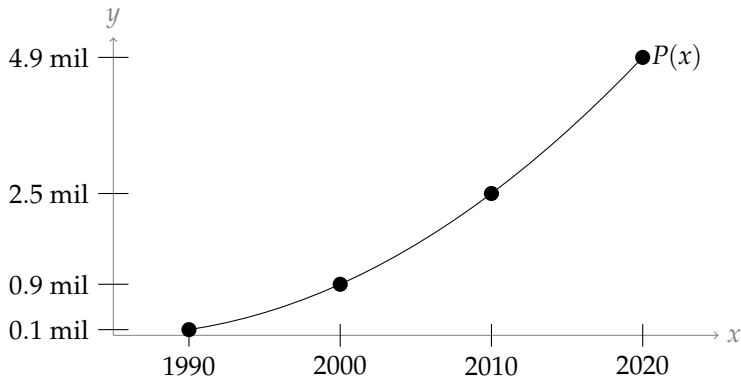


Suppose the population of a small country is given in the chart below.



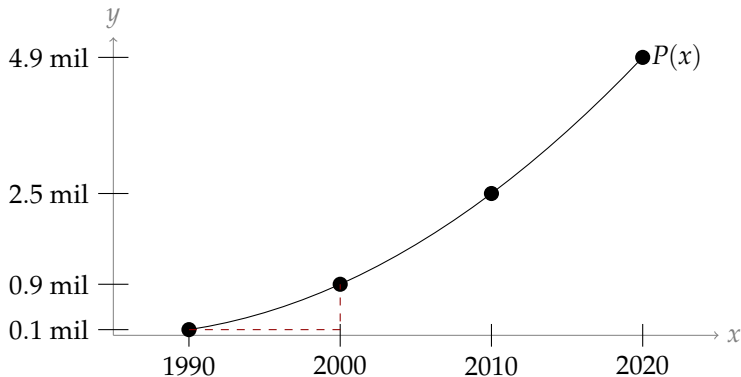
Rate of change $\frac{\Delta \text{pop}}{\Delta \text{time}}$

Suppose the population of a small country is given in the chart below.



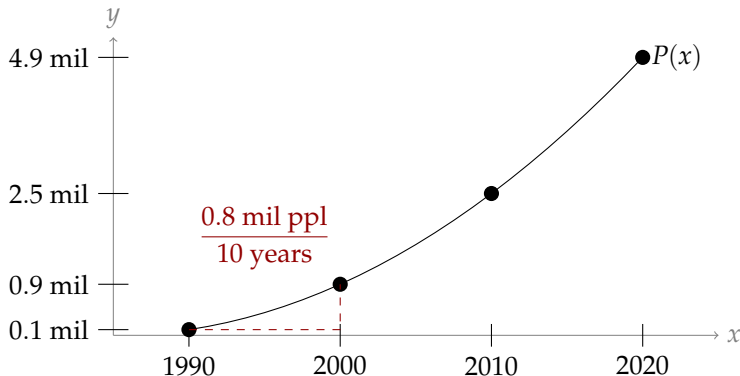
Rate of change $\frac{\Delta \text{pop}}{\Delta \text{time}}$ depends on time interval

Suppose the population of a small country is given in the chart below.



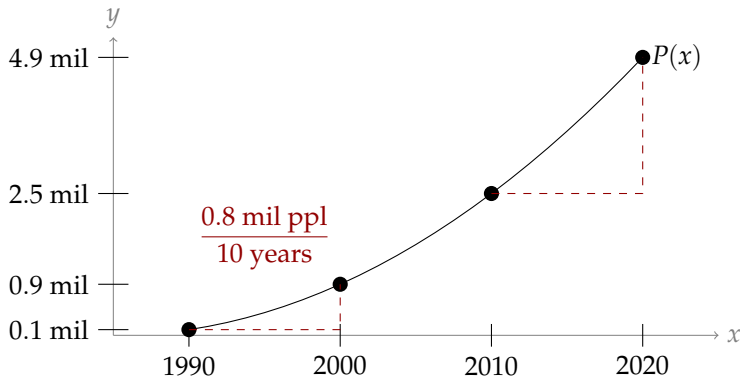
Rate of change $\frac{\Delta \text{pop}}{\Delta \text{time}}$ depends on time interval

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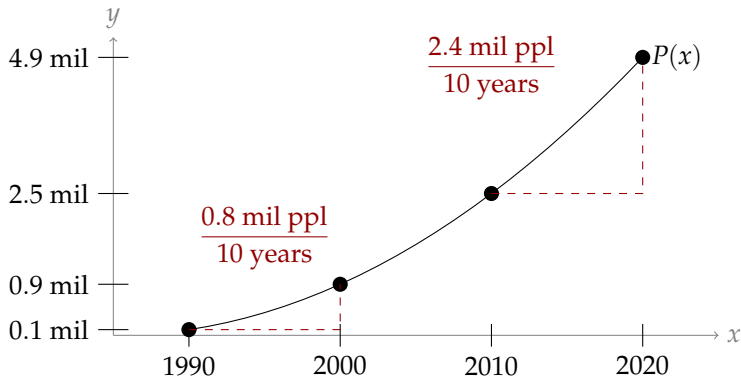
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Rate of change $\frac{\Delta \text{pop}}{\Delta \text{time}}$ depends on time interval

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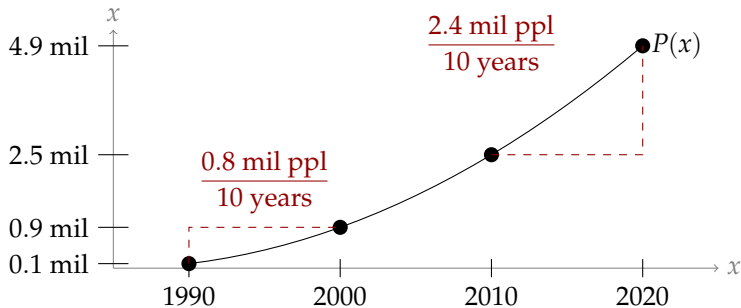


Rate of change $\frac{\Delta \text{pop}}{\Delta \text{time}}$ depends on time interval

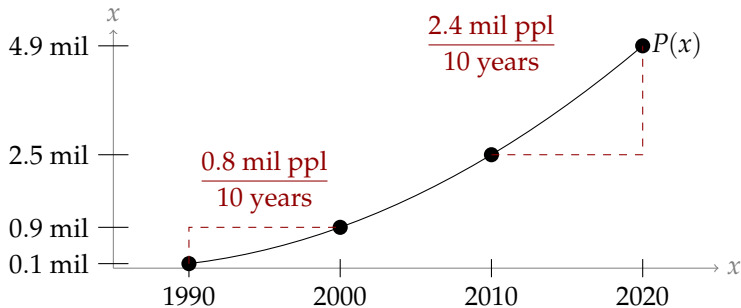
Definition

Let $y = f(x)$ be a curve that passes through (x_1, y_1) and (x_2, y_2) . Then the **average rate of change** of $f(x)$ when $x_1 \leq x \leq x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

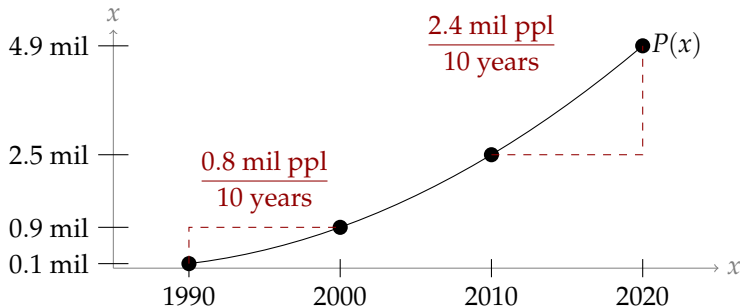


Average rate of change from 1990 to 2000:



Average rate of change from 1990 to 2000:
80,000 people per year.

Average rate of change from 2010 to 2020:



Average rate of change from 1990 to 2000:
80,000 people per year.

Average rate of change from 2010 to 2020:
240,000 people per year.

Average Rate of Change and Slope

The **average rate of change** of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is “change in output” divided by “change in input:”

$$\frac{f(b) - f(a)}{b - a}$$

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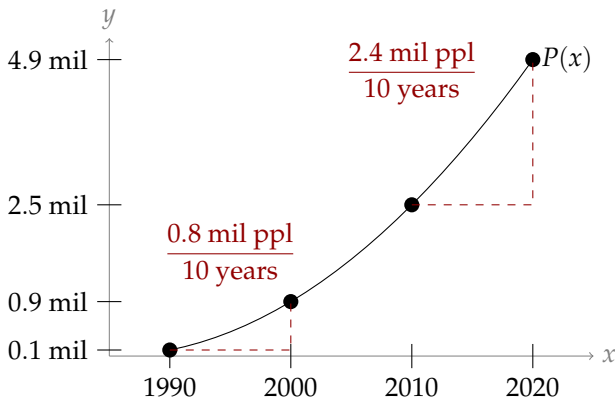
If the function $f(x)$ is a **line**, then the slope of the line is “rise over run,”

$$\frac{f(b) - f(a)}{b - a}$$

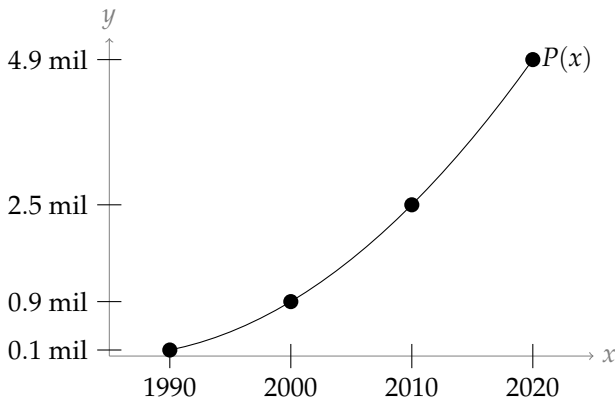
If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

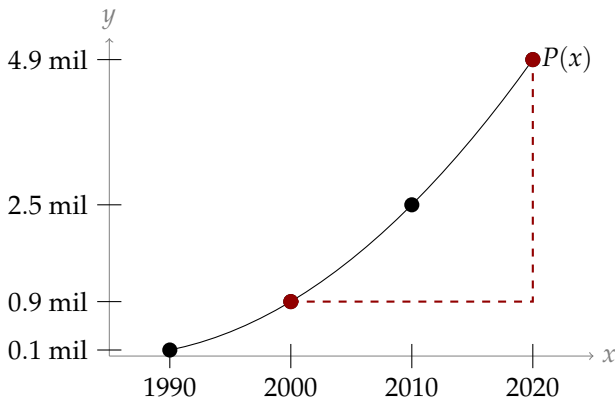
How fast was this population growing in the year 2010? (What was its **instantaneous** rate of change?)



