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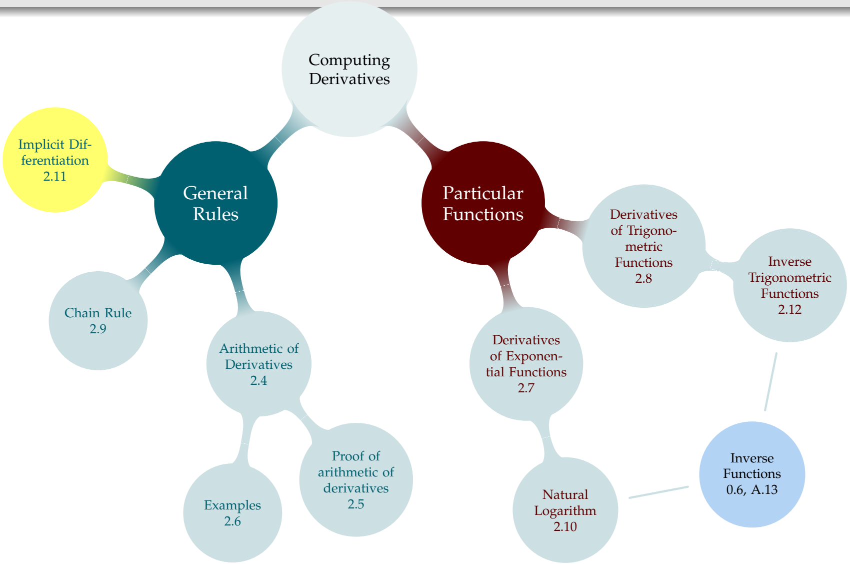

# Implicitly Defined Functions

$$y^2 + x^2 + xy + x^2y = 1$$

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If $x = -3$, what is $y$?

Q



ans

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Which of the following points are on the curve?

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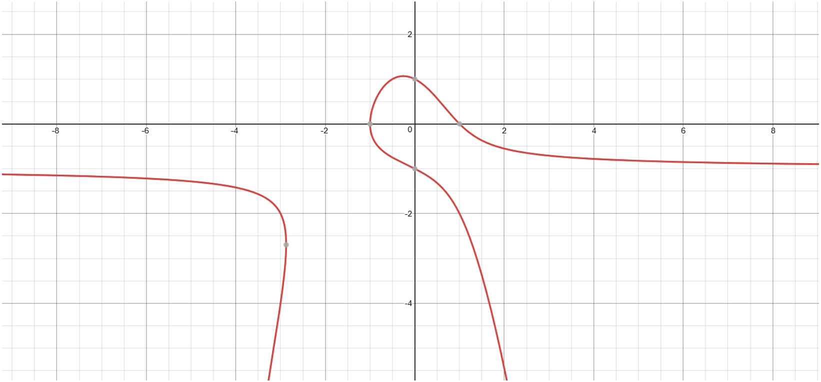
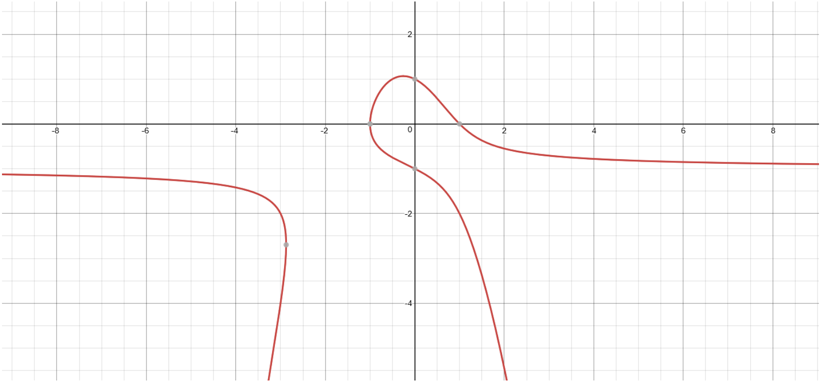
$(0, 1)$ and $(0, -1)$

