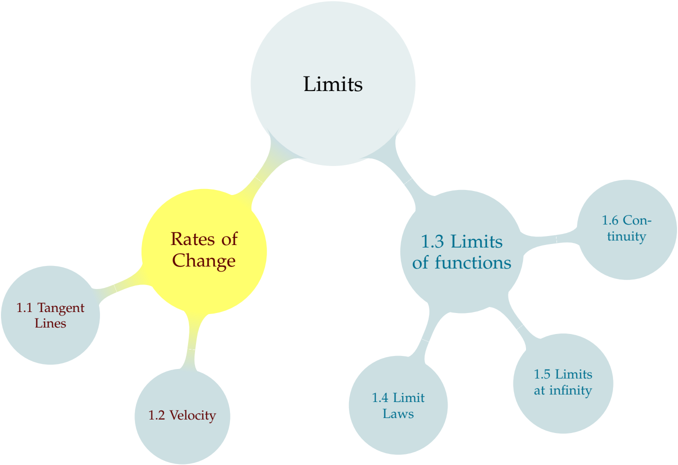
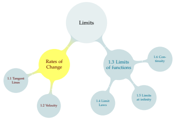
Table of Contents



2022-03-10

Table of Contents

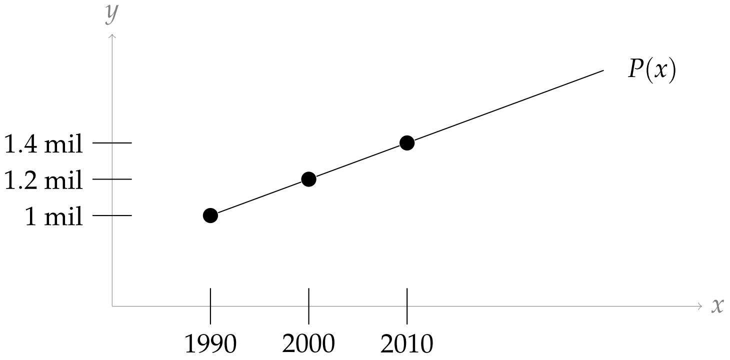


We'll start by introducing "big ideas" before we get to the actual textbook content. We use scenarios that have a lot of intuition behind them, rather than purely abstract examples.

We go through: constant rate of change, non-constant rate of change, average vs continuous rate of change, tangent and secant lines

Rates of Change

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.

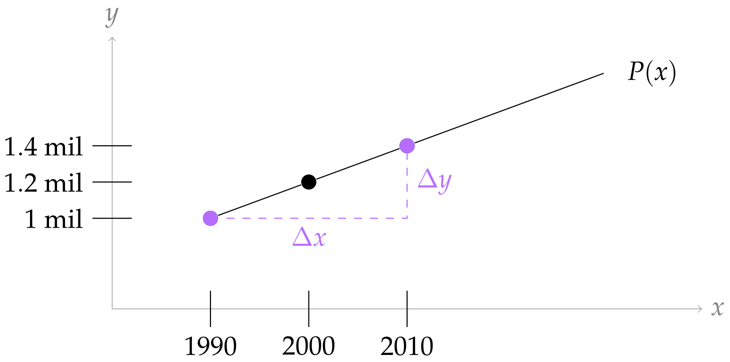


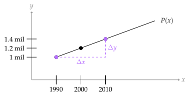
Definition

The **slope** of a line that passes through the points (x_1, y_1) and (x_2, y_2) is "rise over run"

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

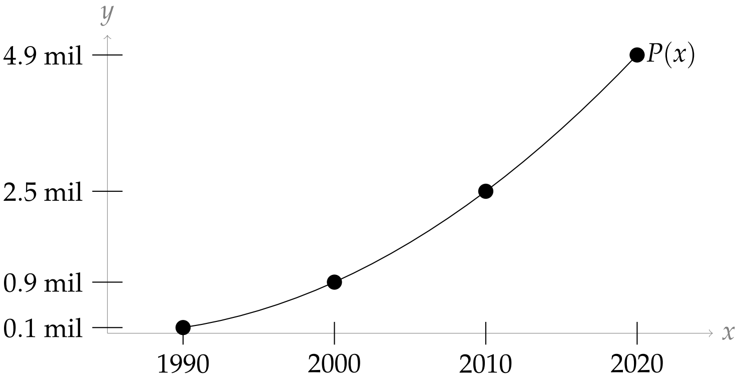
This is also called the **rate of change** of the function.
If a line has equation $y = mx + b$, its slope is m.


y
x
P(x)
1.4 mil
1.2 mil
1 mil
Δy
Δx
1990
2000
2010



We see that the rate of change is the same whether measuring over one year (given information) or two years (calculated)

