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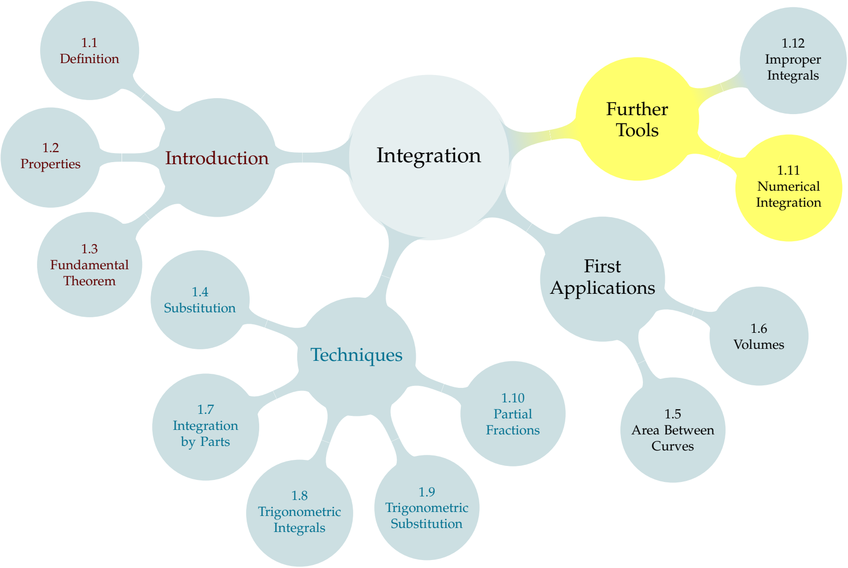

Sometimes, integrals can't be evaluated using the fundamental theorem of calculus:

$$\int_0^1 e^{x^2}\,\mathrm{d}x = ? \qquad \int_0^1 \sin(x^2)\,\mathrm{d}x = ?$$

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$$\int_0^3 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(3) = \ldots ?$$

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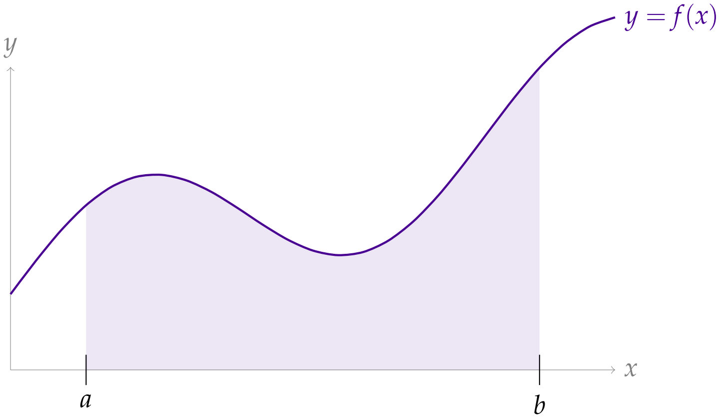
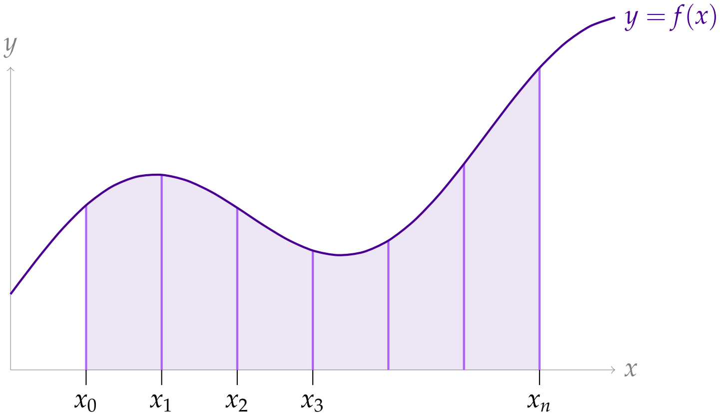
A numerical approximation will give us an approximate number for a definite integral.



We can approximate the area $\int_a^b f(x)\,\mathrm{d}x$ by cutting it into slices and approximating the area of those slices with a simple geometric figure, such as a rectangle, a trapezoid, or a parabola.



