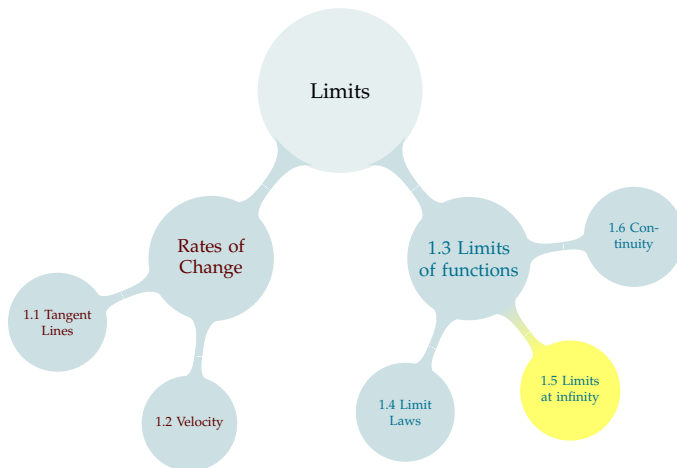


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# END BEHAVIOR

We write:

$$\lim_{x \rightarrow \infty} f(x) = L$$

to express that, as  $x$  grows larger and larger,  $f(x)$  approaches  $L$ .

Similarly, we write:

$$\lim_{x \rightarrow -\infty} f(x) = L$$

to express that, as  $x$  grows more and more strongly negative,  $f(x)$  approaches  $L$ .

# END BEHAVIOR

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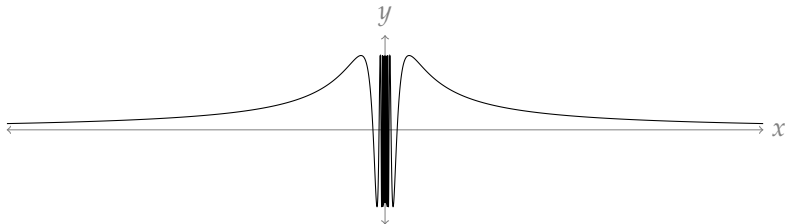
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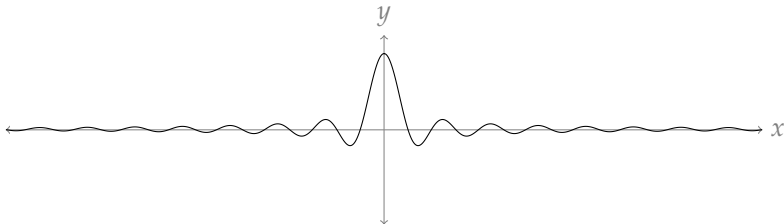
If  $L$  is a number, we call  $y = L$  a **horizontal asymptote** of  $f(x)$ .

## HORIZONTAL ASYMPTOTES



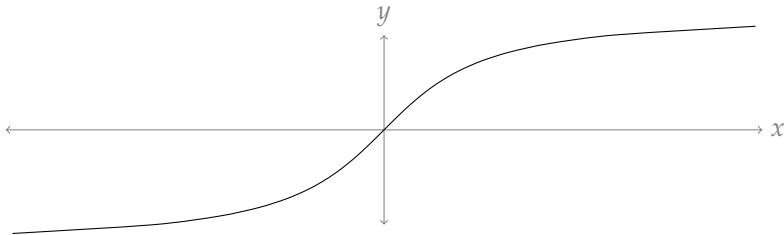
$y = 0$  is a horizontal asymptote for  $y = \sin\left(\frac{1}{x}\right)$

## HORIZONTAL ASYMPTOTES



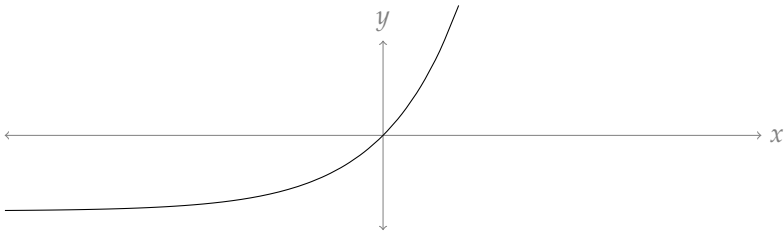
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## HORIZONTAL ASYMPTOTES



$y = \frac{\pi}{2}$  and  $y = -\frac{\pi}{2}$  are horizontal asymptotes for  $y = \arctan(x)$

## HORIZONTAL ASYMPTOTES



$y = -1$  is a horizontal asymptote for  $y = e^x - 1$









# ARITHMETIC WITH LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} \left( x + \frac{x^2}{10} \right) = \infty$$



$$\lim_{x \rightarrow \infty} \left( x - \frac{x^2}{10} \right) = \lim_{x \rightarrow \infty} x \left( 1 - \frac{x}{10} \right) = -\infty$$

$$\lim_{x \rightarrow -\infty} (x^2 + x^3 + x^4) = \lim_{x \rightarrow -\infty} x^4 \left( \frac{1}{x^2} + \frac{1}{x} + 1 \right) = \infty$$

$$\lim_{x \rightarrow -\infty} (x + 13) (x^2 + 13)^{1/3} = -\infty$$

# CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3}$$



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$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3} &= \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^3} \left( \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \right) \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}}{1} = \frac{\lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{2}{x^2} + \lim_{x \rightarrow \infty} \frac{1}{x^3}}{\lim_{x \rightarrow \infty} 1} \\ &= \frac{0 + 0 + 0}{1} = 0 \end{aligned}$$

## CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow -\infty} (x^{7/3} - x^{5/3})$$

# CALCULATING LIMITS AT INFINITY

$$\lim_{x \rightarrow -\infty} (x^{7/3} - x^{5/3})$$

Again: factor out largest power of  $x$ .





# CALCULATING LIMITS AT INFINITY

Suppose the height of a bouncing ball is given by  $h(t) = \frac{\sin(t)+1}{t}$ , for  $t \geq 1$ . What happens to the height over a long period of time?



# CALCULATING LIMITS AT INFINITY



$$\lim_{x \rightarrow \infty} \sqrt{x^4 + x^2 + 1} - \sqrt{x^4 + 3x^2}$$

# CALCULATING LIMITS AT INFINITY



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Multiply function by conjugate:

$$\begin{aligned} & \left( \sqrt{x^4 + x^2 + 1} - \sqrt{x^4 + 3x^2} \right) \left( \frac{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}}{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}} \right) \\ &= \frac{-2x^2 + 1}{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}} \end{aligned}$$

# CALCULATING LIMITS AT INFINITY



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Factor out highest power:  $x^2$  (same as  $\sqrt{x^4}$ )

$$\begin{aligned} & \frac{-2x^2 + 1}{\sqrt{x^4 + x^2 + 1} + \sqrt{x^4 + 3x^2}} \left( \frac{1/x^2}{1/\sqrt{x^4}} \right) \\ &= \frac{-2 + \frac{1}{x^2}}{\sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}} + \sqrt{1 + \frac{3}{x^2}}} \end{aligned}$$



# CALCULATING LIMITS AT INFINITY

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$$\lim_{x \rightarrow \infty} \frac{-2 + \frac{1}{x^2}}{\sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}} + \sqrt{1 + \frac{3}{x^2}}} = \frac{-2 + 0}{\sqrt{1 + 0 + 0} + \sqrt{1 + 0}} = \frac{-2}{2} = -1$$



Now  
You



Evaluate  $\lim_{x \rightarrow -\infty} \frac{\sqrt{3+x^2}}{3x}$





Now  
You



Evaluate  $\lim_{x \rightarrow -\infty} \frac{\sqrt{3+x^2}}{3x}$

We factor out the largest power of the denominator, which is  $x$ .

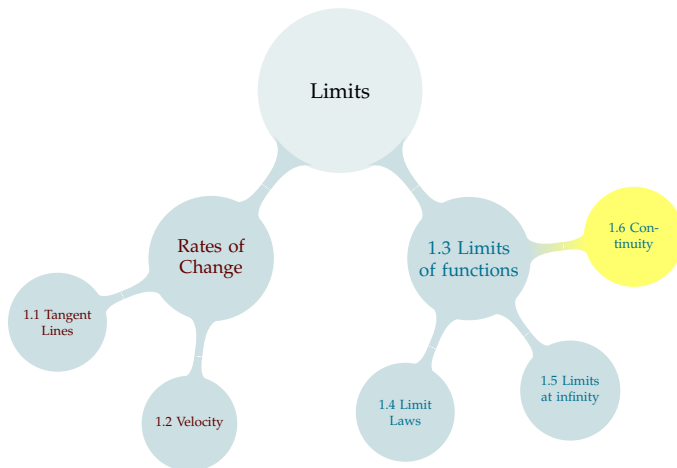
$$\lim_{x \rightarrow -\infty} \frac{\sqrt{3+x^2}}{3x} \left( \frac{1/x}{1/x} \right) = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{3+x^2}}{x}}{3}$$

When  $x < 0$ ,  $\sqrt{x^2} = |x| = -x$

$$\begin{aligned} &= \lim_{x \rightarrow -\infty} \frac{1}{3} \frac{\sqrt{3+x^2}}{-\sqrt{x^2}} \\ &= \lim_{x \rightarrow -\infty} -\frac{1}{3} \sqrt{\frac{3+x^2}{x^2}} \\ &= \lim_{x \rightarrow -\infty} -\frac{1}{3} \sqrt{\frac{3}{x^2} + 1} \\ &= -\frac{1}{3} \end{aligned}$$



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# CONTINUITY

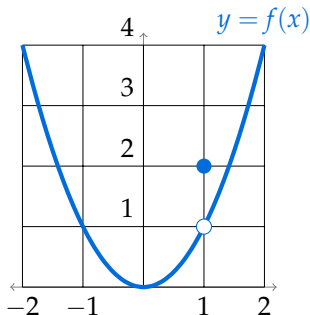
## Definition 1.6.1

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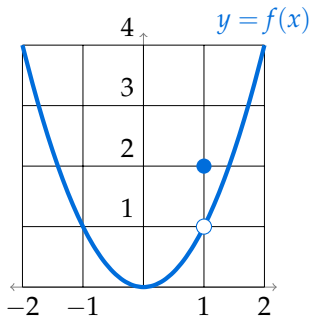
Does  $f(x)$  exist at  $x = 1$ ?

