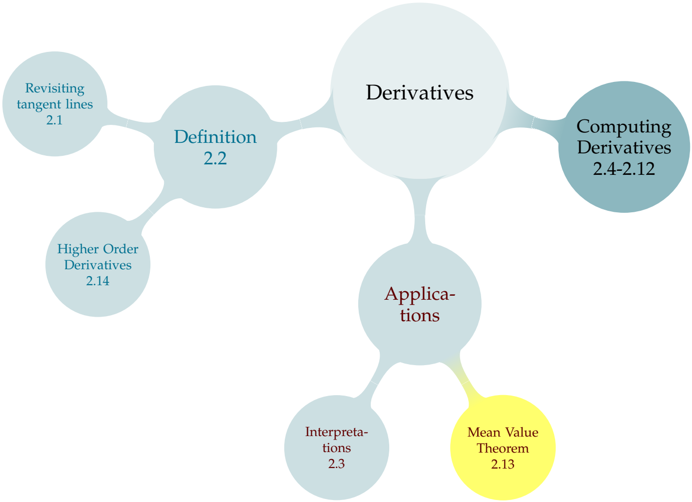
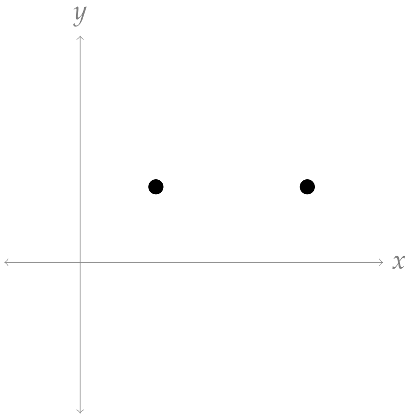
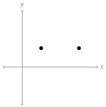
Table of Contents



Rolle's Theorem



ROLLE'S THEOREM



Draw several functions passing through these two points. Start with continuous, differentiable, and point out there is always a flat spot. Ask students: *must* there always be a flat spot? Give examples with corners, discontinuities.

Rolle's Theorem – Theorem 2.13.1

Let a and b be real numbers, with $a < b$. And let f be a function with the properties:

- $f(x)$ is continuous for every x with $a \leq x \leq b$;
- $f(x)$ is differentiable when $a < x < b$;
- and $f(a) = f(b)$.

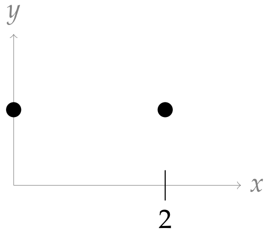
Then there exists a number c with $a < c < b$ such that

$$f'(c) = 0.$$

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

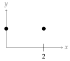
Example: Let $f(x) = x^3 - 2x^2 + 1$, and observe $f(2) = f(0) = 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere.



Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

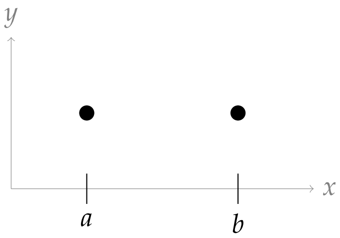
Example: Let $f(x) = x^3 - 2x^2 + 1$, and observe $f(2) = f(0) = 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere.



By Rolle's Theorem, we are guaranteed that $f'(x) = 0$ for *some* x in the interval $(0, 2)$. Which x fulfils this guarantee?

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.



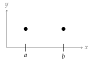
Suppose $a < b$ and $f(a) = f(b)$, $f(x)$ is continuous over $[a, b]$, and $f(x)$ is differentiable over (a, b).

How many different values of x between a and b have $f'(x) = 0$?

A. 0 or 1
B. 1
C. 0, 1, or more
D. 1 or more
E. I'm not sure

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.



Suppose $a < b$ and $f(a) = f(b)$, $f(x)$ is continuous over $[a, b]$, and $f(x)$ is differentiable over (a, b).

