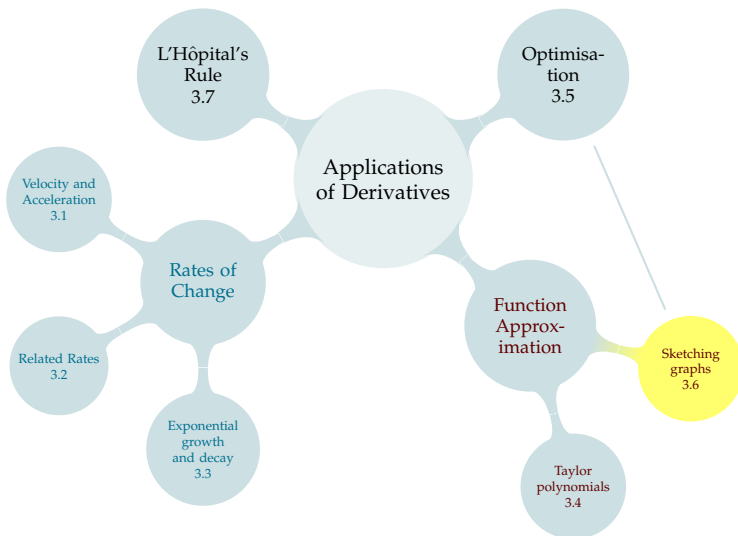


# TABLE OF CONTENTS



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Where might you expect  $f(x)$  to have a vertical asymptote? What does the function look like nearby?

(Recall: a vertical asymptote occurs at  $x = a$  if the function has an infinite discontinuity at  $a$ . That is,  $\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$ .)

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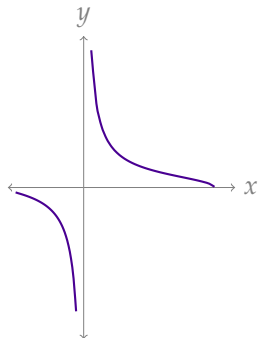
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Where is  $f(x) = 0$ ?

What happens to  $f(x)$  near its other endpoint,  $x = -1$ ?





# CURVE SKETCHING

Good things to check:

- Domain
- Vertical asymptotes:  $\lim_{x \rightarrow a} f(x) = \pm\infty$
- Intercepts:  $x = 0, f(x) = 0$
- Horizontal asymptotes and end behavior:  $\lim_{x \rightarrow \pm\infty} f(x)$



# CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

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# CURVE SKETCHING

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$$f(x) = \frac{x-2}{(x+3)^2}$$

- ▶ Domain:  $x \neq -3$
- ▶ Vertical asymptote:  $x = -3$
- ▶ Intercepts:  $(2, 0)$ ,  $(0, -\frac{2}{9})$
- ▶ Horizontal asymptote:  $y = 0$  in both directions

<https://www.desmos.com/calculator/hyzl5cyq7i>

# CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

# CURVE SKETCHING

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- ▶ Domain:  $x \neq 0, 5$
- ▶ Vertical asymptotes:  $x = 0, x = 5$
- ▶ Intercepts:  $(-2, 0), (3, 0)$
- ▶ Horizontal asymptote: none

<https://www.desmos.com/calculator/ploa0q7bxn>

# FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

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Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

**Domain**            all real numbers

**Intercepts**        Factor  $f(x) = x^2(\frac{1}{2}x^2 - \frac{4}{3}x - 15)$ . Intercept at the origin; use the quadratic formula to find  $x$ -intercepts at  $x = \frac{4 \pm \sqrt{286}}{3}$ , so  $x \approx 7$  and  $x \approx -4.3$ .

**End  
behaviour**         $\lim_{x \rightarrow \infty} f(x) = \infty$     and     $\lim_{x \rightarrow -\infty} f(x) = \infty$

This is the information we've already talked about gathering. Now let's add the first derivative.

# FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

$$\begin{aligned} f'(x) &= 2x^3 - 4x^2 - 30x \\ &= 2x(x^2 - 2x - 15) \\ &= 2x(x - 5)(x + 3) \end{aligned}$$

So, critical points are  $x = 0$ ,  $x = -3$ , and  $x = 5$ . No singular points. At the critical points  $f(0) = 0$ ,  $f(-3) = -58.5$ ,  $f(5) = -229.1\bar{6}$ .

$x \approx -4.3$	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$0 < x < 5$	$x = 5$	$x > 5$	$x \approx 7$
$f(x) = 0$	$f' < 0$	CP	$f' > 0$	CP	$f' < 0$	CP	$f' > 0$	$f(x) = 0$
intercept	decr	loc min	incr	loc max	decr	loc min	incr	intercept

This gives us enough information to draw a skeleton.

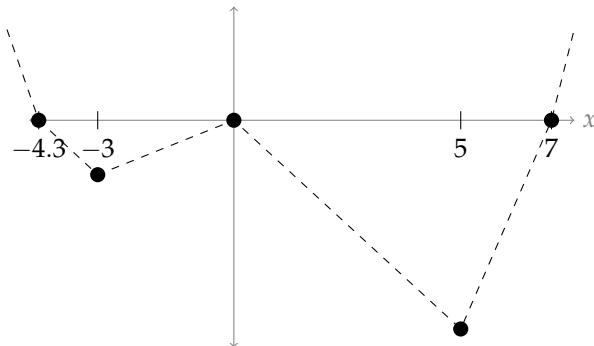




# FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



<https://www.desmos.com/calculator/lxdlgmhns1>

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

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$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

**Domain**                      all real numbers.

**End  
behaviour**                       $\lim_{x \rightarrow -\infty} f(x) = -\infty$     and     $\lim_{x \rightarrow \infty} f(x) = \infty$

**Intercepts**

$$f(0) = 24$$

$$\begin{aligned} f(x) &= \frac{1}{3}x^2(x+6) + 4(x+6) \\ &= \left(\frac{1}{3}x^2 + 4\right)(x+6) \end{aligned}$$

Only two intercepts:  $(0, 24)$  and  $(-6, 0)$ .

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

**Critical  
points**

$$f'(x) = x^2 + 4x + 4 = (x + 2)^2 \quad \text{so } x = -2 \text{ is the location of the only critical point.}$$
$$f(-2) = 24 - \frac{8}{3} = 21 + \frac{1}{3}.$$

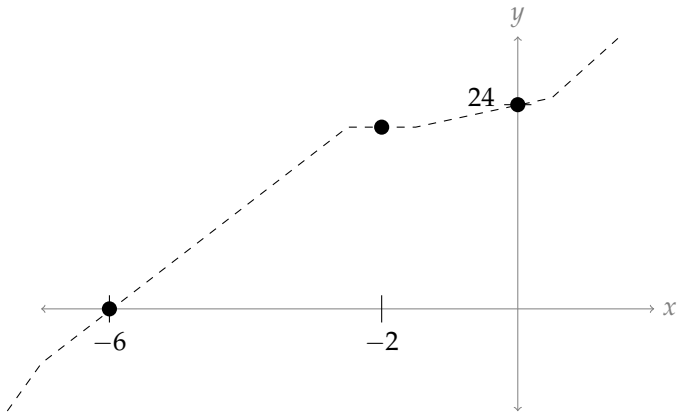
**Increasing,  
decreasing**

$$f'(x) > 0 \text{ except at } x = -2, \text{ so apart from the critical point, } f(x) \text{ is increasing}$$

This is enough for us to draw a skeleton.

What does the graph of the following function look like?

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$$f(x) = e^{\frac{x+1}{x-1}}$$

**Domain**  $x \neq 1$

We need to consider what happens as  $x$  approaches 1 from the left and the right.

**Vertical asymptotes**

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty \implies \lim_{x \rightarrow 1^-} f(x) = \lim_{A \rightarrow -\infty} e^A = 0$$

$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \infty \implies \lim_{x \rightarrow 1^+} f(x) = \lim_{A \rightarrow +\infty} e^A = \infty$$

**End behaviour**

$$\lim_{x \rightarrow \pm\infty} \frac{x+1}{x-1} = 1, \text{ so } \lim_{x \rightarrow \pm\infty} f(x) = e$$

**Intercepts**  $f(0) = \frac{1}{e}$ ; there are no roots



What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

**Critical  
points**

$$f'(x) = e^{\frac{x+1}{x-1}} \left( \frac{-2}{(x-1)^2} \right) \quad \text{no critical points}$$

**Increasing,  
decreasing**

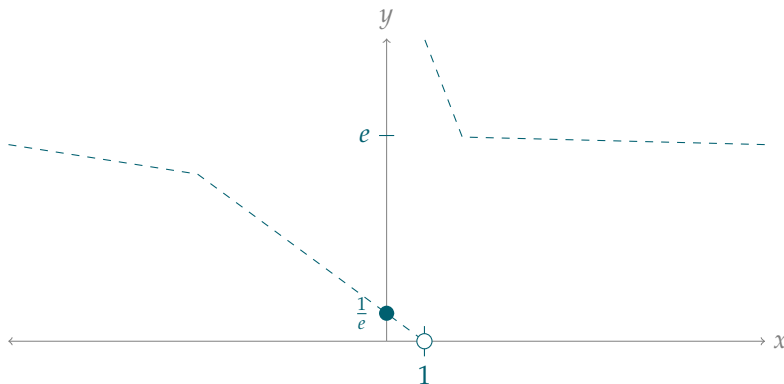
$f(x)$  is decreasing everywhere it is defined

This information is enough to draw a skeleton.



What does the graph of the following function look like?

