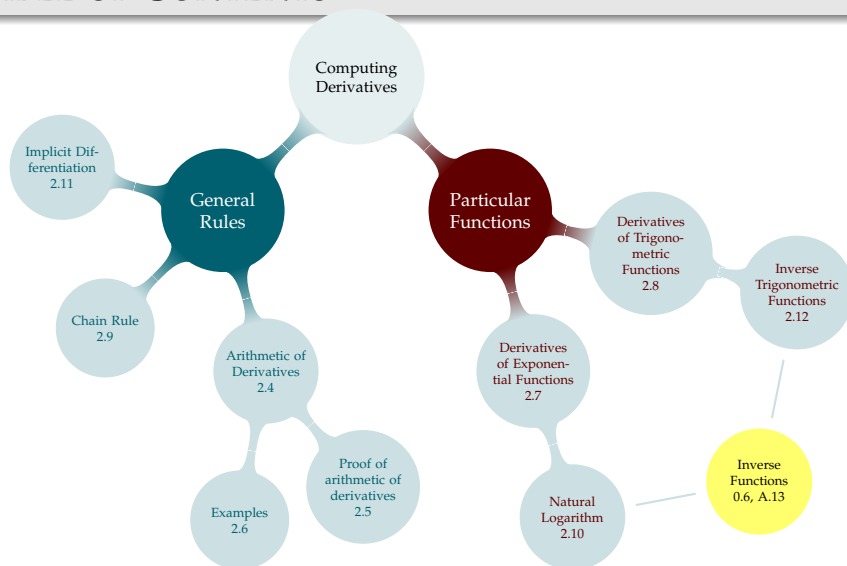


TABLE OF CONTENTS



INVERTIBILITY GAME

- ▶ A function $y = f(x)$ is known to both players
- ▶ **Player A** chooses a secret value x in the domain of $f(x)$
- ▶ **Player A** tells **Player B** what $f(x)$ is
- ▶ **Player B** tries to guess **Player A**'s x -value.

