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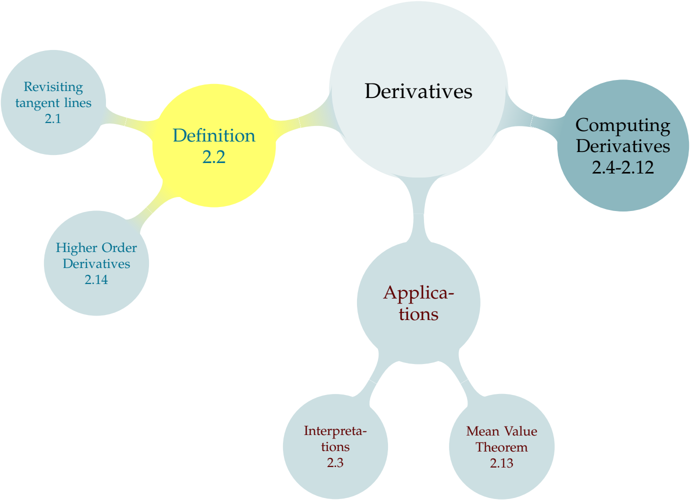


# Derivative at a Point

## Definition 2.2.1

Given a function $f(x)$ and a point $a$, the slope of the tangent line to $f(x)$ at $a$ is the derivative of $f$ at $a$, written $f'(a)$.

So, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$f'(a)$ is also the instantaneous rate of change of $f$ at $a$.

## Derivative

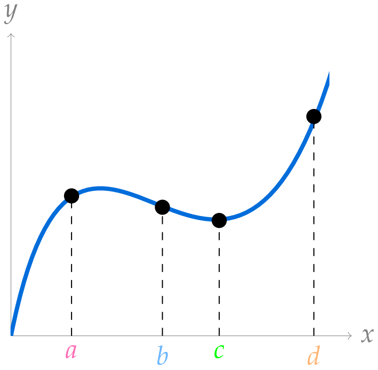
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then $f$ is increasing at $a$. Its graph "points up."

If $f'(a) < 0$, then $f$ is decreasing at $a$. Its graph "points down."

If $f'(a) = 0$, then $f$ looks constant or flat at $a$.

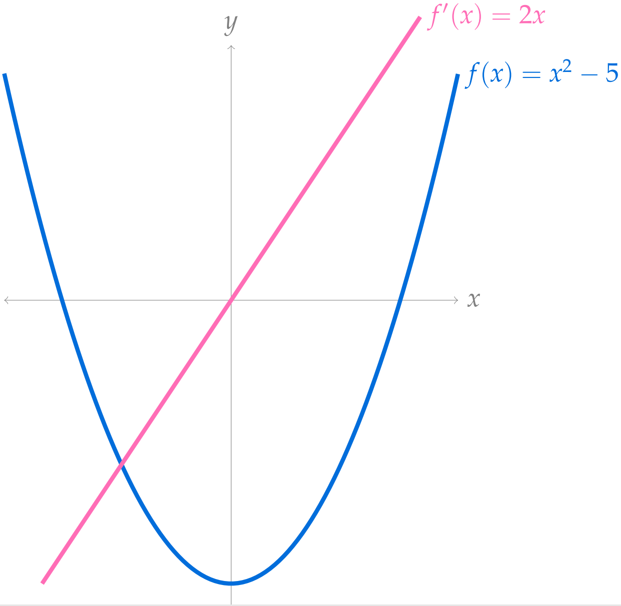
# Practice: Increasing and Decreasing




Where is $f'(x) < 0$? Where is $f'(x) > 0$? Where is $f'(x) \approx 0$?

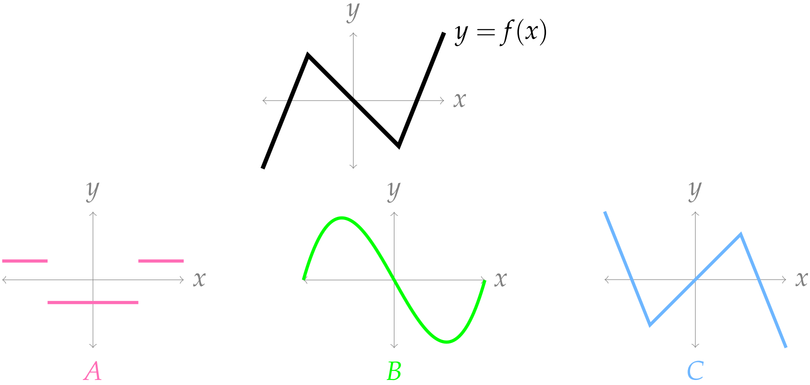
Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.

Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$. We can also find its derivative at an arbitrary point $x$:


$f'(x) = 2x$
$f(x) = x^2 - 5$
$y$
$x$

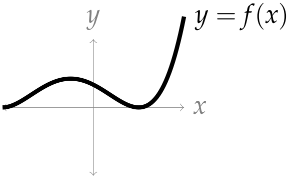
# Increasing and Decreasing

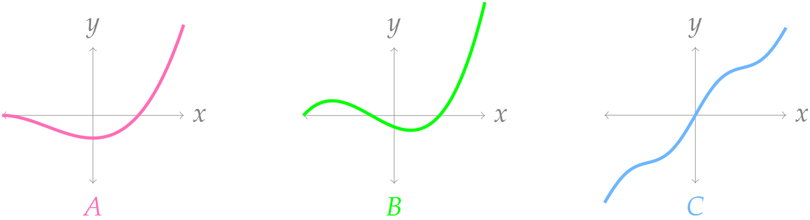
In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



# Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



## Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.
The derivative of $f(x)$ with respect to $x$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that $x$ will be a part of your final expression: this is a function.

