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Finite addition is commutative

$$1 + 2 + 3 + 4 = 4 + 1 + 3 + 2$$

What happens if we re-arrange the terms in a series?

We'll illustrate some possibilities, but first we need to establish some preliminary results.

Preliminary Results

Split up the alternating harmonic series into two series: one with the positive terms, and one with the negative terms.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\cdots$$

$$\begin{aligned}
&-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots\\
=&-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)\\
&=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}
\end{aligned}$$

So, we can make an arbitrarily large negative number by adding up these terms.

$$\begin{aligned}
&1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\\
\geq&\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots
\end{aligned}$$

So, we can make an arbitrarily large positive number by adding up these terms.

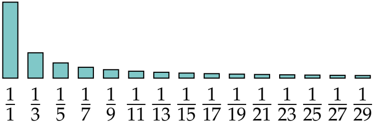
Preliminary Results

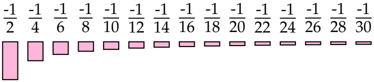
We've shown that the alternating harmonic series converges. We don't have the tools to do it just yet, but later we'll be able to compute what it converges to:

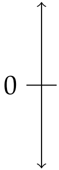
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$$

Surprising fact: if we reorder the terms of the series carefully, we can make a new series adding up to any number we want.

Rearrange the alternating harmonic series to sum to 0.

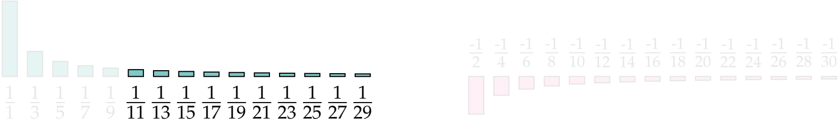


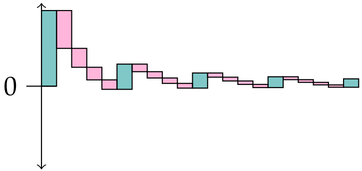




- Add positive terms until the partial sum is greater than 0.
- Add negative terms until the partial sum is less than 0.
- Repeat.

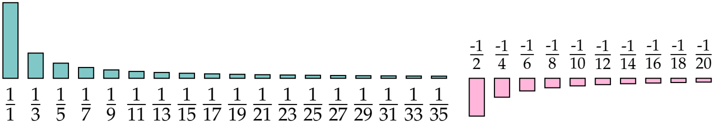
Rearrange the alternating harmonic series to sum to 0.





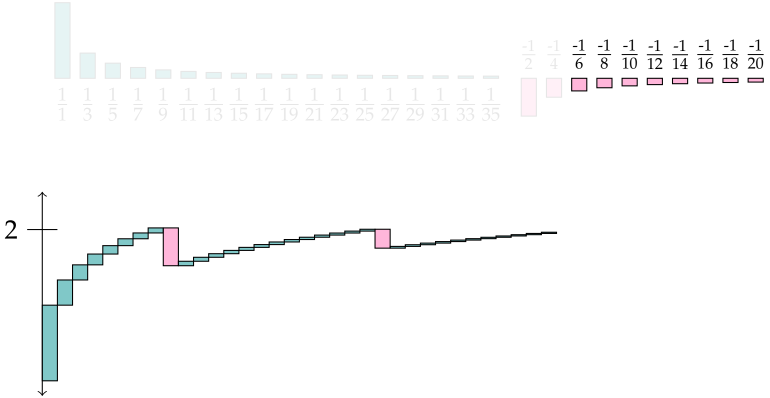
- Add positive terms until the partial sum is greater than 0.
- Add negative terms until the partial sum is less than 0.
- Repeat.

Rearrange the alternating harmonic series to sum to 2.



- Add positive terms until the partial sum is greater than 2.
- Add negative terms until the partial sum is less than 2.
- Repeat.

Rearrange the alternating harmonic series to sum to 2.



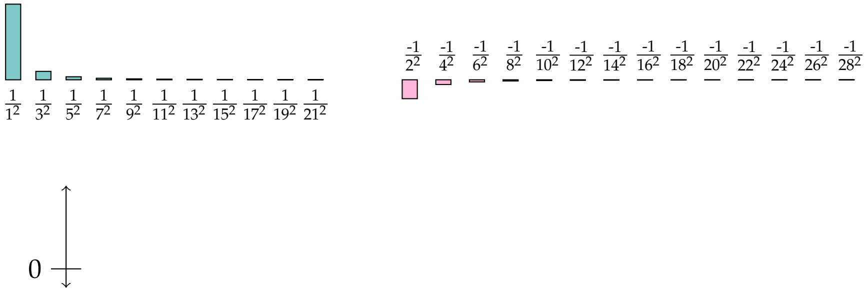
- Add positive terms until the partial sum is greater than 2.
- Add negative terms until the partial sum is less than 2.
- Repeat.

In fact: you can reorder *any* conditionally convergent series to

- add up to *any* number, or
- diverge to infinity, or
- diverge to negative infinity.

This doesn't work with absolutely convergent series.

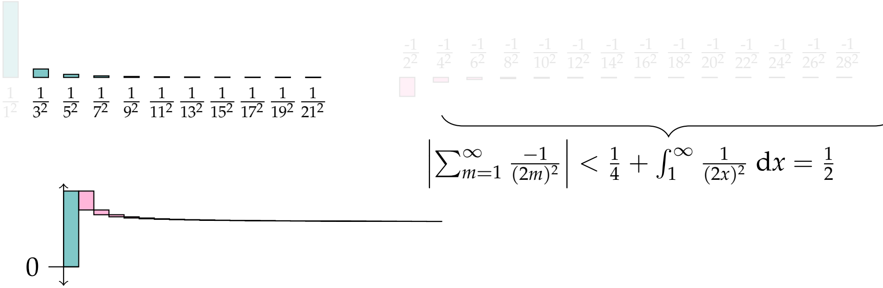
Let's try to rearrange the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to add up to 0:



- Add positive terms until the partial sum is greater than 0.
- Add negative terms (those with $n = 2m$, $m = 1, 2, 3, \cdots$) until the partial sum is less than 0.
- Repeat.

This doesn't work with absolutely convergent series.

Let's try to rearrange the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to add up to 0:



$$\left|\sum_{m=1}^{\infty} \frac{-1}{(2m)^2}\right| < \frac{1}{4} + \int_1^{\infty} \frac{1}{(2x)^2}\,\mathrm{d}x = \frac{1}{2}$$

- Add positive terms until the partial sum is greater than 0.
- Add negative terms (those with $n = 2m$, $m = 1, 2, 3, \cdots$) until the partial sum is less than 0.
- Repeat.

In fact: you can reorder *any* conditionally convergent series to

- add up to *any* number, or
- diverge to infinity, or
- diverge to negative infinity.

Changing the order of terms in an absolutely convergent series does not change its value.