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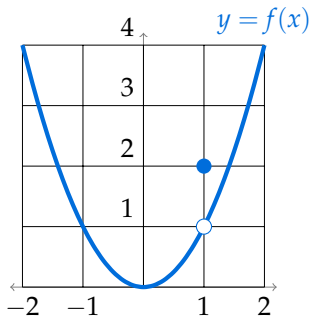


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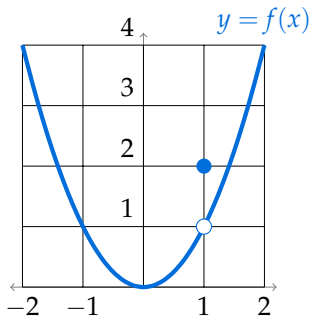
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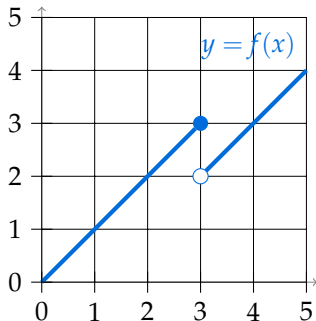
This kind of discontinuity is called **removable**.





## Definitions 1.6.1 and 1.6.2

A function  $f(x)$  is continuous at a point  $a$  if  $\lim_{x \rightarrow a} f(x)$  exists AND is equal to  $f(a)$ .

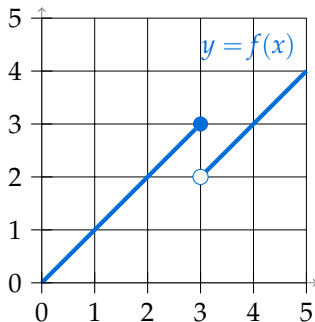


Is  $f(x)$  continuous at  $x = 3$ ? **No.**

This kind of discontinuity is called a **jump**.

## Definitions 1.6.1 and 1.6.2

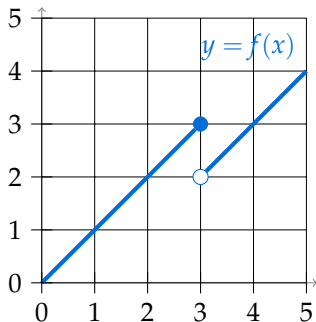
A function  $f(x)$  is continuous **from the left** at a point  $a$  if  $\lim_{x \rightarrow a^-} f(x)$  exists AND is equal to  $f(a)$ .



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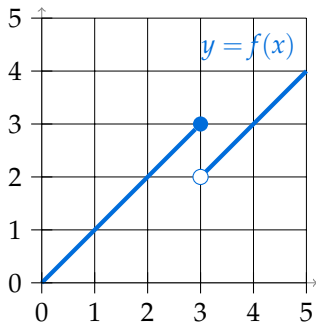
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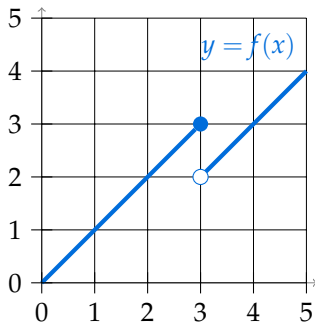
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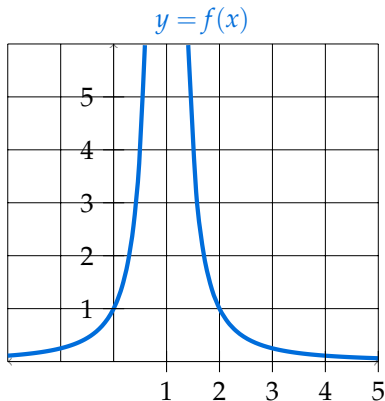
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Is  $f(x)$  continuous from the right at  $x = 3$ ? **No.**

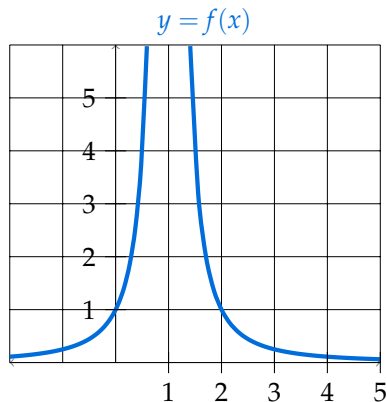
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Since no one-sided limits exist at  $x = 1$ , there's no hope for continuity there – not even “from the left” or “from the right.”

This is called an **infinite discontinuity**





# CONTINUOUS FUNCTIONS

Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point **in their domain**.

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$$f(x) = \frac{x^2}{2x - 10} - \left( \frac{x^2 + 2x - 1}{x - 1} + \frac{\sqrt[5]{25 - x} - \frac{1}{x}}{x + 2} \right)^{1/3}$$

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$f(x)$  is continuous at every real number **except** 5, 1, 0, and  $-2$ .



# CONTINUOUS FUNCTIONS

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We say  $f(x)$  is **continuous over  $(a, b)$**  if it is continuous at every point in  $(a, b)$ . So,  $f(x)$  is **continuous over its domain**,  $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, 5) \cup (5, \infty)$ .

Functions of the following types are continuous over their domains:

- polynomials and rationals
- roots and powers
- trig functions and their inverses
- exponential and logarithm
- The products, sums, differences, quotients, powers, and compositions of continuous functions

Where is the following function continuous?

$$f(x) = \left( \frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

Where is the following function continuous?

$$f(x) = \left( \frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

Over its domain:  $[0, 2) \cup (2, \infty)$ .





# A TECHNICAL POINT

## Definition 1.6.3

A function  $f(x)$  is continuous on the closed interval  $[a, b]$  if:

- ▶  $f(x)$  is continuous over  $(a, b)$ , and
- ▶  $f(x)$  is continuous from the **left** at  $a$ , and
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# A TECHNICAL POINT

## Definition 1.6.3

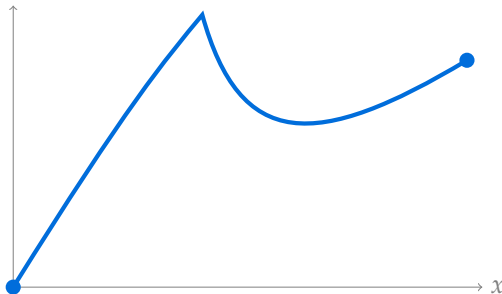
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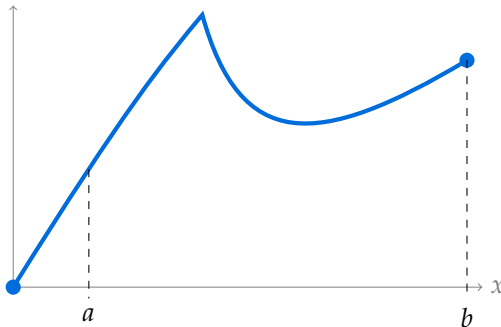
## Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let  $a < b$  and let  $f(x)$  be continuous over  $[a, b]$ . If  $y$  is any number between  $f(a)$  and  $f(b)$ , then there exists  $c$  in  $(a, b)$  such that  $f(c) = y$ .



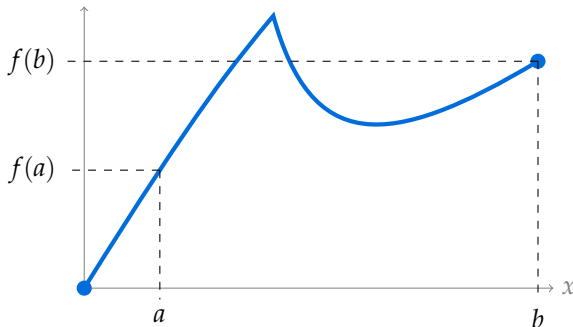
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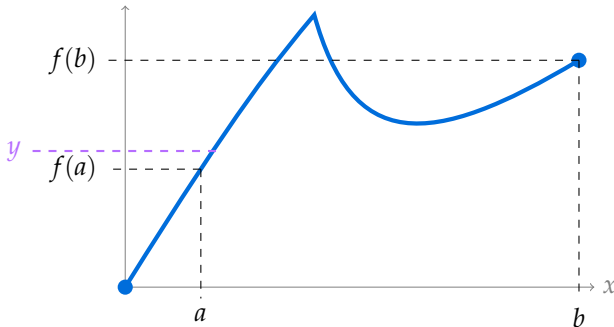
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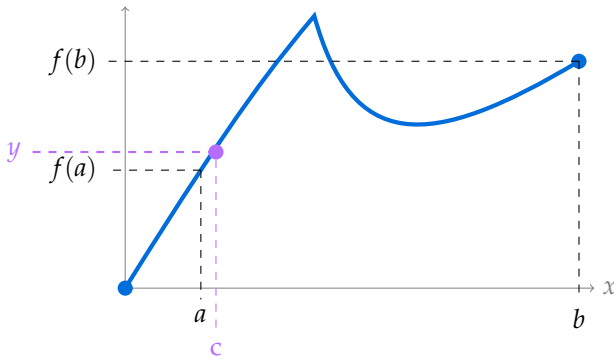
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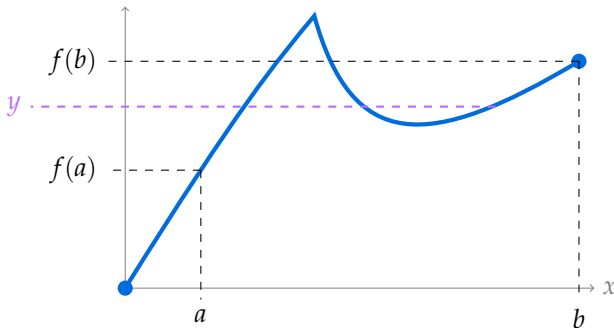
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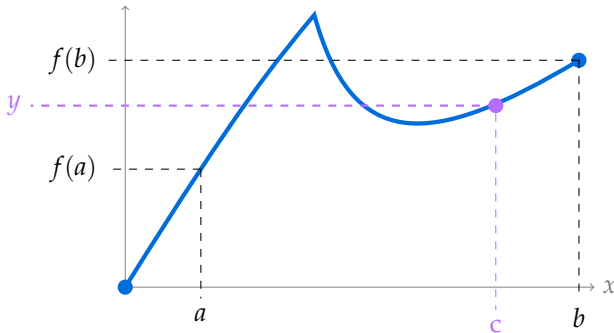
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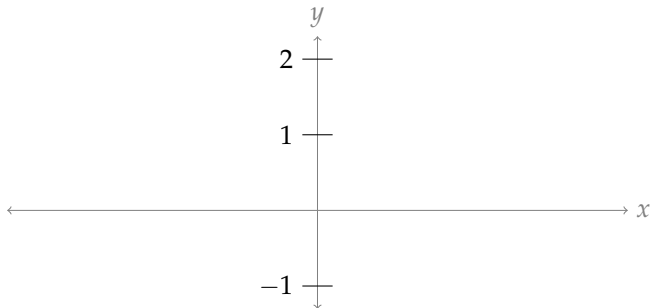


## USING IVT TO FIND ROOTS: “BISECTION METHOD”

Let  $f(x) = x^5 - 2x^4 + 2$ . Find any value  $x$  for which  $f(x) = 0$ .