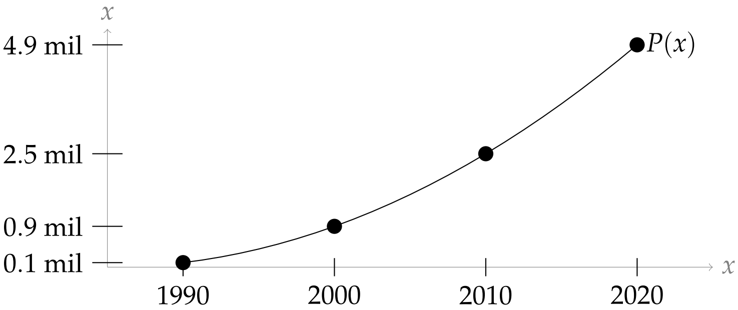
Suppose the population of a small country is given in the chart below.



Definition

Let $y = f(x)$ be a curve that passes through (x_1, y_1) and (x_2, y_2). Then the **average rate of change** of $f(x)$ when $x_1 \leq x \leq x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$


x
4.9 mil
2.5 mil
0.9 mil
0.1 mil
P(x)
1990
2000
2010
2020
x

Average Rate of Change and Slope

The average rate of change of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is "change in output" divided by "change in input:"

$$\frac{f(b) - f(a)}{b - a}$$

If the function $f(x)$ is a line, then the slope of the line is "rise over run,"

$$\frac{f(b) - f(a)}{b - a}$$

If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

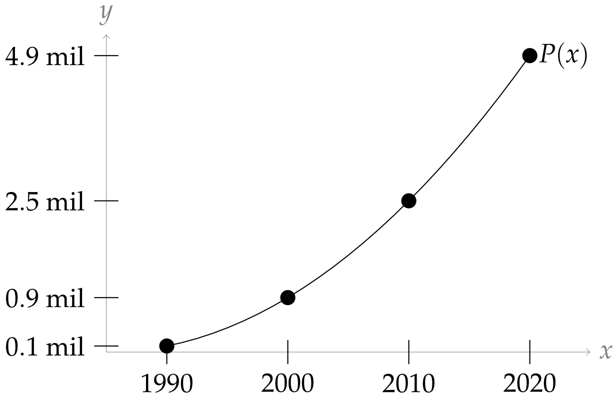
If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

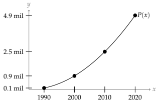
If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

Say this frame out loud while **showing** the previous frame. It sums up our "big ideas" section.
At the end can add: let's think about what slope might mean in the context of rates of change for functions that aren't lines

How fast was this population growing in the year 2010? (What was its instantaneous rate of change?)



How fast was this population growing in the year 2010? (What was its instantaneous rate of change?)

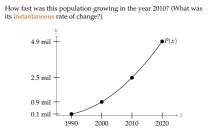


Mention it was slower before 2010, and faster after 2010"We can start by looking at an interval around 2010. " show ± 10 as an example that will make students uncomfortable. Note that the average rate of change is the slope of the line between the two points, but the line and the curve look pretty different. Then move to ± 1. "This is getting hard to see, so let's zoom in." Same, now note the line and the curve are getting closer.

"But we don't care about 2009, or 2019. So let's just look at Jan 2010 to Dec 2010." More zooming. Note it's basically a line now.

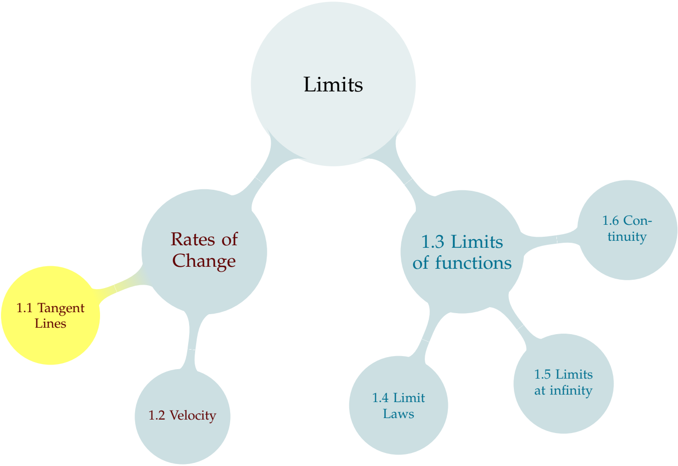
2022-03-10

Big Ideas



So these are the big ideas we'll be thinking about: rates of change, measured as average over a large interval or over a very small one.
In the case of a line, average rate of change is the slope. In the case or a curve, sometimes we can zoom in close and the curve looks like a line. So the average rate of change over a small interval will look something like the slope of a line.

Table of Contents



Definition

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q.

