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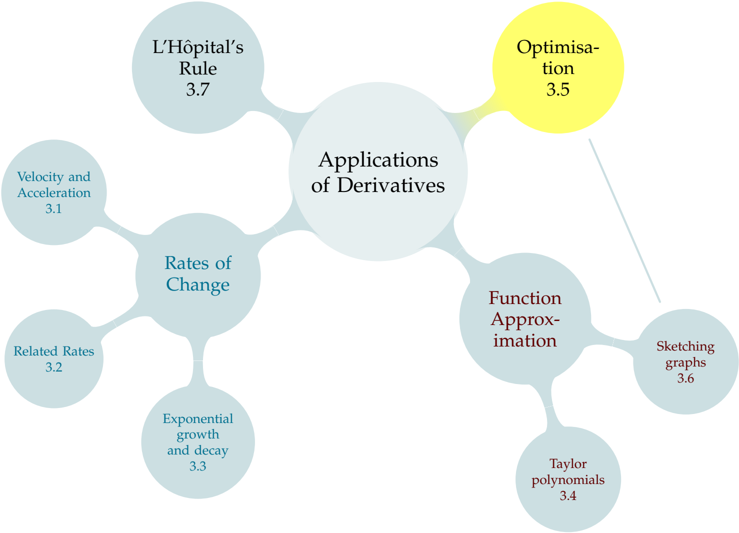


Optimisation:
finding the biggest/smallest/highest/lowest, etc.

Lots of non-standard problems! Opportunities to work on your problem-solving skills.

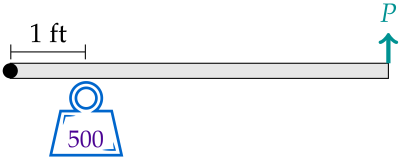
We start with some problems that someone might want to solve, only by way of motivation. We won't actually solve them here.

Optimisation:
finding the biggest/smallest/highest/lowest, etc.

Lots of non-standard problems! Opportunities to work on your problem-solving skills.

Engineering Design Example

A lever of density 3 lbs/ft is being used to lift a 500-pound weight, attached one foot from the fixed point.



For an L-foot-long lever, the force P required to lift the system satisfies

$$500(1) + 3L\left(\frac{L}{2}\right) - PL = 0$$

What length of lever will require the least amount of force to lift?

Source: Drexel (2006)

Medical Dosing Example

Let D be the size of a dose, α be the absorption rate, and β the elimination rate of a drug.
Caffeine is absorbed and eliminated by first-order kinetics. Its blood concentration over time is modelled as

$$c(t) = \frac{D}{1 - \beta/\alpha}\left(e^{-\beta t} - e^{-\alpha t}\right)$$

Will the blood concentration reach a toxic level?

Source (including links to a study): Vectornaut (2015)

Circuit Example

When a critically damped RLC circuit is connected to a voltage source, the current I in the circuit varies with time according to the equation

$$I(t) = \left(\frac{V}{L}\right) te^{-\frac{Rt}{2L}}$$

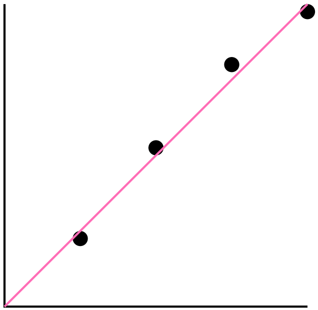
where V is the applied voltage, L is the inductance, and R is the resistance (all of which are constant).

We need to choose wires that will be able to safely carry the current at all times.

Source: Belk (2014)

Least Squares Example

You have a lot of data that more-or-less resembles a line.
Which line does it most resemble?



Extrema – Definition 3.5.3

Let $a \leq b$ and let the function $f(x)$ be defined for all x in the interval $[a, b]$. Let $a \leq c \leq b$.

- We say that $f(x)$ has a global / absolute minimum at $x = c$ if $f(x) \geq f(c)$ for all $a \leq x \leq b$.
- We say that $f(x)$ has a global / absolute maximum at $x = c$ if $f(x) \leq f(c)$ for all $a \leq x \leq b$.

Now let $a < c < b$.

- We say that $f(x)$ has a local minimum at $x = c$ if there are a' and b' obeying $a \leq a' < c < b' \leq b$ such that $f(x) \geq f(c)$ for all x obeying $a' < c < b'$.
- We say that $f(x)$ has a local maximum at $x = c$ if there are a' and b' obeying $a \leq a' < c < b' \leq b$ such that $f(x) \leq f(c)$ for all x obeying $a' < c < b'$.