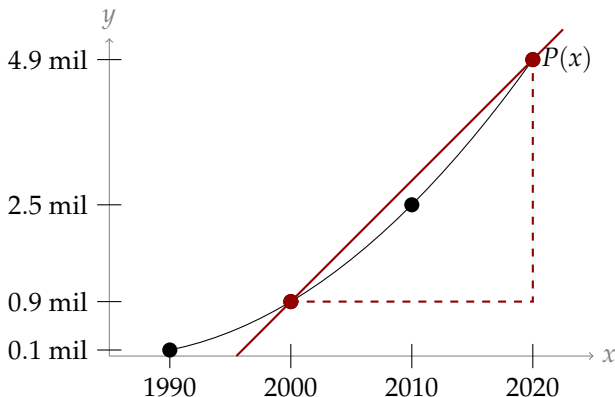
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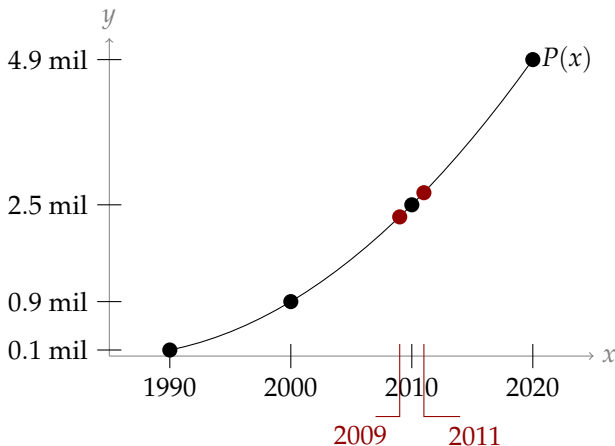
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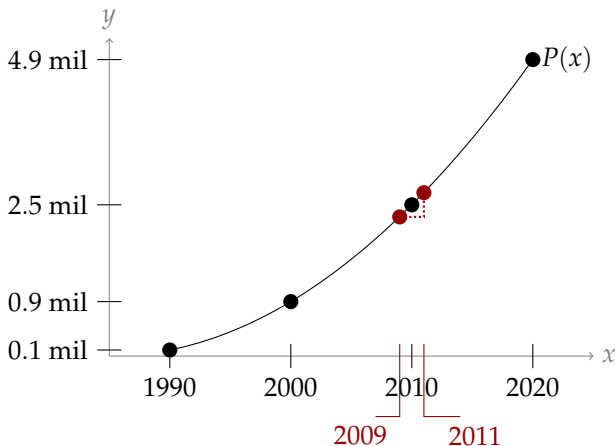
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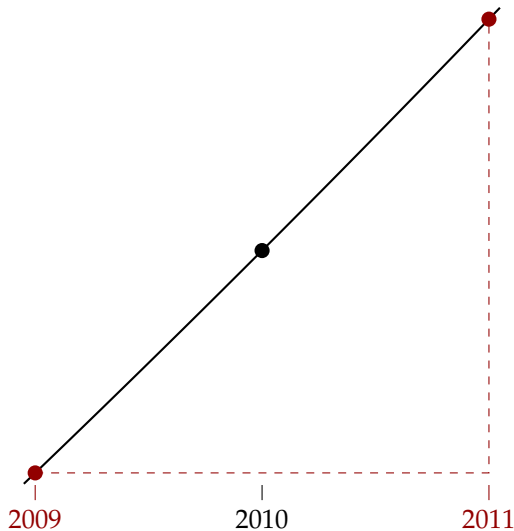
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How fast was this population growing in the year 2010?



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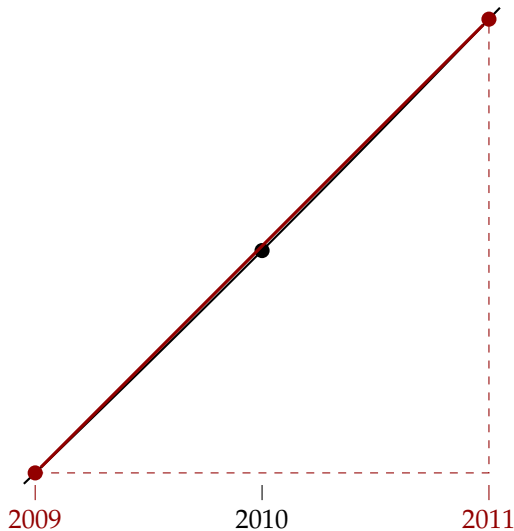
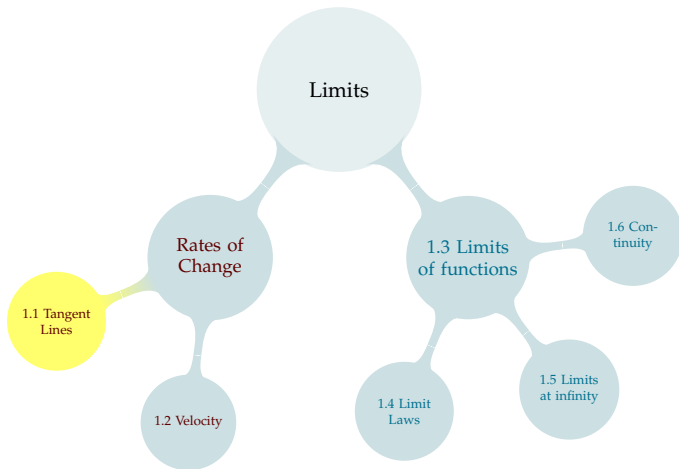
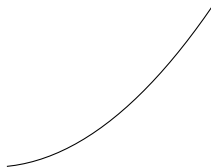


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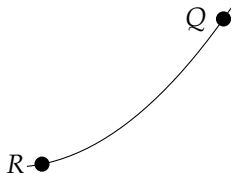
Definition

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q .



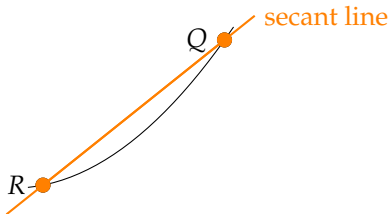
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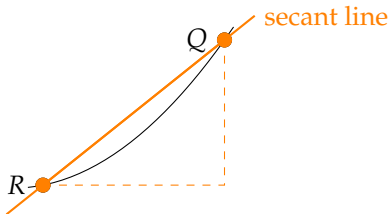
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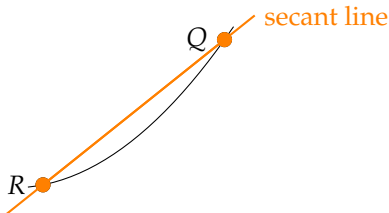
We call the slope of the secant line the **average rate of change of $f(x)$ from R to Q** .



Definition

The **tangent line** to the curve $y = f(x)$ at point P is a line that

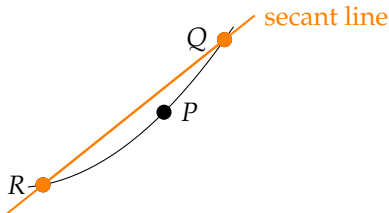
- passes through P and
- has the same slope as $f(x)$ at P .



Definition

The **tangent line** to the curve $y = f(x)$ at point P is a line that

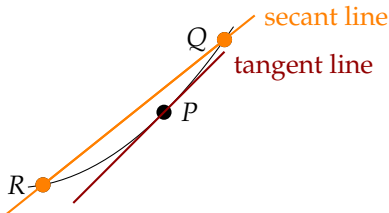
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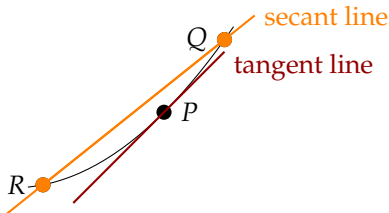


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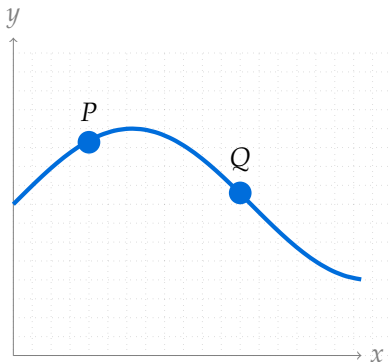
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- passes through P and
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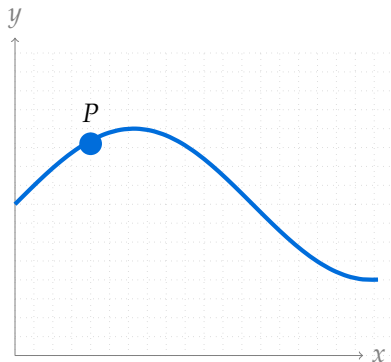
We call the slope of the tangent line the **instantaneous rate of change of $f(x)$ at P** .



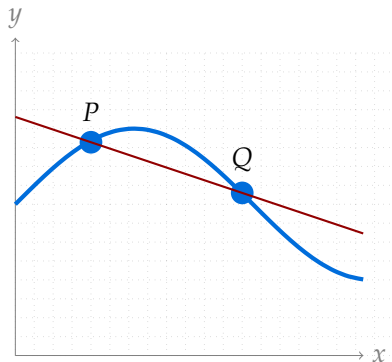
On the graph below, draw the secant line to the curve through points P and Q .



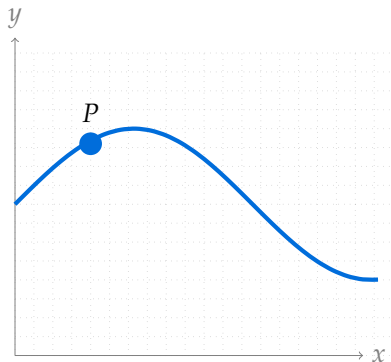
On the graph below, draw the tangent line to the curve at point P .



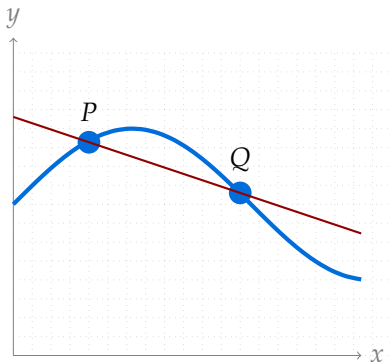
On the graph below, draw the secant line to the curve through points P and Q .



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On the graph below, draw the tangent line to the curve at point P .