If $x = -3$, what is $y$? $y = -2$ and $y = -4$

$$y^2 + x^2 + xy + x^2y = 1$$

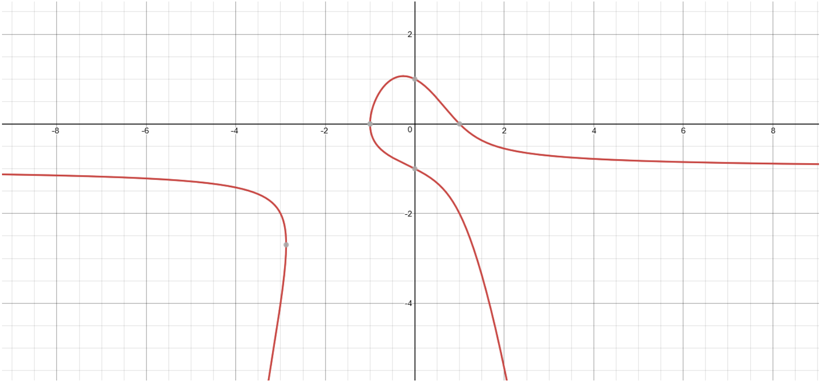


$$y^2 + x^2 + xy + x^2y = 1$$




Still has a slope: $\frac{\Delta y}{\Delta x}$

$$y^2 + x^2 + xy + x^2y = 1$$




Still has a slope: $\frac{\Delta y}{\Delta x}$
**Locally**, $y$ is still a function of $x$.

$$y^2 + x^2 + xy + x^2y = 1$$

Consider $y$ as a function of $x$. Can we find $\frac{\mathrm{d}y}{\mathrm{d}x}$?

$\frac{\mathrm{d}}{\mathrm{d}x}[y] =$



ans

$$y^2 + x^2 + xy + x^2y = 1$$

Consider $y$ as a function of $x$. Can we find $\frac{\mathrm{d}y}{\mathrm{d}x}$?

$\frac{\mathrm{d}}{\mathrm{d}x}[y] = \frac{\mathrm{d}y}{\mathrm{d}x} = y'$

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$$\frac{\mathrm{d}}{\mathrm{d}x}[y] = \frac{\mathrm{d}y}{\mathrm{d}x} = y' \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x}[x] =$$

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$$y^2 + x^2 + xy + x^2y = 1$$

Consider $y$ as a function of $x$. Can we find $\frac{dy}{dx}$?

$\frac{d}{dx}[y] = \frac{dy}{dx} = y'$ $\qquad$ $\frac{d}{dx}[x] = 1$ $\qquad$ $\frac{d}{dx}[1] =$

$$y^2 + x^2 + xy + x^2y = 1$$

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$$\frac{d}{dx}[y] = \frac{dy}{dx} = y' \qquad \frac{d}{dx}[x] = 1 \qquad \frac{d}{dx}[1] = 0$$

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Consider $y$ as a function of $x$. Can we find $\frac{\mathrm{d}y}{\mathrm{d}x}$?

$$\frac{\mathrm{d}}{\mathrm{d}x}[y] = \frac{\mathrm{d}y}{\mathrm{d}x} = y' \qquad \frac{\mathrm{d}}{\mathrm{d}x}[x] = 1 \qquad \frac{\mathrm{d}}{\mathrm{d}x}[1] = 0$$

Differentiate both sides with respect to $x$.

$$\begin{aligned} 0 &= 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2x + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + (1)y\right) + \left(x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy\right) \\ 0 &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(2y + x + x^2\right) + (2x + y + 2xy) \\ -(2x + y + 2xy) &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(2y + x + x^2\right) \\ -\frac{2x + y + 2xy}{2y + x + x^2} &= \frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

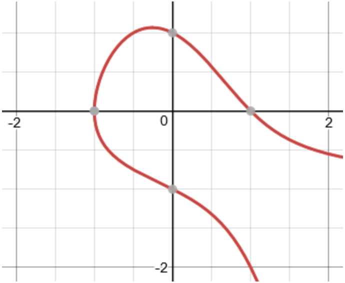
$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily, $\frac{dy}{dx}$ depends on **both** $y$ and $x$. Why?





ans

$$y^2 + x^2 + xy + x^2y = 1$$

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Necessarily, $\frac{\mathrm{d}y}{\mathrm{d}x}$ depends on **both** $y$ and $x$. Why?

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,0)} = \qquad\qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,-2)} =$$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily, $\frac{\mathrm{d}y}{\mathrm{d}x}$ depends on **both** $y$ and $x$. Why?

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,0)} = -\frac{2(1) + 0 + 2(1)(0)}{2(0) + 1 + 1} = -\frac{2}{2} = -1$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,-2)} = -\frac{2(1) - 2 + 2(1)(-2)}{2(-2) + 1 + 1} = -2$$

$$y^2 + x^2 + xy + x^2y = 1$$

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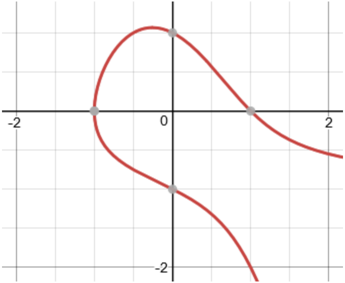
Points with the same $x$-value may have different slopes. We need both the $x$-value and the $y$-value to figure out which point we're talking about.