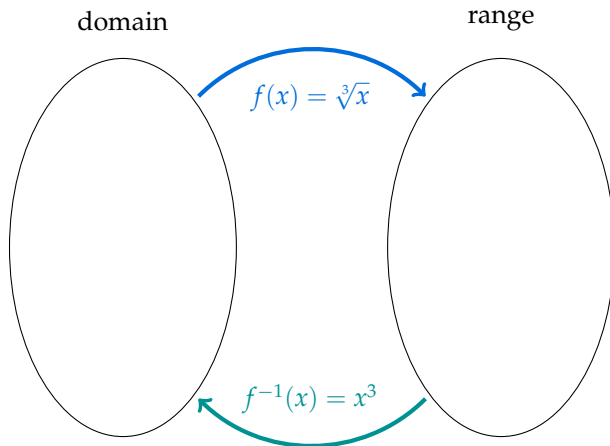
Round 1: $f(x) = 2x$

Round 2: $f(x) = \sqrt[3]{x}$

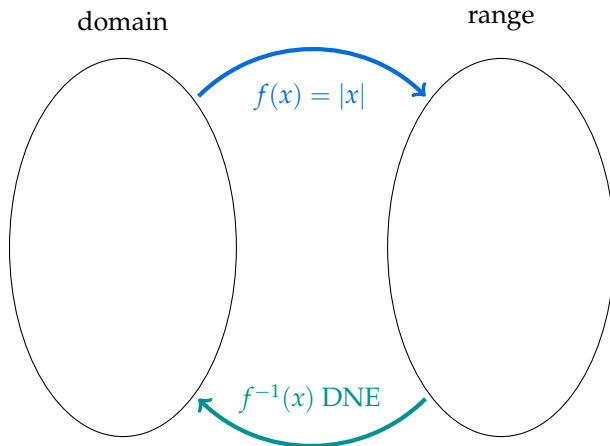
Round 3: $f(x) = |x|$

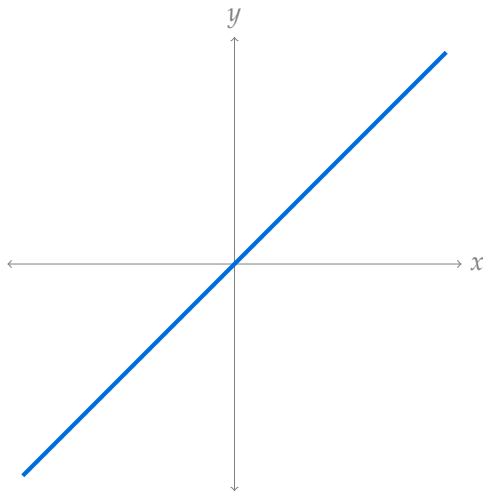
Round 4: $f(x) = \sin x$

FUNCTIONS ARE MAPS



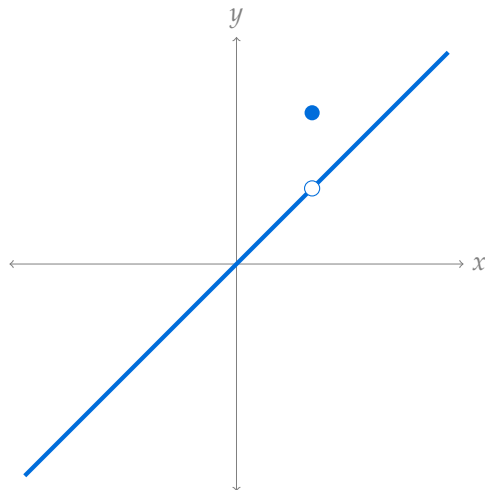
FUNCTIONS ARE MAPS





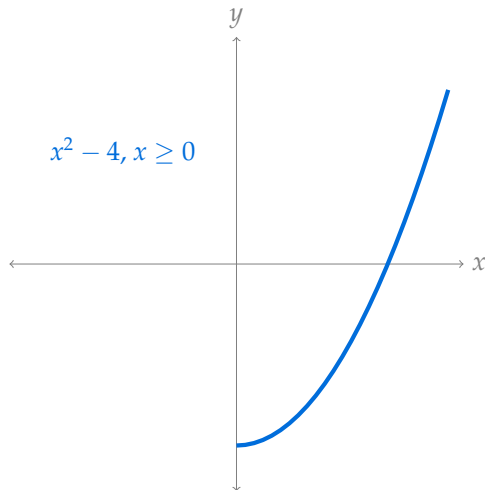
A. invertible

B. not invertible



A. invertible

B. not invertible



A. invertible

B. not invertible

RELATIONSHIP BETWEEN $f(x)$ AND $f^{-1}(x)$

Let f be an invertible function.

What is $f^{-1}(f(x))$?

- A. x
- B. 1
- C. 0
- D. not sure

Invertibility

In order for a function to be invertible, different x values cannot map to the same y value.

We call such a function **one-to-one**, or **injective**.

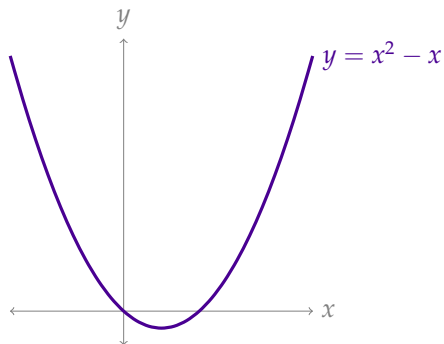
Suppose $f(x) = \sqrt[3]{19 + x^3}$. What is $f^{-1}(3)$? (simplify your answer)

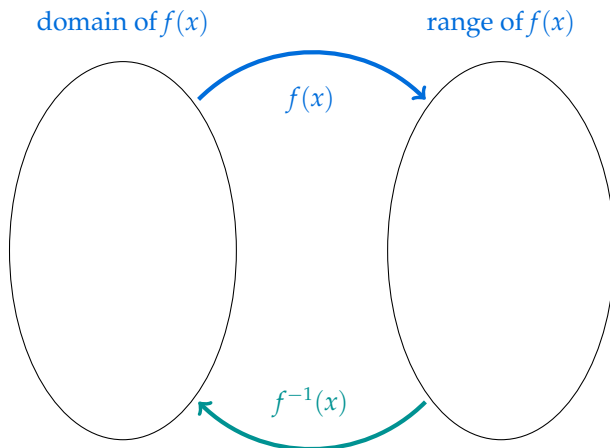
What is $f^{-1}(10)$? (do not simplify)

