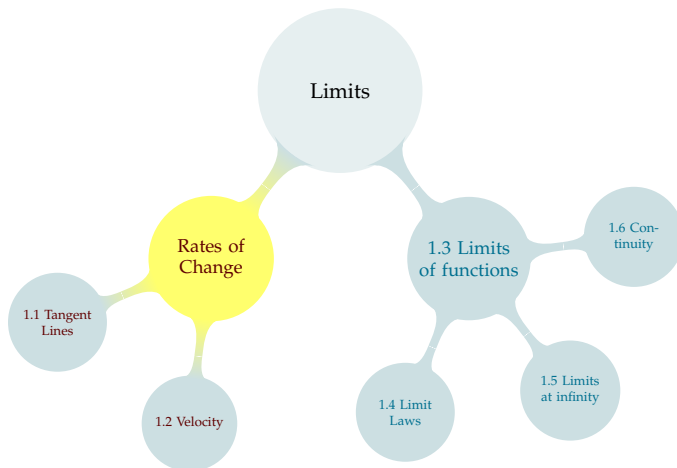
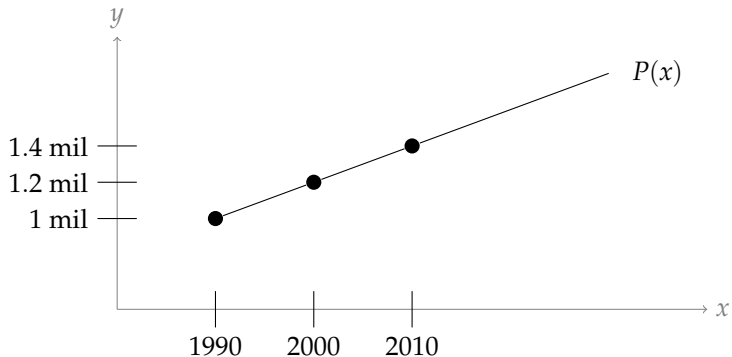


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# RATES OF CHANGE

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.



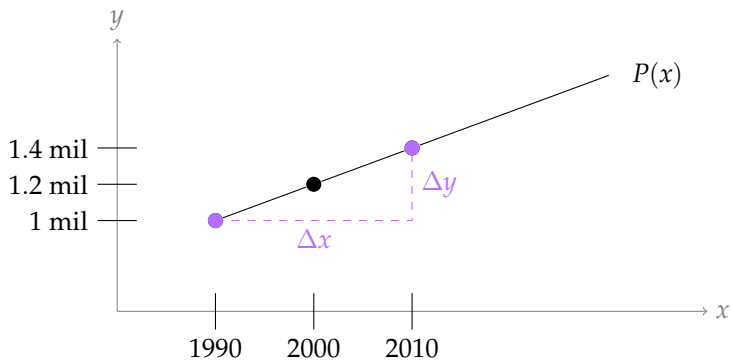
## Definition

The **slope** of a line that passes through the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is “rise over run”

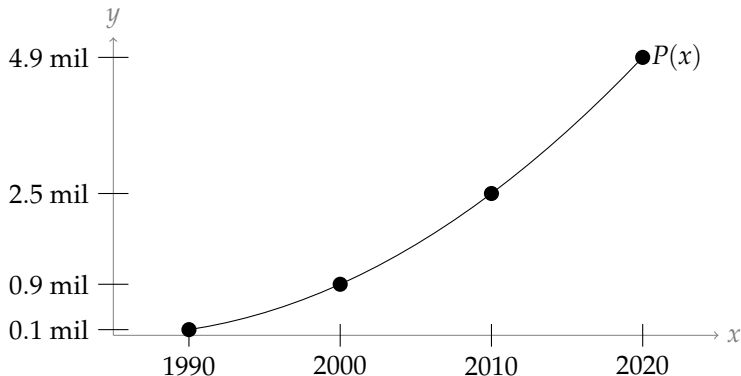
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is also called the **rate of change** of the function.

If a line has equation  $y = mx + b$ , its slope is  $m$ .



