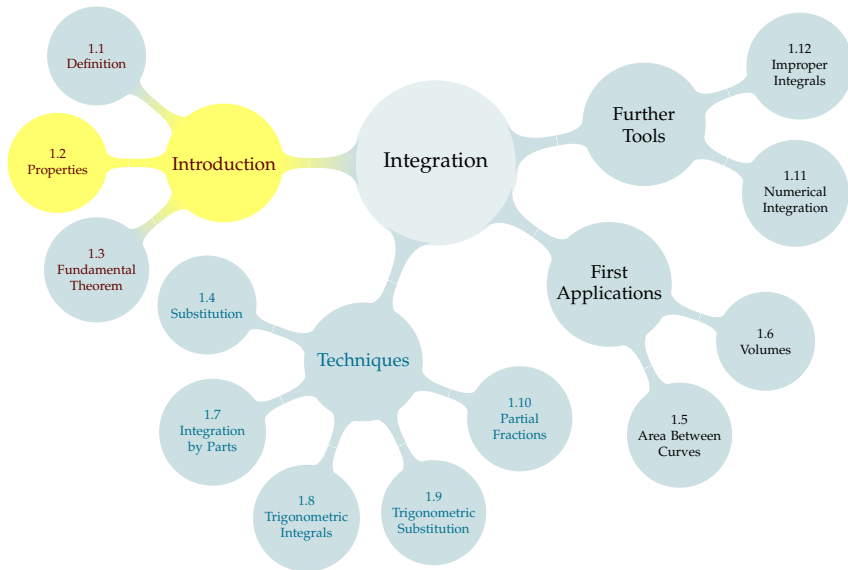


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We defined the definite integral using a limit and a sum.

## Definition

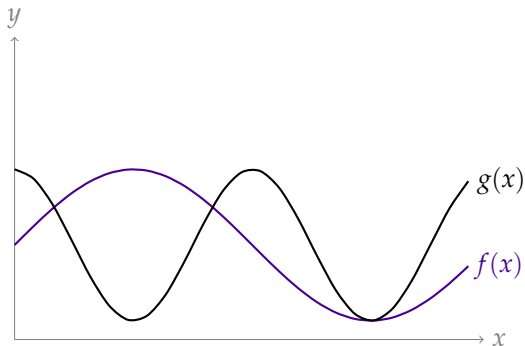
Let  $a$  and  $b$  be two real numbers and let  $f(x)$  be a function that is defined for all  $x$  between  $a$  and  $b$ . Then we define  $\Delta x = \frac{b-a}{N}$  and

$$\int_a^b f(x) \, dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_{i,N}^*) \cdot \Delta x$$

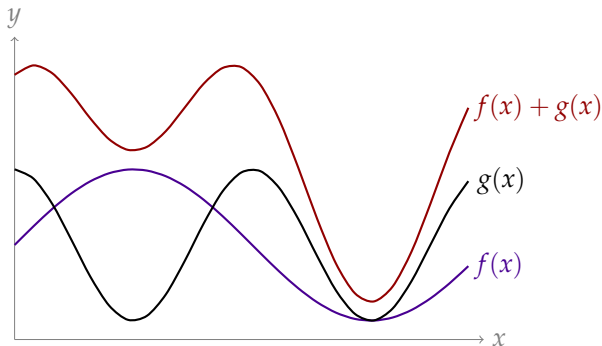
when the limit exists and when the choice of  $x_{i,N}^*$  in the  $i^{\text{th}}$  interval doesn't matter.

Many of the operations that work nicely with sums and limits will also work nicely with integrals.

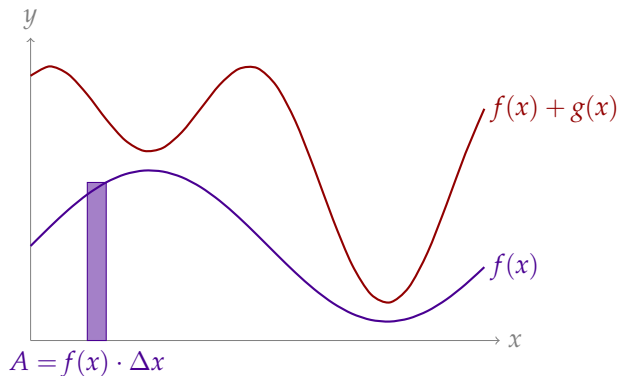
# ADDING (AND SUBTRACTING) FUNCTIONS



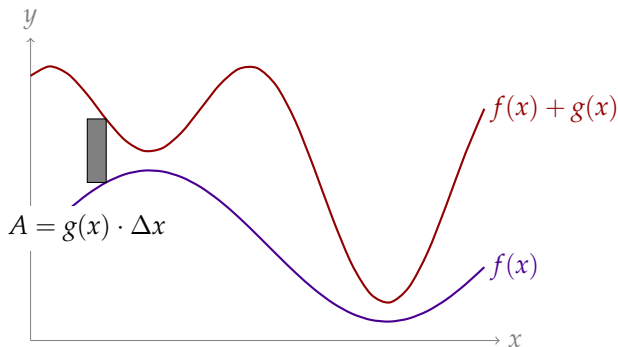
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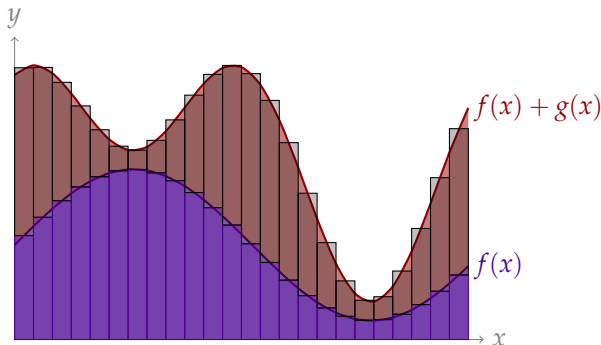
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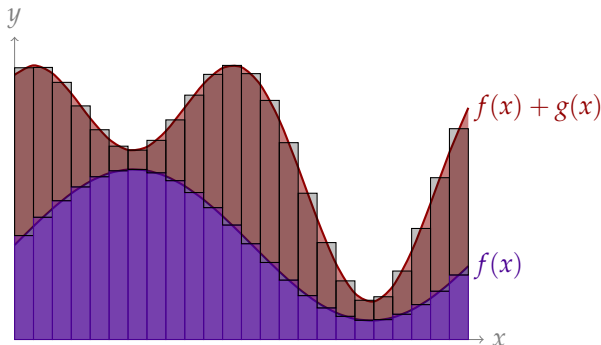


