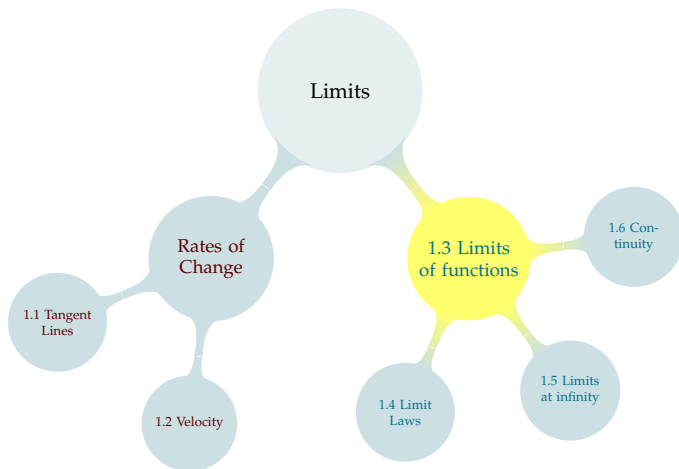


TABLE OF CONTENTS



Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \rightarrow a} f(x) = L$$

where a and L are real numbers

We read the above as “the limit as x goes to a of $f(x)$ is L .”

Its meaning is: as x gets very close to (but not equal to) a , $f(x)$ gets very close to L .

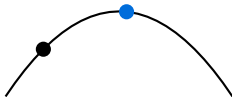
FINDING SLOPES OF TANGENT LINES

We NEED limits to find slopes of tangent lines.



FINDING SLOPES OF TANGENT LINES

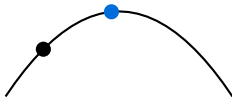
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Slope of secant line: $\frac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.

FINDING SLOPES OF TANGENT LINES

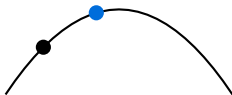
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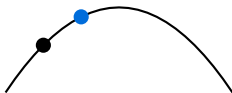
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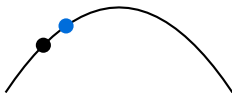
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FINDING SLOPES OF TANGENT LINES



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If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

FINDING SLOPES OF TANGENT LINES



We NEED limits to find slopes of tangent lines.



Slope of secant line: $\frac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.

Slope of tangent line: *can't do the same way*.

If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

$$\lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$$

EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \rightarrow 1} f(x)$.

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What is $f(1)$?

EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \rightarrow 1} f(x)$.

What is $f(1)$? **DNE** (can't divide by zero)

EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \rightarrow 1} f(x)$.

Use the tables below to guess $\lim_{x \rightarrow 1} f(x)$

x	$f(x)$	x	$f(x)$
0.9	3.61	1.1	4.41
0.99	3.9601	1.01	4.0401
0.999	3.99600	1.001	4.00400
0.9999	3.99960	1.0001	4.00040

EVALUATING LIMITS

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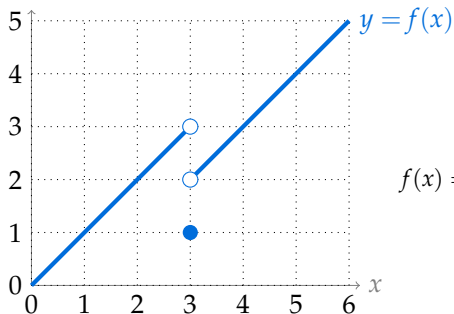
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0.9999	3.99960	1.0001	4.00040

$$\lim_{x \rightarrow 1} f(x) = 4$$



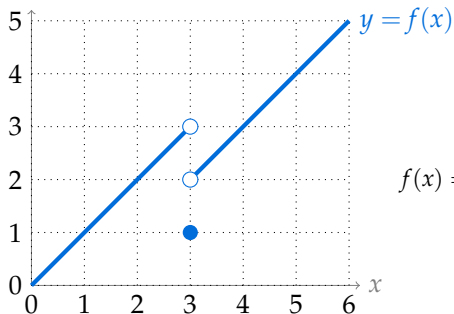
ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim_{x \rightarrow 3} f(x)$ should be?

ONE-SIDED LIMITS

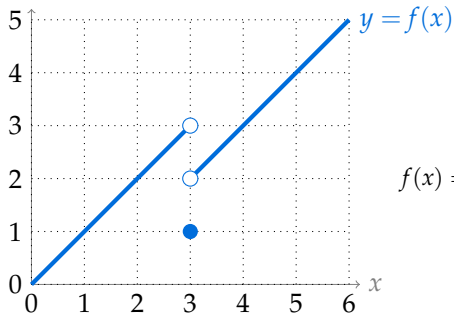


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ONE-SIDED LIMITS

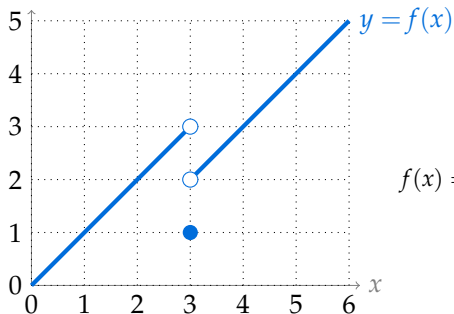


$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate: $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}}$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}}$

