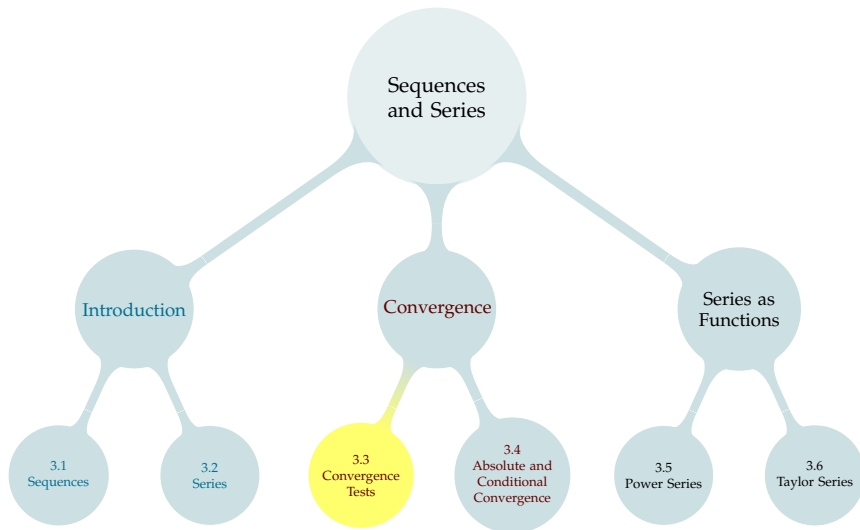


TABLE OF CONTENTS



REVIEW

$$\text{Let } S_N = \sum_{n=1}^N a_n.$$

Simplify: $S_N - S_{N-1}$.

(This will come in handy soon.)

REVIEW

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(This will come in handy soon.)

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N$$

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REVIEW

$$\text{Let } S_N = \sum_{n=1}^N a_n.$$

Simplify: $S_N - S_{N-1}$.

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$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N$$

$$S_{N-1} = a_1 + a_2 + a_3 + \cdots + a_{N-1}$$

$$S_N - S_{N-1} = a_N$$

ALTERNATING SERIES

Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

is alternating if every $A_n \geq 0$.

Alternating series:

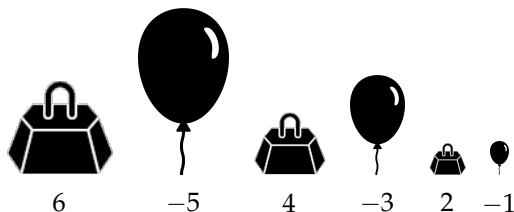
▶ $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$

▶ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

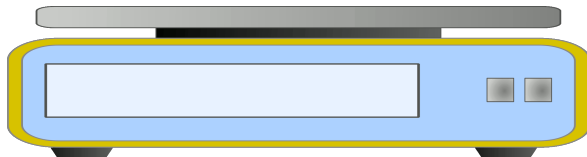
Not alternating:

▶ $\cos(1) + \cos(2) + \cos(3) + \cdots$

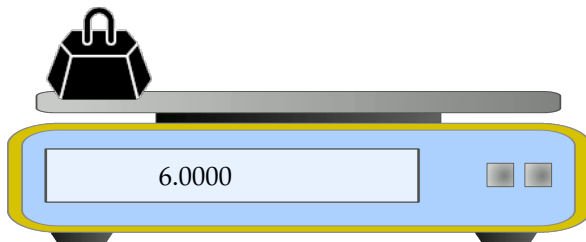
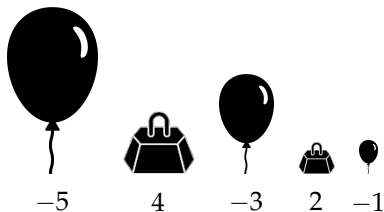
▶ $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$



Note: these terms alternate signs, **and** their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$

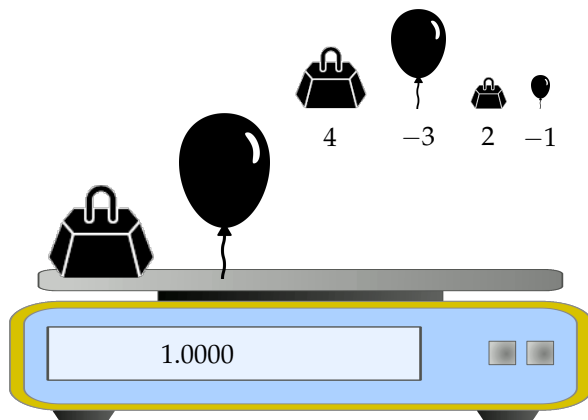


$$S_1 = 6.0000$$



$$S_1 = 6.0000$$

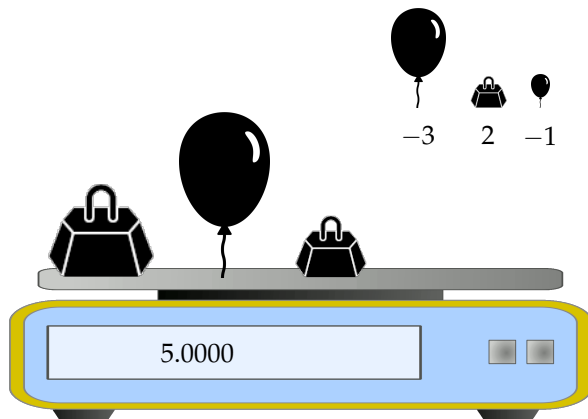
$$S_2 = 1.0000$$



$$S_1 = 6.0000$$

$$S_2 = 1.0000$$

$$S_3 = 5.0000$$



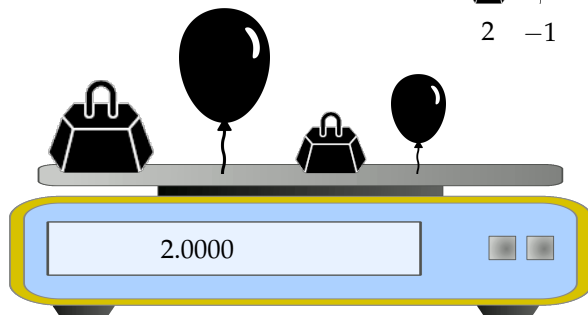
$$S_1 = 6.0000$$

$$S_2 = 1.0000$$

$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$\begin{array}{cc} \text{bag} & \text{balloon} \\ 2 & -1 \end{array}$$



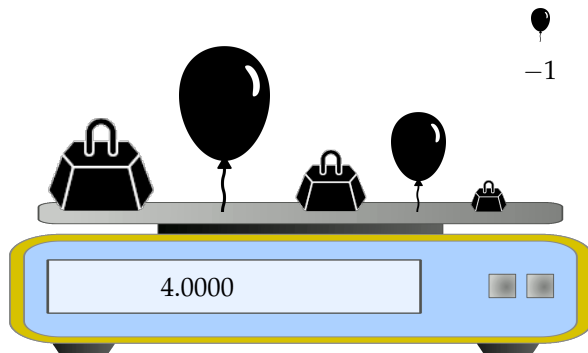
$$S_1 = 6.0000$$

$$S_2 = 1.0000$$

$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$



$$S_1 = 6.0000$$

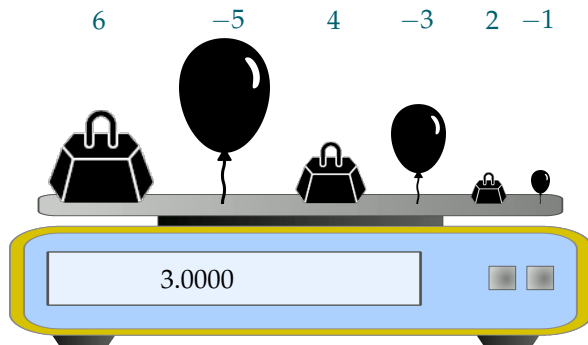
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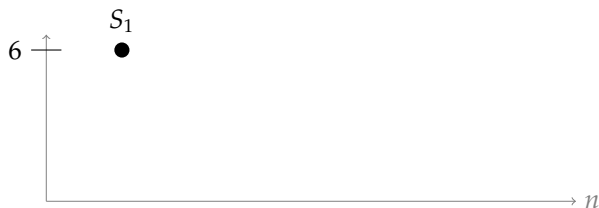
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





