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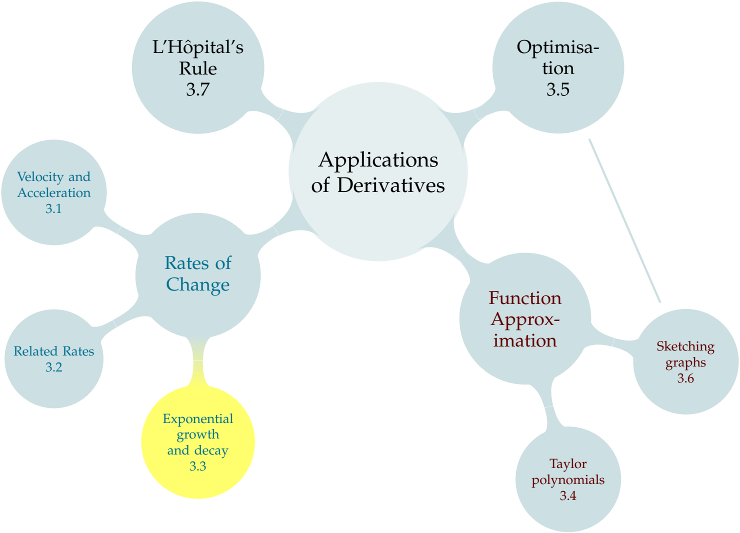


# Radioactive Decay

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

## Differential Equation

Let $Q = Q(t)$ be the amount of a radioactive substance at time $t$. Then for some positive constant $k$:

$$\frac{dQ}{dt} = -kQ$$

## Solution – Theorem 3.3.2

Let $\boxed{Q(t) = Ce^{-kt}}$, where $k$ and $C$ are constants. Then:

# Radioactive Decay

## Quantity of a Radioactive Isotope

$$Q(t) = Ce^{-kt}$$

$Q(t)$: quantity at time $t$

What is the sign of $Q(t)$?

A. positive or zero
B. negative or zero
C. could be either
D. I don't know

What is the sign of $C$?

A. positive or zero
B. negative or zero
C. could be either
D. I don't know

## Seaborgium Decay

The amount of $^{266}Sg$ (Seaborgium-266) in a sample at time $t$ (measured in seconds) is given by

$$Q(t) = Ce^{-kt}$$

Let's approximate the half life of $^{266}Sg$ as 30 seconds. That is, every 30 seconds, the size of the sample halves.

What are $C$ and $k$?

A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope. In the year 2005, it is tested and found to contain only 2 units of that same isotope. How many units of the isotope were present in the year 2000?

# $Q'(t) = kQ(t)$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

The rate of growth of a population in a given time interval is propotional to the number of individuals in the population, when the population has ample resources.

The amount of interest a bank account accrues in a given time interval is proportional to the balance in that bank account.

## Exponential Growth – Theorem 3.3.2

Let $Q = Q(t)$ satisfy:

$$\frac{dQ}{dt} = kQ$$

for some constant $k$. Then for some constant $C = Q(0)$,

$$Q(t) = Ce^{kt}$$

Suppose $y(t)$ is a function with the properties that

$$\frac{dy}{dt} + 3y = 0 \quad \text{and} \quad y(1) = 2.$$

What is $y(t)$?

# Population Growth

Suppose a petri dish starts with a culture of 100 bacteria cells and a limited amount of food and space. The population of the culture at different times is given in the table below. At approximately what time did the culture start to show signs of limited resources?

| time | population |
|---|---|
| 0 | 100 |
| 1 | 1000 |
| 3 | 100000 |
| 5 | 1000000 |

# Flu Season

The CDC keeps records (link) on the number of flu cases in the US by week. At the start of the flu season, the 40th week of 2014, there are 100 cases of a particular strain. Five weeks later (at week 45), there are 506 cases. What do you think was the first week to have 5,000 cases? What about 10,000 cases?

## Newton's Law of Cooling – Equation 3.3.7

The rate of change of temperature of an object is proportional to the difference in temperature between that object and its surroundings.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

where $T(t)$ is the temperature of the object at time $t$, $A$ is the (constant) ambient temperature of the surroundings, and $K$ is some constant depending on the object.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, $A$ is the ambient temperature, $K$ is some constant.

What is true of $K$?

A. $K \geq 0$

B. $K \leq 0$

C. $K = 0$

D. $K$ could be positive, negative, or zero, depending on the object

E. I don't know

## Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, $A$ is the ambient temperature, and $K$ is some constant.

$$T(t) = [T(0) - A]e^{Kt} + A$$

is the only function satisfying Newton's Law of Cooling

If $T(10) < A$, then:

A. $K > 0$
B. $T(0) > 0$
C. $T(0) > A$
D. $T(0) < A$

Evaluate $\displaystyle\lim_{t\to\infty} T(t)$.

