Table of Contents

- Sequences and Series
  - Introduction
    - 3.1 Sequences
    - 3.2 Series
  - Convergence
    - 3.3 Convergence Tests
    - 3.4 Absolute and Conditional Convergence
  - Series as Functions
    - 3.5 Power Series
    - 3.6 Taylor Series

We can imagine the list of numbers below carrying on forever:

$$\begin{array}{l} 0.1 \\ 0.01 \\ 0.001 \\ 0.0001 \\ 0.00001 \\ \vdots \end{array}$$

A sequence is a list of infinitely many numbers with a specified order.

We can imagine the list of numbers below carrying on forever:

$$\begin{aligned} a_1 &= 0.1 \\ a_2 &= 0.01 \\ a_3 &= 0.001 \\ a_4 &= 0.0001 \\ a_5 &= 0.00001 \\ &\vdots \end{aligned}$$

A sequence is a list of infinitely many numbers with a specified order. It is denoted $\{a_1,\ a_2,\ \cdots, a_n,\ \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

We can imagine the list of numbers below carrying on forever:

$$
\begin{array}{rl}
 & 0.1 \\
+ & 0.01 \\
+ & 0.001 \\
+ & 0.0001 \\
+ & 0.00001 \\
 & \vdots \\
\hline
\end{array}
$$

A sequence is a list of infinitely many numbers with a specified order.
It is denoted $\{a_1,\ a_2,\ \cdots,a_n,\ \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.
Imagine *adding up* this sequence of numbers.

We can imagine the list of numbers below carrying on forever:

$$
\begin{array}{cl}
 & 0.1 \\
+ & 0.01 \\
+ & 0.001 \\
+ & 0.0001 \\
+ & 0.00001 \\
 & \vdots \\
\hline
 & 0.11111\cdots
\end{array}
$$

A sequence is a list of infinitely many numbers with a specified order.
It is denoted $\{a_1,\ a_2,\ \cdots,a_n,\ \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.
Imagine *adding up* this sequence of numbers.
A series is a sum $a_1 + a_2 + \cdots + a_n + \cdots$ of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- What does it mean to add up infinitely many things?
- Should infinitely many things add up to an infinitely large number?
- Does the order in which the numbers are added matter?
- Can we add up infinitely many functions, instead of just infinitely many numbers?

Sequence

A sequence is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)

Sequence

A sequence is a list of infinitely many numbers with a specified order. It is denoted $\{a_1, \ a_2, \ a_3, \ \cdots, a_n \ \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

Sequence

A sequence is a list of infinitely many numbers with a specified order. It is denoted $\{a_1,\ a_2,\ a_3,\ \cdots, a_n\ \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.

Sequence

A sequence is a list of infinitely many numbers with a specified order. It is denoted $\{a_1, a_2, a_3, \cdots, a_n \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.
- ∞: there is no end to our sequence.

Sequence

A sequence is a list of infinitely many numbers with a specified order. It is denoted $\{a_1,\ a_2,\ a_3,\ \cdots, a_n\ \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.
- ∞: there is no end to our sequence.
- $\frac{1}{n}$: this tells us the value of a_n.

Sequence

A sequence is a list of infinitely many numbers with a specified order. It is denoted $\{a_1,\ a_2,\ a_3,\ \cdots, a_n\ \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.
- ∞: there is no end to our sequence.
- $\frac{1}{n}$: this tells us the value of a_n.
- Often we omit the limits and even the brackets, writing $a_n = \frac{1}{n}$.

Sequence Notation

For convenience, we write a_1 for the first term of a sequence, a_2 for the second term, etc.

