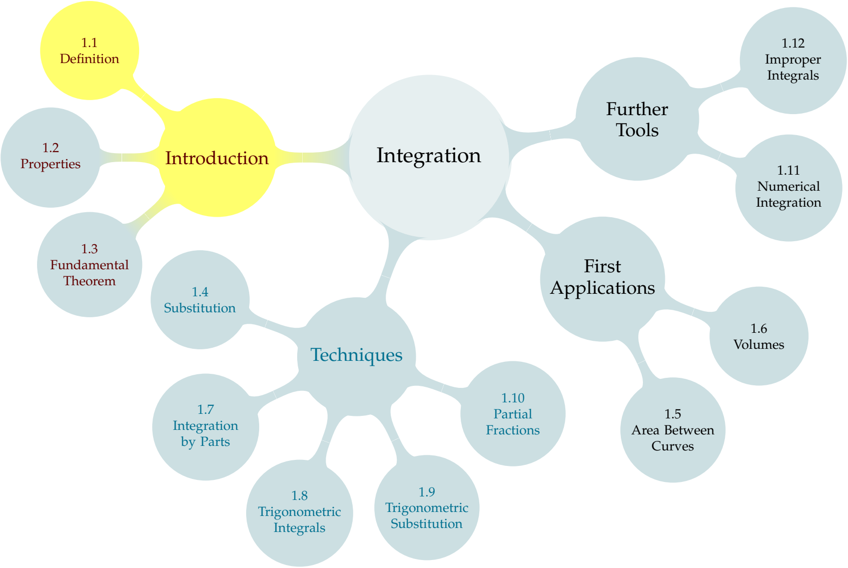
Table of Contents



Calculus is build on two operations: differentiation and integration.

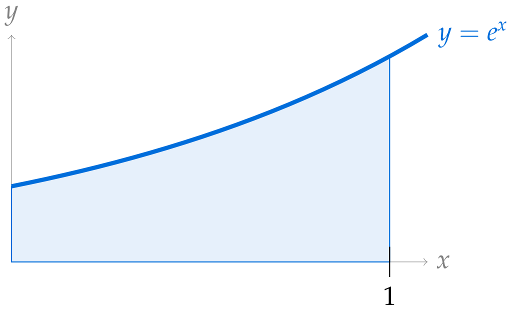
Differentiation

- Slope of a line
- Rate of change
- Optimization
- Numerical Approximations

Integration

- Area under a curve
- “Reverse” of differentiation
- Solving differential equations
- Calculate net change from rate of change
- Volume of solids
- Work (in the physics sense)

Approximate the area of the shaded region using rectangles.



We're going to be doing a lot of adding.

Summation (Sigma) Notation

$$\sum_{i=a}^{b} f(i)$$

- a, b (integers with $a \leq b$) "bounds"
- i "index:" integer which runs from a to b
- $f(i)$ "summands:" compute for every i, add

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

Sigma Notation

Expand $\displaystyle\sum_{i=2}^{4}(2i+5)$.

$$\sum_{i=2}^{4}(2i+5) = \underbrace{(2\cdot 2+5)}_{i=2}+\underbrace{(2\cdot 3+5)}_{i=3}+\underbrace{(2\cdot 4+5)}_{i=4}$$

$$= 9+11+13 = 33$$

Sigma Notation

$$\text{Expand } \sum_{i=1}^{4}(i+(i-1)^2).$$

$$\begin{aligned} &= \underbrace{(1+0^2)}_{i=1} + \underbrace{(2+1^2)}_{i=2} + \underbrace{(3+2^2)}_{i=3} + \underbrace{(4+3^2)}_{i=4} \\ &= 1+3+7+13 = 24 \end{aligned}$$

Write the following expressions in sigma notation:

- $3+4+5+6+7$

  $\sum_{i=3}^{7} i$ and $\sum_{i=1}^{5}(i+2)$ are two options (others are possible)

- $8+8+8+8+8$

  $\sum_{i=1}^{5} 8$ is one way (others are possible)

- $1+(-2)+4+(-8)+16$

  $\sum_{i=0}^{4}(-2)^i$ is one way (others are possible)

Q

Arithmetic of Summation Notation

Let c be a constant.

- Adding constants: $\displaystyle\sum_{i=1}^{10} c =$
- Factoring constants: $\displaystyle\sum_{i=1}^{10} 5(i^2) =$
- Addition is Commutative: $\displaystyle\sum_{i=1}^{10} (i + i^2) =$

Arithmetic of Summation Notation

Let c be a constant.