What is $f^{-1}(x)$?

$$\text{Let } f(x) = x^2 - x.$$

1. Sketch a graph of $f(x)$, and choose a (large) domain over which it is invertible.
2. For the domain you chose, evaluate $f^{-1}(20)$.
3. For the domain you chose, evaluate $f^{-1}(x)$.
4. What are the domain and range of $f^{-1}(x)$? What are the (restricted) domain and range of $f(x)$?





INVERTIBILITY GAME: $f(x) = e^x$

 $f^{-1}(x) = \log_e x$

- ▶ I'm thinking of an x . Your clue: $f(x) = e$. What is my x ?
- ▶ I'm thinking of an x . Your clue: $f(x) = 1$. What is my x ?
- ▶ I'm thinking of an x . Your clue: $f(x) = \frac{1}{e}$. What is my x ?
- ▶ I'm thinking of an x . Your clue: $f(x) = e^3$. What is my x ?
- ▶ I'm thinking of an x . Your clue: $f(x) = 0$. What is my x ?

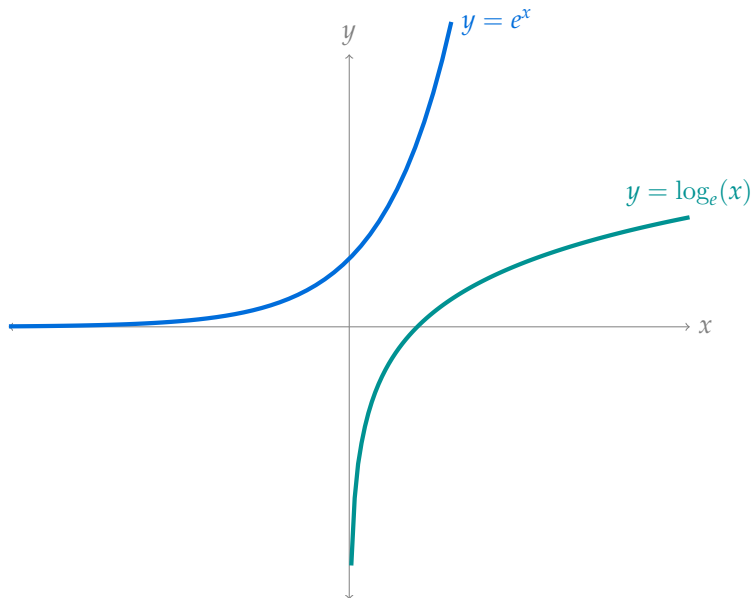
1. Suppose $0 < x < 1$. Then $\log_e(x)$ is...
2. Suppose $-1 < x < 0$. Then $\log_e(x)$ is...
3. Suppose $e < x$. Then $\log_e(x)$ is...
 - A. positive
 - B. negative
 - C. greater than one
 - D. less than one
 - E. undefined

EXPONENTS AND LOGARITHMS

$$f(x) = e^x$$

$$f^{-1}(x) = \log_e(x) = \ln(x) = \text{log}(x)$$

x	e^x	$e \text{ fact} \leftrightarrow \log_e \text{ fact}$	x	$\log_e(x)$
0	1			
1	e			
-1	$\frac{1}{e}$			
n	e^n			



LOGS OF OTHER BASES: $\log_n(x)$ IS THE INVERSE OF n^x

$$\log_{10} 10^8 =$$

- A. 0
- B. 8
- C. 10
- D. other

$$\log_2 16 =$$

- A. 1
- B. 2
- C. 3
- D. other

Logarithm Rules

Let A and B be positive, and let n be any real number.

