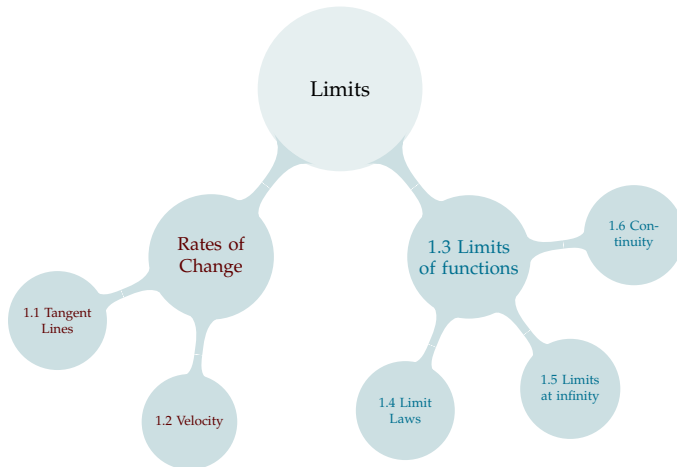


# TABLE OF CONTENTS

## 1.7 (Optional) Making the Informal a Little More Formal



Now that we've seen the limits of functions as  $x$  goes to positive and negative infinity, let's look at limits as  $x$  approaches a real number.

## └ 1.7 (Optional) Making the Informal a Little More Formal

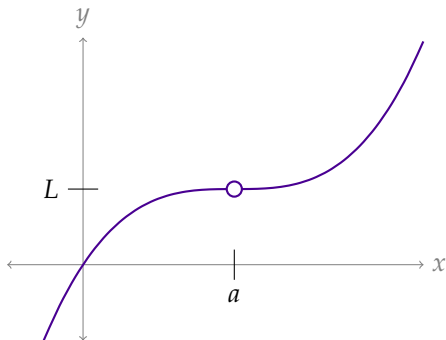
Now that we've seen the limits of functions as  $x$  goes to positive and negative infinity, let's look at limits as  $x$  approaches a real number.

The actual computations for limits as  $x$  goes to infinity are generally easier, so I like to teach 1.8 before 1.7.

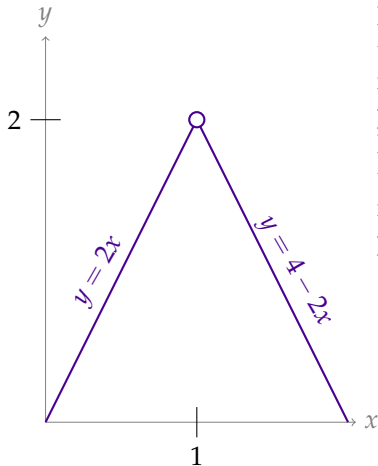
A lot of the same language from the canning analogy can be re-used here:  $\epsilon$  as error, for instance.

$$\lim_{x \rightarrow a} f(x) = L$$

Informally: If  $x$  is close enough (but not equal to)  $a$ , then  $y$  is close enough to  $L$ .



Let  $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 - 2x & \text{if } x > 1 \end{cases}$ . Then  $\lim_{x \rightarrow 1} |f(x)| = 2$ .



Find a positive number  $\delta$  such that  $|f(x) - 2| < \frac{1}{2}$  for all  $x$  in the interval  $(1 - \delta, 1 + \delta)$ , except possibly  $x = 1$ .

Find a positive number  $\delta$  such that  $|f(x) - 2| < \frac{1}{4}$  for all  $x$  in the interval  $(1 - \delta, 1 + \delta)$ , except possibly  $x = 1$ .

