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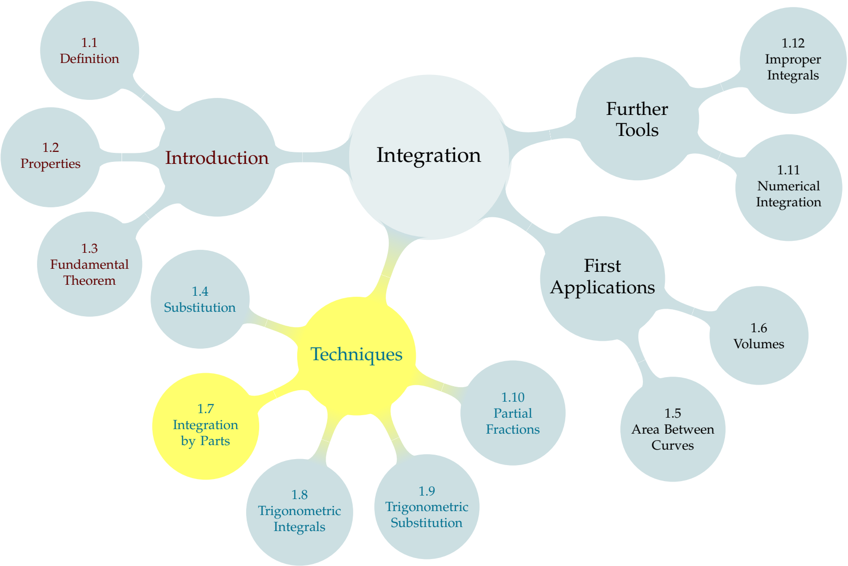

Reverse the Product Rule

Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{u(x)\cdot v(x)\right\} = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

Related fact:

$$\int \left[u'(x)\cdot v(x) + u(x)\cdot v'(x)\right]\mathrm{d}x = u(x)\cdot v(x) + C$$

Rearrange:

$$\implies \int \left[u'(x)v(x)\right]\mathrm{d}x + \int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) + C$$

$$\implies \int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

Integration by Parts

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x$$

Example: $\int xe^x \mathrm{d}x$

Let $u(x) = x$ and $v'(x) = e^x$. (We'll talk **later** about choosing these)
Then $u'(x) = 1$ and $v(x) = e^x$.

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x$$

$$\int \left[xe^x\right]\mathrm{d}x = x(e^x) - \int \left[(e^x)1\right]\mathrm{d}x + C$$

$$\int xe^x = xe^x - \int (e^x)\mathrm{d}x + C$$

$$= xe^x - e^x + C$$

Check Our Work

In the previous slide, we evaluated

$$\int xe^x \mathrm{d}x = xe^x - e^x + C$$

for some constant C. We can check that this is correct by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{xe^x - e^x + C\right\} = (xe^x + e^x) - e^x = xe^x$$

We used the product rule to differentiate. Remember integration by parts helps us to reverse the product rule.

Integration by Parts (IBP): A Closer Look

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\underbrace{\int xe^x\mathrm{d}x}_{\text{How to integrate??}} = x(e^x) - \underbrace{1\int e^x\mathrm{d}x}_{\text{Easy to integrate!}} + C$$

We start and end with an integral. IBP is only useful if the new integral is somehow an improvement.

We differentiate the function we choose as $u(x)$, and antidifferentiate the function we choose as $v'(x)$

Choosing $u(x)$ and $v(x)$

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int \left[x \sin x\right]\mathrm{d}x =$$

Option A:

$$u(x) = x \qquad u'(x) = 1$$
$$v'(x) = \sin x \qquad v(x) = -\cos x$$
$$\to \int -\cos x \cdot 1 \, \mathrm{d}x$$

Option B:

$$u(x) = \sin x \qquad u'(x) = \cos x$$
$$v'(x) = x \qquad v(x) = \frac{1}{2}x^2$$
$$\to \int \frac{1}{2}x^2 \cdot \cos x \, \mathrm{d}x$$

Option A:

$$\int x \sin x \, \mathrm{d}x = x(-\cos x) - \int -\cos x \, \mathrm{d}x = -x\cos x + \sin x + C$$

Fine Print: We can choose <u>any</u> antiderivative of $v'(x)$ to be $v(x)$. So, we omit "+C."

