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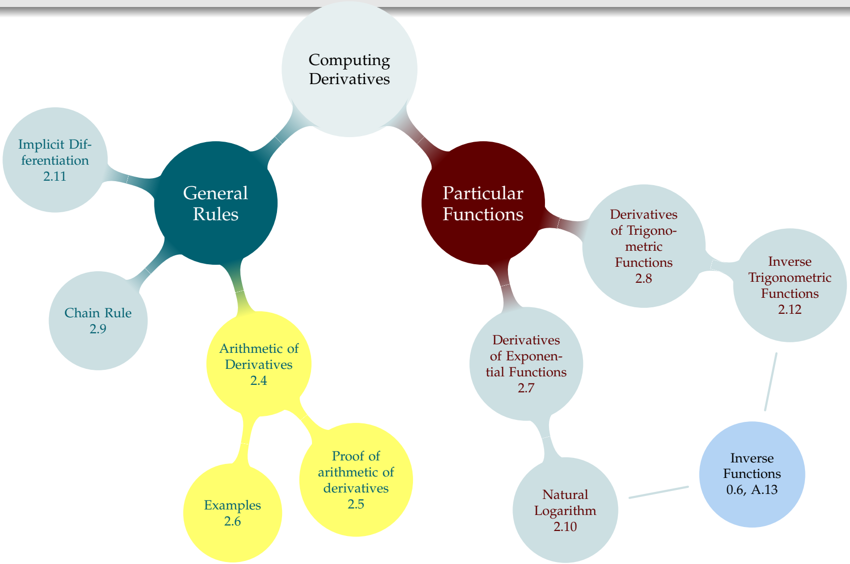

Sections 2.4 (*Arithmetic of Derivatives – A Differentiation Toolbox*), 2.5 (*Proofs of the Arithmetic of Derivatives*), and 2.6 (*Using the Arithmetic of Derivatives – Examples*) are closely linked in content. Content from these sections does not necessarily appear in order in this file.

- Content from Section 2.4 can be found in the following locations:
  - 22: derivative of sum and difference
  - 32: Theorem 2.4.3, the product rule
  - 40: Theorem 2.4.5, the quotient rule
- Content from Section 2.5 can be found in the following locations:
  - 18, 20: proof of the linearity of differentiation
  - 31: proof of the product rule
- Content from Section 2.6 can be found in the following locations:
  - 23, 25: linearity of differentiation
  - 35, 37: product rule
  - 38: Example 2.6.6, product of three functions
  - 41, 43: quotient rule
  - 53: Lemma 2.6.9, derivative of x^n.
  - 70, 74: power rule
  - 45, 49, 51, 72: various rules

Derivatives of Lines

$$f(x) = 2x - 15$$

The equation of the tangent line to $f(x)$ at $x = 100$ is:

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A. 0 B. 1 C. 2 D. -15 E. -13

Q

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$f'(1) =$

A. 0 B. 1 C. 2 D. -15 E. -13

$f'(5) =$

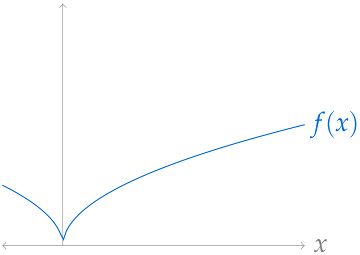
$f'(-13) =$

$$g(x) = 13$$

$g'(1) =$

A. 0 B. 1 C. 2 D. 13

Adding a Constant



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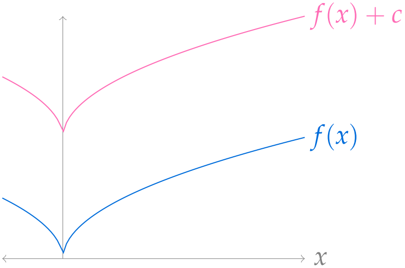
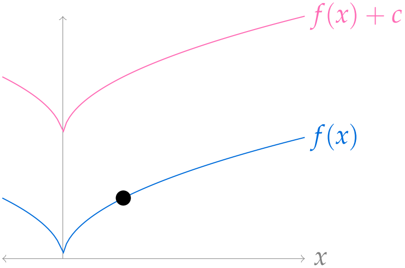


Adding a Constant



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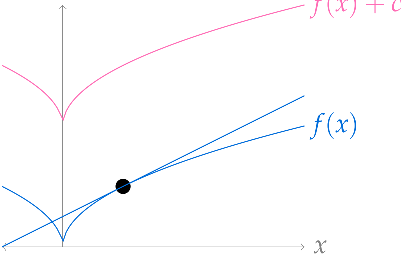