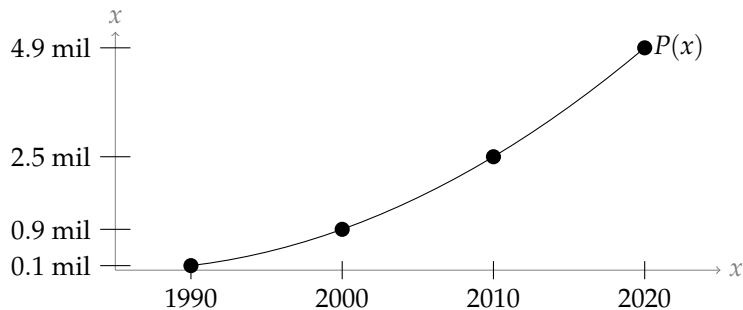
Suppose the population of a small country is given in the chart below.



## Definition

Let  $y = f(x)$  be a curve that passes through  $(x_1, y_1)$  and  $(x_2, y_2)$ . Then the **average rate of change** of  $f(x)$  when  $x_1 \leq x \leq x_2$  is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



## Average Rate of Change and Slope

The **average rate of change** of a function  $f(x)$  on the interval  $[a, b]$  (where  $a \neq b$ ) is “change in output” divided by “change in input:”

$$\frac{f(b) - f(a)}{b - a}$$

If the function  $f(x)$  is a **line**, then the slope of the line is “rise over run,”

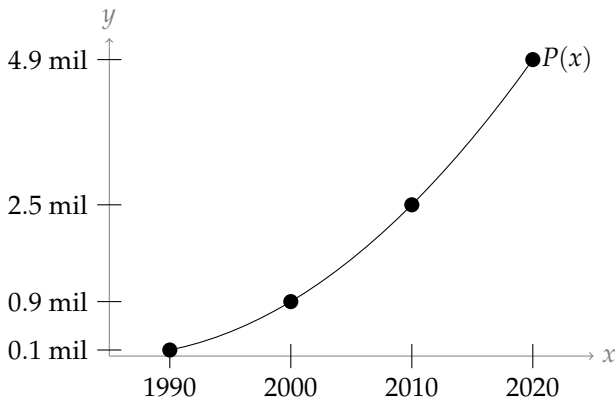
$$\frac{f(b) - f(a)}{b - a}$$



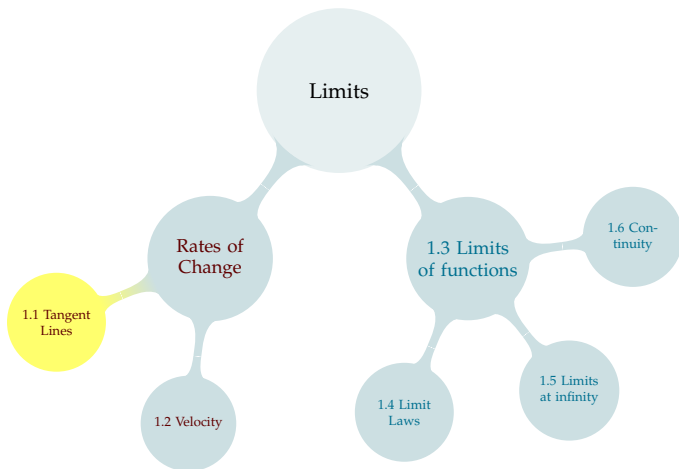
If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

How fast was this population growing in the year 2010? (What was its **instantaneous** rate of change?)



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## Definition

The **secant line** to the curve  $y = f(x)$  through points  $R$  and  $Q$  is a line that passes through  $R$  and  $Q$ .

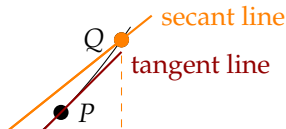
We call the slope of the secant line the **average rate of change of  $f(x)$  from  $R$  to  $Q$** .

