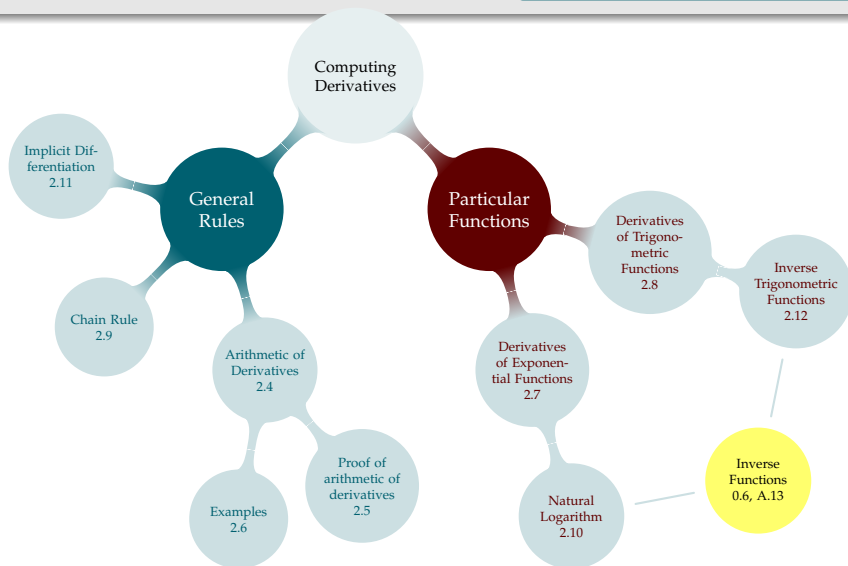


# TABLE OF CONTENTS

▶ SKIP REVIEW OF INVERSE FUNCTIONS AND LOGARITHMS



# INVERTIBILITY GAME

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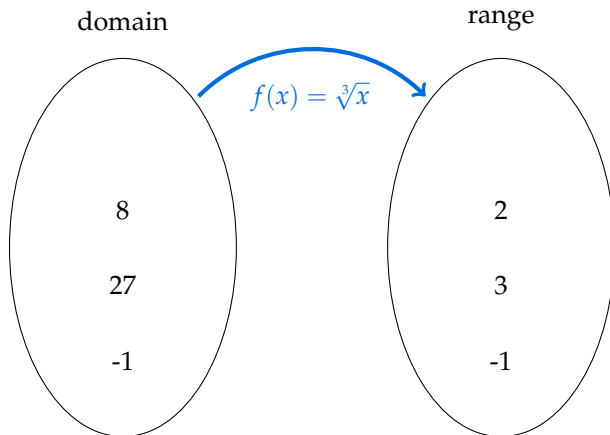
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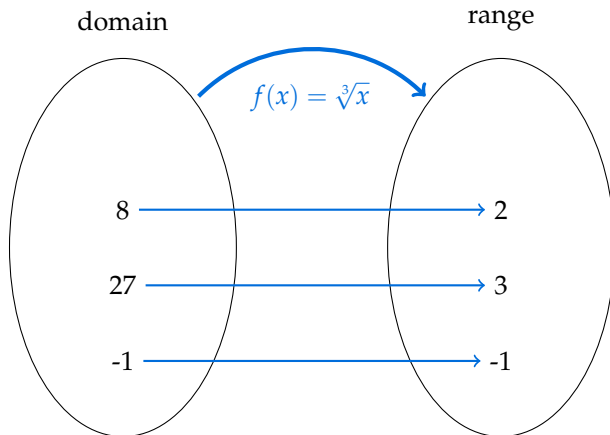
**Round 3:**  $f(x) = |x|$

**Round 4:**  $f(x) = \sin x$

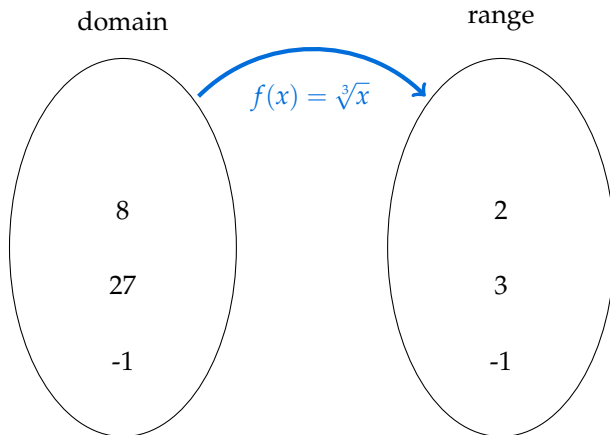
# FUNCTIONS ARE MAPS



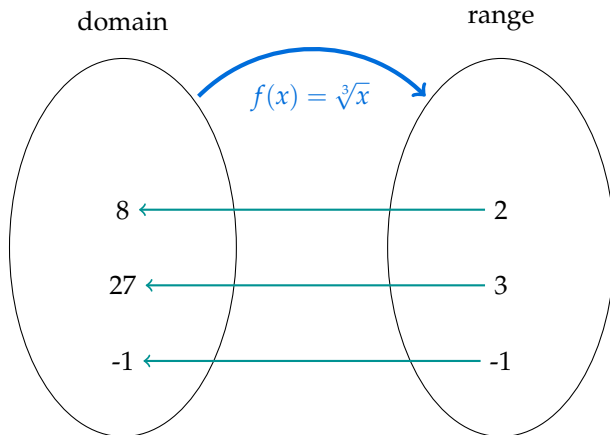
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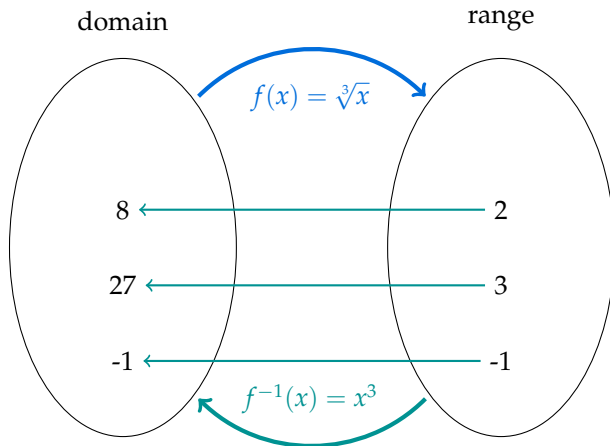
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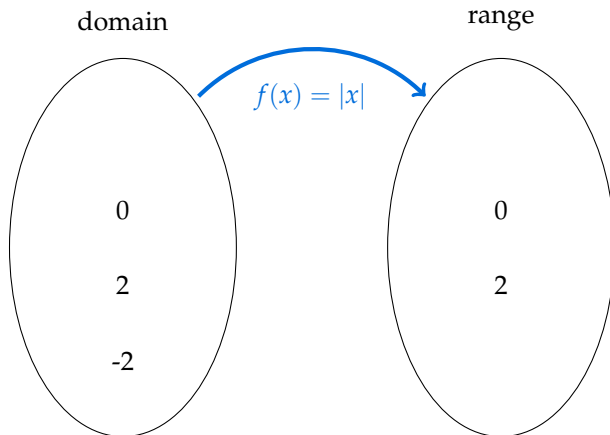
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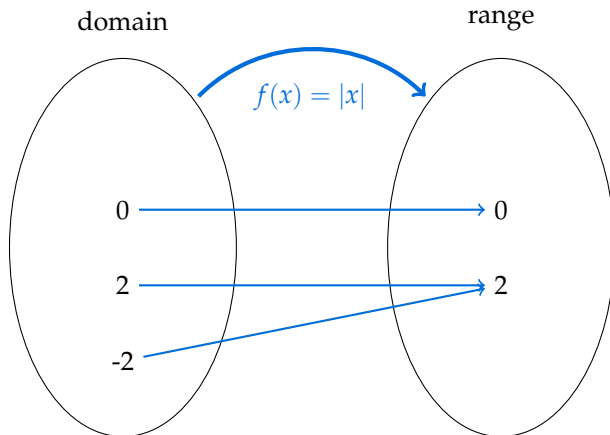


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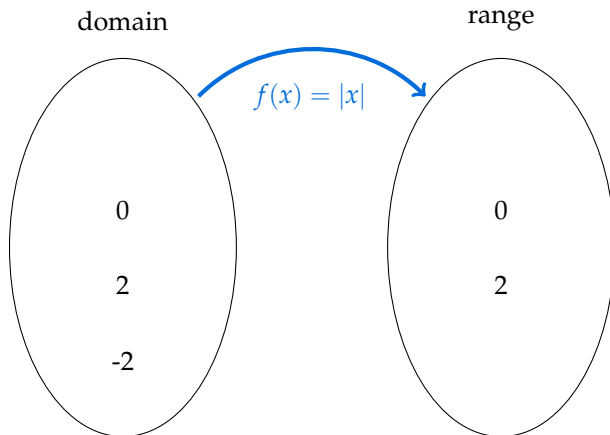




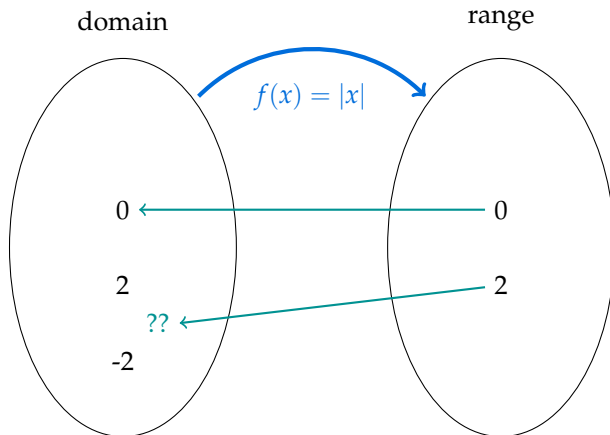
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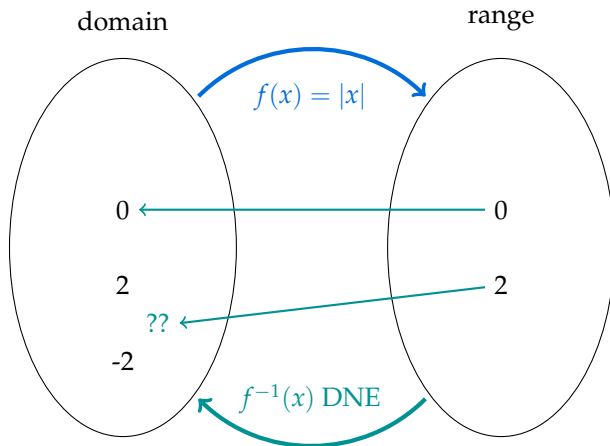
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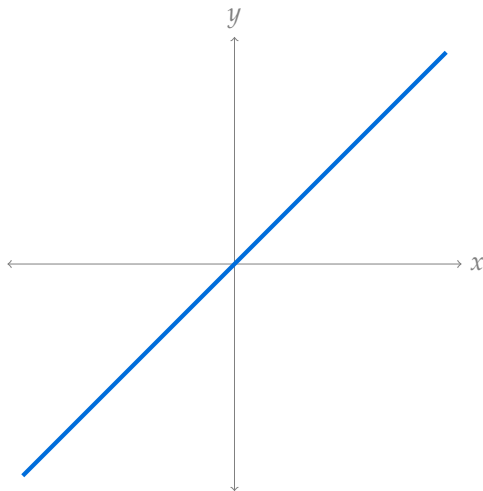


