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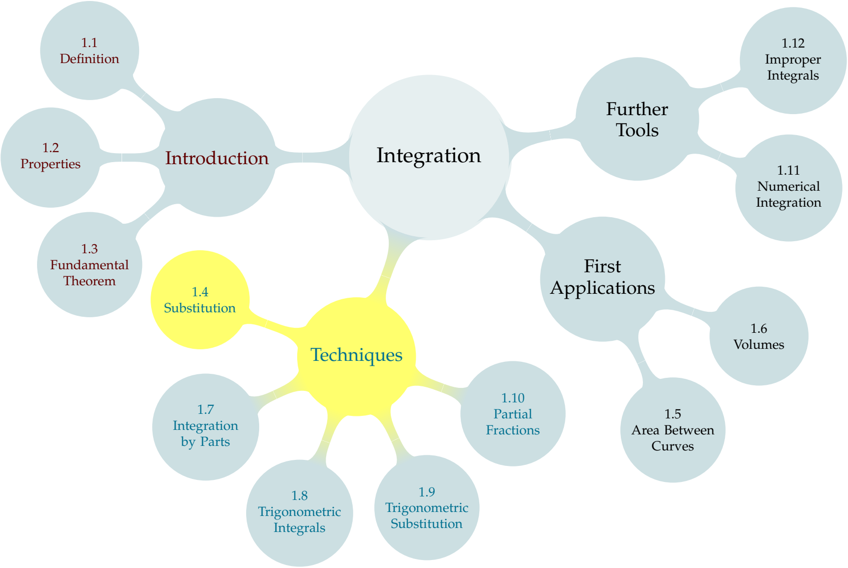

Antiderivatives

Fact:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sin\left(x^2+x\right)\right\} =$$

Related Fact:

$$\int (2x+1)\cos(x^2+x)\,\mathrm{d}x =$$

ANTIDERIVATIVES

Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sin\left(\underbrace{x^2+x}_{\text{inside function}}\right)\right\} = \left(\underbrace{2x+1}_{\substack{\text{derivative of}\\ \text{inside function}}}\right)\cos\left(\underbrace{x^2+x}_{\text{inside function}}\right)$$

Hallmark of the chain rule: an "inside" function, with that function's derivative multiplied.

Solve by Inspection

$$\int 2xe^{x^2+1}\,\mathrm{d}x = e^{x^2+1} + C$$

$$\int \frac{1}{x}\cos(\log x)\,\mathrm{d}x = \sin(\log x) + C$$

$$\int 3(\sin x + 1)^2 \cos x\,\mathrm{d}x = (\sin x + 1)^3 + C$$

(Look for an "inside" function, with its derivative multiplied.)

Undoing the Chain Rule

Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(u(x))\} = f'(u(x)) \cdot u'(x)$$

(Here, $u(x)$ is our "inside function")

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x)\, \mathrm{d}x = f(u(x)) + C$$

Undoing the Chain Rule

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x)\,\mathrm{d}x = f(u(x)) + C$$

Shorthand: call $u(x)$ simply u;
since $\frac{\mathrm{d}u}{\mathrm{d}x} = u'(x)$, call $u'(x)\,\mathrm{d}x$ simply $\mathrm{d}u$.

$$\int f'(u(x)) \cdot u'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u\Big|_{u=u(x)} = f\big(u(x)\big) + C$$

This is the substitution rule.