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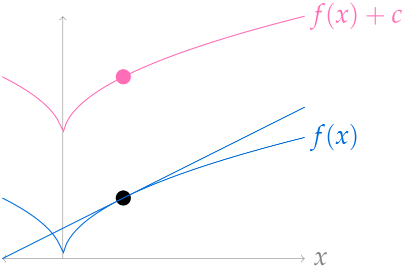

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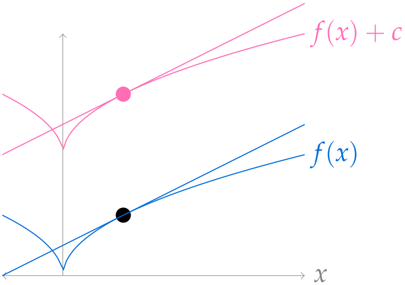

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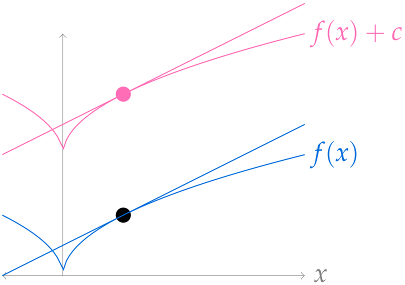

Adding or subtracting a constant to a function does not change its derivative.

Adding a Constant

Adding or subtracting a constant to a function does not change its derivative.

We saw

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3-0.8t^2\right)\Big|_{t=1} = -1.6$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(10-0.8t^2\right)\Big|_{t=1} =$$

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Differentiating Sums

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{f(x)+g(x)\right\} =$$

DIFFERENTIATING SUMS

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+g(x)\} &= \lim_{h\to 0}\left[\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\right] \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)+g(x+h)+g(x)}{h}\right] \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] \\
&= f'(x)+g'(x)
\end{aligned}
$$

Constant Multiple of a Function

Let a be a constant.

$$\frac{d}{dx}\{a \cdot f(x)\} =$$

Constant Multiple of a Function

Let a be a constant.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{a \cdot f(x)\right\} &= \lim_{h\to 0}\left[\frac{a \cdot f(x+h) - a \cdot f(x)}{h}\right] \\
&= \lim_{h\to 0}\left[a \cdot \frac{f(x+h) - f(x)}{h}\right] \\
&= a \cdot \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \\
&= a \cdot f'(x)
\end{aligned}$$

Rules – Lemma 2.4.1

Suppose $f(x)$ and $g(x)$ are differentiable, and let c be a constant number. Then:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+g(x)\} = f'(x)+g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)-g(x)\} = f'(x)-g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{cf(x)\} = cf'(x)$

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For instance: let $f(x) = 10\left((2x-15)+13-\sqrt{x}\right)$. Then $f'(x) =$

Rules – Lemma 2.4.1

Suppose $f(x)$ and $g(x)$ are differentiable, and let c be a constant number. Then:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+g(x)\} = f'(x)+g'(x)$ ←Add a constant: no change
- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)-g(x)\} = f'(x)-g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{cf(x)\} = cf'(x)$ ← Multiply by a constant: keep the constant

For instance: let $f(x) = 10\left((2x-15)+13-\sqrt{x}\right)$. Then $f'(x) = 10\left((2)+0-\frac{1}{2\sqrt{x}}\right)$.

Now You

Suppose $f'(x) = 3x$, $g'(x) = -x^2$, and $h'(x) = 5$.

Calculate:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{f(x) + 5g(x) - h(x) + 22\right\}$$

A. $3x - 5x^2$

B. $3x - 5x^2 - 5$

C. $3x - 5x^2 - 5 + 22$

D. none of the above

Now You

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Derivatives of Products

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x\} =$$

Derivatives of Products

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x\} = 1$$

Derivatives of Products

$$\frac{d}{dx}\{x\} = 1$$

True or False:

$$\begin{aligned}\frac{d}{dx}\{2x\} &= \frac{d}{dx}\{x+x\}\\ &= [1]+[1]\\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{d}{dx}\{x^2\} &= \frac{d}{dx}\{x\cdot x\}\\ &= [1]\cdot[1]\\ &= 1\end{aligned}$$

What to do with Products?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x. What about $f(x)g(x)$?

$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} =$

What to do with Products?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x. What about $f(x)g(x)$?

