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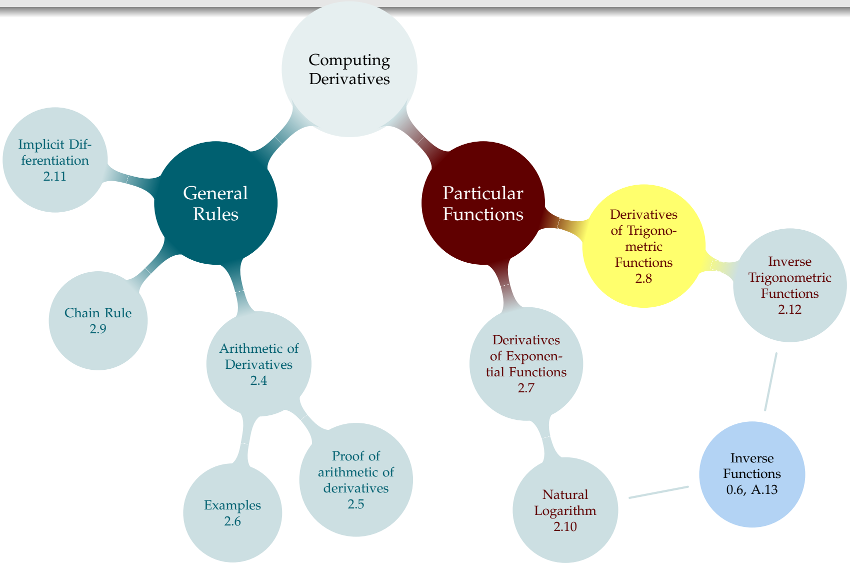

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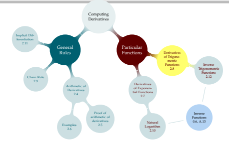
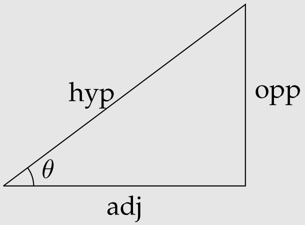


Students often feel stressed because they don't know which facts from trigonometry class they are responsible for. So, we start with a quick overview – less to teach these things than to list what they are responsible for knowing

Basic Trig Functions



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$

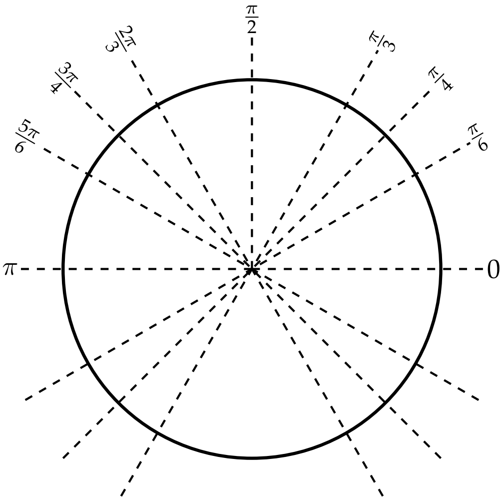
$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

COMMONLY USED FACTS

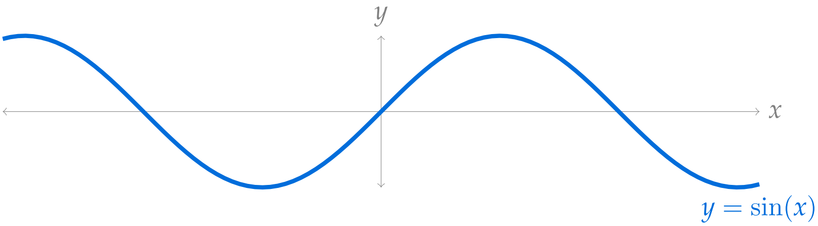
- Graphs of sine, cosine, tangent
- Sine, cosine, and tangent of reference angles: $0, \dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\pi}{2}$
- How to use reference angles to find sine, cosine and tangent of other angles
- Identities: $\sin^2 x + \cos^2 x = 1;\qquad \tan^2 x + 1 - \sec^2 x;$
  $\sin^2 x = \dfrac{1-\cos(2x)}{2};\qquad \cos^2 x = \dfrac{1+\cos 2x}{2}$
- Conversion between radians and degrees

CLP-1 has an appendix on high school trigonometry that you should be familiar with.

