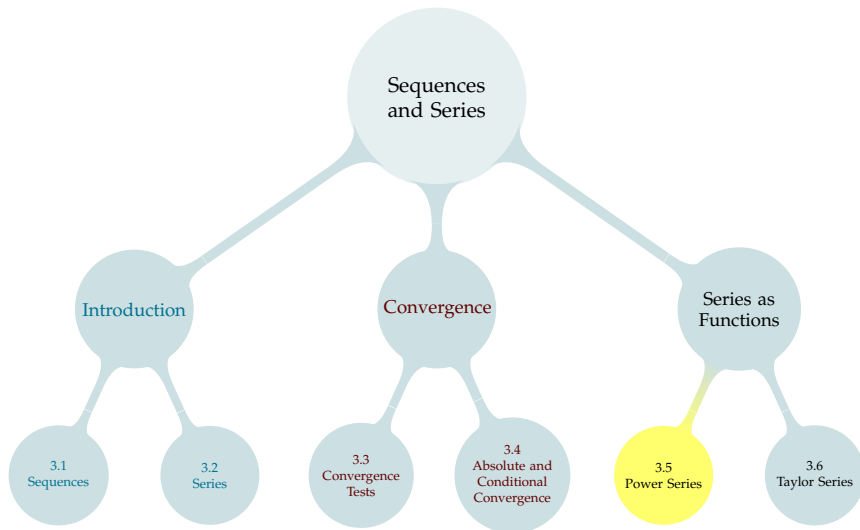


# TABLE OF CONTENTS



Recall the geometric series: for a constant  $r$ , with  $|r| < 1$ :

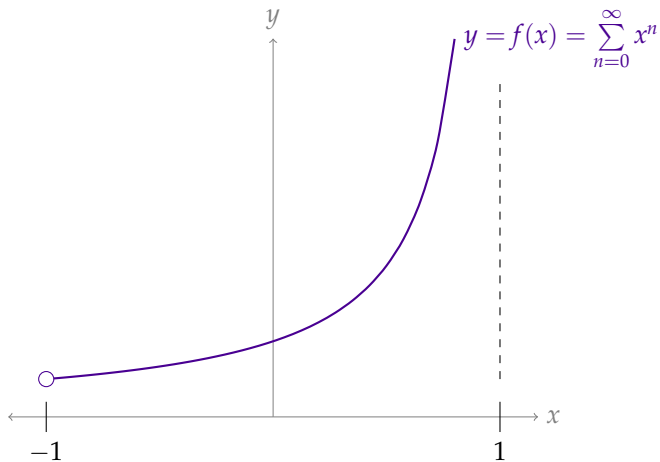
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

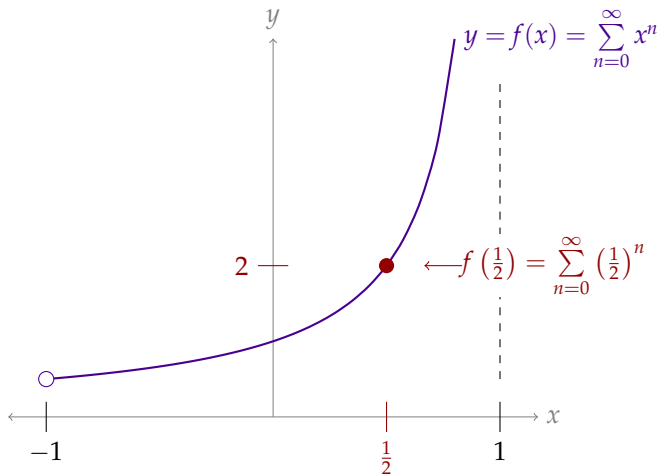
We can think of this as a function. If we set

