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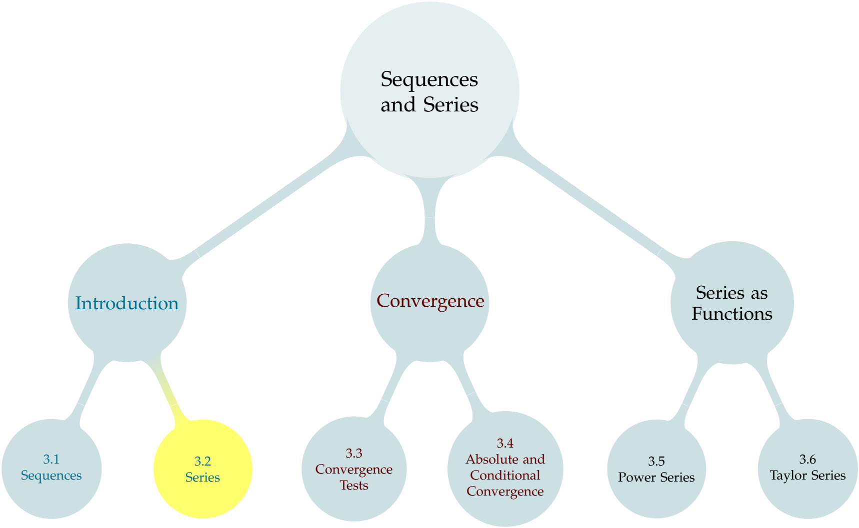


Sequences and Series

A sequence is a list of numbers
A series is the sum of such a list.

Sequences and Series

Sequence

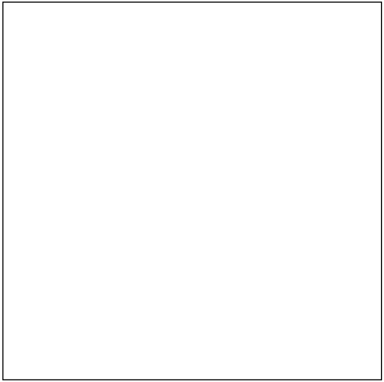
List of numbers,

approaching

Series

Sum of numbers,

approaching



Square of Area 1

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

(a more rigorous definition will be discussed soon)

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(C \cdot a_n) =$$

A. $\displaystyle\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\displaystyle\sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$

C. $\displaystyle C \sum_{n=1}^{\infty} a_n$

D. $\displaystyle a_n \sum_{n=1}^{\infty} C$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty}(a_n + b_n) =$$

A. $\displaystyle\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$

B. $\displaystyle\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $\displaystyle a_n + \sum_{n=1}^{\infty} b_n$

D. $\displaystyle a_n \sum_{n=1}^{\infty} b_n$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

A. $\displaystyle\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\displaystyle\left(\sum_{n=1}^{\infty} a_n\right)^C$

C. $\displaystyle C^n \sum_{n=1}^{\infty} a_n$

D. $\displaystyle\sum_{n=1}^{\infty} C(a_n)^{C-1}$

E. none of the above

Series Philosophy

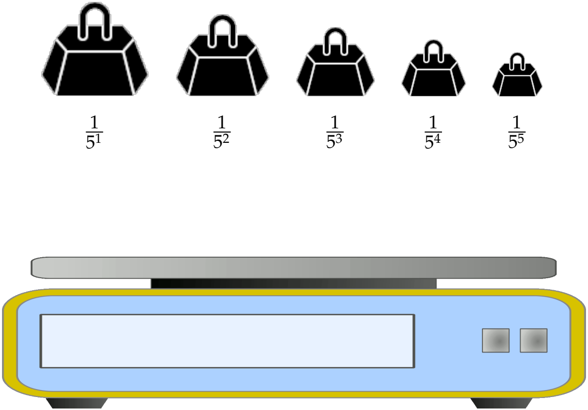
What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

We need an unambiguous definition.

HOW CAN WE ADD UP INFINITELY MANY THINGS?

SEQUENCE OF PARTIAL SUMS



$$S_1 = 0.2000$$

$$S_2 = 0.2400$$

$$S_3 = 0.2480$$

$$S_4 = 0.2496$$

$$S_5 = 0.2499$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad\qquad S_1 = 0.2$$
$$a_2 = \frac{1}{5^2} = 0.04 \qquad\qquad S_2 = 0.24$$
$$a_3 = \frac{1}{5^3} = 0.008 \qquad\qquad S_3 = 0.248$$
$$a_4 = \frac{1}{5^4} = 0.0016 \qquad\qquad S_4 = 0.2496$$
$$a_5 = \frac{1}{5^5} = 0.00032 \qquad\qquad S_5 = 0.24992$$

