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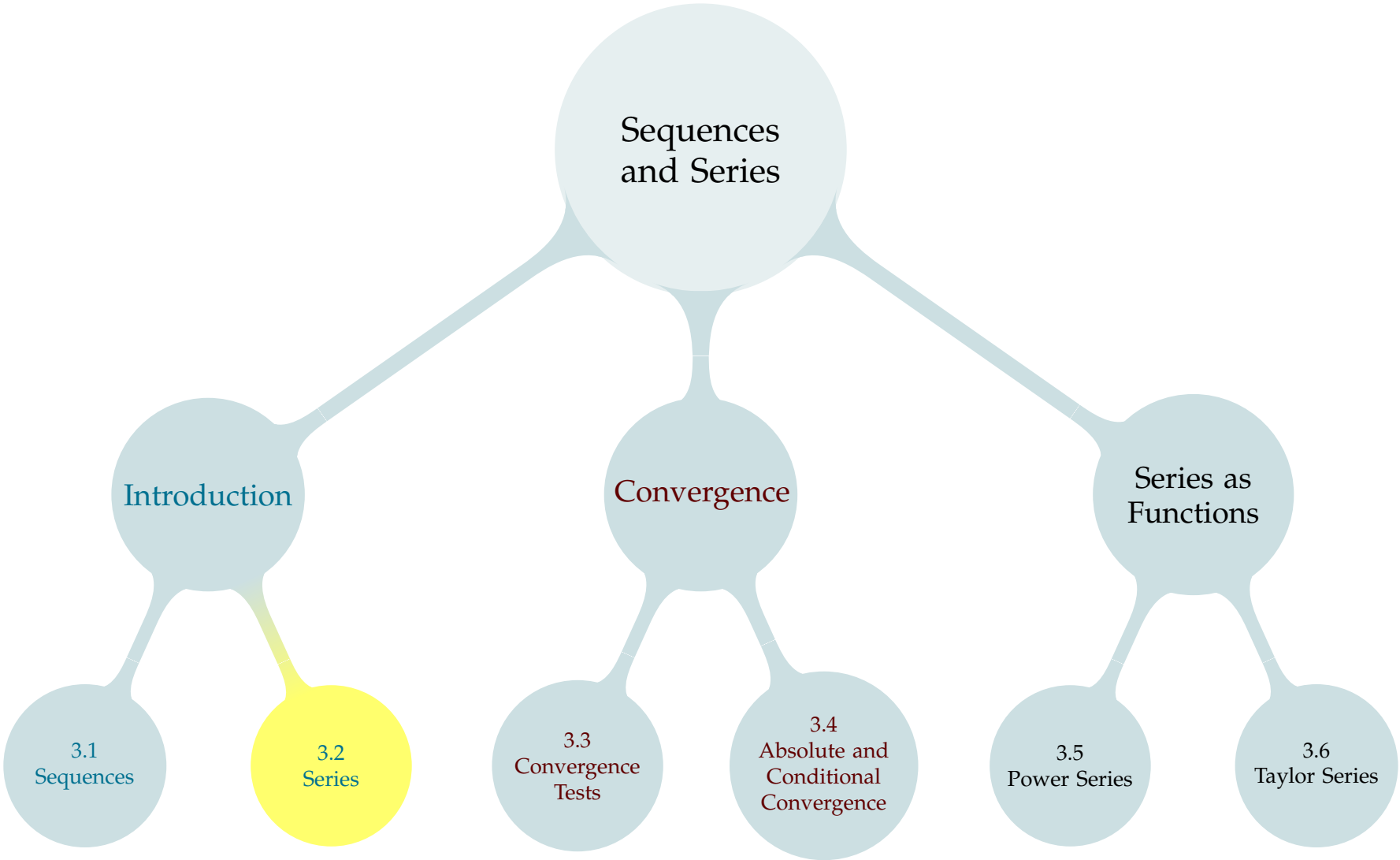
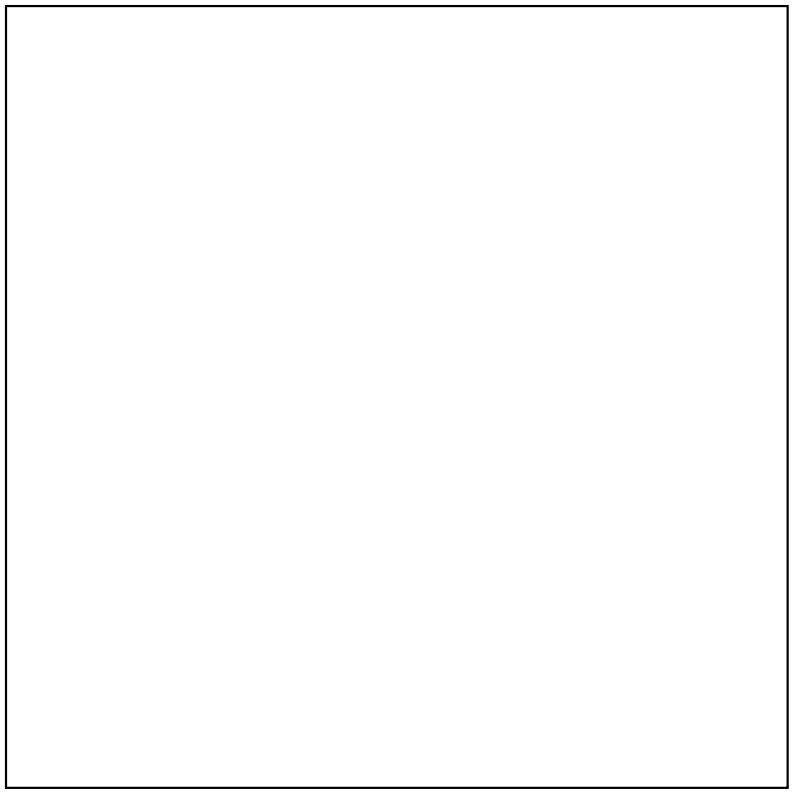


Sequences and Series

A sequence is a list of numbers
A series is the sum of such a list.

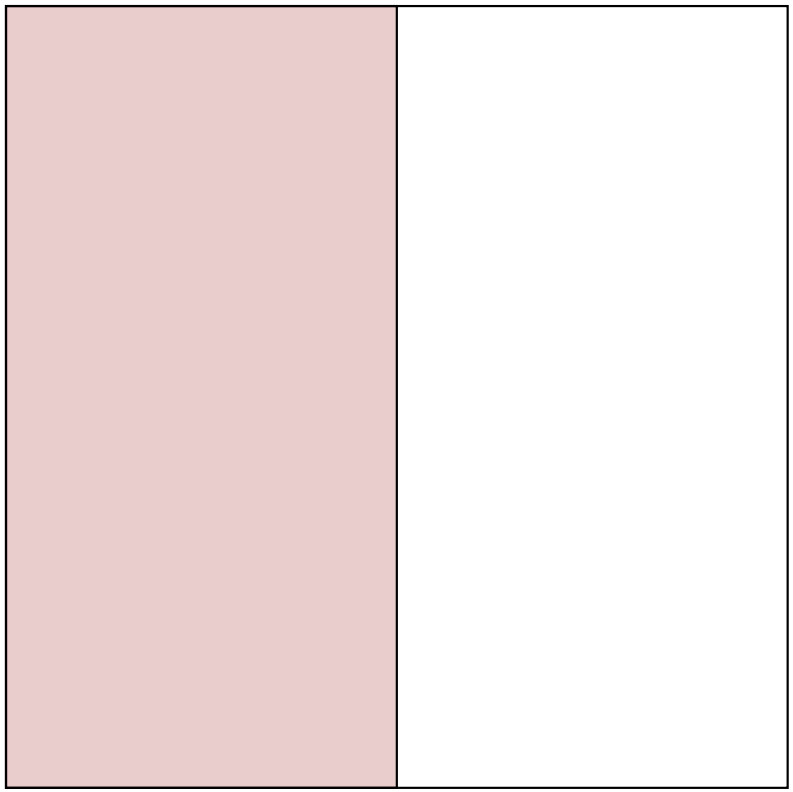
Sequences and Series



Square of Area 1

Sequences and Series

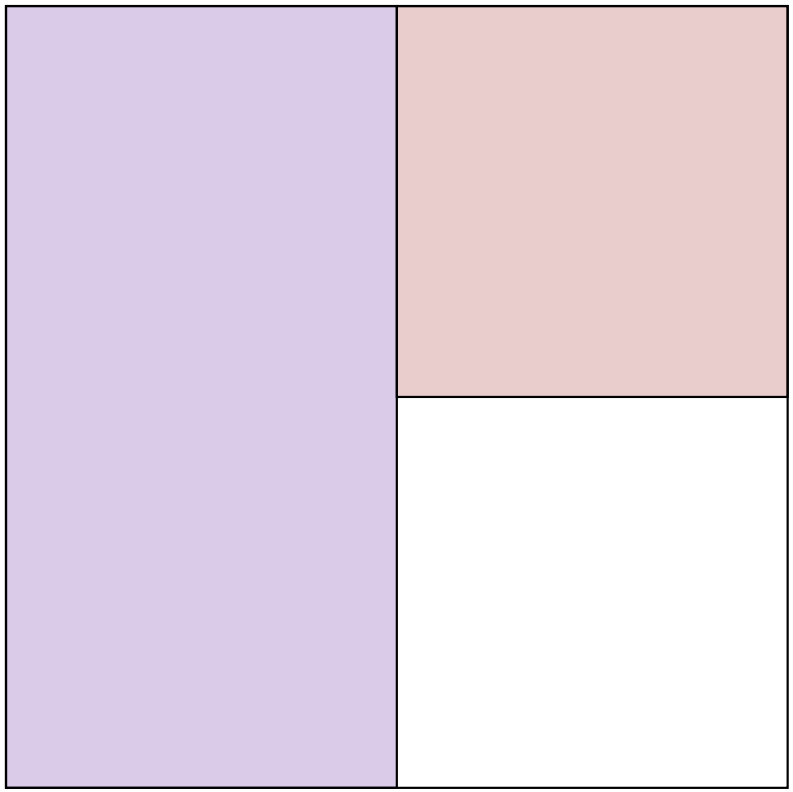
Size of Tiles: $\frac{1}{2}$



Covered Area: $\frac{1}{2}$

Sequences and Series

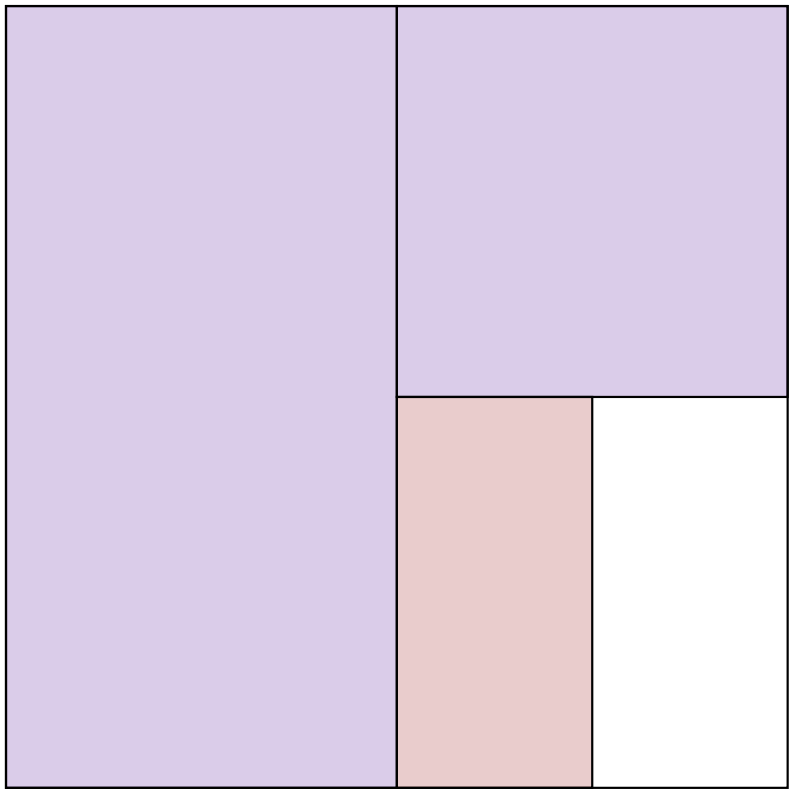
Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}$

Sequences and Series

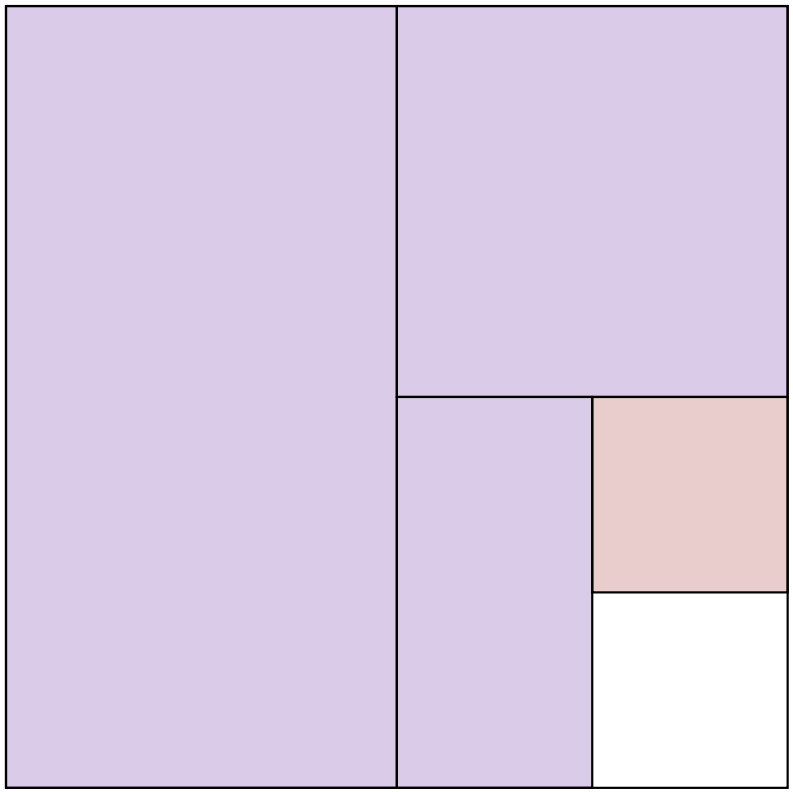
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}$

Sequences and Series

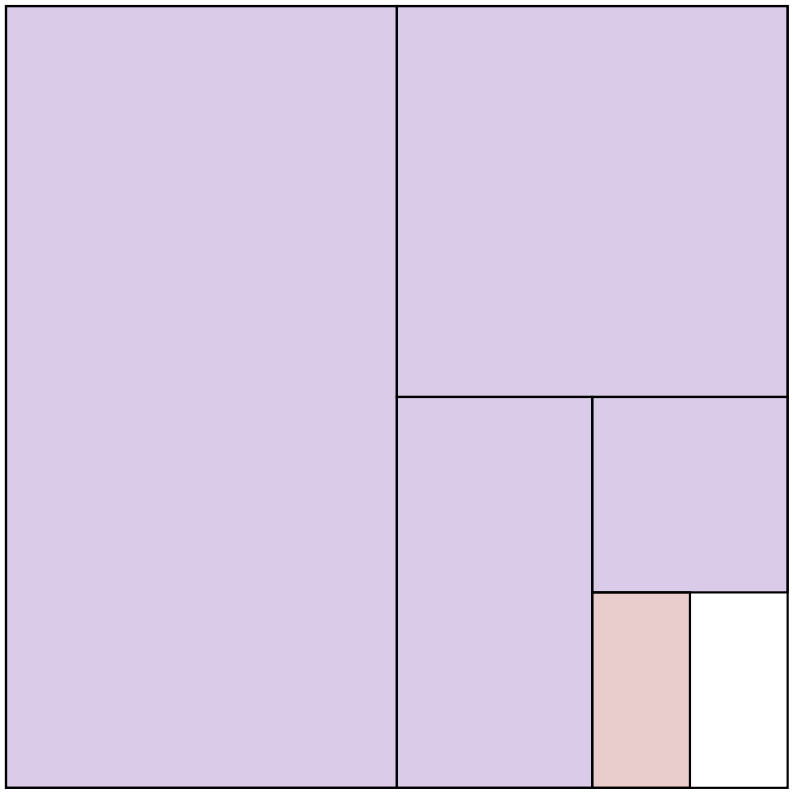
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$

Sequences and Series

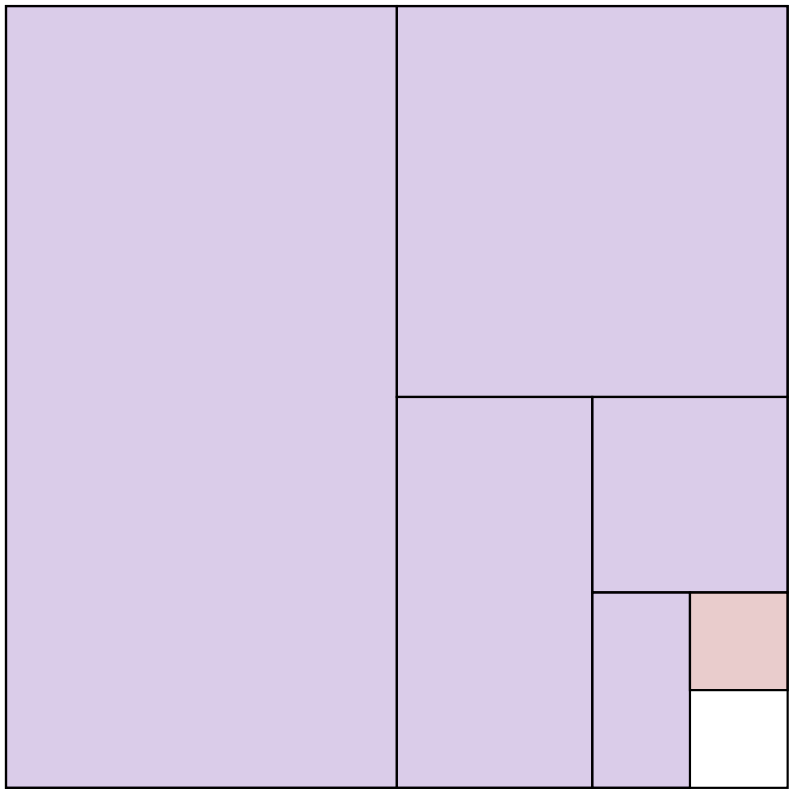
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}$

Sequences and Series

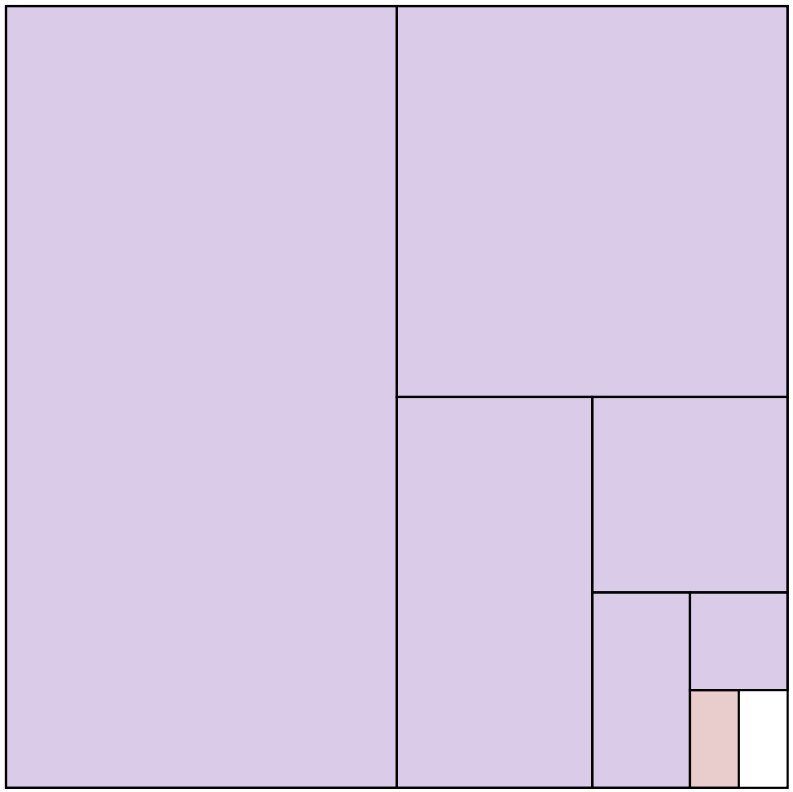
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}$

Sequences and Series

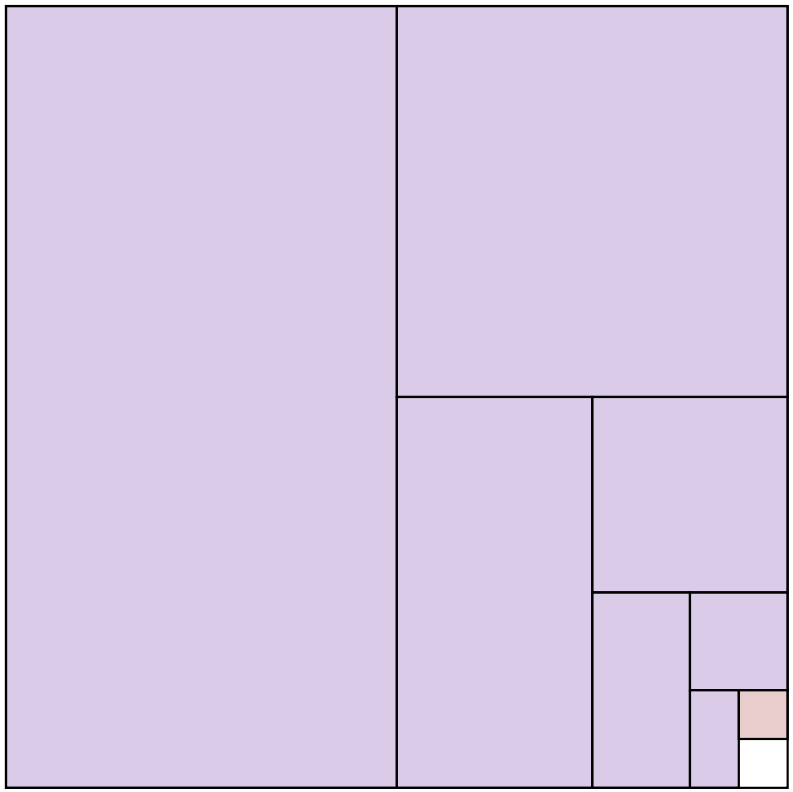
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}$

Sequences and Series

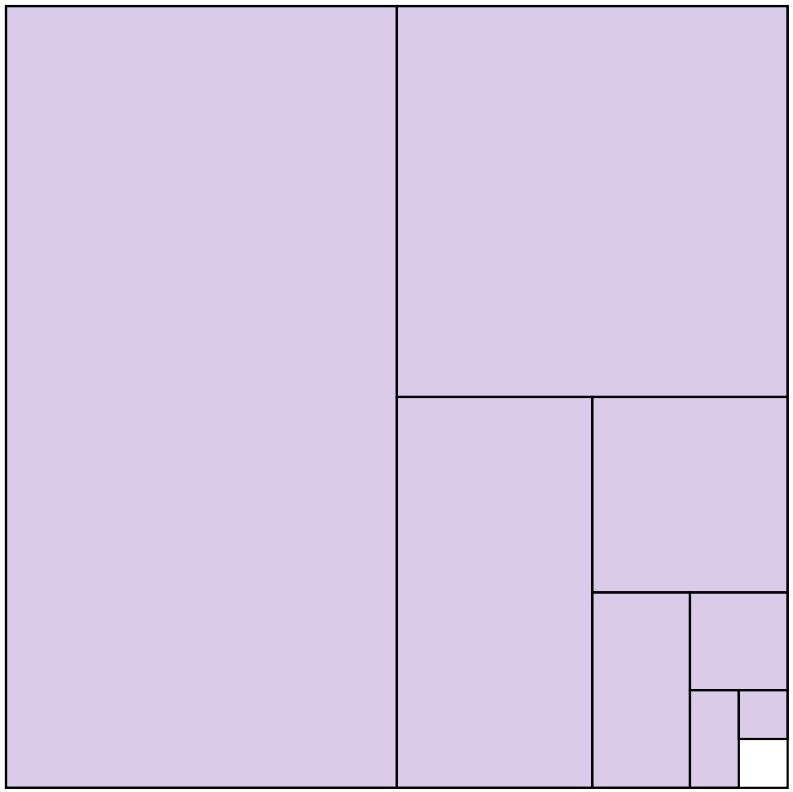
Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}$



Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}$

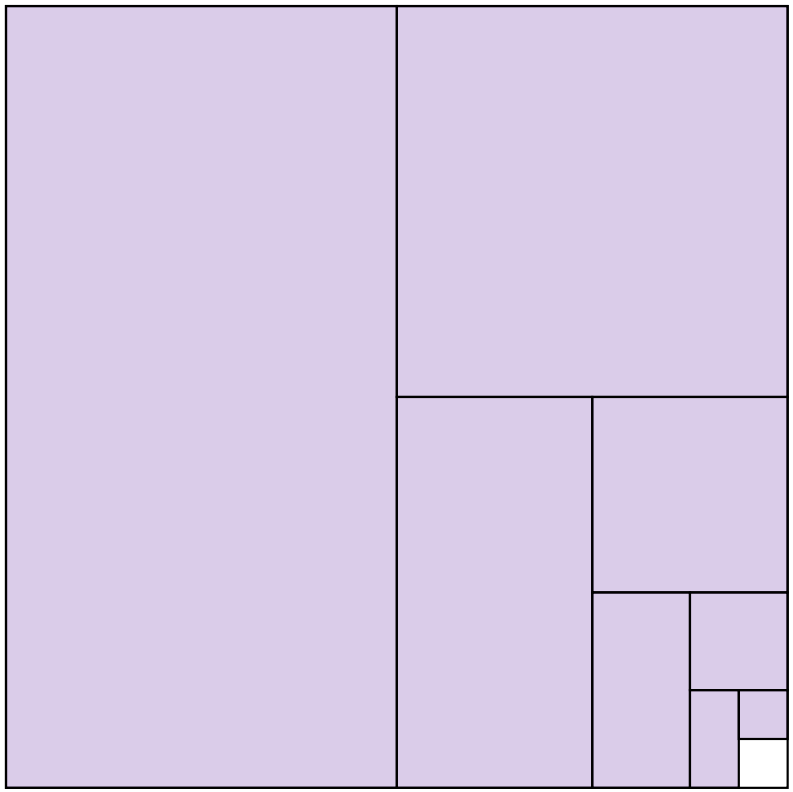


Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \cdots$

Sequence



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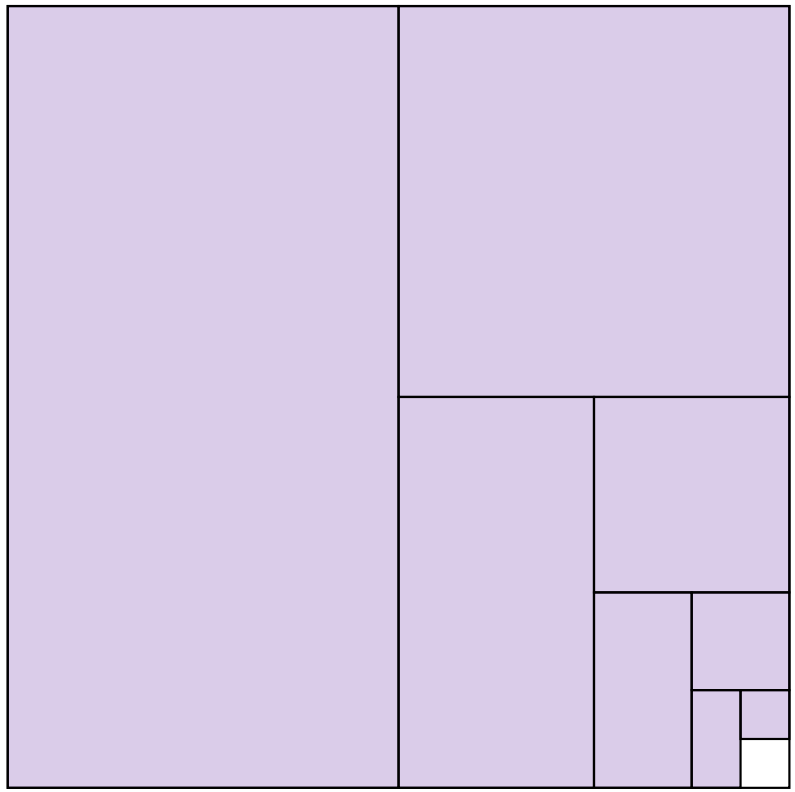
Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \cdots$

Sequence

List of numbers,

approaching



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

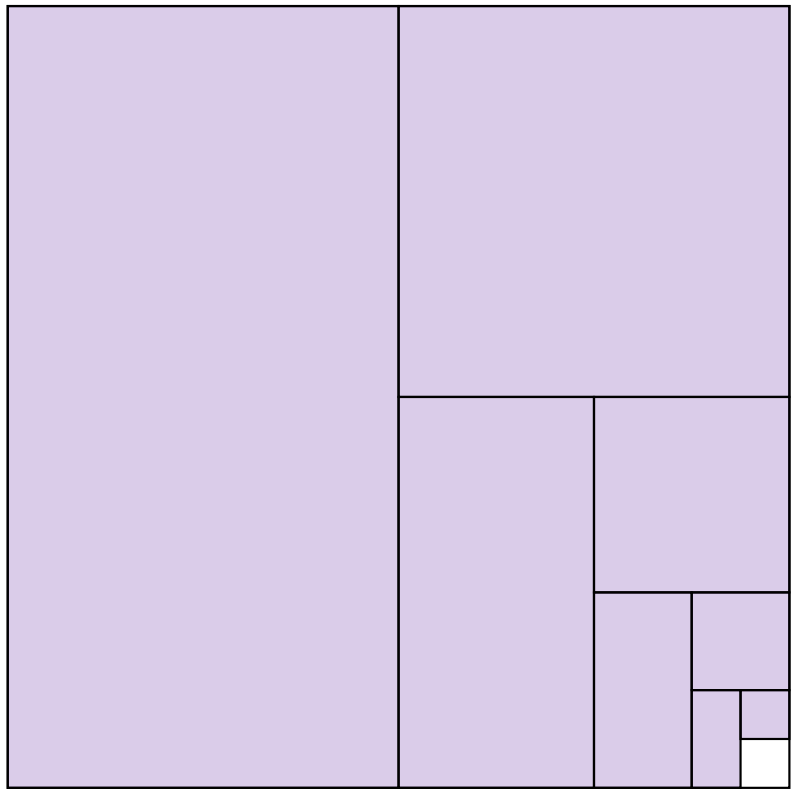
Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \cdots$

Sequence

List of numbers,

approaching **zero**.



