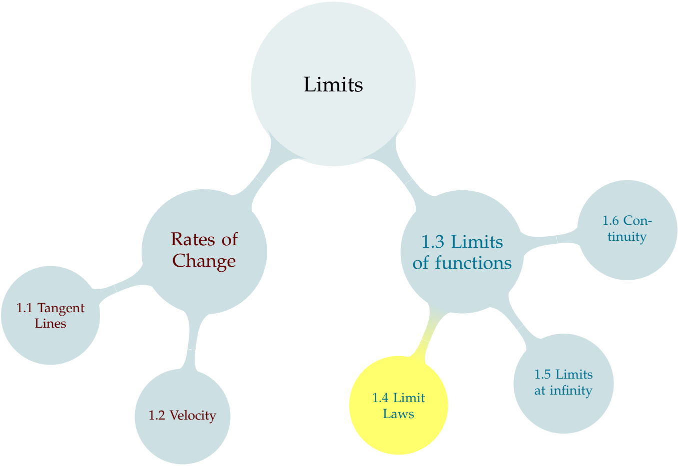
Table of Contents



Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

$$\lim_{x \to a} f(x) = f(a).$$

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Can't find in the same way: 3 not in domain

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x\to a}(f(x) + g(x)) = F + G$
- $\lim_{x\to a}(f(x) - g(x)) = F - G$
- $\lim_{x\to a}(f(x)g(x)) = FG$
- $\lim_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

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Calculate: $\lim_{x\to 1}\left[\frac{2x+4}{x+2} + 13\left(\frac{x+5}{3x}\right)\left(\frac{x^2}{2x-1}\right)\right]$

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Calculate: $\lim\limits_{x\to 1}\left[\dfrac{2x+4}{x+2}+13\left(\dfrac{x+5}{3x}\right)\left(\dfrac{x^2}{2x-1}\right)\right]$

$$=\lim_{x\to 1}\left(\frac{2x+4}{x+2}\right)+\left(\lim_{x\to 1}13\right)\left(\lim_{x\to 1}\frac{x+5}{3x}\right)\left(\lim_{x\to 1}\frac{x^2}{2x-1}\right)$$

$$=\left(\frac{2(1)+4}{1+2}\right)+(13)\left(\frac{(1)+5}{3(1)}\right)\left(\frac{1^2}{2(1)-1}\right)$$

$$=(2)+13(2)(1)$$

$$=28$$

Limits involving Powers and Roots

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

ans

Limits involving Powers and Roots

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E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

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Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x \to a} f(x) = F$ (where F is a real number), then:

$$\lim_{x \to a} (f(x))^n = F^n.$$

Furthermore, unless n is even and F is negative,

$$\lim_{x \to a} (f(x))^{1/n} = F^{1/n}$$

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$$\lim_{x\to 4}(x+5)^{1/2}$$

ans

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$$\lim_{x\to a} \left(f(x)\right)^{1/n} = F^{1/n}$$

$$\lim_{x\to 4}(x+5)^{1/2} = \left[\lim_{x\to 4}(x+5)\right]^{1/2} = 9^{1/2} = 3$$

Cautionary Tales

- $\lim\limits_{x \to 0} \dfrac{(5+x)^2 - 25}{x}$

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- $\lim_{x \to 3} \left(\frac{x-6}{3}\right)^{1/8}$

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- $\lim_{x\to 0} \dfrac{32}{x}$

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- $\lim_{x\to 5} \left(x^2+2\right)^{1/3}$

Cautionary Tales

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- $\lim_{x \to 3} \left(\frac{x-6}{3}\right)^{1/8} \to \sqrt[8]{-1}$; danger danger
- $\lim_{x \to 0} \frac{32}{x} \to \frac{32}{0}$; this expression is meaningless
- $\lim_{x \to 5} \left(x^2+2\right)^{1/3} = (5^2+2)^{1/3} = \sqrt[3]{27} = 3$

Suppose you want to evaluate $\lim_{x\to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x\to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x\to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x\to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x\to 1} f(x)$ does not exist.

E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

Q

ans

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E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

A

Which of the following statements is true about $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 0}{0^3 - 0^2 + 0} = \dfrac{0}{0}$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Q

Which of the following statements is true about $\lim\limits_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x}$?

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A

Which of the following statements is true about $\lim_{x\to1} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x\to1} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

C Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

D Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

Q

ans

Which of the following statements is true about $\lim\limits_{x \to 1} \dfrac{\sin x}{x^3 - x^2 + x}$?

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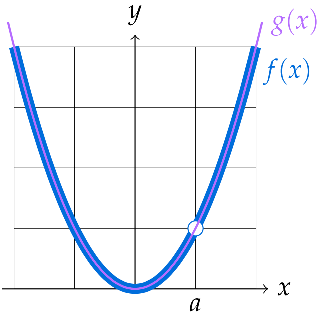
D Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

A

Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim_{x \to a} g(x)$ exists, and $f(x) = g(x)$ when x is close to a (but not necessarily equal to a).

Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.



Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 + x^2 - x - 1}{x - 1}$.

