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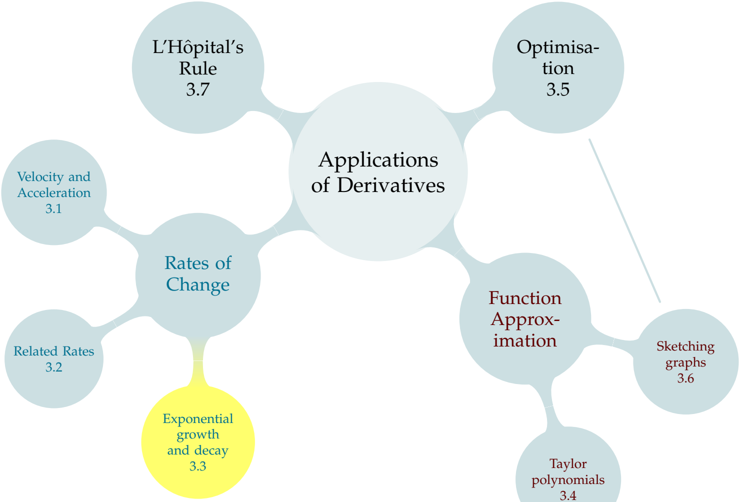

RADIOACTIVE DECAY

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Let $\boxed{Q(t) = Ce^{-kt}}$, where k and C are constants. Then:

$$\frac{dQ}{dt}(t) = C \cdot e^{-kt} \cdot (-k) = -kCe^{-kt} = -kQ(t)$$

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Quantity of a Radioactive Isotope

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What is the sign of $Q(t)$?

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B. negative or zero
C. could be either
D. I don't know

Q

ans

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What is the sign of C?

A. positive or zero
B. negative or zero
C. could be either
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Q

ans

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B. negative or zero
C. could be either
D. I don't know

A

Seaborgium Decay

The amount of ${}^{266}Sg$ (Seaborgium-266) in a sample at time t (measured in seconds) is given by

$$Q(t) = Ce^{-kt}$$

Let's approximate the half life of ${}^{266}Sg$ as 30 seconds. That is, every 30 seconds, the size of the sample halves.

What are C and k?

Q

ans

$$Q(t) = Ce^{-kt}$$

Every 30 seconds, the size of the sample halves. What are C and k?

Q

$$Q(t) = Ce^{-kt}$$

Every 30 seconds, the size of the sample halves. What are C and k?

(1) $Q(0)$ is the amount of ${}^{266}Sg$ at time 0: usually, the initial sample size.
C is the quantity at time 0

(2) The half-life of ${}^{266}Sg$ is 30 seconds. So, if we're measuring t in seconds, $Q(30) = \frac{1}{2}Q(0)$.

$$\frac{1}{2}C = \frac{1}{2}Q(0) = Q(30) = Ce^{-30k}$$

$$k = \frac{\log 2}{30}$$

A

A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope. In the year 2005, it is tested and found to contain only 2 units of that same isotope. How many units of the isotope were present in the year 2000?

Q

ans

The quantity of the isotope t years after 2000 is given by

$$Q(t) = Ce^{-kt}$$

where $C = Q(0)$ is the amount in the initial sample. Then the question asks us to solve for C, given

$$10 = Q(2) = Ce^{-2k} \quad \text{and} \quad 2 = Q(5) = Ce^{-5k}$$

Then

$$C = 10e^{2k} = 2e^{5k}$$

$$e^{3k} = 5 \qquad \implies \qquad e^{k} = \sqrt[3]{5}$$

$$C = 10e^{2k} = 10\left(\sqrt[3]{5}\right)^2 \approx 29$$

$Q'(t) = kQ(t)$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

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The rate of growth of a population in a given time interval is propotional to the number of individuals in the population, when the population has ample resources.

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The amount of interest a bank account accrues in a given time interval is proportional to the balance in that bank account.

Exponential Growth – Theorem 3.3.2

Let $Q = Q(t)$ satisfy:

$$\frac{dQ}{dt} = kQ$$

for some constant k. Then for some constant $C = Q(0)$,

$$Q(t) = Ce^{kt}$$

Exponential Growth – Theorem 3.3.2

Let $Q = Q(t)$ satisfy:

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for some constant k. Then for some constant $C = Q(0)$,

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Suppose $y(t)$ is a function with the properties that

$$\frac{dy}{dt} + 3y = 0 \quad \text{and} \quad y(1) = 2.$$

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$\frac{\mathrm{d}y}{\mathrm{d}t} = -3y$, so $y(t) = Ce^{-3t}$ by the result above.

To solve for C, we set $t = 1$:

$$2 = y(1) = Ce^{-3} \implies C = 2e^3$$

So,

$$y(t) = 2e^3 \cdot e^{-3t} = 2e^{3(1-t)}$$

Population Growth

Suppose a petri dish starts with a culture of 100 bacteria cells and a limited amount of food and space. The population of the culture at different times is given in the table below. At approximately what time did the culture start to show signs of limited resources?

time	population
0	100
1	1000
3	100000
5	1000000

Q

ans

Population Growth

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time	population
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3	100000
5	1000000

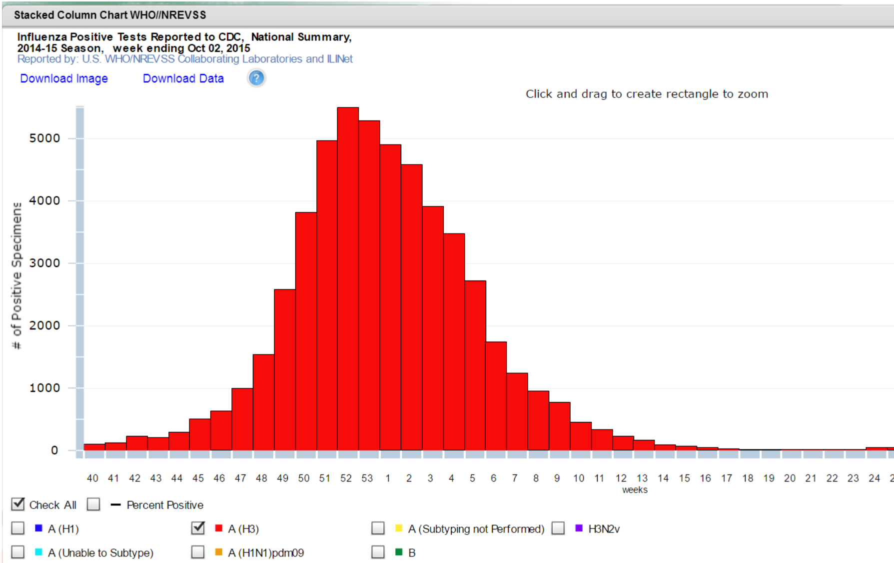
All the populations before $t = 5$ follow $B(t) = 100 \cdot 10^t = 100e^{t \log 10}$. At $t = 5$ they do not; so some time between $t = 3$ and $t = 5$, the bacteria started reproducing at a slower rate.

FLU SEASON

The CDC keeps records (link) on the number of flu cases in the US by week. At the start of the flu season, the 40th week of 2014, there are 100 cases of a particular strain. Five weeks later (at week 45), there are 506 cases. What do you think was the first week to have 5,000 cases? What about 10,000 cases?

Q

FLU SEASON


Stacked Column Chart WHO//NREVSS
Influenza Positive Tests Reported to CDC, National Summary,
2014-15 Season, week ending Oct 02, 2015
Reported by: U.S. WHO/NREVSS Collaborating Laboratories and ILINet
Download Image
Download Data
Click and drag to create rectangle to zoom
# of Positive Specimens
0
1000
2000
3000
4000
5000
40 41 42 43 44 45 46 47 48 49 50 51 52 53 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
weeks
Check All
Percent Positive
A (H1)
A (H3)
A (Subtyping not Performed)
H3N2v
A (Unable to Subtype)
A (H1N1)pdm09
B

Flu Season

Let $t = 0$ be the 40th week of 2014. Then we can model the spread of the virus like so:

$$P(t) = 100e^{kt}$$

We have one other data point: $506 = P(5) = 100e^{5k}$, so we get $e^k = 5.06^{1/5}$. Now our equation is:

$$P(t) = 100(5.06)^{t/5}$$

We set it equal to 5000 and solve: $5000 = 100(5.06)^{t/5}$ implies

$$(5.06)^{t/5} = 50 \implies \frac{t}{5}\log(5.06) = \log 50 \implies t = \frac{5\log 50}{\log(5.06)} \approx 12.06$$

Data from the CDC says Week 51 ($t = 11$) had 4972 cases, and Week 52 ($t = 12$) had 5498 cases.

Using the same formula, $10000 = 100(5.06)^{t/5}$ yields $t = \frac{5\log 100}{\log 5.06} \approx 14.2$ weeks; but the data shows that the flu season peaked with around 5,000 cases a week, and never got much higher.

Newton's Law of Cooling – Equation 3.3.7

The rate of change of temperature of an object is proportional to the difference in temperature between that object and its surroundings.