ONE-SIDED LIMITS

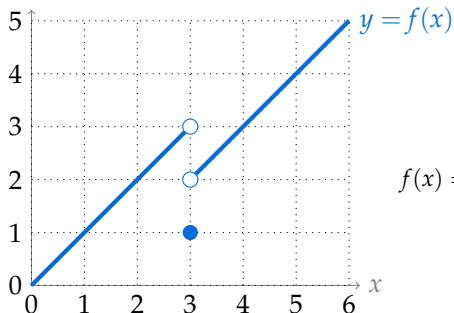


$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate: $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}} = 3$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}}$

ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

Evaluate: $\underbrace{\lim_{x \rightarrow 3^-} f(x)}_{\text{from the left}} = 3$

$\underbrace{\lim_{x \rightarrow 3^+} f(x)}_{\text{from the right}} = 2$



Definition 1.3.7

The limit as x goes to a **from the left** of $f(x)$ is written

$$\lim_{x \rightarrow a^-} f(x)$$

We only consider values of x that are **less than** a .

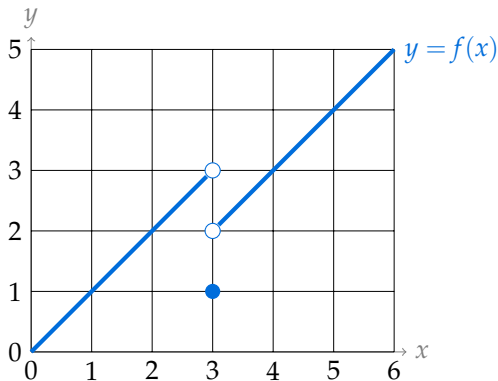
The limit as x goes to a **from the right** of $f(x)$ is written

$$\lim_{x \rightarrow a^+} f(x)$$

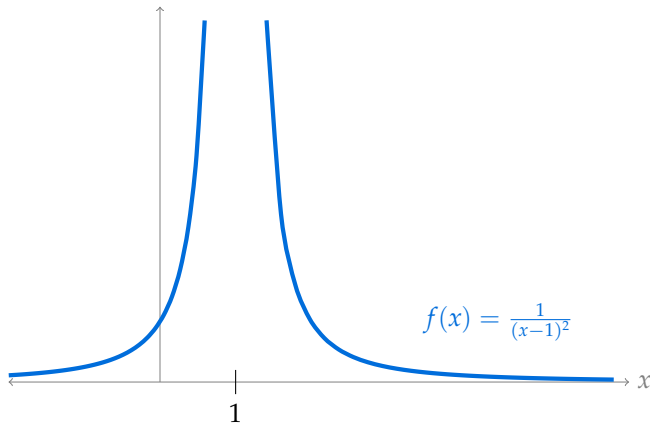
We only consider values of x **greater than** a .

Theorem 1.3.8

In order for $\lim_{x \rightarrow a} f(x)$ to exist, both one-sided limits must exist and be equal.

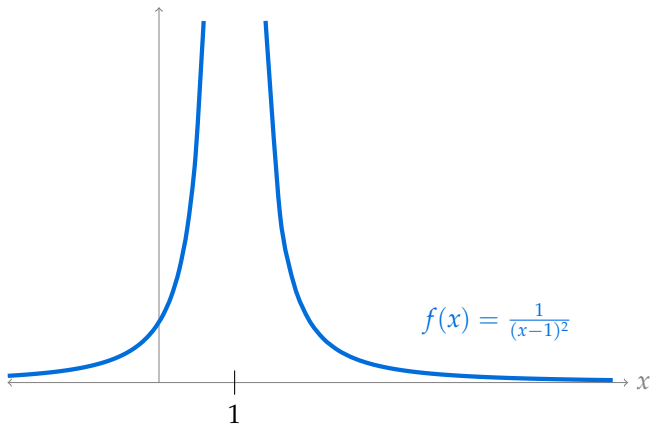


Consider the function $f(x) = \frac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ **not** defined?



