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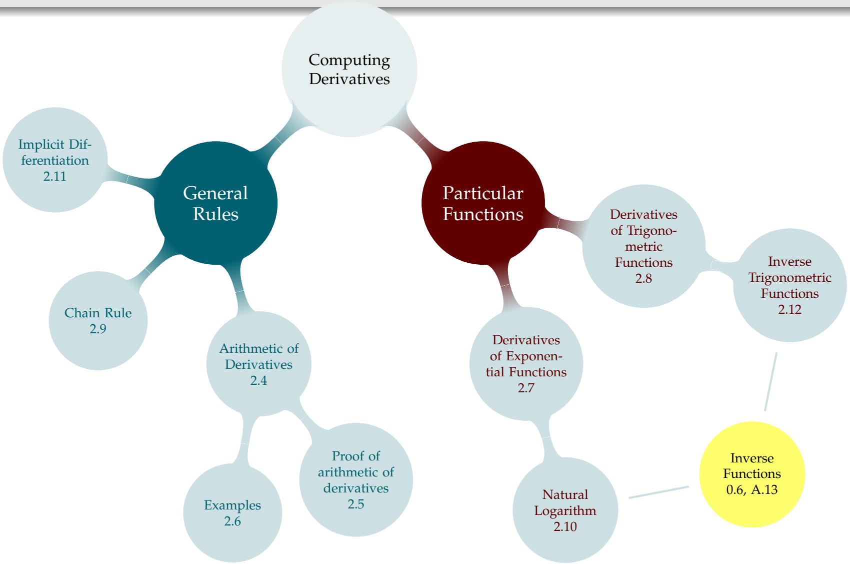


# Invertibility Game

- A function $y = f(x)$ is known to both players
- Player A chooses a secret value $x$ in the domain of $f(x)$
- Player A tells Player B what $f(x)$ is
- Player B tries to guess Player A's $x$-value.

