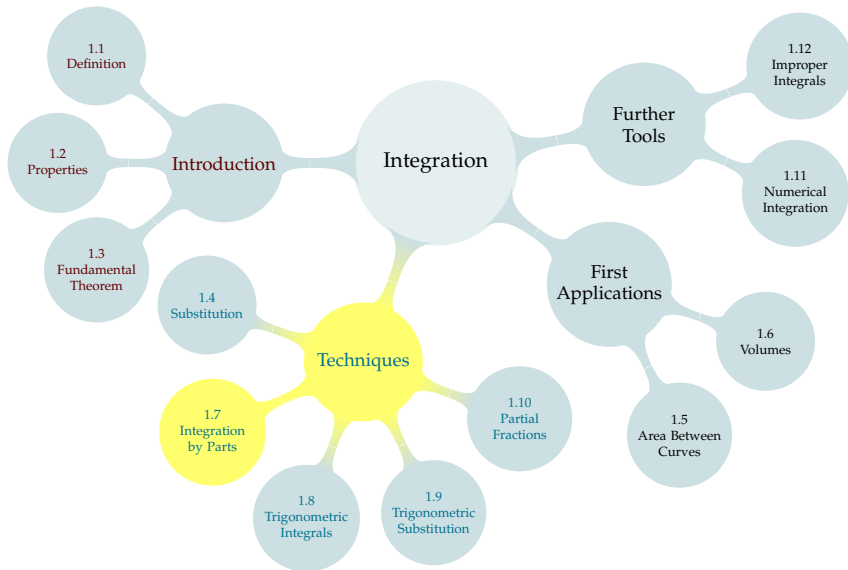


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REVERSE THE PRODUCT RULE

Product Rule:

$$\frac{d}{dx} \{u(x) \cdot v(x)\} = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

REVERSE THE PRODUCT RULE

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Related fact:

$$\int \left[u'(x) \cdot v(x) + u(x) \cdot v'(x) \right] dx = u(x) \cdot v(x) + C$$

REVERSE THE PRODUCT RULE

Product Rule:

$$\frac{d}{dx} \{u(x) \cdot v(x)\} = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Related fact:

$$\int [u'(x) \cdot v(x) + u(x) \cdot v'(x)] dx = u(x) \cdot v(x) + C$$

Rearrange:

$$\Rightarrow \int [u'(x)v(x)] dx + \int [u(x)v'(x)] dx = u(x)v(x) + C$$

$$\Rightarrow \int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$

INTEGRATION BY PARTS

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx$$

Example: $\int x e^x dx$

INTEGRATION BY PARTS

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx$$

Example: $\int xe^x dx$

Let $u(x) = x$ and $v'(x) = e^x$. (We'll talk **later** about choosing these)

Then $u'(x) = 1$ and $v(x) = e^x$.

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx$$

$$\int \left[xe^x \right] dx = x(e^x) - \int \left[(e^x)1 \right] dx + C$$

$$\int xe^x = xe^x - \int (e^x) dx + C$$

$$= xe^x - e^x + C$$

CHECK OUR WORK

In the previous slide, we evaluated

$$\int xe^x dx = xe^x - e^x + C$$

for some constant C . We can check that this is correct by differentiating.

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$$\int xe^x dx = xe^x - e^x + C$$

for some constant C . We can check that this is correct by differentiating.

$$\frac{d}{dx} \{ xe^x - e^x + C \} = (xe^x + e^x) - e^x = xe^x$$

We used the product rule to differentiate. Remember integration by parts helps us to reverse the product rule.

INTEGRATION BY PARTS (IBP): A CLOSER LOOK

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\underbrace{\int x e^x dx}_{\text{How to integrate??}} = x(e^x) - \underbrace{1 \int e^x dx}_{\text{Easy to integrate!}} + C$$

We start and end with an integral. IBP is only useful if the new integral is somehow an improvement.

We **differentiate** the function we choose as $u(x)$, and **antidifferentiate** the function we choose as $v'(x)$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[x \sin x \right] dx =$$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[x \sin x \right] dx =$$

Option A:

$$\begin{array}{l} u(x) = x \\ v'(x) = \sin x \end{array} \quad \left| \right.$$

Option B:

$$\begin{array}{l} u(x) = \sin x \\ v'(x) = x \end{array} \quad \left| \right.$$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[x \sin x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x & u'(x) = 1 \\ v'(x) = \sin x & v(x) = -\cos x \end{array}$$

Option B:

$$\begin{array}{l|l} u(x) = \sin x & \\ v'(x) = x & \end{array}$$

Fine Print: We can choose any antiderivative of $v'(x)$ to be $v(x)$. So, we omit “+C.”



CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

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Option A:

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Option B:

$$\begin{array}{l|l} u(x) = \sin x & u'(x) = \cos x \\ v'(x) = x & v(x) = \frac{1}{2}x^2 \end{array}$$

Fine Print: We can choose any antiderivative of $v'(x)$ to be $v(x)$. So, we omit "+C."

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[x \sin x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x & u'(x) = 1 \\ v'(x) = \sin x & v(x) = -\cos x \end{array}$$

$$\rightarrow \int -\cos x \cdot 1 \, dx$$

Option B:

$$\begin{array}{l|l} u(x) = \sin x & u'(x) = \cos x \\ v'(x) = x & v(x) = \frac{1}{2}x^2 \end{array}$$

$$\rightarrow \int \frac{1}{2}x^2 \cdot \cos x \, dx$$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[x \sin x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x & u'(x) = 1 \\ v'(x) = \sin x & v(x) = -\cos x \end{array}$$

$$\rightarrow \int -\cos x \cdot 1 \, dx$$

Option B:

$$\begin{array}{l|l} u(x) = \sin x & u'(x) = \cos x \\ v'(x) = x & v(x) = \frac{1}{2}x^2 \end{array}$$

$$\rightarrow \int \frac{1}{2}x^2 \cdot \cos x \, dx$$

Option A:

$$\int x \sin x \, dx = x(-\cos x) - \int -\cos x \, dx = -x \cos x + \sin x + C$$

CHECK OUR WORK

To check our work, we can calculate $\frac{d}{dx} \left\{ -x \cos x + \sin x + C \right\}$. It should work out to be $x \sin x$.



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To check our work, we can calculate $\frac{d}{dx} \left\{ -x \cos x + \sin x + C \right\}$. It should work out to be $x \sin x$.

$$\frac{d}{dx} \left\{ -x \cos x + \sin x + C \right\} = (-x)(-\sin x) + (\cos x)(-1) + \cos x = x \sin x$$

Our answer works!



CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[x^2 \log x \right] dx =$$



CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[x^2 \log x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x^2 & \\ v'(x) = \log x & \end{array}$$

Option B:

$$\begin{array}{l|l} u(x) = \log x & \\ v'(x) = x^2 & \end{array}$$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[x^2 \log x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x^2 & u'(x) = 2x \\ v'(x) = \log x & v(x) = ?? \end{array}$$

Option B:

$$\begin{array}{l|l} u(x) = \log x & \\ v'(x) = x^2 & \end{array}$$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

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Option A:

$$\begin{array}{l|l} u(x) = x^2 & u'(x) = 2x \\ v'(x) = \log x & v(x) = ?? \end{array}$$

Option B:

$$\begin{array}{l|l} u(x) = \log x & u'(x) = \frac{1}{x} \\ v'(x) = x^2 & v(x) = \frac{1}{3}x^3 \end{array}$$

CHOOSING $u(x)$ AND $v(x)$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[x^2 \log x \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x^2 & u'(x) = 2x \\ v'(x) = \log x & v(x) = ?? \end{array} \rightarrow \int ?? \cdot 2x \, dx$$

Option B:

$$\begin{array}{l|l} u(x) = \log x & u'(x) = \frac{1}{x} \\ v'(x) = x^2 & v(x) = \frac{1}{3}x^3 \end{array} \rightarrow \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$$

CHOOSING $u(x)$ AND $v(x)$

$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$

$$\int [x^2 \log x] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = x^2 & u'(x) = 2x \\ v'(x) = \log x & v(x) = ?? \end{array}$$

$$\rightarrow \int ?? \cdot 2x \, dx$$

Option B:

$$\begin{array}{l|l} u(x) = \log x & u'(x) = \frac{1}{x} \\ v'(x) = x^2 & v(x) = \frac{1}{3}x^3 \end{array}$$

$$\rightarrow \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$$

Option B:

$$\begin{aligned} \int x^2 \log x \, dx &= \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx \\ &= \frac{1}{3}x^3 \log x - \frac{1}{3} \int x^2 \, dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \end{aligned}$$

CHECK OUR WORK

To check our work, we can calculate $\frac{d}{dx} \left\{ \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \right\}$. It should work out to be $x^2 \log x$.