- Adding constants: $\sum_{i=1}^{10} c = 10c$
- Factoring constants: $\sum_{i=1}^{10} 5(i^2) = 5\sum_{i=1}^{10}(i^2)$
- Addition is Commutative: $\sum_{i=1}^{10}(i+i^2) = \left(\sum_{i=1}^{10} i\right) + \left(\sum_{i=1}^{10} i^2\right)$

COMMON SUMS

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\sum_{i=1}^{13}(i^2 + i^3) = \sum_{i=1}^{13} i^2 + \sum_{i=1}^{13} i^3 = \frac{13(14)(27)}{6} + \frac{13^2(14^2)}{4}$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\displaystyle\sum_{i=1}^{50}(1-i^2) = \sum_{i=1}^{50} 1 - \sum_{i=1}^{50} i^2 = 50 - \frac{50(51)(101)}{6}$

(Optional) Proof of a Common Sum

Here is a derivation of $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$:

$$A = 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + \cancel{r^n}$$

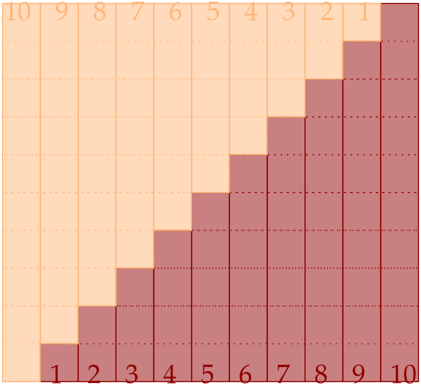
$$rA = \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + \cancel{r^n} + r^{n+1}$$

subtract $\quad A - rA = 1 - r^{n+1}$

divide across $\quad A = \dfrac{1-r^{n+1}}{1-r}$

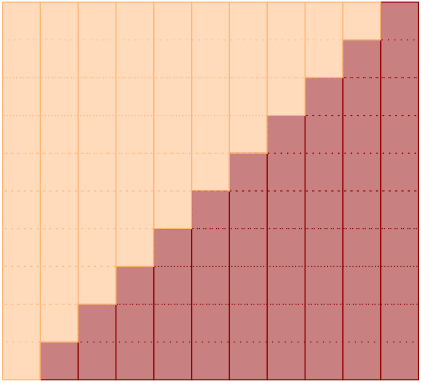
(Optional) Proof of Another Common Sum

$$\sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 = \frac{10 \cdot 11}{2}$$



(Optional) Proof of a Common Sum

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n \cdot (n+1)}{2}$$



The purpose of these sums is to describe areas.

Notation

The symbol

$$\int_a^b f(x)\,\mathrm{d}x$$

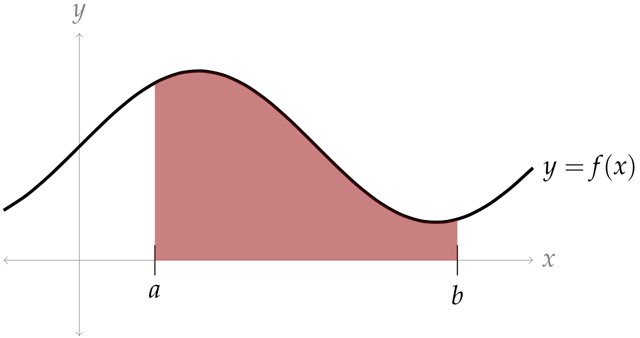
is read "the definite integral of the function $f(x)$ from a to b."

- $f(x)$: integrand
- a and b: limits of integration
- $\mathrm{d}x$: differential

If $f(x) \geq 0$ and $a \leq b$, one interpretation of

$$\int_a^b f(x)\,\mathrm{d}x$$

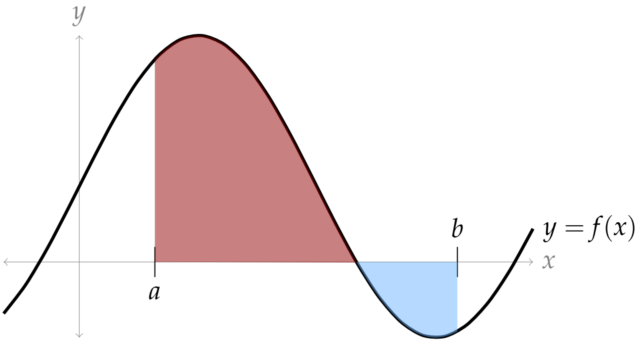
is "the area of the region bounded above by $y = f(x)$, below by $y = 0$, to the left by $x = a$, and to the right by $x = b$."



If ~~$f(x) \geq 0$ and~~ $a \leq b$, one interpretation of

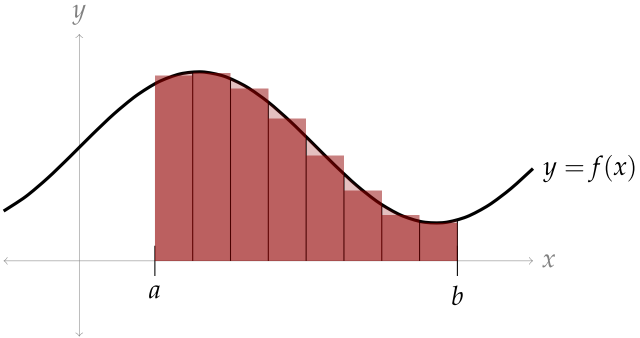
$$\int_a^b f(x)\,\mathrm{d}x$$

is the signed area of the region between $y = f(x)$ and $y = 0$, from $x = a$ to $x = b$. Area above the axis is positive, and area below it is negative.



