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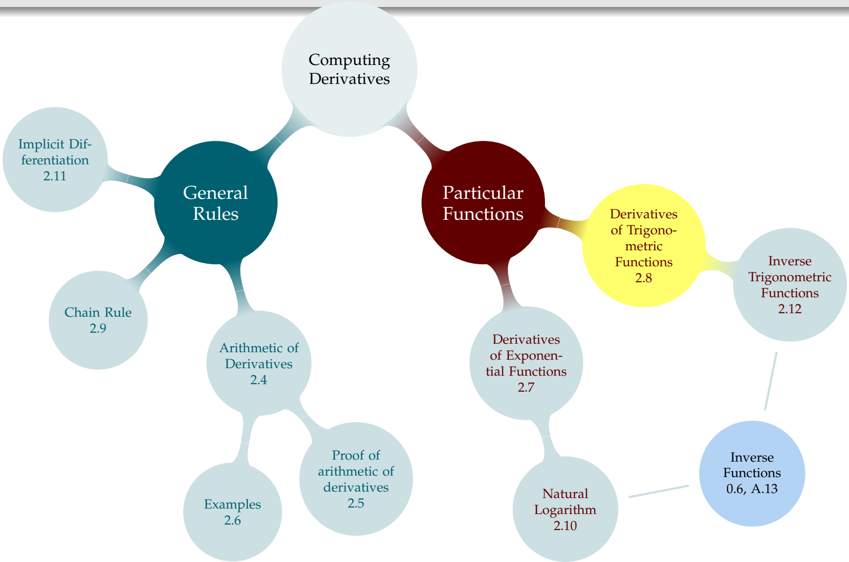
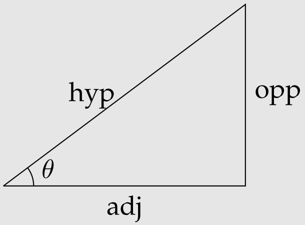
SKIP TRIG REVIEW



## Basic Trig Functions




$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

# Commonly used facts

- Graphs of sine, cosine, tangent

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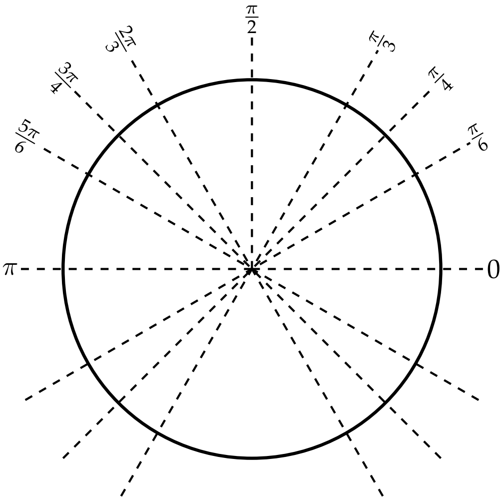
- Conversion between radians and degrees

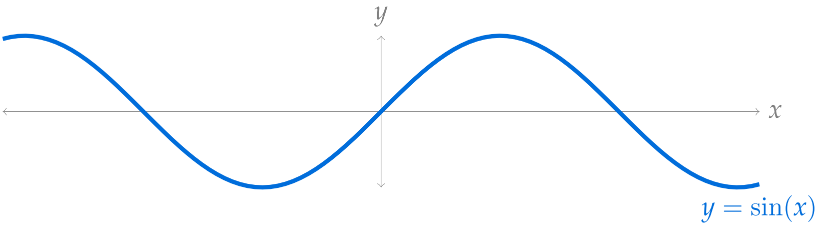
CLP-1 has an appendix on high school trigonometry that you should be familiar with.

# Reference Angles



# Derivative of Sine

SKIP PROOFS OF SINE AND COSINE DERIVATIVES



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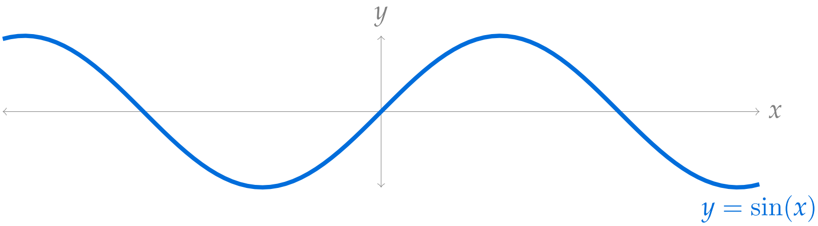


Consider the derivative of $f(x) = \sin(x)$.

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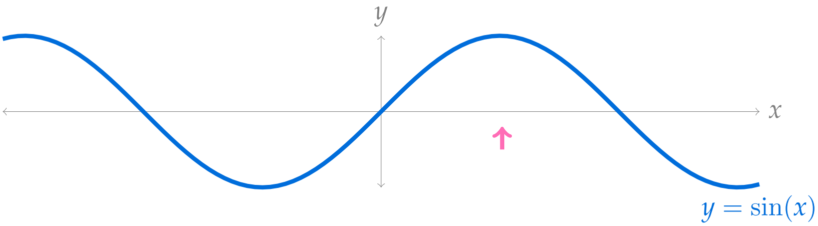


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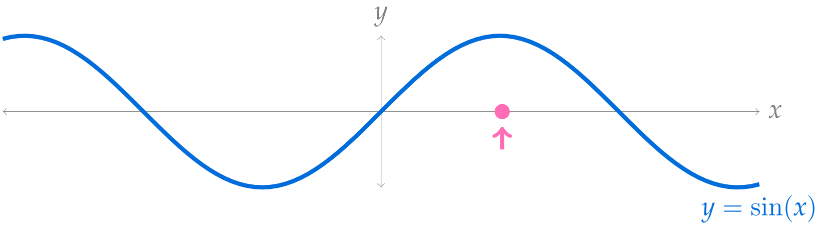


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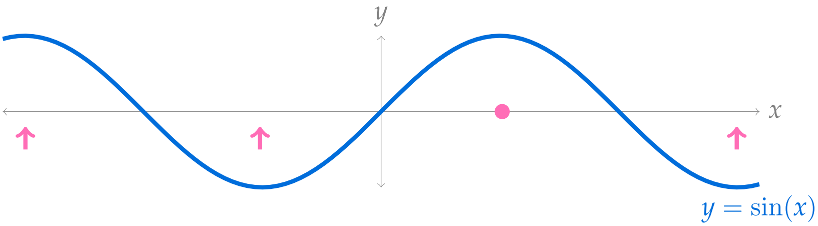


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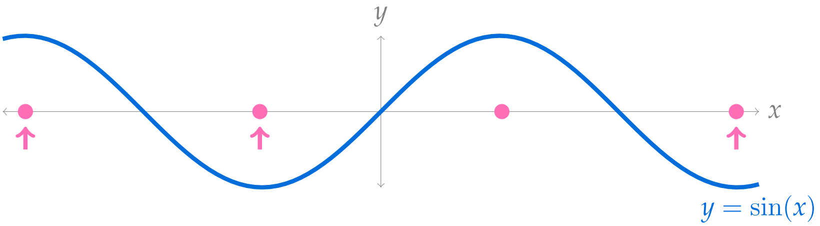


