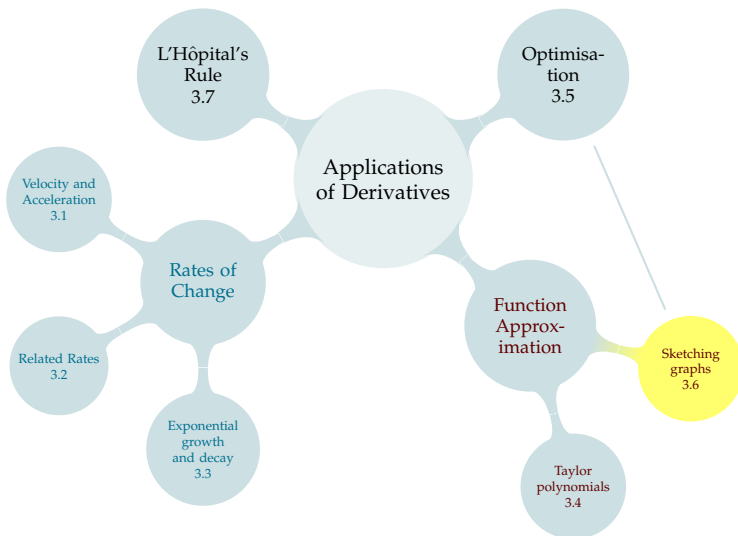


TABLE OF CONTENTS



CURVE SKETCHING

Review: find the domain of the following function.

$$f(x) = \frac{\sqrt{3 - x^2}}{\log(x + 1)}$$

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Where might you expect $f(x)$ to have a vertical asymptote? What does the function look like nearby?

(Recall: a vertical asymptote occurs at $x = a$ if the function has an infinite discontinuity at a . That is, $\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$.)

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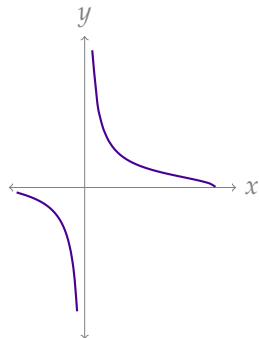
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Where is $f(x) = 0$?

What happens to $f(x)$ near its other endpoint, $x = -1$?





CURVE SKETCHING

Good things to check:

- Domain
- Vertical asymptotes: $\lim_{x \rightarrow a} f(x) = \pm\infty$
- Intercepts: $x = 0, f(x) = 0$
- Horizontal asymptotes and end behavior: $\lim_{x \rightarrow \pm\infty} f(x)$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x - 2}{(x + 3)^2}$$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

- ▶ Domain: $x \neq -3$
- ▶ Vertical asymptote: $x = -3$
- ▶ Intercepts: $(2, 0)$, $(0, -\frac{2}{9})$
- ▶ Horizontal asymptote: $y = 0$ in both directions

<https://www.desmos.com/calculator/hyzl5cyq7i>

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

- ▶ Domain: $x \neq 0, 5$
- ▶ Vertical asymptotes: $x = 0, x = 5$
- ▶ Intercepts: $(-2, 0), (3, 0)$
- ▶ Horizontal asymptote: none

<https://www.desmos.com/calculator/ploa0q7bxn>

FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

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Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

Domain all real numbers

Intercepts Factor $f(x) = x^2(\frac{1}{2}x^2 - \frac{4}{3}x - 15)$. Intercept at the origin; use the quadratic formula to find x -intercepts at $x = \frac{4 \pm \sqrt{286}}{3}$, so $x \approx 7$ and $x \approx -4.3$.

**End
behaviour** $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$

This is the information we've already talked about gathering. Now let's add the first derivative.

FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

$$\begin{aligned} f'(x) &= 2x^3 - 4x^2 - 30x \\ &= 2x(x^2 - 2x - 15) \\ &= 2x(x - 5)(x + 3) \end{aligned}$$

So, critical points are $x = 0$, $x = -3$, and $x = 5$. No singular points. At the critical points $f(0) = 0$, $f(-3) = -58.5$, $f(5) = -229.1\bar{6}$.

$x \approx -4.3$	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$0 < x < 5$	$x = 5$	$x > 5$	$x \approx 7$
$f(x) = 0$	$f' < 0$	CP	$f' > 0$	CP	$f' < 0$	CP	$f' > 0$	$f(x) = 0$
intercept	decr	loc min	incr	loc max	decr	loc min	incr	intercept

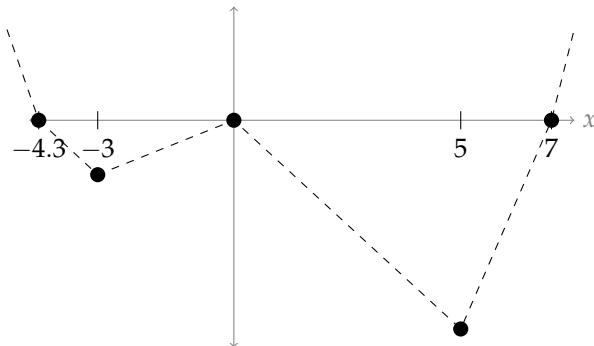
This gives us enough information to draw a skeleton.



FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



<https://www.desmos.com/calculator/lxdlgmhns1>

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

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$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

Domain

all real numbers.

**End
behaviour**

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x) = \infty$$

Intercepts

$$f(0) = 24$$

$$\begin{aligned} f(x) &= \frac{1}{3}x^2(x+6) + 4(x+6) \\ &= \left(\frac{1}{3}x^2 + 4\right)(x+6) \end{aligned}$$

Only two intercepts: $(0, 24)$ and $(-6, 0)$.

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

**Critical
points**

$$f'(x) = x^2 + 4x + 4 = (x + 2)^2 \quad \text{so } x = -2 \text{ is the location of the only critical point.}$$
$$f(-2) = 24 - \frac{8}{3} = 21 + \frac{1}{3}.$$

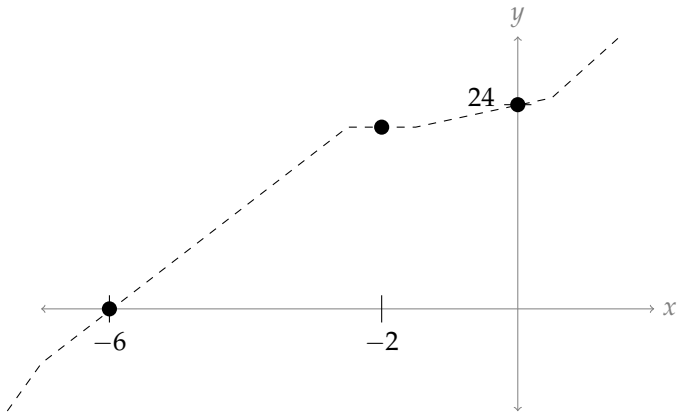
**Increasing,
decreasing**

$$f'(x) > 0 \text{ except at } x = -2, \text{ so apart from the critical point, } f(x) \text{ is increasing}$$

This is enough for us to draw a skeleton.

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$



<https://www.desmos.com/calculator/xum0mstmiv>

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What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

Domain $x \neq 1$

We need to consider what happens as x approaches 1 from the left and the right.

Vertical asymptotes

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty \implies \lim_{x \rightarrow 1^-} f(x) = \lim_{A \rightarrow -\infty} e^A = 0$$

$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = \infty \implies \lim_{x \rightarrow 1^+} f(x) = \lim_{A \rightarrow +\infty} e^A = \infty$$

End behaviour

$$\lim_{x \rightarrow \pm\infty} \frac{x+1}{x-1} = 1, \text{ so } \lim_{x \rightarrow \pm\infty} f(x) = e$$

Intercepts $f(0) = \frac{1}{e}$; there are no roots



What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

**Critical
points**

$$f'(x) = e^{\frac{x+1}{x-1}} \left(\frac{-2}{(x-1)^2} \right) \quad \text{no critical points}$$

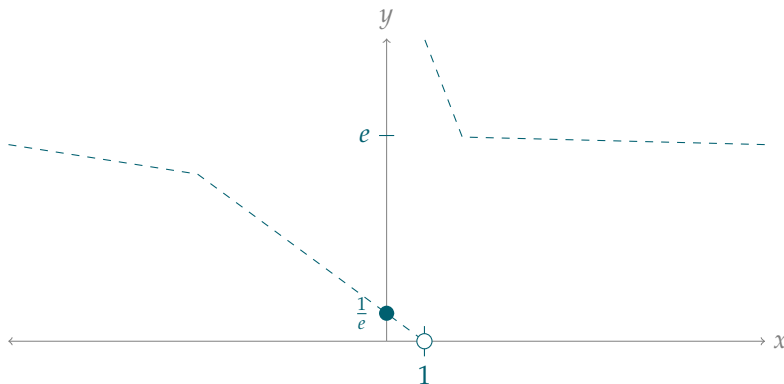
**Increasing,
decreasing**

$f(x)$ is decreasing everywhere it is defined

This information is enough to draw a skeleton.

