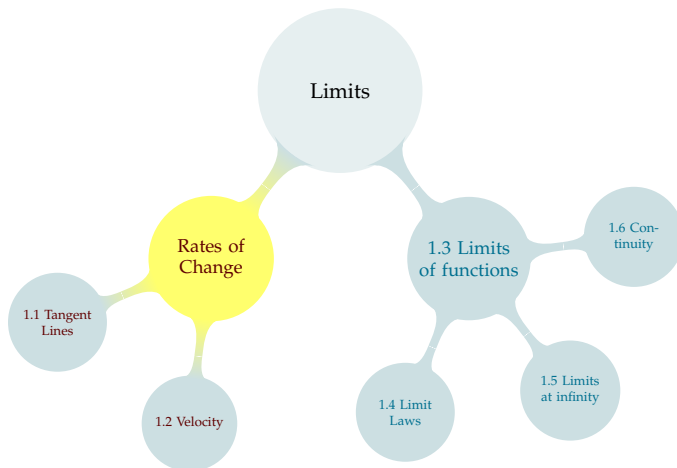
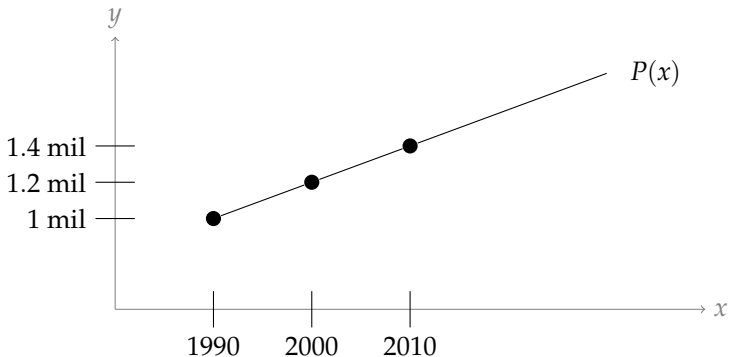


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RATES OF CHANGE

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.



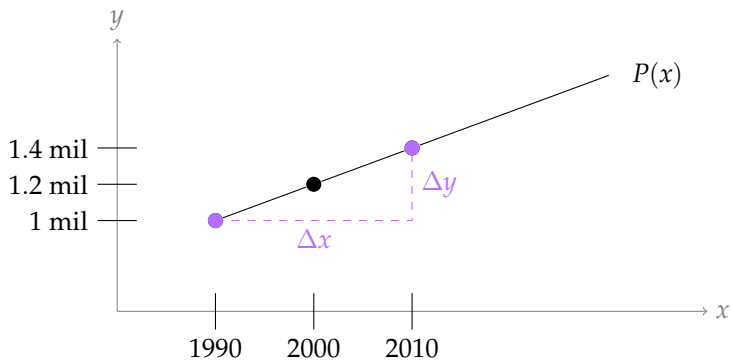
Definition

The **slope** of a line that passes through the points (x_1, y_1) and (x_2, y_2) is “rise over run”

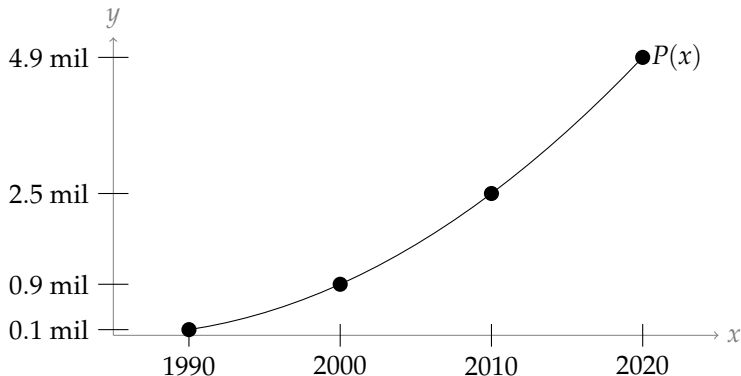
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is also called the **rate of change** of the function.

If a line has equation $y = mx + b$, its slope is m .



