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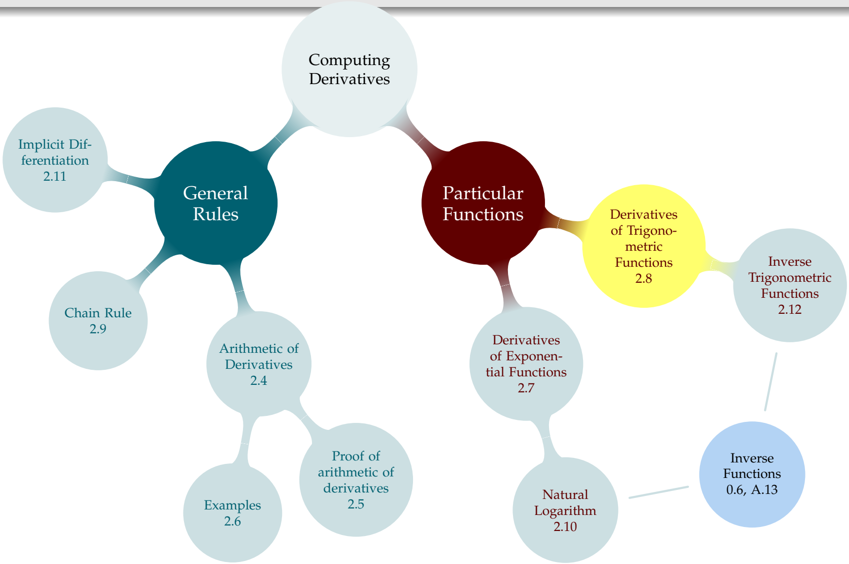
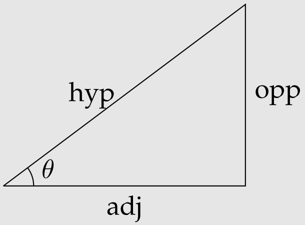
SKIP TRIG REVIEW



Basic Trig Functions



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Commonly used facts

- Graphs of sine, cosine, tangent

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- Identities: $\sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x;$
  $\sin^2 x = \dfrac{1 - \cos(2x)}{2}; \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

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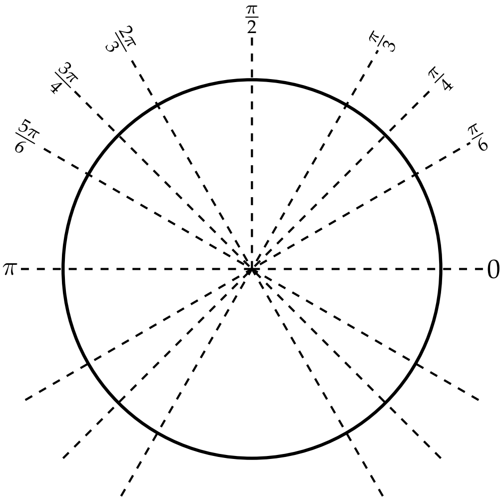
- Conversion between radians and degrees

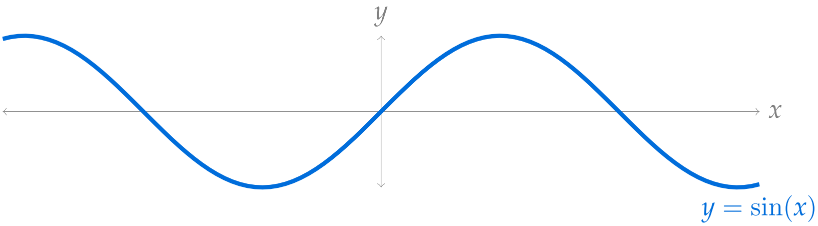
CLP-1 has an appendix on high school trigonometry that you should be familiar with.

Reference Angles



Derivative of Sine

SKIP PROOFS OF SINE AND COSINE DERIVATIVES



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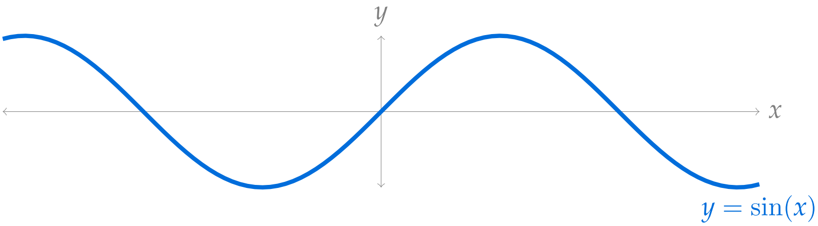

Consider the derivative of $f(x) = \sin(x)$.

Derivative of Sine

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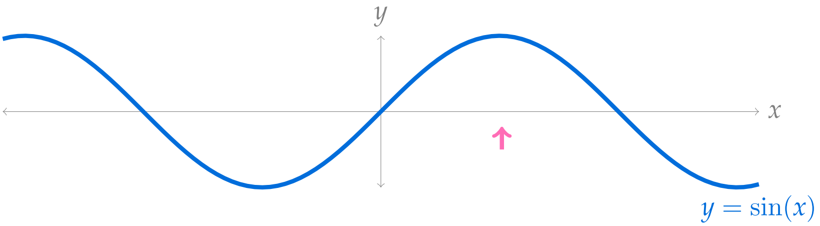

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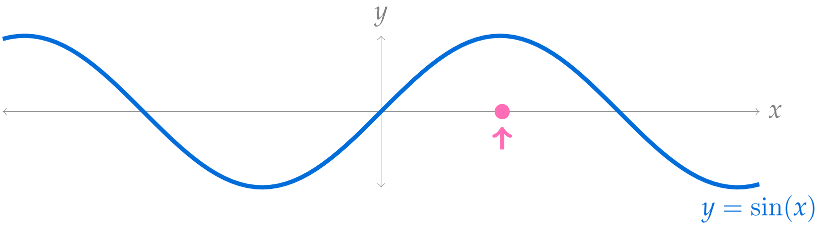

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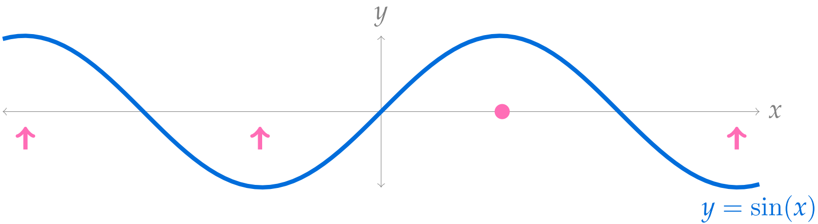

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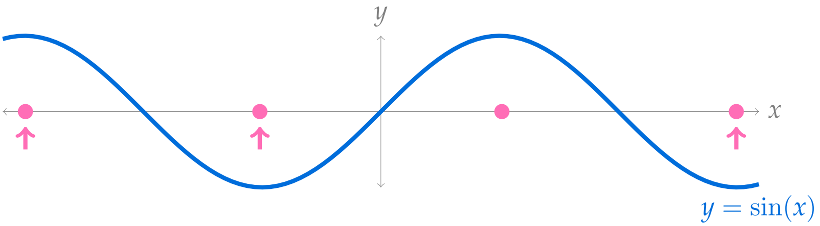

