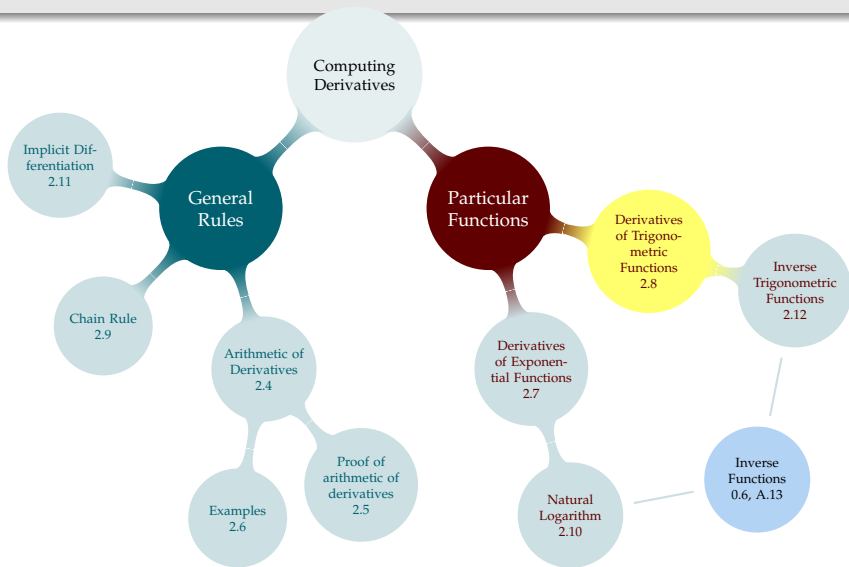
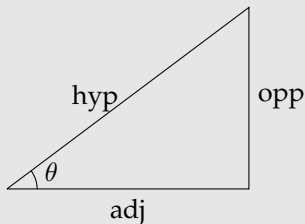


TABLE OF CONTENTS



Basic Trig Functions



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

COMMONLY USED FACTS

- Graphs of sine, cosine, tangent

COMMONLY USED FACTS

- ▶ Graphs of sine, cosine, tangent
- ▶ Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$

COMMONLY USED FACTS

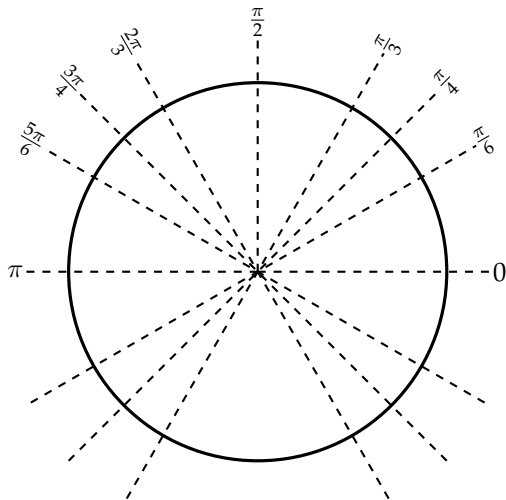
- ▶ Graphs of sine, cosine, tangent
- ▶ Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- ▶ How to use reference angles to find sine, cosine and tangent of other angles

COMMONLY USED FACTS

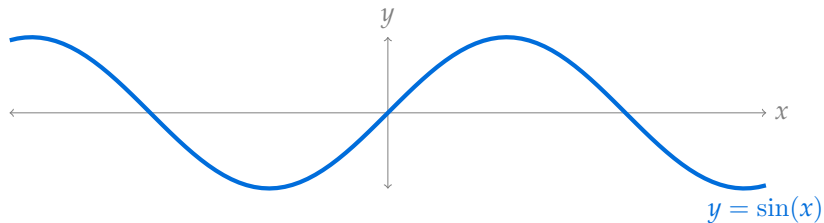
- ▶ Graphs of sine, cosine, tangent
- ▶ Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- ▶ How to use reference angles to find sine, cosine and tangent of other angles
- ▶ Identities: $\sin^2 x + \cos^2 x = 1; \quad \tan^2 x + 1 = \sec^2 x;$
 $\sin^2 x = \frac{1 - \cos(2x)}{2}; \quad \cos^2 x = \frac{1 + \cos 2x}{2}$

- CLP-1 has an appendix on high school trigonometry that you should be familiar with.

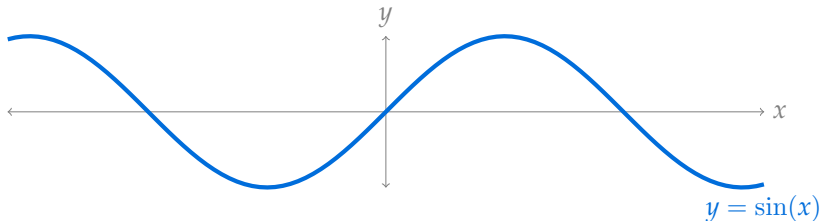
REFERENCE ANGLES



DERIVATIVE OF SINE

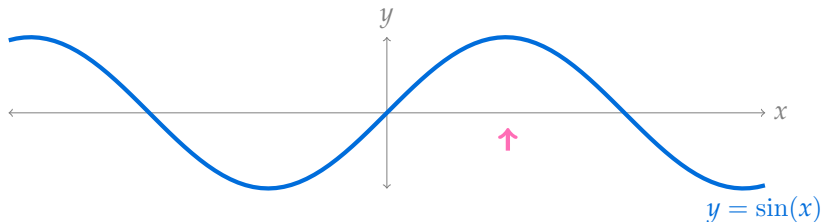
[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