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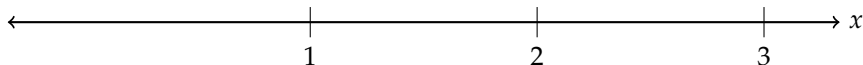
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# SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

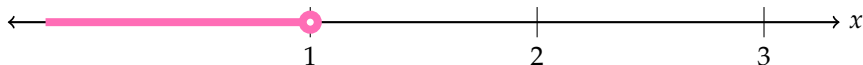
$$f(x) = (x - 1)(x - 2)(x - 3)$$



# SIGNS OF FACTORED FUNCTIONS

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$$f(x) = \underbrace{(x-1)}_{-} (x-2) (x-3)$$



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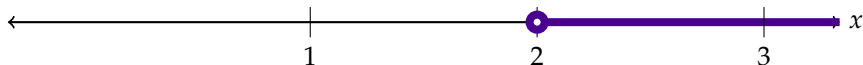


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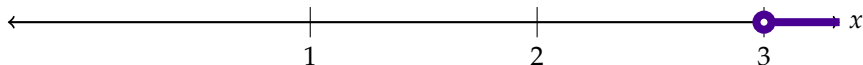
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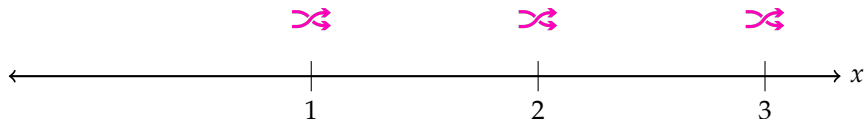




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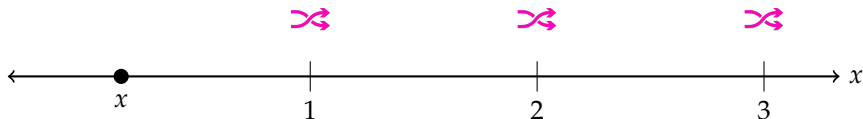


Sign of entire function:

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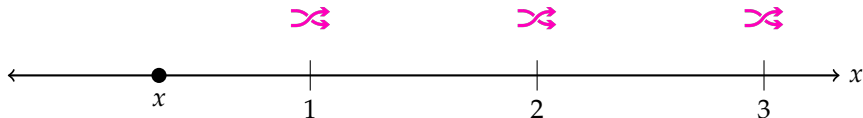
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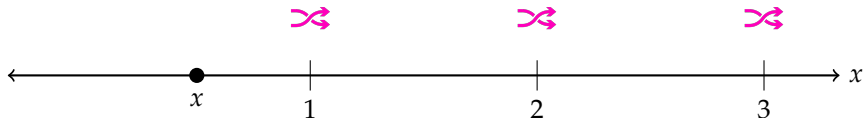
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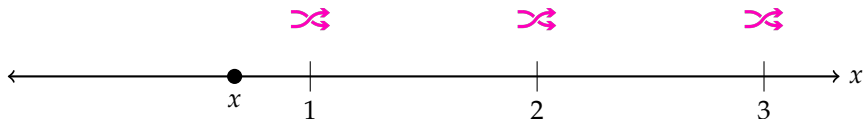
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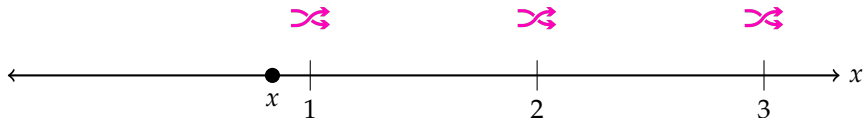
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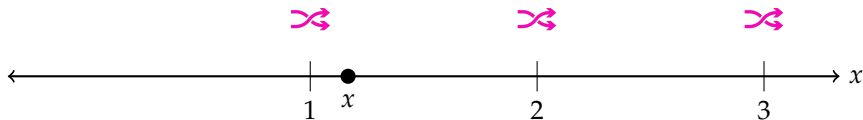
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▶ SKIP SIGN CHANGES

$$f(x) = \begin{array}{ccc} (x-1) & (x-2) & (x-3) \\ + & - & - \end{array}$$



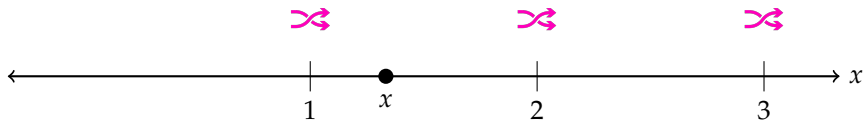
Sign of entire function:

+

# SIGNS OF FACTORED FUNCTIONS

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$$f(x) = \begin{matrix} (x-1) & (x-2) & (x-3) \\ + & - & - \end{matrix}$$



Sign of entire function:

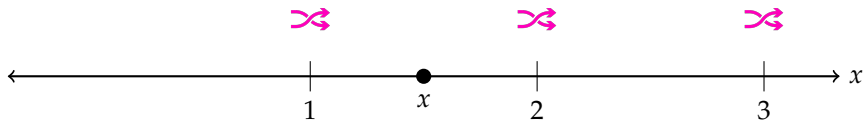
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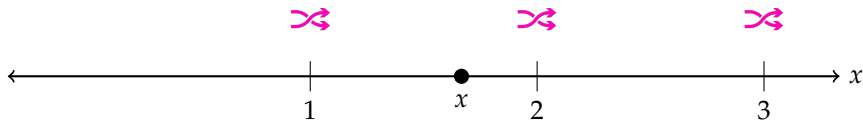
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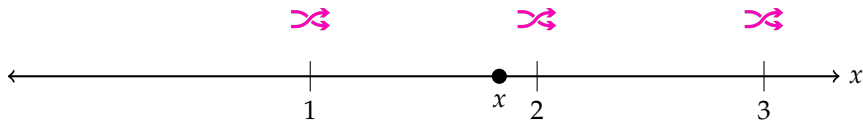
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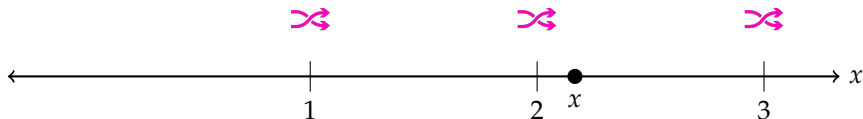
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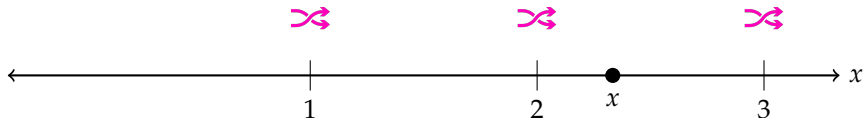
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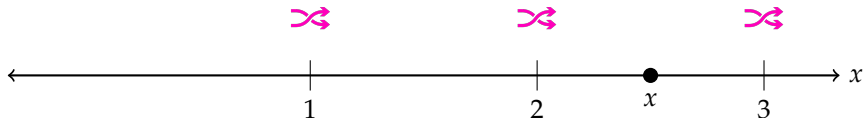
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$$f(x) = \begin{matrix} (x-1) & (x-2) & (x-3) \\ + & + & - \end{matrix}$$



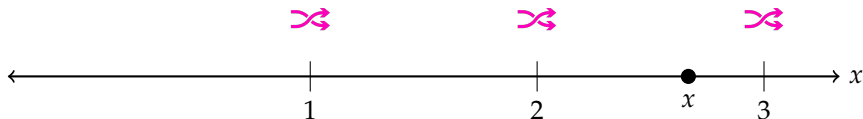
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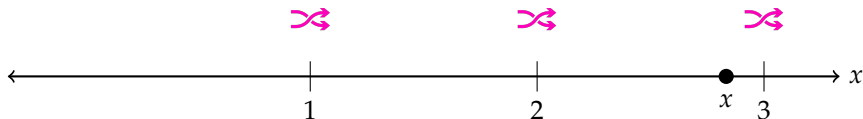
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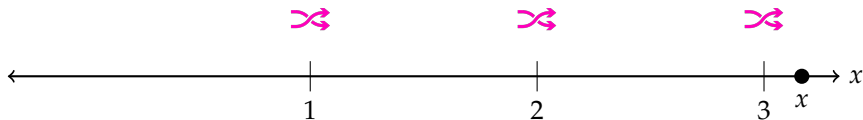
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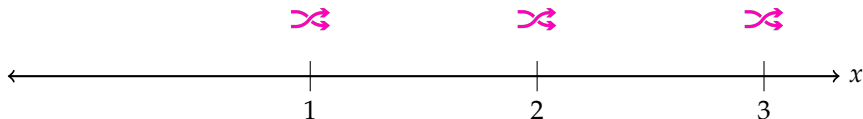
Sign of entire function:

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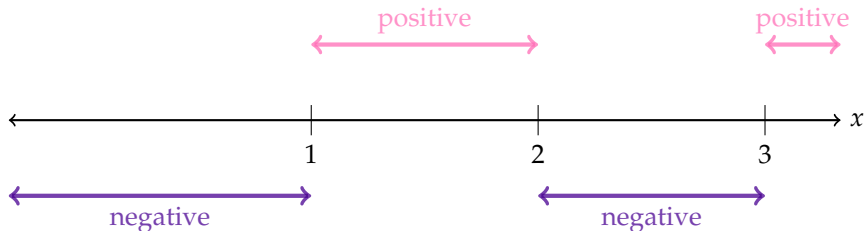


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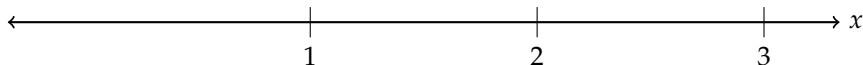


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[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 1) (x - 2)^2 (x - 3)$$

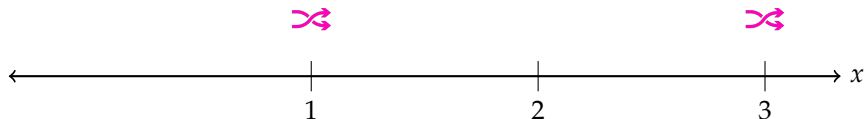


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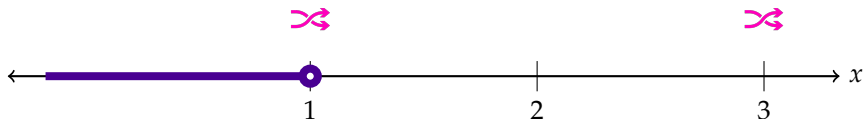


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$$f(x) = \underbrace{(x-1)}_{-} \underbrace{(x-2)^2}_{+} \underbrace{(x-3)}_{-}$$



Sign of entire function:

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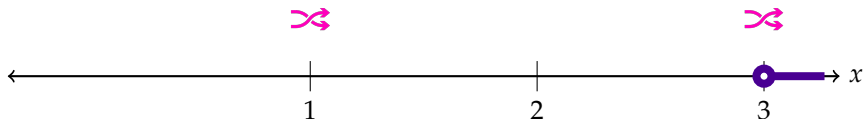


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Sign of entire function:

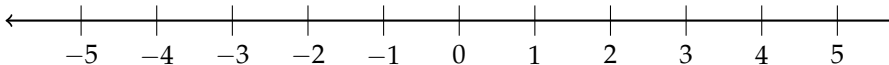


# SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

Where is  $f(x)$  positive? Where is it negative?

