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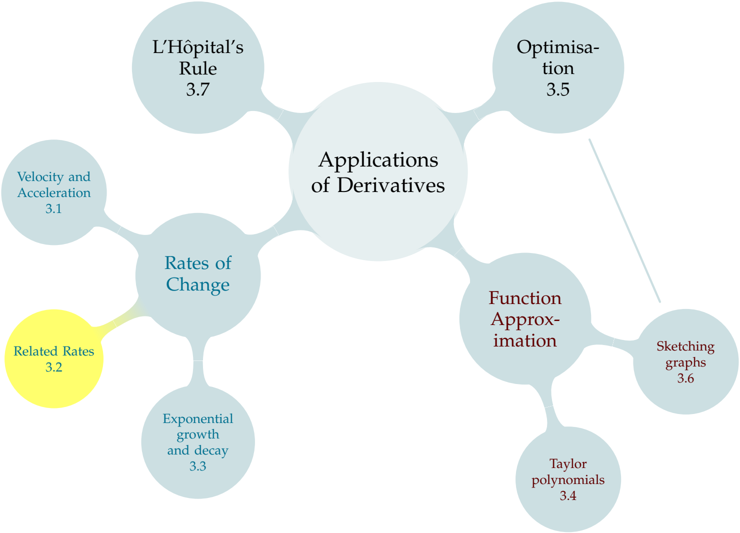

Related Rates - Introduction

"Related rates" problems involve finding the rate of change of one quantity, based on the rate of change of a related quantity.

Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3.$$

If $\frac{dP}{dt}(t) = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{dQ}{dt}$ at that time?

Q

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If $\frac{dP}{dt}(t) = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{dQ}{dt}$ at that time?

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of $3P(t)^2 = 2Q(t)^2 + Q(t) + 3$.

$$6P(t)\frac{dP}{dt}(t) = 4Q(t)\frac{dQ}{dt}(t) + \frac{dQ}{dt}(t)$$

Then when $\frac{dP}{dt}(t) = 5$, $P(t) = 1$ and $Q(t) = 0$,

$$6(1)(5) = 4(0)\frac{dQ}{dt} + \frac{dQ}{dt}$$
$$30 = \frac{dQ}{dt}$$

Related rates problems often involve some kind of geometric or trigonometric modeling

A garden hose can pump out a cubic meter of water in about 20 minutes. Suppose you're filling up a rectangular backyard pool, 3 meters wide and 6 meters long, with a garden hose. How fast is the water rising?

Q

ans

We know the rate of change over time of the volume of water, and we want to know the rate of change over time of the height of the water.

Let V be the volume of water in the cell, and let h be the height of water. Then we relate V and h:

$$V = 3 \cdot 6 \cdot h = 18h$$

where V is measured in cubic meters, and h is measured in meters. Then we differentiate both sides with respect to t:

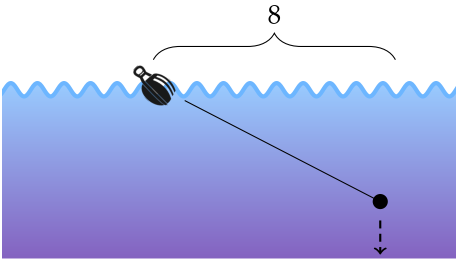
$$\frac{dV}{dt} = 18\frac{dh}{dt}$$

Now, $\frac{1}{20} = 18\frac{dh}{dt}$, so the water level is rising at about $\frac{1}{20 \cdot 18} = \frac{1}{360}$ meters per minute, or something less than a third of a centimeter per minute. It's going to take a long time to fill up the pool.

Solving Related Rates

1. Draw a Picture
2. Write what you know, and what you want to know. Note units.
3. Relate all your relevant variables in one equation.
4. Differentiate both sides (with respect to the appropriate variable!)
5. Solve for what you want.