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$$\frac{dT}{dt}(t) = K[T(t) - A]$$

where $T(t)$ is the temperature of the object at time t, A is the (constant) ambient temperature of the surroundings, and K is some constant depending on the object.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, K is some constant.

What is true of K?

A. $K \geq 0$

B. $K \leq 0$

C. $K = 0$

D. K could be positive, negative, or zero, depending on the object

E. I don't know

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is the only function satisfying Newton's Law of Cooling

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is the only function satisfying Newton's Law of Cooling

If $T(10) < A$, then:

A. $K > 0$

B. $T(0) > 0$

C. $T(0) > A$

D. $T(0) < A$

Q

ans

Newton's Law of Cooling – Equation 3.3.7

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D. $T(0) < A$

Evaluate $\lim_{t\to\infty} T(t)$.

A. A

B. 0

C. ∞

D. $T(0)$

Newton's Law of Cooling – Equation 3.3.7

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Evaluate $\lim_{t\to\infty} T(t)$.

A. A
B. 0
C. ∞
D. $T(0)$

What assumptions are we making that might not square with the real world?

Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt} = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, and K is some constant.

Temperature of a Cooling Body – Corollary 3.3.8

$$T(t) = [T(0) - A]e^{Kt} + A$$

Q

ans

A farrier forms a horseshoe heated to 400° C, then dunks it in a river at room-temperature (25° C). The water boils for 30 seconds. The horseshoe is safe for the horse when it's 40° C. When can the farrier put on the horseshoe?

Q

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$$T(t) = [T(0) - A]e^{Kt} + A$$

Q

We know: $T(0) = 400$, $T(30) = 100$, and $A = 25$. We want to find K.

$$100 = T(30) = [T(0) - A]e^{30K} + A = 375e^{30K} + 25$$
$$\Rightarrow 75 = 375e^{30K} \Rightarrow \frac{1}{5} = e^{30K} \Rightarrow K = \frac{-\log 5}{30}$$

Now, we set $T(t) = 40$ and solve for t:

$$\begin{aligned}
40 &= T(t) = 375e^{\frac{-\log 5}{30}t} + 25 \\
15 &= 375e^{\frac{-log5}{30}t} = 375 \cdot 5^{-t/30} \\
\frac{1}{25} &= 5^{-t/30} \\
25 &= 5^{t/30} \\
2 &= t/30
\end{aligned}$$

So the farrier can put the shoe on after 60 seconds in the water.

A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°, and after 20 minutes, the tea is 25°. What is the temperature outside?

Q

$T(0) = 100$, so

$$T(10) = [100 - A]e^{10K} + A = 100e^{10K} + A\left(1 - e^{10K}\right) = 40$$
$$T(20) = [100 - A]e^{20K} + A = 100e^{20K} + A\left(1 - e^{20K}\right) = 25$$

Solving both for A, we get $A = \dfrac{40 - 100e^{10K}}{1 - e^{10K}} = \dfrac{25 - 100e^{20K}}{1 - e^{20K}}$

Although this looks complicated, if we set $x = e^{10k}$, it simplifies to something we can easily solve.

$$A = \frac{40 - 100e^{10K}}{1 - e^{10K}} = \frac{25 - 100e^{20K}}{1 - e^{20K}}$$

$$A = \frac{40 - 100x}{1 - x} = \frac{25 - 100x^2}{1 - x^2}$$

$$(40 - 100x)(1 - x^2) = (25 - 100x^2)(1 - x)$$

$$(40 - 100x)(1 + x)(1 - x) = (25 - 100x^2)(1 - x)$$

$$(40 - 100x)(1 + x) = 25 - 100x^2$$

$$40 - 60x - 100x^2 = 25 - 100x^2$$

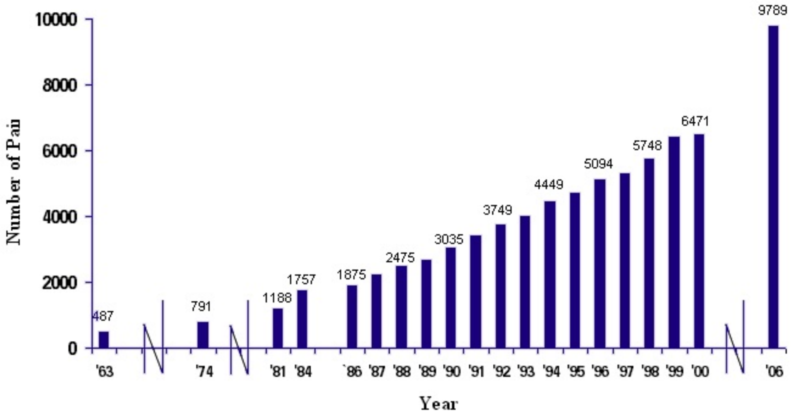
$$40 - 60x = 25$$

$$x = \frac{1}{4}$$

$$A = \frac{40 - 100x}{1 - x} = \frac{40 - \frac{100}{4}}{1 - \frac{1}{4}} = 20$$

It is 20 degrees outside.

In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?


Number of Pai
10000
8000
6000
4000
2000
0
487
791
1188
1757
1875
2475
3035
3749
4449
5094
5748
6471
9789
'63
'74
'81
'84
'86
'87
'88
'89
'90
'91
'92
'93
'94
'95
'96
'97
'98
'99
'00
'06
Year

Since we don't have a better model, let's assume the population P of nesting pairs follows:

$$P(t) = P(0)e^{Kt}$$

for some constant K.

To fit the data we have, let $t = 0$ represent 1963, so $P(0) = 487$. Then

$$4015 = P(30) = 487e^{30K}$$

so $e^K = \left(\frac{4015}{487}\right)^{1/30}$.

Now we use this to predict $P(43)$ (since 2006 is 43 years after 1963) and $P(52)$ (since 2015 is 52 years after 1963).

$$P(43) = 487(e^K)^{43} = 487\left(\frac{4015}{487}\right)^{43/30} \approx 10016$$

So we guess in 2016 there were about 10,016 breeding pairs in the lower 48.

$$P(52) = 487(e^K)^{52} = 487\left(\frac{4015}{487}\right)^{52/30} \approx 18860$$

link: Wood Bison Restoration in Alaska, Alaska Department of Fish and Game

Excerpt:

> *Based on experience with reintroduced populations elsewhere, wood bison would be expected to increase at a rate of 15%-25% annually after becoming established.... With an average annual growth rate of 20%, an initial precalving population of 50 bison would increase to 500 in approximately 13 years.*

Now You

Are they using our same model?

Q

ans

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NOW YOU

Are they using our same model?

Our model gives the same result.

Compound Interest

Suppose you invest \$10,000 in an account that accrues interest each month. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?

Q

ans

Compound Interest

Suppose you invest \$10,000 in an account that accrues interest each month. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?

Compound interest is calculated according to the formula Pe^{rt}, where r is the interest rate and t is time.

Q

ans

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Measuring time in months,

$$\begin{aligned} 10000e^{r\cdot 1} &= 10100 \\ e^r &= \frac{10100}{10000} = 1.01 \\ 10000e^{12r} &= 10000 \cdot (e^r)^{12} = 10000 \cdot 1.01^{12} \approx 11268.25 \end{aligned}$$

Carrying Capacity

For a population of size P with unrestricted access to resources, let β be the average number of offspring each breeding pair produces per generation, where a generation has length t_g. Then $b = \frac{\beta-2}{2t_g}$ is the net birthrate (births minus deaths) per member per unit time. This yields $\frac{dP}{dt}(t) = bP(t)$, hence:

Q

ans

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$$P(t) = P(0)e^{bt}$$

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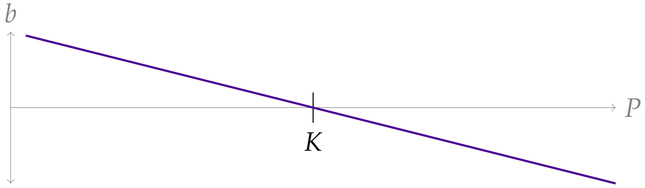
$$P(t) = P(0)e^{bt}$$

But as resources grow scarce, b might change.

Carrying Capacity

b is the net birthrate (births minus deaths) per member per unit time.
If K is the carrying capacity of an ecosystem, we can model
$b = b_0(1 - \frac{P}{K})$.



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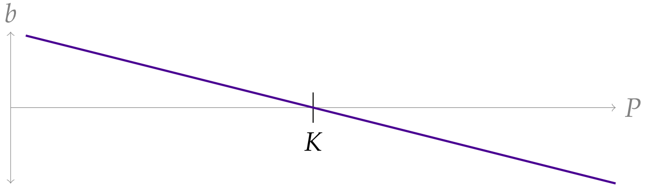

Now You  Describe to your neighbour what the following mean in terms of the model:

- $b > 0$, $b = 0$, $b < 0$
- $P = 0$, $P > 0$, $P < 0$

Carrying Capacity

Then:

$$\frac{dP}{dt}(t) = \underbrace{b_0\left(1 - \frac{P(t)}{K}\right)}_{\text{per capita birthrate}} P(t)$$

Carrying Capacity

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This is an example of a differential equation that we don't have the tools to solve. (If you take more calculus, though, you'll learn how!) It's also an example of a way you might tweak a model so its assumptions better fit what you observe.