We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1}\,\mathrm{d}x$$

Using u as shorthand for x^2+1, and $\mathrm{d}u$ as shorthand for $2x\,\mathrm{d}x$:
$\int 2xe^{x^2+1}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u + C = e^{x^2+1} + C$

$$\int \frac{1}{x}\cos(\log x)\,\mathrm{d}x$$

Using u as shorthand for $\log x$, and $\mathrm{d}u$ as shorthand for $\frac{1}{x}\,\mathrm{d}x$:
$\int \frac{1}{x}\cos(\log x)\,\mathrm{d}x = \int \cos(u)\,\mathrm{d}u = \sin(u) + C = \sin(\log x) + C$

$$\int 3(\sin x + 1)^2 \cos x\,\mathrm{d}x$$

Using u as shorthand for $\sin x + 1$, and $\mathrm{d}u$ as shorthand for $\cos x\,\mathrm{d}x$:
$\int 3(\sin x + 1)^2 \cos x\,\mathrm{d}x = \int 3u^2\,\mathrm{d}u = u^3 + C = (\sin x + 1)^3 + C$

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x =$$

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u\Big|_{u=x^3+1}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u, \quad 3x^2\,\mathrm{d}x \to \mathrm{d}u$

$$\begin{aligned}\int (3x^2)\sin(x^3+1)\,\mathrm{d}x &= \left.\int \sin(u)\,\mathrm{d}u\right|_{u=x^3+1}\\ &= -\cos(u)+C|_{u=x^3+1}\\ &= \cos(x^3+1)+C\end{aligned}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u, \quad 3x^2\,\mathrm{d}x \to \mathrm{d}u$

Warning 1: We don't just change dx to du. We need to couple dx with the derivative of our inside function.
After all, we're undoing the chain rule! We need to have an "inside derivative."

Warning 2: The final answer is a function of x.

We used the substitution rule to conclude

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = -\cos(x^3+1) + C$$

We can check that our antiderivative is correct by differentiating.

We saw:

$$\int 3x^2 \sin(x^3+1)\,\mathrm{d}x = -\cos(x^3+1) + C$$

So, we can evaluate:

$$\int_0^1 3x^2 \sin(x^3+1)\,\mathrm{d}x = -\cos(x^3+1)\Big|_0^1 = \cos(1) - \cos(2)$$

Alternately, we can put in the limits of integration as we substitute. The bounds are originally given as values of x; we simply change them to values of u.

If $u(x) = x^3 + 1$, then $u(0) = 1$ and $u(1) = 2$.

$$\underbrace{\int_0^1}_{x\text{-values}} 3x^2 \sin(x^3+1)\,\mathrm{d}x = \underbrace{\int_1^2}_{u\text{-values}} \sin(u)\,\mathrm{d}u = -\cos(2) + \cos(1)$$

Notation: limits of integration

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x$$

Let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$. Note the limits (or bounds) of integration $\pi/4$ and $\pi/2$ are values of x, not u: they follow the differential, unless otherwise specified.

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x$$

$$\begin{array}{l} x=\frac{\pi}{2} \\ x=\frac{\pi}{4} \end{array}$$

TRUE OR FALSE?

1. Using $u = x^2$,

$$\int e^{x^2}\,dx = \int e^u\,du$$

False: missing $u'(x)$.
$du = (2x\,dx) \neq\ dx$

2. Using $u = x^2 + 1$,

$$\int_0^1 x\sin(x^2+1)\,dx = \int_0^1 \frac{1}{2}\sin u\ \ du$$

False: limits of integration didn't translate.
When $x = 0$, $u = 0^2 + 1 = 1$, and when $x = 1$, $u = 1^2 + 1 = 2$.

Evaluate $\displaystyle\int_0^1 x^7 \left(x^4+1\right)^5 \,\mathrm{d}x$.

$$
\begin{aligned}
u &= x^4+1,\ \mathrm{d}u = 4x^3\,\mathrm{d}x \\
u(0) &= 1,\ u(1) = 2 \\
x^4 &= u-1,\ x^3\,\mathrm{d}x = \tfrac{1}{4}\,\mathrm{d}u \\
\int_0^1 x^7 \left(x^4+1\right)^5 \mathrm{d}x &= \int_0^1 (x^4)\cdot(x^4+1)^5\cdot x^3\,\mathrm{d}x \\
&= \int_1^2 (u-1)\cdot u^5 \cdot \tfrac{1}{4}\,\mathrm{d}u \\
&= \tfrac{1}{4}\int_1^2 (u^6-u^5)\,\mathrm{d}u \\
&= \tfrac{1}{4}\left[\tfrac{1}{7}u^7 - \tfrac{1}{6}u^6\right]_1^2 \\
&= \tfrac{1}{4}\left[\tfrac{2^7}{7} - \tfrac{2^6}{6} - \tfrac{1}{7} + \tfrac{1}{6}\right]
\end{aligned}
$$