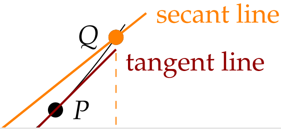
We call the slope of the secant line the **average rate of change of $f(x)$ from R to Q.**

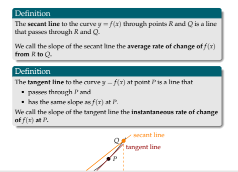
Definition

The **tangent line** to the curve $y = f(x)$ at point P is a line that

- passes through P and
- has the same slope as $f(x)$ at P.

We call the slope of the tangent line the **instantaneous rate of change of $f(x)$ at P.**

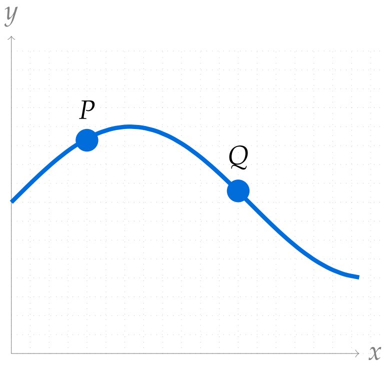




Verbally: we haven't formally defined the slope of a curve yet, but think of it as the slope of the *line* that the curve looks like, if you zoom way in.

When P pops up, draw the tangent by hand before clicking to it, to demonstrate what you mean.

On the graph below, draw the secant line to the curve through points P and Q.



On the graph below, draw the tangent line to the curve at point P.

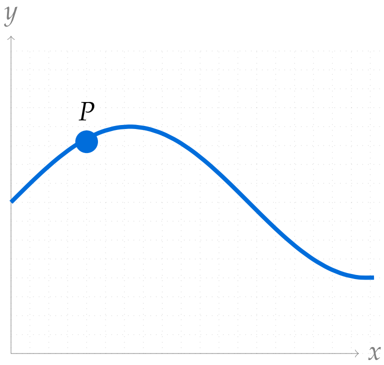
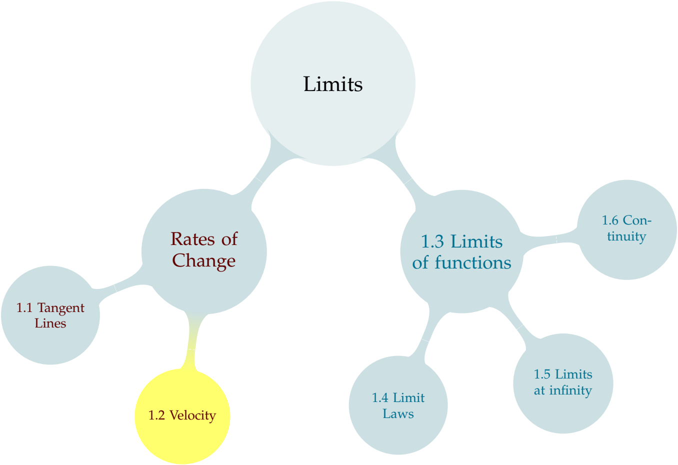
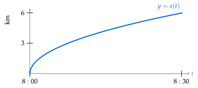


Table of Contents




$y = s(t)$
km
6
3
t
8 : 00
8 : 30



8:07 is highlighted to emphasize that I was not ravelling at a constant speed: half the distance was covered in only 7 minutes, the other half took 23.

I've found that the bike example is a good concept check. When students answer the multiple choice question there's usually a lot who get it wrong, so it's an opportunity to fix misconceptions.

It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$

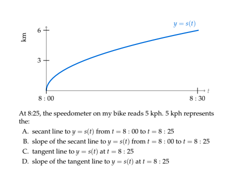
At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

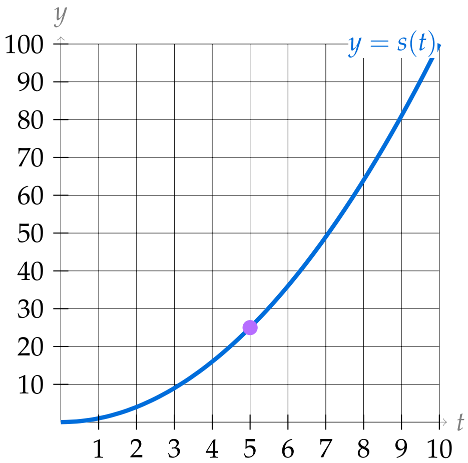
C. tangent line to $y = s(t)$ at $t = 8:25$

D. slope of the tangent line to $y = s(t)$ at $t = 8:25$



Now we move into using limits for instantaneous rates of change. Students are usually tired of the bike by now so we change the example.
For next slide: in "one way," remind verbally that instantaneous rate of change is slope of tangent line. Use a straight edge to draw the tangent and make use of the graph paper to get a decent approximation.
We use y for the vertical axis instead of h because h is used later for something else

Suppose the distance from the ground s (in meters) of a helium-filled balloon at time t over a 10-second interval is given by $s(t) = t^2$. Try to estimate how fast the balloon is rising when $t = 5$.