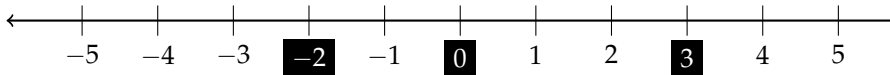


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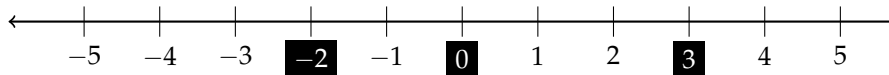


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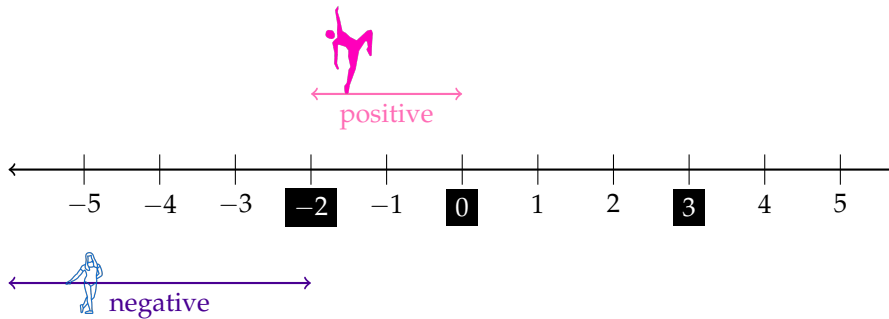


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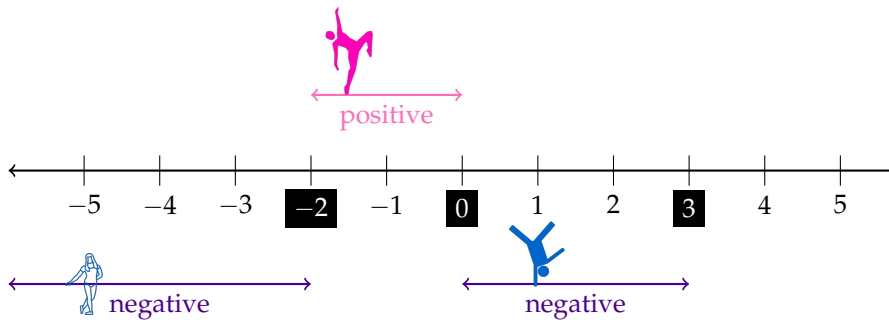


# SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

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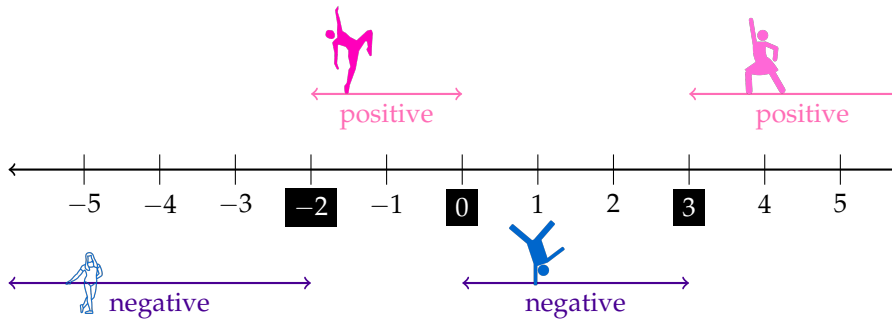


# SIGNS OF FACTORED FUNCTIONS

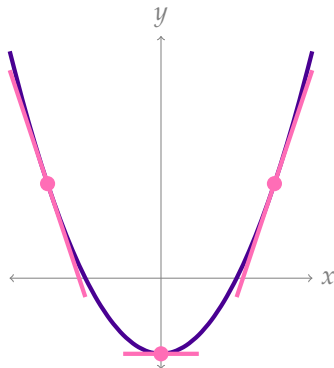
[▶ SKIP SIGN CHANGES](#)

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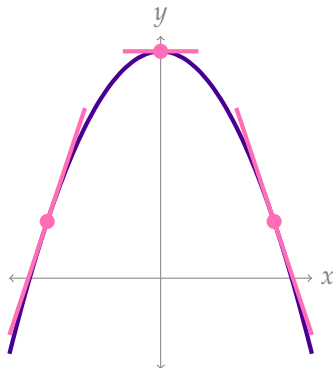
Where is  $f(x)$  positive? Where is it negative?



# CONCAVITY



- ▶ Slopes are increasing
- ▶  $f''(x) > 0$
- ▶ “concave up”
- ▶ tangent line below curve



- ▶ Slopes are decreasing
- ▶  $f''(x) < 0$
- ▶ “concave down”
- ▶ tangent line above curve

3.6.1: Domain, Intercepts, Asymptotes  
○○○○○

3.6.2: First Derivative  
○○○○○

3.6.3: Concavity  
○●○○○○○

3.6.4 : Symmetries  
○○○○○○○○○○○○○○○○○○○

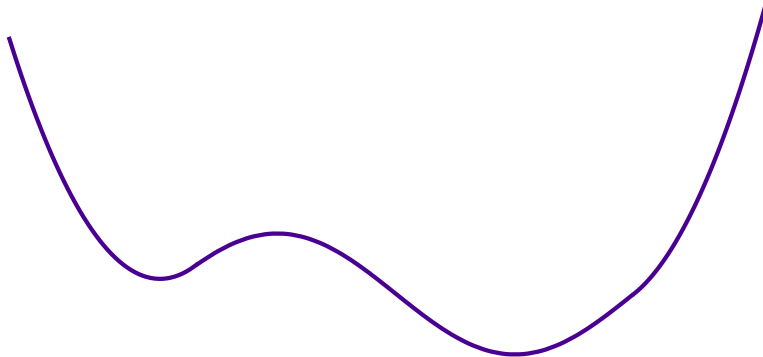
3.6.5: A checklist for sketching  
○○○○○○○○○○○○○○○

