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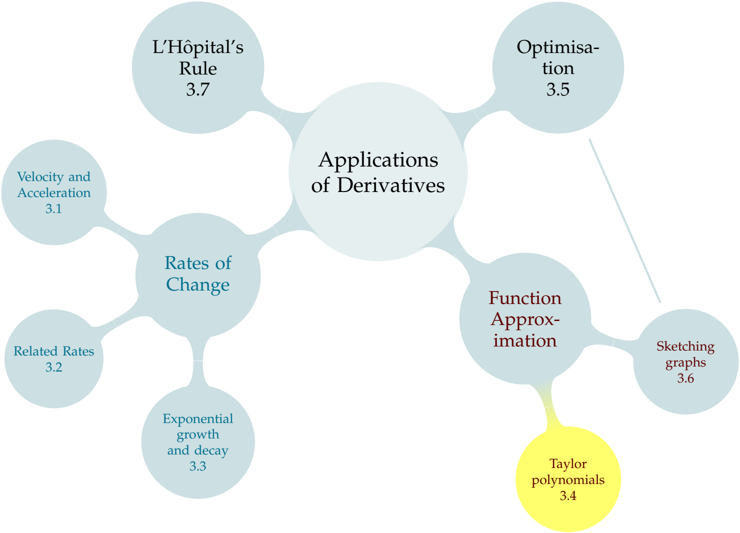
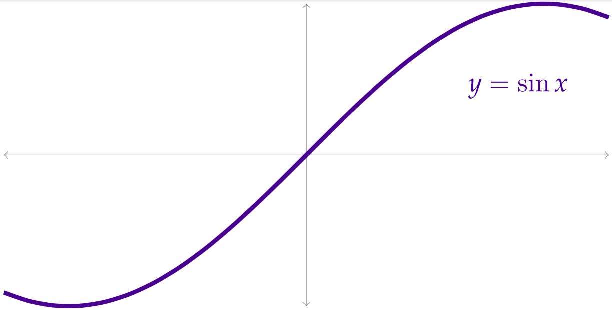


Approximating a Function



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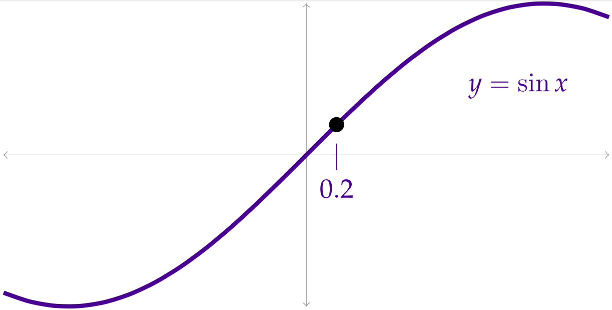

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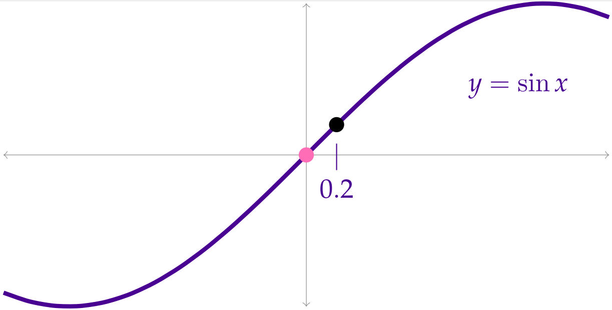

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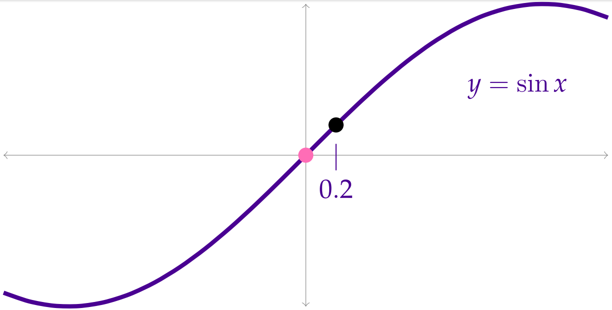
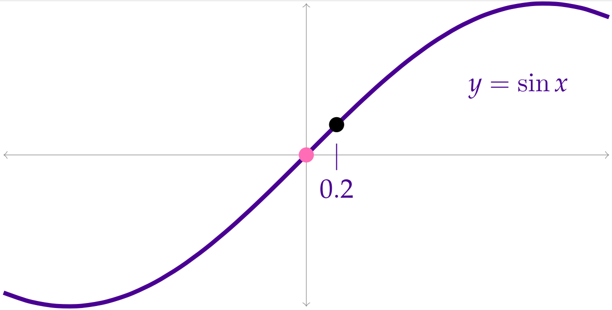


Constant Approximation – Equation 3.4.1

We can approximate $f(x)$ near a point a by

$$f(x) \approx f(a)$$

Approximating a Function



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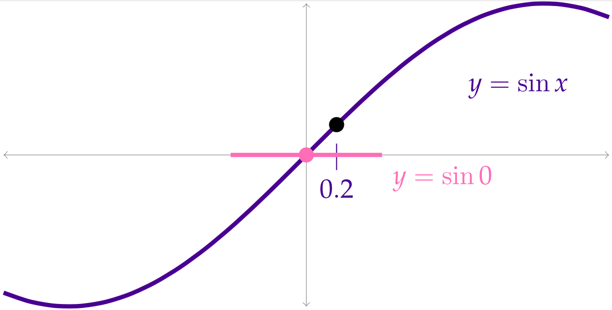
Google: $\sin(0.2) \approx 0.198669...$ Constant approx: $\sin(0.2) \approx 0$

Approximating a Function



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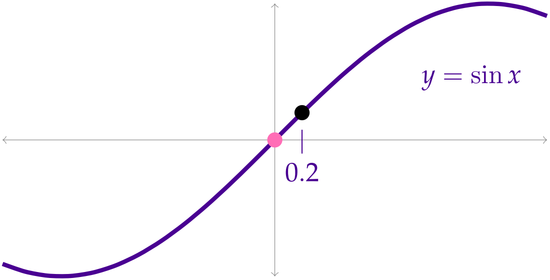

