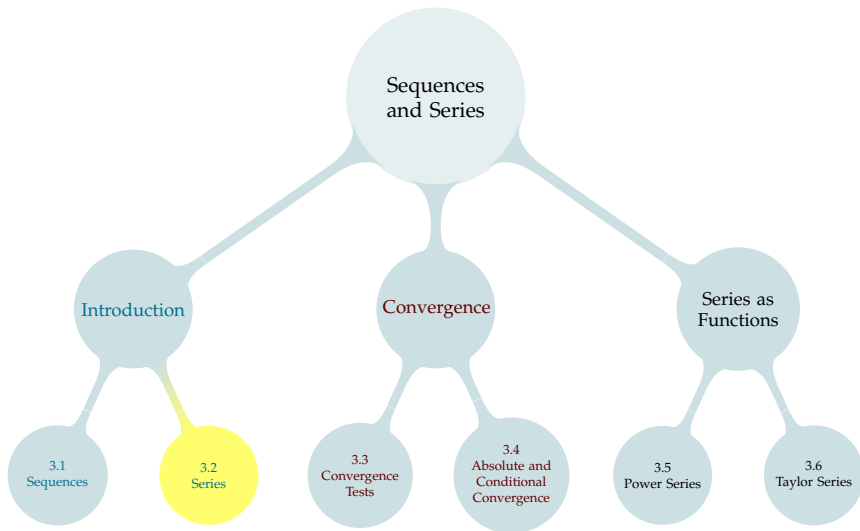


# TABLE OF CONTENTS



# SEQUENCES AND SERIES

A **sequence** is a list of numbers  
A **series** is the sum of such a list.

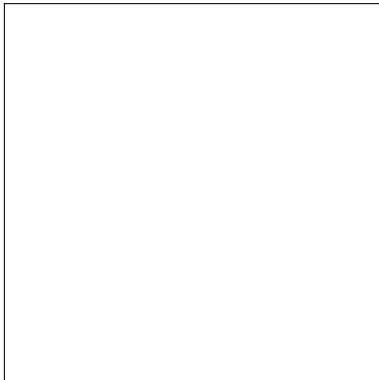
# SEQUENCES AND SERIES

## Sequence

**List of numbers,**  
approaching

Series

Sum of numbers,  
approaching



Square of Area 1

## QUICK REVIEW: SIGMA NOTATION

Recall:

$$\sum_{n=1}^5 \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

(a more rigorous definition will be discussed soon)

Let  $a_n$  and  $b_n$  be sequences, and let  $C$  be a constant.

$$\sum_{n=1}^{\infty} (C \cdot a_n) =$$

$$\text{A. } \sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$$

$$\text{B. } \sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$$

$$\text{C. } C \sum_{n=1}^{\infty} a_n$$

$$\text{D. } a_n \sum_{n=1}^{\infty} C$$

E. none of the above

Let  $a_n$  and  $b_n$  be sequences, and let  $C$  be a constant.

$$\sum_{n=1}^{\infty} (a_n + b_n) =$$

$$\text{A. } \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$$

$$\text{B. } \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

C.  $a_n + \sum_{n=1}^{\infty} b_n$

$$\text{D. } a_n \sum_{n=1}^{\infty} b_n$$

E. none of the above

Let  $a_n$  and  $b_n$  be sequences, and let  $C$  be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

$$\text{A. } \sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$$

B.  $\left(\sum_{n=1}^{\infty} a_n\right)^C$

$$\text{C. } C^n \sum_{n=1}^{\infty} a_n$$

$$\text{D. } \sum_{n=1}^{\infty} C(a_n)^{C-1}$$

E. none of the above

## SERIES PHILOSOPHY

## What does it really mean to add up infinitely many things?

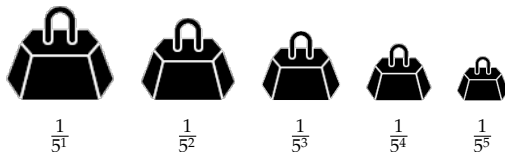
$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

We need an unambiguous definition.