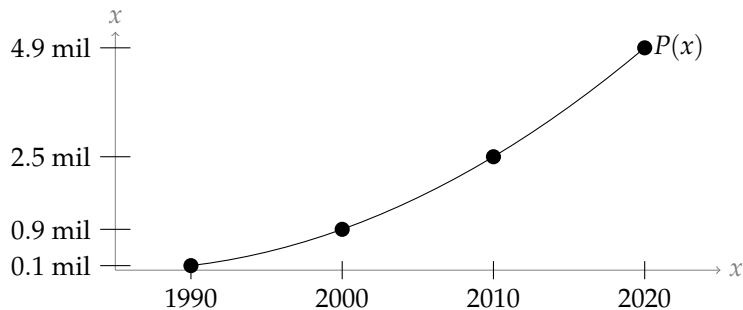
Suppose the population of a small country is given in the chart below.



Definition

Let $y = f(x)$ be a curve that passes through (x_1, y_1) and (x_2, y_2) . Then the **average rate of change** of $f(x)$ when $x_1 \leq x \leq x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Average Rate of Change and Slope

The **average rate of change** of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is “change in output” divided by “change in input:”

$$\frac{f(b) - f(a)}{b - a}$$

If the function $f(x)$ is a **line**, then the slope of the line is “rise over run,”

$$\frac{f(b) - f(a)}{b - a}$$

If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

How fast was this population growing in the year 2010? (What was its **instantaneous** rate of change?)

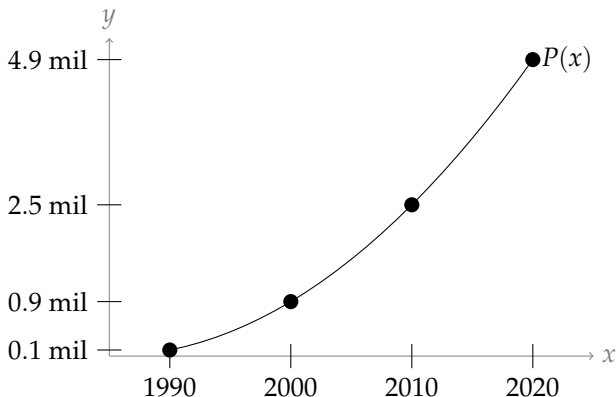
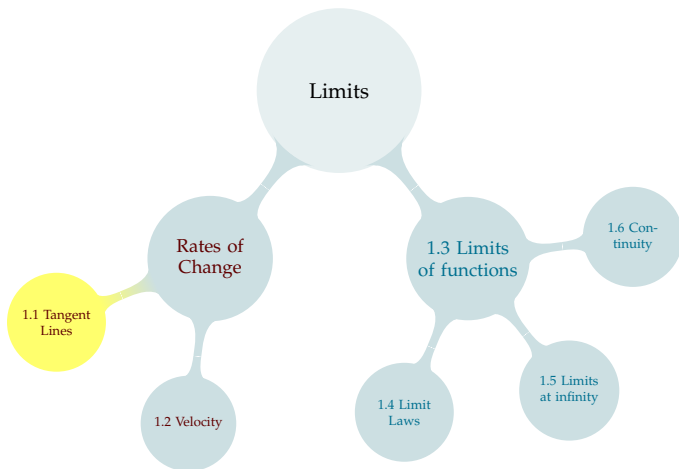


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Definition

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q .

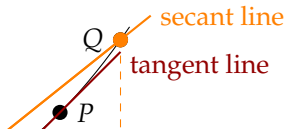
We call the slope of the secant line the **average rate of change of $f(x)$ from R to Q** .

Definition

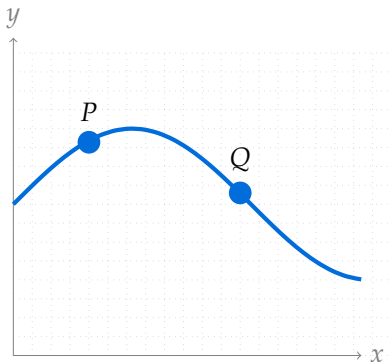
The **tangent line** to the curve $y = f(x)$ at point P is a line that

- passes through P and
- has the same slope as $f(x)$ at P .

We call the slope of the tangent line the **instantaneous rate of change of $f(x)$ at P** .



On the graph below, draw the secant line to the curve through points P and Q .



On the graph below, draw the tangent line to the curve at point P .