Let's look for an algebraic way of determining the velocity of the balloon when $t = 5$.

Our First Limit

Average Velocity, $t = 5$ to $t = 5 + h$:

$$\begin{aligned} \frac{\Delta s}{\Delta t} &= \frac{s(5+h) - s(5)}{h} \\ &= \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \qquad \text{when } h \neq 0 \end{aligned}$$

When h is very small,

$$\text{Vel} \approx 10$$

2022-03-10

1.2 Computing Velocity

Our First Limit

Our First Limit

Average Velocity, $t = 5$ to $t = 5 + h$:

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= \frac{s(5+h) - s(5)}{h} \\ &= \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \qquad \text{when } h \neq 0\end{aligned}$$

When h is very small,

$$\text{Vel} \approx 10$$

Now we recap what we did: the informal calculation first, then using it to introduce limit notation

Limit Notation

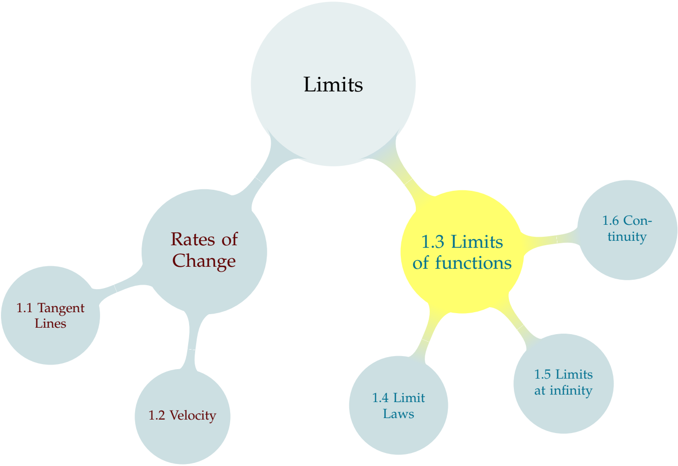
We write:

$$\lim_{h\to 0}(10+h)=10$$

We say: "The limit as h goes to 0 of $(10+h)$ is 10."

It means: As h gets extremely close to 0, $(10+h)$ gets extremely close to 10.

Table of Contents



Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \to a} f(x) = L$$

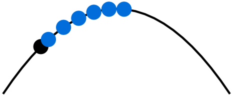
where a and L are real numbers

We read the above as "the limit as x goes to a of $f(x)$ is L."

Its meaning is: as x gets very close to (but not equal to) a, $f(x)$ gets very close to L.

Finding Slopes of Tangent Lines

We NEED limits to find slopes of tangent lines.



Slope of secant line: $\dfrac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.
Slope of tangent line: can't do the same way.

If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

FINDING SLOPES OF TANGENT LINES

We NEED limits to find slopes of tangent lines.

Slope of secant line: $\frac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.
Slope of tangent line: can't do the same way.

If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

Students often forget this: remind them that if we try to take an interval of length 0, it causes us to divide by zero.
This is also a nice example to hearken back to later when someone asks why zero divided by zero isn't just 1."This is fine, this is fine, this is still fine..." (during animation)Here I'm intentionally mixing "slope of tangent line" and "instantaneous rate of change" as an opportunity to remind students verbally that they're the same thing.
Verbally describe during animation how we're taking an average RoC over smaller and smaller intervals, approximating a rate of change. Since we can't actually plug in only one point, we take a limit.
We just showed a limit at x=5; this one only differs because we're using a parameter now. Pause is there before the formula so you can write it out and explain bit by bit.

We NEED limits to find slopes of tangent lines.

Slope of secant line: $\frac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.
Slope of tangent line: can't do the same way.

If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

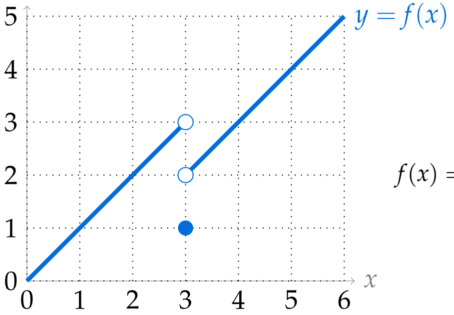
So, limits are going to be an important part of our lives for the next little while. Let's think about a few of them.

Evaluating Limits

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

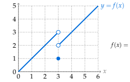
We want to evaluate $\lim_{x \to 1} f(x)$.

One-Sided Limits



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim_{x \to 3} f(x)$ should be?



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim\limits_{x \to 3} f(x)$ should be?

