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Review

Let $S_N = \sum_{n=1}^{N} a_n$.

Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

Alternating Series

Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

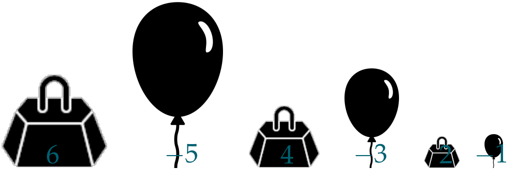
is alternating if every $A_n \geq 0$.

Alternating series:

- $1-2+3-4+5-6+7-8+\cdots$
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Not alternating:

- $\cos(1) + \cos(2) + \cos(3) + \cdots$
- $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$



$6 \quad -5 \quad 4 \quad -3 \quad 2 \quad -1$

$S_1 = 6.0000$

$S_2 = 1.0000$

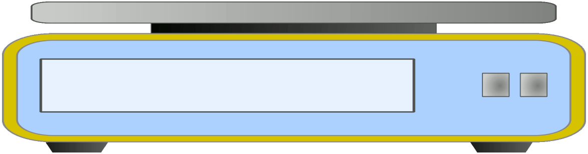
$S_3 = 5.0000$

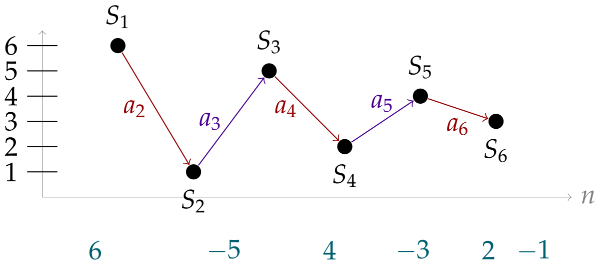
$S_4 = 2.0000$

$S_5 = 4.0000$

$S_6 = 3.0000$

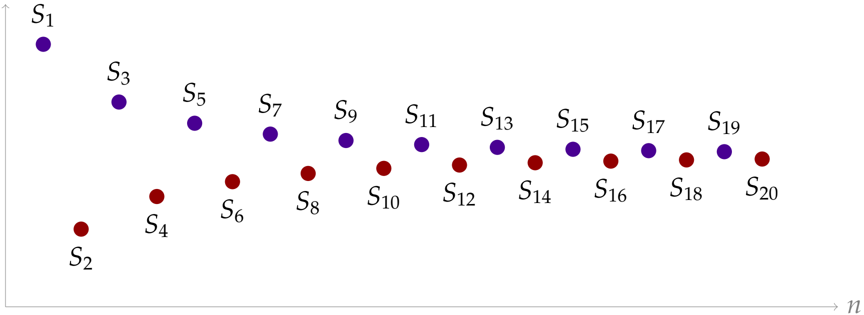
Note: these terms alternate signs, and their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$



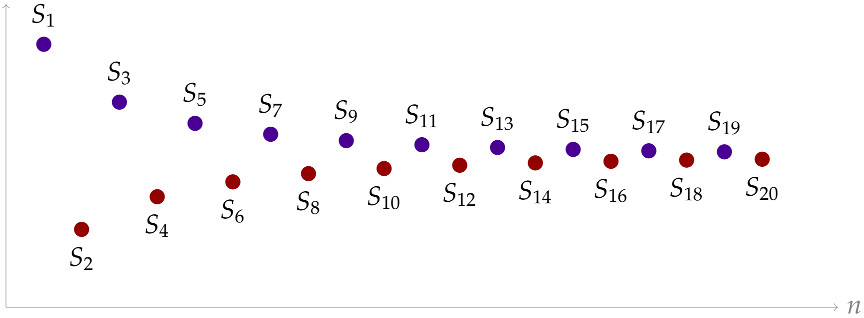


Note: these terms alternate signs, and their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$

Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.



- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.
- For all $n \geq 5$, S_n lies between S_4 and S_5.

The difference between consecutive sums S_n and S_{n-1} is:

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N}(-1)^{n-1}a_n$.