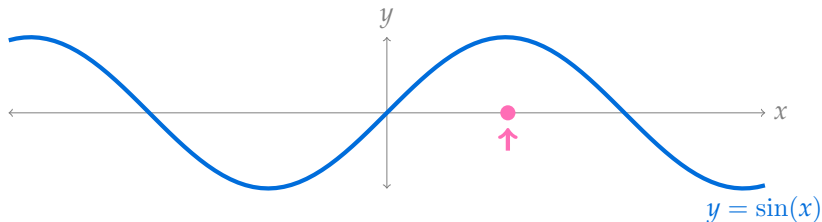
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

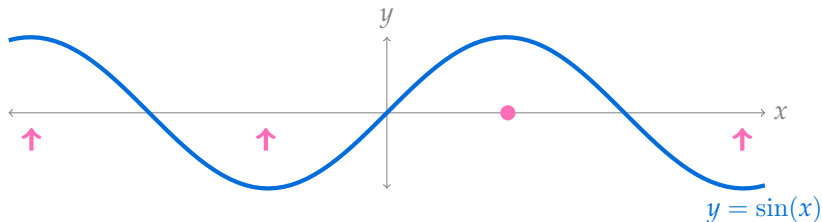
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

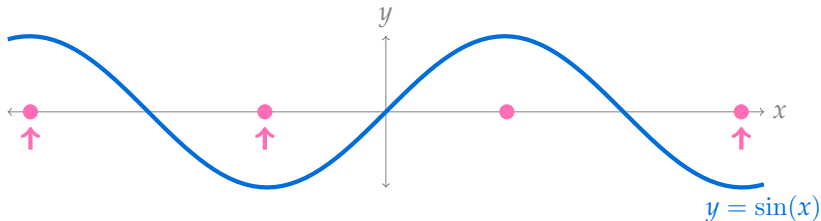
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

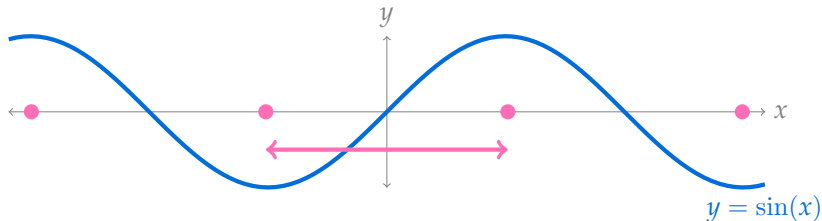
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

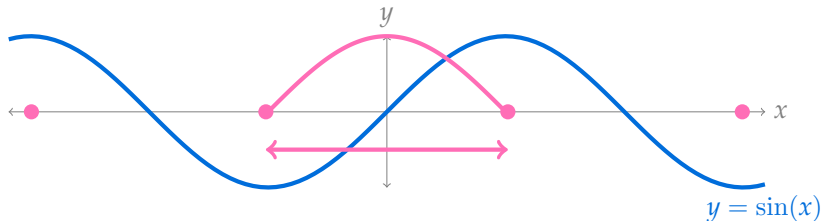
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

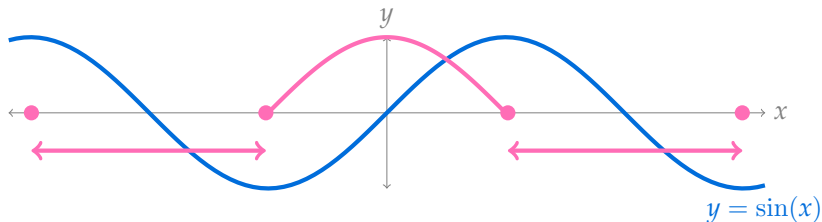
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

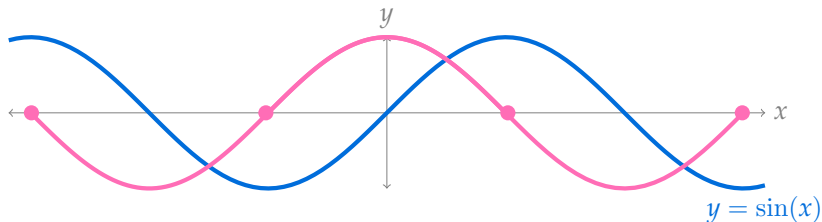
Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

[▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

Consider the derivative of $f(x) = \sin(x)$.

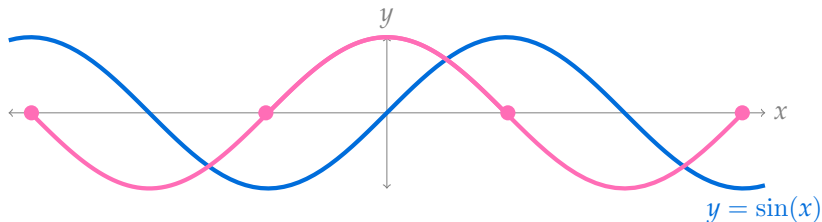
DERIVATIVE OF SINE

[▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES](#)

Consider the derivative of $f(x) = \sin(x)$.

