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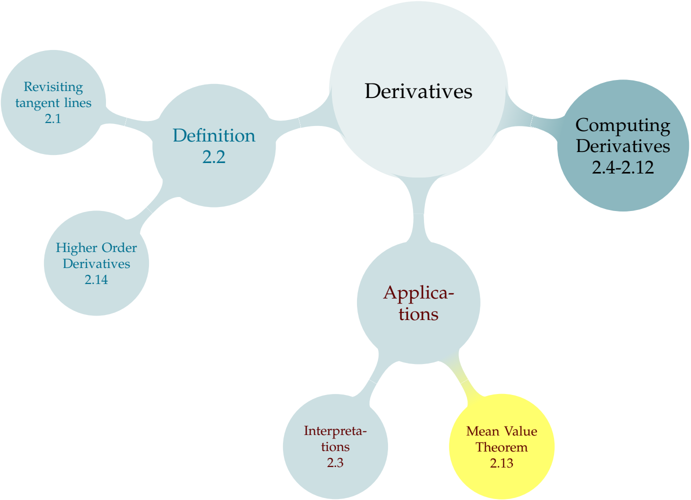
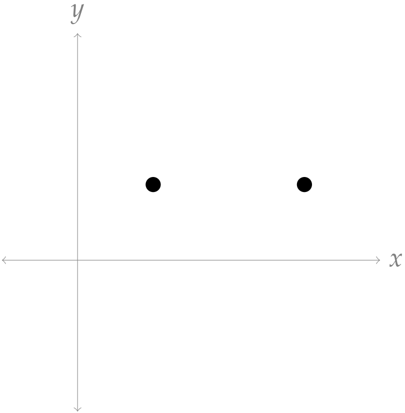


Rolle's Theorem



Rolle's Theorem – Theorem 2.13.1

Let a and b be real numbers, with $a < b$. And let f be a function with the properties:

-
-
- and $f(a) = f(b)$.

Then there exists a number c with $a < c < b$ such that

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Let a and b be real numbers, with $a < b$. And let f be a function with the properties:

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Let a and b be real numbers, with $a < b$. And let f be a function with the properties:

- $f(x)$ is continuous for every x with $a \leq x \leq b$;
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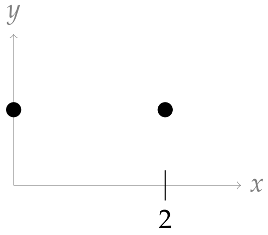
Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Example: Let $f(x) = x^3 - 2x^2 + 1$, and observe $f(2) = f(0) = 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere.



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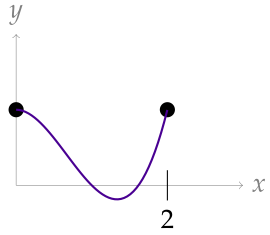

$$\begin{aligned} 0 = f'(x) &= 3x^2 - 4x \\ &= x(3x - 4) \end{aligned}$$

$$x = 0 \text{ and } \boxed{x = \frac{4}{3}}$$

$$f'\left(\frac{4}{3}\right) = 0$$

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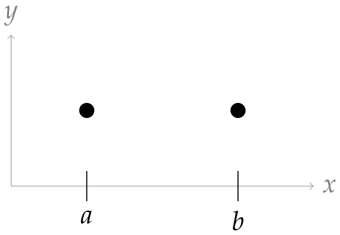

Suppose $a < b$ and $f(a) = f(b)$, $f(x)$ is continuous over $[a, b]$, and $f(x)$ is differentiable over (a, b).

How many different values of x between a and b have $f'(x) = 0$?

A. 0 or 1

B. 1

C. 0, 1, or more

D. 1 or more

E. I'm not sure

Q

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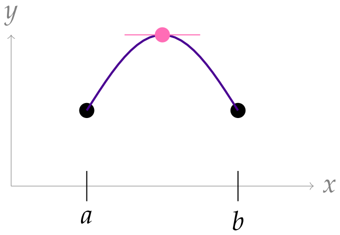

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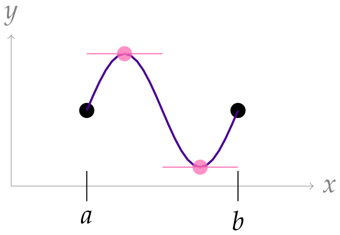

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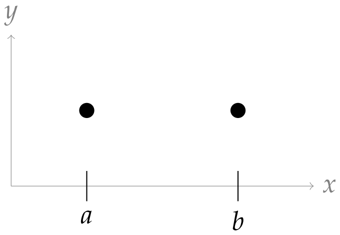

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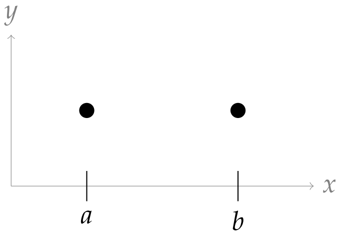

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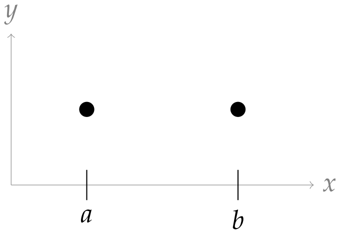

Suppose $a < b$ and $f(a) = f(b)$, $f(x)$ is continuous over $[a, b]$, and $f(x)$ is differentiable over (a, b).

Can f have an infinite number of points where $f'(x) = 0$ between a and b?

A. Sure! 

B. No way! 

C. Only if a and b are infinitely far apart 

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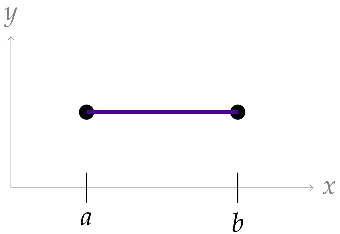

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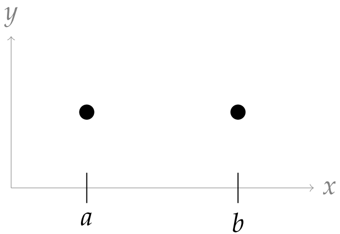

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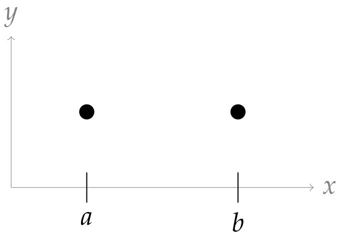

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Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f'(x)$ have?

A. precisely six

B. precisely seven

C. at most seven

D. at least six

Q

ans

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Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x)$ is also continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f''(x)$ have?

A. precisely six
B. precisely five
C. at most five
D. at least five

Q

ans

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Suppose $f(x)$ is continuous and differentiable for all real numbers, and there are precisely three places where $f'(x) = 0$. How many distinct roots does $f(x)$ have?

A. at most three

B. at most four

C. at least three

D. at least four

Q

ans

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Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x)=0$ for precisely three values of x. How many distinct values x exist with $f(x)=17$?

A. at most three

B. at most four

C. at least three

D. at least four

Q

ans

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x) = 0$ for precisely three values of x. How many distinct values x exist with $f(x) = 17$?

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C. at least three

D. at least four

Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

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Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How many roots does $f(x)$ have?