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)\left[g(x+h)-g(x)\right]+g(x)\left[f(x+h)-f(x)\right]}{h}\\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right]\\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right]\\
&= f(x)g'(x)+g(x)f'(x)
\end{aligned}$$

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] =$$

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[x \cdot x\right] = x(1) + x(1) = 2x$$

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

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Example: suppose $f(x) = 3x^2, f'(x) = 6x$, $g(x) = \sin(x), g'(x) = \cos(x).$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[3x^2\sin(x)\right] =$$

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{d}{dx}\left[x^2\right] = \frac{d}{dx}[x \cdot x] = x(1) + x(1) = 2x$$

Example: suppose $f(x) = 3x^2, f'(x) = 6x$, $g(x) = \sin(x), g'(x) = \cos(x)$.

$$\frac{d}{dx}\left[3x^2 \sin(x)\right] = 3x^2 \cdot \cos(x) + \sin(x) \cdot 6x$$

Given $\frac{d}{dx}[2x+5] = 2, \quad \frac{d}{dx}\left[\sin(x^2)\right] = 2x\cos(x^2), \quad \frac{d}{dx}\left[x^2\right] = 2x$

Now You

$$f(x) = (2x+5)\sin(x^2)$$

A. $f'(x) = (2)\left(2x\cos(x^2)\right)(2x)$

B. $f'(x) = (2)\left(2x\cos(x^2)\right)$

C. $f'(x) = (2x+5)(2) + \sin(x^2)\left(2x\cos(x^2)\right)$

D. $f'(x) = (2x+5)\left(2x\cos(x^2)\right) + (2)\sin(x^2)$

E. none of the above

Now You

$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is $f'(x)$?

NOW YOU

$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is $f'(x)$?

$$\begin{aligned}
f(x) &= [a(x)b(x)]\, c(x) \\
f'(x) &= [a(x)b(x)]\, c'(x) + c(x)\frac{\mathrm{d}}{\mathrm{d}x}\{a(x)b(x)\} \\
&= a(x)b(x)c'(x) + c(x)\,[a(x)b'(x) + a'(x)b(x)] \\
&= a'(x)b(x)c(x) + a(x)b'(x)c(x) + a(x)b(x)c'(x)
\end{aligned}$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x+5}{3x-6}\right\} =$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

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Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x+5}{3x-6}\right\} = \frac{(3x-6)(2)-(2x+5)(3)}{(3x-6)^2}$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{5x}{\sqrt{x}-1}\right\} =$$

Q

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{5x}{\sqrt{x}-1}\right\} = \frac{(\sqrt{x}-1)(5) - (5x)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}-1)^2} = \frac{\frac{5}{2}\sqrt{x}-5}{(\sqrt{x}-1)^2}$$

Now You

Differentiate the following.

$$f(x) = 2x + 5$$
$$g(x) = (2x + 5)(3x - 7) + 25$$
$$h(x) = \frac{2x + 5}{8x - 2}$$
$$j(x) = \left(\frac{2x + 5}{8x - 2}\right)^2$$

Rules

Product: $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

$$f(x) = 2x + 5$$

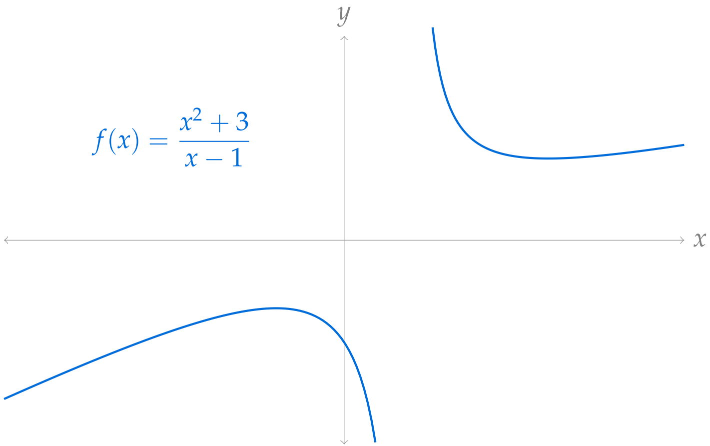
$$g(x) = (2x + 5)(3x - 7) + 25$$

Q

ans

$$h(x) = \frac{2x+5}{8x-2}$$

$$j(x) = \left(\frac{2x+5}{8x-2}\right)^2$$



$$f(x) = \frac{x^2+3}{x-1}$$

For which values of x is the tangent line to the curve horizontal?

A horizontal line has slope 0, and the slope of the tangent line is the function's derivative. So, we should find where the function's derivative is 0.

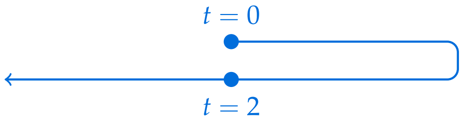
$$f'(x) = \frac{(x-1)(2x) - (x^2+3)(1)}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2}$$

$$\boxed{x = -1, x = 3}$$

The position of an object moving left and right at time t, $t \geq 0$, is given by

$$s(t) = -t^2(t-2)$$

where a positive position means it is to the right of its starting position, and a negative position means it is to the left. First it moves to the right, then it moves left forever.