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$



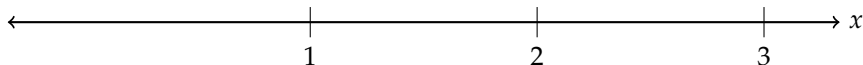
<https://www.desmos.com/calculator/x0cccy1ggj>



SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

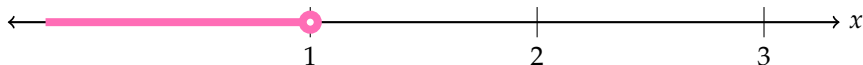
$$f(x) = (x - 1)(x - 2)(x - 3)$$



SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

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SIGNS OF FACTORED FUNCTIONS

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+



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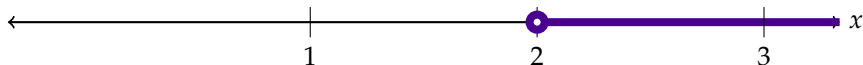


SIGNS OF FACTORED FUNCTIONS

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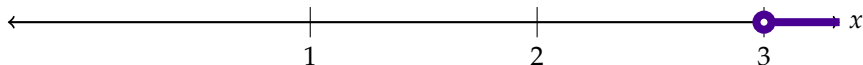
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SIGNS OF FACTORED FUNCTIONS

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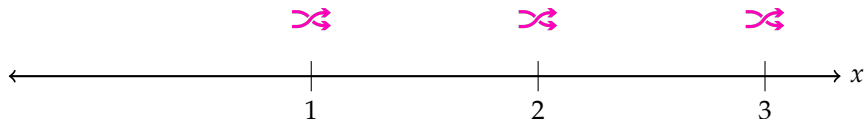
$$f(x) = (x - 1)(x - 2)(x - 3) +$$



SIGNS OF FACTORED FUNCTIONS

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$$f(x) = (x - 1)(x - 2)(x - 3)$$

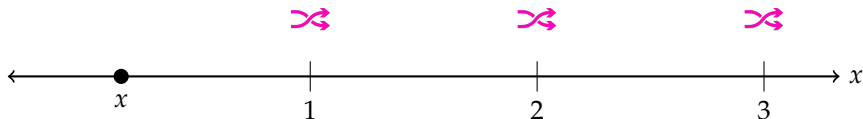


Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

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$$f(x) = \underbrace{(x - 1)} \quad \underbrace{(x - 2)} \quad \underbrace{(x - 3)}$$



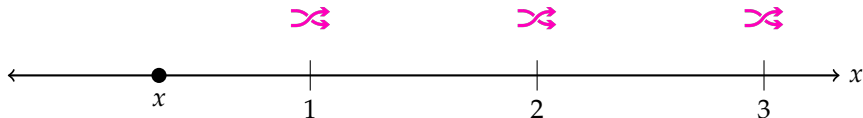
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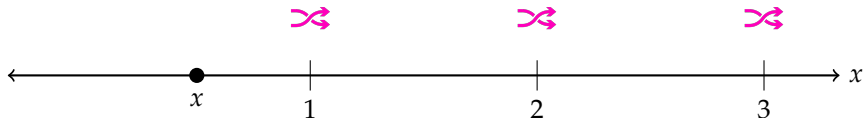
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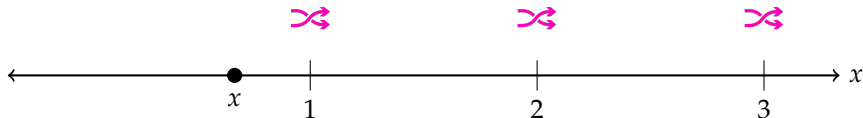
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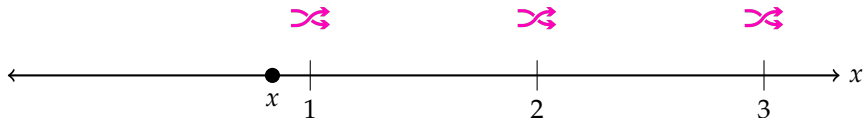
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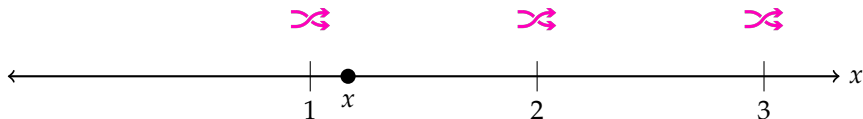
Sign of entire function:

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SIGNS OF FACTORED FUNCTIONS

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$$f(x) = \begin{array}{ccc} (x-1) & (x-2) & (x-3) \\ + & - & - \end{array}$$



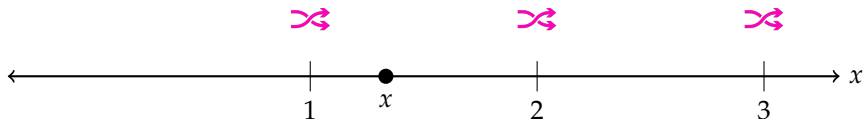
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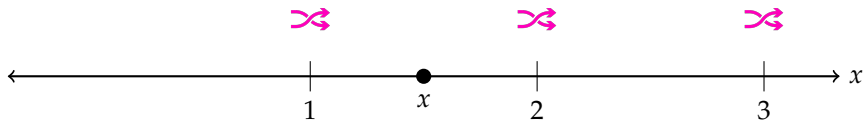
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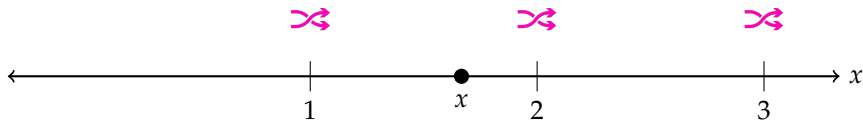
Sign of entire function:

+

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$$f(x) = \underset{+}{(x-1)} \underset{-}{(x-2)} \underset{-}{(x-3)}$$



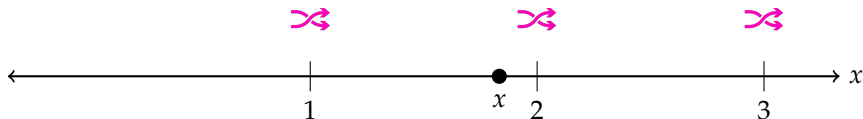
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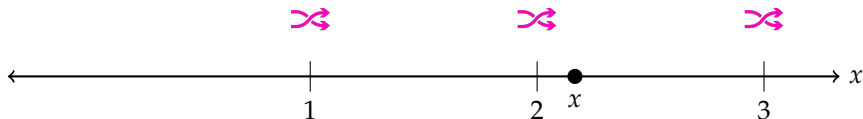
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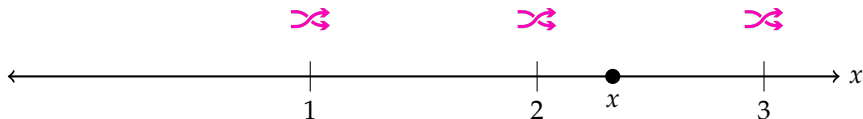
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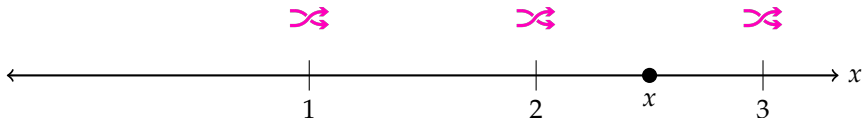
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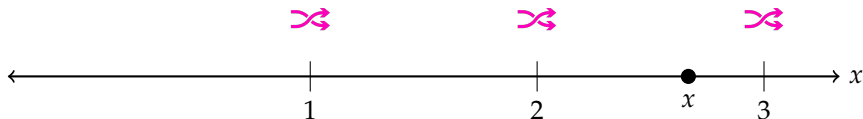
Sign of entire function:

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SIGNS OF FACTORED FUNCTIONS

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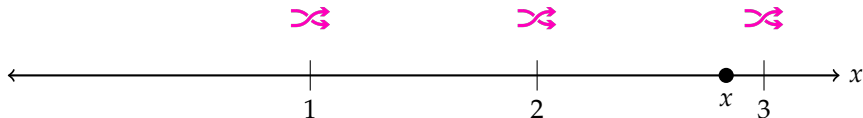
Sign of entire function:

—

SIGNS OF FACTORED FUNCTIONS

▶ SKIP SIGN CHANGES

$$f(x) = \begin{matrix} (x-1) & (x-2) & (x-3) \\ + & + & - \end{matrix}$$



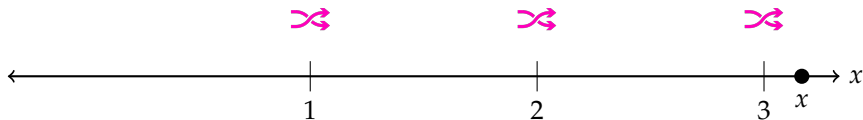
Sign of entire function:

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SIGNS OF FACTORED FUNCTIONS

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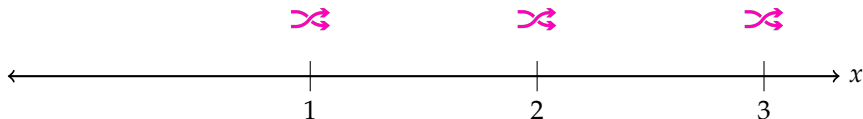
Sign of entire function:

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SIGNS OF FACTORED FUNCTIONS

▶ SKIP SIGN CHANGES

$$f(x) = (x - 1)(x - 2)(x - 3)$$

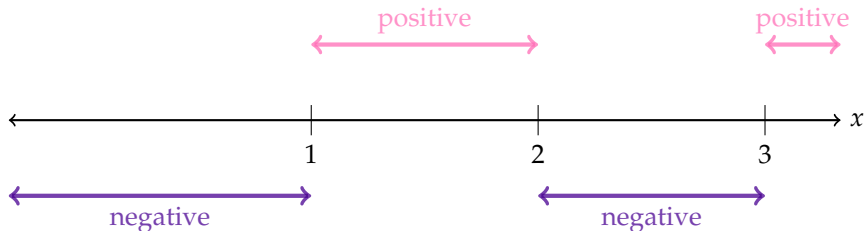


Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

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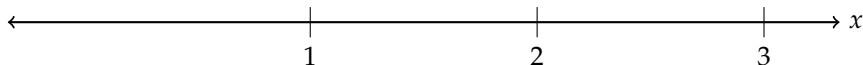


Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 1) (x - 2)^2 (x - 3)$$

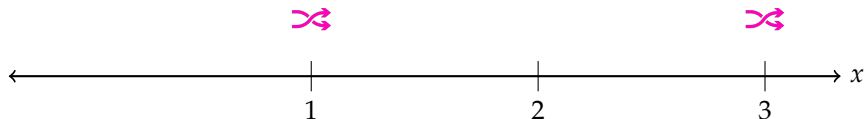


Sign of entire function:

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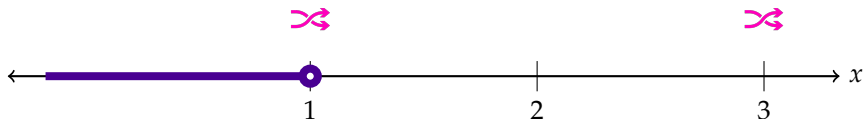


Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

▶ SKIP SIGN CHANGES

$$f(x) = \underset{-}{(x-1)} \underset{+}{(x-2)^2} \underset{-}{(x-3)}$$



Sign of entire function:

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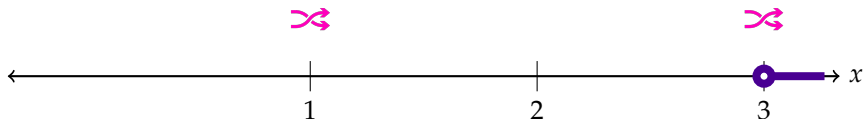


Sign of entire function:

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$$f(x) = \begin{array}{ccccc} (x-1) & & (x-2)^2 & & (x-3) \\ & + & & + & \\ & & & & \end{array}$$



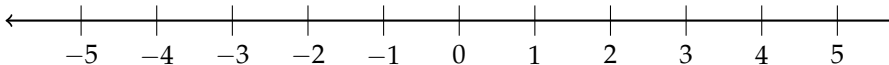
Sign of entire function:

SIGNS OF FACTORED FUNCTIONS

[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

Where is $f(x)$ positive? Where is it negative?