DERIVATIVE OF SINE

▶▶ SKIP PROOFS OF SINE AND COSINE DERIVATIVES



Consider the derivative of $f(x) = \sin(x)$.

$$\frac{d}{dx}\{\sin(x)\} \stackrel{?}{=} \cos(x).$$

$$\frac{d}{dx} \{\sin x\} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\frac{d}{dx} \{\sin x\} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}$$

$$\begin{aligned}\frac{d}{dx} \{\sin x\} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x) \sin(h)}{h}\end{aligned}$$

$$\frac{d}{dx} \{\sin x\} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x) \sin(h)}{h}$$

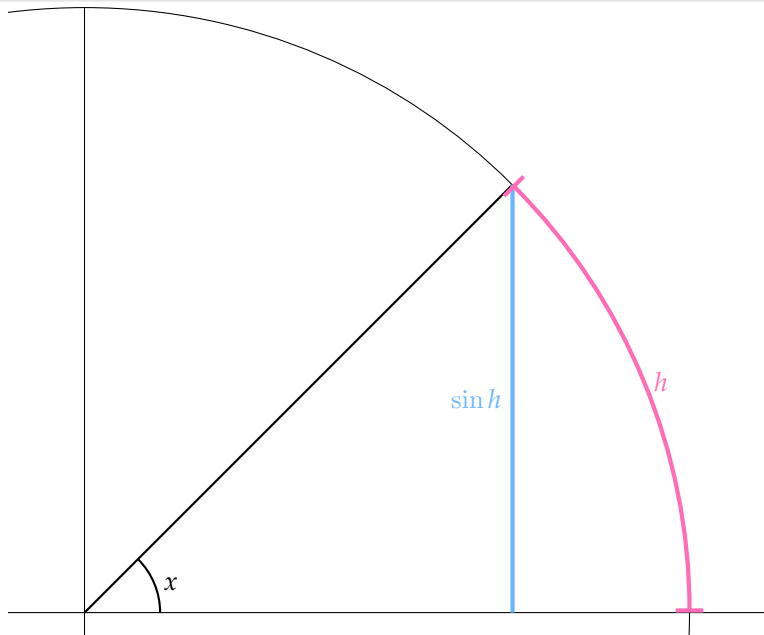
$$= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(0+h) - \cos(0)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

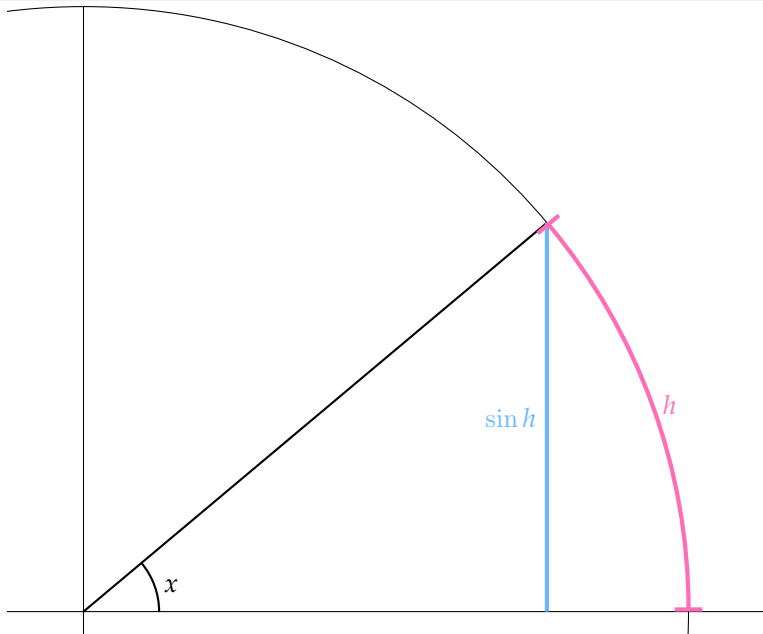
$$\begin{aligned}
\frac{d}{dx} \{\sin x\} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x) \sin(h)}{h} \\
&= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(0+h) - \cos(0)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
&= \sin(x) \frac{d}{dx} \{\cos(x)\} \Big|_{x=0} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}
\end{aligned}$$

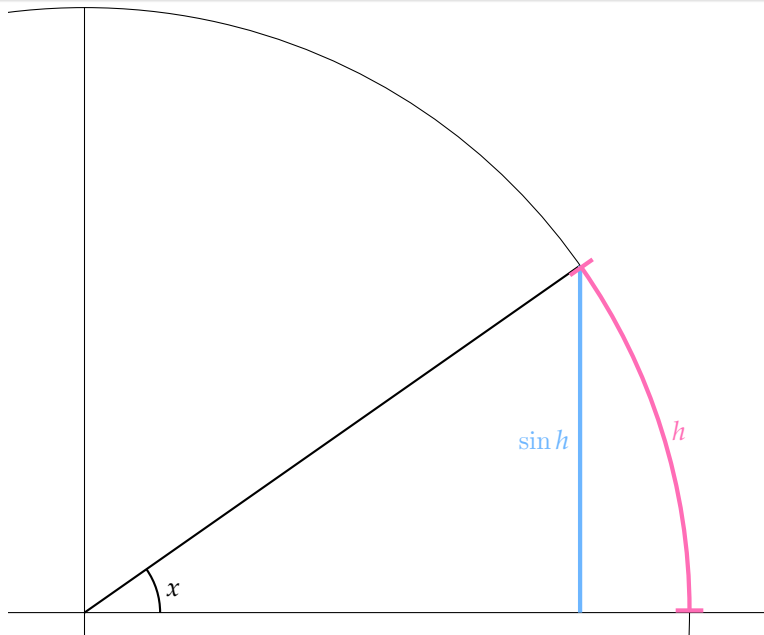
$$\begin{aligned}
\frac{d}{dx} \{\sin x\} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x) \sin(h)}{h} \\
&= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(0+h) - \cos(0)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
&= \sin(x) \frac{d}{dx} \{\cos(x)\} \Big|_{x=0} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \boxed{\cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}}
\end{aligned}$$