2 or more → $f'(c) = 0$ somewhere

0 or 1 → anything can happen

Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

Note that $f(x)$ is continuous and differentiable over all real numbers. So, by Rolle's Theorem, if it has two roots, then $f'(x) = 0$ for some x.

$f'(x) = 3x^2 + 1$, and this is always at least one, so it's never zero. Therefore, by Rolle's Theorem, $f(x)$ does not have two roots; so it has at most one.

$$0 \qquad 1 \qquad \cancel{2} \qquad \cancel{3} \qquad \cancel{4} \qquad \cancel{5} \qquad \cdots$$

Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

Logical Structure:

- If A is true, then B is true.
- B is false.
- Therefore, A is false.

- If $f(x)$ has two (or more) roots, then $f'(x)$ has a root.
- $f'(x)$ does not have a root.
- Therefore, $f(x)$ does not have two (or more) roots.

A

Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How would you show that $f(x)$ has precisely one real root?

Q

Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How would you show that $f(x)$ has precisely one real root?

We know it has 0 or 1 root.

$$\boxed{0 \qquad 1} \qquad \cancel{2} \qquad \cancel{3} \qquad \cancel{4} \qquad \cancel{5} \qquad \cdots$$

We need to show it has "not zero" roots. So, we would find a root.

APPLICATIONS OF ROLLE'S THEOREM

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How would you show that $f(x)$ has precisely one real root?

Inconveniently, it's hard to actually solve $0 = x^3 + x - 1$. So, we use the IVT (Section 1.6).

Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.

Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

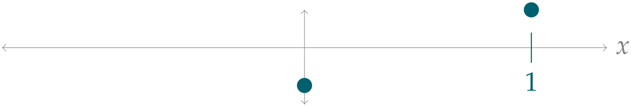
How would you show that $f(x)$ has precisely one real root?

Inconveniently, it's hard to actually solve $0 = x^3 + x - 1$. So, we use the IVT (Section 1.6).

Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.

Since $f(1) > 0$ and $f(0) < 0$, and $f(x)$ is a continuous function, by the IVT $f(x)$ has a root (between 0 and 1).



Use Rolle's Theorem to show that the function
$f(x) = \frac{1}{3}x^3 + 3x^2 + 9x - 3$ has at most two distinct real roots.

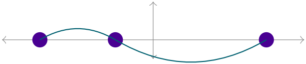
Q

ans

Use Rolle's Theorem to show that the function
$f(x) = \frac{1}{3}x^3 + 3x^2 + 9x - 3$ has at most two distinct real roots.

Again we use the structure:

- If $f(x)$ has three distinct roots, then $f'(x)$ has two (or more) distinct roots.



- $f'(x)$ does not have two (or more) distinct roots.
- Therefore, $f(x)$ does not have three distinct roots.

So, all we need to do is make sure the conditions of Rolle's Theorem are satisfied, and show that $f'(x)$ does not have two (or more) distinct roots.

Use Rolle's Theorem to show that the function $f(x) = \frac{1}{3}x^3 + 3x^2 + 9x - 3$ has at most two distinct real roots.

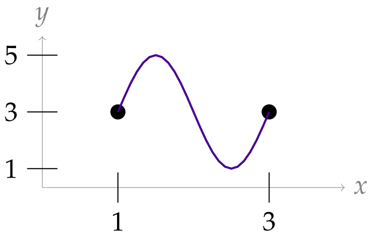
Since $f(x)$ is continuous and differentiable over all real numbers, the conditions of Rolle's Theorem are satisfied.

$f'(x) = x^2 + 6x + 9 = (x+3)^2$, which only has ONE root, namely $x = -3$.

Therefore, $f'(x)$ does not have two distinct roots, so $f(x)$ does not have three distinct roots.

So, $f(x)$ has at most two distinct roots.

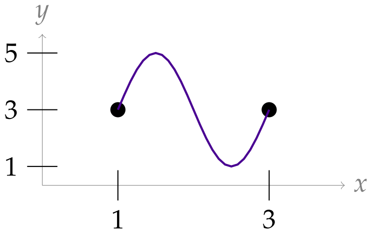
Average Rate of Change



What is the average rate of change of $f(x)$ from $x = 1$ to $x = 3$?

A. 0
B. 1
C. 2
D. 4
E. I'm not sure

Average Rate of Change



$$\frac{\Delta y}{\Delta x} = \frac{3-3}{3-1} = \frac{0}{2} = 0$$

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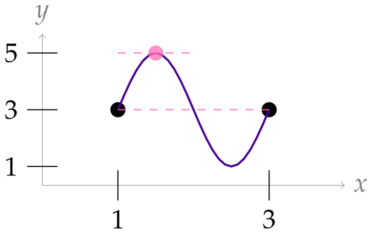

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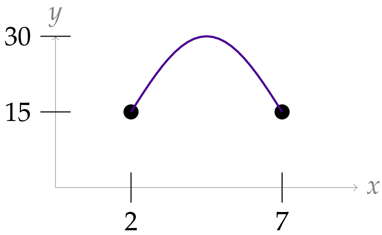


What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

A. 0

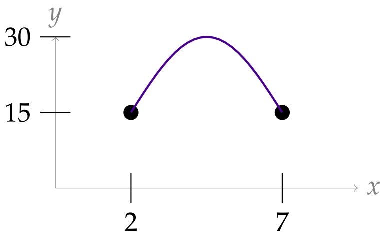
B. 3

C. 5

D. 15

E. I'm not sure

Average Rate of Change



$$\frac{\Delta y}{\Delta x} = \frac{15-15}{7-2} = \frac{0}{5} = 0$$

What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

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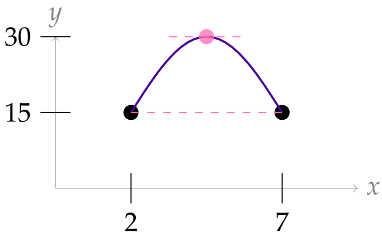
Average Rate of Change



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What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

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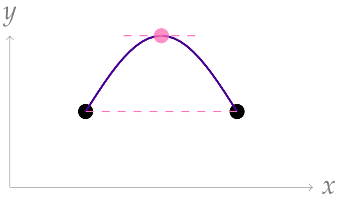
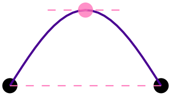
Rolle's Theorem and Average Rate of Change

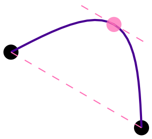
Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on the interval (a, b), and $f(a) = f(b)$. Then there exists a number c strictly between a and b such that

$$f'(c) = 0 = \frac{f(b) - f(a)}{b - a}.$$

So there exists a point where the derivative is the same as the average rate of change.

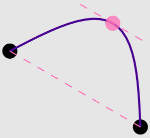

y
x





Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that:



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is: there is some point c in (a, b) where the instantaneous rate of change of the function is equal to the average rate of change of the function on the interval $[a, b]$.

Suppose you are driving along a long, straight highway with no shortcuts. The speed limit is 100 kph. A police officer notices your car going 90 kph, and uploads your plate and the time they saw you to their database. 150 km down this same straight road, 75 minutes later, another police officer notices your car going 85kph, and uploads your plates to the database. Then they pull you over, and give you a speeding ticket. Why were they justified?

