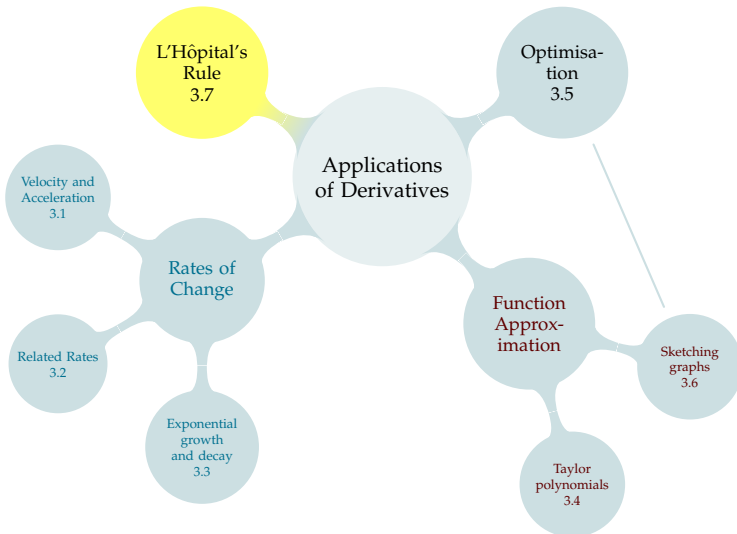


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# BACK TO LIMITS!

$$\lim_{x \rightarrow \infty} \frac{x^2}{5}$$

$$\lim_{x \rightarrow \infty} \frac{5}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{x^2}{5}$$

$$\lim_{x \rightarrow 0} \frac{5}{x^2}$$

## BACK TO LIMITS!

$$\lim_{x \rightarrow \infty} \frac{x^2}{5} = \infty$$

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## Indeterminate Forms – Definition 3.7.1

Suppose  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ . Then the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

is an **indeterminate form** of the type  $\frac{0}{0}$ .

Suppose  $\lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} G(x) = \infty$  (or  $-\infty$ ). Then the limit

$$\lim_{x \rightarrow a} \frac{F(x)}{G(x)}$$

is an **indeterminate form** of the type  $\frac{\infty}{\infty}$ .

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**When you see an indeterminate form, you need to do more work.**



$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$$

indeterminate form of the type  $\frac{0}{0}$

# INDETERMINATE FORMS

$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$$

indeterminate form of the type  $\frac{0}{0}$

To evaluate, factor the top:

$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x \rightarrow 5} \frac{(x - 5)(x + 2)}{x - 5} = \lim_{x \rightarrow 5} x + 2 = \boxed{7}$$

# INDETERMINATE FORMS

$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$$

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$$\lim_{x \rightarrow \infty} \frac{3x^2 - 4x + 2}{8x^2 - 5}$$

indeterminate form of the type  $\frac{\infty}{\infty}$

# INDETERMINATE FORMS

$$\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$$

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$$\lim_{x \rightarrow \infty} \frac{3x^2 - 4x + 2}{8x^2 - 5}$$

indeterminate form of the type  $\frac{\infty}{\infty}$

To evaluate, pull out  $x^2$ :

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 4x + 2}{8x^2 - 5} = \lim_{x \rightarrow \infty} \frac{x^2(3 - \frac{4}{x} + \frac{2}{x^2})}{x^2(8 - \frac{5}{x^2})} = \lim_{x \rightarrow \infty} \frac{3 - \frac{4}{x} + \frac{2}{x^2}}{8 - \frac{5}{x^2}} =$$

$$\lim_{x \rightarrow \infty} \frac{3 - 0 + 0}{8 - 0} = \boxed{\frac{3}{8}}$$

# INDETERMINATE FORMS AND THE DERIVATIVE

$$\lim_{x \rightarrow 0} \frac{3 \sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type  $\frac{0}{0}$



# INDETERMINATE FORMS AND THE DERIVATIVE

$$\lim_{x \rightarrow 0} \frac{3 \sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type  $\frac{0}{0}$

Suppose  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ . Suppose also that  $f$  and  $g$  are continuous and differentiable at  $a$ , and  $g'(a) \neq 0$ . Then:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} \frac{(x - a)^{-1}}{(x - a)^{-1}} \\ &= \lim_{x \rightarrow a} \left( \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \right) = \frac{f'(a)}{g'(a)} \end{aligned}$$