Time permitting, more examples using the substitution rule

$$\text{Evaluate } \int \sin x \cos x \, dx.$$

Let $u = \sin x$, $du = \cos x \, dx$:

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Or, let $u = \cos x$, $du = -\sin x \, dx$:

$$\int \cos x \sin x \, dx = -\int u \, du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$$

Recall $\sin^2 x + \cos^2 x = 1$ for all x, so $\frac{1}{2}\sin^2 x = -\frac{1}{2}\cos^2 x + \frac{1}{2}$. The two answers look different, but they only differ by a constant, which can be absorbed in the arbitrary constant C. If we rename the second C to C' so that the second answer is $-\frac{1}{2}\cos^2 x + C'$, then $C' = C + \frac{1}{2}$.

Check Our Work

We can check that $\int \sin x \cos x \, \mathrm{d}x =$ by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{2}\sin^2 x + C\right\} = \frac{2}{2}\sin x \cdot \cos x = \sin x \cos x$$

Our answer works.

We can check that $\int \sin x \cos x \, \mathrm{d}x =$ by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{-\frac{1}{2}\cos^2 x + C\right\} = -\frac{2}{2}\cos x \cdot (-\sin x) = \sin x \cos x$$

This answer works too.

$$\text{Evaluate } \int \frac{\log x}{3x}\,\mathrm{d}x.$$

Let $u = \log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$:

$$\begin{aligned}\int \frac{\log x}{3} \cdot \frac{1}{x}\,\mathrm{d}x &= \frac{1}{3}\int u \;\mathrm{d}u \\ &= \frac{1}{6}u^2 + C \\ &= \frac{1}{6}\log^2 x + C\end{aligned}$$

Check Our Work

We can check that $\displaystyle\int \frac{\log x}{3x}\,\mathrm{d}x =$ by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{6}\log^2 x + C\right\} = \frac{2}{6}\log x \cdot \frac{1}{x} = \frac{\log x}{3x}$$

Our answer works.

$$\text{Evaluate } \int \frac{e^x}{e^x + 15}\,\mathrm{d}x.$$

Let $u = e^x + 15$, $\mathrm{d}u = e^x\,\mathrm{d}x$

$$\int \frac{e^x}{e^x + 15}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \log|u| + C = \log|e^x + 15| + C$$

In this case, since $e^x + 15 > 0$, the absolute values on $|e^x + 15|$ are optional.

$$\text{Evaluate } \int x^4(x^5+1)^8\,\mathrm{d}x.$$

Let $u = x^5 + 1$, $\mathrm{d}u = 5x^4\,\mathrm{d}x$. Then, $x^4\,\mathrm{d}x = \frac{1}{5}\,\mathrm{d}u$.

$$\begin{aligned}\int x^4(x^5+1)^8\,\mathrm{d}x &= \int \frac{1}{5}(u)^8\,\mathrm{d}u \\ &= \frac{1}{5}\cdot\frac{1}{9}u^9 + C = \frac{1}{45}(x^5+1)^9 + C\end{aligned}$$

QQ

Check Our Work

We can check that $\displaystyle\int \frac{e^x}{e^x+15}\,\mathrm{d}x =$ by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log|e^x+15|+C\right\} = \frac{1}{e^x+15}\cdot e^x = \frac{e^x}{e^x+15}$$

Our answer works.