6

-5

4

-3

2

-1

$$S_1 = 6.0000$$

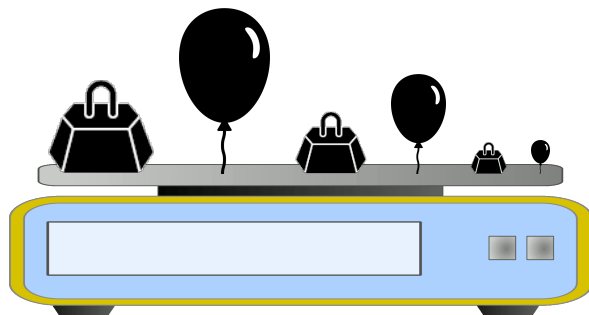
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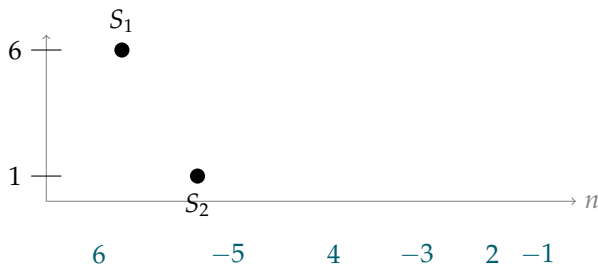
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$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

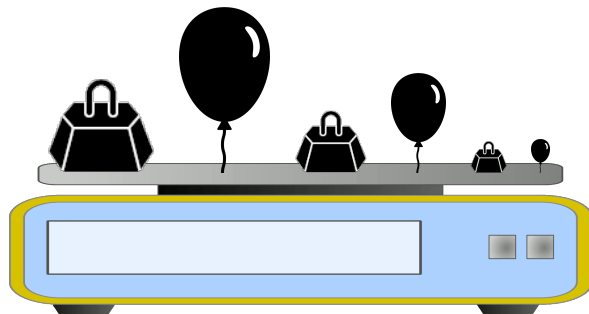
$$S_2 = 1.0000$$

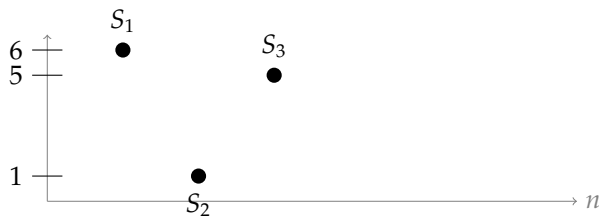
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

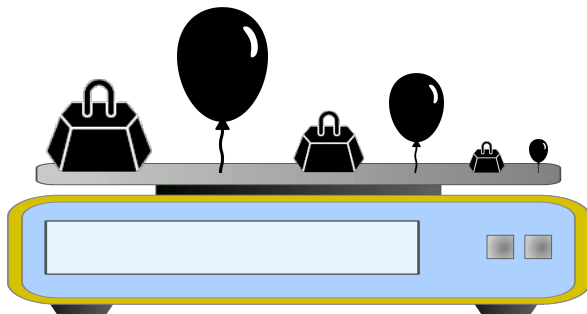
$$S_2 = 1.0000$$

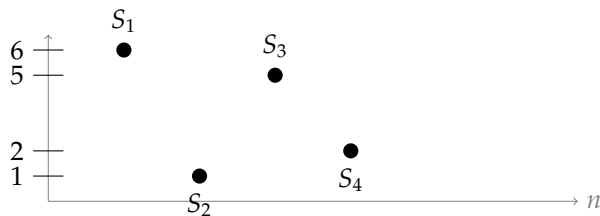
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

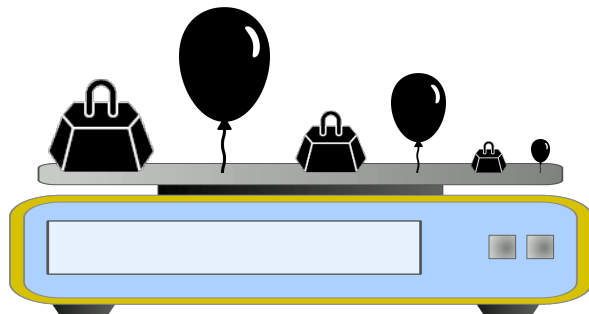
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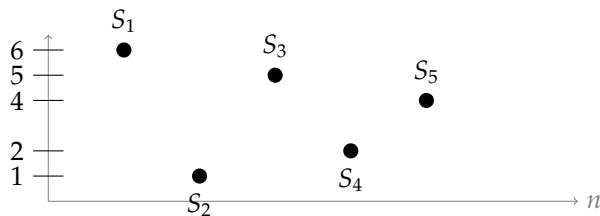
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$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

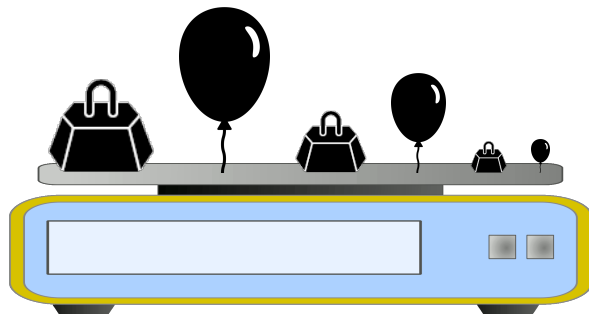
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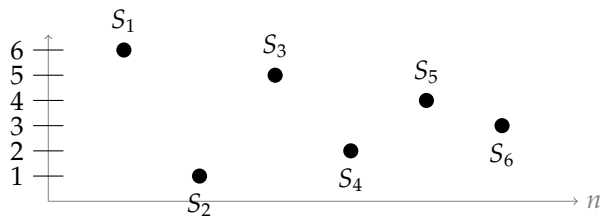
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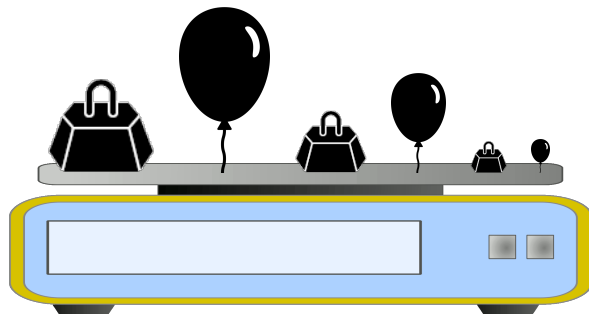
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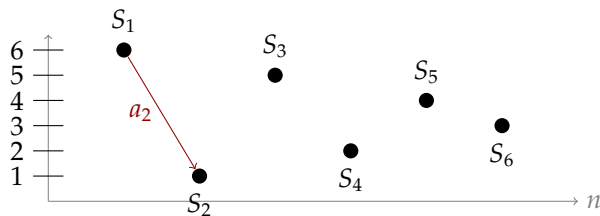
$$S_3 = 5.0000$$

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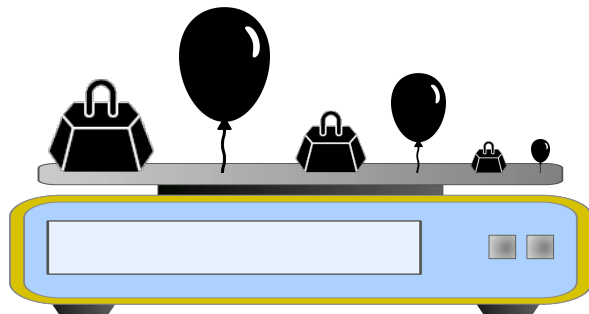
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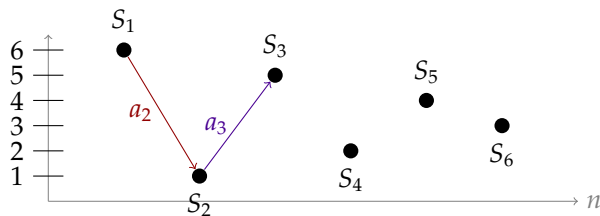
$$S_3 = 5.0000$$

