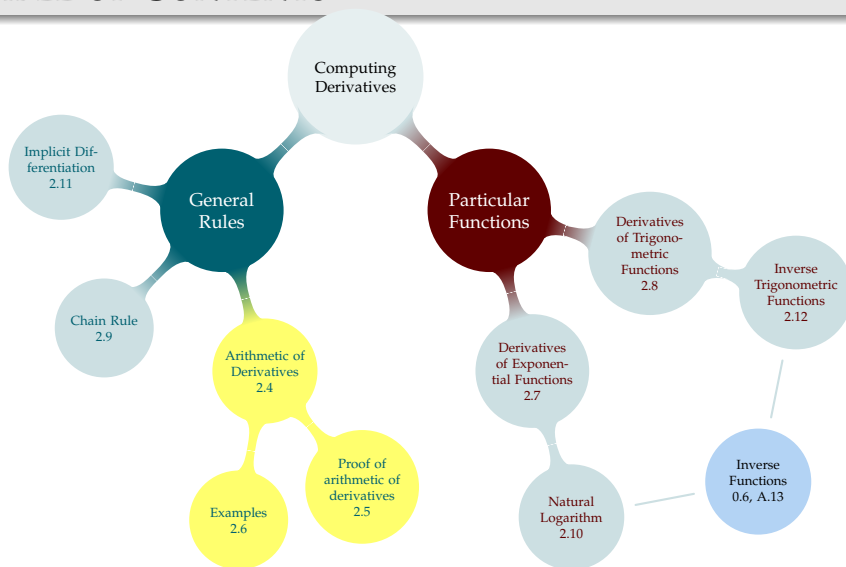


TABLE OF CONTENTS



DERIVATIVES OF LINES

$$f(x) = 2x - 15$$

The equation of the tangent line to $f(x)$ at $x = 100$ is:

$$f'(1) =$$

A. 0

B. 1

C. 2

D. -15

E. -13

$$f'(5) =$$

$$f'(-13) =$$

$$g(x) = 13$$

$$g'(1) =$$

A. 0

B. 1

C. 2

D. 13

ADDING A CONSTANT

Adding or subtracting a constant to a function **does not change its derivative**.

We saw

$$\left. \frac{d}{dx} (3 - 0.8t^2) \right|_{t=1} = -1.6$$

So,

$$\left. \frac{d}{dx} (10 - 0.8t^2) \right|_{t=1} =$$

DIFFERENTIATING SUMS

$$\frac{d}{dx} \{f(x) + g(x)\} =$$

└ 2.4-2.6: Arithmetic of Derivatives

└ Differentiating Sums

$$\frac{d}{dx} [f(x) + g(x)] =$$

This is a really good intro to showing how the definition can be used to prove rules. Alternately put, how rules are just shortcuts for calculations starting with the definition.

CONSTANT MULTIPLE OF A FUNCTION

Let a be a constant.

$$\frac{d}{dx} \{a \cdot f(x)\} =$$

└ 2.4-2.6: Arithmetic of Derivatives

└ Constant Multiple of a Function

Let a be a constant.

$$\frac{d}{dx} [a \cdot f(x)] =$$

Ask students to explain to their neighbours which step wouldn't have worked if a weren't constant

Rules – Lemma 2.4.1

Suppose $f(x)$ and $g(x)$ are differentiable, and let c be a constant number. Then:

- ▶ $\frac{d}{dx} \{f(x) + g(x)\} = f'(x) + g'(x)$
- ▶ $\frac{d}{dx} \{f(x) - g(x)\} = f'(x) - g'(x)$
- ▶ $\frac{d}{dx} \{cf(x)\} = cf'(x)$

For instance: let $f(x) = 10((2x - 15) + 13 - \sqrt{x})$. Then $f'(x) =$

Now
You

Calculate:

Suppose $f'(x) = 3x$, $g'(x) = -x^2$, and $h'(x) = 5$.

$$\frac{d}{dx} \{f(x) + 5g(x) - h(x) + 22\}$$

- A. $3x - 5x^2$
- B. $3x - 5x^2 - 5$
- C. $3x - 5x^2 - 5 + 22$
- D. none of the above

DERIVATIVES OF PRODUCTS

$$\frac{d}{dx}\{x\} = 1$$

True or False:

$$\begin{aligned}\frac{d}{dx}\{2x\} &= \frac{d}{dx}\{x + x\} \\ &= [1] + [1] \\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{d}{dx}\{x^2\} &= \frac{d}{dx}\{x \cdot x\} \\ &= [1] \cdot [1] \\ &= 1\end{aligned}$$

2.4-2.6: Arithmetic of Derivatives

Derivatives of Products

DERIVATIVES OF PRODUCTS

$$\frac{d}{dx}(x) = 1$$

True or False:

$$\begin{aligned}\frac{d}{dx}(2x) &= \frac{d}{dx}(x + x) \\ &= [1] + [1] \\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{d}{dx}(x^2) &= \frac{d}{dx}(x \cdot x) \\ &= [1] \cdot [1] \\ &= 1\end{aligned}$$

Students have seen these rules before and often feel bored here. I like to remind them how weird the rules are. It wasn't obvious that the product rule would be what it is, but they've known it for so long, it *feels* obvious. So I like to show the contrasting patterns below.

WHAT TO DO WITH PRODUCTS?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x . What about $f(x)g(x)$?

└ 2.4-2.6: Arithmetic of Derivatives

└ What to do with Products?

To avoid panic, emphasize that students are not responsible for reproducing this. The rule is surprising, we're just explaining why it's true. Also useful to emphasize that all the "tricks" are just *shortcuts for the definition of the derivative*. Otherwise students often wonder why we bother with the definition at all.