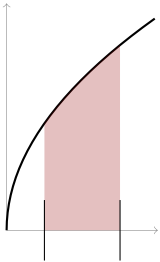
Riemann Sums

A Riemann sum approximates the area under a curve by cutting it into equal-width segments, and approximating each segment as a rectangle.

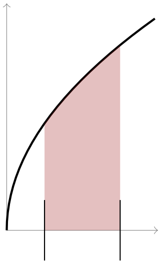


There are different ways to choose the height of each rectangle.

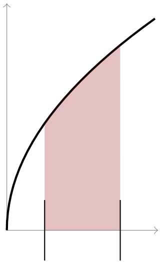
Types of Riemann Sums (RS)



Left RS

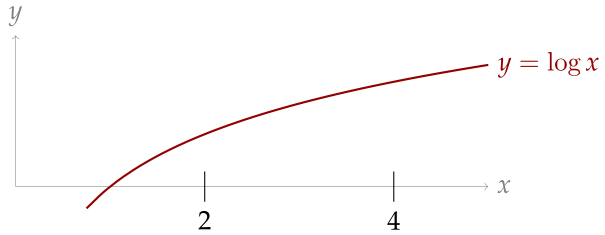


Right RS

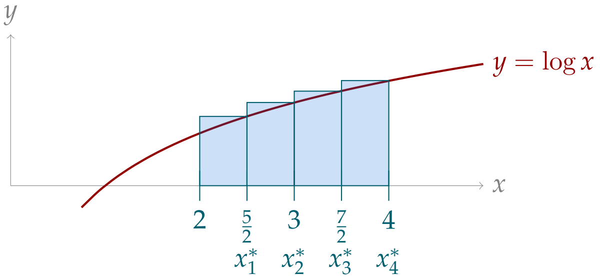


Midpoint RS

Approximate $\int_2^4 \log(x)\,\mathrm{d}x$ using a right Riemann sum with $n = 4$ rectangles. For now, do not use sigma notation.

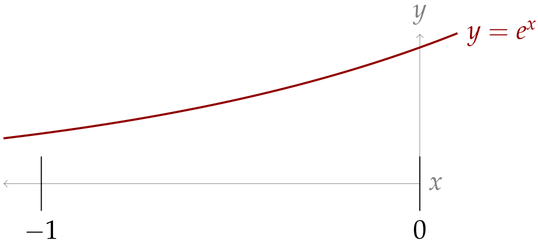


Approximate $\int_2^4 \log(x)\,\mathrm{d}x$ using a right Riemann sum with $n = 4$ rectangles. For now, do not use sigma notation.

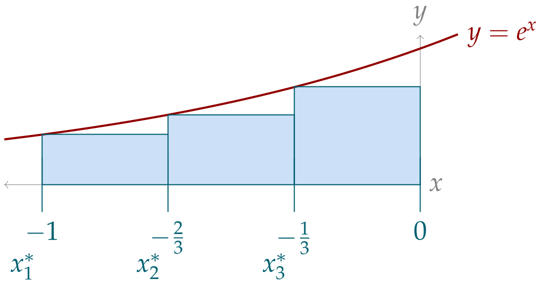


- Width of each rectangle: $\frac{4-2}{4} = \frac{1}{2}$
- Heights taken at right endpoints of rectangles:
  $x_1^* = \frac{5}{2}, x_2^* = 3, x_3^* = \frac{7}{2}, x_4^* = 4$
  $$\int_2^4 \log(x)\,\mathrm{d}x \approx \frac{1}{2}\log\left(\frac{5}{2}\right) + \frac{1}{2}\log(3) + \frac{1}{2}\log\left(\frac{7}{2}\right) + \frac{1}{2}\log(4)$$

Approximate $\int_{-1}^{0} e^x \, dx$ using a left Riemann sum with $n = 3$ rectangles. For now, do not use sigma notation.



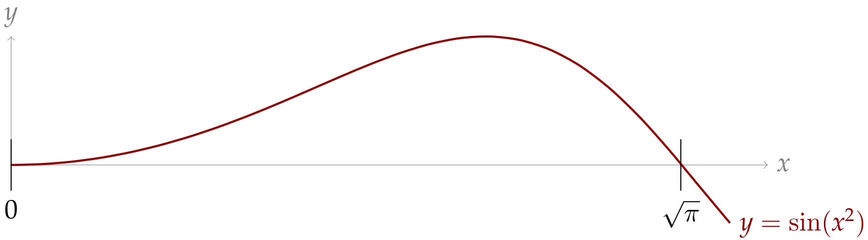
Approximate $\int_{-1}^{0} e^x \, \mathrm{d}x$ using a left Riemann sum with $n = 3$ rectangles. For now, do not use sigma notation.