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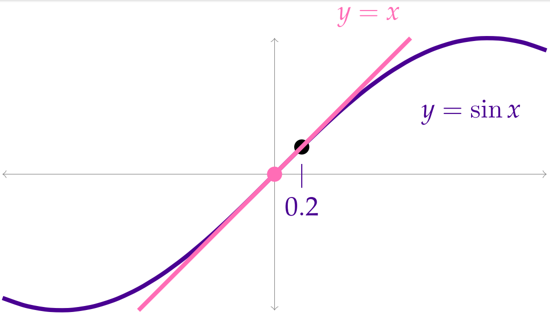

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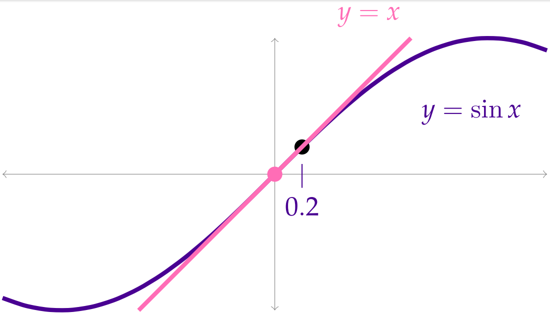
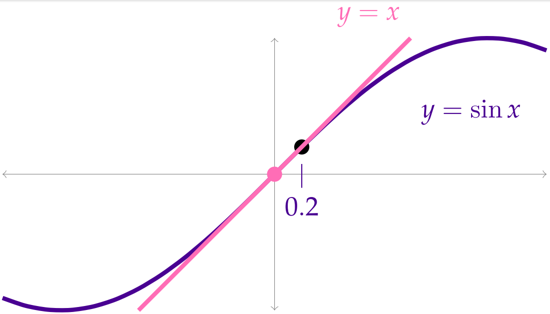


Linear Approximation (Linearization) – Equation 3.4.3

We can approximate $f(x)$ near a point a by the tangent line to $f(x)$ at a, namely

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Approximating a Function



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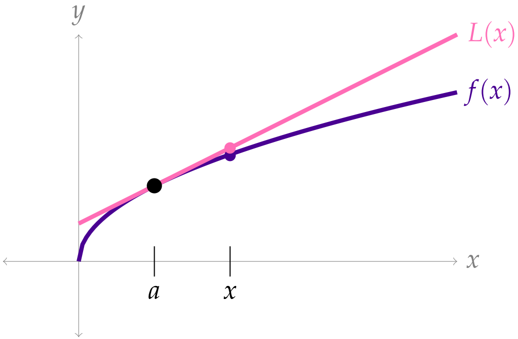
Google:
$\sin(0.2) \approx 0.198669\ldots$

Linear approx:
$\sin(0.2) \approx 0 + 1(0.2 - 0) = 0.2$

To find a linear approximation of $f(x)$ at a particular point x, pick a point a near to x, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$



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Constant approximation: $8.9 \approx 9$, so $f(8.9) \approx f(9) = \boxed{3}$

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Linear approximation: Using $a = 9$,

$$f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$f(8.9) \approx f(9) + f'(9)(8.9-9) = 3 + \frac{1}{6}(-.1)$$

$$f(8.9) \approx 3 - \frac{1}{60} = 2.98\overline{33}$$

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$$f(8.9) \approx 3 - \frac{1}{60} = 2.98\overline{33}$$

Google: $\sqrt{8.9} = 2.98328677804\ldots$

Characteristics of a Good Approximation

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Accurate

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Accurate

Possible to calculate: add, subtract, multiply, divide. Use integers or known constants

Can we compute?

Suppose we want to approximate the value of $\cos(1.5)$. Which of the following linear approximations could we calculate by hand? (You can leave things in terms of π.)

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = 3/2$

C. both

D. neither

Q

ans

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B. tangent line to $f(x) = \cos x$ when $x = 3/2$

C. both

D. neither

We know $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$, so we can easily compute the linear approximation if we centre it at $\pi/2$. However, what kind of ugly number is $\cos(3/2)$?

A

CAN WE COMPUTE?

Which of the following tangent lines is probably the most accurate in approximating $\cos(1.5)$?

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = \pi/4$

C. constant approximation: $\cos 1.5 \approx \cos(\pi/2) = 0$

D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

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C. constant approximation: $\cos 1.5 \approx \cos(\pi/2) = 0$

D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

$\pi/2$ is very close to 1.5.

A

Linear Approximation

Approximate $\sin(3)$ using a linear approximation. You may leave your answer in terms of π.

Q

ans

Linear Approximation

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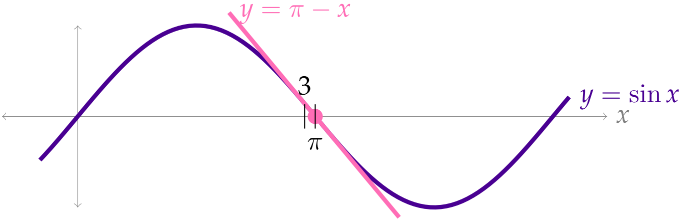

Let $f(x) = \sin x$ and $a = \pi$. Then

$$f(3) \approx f(\pi) + f'(\pi)(3 - \pi) = \sin(\pi) + \cos(\pi)(3 - \pi) = \boxed{\pi - 3} \approx 0.14159$$

Google: $\sin(3) = 0.14112000806...$

Linear Approximation

Approximate $e^{1/10}$ using a linear approximation.

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Q

ans

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If $f(x) = e^x$ and $a = 0$:

$$\begin{aligned} f'(x) &= e^x \\ f(1/10) &\approx f(0) + f'(0)(1/10 - 0) = e^0 + e^0(1/10 - 0) = 1 + 1/10 \\ &= 1.1 \end{aligned}$$

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If $g(x) = x^{1/10}$:

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If $g(x) = x^{1/10}$:
The closest number to e with a simple tenth root is $a = 1$.

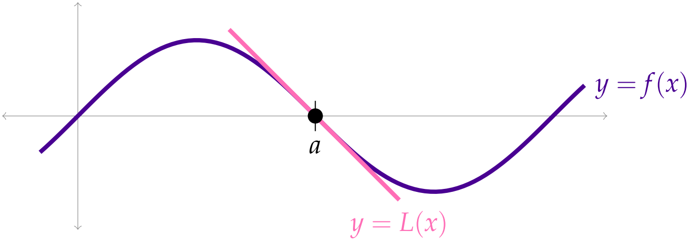
$$\begin{aligned} g'(x) &= \tfrac{1}{10}x^{-9/10} \\ g(e) &\approx g(1) + g'(1)(e-1) = 1 + \tfrac{1}{10}(e-1) = \tfrac{e+9}{10} \end{aligned}$$

... but what's e?
Google: $e^{1/10} = 1.10517091808...$

A

Linear Approximation Wrap-Up

Let $L(x) = f(a) + f'(a)(x-a)$, so $L(x)$ is the linear approximation (linearization) of $f(x)$ at a.