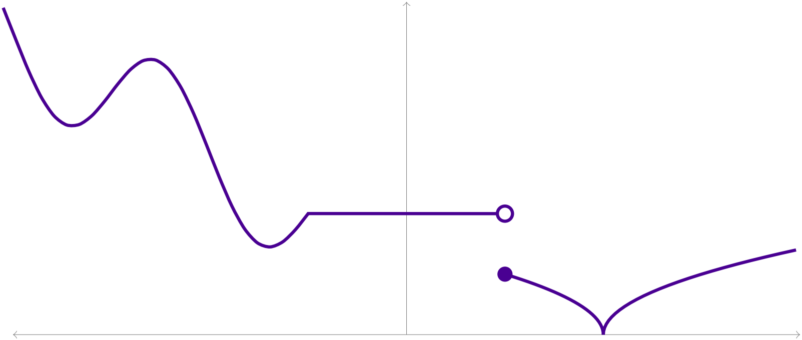
The maxima and minima of a function are called the extrema of that function.

Critical and Singular Points – Definition 3.5.6

Let $f(x)$ be a function and let c be a point in its domain. Then

- If $f'(c)$ exists and is zero we call $x = c$ a critical point of the function, and
- If $f'(c)$ does not exist then we call $x = c$ a singular point of the function.

Anatomy of a Function



c is a critical point if $f'(c) = 0$.

c is a singular point if $f'(c)$ does not exist.

Theorem 3.5.4

If a function $f(x)$ has a local maximum or local minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

Multiple Choice

Suppose $f(x)$ has domain $(-\infty, \infty)$.
If f has a local minimum at $x = 2$, then:

A. $f'(2) = 0$

B. $f'(2)$ DNE

C. $f'(2) = 0$ OR $f'(2)$ DNE

D. $f(2) = 0$

E. Not necessarily any of the above

MULTIPLE CHOICE

Suppose $f(x)$ has domain $(-\infty, \infty)$.
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A. $f'(2) = 0$
B. $f'(2)$ DNE
C. $f'(2) = 0$ OR $f'(2)$ DNE
D. $f(2) = 0$
E. Not necessarily any of the above

"If the derivative exists and is nonzero, then the function is higher on one side of $x = 2$ and lower on the other, so $x = 2$ is neither a max nor a min"

Multiple Choice

Suppose $f(x)$ has domain $(-\infty, \infty)$.
If $f'(5) = 0$, then:

A. $f'(5)$ DNE

B. f has a local maximum at 5

C. f has a local minimum at 5

D. f has a local extremum (maximum or minimum) at 5

E. f may or may not have a local extremum (max or min) at 5

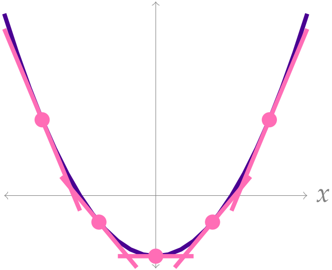
Sketch

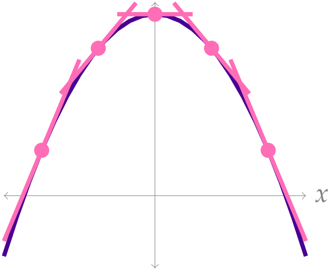
Draw a continuous function $f(x)$ with a local maximum at $x = 3$ and a local minimum at $x = -1$.

Draw a continuous function $f(x)$ with a local maximum at $x = 3$ and a local minimum at $x = -1$, but $f(3) < f(-1)$.

Draw a function $f(x)$ with a singular point at $x = 2$ that is NOT a local maximum, or a local minimum.

SECOND DERIVATIVES





- Is slope increasing, decreasing, or constant?
- Is second derivative positive, negative, or zero?
- Is critical point a local max, local min, or neither?

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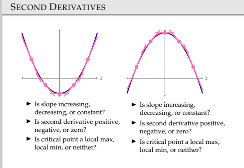

motivating the second derivative test. Nice to point out that in CCU function, at a CP, function changes from decreasing to increasing, hence local min. Students sometime have a hard time with positive/negative slopes, thinking instead in terms of steep and flat. So i like to point out that the slopes go: big negative number, small negative number, zero, small positive number, big positive number.

Suppose $f'(x) = (x+5)^2(x-5)$. Then f has no singular points, and its critical points are ± 5. Identify whether the critical points are local maxima, local minima, or neither.

Second Derivative Test:
Suppose $f'(a) = 0$ and $f''(a) > 0$. Then $x = a$ is a local

Suppose $f'(a) = 0$ and $f''(a) < 0$. Then $x = a$ is a local

2021-08-25

3.5.2: Finding Global Maxima and Minima

Suppose $f'(x) = (x+5)^2(x-5)$. Then f has no singular points, and its critical points are ± 5. Identify whether the critical points are local maxima, local minima, or neither.