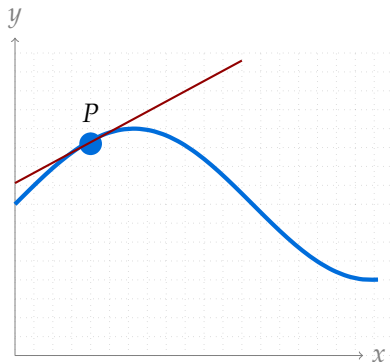
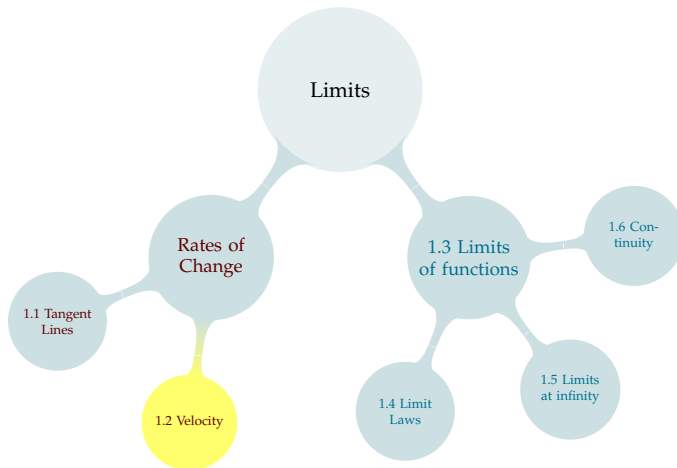
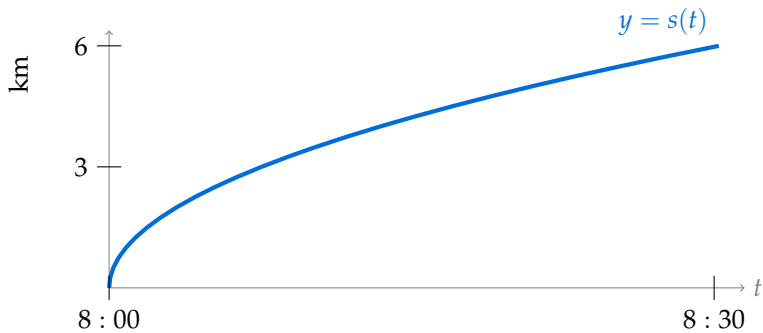
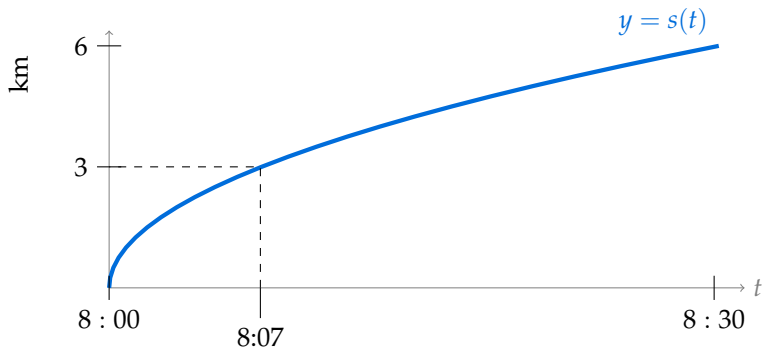
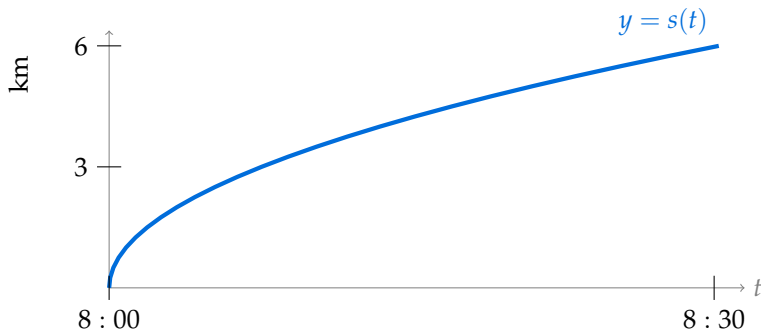


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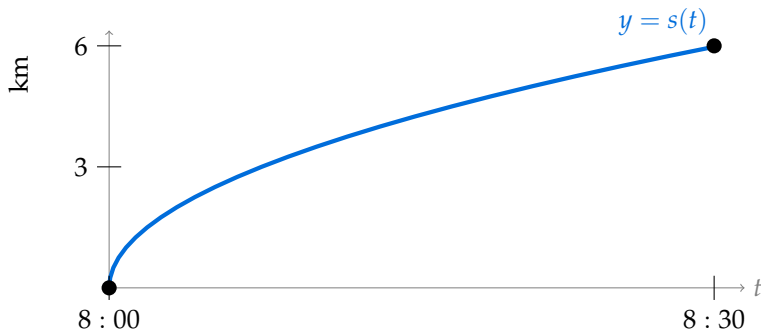






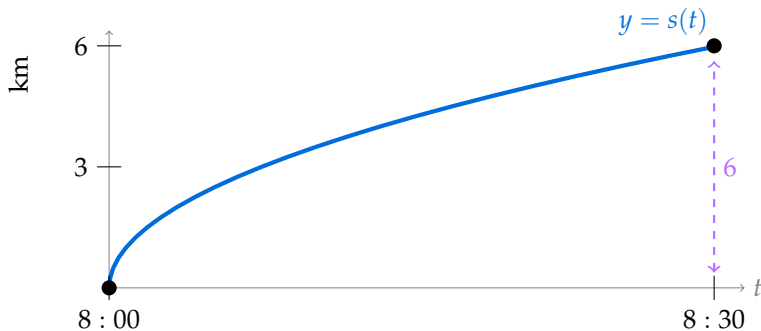
It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

- A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$
- B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$
- C. tangent line to $y = s(t)$ at $t = 8:30$
- D. slope of the tangent line to $y = s(t)$ at $t = 8:30$



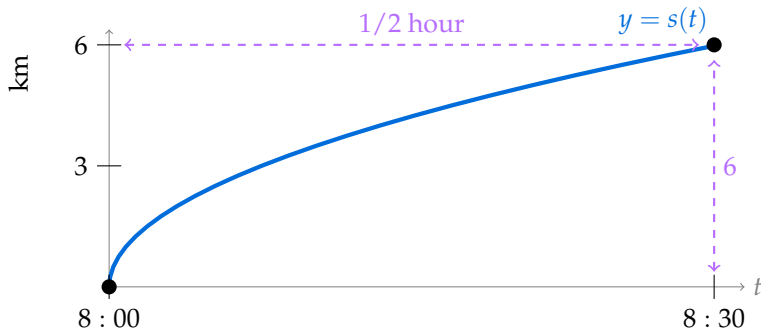
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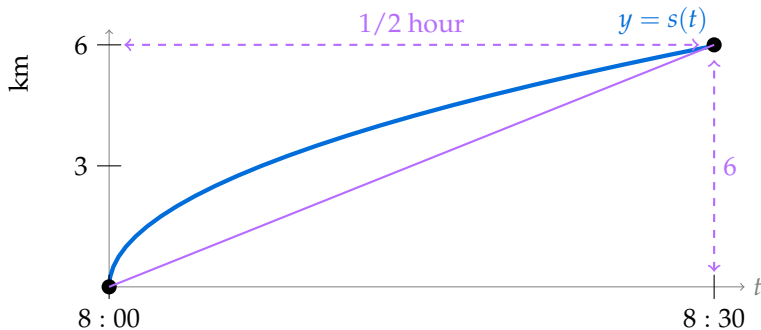
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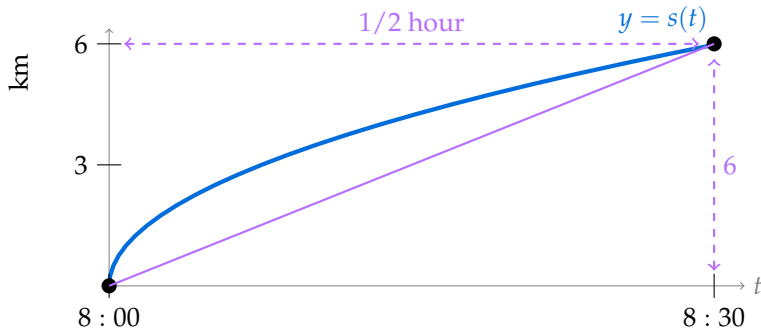
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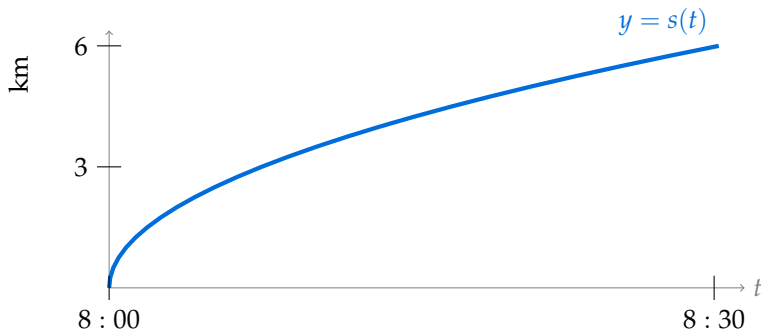
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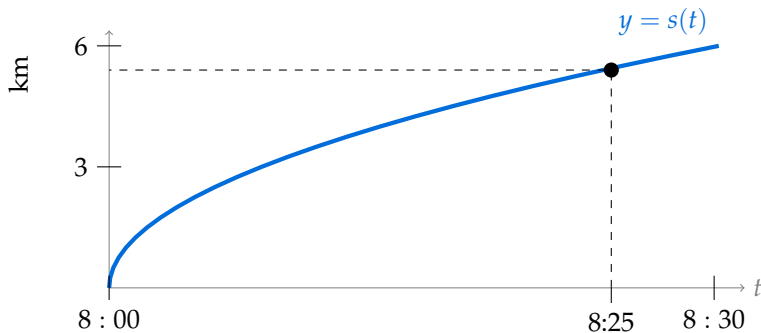
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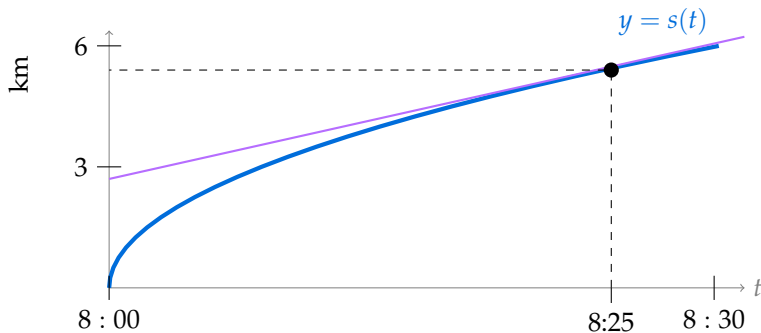
At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

- A. secant line to $y = s(t)$ from $t = 8 : 00$ to $t = 8 : 25$
- B. slope of the secant line to $y = s(t)$ from $t = 8 : 00$ to $t = 8 : 25$
- C. tangent line to $y = s(t)$ at $t = 8 : 25$
- D. slope of the tangent line to $y = s(t)$ at $t = 8 : 25$



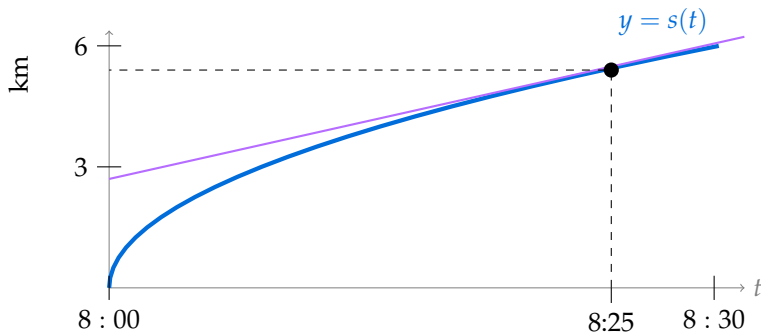
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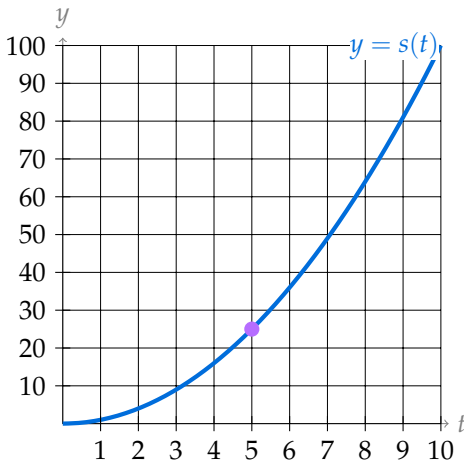