# ADDING (AND SUBTRACTING) FUNCTIONS



$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

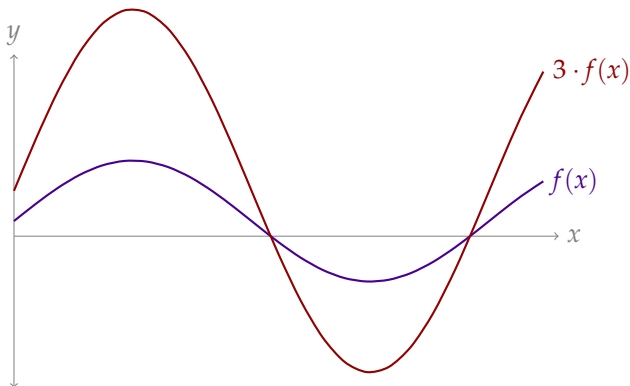
# ADDING (AND SUBTRACTING) FUNCTIONS



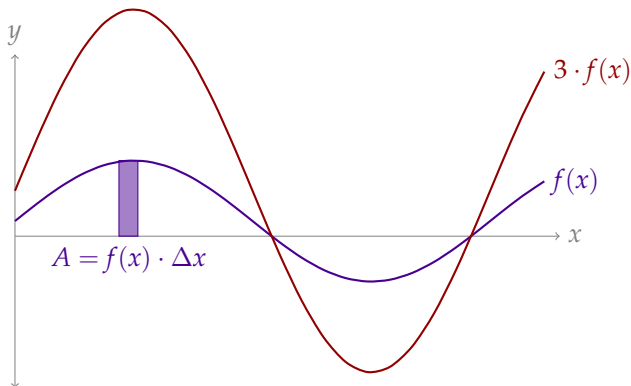
$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$



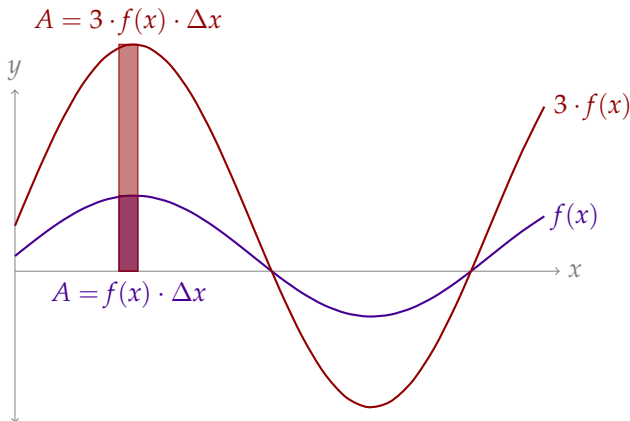
# MULTIPLYING A FUNCTION BY A CONSTANT



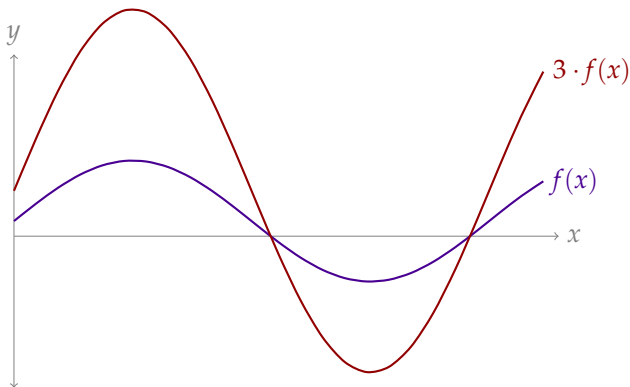
# MULTIPLYING A FUNCTION BY A CONSTANT



# MULTIPLYING A FUNCTION BY A CONSTANT



# MULTIPLYING A FUNCTION BY A CONSTANT



$$\int_a^b c \cdot f(x) \, dx = c \int_a^b f(x) \, dx$$

# ARITHMETIC OF INTEGRATION

When  $a$ ,  $b$ , and  $c$  are real numbers, and the functions  $f(x)$  and  $g(x)$  are integrable on an interval containing  $a$  and  $b$ :

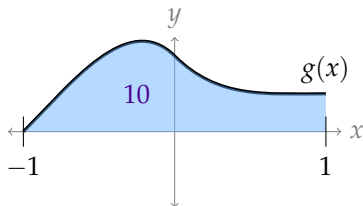
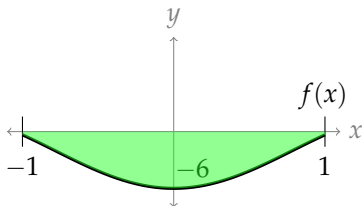
$$(a) \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$$

$$(b) \int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

$$(c) \int_a^b c \cdot f(x) \, dx = c \int_a^b f(x) \, dx \quad \text{when } c \text{ is constant}$$

# ARITHMETIC OF INTEGRATION

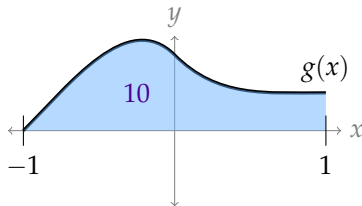
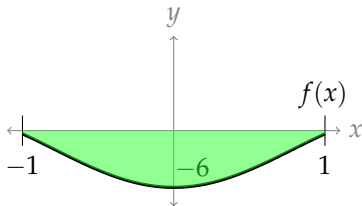
Suppose  $\int_{-1}^1 f(x) \, dx = -6$  and  $\int_{-1}^1 g(x) \, dx = 10$ .