Reference Angles



Derivative of Sine

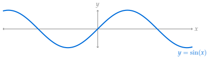


Consider the derivative of $f(x) = \sin(x)$.

2.8: Derivatives of Trig Functions

Derivative of Sine

Derivative of Sine



Consider the derivative of $f(x) = \sin(x)$.

By plotting points we know (easy to find where deriv is zero; then where it's positive/negative/max) it seems *likely* that the deriv of sine is cosine. But this isn't a proof. Nice to mention that when we're using degrees instead of radians, the deriv of sine is NOT cosine. Proof follows but it's basically a TED talk – you won't be assessed on it.

$$\frac{d}{dx}\{\sin x\} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x)\lim_{h\to 0} \frac{\cos(0+h)-\cos(0)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\,\frac{d}{dx}\{\cos(x)\}\Big|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \qquad = \qquad \boxed{\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}}$$

since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.

2.8: Derivatives of Trig Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin x\} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

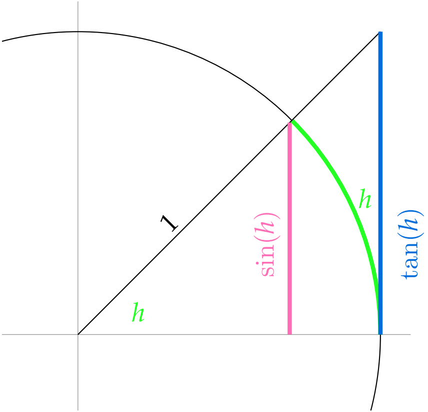
$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x)\lim_{h\to 0}\frac{\cos(0+h)-\cos(0)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\left.\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\}\right|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \qquad = \qquad \boxed{\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}}$$

since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x=0$.

Now, the question becomes: what is that limit? Once again, it's important to note that we are measuring in radians, not degrees.


1
h
$\sin(h)$
h
$\tan(h)$

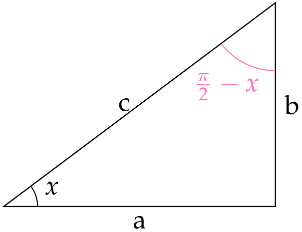
Derivatives of Sine and Cosine

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h} = \cos(x)$$

Derivative of Cosine

Now for the derivative of cos. We already know the derivative of sin, and it is easy to convert between sin and cos using trig identities.



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

When we use radians:

Derivatives of Trig Functions

$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} = -\sin(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} =$

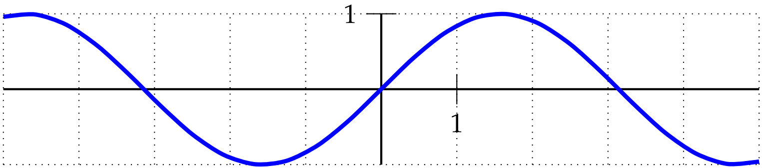
$\frac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} =$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} =$

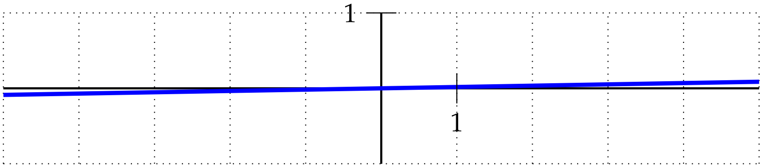
$\frac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} =$

Honorable Mention

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

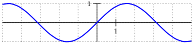


$y = \sin x$, radians

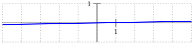


$y = \sin x$, degrees

2.8: Derivatives of Trig Functions

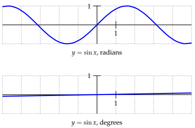


$y = \sin x$, radians



$y = \sin x$, degrees

Remember, these are only true using radians.



