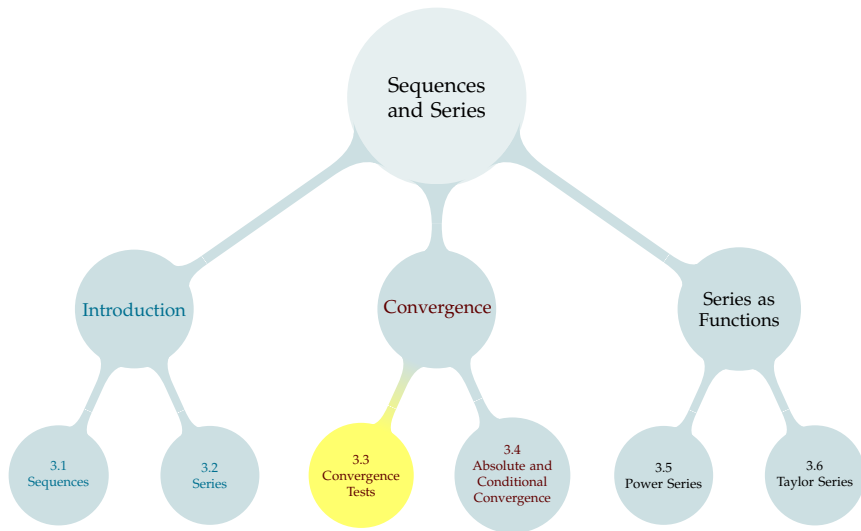


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# REVIEW

Let  $S_N = \sum_{n=1}^N a_n$ .

Simplify:  $S_N - S_{N-1}$ .

(This will come in handy soon.)

# ALTERNATING SERIES

## Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

is alternating if every  $A_n \geq 0$ .

Alternating series:

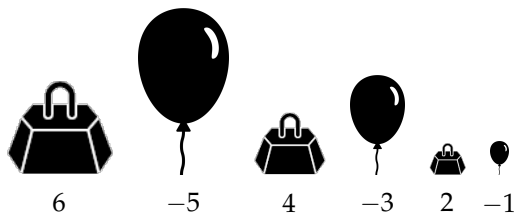
▶  $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$

▶  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Not alternating:

▶  $\cos(1) + \cos(2) + \cos(3) + \cdots$

▶  $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$



$$S_1 = 6.0000$$

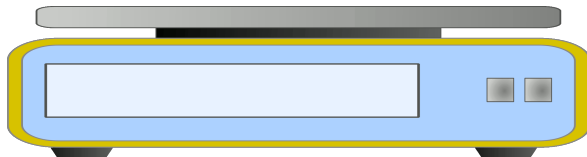
$$S_2 = 1.0000$$

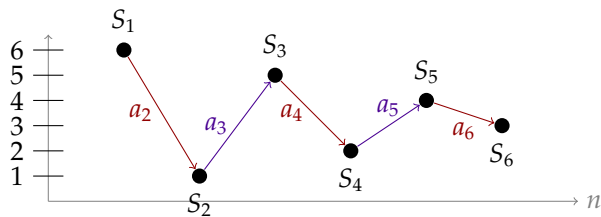
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