## Definition 1.7.1

Let  $a \in \mathbb{R}$  and let  $f(x)$  be a function defined everywhere in a neighbourhood of  $a$ , except possibly at  $a$ . We say that

*the limit as  $x$  approaches  $a$  of  $f(x)$  is  $L$*

and write

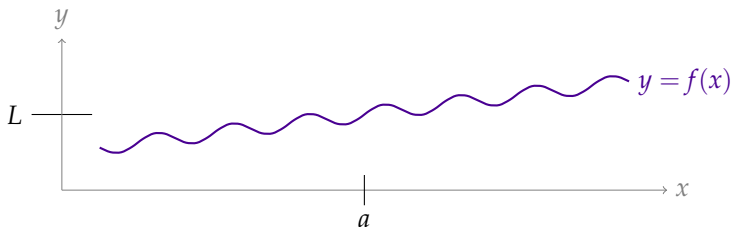
$$\lim_{x \rightarrow a} f(x) = L$$

if and only if for every  $\epsilon > 0$  there exists  $\delta > 0$  so that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta$$

Note that an equivalent way of writing this very last statement is

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$



Let  $a \in \mathbb{R}$  and let  $f(x)$  be a function defined everywhere in a neighbourhood of  $a$ , except possibly at  $a$ .

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Using Definition 1.7.1, prove that  $\lim_{x \rightarrow -1} |x + 1| = 0$ .



## Definition 1.7.1

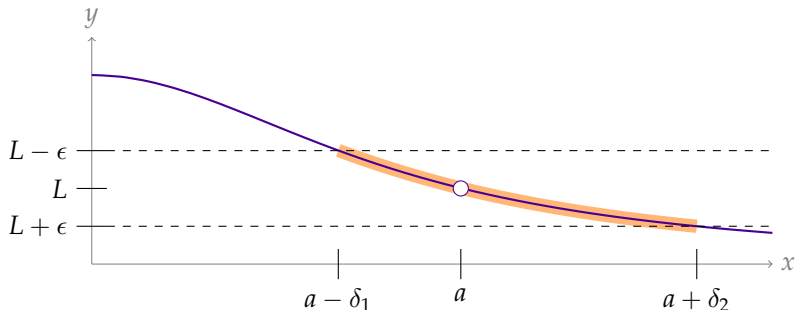
Let  $a \in \mathbb{R}$  and let  $f(x)$  be a function defined everywhere in a neighbourhood of  $a$ , except possibly at  $a$ . We say that  $\lim_{x \rightarrow a} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < \epsilon$ .

$$\text{Let } f(x) = \begin{cases} x + 1 & x < 0 \\ 1 - x^2 & x > 0 \end{cases}.$$

Using Definition 1.7.1, prove that  $\lim_{x \rightarrow 0} f(x) = 1$ .

## GENERAL PRINCIPLES

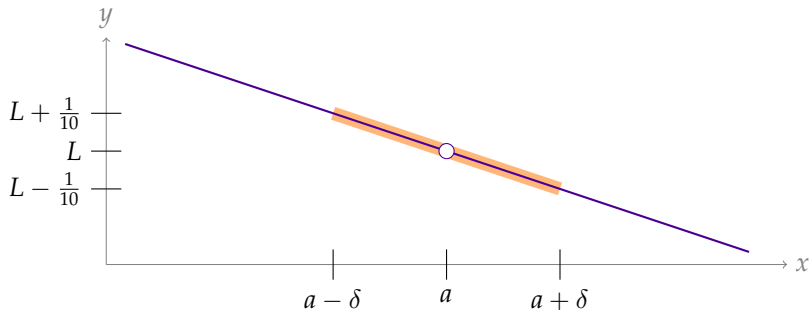
Suppose  $|f(x) - L| < \epsilon$  whenever  $a - \delta_1 < x < a$  and whenever  $a < x < a + \delta_2$ .



Consider values of  $x$  such that  $0 < |x - a| < \min\{\delta_1, \delta_2\}$ .

## GENERAL PRINCIPLES

Suppose  $|f(x) - L| < \frac{1}{10}$  for all  $x$  such that  $0 < |x - a| < \delta$ .



