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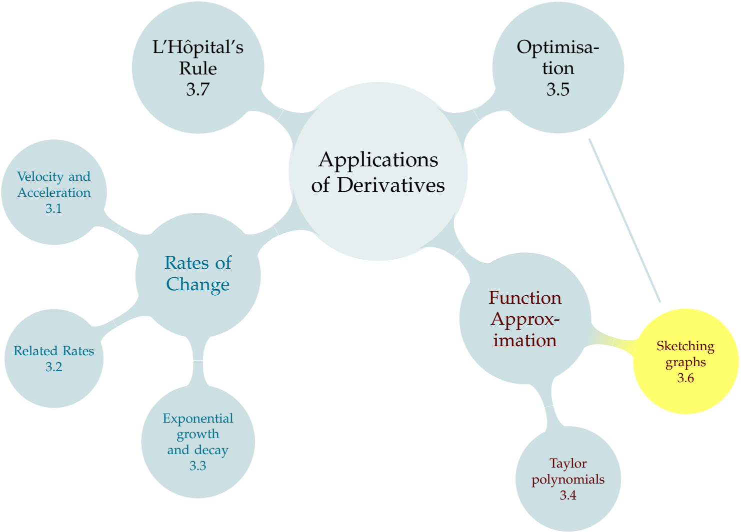

CURVE SKETCHING

Review: find the domain of the following function.

$$f(x) = \frac{\sqrt{3 - x^2}}{\log(x + 1)}$$

Q

ans

Curve Sketching

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$(-1,0) \cup \left(0, \sqrt{3}\right]$

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Where might you expect $f(x)$ to have a vertical asymptote? What does the function look like nearby?
(Recall: a vertical asymptote occurs at $x = a$ if the function has an infinite discontinuity at a. That is, $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$.)

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Where is $f(x) = 0$?

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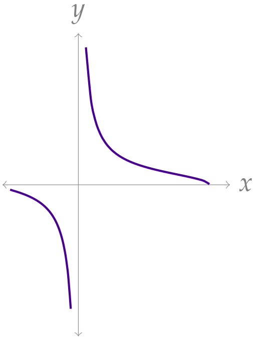
Where is $f(x) = 0$?

What happens to $f(x)$ near its other endpoint, $x = -1$?

Q

ans



y
x

Curve Sketching

Good things to check:

- Domain
- Vertical asymptotes: $\lim_{x\to a} f(x) = \pm\infty$
- Intercepts: $x = 0, f(x) = 0$
- Horizontal asymptotes and end behavior: $\lim_{x\to\pm\infty} f(x)$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

Q

Curve Sketching

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

- Domain: $x \neq -3$
- Vertical asymptote: $x = -3$
- Intercepts: $(2,0)$, $(0, -\frac{2}{9})$
- Horizontal asymptote: $y = 0$ in both directions

https://www.desmos.com/calculator/hyzl5cyq7i

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

Q

ans

Curve Sketching

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

- Domain: $x \neq 0, 5$
- Vertical asymptotes: $x = 0$, $x = 5$
- Intercepts: $(-2, 0)$, $(3, 0)$
- Horizontal asymptote: none

https://www.desmos.com/calculator/ploa0q7bxn

First Derivative

Add complexity: Increasing/decreasing, critical and singular points.

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$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

Q

ans

First Derivative

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

Domain all real numbers

Intercepts Factor $f(x) = x^2\left(\frac{1}{2}x^2 - \frac{4}{3}x - 15\right)$. Intercept at the origin; use the quadratic formula to find x-intercepts at $x = \frac{4\pm\sqrt{286}}{3}$, so $x \approx 7$ and $x \approx -4.3$.

End behaviour $\displaystyle\lim_{x\to\infty} f(x) = \infty$ and $\displaystyle\lim_{x\to-\infty} f(x) = \infty$

This is the information we've already talked about gathering. Now let's add the first derivative.

FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

$$\begin{aligned} f'(x) &= 2x^3 - 4x^2 - 30x \\ &= 2x(x^2 - 2x - 15) \\ &= 2x(x-5)(x+3) \end{aligned}$$

So, critical points are $x = 0$, $x = -3$, and $x = 5$. No singular points. At the critical points $f(0) = 0$, $f(-3) = -58.5$, $f(5) = -229.1\overline{6}$.

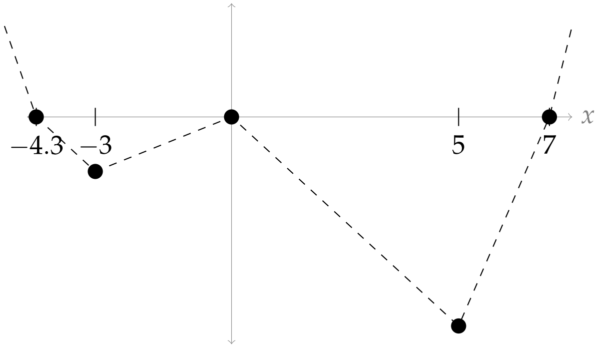
$x \approx -4.3$	$x < -3$	x=-3	$-3 < x < 0$	x=0	$0 < x < 5$	x=5	$x > 5$	$x \approx 7$
$f(x) = 0$	$f' < 0$	CP	$f' > 0$	CP	$f' < 0$	CP	$f' > 0$	f(x)=0
intercept	decr	loc min	incr	loc max	decr	loc min	incr	intercept

This gives us enough information to draw a skeleton.

First Derivative

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



https://www.desmos.com/calculator/lxdlgmhnsl

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

ans

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

Domain all real numbers.

End behaviour $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$

Intercepts

$$f(0) = 24$$

$$f(x) = \frac{1}{3}x^2(x+6) + 4(x+6)$$

$$= \left(\frac{1}{3}x^2 + 4\right)(x+6)$$

Only two intercepts: $(0, 24)$ and $(-6, 0)$.

What does the graph of the following function look like?

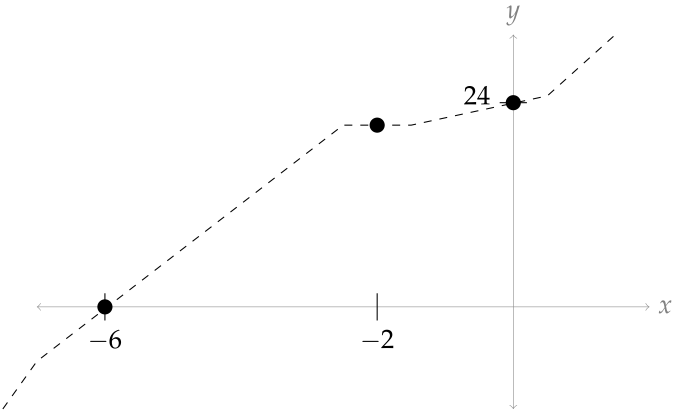
$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

Critical points	$f'(x) = x^2 + 4x + 4 = (x+2)^2$ so $x = -2$ is the location of the only critical point. $f(-2) = 24 - \frac{8}{3} = 21 + \frac{1}{3}$.
Increasing, decreasing	$f'(x) > 0$ except at $x = -2$, so apart from the critical point, $f(x)$ is increasing

This is enough for us to draw a skeleton.

What does the graph of the following function look like?

$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$



https://www.desmos.com/calculator/xum0mstmiv

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

Q

ans

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

Domain $x \neq 1$

We need to consider what happens as x approaches 1 from the left and the right.

Vertical asymptotes

$$\lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty \implies \lim_{x\to 1^-} f(x) = \lim_{A\to -\infty} e^A = 0$$

$$\lim_{x\to 1^+} \frac{x+1}{x-1} = \infty \implies \lim_{x\to 1^+} f(x) = \lim_{A\to +\infty} e^A = \infty$$

End behaviour $\displaystyle\lim_{x\to\pm\infty} \frac{x+1}{x-1} = 1$, so $\displaystyle\lim_{x\to\pm\infty} f(x) = e$

Intercepts $f(0) = \frac{1}{e}$; there are no roots

A

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

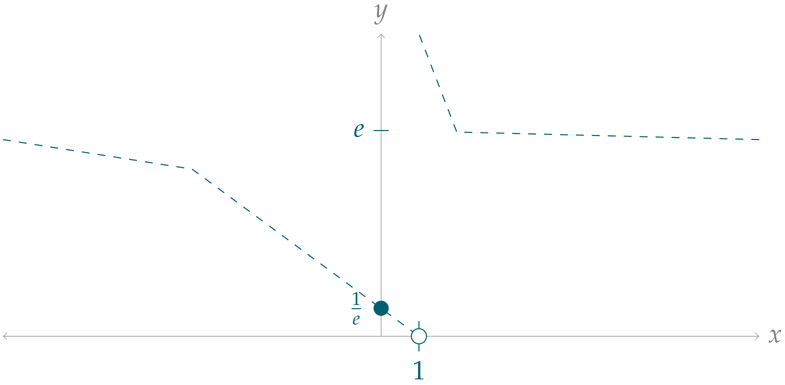
Critical points	$f'(x) = e^{\frac{x+1}{x-1}}\left(\frac{-2}{(x-1)^2}\right)$	no critical points
Increasing, decreasing	$f(x)$ is decreasing everywhere it is defined	

This information is enough to draw a skeleton.