What the limit should be can generate some good class discussions. Can ask to raise hands: Who thinks it's 1/2/3/more than one / none, etc.
That motivates the need for left and right limits. Explain verbally for this slide. It can be helpful to remind students that we needed limits before precisely because there was something fishy going on at a point. So if we were to say that the limit were equal to the value, it wouldn't capture the fishy behaviour. You can also ask – how would you want to describe the behaviour of the function? This motivates our ideas of limits, for students whose instinct is often that the value of the function should equal the limit because that's somehow the important value.

Definition 1.3.7

The limit as x goes to a **from the left** of $f(x)$ is written

$$\lim_{x \to a^-} f(x)$$

We only consider values of x that are less than a.

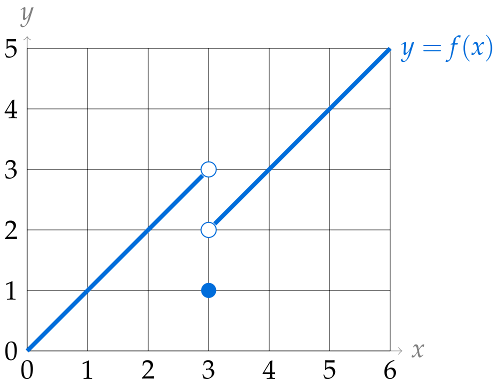
The limit as x goes to a **from the right** of $f(x)$ is written

$$\lim_{x \to a^+} f(x)$$

We only consider values of x greater than a.

Theorem 1.3.8

In order for $\lim_{x \to a} f(x)$ to exist, both one-sided limits must exist and be equal.

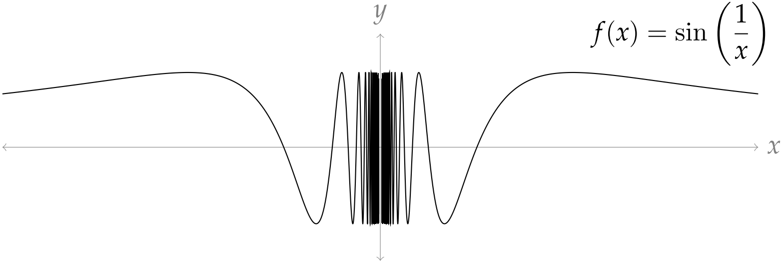


Consider the function $f(x) = \dfrac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ not defined?

Consider the function $f(x) = \frac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ not defined?

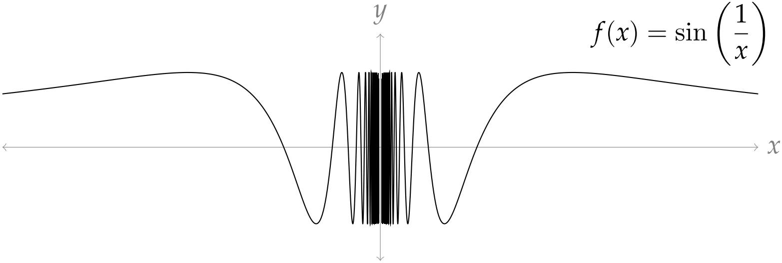
It's nice to explain verbally that the distinction is only really important in their lives for parsing theorems. If something says "if the limit exists...." then it might not apply here. It's still standard to write "equals infinity."

A Stranger Limit Example



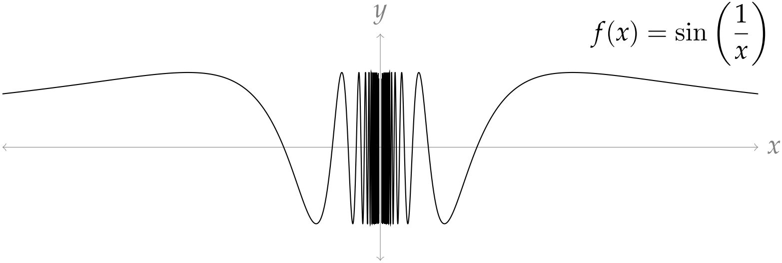
What is $\lim_{x\to\infty} f(x)$?

A Stranger Limit Example



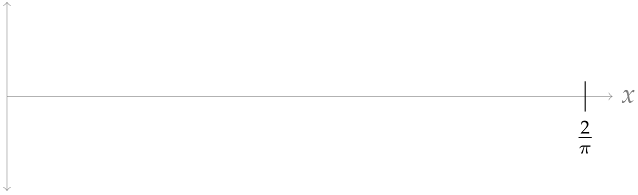
What is $\lim_{x\to 0} f(x)$?