How many different values of x between a and b have $f'(x) = 0$?

A. 0 or 1
B. 1
C. 0, 1, or more
D. 1 or more
E. I'm not sure

If there's not much agreement, flash the two examples, then let students vote again.If most students choose the wrong answer, show them the example, vote again. They'll likely ask whether this is the *only* example. We've seen infinite wiggling before.

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a,b]$, differentiable on (a,b), and let $f(a)=f(b)$. Then there is a number c strictly between a and b such that $f'(c)=0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f'(x)$ have?

A. precisely six

B. precisely seven

C. at most seven

D. at least six

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x)$ is also continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f''(x)$ have?

A. precisely six
B. precisely five
C. at most five
D. at least five

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and there are precisely three places where $f'(x) = 0$. How many distinct roots does $f(x)$ have?

A. at most three

B. at most four

C. at least three

D. at least four

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x) = 0$ for precisely three values of x. How many distinct values x exist with $f(x) = 17$?

A. at most three

B. at most four

C. at least three

D. at least four

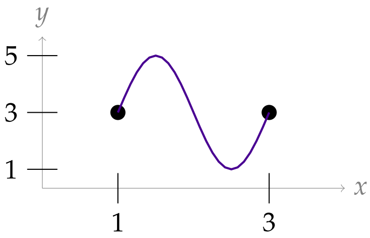
Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How would you show that $f(x)$ has precisely one real root?

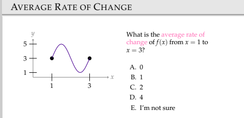
Use Rolle's Theorem to show that the function $f(x) = \frac{1}{3}x^3 + 3x^2 + 9x - 3$ has at most two distinct real roots.

Average Rate of Change



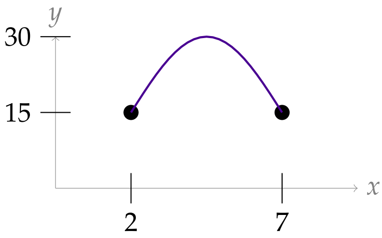
What is the average rate of change of $f(x)$ from $x = 1$ to $x = 3$?

A. 0
B. 1
C. 2
D. 4
E. I'm not sure



Rolle's Theorem tells us there is a spot where the derivative is 0; in this case, that's the same as the average rate of change.

Average Rate of Change



What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

A. 0

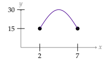
B. 3

C. 5

D. 15

E. I'm not sure

AVERAGE RATE OF CHANGE



What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

A. 0
B. 3
C. 5
D. 15
E. I'm not sure

Want a visual example that's simpler than the last one in order to illustrate Average RoC theorem. This also gives students a second chance to get the average RoC correct.

Rolle's Theorem and Average Rate of Change

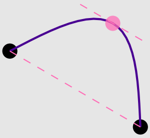
Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on the interval (a, b), and $f(a) = f(b)$. Then there exists a number c strictly between a and b such that

$$f'(c) = 0 = \frac{f(b) - f(a)}{b - a}.$$

So there exists a point where the derivative is the same as the average rate of change.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that:



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is: there is some point c in (a, b) where the instantaneous rate of change of the function is equal to the average rate of change of the function on the interval $[a, b]$.

Suppose you are driving along a long, straight highway with no shortcuts. The speed limit is 100 kph. A police officer notices your car going 90 kph, and uploads your plate and the time they saw you to their database. 150 km down this same straight road, 75 minutes later, another police officer notices your car going 85kph, and uploads your plates to the database. Then they pull you over, and give you a speeding ticket. Why were they justified?

According to this website, Canada geese may fly 1500 miles in a single day under favorable conditions. It also says their top speed is around 70mph. Does this seem like a typo? (If it contradicts the Mean Value Theorem, it's probably a typo.)

The record for fastest wheel-driven land speed is around 700 kph. [1]
However, non-wheel driven cars (such as those powered by jet engines) have achieved higher speeds. [2]
Suppose a driver of a jet-powered car starts a 10km race at 12:00, and finishes at 12:01. Did they beat 700kph?