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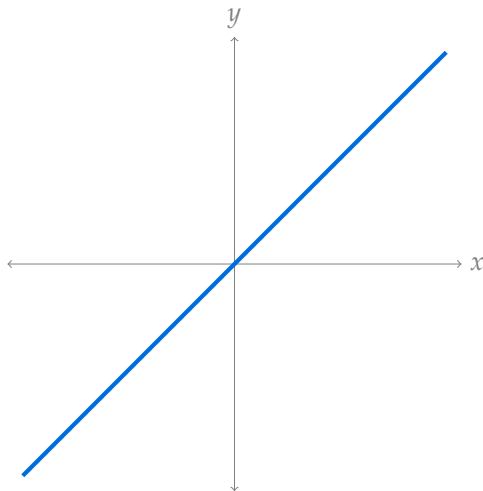
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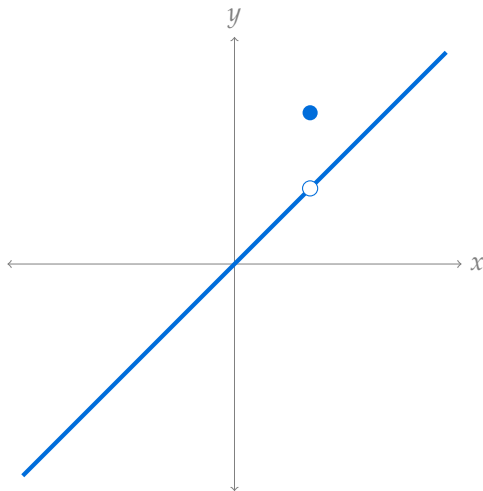
A. invertible

B. not invertible



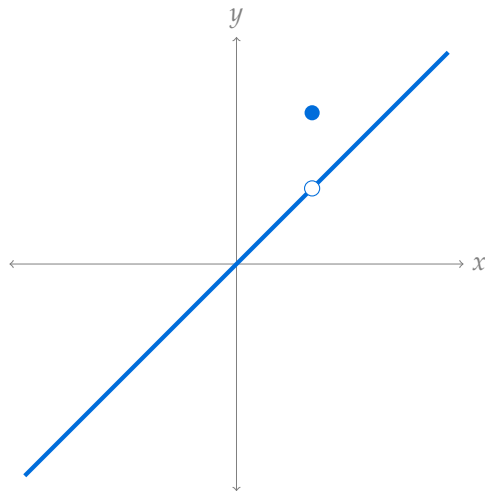
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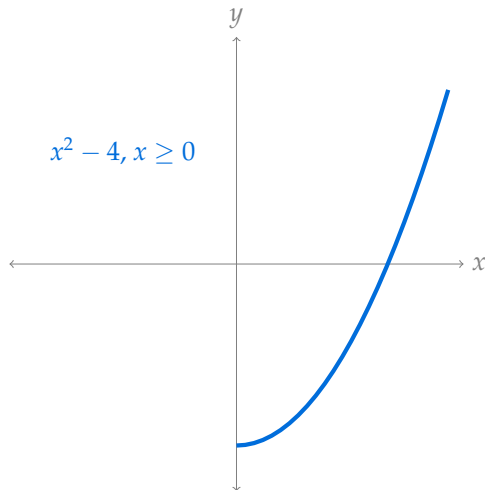
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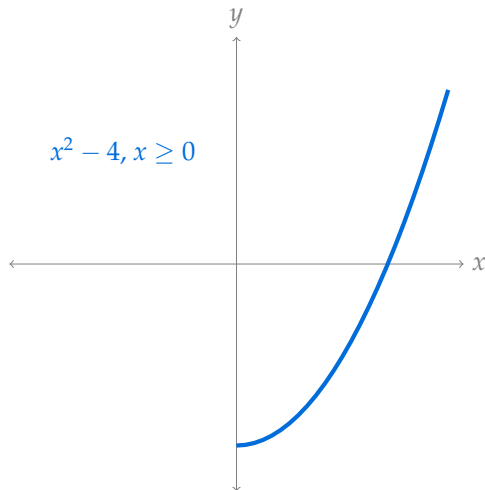
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RELATIONSHIP BETWEEN  $f(x)$  AND  $f^{-1}(x)$ 

Let  $f$  be an invertible function.

What is  $f^{-1}(f(x))$ ?

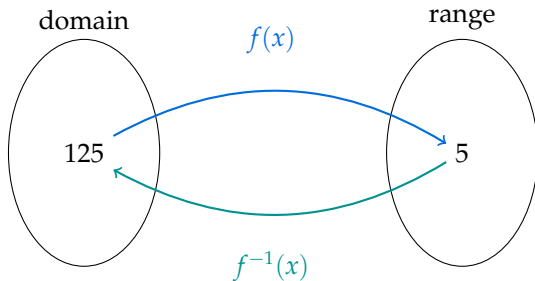
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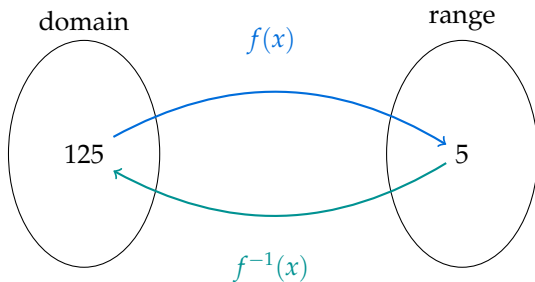


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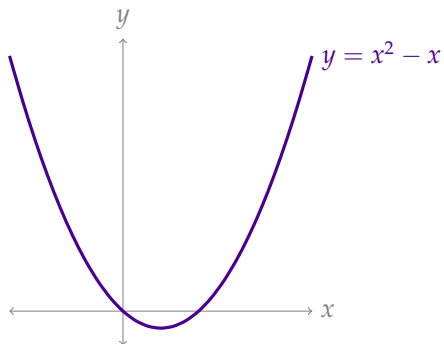
Suppose  $f(x) = \sqrt[3]{19 + x^3}$ . What is  $f^{-1}(3)$ ? (simplify your answer)  
 $f(2) = 3$ , so  $f^{-1}(3) = 2$

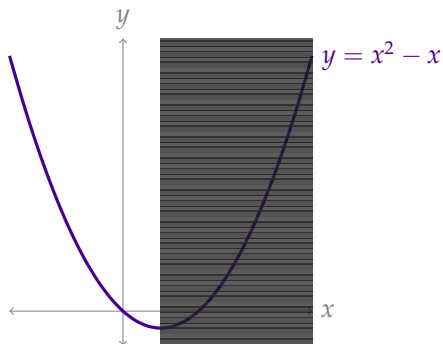
What is  $f^{-1}(10)$ ? (do not simplify)  
 $\sqrt[3]{19 + y^3} = 10$  tells us  $f^{-1}(10) = \sqrt[3]{10^3 - 19}$

What is  $f^{-1}(x)$ ?  
 $\sqrt[3]{19 + y^3} = x$  tells us  $f^{-1}(x) = \sqrt[3]{x^3 - 19}$

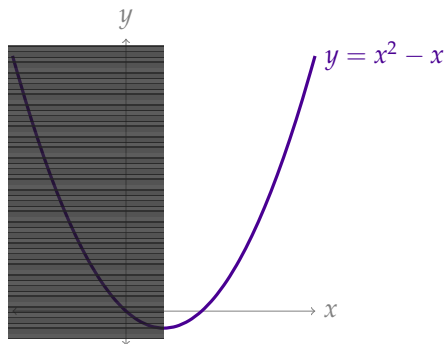
$$\text{Let } f(x) = x^2 - x.$$

1. Sketch a graph of  $f(x)$ , and choose a (large) domain over which it is invertible.
2. For the domain you chose, evaluate  $f^{-1}(20)$ .
3. For the domain you chose, evaluate  $f^{-1}(x)$ .
4. What are the domain and range of  $f^{-1}(x)$ ? What are the (restricted) domain and range of  $f(x)$ ?



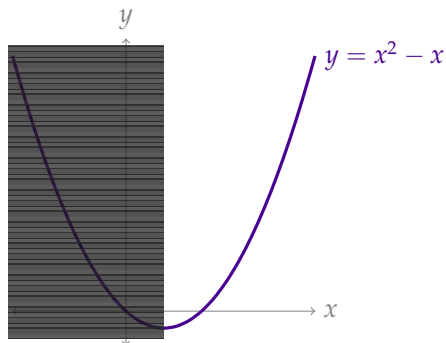


Domain:  $(-\infty, \frac{1}{2}]$



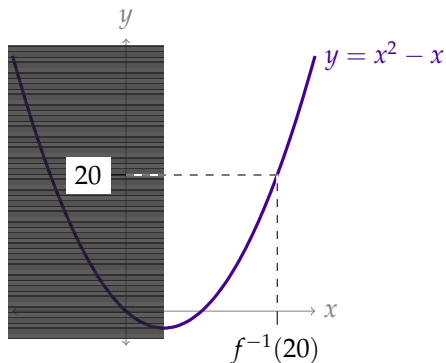
Domain:  $[\frac{1}{2}, \infty)$



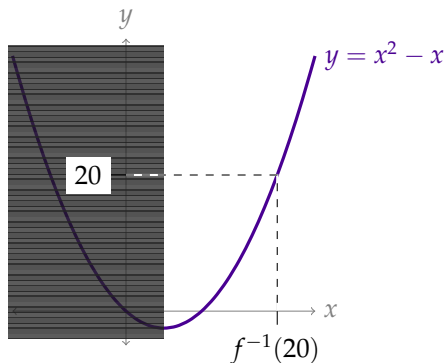


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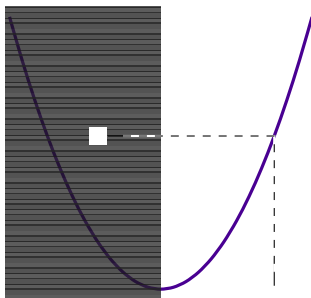


$$f^{-1}(20) = 5$$

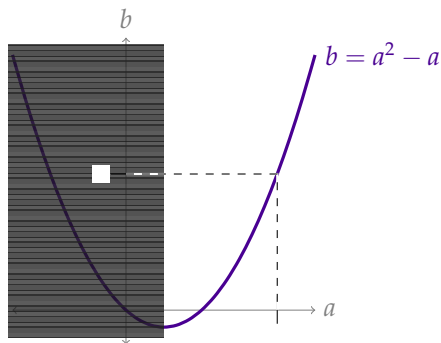
$$20 = x^2 - x$$

$$0 = x^2 - x - 20 = (x - 5)(x + 4)$$

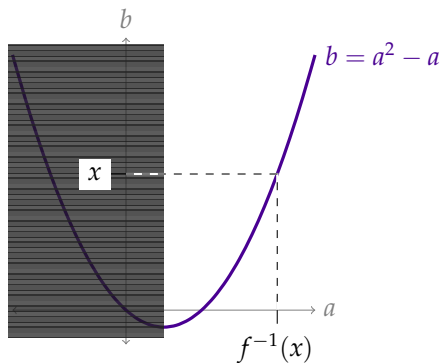
$$x = 5$$



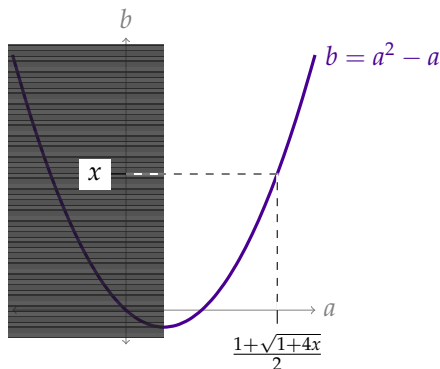
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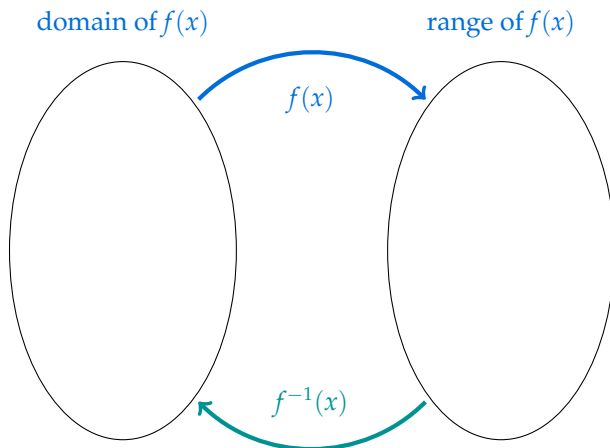
$$f^{-1}(x) = \frac{1 + \sqrt{1 + 4x}}{2}$$