Radiocarbon Dating

Researchers at Charlie Lake in BC have found evidence[1] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago.

Suppose a comparable bone of a bison alive today contains 1μg of ^{14}C. If the half-life of ^{14}C is about 5730 years, roughly how much ^{14}C do you think the researchers found in the sample?

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

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A. About $\frac{1}{10,500}$ μg
B. About $\frac{1}{4}$ μg
C. About $\frac{1}{2}$ μg
D. About 1 μg
E. I'm not sure how to estimate this

Q

Radiocarbon Dating

First, an estimate; 10500 is not so far off from 2(5730), i.e. two half-lives, so we might guess that there is roughly a $(\frac{1}{2})^2 = \frac{1}{4}$ of a microgram left.

We know $Q(t) = Ce^{-kt} = e^{-kt}$ μg. We want to find $Q(10500)$, so we need to solve for k. Since we know the half-life: to do this, solve

$$\frac{1}{2} = e^{-k\cdot 5730} \quad \text{to get} \quad k = \frac{\log 2}{5730}$$

Now:

$$Q(10500) = e^{-\frac{\log 2}{5730}\cdot 10500} = 2^{-\frac{10500}{5730}} \approx 0.28\ \mu\text{g}$$

Suppose a body is discovered at 3:45 pm, in a room held at 20°, and the body's temperature is 27°, not the normal 37°. At 5:45 pm, the temperature of the body has dropped to 25.3°. When did the inhabitant of the body die?

Set our time so that $t = 0$ is 3:45pm and $t = 2$ is 5:45pm. Then $T(0) = 27$, $T(2) = 25.4$, and $A = 20$. Now:

$$T(t) = [27 - 20]e^{Kt} + 20 = 7e^{Kt} + 20$$

Using what we know about 5:45pm:

$$7e^{2K} + 20 = T(2) = 25.3$$

so

$$7e^{2K} = 5.3 \implies e^{2K} = \frac{5.3}{7} \implies e^{K} = \left(\frac{5.3}{7}\right)^{1/2}$$

Now:

$$T(t) = 7e^{Kt} + 20 = 7\left(\frac{5.3}{7}\right)^{t/2} + 20$$

So we set $T(t) = 37$ and solve for t.

$$
\begin{aligned}
7\left(\frac{5.3}{7}\right)^{t/2} + 20 &= 37 \\
7\left(\frac{5.3}{7}\right)^{t/2} &= 17 \\
\left(\frac{5.3}{7}\right)^{t/2} &= \frac{17}{7} \\
\frac{t}{2} &= \frac{\log(17/7)}{\log(5.3/7)} \\
t &= 2\frac{\log(17/7)}{\log(5.3/7)} \approx -6.4
\end{aligned}
$$

So the person died about 6.4 hours before 3:45pm. Now 0.4 hours is 24 minutes. So 6 hours and 24 minutes before 3:45 pm is 6 hours before 3:21pm, which is 9:21 am.

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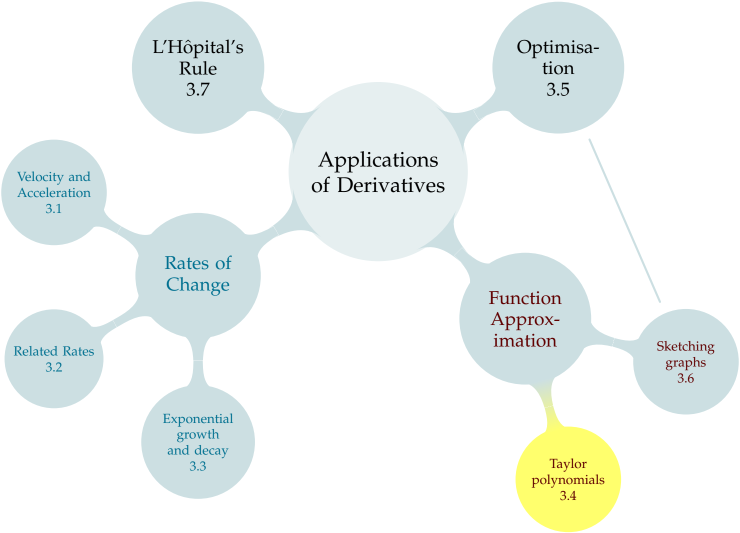
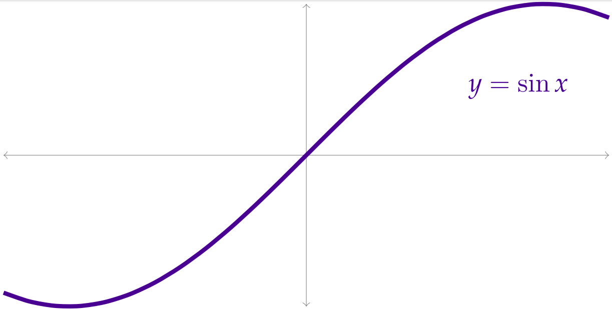


Approximating a Function



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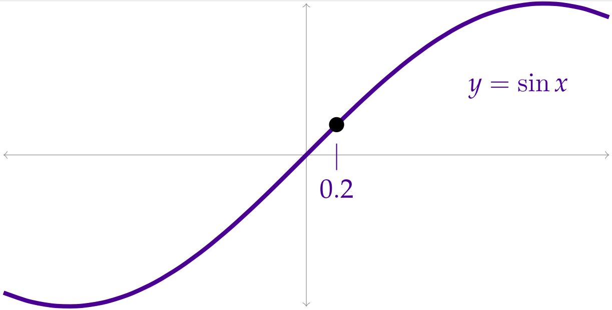

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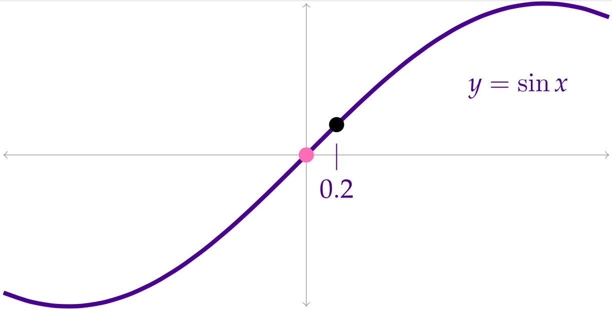

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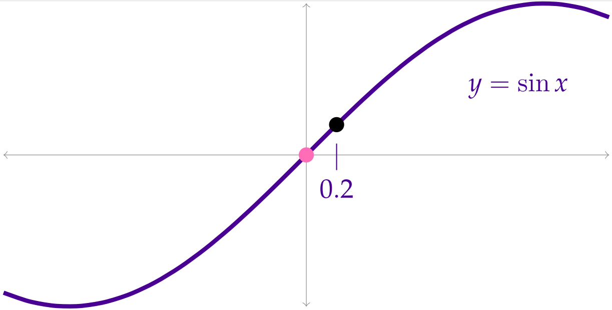
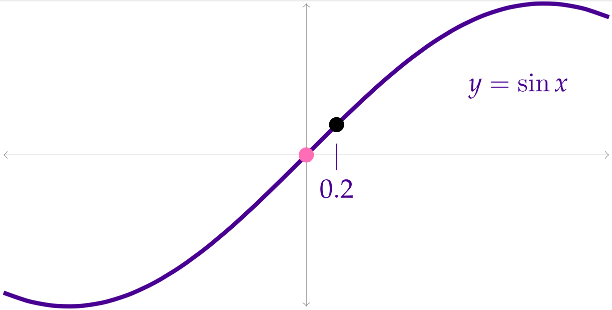


Constant Approximation – Equation 3.4.1

We can approximate $f(x)$ near a point a by

$$f(x) \approx f(a)$$

Approximating a Function



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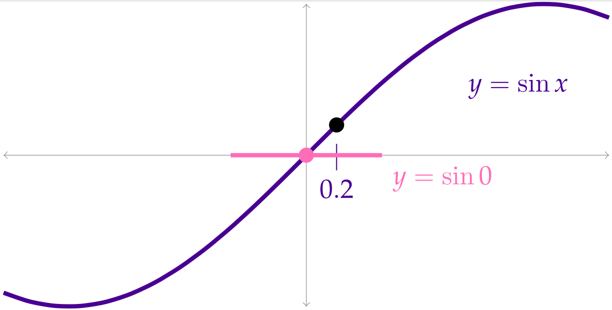
Google: $\sin(0.2) \approx 0.198669...$ Constant approx: $\sin(0.2) \approx 0$

Approximating a Function



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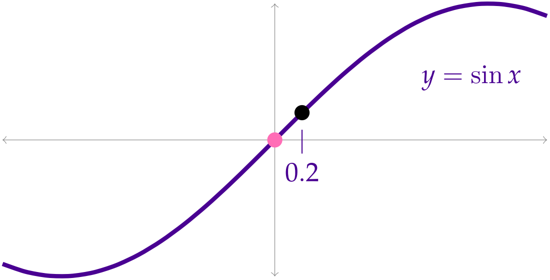