$$\int_{-1}^1 (2f(x) + g(x)) \, dx =$$

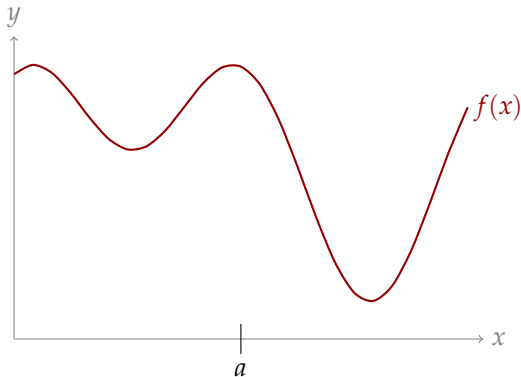
## ARITHMETIC OF INTEGRATION

Suppose  $\int_{-1}^1 f(x) \, dx = -6$  and  $\int_{-1}^1 g(x) \, dx = 10$ .



$$\int_{-1}^1 (2f(x) + g(x)) \, dx = 2 \int_{-1}^1 f(x) \, dx + \int_{-1}^1 g(x) \, dx = 2(-6) + 10 = -2$$

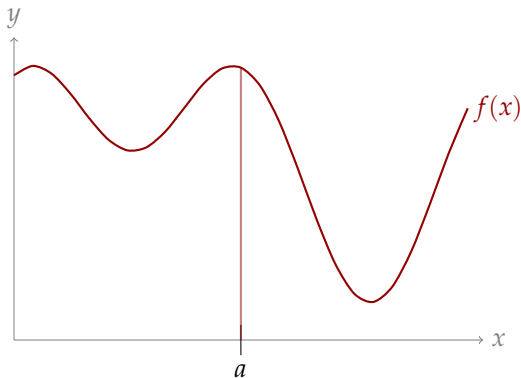
# INTERVAL OF INTEGRATION



$$\int_a^a f(x) \, dx =$$

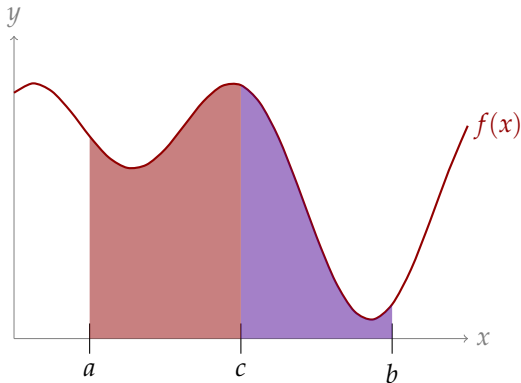


# INTERVAL OF INTEGRATION



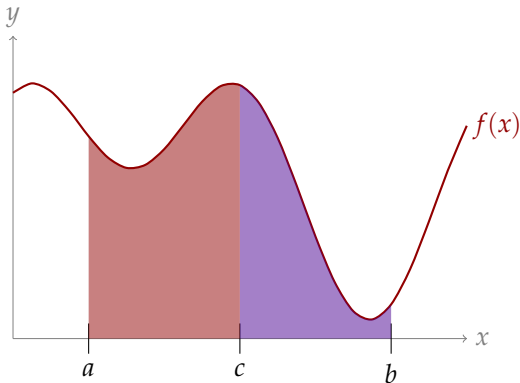
$$\int_a^a f(x) \, dx = 0$$

# INTERVAL OF INTEGRATION



What rule do you think is being illustrated?

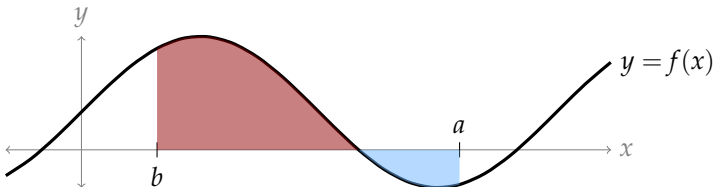
# INTERVAL OF INTEGRATION



What rule do you think is being illustrated?

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

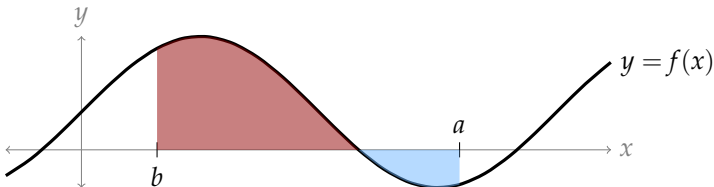
WHAT HAPPENS IN  $\int_a^b f(x) \, dx$  WHEN  $b < a$ ?



$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

This is the definition of a definite integral *whether or not*  $a < b$ .

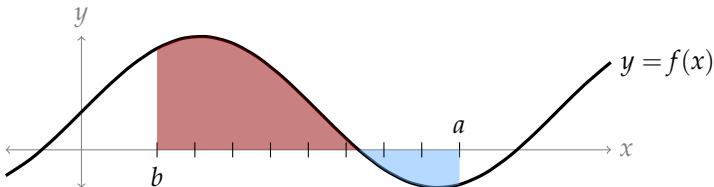
# WHAT HAPPENS IN $\int_a^b f(x) \, dx$ WHEN $b < a$ ?



Choose a number of intervals,  $n$ .

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

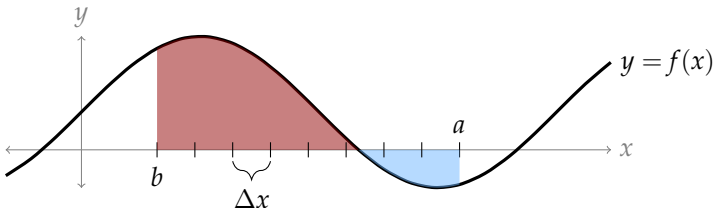
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# WHAT HAPPENS IN $\int_a^b f(x) dx$ WHEN $b < a$ ?

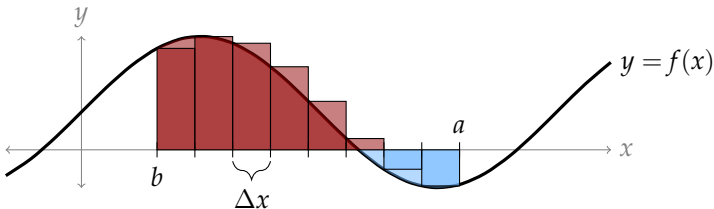


Choose a number of intervals,  $n$ .

The (signed) width of each interval is  $\Delta x = \frac{b-a}{n}$ , which is negative

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

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Choose a number of intervals,  $n$ .