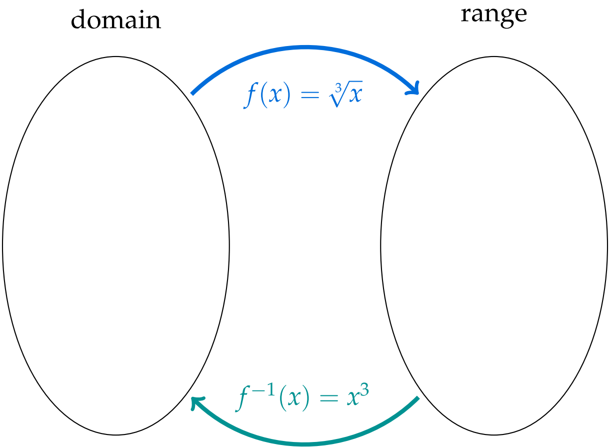
**Round 1:** $f(x) = 2x$

**Round 2:** $f(x) = \sqrt[3]{x}$

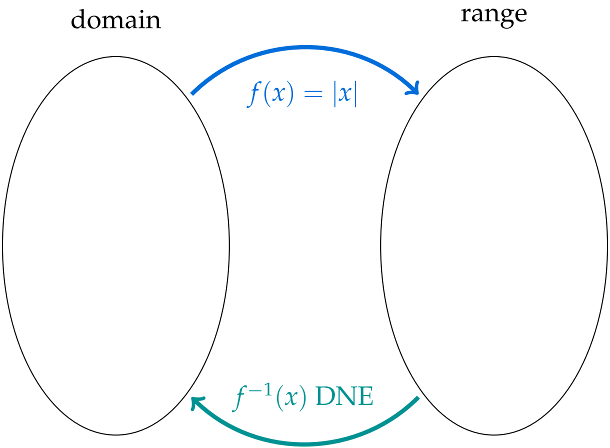
**Round 3:** $f(x) = |x|$

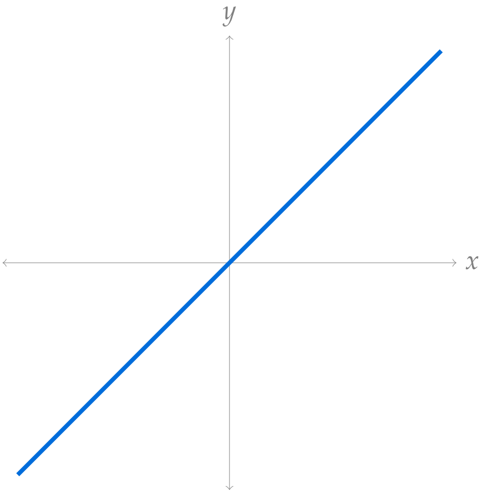
**Round 4:** $f(x) = \sin x$

# Functions are Maps



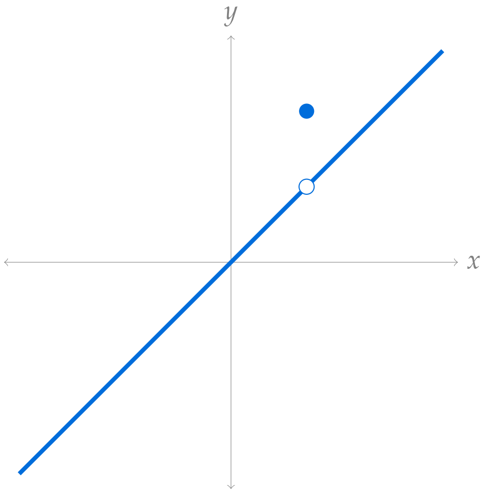
# Functions are Maps





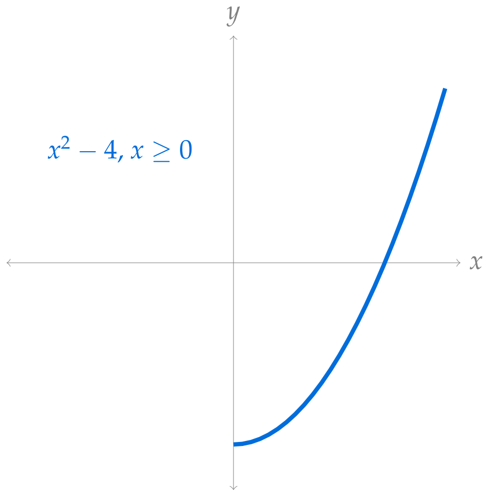

A. invertible

B. not invertible




A. invertible

B. not invertible




A. invertible

B. not invertible

# Relationship between $f(x)$ and $f^{-1}(x)$

Let $f$ be an invertible function.
What is $f^{-1}(f(x))$?

A. $x$

B. 1

C. 0

D. not sure

## Invertibility

In order for a function to be invertible , different $x$ values cannot map to the same $y$ value.
We call such a function **one-to-one**, or **injective**.

Suppose $f(x) = \sqrt[3]{19 + x^3}$. What is $f^{-1}(3)$? (simplify your answer)

What is $f^{-1}(10)$? (do not simplify)

What is $f^{-1}(x)$?

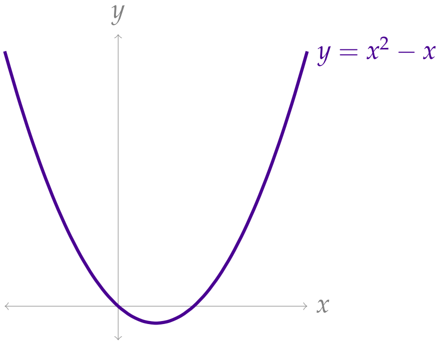
Let $f(x) = x^2 - x$.

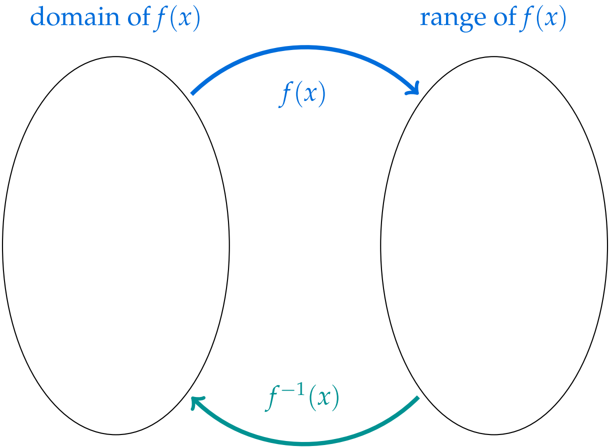
1. Sketch a graph of $f(x)$, and choose a (large) domain over which it is invertible.

2. For the domain you chose, evaluate $f^{-1}(20)$.

3. For the domain you chose, evaluate $f^{-1}(x)$.

4. What are the domain and range of $f^{-1}(x)$? What are the (restricted) domain and range of $f(x)$?


y
$y = x^2 - x$
x


domain of $f(x)$
range of $f(x)$
$f(x)$
$f^{-1}(x)$

# INVERTIBILITY GAME: $f(x) = e^x$ $f^{-1}(x) = \log_e x$

- I'm thinking of an $x$. Your clue: $f(x) = e$. What is my $x$?
- I'm thinking of an $x$. Your clue: $f(x) = 1$. What is my $x$?
- I'm thinking of an $x$. Your clue: $f(x) = \frac{1}{e}$. What is my $x$?
- I'm thinking of an $x$. Your clue: $f(x) = e^3$. What is my $x$?
- I'm thinking of an $x$. Your clue: $f(x) = 0$. What is my $x$?