$$a^2 - a = x, \text{ find } a$$

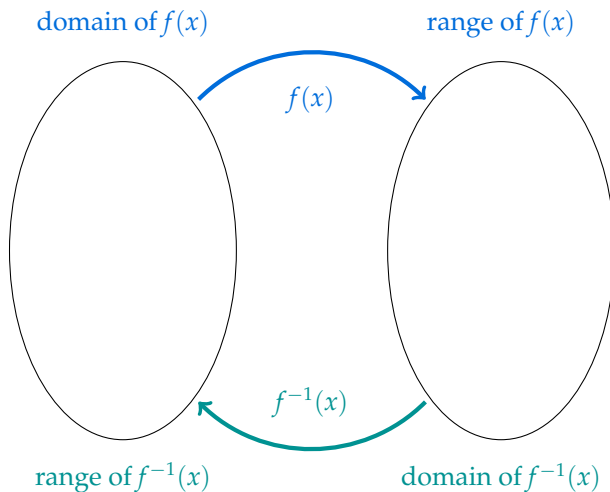
$$a^2 - a - x = 0$$

$$a = \frac{1 \pm \sqrt{1 + 4x}}{2}$$

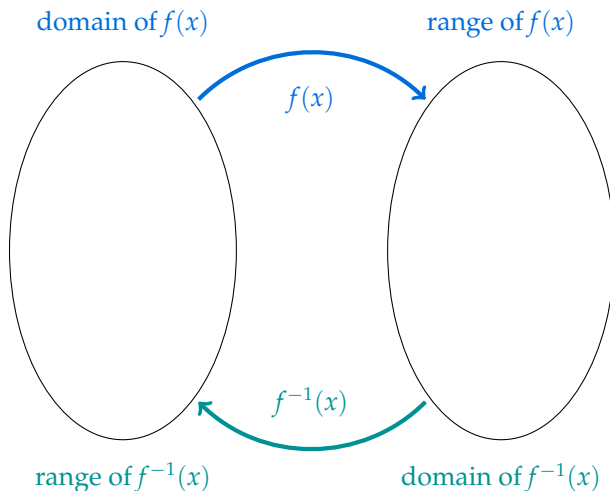
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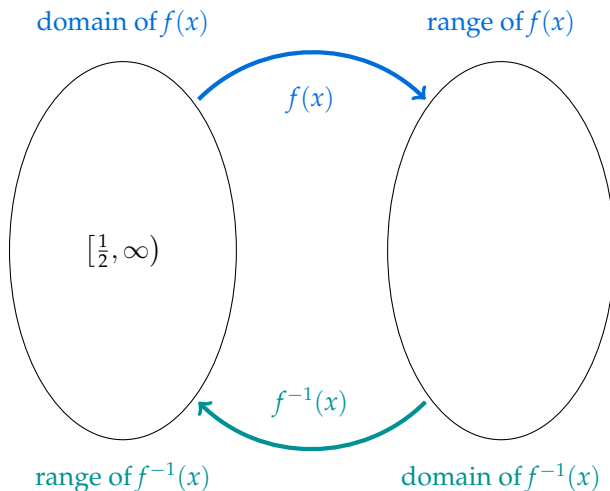




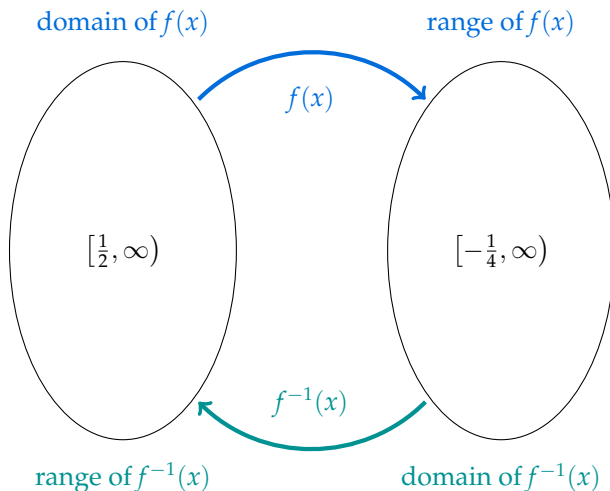
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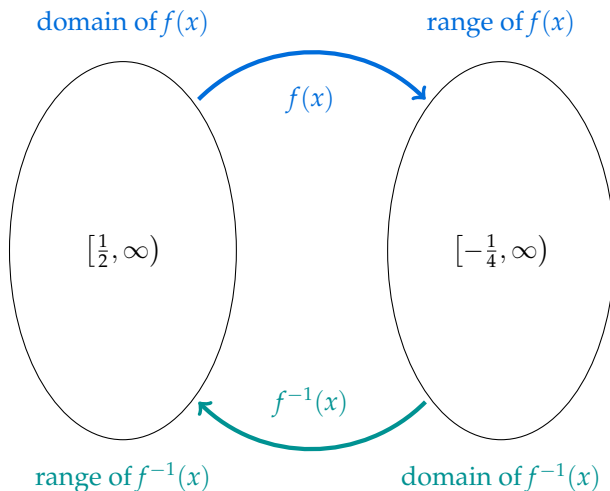


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- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = e^3$ . What is my  $x$ ?

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- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = 0$ . What is my  $x$ ? **Trick question: no  $x$  gives  $f(x) = 0$ .**

INVERTIBILITY GAME:  $f(x) = e^x$

$f^{-1}(x) = \log_e x$

- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = e$ . What is my  $x$ ?  $x = 1$
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- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = \frac{1}{e}$ . What is my  $x$ ?  $x = -1$
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# INVERTIBILITY GAME: $f(x) = e^x$

# $f^{-1}(x) = \log_e x$

- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = e$ . What is my  $x$ ?  $x = 1$   
 $\log_e(e) = 1$
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = 1$ . What is my  $x$ ?  $x = 0$   
 $\log_e(1) = 0$
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = \frac{1}{e}$ . What is my  $x$ ?  $x = -1$   
 $\log_e\left(\frac{1}{e}\right) = -1$
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = e^3$ . What is my  $x$ ?  $x = 3$   
 $\log_e(e^3) = 3$
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = 0$ . What is my  $x$ ? **Trick question: no  $x$  gives  $f(x) = 0$ .**  
 $\log_e(x)$  is undefined at  $x = 0$

1. Suppose  $0 < x < 1$ . Then  $\log_e(x)$  is...
  
2. Suppose  $-1 < x < 0$ . Then  $\log_e(x)$  is...
  
3. Suppose  $e < x$ . Then  $\log_e(x)$  is...
  - A. positive
  - B. negative
  - C. greater than one
  - D. less than one
  - E. undefined

# EXPONENTS AND LOGARITHMS

$$f(x) = e^x$$

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$x$	$e^x$	
0	1	
1	$e$	
-1	$\frac{1}{e}$	
$n$	$e^n$	



# EXPONENTS AND LOGARITHMS

$$f(x) = e^x$$

$$f^{-1}(x) = \log_e(x) = \ln(x) = \text{log}(x)$$

$x$	$e^x$	$e \text{ fact} \leftrightarrow \log_e \text{ fact}$	$x$	$\log_e(x)$
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1	$e$			
-1	$\frac{1}{e}$			
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0	1	$e^0 = 1 \leftrightarrow \log_e(1) = 0$	1	0
1	$e$			
-1	$\frac{1}{e}$			
$n$	$e^n$			

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-1	$\frac{1}{e}$			
$n$	$e^n$			

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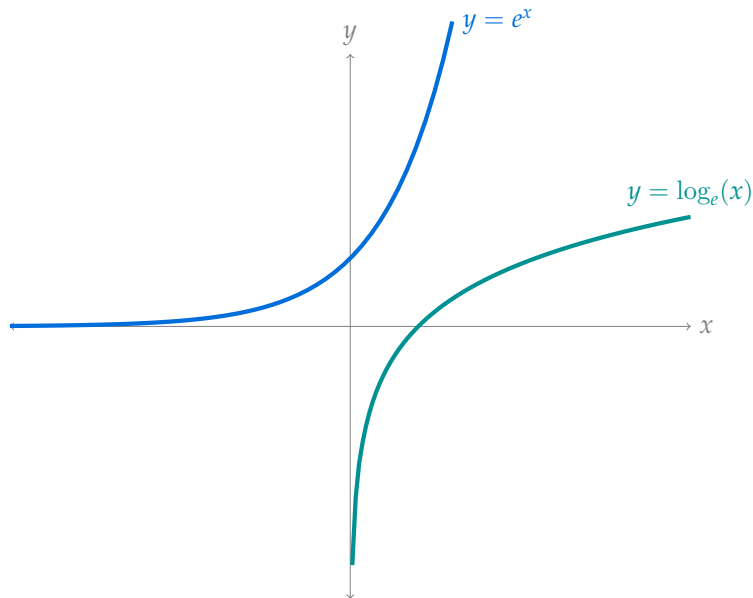
$x$	$e^x$	$e \text{ fact} \leftrightarrow \log_e \text{ fact}$	$x$	$\log_e(x)$
0	1	$e^0 = 1 \leftrightarrow \log_e(1) = 0$	1	0
1	$e$	$e^1 = e \leftrightarrow \log_e(e) = 1$	$e$	1
-1	$\frac{1}{e}$	$e^{-1} = \frac{1}{e} \leftrightarrow \log_e(\frac{1}{e}) = -1$	$\frac{1}{e}$	-1
$n$	$e^n$			

