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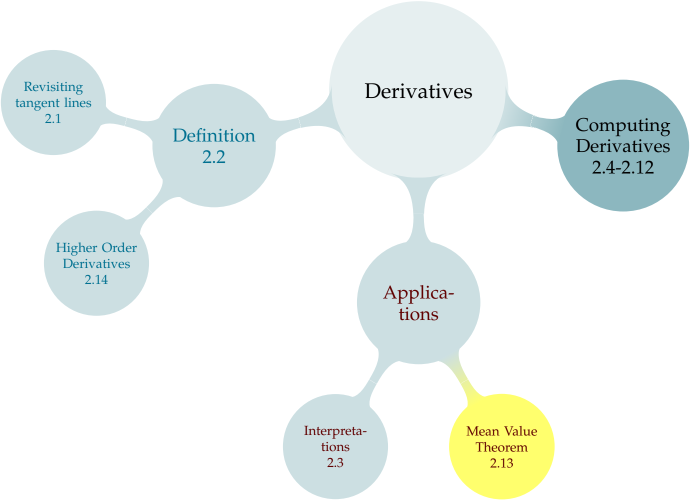
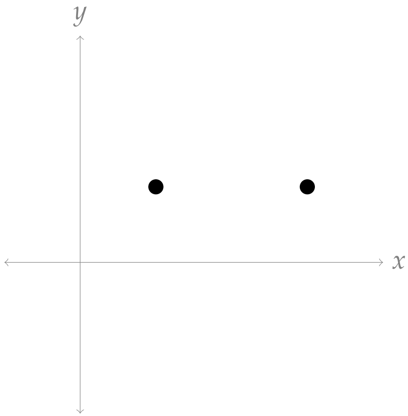


# Rolle's Theorem



## Rolle's Theorem – Theorem 2.13.1

Let $a$ and $b$ be real numbers, with $a < b$. And let $f$ be a function with the properties:

- $f(x)$ is continuous for every $x$ with $a \leq x \leq b$;
- $f(x)$ is differentiable when $a < x < b$;
- and $f(a) = f(b)$.

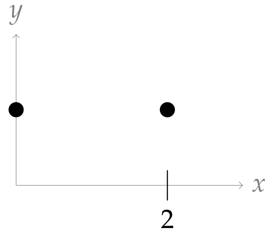
Then there exists a number $c$ with $a < c < b$ such that

$$f'(c) = 0.$$

## Rolle's Theorem – Theorem 2.13.1

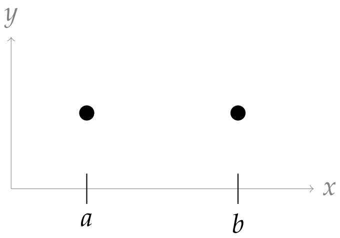
Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on $(a, b)$, and let $f(a) = f(b)$. Then there is a number $c$ strictly between $a$ and $b$ such that $f'(c) = 0$.

Example: Let $f(x) = x^3 - 2x^2 + 1$, and observe $f(2) = f(0) = 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere.



## Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on $(a, b)$, and let $f(a) = f(b)$. Then there is a number $c$ strictly between $a$ and $b$ such that $f'(c) = 0$.




Suppose $a < b$ and $f(a) = f(b)$, $f(x)$ is continuous over $[a, b]$, and $f(x)$ is differentiable over $(a, b)$.

How many different values of $x$ between $a$ and $b$ have $f'(x) = 0$?

A. 0 or 1
B. 1
C. 0, 1, or more
D. 1 or more
E. I'm not sure

## Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on $(a, b)$, and let $f(a) = f(b)$. Then there is a number $c$ strictly between $a$ and $b$ such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f'(x)$ have?

A. precisely six
B. precisely seven
C. at most seven
D. at least six

## Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on $(a, b)$, and let $f(a) = f(b)$. Then there is a number $c$ strictly between $a$ and $b$ such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x)$ is also continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f''(x)$ have?

A. precisely six
B. precisely five
C. at most five
D. at least five

## Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on $(a, b)$, and let $f(a) = f(b)$. Then there is a number $c$ strictly between $a$ and $b$ such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and there are precisely three places where $f'(x) = 0$. How many distinct roots does $f(x)$ have?

A. at most three

B. at most four

C. at least three

D. at least four

## Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on $(a, b)$, and let $f(a) = f(b)$. Then there is a number $c$ strictly between $a$ and $b$ such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x) = 0$ for precisely three values of $x$. How many distinct values $x$ exist with $f(x) = 17$?

A. at most three

B. at most four

C. at least three

D. at least four

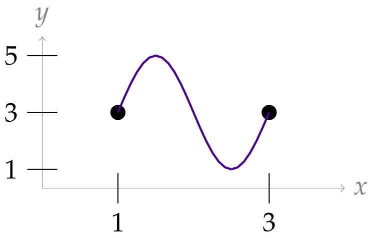
# Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How would you show that $f(x)$ has precisely one real root?

Use Rolle's Theorem to show that the function $f(x) = \frac{1}{3}x^3 + 3x^2 + 9x - 3$ has at most two distinct real roots.

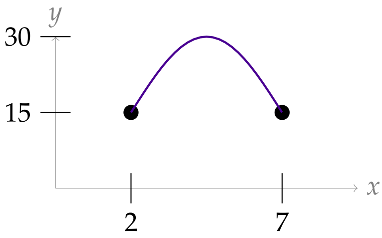
# Average Rate of Change




What is the average rate of change of $f(x)$ from $x = 1$ to $x = 3$?