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

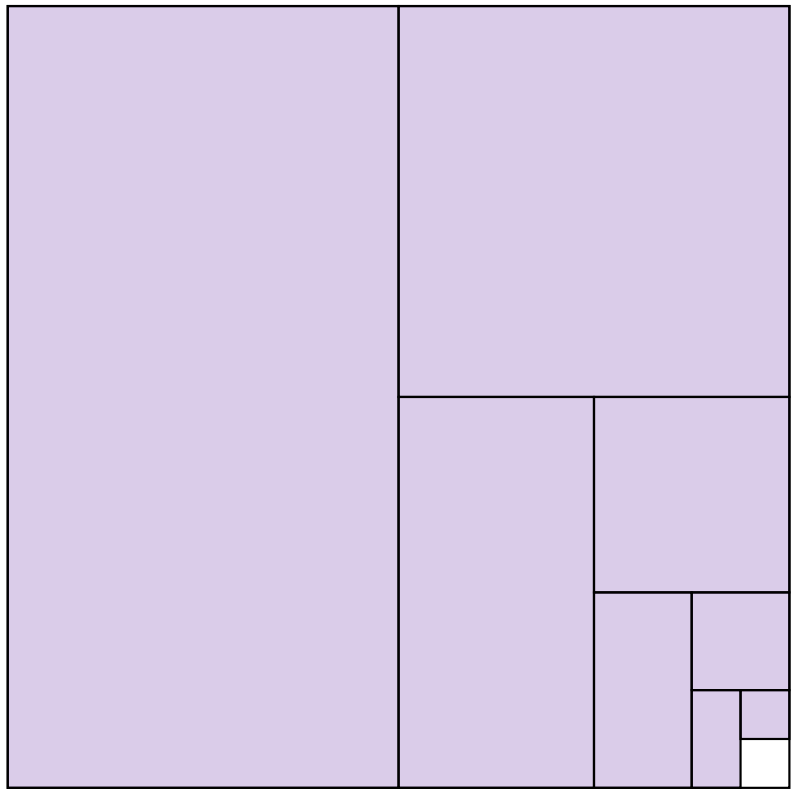
Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \cdots$

Sequence

List of numbers,

approaching **zero**.

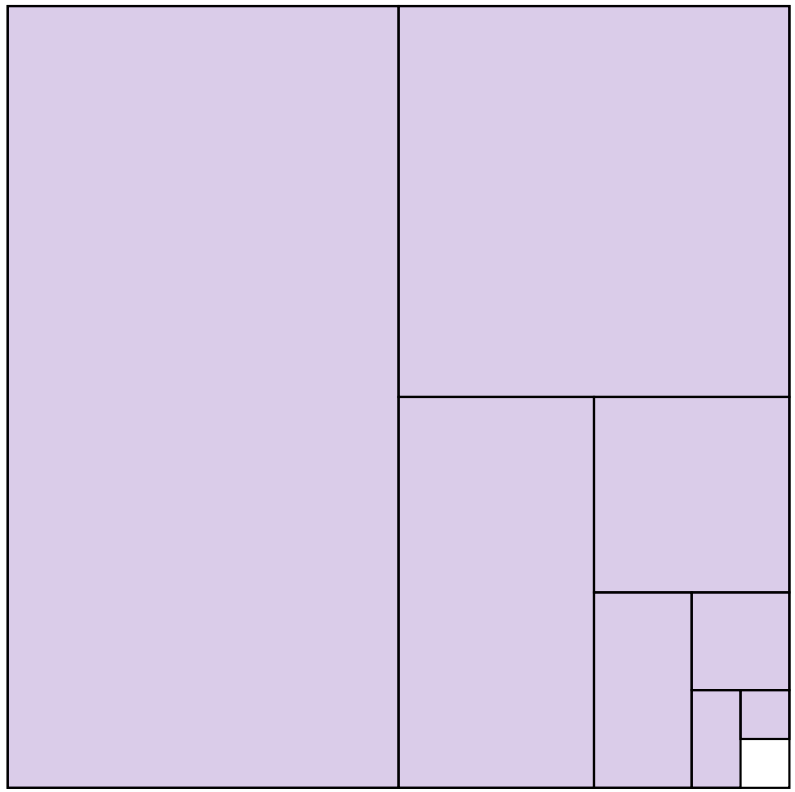


Series

Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}+\cdots$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \cdots$



Sequence

List of numbers,

approaching **zero**.

Series

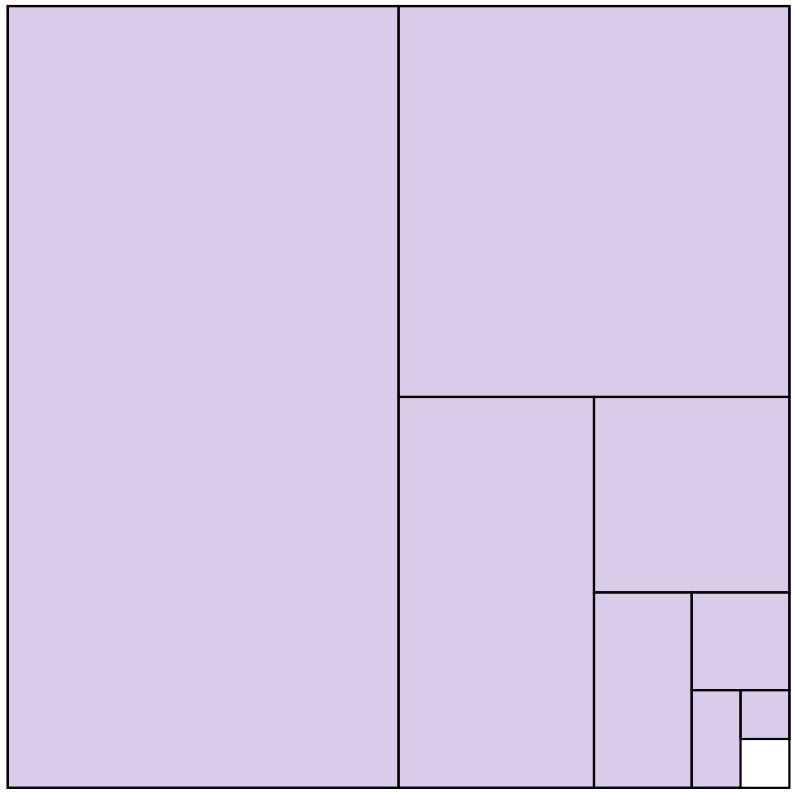
Sum of numbers,

approaching

Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}+\cdots$

Sequences and Series

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \cdots$



Sequence

List of numbers,

approaching **zero**.

Series

Sum of numbers,

approaching **one**.

Covered Area: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}+\cdots$

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =$$

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

(a more rigorous definition will be discussed soon)

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(C \cdot a_n) =$$

A. $\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$

C. $C \sum_{n=1}^{\infty} a_n$

D. $a_n \sum_{n=1}^{\infty} C$

E. none of the above

Q

ans

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(C \cdot a_n) =$$

A. $\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$

C. $C \sum_{n=1}^{\infty} a_n$

D. $a_n \sum_{n=1}^{\infty} C$

E. none of the above

A

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(a_n + b_n) =$$

A. $\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$

B. $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $a_n + \sum_{n=1}^{\infty} b_n$

D. $a_n \sum_{n=1}^{\infty} b_n$

E. none of the above

Q

ans

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(a_n + b_n) =$$

A. $\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$

B. $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $a_n + \sum_{n=1}^{\infty} b_n$

D. $a_n \sum_{n=1}^{\infty} b_n$

E. none of the above

A

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

A. $\displaystyle\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\displaystyle\left(\sum_{n=1}^{\infty} a_n\right)^C$

C. $\displaystyle C^n \sum_{n=1}^{\infty} a_n$

D. $\displaystyle\sum_{n=1}^{\infty} C(a_n)^{C-1}$

E. none of the above

Q

ans

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

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A

Series Philosophy

What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Series Philosophy

What does it really mean to add up infinitely many things?

$$\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\cdots$$

Series Philosophy

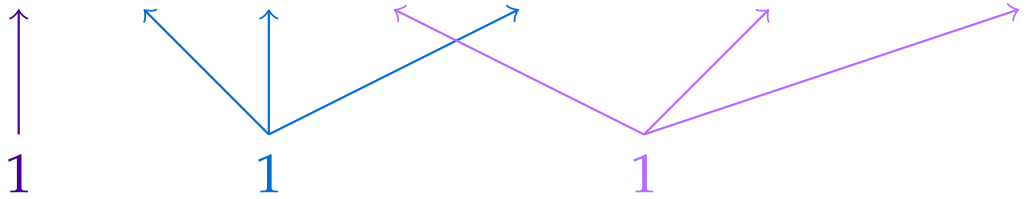
What does it really mean to add up infinitely many things?

$$\underbrace{1}_{1} \underbrace{- 1 + 1}_{0} \underbrace{- 1 + 1}_{0} \underbrace{- 1 + 1}_{0} \underbrace{- 1 + 1}_{0} \underbrace{- 1 + 1}_{0} \underbrace{- 1 + \cdots}_{0}$$

Series Philosophy

What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$



Series Philosophy

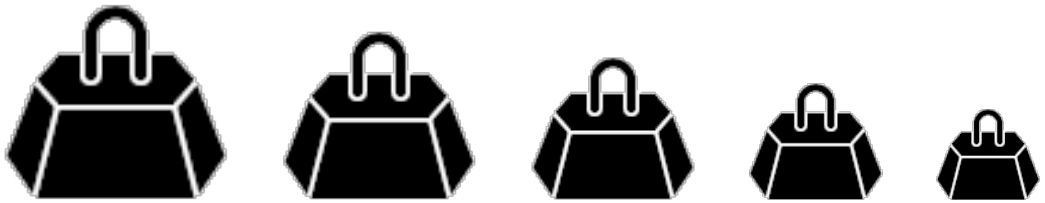
What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

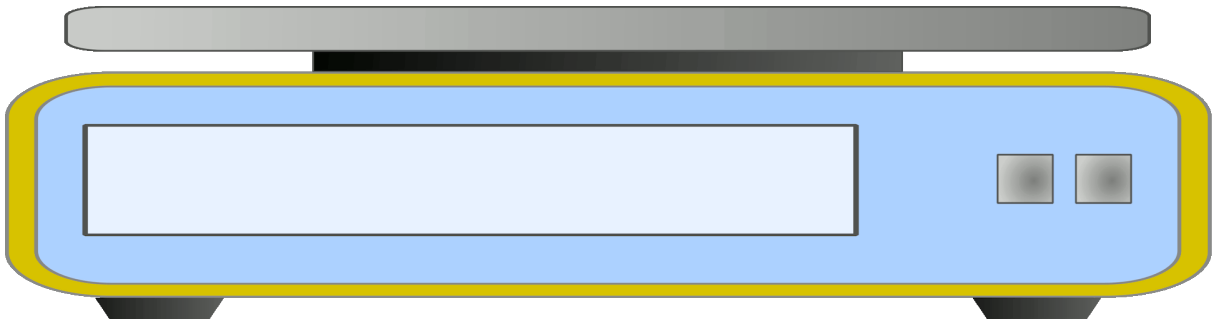
We need an unambiguous definition.

How can we add up infinitely many things?

Sequence of Partial Sums



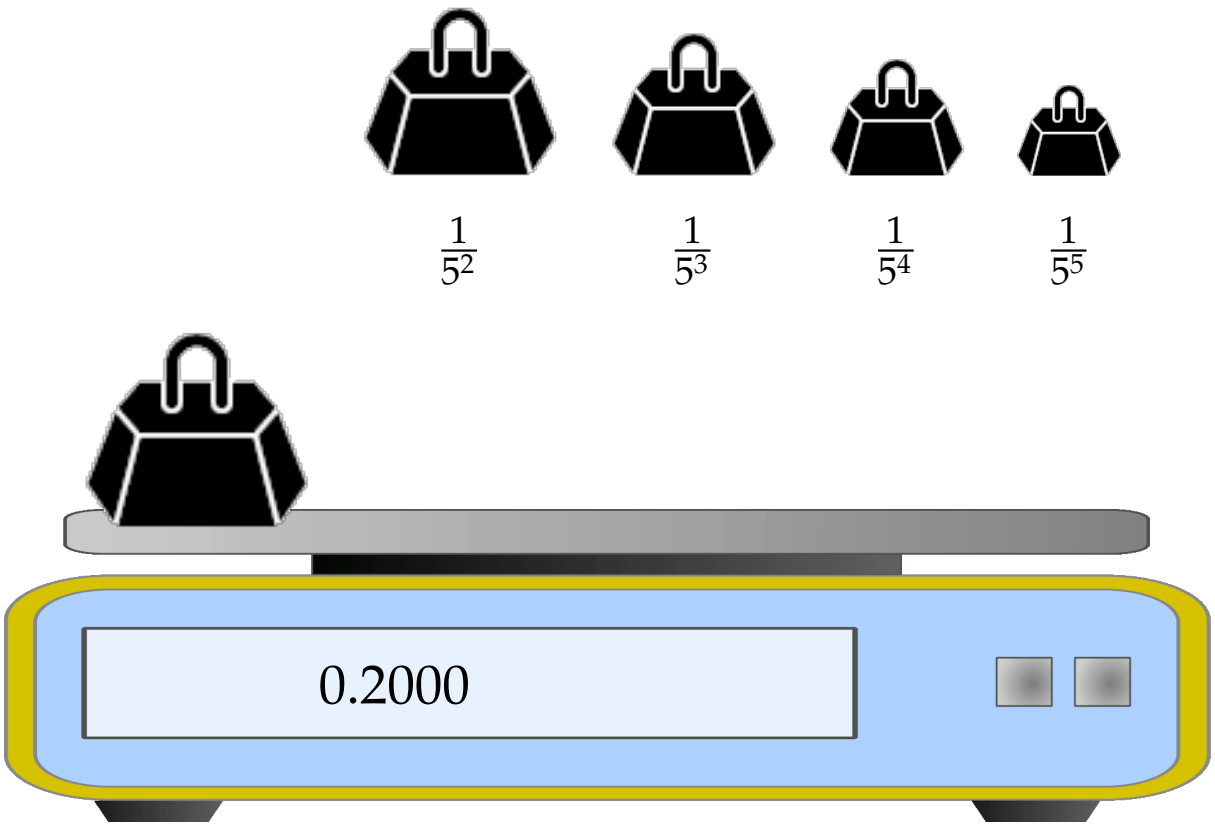
$\frac{1}{5^1}$ $\frac{1}{5^2}$ $\frac{1}{5^3}$ $\frac{1}{5^4}$ $\frac{1}{5^5}$



How can we add up infinitely many things?

Sequence of Partial Sums

$$S_1 = 0.2000$$

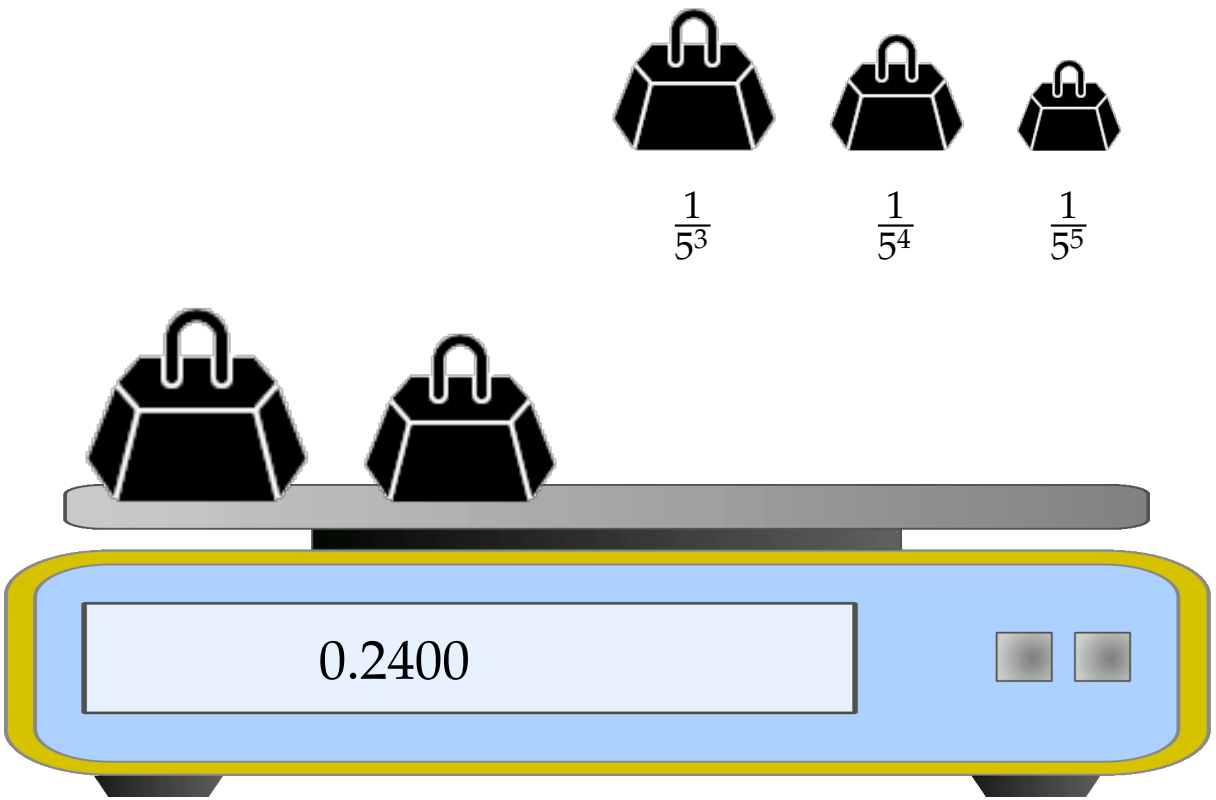


How can we add up infinitely many things?

Sequence of Partial Sums

$S_1 = 0.2000$

$S_2 = 0.2400$



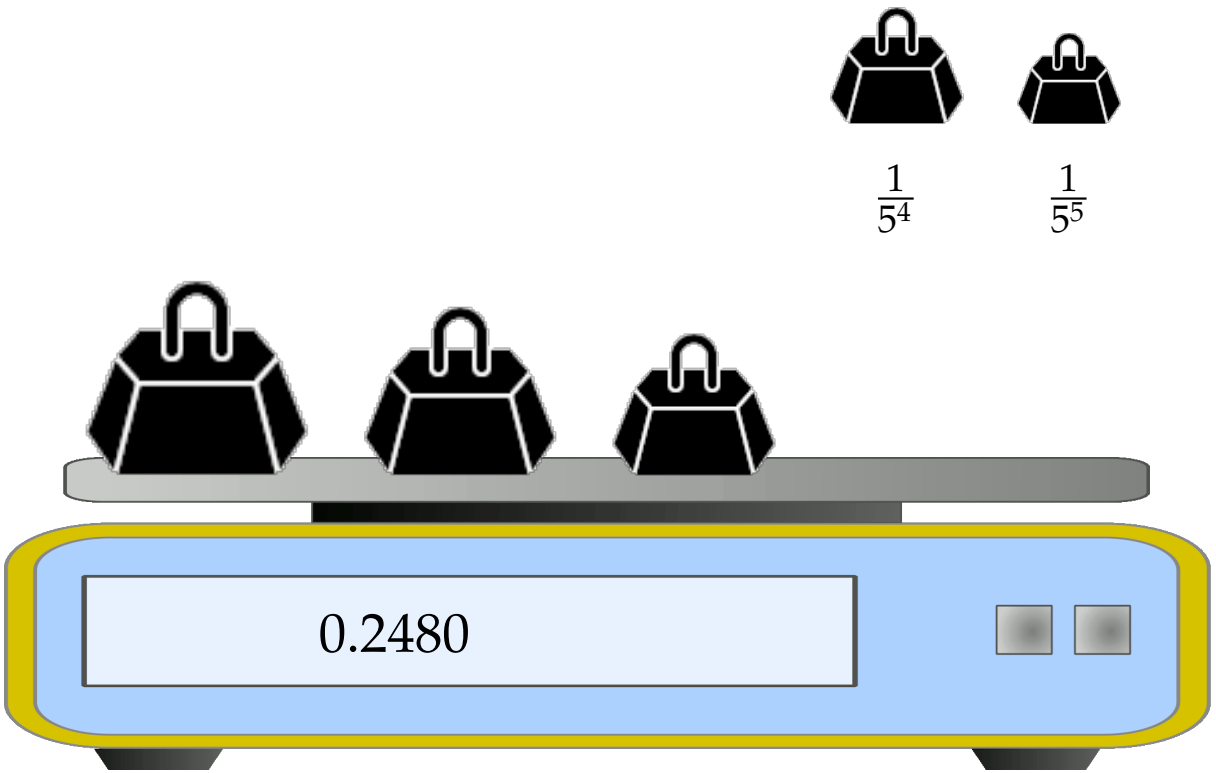
How can we add up infinitely many things?

Sequence of Partial Sums

$$S_1 = 0.2000$$

$$S_2 = 0.2400$$

$$S_3 = 0.2480$$



How can we add up infinitely many things?

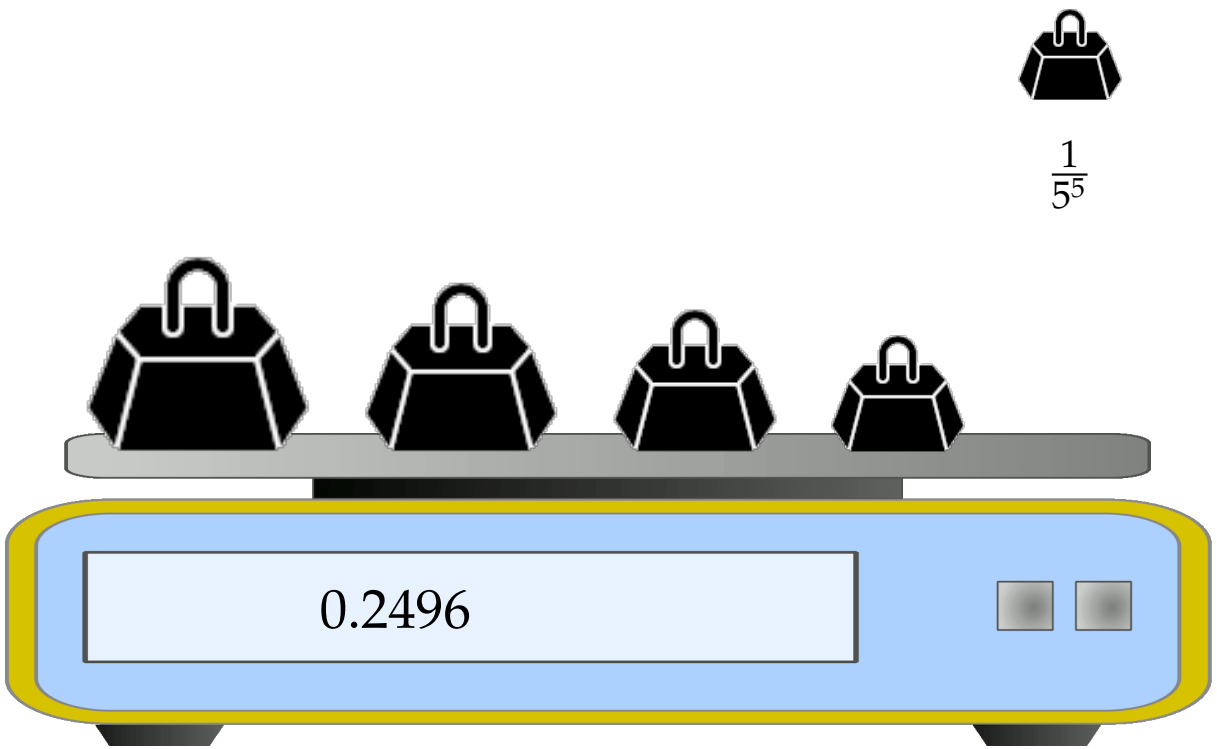
Sequence of Partial Sums

$$S_1 = 0.2000$$

$$S_2 = 0.2400$$

$$S_3 = 0.2480$$

$$S_4 = 0.2496$$



How can we add up infinitely many things?

Sequence of Partial Sums

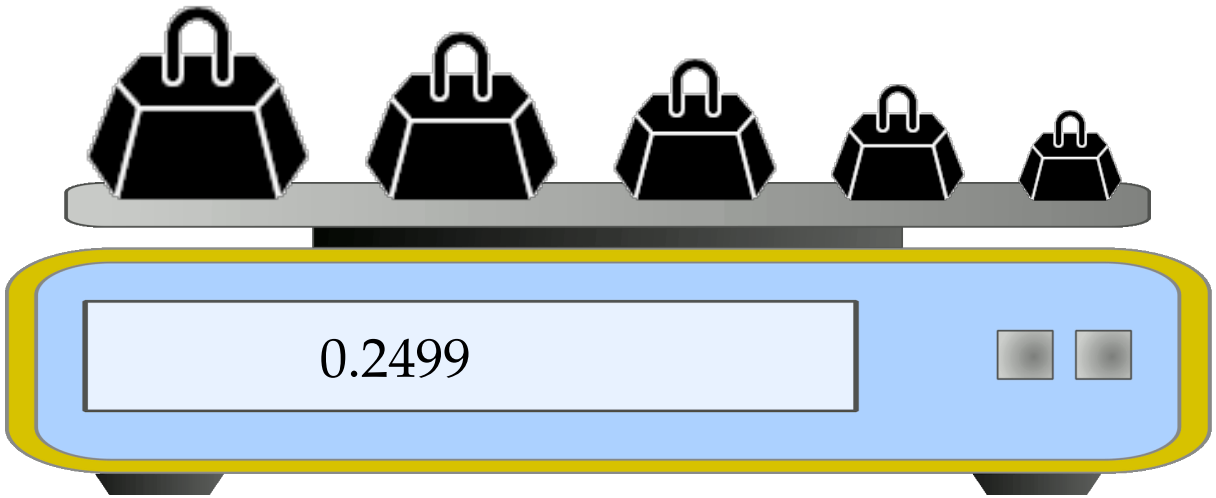
$$S_1 = 0.2000$$

$$S_2 = 0.2400$$

$$S_3 = 0.2480$$

$$S_4 = 0.2496$$

$$S_5 = 0.2499$$



Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2$$
$$a_2 = \frac{1}{5^2} = 0.04$$
$$a_3 = \frac{1}{5^3} = 0.008$$
$$a_4 = \frac{1}{5^4} = 0.0016$$
$$a_5 = \frac{1}{5^5} = 0.00032$$

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$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008$$

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Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

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We define $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$.

