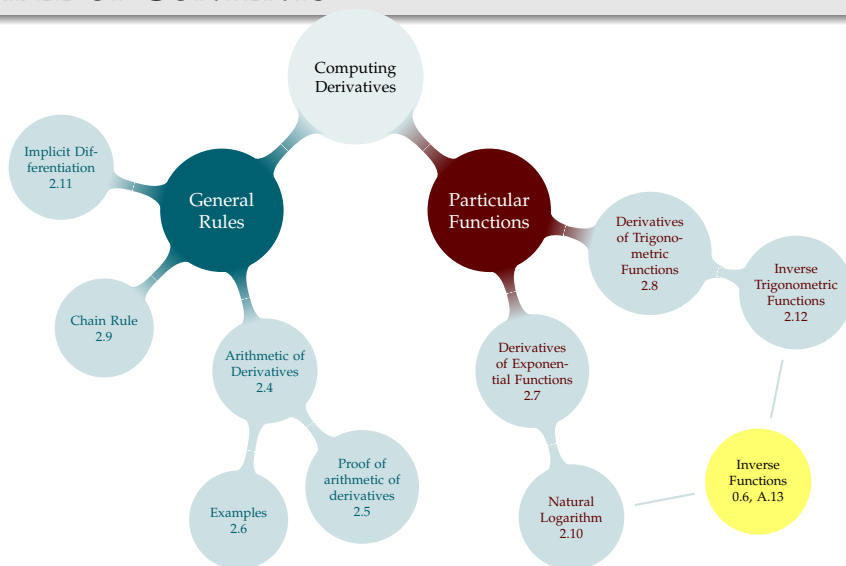


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# INVERTIBILITY GAME

- ▶ A function  $y = f(x)$  is known to both players
- ▶ **Player A** chooses a secret value  $x$  in the domain of  $f(x)$
- ▶ **Player A** tells **Player B** what  $f(x)$  is
- ▶ **Player B** tries to guess **Player A**'s  $x$ -value.