Evaluate $\lim\limits_{x\to 1} \dfrac{x^3+x^2-x-1}{x-1}$.

$$\begin{aligned}\frac{x^3+x^2-x-1}{x-1} &= \frac{(x+1)^2(x-1)}{x-1}\\ &= (x+1)^2 \text{ whenever } x \neq 1\end{aligned}$$

So, $\lim\limits_{x\to 1} \dfrac{x^3+x^2-x-1}{x-1} = \lim\limits_{x\to 1} (x+1)^2 = 4$

Evaluate $\displaystyle\lim_{x\to 5} \frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}$

Q

Evaluate $\lim_{x\to 5} \dfrac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}$

$$\begin{aligned}
\frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5} &= \frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}\left(\frac{\sqrt{x+20}+\sqrt{4x+5}}{\sqrt{x+20}+\sqrt{4x+5}}\right)\\
&= \frac{(x+20)-(4x+5)}{(x-5)(\sqrt{x+20}+\sqrt{4x+5})}\\
&= \frac{-3x+15}{(x-5)(\sqrt{x+20}+\sqrt{4x+5})}\\
&= \frac{-3}{\sqrt{x+20}+\sqrt{4x+5}}
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to 5}\frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5} &= \lim_{x\to 5}\frac{-3}{\sqrt{x+20}+\sqrt{4x+5}}\\
&= \frac{-3}{\sqrt{5+20}+\sqrt{4(5)+5}} = \frac{-3}{10}
\end{aligned}$$

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x \to 1} \left(\sqrt{35 + x^5} + \frac{x-3}{x^2} \right)^3 =$$

ans

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x \to 1} \left(\sqrt{35 + x^5} + \frac{x-3}{x^2} \right)^3 =$$

$$\left(\sqrt{35 + 1^5} + \frac{1-3}{1^2} \right)^3 = 64$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0} \frac{x+7}{\frac{1}{x}-\frac{1}{2x}}$$

ans

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0}\frac{x+7}{\frac{1}{x}-\frac{1}{2x}}=\lim_{x\to 0}\frac{x+7}{\frac{2}{2x}-\frac{1}{2x}}$$

$$=\lim_{x\to 0}\frac{x+7}{\frac{1}{2x}}=\lim_{x\to 0}2x(x+7)=0$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0}\frac{x+7}{\frac{1}{x}-\frac{1}{2x}} = \lim_{x\to 0}\frac{x+7}{\frac{2}{2x}-\frac{1}{2x}}$$

$$= \lim_{x\to 0}\frac{x+7}{\frac{1}{2x}} = \lim_{x\to 0} 2x(x+7) = 0$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

Denominators Approaching Zero

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

$$\lim_{x\to 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{1}{x-1}$$

$$\lim_{x\to 1^+} \frac{1}{x-1}$$

Denominators Approaching Zero

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$$

$$\lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty$$

$$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty$$

Denominators Approaching Zero

Now You

$$\lim_{x \to 2^+} \frac{x}{x^2 - 4}$$

$$\lim_{x \to 2^-} \frac{x}{4 - x^2}$$

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$$

Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4} = \infty$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2} = \infty$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$$

ans

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \leq g(x) \leq h(x)$$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. Then $\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$.

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

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$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Evaluate:

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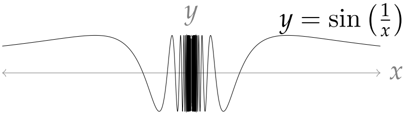
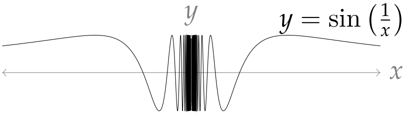
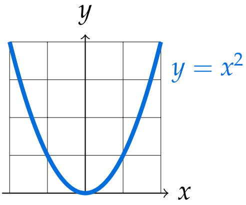


Evaluate:

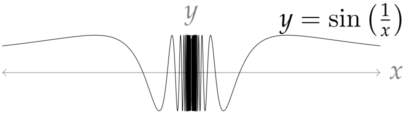
$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

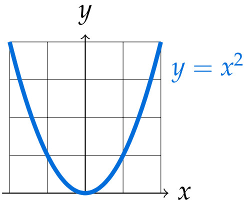


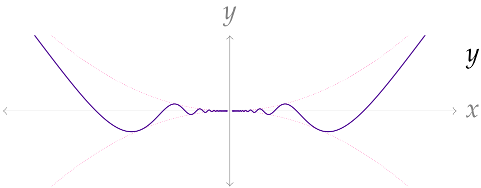


Evaluate:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$







$y = x^2 \sin\left(\frac{1}{x}\right)$

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\begin{aligned}
-1 &\leq \sin\left(\frac{1}{x}\right) \leq 1 \\
\text{so} \quad -x^2 &\leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \\
\text{and also} \quad \lim_{x \to 0} -x^2 &= 0 = \lim_{x \to 0} x^2
\end{aligned}$$

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\begin{aligned}
-1 &\leq \sin\left(\frac{1}{x}\right) \leq 1 \\
\text{so} \quad -x^2 &\leq x^2\sin\left(\frac{1}{x}\right) \leq x^2 \\
\text{and also} \quad \lim_{x\to 0} -x^2 &= 0 = \lim_{x\to 0} x^2
\end{aligned}$$

Therefore, by the Squeeze Theorem, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Included Work

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