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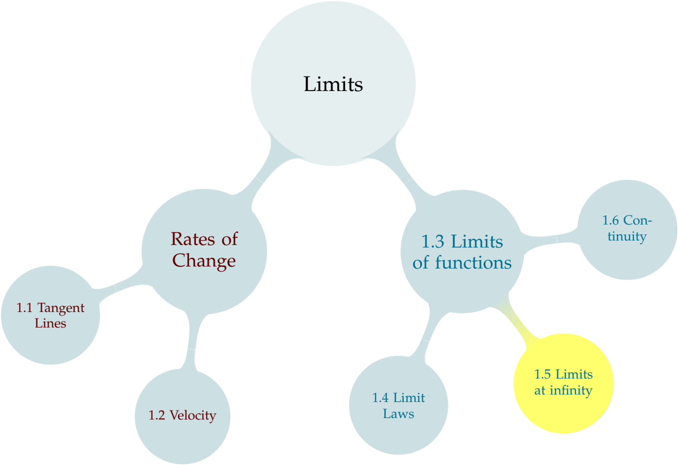


End Behavior

We write:

$$\lim_{x \to \infty} f(x) = L$$

to express that, as x grows larger and larger, $f(x)$ approaches L.

Similarly, we write:

$$\lim_{x \to -\infty} f(x) = L$$

to express that, as x grows more and more strongly negative, $f(x)$ approaches L.

END BEHAVIOR

We write:

$$\lim_{x\to\infty} f(x) = L$$

to express that, as x grows larger and larger, $f(x)$ approaches L.

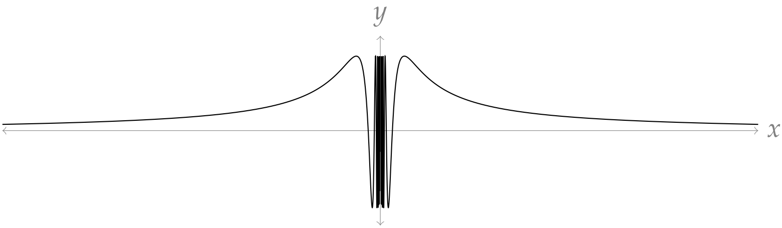
Similarly, we write:

$$\lim_{x\to-\infty} f(x) = L$$

to express that, as x grows more and more strongly negative, $f(x)$ approaches L.

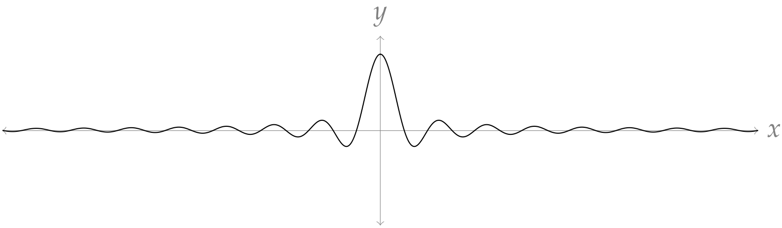
If L is a number, we call $y = L$ a horizontal asymptote of $f(x)$.

HORIZONTAL ASYMPTOTES



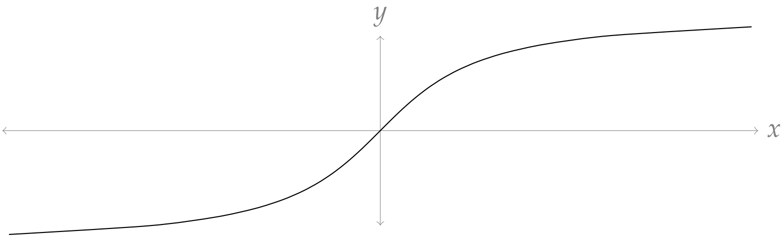
$y = 0$ is a horizontal asymptote for $y = \sin\left(\frac{1}{x}\right)$

Horizontal Asymptotes



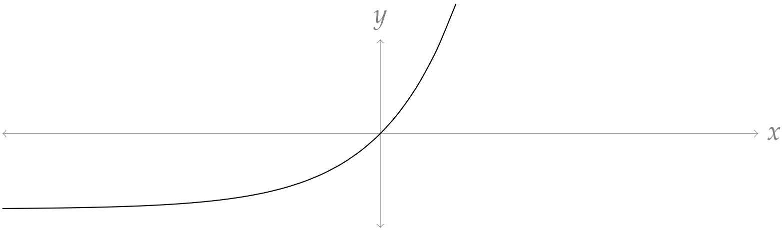
$y = 0$ is a horizontal asymptote for $y = \frac{\sin x}{x}$

Horizontal Asymptotes



$y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes for $y = \arctan(x)$

Horizontal Asymptotes



$y = -1$ is a horizontal asymptote for $y = e^x - 1$

Common Limits at Infinity

$$\lim_{x\to\infty} 13 =$$
$$\lim_{x\to-\infty} 13 =$$

$$\lim_{x\to\infty} x^3 =$$
$$\lim_{x\to-\infty} x^3 =$$

$$\lim_{x\to\infty} \frac{1}{x} =$$
$$\lim_{x\to-\infty} \frac{1}{x} =$$

$$\lim_{x\to-\infty} x^{5/3} =$$
$$\lim_{x\to-\infty} x^{2/3} =$$

$$\lim_{x\to\infty} x^2 =$$
$$\lim_{x\to-\infty} x^2 =$$

Q

ans

Common Limits at Infinity

$$\lim_{x\to\infty} 13 = 13 \qquad \lim_{x\to\infty} x^3 = \infty$$

$$\lim_{x\to-\infty} 13 = 13 \qquad \lim_{x\to-\infty} x^3 = -\infty$$

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} x^{5/3} = -\infty$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} x^{2/3} = \infty$$

$$\lim_{x\to\infty} x^2 = \infty$$

$$\lim_{x\to-\infty} x^2 = \infty$$

Arithmetic with Limits at Infinity

Now You

$$\lim_{x\to\infty}\left(x+\frac{x^2}{10}\right)=$$

$$\lim_{x\to\infty}\left(x-\frac{x^2}{10}\right)=$$

$$\lim_{x\to-\infty}\left(x^2+x^3+x^4\right)=$$

$$\lim_{x\to-\infty}\,(x+13)\left(x^2+13\right)^{1/3}=$$

Q

ans

Arithmetic with Limits at Infinity

Now You

$$\lim_{x\to\infty}\left(x+\frac{x^2}{10}\right)=\infty$$

$$\lim_{x\to\infty}\left(x-\frac{x^2}{10}\right)=\lim_{x\to\infty} x\left(1-\frac{x}{10}\right)=-\infty$$

$$\lim_{x\to-\infty}\left(x^2+x^3+x^4\right)=\lim_{x\to-\infty} x^4\left(\frac{1}{x^2}+\frac{1}{x}+1\right)=\infty$$

$$\lim_{x\to-\infty}(x+13)\left(x^2+13\right)^{1/3}=-\infty$$

Calculating Limits at Infinity

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x^3}$$

Q

Calculating Limits at Infinity

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x^3}$$

Trick: factor out largest power of denominator.

Calculating Limits at Infinity

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x^3}$$

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x^3} = \lim_{x\to\infty} \frac{x^2+2x+1}{x^3}\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}}{1} = \frac{\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{2}{x^2}+\lim\limits_{x\to\infty}\frac{1}{x^3}}{\lim\limits_{x\to\infty} 1}$$

$$= \frac{0+0+0}{1} = 0$$

Calculating Limits at Infinity

$$\lim_{x\to-\infty}\left(x^{7/3}-x^{5/3}\right)$$

Q

Calculating Limits at Infinity

$$\lim_{x\to-\infty}\left(x^{7/3}-x^{5/3}\right)$$

Again: factor out largest power of x.

Q

Calculating Limits at Infinity

$$\lim_{x\to-\infty}\left(x^{7/3}-x^{5/3}\right)$$

Again: factor out largest power of x.

$$\left(x^{7/3}-x^{5/3}\right)=x^{7/3}\left(1-\frac{1}{x^{2/3}}\right)$$

$$\left(\text{note: }\lim_{x\to-\infty}x^{7/3}=-\infty\right)$$

$$\left(\text{note also: }\lim_{x\to-\infty}\left(1-\frac{1}{x^{2/3}}\right)=1\right)$$

So, $\lim_{x\to-\infty}\left(x^{7/3}-x^{5/3}\right)=-\infty$

CALCULATING LIMITS AT INFINITY

Suppose the height of a bouncing ball is given by $h(t) = \frac{\sin(t)+1}{t}$, for $t \geq 1$. What happens to the height over a long period of time?

Q

ans

Calculating Limits at Infinity

Suppose the height of a bouncing ball is given by $h(t) = \frac{\sin(t)+1}{t}$, for $t \geq 1$. What happens to the height over a long period of time?

$$\begin{array}{ccccc} 0 & \leq & \dfrac{\sin(t)+1}{t} & \leq & \dfrac{2}{t} \\ \lim\limits_{t\to\infty} 0 & = & 0 & = & \lim\limits_{t\to\infty} \dfrac{2}{t} \end{array}$$

So, by the Squeeze Theorem,

$$\lim_{t\to\infty} \frac{\sin(t)+1}{t} = 0$$

A

Calculating Limits at Infinity

Now You

$$\lim_{x\to\infty} \sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}$$

Q

ans

Calculating Limits at Infinity

Now You 

$$\lim_{x\to\infty} \sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}$$

Multiply function by conjugate:

$$\left(\sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}\right)\left(\frac{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}\right)$$

$$=\frac{-2x^2+1}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}$$

Calculating Limits at Infinity

Now You

$$\lim_{x\to\infty} \sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}$$

Multiply function by conjugate:

$$\left(\sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}\right)\left(\frac{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}\right)$$

$$=\frac{-2x^2+1}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}$$

Factor out highest power: x^2 (same as $\sqrt{x^4}$)

$$\frac{-2x^2+1}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}\left(\frac{1/x^2}{1/\sqrt{x^4}}\right)$$

$$=\frac{-2+\frac{1}{x^2}}{\sqrt{1+\frac{1}{x^2}+\frac{1}{x^4}}+\sqrt{1+\frac{3}{x^2}}}$$

Calculating Limits at Infinity

Multiply function by conjugate:

$$\left(\sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}\right)\left(\frac{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}\right)$$
$$=\frac{-2x^2+1}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}$$

Factor out highest power: x^2 (same as $\sqrt{x^4}$)

$$\frac{-2x^2+1}{\sqrt{x^4+x^2+1}+\sqrt{x^4+3x^2}}\left(\frac{1/x^2}{1/\sqrt{x^4}}\right)$$
$$=\frac{-2+\frac{1}{x^2}}{\sqrt{1+\frac{1}{x^2}+\frac{1}{x^4}}+\sqrt{1+\frac{3}{x^2}}}$$

$$\lim_{x\to\infty}\frac{-2+\frac{1}{x^2}}{\sqrt{1+\frac{1}{x^2}+\frac{1}{x^4}}+\sqrt{1+\frac{3}{x^2}}}=\frac{-2+0}{\sqrt{1+0+0}+\sqrt{1+0}}=\frac{-2}{2}=-1$$