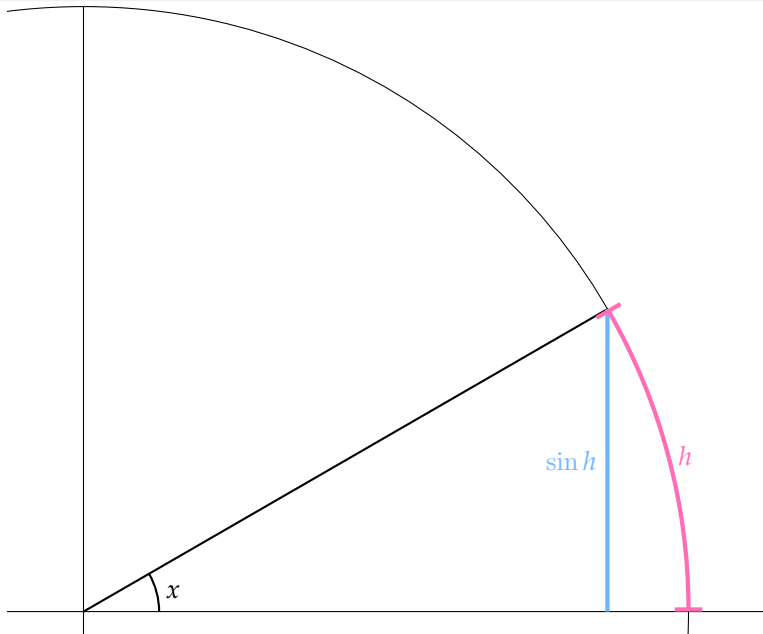
since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.

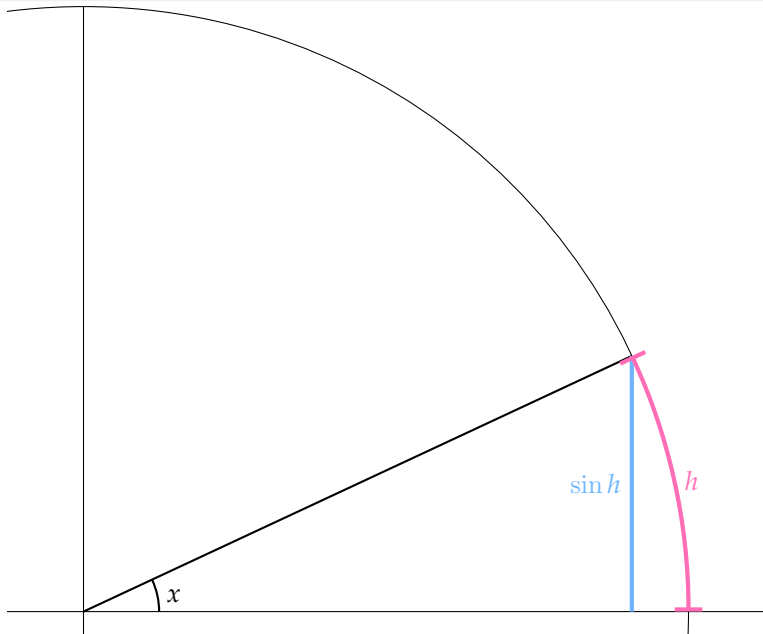
First, we investigate $\lim_{h \rightarrow 0} \frac{\sin h}{h}$ informally.