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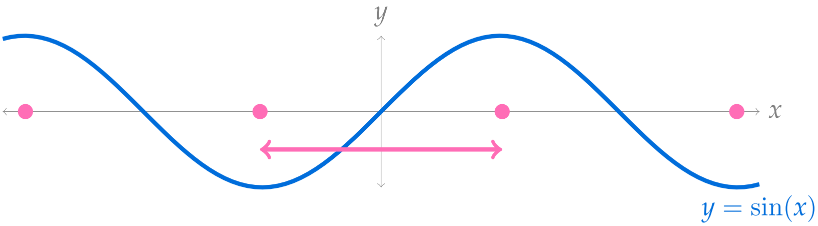


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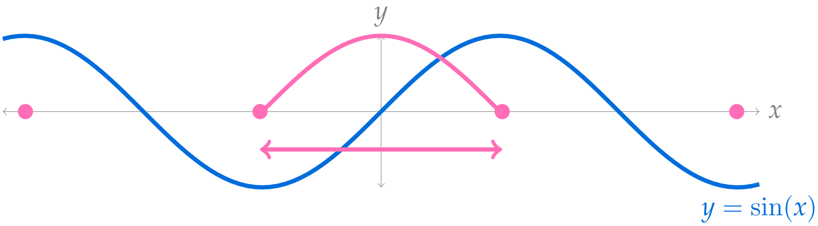
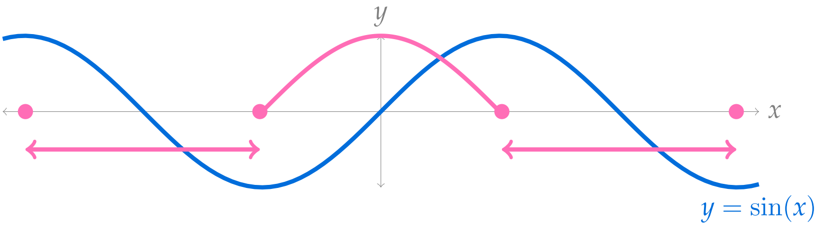



Consider the derivative of $f(x) = \sin(x)$.

# DERIVATIVE OF SINE

SKIP PROOFS OF SINE AND COSINE DERIVATIVES

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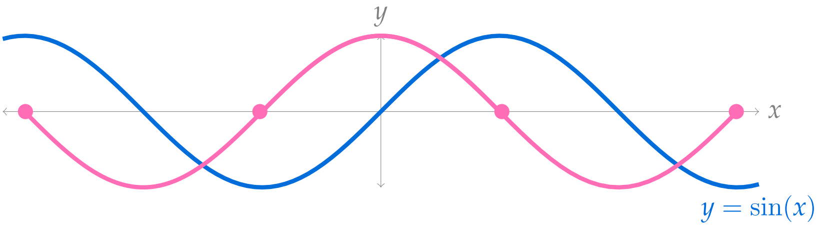


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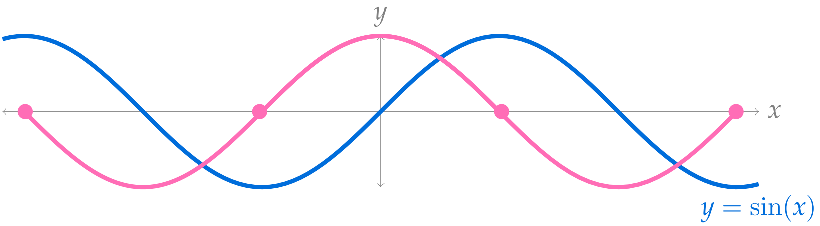


Consider the derivative of $f(x) = \sin(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} \stackrel{?}{=} \cos(x).$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin x\} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin x\} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$\frac{d}{dx}\{\sin x\} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$$

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$$= \sin(x)\left.\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\}\right|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

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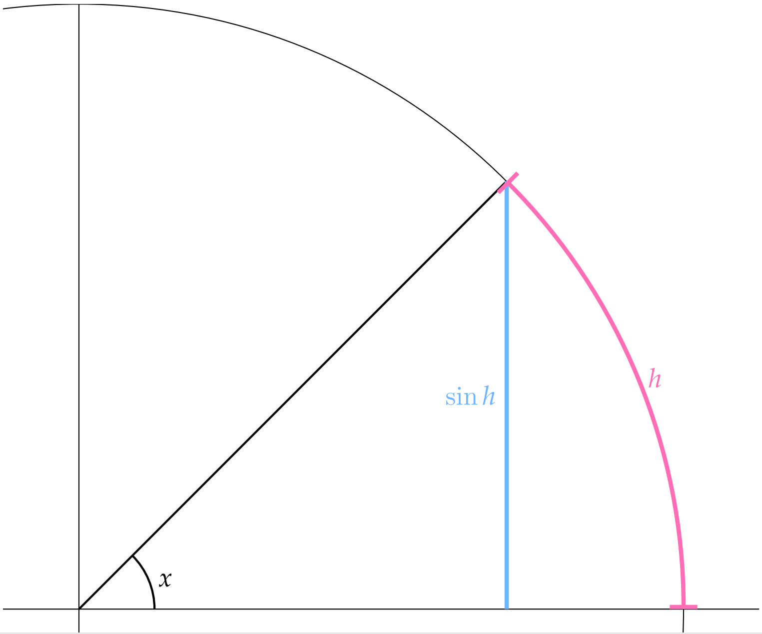
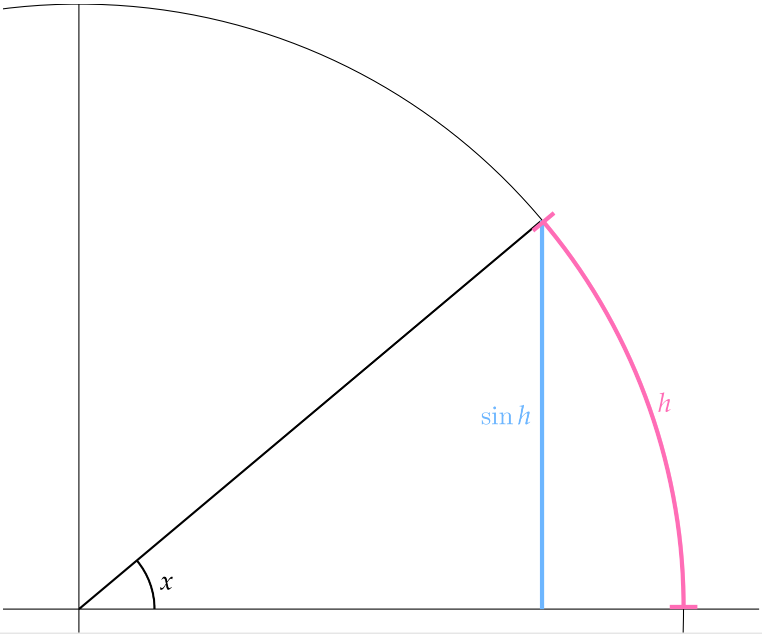
$$= \sin(x)\lim_{h\to 0}\frac{\cos(0+h)-\cos(0)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