# INDETERMINATE FORMS AND THE DERIVATIVE

$$\lim_{x \rightarrow 0} \frac{3 \sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type  $\frac{0}{0}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3 \sin x - x^4}{x^2 + \cos x - e^x} &= \frac{\frac{d}{dx}[3 \sin x - x^4]|_{x=0}}{\frac{d}{dx}[x^2 + \cos x - e^x]|_{x=0}} \\ &= \frac{[3 \cos x - 4x^3]|_{x=0}}{[2x - \sin x - e^x]|_{x=0}} \\ &= \frac{3 - 0}{0 - 0 - 1} = \boxed{-3} \end{aligned}$$



## L'Hôpital's Rule: First Part – Theorem 3.7.2

Let  $f$  and  $g$  be functions such that  $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$ .

If  $f'(a)$  and  $g'(a)$  exist and  $g'(a) \neq 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ .

If  $f$  and  $g$  are differentiable on an open interval containing  $a$ , and if  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  exists, then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .

This works even for  $a = \pm\infty$ .

**Extremely Important Note:**  
L'Hôpital's Rule only works on indeterminate forms.



## L'Hôpital's Rule: Second Part – Theorem 3.7.2

Let  $f$  and  $g$  be functions such that  $\lim_{x \rightarrow a} f(x) = \infty = \lim_{x \rightarrow a} g(x)$ .

If  $f'(a)$  and  $g'(a)$  exist and  $g'(a) \neq 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ .

If  $f$  and  $g$  are differentiable on an open interval containing  $a$ , and if  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  exists, then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .

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**Extremely Important Note:**  
L'Hôpital's Rule only works on indeterminate forms.

Evaluate:

$$\lim_{x \rightarrow 2} \frac{3x \tan(x - 2)}{x - 2}$$

Evaluate:

$$\lim_{x \rightarrow 2} \frac{3x \tan(x-2)}{x-2}$$

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{3x \tan(x-2)}{x-2} & \quad \text{form } \frac{0}{0} \\ \underline{\underline{r_H}} \quad \frac{3 [x \sec^2(x-2) + \tan(x-2)]_{x=2}}{1} \\ &= 3 [2 \sec^2 0 + \tan 0] = \boxed{6} \end{aligned}$$

## LITTLE HARDER

$$\lim_{x \rightarrow 0} \frac{x^4}{e^x - \cos x - x}$$

indeterminate form of the type  $\frac{0}{0}$

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$$\lim_{x \rightarrow 0} \frac{x^4}{e^x - \cos x - x} \stackrel{?}{=} \stackrel{?}{=} \frac{4x^3}{e^x + \sin x - 1} \Big|_{x=0} = \frac{0}{0}$$

oops



## LITTLE HARDER

$$\lim_{x \rightarrow 0} \frac{x^4}{e^x - \cos x - x}$$

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$$\lim_{x \rightarrow 0} \frac{x^4}{e^x - \cos x - x} \stackrel{?}{=} \stackrel{?}{=} \frac{4x^3}{e^x + \sin x - 1} \bigg|_{x=0} = \frac{0}{0}$$

oops

Iterate!

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^4}{e^x - \cos x - x} &\stackrel{\text{rH}}{=} \lim_{x \rightarrow 0} \frac{4x^3}{e^x + \sin x - 1} \\ &\stackrel{\text{rH}}{=} \lim_{x \rightarrow 0} \frac{12x^2}{e^x + \cos x} = \frac{0}{2} = \boxed{0} \end{aligned}$$



Evaluate:

$$\lim_{x \rightarrow \infty} \frac{\log x}{\sqrt{x}}$$

Evaluate:

$$\lim_{x \rightarrow \infty} \frac{\log x}{\sqrt{x}}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\log x}{x} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = \boxed{0} \end{aligned}$$



# OTHER INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} e^{-x} \log x$$

form  $0 \cdot \infty$



## OTHER INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} e^{-x} \log x$$

form  $0 \cdot \infty$ 

$$\lim_{x \rightarrow \infty} e^{-x} \log x = \lim_{x \rightarrow \infty} \frac{\log x}{e^x}$$

form  $\frac{\infty}{\infty}$ 

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{1/x}{e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{xe^x} = \boxed{0}$$



## VOTE VOTE VOTE

Which of the following can you immediately apply L'Hôpital's rule to?