A

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

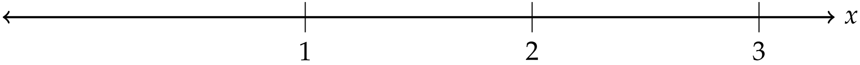


https://www.desmos.com/calculator/x0cccy1ggj

Signs of Factored Functions

SKIP SIGN CHANGES

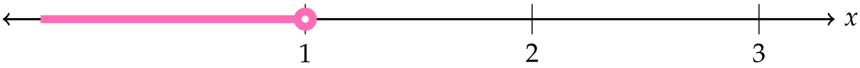
$$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$$



Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \underset{-}{(x-1)} \quad (x-2) \quad (x-3)$$



Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \quad \underset{+}{(x-1)} \quad (x-2) \quad (x-3)$$

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$$

$-$

Signs of Factored Functions

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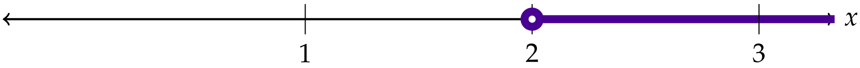
$$f(x) = \quad (x-1) \quad \underset{+}{(x-2)} \quad (x-3)$$



Signs of Factored Functions

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$$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$$

$-$

Signs of Factored Functions

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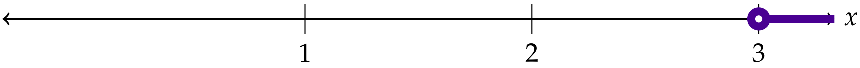
$+$



Signs of Factored Functions

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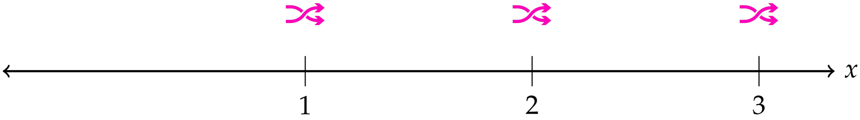
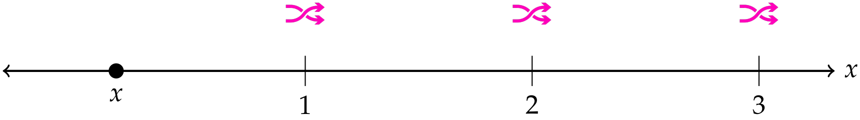


Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \underset{-}{(x-1)} \quad \underset{-}{(x-2)} \quad \underset{-}{(x-3)}$$



Sign of entire function: $\boxed{-}$

Signs of Factored Functions

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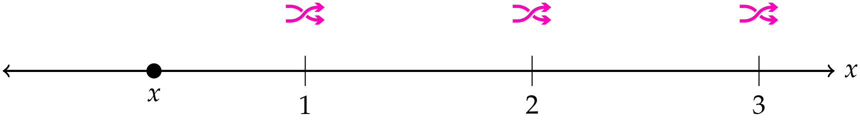

Sign of entire function: $-$

Signs of Factored Functions

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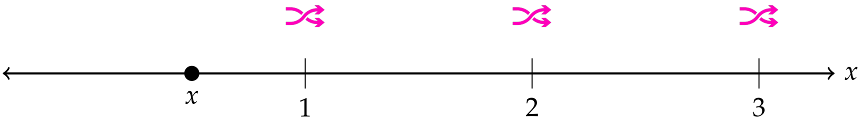

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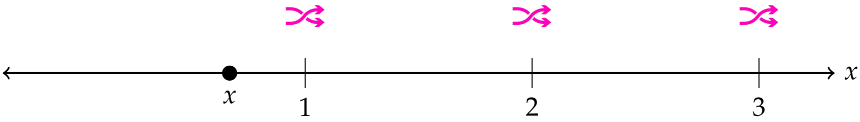

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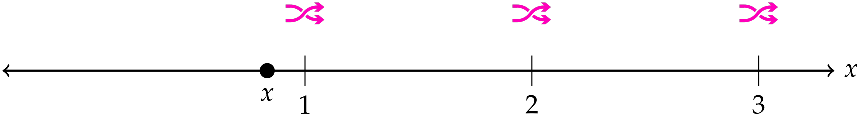

Sign of entire function: $-$

Signs of Factored Functions

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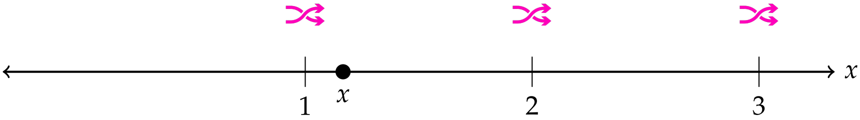
$$f(x) = \underset{+}{(x-1)} \quad \underset{-}{(x-2)} \quad \underset{-}{(x-3)}$$



Sign of entire function: $+$

Signs of Factored Functions

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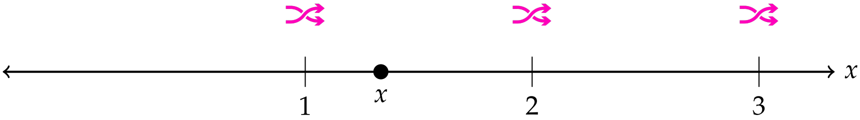

Sign of entire function: $+$

Signs of Factored Functions

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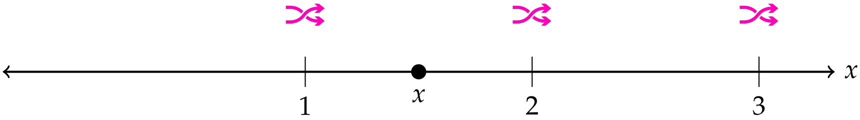

Sign of entire function: $+$

Signs of Factored Functions

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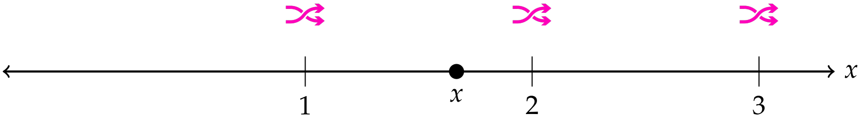

Sign of entire function: $+$

SIGNS OF FACTORED FUNCTIONS

SKIP SIGN CHANGES

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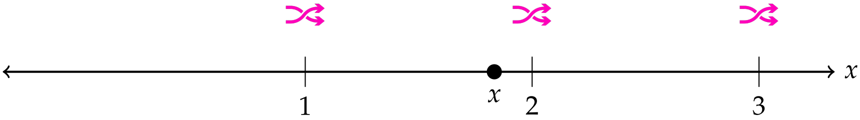

Sign of entire function: $+$

Signs of Factored Functions

SKIP SIGN CHANGES

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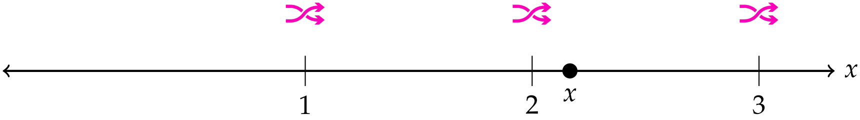
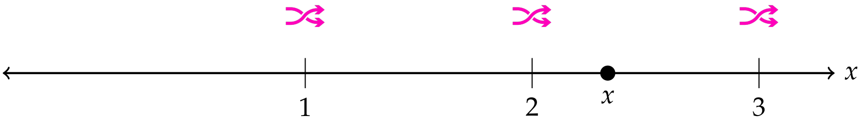


Sign of entire function: $\boxed{-}$

SIGNS OF FACTORED FUNCTIONS

SKIP SIGN CHANGES

$$f(x) = \underset{+}{(x-1)}\ \underset{+}{(x-2)}\ \underset{-}{(x-3)}$$

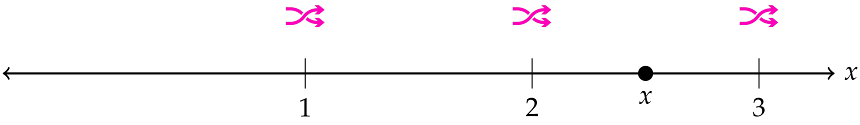


Sign of entire function: $-$

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \begin{array}{ccc} (x-1) & (x-2) & (x-3) \\ + & + & - \end{array}$$

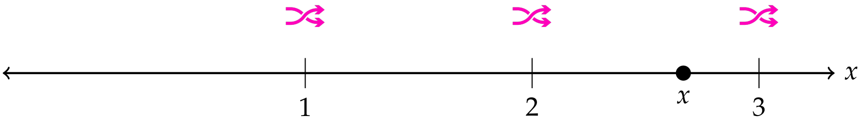


Sign of entire function: $-$

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \underset{+}{(x-1)} \quad \underset{+}{(x-2)} \quad \underset{-}{(x-3)}$$



Sign of entire function: $-$

Signs of Factored Functions

SKIP SIGN CHANGES

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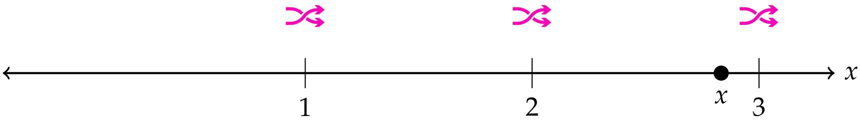

Sign of entire function: $\boxed{-}$

SIGNS OF FACTORED FUNCTIONS

SKIP SIGN CHANGES

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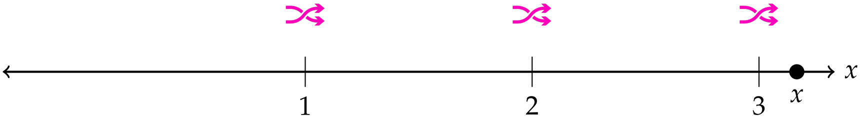
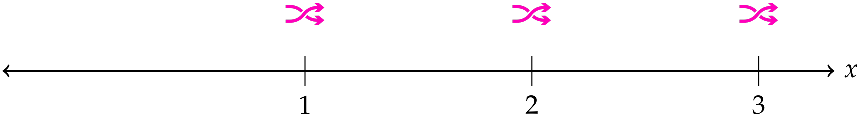


Sign of entire function: $+$

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$$



Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

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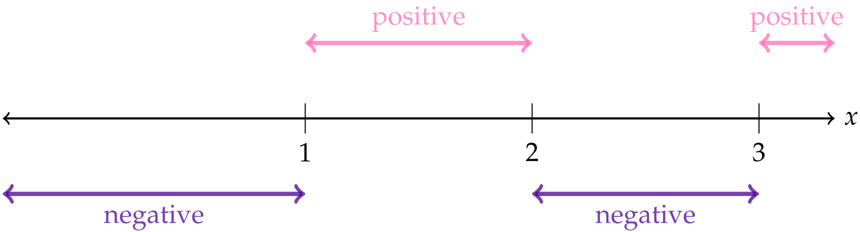
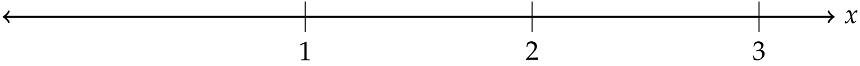


Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \quad (x-1) \quad (x-2)^2 \quad (x-3)$$

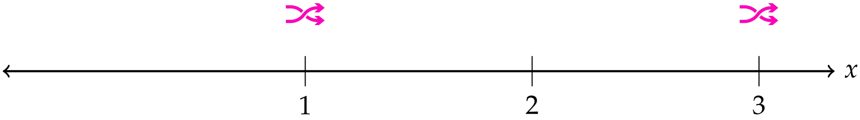


Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = (x-1)\,\underset{+}{(x-2)^2}\,(x-3)$$

