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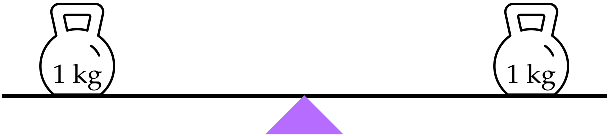

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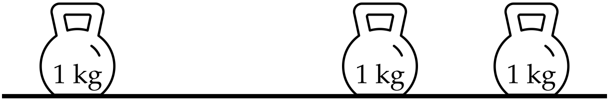

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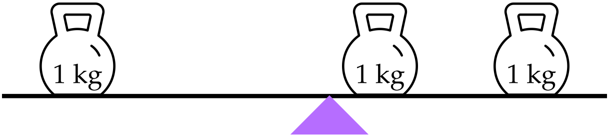

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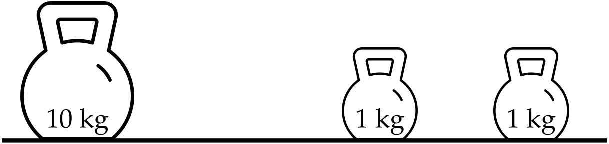

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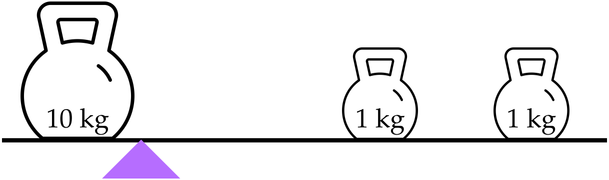

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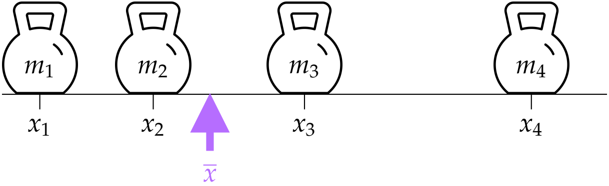


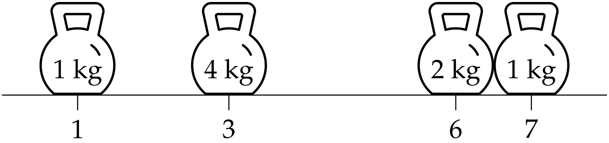
If the body consists of a finite number of masses $m_1, \cdots, m_n$ attached to an infinitely strong, weightless (idealized) rod with mass number i attached at position x_i, then the center of mass is at the (weighted) average value of x:

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}$$

The denominator $m = \sum_{i=1}^n m_i$ is the total mass of the body.

An idealized (weightless, unbending) rod has small masses attached to it at the following locations:

- 1 kg at $x = 1$ metre from the left end
- 4 kg at $x = 3$ metres from the left end
- 2 kg at $x = 6$ metres from the left end
- 1 kg at $x = 7$ metres from the left end



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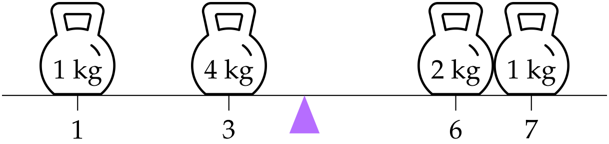
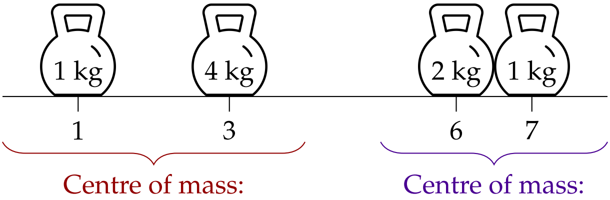


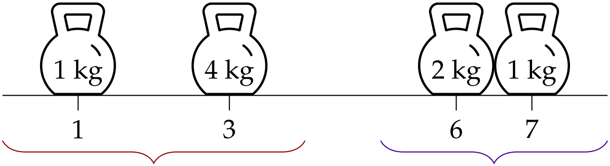
What is the location of its centre of mass?

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{1(1) + 4(3) + 2(6) + 1(7)}{1 + 4 + 2 + 1} = 4$$

So the centre of mass is 4 metres from the left end of the rod.