# USING IVT TO FIND ROOTS: “BISECTION METHOD”

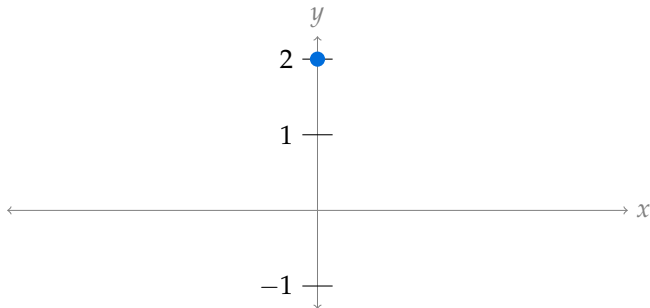
Let  $f(x) = x^5 - 2x^4 + 2$ . Find any value  $x$  for which  $f(x) = 0$ . Let's find some points:



# USING IVT TO FIND ROOTS: "BISECTION METHOD"

Let  $f(x) = x^5 - 2x^4 + 2$ . Find any value  $x$  for which  $f(x) = 0$ . Let's find some points:

$$f(0) = 2$$

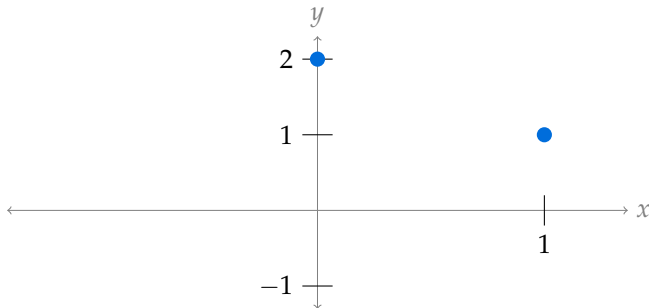


# USING IVT TO FIND ROOTS: “BISECTION METHOD”

Let  $f(x) = x^5 - 2x^4 + 2$ . Find any value  $x$  for which  $f(x) = 0$ . Let's find some points:

$$f(0) = 2$$

$$f(1) = 1$$





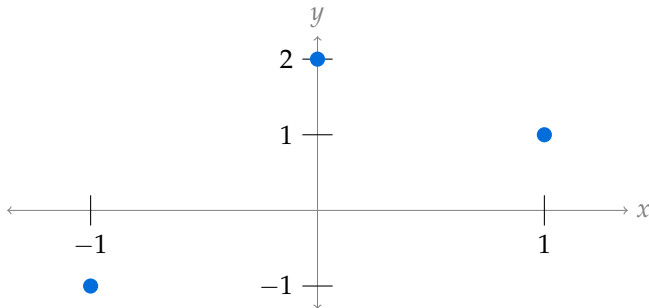
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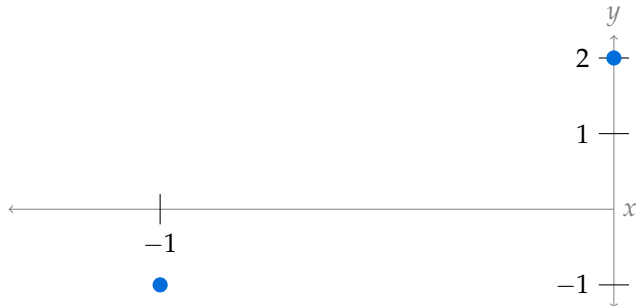
$$f(-1) = -1$$



# USING IVT TO FIND ROOTS: "BISECTION METHOD"

Let  $f(x) = x^5 - 2x^4 + 2$ . Find any value  $x$  for which  $f(x) = 0$ .

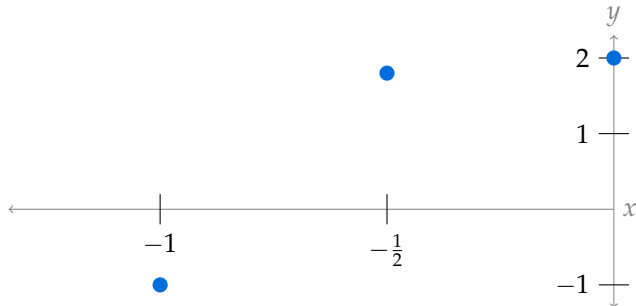
$$f(0) = 2, f(-1) = -1$$



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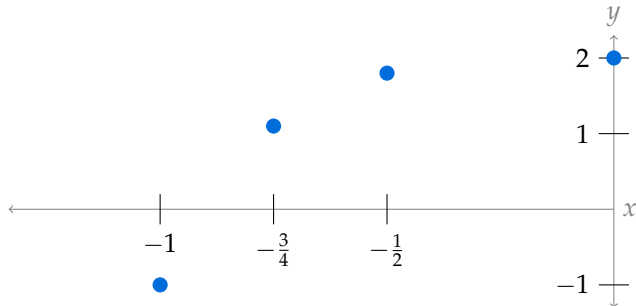
$$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84$$



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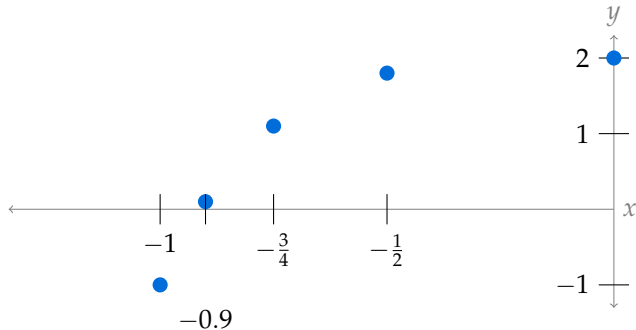
$$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84, f\left(-\frac{3}{4}\right) \approx 1.13$$



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Let  $f(x) = x^5 - 2x^4 + 2$ . Find any value  $x$  for which  $f(x) = 0$ .

$$f(0) = 2, f(-1) = -1, f\left(-\frac{1}{2}\right) \approx 1.84, f\left(-\frac{3}{4}\right) \approx 1.13, f(-.9) = 0.097$$



Use the Intermediate Value Theorem to show that there exists some solution to the equation  $\ln x \cdot e^x = 4$ , and give a reasonable interval where that solution might occur.

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- The function  $f(x) = \ln x \cdot e^x$  is continuous over its domain, which is  $(0, \infty)$ . In particular, then, it is continuous over the interval  $(1, e)$ .
- $f(1) = \ln(1)e = 0 \cdot e = 0$  and  $f(e) = \ln(e) \cdot e^e = e^e$ . Since  $e > 2$ , we know  $f(e) = e^e > 2^2 = 4$ .
- Then 4 is between  $f(1)$  and  $f(e)$ .
- By the Intermediate Value Theorem,  $f(c) = 4$  for some  $c$  in  $(1, e)$ .

NOW  
YOU



Use the Intermediate Value Theorem to give a

reasonable interval where the following is true:  $e^x = \sin(x)$ . (Don't use a calculator – use numbers you can easily evaluate.)



NOW  
YOU



Use the Intermediate Value Theorem to give a

reasonable interval where the following is true:  $e^x = \sin(x)$ . (Don't use a calculator – use numbers you can easily evaluate.)

We can rearrange this: let  $f(x) = e^x - \sin(x)$ , and note  $f(x)$  has roots exactly when  $e^x = \sin(x)$ .

- The function  $f(x) = e^x - \sin x$  is continuous over its domain, which is all real numbers. In particular, then, it is continuous over the interval  $(-\frac{3\pi}{2}, e)$ .
- $f(0) = e^0 - \sin 0 = 1 - 0 = 1 > 0$  and  $f(-\frac{3\pi}{2}) = e^{-\frac{3\pi}{2}} - \sin(-\frac{3\pi}{2}) = e^{-\frac{3\pi}{2}} - 1 < e^0 - 1 = 1 - 1 = 0$ .
- Then 0 is between  $f(0)$  and  $f(-\frac{3\pi}{2})$ .
- By the Intermediate Value Theorem,  $f(c) = 0$  for some  $c$  in  $(-\frac{3\pi}{2}, 0)$ .
- Therefore,  $e^c = \sin c$  for some  $c$  in  $(-\frac{3\pi}{2}, 0)$ .



Is there any value of  $x$  so that  $\sin x = \cos(2x) + \frac{1}{4}$ ?

Is there any value of  $x$  so that  $\sin x = \cos(2x) + \frac{1}{4}$ ?

Yes, somewhere between 0 and  $\frac{\pi}{2}$ .

NOW  
YOU



Is the following reasoning correct?

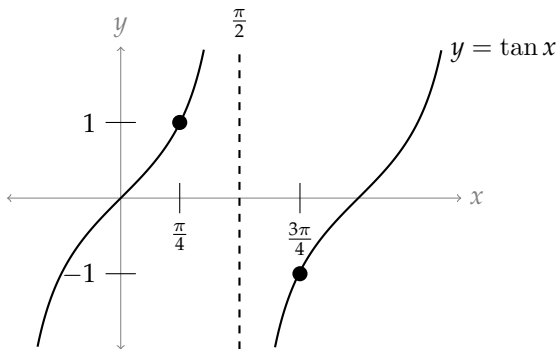
- $f(x) = \tan x$  is continuous over its domain, because it is a trigonometric function.
- In particular,  $f(x)$  is continuous over the interval  $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ .
- $f\left(\frac{\pi}{4}\right) = 1$ , and  $f\left(\frac{3\pi}{4}\right) = -1$ .
- Since  $f\left(\frac{3\pi}{4}\right) < 0 < f\left(\frac{\pi}{4}\right)$ , by the Intermediate Value Theorem, there exists some number  $c$  in the interval  $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$  such that  $f(c) = 0$ .

Now  
You



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false
- $f\left(\frac{\pi}{4}\right) = 1$ , and  $f\left(\frac{3\pi}{4}\right) = -1$ .
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# CONTINUITY

## Section 1.6 Review

Suppose  $f(x)$  is continuous at  $x = 1$ . Does  $f(x)$  have to be defined at  $x = 1$ ?



Suppose  $f(x)$  is continuous at  $x = 1$ . Does  $f(x)$  have to be defined at  $x = 1$ ?

Yes. Since  $f(x)$  is continuous at  $x = 1$ ,  $\lim_{x \rightarrow 1} f(x) = f(1)$ , so  $f(1)$  must exist.

Suppose  $f(x)$  is continuous at  $x = 1$  and  $\lim_{x \rightarrow 1^-} f(x) = 30$ .

True or false:  $\lim_{x \rightarrow 1^+} f(x) = 30$ .

Suppose  $f(x)$  is continuous at  $x = 1$  and  $\lim_{x \rightarrow 1^-} f(x) = 30$ .

True or false:  $\lim_{x \rightarrow 1^+} f(x) = 30$ .