A. 0
B. 1
C. 2
D. 4
E. I'm not sure

# Average Rate of Change




What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

A. 0
B. 3
C. 5
D. 15
E. I'm not sure

## Rolle's Theorem and Average Rate of Change

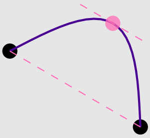
Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on the interval $(a, b)$, and $f(a) = f(b)$. Then there exists a number $c$ strictly between $a$ and $b$ such that

$$f'(c) = 0 = \frac{f(b) - f(a)}{b - a}.$$

So there exists a point where the derivative is the same as the average rate of change.

## Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on $(a, b)$. Then there is a number $c$ strictly between $a$ and $b$ such that:



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is: there is some point $c$ in $(a, b)$ where the instantaneous rate of change of the function is equal to the average rate of change of the function on the interval $[a, b]$.

Suppose you are driving along a long, straight highway with no shortcuts. The speed limit is 100 kph. A police officer notices your car going 90 kph, and uploads your plate and the time they saw you to their database. 150 km down this same straight road, 75 minutes later, another police officer notices your car going 85kph, and uploads your plates to the database. Then they pull you over, and give you a speeding ticket. Why were they justified?

According to this website, Canada geese may fly 1500 miles in a single day under favorable conditions. It also says their top speed is around 70mph. Does this seem like a typo? (If it contradicts the Mean Value Theorem, it's probably a typo.)

The record for fastest wheel-driven land speed is around 700 kph. [1] However, non-wheel driven cars (such as those powered by jet engines) have achieved higher speeds. [2]
Suppose a driver of a jet-powered car starts a 10km race at 12:00, and finishes at 12:01. Did they beat 700kph?

[1] (at time of writing) George Poteet, https://en.wikipedia.org/wiki/Wheel-driven_land_speed_record

[2] https://en.wikipedia.org/wiki/Land_speed_record

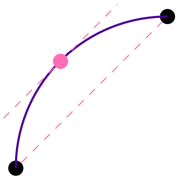
Suppose you want to download a file that is 3000 MB (slightly under 3GB). Your internet provider guarantees you that your download speeds will always be between 1 MBPS (MB per second) and 5 MBPS (because you bought the cheap plan). Using the Mean Value Theorem, give an upper and lower bound for how long the download can take (assuming your providers aren't lying, and your device is performing adequately).

Suppose $1 \leq f'(t) \leq 5$ for all values of $t$, and $f(0) = 0$. What are the possible solutions to $f(t) = 3000$?
Notice: since the derivative exists for all real numbers, $f(x)$ is differentiable and continuous for all real numbers!

## Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over $(a, b)$.

If $f'(x) = 0$ for all $x$ in $(a, b)$, then

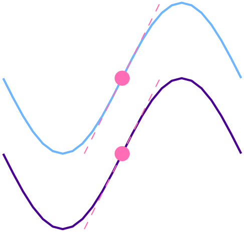


If $f(c) \neq f(d)$, then $\frac{f(d)-f(c)}{d-c} \neq 0$, so $f'(e) \neq 0$ for some $e$.

## Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over $(a, b)$.

If $f'(x) = g'(x)$ for all $x$ in $(a, b)$, then

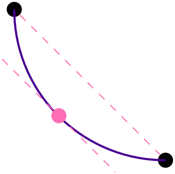


Define a new function $k(x) = f(x) - g(x)$. Then $k'(x) = 0$ everywhere, so (by the last corollary) $k(x) = A$ for some constant $A$.

## Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over $(a, b)$.

If $f'(x) > 0$ for all $x$ in $(a, b)$, then

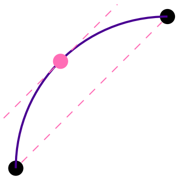


If $f(c) > f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(negative)}}{\text{(positive)}} < 0$. Then $f'(e) < 0$ for some $e$ between $c$ and $d$.

## Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over $(a, b)$.

If $f'(x) < 0$ for all $x$ in $(a, b)$, then



If $f(c) < f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(positive)}}{\text{(positive)}} > 0$. Then $f'(e) > 0$ for some $e$ between $c$ and $d$.

## Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on $(a, b)$. Then there is a number $c$ strictly between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

WARNING: The MVT has two hypotheses.

- $f(x)$ has to be continuous on $[a, b]$.
- $f(x)$ has to be differentiable on $(a, b)$.

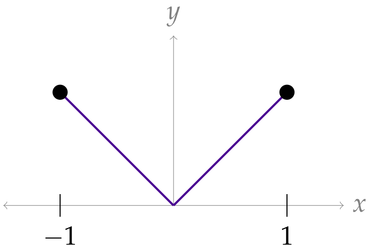
If either of these hypotheses are violated, the conclusion of the MVT can fail. Here are two examples.

## Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on $(a, b)$. Then there is a number $c$ strictly between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Let $a = -1$, $b = 1$ and $f(x) = |x|$.

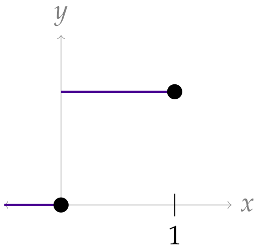


## Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on $(a, b)$. Then there is a number $c$ strictly between $a$ and $b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Let $a = 0$, $b = 1$ and $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$.



## Included Work