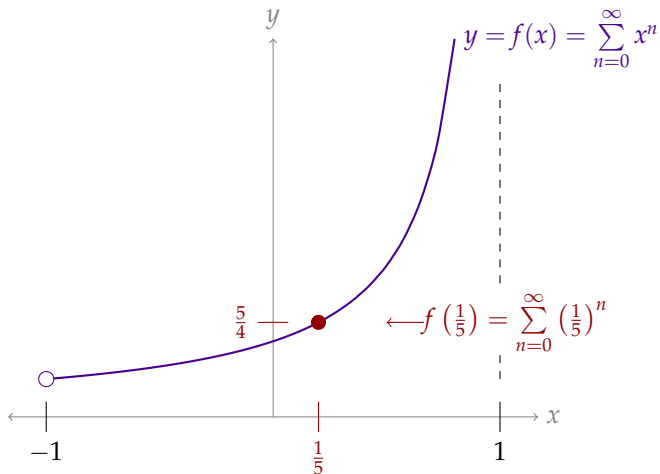
$$f(x) = \sum_{n=0}^{\infty} x^n$$

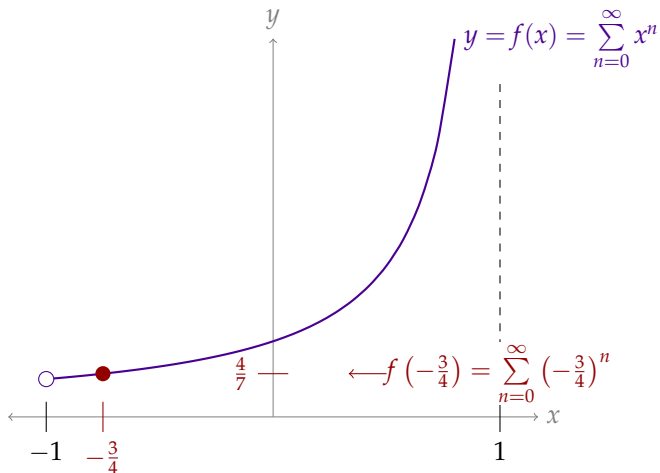
and restrict our domain to  $-1 < x < 1$ , then

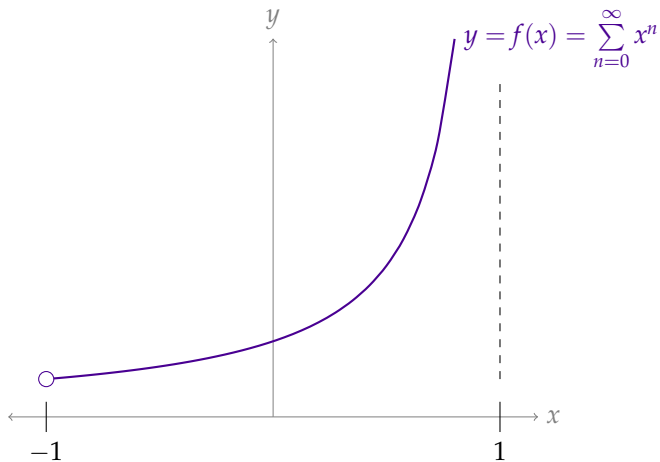
$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$











Why would we ever prefer to write  $\sum_{n=0}^{\infty} x^n$  instead of  $\frac{1}{1-x}$ ?



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The function

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

isn't a polynomial, but in certain ways it behaves like one. For  $|x| < 1$ :

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$$\int \frac{1}{1-x} dx = \int \left( \sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=0}^{\infty} \left( \int x^n dx \right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

## Definition

A series of the form

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

is called a *power series in  $(x-c)$*  or a *power series centered on  $c$* . The numbers  $A_n$  are called the coefficients of the power series.

One often considers power series centered on  $c = 0$  and then the series reduces to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots = \sum_{n=0}^{\infty} A_nx^n$$

$$\sum_{n=0}^{\infty} A_n (x - c)^n = A_0 + A_1(x - c) + A_2(x - c)^2 + A_3(x - c)^3 + \cdots$$

In a power series, we think of the coefficients  $A_n$  as fixed constants, and we think of  $x$  as the variable of a function.

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Evaluate the power series  $\sum_{n=0}^{\infty} A_n(x-c)^n$  when  $x = c$  :





$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

In a power series, we think of the coefficients  $A_n$  as fixed constants, and we think of  $x$  as the variable of a function.

Evaluate the power series  $\sum_{n=0}^{\infty} A_n(x-c)^n$  when  $x = c$  :

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

$$\sum_{n=0}^{\infty} A_n(c-c)^n = A_0 + A_1 \underbrace{(c-c)}_0 + A_2 \underbrace{(c-c)^2}_0 + A_3 \underbrace{(c-c)^3}_0 + \cdots$$

$$= A_0 \quad (\text{In particular, the series converges when } x = c.)$$

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of  $x$  for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

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converges.