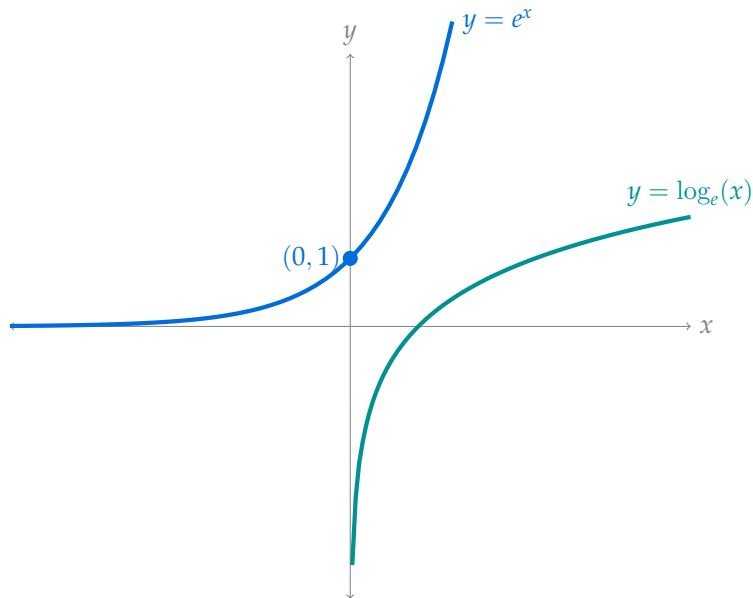
# EXPONENTS AND LOGARITHMS

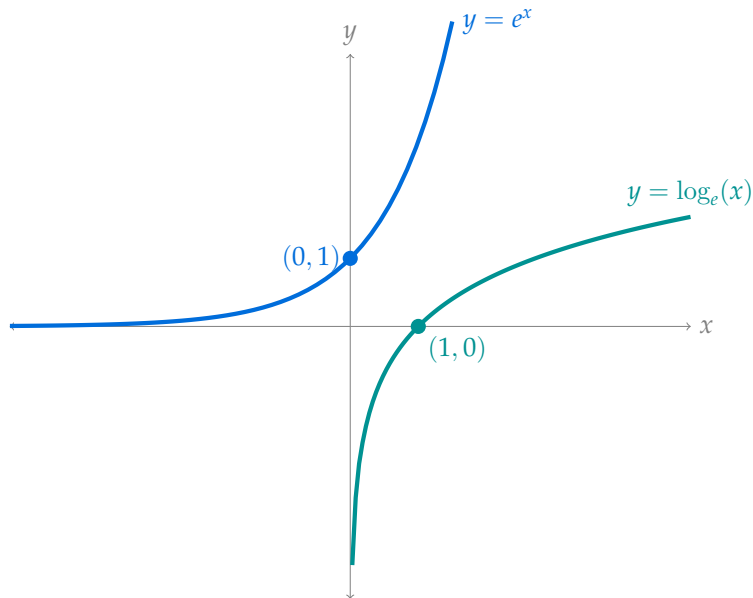
$$f(x) = e^x$$

$$f^{-1}(x) = \log_e(x) = \ln(x) = \text{log}(x)$$

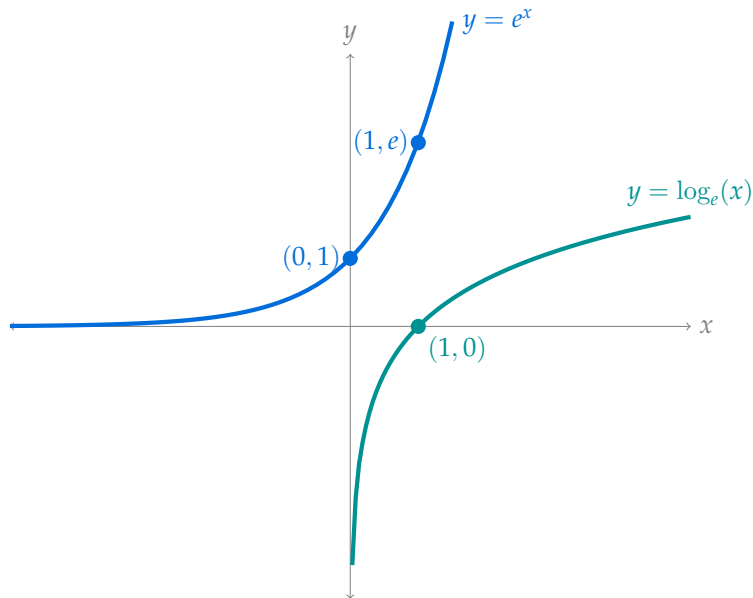
$x$	$e^x$	$e$ fact $\leftrightarrow$ $\log_e$ fact	$x$	$\log_e(x)$
0	1	$e^0 = 1 \leftrightarrow \log_e(1) = 0$	1	0
1	$e$	$e^1 = e \leftrightarrow \log_e(e) = 1$	$e$	1
-1	$\frac{1}{e}$	$e^{-1} = \frac{1}{e} \leftrightarrow \log_e(\frac{1}{e}) = -1$	$\frac{1}{e}$	-1
$n$	$e^n$	$e^n = e^n \leftrightarrow \log_e(e^n) = n$	$e^n$	$n$

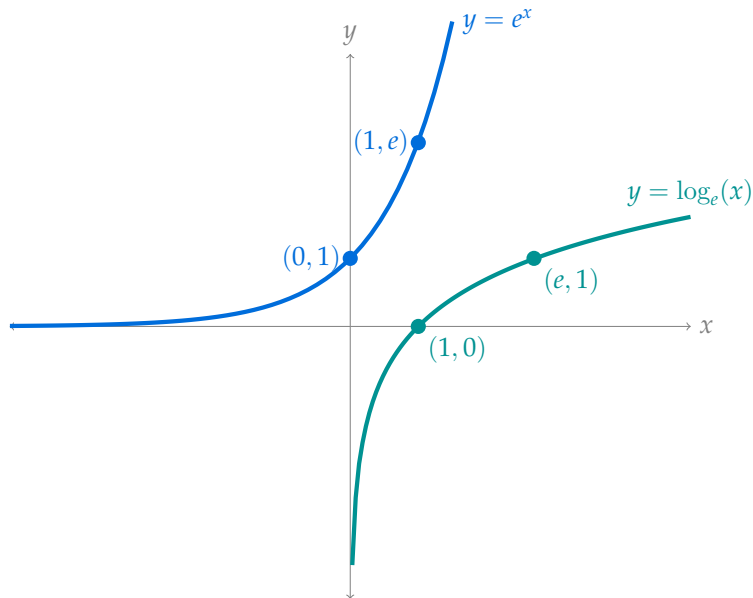


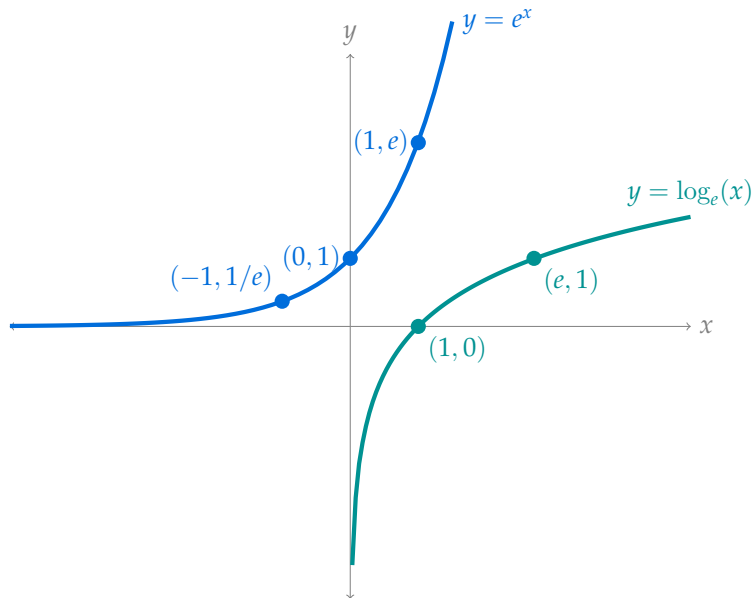


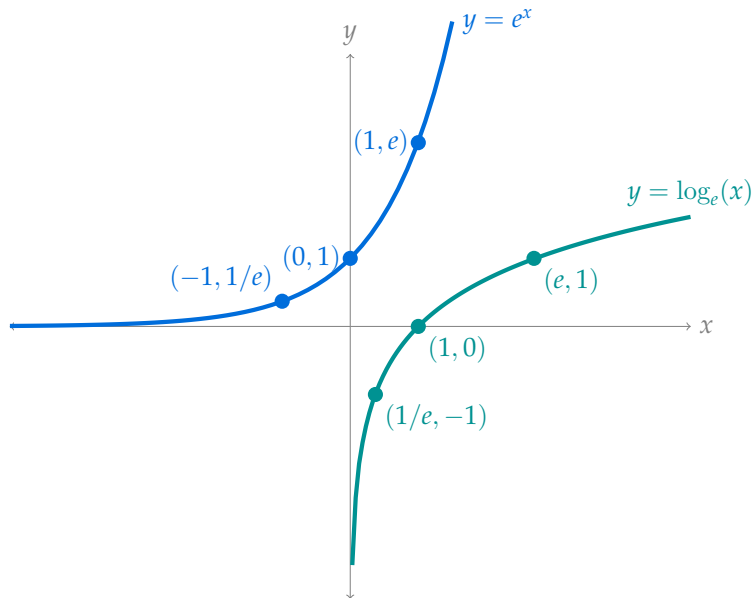


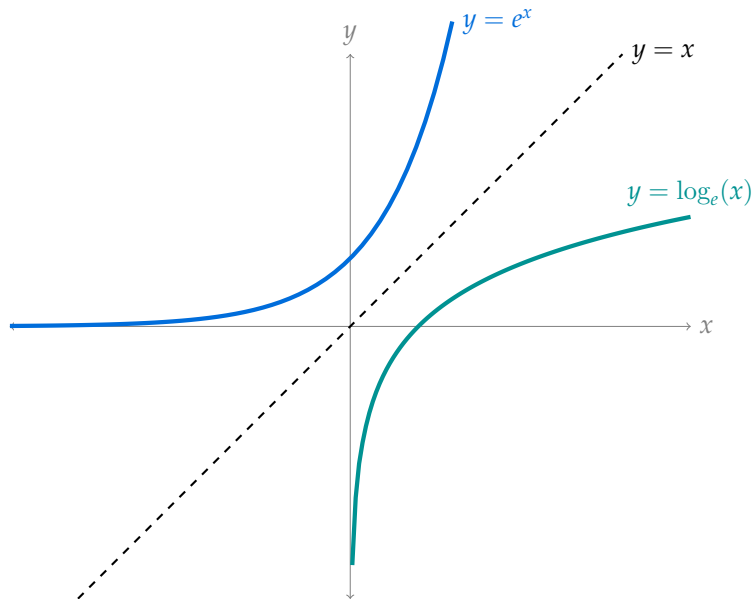












# LOGS OF OTHER BASES: $\log_n(x)$ IS THE INVERSE OF $n^x$

$$\log_{10} 10^8 =$$

A. 0

B. 8

C. 10

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## Logarithm Rules

Let  $A$  and  $B$  be positive, and let  $n$  be any real number.

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Write as a single logarithm:

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2 \log x + \log(10 + x)$$

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$$\begin{aligned} f(x) &= \log\left(\frac{10}{x^2}\right) + 2 \log x + \log(10 + x) \\ &= \log 10 - \log(x^2) + 2 \log x + \log(10 + x) \\ &= \log 10 - 2 \log x + 2 \log x + \log(10 + x) \\ &= \log 10 + \log(10 + x) = \log(10(10 + x)) \\ &= \log(100 + 10x) \end{aligned}$$

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Suppose your calculator can only compute natural logarithms. What would you enter to calculate  $\log_2(57)$ ?

Suppose your calculator can only compute logarithms base 2. What would you enter to calculate  $\log(2)$ ?

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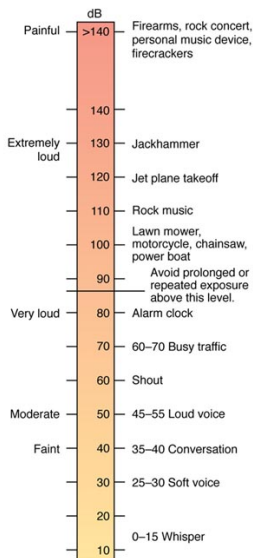
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Decibels: For a particular measure of the power  $P$  of a sound wave, the decibels of that sound is:

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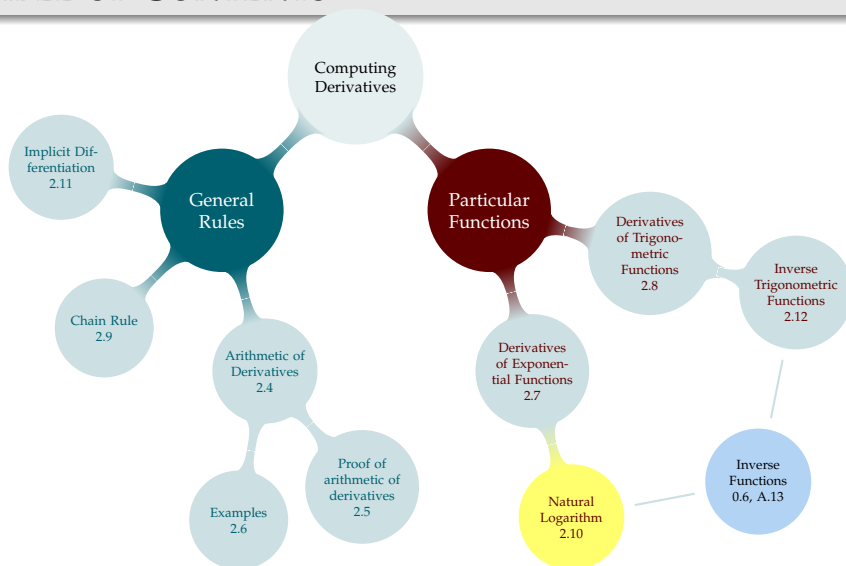
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$$1 = e^{\log_e x} \cdot \frac{d}{dx}\{\log_e x\} = x \cdot \frac{d}{dx}\{\log_e x\}$$

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We use the chain rule:

$$\begin{aligned} \frac{d}{dx} \left\{ \log_e \left| \boxed{x^2 + 1} \right| \right\} &= \frac{1}{x^2 + 1} \cdot (2x) \\ &= \frac{2x}{x^2 + 1} \end{aligned}$$

## Derivatives of Logarithms – Corollary 2.10.6

For  $a > 0$ :

$$\frac{d}{dx} [\log_a |x|] = \frac{1}{x \log a}$$

In particular:

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We use the chain rule:

$$\frac{d}{dx} [\log_e |\boxed{\cot x}|] = \frac{1}{\cot x} \cdot (-\csc^2 x) = \frac{-\csc^2 x}{\cot x}$$

# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

$$\blacktriangleright \log(f \cdot g) = \log f + \log g$$

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# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

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multiplication turns into addition

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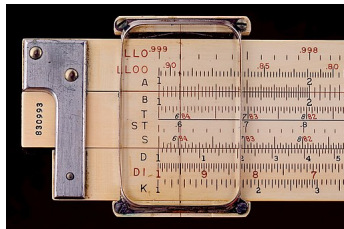
We can exploit these properties to differentiate!