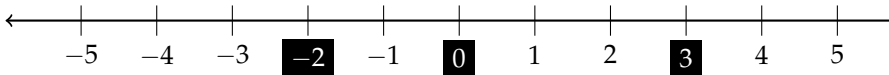


SIGNS OF FACTORED FUNCTIONS

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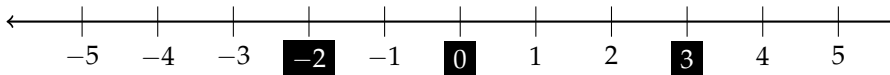


SIGNS OF FACTORED FUNCTIONS

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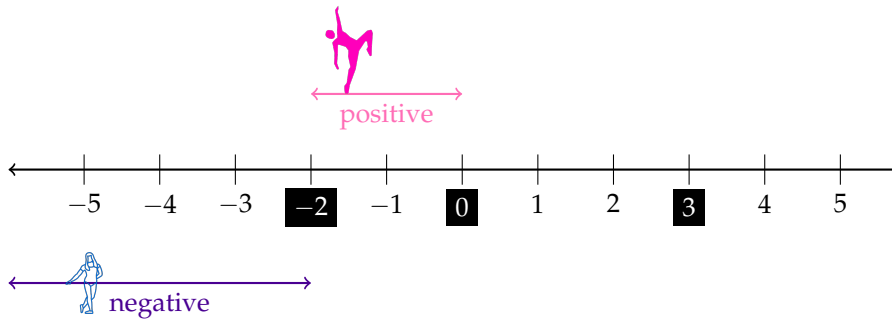


SIGNS OF FACTORED FUNCTIONS

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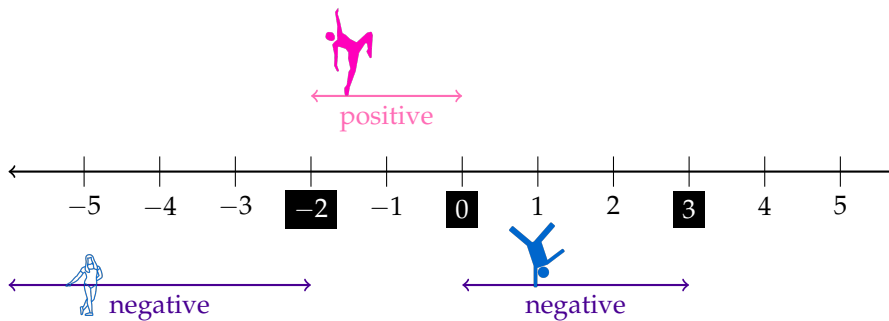


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$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

Where is $f(x)$ positive? Where is it negative?

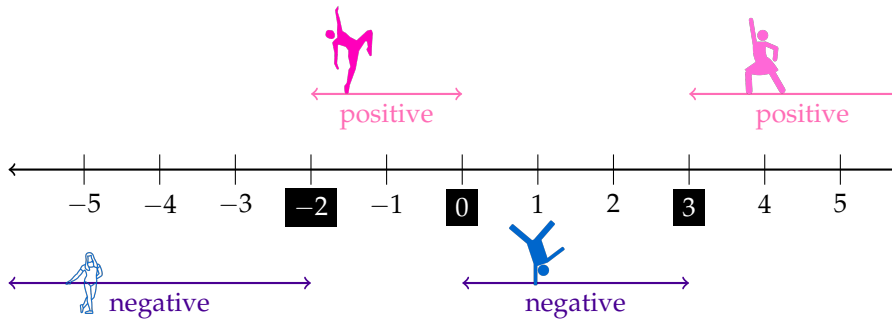


SIGNS OF FACTORED FUNCTIONS

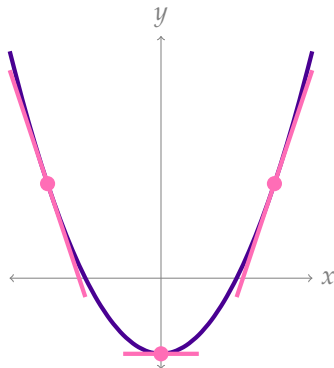
[▶ SKIP SIGN CHANGES](#)

$$f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4$$

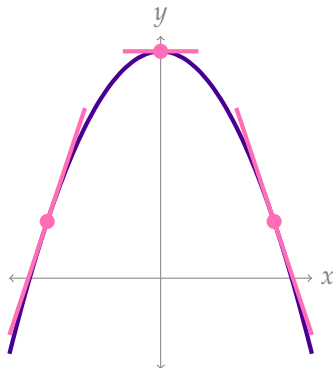
Where is $f(x)$ positive? Where is it negative?



CONCAVITY



- ▶ Slopes are increasing
- ▶ $f''(x) > 0$
- ▶ “concave up”
- ▶ tangent line below curve



- ▶ Slopes are decreasing
- ▶ $f''(x) < 0$
- ▶ “concave down”
- ▶ tangent line above curve

3.6.1: Domain, Intercepts, Asymptotes
○○○○○

3.6.2: First Derivative
○○○○○

3.6.3: Concavity
○●○○○○○

3.6.4 : Symmetries
○○○○○○○○○○○○○○○○○○○

3.6.5: A checklist for sketching
○○○○○○○○○○○○○○○

MNEMONIC

+

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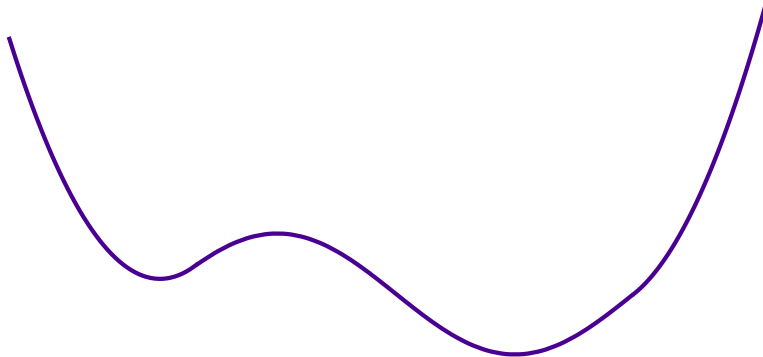


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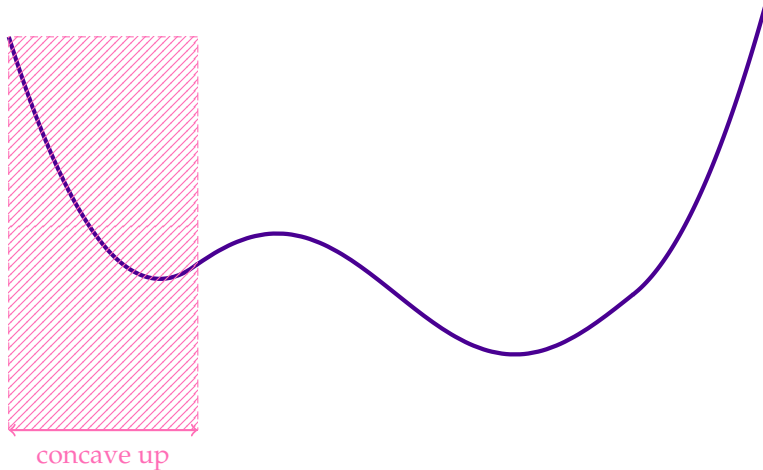
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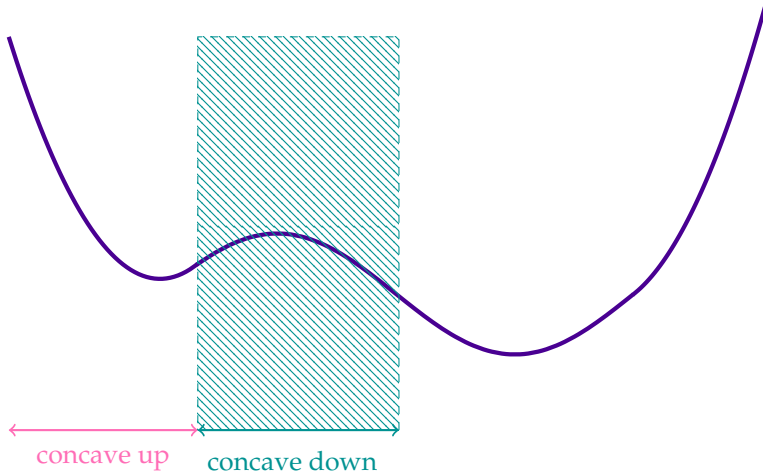
CONCAVITY