[1](at time of writing) George Poteet, https://en.wikipedia.org/wiki/Wheel-driven_land_speed_record

[2]https://en.wikipedia.org/wiki/Land_speed_record

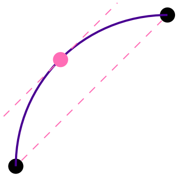
Suppose you want to download a file that is 3000 MB (slightly under 3GB). Your internet provider guarantees you that your download speeds will always be between 1 MBPS (MB per second) and 5 MBPS (because you bought the cheap plan). Using the Mean Value Theorem, give an upper and lower bound for how long the download can take (assuming your providers aren't lying, and your device is performing adequately).

Suppose $1 \leq f'(t) \leq 5$ for all values of t, and $f(0) = 0$. What are the possible solutions to $f(t) = 3000$?
Notice: since the derivative exists for all real numbers, $f(x)$ is differentiable and continuous for all real numbers!

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = 0$ for all x in (a, b), then



If $f(c) \neq f(d)$, then $\frac{f(d)-f(c)}{d-c} \neq 0$, so $f'(e) \neq 0$ for some e.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = 0$ for all x in (a, b), then

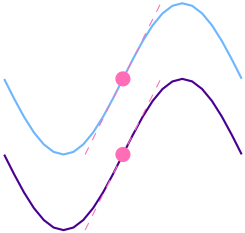
If $f(c) \neq f(d)$, then $\frac{f(d)-f(c)}{d-c} \neq 0$, so $f'(e) \neq 0$ for some e.

Explain what a “corollary” is.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = g'(x)$ for all x in (a, b), then

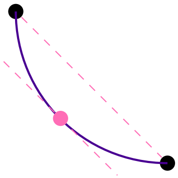


Define a new function $k(x) = f(x) - g(x)$. Then $k'(x) = 0$ everywhere, so (by the last corollary) $k(x) = A$ for some constant A.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a,b]$ and differentiable over (a,b).

If $f'(x) > 0$ for all x in (a,b), then

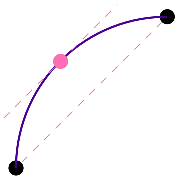


If $f(c) > f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(negative)}}{\text{(positive)}} < 0$. Then $f'(e) < 0$ for some e between c and d.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) < 0$ for all x in (a, b), then



If $f(c) < f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(positive)}}{\text{(positive)}} > 0$. Then $f'(e) > 0$ for some e between c and d.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

WARNING: The MVT has two hypotheses.

- $f(x)$ has to be continuous on $[a, b]$.
- $f(x)$ has to be differentiable on (a, b).

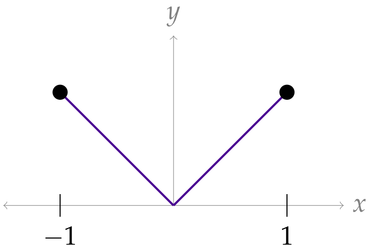
If either of these hypotheses are violated, the conclusion of the MVT can fail. Here are two examples.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Let $a = -1$, $b = 1$ and $f(x) = |x|$.



Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Let $a = 0$, $b = 1$ and $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$.

