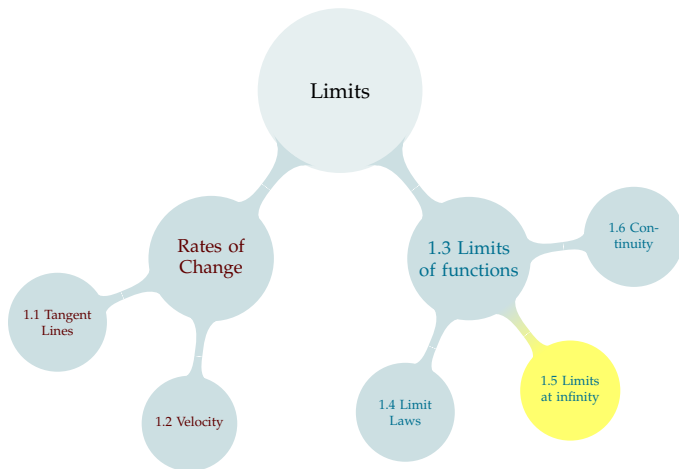


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# END BEHAVIOR

We write:

$$\lim_{x \rightarrow \infty} f(x) = L$$

to express that, as  $x$  grows larger and larger,  $f(x)$  approaches  $L$ .

Similarly, we write:

$$\lim_{x \rightarrow -\infty} f(x) = L$$

to express that, as  $x$  grows more and more strongly negative,  $f(x)$  approaches  $L$ .

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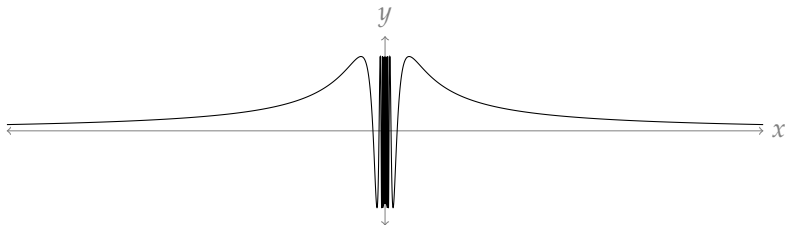
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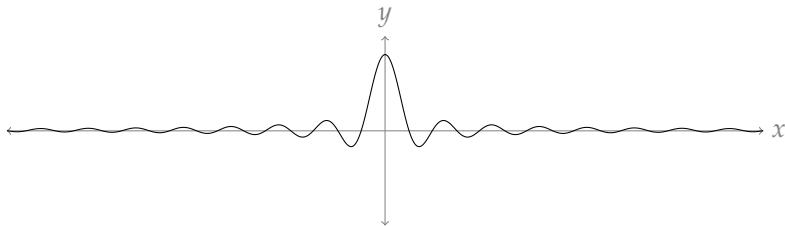
If  $L$  is a number, we call  $y = L$  a **horizontal asymptote** of  $f(x)$ .

# HORIZONTAL ASYMPTOTES



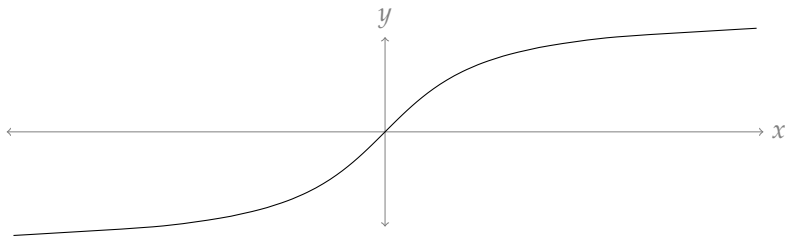
$y = 0$  is a horizontal asymptote for  $y = \sin\left(\frac{1}{x}\right)$

# HORIZONTAL ASYMPTOTES



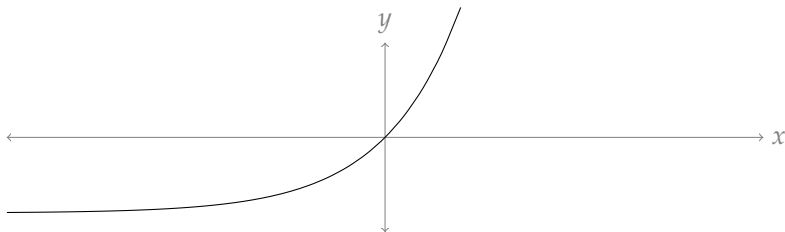
$y = 0$  is a horizontal asymptote for  $y = \frac{\sin x}{x}$

# HORIZONTAL ASYMPTOTES



$y = \frac{\pi}{2}$  and  $y = -\frac{\pi}{2}$  are horizontal asymptotes for  $y = \arctan(x)$

# HORIZONTAL ASYMPTOTES



$y = -1$  is a horizontal asymptote for  $y = e^x - 1$

## COMMON LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} 13 =$$

$$\lim_{x \rightarrow -\infty} 13 =$$

$$\lim_{x \rightarrow \infty} x^3 =$$

$$\lim_{x \rightarrow -\infty} x^3 =$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} =$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} =$$

$$\lim_{x \rightarrow -\infty} x^{5/3} =$$

$$\lim_{x \rightarrow -\infty} x^{2/3} =$$

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## COMMON LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} 13 = 13$$

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$$\lim_{x \rightarrow \infty} x^2 = \infty$$