What is the farthest point to the right that the object reaches?

When the object turns to come back around, $s'(t) = 0$. If we can find the value of t that makes this true, then we plug it in to $s(t)$ to find the farthest to the right reached by the object.

$$s'(t) = [-t^2](1) + (-2t)(t-2) = -3t^2 + 4t = t(4-3t)$$

So, the object turns around when $t = \frac{4}{3}$.

Its position at that time is $s\left(\frac{4}{3}\right) = \frac{32}{27}$ units to the right of its starting position.

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\}$$

function	derivative
x	1

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$

function	derivative
x	1

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

function	derivative
x	1
x^2	$2x$

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^3\}$$

function	derivative
x	1
x^2	$2x$

More About the Product Rule

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$

function	derivative
x	1
x^2	$2x$

More About the Product Rule

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1)$$

function	derivative
x	1
x^2	$2x$

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^3\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^3\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^4\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^3\}$$

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^3\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^4\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^3\}$$
$$= x(3x^2) + x^3(1)$$

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$

More About the Product Rule

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{d}{dx}\{x^4\} = \frac{d}{dx}\{x \cdot x^3\}$$
$$= x(3x^2) + x^3(1) = 4x^3$$

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^3\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^4\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x^3\}$$
$$= x(3x^2) + x^3(1) = 4x^3$$

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{30}	

More About the Product Rule

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{d}{dx}\{x^4\} = \frac{d}{dx}\{x \cdot x^3\}$$
$$= x(3x^2) + x^3(1) = 4x^3$$

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{30}	$30x^{29}$

More About the Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x(1) + x(1)$$
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Where are these functions defined?

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
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CAUTIONARY TALE

WITH *functions* RAISED TO A POWER, IT'S MORE COMPLICATED.

Differentiate $(2x+1)^2$

Cautionary Tale

With *functions* raised to a power, it's more complicated.

Differentiate $(2x+1)^2$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{(2x+1)^2\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{(2x+1)(2x+1)\right\}\\ &= (2x+1)(2)+(2x+1)(2)\\ &= 4(2x+1)\end{aligned}$$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

$$\frac{\mathrm{d}}{\mathrm{d}x}\{3x^5 + 7x^2 - x + 15\} =$$

Q

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left\{3x^5 + 7x^2 - x + 15\right\}\\ &= 3 \cdot 5x^4 + 7 \cdot 2x - 1\\ &= 15x^4 + 14x - 1\end{aligned}$$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Differentiate $\dfrac{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{2x+5}$

Q

Power Rule – Corollary 2.6.17

$\frac{d}{dx}\{x^a\} = ax^{a-1}$ (where defined)

Differentiate $\dfrac{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{2x+5}$

$$\frac{d}{dx}\left\{\frac{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{2x+5}\right\}$$

$$= \frac{(2x+5)\cdot\frac{d}{dx}\left\{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})\right\} - (x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})(2)}{(2x+5)^2}$$

$$= \frac{(2x+5)\left[(x^4+1)\left(\frac{1}{3}x^{-2/3}+\frac{1}{4}x^{-3/4}\right)+4x^3(\sqrt[3]{x}+\sqrt[4]{x})\right]}{(2x+5)^2}$$

$$- \frac{2(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{(2x+5)^2}$$

Suppose a motorist is driving their car, and their position is given by $s(t) = 10t^3 - 90t^2 + 180t$ kilometres. At $t = 1$ (t measured in hours), a police officer notices they are driving erratically. The motorist claims to have simply suffered a lack of attention: they were in the act of pressing the brakes even as the officer noticed their speed.

At $t = 1$, how fast was the motorist going, and were they pressing the gas or the brake?

Challenge: What about $t = 2$?

Velocity is the rate of change of position, so the velocity of the car is given by:

$$v(t) = s'(t) = 30t^2 - 180t + 180$$

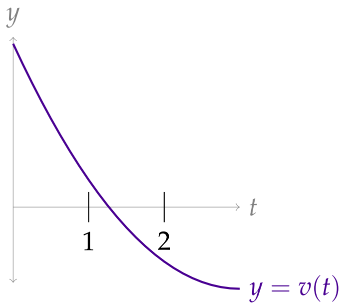
When $t = 1$, $v(1) = 30$, so the motorist was going 30 kph.

$$v'(t) = 60t - 180$$

When $t = 1$, the velocity of the car was changing by $v'(t) = -120$ kph per hour. Since the velocity was positive, but its rate of change is negative, the car was slowing down, i.e. decelerating, when $t = 1$.