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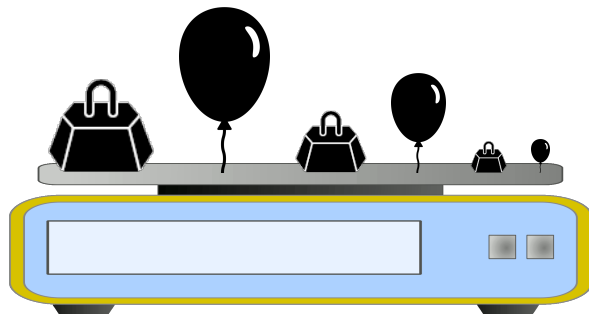
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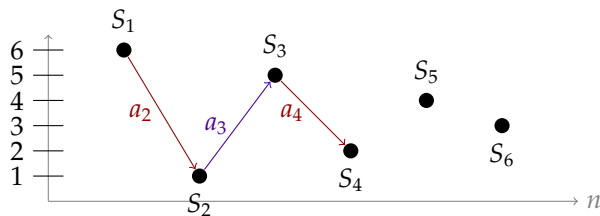
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





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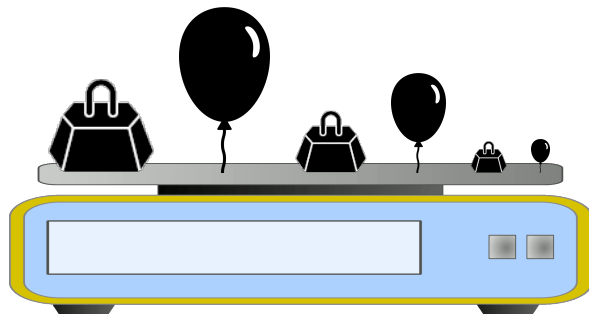
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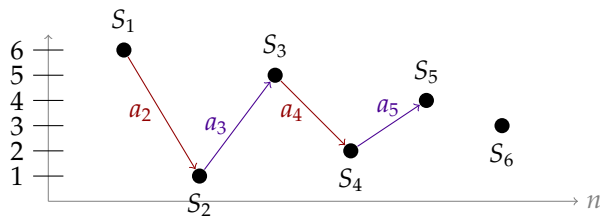
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$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

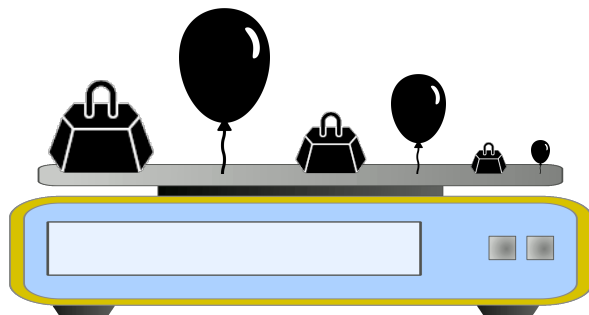
$$S_2 = 1.0000$$

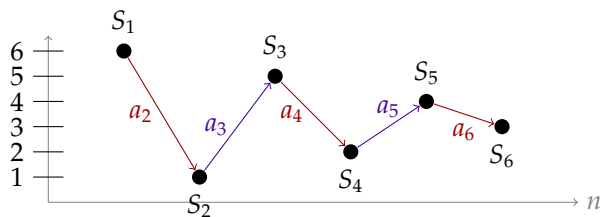
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

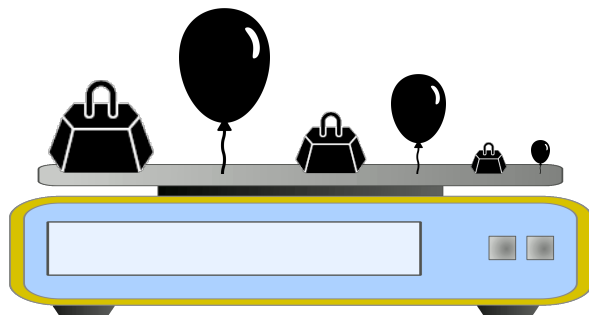
$$S_2 = 1.0000$$

$$S_3 = 5.0000$$

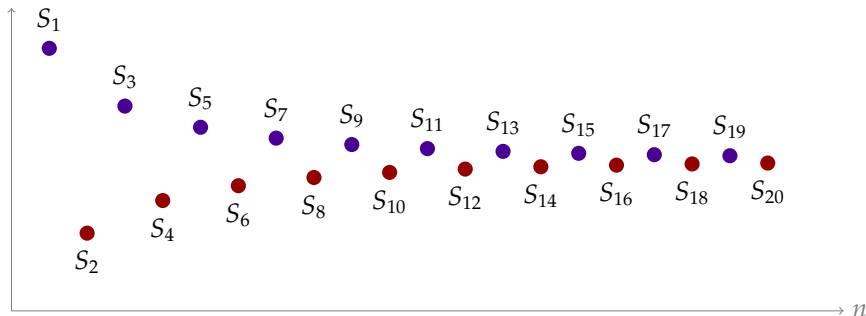
$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$

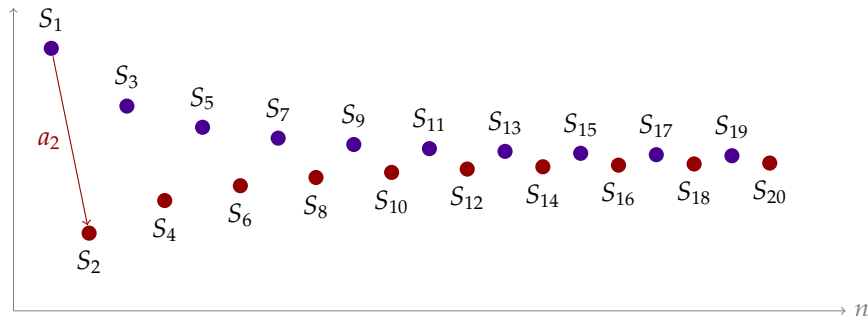


Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, **decreasing** terms and with $\lim_{n \rightarrow \infty} a_n = 0$.



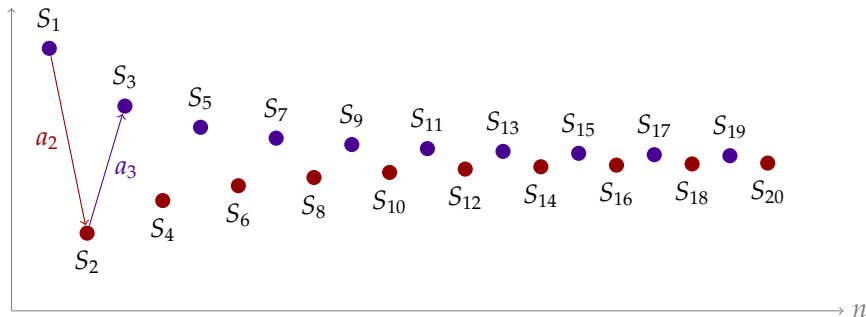
Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.

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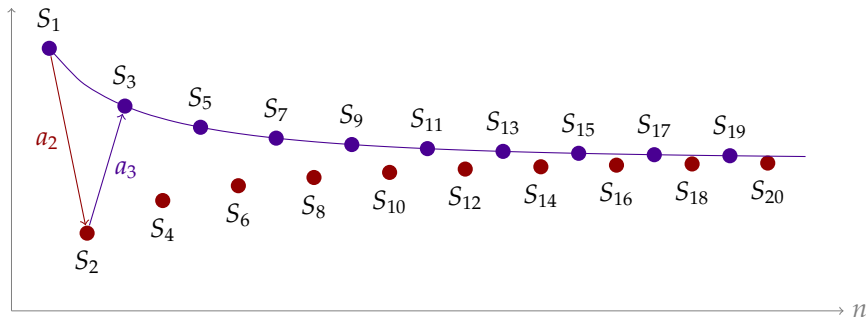
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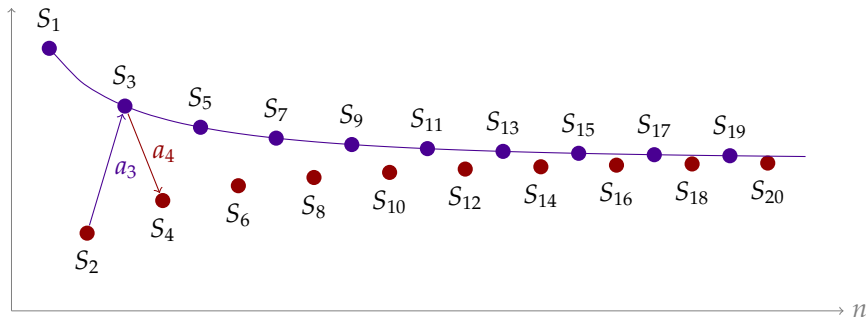
Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, **decreasing** terms and with $\lim_{n \rightarrow \infty} a_n = 0$.



Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.

Odd-indexed partial sums are decreasing.

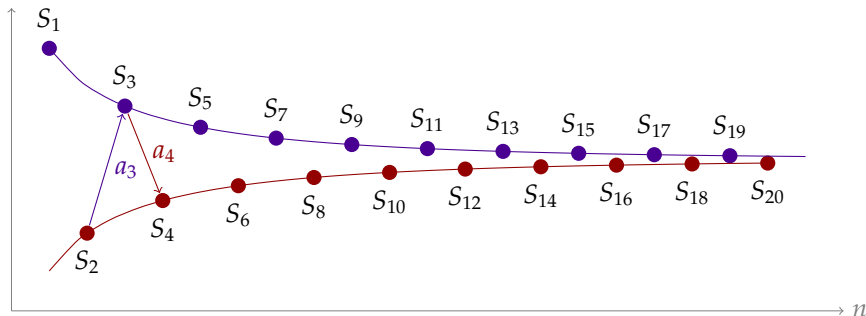
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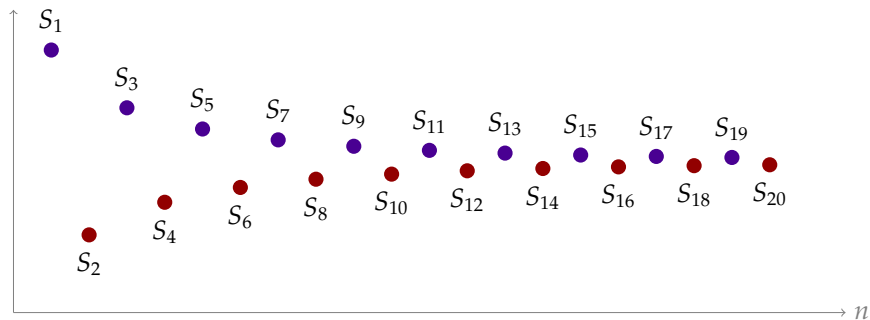


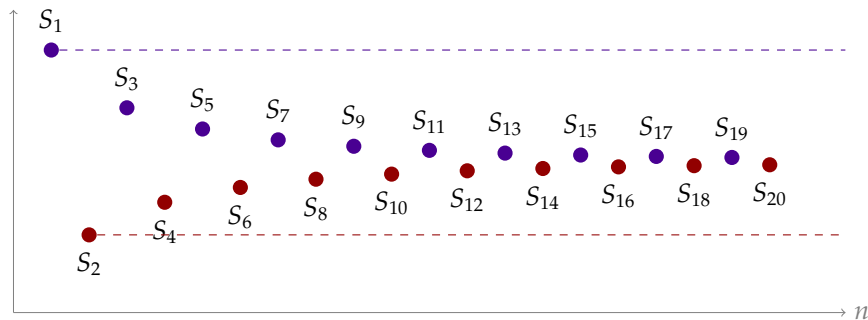
Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.

Odd-indexed partial sums are decreasing.

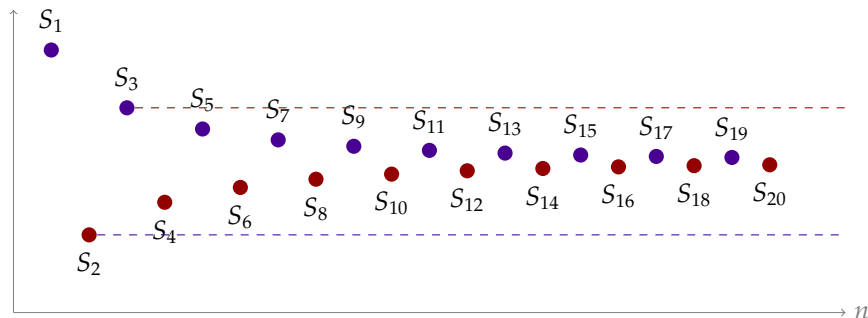
Since $a_3 > a_4$, we have $a_1 - a_2 + (a_3 - a_4) > a_1 - a_2$, so $S_4 > S_2$.

Even-indexed partial sums are increasing.

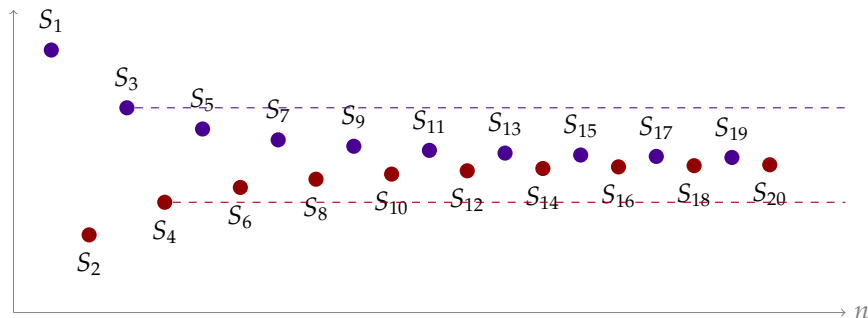




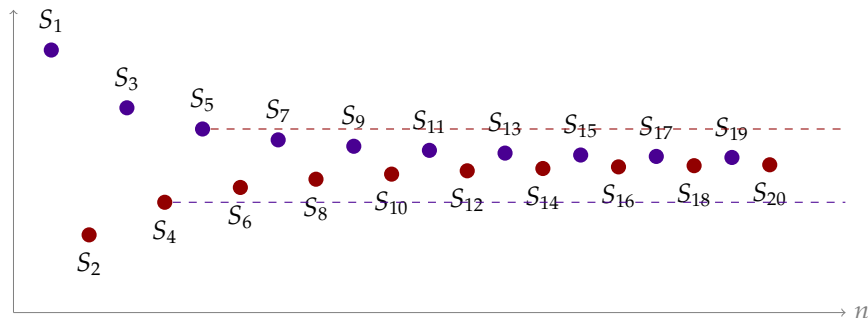
► For all $n \geq 2$, S_n lies between S_1 and S_2 .



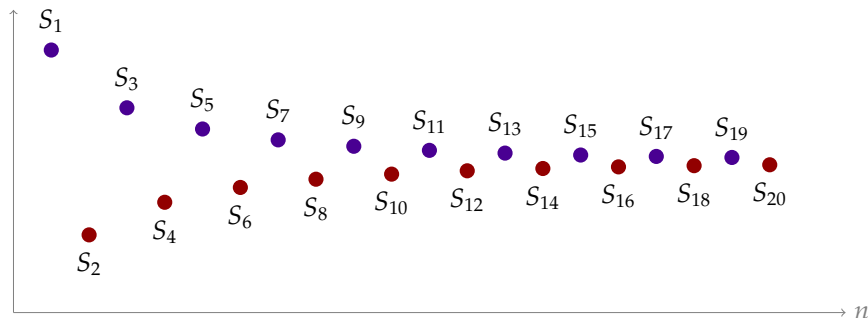
- ▶ For all $n \geq 2$, S_n lies between S_1 and S_2 .
- ▶ For all $n \geq 3$, S_n lies between S_2 and S_3 .



- ▶ For all $n \geq 2$, S_n lies between S_1 and S_2 .
- ▶ For all $n \geq 3$, S_n lies between S_2 and S_3 .
- ▶ For all $n \geq 4$, S_n lies between S_3 and S_4 .

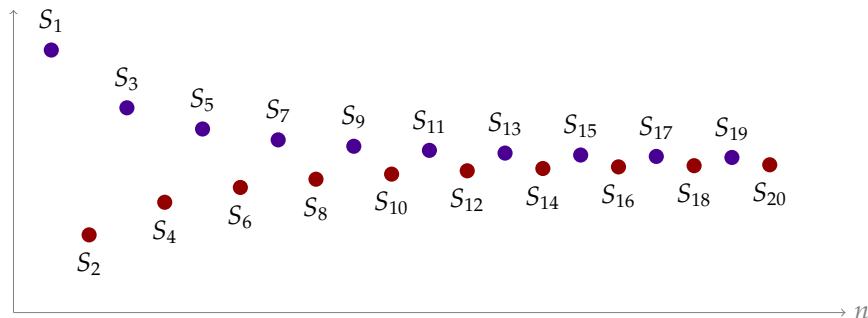


- ▶ For all $n \geq 2$, S_n lies between S_1 and S_2 .
- ▶ For all $n \geq 3$, S_n lies between S_2 and S_3 .
- ▶ For all $n \geq 4$, S_n lies between S_3 and S_4 .
- ▶ For all $n \geq 5$, S_n lies between S_4 and S_5 .