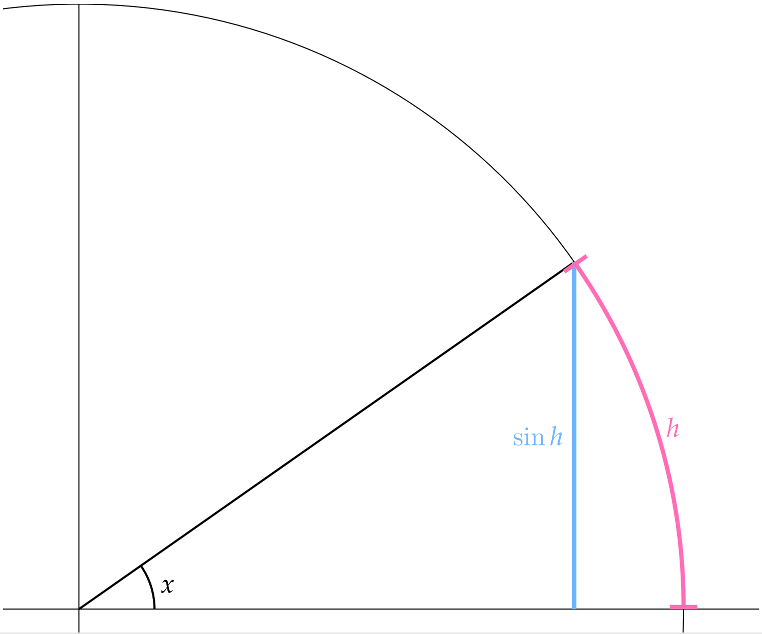
$$= \sin(x)\left.\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\}\right|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \qquad = \qquad \boxed{\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}}$$

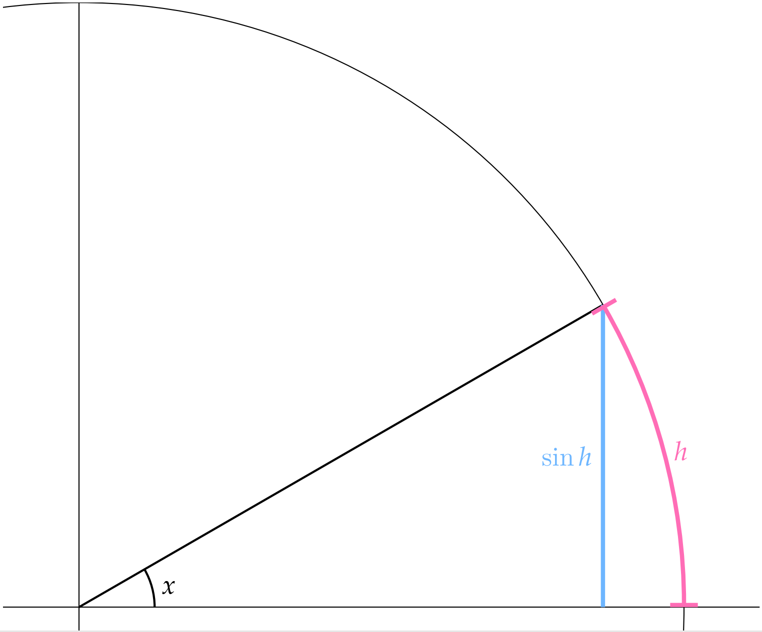
since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.

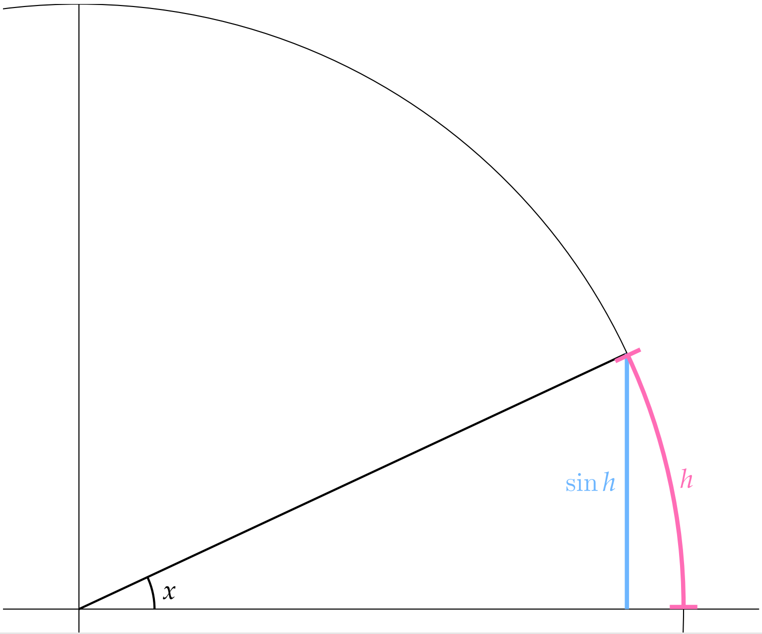
First, we investigate $\lim_{h \to 0} \frac{\sin h}{h}$ informally.