## Definition

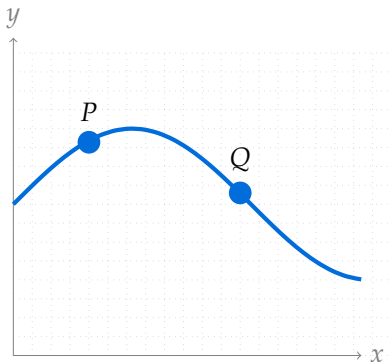
The **tangent line** to the curve  $y = f(x)$  at point  $P$  is a line that

- passes through  $P$  and
- has the same slope as  $f(x)$  at  $P$ .

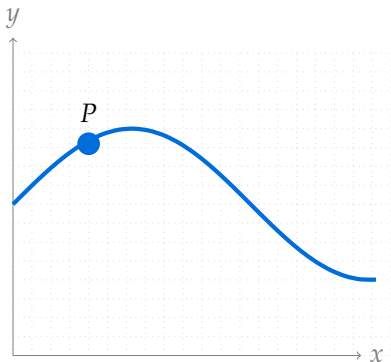
We call the slope of the tangent line the **instantaneous rate of change of  $f(x)$  at  $P$** .



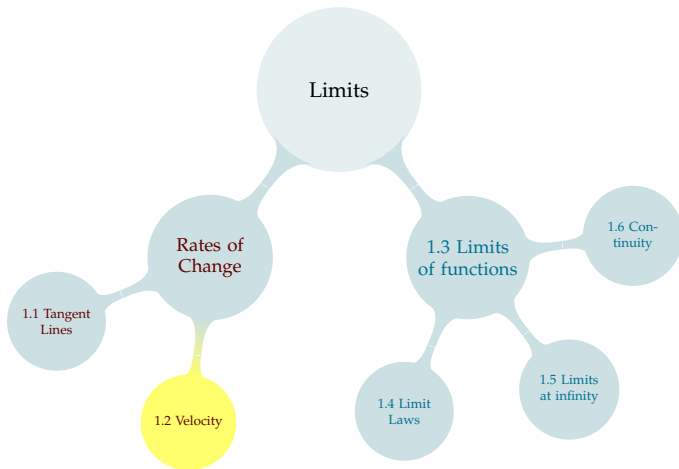
On the graph below, draw the secant line to the curve through points  $P$  and  $Q$ .

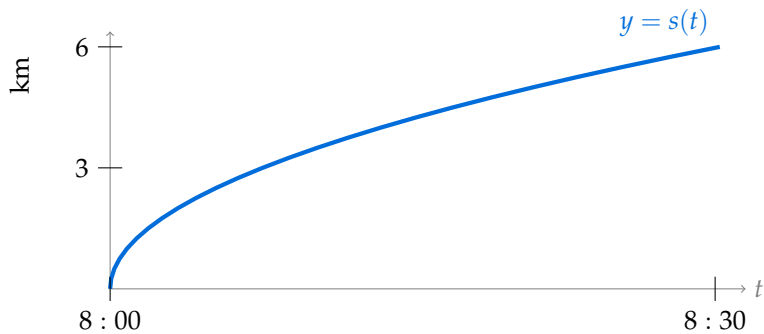


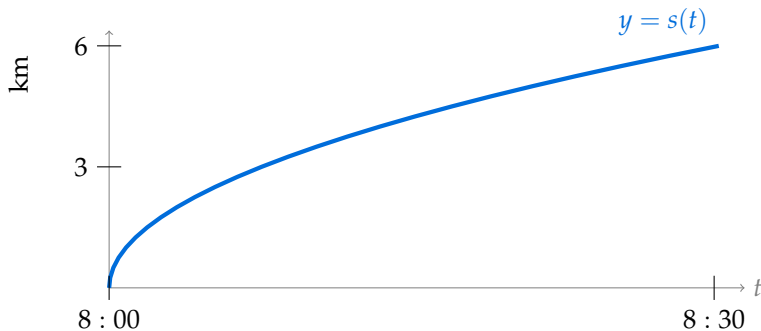
On the graph below, draw the tangent line to the curve at point  $P$ .



