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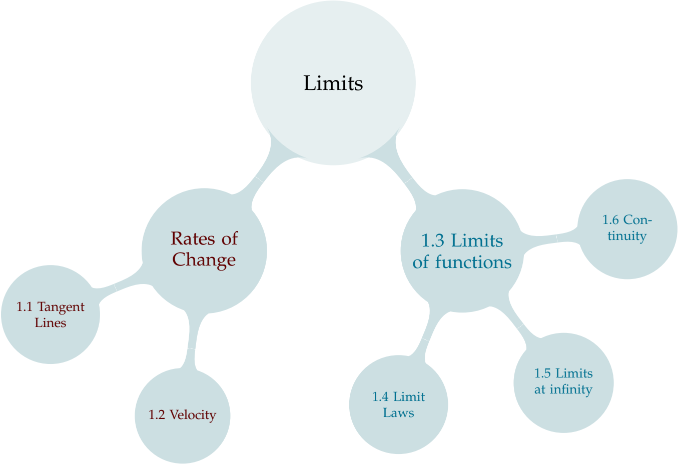
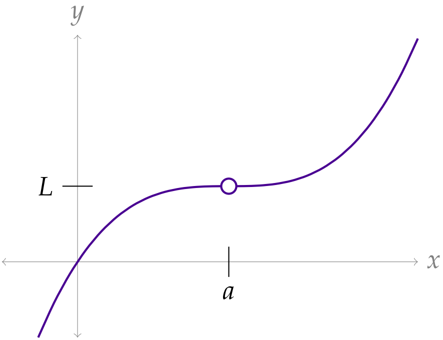
1.7 (Optional) Making the Informal a Little More Formal



Now that we've seen the limits of functions as x goes to positive and negative infinity, let's look at limits as x approaches a real number.

$$\lim_{x \to a} f(x) = L$$

Informally: If x is close enough (but not equal to) a, then y is close enough to L.



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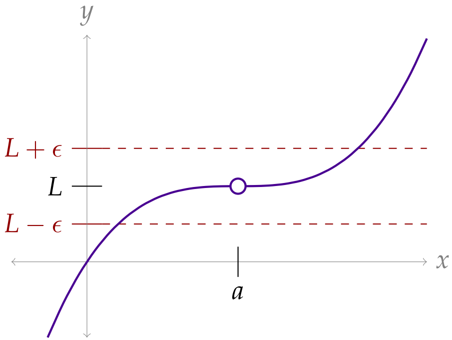

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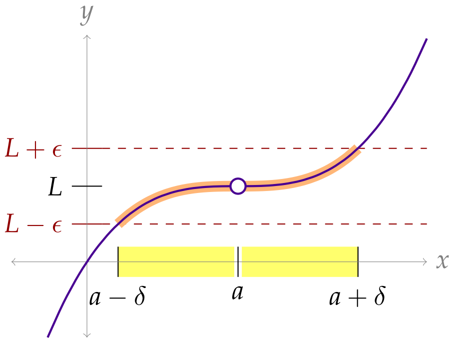

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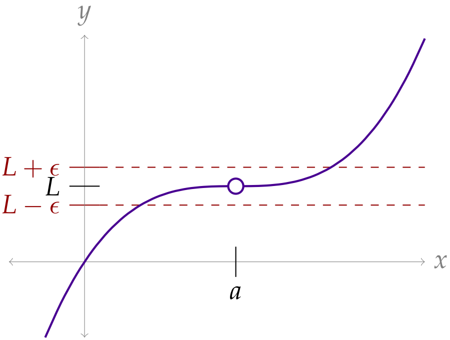

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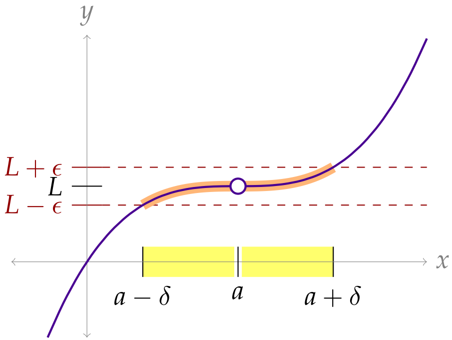

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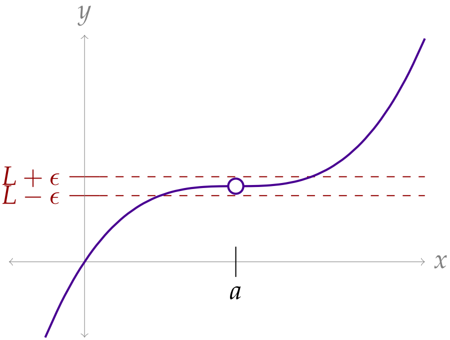

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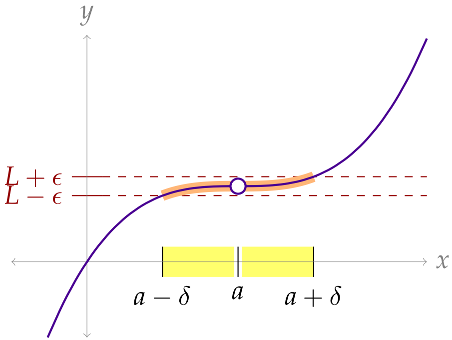