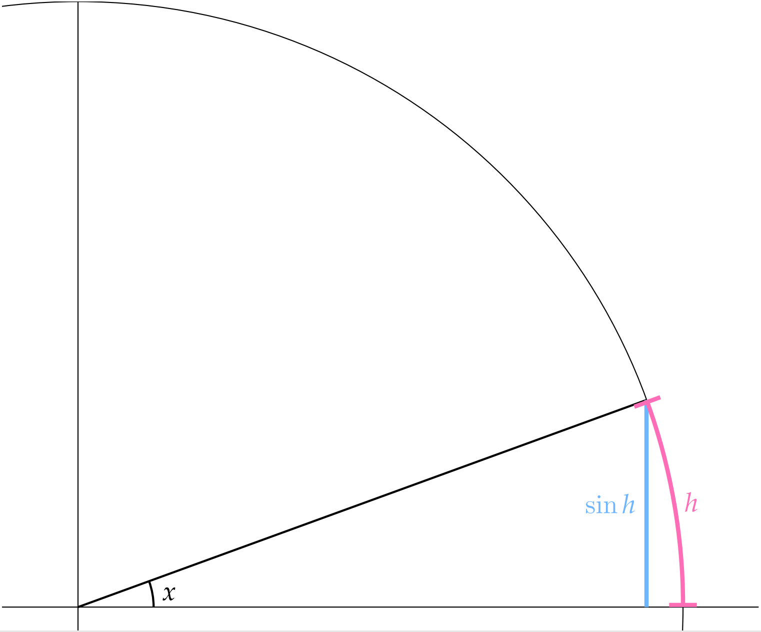

$\sin h$
$h$
$x$

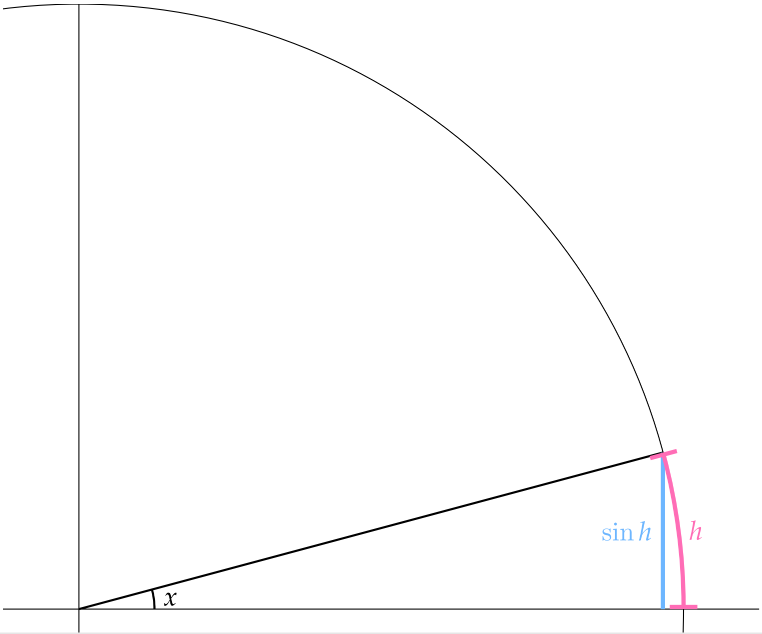

$\sin h$
$h$
$x$

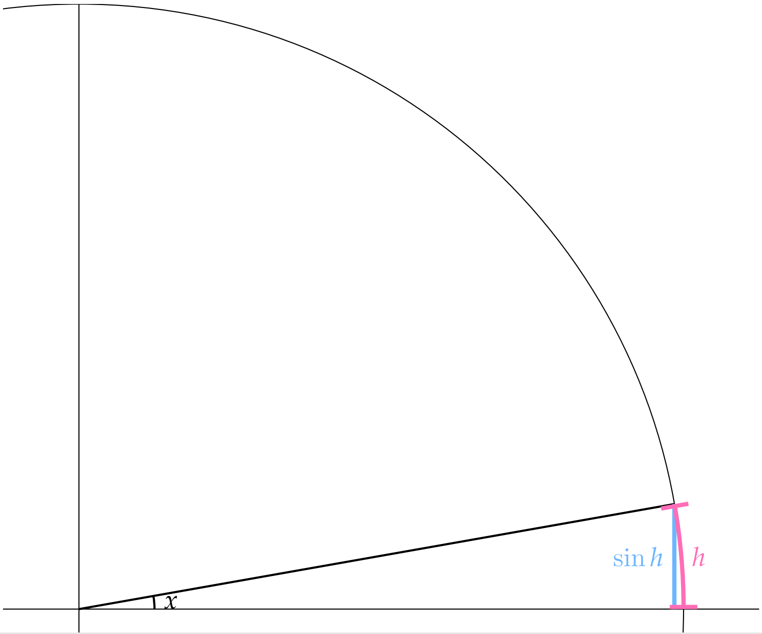

$\sin h$
$h$
$x$


$x$
$\sin h$
$h$


$x$
$\sin h$
$h$


sin h
h
x

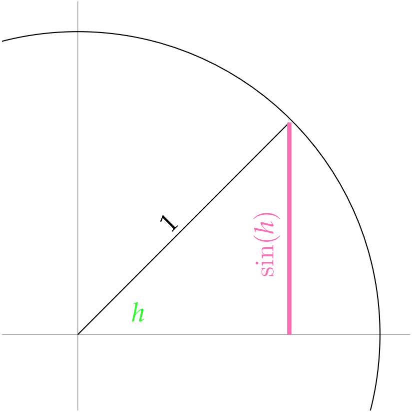

$x$
$\sin h$
$h$

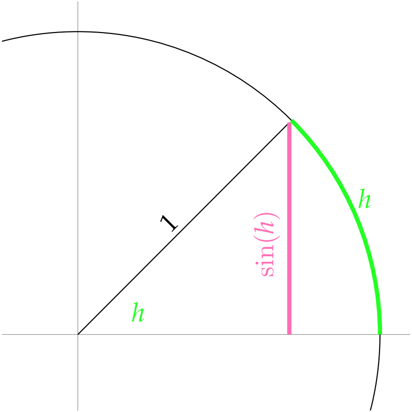

x
sin h
h

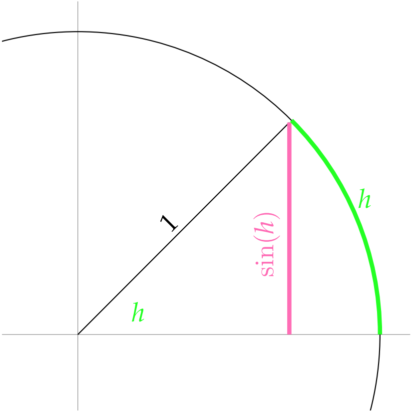
It seems $\sin h \approx h$ when $h \approx 0$, so $\lim_{h \to 0} \frac{\sin h}{h} \stackrel{?}{=} 1$.

We can prove this more formally using the Squeeze Theorem and more trigonometry. We will first prove that $\frac{\sin(h)}{h} \leq 1$ and then we will prove that $\frac{\sin(h)}{h} \geq \cos(h)$. Then we will apply the Squeeze Theorem.

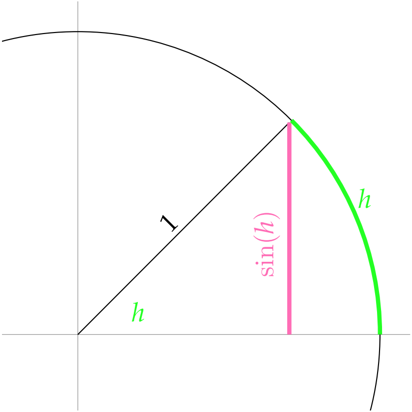
Here is the proof that $\frac{\sin(h)}{h} \leq 1$.


1
$\sin(h)$
$h$


1
$h$
$\sin(h)$
$h$

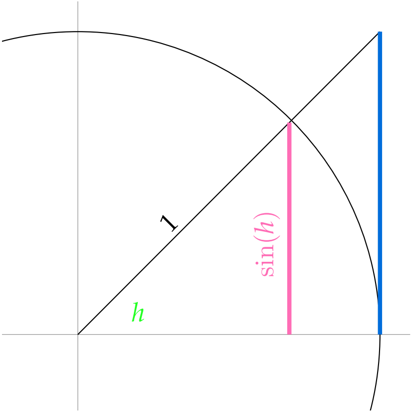


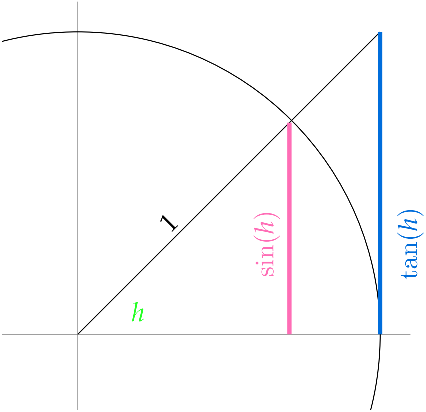

$\sin(h) \leq h$

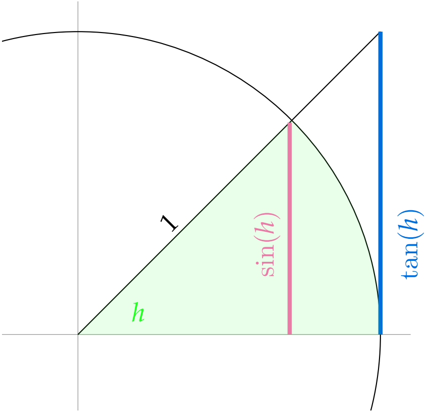


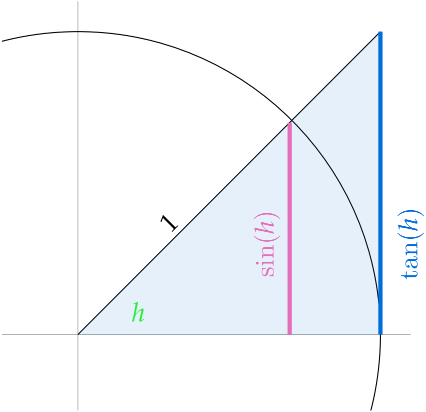

$\sin(h) \le h$ so $\boxed{\dfrac{\sin(h)}{h} \le 1}$

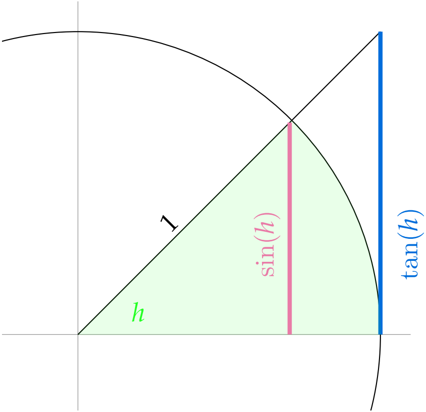
Now for the proof that $\dfrac{\sin(h)}{h} \ge \cos(h)$.