Q

Check Our Work

To check our work, we can calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{-x\cos x+\sin x+C\right\}$. It should work out to be $x\sin x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{-x\cos x+\sin x+C\right\}=(-x)(-\sin x)+(\cos x)(-1)+\cos x=x\sin x$$

Our answer works!

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x)\right] \mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right] \mathrm{d}x + C$$

$$\int \left[x^2 \log x\right] \mathrm{d}x =$$

Option A:

$u(x) = x^2$ | $u'(x) = 2x$

$v'(x) = \log x$ | $v(x) = ??$

$\rightarrow \int ?? \cdot 2x \, \mathrm{d}x$

Option B:

$u(x) = \log x$ | $u'(x) = \frac{1}{x}$

$v'(x) = x^2$ | $v(x) = \frac{1}{3}x^3$

$\rightarrow \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, \mathrm{d}x$

Option B:

$$\int x^2 \log x \, \mathrm{d}x = \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, \mathrm{d}x$$

$$= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2 \, \mathrm{d}x = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C$$

Q

Check Our Work

To check our work, we can calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{3}x^3\log x - \frac{1}{9}x^3 + C\right\}$. It should work out to be $x^2\log x$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{3}x^3\log x - \frac{1}{9}x^3 + C\right\} &= x^2\log x + \frac{1}{3}x^3\cdot\frac{1}{x} - \frac{3}{9}x^2 \\ &= x^2\log x\end{aligned}$$

Our answer works.

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int \left[\tfrac{1}{2}xe^{6x}\right]\mathrm{d}x =$$

Option A:

$$\begin{array}{l|l} u(x) = \tfrac{1}{2}x & u'(x) = \tfrac{1}{2} \\ v'(x) = e^{6x} & v(x) = \tfrac{1}{6}e^{6x} \end{array}$$

$$\to \int \tfrac{1}{6}e^{6x} \cdot \tfrac{1}{2}\ \mathrm{d}x$$

Option B:

$$\begin{array}{l|l} u(x) = e^{6x} & u'(x) = 6e^{6x} \\ v'(x) = \tfrac{1}{2}x & v(x) = \tfrac{1}{4}x^2 \end{array}$$

$$\to \int \tfrac{1}{4}x^2 \cdot 6e^{6x}\ \mathrm{d}x$$

Option A:

$$\int \tfrac{1}{2}x \cdot e^{6x}\ \mathrm{d}x = \frac{1}{2}x \cdot \frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x} \cdot \frac{1}{2}\ \mathrm{d}x$$

$$= \frac{1}{12}xe^{6x} - \frac{1}{12}\int e^{6x}\mathrm{d}x = \frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C$$

Check Our Work

We check that $\int \left[\frac{1}{2}xe^{6x}\right] \mathrm{d}x = \frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C$ by differentiating.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C\right\} &= \frac{1}{12}x \cdot 6e^{6x} + e^{6x} \cdot \frac{1}{12} - \frac{6}{72}e^{6x} \\
&= \frac{1}{2}xe^{6x} + \frac{1}{12}e^{6x} - \frac{1}{12}e^{6x} \\
&= \frac{1}{2}xe^{6x}
\end{aligned}$$

Our answer works.

Mnemonic

$$\int \left[u(x)v'(x) \right] \mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x) \right] \mathrm{d}x + C$$

$$\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u + C$$

We abbreviate:

- $u(x) \to u$
- $u'(x)\,\mathrm{d}x \to \mathrm{d}u$
- $v(x) \to v$
- $v'(x)\,\mathrm{d}x \to \mathrm{d}v$

CHOOSING u, dv IN YOUR HEAD

Choose u and dv to evaluate the integral below:

$$\int (3t+5)\cos(t/4)\mathrm{d}t$$

Thoughts: $\displaystyle\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

u gets differentiated, and dv gets antidifferentiated.