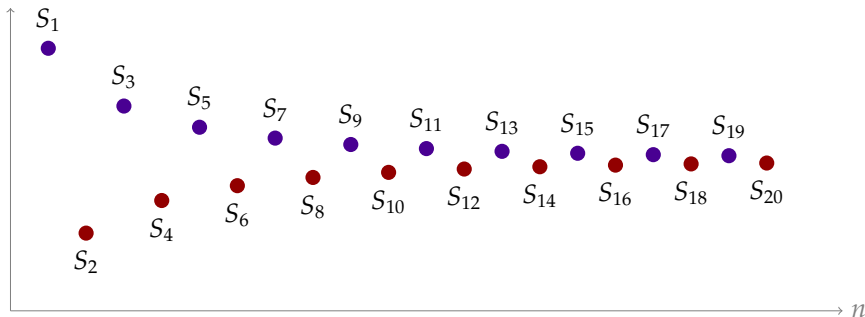
$$S_5 = 4.0000$$

$$S_6 = 3.0000$$

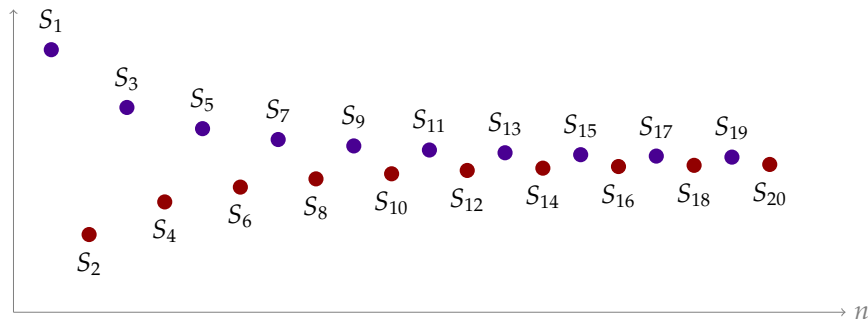




Consider an alternating series  $a_1 - a_2 + a_3 - a_4 + \cdots$ , where  $\{a_n\}$  is a sequence with positive, **decreasing** terms and with  $\lim_{n \rightarrow \infty} a_n = 0$ .



Since  $a_2 > a_3$ , we have  $a_1 - (a_2 - a_3) < a_1$ , so  $S_3 < S_1$ .



- ▶ For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .
- ▶ For all  $n \geq 3$ ,  $S_n$  lies between  $S_2$  and  $S_3$ .
- ▶ For all  $n \geq 4$ ,  $S_n$  lies between  $S_3$  and  $S_4$ .
- ▶ For all  $n \geq 5$ ,  $S_n$  lies between  $S_4$  and  $S_5$ .

The difference between consecutive sums  $S_n$  and  $S_{n-1}$  is:

## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys

- (i)  $a_n \geq 0$  for all  $n \geq 1$ ;
- (ii)  $a_{n+1} \leq a_n$  for all  $n \geq 1$  (i.e. the sequence is monotone decreasing);
- (iii) and  $\lim_{n \rightarrow \infty} a_n = 0$ .

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number  $N$ ,  $S - S_N$  is between 0 and (the first dropped term)  $(-1)^N a_{N+1}$ . Here  $S_N$  is, as previously, the  $N^{\text{th}}$  partial sum  $\sum_{n=1}^N (-1)^{n-1} a_n$ .



## Alternating Series Test (abridged)

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys

- (i)  $a_n \geq 0$  for all  $n \geq 1$ ;
- (ii)  $a_{n+1} \leq a_n$  for all  $n \geq 1$  (i.e. the sequence is monotone decreasing);
- (iii) and  $\lim_{n \rightarrow \infty} a_n = 0$ .

Then

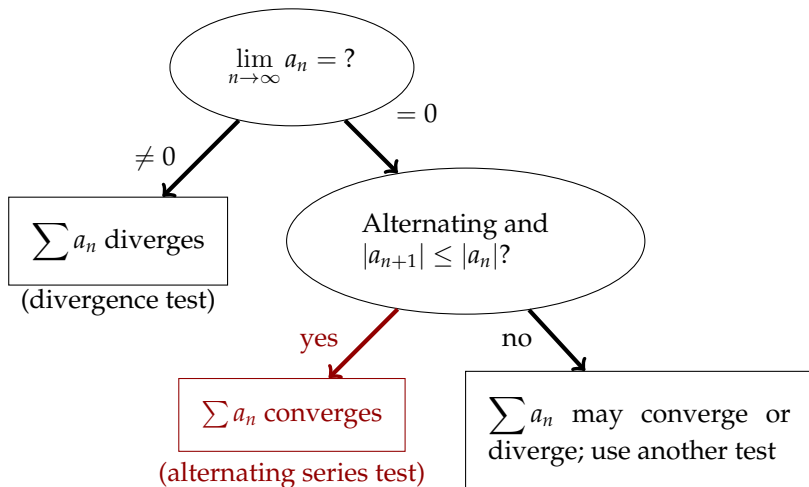
$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

► True or false: the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  converges.