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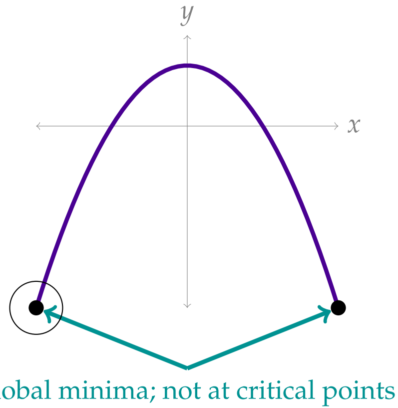
Suppose $f'(a) = 0$ and $f''(a) > 0$. Then $x = a$ is a local

Suppose $f'(a) = 0$ and $f''(a) < 0$. Then $x = a$ is a local

Mnemonic: happy face and frowny face

ENDPOINTS



Theorems 3.5.11 and 3.5.12

A function that is continuous on the interval $[a, b]$ (where a and b are real numbers–not infinite) has a global max and min, and they occur

DETERMINING EXTREMA

To find local extrema:

- Could be at
- Could be at
- At these points, check whether there is some interval around x where $f(x)$ is no larger than the other numbers, or no smaller. (A sketch helps. The signs of the derivatives on either side of x are also a clue.)

To find global extrema:

- Could be at
- Could be at
- Could be at

- Check the value of the function at all of these, and compare.

Find All Extrema[1]:

$$f(x) = x^3 - 3x$$

[1] Extrema: local and global maxima and minima

Find All Extrema

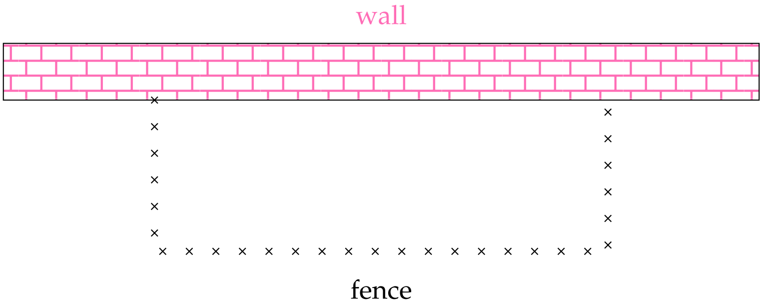
$$f(x) = \sqrt[3]{x^2 - 64}, \quad x \text{ in } [-1, 10]$$

Find the largest and smallest value of $f(x) = x^4 - 18x^2$.

Find the largest and smallest values of $f(x) = \sin^2 x - \cos x$.

Max/min word problems

A rancher wants to build a rectangular pen, using an existing wall for one side of the pen, and using 100m of fencing for the other three sides. What are the dimensions of the pen built this way that has the largest area?



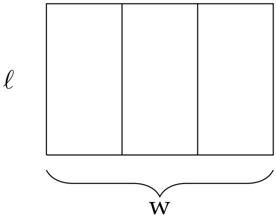
General Idea

We know how to find the global extrema of a function over an interval.

Problems often involve multiple variables, but we can only deal with functions of one variable.

Find all the variables in terms of ONE variable, so we can find extrema.

You want to build a pen, as shown below, in the shape of a rectangle with two interior divisions. If you have 1000m of fencing, what is the greatest area you can enclose?

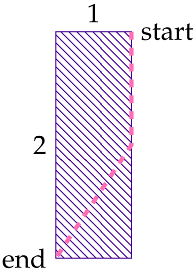


Suppose you want to make a rectangle with perimeter 400. What dimensions give you the maximum area?

Suppose you want to make a rectangle with perimeter 400. What dimensions give you the maximum area?

There are some nice arguments from symmetry that mean you can do this in your head

You are standing on the bank of a river that is 1km wide, and you want to reach the opposite side, two km down the river. You can paddle 3 kilometres per hour, and walk 6 kph while carrying your boat. What route takes you to your desired destination in the least amount of time?



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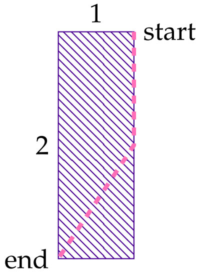
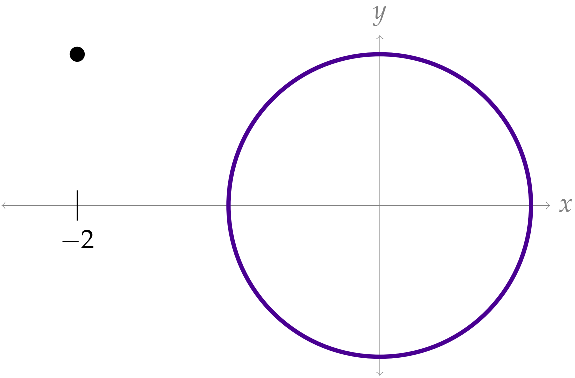


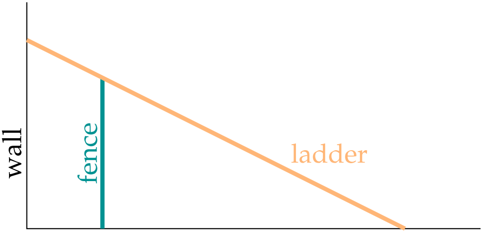
Let C be the circle given by $x^2 + y^2 = 1$. What is the closest point on C to the point $(-2, 1)$?