1
$\sin(h)$
$h$

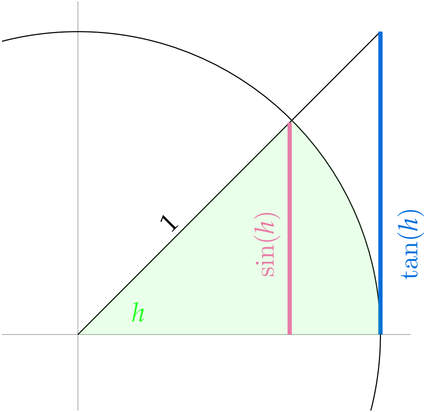

1
$h$
$\sin(h)$
$\tan(h)$


1
$h$
$\sin(h)$
$\tan(h)$

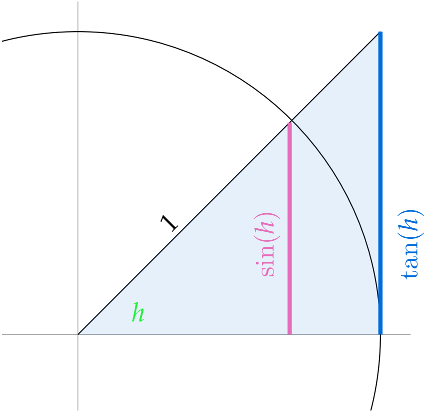

1
$h$
$\sin(h)$
$\tan(h)$




green area:

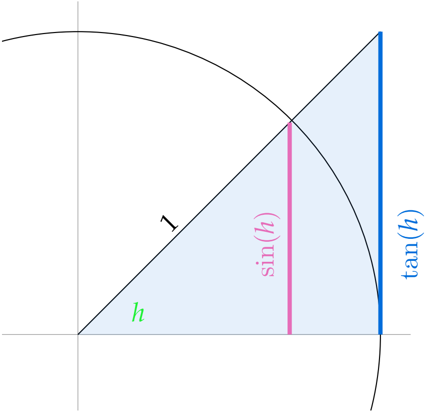



green area: $\frac{h}{2}$



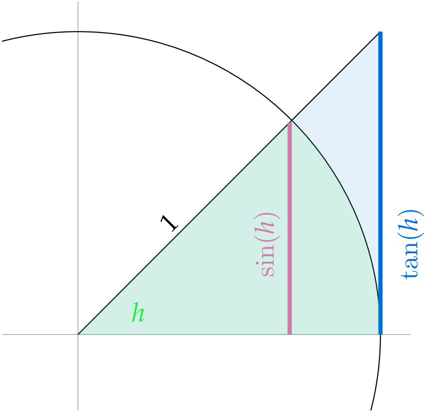

green area: $\dfrac{h}{2}$

Blue area:




green area: $\dfrac{h}{2}$

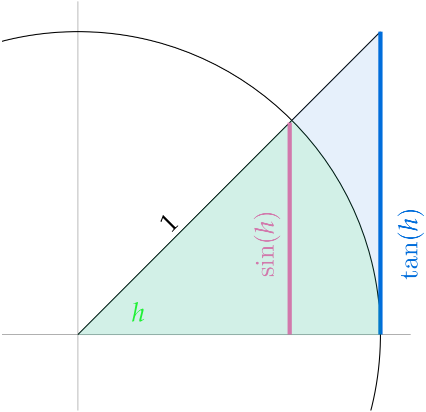
Blue area: $\dfrac{\tan h}{2}$




green area: $\frac{h}{2}$

$$\frac{h}{2} \leq \frac{\tan(h)}{2}$$

Blue area: $\frac{\tan h}{2}$




green area: $\dfrac{h}{2}$

$$\frac{h}{2} \le \frac{\tan(h)}{2}$$

Blue area: $\dfrac{\tan h}{2}$

$$\boxed{\cos(h) \le \frac{\sin(h)}{h}}$$

We are now ready for the Squeeze Theorem. We have

$$\cos h \qquad \leq \qquad \frac{\sin h}{h} \qquad \leq \qquad 1$$

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$$\cos h \leq \frac{\sin h}{h} \leq 1$$

$$\lim_{h\to 0} \cos h = 1 \qquad \lim_{h\to 0} 1 = 1$$

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$$\begin{array}{ccccc} \cos h & \leq & \dfrac{\sin h}{h} & \leq & 1 \\ \lim_{h\to 0} \cos h = 1 & & & & \lim_{h\to 0} 1 = 1 \end{array}$$

By the Squeeze Theorem,

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$$\cos h \qquad \leq \qquad \frac{\sin h}{h} \qquad \leq \qquad 1$$

$$\lim_{h\to 0} \cos h = 1 \qquad\qquad\qquad\qquad \lim_{h\to 0} 1 = 1$$

By the Squeeze Theorem,

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

# Derivatives of Sine and Cosine

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} =$$

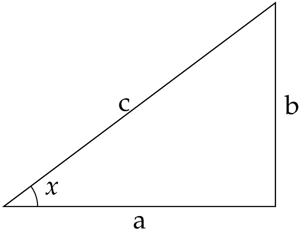
# Derivatives of Sine and Cosine

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x)$$

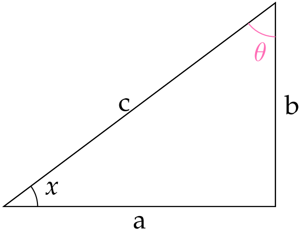
# Derivative of Cosine

Now for the derivative of $\cos$. We already know the derivative of $\sin$, and it is easy to convert between $\sin$ and $\cos$ using trig identities.



# Derivative of Cosine

Now for the derivative of cos. We already know the derivative of sin, and it is easy to convert between sin and cos using trig identities.



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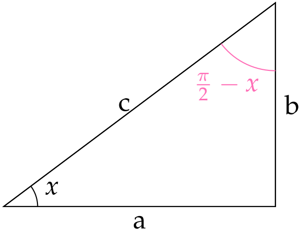
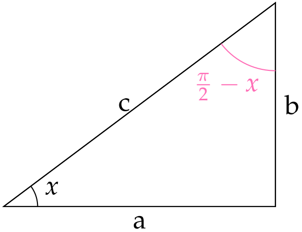


# Derivative of Cosine

Now for the derivative of cos. We already know the derivative of sin, and it is easy to convert between sin and cos using trig identities.