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{d}{dx} [x^2] =$$

Example: suppose $f(x) = 3x^2$, $f'(x) = 6x$, $g(x) = \sin(x)$, $g'(x) = \cos(x)$.

$$\frac{d}{dx} [3x^2 \sin(x)] =$$

Given $\frac{d}{dx} [2x + 5] = 2$, $\frac{d}{dx} [\sin(x^2)] = 2x \cos(x^2)$, $\frac{d}{dx} [x^2] = 2x$

NOW
YOU



$$f(x) = (2x + 5) \sin(x^2)$$

- A. $f'(x) = (2) (2x \cos(x^2)) (2x)$
- B. $f'(x) = (2) (2x \cos(x^2))$
- C. $f'(x) = (2x + 5)(2) + \sin(x^2) (2x \cos(x^2))$
- D. $f'(x) = (2x + 5) (2x \cos(x^2)) + (2) \sin(x^2)$
- E. none of the above

NOW
YOU



$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is $f'(x)$?

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{d}{dx} \left\{ \frac{2x + 5}{3x - 6} \right\} =$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{d}{dx} \left\{ \frac{5x}{\sqrt{x} - 1} \right\} =$$

NOW
YOU



Differentiate the following.

$$f(x) = 2x + 5$$

$$g(x) = (2x + 5)(3x - 7) + 25$$

$$h(x) = \frac{2x + 5}{8x - 2}$$



$$j(x) = \left(\frac{2x + 5}{8x - 2} \right)^2$$

Rules

Product: $\frac{d}{dx} \{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

2.4-2.6: Arithmetic of Derivatives

NUM  DIFF Differentiate the following.
YOU 

$$f(x) = 2x + 5$$

$$g(x) = (2x + 5)(3x - 7) + 25$$

$$h(x) = \frac{2x + 5}{8x - 2}$$

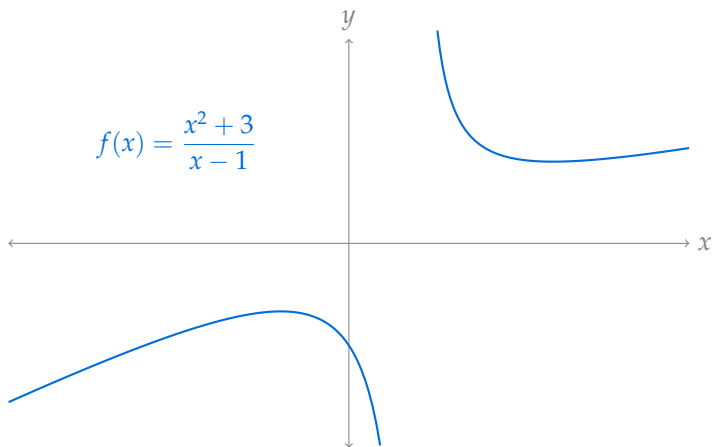
$$j(x) = \left(\frac{2x + 5}{8x - 2} \right)^2$$

Rules

Product: $\frac{d}{dx} \{ f(x)g(x) \} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

Students probably remember power and chain rules from high school, but these can be done without power and chain rules. The last example is there to give the faster workers something to work on, while the slower workers get quality time with the foundational examples.

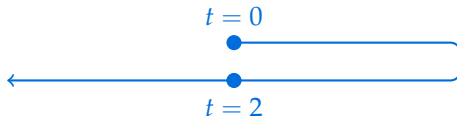


For which values of x is the tangent line to the curve horizontal?

The position of an object moving left and right at time t , $t \geq 0$, is given by

$$s(t) = -t^2(t - 2)$$

where a positive position means it is to the right of its starting position, and a negative position means it is to the left. First it moves to the right, then it moves left forever.



What is the farthest point to the right that the object reaches?

MORE ABOUT THE PRODUCT RULE

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1) \\ = 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\} \\ = (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{d}{dx}\{x^4\} = \frac{d}{dx}\{x \cdot x^3\} \\ = x(3x^2) + x^3(1) = 4x^3$$

Where are these functions defined?

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{30}	$30x^{29}$
x^n	nx^{n-1}

CAUTIONARY TALE

WITH *functions* RAISED TO A POWER, IT'S MORE COMPLICATED.

Differentiate $(2x + 1)^2$

Power Rule – Corollary 2.6.17

$$\frac{d}{dx}\{x^a\} = ax^{a-1} \text{ (where defined)}$$

$$\frac{d}{dx}\{3x^5 + 7x^2 - x + 15\} =$$

Power Rule – Corollary 2.6.17

$$\frac{d}{dx}\{x^a\} = ax^{a-1} \text{ (where defined)}$$

Differentiate $\frac{(x^4 + 1)(\sqrt[3]{x} + \sqrt[4]{x})}{2x + 5}$

Power Rule – Corollary 2.6.17

$$\frac{d}{dx}\{x^a\} = ax^{a-1} \text{ (where defined)}$$

Suppose a motorist is driving their car, and their position is given by $s(t) = 10t^3 - 90t^2 + 180t$ kilometres. At $t = 1$ (t measured in hours), a police officer notices they are driving erratically. The motorist claims to have simply suffered a lack of attention: they were in the act of pressing the brakes even as the officer noticed their speed.

At $t = 1$, how fast was the motorist going, and were they pressing the gas or the brake?

Challenge: What about $t = 2$?

Power Rule – Corollary 2.6.17

$$\frac{d}{dx}\{x^a\} = ax^{a-1} \text{ (where defined)}$$

Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.

Suppose you are winding twine into a gigantic twine ball, filming the process, and trying to make a viral video. You can wrap one cubic meter of twine per hour. (In other words, when we have V cubic meters of twine, we're at time V hours.) How fast is the radius of your spherical twine ball increasing?

Included Work



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