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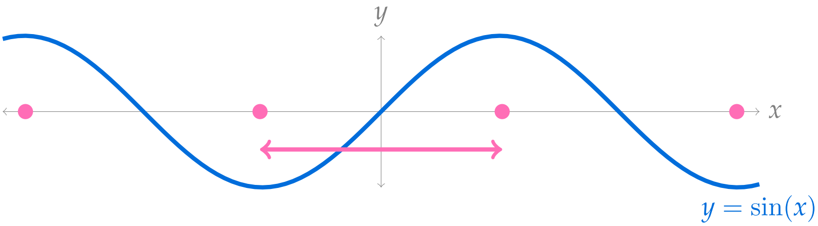

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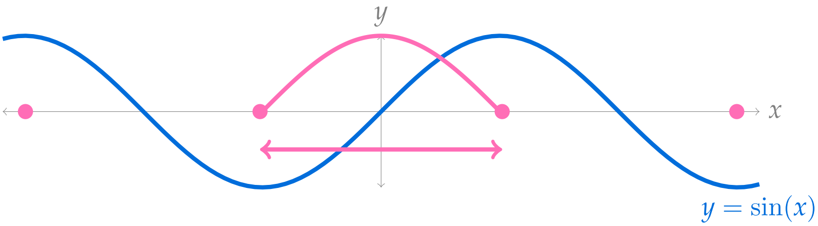

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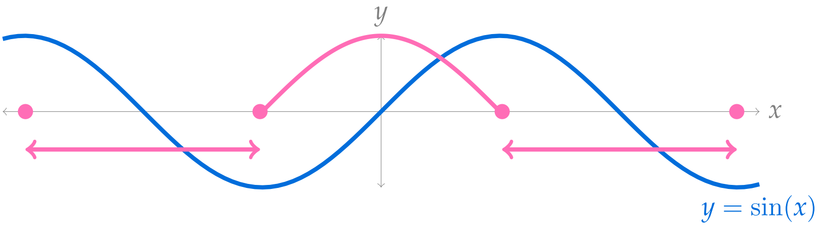
▸▸ SKIP PROOFS OF SINE AND COSINE DERIVATIVES



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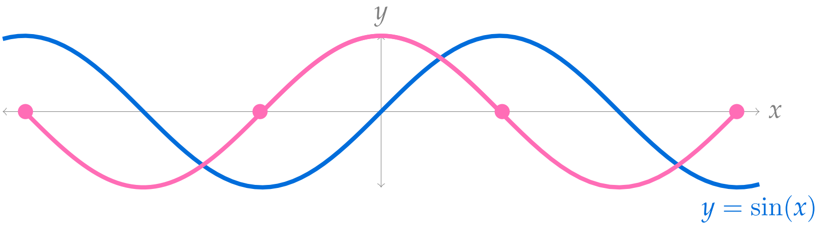

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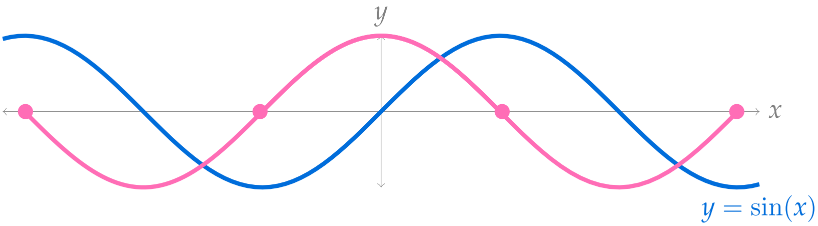

Consider the derivative of $f(x) = \sin(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} \stackrel{?}{=} \cos(x).$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin x\} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin x\} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin x\} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

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$$= \sin(x)\,\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\}\Big|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

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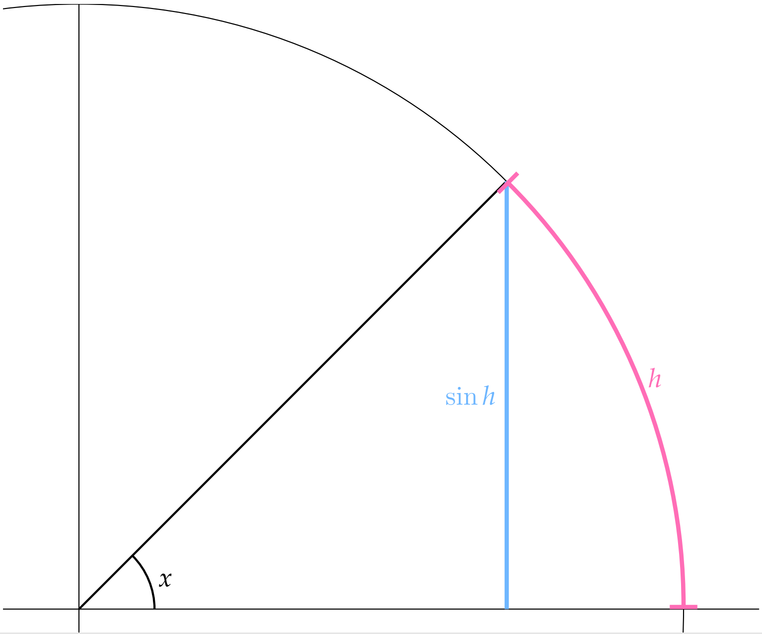
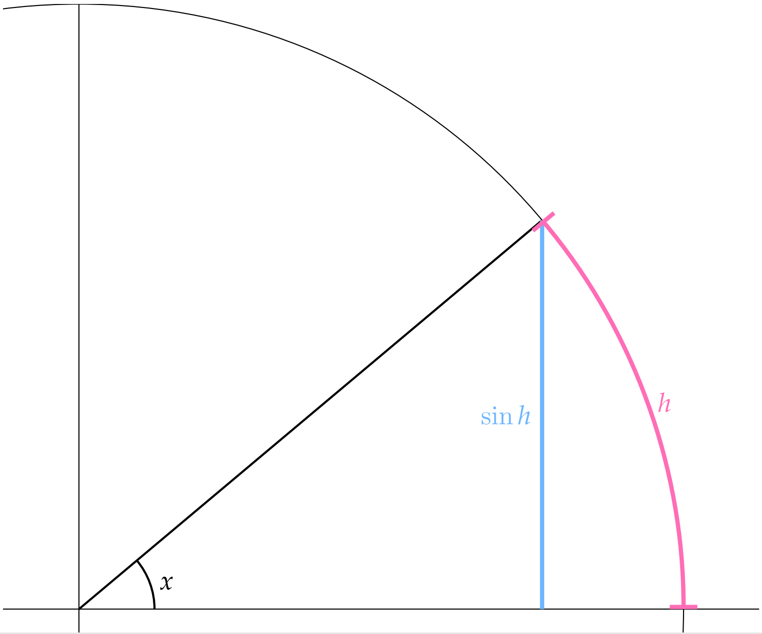
$$= \sin(x)\lim_{h\to 0}\frac{\cos(0+h)-\cos(0)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

