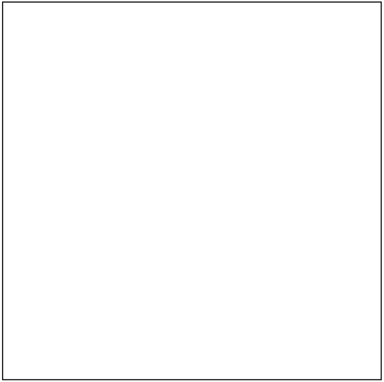
Table of Contents

Sequences and Series

A sequence is a list of numbers
A series is the sum of such a list.

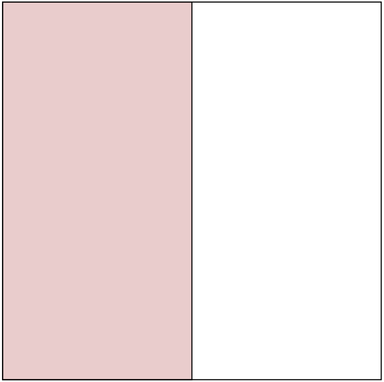
Sequences and Series



Square of Area 1

Sequences and Series

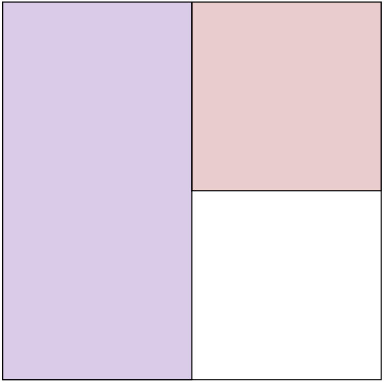
Size of Tiles: $\frac{1}{2}$



Covered Area: $\frac{1}{2}$

SEQUENCES AND SERIES

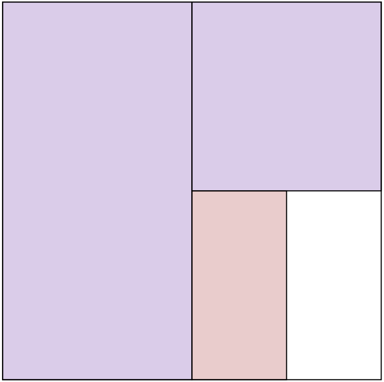
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2}$

Sequences and Series

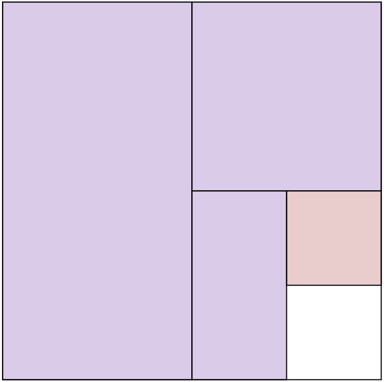
Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$

SEQUENCES AND SERIES

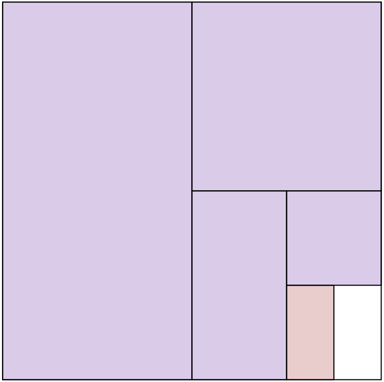
Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

Sequences and Series

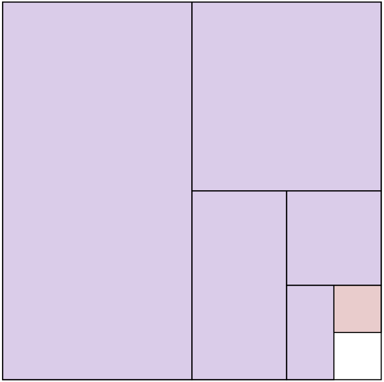
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$

Sequences and Series

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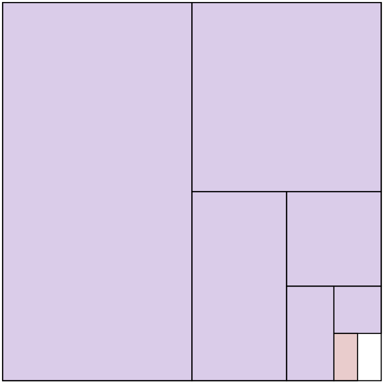

Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}$

Sequences and Series

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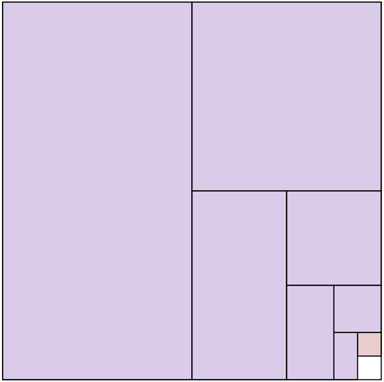
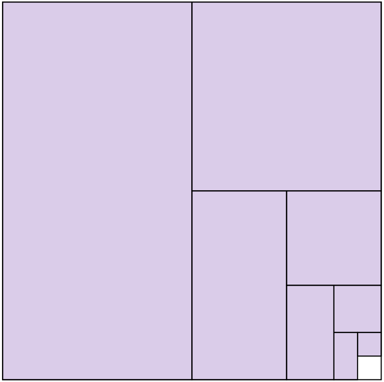


Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8}$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9}$

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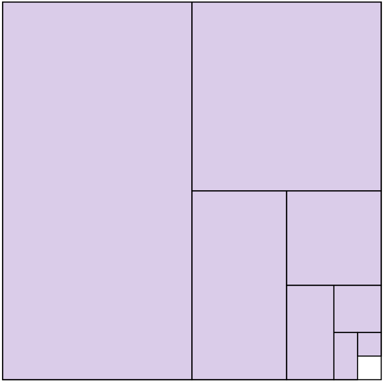
Sequence



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

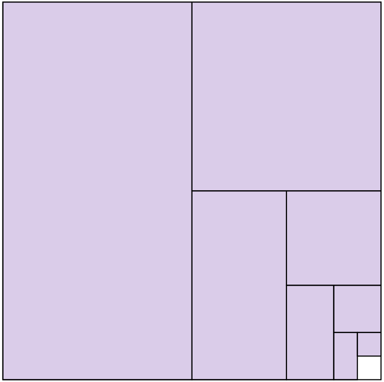
Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$

Sequence

List of numbers,

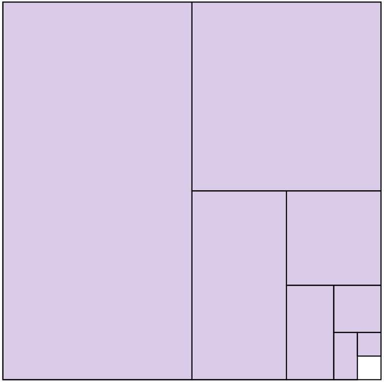
approaching



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \ldots$



Sequence

List of numbers,

approaching **zero**.

Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

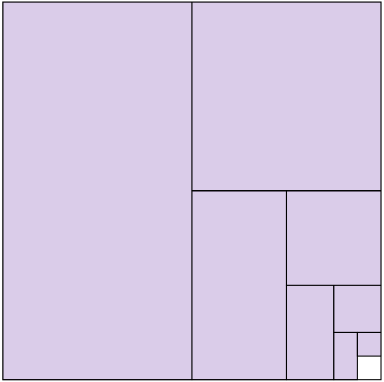
Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \ldots$

Sequence

List of numbers,

approaching **zero**.

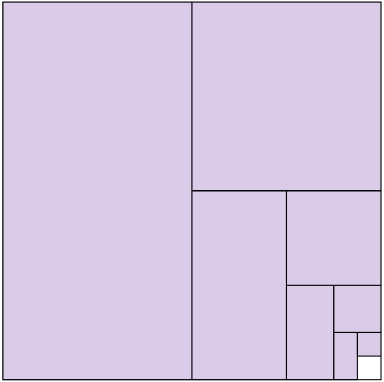


Series

Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}+\cdots$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$



Sequence

List of numbers,

approaching **zero**.

Series

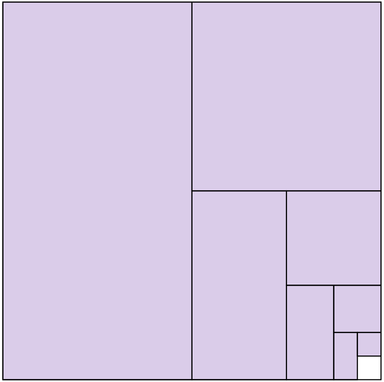
Sum of numbers,

approaching

Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}+\cdots$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \ldots$



$\boxed{\text{Sequence}}$

List of numbers,

approaching **zero**.

$\boxed{\text{Series}}$

Sum of numbers,

approaching **one**.

Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =$$

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

(a more rigorous definition will be discussed soon)

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(C \cdot a_n) =$$

A. $\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$

C. $C \sum_{n=1}^{\infty} a_n$

D. $a_n \sum_{n=1}^{\infty} C$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(a_n + b_n) =$$

A. $\displaystyle\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$

B. $\displaystyle\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $\displaystyle a_n + \sum_{n=1}^{\infty} b_n$

D. $\displaystyle a_n \sum_{n=1}^{\infty} b_n$

E. none of the above

Q

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(a_n)^C =$$

A. $\displaystyle\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\displaystyle\left(\sum_{n=1}^{\infty} a_n\right)^C$

C. $\displaystyle C^n \sum_{n=1}^{\infty} a_n$

D. $\displaystyle\sum_{n=1}^{\infty} C(a_n)^{C-1}$

E. none of the above

Q

SERIES PHILOSOPHY

What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Series Philosophy

What does it really mean to add up infinitely many things?

$$\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\cdots$$

Series Philosophy

What does it really mean to add up infinitely many things?

$$\underbrace{1}_{1}\;\underbrace{-\,1+1}_{0}\;\underbrace{-\,1+1}_{0}\;\underbrace{-\,1+1}_{0}\;\underbrace{-\,1+1}_{0}\;\underbrace{-\,1+1}_{0}\;\underbrace{-\,1+1}_{0}\;-\,1+\cdots$$

Series Philosophy

What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

$$1 \qquad 1 \qquad 1$$

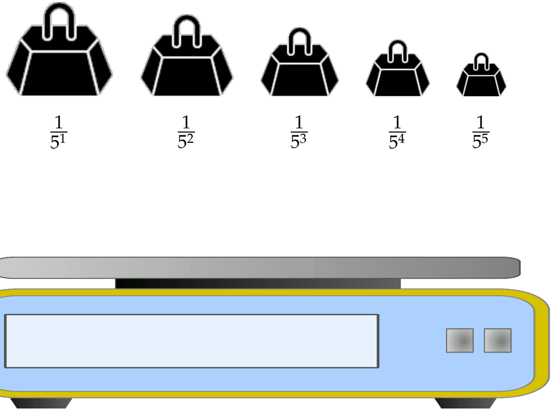
Series Philosophy

What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

We need an unambiguous definition.

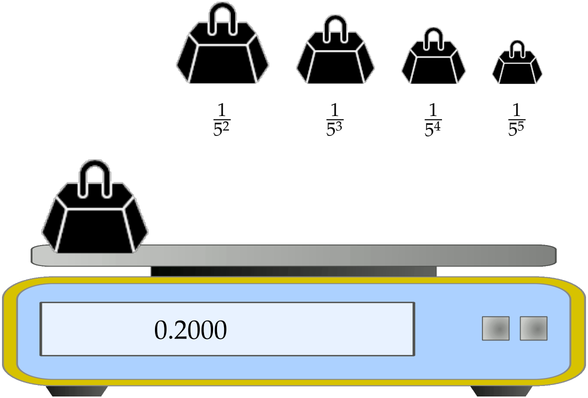
How can we add up infinitely many things?
Sequence of Partial Sums



How can we add up infinitely many things?

Sequence of Partial Sums

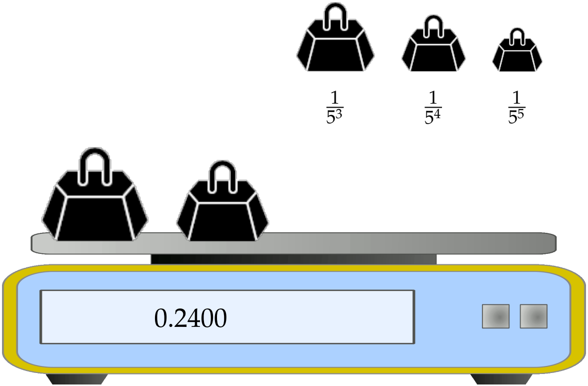
$$S_1 = 0.2000$$



How can we add up infinitely many things?
Sequence of Partial Sums

$S_1 = 0.2000$

$S_2 = 0.2400$



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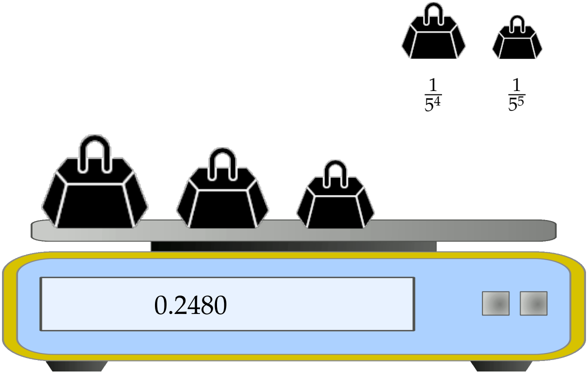
Sequence of Partial Sums



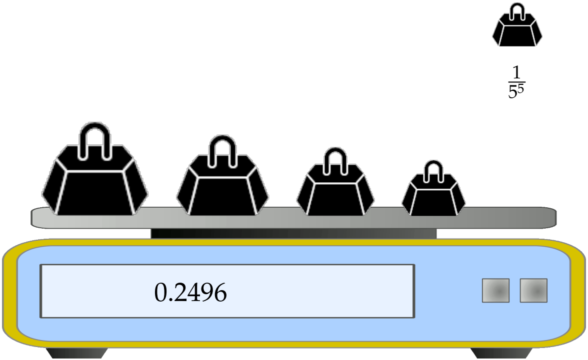
$S_1 = 0.2000$

$S_2 = 0.2400$

$S_3 = 0.2480$

How can we add up infinitely many things?

Sequence of Partial Sums



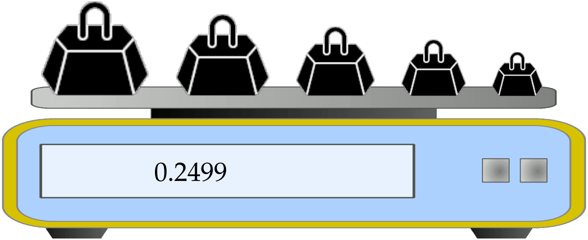
$S_1 = 0.2000$

$S_2 = 0.2400$

$S_3 = 0.2480$

$S_4 = 0.2496$

How can we add up infinitely many things?
Sequence of Partial Sums



$S_1 = 0.2000$

$S_2 = 0.2400$

$S_3 = 0.2480$

$S_4 = 0.2496$

$S_5 = 0.2499$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$\begin{aligned} a_1 &= \frac{1}{5} = 0.2 \\ a_2 &= \frac{1}{5^2} = 0.04 \\ a_3 &= \frac{1}{5^3} = 0.008 \\ a_4 &= \frac{1}{5^4} = 0.0016 \\ a_5 &= \frac{1}{5^5} = 0.00032 \end{aligned}$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

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$$a_2 = \frac{1}{5^2} = 0.04$$

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$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

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Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

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$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

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Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$

$$a_5 = \frac{1}{5^5} = 0.00032 \qquad S_5 = 0.24992$$

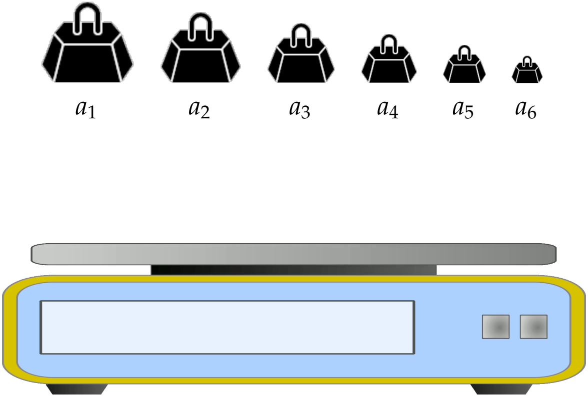
We define $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N.$

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$
$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$
$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$
$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$
$$a_5 = \frac{1}{5^5} = 0.00032 \qquad S_5 = 0.24992$$
$$a_6 = \frac{1}{5^6} = 0.000064 \qquad S_6 = 0.249984$$
$$a_7 = \frac{1}{5^7} = 0.0000128 \qquad S_7 = 0.2499968$$
$$a_8 = \frac{1}{5^8} = 0.00000256 \qquad S_8 = 0.24999936$$

From the sequence of partial sums, we guess

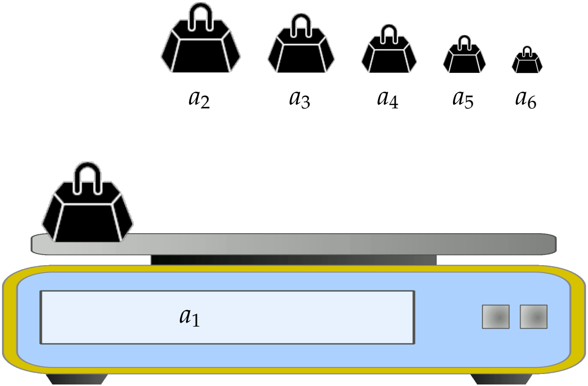
$$\sum_{n=1}^{\infty} \qquad = \qquad \lim_{N\to\infty} S_N \qquad =$$

Notation: $S_N = \sum_{n=1}^{N} a_n$



NOTATION: $S_N = \sum_{n=1}^{N} a_n$

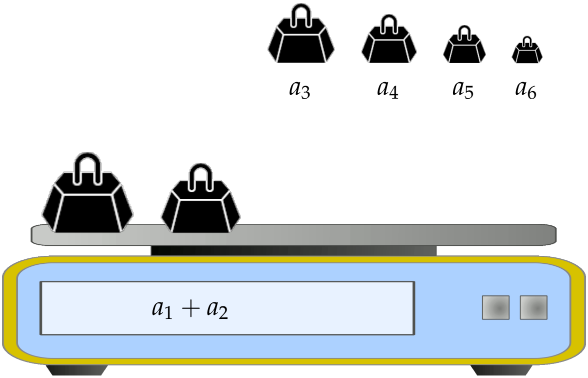
$S_1 = a_1$



Notation: $S_N = \sum_{n=1}^{N} a_n$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$



