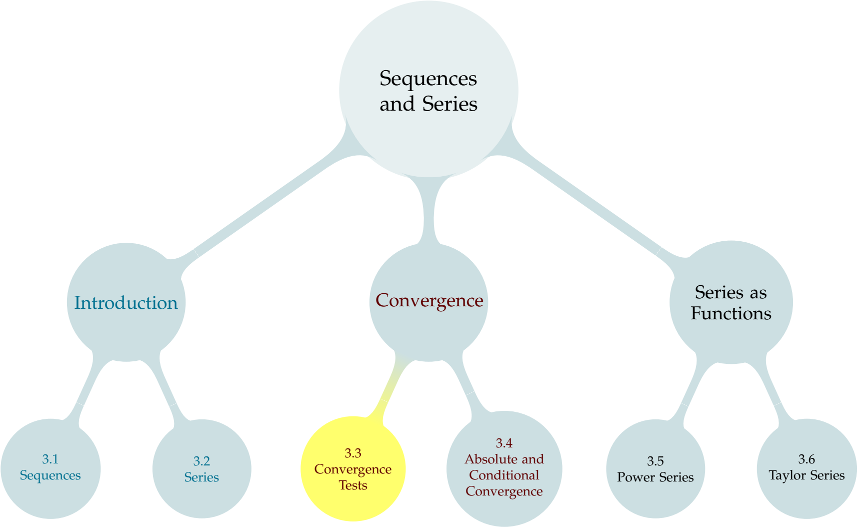
Table of Contents



For a convergent geometric or telescoping series, we can easily determine what the series converges *to*.

For other types of series, finding out what the series converges to can be very difficult. It is often necessary to resort to approximating the full sum by, for example, using a computer to find the sum of the first N terms, for some large N. But before we even try to do that, we should at least know *whether or not the series converges*.

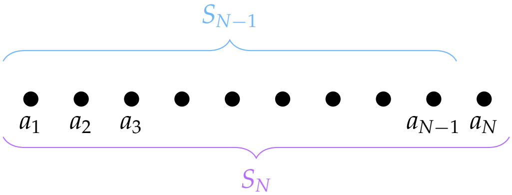
Suppose a series $\sum_{n=1}^{\infty} a_n$ converges to a limit L. Let $S_N = \sum_{n=1}^{N} a_n$.

$$\lim_{N\to\infty} S_N = L$$

$$\lim_{N\to\infty} S_{N-1} = L$$

$$\lim_{N\to\infty} \left[S_N - S_{N-1}\right] = L - L = 0$$

$$\lim_{N\to\infty} a_N = 0$$



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$$\lim_{N\to\infty} a_N = 0$$

$$\underbrace{\overbrace{a_1 \quad a_2 \quad a_3 \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad a_{N-1}}^{S_{N-1}} \quad a_N}_{S_N}$$

Every convergent series has its N^{th} term, a_N, tending to 0 as $N \to \infty$.

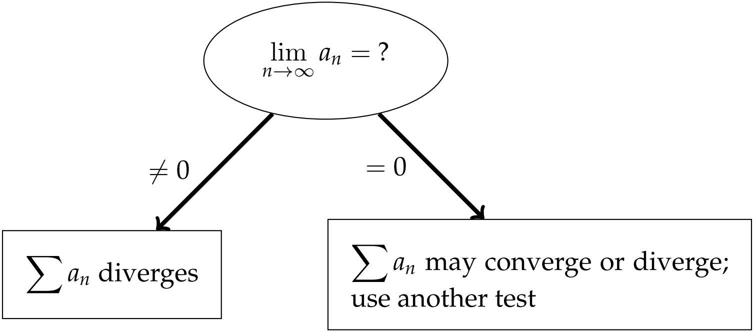
Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ fails to converge to zero as $n \to \infty$, then the series $\sum_{n=c}^{\infty} a_n$ diverges.