A.  $\frac{e^x}{2e^x + 1}$

B.  $\lim_{x \rightarrow 0} \frac{e^x}{2e^x + 1}$

C.  $\lim_{x \rightarrow \infty} \frac{e^x}{2e^x + 1}$

D.  $\lim_{x \rightarrow \infty} e^{-x}(2e^x + 1)$

E.  $\lim_{x \rightarrow 0} \frac{e^x}{x^2}$

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D.  $\lim_{x \rightarrow \infty} e^{-x}(2e^x + 1)$

E.  $\lim_{x \rightarrow 0} \frac{e^x}{x^2}$

## VOTEY McVOTEFACE

Suppose you want to use L'Hôpital's rule to evaluate  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ , which has the form  $\frac{0}{0}$ . How does the quotient rule fit into this problem?

- A. You should use the quotient rule because the function you are differentiating is a quotient.
- B. You will not use the quotient rule because you differentiate the numerator and the denominator separately
- C. You may use the quotient rule because perhaps  $f(x)$  or  $g(x)$  is itself in the form of a quotient
- D. You will not use L'Hôpital's rule because  $\frac{0}{0}$  is not an appropriate indeterminate form
- E. You will not use L'Hôpital's rule because, since the top has limit zero, the whole function has limit 0

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- E. You will not use L'Hôpital's rule because, since the top has limit zero, the whole function has limit 0

# MORE QUESTIONS

Which of the following is NOT an indeterminate form?

- A.  $\frac{\infty}{\infty}$  for example,  $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$
- B.  $\frac{0}{0}$  for example,  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$
- C.  $\frac{0}{\infty}$  for example,  $\lim_{x \rightarrow 0^+} \frac{x}{\log x}$
- D.  $0 \cdot \infty$  for example,  $\lim_{x \rightarrow \infty} x(\arctan(x) - \pi/2)$
- E. all of the above are indeterminate forms

# MORE QUESTIONS

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- C.  $\frac{0}{\infty}$  for example,  $\lim_{x \rightarrow 0^+} \frac{x}{\log x} = 0$
- D.  $0 \cdot \infty$  for example,  $\lim_{x \rightarrow \infty} x(\arctan(x) - \pi/2)$
- E. all of the above are indeterminate forms



# I HAVE SO MANY QUESTIONS

Which of the following is NOT an indeterminate form?

A.  $1^\infty$  for example,  $\lim_{x \rightarrow \infty} \left( \frac{x+1}{x} \right)^x$

B.  $0^\infty$  for example,  $\lim_{x \rightarrow \infty} \left( \frac{1}{x} \right)^x$

C.  $\infty^0$  for example,  $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$

D.  $0^0$  for example,  $\lim_{x \rightarrow 0^+} x^x$

E. all of the above are indeterminate forms

F. none of the above are indeterminate forms

# I HAVE SO MANY QUESTIONS

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B.  $0^\infty$  for example,  $\lim_{x \rightarrow \infty} \left( \frac{1}{x} \right)^x = 0$

C.  $\infty^0$  for example,  $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$

D.  $0^0$  for example,  $\lim_{x \rightarrow 0^+} x^x$

E. all of the above are indeterminate forms

F. none of the above are indeterminate forms



# EXPONENTIAL INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} x^{1/x}$$

## EXPONENTIAL INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} x^{1/x}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} x^{1/x} &= \lim_{x \rightarrow \infty} e^{(\log(x^{1/x}))} \\ &= \lim_{x \rightarrow \infty} e^{\left(\frac{\log x}{x}\right)} \\ &= e^{\left(\lim_{x \rightarrow \infty} \frac{\log x}{x}\right)} \\ &\stackrel{\text{L'H}}{=} e^{\left(\lim_{x \rightarrow \infty} \frac{1}{x}\right)} \\ &= e^0 = \boxed{1} \end{aligned}$$