This looks somewhat like a geometric series, but not exactly, so the ratio test is a good option.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| \left( \frac{n}{n+1} \right) \\ &= \lim_{n \rightarrow \infty} |x| \left( \frac{n}{n+1} \right) = |x| \end{aligned}$$

So the series converges when  $|x| < 1$  and diverges when  $|x| > 1$ .

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of  $x$  for which the power series

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converges.

When  $x = 1$ , we have the harmonic series, which diverges. When  $x = -1$ , we have the alternating harmonic series, which converges.

So, all together, the series converges when  $-1 \leq x < 1$ , and diverges everywhere else.



A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of  $x$  for which the power series

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converges.

## Definition

Consider the power series

$$\sum_{n=0}^{\infty} A_n(x - c)^n.$$

The set of real  $x$ -values for which it converges is called the interval of convergence of the series.

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots .$$

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots$$

This still looks somewhat like a geometric series, so the ratio test is a still good option to start.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{2^n(x-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| \left( \frac{2^{n+1}}{2^n} \right) \\ &= 2|x-1| \end{aligned}$$

So we see that the series converges when  $|x-1| < \frac{1}{2}$  and diverges when  $|x-1| > \frac{1}{2}$ .

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots$$

When  $x-1 = -\frac{1}{2}$ , i.e.  $x = \frac{1}{2}$ , our series is

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2} - 1\right)^n = \sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$$

When  $x-1 = \frac{1}{2}$ , i.e.  $x = \frac{3}{2}$ , our series is

$$\sum_{n=0}^{\infty} 2^n \left(\frac{3}{2} - 1\right)^n = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1$$

In both cases, the series diverge by the divergence test. All together, the interval of convergence is  $\frac{1}{2} < x < \frac{3}{2}$ .





What happens if we apply the ratio test to a generic power series,

$$\sum_{n=0}^{\infty} A_n(x - c)^n?$$

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$$\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}(x - c)^{n+1}}{A_n(x - c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} (x - c) \right| = |x - c| \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right|$$

- ▶ If  $\left| \frac{A_{n+1}}{A_n} \right|$  does not approach a limit as  $n \rightarrow \infty$ , the ratio test tells us nothing. (We should try other tests.)
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = 0$ , then
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \infty$ , then
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = A$  for some real number  $A$ , then

What happens if we apply the ratio test to a generic power series,

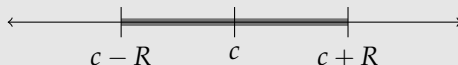
$$\sum_{n=0}^{\infty} A_n(x - c)^n?$$

$$\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}(x - c)^{n+1}}{A_n(x - c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} (x - c) \right| = |x - c| \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right|$$

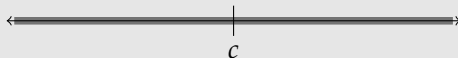
- ▶ If  $\left| \frac{A_{n+1}}{A_n} \right|$  does not approach a limit as  $n \rightarrow \infty$ , the ratio test tells us nothing. (We should try other tests.)
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = 0$ , then the series converges for all  $x$ .
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \infty$ , then the series converges when  $x = c$ , and diverges otherwise.
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = A$  for some real number  $A$ , then the series converges when  $|x - c| < \frac{1}{A}$ , and diverges for  $|x - c| > \frac{1}{A}$ . The cases  $|x - c| = \frac{1}{A}$  need further inspection.

## Definition: Radius of Convergence

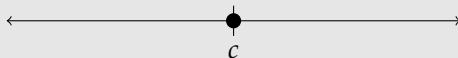
- (a) Let  $0 < R < \infty$ . If  $\sum_{n=0}^{\infty} A_n(x - c)^n$  converges for  $|x - c| < R$ , and diverges for  $|x - c| > R$ , then we say that the series has radius of convergence  $R$ .



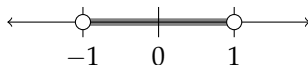
- (b) If  $\sum_{n=0}^{\infty} A_n(x - c)^n$  converges for every number  $x$ , we say that the series has an infinite radius of convergence.



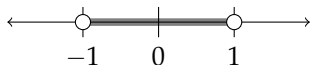
- (c) If  $\sum_{n=0}^{\infty} A_n(x - c)^n$  diverges for every  $x \neq c$ , we say that the series has radius of convergence zero.