True. Since  $f(x)$  is continuous at  $x = 1$ ,  $\lim_{x \rightarrow 1} f(x) = f(1)$ , so  $\lim_{x \rightarrow 1} f(x)$  must exist. That means both one-sided limits exist, and are equal to each other.

Suppose  $f(x)$  is continuous at  $x = 1$  and  $f(1) = 22$ . What is  $\lim_{x \rightarrow 1} f(x)$ ?

Suppose  $f(x)$  is continuous at  $x = 1$  and  $f(1) = 22$ . What is  $\lim_{x \rightarrow 1} f(x)$ ?

$$22 = f(1) = \lim_{x \rightarrow 1} f(x).$$

Suppose  $\lim_{x \rightarrow 1} f(x) = 2$ . Must it be true that  $f(1) = 2$ ?

Suppose  $\lim_{x \rightarrow 1} f(x) = 2$ . Must it be true that  $f(1) = 2$ ?

No. In order to determine the limit as  $x$  goes to 1, we ignore  $f(1)$ .  
(Perhaps  $f(x)$  is not even defined at 1.)



$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of  $a$  is  $f(x)$  continuous?



$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of  $a$  is  $f(x)$  continuous?

We need  $ax^2 = 3x$  when  $x = 1$ , so  $a = 3$ .

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of  $a$  is  $f(x)$  continuous at  $x = -\sqrt{3}$ ?

By the definition of continuity, if  $f(x)$  is continuous at  $x = -\sqrt{3}$ , then  $f(-\sqrt{3}) = \lim_{x \rightarrow -\sqrt{3}} f(x)$ . Note  $f(-\sqrt{3}) = a$ , and when  $x$  is close to (but not equal to)  $-\sqrt{3}$ , then  $f(x) = \frac{\sqrt{3}x+3}{x^2-3}$ .

$$f(-\sqrt{3}) = \lim_{x \rightarrow -\sqrt{3}} f(x)$$

$$\begin{aligned} a &= \lim_{x \rightarrow -\sqrt{3}} \frac{\sqrt{3}x+3}{x^2-3} = \lim_{x \rightarrow -\sqrt{3}} \frac{\sqrt{3}(x+\sqrt{3})}{(x+\sqrt{3})(x-\sqrt{3})} \\ &= \lim_{x \rightarrow -\sqrt{3}} \frac{\sqrt{3}}{x-\sqrt{3}} = \frac{\sqrt{3}}{-\sqrt{3}-\sqrt{3}} = -\frac{1}{2} \end{aligned}$$

So we can use  $a = -\frac{1}{2}$  to make  $f(x)$  continuous at  $x = -\sqrt{3}$ .

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of  $a$  is  $f(x)$  continuous at  $x = \sqrt{3}$ ?

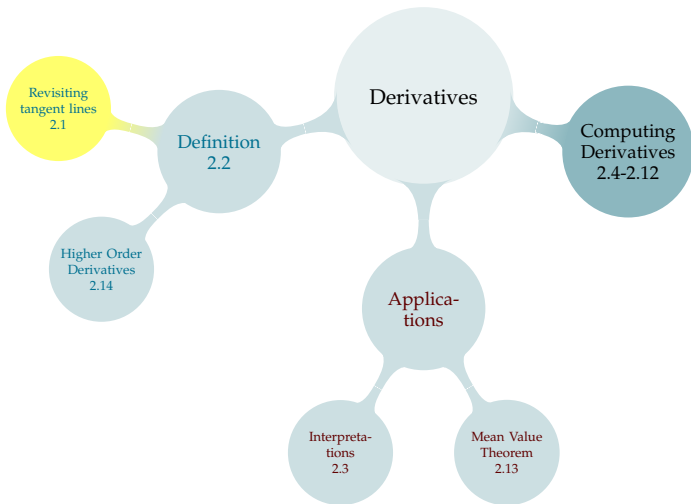
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For which value(s) of  $a$  is  $f(x)$  continuous at  $x = \sqrt{3}$ ?

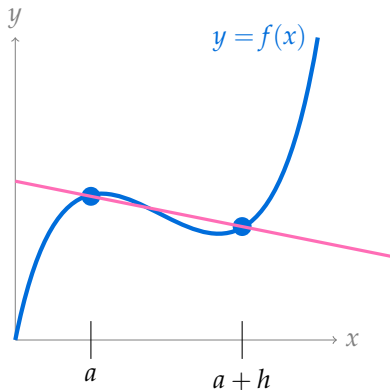
By the definition of continuity, if  $f(x)$  is continuous at  $x = \sqrt{3}$ , then  $f(\sqrt{3}) = \lim_{x \rightarrow \sqrt{3}} f(x)$ . When  $x$  is close to (but not equal to)  $\sqrt{3}$ , then

$f(x) = \frac{\sqrt{3}x+3}{x^2-3}$ . However, as  $x$  approaches  $\sqrt{3}$ , the denominator of this expression gets closer and closer to zero, while the top gets closer and closer to 6. So, this limit does not exist. Therefore, no value of  $a$  will make  $f(x)$  continuous at  $x = \sqrt{3}$ .

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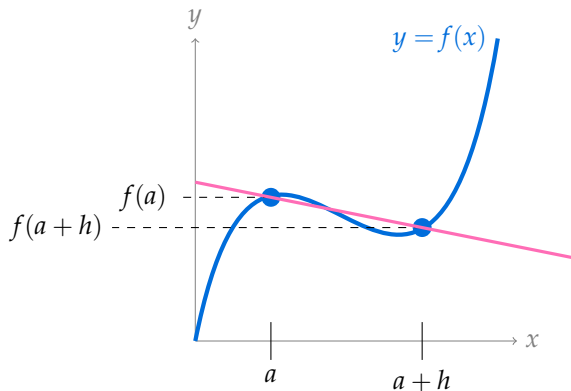




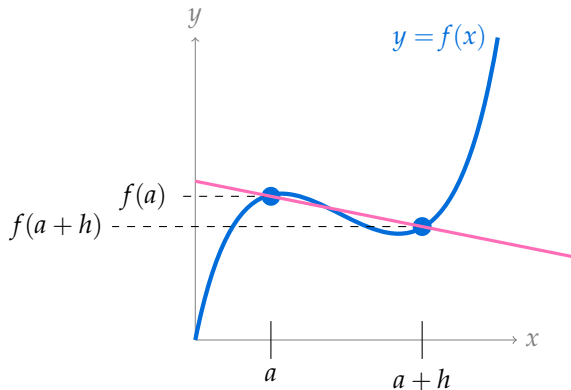


Slope of secant line:

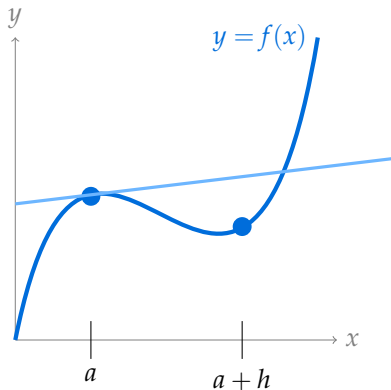




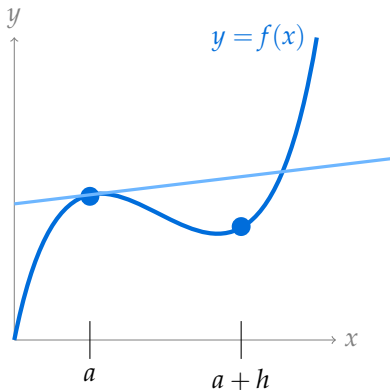
Slope of secant line:



Slope of secant line:  $\frac{f(a+h)-f(a)}{h}$

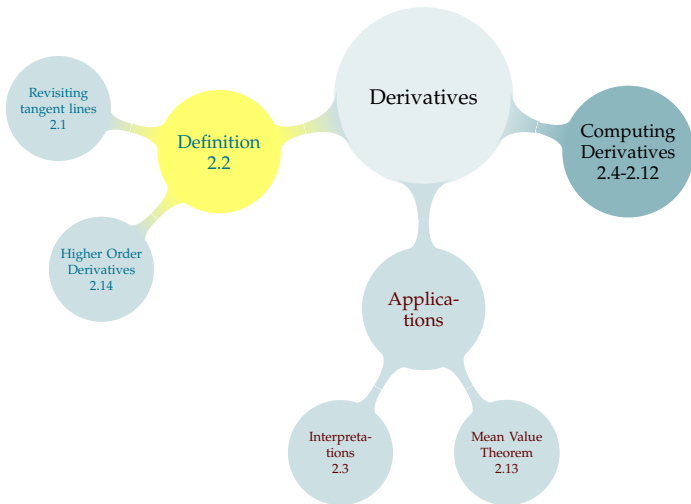


Slope of tangent line:



Slope of tangent line:  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

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# DERIVATIVE AT A POINT

## Definition 2.2.1

Given a function  $f(x)$  and a point  $a$ , the slope of the tangent line to  $f(x)$  at  $a$  is the **derivative of  $f$  at  $a$** , written  $f'(a)$ .

$$\text{So, } f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

$f'(a)$  is also the **instantaneous rate of change of  $f$  at  $a$** .











## Derivative

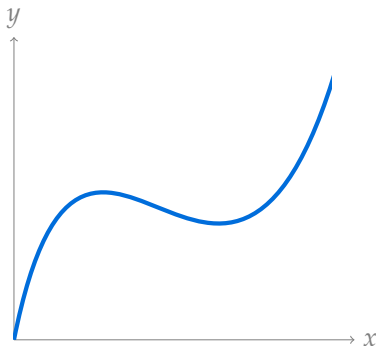
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If  $f'(a) > 0$ , then  $f$  is **increasing** at  $a$ . Its graph “points up.”

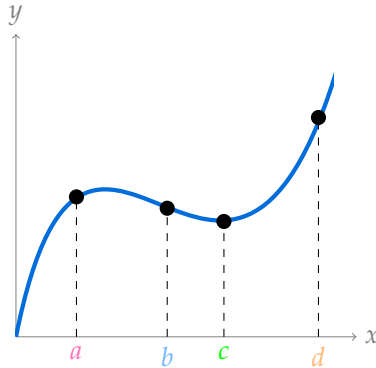
If  $f'(a) < 0$ , then  $f$  is **decreasing** at  $a$ . Its graph “points down.”

If  $f'(a) = 0$ , then  $f$  looks **constant** or **flat** at  $a$ .