# HOW CAN WE ADD UP INFINITELY MANY THINGS?

SEQUENCE OF PARTIAL SUMS



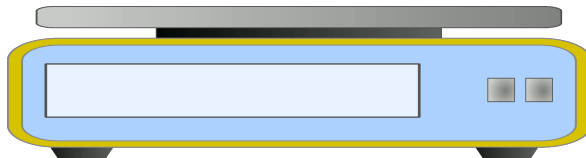
$$S_1 = 0.2000$$

$$S_2 = 0.2400$$

$$S_3 = 0.2480$$

$$S_4 = 0.2496$$

$$S_5 = 0.2499$$



**Partial sums** let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$

$$a_5 = \frac{1}{5^5} = 0.00032 \qquad S_5 = 0.24992$$

We define  $\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n = \lim_{N \rightarrow \infty} S_N$ .

$$a_1 = \frac{1}{5} = 0.2$$

$$S_1 = 0.2$$

$$a_5 = \frac{1}{5^5} = 0.00032 \quad S_5 = 0.24992$$

$$a_2 = \frac{1}{5^2} = 0.04$$

$$S_2 = 0.24$$

$$a_6 = \frac{1}{5^6} = 0.000064 \quad S_6 = 0.249984$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$S_3 = 0.248$$

$$a_7 = \frac{1}{5^7} = 0.0000128 \quad S_7 = 0.2499968$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$S_4 = 0.2496$$

$$a_8 = \frac{1}{5^8} = 0.00000256 \quad S_8 = 0.24999936$$

From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N =$$

NOTATION:  $S_N = \sum_{n=1}^N a_n$



$$S_1 = a_1$$

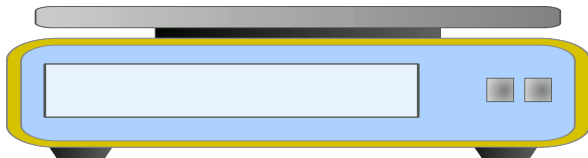
$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

$$S_5 = a_1 + \cdots + a_5$$

$$S_6 = a_1 + \cdots + a_6$$



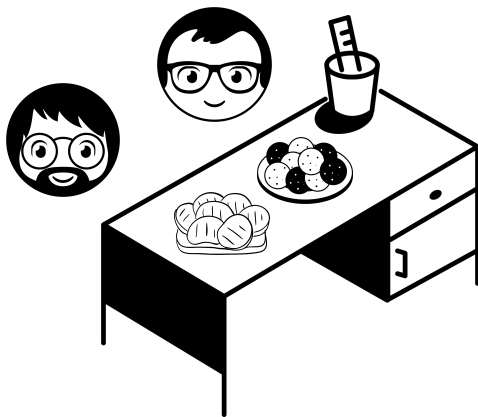
# NOTATION PRACTICE

Suppose  $\sum_{n=1}^{\infty} a_n$  has partial sums  $S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$ .

► Evaluate  $\sum_{n=1}^{100} a_n$ .

► Evaluate  $\sum_{n=1}^{\infty} a_n$ .

# NOTATION PRACTICE

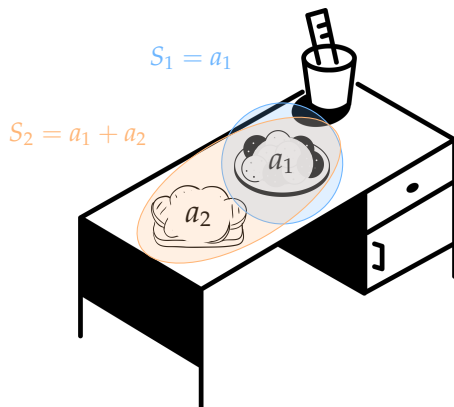


Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

# NOTATION PRACTICE



Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

## NOTATION PRACTICE

Suppose  $\sum_{n=1}^{\infty} a_n$  has partial sums  $S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$ .

- Find  $a_1$ .
- Give an explicit expression for  $a_n$ , when  $n > 1$ .