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$




Points with the same $x$-value may have different slopes. We need both the $x$-value and the $y$-value to figure out which point we're talking about.

NOW YOU

Suppose $x^4y + y^4x = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

Q



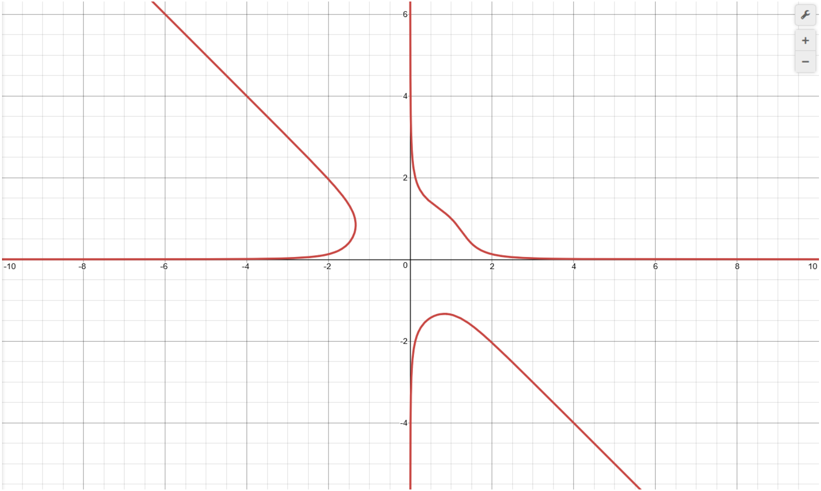
ans

**Now You** Suppose $x^4y + y^4x = 2$. Find $\frac{dy}{dx}$ at the point $(1,1)$.

$$\begin{aligned} x^4y(x) + y(x)^4x &= 2 \\ 4x^3y(x) + x^4\frac{dy}{dx}(x) + y(x)^4 + 4y(x)^3\frac{dy}{dx}(x)\,x &= 0 \end{aligned}$$

We may only replace variables with constants *after* differentiating. When $x = 1$ and $y(1) = 1$,

$$\begin{aligned} 4(1)^3y(1) + (1)^4\frac{dy}{dx}(1) + y(1)^4 + 4y(1)^3\frac{dy}{dx}(1) &= 0 \\ 4 + \frac{dy}{dx}(1) + 1 + 4\frac{dy}{dx}(1) &= 0 \\ 5\frac{dy}{dx}(1) &= -5 \\ \frac{dy}{dx}(1) &= -1 \end{aligned}$$




$$x^4y + y^4x = 2$$

**Now You** Suppose $\dfrac{3y^2+2y+y^3}{x^2+1}=x$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x=0$, and the equations of the associated tangent line(s).

**Now You** Suppose $\dfrac{3y^2+2y+y^3}{x^2+1}=x$. Find $\dfrac{dy}{dx}$ when $x=0$, and the equations of the associated tangent line(s).

To avoid the quotient rule, we start by simplifying our expression.

$$\frac{3y(x)^2+2y(x)+y(x)^3}{x^2+1}=x$$

$$3y(x)^2+2y(x)+y(x)^3=x^3+x$$

$$6y(x)\frac{dy}{dx}(x)+2\frac{dy}{dx}(x)+3y(x)^2\frac{dy}{dx}(x)=3x^2+1$$

When $x=0$:

$$\frac{dy}{dx}(0)=\frac{1}{6y(0)+2+3y(0)^2}$$

We need to know $y$ to find $\frac{dy}{dx}$. We want all points where $x = 0$.

$$\begin{aligned} 3y(0)^2 + 2y(0) + y(0)^3 &= 0 \\ y(0)\left(y(0)^2 + 3y(0) + 2\right) &= 0 \\ y(0)\left(y(0) + 1\right)\left(y(0) + 2\right) &= 0 \\ y(0) = 0,\ y(0) = -1,\ y(0) &= -2 \end{aligned}$$