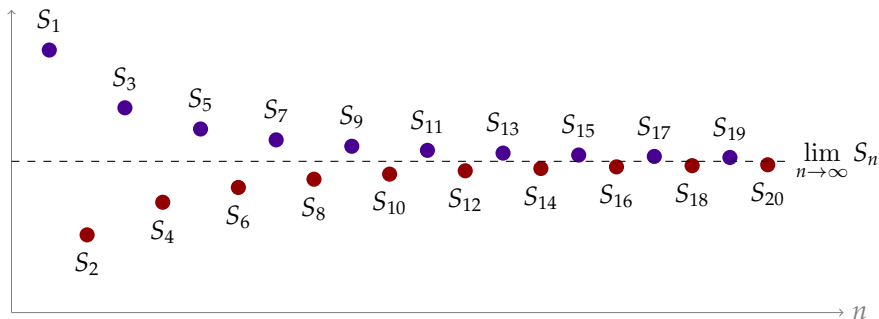
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- ▶ For all $n \geq 3$, S_n lies between S_2 and S_3 .
- ▶ For all $n \geq 4$, S_n lies between S_3 and S_4 .
- ▶ For all $n \geq 5$, S_n lies between S_4 and S_5 .

The difference between consecutive sums S_n and S_{n-1} is:



- For all $n \geq 2$, S_n lies between S_1 and S_2 .
- For all $n \geq 3$, S_n lies between S_2 and S_3 .
- For all $n \geq 4$, S_n lies between S_3 and S_4 .
- For all $n \geq 5$, S_n lies between S_4 and S_5 .

The difference between consecutive sums S_n and S_{n-1} is:
 $|a_n|$, which approaches 0.



- ▶ For all $n \geq 2$, S_n lies between S_1 and S_2 .
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The difference between consecutive sums S_n and S_{n-1} is:
 $|a_n|$, which approaches 0.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

- (i) $a_n \geq 0$ for all $n \geq 1$;
- (ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);
- (iii) and $\lim_{n \rightarrow \infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N , $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^N (-1)^{n-1} a_n$.

Alternating Series Test (abridged)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

- (i) $a_n \geq 0$ for all $n \geq 1$;
- (ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);
- (iii) and $\lim_{n \rightarrow \infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

► True or false: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

► True or false: the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Let $a_n = \frac{1}{n}$.

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(i) $a_n \geq 0$

(ii) $a_{n+1} \leq a_n$

(iii) $\lim_{n \rightarrow \infty} a_n = 0$

Let $a_n = \frac{1}{n}$.

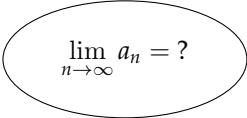
- (i) $a_n \geq 0$
- (ii) $a_{n+1} \leq a_n$
- (iii) $\lim_{n \rightarrow \infty} a_n = 0$

- We've already seen that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges. That is,

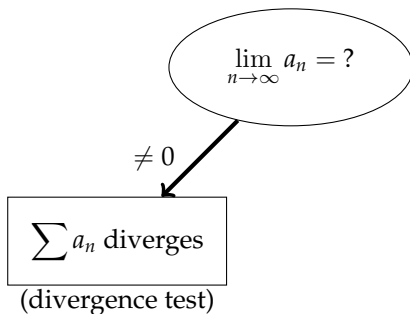
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

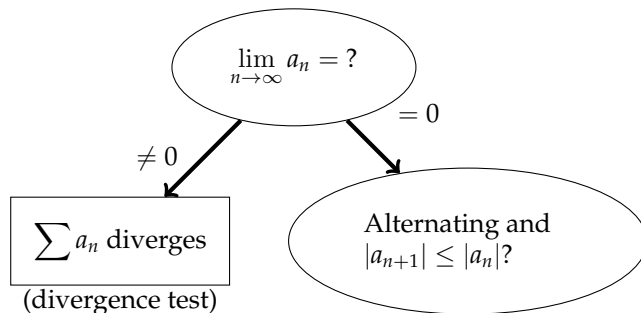
DIVERGENCE TEST + ALTERNATING SERIES TEST


$$\lim_{n \rightarrow \infty} a_n = ?$$

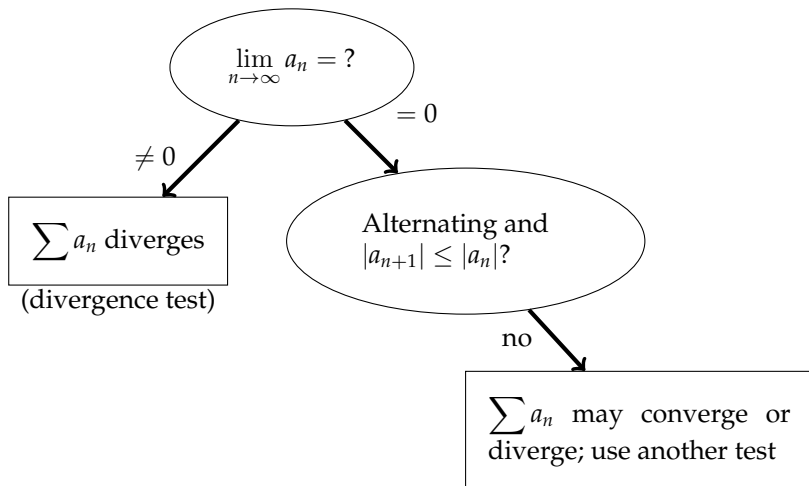
DIVERGENCE TEST + ALTERNATING SERIES TEST



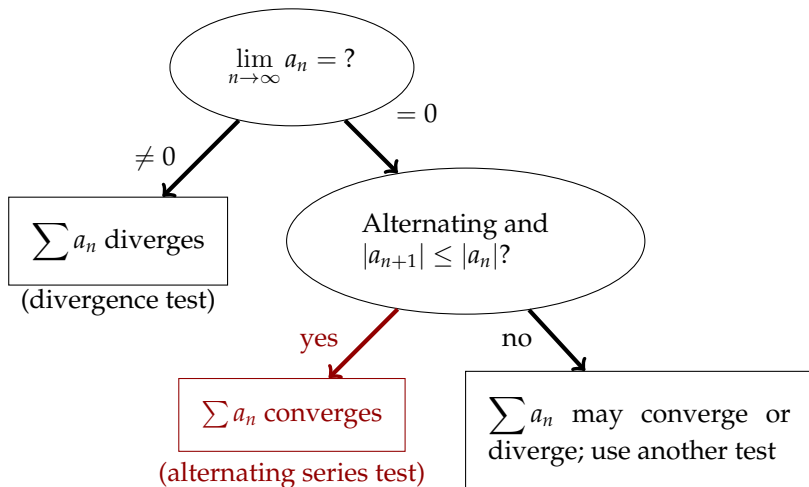
DIVERGENCE TEST + ALTERNATING SERIES TEST



DIVERGENCE TEST + ALTERNATING SERIES TEST



DIVERGENCE TEST + ALTERNATING SERIES TEST



Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n \rightarrow \infty} a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698$.

How close is that to the value $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

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How close is that to the value $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

$$\frac{-1}{100} = \frac{(-1)^{100-1}}{100} \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{99} \frac{(-1)^n}{n} \leq 0.$$

That is, the actual series has a sum in the interval $[0.688, 0.698]$.

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Using a computer, you find $\sum_{n=1}^{19} (-1)^{n-1} \frac{n^2}{n^2 + 1} \approx 0.6347$.

How close is that to the value $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2 + 1}$?

Alternating Series Test

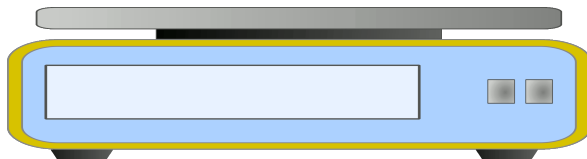
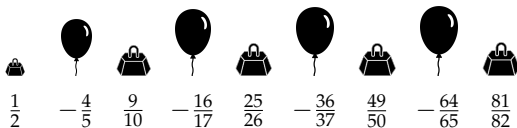
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Not close at all: the series is divergent (which we can see by the divergence test).