Can go through tangent, but no need to do the rest. Just make sure students know how they could be computed from sines and cosines, should they so desire

Other Trig Functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Other Trig Functions

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}
\frac{d}{dx}[\sec(x)] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] \\
&= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} \\
&= \sec(x)\tan(x)
\end{aligned}$$

Other Trig Functions

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}
\frac{d}{dx}[\csc(x)] &= \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] \\
&= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin(x)}\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x)
\end{aligned}$$

Other Trig Functions

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\cot(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{\sin(x)}\right] \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}$$

MEMORIZE

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} = \sec(x)\tan(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} = -\sin(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} = -\csc(x)\cot(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} = \sec^2(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.

Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Practice and Review

Practice and Review

Not to do in class (unless you have extra time to fill)

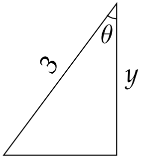
$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

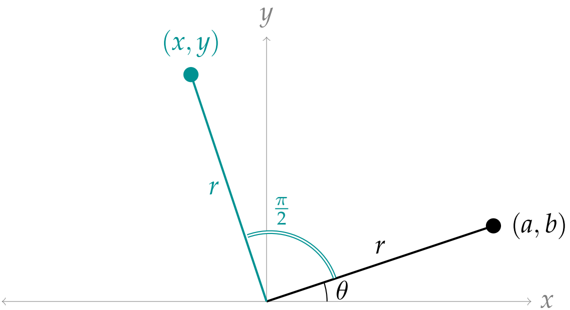
$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x-a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ?
When is the top of the ladder sinking the fastest? The slowest?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y)? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ?



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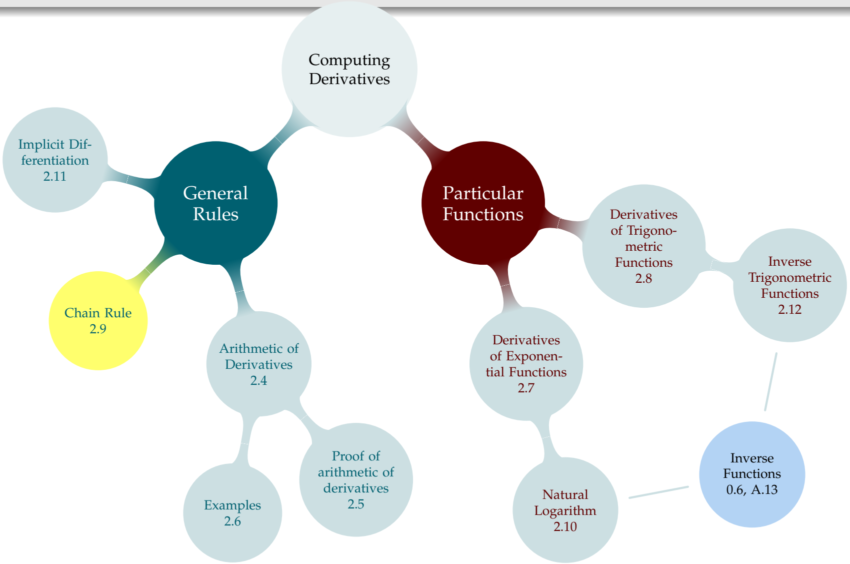
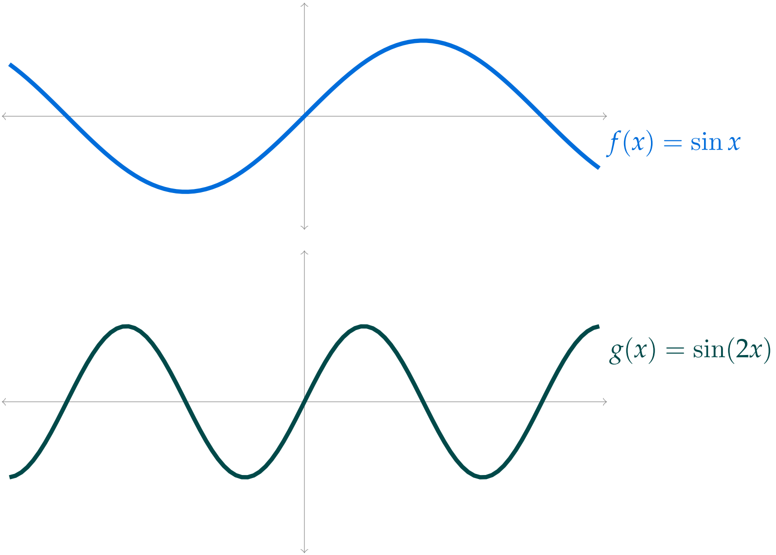
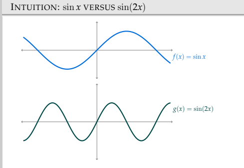