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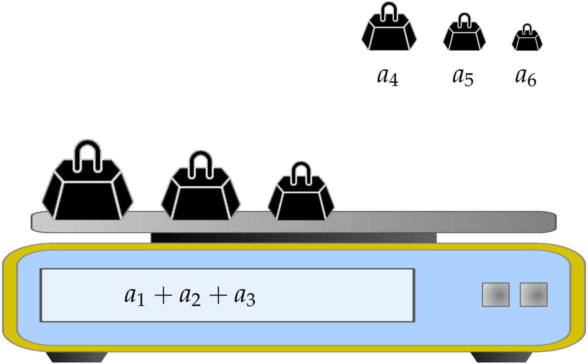
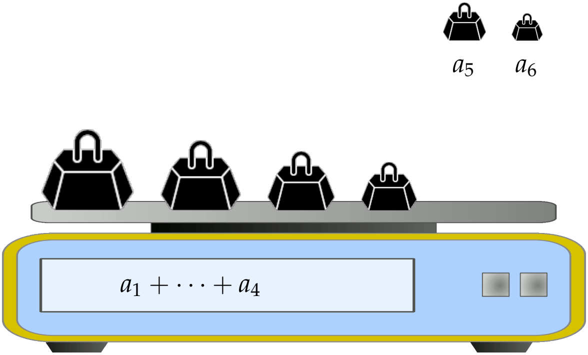
$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$



NOTATION: $S_N = \sum_{n=1}^{N} a_n$



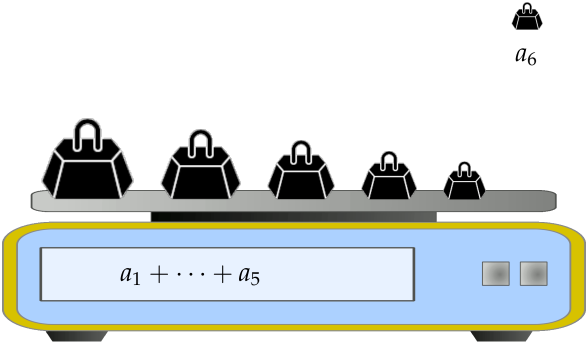
$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

NOTATION: $S_N = \sum_{n=1}^{N} a_n$



$$S_1 = a_1$$

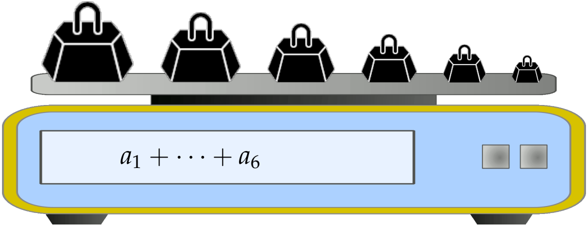
$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

$$S_5 = a_1 + \cdots + a_5$$

NOTATION: $S_N = \sum_{n=1}^{N} a_n$



$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

$$S_5 = a_1 + \cdots + a_5$$

$$S_6 = a_1 + \cdots + a_6$$

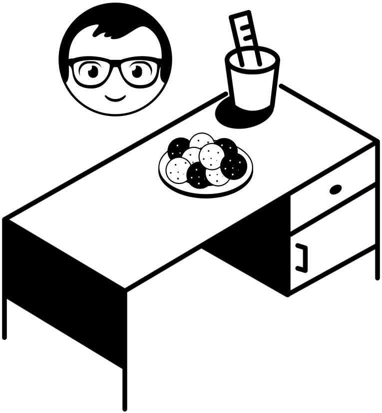
Notation Practice

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Evaluate $\sum_{n=1}^{100} a_n$.
- Evaluate $\sum_{n=1}^{\infty} a_n$.

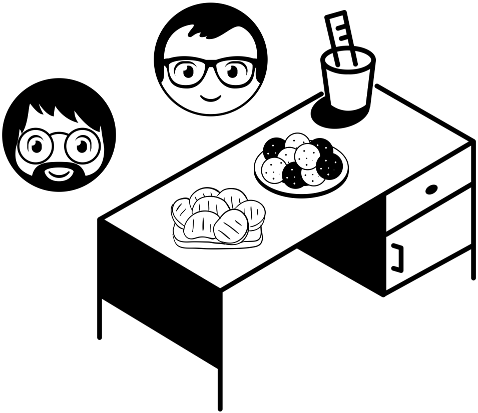
Q

Notation Practice



Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Notation Practice

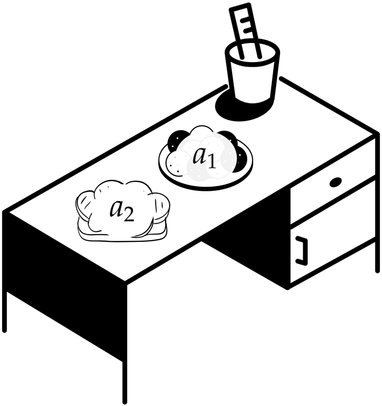


Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

Notation Practice



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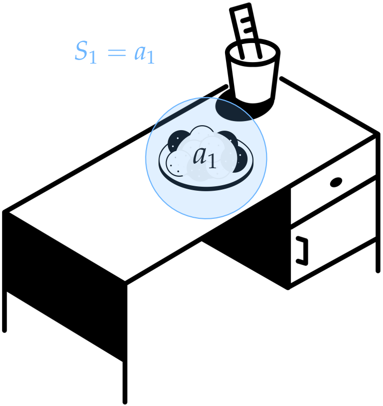
Notation Practice



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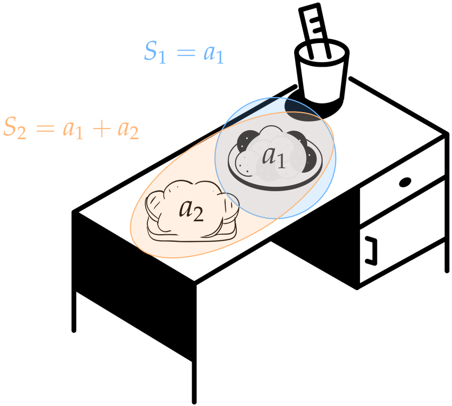
Notation Practice



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How many cookies did each person bring?

NOTATION PRACTICE

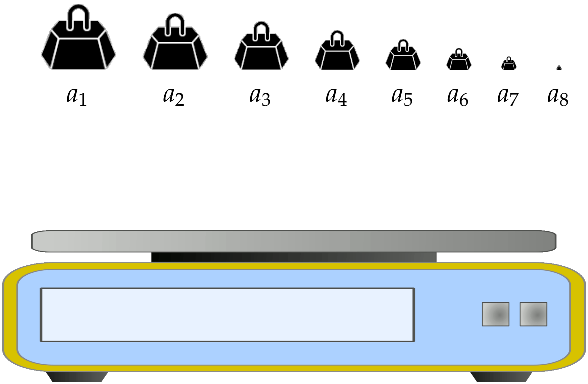
Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \displaystyle\sum_{n=1}^{N} a_n = \frac{N}{N+1}$.

- Find a_1.

- Give an explicit expression for a_n, when $n > 1$.

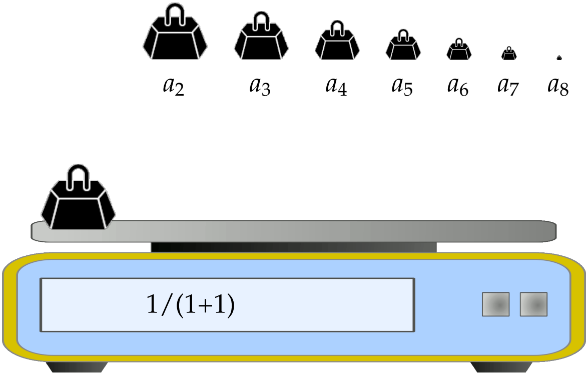
Q

$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$



$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$

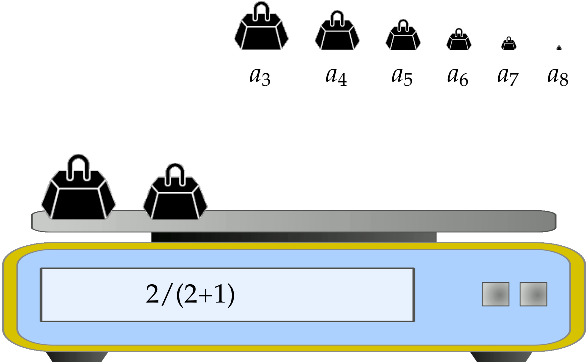
$S_1 = 1/(1+1)$



$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$

$S_1 = 1/(1+1)$

$S_2 = 2/(2+1)$

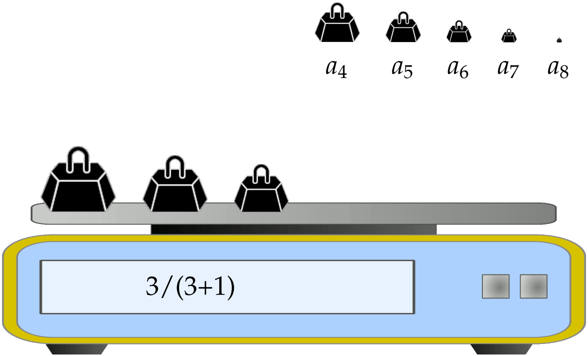


$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$

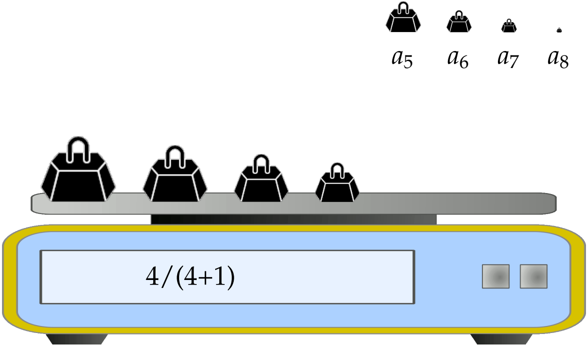
$S_1 = 1/(1+1)$

$S_2 = 2/(2+1)$

$S_3 = 3/(3+1)$



$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$



$S_1 = 1/(1+1)$

$S_2 = 2/(2+1)$

$S_3 = 3/(3+1)$

$S_4 = 4/(4+1)$

Definition

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^{N} a_n.$$

The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \to S$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Geometric Series

Let a and r be two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + ar^3 + \cdots$$

is called the **geometric series** with first term a and ratio r.