A. $A$
B. $0$
C. $\infty$
D. $T(0)$

What assumptions are we making that might not square with the real world?

## Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt} = K[T(t) - A]$$

$T(t)$ is the temperature of the object, $A$ is the ambient temperature, and $K$ is some constant.

## Temperature of a Cooling Body – Corollary 3.3.8

$$T(t) = [T(0) - A]e^{Kt} + A$$

A farrier forms a horseshoe heated to 400° C, then dunks it in a river at room-temperature (25° C). The water boils for 30 seconds. The horseshoe is safe for the horse when it's 40° C. When can the farrier put on the horseshoe?

$$T(t) = [T(0) - A]e^{Kt} + A$$

A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°, and after 20 minutes, the tea is 25°. What is the temperature outside?

In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?

link: Wood Bison Restoration in Alaska, Alaska Department of Fish and Game

Excerpt:

> *Based on experience with reintroduced populations elsewhere, wood bison would be expected to increase at a rate of 15%-25% annually after becoming established.... With an average annual growth rate of 20%, an initial precalving population of 50 bison would increase to 500 in approximately 13 years.*

Now You

Are they using our same model?

# Compound Interest

Suppose you invest $10,000 in an account that accrues interest each month. After one month, your balance (with interest) is $10,100. How much money will be in your account after a year?

Compound interest is calculated according to the formula $Pe^{rt}$, where $r$ is the interest rate and $t$ is time.

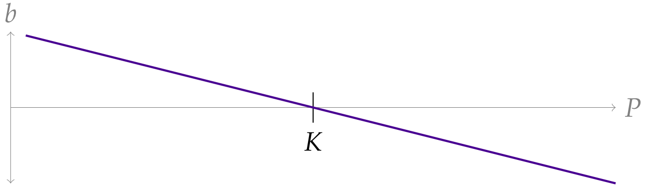
# Carrying Capacity

For a population of size $P$ with unrestricted access to resources, let $\beta$ be the average number of offspring each breeding pair produces per generation, where a generation has length $t_g$. Then $b = \frac{\beta - 2}{2t_g}$ is the net birthrate (births minus deaths) per member per unit time. This yields $\frac{dP}{dt}(t) = bP(t)$, hence:

But as resources grow scarce, $b$ might change.

# Carrying Capacity

$b$ is the net birthrate (births minus deaths) per member per unit time. If $K$ is the carrying capacity of an ecosystem, we can model $b = b_0(1 - \frac{P}{K})$.




Now You

Describe to your neighbour what the following mean in terms of the model:

- $b > 0$, $b = 0$, $b < 0$
- $P = 0$, $P > 0$, $P < 0$

# CARRYING CAPACITY

Then:

$$\frac{dP}{dt}(t) = \underbrace{b_0\left(1 - \frac{P(t)}{K}\right)}_{\text{per capita birthrate}} P(t)$$

This is an example of a differential equation that we don't have the tools to solve. (If you take more calculus, though, you'll learn how!) It's also an example of a way you might tweak a model so its assumptions better fit what you observe.

# Radiocarbon Dating

Researchers at Charlie Lake in BC have found evidence[1] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago.

Suppose a comparable bone of a bison alive today contains $1\mu$g of $^{14}C$. If the half-life of $^{14}C$ is about 5730 years, roughly how much $^{14}C$ do you think the researchers found in the sample?

A. About $\frac{1}{10,500}$ $\mu$g

B. About $\frac{1}{4}$ $\mu$g

C. About $\frac{1}{2}$ $\mu$g

D. About 1 $\mu$g

E. I'm not sure how to estimate this

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

Suppose a body is discovered at 3:45 pm, in a room held at $20^\circ$, and the body's temperature is $27^\circ$, not the normal $37^\circ$. At 5:45 pm, the temperature of the body has dropped to $25.3^\circ$. When did the inhabitant of the body die?