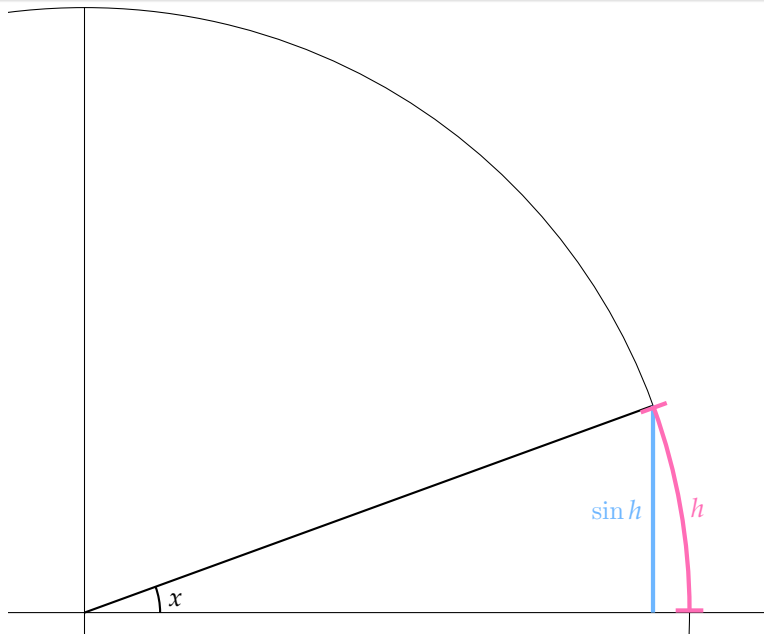


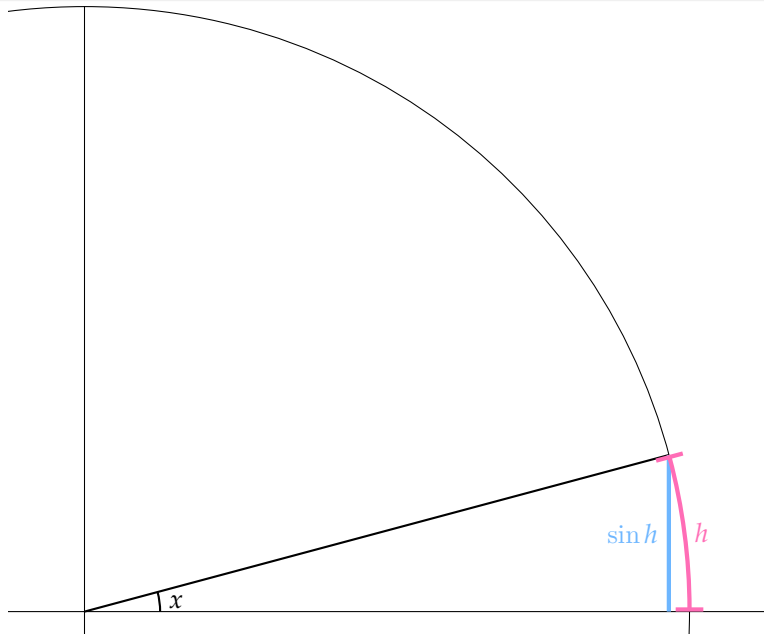


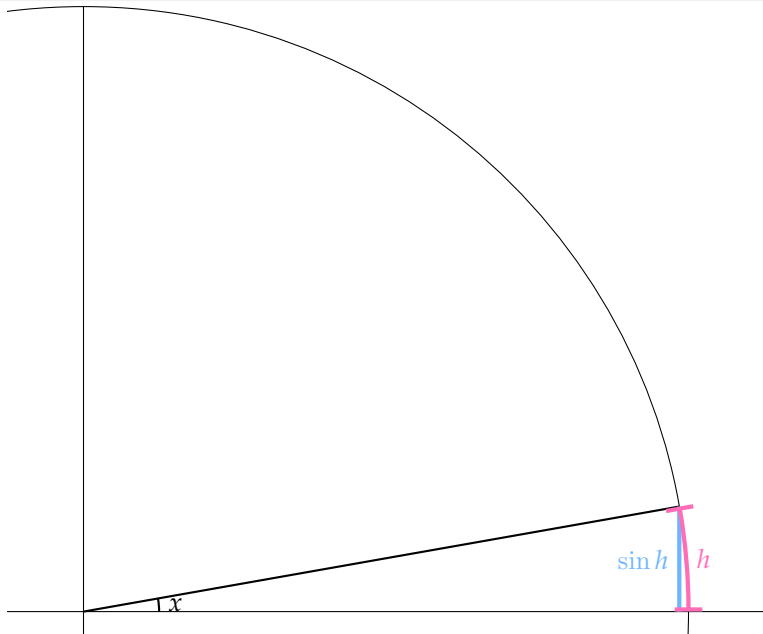








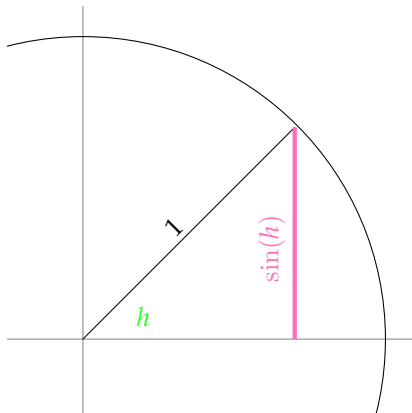


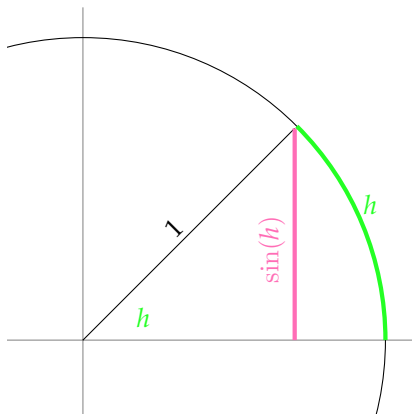


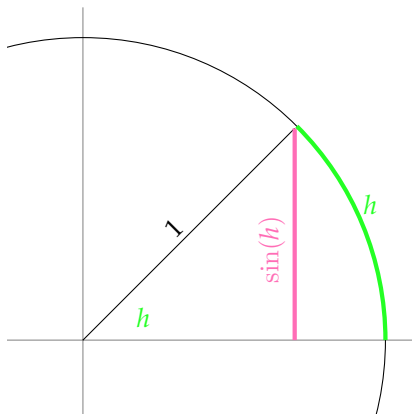
It seems $\sin h \approx h$ when $h \approx 0$, so $\lim_{h \rightarrow 0} \frac{\sin h}{h} \stackrel{?}{=} 1$.

We can prove this more formally using the Squeeze Theorem and more trigonometry. We will first prove that $\frac{\sin(h)}{h} \leq 1$ and then we will prove that $\frac{\sin(h)}{h} \geq \cos(h)$. Then we will apply the Squeeze Theorem.

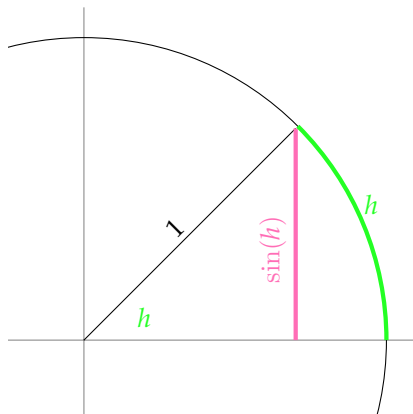
Here is the proof that $\frac{\sin(h)}{h} \leq 1$.