# Logarithmic Differentiation

In general, if  $f(x) \neq 0$ ,  $\frac{d}{dx} [\log |f(x)|] = \frac{f'(x)}{f(x)}$ .

$$f(x) = \left( \frac{(2x+5)^4(x^2+1)}{x+3} \right)^5$$

Find  $f'(x)$ .



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$$\log(f(x)) = \log \left[ \left( \frac{(2x+5)^4(x^2+1)}{x+3} \right)^5 \right]$$

$$= 5 \log \left[ \frac{(2x+5)^4(x^2+1)}{x+3} \right]$$

$$= 5 \left[ 4 \log(2x+5) + \log(x^2+1) - \log(x+3) \right]$$

$$\frac{f'(x)}{f(x)} = 5 \left[ 4 \frac{2}{2x+5} + \frac{2x}{x^2+1} - \frac{1}{x+3} \right]$$

$$f'(x) = \left( \frac{(2x+5)^4(x^2+1)}{x+3} \right)^5 \cdot 5 \left[ 4 \frac{2}{2x+5} + \frac{2x}{x^2+1} - \frac{1}{x+3} \right]$$

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$$= x \log x$$

$$\frac{f'(x)}{f(x)} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$= 1 + \log x$$

$$f'(x) = x^x [1 + \log x]$$

# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

Differentiate:

$$f(x) = \left( \frac{(x^{15} - 9x^2)^{10}(x + x^2 + 1)}{(x^7 + 7)(x + 1)(x + 2)(x + 3)} \right)^5$$



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$$= 5 \log \left[ \frac{(x^{15} - 9x^2)^{10}(x + x^2 + 1)}{(x^7 + 7)(x + 1)(x + 2)(x + 3)} \right]$$

$$= 5 \left[ 10 \log(x^{15} - 9x^2) + \log(x + x^2 + 1) - \log(x^7 + 7) - \log(x + 1) - \log(x + 2) - \log(x + 3) \right]$$

$$\frac{f'(x)}{f(x)} = 5 \left[ 10 \frac{15x^{14} - 18x}{x^{15} - 9x^2} + \frac{1 + 2x}{x + x^2 + 1} - \frac{7x^6}{x^7 + 7} - \frac{1}{x + 1} - \frac{1}{x + 2} - \frac{1}{x + 3} \right]$$

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$$\log \left| \frac{(x^8 - e^x)(\sqrt{x} + 5)}{\csc^5 x} \right| = \log |f(x)|$$

$$\left[ \log |x^8 - e^x| + \log |x^{1/2} + 5| - 5 \log |\csc x| \right] = \log |f(x)|$$

$$\frac{d}{dx} \left[ \log |x^8 - e^x| + \log |x^{1/2} + 5| - 5 \log |\csc x| \right] = \frac{d}{dx} \log |f(x)|$$

$$\frac{8x^7 - e^x}{x^8 - e^x} + \frac{\frac{1}{2}x^{-1/2}}{x^{1/2} + 5} - 5 \frac{-\csc x \cot x}{\csc x} = \frac{f'(x)}{f(x)}$$

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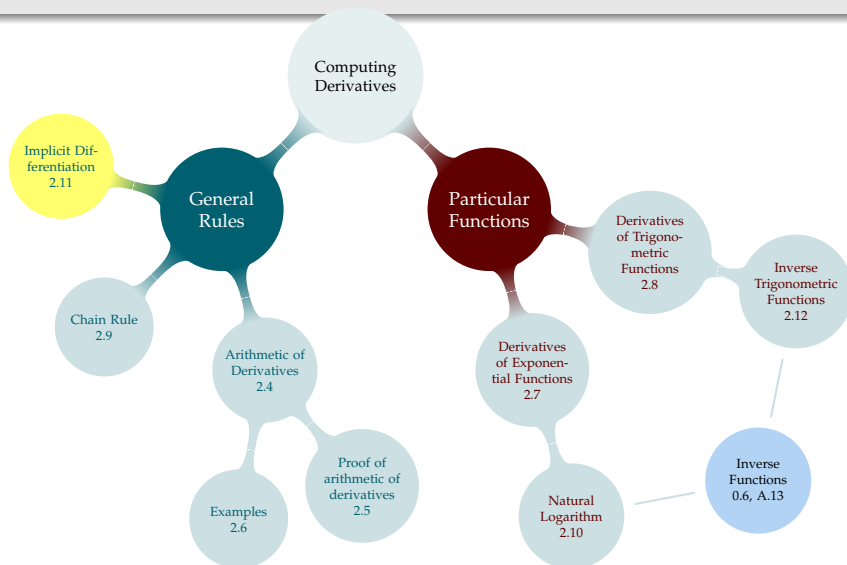
$$\frac{2x}{x^2 + 17} + \frac{160x^4}{32x^5 - 8} + 4 \frac{98x^{97} - 57x^{56} + 64x}{x^{98} - x^{57} + 32x^2} + \frac{320x^9 - 320x^{31}}{32x^{10} - 10x^{32}} = \frac{f'(x)}{f(x)}$$

$$\left( (x^2 + 17)(32x^5 - 8)(x^{98} - x^{57} + 32x^2)^4(32x^{10} - 10x^{32}) \right).$$

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If  $x = -3$ , what is  $y$ ?

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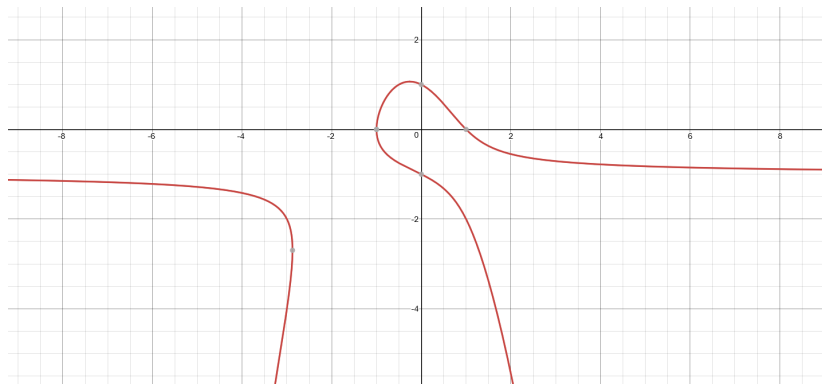
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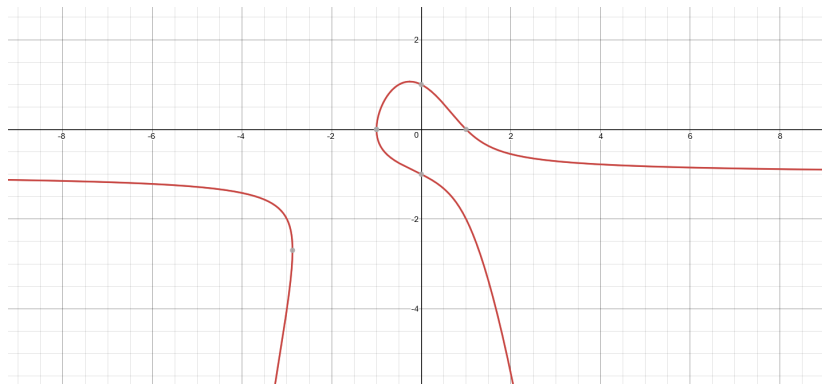
$(0, 1)$  and  $(0, -1)$

If  $x = -3$ , what is  $y$ ?  $y = -2$  and  $y = -4$

$$y^2 + x^2 + xy + x^2y = 1$$

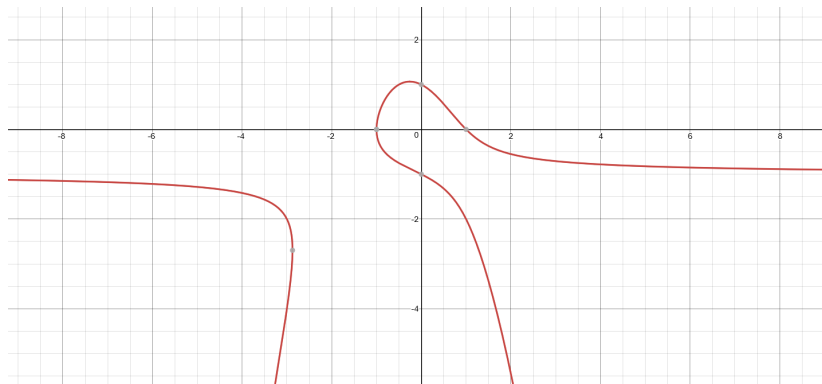


$$y^2 + x^2 + xy + x^2y = 1$$



Still has a slope:  $\frac{\Delta y}{\Delta x}$

$$y^2 + x^2 + xy + x^2y = 1$$



Still has a slope:  $\frac{\Delta y}{\Delta x}$

**Locally**,  $y$  is still a function of  $x$ .

$$y^2 + x^2 + xy + x^2y = 1$$

Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

$$\frac{d}{dx}[y] =$$

$$y^2 + x^2 + xy + x^2y = 1$$

Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

$$\frac{d}{dx}[y] = \frac{dy}{dx} = y'$$



$$y^2 + x^2 + xy + x^2y = 1$$

Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

$$\frac{d}{dx}[y] = \frac{dy}{dx} = y' \qquad \frac{d}{dx}[x] =$$

$$y^2 + x^2 + xy + x^2y = 1$$

Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

$$\frac{d}{dx}[y] = \frac{dy}{dx} = y'$$

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Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

$$\frac{d}{dx}[y] = \frac{dy}{dx} = y'$$

$$\frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[1] =$$

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Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

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Consider  $y$  as a function of  $x$ . Can we find  $\frac{dy}{dx}$ ?

$$\frac{d}{dx}[y] = \frac{dy}{dx} = y'$$

$$\frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[1] = 0$$

Differentiate both sides with respect to  $x$ .

$$0 = 2y \frac{dy}{dx} + 2x + \left( x \frac{dy}{dx} + (1)y \right) + \left( x^2 \frac{dy}{dx} + 2xy \right)$$

$$0 = \frac{dy}{dx} (2y + x + x^2) + (2x + y + 2xy)$$

$$- (2x + y + 2xy) = \frac{dy}{dx} (2y + x + x^2)$$

$$-\frac{2x + y + 2xy}{2y + x + x^2} = \frac{dy}{dx}$$

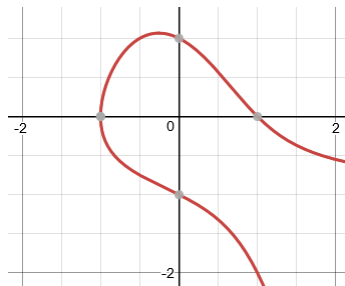
$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily,  $\frac{dy}{dx}$  depends on **both**  $y$  and  $x$ . Why?



$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily,  $\frac{dy}{dx}$  depends on **both**  $y$  and  $x$ . Why?

$$\left. \frac{dy}{dx} \right|_{(1,0)} =$$

$$\left. \frac{dy}{dx} \right|_{(1,-2)} =$$



$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily,  $\frac{dy}{dx}$  depends on **both**  $y$  and  $x$ . Why?

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(1,0)} &= -\frac{2(1) + 0 + 2(1)(0)}{2(0) + 1 + 1} \\ &= -\frac{2}{2} = -1\end{aligned}$$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(1,-2)} &= -\frac{2(1) - 2 + 2(1)(-2)}{2(-2) + 1 + 1} \\ &= -2\end{aligned}$$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily,  $\frac{dy}{dx}$  depends on **both**  $y$  and  $x$ . Why?

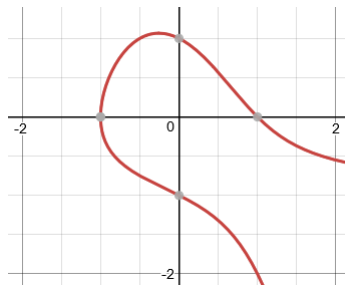
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Points with the same  $x$ -value may have different slopes. We need both the  $x$ -value and the  $y$ -value to figure out which point we're talking about.