In the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots,$
a_3 is another name for

Sometimes we can find a rule for a sequence.
In the above sequence, $a_n =$

We can write $\{a_n\}_{n=1}^{\infty} =$

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to **converge** to the limit A if a_n approaches A as n tends to infinity. If so, we write

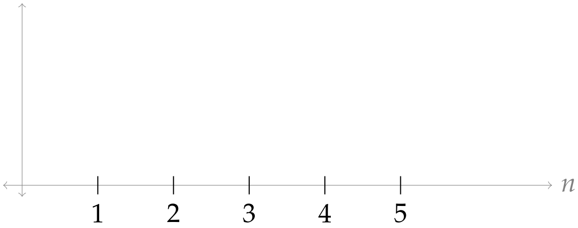
$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

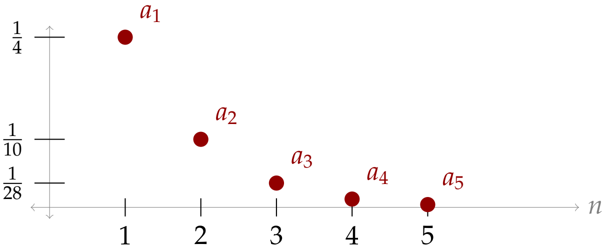
- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
  This sequence
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
  This sequence
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)
  This sequence

Does the sequence $a_n = \dfrac{n}{2n+1}$ converge or diverge?

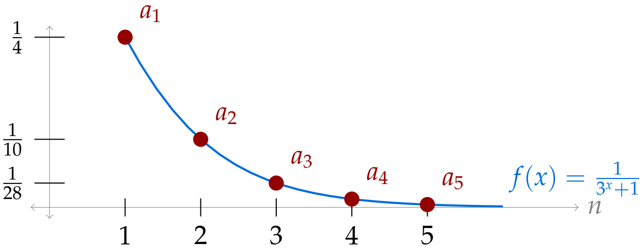
Consider the sequence $a_n = \dfrac{1}{3^n + 1}$. $\qquad \lim\limits_{n\to\infty} a_n =$



Consider the sequence $a_n = \dfrac{1}{3^n + 1}$. $\qquad \lim\limits_{n \to \infty} a_n =$



Consider the sequence $a_n = \dfrac{1}{3^n + 1}$. $\qquad \lim_{n\to\infty} a_n =$



Theorem 3.1.6

If

$$\lim_{x\to\infty} f(x) = L$$

and if $a_n = f(n)$ for all positive integers n, then

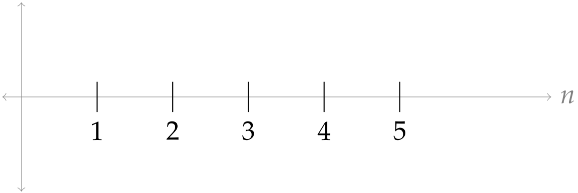
$$\lim_{n\to\infty} a_n = L$$

Cautionary Tale

Consider the sequence $b_n = \sin(\pi n) =$

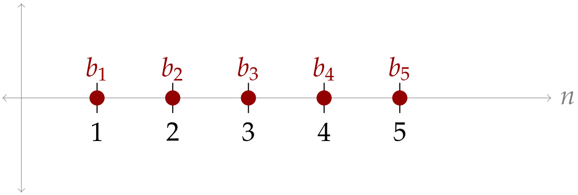
Cautionary Tale

Consider the sequence $b_n = \sin(\pi n) =$



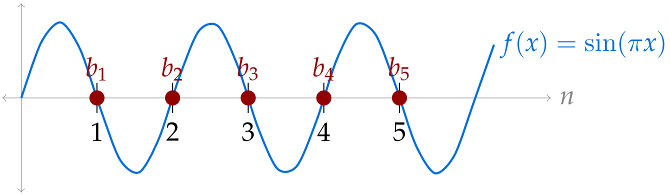
Cautionary Tale

Consider the sequence $b_n = \sin(\pi n) =$



Cautionary Tale

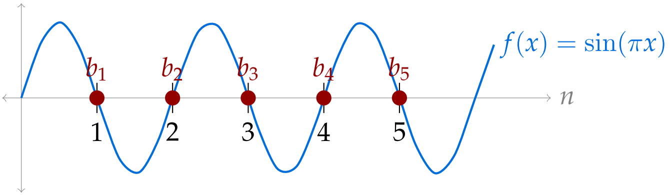
Consider the sequence $b_n = \sin(\pi n) =$