We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

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We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

Another useful way of identifying geometric series is to determine whether all pairs of consecutive terms have the same ratio.

- Geometric: $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$
- Geometric: $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- Not geometric: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

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$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

$$rS_N =$$

Consider the partial sum S_N of a geometric series:

$$\begin{aligned} S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\ rS_N &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1} \end{aligned}$$

Consider the partial sum S_N of a geometric series:

$$
\begin{aligned}
S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\
rS_N &= \quad\;\; ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1} \\
rS_N - S_N &=
\end{aligned}
$$

Consider the partial sum S_N of a geometric series:

$$\begin{aligned} S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\ rS_N &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1} \\ rS_N - S_N &= -a \qquad\qquad + \qquad\qquad ar^{N+1} \end{aligned}$$

Consider the partial sum S_N of a geometric series:

$$\begin{aligned} S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\ rS_N &= \quad ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1} \\ rS_N - S_N &= -a \qquad + \qquad ar^{N+1} \\ S_N(r-1) &= ar^{N+1} - a \end{aligned}$$

If $r \neq 1$, then

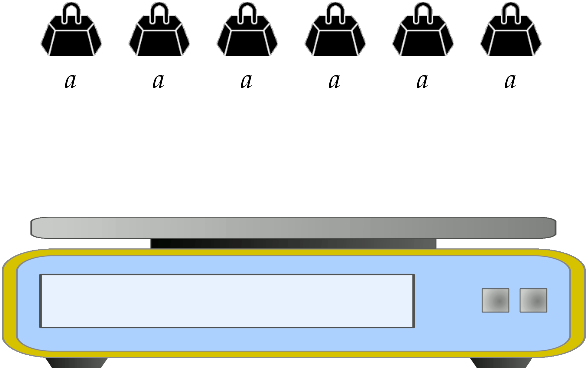
$$S_N = \frac{ar^{N+1} - a}{r-1} = a\frac{r^{N+1} - 1}{r-1}$$

Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

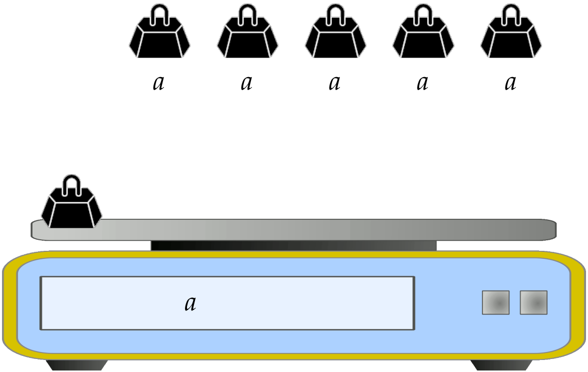
- If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a\dfrac{r^{N+1} - 1}{r - 1}$.
- If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N =$
- If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n =$
- If $r = 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$
- If $r = -1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$
- If $|r| > 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$

$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$



$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

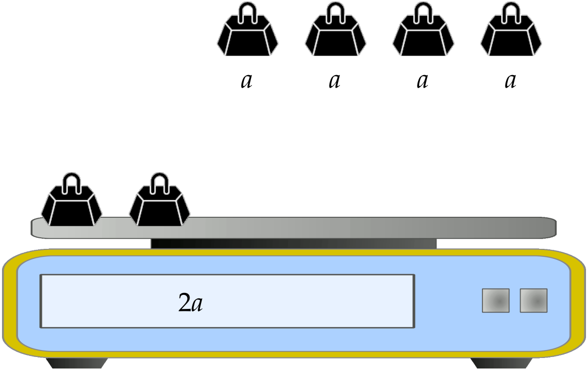
$$S_0 = a$$



$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 2a$$

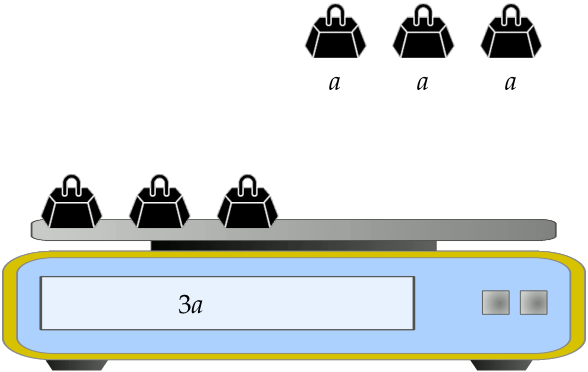


$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 2a$$

$$S_2 = 3a$$



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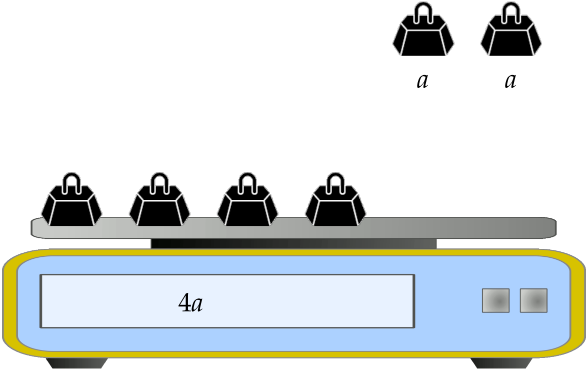


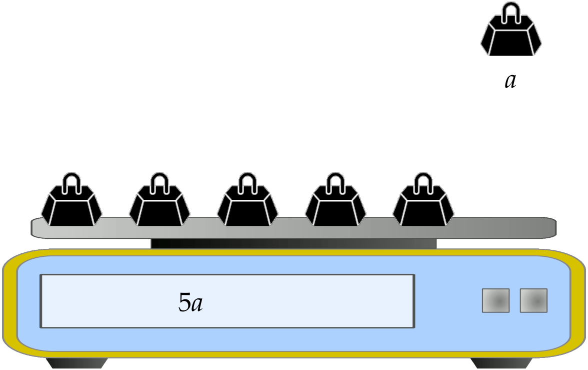
$S_0 = a$

$S_1 = 2a$

$S_2 = 3a$

$S_3 = 4a$

$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$



$$S_0 = a$$

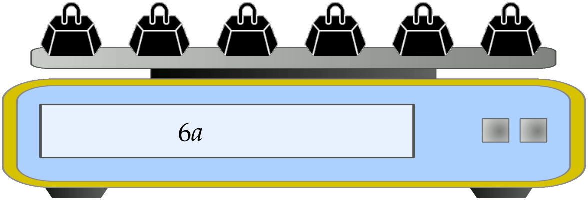
$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$

$$S_4 = 5a$$

$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$



$S_0 = a$

$S_1 = 2a$

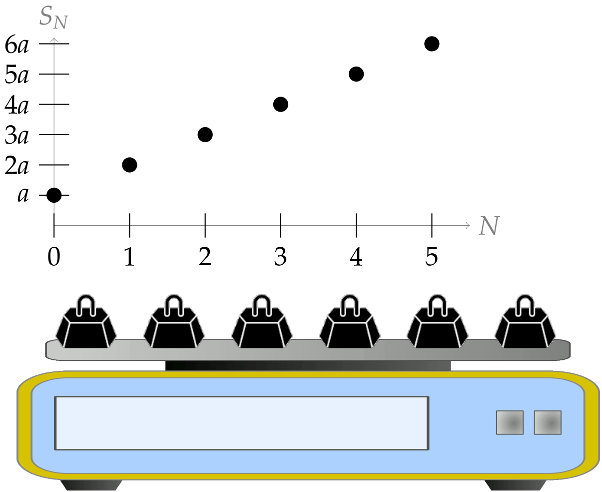
$S_2 = 3a$

$S_3 = 4a$

$S_4 = 5a$

$S_5 = 6a$

$\sum_{n=0}^{\infty} ar^n,\ r = 1,\ a \neq 0$



$S_0 = a$

$S_1 = 2a$

$S_2 = 3a$

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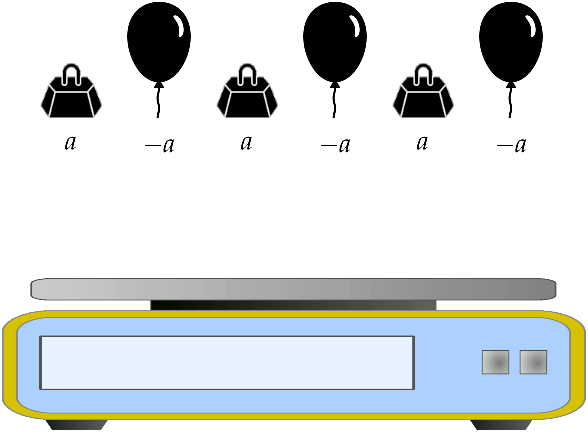
$S_4 = 5a$