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$$S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$$


 $a_1$ 

 $a_2$ 

 $a_3$ 

 $a_4$ 

 $a_5$ 

 $a_6$ 

 $a_7$ 

 $a_8$ 

$$S_1 = 1/(1+1)$$

$$S_2 = 2/(2+1)$$

$$S_3 = 3/(3+1)$$

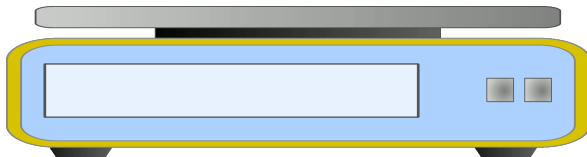
$$S_4 = 4/(4+1)$$

$$S_5 = 5/(5+1)$$

$$S_6 = 6/(6+1)$$

$$S_7 = 7/(7+1)$$

$$S_8 = 8/(8+1)$$



## Definition

The  $N^{\text{th}}$  partial sum of the series  $\sum_{n=1}^{\infty} a_n$  is the sum of its first  $N$  terms

$$S_N = \sum_{n=1}^N a_n.$$

The partial sums form a sequence  $\{S_N\}_{N=1}^{\infty}$ . If this sequence of partial sums converges  $S_N \rightarrow S$  as  $N \rightarrow \infty$  then we say that the series  $\sum_{n=1}^{\infty} a_n$  converges to  $S$  and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

## Geometric Series

Let  $a$  and  $r$  be two fixed real numbers with  $a \neq 0$ . The series

$$a + ar + ar^2 + ar^3 + \cdots$$

is called the **geometric series** with first term  $a$  and ratio  $r$ .

We call  $r$  the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

Another useful way of identifying geometric series is to determine whether all pairs of consecutive terms have the same ratio.

- ▶ Geometric:  $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$
- ▶ Geometric:  $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- ▶ Not geometric:  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

Consider the partial sum  $S_N$  of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

$$rS_N =$$

$$rS_N - S_N =$$

$$S_N(r - 1) = ar^{N+1} - a$$

If  $r \neq 1$ , then

$$S_N = \frac{ar^{N+1} - a}{r - 1} = a \frac{r^{N+1} - 1}{r - 1}$$

## Geometric Series and Partial Sums

Let  $a$  and  $r$  be constants with  $a \neq 0$ , and let  $N$  be a natural number.

► If  $r \neq 1$ , then  $a + ar + ar^2 + ar^3 + \cdots + ar^N = a \frac{r^{N+1} - 1}{r - 1}.$

► If  $r = 1$ , then  $a + ar + ar^2 + ar^3 + \cdots + ar^N =$

► If  $|r| < 1$ , then  $\sum_{n=0}^{\infty} ar^n =$

► If  $r = 1$ , then  $\sum_{n=0}^{\infty} ar^n$

► If  $r = -1$ , then  $\sum_{n=0}^{\infty} ar^n$

► If  $|r| > 1$ , then  $\sum_{n=0}^{\infty} ar^n$

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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$



$$S_0 = a$$

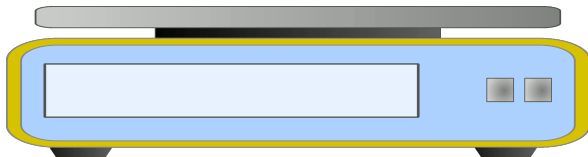
$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$

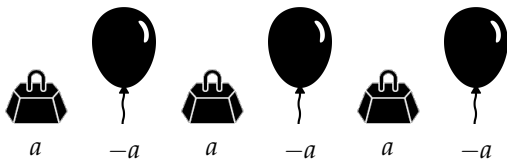
$$S_4 = 5a$$

$$S_5 = 6a$$



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$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$$S_0 = a$$

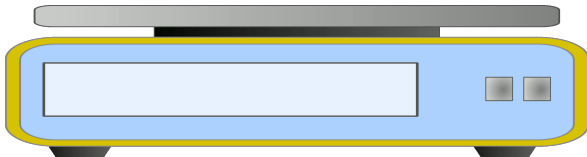
$$S_1 = 0$$

$$S_2 = a$$

$$S_3 = 0$$

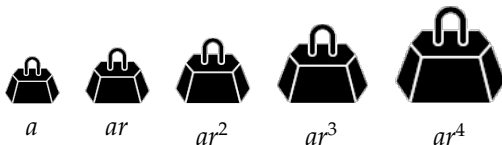
$$S_4 = a$$

$$S_5 = 0$$



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$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$