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- D. slope of the tangent line to $y = s(t)$ at $t = 8 : 25$

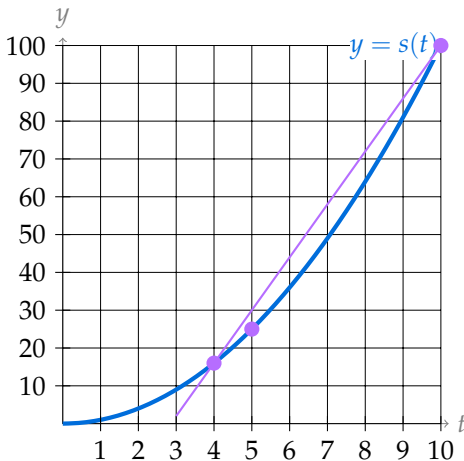
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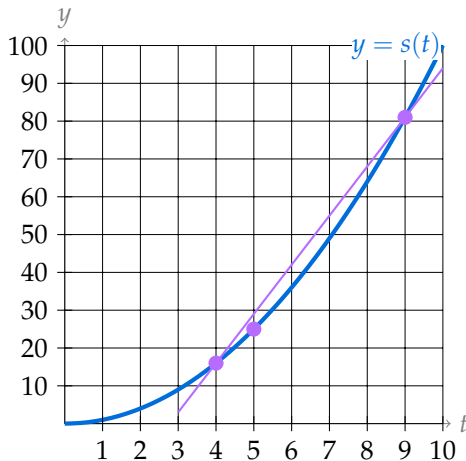
One way:
Estimate the
slope of the
tangent line to
the curve

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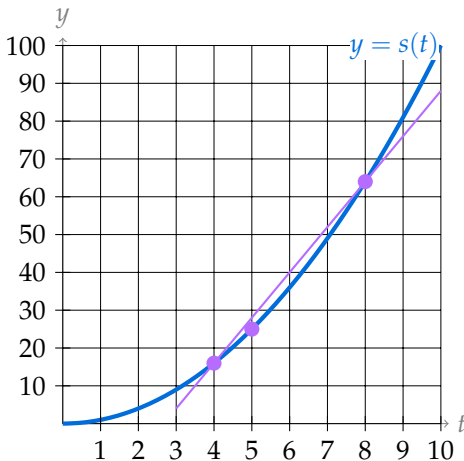
Another way:
Calculate
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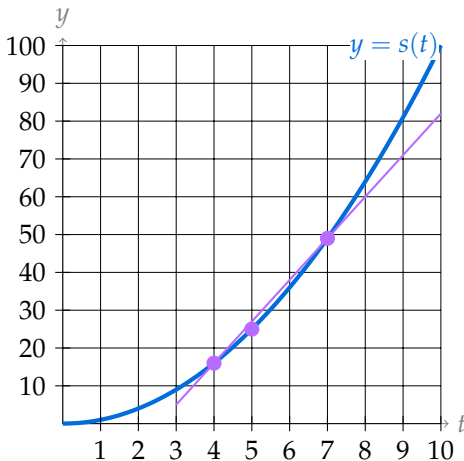
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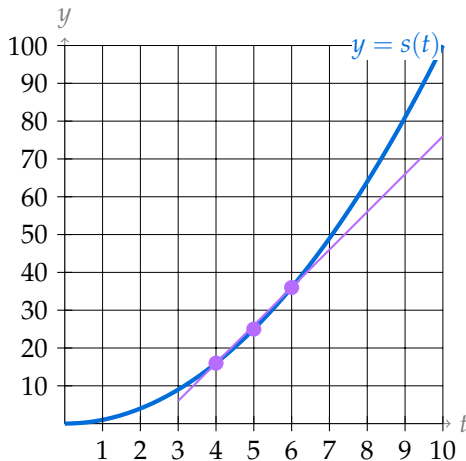
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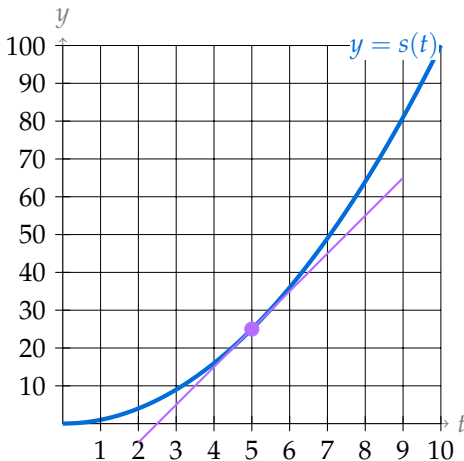
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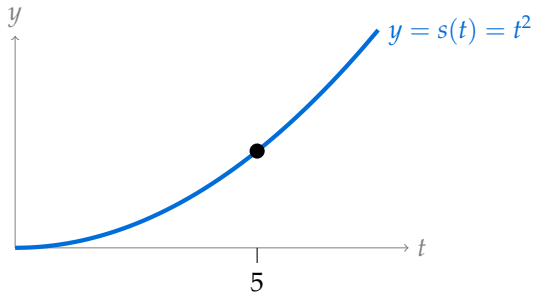
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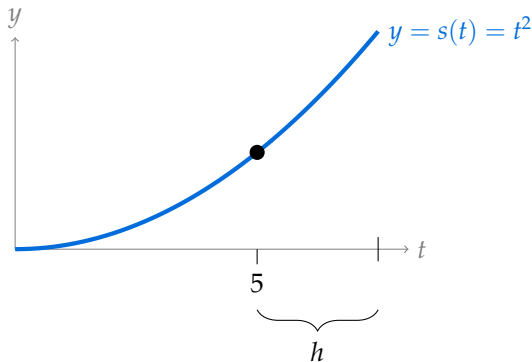


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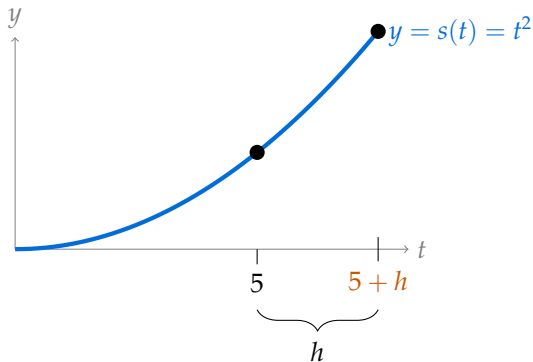
Let's look for an algebraic way of determining the velocity of the balloon when $t = 5$.



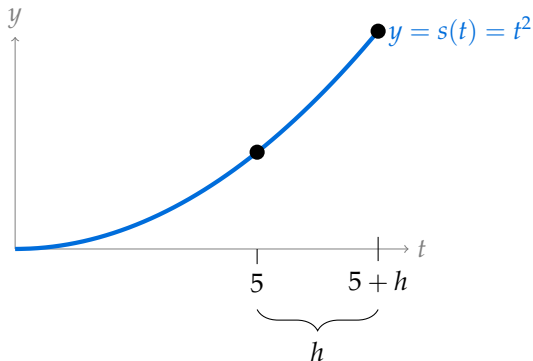
Suppose the interval $[5, \quad]$ has length h . What is the right endpoint of the interval?



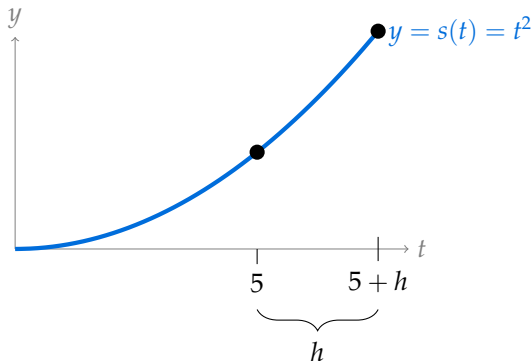
Suppose the interval $[5, \]$ has length h . What is the right endpoint of the interval?



Write the equation for the average (vertical) velocity from $t = 5$ to $t = 5 + h$.

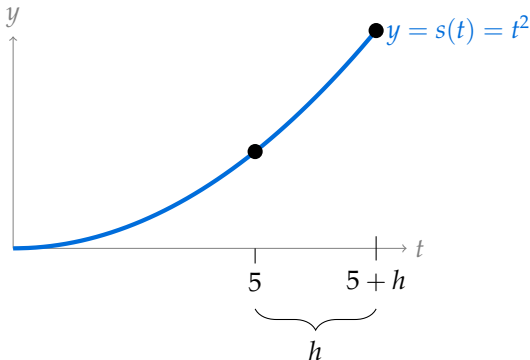


Write the equation for the average (vertical) velocity from $t = 5$ to $t = 5 + h$.



$$\text{vel} = \frac{\Delta \text{ height}}{\Delta \text{ time}} = \frac{s(5+h) - s(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

What happens to the velocity when h is very, very small?



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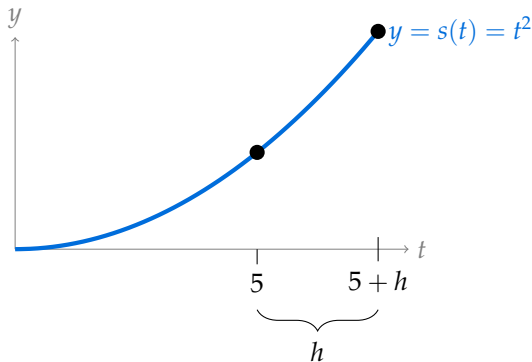
$$\begin{aligned}\text{vel} &= \frac{\Delta \text{ height}}{\Delta \text{ time}} = \frac{s(5+h) - s(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \text{ when } h \neq 0\end{aligned}$$

When h is very small,

$$\approx 10$$



What do you think is the slope of the tangent line to the graph when $t = 5$?