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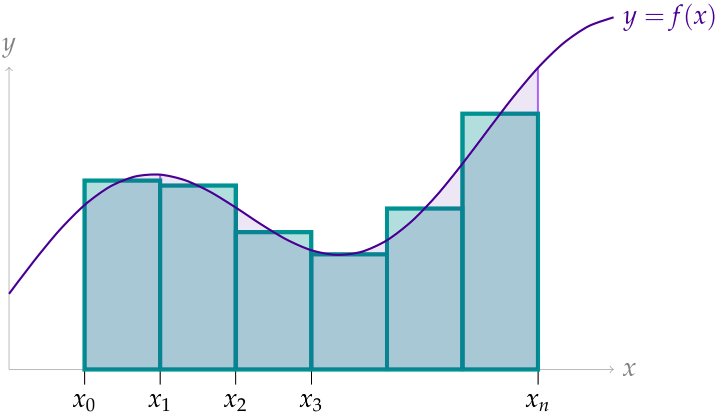

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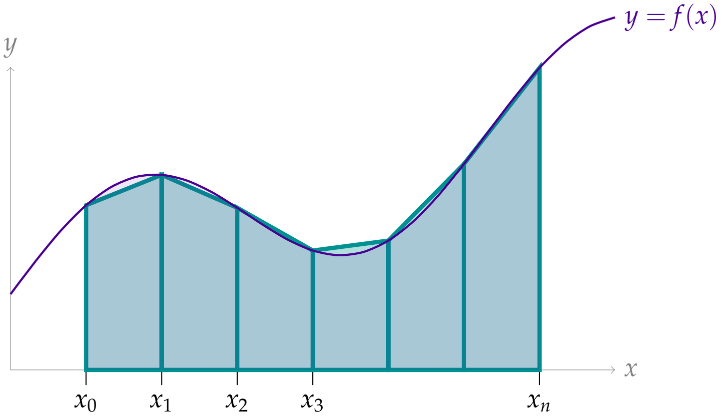

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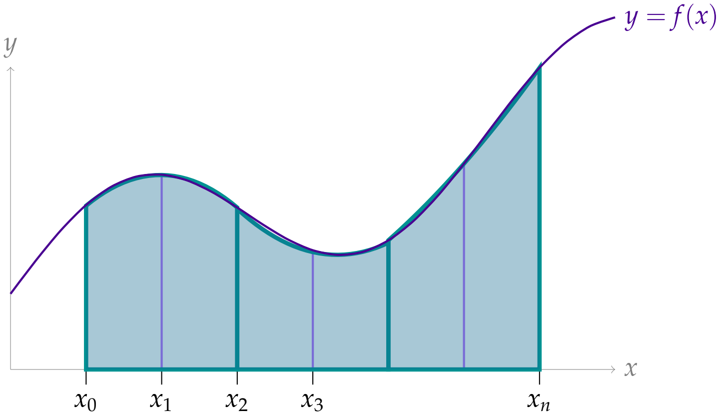
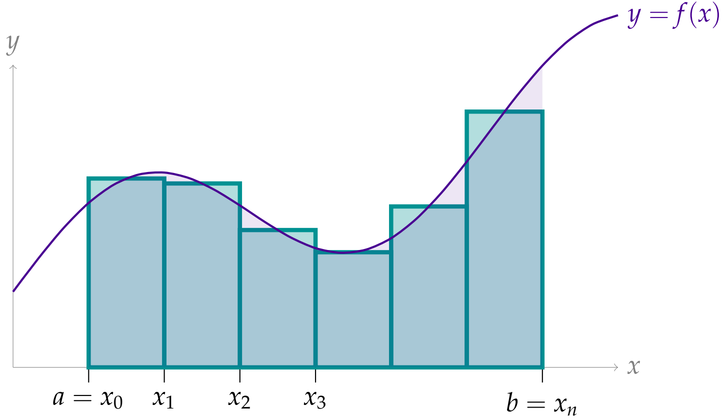


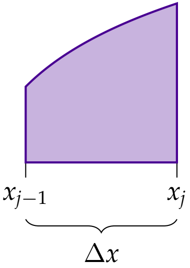
We can approximate the area $\int_a^b f(x)\,\mathrm{d}x$ by cutting it into slices and approximating the area of those slices with a simple geometric figure, such as a rectangle, a trapezoid, or a parabola.