We can also group the masses, and treat them as single points of mass at their centres of gravity, without affecting the centre of gravity of the entire object.





Centre of mass:

$$\frac{1(1)+4(3)}{1+4} = \frac{13}{5}$$

Centre of mass:

$$\frac{2(6)+1(7)}{2+1} = \frac{19}{3}$$

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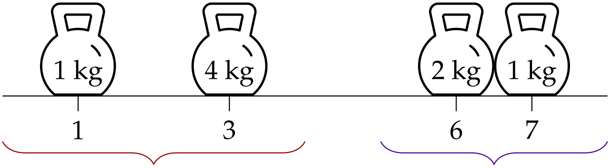

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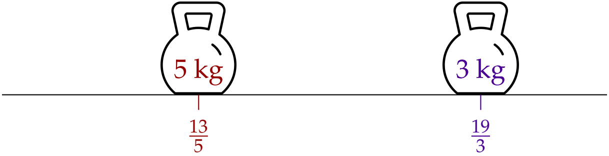

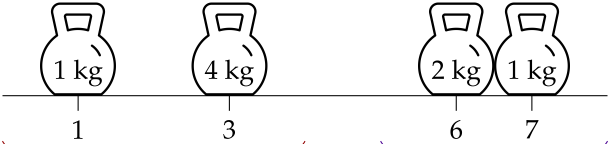

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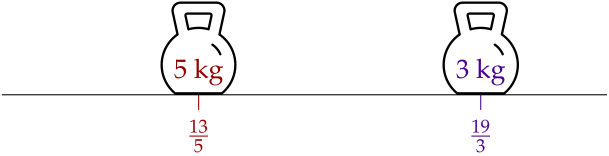
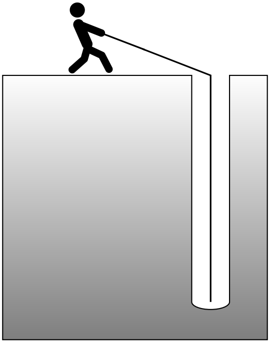


Centre of mass of second rod: $\bar{x} = \frac{5\left(\frac{13}{5}\right)+3\left(\frac{19}{3}\right)}{5+3} = 4$

Sometimes we can simplify a physical calculation by treating an object as a point particle located at its centre of mass.
When we were learning about work, we found the following:



A cable dangles in a hole. The cable is 10 metres long, and has a mass of 5 kg. Its density is constant. We found that the work required to pull the cable out of the hole was

$$25g\ J$$

where g is the acceleration due to gravity.

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Since the cable has constant density, it should "balance" at its centre (if it were rigid), so its centre of mass starts 5 metres below the ground. It ends up on the ground. If we treat the cable as a point particle of mass 5 kg, moving against gravity for a distance of 5 metres, we find the work done to be

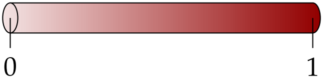
$$\text{Work} = \text{Force} \cdot \text{distance} = \left(5\,\text{kg} \cdot g\,\frac{\text{m}}{\text{s}^2}\right) \cdot 5\,\text{m} = 25g\ J$$

This is much easier than our original calculation.

Consider a metre-long rod that is denser on one end than the other, with density

$$\rho(x) = (2x+1)\ \frac{\text{kg}}{\text{m}}$$

at a position x metres from its left end.



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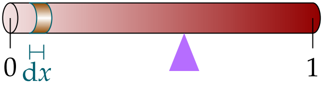

What is its centre of mass?

We can use our usual slicing-up procedure. Consider slicing the rod into tiny cross-sections, each with width $\mathrm{d}x$. Then a cross-section at position x is approximately a point mass with position x and mass $\rho(x)\,\mathrm{d}x = (2x+1)\,\mathrm{d}x$. So, using integrals to add up the contributions from the different slices, the centre of mass is:

$$\bar{x} = \frac{\int_0^1 x(2x+1)\,\mathrm{d}x}{\int_0^1 (2x+1)\,\mathrm{d}x} = \frac{\left[\frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1}{\left[x^2+x\right]_0^1} = \frac{7/6}{2} = \frac{7}{12}$$

If a body consists of mass distributed continuously along a straight line, say with mass density $\rho(x)$kg/m and with x running from a to b, rather than consisting of a finite number of point masses, the formula for the center of mass becomes

$$\bar{x} = \frac{\int_a^b x\,\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$$

Think of $\rho(x)\,\mathrm{d}x$ as the mass of the “almost point particle” between x and $x + \mathrm{d}x$.