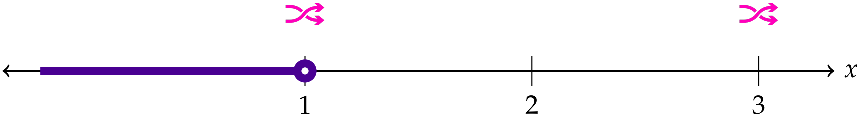


Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \underset{-}{(x-1)} \quad \underset{+}{(x-2)^2} \quad \underset{-}{(x-3)}$$



Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

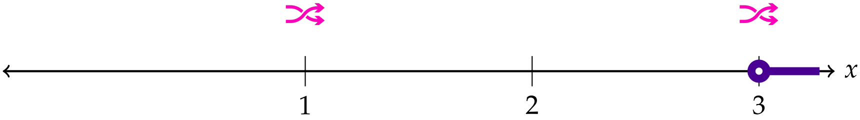
$$f(x) = \underset{+}{(x-1)} \quad \underset{+}{(x-2)^2} \quad \underset{-}{(x-3)}$$

Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = \underset{+}{(x-1)} \quad \underset{+}{(x-2)^2} \quad \underset{+}{(x-3)}$$



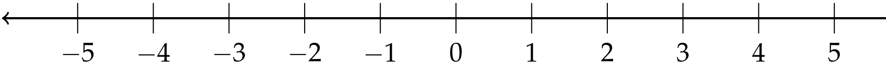
Sign of entire function:

Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = (x-3)(x-1)^2x(x+2)^3(x+5)^4$$

Where is $f(x)$ positive? Where is it negative?



Signs of Factored Functions

▸▸ SKIP SIGN CHANGES

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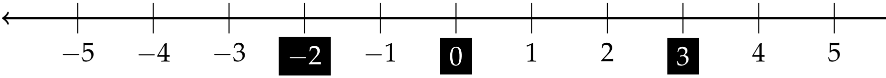
Where is $f(x)$ positive? Where is it negative?

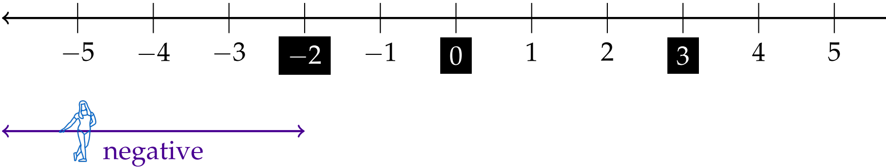


Signs of Factored Functions

SKIP SIGN CHANGES

$$f(x) = (x-3)(x-1)^2x(x+2)^3(x+5)^4$$

Where is $f(x)$ positive? Where is it negative?



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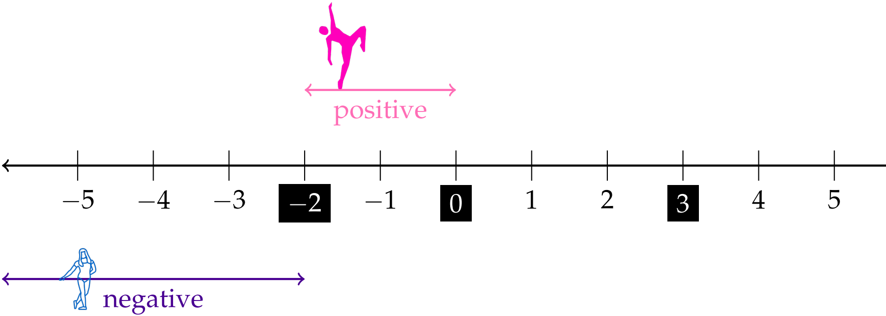

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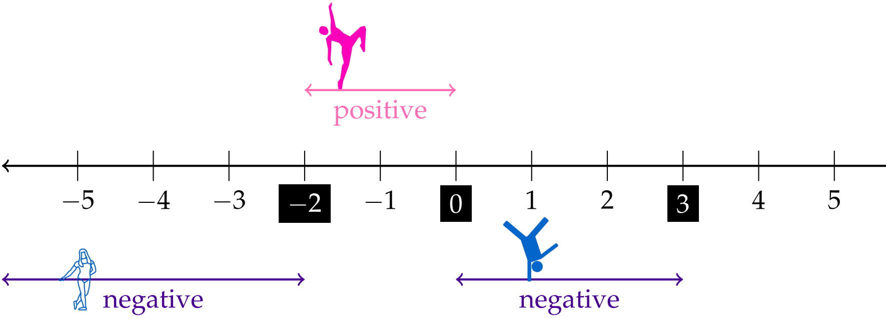

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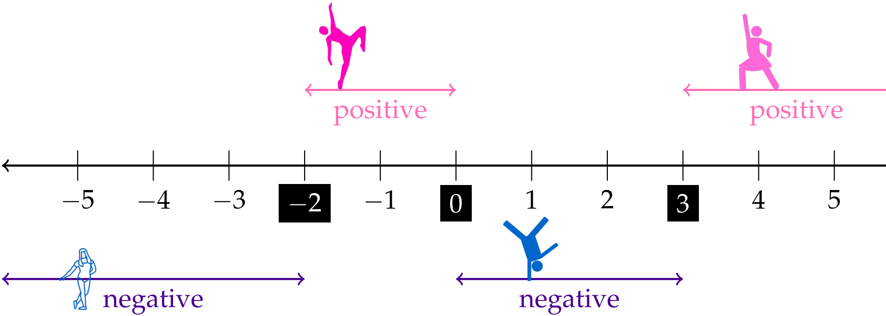
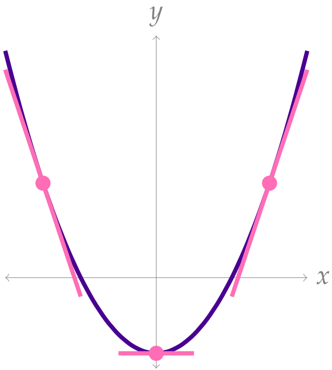
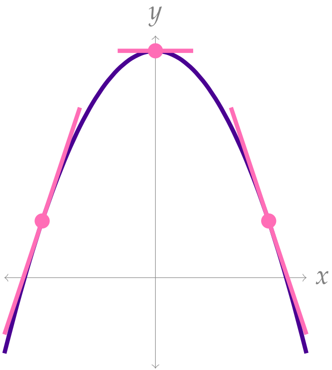


Concavity



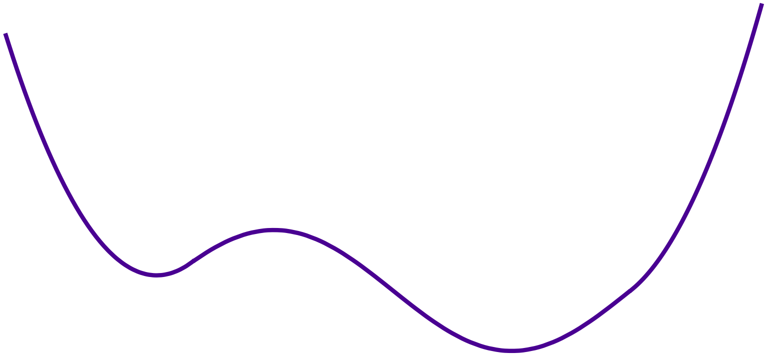
- Slopes are increasing
- $f''(x) > 0$
- "concave up"
- tangent line below curve



- Slopes are decreasing
- $f''(x) < 0$
- "concave down"
- tangent line above curve

MNEMONIC

Concavity



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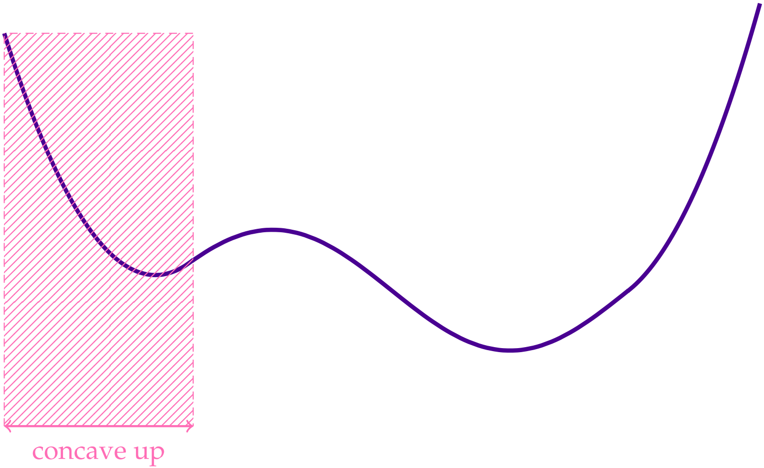

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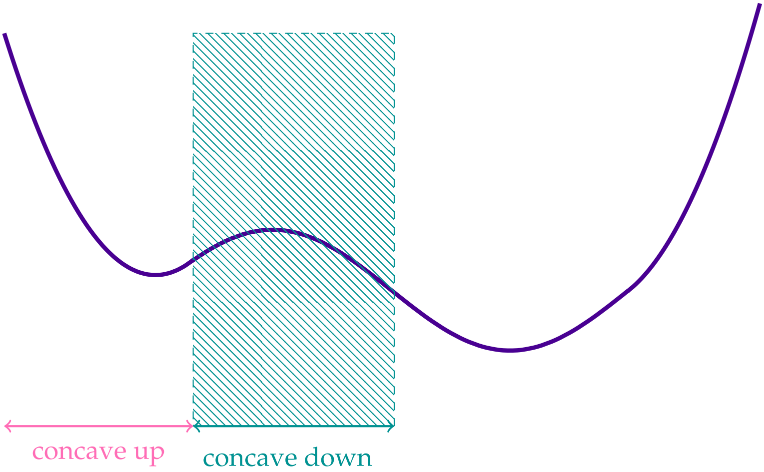
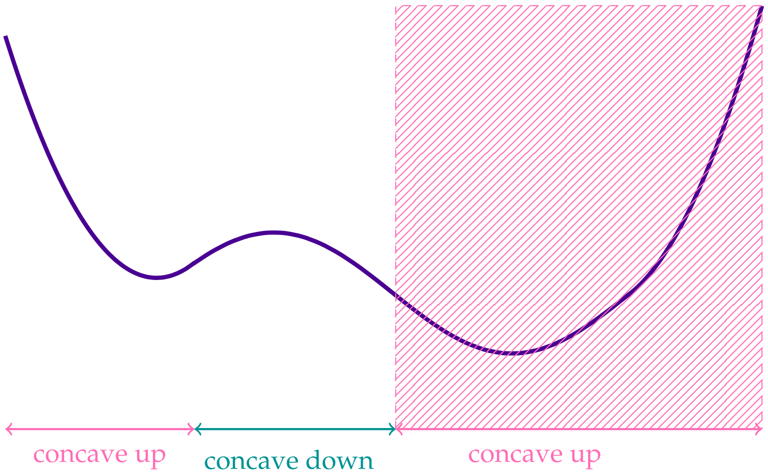


