

Definition 2.2.1

$$\text{So, } f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

$f'(a)$ is also the **instantaneous rate of change** of f at a .

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

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$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then f is **increasing** at a . Its graph “points up.”

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) < 0$, then f is _____ at a .

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

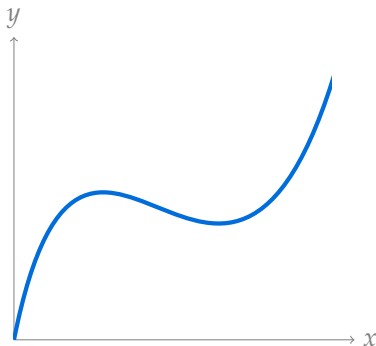
If $f'(a) < 0$, then f is **decreasing** at a . Its graph “points down.”

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

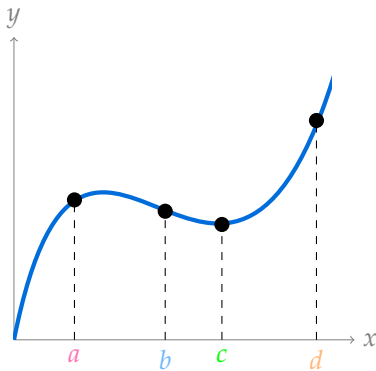
If $f'(a) < 0$, then f is **decreasing** at a . Its graph “points down.”

If $f'(a) = 0$, then f looks **constant** or **flat** at a .

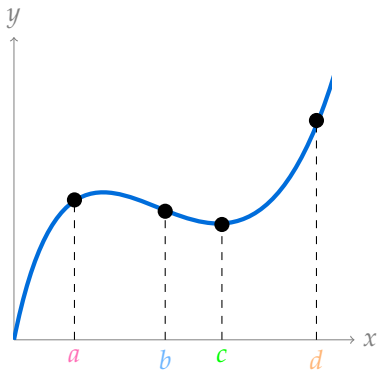
PRACTICE: INCREASING AND DECREASING



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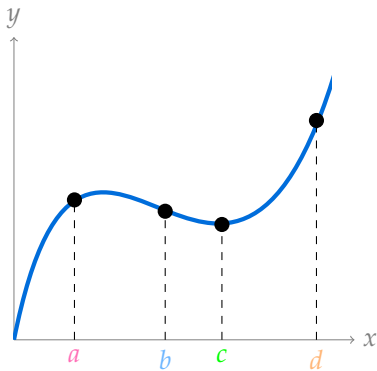


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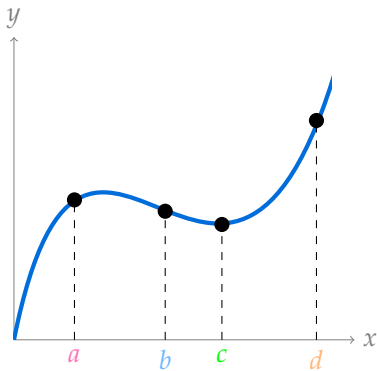
Where is $f'(x) < 0$?

PRACTICE: INCREASING AND DECREASING



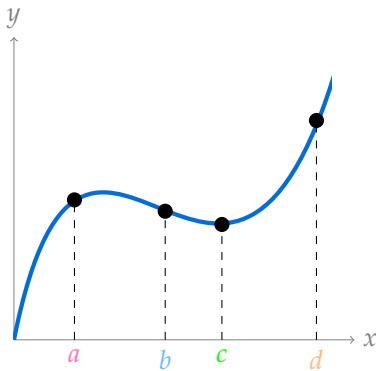
Where is $f'(x) < 0$? $f'(b) < 0$

PRACTICE: INCREASING AND DECREASING



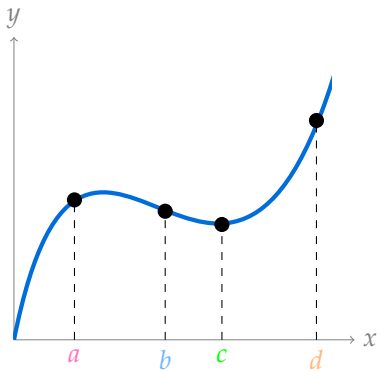
Where is $f'(x) > 0$?