$$\log(A \cdot B) = \log(A) + \log(B)$$

Proof: $\log(A \cdot B) = \log(e^{\log A} e^{\log B}) = \log(e^{\log A + \log B}) = \log(A) + \log(B)$

$$\log(A/B) = \log(A) - \log(B)$$

Proof: $\log(A/B) = \log\left(\frac{e^{\log A}}{e^{\log B}}\right) = \log(e^{\log A - \log B}) = \log A - \log B$

$$\log(A^n) = n \log(A)$$

Proof: $\log(A^n) = \log\left((e^{\log A})^n\right) = \log(e^{n \log A}) = n \log A$

Logarithm Rules

Let A and B be positive, and let n be any real number.

$$\log(A \cdot B) = \log(A) + \log(B)$$

$$\log(A/B) = \log(A) - \log(B)$$

$$\log(A^n) = n \log(A)$$

Write as a single logarithm:

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2 \log x + \log(10 + x)$$

BASE CHANGE

$$\text{Fact: } b^{\log_b(a)} = a$$

$$\Rightarrow \log(b^{\log_b(a)}) = \log(a)$$

$$\Rightarrow \log_b(a) \log(b) = \log(a)$$

$$\Rightarrow \log_b(a) = \frac{\log(a)}{\log(b)}$$

In general, for positive a , b , and c :

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

In general, for positive a , b , and c :

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$$

Suppose your calculator can only compute logarithms base 10. What would you enter to calculate $\log(17)$?

Suppose your calculator can only compute natural logarithms. What would you enter to calculate $\log_2(57)$?

Suppose your calculator can only compute logarithms base 2. What would you enter to calculate $\log(2)$?

Decibels: For a particular measure of the power P of a sound wave, the decibels of that sound is:

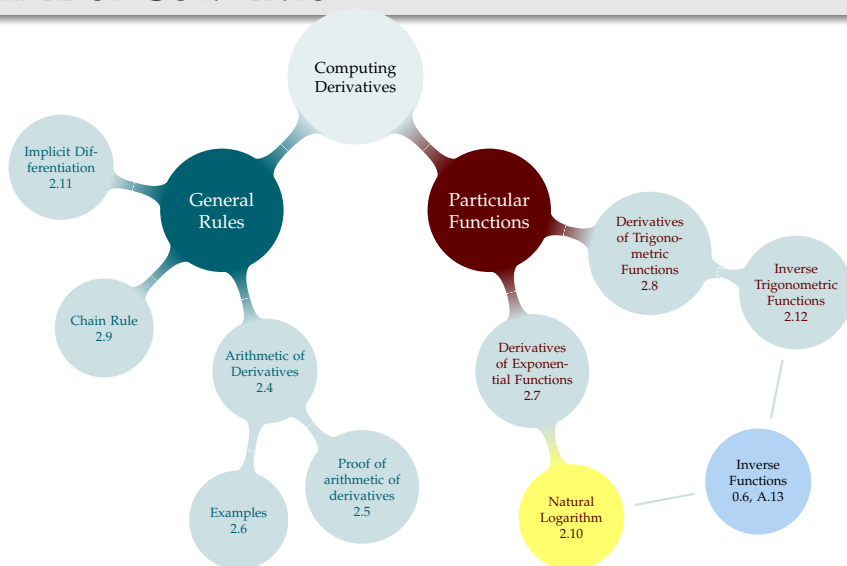
$$10 \log_{10}(P)$$

So, every ten decibels corresponds to a sound being ten **times** louder.

A lawnmower emits a 100dB sound. How much sound will two lawnmowers make?

- A. 100 dB
- B. 110 dB
- C. 200 dB
- D. other

TABLE OF CONTENTS



DIFFERENTIATING THE NATURAL LOGARITHM

Calculate $\frac{d}{dx} \{\log_e x\}$.