CONCAVITY



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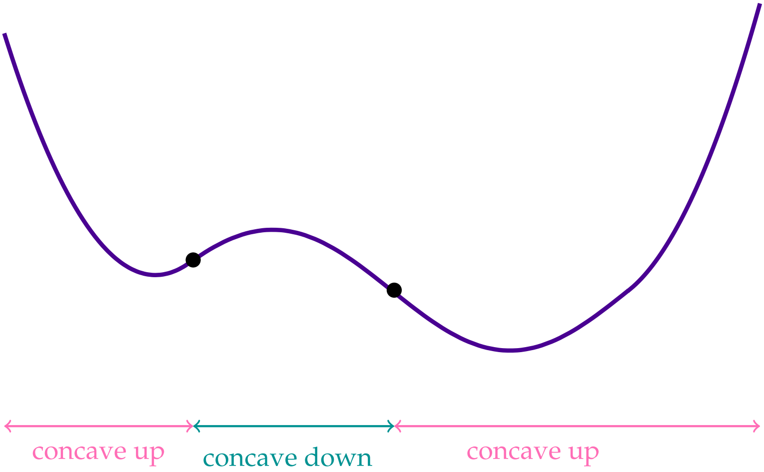

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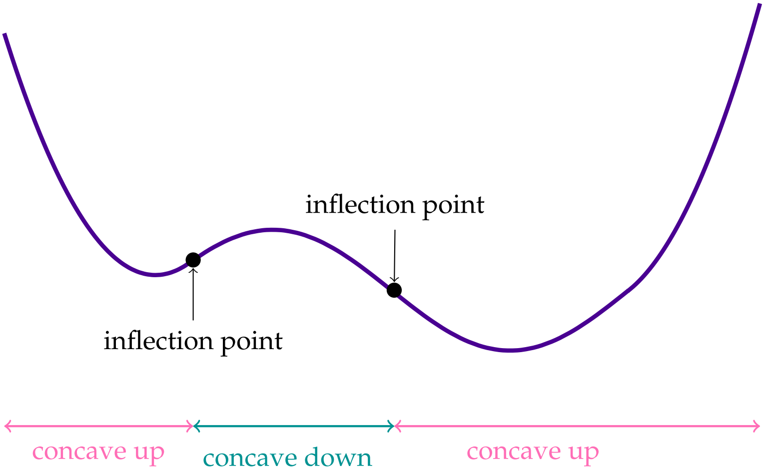
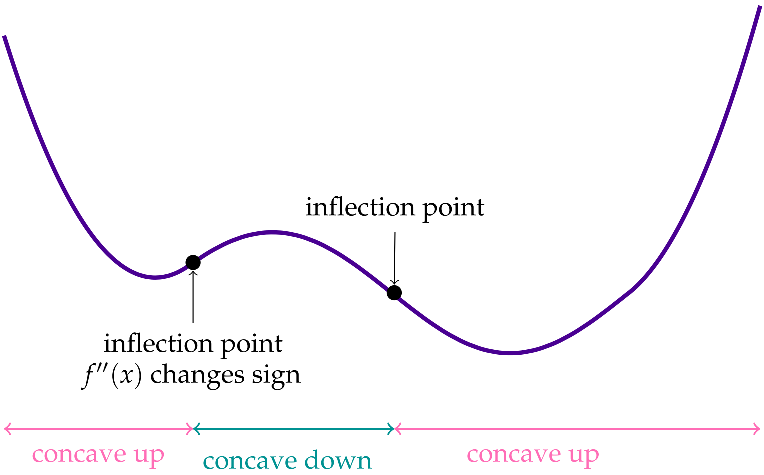
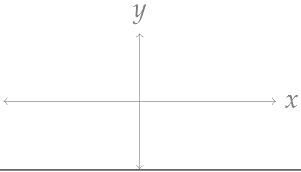
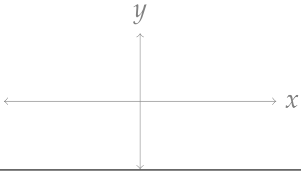
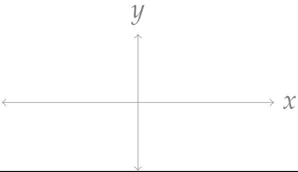
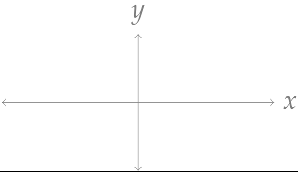

inflection point
inflection point
concave up
concave down
concave up

Concavity


inflection point
inflection point
$f''(x)$ changes sign
concave up
concave down
concave up

Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

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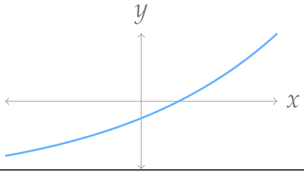
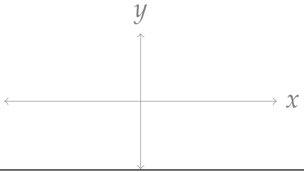
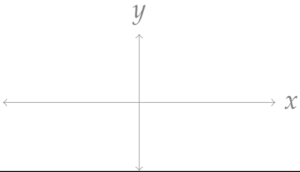
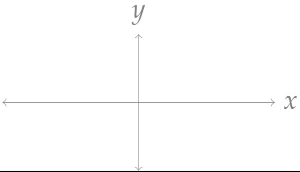
increasing		
decreasing		

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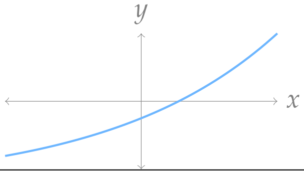
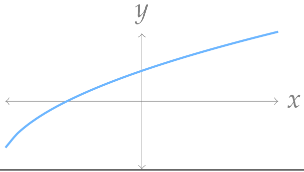
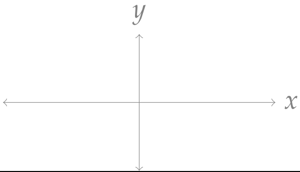
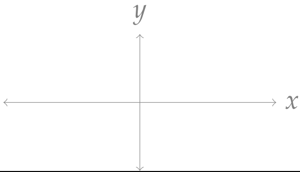
increasing		
decreasing		

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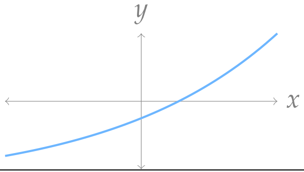
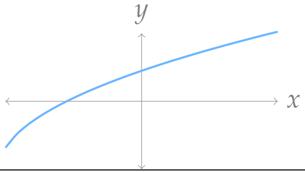
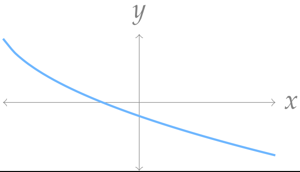
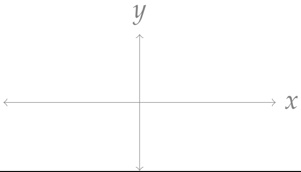
increasing		
decreasing		

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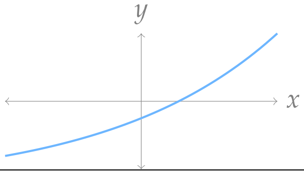
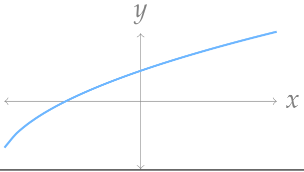
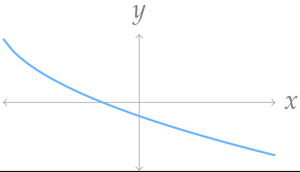
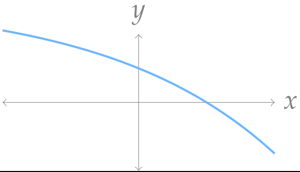
increasing	 	
decreasing	 	

POLL QUESTIONS

Describe the concavity of the function $f(x) = e^x$.

A. concave up
B. concave down
C. concave up for $x < 0$; concave down for $x > 0$
D. concave down for $x < 0$; concave up for $x > 0$
E. I'm not sure

Q

ans

Poll Questions

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Is it possible to be concave up and decreasing?

A. Yes B. No C. I'm not sure

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A. $f'(0) < f'(1)$
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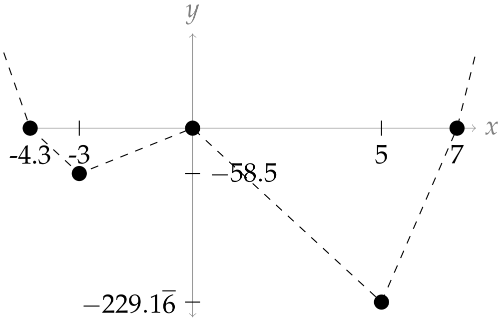
A

REVISITING A PREVIOUS EXAMPLE

original example

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$



Q

ans

Revisiting a previous example

original example

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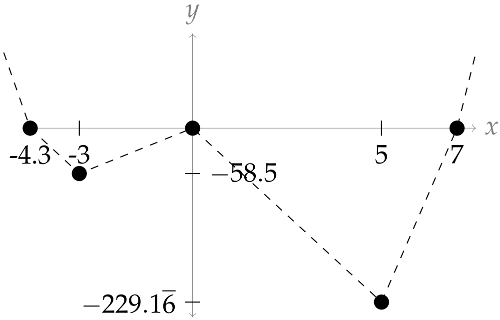

$$f''(x) = 6x^2 - 8x - 30 = 2(x-3)(3x+5)$$

Revisiting a previous example

original example

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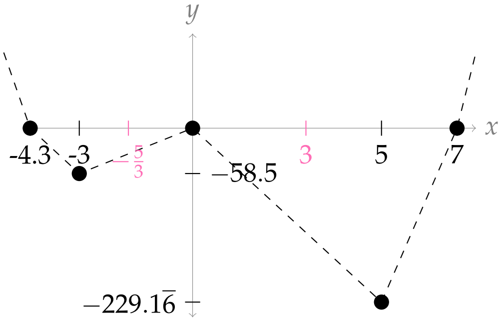