Evaluate, using IBP or Substitution

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u + C$$

- $\displaystyle\int xe^{x^2}\,\mathrm{d}x$
- $\displaystyle\int x^2e^x\,\mathrm{d}x$
- $\displaystyle\int e^{x+e^x}\,\mathrm{d}x$

$$\text{(sub)}\quad \int xe^{x^2}\,\mathrm{d}x = \int \frac{1}{2}e^u\,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

$$\text{(IBP)}\quad \int \underbrace{x^2}_{u}\underbrace{e^x\,\mathrm{d}x}_{\mathrm{d}v} = x^2\cdot e^x - \int e^x\cdot 2x\,\mathrm{d}x$$

$$= x^2e^x - 2\int \underbrace{x}_{u}\underbrace{e^x\,\mathrm{d}x}_{\mathrm{d}v} = x^2e^x - 2\left[xe^x - \int e^x\,\mathrm{d}x\right]$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

Definite Integration by Parts

Method 1: Antidifferentiate first, then plug in limits of integration.

Method 2: Plug as you go.

Evaluate $\displaystyle\int_1^e \log^2 x \,\mathrm{d}x$

Method 1:

Let $u = \log^2 x$, $\mathrm{d}v = 1\mathrm{d}x$; $\mathrm{d}u = 2\log x \cdot \frac{1}{x}\mathrm{d}x$, $v = x$

$$\int \log^2 x \,\mathrm{d}x = x\log^2 x - \int 2\log x \,\mathrm{d}x$$

Now let $u = \log x$, $\mathrm{d}v = 2\mathrm{d}x$; $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, $v = 2x$

$$= x\log^2 x - \left[2x\log x - \int 2\mathrm{d}x\right] = x\log^2 x - 2x\log x + 2x + C$$

$$\int_1^e \log^2 x \,\mathrm{d}x = \Big[x\log^2 x - 2x\log x + 2x + C\Big]_1^e$$

$$= (e - 2e + 2e + C) - (0 - 0 + 2 + C) = e - 2$$

Method 2:

Let $u = \log^2 x$, $\mathrm{d}v = 1\mathrm{d}x$; $\mathrm{d}u = 2\log x \cdot \frac{1}{x}\mathrm{d}x$, $v = x$

$$\int_1^e \log^2 x \,\mathrm{d}x = \Big[x\log^2 x\Big]_1^e - \int_1^e 2\log x \,\mathrm{d}x = (e-0) - \int_1^e 2\log x \,\mathrm{d}x$$

Special Technique: $v'(x) = 1$

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u + C$$

Evaluate $\int \log x \, \mathrm{d}x$ using integration by parts.

$$\begin{aligned}
\int \log x \, \mathrm{d}x &= \int \underbrace{\log x}_{u} \cdot \underbrace{1 \, \mathrm{d}x}_{\mathrm{d}v} \\
&= \log x \cdot x - \int x \cdot \frac{1}{x} \, \mathrm{d}x \\
&= x \log x - \int 1 \, \mathrm{d}x = x \log x - x + C
\end{aligned}$$

Check Our Work

Let's check that $\int \log x \, \mathrm{d}x = x \log x - x + C$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x \log x - x + C\right\} = x \cdot \frac{1}{x} + \log x - 1 = 1 + \log x - 1 = \log x$$

So we have indeed found the antiderivative of $\log x$.

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u + C$$

Evaluate $\displaystyle\int \arctan x \, \mathrm{d}x$ using integration by parts.

Hint: $\arctan x = (\arctan x)(1)$, and $\frac{\mathrm{d}}{\mathrm{d}x}\left\{ \arctan x \right\} = \dfrac{1}{1+x^2}$

$$\int \underbrace{\arctan x}_{u} \cdot \underbrace{1 \, \mathrm{d}x}_{\mathrm{d}v} = \arctan x \cdot x - \int x \cdot \frac{1}{1+x^2} \, \mathrm{d}x$$

Set $s = 1 + x^2$, $\mathrm{d}s = 2x \, \mathrm{d}x$.

$$= x \arctan x - \frac{1}{2} \int \frac{1}{s} \, \mathrm{d}s$$

$$= x \arctan x - \frac{1}{2} \log |1 + x^2| + C$$

Q

Check Our Work

Let's check that $\displaystyle\int \arctan x \, dx = x \arctan x - \frac{1}{2}\log|1+x^2| + C$.

$$\begin{aligned}\frac{d}{dx}\left\{x\arctan x - \frac{1}{2}\log|1+x^2| + C\right\} &= x \cdot \frac{1}{1+x^2} + \arctan x - \frac{1}{2}\cdot\frac{2x}{1+x^2}\\ &= \frac{x}{1+x^2} + \arctan x - \frac{x}{1+x^2}\\ &= \arctan x\end{aligned}$$

So we have indeed found the antiderivative of $\arctan x$.