One Weird Trick:

$$x = e^{\log_e x}$$

$$\frac{d}{dx} \{x\} = \frac{d}{dx} \{e^{\log_e x}\}$$

$$1 = e^{\log_e x} \cdot \frac{d}{dx} \{\log_e x\} = x \cdot \frac{d}{dx} \{\log_e x\}$$

$$\frac{1}{x} = \frac{d}{dx} \{\log_e x\}$$

Derivative of Natural Logarithm

$$\frac{d}{dx} \{\log_e |x|\} = \frac{1}{x} \quad (x \neq 0)$$

Differentiate: $f(x) = \log_e |x^2 + 1|$

Derivatives of Logarithms – Corollary 2.10.6

For $a > 0$:

$$\frac{d}{dx} [\log_a |x|] = \frac{1}{x \log a}$$

In particular:

$$\frac{d}{dx} [\log |x|] = \frac{1}{x}$$

Differentiate: $f(x) = \log_e |\cot x|$

LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

► $\log(f \cdot g) = \log f + \log g$

multiplication turns into addition

► $\log\left(\frac{f}{g}\right) = \log f - \log g$

division turns into subtraction

► $\log(f^g) = g \log f$

exponentiation turns into multiplication

We can exploit these properties to differentiate!

Logarithmic Differentiation

In general, if $f(x) \neq 0$, $\frac{d}{dx} [\log |f(x)|] = \frac{f'(x)}{f(x)}$.

$$f(x) = \left(\frac{(2x+5)^4(x^2+1)}{x+3} \right)^5$$

Find $f'(x)$.

LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

$$f(x) = \left(\frac{(2x+5)^4(x^2+1)}{x+3} \right)^5$$

LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

Differentiate:

$$f(x) = x^x$$

LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

Differentiate:

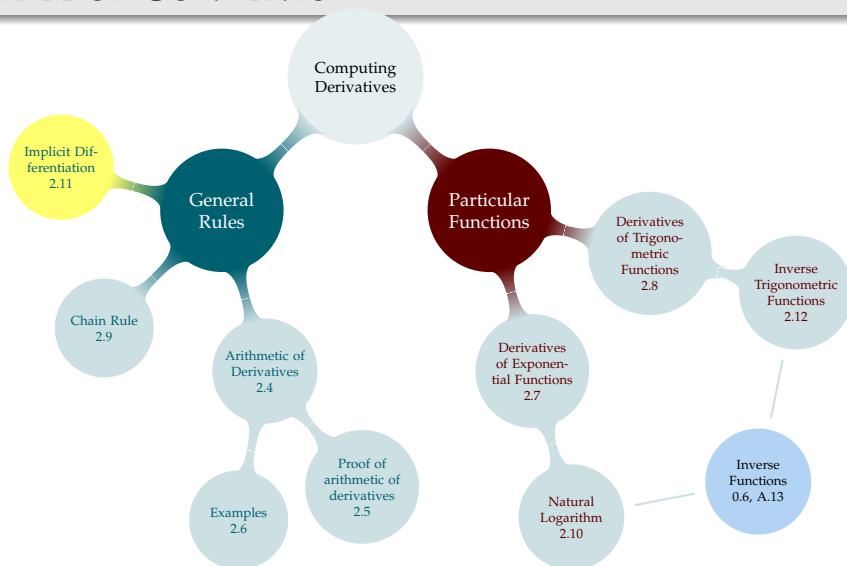
$$f(x) = \left(\frac{(x^{15} - 9x^2)^{10}(x + x^2 + 1)}{(x^7 + 7)(x + 1)(x + 2)(x + 3)} \right)^5$$

$$f(x) = \frac{(x^8 - e^x)(\sqrt{x} + 5)}{\csc^5 x}$$

$$f(x) = (x^2 + 17)(32x^5 - 8)(x^{98} - x^{57} + 32x^2)^4(32x^{10} - 10x^{32})$$

Find $f'(x)$.