$v(2) = -60$, so the motorist is driving backwards at 60 kph when $t = 2$. Also $v'(2) = -60$ so the motorist's velocity is becoming increasingly more negative. That is, the motorist is going backwards faster and faster, and so is accelerating.

The contrast between $t = 1$ and $t = 2$ can be a subtle point. To help illustrate it, consider the graph below.



The vertical axis corresponds to velocity, so the car is stopped when the curve crosses the t-axis. At $t = 1$, the curve is "heading towards" its t-intercept. So the velocity is approaching 0. At $t = 2$, it is "heading away" from its t-intercept. The velocity is moving away from 0.

Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.
Suppose you are winding twine into a gigantic twine ball, filming the process, and trying to make a viral video. You can wrap one cubic meter of twine per hour. (In other words, when we have V cubic meters of twine, we're at time V hours.) How fast is the radius of your spherical twine ball increasing?

Q

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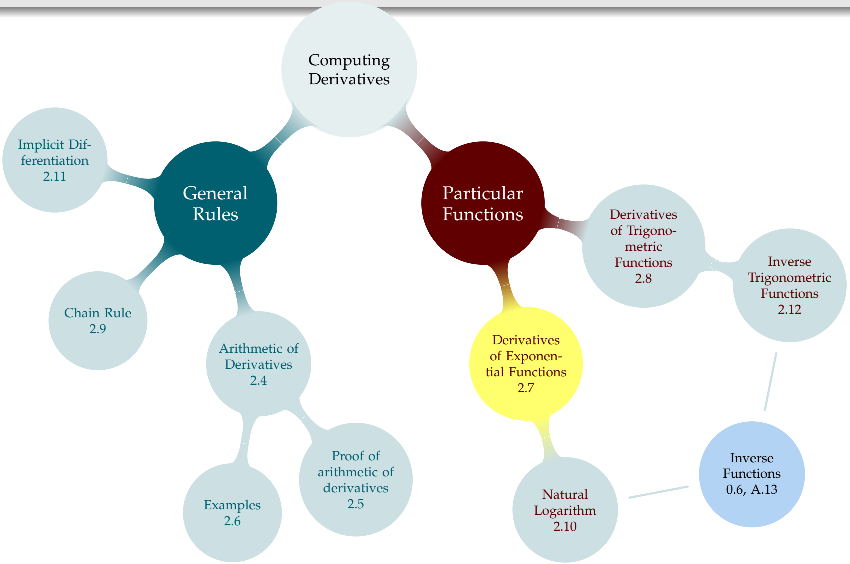
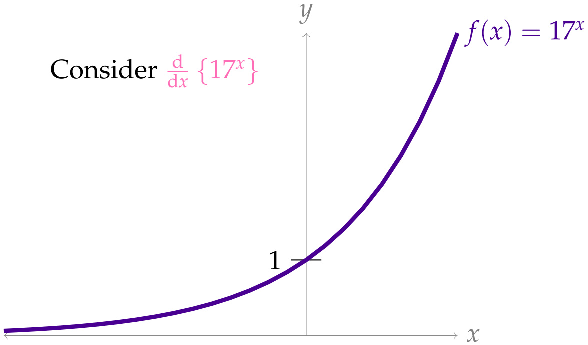
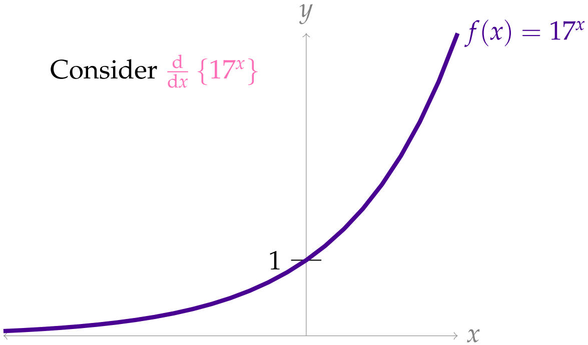


Exponential Functions

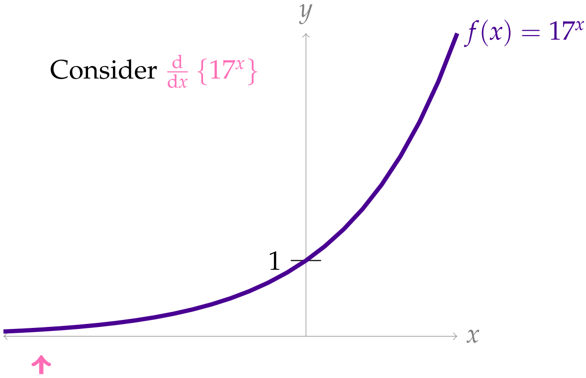


Exponential Functions



$f(x)$ is always increasing, so $f'(x)$ is always positive.