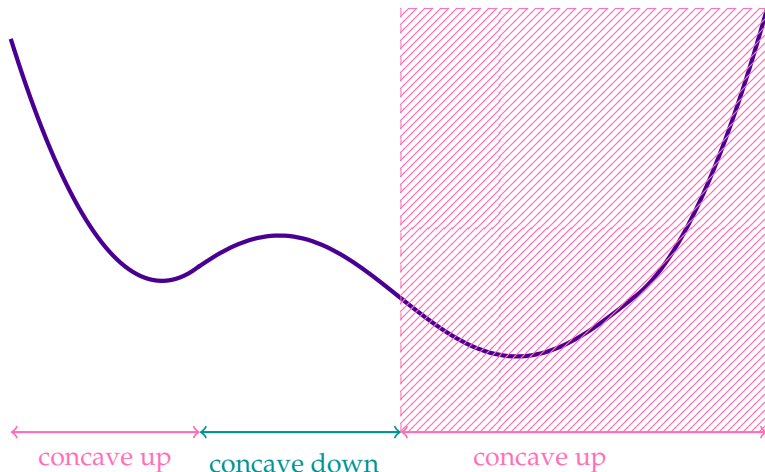
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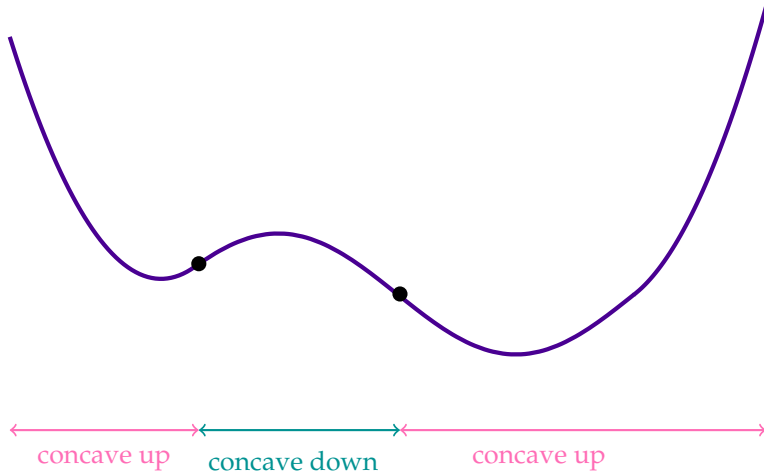
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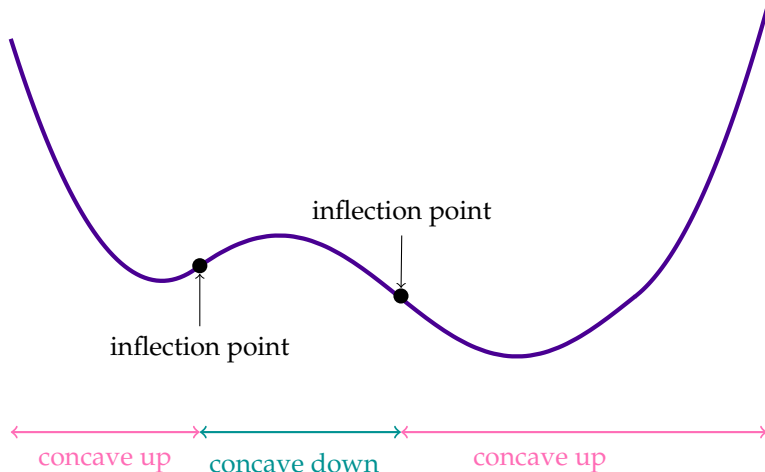
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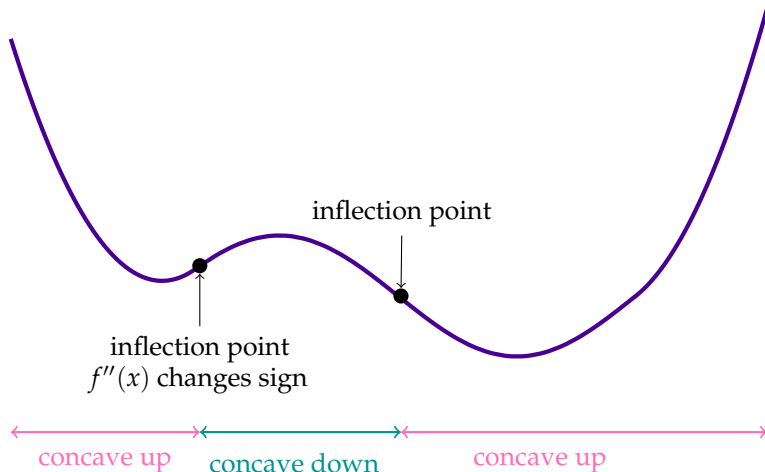
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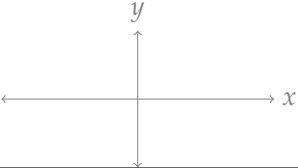
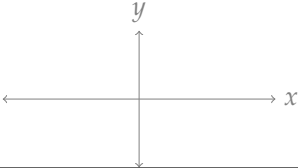
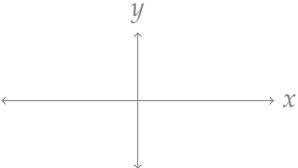
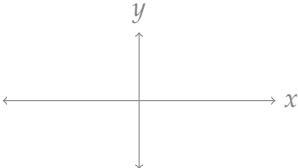
CONCAVITY



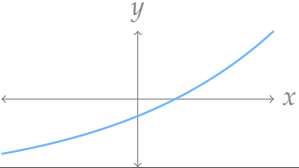
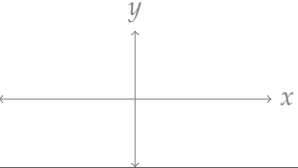
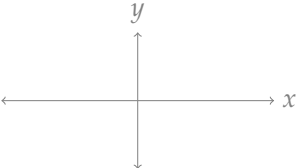
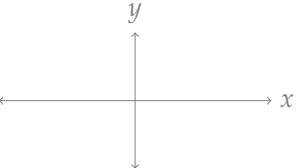
CONCAVITY



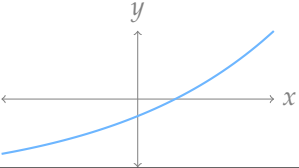
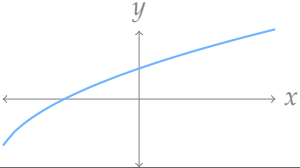
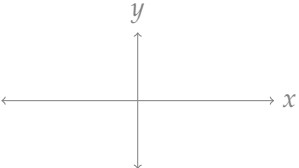
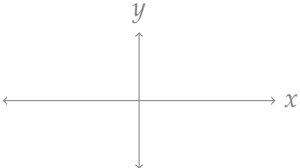
Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

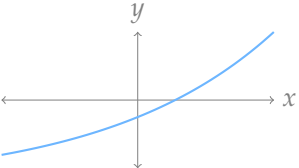
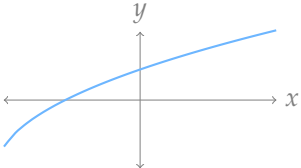
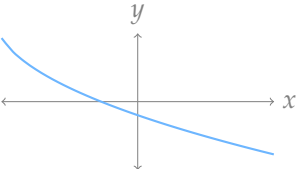
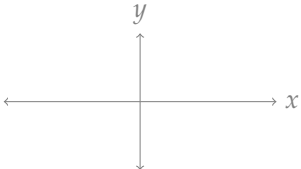
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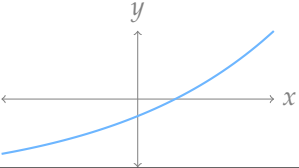
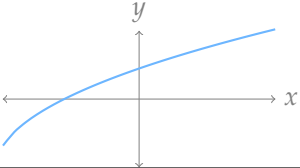
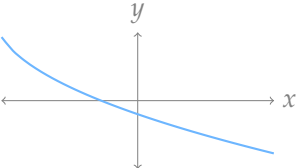
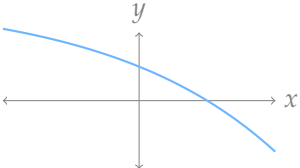
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	concave up	concave down
increasing		
decreasing		

Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

POLL QUESTIONS

Describe the concavity of the function $f(x) = e^x$.

- A. concave up
- B. concave down
- C. concave up for $x < 0$; concave down for $x > 0$
- D. concave down for $x < 0$; concave up for $x > 0$
- E. I'm not sure

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Is it possible to be concave up and decreasing?

A. Yes

B. No

C. I'm not sure

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Is it possible to be concave up and decreasing?

- A. Yes
- B. No
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Suppose a function $f(x)$ is defined for all real numbers, and is concave up on the interval $[0, 1]$. Which of the following must be true?

- A. $f'(0) < f'(1)$
- B. $f'(0) > f'(1)$
- C. $f'(0)$ is positive
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Is it possible to be concave up and decreasing?

- A. Yes
- B. No
- C. I'm not sure

Suppose a function $f(x)$ is defined for all real numbers, and is concave up on the interval $[0, 1]$. Which of the following must be true?

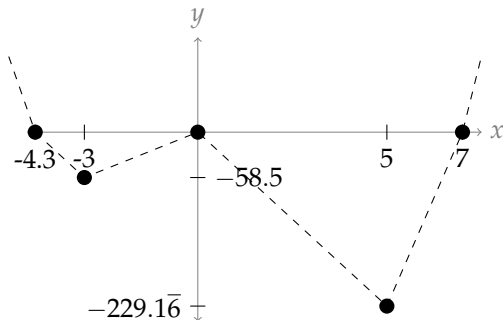
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- B. $f'(0) > f'(1)$
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REVISITING A PREVIOUS EXAMPLE

◀ original example

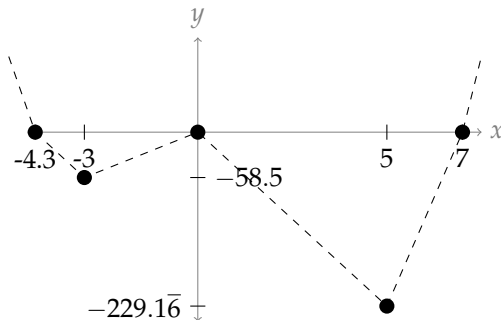
$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



REVISITING A PREVIOUS EXAMPLE

[◀ original example](#)