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$



Points with the same  $x$ -value may have different slopes. We need both the  $x$ -value and the  $y$ -value to figure out which point we're talking about.

Now  
You



Suppose  $x^4y + y^4x = 2$ . Find  $\frac{dy}{dx}$  at the point  $(1, 1)$ .

NOW  
YOU



Suppose  $x^4y + y^4x = 2$ . Find  $\frac{dy}{dx}$  at the point  $(1, 1)$ .

$$x^4y(x) + y(x)^4x = 2$$

$$4x^3y(x) + x^4\frac{dy}{dx}(x) + y(x)^4 + 4y(x)^3\frac{dy}{dx}(x)x = 0$$

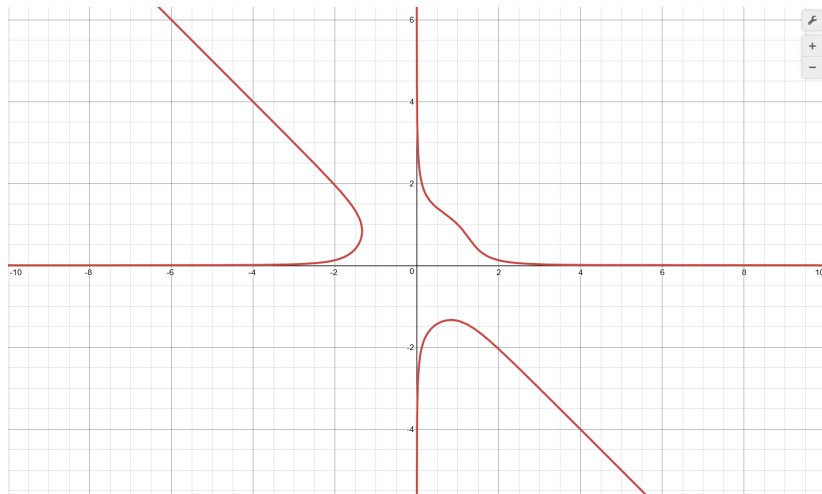
We may only replace variables with constants *after* differentiating.  
When  $x = 1$  and  $y(1) = 1$ ,

$$4(1)^3y(1) + (1)^4\frac{dy}{dx}(1) + y(1)^4 + 4y(1)^3\frac{dy}{dx}(1) = 0$$

$$4 + \frac{dy}{dx}(1) + 1 + 4\frac{dy}{dx}(1) = 0$$

$$5\frac{dy}{dx}(1) = -5$$

$$\frac{dy}{dx}(1) = -1$$



$$x^4y + y^4x = 2$$

NOW  
YOU



Suppose  $\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$ . Find  $\frac{dy}{dx}$  when  $x = 0$ , and the equations of the associated tangent line(s).

NOW  
YOU



Suppose  $\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$ . Find  $\frac{dy}{dx}$  when  $x = 0$ , and the equations of the associated tangent line(s).

To avoid the quotient rule, we start by simplifying our expression.

$$\frac{3y(x)^2 + 2y(x) + y(x)^3}{x^2 + 1} = x$$

$$3y(x)^2 + 2y(x) + y(x)^3 = x^3 + x$$

$$6y(x) \frac{dy}{dx}(x) + 2 \frac{dy}{dx}(x) + 3y(x)^2 \frac{dy}{dx}(x) = 3x^2 + 1$$

When  $x = 0$ :

$$\frac{dy}{dx}(0) = \frac{1}{6y(0) + 2 + 3y(0)^2}$$



We need to know  $y$  to find  $\frac{dy}{dx}$ . We want all points where  $x = 0$ .

$$3y(0)^2 + 2y(0) + y(0)^3 = 0$$

$$y(0)(y(0)^2 + 3y(0) + 2) = 0$$

$$y(0)(y(0) + 1)(y(0) + 2) = 0$$

$$y(0) = 0, y(0) = -1, y(0) = -2$$

$$\frac{dy}{dx} = \frac{1}{6y + 2 + 3y^2}$$

$$(0, 0)$$

$$(0, -1)$$

$$(0, -2)$$

$$\left. \frac{dy}{dx} \right|_{(0,0)} = \frac{1}{2}$$

$$y - 0 = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x$$

$$\left. \frac{dy}{dx} \right|_{(0,-1)} = \frac{1}{-6+2+3}$$

$$= -1$$

$$\left. \frac{dy}{dx} \right|_{(0,-2)} = \frac{1}{-12+2+12}$$

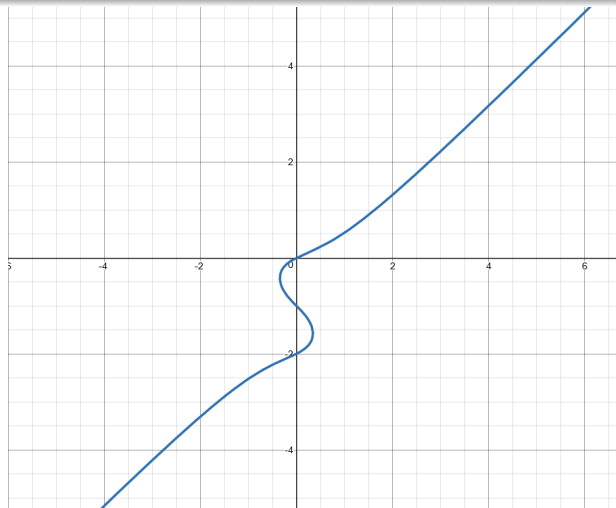
$$= \frac{1}{2}$$

$$y - (-1) = -1(x - 0)$$

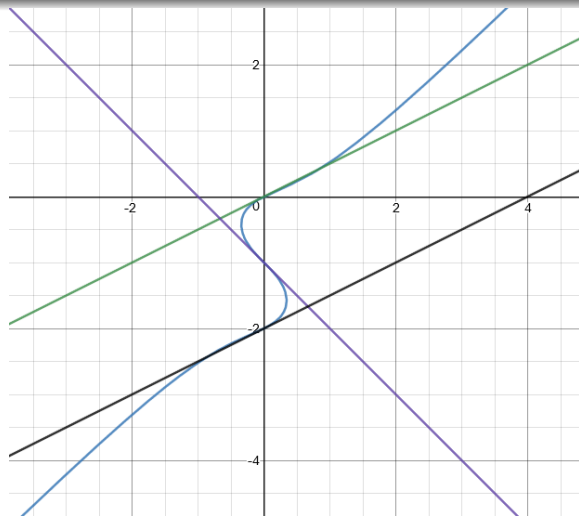
$$y = -x - 1$$

$$y - (-2) = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x - 2$$



$$\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$$



$$\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$$

Use implicit differentiation to differentiate  $\log(x)$ ,  $x > 0$ .

Use implicit differentiation to differentiate  $\log(x)$ ,  $x > 0$ .

$$\log x = y(x)$$

$$x = e^{y(x)}$$

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$$\log x = y(x)$$

$$x = e^{y(x)}$$

$$1 = e^{y(x)} \cdot \frac{dy}{dx}(x)$$

$$\frac{dy}{dx}(x) = \frac{1}{e^{y(x)}} = \frac{1}{x}$$

Use implicit differentiation to differentiate  $\log(x)$ ,  $x > 0$ .

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Use implicit differentiation to differentiate  $\log|x|$ ,  $x < 0$ .



Use implicit differentiation to differentiate  $\log(x)$ ,  $x > 0$ .

Use implicit differentiation to differentiate  $\log|x|$ ,  $x < 0$ .

Use implicit differentiation to differentiate  $\log(x)$ ,  $x > 0$ .

Use implicit differentiation to differentiate  $\log|x|$ ,  $x < 0$ .

$$\log|x| = y(x)$$

$$\log(-x) = y(x)$$

$$-x = e^{y(x)}$$

$$-1 = e^{y(x)} \cdot \frac{dy}{dx}(x)$$

$$\frac{dy}{dx}(x) = \frac{-1}{e^{y(x)}} = \frac{-1}{-x} = \frac{1}{x}$$

Use implicit differentiation to differentiate  $\log_a(x)$ , where  $a > 0$  is a constant and  $x > 0$ .

Use implicit differentiation to differentiate  $\log_a |x|$ ,  $a > 0$ .

Use implicit differentiation to differentiate  $\log_a(x)$ , where  $a > 0$  is a constant and  $x > 0$ .

Use implicit differentiation to differentiate  $\log_a(x)$ , where  $a > 0$  is a constant and  $x > 0$ .

$$\log_a x = y(x)$$

$$x = a^{y(x)}$$

$$1 = a^{y(x)} \cdot \log_e a \cdot \frac{dy}{dx}(x)$$

$$\frac{dy}{dx}(x) = \frac{1}{a^{y(x)} \cdot \log_e a} = \frac{1}{x \log_e a}$$

Use implicit differentiation to differentiate  $\log_a |x|$ ,  $a > 0$ .

Use implicit differentiation to differentiate  $\log_a |x|$ ,  $a > 0$ .

If  $x > 0$ , it's what we just computed. So assume  $x < 0$ .

$$\log_a |x| = y(x)$$

$$\log_a (-x) = y(x)$$

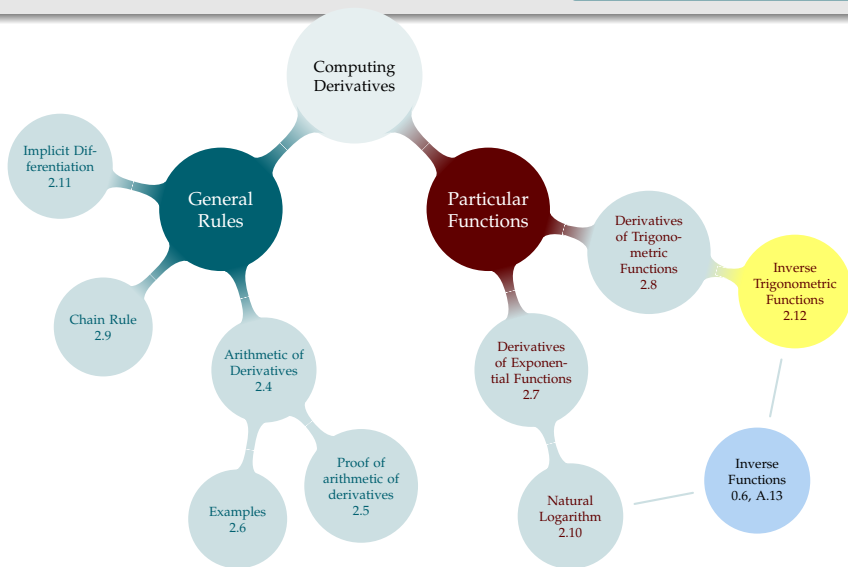
$$-x = a^{y(x)}$$

$$-1 = a^{y(x)} \cdot \log_e a \cdot \frac{dy}{dx}(x)$$

$$\frac{dy}{dx}(x) = \frac{-1}{a^{y(x)} \cdot \log_e a} = \frac{-1}{x \log_e a}$$

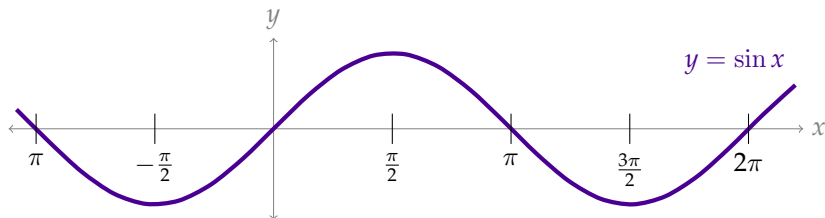
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► SKIP DEFINITIONS OF INVERSE TRIG FUNCTIONS

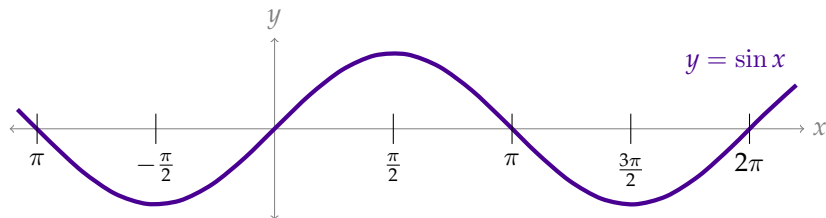




# INVERTIBILITY GAME

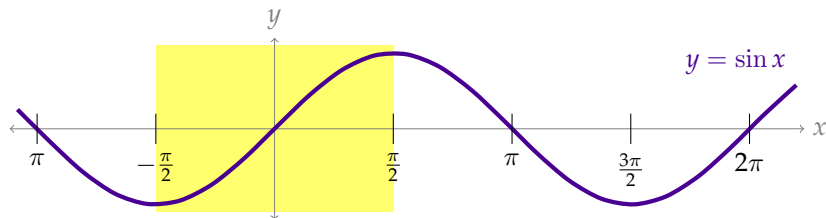


# INVERTIBILITY GAME



I'm thinking of a number  $x$ . Your hint:  $\sin(x) = 0$ . What number am I thinking of?

