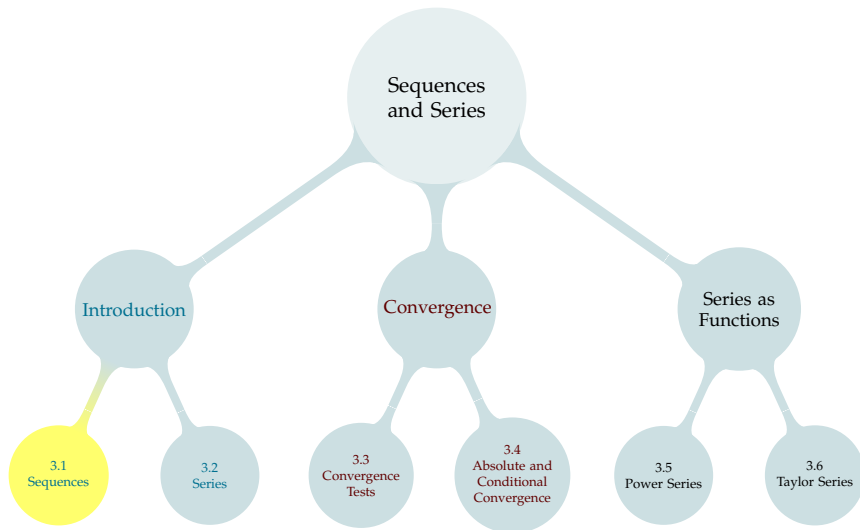


TABLE OF CONTENTS



We can imagine the list of numbers below carrying on forever:

$$a_1 = 0.1$$

$$a_2 = 0.01$$

$$a_3 = 0.001$$

$$a_4 = 0.0001$$

$$a_5 = 0.00001$$

$$\vdots$$

A **sequence** is a list of infinitely many numbers with a specified order.

It is denoted $\{a_1, a_2, \dots, a_n, \dots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

Imagine *adding up* this sequence of numbers.

A **series** is a sum $a_1 + a_2 + \dots + a_n + \dots$ of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- ▶ What does it mean to add up infinitely many things?
- ▶ Should infinitely many things add up to an infinitely large number?
- ▶ Does the order in which the numbers are added matter?
- ▶ Can we add up infinitely many functions, instead of just infinitely many numbers?

Sequence

A **sequence** is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- ▶ $\{1, 2, 3, 4, 5, 6, 7, 8, \dots\}$ (natural numbers)
- ▶ $\{3, 1, 4, 1, 5, 9, 2, 6, \dots\}$ (digits of π)
- ▶ $\{1, -1, 1, -1, 1, \dots\}$ (powers of -1 : $(-1)^0, (-1)^1, (-1)^2$, etc.)

Sequence

A **sequence** is a list of infinitely many numbers with a specified order. It is denoted $\{a_1, a_2, a_3, \dots, a_n, \dots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

- ▶ $n = 1$: this is the index of the first term of our sequence.
Sometimes it's 0, sometimes something else, for example a year.
- ▶ ∞ : there is no end to our sequence.
- ▶ $\frac{1}{n}$: this tells us the value of a_n .
- ▶ Often we omit the limits and even the brackets, writing $a_n = \frac{1}{n}$.

SEQUENCE NOTATION

For convenience, we write a_1 for the first term of a sequence, a_2 for the second term, etc.

In the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$,
 a_3 is another name for

Sometimes we can find a rule for a sequence.
In the above sequence, $a_n =$

We can write $\{a_n\}_{n=1}^{\infty} =$

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n “settles down” to some value as n tends to infinity.

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to **converge** to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n \rightarrow \infty} a_n = A \quad \text{or} \quad a_n \rightarrow A \text{ as } n \rightarrow \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

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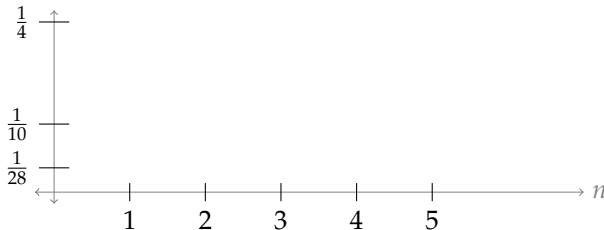
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This sequence

Does the sequence $a_n = \frac{n}{2n+1}$ converge or diverge?