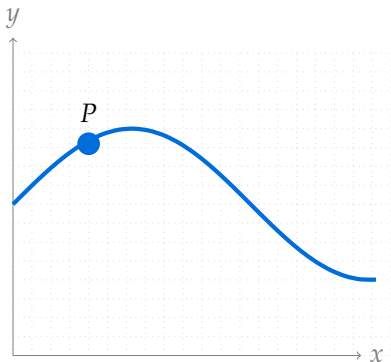
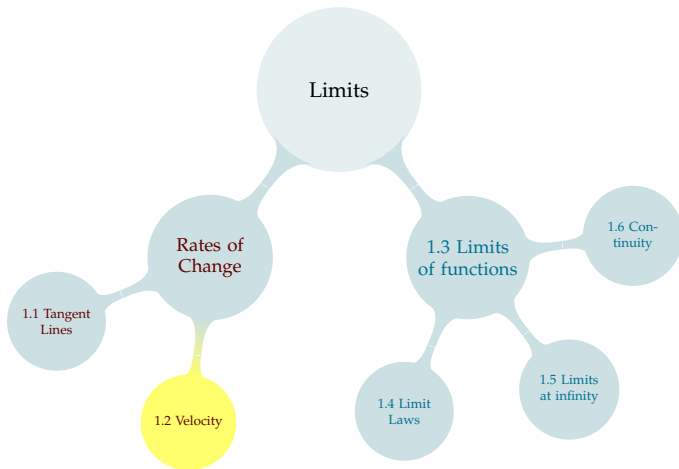
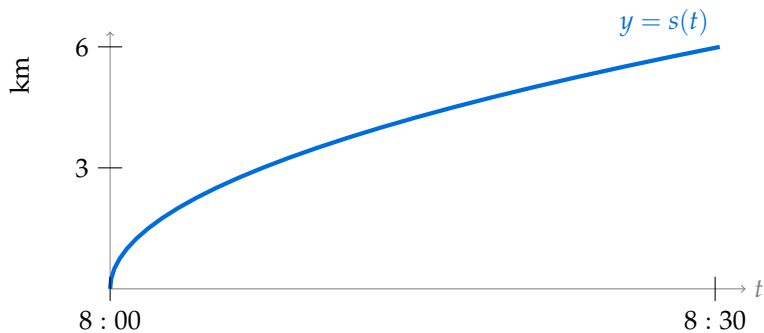
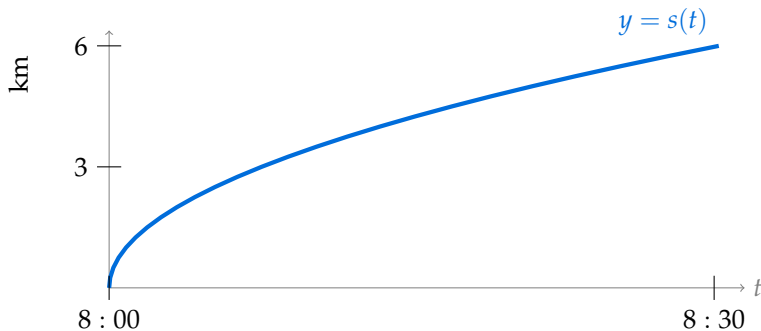


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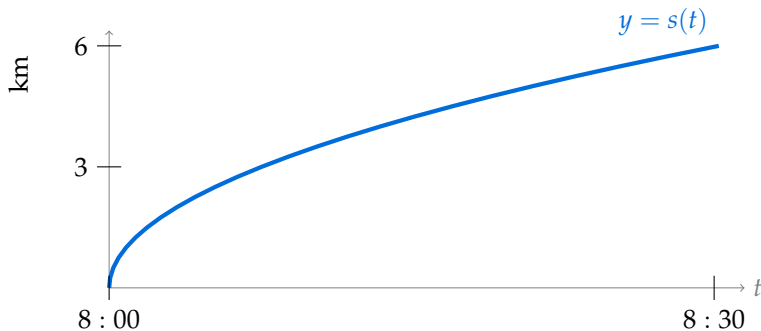