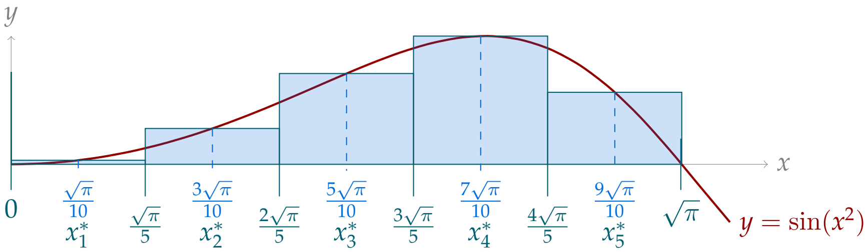
- Width of each rectangle: $\frac{0-(-1)}{3} = \frac{1}{3}$
- Heights taken at left endpoints of rectangles: $x_1^* = -1, x_2^* = -\frac{2}{3}, x_3^* = -\frac{1}{3}$

$$\int_{-1}^{0} e^x \, \mathrm{d}x \approx \frac{1}{3}e^{-1} + \frac{1}{3}e^{-2/3} + \frac{1}{3}e^{-1/3}$$

Approximate $\int_0^{\sqrt{\pi}} \sin\left(x^2\right) \mathrm{d}x$ using a midpoint Riemann sum with $n = 5$ rectangles. For now, do not use sigma notation.



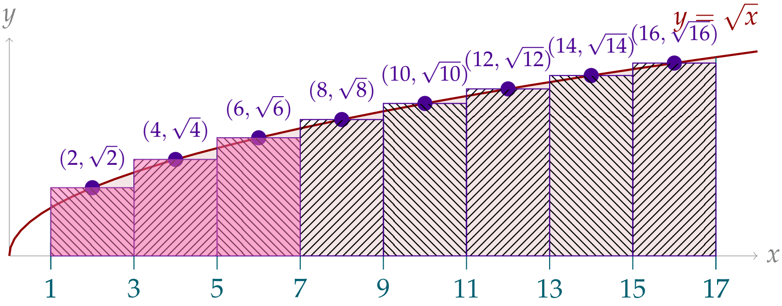
Approximate $\int_0^{\sqrt{\pi}} \sin\left(x^2\right) \mathrm{d}x$ using a midpoint Riemann sum with $n = 5$ rectangles. For now, do not use sigma notation.



- Width of each rectangle: $\frac{\sqrt{\pi}-0}{5} = \frac{\sqrt{\pi}}{5}$
- Heights taken at midpoints of rectangles:
  $x_1^* = \frac{\sqrt{\pi}}{10}$, $x_2^* = \frac{3\sqrt{\pi}}{10}$, $x_3^* = \frac{5\sqrt{\pi}}{10}$, $x_4^* = \frac{7\sqrt{\pi}}{10}$, $x_5^* = \frac{9\sqrt{\pi}}{10}$

$$\frac{\sqrt{\pi}}{5}\left[\sin\left(\frac{\pi}{100}\right) + \sin\left(\frac{9\pi}{100}\right) + \sin\left(\frac{25\pi}{100}\right) + \sin\left(\frac{49\pi}{100}\right) + \sin\left(\frac{81\pi}{100}\right)\right]$$

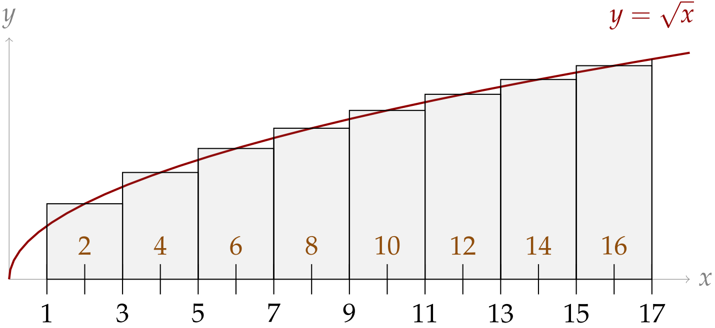
Approximate $\int_1^{17} \sqrt{x}\, dx$ using a midpoint Riemann sum with 8 rectangles. Write the result in sigma notation.