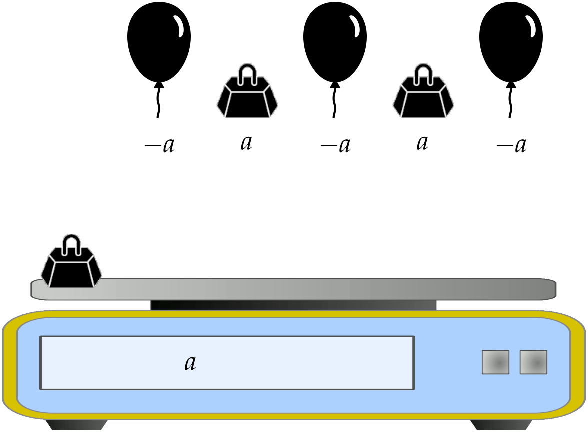
$S_5 = 6a$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

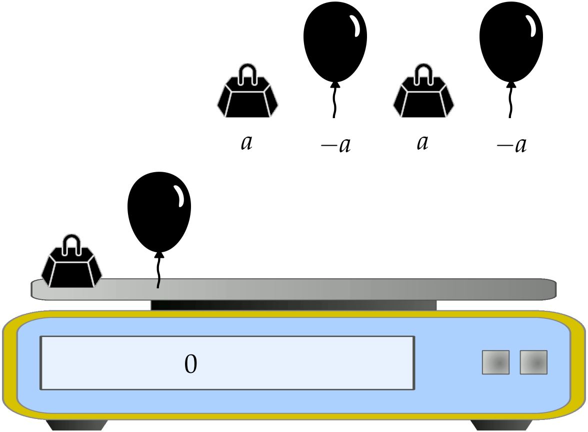


$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$

$S_0 = a$



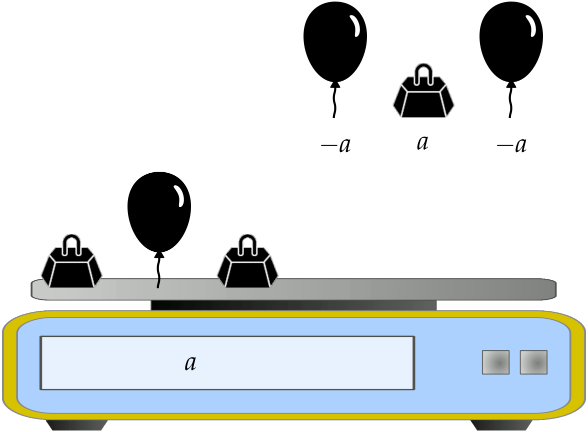
$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$S_0 = a$

$S_1 = 0$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

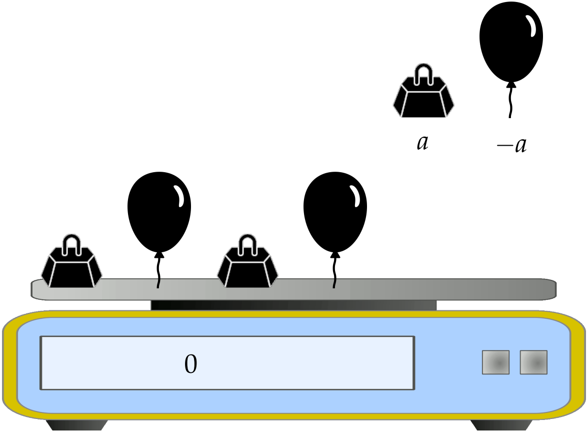


$S_0 = a$

$S_1 = 0$

$S_2 = a$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$S_0 = a$

$S_1 = 0$

$S_2 = a$

$S_3 = 0$

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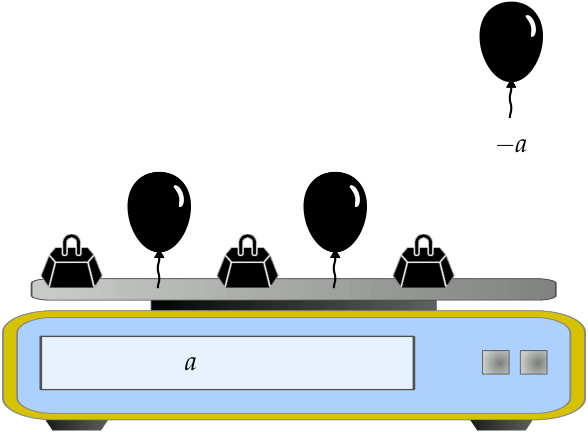


$S_0 = a$

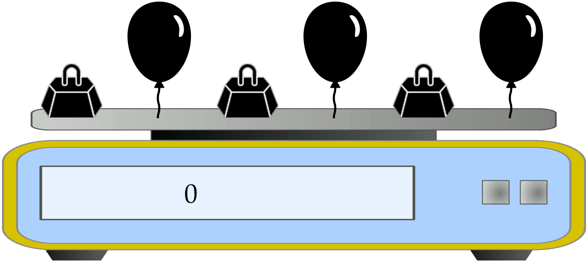
$S_1 = 0$

$S_2 = a$

$S_3 = 0$

$S_4 = a$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$S_0 = a$

$S_1 = 0$

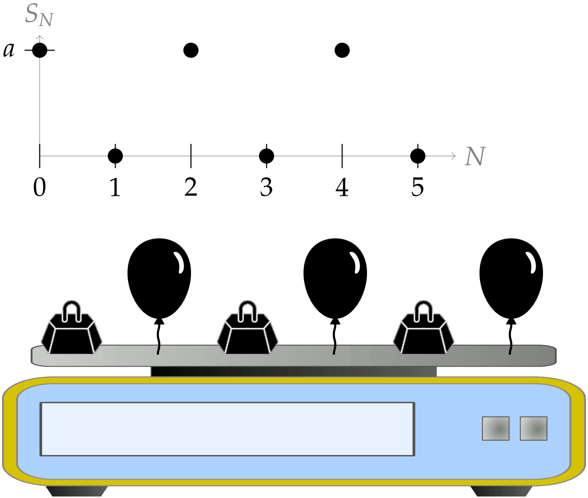
$S_2 = a$

$S_3 = 0$

$S_4 = a$

$S_5 = 0$

$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$



$S_0 = a$

$S_1 = 0$

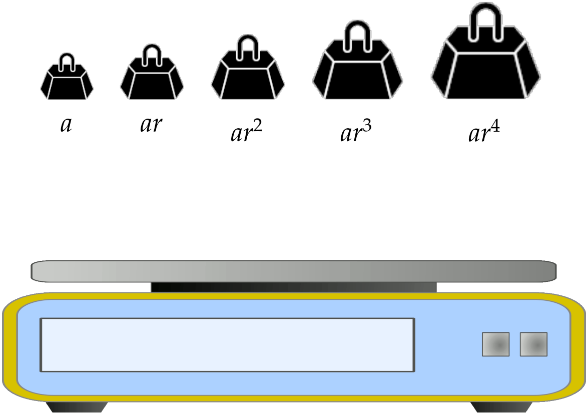
$S_2 = a$

$S_3 = 0$

$S_4 = a$

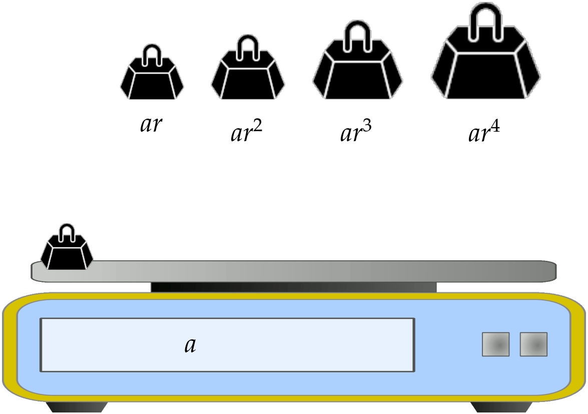
$S_5 = 0$

$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$



$\sum_{n=0}^{\infty} ar^n,\ r > 1,\ a \neq 0$

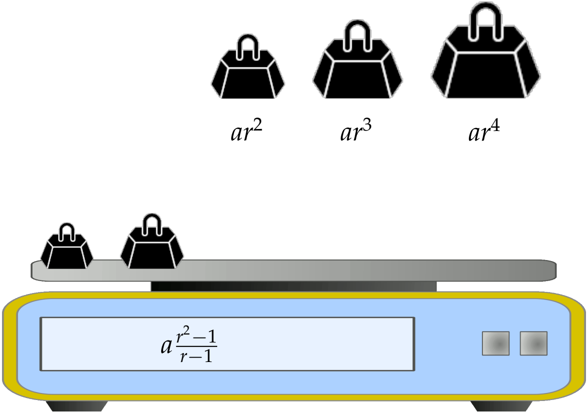
$S_0 = a$



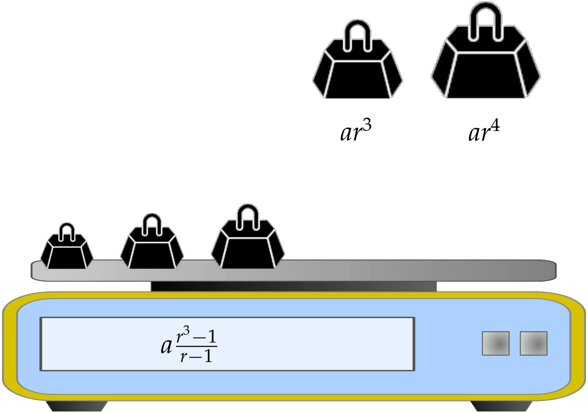
$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$



