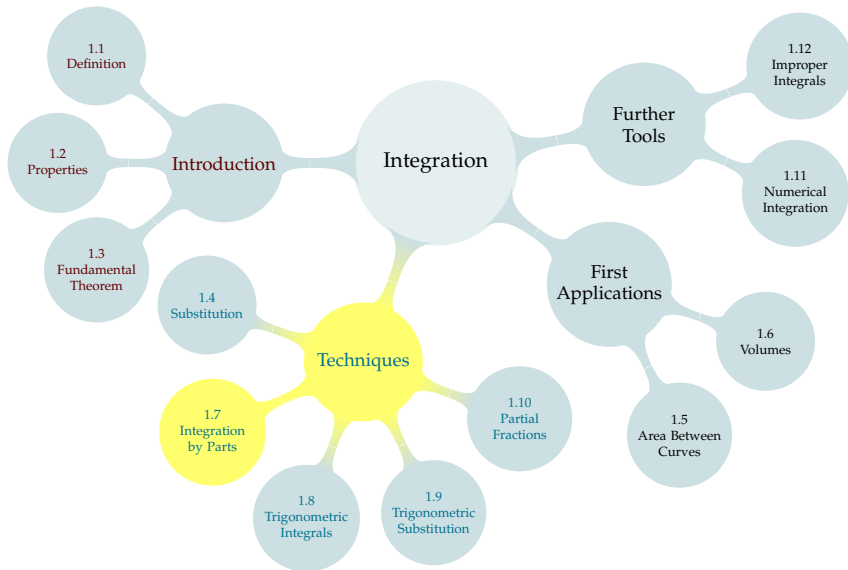


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# REVERSE THE PRODUCT RULE

Product Rule:

$$\frac{d}{dx} \{u(x) \cdot v(x)\} = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

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Product Rule:

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Related fact:

$$\int \left[ u'(x) \cdot v(x) + u(x) \cdot v'(x) \right] dx = u(x) \cdot v(x) + C$$

# REVERSE THE PRODUCT RULE

Product Rule:

$$\frac{d}{dx} \{u(x) \cdot v(x)\} = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Related fact:

$$\int [u'(x) \cdot v(x) + u(x) \cdot v'(x)] dx = u(x) \cdot v(x) + C$$

Rearrange:

$$\Rightarrow \int [u'(x)v(x)] dx + \int [u(x)v'(x)] dx = u(x)v(x) + C$$

$$\Rightarrow \int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$

# INTEGRATION BY PARTS

$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx$$

Example:  $\int x e^x dx$

# CHECK OUR WORK

In the previous slide, we evaluated

$$\int xe^x dx = xe^x - e^x + C$$

for some constant  $C$ . We can check that this is correct by differentiating.

# INTEGRATION BY PARTS (IBP): A CLOSER LOOK

$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx + C$$

$$\underbrace{\int x e^x dx}_{\text{How to integrate??}} = x(e^x) - \underbrace{1 \int e^x dx}_{\text{Easy to integrate!}} + C$$

We start and end with an integral. IBP is only useful if the new integral is somehow an improvement.

We **differentiate** the function we choose as  $u(x)$ , and **antidifferentiate** the function we choose as  $v'(x)$

CHOOSING  $u(x)$  AND  $v(x)$ 

$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx + C$$
$$\int \left[ x \sin x \right] dx =$$



CHOOSING  $u(x)$  AND  $v(x)$ 

$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx + C$$
$$\int \left[ x \sin x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x & \\ v'(x) = \sin x & \end{array}$$

Option B:

$$\begin{array}{l|l} u(x) = \sin x & \\ v'(x) = x & \end{array}$$

# CHECK OUR WORK

To check our work, we can calculate  $\frac{d}{dx} \left\{ -x \cos x + \sin x + C \right\}$ . It should work out to be  $x \sin x$ .