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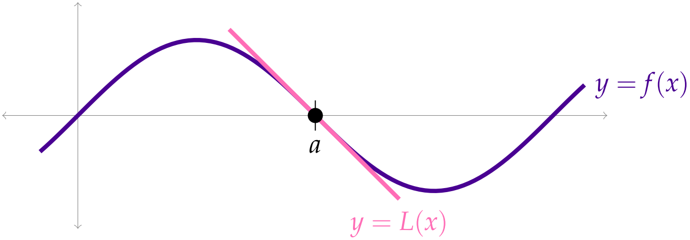
What is $L(a)$?



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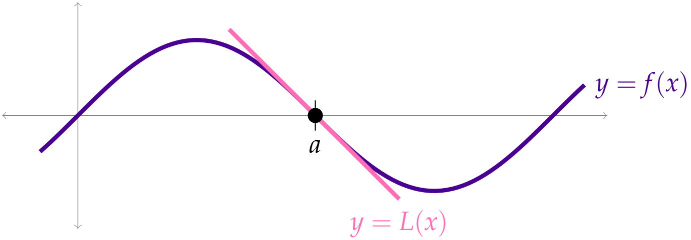

Q

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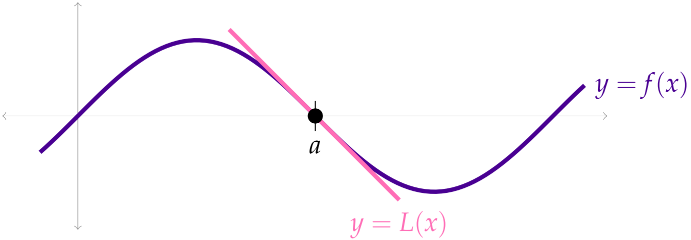

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What is $L(a)$? $L(a) = f(a)$

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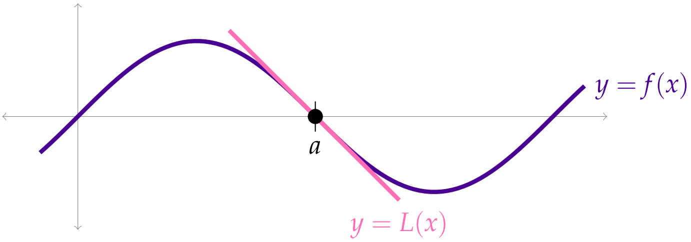

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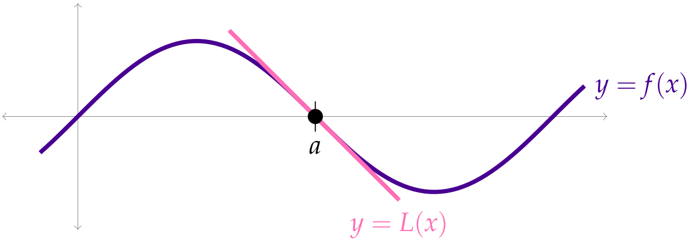


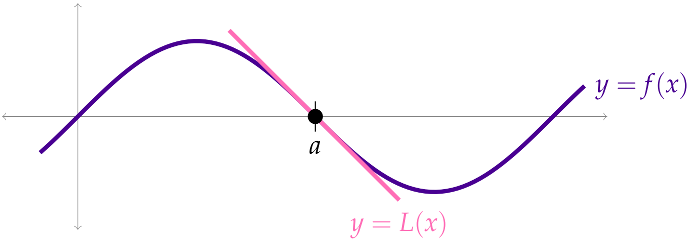
LINEAR APPROXIMATION WRAP-UP

Let $L(x) = f(a) + f'(a)(x-a)$, so $L(x)$ is the linear approximation (linearization) of $f(x)$ at a.

What is $L(a)$? $L(a) = f(a)$

What is $L'(a)$? $L'(a) = f'(a)$

What is $L''(a)$? (Recall $L''(x)$ is the derivative of $L'(x)$.) $L''(a) = 0$



Linear Approximation Wrap-Up

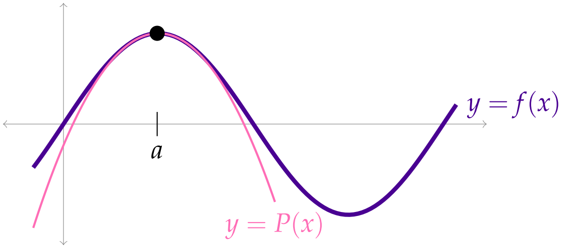
Let $L(x)$ be a linear approximation of $f(x)$.

$f(a)$	$L(a)$	same
$f'(a)$	$L'(a)$	same
$f''(a)$	$L''(a)$	different[1]

[1] unless $f''(a) = 0$

Quadratic Approximation

Imagine we approximate $f(x)$ at $x = a$ with a parabola, $P(x)$.



QUADRATIC APPROXIMATION

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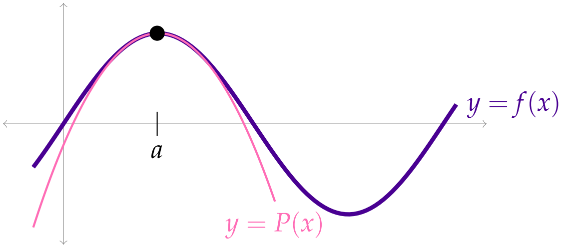

Then we could ensure:

$$P(a) = f(a), \qquad P'(a) = f'(a), \qquad \textbf{and} \qquad P''(a) = f''(a).$$

Quadratic Approximation

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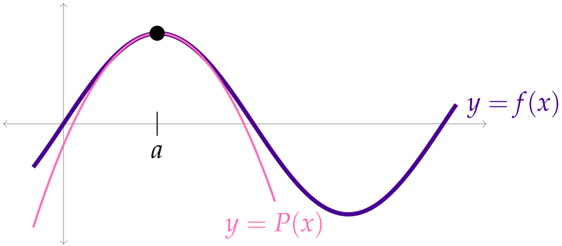