CHECK OUR WORK

To check our work, we can calculate $\frac{d}{dx} \left\{ \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \right\}$. It should work out to be $x^2 \log x$.

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \right\} &= x^2 \log x + \frac{1}{3}x^3 \cdot \frac{1}{x} - \frac{3}{9}x^2 \\ &= x^2 \log x \end{aligned}$$

Our answer works.



$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[\frac{1}{2}xe^{6x} \right] dx =$$

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Option A:

$$\left. \begin{array}{l} u(x) = \frac{1}{2}x \\ v'(x) = e^{6x} \end{array} \right|$$

Option B:

$$\left. \begin{array}{l} u(x) = e^{6x} \\ v'(x) = \frac{1}{2}x \end{array} \right|$$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[\frac{1}{2}xe^{6x} \right] dx =$$

Option A:

$$\left. \begin{array}{l} u(x) = \frac{1}{2}x \\ v'(x) = e^{6x} \end{array} \right| \begin{array}{l} u'(x) = \frac{1}{2} \\ v(x) = \frac{1}{6}e^{6x} \end{array}$$

Option B:

$$\left. \begin{array}{l} u(x) = e^{6x} \\ v'(x) = \frac{1}{2}x \end{array} \right|$$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[\frac{1}{2}xe^{6x} \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = \frac{1}{2}x & u'(x) = \frac{1}{2} \\ v'(x) = e^{6x} & v(x) = \frac{1}{6}e^{6x} \end{array}$$

Option B:

$$\begin{array}{l|l} u(x) = e^{6x} & u'(x) = 6e^{6x} \\ v'(x) = \frac{1}{2}x & v(x) = \frac{1}{4}x^2 \end{array}$$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int \left[\frac{1}{2}xe^{6x} \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = \frac{1}{2}x & u'(x) = \frac{1}{2} \\ v'(x) = e^{6x} & v(x) = \frac{1}{6}e^{6x} \end{array}$$
$$\rightarrow \int \frac{1}{6}e^{6x} \cdot \frac{1}{2} dx$$

Option B:

$$\begin{array}{l|l} u(x) = e^{6x} & u'(x) = 6e^{6x} \\ v'(x) = \frac{1}{2}x & v(x) = \frac{1}{4}x^2 \end{array}$$
$$\rightarrow \int \frac{1}{4}x^2 \cdot 6e^{6x} dx$$

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$

$$\int \left[\frac{1}{2}xe^{6x} \right] dx =$$

Option A:

$$\begin{array}{l|l} u(x) = \frac{1}{2}x & u'(x) = \frac{1}{2} \\ v'(x) = e^{6x} & v(x) = \frac{1}{6}e^{6x} \end{array}$$

$$\rightarrow \int \frac{1}{6}e^{6x} \cdot \frac{1}{2} dx$$

Option B:

$$\begin{array}{l|l} u(x) = e^{6x} & u'(x) = 6e^{6x} \\ v'(x) = \frac{1}{2}x & v(x) = \frac{1}{4}x^2 \end{array}$$

$$\rightarrow \int \frac{1}{4}x^2 \cdot 6e^{6x} dx$$

Option A:

$$\begin{aligned} \int \frac{1}{2}x \cdot e^{6x} dx &= \frac{1}{2}x \cdot \frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x} \cdot \frac{1}{2} dx \\ &= \frac{1}{12}xe^{6x} - \frac{1}{12} \int e^{6x} dx = \frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C \end{aligned}$$



CHECK OUR WORK

We check that $\int \left[\frac{1}{2} x e^{6x} \right] dx = \frac{1}{12} x e^{6x} - \frac{1}{72} e^{6x} + C$ by differentiating.



CHECK OUR WORK

We check that $\int \left[\frac{1}{2} x e^{6x} \right] dx = \frac{1}{12} x e^{6x} - \frac{1}{72} e^{6x} + C$ by differentiating.

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{1}{12} x e^{6x} - \frac{1}{72} e^{6x} + C \right\} &= \frac{1}{12} x \cdot 6e^{6x} + e^{6x} \cdot \frac{1}{12} - \frac{6}{72} e^{6x} \\ &= \frac{1}{2} x e^{6x} + \frac{1}{12} e^{6x} - \frac{1}{12} e^{6x} \\ &= \frac{1}{2} x e^{6x} \end{aligned}$$

Our answer works.