$\sum_{n=0}^{\infty} ar^n, \, r > 1, \, a \neq 0$

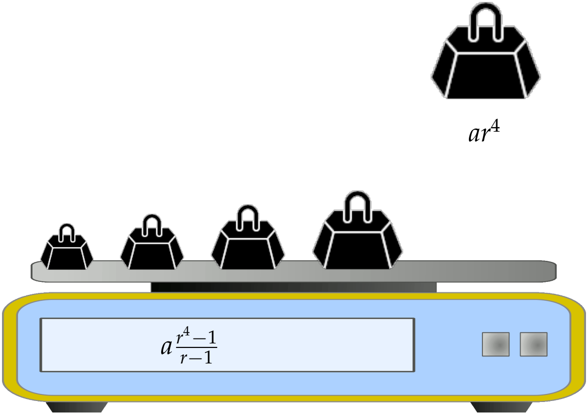


$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$\sum_{n=0}^{\infty} ar^n,\ r > 1,\ a \neq 0$



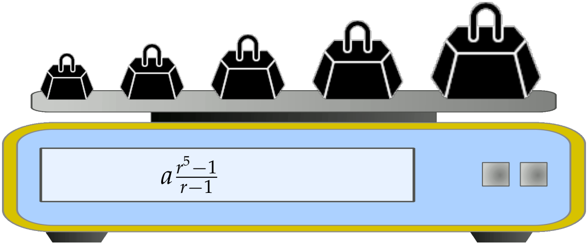
$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$



$$S_0 = a$$

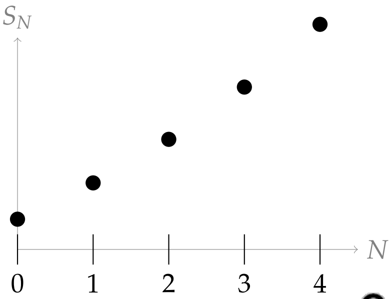
$$S_1 = a\frac{r^2-1}{r-1}$$

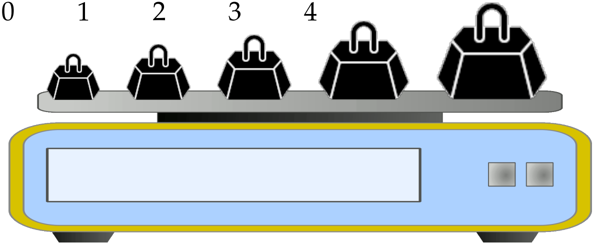
$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$





$$S_0 = a$$

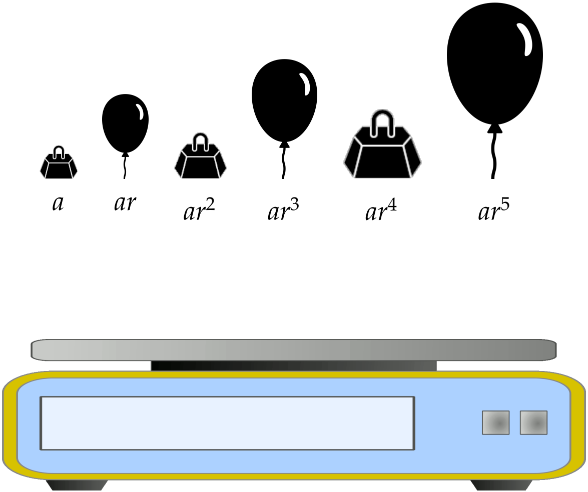
$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

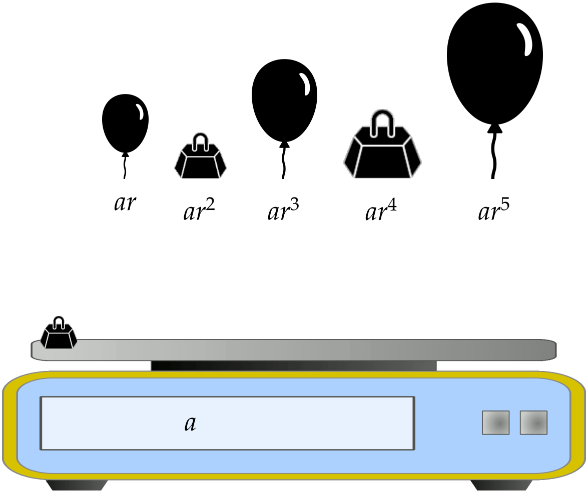
$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n,\ r < -1,\ a \neq 0$$

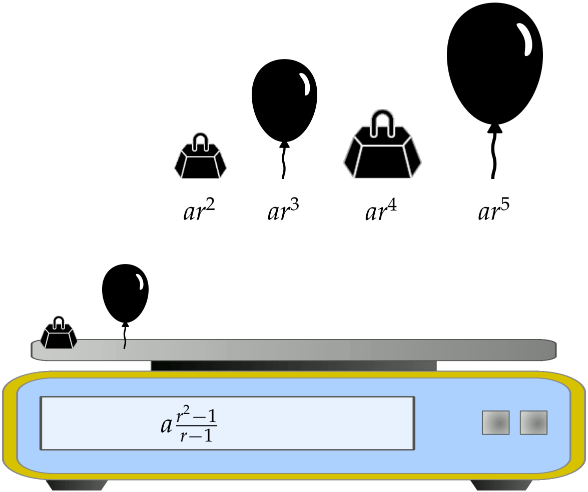


$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$S_0 = a$

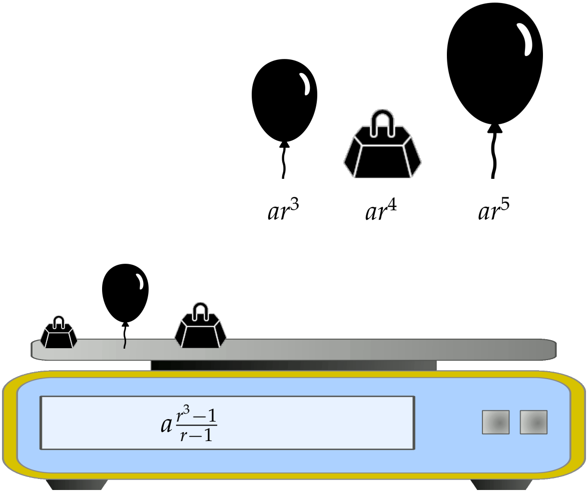
$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$



$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$

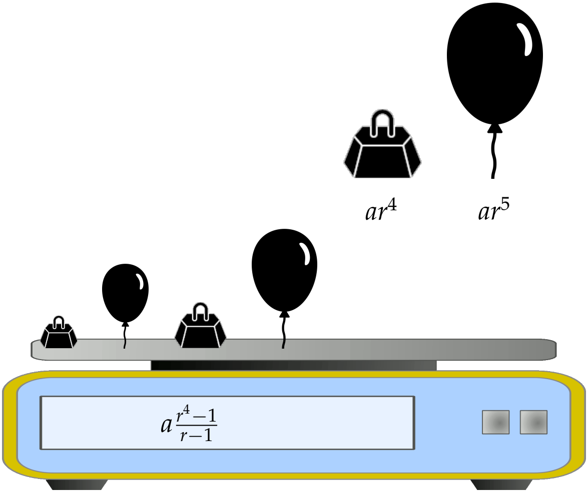


$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$



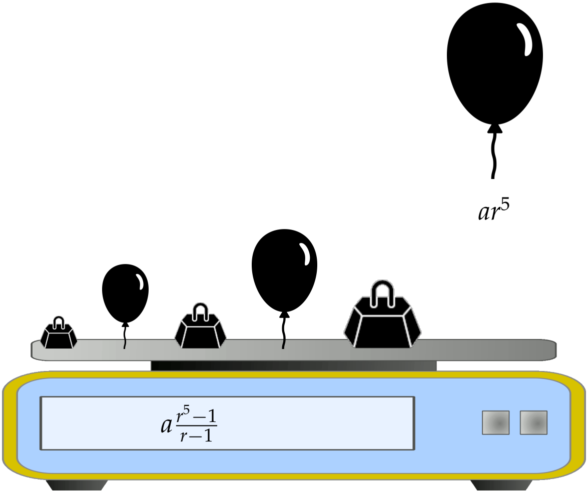
$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$



$$S_0 = a$$

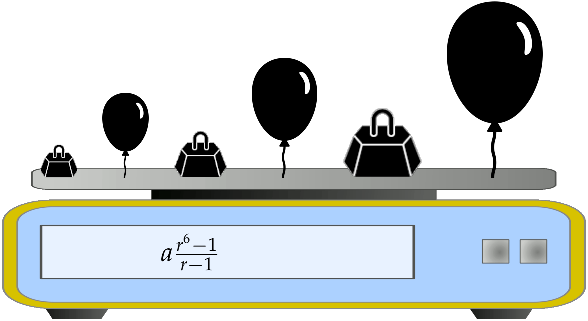
$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$\sum_{n=0}^{\infty} ar^n,\ r < -1,\ a \neq 0$



