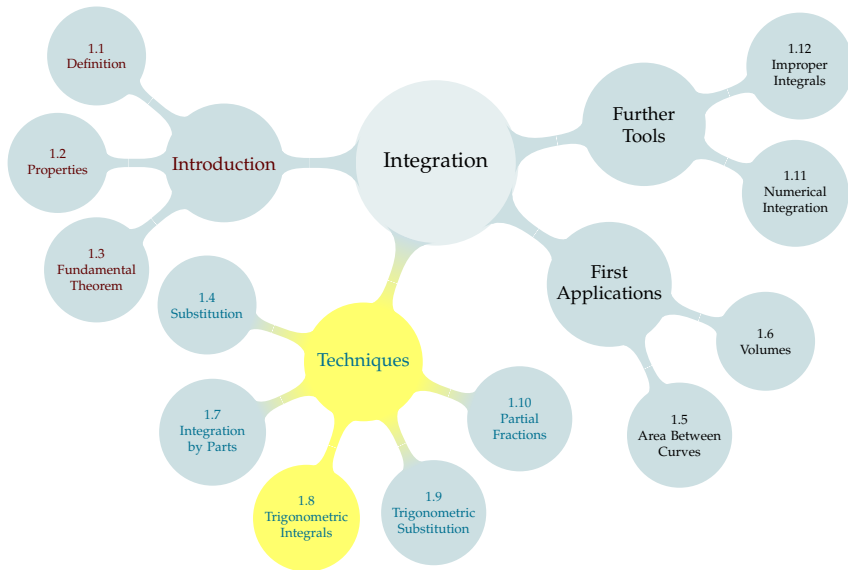


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# 1.8 TRIGONOMETRIC INTEGRALS

Recall:

- ▶  $\sin^2 x + \cos^2 x = 1$
- ▶  $\tan^2 x + 1 = \sec^2 x$
- ▶  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- ▶  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- ▶  $\sin(2x) = 2 \sin x \cos x$

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin x \cos x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin x \cos x dx =$$

$$\int \sin^{10} x \cos x dx =$$

# CHECK OUR WORK

If we are correct that  $\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^2 x}{2} + C \right\} = \sin x \cos x$ .

# CHECK OUR WORK

If we are correct that  $\int \sin^{10} x \cos x dx = \frac{\sin^{11} x}{11} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} + C \right\} = \sin^{10} x \cos x$ .

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x dx =$$

# CHECK OUR WORK

If we are correct that  $\int \sin^{\pi+1} x \cos x dx = \frac{\sin^{\pi+2} x}{\pi+2} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{\pi+2} x}{\pi+2} + C \right\} = \sin^{\pi+1} x \cos x$ .





# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$ .

$$\int \sin^{10} x \cos^3 x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$ .

Use  $\sin^2 x + \cos^2 x = 1$ .

$$\int \sin^{10} x \cos^3 x dx =$$

## CHECK OUR WORK

If we are correct that  $\int \sin^{10} x \cos^3 x dx = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C \right\} = \sin^{10} x \cos^3 x$ .

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin^5 x \cos^4 x dx =$$

# CHECK OUR WORK

If we are correct that

$$\int \sin^5 x \cos^4 x dx = -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C, \text{ then it should}$$

$$\text{be true that } \frac{d}{dx} \left\{ -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C \right\} = \sin^5 x \cos^4 x.$$

GENERALIZE:  $\int \sin^m x \cos^n bx dx$

To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

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To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

- ▶ We need to **reserve** one  **$\cos x$**  for the differential.
- ▶ We need to **convert** the remaining  **$\cos^{n-1} x$**  to  **$\sin x$**  terms.



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- ▶ We convert using  **$\cos^2 x = 1 - \sin^2 x$** . To avoid square roots, that means  $n - 1$  should be **even when we convert**.

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To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

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- ▶ **So, we can use this substitution when the original power of cosine,  $n$ , is ODD: one cosine goes to the differential, the rest are converted to sines.**

GENERALIZE:  $\int \sin^m x \cos^n x dx$

To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

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To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

- ▶ We need to **reserve** one  $\sin x$  for the differential.
- ▶ We need to **convert** the remaining  $\sin^{m-1} x$  to  $\cos x$  terms.
- ▶ We convert using  $\sin^2 x = 1 - \cos^2 x$ . To avoid square roots, that means  $m - 1$  should be **even when we convert**.
- ▶ **So, we can use this substitution when the original power of sine,  $m$ , is ODD: one sine goes to the differential, the rest are converted to cosines.**

# MNEMONIC: "ODD ONE OUT"

$$\text{Integrating } \int \sin^m x \cos^n x dx$$

If you want to use  $u = \sin x$ , there should be an odd power of **cosine**.

If you want to use  $u = \cos x$ , there should be an odd power of **sine**.



Carry out a suitable substitution (but do not evaluate the resulting integral):

►  $\int \sin^4 x \cos^7 x dx$

►  $\int \sin^7 x \cos^4 x dx$

►  $\int \sin^7 x \cos^7 x dx$

To evaluate  $\int \sin^m x \cos^n x dx$ , we use:

- ▶  $u = \sin x$  if  $n$  is odd, and/or
- ▶  $u = \cos x$  if  $m$  is odd

What if  $n$  and  $m$  are both even?

To evaluate  $\int \sin^m x \cos^n x dx$ , we use:

- ▶  $u = \sin x$  if  $n$  is odd, and/or
- ▶  $u = \cos x$  if  $m$  is odd

What if  $n$  and  $m$  are both even?

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \sin^2 x dx =$$

# CHECK OUR WORK

We check that  $\int \sin^2 x dx = \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C$  by differentiating:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Evaluate  $\int \sin^4 x dx$ .