PRACTICE: INCREASING AND DECREASING



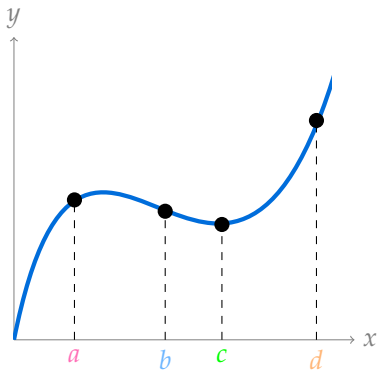
Where is $f'(x) > 0$? $f'(a) > 0$ and $f'(d) > 0$

PRACTICE: INCREASING AND DECREASING



Where is $f'(x) \approx 0$?

PRACTICE: INCREASING AND DECREASING



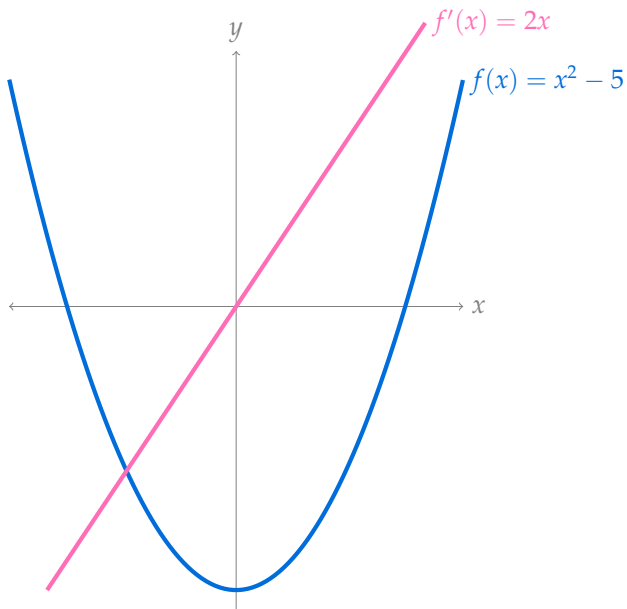
Where is $f'(x) \approx 0$? $f'(c) \approx 0$

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{((3+h)^2 - 5) - (3^2 - 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(9 + 6h + h^2 - 5) - 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 6h}{h} \\ &= \lim_{h \rightarrow 0} h + 6 = 6 \end{aligned}$$

Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$.
We can also find its derivative at an arbitrary point x :

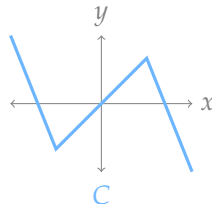
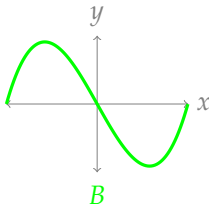
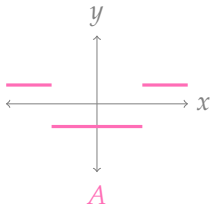
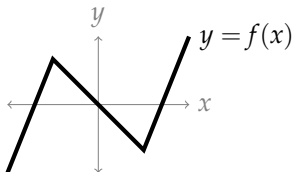
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We can also find its derivative at an arbitrary point x :

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 5 - (x^2 - 5)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\
 &= \lim_{h \rightarrow 0} 2x + h = 2x \quad (\text{In particular, } f'(3) = 6.)
 \end{aligned}$$



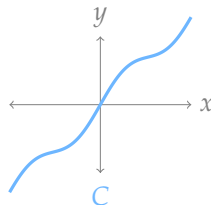
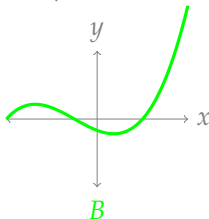
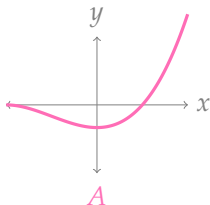
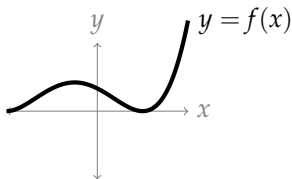
INCREASING AND DECREASING

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



INCREASING AND DECREASING

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Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.

The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that x will be a part of your final expression: this is a **function**.

If $f'(x)$ exists for all x in an interval (a, b) , we say that f is **differentiable on (a, b)** .