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

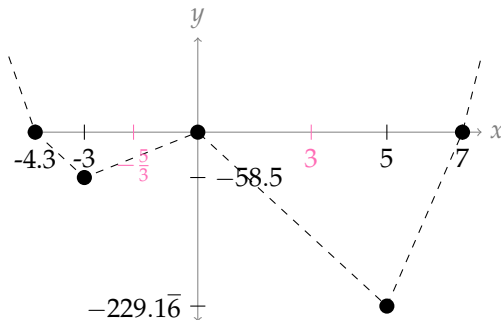


$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

REVISITING A PREVIOUS EXAMPLE

◀ original example

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

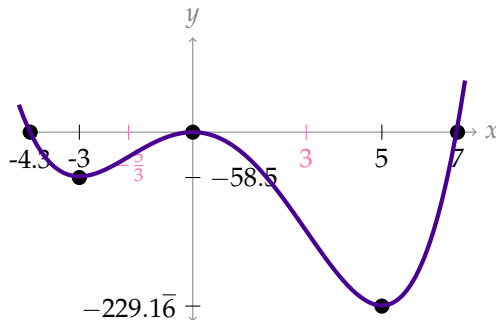


$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

REVISITING A PREVIOUS EXAMPLE

[◀ original example](#)

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$$f''(x) = 6x^2 - 8x - 30 = 2(x - 3)(3x + 5)$$

Sketch:

$$f(x) = x^5 - 15x^3$$

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$$f(x) = x^5 - 15x^3$$

Domain	Defined and differentiable for all real numbers.
Intercepts	$f(x) = x^3(x^2 - 15)$: Roots are at $x = 0$ and $x = \pm\sqrt{15} \approx \pm 4$
End behaviour	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$
Critical points	$f'(x) = 5x^4 - 45x^2 = 5x^2(x^2 - 9)$. So the critical points are $x = 0, x = \pm 3$.
Increasing, decreasing	Increasing on $(-\infty, -3)$, decreasing on $(-3, 0)$ and $(0, 3)$, increasing on $(3, \infty)$
Local extrema	From intervals of increase and decrease: local max at $x = -3$ and local min at $x = 3$

Sketch:

$$f(x) = x^5 - 15x^3$$

Concavity

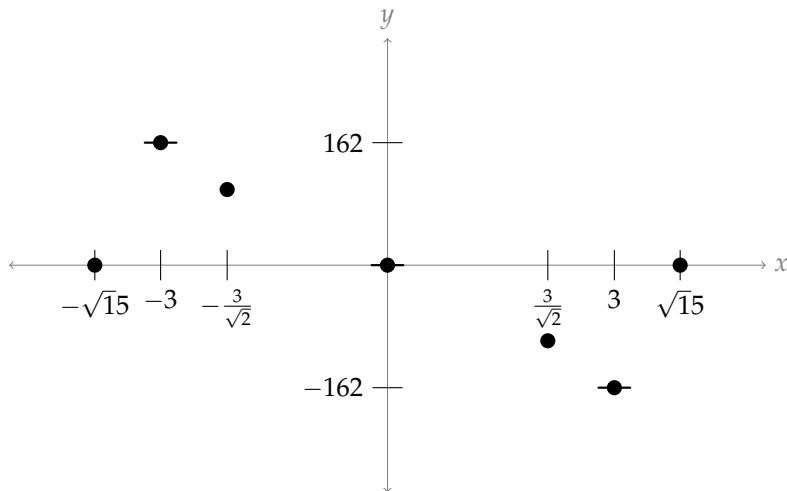
$f''(x) = 20x^3 - 90x = 10x(2x^2 - 9) = 0$ for $x = 0$ and $x = \pm \frac{3}{\sqrt{2}} \approx \pm 2.12$. All of these are inflection points; concave down $(-\infty, -\frac{3}{\sqrt{2}})$, concave up $(-\frac{3}{\sqrt{2}}, 0)$, concave down $(0, \frac{3}{\sqrt{2}})$, and concave up $(\frac{3}{\sqrt{2}}, \infty)$.

y-values of notable points

$f(3) = -162, f(-3) = 162, f(-3/\sqrt{2}) \approx 100, f(3/\sqrt{2}) \approx -100$

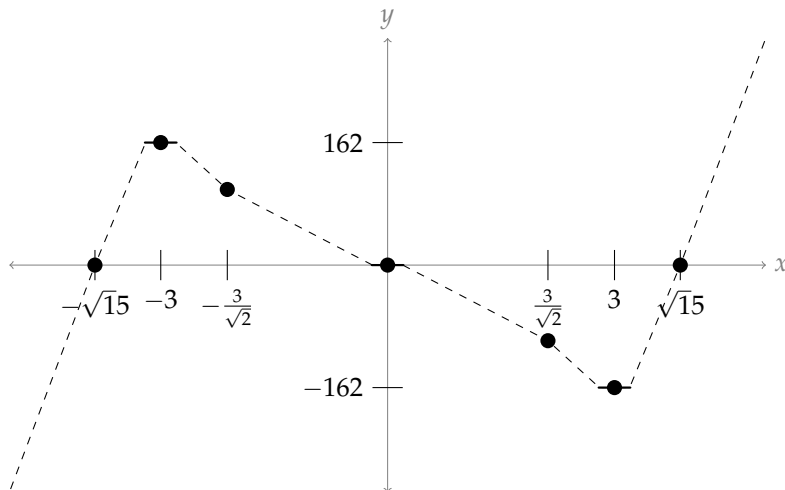
Sketch:

$$f(x) = x^5 - 15x^3$$



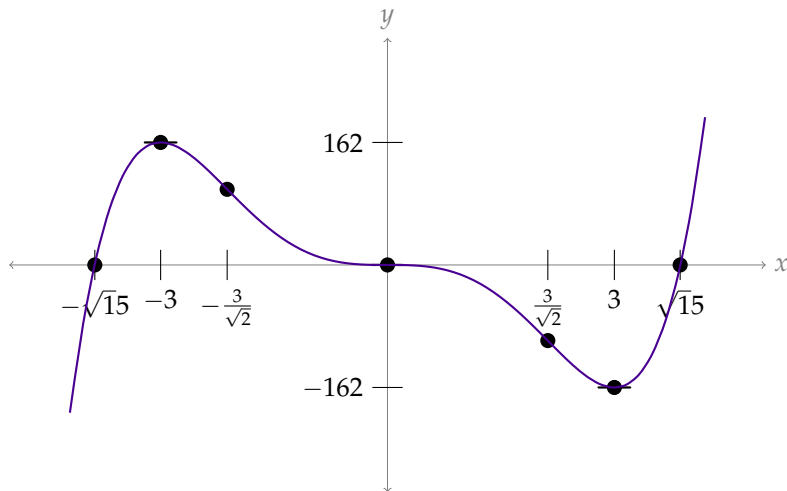
Sketch:

$$f(x) = x^5 - 15x^3$$



Sketch:

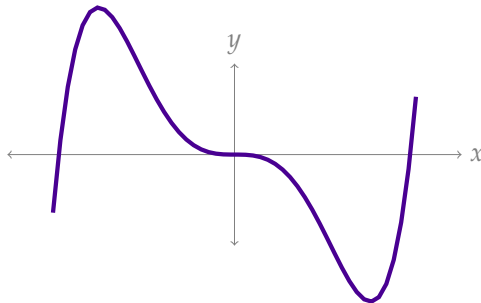
$$f(x) = x^5 - 15x^3$$



<https://www.desmos.com/calculator/uoi6nmgr8>

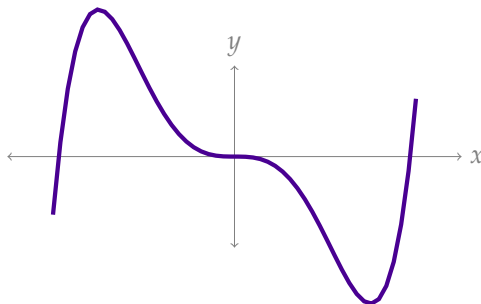


EVEN AND ODD FUNCTIONS



$$f(x) = x^5 - 15x^3$$

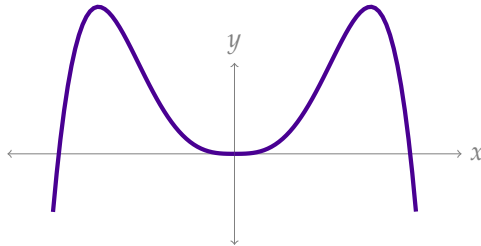
EVEN AND ODD FUNCTIONS



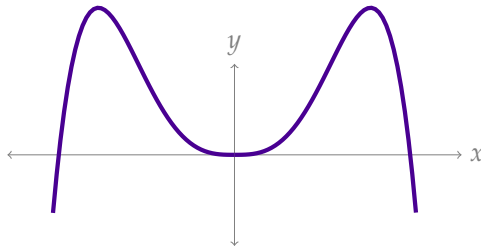
$$f(x) = x^5 - 15x^3$$

odd function

EVEN AND ODD FUNCTIONS



EVEN AND ODD FUNCTIONS

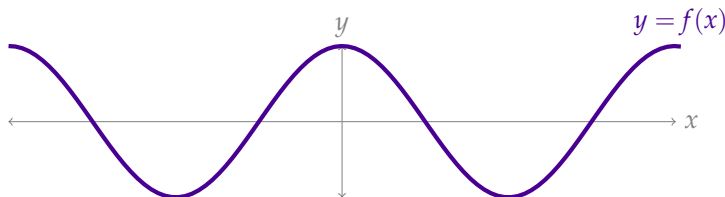


even function

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

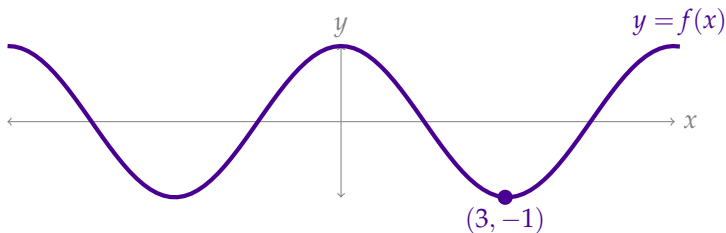
$$f(-x) = f(x)$$



Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

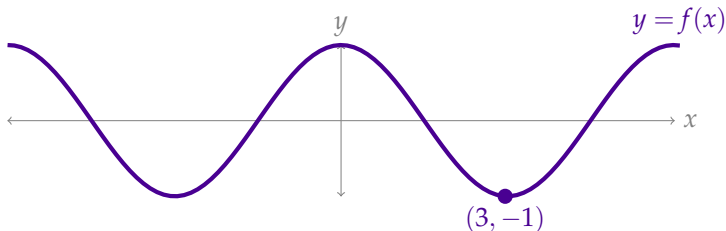


Suppose $f(3) = -1$.