First $i = 1$	Second $i = 2$	Third $i = 3$	$\cdots i$
Base: 2 Height: $\sqrt{2}$	Base: 2 Height: $\sqrt{4}$	Base: 2 Height: $\sqrt{6}$	$\cdots$ Base: 2 $\cdots$ Height: $\sqrt{2i}$

The

$$\sum_{i=1}^{8} 2\sqrt{2i} = \underbrace{2\sqrt{2}}_{i=1} + \underbrace{2\sqrt{4}}_{i=2} + \underbrace{2\sqrt{6}}_{i=3} + \underbrace{2\sqrt{8}}_{i=4} + \underbrace{2\sqrt{10}}_{i=5} + \underbrace{2\sqrt{12}}_{i=6} + \underbrace{2\sqrt{14}}_{i=7} + \underbrace{2\sqrt{16}}_{i=8}$$

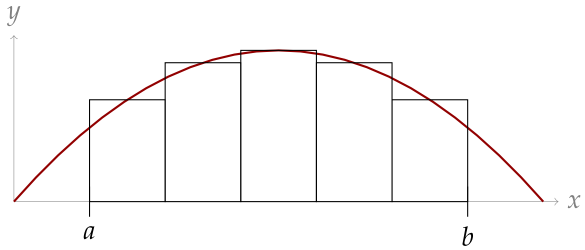


Riemann sum with n rectangles

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} \Delta x \cdot f(x_{i,n}^*)$$

where $\Delta x = \frac{b-a}{n}$ and $x_{i,n}^*$ is an x-value in the ith rectangle.

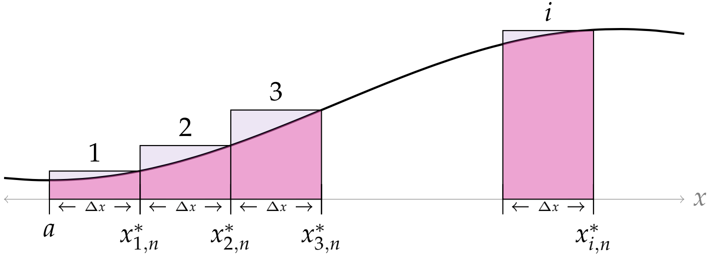
$$\sum_{i=1}^{n} \Delta x \cdot f(x_{i,n}^*) = \Delta x \cdot f\left(x_{1,n}^*\right) + \Delta x \cdot f\left(x_{2,n}^*\right) + \Delta x \cdot f\left(x_{3,n}^*\right) + \cdots + \Delta x \cdot f\left(x_{n,n}^*\right)$$



Right Riemann sum with n rectangles

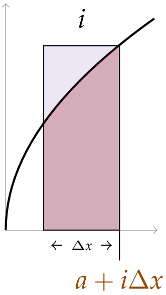
$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \Delta x \cdot f\left(x_{i,n}^* a + i\Delta x\right)$$

where $\Delta x = \frac{b-a}{n}$ and $x_{i,n}^* =$

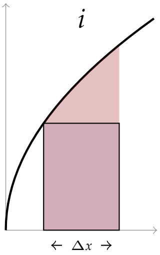


Types of Riemann Sums (RS)

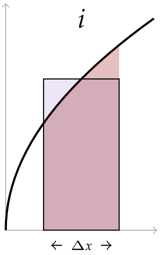
What height would you choose for the ith rectangle?



Right RS



Left RS



Midpoint RS

Riemann sums with n rectangles. Let $\Delta x = \frac{b-a}{n}$

The right Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + i\Delta x\right)$$

The left Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + (i-1)\Delta x\right)$$

The midpoint Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$$

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The right Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + i\Delta x\right)$$

Give a right Riemann Sum for the area under the curve $y = x^2 - x$ from $a = 1$ to $b = 6$ using $n = 1000$ intervals.

$$\sum_{n=1}^{1000} \frac{5}{1000} \left[\left(1 + \frac{5}{1000} i\right)^2 - \left(1 + \frac{5}{1000} i\right) \right]$$

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The midpoint Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$$

Give a midpoint Riemann Sum for the area under the curve $y = 5x - x^2$ from $a = 6$ to $b = 9$ using $n = 1000$ intervals.

$$\sum_{n=1}^{1000} \frac{3}{1000}\left[5\left(6 + \frac{3}{1000}(i - 1/2)\right) - \left(6 + \frac{3}{1000}(i - 1/2)\right)^2\right]$$

Q

Evaluating Riemann Sums

SKIP RIEMANN EVALUATIONS

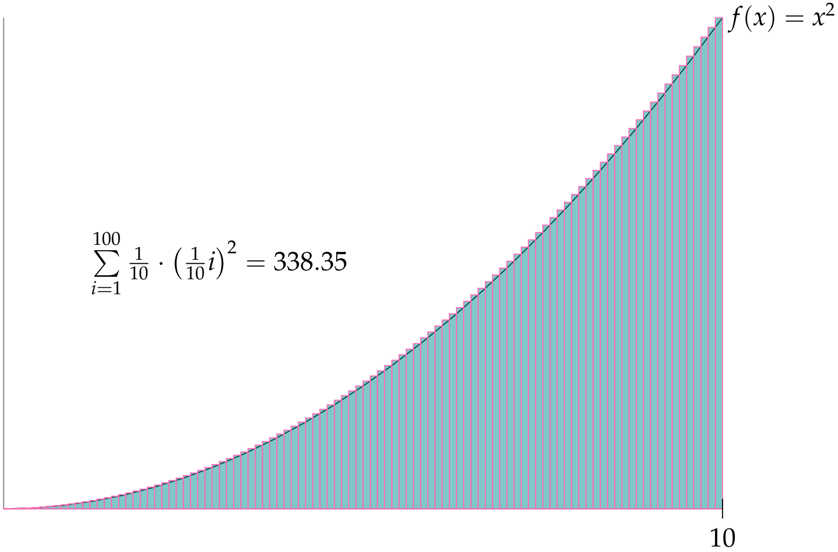
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^2$ from $a = 0$ to $b = 10$, $n = 100$:

$$\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) = \sum_{i=1}^{100} \frac{10}{100} \cdot \left(0 + \frac{10}{100} i\right)^2$$

$$= \sum_{i=1}^{100} \frac{1}{10} \cdot \left(\frac{1}{10} i\right)^2 = \frac{1}{10} \sum_{i=1}^{100} \frac{1}{100} i^2$$

$$= \frac{1}{1000} \sum_{i=1}^{100} i^2 = \frac{1}{1000} \frac{100(101)(201)}{6} = \frac{101 \cdot 201}{60}$$

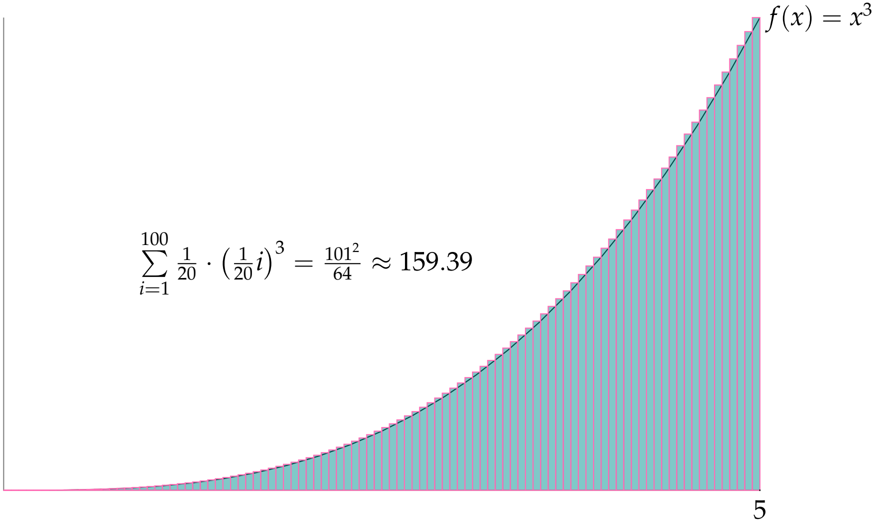

$f(x) = x^2$
$\sum_{i=1}^{100} \frac{1}{10} \cdot \left(\frac{1}{10} i\right)^2 = 338.35$
10

Evaluating Riemann Sums in Sigma Notation

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^3$ from $a = 0$ to $b = 5$, $n = 100$:

$$\begin{aligned}\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) &= \sum_{i=1}^{100} \frac{5}{100} \cdot \left(0 + \frac{5}{100} i\right)^3 \\ &= \sum_{i=1}^{100} \frac{1}{20} \cdot \left(\frac{1}{20} i\right)^3 = \frac{1}{20} \sum_{i=1}^{100} \frac{1}{20^3} i^3 \\ &= \frac{1}{20^4} \sum_{i=1}^{100} i^3 = \frac{1}{20^4} \frac{100^2(101^2)}{4} = \frac{101^2}{64}\end{aligned}$$



$$\sum_{i=1}^{100} \frac{1}{20} \cdot \left(\frac{1}{20} i\right)^3 = \frac{101^2}{64} \approx 159.39$$

Definition

Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define $\Delta x = \frac{b-a}{N}$ and

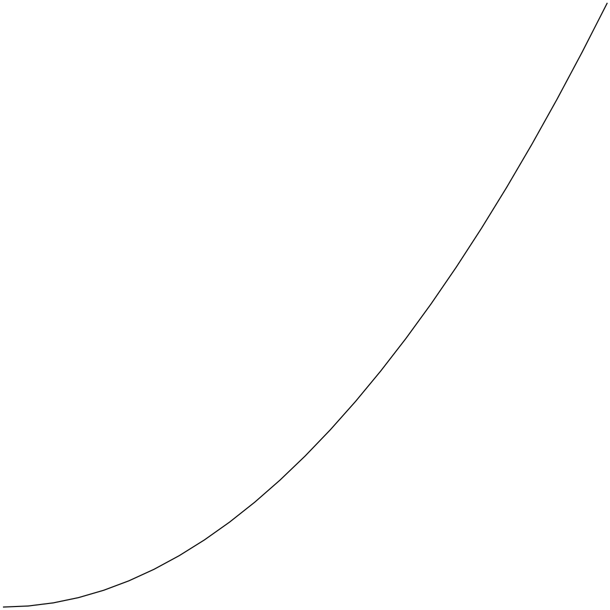
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_{i,N}^*) \cdot \Delta x$$

when the limit exists and when the choice of $x_{i,N}^*$ in the i^{th} interval doesn't matter.