Notation 2.2.8

The “prime” notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{df}{dx}$$

$$\frac{df}{dx}(a)$$

$$\frac{d}{dx}f(x)$$

$$\frac{d}{dx}f(x)\Big|_{x=a}$$

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The “prime” notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibnitz notation:

$$\frac{df}{dx}$$

function

$$\frac{df}{dx}(a)$$

number

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$$\frac{d}{dx}f(x)\Big|_{x=a}$$

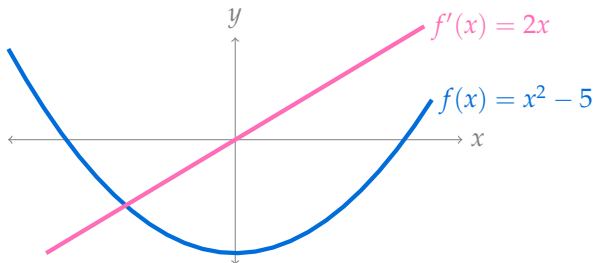
number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibnitz Notation:

$$\frac{df}{dx} = \qquad \frac{df}{dx}(3) = \qquad \frac{d}{dx}f(x) = \qquad \left. \frac{d}{dx}f(x) \right|_{x=3} =$$

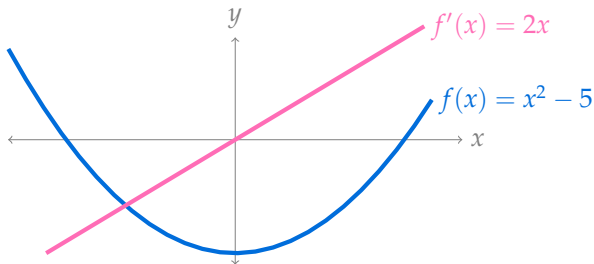


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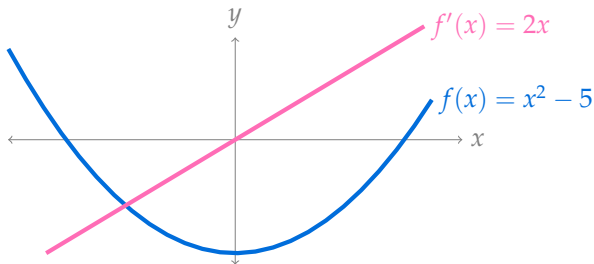


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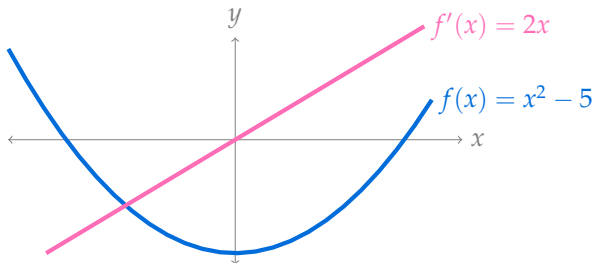


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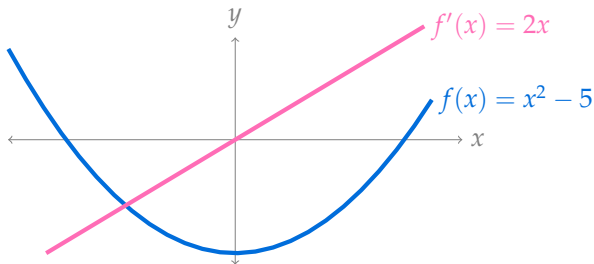


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Alternate Definition – Definition 2.2.1

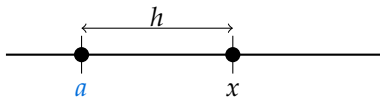
Calculating

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

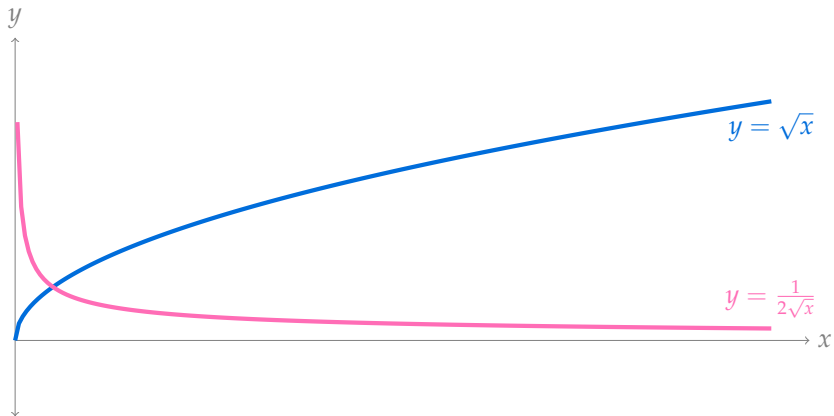
Notice in these scenarios, $h = x - a$.