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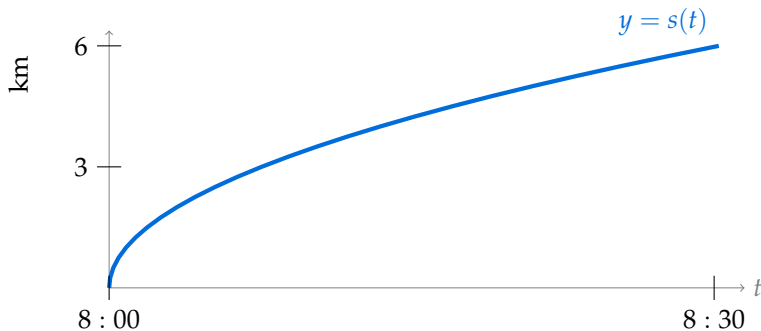




It took  $\frac{1}{2}$  hour to bike 6 km. 12 kph represents the:

- A. secant line to  $y = s(t)$  from  $t = 8 : 00$  to  $t = 8 : 30$
- B. slope of the secant line to  $y = s(t)$  from  $t = 8 : 00$  to  $t = 8 : 30$
- C. tangent line to  $y = s(t)$  at  $t = 8 : 30$
- D. slope of the tangent line to  $y = s(t)$  at  $t = 8 : 30$

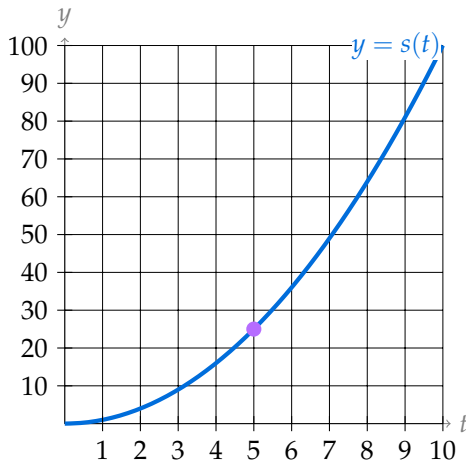




At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

- A. secant line to  $y = s(t)$  from  $t = 8:00$  to  $t = 8:25$
- B. slope of the secant line to  $y = s(t)$  from  $t = 8:00$  to  $t = 8:25$
- C. tangent line to  $y = s(t)$  at  $t = 8:25$
- D. slope of the tangent line to  $y = s(t)$  at  $t = 8:25$

Suppose the distance from the ground  $s$  (in meters) of a helium-filled balloon at time  $t$  over a 10-second interval is given by  $s(t) = t^2$ . Try to estimate how fast the balloon is rising when  $t = 5$ .



Let's look for an algebraic way of determining the velocity of the balloon when  $t = 5$ .

# OUR FIRST LIMIT

Average Velocity,  $t = 5$  to  $t = 5 + h$ :

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= \frac{s(5+h) - s(5)}{h} \\ &= \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \quad \text{when } h \neq 0\end{aligned}$$

When  $h$  is very small,

$$\text{Vel} \approx 10$$

# LIMIT NOTATION

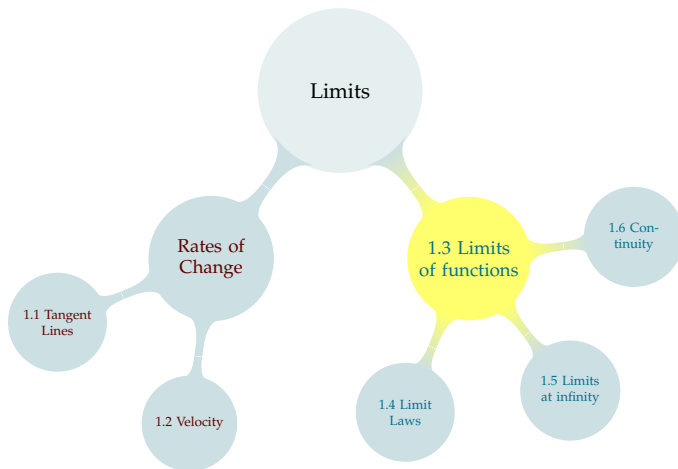
We write:

$$\lim_{h \rightarrow 0} (10 + h) = 10$$

We say: “The limit as  $h$  goes to 0 of  $(10 + h)$  is 10.”

It means: As  $h$  gets extremely close to 0,  $(10 + h)$  gets extremely close to 10.