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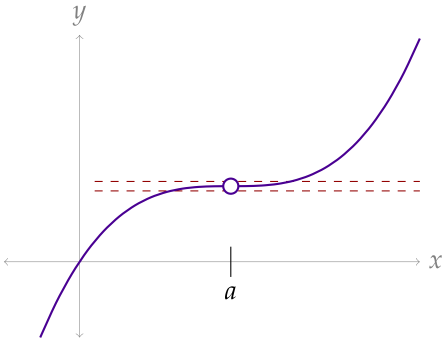

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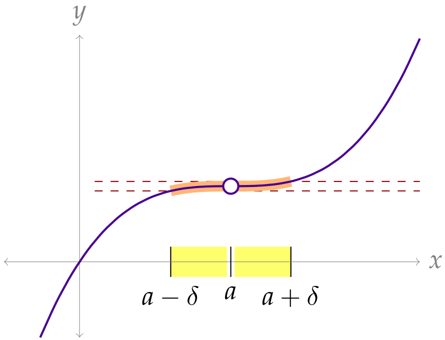
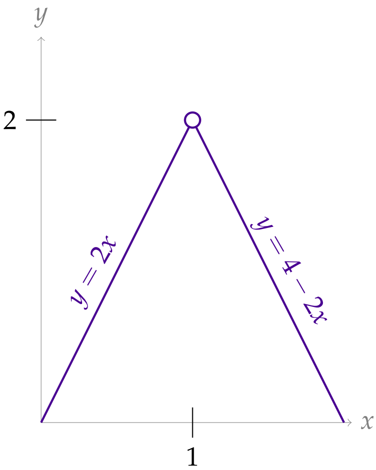


Let $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$. Then $\lim_{x \to 1} |x| = 2$.



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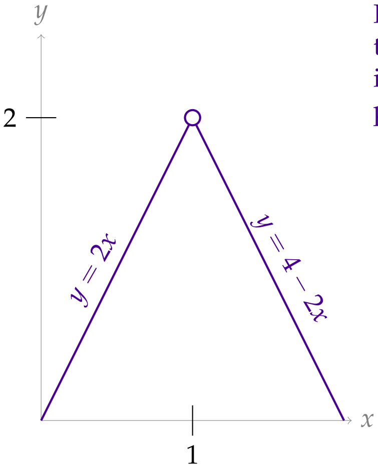
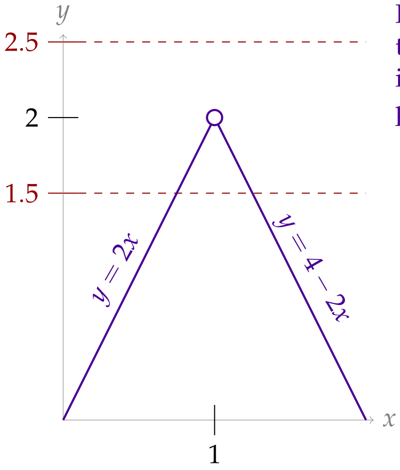


Find a positive number δ such that $|f(x) - 2| < \frac{1}{2}$ for all x in the interval $(1-\delta, 1+\delta)$, except possibly $x = 1$.

ans

Let $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$. Then $\lim_{x \to 1} |x| = 2$.



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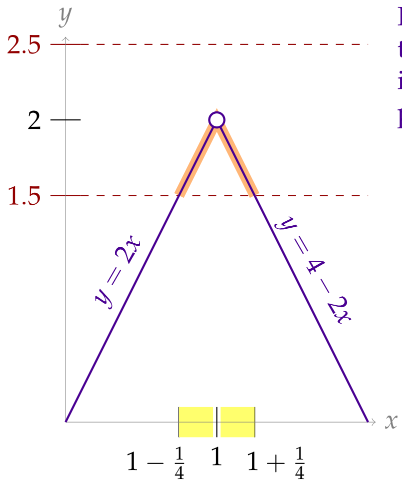

Find a positive number δ such that $|f(x) - 2| < \frac{1}{2}$ for all x in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

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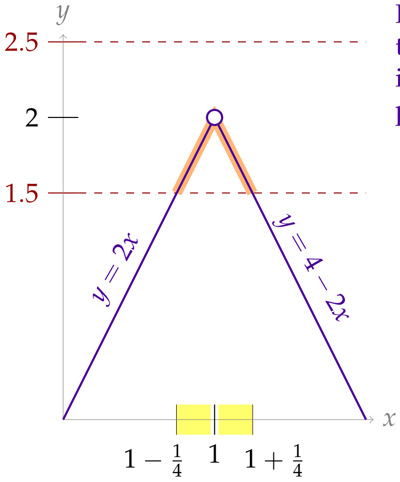

Find a positive number δ such that $|f(x) - 2| < \frac{1}{2}$ for all x in the interval $(1-\delta, 1+\delta)$, except possibly $x = 1$.

$$\delta = \frac{1}{4}$$

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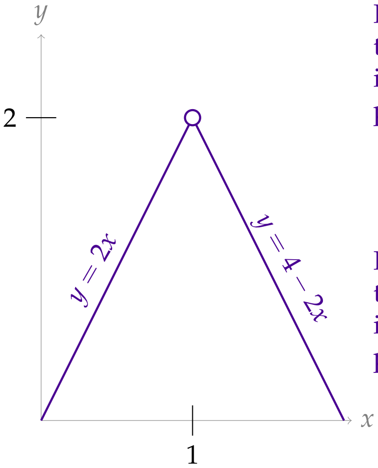