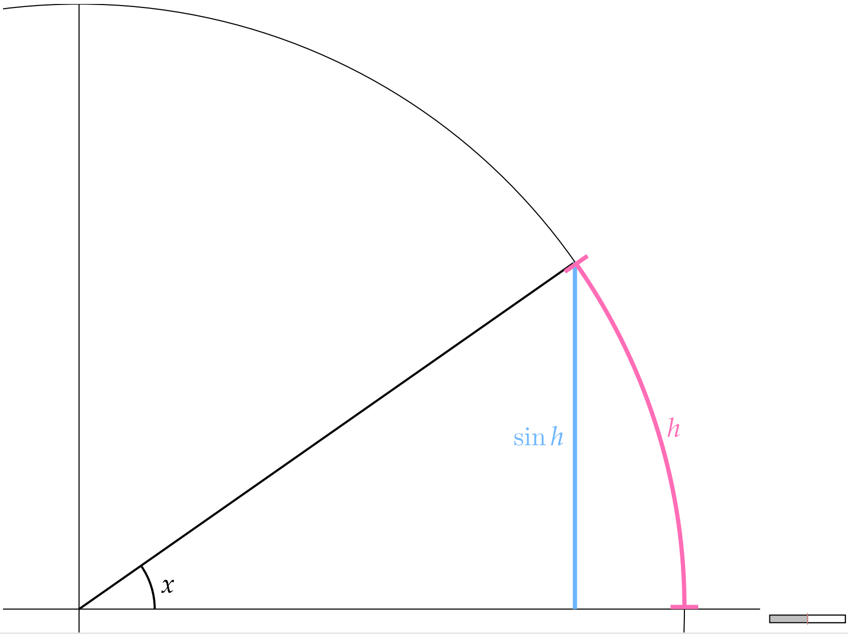
$$= \sin(x)\left.\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\}\right|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} = \boxed{\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}}$$

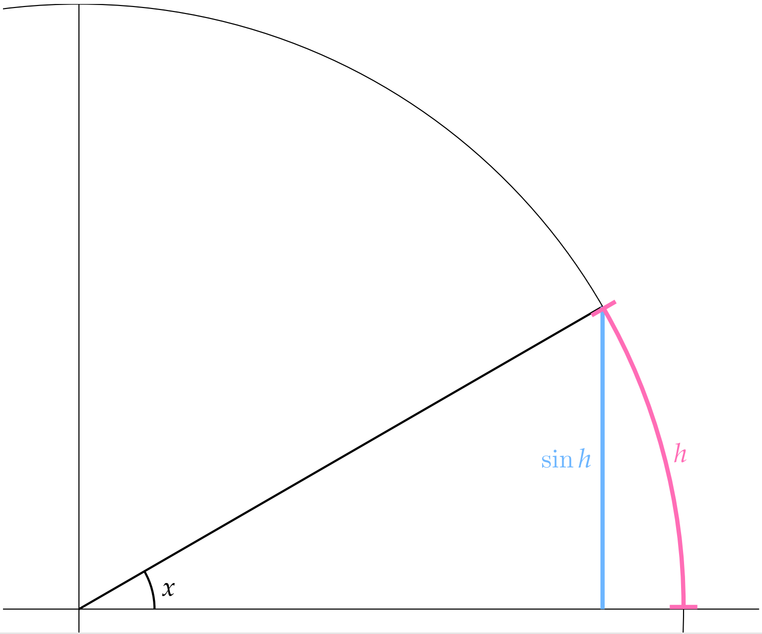
since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.

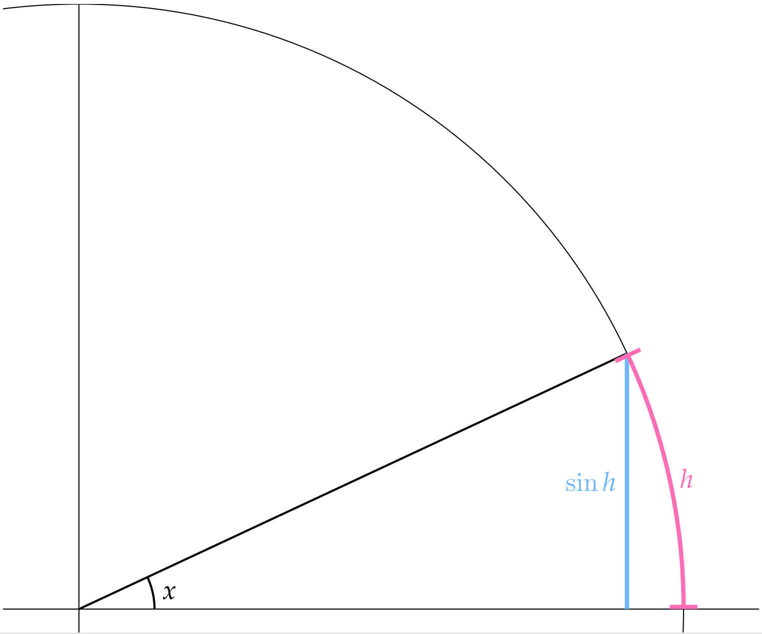
First, we investigate $\lim_{h\to 0} \frac{\sin h}{h}$ informally.