Based on the graph below, what would you like to write for:

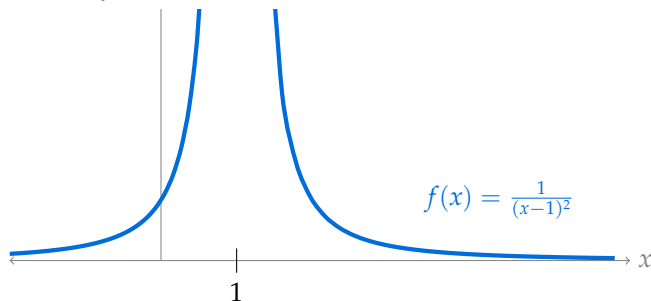
$$\lim_{x \rightarrow 1} f(x) =$$



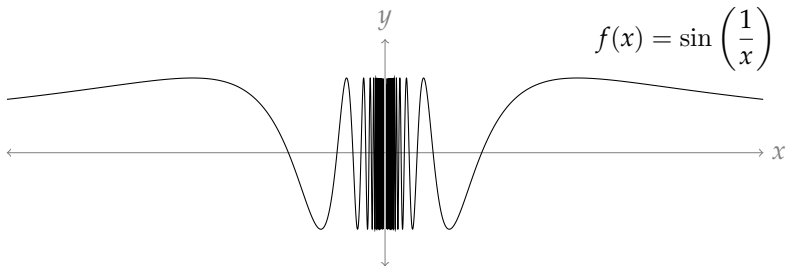
Based on the graph below, what would you like to write for:

$$\lim_{x \rightarrow 1} f(x) = \infty$$

A subtle point: we say that this limit **does not exist**. It “does not exist” in a way that we can, nonetheless, describe.

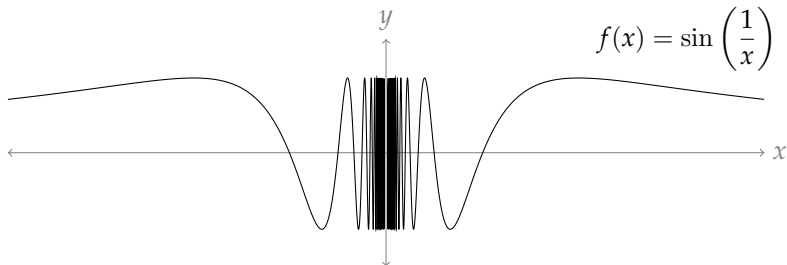


A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow \infty} f(x)$?

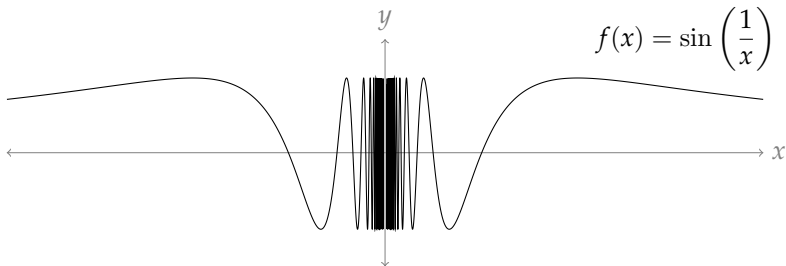
A STRANGER LIMIT EXAMPLE



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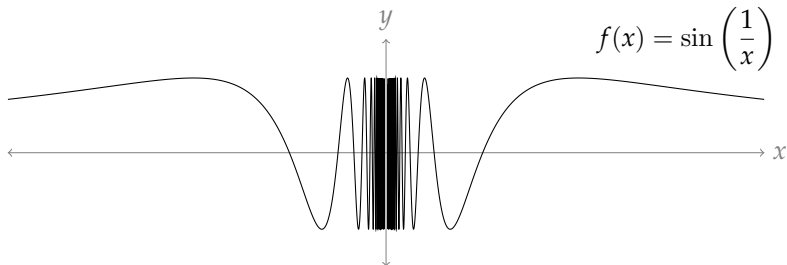
$$\lim_{x \rightarrow \infty} f(x) = 0$$

A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow 0} f(x)$?

A STRANGER LIMIT EXAMPLE

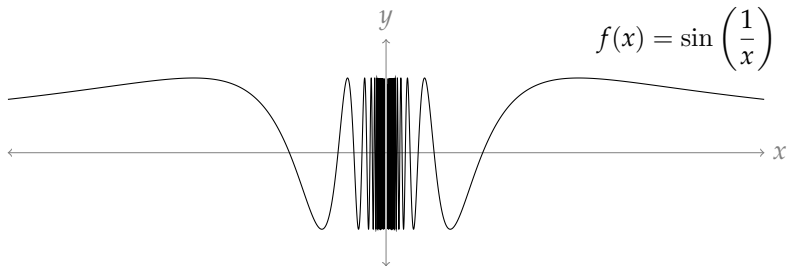


What is $\lim_{x \rightarrow 0} f(x)$?

$\lim_{x \rightarrow 0} f(x)$ does not exist.

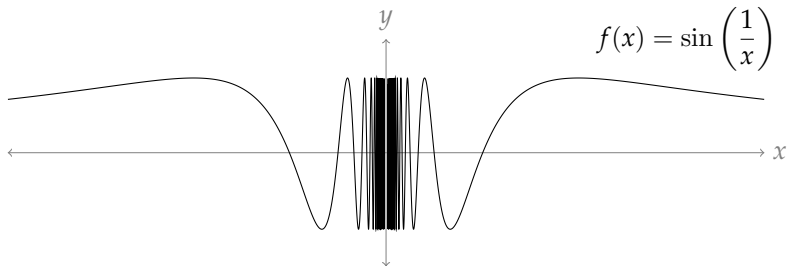
We can call this behaviour “infinite wiggling.”