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Q

ans

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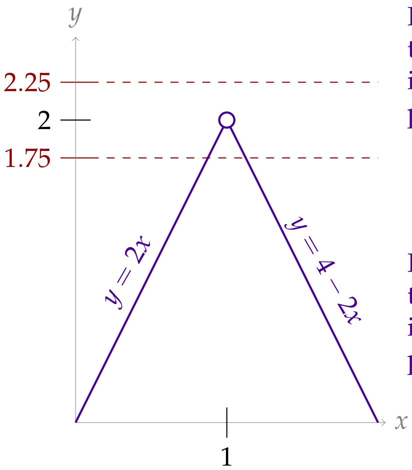

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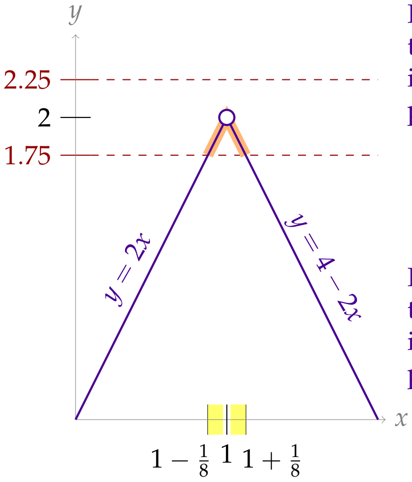

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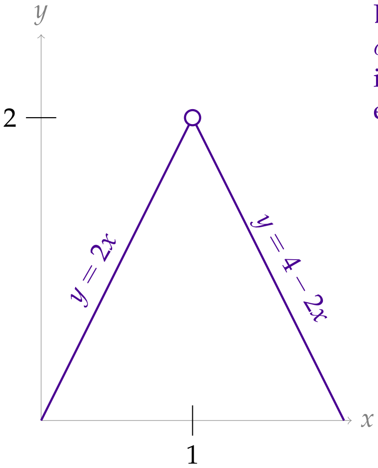
$$\delta = \frac{1}{8}$$

Let $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$. Then $\lim_{x\to 1} |x| = 2$.

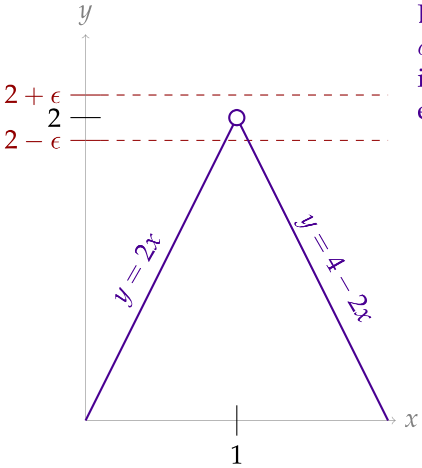


Let $\epsilon > 0$. Find a positive number δ such that $|f(x) - 2| < \epsilon$ for all x in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

Q

ans

$$\text{Let } f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}. \quad \text{Then } \lim_{x \to 1} |x| = 2.$$



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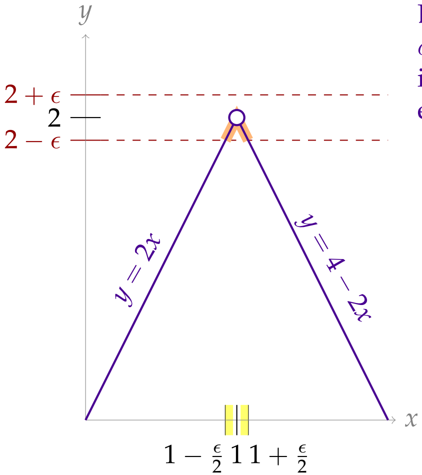


Let $\epsilon > 0$. Find a positive number δ such that $|f(x) - 2| < \epsilon$ for all x in the interval $(1 - \delta, 1 + \delta)$, except possibly $x = 1$.

$$\delta = \frac{\epsilon}{2}$$

Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that

the limit as x approaches a of $f(x)$ is L

and write

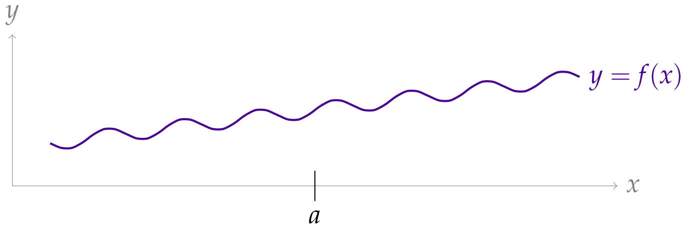
$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Note that an equivalent way of writing this very last statement is

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$



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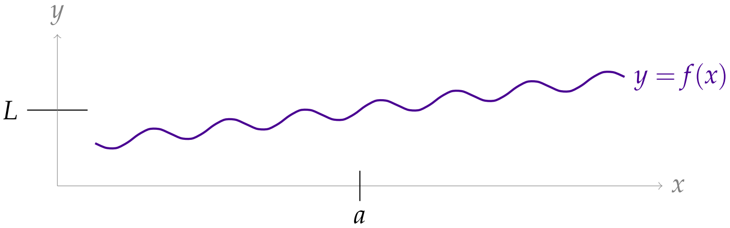