$\sum$, $\int$ both stand for "sum"

Δx, $\mathrm{d}x$ are tiny pieces of the x-axis, the bases of our very skinny rectangles

It's understood we're taking a limit as N goes to infinity, so we don't bother specifying N (or each location where we find our height) in the second notation.



$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

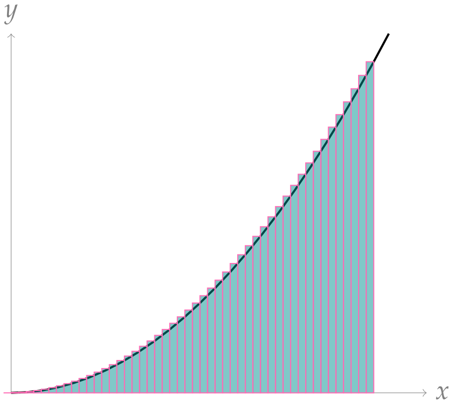
Give the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ with n slices, and simplify:

$$\begin{aligned}
\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) &= \sum_{i=1}^{n} \frac{5}{n} \cdot \left(\frac{5}{n} i\right)^2 = \sum_{i=1}^{n} \frac{125}{n^3} i^2 \\
&= \frac{125}{n^3} \left[\sum_{i=1}^{n} i^2\right] = \frac{125}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{125}{n^2} \left(\frac{(n+1)(2n+1)}{6}\right) = \frac{125}{6} \left(\frac{2n^2 + 3n + 1}{n^2}\right)
\end{aligned}$$

We found the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ using n slices:

$$\frac{125}{6} \cdot \frac{2n^2 + 3n + 1}{n^2}$$

Use it to find the exact area under the curve.



$$\int^5 x^2\, dx = \lim \left[\frac{125}{6} \cdot \frac{2n^2 + 3n + 1}{n^2}\right]$$

Refresher: Limits of Rational Functions

$$\lim_{n\to\infty} \frac{n^2+2n+15}{3n^2-9n+5} = \lim_{n\to\infty} \frac{1+2/n+15/n^2}{3-9/n+5/n^2} = \frac{1}{3}$$

When the degree of the top and bottom are the same, the limit as n goes to infinity is the ratio of the leading coefficients.

$$\lim_{n\to\infty} \frac{n^2+2n+15}{3n^3-9n+5} = \lim_{n\to\infty} \frac{1/n+2/n^2+15/n^3}{3-9/n^2+5/n^3} = 0$$

When the degree of the top is smaller than the degree of the bottom, the limit as n goes to infinity is 0.

$$\lim_{n\to\infty} \frac{n^3+2n+15}{3n^2-9n+5} = \lim_{n\to\infty} \frac{n+2/n+15/n^2}{3-9/n+5/n^2} = \infty$$

When the degree of the top is larger than the degree of the bottom, the limit as n goes to infinity is positive or negative infinity.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Evaluate $\int_0^1 x^2\, dx$ exactly using midpoint Riemann sums.

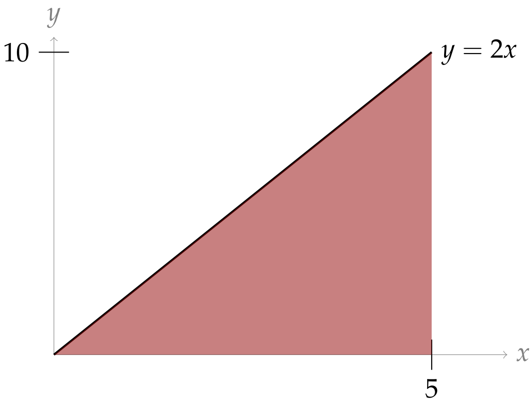
$$\begin{aligned}
\sum_{i=1}^{n} \Delta x \cdot \left(\left(i-\frac{1}{2}\right)\Delta x\right)^2 &= \frac{1}{n^3}\sum_{i=1}^{n}\left(i^2 - i + \frac{1}{4}\right) = \frac{1}{n^3}\left[\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i + \sum_{i=1}^{n}\frac{1}{4}\right] \\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{1}{4}n\right] \\
&= \frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n^2} + \frac{1}{4n^2}
\end{aligned}$$

Exact area under the curve:

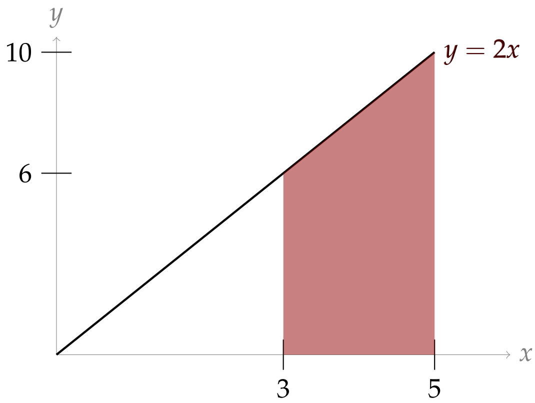
$$\lim_{n\to\infty}\left[\frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n^2} + \frac{1}{4n^2}\right] = \frac{2}{6} - 0 + 0 = \frac{1}{3}$$

Let's see some special cases where we can use geometry to evaluate integrals without Riemann sums.