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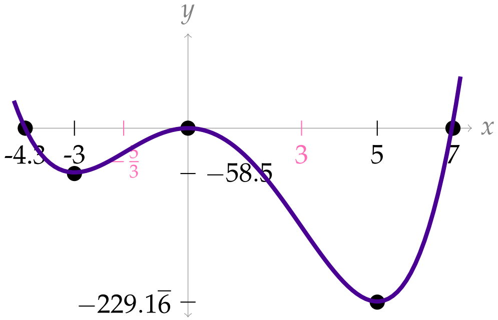


$$f''(x) = 6x^2 - 8x - 30 = 2(x-3)(3x+5)$$

Sketch:

$$f(x) = x^5 - 15x^3$$

A

ans

Sketch:

$$f(x) = x^5 - 15x^3$$

Domain	Defined and differentiable for all real numbers.
Intercepts	$f(x) = x^3(x^2 - 15)$: Roots are at $x = 0$ and $x = \pm\sqrt{15} \approx \pm 4$
End behaviour	$\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$
Critical points	$f'(x) = 5x^4 - 45x^2 = 5x^2\left(x^2 - 9\right)$. So the critical points are $x = 0$, $x = \pm 3$.
Increasing, decreasing	Increasing on $(-\infty, -3)$, decreasing on $(-3, 0)$ and $(0, 3)$, increasing on $(3, \infty)$
Local extrema	From intervals of increase and decrease: local max at $x = -3$ and local min at $x = 3$

Q

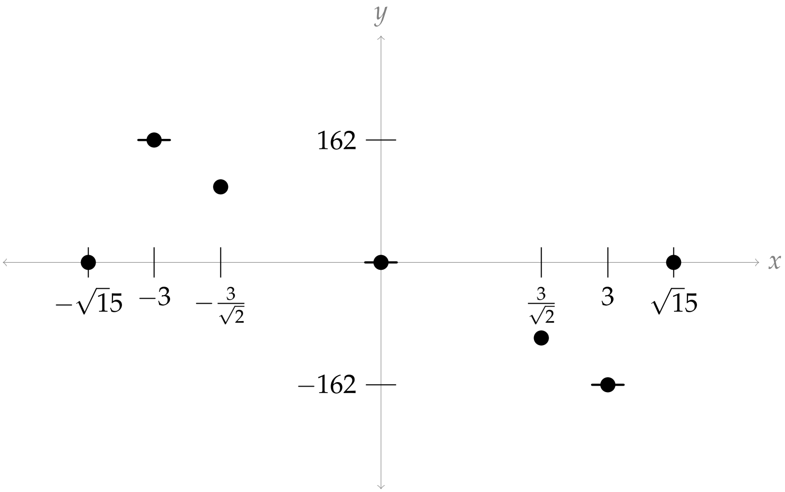
Sketch:

$$f(x) = x^5 - 15x^3$$

Concavity	$f''(x) = 20x^3 - 90x = 10x(2x^2 - 9) = 0$ for $x = 0$ and $x = \pm\frac{3}{\sqrt{2}} \approx \pm 2$. All of these are inflection points; concave down $(-\infty, -\frac{3}{\sqrt{2}})$, concave up $(-\frac{3}{\sqrt{2}}, 0)$, concave down $(0, \frac{3}{\sqrt{2}})$, and concave up $(\frac{3}{\sqrt{2}}, \infty)$.
***y*-values of notable points**	$f(3) = -162, f(-3) = 162, f(-3/\sqrt{2}) \approx 100, f(3/\sqrt{2}) \approx -100$

Sketch:

$$f(x) = x^5 - 15x^3$$



Sketch:

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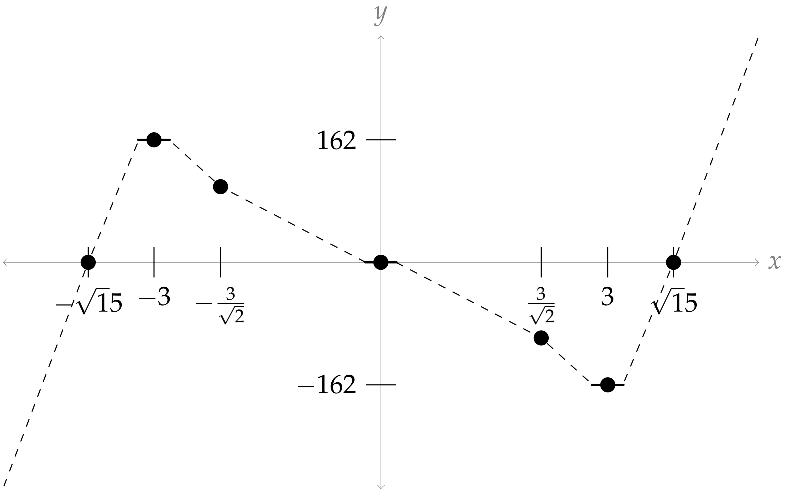

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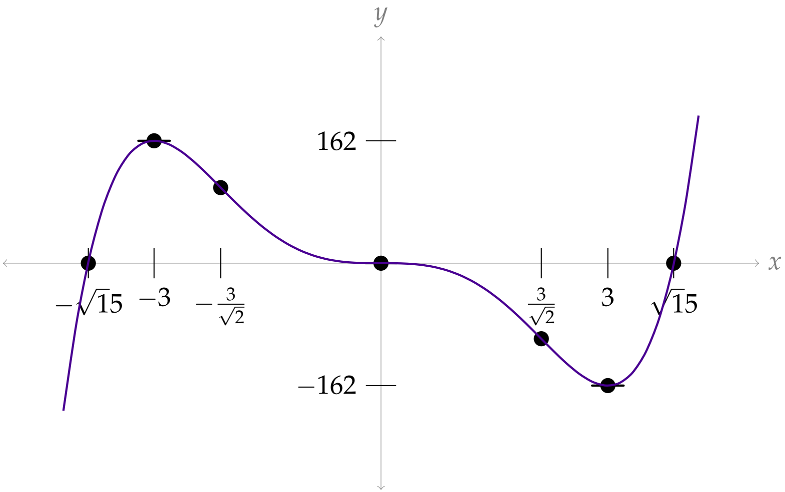
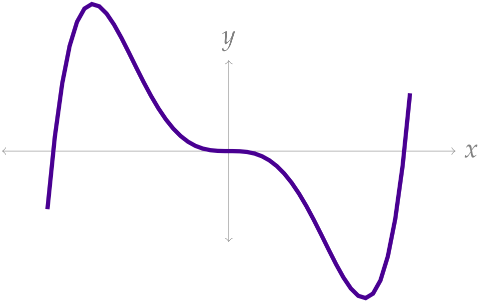


https://www.desmos.com/calculator/uoii6nmgr8

Even and Odd Functions



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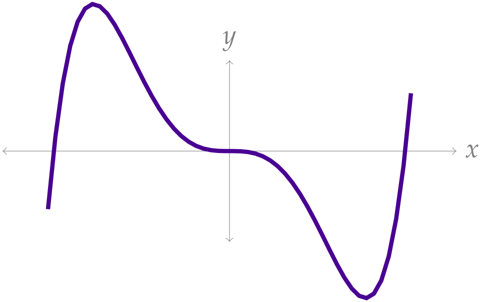
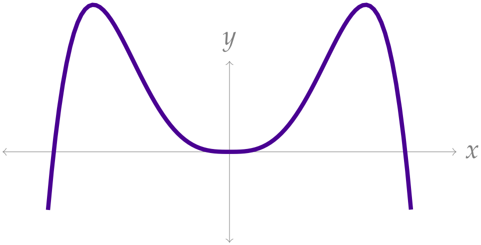
Even and Odd Functions



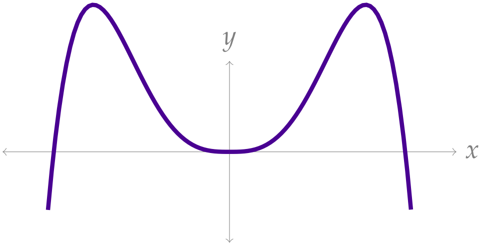
$$f(x) = x^5 - 15x^3$$

odd function

Even and Odd Functions



Even and Odd Functions

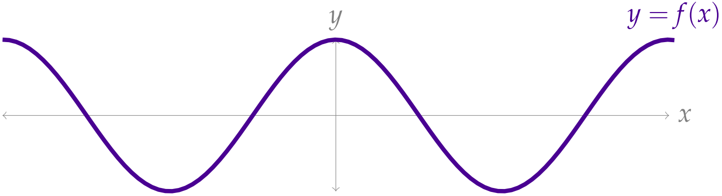


even function

Even Function – Definition 3.6.6

A function $f(x)$ is even if, for all x in its domain,

$$f(-x) = f(x)$$



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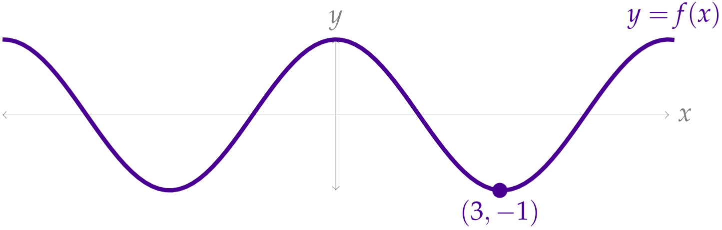

Suppose $f(3) = -1$.

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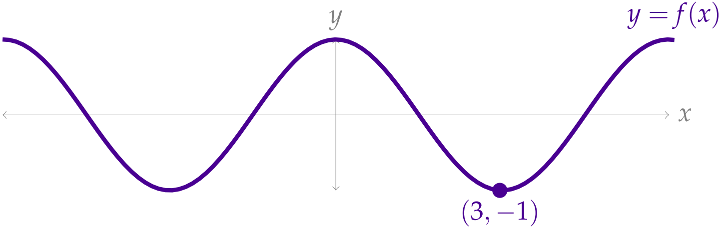

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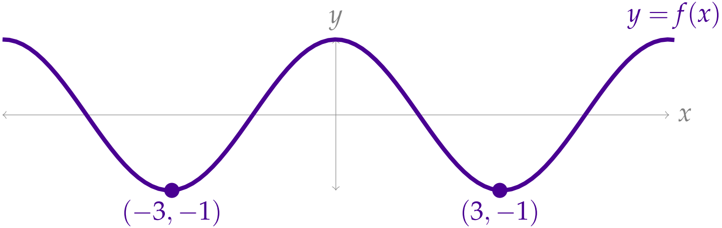

Suppose $f(3) = -1$. Then $f(-3) = -1$ also.