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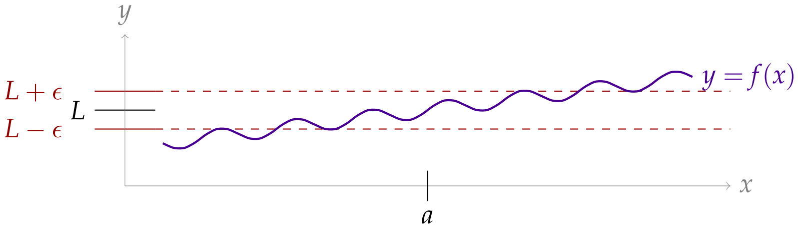

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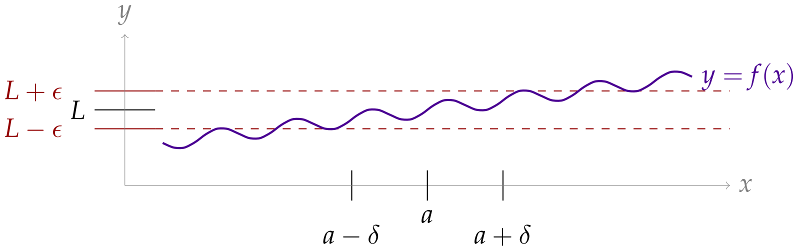

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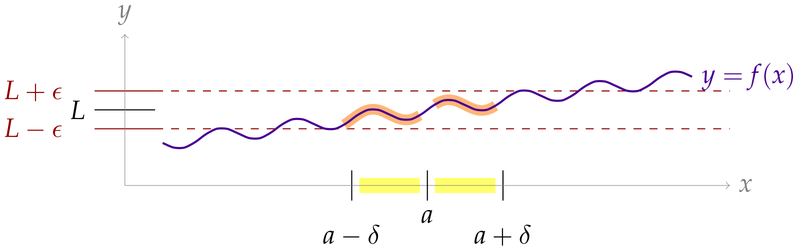

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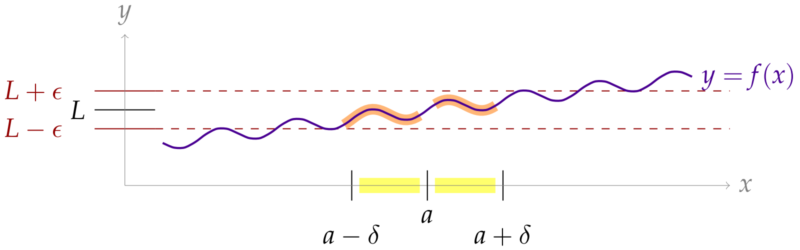
if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$



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Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Using Definition 1.7.1, prove that $\lim_{x \to -1} |x + 1| = 0$.

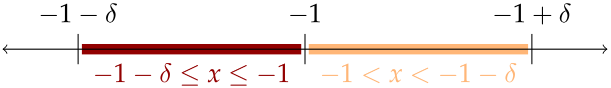
Q

ans

Using Definition 1.7.1, prove that $\lim\limits_{x\to-1} |x+1| = 0$.

By inspection (look at the graph of $y = |x+1|$), we should use $\delta = \epsilon$.

Proof: Let $f(x) = |x+1|$ and for any positive ϵ, let $\delta = \epsilon$.



If $\quad -1 < x < -1+\delta$:

$$|f(x) - 0| = ||x+1| - 0| = |x+1| = x+1 < (-1+\delta)+1 = \delta = \epsilon$$

If $\quad -1-\delta < x < -1$:

$$|f(x) - 0| = ||x+1| - 0| = |x+1| = -x-1 < -(-1-\delta)-1 = \delta = \epsilon$$

So if $0 < |x-(-1)| < \delta$, then $|f(x) - 0| < \epsilon$. That is, $\lim\limits_{x\to-1} f(x) = 0$. $\square$

Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Let $f(x) = \begin{cases} x + 1 & x < 0 \\ 1 - x^2 & x > 0 \end{cases}$.

Using Definition 1.7.1, prove that $\lim_{x \to 0} f(x) = 1$.

Q

ans

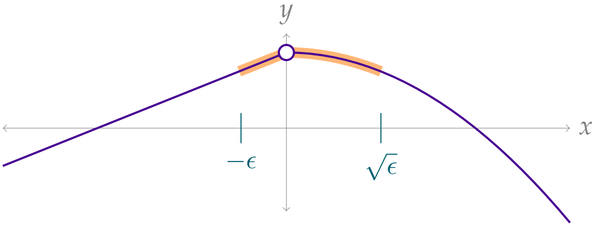
First, we need to find δ for any given ϵ. Suppose $x > 0$ and $|f(x) - 1| < \epsilon$:

$$\begin{aligned} |f(x) - 1| &< \epsilon \\ |1 - x^2 - 1| &< \epsilon \\ x^2 &< \epsilon \\ 0 < x &< \sqrt{\epsilon} \end{aligned}$$

Now, suppose $x < 0$ and $|f(x) - 1| < \epsilon$:

$$\begin{aligned} |f(x) - 1| &< \epsilon \\ |x + 1 - 1| &< \epsilon \\ |x| &< \epsilon \\ -x &< \epsilon \\ 0 > x &> -\epsilon \end{aligned}$$

Now we know the interval over which $|f(x) - 1| < \epsilon$, but it's actually more information than we need. We don't need the exact interval; we just need some value of δ such that $0 < |x - 1| < \delta$ guarantees $|f(x) - 1| < \epsilon$.