We can check that $\displaystyle\int x^4(x^5+1)^8\,\mathrm{d}x =$ by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{45}(x^5+1)^9+C\right\} = \frac{9}{45}(x^5+1)^8\cdot 5x^4 = (x^5+1)^8x^4$$

Our answer works.

Evaluate $\displaystyle\int_4^8 \frac{s}{s-3}\,\mathrm{d}s$. Be careful to use correct notation.

Let $u = s-3$, $\mathrm{d}u = \mathrm{d}s$.
Then $s = u+3,\ u(4) = 1$ and $u(8) = 5$.

$$\begin{aligned}
\int_4^8 \frac{s}{s-3}\,\mathrm{d}s &= \int_1^5 \frac{u+3}{u}\,\mathrm{d}u \\
&= \int_1^5 \left(1+\frac{3}{u}\right)\,\mathrm{d}u \\
&= \left[u + 3\log|u|\right]_1^5 \\
&= [5+3\log 5] - [1+3\log 1] \\
&= 4+3\log 5
\end{aligned}$$

Evaluate $\displaystyle\int x^9(x^5+1)^8\,\mathrm{d}x.$

Let $u = x^5+1$, $\mathrm{d}u = 5x^4\,\mathrm{d}x$.
Then $x^4\,\mathrm{d}x = \frac{1}{5}\,\mathrm{d}u$, $x^5 = u-1$.

$$\begin{aligned}
\int x^9(x^5+1)^8\,\mathrm{d}x &= \int (x^4)\cdot(x^5)\cdot(x^5+1)^8\,\mathrm{d}x \\
&= \int \left(\frac{1}{5}\right)\cdot(u-1)\cdot u^8\,\mathrm{d}u = \frac{1}{5}\int (u^9-u^8)\,\mathrm{d}u \\
&= \frac{1}{5}\left[\frac{1}{10}u^{10} - \frac{1}{9}u^9\right] + C \\
&= \frac{1}{5}\left[\frac{(x^5+1)^{10}}{10} - \frac{(x^5+1)^9}{9}\right] + C
\end{aligned}$$

Q

Check Our Work

We can check that $\int x^9(x^5+1)^8 \, \mathrm{d}x =$

by differentiating.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{5}\left[\frac{(x^5+1)^{10}}{10}-\frac{(x^5+1)^9}{9}\right]+C\right\} \\
&\quad=\frac{1}{5}\left[(x^5+1)^9\cdot 5x^4-(x^5+1)^8\cdot 5x^4\right] \\
&\quad=x^4(x^5+1)^9-x^4(x^5+1)^8 \\
&\quad=x^4(x^5+1)^8\left[(x^5+1)-1\right] \\
&\quad=x^4(x^5+1)^8[x^5] \\
&\quad=x^9(x^5+1)^8
\end{aligned}
$$

Our answer works.

Particularly Tricky Substitution

Evaluate $\displaystyle\int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x.$

Let $u = e^x$, $\mathrm{d}u = e^x\,\mathrm{d}x$. Then $\mathrm{d}x = \frac{\mathrm{d}u}{e^x} = \frac{\mathrm{d}u}{u}$.

$$\begin{aligned}
\int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x &= \int \frac{1}{u + \frac{1}{u}}\,\frac{\mathrm{d}u}{u} \\
&= \int \frac{1}{u^2 + 1}\,\mathrm{d}u \\
&= \arctan(u) + C \\
&= \arctan(e^x) + C
\end{aligned}$$

Check Our Work

We can check that $\int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x =$ by differentiating.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\arctan(e^x) + C\right\} &= \frac{1}{(e^x)^2 + 1} \cdot e^x \\ &= \frac{e^x}{(e^x)^2 + 1} \\ &= \frac{e^x}{(e^x)^2 + 1} \cdot \frac{e^{-x}}{e^{-x}} \\ &= \frac{1}{e^x + e^{-x}}\end{aligned}$$

Our answer works.