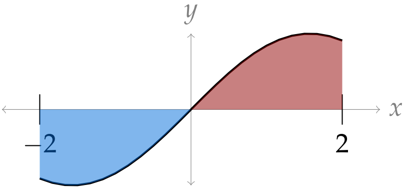
$$\int_0^5 2x\,dx = \frac{1}{2}(5)(10) = 25$$



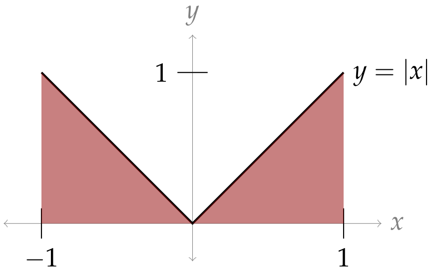
$$\int_3^5 2x\,\mathrm{d}x = \frac{1}{2}(5)(10) - \frac{1}{2}(3)(6) = 25 - 9 = 16$$



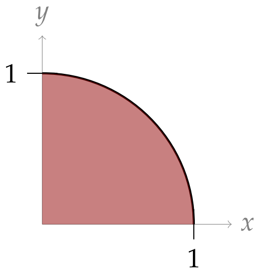
$$\int_{-2}^{2} \sin x \, \mathrm{d}x = -A + A = 0$$



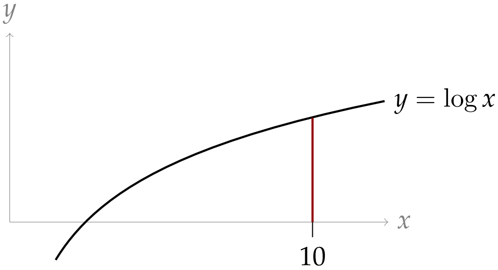
$$\int_{-1}^{1} |x|\,\mathrm{d}x = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1$$



$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}(\pi \cdot 1^2) = \frac{\pi}{4}$$



$$\int_{10}^{10} \log x \, \mathrm{d}x = 0$$



A car travelling down a straight highway records the following measurements:

Time	12:00	12:10	12:20	12:30	12:40	12:50	1:00
Speed (kph)	80	100	100	90	90	75	100

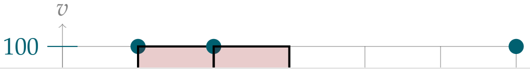
Approximately how far did the car travel from 12:00 to 1:00?

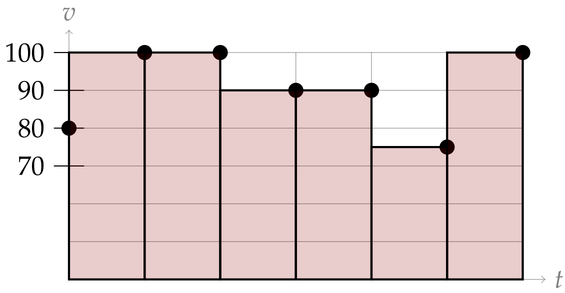
We don't know the speed of the car over the entire hour, so the best we can do is to use the measured speeds as approximations for the speeds the car travelled over 10-minute intervals.

We can use left, right, and midpoint Riemann sums. Note that there are only six 10-minute intervals, but we know seven points. For a midpoint Riemann sum, since we need to know the speed at the midpoint of the interval, we can only use three intervals, not six.

Finally, note that 10 minutes is $\frac{1}{6}$ of an hour, and 20 minutes is $\frac{1}{3}$ of an hour.

Left RS: $\underbrace{80 \cdot \frac{1}{6}}_{12:00-12:10} + \underbrace{100 \cdot \frac{1}{6}}_{12:10-12:20} + \underbrace{100 \cdot \frac{1}{6}}_{12:20-12:30} + \underbrace{90 \cdot \frac{1}{6}}_{12:30-12:40} + \underbrace{90 \cdot \frac{1}{6}}_{12:40-12:50} + \underbrace{75 \cdot \frac{1}{6}}_{12:50-1:00}$





The computation

$$\text{distance} = \text{rate} \times \text{time}$$

looks a lot like the computation

$$\text{area} = \text{base} \times \text{height}$$

for a rectangle. This gives us another interpretation for an integral.

Another interpretation of the integral

Let $x(t)$ be the position of an object moving along the x-axis at time t, and let $v(t) = x'(t)$ be its velocity. Then for all $b > a$,

$$x(b) - x(a) = \int_a^b v(t)\,\mathrm{d}t$$

That is, $\int_a^b v(t)\,\mathrm{d}t$ gives the *net distance* moved by the object from time a to time b.