The midpoint rule approximates $\int_a^b f(x)\,\mathrm{d}x$ as its midpoint Riemann sum with n intervals.



Approximate the area under the curve $y = f(x)$ from $x = x_{j-1}$ to $x = x_j$ with a rectangle.

To make our writing cleaner, let $\overline{x_j} = \frac{x_{j-1}+x_j}{2}$



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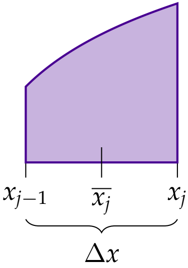

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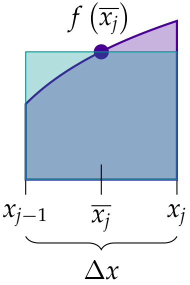

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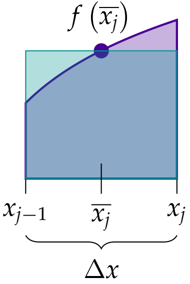


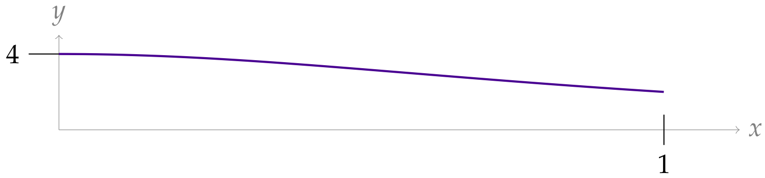
Midpoint Rule

The midpoint rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \left[f\left(\overline{x_1}\right) + f\left(\overline{x_2}\right) + \cdots + f\left(\overline{x_n}\right)\right]\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_j = a + j\Delta x$

Approximate $\displaystyle\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$ using the midpoint rule and $n = 4$ slices. Leave your answer in calculator-ready form.



Error

$$\pi = \int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$

$$\approx 3.14680$$

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Error:

$$|\text{exact} - \text{approximate}|$$

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Error:

$$|\text{exact} - \text{approximate}| \approx |3.14159 - 3.14680| = 0.00521$$

Error

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Error:

$$|\text{exact} - \text{approximate}| \approx |3.14159 - 3.14680| = 0.00521$$

Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right|$$

Error

$$\pi = \int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$
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$$|\text{exact} - \text{approximate}| \approx |3.14159 - 3.14680| = 0.00521$$

Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right| \approx \frac{0.00521}{3.14159} \approx 0.001658$$

Error

$$\pi = \int_0^1 \frac{4}{1+x^2}\,dx \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$
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Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right| \approx \frac{0.00521}{3.14159} \approx 0.001658$$

Percent error:

$$100\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right|$$

Error

$$\pi = \int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$
$$\approx 3.14680$$

Error:

$$|\text{exact} - \text{approximate}| \approx |3.14159 - 3.14680| = 0.00521$$

Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right| \approx \frac{0.00521}{3.14159} \approx 0.001658$$

Percent error:

$$100\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right| \approx 100(0.001658) = 0.1658\%$$

Error

A numerical approximation will give us an approximate value for a definite integral.
This is most useful if we know something about its accuracy.