# MNEMONIC

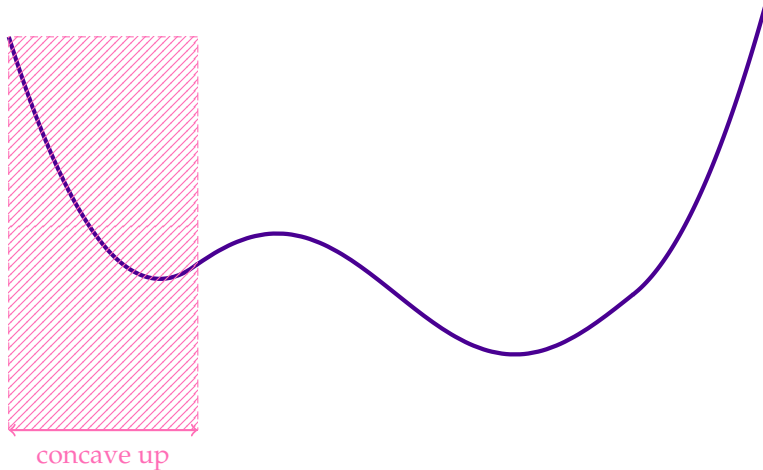




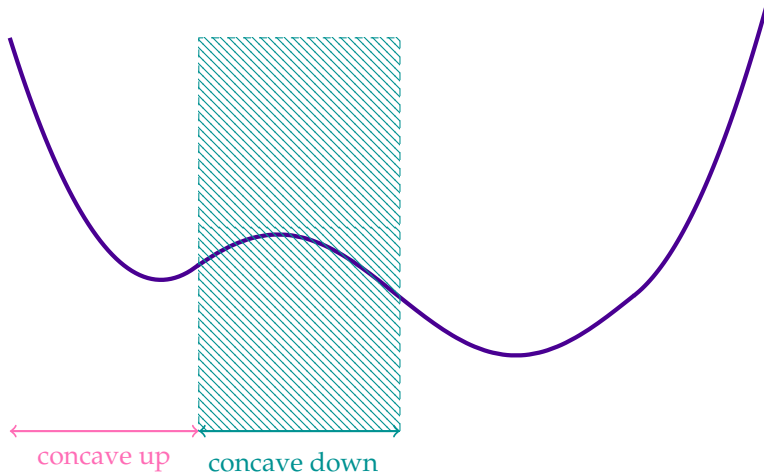
# CONCAVITY



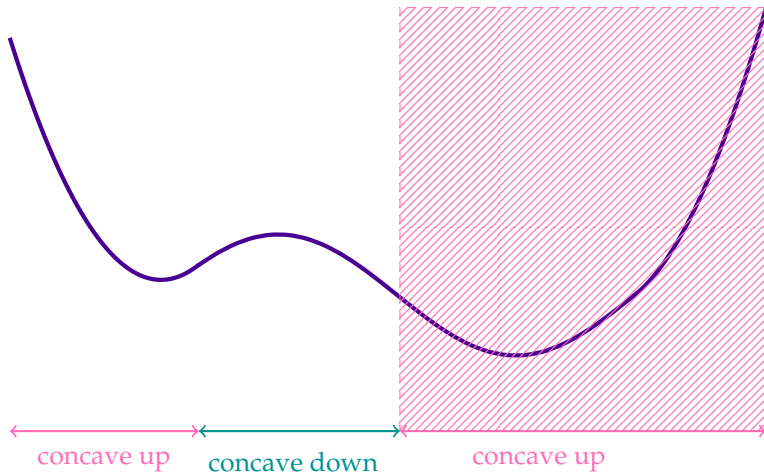
# CONCAVITY



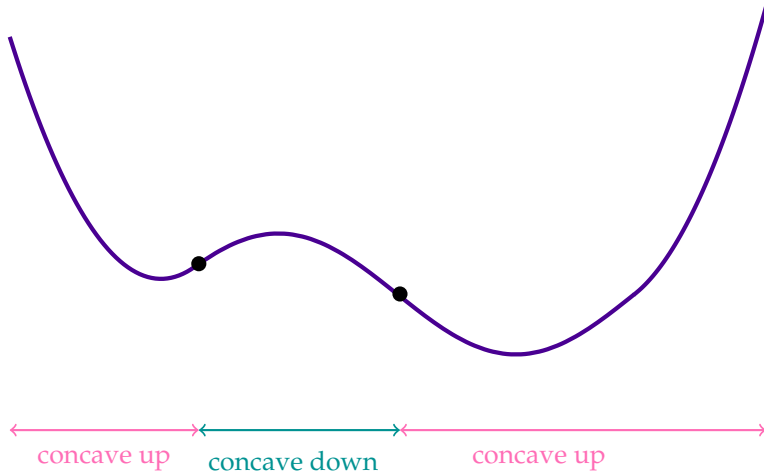
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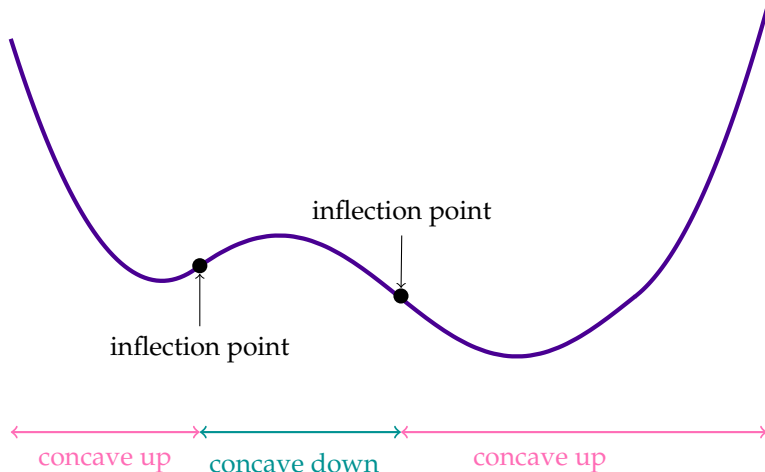
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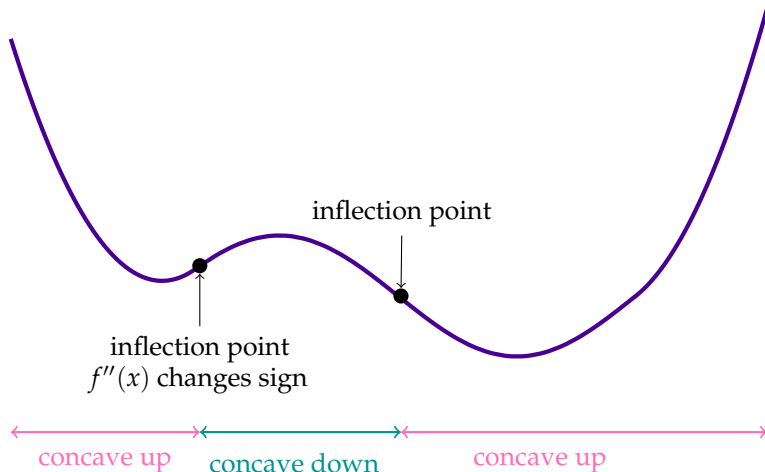
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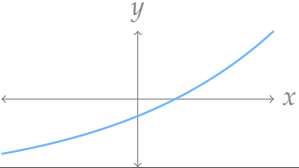
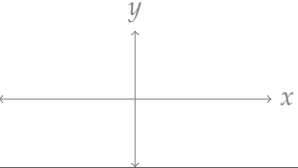
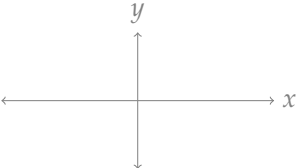
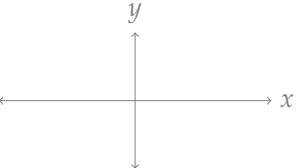


Sketch graphs with the following properties, or explain that none exist.

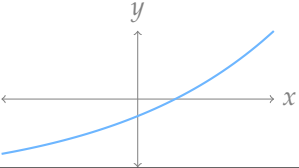
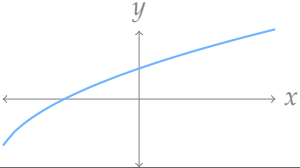
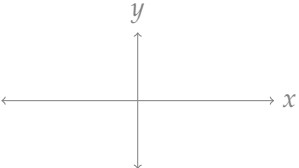
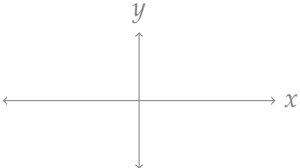
	concave up	concave down
increasing		
decreasing		



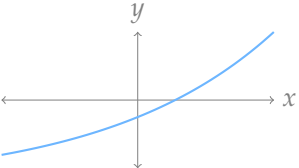
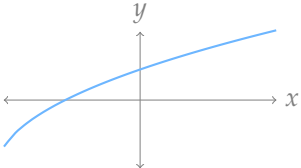
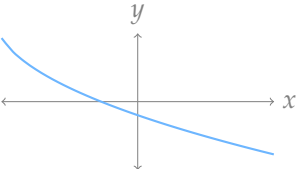
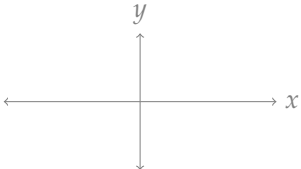
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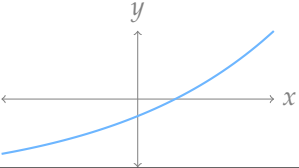
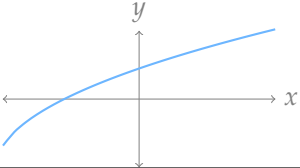
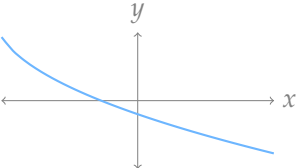
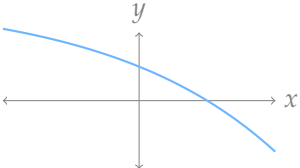
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# POLL QUESTIONS

Describe the concavity of the function  $f(x) = e^x$ .

- A. concave up
- B. concave down
- C. concave up for  $x < 0$ ; concave down for  $x > 0$
- D. concave down for  $x < 0$ ; concave up for  $x > 0$
- E. I'm not sure

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Is it possible to be concave up and decreasing?

A. Yes

B. No

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Suppose a function  $f(x)$  is defined for all real numbers, and is concave up on the interval  $[0, 1]$ . Which of the following must be true?

- A.  $f'(0) < f'(1)$
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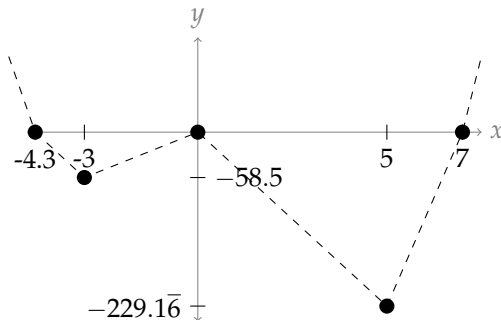
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# REVISITING A PREVIOUS EXAMPLE

◀ original example

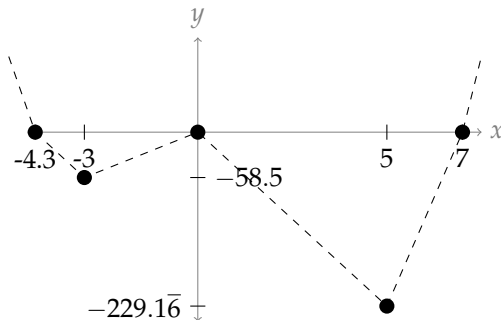
$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



# REVISITING A PREVIOUS EXAMPLE

[◀ original example](#)

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

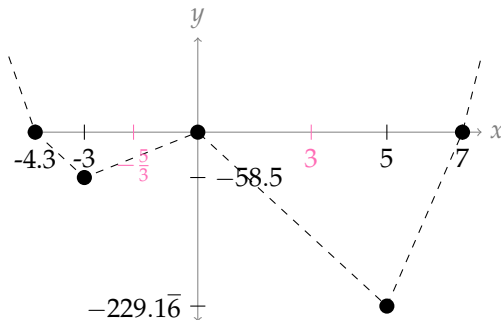


$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

# REVISITING A PREVIOUS EXAMPLE

◀ original example

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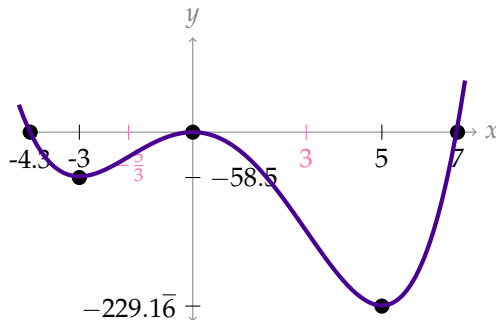


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$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

Sketch:

$$f(x) = x^5 - 15x^3$$

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<b>Domain</b>	Defined and differentiable for all real numbers.
<b>Intercepts</b>	$f(x) = x^3(x^2 - 15)$ : Roots are at $x = 0$ and $x = \pm\sqrt{15} \approx \pm 4$
<b>End behaviour</b>	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$
<b>Critical points</b>	$f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9)$ . So the critical points are $x = 0, x = \pm 3$ .
<b>Increasing, decreasing</b>	Increasing on $(-\infty, -3)$ , decreasing on $(-3, 0)$ and $(0, 3)$ , increasing on $(3, \infty)$
<b>Local extrema</b>	From intervals of increase and decrease: local max at $x = -3$ and local min at $x = 3$