Setting $\mathrm{d}v = 1\,\mathrm{d}x$ is a very specific technique. You'll probably only see it in situations integrating logarithms and inverse trigonometric functions.

$$\int \log x\,\mathrm{d}x, \qquad \int \arcsin x\,\mathrm{d}x, \qquad \int \arccos x\,\mathrm{d}x, \qquad \int \arctan x\,\mathrm{d}x, \qquad \text{etc.}$$

Evaluate $\int e^x \cos x \, dx$ using integration by parts.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$:

$$= e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, dx\right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

Integrating around in a circle

We can use this technique to antidifferentiate products of two functions that almost, but don't quite, stay the same under (anti)differentiation.

Use integration by parts a number of times, ending up with an expression involving (a scalar multiple of) the original integral.

To do this, be consistent with your choice of u and $\mathrm{d}v$.

Evaluate $\int \cos(\log x)\,\mathrm{d}x$.

Let $u = \cos(\log x)$, $\mathrm{d}v = \mathrm{d}x$; then $\mathrm{d}u = -\frac{\sin(\log x)}{x}\mathrm{d}x$, $v = x$

$$\int \cos(\log x)\mathrm{d}x = x\cos(\log x) - \int \left(-\frac{\sin(\log x)}{x}\right) x\,\mathrm{d}x$$

$$= x\cos(\log x) + \int \sin(\log x)\,\mathrm{d}x$$

Let $u = \sin(\log x)$, $\mathrm{d}v = \mathrm{d}x$; then $\mathrm{d}u = \frac{\cos(\log x)}{x}$, $v = x$

$$= x\cos(\log x) + x\sin(\log x) - \int \cos(\log x)\,\mathrm{d}x$$

So, $\quad 2\int \cos(\log x)\,\mathrm{d}x = x\cos(\log x) + x\sin(\log x)$

$$\int \cos(\log x)\,\mathrm{d}x = \frac{x}{2}\left[\cos(\log x) + \sin(\log x)\right] + C$$

Q

Check Our Work

We check that $\displaystyle\int \cos(\log x)\mathrm{d}x = \frac{x}{2}\left[\cos(\log x) + \sin(\log x)\right] + C$ by differentiating.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x}{2}\left[\cos(\log x) + \sin(\log x)\right] + C\right\} \\
&\quad= \frac{x}{2}\left[\frac{-\sin(\log x)}{x} + \frac{\cos(\log x)}{x}\right] + \frac{1}{2}\left[\cos(\log x) + \sin(\log x)\right] \\
&\quad= -\frac{1}{2}\sin(\log x) + \frac{1}{2}\cos(\log x) + \frac{1}{2}\cos(\log x) + \frac{1}{2}\sin(\log x) \\
&\quad= \cos(\log x)
\end{aligned}$$

Our answer works.

Evaluate $\displaystyle\int e^{2x}\sin x\,dx$ using integration by parts.

Let $u = e^{2x}$ and $dv = \sin x\,dx$. Then $du = 2e^{2x}\,dx$ and $v = -\cos x$.

$$\begin{aligned}\int e^{2x}\sin x\,dx &= e^{2x}(-\cos x) - \int(-\cos x)2e^{2x}\,dx\\ &= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx\end{aligned}$$

Let $u = e^{2x}$ and $dv = \cos x\,dx$. Then $du = 2e^{2x}\,dx$ and $v = \sin x$

$$\begin{aligned}\int e^{2x}\sin x\,dx &= -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right]\\ &= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx\end{aligned}$$

$$5\int e^{2x}\sin x\,dx = -e^{2x}(\cos x - 2\sin x)$$

$$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

Q

CHECK OUR WORK

We can check our work by differentiating $\frac{1}{5}e^{2x}[2\sin x - \cos x] + C$.
We should end up with $e^{2x}\sin x$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{5}e^{2x}(2\sin x - \cos x) + C\right\} &= \frac{1}{5}e^{2x}(2\cos x + \sin x) + \frac{2}{5}e^{2x}(2\sin x - \cos x)\\ &= \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + \frac{4}{5}e^{2x}\sin x - \frac{2}{5}e^{2x}\cos x\\ &= e^{2x}\sin x\end{aligned}$$

Our answer, strange though it looks, is the correct antiderivative.