The (signed) width of each interval is  $\Delta x = \frac{b-a}{n}$ , **which is negative**

$$\begin{aligned}\int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i,n}^*) \cdot \frac{b-a}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i,n}^*) \left( -\frac{a-b}{n} \right) = - \int_b^a f(x) \, dx\end{aligned}$$



# PROPERTY OF DEFINITE INTEGRALS

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

As strictly a measure of area, not usually a super useful fact – but helps later when we do arithmetic with integrals.

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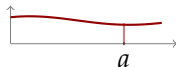
As strictly a measure of area, not usually a super useful fact – but helps later when we do arithmetic with integrals.

It's also useful that the definition works without having to worry about which limit of integration ( $a$  or  $b$ ) is larger.

# ARITHMETIC FOR DOMAIN OF INTEGRATION

When  $a$ ,  $b$ , and  $c$  are constants, and  $f(x)$  is integrable over a domain containing all three:

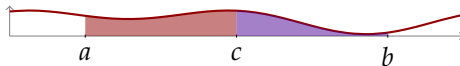
$$(a) \int_a^a f(x) dx = 0$$



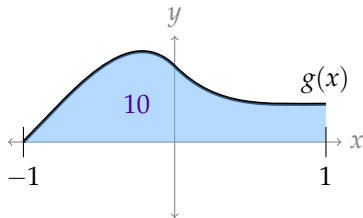
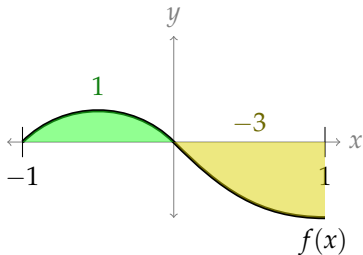
$$(b) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\Delta x = \frac{b-a}{n} = -\frac{a-b}{n}$$

$$(c) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \text{ for constant } c$$

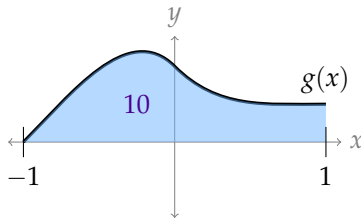
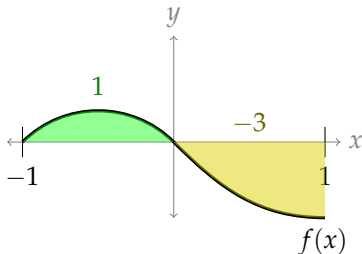


Suppose  $\int_{-1}^0 f(x) \, dx = 1$ ,  $\int_0^1 f(x) \, dx = -3$ , and  $\int_{-1}^1 g(x) \, dx = 10$ .



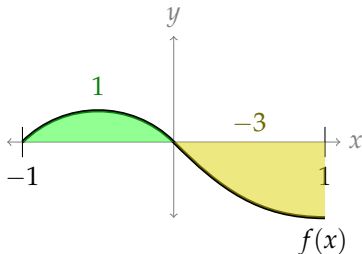
$$\int_{-1}^1 (2f(x) + g(x)) \, dx =$$

Suppose  $\int_{-1}^0 f(x) \, dx = 1$ ,  $\int_0^1 f(x) \, dx = -3$ , and  $\int_{-1}^1 g(x) \, dx = 10$ .



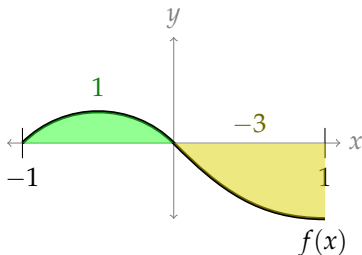
$$\begin{aligned} \int_{-1}^1 (2f(x) + g(x)) \, dx &= 2 \left[ \int_{-1}^0 f(x) \, dx + \int_0^1 f(x) \, dx \right] + \int_{-1}^1 g(x) \, dx \\ &= 2[1 - 3] + 10 = 6 \end{aligned}$$

Suppose  $\int_{-1}^0 f(x) \, dx = 1$  and  $\int_0^1 f(x) \, dx = -3$ .



$$\int_{-1}^3 f(x) \, dx + \int_3^0 f(x) \, dx =$$

Suppose  $\int_{-1}^0 f(x) \, dx = 1$  and  $\int_0^1 f(x) \, dx = -3$ .

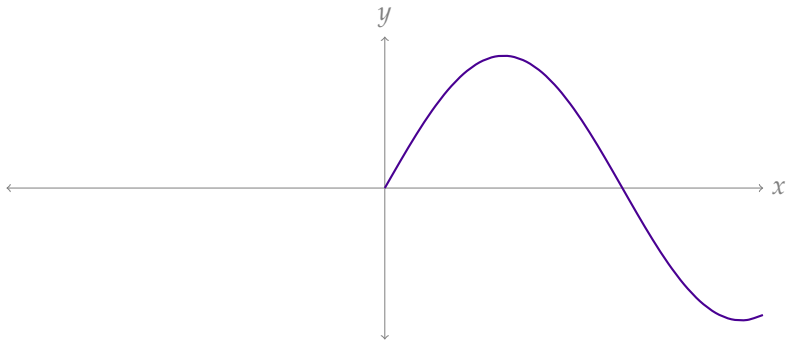


$$\int_{-1}^3 f(x) \, dx + \int_3^0 f(x) \, dx = \int_{-1}^0 f(x) \, dx = 1$$

## Even and Odd Functions

Let  $f(x)$  be a function.

- ▶ We say  $f(x)$  is **even** when  $f(x) = f(-x)$  for all  $x$ , and
- ▶ we say  $f(x)$  is **odd** when  $f(x) = -f(-x)$  for all  $x$ .

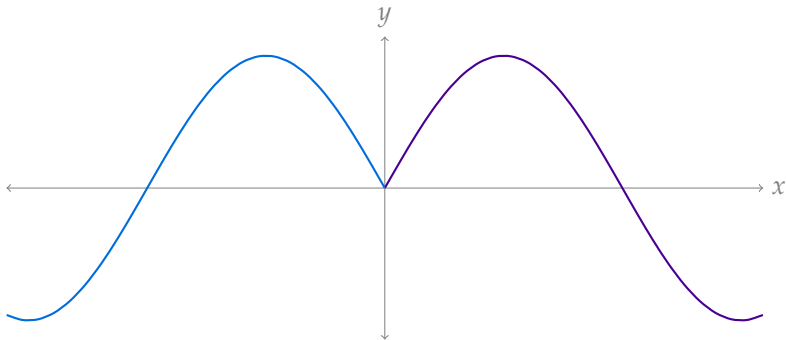




## Even and Odd Functions

Let  $f(x)$  be a function.