A

Now You

Evaluate $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{3+x^2}}{3x}$

Q

ans

Now You

Evaluate $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{3+x^2}}{3x}$

We factor out the largest power of the denominator, which is is x.

$$\lim_{x\to-\infty}\frac{\sqrt{3+x^2}}{3x}\left(\frac{1/x}{1/x}\right)=\lim_{x\to-\infty}\frac{\frac{\sqrt{3+x^2}}{x}}{3}$$

When $x<0$, $\sqrt{x^2}=|x|=-x$

$$\begin{aligned}
&=\lim_{x\to-\infty}\frac{1}{3}\frac{\sqrt{3+x^2}}{-\sqrt{x^2}}\\
&=\lim_{x\to-\infty}-\frac{1}{3}\sqrt{\frac{3+x^2}{x^2}}\\
&=\lim_{x\to-\infty}-\frac{1}{3}\sqrt{\frac{3}{x^2}+1}\\
&=-\frac{1}{3}
\end{aligned}$$

A

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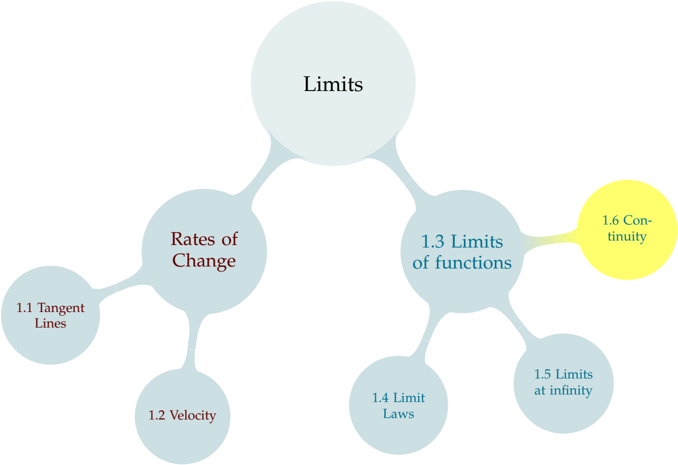

Continuity

Definition 1.6.1

A function $f(x)$ is continuous at a point a if $\lim_{x \to a} f(x)$ exists AND is equal to $f(a)$.

CONTINUITY

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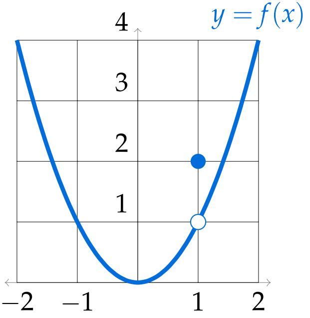
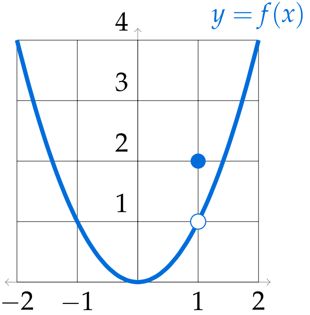


Does $f(x)$ exist at $x = 1$?
Is $f(x)$ continuous at $x = 1$?

Continuity

Definition 1.6.1

A function $f(x)$ is continuous at a point a if $\lim\limits_{x\to a} f(x)$ exists AND is equal to $f(a)$.



Does $f(x)$ exist at $x = 1$? Yes.
Is $f(x)$ continuous at $x = 1$?

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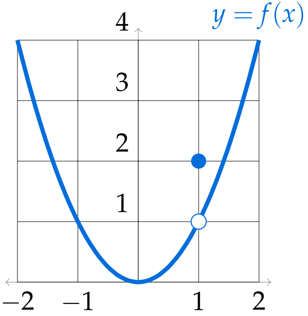

Does $f(x)$ exist at $x = 1$? Yes.
Is $f(x)$ continuous at $x = 1$? No.

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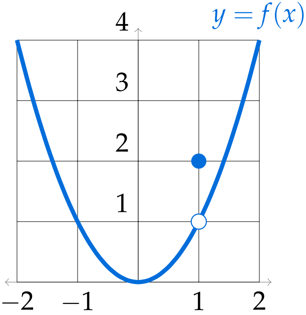
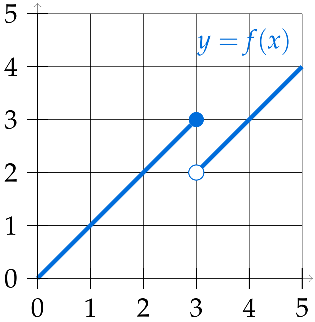


Does $f(x)$ exist at $x = 1$? Yes.
Is $f(x)$ continuous at $x = 1$? No.

This kind of discontinuity is called removable.

Definitions 1.6.1 and 1.6.2

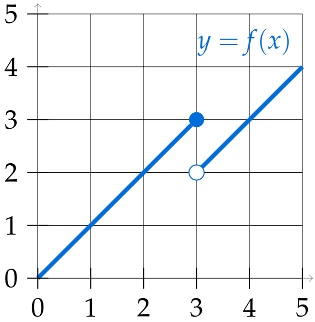
A function $f(x)$ is continuous at a point a if $\lim_{x \to a} f(x)$ exists AND is equal to $f(a)$.



Is $f(x)$ continuous at $x = 3$?

Definitions 1.6.1 and 1.6.2

A function $f(x)$ is continuous at a point a if $\lim_{x\to a} f(x)$ exists AND is equal to $f(a)$.

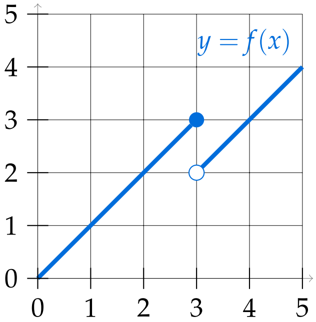


Is $f(x)$ continuous at $x = 3$? No.

This kind of discontinuity is called a jump.

Definitions 1.6.1 and 1.6.2

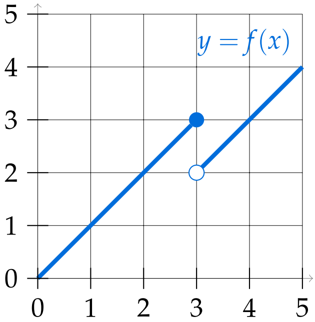
A function $f(x)$ is continuous from the left at a point a if $\lim\limits_{x \to a^-} f(x)$ exists AND is equal to $f(a)$.



Is $f(x)$ continuous at $x = 3$? No.

Definitions 1.6.1 and 1.6.2

A function $f(x)$ is continuous from the left at a point a if $\lim\limits_{x \to a^-} f(x)$ exists AND is equal to $f(a)$.



Is $f(x)$ continuous at $x = 3$? No.

Is $f(x)$ continuous from the left at $x = 3$?

Is $f(x)$ continuous from the right at $x = 3$?

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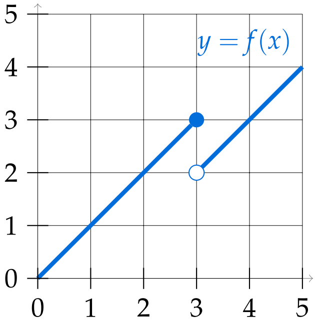

Is $f(x)$ continuous at $x = 3$? No.

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Is $f(x)$ continuous from the right at $x = 3$?

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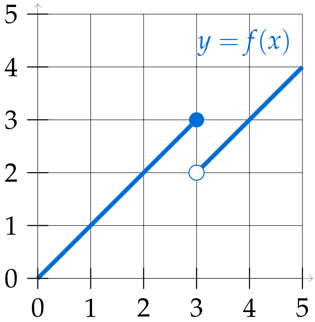

Is $f(x)$ continuous at $x = 3$? No.

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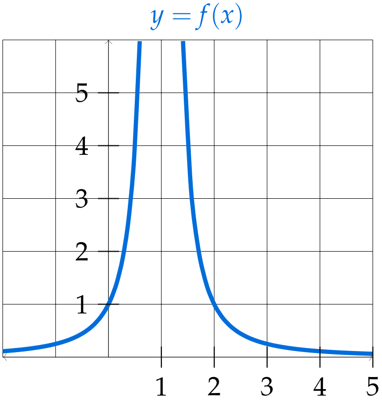
Is $f(x)$ continuous from the right at $x = 3$? No.

Definition

A function $f(x)$ is continuous at a point a if $\lim_{x\to a} f(x)$ exists AND is equal to $f(a)$.



Definition

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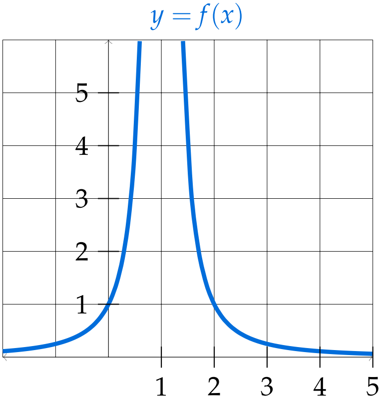

Since no one-sided limits exist at $x=1$, there's no hope for continuity there – not even "from the left" or "from the right."

This is called an infinite discontinuity

Definition

A function $f(x)$ is continuous at a point a if $\lim\limits_{x \to a} f(x)$ exists AND is equal to $f(a)$.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ continuous at 0?

Continuous Functions

Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point in their domain.

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Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point in their domain.

$$f(x) = \frac{x^2}{2x-10} - \left(\frac{x^2+2x-1}{x-1} + \frac{\sqrt[5]{25-x} - \frac{1}{x}}{x+2} \right)^{1/3}$$

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$f(x)$ is continuous at every real number except $5, 1, 0,$ and -2.

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A continuous function is continuous for every point in $\mathbb{R}$.

Continuous Functions

Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point in their domain.

$$f(x) = \frac{x^2}{2x-10} - \left(\frac{x^2+2x-1}{x-1} + \frac{\sqrt[5]{25-x} - \frac{1}{x}}{x+2} \right)^{1/3}$$

We say $f(x)$ is continuous over (a, b) if it is continuous at every point in (a, b). So, $f(x)$ is continuous over its domain, $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, 5) \cup (5, \infty)$.

Common Functions – Theorem 1.6.8

Functions of the following types are continuous over their domains:

- polynomials and rationals
- roots and powers
- trig functions and their inverses
- exponential and logarithm
- The products, sums, differences, quotients, powers, and compositions of continuous functions

Where is the following function continuous?

$$f(x) = \left(\frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

Where is the following function continuous?

$$f(x) = \left(\frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

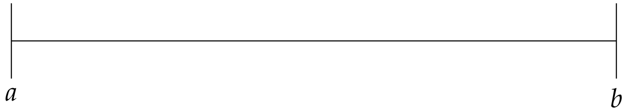
Over its domain: $[0, 2) \cup (2, \infty)$.