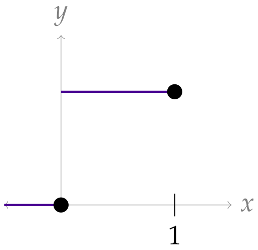
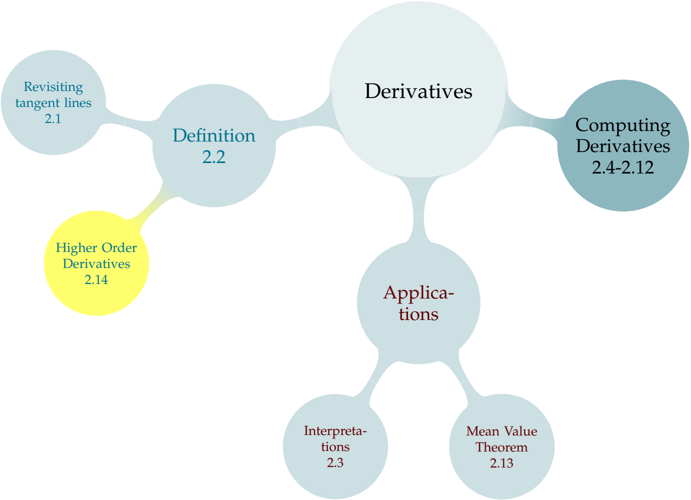


Table of Contents



Higher Order Derivatives

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}[x^5 - 2x^2 + 3]\right]$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^5 - 2x^2 + 3] =$$

Notation 2.14.1

The derivative of a derivative is called the **second derivative**, written

$$f''(x) \qquad \text{or} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x)$$

Similarly, the derivative of a second derivative is a third derivative, etc.

Notation 2.14.1

- $f''(x)$ and $f^{(2)}(x)$ and $\frac{d^2f}{dx^2}(x)$ all mean $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$
- $f'''(x)$ and $f^{(3)}(x)$ and $\frac{d^3f}{dx^3}(x)$ all mean $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)\right)$
- $f^{(4)}(x)$ and $\frac{d^4f}{dx^4}(x)$ both mean $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)\right)\right)$
- and so on.

Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

The position of an object at time t is given by $s(t) = t(5-t)$. *Time is measured in seconds, and position is measured in metres.*

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$? Include units.
3. What is the acceleration of the object when $t = 1$? Include units.

Concept Check

True or False: If $f'(1) = 18$, then $f''(1) = 0$, since the $\frac{\mathrm{d}}{\mathrm{d}x}\{18\} = 0$.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$

B. $f'(0) = 0$

C. $f''(0) = 0$

D. $f'''(0) = 0$

E. $f^{(4)}(0) = 0$

Which of the following is always true of a CUBIC polynomial $f(x)$?

A. $f(0) = 0$

B. $f'(0) = 0$

C. $f''(0) = 0$

D. $f'''(0) = 0$

E. $f^{(4)}(0) = 0$

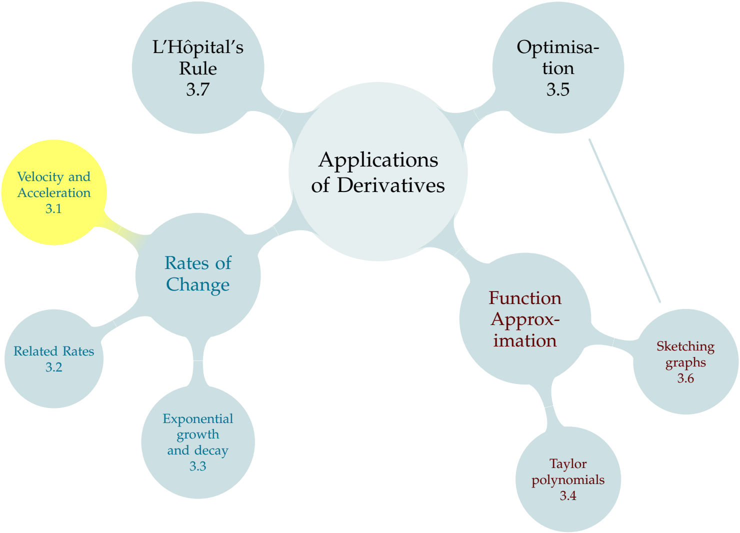
IMPLICIT DIFFERENTIATION

Suppose $y(x)$ is a function such that

$$y(x) = y^3x + x^2 - 1$$

Find $y''(x)$ at the point $(-2, 1)$.