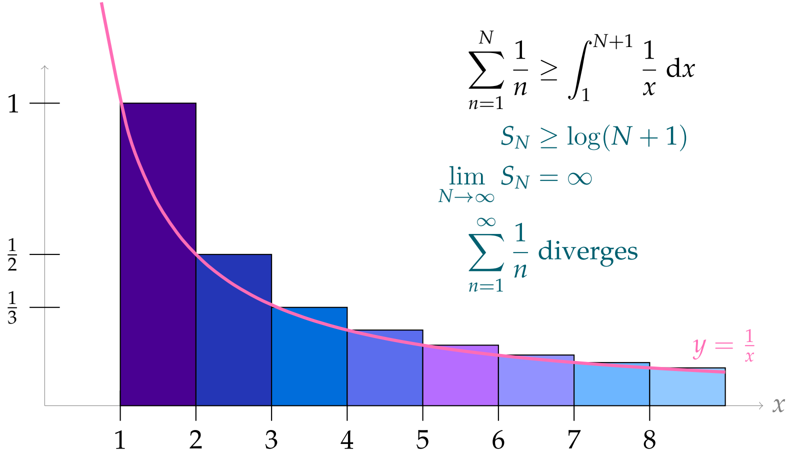
Do the following series diverge?

- $\displaystyle\sum_{n=0}^{\infty}(-1)^n$ yes, it diverges
- $\displaystyle\sum_{n=10}^{\infty}\left(\frac{1}{10}+\frac{1}{2^n}\right)$ yes, it diverges
- $\displaystyle\sum_{n=15}^{\infty}\frac{e^n}{2e^n-1}$ yes, it diverges
- $\displaystyle\sum_{n=15}^{\infty}\frac{1}{n}$ at this point, unclear: maybe, maybe not

Using the Divergence Test for $\sum a_n$



Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$



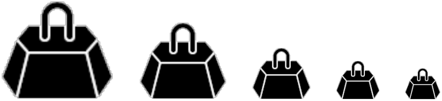
$$\sum_{n=1}^{N} \frac{1}{n} \geq \int_1^{N+1} \frac{1}{x}\,\mathrm{d}x$$

$$S_N \geq \log(N+1)$$

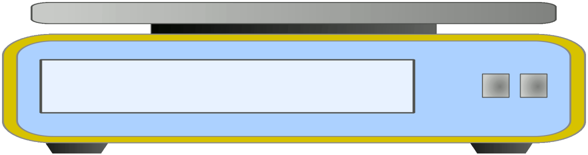
$$\lim_{N\to\infty} S_N = \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES



$1 \qquad \frac{1}{2} \qquad \frac{1}{3} \qquad \frac{1}{4} \qquad \frac{1}{5}$



$$S_1 = 1.0000$$

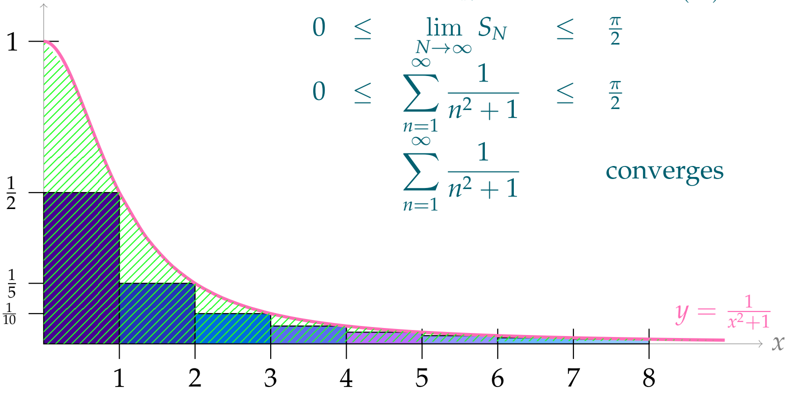
$$S_2 = 1.5000$$

$$S_3 = 1.8333$$

$$S_4 = 2.0833$$

$$S_5 = 2.2833$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$



$$0 \leq \sum_{n=1}^{N} \frac{1}{n^2+1} \leq \int_0^N \frac{1}{x^2+1}\,\mathrm{d}x$$