A Technical Point

Definition 1.6.3

A function $f(x)$ is continuous on the closed interval $[a,b]$ if:

- $f(x)$ is continuous over (a,b), and
- $f(x)$ is continuous from the left at , and
- $f(x)$ is continuous from the right at



A Technical Point

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- $f(x)$ is continuous from the left at , and
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A Technical Point

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- $f(x)$ is continuous from the left at b, and
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A Technical Point

Definition 1.6.3

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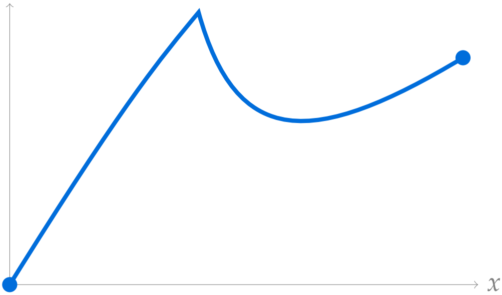
- $f(x)$ is continuous from the left at b, and
- $f(x)$ is continuous from the right at a

Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.



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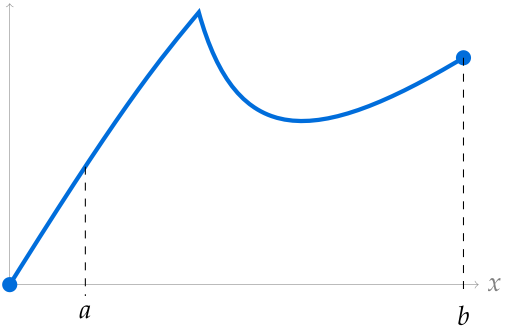

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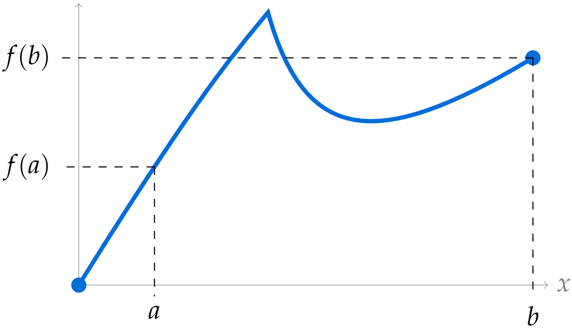

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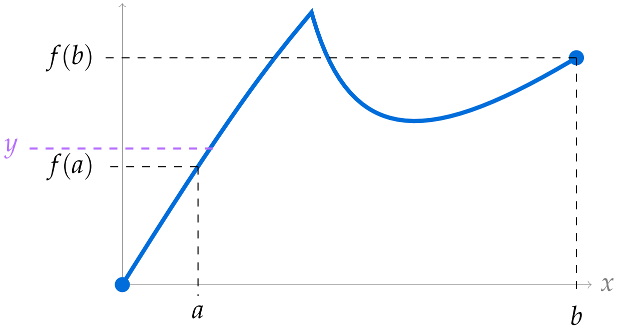

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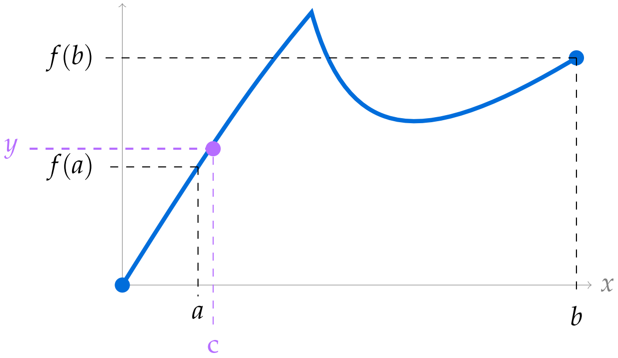

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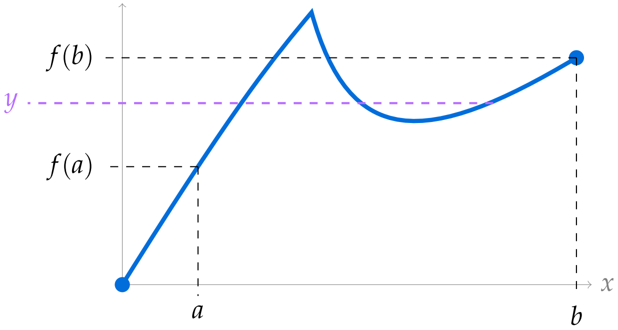

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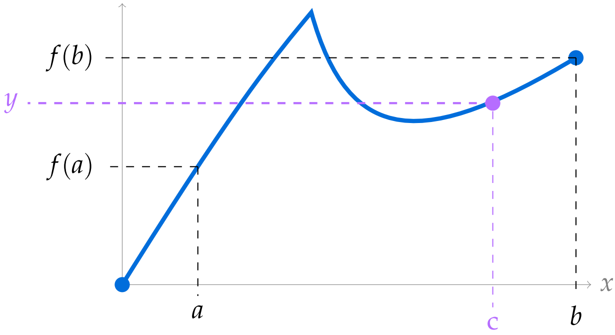

Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.

Suppose your favourite number is 45.54. At noon, your car is parked, and at 1pm you're driving 100kph. By the Intermediate Value Theorem, at some point between noon and 1pm you were going exactly 45.54 kph.

Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$.

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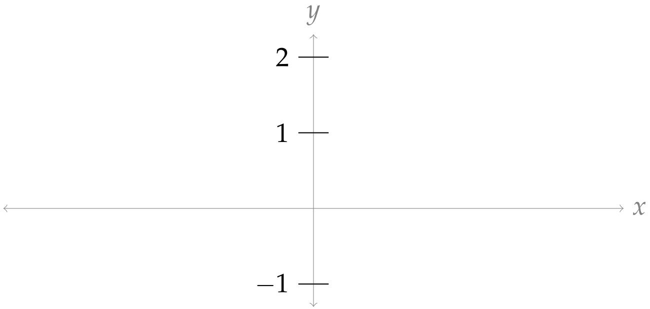
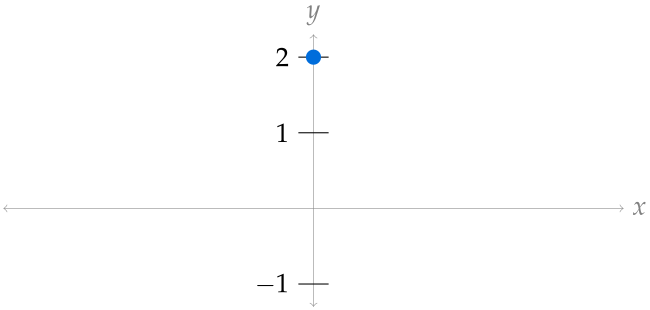


Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$. Let's find some points:

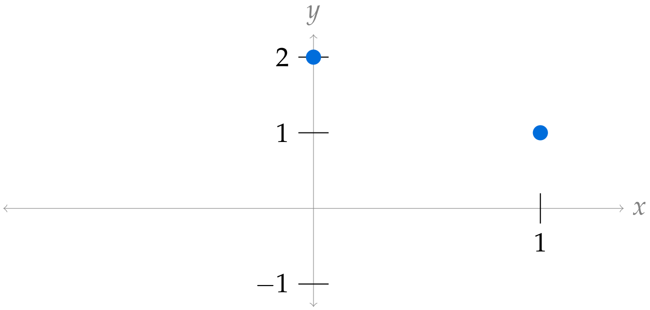
$f(0) = 2$



Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$. Let's find some points:

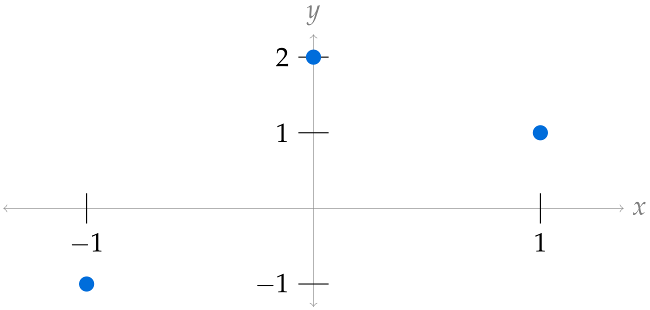
$f(0) = 2$ $f(1) = 1$



USING IVT TO FIND ROOTS: "BISECTION METHOD"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$. Let's find some points:

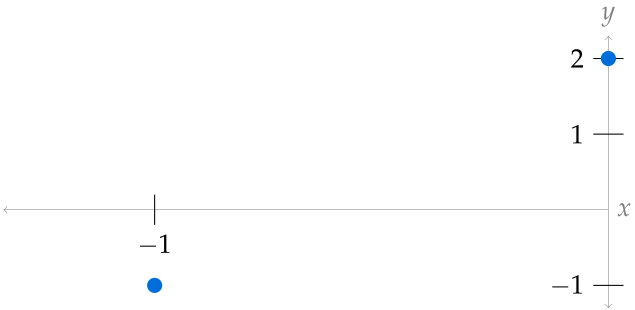
$f(0) = 2$ $\quad$ $f(1) = 1$ $\quad$ $f(-1) = -1$



Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$.

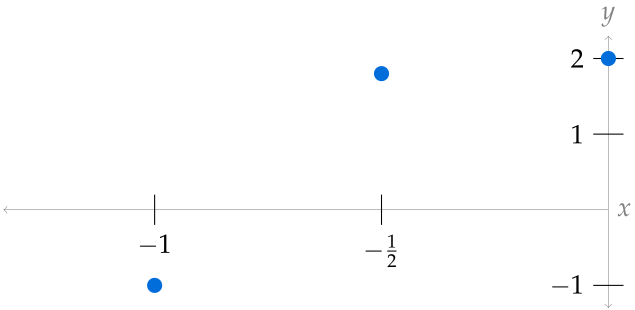
$f(0) = 2, f(-1) = -1$



Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$.

$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84$



Using IVT to Find Roots: "Bisection Method"

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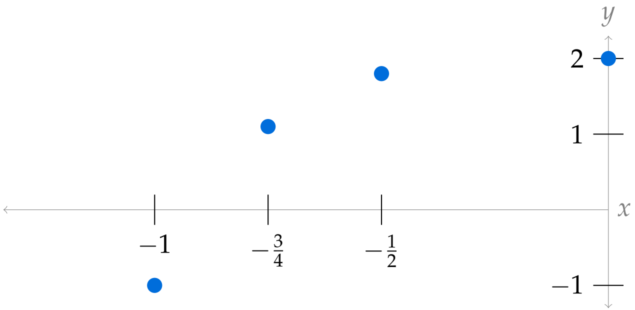
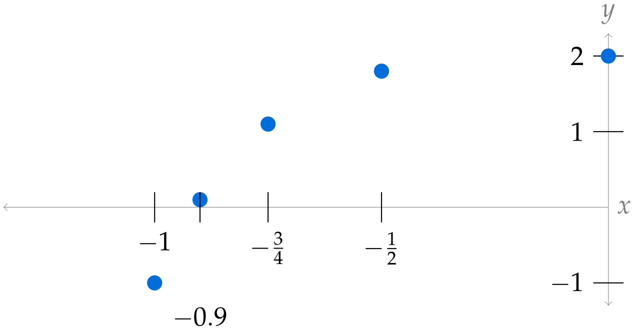


Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$.

$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84, f\left(-\frac{3}{4}\right) \approx 1.13, f(-.9) = 0.097$



Use the Intermediate Value Theorem to show that there exists some solution to the equation $\ln x \cdot e^x = 4$, and give a reasonable interval where that solution might occur.

Use the Intermediate Value Theorem to show that there exists some solution to the equation $\ln x \cdot e^x = 4$, and give a reasonable interval where that solution might occur.

