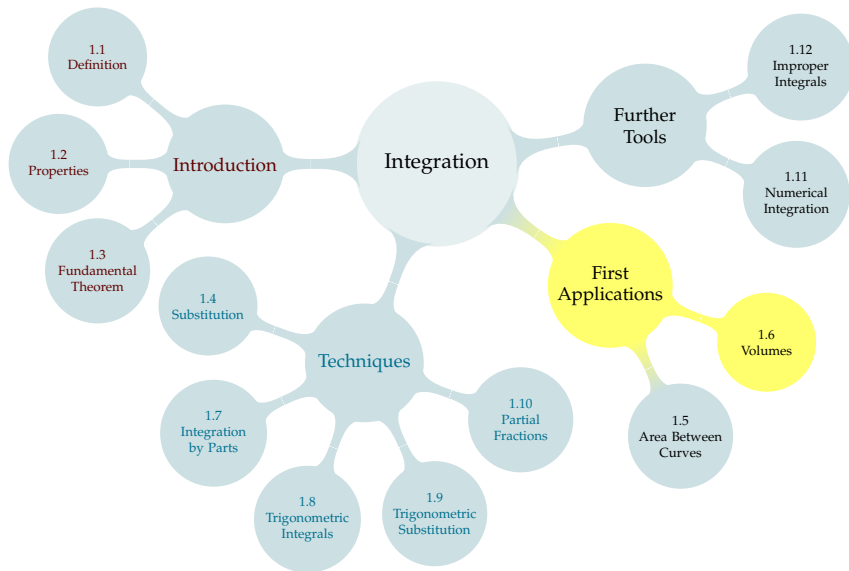
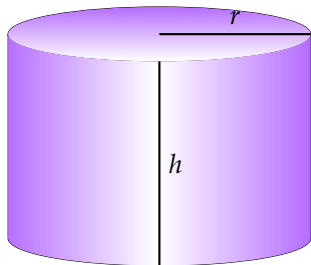


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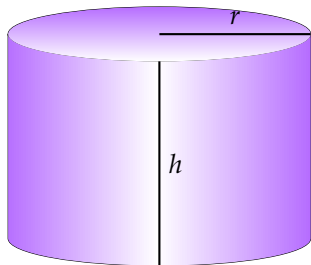


QUICK REFRESHER: VOLUMES OF CYLINDERS



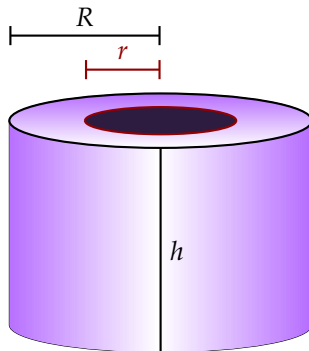
The volume of a cylinder with radius r and height h is:

QUICK REFRESHER: VOLUMES OF CYLINDERS



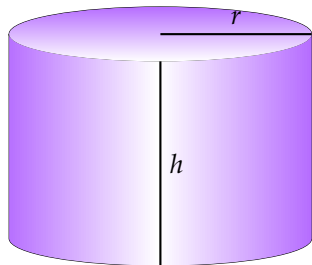
The volume of a cylinder with radius r and height h is:

$$\pi r^2 h$$



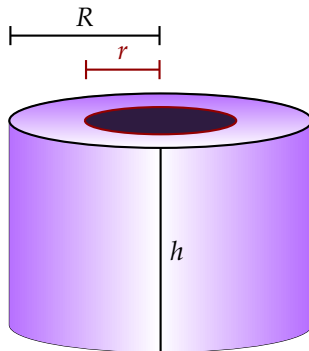
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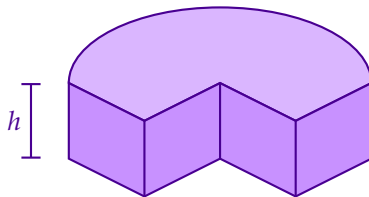
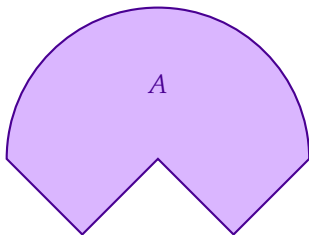


The volume of a washer, with outer radius R , inner radius r , and height h is:

$$(\pi R^2 h - \pi r^2 h) = \pi h (R^2 - r^2)$$

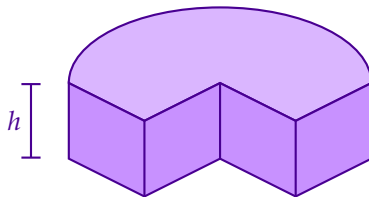
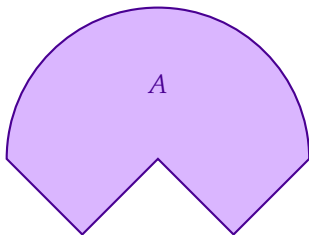
QUICK REFRESHER: VOLUMES OF CYLINDERS