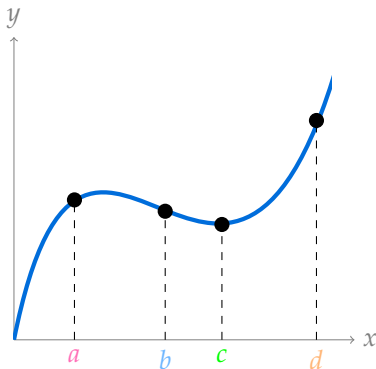
## PRACTICE: INCREASING AND DECREASING



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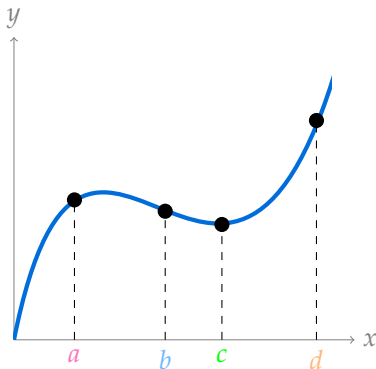


# PRACTICE: INCREASING AND DECREASING



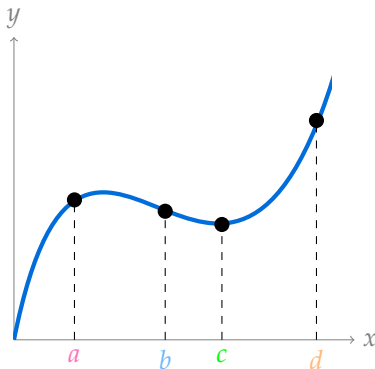
Where is  $f'(x) < 0$ ?

# PRACTICE: INCREASING AND DECREASING



Where is  $f'(x) < 0$ ?  $f'(b) < 0$

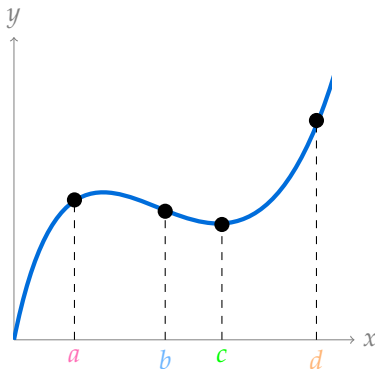
## PRACTICE: INCREASING AND DECREASING



Where is  $f'(x) > 0$ ?

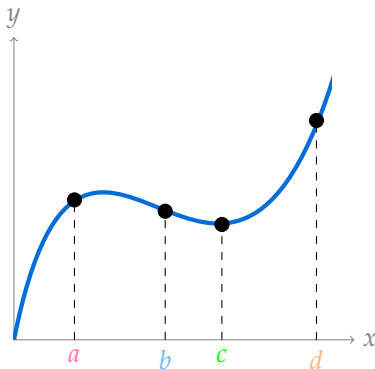


## PRACTICE: INCREASING AND DECREASING



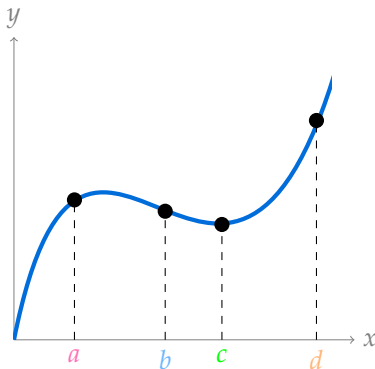
Where is  $f'(x) > 0$ ?  $f'(a) > 0$  and  $f'(d) > 0$

## PRACTICE: INCREASING AND DECREASING



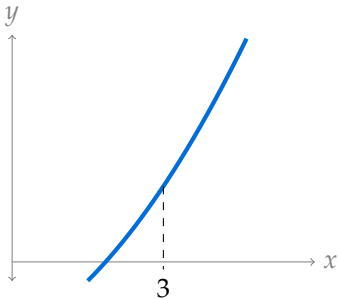
Where is  $f'(x) \approx 0$ ?

## PRACTICE: INCREASING AND DECREASING



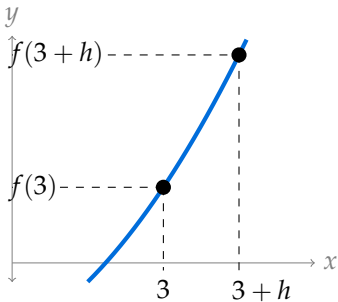
Where is  $f'(x) \approx 0$ ?  $f'(c) \approx 0$

Use the definition of the derivative to find the slope of the tangent line to  $f(x) = x^2 - 5$  at the point  $x = 3$ .

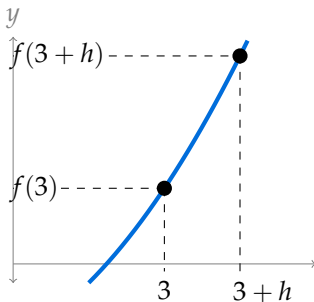


A graph of a function  $y = f(x)$  is shown. The x-axis is labeled  $x$  and the y-axis is labeled  $y$ . A blue curve represents the function. Two points on the x-axis are marked:  $3$  and  $3+h$ . Vertical dashed lines extend from these points on the x-axis up to the curve, indicating the interval  $[3, 3+h]$  on the x-axis.

Use the definition of the derivative to find the slope of the tangent line to  $f(x) = x^2 - 5$  at the point  $x = 3$ .



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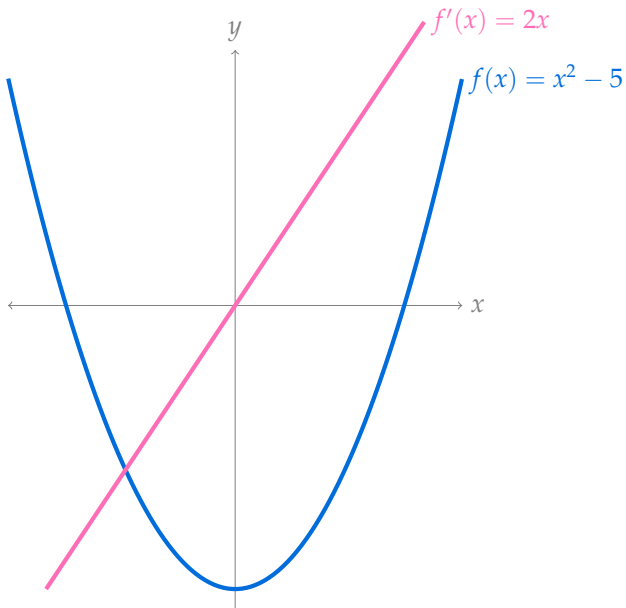
$$\begin{aligned}
 f'(3) &= \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{((3 + h)^2 - 5) - (3^2 - 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(9 + 6h + h^2 - 5) - 4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 + 6h}{h} \\
 &= \lim_{h \rightarrow 0} h + 6 = 6
 \end{aligned}$$

Let's keep the function  $f(x) = x^2 - 5$ . We just showed  $f'(3) = 6$ .  
 We can also find its derivative at an arbitrary point  $x$ :



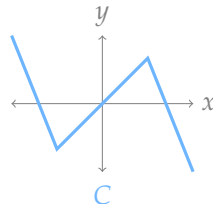
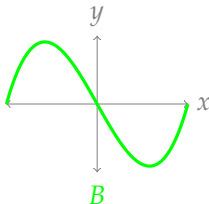
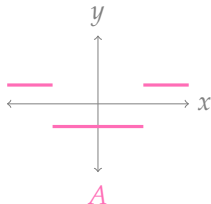
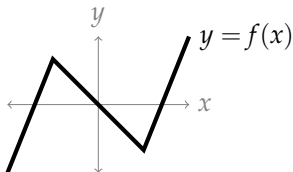
Let's keep the function  $f(x) = x^2 - 5$ . We just showed  $f'(3) = 6$ .  
We can also find its derivative at an arbitrary point  $x$ :

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 5 - (x^2 - 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\
 &= \lim_{h \rightarrow 0} 2x + h = 2x \quad (\text{In particular, } f'(3) = 6.)
 \end{aligned}$$



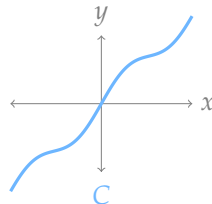
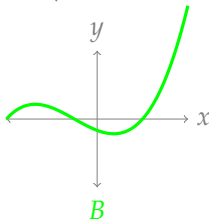
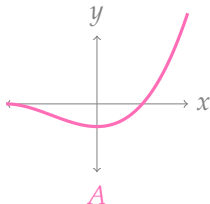
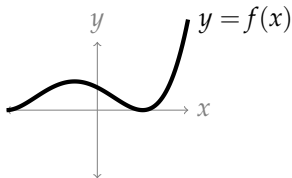
# INCREASING AND DECREASING

In black is the curve  $y = f(x)$ . Which of the coloured curves corresponds to  $y = f'(x)$ ?



# INCREASING AND DECREASING

In black is the curve  $y = f(x)$ . Which of the coloured curves corresponds to  $y = f'(x)$ ?



## Derivative as a Function – Definition 2.2.6

Let  $f(x)$  be a function.

The derivative of  $f(x)$  with respect to  $x$  is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that  $x$  will be a part of your final expression: this is a **function**.

If  $f'(x)$  exists for all  $x$  in an interval  $(a, b)$ , we say that  $f$  is **differentiable on  $(a, b)$** .

## Notation 2.2.8

The “prime” notation  $f'(x)$  and  $f'(a)$  is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{df}{dx}$$

$$\frac{df}{dx}(a)$$

$$\frac{d}{dx}f(x)$$

$$\frac{d}{dx}f(x)\Big|_{x=a}$$

## Notation 2.2.8

The “prime” notation  $f'(x)$  and  $f'(a)$  is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{df}{dx}$$

function

$$\frac{df}{dx}(a)$$

number

$$\frac{d}{dx}f(x)$$

function

$$\frac{d}{dx}f(x)\Big|_{x=a}$$

number

## Newtonian Notation:

$$f(x) = x^2 + 5$$

$$f'(x) = 2x$$

$$f'(3) = 6$$

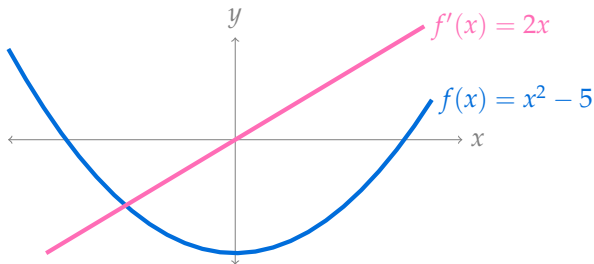
## Leibnitz Notation:

$$\frac{df}{dx} =$$

$$\frac{df}{dx}(3) =$$

$$\frac{d}{dx}f(x) =$$

$$\left. \frac{d}{dx}f(x) \right|_{x=3} =$$





## Newtonian Notation:

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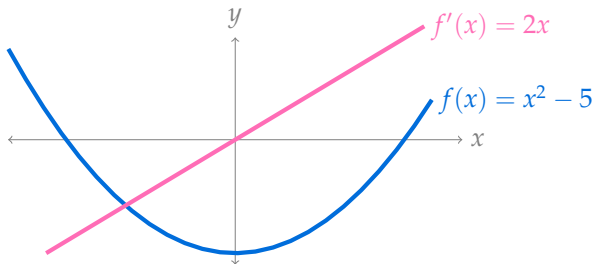
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$$\frac{df}{dx} = 2x$$