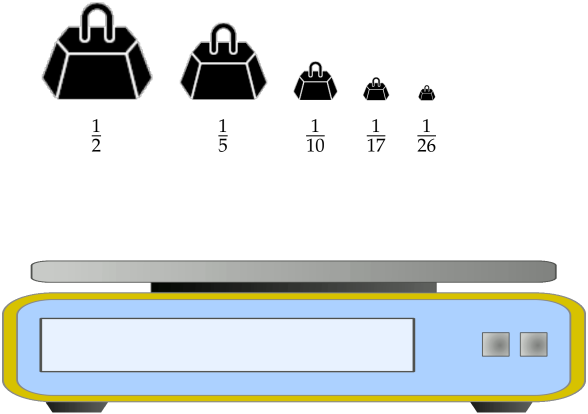
$$0 \leq S_N \leq \arctan(N)$$

$$0 \leq \lim_{N\to\infty} S_N \leq \frac{\pi}{2}$$

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \quad \text{converges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$ CONVERGES



$S_1 = 0.5000$

$S_2 = 0.7000$

$S_3 = 0.8000$

$S_4 = 0.8588$

$S_5 = 0.8973$

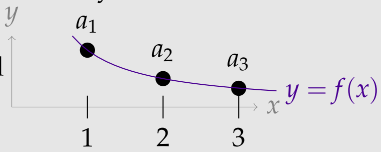
Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

(i) $f(x) \geq 0$ for all $x \geq N_0$ and

(ii) $f(x)$ decreases as x increases and

(iii) $f(n) = a_n$ for all $n \geq N_0$.



Then

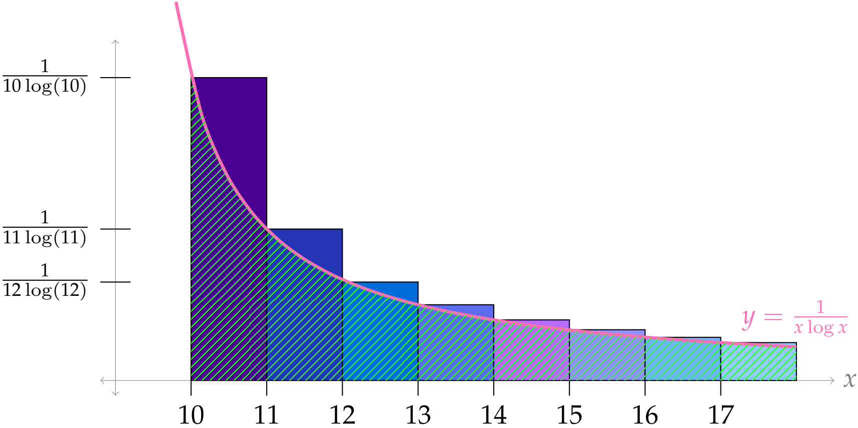
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_{N}^{\infty} f(x)\,\mathrm{d}x \qquad \text{for all } N \geq N_0$$

Does the series $\sum_{n=10}^{\infty} \frac{1}{n \log n}$ converge or diverge?

Does the series $\sum_{n=10}^{\infty} \frac{1}{n \log n}$ converge or diverge?

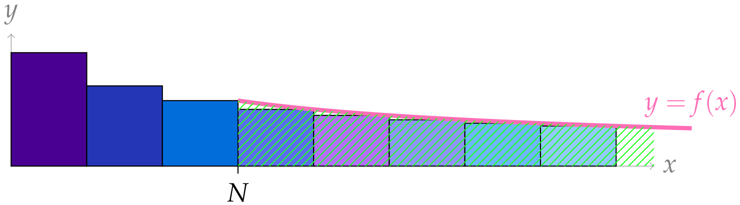


$$\int_{10}^{\infty} \frac{1}{x \log x} \, \mathrm{d}x = \infty$$

Integral Test, abridged

... When the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_N^{\infty} f(x)\, \mathrm{d}x$$



$$\underbrace{\sum_{n=1}^{N} a_n}_{} \qquad \underbrace{\sum_{n=N+1}^{\infty} a_n}_{} \quad \text{(truncation error)}$$

Integral Test, abridged

When the series converges, the truncation error satisfies

$$0 \le \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \le \int_N^{\infty} f(x)\, \mathrm{d}x$$

We already decided that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Suppose we had a computer add up the terms $n = 1$ through $n = 100$.