$P(x) = A + Bx + Cx^2$	$P(a) = A + Ba + Ca^2$	$f(a)$
$P'(x) = \quad B + 2Cx$	$P'(a) = \quad B + 2Ca$	$f'(a)$
$P''(x) = \quad 2C$	$P''(a) = \quad 2C$	$f''(a)$

Quadratic Approximation

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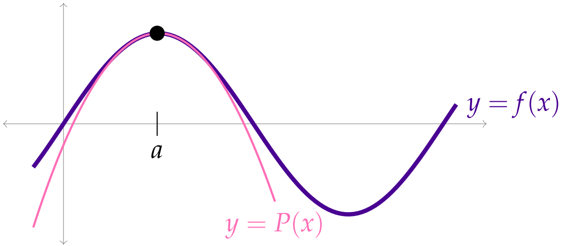

Solving $2C = f''(a)$ for C, and then solving $B + 2Ca = f'(a)$ for B, and then solving $A + Ba + Ca^2 = f(a)$ for A and then substituting back into $P(x) = A + Bx + Cx^2$ gives

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

	Constant	Linear	Quadratic
Function value matches at $x = a$	✓	✓	✓
First derivative matches at $x = a$	×	✓	✓
Second derivative matches at $x = a$	×	×	✓

Constant: $f(x) \approx f(a)$

Linear: $f(x) \approx f(a) + f'(a)(x-a)$

Quadratic: $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

We use $f(x) = \log x$ and $a = 1$. Then $f'(x) = x^{-1}$ and $f''(x) = -x^{-2}$, so $f(a) = 0$, $f'(a) = 1$, and $f''(a) = -1$. Now:

$$\begin{aligned} f(1.1) &\approx f(a) + f'(a)(1.1-a) + \frac{1}{2}f''(a)(1.1-a)^2 \\ &= 0 + 1(1.1-1) + \frac{1}{2}(-1)(1.1-1)^2 \\ &= 0.1 - \frac{1}{200} = \frac{20}{200} - \frac{1}{200} = \frac{19}{200} = \frac{9.5}{100} = 0.095 \end{aligned}$$

Google: $\log(1.1) = 0.0953101798\ldots$

QUADRATIC APPROXIMATION

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.

Q

ans

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.

We use $f(x) = x^{1/3}$ and $a = 27$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = \frac{-2}{9}x^{-5/3}$. So, $f(a) = 3, f'(a) = \frac{1}{3^3}$, and $f''(a) = \frac{-2}{3^7}$.

$$\begin{aligned} f(28) &\approx f(27) + f'(27)(28-27) + \frac{1}{2}f''(27)(28-27)^2 \\ &= 3 + \frac{1}{3^3}(1) + \frac{-1}{3^7}(1^2) \\ &= 3 + \frac{1}{3^3} - \frac{1}{3^7} \\ &= 3.03657978967\ldots \end{aligned}$$

$$\textit{Google}: \sqrt[3]{28} = 3.03658897188\ldots$$

Determine what $f(x)$ and a should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$

$e^{-1/30}$

$\sqrt[5]{30}$

$(2.01)^6$

Q

ans

Determine what $f(x)$ and a should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$ $\quad f(x) = \log(x), a = 1$

$e^{-1/30}$ $\quad f(x) = e^x, a = 0$

$\sqrt[5]{30}$ $\quad f(x) = \sqrt[5]{x}, a = 32 = 2^5$

$(2.01)^6$ $\quad f(x) = x^6, a = 2$

It is possible to compute the last one without an approximation, but an approximation might save time while being sufficiently accurate for your purposes.

	Constant	Linear	Quadratic	degree n
match $f(a)$	✓	✓	✓	✓
match $f'(a)$	×	✓	✓	✓
match $f''(a)$	×	×	✓	✓
...				
match $f^{(n)}(a)$	×	×	×	✓
match $f^{(n+1)}(a)$	×	×	×	×

Constant:

$$f(x) \approx f(a)$$

Linear:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Quadratic:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Degree-n:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$$

Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

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- a, b (integers) "bounds"

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- i "index": runs over integers from a to b

Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

- a, b (integers) “bounds”
- i “index”: runs over integers from a to b
- $f(i)$ “summand”: compute for every i, add

Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

Q

ans

Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

$$\sum_{i=2}^{4}(2i+5) = \underbrace{(2\cdot 2+5)}_{i=2} + \underbrace{(2\cdot 3+5)}_{i=3} + \underbrace{(2\cdot 4+5)}_{i=4}$$
$$= 9 + 11 + 13 = 33$$

Sigma Notation

$$\sum_{i=1}^{4}(i+(i-1)^2)$$

Q

ans

Sigma Notation

$$\sum_{i=1}^{4}(i+(i-1)^2)$$

$$= \underbrace{(1+0^2)}_{i=1} + \underbrace{(2+1^2)}_{i=2} + \underbrace{(3+2^2)}_{i=3} + \underbrace{(4+3^2)}_{i=4}$$

$$= 1+3+7+13 = 24$$

Write the following expressions in sigma notation:

1. $3 + 4 + 5 + 6 + 7$
2. $8 + 8 + 8 + 8 + 8$
3. $1 + (-2) + 4 + (-8) + 16$

Q

ans

Factorial – Definition 3.4.9

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For a natural number n, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

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Taylor Polynomial – Definition 3.4.11

Given a function $f(x)$ that is differentiable n times at a point a, the n-th degree **Taylor polynomial** for $f(x)$ about a is

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, we also call it a **Maclaurin polynomial**.

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=$$

ans

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \underbrace{f(a)}_{k=0} + \underbrace{f'(a)(x-a)}_{k=1} + \underbrace{\frac{1}{2!}f''(a)(x-a)^2}_{k=2} +$$

$$\underbrace{\frac{1}{3!}f'''(a)(x-a)^3}_{k=3} + \underbrace{\frac{1}{4!}f^{(4)}(a)(x-a)^4}_{k=4} +$$

$$\cdots + \underbrace{\frac{1}{n!}f^{(n)}(a)(x-a)^n}_{k=n}$$

Small degree Taylor polynomials

$$T_0(a) = \sum_{k=0}^{0} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a)$$

The 0th degree Taylor polynomial is the constant approximation

Small degree Taylor polynomials

$$T_1(a) = \sum_{k=0}^{1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a)$$

The 1st degree Taylor polynomial is the linear approximation

Small degree Taylor polynomials

$$
\begin{aligned}
T_2(a) &= \sum_{k=0}^{2} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2
\end{aligned}
$$

The 2nd degree Taylor polynomial is the quadratic approximation

Small degree Taylor polynomials

$$
\begin{aligned}
T_3(a) &= \sum_{k=0}^{3} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3
\end{aligned}
$$

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Maclaurin[2] polynomial for e^x.