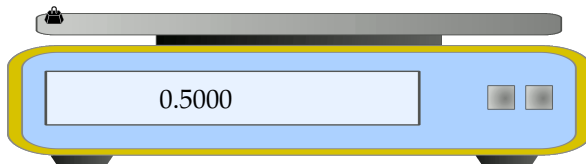
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$








							
$-\frac{4}{5}$	$\frac{9}{10}$	$-\frac{16}{17}$	$\frac{25}{26}$	$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$

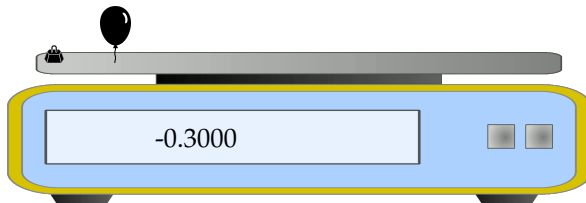


$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

						
$\frac{9}{10}$	$-\frac{16}{17}$	$\frac{25}{26}$	$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$









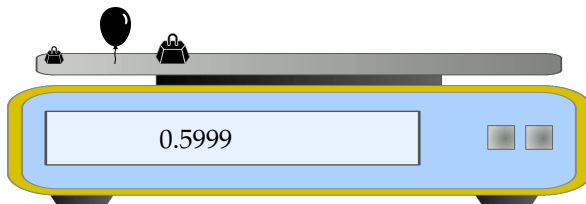
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$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

					
$-\frac{16}{17}$	$\frac{25}{26}$	$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$








$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

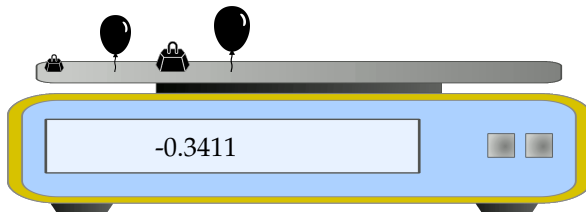
$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

				
$\frac{25}{26}$	$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

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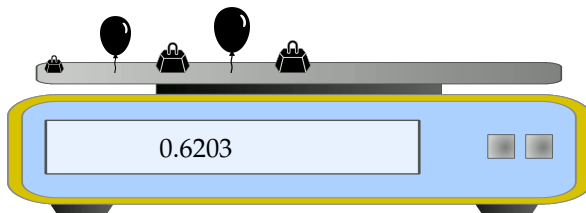
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

●	●	●	●
- $\frac{36}{37}$	$\frac{49}{50}$	- $\frac{64}{65}$	$\frac{81}{82}$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

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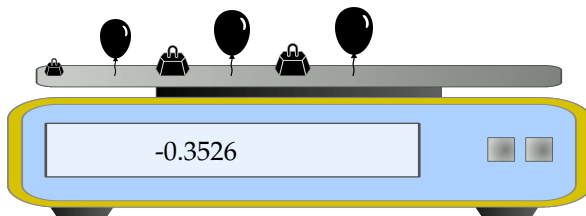
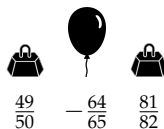
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

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$$S_6 = -0.3526$$



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$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

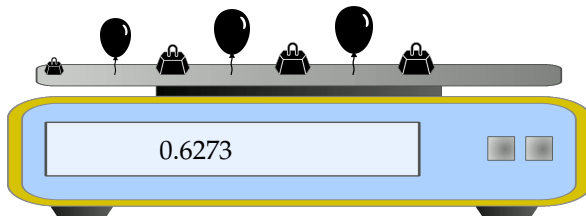
$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$$-\frac{64}{65} \quad \frac{81}{82}$$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

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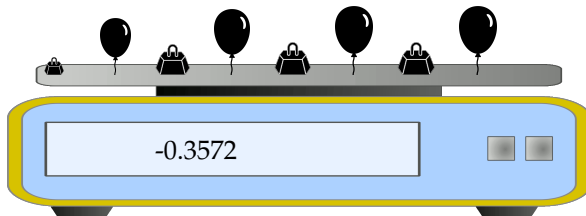
$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

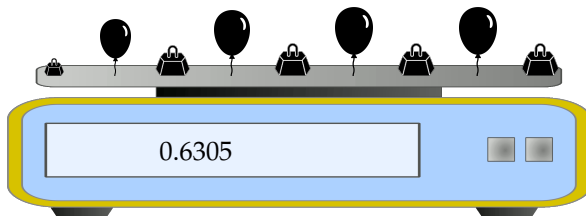
$$S_7 = 0.6273$$

$$S_8 = -0.3572$$



$$\frac{81}{82}$$


$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$



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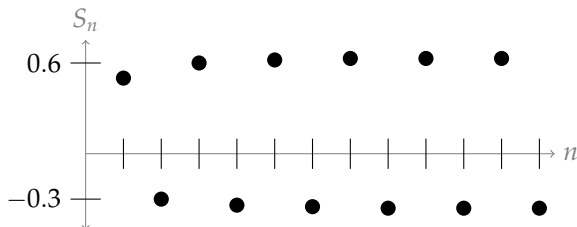
$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$$S_8 = -0.3572$$

$$S_9 = 0.6305$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$



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$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

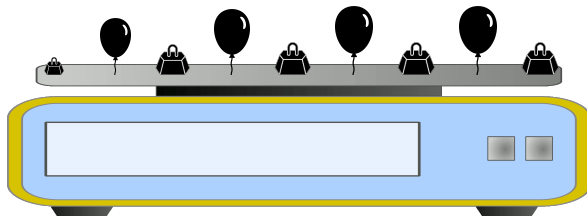
$$S_5 = 0.6203$$

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$$S_7 = 0.6273$$

$$S_8 = -0.3572$$

$$S_9 = 0.6305$$



Recall for a geometric series, the **ratios of consecutive terms** is constant.

$$\times \frac{1}{2}$$


$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \dots$$

If that ratio has magnitude **less than one**, then the series converges.
If the ratio has magnitude **greater than one**, the series diverges.

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$$\begin{array}{c} \times \frac{1}{2} \quad \times \frac{1}{2} \\ \curvearrowright \quad \curvearrowright \end{array}$$

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 \frown \quad \frown \quad \frown \\
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 & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} \\
 & \frown & & \frown & & \frown & & \frown \\
 & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
 \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} \cdots
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 \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \underbrace{\hspace{1.5cm}} \\
 \frac{1/4}{1/2} =
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 \text{—————} \rightarrow \text{—————} \rightarrow \text{—————} \rightarrow \text{—————} \rightarrow \\
 \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \\
 \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \\
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 \frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} = \frac{1/32}{1/16} = \frac{1}{2}
 \end{array}$$

If that ratio has magnitude **less than one**, then the series converges.
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For series convergence, we are concerned with what happens to terms a_n when n is sufficiently large.

Suppose for a sequence a_n , $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$ for some constant L .

$$\underbrace{a_n + a_{n+1}}_{\frac{a_{n+1}}{a_n} \approx} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots$$

Like in a geometric series:

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$$\begin{array}{ccccccc}
 a_n & + & a_{n+1} & + & a_{n+2} & + & a_{n+3} & + & a_{n+4} & + & \cdots \\
 \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & & & \\
 \frac{a_{n+1}}{a_n} & \approx & \frac{a_{n+2}}{a_{n+1}} & \approx & \frac{a_{n+3}}{a_{n+2}} & \approx & \frac{a_{n+4}}{a_{n+3}} & \approx & & &
 \end{array}$$

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Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

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Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

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$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{3}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, by the ratio test, $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges.

REMARK

The series we just considered, $\sum_{n=1}^{\infty} \frac{n}{3^n}$, looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

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► Integral test:

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- ▶ Integral test: $\int \frac{x}{3^x} dx$ can be evaluated using integration by parts.
- ▶ Comparison test:

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 - ▶ $\sum \frac{1}{3^n}$ is not a valid comparison series, nor is $\sum n$.