$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

# Derivative of Cosine

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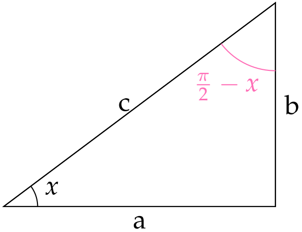



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

$$\frac{d}{dx}[\cos(x)] = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = -\frac{d}{dx}\left[\sin\left(x - \frac{\pi}{2}\right)\right] = -\cos\left(x - \frac{\pi}{2}\right) = -\sin x$$

since $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

When we use radians:

## Derivatives of Trig Functions

$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} = -\sin(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} =$

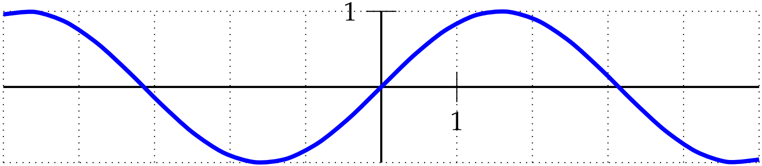
$\frac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} =$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} =$

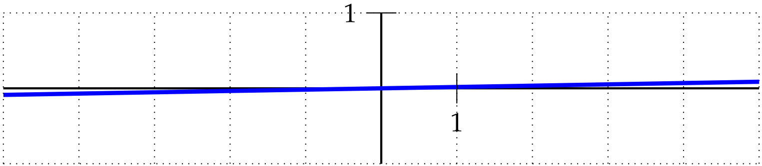
$\frac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} =$

## Honorable Mention

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$




$y = \sin x$, radians




$y = \sin x$, degrees

# Other Trig Functions

Skip proofs of other trig derivatives

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

# Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\tan(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin(x)}{\cos(x)}\right] \\
&= \frac{\cos(x)\cos(x) - \sin(x)[-\sin(x)]}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x)
\end{aligned}$$

# Other Trig Functions

Skip proofs of other trig derivatives

$$\sec(x) = \frac{1}{\cos(x)}$$

# Other Trig Functions

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$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}
\frac{d}{dx}[\sec(x)] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] \\
&= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} \\
&= \sec(x)\tan(x)
\end{aligned}$$

# Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\csc(x) = \frac{1}{\sin(x)}$$

# Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}
\frac{d}{dx}[\csc(x)] &= \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] \\
&= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin(x)}\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x)
\end{aligned}$$

# Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

# Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\cot(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{\sin(x)}\right] \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}$$

# Memorize

$$\frac{d}{dx}\{\sin(x)\} = \cos(x)$$
$$\frac{d}{dx}\{\cos(x)\} = -\sin(x)$$
$$\frac{d}{dx}\{\tan(x)\} = \sec^2(x)$$
$$\frac{d}{dx}\{\sec(x)\} = \sec(x)\tan(x)$$
$$\frac{d}{dx}\{\csc(x)\} = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

NOW YOU

1. Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the **definition of the derivative** to find $f'(0)$.

2. Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.

3. Let $h(x) = \begin{cases} \frac{\sin x}{x}, & x < 0 \\ \frac{ax+b}{\cos x}, & x \geq 0 \end{cases}$
   Which values of $a$ and $b$ make $h(x)$ continuous at $x = 0$?

Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.


ans

Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$

Which values of $a$ and $b$ make $h(x)$ continuous at $x = 0$?

Q

Practice and Review

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x-a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of $a$ makes $g(x)$ continuous at $x = 0$?

Q

We don't have rules for differentiating $f(x)$ at $x = 0$, so we have to fall back on the definition of the derivative.

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \cos\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$$

Since the limit exists, $f(x)$ is differentiable at 0.

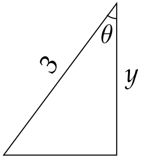
By the definition of continuity, $g(x)$ is continuous at $x = 0$ if

$$\lim_{x\to 0} g(x) = g(0)$$

- $g(0) = (0 - a)^2 = a^2$
- $\displaystyle\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} e^{\frac{\sin x}{x}} = e^1 = e$
- $\displaystyle\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x - a)^2 = a^2$

In order for $g(x)$ to be continuous, we need $a^2 = e$. That is, $a = \sqrt{e}$ or $a = -\sqrt{e}$.

A ladder 3 meters long rests against a vertical wall. Let $\theta$ be the angle between the top of the ladder and the wall, measured in radians, and let $y$ be the height of the top of the ladder. If the ladder slides away from the wall, how fast does $y$ change with respect to $\theta$?
When is the top of the ladder sinking the fastest? The slowest?




Q

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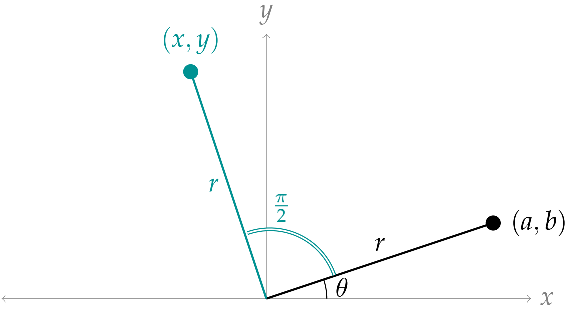
When is the top of the ladder sinking the fastest? The slowest?

We want to find how fast $y$ is changing with respect to $\theta$, so we want $\frac{dy}{d\theta}$, or $y'(\theta)$. To calculate that, we need to find $y$ as a function of $\theta$.
Note that the ladder forms a right triangle with the wall, and $y$ is the side adjacent to $\theta$, while 3 is the hypotenuse. So, $\cos(\theta) = \frac{y}{3}$, hence $y = 3\cos(\theta)$. Now we differentiate, and see

$$\frac{dy}{d\theta} = -3\sin(\theta)$$

To answer the other questions, note that $\theta$ never gets larger than $\pi/2$, since at that point the ladder is lying on the ground. When $0 \leq \theta \leq \pi/2$, the smaller $\theta$ gives the smaller rate of change (in absolute value); so the top of the ladder is sinking slowly at first, then faster and faster, fastest just as it hits the ground.

Suppose a point in the plane that is $r$ centimetres from the origin, at an angle of $\theta$ $(0 \leq \theta \leq \frac{\pi}{2})$, is rotated $\pi/2$ radians. What is its new coordinate $(x, y)$? If the point rotates at a constant rate of $a$ radians per second, when is the $x$ coordinate changing fastest and slowest with respect to $\theta$?