A weight is attached to a rope, which is attached to a pulley on a boat, at water level. The weight is taken 8 (horizontal) metres from its attachment point on the boat, then dropped in the water.
The weight sinks straight down. The rope stays taught as it is let out at a constant rate of one metre per second, and two seconds have passed. How fast is the weight descending?



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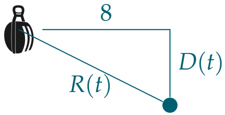
1. Draw a Picture



2. Write what we know, and what we want to know.

We know: $\frac{\mathrm{d}R}{\mathrm{d}t} = 1\frac{\mathrm{m}}{\mathrm{s}}$, $t = 2$. We want to know $\left.\frac{\mathrm{d}D}{\mathrm{d}t}\right|_{t=2}$.

3. Relate all relevant variables in one equation.

$$64 + D^2(t) = R^2(t)$$

4. Differentiate with respect to the appropriate variable. With t as our variable, we need to use the chain rule:

$$64 + D^2(t) = R^2(t)$$
$$0 + 2D(t) \cdot \frac{\mathrm{d}D}{\mathrm{d}t}(t) = 2R(t) \cdot \frac{\mathrm{d}R}{\mathrm{d}t}(t)$$

5. Solve for what you want.

$$\frac{\mathrm{d}D}{\mathrm{d}t}(t) = \frac{R(t) \cdot \frac{\mathrm{d}R}{\mathrm{d}t}(t)}{D(t)}$$
$$D'(2) = \frac{R(2) \cdot R'(2)}{D(2)}$$

$R(0) = 8$ and the rope is being let out at 1 metre per second, so $R(2) = 8 + 2 = 10$. Then, using the Pythagorean Theorem, $D(2) = \sqrt{10^2 - 8^2} = 6$

$$D'(2) = \frac{10 \cdot 1}{6} = \frac{5}{3}$$

So, the weight is sinking at $\frac{5}{3}$ metres per second.

You are pouring water through a funnel with an extremely small hole. The funnel lets water out at 100mL per second, and you are pouring water into the funnel at 300mL per second. The funnel is shaped like a cone with height 20 cm and with the diameter at the top also 20 cm. (Ignore the hole in the bottom.) How fast is the height of the water in the funnel rising when it is 10 cm high?

A cone with radius r and height h has volume $\frac{\pi}{3}r^2h$.

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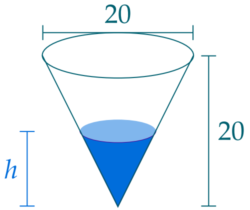
1. Draw a Picture



2. Write what we know, and what we want to know.

The water in the funnel is in the shape of a cone. Let that cone have height $h(t)$ and radius $r(t)$, in cm, and volume $V(t)$, in mL (i.e. cm^3).

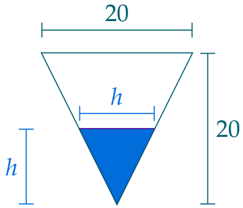
We know
$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = 300 - 100 = 200$ mL/sec.

We want to know $\frac{\mathrm{d}h}{\mathrm{d}t}(t)$ when $h(t) = 10$.

3. Relate all relevant variables in one equation.
The volume of water in the funnel is $V(t) = \frac{\pi}{3}r^2(t) \cdot h(t)$
We're given information about volume, and asked about height. Radius is sort of ... in the way. Since it isn't "relevant" variable, let's figure out how to get rid of it.

A vertical cross-section of a cone is an isosceles triangle. We see that the funnel has a diameter equal to its height. The water makes a similar triangle, so its diameter will also be equal to its height. That is,

$$r(t) = \frac{1}{2}h(t)$$



So $V(t) = \frac{\pi}{3}\left(\frac{h(t)}{2}\right)^2 \cdot h(t)$. Simplifying $V(t) = \frac{\pi}{12}h^3(t)$

4. Differentiate with respect to the appropriate variable, namely t.

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = \frac{\pi}{4}h^2(t) \cdot \frac{\mathrm{d}h}{\mathrm{d}t}(t)$$