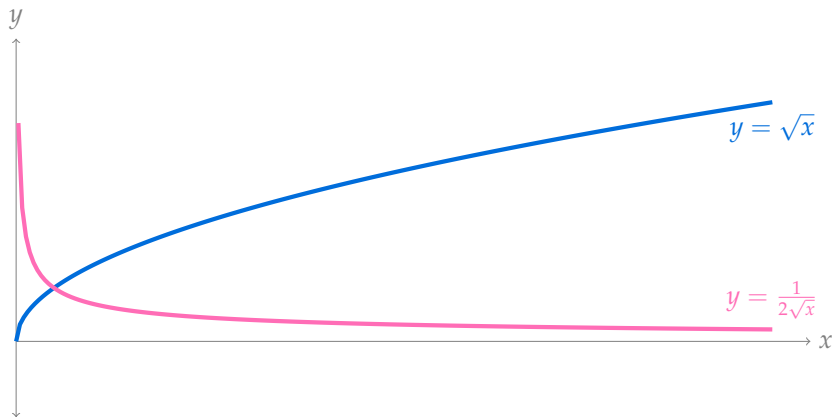


Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.

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$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

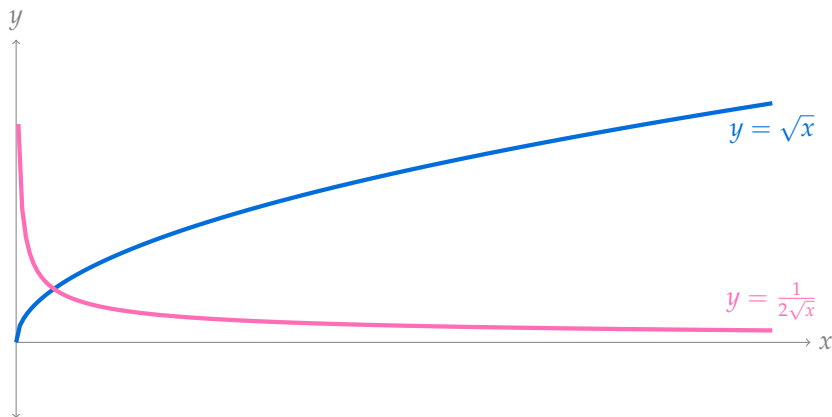




Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} =$$

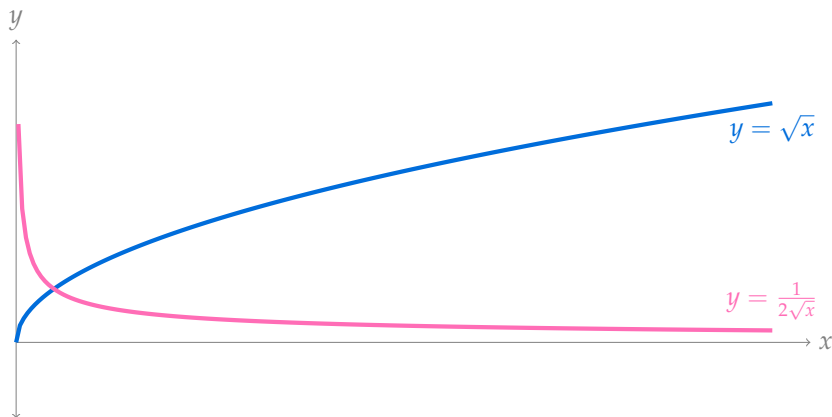
$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} =$$



Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

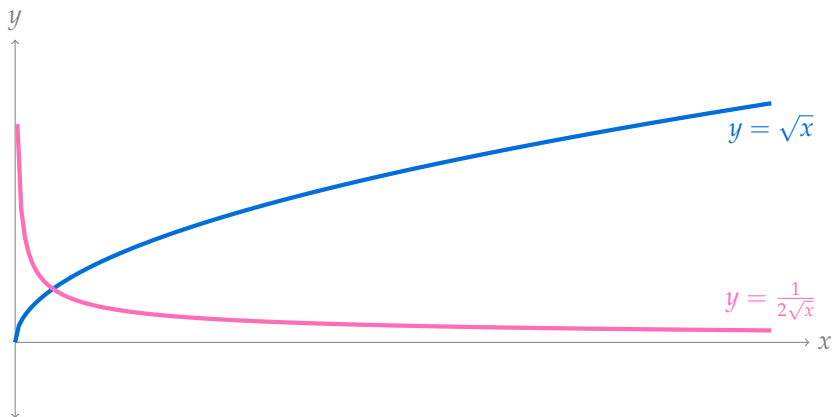
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Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$



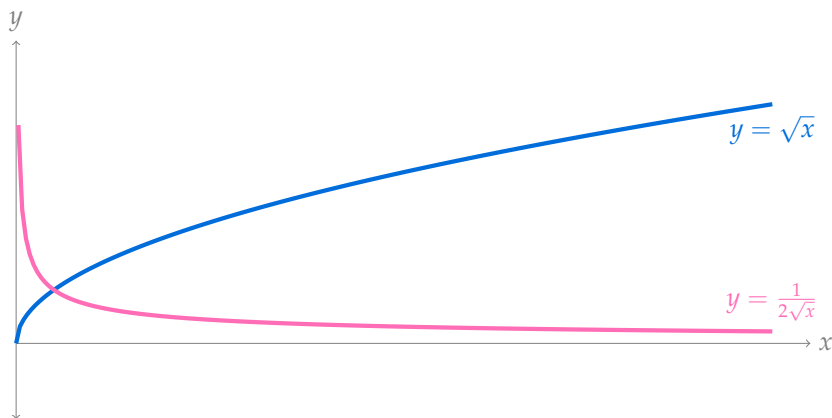
Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$

$$\lim_{x \rightarrow 0^+} \sqrt{x} =$$

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Review:

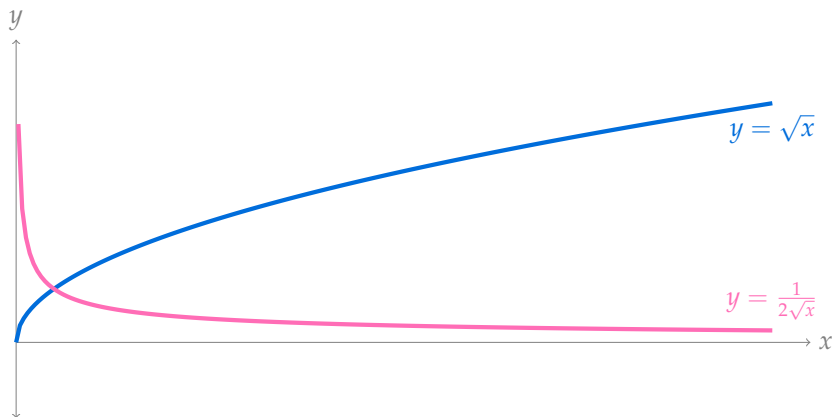
$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

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Review:

$$\lim_{x \rightarrow \infty} \sqrt{x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} = 0$$

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0$$

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Now
You



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{1}{x} \right\}.$$

Now
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Using the definition of the derivative, calculate

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$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{x} \right] &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \end{aligned}$$

NOW
YOU



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{2x}{x+1} \right\}.$$

NOW
YOU



Using the definition of the derivative, calculate

$$\frac{d}{dx} \left\{ \frac{1}{\sqrt{x^2 + x}} \right\}.$$

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$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{2(x+h)}{x+h+1} - \frac{2x}{x+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2(x+h)(x+1)}{(x+h+1)(x+1)} - \frac{2x(x+h+1)}{(x+1)(x+h+1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2}{h} \left(\frac{(x^2 + x + xh + h) - (x^2 + xh + x)}{(x+h+1)(x+1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2}{h} \left(\frac{h}{(x+h+1)(x+1)} \right) \\
 &= \lim_{h \rightarrow 0} \frac{2}{(x+h+1)(x+1)} = \frac{2}{(x+1)^2}
 \end{aligned}$$