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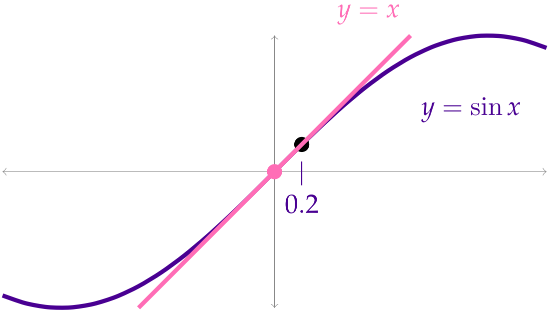

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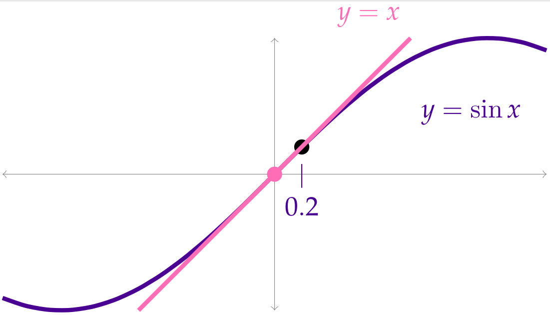

Linear Approximation (Linearization) – Equation 3.4.3

We can approximate $f(x)$ near a point a by the tangent line to $f(x)$ at a, namely

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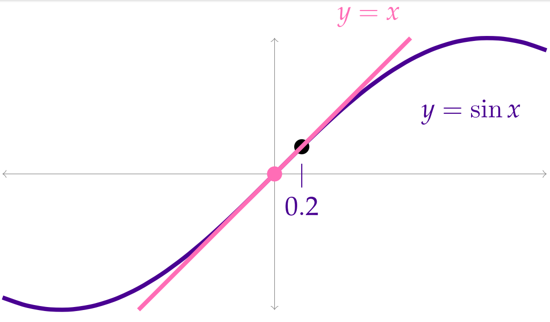
Approximating a Function



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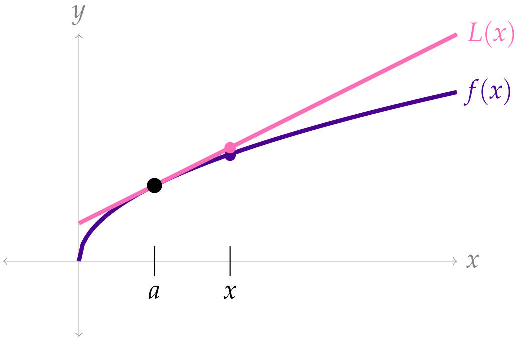
Google:
$\sin(0.2) \approx 0.198669\ldots$

Linear approx:
$\sin(0.2) \approx 0 + 1(0.2 - 0) = 0.2$

To find a linear approximation of $f(x)$ at a particular point x, pick a point a near to x, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$



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Linear approximation: Using $a = 9$,

$$f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$f(8.9) \approx f(9) + f'(9)(8.9-9) = 3 + \frac{1}{6}(-.1)$$

$$f(8.9) \approx 3 - \frac{1}{60} = 2.98\overline{33}$$

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Google: $\sqrt{8.9} = 2.98328677804...$

CHARACTERISTICS OF A GOOD APPROXIMATION

Characteristics of a Good Approximation

Accurate

Characteristics of a Good Approximation

Accurate

Possible to calculate: add, subtract, multiply, divide. Use integers or known constants

Can we compute?

Suppose we want to approximate the value of $\cos(1.5)$. Which of the following linear approximations could we calculate by hand? (You can leave things in terms of π.)

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = 3/2$

C. both

D. neither

Q

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C. both

D. neither

We know $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$, so we can easily compute the linear approximation if we centre it at $\pi/2$. However, what kind of ugly number is $\cos(3/2)$?

A

Can we compute?

Which of the following tangent lines is probably the most accurate in approximating $\cos(1.5)$?

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = \pi/4$

C. constant approximation: $\cos 1.5 \approx \cos(\pi/2) = 0$

D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

Q

ans

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D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

$\pi/2$ is very close to 1.5.

A

Linear Approximation

Approximate $\sin(3)$ using a linear approximation. You may leave your answer in terms of π.

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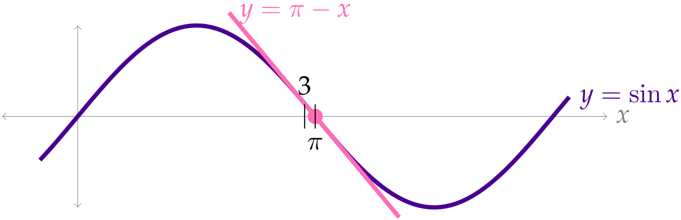

Let $f(x) = \sin x$ and $a = \pi$. Then
$f(3) \approx f(\pi) + f'(\pi)(3-\pi) = \sin(\pi) + \cos(\pi)(3-\pi) = \boxed{\pi - 3} \approx 0.14159$

Google: $\sin(3) = 0.14112000806...$

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$$\begin{aligned} f'(x) &= e^x \\ f(1/10) &\approx f(0) + f'(0)(1/10 - 0) = e^0 + e^0(1/10 - 0) = 1 + 1/10 \\ &= 1.1 \end{aligned}$$

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Google: $e^{1/10} = 1.10517091808...$

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If $g(x) = x^{1/10}$:

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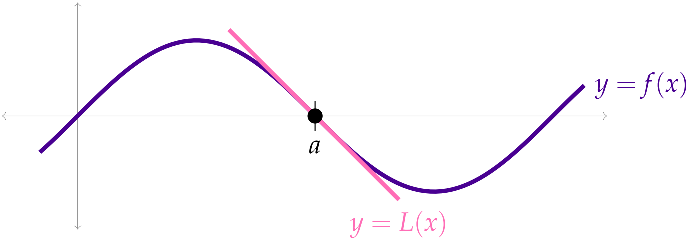
If $g(x) = x^{1/10}$:
The closest number to e with a simple tenth root is $a = 1$.

$$\begin{aligned} g'(x) &= \tfrac{1}{10}x^{-9/10} \\ g(e) &\approx g(1) + g'(1)(e-1) = 1 + \tfrac{1}{10}(e-1) = \tfrac{e+9}{10} \end{aligned}$$

... but what's e?
Google: $e^{1/10} = 1.10517091808...$

Linear Approximation Wrap-Up

Let $L(x) = f(a) + f'(a)(x-a)$, so $L(x)$ is the linear approximation (linearization) of $f(x)$ at a.



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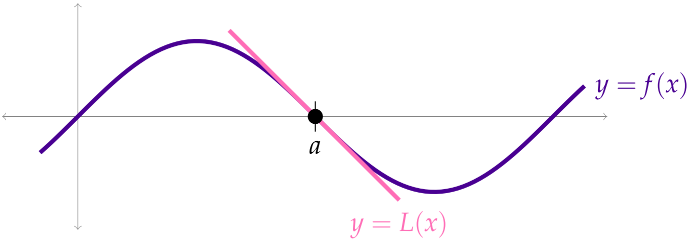

Q

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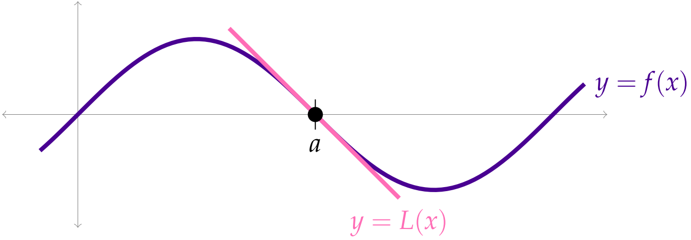

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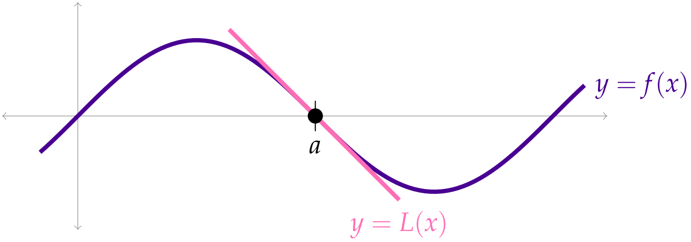

Q

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Linear Approximation Wrap-Up

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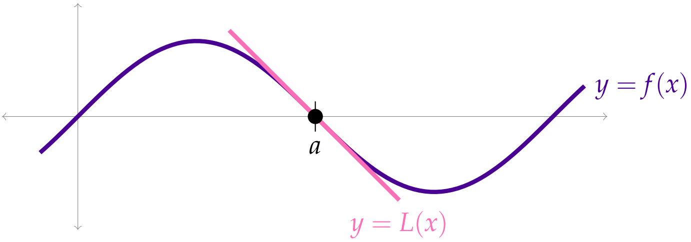

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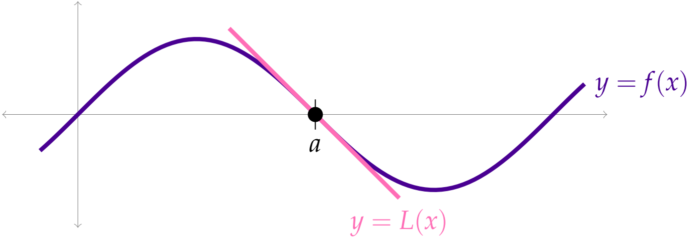