A STRANGER LIMIT EXAMPLE



What is $\lim_{x \rightarrow \pi} f(x)$?

A STRANGER LIMIT EXAMPLE

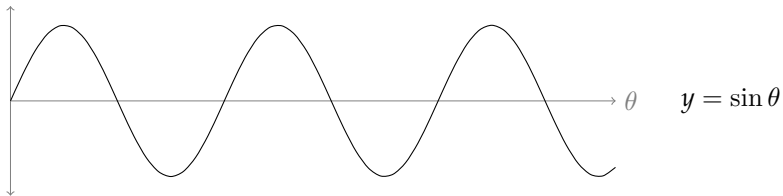


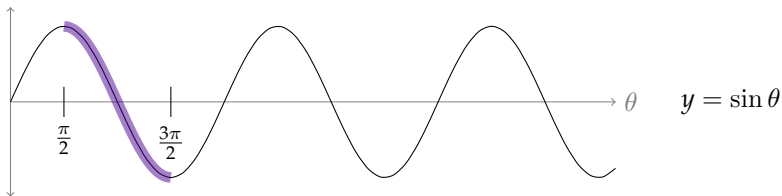
What is $\lim_{x \rightarrow \pi} f(x)$?

$$\lim_{x \rightarrow \pi} f(x) = \sin\left(\frac{1}{\pi}\right)$$

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

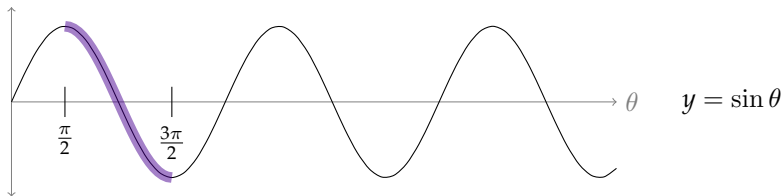
▶ SKIP SKETCHING



OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$ [▶ SKIP SKETCHING](#)

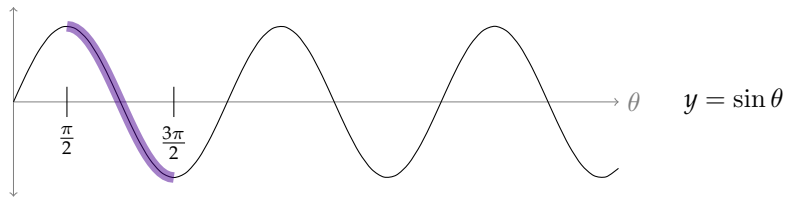
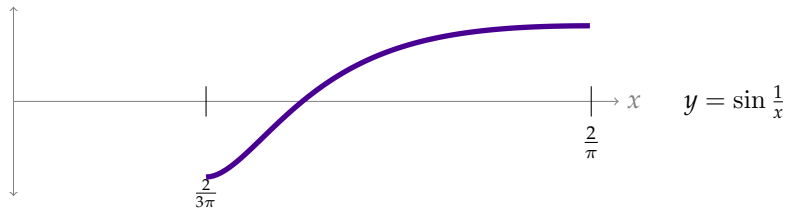
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▶ SKIP SKETCHING



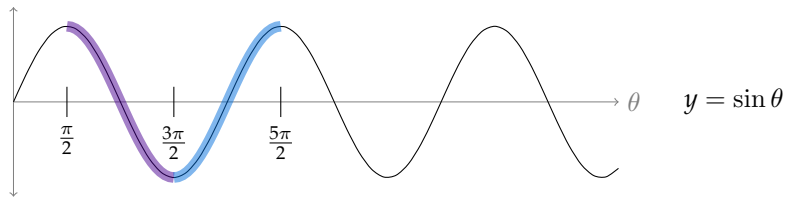
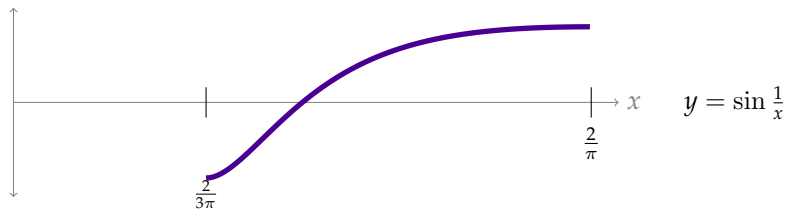
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▶ SKIP SKETCHING



