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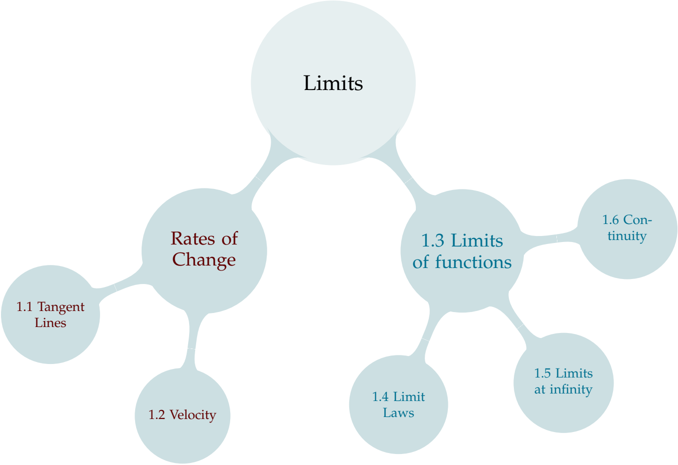
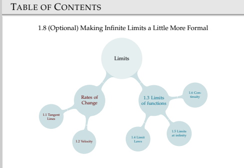
1.8 (Optional) Making Infinite Limits a Little More Formal

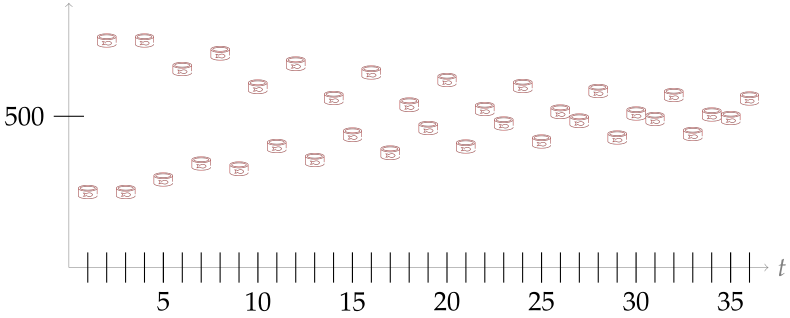




One conceptual hurdle I've noticed with students is that the epsilon-delta definitions just have a lot of variables, and it's tough to remember which is which. I've had success in the past with the model below, I think because it disambiguates the parameters. Rather than an abstract list of variable names, it gives us a story with pieces that are easy to distinguish. "The amount you put in the can," "the amount your boss wants you to put in a can," "the time you've been working," and "the error in your can weight" are easier to keep straight then "$f(x)$", "L", "x," and "ϵ".

Because limits at infinity have fewer moving parts (notably: you only have to worry about one "side), I like to teach 1.8 *before* 1.7.

You work at a salmon cannery, putting salmon into cans. Each can is supposed to contain the amount of salmon shown on the label, but some error is allowable. As you work longer, and get more experience, the amount of error you are allowed to have gets smaller.



Was there a time after which your error always *less than* 250 g?

1.8 (Optional) Making Infinite Limits a Little More Formal

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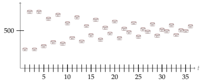
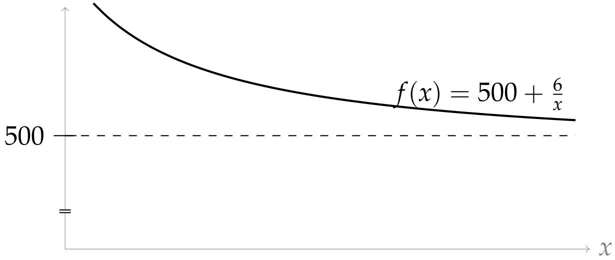


Was there a time after which your error always *less than* 250 g?

Emphasize: we don't really care exactly what that time was, we only need to know that it existsGood to point out that earlier, you were sometimes *but not always* within the specified error tolerance. Non-monotone convergence is a common conceptual sticking point.

Suppose the amount of salmon that is supposed to be in a can is 500 g. The amount of salmon you put into a can at time x is $500 + \frac{6}{x}$.

You need to reassure your boss that, after some time, your error is never more than 3 g. Find such a time.



No matter how exacting your boss is, if they give you a non-zero error allowance, you can *always* schedule a time after which you will meet their standards.

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that

the limit as x approaches ∞ of $f(x)$ is L

and write

$$\lim_{x \to \infty} f(x) = L$$

if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.
Similarly we write

$$\lim_{x \to -\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

Let f be a function defined on the whole real line.

$f(x)$: actual can weights

We say that "the limit as x approaches ∞ of $f(x)$ is L" and write

L: weight on the label that you want to match

$$\lim_{x\to\infty} f(x) = L$$

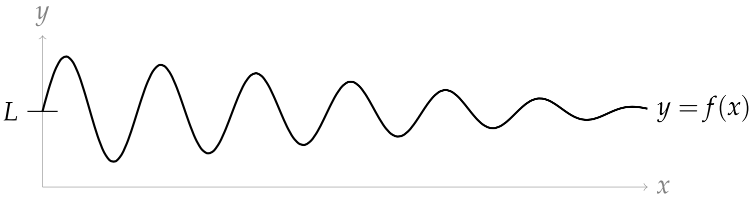
if and only if for every $\epsilon > 0$

ϵ: amount of allowable error

there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

M: time after which your weights are always off by less than ϵ

$|f(x) - L|$: error (difference between actual amount and label)



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Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2}{x} + 1\right] = 1$.

