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Warmup

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x$.

$$\begin{aligned}\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x &= \int_3^7 \frac{1}{\sqrt{(x+1)^2}}\,\mathrm{d}x \\ &= \int_3^7 \frac{1}{|x+1|}\,\mathrm{d}x\end{aligned}$$

When $3 \le x \le 7$, we have $|x+1| = x+1$.

$$\begin{aligned}&= \int_3^7 \frac{1}{x+1}\,\mathrm{d}x \\ &= \left[\log|x+1|\right]_3^7 \\ &= \log 8 - \log 4 = \log 2\end{aligned}$$

Idea: $\sqrt{(\text{something})^2} = |\text{something}|$. We cancelled off the square root.

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$.

We still want to cancel off the square root, but x^2+1 is not obviously of the form $(\text{something})^2$.
Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$.

$$\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \int \frac{1}{\sqrt{\tan^2\theta+1}}\sec^2\theta\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\mathrm{d}\theta$$
$$= \int \frac{\sec^2\theta}{\sec\theta}\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \log|\sec\theta + \tan\theta| + C$$

We need to get these back in terms of x. From our substitution, we know $\tan\theta = x$. From simplifying our denominator, we also know $\sec\theta = \sqrt{x^2+1}$.

$$= \log\left|\sqrt{x^2+1} + x\right| + C$$

Same idea: $\sqrt{(\text{something})^2} = |\text{something}|$; cancel off the square root.

CHECK OUR WORK

Let's verify that $\int \frac{1}{\sqrt{x^2+1}} =$.
Seems improbable, right?

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log\left|\sqrt{x^2+1}+x\right|+C\right] = \frac{1}{\sqrt{x^2+1}+x}\cdot\left(\frac{2x}{2\sqrt{x^2+1}}+1\right)$$

$$= \frac{x+\sqrt{x^2+1}}{(\sqrt{x^2+1}+x)\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

So, our answer works!

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into
- After integrating, convert back to the original variable (possibly using a triangle–more details later)

Focus on the Algebra

$1-\sin^2\theta=\cos^2\theta$ $\quad 1+\tan^2\theta=\sec^2\theta$ $\quad \sec^2\theta-1=\tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2-1}$
  Let $x=\sec\theta$, so $\sqrt{x^2-1}$ becomes $\sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=|\tan\theta|$
- $\sqrt{x^2+1}$
  Let $x=\tan\theta$, so $\sqrt{x^2+1}$ becomes $\sqrt{\tan^2\theta+1}=\sqrt{\sec^2\theta}=|\sec\theta|$
- $\sqrt{1-x^2}$
  Let $x=\sin\theta$ so $\sqrt{1-x^2}$ becomes $\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=|\cos\theta|$
  (Alternately, $x=\cos\theta$ works as well)

Focus on the Algebra

$1-\sin^2\theta=\cos^2\theta$ $\qquad 1+\tan^2\theta=\sec^2\theta$ $\qquad \sec^2\theta-1=\tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2+7}$
  Adjust a given identity by multiplying both sides by 7:
  $7\tan^2\theta+7=7\sec^2\theta$. Now we see we want $x^2=7\tan^2\theta$. That is, $x=\sqrt{7}\tan\theta$:
  $\sqrt{x^2+7}=\sqrt{7\tan^2\theta+7}=\sqrt{7(\sec^2\theta)}=\sqrt{7}\,|\sec\theta|$

- $\sqrt{3-2x^2}$
  Adjust a given identity by multiplying both sides by 3:
  $3-3\sin^2\theta=3\cos^2\theta$. Now we see we want $2x^2=3\sin^2\theta$, so $x=\sqrt{\frac{3}{2}}\sin\theta$:
  $\sqrt{3-2x^2}=\sqrt{3-2\left(\frac{3}{2}\sin^2\theta\right)}=\sqrt{3-3\sin^2\theta}=\sqrt{3\cos^2\theta}=\sqrt{3}\,|\cos\theta|$