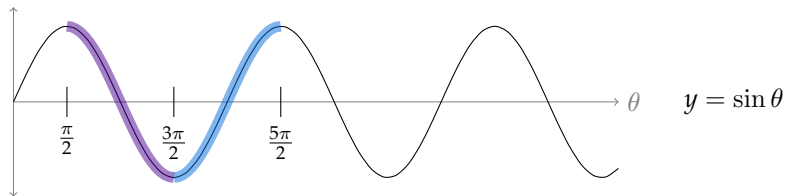
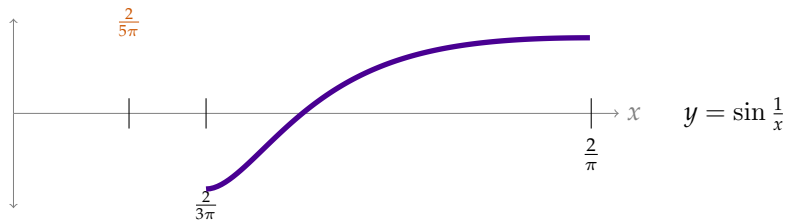
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▶ SKIP SKETCHING



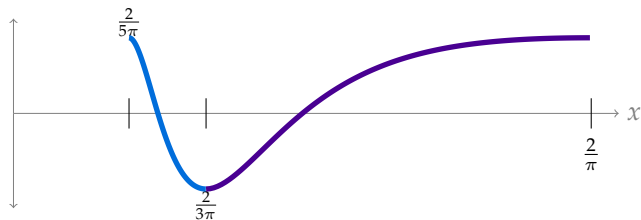
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▶ SKIP SKETCHING

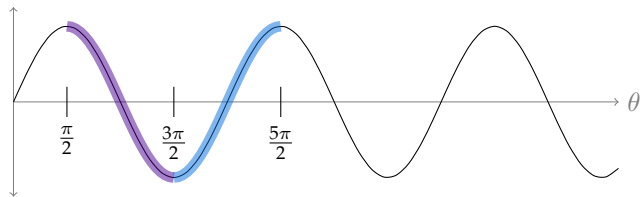


OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

▶ SKIP SKETCHING



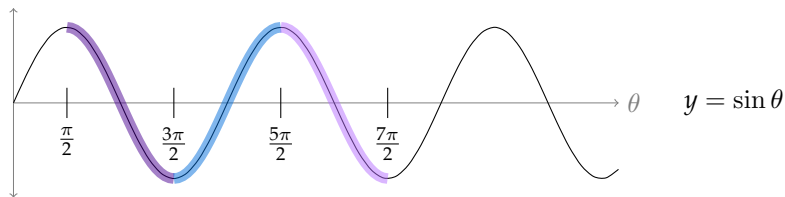
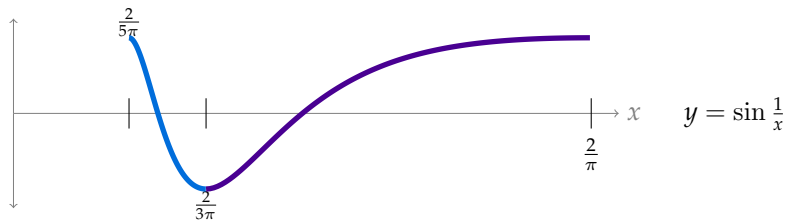
$$y = \sin \frac{1}{x}$$



$$y = \sin \theta$$

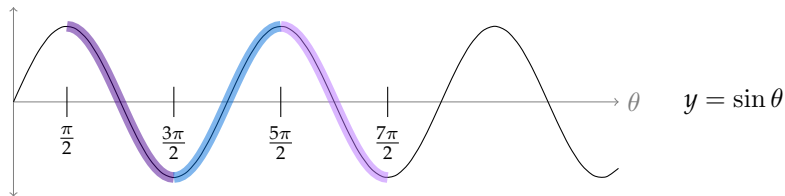
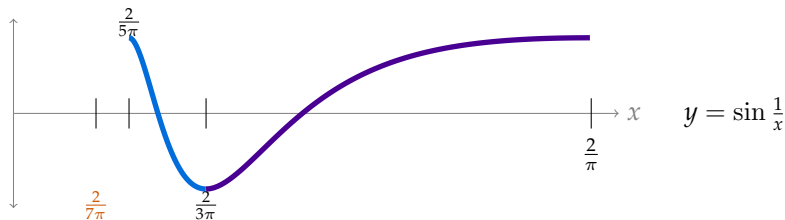
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▶ SKIP SKETCHING