MNEMONIC

$$\int \left[u(x)v'(x) \right] dx = u(x)v(x) - \int \left[v(x)u'(x) \right] dx + C$$
$$\int u \, dv = uv - \int v \, du + C$$

We abbreviate:

- ▶ $u(x) \rightarrow u$
- ▶ $u'(x) \, dx \rightarrow du$
- ▶ $v(x) \rightarrow v$
- ▶ $v'(x) \, dx \rightarrow dv$

CHOOSING u , dv IN YOUR HEAD

Choose u and dv to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

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Thoughts: $\int u dv = uv - \int v du$
 u gets differentiated, and dv gets antidifferentiated.

CHOOSING u , dv IN YOUR HEAD

Choose u and dv to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts: $\int u dv = uv - \int v du$

u gets differentiated, and dv gets antidifferentiated.

- If we differentiate OR antidifferentiate $\cos(t/4)$ we get almost the same thing (they only differ by a constant).

CHOOSING u , dv IN YOUR HEAD

Choose u and dv to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts: $\int u dv = uv - \int v du$

u gets differentiated, and dv gets antidifferentiated.

- ▶ If we differentiate OR antidifferentiate $\cos(t/4)$ we get almost the same thing (they only differ by a constant).
- ▶ If we differentiate $3t + 5$, we get something simpler: a constant. If we antidifferentiate $3t + 4$, we get something more complicated—a quadratic.

CHOOSING u , dv IN YOUR HEAD

Choose u and dv to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts: $\int u dv = uv - \int v du$

u gets differentiated, and dv gets antiderivative.

- ▶ If we differentiate OR antiderivative $\cos(t/4)$ we get almost the same thing (they only differ by a constant).
- ▶ If we differentiate $3t + 5$, we get something simpler: a constant. If we antiderivative $3t + 4$, we get something more complicated—a quadratic.

Simpler is better: we want to differentiate $3t + 5$, which means we choose it to be u . Then $dv = \cos(t/4) dt$.

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u}v - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

► $\int x e^{x^2} dx$

► $\int x^2 e^x dx$

► $\int e^{x+e^x} dx$

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

► $\int x e^{x^2} dx$

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

► $\int x^2 e^x dx$

Evaluate, using IBP or Substitution

$$\int \textcolor{blue}{u} \textcolor{red}{d}v = \textcolor{blue}{u} \textcolor{red}{v} - \int \textcolor{red}{v} \textcolor{blue}{d}u + C$$

► $\int e^{x+e^x} dx$

$$\text{(sub)} \quad \int x e^{x^2} dx = \int \frac{1}{2} e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

$$\begin{aligned} \text{(IBP)} \quad \int \underbrace{x^2}_u \underbrace{e^x dx}_{dv} &= x^2 \cdot e^x - \int e^x \cdot 2x dx \\ &= x^2 e^x - 2 \int \underbrace{x}_u \underbrace{e^x dx}_{dv} = x^2 e^x - 2 \left[x e^x - \int e^x dx \right] \\ &= x^2 e^x - 2x e^x + 2e^x + C \end{aligned}$$

$$\text{(sub)} \quad \int e^{x+e^x} dx = \int e^{e^x} \cdot e^x dx = \int e^u du = e^u + C = e^{e^x} + C$$

DEFINITE INTEGRATION BY PARTS

Method 1: Antidifferentiate first, then plug in limits of integration.

Method 2: Plug as you go.

Evaluate $\int_1^e \log^2 x \, dx$

Evaluate $\int_1^e \log^2 x \, dx$

Method 1:

Let $u = \log^2 x$, $dv = 1dx$; $du = 2 \log x \cdot \frac{1}{x} dx$, $v = x$

$$\int \log^2 x \, dx = x \log^2 x - \int 2 \log x \, dx$$

Now let $u = \log x$, $dv = 2dx$; $du = \frac{1}{x} dx$, $v = 2x$

$$= x \log^2 x - \left[2x \log x - \int 2dx \right] = x \log^2 x - 2x \log x + 2x + C$$

$$\begin{aligned} \int_1^e \log^2 x \, dx &= \left[x \log^2 x - 2x \log x + 2x + C \right]_1^e \\ &= (e - 2e + 2e + C) - (0 - 0 + 2 + C) = e - 2 \end{aligned}$$