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From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} \qquad = \qquad \lim_{N\to\infty} S_N \qquad =$$

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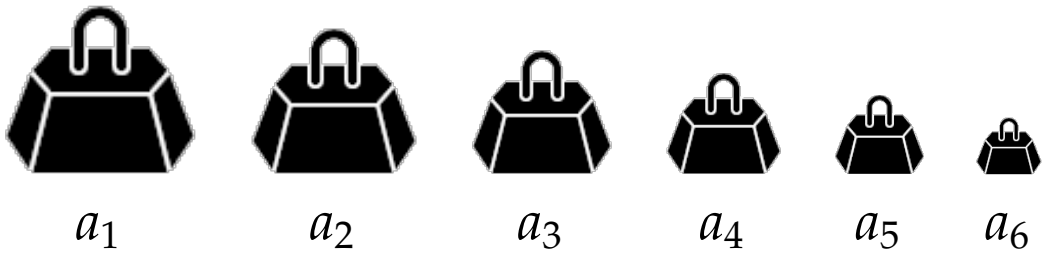
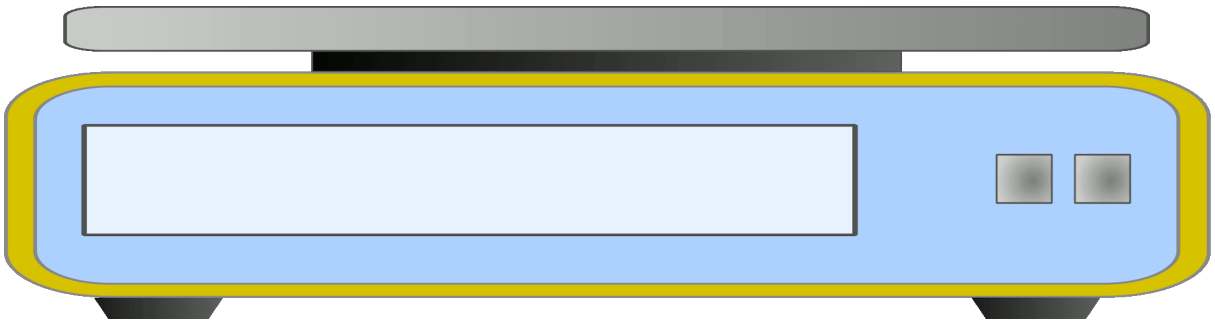
From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \lim_{N\to\infty} S_N = \frac{1}{4}$$

Notation: $S_N = \sum_{n=1}^{N} a_n$





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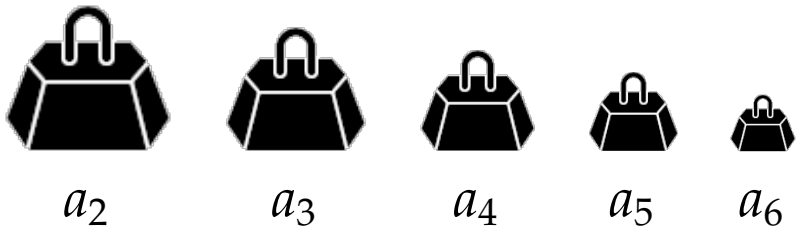
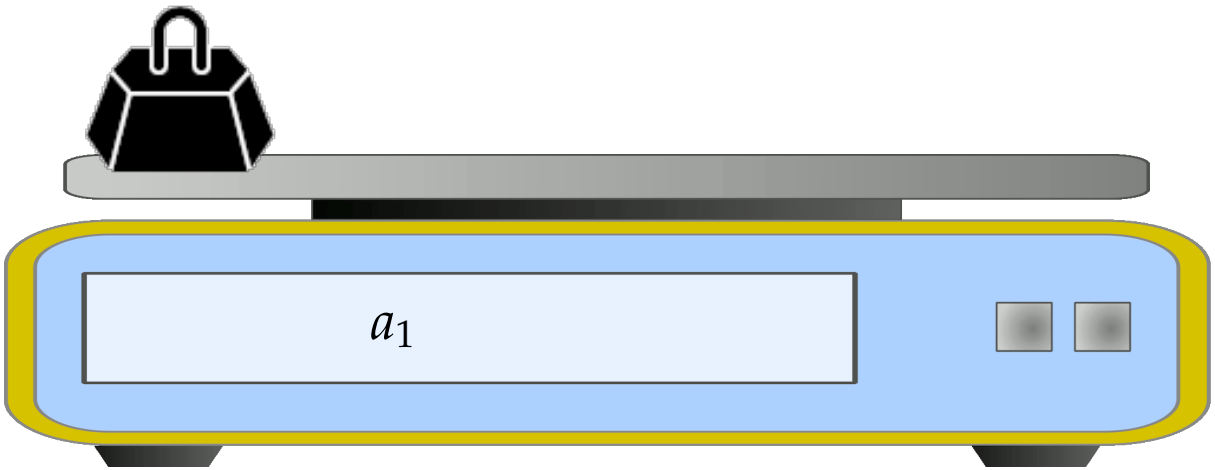
$S_1 = a_1$





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$$S_1 = a_1$$

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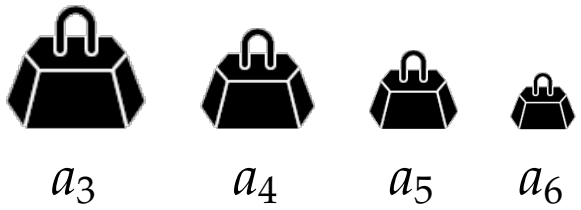

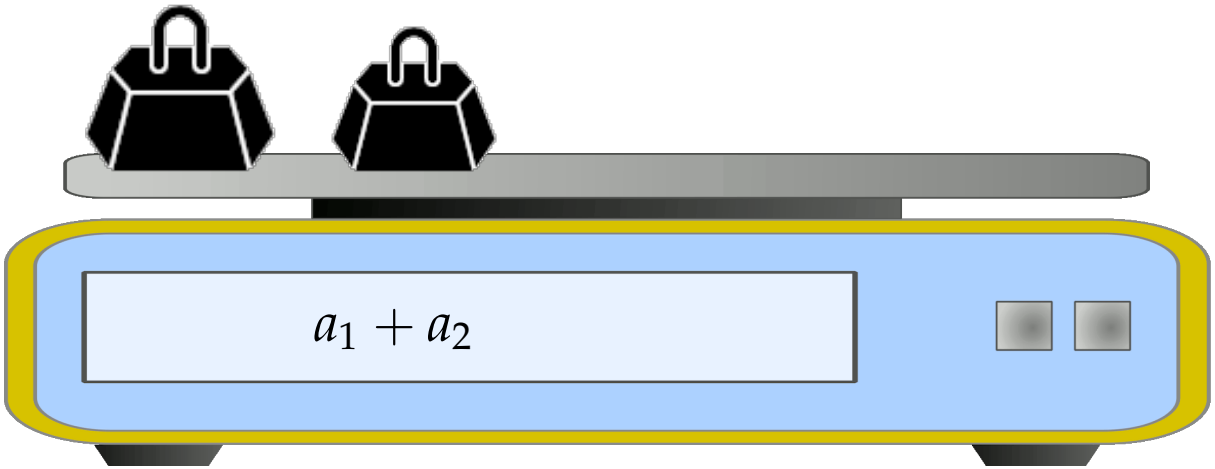

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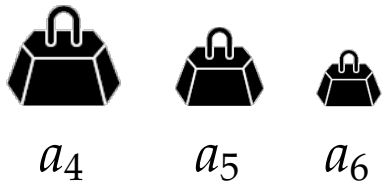

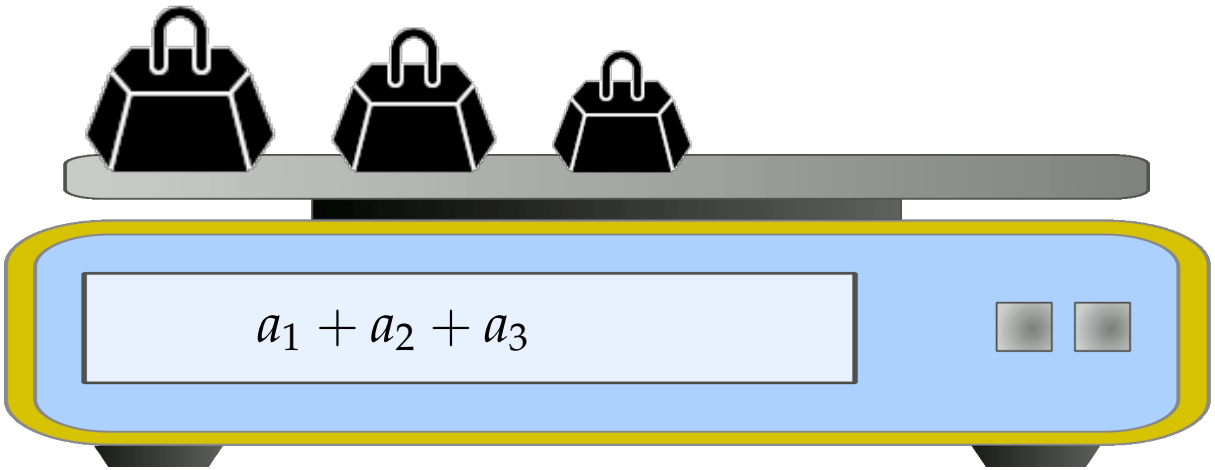

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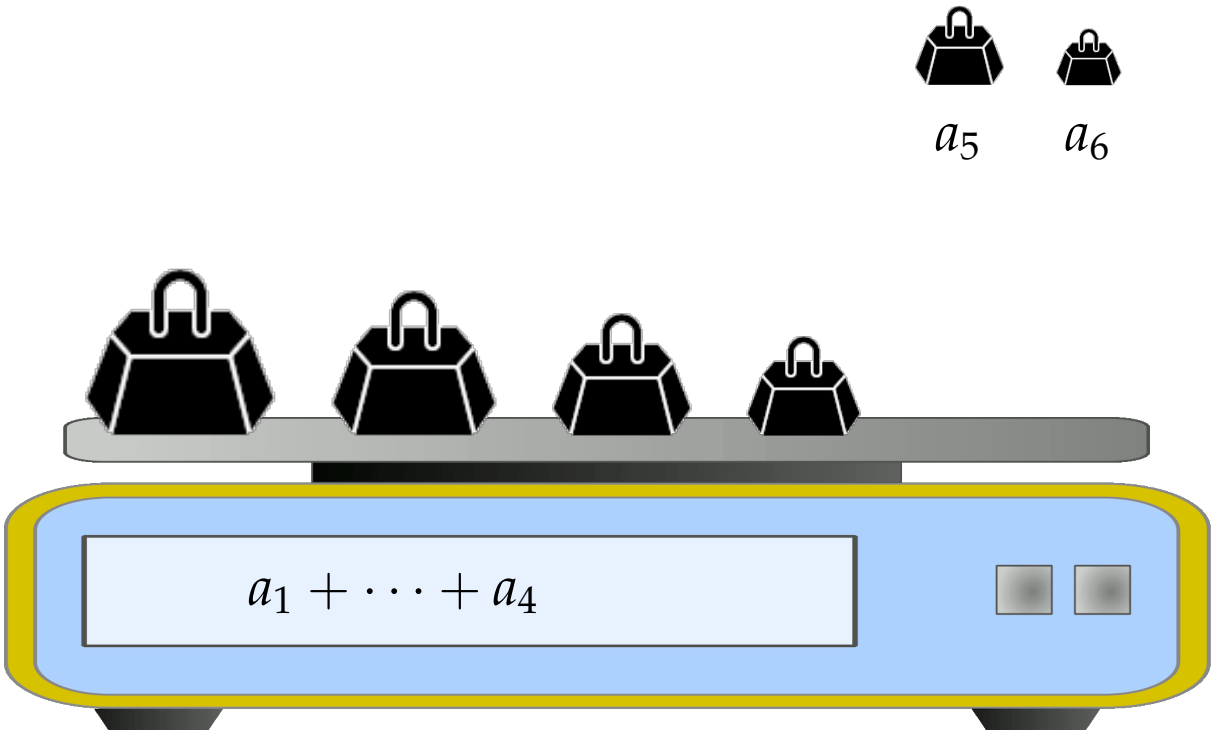

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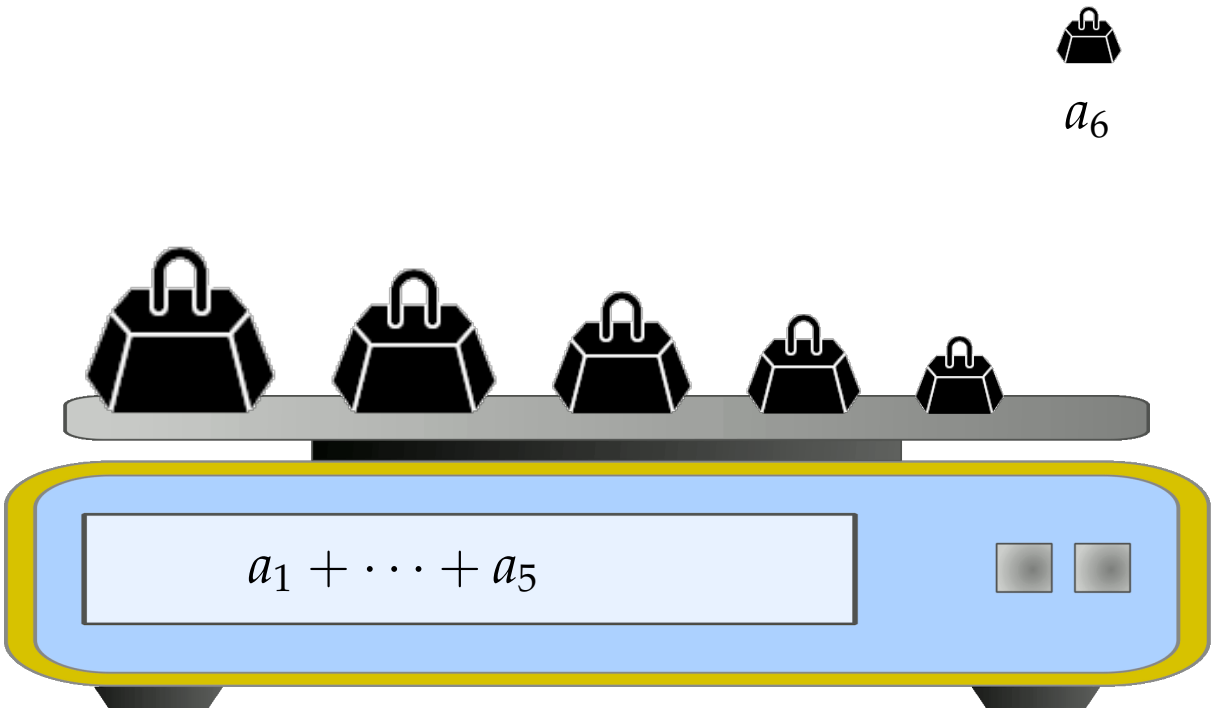

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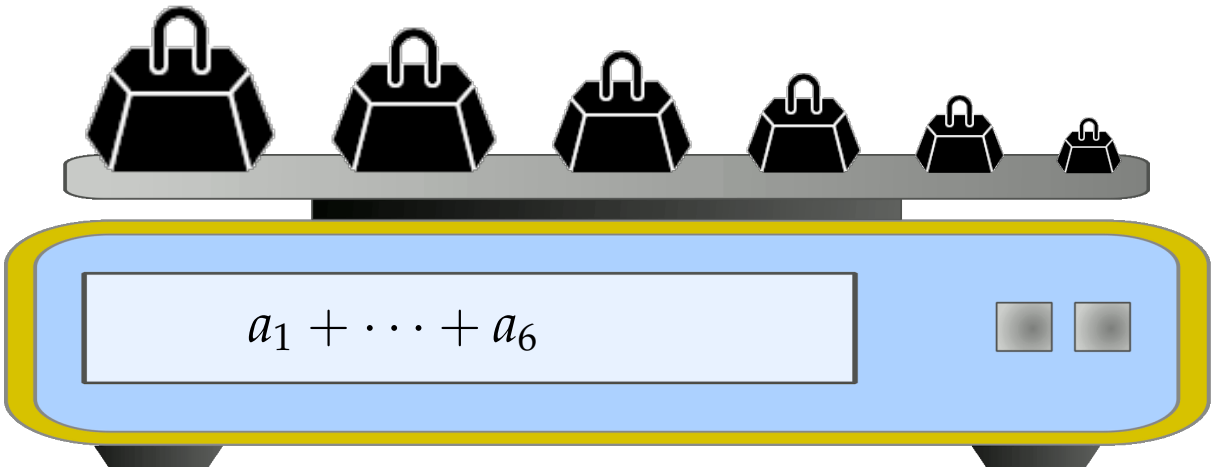

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Notation Practice

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Evaluate $\sum_{n=1}^{100} a_n$.

- Evaluate $\sum_{n=1}^{\infty} a_n$.

Q

ans

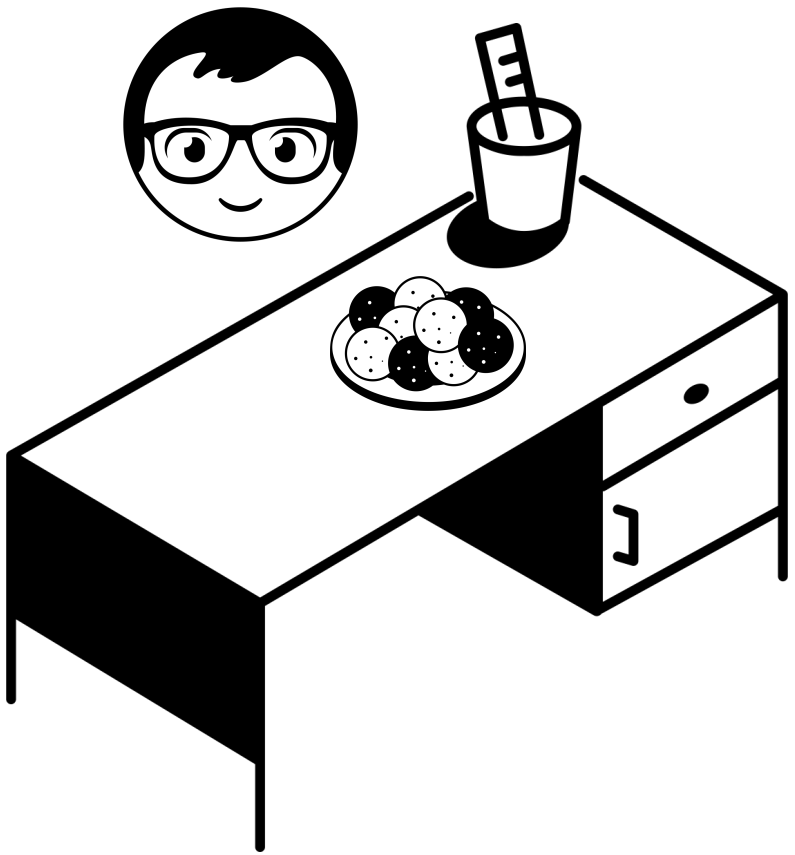
Notation Practice

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Evaluate $\sum_{n=1}^{100} a_n$. $\qquad \sum_{n=1}^{100} a_n = S_{100} = \dfrac{100}{101}$

- Evaluate $\sum_{n=1}^{\infty} a_n$. $\qquad \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \dfrac{N}{N+1} = 1$

Notation Practice



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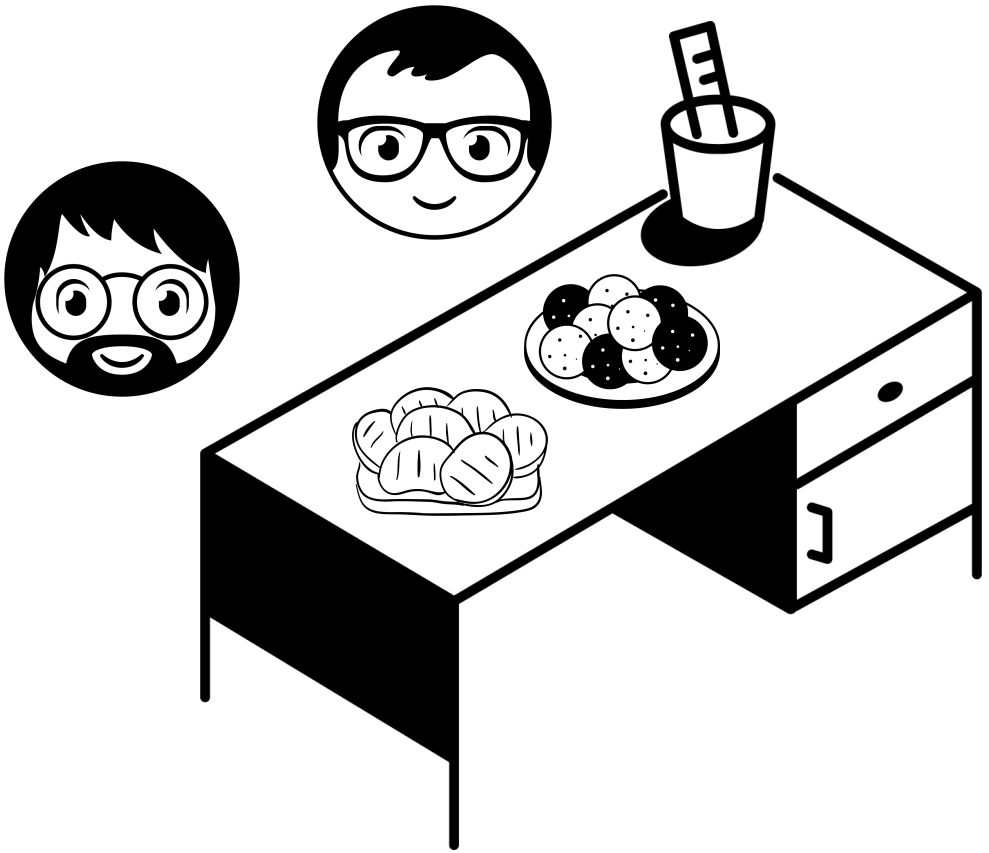
Notation Practice



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Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

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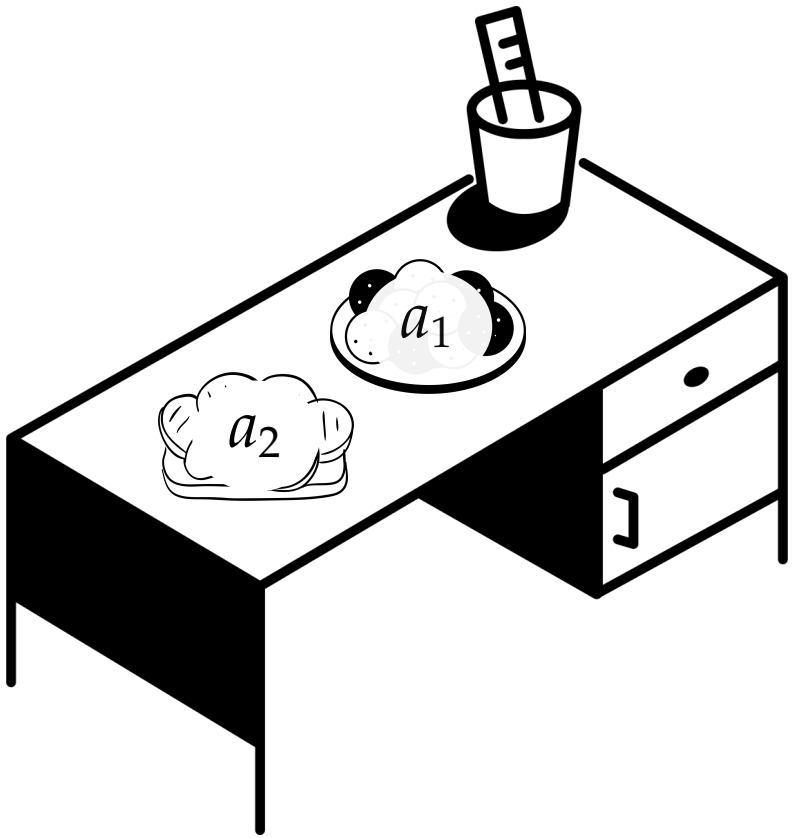
Notation Practice



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Andrew brought 10, and Joel brought $19 - 10 = 9$.

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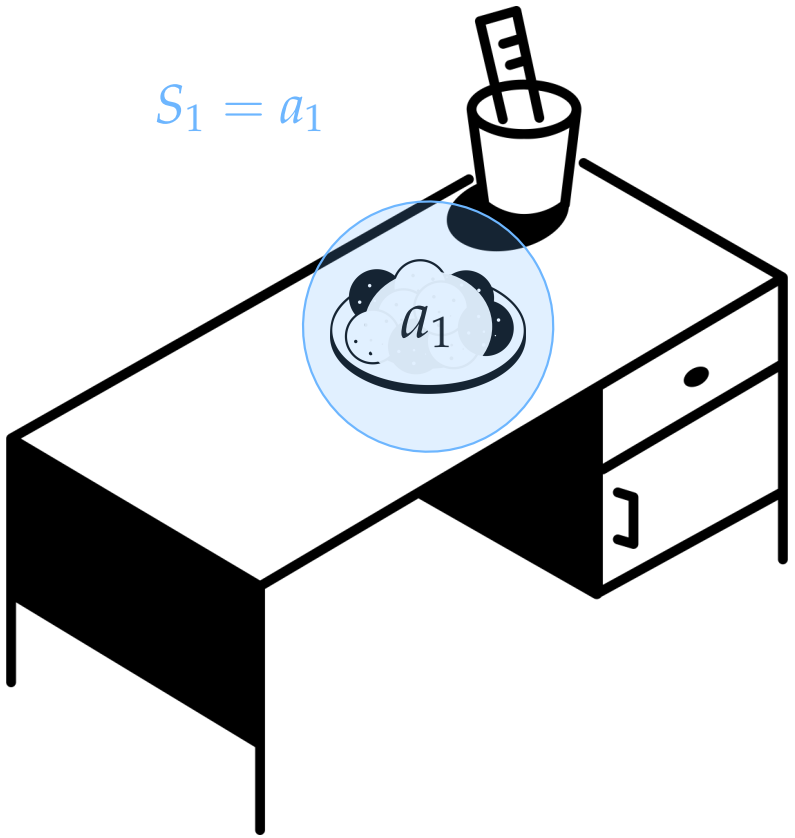

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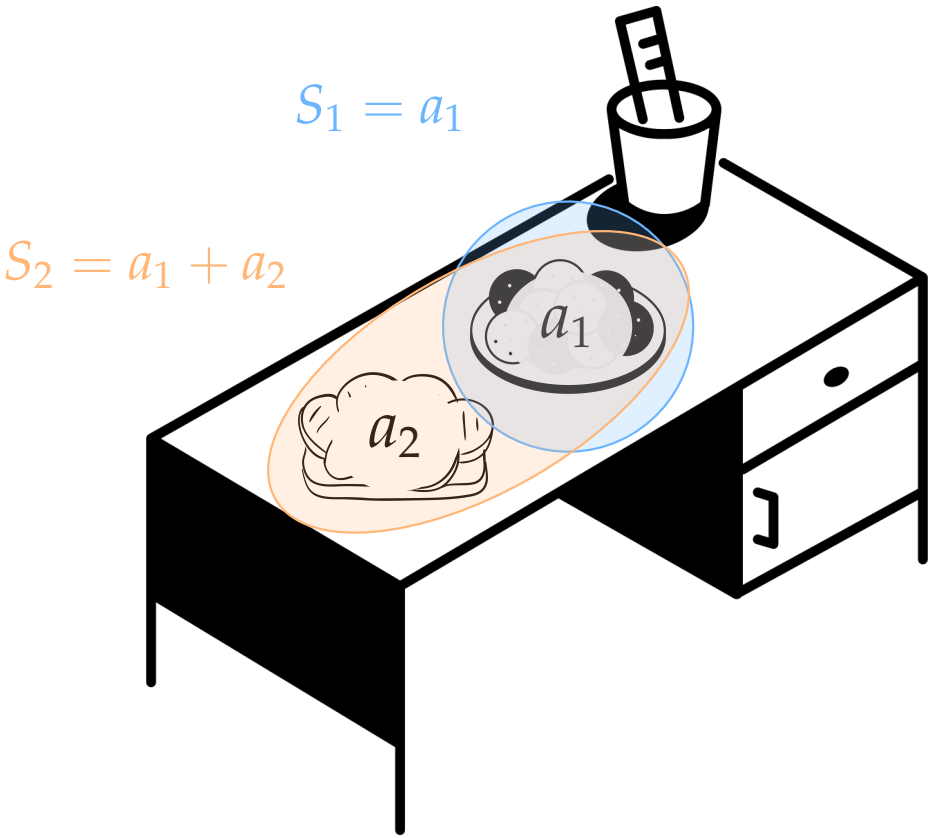

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Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Find a_1.

- Give an explicit expression for a_n, when $n > 1$.