# EXPONENTIAL INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{3x}$$

## EXPONENTIAL INDETERMINATE FORMS

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{3x}$$

First we calculate:

$$\begin{aligned} \lim_{x \rightarrow \infty} \log \left( \left(1 + \frac{2}{x}\right)^{3x} \right) &= \lim_{x \rightarrow \infty} 3x \log \left(1 + \frac{2}{x}\right) \\ &= \lim_{x \rightarrow \infty} \frac{3 \log \left(1 + \frac{2}{x}\right)}{x^{-1}} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{3 \left( \frac{-2x^{-2}}{1+2/x} \right)}{-x^{-2}} \\ &= \lim_{x \rightarrow \infty} \frac{6}{1 + 2/x} = 6 \end{aligned}$$

So, now:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{3x} = \boxed{e^6}$$



Evaluate:

$$\lim_{x \rightarrow \infty} \frac{\log x}{\log \sqrt{x}}$$

$$\lim_{x \rightarrow \infty} (\log x)^{\sqrt{x}}$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x}$$



Evaluate:

$$\lim_{x \rightarrow \infty} \frac{\log x}{\log \sqrt{x}}$$

Easier to simplify first.

$$\lim_{x \rightarrow \infty} (\log x)^{\sqrt{x}}$$

Not an indeterminate form: huge number to a huge power. Limit is infinity.

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x}$$

$$\text{L'Hôpital: } \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$





# MORE EXAMPLES

$$\lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} - \sqrt{x^2 + x}$$

$$\lim_{x \rightarrow 0} \sqrt[x^2]{\sin^2 x}$$

$$\lim_{x \rightarrow 0} \sqrt[x^2]{\cos x}$$



$$\lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} - \sqrt{x^2 + x}$$

$\lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} - \sqrt{x^2 + x}$  has the indeterminate form  $\infty - \infty$ . To get a better idea of what's going on, let's rationalize.

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \sqrt{2x^2 + 1} - \sqrt{x^2 + x} \\
 &= \lim_{x \rightarrow \infty} \left( \sqrt{2x^2 + 1} - \sqrt{x^2 + x} \right) \left( \frac{\sqrt{2x^2 + 1} + \sqrt{x^2 + x}}{\sqrt{2x^2 + 1} + \sqrt{x^2 + x}} \right) \\
 &= \lim_{x \rightarrow \infty} \frac{(2x^2 + 1) - (x^2 + x)}{\sqrt{2x^2 + 1} + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{x^2 - x + 1}{\sqrt{2x^2 + 1} + \sqrt{x^2 + x}}
 \end{aligned}$$

Here, we have the indeterminate form  $\frac{\infty}{\infty}$ , so l'Hôpital's Rule applies. However, if we try to use it here, we quickly get a huge mess. Instead, remember how we dealt with these kinds of limits in the past: factor out the highest power of the denominator, which is  $x$ .



$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{x^2 - x + 1}{\sqrt{2x^2 + 1} + \sqrt{x^2 + x}} &= \lim_{x \rightarrow \infty} \frac{x \left( x - 1 + \frac{1}{x} \right)}{\sqrt{x^2 \left( 2 + \frac{1}{x^2} \right)} + \sqrt{x^2 \left( 1 + \frac{1}{x} \right)}} \\
 &= \lim_{x \rightarrow \infty} \frac{x \left( x - 1 + \frac{1}{x} \right)}{x \left( \sqrt{2 + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x}} \right)} \\
 &= \lim_{x \rightarrow \infty} \frac{x - 1 + \frac{1}{x}}{\underbrace{\sqrt{2 + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x}}}_{\substack{\text{num} \rightarrow \infty \\ \text{den} \rightarrow \sqrt{2} + 1}}} \\
 &= \infty
 \end{aligned}$$

$$\lim_{x \rightarrow 0} \sqrt{x^2 \sin^2 x}$$



$$\lim_{x \rightarrow 0} \sqrt{x^2 \sin^2 x}$$

$\lim_{x \rightarrow 0} \sin^2 x = 0$ , and  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ , so we have the form  $0^\infty$ . (Note that  $\sin^2 x$  is positive, so our root is defined.) This is not an indeterminate form:  $\lim_{x \rightarrow 0} \sqrt{x^2 \sin^2 x} = 0$ .