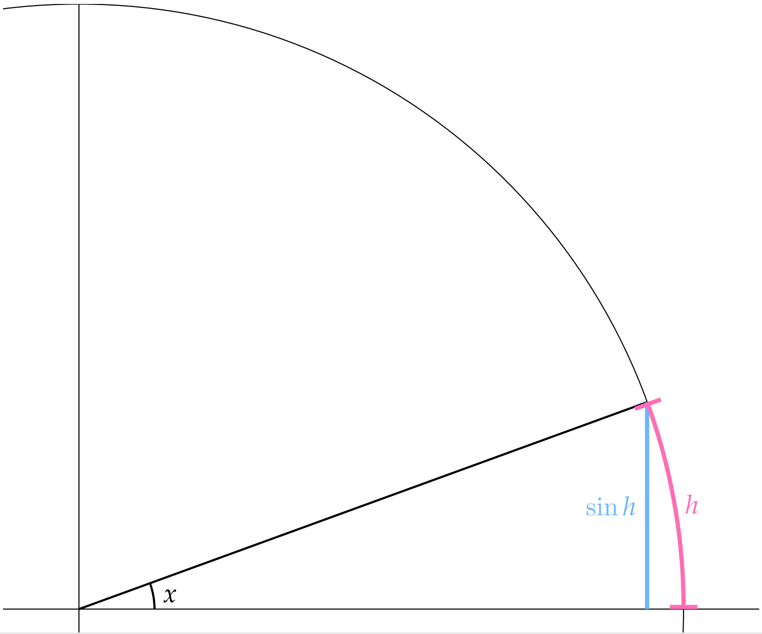

$\sin h$
h
x

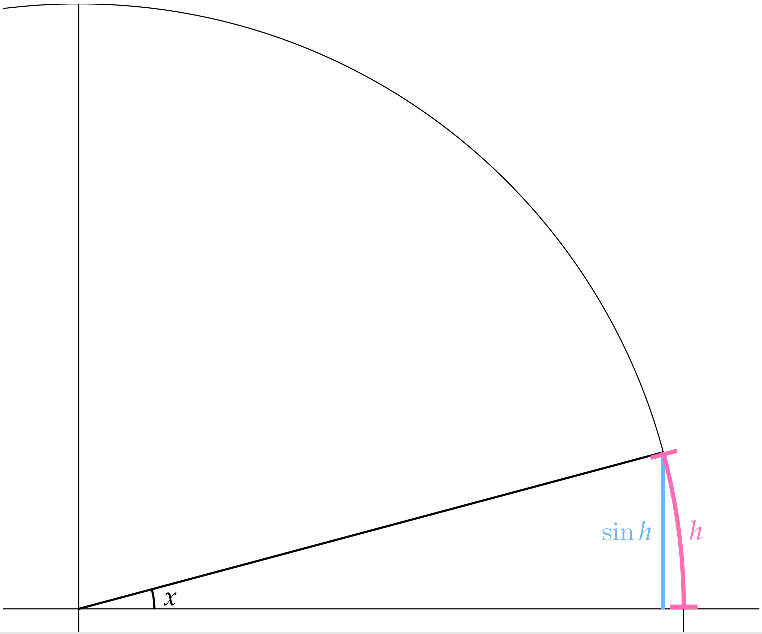

$\sin h$
h
x

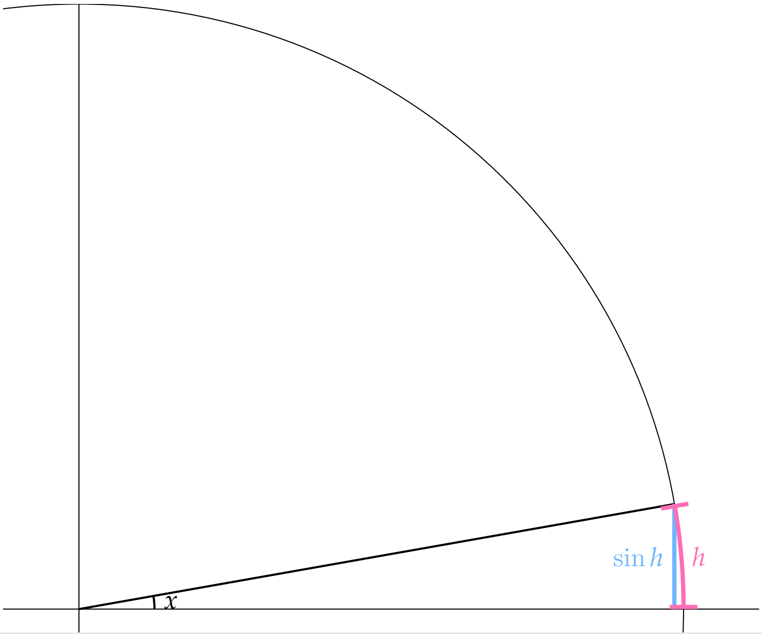

$\sin h$
h
x


x
$\sin h$
h


x
$\sin h$
h


$\sin h$
h
x

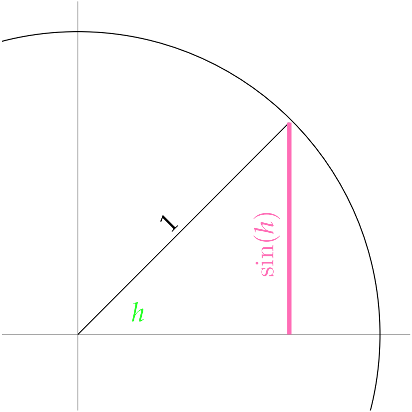

x
$\sin h$
h

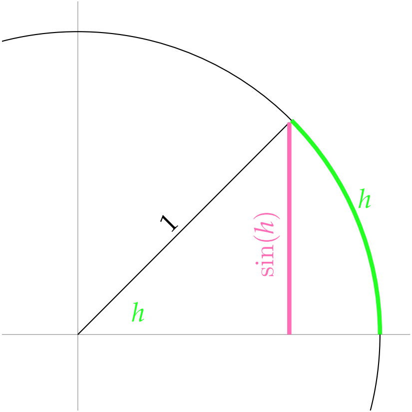

x
$\sin h$
h

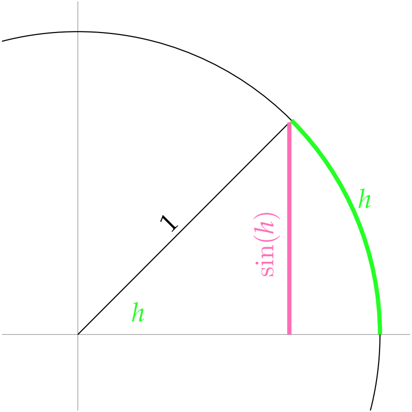
It seems $\sin h \approx h$ when $h \approx 0$, so $\lim_{h\to 0} \frac{\sin h}{h} \stackrel{?}{=} 1$.

We can prove this more formally using the Squeeze Theorem and more trigonometry. We will first prove that $\frac{\sin(h)}{h} \leq 1$ and then we will prove that $\frac{\sin(h)}{h} \geq \cos(h)$. Then we will apply the Squeeze Theorem.

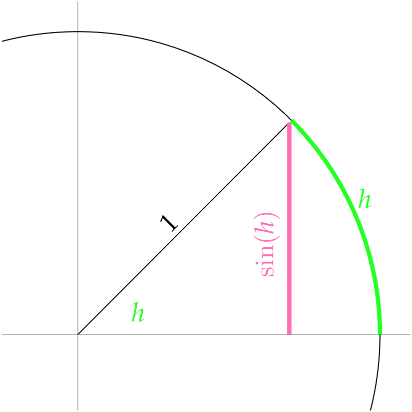
Here is the proof that $\frac{\sin(h)}{h} \leq 1$.


1
$\sin(h)$
h


1
$\sin(h)$
h
h

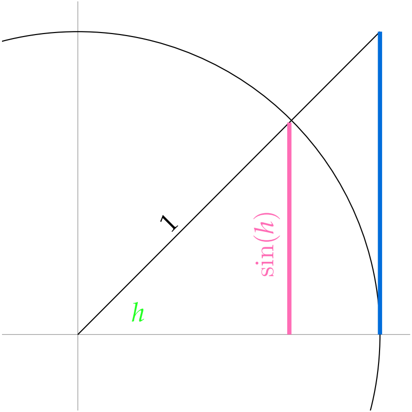


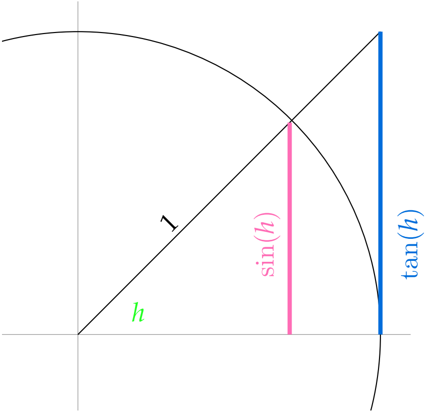
$\sin(h) \leq h$

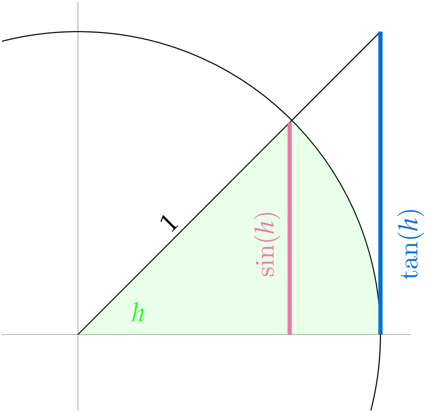


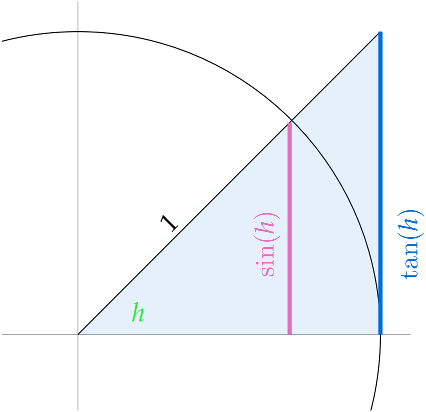
$\sin(h) \leq h$ so $\boxed{\dfrac{\sin(h)}{h} \leq 1}$

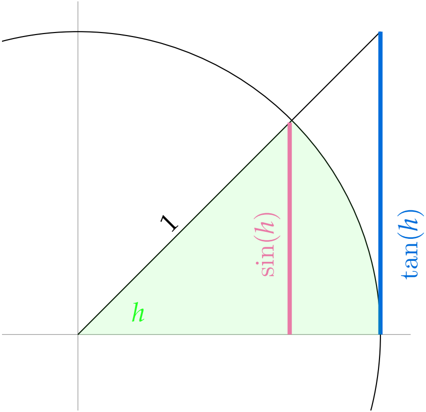
Now for the proof that $\dfrac{\sin(h)}{h} \geq \cos(h)$.