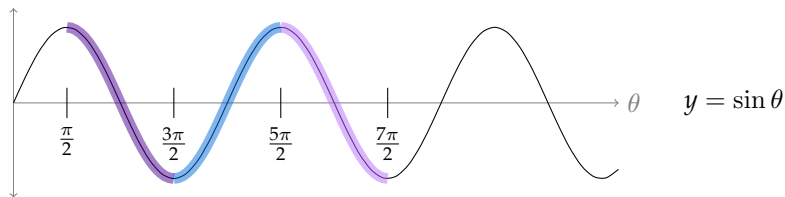
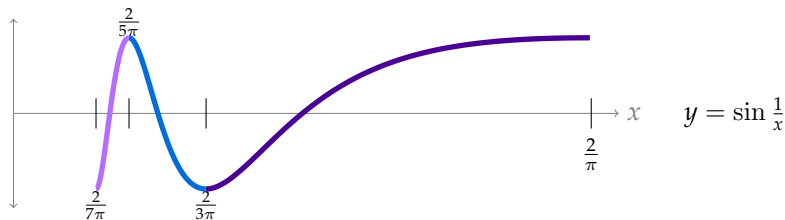
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▶ SKIP SKETCHING



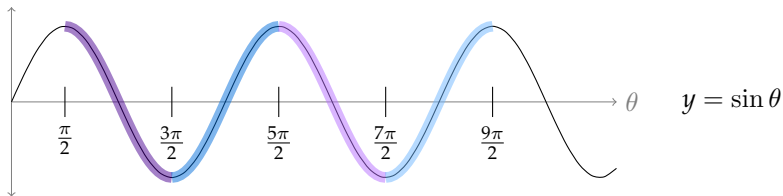
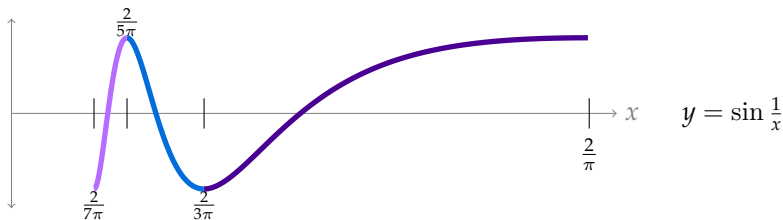
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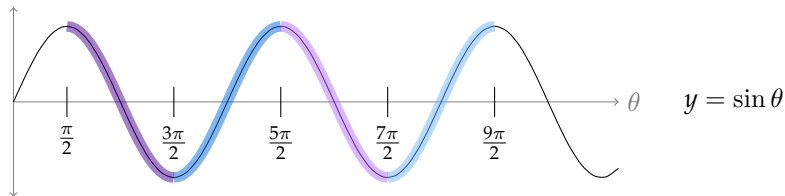
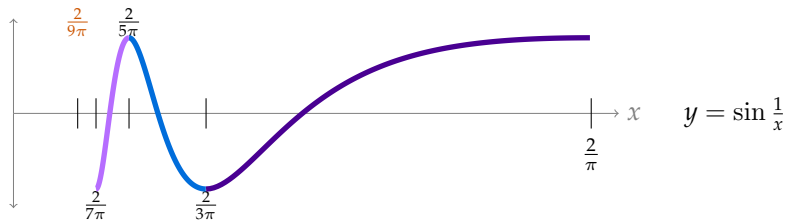
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▶ SKIP SKETCHING



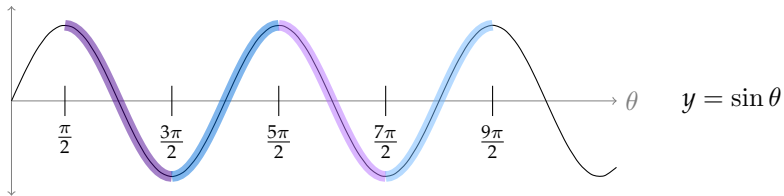
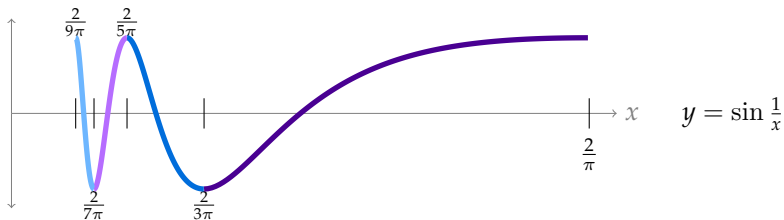
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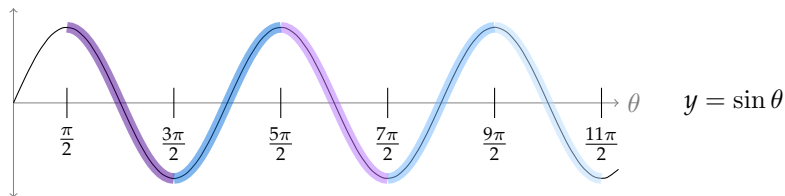
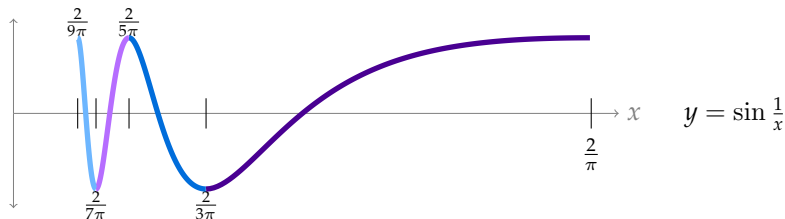
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▶ SKIP SKETCHING