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## Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \rightarrow a} f(x) = L$$

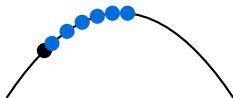
where  $a$  and  $L$  are real numbers

We read the above as “the limit as  $x$  goes to  $a$  of  $f(x)$  is  $L$ .”

Its meaning is: as  $x$  gets very close to (but not equal to)  $a$ ,  $f(x)$  gets very close to  $L$ .

# FINDING SLOPES OF TANGENT LINES

We NEED limits to find slopes of tangent lines.



Slope of secant line:  $\frac{\Delta y}{\Delta x}$ ,  $\Delta x \neq 0$ .

Slope of tangent line: can't do the same way.

If the position of an object at time  $t$  is given by  $s(t)$ , then its instantaneous velocity is given by

$$\lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$$

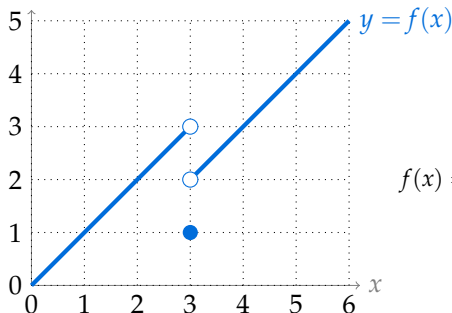


# EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate  $\lim_{x \rightarrow 1} f(x)$ .

# ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think  $\lim_{x \rightarrow 3} f(x)$  should be?

## Definition 1.3.7

The limit as  $x$  goes to  $a$  **from the left** of  $f(x)$  is written

$$\lim_{x \rightarrow a^-} f(x)$$

We only consider values of  $x$  that are **less than**  $a$ .

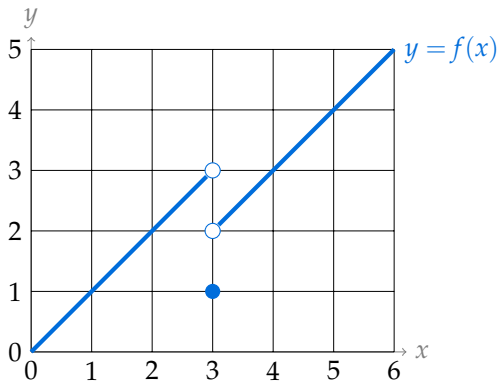
The limit as  $x$  goes to  $a$  **from the right** of  $f(x)$  is written

$$\lim_{x \rightarrow a^+} f(x)$$

We only consider values of  $x$  **greater than**  $a$ .

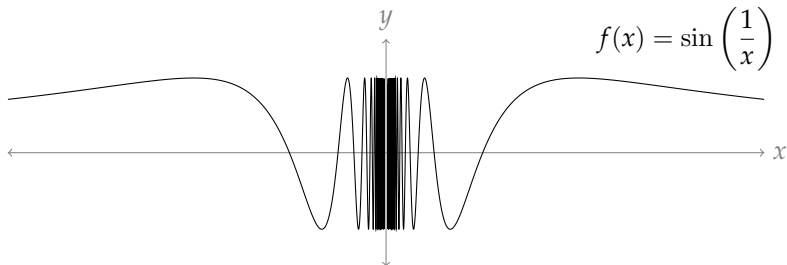
## Theorem 1.3.8

In order for  $\lim_{x \rightarrow a} f(x)$  to exist, both one-sided limits must exist and be equal.



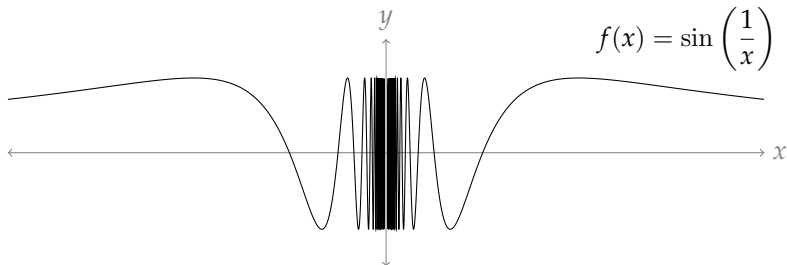
Consider the function  $f(x) = \frac{1}{(x-1)^2}$ . For what value(s) of  $x$  is  $f(x)$  **not** defined?