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 - ▶ Because $n < 2^n$ for all $n \geq 1$, the series $\sum \left(\frac{2}{3}\right)^n$ will work.
- ▶ The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$



$$\frac{1}{3}$$



$$\frac{2}{3^2}$$



$$\frac{3}{3^3}$$



$$\frac{4}{3^4}$$



$$\frac{5}{3^5}$$



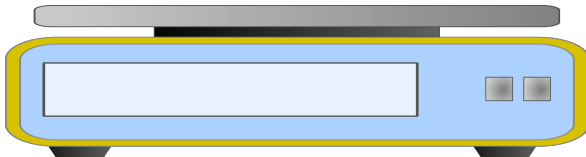
$$\frac{6}{3^6}$$



$$\frac{7}{3^7}$$

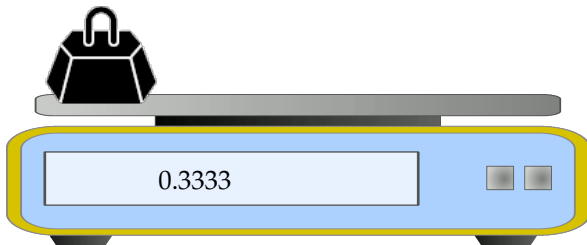


$$\frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

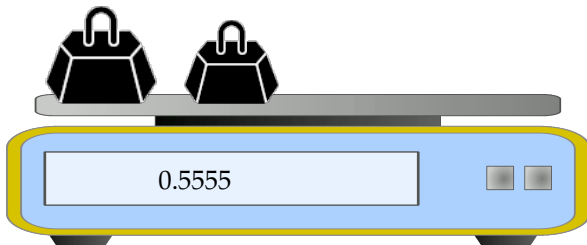
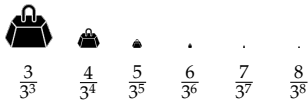
$$S_1 = 0.3333$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$








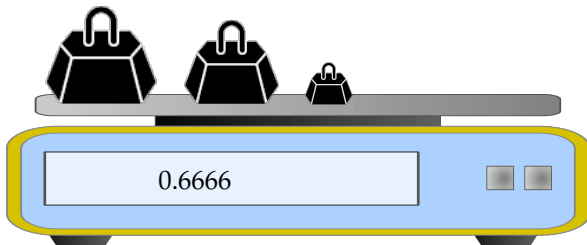
$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

				
$\frac{4}{3^4}$	$\frac{5}{3^5}$	$\frac{6}{3^6}$	$\frac{7}{3^7}$	$\frac{8}{3^8}$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

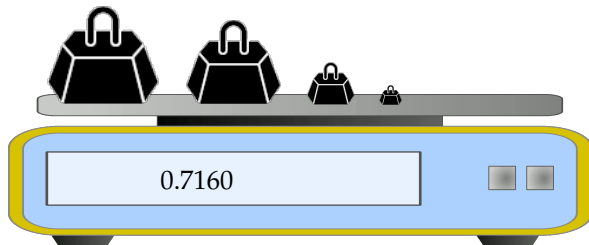
$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$\frac{5}{3^5} \quad \frac{6}{3^6} \quad \frac{7}{3^7} \quad \frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

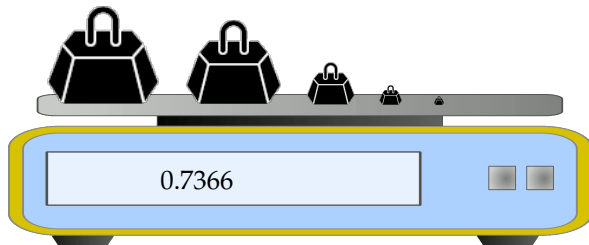
$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$\frac{6}{3^6} \quad \frac{7}{3^7} \quad \frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

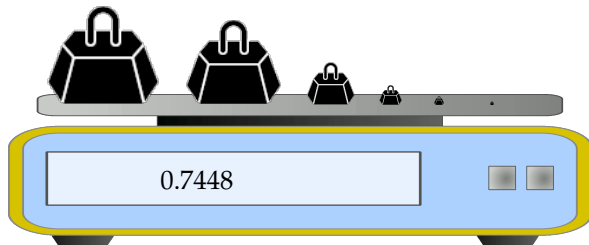
$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$\frac{7}{3^7} \quad \frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

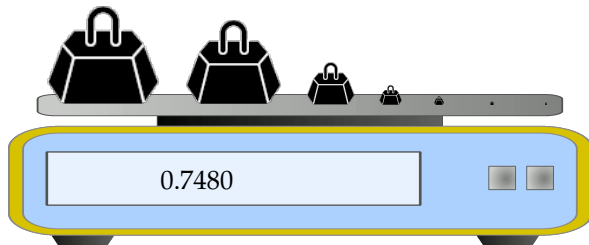
$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$S_7 = 0.7480$$

$$\frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

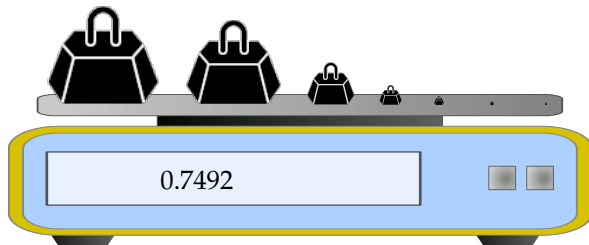
$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$S_7 = 0.7480$$

$$S_8 = 0.7492$$



Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Let a and x be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of a and x .)

$$\sum_{n=1}^{\infty} anx^{n-1}$$

$$\sum_{n=1}^{\infty} anx^{n-1}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{a(n+1)x^n}{anx^{n-1}} \right| = \left| \left(\frac{n+1}{n} \right) x \right| = \left(1 + \frac{1}{n} \right) |x|$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. For the cases $x = \pm 1$, the ratio test is inconclusive, so we'll need another test. Fortunately, the divergence test makes things quick.

For $x = 1$: $\lim_{n \rightarrow \infty} an(1)^{n-1} = \lim_{n \rightarrow \infty} an \neq 0$

For $x = -1$: $\lim_{n \rightarrow \infty} an(-1)^{n-1} \neq 0$

All together, for any nonzero a , the series diverges when $|x| \geq 1$ and converges when $|x| < 1$.



Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x .)

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x .)

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{(-3)^{n+1} \sqrt{n+2}}{2(n+1)+3} x^{n+1}}{\frac{(-3)^n \sqrt{n+1}}{2n+3} x^n} \right| = \left| \frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{2n+3}{2n+5} \cdot \frac{x^{n+1}}{x^n} \right| \\ &= 3 \cdot \sqrt{\frac{n+2}{n+1}} \cdot \left(\frac{2n+3}{2n+5} \right) \cdot |x| = 3 \sqrt{\frac{1+2/n}{1+1/n}} \cdot \left(\frac{2+3/n}{2+5/n} \right) \cdot |x| \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 3 \sqrt{\frac{1}{1}} \left(\frac{2}{2} \right) |x| = 3|x|$$

So the series converges when $3|x| < 1$ and diverges when $3|x| > 1$.

So for $|x| < \frac{1}{3}$, the series converges, and for $|x| > \frac{1}{3}$, it diverges.



Consider $x = \frac{1}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}}{2n+3}$$

This is an alternating series. Let's use the alternating series test.

- (i) $a_n = \frac{\sqrt{n+1}}{2n+3} \geq 0$ for all $n \geq 1$,
- (ii) To show that a_n is monotonically decreasing, consider the derivative of $f(t) = \frac{\sqrt{t+1}}{2t+3}$:

$$\begin{aligned} f'(t) &= \frac{(2t+3) \frac{1}{2\sqrt{t+1}} - \sqrt{t+1}(2)}{(2t+3)^2} \left(\frac{\sqrt{t+1}}{\sqrt{t+1}} \right) \\ &= \frac{\left(t + \frac{3}{2}\right) - (t+1)(2)}{(2t+3)^2 \sqrt{t+1}} = \frac{-t - \frac{1}{2}}{(2t+3)^2 \sqrt{t+1}} \end{aligned}$$

Since $f'(t) < 0$ for all $t > 0$, we see it is a decreasing function on that domain, so $a_{n+1} < a_n$ for all $n \geq 1$.

- (iii) $\lim_{n \rightarrow \infty} a_n = 0$

So, our series converges by the alternating series test when $x = \frac{1}{3}$.

Finally, consider $x = -\frac{1}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3} \frac{(-3)^n}{(-3)^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3}$$

We will use the limit comparison test, with comparison series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n+1}}{2n+3}}{\frac{1}{\sqrt{n}}} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n+1} \sqrt{n}}{2n+3} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+n}}{2n+3} \left(\frac{1/n}{1/n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{1+1/n}}{2+3/n} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2} \end{aligned}$$

Since $\frac{1}{2}$ is a nonzero constant, and since $\sum \frac{1}{\sqrt{n}}$ diverges (by the p -test), our series diverges as well.

All together, the original series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, and diverges otherwise.