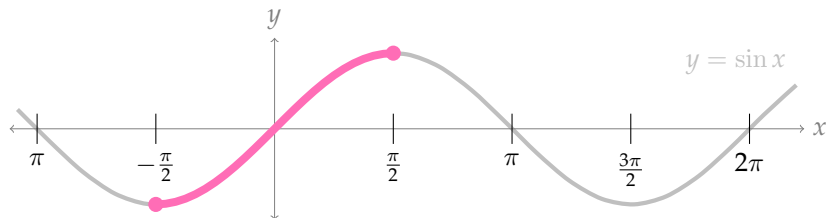
# INVERTIBILITY GAME



I'm thinking of a number  $x$ . Your hint:  $\sin(x) = 0$ . What number am I thinking of?

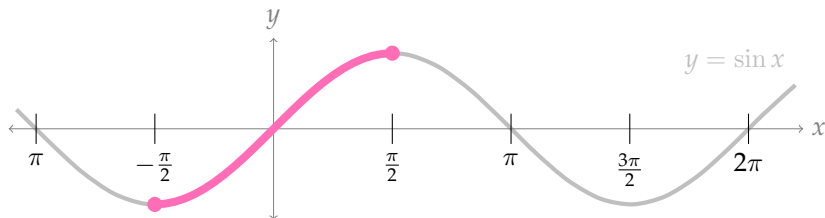
I'm thinking of a number  $x$ , and  $x$  is between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ . Your hint:  $\sin(x) = 0$ . What number am I thinking of?

# ARCSINE



$\arcsin(x)$  is the inverse of  $\sin x$  restricted to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$

# ARCSINE



$\arcsin(x)$  is the inverse of  $\sin x$  restricted to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin x$  is the (unique) number  $\theta$  such that:

- ▶  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , and
- ▶  $\sin \theta = x$

# ARCSINE

Reference Angles:

$\theta$	$\sin \theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

## ARCSINE

Reference Angles:

$\theta$	$\sin \theta$
0	0
$-\frac{\pi}{6}$	$-\frac{1}{2}$
$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$-\frac{\pi}{2}$	-1

## ARCSINE

Reference Angles:

►  $\arcsin(0)$ 

$\theta$	$\sin \theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1



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Reference Angles:

►  $\arcsin(0) = 0$

$\theta$	$\sin \theta$
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$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

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Reference Angles:

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$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

►  $\arcsin(0) = 0$

►  $\arcsin\left(\frac{1}{\sqrt{2}}\right)$

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Reference Angles:

$\theta$	$\sin \theta$
0	0
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$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

►  $\arcsin(0) = 0$

►  $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

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Reference Angles:

$\theta$	$\sin \theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
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$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

►  $\arcsin(0) = 0$

►  $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

►  $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

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Reference Angles:

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# ARCSINE

Reference Angles:

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►  $\arcsin\left(\frac{\pi}{2}\right)$

# ARCSINE

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►  $\arcsin(0) = 0$

►  $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

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►  $\arcsin\left(\frac{\pi}{2}\right)$  undefined

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$\theta$	$\sin \theta$
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$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

►  $\arcsin(0) = 0$

►  $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

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►  $\arcsin\left(\frac{\pi}{2}\right)$  undefined

►  $\arcsin\left(\frac{\pi}{4}\right)$



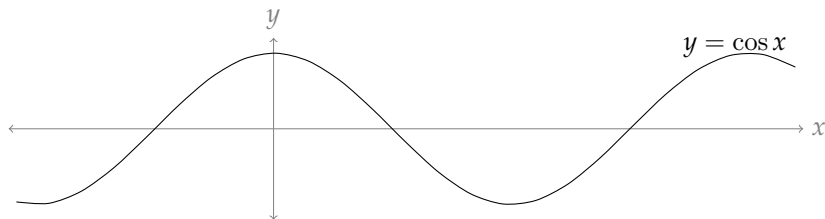
# ARCSINE

Reference Angles:

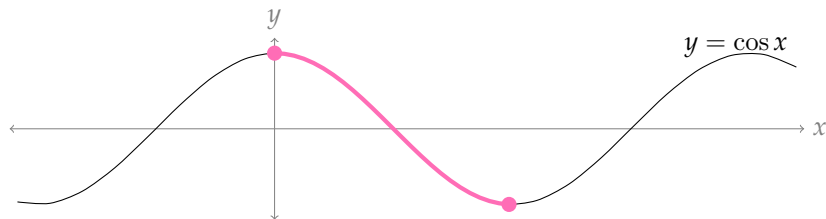
$\theta$	$\sin \theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

- ▶  $\arcsin(0) = 0$
- ▶  $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$
- ▶  $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$
- ▶  $\arcsin\left(\frac{\pi}{2}\right)$  undefined
- ▶  $\arcsin\left(\frac{\pi}{4}\right)$  defined, but we haven't covered tools (yet) to figure it out

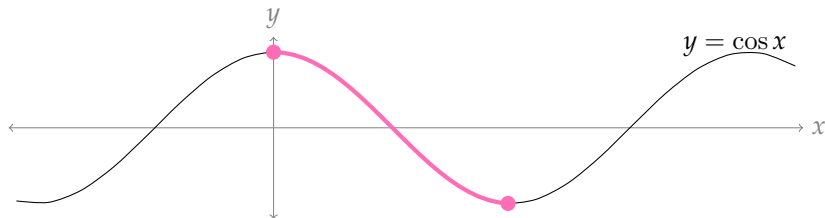
# ARCCOSINE



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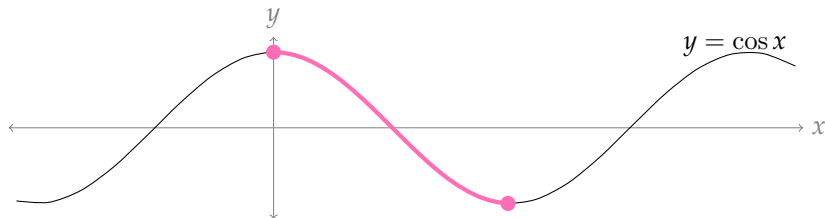


$\arccos(x)$  is the inverse of  $\cos x$  restricted to  $[0, \pi]$ .

$\arccos(x)$  is the (unique) number  $\theta$  such that:

- ▶  $\cos(\theta) = x$  and
- ▶  $0 \leq \theta \leq \pi$

# ARCCOSINE

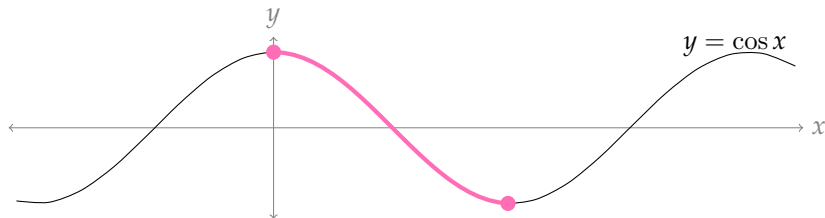


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# ARCCOSINE

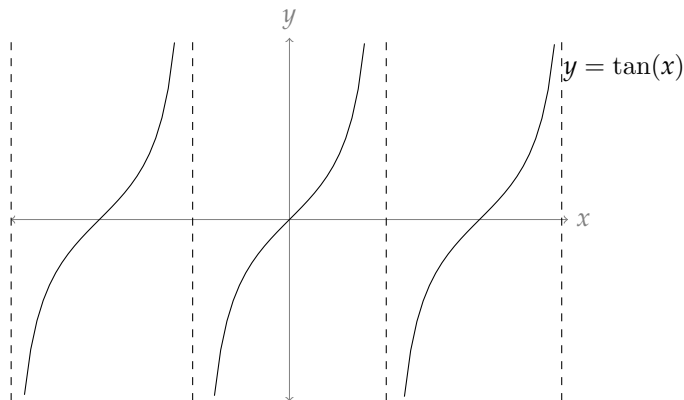


$\arccos(x)$  is the inverse of  $\cos x$  restricted to  $[0, \pi]$ .

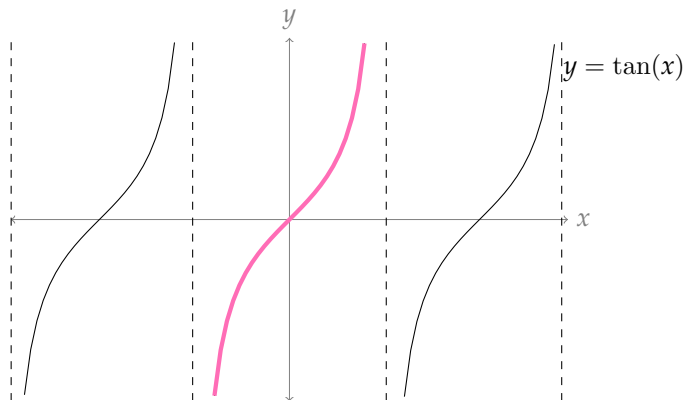
$\arccos(x)$  is the (unique) number  $\theta$  such that:

- ▶  $\cos(\theta) = x$  and ←←← inverse
- ▶  $0 \leq \theta \leq \pi$  ←←← inverse exists

# ARCTANGENT

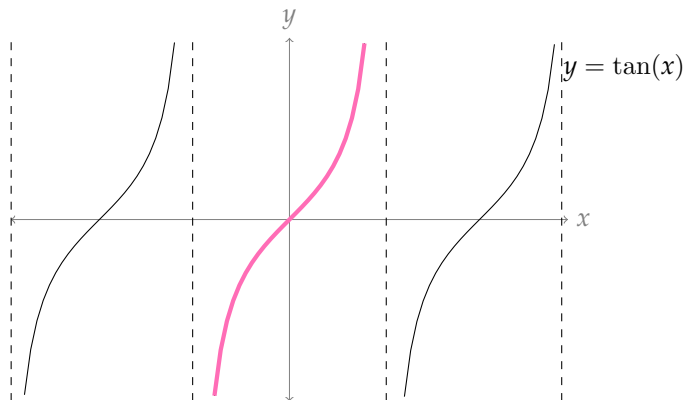


# ARCTANGENT





# ARCTANGENT



$\arctan(x) = \theta$  means:

- (1)  $\tan(\theta) = x$  and
- (2)  $-\pi/2 < \theta < \pi/2$

# ARCSECANT, ARCSINE, AND ARCCOTANGENT

$$\operatorname{arcsec}(x) =$$

# ARCSECANT, ARCSINE, AND ARCCOTANGENT

$$\operatorname{arcsec}(x) =$$

$$\operatorname{arcsec}(x) = y$$

$$\sec y = x$$

$$\frac{1}{\cos y} = x$$

$$\cos y = \frac{1}{x}$$

$$y = \arccos\left(\frac{1}{x}\right)$$

$$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

# ARCSECANT, ARCSINE, AND ARCCOTANGENT

$$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

$$\operatorname{arcsec}(x) = y$$

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# ARCSECANT, ARCSINE, AND ARCCOTANGENT