1. Suppose $0 < x < 1$. Then $\log_e(x)$ is...

2. Suppose $-1 < x < 0$. Then $\log_e(x)$ is...

3. Suppose $e < x$. Then $\log_e(x)$ is...

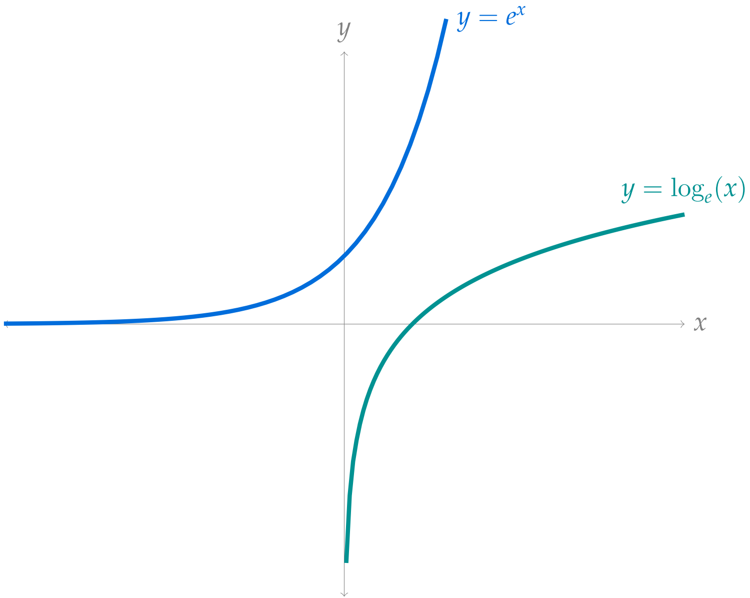
A. positive
B. negative
C. greater than one
D. less than one
E. undefined

# Exponents and Logarithms

$f(x) = e^x$

$f^{-1}(x) = \log_e(x) = \ln(x) = \log(x)$

| $x$ | $e^x$ | $e$ fact $\leftrightarrow$ $\log_e$ fact | $x$ | $\log_e(x)$ |
|---|---|---|---|---|
| $0$ | $1$ | | | |
| $1$ | $e$ | | | |
| $-1$ | $\frac{1}{e}$ | | | |
| $n$ | $e^n$ | | | |


$y = e^x$
$y$
$y = \log_e(x)$
$x$

# Logs of Other Bases: $\log_n(x)$ is the inverse of $n^x$

$\log_{10} 10^8 =$

A. 0

B. 8

C. 10

D. other

$\log_2 16 =$

A. 1

B. 2

C. 3

D. other

## Logarithm Rules

Let $A$ and $B$ be positive, and let $n$ be any real number.

$\log(A \cdot B) = \log(A) + \log(B)$

Proof: $\log(A \cdot B) = \log(e^{\log A} e^{\log B}) = \log(e^{\log A + \log B}) = \log(A) + \log(B)$

$\log(A/B) = \log(A) - \log(B)$

Proof: $\log(A/B) = \log\left(\frac{e^{\log A}}{e^{\log B}}\right) = \log(e^{\log A - \log B}) = \log A - \log B$

$\log(A^n) = n \log(A)$

Proof: $\log(A^n) = \log\left(\left(e^{\log A}\right)^n\right) = \log\left(e^{n \log A}\right) = n \log A$

## Logarithm Rules

Let $A$ and $B$ be positive, and let $n$ be any real number.

$\log(A \cdot B) = \log(A) + \log(B)$

$\log(A/B) = \log(A) - \log(B)$

$\log(A^n) = n\log(A)$

Write as a single logarithm:

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2\log x + \log(10 + x)$$

# Base Change

Fact: $b^{\log_b(a)} = a$

$$\Rightarrow \log\left(b^{\log_b(a)}\right) = \log(a)$$

$$\Rightarrow \log_b(a)\log(b) = \log(a)$$

$$\Rightarrow \log_b(a) = \frac{\log(a)}{\log(b)}$$

In general, for positive $a$, $b$, and $c$:

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

In general, for positive $a$, $b$, and $c$:

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

Suppose your calculator can only compute logarithms base 10. What would you enter to calculate $\log(17)$?