OUR FIRST LIMIT

Average Velocity, $t = 5$ to $t = 5 + h$:

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LIMIT NOTATION

We write:

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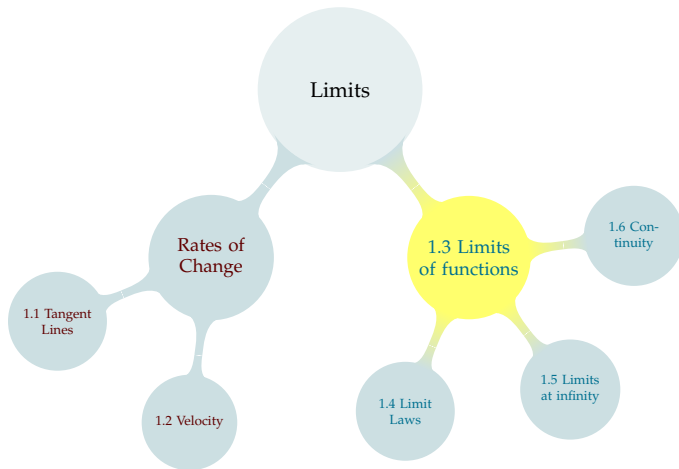
We write:

$$\lim_{h \rightarrow 0} (10 + h) = 10$$

We say: “The limit as h goes to 0 of $(10 + h)$ is 10.”

It means: As h gets extremely close to 0, $(10 + h)$ gets extremely close to 10.

TABLE OF CONTENTS



Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \rightarrow a} f(x) = L$$

where a and L are real numbers

We read the above as “the limit as x goes to a of $f(x)$ is L .”

Its meaning is: as x gets very close to (but not equal to) a , $f(x)$ gets very close to L .

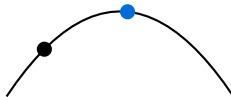
FINDING SLOPES OF TANGENT LINES

We NEED limits to find slopes of tangent lines.



FINDING SLOPES OF TANGENT LINES

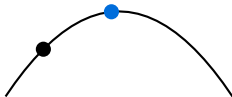
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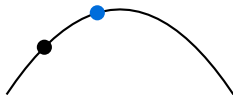
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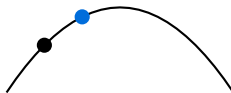
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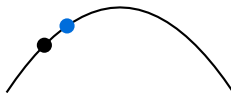
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If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

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EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \rightarrow 1} f(x)$.

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What is $f(1)$?

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$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

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What is $f(1)$? **DNE** (can't divide by zero)

EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \rightarrow 1} f(x)$.

Use the tables below to guess $\lim_{x \rightarrow 1} f(x)$

x	$f(x)$
0.9	3.61
0.99	3.9601
0.999	3.99600
0.9999	3.99960

x	$f(x)$
1.1	4.41
1.01	4.0401
1.001	4.00400
1.0001	4.00040

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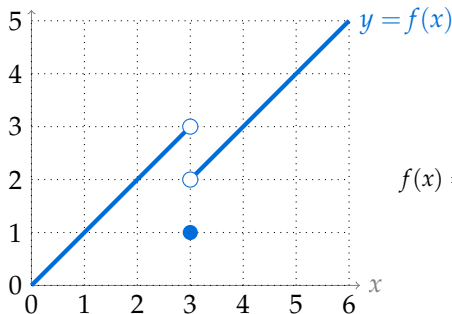
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0.999	3.99600	1.001	4.00400
0.9999	3.99960	1.0001	4.00040

$$\lim_{x \rightarrow 1} f(x) = 4$$



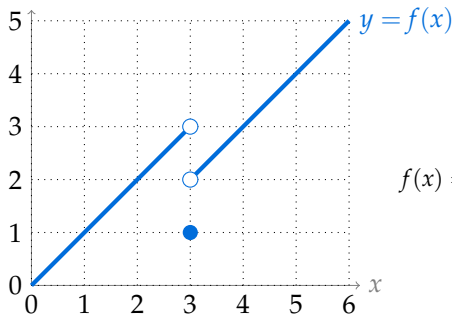
ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim_{x \rightarrow 3} f(x)$ should be?

ONE-SIDED LIMITS

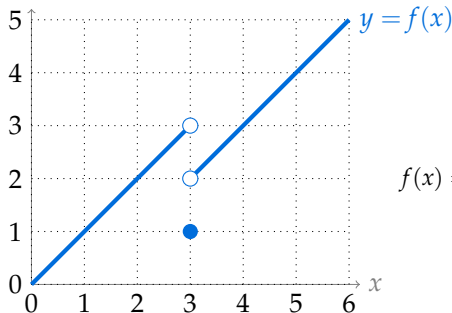


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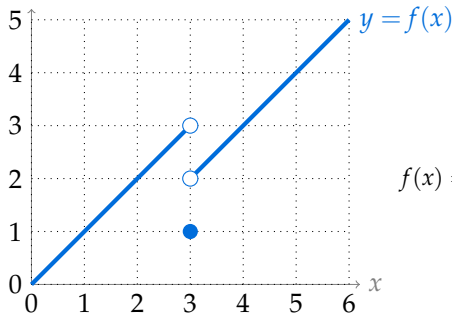
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Evaluate: $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}}$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}}$



ONE-SIDED LIMITS



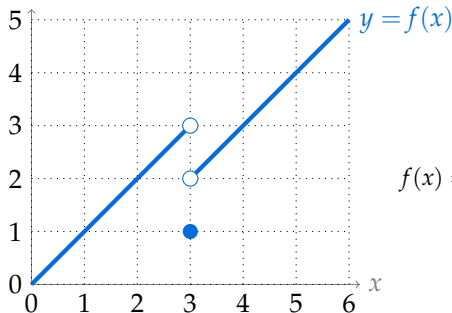
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Evaluate: $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}} = 3$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}}$



ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate: $\lim_{x \rightarrow 3^-} f(x) = 3$
 from the left

$\lim_{x \rightarrow 3^+} f(x) = 2$
 from the right



Definition 1.3.7

The limit as x goes to a **from the left** of $f(x)$ is written

$$\lim_{x \rightarrow a^-} f(x)$$

We only consider values of x that are **less than** a .

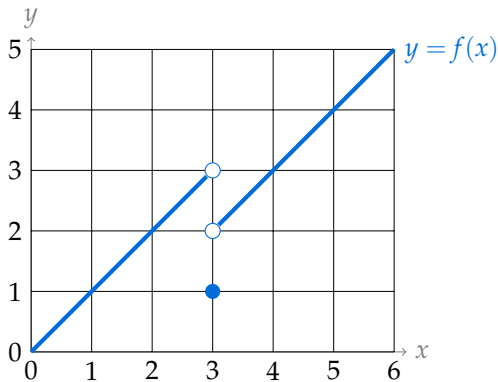
The limit as x goes to a **from the right** of $f(x)$ is written

$$\lim_{x \rightarrow a^+} f(x)$$

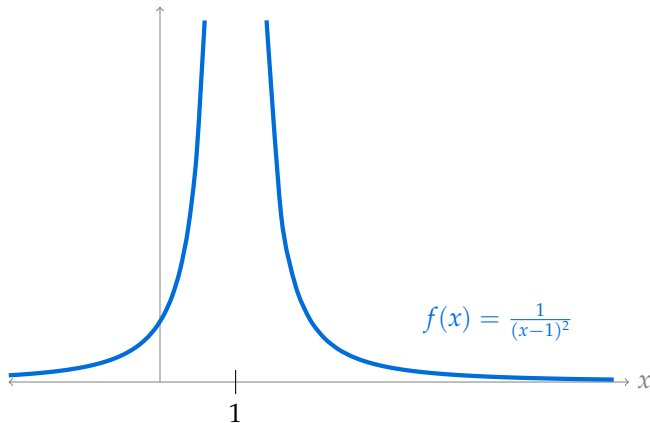
We only consider values of x **greater than** a .

Theorem 1.3.8

In order for $\lim_{x \rightarrow a} f(x)$ to exist, both one-sided limits must exist and be equal.

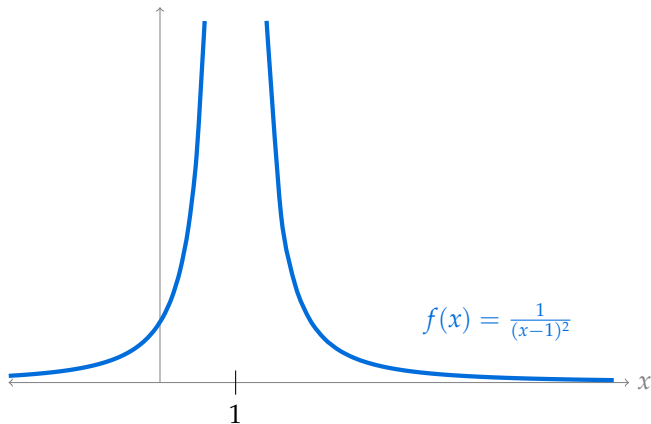


Consider the function $f(x) = \frac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ **not** defined?



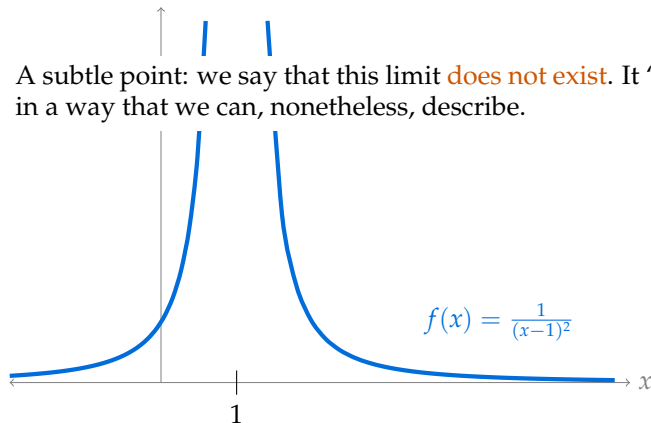
Based on the graph below, what would you like to write for:

$$\lim_{x \rightarrow 1} f(x) =$$

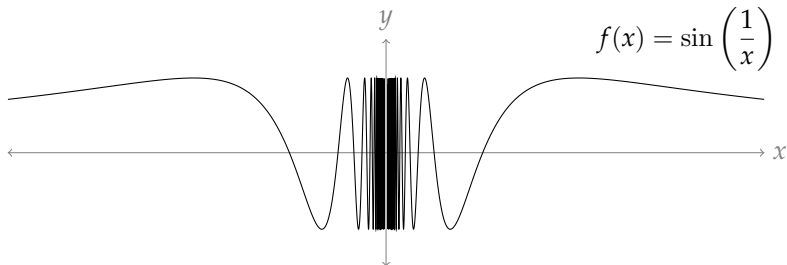


Based on the graph below, what would you like to write for:

$$\lim_{x \rightarrow 1} f(x) = \infty$$

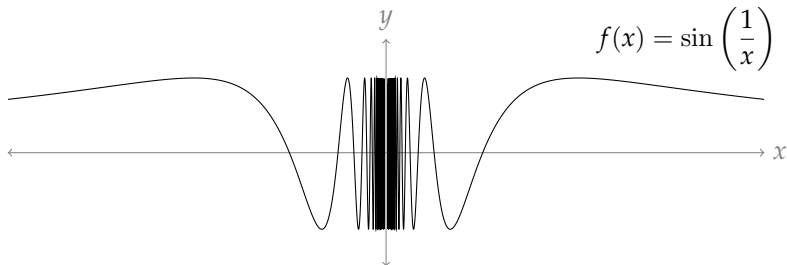


A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow \infty} f(x)$?