[2] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

ans

Find the 7th degree Maclaurin[2] polynomial for e^x.

Let $f(x) = e^x$. Then every derivative of e^x is just e^x, and $e^0 = 1$. So:

$$\begin{aligned} T_7(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{7!}f^{(7)}(0)(x-0)^7 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\ &= \sum_{k=0}^{7} \frac{x^k}{k!} \end{aligned}$$

e^x approximations - link

[2] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

ans

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

$$\begin{array}{llll}
f(x) = \sin x & f(0) = 0 & f^{(4)}(0) = 0 & f^{(8)}(0) = 0 \\
f'(x) = \cos x & f'(0) = 1 & f^{(5)}(0) = 1 & \\
f''(x) = -\sin x & f''(0) = 0 & f^{(6)}(0) = 0 & \\
f'''(x) = -\cos x & f'''(0) = -1 & f^{(7)}(0) = -1 &
\end{array}$$

$$\begin{aligned}
T_8(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{8!}f^{(8)}(0)(x-0)^8 \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
&= \sum_{k=0}^{3} \frac{x^{2k+1}}{(2k+1)!}
\end{aligned}$$

Link: sine approximations

Now You

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

$$\begin{array}{llll}
f(x) = \log x & f(1) = 0 & f^{(4)}(x) = -3!x^{-4} & f^{(4)}(1) = -3! \\
f'(x) = x^{-1} & f'(1) = 1 & f^{(5)}(x) = 4!x^{-5} & f^{(5)}(1) = 4! \\
f''(x) = -x^{-2} & f''(1) = -1 & f^{(6)}(x) = -5!x^{-6} & f^{(6)}(1) = -5! \\
f'''(x) = 2x^{-3} & f'''(1) = 2 & f^{(7)}(x) = 6!x^{-7} & f^{(7)}(1) = 6!
\end{array}$$

$$\begin{aligned}
T_8(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \cdots + \frac{1}{7!}f^{(7)}(1)(x-1)^7 \\
&= 0 + (1)(x-1) + (-1)\frac{1}{2}(x-1)^2 + (2)\frac{1}{3!}(x-1)^3 - 3!\frac{1}{4!}(x-1)^4 \\
&\quad + 4!\frac{1}{5!}(x-1)^5 - 5!\frac{1}{6!}(x-1)^6 + 6!\frac{1}{7!}(x-1)^7 \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7!} \\
&= \sum_{k=1}^{7}(-1)^{k+1}\frac{(x-1)^k}{k}
\end{aligned}$$

log approximations

A

skip Δx notation

Notation 3.4.18

Let x, y be variables related such that $y = f(x)$. Then we denote a small change in the variable x by Δx (read as "delta x"). The corresponding small change in the variable y is denoted Δy (read as "delta y").

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let $y = f(x)$ be the amount of water needed to produce x apples in an orchard.
A farmer wants to know how a much water is needed to increase their crop yield.

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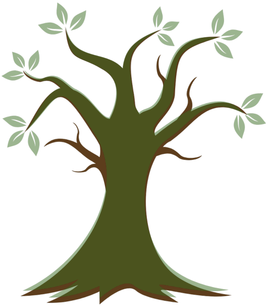
A farmer wants to know how a much water is needed to increase their crop yield. Δx is shorthand for some change in the number of apples, and Δy is shorthand for some change in the amount of water.



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- Consider changing the number of apples grown from a to $a + \Delta x$

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A farmer wants to know how a much water is needed to increase their crop yield. Δx is shorthand for some change in the number of apples, and Δy is shorthand for some change in the amount of water.

- Consider changing the number of apples grown from a to $a + \Delta x$
- Then the change in water requirements goes from $y = f(a)$ to $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximation of Δy

- Using a linear approximation, setting $x = a + \Delta x$:

LINEAR APPROXIMATION OF Δy

- Using a linear approximation, setting $x = a + \Delta x$:

$$
\begin{aligned}
f(x) &\approx f(a) + f'(a)(x-a) && \text{linear approximation} \\
f(a+\Delta x) &\approx f(a) + f'(a)(\Delta x) && \text{set } x = a + \Delta x \\
\Delta y = f(a+\Delta x) - f(a) &\approx f'(a)\Delta x && \text{subtract } f(a) \text{ both sides}
\end{aligned}
$$

Linear Approximation of Δy

- Using a linear approximation, setting $x = a + \Delta x$:

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Linear Approximation of Δy (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

Linear Approximation of Δy

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Linear Approximation of Δy (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

If we set $\Delta x = 1$, then $\Delta y \approx f'(a)$. So, if we want to produce $a+1$ apples instead of a apples, the extra water needed for that one extra apple is about $f'(a)$. We call this the *marginal* water cost of the apple.

Quadratic Approximation of Δy

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

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$$\begin{aligned}
f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\
f(a+\Delta x) &\approx f(a) + f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2 \\
f(a+\Delta x) - f(a) &\approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2 \\
\Delta y &\approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2
\end{aligned}$$

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If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

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f(a+\Delta x) - f(a) &\approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2 \\
\Delta y &\approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2
\end{aligned}$$

Quadratic Approximation of Δy (Equation 3.4.21)

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and π.

Q

ans

skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and π.

All our derivatives were based on radians, so first, let's do a conversion:

$$65 \text{ degrees} \cdot \left(\frac{2\pi \text{ radians}}{360 \text{ degrees}}\right) = \frac{13\pi}{36} \text{ radians}$$

$\frac{13\pi}{36}$ is pretty close to $\frac{\pi}{3}$ (and 65 is pretty close to 60), so we centre our approximation at $a = \frac{\pi}{3}$ (or $60°$). This is the closest reference angle to our desired angle.