Alternating Series Test (abridged)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

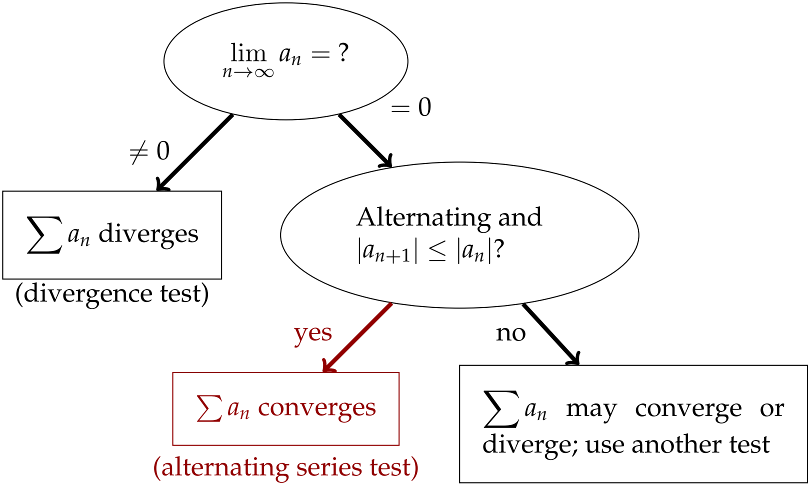
Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

- True or false: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.
- True or false: the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Divergence Test + Alternating Series Test



Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim\limits_{n\to\infty} a_n = 0$. Then $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\displaystyle\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698.$

How close is that to the value $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

$$\frac{-1}{100} = \frac{(-1)^{100-1}}{100} \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{99} \frac{(-1)^n}{n} \leq 0.$$

That is, the actual series has a sum in the interval $[0.688, 0.698]$.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim\limits_{n\to\infty} a_n = 0$. Then $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\displaystyle\sum_{n=1}^{19}(-1)^{n-1}\frac{n^2}{n^2+1} \approx 0.6347$.

How close is that to the value $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$?

Not close at all: the series is divergent (which we can see by the divergence test).

Recall for a geometric series, the ratios of consecutive terms is constant.

$$\frac{1}{2} \xrightarrow{\times\frac{1}{2}} \frac{1}{4} \xrightarrow{\times\frac{1}{2}} \frac{1}{8} \xrightarrow{\times\frac{1}{2}} \frac{1}{16} \xrightarrow{\times\frac{1}{2}} \frac{1}{32}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

$$\frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} = \frac{1/32}{1/16} = \frac{1}{2}$$

If that ratio has magnitude less then one, then the series converges.
If the ratio has magnitude greater than one, the series diverges.

For series convergence, we are concerned with what happens to terms a_n when n is sufficiently large.
Suppose for a sequence a_n, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ for some constant L.

$$\underbrace{a_n \quad + \quad a_{n+1}}_{\frac{a_{n+1}}{a_n}} \underbrace{\quad + \quad a_{n+2}}_{\approx \frac{a_{n+2}}{a_{n+1}}} \underbrace{\quad + \quad a_{n+3}}_{\approx \frac{a_{n+3}}{a_{n+2}}} \underbrace{\quad + \quad a_{n+4}}_{\approx \frac{a_{n+4}}{a_{n+3}}} \underbrace{\quad + \quad \cdots}_{\approx \frac{a_{n+5}}{a_{n+4}}} \approx L$$

Like in a geometric series:

If L has magnitude less then one, then the series converges.
If L has magnitude greater than one, the series diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges or diverges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{3}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$$

REMARK

The series we just considered, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$, looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

We could have used other tests, but ratio was probably the easiest.

- Integral test: $\displaystyle\int \frac{x}{3^x}\,\mathrm{d}x$ can be evaluated using integration by parts.
- Comparison test:
  - $\sum \frac{1}{3^n}$ is not a valid comparison series, nor is $\sum n$.
  - Because $n < 2^n$ for all $n \geq 1$, the series $\sum \left(\frac{2}{3}\right)^n$ will work.
- The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Let a and x be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of a and x.)

Q

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}\sqrt{n+2}}{2(n+1)+3}x^{n+1}}{\frac{(-3)^n\sqrt{n+1}}{2n+3}x^n}\right| = \left|\frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{2n+3}{2n+5} \cdot \frac{x^{n+1}}{x^n}\right|$$

$$= 3 \cdot \sqrt{\frac{n+2}{n+1}} \cdot \left(\frac{2n+3}{2n+5}\right) \cdot |x| = 3\sqrt{\frac{1+2/n}{1+1/n}} \cdot \left(\frac{2+3/n}{2+5/n}\right) \cdot |x|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3\sqrt{\frac{1}{1}}\left(\frac{2}{2}\right)|x| = 3|x|$$

So the series converges when $3|x| < 1$ and diverges when $3|x| > 1$.
So for $|x| < \frac{1}{3}$, the series converges, and for $|x| > \frac{1}{3}$, it diverges.