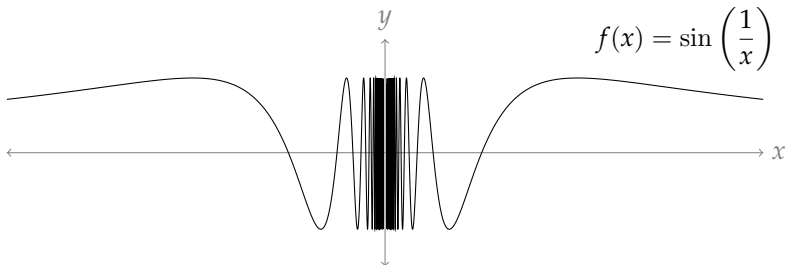
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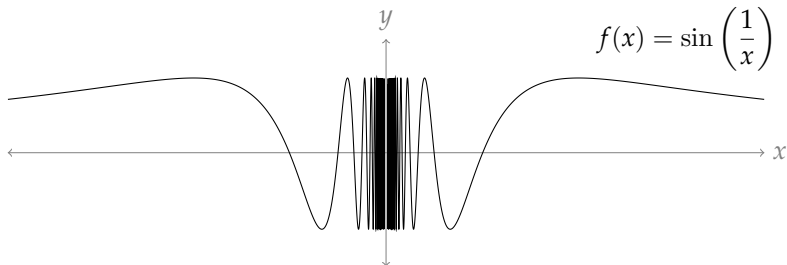
$$\lim_{x \rightarrow \infty} f(x) = 0$$

A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow 0} f(x)$?

A STRANGER LIMIT EXAMPLE

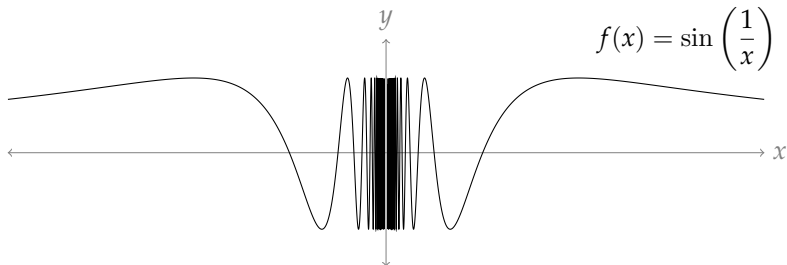


What is $\lim_{x \rightarrow 0} f(x)$?

$\lim_{x \rightarrow 0} f(x)$ does not exist.

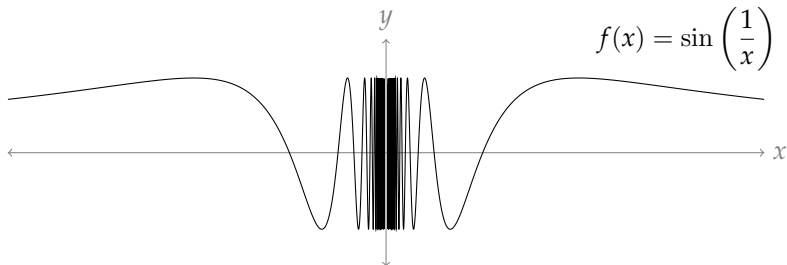
We can call this behaviour “infinite wiggling.”

A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow \pi} f(x)$?

A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow \pi} f(x)$?

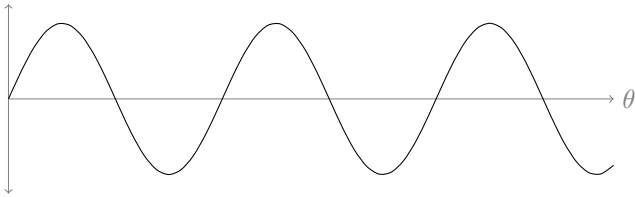
$$\lim_{x \rightarrow \pi} f(x) = \sin\left(\frac{1}{\pi}\right)$$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

▶ SKIP SKETCHING

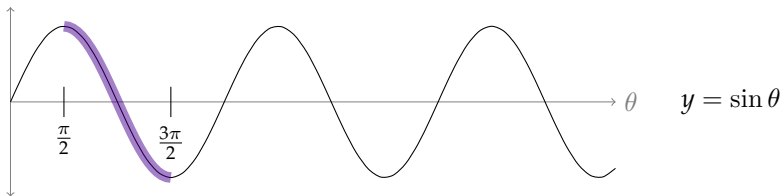


$$y = \sin \frac{1}{x}$$

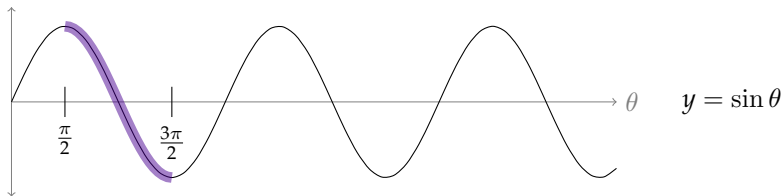


$$y = \sin \theta$$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

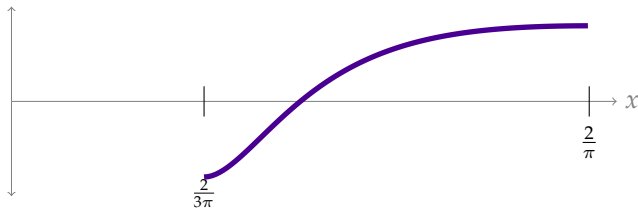
[▶ SKIP SKETCHING](#)


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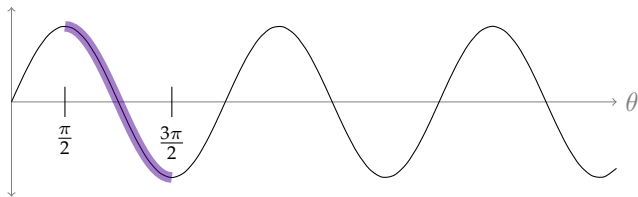
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OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

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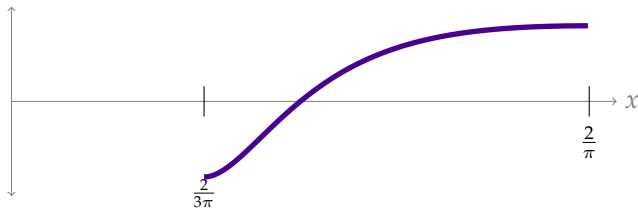
$$y = \sin \frac{1}{x}$$



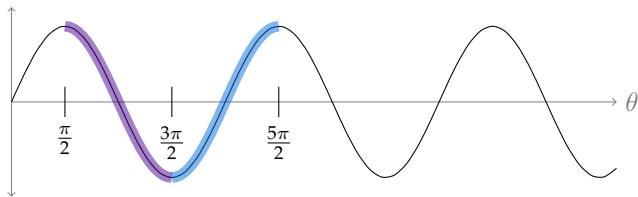
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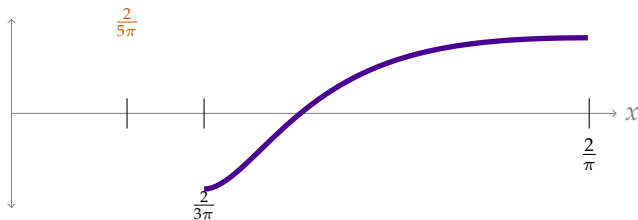
$$y = \sin \frac{1}{x}$$



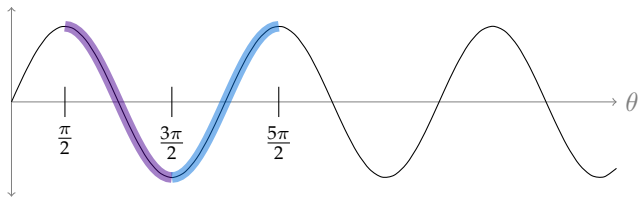
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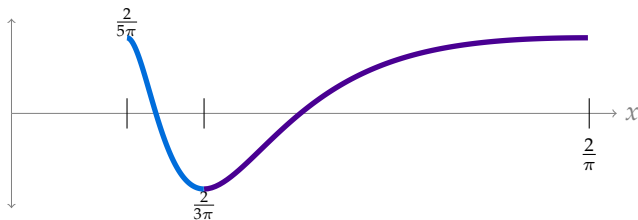
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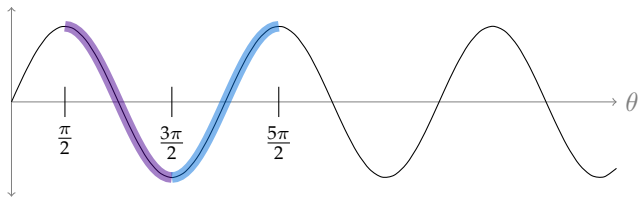
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▶ SKIP SKETCHING



