# Table of Contents

## Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

- the total error introduced by the midpoint rule is bounded by $\dfrac{M}{24}\dfrac{(b-a)^3}{n^2}$,
- the total error introduced by the trapezoidal rule is bounded by $\dfrac{M}{12}\dfrac{(b-a)^3}{n^2}$, and
- the total error introduced by Simpson's rule is bounded by $\dfrac{L}{180}\dfrac{(b-a)^5}{n^4}$

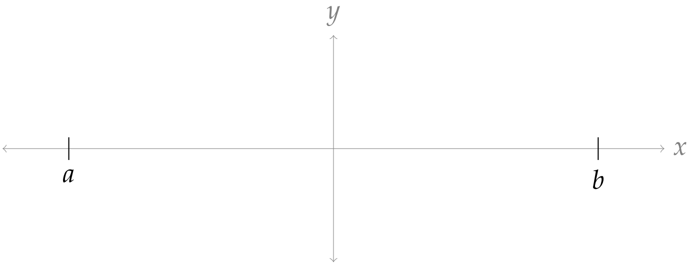
when approximating $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

# WHY THE *second* DERIVATIVE?

The midpoint rule gives the exact area under the curve for

$$f(x) = ax + b$$

when $a$ and $b$ are any constants.




The first derivative can be large without causing a large error.

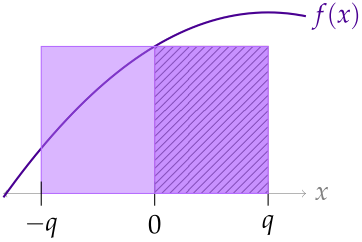
## Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

- the total error introduced by the midpoint rule is bounded by $\dfrac{M}{24}\dfrac{(b-a)^3}{n^2}$,
- the total error introduced by the trapezoidal rule is bounded by $\dfrac{M}{12}\dfrac{(b-a)^3}{n^2}$, and
- the total error introduced by Simpson's rule is bounded by $\dfrac{L}{180}\dfrac{(b-a)^5}{n^4}$

when approximating $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

We'll start small: let's consider one-half of a single interval being approximated using the midpoint rule.
To avoid messiness, let's also consider a simplified location:




We want to relate the actual area of this half-slice to its approximate area:

$$\int_0^q f(x)\,\mathrm{d}x \approx q \cdot f(0)$$

$$\int_0^q f(x)\,\mathrm{d}x \approx q \cdot f(0)$$

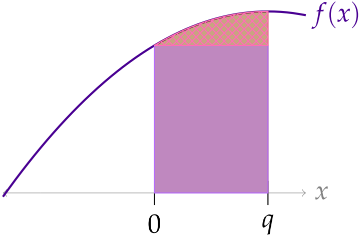
If you squint just right, the right-hand side looks a bit like the "$u \cdot v$" term from integration by parts, where $u = f(x)$ and $\mathrm{d}v = \mathrm{d}x$.

- Set $u = f(x)$ and $\mathrm{d}v = \mathrm{d}x$, so $\mathrm{d}u = f'(x)\,\mathrm{d}x$.
  We choose $v(x) = x - q$, so that $f(v(q)) = f(0)$.

$$\begin{aligned} \int_0^q f(x)\,\mathrm{d}x &= \big[(x-q)f(x)\big]_0^q - \int_0^q (x-q)f'(x)\,\mathrm{d}x \\ &= q \cdot f(0) - \int_0^q (x-q)f'(x)\,\mathrm{d}x \end{aligned}$$

- We know something about the second derivative, not the first, so repeat: set $u = f'(x)$, $\mathrm{d}v = (x-q)\,\mathrm{d}x$; $\mathrm{d}u = f''(x)\,\mathrm{d}x$, $v = \frac{(x-q)^2}{2}$

$$\int_0^q f(x)\,\mathrm{d}x = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$




$$\underbrace{\int_0^q f(x)\,\mathrm{d}x}_{\text{exact}} \quad = \quad \underbrace{q \cdot f(0)}_{\text{approximate}} \quad + \quad \underbrace{\frac{q^2}{2}\cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x}_{\pm\ \text{error}}$$