Q

ans

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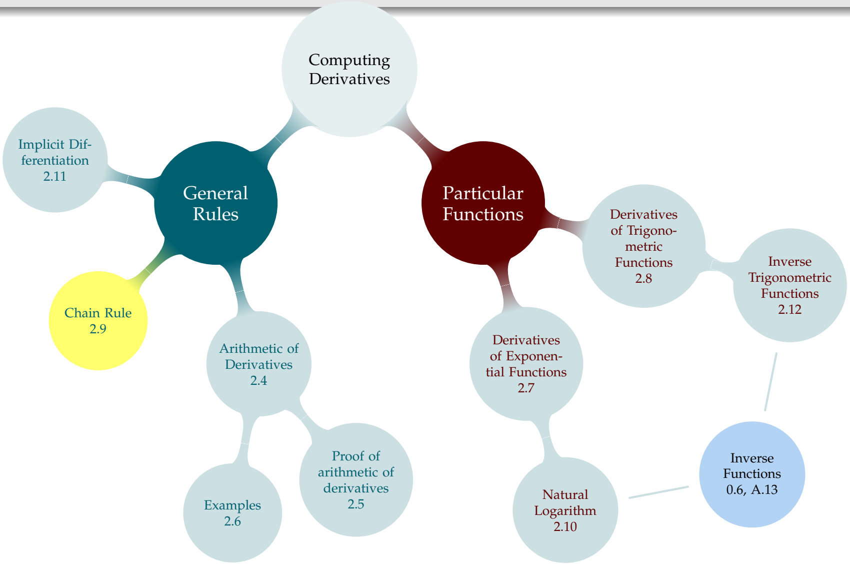
$$x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin(\theta)$$

and

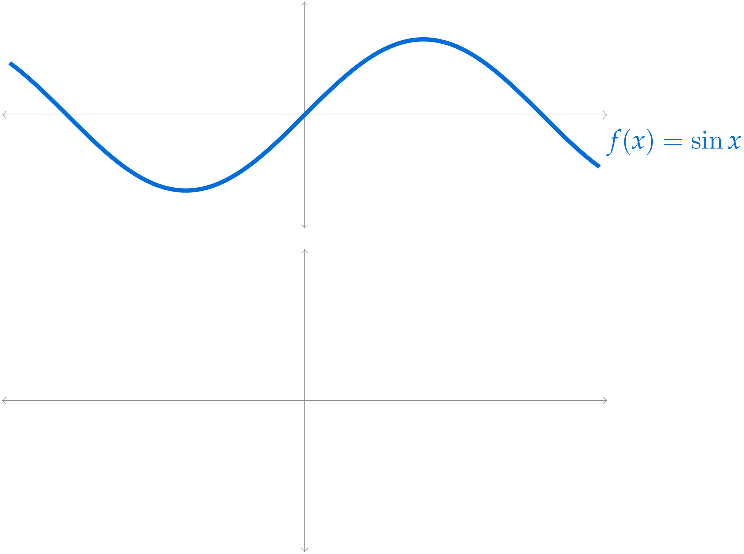
$$y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos(\theta)$$

To find how fast $x$ is changing with respect to $\theta$, we take $x'(\theta) = -r\cos(\theta)$. We see that when $\theta = 0$, $x$ changes a lot when $\theta$ changes; and when $\theta = \pi/2$, $x$ only changes a little when $\theta$ changes.

# Table of Contents



# INTUITION: $\sin x$ VERSUS $\sin(2x)$



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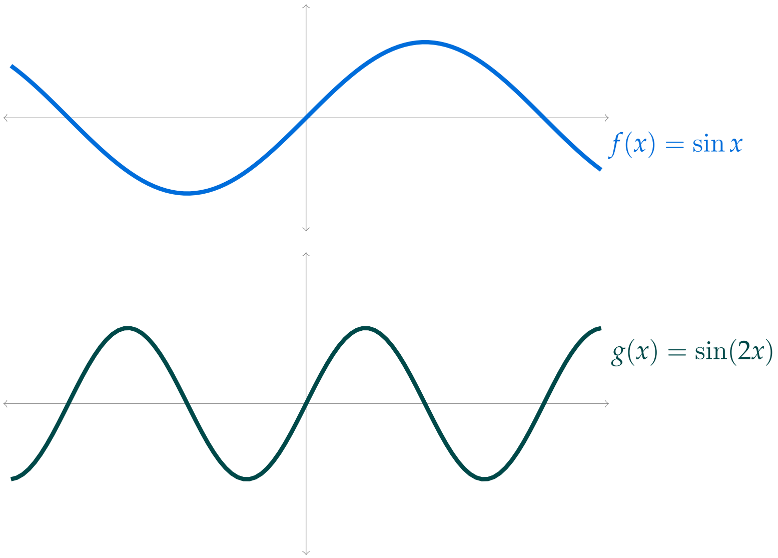
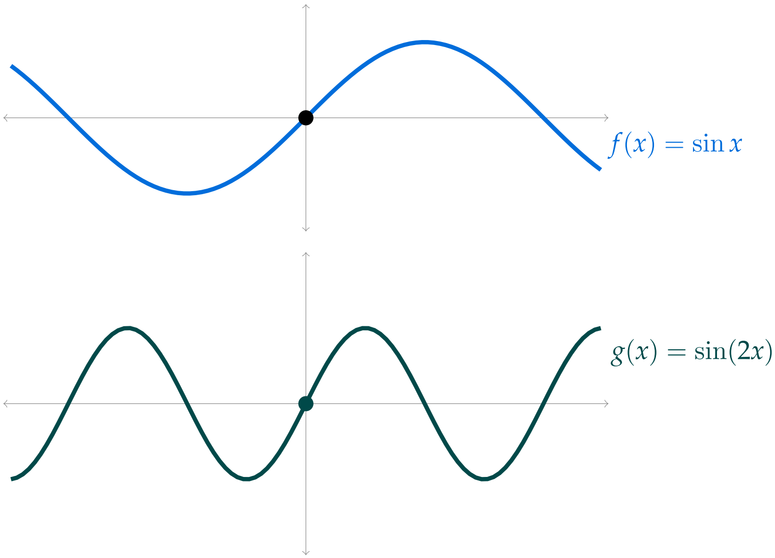


# Intuition: $\sin x$ versus $\sin(2x)$



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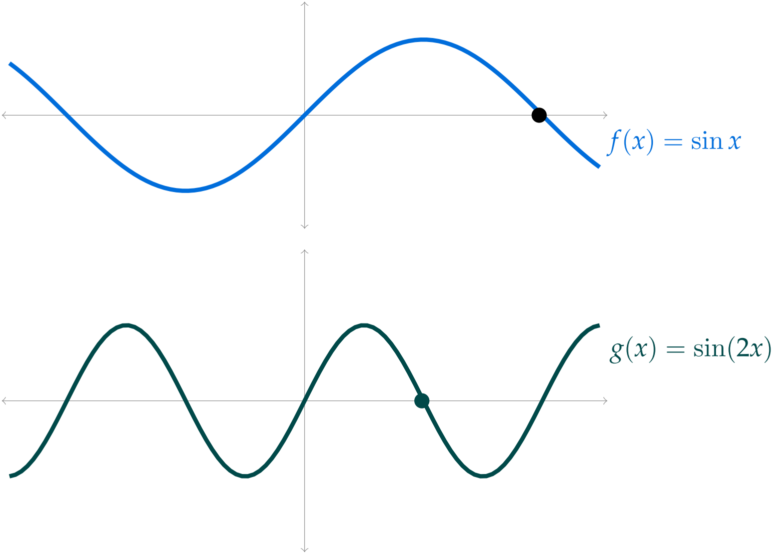

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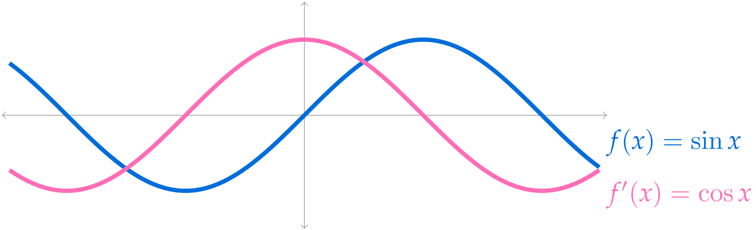