Exponential Functions



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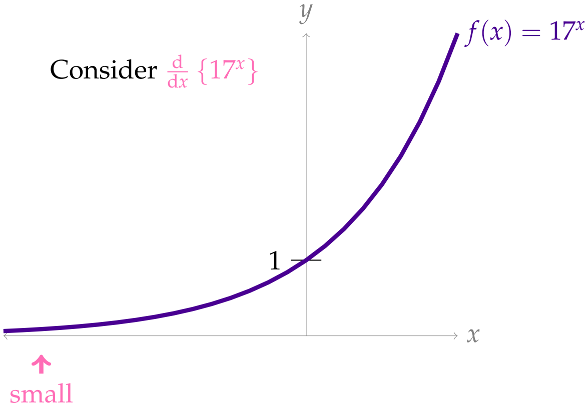
Exponential Functions



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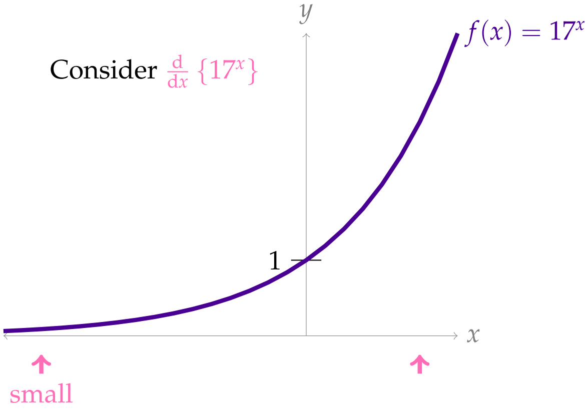
Exponential Functions



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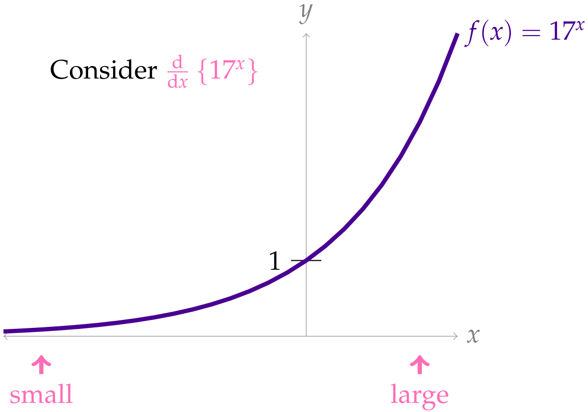
EXPONENTIAL FUNCTIONS



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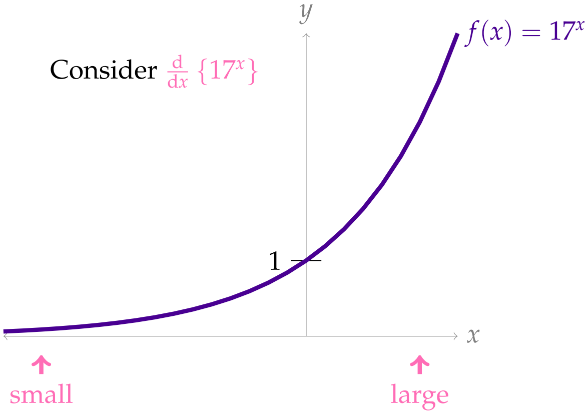
Exponential Functions



$f(x)$ is always increasing, so $f'(x)$ is always positive.
$f'(x)$ might look similar to $f(x)$.

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} =$$

Exponential Functions

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} &= \lim_{h\to 0}\frac{17^{x+h}-17^x}{h}\\
&= \lim_{h\to 0}\frac{17^x 17^h-17^x}{h}\\
&= \lim_{h\to 0}\frac{17^x(17^h-1)}{h}\\
&= 17^x\lim_{h\to 0}\frac{(17^h-1)}{h}\\
&= 17^x(\text{ times a constant })
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h - 1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0}\frac{17^h - 1}{h} = 0$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be 0, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h - 1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0}\frac{17^h - 1}{h} = \infty$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be ∞, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

How could we find out what $\displaystyle\lim_{h\to 0}\frac{(17^h-1)}{h}$ is?