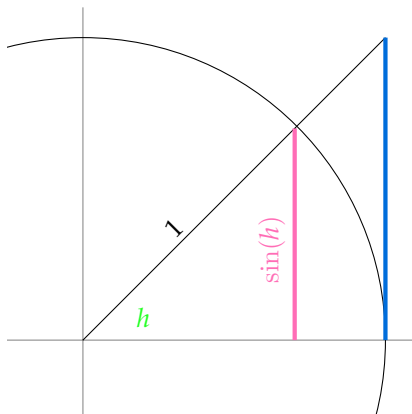


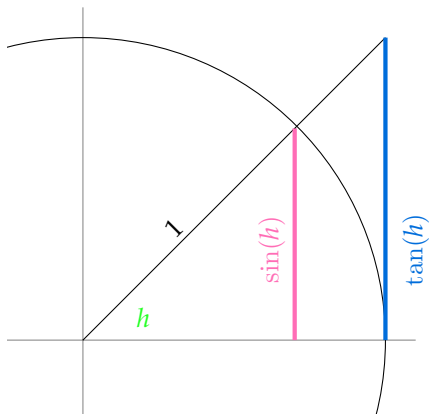
$$\sin(h) \leq h$$

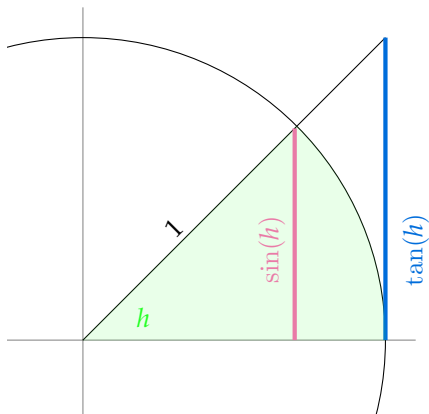


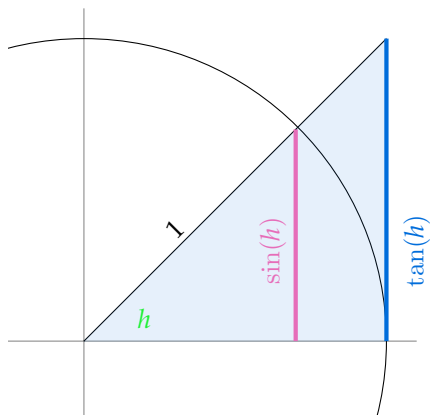
$$\sin(h) < h \text{ so } \boxed{\frac{\sin(h)}{h} < 1}$$

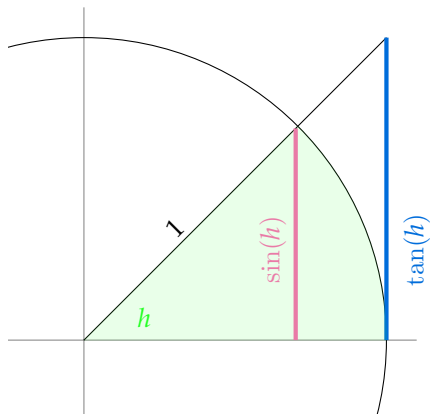
Now for the proof that $\frac{\sin(h)}{h} \geq \cos(h)$.



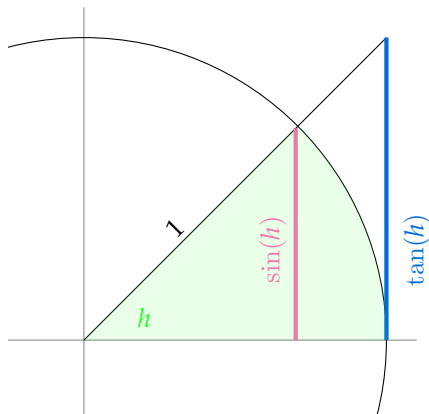




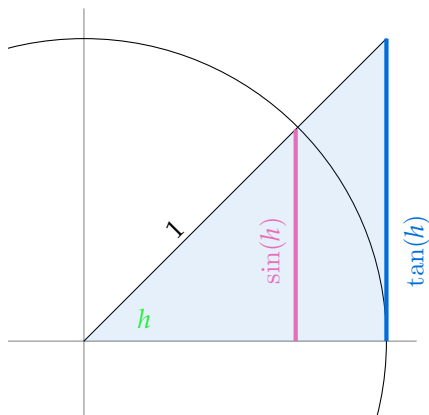




green area:

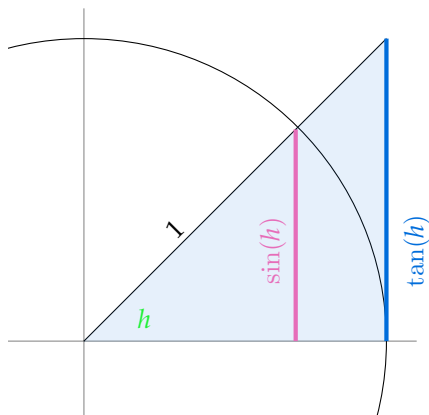


green area: $\frac{h}{2}$



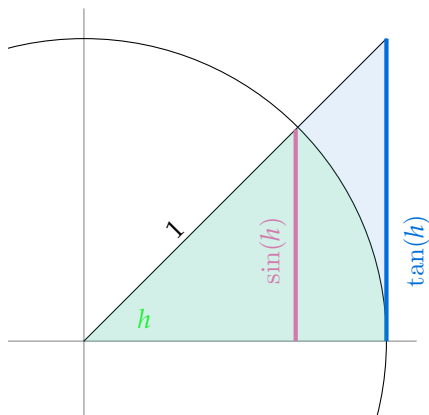
green area: $\frac{h}{2}$

Blue area:



green area: $\frac{h}{2}$

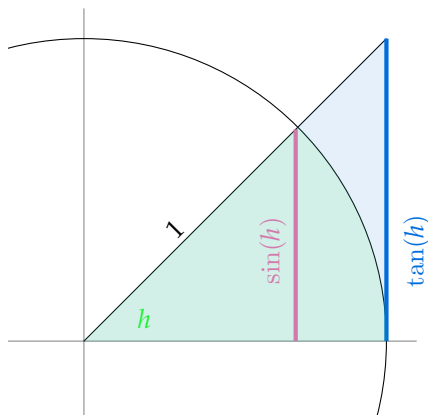
Blue area: $\frac{\tan h}{2}$



green area: $\frac{h}{2}$

$$\frac{h}{2} \leq \frac{\tan(h)}{2}$$

Blue area: $\frac{\tan h}{2}$



green area: $\frac{h}{2}$

$$\frac{h}{2} \leq \frac{\tan(h)}{2}$$

Blue area: $\frac{\tan h}{2}$

$$\cos(h) \leq \frac{\sin(h)}{h}$$

We are now ready for the Squeeze Theorem. We have

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

We are now ready for the Squeeze Theorem. We have

$$\begin{array}{ccccc} \cos h & & \leq & & \frac{\sin h}{h} & & \leq & & 1 \\ \lim_{h \rightarrow 0} \cos h = 1 & & & & & & & & \lim_{h \rightarrow 0} 1 = 1 \end{array}$$

We are now ready for the Squeeze Theorem. We have

$$\begin{array}{ccccc} \cos h & \leq & \frac{\sin h}{h} & \leq & 1 \\ \lim_{h \rightarrow 0} \cos h = 1 & & & & \lim_{h \rightarrow 0} 1 = 1 \end{array}$$

By the Squeeze Theorem,

We are now ready for the Squeeze Theorem. We have

$$\begin{array}{ccccc} \cos h & & \leq & \frac{\sin h}{h} & \leq & 1 \\ \lim_{h \rightarrow 0} \cos h = 1 & & & & & \lim_{h \rightarrow 0} 1 = 1 \end{array}$$

By the Squeeze Theorem,

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

DERIVATIVES OF SINE AND COSINE

From before,

$$\frac{d}{dx}\{\sin(x)\} = \cos(x) \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h} =$$

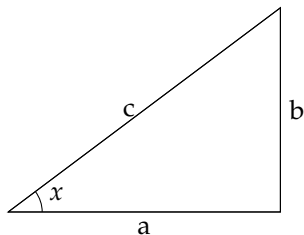
DERIVATIVES OF SINE AND COSINE

From before,

$$\frac{d}{dx}\{\sin(x)\} = \cos(x) \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \cos(x)$$

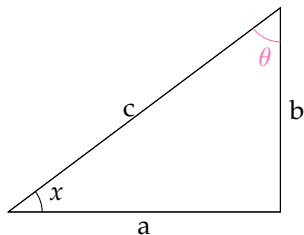
DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.



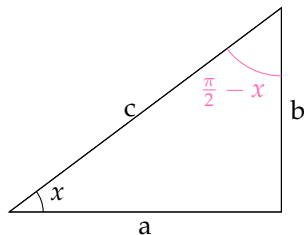
DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.



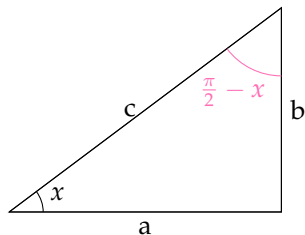
DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.



DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.

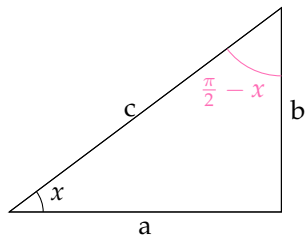


$$\sin x = \frac{b}{c} = \cos \left(\frac{\pi}{2} - x \right)$$

$$\cos x = \frac{a}{c} = \sin \left(\frac{\pi}{2} - x \right)$$

DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.



$$\sin x = \frac{b}{c} = \cos \left(\frac{\pi}{2} - x \right)$$

$$\cos x = \frac{a}{c} = \sin \left(\frac{\pi}{2} - x \right)$$

$$\frac{d}{dx} [\cos(x)] = \frac{d}{dx} \left[\sin \left(\frac{\pi}{2} - x \right) \right] = -\frac{d}{dx} \left[\sin \left(x - \frac{\pi}{2} \right) \right] = -\cos \left(x - \frac{\pi}{2} \right) = -\sin x$$

since $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

When we use radians:

Derivatives of Trig Functions

$$\frac{d}{dx} \{\sin(x)\} = \cos(x)$$

$$\frac{d}{dx} \{\cos(x)\} = -\sin(x)$$

$$\frac{d}{dx} \{\tan(x)\} =$$

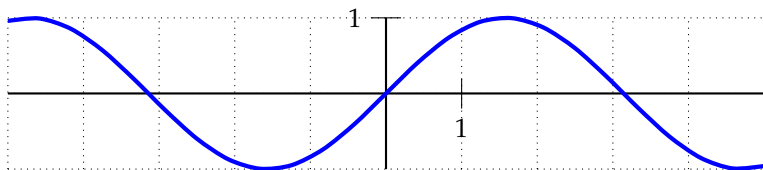
$$\frac{d}{dx} \{\sec(x)\} =$$

$$\frac{d}{dx} \{\csc(x)\} =$$

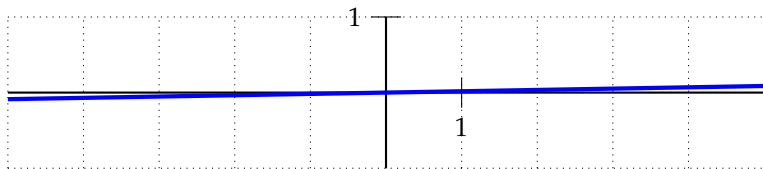
$$\frac{d}{dx} \{\cot(x)\} =$$

Honorable Mention

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



$$y = \sin x, \text{ radians}$$



$$y = \sin x, \text{ degrees}$$

OTHER TRIG FUNCTIONS

[▶ SKIP PROOFS OF OTHER TRIG DERIVATIVES](#)

