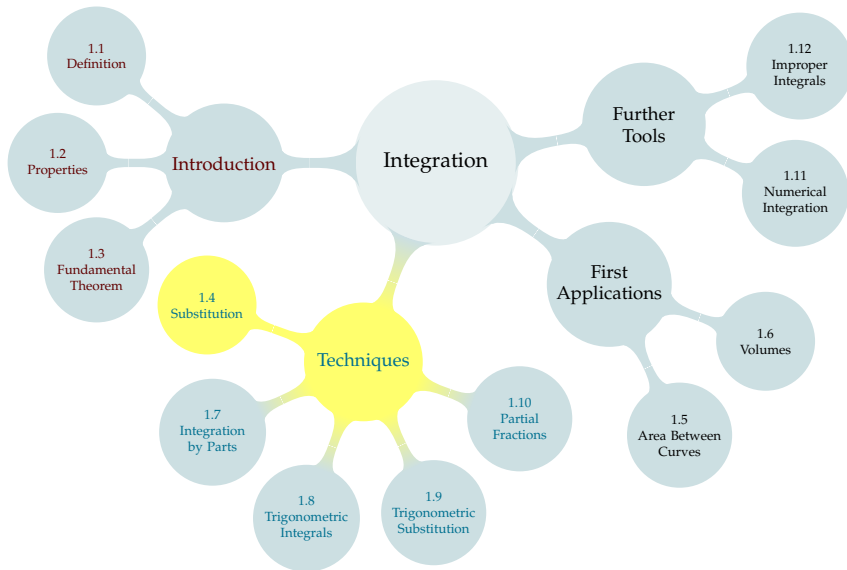


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ANTIDERIVATIVES

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Hard to guess the antiderivative without seeing the derivative first!

ANTIDERIVATIVES

Chain Rule:

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Hallmark of the chain rule: an “inside” function, with that function’s derivative multiplied.

SOLVE BY INSPECTION

$$\int 2xe^{x^2+1} dx$$

$$\int \frac{1}{x} \cos(\log x) dx$$

$$\int 3(\sin x + 1)^2 \cos x dx$$

(Look for an “inside” function, with its derivative multiplied.)

UNDOING THE CHAIN RULE

Chain Rule:

$$\frac{d}{dx} \{f(u(x))\} = f'(u(x)) \cdot u'(x)$$

(Here, $u(x)$ is our “inside function”)

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x) \, dx = f(u(x)) + C$$

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Shorthand: call $u(x)$ simply u ;
since $\frac{du}{dx} = u'(x)$, call $u'(x) \, dx$ simply du .

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$$\int f'(u(x)) \cdot u'(x) \, dx = \int f'(u) \, du \Big|_{u=u(x)} = f(u(x)) + C$$

This is the **substitution rule**.

We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1} dx$$

$$\int \frac{1}{x} \cos(\log x) dx$$

$$\int 3(\sin x + 1)^2 \cos x dx$$



$$\int (3x^2) \sin(x^3 + 1) \, dx =$$

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“Inside” function: $x^3 + 1$.

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“Inside” function: $x^3 + 1$. Its derivative: $3x^2$

Shorthand: $x^3 + 1 \rightarrow u$, $3x^2 \, dx \rightarrow du$

$$\int (3x^2) \sin(x^3 + 1) \, dx = \int \sin(u) \, du \Big|_{u=x^3+1}$$

“Inside” function: $x^3 + 1$. Its derivative: $3x^2$

Shorthand: $x^3 + 1 \rightarrow u$, $3x^2 \, dx \rightarrow du$

$$\begin{aligned}\int (3x^2) \sin(x^3 + 1) \, dx &= \int \sin(u) \, du \Big|_{u=x^3+1} \\ &= -\cos(u) + C \Big|_{u=x^3+1}\end{aligned}$$

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“Inside” function: $x^3 + 1$. Its derivative: $3x^2$

Shorthand: $x^3 + 1 \rightarrow u$, $3x^2 dx \rightarrow du$

Warning 1: We don’t just change dx to du . We need to couple dx with the derivative of our inside function.