A Stranger Limit Example

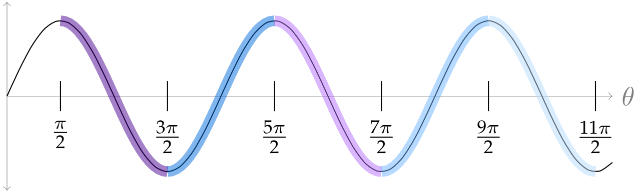


What is $\lim_{x\to\pi} f(x)$?

Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$



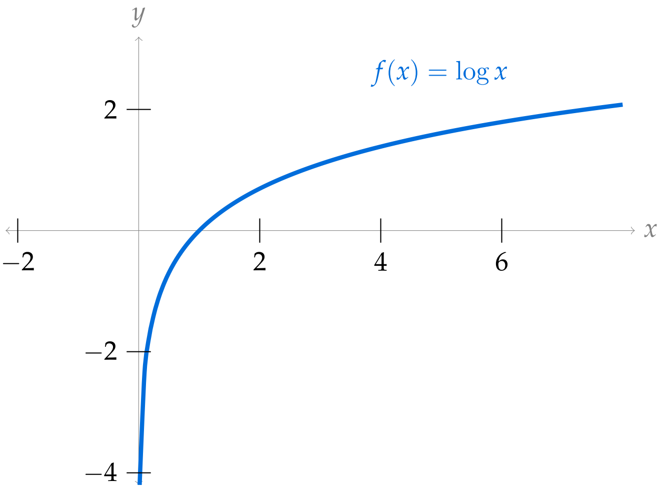
$y = \sin\frac{1}{x}$



$y = \sin\theta$

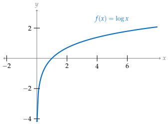
Limits and the Natural Logarithm

Where is $f(x)$ defined, and where is it not defined?



Limits and the Natural Logarithm

Where is $f(x)$ defined, and where is it not defined?

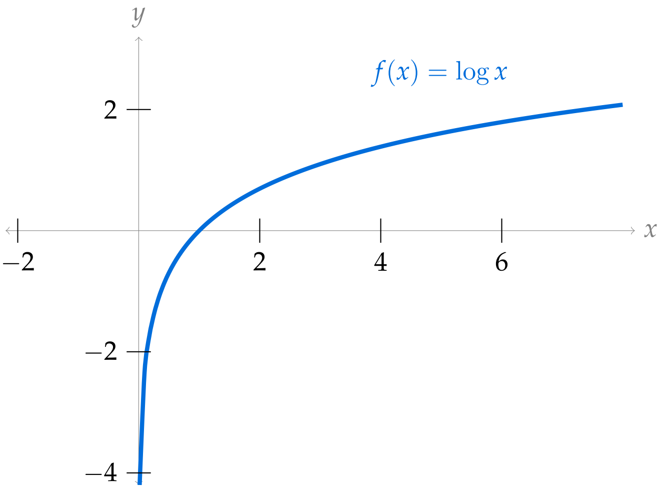


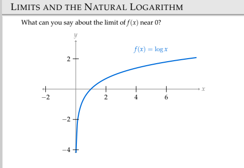
We put an example between this and the last limit going to infinite intentionally, to allow students' short-term memory to clear a little. Now when we ask whether this limit exists, we get to reinforce rather than just reiterate.

The question is also asked a little vaguely, since we **can't** say $\lim\limits_{x\to 0} = -\infty$. You can ask students to describe the behaviour to their neighbours, and prime the pump a little with a reminder about one-sided limits. Once the "limit going to infinity" business is a little clearer in students' minds, we can add this subtlety: that some kinds of limits don't exist in ways we don't have convenient notation for.

Limits and the Natural Logarithm

What can you say about the limit of $f(x)$ near 0?

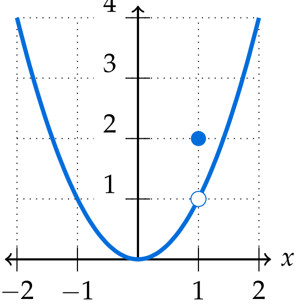




We put an example between this and the last limit going to infinite intentionally, to allow students' short-term memory to clear a little. Now when we ask whether this limit exists, we get to reinforce rather than just reiterate.

The question is also asked a little vaguely, since we **can't** say $\lim_{x \to 0} = -\infty$. You can ask students to describe the behaviour to their neighbours, and prime the pump a little with a reminder about one-sided limits. Once the "limit going to infinity" business is a little clearer in students' minds, we can add this subtlety: that some kinds of limits don't exist in ways we don't have convenient notation for.

$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x\to 1} f(x)$?

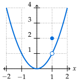
A. $\lim_{x\to 1} f(x) = 2$

B. $\lim_{x\to 1} f(x) = 1$

C. $\lim_{x\to 1} f(x)$ DNE

D. none of the above

$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x\to 1} f(x)$?