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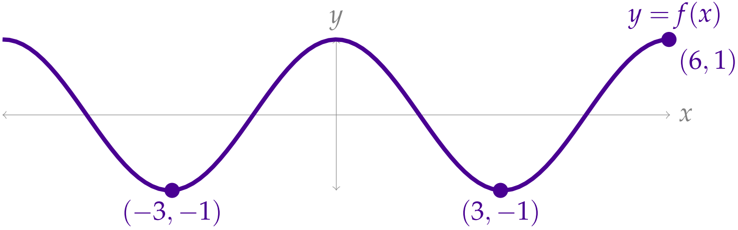

Suppose $f(3) = -1$. Then $f(-3) = -1$ also.
Suppose $f(6) = 1$.

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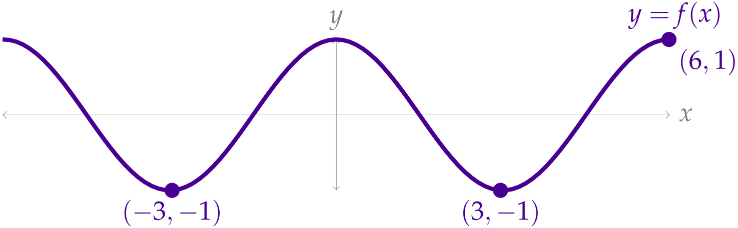

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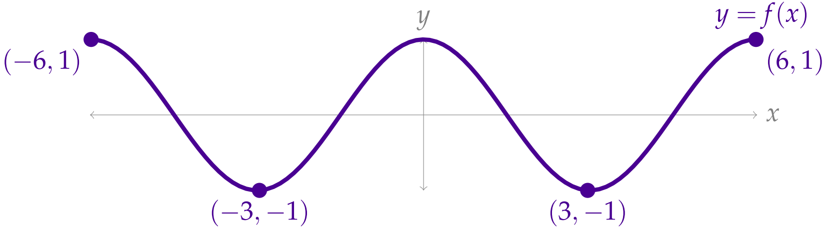

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Even Functions

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Examples:

Even Functions

Even Function – Definition 3.6.6

A function $f(x)$ is even if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:
$f(x) = x^2$

Even Functions

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Examples:
$f(x) = x^2$
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Even Functions

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Examples:
$f(x) = x^2$
$f(x) = x^4$
$f(x) = \cos(x)$

Even Functions

Even Function – Definition 3.6.6

A function $f(x)$ is even if, for all x in its domain,

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Examples:

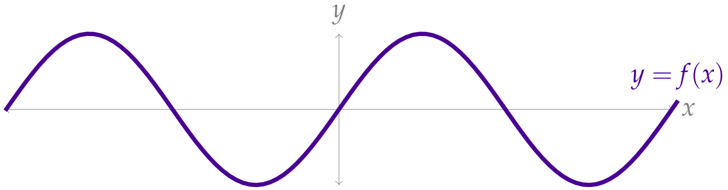
$f(x) = x^2$

$f(x) = x^4$

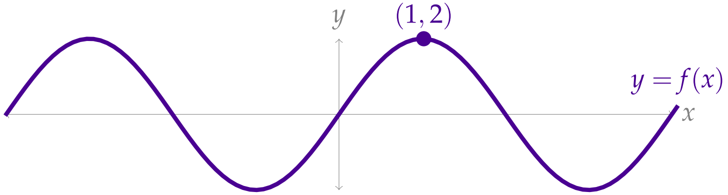
$f(x) = \cos(x)$

$$f(x) = \frac{x^4 + \cos(x)}{x^{16} + 7}$$

Odd Functions

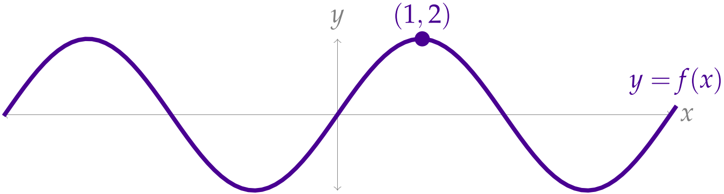


Odd Functions



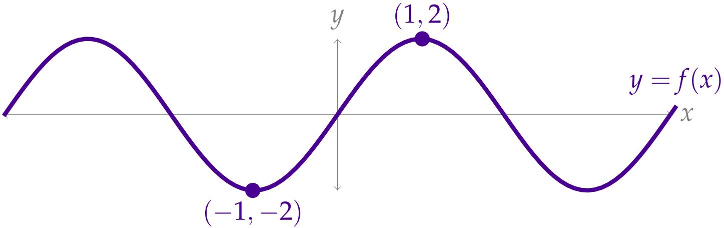
Suppose $f(1)=2$.

ODD FUNCTIONS



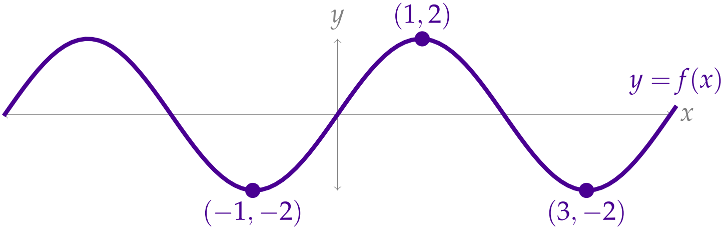
Suppose $f(1) = 2$. Then $f(-1) =$

ODD FUNCTIONS



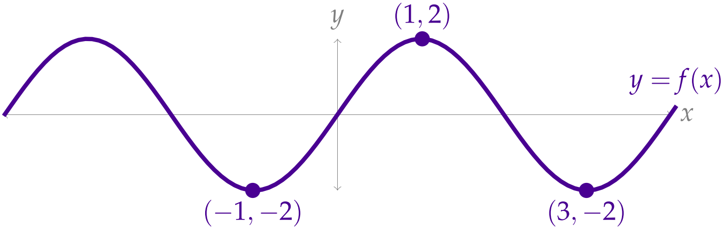
Suppose $f(1) = 2$. Then $f(-1) = -2$.

ODD FUNCTIONS



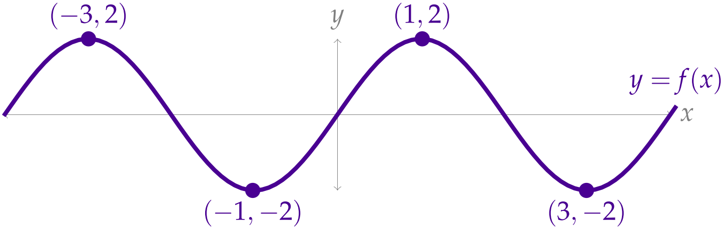
Suppose $f(1) = 2$. Then $f(-1) = -2$.
Suppose $f(3) = -2$.

ODD FUNCTIONS



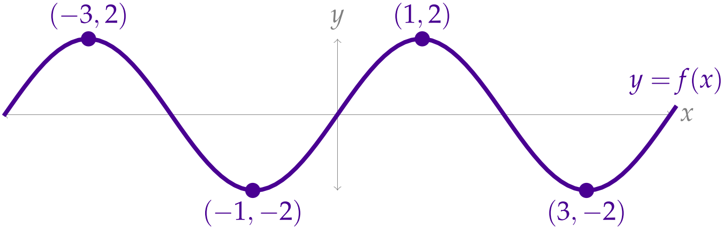
Suppose $f(1) = 2$. Then $f(-1) = -2$.
Suppose $f(3) = -2$. Then $f(-3) =$

Odd Functions



Suppose $f(1) = 2$. Then $f(-1) = -2$.
Suppose $f(3) = -2$. Then $f(-3) = 2$.

Odd Functions

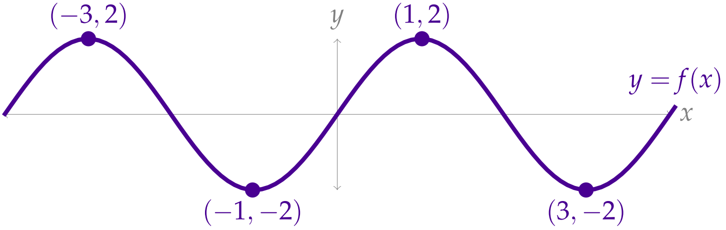


Suppose $f(1) = 2$. Then $f(-1) = -2$.
Suppose $f(3) = -2$. Then $f(-3) = 2$.

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

ODD FUNCTIONS



Suppose $f(1)=2$. Then $f(-1)=-2$.
Suppose $f(3)=-2$. Then $f(-3)=2$.

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x)=-f(x)$$

Odd Functions

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

Odd Functions

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$f(x) = x$

ODD FUNCTIONS

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:
$f(x) = x$
$f(x) = x^3$

Odd Functions

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$f(x) = x$
$f(x) = x^3$
$f(x) = \sin(x)$

Odd Functions

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Examples:

$f(x) = x$

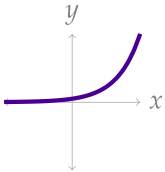
$f(x) = x^3$

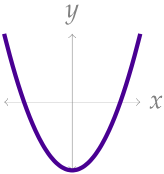
$f(x) = \sin(x)$

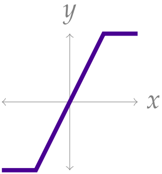
$f(x) = \dfrac{x(1+x^2)}{x^2+5}$

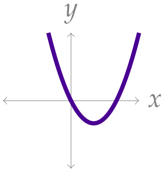
Poll Tiiime

Pick out the odd function.

A: 

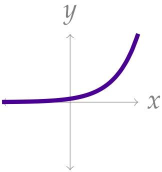
B: 

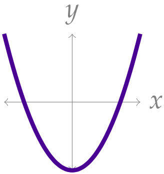
C: 

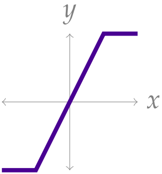
D: 

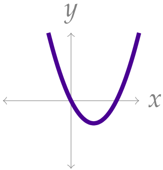
Poll Tiiime

Pick out the odd function.