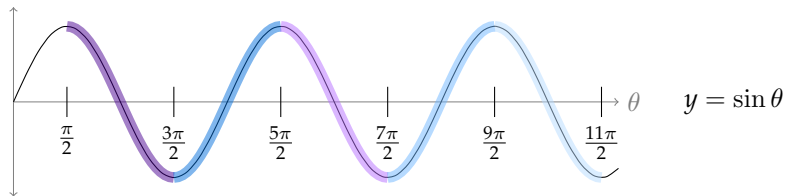
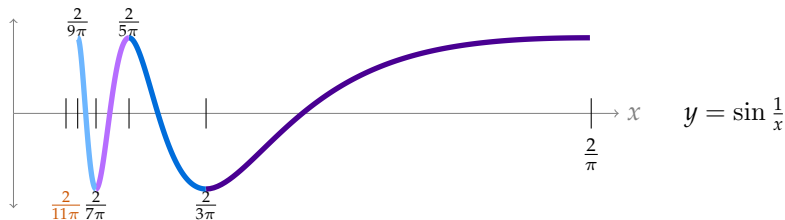
OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

▶ SKIP SKETCHING



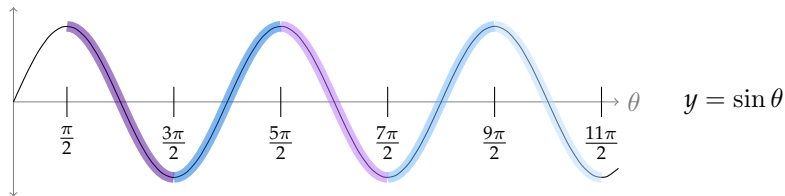
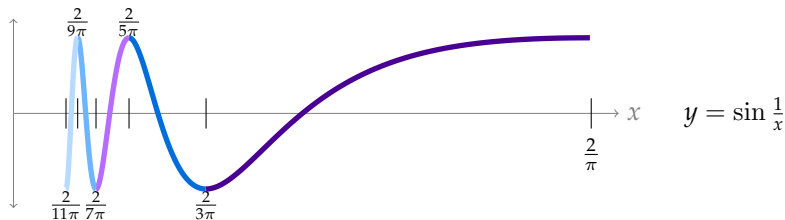
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▶ SKIP SKETCHING



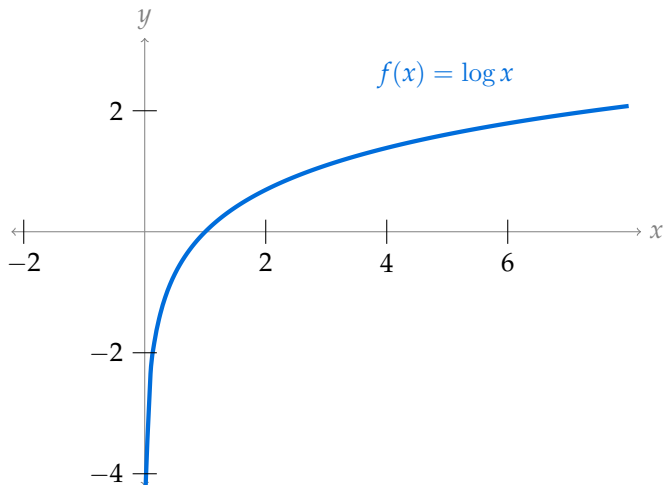
OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$

▶ SKIP SKETCHING



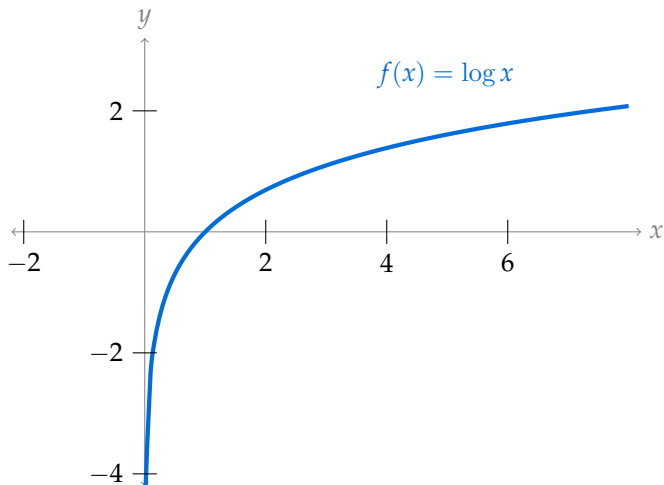
LIMITS AND THE NATURAL LOGARITHM

Where is $f(x)$ defined, and where is it not defined?