Sketch:

$$f(x) = x^5 - 15x^3$$

**Concavity**

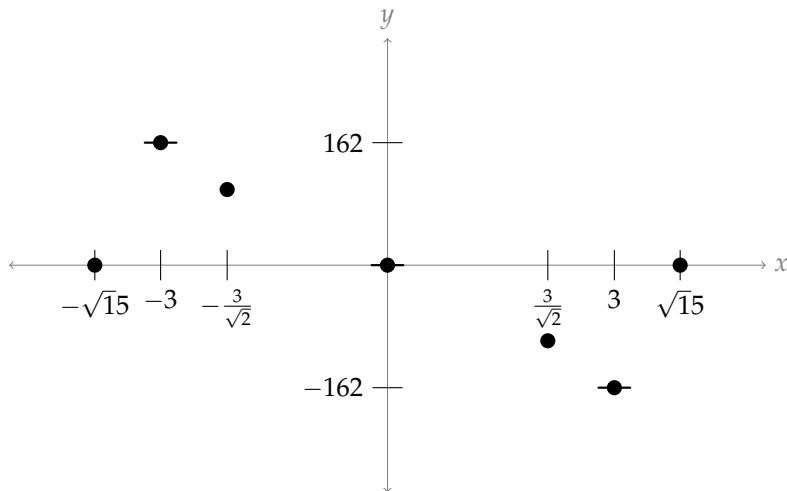
$f''(x) = 20x^3 - 90x = 10x(2x^2 - 9) = 0$  for  $x = 0$  and  $x = \pm \frac{3}{\sqrt{2}} \approx \pm 2.12$ . All of these are inflection points; concave down  $(-\infty, -\frac{3}{\sqrt{2}})$ , concave up  $(-\frac{3}{\sqrt{2}}, 0)$ , concave down  $(0, \frac{3}{\sqrt{2}})$ , and concave up  $(\frac{3}{\sqrt{2}}, \infty)$ .

**y-values of notable points**

$f(3) = -162, f(-3) = 162, f(-3/\sqrt{2}) \approx 100, f(3/\sqrt{2}) \approx -100$

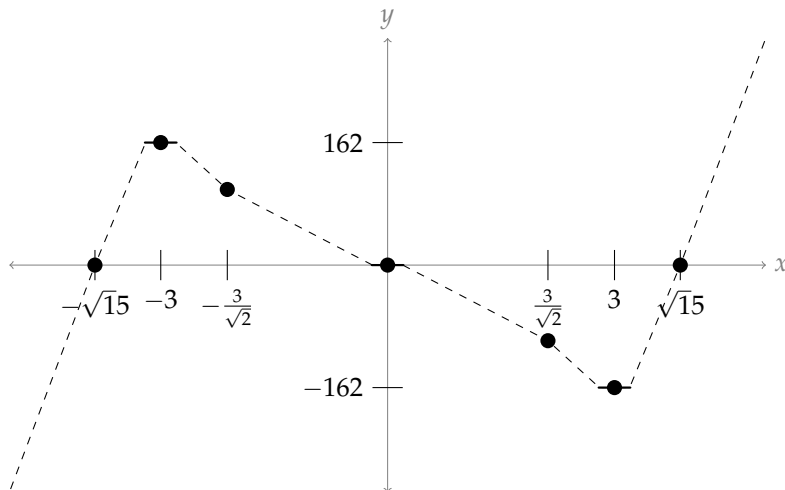
Sketch:

$$f(x) = x^5 - 15x^3$$



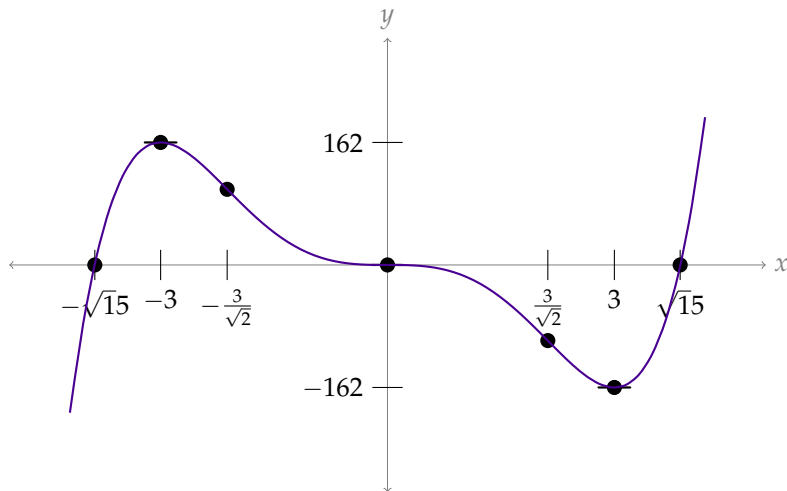
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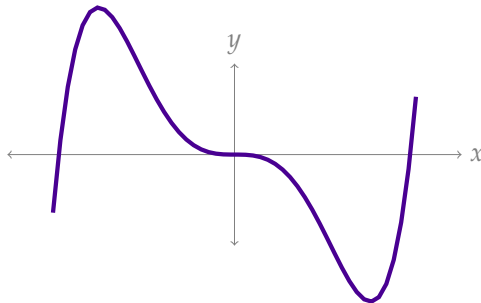
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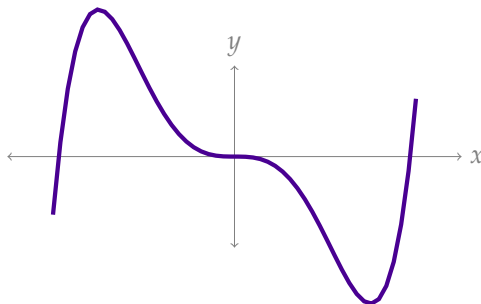
<https://www.desmos.com/calculator/uoi6nmgr8>

# EVEN AND ODD FUNCTIONS



$$f(x) = x^5 - 15x^3$$

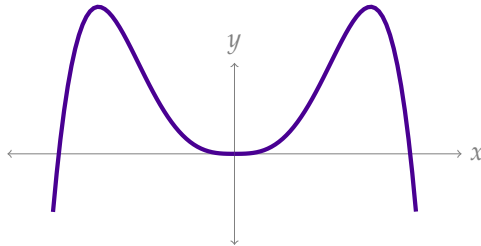
# EVEN AND ODD FUNCTIONS



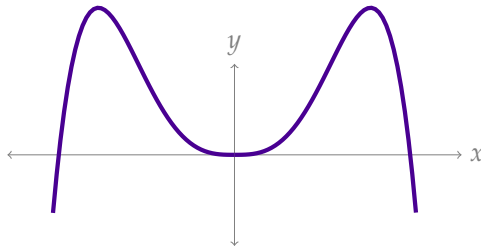
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odd function

# EVEN AND ODD FUNCTIONS



# EVEN AND ODD FUNCTIONS



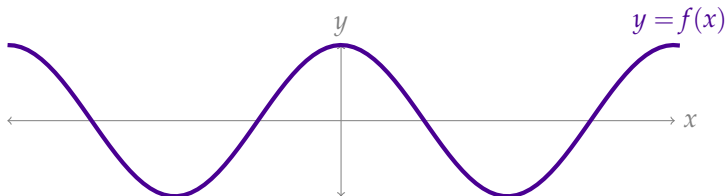
even function



## Even Function – Definition 3.6.6

A function  $f(x)$  is **even** if, for all  $x$  in its domain,

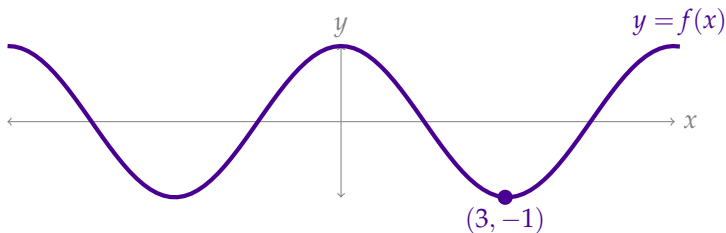
$$f(-x) = f(x)$$



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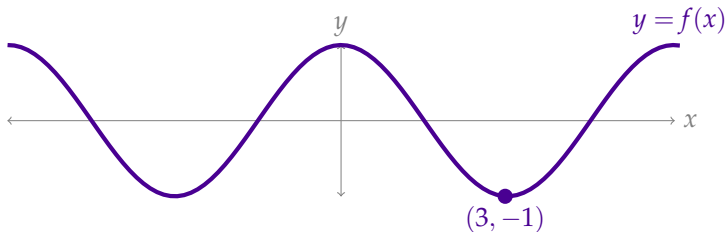


Suppose  $f(3) = -1$ .

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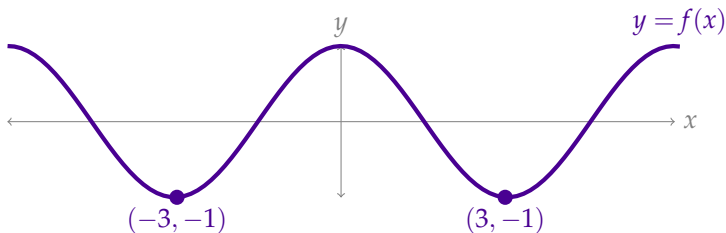


Suppose  $f(3) = -1$ . Then  $f(-3) =$

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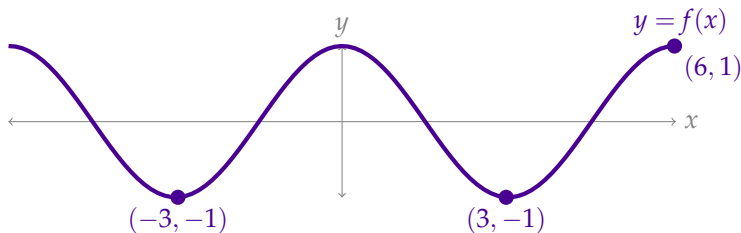


Suppose  $f(3) = -1$ . Then  $f(-3) = -1$  also.