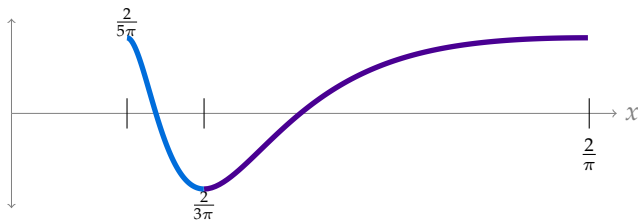
$$y = \sin \frac{1}{x}$$



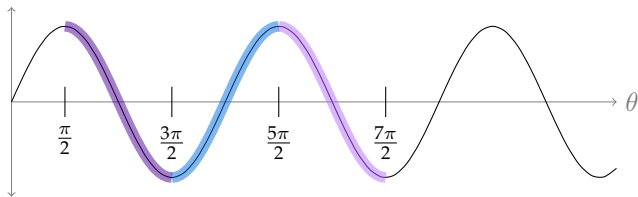
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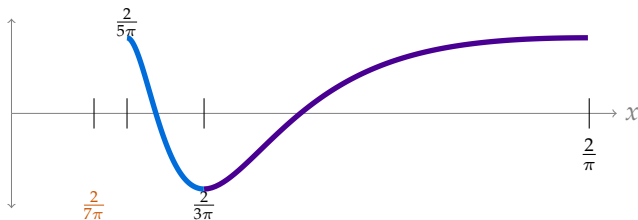
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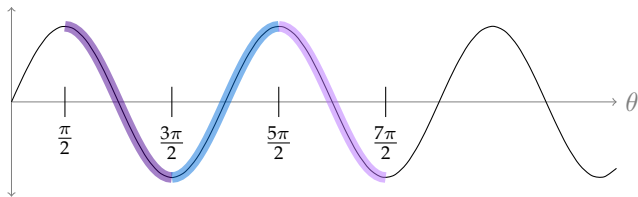
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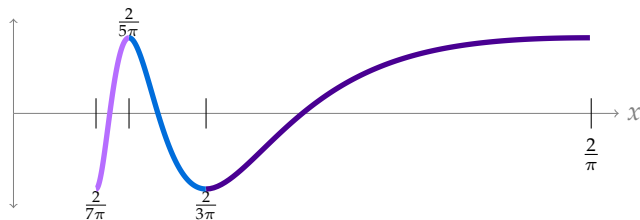
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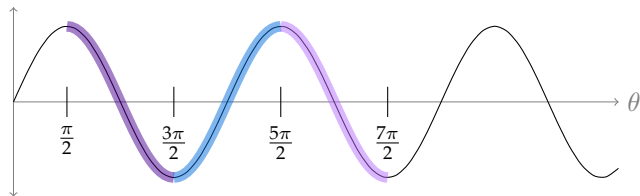
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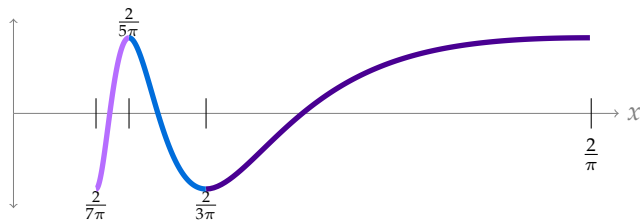
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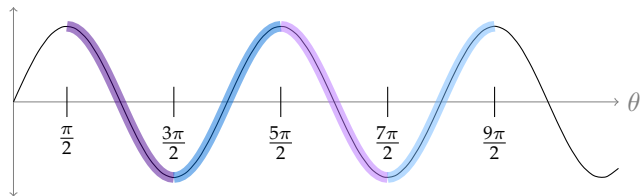
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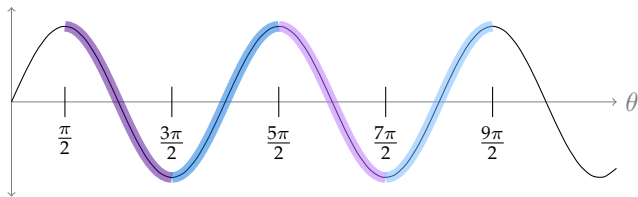
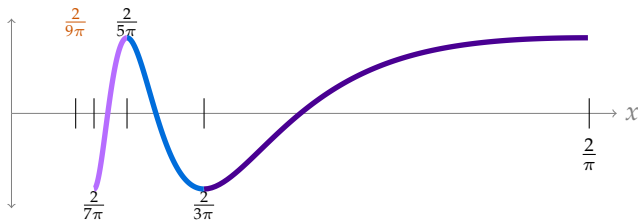
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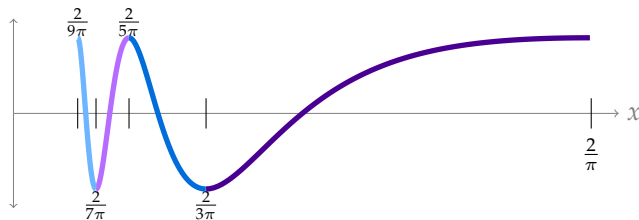
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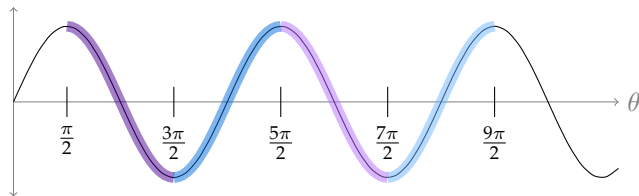


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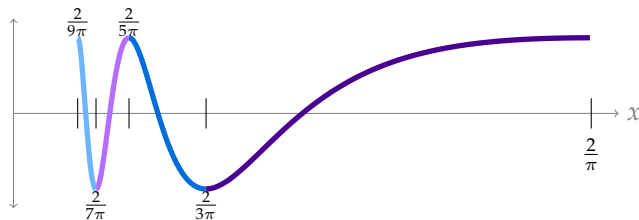
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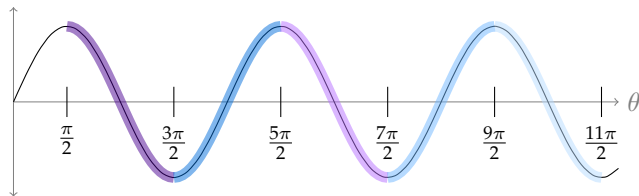
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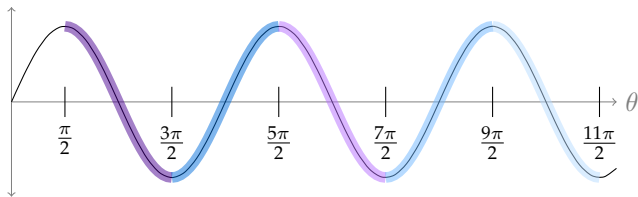
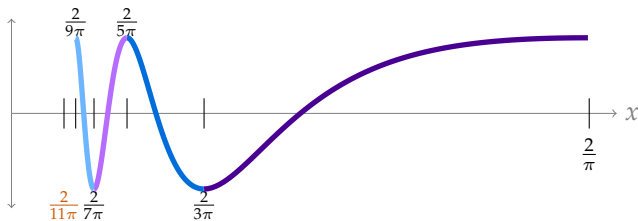
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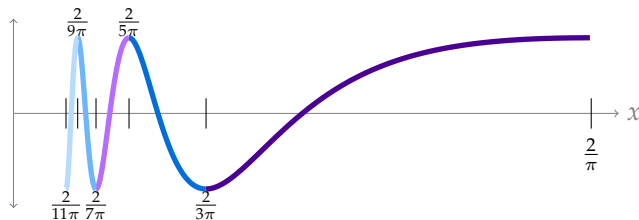
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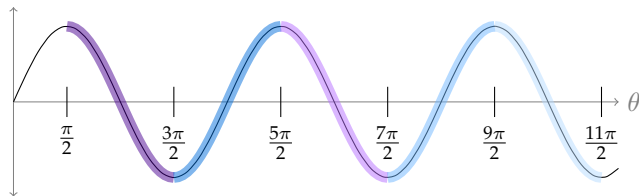


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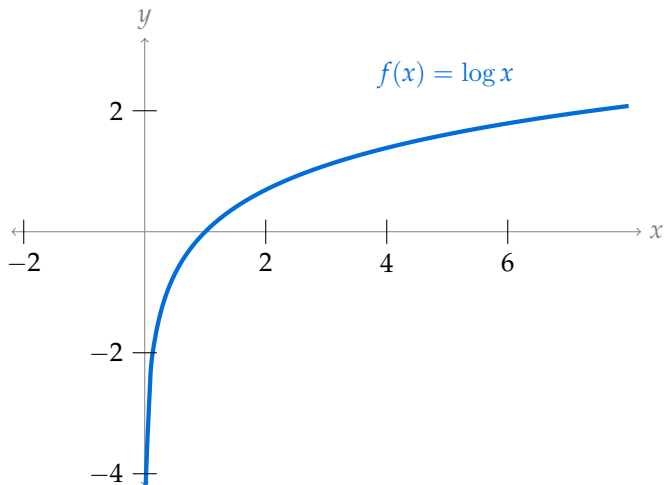
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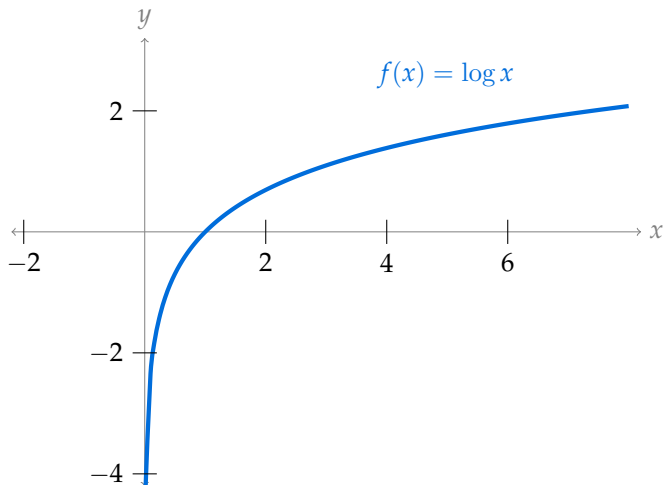
LIMITS AND THE NATURAL LOGARITHM

Where is $f(x)$ defined, and where is it not defined?



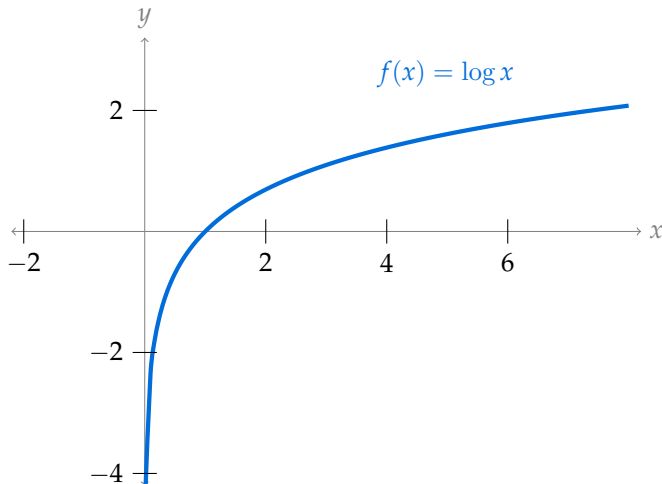
LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of $f(x)$ near 0?



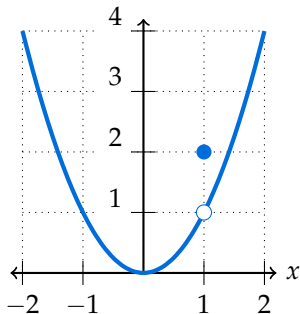
LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of $f(x)$ near 0? $\lim_{x \rightarrow 0^+} \log(x) = -\infty$



Section 1.3 Review

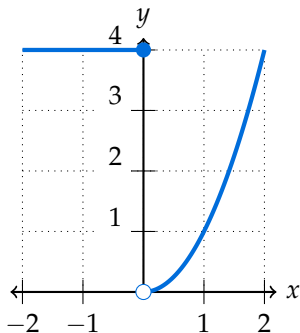
$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x \rightarrow 1} f(x)$?

- A. $\lim_{x \rightarrow 1} f(x) = 2$
- B. $\lim_{x \rightarrow 1} f(x) = 1$
- C. $\lim_{x \rightarrow 1} f(x)$ DNE
- D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0} f(x)$?

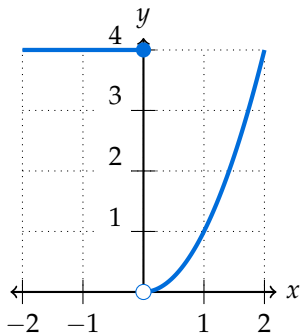
A. $\lim_{x \rightarrow 0} f(x) = 4$

B. $\lim_{x \rightarrow 0} f(x) = 0$

C. $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0} f(x)$?

A. $\lim_{x \rightarrow 0} f(x) = 4$

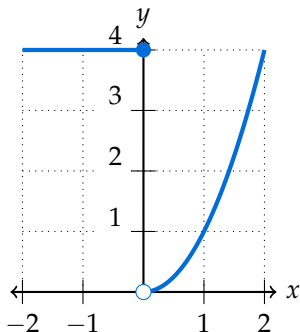
B. $\lim_{x \rightarrow 0} f(x) = 0$

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D. none of the above

$\lim_{x \rightarrow 0} f(x)$ DNE

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What is $\lim_{x \rightarrow 0^+} f(x)$?