► True or false: the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges.

# DIVERGENCE TEST + ALTERNATING SERIES TEST



## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys  $a_n \geq 0$  for all  $n \geq 1$ ;  $a_{n+1} \leq a_n$  for all  $n \geq 1$ ; and  $\lim_{n \rightarrow \infty} a_n = 0$ . Then  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$  converges and  $S - S_N$  is between 0 and  $(-1)^N a_{N+1}$ .

Using a computer, you find  $\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698$ .

How close is that to the value  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ ?

## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys  $a_n \geq 0$  for all  $n \geq 1$ ;  $a_{n+1} \leq a_n$  for all  $n \geq 1$ ; and  $\lim_{n \rightarrow \infty} a_n = 0$ . Then  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$  converges and  $S - S_N$  is between 0 and  $(-1)^N a_{N+1}$ .

Using a computer, you find  $\sum_{n=1}^{19} (-1)^{n-1} \frac{n^2}{n^2 + 1} \approx 0.6347$ .

How close is that to the value  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2 + 1}$ ?

Recall for a geometric series, the **ratios of consecutive terms** is constant.

$$\begin{array}{c}
 \times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2} \\
 \text{~~~~~} \rightarrow \text{~~~~~} \rightarrow \text{~~~~~} \rightarrow \text{~~~~~} \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \\
 \frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} = \frac{1/32}{1/16} = \frac{1}{2}
 \end{array}$$

If that ratio has magnitude **less than one**, then the series converges.  
 If the ratio has magnitude **greater than one**, the series diverges.

For series convergence, we are concerned with what happens to terms  $a_n$  when  $n$  is sufficiently large.

Suppose for a sequence  $a_n$ ,  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$  for some constant  $L$ .

$$\underbrace{a_n + a_{n+1}}_{\frac{a_{n+1}}{a_n} \approx} + \underbrace{a_{n+1} + a_{n+2}}_{\frac{a_{n+2}}{a_{n+1}} \approx} + \underbrace{a_{n+2} + a_{n+3}}_{\frac{a_{n+3}}{a_{n+2}} \approx} + \underbrace{a_{n+3} + a_{n+4}}_{\frac{a_{n+4}}{a_{n+3}} \approx} + \underbrace{a_{n+4} + \cdots}_{\frac{a_{n+5}}{a_{n+4}} \approx} \approx L$$

Like in a geometric series:

If  $L$  has magnitude **less than one**, then the series converges.

If  $L$  has magnitude **greater than one**, the series diverges.

## Ratio Test

Let  $N$  be any positive integer and assume that  $a_n \neq 0$  for all  $n \geq N$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

## Ratio Test

Let  $N$  be any positive integer and assume that  $a_n \neq 0$  for all  $n \geq N$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges or diverges.



## REMARK

The series we just considered,  $\sum_{n=1}^{\infty} \frac{n}{3^n}$ , looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

We could have used other tests, but ratio was probably the easiest.

- ▶ Integral test:  $\int \frac{x}{3^x} dx$  can be evaluated using integration by parts.
- ▶ Comparison test:
  - ▶  $\sum \frac{1}{3^n}$  is not a valid comparison series, nor is  $\sum n$ .
  - ▶ Because  $n < 2^n$  for all  $n \geq 1$ , the series  $\sum \left(\frac{2}{3}\right)^n$  will work.
- ▶ The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

## Ratio Test

Let  $N$  be any positive integer and assume that  $a_n \neq 0$  for all  $n \geq N$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

Let  $a$  and  $x$  be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of  $a$  and  $x$ .)

Let  $x$  be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of  $x$ .)



# FILL IN THE BLANKS

## Divergence Test

If the sequence  $\{a_n\}_{n=c}^{\infty}$    
 then the series  $\sum_{n=c}^{\infty} a_n$  diverges.

## Ratio Test

Let  $N$  be any positive integer and assume that  $a_n \neq 0$  for all  $n \geq N$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  , then  $\sum_{n=1}^{\infty} a_n$  converges.