A: approximation E: exact number

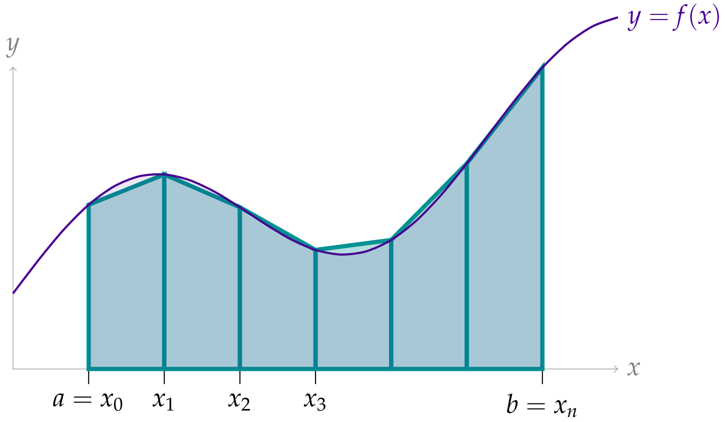
Error: $|A-E|$

Relative Error: $\left|\frac{A-E}{E}\right|$

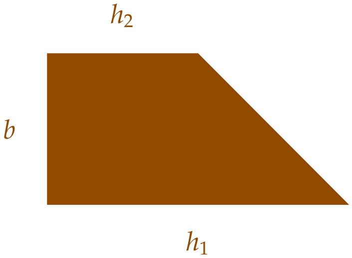
Percent Error: $100\left|\frac{A-E}{E}\right|$

We will discuss error more after we've learned the three approximation rules. For now, we're using error to illustrate that our methods have the potential to produce reasonable approximations without too much work.

The trapezoidal rule approximates each slice of $\int_a^b f(x)\,dx$ with a trapezoid.



Recall the area of a right trapezoid with base b and heights h_1 and h_2:



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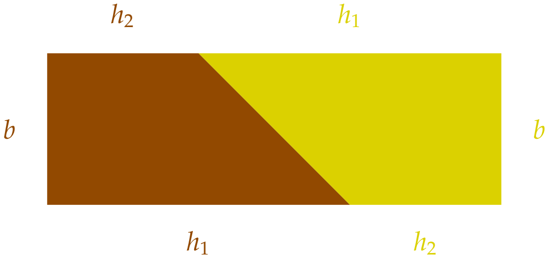

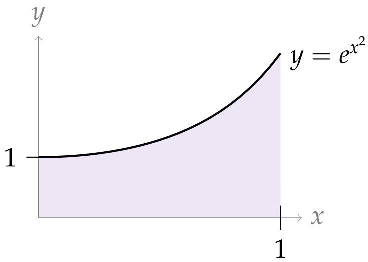

Trapezoid area: $\frac{\text{base}}{2}(h_1 + h_2)$

Approximate $\displaystyle\int_0^1 e^{x^2}\,\mathrm{d}x$ using $n = 4$ trapezoids.
Leave your answer in calculator-ready form.

Trapezoidal Rule

The trapezoidal rule approximation is

$$\int_a^b f(x)\mathrm{d}x \approx \Delta x \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

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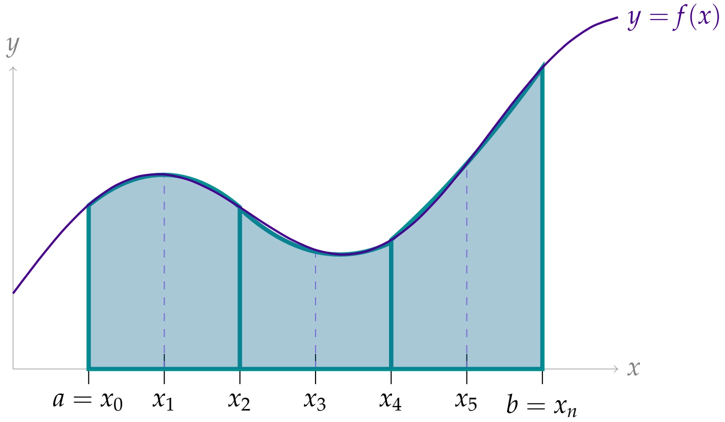
where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

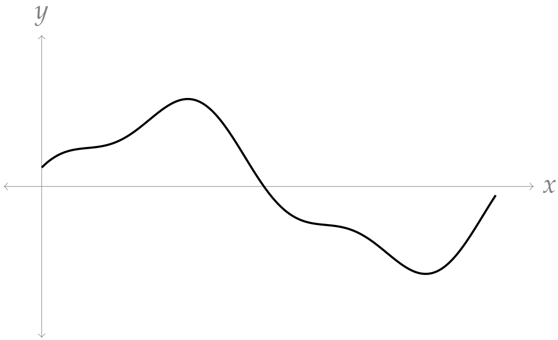
Using $n = 3$ trapezoids, approximate $\displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x$.

Simpson's rule approximates each *pair of slices* of $\int_a^b f(x)\,\mathrm{d}x$ with a parabola.