More generally, if we have a shape of area A , and we extrude it into a solid of height h , the resulting solid has volume:

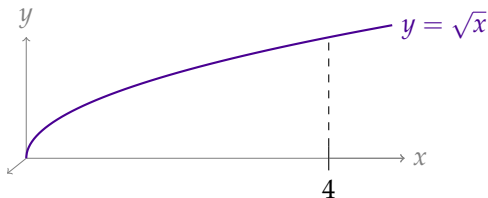


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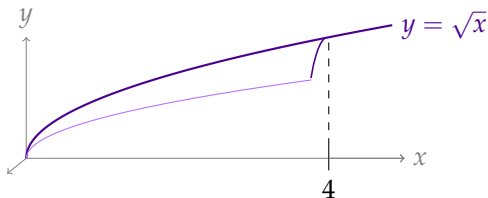
More generally, if we have a shape of area A , and we extrude it into a solid of height h , the resulting solid has volume: Ah



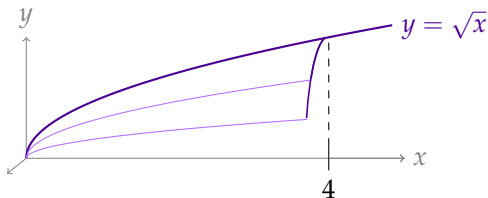
Consider the volume, V , enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x -axis.



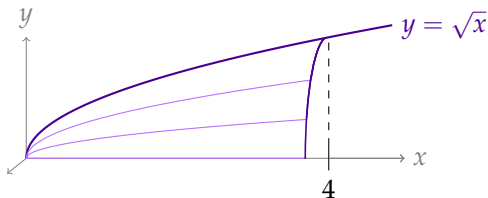
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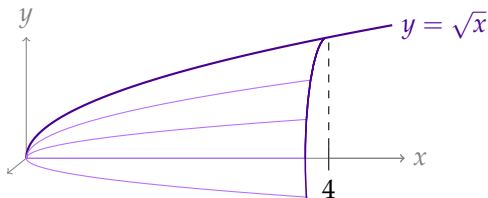
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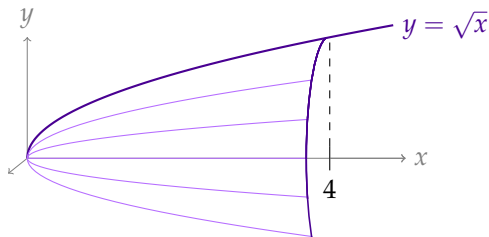
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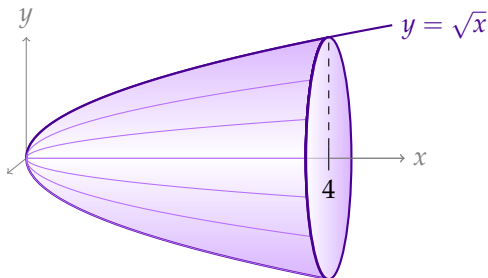
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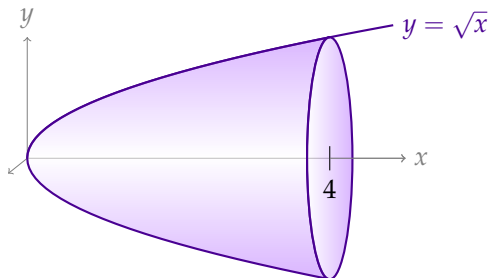
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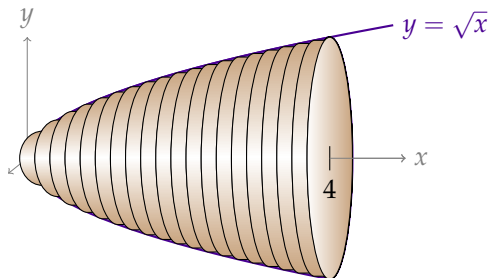
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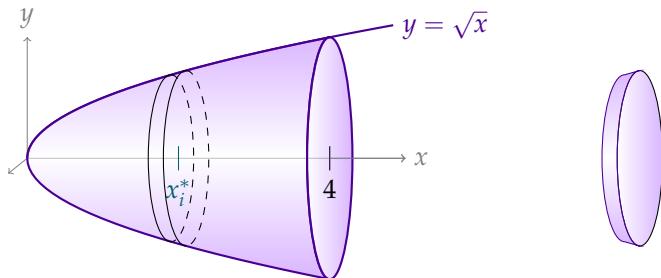


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We cut the solid into slices, and approximate the volume of each slice.