5. Solve for what you want.

$$\frac{\pi}{4}h^2(t) \cdot \frac{\mathrm{d}h}{\mathrm{d}t}(t) = \frac{\mathrm{d}V}{\mathrm{d}t}(t) = 200$$
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{800}{\pi h^2(t)}$$
$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{h=10} = \frac{800}{\pi(10^2)} = \frac{8}{\pi}\,\mathrm{cm/sec}$$

A sprinkler is 3m from a long, straight wall. The sprinkler sprays water in a circle, making three revolutions per minute. Let P be the point on the wall closest to the sprinkler. The water hits the wall at some spot, and that spot moves as the sprinkler rotates. When the spot where the water hits the wall is 1m away from P, how fast is the spot moving horizontally?
(You may assume the water travels from the sprinkler to the wall instantaneously.)

Q

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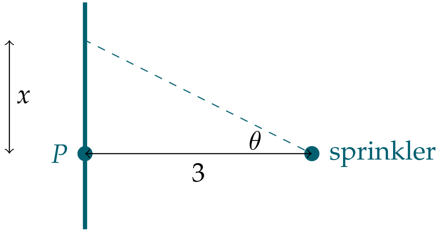
Q



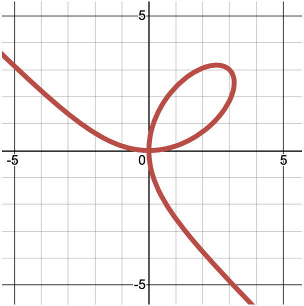
Given the labels in the picture above, we know $\frac{d\theta}{dt} = 3(2\pi)$ radians per minute, and we can relate x to θ by $\tan\big(\theta(t)\big) = \frac{x(t)}{3}$.
Then differentiate both sides with respect to t to get:

$$\sec^2\theta(t)\cdot\frac{d\theta}{dt}(t) = \frac{1}{3}\frac{dx}{dt}(t)$$

We only need to find $\sec\theta$, which we get from our triangle. When $x = 1$, we have $\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{3^2+1^2}}{3}$, so

$$\frac{dx}{dt} = 3\left(\frac{\sqrt{10}}{3}\right)^2\cdot 6\pi = 20\pi \text{ meters per minute}$$

A roller coaster has a track shaped in part like the folium of Descartes: $x^3 + y^3 = 6xy$. When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?



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We know $\frac{dx}{dt} = -2$ when $x = y = 3$, and we would like to find $\frac{dy}{dt}$ at the same time. The relationship between x and y is given.

$$x(t)^3 + y(t)^3 = 6x(t)y(t)$$

We differentiate.

$$3x(t)^2 \frac{dx}{dt}(t) + 3y(t)^2 \frac{dy}{dt}(t) = 6\left(x(t)\frac{dy}{dt}(t) + y(t)\frac{dx}{dt}(t)\right)$$

When $x(t) = y(t) = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = -2$,

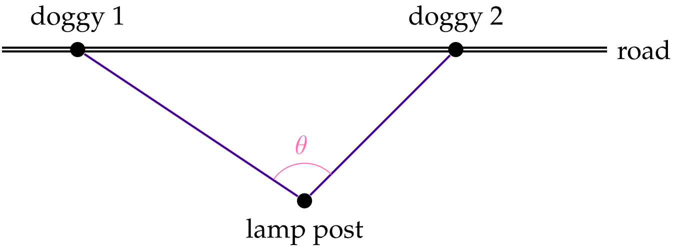
$$3(9)(-2) + 3(9)\frac{dy}{dt} = 6\left(3\frac{dy}{dt} + 3(-2)\right)$$

$$\frac{dy}{dt} = 2$$

So, the roller coaster is moving 2 units per second in the positive y direction.

A

Two dogs are tied with elastic leashes to a lamp post that is 2 metres from a straight road. At first, both dogs are on the road, at the closest part of the road to the lamp post. Then, they start running in opposite directions: one dog runs 3 metres per second, and the other runs 2 metres per second. After one second of running, how fast is the angle made by the two leashes increasing?