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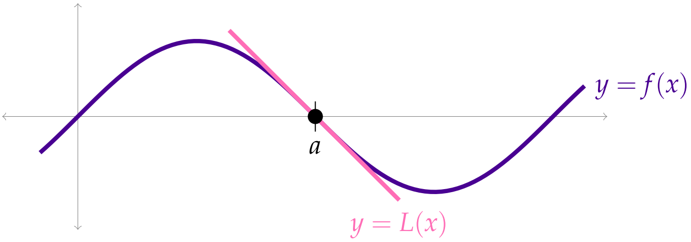
What is $L''(a)$? (Recall $L''(x)$ is the derivative of $L'(x)$.) $L''(a) = 0$



Linear Approximation Wrap-Up

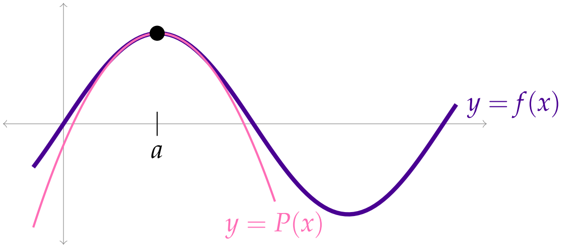
Let $L(x)$ be a linear approximation of $f(x)$.

$f(a)$	$L(a)$	same
$f'(a)$	$L'(a)$	same
$f''(a)$	$L''(a)$	different[1]

[1] unless $f''(a) = 0$

Quadratic Approximation

Imagine we approximate $f(x)$ at $x = a$ with a parabola, $P(x)$.



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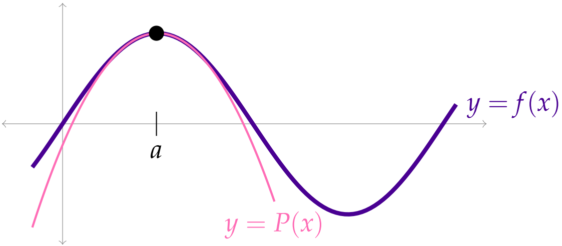

Then we could ensure:

$$P(a) = f(a), \qquad P'(a) = f'(a), \qquad \textbf{and} \qquad P''(a) = f''(a).$$

Quadratic Approximation

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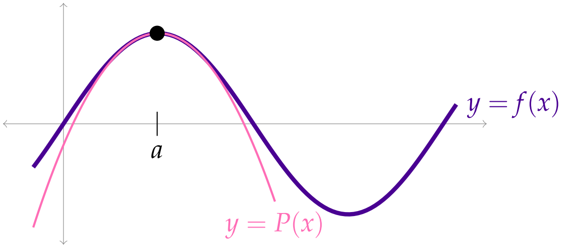

$P(x) = A + Bx + Cx^2$	$P(a) = A + Ba + Ca^2$	$f(a)$
$P'(x) = \quad B + 2Cx$	$P'(a) = \quad B + 2Ca$	$f'(a)$
$P''(x) = \quad 2C$	$P''(a) = \quad 2C$	$f''(a)$

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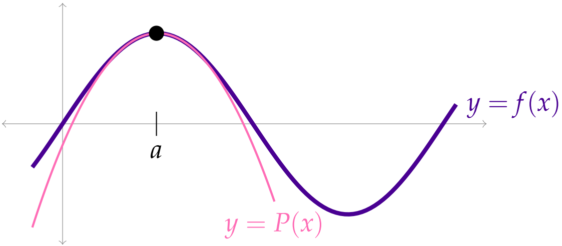

Solving $2C = f''(a)$ for C, and then solving $B + 2Ca = f'(a)$ for B, and then solving $A + Ba + Ca^2 = f(a)$ for A and then substituting back into $P(x) = A + Bx + Cx^2$ gives

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

	Constant	Linear	Quadratic
Function value matches at $x = a$	✓	✓	✓
First derivative matches at $x = a$	×	✓	✓
Second derivative matches at $x = a$	×	×	✓

Constant:	$f(x) \approx f(a)$
Linear:	$f(x) \approx f(a) + f'(a)(x-a)$
Quadratic:	$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

Q

ans

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

We use $f(x) = \log x$ and $a = 1$. Then $f'(x) = x^{-1}$ and $f''(x) = -x^{-2}$, so $f(a) = 0$, $f'(a) = 1$, and $f''(a) = -1$. Now:

$$\begin{aligned} f(1.1) &\approx f(a) + f'(a)(1.1-a) + \frac{1}{2}f''(a)(1.1-a)^2 \\ &= 0 + 1(1.1-1) + \frac{1}{2}(-1)(1.1-1)^2 \\ &= 0.1 - \frac{1}{200} = \frac{20}{200} - \frac{1}{200} = \frac{19}{200} = \frac{9.5}{100} = 0.095 \end{aligned}$$

Google: $\log(1.1) = 0.0953101798\ldots$

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.

Q

ans

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.

We use $f(x) = x^{1/3}$ and $a = 27$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = \frac{-2}{9}x^{-5/3}$. So, $f(a) = 3, f'(a) = \frac{1}{3^3}$, and $f''(a) = \frac{-2}{3^7}$.

$$\begin{aligned} f(28) &\approx f(27) + f'(27)(28-27) + \frac{1}{2}f''(27)(28-27)^2 \\ &= 3 + \frac{1}{3^3}(1) + \frac{-1}{3^7}(1^2) \\ &= 3 + \frac{1}{3^3} - \frac{1}{3^7} \\ &= 3.03657978967\ldots \end{aligned}$$

$$\textit{Google}: \sqrt[3]{28} = 3.03658897188\ldots$$

Determine what $f(x)$ and a should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$

$e^{-1/30}$

$\sqrt[5]{30}$

$(2.01)^6$

Q

ans

Determine what $f(x)$ and a should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$ $\qquad f(x) = \log(x), a = 1$

$e^{-1/30}$ $\qquad f(x) = e^x, a = 0$

$\sqrt[5]{30}$ $\qquad f(x) = \sqrt[5]{x}, a = 32 = 2^5$

$(2.01)^6$ $\qquad f(x) = x^6, a = 2$

It is possible to compute the last one without an approximation, but an approximation might save time while being sufficiently accurate for your purposes.

	Constant	Linear	Quadratic	degree n
match $f(a)$	✓	✓	✓	✓
match $f'(a)$	×	✓	✓	✓
match $f''(a)$	×	×	✓	✓
...				
match $f^{(n)}(a)$	×	×	×	✓
match $f^{(n+1)}(a)$	×	×	×	×

Constant:

$$f(x) \approx f(a)$$

Linear:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Quadratic:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Degree-n:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$$

Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

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$$\sum_{i=a}^{b} f(i)$$

- a, b (integers) "bounds"

Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

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$$\sum_{i=a}^{b} f(i)$$

- a, b (integers) "bounds"
- i "index": runs over integers from a to b
- $f(i)$ "summand": compute for every i, add

Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

Q

ans

Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

$$\sum_{i=2}^{4}(2i+5) = \underbrace{(2\cdot 2+5)}_{i=2} + \underbrace{(2\cdot 3+5)}_{i=3} + \underbrace{(2\cdot 4+5)}_{i=4}$$
$$= 9 + 11 + 13 = 33$$

Sigma Notation

$$\sum_{i=1}^{4}(i + (i-1)^2)$$

Q

ans

Sigma Notation

$$\sum_{i=1}^{4}(i+(i-1)^2)$$

$$= \underbrace{(1+0^2)}_{i=1} + \underbrace{(2+1^2)}_{i=2} + \underbrace{(3+2^2)}_{i=3} + \underbrace{(4+3^2)}_{i=4}$$

$$= 1+3+7+13 = 24$$

Write the following expressions in sigma notation:

1. $3 + 4 + 5 + 6 + 7$
2. $8 + 8 + 8 + 8 + 8$
3. $1 + (-2) + 4 + (-8) + 16$

Q

~~ans~~

Factorial – Definition 3.4.9

We read "n!" as "n factorial."
For a natural number n, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

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Taylor Polynomial – Definition 3.4.11

Given a function $f(x)$ that is differentiable n times at a point a, the n-th degree **Taylor polynomial** for $f(x)$ about a is

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, we also call it a **Maclaurin polynomial**.

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=$$

ans

$$
\begin{aligned}
T_n(a) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= \underbrace{f(a)}_{k=0} + \underbrace{f'(a)(x-a)}_{k=1} + \underbrace{\frac{1}{2!}f''(a)(x-a)^2}_{k=2} + \\
&\quad \underbrace{\frac{1}{3!}f'''(a)(x-a)^3}_{k=3} + \underbrace{\frac{1}{4!}f^{(4)}(a)(x-a)^4}_{k=4} + \\
&\quad \cdots + \underbrace{\frac{1}{n!}f^{(n)}(a)(x-a)^n}_{k=n}
\end{aligned}
$$

Small degree Taylor polynomials

$$
\begin{aligned}
T_0(a) &= \sum_{k=0}^{0} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a)
\end{aligned}
$$

The 0th degree Taylor polynomial is the constant approximation

Small degree Taylor polynomials

$$T_1(a) = \sum_{k=0}^{1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a)$$

The 1st degree Taylor polynomial is the linear approximation

Small degree Taylor polynomials

$$
\begin{aligned}
T_2(a) &= \sum_{k=0}^{2} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2
\end{aligned}
$$

The 2nd degree Taylor polynomial is the quadratic approximation

Small degree Taylor polynomials

$$
\begin{aligned}
T_3(a) &= \sum_{k=0}^{3} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3
\end{aligned}
$$

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Maclaurin[2] polynomial for e^x.