Use the integral test to bound the error, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} - \sum_{n=1}^{100} \frac{1}{n^2+1}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} - \sum_{n=1}^{100} \frac{1}{n^2+1} \le \int_{100}^{\infty} \frac{1}{x^2+1}\, \mathrm{d}x$$

$$= \lim_{b\to\infty} \left[\arctan(b) - \arctan(100)\right] = \frac{\pi}{2} - \arctan(100) \approx 0.01$$

By computer, $\displaystyle\sum_{n=1}^{100} \frac{1}{n^2+1} \approx 1.0667$. Using the truncation error of about 0.01, give a (small) range of possible values for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

$$\begin{array}{ccccc}
0 & \le & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} - \sum_{n=1}^{100} \frac{1}{n^2+1} & \le & \displaystyle\int_{100}^{\infty} \frac{1}{x^2+1}\,dx \\
0 & \le & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} - 1.0667 & \le & 0.01 \\
1.0667 & \le & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} & \le & 1.0767
\end{array}$$

p-TEST

Let p be a positive constant. When we talked about improper integrals, we showed:

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Set $f(x) = \dfrac{1}{x^p}$.

(i) $f(x) \geq 0$ for all $x \geq 1$, and

(ii) $f(x)$ decreases as x increases

$$\sum_{n=1}^\infty \frac{1}{n^p}\,\mathrm{d}x \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}\,.$$

By the *p*-test, we know this series

How many terms should we add up to approximate the series to within an error of no more than 0.02?

$$\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^3} \le \int_N^{\infty} \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_N^b = \frac{1}{2N^2}$$

$$\frac{1}{2N^2} \le \frac{2}{100} \implies N \ge 5$$

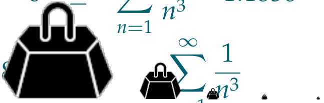
5 terms will suffice.

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \text{ converges to within 0.02 of } \sum_{n=1}^{5}\frac{1}{n^3}.$$

$$0 \le \sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{5}\frac{1}{n^3} \le 0.02$$

$$0 \le \sum_{n=1}^{\infty}\frac{1}{n^3} - 1.1856 \le 0.02$$

$$1.18 \quad \sum_{n=1}^{\infty}\frac{1}{n^3} \quad \le 1.2056$$



$1 \qquad \frac{1}{2^3} \qquad \frac{1}{3^3} \qquad \frac{1}{4^3} \qquad \frac{1}{5^3}$

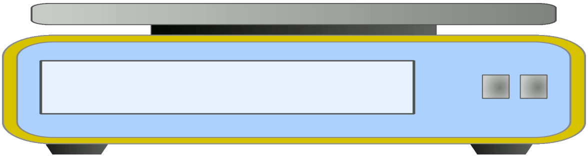
$S_1 = 1.0000$

$S_2 = 1.1250$

$S_3 = 1.1620$

$S_4 = 1.1776$

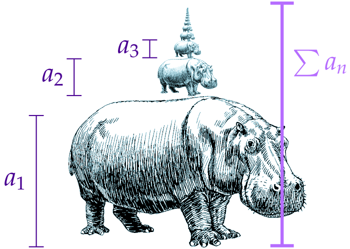
$S_5 = 1.1856$



Observation

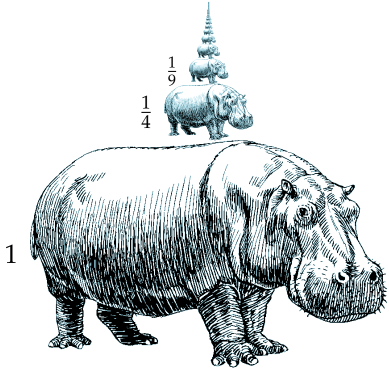
In a series with **positive** terms, the series either **converges**, or **diverges to infinity**.

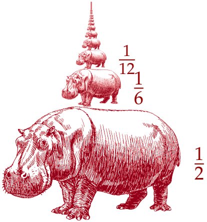
If terms are "too big," series will diverge.