If $0 < |x - 1| < \min\{\epsilon, \sqrt{\epsilon}\}$, then $0 < |x - 1| < \epsilon$ and $0 < |x - 1| < \sqrt{\epsilon}$ are both true. So we set $\delta = \min\{\epsilon, \sqrt{\epsilon}\}$. (For $\epsilon < 1$, that is $\delta = \epsilon$.)

Proof: $f(x) = \begin{cases} x+1 & x<0 \\ 1-x^2 & x>0 \end{cases}$.

For any $\epsilon > 0$, let $\delta = \min\{\epsilon, \sqrt{\epsilon}\}$. Suppose $0 < |x-0| < \delta$.

- If $x > 0$, then

$$\begin{aligned} |f(x)-1| &= |(1-x^2)-1| = |-x^2| = x^2 \\ &< \delta^2 \leq \sqrt{\epsilon}^2 = \epsilon \end{aligned}$$

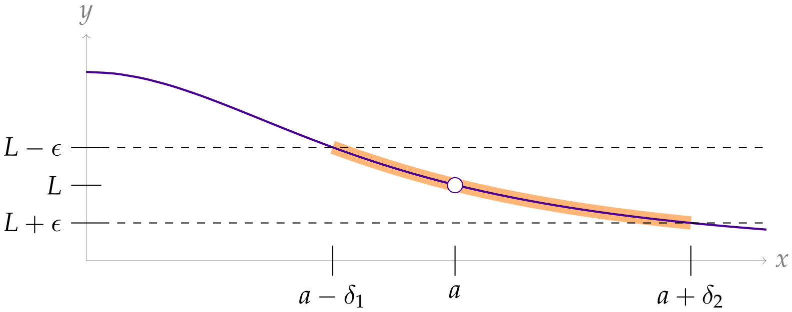
- If $x < 0$, then

$$\begin{aligned} |f(x)-1| &= |(x+1)-1| = |x| \\ &< \delta \leq \epsilon \end{aligned}$$

So whenever $0 < |x-0| < \delta$, then $|f(x)-1| < \epsilon$. So $\lim_{x\to 0} f(x) = 1$. $\square$

General Principles

Suppose $|f(x) - L| < \epsilon$ whenever $a - \delta_1 < x < a$ and whenever $a < x < a + \delta_2$.



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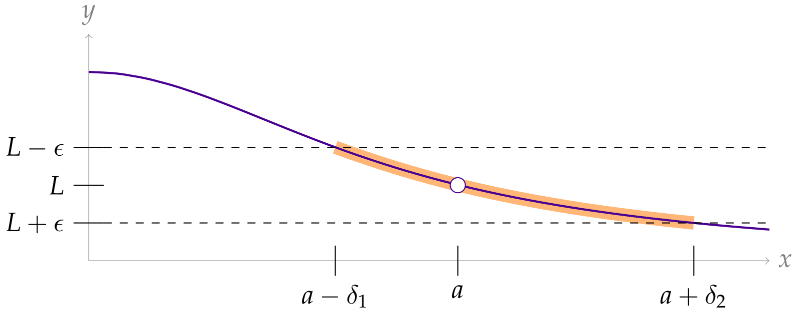

Consider values of x such that $0 < |x - a| < \min\{\delta_1, \delta_2\}$.

General Principles

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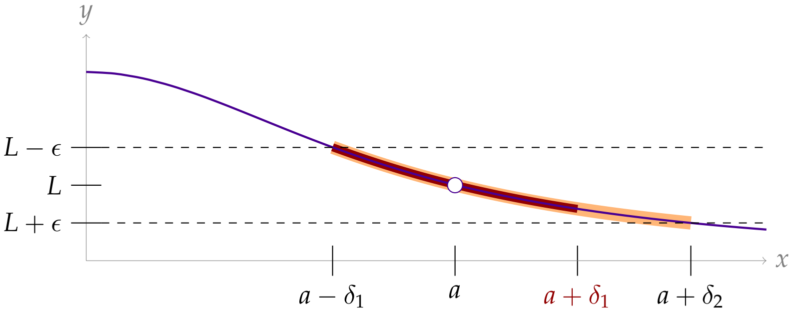