You travelled 150 km in 75 minutes. Since a moving car has a position that is continuous and differentiable, the MVT tells us that at some point, your instantaneous velocity was $\frac{150}{75}$ kilometers per minute, which works out to $\frac{150 \cdot 60}{75} = 120$ kph. So even though you weren't speeding when the officers saw you, you were definitely speeding some time in between.

Alternately, if you were going at most 100kph, then you would need at least 90 minutes to travel 150 kilometers.

According to this website, Canada geese may fly 1500 miles in a single day under favorable conditions. It also says their top speed is around 70mph. Does this seem like a typo? (If it contradicts the Mean Value Theorem, it's probably a typo.)

We can assume that the position of a goose is continuous and differentiable. Then the MVT tells us that a goose that travels 1500 miles in a day (24 hours) achieves, at some instant, a speed of $\frac{1500}{24}$ mph. Since $\frac{1500}{24} = 62.5$, these two facts seem compatible (and amazing!).

The record for fastest wheel-driven land speed is around 700 kph. [1] However, non-wheel driven cars (such as those powered by jet engines) have achieved higher speeds. [2]

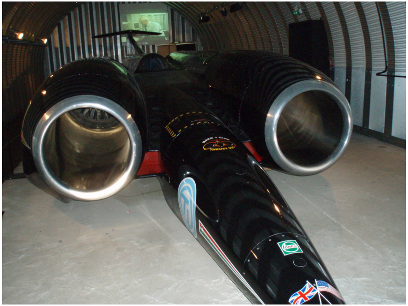
Suppose a driver of a jet-powered car starts a 10km race at 12:00, and finishes at 12:01. Did they beat 700kph?

[1] (at time of writing) George Poteet, https://en.wikipedia.org/wiki/Wheel-driven_land_speed_record

[2] https://en.wikipedia.org/wiki/Land_speed_record

ans

Maybe, but not necessarily. We are only guaranteed by the MVT that at some point they reached the following speed: $\frac{10}{(1/60)} = 600$ kph.



Suppose you want to download a file that is 3000 MB (slightly under 3GB). Your internet provider guarantees you that your download speeds will always be between 1 MBPS (MB per second) and 5 MBPS (because you bought the cheap plan). Using the Mean Value Theorem, give an upper and lower bound for how long the download can take (assuming your providers aren't lying, and your device is performing adequately).

We assume the download is continuous and differentiable, so we can use the MVT.
Let T be the time (in seconds) the download takes. The MVT tells us that at some point, our speed was exactly $\frac{3000}{T}$, so it must be true that

$$1 \leq \frac{3000}{T} \leq 5$$

So, $\frac{3000}{5} \leq T \leq 3000$. That is, T is between 600 and 3000 seconds, or between 10 and 50 minutes.

Suppose $1 \leq f'(t) \leq 5$ for all values of t, and $f(0) = 0$. What are the possible solutions to $f(t) = 3000$?

Q

Since f is continuous and differentiable, we can use the MVT.

$$\frac{f(t)-f(0)}{t-0}=\frac{3000}{t}=f'(c)$$

for some value c between 0 and t.
So,

$$1 \leq \frac{3000}{t} \leq 5$$

hence

$$600 \leq t \leq 3000$$

Corollary to the MVT

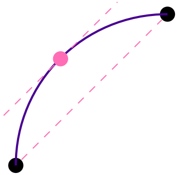
Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = 0$ for all x in (a, b), then

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a,b]$ and differentiable over (a,b).

If $f'(x) = 0$ for all x in (a,b), then $f(x)$ is constant in that interval. That is, $f(c) = f(d)$ for all c,d in $[a,b]$.



If $f(c) \neq f(d)$, then $\frac{f(d)-f(c)}{d-c} \neq 0$, so $f'(e) \neq 0$ for some e.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

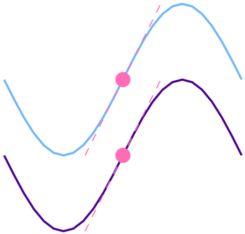
If $f'(x) = g'(x)$ for all x in (a, b), then

Q

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) = g(x) + A$ for some constant value A.



Define a new function $k(x) = f(x) - g(x)$. Then $k'(x) = 0$ everywhere, so (by the last corollary) $k(x) = A$ for some constant A.

A

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) > 0$ for all x in (a, b), then

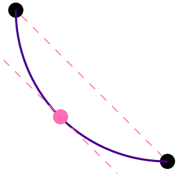
Q

ans

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) > 0$ for all x in (a, b), then $f(x)$ is increasing. That is, $f(c) < f(d)$ for all $c < d$ in $[a, b]$.



If $f(c) > f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(negative)}}{\text{(positive)}} < 0$. Then $f'(e) < 0$ for some e between c and d.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

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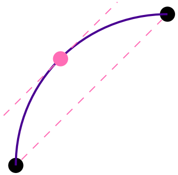
Q

ans

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) < 0$ for all x in (a, b), then $f(x)$ is decreasing. That is, $f(d) < f(c)$ for all $c < d$ in $[a, b]$.



If $f(c) < f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(positive)}}{\text{(positive)}} > 0$. Then $f'(e) > 0$ for some e between c and d.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

WARNING: The MVT has two hypotheses.

- $f(x)$ has to be continuous on $[a, b]$.
- $f(x)$ has to be differentiable on (a, b).

If either of these hypotheses are violated, the conclusion of the MVT can fail. Here are two examples.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

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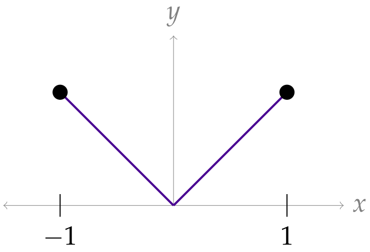
Example: Let $a = -1$, $b = 1$ and $f(x) = |x|$.



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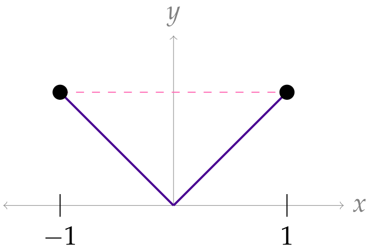

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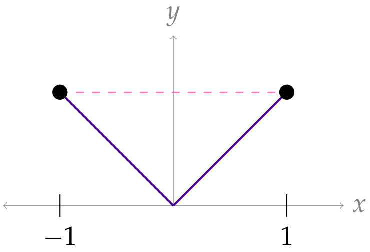

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

$f'(x)$ is never $0 = \frac{f(b)-f(a)}{b-a}$.