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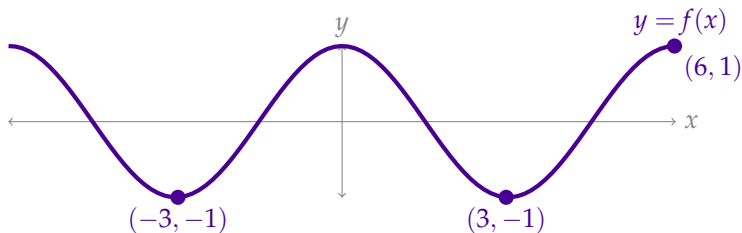
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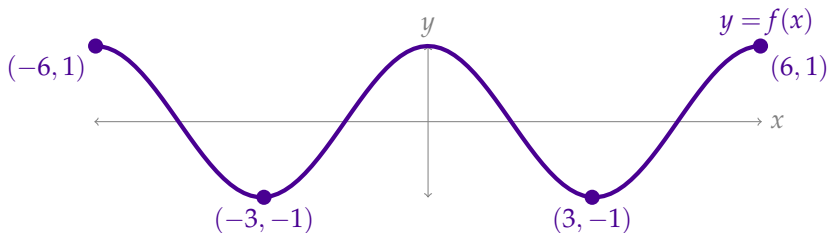
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# EVEN FUNCTIONS

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Examples:



# EVEN FUNCTIONS

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$$f(x) = x^4$$

$$f(x) = \cos(x)$$

# EVEN FUNCTIONS

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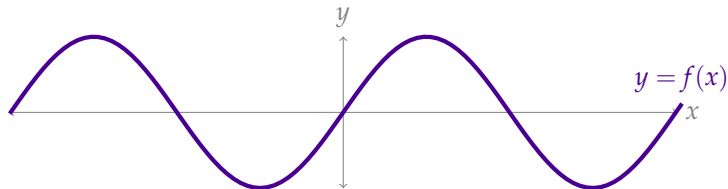
$$f(x) = x^2$$

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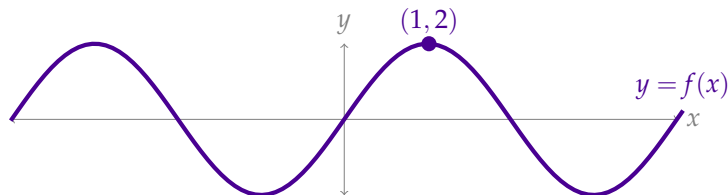
$$f(x) = \cos(x)$$

$$f(x) = \frac{x^4 + \cos(x)}{x^{16} + 7}$$

# ODD FUNCTIONS

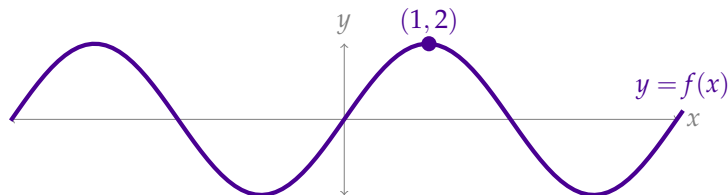


# ODD FUNCTIONS



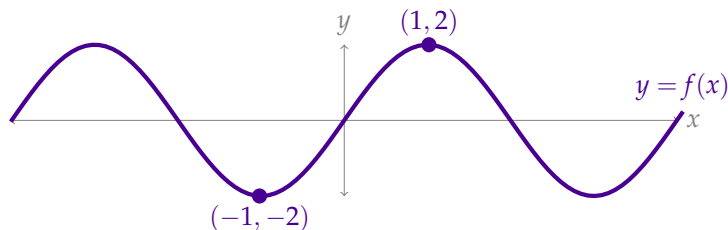
Suppose  $f(1) = 2$ .

# ODD FUNCTIONS



Suppose  $f(1) = 2$ . Then  $f(-1) =$

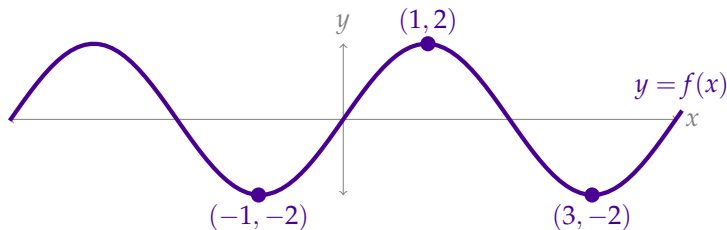
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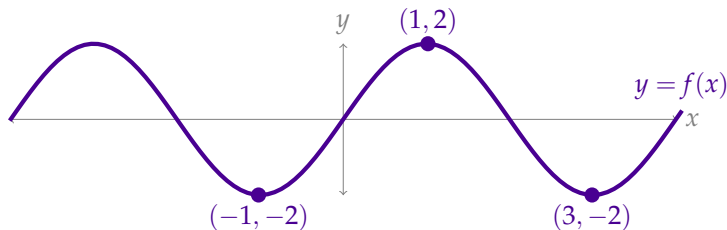
# ODD FUNCTIONS



Suppose  $f(1) = 2$ . Then  $f(-1) = -2$ .

Suppose  $f(3) = -2$ .

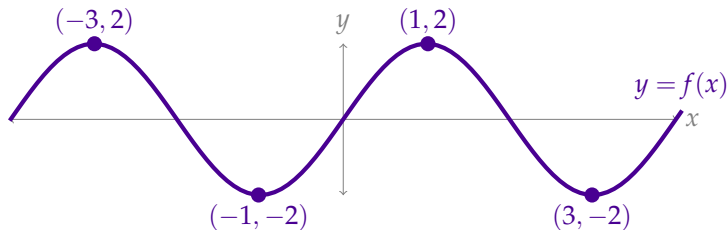
# ODD FUNCTIONS



Suppose  $f(1) = 2$ . Then  $f(-1) = -2$ .

Suppose  $f(3) = -2$ . Then  $f(-3) =$

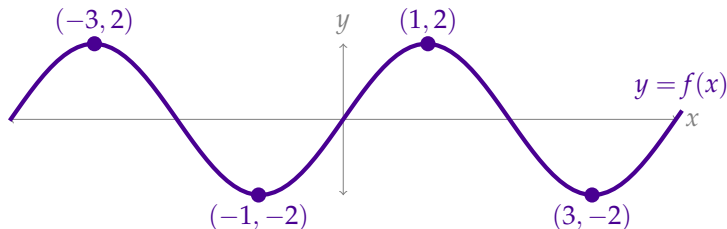
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# ODD FUNCTIONS



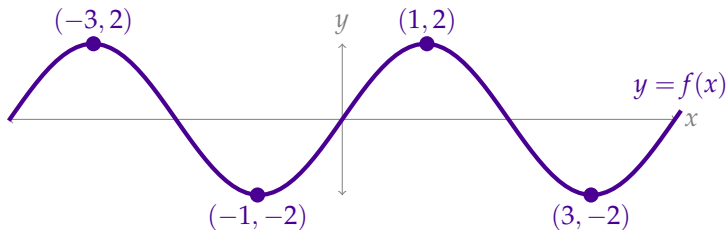
Suppose  $f(1) = 2$ . Then  $f(-1) = -2$ .

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## Odd Function – Definition 3.6.7

A function  $f(x)$  is **odd** if, for all  $x$  in its domain,

# ODD FUNCTIONS



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Examples:

# ODD FUNCTIONS

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A function  $f(x)$  is **odd** if, for all  $x$  in its domain,

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Examples:

$$f(x) = x$$

# ODD FUNCTIONS

## Odd Function – Definition 3.6.7

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Examples:

$$f(x) = x$$

$$f(x) = x^3$$



# ODD FUNCTIONS

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A function  $f(x)$  is **odd** if, for all  $x$  in its domain,

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Examples:

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$$f(x) = \sin(x)$$

# ODD FUNCTIONS

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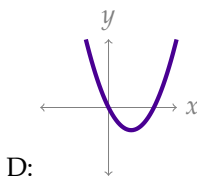
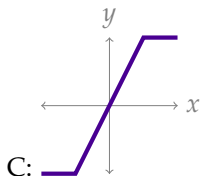
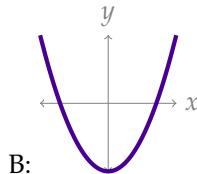
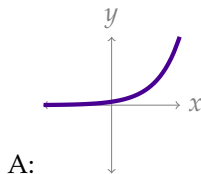
$$f(x) = x^3$$

$$f(x) = \sin(x)$$

$$f(x) = \frac{x(1 + x^2)}{x^2 + 5}$$

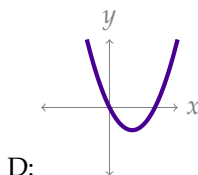
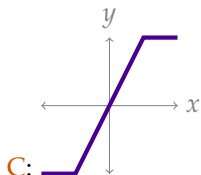
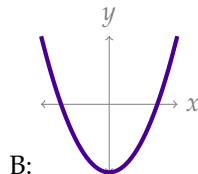
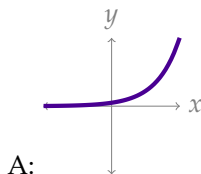
# POLL TIME

Pick out the **odd** function.



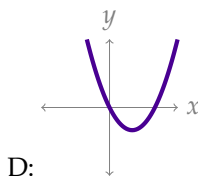
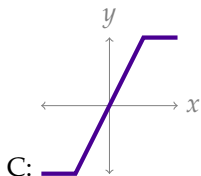
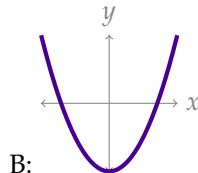
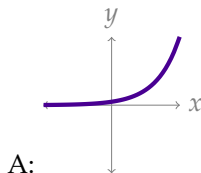
# POLL TIME

Pick out the **odd** function.