It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

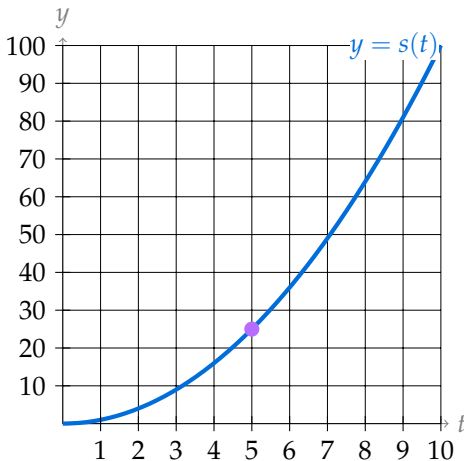
- A. secant line to $y = s(t)$ from $t = 8 : 00$ to $t = 8 : 30$
- B. slope of the secant line to $y = s(t)$ from $t = 8 : 00$ to $t = 8 : 30$
- C. tangent line to $y = s(t)$ at $t = 8 : 30$
- D. slope of the tangent line to $y = s(t)$ at $t = 8 : 30$



At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

- A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
- B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
- C. tangent line to $y = s(t)$ at $t = 8:25$
- D. slope of the tangent line to $y = s(t)$ at $t = 8:25$

Suppose the distance from the ground s (in meters) of a helium-filled balloon at time t over a 10-second interval is given by $s(t) = t^2$. Try to estimate how fast the balloon is rising when $t = 5$.



Let's look for an algebraic way of determining the velocity of the balloon when $t = 5$.

OUR FIRST LIMIT

Average Velocity, $t = 5$ to $t = 5 + h$:

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= \frac{s(5+h) - s(5)}{h} \\ &= \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \quad \text{when } h \neq 0\end{aligned}$$

When h is very small,

$$\text{Vel} \approx 10$$

LIMIT NOTATION

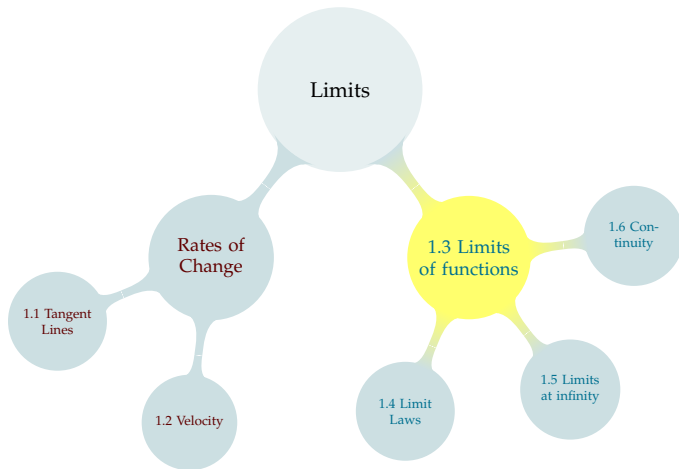
We write:

$$\lim_{h \rightarrow 0} (10 + h) = 10$$

We say: “The limit as h goes to 0 of $(10 + h)$ is 10.”

It means: As h gets extremely close to 0, $(10 + h)$ gets extremely close to 10.

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Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \rightarrow a} f(x) = L$$

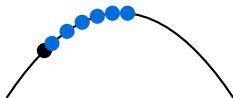
where a and L are real numbers

We read the above as “the limit as x goes to a of $f(x)$ is L .”

Its meaning is: as x gets very close to (but not equal to) a , $f(x)$ gets very close to L .

FINDING SLOPES OF TANGENT LINES

We NEED limits to find slopes of tangent lines.



Slope of secant line: $\frac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.

Slope of tangent line: can't do the same way.

If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

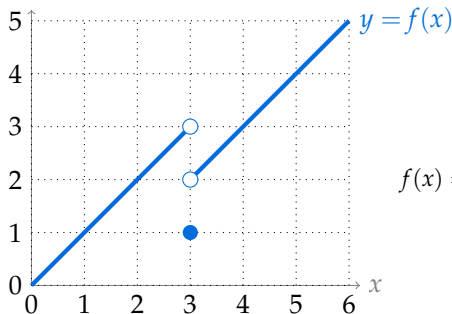
$$\lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$$

EVALUATING LIMITS

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \rightarrow 1} f(x)$.

ONE-SIDED LIMITS



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim_{x \rightarrow 3} f(x)$ should be?

Definition 1.3.7

The limit as x goes to a **from the left** of $f(x)$ is written

$$\lim_{x \rightarrow a^-} f(x)$$

We only consider values of x that are **less than** a .