Evaluate $\int_1^e \log^2 x \, dx$

Method 2:

Let $u = \log^2 x$, $dv = 1 \, dx$; $du = 2 \log x \cdot \frac{1}{x} \, dx$, $v = x$

$$\int_1^e \log^2 x \, dx = \left[x \log^2 x \right]_1^e - \int_1^e 2 \log x \, dx = (e - 0) - \int_1^e 2 \log x \, dx$$

Now let $u = \log x$, $dv = 2 \, dx$; $du = \frac{1}{x} \, dx$, $v = 2x$

$$\begin{aligned} &= e - \left[\left[2x \log x \right]_1^e - \int_1^e 2 \, dx \right] = e - (2e - 0) + [2x]_1^e \\ &= e - 2e + 2e - 2 = e - 2 \end{aligned}$$

SPECIAL TECHNIQUE: $v'(x) = 1$

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \log x \, dx$ using integration by parts.

SPECIAL TECHNIQUE: $v'(x) = 1$

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \log x \, dx$ using integration by parts.

Hint: $\log x = (\log x)(1)$.

SPECIAL TECHNIQUE: $v'(x) = 1$

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \log x \, dx$ using integration by parts.

Hint: $\log x = (\log x)(1)$.

$$\begin{aligned}\int \log x \, dx &= \int \underbrace{\log x}_u \cdot \underbrace{1 \, dx}_{dv} \\ &= \log x \cdot x - \int x \cdot \frac{1}{x} \, dx \\ &= x \log x - \int 1 \, dx = x \log x - x + C\end{aligned}$$

CHECK OUR WORK

Let's check that $\int \log x \, dx = x \log x - x + C$.

CHECK OUR WORK

Let's check that $\int \log x \, dx = x \log x - x + C$.

$$\frac{d}{dx} \left\{ x \log x - x + C \right\} = x \cdot \frac{1}{x} + \log x - 1 = 1 + \log x - 1 = \log x$$

So we have indeed found the antiderivative of $\log x$.

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \arctan x \, dx$ using integration by parts.

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \arctan x \, dx$ using integration by parts.

Hint: $\arctan x = (\arctan x)(1)$, and $\frac{d}{dx} \{ \arctan x \} = \frac{1}{1+x^2}$

$$\int \underbrace{\arctan x}_u \cdot \underbrace{1 \, dx}_{dv} = \arctan x \cdot x - \int x \cdot \frac{1}{1+x^2} \, dx$$

Set $s = 1 + x^2$, $ds = 2x \, dx$.

$$\begin{aligned} &= x \arctan x - \frac{1}{2} \int \frac{1}{s} \, ds \\ &= x \arctan x - \frac{1}{2} \log |1 + x^2| + C \end{aligned}$$

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \arctan x \, dx$ using integration by parts.

Hint: $\arctan x = (\arctan x)(1)$, and $\frac{d}{dx} \{ \arctan x \} = \frac{1}{1+x^2}$

$$\int \underbrace{\arctan x}_u \cdot \underbrace{1 \, dx}_{dv} = \arctan x \cdot x - \int x \cdot \frac{1}{1+x^2} \, dx$$

Set $s = 1 + x^2$, $ds = 2x \, dx$.

$$\begin{aligned} &= x \arctan x - \frac{1}{2} \int \frac{1}{s} \, ds \\ &= x \arctan x - \frac{1}{2} \log |1 + x^2| + C \end{aligned}$$

CHECK OUR WORK

Let's check that $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \log |1 + x^2| + C$.

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$$\begin{aligned} \frac{d}{dx} \left\{ x \arctan x - \frac{1}{2} \log |1 + x^2| + C \right\} &= x \cdot \frac{1}{1 + x^2} + \arctan x - \frac{1}{2} \cdot \frac{2x}{1 + x^2} \\ &= \frac{x}{1 + x^2} + \arctan x - \frac{x}{1 + x^2} \\ &= \arctan x \end{aligned}$$

So we have indeed found the antiderivative of $\arctan x$.

Setting $dv = 1 \, dx$ is a very specific technique. You'll probably only see it in situations integrating logarithms and inverse trigonometric functions.

$$\int \log x \, dx, \quad \int \arcsin x \, dx, \quad \int \arccos x \, dx, \quad \int \arctan x \, dx, \quad \text{etc.}$$

Evaluate $\int e^x \cos x \, dx$ using integration by parts.