$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccsc}(x) = y$$

$$\csc y = x$$

$$\frac{1}{\sin y} = x$$

$$\sin y = \frac{1}{x}$$

$$y = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

$$\operatorname{arccot}(x) = y$$

$$\cot y = x$$

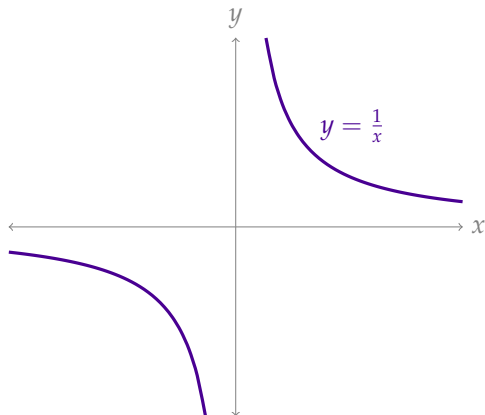
$$\frac{1}{\tan y} = x$$

$$\tan y = \frac{1}{x}$$

$$y = \arctan\left(\frac{1}{x}\right)$$

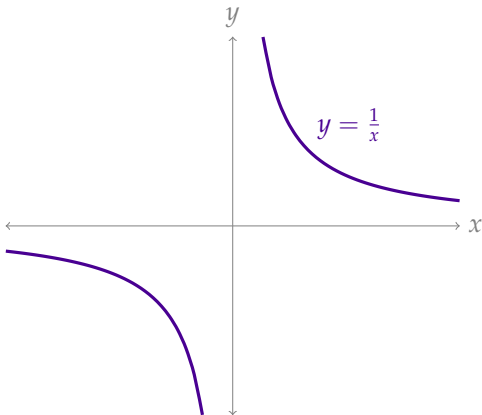
$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

$$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$



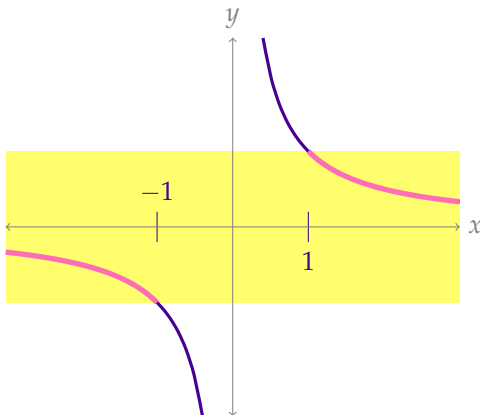
$$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

The domain of  $\arccos(y)$  is  $-1 \leq y \leq 1$ , so the domain of  $\operatorname{arcsec}(y)$  is



$$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

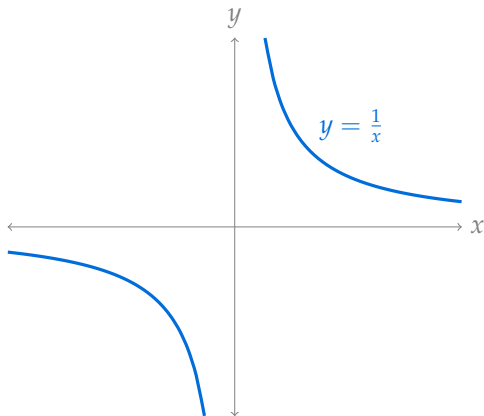
The domain of  $\arccos(y)$  is  $-1 \leq y \leq 1$ , so the domain of  $\operatorname{arcsec}(y)$  is



$$(-\infty, -1] \cup [1, \infty).$$

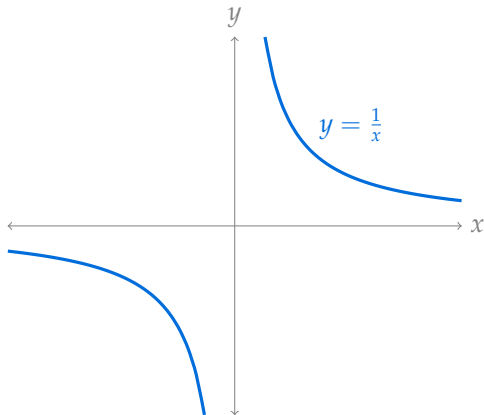


$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$



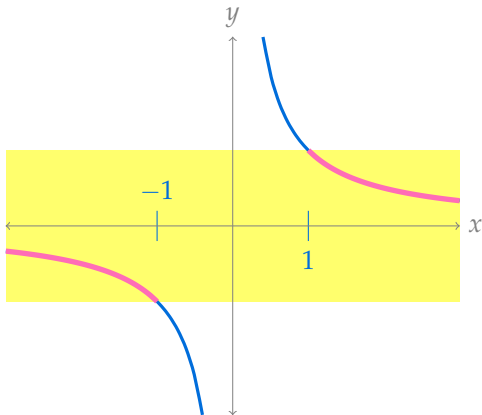
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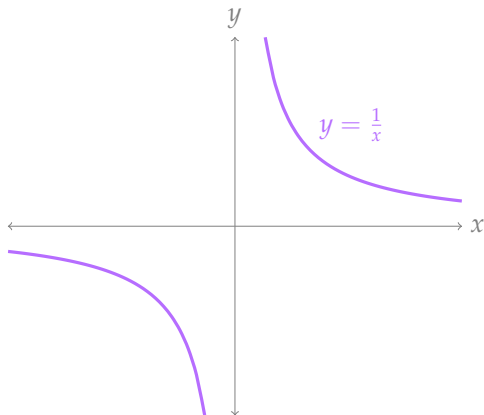
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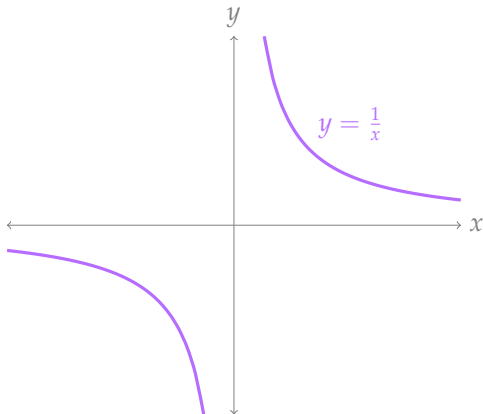
$$(-\infty, -1] \cup [1, \infty).$$

$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$



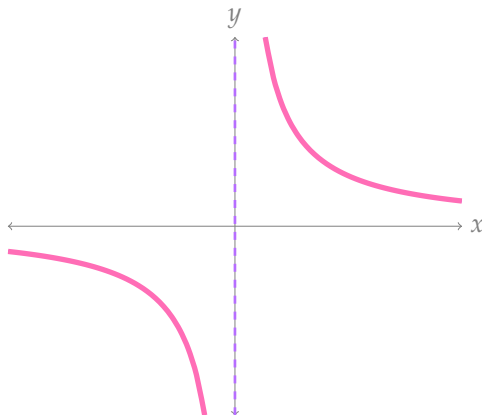
$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Domain of  $\arctan(x)$  is all real numbers, so the domain of  $\operatorname{arccot}(x)$  is



$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Domain of  $\arctan(x)$  is all real numbers, so the domain of  $\operatorname{arccot}(x)$  is



$$(-\infty, 0) \cup (0, \infty).$$

$$y = \arcsin x$$

Find  $\frac{dy}{dx}$ .

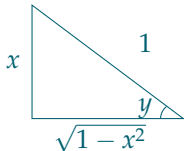
$$y(x) = \arcsin x$$

$$x = \sin y(x)$$

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y(x)]$$

$$1 = \cos y(x) \cdot \frac{dy}{dx}(x)$$

$$\begin{aligned} \frac{dy}{dx}(x) &= \frac{1}{\cos y(x)} \\ &= \frac{\text{hyp}}{\text{adj}} \\ &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$





$$y = \arctan x$$

Find  $\frac{dy}{dx}$ .

$$y(x) = \arctan x$$

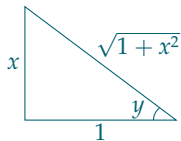
$$x = \tan y(x)$$

$$\frac{d}{dx}[x] = \frac{d}{dx}[\tan y(x)]$$

$$1 = \sec^2 y(x) \cdot \frac{dy}{dx}(x)$$

$$\frac{dy}{dx}(x) = \cos^2 y(x)$$

$$\begin{aligned} \frac{dy}{dx}(x) &= \left( \frac{\text{adj}}{\text{hyp}} \right)^2 = \left( \frac{1}{\sqrt{1+x^2}} \right)^2 \\ &= \frac{1}{1+x^2} \end{aligned}$$



$$y = \arccos x$$

Find  $\frac{dy}{dx}$ .

$$y(x) = \arccos x$$

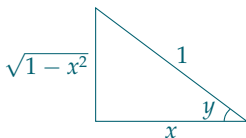
$$x = \cos y(x)$$

$$\frac{d}{dx}[x] = \frac{d}{dx}[\cos y(x)]$$

$$1 = -\sin y(x) \cdot \frac{dy}{dx}(x)$$

$$\frac{dy}{dx}(x) = \frac{-1}{\sin y(x)}$$

$$\frac{dy}{dx}(x) = \frac{-\text{hyp}}{\text{opp}} = \frac{-1}{\sqrt{1-x^2}}$$



To differentiate arcsecant, arccosecant, and arccotangent, you can use the chain rule!

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$$\frac{d}{dx} [\operatorname{arccsc}(x)] = \frac{d}{dx} \left[ \arcsin \left( \frac{1}{x} \right) \right] = \frac{d}{dx} [\arcsin (x^{-1})]$$

To differentiate arcsecant, arccosecant, and arccotangent, you can use the chain rule!

$$\frac{d}{dx} [\operatorname{arccsc}(x)] = \frac{d}{dx} \left[ \arcsin \left( \frac{1}{x} \right) \right] = \frac{d}{dx} [\arcsin (x^{-1})]$$

$$\begin{aligned} \frac{d}{dx} \left[ \arcsin \left( \boxed{x^{-1}} \right) \right] &= \frac{1}{\sqrt{1 - \left( \boxed{x^{-1}} \right)^2}} \cdot \boxed{\left( -x^{-2} \right)} = \frac{-1}{x^2 \sqrt{1 - x^{-2}}} \\ &= \frac{-1}{\sqrt{x^4} \sqrt{1 - x^{-2}}} = \frac{-1}{\sqrt{x^2} \sqrt{x^2} \sqrt{1 - x^{-2}}} = \frac{-1}{\sqrt{x^2} \sqrt{x^2 - 1}} = \frac{-1}{|x| \sqrt{1 - x^2}} \end{aligned}$$

## Derivatives of Inverse Trigonometric Functions – Theorem 2.12.7

Memorize:

$$\begin{aligned}\frac{d}{dx}[\arcsin x] &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}[\arccos x] &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}[\arctan x] &= \frac{1}{1+x^2}\end{aligned}$$

Be able to derive:

$$\begin{aligned}\frac{d}{dx}[\operatorname{arccsc} x] &= -\frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}[\operatorname{arcsec} x] &= \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}[\operatorname{arccot} x] &= -\frac{1}{1+x^2}\end{aligned}$$



## Included Work



Anonymous. (2012) [Decibel Scale of Frequently Heard Sounds](#).  
Biology Forums. <http://biology-forums.com/index.php?action=gallery;sa=view;id=6156> (accessed 7 October 2015) , 97



'Brain' by Eucalyp is licensed under [CC BY 3.0](#) (accessed 8 June 2021), 148, 149, 151, 152



'Slide rule – KEUFFEL & ESSER CO. N.Y.' by s58y is licensed under [CC BY 2.0](#) (accessed 20 July 2021) , 115



screenshot of graph using Desmos Graphing Calculator,  
<https://www.desmos.com/calculator> (accessed 19 October 2017), 143, 147



screenshot of graph using Desmos Graphing Calculator,  
<https://www.desmos.com/calculator> (accessed 19 October 2017), 132–134



screenshot of graph using Desmos Graphing Calculator,  
<https://www.desmos.com/calculator> (accessed 24 October 2018), 150



screenshot of graph using Desmos Graphing Calculator,  
<https://www.desmos.com/calculator>, (accessed 24 October 2018), 155



screenshot of graph using Desmos Graphing Calculator,  
<https://www.desmos.com/calculator> (accessed 24 October 2018), 156



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160, 163