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$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{(x+h)^2 + x+h}} - \frac{1}{\sqrt{x^2 + x}}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{x^2 + x}}{\sqrt{(x^2 + h)^2 + x+h} \sqrt{x^2 + x}} - \frac{\sqrt{(x+h)^2 + x+h}}{\sqrt{(x^2 + h)^2 + x+h} \sqrt{x^2 + x}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{x^2 + x} - \sqrt{(x+h)^2 + x+h}}{\sqrt{(x^2 + h)^2 + x+h} \sqrt{x^2 + x}} \right) \left(\frac{\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}}{\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(x^2 + x) - [(x+h)^2 + x+h]}{\sqrt{(x^2 + h)^2 + x+h} \sqrt{x^2 + x} [\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}]} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-(2xh + h^2 + h)}{\sqrt{(x^2 + h)^2 + x+h} \sqrt{x^2 + x} [\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}]} \right) \\
 &= \lim_{h \rightarrow 0} \frac{-(2x + h + 1)}{\sqrt{(x^2 + h)^2 + x+h} \sqrt{x^2 + x} [\sqrt{x^2 + x} + \sqrt{(x+h)^2 + x+h}]} = \frac{-(2x+1)}{2(x^2 + x)^{3/2}}
 \end{aligned}$$



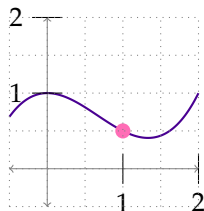
Memorize

The derivative of a function f at a point a is given by the following limit, if it exists:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

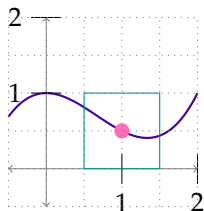
ZOOMING IN

For a smooth function, if we zoom in at a point, we see a line:



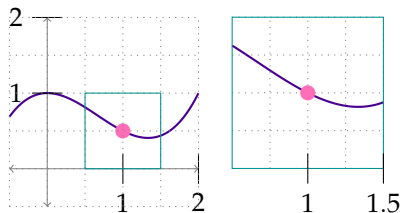
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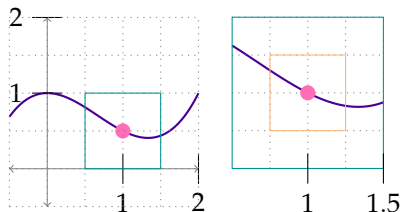
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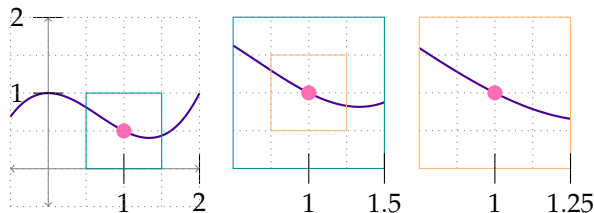
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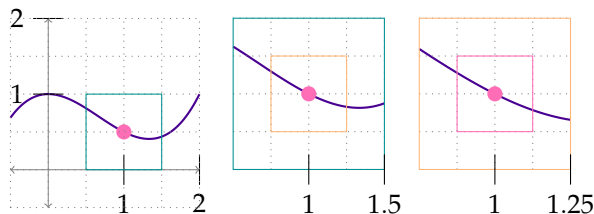
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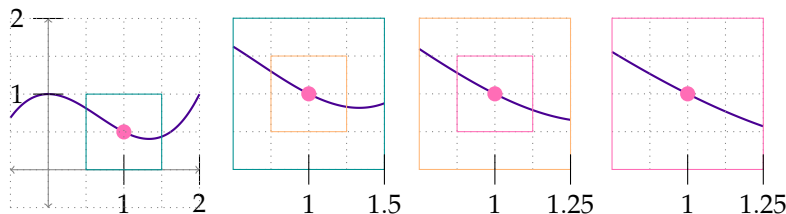
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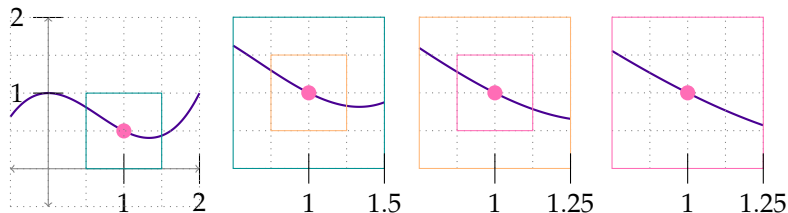
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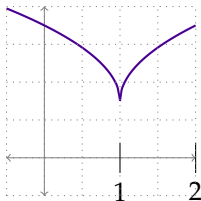
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