Consider the sequence $a_n = \frac{1}{3^n + 1}$. $\lim_{n \rightarrow \infty} a_n =$



Theorem 3.1.6

If $\lim_{x \rightarrow \infty} f(x) = L$

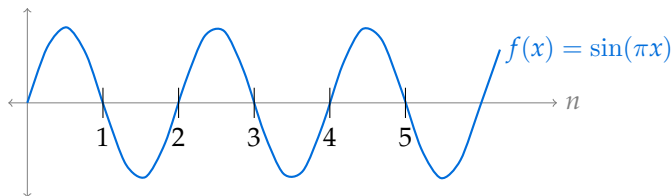
and if $a_n = f(n)$ for all positive integers n , then

$$\lim_{n \rightarrow \infty} a_n = L$$

CAUTIONARY TALE

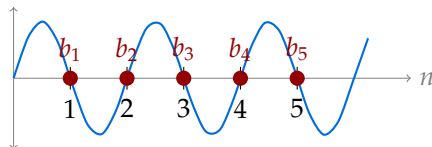
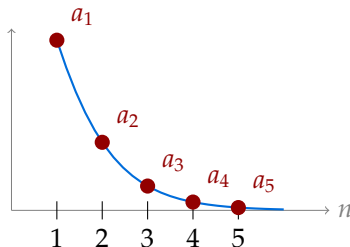
Consider the sequence $b_n = \sin(\pi n) =$

$$\lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} f(x)$$



Theorem

If $\lim_{x \rightarrow \infty} f(x) = L$ and if $a_n = f(n)$ for all natural n , then $\lim_{n \rightarrow \infty} a_n = L$.



Arithmetic of Limits

Let A , B and C be real numbers and let the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that

$$\lim_{n \rightarrow \infty} a_n = A$$

$$\lim_{n \rightarrow \infty} b_n = B$$

Then the following limits hold.

(a) $\lim_{n \rightarrow \infty} [a_n + b_n] = A + B$

(b) $\lim_{n \rightarrow \infty} [a_n - b_n] = A - B$

(c) $\lim_{n \rightarrow \infty} C a_n = C A.$

(d) $\lim_{n \rightarrow \infty} a_n b_n = A B$

(e) If $B \neq 0$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$

Evaluate the following limits:

$$\blacktriangleright \lim_{n \rightarrow \infty} e^{-n} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1+n}{n} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1}{n^2} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} 2n^2 =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) (2n^2) =$$

Continuous functions of limits

If $\lim_{n \rightarrow \infty} a_n = L$ and if the function $g(x)$ is continuous at L , then

$$\lim_{n \rightarrow \infty} g(a_n) = g(L)$$

Evaluate $\lim_{n \rightarrow \infty} \left[\sin \left(\frac{\pi n}{2n+1} \right) \right]$

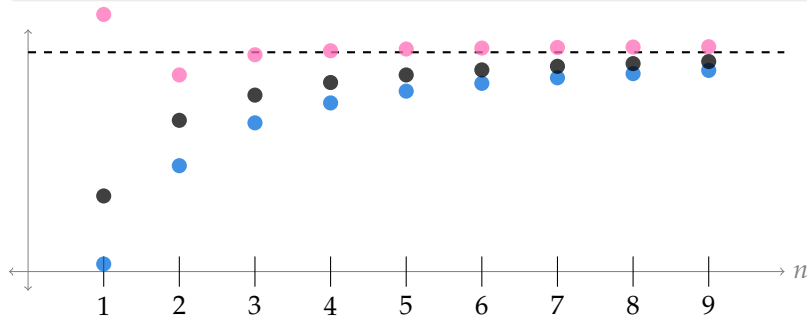
Squeeze Theorem

If $a_n \leq c_n \leq b_n$ for all sufficiently large natural numbers n , and if

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$$

then

$$\lim_{n \rightarrow \infty} c_n = L$$



Evaluate

$$\lim_{n \rightarrow \infty} \left(\frac{2n + \cos n}{n + 1} \right)$$

Let $a_n = (-n)^{-n}$. Evaluate $\lim_{n \rightarrow \infty} a_n$.