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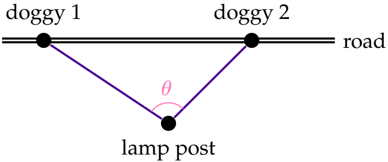

Q

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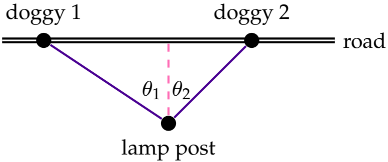


The angle θ made by the leashes is made up of $\theta_1 + \theta_2$, as shown on either side of the dashed line above.

$$\tan\left(\theta_1(t)\right) = \frac{3t}{2} \implies \sec^2\left(\theta_1(t)\right) \cdot \theta_1'(t) = \frac{3}{2} \implies \theta_1'(t) = \frac{3}{2}\cos^2\left(\theta_1(t)\right)$$

$$\tan\left(\theta_2(t)\right) = \frac{2t}{2} \implies \sec^2\left(\theta_2(t)\right) \cdot \theta_2'(t) = 1 \implies \theta_2'(t) = \cos^2\left(\theta_2(t)\right)$$

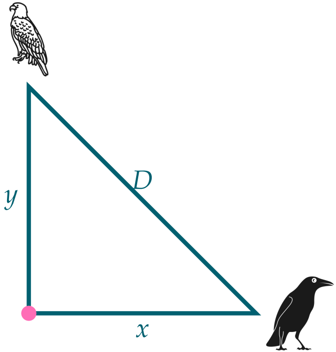
At $t = 1$, doggy 1 is three metres away from its starting position and doggy 2 is two metres away, so $\cos\theta_1 = \frac{2}{\sqrt{2^2+3^2}} = \frac{2}{\sqrt{13}}$ and $\cos\theta_2 = \frac{2}{\sqrt{2^2+2^2}} = \frac{1}{\sqrt{2}}$ and

$$\theta_1' = \frac{3}{2}\cos^2\theta_1 = \frac{6}{13} \qquad \theta_2' = \cos^2\theta_1 = \frac{1}{2}$$

So, all together, $\theta' = \theta_1' + \theta_2' = \frac{6}{13} + \frac{1}{2} = \frac{25}{26}$ radians per second.

A

A crow is one kilometre due east of the math building, heading east at 5 kph. An eagle is two kilometres due north of the math building, heading north at 7kph. How fast is the distance between the two birds increasing at this instant?



We relate all the variables:

$$D(t)^2 = x(t)^2 + y(t)^2$$

Differentiate with respect to time:

$$2D(t)\frac{dD}{dt}(t) = 2x(t)\frac{dx}{dt}(t) + 2y(t)\frac{dy}{dt}(t)$$

At the time of interest $x = 1$, $x' = 5$, $y = 2$, $y' = 7$ and

$$D = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

So $2\sqrt{5}\frac{dD}{dt} = 2(1)(5) + 2(2)(7)$.
Then their distance is increasing at $\frac{19}{\sqrt{5}} \approx 8.5$ kph.

A triangle has one side that is 1cm long, and another side that is 2cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.

Q

ans

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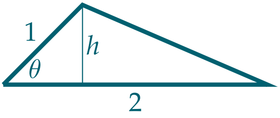

$\sin\theta(t) = \frac{opp}{hyp} = \frac{h(t)}{1} = h(t)$, and
$A(t) = \frac{1}{2} b\, h(t)$,

So

$$A(t) = \tfrac{1}{2}(2)\sin\theta(t) = \sin\theta(t)$$

$$\frac{dA}{dt}(t) = \cos\theta(t) \cdot \frac{d\theta}{dt}(t)$$

When $\theta = \pi/4$,

$$\frac{dA}{dt} = \cos\left(\frac{\pi}{4}\right)(1.5) = \frac{3}{2\sqrt{2}} \frac{\text{cm}^2}{\text{sec}}$$

A

Included Work

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