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## ARITHMETIC WITH LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} \left( x + \frac{x^2}{10} \right) =$$



$$\lim_{x \rightarrow \infty} \left( x - \frac{x^2}{10} \right) =$$

$$\lim_{x \rightarrow -\infty} (x^2 + x^3 + x^4) =$$

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## ARITHMETIC WITH LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} \left( x + \frac{x^2}{10} \right) = \infty$$



$$\lim_{x \rightarrow \infty} \left( x - \frac{x^2}{10} \right) = \lim_{x \rightarrow \infty} x \left( 1 - \frac{x}{10} \right) = -\infty$$

$$\lim_{x \rightarrow -\infty} (x^2 + x^3 + x^4) = \lim_{x \rightarrow -\infty} x^4 \left( \frac{1}{x^2} + \frac{1}{x} + 1 \right) = \infty$$

$$\lim_{x \rightarrow -\infty} (x + 13) (x^2 + 13)^{1/3} = -\infty$$

## CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3}$$

## CALCULATING LIMITS AT INFINITY

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Trick: factor out largest power of denominator.

## CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3}$$

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3} &= \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3} \left( \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \right) \\&= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}}{1} = \frac{\lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{2}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^3}}{\lim_{x \rightarrow \infty} 1} \\&= \frac{0 + 0 + 0}{1} = 0\end{aligned}$$

## CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow -\infty} (x^{7/3} - x^{5/3})$$

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Again: factor out largest power of  $x$ .



# CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow -\infty} (x^{7/3} - x^{5/3})$$

Again: factor out largest power of  $x$ .

$$(x^{7/3} - x^{5/3}) = x^{7/3} \left( 1 - \frac{1}{x^{2/3}} \right)$$

$$\left( \text{note: } \lim_{x \rightarrow -\infty} x^{7/3} = -\infty \right)$$

$$\left( \text{note also: } \lim_{x \rightarrow -\infty} \left( 1 - \frac{1}{x^{2/3}} \right) = 1 \right)$$

$$\text{So, } \lim_{x \rightarrow -\infty} (x^{7/3} - x^{5/3}) = -\infty$$



# CALCULATING LIMITS AT INFINITY

Suppose the height of a bouncing ball is given by  $h(t) = \frac{\sin(t)+1}{t}$ , for  $t \geq 1$ . What happens to the height over a long period of time?

# CALCULATING LIMITS AT INFINITY

Suppose the height of a bouncing ball is given by  $h(t) = \frac{\sin(t)+1}{t}$ , for  $t \geq 1$ . What happens to the height over a long period of time?

$$\begin{array}{ccccccc}
 0 & & \leq & & \frac{\sin(t)+1}{t} & & \leq & & \frac{2}{t} \\
 \lim_{t \rightarrow \infty} 0 & & = & & 0 & & = & & \lim_{t \rightarrow \infty} \frac{2}{t}
 \end{array}$$

So, by the Squeeze Theorem,

$$\lim_{t \rightarrow \infty} \frac{\sin(t)+1}{t} = 0$$



# CALCULATING LIMITS AT INFINITY



$$\lim_{x \rightarrow \infty} \sqrt{x^4 + x^2 + 1} - \sqrt{x^4 + 3x^2}$$

# CALCULATING LIMITS AT INFINITY



$$\lim_{x \rightarrow \infty} \sqrt{x^4 + x^2 + 1} - \sqrt{x^4 + 3x^2}$$

Multiply function by conjugate:

$$\begin{aligned} & \left( \sqrt{x^4 + x^2 + 1} - \sqrt{x^4 + 3x^2} \right) \left( \frac{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}}{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}} \right) \\ &= \frac{-2x^2 + 1}{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}} \end{aligned}$$

# CALCULATING LIMITS AT INFINITY



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Factor out highest power:  $x^2$  (same as  $\sqrt{x^4}$ )

$$\begin{aligned} & \frac{-2x^2 + 1}{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}} \left( \frac{1/x^2}{1/\sqrt{x^4}} \right) \\ &= \frac{-2 + \frac{1}{x^2}}{\sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}} + \sqrt{1 + \frac{3}{x^2}}} \end{aligned}$$



# CALCULATING LIMITS AT INFINITY

Multiply function by conjugate:

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$$\lim_{x \rightarrow \infty} \frac{-2 + \frac{1}{x^2}}{\sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}} + \sqrt{1 + \frac{3}{x^2}}} = \frac{-2 + 0}{\sqrt{1 + 0 + 0} + \sqrt{1 + 0}} = \frac{-2}{2} = -1$$



NOW  
YOU



Evaluate  $\lim_{x \rightarrow -\infty} \frac{\sqrt{3+x^2}}{3x}$





NOW  
YOU



Evaluate  $\lim_{x \rightarrow -\infty} \frac{\sqrt{3+x^2}}{3x}$

We factor out the largest power of the denominator, which is  $x$ .

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{3+x^2}}{3x} \left( \frac{1/x}{1/x} \right) = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{3+x^2}}{x}}{3}$$

When  $x < 0$ ,  $\sqrt{x^2} = |x| = -x$

$$\begin{aligned} &= \lim_{x \rightarrow -\infty} \frac{1}{3} \frac{\sqrt{3+x^2}}{-\sqrt{x^2}} \\ &= \lim_{x \rightarrow -\infty} -\frac{1}{3} \sqrt{\frac{3+x^2}{x^2}} \\ &= \lim_{x \rightarrow -\infty} -\frac{1}{3} \sqrt{\frac{3}{x^2} + 1} \\ &= -\frac{1}{3} \end{aligned}$$



## Included Work



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