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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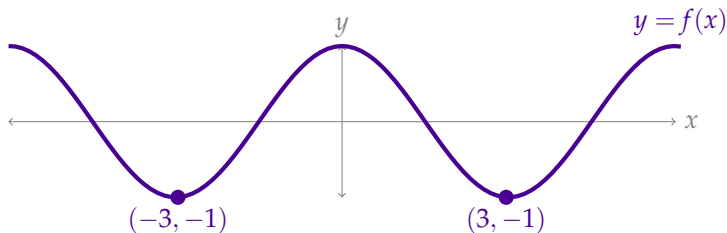


Suppose $f(3) = -1$. Then $f(-3) =$

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

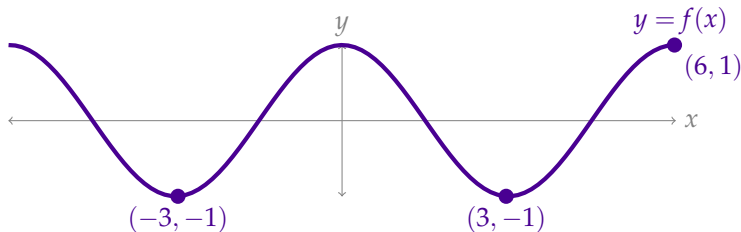


Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$



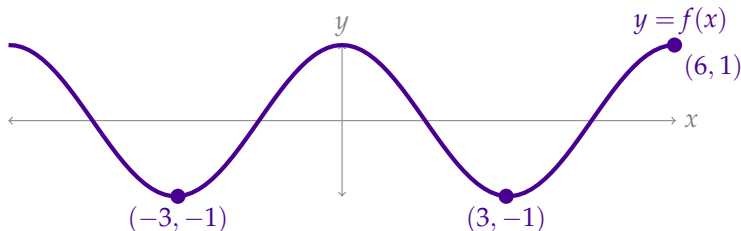
Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Suppose $f(6) = 1$.

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$



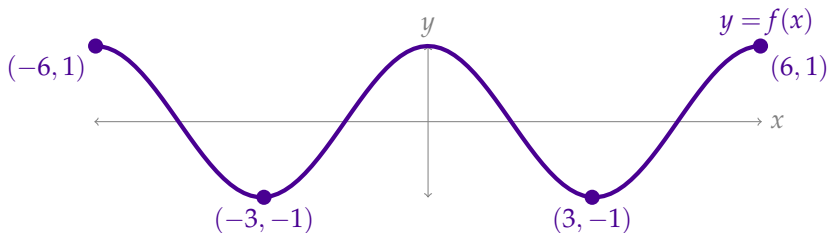
Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Suppose $f(6) = 1$. Then $f(-6) =$

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$



Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

Suppose $f(6) = 1$. Then $f(-6) = 1$ also.

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:

$$f(x) = x^2$$

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:

$$f(x) = x^2$$

$$f(x) = x^4$$

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

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Examples:

$$f(x) = x^2$$

$$f(x) = x^4$$

$$f(x) = \cos(x)$$

EVEN FUNCTIONS

Even Function – Definition 3.6.5

A function $f(x)$ is **even** if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:

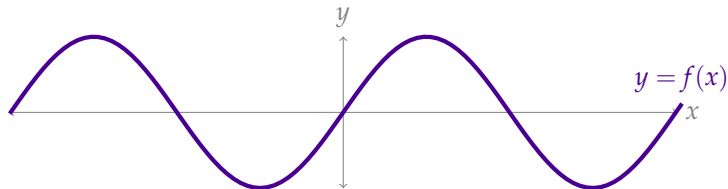
$$f(x) = x^2$$

$$f(x) = x^4$$

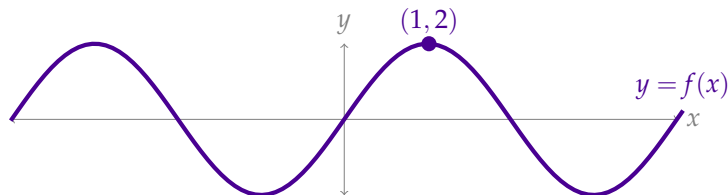
$$f(x) = \cos(x)$$

$$f(x) = \frac{x^4 + \cos(x)}{x^{16} + 7}$$

ODD FUNCTIONS

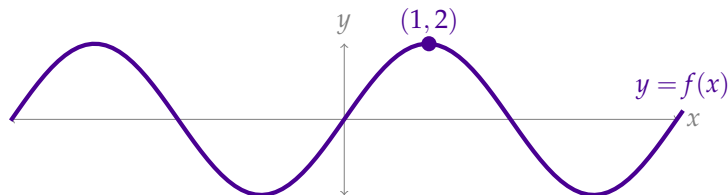


ODD FUNCTIONS



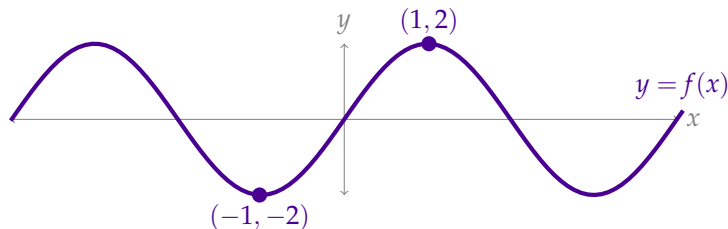
Suppose $f(1) = 2$.

ODD FUNCTIONS



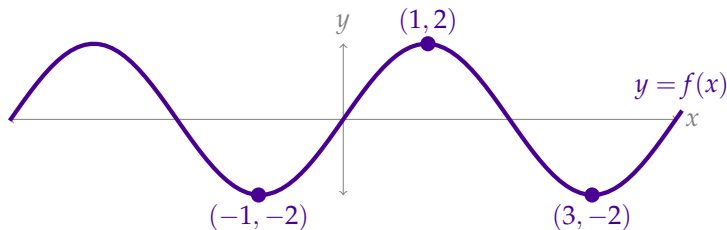
Suppose $f(1) = 2$. Then $f(-1) =$

ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

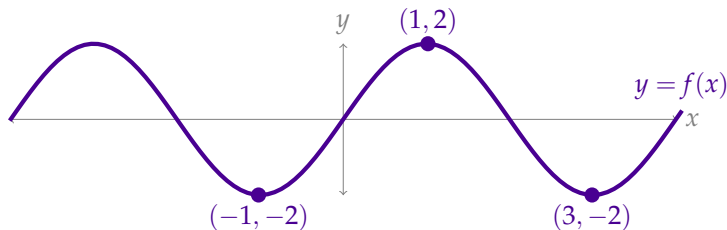
ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$.

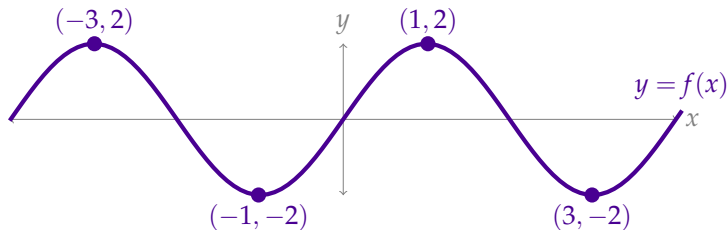
ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) =$

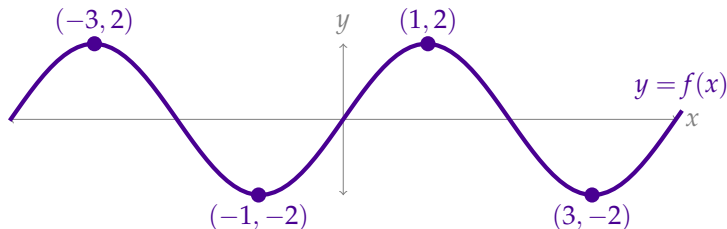
ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) = 2$.

ODD FUNCTIONS



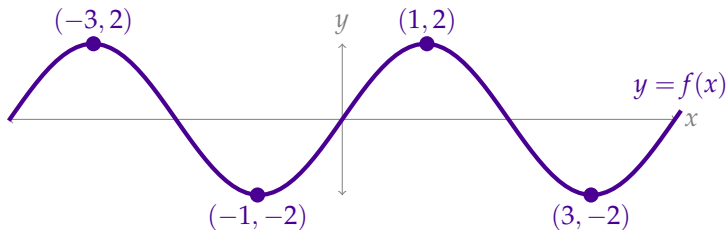
Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) = 2$.

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

ODD FUNCTIONS



Suppose $f(1) = 2$. Then $f(-1) = -2$.

Suppose $f(3) = -2$. Then $f(-3) = 2$.

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$$f(x) = x$$

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$$f(x) = x$$

$$f(x) = x^3$$

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$$f(x) = x$$

$$f(x) = x^3$$

$$f(x) = \sin(x)$$

ODD FUNCTIONS

Odd Function – Definition 3.6.6

A function $f(x)$ is **odd** if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$$f(x) = x$$

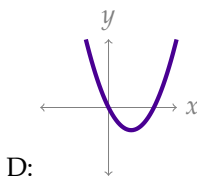
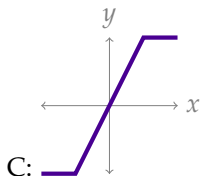
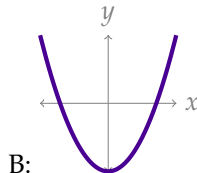
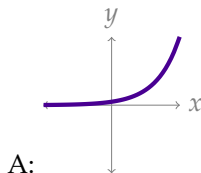
$$f(x) = x^3$$

$$f(x) = \sin(x)$$

$$f(x) = \frac{x(1 + x^2)}{x^2 + 5}$$

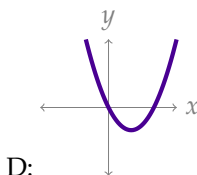
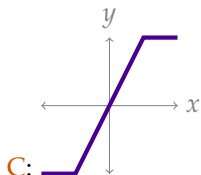
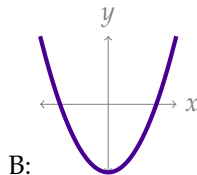
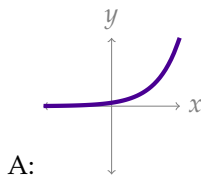
POLL TIME

Pick out the **odd** function.