Centre of Mass

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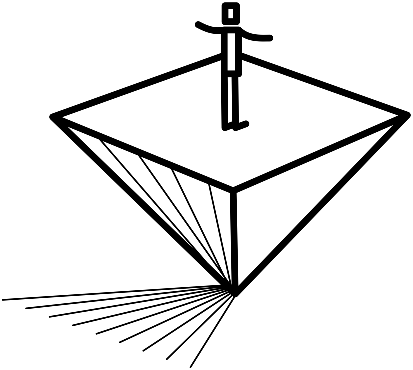
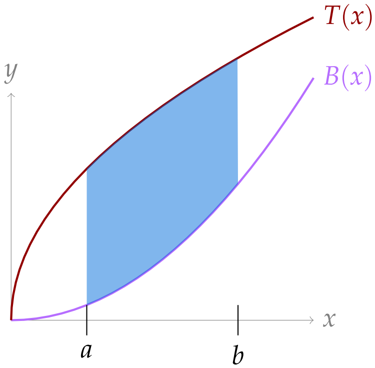
Centre of mass isn't just for linear solids: it applies to 2- and 3-dimensional objects as well.



Consider a flat metal plate of uniform density, whose shape is the area below curve $y = T(x)$ and above curve $y = B(x)$, from $x = a$ to $x = b$.



The centre of mass will be a point in the xy-plane, $(\bar{x}, \bar{y})$.

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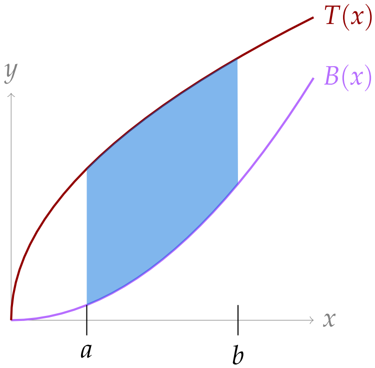

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To find $\bar{x}$ and $\bar{y}$, we will treat vertical slices as point particles.

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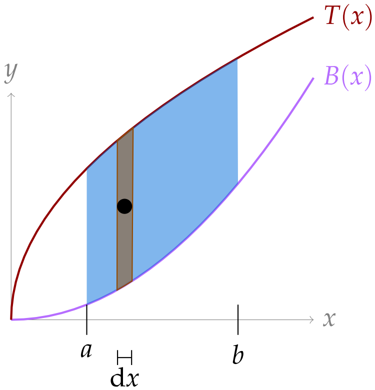

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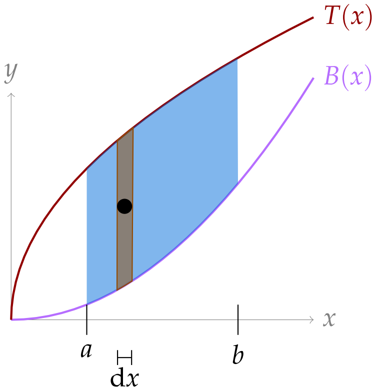
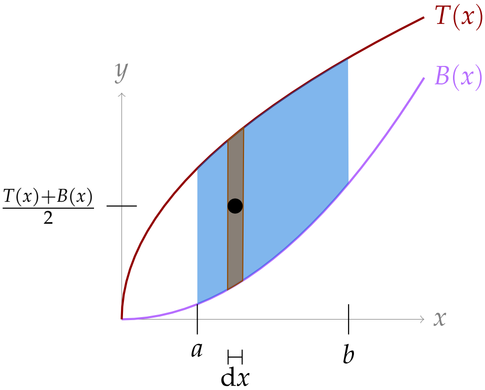