After all, we’re undoing the chain rule! We need to have an “inside derivative.”

$$\begin{aligned}\int (3x^2) \sin(x^3 + 1) \, dx &= \int \sin(u) \, du \Big|_{u=x^3+1} \\ &= -\cos(u) + C \Big|_{u=x^3+1} \\ &= \cos(x^3 + 1) + C\end{aligned}$$

“Inside” function: $x^3 + 1$. Its derivative: $3x^2$

Shorthand: $x^3 + 1 \rightarrow u$, $3x^2 \, dx \rightarrow du$

Warning 2: The final answer is a function of x .

We used the substitution rule to conclude

$$\int (3x^2) \sin(x^3 + 1) \, dx = -\cos(x^3 + 1) + C$$

We can check that our antiderivative is correct by differentiating.

We saw:

$$\int 3x^2 \sin(x^3 + 1) \, dx = -\cos(x^3 + 1) + C$$

So, we can evaluate:

$$\int_0^1 3x^2 \sin(x^3 + 1) \, dx$$

NOTATION: LIMITS OF INTEGRATION

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

NOTATION: LIMITS OF INTEGRATION

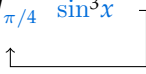
$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

Let $u = \sin x$, $du = \cos x dx$. Note the limits (or bounds) of integration $\pi/4$ and $\pi/2$ are values of x , not u : they follow the differential, unless otherwise specified.

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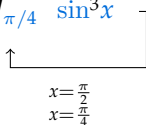
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$x = \frac{\pi}{2}$
 $x = \frac{\pi}{4}$

NOTATION: LIMITS OF INTEGRATION

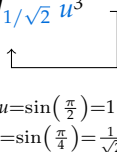
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$$\int_{1/\sqrt{2}}^1 \frac{1}{u^3} du$$


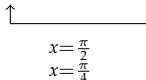
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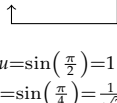
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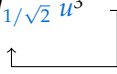
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different

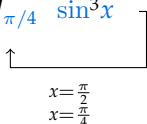
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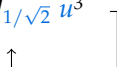
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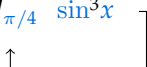
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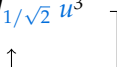
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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx = \int_{x=\pi/4}^{x=\pi/2} \frac{1}{u^3} du = \int_{1/\sqrt{2}}^1 \frac{1}{u^3} du$$

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not standard, but OK

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

Let $u = \sin x$, $du = \cos x dx$.

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx = \int_{1/\sqrt{2}}^1 \frac{1}{u^3} du$$

TRUE OR FALSE?

1. Using $u = x^2$,

$$\int e^{x^2} dx = \int e^u du$$

2. Using $u = x^2 + 1$,

$$\int_0^1 x \sin(x^2 + 1) dx = \int_0^1 \frac{1}{2} \sin u du$$

Evaluate $\int_0^1 x^7 (x^4 + 1)^5 dx$.

Time permitting, more examples using the substitution rule

Evaluate $\int \sin x \cos x \, dx$.

CHECK OUR WORK

We can check that $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$ by differentiating.

We can check that $\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + C$ by differentiating.

Evaluate $\int \frac{\log x}{3x} dx$.

CHECK OUR WORK

We can check that $\int \frac{\log x}{3x} dx = \frac{1}{6} \log^2 x + C$ by differentiating.

Evaluate $\int \frac{e^x}{e^x + 15} dx$.

Evaluate $\int x^4(x^5 + 1)^8 dx$.



CHECK OUR WORK

We can check that $\int \frac{e^x}{e^x + 15} dx = \log |e^x + 15| + C$ by differentiating.

We can check that $\int x^4(x^5 + 1)^8 dx = \frac{1}{45}(x^5 + 1)^9 + C$ by differentiating.

Evaluate $\int_4^8 \frac{s}{s-3} ds$. Be careful to use correct notation.

Evaluate $\int x^9(x^5 + 1)^8 dx$.

CHECK OUR WORK

We can check that $\int x^9(x^5 + 1)^8 \, dx = \frac{1}{5} \left[\frac{(x^5 + 1)^{10}}{10} - \frac{(x^5 + 1)^9}{9} \right] + C$
by differentiating.

PARTICULARLY TRICKY SUBSTITUTION

Evaluate $\int \frac{1}{e^x + e^{-x}} dx$.



CHECK OUR WORK

We can check that $\int \frac{1}{e^x + e^{-x}} dx = \arctan(e^x) + C$ by differentiating.

Included Work



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