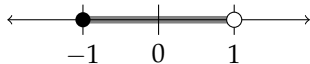
- We saw that  $\sum_{n=0}^{\infty} x^n$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series has radius of convergence  $R =$



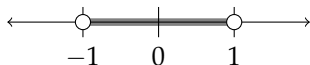
- We saw that  $\sum_{n=0}^{\infty} x^n$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series has radius of convergence  $R = 1$ .



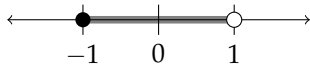
- We saw that  $\sum_{n=1}^{\infty} \frac{x^n}{n}$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series also has radius of convergence  $R =$



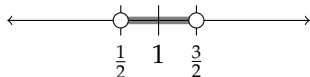
- We saw that  $\sum_{n=0}^{\infty} x^n$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series has radius of convergence  $R = 1$ .



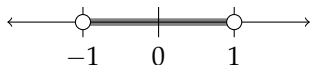
- We saw that  $\sum_{n=1}^{\infty} \frac{x^n}{n}$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series also has radius of convergence  $R = 1$ .



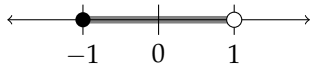
- We saw that  $\sum_{n=1}^{\infty} 2^n(x-1)^n$  converges when  $|x-1| < \frac{1}{2}$  and diverges when  $|x-1| > \frac{1}{2}$ , so this series has radius of convergence  $R =$



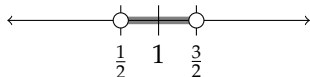
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- We saw that  $\sum_{n=1}^{\infty} \frac{x^n}{n}$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series also has radius of convergence  $R = 1$ .



- We saw that  $\sum_{n=1}^{\infty} 2^n(x-1)^n$  converges when  $|x-1| < \frac{1}{2}$  and diverges when  $|x-1| > \frac{1}{2}$ , so this series has radius of convergence  $R = \frac{1}{2}$ .





What is the radius of convergence for the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  ?

*Recall:*  $n! = (n)(n-1)(n-2) \cdots (2)(1)$ .

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*Recall:*  $n! = (n)(n-1)(n-2) \cdots (2)(1)$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} |x| \frac{(n)(n-1)(n-2) \cdots (2)(1)}{(n+1)(n)(n-1)(n-2) \cdots (2)(1)} \\ &= \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 \end{aligned}$$

For every real  $x$ , the limit is less than one, so the series converges. That is, its radius of convergence is  $\infty$ .



What is the radius of convergence for the series  $\sum_{n=0}^{\infty} n! \cdot (x - 3)^n$  ?

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$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-3)^{n+1}}{(n!)(x-3)^n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n)(n-1)(n-2) \cdots (2)(1)}{(n)(n-1)(n-2) \cdots (2)(1)} |x-3| \\ &= \lim_{n \rightarrow \infty} (n+1) |x-3| \end{aligned}$$

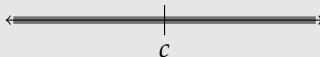
For every real  $x$  except  $x = 3$ , the limit is greater than one, so the series diverges. The series only converges at  $x = 3$ . That is, its radius of convergence is 0.



## Theorem

Given a power series (say with centre  $c$ ), one of the following holds.

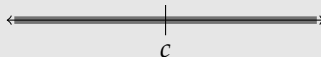
- (a) The power series converges for every number  $x$ . In this case we say that the radius of convergence is  $\infty$ .



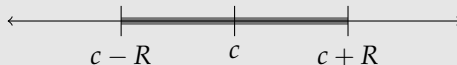
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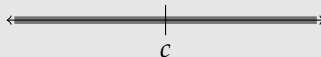
- (b) There is a number  $0 < R < \infty$  such that the series converges for  $|x - c| < R$  and diverges for  $|x - c| > R$ . Then  $R$  is called the radius of convergence.



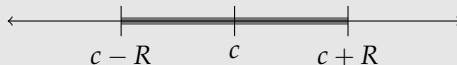
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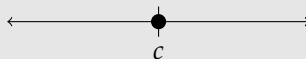
- (a) The power series converges for every number  $x$ . In this case we say that the radius of convergence is  $\infty$ .



- (b) There is a number  $0 < R < \infty$  such that the series converges for  $|x - c| < R$  and diverges for  $|x - c| > R$ . Then  $R$  is called the radius of convergence.