1
$\sin(h)$
h

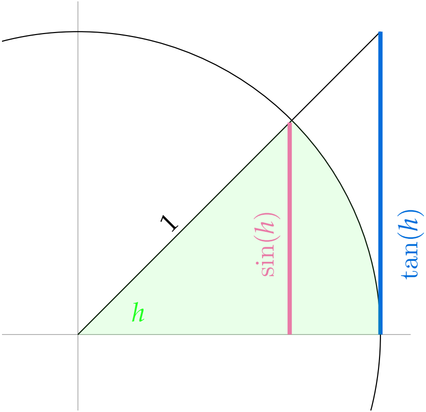

1
h
sin(h)
tan(h)


1
h
$\sin(h)$
$\tan(h)$

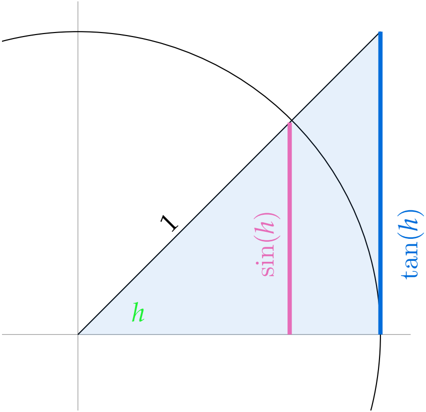

1
h
$\sin(h)$
$\tan(h)$



green area:

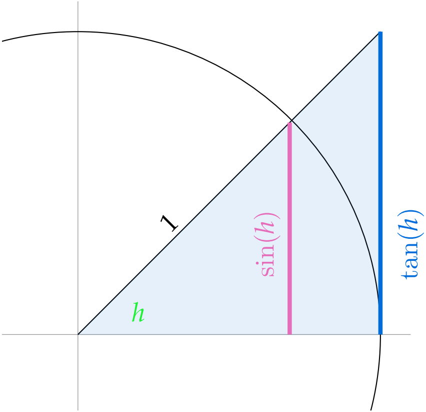


green area: $\dfrac{h}{2}$



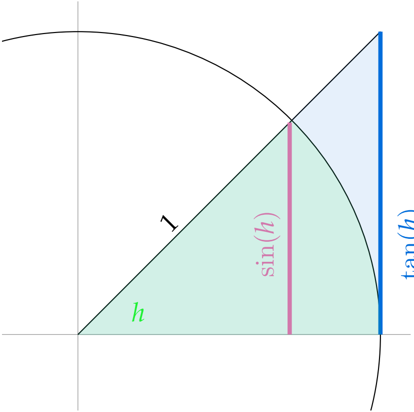
green area: $\dfrac{h}{2}$

Blue area:

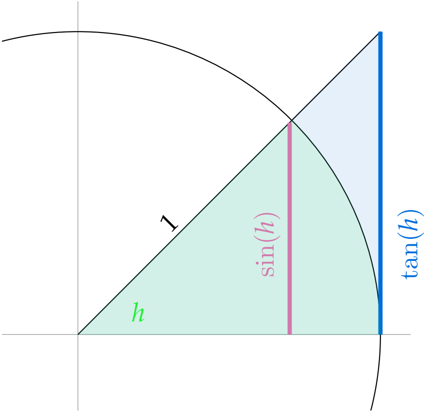


green area: $\dfrac{h}{2}$

Blue area: $\dfrac{\tan h}{2}$



green area: $\frac{h}{2}$ $\qquad$ $\frac{h}{2} \le \frac{\tan(h)}{2}$ $\qquad$ Blue area: $\frac{\tan h}{2}$



green area: $\frac{h}{2}$

$$\frac{h}{2} \leq \frac{\tan(h)}{2}$$

Blue area: $\frac{\tan h}{2}$

$$\boxed{\cos(h) \leq \frac{\sin(h)}{h}}$$

We are now ready for the Squeeze Theorem. We have

$$\cos h \leq \frac{\sin h}{h} \leq 1$$

We are now ready for the Squeeze Theorem. We have

$$\begin{array}{ccccc} \cos h & \leq & \dfrac{\sin h}{h} & \leq & 1 \\ \lim\limits_{h\to 0} \cos h = 1 & & & & \lim\limits_{h\to 0} 1 = 1 \end{array}$$

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$$\cos h \leq \frac{\sin h}{h} \leq 1$$

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By the Squeeze Theorem,

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

DERIVATIVES OF SINE AND COSINE

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} =$$

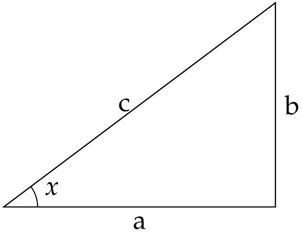
Derivatives of Sine and Cosine

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x)$$

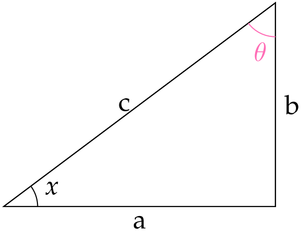
Derivative of Cosine

Now for the derivative of $\cos$. We already know the derivative of $\sin$, and it is easy to convert between $\sin$ and $\cos$ using trig identities.



Derivative of Cosine

Now for the derivative of cos. We already know the derivative of sin, and it is easy to convert between sin and cos using trig identities.



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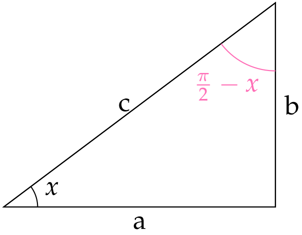
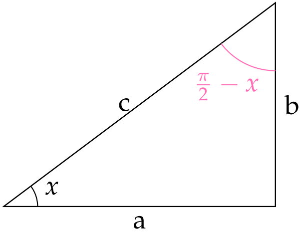


Derivative of Cosine

Now for the derivative of cos. We already know the derivative of sin, and it is easy to convert between sin and cos using trig identities.