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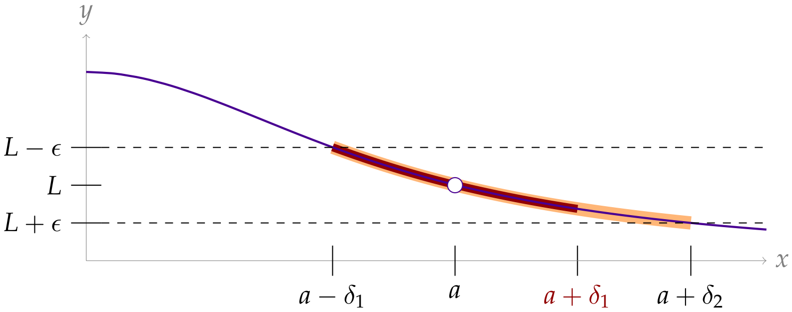
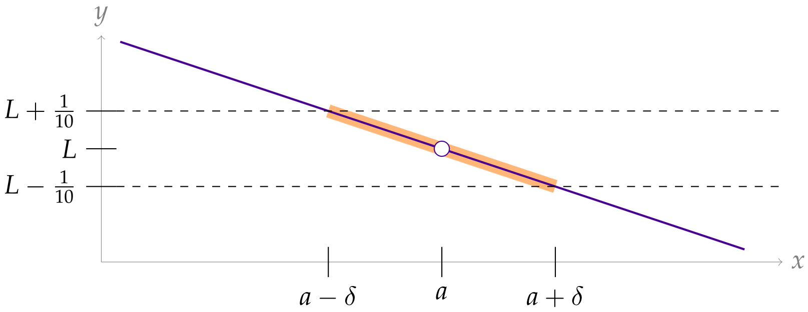


Consider values of x such that $0 < |x - a| < \min\{\delta_1, \delta_2\}$. For these values, it is (still) the case that $|f(x) - L| < \epsilon$.

General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.



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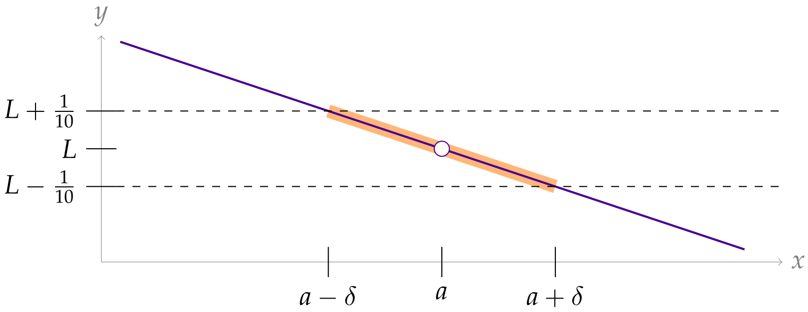
General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.



Can you give values of x where $|f(x) - L| < \frac{1}{5}$?

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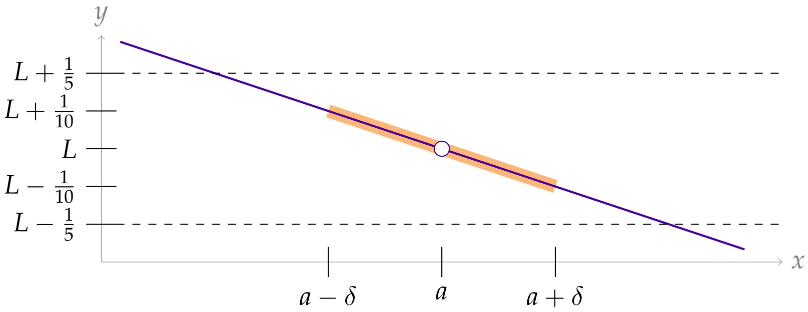
General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.



Can you give values of x where $|f(x) - L| < \frac{1}{5}$?

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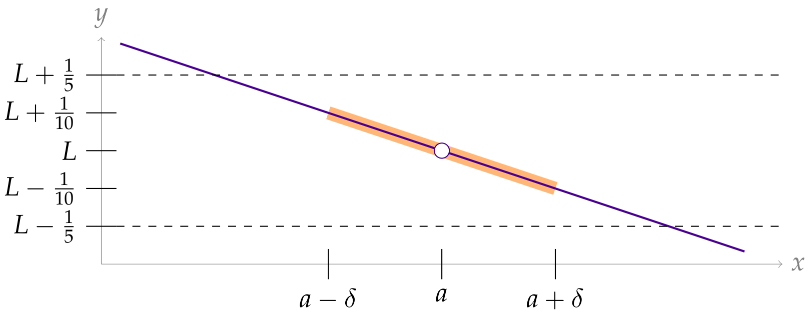
General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.
Then also $|f(x) - L| < \frac{1}{5}$ for all x such that $0 < |x - a| < \delta$.

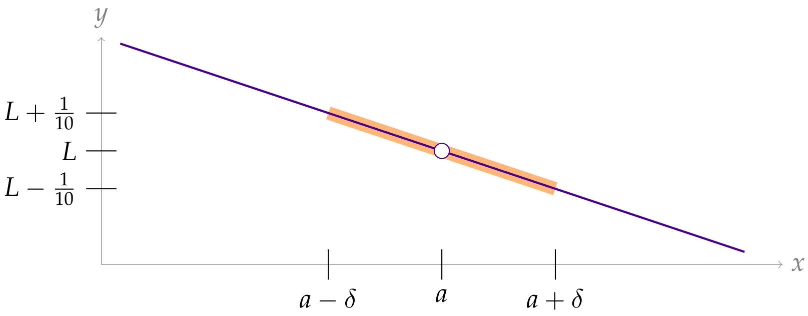


Can you give values of x where $|f(x) - L| < \frac{1}{5}$?

skip ϵ small

General Principles

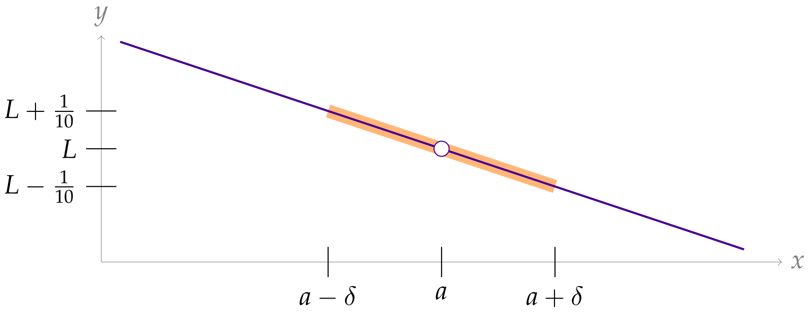
Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.
Then also $|f(x) - L| < \frac{1}{2}$ for all x such that $0 < |x - a| < \delta$.



skip ϵ small

General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.
Then also $|f(x) - L| < 1$ for all x such that $0 < |x - a| < \delta$.