We will need the first two derivatives of $f(x) = \tan x$ at $x = \frac{\pi}{3}$.

$$f(x) = \tan x \qquad f\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$f'(x) = \sec^2 x = \frac{1}{\cos^2 x} \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{(1/2)^2} = 4$$

$$f''(x) = \frac{2\sin x}{\cos^3 x} \qquad f''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{(1/2)^3} = 8\sqrt{3}$$

Constant: $f(x) \approx f(a)$

$$f\left(\frac{13\pi}{36}\right) \approx f\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Linear: $f(x) \approx f(a) + f'(a)(x - a)$

$$f\left(\frac{13\pi}{36}\right) \approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)$$
$$= \sqrt{3} + 4\left(\frac{\pi}{36}\right)$$

Quadratic: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

$$\begin{aligned} f\left(\frac{13\pi}{36}\right) &\approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right) + \frac{1}{2}f''\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)^2 \\ &= \sqrt{3} + 4\left(\frac{\pi}{36}\right) + \frac{1}{2}(8\sqrt{3})\left(\frac{\pi}{36}\right)^2 \\ &= \sqrt{3} + \frac{\pi}{9} + \frac{4\sqrt{3}\pi^2}{6^4} \end{aligned}$$

type	approx	decimal
constant	$\sqrt{3}$	1.732...
linear	$\sqrt{3} + \frac{\pi}{9}$	2.081...
quadratic	$\sqrt{3} + \frac{\pi}{9} + \frac{4\sqrt{3}\pi^2}{6^4}$	2.134...
actual	–	2.145...

You measure an angle $x \approx \frac{\pi}{2}$, and use it to calculate $y = \sin x \approx 1$. However, you suspect the angle was not *exactly* equal to $\frac{\pi}{2}$, which means the actual value y is slightly *less than* 1. In order for your value of y to have an error of no more than $\frac{1}{200}$, how accurate does your measurement of θ have to be?

Q

ans

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Let the actual angle x be $x_0 + \Delta x$ with $x_0 = \frac{\pi}{2}$, so Δx is the error in your measurement. Then let $y_0 = \sin x_0 = 1$ and $y = \sin x$. Then the error in y is $\Delta y = y - y_0$.
We want to solve

$$-\frac{1}{200} = \Delta y = y - y_0 = \sin x - \sin x_0 = \sin\left(\frac{\pi}{2} + \Delta x\right) - 1$$

for (the maximum allowed) Δx.

We'll show two solutions.

Solution 1

$$\begin{aligned}
-\frac{1}{200} &= \sin\left(\frac{\pi}{2}+\Delta x\right)-1\\
\frac{199}{200} &= \sin\left(\frac{\pi}{2}+\Delta x\right)\\
\arcsin\left(\frac{199}{200}\right) &= \frac{\pi}{2}+\Delta x\\
\Delta x &= \arcsin\left(\frac{199}{200}\right)-\frac{\pi}{2}\\
\Delta x &\approx -0.10004
\end{aligned}$$

So, the error in the measurement should not be more than about ± 0.10004.
The problem with that calculation is the last step – we had to find the arcsin of some crazy number, so we needed a pretty sophisticated approximation. Solution 2 will also use an approximation, but a much simpler one.

Solution 2 We will replace $f(x) = \sin x$ with its quadratic approximation centred at $a = \frac{\pi}{2}$:

$$\begin{aligned}\sin x &\approx \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\cdot\left(x-\frac{\pi}{2}\right) - \frac{1}{2}\sin\frac{\pi}{2}\cdot\left(x-\frac{\pi}{2}\right)^2 \\ &= 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2\end{aligned}$$

Now, we adjust the function from the last slide:

$$\begin{aligned} -\frac{1}{200} &= \sin\left(\frac{\pi}{2}+\Delta x\right) - 1 \\ \implies \quad -\frac{1}{200} &\approx 1 - \frac{1}{2}\left(\frac{\pi}{2}+\Delta x - \frac{\pi}{2}\right)^2 - 1 \\ &= -\frac{1}{2}(\Delta x)^2 \\ \frac{1}{100} &\approx \frac{1}{(\Delta x)^2} \\ \Delta x &\approx \pm\frac{1}{10}\end{aligned}$$

So, the error in the measurement should not be more than about $\pm\frac{1}{10}$.
(This accords quite well with Solution 1 – no arcsine required.)

Definition 3.4.25

Let Q_0 be the exact value of a quantity and let $Q_0 + \Delta Q$ be the measured value. We call

$$|\Delta Q|$$

the absolute error of the measurement, and

$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

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Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

Definition 3.4.25

Let Q_0 be the exact value of a quantity and let $Q_0 + \Delta Q$ be the measured value. We call

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the absolute error of the measurement, and

$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

The absolute error of the labelling is 2, and the percent error is $100\frac{2}{502} = 0.4$, or less than one-half of one percent.

Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

Q

ans

Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

Let x_0 be the actual value of the angle, x be the measured angle, and $\Delta x = x - x_0$. Then let $y(x) = \sin x$ (the computed y) and $y_0 = \sin x_0$ (the actual y), with $\Delta y = y - y_0$.
Using the linear approximation $y(x_0 + \Delta x) \approx y(x_0) + y'(x_0)(\Delta x)$:

$$\Delta y = y(x_0 + \Delta x) - y(x_0) \approx y'(x_0)\Delta x = \cos x_0 \cdot \Delta x$$

$$\text{Note: } 1 = 100\frac{|\Delta x|}{x_0} \implies \Delta x = \pm\frac{x_0}{100}$$

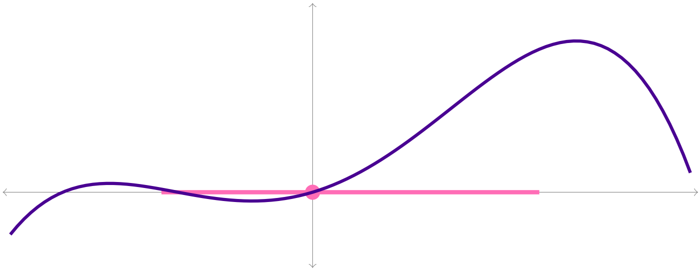
$$\implies \Delta y \approx \pm\frac{x_0 \cos x_0}{100}$$

$$\implies 100\frac{|\Delta y|}{y_0} \approx 100\frac{\frac{|x_0 \cos x_0|}{100}}{y_0} = \frac{x_0|\cos x_0|}{y_0} = \frac{x_0|\cos x_0|}{\sin x_0}$$

Note that when $x_0 \approx \frac{\pi}{2}$, this percentage error, $\frac{x_0|\cos x_0|}{\sin x_0}$, is close to 0; when $x_0 \approx 0$, it is about 1. (For the second fact, remember $\lim_{x\to 0} \frac{\sin x}{x} = 1$.)

A

Error: what "causes" error in an estimation?