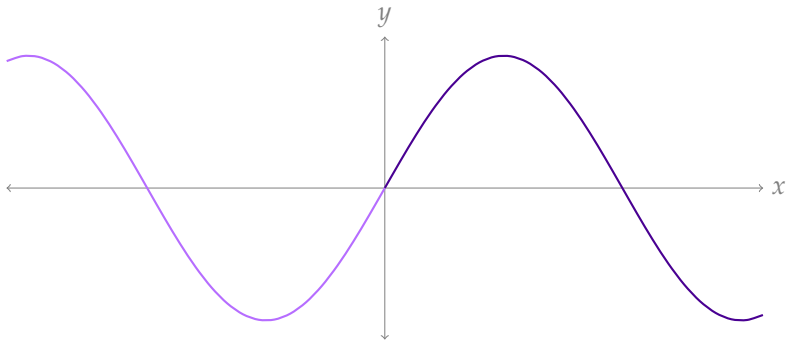
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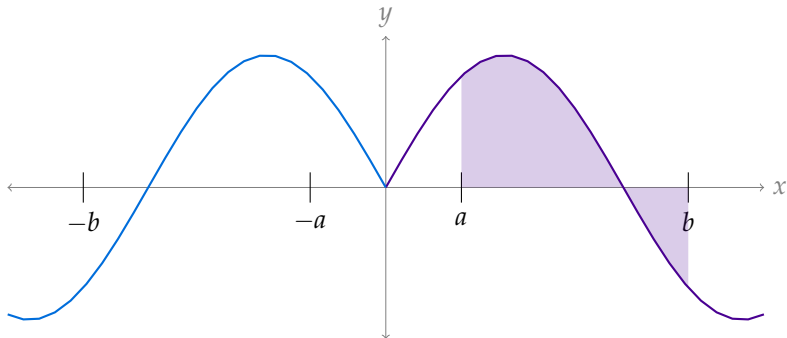
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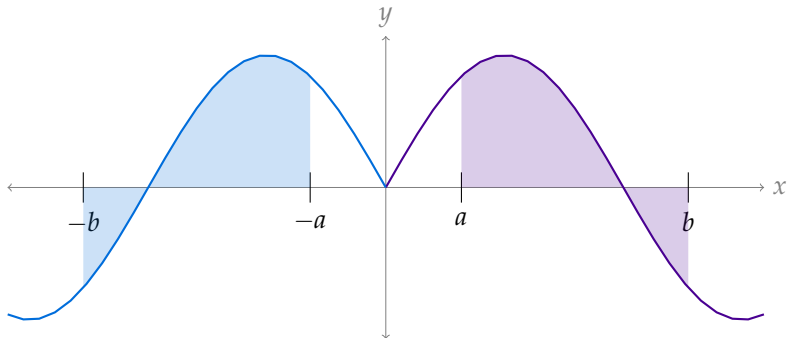
# INTEGRALS OF EVEN FUNCTIONS



Suppose  $f(x)$  is **even**. Then

$$\int_a^b f(x) \, dx =$$

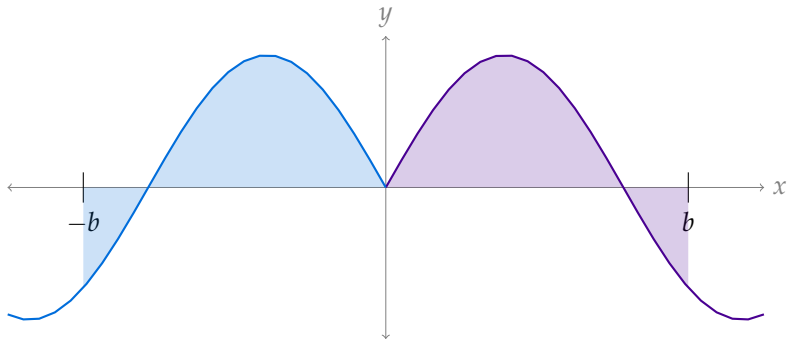
# INTEGRALS OF EVEN FUNCTIONS



Suppose  $f(x)$  is **even**. Then

$$\int_a^b f(x) \, dx = \int_{-b}^{-a} f(x) \, dx$$

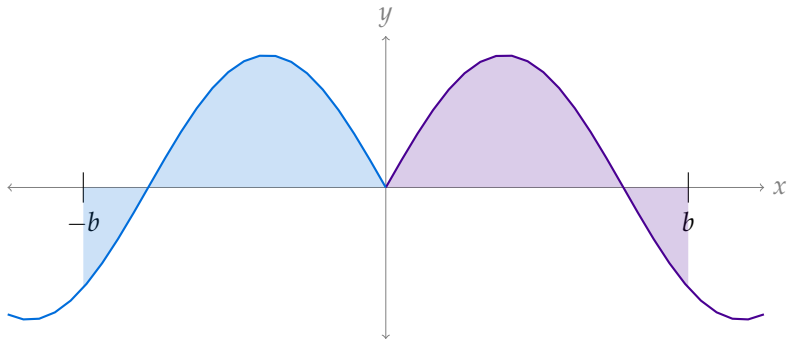
# INTEGRALS OF EVEN FUNCTIONS



Suppose  $f(x)$  is **even**. Then

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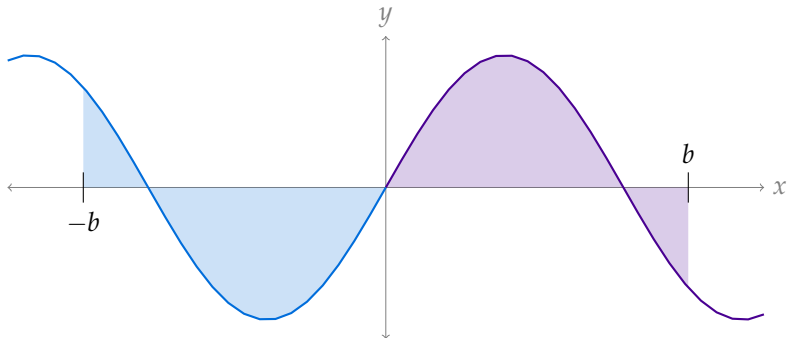
# INTEGRALS OF EVEN FUNCTIONS