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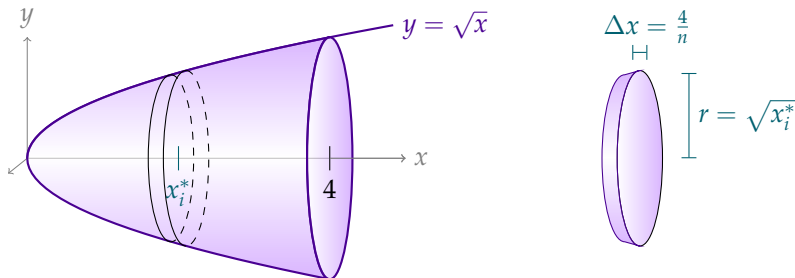


We cut the solid into slices, and approximate the volume of each slice. Each thin slice is *approximately* a cylinder.

If we use n slices, the width of each is:

The radius of the slice at $x = x_i^*$ is:

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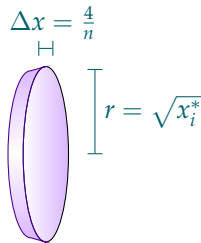
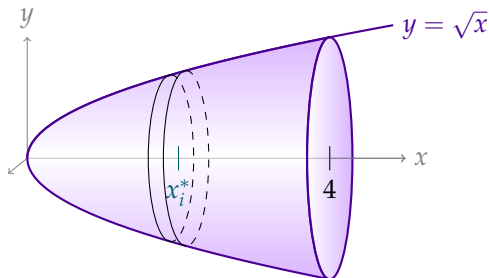


We cut the solid into slices, and approximate the volume of each slice. Each thin slice is *approximately* a cylinder.

If we use n slices, the width of each is: $\frac{4}{n}$.

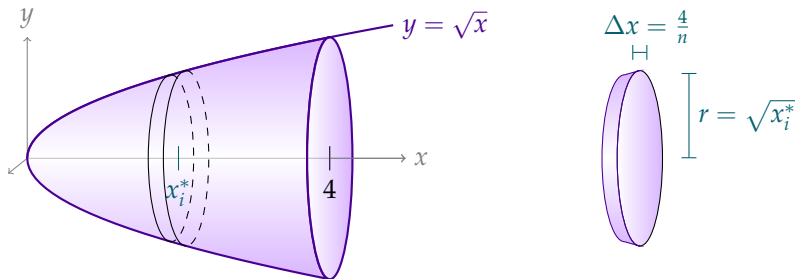
The radius of the slice at $x = x_i^*$ is: $\sqrt{x_i^*}$.

Consider the volume, V , enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x -axis.



$$V \approx \sum_{i=1}^n (\text{volume of each slice})$$

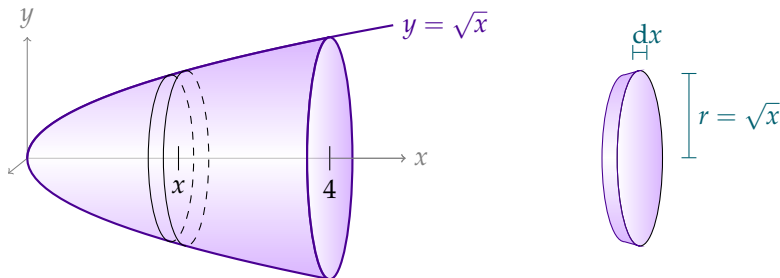
Consider the volume, V , enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x -axis.



$$V \approx \sum_{i=1}^n (\text{volume of each slice}) = \sum_{i=1}^n \pi (\sqrt{x_i^*})^2 \frac{4}{n} = \sum_{i=1}^n \underbrace{\pi x_i^*}_{f(x_i^*)} \underbrace{\frac{4}{n}}_{\Delta x}$$

This is a Riemann sum for $\int_0^4 \pi x \, dx$.

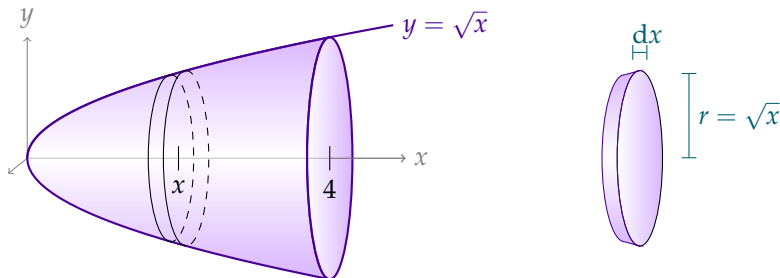
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Informally, we think of one slice, at position x , as having thickness dx . So, we can write the volume of this slice as:

Summing up the volumes of slices from $x = 0$ to $x = 4$, our total volume is:

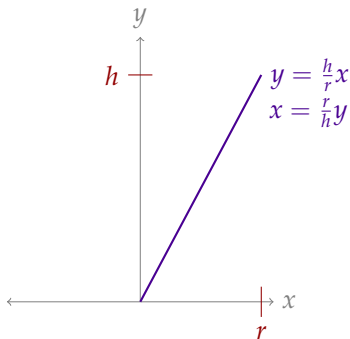
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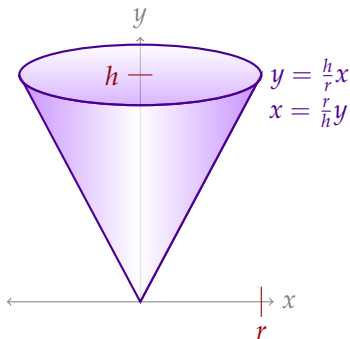
Summing up the volumes of slices from $x = 0$ to $x = 4$, our total volume is:

$$\int_0^4 \pi x \, dx = \left[\frac{\pi}{2} x^2 \right]_0^4 = 8\pi$$



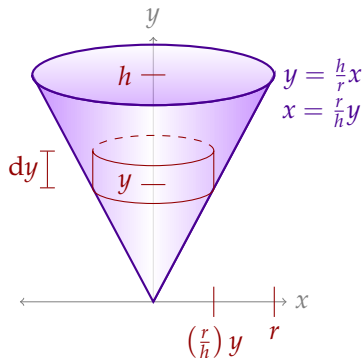
Let h and r be positive constants.

1. What familiar solid results from rotating the line segment from $(0,0)$ to (r, h) around the y -axis?



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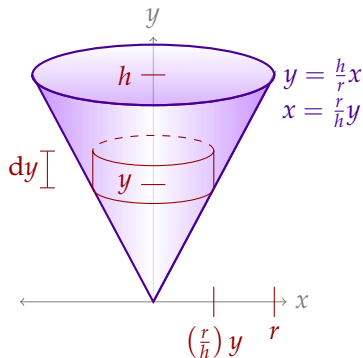
1. What familiar solid results from rotating the line segment from $(0,0)$ to (r,h) around the y -axis?
2. In the informal manner of the last example, describe the volume of a horizontal slice of the cone taken at height y .



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3. What is the volume of the entire cone?

Slice volume: $\pi \left(\frac{r}{h}y\right)^2 dy$



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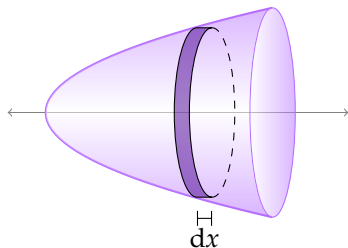
Slice volume: $\pi \left(\frac{r}{h}y\right)^2 dy$

Cone volume: $\int_0^h \pi \left(\frac{r}{h}y\right)^2 dy = \left[\frac{\pi r^2}{3h^2} y^3 \right]_{y=0}^{y=h} = \frac{\pi r^2}{3h^2} (h^3 - 0) = \frac{\pi}{3} r^2 h$

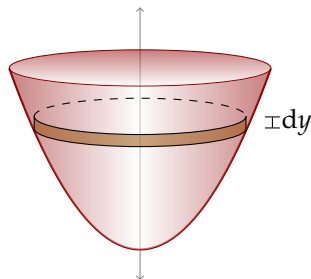
Observation

When we rotated around the **horizontal** axis, the width of our cylindrical slices was dx , and our integrand was written in terms of x .

When we rotated around the **vertical** axis, the width of our cylindrical slices was dy , and we integrated in terms of y .

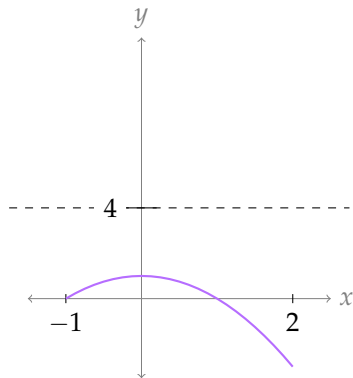


Vertical slices are approximately cylinders



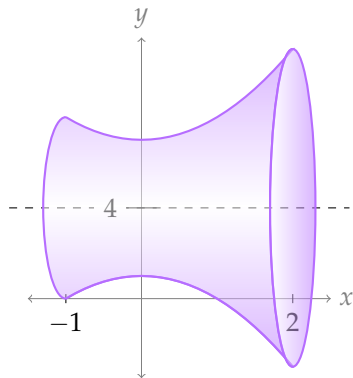
Horizontal slices are approximately cylinders

In this question, we will find the volume enclosed by rotating the curve $y = 1 - x^2$, from $x = -1$ to $x = 2$, about the line $y = 4$.