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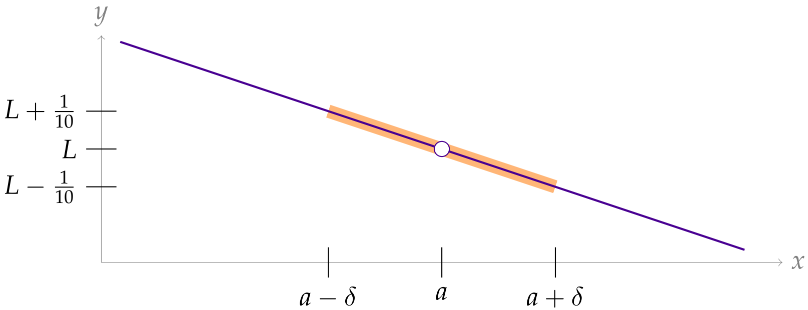
General Principles

Suppose $|f(x) - L| < \frac{1}{10}$ for all x such that $0 < |x - a| < \delta$.
Then also $|f(x) - L| < 100$ for all x such that $0 < |x - a| < \delta$.



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General Principles

Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of a, except possibly at a. We say that $\lim_{x \to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

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It is enough to show that for every ϵ such that $0 < \epsilon < c$ (where c is some constant) there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.
That means it doesn't hurt your proof if you say something like "we assume $\epsilon < 1$".

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That means it doesn't hurt your proof if you say something like "we assume $\epsilon < 1$".

In a previous example, we chose

$$\delta = \min\{\epsilon, \sqrt{\epsilon}\}$$

It would be OK to say "we can assume $\epsilon < 1$; set $\delta = \epsilon$."

Definition 1.7.1

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Using Definition 1.7.1, prove that $\lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$.

Q

ans

We start, as usual, by finding δ. Note

$$\frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2}$$

whenever $x \neq 2$.

$$\left|\frac{1}{x+2} - \frac{1}{4}\right| < \epsilon$$
$$-\epsilon < \frac{1}{x+2} - \frac{1}{4} < \epsilon$$
$$\frac{1}{4} - \epsilon < \frac{1}{x+2} < \frac{1}{4} + \epsilon$$
$$\frac{1-4\epsilon}{4} < \frac{1}{x+2} < \frac{1+4\epsilon}{4}$$
$$\frac{4}{1-4\epsilon} > x+2 > \frac{4}{1+4\epsilon}$$
$$\frac{4}{1-4\epsilon} - 2 > x > \frac{4}{1+4\epsilon} - 2$$

$$\frac{2+8\epsilon}{1-4\epsilon} > x > \frac{2-8\epsilon}{1+4\epsilon}$$

We want our bounds to look like $2-\delta_1$ and $2+\delta_2$.

$$\frac{2-8\epsilon+16\epsilon}{1-4\epsilon} > x > \frac{2+8\epsilon-16\epsilon}{1+4\epsilon}$$
$$2 + \frac{16\epsilon}{1-4\epsilon} > x > 2 - \frac{16\epsilon}{1+4\epsilon}$$

For x in the interval found, $\left|f(x) - \frac{1}{4}\right| < \epsilon$. The interval is not exactly in the form $2-\delta < x < 2+\delta$, but it's close. Remember a smaller interval will also have the property $\left|f(x) - \frac{1}{4}\right| < \epsilon$. So, set

$$\delta = \min\left\{\frac{16\epsilon}{1-4\epsilon}, \frac{16\epsilon}{1+4\epsilon}\right\} = \frac{16\epsilon}{1+4\epsilon}.$$

A

Proof: For any $\epsilon > 0$, let $\delta = \frac{16\epsilon}{1+4\epsilon}$. Suppose $0 < |x-2| < \delta$. Note that since $x \neq 2$, we have $f(x) = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2}$.