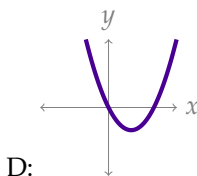
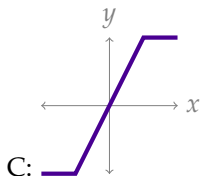
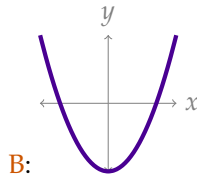
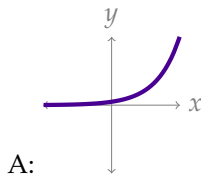
# POLL TIME

Pick out the **even** function.



# POLL TIME

Pick out the **even** function.



# EVEN MORE POLL TIIIIIME

Suppose  $f(x)$  is an **odd** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

- A.  $f(0) = f(-0)$
- B.  $f(0) = -f(0)$
- C.  $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true



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Suppose  $f(x)$  is an **odd** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

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B.  $f(0) = -f(0)$

C.  $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true





# EVEN MORE POLL TIIIIIME

Suppose  $f(x)$  is an **odd** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

- A.  $f(0) = f(-0) \leftarrow$  true but uninteresting, for all functions
- B.  $f(0) = -f(0)$
- C.  $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true

# EVEN MORE POLL TIIIIIME

Suppose  $f(x)$  is an **odd** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

- A.  $f(0) = f(-0) \leftarrow$  true but uninteresting, for all functions
- B.  $f(0) = -f(0) \leftarrow$  only possible for  $f(0) = 0$
- C.  $f(0) = 0$
- D. all of the above are true
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Suppose  $f(x)$  is an **odd** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

- A.  $f(0) = f(-0) \leftarrow$  true but uninteresting, for all functions
- B.  $f(0) = -f(0) \leftarrow$  only possible for  $f(0) = 0$
- C.  $f(0) = 0 \leftarrow$  this is equivalent to the choice above
- D. all of the above are true
- E. none of the above are necessarily true

# EVEN MORE AND MORE POLL TIIIIIME

Suppose  $f(x)$  is an **even** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

- A.  $f(0) = f(-0)$
- B.  $f(0) = -f(0)$
- C.  $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true

# EVEN MORE AND MORE POLL TIIIIIME

Suppose  $f(x)$  is an **even** function, continuous, defined for all real numbers. What is  $f(0)$ ? Pick the best answer.

- A.  $f(0) = f(-0)$
- B.  $f(0) = -f(0)$
- C.  $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true



## OK OK... LAST ONE

Suppose  $f(x)$  is an **even** function, differentiable for all real numbers.  
What can we say about  $f'(x)$ ?

- A.  $f'(x)$  is also even
- B.  $f'(x)$  is odd
- C.  $f'(x)$  is constant
- D. all of the above are true
- E. none of the above are necessarily true



## OK OK... LAST ONE

Suppose  $f(x)$  is an **even** function, differentiable for all real numbers.  
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# PERIODICITY

## Periodic – Definition 3.6.10

A function is **periodic** with period  $P > 0$  if

$$f(x) = f(x + P)$$

whenever  $x$  and  $x + P$  are in the domain of  $f$ , and  $P$  is the smallest such (positive) number

Examples:  $\sin(x)$ ,  $\cos(x)$  both have period  $2\pi$ ;  $\tan(x)$  has period  $\pi$ .



Ignoring concavity, sketch  $f(x) = \sin(\sin x)$ .

Challenge: ignoring exact locations of extrema, sketch  $g(x) = \sin(2\pi \sin x)$ .

$$f(x) = \sin(\sin x)$$

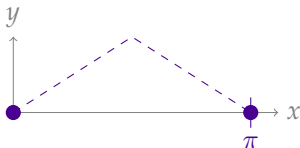
$$f(x) = \sin(\sin x)$$

Sin is periodic; since  $\sin x = \sin(2\pi + x)$ , then  $\sin(\sin x) = \sin(\sin(2\pi + x))$ , so  $f(x)$  is also periodic. It suffices to sketch  $f(x)$  for an interval of length  $2\pi$ , because any such segment will repeat.

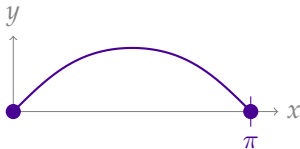
Since the function is also odd, if we sketch it on the interval  $[0, \pi]$ , then we can extrapolate to the interval  $[-\pi, 0]$ . So we consider the interval  $[0, \pi]$ .

- ▶ Intercepts:  $(0, 0), (0, \pi)$
- ▶ First derivative:  $f'(x) = \cos(\sin x) \cdot \cos(x)$ 
  - ▶ For  $0 \leq x \leq \pi$ , we have  $0 \leq \sin x \leq 1 < \frac{\pi}{2}$  and hence  $0 < \cos(\sin x) \leq 1$ .
  - ▶ CP:  $x = \frac{\pi}{2}$
  - ▶ increasing:  $(0, \pi/2)$
  - ▶ decreasing:  $(\pi/2, 0)$

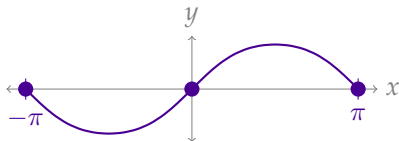
1. Interval  $(0, \pi)$  skeleton, based on above work:



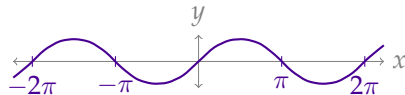
2. Make into a smooth curve:



3. Use odd symmetry to get interval  $[-\pi, \pi]$



4. Use periodicity



» skip  $g(x)$ 

$$g(x) = \sin(2\pi \sin x)$$

» skip  $g(x)$ 

$$g(x) = \sin(2\pi \sin x)$$

Sin is periodic; since  $2\pi \sin x = 2\pi \sin(2\pi + x)$ , then

$$\sin(2\pi \sin x) = \sin(2\pi \sin(2\pi + x))$$

so  $g(x)$  is also periodic. It suffices to sketch  $g(x)$  for an interval of length  $2\pi$ , because any such segment will repeat.

Note

$$\begin{aligned} g(-x) &= \sin(2\pi \sin(-x)) = \sin((2\pi)(-\sin x)) \\ &= \sin(-2\pi \sin x) = -\sin(2\pi \sin x) = -g(x) \end{aligned}$$

so  $g(x)$  is odd. If we sketch it on the interval  $[0, \pi]$ , then we can extrapolate to the interval  $[-\pi, 0]$ . So we consider the interval  $[0, \pi]$ .

» skip  $g(x)$ Intercepts in  $[0, \pi]$ :

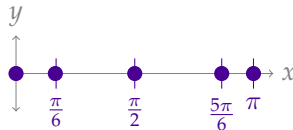
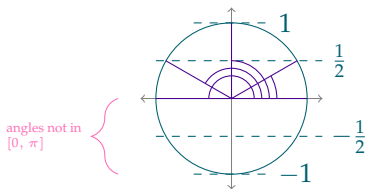
$$g(0) = 0$$

$$0 = g(0) = \sin(2\pi \sin x)$$

$$\Rightarrow 2\pi \sin x \in \{0, \pm\pi, \pm2\pi, \pm3\pi, \dots\}$$

$$\Rightarrow \sin x \in \left\{0, \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \dots\right\}$$

$$\Rightarrow x \in \left\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi\right\}$$

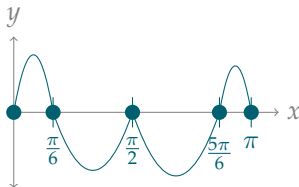


» skip  $g(x)$ 

Now let's consider the sign of  $g(x)$  between the intercepts. Since  $g(x)$  isn't given as a factored product, our old shortcut isn't so useful.

interval	$(0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{5\pi}{6})$	$(\frac{5\pi}{6}, \pi)$
range of $\sin x$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, 1)$	$(0, \frac{1}{2})$
range of $2\pi \sin x$	$(0, \pi)$	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(0, \pi)$
sign of $\sin(2\pi \sin x)$	+	−	−	+

So, a rough sketch on the interval  $[0, \pi]$  is:

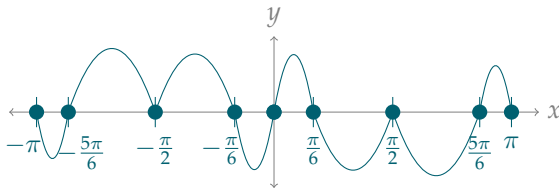


Yes, this is a rough sketch. The curve should be smooth at  $\frac{\pi}{2}$ .

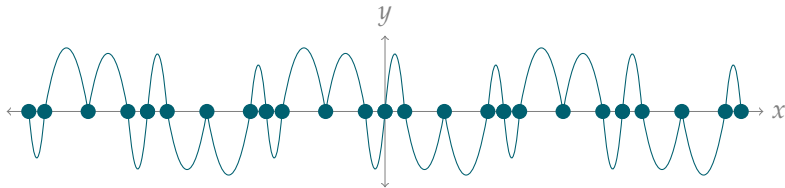


Use odd symmetry, we sketch the interval  $[-\pi, \pi]$ :

» skip  $g(x)$



Using periodicity:



# LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

# LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2 - 64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2 + 64)}{3(x^2 - 64)^{5/3}}$$

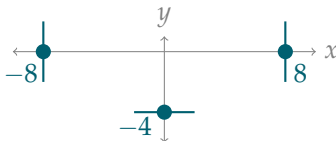
**Domain** all real numbers

**End behaviour**  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow -\infty} f(x) = \infty$

**Intercepts**  $(0, -4), (\pm 8, 0)$

**Critical point**  $(0, -4)$

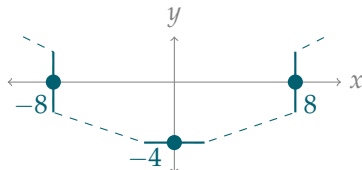
**Singular points**  $(-8, 0), (8, 0)$   
Near the singular points,  $f'(x)$  gets very large, so  $f(x)$  looks vertical.



**Increasing,  
decreasing**

Numerator of  $f'(x)$  is positive on  $(0, \infty)$  and negative on  $(-\infty, 0)$ . Denominator is positive where it exists. So  $f(x)$  is decreasing on  $(-\infty, 0)$  (where it is differentiable) and increasing on  $(0, \infty)$  (where it is differentiable).