TABLE OF CONTENTS



IMPLICITLY DEFINED FUNCTIONS

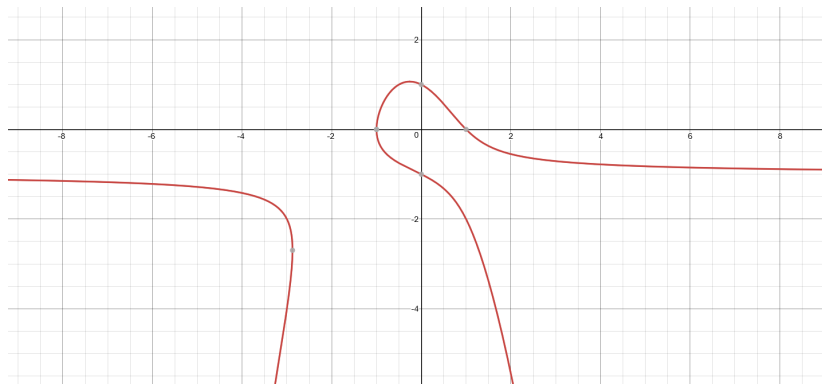
$$y^2 + x^2 + xy + x^2y = 1$$

Which of the following points are on the curve?

$(0, 1)$, $(0, -1)$, $(0, 0)$, $(1, 1)$

If $x = -3$, what is y ?

$$y^2 + x^2 + xy + x^2y = 1$$



Still has a slope: $\frac{\Delta y}{\Delta x}$

Locally, y is still a function of x .

$$y^2 + x^2 + xy + x^2y = 1$$

Consider y as a function of x . Can we find $\frac{dy}{dx}$?

$$\frac{d}{dx}[y] =$$

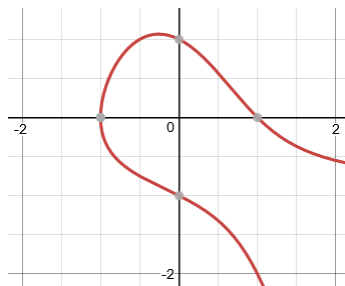
$$\frac{d}{dx}[x] =$$

$$\frac{d}{dx}[1] =$$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily, $\frac{dy}{dx}$ depends on **both** y and x . Why?



NOW
YOU



Suppose $x^4y + y^4x = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

NOW
YOU



Suppose $\frac{3y^2 + 2y + y^3}{x^2 + 1} = x$. Find $\frac{dy}{dx}$ when $x = 0$, and the equations of the associated tangent line(s).

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

$$\log x = y(x)$$

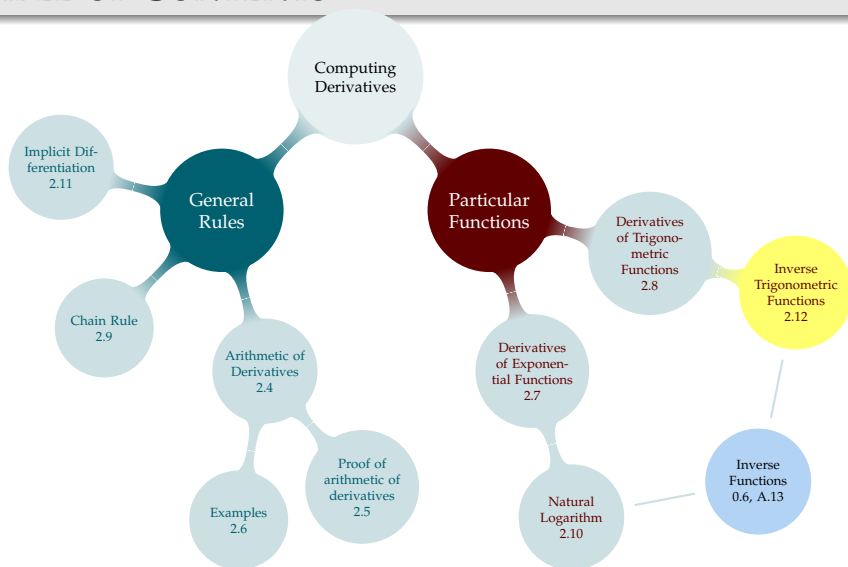
$$x = e^{y(x)}$$

Use implicit differentiation to differentiate $\log|x|$, $x < 0$.