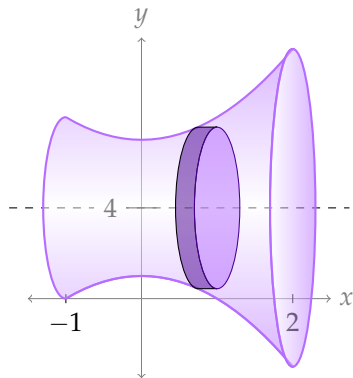
1. Sketch the surface traced out by the rotating curve.
2. Sketch a cylindrical slice. (Consider: will it be horizontal or vertical?)
3. Give the volume of your slice. Use dx or dy for the width, as appropriate.
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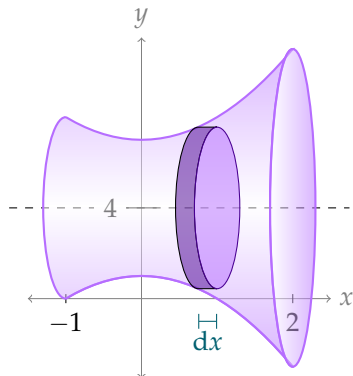
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$$\text{Slice volume: } \pi \underbrace{(4 - (1 - x^2))^2}_{\text{radius}^2} dx = \pi(3 + x^2)^2 dx$$

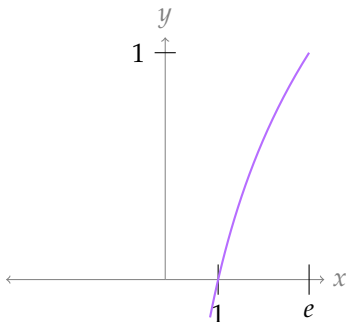
To find the volume of the entire object, we “add up” the slices from $x = -1$ to $x = 2$ by integrating.

$$\int_{-1}^2 \pi(3 + x^2)^2 dx =$$

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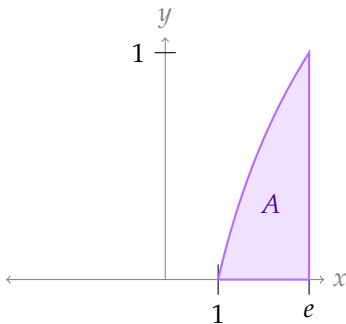
$$\begin{aligned}\int_{-1}^2 \pi(3+x^2)^2 dx &= \pi \int_{-1}^2 (9+6x^2+x^4) dx \\&= \pi \left[9x + 2x^3 + \frac{1}{5}x^5 \right]_{-1}^2 \\&= \pi \left[\left(18 + 16 + \frac{32}{5} \right) - \left(-9 - 2 - \frac{1}{5} \right) \right] \\&= \pi \left[\left(40 + \frac{2}{5} \right) + \left(11 + \frac{1}{5} \right) \right] \\&= 51.6\pi\end{aligned}$$

Let A be the area between the curve $y = \log x$ and the x -axis, from $(1, 0)$ to $(e, 1)$. In this question, we will consider the volume of the solid formed by rotating A about the y -axis.



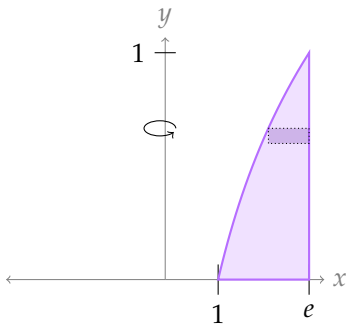
1. Sketch A .
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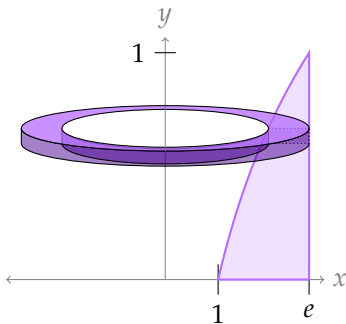
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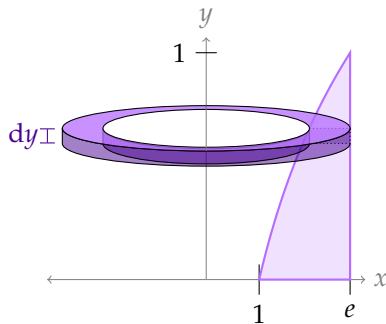
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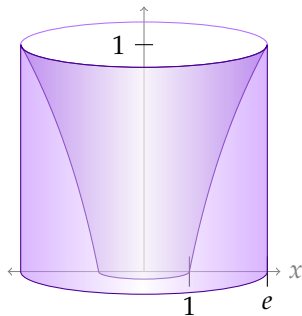
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The outer radius is e , while the inner radius at height y is $x = e^y$.