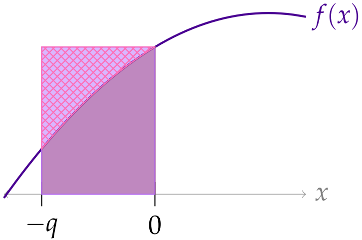
Repeat for the other half of the slice:

$$\int_{-q}^{0} \underbrace{f(x)}_{u} \underbrace{\mathrm{d}x}_{\mathrm{d}v} = \Big[\underbrace{f(x)}_{u} \cdot \underbrace{(x+q)}_{v}\Big]_{-q}^{0} - \int_{-q}^{0} \underbrace{(x+q)}_{v} \cdot \underbrace{f'(x)\,\mathrm{d}x}_{\mathrm{d}u}$$

$$= q \cdot f(0) - \int_{-q}^{0} \underbrace{f'(x)}_{\hat{u}} \cdot \underbrace{(x+q)\,\mathrm{d}x}_{\mathrm{d}\hat{v}}$$

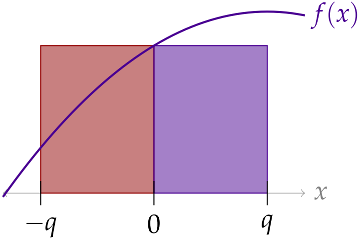
$$= q \cdot f(0) - \left[\underbrace{f'(x)}_{\hat{u}} \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}}\right]_{-q}^{0} + \int_{-q}^{0} \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}} \underbrace{f''(x)\,\mathrm{d}x}_{\mathrm{d}\hat{u}}$$

$$= q \cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x$$




$$\int_{-q}^{0} f(x)\,\mathrm{d}x \quad = \quad q \cdot f(0) \quad - \quad \frac{q^2}{2} \cdot f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x$$

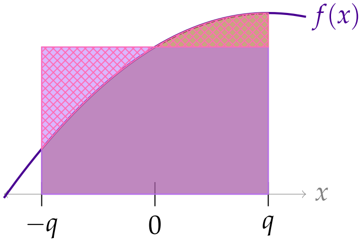
exact — approximate — $\pm$ error




$$\int_{-q}^{0} f(x)\,\mathrm{d}x = q\cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x$$

$$\int_{0}^{q} f(x)\,\mathrm{d}x = q\cdot f(0) + \frac{q^2}{2}\cdot f'(0) + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$

$$\int_{-q}^{q} f(x)\,\mathrm{d}x = 2q\cdot f(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$




$$\int_{-q}^{q} f(x)\,\mathrm{d}x \quad = \quad 2q \cdot f(0) + \quad \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$

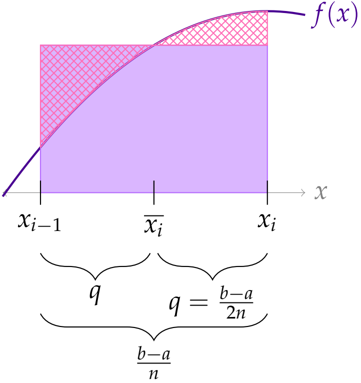
exact     approximate     ± error

We re-arrange to write the error as the difference between the actual area of one slice and its rectangular approximation.

$$\int_{-q}^{q} f(x)\,\mathrm{d}x - 2q \cdot f(0) = \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$

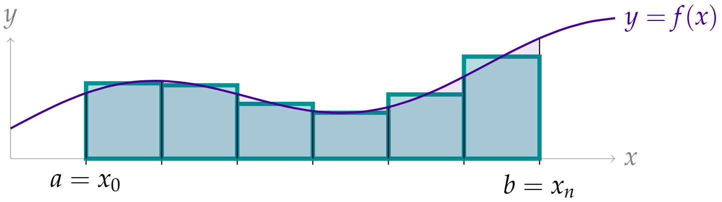
$$\begin{aligned}
\text{error} &= \left| \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x \right| \\
&\leq \left| \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x \right| + \left| \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x \right| \\
&\leq \int_{-q}^{0} \frac{(x+q)^2}{2} M\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} M\,\mathrm{d}x \\
&= M \left[ \frac{(x+q)^3}{6} \right]_{-q}^{0} + M \left[ \frac{(x-q)^3}{6} \right]_{0}^{q} \\
&= \frac{M \cdot q^3}{3}
\end{aligned}$$

Now we can bound the error of a single slice:




$$\left| \int_{-q}^{q} f(x)\,\mathrm{d}x - 2q \cdot f(0) \right| \leq \frac{M}{3} \cdot q^3$$

$$\left| \int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x - \frac{b-a}{n} \cdot f\left(\overline{x_i}\right) \right| \leq \frac{M}{3} \left( \frac{b-a}{2n} \right)^3 = \frac{M}{24} \frac{(b-a)^3}{n^3}$$




- The error in each slice is at most $\frac{M}{24}\frac{(b-a)^3}{n^3}$
- There are $n$ slices
- The overall error is at most $n \cdot \frac{M}{24}\frac{(b-a)^3}{n^3} = \frac{M}{24}\frac{(b-a)^3}{n^2}$