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

Derivative of Cosine

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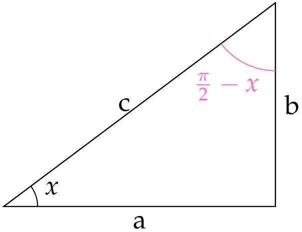


$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

$$\frac{d}{dx}\left[\cos(x)\right] = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = -\frac{d}{dx}\left[\sin\left(x - \frac{\pi}{2}\right)\right] = -\cos\left(x - \frac{\pi}{2}\right) = -\sin x$$

since $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

When we use radians:

Derivatives of Trig Functions

$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} = -\sin(x)$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} =$

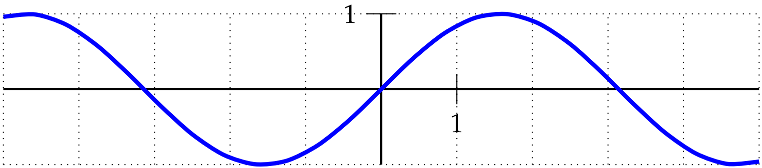
$\frac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} =$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} =$

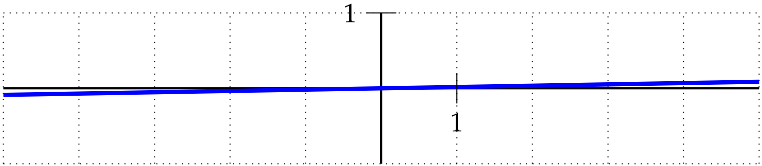
$\frac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} =$

Honorable Mention

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



$y = \sin x$, radians



$y = \sin x$, degrees

Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\tan(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin(x)}{\cos(x)}\right] \\
&= \frac{\cos(x)\cos(x) - \sin(x)[-\sin(x)]}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x)
\end{aligned}$$

Other Trig Functions

Skip proofs of other trig derivatives

$$\sec(x) = \frac{1}{\cos(x)}$$

Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}\frac{d}{dx}[\sec(x)] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] \\ &= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} \\ &= \sec(x)\tan(x)\end{aligned}$$

Other Trig Functions

Skip proofs of other trig derivatives

$$\csc(x) = \frac{1}{\sin(x)}$$

Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\csc(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sin(x)}\right] \\
&= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin(x)}\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x)
\end{aligned}$$

Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Other Trig Functions

SKIP PROOFS OF OTHER TRIG DERIVATIVES

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\cot(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{\sin(x)}\right] \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}$$

MEMORIZE

$$\frac{d}{dx}\{\sin(x)\} = \cos(x) \qquad \frac{d}{dx}\{\sec(x)\} = \sec(x)\tan(x)$$

$$\frac{d}{dx}\{\cos(x)\} = -\sin(x) \qquad \frac{d}{dx}\{\csc(x)\} = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\{\tan(x)\} = \sec^2(x) \qquad \frac{d}{dx}\{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Now You

1. Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the **definition of the derivative** to find $f'(0)$.

2. Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.

3. Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$
   Which values of a and b make $h(x)$ continuous at $x = 0$?

Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.

Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Q

ans

Practice and Review

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x-a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

We don't have rules for differentiating $f(x)$ at $x = 0$, so we have to fall back on the definition of the derivative.

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \cos\left(\frac{1}{h}\right) - 0}{h}$$
$$= \lim_{h\to 0} h \cos\left(\frac{1}{h}\right) = 0$$

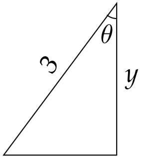
Since the limit exists, $f(x)$ is differentiable at 0.
By the definition of continuity, $g(x)$ is continuous at $x = 0$ if

$$\lim_{x\to 0} g(x) = g(0)$$

- $g(0) = (0 - a)^2 = a^2$
- $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} e^{\frac{\sin x}{x}} = e^1 = e$
- $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} (x - a)^2 = a^2$

In order for $g(x)$ to be continuous, we need $a^2 = e$. That is, $a = \sqrt{e}$ or $a = -\sqrt{e}$.

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ?
When is the top of the ladder sinking the fastest? The slowest?



Q

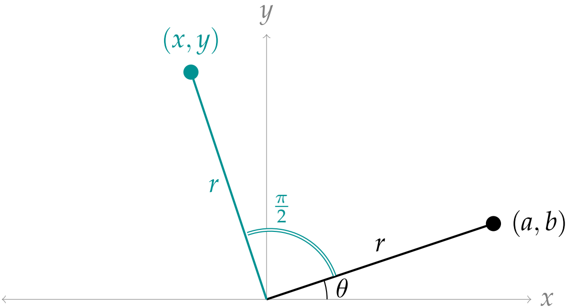
A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ?
When is the top of the ladder sinking the fastest? The slowest?

We want to find how fast y is changing with respect to θ, so we want $\frac{dy}{d\theta}$, or $y'(\theta)$. To calculate that, we need to find y as a function of θ. Note that the ladder forms a right triangle with the wall, and y is the side adjacent to θ, while 3 is the hypotenuse. So, $\cos(\theta) = \frac{y}{3}$, hence $y = 3\cos(\theta)$. Now we differentiate, and see

$$\frac{dy}{d\theta} = -3\sin(\theta)$$

To answer the other questions, note that θ never gets larger than $\pi/2$, since at that point the ladder is lying on the ground. When $0 \leq \theta \leq \pi/2$, the smaller θ gives the smaller rate of change (in absolute value); so the top of the ladder is sinking slowly at first, then faster and faster, fastest just as it hits the ground.

Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y)? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y)? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ?

$$x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin(\theta)$$

and

$$y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos(\theta)$$

To find how fast x is changing with respect to θ, we take $x'(\theta) = -r\cos(\theta)$. We see that when $\theta = 0$, x changes a lot when θ changes; and when $\theta = \pi/2$, x only changes a little when θ changes.

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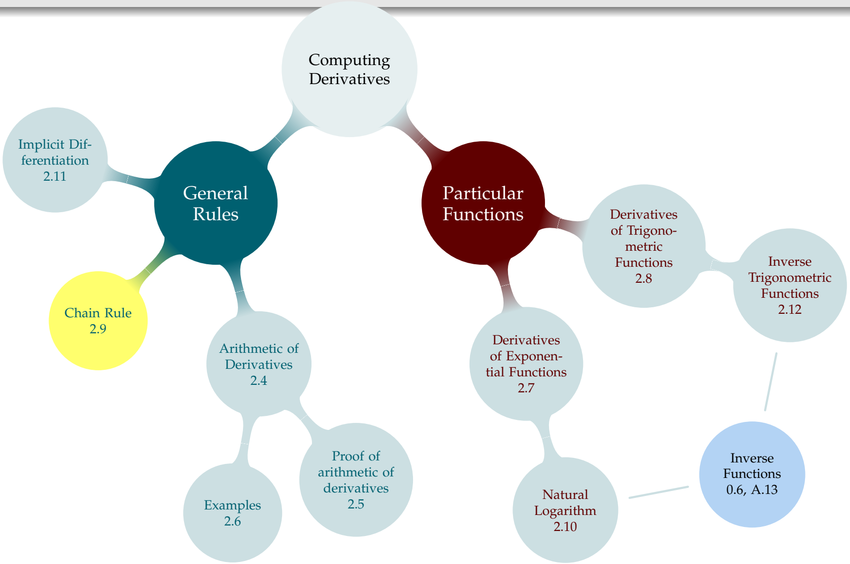
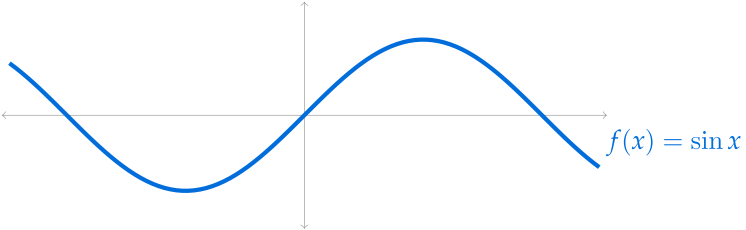
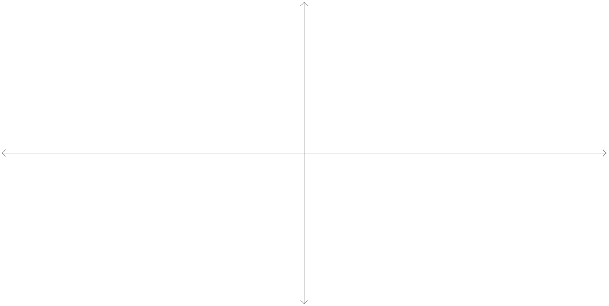