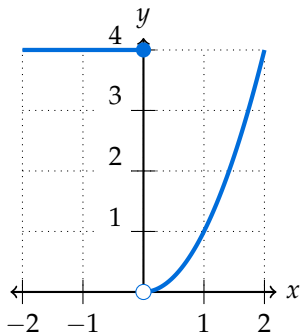
A. $\lim_{x \rightarrow 0} f(x) = 4$

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D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



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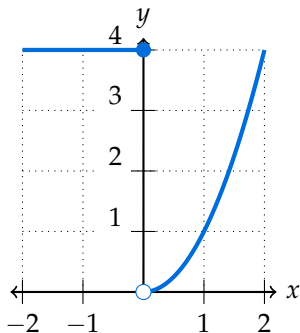
B. $\lim_{x \rightarrow 0^+} f(x) = 0$

C. $\lim_{x \rightarrow 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

What is $f(0)$?



Suppose $\lim_{x \rightarrow 3^-} f(x) = 1$ and $\lim_{x \rightarrow 3^+} f(x) = 1.5$.

Does $\lim_{x \rightarrow 3} f(x)$ exist?

- A. Yes, certainly, because the limits from both sides exist.
- B. No, never, because the limit from the left is not the same as the limit from the right.
- C. Can't tell. For some functions it might exist, for others not.

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Suppose $\lim_{x \rightarrow 3^-} f(x) = 22 = \lim_{x \rightarrow 3^+} f(x)$.

Does $\lim_{x \rightarrow 3} f(x)$ exist?

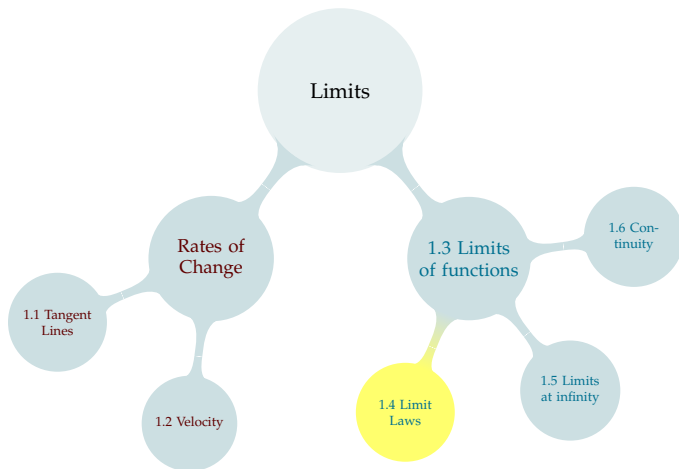
- A. Yes, certainly, because the limits from both sides exist and are equal to each other.
- B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.
- C. Can't tell. We need to know the value of the function at $x = 3$.

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TABLE OF CONTENTS



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Can't find in the same way: 3 not in domain

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x \rightarrow a} f(x) = F$ and $\lim_{x \rightarrow a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$
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Calculate: $\lim_{x \rightarrow 1} \left[\frac{2x+4}{x+2} + 13 \left(\frac{x+5}{3x} \right) \left(\frac{x^2}{2x-1} \right) \right]$

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$$= \lim_{x \rightarrow 1} \left(\frac{2x+4}{x+2} \right) + \left(\lim_{x \rightarrow 1} 13 \right) \left(\lim_{x \rightarrow 1} \frac{x+5}{3x} \right) \left(\lim_{x \rightarrow 1} \frac{x^2}{2x-1} \right)$$

$$= \left(\frac{2(1)+4}{1+2} \right) + (13) \left(\frac{(1)+5}{3(1)} \right) \left(\frac{1^2}{2(1)-1} \right)$$

$$= (2) + 13(2)(1)$$

$$= 28$$

LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$

B. $(-4)^{\frac{1}{2}}$

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If n is a positive integer, and $\lim_{x \rightarrow a} f(x) = F$ (where F is a real number), then:

$$\lim_{x \rightarrow a} (f(x))^n = F^n.$$

Furthermore, **unless** n is even and F is negative,

$$\lim_{x \rightarrow a} (f(x))^{1/n} = F^{1/n}$$

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$$\lim_{x \rightarrow 4} (x + 5)^{1/2} = \left[\lim_{x \rightarrow 4} (x + 5) \right]^{1/2} = 9^{1/2} = 3$$

CAUTIONARY TALES

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{(5+x)^2 - 25}{x}$$

CAUTIONARY TALES

► $\lim_{x \rightarrow 0} \frac{(5+x)^2 - 25}{x} \rightarrow \frac{0}{0}; \text{ need another way}$

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Suppose you want to evaluate $\lim_{x \rightarrow 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

- A $\lim_{x \rightarrow 1} f(x)$ may exist, and it may not exist.
- B We can find $\lim_{x \rightarrow 1} f(x)$ by plugging in 1 to $f(x)$.
- C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x \rightarrow 1} f(x)$.
- D Since $f(1)$ doesn't exist, automatically we know $\lim_{x \rightarrow 1} f(x)$ does not exist.
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Which of the following statements is true about $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 0}{0^3 - 0^2 + 0} = \frac{0}{0}$

B Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\frac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\frac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

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Which of the following statements is true about $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

C Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

D Since the numerator and denominator of $\frac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

Which of the following statements is true about $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

C Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

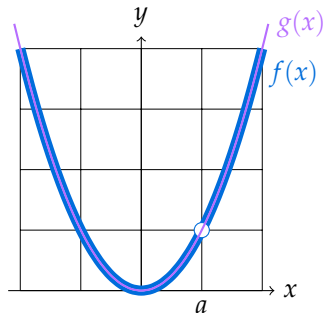
D Since the numerator and denominator of $\frac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.



Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim_{x \rightarrow a} g(x)$ exists, and $f(x) = g(x)$
when x is close to a (but not necessarily equal to a).

Then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$.



Evaluate $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x - 1}$.

Evaluate $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x - 1}$.

$$\begin{aligned} \frac{x^3 + x^2 - x - 1}{x - 1} &= \frac{(x + 1)^2(x - 1)}{x - 1} \\ &= (x + 1)^2 \text{ whenever } x \neq 1 \end{aligned}$$

$$\text{So, } \lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x - 1} = \lim_{x \rightarrow 1} (x + 1)^2 = 4$$

Evaluate $\lim_{x \rightarrow 5} \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$

Evaluate $\lim_{x \rightarrow 5} \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$

$$\begin{aligned}
 \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5} &= \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5} \left(\frac{\sqrt{x+20} + \sqrt{4x+5}}{\sqrt{x+20} + \sqrt{4x+5}} \right) \\
 &= \frac{(x+20) - (4x+5)}{(x-5)(\sqrt{x+20} + \sqrt{4x+5})} \\
 &= \frac{-3x+15}{(x-5)(\sqrt{x+20} + \sqrt{4x+5})} \\
 &= \frac{-3}{\sqrt{x+20} + \sqrt{4x+5}}
 \end{aligned}$$

So,

$$\begin{aligned}
 \lim_{x \rightarrow 5} \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5} &= \lim_{x \rightarrow 5} \frac{-3}{\sqrt{x+20} + \sqrt{4x+5}} \\
 &= \frac{-3}{\sqrt{5+20} + \sqrt{4(5)+5}} = \frac{-3}{10}
 \end{aligned}$$

A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can **directly substitute** (plug in). If your function is made up of the **sum, difference, product, quotient, or power of polynomials**, you can do this **provided** the function exists where you're taking the limit.

$$\lim_{x \rightarrow 1} \left(\sqrt{35 + x^5} + \frac{x - 3}{x^2} \right)^3 =$$

A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can **directly substitute** (plug in). If your function is made up of the **sum, difference, product, quotient, or power of polynomials**, you can do this **provided** the function exists where you're taking the limit.

$$\begin{aligned}\lim_{x \rightarrow 1} \left(\sqrt{35 + x^5} + \frac{x - 3}{x^2} \right)^3 &= \\ \left(\sqrt{35 + 1^5} + \frac{1 - 3}{1^2} \right)^3 &= 64\end{aligned}$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to **simplify and cancel**.

$$\lim_{x \rightarrow 0} \frac{x + 7}{\frac{1}{x} - \frac{1}{2x}}$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to **simplify and cancel**.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x+7}{\frac{1}{x} - \frac{1}{2x}} &= \lim_{x \rightarrow 0} \frac{x+7}{\frac{2}{2x} - \frac{1}{2x}} \\ &= \lim_{x \rightarrow 0} \frac{x+7}{\frac{1}{2x}} = \lim_{x \rightarrow 0} 2x(x+7) = 0\end{aligned}$$

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Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

DENOMINATORS APPROACHING ZERO

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}$$

DENOMINATORS APPROACHING ZERO

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$$

$$\lim_{x \rightarrow 1} \frac{-1}{(x-1)^2} = -\infty$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$$

DENOMINATORS APPROACHING ZERO

Now
You



$$\lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4}$$

$$\lim_{x \rightarrow 2^-} \frac{x}{4 - x^2}$$

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$$

DENOMINATORS APPROACHING ZERO

Now
You



$$\lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{x}{4 - x^2} = \infty$$

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \frac{1}{4}$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a , we have functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \leq g(x) \leq h(x)$$

and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$. Then $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x)$.

Squeeze Theorem – Theorem 1.4.17

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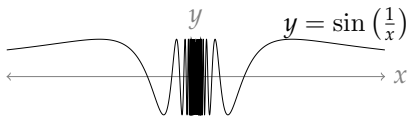
$$\lim_{x \rightarrow 0} x^2 \sin \left(\frac{1}{x} \right)$$

Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

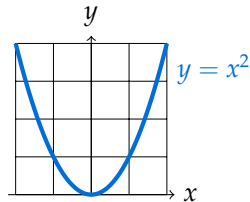
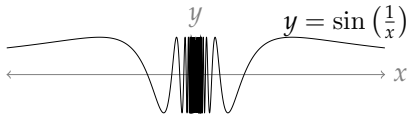
Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$



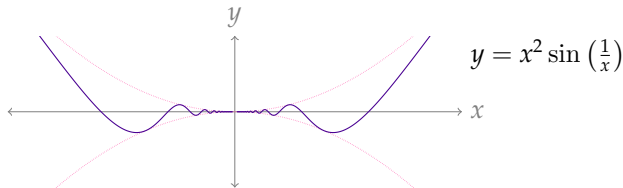
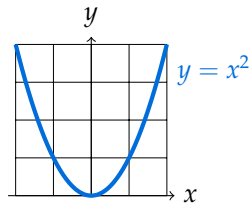
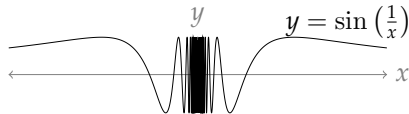
Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$



Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$



$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\text{so } -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\text{and also } \lim_{x \rightarrow 0} -x^2 = 0 = \lim_{x \rightarrow 0} x^2$$

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Therefore, by the Squeeze Theorem, $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Included Work



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