[2] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Maclaurin[2] polynomial for e^x.

Let $f(x) = e^x$. Then every derivative of e^x is just e^x, and $e^0 = 1$. So:

$$\begin{aligned} T_7(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{7!}f^{(7)}(0)(x-0)^7 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\ &= \sum_{k=0}^{7} \frac{x^k}{k!} \end{aligned}$$

e^x approximations - link

[2] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

Q

ans

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

$$\begin{array}{llll}
f(x) = \sin x & f(0) = 0 & f^{(4)}(0) = 0 & f^{(8)}(0) = 0 \\
f'(x) = \cos x & f'(0) = 1 & f^{(5)}(0) = 1 & \\
f''(x) = -\sin x & f''(0) = 0 & f^{(6)}(0) = 0 & \\
f'''(x) = -\cos x & f'''(0) = -1 & f^{(7)}(0) = -1 &
\end{array}$$

$$\begin{aligned}
T_8(x) &= f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2 + \cdots + \frac{1}{8!}f^{(8)}(0)(x-0)^8 \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
&= \sum_{k=0}^{3} \frac{x^{2k+1}}{(2k+1)!}
\end{aligned}$$

Link: sine approximations

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Now You

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

$$T_n(a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

$$\begin{array}{llll}
f(x) = \log x & f(1) = 0 & f^{(4)}(x) = -3!x^{-4} & f^{(4)}(1) = -3! \\
f'(x) = x^{-1} & f'(1) = 1 & f^{(5)}(x) = 4!x^{-5} & f^{(5)}(1) = 4! \\
f''(x) = -x^{-2} & f''(1) = -1 & f^{(6)}(x) = -5!x^{-6} & f^{(6)}(1) = -5! \\
f'''(x) = 2x^{-3} & f'''(1) = 2 & f^{(7)}(x) = 6!x^{-7} & f^{(7)}(1) = 6!
\end{array}$$

$$\begin{aligned}
T_8(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \cdots + \frac{1}{7!}f^{(7)}(1)(x-1)^7 \\
&= 0 + (1)(x-1) + (-1)\frac{1}{2}(x-1)^2 + (2)\frac{1}{3!}(x-1)^3 - 3!\frac{1}{4!}(x-1)^4 \\
&\quad + 4!\frac{1}{5!}(x-1)^5 - 5!\frac{1}{6!}(x-1)^6 + 6!\frac{1}{7!}(x-1)^7 \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7!} \\
&= \sum_{k=1}^{7} (-1)^{k+1} \frac{(x-1)^k}{k}
\end{aligned}$$

log approximations

A

skip Δx notation

Notation 3.4.18

Let x, y be variables related such that $y = f(x)$. Then we denote a small change in the variable x by Δx (read as "delta x"). The corresponding small change in the variable y is denoted Δy (read as "delta y").

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let $y = f(x)$ be the amount of water needed to produce x apples in an orchard.

A farmer wants to know how a much water is needed to increase their crop yield.

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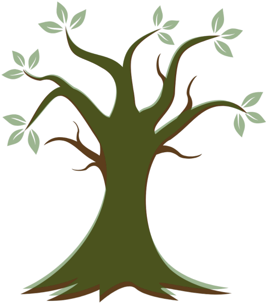

Let $y = f(x)$ be the amount of water needed to produce x apples in an orchard.
A farmer wants to know how a much water is needed to increase their crop yield. Δx is shorthand for some change in the number of apples, and Δy is shorthand for some change in the amount of water.

- Consider changing the number of apples grown from a to $a + \Delta x$

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A farmer wants to know how a much water is needed to increase their crop yield. Δx is shorthand for some change in the number of apples, and Δy is shorthand for some change in the amount of water.

- Consider changing the number of apples grown from a to $a + \Delta x$
- Then the change in water requirements goes from $y = f(a)$ to $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

LINEAR APPROXIMATION OF Δy

- Using a linear approximation, setting $x = a + \Delta x$:

Linear Approximation of Δy

- Using a linear approximation, setting $x = a + \Delta x$:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) && \text{linear approximation} \\ f(a+\Delta x) &\approx f(a) + f'(a)(\Delta x) && \text{set } x = a + \Delta x \\ \Delta y = f(a+\Delta x) - f(a) &\approx f'(a)\Delta x && \text{subtract } f(a) \text{ both sides} \end{aligned}$$

Linear Approximation of Δy

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f(x) &\approx f(a) + f'(a)(x-a) && \text{linear approximation}\\
f(a+\Delta x) &\approx f(a) + f'(a)(\Delta x) && \text{set } x = a + \Delta x\\
\Delta y = f(a+\Delta x) - f(a) &\approx f'(a)\Delta x && \text{subtract } f(a) \text{ both sides}
\end{aligned}$$

Linear Approximation of Δy (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

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Linear Approximation of Δy (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

If we set $\Delta x = 1$, then $\Delta y \approx f'(a)$. So, if we want to produce $a+1$ apples instead of a apples, the extra water needed for that one extra apple is about $f'(a)$. We call this the *marginal* water cost of the apple.

Quadratic Approximation of Δy

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

Quadratic Approximation of Δy

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$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ f(a+\Delta x) &\approx f(a) + f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2 \\ f(a+\Delta x) - f(a) &\approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2 \\ \Delta y &\approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2 \end{aligned}$$

Quadratic Approximation of Δy

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

$$f(a+\Delta x) - f(a) \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

Quadratic Approximation of Δy (Equation 3.4.21)

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and π.

Q

ans

▸▸ skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and π.

All our derivatives were based on radians, so first, let's do a conversion:

$$65 \text{ degrees} \cdot \left(\frac{2\pi \text{ radians}}{360 \text{ degrees}}\right) = \frac{13\pi}{36} \text{ radians}$$

$\frac{13\pi}{36}$ is pretty close to $\frac{\pi}{3}$ (and 65 is pretty close to 60), so we centre our approximation at $a = \frac{\pi}{3}$ (or $60°$). This is the closest reference angle to our desired angle.

We will need the first two derivatives of $f(x) = \tan x$ at $x = \frac{\pi}{3}$.

$$f(x) = \tan x \qquad f\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$f'(x) = \sec^2 x = \frac{1}{\cos^2 x} \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{(1/2)^2} = 4$$

$$f''(x) = \frac{2\sin x}{\cos^3 x} \qquad f''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{(1/2)^3} = 8\sqrt{3}$$

Constant: $f(x) \approx f(a)$

$$f\left(\frac{13\pi}{36}\right) \approx f\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Linear: $f(x) \approx f(a) + f'(a)(x-a)$

$$f\left(\frac{13\pi}{36}\right) \approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)$$
$$= \sqrt{3} + 4\left(\frac{\pi}{36}\right)$$

Quadratic: $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

$$f\left(\frac{13\pi}{36}\right) \approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right) + \frac{1}{2}f''\left(\frac{\pi}{3}\right)\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)^2$$

$$= \sqrt{3} + 4\left(\frac{\pi}{36}\right) + \frac{1}{2}(8\sqrt{3})\left(\frac{\pi}{36}\right)^2$$

$$= \sqrt{3} + \frac{\pi}{9} + \frac{4\sqrt{3}\pi^2}{6^4}$$

type	approx	decimal
constant	$\sqrt{3}$	1.732...
linear	$\sqrt{3} + \frac{\pi}{9}$	2.081...
quadratic	$\sqrt{3} + \frac{\pi}{9} + \frac{4\sqrt{3}\pi^2}{6^4}$	2.134...
actual	–	2.145...

You measure an angle $x \approx \frac{\pi}{2}$, and use it to calculate $y = \sin x \approx 1$. However, you suspect the angle was not *exactly* equal to $\frac{\pi}{2}$, which means the actual value y is slightly *less than* 1. In order for your value of y to have an error of no more than $\frac{1}{200}$, how accurate does your measurement of θ have to be?

Q

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Let the actual angle x be $x_0 + \Delta x$ with $x_0 = \frac{\pi}{2}$, so Δx is the error in your measurement. Then let $y_0 = \sin x_0 = 1$ and $y = \sin x$. Then the error in y is $\Delta y = y - y_0$.
We want to solve

$$-\frac{1}{200} = \Delta y = y - y_0 = \sin x - \sin x_0 = \sin\left(\frac{\pi}{2} + \Delta x\right) - 1$$

for (the maximum allowed) Δx.

We'll show two solutions.