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

How could we find out what $\lim_{h\to 0}\frac{(17^h-1)}{h}$ is?

h	$\frac{17^h-1}{h}$
0.001	2.83723068608
0.00001	2.83325347992
0.0000001	2.83321374583
0.000000001	2.83321344163

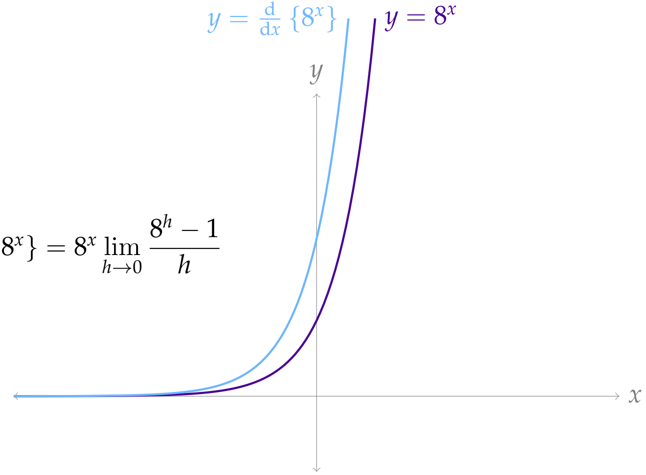
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} &= \lim_{h\to 0} \frac{17^{x+h} - 17^x}{h} \\
&= \lim_{h\to 0} \frac{17^x 17^h - 17^x}{h} \\
&= \lim_{h\to 0} \frac{17^x(17^h - 1)}{h} \\
&= 17^x \lim_{h\to 0} \frac{(17^h - 1)}{h}
\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} &= \lim_{h\to 0}\frac{17^{x+h}-17^x}{h}\\ &= \lim_{h\to 0}\frac{17^x 17^h-17^x}{h}\\ &= \lim_{h\to 0}\frac{17^x(17^h-1)}{h}\\ &= 17^x\lim_{h\to 0}\frac{(17^h-1)}{h}\end{aligned}$$

In general, for any positive number a,

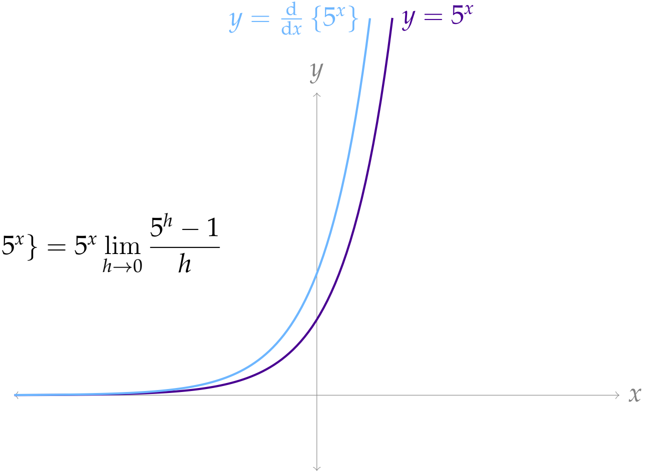
$$\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x\lim_{h\to 0}\frac{a^h-1}{h}$$

Exponential Functions



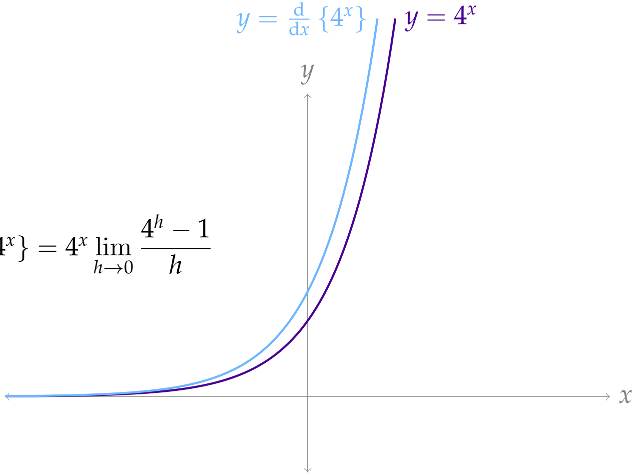
$$\frac{\mathrm{d}}{\mathrm{d}x}\{8^x\} = 8^x \lim_{h\to 0} \frac{8^h - 1}{h}$$