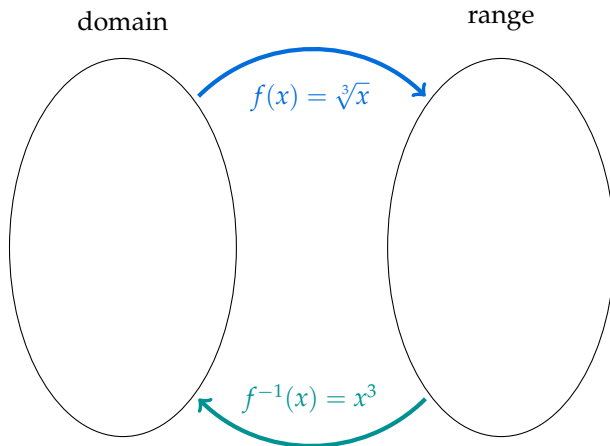
**Round 1:**  $f(x) = 2x$

**Round 2:**  $f(x) = \sqrt[3]{x}$

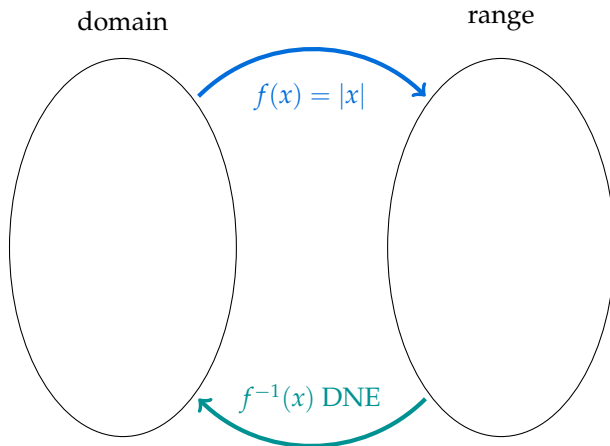
**Round 3:**  $f(x) = |x|$

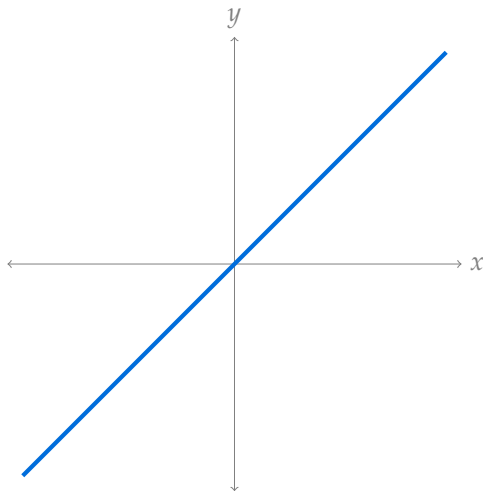
**Round 4:**  $f(x) = \sin x$

# FUNCTIONS ARE MAPS



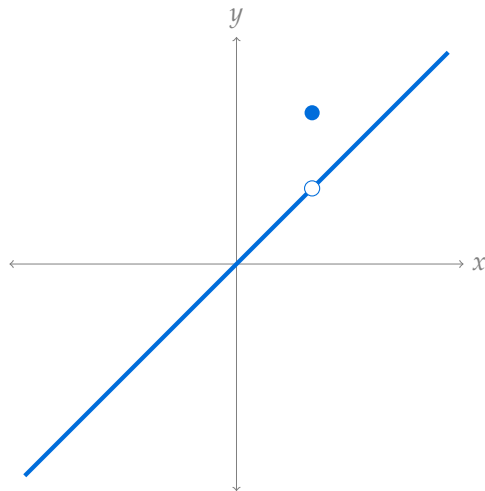
# FUNCTIONS ARE MAPS





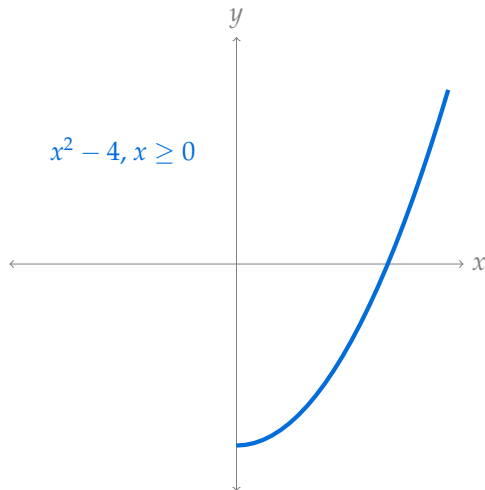
A. invertible

B. not invertible



A. invertible

B. not invertible



A. invertible

B. not invertible

# RELATIONSHIP BETWEEN $f(x)$ AND $f^{-1}(x)$

Let  $f$  be an invertible function.

What is  $f^{-1}(f(x))$ ?

- A.  $x$
- B. 1
- C. 0
- D. not sure



## Invertibility

In order for a function to be invertible, different  $x$  values cannot map to the same  $y$  value.

We call such a function **one-to-one**, or **injective**.

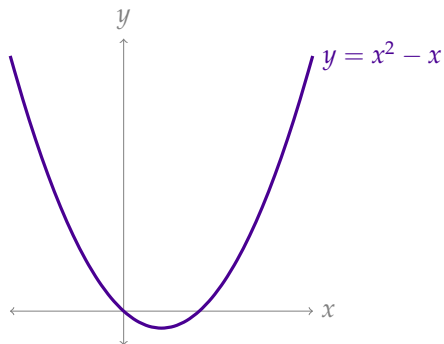
Suppose  $f(x) = \sqrt[3]{19 + x^3}$ . What is  $f^{-1}(3)$ ? (simplify your answer)

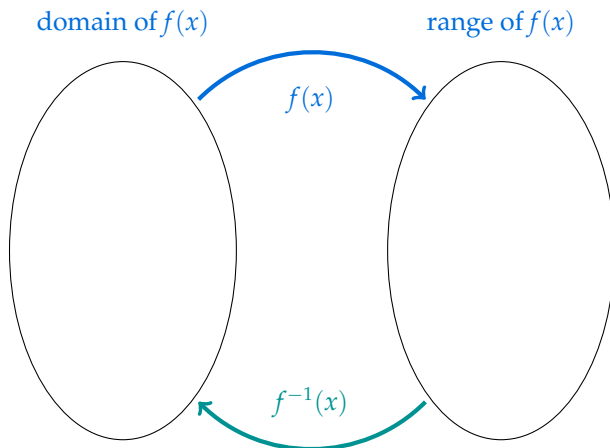
What is  $f^{-1}(10)$ ? (do not simplify)

What is  $f^{-1}(x)$ ?

$$\text{Let } f(x) = x^2 - x.$$

1. Sketch a graph of  $f(x)$ , and choose a (large) domain over which it is invertible.
2. For the domain you chose, evaluate  $f^{-1}(20)$ .
3. For the domain you chose, evaluate  $f^{-1}(x)$ .
4. What are the domain and range of  $f^{-1}(x)$ ? What are the (restricted) domain and range of  $f(x)$ ?