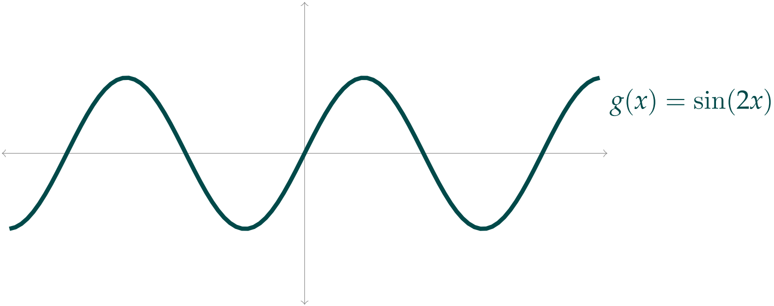

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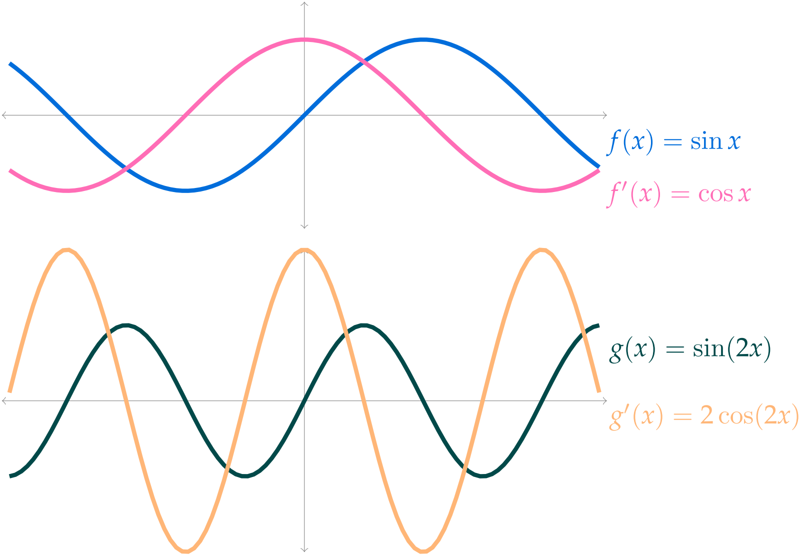

# Compound Functions

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*. Available from https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp

# Kelp Population

| | |
|---|---|
| $k$ | kelp population |
| $u$ | urchin population |
| $o$ | otter population |

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Should $\frac{\mathrm{d}}{\mathrm{d}o}k\big(u(o)\big)$ be positive or negative?
A. positive $\qquad$ B. negative $\qquad$ C. I'm not sure

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# Differentiating Compound Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} =$$

# Differentiating Compound Functions

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\left(\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right)\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f\left(\boxed{g(x+h)}\right)-f\left(\boxed{g(x)}\right)}{\boxed{g(x+h)}-\boxed{g(x)}}\cdot g'(x)
\end{aligned}$$

Set $H=g(x+h)-g(x)$. As $h\to 0$, we also have $H\to 0$. So

$$\begin{aligned}
&= \lim_{H\to 0}\frac{f(g(x)+H)-f(g(x))}{H}\cdot g'(x)\\
&= f'(g(x))\cdot g'(x)
\end{aligned}$$

# Chain Rule

## Chain Rule – Theorem 2.9.3

Suppose $f$ and $g$ are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))\,g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

In the case of kelp, $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o)) = \frac{\mathrm{d}k}{\mathrm{d}u}(u(o))\,\frac{\mathrm{d}u}{\mathrm{d}o}(o)$

## Chain Rule

Suppose $f$ and $g$ are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f\left(g(x)\right)\} = f'\left(g(x)\right) g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}\left(g(x)\right)\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

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Example: suppose $F(x) = \sin(e^x + x^2)$.

We can differentiate $\sin(x)$, so let's set $g(x) = e^x + x^2$ and $f(g) = \sin(g)$. Then $F(x) = f(g(x))$.

$g'(x) = e^x + 2x$ and $\frac{\mathrm{d}f}{\mathrm{d}g}(g) = \cos(g)$ and
$\frac{\mathrm{d}f}{\mathrm{d}g}(g(x)) = \frac{\mathrm{d}f}{\mathrm{d}g}\left(\boxed{e^x + x^2}\right) = \cos\left(\boxed{e^x + x^2}\right)$

So, $F'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x) = \cos(e^x + x^2)\,(e^x + 2x)$

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$


Q
ans

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$

$$\begin{aligned} F'(v) &= 6\left(\boxed{\frac{v}{v^3+1}}\right)^5 \cdot \frac{(v^3+1)(1)-(v)(3v^2)}{(v^3+1)^2} \\ &= 6\left(\boxed{\frac{v}{v^3+1}}\right)^5 \cdot \frac{-2v^3+1}{(v^3+1)^2} \end{aligned}$$

**Now You**

Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

**Now You**

Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t}u\big(o(t)\big)$.
*Note:* your answer should depend only on $t$: not $o$.

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$$f(x) = (\boxed{10^x + \csc x})^{1/2}$$

Using the chain rule,

$$\begin{aligned} f'(x) &= \frac{1}{2}(\boxed{10^x + \csc x})^{-1/2}(10^x \log_e 10 - \csc x \cot x) \\ &= \frac{10^x \log_e 10 - \csc x \cot x}{2\sqrt{10^x + \csc x}} \end{aligned}$$

**NOW YOU** Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t}u\big(o(t)\big)$.
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*Note:* your answer should depend only on $t$: not $o$.

$$\begin{aligned}
o'(t) &= e^t \\
u'(o) &= \frac{(o+\sin o)(0) - (1)(1+\cos o)}{(o+\sin o)^2} \\
&= \frac{-(1+\cos o)}{(o+\sin o)^2} \\
\frac{d}{dt}u(o(t)) &= u'(o(t))\ o'(t) \\
&= -e^t\left(\frac{1+\cos(o(t))}{\left[o(t)+\sin(o(t))\right]^2}\right) \\
&= -e^t\left(\frac{1+\cos(e^t)}{\left[e^t+\sin(e^t)\right]^2}\right)
\end{aligned}$$

# More Examples

Now You

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\dfrac{1}{x}\right)\right\}$

Now You

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\dfrac{1}{x}\right)\right\}$

Q

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\dfrac{1}{x}\right)\right\}$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\right\}\\
&=2x+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\tan\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\boxed{x^2+\frac{1}{x}}\right\}\\
&=2x+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\tan\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\boxed{x^2+x^{-1}}\right\}\\
&=2x+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\tan\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\left(2x-x^{-2}\right)
\end{aligned}$$

Notice: That first term, $2x$, is not multiplied by anything else.

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

Q

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(x+\left(x+x^{-1}\right)^{-1}\right)^{-1}\right\}$$

$$= -\left(\boxed{x+\left(x+x^{-1}\right)^{-1}}\right)^{-2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\boxed{x+\left(x+x^{-1}\right)^{-1}}\right\}$$

$$= -\left(\boxed{x+\left(x+x^{-1}\right)^{-1}}\right)^{-2}\cdot\left[1+(-1)\left(\boxed{x+x^{-1}}\right)^{-2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\boxed{x+x^{-1}}\right\}\right]$$

$$= -\left(\boxed{x+\left(x+x^{-1}\right)^{-1}}\right)^{-2}\cdot\left[1+(-1)\left(\boxed{x+x^{-1}}\right)^{-2}\cdot\left(1-x^{-2}\right)\right]$$

# Included Work

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