Slice volume at height y : $\pi \left(e^2 - (e^y)^2 \right) dy = \pi \left(e^2 - e^{2y} \right) dy$

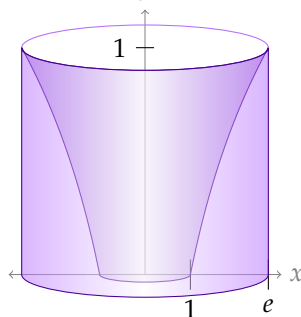
To find the volume of the entire object, we “add up” the slices from $y = 0$ to $y = 1$ by integrating.

$$\int_0^1 \pi (e^2 - e^{2y}) dy =$$



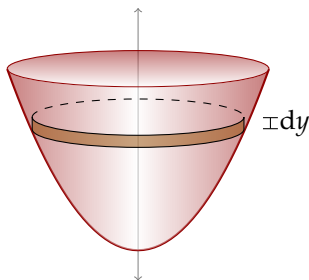
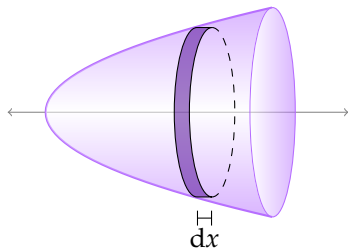
To find the volume of the entire object, we “add up” the slices from $y = 0$ to $y = 1$ by integrating.

Below we use the substitution rule with $u = 2y$ and $du = 2dy$. With practice, you’ll probably be able to do this substitution in your head, but we have written it out for clarity



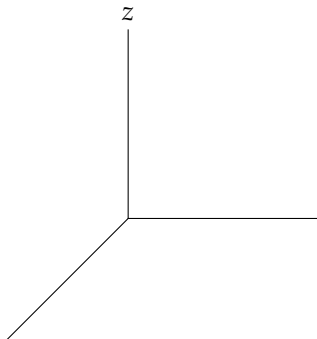
$$\begin{aligned}
 \int_0^1 \pi (e^2 - e^{2y}) dy &= \pi \int_{u(0)}^{u(1)} (e^2 - e^u) \cdot \frac{1}{2} du \\
 &= \frac{\pi}{2} \int_0^2 (e^2 - e^u) du \\
 &= \frac{\pi}{2} [e^2 u - e^u]_0^2 \\
 &= \frac{\pi}{2} [(2e^2 - e^2) - (0 - 1)] \\
 &= \frac{\pi}{2} [e^2 + 1]
 \end{aligned}$$

So far, we've found the volume of solids formed by rotating a curve. When a point rotates about a fixed centre, the result is a circle, so we could slice those solids up into pieces that are approximately cylinders.



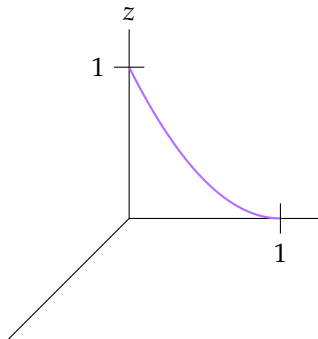
We can find the volumes of other shapes, as long as we can find the areas of their cross-sections.

The corner of a room is sealed off as follows:



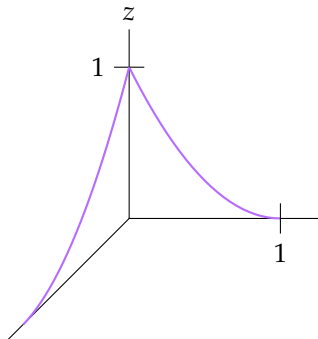
The corner of a room is sealed off as follows:

On both walls, a parabola of the form $z = (x - 1)^2$ is drawn, where z is the vertical axis and x is the horizontal. They start one metre above the corner, and end one metre to the side of the corner.



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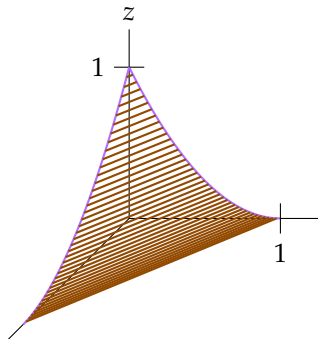
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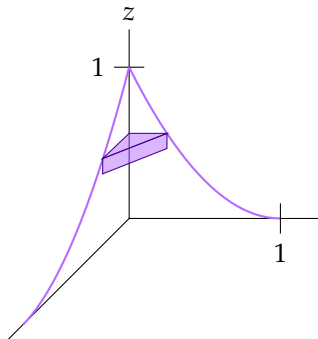
Taught ropes are strung *horizontally* from one parabola to the other, so the horizontal cross-sections are right triangles.



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On both walls, a parabola of the form $z = (x - 1)^2$ is drawn, where z is the vertical axis and x is the horizontal. They start one metre above the corner, and end one metre to the side of the corner.