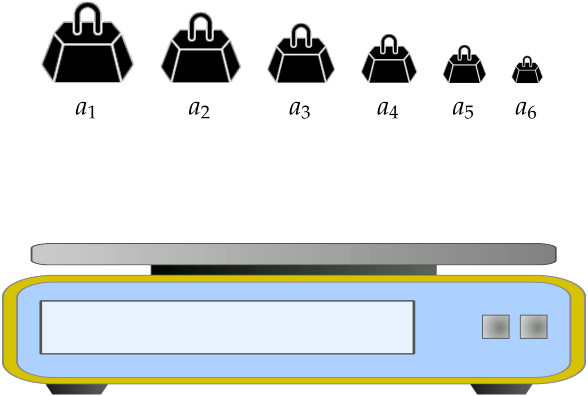
We define $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N.$

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$
$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$
$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$
$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$
$$a_5 = \frac{1}{5^5} = 0.00032 \qquad S_5 = 0.24992$$
$$a_6 = \frac{1}{5^6} = 0.000064 \qquad S_6 = 0.249984$$
$$a_7 = \frac{1}{5^7} = 0.0000128 \qquad S_7 = 0.2499968$$
$$a_8 = \frac{1}{5^8} = 0.00000256 \qquad S_8 = 0.24999936$$

From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \lim_{N\to\infty} S_N = \frac{1}{4}$$

NOTATION: $S_N = \sum_{n=1}^{N} a_n$



$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

$$S_5 = a_1 + \cdots + a_5$$

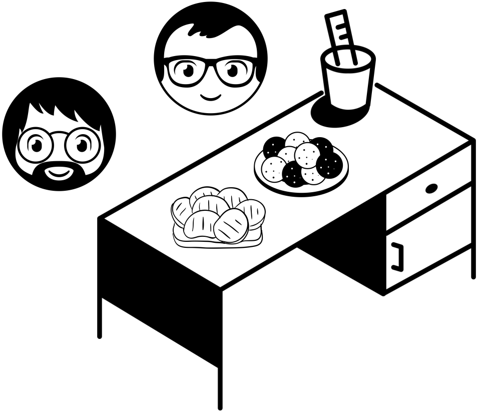
$$S_6 = a_1 + \cdots + a_6$$

Notation Practice

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$.

- Evaluate $\sum_{n=1}^{100} a_n$. $\quad \sum_{n=1}^{100} a_n = S_{100} = \frac{100}{101}$
- Evaluate $\sum_{n=1}^{\infty} a_n$. $\quad \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N}{N+1} = 1$

Notation Practice



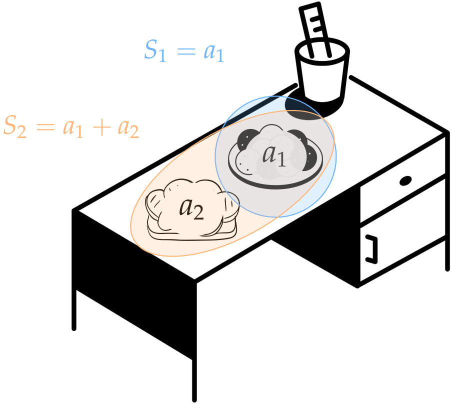
Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

Andrew brought 10, and Joel brought $19 - 10 = 9$.

Notation Practice



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How many cookies did each person bring?

Andrew brought 10, and Joel brought $19 - 10 = 9$.

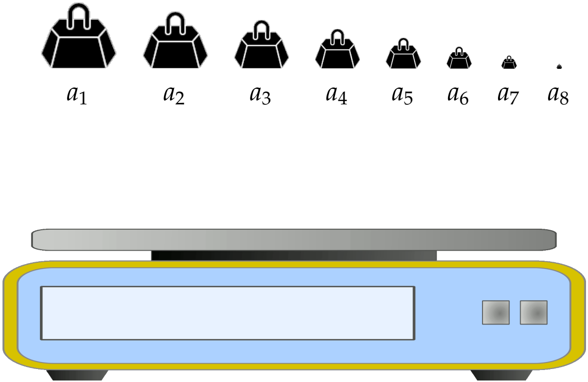
Notation Practice

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Find a_1. $a_1 = \sum_{n=1}^{1} a_n = S_1 = \dfrac{1}{2}$
- Give an explicit expression for a_n, when $n > 1$.

$$a_n = \left(\sum_{k=1}^{n} a_k\right) - \left(\sum_{k=1}^{n-1} a_k\right) = S_n - S_{n-1}$$
$$= \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$$

$$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$$



$S_1 = 1/(1+1)$

$S_2 = 2/(2+1)$

$S_3 = 3/(3+1)$

$S_4 = 4/(4+1)$

$S_5 = 5/(5+1)$

$S_6 = 6/(6+1)$

$S_7 = 7/(7+1)$

$S_8 = 8/(8+1)$

Definition

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^{N} a_n.$$

The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \to S$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Geometric Series

Let a and r be two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + ar^3 + \cdots$$

is called the **geometric series** with first term a and ratio r.