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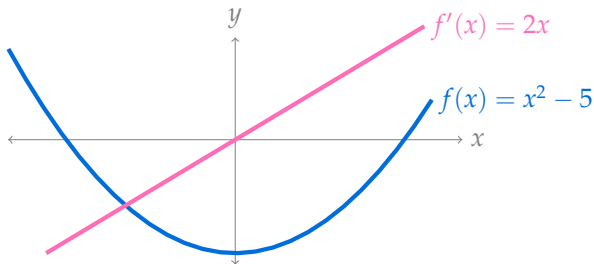
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## Newtonian Notation:

$$f(x) = x^2 + 5$$

$$f'(x) = 2x$$

$$f'(3) = 6$$

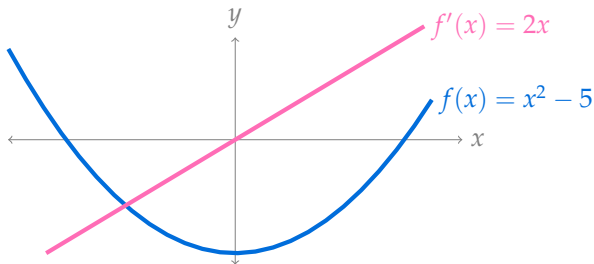
## Leibnitz Notation:

$$\frac{df}{dx} = 2x$$

$$\frac{df}{dx}(3) = 6$$

$$\frac{d}{dx}f(x) = 2x$$

$$\frac{d}{dx}f(x) \Big|_{x=3} =$$

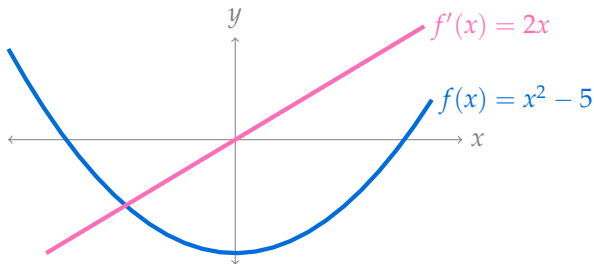


## Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

## Leibnitz Notation:

$$\frac{df}{dx} = 2x \qquad \frac{df}{dx}(3) = 6 \qquad \frac{d}{dx}f(x) = 2x \qquad \left. \frac{d}{dx}f(x) \right|_{x=3} = 6$$



## Alternate Definition – Definition 2.2.1

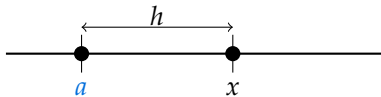
Calculating

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

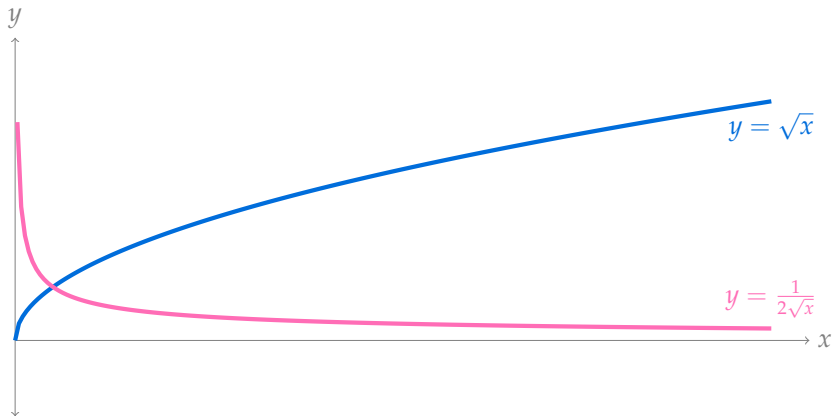
Notice in these scenarios,  $h = x - a$ .



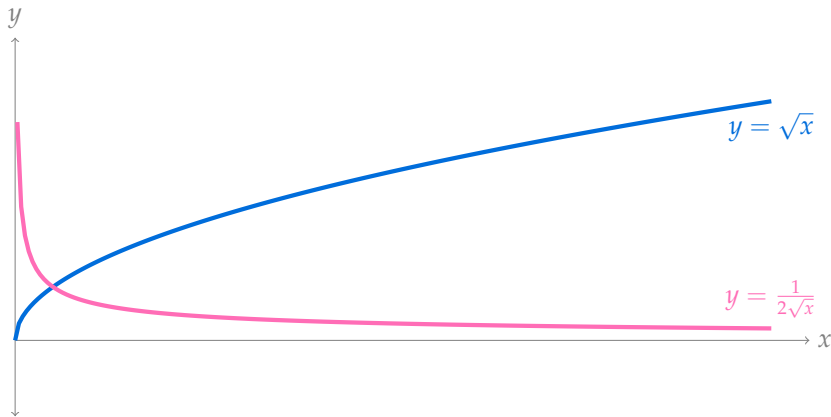
Let  $f(x) = \sqrt{x}$ . Using the definition of a derivative, calculate  $f'(x)$ .

Let  $f(x) = \sqrt{x}$ . Using the definition of a derivative, calculate  $f'(x)$ .

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left( \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$



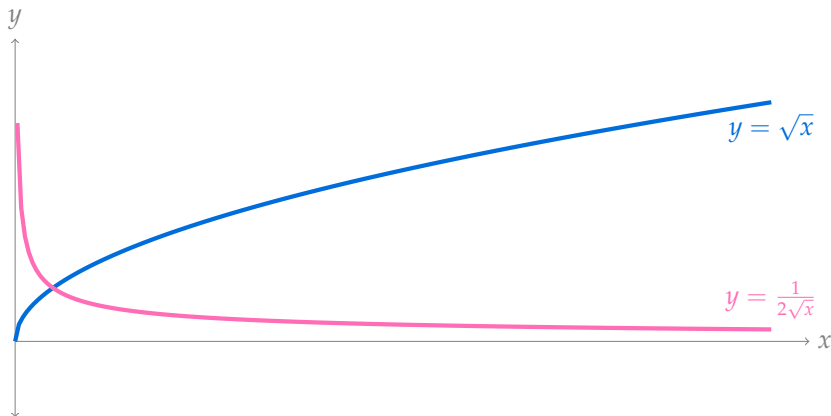




Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} =$$

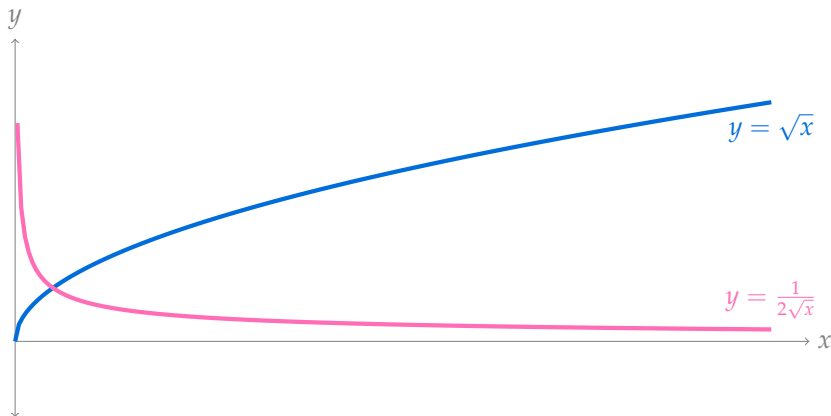
$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} =$$



Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

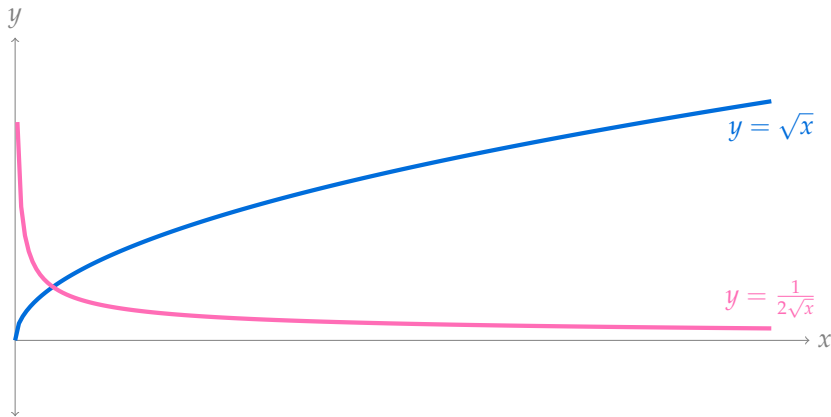
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Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$



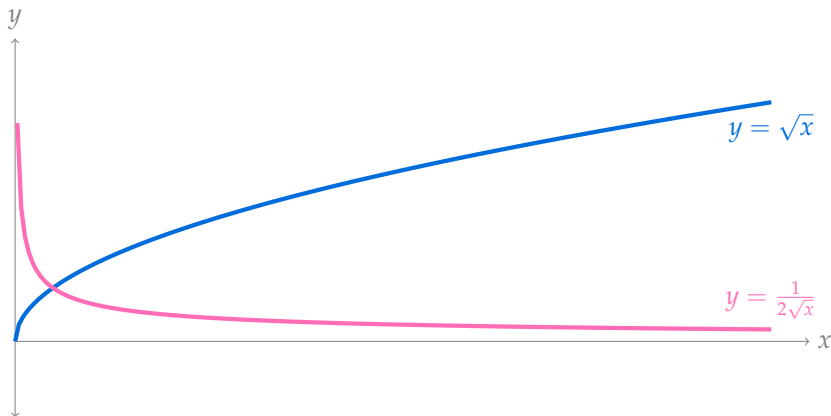
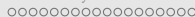
Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$

$$\lim_{x \rightarrow 0^+} \sqrt{x} =$$

$$\lim_{x \rightarrow 0^+} \frac{1}{2\sqrt{x}} =$$



Review:

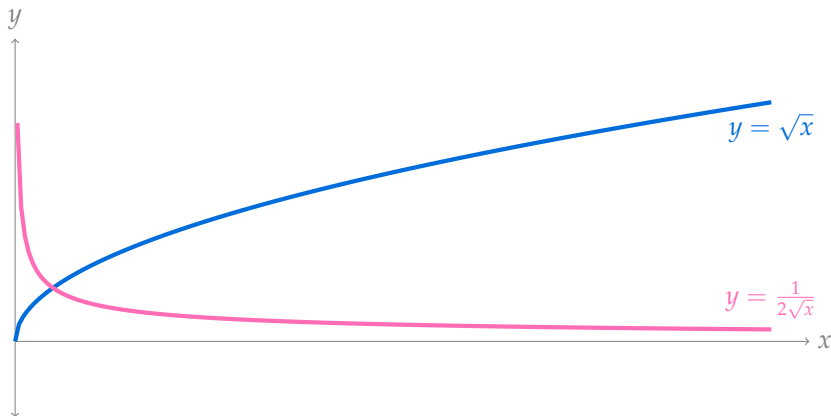
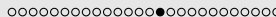
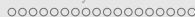
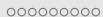
$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

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$$\lim_{x \rightarrow 0^+} \frac{1}{2\sqrt{x}} = \infty$$





Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0$$

$$\lim_{x \rightarrow 0^+} \frac{1}{2\sqrt{x}} = \infty$$



Now  
You



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{1}{x} \right\}.$$

Now  
You



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{1}{x} \right\}.$$

$$\begin{aligned} \frac{d}{dx} \left[ \frac{1}{x} \right] &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \end{aligned}$$



NOW  
YOU



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{2x}{x+1} \right\}.$$

NOW  
YOU



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{1}{\sqrt{x^2 + x}} \right\}.$$

Using the definition of the derivative, calculate  $\frac{d}{dx} \left\{ \frac{2x}{x+1} \right\}$ .