Table of Contents



The position of a unicyclist along a tightrope is given by

$$s(t) = t^3 - 3t^2 - 9t + 10$$

where $s(t)$ gives the distance in meters to the right of the middle of the tightrope, and t is measured in seconds, $-2 \leq t \leq 4$.

Describe the unicyclist's motion: when they are moving right or left; when they are moving fastest and slowest; and how far to the right or left of centre they travel.

A solution in a beaker is undergoing a chemical reaction, and its temperature (in degrees Celsius) at t seconds from noon is given by

$$T(t) = t^3 + 3t^2 + 4t - 273$$

1. When is the reaction increasing the temperature, and when is it decreasing the temperature?
2. What is the slowest rate of change of the temperature?

You roll a magnetic marble across the floor towards a metal fridge, giving it an initial velocity of 50 centimetres per second. The magnet imparts an acceleration on the magnet of 1 meter per second per second. If the magnet hits the fridge after 2 seconds, how far away was it when you rolled it?

The deceleration of a particular car while braking is 9 m/s^2.

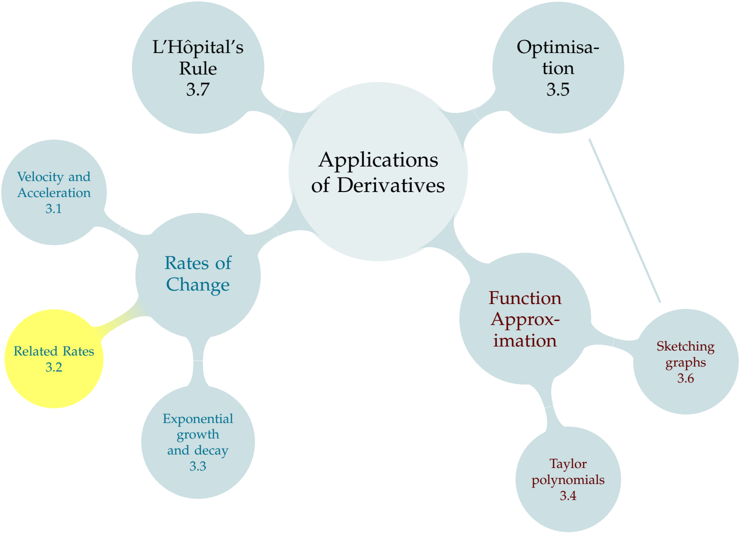
1. Suppose the car needs to stop in 30m. How fast can it be going? (Give your answer in kph.)

2. Suppose the car needs to stop in 50m. How fast can it be going? (Give your answer in kph.)

Suppose your brakes decelerate your car at a constant rate. That is, d meters per second per second, for some constant d.

Is it true that if you double your speed, you double your stopping time?

Table of Contents



Related Rates - Introduction

"Related rates" problems involve finding the rate of change of one quantity, based on the rate of change of a related quantity.

Related Rates - Introduction

“Related rates” problems involve finding the rate of change of one quantity, based on the rate of change of a related quantity.

As usual, there are more examples here than you’ll probably get to in class.

Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3.$$

If $\frac{dP}{dt}(t) = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{dQ}{dt}$ at that time?

Related rates problems often involve some kind of geometric or trigonometric modeling

A garden hose can pump out a cubic meter of water in about 20 minutes. Suppose you're filling up a rectangular backyard pool, 3 meters wide and 6 meters long, with a garden hose. How fast is the water rising?

Solving Related Rates

1. Draw a Picture
2. Write what you know, and what you want to know. Note units.
3. Relate all your relevant variables in one equation.
4. Differentiate both sides (with respect to the appropriate variable!)
5. Solve for what you want.