- The function $f(x) = \ln x \cdot e^x$ is continuous over its domain, which is $(0, \infty)$. In particular, then, it is continuous over the interval $(1, e)$.
- $f(1) = \ln(1)e = 0 \cdot e = 0$ and $f(e) = \ln(e) \cdot e^e = e^e$. Since $e > 2$, we know $f(e) = e^e > 2^2 = 4$.
- Then 4 is between $f(1)$ and $f(e)$.
- By the Intermediate Value Theorem, $f(c) = 4$ for some c in $(1, e)$.

NOW YOU Use the Intermediate Value Theorem to give a reasonable interval where the following is true: $e^x = \sin(x)$. (Don't use a calculator – use numbers you can easily evaluate.)

Q

ans

Now You Use the Intermediate Value Theorem to give a reasonable interval where the following is true: $e^x = \sin(x)$. (Don't use a calculator – use numbers you can easily evaluate.)

We can rearrange this: let $f(x) = e^x - \sin(x)$, and note $f(x)$ has roots exactly when $e^x = \sin(x)$.

- The function $f(x) = e^x - \sin x$ is continuous over its domain, which is all real numbers. In particular, then, it is continuous over the interval $\left(-\frac{3\pi}{2}, e\right)$.
- $f(0) = e^0 - \sin 0 = 1 - 0 = 1 > 0$ and $f\left(-\frac{3\pi}{2}\right) = e^{-\frac{3\pi}{2}} - \sin\left(\frac{-3\pi}{2}\right) = e^{-\frac{3\pi}{2}} - 1 < e^0 - 1 = 1 - 1 = 0$.
- Then 0 is between $f(0)$ and $f\left(-\frac{3\pi}{2}\right)$.
- By the Intermediate Value Theorem, $f(c) = 0$ for some c in $\left(-\frac{3\pi}{2}, 0\right)$.
- Therefore, $e^c = \sin c$ for some c in $\left(-\frac{3\pi}{2}, 0\right)$.

NOW YOU

Is there any value of x so that $\sin x = \cos(2x) + \frac{1}{4}$?

Q

ans

Now You

Is there any value of x so that $\sin x = \cos(2x) + \frac{1}{4}$?

Yes, somewhere between 0 and $\frac{\pi}{2}$.

A

NOW YOU

Is the following reasoning correct?

- $f(x) = \tan x$ is continuous over its domain, because it is a trigonometric function.

- In particular, $f(x)$ is continuous over the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

- $f\left(\frac{\pi}{4}\right) = 1$, and $f\left(\frac{3\pi}{4}\right) = -1$.

- Since $f\left(\frac{3\pi}{4}\right) < 0 < f\left(\frac{\pi}{4}\right)$, by the Intermediate Value Theorem, there exists some number c in the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ such that $f(c) = 0$.

Now You 

Is the following reasoning correct?

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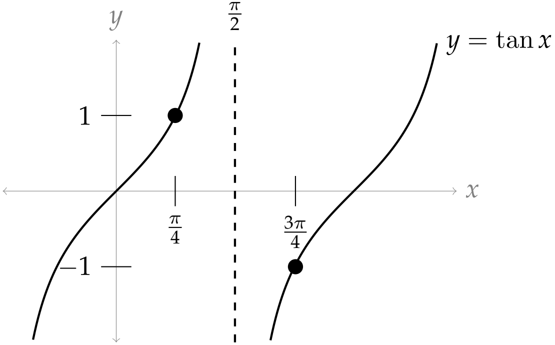

y
$\frac{\pi}{2}$
$y = \tan x$
1
-1
$\frac{\pi}{4}$
$\frac{3\pi}{4}$
x

CONTINUITY

Section 1.6 Review

Suppose $f(x)$ is continuous at $x = 1$. Does $f(x)$ have to be defined at $x = 1$?

ans

Suppose $f(x)$ is continuous at $x = 1$. Does $f(x)$ have to be defined at $x = 1$?

Yes. Since $f(x)$ is continuous at $x = 1$, $\lim_{x\to1} f(x) = f(1)$, so $f(1)$ must exist.

Suppose $f(x)$ is continuous at $x = 1$ and $\lim_{x \to 1^-} f(x) = 30$.

True or false: $\lim_{x \to 1^+} f(x) = 30$.

Suppose $f(x)$ is continuous at $x = 1$ and $\lim\limits_{x\to 1^-} f(x) = 30$.

True or false: $\lim\limits_{x\to 1^+} f(x) = 30$.

True. Since $f(x)$ is continuous at $x = 1$, $\lim\limits_{x\to 1} f(x) = f(1)$, so $\lim\limits_{x\to 1} f(x)$ must exist. That means both one-sided limits exist, and are equal to each other.

Suppose $f(x)$ is continuous at $x = 1$ and $f(1) = 22$. What is $\lim\limits_{x \to 1} f(x)$?

Q

ans

Suppose $f(x)$ is continuous at $x = 1$ and $f(1) = 22$. What is $\lim_{x\to 1} f(x)$?

$22 = f(1) = \lim_{x\to 1} f(x)$.

Suppose $\lim_{x\to 1} f(x) = 2$. Must it be true that $f(1) = 2$?

Q

ans

Suppose $\lim_{x \to 1} f(x) = 2$. Must it be true that $f(1) = 2$?

No. In order to determine the limit as x goes to 1, we ignore $f(1)$. (Perhaps $f(x)$ is not even defined at 1.)

$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of a is $f(x)$ continuous?

$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of a is $f(x)$ continuous?

We need $ax^2 = 3x$ when $x = 1$, so $a = 3$.

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = -\sqrt{3}$?

Q

ans

By the definition of continuity, if $f(x)$ is continuous at $x=-\sqrt{3}$, then $f(-\sqrt{3})=\lim\limits_{x\to-\sqrt{3}} f(x)$. Note $f(-\sqrt{3})=a$, and when x is close to (but not equal to) $-\sqrt{3}$, then $f(x)=\frac{\sqrt{3}x+3}{x^2-3}$.

$$
\begin{aligned}
f(-\sqrt{3}) &= \lim_{x\to-\sqrt{3}} f(x) \\
a &= \lim_{x\to-\sqrt{3}} \frac{\sqrt{3}x+3}{x^2-3} = \lim_{x\to-\sqrt{3}} \frac{\sqrt{3}(x+\sqrt{3})}{(x+\sqrt{3})(x-\sqrt{3})} \\
&= \lim_{x\to-\sqrt{3}} \frac{\sqrt{3}}{x-\sqrt{3}} = \frac{\sqrt{3}}{-\sqrt{3}-\sqrt{3}} = -\frac{1}{2}
\end{aligned}
$$

So we can use $a=-\frac{1}{2}$ to make $f(x)$ continuous at $x=-\sqrt{3}$.

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = \sqrt{3}$?

Q

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = \sqrt{3}$?

By the definition of continuity, if $f(x)$ is continuous at $x = \sqrt{3}$, then $f(\sqrt{3}) = \lim\limits_{x \to \sqrt{3}} f(x)$. When x is close to (but not equal to) $\sqrt{3}$, then $f(x) = \frac{\sqrt{3}x+3}{x^2-3}$. However, as x approaches $\sqrt{3}$, the denominator of this expression gets closer and closer to zero, while the top gets closer and closer to 6. So, this limit does not exist. Therefore, no value of a will make $f(x)$ continuous at $x = \sqrt{3}$.

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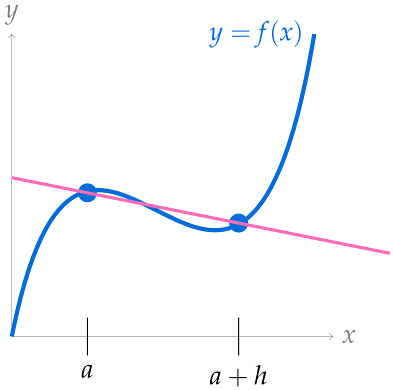
Slope of Secant and Tangent Line

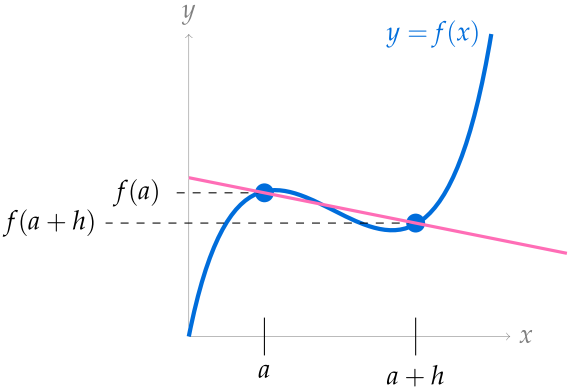
Slope

Recall, the slope of a line is given by any of the following:

$$\frac{\text{rise}}{\text{run}} \qquad \frac{\Delta y}{\Delta x} \qquad \frac{y_2 - y_1}{x_2 - x_1}$$



Slope of secant line:



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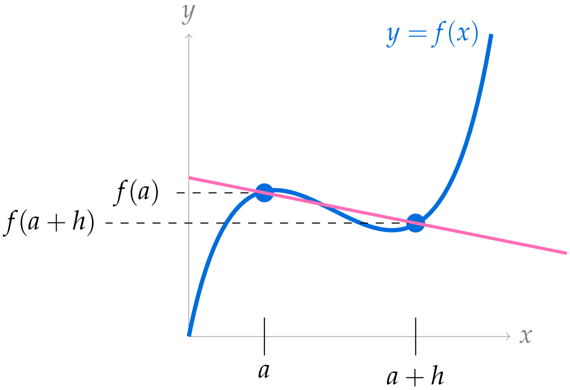
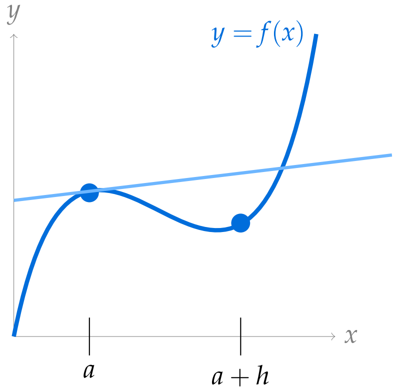
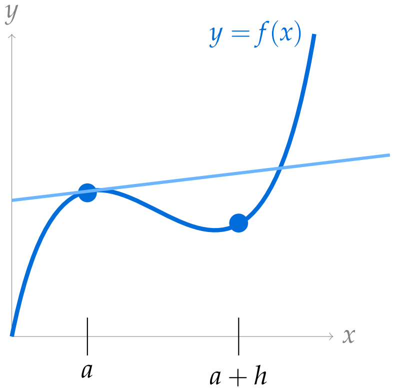


Slope of secant line: $\frac{f(a+h)-f(a)}{h}$



Slope of tangent line:



Slope of tangent line: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

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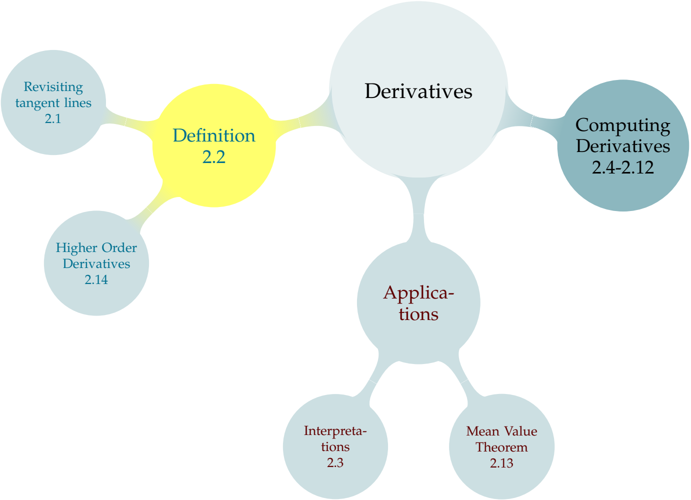

Derivative at a Point

Definition 2.2.1

Given a function $f(x)$ and a point a, the slope of the tangent line to $f(x)$ at a is the derivative of f at a, written $f'(a)$.

So, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$f'(a)$ is also the instantaneous rate of change of f at a.

Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then f is at a.

Derivative

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If $f'(a) > 0$, then f is increasing at a. Its graph "points up."

Derivative

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If $f'(a) > 0$, then f is increasing at a. Its graph “points up.”

If $f'(a) < 0$, then f is at a.

Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then f is increasing at a. Its graph "points up."

If $f'(a) < 0$, then f is decreasing at a. Its graph "points down."

Derivative

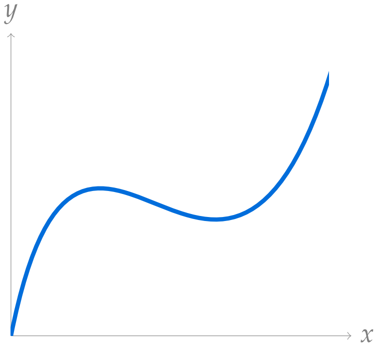
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then f is increasing at a. Its graph "points up."

If $f'(a) < 0$, then f is decreasing at a. Its graph "points down."

If $f'(a) = 0$, then f looks constant or flat at a.

Practice: Increasing and Decreasing



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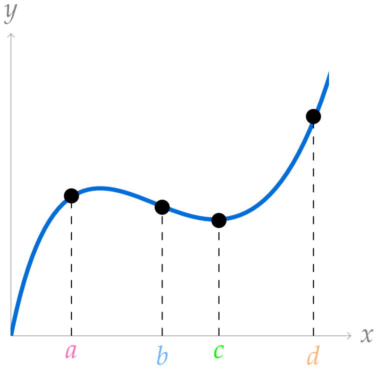
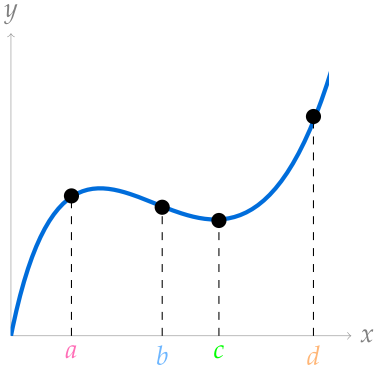


Practice: Increasing and Decreasing

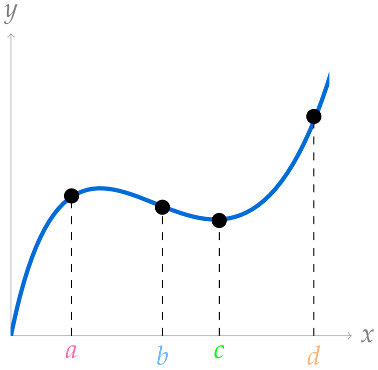


Where is $f'(x) < 0$?

Q

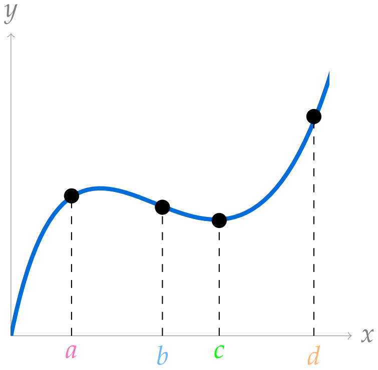
ans

Practice: Increasing and Decreasing



Where is $f'(x) < 0$? $f'(b) < 0$

Practice: Increasing and Decreasing

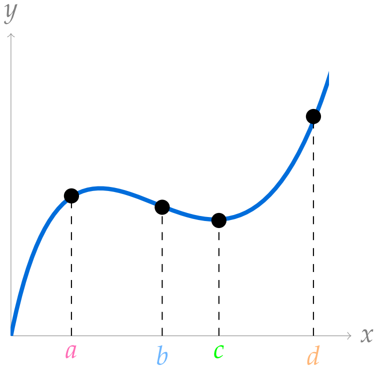


Where is $f'(x) > 0$?

Q

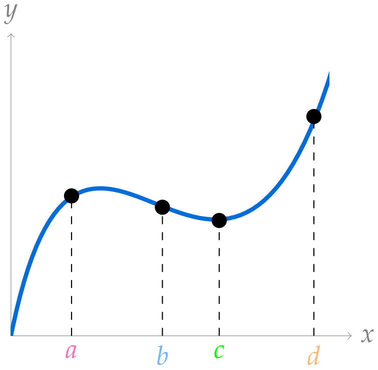
ans

Practice: Increasing and Decreasing



Where is $f'(x) > 0$? $f'(a) > 0$ and $f'(d) > 0$

Practice: Increasing and Decreasing

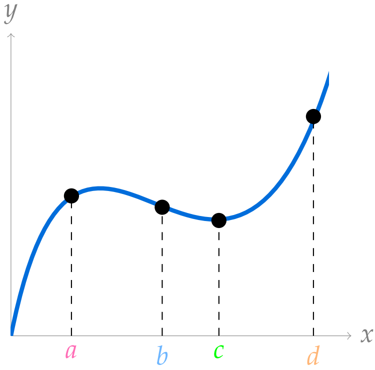


Where is $f'(x) \approx 0$?

Q

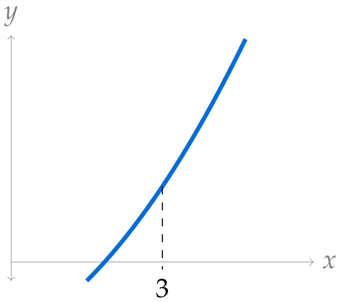
ans

Practice: Increasing and Decreasing



Where is $f'(x) \approx 0$? $f'(c) \approx 0$

Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.



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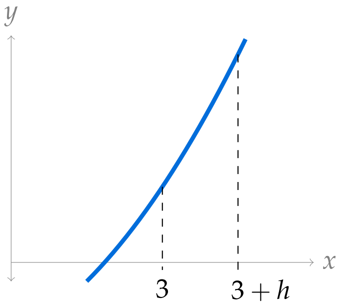

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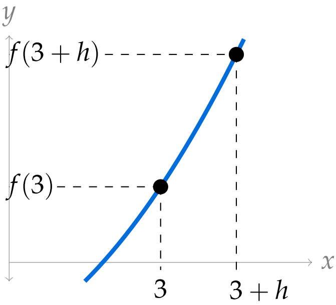

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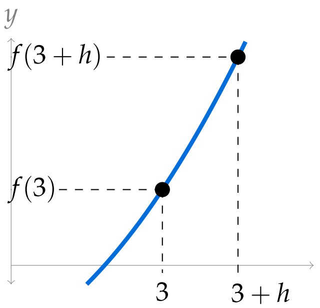


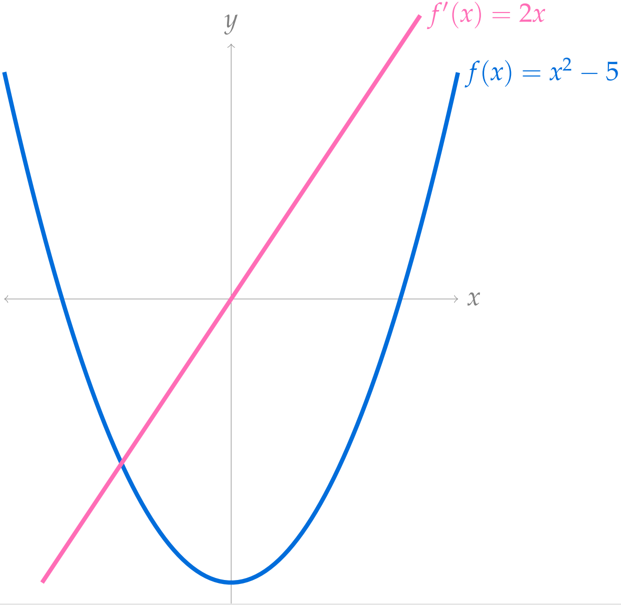
$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{\left((3+h)^2 - 5\right) - (3^2 - 5)}{h} \\ &= \lim_{h\to 0} \frac{\left(9 + 6h + h^2 - 5\right) - 4}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 6h}{h} \\ &= \lim_{h\to 0} h + 6 = 6 \end{aligned}$$

Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$. We can also find its derivative at an arbitrary point x:

ans

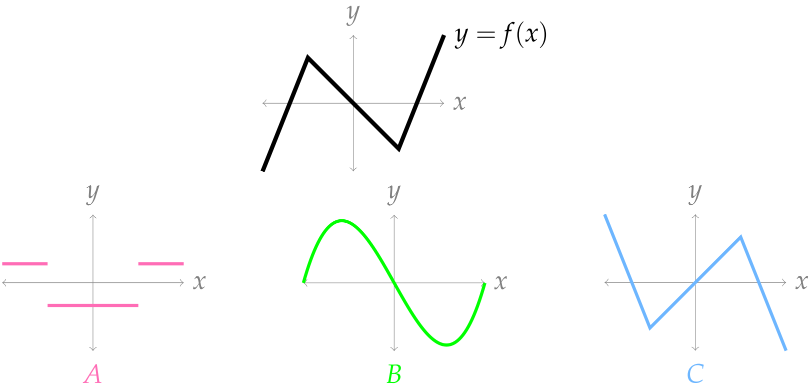
Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$. We can also find its derivative at an arbitrary point x:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 5 - (x^2 - 5)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h = 2x \qquad \text{(In particular, } f'(3) = 6.\text{)}
\end{aligned}$$


$f'(x) = 2x$
$f(x) = x^2 - 5$
y
x

Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?

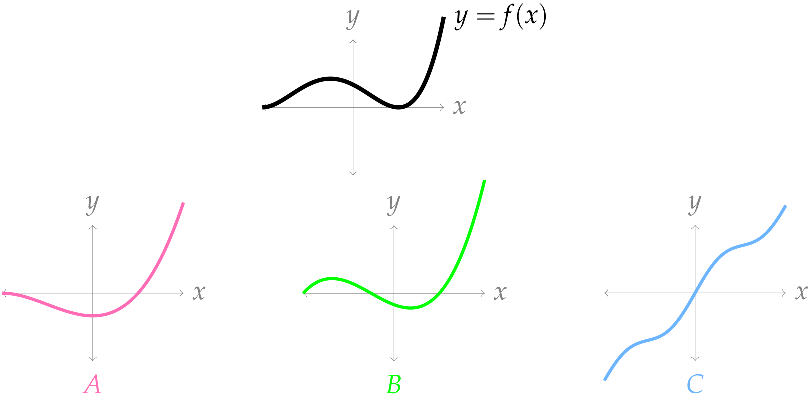


Q

ans

Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.
The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that x will be a part of your final expression: this is a function.