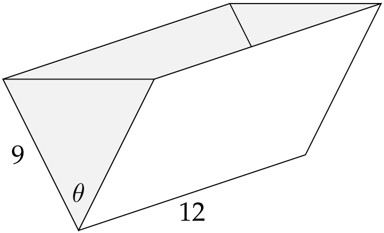
Suppose you want to manufacture a closed cylindrical can on the cheap. If the can should have a volume of one litre (1000 cm^3), what is the smallest surface area it can have?

A cylindrical can is to hold 20π cubic metres. The material for the top and bottom costs \$10 per square metre, and material for the side costs \$8 per square metre. Find the radius r and height h of the most economical can.

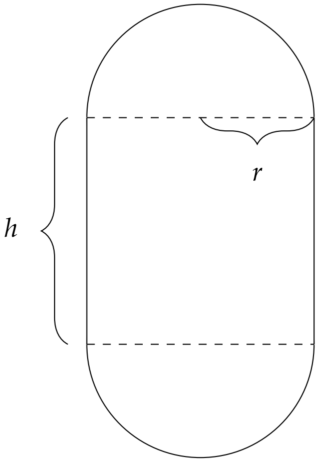
Suppose a 2-metre high fence stands 1 metre away from a high wall. What is the shortest ladder that will reach over the fence to the wall?



Suppose a file folder is 12 inches long and 9 inches wide. You want to make a box by opening the folder and capping the ends. What angle should you open the folder to, to make the box with the greatest volume?

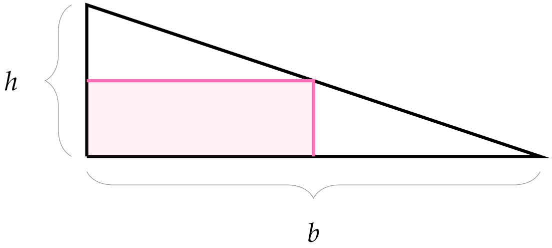


We want to bend a piece of wire into the perimeter of the shape shown below: a rectangle of height h and width $2r$, with a half circle of radius r on the top and bottom.



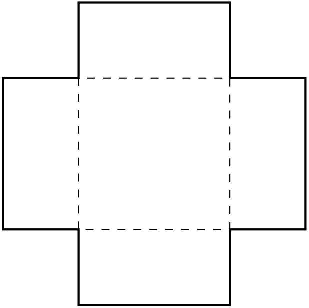
If you only have 100cm of wire, what values of r and h give the largest enclosed area?

Suppose we take a right triangle, with height h and base b. We inscribe a rectangle in it that shares a right angle, as shown below. What are the dimensions of the rectangle with the biggest area?



ACTIVITY

By cutting out squares from the corners, turn a piece of paper into an open-topped box that holds a lot of beans.



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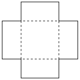

I actually did this in class. It was pretty fun and it fits into about the last 15 minutes. I made bags of beans that fit in an optimal box, and students tried to make boxes that could fit them all. Almost every group got it. Bring paper, scissors, tape, rulers (printed on paper is fine), beans, and books.

Using printer paper, the boxes are not sturdy enough to fill with beans. I brought four big books to buttress the edges of each box as they were being filled. It turns out that boxes with really different dimensions from optimal can hold almost all the beans, which leads to another discussion: the curve at its maximum is fairly shallow, rather than steep.

You could probably do this without cutting: just fold the corners over. I haven't tried it that way. You could also give paper that had ruler marks printed on the four sides.

Included Work

Belk, J. (13 April 2014). *Bad Optimization Problems. I thought that Jack M made an interesting comment about this question*. [Comment on the online forum post *Optimization problems that today's students might actually encounter?*]. Stackexchange. https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter (accessed October 2019 or earlier), 6

 'Weight' by Bakunetsu Kaito is licensed under CC BY 3.0 (accessed 16 July 2021) , 4

Author unknown. Optimization Problems. (2006). *Drexel University Department of Mathematics, Calculus I Home Page Spring 2006, Calc 1 Spring lecture 6.* https://www.math.drexel.edu/%7Ejwd25/CALC1_SPRING_06/lectures/lecture9.html (accessed October 2019 or earlier), 4

Vectornaut. (10 May 2015). *When someone swallows a dose of a drug, it doesn't go into their bloodstream all at once.* [Comment on the online forum post *Optimization problems that today's students might actually encounter?*]. Stackexchange. https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter (accessed October 2019 or earlier), 5