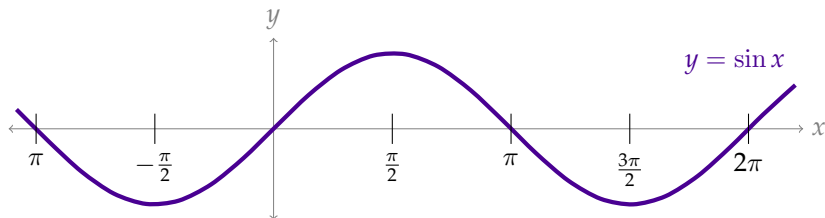
Use implicit differentiation to differentiate $\log_a(x)$, where $a > 0$ is a constant and $x > 0$.

Use implicit differentiation to differentiate $\log_a |x|$, $a > 0$.

TABLE OF CONTENTS



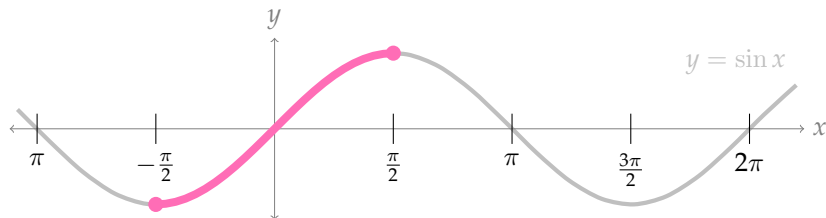
INVERTIBILITY GAME



I'm thinking of a number x . Your hint: $\sin(x) = 0$. What number am I thinking of?

I'm thinking of a number x , and x is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Your hint: $\sin(x) = 0$. What number am I thinking of?

ARCSINE



$\arcsin(x)$ is the inverse of $\sin x$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin x$ is the (unique) number θ such that:

- ▶ $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and
- ▶ $\sin \theta = x$

ARCSINE

Reference Angles:

θ	$\sin \theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

► $\arcsin(0)$

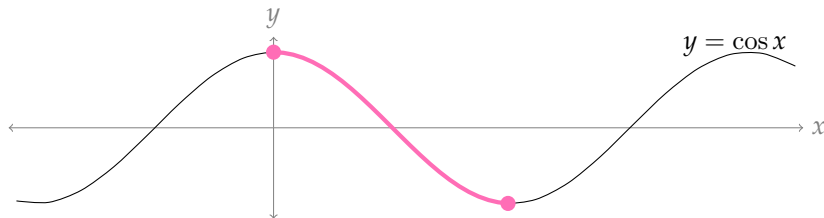
► $\arcsin\left(\frac{1}{\sqrt{2}}\right)$

► $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

► $\arcsin\left(\frac{\pi}{2}\right)$

► $\arcsin\left(\frac{\pi}{4}\right)$

ARCCOSINE

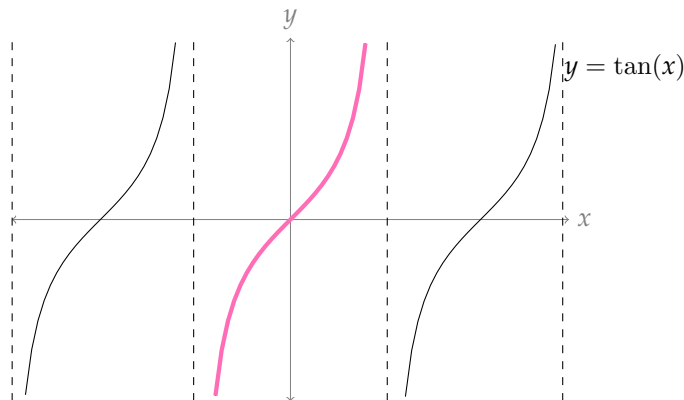


$\arccos(x)$ is the inverse of $\cos x$ restricted to $[0, \pi]$.