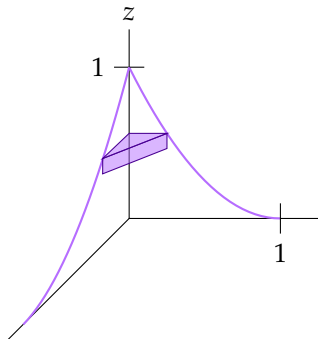
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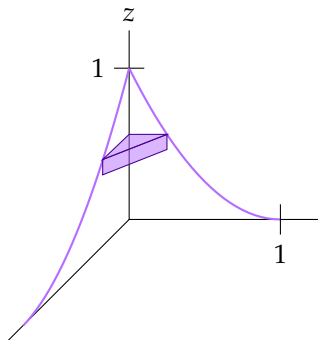
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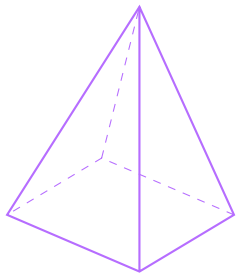
At height z , the cross-section is a right triangle. Its side length is the x -value on the parabola.

Solving $z = (x - 1)^2$ for x , we find $x = \sqrt{z} + 1$.

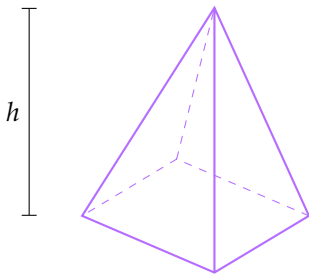
So, the area of a cross-section at height z is $\frac{1}{2} (\sqrt{z} + 1)^2$. We call its width dz .

All together, the enclosed volume is $\int_0^1 \frac{1}{2} (z + 2\sqrt{z} + 1) dz = \frac{17}{12}$ cubic metres.

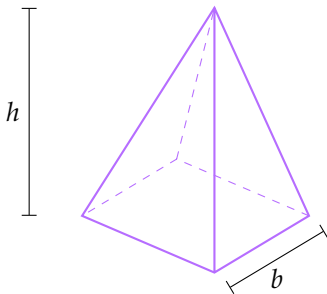
A pyramid with height h metres has a square base with side-length b metres. At an elevation of y metres above the base, $0 \leq y \leq h$, the cross-section of the pyramid is a square with side-length $\frac{b}{h}(h - y)$. What is the volume of the pyramid?



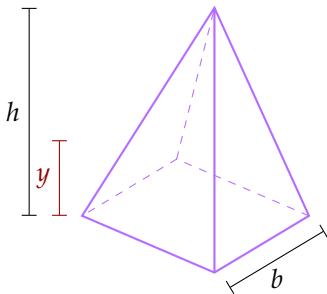
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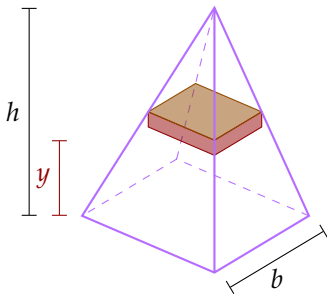
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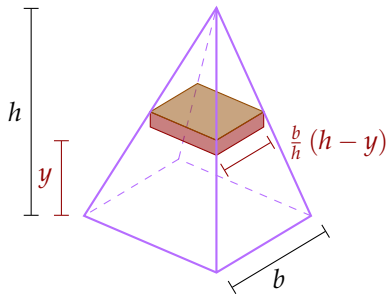
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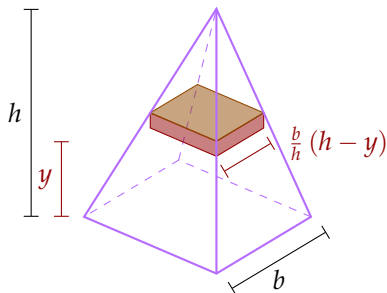
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The area of the square cross-section at height y is

$$\left[\frac{b}{h}(h - y) \right]^2 = \frac{b^2}{h^2} (h^2 - 2hy + y^2).$$

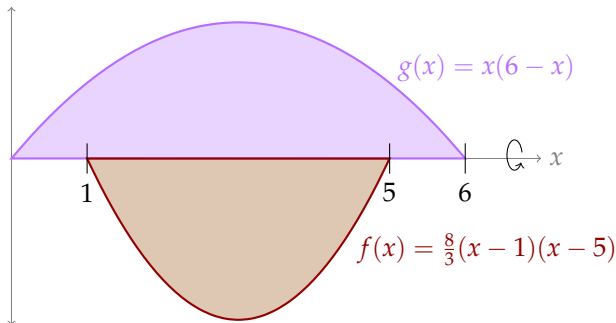
If we give a horizontal slice width dy , then the slice volume is $\frac{b^2}{h^2} (h^2 - 2hy + y^2) dy$. So, the total volume of the pyramid is

$$\begin{aligned} & \int_0^h \frac{b^2}{h^2} (h^2 - 2hy + y^2) dy \\ &= \frac{b^2}{h^2} \left[h^2 y - hy^2 + \frac{1}{3} y^3 \right]_{y=0}^{y=h} \\ &= \frac{b^2}{h^2} \left[h^3 - h^3 + \frac{1}{3} h^3 \right] = \frac{1}{3} b^2 h \end{aligned}$$