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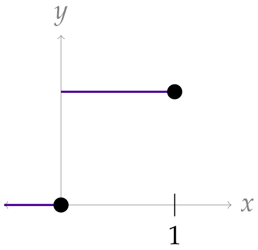

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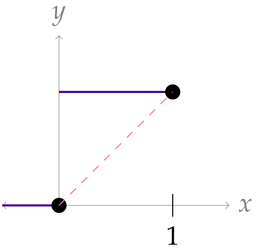

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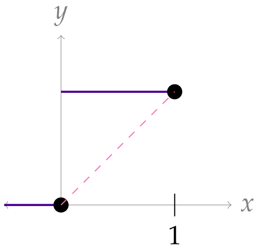

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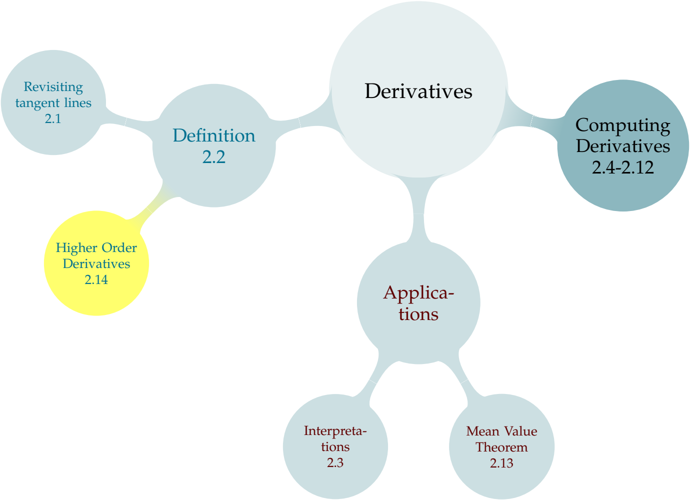

Higher Order Derivatives

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}[x^5 - 2x^2 + 3]\right]$

ans

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$$=20x^3-4$$

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Notation 2.14.1

The derivative of a derivative is called the **second derivative**, written

$$f''(x) \qquad \text{or} \qquad \frac{d^2y}{dx^2}(x)$$

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Similarly, the derivative of a second derivative is a third derivative, etc.

Notation 2.14.1

- $f''(x)$ and $f^{(2)}(x)$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$ all mean $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$
- $f'''(x)$ and $f^{(3)}(x)$ and $\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(x)$ all mean $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\right)$
- $f^{(4)}(x)$ and $\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}(x)$ both mean $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\right)\right)$
- and so on.

Typical Example: Acceleration

- Velocity: rate of change of position

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The position of an object at time t is given by $s(t) = t(5-t)$. *Time is measured in seconds, and position is measured in metres.*

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$? Include units.
3. What is the acceleration of the object when $t = 1$? Include units.

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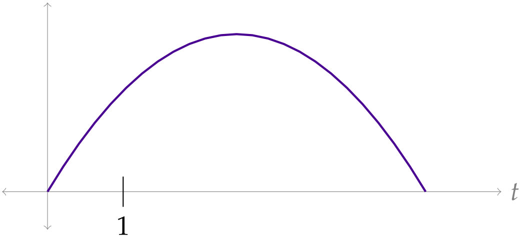

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$$s(t) = t(5-t) = 5t - t^2$$
$$s'(t) = 5 - 2t$$
$$s'(1) = 5 - 2(1) = 3 = \text{vel}$$

Units of velocity: $\frac{\Delta s}{\Delta t} = \frac{m}{s}$ Units of acceleration: $\frac{\Delta s'}{\Delta t} = \frac{m/s}{s} = \frac{m}{s^2}$

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Units of velocity: $\frac{\Delta s}{\Delta t} = \frac{m}{s}$ Units of acceleration: $\frac{\Delta s'}{\Delta t} = \frac{m/s}{s} = \frac{m}{s^2}$

Concept Check

True or False: If $f'(1) = 18$, then $f''(1) = 0$, since the $\frac{\mathrm{d}}{\mathrm{d}x}\{18\} = 0$.

Q

ans

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False: for example, $f(x) = 9x^2$.

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Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$
B. $f'(0) = 0$
C. $f''(0) = 0$
D. $f'''(0) = 0$
E. $f^{(4)}(0) = 0$

Q

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Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$ $\quad f(x) = ax^2 + bx + c$
B. $f'(0) = 0$ $\quad f'(x) = 2ax + b$
C. $f''(0) = 0$ $\quad f''(x) = 2a$
D. $f'''(0) = 0$ $\quad f'''(x) = 0$ ✓
E. $f^{(4)}(0) = 0$ $\quad f^{(4)}(x) = 0$ ✓

Concept Check

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Which of the following is always true of a CUBIC polynomial $f(x)$?

A. $f(0) = 0$ $\quad f(x) = ax^3 + bx^2 + cx + d$
B. $f'(0) = 0$ $\quad f'(x) = 3ax^2 + 2bx + c$
C. $f''(0) = 0$ $\quad f''(x) = 6ax + 2b$
D. $f'''(0) = 0$ $\quad f'''(x) = 6a$
E. $f^{(4)}(0) = 0$ $\quad f^{(4)}(x) = 0$ ✓

A

Implicit Differentiation

Suppose $y(x)$ is a function such that

$$y(x) = y^3x + x^2 - 1$$

Find $y''(x)$ at the point $(-2, 1)$.

Implicit Differentiation

Suppose $y(x)$ is a function such that

$$y(x) = y^3x + x^2 - 1$$

Find $y''(x)$ at the point $(-2, 1)$. We start by differentiating both sides of the function. Remember that y is a function, not a variable.

$$\begin{aligned} y(x) &= y(x)^3x + x^2 + 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x) &\stackrel{\text{prod}}{=} y(x)^3 + 3xy(x)^2\frac{\mathrm{d}y}{\mathrm{d}x}(x) + 2x \qquad (*) \end{aligned}$$

Let's differentiate both sides again. Remember we have a rule for the product of three functions.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3\left(y^2\frac{\mathrm{d}y}{\mathrm{d}x} + x \cdot 2y\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + xy^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + 2 \qquad (**)$$

When $x=-2$ and $y=1$, using $(*)$, we find

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(-2,1)} = 1^3 + 3(-2)(1^2)\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(-2,1)} + 2(-2) = -3 - 6\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(-2,1)}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(-2,1)} = -\frac{3}{7}$$

We set $x=-2$, $y=1$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{7}$ in equation $(**)$. Now by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, we actually mean $\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{(-2,1)}$, but to avoid clutter we don't write it that way until the end.

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 3(1)\left(-\frac{3}{7}\right) + 3\left((1)\left(-\frac{3}{7}\right) + (-2)\cdot 2(1)\left(-\frac{3}{7}\right)\cdot\left(-\frac{3}{7}\right) + (-2)(1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + 2\\
&= -\frac{9}{7} + 3\left(-\frac{3}{7} - \frac{36}{49} - 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + 2\\
&= \left(2 - \frac{18}{7} - \frac{108}{49}\right) - 6\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\\
7\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{136}{7^2}\\
\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{(-2,1)} &= -\frac{136}{7^3}
\end{aligned}$$

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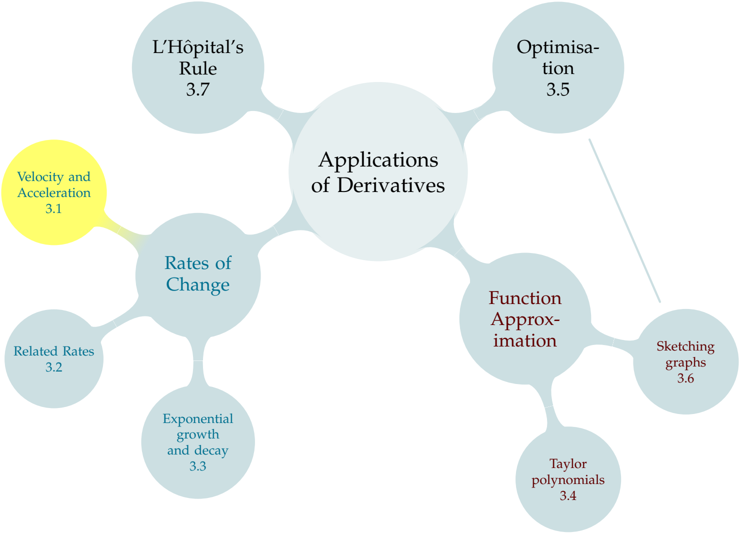


The position of a unicyclist along a tightrope is given by

$$s(t) = t^3 - 3t^2 - 9t + 10$$

where $s(t)$ gives the distance in meters to the right of the middle of the tightrope, and t is measured in seconds, $-2 \leq t \leq 4$.