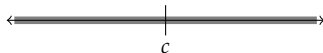
- (c) The series converges for  $x = c$  and diverges for all  $x \neq c$ . In this case, we say that the radius of convergence is 0.



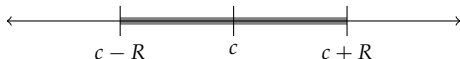
We are told that a certain power series with centre  $c = 3$  converges at  $x = 4$  and diverges at  $x = 1$ . What else can we say about the convergence or divergence of the series for other values of  $x$ ?

Given a power series (say with centre  $c$ ), one of the following holds.

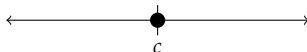
- (a) The power series converges for every number  $x$ . In this case we say that the radius of convergence is  $\infty$ .



- (b) There is a number  $0 < R < \infty$  such that the series converges for  $|x - c| < R$  and diverges for  $|x - c| > R$ . Then  $R$  is called the radius of convergence.



- (c) The series converges for  $x = c$  and diverges for all  $x \neq c$ . In this case, we say that the radius of convergence is 0.





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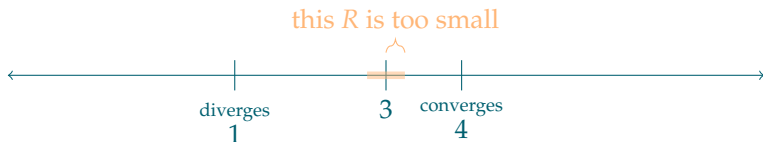


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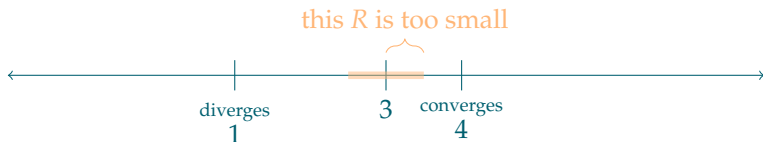


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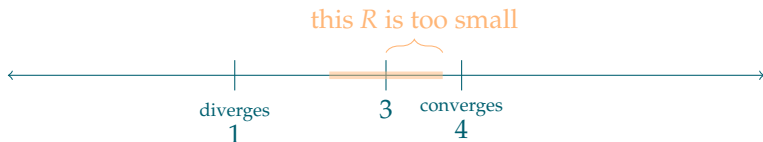


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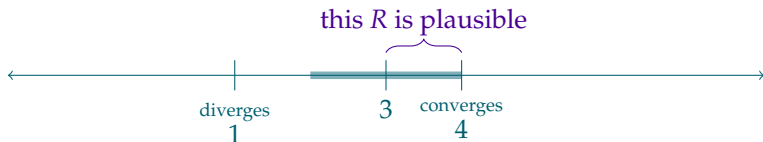


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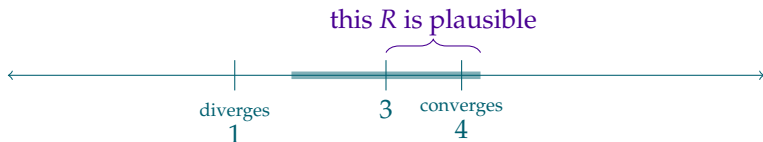


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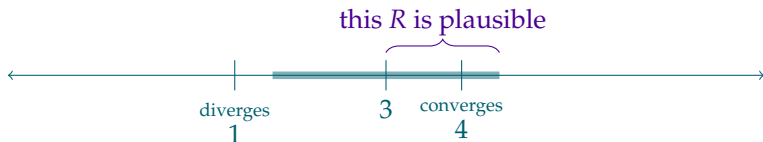


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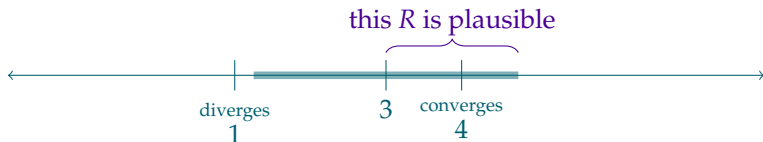
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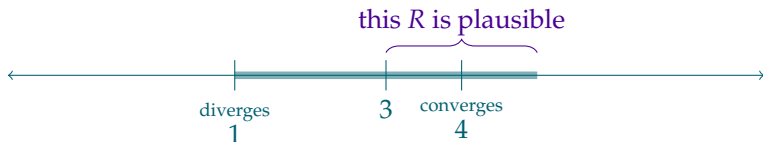