Suppose  $f(x)$  is **even**. Then

$$\int_{-b}^b f(x) \, dx = 2 \int_0^b f(x) \, dx$$

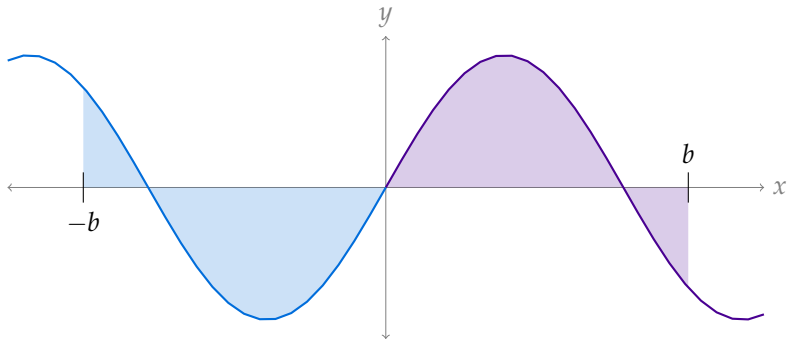
# INTEGRALS OF ODD FUNCTIONS



Suppose  $f(x)$  is **odd**. Then

$$\int_{-b}^b f(x) \, dx =$$

# INTEGRALS OF ODD FUNCTIONS

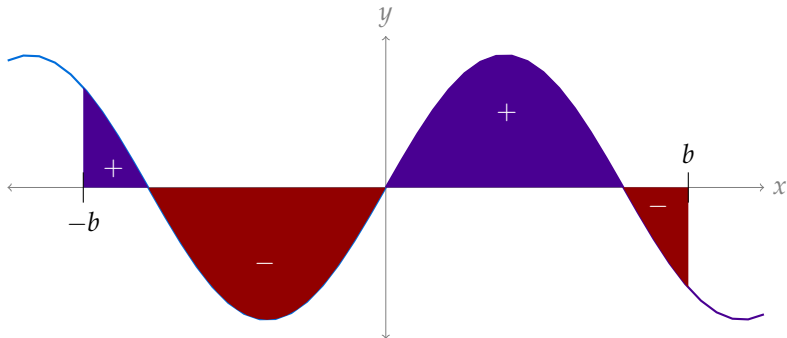


Suppose  $f(x)$  is **odd**. Then

$$\int_{-b}^b f(x) \, dx = 0$$



# INTEGRALS OF ODD FUNCTIONS



Suppose  $f(x)$  is odd. Then

$$\int_{-b}^b f(x) \, dx = 0$$

## Theorem 1.2.11 (Even and Odd)

Let  $a > 0$ .

(a) If  $f(x)$  is an **even** function, then

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$$

(b) If  $f(x)$  is an **odd** function, then

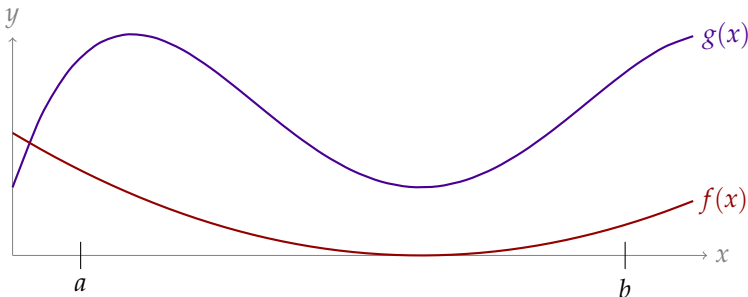
$$\int_{-a}^a f(x) \, dx = 0$$

## Integral Inequality

Let  $a \leq b$  be real numbers and let the functions  $f(x)$  and  $g(x)$  be integrable on the interval  $a \leq x \leq b$ .

If  $f(x) \leq g(x)$  for all  $a \leq x \leq b$ , then

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$

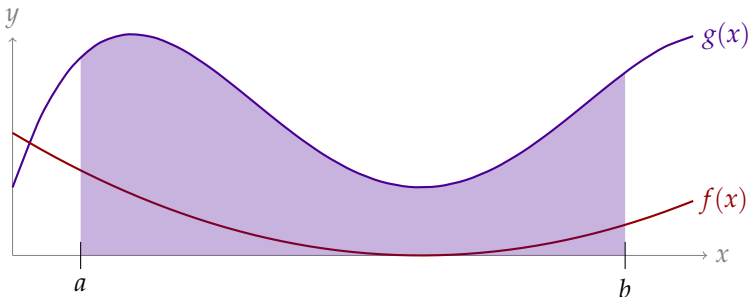


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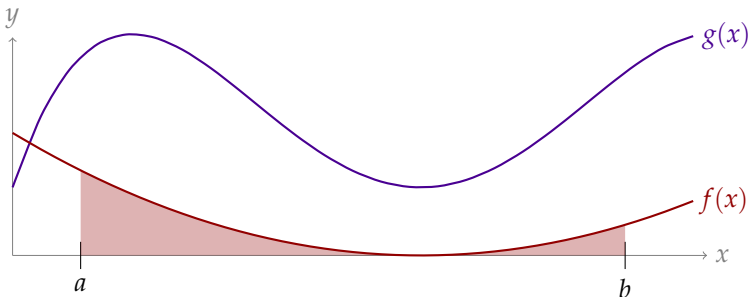


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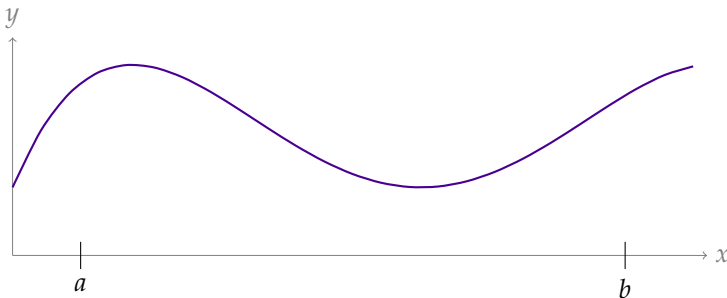


## Integral Inequality

Let  $a \leq b$  and  $m \leq M$  be real numbers and let the function  $f(x)$  be integrable on the interval  $a \leq x \leq b$ .