Fill in in the blanks

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ $\boxed{}$
then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \boxed{}$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \boxed{}$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

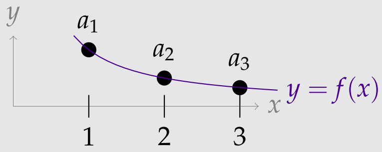
Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

(i) $\boxed{}$ and

(ii) $\boxed{}$ and

(iii) $f(n) = a_n$ for all $n \geq N_0$.



Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_N^{\infty} f(x)\,\mathrm{d}x \qquad \text{for all } N \geq N_0$$

Fill in in the blanks

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \boxed{} Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \boxed{} Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Fill in in the blanks

Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $\boxed{}$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $\boxed{}$

(ii) $a_{n+1} \le a_n$ for all $n \ge 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\boxed{}$

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

List of convergence tests

Divergence Test

When the n^{th} term in the series *fails* to converge to zero as n tends to infinity.
This is a good first thing to check: if it works, it's quick, but it doesn't always work.

Alternating Series Test

- successive terms in the series alternate in sign
- don't forget to check that successive terms decrease in magnitude and tend to zero as n tends to infinity

Integral Test

- works well when, if you substitute x for n in the n^{th} term you get a function, $f(x)$, that you can easily integrate
- don't forget to check that $f(x) \geq 0$ and that $f(x)$ decreases as x increases

List of convergence tests

Ratio Test

- works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$
- this often happens when a_n contains powers, like 7^n, or factorials, like $n!$
- don't forget that $L = 1$ tells you nothing about the convergence/divergence of the series

Comparison Test and Limit Comparison Test

- Comparison test lets you ignore pieces of a function that feel extraneous (like replacing n^2+1 with n^2) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- Limit comparison works well when, for very large n, the n^{th} term a_n is approximately the same as a simpler, nonnegative term b_n

- The integral test gave us the p-test. When you're looking for comparison series, p-series $\sum \frac{1}{n^p}$ are often good choices, because their convergence or divergence is so easy to ascertain.

- Geometric series have the form $\sum a \cdot r^n$ for some nonzero constants a and r. The magnitude of r is all you need to know to deicide whether they converge or diverge, so these are also common comparison series.

- Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

The **divergence test** is inconclusive, because $\lim_{n\to\infty} \frac{\cos n}{2^n} = 0$ (which you can show with the squeeze theorem).
The **integral test** doesn't apply, because $f(x) = \frac{\cos x}{2^x}$ is not always positive (and not decreasing).
The **alternating series test** doesn't apply because the signs of the series do not strictly alternate every term.
The **ratio test** does not apply, because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

Comparison test: Let $a_n = \frac{\cos n}{2^n}$. Note $|a_n| \le \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a geometric sum with ratio of consecutive terms $\frac{1}{2}$).

So by the comparison test, $\sum^{\infty} \frac{\cos n}{2^n}$ converges.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

The **alternating series test** doesn't apply because the signs of the series do not alternate.

The **integral test** doesn't apply $f(x) = \frac{2^x \cdot x^2}{(x+5)^5}$ is not a decreasing function.

Divergence test: $\lim_{n\to\infty} \frac{2^n \cdot n^2}{(n+5)^5} = \infty$ (which you can see because the numerator is larger than a power function; the denominator is a polynomial; and power functions grow faster than polynomials), so the series diverges by the divergence test.

This is the fastest option, but not the only one.

Ratio test:

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

The **divergence test** is inconclusive because $\displaystyle\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n} = 0$.

The **alternating series test** does not apply because we are not considering an alternating series.

The **integral test** won't work for us because $\int_1^\infty \frac{1}{x}\sin\left(\frac{1}{x}\right)\, dx$ cannot be evaluated with techniques we've learned in class so far.

The **ratio test** is inconclusive because $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+1}\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}\sin\left(\frac{1}{n}\right)} = 1$:

Set $x = \frac{1}{n+1}$. Then $\frac{1}{n} = \frac{x}{1-x}$:

Included Work

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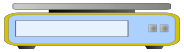 ‘Waage/Libra’ by B. Lachner is in the public domain (accessed April 2021, edited), 4

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