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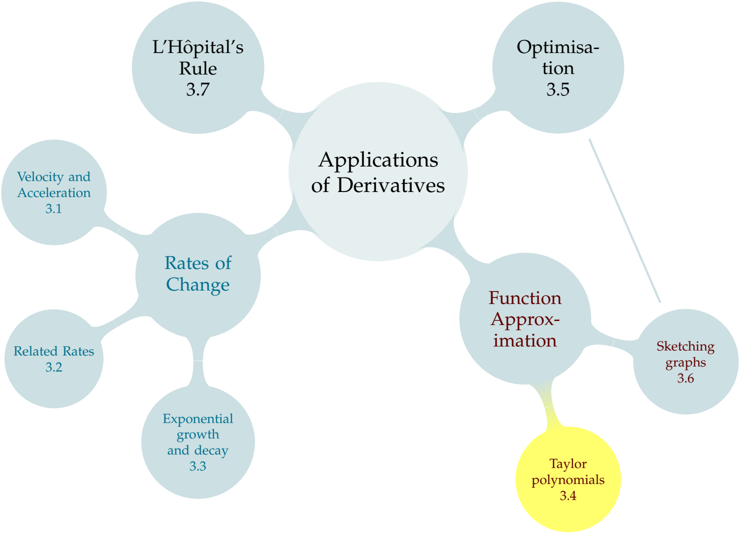
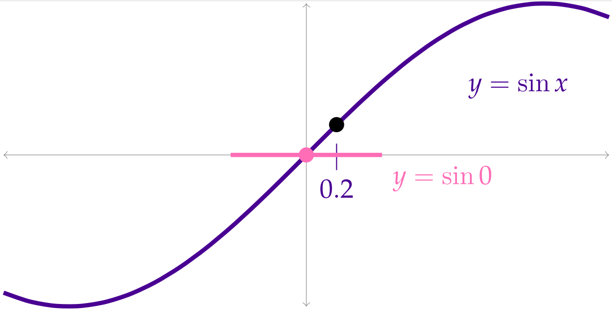


# Approximating a Function

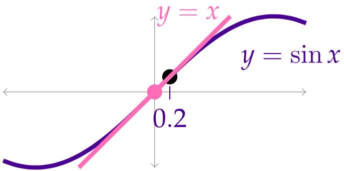



## Constant Approximation – Equation 3.4.1

We can approximate $f(x)$ near a point $a$ by

$$f(x) \approx f(a)$$

# Approximating a Function



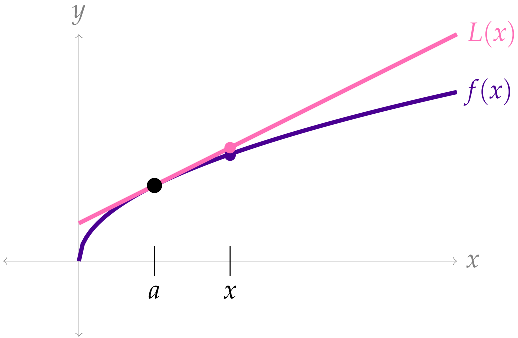

## Linear Approximation (Linearization) – Equation 3.4.3

We can approximate $f(x)$ near a point $a$ by the tangent line to $f(x)$ at $a$, namely

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

To find a linear approximation of $f(x)$ at a particular point $x$, pick a point $a$ near to $x$, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$



To find a linear approximation of $f(x)$ at a particular point $x$, pick a point $a$ near to $x$, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

Let $f(x) = \sqrt{x}$. Approximate $f(8.9)$.

# CAN WE COMPUTE?

Suppose we want to approximate the value of $\cos(1.5)$. Which of the following linear approximations could we calculate by hand? (You can leave things in terms of $\pi$.)

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = 3/2$

C. both

D. neither

# CAN WE COMPUTE?

Which of the following tangent lines is probably the most accurate in approximating $\cos(1.5)$?

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = \pi/4$

C. constant approximation: $\cos 1.5 \approx \cos(\pi/2) = 0$

D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

# LINEAR APPROXIMATION

Approximate $\sin(3)$ using a linear approximation. You may leave your answer in terms of $\pi$.

# Linear Approximation

Approximate $e^{1/10}$ using a linear approximation.
If $f(x) = e^x$ and $a = 0$ :

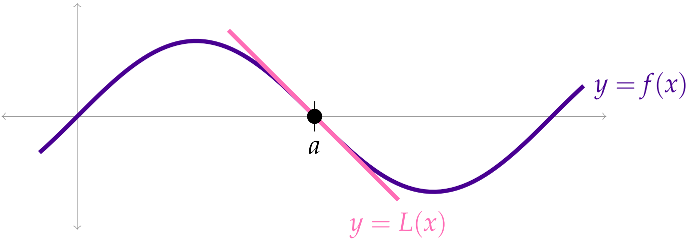
# Linear Approximation Wrap-Up

Let $L(x) = f(a) + f'(a)(x - a)$, so $L(x)$ is the linear approximation (linearization) of $f(x)$ at $a$.

What is $L(a)$?

What is $L'(a)$?

What is $L''(a)$? (Recall $L''(x)$ is the derivative of $L'(x)$.)