$\arccos(x)$ is the (unique) number θ such that:

- ▶ $\cos(\theta) = x$ and
- ▶ $0 \leq \theta \leq \pi$

ARCTANGENT



$\arctan(x) = \theta$ means:

- (1) $\tan(\theta) = x$ and
- (2) $-\pi/2 < \theta < \pi/2$

ARCSECANT, ARCSINE, AND ARCCOTANGENT

$$\operatorname{arcsec}(x) =$$

ARCSECANT, ARCSINE, AND ARCCOTANGENT

$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccsc}(x) = y$$

$$\csc y = x$$

$$\frac{1}{\sin y} = x$$

$$\sin y = \frac{1}{x}$$

$$y = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

$$\operatorname{arccot}(x) = y$$

$$\cot y = x$$

$$\frac{1}{\tan y} = x$$

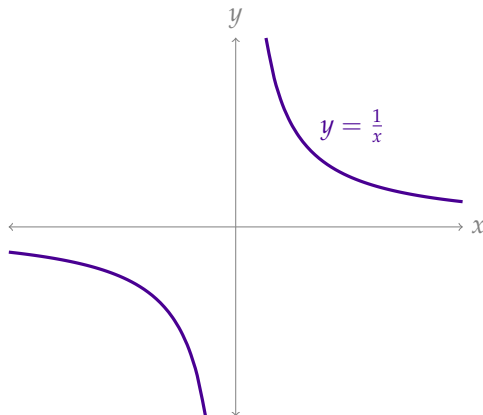
$$\tan y = \frac{1}{x}$$

$$y = \arctan\left(\frac{1}{x}\right)$$

$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

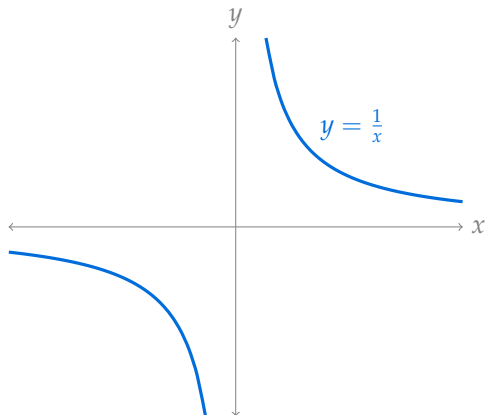
$$\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

The domain of $\arccos(y)$ is $-1 \leq y \leq 1$, so the domain of $\operatorname{arcsec}(y)$ is



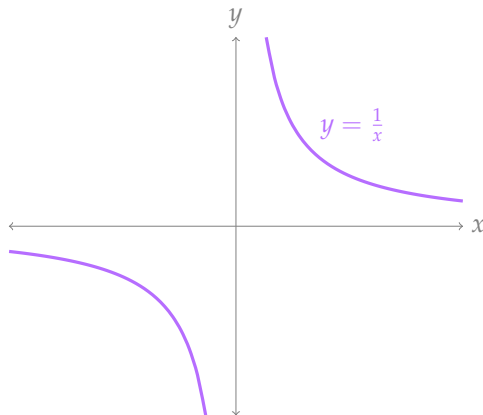
$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

Domain of $\arcsin(y)$ is $-1 \leq y \leq 1$, so the domain of $\operatorname{arccsc}(x)$ is



$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Domain of $\arctan(x)$ is all real numbers, so the domain of $\operatorname{arccot}(x)$ is



$$y = \arcsin x$$

Find $\frac{dy}{dx}$.

$$y = \arctan x$$

Find $\frac{dy}{dx}$.

$$y = \arccos x$$

Find $\frac{dy}{dx}$.

To differentiate arcsecant, arccosecant, and arccotangent, you can use the chain rule!

$$\frac{d}{dx} [\operatorname{arccsc}(x)] = \frac{d}{dx} \left[\arcsin \left(\frac{1}{x} \right) \right] = \frac{d}{dx} [\arcsin (x^{-1})]$$

Derivatives of Inverse Trigonometric Functions – Theorem 2.12.7

Memorize:

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{1+x^2}$$

Be able to derive:

$$\frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$$

Included Work



'Brain' by Eucalyp is licensed under [CC BY 3.0](#) (accessed 8 June 2021), 39, 40



screenshot of graph using Desmos Graphing Calculator,

<https://www.desmos.com/calculator> (accessed 19 October 2017), 38



screenshot of graph using Desmos Graphing Calculator,

<https://www.desmos.com/calculator> (accessed 19 October 2017), 36