A: 

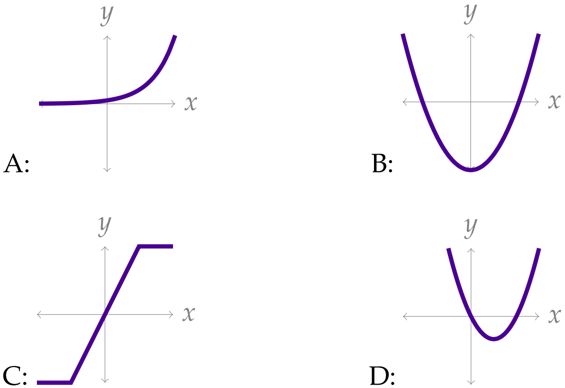
B: 

C: 

D: 

POLL TIIIME

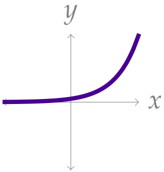
Pick out the even function.

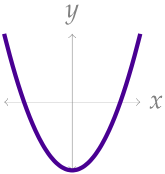


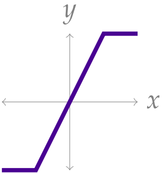
ans

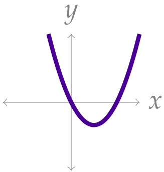
Poll Tiiime

Pick out the even function.

A:


B:


C:


D:


EVEN MORE POLL TIIIIIME

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

Q

ans

Even more Poll tiiiiime

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

A

Even more Poll tiiiiime

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$ $\leftarrow$ true but uninteresting, for all functions

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

Even more Poll tiiiiime

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0) \leftarrow$ true but uninteresting, for all functions

B. $f(0) = -f(0) \leftarrow$ only possible for $f(0) = 0$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

Even more Poll tiiiiime

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$ $\leftarrow$ true but uninteresting, for all functions

B. $f(0) = -f(0)$ $\leftarrow$ only possible for $f(0) = 0$

C. $f(0) = 0$ $\leftarrow$ this is equivalent to the choice above

D. all of the above are true

E. none of the above are necessarily true

Even more and more Poll tiiiiime

Suppose $f(x)$ is an even function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

Q

ans

EVEN MORE AND MORE POLL TIIIIIME

Suppose $f(x)$ is an even function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$

B. $f(0) = -f(0)$

C. $f(0) = 0$

D. all of the above are true

E. none of the above are necessarily true

A

OK OK... LAST ONE

Suppose $f(x)$ is an even function, differentiable for all real numbers. What can we say about $f'(x)$?

A. $f'(x)$ is also even

B. $f'(x)$ is odd

C. $f'(x)$ is constant

D. all of the above are true

E. none of the above are necessarily true

Q

ans

OK OK... LAST ONE

Suppose $f(x)$ is an even function, differentiable for all real numbers. What can we say about $f'(x)$?

A. $f'(x)$ is also even

B. $f'(x)$ is odd

C. $f'(x)$ is constant

D. all of the above are true

E. none of the above are necessarily true

A

PERIODICITY

Periodic – Definition 3.6.10

A function is periodic with period $P > 0$ if

$$f(x) = f(x + P)$$

whenever x and $x + P$ are in the domain of f, and P is the smallest such (positive) number

Examples: $\sin(x)$, $\cos(x)$ both have period 2π; $\tan(x)$ has period π.

Ignoring concavity, sketch $f(x) = \sin(\sin x)$.

Challenge: ignoring exact locations of extrema, sketch $g(x) = \sin(2\pi \sin x)$.

$$f(x) = \sin(\sin x)$$

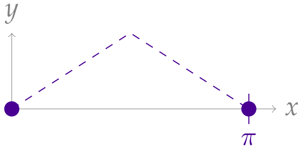
$$f(x) = \sin(\sin x)$$

Sin is periodic; since $\sin x = \sin(2\pi + x)$, then $\sin(\sin x) = \sin(\sin(2\pi + x))$, so $f(x)$ is also periodic. It suffices to sketch $f(x)$ for an interval of length 2π, because any such segment will repeat.

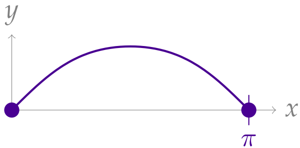
Since the function is also odd, if we sketch it on the interval $[0, \pi]$, then we can extrapolate to the interval $[-\pi, 0]$. So we consider the interval $[0, \pi]$.

- Intercepts: $(0, 0)$, $(0, \pi)$
- First derivative: $f'(x) = \cos(\sin x) \cdot \cos(x)$
  - For $0 \le x \le \pi$, we have $0 \le \sin x \le 1 < \frac{\pi}{2}$ and hence $0 < \cos(\sin x) \le 1$.
  - CP: $x = \frac{\pi}{2}$
  - increasing: $(0, \pi/2)$
  - decreasing: $(\pi/2, 0)$

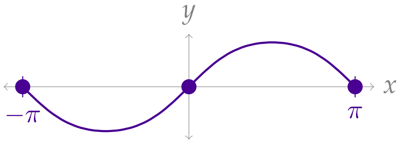
1. Interval $(0, \pi)$ skeleton, based on above work:



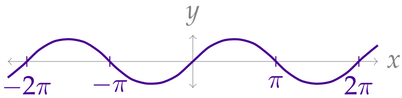
2. Make into a smooth curve:



3. Use odd symmetry to get interval $[-\pi, \pi]$



4. Use periodicity



skip $g(x)$

$$g(x) = \sin(2\pi \sin x)$$

A

skip $g(x)$

$$g(x) = \sin(2\pi \sin x)$$

Sin is periodic; since $2\pi \sin x = 2\pi \sin(2\pi + x)$, then

$$\sin(2\pi \sin x) = \sin(2\pi \sin(2\pi + x))$$

so $g(x)$ is also periodic. It suffices to sketch $g(x)$ for an interval of length 2π, because any such segment will repeat.

Note

$$\begin{aligned} g(-x) &= \sin\left(2\pi \sin(-x)\right) = \sin\left((2\pi)(-\sin x)\right) \\ &= \sin(-2\pi \sin x) = -\sin(2\pi \sin x) = -g(x) \end{aligned}$$

so $g(x)$ is odd. If we sketch it on the interval $[0, \pi]$, then we can extrapolate to the interval $[-\pi, 0]$. So we consider the interval $[0, \pi]$.

skip $g(x)$

Intercepts in $[0, \pi]$:

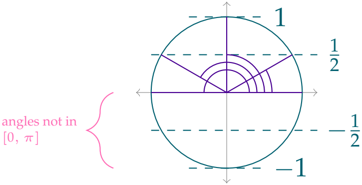
$$g(0) = 0$$

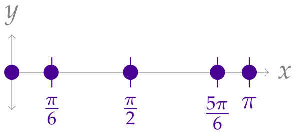
$$0 = g(0) = \sin(2\pi \sin x)$$

$$\implies 2\pi \sin x \in \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$$

$$\implies \sin x \in \left\{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots\right\}$$

$$\implies x \in \left\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi\right\}$$



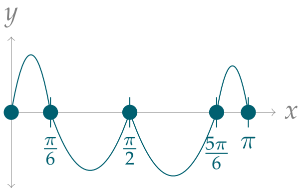


skip $g(x)$

Now let's consider the sign of $g(x)$ between the intercepts. Since $g(x)$ isn't given as a factored product, our old shortcut isn't so useful.

interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \pi\right)$
range of $\sin x$	$\left(0, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 1\right)$	$\left(0, \frac{1}{2}\right)$
range of $2\pi \sin x$	$(0, \pi)$	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(0, \pi)$
sign of $\sin(2\pi \sin x)$	$+$	$-$	$-$	$+$

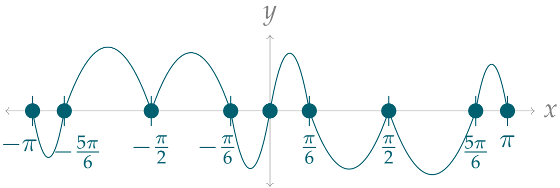
So, a rough sketch on the interval $[0, \pi]$ is:



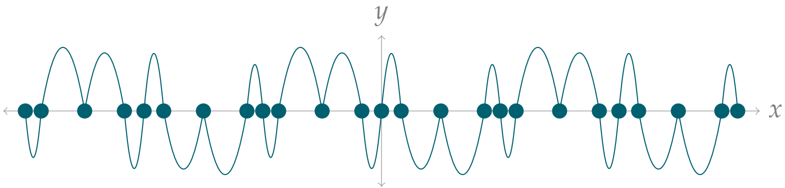
Yes, this is a rough sketch. The curve should be smooth at $\frac{\pi}{2}$.

skip $g(x)$

Use odd symmetry, we sketch the interval $[-\pi, \pi]$:



Using periodicity:



Let's Graph

$$f(x) = (x^2 - 64)^{1/3}$$

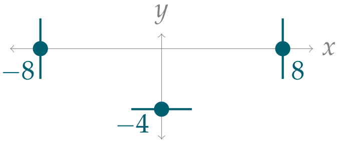
LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2 - 64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2 + 64)}{3(x^2 - 64)^{5/3}}$$

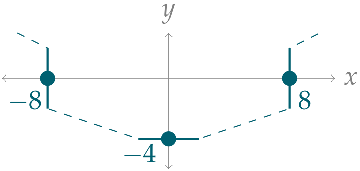
Domain	all real numbers
End behaviour	$\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$
Intercepts	$(0, -4)$, $(\pm 8, 0)$
Critical point	$(0, -4)$
Singular points	$(-8, 0)$, $(8, 0)$ Near the singular points, $f'(x)$ gets very large, so $f(x)$ looks vertical.



Increasing, decreasing

Numerator of $f'(x)$ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$. Denominator is positive where it exists. So $f(x)$ is decreasing on $(-\infty, 0)$ (where it is differentiable) and increasing on $(0, \infty)$ (where it is differentiable).

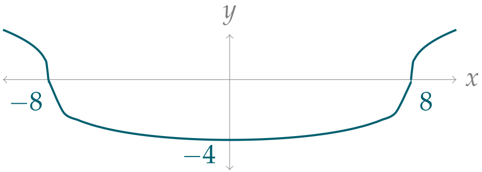
Skeleton



Concavity

Numerator of second derivative is negative everywhere. Denominator is positive (so $f''(x)$ is negative) on $(-\infty, -8) \cup (8, \infty)$ and denominator is negative (so $f''(x)$ is positive) on $(-8, 8)$.