Solution 1

$$\begin{aligned}
-\frac{1}{200} &= \sin\left(\frac{\pi}{2}+\Delta x\right)-1\\
\frac{199}{200} &= \sin\left(\frac{\pi}{2}+\Delta x\right)\\
\arcsin\left(\frac{199}{200}\right) &= \frac{\pi}{2}+\Delta x\\
\Delta x &= \arcsin\left(\frac{199}{200}\right)-\frac{\pi}{2}\\
\Delta x &\approx -0.10004
\end{aligned}$$

So, the error in the measurement should not be more than about ± 0.10004.
The problem with that calculation is the last step – we had to find the arcsin of some crazy number, so we needed a pretty sophisticated approximation. Solution 2 will also use an approximation, but a much simpler one.

Solution 2 We will replace $f(x) = \sin x$ with its quadratic approximation centred at $a = \frac{\pi}{2}$:

$$\begin{aligned}\sin x &\approx \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\cdot\left(x - \frac{\pi}{2}\right) - \frac{1}{2}\sin\frac{\pi}{2}\cdot\left(x-\frac{\pi}{2}\right)^2 \\ &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2\end{aligned}$$

Now, we adjust the function from the last slide:

$$\begin{aligned} -\frac{1}{200} &= \sin\left(\frac{\pi}{2} + \Delta x\right) - 1 \\ \implies -\frac{1}{200} &\approx 1 - \frac{1}{2}\left(\frac{\pi}{2} + \Delta x - \frac{\pi}{2}\right)^2 - 1 \\ &= -\frac{1}{2}(\Delta x)^2 \\ \frac{1}{100} &\approx \frac{1}{(\Delta x)^2} \\ \Delta x &\approx \pm\frac{1}{10}\end{aligned}$$

So, the error in the measurement should not be more than about $\pm\frac{1}{10}$.
(This accords quite well with Solution 1 – no arcsine required.)

Definition 3.4.25

Let Q_0 be the exact value of a quantity and let $Q_0 + \Delta Q$ be the measured value. We call

$$|\Delta Q|$$

the absolute error of the measurement, and

$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

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Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

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$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

The absolute error of the labelling is 2, and the percent error is $100\frac{2}{502} = 0.4$, or less than one-half of one percent.

Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

Q

ans

Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

Let x_0 be the actual value of the angle, x be the measured angle, and $\Delta x = x - x_0$. Then let $y(x) = \sin x$ (the computed y) and $y_0 = \sin x_0$ (the actual y), with $\Delta y = y - y_0$.
Using the linear approximation $y(x_0 + \Delta x) \approx y(x_0) + y'(x_0)(\Delta x)$:

$$\Delta y = y(x_0 + \Delta x) - y(x_0) \approx y'(x_0)\Delta x = \cos x_0 \cdot \Delta x$$

$$\text{Note: } 1 = 100\frac{|\Delta x|}{x_0} \implies \Delta x = \pm\frac{x_0}{100}$$

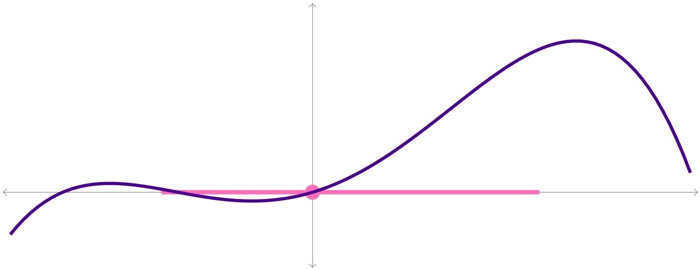
$$\implies \Delta y \approx \pm\frac{x_0 \cos x_0}{100}$$

$$\implies 100\frac{|\Delta y|}{y_0} \approx 100\frac{\frac{|x_0 \cos x_0|}{100}}{y_0} = \frac{x_0|\cos x_0|}{y_0} = \frac{x_0|\cos x_0|}{\sin x_0}$$

Note that when $x_0 \approx \frac{\pi}{2}$, this percentage error, $\frac{x_0|\cos x_0|}{\sin x_0}$, is close to 0; when $x_0 \approx 0$, it is about 1. (For the second fact, remember $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.)

A

Error: what "causes" error in an estimation?



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

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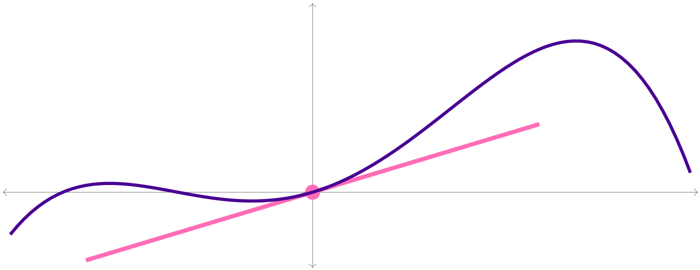
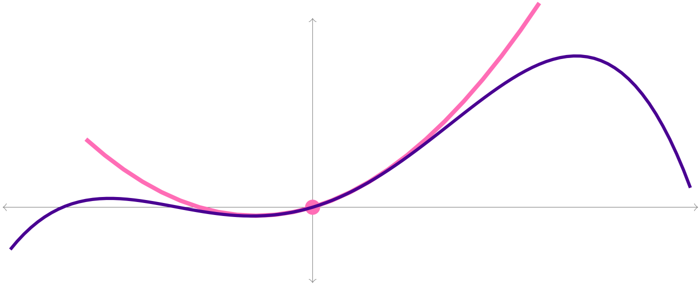


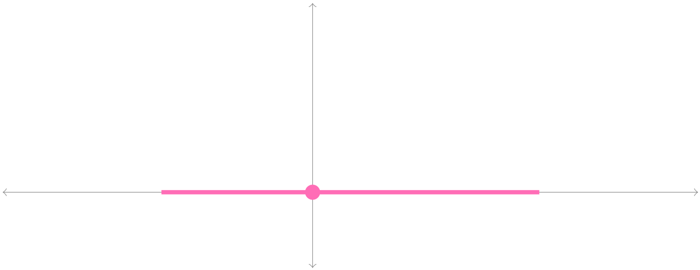
Linear approximation: We assume the function changes at a constant rate, but in fact the function changes at different rates (its second derivative is not always zero).

Error: what "causes" error in an estimation?



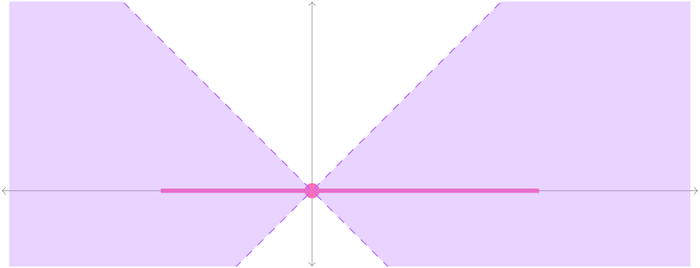
Quadratic approximation: We assume the function's derivative changes at a constant rate, but in fact the function's derivative changes at different rates (its third derivative is not always zero).

Controlling the "cause" of the error



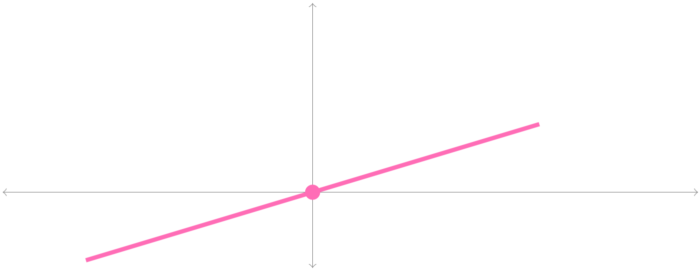
Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).
BUT: suppose we know the max and min values of the function's slope.

CONTROLLING THE "CAUSE" OF THE ERROR



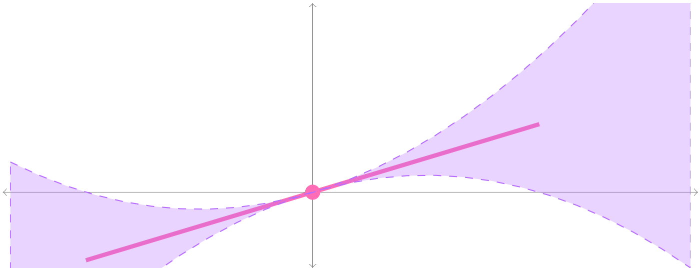
Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero). BUT: suppose we know the max and min values of the function's slope.

Controlling the “cause” of the error



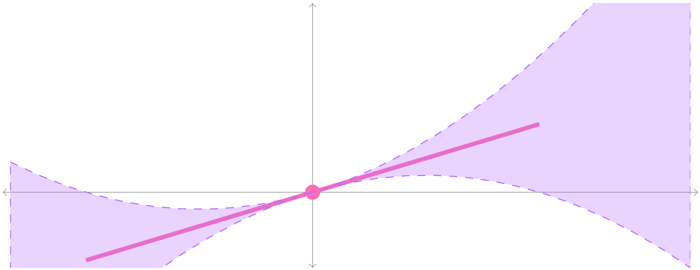
Linear approximation: We assume the function changes at a constant rate, but in fact the function changes at different rates (its first derivative is not always zero). BUT: suppose we know the max and min values of the function’s second derivative.