Cautionary Tale

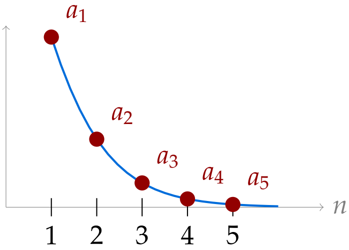
Consider the sequence $b_n = \sin(\pi n) =$

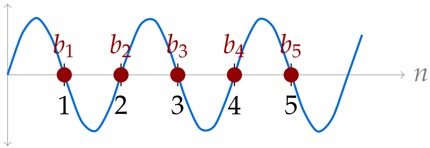
$\lim_{n\to\infty} b_n =$ $\lim_{x\to\infty} f(x)$



Theorem

If $\lim_{x \to \infty} f(x) = L$ and if $a_n = f(n)$ for all natural n, then $\lim_{n \to \infty} a_n = L$.





Arithmetic of Limits

Let A, B and C be real numbers and let the two sequences $\left\{a_n\right\}_{n=1}^{\infty}$ and $\left\{b_n\right\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that

$$\lim_{n\to\infty} a_n = A \qquad \lim_{n\to\infty} b_n = B$$

Then the following limits hold.

(a) $\lim_{n\to\infty} \left[a_n + b_n\right] = A + B$

(b) $\lim_{n\to\infty} \left[a_n - b_n\right] = A - B$

(c) $\lim_{n\to\infty} C a_n = CA.$

(d) $\lim_{n\to\infty} a_n \, b_n = A \, B$

(e) If $B \neq 0$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$

Evaluate the following limits:

- $\lim\limits_{n\to\infty} e^{-n} =$
- $\lim\limits_{n\to\infty} \frac{1+n}{n} =$
- $\lim\limits_{n\to\infty} \frac{1}{n^2} =$
- $\lim\limits_{n\to\infty} 2n^2 =$
- $\lim\limits_{n\to\infty} \left(\frac{1}{n^2}\right)\left(2n^2\right) =$

Continuous functions of limits

If $\lim_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Evaluate $\lim_{n\to\infty} \left[\sin\left(\frac{\pi n}{2n+1}\right)\right]$

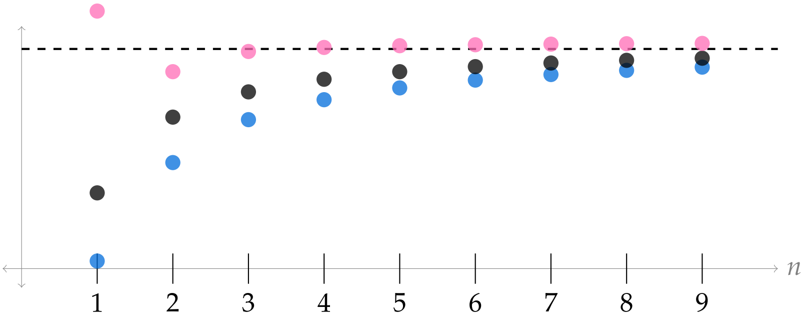
Squeeze Theorem

If $a_n \leq c_n \leq b_n$ for all sufficiently large natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$



Evaluate

$$\lim_{n\to\infty}\left(\frac{2n+\cos n}{n+1}\right)$$

Let $a_n = (-n)^{-n}$. Evaluate $\lim_{n\to\infty} a_n$.

Included Work

'Notebook' by Iconic is licensed under CC BY 3.0 (accessed 9 June 2021, modified), 15, 27