# A STRANGER LIMIT EXAMPLE



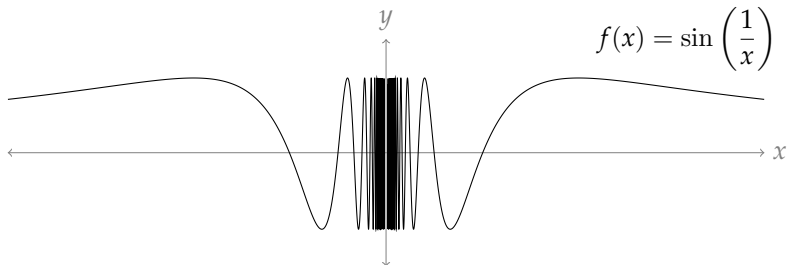
What is  $\lim_{x \rightarrow \infty} f(x)$ ?

# A STRANGER LIMIT EXAMPLE



What is  $\lim_{x \rightarrow 0} f(x)$  ?

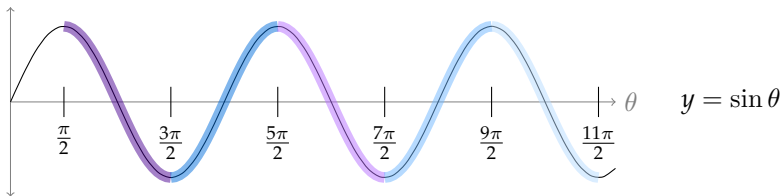
# A STRANGER LIMIT EXAMPLE



What is  $\lim_{x \rightarrow \pi} f(x)$  ?

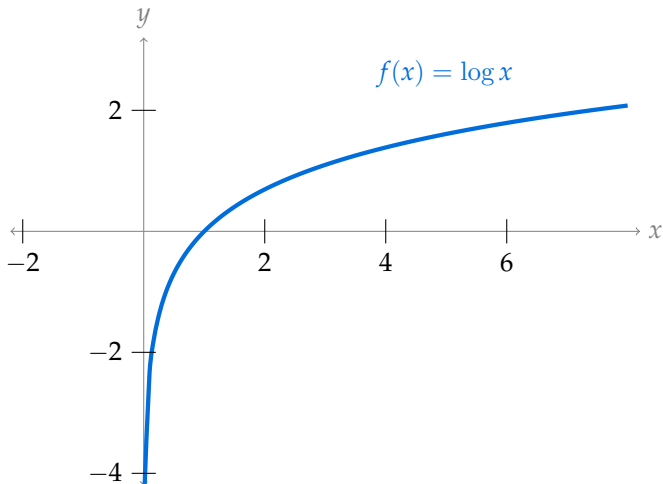


# OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$



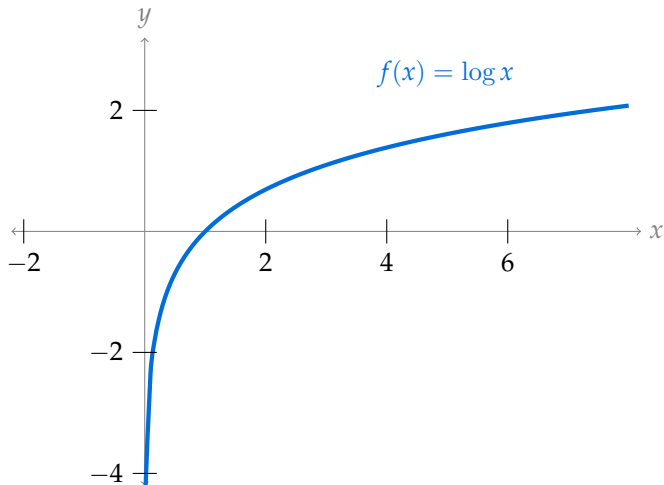
# LIMITS AND THE NATURAL LOGARITHM

Where is  $f(x)$  defined, and where is it not defined?