# CHECK OUR WORK

We want to check that  $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$

Recall:

- ▶  $\frac{d}{dx} \{\tan x\} = \sec^2 x$
- ▶  $\frac{d}{dx} \{\sec x\} = \sec x \tan x$
- ▶  $\tan^2 x + 1 = \sec^2 x$



$$\int \tan x \, dx =$$

# CHECK OUR WORK

Let's check that  $\int \tan x dx = \log |\sec x| + C$  by differentiating.

Optional: A nifty trick – you won't be expected to come up with it.  
There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx =$$

Optional: A nifty trick – you won't be expected to come up with it.  
There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx = \int \sec x \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

Useful integrals:

$$\blacktriangleright \int \tan x \, dx = \log |\sec x| + C$$

$$\blacktriangleright \int \sec x \, dx = \log |\sec x + \tan x| + C$$

$$1. \int \sec x \tan x \, dx =$$

$$2. \int \sec^2 x \, dx =$$

$$3. \int \tan x \, dx =$$

$$4. \int \sec x \, dx =$$

Evaluate using the substitution rule:

$$\int \tan^5 x \sec^2 x dx =$$

$$\int \sec^4 x (\sec x \tan x) dx =$$

# CHECK OUR WORK

Let's check that  $\int \tan^5 x \sec^2 x dx = \frac{1}{6} \tan^6 x + C$  by differentiating.



# CHECK OUR WORK

Let's check that  $\int \sec^4 x (\sec x \tan x) dx = \frac{1}{5} \sec^5 x + C$  by differentiating.

Evaluate using the identity  $\sec^2 x = 1 + \tan^2 x$

$$\int \tan^4 x \sec^6 x dx =$$

$$\int \tan^3 x \sec^5 x dx =$$

# CHECK OUR WORK

Let's check that  $\int \tan^4 x \sec^6 x dx = \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$

# CHECK OUR WORK

Let's check that  $\int \tan^3 x \sec^5 x dx = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C.$

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- Reserve  $\sec x \tan x$  for the differential.

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- ▶ Reserve  $\sec x \tan x$  for the differential.
- ▶ From the remaining  $\tan^{m-1} x \sec^{n-1} x$ , convert all tangents to secants using  $\tan^2 x + 1 = \sec^2 x$ .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- ▶ Reserve  $\sec x \tan x$  for the differential.  
( $m, n$  should each be at least 1)
- ▶ From the remaining  $\tan^{m-1} x \sec^{n-1} x$ , convert all tangents to secants using  $\tan^2 x + 1 = \sec^2 x$ .



# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- ▶ Reserve  $\sec x \tan x$  for the differential.  
( $m, n$  should each be at least 1)
- ▶ From the remaining  $\tan^{m-1} x \sec^{n-1} x$ , convert all tangents to secants using  $\tan^2 x + 1 = \sec^2 x$ .  
( $m - 1$  should be even, to avoid square roots)

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- ▶ Reserve  $\sec x \tan x$  for the differential.  
 ( $m, n$  should each be at least 1)
- ▶ From the remaining  $\tan^{m-1} x \sec^{n-1} x$ , convert all tangents to secants using  $\tan^2 x + 1 = \sec^2 x$ .  
 ( $m - 1$  should be even, to avoid square roots)

To use the substitution  $u = \sec x$ ,  $du = \sec x \tan x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be , and  $m$  should be .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  for the differential.
- ▶ From the remaining terms, convert all  to  using  $\tan^2 x + 1 = \sec^2 x$ .

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate  $\int \tan^m x \sec^n x dx$ ,  $n$  should be .

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$

Choose a substitution for the integrals below.

▶  $\int \sec^2 x \tan^3 x dx$

▶  $\int \sec^2 x \tan^2 x dx$

▶  $\int \sec^3 x \tan^3 x dx$

$$\int \sec^2 x \tan^3 x dx$$

$$\int \sec^2 x \tan^2 x dx$$

$$\int \sec^3 x \tan^3 x dx$$

Evaluate  $\int \tan^3 x dx$



Evaluate  $\int \tan^3 x dx = \int \frac{\sin^3 x}{\cos^3 x} dx$

# CHECK OUR WORK

Let's check that  $\int \tan^3 x dx = \log |\cos x| + \frac{1}{2} \sec^2 x + C$ . by  
differentiating.

Generalizing the last example:

$$\int \tan^m x \sec^n x dx =$$

Generalizing the last example:

$$\begin{aligned}
 \int \tan^m x \sec^n x dx &= \int \left( \frac{\sin x}{\cos x} \right)^m \left( \frac{1}{\cos x} \right)^n dx \\
 &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\
 &= \int \left( \frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx
 \end{aligned}$$

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \left( \frac{\sin x}{\cos x} \right)^m \left( \frac{1}{\cos x} \right)^n dx \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\ &= \int \left( \frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx\end{aligned}$$

To use  $u = \cos x$ ,  $du = -\sin x dx$ :

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \left( \frac{\sin x}{\cos x} \right)^m \left( \frac{1}{\cos x} \right)^n dx \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\ &= \int \left( \frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx\end{aligned}$$

To use  $u = \cos x$ ,  $du = -\sin x dx$ : we will convert  $\sin^{m-1}(x)$  into cosines, so  $m - 1$  must be even, so  $m$  must be odd.

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd

Evaluate  $\int \tan^2 x dx$

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd

Evaluate  $\int \tan^2 x dx$



## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd

Evaluate  $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:


- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd
- ▶  $u = \tan x$  if  $m$  is even and  $n = 0$   
(after using  $\tan^2 x = \sec^2 x - 1$ , maybe several times)

Evaluate  $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

## Included Work

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