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

OTHER TRIG FUNCTIONS

▶ SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\begin{aligned}\frac{d}{dx}[\tan(x)] &= \frac{d}{dx} \left[\frac{\sin(x)}{\cos(x)} \right] \\ &= \frac{\cos(x) \cos(x) - \sin(x)[- \sin(x)]}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} = \sec^2(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

[▶ SKIP PROOFS OF OTHER TRIG DERIVATIVES](#)

$$\sec(x) = \frac{1}{\cos(x)}$$

OTHER TRIG FUNCTIONS

►► SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}\frac{d}{dx}[\sec(x)] &= \frac{d}{dx} \left[\frac{1}{\cos(x)} \right] \\ &= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} \\ &= \sec(x) \tan(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

▶▶ SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\csc(x) = \frac{1}{\sin(x)}$$

OTHER TRIG FUNCTIONS

►► SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}\frac{d}{dx}[\csc(x)] &= \frac{d}{dx} \left[\frac{1}{\sin(x)} \right] \\ &= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\ &= \frac{-\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin(x)} \frac{\cos(x)}{\sin(x)} \\ &= -\csc(x) \cot(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

[▶ SKIP PROOFS OF OTHER TRIG DERIVATIVES](#)

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

OTHER TRIG FUNCTIONS

▶ SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}\frac{d}{dx}[\cot(x)] &= \frac{d}{dx} \left[\frac{\cos(x)}{\sin(x)} \right] \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin^2(x)} \\ &= -\csc^2(x)\end{aligned}$$

MEMORIZE

$$\frac{d}{dx}\{\sin(x)\} = \cos(x)$$

$$\frac{d}{dx}\{\cos(x)\} = -\sin(x)$$

$$\frac{d}{dx}\{\tan(x)\} = \sec^2(x)$$

$$\frac{d}{dx}\{\sec(x)\} = \sec(x)\tan(x)$$

$$\frac{d}{dx}\{\csc(x)\} = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



1. Let $f(x) = \frac{x \tan(x^2 + 7)}{15e^x}$. Use the **definition of the derivative** to find $f'(0)$.
2. Differentiate $(e^x + \cot x)(5x^6 - \csc x)$.
3. Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$
Which values of a and b make $h(x)$ continuous at $x = 0$?

Let $f(x) = \frac{x \tan(x^2 + 7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)(5x^6 - \csc x)$.

$$\text{Let } h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Practice and Review

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x - a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

We don't have rules for differentiating $f(x)$ at $x = 0$, so we have to fall back on the definition of the derivative.

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \cos\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h \rightarrow 0} h \cos\left(\frac{1}{h}\right) = 0 \end{aligned}$$

Since the limit exists, $f(x)$ is differentiable at 0.

By the definition of continuity, $g(x)$ is continuous at $x = 0$ if

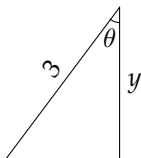
$$\lim_{x \rightarrow 0} g(x) = g(0)$$

- ▶ $g(0) = (0 - a)^2 = a^2$
- ▶ $\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} e^{\frac{\sin x}{x}} = e^1 = e$
- ▶ $\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x - a)^2 = a^2$

In order for $g(x)$ to be continuous, we need $a^2 = e$. That is, $a = \sqrt{e}$ or $a = -\sqrt{e}$.

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ ?

When is the top of the ladder sinking the fastest? The slowest?



A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ ?

When is the top of the ladder sinking the fastest? The slowest?

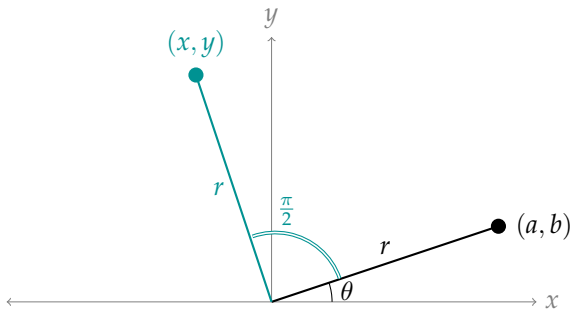
We want to find how fast y is changing with respect to θ , so we want $\frac{dy}{d\theta}$, or $y'(\theta)$. To calculate that, we need to find y as a function of θ . Note that the ladder forms a right triangle with the wall, and y is the side adjacent to θ , while 3 is the hypotenuse. So, $\cos(\theta) = \frac{y}{3}$, hence $y = 3 \cos(\theta)$. Now we differentiate, and see

$$\frac{dy}{d\theta} = -3 \sin(\theta)$$

To answer the other questions, note that θ never gets larger than $\pi/2$, since at that point the ladder is lying on the ground. When $0 \leq \theta \leq \pi/2$, the smaller θ gives the smaller rate of change (in absolute value); so the top of the ladder is sinking slowly at first, then faster and faster, fastest just as it hits the ground.



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y) ? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ ?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y) ? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ ?

$$x = r \cos \left(\theta + \frac{\pi}{2} \right) = -r \sin(\theta)$$

and

$$y = r \sin \left(\theta + \frac{\pi}{2} \right) = r \cos(\theta)$$

To find how fast x is changing with respect to θ , we take $x'(\theta) = -r \cos(\theta)$. We see that when $\theta = 0$, x changes a lot when θ changes; and when $\theta = \pi/2$, x only changes a little when θ changes.

Included Work



'Brain' by [Eucalyp](#) is licensed under [CC BY 3.0](#) (accessed 8 June 2021), 72



'Notebook' by [Iconic](#) is licensed under [CC BY 3.0](#) (accessed 9 June 2021), 72