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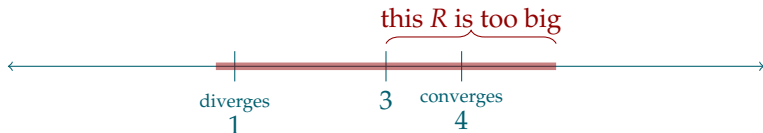


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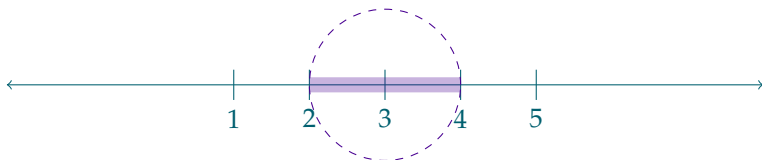
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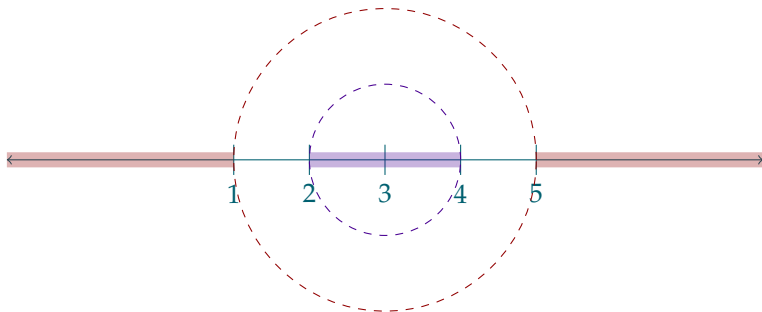
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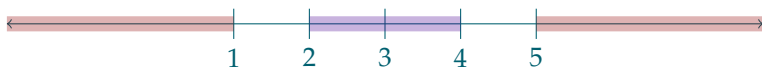
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- ▶ We do not know whether the series converges for other  $x$ .

## Operations on Power Series

Assume that the functions  $f(x)$  and  $g(x)$  are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x - c)^n$$

for all  $x$  obeying  $|x - c| < R$ . Let  $K$  be a constant. Then:

$$f(x) + g(x) = \sum_{n=0}^{\infty} [A_n + B_n] (x - c)^n$$

$$Kf(x) = \sum_{n=0}^{\infty} K A_n (x - c)^n$$

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for all  $x$  obeying  $|x - c| < R$ . Then:

$$\begin{aligned} (x - c)^N f(x) &= \sum_{n=0}^{\infty} A_n (x - c)^{n+N} \quad \text{for any integer } N \geq 1 \\ &= \sum_{k=N}^{\infty} A_{k-N} (x - c)^k \quad \text{where } k = n + N \end{aligned}$$

for all  $x$  obeying  $|x - c| < R$ .



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for all  $x$  obeying  $|x - c| < R$ . Then:

$$f'(x) = \sum_{n=0}^{\infty} A_n n (x - c)^{n-1} = \sum_{n=1}^{\infty} A_n n (x - c)^{n-1}$$

$$\int_c^x f(t) \, dt = \sum_{n=0}^{\infty} A_n \frac{(x - c)^{n+1}}{n + 1}$$

$$\int f(x) \, dx = \left[ \sum_{n=0}^{\infty} A_n \frac{(x - c)^{n+1}}{n + 1} \right] + C \quad \text{with } C \text{ an arbitrary constant}$$

for all  $x$  obeying  $|x - c| < R$ .

## Operations on Power Series

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for all  $x$  obeying  $|x-c| < R$ .

Differentiating, antidifferentiating, multiplying by a nonzero constant, and multiplying by a positive power of  $(x-c)$  do not change the radius of convergence of  $f(x)$  (although they may change the interval of convergence).

Given that  $\frac{d}{dx} \left\{ \frac{1}{1-x} \right\} = \frac{1}{(1-x)^2}$ , find a power series representation for  $\frac{1}{(1-x)^2}$  when  $|x| < 1$ .

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$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx} \left\{ \frac{1}{1-x} \right\} \\ &= \frac{d}{dx} \left\{ \sum_{n=0}^{\infty} x^n \right\} \\ &= \sum_{n=0}^{\infty} \left( \frac{d}{dx} \{x^n\} \right) \\ &= \sum_{n=0}^{\infty} nx^{n-1} \\ &= \sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

Find a power series representation for  $\log(1 + x)$  when  $|x| < 1$ .