INTUITION: $\sin x$ VERSUS $\sin(2x)$





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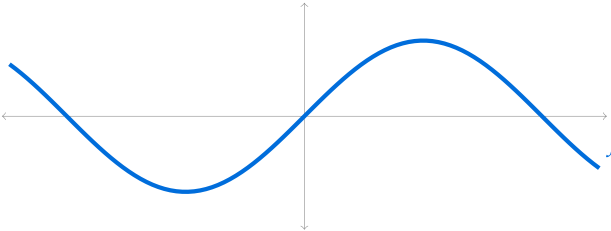
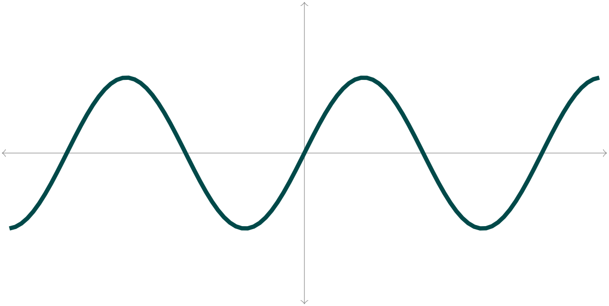
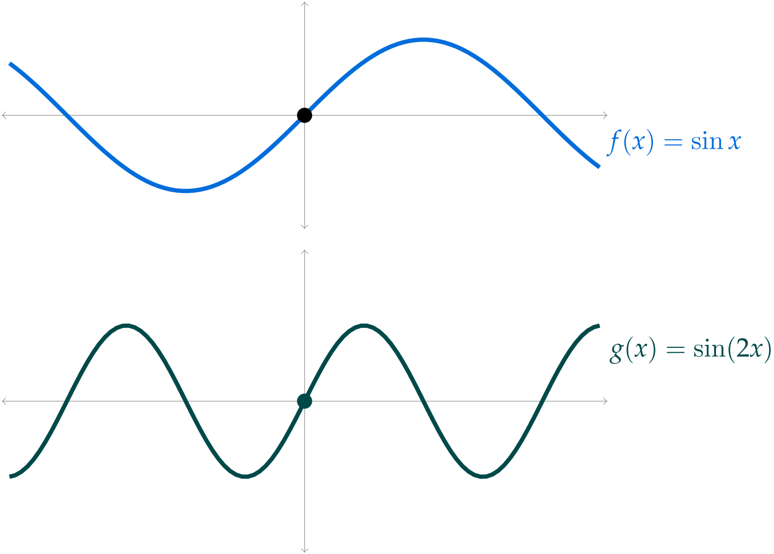


$f(x) = \sin x$



$g(x) = \sin(2x)$

INTUITION: $\sin x$ VERSUS $\sin(2x)$



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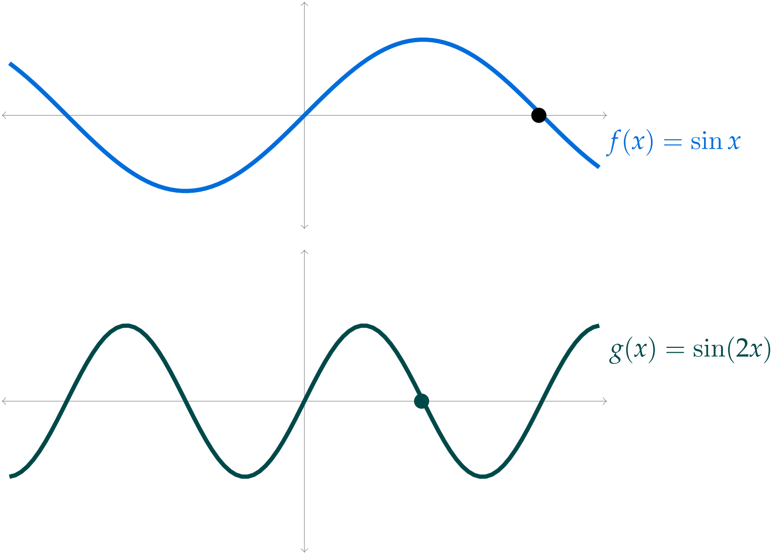

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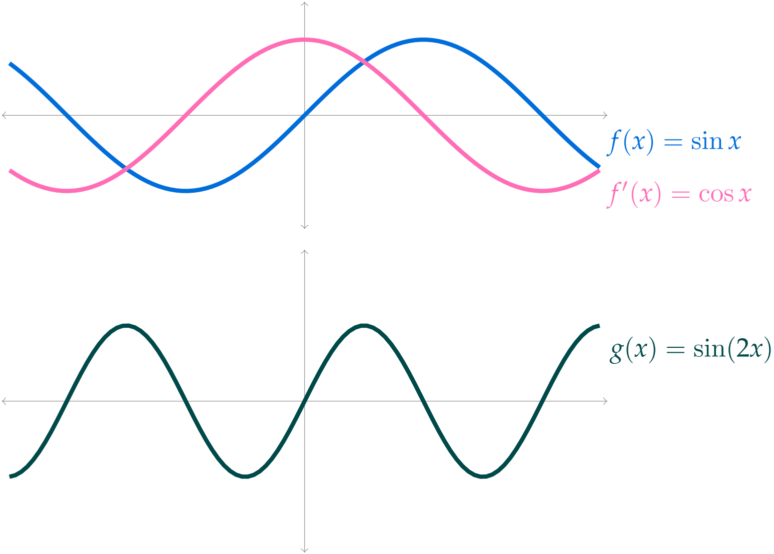

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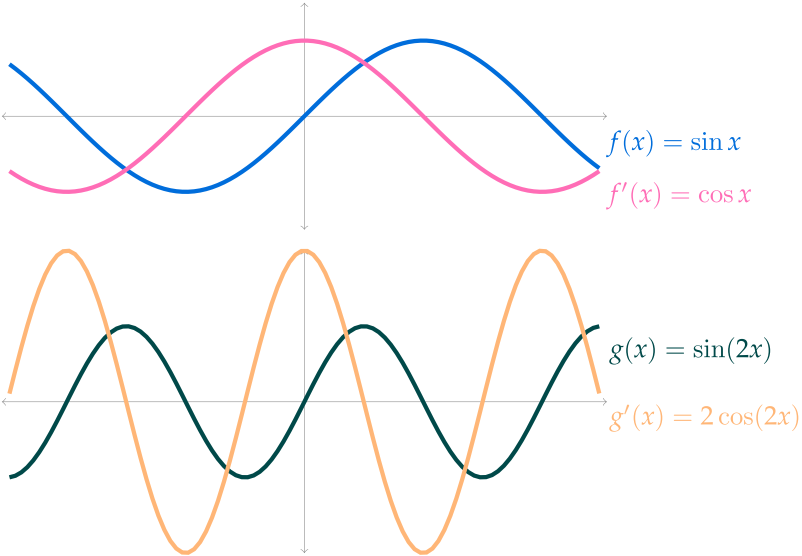

COMPOUND FUNCTIONS

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*. Available from https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp

Kelp Population

k	kelp population
u	urchin population
o	otter population

KELP POPULATION

k	kelp population
u	urchin population
o	otter population

$k(u)$

Kelp Population

k	kelp population
u	urchin population
o	otter population

$k(u)$ $\qquad$ $k(u(o))$

Kelp Population

k	kelp population
u	urchin population
o	otter population
p	public policy

$k(u)$ $\qquad$ $k(u(o))$ $\qquad$ $k(u(o(p)))$

KELP POPULATION

k	kelp population
u	urchin population
o	otter population
p	public policy

$k(u)$ $\quad$ $k(u(o))$ $\quad$ $k(u(o(p)))$

These are examples of compound functions.