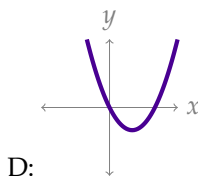
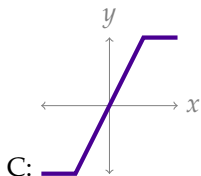
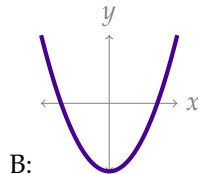
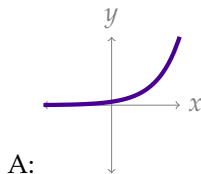
POLL TIME

Pick out the **odd** function.



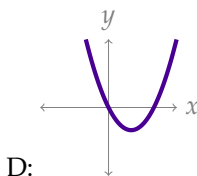
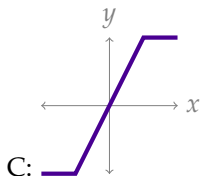
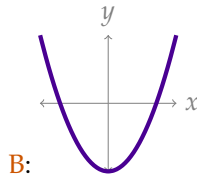
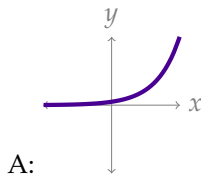
POLL TIME

Pick out the **even** function.



POLL TIME

Pick out the **even** function.



EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0)$
- B. $f(0) = -f(0)$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true



EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions
- B. $f(0) = -f(0)$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true

EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions
- B. $f(0) = -f(0) \leftarrow$ only possible for $f(0) = 0$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true

EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an **odd** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions
- B. $f(0) = -f(0) \leftarrow$ only possible for $f(0) = 0$
- C. $f(0) = 0 \leftarrow$ this is equivalent to the choice above
- D. all of the above are true
- E. none of the above are necessarily true

EVEN MORE AND MORE POLL TIIIIIME

Suppose $f(x)$ is an **even** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0)$
- B. $f(0) = -f(0)$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true

EVEN MORE AND MORE POLL TIIIIIME

Suppose $f(x)$ is an **even** function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

- A. $f(0) = f(-0)$
- B. $f(0) = -f(0)$
- C. $f(0) = 0$
- D. all of the above are true
- E. none of the above are necessarily true



OK OK... LAST ONE

Suppose $f(x)$ is an **even** function, differentiable for all real numbers.
What can we say about $f'(x)$?

- A. $f'(x)$ is also even
- B. $f'(x)$ is odd
- C. $f'(x)$ is constant
- D. all of the above are true
- E. none of the above are necessarily true

OK OK... LAST ONE

Suppose $f(x)$ is an **even** function, differentiable for all real numbers.
What can we say about $f'(x)$?

- A. $f'(x)$ is also even
- B. $f'(x)$ is odd
- C. $f'(x)$ is constant
- D. all of the above are true
- E. none of the above are necessarily true



PERIODICITY

Periodic – Definition 3.6.9

A function is **periodic** with period $P > 0$ if

$$f(x) = f(x + P)$$

whenever x and $x + P$ are in the domain of f , and P is the smallest such (positive) number

Examples: $\sin(x)$, $\cos(x)$ both have period 2π ; $\tan(x)$ has period π .

Ignoring concavity, sketch $f(x) = \sin(\sin x)$.

Challenge: ignoring exact locations of extrema, sketch $g(x) = \sin(2\pi \sin x)$.

$$f(x) = \sin(\sin x)$$

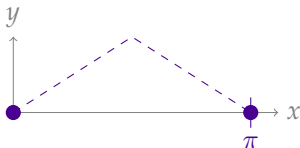
$$f(x) = \sin(\sin x)$$

Sin is periodic; since $\sin x = \sin(2\pi + x)$, then $\sin(\sin x) = \sin(\sin(2\pi + x))$, so $f(x)$ is also periodic. It suffices to sketch $f(x)$ for an interval of length 2π , because any such segment will repeat.

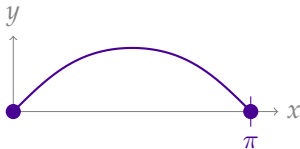
Since the function is also odd, if we sketch it on the interval $[0, \pi]$, then we can extrapolate to the interval $[-\pi, 0]$. So we consider the interval $[0, \pi]$.

- ▶ Intercepts: $(0, 0), (0, \pi)$
- ▶ First derivative: $f'(x) = \cos(\sin x) \cdot \cos(x)$
 - ▶ For $0 \leq x \leq \pi$, we have $0 \leq \sin x \leq 1 < \frac{\pi}{2}$ and hence $0 < \cos(\sin x) \leq 1$.
 - ▶ CP: $x = \frac{\pi}{2}$
 - ▶ increasing: $(0, \pi/2)$
 - ▶ decreasing: $(\pi/2, 0)$

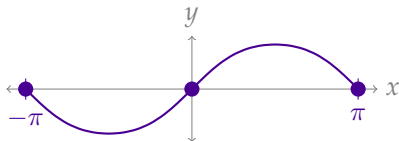
1. Interval $(0, \pi)$ skeleton, based on above work:



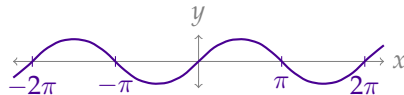
2. Make into a smooth curve:



3. Use odd symmetry to get interval $[-\pi, \pi]$



4. Use periodicity



» skip $g(x)$

$$g(x) = \sin(2\pi \sin x)$$

» skip $g(x)$

$$g(x) = \sin(2\pi \sin x)$$

Sin is periodic; since $2\pi \sin x = 2\pi \sin(2\pi + x)$, then

$$\sin(2\pi \sin x) = \sin(2\pi \sin(2\pi + x))$$

so $g(x)$ is also periodic. It suffices to sketch $g(x)$ for an interval of length 2π , because any such segment will repeat.

Note

$$\begin{aligned} g(-x) &= \sin(2\pi \sin(-x)) = \sin((2\pi)(-\sin x)) \\ &= \sin(-2\pi \sin x) = -\sin(2\pi \sin x) = -g(x) \end{aligned}$$

so $g(x)$ is odd. If we sketch it on the interval $[0, \pi]$, then we can extrapolate to the interval $[-\pi, 0]$. So we consider the interval $[0, \pi]$.

» skip $g(x)$ Intercepts in $[0, \pi]$:

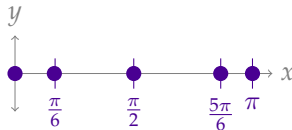
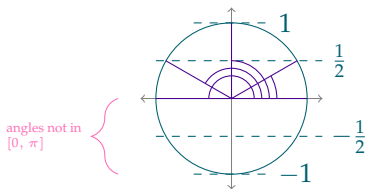
$$g(0) = 0$$

$$0 = g(0) = \sin(2\pi \sin x)$$

$$\Rightarrow 2\pi \sin x \in \{0, \pm\pi, \pm2\pi, \pm3\pi, \dots\}$$

$$\Rightarrow \sin x \in \left\{0, \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \dots\right\}$$

$$\Rightarrow x \in \left\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi\right\}$$

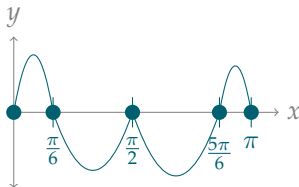


» skip $g(x)$

Now let's consider the sign of $g(x)$ between the intercepts. Since $g(x)$ isn't given as a factored product, our old shortcut isn't so useful.

interval	$(0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{5\pi}{6})$	$(\frac{5\pi}{6}, \pi)$
range of $\sin x$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, 1)$	$(0, \frac{1}{2})$
range of $2\pi \sin x$	$(0, \pi)$	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(0, \pi)$
sign of $\sin(2\pi \sin x)$	+	−	−	+

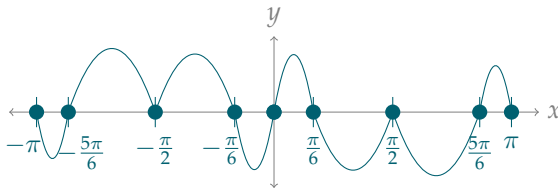
So, a rough sketch on the interval $[0, \pi]$ is:



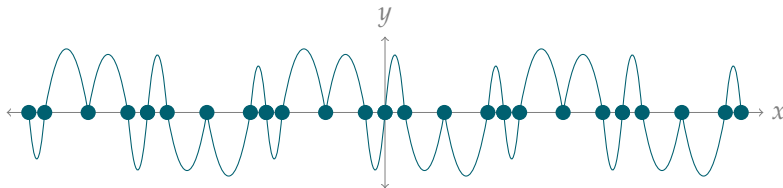
Yes, this is a rough sketch. The curve should be smooth at $\frac{\pi}{2}$.

[» skip \$g\(x\)\$](#)

Use odd symmetry, we sketch the interval $[-\pi, \pi]$:



Using periodicity:



LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2 - 64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2 + 64)}{3(x^2 - 64)^{5/3}}$$

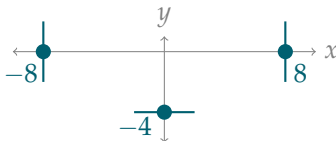
Domain all real numbers

End behaviour $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$

Intercepts $(0, -4), (\pm 8, 0)$

Critical point $(0, -4)$

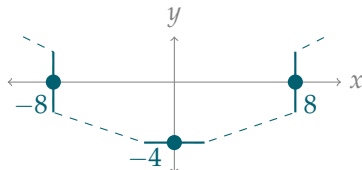
Singular points $(-8, 0), (8, 0)$
Near the singular points, $f'(x)$ gets very large, so $f(x)$ looks vertical.



**Increasing,
decreasing**

Numerator of $f'(x)$ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$. Denominator is positive where it exists. So $f(x)$ is decreasing on $(-\infty, 0)$ (where it is differentiable) and increasing on $(0, \infty)$ (where it is differentiable).