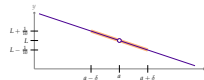
Can you give values of  $x$  where  $|f(x) - L| < \frac{1}{5}$ ?

# 1.7 (Optional) Making the Informal a Little More Formal

## General Principles

### GENERAL PRINCIPLES

Suppose  $|f(x) - L| < \frac{\epsilon}{2}$  for all  $x$  such that  $0 < |x - a| < \delta$ .



Can you give values of  $x$  where  $|f(x) - L| < \frac{\epsilon}{2}$ ?

SKIP

WLOG prove only for small epsilon. This doesn't really come up so often at this level, which is why there's a skip button.

## GENERAL PRINCIPLES

## Definition 1.7.1

Let  $a \in \mathbb{R}$  and let  $f(x)$  be a function defined everywhere in a neighbourhood of  $a$ , except possibly at  $a$ . We say that  $\lim_{x \rightarrow a} f(x) = L$  if and only if **for every**  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < \epsilon$ .

It is enough to show that **for every**  $\epsilon$  such that  $0 < \epsilon < c$  (where  $c$  is some constant) there exists  $\delta > 0$  so that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < \epsilon$ .

That means it doesn't hurt your proof if you say something like "we assume  $\epsilon < 1$ ".

In a previous example, we chose

$$\delta = \min\{\epsilon, \sqrt{\epsilon}\}$$

It would be OK to say "we can assume  $\epsilon < 1$ ; set  $\delta = \epsilon$ ."

## Definition 1.7.1

Let  $a \in \mathbb{R}$  and let  $f(x)$  be a function defined everywhere in a neighbourhood of  $a$ , except possibly at  $a$ . We say that  $\lim_{x \rightarrow a} f(x) = L$  if and only if for every  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $0 < |x - a| < \delta$  then  $|f(x) - L| < \epsilon$ .

Using Definition 1.7.1, prove that  $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \frac{1}{4}$ .

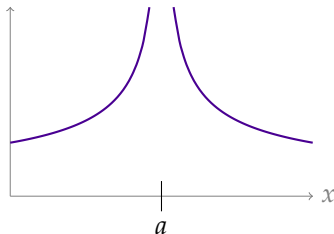
## INFINITE LIMITS

## Definition 1.8.1 (b)

Let  $a$  be a real number and  $f(x)$  be a function defined for all  $x \neq a$ . We write

$$\lim_{x \rightarrow a} f(x) = \infty$$

if and only if for every  $P > 0$  there exists  $\delta > 0$  so that  $f(x) > P$  whenever  $0 < |x - a| < \delta$ .

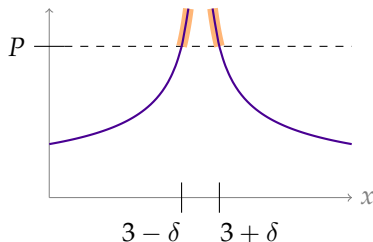


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Let  $f(x) = \frac{1}{(x-3)^2}$ . Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow 3} f(x) = \infty$$



# 1.7 (Optional) Making the Informal a Little More Formal

## Definition 1.8.1 (b)

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Let  $f(x) = \frac{1}{x-a}$ . Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow a} f(x) = \infty$$

The generic picture is kept on the left, but it's nice to mention that it is, indeed, generic. In particular, the picture won't fit for  $f(x) = \frac{1}{x-2}$ .

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if and only if for every  $P > 0$  there exists  $\delta > 0$  so that  $f(x) > P$  whenever  $0 < |x - a| < \delta$ .

Let  $f(x) = \frac{1}{x-2}$ . Using Definition 1.8.1, prove or disprove that

$$\lim_{x \rightarrow 2} f(x) = \infty$$

## 1.7 (Optional) Making the Informal a Little More Formal

### Definition 1.8.1 (b)

Let  $a$  be a real number and  $f(x)$  be a function defined for all  $x \neq a$ . We write

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