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

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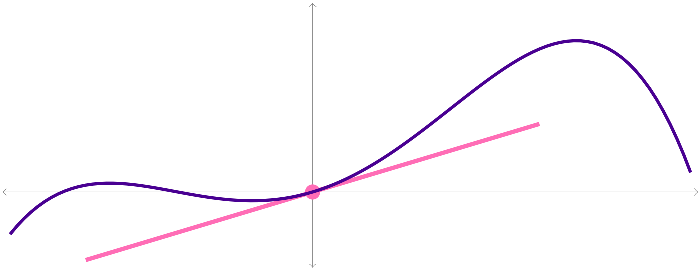
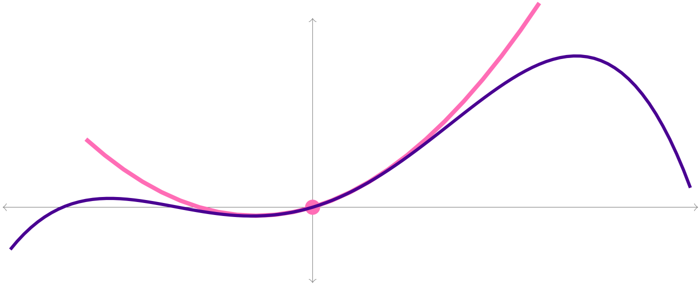


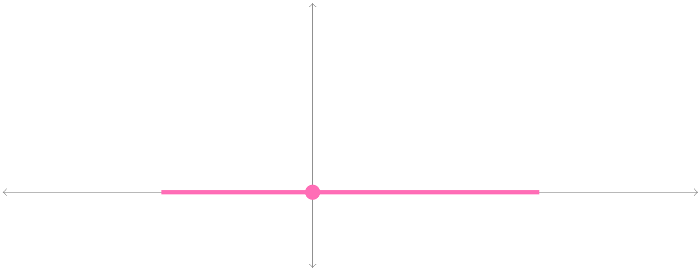
Linear approximation: We assume the function changes at a constant rate, but in fact the function changes at different rates (its second derivative is not always zero).

ERROR: WHAT "CAUSES" ERROR IN AN ESTIMATION?



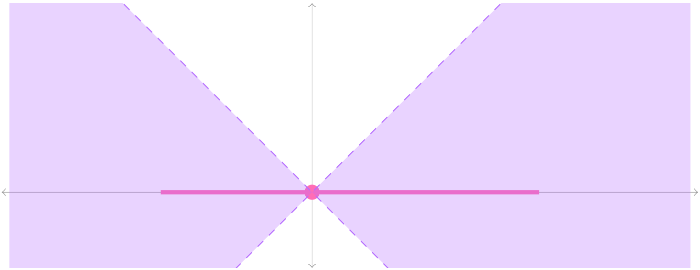
Quadratic approximation: We assume the function's derivative changes at a constant rate, but in fact the function's derivative changes at different rates (its third derivative is not always zero).

CONTROLLING THE "CAUSE" OF THE ERROR



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero). BUT: suppose we know the max and min values of the function's slope.

Controlling the "cause" of the error



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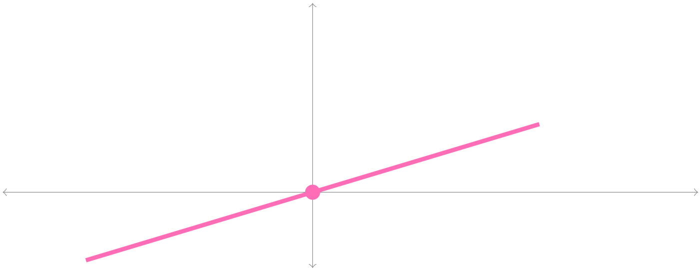
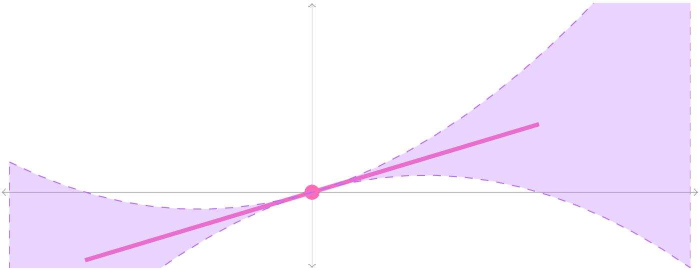


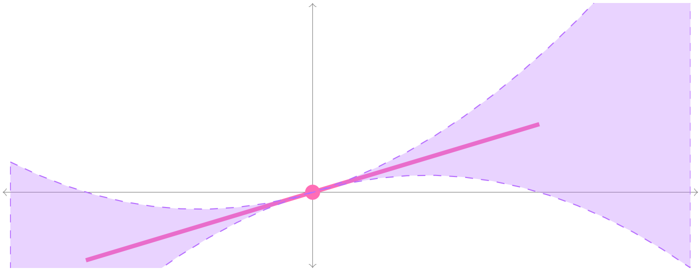
Linear approximation: We assume the function changes at a constant rate, but in fact the function changes at different rates (its first derivative is not always zero). BUT: suppose we know the max and min values of the function's second derivative.

Controlling the "cause" of the error



Linear approximation: We assume the function changes at a constant rate, but in fact the function changes at different rates (its first derivative is not always zero). BUT: suppose we know the max and min values of the function's second derivative.

Controlling the "cause" of the error



In general, if the "thing that causes the error" is big, then our error is big. We find the largest and smallest possible errors.

Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

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The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

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$$\begin{aligned}
T_3(x) &= f(0) + f'(0)(x-0) + \tfrac{1}{2!}f''(0)(x-0)^2 + \tfrac{1}{3!}f'''(0)(x-0)^3 \\
&= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}
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Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

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For some c in $(0\,,\,0.1)$:

$$\underbrace{f(0.1) - T_3(0.1)}_{\text{error}} = \frac{1}{4!}f^{(4)}(c)(.1-0)^4$$

$$= \frac{1}{4!}(0.0001)e^c$$

For c in $(0\,,\,0.1)$, $1 \le e^c < e^1 < 3$, so

$$\frac{1}{4!}(0.0001)1 \le \underbrace{f(.1) - T_3(.1)}_{\text{error}}$$

$$\le \frac{1}{4!}(0.0001)3$$

$$4.2 \times 10^{-6} \le \underbrace{f(0.1) - T_3(0.1)}_{\text{error}} \le 1.3 \times 10^{-5}$$

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$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could the magnitude of the error be if we approximate $\cos(2)$?