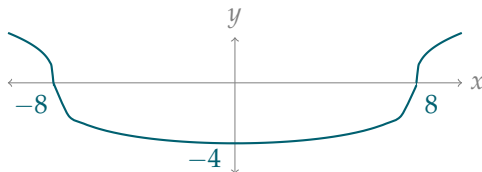
Skeleton



Concavity

Numerator of second derivative is negative everywhere. Denominator is positive (so $f''(x)$ is negative) on $(-\infty, -8) \cup (8, \infty)$ and denominator is negative (so $f''(x)$ is positive) on $(-8, 8)$.

So our function is concave up on $(-8, 8)$ and concave down on $(-\infty, -8) \cup (8, \infty)$.



LET'S GRAPH

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

LET'S GRAPH

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

Note: for $x \neq -1$, $f(x) = \frac{x(x + 1)}{(x + 1)(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^2}$

LET'S GRAPH

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Note: for $x \neq -1$, $f(x) = \frac{x(x + 1)}{(x + 1)(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^2}$

$$g(x) := \frac{x}{(x^2 + 1)^2}$$

LET'S GRAPH

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

Note: for $x \neq -1$, $f(x) = \frac{x(x + 1)}{(x + 1)(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^2}$

$$g(x) := \frac{x}{(x^2 + 1)^2}$$

$$g'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3}$$

$$g''(x) = \frac{12x(x^2 - 1)}{(x^2 + 1)^4}$$

$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

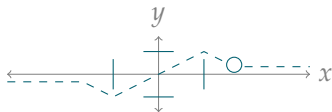
$$f(x) = \frac{x^2 + x}{(x + 1)(x^2 + 1)^2}$$

When $x \neq 1$, $f(x) = g(x)$. So, $f(x)$ looks like $g(x)$ except it has a removable discontinuity (hole) at $x = 1$. Let's graph

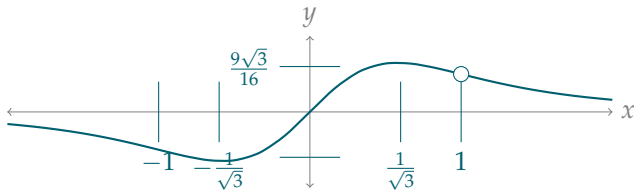
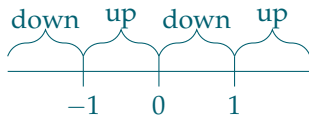
$$g(x) = \frac{x}{(x^2 + 1)^2}$$

- ▶ Domain: all real numbers
- ▶ HA: $y = 0$ on both sides
- ▶ VA: none
- ▶ Intercepts: $(0, 0)$
- ▶ Odd symmetry
- ▶ CP: $x = \pm \frac{1}{\sqrt{3}}$; associated points: $\left(-\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{16}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{16}\right)$

- Increasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$



- g'' is given (nearly) factored, so we can see its sign changes at $x = -1, 0, 1$. Concavity:



LET'S GRAPH

$$f(x) = x(x-1)^{2/3}$$

- $f'(x) = \frac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \frac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

LET'S GRAPH

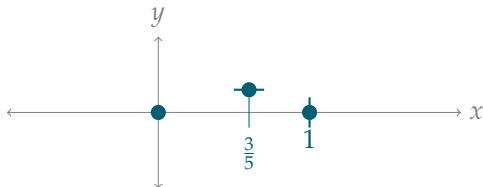
$$f(x) = x(x-1)^{2/3}$$

- $f'(x) = \frac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \frac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

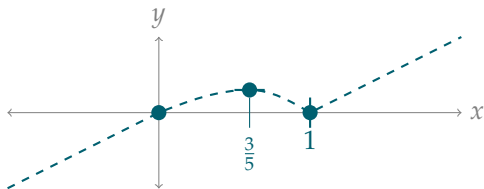
► $f(3/5) \approx 0.3$

► $f(6/5) \approx 0.4$

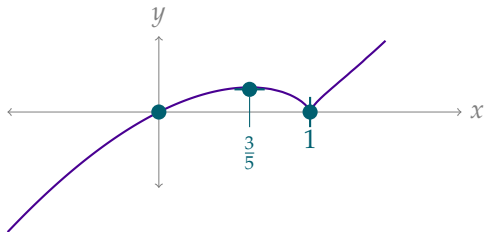
- ▶ Domain: all reals
- ▶ VA: none
- ▶ HA: none
- ▶ Intercepts: $(0, 0)$, $(1, 0)$
- ▶ Symmetry: not even, not odd, not periodic
- ▶ CP: $x = \frac{3}{5}$ ($y \approx 0.3$); SP $x = 1$ ($y = 0$).
Near the SP, first deriv is very large, so function looks vertical.



- First derivative changes sign at $x = \frac{3}{5}$ and $x = 1$. Function is increasing on $(-\infty, \frac{3}{5}) \cup (1, \infty)$ and decreasing on $(\frac{3}{5}, 1)$.



- Second derivative changes sign at $x = \frac{5}{6}$. (Note the denominator has an even power). Function is concave down on $(-\infty, \frac{5}{6})$ and concave up on $(\frac{5}{6}, \infty)$.



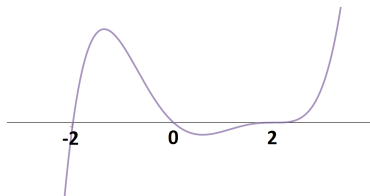
Ch 3.6 Review: matching

MATCH THE FUNCTION TO ITS GRAPH

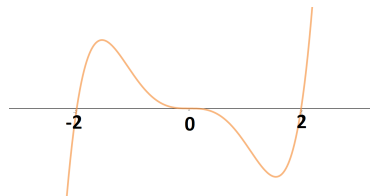
A. $f(x) = x^3(x+2)(x-2) = x^5 - 4x^3$

B. $f(x) = x(x+2)^3(x-2) = x^5 + 4x^4 - 16x^2 - 16x$

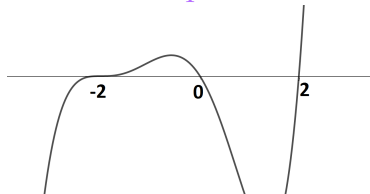
C. $f(x) = x(x+2)(x-2)^3 = x^5 - 4x^4 + 16x^2 - 16x$



I



III



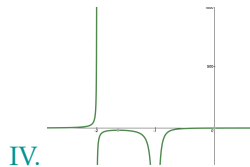
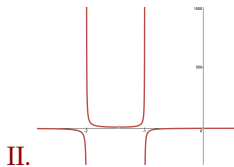
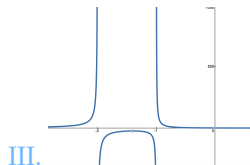
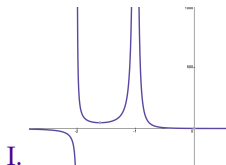
II

A. $f(x) = \frac{x-1}{(x+1)(x+2)}$

B. $f(x) = \frac{(x-1)^2}{(x+1)(x+2)}$

C. $f(x) = \frac{x-1}{(x+1)^2(x+2)}$

D. $f(x) = \frac{(x-1)^2}{(x+1)^2(x+2)}$



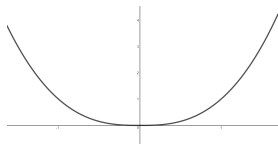
MATCH THE FUNCTION TO ITS GRAPH

A. $f(x) = |x|^e$

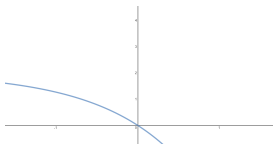
B. $f(x) = e^{|x|}$

C. $f(x) = e^{x^2}$

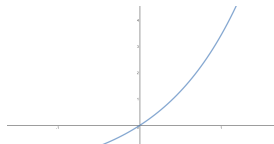
D. $f(x) = e^{x^4-x}$



I



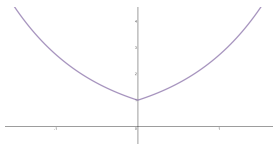
II



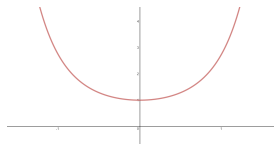
III



IV



V



VI

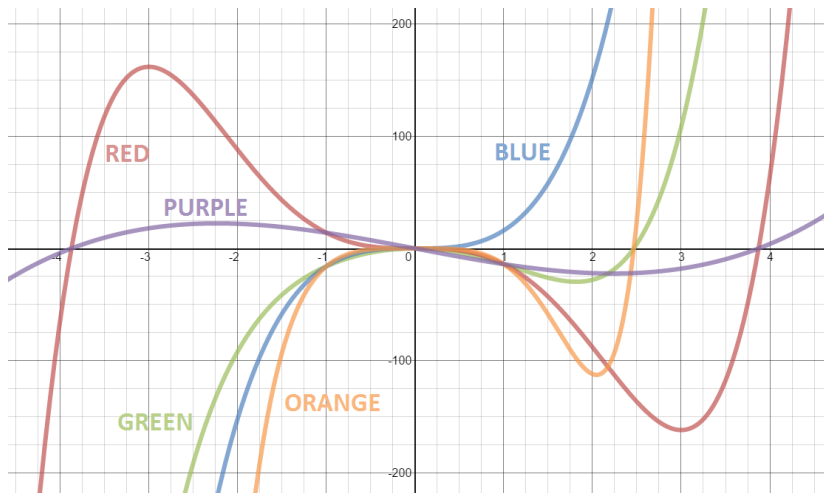
A. $f(x) = x^5 + 15x^3$

B. $f(x) = x^5 - 15x^3$

C. $f(x) = x^5 - 15x^2$

D. $f(x) = x^3 - 15x$

E. $f(x) = x^7 - 15x^4$



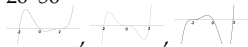
Included Work



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screenshots of graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator> (accessed 13 November 2015), 163



screenshots of graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator>, (accessed 13 November 2015), 165



screenshots from graphs generated using Desmos Graphing Calculator <https://www.desmos.com/calculator> (accessed 16 July 2021), 164



screenshot from graphs generated using Desmos Graphing Calculator
<https://www.desmos.com/calculator>, with text added (accessed 13 November 2015), 166



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151–154