CHOOSING  $u(x)$  AND  $v(x)$ 

$$\int \left[ \textcolor{blue}{u}(x) \textcolor{red}{v}'(x) \right] dx = \textcolor{blue}{u}(x) \textcolor{red}{v}(x) - \int \left[ \textcolor{red}{v}(x) \textcolor{blue}{u}'(x) \right] dx + C$$
$$\int \left[ x^2 \log x \right] dx =$$



CHOOSING  $u(x)$  AND  $v(x)$ 

$$\int \left[ \textcolor{blue}{u}(x) \textcolor{red}{v}'(x) \right] dx = \textcolor{blue}{u}(x) v(x) - \int \left[ \textcolor{red}{v}(x) \textcolor{blue}{u}'(x) \right] dx + C$$
$$\int \left[ x^2 \log x \right] dx =$$

Option A:

$$\begin{array}{l|l} \textcolor{blue}{u}(x) = x^2 & \\ \textcolor{red}{v}'(x) = \log x & \end{array}$$

Option B:

$$\begin{array}{l|l} \textcolor{blue}{u}(x) = \log x & \\ \textcolor{red}{v}'(x) = x^2 & \end{array}$$

# CHECK OUR WORK

To check our work, we can calculate  $\frac{d}{dx} \left\{ \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \right\}$ . It should work out to be  $x^2 \log x$ .

$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx + C$$
$$\int \left[ \frac{1}{2}xe^{6x} \right] dx =$$



$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx + C$$
$$\int \left[ \frac{1}{2}xe^{6x} \right] dx =$$

Option A:

$$\left. \begin{array}{l} u(x) = \frac{1}{2}x \\ v'(x) = e^{6x} \end{array} \right|$$

Option B:

$$\left. \begin{array}{l} u(x) = e^{6x} \\ v'(x) = \frac{1}{2}x \end{array} \right|$$



# CHECK OUR WORK

We check that  $\int \left[ \frac{1}{2} x e^{6x} \right] dx = \frac{1}{12} x e^{6x} - \frac{1}{72} e^{6x} + C$  by differentiating.



## MNEMONIC

$$\int \left[ u(x)v'(x) \right] dx = u(x)v(x) - \int \left[ v(x)u'(x) \right] dx + C$$
$$\int u \, dv = uv - \int v \, du + C$$

We abbreviate:

- ▶  $u(x) \rightarrow u$
- ▶  $u'(x) \, dx \rightarrow du$
- ▶  $v(x) \rightarrow v$
- ▶  $v'(x) \, dx \rightarrow dv$

# CHOOSING $u$ , $dv$ IN YOUR HEAD

Choose  $u$  and  $dv$  to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

# CHOOSING $u$ , $dv$ IN YOUR HEAD

Choose  $u$  and  $dv$  to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts:  $\int \textcolor{blue}{u} \textcolor{red}{dv} = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{v} \textcolor{blue}{du}$   
 $\textcolor{blue}{u}$  gets differentiated, and  $\textcolor{red}{dv}$  gets antidifferentiated.

# CHOOSING $u$ , $dv$ IN YOUR HEAD

Choose  $u$  and  $dv$  to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts:  $\int u dv = uv - \int v du$

$u$  gets differentiated, and  $dv$  gets antidifferentiated.

- If we differentiate OR antidifferentiate  $\cos(t/4)$  we get almost the same thing (they only differ by a constant).

# CHOOSING $u$ , $dv$ IN YOUR HEAD

Choose  $u$  and  $dv$  to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts:  $\int u dv = uv - \int v du$

$u$  gets differentiated, and  $dv$  gets antidifferentiated.

- ▶ If we differentiate OR antidifferentiate  $\cos(t/4)$  we get almost the same thing (they only differ by a constant).
- ▶ If we differentiate  $3t + 5$ , we get something simpler: a constant. If we antidifferentiate  $3t + 4$ , we get something more complicated—a quadratic.