Definition 1.8.1 (a)

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Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2}{x} + 1\right] = 1$.

For easy questions, students may wonder what the use of the definition is. Point out in later examples that it clears up ambiguity when the function is less obvious.Two things to note. One, that we have to find M, and that part is separate from the actual proof we write down. Two, we get to assume x is large (so in this case, positive) because we only care about values larger than M. Remind students our cannery boss didn't care about our initial novice imprecision.

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Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[5e^{-x}\right] = 0$

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Now You

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{\sin x}{x}\right] = 0$

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Now You

Prove, using Definition 1.8.1, that $\lim\limits_{x\to\infty} \left[\dfrac{2x^2}{x^2+1}\right] = 2$

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Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Now You: Prove, using Definition 1.8.1, that $\lim_{x\to\infty} \left[\frac{2x^2}{x^2+1}\right] = 2$

For easy questions, students may wonder what the use of the definition is. Point out in later examples that it clears up ambiguity when the function is less obvious.Two things to note. One, that we have to find M, and that part is separate from the actual proof we write down. Two, we get to assume x is large (so in this case, positive) because we only care about values larger than M. Remind students our cannery boss didn't care about our initial novice imprecision.

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Prove, using Definition 1.8.1, that $\lim_{x\to\infty} 5 = 5$

Definition 1.8.1 (a)

Let f be a function defined on the whole real line. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M \in \mathbb{R}$ so that $|f(x) - L| < \epsilon$ whenever $x > M$.

Prove, using Definition 1.8.1, that $\lim_{x\to\infty} 5 = 5$

For easy questions, students may wonder what the use of the definition is. Point out in later examples that it clears up ambiguity when the function is less obvious.Two things to note. One, that we have to find M, and that part is separate from the actual proof we write down. Two, we get to assume x is large (so in this case, positive) because we only care about values larger than M. Remind students our cannery boss didn't care about our initial novice imprecision.

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True or False? $\lim_{x\to\infty} \sin x = 0$

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Definition 1.8.1 (a)

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True or False? $\lim_{x\to\infty} \sin x = 0$

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Useful General Principles

When we showed $\lim_{x\to\infty}\left[\frac{\sin x}{x}\right]=0$, we chose M using:

$$\left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right| = \frac{1}{x} < \epsilon$$

- $\left|\frac{1}{x}\right| = \frac{1}{x}$ only when x is positive. We want to show that an inequality holds for *large enough* values of x, so if it helps our cause, we can say "make sure x is larger than *blah*." Then we just choose M to be at least that number *blah*.
- If $a < b < c$, then $a < c$. So if you want to solve $a < c$, but it's too hard to find *exactly* when that's true, see whether you can replace a with a larger, easier expression b.
  That's what we did when we said $\left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$.

LIMIT AS x GOES TO NEGATIVE INFINITY

Definition 1.8.1 (a)

We write

$$\lim_{x \to -\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

Use Definition 1.8.1 to prove $\displaystyle\lim_{x \to -\infty} \frac{x^3}{x^3 + 1} = 1$

Limit as x goes to negative infinity

Definition 1.8.1 (a)

We write

$$\lim_{x \to -\infty} f(x) = K$$

if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ so that $|f(x) - K| < \epsilon$ whenever $x < N$.

Use Definition 1.8.1 to prove $\displaystyle\lim_{x \to -\infty} \frac{\cos x}{\sqrt{x^4 + x^2 + 1}} = 0$

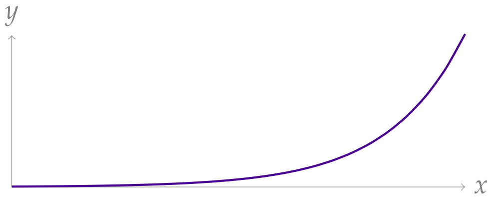
Infinite Limits

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write

$$\lim_{x\to\infty} f(x) = \infty$$

if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.



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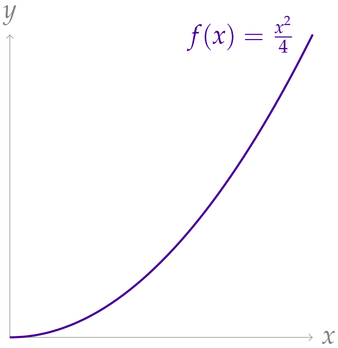

Let
$P = 1P = 2P = 1\,000\,000P > 0.$
Find $M > 0$ so that $f(x) > P$ whenever $x > M$.

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} \sqrt[3]{x} = \infty$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} x(\sin x + 1) = \infty$$

Definition 1.8.1 (c)

Let f be a function defined on the whole real line. We write $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $P > 0$ there exists $M > 0$ so that $f(x) > P$ whenever $x > M$.

Using definition 1.8.1, prove or disprove the following:

$$\lim_{x\to\infty} x(\sin x + 2) = \infty$$

Included Work

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