Exponential Functions



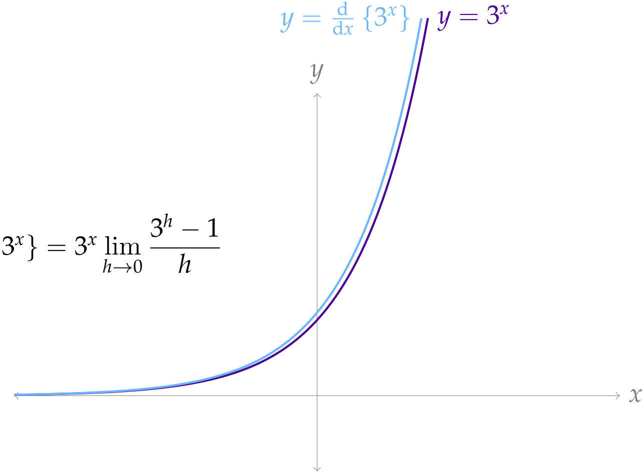
$$\frac{d}{dx}\{5^x\} = 5^x \lim_{h\to 0} \frac{5^h - 1}{h}$$

Exponential Functions



$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{4^x\right\} = 4^x \lim_{h\to 0} \frac{4^h - 1}{h}$$

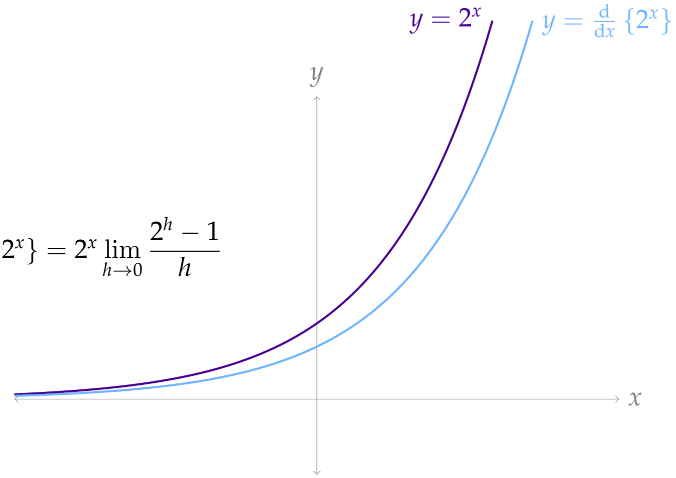
Exponential Functions



$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{3^x\right\} = 3^x \lim_{h\to 0}\frac{3^h - 1}{h}$$

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{2^x\right\} = 2^x \lim_{h\to 0} \frac{2^h - 1}{h}$$



In general, for any positive number a, $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \lim_{h\to 0} \dfrac{a^h - 1}{h}$

In general, for any positive number a, $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \lim_{h\to 0} \dfrac{a^h - 1}{h}$

Euler's Number – Theorem 2.7.4

We define e to be the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

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Euler's Number – Theorem 2.7.4

We define e to be the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$e \approx 2.7182818284590452353602874713526624\ldots$ (Wikipedia)

Theorem 2.7.4 and Corollary 2.10.6

Using this definition of e,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} = e^x \underbrace{\lim_{h\to 0}\frac{e^h-1}{h}}_{1} = e^x$$

Theorem 2.7.4 and Corollary 2.10.6

Using this definition of e,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} = e^x \underbrace{\lim_{h\to 0}\frac{e^h - 1}{h}}_{1} = e^x$$

In general, $\lim_{h\to 0}\frac{a^h - 1}{h} = \log_e(a)$, so $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \log_e(a)$

That $\lim_{h\to 0}\frac{a^h - 1}{h} = \log_e(a)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \log_e(a)$ are consequences of

$$a^x = \left(e^{\log_e(a)}\right)^x = e^{x\log_e(a)}$$

For the details, see the end of Section 2.7.

Things to Have Memorized

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Let $f(x) = \dfrac{e^x}{3x^5}$. When is the tangent line to $f(x)$ horizontal?

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Horizontal tangent line $\Leftrightarrow$ slope of tangent line is zero $\Leftrightarrow f'(x) = 0$

$$0 = f'(x) = \frac{3x^5e^x - e^x(15x^4)}{(3x^5)^2} = \left(\frac{e^x}{9x^{10}}\right)\left(3x^4\right)(x-5)$$

$$x = 0 \text{ or } x = 5$$

But, since $f(x)$ is not defined at zero, the tangent line is only horizontal at

$$x = 5$$

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{3x}\right\}$

~~ans~~

Suppose the deficit, in millions, of a fictitious country is given by

$$f(x) = e^x(4x^3 - 12x^2 + 14x - 4)$$

where x is the number of years since the current leader took office. Suppose the leader has been in power for exactly two years.

1. Is the deficit increasing or decreasing?

Suppose the deficit, in millions, of a fictitious country is given by

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where x is the number of years since the current leader took office. Suppose the leader has been in power for exactly two years.

1. Is the deficit increasing or decreasing?

2. Is the rate at which the deficit is growing increasing or decreasing?

Included Work

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