Evaluate $\int e^x \cos x \, dx$ using integration by parts.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$:

$$= e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2} (e^x \sin x + e^x \cos x) + C$$

INTEGRATING AROUND IN A CIRCLE

We can use this technique to antidifferentiate products of two functions that almost, but don't quite, stay the same under (anti)differentiation.

Use integration by parts a number of times, ending up with an expression involving (a scalar multiple of) the original integral.

To do this, **be consistent** with your choice of u and dv .

Evaluate $\int e^x \cos x \, dx$ using integration by parts.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$:

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Evaluate $\int \cos(\log x) \, dx$.

Evaluate $\int \cos(\log x) \, dx$.

Let $u = \cos(\log x)$, $dv = dx$; then $du = -\frac{\sin(\log x)}{x} dx$, $v = x$

$$\begin{aligned}\int \cos(\log x) \, dx &= x \cos(\log x) - \int \left(-\frac{\sin(\log x)}{x} \right) x \, dx \\ &= x \cos(\log x) + \int \sin(\log x) \, dx\end{aligned}$$

Let $u = \sin(\log x)$, $dv = dx$; then $du = \frac{\cos(\log x)}{x} dx$, $v = x$

$$= x \cos(\log x) + x \sin(\log x) - \int \cos(\log x) \, dx$$

$$\text{So, } 2 \int \cos(\log x) \, dx = x \cos(\log x) + x \sin(\log x)$$

$$\int \cos(\log x) \, dx = \frac{x}{2} [\cos(\log x) + \sin(\log x)] + C$$

CHECK OUR WORK

We check that $\int \cos(\log x) dx = \frac{x}{2} [\cos(\log x) + \sin(\log x)] + C$ by differentiating.

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We check that $\int \cos(\log x) dx = \frac{x}{2} [\cos(\log x) + \sin(\log x)] + C$ by differentiating.

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{x}{2} [\cos(\log x) + \sin(\log x)] + C \right\} \\ &= \frac{x}{2} \left[\frac{-\sin(\log x)}{x} + \frac{\cos(\log x)}{x} \right] + \frac{1}{2} [\cos(\log x) + \sin(\log x)] \\ &= -\frac{1}{2} \sin(\log x) + \frac{1}{2} \cos(\log x) + \frac{1}{2} \cos(\log x) + \frac{1}{2} \sin(\log x) \\ &= \cos(\log x) \end{aligned}$$

Our answer works.

Evaluate $\int e^{2x} \sin x \, dx$ using integration by parts.

Evaluate $\int e^{2x} \sin x \, dx$ using integration by parts.

Let $u = e^{2x}$ and $dv = \sin x \, dx$. Then $du = 2e^{2x} \, dx$ and $v = -\cos x$.

$$\begin{aligned}\int e^{2x} \sin x \, dx &= e^{2x}(-\cos x) - \int (-\cos x)2e^{2x} \, dx \\ &= -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx\end{aligned}$$

Let $u = e^{2x}$ and $dv = \cos x \, dx$. Then $du = 2e^{2x} \, dx$ and $v = \sin x$

$$\begin{aligned}\int e^{2x} \sin x \, dx &= -e^{2x} \cos x + 2 \left[e^{2x} \sin x - \int 2e^{2x} \sin x \, dx \right] \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx\end{aligned}$$

$$5 \int e^{2x} \sin x \, dx = -e^{2x}(\cos x - 2 \sin x)$$

$$\int e^{2x} \sin x \, dx = \frac{e^{2x}}{5}(2 \sin x - \cos x) + C$$



CHECK OUR WORK

We can check our work by differentiating $\frac{1}{5}e^{2x}[2\sin x - \cos x] + C$.
We should end up with $e^{2x}\sin x$.



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We should end up with $e^{2x} \sin x$.

$$\begin{aligned}\frac{d}{dx} \left\{ \frac{1}{5}e^{2x}(2\sin x - \cos x) + C \right\} &= \frac{1}{5}e^{2x}(2\cos x + \sin x) + \frac{2}{5}e^{2x}(2\sin x - \cos x) \\ &= \frac{2}{5}e^{2x} \cos x + \frac{1}{5}e^{2x} \sin x + \frac{4}{5}e^{2x} \sin x - \frac{2}{5}e^{2x} \cos x \\ &= e^{2x} \sin x\end{aligned}$$

Our answer, strange though it looks, is the correct antiderivative.



Included Work



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