If  $m \leq f(x) \leq M$  for all  $a \leq x \leq b$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

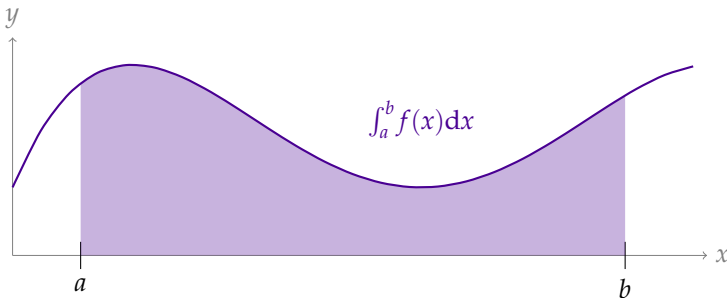


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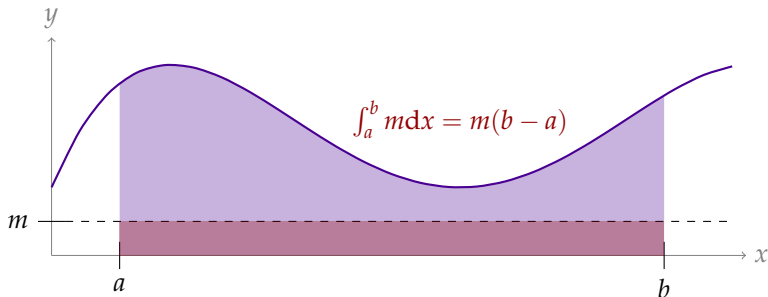


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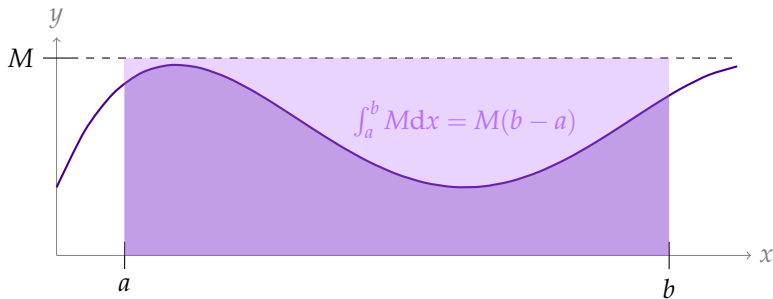


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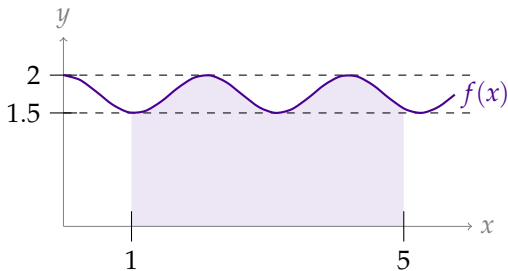
If  $m \leq f(x) \leq M$  for all  $a \leq x \leq b$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$



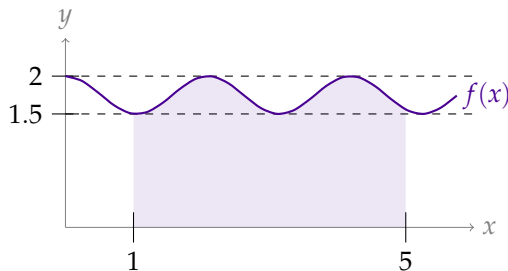
Find a lower bound  $c$  and an upper bound  $d$  such that

$$c \leq \int_1^5 f(x) \, dx \leq d$$



Find a lower bound  $c$  and an upper bound  $d$  such that

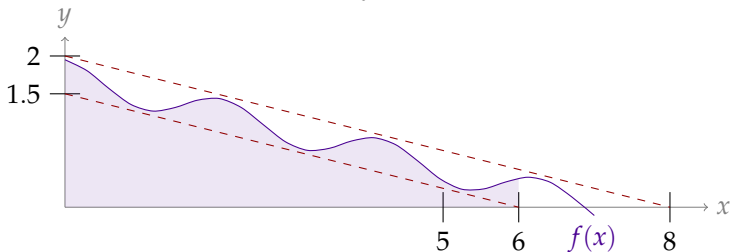
$$c \leq \int_1^5 f(x) \, dx \leq d$$



$$1.5 \leq f(x) \leq 2 \implies \overbrace{1.5(5-1)}^6 \leq \int_1^5 f(x) \, dx \leq \overbrace{2(5-1)}^8$$

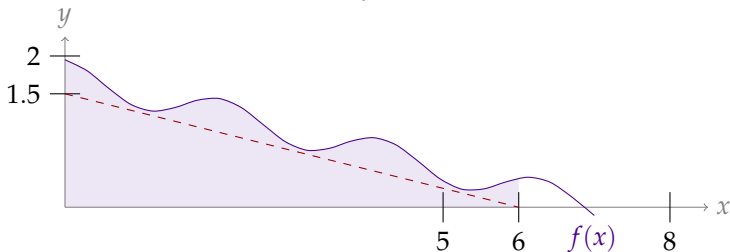
Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

$$c \leq \int_0^6 f(x) \, dx \leq d$$



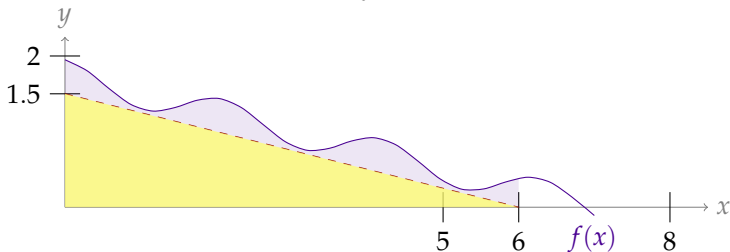
Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

$$c \leq \int_0^6 f(x) \, dx \leq d$$



Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

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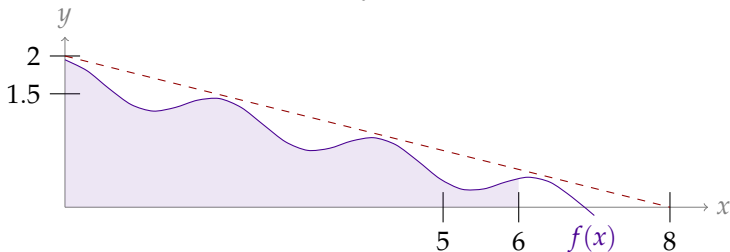