Closer look at absolute values

SKIP CLOSER LOOK

Consider the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?
If $x = 0$, then $\sin\theta = 0$, but there are infinitely many values of θ that could make this true. To use the substitution $x = \sin\theta$, we need the function $x = \sin\theta$ to be invertible. That way, we can unambiguously convert between x and θ. With that in mind, we'll actually set $\theta = \arcsin x$. Now θ is restricted to the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_{\arcsin 0}^{\arcsin 1} \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cdot\cos\theta\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{2}} |\cos\theta|\cdot\cos\theta\,\mathrm{d}\theta$$

For $0 \leq \theta \leq \frac{\pi}{2}$, we have $\cos\theta \geq 0$, so $|\cos\theta| = \cos\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

Closer look at absolute values

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

Absolute Values

From now on, we will assume:

- With the substitution $x = a\sin\theta$ for $\sqrt{a^2 - x^2}$, $|\cos\theta| = \cos\theta$
- With the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, $|\sec\theta| = \sec\theta$

Identities

$$1-\sin^2\theta=\cos^2\theta \qquad \sin(2\theta)=2\sin\theta\cos\theta$$

$$1+\tan^2\theta=\sec^2\theta \qquad \sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta-1=\tan^2\theta \qquad \cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

Let $x=\tan\theta$, $\mathrm{d}x=\sec^2\theta\,\mathrm{d}\theta$. When $x=0$, then $\theta=\arctan 0=0$; when $x=1$, then $\theta=\arctan 1=\frac{\pi}{4}$.

$$\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x=\int_{\theta=0}^{\theta=\pi/4}\frac{1}{\sqrt{1+\tan^2\theta}^3}\sec^2\theta\,\mathrm{d}\theta$$

$$=\int_0^{\pi/4}\frac{\sec^2\theta}{\sqrt{\sec^2\theta}^3}\mathrm{d}\theta=\int_0^{\pi/4}\frac{\sec^2\theta}{|\sec\theta|^3}\mathrm{d}\theta$$

Identities

$$1-\sin^2\theta=\cos^2\theta \qquad \sin(2\theta)=\cos\theta$$

$$1+\tan^2\theta=\sec^2\theta \qquad \sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta-1=\tan^2\theta \qquad \cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \sqrt{1-4x^2}\,\mathrm{d}x$

Under the square root, we have "one minus a term with a variable," which matches the identity $1-\sin^2\theta$. So, we want $4x^2$ to become $\sin^2\theta$. That is, $x=\frac{1}{2}\sin\theta$. Then $\mathrm{d}x=\frac{1}{2}\cos\theta\,\mathrm{d}\theta$.

$$\int\sqrt{1-4x^2}\,\mathrm{d}x=\int\sqrt{1-4\left(\frac{1}{2}\sin\theta\right)^2}\cdot\frac{1}{2}\cos\theta\,\mathrm{d}\theta$$

$$=\frac{1}{2}\int\sqrt{1-\sin^2\theta}\cdot\cos\theta\mathrm{d}\theta=\frac{1}{2}\int\sqrt{\cos^2\theta}\cdot\cos\theta\mathrm{d}\theta$$

Check Our Work

In the last example, we computed

$$\int \sqrt{1-4x^2}\,\mathrm{d}x =$$

To check, we differentiate.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{4}\left(\arcsin(2x)+2x\sqrt{1-4x^2}\right)+C\right\}\\
=&\frac{1}{4}\left(\frac{2}{\sqrt{1-(2x)^2}}+2x\frac{-8x}{2\sqrt{1-4x^2}}+2\sqrt{1-4x^2}\right)\\
=&\frac{1}{4}\left(\frac{2}{\sqrt{1-4x^2}}-\frac{8x^2}{\sqrt{1-4x^2}}+\frac{2(1-4x^2)}{\sqrt{1-4x^2}}\right)\\
&=\frac{1}{4}\left(\frac{2-8x^2+2-8x^2}{\sqrt{1-4x^2}}\right)=\frac{1-4x^2}{\sqrt{1-4x^2}}=\sqrt{1-4x^2} \qquad \checkmark
\end{aligned}$$

Identities

$$1-\sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1+\tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$

We use the substitution $x=\sec\theta$, $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$.