So our function is concave up on $(-8, 8)$ and concave down on $(-\infty, -8) \cup (8, \infty)$.



Let's Graph

$$f(x) = \frac{x^2 + x}{(x+1)(x^2+1)^2}$$

Q

Let's Graph

$$f(x) = \frac{x^2+x}{(x+1)(x^2+1)^2}$$

Note: for $x \neq -1, f(x) = \dfrac{x(x+1)}{(x+1)(x^2+1)^2} = \dfrac{x}{(x^2+1)^2}$

Let's Graph

$$f(x) = \frac{x^2 + x}{(x+1)(x^2+1)^2}$$

Note: for $x \neq -1, f(x) = \dfrac{x(x+1)}{(x+1)(x^2+1)^2} = \dfrac{x}{(x^2+1)^2}$

$$g(x) := \frac{x}{(x^2+1)^2}$$

Let's Graph

$$f(x) = \frac{x^2 + x}{(x+1)(x^2+1)^2}$$

Note: for $x \neq -1, f(x) = \dfrac{x(x+1)}{(x+1)(x^2+1)^2} = \dfrac{x}{(x^2+1)^2}$

$$g(x) := \frac{x}{(x^2+1)^2}$$

$$g'(x) = \frac{1-3x^2}{(x^2+1)^3}$$

$$g''(x) = \frac{12x(x^2-1)}{(x^2+1)^4}$$

$$f(x) = \frac{x^2 + x}{(x+1)(x^2+1)^2}$$

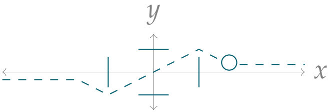
$$f(x) = \frac{x^2 + x}{(x+1)(x^2+1)^2}$$

When $x \neq 1$, $f(x) = g(x)$. So, $f(x)$ looks like $g(x)$ except it has a removable discontinuity (hole) at $x = 1$. Let's graph

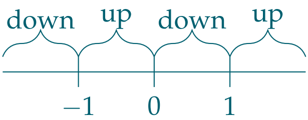
$$g(x) = \frac{x}{(x^2+1)^2}$$

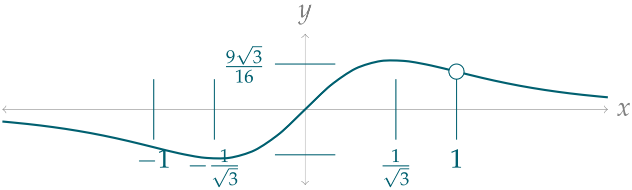
- Domain: all real numbers
- HA: $y = 0$ on both sides
- VA: none
- Intercepts: $(0, 0)$
- Odd symmetry
- CP: $x = \pm\frac{1}{\sqrt{3}}$; associated points: $\left(-\frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{16}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{16}\right)$

- Increasing on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$



- g'' is given (nearly) factored, so we can see its sign changes at $x = -1, 0, 1$. Concavity:





Let's Graph

$$f(x) = x(x-1)^{2/3}$$

- $f'(x) = \dfrac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \dfrac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

Q

ans

Let's Graph

$$f(x) = x(x-1)^{2/3}$$

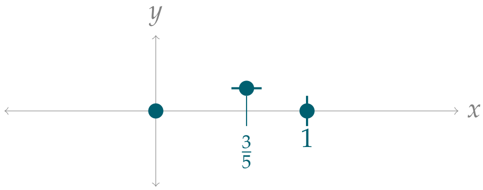
- $f'(x) = \dfrac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \dfrac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

- $f(3/5) \approx 0.3$
- $f(6/5) \approx 0.4$

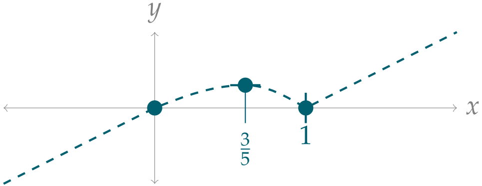
Q

ans

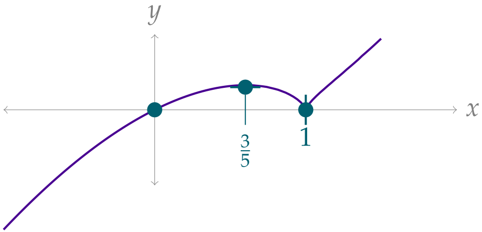
- Domain: all reals
- VA: none
- HA: none
- Intercepts: $(0, 0)$, $(1, 0)$
- Symmetry: not even, not odd, not periodic
- CP: $x = \frac{3}{5}$ $(y \approx 0.3)$; SP $x = 1$ $(y = 0)$.
  Near the SP, first deriv is very large, so function looks vertical.



- First derivative changes sign at $x = \frac{3}{5}$ and $x = 1$. Function is increasing on $\left(-\infty, \frac{3}{5}\right) \cup (1, \infty)$ and decreasing on $\left(\frac{3}{5}, 1\right)$.



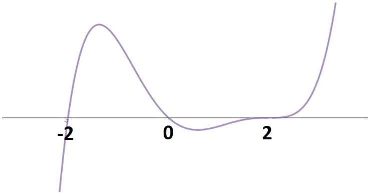
- Second derivative changes sign at $x = \frac{5}{6}$. (Note the denominator has an even power). Function is concave down on $\left(-\infty, \frac{5}{6}\right)$ and concave up on $\left(\frac{5}{6}, \infty\right)$.



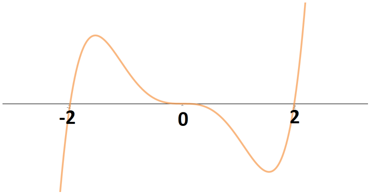
Ch 3.6 Review: matching

MATCH THE FUNCTION TO ITS GRAPH

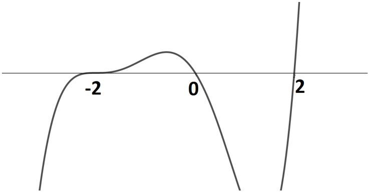
A. $f(x) = x^3(x+2)(x-2) = x^5 - 4x^3$
B. $f(x) = x(x+2)^3(x-2) = x^5 + 4x^4 - 16x^2 - 16x$
C. $f(x) = x(x+2)(x-2)^3 = x^5 - 4x^4 + 16x^2 - 16x$



I



III



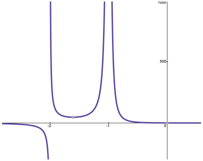
II

A. $f(x) = \dfrac{x-1}{(x+1)(x+2)}$

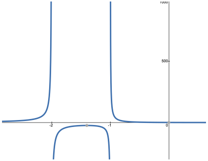
B. $f(x) = \dfrac{(x-1)^2}{(x+1)(x+2)}$

C. $f(x) = \dfrac{x-1}{(x+1)^2(x+2)}$

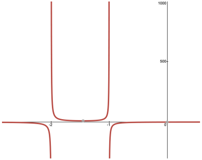
D. $f(x) = \dfrac{(x-1)^2}{(x+1)^2(x+2)}$



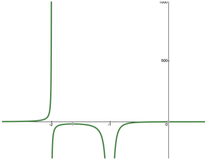
I.



III.



II.

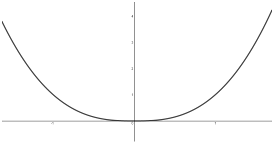


IV.

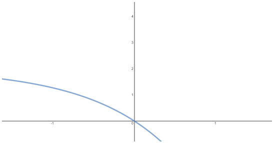
Q

Match the Function to its Graph

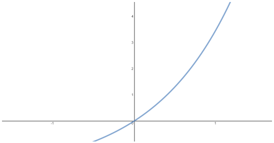
A. $f(x) = |x|^e$ B. $f(x) = e^{|x|}$ C. $f(x) = e^{x^2}$ D. $f(x) = e^{x^4 - x}$



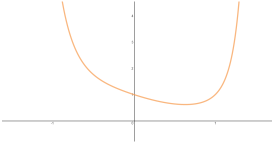
I



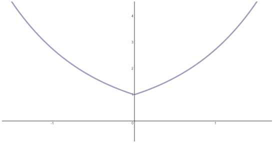
II



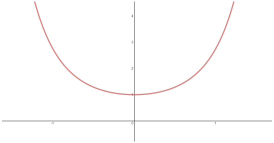
III



IV

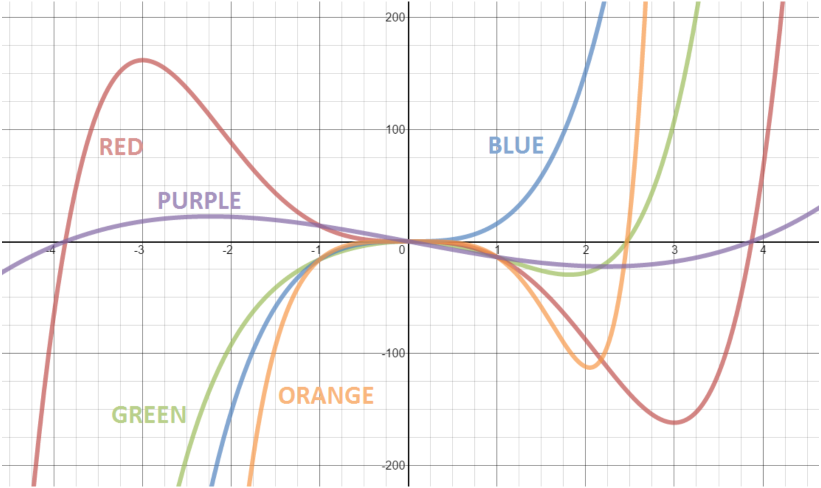


V



VI

A. $f(x) = x^5 + 15x^3$ B. $f(x) = x^5 - 15x^3$ C. $f(x) = x^5 - 15x^2$
D. $f(x) = x^3 - 15x$ E. $f(x) = x^7 - 15x^4$



Q

ans

Included Work

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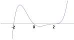
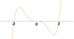
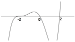
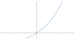
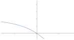
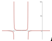
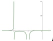
 'switch' by Anthony Ledoux is licensed under CC BY 3.0 (accessed 3 Oct 2018), 26–56

, ,  screenshots of graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator (accessed 13 November 2015), 163

, , , , ,  screenshots of graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator, (accessed 13 November 2015), 165

, , ,  screenshots from graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator (accessed 16 July 2021), 164