INVERTIBILITY GAME:  $f(x) = e^x$

$$f^{-1}(x) = \log_e x$$

- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = e$ . What is my  $x$ ?
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = 1$ . What is my  $x$ ?
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = \frac{1}{e}$ . What is my  $x$ ?
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = e^3$ . What is my  $x$ ?
- ▶ I'm thinking of an  $x$ . Your clue:  $f(x) = 0$ . What is my  $x$ ?

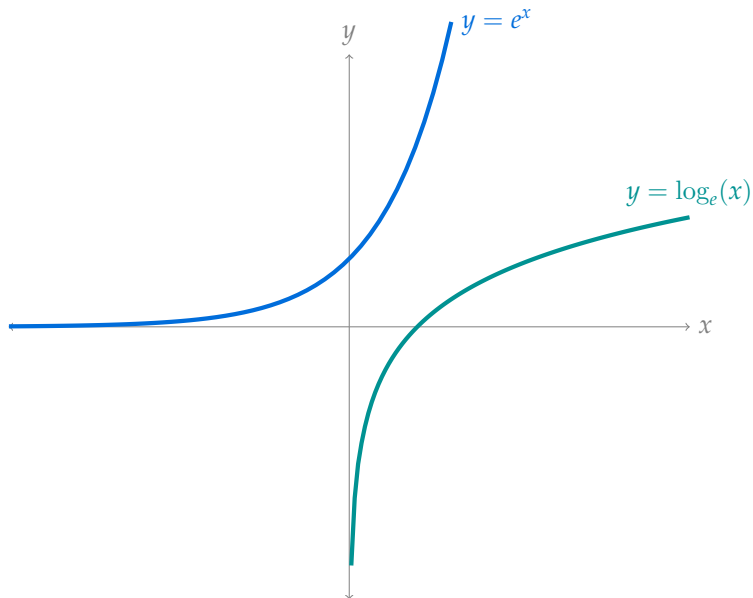
1. Suppose  $0 < x < 1$ . Then  $\log_e(x)$  is...
2. Suppose  $-1 < x < 0$ . Then  $\log_e(x)$  is...
3. Suppose  $e < x$ . Then  $\log_e(x)$  is...
  - A. positive
  - B. negative
  - C. greater than one
  - D. less than one
  - E. undefined

# EXPONENTS AND LOGARITHMS

$$f(x) = e^x$$

$$f^{-1}(x) = \log_e(x) = \ln(x) = \text{log}(x)$$

$x$	$e^x$	$e \text{ fact} \leftrightarrow \log_e \text{ fact}$	$x$	$\log_e(x)$
0	1			
1	$e$			
-1	$\frac{1}{e}$			
$n$	$e^n$			





# LOGS OF OTHER BASES: $\log_n(x)$ IS THE INVERSE OF $n^x$

$$\log_{10} 10^8 =$$

- A. 0
- B. 8
- C. 10
- D. other

$$\log_2 16 =$$

- A. 1
- B. 2
- C. 3
- D. other

## Logarithm Rules

Let  $A$  and  $B$  be positive, and let  $n$  be any real number.

$$\log(A \cdot B) = \log(A) + \log(B)$$

$$\text{Proof: } \log(A \cdot B) = \log(e^{\log A} e^{\log B}) = \log(e^{\log A + \log B}) = \log(A) + \log(B)$$

$$\log(A/B) = \log(A) - \log(B)$$

$$\text{Proof: } \log(A/B) = \log\left(\frac{e^{\log A}}{e^{\log B}}\right) = \log(e^{\log A - \log B}) = \log A - \log B$$

$$\log(A^n) = n \log(A)$$

$$\text{Proof: } \log(A^n) = \log\left((e^{\log A})^n\right) = \log(e^{n \log A}) = n \log A$$

## Logarithm Rules

Let  $A$  and  $B$  be positive, and let  $n$  be any real number.

$$\log(A \cdot B) = \log(A) + \log(B)$$

$$\log(A/B) = \log(A) - \log(B)$$

$$\log(A^n) = n \log(A)$$

Write as a single logarithm:

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2 \log x + \log(10 + x)$$

# BASE CHANGE

Fact:  $b^{\log_b(a)} = a$

$$\Rightarrow \log(b^{\log_b(a)}) = \log(a)$$

$$\Rightarrow \log_b(a) \log(b) = \log(a)$$

$$\Rightarrow \log_b(a) = \frac{\log(a)}{\log(b)}$$

In general, for positive  $a$ ,  $b$ , and  $c$ :

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

In general, for positive  $a$ ,  $b$ , and  $c$ :

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$$

Suppose your calculator can only compute logarithms base 10. What would you enter to calculate  $\log(17)$ ?

Suppose your calculator can only compute natural logarithms. What would you enter to calculate  $\log_2(57)$ ?