FILL IN THE BLANKS

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$
 then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

FILL IN THE BLANKS

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If the sequence $\{a_n\}_{n=c}^{\infty}$ fails to converge to zero as $n \rightarrow \infty$,
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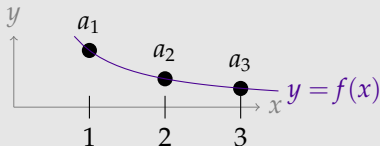
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Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

- (i) and
 (ii) and
 (iii) $f(n) = a_n$ for all $n \geq N_0$.

Then



$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x) \, dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

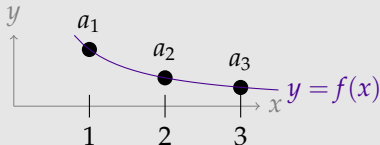
$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n \leq \int_N^{\infty} f(x) \, dx \quad \text{for all } N \geq N_0$$

Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

- (i) $f(x) \geq 0$ for all $x \geq N_0$ and
- (ii) and
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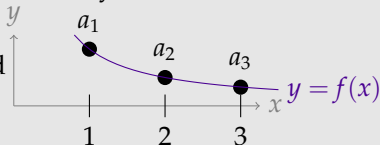
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- (ii) $f(x)$ decreases as x increases and
- (iii) $f(n) = a_n$ for all $n \geq N_0$.

Then



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FILL IN THE BLANKS

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \square Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \square Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

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FILL IN THE BLANKS

Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n . Assume that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

exists.

- (a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.
- (b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if , then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

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Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i)

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N , $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^N (-1)^{n-1} a_n$.

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Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

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LIST OF CONVERGENCE TESTS

Divergence Test

When the n^{th} term in the series *fails* to converge to zero as n tends to infinity.

This is a good first thing to check: if it works, it's quick, but it doesn't always work.

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Integral Test

- ▶ works well when, if you substitute x for n in the n^{th} term you get a function, $f(x)$, that you can easily integrate
- ▶ don't forget to check that $f(x) \geq 0$ and that $f(x)$ decreases as x increases

LIST OF CONVERGENCE TESTS

Ratio Test

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- ▶ works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$
- ▶ this often happens when a_n contains powers, like 7^n , or factorials, like $n!$
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Comparison Test and Limit Comparison Test

- ▶ Comparison test lets you ignore pieces of a function that feel extraneous (like replacing $n^2 + 1$ with n^2) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- ▶ Limit comparison works well when, for very large n , the n^{th} term a_n is approximately the same as a simpler, nonnegative term b_n

- The integral test gave us the p -test. When you're looking for comparison series, p -series $\sum \frac{1}{n^p}$ are often good choices, because their convergence or divergence is so easy to ascertain.

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- ▶ Geometric series have the form $\sum a \cdot r^n$ for some nonzero constants a and r . The magnitude of r is all you need to know to decide whether they converge or diverge, so these are also common comparison series.
- ▶ Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

The **divergence test** is inconclusive, because $\lim_{n \rightarrow \infty} \frac{\cos n}{2^n} = 0$ (which you can show with the squeeze theorem).

The **integral test** doesn't apply, because $f(x) = \frac{\cos x}{2^x}$ is not always positive (and not decreasing).

The **alternating series test** doesn't apply because the signs of the series do not strictly alternate every term.

The **ratio test** does not apply, because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ does not exist.

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

Comparison test: Let $a_n = \frac{\cos n}{2^n}$. Note $|a_n| \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a geometric sum with ratio of consecutive terms $\frac{1}{2}$).

So by the comparison test, $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges.

Limit comparison: Set $a_n = \frac{\cos n}{2^n}$ and $b_n = \left(\frac{2}{3}\right)^n$. Then

$$\frac{a_n}{b_n} = \frac{\frac{\cos n}{2^n}}{\frac{2^n}{3^n}} = \left(\frac{3}{4}\right)^n \cos n$$

$$-\left(\frac{3}{4}\right)^n \leq \left(\frac{3}{4}\right)^n \cos n \leq \left(\frac{3}{4}\right)^n, \text{ and } \lim_{n \rightarrow \infty} -\left(\frac{3}{4}\right)^n = \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0$$

So, by the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

Since $\sum_{n=1}^{\infty} b_n$ converges, by the limit comparison theorem, $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges as well.

Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

The **alternating series test** doesn't apply because the signs of the series do not alternate.

The **integral test** doesn't apply $f(x) = \frac{2^x \cdot x^2}{(x+5)^5}$ is not a decreasing function.

Divergence test: $\lim_{n \rightarrow \infty} \frac{2^n \cdot n^2}{(n+5)^5} = \infty$ (which you can see because the numerator is larger than a power function; the denominator is a polynomial; and power functions grow faster than polynomials), so the series diverges by the divergence test.

This is the fastest option, but not the only one.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Ratio test:

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{\frac{2^{n+1} \cdot (n+1)^2}{(n+1+5)^5}}{\frac{2^n \cdot n^2}{(n+5)^5}} = \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{(n+5)^5}{(n+6)^5} \\ &= 2 \left(1 + \frac{1}{n}\right)^2 \left(1 - \frac{1}{n+6}\right)^5 \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= 2(1)^2(1)^5 = 2 \end{aligned}$$

So, the limit of the ratio of consecutive terms is greater than 1.

Therefore $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges by the ratio test.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Comparison test: Since power functions grow faster than polynomials, for large values of n , $2^n > (n+5)^5$, so $\frac{2^n}{(n+5)^5} > 1$. Then, for large enough n ,

$$\frac{2^n \cdot n^2}{(n+5)^5} > n^2.$$

By the divergence test, $\sum_{n=1}^{\infty} n^2$ diverges. So by the comparison test, $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges as well.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Limit comparison: Set $a_n = \frac{2^n \cdot n^2}{(n+5)^5}$ and $b_n = \frac{2^n}{n^3}$.

Then

$$\frac{a_n}{b_n} = \frac{\frac{2^n \cdot n^2}{(n+5)^5}}{\frac{2^n}{n^3}} = \frac{n^5}{(n+5)^5} = \left(1 - \frac{5}{n+5}\right)^5$$

$$\text{So, } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1^5 = 1$$

Note that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges. (You can show this using the tests we've already used on the original series, so this method isn't really an improvement.) Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists and is nonzero, by the limit

comparison theorem, $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges.

Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin \theta \leq \theta$.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin \theta \leq \theta$.

The **divergence test** is inconclusive because $\lim_{n \rightarrow \infty} \frac{\sin(\frac{1}{n})}{n} = 0$.

The **alternating series test** does not apply because we are not considering an alternating series.

The **integral test** won't work for us because $\int_1^{\infty} \frac{1}{x} \sin\left(\frac{1}{x}\right) dx$ cannot be evaluated with techniques we've learned in class so far.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin \theta \leq \theta$.

The **ratio test** is inconclusive because $\lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} \sin\left(\frac{1}{n+1}\right)}{\frac{1}{n} \sin\left(\frac{1}{n}\right)} = 1$:

Set $x = \frac{1}{n+1}$. Then $\frac{1}{n} = \frac{x}{1-x}$:

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}} = \lim_{x \rightarrow 0^+} \frac{\sin x}{\frac{x}{1-x}} = \lim_{x \rightarrow 0^+} (1-x) \frac{\sin x}{x} = 1 \cdot 1 = 1$$

Set $y = \frac{1}{n}$. Then $\frac{1}{n+1} = \frac{y}{1+y}$:

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n+1}} = \lim_{y \rightarrow 0^+} \frac{\sin y}{\frac{y}{1+y}} = \lim_{y \rightarrow 0^+} (1+y) \frac{\sin y}{y} = 1 \cdot 1 = 1$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} \sin\left(\frac{1}{n+1}\right)}{\frac{1}{n} \sin\left(\frac{1}{n}\right)} = 1$$

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin \theta \leq \theta$.

Comparison test: For $n \geq 1$, $\frac{1}{n} > 0$. Then setting $\theta = \frac{1}{n}$ in the hint, $\sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$. Furthermore, $0 < \frac{1}{n} < \pi$, so $\sin\left(\frac{1}{n}\right) > 0$.

$$0 < \frac{1}{n} \sin\left(\frac{1}{n}\right) \leq \frac{1}{n} \left(\frac{1}{n}\right) = \frac{1}{n^2}$$

The p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges as well.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin \theta \leq \theta$.

Limit comparison: Set $a_n = \frac{1}{n} \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sin\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$


Setting $x = \frac{1}{n}$,


$$= \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$


The p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges as well.



Included Work

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