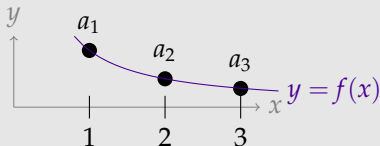
(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

## Integral Test

Let  $N_0$  be any natural number. If  $f(x)$  is a function which is defined and continuous for all  $x \geq N_0$  and which obeys

- (i)  and  
 (ii)  and  
 (iii)  $f(n) = a_n$  for all  $n \geq N_0$ .

Then



$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x) \, dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n \leq \int_N^{\infty} f(x) \, dx \quad \text{for all } N \geq N_0$$

# FILL IN THE BLANKS

## The Comparison Test

Let  $N_0$  be a natural number and let  $K > 0$ .

(a) If  $|a_n| \square Kc_n$  for all  $n \geq N_0$  and  $\sum_{n=0}^{\infty} c_n$  converges, then  $\sum_{n=0}^{\infty} a_n$  converges.

(b) If  $a_n \square Kd_n \geq 0$  for all  $n \geq N_0$  and  $\sum_{n=0}^{\infty} d_n$  diverges, then  $\sum_{n=0}^{\infty} a_n$  diverges.

# FILL IN IN THE BLANKS

## Limit Comparison Theorem

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two series with  $b_n > 0$  for all  $n$ . Assume that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

exists.

- (a) If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges too.
- (b) If  $L \neq 0$  and  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges too.

In particular, if , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges.

## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys

(i)

(ii)  $a_{n+1} \leq a_n$  for all  $n \geq 1$  (i.e. the sequence is monotone decreasing);

(iii) and

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number  $N$ ,  $S - S_N$  is between 0 and (the first dropped term)  $(-1)^N a_{N+1}$ . Here  $S_N$  is, as previously, the  $N^{\text{th}}$  partial sum  $\sum_{n=1}^N (-1)^{n-1} a_n$ .



# LIST OF CONVERGENCE TESTS

## Divergence Test

When the  $n^{\text{th}}$  term in the series *fails* to converge to zero as  $n$  tends to infinity.

This is a good first thing to check: if it works, it's quick, but it doesn't always work.

## Alternating Series Test

- ▶ successive terms in the series alternate in sign
- ▶ don't forget to check that successive terms decrease in magnitude and tend to zero as  $n$  tends to infinity

## Integral Test

- ▶ works well when, if you substitute  $x$  for  $n$  in the  $n^{\text{th}}$  term you get a function,  $f(x)$ , that you can easily integrate
- ▶ don't forget to check that  $f(x) \geq 0$  and that  $f(x)$  decreases as  $x$  increases

# LIST OF CONVERGENCE TESTS

## Ratio Test

- ▶ works well when  $\frac{a_{n+1}}{a_n}$  simplifies enough that you can easily compute  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$
- ▶ this often happens when  $a_n$  contains powers, like  $7^n$ , or factorials, like  $n!$
- ▶ don't forget that  $L = 1$  tells you nothing about the convergence/divergence of the series

## Comparison Test and Limit Comparison Test

- ▶ Comparison test lets you ignore pieces of a function that feel extraneous (like replacing  $n^2 + 1$  with  $n^2$ ) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- ▶ Limit comparison works well when, for very large  $n$ , the  $n^{\text{th}}$  term  $a_n$  is approximately the same as a simpler, nonnegative term  $b_n$

- ▶ The integral test gave us the  $p$ -test. When you're looking for comparison series,  $p$ -series  $\sum \frac{1}{n^p}$  are often good choices, because their convergence or divergence is so easy to ascertain.
- ▶ Geometric series have the form  $\sum a \cdot r^n$  for some nonzero constants  $a$  and  $r$ . The magnitude of  $r$  is all you need to know to decide whether they converge or diverge, so these are also common comparison series.
- ▶ Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

## Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series  $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$  converges or diverges.

## Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$  converges or diverges.

## Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series  $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$  converges or diverges.

*Hint:* If  $\theta \geq 0$  then  $\sin \theta \leq \theta$ .

## Included Work

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