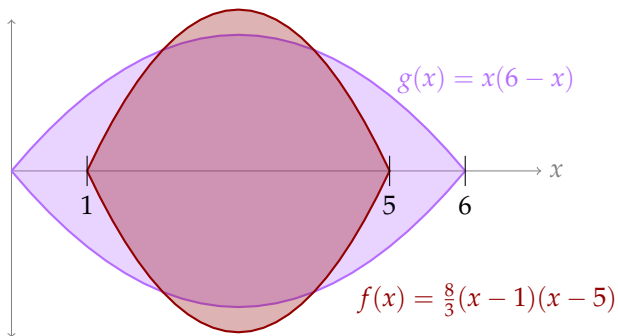
OPTIONAL: CHALLENGE QUESTION

A paddle fixed to the x -axis has two flat blades. One blade is in the shape of $f(x) = \frac{8}{3}(x-1)(x-5)$, from $x = 1$ to $x = 5$. The other blade is in the shape of $g(x) = x(6-x)$, $0 \leq x \leq 6$. The paddle turns through a gelatinous fluid, scraping out a hollow cavity as it turns. What is the volume of this cavity?

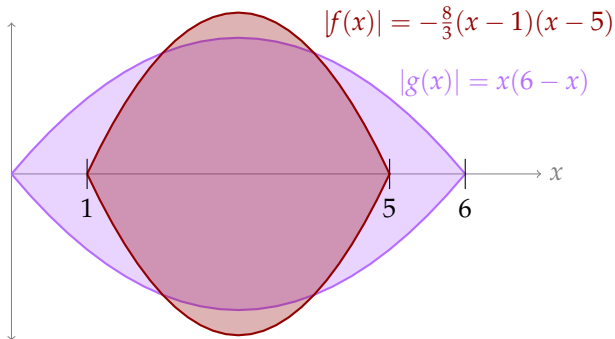
You may leave your answer as an integral, or sum of integrals.



The size of the cavity at a point x along the paddle is determined by the **larger** of $|f(x)|$ and $|g(x)|$.



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Let's find where $|f(x)| = |g(x)|$:

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$$x(6 - x) = -\frac{8}{3}(x - 1)(x - 5)$$

$$6x - x^2 = -\frac{8}{3}(x^2 - 6x + 5) = -\frac{8}{3}x^2 + 16x - \frac{40}{3}$$

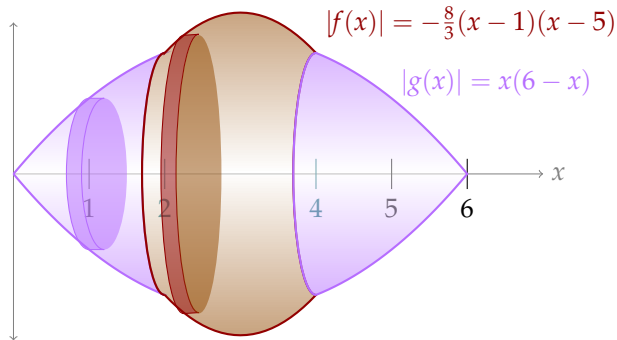
$$\frac{5}{3}x^2 - 10x + \frac{40}{3} = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$x = 2, x = 4$$

The size of the cavity at a point x along the paddle is determined by the **larger** of $|f(x)|$ and $|g(x)|$.



The radius of a cylindrical slice is $|g(x)| = x(6-x)$ when $0 < x < 2$ and $4 < x < 6$, and the radius is $|f(x)| = -\frac{8}{3}(x-1)(x-5)$ when $2 < x < 4$.

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The radius of a cylindrical slice is $|g(x)| = x(6 - x)$ when $0 < x < 2$ and $4 < x < 6$, and the radius is $|f(x)| = -\frac{8}{3}(x - 1)(x - 5)$ when $2 < x < 4$.

$|f(x)|^2 = [f(x)]^2$, so we can drop our absolute values in this step.

$$\begin{aligned} \text{Volume} &= \int_0^2 \pi (6x - x^2)^2 dx + \int_2^4 \pi \left(\frac{8}{3} (x^2 - 6x + 5) \right)^2 dx \\ &\quad + \int_4^6 \pi (6x - x^2)^2 dx \end{aligned}$$

We could make our calculation slightly shorter by noting that the shape is symmetric to the left and right of $x = 3$.

$$= 2 \underbrace{\left[\int_0^2 \pi (6x - x^2)^2 dx + \int_2^3 \pi \left(\frac{8}{3} (x^2 - 6x + 5) \right)^2 dx \right]}_{\text{Volume of half the object, } 0 \leq x \leq 3}$$

Included Work

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