If $f'(x)$ exists for all x in an interval (a, b), we say that f is differentiable on (a, b).

Notation 2.2.8

The "prime" notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(a) \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)\bigg|_{x=a}$$

Notation 2.2.8

The "prime" notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibniz notation:

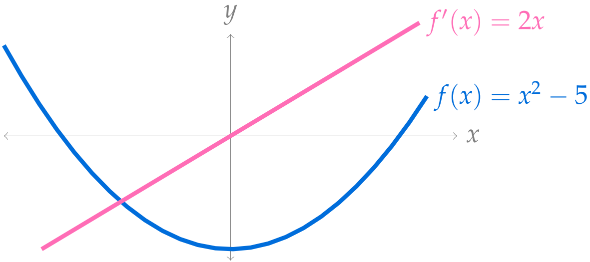
$\dfrac{\mathrm{d}f}{\mathrm{d}x}$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$	$\left.\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)\right\vert_{x=a}$
function	number	function	number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibniz Notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(3) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)\Big|_{x=3} =$$



Newtonian Notation:

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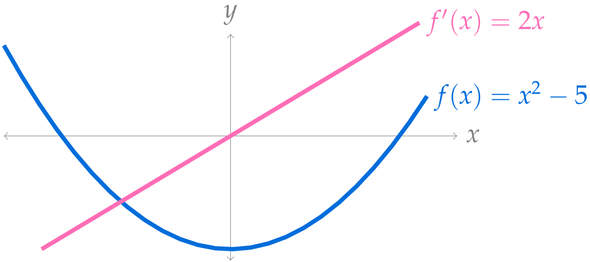

ans

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

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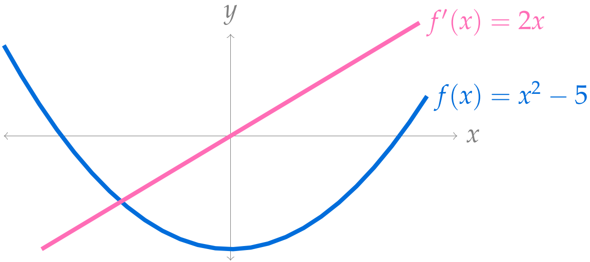
$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2x \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(3) = 6 \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \qquad \left.\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right|_{x=3} =$$



ans

Newtonian Notation:

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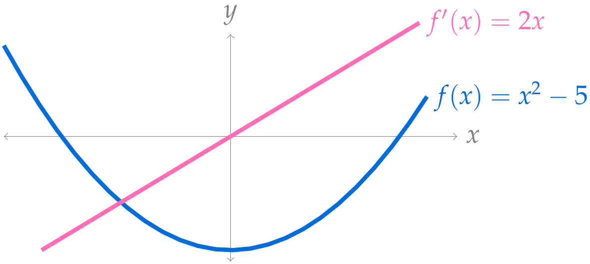

Newtonian Notation:

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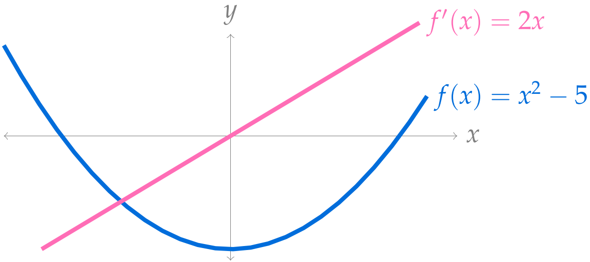


Alternate Definition – Definition 2.2.1

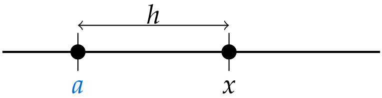
Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

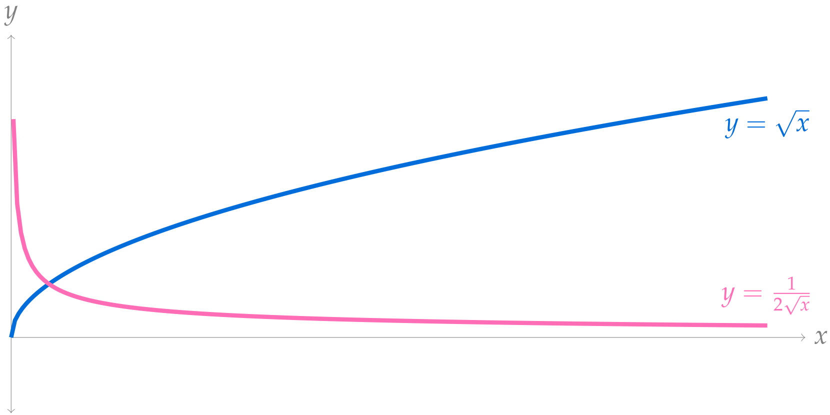
Notice in these scenarios, $h = x - a$.

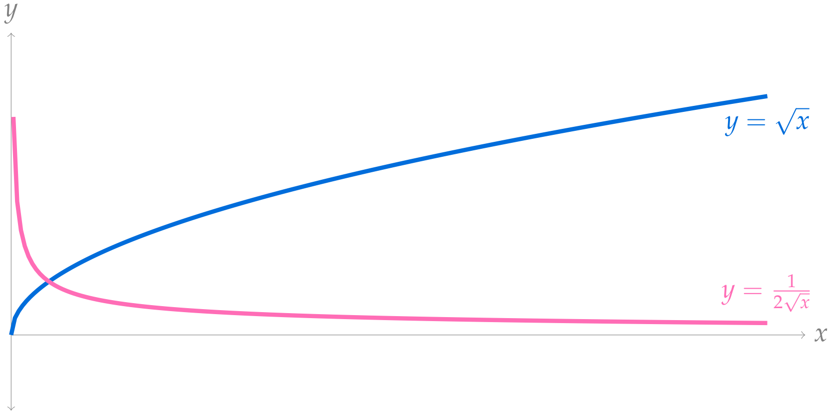


Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.

Let $f(x)=\sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.

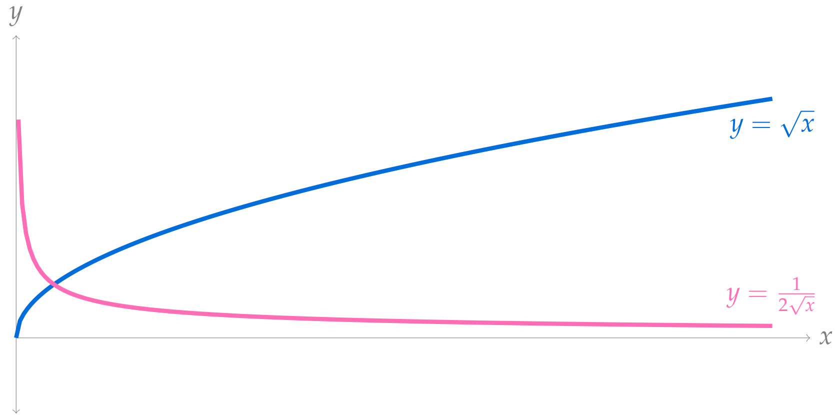
$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\\
&= \lim_{h\to 0}\frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}
\end{aligned}$$


y
x
$y = \sqrt{x}$
$y = \frac{1}{2\sqrt{x}}$

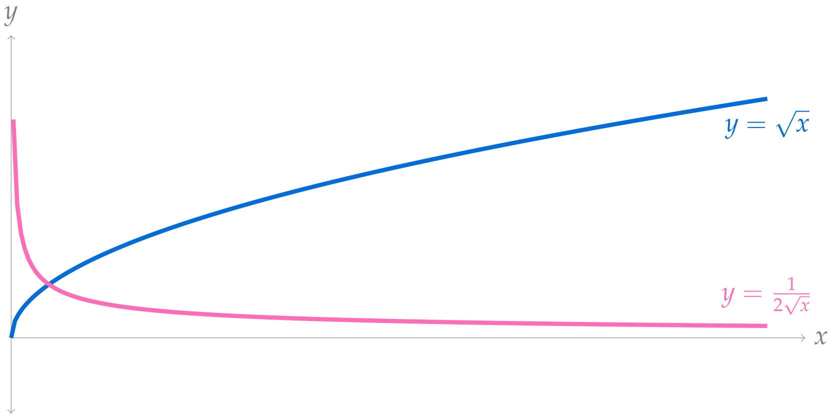


Review: $\lim_{x\to\infty}\sqrt{x}=$ $\lim_{x\to\infty}\frac{1}{2\sqrt{x}}=$

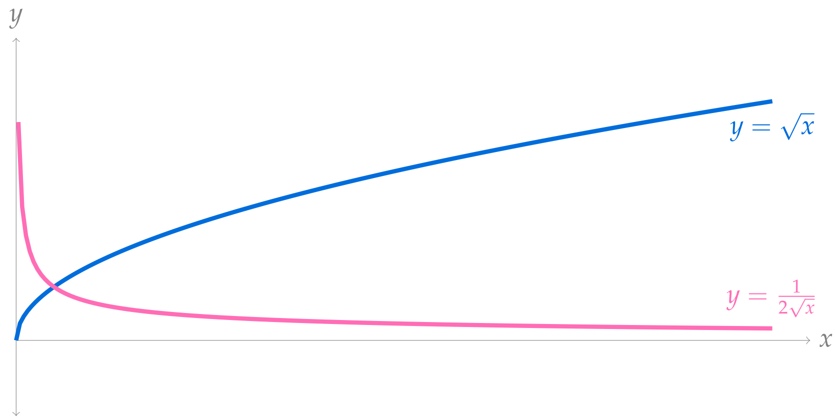
ans



$\boxed{\text{Review:}}$ $\displaystyle\lim_{x\to\infty} \sqrt{x} = \infty$ $\displaystyle\lim_{x\to\infty} \frac{1}{2\sqrt{x}} =$



$\boxed{\text{Review:}}$ $\lim_{x\to\infty} \sqrt{x} = \infty$ $\lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$

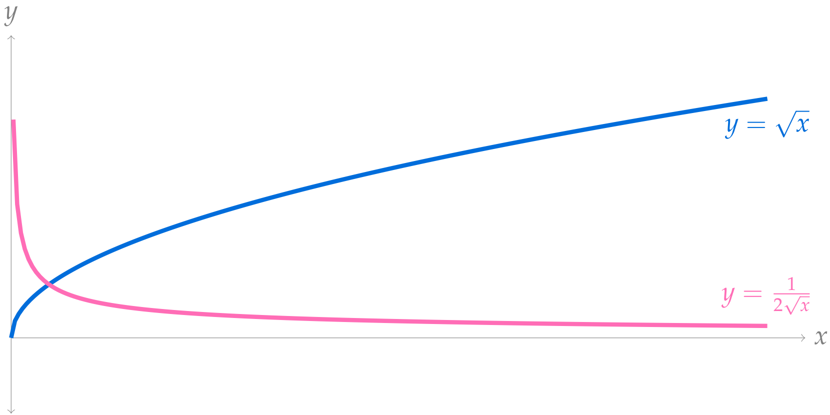


$\boxed{\text{Review:}}$ $\lim_{x\to\infty}\sqrt{x}=\infty$ $\lim_{x\to\infty}\frac{1}{2\sqrt{x}}=0$