Using the definition of the derivative, calculate  $\frac{d}{dx} \left\{ \frac{2x}{x+1} \right\}$ .

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{2(x+h)}{x+h+1} - \frac{2x}{x+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{2(x+h)(x+1)}{(x+h+1)(x+1)} - \frac{2x(x+h+1)}{(x+1)(x+h+1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2}{h} \left( \frac{(x^2 + x + xh + h) - (x^2 + xh + x)}{(x+h+1)(x+1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2}{h} \left( \frac{h}{(x+h+1)(x+1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2}{(x+h+1)(x+1)} = \frac{2}{(x+1)^2}
 \end{aligned}$$

Using the definition of the derivative, calculate  $\frac{d}{dx} \left\{ \frac{1}{\sqrt{x^2 + x}} \right\}$ .

Using the definition of the derivative, calculate  $\frac{d}{dx} \left\{ \frac{1}{\sqrt{x^2 + x}} \right\}$ .

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{(x+h)^2 + x+h}} - \frac{1}{\sqrt{x^2 + x}}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{\sqrt{x^2 + x}}{\sqrt{(x+h)^2 + x+h} \sqrt{x^2 + x}} - \frac{\sqrt{(x+h)^2 + x+h}}{\sqrt{(x+h)^2 + x+h} \sqrt{x^2 + x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{\sqrt{x^2 + x} - \sqrt{(x+h)^2 + x+h}}{\sqrt{(x+h)^2 + x+h} \sqrt{x^2 + x}} \right) \left( \frac{\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}}{\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{(x^2 + x) - [(x+h)^2 + x+h]}{\sqrt{(x+h)^2 + x+h} \sqrt{x^2 + x} [\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}]} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-(2xh + h^2 + h)}{\sqrt{(x+h)^2 + x+h} \sqrt{x^2 + x} [\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}]} \right) \\
 &= \lim_{h \rightarrow 0} \frac{-(2x + h + 1)}{\sqrt{(x+h)^2 + x+h} \sqrt{x^2 + x} [\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}]} = \frac{-(2x+1)}{2(x^2+x)^{3/2}}
 \end{aligned}$$

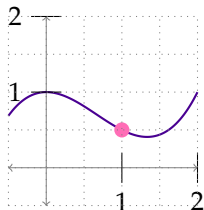
## Memorize

The derivative of a function  $f$  at a point  $a$  is given by the following limit, if it exists:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

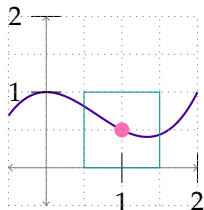
# ZOOMING IN

For a smooth function, if we zoom in at a point, we see a line:



# ZOOMING IN

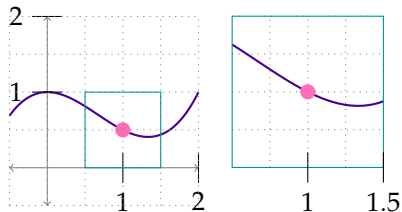
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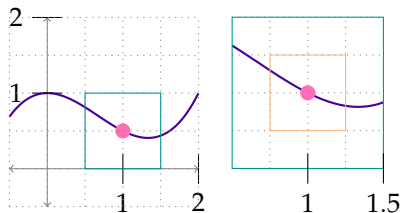
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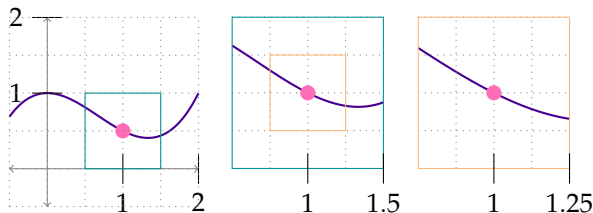
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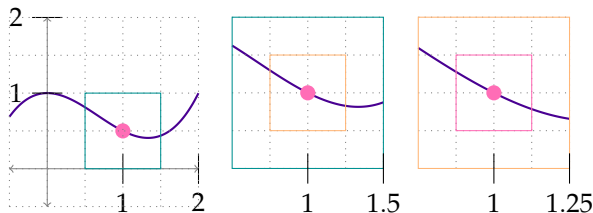
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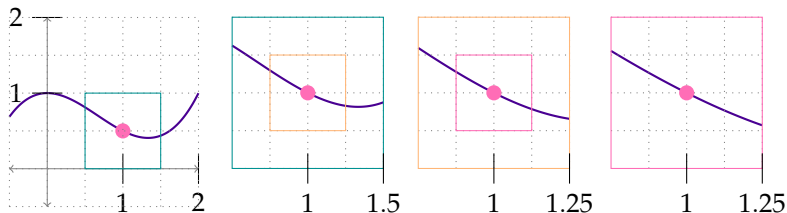
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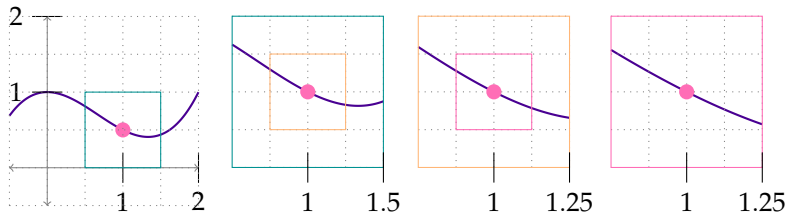
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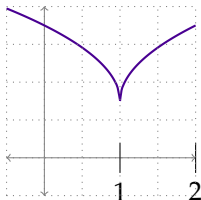
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

# ZOOMING IN ON FUNCTIONS THAT AREN'T SMOOTH

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

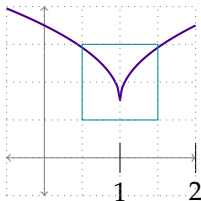
Cusp:



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Cusp:

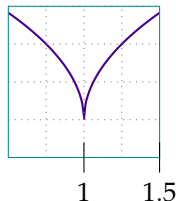
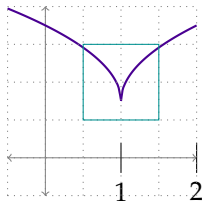




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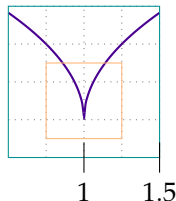
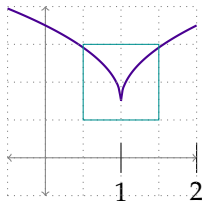
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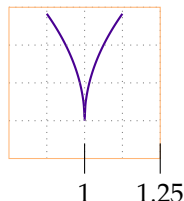
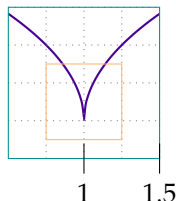
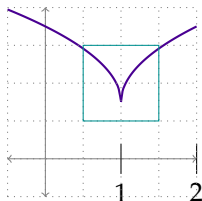
Cusp:



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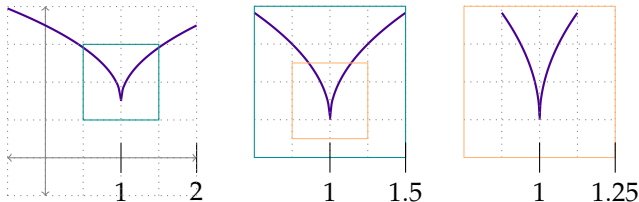
Cusp:



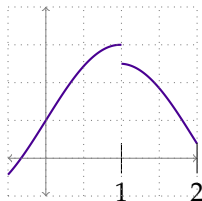
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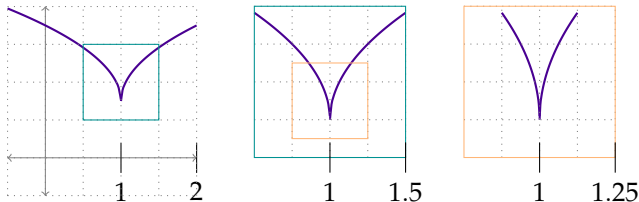
Discontinuity:



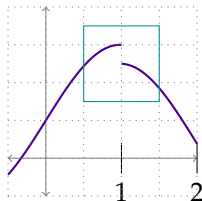
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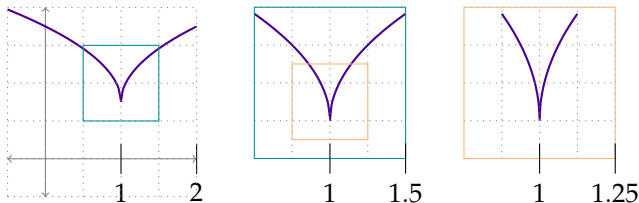
Discontinuity:



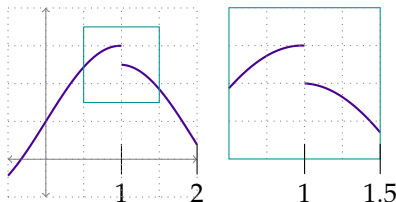
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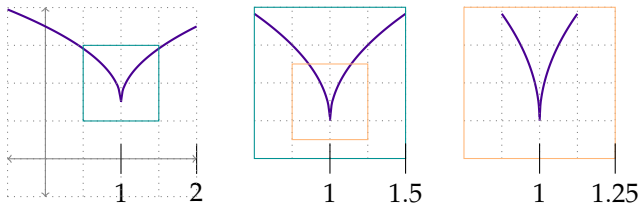
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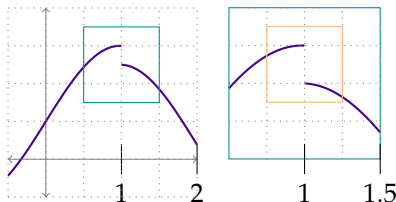
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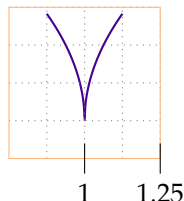
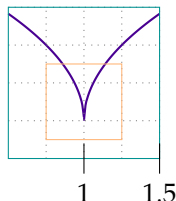
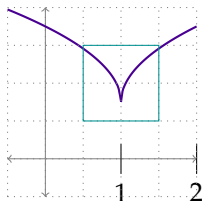
Discontinuity:



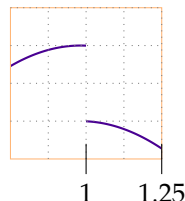
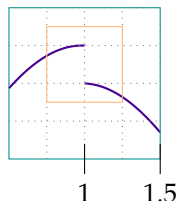
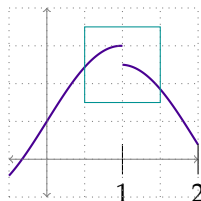
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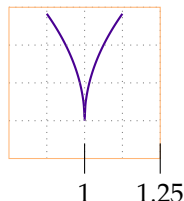
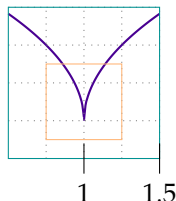
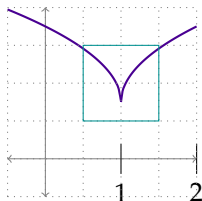




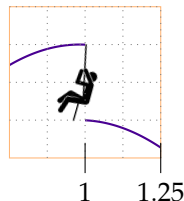
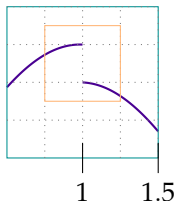
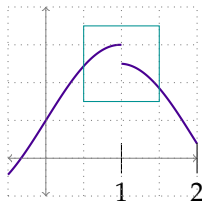
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For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



Discontinuity:



## Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios,  $h = x - a$ .

The derivative of  $f(x)$  **does not exist** at  $x = a$  if

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

does not exist.

Note this is the slope of the tangent line to  $y = f(x)$  at  $x = a$ ,  $\frac{\Delta y}{\Delta x}$ .

# WHEN DERIVATIVES DON'T EXIST

What happens if we try to calculate a derivative where none exists?

Find the derivative of  $f(x) = x^{1/3}$  at  $x = 0$ .

# WHEN DERIVATIVES DON'T EXIST

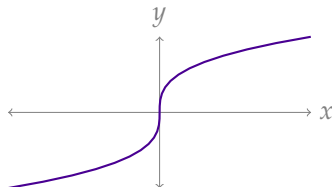
What happens if we try to calculate a derivative where none exists?

Find the derivative of  $f(x) = x^{1/3}$  at  $x = 0$ .

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty \end{aligned}$$

Since the limit does not exist, we conclude  $f'(x)$  is not defined at  $x = 0$ .

We can go a little farther: since the limit goes to infinity, the graph  $y = f(x)$  looks vertical at  $x = 0$ .



## Theorem 2.2.14

If the function  $f(x)$  is differentiable at  $x = a$ , then  $f(x)$  is also continuous at  $x = a$ .

Proof:

## Theorem 2.2.14

If the function  $f(x)$  is differentiable at  $x = a$ , then  $f(x)$  is also continuous at  $x = a$ .

Proof: If  $f'(a)$  exists, that means:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists} \\ \implies \lim_{h \rightarrow 0} \left[ h \cdot \frac{f(a+h) - f(a)}{h} \right] &= \left[ \lim_{h \rightarrow 0} h \right] \cdot \left[ \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right] \\ \implies \lim_{h \rightarrow 0} \left[ h \cdot \frac{f(a+h) - f(a)}{h} \right] &= 0 \\ \implies \lim_{h \rightarrow 0} [f(a+h) - f(a)] &= 0 \\ \implies \lim_{h \rightarrow 0} f(a+h) &= f(a) \end{aligned}$$

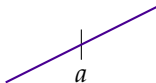
and that is the definition of  $f(x)$  being continuous at  $x = a$ .

Let  $f(x)$  be a function and let  $a$  be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

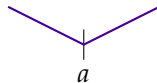
$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ not differentiable at $x = a$
$f(x)$ not continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ not continuous at $x = a$ $f(x)$ not differentiable at $x = a$

Let  $f(x)$  be a function and let  $a$  be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

$f(x)$  continuous at  $x = a$   
 $f(x)$  differentiable at  $x = a$



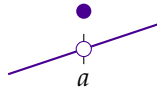
$f(x)$  continuous at  $x = a$   
 $f(x)$  not differentiable at  $x = a$



$f(x)$  not continuous at  $x = a$   
 $f(x)$  differentiable at  $x = a$

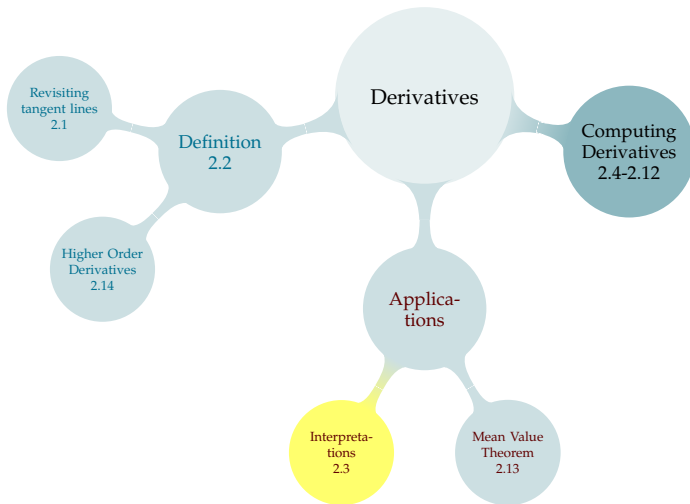
impossible

$f(x)$  not continuous at  $x = a$   
 $f(x)$  not differentiable at  $x = a$





# TABLE OF CONTENTS



## Interpreting the Derivative

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At midnight of January 1, 2012, the world population was increasing at a rate of 156 people each minute

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How fast your profit is increasing as you sell more widgets, measured in dollars per widget, at the time you sell Widget #100. So, roughly the profit earned from the sale of the 101st widget.



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Suppose  $h(t)$  gives the height of a rocket  $t$  seconds after liftoff. What is the interpretation of  $h'(t)$ ?







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The rate (measured in molecules per second) at which the number of molecules of a certain type is changing. Roughly, how many molecules of that type are being added (or taken away, if negative) per second at time  $t$ .



## Interpreting the Derivative

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Suppose  $G(w)$  gives the diameter in millimetres of steel wire needed to safely support a load of  $w$  kg. Suppose further that  $G'(100) = 0.01$ . How do you interpret  $G'(100) = 0.01$ ?





A paper<sup>1</sup> on the impacts of various factors in average life expectancy contains the following:

*The only statistically significant variable in the model is physician density. The coefficient for this variable 20.67 indicating that a one unit increase in physician density leads to a 20.67 unit increase in life expectancy. This variable is also statistically significant at the 1% level demonstrating that this variable is very strongly and positively correlated with quality of healthcare received. This denotes that access to healthcare is very impactful in terms of increasing the quality of health in the country.*

---

<sup>1</sup>Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami, *The Effect of National Healthcare Expenditure on Life Expectancy*, page 12.

Remark: physician density is measured as number of doctors per 1000 members of the population.

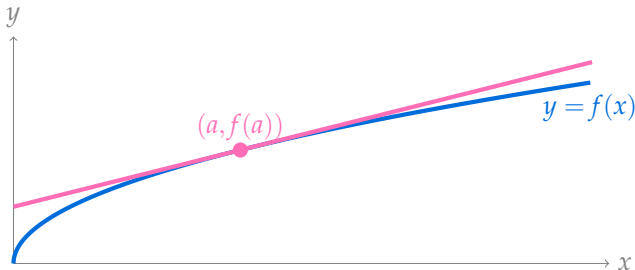
If  $L(p)$  is the average life expectancy in an area with a density  $p$  of physicians, write the statement as a derivative: “a one unit increase in physician density leads to a 20.67 unit increase in life expectancy.”

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$$L'(p) = 20.67$$

# EQUATION OF THE TANGENT LINE

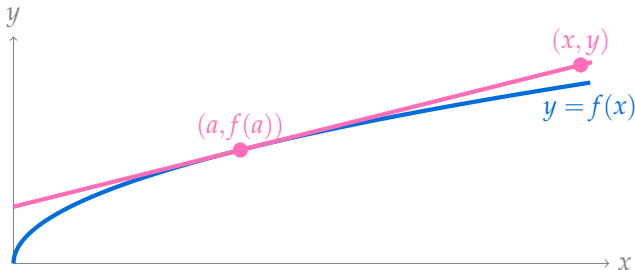
The **tangent line** to  $f(x)$  at  $a$  has slope  $f'(a)$  and passes through the point  $(a, f(a))$ .





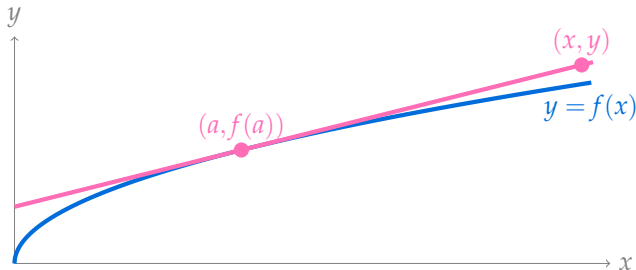
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Slope of the tangent line:

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Rearranging:  $y - f(a) = f'(a)(x - a)$  (equation of tangent line)

## Tangent Line Equation – Theorem 2.3.2

The tangent line to the function  $f(x)$  at point  $a$  is:

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$$a = 9, \quad f(a) = 3, \quad f'(a) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$y - 3 = \frac{1}{6}(x - 9)$$





Now  
You



Let  $s(t) = 3 - 0.8t^2$ . Then  $s'(t) = -1.6t$ . Find the equation for the tangent line to the function  $s(t)$  when  $t = 1$ .

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$$a = 1, \quad s(a) = 2.2, \quad s'(a) = -1.6$$

$$y - 2.2 = -1.6(x - 1)$$

## Included Work



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Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami. (2014). The Effect of National Healthcare Expenditure on Life Expectancy, page 12. *College of Liberal Arts - Ivan Allen College (IAC), School of Economics: Econometric Analysis Undergraduate Research Papers*. <https://smartech.gatech.edu/handle/1853/51648> (accessed July 2021), 189