A. $\lim_{x\to 1} f(x) = 2$

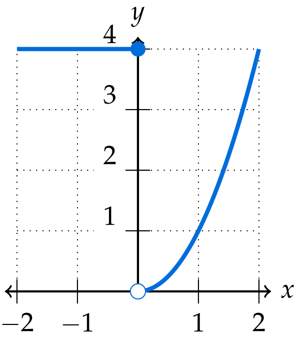
B. $\lim_{x\to 1} f(x) = 1$

C. $\lim_{x\to 1} f(x)$ DNE

D. none of the above

Often here students need reminding that the limit doesn't necessarily equal the function. I like to point out that we can already write $f(1) = 2$ to describe what happens at that point, so there's no point to defining a limit as just another way of saying that information. The limit gives us *different* information. It's also nice to point out why the limit isn't DNE. We motivated limits with slopes of tangent lines, where we were always examining functions with some weird issue at a point. If there weren't that weird issue, we wouldn't really need limits; so it doesn't make sense for a removable discontinuity to lead to a limit not existing.

$$f(x) = \begin{cases} 4 & x \le 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x\to 0} f(x)$? What is $\lim_{x\to 0^+} f(x)$? What is $f(0)$?

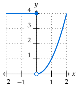
A. $\lim_{x\to 0^+} f(x) = 4$

B. $\lim_{x\to 0^+} f(x) = 0$

C. $\lim_{x\to 0^+} f(x) = \begin{cases} 4 & x \le 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

$$f(x) = \begin{cases} 4 & x \le 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x\to 0} f(x)$? What is $\lim_{x\to 0^+} f(x)$? What is $f(0)$?

A. $\lim_{x\to 0^+} f(x) = 4$

B. $\lim_{x\to 0^+} f(x) = 0$

C. $\lim_{x\to 0^+} f(x) = \begin{cases} 4 & x \le 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

"The essence of option C is captured in one-sided limits, so it's there in spirit, that's just not how we write it."

Suppose $\lim_{x \to 3^-} f(x) = 1$ and $\lim_{x \to 3^+} f(x) = 1.5$.

Does $\lim_{x \to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist.

B. No, never, because the limit from the left is not the same as the limit from the right.

C. Can't tell. For some functions is might exist, for others not.

Suppose $\lim\limits_{x\to 3^-} f(x) = 22 = \lim\limits_{x\to 3^+} f(x)$.

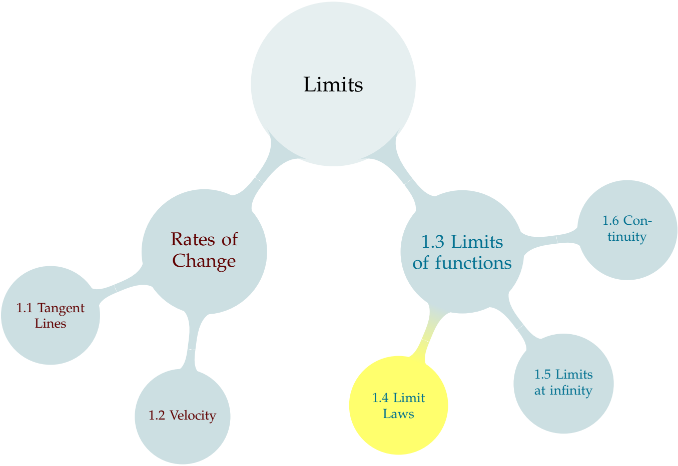
Does $\lim\limits_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist and are equal to each other.

B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.

C. Can't tell. We need to know the value of the function at $x = 3$.

Table of Contents



Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

$$\lim_{x \to a} f(x) = f(a).$$

Calculate: $\lim\limits_{x \to 3} \left(\dfrac{x^2 - 9}{x + 3} \right)$

Calculate: $\lim\limits_{x \to 3} \left(\dfrac{x^2 - 9}{x - 3} \right)$

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x\to a}(f(x) + g(x)) = F + G$
- $\lim_{x\to a}(f(x) - g(x)) = F - G$
- $\lim_{x\to a}(f(x)g(x)) = FG$
- $\lim_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

Calculate: $\lim_{x\to 1}\left[\frac{2x+4}{x+2} + 13\left(\frac{x+5}{3x}\right)\left(\frac{x^2}{2x-1}\right)\right]$

Limits involving Powers and Roots

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

Limits involving Powers and Roots

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

Raise your hand if you think A is real / not; Raise your hand if you think B is real/not; etc.Ask students to turn to their neighbours and describe a rule for when A^B is real, and when it is not.

Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x \to a} f(x) = F$ (where F is a real number), then:

$$\lim_{x \to a} (f(x))^n = F^n.$$

Furthermore, unless n is even and F is negative,

$$\lim_{x \to a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x \to 4} (x+5)^{1/2}$$

Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x\to a} f(x) = F$ (where F is a real number), then:

$$\lim_{x\to a} (f(x))^n = F^n.$$

Furthermore, unless n is even and F is negative,

$$\lim_{x\to a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x\to 4} (x+5)^{1/2}$$

Takeaway: when calculating limits, you can start by trying to "plug in." But, the MINUTE you divide by zero, or see $0/0$, or if infinity shows up anywhere, YOU NEED TO DO SOMETHING ELSE

Cautionary Tales

- $\displaystyle \lim_{x\to 0} \frac{(5+x)^2-25}{x}$
- $\displaystyle \lim_{x\to 3} \left(\frac{x-6}{3}\right)^{1/8}$
- $\displaystyle \lim_{x\to 0} \frac{32}{x}$
- $\displaystyle \lim_{x\to 5} \left(x^2+2\right)^{1/3}$

Suppose you want to evaluate $\lim_{x\to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x\to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x\to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x\to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x\to 1} f(x)$ does not exist.

E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

Suppose you want to evaluate $\lim_{x\to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x\to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x\to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x\to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x\to 1} f(x)$ does not exist.

E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

We're identifying things that make limits harder to find. A limit being hard to find is not the same as the limit not existing, it just means you have to look harder. In a moment, we'll talk about what to do in these cases.

Which of the following statements is true about $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 0}{0^3 - 0^2 + 0} = \dfrac{0}{0}$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about $\displaystyle\lim_{x\to 1} \frac{\sin x}{x^3 - x^2 + x}$?

A $\displaystyle\lim_{x\to 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

C Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

D Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

Which of the following statements is true about $\lim_{x \to 1} \frac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x \to 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

C Since the function $\frac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

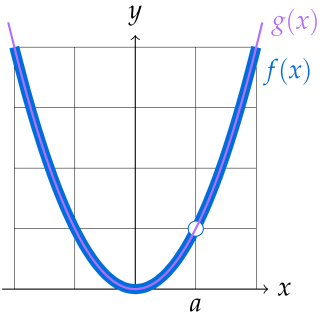
D Since the numerator and denominator of $\frac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

We're identifying things that make limits harder to find. A limit being hard to find is not the same as the limit not existing. We'll talk now about more things you can do to evaluate a limit in these trickier situations.

Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim_{x \to a} g(x)$ exists, and $f(x) = g(x)$ when x is close to a (but not necessarily equal to a).

Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.



Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 + x^2 - x - 1}{x - 1}$.

Evaluate $\displaystyle\lim_{x\to 5} \frac{\sqrt{x+20}-\sqrt{4x+5}}{x-5}$

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

Verbally: Separate piecewise functions from everything else. It's likely that the only times they'll see functions with discontinuities inside their domains, they will be written piecewise.

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0} \frac{x+7}{\frac{1}{x}-\frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

Denominators Approaching Zero

$$\lim_{x\to1} \frac{1}{(x-1)^2}$$

$$\lim_{x\to1} \frac{-1}{(x-1)^2}$$

$$\lim_{x\to1^-} \frac{1}{x-1}$$

$$\lim_{x\to1^+} \frac{1}{x-1}$$

DENOMINATORS APPROACHING ZERO

$$\lim_{x\to1}\frac{1}{(x-1)^2}$$

$$\lim_{x\to1}\frac{-1}{(x-1)^2}$$

$$\lim_{x\to1^-}\frac{1}{x-1}$$

$$\lim_{x\to1^+}\frac{1}{x-1}$$

Good to remind students about division at this point. A small number goes into any number lots of times: if I have one cake, and I cut it into tiny pieces, I get a lot of pieces. They often think these limits have some kind of unknowable magic to them, so it's good to bring them back to a place where things make intuitive sense.

Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4}$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2}$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \leq g(x) \leq h(x)$$

and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x)$. Then $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} f(x)$.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \le g(x) \le h(x)$$

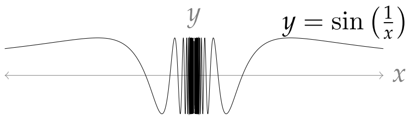
and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

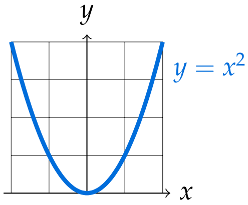
$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Let's start by graphing the function

Evaluate:

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$





$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \qquad \leq \qquad \sin\left(\frac{1}{x}\right) \qquad \leq \qquad 1$$

Included Work

 'Brain' by Eucalyp is licensed under CC BY 3.0 (accessed 8 June 2021), 78

 'Dog' by Vladimir Belochkin is licensed under CC BY 3.0 (accessed 17 June 2021), 33