ZOOMING IN ON FUNCTIONS THAT AREN'T SMOOTH

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

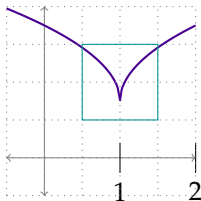
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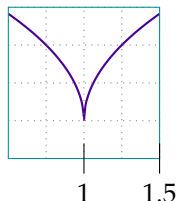
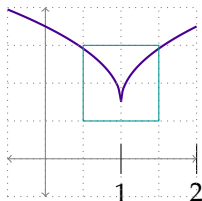
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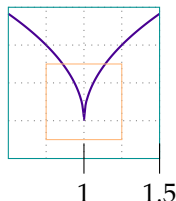
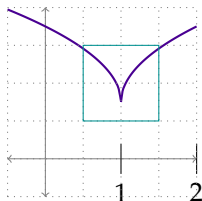
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ZOOMING IN ON FUNCTIONS THAT AREN'T SMOOTH

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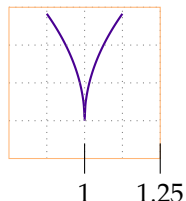
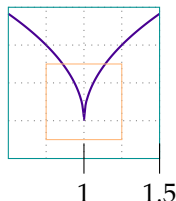
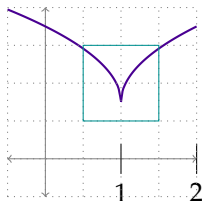
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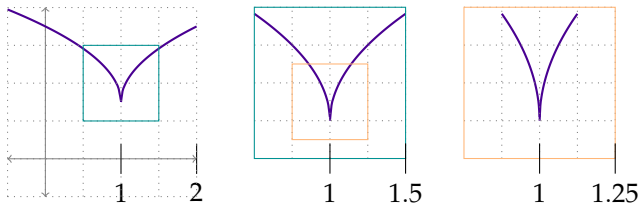
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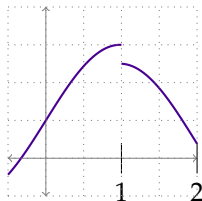
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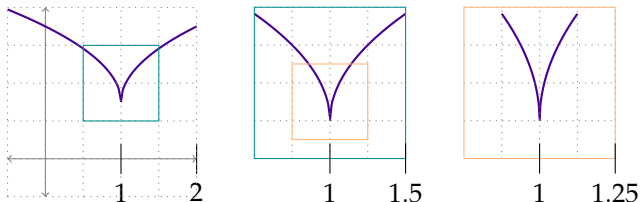
Discontinuity:



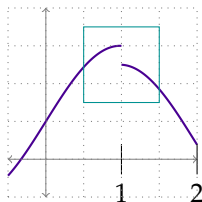
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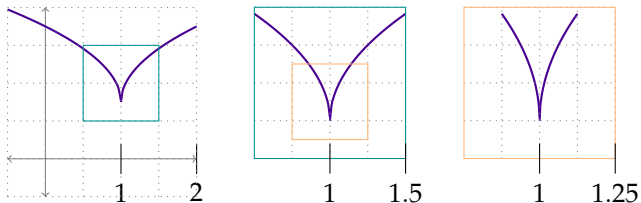
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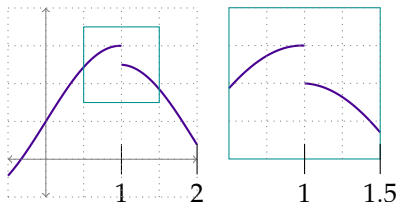
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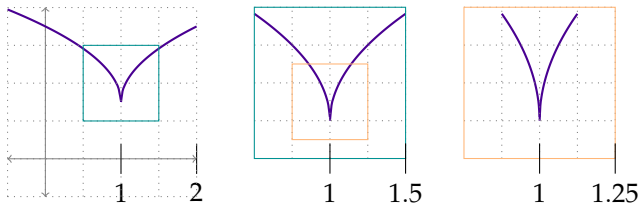
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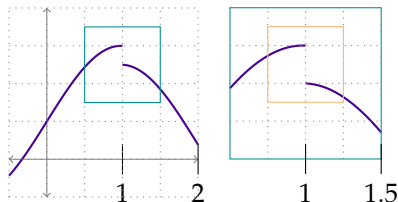
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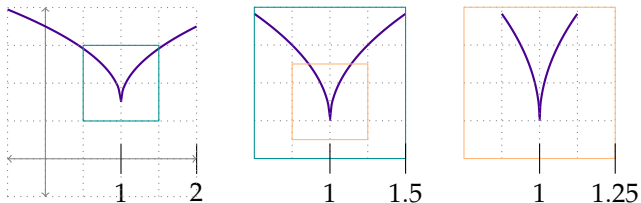
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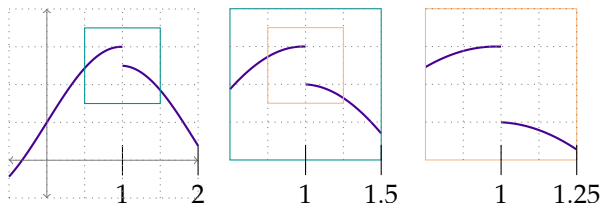
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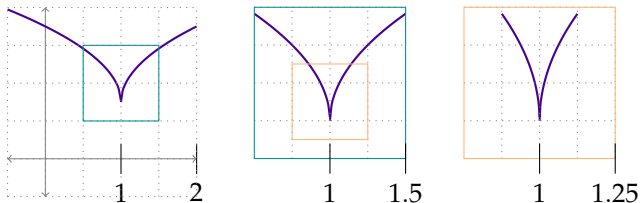
Discontinuity:



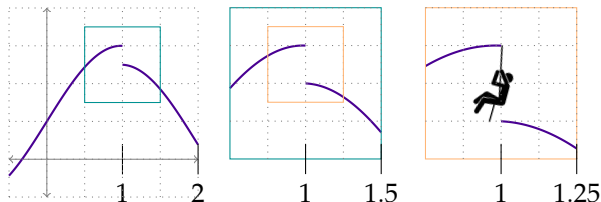
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Cusp:



Discontinuity:



Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ **does not exist** at $x = a$ if

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

does not exist.

Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

WHEN DERIVATIVES DON'T EXIST

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

WHEN DERIVATIVES DON'T EXIST

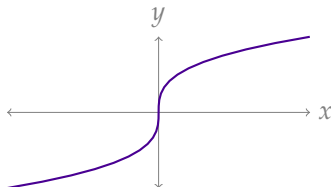
What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty \end{aligned}$$

Since the limit does not exist, we conclude $f'(x)$ is not defined at $x = 0$.

We can go a little farther: since the limit goes to infinity, the graph $y = f(x)$ looks vertical at $x = 0$.



Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof: If $f'(a)$ exists, that means:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists} \\ \implies \lim_{h \rightarrow 0} \left[h \cdot \frac{f(a+h) - f(a)}{h} \right] &= \left[\lim_{h \rightarrow 0} h \right] \cdot \left[\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right] \\ \implies \lim_{h \rightarrow 0} \left[h \cdot \frac{f(a+h) - f(a)}{h} \right] &= 0 \\ \implies \lim_{h \rightarrow 0} [f(a+h) - f(a)] &= 0 \\ \implies \lim_{h \rightarrow 0} f(a+h) &= f(a) \end{aligned}$$

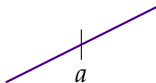
and that is the definition of $f(x)$ being continuous at $x = a$.

Let $f(x)$ be a function and let a be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

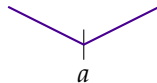
$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ not differentiable at $x = a$
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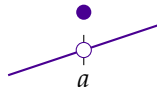
$f(x)$ continuous at $x = a$
 $f(x)$ not differentiable at $x = a$



$f(x)$ not continuous at $x = a$
 $f(x)$ differentiable at $x = a$

impossible

$f(x)$ not continuous at $x = a$
 $f(x)$ not differentiable at $x = a$



Included Work



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