$$\lim_{x \rightarrow 0} \sqrt[x^2]{\cos x}$$



$$\lim_{x \rightarrow 0} \sqrt[x^2]{\cos x}$$

$\lim_{x \rightarrow 0} \cos x = 1$  and  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ , so  $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$  has the indeterminate form  $1^\infty$ . We want to use l'Hôpital, but we need to get our function into a fractional indeterminate form. So, we'll use a logarithm.

$$y := (\cos x)^{\frac{1}{x^2}}$$

$$\log y = \log \left( (\cos x)^{\frac{1}{x^2}} \right) = \frac{1}{x^2} \log(\cos x) = \frac{\log \cos x}{x^2}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \log y &= \lim_{x \rightarrow 0} \underbrace{\frac{\log \cos x}{x^2}}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}} = \lim_{x \rightarrow 0} \frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x \rightarrow 0} \underbrace{\frac{-\tan x}{2x}}_{\substack{\text{num} \rightarrow 0 \\ \text{den} \rightarrow 0}} = \lim_{x \rightarrow 0} \frac{-\sec^2 x}{2} \\ &= \lim_{x \rightarrow 0} \frac{-1}{2 \cos^2 x} = -\frac{1}{2} \end{aligned}$$

$$\text{Therefore, } \lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} e^{\log y} = e^{-1/2} = \frac{1}{\sqrt{e}}$$





Sketch the graph of  $f(x) = x \log x$ .

Note: when you want to know  $\lim_{x \rightarrow 0} f(x)$ , you'll need to use L'Hôpital.

Evaluate  $\lim_{x \rightarrow 0^+} (\csc x)^x$



$$f(x) = x \log x$$



$$f(x) = x \log x$$

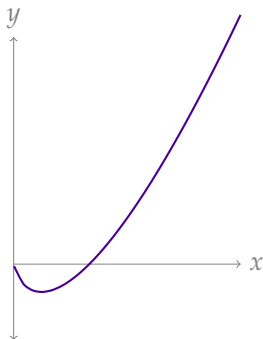
- ▶ Domain:  $x > 0$
- ▶ HA: none
- ▶ Intercepts:  $(1, 0)$
- ▶  $y = \log x$  has a VA at  $x = 0$ . For our function, let's see what its behaviour is near 0:

$$\lim_{x \rightarrow 0^+} x \log x = \lim_{x \rightarrow 0^+} \frac{\log x}{1/x} \quad \text{form } \frac{\infty}{\infty}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0$$

- ▶  $f'(x) = 1 + \log x$ ;  
 CP at  $x = \frac{1}{e}$  (and  $y = -\frac{1}{e}$ );  
 As  $x$  gets close to 0,  $f'(x)$  goes to negative infinity, so near  $x = 0$  our line looks vertical.  
 Decreasing on  $(0, \frac{1}{e})$  and increasing on  $(1/e, \infty)$
- ▶  $f''(x) = \frac{1}{x}$ ; concave up on entire domain





Evaluate  $\lim_{x \rightarrow 0^+} (\csc x)^x$



Evaluate  $\lim_{x \rightarrow 0^+} (\csc x)^x$

Indeterminate form:  $\infty^0$

$$y = (\csc x)^x$$

$$\log y = x \log(\csc x)$$

$$\lim_{x \rightarrow 0^+} \log y = \lim_{x \rightarrow 0^+} x \log(\csc x)$$

$$= \lim_{x \rightarrow 0^+} x \log \left( \frac{1}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0^+} -x \log(\sin x)$$

indeterminate form  $0 \cdot \infty$

$$= \lim_{x \rightarrow 0^+} \frac{\log(\sin x)}{-1/x}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{-1/x^2}$$

$$= \lim_{x \rightarrow 0^+} \frac{-x^2}{\tan x}$$

indeterminate form  $\frac{0}{0}$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{-2x}{\sec^2 x}$$

$$= \frac{0}{1} = 0$$

conclusion

$$\lim_{x \rightarrow 0^+} \log y = 0$$

$$\lim_{x \rightarrow 0^+} y = e^0 = \boxed{1}$$



## Included Work



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