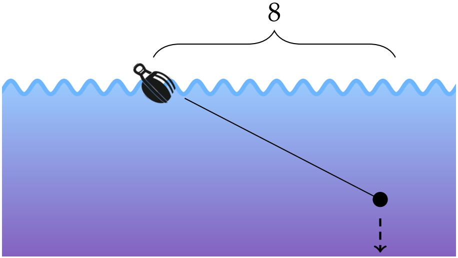
Solving Related Rates

1. Draw a Picture
2. Write what you know, and what you want to know. Note units.
3. Relate all your relevant variables in one equation.
4. Differentiate both sides (with respect to the appropriate variable!)
5. Solve for what you want.

Repeat in class: "If we know how to things are related, we can differentiate to find how their derivatives are related."

For the first few examples, the answers are given with this structure. After that, it's more condensed. Students will likely appreciate if you (at least verbally) continue mentioning these steps, but at some point it's good to leave them and let students develop a little more flexibility and confidence.

A weight is attached to a rope, which is attached to a pulley on a boat, at water level. The weight is taken 8 (horizontal) metres from its attachment point on the boat, then dropped in the water.
The weight sinks straight down. The rope stays taught as it is let out at a constant rate of one metre per second, and two seconds have passed. How fast is the weight descending?

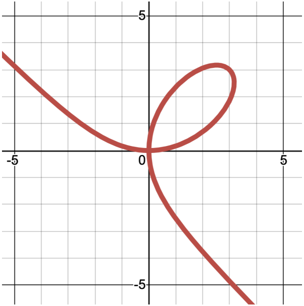


You are pouring water through a funnel with an extremely small hole. The funnel lets water out at 100mL per second, and you are pouring water into the funnel at 300mL per second. The funnel is shaped like a cone with height 20 cm and with the diameter at the top also 20 cm. (Ignore the hole in the bottom.) How fast is the height of the water in the funnel rising when it is 10 cm high?

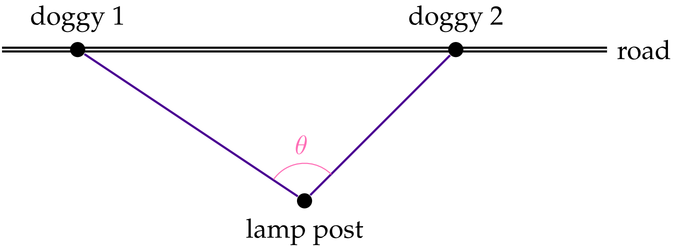
A cone with radius r and height h has volume $\frac{\pi}{3}r^2h$.

A sprinkler is 3m from a long, straight wall. The sprinkler sprays water in a circle, making three revolutions per minute. Let P be the point on the wall closest to the sprinkler. The water hits the wall at some spot, and that spot moves as the sprinkler rotates. When the spot where the water hits the wall is 1m away from P, how fast is the spot moving horizontally?
(You may assume the water travels from the sprinkler to the wall instantaneously.)

A roller coaster has a track shaped in part like the folium of Descartes: $x^3 + y^3 = 6xy$. When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?



Two dogs are tied with elastic leashes to a lamp post that is 2 metres from a straight road. At first, both dogs are on the road, at the closest part of the road to the lamp post. Then, they start running in opposite directions: one dog runs 3 metres per second, and the other runs 2 metres per second. After one second of running, how fast is the angle made by the two leashes increasing?



A crow is one kilometre due east of the math building, heading east at 5 kph. An eagle is two kilometres due north of the math building, heading north at 7kph. How fast is the distance between the two birds increasing at this instant?

A triangle has one side that is 1cm long, and another side that is 2cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.

Included Work

 ‘Gears’ by Dan Hetteix is licensed under CC BY 3.0 (accessed 7 July 2021), 48

 ‘Speedometer’ by Serhii Smirnov is licensed under CC BY 3.0 (accessed 6 July 2021), 22

 ‘Switch’ by K Staelin is in the public domain (accessed 7 July 2021), 54

 screenshot of graph using Desmos Graphing Calculator, `https://www.desmos.com/calculator` (accessed 7 July 2021), 57

 ‘Goose’ by Mary B is licensed under CC BY 3.0 (accessed 6 July 2021), 23