Q

ans

Notation Practice

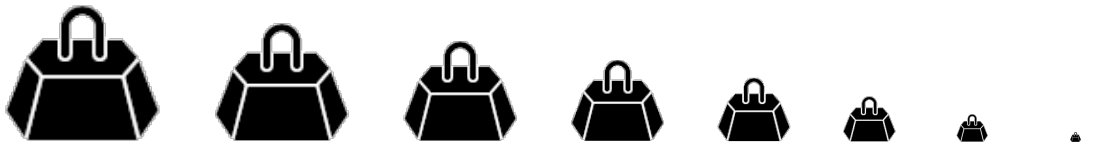
Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Find a_1. $\quad a_1 = \sum_{n=1}^{1} a_n = S_1 = \dfrac{1}{2}$

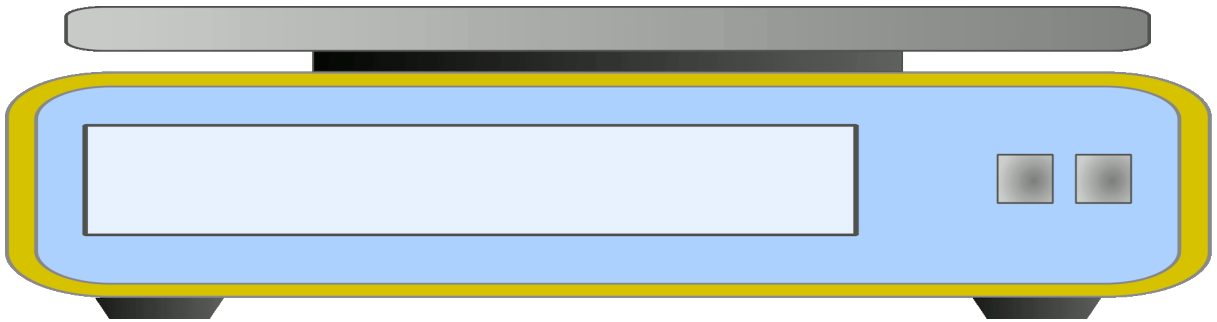
- Give an explicit expression for a_n, when $n > 1$.

$$a_n = \left(\sum_{k=1}^{n} a_k\right) - \left(\sum_{k=1}^{n-1} a_k\right) = S_n - S_{n-1}$$
$$= \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$$

$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$

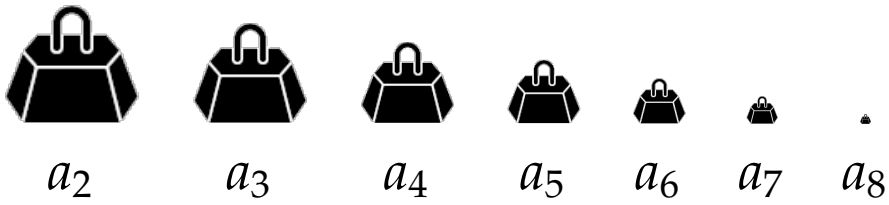


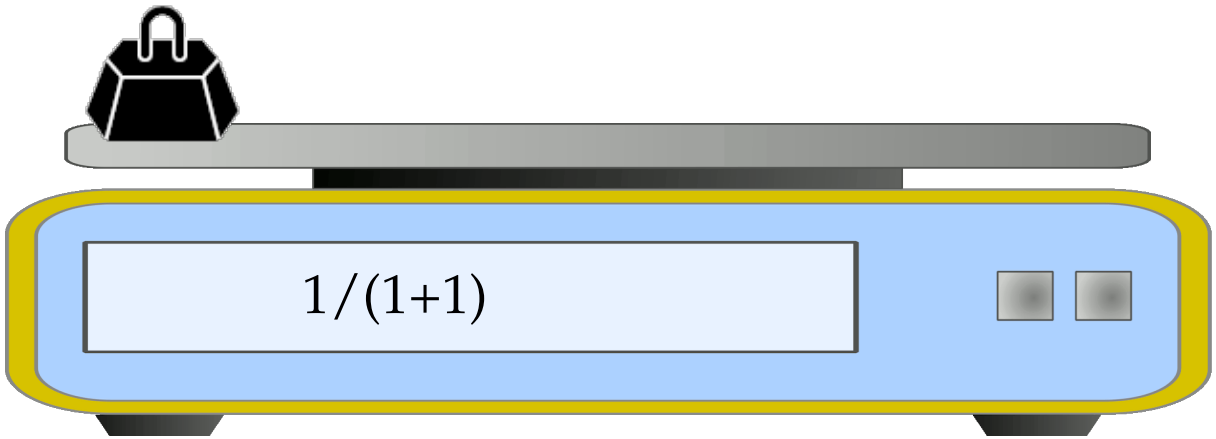
a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8



$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$

$S_1 = 1/(1+1)$





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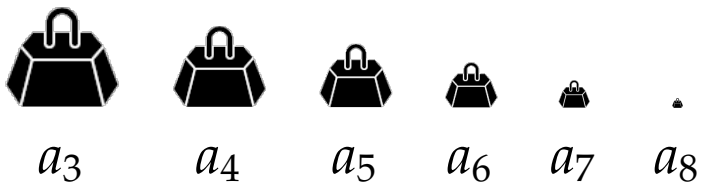

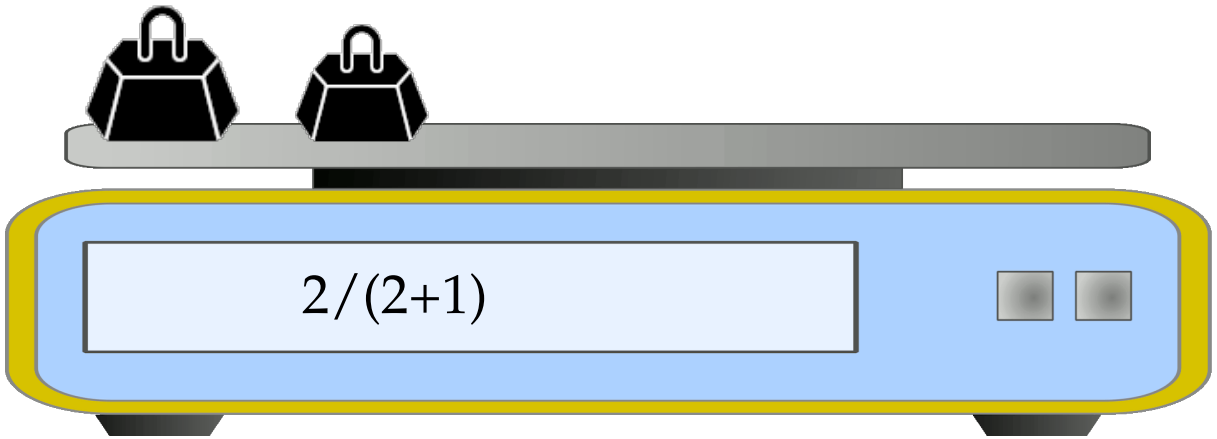

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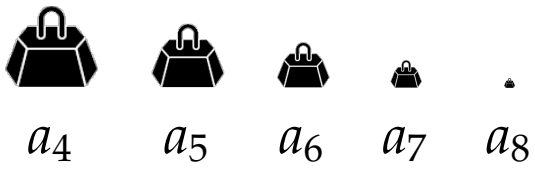

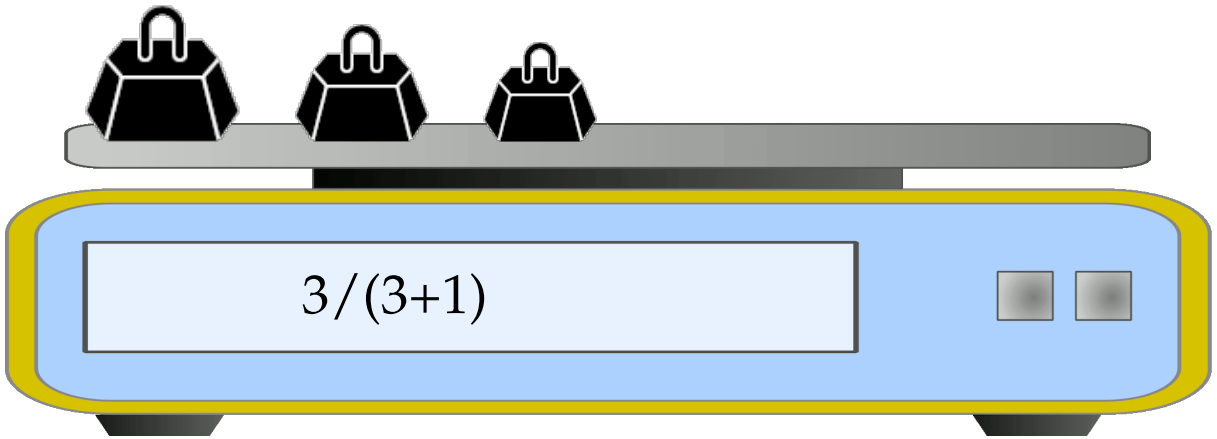

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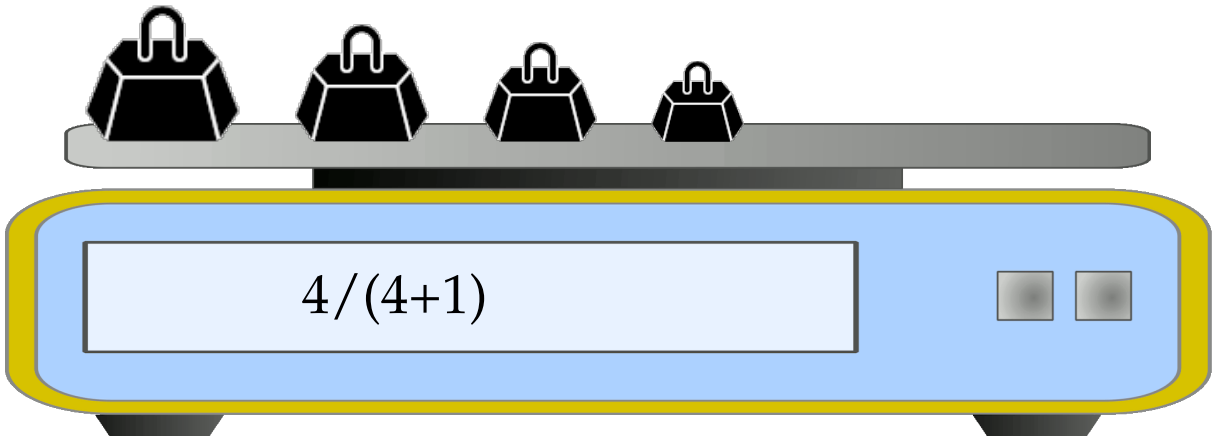

Definition

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^{N} a_n.$$

The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \to S$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Geometric Series

Let a and r be two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + ar^3 + \cdots$$

is called the **geometric series** with first term a and ratio r.

We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

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Another useful way of identifying geometric series is to determine whether all pairs of consecutive terms have the same ratio.

- Geometric: $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$
- Geometric: $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- Not geometric: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

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$$\begin{aligned} S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\ rS_N &= \quad ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1} \\ rS_N - S_N &= \end{aligned}$$

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If $r \neq 1$, then

$$S_N = \frac{ar^{N+1} - a}{r-1} = a\frac{r^{N+1} - 1}{r-1}$$

Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

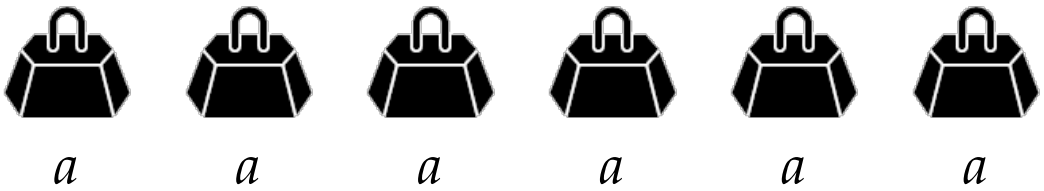
- If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a\dfrac{r^{N+1} - 1}{r - 1}$.
- If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N =$
- If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n =$
- If $r = 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$
- If $r = -1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$
- If $|r| > 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$

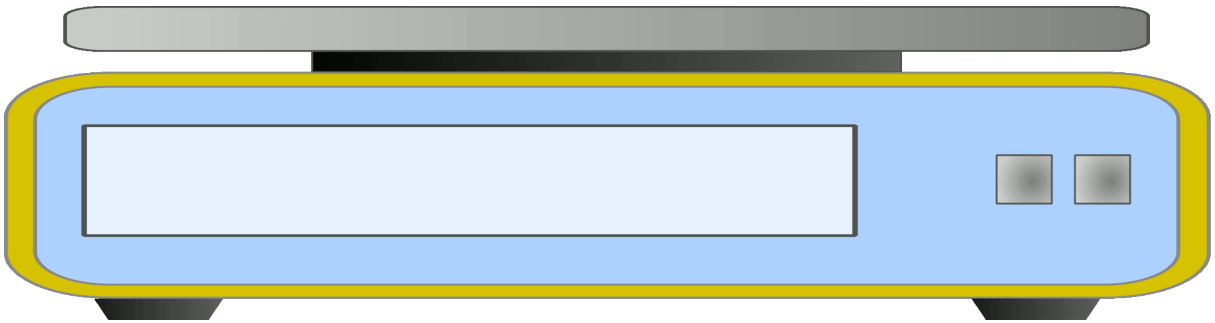
Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

- If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a\dfrac{r^{N+1} - 1}{r - 1}$.
- If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = (N + 1)a$.
- If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} a\frac{r^{N+1} - 1}{r - 1} = a\frac{1}{1 - r}$
- If $r = 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges
- If $r = -1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges
- If $|r| > 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges

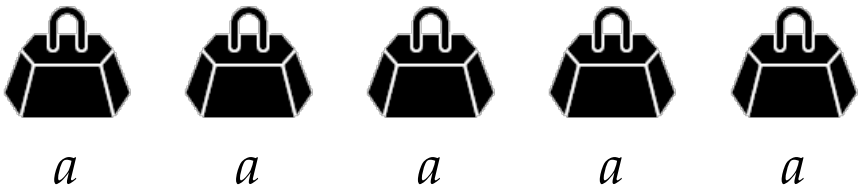
$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

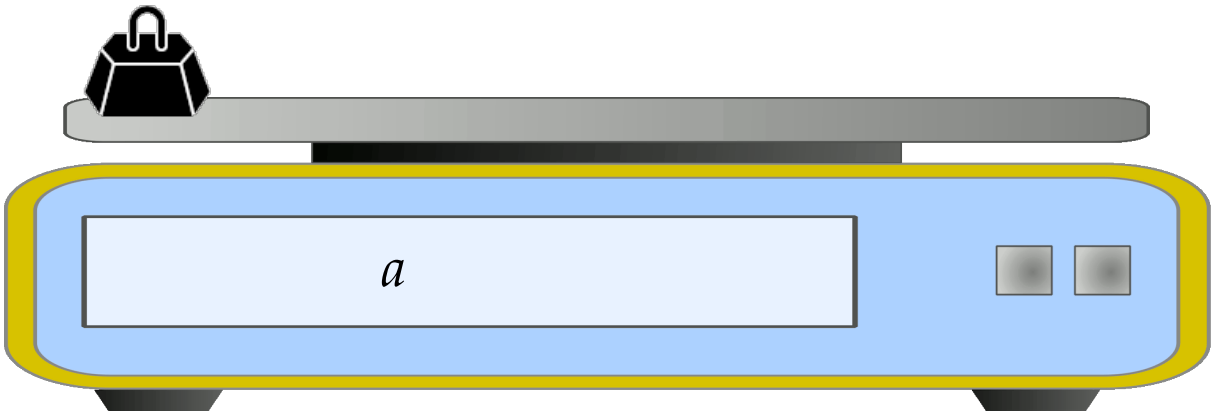




$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$S_0 = a$

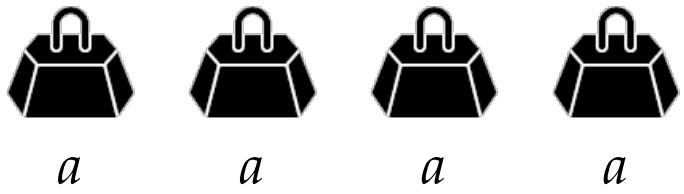


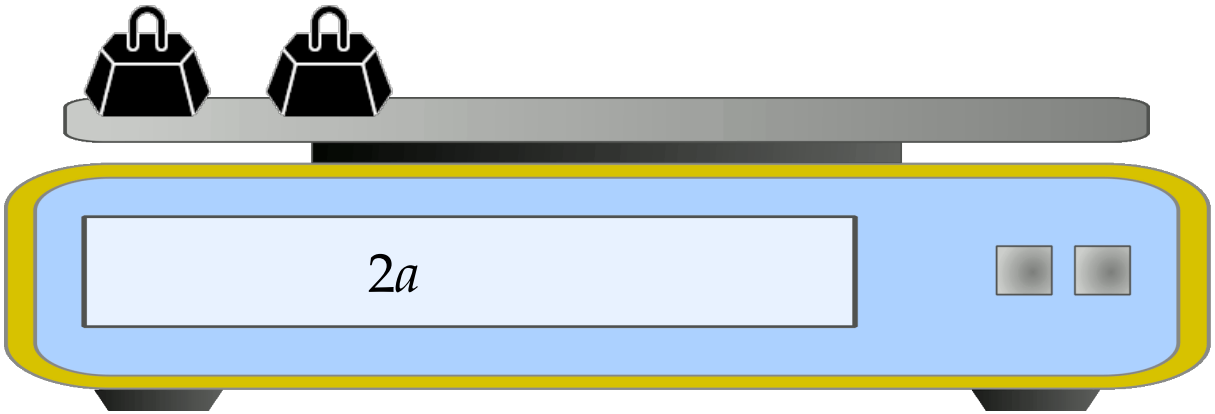


$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$S_0 = a$

$S_1 = 2a$



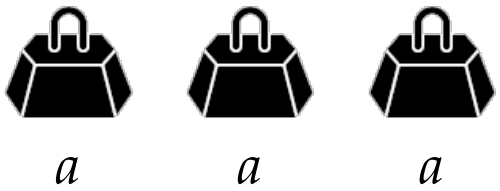


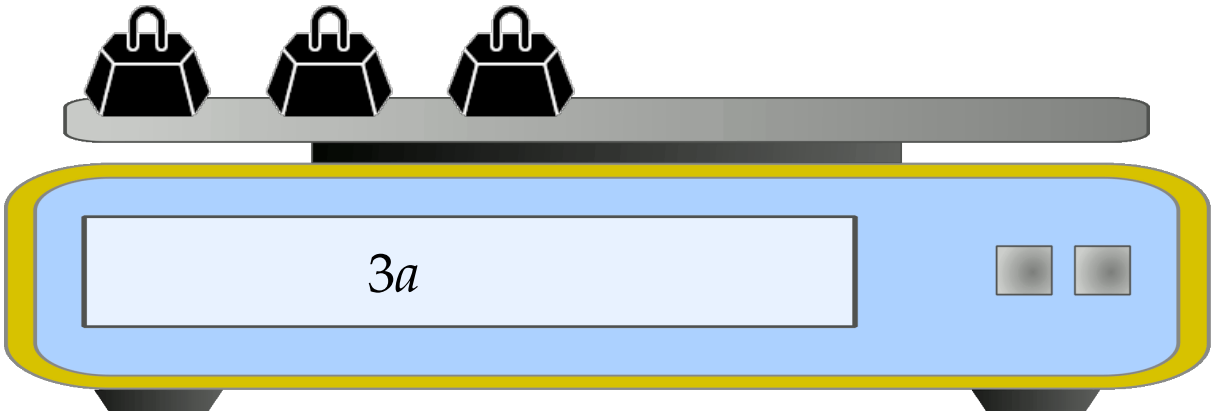
$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$S_0 = a$

$S_1 = 2a$

$S_2 = 3a$





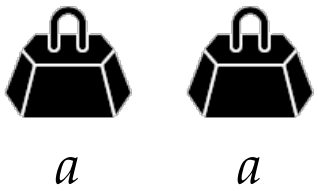
$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

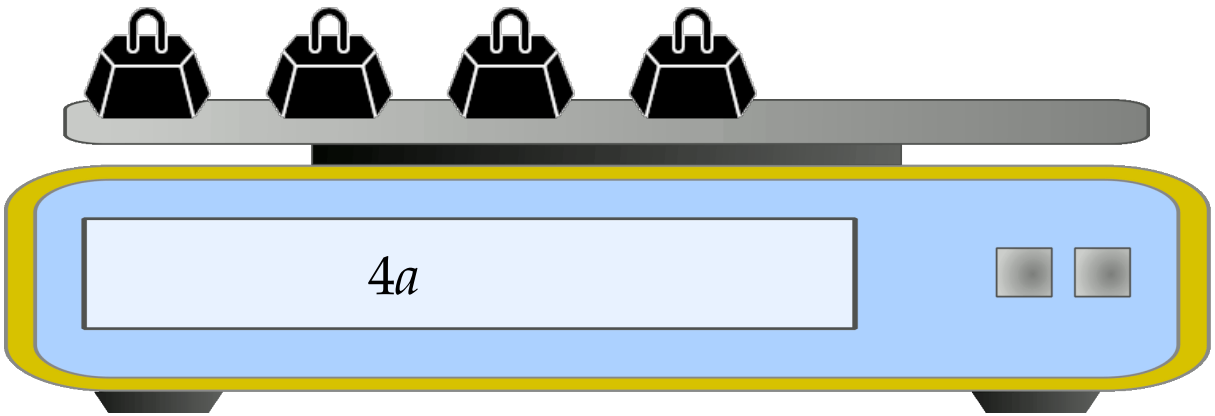
$$S_0 = a$$

$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$





$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

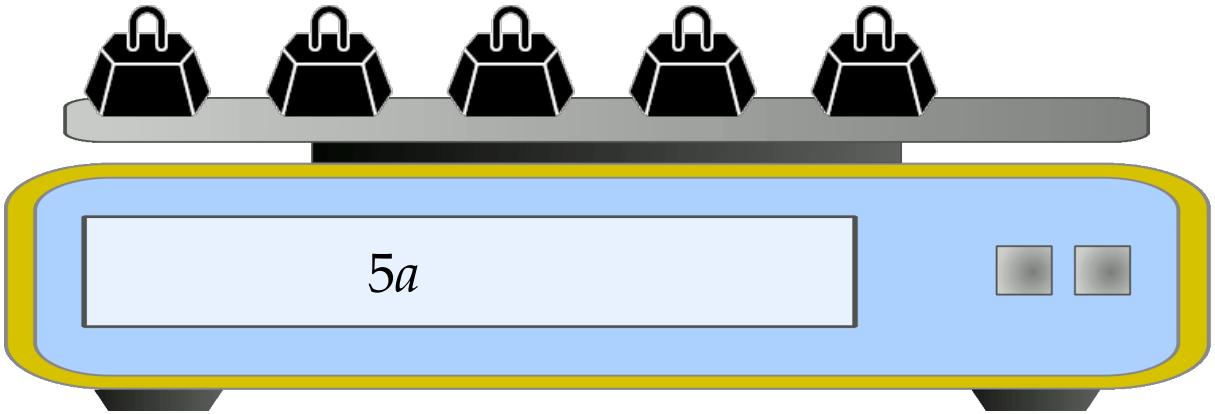
$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$

$$S_4 = 5a$$

a



$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

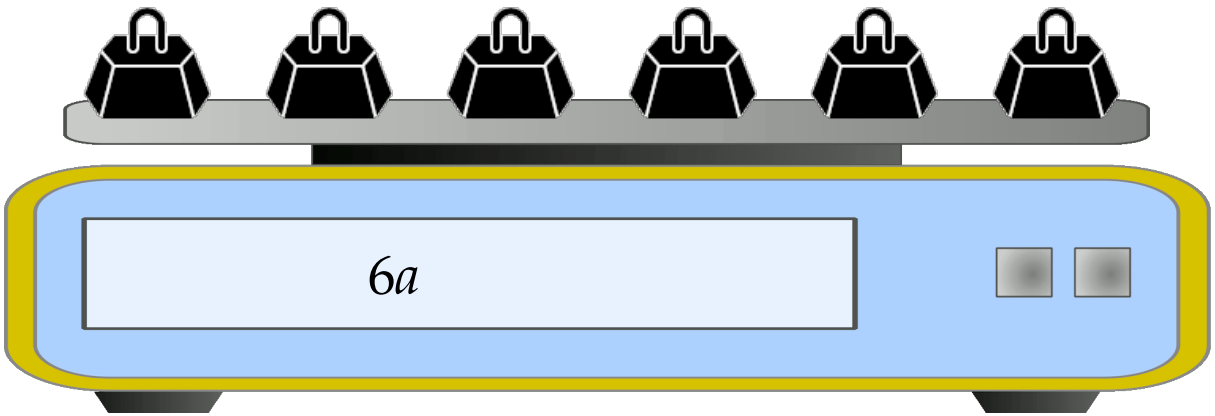
$$S_1 = 2a$$

$$S_2 = 3a$$

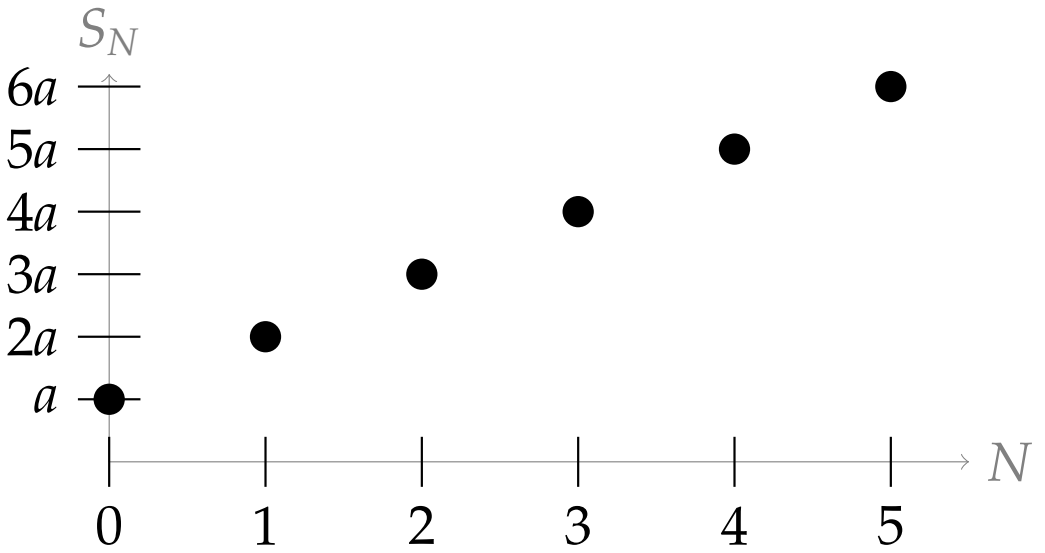
$$S_3 = 4a$$

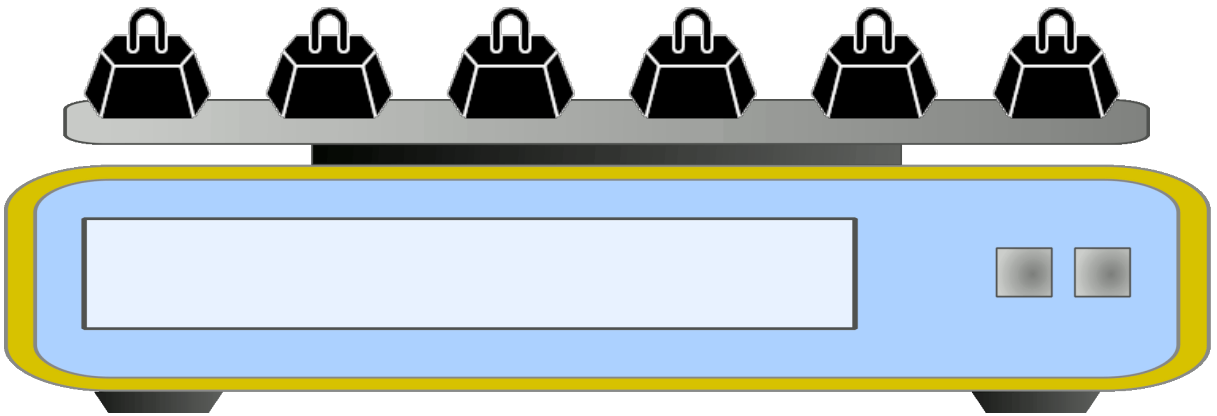
$$S_4 = 5a$$

$$S_5 = 6a$$



$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$





$$S_0 = a$$

$$S_1 = 2a$$

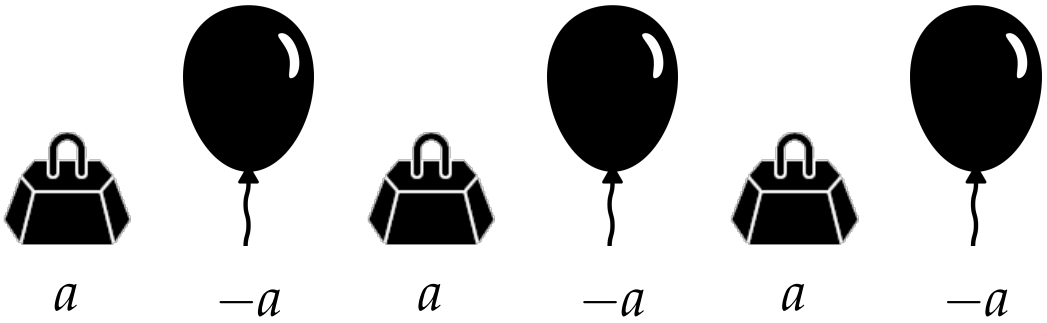
$$S_2 = 3a$$

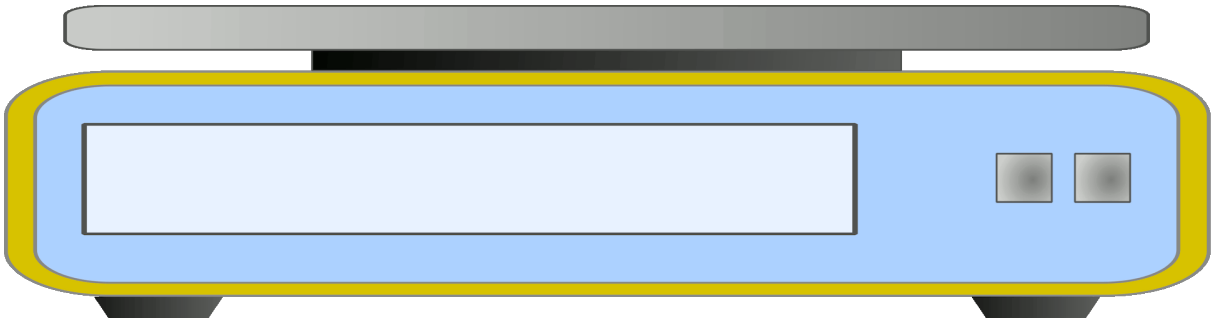
$$S_3 = 4a$$

$$S_4 = 5a$$

$$S_5 = 6a$$

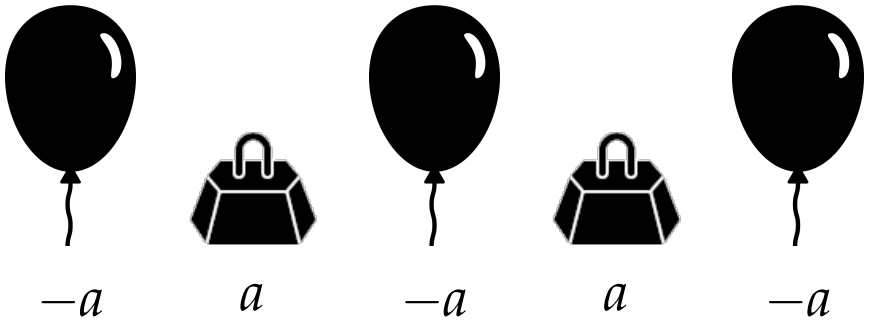
$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