$\lim_{x\to 0^+}\sqrt{x}=$ $\lim_{x\to 0^+}\frac{1}{2\sqrt{x}}=$

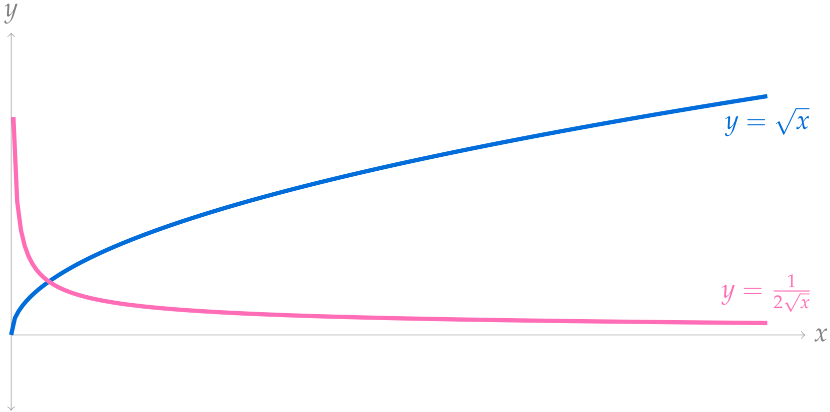
Q

ans



$\boxed{\text{Review:}}$ $\lim_{x\to\infty}\sqrt{x}=\infty$ $\lim_{x\to\infty}\frac{1}{2\sqrt{x}}=0$

$\lim_{x\to 0^+}\sqrt{x}=0$ $\lim_{x\to 0^+}\frac{1}{2\sqrt{x}}=$



$\boxed{\text{Review:}}$ $\lim\limits_{x\to\infty}\sqrt{x}=\infty$ $\lim\limits_{x\to\infty}\dfrac{1}{2\sqrt{x}}=0$

$\lim\limits_{x\to0^+}\sqrt{x}=0$ $\lim\limits_{x\to0^+}\dfrac{1}{2\sqrt{x}}=\infty$

Now You

Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\right\}.$$

Q

Now You

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x}\right\}$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\right] &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}\end{aligned}$$

Now You Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x}{x+1}\right\}.$$

Now You Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x^2+x}}\right\}.$$

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{2x}{x+1}\right\}$.

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{2x}{x+1}\right\}$.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{2(x+h)}{x+h+1}-\frac{2x}{x+1}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{2(x+h)(x+1)}{(x+h+1)(x+1)}-\frac{2x(x+h+1)}{(x+1)(x+h+1)}\right)\\
&= \lim_{h\to 0}\frac{2}{h}\left(\frac{(x^2+x+xh+h)-(x^2+xh+x)}{(x+h+1)(x+1)}\right)\\
&= \lim_{h\to 0}\frac{2}{h}\left(\frac{h}{(x+h+1)(x+1)}\right)\\
&= \lim_{h\to 0}\frac{2}{(x+h+1)(x+1)} = \frac{2}{(x+1)^2}
\end{aligned}$$

A

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{\sqrt{x^2+x}}\right\}$.

Q

ans

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{\sqrt{x^2+x}}\right\}$.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{(x+h)^2+x+h}}-\frac{1}{\sqrt{x^2+x}}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{x^2+x}}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}}-\frac{\sqrt{(x+h)^2+x+h}}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sqrt{x^2+x}-\sqrt{(x+h)^2+x+h}}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}}\right)\left(\frac{\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}}{\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{(x^2+x)-[(x+h)^2+x+h]}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}\left[\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}\right]}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{-(2xh+h^2+h)}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}\left[\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}\right]}\right)$$

$$=\lim_{h\to 0}\frac{-(2x+h+1)}{\sqrt{(x^2+h)^2+x+h}\sqrt{x^2+x}\left[\sqrt{x^2+x}+\sqrt{(x+h)^2+x+h}\right]}=\frac{-(2x+1)}{2(x^2+x)^{3/2}}$$

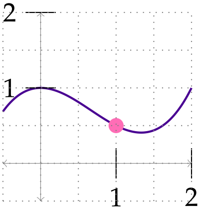
Memorize

The derivative of a function f at a point a is given by the following limit, if it exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Zooming In

For a smooth function, if we zoom in at a point, we see a line:



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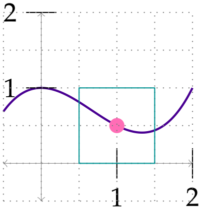

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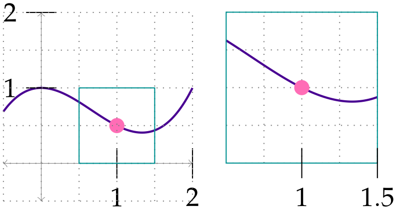

Zooming In

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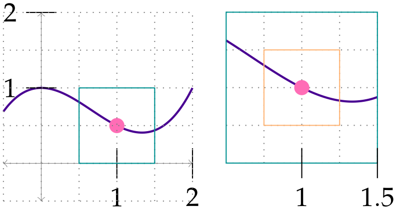

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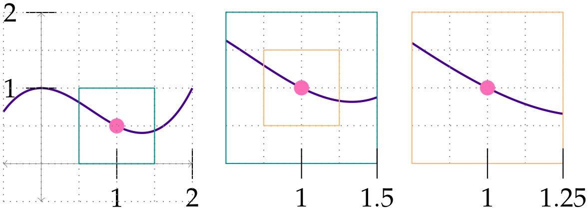

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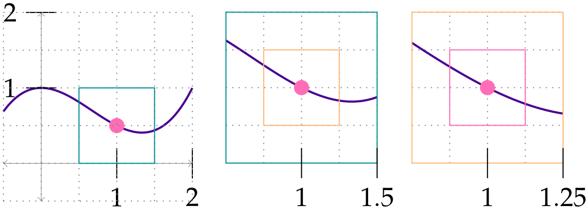

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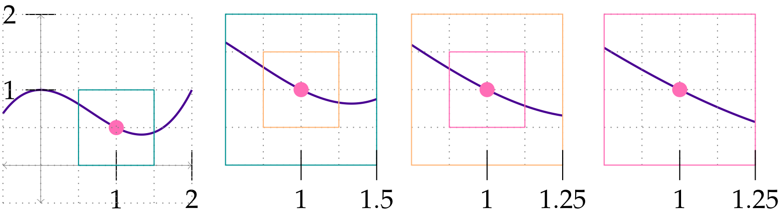

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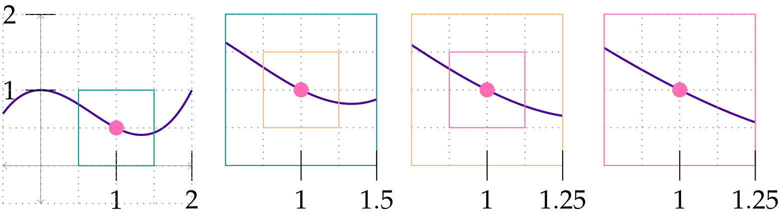


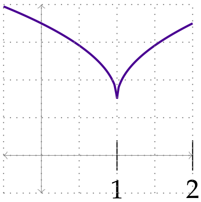
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



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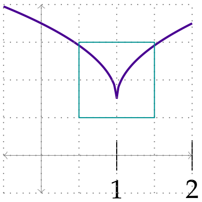

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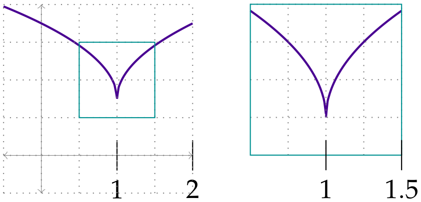

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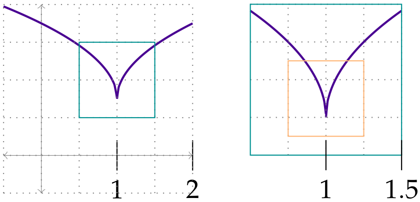

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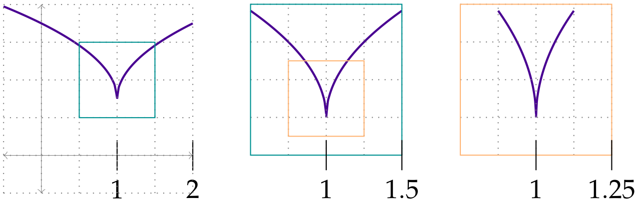

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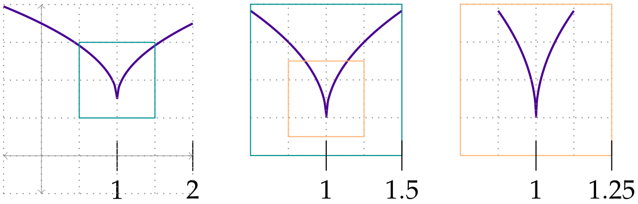
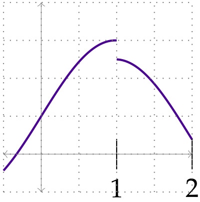


Discontinuity:



Zooming in on functions that aren't smooth

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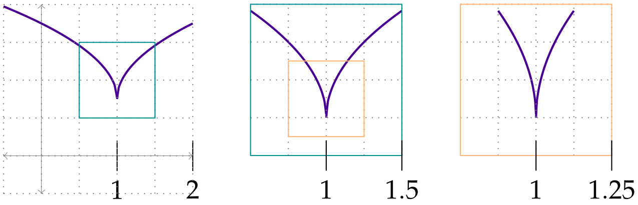
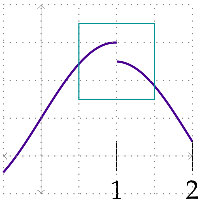


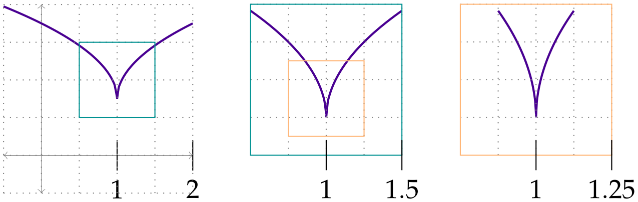
Discontinuity:



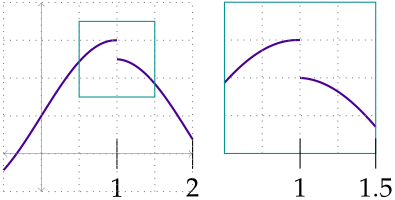
ZOOMING IN ON FUNCTIONS THAT AREN'T SMOOTH

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



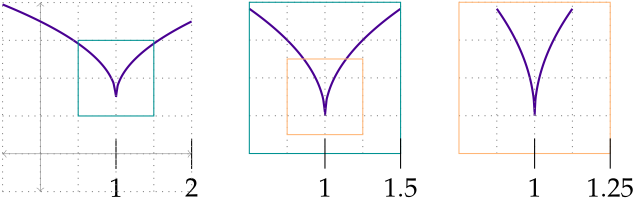
Discontinuity:



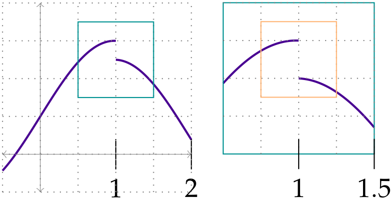
Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



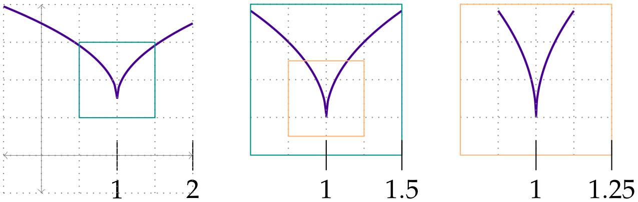
Discontinuity:



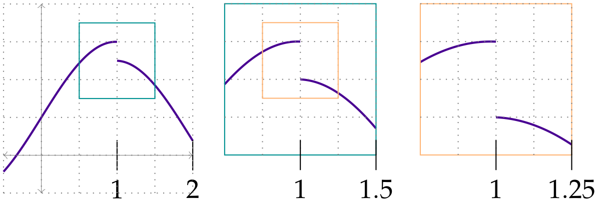
Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:

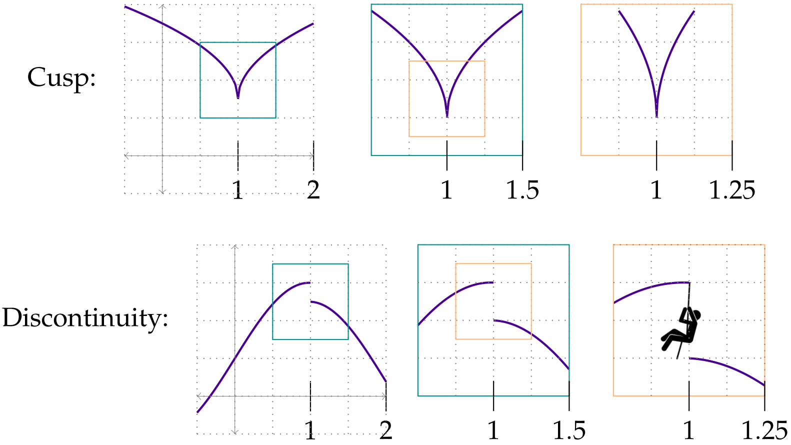


Discontinuity:



Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.



Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ does not exist at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

does not exist.
Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

When Derivatives Don't Exist

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

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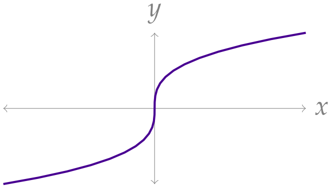
Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty \end{aligned}$$

Since the limit does not exist, we conclude $f'(x)$ is not defined at $x = 0$.

We can go a little farther: since the limit goes to infinity, the graph $y = f(x)$ looks vertical at $x = 0$.



Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof: If $f'(a)$ exists, that means:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad \text{exists}$$

$$\implies \lim_{h\to 0}\left[h \cdot \frac{f(a+h)-f(a)}{h}\right] = \left[\lim_{h\to 0} h\right] \cdot \left[\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right]$$

$$\implies \lim_{h\to 0}\left[h \cdot \frac{f(a+h)-f(a)}{h}\right] = 0$$

$$\implies \lim_{h\to 0}\left[f(a+h)-f(a)\right] = 0$$

$$\implies \lim_{h\to 0} f(a+h) = f(a)$$

and that is the definition of $f(x)$ being continuous at $x = a$.

Let $f(x)$ be a function and let a be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

$f(x)$ continuous at $x=a$ $f(x)$ differentiable at $x=a$	$f(x)$ continuous at $x=a$ $f(x)$ not differentiable at $x=a$
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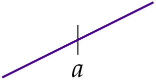
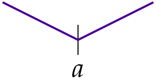
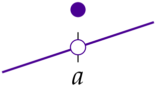
$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$  	$f(x)$ continuous at $x = a$ $f(x)$ not differentiable at $x = a$ 
$f(x)$ not continuous at $x = a$ $f(x)$ differentiable at $x = a$ impossible	$f(x)$ not continuous at $x = a$ $f(x)$ not differentiable at $x = a$ 

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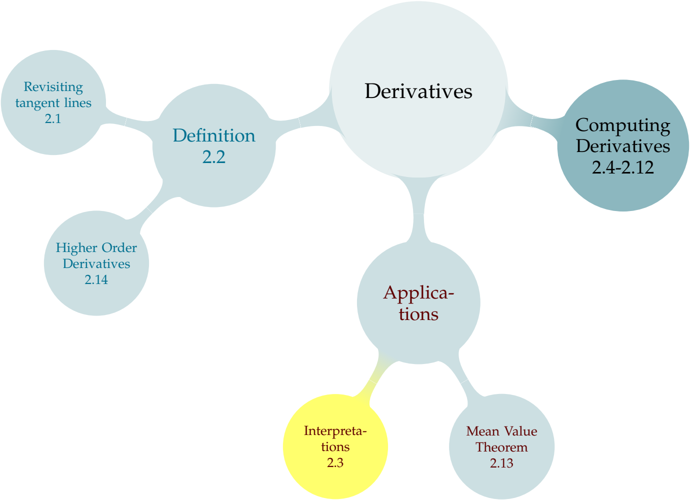

Interpreting the Derivative

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Suppose $P(t)$ gives the number of people in the world at t minutes past midnight, January 1, 2012. Suppose further that $P'(0) = 156$. How do you interpret $P'(0) = 156$?

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Suppose $P(t)$ gives the number of people in the world at t minutes past midnight, January 1, 2012. Suppose further that $P'(0) = 156$. How do you interpret $P'(0) = 156$?

At midnight of January 1, 2012, the world population was increasing at a rate of 156 people each minute

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Suppose $P(n)$ gives the total profit, in dollars, earned by selling n widgets. How do you interpret $P'(100)$?

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How fast your profit is increasing as you sell more widgets, measured in dollars per widget, at the time you sell Widget #100. So, roughly the profit earned from the sale of the 101st widget.

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Suppose $h(t)$ gives the height of a rocket t seconds after liftoff. What is the interpretation of $h'(t)$?

Q

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Suppose $h(t)$ gives the height of a rocket t seconds after liftoff. What is the interpretation of $h'(t)$?

The speed at which the rocket is rising at time t.

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The rate (measured in molecules per second) at which the number of molecules of a certain type is changing. Roughly, how many molecules of that type are being added (or taken away, if negative) per second at time t.

A

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $G(w)$ gives the diameter in millimetres of steel wire needed to safely support a load of w kg. Suppose further that $G'(100) = 0.01$. How do you interpret $G'(100) = 0.01$?

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When your load is about 100 kg, you need to increase the diameter of your wire by about 0.01 mm for each kg increase in your load.

A

A paper[1] on the impacts of various factors in average life expectancy contains the following:

> *The only statistically significant variable in the model is physician density. The coefficient for this variable 20.67 indicating that a one unit increase in physician density leads to a 20.67 unit increase in life expectancy. This variable is also statistically significant at the 1% level demonstrating that this variable is very strongly and positively correlated with quality of healthcare received. This denotes that access to healthcare is very impactful in terms of increasing the quality of health in the country.*

[1] Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami, *The Effect of National Healthcare Expenditure on Life Expectancy*, page 12.

Remark: physician density is measured as number of doctors per 1000 members of the population.

If $L(p)$ is the average life expectancy in an area with a density p of physicians, write the statement as a derivative: “a one unit increase in physician density leads to a 20.67 unit increase in life expectancy.”

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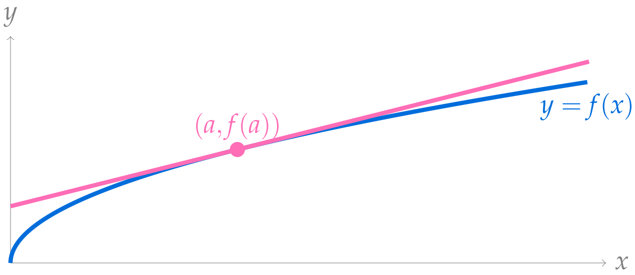
$L'(p) = 20.67$

Equation of the Tangent Line

The tangent line to $f(x)$ at a has slope $f'(a)$ and passes through the point $(a, f(a))$.



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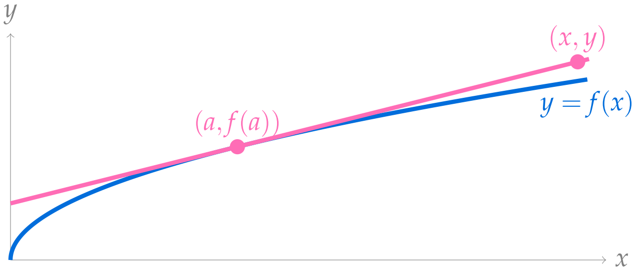

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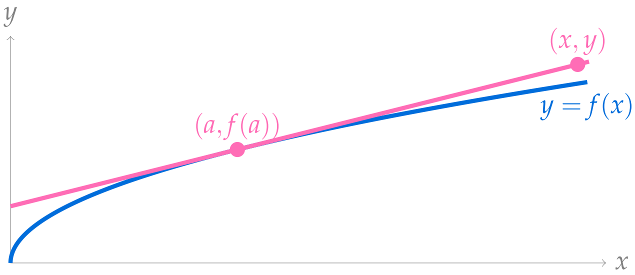

Slope of the tangent line:

$$f'(a) = \frac{\text{rise}}{\text{run}} = \frac{y - f(a)}{x - a}$$

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Slope of the tangent line:

$$f'(a) = \frac{\text{rise}}{\text{run}} = \frac{y - f(a)}{x - a}$$

Rearranging: $y - f(a) = f'(a)(x - a)$ (equation of tangent line)

Tangent Line Equation – Theorem 2.3.2

The tangent line to the function $f(x)$ at point a is:

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In general, a line with slope m passing through point (x_1, y_1) has the equation:

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Find the equation of the tangent line to the curve $f(x) = \sqrt{x}$ at $x = 9$. (Recall $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}$).

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$$a = 9, \quad f(a) = 3, \quad f'(a) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$y - 3 = \frac{1}{6}(x - 9)$$

Memorize

The tangent line to the function $f(x)$ at point a is:

$$(y - f(a)) = f'(a)(x - a)$$

NOW YOU

Let $s(t) = 3 - 0.8t^2$. Then $s'(t) = -1.6t$. Find the equation for the tangent line to the function $s(t)$ when $t = 1$.

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$$a = 1, \quad s(a) = 2.2, \quad s'(a) = -1.6$$
$$y - 2.2 = -1.6(x - 1)$$

Included Work

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 'Brain' by Eucalyp is licensed under CC BY 3.0 (accessed 8 June 2021), 10, 11, 20–22, 24, 25, 72–77, 143–145, 202, 203

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Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami. (2014). The Effect of National Healthcare Expenditure on Life Expectancy, page 12. *College of Liberal Arts - Ivan Allen College (IAC), School of Economics: Econometric Analysis Undergraduate Research Papers*. https://smartech.gatech.edu/handle/1853/51648 (accessed July 2021), 189