Suppose your calculator can only compute logarithms base 2. What would you enter to calculate  $\log(2)$ ?

Decibels: For a particular measure of the power  $P$  of a sound wave, the decibels of that sound is:

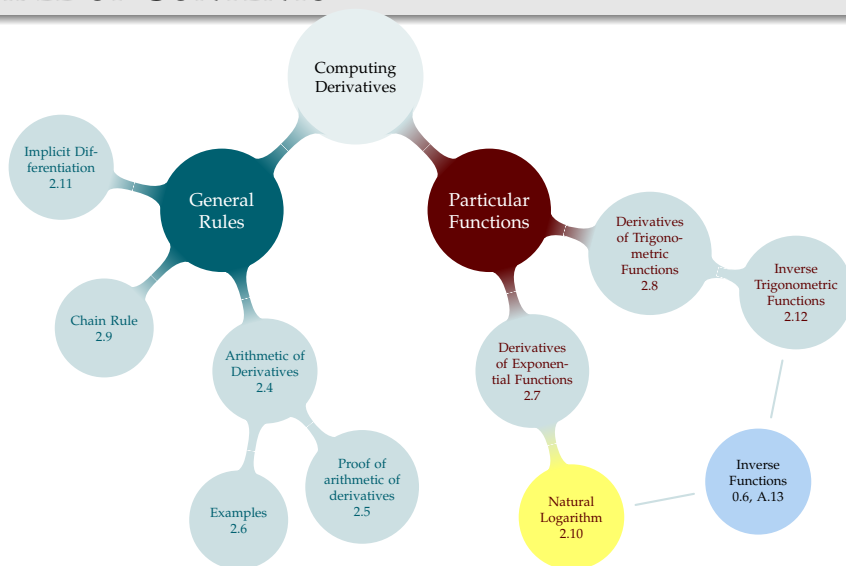
$$10 \log_{10}(P)$$

So, every ten decibels corresponds to a sound being ten **times** louder.

A lawnmower emits a 100dB sound. How much sound will two lawnmowers make?

- A. 100 dB
- B. 110 dB
- C. 200 dB
- D. other

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# DIFFERENTIATING THE NATURAL LOGARITHM

Calculate  $\frac{d}{dx}\{\log_e x\}$ .

One Weird Trick:

$$x = e^{\log_e x}$$

$$\frac{d}{dx}\{x\} = \frac{d}{dx}\{e^{\log_e x}\}$$

$$1 = e^{\log_e x} \cdot \frac{d}{dx}\{\log_e x\} = x \cdot \frac{d}{dx}\{\log_e x\}$$

$$\frac{1}{x} = \frac{d}{dx}\{\log_e x\}$$



## Derivative of Natural Logarithm

$$\frac{d}{dx} \{\log_e |x|\} = \frac{1}{x} \quad (x \neq 0)$$

Differentiate:  $f(x) = \log_e |x^2 + 1|$

## Derivatives of Logarithms – Corollary 2.10.6

For  $a > 0$ :

$$\frac{d}{dx} [\log_a |x|] = \frac{1}{x \log a}$$

In particular:

$$\frac{d}{dx} [\log |x|] = \frac{1}{x}$$

Differentiate:  $f(x) = \log_e |\cot x|$

# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

►  $\log(f \cdot g) = \log f + \log g$

multiplication turns into addition

►  $\log\left(\frac{f}{g}\right) = \log f - \log g$

division turns into subtraction

►  $\log(f^g) = g \log f$

exponentiation turns into multiplication

We can exploit these properties to differentiate!

## Logarithmic Differentiation

In general, if  $f(x) \neq 0$ ,  $\frac{d}{dx} [\log |f(x)|] = \frac{f'(x)}{f(x)}$ .

$$f(x) = \left( \frac{(2x+5)^4(x^2+1)}{x+3} \right)^5$$

Find  $f'(x)$ .

# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

$$f(x) = \left( \frac{(2x+5)^4(x^2+1)}{x+3} \right)^5$$

# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

Differentiate:

$$f(x) = x^x$$

# LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

Differentiate:

$$f(x) = \left( \frac{(x^{15} - 9x^2)^{10}(x + x^2 + 1)}{(x^7 + 7)(x + 1)(x + 2)(x + 3)} \right)^5$$

$$f(x) = \frac{(x^8 - e^x)(\sqrt{x} + 5)}{\csc^5 x}$$



$$f(x) = (x^2 + 17)(32x^5 - 8)(x^{98} - x^{57} + 32x^2)^4(32x^{10} - 10x^{32})$$

Find  $f'(x)$ .