Simpson's Rule

Add up parabolas.



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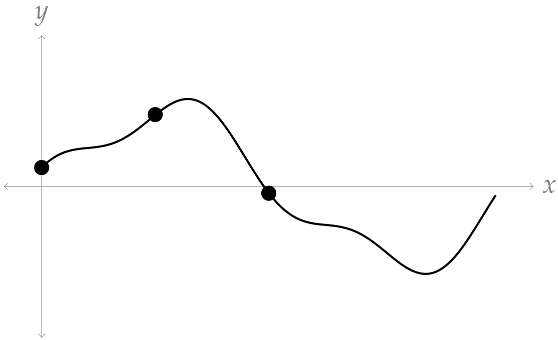

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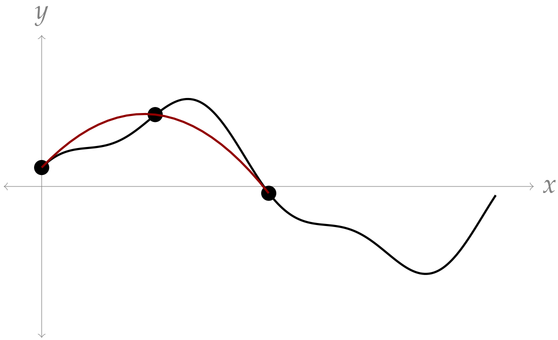

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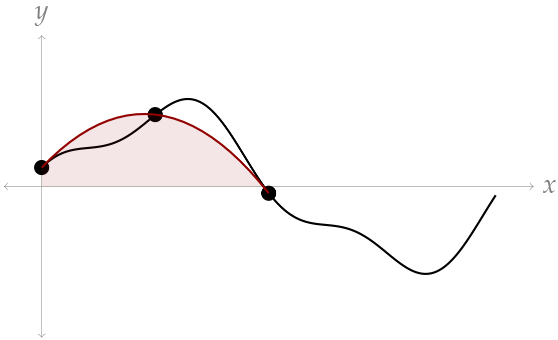

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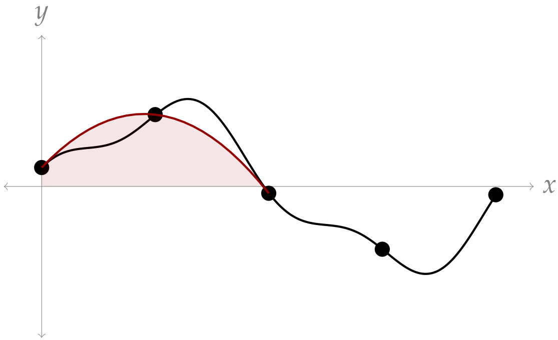

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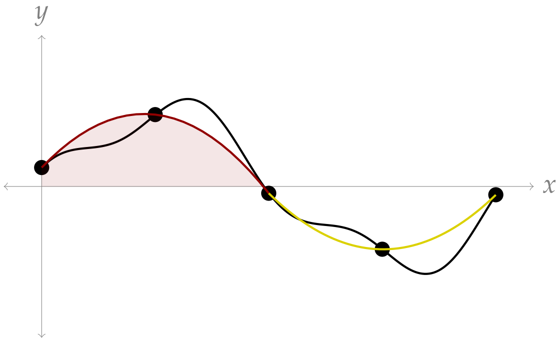

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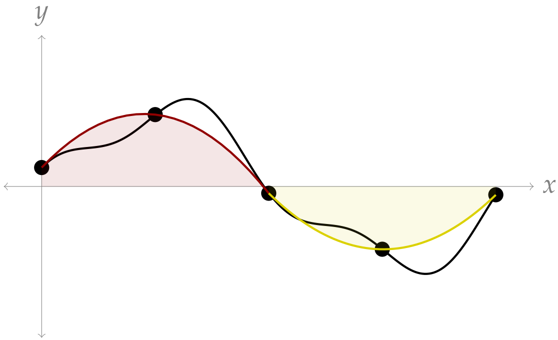
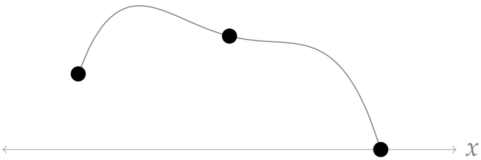


Simpson's Rule Derivation

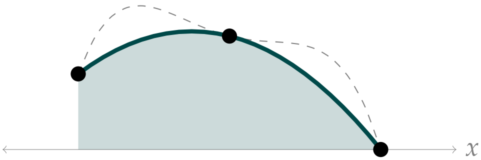
SKIP DERIVATION OF SIMPSON'S RULE



What is the area under the parabola passing through three specified points?