If ρ is the density of the plate, so that a slice of width $\mathrm{d}x$ and height $h = T(x) - B(x)$ has mass $\rho\, h\, \mathrm{d}x = \rho\big(T(x) - B(x)\big)\, \mathrm{d}x$, then:

$$\bar{x} = \frac{\int_a^b \rho(T(x) - B(x)) \cdot x \, \mathrm{d}x}{\int_a^b \rho(T(x) - B(x)) \, \mathrm{d}x} = \frac{\int_a^b (T(x) - B(x)) \cdot x \, \mathrm{d}x}{\int_a^b (T(x) - B(x)) \, \mathrm{d}x}$$

The centre of mass will be a point in the xy-plane, $(\bar{x}, \bar{y})$.
To find $\bar{x}$ and $\bar{y}$, we will treat vertical slices as point particles.

To find $\bar{y}$, note that the y-coordinate of the centre of mass of a slice that is almost a rectangle, and has uniform density, will be halfway up the slice, at $\frac{T(x)+B(x)}{2}$.



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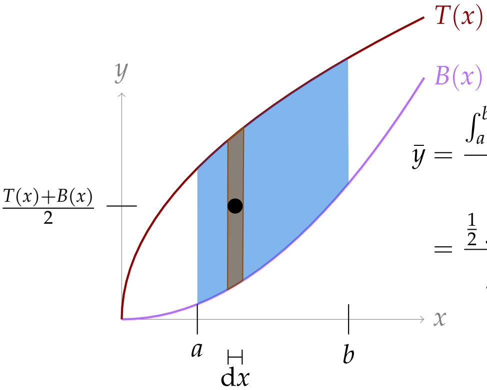
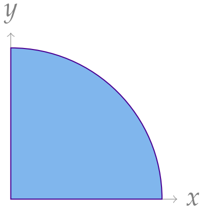


$$\bar{y} = \frac{\int_a^b \left(\frac{T(x)+B(x)}{2}\right) \cdot \rho(T(x) - B(x))\, \mathrm{d}x}{\int_a^b \rho(T(x) - B(x))\, \mathrm{d}x}$$

$$= \frac{\frac{1}{2}\int_a^b (T^2(x) - B^2(x))\, \mathrm{d}x}{\int_a^b (T(x) - B(x))\, \mathrm{d}x}$$

Find the centre of mass (centroid) of the quarter circular unit disk $x \geq 0, y \geq 0, x^2 + y^2 \leq 1$.



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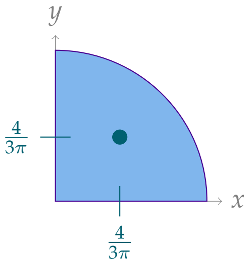


By symmetry, $\bar{x} = \bar{y}$. Using the equations we developed above with top $y = T(x) = \sqrt{1-x^2}$ and bottom $y = B(x) = 0$:

$$\bar{x} = \frac{\int_0^1 (\sqrt{1-x^2} - 0) \cdot x \, \mathrm{d}x}{\int_0^1 (\sqrt{1-x^2} - 0) \, \mathrm{d}x}$$

For the integral in the numerator, let $u = 1 - x^2$, $\mathrm{d}u = -2x \, \mathrm{d}x$. The denominator is the area of the quarter unit circle.

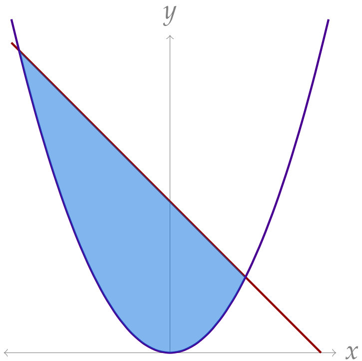
$$= \frac{\int_1^0 -\frac{1}{2} u^{1/2} \, \mathrm{d}u}{\frac{\pi}{4}}$$

$$= \frac{2}{\pi} \int_0^1 u^{1/2} \, \mathrm{d}u$$

$$= \frac{2}{\pi} \left[\frac{2}{3} u^{3/2} \right]_0^1 = \frac{4}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}, \frac{4}{3\pi} \right)$$

Find the centre of mass (centroid) of a plate of uniform density in the shape of the finite area enclosed by the functions $y = T(x) = 2 - x$ and $y = B(x) = x^2$.