- If $x > 2$, then $2 < x < 2+\delta$:

$$\begin{aligned}\left|f(x) - \frac{1}{4}\right| &= \left|\frac{1}{(x+2)} - \frac{1}{4}\right| = \frac{1}{4} - \frac{1}{(x+2)} \\ &< \frac{1}{4} - \frac{1}{(2+\delta)+2} = \frac{1}{4} - \frac{1}{4+\delta} \\ &= \frac{1}{4} - \frac{1}{4+\frac{16\epsilon}{1+4\epsilon}} < \frac{1}{4} - \frac{1}{4+\frac{16\epsilon}{1-4\epsilon}}\end{aligned}$$

(using the fact that $\delta = \frac{16\epsilon}{1+4\epsilon} < \frac{16\epsilon}{1-4\epsilon}$)

$$= \frac{1}{4} - \frac{1}{\frac{4-16\epsilon+16\epsilon}{1-4\epsilon}} = \frac{1}{4} - \frac{1-4\epsilon}{4} = \frac{4\epsilon}{4} = \epsilon$$

- If $x < 2$, then $2 - \delta < x < 2$:

$$\begin{aligned}
\left| f(x) - \frac{1}{4} \right| &= \left| \frac{1}{(x+2)} - \frac{1}{4} \right| = \frac{1}{(x+2)} - \frac{1}{4} \\
&< \frac{1}{(2-\delta)+2} - \frac{1}{4} = \frac{1}{4-\delta} - \frac{1}{4} \\
&= \frac{1}{4 - \frac{16\epsilon}{1+4\epsilon}} - \frac{1}{4} = \frac{1}{\frac{4+16\epsilon-16\epsilon}{1+4\epsilon}} - \frac{1}{4} \\
&= \frac{1+4\epsilon}{4} - \frac{1}{4} = \epsilon
\end{aligned}$$

We have shown that whenever $0 < |x-2| < \delta$, then $\left|f(x) - \frac{1}{4}\right| < \epsilon$. So, $\lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$.

□

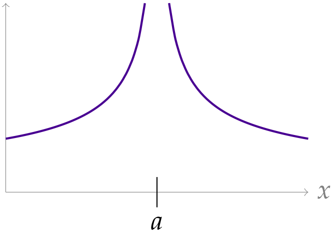
Infinite Limits

Definition 1.8.1 (b)

Let a be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every $P > 0$ there exists $\delta > 0$ so that $f(x) > P$ whenever $0 < |x - a| < \delta$.



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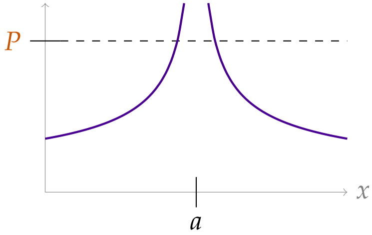

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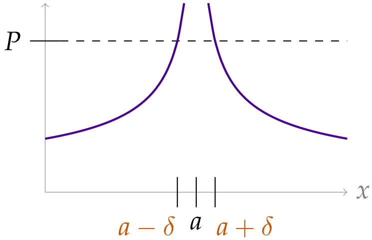

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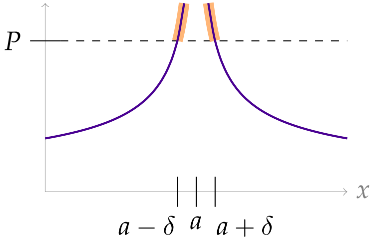

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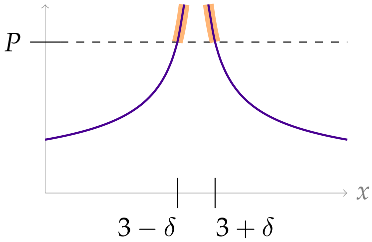

Let $f(x) = \frac{1}{(x-3)^2}$. Using Definition 1.8.1, prove or disprove that

$$\lim_{x \to 3} f(x) = \infty$$

Q

ans

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First, we find δ. Let $P > 0$.

$$\begin{aligned} f(x) &> P \\ \frac{1}{(x-3)^2} &> P \\ (x-3)^2 &< \frac{1}{P} \\ |x-3| &< \frac{1}{\sqrt{P}} \end{aligned}$$

So we choose $\delta = \frac{1}{\sqrt{P}}$.

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if and only if for every $P > 0$ there exists $\delta > 0$ so that $f(x) > P$ whenever $0 < |x - a| < \delta$.

Proof: For any $P > 0$, set $\delta = \frac{1}{\sqrt{P}}$. If $0 < |x - 3|$, then $x \neq 3$, so $f(x)$ exists. If, furthermore, $|x - 3| < \delta$, then:

$$\begin{aligned} f(x) &= \frac{1}{(x-3)^2} \\ &> \frac{1}{\delta^2} = \frac{1}{\left(\frac{1}{\sqrt{P}}\right)^2} = P \end{aligned}$$

So, $\displaystyle\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$.

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Let a be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

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Let $f(x) = \frac{1}{x-2}$. Using Definition 1.8.1, prove or disprove that

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Note that $x - 2 < 0$ when $x < 2$. So for any $P > 0$, whenever $x < 2$, we have $f(x) < P$. That tells us that $\lim_{x \to 2} f(x) \neq \infty$.

Proof: Let $P = 1$. For any $\delta > 0$, set $x_0 = 2 - \frac{\delta}{2}$. Then $0 < |x_0 - 2| < \delta$, but since $x < 2$, we have $\frac{1}{x-2} < 0 < P$. That is, there does not exist any $\delta > 0$ such that $f(x) > P$ whenever $0 < |x - 2| < \delta$. Therefore $\lim_{x \to 2} f(x) \neq \infty$.

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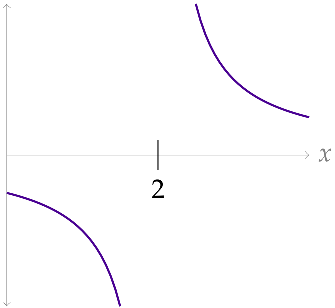

Included Work

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