The area under the curve is no smaller than the area of the highlighted triangle.

$$\int_0^6 (\text{dashed line}) \, dx = \frac{1}{2} \cdot \frac{3}{2} \cdot 6 = \frac{9}{2} \leq \int_0^6 f(x) \, dx$$

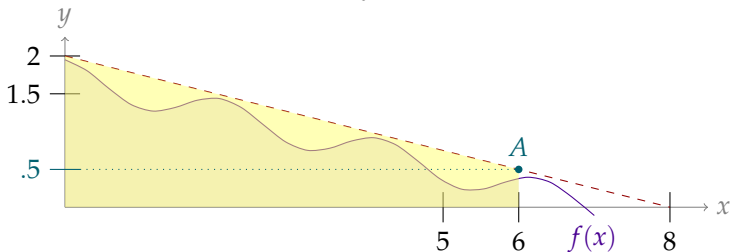
Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

$$c \leq \int_0^6 f(x) \, dx \leq d$$



Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

$$c \leq \int_0^6 f(x) \, dx \leq d$$

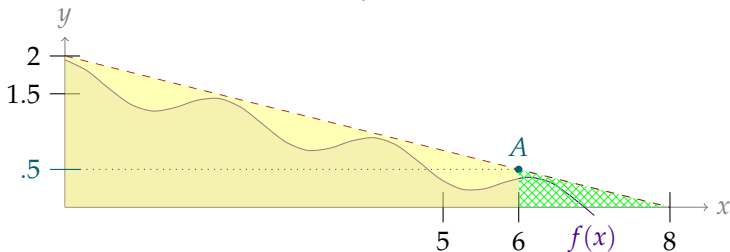


The area under the curve is not greater than the area under the solid yellow trapezoid. Because the dashed line has slope  $-\frac{1}{4}$ , the  $y$ -coordinate of point  $A$  is  $\frac{1}{2}$ .



Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

$$c \leq \int_0^6 f(x) \, dx \leq d$$

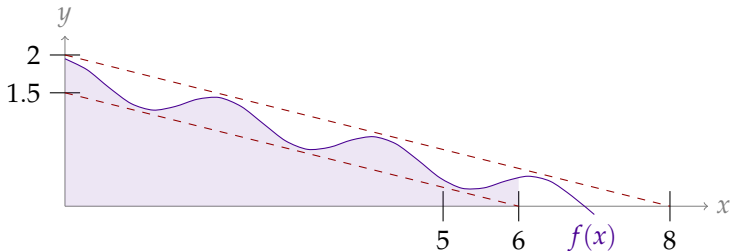


We can compute the area of the trapezoid as the difference in the area of the triangle under the dotted line, and the green cross-hatched triangle.

$$\int_0^6 f(x) \, dx \leq \int_0^6 (\text{dashed line}) \, dx = \frac{1}{2}(8)(2) - \frac{1}{2}(2)\frac{1}{2} = \frac{15}{2}$$

Find a lower bound  $c$  and an upper bound  $d$  such that  $d - c \leq 3$  and

$$c \leq \int_0^6 f(x) \, dx \leq d$$



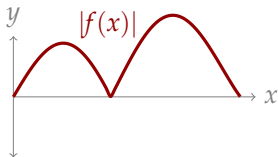
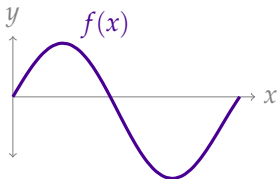
$$\frac{9}{2} \leq \int_0^6 f(x) \, dx \leq \frac{15}{2}$$

Note  $\frac{15}{2} - \frac{9}{2} = 3$ , as required.

(Many bounds of the integral are possible, but looser bounds won't satisfy  $d - c = 3$ .)

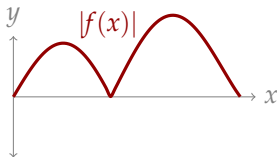
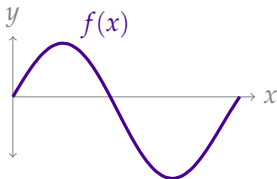
# ABSOLUTE VALUES

$$f(x) \leq |f(x)| \text{ for any } f(x)$$



# ABSOLUTE VALUES

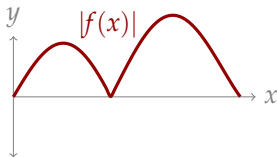
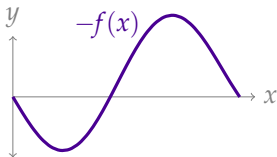
$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$



# ABSOLUTE VALUES

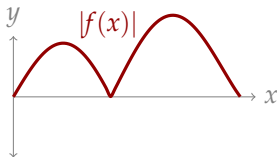
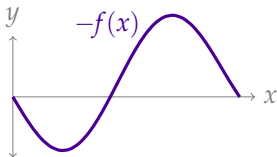
$$f(x) \leq |f(x)| \text{ for any } f(x)$$

$$-f(x) \leq |f(x)| \text{ for any } f(x)$$



# ABSOLUTE VALUES

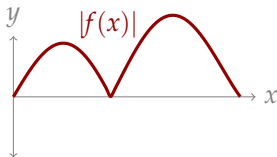
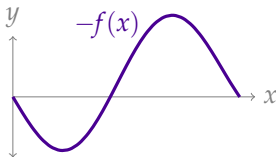
$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$



$$\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx \quad \text{and} \quad \int_a^b -f(x) \, dx \leq \int_a^b |f(x)| \, dx$$

# ABSOLUTE VALUES

$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$




$$\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx \quad \text{and} \quad \int_a^b -f(x) \, dx \leq \int_a^b |f(x)| \, dx$$

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx$$

because  $\left| \int_a^b f(x) \, dx \right|$  is either  $\int_a^b f(x) \, dx$  or  $-\int_a^b f(x) \, dx$ .

## Included Work

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