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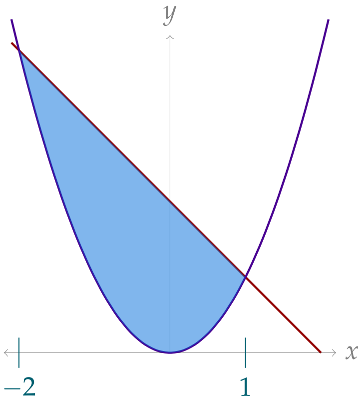

First, we find where the curves intersect.

$$\begin{aligned} x^2 &= 2 - x \\ x^2 + x - 2 &= 0 \\ (x-1)(x+2) &= 0 \\ x &= -2,\ x = 1 \end{aligned}$$

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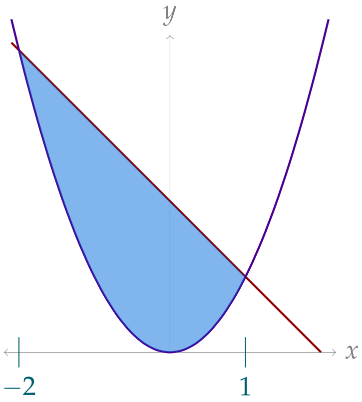

The denominator is the same in our $\bar{x}$ and $\bar{y}$ calculations, so let's find that next.

$$\int_{-2}^{1} (T(x) - B(x))\,\mathrm{d}x = \int_{-2}^{1} (2 - x - x^2)\,\mathrm{d}x$$

$$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= \frac{9}{2}$$

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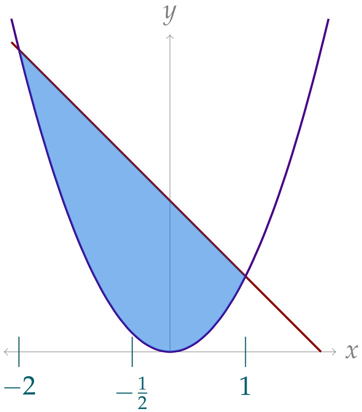
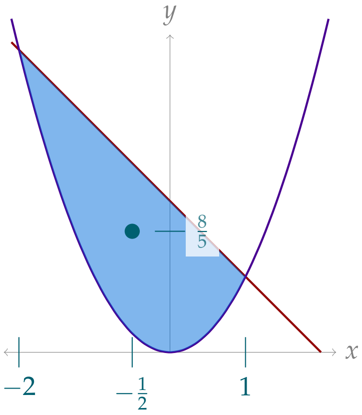


$$\begin{aligned}
\bar{x} &= \frac{\int_{-2}^{1} \left(2 - x - x^2\right) x \, \mathrm{d}x}{\int_{-2}^{1} \left(2 - x - x^2\right) \, \mathrm{d}x} \\
&= \frac{\int_{-2}^{1} \left(2x - x^2 - x^3\right) \, \mathrm{d}x}{\frac{9}{2}} \\
&= \frac{2}{9} \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{-2}^{1} \\
&= \frac{2}{9} \left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(4 + \frac{8}{3} - 4\right)\right] \\
&= -\frac{1}{2}
\end{aligned}$$

Find the centre of mass (centroid) of a plate of uniform density in the shape of the finite area enclosed by the functions $y = T(x) = 2 - x$ and $y = B(x) = x^2$.



$$
\begin{aligned}
\bar{y} &= \frac{\frac{1}{2}\int_{-2}^{1}\left((2-x)^2 - (x^2)^2\right)x\,\mathrm{d}x}{\int_{-2}^{1}\left(2-x-x^2\right)\,\mathrm{d}x} \\
&= \frac{\frac{1}{2}\int_{-2}^{1}\left(4-4x+x^2-x^4\right)\mathrm{d}x}{\frac{9}{2}} \\
&= \frac{1}{9}\left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^{1} \\
&= \frac{1}{9}\Bigg[\left(4-2+\frac{1}{3}-\frac{1}{5}\right) \\
&\qquad - \left(-8-8-\frac{8}{3}+\frac{32}{5}\right)\Bigg] = \frac{8}{5}
\end{aligned}
$$

Included Work

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