# CHOOSING $u$ , $dv$ IN YOUR HEAD

Choose  $u$  and  $dv$  to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts:  $\int u dv = uv - \int v du$

$u$  gets differentiated, and  $dv$  gets antiderivative.

- ▶ If we differentiate OR antiderivative  $\cos(t/4)$  we get almost the same thing (they only differ by a constant).
- ▶ If we differentiate  $3t + 5$ , we get something simpler: a constant. If we antiderivative  $3t + 4$ , we get something more complicated—a quadratic.

Simpler is better: we want to differentiate  $3t + 5$ , which means we choose it to be  $u$ . Then  $dv = \cos(t/4) dt$ .

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u}v - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

►  $\int x e^{x^2} dx$

►  $\int x^2 e^x dx$

►  $\int e^{x+e^x} dx$

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

►  $\int x e^{x^2} dx$



Evaluate, using IBP or Substitution

$$\int u \, dv = uv - \int v \, du + C$$

►  $\int x^2 e^x \, dx$

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

►  $\int e^{x+e^x} dx$

# DEFINITE INTEGRATION BY PARTS

Method 1: Antidifferentiate first, then plug in limits of integration.

Method 2: Plug as you go.

Evaluate  $\int_1^e \log^2 x \, dx$

# SPECIAL TECHNIQUE: $v'(x) = 1$

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate  $\int \log x \, dx$  using integration by parts.

# CHECK OUR WORK

Let's check that  $\int \log x \, dx = x \log x - x + C$ .

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate  $\int \arctan x \, dx$  using integration by parts.

# CHECK OUR WORK

Let's check that  $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \log |1 + x^2| + C$ .



Setting  $dv = 1 \, dx$  is a very specific technique. You'll probably only see it in situations integrating logarithms and inverse trigonometric functions.

$$\int \log x \, dx, \quad \int \arcsin x \, dx, \quad \int \arccos x \, dx, \quad \int \arctan x \, dx, \quad \text{etc.}$$

Evaluate  $\int e^x \cos x \, dx$  using integration by parts.

Evaluate  $\int e^x \cos x \, dx$  using integration by parts.

Let  $u = e^x$  and  $dv = \cos x \, dx$ . Then  $du = e^x \, dx$  and  $v = \sin x$ :

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let  $u = e^x$  and  $dv = \sin x \, dx$ . Then  $du = e^x \, dx$  and  $v = -\cos x$ :

$$= e^x \sin x - \left[ -e^x \cos x - \int -e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2} (e^x \sin x + e^x \cos x) + C$$

## INTEGRATING AROUND IN A CIRCLE

We can use this technique to antidifferentiate products of two functions that almost, but don't quite, stay the same under (anti)differentiation.

Use integration by parts a number of times, ending up with an expression involving (a scalar multiple of) the original integral.

To do this, **be consistent** with your choice of  $u$  and  $dv$ .

Evaluate  $\int e^x \cos x \, dx$  using integration by parts.

Let  $u = e^x$  and  $dv = \cos x \, dx$ . Then  $du = e^x \, dx$  and  $v = \sin x$ :

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let  $u = e^x$  and  $dv = \sin x \, dx$ . Then  $du = e^x \, dx$  and  $v = -\cos x$ :

$$= e^x \sin x - \left[ -e^x \cos x - \int -e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2} (e^x \sin x + e^x \cos x) + C$$

Evaluate  $\int \cos(\log x) \, dx$ .

# CHECK OUR WORK

We check that  $\int \cos(\log x) dx = \frac{x}{2} [\cos(\log x) + \sin(\log x)] + C$  by differentiating.

Evaluate  $\int e^{2x} \sin x \, dx$  using integration by parts.



# CHECK OUR WORK

We can check our work by differentiating  $\frac{1}{5}e^{2x}[2\sin x - \cos x] + C$ .  
We should end up with  $e^{2x}\sin x$ .

## Included Work



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