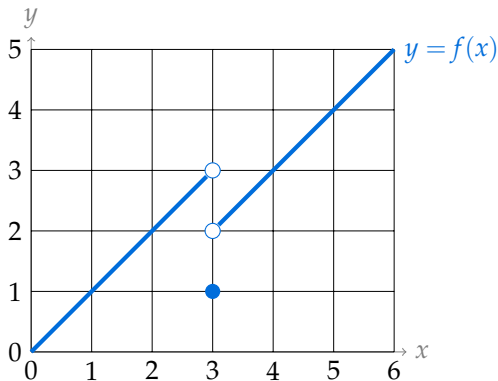
The limit as x goes to a **from the right** of $f(x)$ is written

$$\lim_{x \rightarrow a^+} f(x)$$

We only consider values of x **greater than** a .

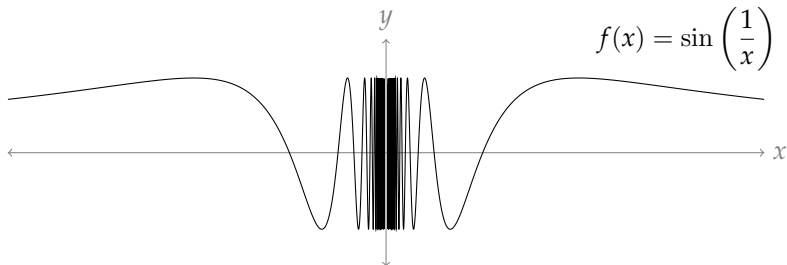
Theorem 1.3.8

In order for $\lim_{x \rightarrow a} f(x)$ to exist, both one-sided limits must exist and be equal.



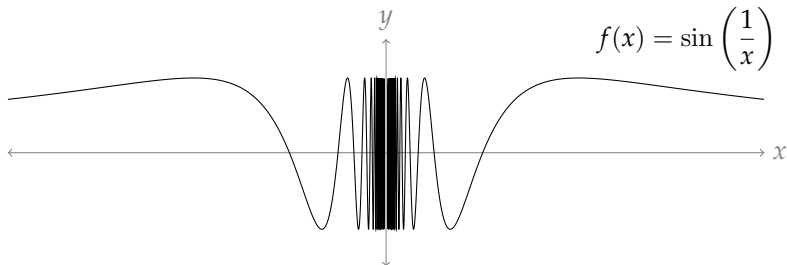
Consider the function $f(x) = \frac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ **not** defined?

A STRANGER LIMIT EXAMPLE



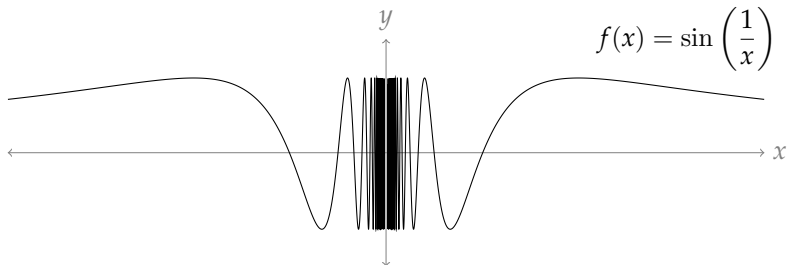
What is $\lim_{x \rightarrow \infty} f(x)$?

A STRANGER LIMIT EXAMPLE



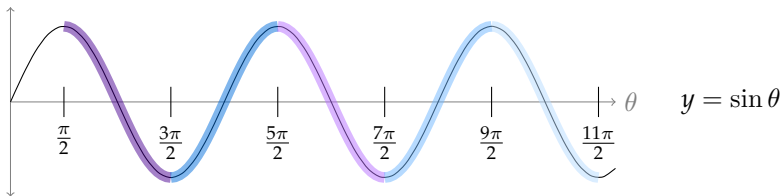
What is $\lim_{x \rightarrow 0} f(x)$?

A STRANGER LIMIT EXAMPLE



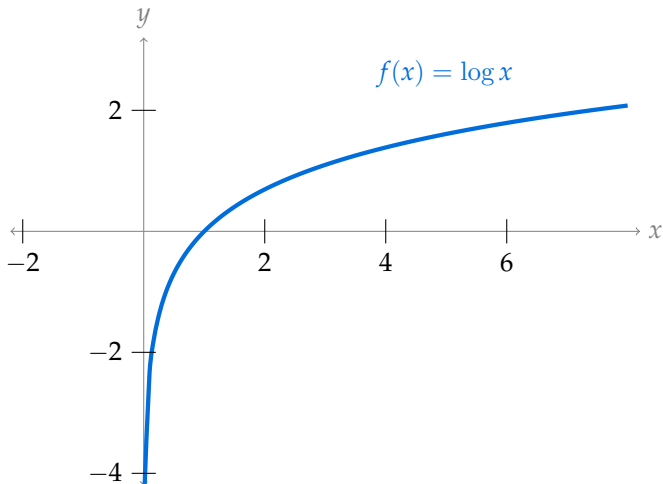
What is $\lim_{x \rightarrow \pi} f(x)$?