Simpson's Rule Derivation

Skip derivation of Simpson's rule



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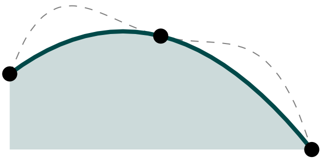
SIMPSON'S RULE DERIVATION

SKIP DERIVATION OF SIMPSON'S RULE



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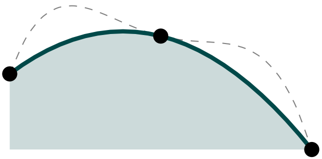
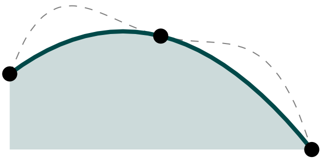


What is the area under the parabola passing through three specified points?

Parabola: $Ax^2 + Bx + C$

Simpson's Rule Derivation

Skip derivation of Simpson's rule



What is the area under the parabola passing through three specified points?

Parabola: $Ax^2 + Bx + C$

Area: $\int_{-h}^{h} (Ax^2 + Bx + C)\mathrm{d}x = \frac{h}{3}\left(2Ah^2 + 6C\right)$

Find

$$\frac{h}{3}\left(2Ah^2 + 6C\right)$$

for A, B, and C such that

$$Ah^2 - Bh + C = f(-h) \tag{E1}$$

$$C = f(0) \tag{E2}$$

$$Ah^2 + Bh + C = f(h) \tag{E3}$$

Try $(E1) + 4(E2) + (E3)$:

Find

$$\frac{h}{3}\left(2Ah^2 + 6C\right)$$

for A, B, and C such that

$$Ah^2 - Bh + C = f(-h) \qquad \text{(E1)}$$

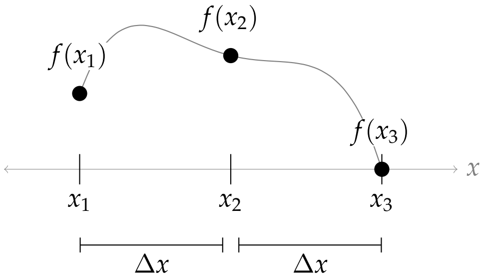
$$C = f(0) \qquad \text{(E2)}$$

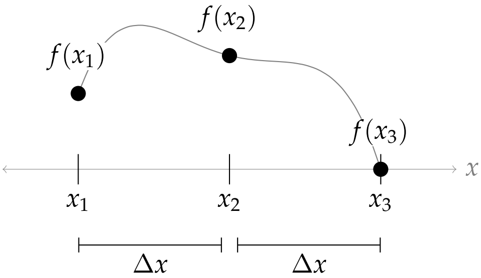
$$Ah^2 + Bh + C = f(h) \qquad \text{(E3)}$$

Try $(E1) + 4(E2) + (E3)$:

$$2Ah^2 + 6C = f(-h) + 4f(0) + f(h)$$

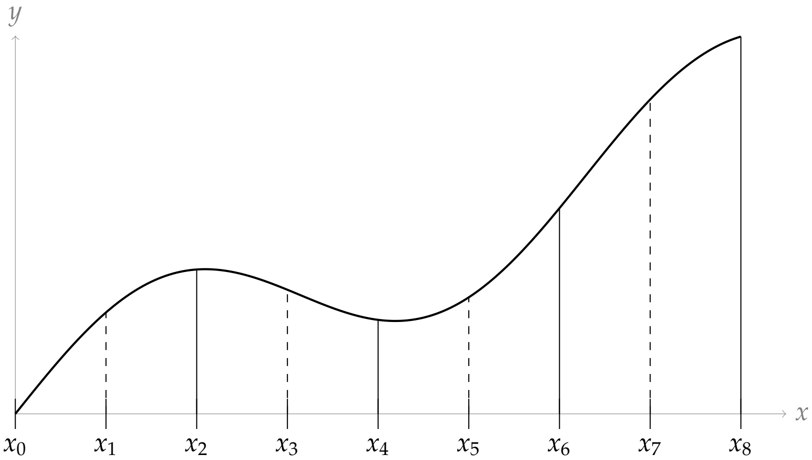
$$\text{Area} = \frac{h}{3}\left(2Ah^2 + 6C\right) = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right)$$


$f(x_1)$
$f(x_2)$
$f(x_3)$
x
x_1
x_2
x_3
Δx
Δx



Area under parabola:

$$\frac{\Delta x}{3}\Big(f(x_1) + 4f(x_2) + f(x_3)\Big)$$


y
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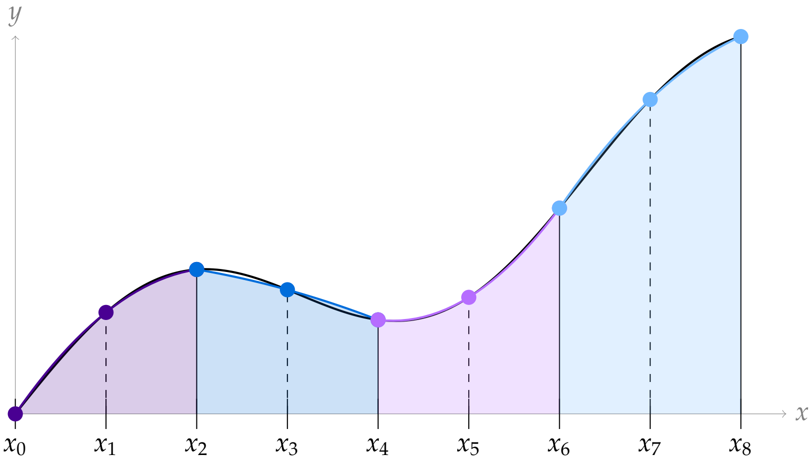
y
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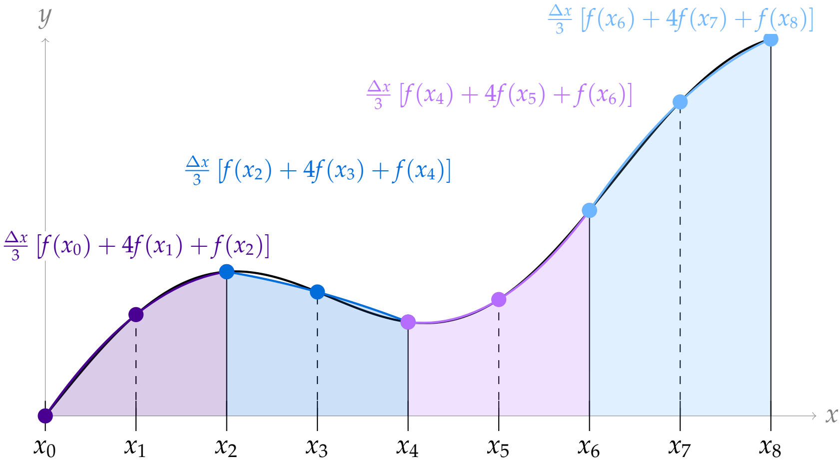
$\frac{\Delta x}{3}\left[f(x_6)+4f(x_7)+f(x_8)\right]$
$\frac{\Delta x}{3}\left[f(x_4)+4f(x_5)+f(x_6)\right]$
$\frac{\Delta x}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$
$\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$
y
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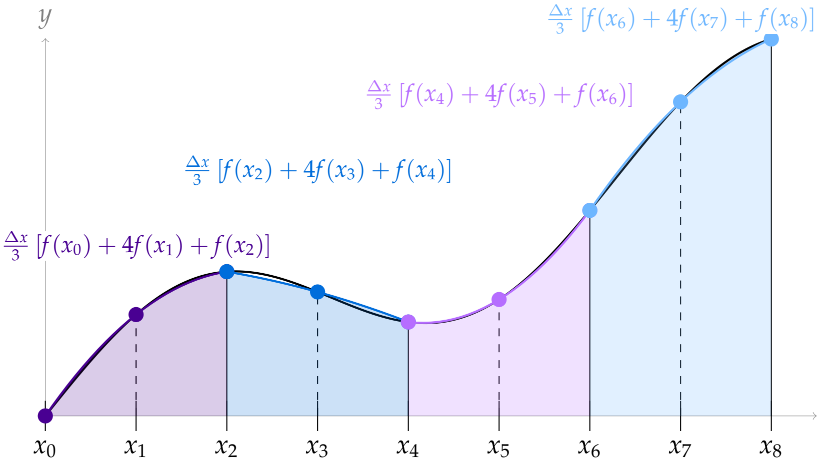