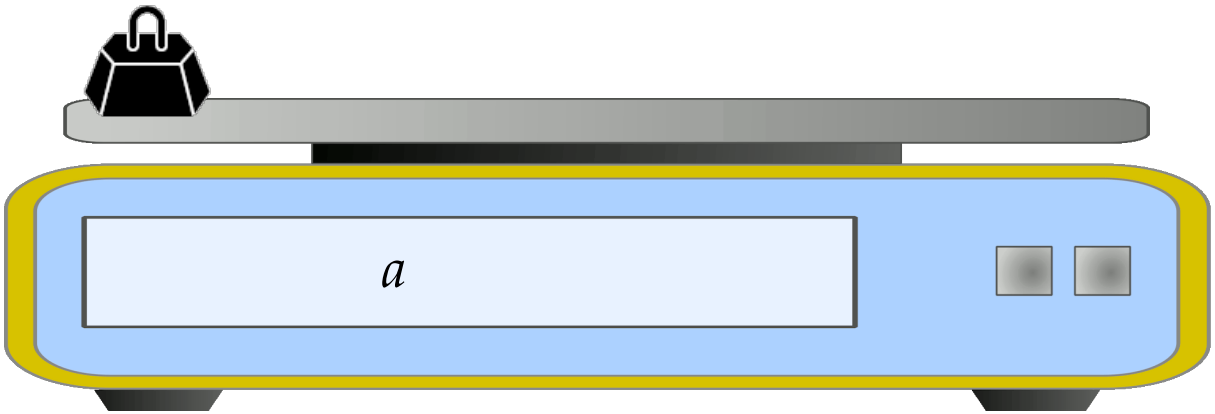




$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

$S_0 = a$

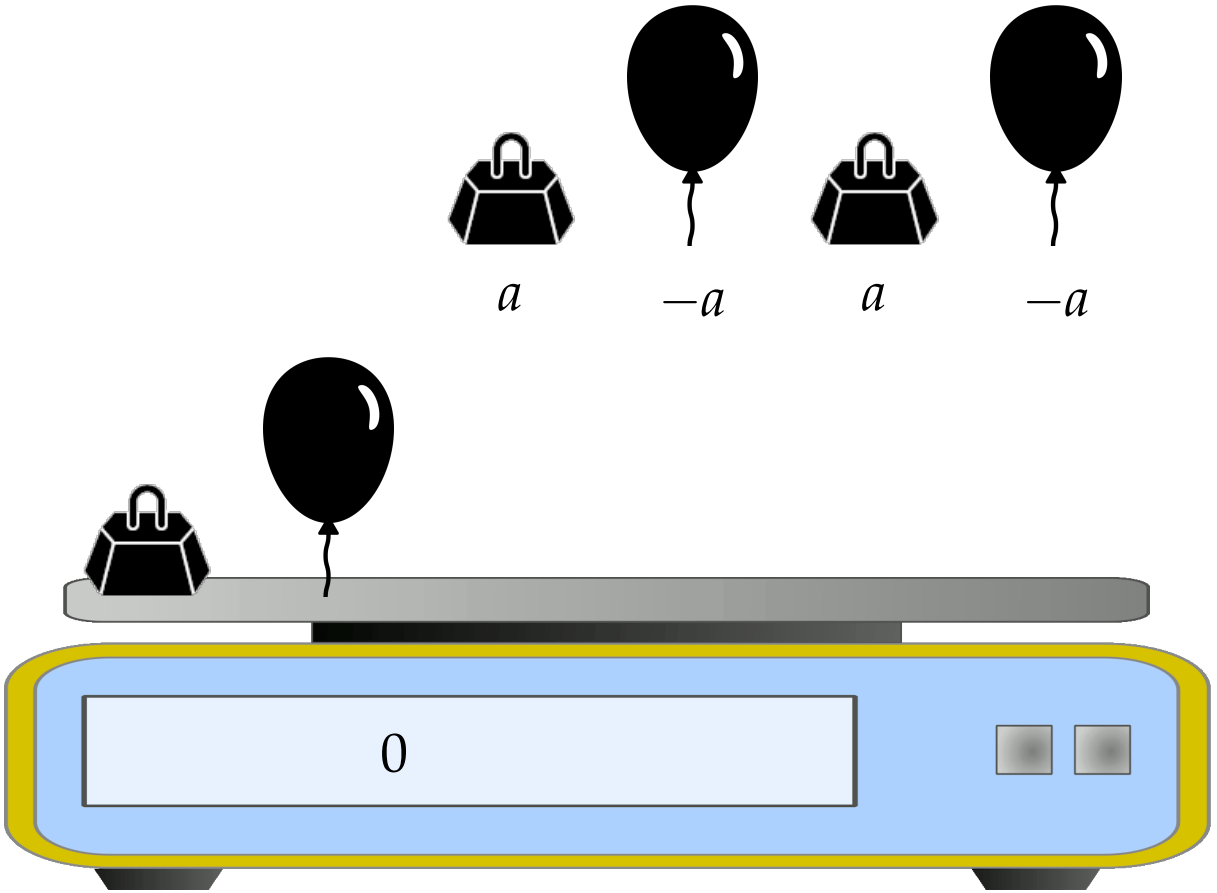




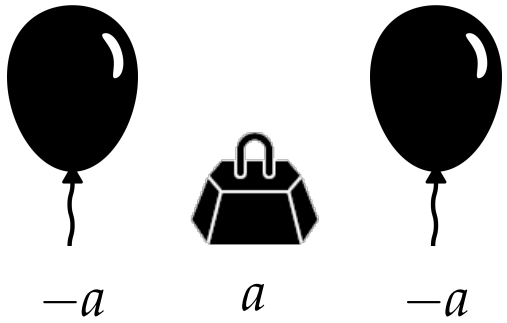
$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

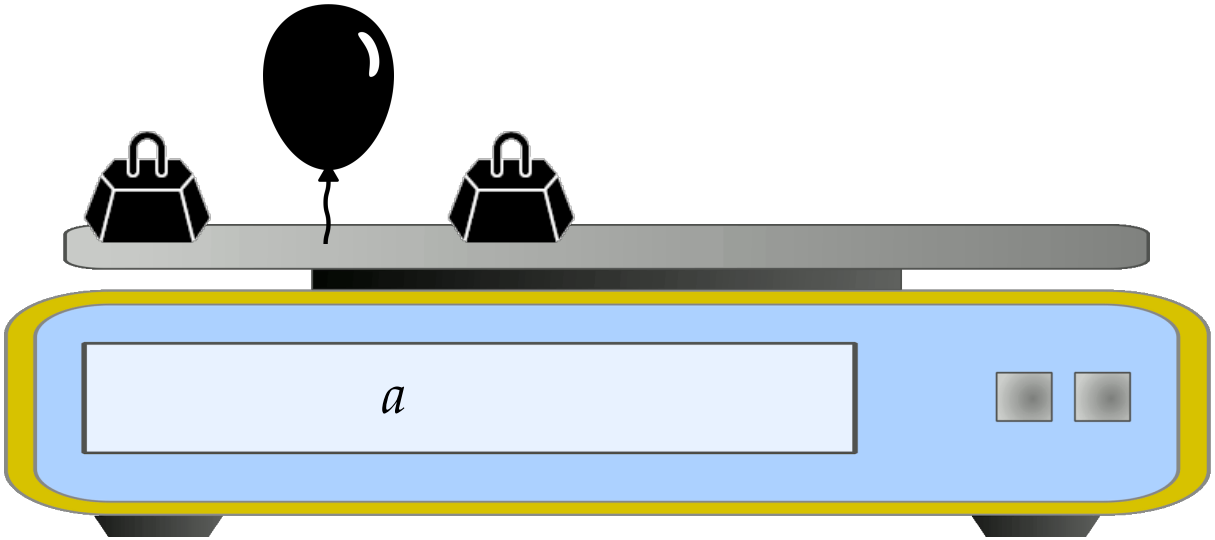
$S_0 = a$

$S_1 = 0$



$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



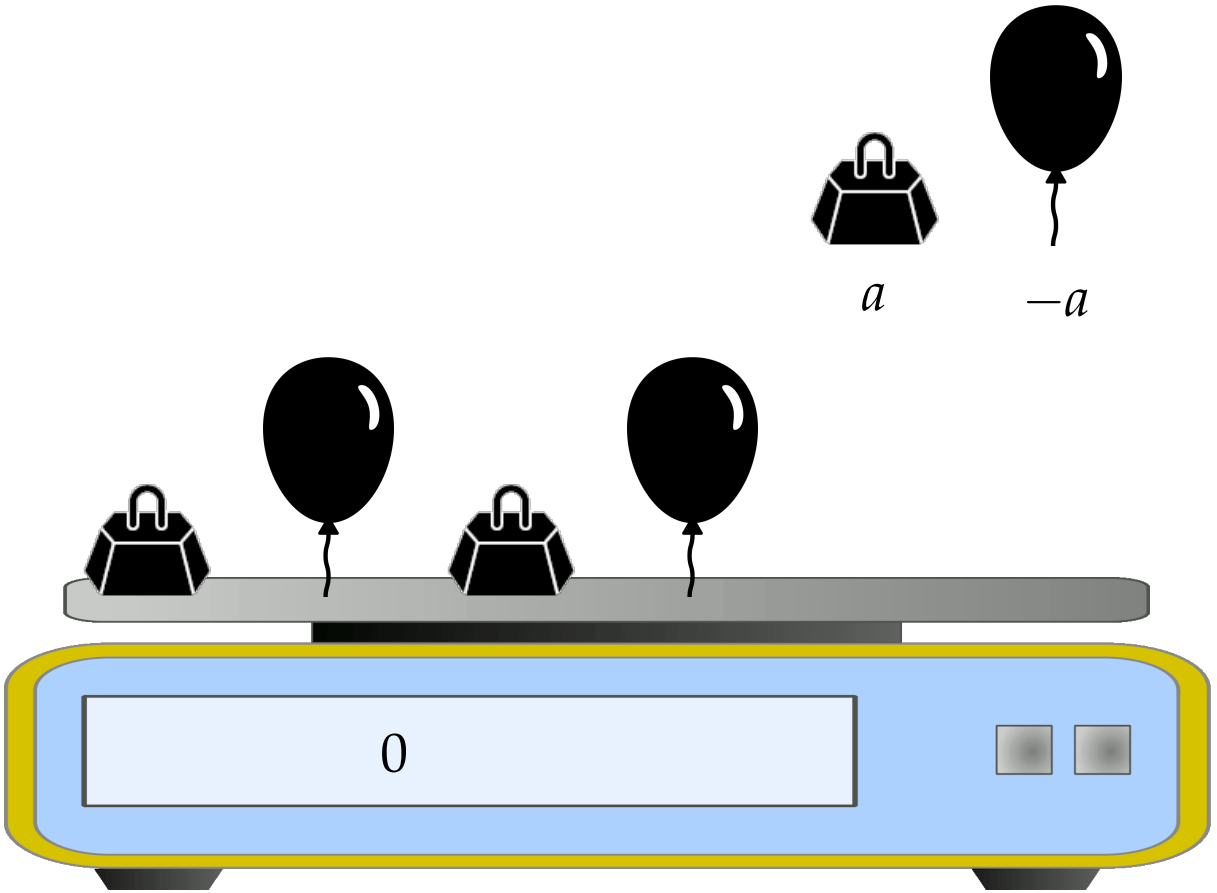


$$S_0 = a$$

$$S_1 = 0$$

$$S_2 = a$$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



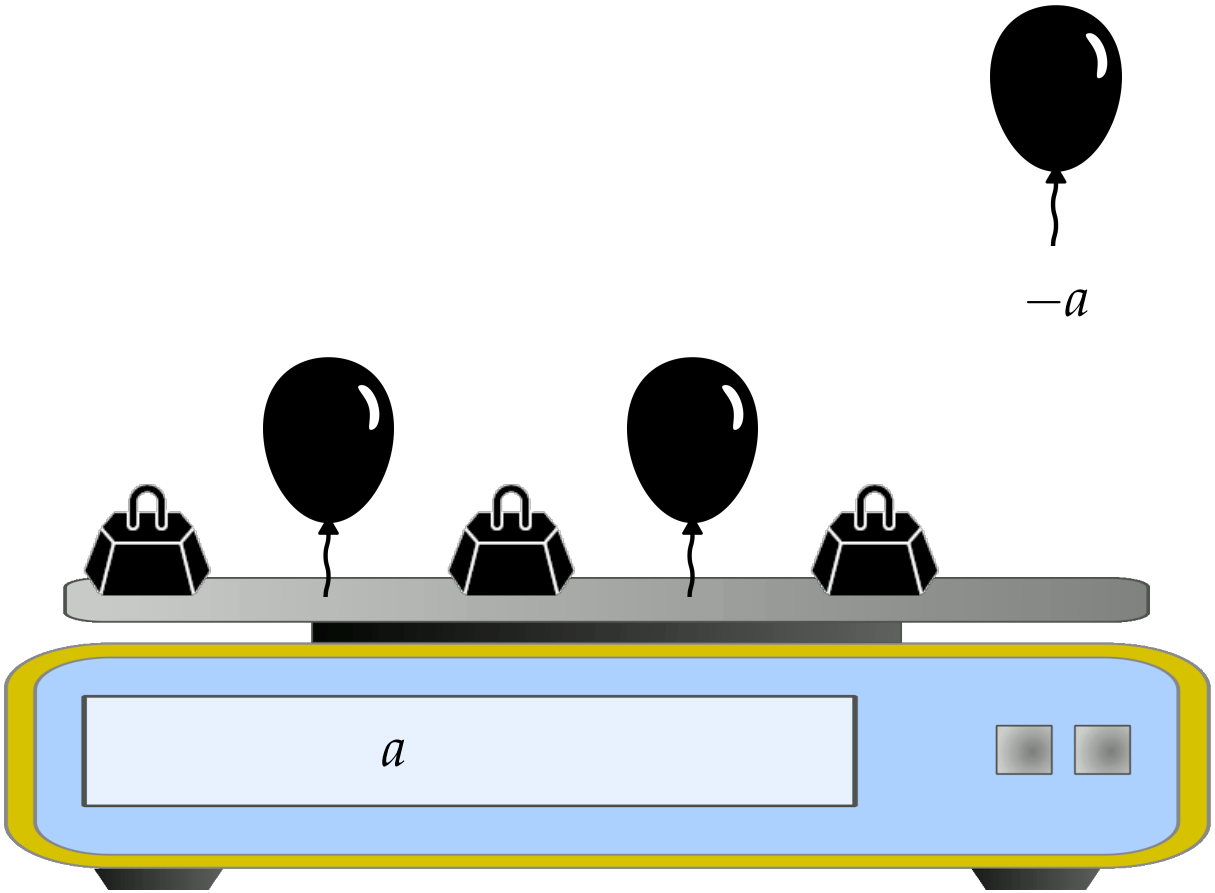
$S_0 = a$

$S_1 = 0$

$S_2 = a$

$S_3 = 0$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$S_0 = a$

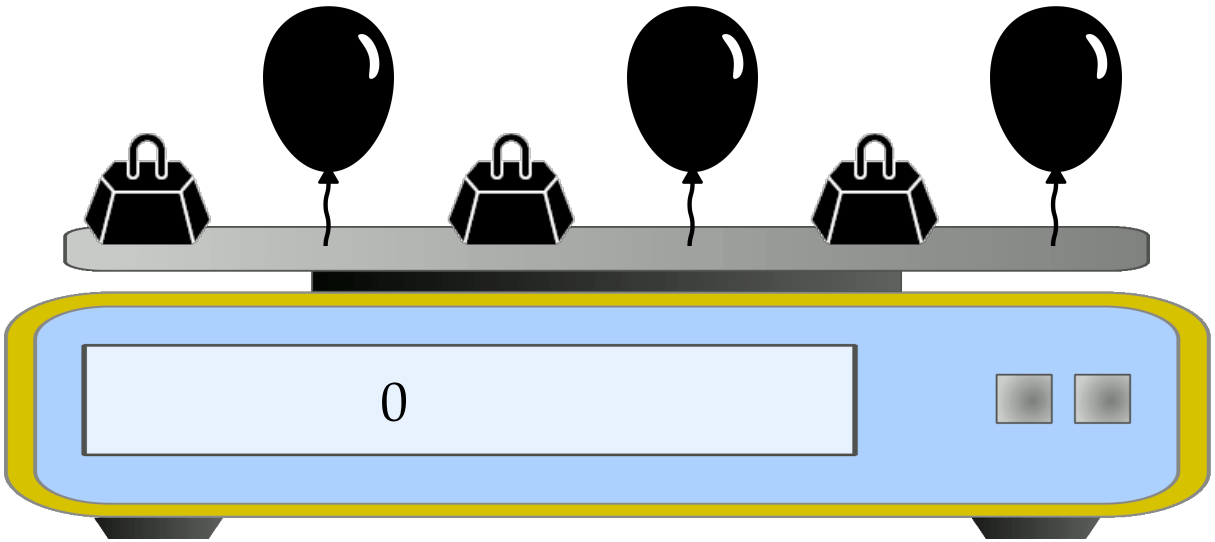
$S_1 = 0$

$S_2 = a$

$S_3 = 0$

$S_4 = a$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$S_0 = a$

$S_1 = 0$

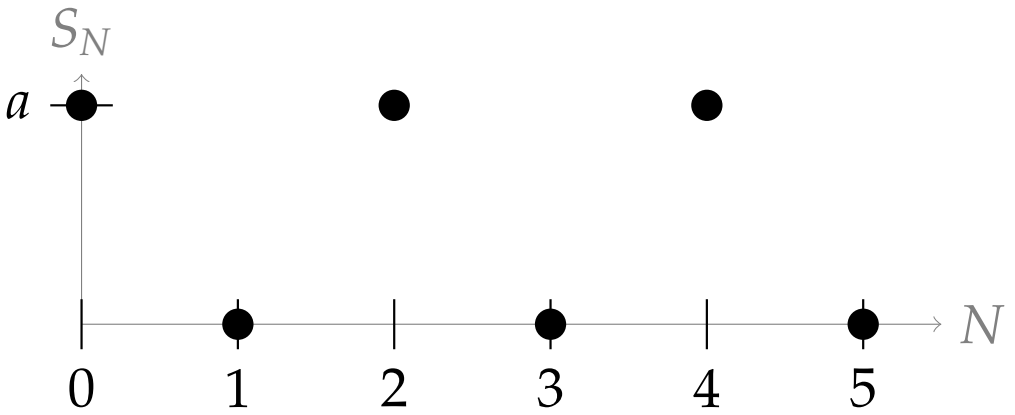
$S_2 = a$

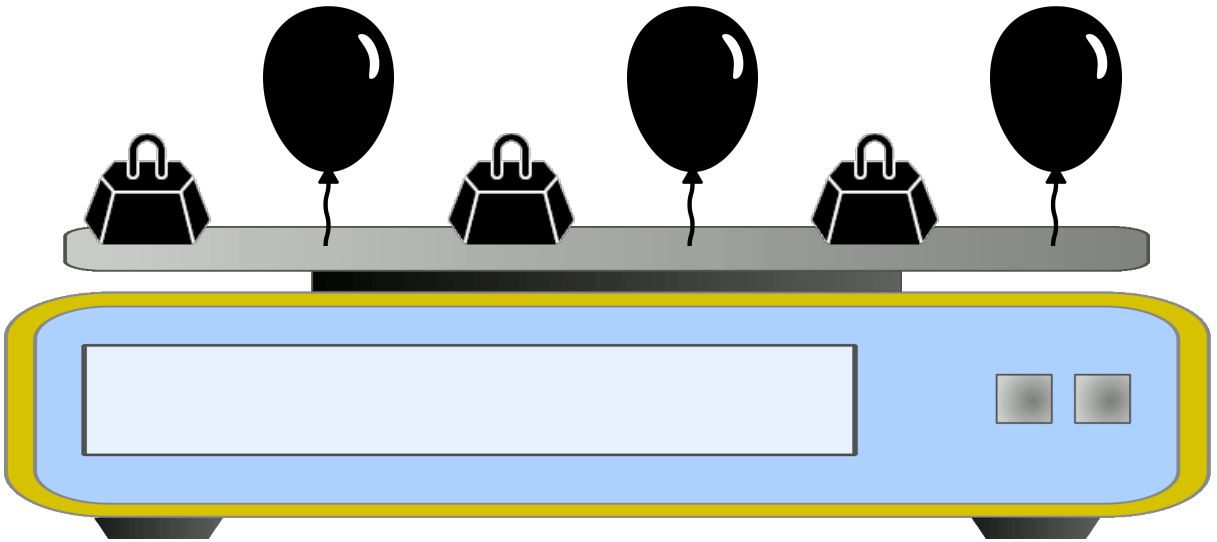
$S_3 = 0$

$S_4 = a$

$S_5 = 0$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$





$S_0 = a$

$S_1 = 0$

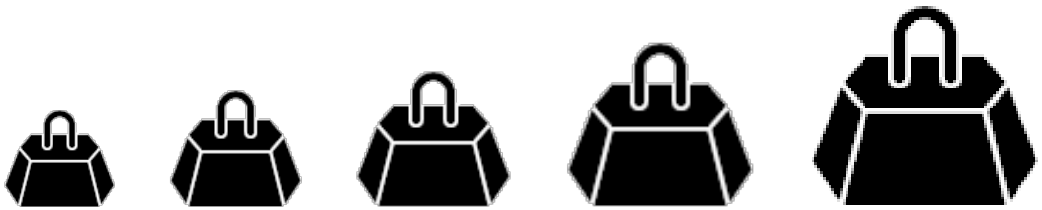
$S_2 = a$

$S_3 = 0$

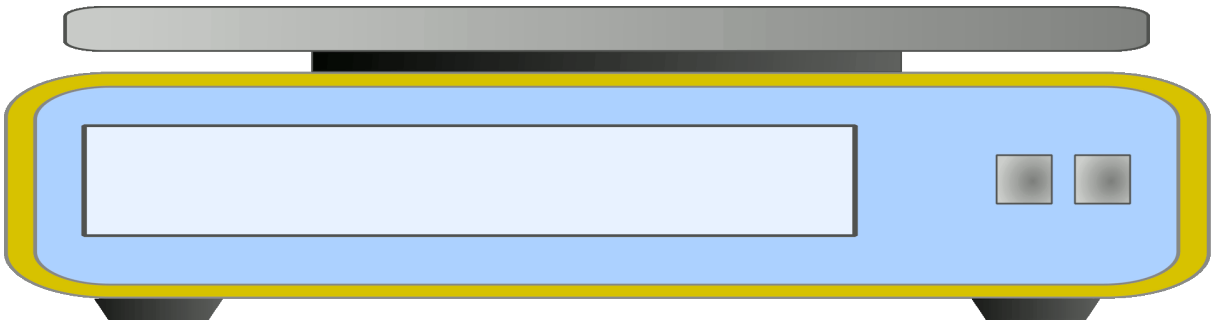
$S_4 = a$

$S_5 = 0$

$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

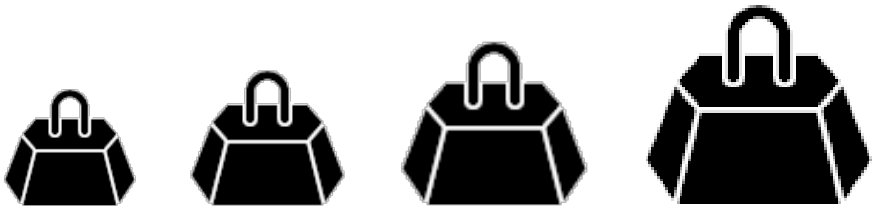


a ar ar^2 ar^3 ar^4

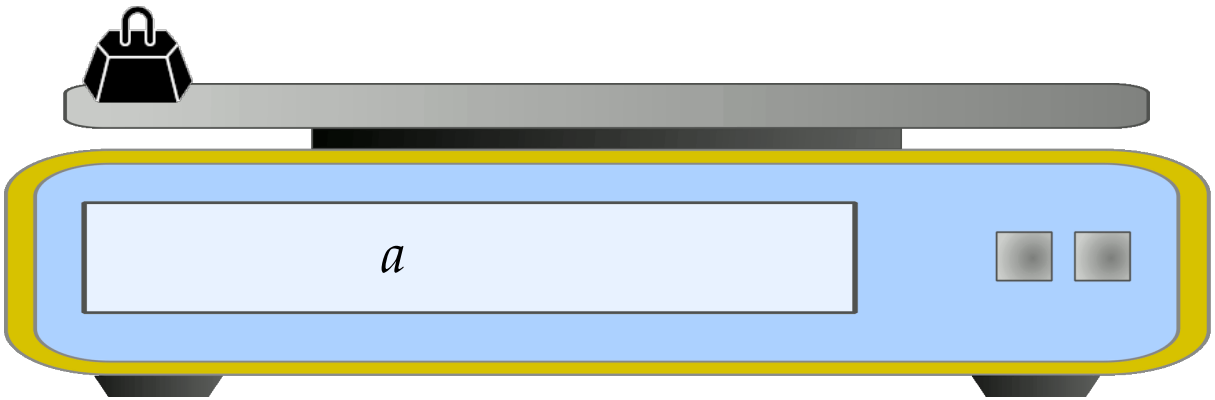


$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

$S_0 = a$



ar ar^2 ar^3 ar^4

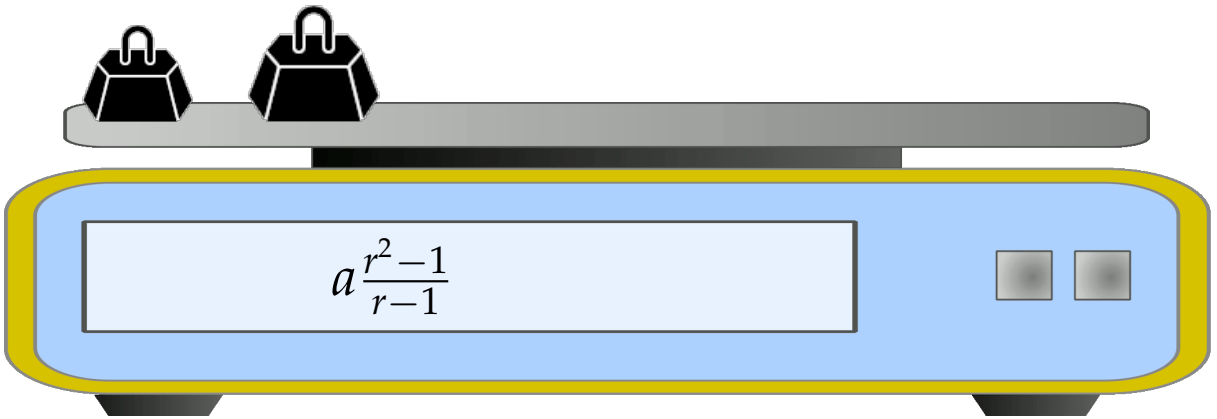


$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

ar^2 ar^3 ar^4

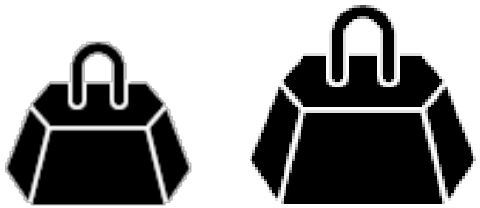


$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

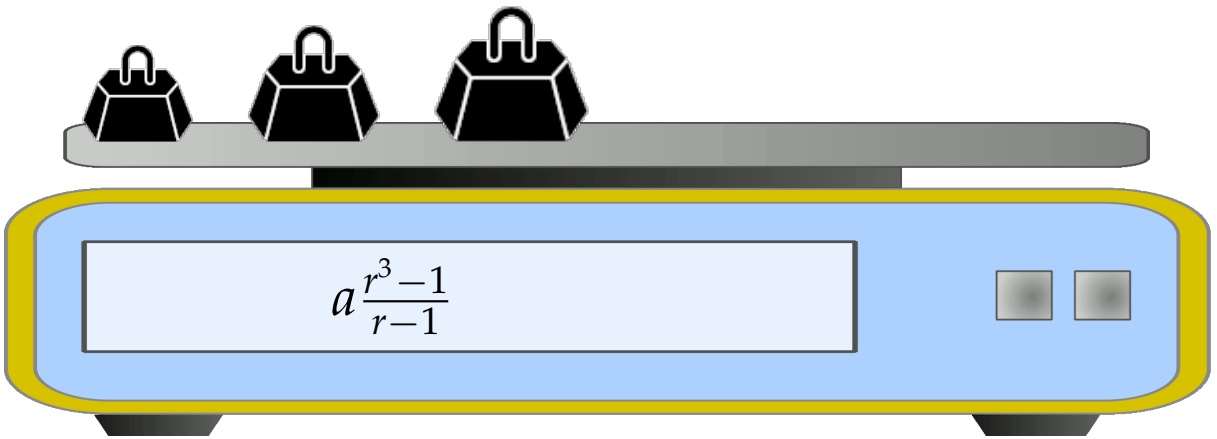
$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$



ar^3 ar^4



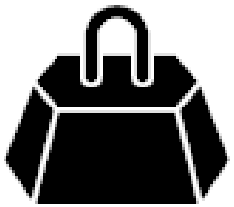
$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

$$S_0 = a$$

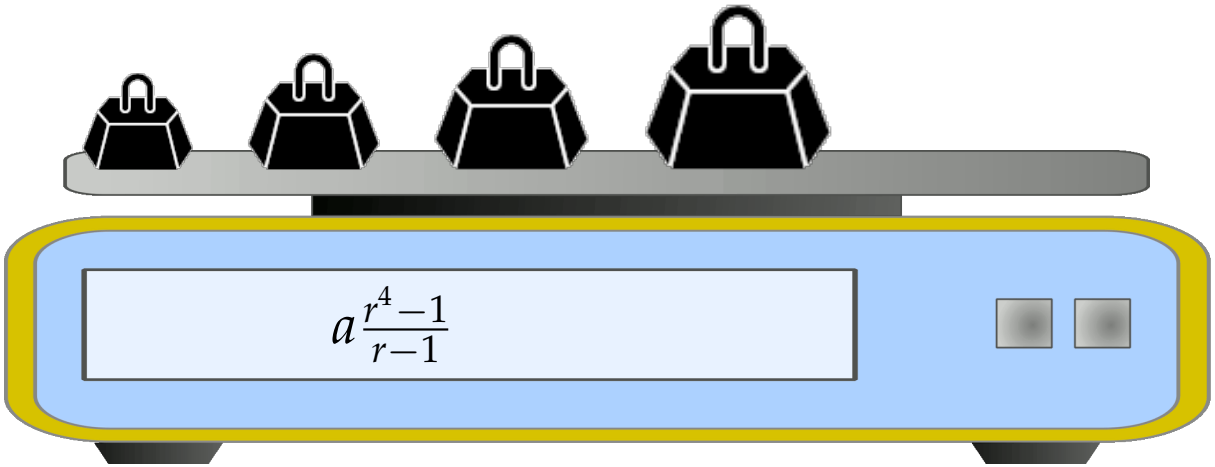
$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$



ar^4



$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

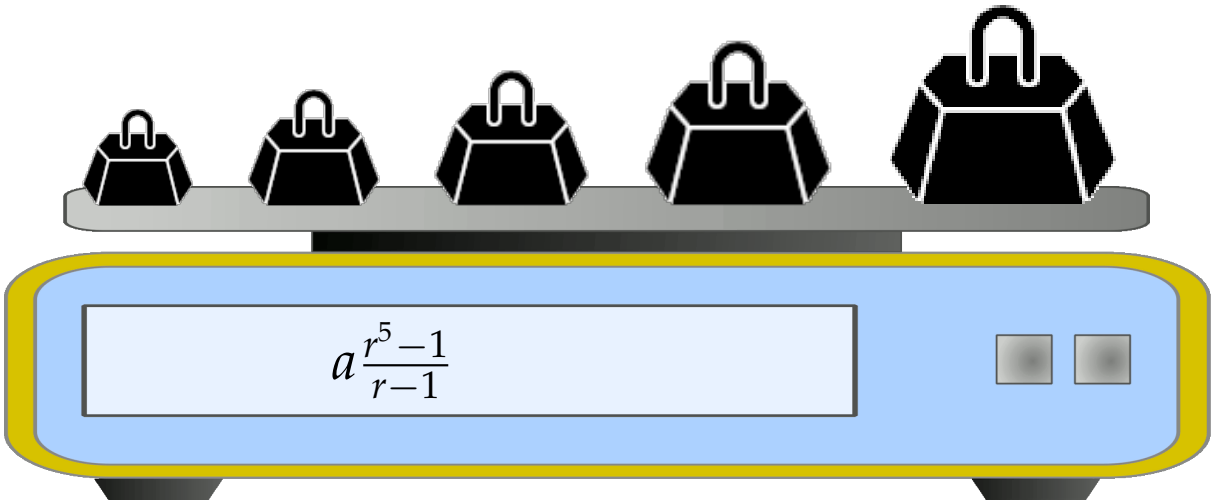
$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