Describe the unicyclist's motion: when they are moving right or left; when they are moving fastest and slowest; and how far to the right or left of centre they travel.

Q

The position of a unicyclist along a tightrope is given by

$$s(t) = t^3 - 3t^2 - 9t + 10$$

The velocity of the unicyclist is given by

$$s'(t) = 3t^2 - 6t - 9 = 3(t-3)(t+1)$$

Let's decide where this is positive and negative. It's a parabola pointing up, with zeroes at $t = -1$ and $t = 3$.

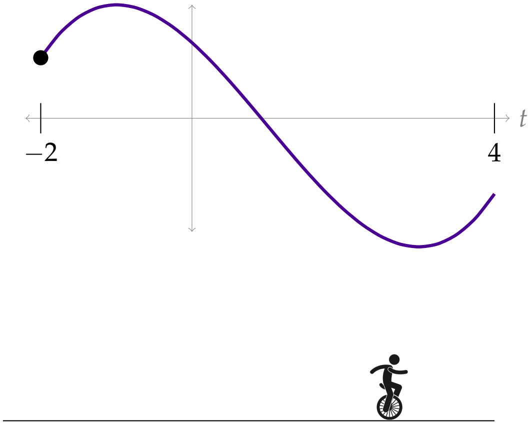
- So, $s'(t)$ is positive when $-2 \leq t < -1$ (so that $t - 3 < 0$ and $t + 1 < 0$) and $3 < t \leq 4$ (so that $t - 3 > 0$ and $t + 1 > 0$), so these are the times when the unicyclist is moving right.
- They are moving left when $-1 < t < 3$ (so that $t - 3 < 0$ and $t + 1 > 0$).

The fastest leftward speed corresponds to the minimum of $s'(t)$. This occurs at the "bottom" of the parabola; to find where this bottom is, we can either remember that parabolas are symmetric (so it occurs halfway between -1 and 3) or we can notice that the minimum occurs when $s'(t)$ is not decreasing any more, but not yet increasing: when $s''(t) = 0$. Since $s''(t) = 6t - 6$, this happens at $t = 1$, and at $t = 1$, $s'(t) = -12$, so the unicyclist's fastest leftward speed is 12 m/s.
Their fastest rightward speed will happen at the time farthest from $t = 1$, the bottom of the parabola. Their rightward speed is 15m/s when $t = -2$ and $t = 4$, and this is their fastest rightward speed.

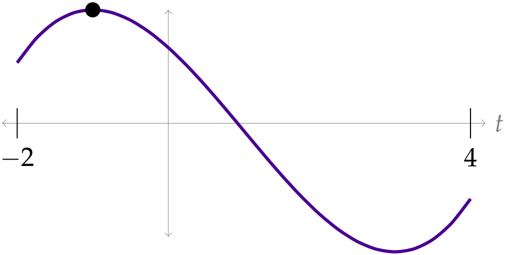
They are moving the slowest when they switch from left to right motion; at these times, their instantaneous rate of change is zero, and these occur at $t = -1$ and $t = 3$.

It remains to determine how far left and right the unicyclist travels.

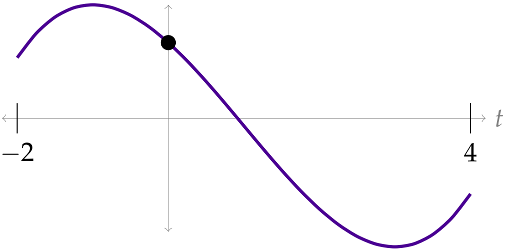
- $s(-2) = 8$, so they start 8 meters to the right of centre;
- they continue travelling rightward until $t = -1$. $s(-1) = 15$, so when they turn, they are 15 meters to the right of centre.
- Then they go left until $t = 3$. $s(3) = -17$, so they travel leftward until they are 17 meters to the left of centre.
- Then they turn again, and $s(4) = -10$, so they end up 10 meters to the left of centre.



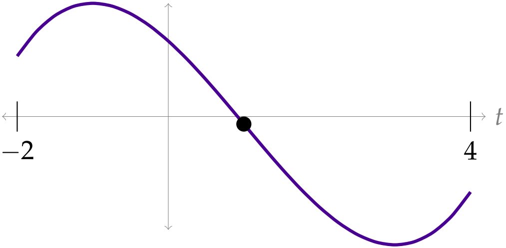
$t = -2$



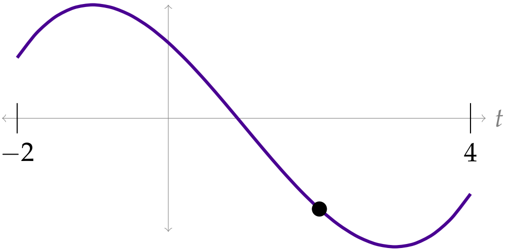
$$t = -1$$



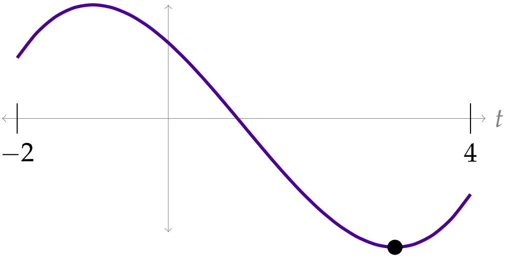
$t = 0$



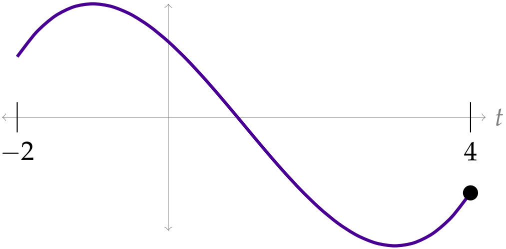
$t = 1$



$t = 2$



$t = 3$



$t = 4$

A solution in a beaker is undergoing a chemical reaction, and its temperature (in degrees Celsius) at t seconds from noon is given by

$$T(t) = t^3 + 3t^2 + 4t - 273$$

1. When is the reaction increasing the temperature, and when is it decreasing the temperature?
2. What is the slowest rate of change of the temperature?

Q

ans

A solution in a beaker is undergoing a chemical reaction, and its temperature (in degrees Celsius) at t seconds from noon is given by

$$T(t) = t^3 + 3t^2 + 4t - 273$$

Q

ans

1. The temperature is always increasing. To see that, remember that a positive derivative means an increasing temperature, and a negative derivative means a decreasing temperature.