Find a power series representation for  $\log(1+x)$  when  $|x| < 1$ .

First, note  $\frac{d}{dx}\{\log(1+x)\} = \frac{1}{1+x}$ . Our plan is to antidifferentiate a power series representation of  $\frac{1}{1+x}$ . For  $|x| < 1$ :

$$\begin{aligned}\frac{1}{1+x} &= \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \\ \int \frac{1}{1+x} dx &= \int \left( \sum_{n=0}^{\infty} (-1)^n x^n \right) dx \\ &= \sum_{n=0}^{\infty} \left( \int (-1)^n x^n dx \right)\end{aligned}$$

So, for some constant  $C$ ,

$$\log(1+x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$



Find a power series representation for  $\log(1 + x)$  when  $|x| < 1$ .

To find  $C$ , let's plug in a value for  $x$  where both sides of the equation are easy to evaluate:  $x = 0$ .

$$\log(1 + 0) = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{0^n}{n}$$

$$0 = C$$

$$\text{So, } \log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

when  $|x| < 1$ .



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First, note  $\frac{d}{dx} \{\arctan x\} = \frac{1}{1+x^2}$ . To obtain a power series representation of  $\frac{1}{1+x^2}$ , we'll substitute into the geometric series.

Let  $y = -x^2$  with  $|y| < 1$ . Then:

$$\begin{aligned}\frac{1}{1-y} &= \sum_{n=0}^{\infty} y^n \\ \Rightarrow \frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \\ \Rightarrow \int \frac{1}{1+x^2} dx &= \int \left( \sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = \sum_{n=0}^{\infty} \left( \int (-1)^n x^{2n} dx \right) \\ \Rightarrow \arctan x &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

for some constant  $C$ .



Find a power series representation for  $\arctan(x)$  when  $|x| < 1$ .

To find  $C$ , we'll plug in  $x = 0$ , which makes both sides of the last equation easy to evaluate.

$$\arctan 0 = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$0 = C$$

$$\text{So, } \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

when  $|x| < 1$ , i.e. when  $|x| < 1$ .



## Substituting in a Power Series

Assume that the function  $f(x)$  is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all  $x$  in the interval  $I$ . Also let  $K$  and  $k$  be real constants. Then

$$f(Kx^k) = \sum_{n=0}^{\infty} A_n K^n x^{kn}$$

whenever  $Kx^k$  is in  $I$ . In particular, if  $\sum_{n=0}^{\infty} A_n x^n$  has radius of convergence  $R$ ,  $K$  is nonzero and  $k$  is a natural number, then  $\sum_{n=0}^{\infty} A_n K^n x^{kn}$  has radius of convergence  $\sqrt[k]{R/|K|}$ .

Find a power series representation for  $\frac{1}{5-x}$  with centre 3.



Find a power series representation for  $\frac{1}{5-x}$  with centre 3.

We know that  $\frac{1}{1-(x-3)} = \sum_{n=0}^{\infty} (x-3)^n$  when  $|x-3| < 1$ . To take advantage of our ability to substitute into power functions, we'd like to write  $\frac{1}{5-x}$  in the form  $\frac{1}{1-K(x-3)^k}$  for some constant  $K$  and some whole number  $k$ .

$$\frac{1}{5-x} = \frac{1}{2-(x-3)} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{x-3}{2}\right)}$$

Set  $y = \frac{x-3}{2}$ . When  $|y| < 1$ :

$$\begin{aligned} \frac{1}{2} \cdot \frac{1}{1-y} &= \frac{1}{2} \sum_{n=0}^{\infty} y^n \\ \Rightarrow \frac{1}{2} \cdot \frac{1}{1-\left(\frac{x-3}{2}\right)} &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x-3}{2}\right)^n \\ \Rightarrow \frac{1}{5-x} &= \sum_{n=0}^{\infty} \frac{(x-3)^n}{2^{n+1}}. \end{aligned}$$



Find a power series representation for  $\frac{1}{5-x}$  with centre 3.

The series converges when:

$$\begin{aligned} |y| &< 1 \\ \left| \frac{x-3}{2} \right| &< 1 \\ |x-3| &< 2 \end{aligned}$$

So the radius of convergence of our series is 2.



## Included Work



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