A

$$\frac{dy}{dx} = \frac{1}{6y + 2 + 3y^2}$$

$(0, 0)$

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{1}{2}$$

$$y - 0 = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x$$

$(0, -1)$

$$\left.\frac{dy}{dx}\right|_{(0,-1)} = \frac{1}{-6+2+3} = -1$$

$$y - (-1) = -1(x - 0)$$

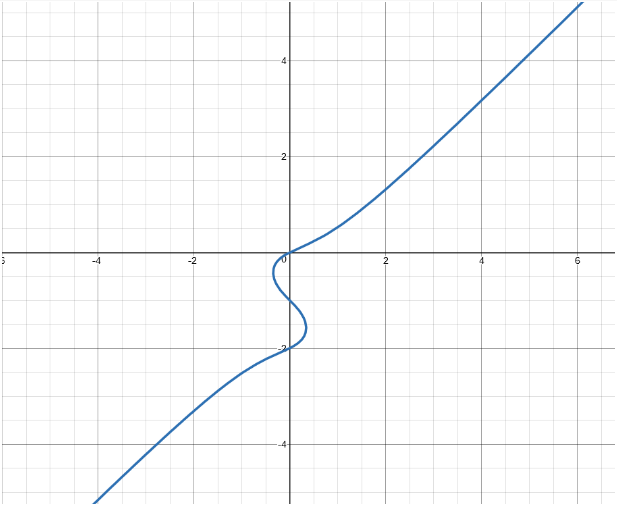
$$y = -x - 1$$

$(0, -2)$

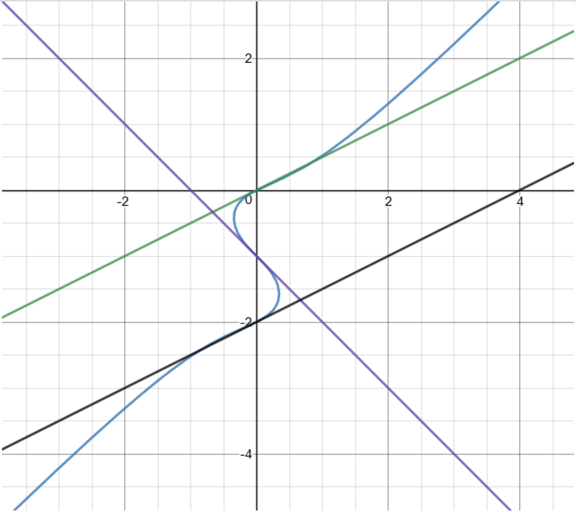
$$\left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{1}{-12+2+12} = \frac{1}{2}$$

$$y - (-2) = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x - 2$$




$$\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$$




$$\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$$

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

$$\log x = y(x)$$
$$x = e^{y(x)}$$


ans

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

$$\begin{aligned}
\log x &= y(x) \\
x &= e^{y(x)} \\
1 &= e^{y(x)} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}(x) \\
\frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{1}{e^{y(x)}} = \frac{1}{x}
\end{aligned}$$

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

$$\begin{aligned} \log x &= y(x) \\ x &= e^{y(x)} \\ 1 &= e^{y(x)} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}(x) \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{1}{e^{y(x)}} = \frac{1}{x} \end{aligned}$$

Use implicit differentiation to differentiate $\log|x|$, $x < 0$.

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

Use implicit differentiation to differentiate $\log|x|$, $x < 0$.

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

Use implicit differentiation to differentiate $\log|x|$, $x < 0$.

$$\begin{aligned}
\log|x| &= y(x) \\
\log(-x) &= y(x) \\
-x &= e^{y(x)} \\
-1 = e^{y(x)} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}(x) & \\
\frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{-1}{e^{y(x)}} = \frac{-1}{-x} = \frac{1}{x}
\end{aligned}$$

Use implicit differentiation to differentiate $\log_a(x)$, where $a > 0$ is a constant and $x > 0$.

Use implicit differentiation to differentiate $\log_a|x|$, $a > 0$.

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Use implicit differentiation to differentiate $\log_a(x)$, where $a > 0$ is a constant and $x > 0$.

$$\begin{aligned}
\log_a x &= y(x) \\
x &= a^{y(x)} \\
1 &= a^{y(x)} \cdot \log_e a \cdot \frac{\mathrm{d}y}{\mathrm{d}x}(x) \\
\frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{1}{a^{y(x)} \cdot \log_e a} = \frac{1}{x \log_e a}
\end{aligned}$$

Use implicit differentiation to differentiate $\log_a |x|$, $a > 0$.

Use implicit differentiation to differentiate $\log_a |x|$, $a > 0$.

If $x > 0$, it's what we just computed. So assume $x < 0$.

$$\begin{aligned}
\log_a |x| &= y(x) \\
\log_a(-x) &= y(x) \\
-x &= a^{y(x)} \\
-1 &= a^{y(x)} \cdot \log_e a \cdot \frac{\mathrm{d}y}{\mathrm{d}x}(x) \\
\frac{\mathrm{d}y}{\mathrm{d}x}(x) &= \frac{-1}{a^{y(x)} \cdot \log_e a} = \frac{-1}{x \log_e a}
\end{aligned}$$

# Included Work

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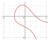
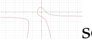
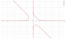
 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 19 October 2017), 18, 22

 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 19 October 2017), 7–9

 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 24 October 2018), 25

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