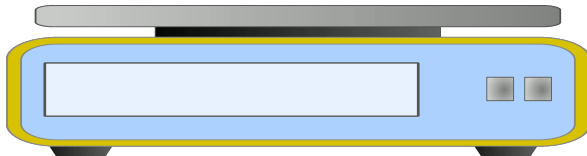
$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$

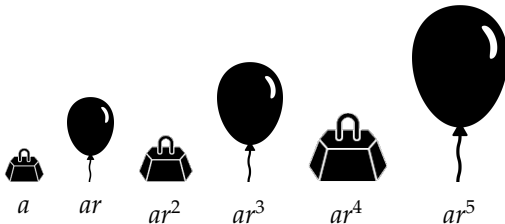
$$S_4 = a \frac{r^5 - 1}{r - 1}$$





oooooooooooooooooooooooo●oooooooo

$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

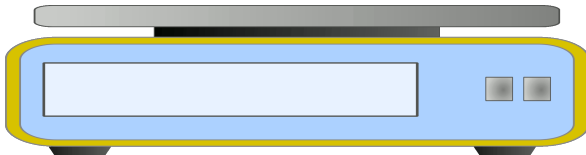
$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$

$$S_4 = a \frac{r^5 - 1}{r - 1}$$

$$S_5 = a \frac{r^6 - 1}{r - 1}$$



# GEOMETRIC SERIES

New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half.

- ▶ Each of the first 210,000 solutions can produce 50 bitcoins.
- ▶ Each of the next 210,000 solutions can produce  $\frac{50}{2}$  bitcoins.
- ▶ Each of the next 210,000 solutions can produce  $\frac{50}{2^2}$  bitcoins.
- ▶ Each of the next 210,000 solutions can produce  $\frac{50}{2^3}$  bitcoins.

Assume that this continues forever, and that bitcoins are infinitely divisible.<sup>1</sup> How many bitcoins can possibly be produced?

---

<sup>1</sup>Actually the smallest allowed division of a bitcoin is  $10^{-8}$ .

$$\sum_{n=0}^{\infty} 210\,000 \left( \frac{50}{2^n} \right) = 21\,000\,000$$



10 500 000



5 250 000



2 625 000



1 312 500



656 250

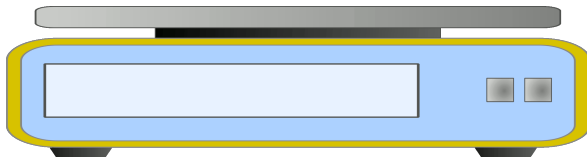
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

$$S_4 = 20\,343\,750$$



## Arithmetic of Series

Let  $S$ ,  $T$ , and  $C$  be real numbers. Let the two series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  converge to  $S$  and  $T$  respectively. Then

$$\sum_{n=1}^{\infty} [a_n + b_n] = S + T$$

$$\sum_{n=1}^{\infty} [a_n - b_n] = S - T$$

$$\sum_{n=1}^{\infty} [Ca_n] = CS$$

## Geometric Series and Partial Sums

Let  $a$  and  $r$  be fixed numbers, and let  $N$  be a positive integer. Then

$$\sum_{n=0}^N ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate  $\sum_{n=0}^{\infty} \left( \frac{2}{3^n} + \frac{4}{5^n} \right)$

## Geometric Series and Partial Sums

Let  $a$  and  $r$  be fixed numbers, and let  $N$  be a positive integer. Then

$$\sum_{n=0}^N ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate  $\sum_{n=6}^{\infty} \left( \frac{3^{n-1}}{5^{2n}} \right)$

## Geometric Series and Partial Sums

Let  $a$  and  $r$  be fixed numbers, and let  $N$  be a positive integer. Then

$$\sum_{n=0}^N ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate  $\sum_{n=0}^{\infty} \left( \frac{2^{2n}}{3^n} \right)$

## TELESCOPING SUMS

Evaluate  $\sum_{n=1}^{800} \left( \frac{1}{n} - \frac{1}{n+1} \right).$


Evaluate  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right).$





Evaluate  $\sum_{n=1}^{1000} \log \left( \frac{n+1}{n} \right)$ .

Evaluate  $\sum_{n=1}^{\infty} \log \left( \frac{n+1}{n} \right)$ .

## Included Work


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