$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

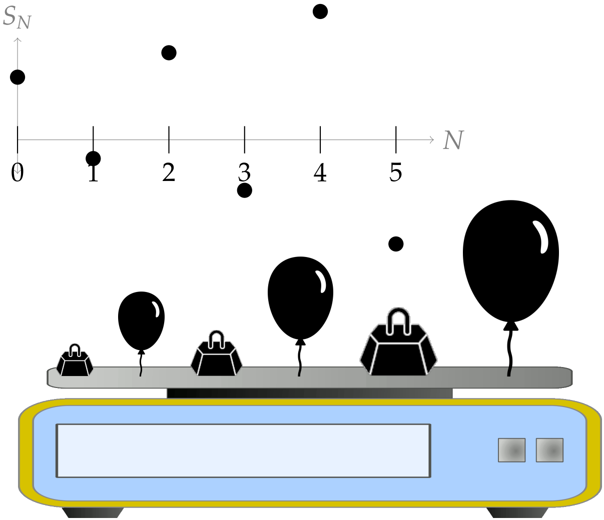
$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$S_5 = a\frac{r^6-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$S_5 = a\frac{r^6-1}{r-1}$$

GEOMETRIC SERIES

New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half.

- Each of the first 210,000 solutions can produce 50 bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^3}$ bitcoins.

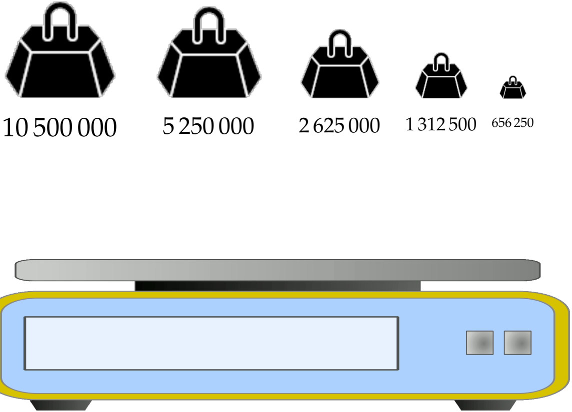
Assume that this continues forever, and that bitcoins are infinitely divisible.[1] How many bitcoins can possibly be produced?

[1] Actually the smallest allowed division of a bitcoin is 10^{-8}.

GEOMETRIC SERIES

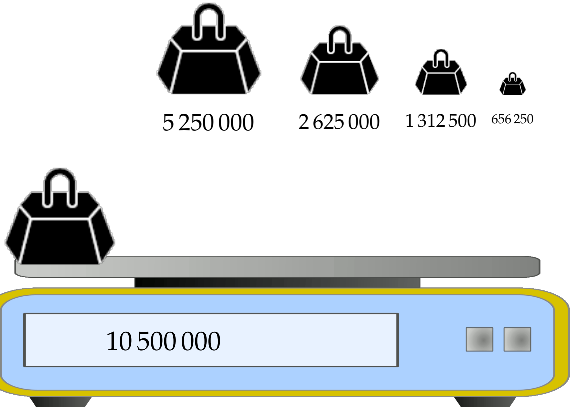
New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half. How many bitcoins can possibly be produced?

$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$



$$\sum_{n=0}^{\infty} 210\,000\left(\frac{50}{2^n}\right) = 21\,000\,000$$

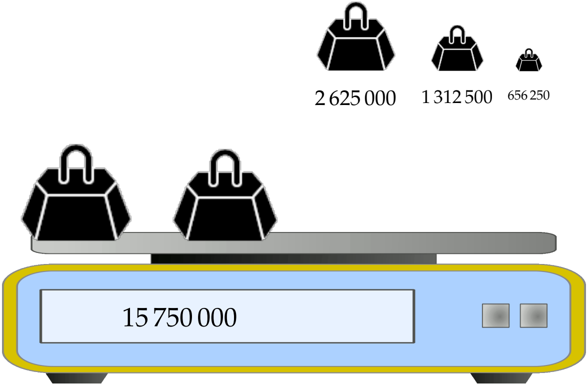
$$S_0 = 10\,500\,000$$



$$\sum_{n=0}^{\infty} 210\,000\left(\frac{50}{2^n}\right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

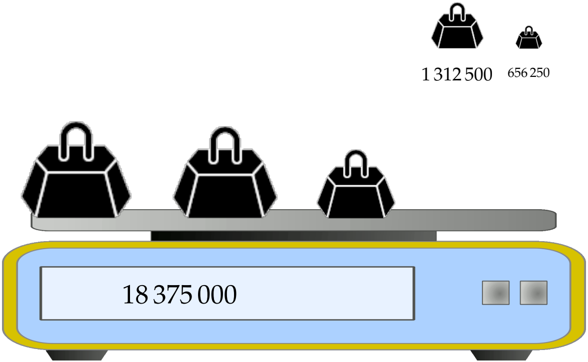


$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$

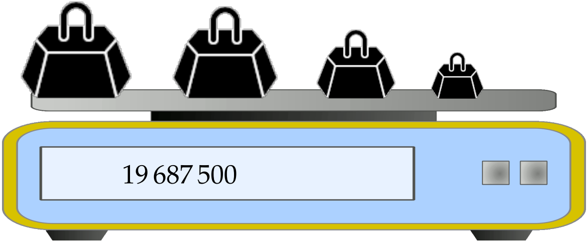
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

656 250



$$\sum_{n=0}^{\infty} 210\,000\left(\frac{50}{2^n}\right) = 21\,000\,000$$

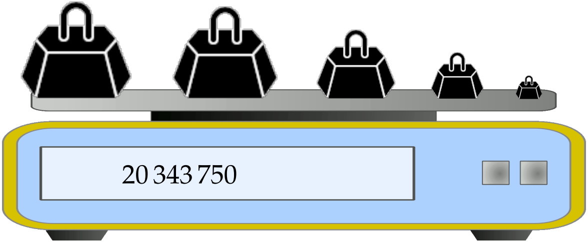
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

$$S_4 = 20\,343\,750$$



Arithmetic of Series

Let S, T, and C be real numbers. Let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T respectively. Then

$$\sum_{n=1}^{\infty}[a_n + b_n] = S + T$$

$$\sum_{n=1}^{\infty}[a_n - b_n] = S - T$$

$$\sum_{n=1}^{\infty}[Ca_n] = CS$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right)$

$$\sum_{n=0}^{\infty}\left(\frac{2}{3^n}+\frac{4}{5^n}\right)=$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right)$

$$\sum_{n=6}^{\infty}\left(\frac{3^{n-1}}{5^{2n}}\right) =$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

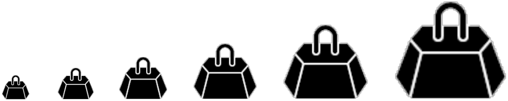
Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n}\right)$

Q

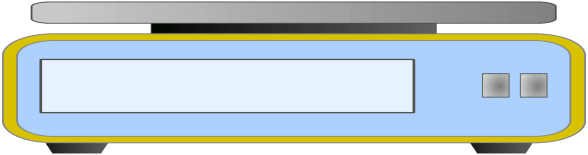
$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) =$$

Q

$$\sum_{n=0}^{\infty}\left(\frac{2^{2n}}{3^n}\right) \text{ diverges}$$

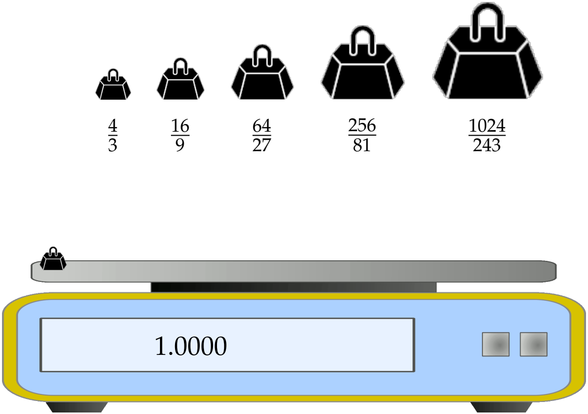


$1 \qquad \frac{4}{3} \qquad \frac{16}{9} \qquad \frac{64}{27} \qquad \frac{256}{81} \qquad \frac{1024}{243}$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$$S_0 = 1.0000$$

