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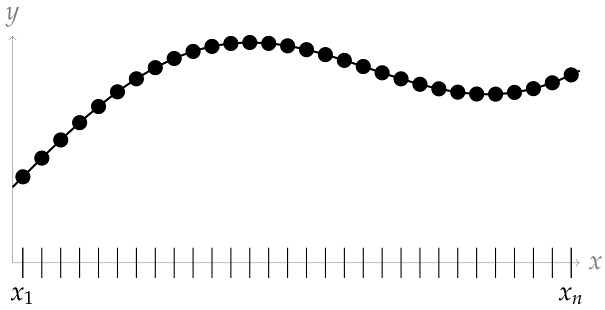


$$\text{Average} \approx \frac{f(x_1) + \cdots + f(x_n)}{n}$$

$$\text{Average} = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=1}^{n} f(x_i) \right] = \lim_{n\to\infty} \left[\frac{(b-a)}{(b-a)n} \sum_{i=1}^{n} f(x_i) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x \right] = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x$$

Average

Let $f(x)$ be an integrable function defined on the interval $a \leq x \leq b$. The average value of f on that interval is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

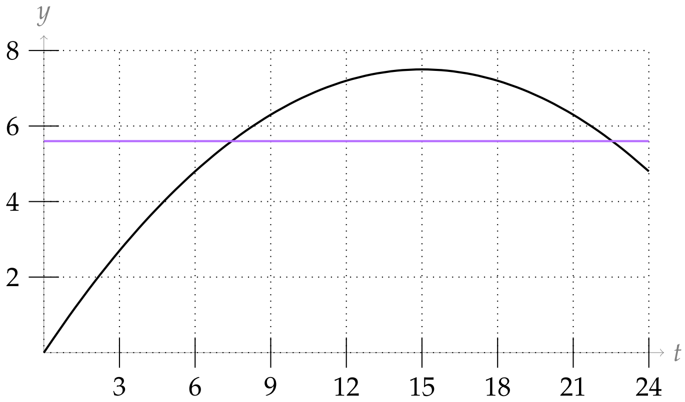
The temperature in a certain city at time t (measured in hours past midnight) is given by

$$T(t) = t - \frac{t^2}{30}$$

What was the average temperature of one day (from $t = 0$ to $t = 24$)?

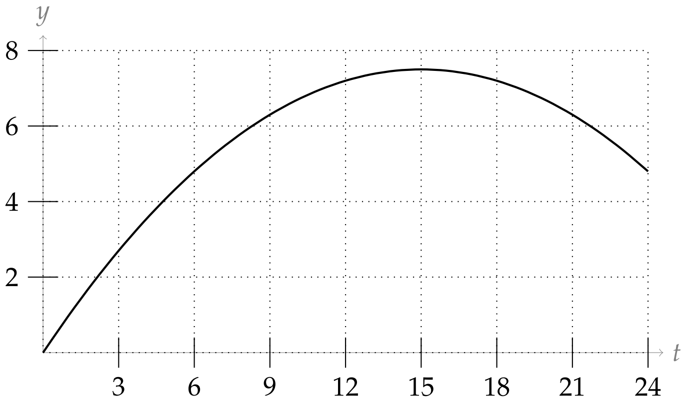
$$\begin{aligned}\text{Average} &= \frac{1}{24}\int_0^{24}\left[t - \frac{t^2}{30}\right]\mathrm{d}t \\ &= \frac{1}{24}\left[\frac{t^2}{2} - \frac{t^3}{90}\right]_0^{24}\end{aligned}$$

Let's check that our answer makes some intuitive sense.

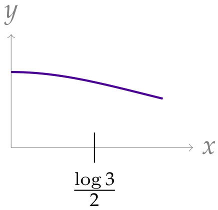


Since the temperature is always between 0 and 8, we expect the average to be between 0 and 8

Let's also recall the motivation for our definition



Find the average value of the function $f(x) = \dfrac{e^x}{e^{2x}+1}$ over the interval $\left[0, \frac{\log 3}{2}\right]$.



Let $u(x) = e^x$. Then $u(0) = 1$ and $u\left(\frac{\log 3}{2}\right) = e^{\frac{\log 3}{2}} = 3^{1/2} = \sqrt{3}$.

$$\frac{1}{\frac{\log 3}{2} - 0} \cdot \int_0^{\frac{\log 3}{2}} \frac{e^x}{e^{2x}+1}\,\mathrm{d}x$$

$$= \frac{2}{\log 3} \int_1^{\sqrt{3}} \frac{1}{u^2+1} \mathrm{d}u$$

$$= \frac{2}{\log 3} \left[\arctan(\sqrt{3}) - \arctan(1)\right]$$

$$= \frac{2}{\log 3} \left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{6 \log 3} \approx 0.477$$

Average Velocity

Let $x(t)$ be the position at time t of a car moving along the x-axis. The velocity of the car at time t is the derivative $v(t) = x'(t)$. The average velocity of the car over the time interval $a \le t \le b$ is:

$$v_{\mathrm{ave}} = \frac{1}{b-a} \int_a^b v(t)\,\mathrm{d}t = \frac{1}{b-a} \int_a^b x'(t)\,\mathrm{d}t = \frac{x(b) - x(a)}{b-a}$$

That is: $\dfrac{\text{change in distance}}{\text{change in time}}$

Notice that this is exactly the formula we used way back at the start of your differential calculus class to help introduce the idea of the derivative. Of course this is a very circuitous way to get to this formula — but it is reassuring that we get the same answer.