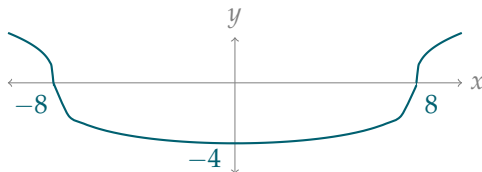
**Skeleton**



## Concavity

Numerator of second derivative is negative everywhere. Denominator is positive (so  $f''(x)$  is negative) on  $(-\infty, -8) \cup (8, \infty)$  and denominator is negative (so  $f''(x)$  is positive) on  $(-8, 8)$ .

So our function is concave up on  $(-8, 8)$  and concave down on  $(-\infty, -8) \cup (8, \infty)$ .



# LET'S GRAPH

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

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Note: for  $x \neq -1$ ,  $f(x) = \frac{x(x + 1)}{(x + 1)(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^2}$



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$$g(x) := \frac{x}{(x^2 + 1)^2}$$

$$g'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3}$$

$$g''(x) = \frac{12x(x^2 - 1)}{(x^2 + 1)^4}$$

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

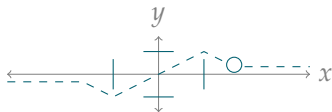
$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

When  $x \neq 1$ ,  $f(x) = g(x)$ . So,  $f(x)$  looks like  $g(x)$  except it has a removable discontinuity (hole) at  $x = 1$ . Let's graph

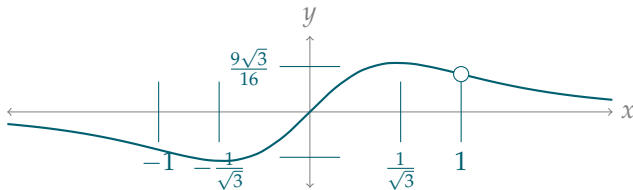
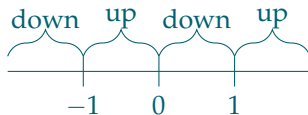
$$g(x) = \frac{x}{(x^2 + 1)^2}$$

- ▶ Domain: all real numbers
- ▶ HA:  $y = 0$  on both sides
- ▶ VA: none
- ▶ Intercepts:  $(0, 0)$
- ▶ Odd symmetry
- ▶ CP:  $x = \pm \frac{1}{\sqrt{3}}$ ; associated points:  $\left(-\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{16}\right)$  and  $\left(\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{16}\right)$

- Increasing on  $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$  and decreasing on  $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$



- $g''$  is given (nearly) factored, so we can see its sign changes at  $x = -1, 0, 1$ . Concavity:



## LET'S GRAPH

$$f(x) = x(x-1)^{2/3}$$

- $f'(x) = \frac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \frac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

## LET'S GRAPH

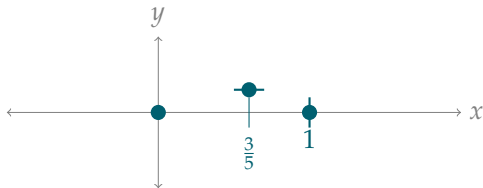
$$f(x) = x(x-1)^{2/3}$$

- $f'(x) = \frac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \frac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

►  $f(3/5) \approx 0.3$

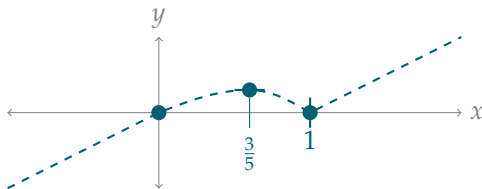
►  $f(6/5) \approx 0.4$

- ▶ Domain: all reals
- ▶ VA: none
- ▶ HA: none
- ▶ Intercepts:  $(0, 0)$ ,  $(1, 0)$
- ▶ Symmetry: not even, not odd, not periodic
- ▶ CP:  $x = \frac{3}{5}$  ( $y \approx 0.3$ ); SP  $x = 1$  ( $y = 0$ ).  
Near the SP, first deriv is very large, so function looks vertical.

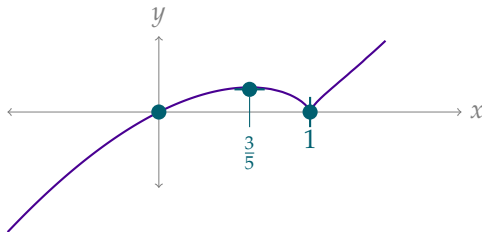




- First derivative changes sign at  $x = \frac{3}{5}$  and  $x = 1$ . Function is increasing on  $(-\infty, \frac{3}{5}) \cup (1, \infty)$  and decreasing on  $(\frac{3}{5}, 1)$ .



- Second derivative changes sign at  $x = \frac{5}{6}$ . (Note the denominator has an even power). Function is concave down on  $(-\infty, \frac{5}{6})$  and concave up on  $(\frac{5}{6}, \infty)$ .



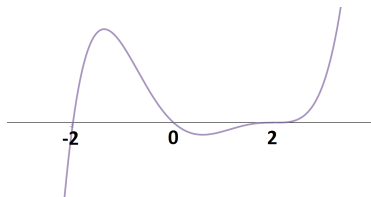
## Ch 3.6 Review: matching

# MATCH THE FUNCTION TO ITS GRAPH

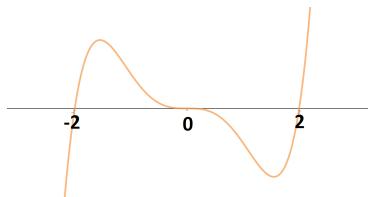
A.  $f(x) = x^3(x+2)(x-2) = x^5 - 4x^3$

B.  $f(x) = x(x+2)^3(x-2) = x^5 + 4x^4 - 16x^2 - 16x$

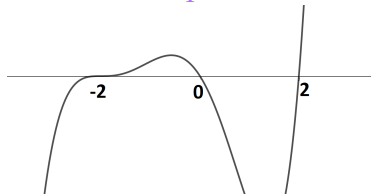
C.  $f(x) = x(x+2)(x-2)^3 = x^5 - 4x^4 + 16x^2 - 16x$



I



III



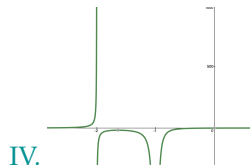
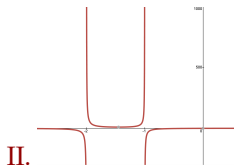
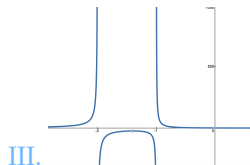
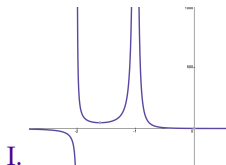
II

A.  $f(x) = \frac{x-1}{(x+1)(x+2)}$

B.  $f(x) = \frac{(x-1)^2}{(x+1)(x+2)}$

C.  $f(x) = \frac{x-1}{(x+1)^2(x+2)}$

D.  $f(x) = \frac{(x-1)^2}{(x+1)^2(x+2)}$



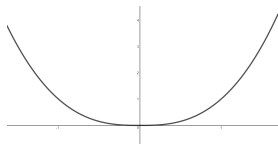
# MATCH THE FUNCTION TO ITS GRAPH

A.  $f(x) = |x|^e$

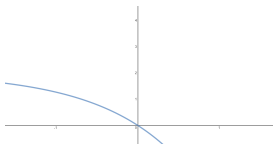
B.  $f(x) = e^{|x|}$

C.  $f(x) = e^{x^2}$

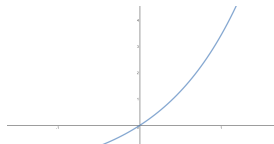
D.  $f(x) = e^{x^4-x}$



I



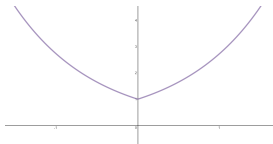
II



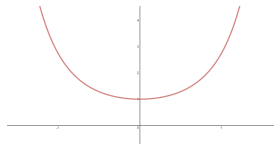
III



IV



V



VI

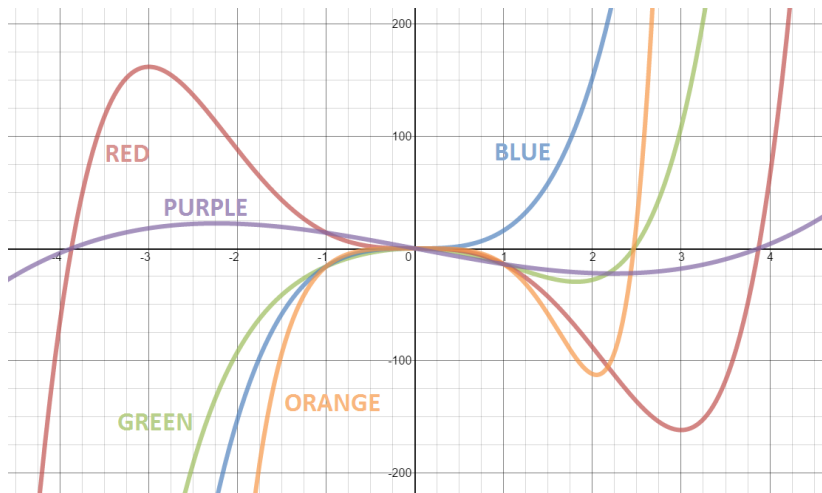
A.  $f(x) = x^5 + 15x^3$

B.  $f(x) = x^5 - 15x^3$

C.  $f(x) = x^5 - 15x^2$

D.  $f(x) = x^3 - 15x$

E.  $f(x) = x^7 - 15x^4$



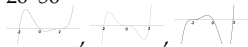
## Included Work



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screenshots of graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator> (accessed 13 November 2015), 163



screenshots of graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator>, (accessed 13 November 2015), 165



screenshots from graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator> (accessed 16 July 2021), 164



screenshot from graphs generated using Desmos Graphing Calculator  
<https://www.desmos.com/calculator>, with text added (accessed 13 November 2015), 166



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151–154