# LINEAR APPROXIMATION WRAP-UP

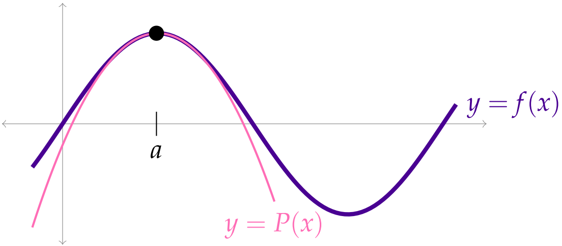
Let $L(x)$ be a linear approximation of $f(x)$.

| | | |
|---|---|---|
| $f(a)$ | $L(a)$ | same |
| $f'(a)$ | $L'(a)$ | same |
| $f''(a)$ | $L''(a)$ | different[2] |

[2] unless $f''(a) = 0$

# Quadratic Approximation

Imagine we approximate $f(x)$ at $x = a$ with a parabola, $P(x)$.



| | Constant | Linear | Quadratic |
|---|---|---|---|
| Function value matches at $x = a$ | ✓ | ✓ | ✓ |
| First derivative matches at $x = a$ | × | ✓ | ✓ |
| Second derivative matches at $x = a$ | × | × | ✓ |

| | |
|---|---|
| Constant: | $f(x) \approx f(a)$ |
| Linear: | $f(x) \approx f(a) + f'(a)(x-a)$ |
| Quadratic: | $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$ |

# Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

# QUADRATIC APPROXIMATION

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
*You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.*

Determine what $f(x)$ and $a$ should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$

$e^{-1/30}$

$\sqrt[5]{30}$

$(2.01)^6$

| | Constant | Linear | Quadratic | degree $n$ |
|---|---|---|---|---|
| match $f(a)$ | ✓ | ✓ | ✓ | ✓ |
| match $f'(a)$ | × | ✓ | ✓ | ✓ |
| match $f''(a)$ | × | × | ✓ | ✓ |
| ... | | | | |
| match $f^{(n)}(a)$ | × | × | × | ✓ |
| match $f^{(n+1)}(a)$ | × | × | × | × |

Constant:

$f(x) \approx f(a)$

Linear:

$f(x) \approx f(a) + f'(a)(x-a)$

Quadratic:

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Degree-$n$:

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$

# Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

- $a, b$ (integers) "bounds"
- $i$ "index": runs over integers from $a$ to $b$
- $f(i)$ "summand": compute for every $i$, add

# Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

# Sigma Notation

$$\sum_{i=1}^{4}(i+(i-1)^2)$$

Write the following expressions in sigma notation:

1. $3+4+5+6+7$
2. $8+8+8+8+8$
3. $1+(-2)+4+(-8)+16$

## Factorial – Definition 3.4.9

We read "$n!$" as "$n$ factorial."
For a natural number $n$, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.
By convention, $0! = 1$.

We write $f^{(n)}(x)$ to mean the $n^{\text{th}}$ derivative of $f(x)$. By convention, $f^{(0)}(x) = f(x)$.

## Taylor Polynomial – Definition 3.4.11

Given a function $f(x)$ that is differentiable $n$ times at a point $a$, the $n$-th degree **Taylor polynomial** for $f(x)$ about $a$ is

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, we also call it a **Maclaurin polynomial**.

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=$$

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Maclaurin[3] polynomial for $e^x$.

[3] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

Now You

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

skip $\Delta x$ notation

## Notation 3.4.18

Let $x, y$ be variables related such that $y = f(x)$. Then we denote a small change in the variable $x$ by $\Delta x$ (read as "delta $x$"). The corresponding small change in the variable $y$ is denoted $\Delta y$ (read as "delta $y$").

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let $y = f(x)$ be the amount of water needed to produce $x$ apples in an orchard.
A farmer wants to know how a much water is needed to increase their crop yield. $\Delta x$ is shorthand for some change in the number of apples, and $\Delta y$ is shorthand for some change in the amount of water.

- Consider changing the number of apples grown from $a$ to $a + \Delta x$
- Then the change in water requirements goes from $y = f(a)$ to $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

# Linear Approximation of $\Delta y$

- Using a linear approximation, setting $x = a + \Delta x$:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) && \text{linear approximation} \\ f(a+\Delta x) &\approx f(a) + f'(a)(\Delta x) && \text{set } x = a + \Delta x \\ \Delta y = f(a+\Delta x) - f(a) &\approx f'(a)\Delta x && \text{subtract } f(a) \text{ both sides} \end{aligned}$$

**Linear Approximation of $\Delta y$ (Equation 3.4.20)**

$$\Delta y \approx f'(a)\Delta x$$

If we set $\Delta x = 1$, then $\Delta y \approx f'(a)$. So, if we want to produce $a+1$ apples instead of $a$ apples, the extra water needed for that one extra apple is about $f'(a)$. We call this the *marginal* water cost of the apple.