$$\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+2f(x_6)+4f(x_7)+f(x_8)\right]$$

Simpson's Rule

The Simpson's rule approximation is $\displaystyle\int_a^b f(x)\mathrm{d}x \approx$

$$\frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]$$

where n is even, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$

(We'll call n the number of slices; some people call $n/2$ the number of slices, because that's the number of approximating parabolas.)

Simpson's Rule

The Simpson's rule approximation is $\int_a^b f(x)\mathrm{d}x \approx$

$$\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

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Simpson's Rule

The Simpson's rule approximation is $\int_a^b f(x)\mathrm{d}x \approx$

$$\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$

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where n is even, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$

Using Simpson's rule and $n = 8$ (i.e. 4 parabolas),

approximate $\displaystyle\int_1^{17} \frac{1}{x}\,\mathrm{d}x$. Leave your answer in calculator-ready form.

The instantaneous electricity use rate (kW/hr) of a factory is measured throughout the day.

time	12:00	1:00	2:00	3:00	4:00	5:00	6:00	7:00	8:00
rate	100	200	150	400	300	300	200	100	150

Use Simpson's Rule to approximate the total amount of electricity you used from noon to 8:00.

Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

- the total error introduced by the midpoint rule is bounded by $\frac{M}{24}\frac{(b-a)^3}{n^2}$,
- the total error introduced by the trapezoidal rule is bounded by $\frac{M}{12}\frac{(b-a)^3}{n^2}$, and
- the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$

when approximating $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$. Then the total error introduced by the midpoint rule is bounded by $\dfrac{M}{24}\dfrac{(b-a)^3}{n^2}$ when approximating $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Suppose we approximate $\displaystyle\int_0^3 \sin(x)\,\mathrm{d}x$ using the midpoint rule and $n = 6$ intervals. Give an upper bound of the resulting error.

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,\mathrm{d}x$.

Suppose we approximate $\int_2^3 \frac{1}{x}\mathrm{d}x$ using Simpson's rule with $n = 10$ slices (5 parabolas). Give an upper bound of the resulting error.

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180} \frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,\mathrm{d}x$.

We will approximate $\int_0^{1/2} e^{x^2}\,\mathrm{d}x$ using Simpson's rule, and we need our error to be no more than $\frac{1}{10\,000}$. How many intervals will suffice?

You may use, without proof:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{e^{x^2}\right\} = 4e^{x^2}\left(4x^4 + 12x^2 + 3\right) \qquad \frac{25\sqrt[4]{e}}{180 \cdot 2^5} < \frac{1}{3^4}$$

$$\int_0^{1/2} e^{x^2}\,\mathrm{d}x \qquad \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{e^{x^2}\right\} = 4e^{x^2}\left(4x^4 + 12x^2 + 3\right) \qquad \frac{25\sqrt[4]{e}}{180 \cdot 2^5} < \frac{1}{3^4}$$

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\dfrac{L}{180}\dfrac{(b-a)^5}{n^4}$ when approximating $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

It can be shown that the fourth derivative of $\frac{1}{x^2+1}$ has absolute value at most 24 for all real numbers x. Using this information, find a rational number approximating $\arctan(2)$ with an error of no more than $\frac{2^6}{3\cdot 5^5} \approx 0.007$.

First, we'll set up our integral:

Included Work

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