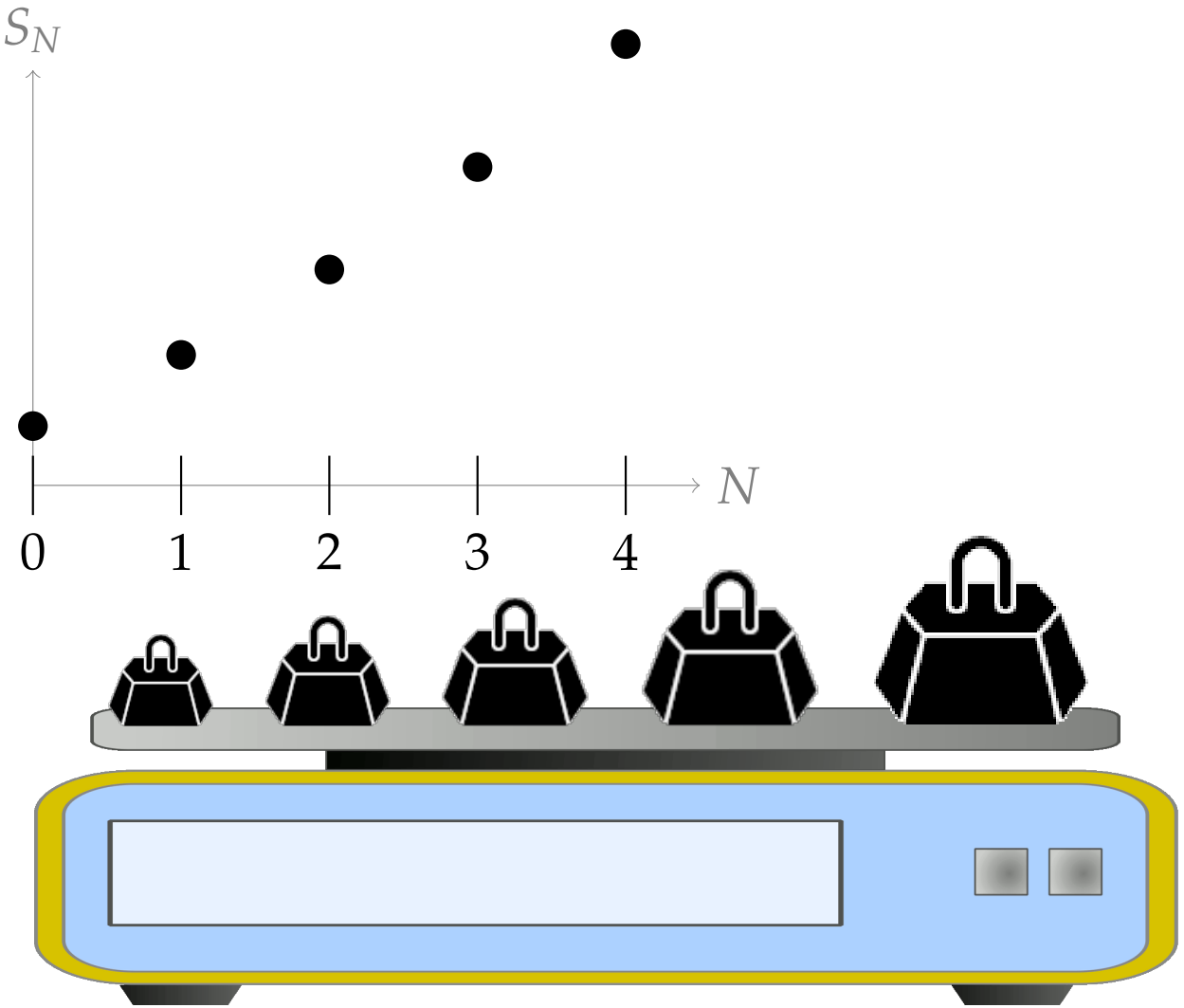
$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$



$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$



$$S_0 = a$$

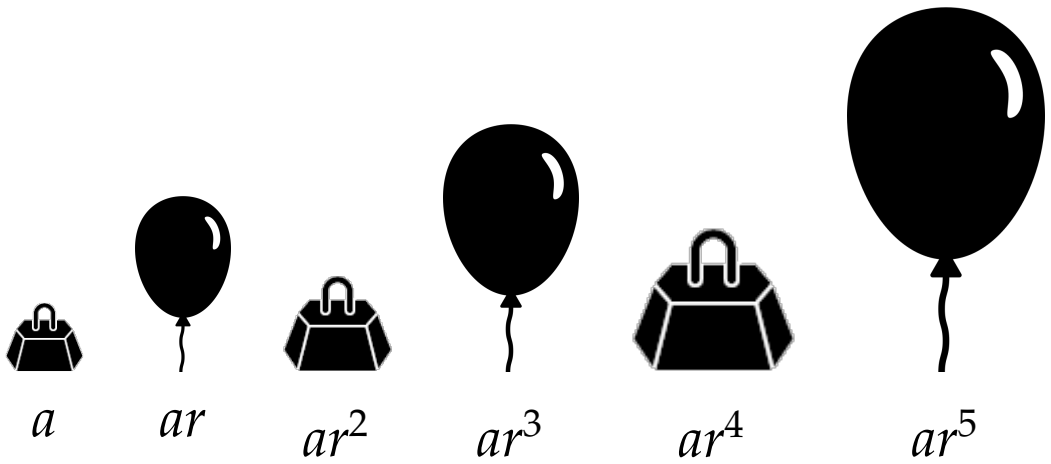
$$S_1 = a\frac{r^2-1}{r-1}$$

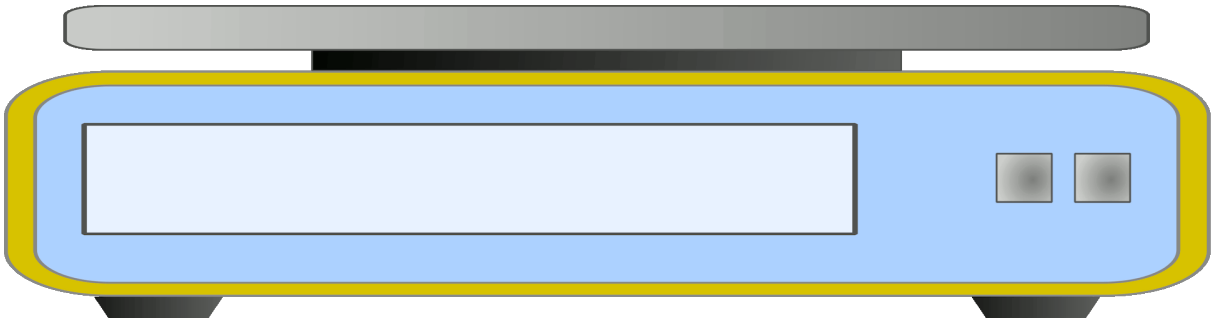
$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

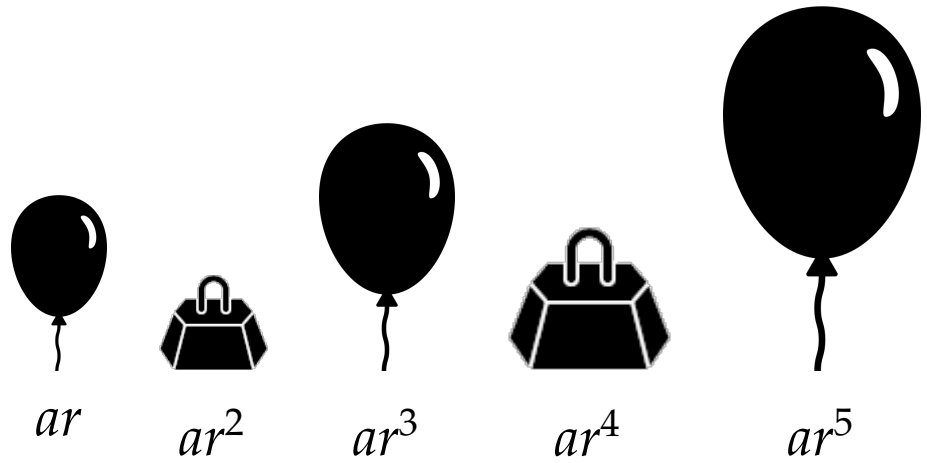
$$S_4 = a\frac{r^5-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$

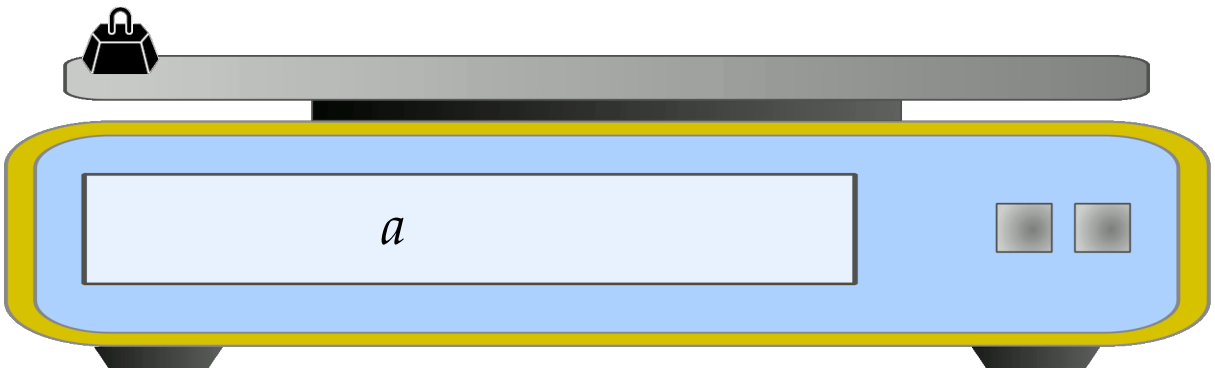




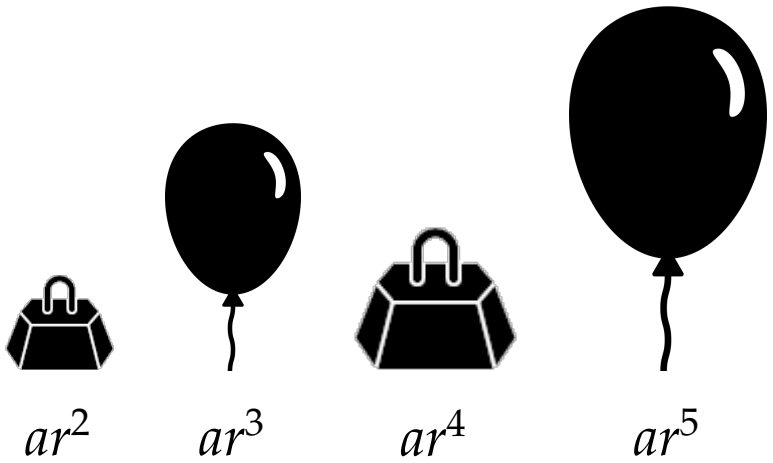
$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$S_0 = a$

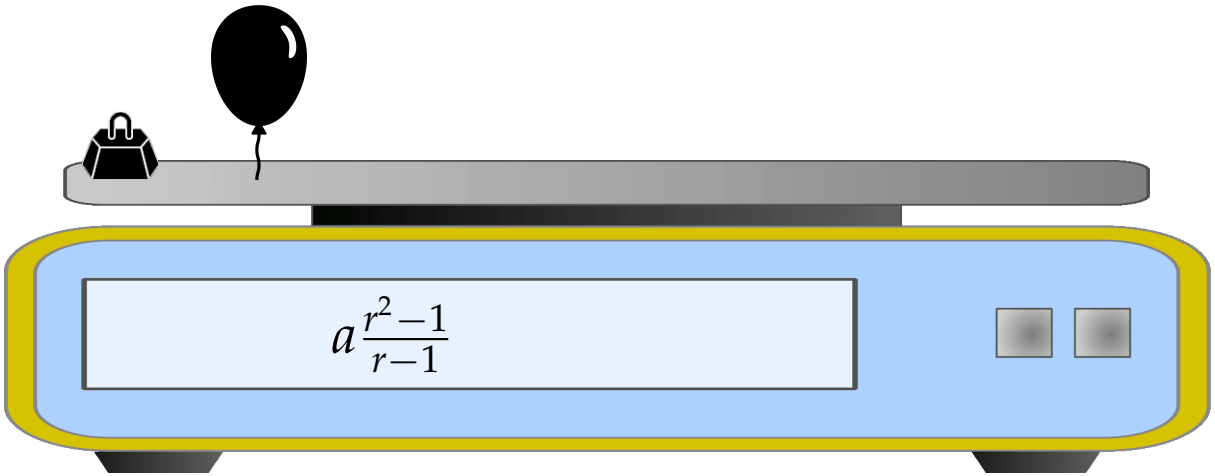


$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$

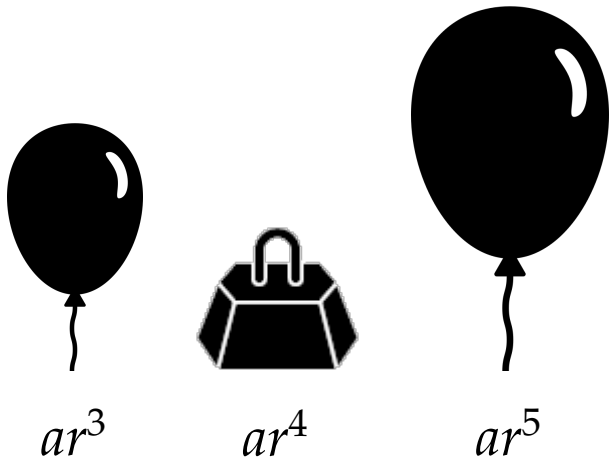


$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$



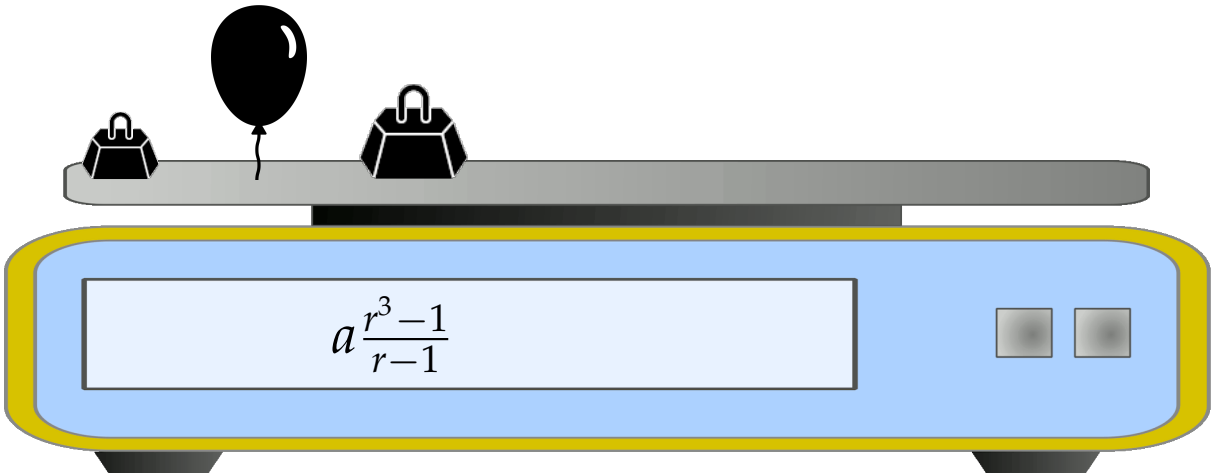
$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



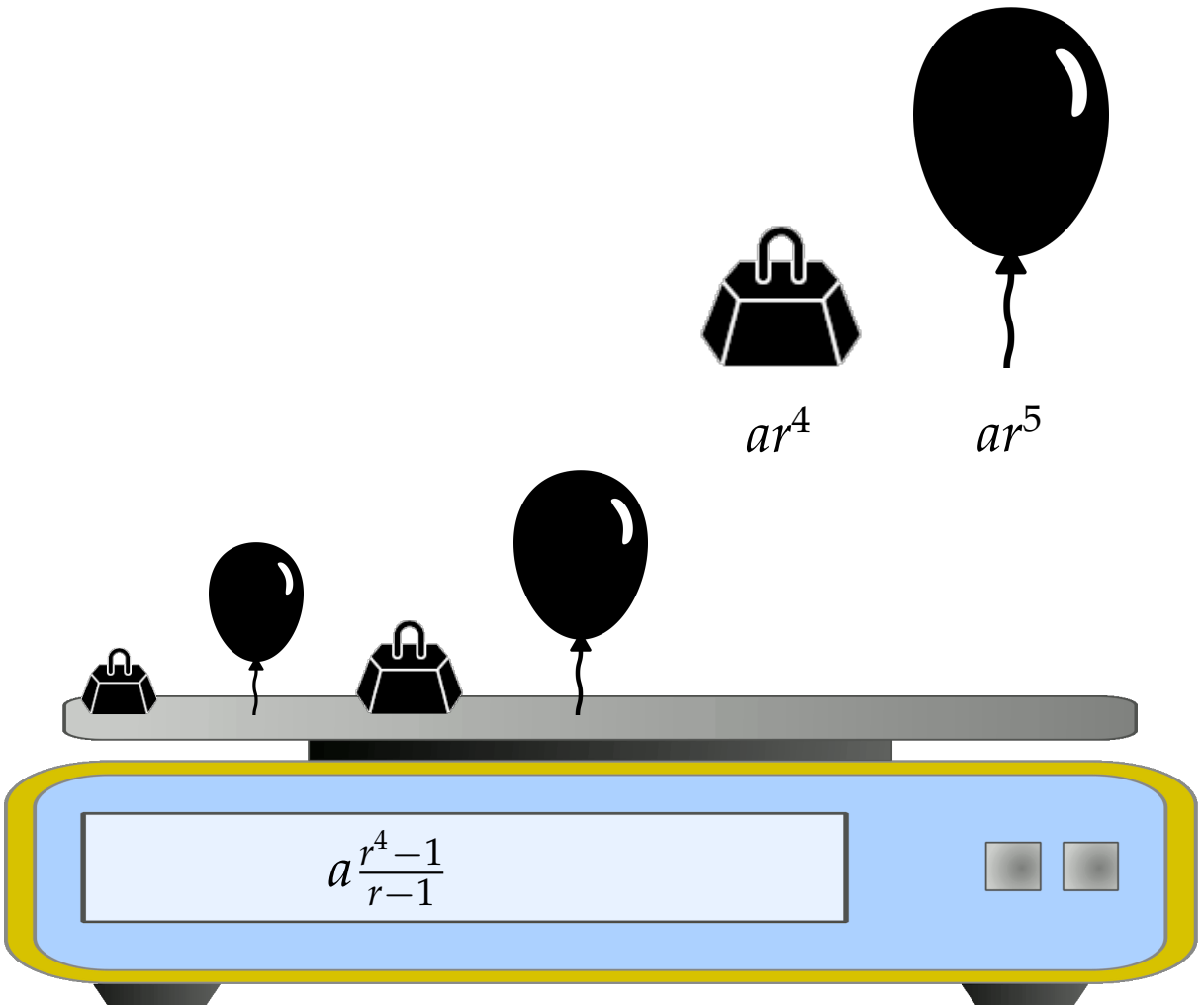
$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$



$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



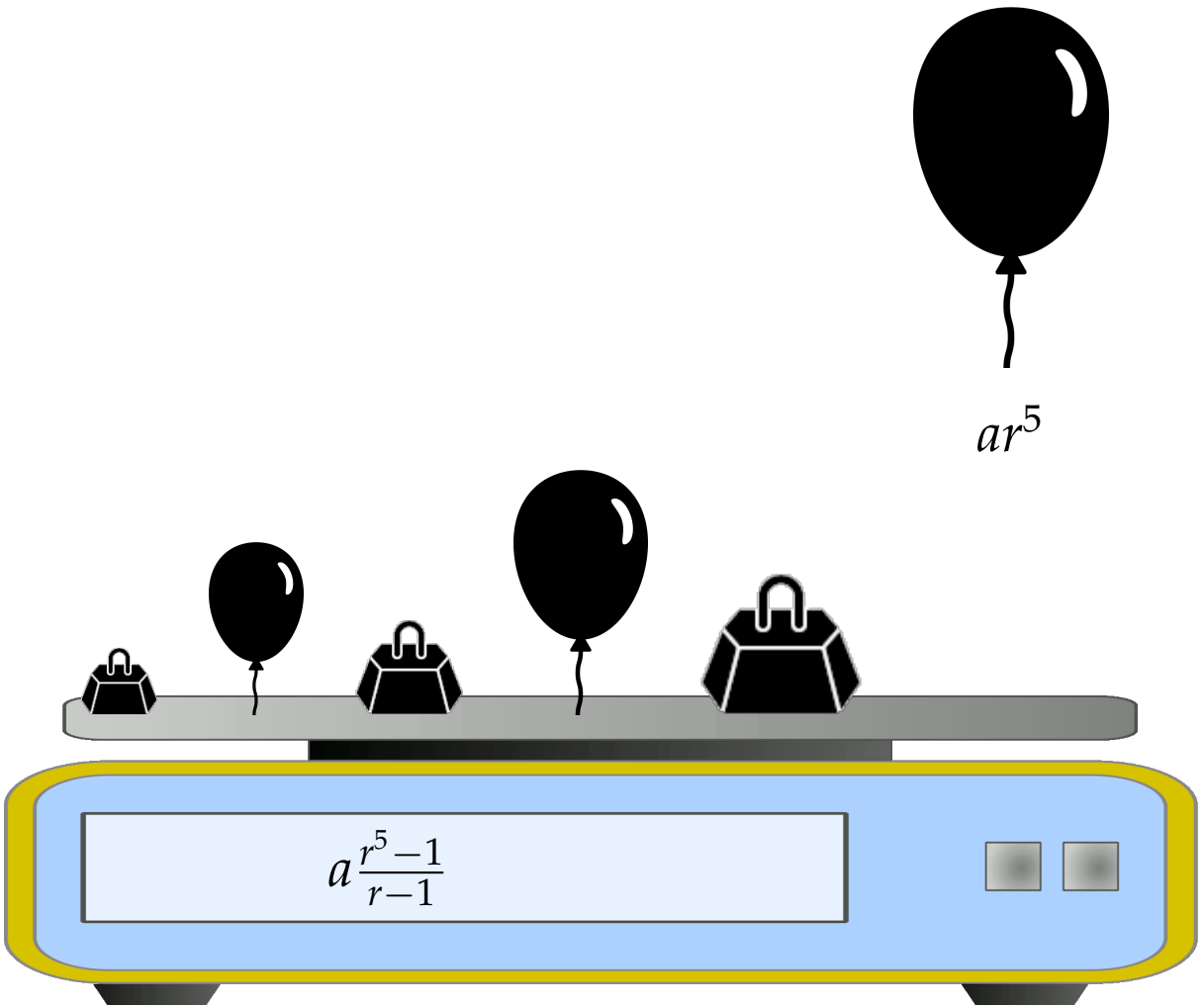
$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



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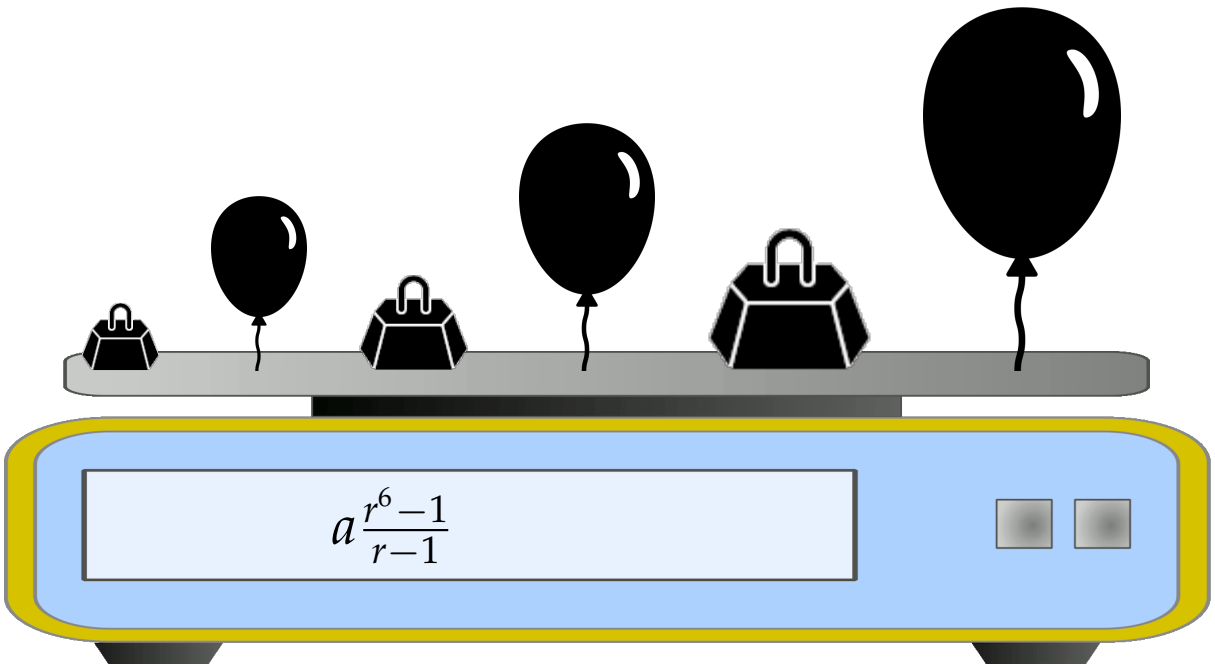


$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

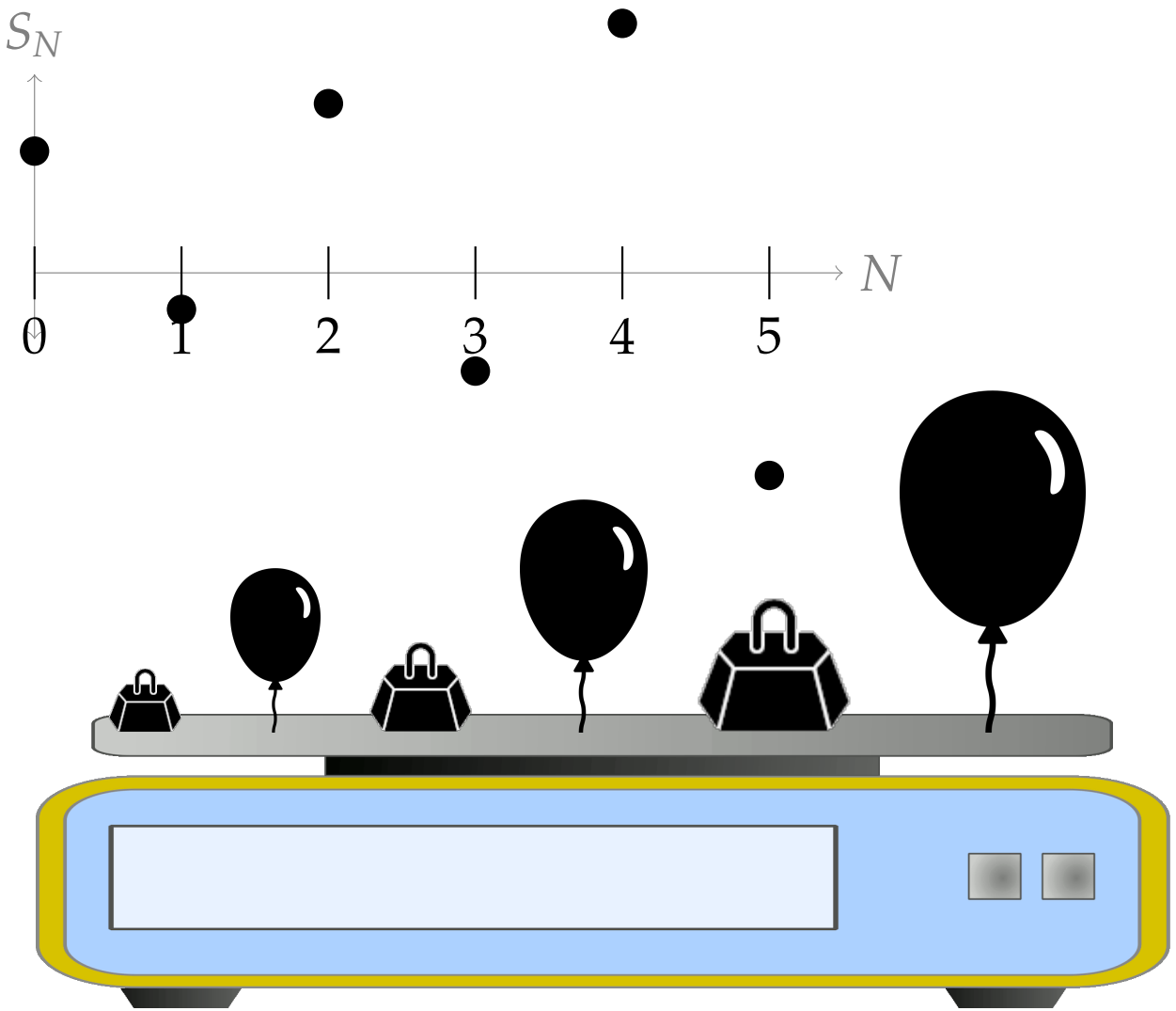
$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$S_5 = a\frac{r^6-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$S_5 = a\frac{r^6-1}{r-1}$$

Geometric Series

New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half.