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For some c in $(\pi/2, 2)$:
$$\underbrace{f(2) - T_5(2)}_{error} = \frac{1}{6!}f^{(6)}(c)(2-\pi/2)^6$$

Note $f^{(6)}(x)$ is going to be plus or minus sine or cosine, so $-1 \le f^{(6)}(c) \le 1$. Also, $0 < 2 - \pi/2 < 1$. Now:

$$\frac{-1}{6!} = \frac{1}{6!}(-1)(1)^6 \le f(2) - T_5(2) \le \frac{1}{6!}(1)(1)^6 = \frac{1}{6!}$$

And $\frac{1}{6!} \approx 0.0014$. Be very careful with positives and negatives here :)

We don't actually have to compute $T_5(x)$, but if you want to as an exercise, click here to see the result.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use a third degree Taylor polynomial centred at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for our error.

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For some c in $(4, 4.1)$,
$|f(4.1) - T_3(4.1)| = \left|\frac{1}{4!}f^{(4)}(c)(4.1-4)^4\right| = \frac{0.1^4}{4!}\left|f^{(4)}(c)\right|$

So, let's investigate $f^{(4)}(c)$. First we find that the fourth derivative of $f(x) = x^{1/2}$ is $f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$. So, for c in $(4.1, 4)$, we have
$|f^{(4)}(c)| = \left|\frac{-15}{16\sqrt{c^7}}\right| = \frac{15}{16\sqrt{c^7}} \leq \frac{15}{16(\sqrt{4})^7} = \frac{15}{16\cdot 2^7}$

So, the error is bounded by:
$|f(4.1) - T_3(4.1)| \leq \frac{0.1^4}{4!} \cdot \frac{15}{16\cdot 2^7} \approx 0.00000003$

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose you want to approximate the value of e, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound your error.

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For some c in $(0, 1)$:
$|f(1) - T_4(1)| = \left|\frac{1}{5!}f^{(5)}(c)(1-0)^5\right| = \frac{1}{5!}e^c \leq \frac{1}{5!}e^1 < \frac{3}{5!} = 0.025$

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?

WHICH DEGREE?

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

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We want the magnitude of the error, so let's deal with absolute values. For some c in $(3, \pi)$:

$$|f(3) - T_n(3)| = \left|\frac{1}{(n+1)!} f^{(n+1)}(c)(3-\pi)^{n+1}\right| = \frac{|3-\pi|^{n+1}}{(n+1)!}\left|f^{(n+1)}(c)\right|$$
$$\leq \frac{(0.2)^{n+1}}{(n+1)!}(1) = \frac{0.2^{n+1}}{(n+1)!}$$

If we plug in $n = 2$, we get $\frac{0.2^{n+1}}{(n+1)!} = 0.00133...$, which is not SMALLER than 0.001. If we plug in $n = 3$, we get $\frac{0.2^{n+1}}{(n+1)!} = 0.000066...$ which IS smaller than 0.001. So we have to use the degree 3 Taylor polynomial.

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The magnitude of the error means its absolute value. Our error is, for some c in $(0, 5)$:

$$f(5) - T_n(5) = \frac{1}{(n+1)!}f^{(n+1)}(c)(5-0)^{n+1} = \frac{1}{(n+1)!}e^c 5^{n+1}$$

We can bound e^c for c in $(0, 5)$ by $1 = e^0 < e^c < e^5 < 3^5$. So now:

$$0 \le f(5) - T_n(5) \le \frac{1}{(n+1)!} \cdot 3^5 \cdot 5^{n+1}$$

We want $\frac{1}{(n+1)!} \cdot 3^5 \cdot 5^{n+1} < 0.001$, and by plugging in different values of n, we find the smallest n that makes the inequality true is $n = 21$. So we can use the 21st-degree Maclaurin polynomial and get our desired error.

Which Degree?

Suppose you want to approximate $\log \frac{4}{3}$ using a Taylor polynomial of $f(x) = \log x$ centred at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

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Your error will be: $f(3) - T_n(3) = \frac{1}{(n+1)!} f^{(n)}(c)\left(\frac{4}{3} - 1\right)^{n+1}$ for some c in $(1, 4/3)$. So, we need to bound $f^{(n)}(c)$. By writing out a number of derivatives of natural log, we notice that for $n \geq 1$, $f^{(n)}(c) = (-1)^{n-1}(n-1)!c^{-n}$. So, $f^{(n+1)}(c) = (-1)^n n! c^{-(n+1)}$. For c in $(1, 4/3)$:

$$\frac{n!}{(4/3)^{n+1}} \leq |f^{(n+1)}(c)| \leq \frac{n!}{1^{n+1}} = n!$$

Now for the error:

$$|f(3) - T_n(3)| \leq \frac{1}{(n+1)!} \cdot n! \cdot \left(\frac{4}{3} - 1\right)^{n+1} = \frac{1}{(n+1)3^{n+1}}$$

Setting this < 0.001, we find by plugging in values of n that $n = 4$ is the smallest n that makes the inequality true. So, using $T_4(x)$ will give us our desired error.

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

Taylor's Theorem tells us that, for some c in $(81, 81.2)$:

$$f(81.2) - T_2(81.2) = \frac{1}{3!}f^{(3)}(c)(81.2-81)^3 = \frac{1}{6}\cdot\left(\frac{1}{5}\right)^3 f^{(3)}(c) = \frac{1}{6\cdot 5^3}f^{(3)}(c)$$

So, we should probably find out what $f^{(3)}(x)$ is. Since $f(x) = x^{1/4}$, it's not too hard to figure out $f'''(x) = \frac{21}{4^3}x^{-11/4}$. So,

$$f(81.2) - T_2(81.2) = \frac{1}{6\cdot 5^3}\cdot\frac{21}{4^3c^{11/4}} = \frac{7}{2\cdot 4^3\cdot 5^3\cdot c^{11/4}}$$

Now our job is to bound this, and we should use reasonable numbers.

$$81 \le c \le 81.2$$

$$3 = 81^{1/4} \le c^{1/4} \le 81.2^{1/4} < 4$$

$$0 < \frac{7}{2\cdot 4^3\cdot 5^3\cdot 4^{11}} \le \frac{7}{2\cdot 4^3\cdot 5^3\cdot c^{11/4}} \le \frac{7}{2\cdot 4^3\cdot 5^3\cdot 3^{11}} \le 0.0000000025$$

Since $f(x) - T_2(x)$ is positive, $T_2(x)$ is an underestimate.

Included Work