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WARMUP

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x.$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$.

Check Our Work

Let's verify that $\int \frac{1}{\sqrt{x^2+1}} =$.

Seems improbable, right?

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1 - x^2}$ into
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1 + x^2}$ into
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2 - 1}$ into
- After integrating, convert back to the original variable (possibly using a triangle–more details later)

Focus on the Algebra

$1-\sin^2\theta = \cos^2\theta$ $\quad$ $1+\tan^2\theta = \sec^2\theta$ $\quad$ $\sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2-1}$
- $\sqrt{x^2+1}$
- $\sqrt{1-x^2}$

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta$ $\qquad$ $1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $\sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 + 7}$
- $\sqrt{3 - 2x^2}$

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

Consider the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta \, \mathrm{d}\theta$ for the integral:

$$\int_0^1 \sqrt{1 - x^2} \, \mathrm{d}x$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

Now, consider the substitution $x = a \tan\theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

Closer look at absolute values

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

Absolute Values

From now on, we will assume:

- With the substitution $x = a\sin\theta$ for $\sqrt{a^2 - x^2}$, $|\cos\theta| = \cos\theta$
- With the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, $|\sec\theta| = \sec\theta$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \sqrt{1 - 4x^2}\, \mathrm{d}x$

Check Our Work

In the last example, we computed

$$\int \sqrt{1 - 4x^2}\, \mathrm{d}x =$$

To check, we differentiate.

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}}\,\mathrm{d}x$

Check Our Work

Let's check our result, $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x =$

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3-x^2+2x}$.

Identities have two "parts" that turn into one part:

- $1-\sin^2\theta = \cos^2\theta$
- $1+\tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.
Fact: $3-x^2+2x = 4-(x-1)^2$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b:

2. Solve for c:

3. All together:

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\, \mathrm{d}x$.

Identities have two “parts” that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

One of those parts is a constant, and one is squared.

Evaluate $\displaystyle\int \frac{x^2-6x+9}{\sqrt{6x-x^2}}\,\mathrm{d}x = \int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x.$

Check Our Work

Let's verify that

$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} =$$