# Quadratic Approximation of $\Delta y$

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

**Quadratic Approximation of $\Delta y$ (Equation 3.4.21)**

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and $\pi$.

You measure an angle $x \approx \frac{\pi}{2}$, and use it to calculate $y = \sin x \approx 1$. However, you suspect the angle was not *exactly* equal to $\frac{\pi}{2}$, which means the actual value $y$ is slightly *less than* 1. In order for your value of $y$ to have an error of no more than $\frac{1}{200}$, how accurate does your measurement of $\theta$ have to be?

## Definition 3.4.25

Let $Q_0$ be the exact value of a quantity and let $Q_0 + \Delta Q$ be the measured value. We call

$$|\Delta Q|$$

the absolute error of the measurement, and

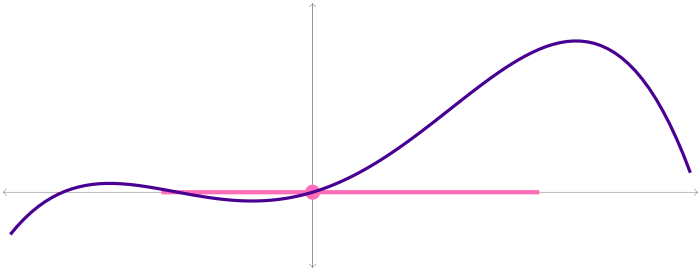
$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

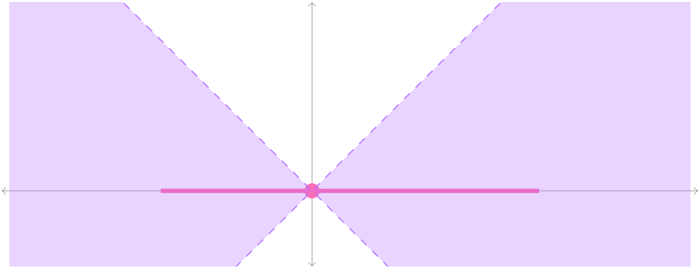
Once again, you find yourself in the position of measuring an angle $x$, which you use to compute $y = \sin x$. Let's say both $x$ and $y$ are positive. If your percentage error in measuring $x$ is at most 1%, what is the corresponding maximum percentage error in $y$?
Use a linear approximation.

# Error: what "causes" error in an estimation?



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

# CONTROLLING THE "CAUSE" OF THE ERROR



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero). BUT: suppose we know the max and min values of the function's slope.

## Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

## Taylor's Theorem – Equation 3.4.33

For some $c$ strictly between $x$ and $a$,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

Third degree Maclaurin polynomial for $f(x) = e^x$:

$$\begin{aligned} T_3(x) &= f(0) + f'(0)(x-0) + \tfrac{1}{2!}f''(0)(x-0)^2 + \tfrac{1}{3!}f'''(0)(x-0)^3 \\ &= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned}$$

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

## Taylor's Theorem – Equation 3.4.33

For some $c$ strictly between $x$ and $a$,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

## Taylor's Theorem – Equation 3.4.33

For some $c$ strictly between $x$ and $a$,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could the magnitude of the error be if we approximate $\cos(2)$?

## Taylor's Theorem – Equation 3.4.33

For some $c$ strictly between $x$ and $a$,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use a third degree Taylor polynomial centred at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for our error.

## Taylor's Theorem – Equation 3.4.33

For some $c$ strictly between $x$ and $a$,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose you want to approximate the value of $e$, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound your error.

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?

# Which Degree?

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

# WHICH DEGREE?

Suppose you want to approximate $e^5$ using a Maclaurin polynomial of $f(x) = e^x$. If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

# WHICH DEGREE?

Suppose you want to approximate $\log \frac{4}{3}$ using a Taylor polynomial of $f(x) = \log x$ centred at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

# WHICH DEGREE?

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

# Included Work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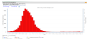
 'old tree' by FayraLovers is licensed under CC BY 3.0 (accessed 21 July 2021), 53

 U.S. WHO/NREVSS Collaborating Laboratories and ILNet. 'Stacked Column Chart WHO//NREVSS' Centers for Disease Control and Prevention. No longer available from http://gis.cdc.gov/grasp/fluview/fluportaldashboard.html (accessed 20 October 2015), 9

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