- Each of the first 210,000 solutions can produce 50 bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^3}$ bitcoins.

Assume that this continues forever, and that bitcoins are infinitely divisible.[1] How many bitcoins can possibly be produced?

[1] Actually the smallest allowed division of a bitcoin is 10^{-8}.

ans

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Assume that this continues forever, and that bitcoins are infinitely divisible.[1] How many bitcoins can possibly be produced?

We start by writing the total number of bitcoin produced as a series. Since we want to know an upper bound, we'll assume that infinite solutions can be found and used to make bitcoin.

$$210\,000(50) + 210\,000\left(\frac{50}{2}\right) + 210\,000\left(\frac{50}{2^2}\right) + \cdots = \sum_{n=0}^{\infty} (210\,000)\left(\frac{50}{2^n}\right)$$

[1] Actually the smallest allowed division of a bitcoin is 10^{-8}.

Geometric Series

$$\sum_{n=0}^{\infty} (210\,000) \left(\frac{50}{2^n} \right) =$$

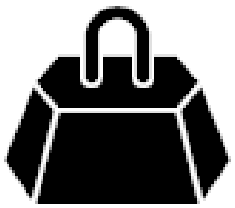
ans

Geometric Series

$$\begin{aligned}\sum_{n=0}^{\infty}(210\,000)\left(\frac{50}{2^n}\right) &= \sum_{n=0}^{\infty}(210\,000\cdot 50)\left(\frac{1}{2}\right)^n \\ &= (210\,000\cdot 50)\frac{1}{1-\frac{1}{2}} \\ &= (210\,000\cdot 50)(2) \\ &= 21\,000\,000\end{aligned}$$

So there will never be more than 21,000,000 bitcoins produced this way.

$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$



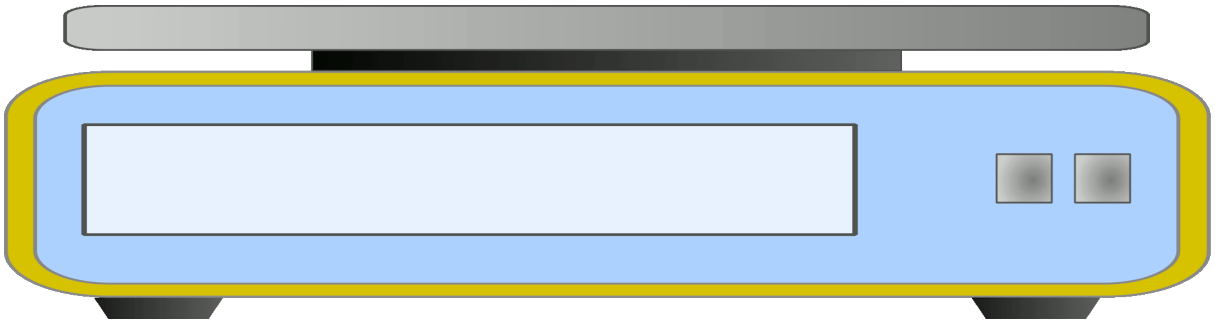
10 500 000

5 250 000

2 625 000

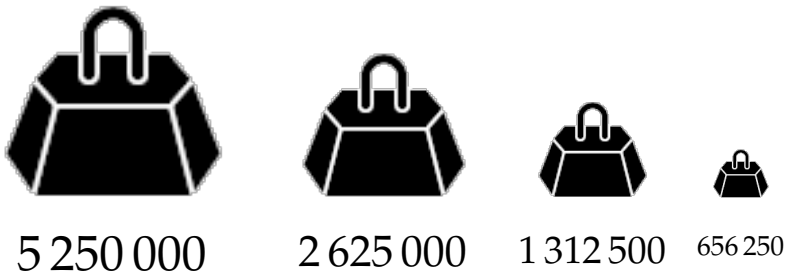
1 312 500

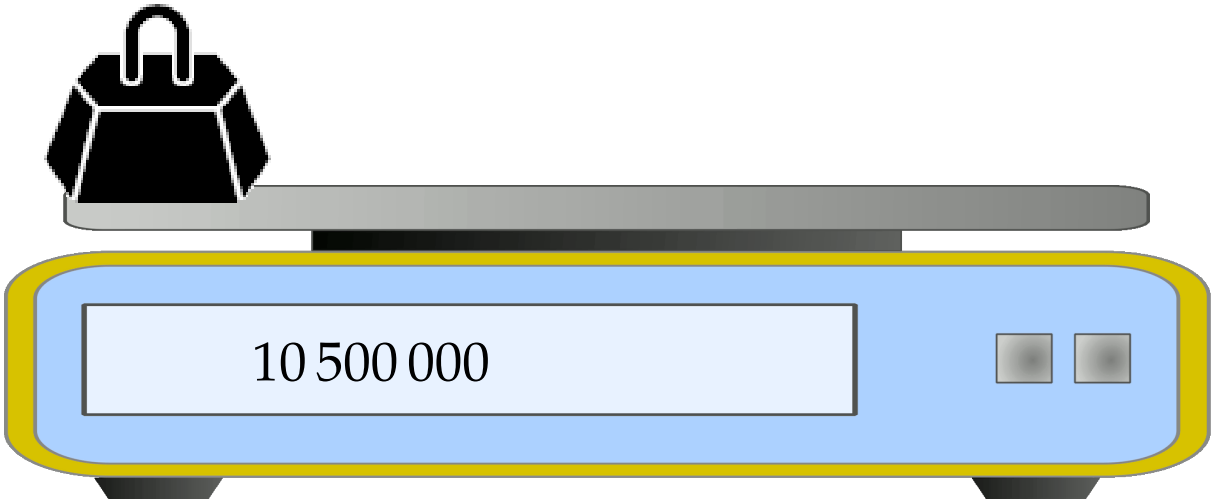
656 250



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

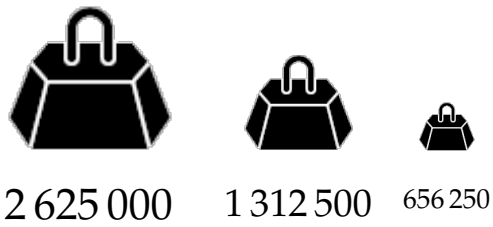


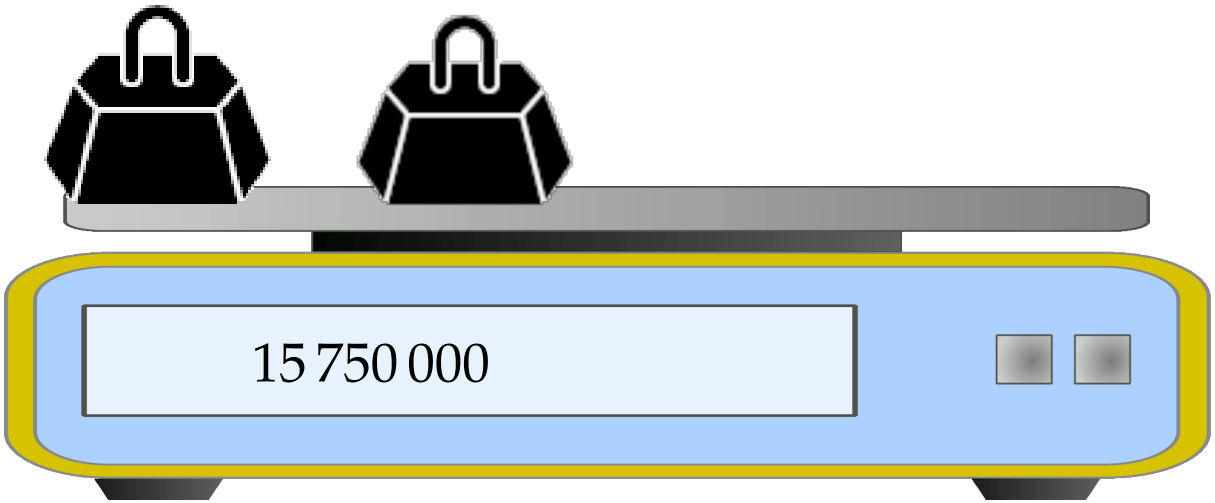


$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$





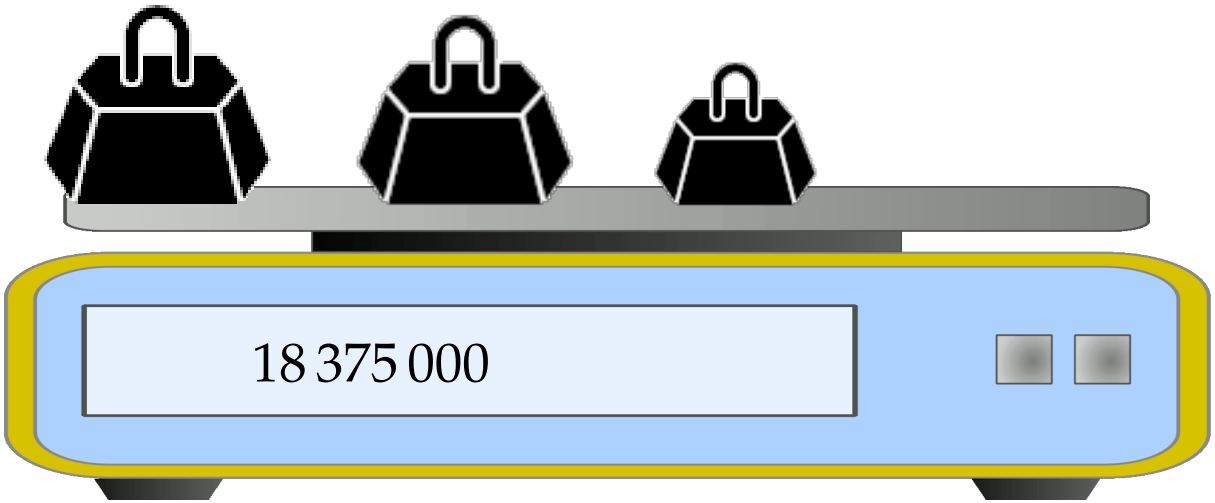
$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

1 312 500 656 250



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

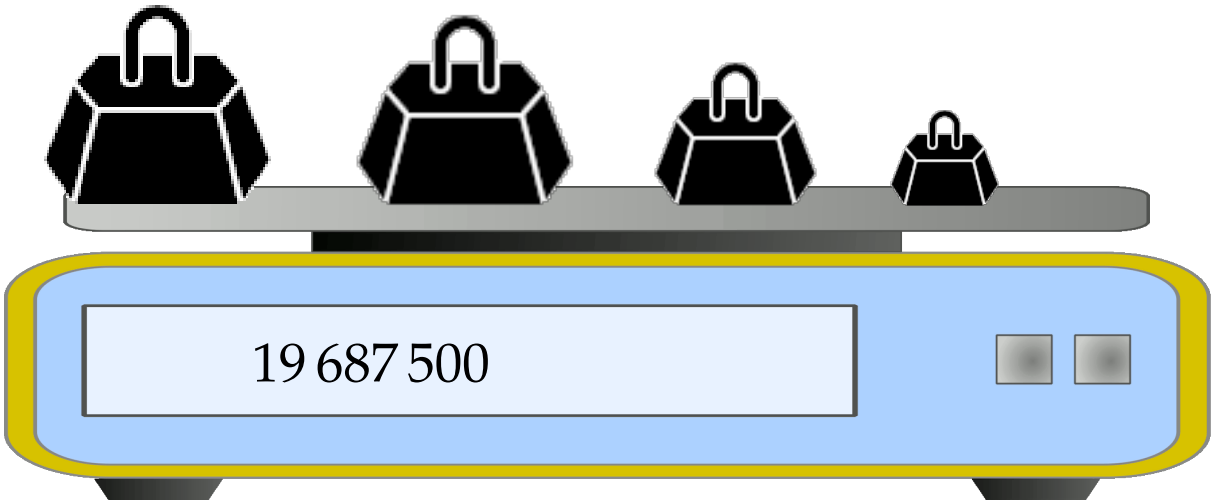
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

656 250



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

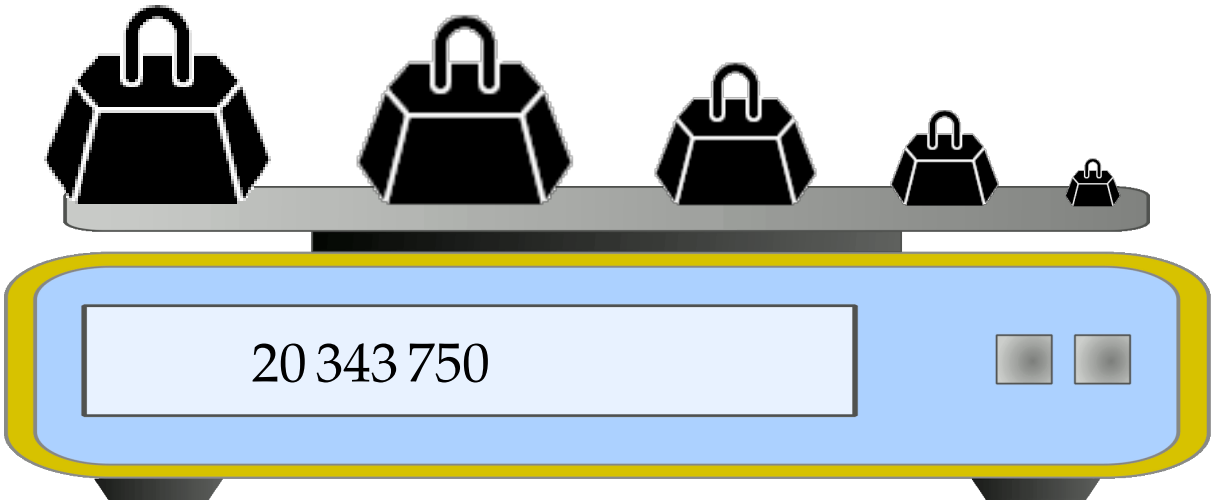
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

$$S_4 = 20\,343\,750$$



Arithmetic of Series

Let S, T, and C be real numbers. Let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T respectively. Then

$$\sum_{n=1}^{\infty}[a_n + b_n] = S + T$$

$$\sum_{n=1}^{\infty}[a_n - b_n] = S - T$$

$$\sum_{n=1}^{\infty}[Ca_n] = CS$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right)$

$$\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right) =$$

ans

$$\sum_{n=0}^{\infty}\left(\frac{2}{3^n}+\frac{4}{5^n}\right)=\sum_{n=0}^{\infty}\frac{2}{3^n}+\sum_{n=0}^{\infty}\frac{4}{5^n}$$

$$=\sum_{n=0}^{\infty}2\left(\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}4\left(\frac{1}{5}\right)^n$$

$$=\frac{2}{1-\frac{1}{3}}+\frac{4}{1-\frac{1}{5}}$$

$$=\frac{2}{2/3}+\frac{4}{4/5}$$

$$=3+5=8$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right)$

$$\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right) =$$

ans

$$\sum_{n=6}^{\infty}\left(\frac{3^{n-1}}{5^{2n}}\right)=\sum_{n=6}^{\infty}\frac{1}{3}\left(\frac{3^{n}}{5^{2n}}\right)=\sum_{n=6}^{\infty}\frac{1}{3}\left(\frac{3^{n}}{25^{n}}\right)=\sum_{n=6}^{\infty}\frac{1}{3}\left(\frac{3}{25}\right)^{n}$$

Set $k=n-6$. Then $n=k+6$ and when $n=6$, $k=0$.

$$=\sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{3}{25}\right)^{k+6}=\sum_{k=1}^{\infty}\frac{1}{3}\left(\frac{3}{25}\right)^{6}\left(\frac{3}{25}\right)^{k}$$

$$=\frac{1}{3}\left(\frac{3}{25}\right)^{6}\cdot\frac{1}{1-3/25}=\frac{3^{5}}{25^{6}}\cdot\frac{25}{22}=\frac{3^{5}}{25^{5}\cdot 22}$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n}\right)$

$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) =$$

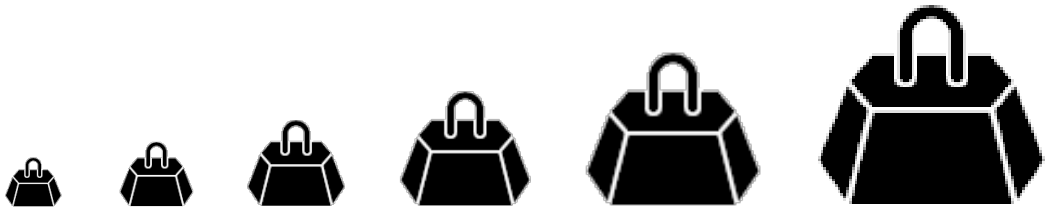
A

ans

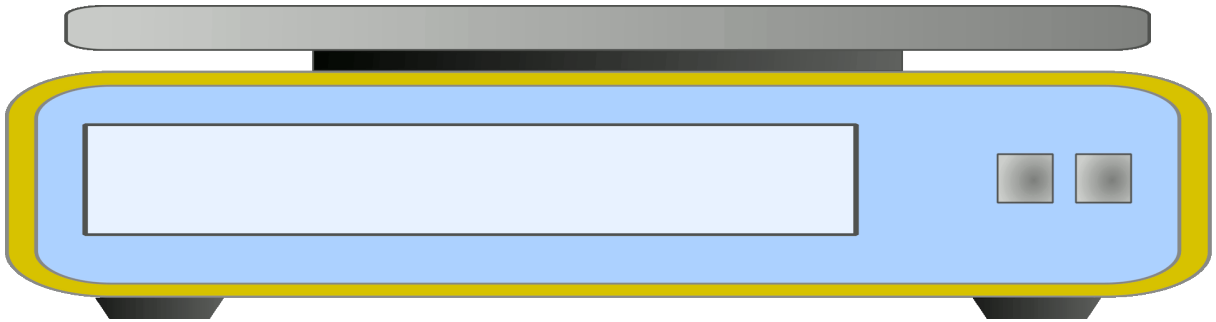
$$\sum_{n=0}^{\infty}\left(\frac{2^{2n}}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{4^n}{3^n}\right)$$
$$= \sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$$

Since $\left|\frac{4}{3}\right| > 1$, the series diverges.

$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

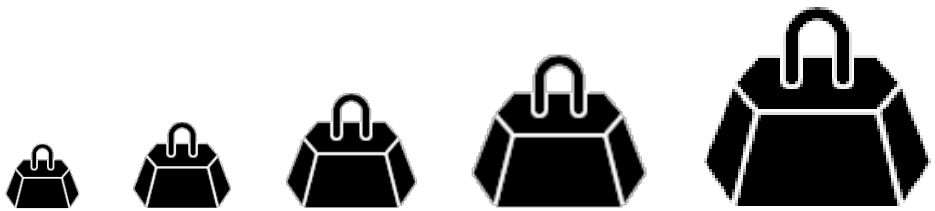


1 $\frac{4}{3}$ $\frac{16}{9}$ $\frac{64}{27}$ $\frac{256}{81}$ $\frac{1024}{243}$

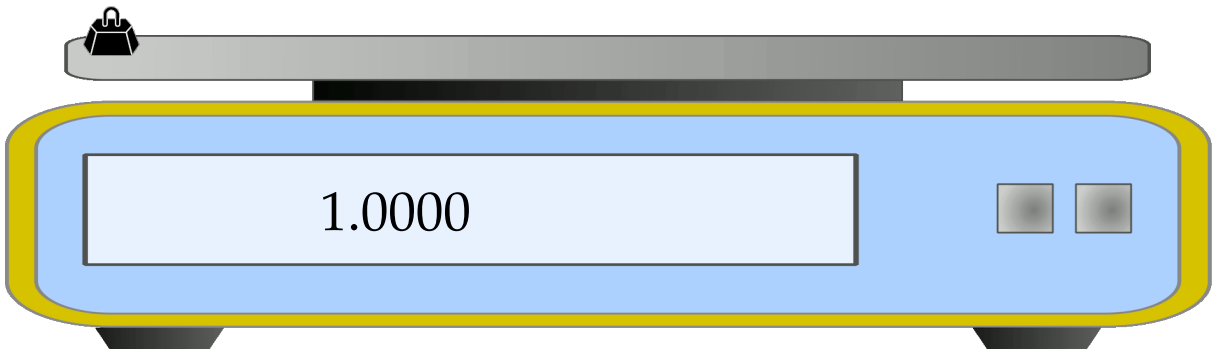


$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$S_0 = 1.0000$



$\frac{4}{3}$ $\frac{16}{9}$ $\frac{64}{27}$ $\frac{256}{81}$ $\frac{1024}{243}$

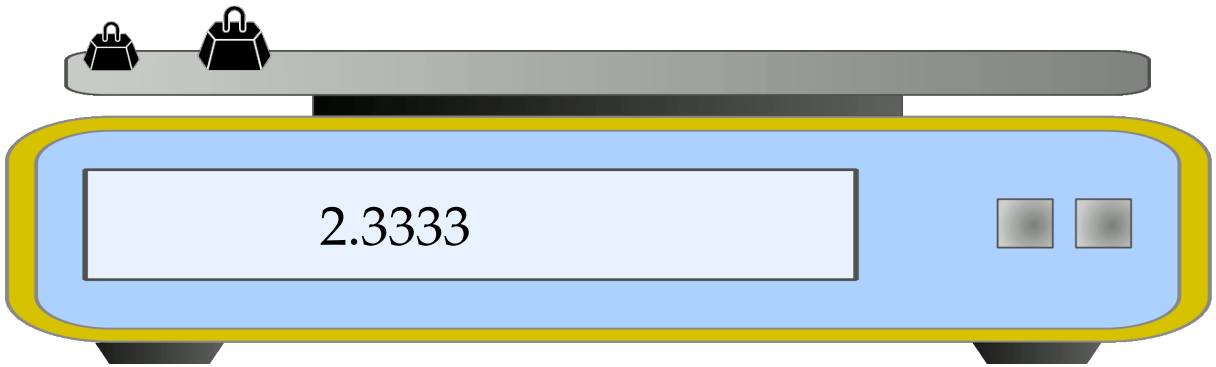


$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$S_0 = 1.0000$

$S_1 = 2.3333$

$\frac{16}{9}$ $\frac{64}{27}$ $\frac{256}{81}$ $\frac{1024}{243}$



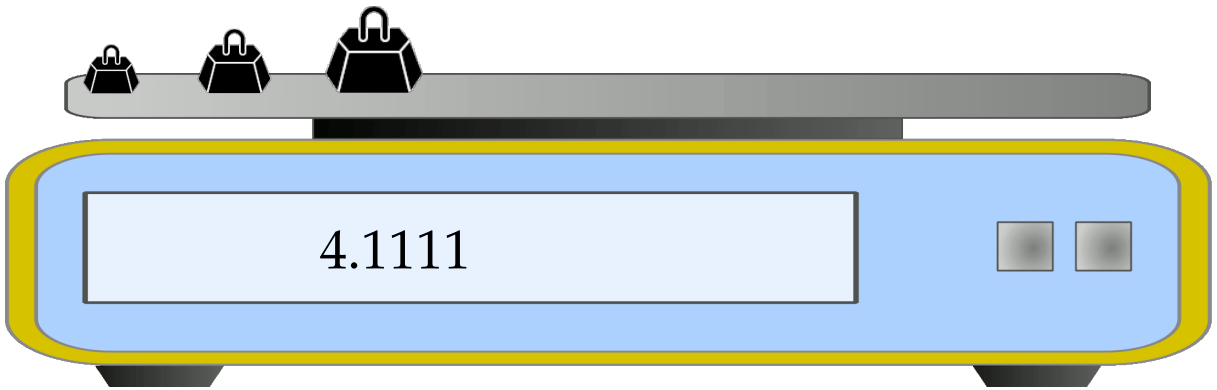
$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$S_0 = 1.0000$

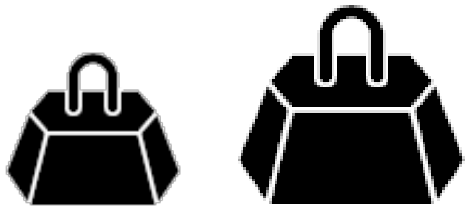
$S_1 = 2.3333$

$S_2 = 4.1111$

$\frac{64}{27}$ $\frac{256}{81}$ $\frac{1024}{243}$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$