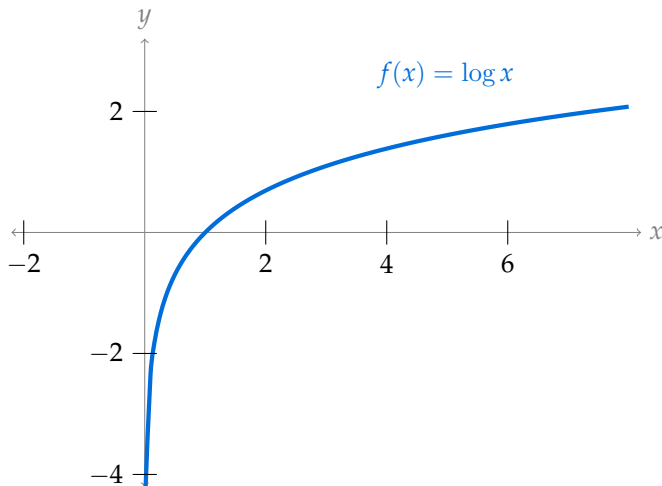
LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of $f(x)$ near 0?



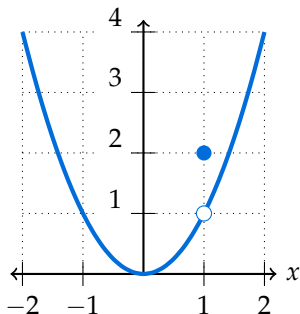
LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of $f(x)$ near 0? $\lim_{x \rightarrow 0^+} \log(x) = -\infty$



Section 1.3 Review

$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x \rightarrow 1} f(x)$?

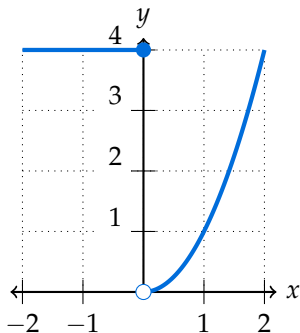
A. $\lim_{x \rightarrow 1} f(x) = 2$

B. $\lim_{x \rightarrow 1} f(x) = 1$

C. $\lim_{x \rightarrow 1} f(x)$ DNE

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0} f(x)$?

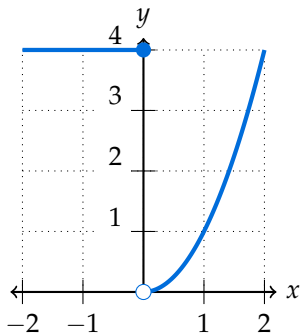
A. $\lim_{x \rightarrow 0} f(x) = 4$

B. $\lim_{x \rightarrow 0} f(x) = 0$

C. $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0} f(x)$?

A. $\lim_{x \rightarrow 0} f(x) = 4$

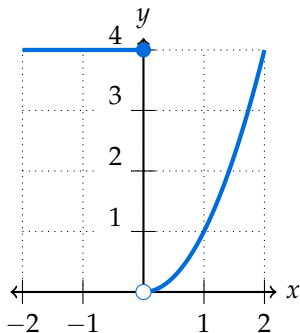
B. $\lim_{x \rightarrow 0} f(x) = 0$

C. $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$\lim_{x \rightarrow 0} f(x)$ DNE

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0^+} f(x)$?

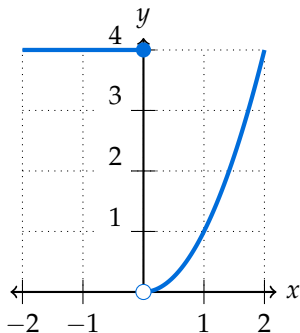
A. $\lim_{x \rightarrow 0} f(x) = 4$

B. $\lim_{x \rightarrow 0} f(x) = 0$

C. $\lim_{x \rightarrow 0} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0^+} f(x)$?

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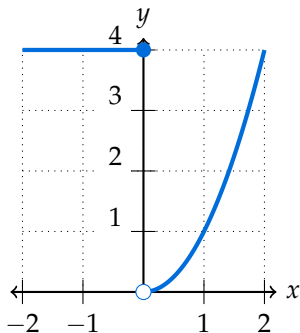
B. $\lim_{x \rightarrow 0^+} f(x) = 0$

C. $\lim_{x \rightarrow 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

What is $f(0)$?



Suppose $\lim_{x \rightarrow 3^-} f(x) = 1$ and $\lim_{x \rightarrow 3^+} f(x) = 1.5$.

Does $\lim_{x \rightarrow 3} f(x)$ exist?

- A. Yes, certainly, because the limits from both sides exist.
- B. No, never, because the limit from the left is not the same as the limit from the right.
- C. Can't tell. For some functions it might exist, for others not.

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Suppose $\lim_{x \rightarrow 3^-} f(x) = 22 = \lim_{x \rightarrow 3^+} f(x)$.

Does $\lim_{x \rightarrow 3} f(x)$ exist?

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- B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.
- C. Can't tell. We need to know the value of the function at $x = 3$.

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Included Work



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3-11