Kelp Population

k	kelp population
u	urchin population
o	otter population
p	public policy

$k(u)$ $\qquad$ $k(u(o))$ $\qquad$ $k(u(o(p)))$

These are examples of compound functions.

Should $\frac{\mathrm{d}}{\mathrm{d}o}k\big(u(o)\big)$ be positive or negative?

A. positive B. negative C. I'm not sure

Kelp Population

k	kelp population
u	urchin population
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$k(u)$ $\qquad$ $k(u(o))$ $\qquad$ $k(u(o(p)))$

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Should $\frac{\mathrm{d}}{\mathrm{d}o}k\big(u(o)\big)$ be positive or negative?

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Kelp Population

k	kelp population
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$k(u)$ $\quad$ $k(u(o))$ $\quad$ $k(u(o(p)))$

These are examples of compound functions.

Should $\frac{d}{do}k(u(o))$ be positive or negative?
A. positive $\quad$ B. negative $\quad$ C. I'm not sure

Should $k'(u)$ be positive or negative?
A. positive $\quad$ B. negative $\quad$ C. I'm not sure

KELP POPULATION

k	kelp population
u	urchin population
o	otter population
p	public policy

$k(u)$ $\quad$ $k(u(o))$ $\quad$ $k(u(o(p)))$

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Should $\frac{\mathrm{d}}{\mathrm{d}o}k\big(u(o)\big)$ be positive or negative?

A. positive $\quad$ B. negative $\quad$ C. I'm not sure

Should $k'(u)$ be positive or negative?

A. positive $\quad$ B. negative $\quad$ C. I'm not sure

Differentiating Compound Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} =$$

Differentiating Compound Functions

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\left(\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right)\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f\left(\boxed{g(x+h)}\right)-f\left(\boxed{g(x)}\right)}{\boxed{g(x+h)}-\boxed{g(x)}}\cdot g'(x)
\end{aligned}$$

Set $H = g(x+h)-g(x)$. As $h\to 0$, we also have $H \to 0$. So

$$\begin{aligned}
&= \lim_{H\to 0}\frac{f(g(x)+H)-f(g(x))}{H}\cdot g'(x)\\
&= f'(g(x))\cdot g'(x)
\end{aligned}$$

Chain Rule

Chain Rule – Theorem 2.9.3

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))\,g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

In the case of kelp, $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o)) = \frac{\mathrm{d}k}{\mathrm{d}u}(u(o))\,\frac{\mathrm{d}u}{\mathrm{d}o}(o)$

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f\left(g(x)\right)\} = f'\left(g(x)\right)g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}\left(g(x)\right)\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))\, g'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

We can differentiate $\sin(x)$, so let's set $g(x) = e^x + x^2$ and $f(g) = \sin(g)$. Then $F(x) = f(g(x))$.

$g'(x) = e^x + 2x$ and $\frac{\mathrm{d}f}{\mathrm{d}g}(g) = \cos(g)$ and
$\frac{\mathrm{d}f}{\mathrm{d}g}(g(x)) = \frac{\mathrm{d}f}{\mathrm{d}g}\left(\boxed{e^x + x^2}\right) = \cos\left(\boxed{e^x + x^2}\right)$

So, $F'(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x) = \cos(e^x + x^2)\,(e^x + 2x)$

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$

Q

ans

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$

$$F'(v) = 6\left(\boxed{\frac{v}{v^3+1}}\right)^5 \cdot \frac{(v^3+1)(1)-(v)(3v^2)}{(v^3+1)^2}$$

$$= 6\left(\boxed{\frac{v}{v^3+1}}\right)^5 \cdot \frac{-2v^3+1}{(v^3+1)^2}$$

Now You

Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

Now You

Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t} u(o(t))$.
Note: your answer should depend only on t: not o.

Now You

Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

Q

Now You Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

$$f(x) = (\boxed{10^x + \csc x})^{1/2}$$

Using the chain rule,

$$\begin{aligned} f'(x) &= \frac{1}{2}(\boxed{10^x + \csc x})^{-1/2}(10^x \log_e 10 - \csc x \cot x) \\ &= \frac{10^x \log_e 10 - \csc x \cot x}{2\sqrt{10^x + \csc x}} \end{aligned}$$

Now You Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t}u\big(o(t)\big)$.
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Note: your answer should depend only on t: not o.

$$
\begin{aligned}
o'(t) &= e^t \\
u'(o) &= \frac{(o+\sin o)(0) - (1)(1+\cos o)}{(o+\sin o)^2} \\
&= \frac{-(1+\cos o)}{(o+\sin o)^2} \\
\frac{\mathrm{d}}{\mathrm{d}t} u\big(o(t)\big) &= u'\big(o(t)\big)\; o'(t) \\
&= -e^t \left(\frac{1+\cos\big(o(t)\big)}{\big[o(t)+\sin(o(t))\big]^2} \right) \\
&= -e^t \left(\frac{1+\cos(e^t)}{\big[e^t+\sin(e^t)\big]^2} \right)
\end{aligned}
$$

A

More Examples

Now You Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\dfrac{1}{x}\right)\right\}$

Now You Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2 + \sec\left(x^2 + \dfrac{1}{x}\right)\right\}$

Q

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(x^2+\dfrac{1}{x}\right)\right\}$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\right\}\\
&\quad=2x+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\tan\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\boxed{x^2+\frac{1}{x}}\right\}\\
&\quad=2x+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\tan\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\boxed{x^2+x^{-1}}\right\}\\
&\quad=2x+\sec\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\tan\left(\boxed{x^2+\frac{1}{x}}\right)\cdot\left(2x-x^{-2}\right)
\end{aligned}$$

Notice: That first term, $2x$, is not multiplied by anything else.

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x+\frac{1}{x+\frac{1}{x}}}\right\}$

Q

Evaluate $\dfrac{d}{dx}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

$$\frac{d}{dx}\left\{\frac{1}{x+\frac{1}{x+\frac{1}{x}}}\right\} = \frac{d}{dx}\left\{\left(x+\left(x+x^{-1}\right)^{-1}\right)^{-1}\right\}$$

$$= -\left(\boxed{x+\left(x+x^{-1}\right)^{-1}}\right)^{-2}\cdot\frac{d}{dx}\left\{\boxed{x+\left(x+x^{-1}\right)^{-1}}\right\}$$

$$= -\left(\boxed{x+\left(x+x^{-1}\right)^{-1}}\right)^{-2}\cdot\left[1+(-1)\left(\boxed{x+x^{-1}}\right)^{-2}\cdot\frac{d}{dx}\left\{\boxed{x+x^{-1}}\right\}\right]$$

$$= -\left(\boxed{x+\left(x+x^{-1}\right)^{-1}}\right)^{-2}\cdot\left[1+(-1)\left(\boxed{x+x^{-1}}\right)^{-2}\cdot\left(1-x^{-2}\right)\right]$$

A

Included Work

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