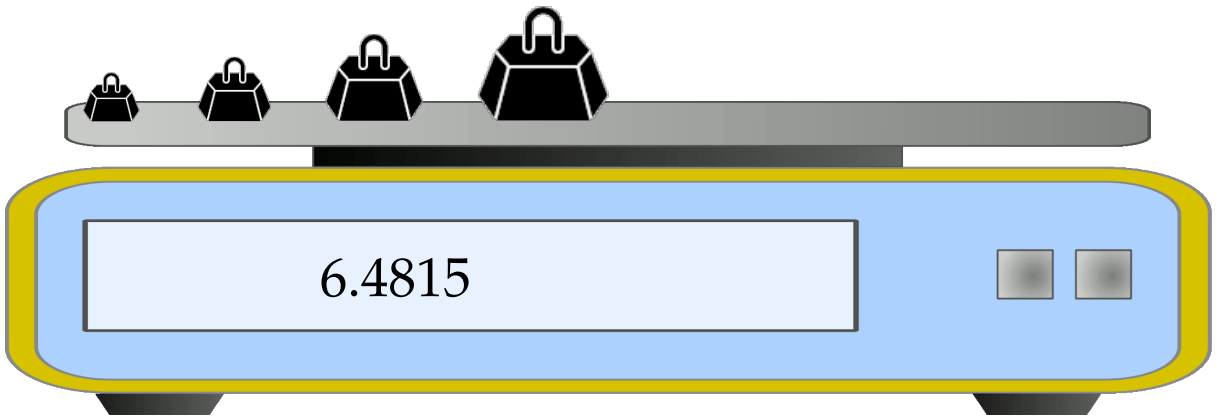
$\frac{256}{81}$ $\frac{1024}{243}$

$$S_0 = 1.0000$$

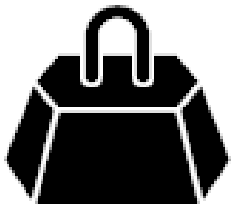
$$S_1 = 2.3333$$

$$S_2 = 4.1111$$

$$S_3 = 6.4815$$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$



$\frac{1024}{243}$

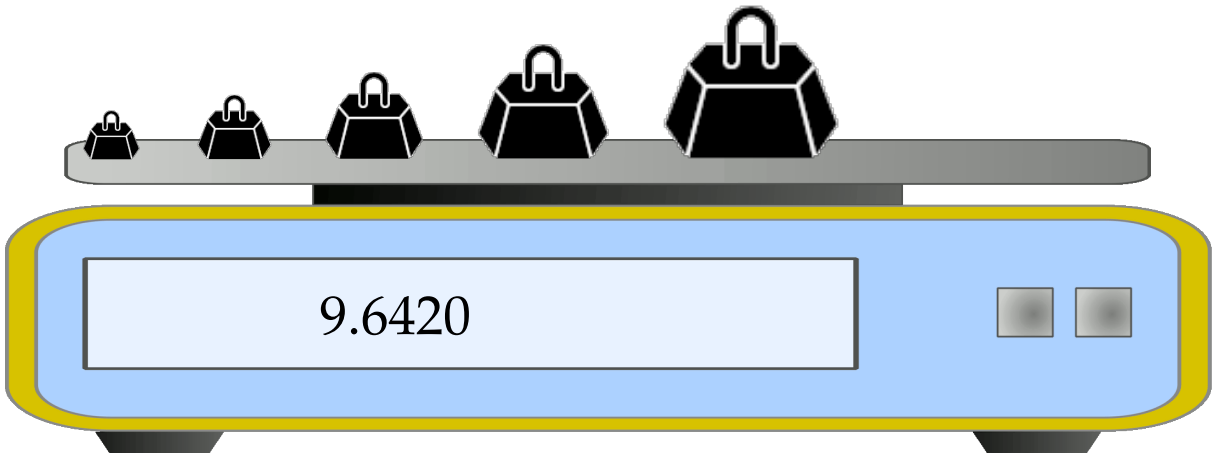
$S_0 = 1.0000$

$S_1 = 2.3333$

$S_2 = 4.1111$

$S_3 = 6.4815$

$S_4 = 9.6420$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$S_0 = 1.0000$

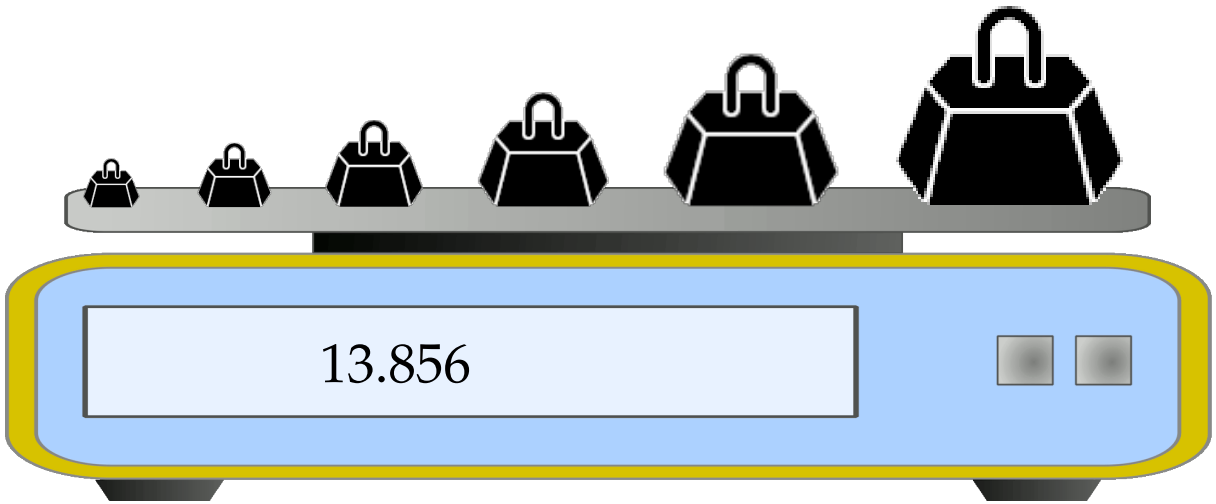
$S_1 = 2.3333$

$S_2 = 4.1111$

$S_3 = 6.4815$

$S_4 = 9.6420$

$S_5 = 13.856$



Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

Q

ans

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
a_1 &: \quad \tfrac{1}{1} \quad - \quad \tfrac{1}{2} \\
a_2 &: \quad \tfrac{1}{2} \quad - \quad \tfrac{1}{3} \\
a_3 &: \quad \tfrac{1}{3} \quad - \quad \tfrac{1}{4} \\
a_4 &: \quad \tfrac{1}{4} \quad - \quad \tfrac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \tfrac{1}{N-1} \quad - \quad \tfrac{1}{N} \\
a_N &: \quad \tfrac{1}{N} \quad - \quad \tfrac{1}{N+1}
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \quad \tfrac{1}{1} \quad - \quad \tfrac{1}{2} \\
S_2 &= \\
S_3 &= \\
S_4 &= \\
&\vdots \\
S_N &=
\end{aligned}
$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

Q

ans

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
a_1 &: \quad \frac{1}{1} - \frac{1}{2} \\
a_2 &: \quad \frac{1}{2} - \frac{1}{3} \\
a_3 &: \quad \frac{1}{3} - \frac{1}{4} \\
a_4 &: \quad \frac{1}{4} - \frac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \frac{1}{N-1} - \frac{1}{N} \\
a_N &: \quad \frac{1}{N} - \frac{1}{N+1}
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \frac{1}{1} - \frac{1}{2} \\
S_2 &= \frac{1}{1} - \frac{1}{3} \\
S_3 &= \\
S_4 &= \\
&\vdots \\
S_N &=
\end{aligned}
$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
a_1 &: \quad \tfrac{1}{1} - \tfrac{1}{2} \\
a_2 &: \quad \tfrac{1}{2} - \tfrac{1}{3} \\
a_3 &: \quad \tfrac{1}{3} - \tfrac{1}{4} \\
a_4 &: \quad \tfrac{1}{4} - \tfrac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \tfrac{1}{N-1} - \tfrac{1}{N} \\
a_N &: \quad \tfrac{1}{N} - \tfrac{1}{N+1}
\end{aligned}
$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
S_1 &= \tfrac{1}{1} - \tfrac{1}{2} \\
S_2 &= \tfrac{1}{1} - \tfrac{1}{3} \\
S_3 &= \tfrac{1}{1} - \tfrac{1}{4} \\
S_4 &= \\
&\vdots \\
S_N &=
\end{aligned}
$$

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
a_1 &: \quad \tfrac{1}{1} - \tfrac{1}{2} \\
a_2 &: \quad \tfrac{1}{2} - \tfrac{1}{3} \\
a_3 &: \quad \tfrac{1}{3} - \tfrac{1}{4} \\
a_4 &: \quad \tfrac{1}{4} - \tfrac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \tfrac{1}{N-1} - \tfrac{1}{N} \\
a_N &: \quad \tfrac{1}{N} - \tfrac{1}{N+1}
\end{aligned}
$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
S_1 &= \tfrac{1}{1} - \tfrac{1}{2} \\
S_2 &= \tfrac{1}{1} - \tfrac{1}{3} \\
S_3 &= \tfrac{1}{1} - \tfrac{1}{4} \\
S_4 &= \tfrac{1}{1} - \tfrac{1}{5} \\
&\vdots \\
S_N &=
\end{aligned}
$$

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
a_1 &: \quad \tfrac{1}{1} - \tfrac{1}{2} \\
a_2 &: \quad \tfrac{1}{2} - \tfrac{1}{3} \\
a_3 &: \quad \tfrac{1}{3} - \tfrac{1}{4} \\
a_4 &: \quad \tfrac{1}{4} - \tfrac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \tfrac{1}{N-1} - \tfrac{1}{N} \\
a_N &: \quad \tfrac{1}{N} - \tfrac{1}{N+1}
\end{aligned}
$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
S_1 &= \tfrac{1}{1} - \tfrac{1}{2} \\
S_2 &= \tfrac{1}{1} - \tfrac{1}{3} \\
S_3 &= \tfrac{1}{1} - \tfrac{1}{4} \\
S_4 &= \tfrac{1}{1} - \tfrac{1}{5} \\
&\vdots \\
S_N &=
\end{aligned}
$$

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$
\begin{aligned}
a_1 &: \quad \tfrac{1}{1} - \tfrac{1}{2} \\
a_2 &: \quad \tfrac{1}{2} - \tfrac{1}{3} \\
a_3 &: \quad \tfrac{1}{3} - \tfrac{1}{4} \\
a_4 &: \quad \tfrac{1}{4} - \tfrac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \tfrac{1}{N-1} - \tfrac{1}{N} \\
a_N &: \quad \tfrac{1}{N} - \tfrac{1}{N+1}
\end{aligned}
\qquad
\begin{aligned}
S_1 &= \tfrac{1}{1} - \tfrac{1}{2} \\
S_2 &= \tfrac{1}{1} - \tfrac{1}{3} \\
S_3 &= \tfrac{1}{1} - \tfrac{1}{4} \\
S_4 &= \tfrac{1}{1} - \tfrac{1}{5} \\
&\vdots \\
S_N &= \tfrac{1}{1} - \tfrac{1}{N+1} = \tfrac{N}{N+1}
\end{aligned}
$$

Q

ans

Telescoping Sums

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{aligned}
a_1 &: \quad \tfrac{1}{1} - \tfrac{1}{2} \\
a_2 &: \quad \tfrac{1}{2} - \tfrac{1}{3} \\
a_3 &: \quad \tfrac{1}{3} - \tfrac{1}{4} \\
a_4 &: \quad \tfrac{1}{4} - \tfrac{1}{5} \\
&\vdots \\
a_{N-1} &: \quad \tfrac{1}{N-1} - \tfrac{1}{N} \\
a_N &: \quad \tfrac{1}{N} - \tfrac{1}{N+1}
\end{aligned}$$

$$\begin{aligned}
S_1 &= \tfrac{1}{1} - \tfrac{1}{2} \\
S_2 &= \tfrac{1}{1} - \tfrac{1}{3} \\
S_3 &= \tfrac{1}{1} - \tfrac{1}{4} \\
S_4 &= \tfrac{1}{1} - \tfrac{1}{5} \\
&\vdots \\
S_N &= \tfrac{1}{1} - \tfrac{1}{N+1} = \tfrac{N}{N+1}
\end{aligned}$$

$$\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right) = S_{800} = \frac{800}{801}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{N \to \infty} S_N = 1$$

Evaluate $\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right)$.

Evaluate $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$.

Evaluate $\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right)$.

Evaluate $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$.

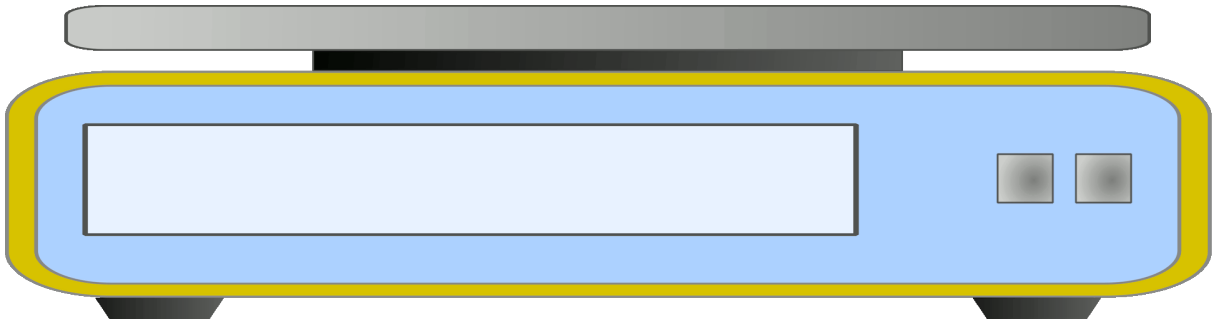
$$
\begin{array}{lcll}
a_1 & : & \log(2) - \cancel{\log(1)} & S_1 = \log(2) \\
a_2 & : & \log(3) - \log(2) & S_2 = \log(3) \\
a_3 & : & \log(4) - \log(3) & S_3 = \log(4) \\
\vdots & & & \\
a_{n-1} & : & \log(n) - \log(n-1) & \\
a_n & : & \log(n+1) - \log(n) & S_n = \log(n+1)
\end{array}
$$

So, $\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right) = S_{1000} = \log(1001)$ and

$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \log(n+1) = \infty$

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

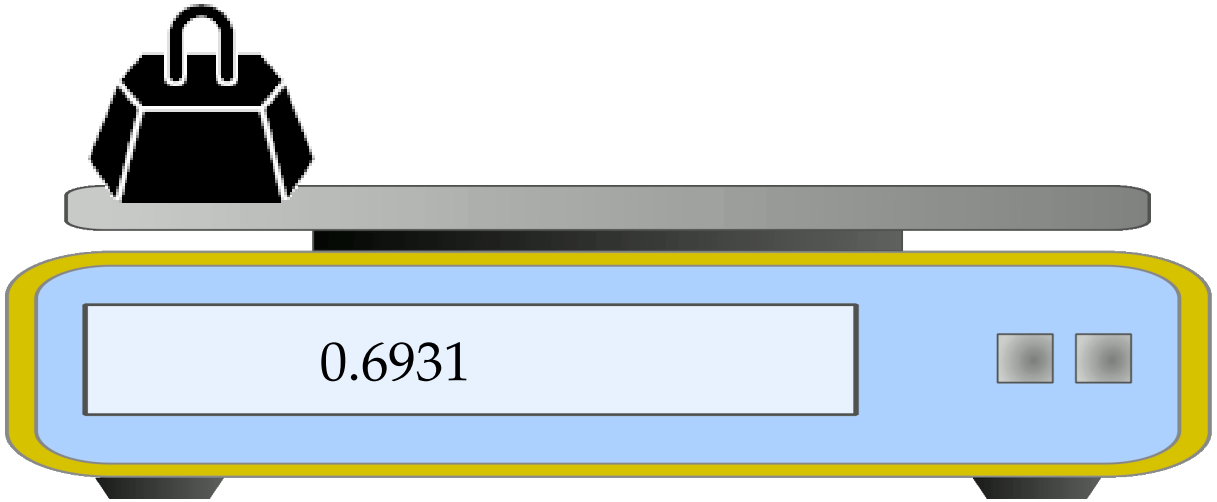
$\log 2$ $\log \frac{3}{2}$ $\log \frac{4}{3}$ $\log \frac{5}{4}$ $\log \frac{6}{5}$ $\log \frac{7}{6}$ $\log \frac{8}{7}$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

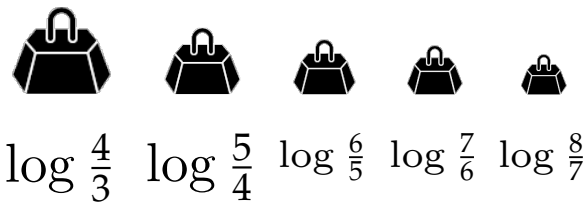
$\log \frac{3}{2}$ $\log \frac{4}{3}$ $\log \frac{5}{4}$ $\log \frac{6}{5}$ $\log \frac{7}{6}$ $\log \frac{8}{7}$

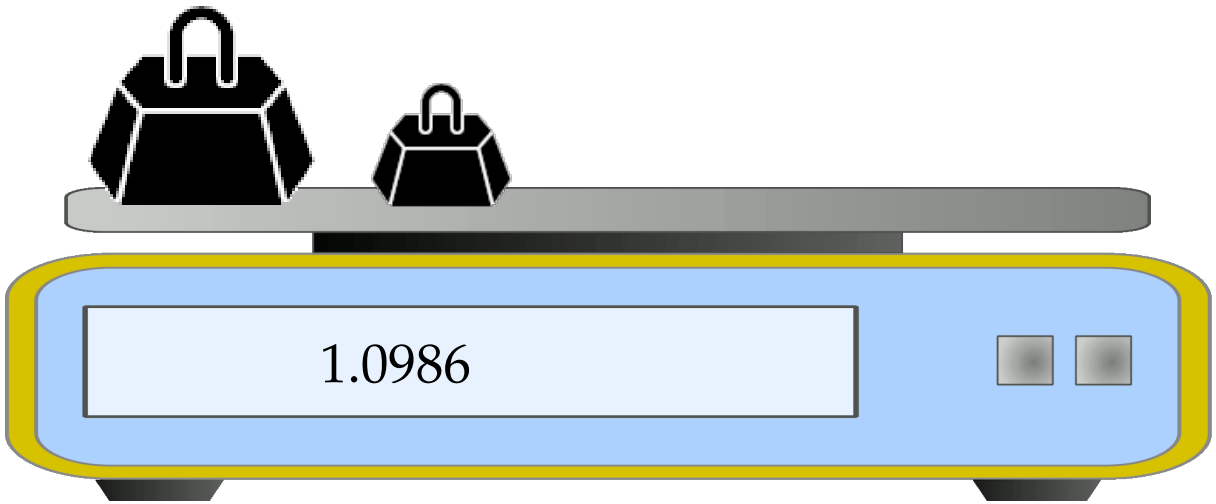


$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$





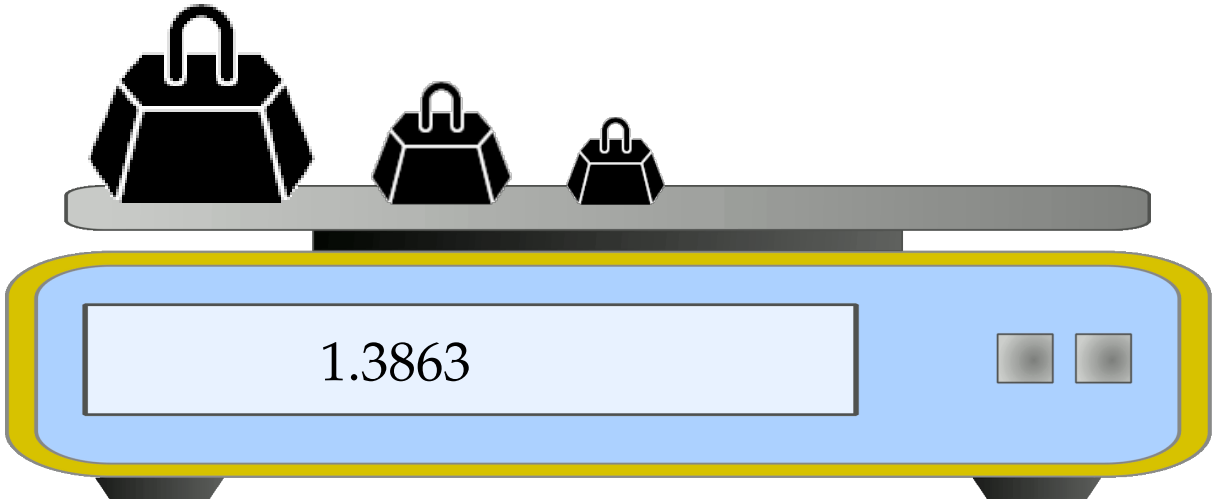
$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

$\log \frac{5}{4}$ $\log \frac{6}{5}$ $\log \frac{7}{6}$ $\log \frac{8}{7}$



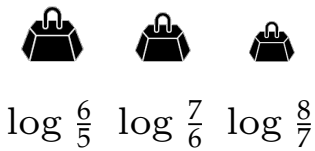
$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

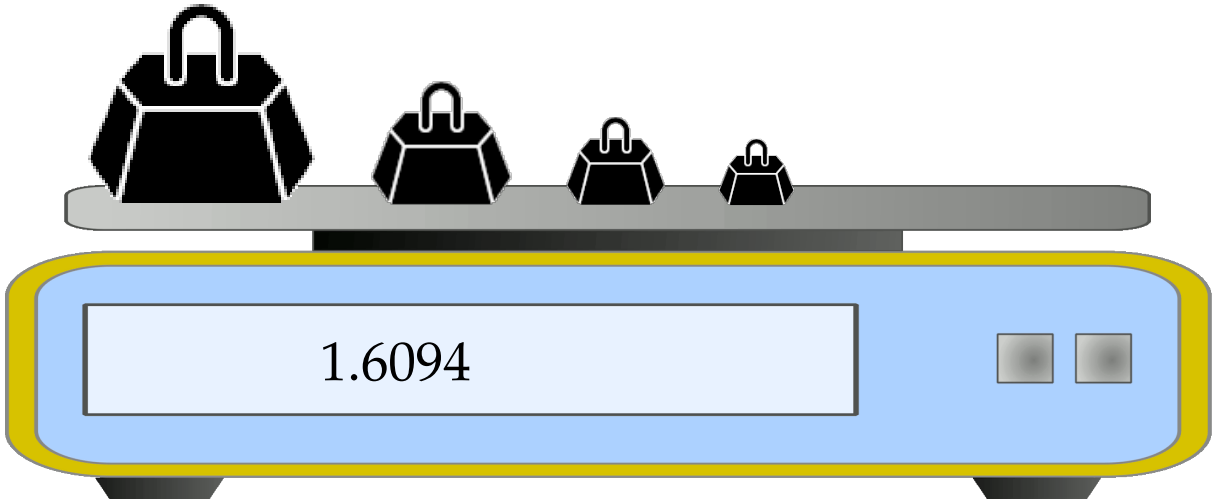
$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

$$S_4 = 1.6094$$





$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

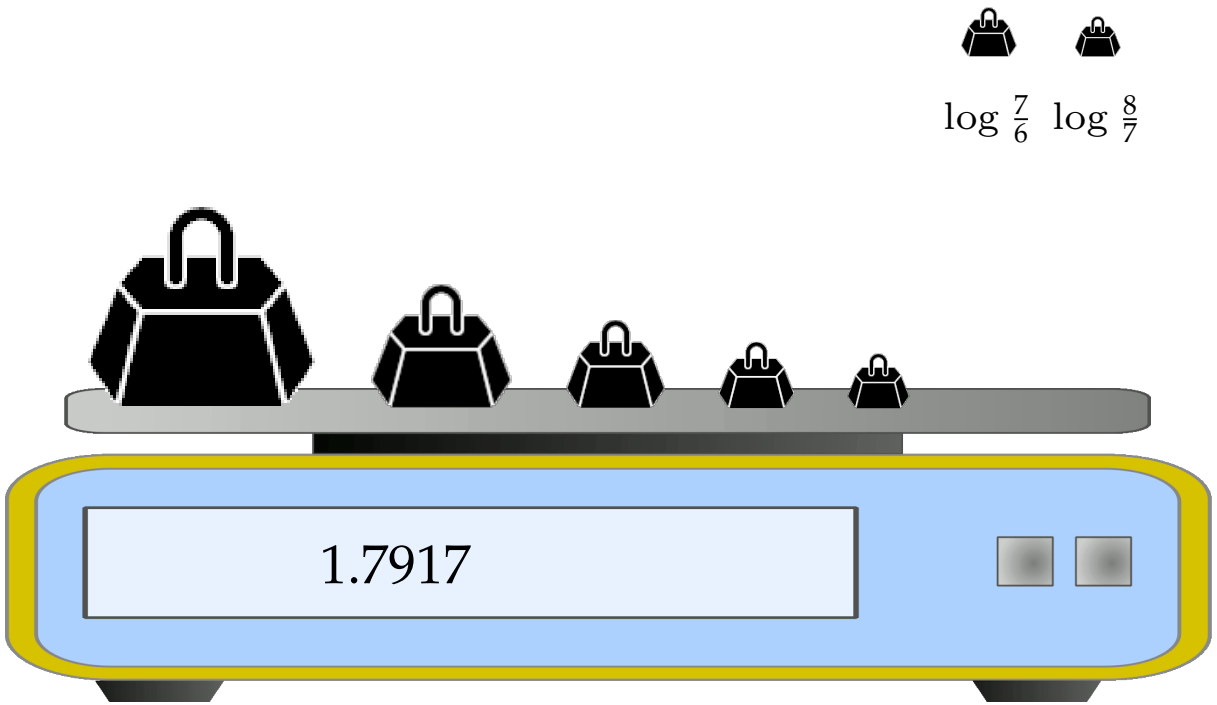
$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

$$S_4 = 1.6094$$

$$S_5 = 1.7917$$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

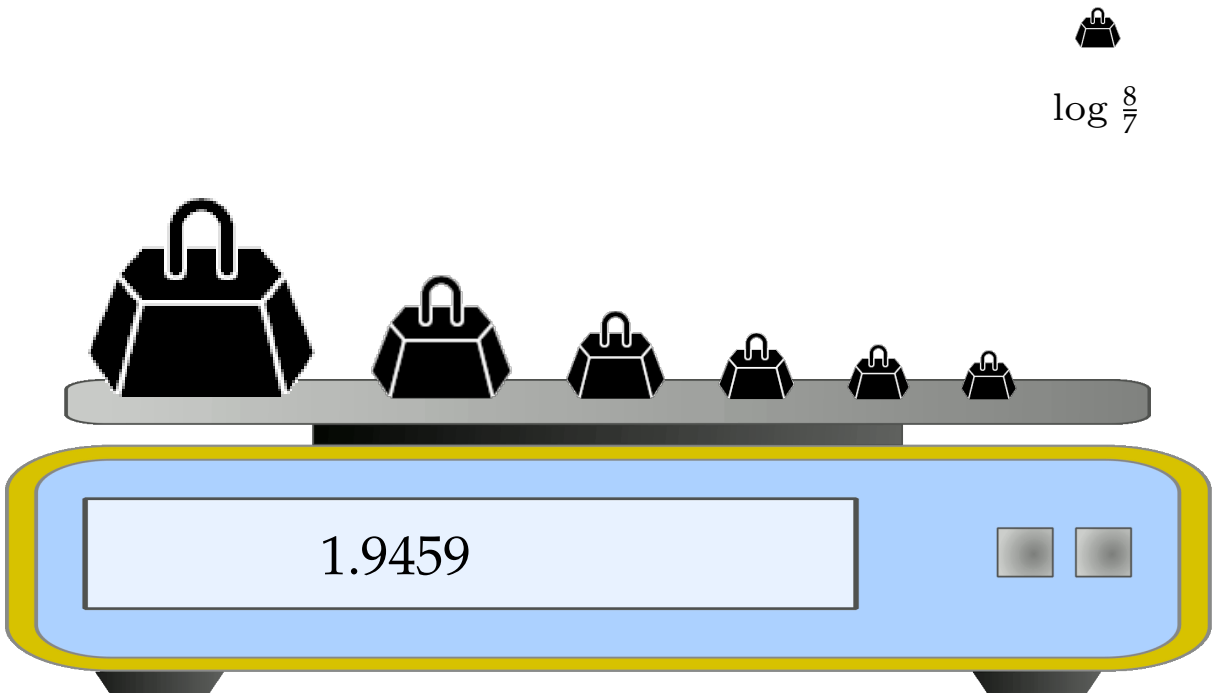
$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

$$S_4 = 1.6094$$

$$S_5 = 1.7917$$

$$S_6 = 1.9459$$



$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

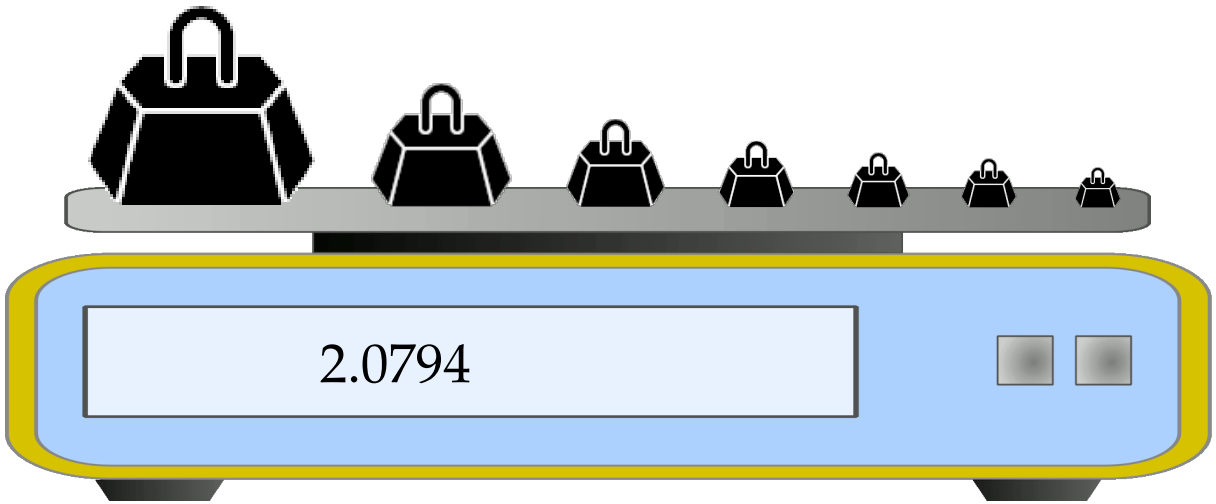
$$S_3 = 1.3863$$

$$S_4 = 1.6094$$

$$S_5 = 1.7917$$

$$S_6 = 1.9459$$

$$S_7 = 2.0794$$



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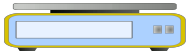 ‘Waage/Libra’ by B. Lachner is in the public domain (accessed April 2021, edited), 33–38, 52–58, 68–72, 84–114, 119–124, 135–141, 152–159

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