We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

Another useful way of identifying geometric series is to determine whether all pairs of consecutive terms have the same ratio.

- Geometric: $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$
- Geometric: $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- Not geometric: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

Consider the partial sum S_N of a geometric series:

$$\begin{aligned} S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\ rS_N &= \\ rS_N - S_N &= \\ S_N(r-1) &= ar^{N+1} - a \end{aligned}$$

If $r \neq 1$, then

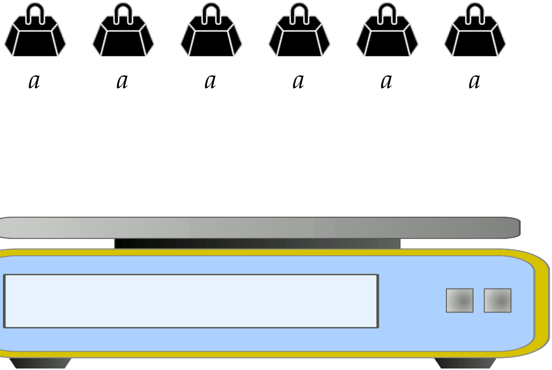
$$S_N = \frac{ar^{N+1} - a}{r-1} = a\frac{r^{N+1} - 1}{r-1}$$

Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

- If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a\dfrac{r^{N+1} - 1}{r - 1}$.
- If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = (N+1)a$.
- If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n = \lim_{N \to \infty} a\frac{r^{N+1} - 1}{r - 1} = a\frac{1}{1 - r}$
- If $r = 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges
- If $r = -1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges
- If $|r| > 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ diverges

$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$



$$S_0 = a$$

$$S_1 = 2a$$

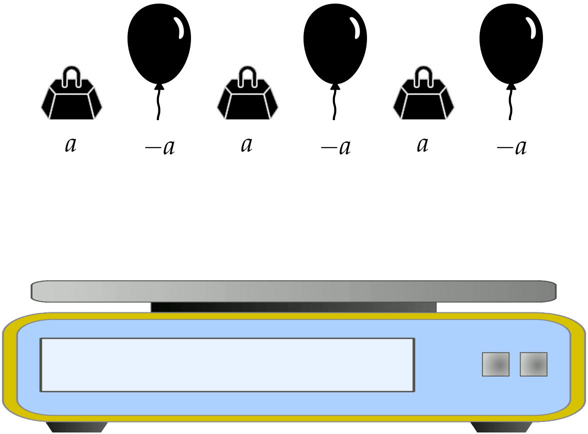
$$S_2 = 3a$$

$$S_3 = 4a$$

$$S_4 = 5a$$

$$S_5 = 6a$$

$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$S_0 = a$

$S_1 = 0$

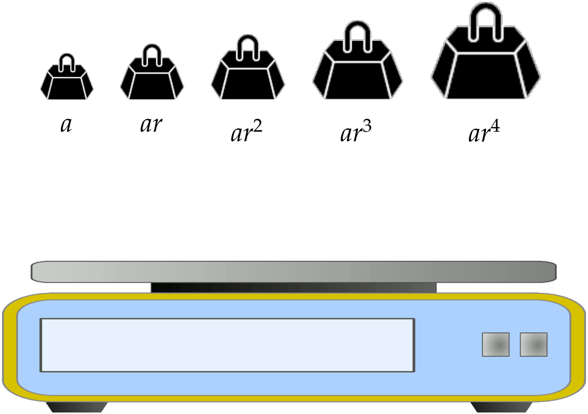
$S_2 = a$

$S_3 = 0$

$S_4 = a$

$S_5 = 0$

$$\sum_{n=0}^{\infty} ar^n,\ r > 1,\ a \neq 0$$



$$S_0 = a$$

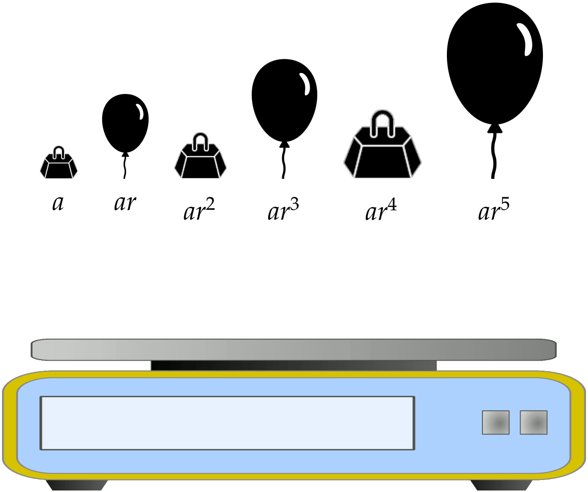
$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a\frac{r^2-1}{r-1}$$

$$S_2 = a\frac{r^3-1}{r-1}$$

$$S_3 = a\frac{r^4-1}{r-1}$$

$$S_4 = a\frac{r^5-1}{r-1}$$

$$S_5 = a\frac{r^6-1}{r-1}$$

Geometric Series

New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half.