Suppose your calculator can only compute natural logarithms. What would you enter to calculate $\log_2(57)$?

Suppose your calculator can only compute logarithms base 2. What would you enter to calculate $\log(2)$?

Decibels: For a particular measure of the power $P$ of a sound wave, the decibels of that sound is:

$$10\log_{10}(P)$$

So, every ten decibels corresponds to a sound being ten **times** louder.

A lawnmower emits a 100dB sound. How much sound will two lawnmowers make?

A. 100 dB
B. 110 dB
C. 200 dB
D. other

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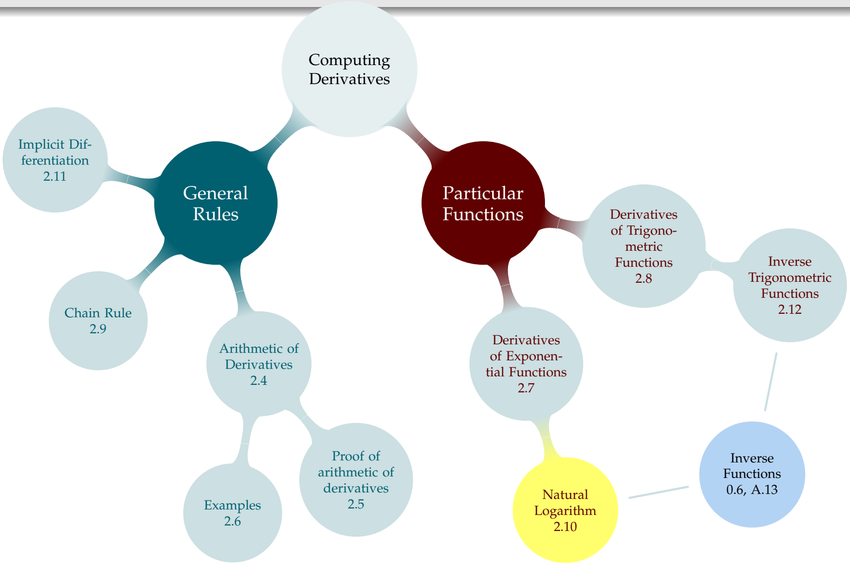

# DIFFERENTIATING THE NATURAL LOGARITHM

Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\}$.
One Weird Trick:

$$\begin{aligned}
x &= e^{\log_e x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\{x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{\log_e x}\right\} \\
1 &= e^{\log_e x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} = x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} \\
\frac{1}{x} &= \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\}
\end{aligned}$$

## Derivative of Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log_e |x|\right\} = \frac{1}{x} \qquad (x \neq 0)$$

Differentiate: $f(x) = \log_e |x^2 + 1|$

## Derivatives of Logarithms – Corollary 2.10.6

For $a > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a |x|] = \frac{1}{x \log a}$$

In particular:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log |x|] = \frac{1}{x}$$

Differentiate: $f(x) = \log_e |\cot x|$

# Logarithmic Differentiation - A Fancy Trick

- $\log(f \cdot g) = \log f + \log g$

  multiplication turns into addition

- $\log\left(\frac{f}{g}\right) = \log f - \log g$

  division turns into subtraction

- $\log\left(f^g\right) = g \log f$

  exponentiation turns into multiplication

  We can exploit these properties to differentiate!

## Logarithmic Differentiation

In general, if $f(x) \neq 0$, $\frac{\mathrm{d}}{\mathrm{d}x}\left[\log|f(x)|\right] = \frac{f'(x)}{f(x)}$.

$$f(x) = \left(\frac{(2x+5)^4(x^2+1)}{x+3}\right)^5$$

Find $f'(x)$.

# Logarithmic Differentiation - A Fancy Trick

$$f(x) = \left( \frac{(2x+5)^4 (x^2+1)}{x+3} \right)^5$$

# Logarithmic Differentiation - A Fancy Trick

Differentiate:

$$f(x) = x^x$$

# Logarithmic Differentiation - A Fancy Trick

Differentiate:

$$f(x) = \left( \frac{(x^{15} - 9x^2)^{10}(x + x^2 + 1)}{(x^7 + 7)(x + 1)(x + 2)(x + 3)} \right)^5$$

$$f(x) = \frac{(x^8 - e^x)(\sqrt{x} + 5)}{\csc^5 x}$$

$$f(x) = (x^2 + 17)(32x^5 - 8)(x^{98} - x^{57} + 32x^2)^4(32x^{10} - 10x^{32})$$

Find $f'(x)$.