OPTIONAL: SKETCHING $f(x) = \sin\left(\frac{1}{x}\right)$



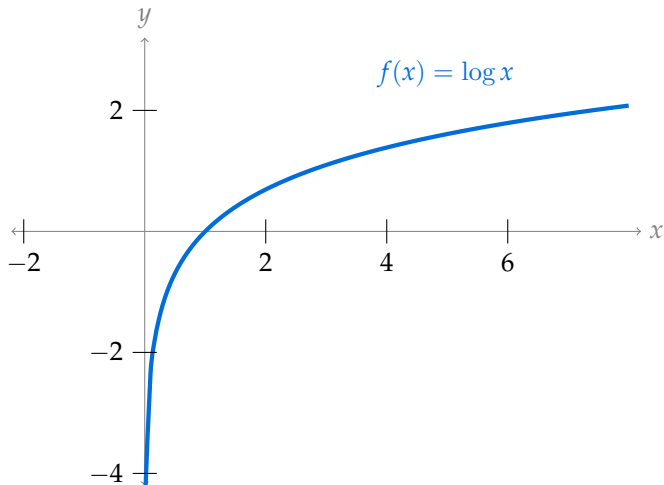
LIMITS AND THE NATURAL LOGARITHM

Where is $f(x)$ defined, and where is it not defined?

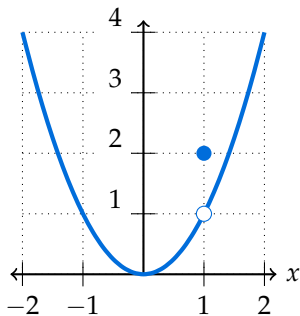


LIMITS AND THE NATURAL LOGARITHM

What can you say about the limit of $f(x)$ near 0?



$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x \rightarrow 1} f(x)$?

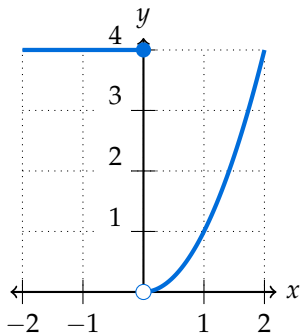
A. $\lim_{x \rightarrow 1} f(x) = 2$

B. $\lim_{x \rightarrow 1} f(x) = 1$

C. $\lim_{x \rightarrow 1} f(x)$ DNE

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x \rightarrow 0} f(x)$? What is $\lim_{x \rightarrow 0^+} f(x)$? What is $f(0)$?

A. $\lim_{x \rightarrow 0^+} f(x) = 4$

B. $\lim_{x \rightarrow 0^+} f(x) = 0$

C. $\lim_{x \rightarrow 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

Suppose $\lim_{x \rightarrow 3^-} f(x) = 1$ and $\lim_{x \rightarrow 3^+} f(x) = 1.5$.

Does $\lim_{x \rightarrow 3} f(x)$ exist?

- A. Yes, certainly, because the limits from both sides exist.
- B. No, never, because the limit from the left is not the same as the limit from the right.
- C. Can't tell. For some functions it might exist, for others not.

Suppose $\lim_{x \rightarrow 3^-} f(x) = 22 = \lim_{x \rightarrow 3^+} f(x)$.

Does $\lim_{x \rightarrow 3} f(x)$ exist?

- A. Yes, certainly, because the limits from both sides exist and are equal to each other.
- B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.
- C. Can't tell. We need to know the value of the function at $x = 3$.