- Each of the first 210,000 solutions can produce 50 bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^3}$ bitcoins.

Assume that this continues forever, and that bitcoins are infinitely divisible.[1] How many bitcoins can possibly be produced?

We start by writing the total number of bitcoin produced as a series. Since we want to know an upper bound, we'll assume that infinite solutions can be found and used to make bitcoin.

$$210\,000(50)+210\,000\left(\frac{50}{2}\right)+210\,000\left(\frac{50}{2^2}\right)+\cdots=\sum_{n=0}^{\infty}(210\,000)\left(\frac{50}{2^n}\right)$$

$$\sum^{\infty}(210\,000)\left(\frac{50}{2^n}\right)=\sum^{\infty}(210\,000\cdot 50)\left(\frac{1}{2}\right)^n$$

$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n}\right) = 21\,000\,000$$

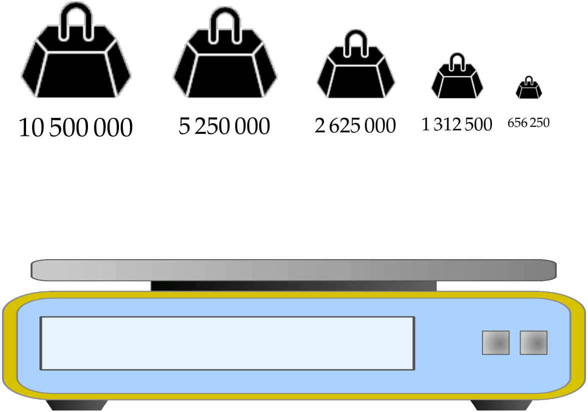
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

$$S_4 = 20\,343\,750$$



Arithmetic of Series

Let S, T, and C be real numbers. Let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T respectively. Then

$$\sum_{n=1}^{\infty} [a_n + b_n] = S + T$$

$$\sum_{n=1}^{\infty} [a_n - b_n] = S - T$$

$$\sum_{n=1}^{\infty} [Ca_n] = CS$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right)$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right)$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right)$

Telescoping Sums

Evaluate $\displaystyle\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

$$\begin{array}{rccc}
a_1: & \frac{1}{1} & - & \frac{1}{2} \\
a_2: & \frac{1}{2} & - & \frac{1}{3} \\
a_3: & \frac{1}{3} & - & \frac{1}{4} \\
a_4: & \frac{1}{4} & - & \frac{1}{5} \\
\vdots & & & \\
a_{N-1}: & \frac{1}{N-1} & - & \frac{1}{N} \\
a_N: & \frac{1}{N} & - & \frac{1}{N+1}
\end{array}$$

Evaluate $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

$$\begin{array}{rcccc}
S_1 = & \frac{1}{1} & - & \frac{1}{2} & \\
S_2 = & \frac{1}{1} & - & \frac{1}{3} & \\
S_3 = & \frac{1}{1} & - & \frac{1}{4} & \\
S_4 = & \frac{1}{1} & - & \frac{1}{5} & \\
\vdots & & & & \\
S_N = & \frac{1}{1} & - & \frac{1}{N+1} & = \frac{N}{N+1}
\end{array}$$

$$\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right) = S_{800} = \frac{800}{801}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{N\to\infty} S_N = 1$$

Q

Evaluate $\displaystyle\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right)$.

Evaluate $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$.

$$
\begin{array}{lcl}
a_1 & : & \log(2) - \cancel{\log(1)} \\
a_2 & : & \log(3) - \log(2) \\
a_3 & : & \log(4) - \log(3) \\
\vdots & & \\
a_{n-1} & : & \log(n) - \log(n-1) \\
a_n & : & \log(n+1) - \log(n)
\end{array}
\qquad
\begin{array}{lcl}
S_1 & = & \log(2) \\
S_2 & = & \log(3) \\
S_3 & = & \log(4) \\
 & & \\
 & & \\
S_n & = & \log(n+1)
\end{array}
$$

So, $\displaystyle\sum_{n=1}^{1000} \log\left(\frac{n+1}{n}\right) = S_{1000} = \log(1001)$ and

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \log(n+1) = \infty$$

Included Work

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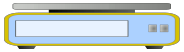 ‘Waage/Libra’ by B. Lachner is in the public domain (accessed April 2021, edited), 9, 12, 17, 22–25, 27

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