$\sum \frac{1}{n^2}$ converges

$\sum \frac{1}{n^2+n}$ converges, too





Terms are "small enough" for sum to converge

Terms are also "small enough" for sum to converge

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \leq Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \geq Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n-0.1}$.

- We know $0 < \frac{1}{n} < \frac{1}{n-0.1}$
- We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)
- So, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n-0.1}$ diverges as well.

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n + \cos n}{n^3 - 1/3}$ converge or diverge?

Step 1: Intuition.

When n is very large, we expect:

- $n + \cos n \approx n$
- $n^3 + \frac{1}{3} \approx n^3$
- So, we expect $\dfrac{n + \cos n}{n^3 - 1/3} \approx \dfrac{n}{n^3} = \dfrac{1}{n^2}$.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ ... converges (by the p-test),

we expect $\displaystyle\sum_{n=1}^{\infty} \frac{n + \cos n}{n^3 - 1/3}$ to also .... converge.

Does the series $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3 - 1/3}$ converge or diverge?

Step 2: Choose comparison series.

The Comparison Test, abridged

Let N_0 be a natural number and let $K > 0$.

If $|a_n| \le K c_n$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

To show that original series converges, we should find a comparison series that also converges and whose terms (times some positive constant) are larger than the original terms. *There are many possibilities.* For $n \ge 1$,

- $n + \cos n < n + n = 2n$
- $n^3 - \frac{1}{3} > n^3 - \frac{n^3}{2} = \frac{1}{2}n^3$
- So $\frac{n+\cos n}{n^3 - 1/3} < \frac{2n}{\frac{1}{2}n^3} = 4 \cdot \frac{1}{n^2}$

Does the series $\sum_{n=1}^{\infty} \frac{n + \cos n}{n^3 - 1/3}$ converge or diverge?

Step 3: Verify.

The Comparison Test, abridged

Let N_0 be a natural number and let $K > 0$.

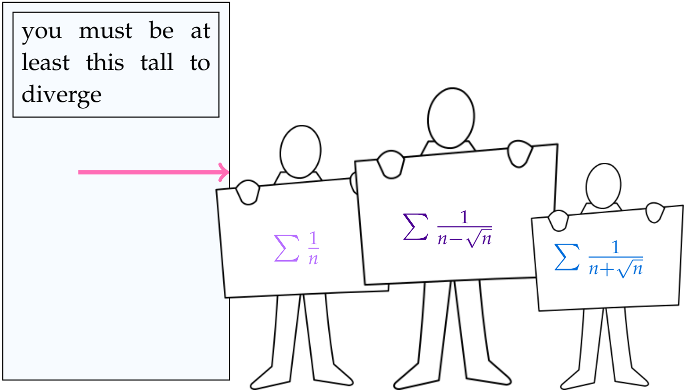
If $|a_n| \le K c_n$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

Set $c_n = \frac{1}{n^2}$ and $K = 4$. Note $\sum_{n=1}^{\infty} c_n$ converges.

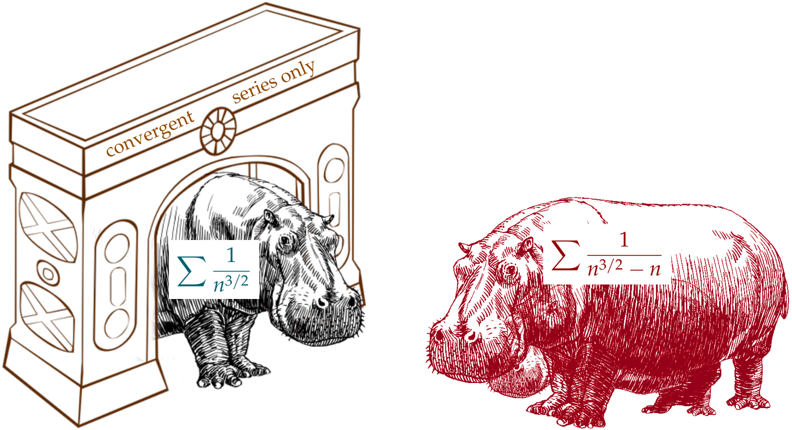
Note also $\left|\frac{n+\cos n}{n^3-1/3}\right| < \frac{n+n}{n^3-\frac{n^3}{2}} = 4 \cdot \frac{1}{n^2}$ for all $n \ge 1$.