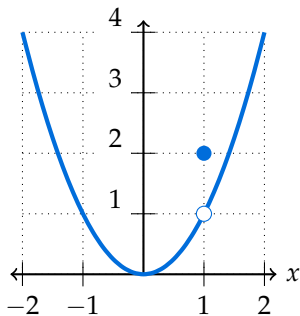


# LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of  $f(x)$  near 0?



$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is  $\lim_{x \rightarrow 1} f(x)$ ?

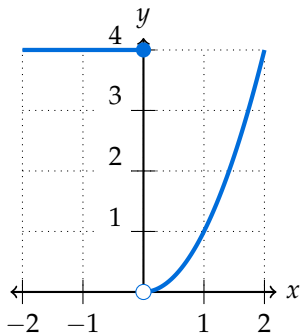
A.  $\lim_{x \rightarrow 1} f(x) = 2$

B.  $\lim_{x \rightarrow 1} f(x) = 1$

C.  $\lim_{x \rightarrow 1} f(x)$  DNE

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is  $\lim_{x \rightarrow 0} f(x)$ ? What is  $\lim_{x \rightarrow 0^+} f(x)$ ? What is  $f(0)$ ?

A.  $\lim_{x \rightarrow 0^+} f(x) = 4$

B.  $\lim_{x \rightarrow 0^+} f(x) = 0$

C.  $\lim_{x \rightarrow 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

Suppose  $\lim_{x \rightarrow 3^-} f(x) = 1$  and  $\lim_{x \rightarrow 3^+} f(x) = 1.5$ .

Does  $\lim_{x \rightarrow 3} f(x)$  exist?

- A. Yes, certainly, because the limits from both sides exist.
- B. No, never, because the limit from the left is not the same as the limit from the right.
- C. Can't tell. For some functions it might exist, for others not.

Suppose  $\lim_{x \rightarrow 3^-} f(x) = 22 = \lim_{x \rightarrow 3^+} f(x)$ .

Does  $\lim_{x \rightarrow 3} f(x)$  exist?

- A. Yes, certainly, because the limits from both sides exist and are equal to each other.
- B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.
- C. Can't tell. We need to know the value of the function at  $x = 3$ .





# CALCULATING LIMITS IN SIMPLE SITUATIONS

## Direct Substitution – Theorem 1.4.10

If  $f(x)$  is a polynomial or rational function, and  $a$  is in the domain of  $f$ , then:

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Calculate:  $\lim_{x \rightarrow 3} \left( \frac{x^2 - 9}{x + 3} \right)$

Calculate:  $\lim_{x \rightarrow 3} \left( \frac{x^2 - 9}{x - 3} \right)$

## Algebra with Limits: Theorem 1.4.2

Suppose  $\lim_{x \rightarrow a} f(x) = F$  and  $\lim_{x \rightarrow a} g(x) = G$ , where  $F$  and  $G$  are both real numbers. Then:

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$
- $\lim_{x \rightarrow a} (f(x)g(x)) = FG$
- $\lim_{x \rightarrow a} (f(x)/g(x)) = F/G$  provided  $G \neq 0$

Calculate:  $\lim_{x \rightarrow 1} \left[ \frac{2x+4}{x+2} + 13 \left( \frac{x+5}{3x} \right) \left( \frac{x^2}{2x-1} \right) \right]$

# LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A.  $4^{\frac{1}{2}}$

B.  $(-4)^{\frac{1}{2}}$

C.  $4^{-\frac{1}{2}}$

D.  $(-4)^{-\frac{1}{2}}$

E.  $8^{1/3}$

F.  $(-8)^{1/3}$

G.  $8^{-1/3}$

H.  $(-8)^{-1/3}$

## Powers of Limits – Theorem 1.4.8

If  $n$  is a positive integer, and  $\lim_{x \rightarrow a} f(x) = F$  (where  $F$  is a real number), then:

$$\lim_{x \rightarrow a} (f(x))^n = F^n.$$

Furthermore, **unless**  $n$  is even and  $F$  is negative,

$$\lim_{x \rightarrow a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x \rightarrow 4} (x + 5)^{1/2}$$

# CAUTIONARY TALES

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{(5+x)^2 - 25}{x}$$

$$\blacktriangleright \lim_{x \rightarrow 3} \left( \frac{x-6}{3} \right)^{1/8}$$

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{32}{x}$$

$$\blacktriangleright \lim_{x \rightarrow 5} (x^2 + 2)^{1/3}$$

Suppose you want to evaluate  $\lim_{x \rightarrow 1} f(x)$ , but  $f(1)$  doesn't exist. What does that tell you?