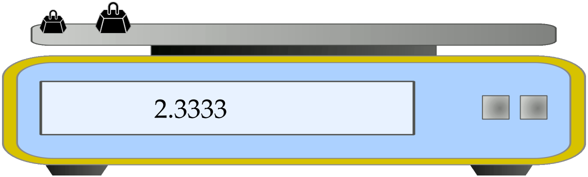


$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$$S_0 = 1.0000$$

$$S_1 = 2.3333$$

$\frac{16}{9}$ $\frac{64}{27}$ $\frac{256}{81}$ $\frac{1024}{243}$

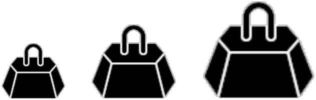


$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

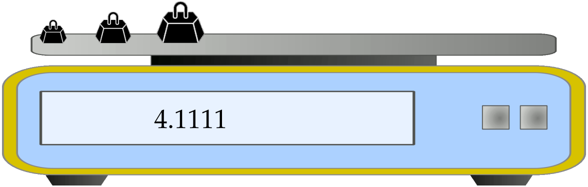
$$S_0 = 1.0000$$

$$S_1 = 2.3333$$

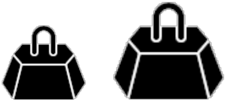
$$S_2 = 4.1111$$



$\frac{64}{27}$ $\frac{256}{81}$ $\frac{1024}{243}$



$$\sum_{n=0}^{\infty}\left(\frac{2^{2n}}{3^n}\right) \text{ diverges}$$



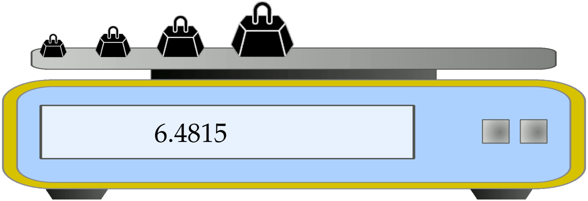
$\frac{256}{81}$ $\frac{1024}{243}$

$S_0 = 1.0000$

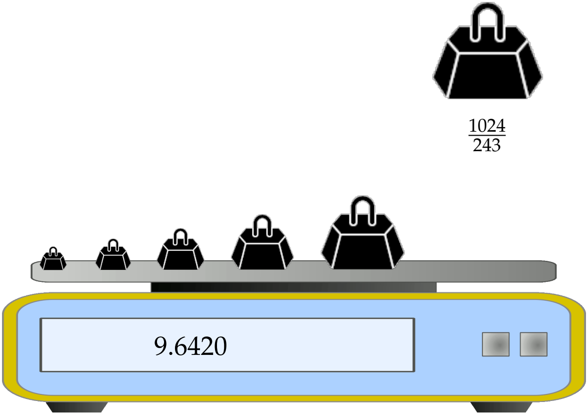
$S_1 = 2.3333$

$S_2 = 4.1111$

$S_3 = 6.4815$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$



$$S_0 = 1.0000$$

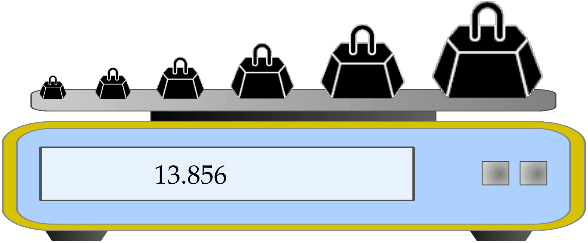
$$S_1 = 2.3333$$

$$S_2 = 4.1111$$

$$S_3 = 6.4815$$

$$S_4 = 9.6420$$

$$\sum_{n=0}^{\infty}\left(\frac{2^{2n}}{3^n}\right) \text{ diverges}$$



$S_0 = 1.0000$

$S_1 = 2.3333$

$S_2 = 4.1111$

$S_3 = 6.4815$

$S_4 = 9.6420$

$S_5 = 13.856$

Telescoping Sums

Evaluate $\displaystyle\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Evaluate $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

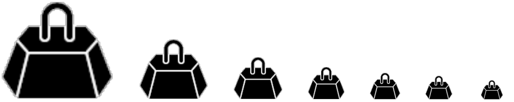
Evaluate $\displaystyle\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right)$.

Evaluate $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$.

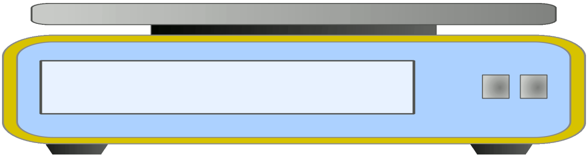
Evaluate $\displaystyle\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right)$.

Evaluate $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$.

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

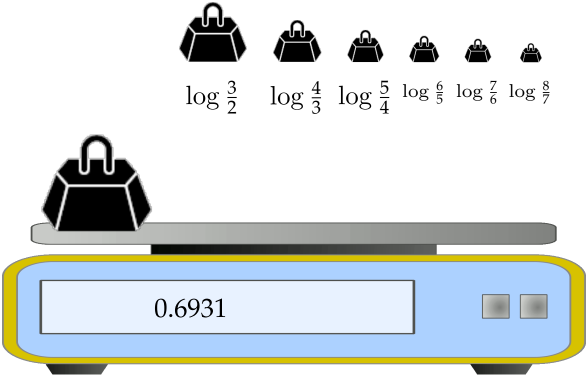


$\log 2$ $\quad \log \frac{3}{2}$ $\quad \log \frac{4}{3}$ $\quad \log \frac{5}{4}$ $\quad \log \frac{6}{5}$ $\quad \log \frac{7}{6}$ $\quad \log \frac{8}{7}$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

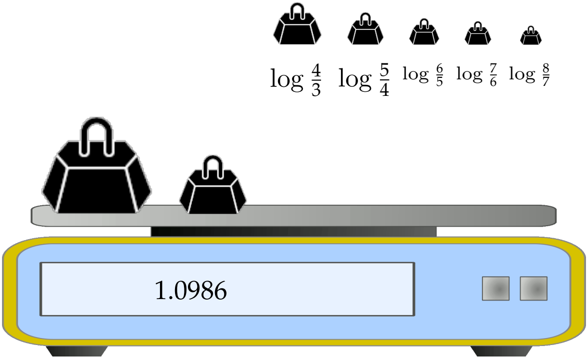
$$S_1 = 0.6931$$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

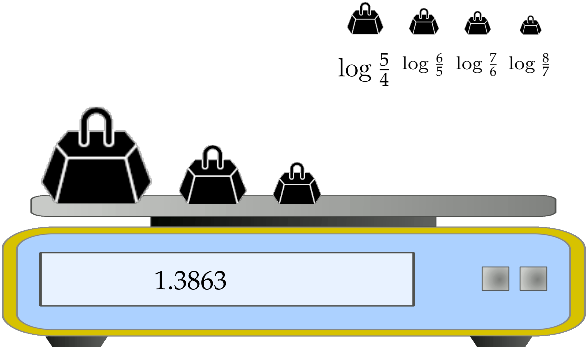


$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$S_1 = 0.6931$

$S_2 = 1.0986$

$S_3 = 1.3863$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

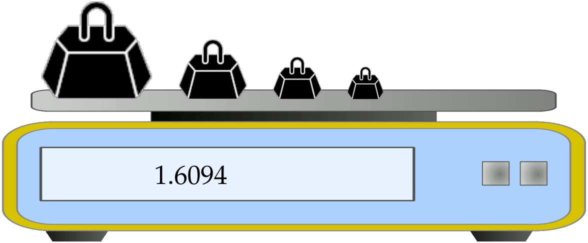
$S_1 = 0.6931$

$S_2 = 1.0986$

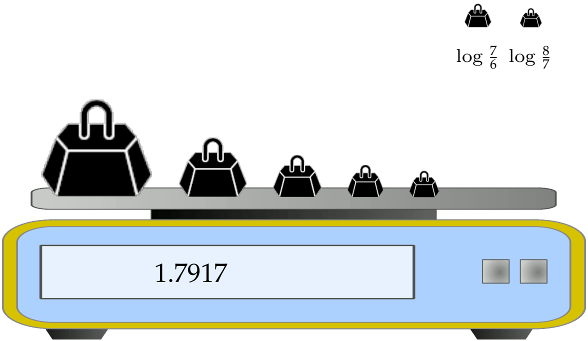
$S_3 = 1.3863$

$S_4 = 1.6094$

$\log \frac{6}{5}$ $\log \frac{7}{6}$ $\log \frac{8}{7}$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$



$S_1 = 0.6931$

$S_2 = 1.0986$

$S_3 = 1.3863$

$S_4 = 1.6094$

$S_5 = 1.7917$

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

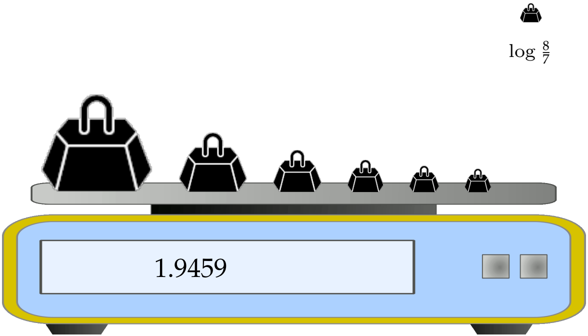
$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

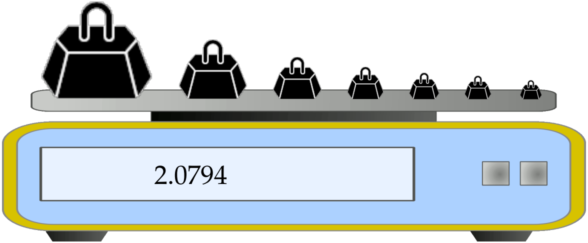
$$S_4 = 1.6094$$

$$S_5 = 1.7917$$

$$S_6 = 1.9459$$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$



$S_1 = 0.6931$

$S_2 = 1.0986$

$S_3 = 1.3863$

$S_4 = 1.6094$

$S_5 = 1.7917$

$S_6 = 1.9459$

$S_7 = 2.0794$

Included Work

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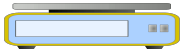 'Waage/Libra' by B. Lachner is in the public domain (accessed April 2021, edited), 30–35, 48–54, 62–66, 77–107, 110–115, 123–129, 133–140

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