$T'(t) = 3t^2 + 6t + 4$. If we try to set $T'(t) = 0$, using the quadratic formula, we find the roots are $t = \dfrac{-6 \pm \sqrt{36 - 4(3)(4)}}{6}$, which are not real numbers. So, $T'(t)$ is never zero. Since $T'(t)$ is a parabola pointing up, that means it is always positive, so the temperature is always increasing.

2. To find when it is increasing the slowest, we need to find the minimum value of its rate of change: the minimum value of $T'(t)$. Since $T'(t)$ is a parabola pointing up, its minimum occurs when it's done decreasing but not yet increasing: when the derivative of $T'(t)$ is zero.

$T''(t) = 6t + 6$, so its derivative is zero at $t = -1$. Then the temperature is changing at $T'(-1) = 3(-1)^2 + 6(-1) + 4 = 1$ degree per second.

You roll a magnetic marble across the floor towards a metal fridge, giving it an initial velocity of 50 centimetres per second. The magnet imparts an acceleration on the magnet of 1 meter per second per second. If the magnet hits the fridge after 2 seconds, how far away was it when you rolled it?

Q

We want to know position and velocity, and we know acceleration. Let's call velocity $v(t)$, where t is measured in seconds and we start pushing the marble at $t = 0$. Let's call position $s(t)$. We need some frame of reference for $s(t)$, so let's impose an axis so that $s(0) := 0$.

Since acceleration is the derivative of velocity, we know $v'(t) = 1$. Anything with a constant slope is a straight line, so $v(t) = t + v_0$ for some constant v_0. Since we start pushing the ball with velocity 1/2 m/s, we must have $v(0) = 1/2$, so

$$v(t) = t + 1/2.$$

Now consider $s(t)$. Note $s'(t) = v(t) = t + 1/2$. We can guess what function $s(t)$ gives us such a derivative. The 1/2 part we can get from $1/2t + s_0$ for some constant s_0. We see that $\frac{d}{dt}[\frac{1}{2}t^2] = t$, so $s(t) = \frac{1}{2}t^2 + \frac{1}{2}t + s_0$. Since we defined $s(0) = 0$, we need $s_0 = 0$ so that

$$s(t) = \frac{1}{2}t^2 + \frac{1}{2}t.$$

Since the ball hits the fridge after 2 seconds, it moved from $s(0) = 0$ to position $s(2) = \frac{1}{2}2^2 + \frac{1}{2}(2) = 3$. So, the fridge is three meters from the initial position of the magnet.

A

The deceleration of a particular car while braking is 9 m/s^2.

1. Suppose the car needs to stop in 30m. How fast can it be going?
(Give your answer in kph.)

2. Suppose the car needs to stop in 50m. How fast can it be going?
(Give your answer in kph.)

Q

ans

The deceleration of a particular car while braking is 9 m/s^2.

1. Suppose the car needs to stop in 30m. How fast can it be going?

Q

The deceleration of a particular car while braking is 9 m/s^2.

1. Suppose the car needs to stop in 30m. How fast can it be going?

Suppose the car is traveling at v_0 kph, and brakes at $t = 0$. Then

$$v(t) = v_0 - 9t$$

so it stops at $t_{stop} = \frac{v_0}{9}$. We need an expression for how far it has traveled. Let $s(t)$ be its position at t seconds after braking, with $s(0) = 0$. Recall $s'(t) = v(t) = v_0 - 9t$. So, by guessing,

$$s(t) = v_0 t - \frac{9}{2}t^2.$$

Then when the car stops, it has traveled $s(t_{stop})$ meters.

$$s(t_{stop}) = s\left(\frac{v_0}{9}\right) = v_0\left(\frac{v_0}{9}\right) - \frac{9}{2}\left(\frac{v_0}{9}\right)^2 = \frac{v_0^2}{18}$$

The deceleration of a particular car while braking is 9 m/s^2.

1. Suppose the car needs to stop in 30m. How fast can it be going?

So, to stop in 30 m, we solve

$$\frac{v_0^2}{18} = 30$$

which tells us the car can only travel at most about 23.238 m/s. We convert to kph:

$$23.238\frac{m}{s}\left(\frac{1\text{ km}}{1000m}\right)\left(\frac{3600s}{1\text{hr}}\right) \approx 84\text{kph}$$

2. Suppose the car needs to stop in 50m. How fast can it be going?

Q

ans

2. Suppose the car needs to stop in 50m. How fast can it be going?

To stop in 50 m, we solve

$$\frac{v_0^2}{18} = 50$$

which yields $v_0 = 30m/s$, or

$$30\frac{m}{s}\left(\frac{1\text{ km}}{1000m}\right)\left(\frac{3600s}{1\text{hr}}\right) = 108\text{kph}$$

Suppose your brakes decelerate your car at a constant rate. That is, d meters per second per second, for some constant d.

Suppose your brakes decelerate your car at a constant rate. That is, d meters per second per second, for some constant d.

Is it true that if you double your speed, you double your stopping time?

Q

ans

Is is true that if you double your speed, you double your stopping distance?

You double your stopping time, but *quadruple* your stopping distance.

As in the last example, if you brake at time $t = 0$ from a speed of v_0 m/s, then $v(t) = v_0 - dt$ gives your velocity while decelerating. Then the time you stop is when $v(t) = 0$, and this occurs at $t_{stop} = \frac{v_0}{d}$.

Notice: if we replace v_0 with $2v_0$, then t_{stop} doubles. So doubling your speed does indeed double your stopping time.

Your position while stopping is given by $s(t) = v_0t - \frac{d}{2}t^2$. Your stopping distance is $s(t_{stop}) = s\left(\frac{v_0}{d}\right) = v_0\left(\frac{v_0}{d}\right) - \frac{d}{2}\left(\frac{v_0}{d}\right)^2 = \frac{v_0^2}{2d}$. So if you replace v_0 with $2v_0$, your stopping distance goes up by a factor of 4: it quadruples.

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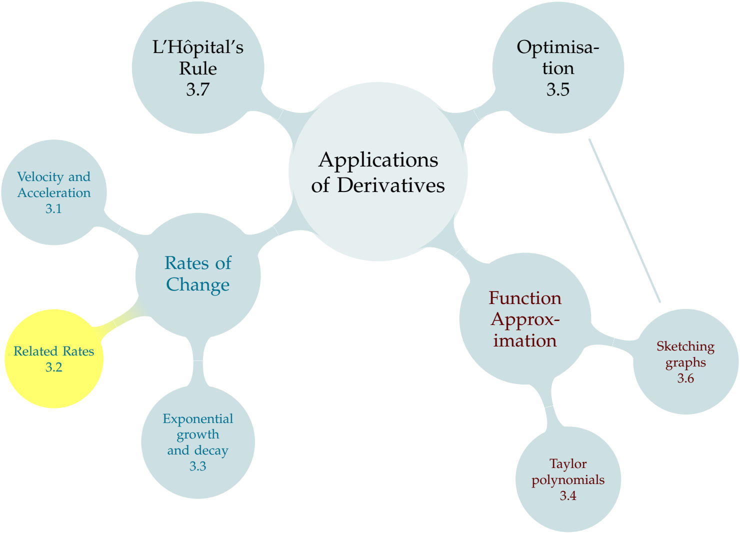


Related Rates - Introduction

"Related rates" problems involve finding the rate of change of one quantity, based on the rate of change of a related quantity.

Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3.$$

If $\frac{dP}{dt}(t) = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{dQ}{dt}$ at that time?

Q

ans

Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3.$$

If $\dfrac{dP}{dt}(t) = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\dfrac{dQ}{dt}$ at that time?

Apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of $3P(t)^2 = 2Q(t)^2 + Q(t) + 3$.

$$6P(t)\frac{dP}{dt}(t) = 4Q(t)\frac{dQ}{dt}(t) + \frac{dQ}{dt}(t)$$

Then when $\dfrac{dP}{dt}(t) = 5$, $P(t) = 1$ and $Q(t) = 0$,

$$6(1)(5) = 4(0)\frac{dQ}{dt} + \frac{dQ}{dt}$$

$$30 = \frac{dQ}{dt}$$

Related rates problems often involve some kind of geometric or trigonometric modeling

A garden hose can pump out a cubic meter of water in about 20 minutes. Suppose you're filling up a rectangular backyard pool, 3 meters wide and 6 meters long, with a garden hose. How fast is the water rising?

We know the rate of change over time of the volume of water, and we want to know the rate of change over time of the height of the water.

Let V be the volume of water in the cell, and let h be the height of water. Then we relate V and h:

$$V = 3 \cdot 6 \cdot h = 18h$$

where V is measured in cubic meters, and h is measured in meters. Then we differentiate both sides with respect to t:

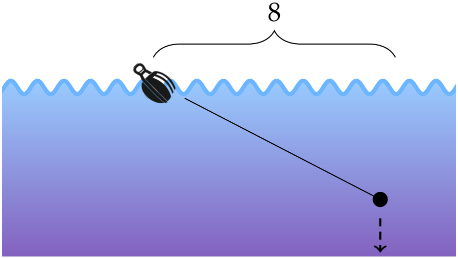
$$\frac{dV}{dt} = 18\frac{dh}{dt}$$

Now, $\frac{1}{20} = 18\frac{dh}{dt}$, so the water level is rising at about $\frac{1}{20 \cdot 18} = \frac{1}{360}$ meters per minute, or something less than a third of a centimeter per minute. It's going to take a long time to fill up the pool.

Solving Related Rates

1. Draw a Picture
2. Write what you know, and what you want to know. Note units.
3. Relate all your relevant variables in one equation.
4. Differentiate both sides (with respect to the appropriate variable!)
5. Solve for what you want.

A weight is attached to a rope, which is attached to a pulley on a boat, at water level. The weight is taken 8 (horizontal) metres from its attachment point on the boat, then dropped in the water.
The weight sinks straight down. The rope stays taught as it is let out at a constant rate of one metre per second, and two seconds have passed. How fast is the weight descending?



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Q

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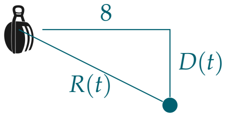

2. Write what we know, and what we want to know.
We know: $\frac{\mathrm{d}R}{\mathrm{d}t} = 1\frac{\mathrm{m}}{\mathrm{s}}$, $t = 2$. We want to know $\left.\frac{\mathrm{d}D}{\mathrm{d}t}\right|_{t=2}$.

3. Relate all relevant variables in one equation.

$$64 + D^2(t) = R^2(t)$$

4. Differentiate with respect to the appropriate variable. With t as our variable, we need to use the chain rule:

$$64 + D^2(t) = R^2(t)$$
$$0 + 2D(t) \cdot \frac{dD}{dt}(t) = 2R(t) \cdot \frac{dR}{dt}(t)$$

5. Solve for what you want.

$$\frac{dD}{dt}(t) = \frac{R(t) \cdot \frac{dR}{dt}(t)}{D(t)}$$
$$D'(2) = \frac{R(2) \cdot R'(2)}{D(2)}$$

$R(0) = 8$ and the rope is being let out at 1 metre per second, so $R(2) = 8 + 2 = 10$. Then, using the Pythagorean Theorem, $D(2) = \sqrt{10^2 - 8^2} = 6$

$$D'(2) = \frac{10 \cdot 1}{6} = \frac{5}{3}$$

So, the weight is sinking at $\frac{5}{3}$ metres per second.

You are pouring water through a funnel with an extremely small hole. The funnel lets water out at 100mL per second, and you are pouring water into the funnel at 300mL per second. The funnel is shaped like a cone with height 20 cm and with the diameter at the top also 20 cm. (Ignore the hole in the bottom.) How fast is the height of the water in the funnel rising when it is 10 cm high?

A cone with radius r and height h has volume $\frac{\pi}{3}r^2h$.

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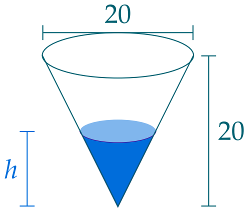
1. Draw a Picture



2. Write what we know, and what we want to know.

The water in the funnel is in the shape of a cone. Let that cone have height $h(t)$ and radius $r(t)$, in cm, and volume $V(t)$, in mL (i.e. cm^3).

We know
$\frac{dV}{dt}(t) = 300 - 100 = 200$ mL/sec.

We want to know $\frac{dh}{dt}(t)$ when $h(t) = 10$.

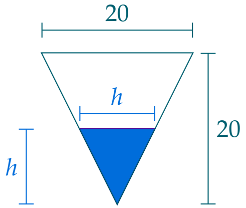
3. Relate all relevant variables in one equation.

The volume of water in the funnel is $V(t) = \frac{\pi}{3}r^2(t) \cdot h(t)$

We're given information about volume, and asked about height. Radius is sort of ... in the way. Since it isn't "relevant" variable, let's figure out how to get rid of it.

A vertical cross-section of a cone is an isosceles triangle. We see that the funnel has a diameter equal to its height. The water makes a similar triangle, so its diameter will also be equal to its height. That is,

$$r(t) = \frac{1}{2}h(t)$$



So $V(t) = \frac{\pi}{3}\left(\frac{h(t)}{2}\right)^2 \cdot h(t)$. Simplifying $V(t) = \frac{\pi}{12}h^3(t)$

4. Differentiate with respect to the appropriate variable, namely t.

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = \frac{\pi}{4}h^2(t) \cdot \frac{\mathrm{d}h}{\mathrm{d}t}(t)$$

5. Solve for what you want.

$$\begin{aligned} \frac{\pi}{4}h^2(t) \cdot \frac{\mathrm{d}h}{\mathrm{d}t}(t) = \frac{\mathrm{d}V}{\mathrm{d}t}(t) &= 200 \\ \frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{800}{\pi h^2(t)} \\ \left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{h=10} &= \frac{800}{\pi(10^2)} = \frac{8}{\pi}\text{cm/sec} \end{aligned}$$

A sprinkler is 3m from a long, straight wall. The sprinkler sprays water in a circle, making three revolutions per minute. Let P be the point on the wall closest to the sprinkler. The water hits the wall at some spot, and that spot moves as the sprinkler rotates. When the spot where the water hits the wall is 1m away from P, how fast is the spot moving horizontally?
(You may assume the water travels from the sprinkler to the wall instantaneously.)