Controlling the "cause" of the error



Linear approximation: We assume the function changes at a constant rate, but in fact the function changes at different rates (its first derivative is not always zero). BUT: suppose we know the max and min values of the function's second derivative.

Controlling the "cause" of the error



In general, if the "thing that causes the error" is big, then our error is big. We find the largest and smallest possible errors.

Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

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The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

Third degree Maclaurin polynomial for $f(x) = e^x$:

Third degree Maclaurin polynomial for $f(x) = e^x$:

$$\begin{aligned} T_3(x) &= f(0) + f'(0)(x-0) + \tfrac{1}{2!}f''(0)(x-0)^2 + \tfrac{1}{3!}f'''(0)(x-0)^3 \\ &= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned}$$

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Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

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For some c in $(0\,,\,0.1)$:

$$\underbrace{f(0.1) - T_3(0.1)}_{\text{error}} = \frac{1}{4!} f^{(4)}(c)(.1-0)^4$$

$$= \frac{1}{4!}(0.0001)e^c$$

For c in $(0\,,\,0.1)$, $1 \le e^c < e^1 < 3$, so

$$\frac{1}{4!}(0.0001)1 \le \underbrace{f(.1) - T_3(.1)}_{\text{error}}$$

$$\le \frac{1}{4!}(0.0001)3$$

$$4.2 \times 10^{-6} \le \underbrace{f(0.1) - T_3(0.1)}_{\text{error}} \le 1.3 \times 10^{-5}$$

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could the magnitude of the error be if we approximate $\cos(2)$?

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could the magnitude of the error be if we approximate $\cos(2)$?

For some c in $(\pi/2, 2)$: $$\underbrace{f(2) - T_5(2)}_{error} = \frac{1}{6!}f^{(6)}(c)(2-\pi/2)^6$$

Note $f^{(6)}(x)$ is going to be plus or minus sine or cosine, so $-1 \le f^{(6)}(c) \le 1$. Also, $0 < 2 - \pi/2 < 1$. Now:

$$\frac{-1}{6!} = \frac{1}{6!}(-1)(1)^6 \le f(2) - T_5(2) \le \frac{1}{6!}(1)(1)^6 = \frac{1}{6!}$$

And $\frac{1}{6!} \approx 0.0014$. Be very careful with positives and negatives here :)

We don't actually have to compute $T_5(x)$, but if you want to as an exercise, click here to see the result.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use a third degree Taylor polynomial centred at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for our error.

Q

Suppose we use a third degree Taylor polynomial centred at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for our error.

For some c in $(4, 4.1)$,
$|f(4.1) - T_3(4.1)| = \left|\frac{1}{4!}f^{(4)}(c)(4.1-4)^4\right| = \frac{0.1^4}{4!}\left|f^{(4)}(c)\right|$

So, let's investigate $f^{(4)}(c)$. First we find that the fourth derivative of $f(x) = x^{1/2}$ is $f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$. So, for c in $(4.1, 4)$, we have
$|f^{(4)}(c)| = \left|\frac{-15}{16\sqrt{c^7}}\right| = \frac{15}{16\sqrt{c^7}} \leq \frac{15}{16(\sqrt{4})^7} = \frac{15}{16\cdot 2^7}$

So, the error is bounded by:
$|f(4.1) - T_3(4.1)| \leq \frac{0.1^4}{4!} \cdot \frac{15}{16\cdot 2^7} \approx 0.00000003$

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose you want to approximate the value of e, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound your error.

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For some c in $(0, 1)$:
$|f(1) - T_4(1)| = \left|\frac{1}{5!}f^{(5)}(c)(1-0)^5\right| = \frac{1}{5!}e^c \leq \frac{1}{5!}e^1 < \frac{3}{5!} = 0.025$

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?

WHICH DEGREE?

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

WHICH DEGREE?

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

We want the magnitude of the error, so let's deal with absolute values. For some c in $(3, \pi)$:

$$|f(3) - T_n(3)| = \left|\frac{1}{(n+1)!}f^{(n+1)}(c)(3-\pi)^{n+1}\right| = \frac{|3-\pi|^{n+1}}{(n+1)!}\left|f^{(n+1)}(c)\right|$$
$$\leq \frac{(0.2)^{n+1}}{(n+1)!}(1) = \frac{0.2^{n+1}}{(n+1)!}$$

If we plug in $n = 2$, we get $\frac{0.2^{n+1}}{(n+1)!} = 0.00133...$, which is not SMALLER than 0.001. If we plug in $n = 3$, we get $\frac{0.2^{n+1}}{(n+1)!} = 0.000066...$ which IS smaller than 0.001. So we have to use the degree 3 Taylor polynomial.

WHICH DEGREE?

Suppose you want to approximate e^5 using a Maclaurin polynomial of $f(x) = e^x$. If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

Which Degree?

Suppose you want to approximate e^5 using a Maclaurin polynomial of $f(x) = e^x$. If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

The magnitude of the error means its absolute value. Our error is, for some c in $(0, 5)$:

$$f(5) - T_n(5) = \frac{1}{(n+1)!} f^{(n+1)}(c)(5-0)^{n+1} = \frac{1}{(n+1)!} e^c 5^{n+1}$$

We can bound e^c for c in $(0, 5)$ by $1 = e^0 < e^c < e^5 < 3^5$. So now:

$$0 \le f(5) - T_n(5) \le \frac{1}{(n+1)!} \cdot 3^5 \cdot 5^{n+1}$$

We want $\frac{1}{(n+1)!} \cdot 3^5 \cdot 5^{n+1} < 0.001$, and by plugging in different values of n, we find the smallest n that makes the inequality true is $n = 21$. So we can use the 21st-degree Maclaurin polynomial and get our desired error.

WHICH DEGREE?

Suppose you want to approximate $\log \frac{4}{3}$ using a Taylor polynomial of $f(x) = \log x$ centred at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

WHICH DEGREE?

Suppose you want to approximate $\log \frac{4}{3}$ using a Taylor polynomial of $f(x) = \log x$ centred at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Your error will be: $f(3) - T_n(3) = \frac{1}{(n+1)!} f^{(n)}(c)\left(\frac{4}{3} - 1\right)^{n+1}$ for some c in $(1, 4/3)$. So, we need to bound $f^{(n)}(c)$. By writing out a number of derivatives of natural log, we notice that for $n \geq 1$, $f^{(n)}(c) = (-1)^{n-1}(n-1)!c^{-n}$. So, $f^{(n+1)}(c) = (-1)^n n! c^{-(n+1)}$. For c in $(1, 4/3)$:

$$\frac{n!}{(4/3)^{n+1}} \leq |f^{(n+1)}(c)| \leq \frac{n!}{1^{n+1}} = n!$$

Now for the error:

$$|f(3) - T_n(3)| \leq \frac{1}{(n+1)!} \cdot n! \cdot \left(\frac{4}{3} - 1\right)^{n+1} = \frac{1}{(n+1)3^{n+1}}$$

Setting this < 0.001, we find by plugging in values of n that $n = 4$ is the smallest n that makes the inequality true. So, using $T_4(x)$ will give us our desired error.

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

Q

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

Taylor's Theorem tells us that, for some c in $(81, 81.2)$:

$$f(81.2) - T_2(81.2) = \frac{1}{3!}f^{(3)}(c)(81.2-81)^3 = \frac{1}{6}\cdot\left(\frac{1}{5}\right)^3 f^{(3)}(c) = \frac{1}{6\cdot 5^3}f^{(3)}(c)$$

So, we should probably find out what $f^{(3)}(x)$ is. Since $f(x) = x^{1/4}$, it's not too hard to figure out $f'''(x) = \frac{21}{4^3}x^{-11/4}$. So,

$$f(81.2) - T_2(81.2) = \frac{1}{6\cdot 5^3}\cdot\frac{21}{4^3c^{11/4}} = \frac{7}{2\cdot 4^3\cdot 5^3\cdot c^{11/4}}$$

Now our job is to bound this, and we should use reasonable numbers.

$$81 \le c \le 81.2$$

$$3 = 81^{1/4} \le c^{1/4} \le 81.2^{1/4} < 4$$

$$0 < \frac{7}{2\cdot 4^3\cdot 5^3\cdot 4^{11}} \le \frac{7}{2\cdot 4^3\cdot 5^3\cdot c^{11/4}} \le \frac{7}{2\cdot 4^3\cdot 5^3\cdot 3^{11}} \le 0.0000000025$$

Since $f(x) - T_2(x)$ is positive, $T_2(x)$ is an underestimate.

Included Work

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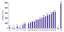 'Chart of Bald Eagle Breeding Pairs in Lower 48 States'. Eagle nesting data, US Fish and Wildlife Service Midwest Region (accessed 19 October 2016), 44

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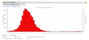 U.S. WHO/NREVSS Collaborating Laboratories and ILNet. 'Stacked Column Chart WHO//NREVSS' Centers for Disease Control and Prevention. No longer available from http://gis.cdc.gov/grasp/fluview/fluportaldashboard.html (accessed 20 October 2015), 26

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