CALCULATING LIMITS IN SIMPLE SITUATIONS

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f , then:

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Calculate: $\lim_{x \rightarrow 3} \left(\frac{x^2 - 9}{x + 3} \right)$

Calculate: $\lim_{x \rightarrow 3} \left(\frac{x^2 - 9}{x - 3} \right)$

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x \rightarrow a} f(x) = F$ and $\lim_{x \rightarrow a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x \rightarrow a} (f(x) + g(x)) = F + G$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = F - G$
- $\lim_{x \rightarrow a} (f(x)g(x)) = FG$
- $\lim_{x \rightarrow a} (f(x)/g(x)) = F/G$ provided $G \neq 0$

Calculate: $\lim_{x \rightarrow 1} \left[\frac{2x+4}{x+2} + 13 \left(\frac{x+5}{3x} \right) \left(\frac{x^2}{2x-1} \right) \right]$

LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$

B. $(-4)^{\frac{1}{2}}$

C. $4^{-\frac{1}{2}}$

D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$

F. $(-8)^{1/3}$

G. $8^{-1/3}$

H. $(-8)^{-1/3}$

Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x \rightarrow a} f(x) = F$ (where F is a real number), then:

$$\lim_{x \rightarrow a} (f(x))^n = F^n.$$

Furthermore, **unless** n is even and F is negative,

$$\lim_{x \rightarrow a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x \rightarrow 4} (x + 5)^{1/2}$$

CAUTIONARY TALES

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{(5+x)^2 - 25}{x}$$

$$\blacktriangleright \lim_{x \rightarrow 3} \left(\frac{x-6}{3} \right)^{1/8}$$

$$\blacktriangleright \lim_{x \rightarrow 0} \frac{32}{x}$$

$$\blacktriangleright \lim_{x \rightarrow 5} (x^2 + 2)^{1/3}$$

Suppose you want to evaluate $\lim_{x \rightarrow 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

- A $\lim_{x \rightarrow 1} f(x)$ may exist, and it may not exist.
- B We can find $\lim_{x \rightarrow 1} f(x)$ by plugging in 1 to $f(x)$.
- C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x \rightarrow 1} f(x)$.
- D Since $f(1)$ doesn't exist, automatically we know $\lim_{x \rightarrow 1} f(x)$ does not exist.
- E $\lim_{x \rightarrow 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

Which of the following statements is true about $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \rightarrow 0} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 0}{0^3 - 0^2 + 0} = \frac{0}{0}$

B Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\frac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\frac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \rightarrow 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

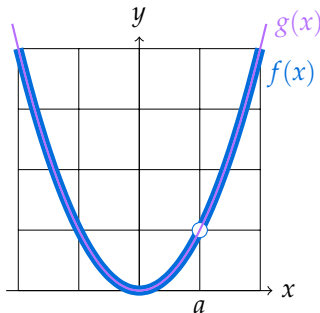
C Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

D Since the numerator and denominator of $\frac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim_{x \rightarrow a} g(x)$ exists, and $f(x) = g(x)$
when x is close to a (but not necessarily equal to a).

Then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$.



Evaluate $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x - 1}$.

Evaluate $\lim_{x \rightarrow 5} \frac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$

A FEW STRATEGIES FOR CALCULATING LIMITS

First, hope that you can **directly substitute** (plug in). If your function is made up of the **sum, difference, product, quotient, or power of polynomials**, you can do this **provided** the function exists where you're taking the limit.

$$\lim_{x \rightarrow 1} \left(\sqrt{35 + x^5} + \frac{x - 3}{x^2} \right)^3 =$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to **simplify and cancel**.

$$\lim_{x \rightarrow 0} \frac{x + 7}{\frac{1}{x} - \frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

DENOMINATORS APPROACHING ZERO

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1}$$

DENOMINATORS APPROACHING ZERO

Now
You



$$\lim_{x \rightarrow 2^+} \frac{x}{x^2 - 4}$$

$$\lim_{x \rightarrow 2^-} \frac{x}{4 - x^2}$$

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4}$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a , we have functions $f(x)$, $g(x)$, and $h(x)$ so that

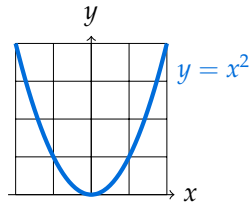
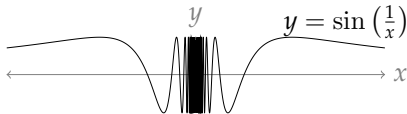
$$f(x) \leq g(x) \leq h(x)$$

and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$. Then $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x)$.

$$\lim_{x \rightarrow 0} x^2 \sin \left(\frac{1}{x} \right)$$

Evaluate:

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$



$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Included Work



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