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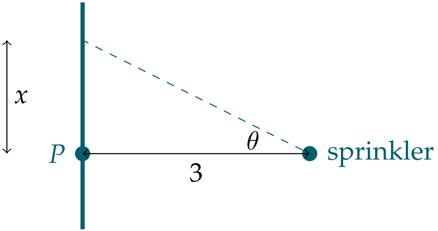


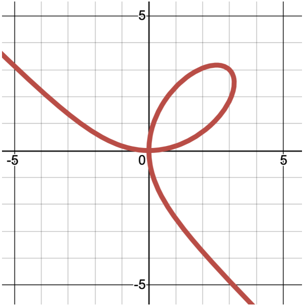
Given the labels in the picture above, we know $\frac{d\theta}{dt} = 3(2\pi)$ radians per minute, and we can relate x to θ by $\tan\left(\theta(t)\right) = \frac{x(t)}{3}$.
Then differentiate both sides with respect to t to get:

$$\sec^2 \theta(t) \cdot \frac{d\theta}{dt}(t) = \frac{1}{3}\frac{dx}{dt}(t)$$

We only need to find $\sec\theta$, which we get from our triangle. When $x = 1$, we have $\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{3^2+1^2}}{3}$, so

$$\frac{dx}{dt} = 3\left(\frac{\sqrt{10}}{3}\right)^2 \cdot 6\pi = 20\pi \text{ meters per minute}$$

A roller coaster has a track shaped in part like the folium of Descartes: $x^3 + y^3 = 6xy$. When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?



Q

ans

A roller coaster has a track shaped in part like the folium of Descartes: $x^3 + y^3 = 6xy$. When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?

We know $\frac{dx}{dt} = -2$ when $x = y = 3$, and we would like to find $\frac{dy}{dt}$ at the same time. The relationship between x and y is given.

$$x(t)^3 + y(t)^3 = 6x(t)y(t)$$

We differentiate.

$$3x(t)^2\frac{dx}{dt}(t) + 3y(t)^2\frac{dy}{dt}(t) = 6\left(x(t)\frac{dy}{dt}(t) + y(t)\frac{dx}{dt}(t)\right)$$

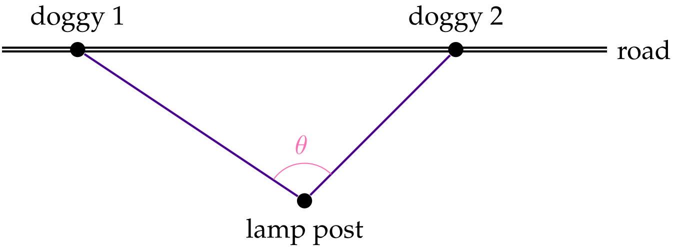
When $x(t) = y(t) = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = -2$,

$$3(9)(-2) + 3(9)\frac{dy}{dt} = 6\left(3\frac{dy}{dt} + 3(-2)\right)$$

$$\frac{dy}{dt} = 2$$

So, the roller coaster is moving 2 units per second in the positive y direction.

Two dogs are tied with elastic leashes to a lamp post that is 2 metres from a straight road. At first, both dogs are on the road, at the closest part of the road to the lamp post. Then, they start running in opposite directions: one dog runs 3 metres per second, and the other runs 2 metres per second. After one second of running, how fast is the angle made by the two leashes increasing?



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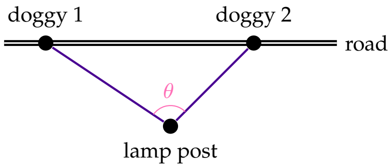

ans

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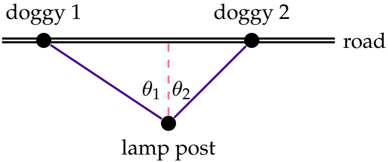


The angle θ made by the leashes is made up of $\theta_1 + \theta_2$, as shown on either side of the dashed line above.

$$\tan\left(\theta_1(t)\right) = \frac{3t}{2} \implies \sec^2\left(\theta_1(t)\right)\cdot\theta_1'(t) = \frac{3}{2} \implies \theta_1'(t) = \frac{3}{2}\cos^2\left(\theta_1(t)\right)$$

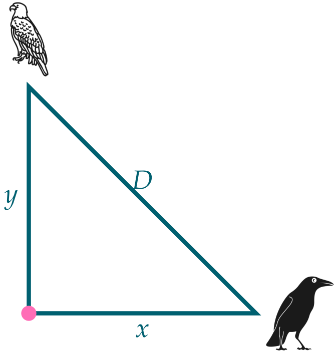
$$\tan\left(\theta_2(t)\right) = \frac{2t}{2} \implies \sec^2\left(\theta_2(t)\right)\cdot\theta_2'(t) = 1 \implies \theta_2'(t) = \cos^2\left(\theta_2(t)\right)$$

At $t = 1$, doggy 1 is three metres away from its starting position and doggy 2 is two metres away, so $\cos\theta_1 = \frac{2}{\sqrt{2^2+3^2}} = \frac{2}{\sqrt{13}}$ and $\cos\theta_2 = \frac{2}{\sqrt{2^2+2^2}} = \frac{1}{\sqrt{2}}$ and

$$\theta_1' = \frac{3}{2}\cos^2\theta_1 = \frac{6}{13} \qquad \theta_2' = \cos^2\theta_1 = \frac{1}{2}$$

So, all together, $\theta' = \theta_1' + \theta_2' = \frac{6}{13} + \frac{1}{2} = \frac{25}{26}$ radians per second.

A crow is one kilometre due east of the math building, heading east at 5 kph. An eagle is two kilometres due north of the math building, heading north at 7kph. How fast is the distance between the two birds increasing at this instant?



We relate all the variables:

$$D(t)^2 = x(t)^2 + y(t)^2$$

Differentiate with respect to time:

$$2D(t)\frac{dD}{dt}(t) = 2x(t)\frac{dx}{dt}(t) + 2y(t)\frac{dy}{dt}(t)$$

At the time of interest $x = 1$, $x' = 5$, $y = 2$, $y' = 7$ and

$$D = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

So $2\sqrt{5}\frac{dD}{dt} = 2(1)(5) + 2(2)(7)$. Then their distance is increasing at $\frac{19}{\sqrt{5}} \approx 8.5$ kph.

A triangle has one side that is 1cm long, and another side that is 2cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.

Q

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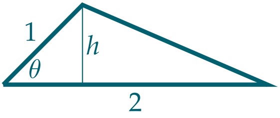
A triangle has one side that is 1cm long, and another side that is 2cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.



$\sin\theta(t) = \frac{opp}{hyp} = \frac{h(t)}{1} = h(t)$, and
$A(t) = \frac{1}{2} b\, h(t)$,

So

$$A(t) = \tfrac{1}{2}(2)\sin\theta(t) = \sin\theta(t)$$

$$\frac{dA}{dt}(t) = \cos\theta(t) \cdot \frac{d\theta}{dt}(t)$$

When $\theta = \pi/4$,

$$\frac{dA}{dt} = \cos\left(\frac{\pi}{4}\right)(1.5) = \frac{3}{2\sqrt{2}}\,\frac{\text{cm}^2}{\text{sec}}$$

Included Work

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