- A  $\lim_{x \rightarrow 1} f(x)$  may exist, and it may not exist.
- B We can find  $\lim_{x \rightarrow 1} f(x)$  by plugging in 1 to  $f(x)$ .
- C Since  $f(1)$  doesn't exist, it is not meaningful to talk about  $\lim_{x \rightarrow 1} f(x)$ .
- D Since  $f(1)$  doesn't exist, automatically we know  $\lim_{x \rightarrow 1} f(x)$  does not exist.
- E  $\lim_{x \rightarrow 1} f(x)$  does not exist if we are "dividing by zero," but may exist otherwise.

Which of the following statements is true about  $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x}$ ?

A  $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 0}{0^3 - 0^2 + 0} = \frac{0}{0}$

B Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of  $\frac{\sin x}{x^3 - x^2 + x}$  are both 0 when  $x = 0$ , the limit exists.

D Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not defined at 0, plugging in  $x = 0$  will not tell us the limit.

E Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about  $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x}$ ?

A  $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not rational, its limit at 1 does not exist.

C Since the function  $\frac{\sin x}{x^3 - x^2 + x}$  is not defined at 1, plugging in  $x = 1$  will not tell us the limit.

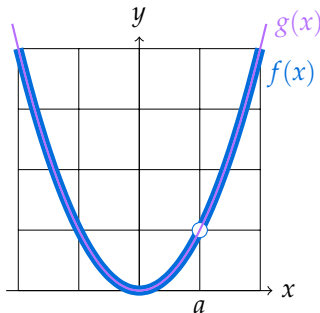
D Since the numerator and denominator of  $\frac{\sin x}{x^3 - x^2 + x}$  are both 0 when  $x = 1$ , the limit exists.



## Functions that Differ at a Single Point – Theorem 1.4.12

Suppose  $\lim_{x \rightarrow a} g(x)$  exists, and  $f(x) = g(x)$   
when  $x$  is close to  $a$  (but not necessarily equal to  $a$ ).

Then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ .



Evaluate  $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x - 1}$ .

Evaluate  $\lim_{x \rightarrow 5} \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$

# A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can **directly substitute** (plug in). If your function is made up of the **sum, difference, product, quotient, or power of polynomials**, you can do this **provided** the function exists where you're taking the limit.

$$\lim_{x \rightarrow 1} \left( \sqrt{35 + x^5} + \frac{x - 3}{x^2} \right)^3 =$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to **simplify and cancel**.

$$\lim_{x \rightarrow 0} \frac{x + 7}{\frac{1}{x} - \frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

# DENOMINATORS APPROACHING ZERO

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}$$

# DENOMINATORS APPROACHING ZERO

NOW  
YOU



$$\lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4}$$

$$\lim_{x \rightarrow 2^-} \frac{x}{4 - x^2}$$

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$$

## Squeeze Theorem – Theorem 1.4.17

Suppose, when  $x$  is near (but not necessarily equal to)  $a$ , we have functions  $f(x)$ ,  $g(x)$ , and  $h(x)$  so that

$$f(x) \leq g(x) \leq h(x)$$

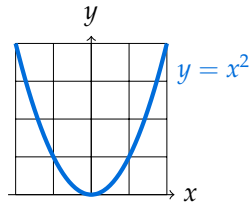
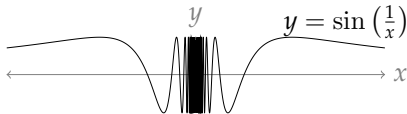
and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$ . Then  $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x)$ .

$$\lim_{x \rightarrow 0} x^2 \sin \left( \frac{1}{x} \right)$$



Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$



$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

## Included Work



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