By the comparison test, $\sum_{n=1}^{\infty} \frac{n + \cos n}{n^3 - 1/3}$ converges.

For the comparison test as we have seen it so far, to conclude that a given series diverges, we have to find a divergent comparison series whose terms are smaller than (a positive multiple of) those of our original series .



For the comparison test as we've seen it so far, to conclude that a given series converges, we have to find a convergent comparison series whose terms are larger than (a positive multiple of) those of our original series .



Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

- For large n, $a_n \approx L \cdot b_n$;
- so we expect $\sum a_n$ to behave roughly like $\sum (L \cdot b_n)$;
- and since $L \neq 0$, we expect $\sum (L \cdot b_n)$ to converge if and only if $\sum b_n$ converges.

By the p-test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

Can we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-n+1}$ also converges?

$$a_n = \frac{1}{n^{3/2}} \qquad b_n = \frac{1}{n^{3/2}-n+1}$$

$$\frac{a_n}{b_n} = \frac{n^{3/2}-n+1}{n^{3/2}} = 1 - \frac{1}{\sqrt{n}} + \frac{1}{n^{3/2}}$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = 1 - 0 + 0 = 1$$

Since L is a nonzero real number, the two series either both converge or both diverge. By the p-test, $\sum \frac{1}{n^{3/2}}$ converges. So, by the limit comparison test, $\sum \frac{1}{n^{3/2}-n+1}$ also converges.

Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$ converge or diverge?

Step 1: Intuition

For large n,

$$\frac{\sqrt{n+1}}{n^2-2n+3} \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

So, we'll use $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ as our comparison series. Since this converges, we expect our original series to converge as well.

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$ converge or diverge?

Step 2: Verify Intuition

Let $a_n = \frac{\sqrt{n+1}}{n^2-2n+3}$ and $b_n = \frac{1}{n^{3/2}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{\sqrt{n}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}\cdot\frac{1}{\sqrt{n}}}{(n^2-2n+3)\cdot\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1-\frac{2}{n}+\frac{3}{n^2}}$$

$$= \frac{\sqrt{1+0}}{1+0+0} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (by the p-test), the original series converges as well, by the Limit Comparison Theorem.

Comparison Strategies

- Before you can use either comparison test, you need to guess a series to compare.
- The series you guess should be easy to deal with.
  - p-series
  - geometric series
- Common guess (especially if monotone): consider "largest" piece of numerator and denominator
  (constant) $<$ (logarithm) $<$ (polynomial) $<$ (exponential)
- After you guess a comparison series, **show it works** by finding the correct inequality (comparison test), or computing the limit of the ratio (limit comparison test).

Choose a Series to Compare

$$\sum_{n=1}^{\infty} \frac{3n}{n^2+1}$$

One option: $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{n^2} = \sum_{n=1}^{\infty} \frac{3}{n}$

$$\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^5-n}$$

One option: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

$$\sum_{k=1}^{\infty} \frac{k(2+\sin k)}{k^{\sqrt{2}}}$$

One option: $\displaystyle\sum_{k=1}^{\infty} \frac{2k}{k^{\sqrt{2}}} = \sum_{k=1}^{\infty} \frac{2}{k^{\sqrt{2}-1}}$

$$\sum_{m=1}^{\infty} \frac{3m+\sin\sqrt{m}}{m^2}$$

One option: $\displaystyle\sum_{m=1}^{\infty} \frac{3m}{m^2} = \sum_{m=1}^{\infty} \frac{3}{m}$

Included Work

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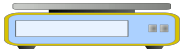
 Hippopotamus vector image is in the Public Domain (accessed January 2021), 20, 21, 27

 ‘Waage/Libra’ by B. Lachner is in the public domain (accessed April 2021, edited), 8, 10, 19

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