Intuition: $\sin x$ versus $\sin(2x)$





Intuition: sin 2x changes its y-values "twice as fast" as sin x, making it "twice as steep." So it's not enough to differentiate the outside function – something else has to happen.

Compound Functions

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*. Available from https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*. Available from https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp

Chain rule works on functions-of-functions; otters/urchins/kelp are a nice example

KELP POPULATION

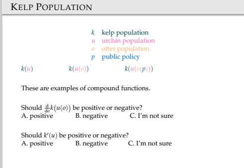
k	kelp population
u	urchin population
o	otter population
p	public policy

$k(u)$ $\qquad$ $k(u(o))$ $\qquad$ $k(u(o(p)))$

These are examples of compound functions.

Should $\frac{\mathrm{d}}{\mathrm{d}o}k\big(u(o)\big)$ be positive or negative?
A. positive $\qquad$ B. negative $\qquad$ C. I'm not sure

Should $k'(u)$ be positive or negative?
A. positive $\qquad$ B. negative $\qquad$ C. I'm not sure



It's nice to show an example of a function whose "derivative" can be two different things (depending on the variable). Now that our heads are in "function of a function" territory, chain rule. I usually just flash this slide to emphasize that all these rules are shorthand for a calculation using the definition of a derivative.

DIFFERENTIATING COMPOUND FUNCTIONS

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\left(\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right)\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f\left(\boxed{g(x+h)}\right)-f\left(\boxed{g(x)}\right)}{\boxed{g(x+h)}-\boxed{g(x)}}\cdot g'(x)
\end{aligned}
$$

Set $H = g(x+h)-g(x)$. As $h\to 0$, we also have $H\to 0$. So

$$
\begin{aligned}
&= \lim_{H\to 0}\frac{f(g(x)+H)-f(g(x))}{H}\cdot g'(x)\\
&= f'(g(x))\cdot g'(x)
\end{aligned}
$$

Chain Rule

Chain Rule – Theorem 2.9.3

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))\, g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

In the case of kelp, $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o)) = \frac{\mathrm{d}k}{\mathrm{d}u}(u(o))\,\frac{\mathrm{d}u}{\mathrm{d}o}(o)$

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f\left(g(x)\right)\} = f'\left(g(x)\right) g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}\left(g(x)\right)\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))\,g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

I generally put the “inside” function in a box, to emphasize we’re treating the whole thing as one variable

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$

Now You

Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

Now You Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t}u\big(o(t)\big)$.
Note: your answer should depend only on t: not o.

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\dfrac{1}{x}\right)\right\}$

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

Included Work

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