If $f'(x)$ exists for all $x$ in an interval $(a, b)$, we say that $f$ is differentiable on $(a, b)$.

## Notation 2.2.8

The "prime" notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibniz notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(a) \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)\bigg|_{x=a}$$

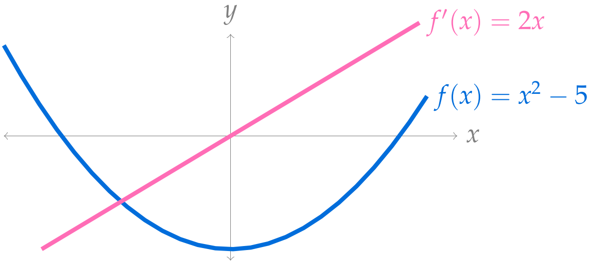
function    number    function    number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibnitz Notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(3) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)\bigg|_{x=3} =$$



## Alternate Definition – Definition 2.2.1

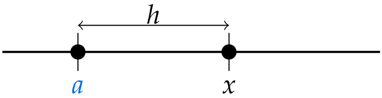
Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

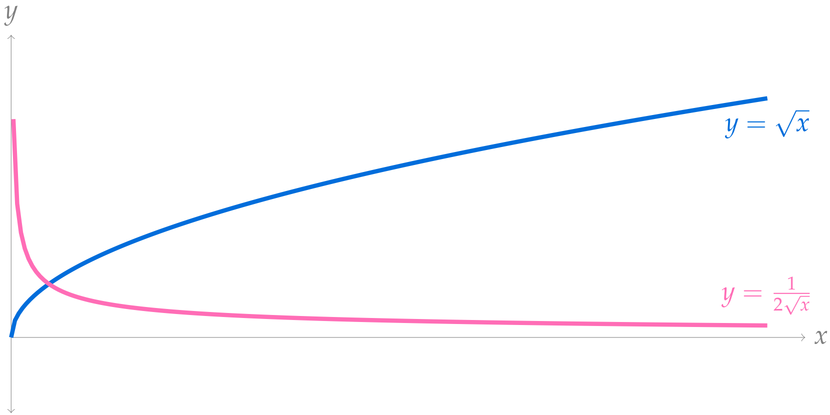
is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.



Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.




$\boxed{\text{Review:}}$ $\lim_{x\to\infty} \sqrt{x} =$ $\lim_{x\to\infty} \frac{1}{2\sqrt{x}} =$

$\lim_{x\to 0^+} \sqrt{x} =$ $\lim_{x\to 0^+} \frac{1}{2\sqrt{x}} =$

NOW YOU

Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\right\}.$$

Using the definition of the derivative, calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x}{x+1}\right\}$.

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{\sqrt{x^2+x}}\right\}$.

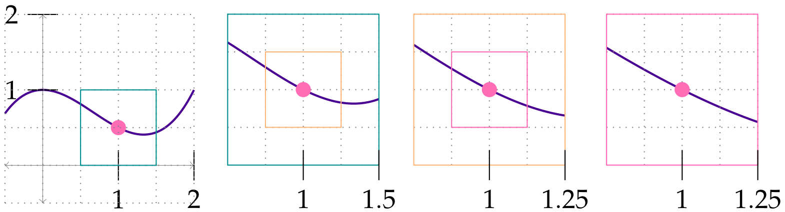
## Memorize

The derivative of a function $f$ at a point $a$ is given by the following limit, if it exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

# Zooming In

For a smooth function, if we zoom in at a point, we see a line:



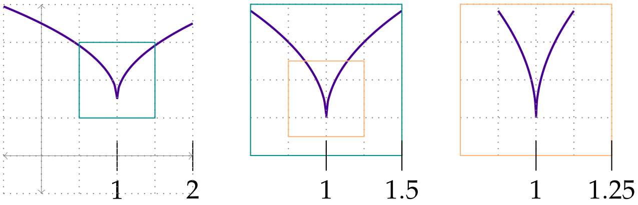

In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

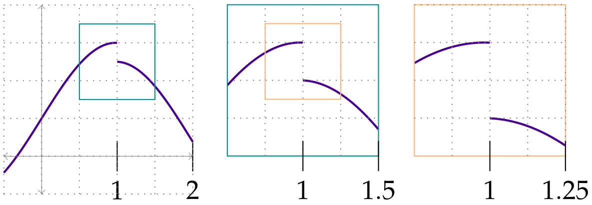
# Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:




Discontinuity:



## Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ does not exist at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

does not exist.
Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

# WHEN DERIVATIVES DON'T EXIST

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

WHEN DERIVATIVES DON'T EXIST

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

As usual, it's nice to reassure students that we did not need to know the graph of this function to answer the question. Otherwise, they might unduly worry.

## Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Let $f(x)$ be a function and let $a$ be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

| | |
|---|---|
| $f(x)$ continuous at $x = a$<br>$f(x)$ differentiable at $x = a$ | $f(x)$ continuous at $x = a$<br>$f(x)$ differentiable at $x = a$ |
| $f(x)$ continuous at $x = a$<br>$f(x)$ differentiable at $x = a$ | $f(x)$ continuous at $x = a$<br>$f(x)$ differentiable at $x = a$ |

# Included Work

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