To make the substitution work, we're actually taking $\theta = \arccos\left(\frac{1}{x}\right)$, and so $0 \le \theta \le \pi$.

Note that the integrand exists on the intervals $x<-1$ and $x>1$.

- When $x>1$, then $0<\frac{1}{x}<1$, so $0<\arccos\left(\frac{1}{x}\right)<\frac{\pi}{2}$.
  That is, $0<\theta<\frac{\pi}{2}$, so $|\tan\theta| = \tan\theta$.
- When $x<-1$, then $-1<\frac{1}{x}<0$, so $\frac{\pi}{2}<\arccos\left(\frac{1}{x}\right)<\pi$.
  That is, $\frac{\pi}{2}<\theta<\pi$, so $|\tan\theta| = -\tan\theta$.

CHECK OUR WORK

Let's check our result, $\int \frac{1}{\sqrt{x^2-1}} \mathrm{d}x =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log\left|x+\sqrt{x^2-1}\right|+C\right\} = \frac{1+\frac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$

$$= \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}\right) = \frac{(\sqrt{x^2-1}+x)}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

So, our answer works.

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$ $\qquad 4 - 4\sin^2\theta = 4\cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.
Fact: $3 - x^2 + 2x = 4 - (x-1)^2$

$$\sqrt{3 - x^2 + 2x} = \sqrt{4 - (x-1)^2}$$

We want $(x-1)^2 = 4\sin^2\theta$, so let $(x-1) = 2\sin\theta$

$$= \sqrt{4 - 4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b:

2. Solve for c:

3. All together:

Evaluate $\displaystyle\int \frac{x^2-6x+9}{\sqrt{6x-x^2}}\,\mathrm{d}x$.

Identities have two "parts" that turn into one part:

- $1-\sin^2\theta=\cos^2\theta$
- $1+\tan^2\theta=\sec^2\theta$
- $\sec^2\theta-1=\tan^2\theta$

One of those parts is a constant, and one is squared.
Write $6x-x^2$ as $c-(x+b)^2$.

$$c-(x+b)^2=(c-b^2)-x^2-2bx$$

$$6x=-2bx \implies b=-3$$

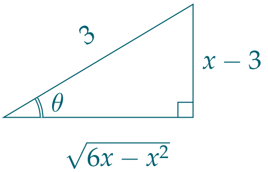
$$0=c-b^2=c-9 \implies c=9$$

$$6x-x^2=9-(x-3)^2$$

Evaluate $\displaystyle\int \frac{x^2-6x+9}{\sqrt{6x-x^2}}\,\mathrm{d}x = \int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x.$

We use the identity $9-9\sin^2\theta = 9\cos^2\theta$.
We want $(x-3)^2 = 9\sin^2\theta$, so take $(x-3) = 3\sin\theta$, $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x &= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,\mathrm{d}\theta \\
&= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}}\,3\cos\theta\,\mathrm{d}\theta = \int 9\sin^2\theta\,\mathrm{d}\theta \\
&= \frac{9}{2}\int (1-\cos 2\theta)\,\mathrm{d}\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + C \\
&= \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C
\end{aligned}$$



Check Our Work

Let's verify that

$$\int \frac{x^2-6x+9}{\sqrt{6x-x^2}} =$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right)-\frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right)+C\right\}\\
&\quad=\frac{9}{2}\left(\frac{1/3}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}-\frac{x-3}{3}\cdot\frac{3-x}{3\sqrt{6x-x^2}}-\frac{1}{9}\sqrt{6x-x^2}\right)\\
&\quad=\frac{9}{2}\left(\frac{9}{9\sqrt{6x-x^2}}-\frac{6x-x^2-9}{9\sqrt{6x-x^2}}-\frac{6x-x^2}{9\sqrt{6x-x^2}}\right)\\
&\quad=\frac{9-6x+x^2}{\sqrt{6x-x^2}}
\end{aligned}$$

So, our answer works.