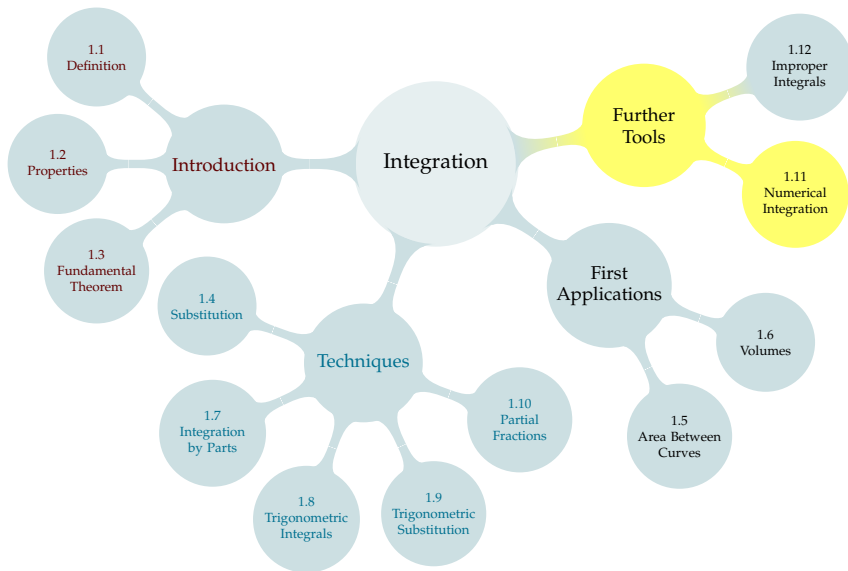# TABLE OF CONTENTS

Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

▶ the total error introduced by the midpoint rule is bounded by
$$\frac{M}{24} \frac{(b-a)^3}{n^2},$$

▶ the total error introduced by the trapezoidal rule is bounded by
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▶ the total error introduced by Simpson's rule is bounded by
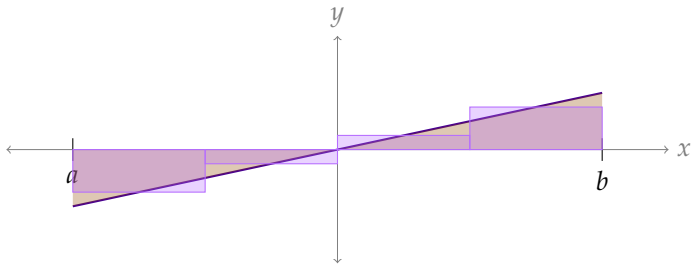$$\frac{L}{180} \frac{(b-a)^5}{n^4}$$

when approximating $\int_a^b f(x) \, dx$.

WHY THE *second* DERIVATIVE?

The midpoint rule gives the exact area under the curve for

$$f(x) = ax + b$$

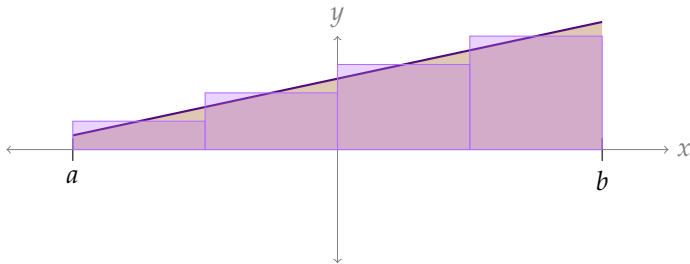when a and b are any constants.

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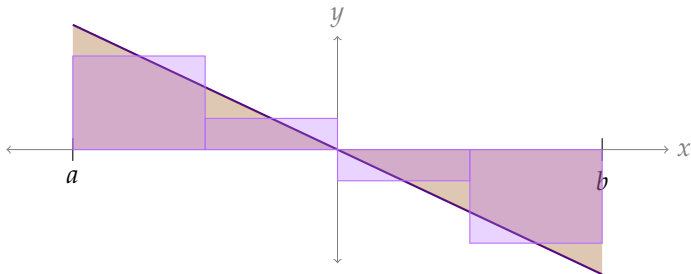
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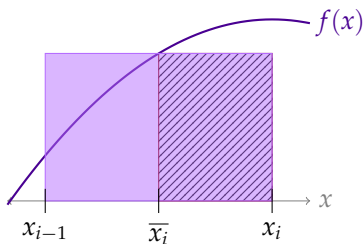
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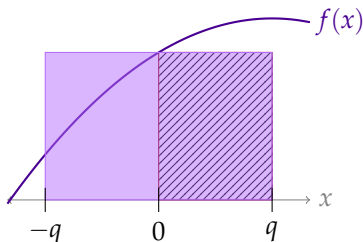 $$\frac{L}{180} \frac{(b-a)^5}{n^4}$$

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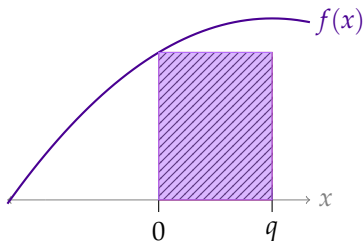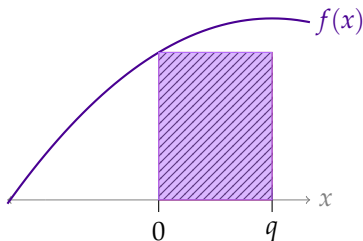To avoid messiness, let's also consider a simplified location:

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▶ Set $u = f(x)$ and $\mathrm{d}v = \mathrm{d}x$, so $\mathrm{d}u = f'(x)\, \mathrm{d}x$.
We choose $v(x) = x - q$, so that $f(v(q)) = f(0)$.

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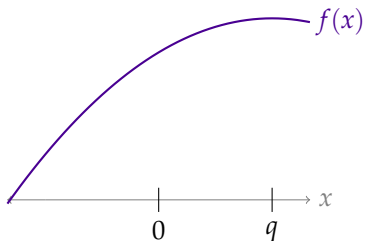
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- We know something about the second derivative, not the first, so repeat: set $u = f'(x)$, $\mathrm{d}v = (x - q)\, \mathrm{d}x$; $\mathrm{d}u = f''(x)\, \mathrm{d}x$, $v = \frac{(x-q)^2}{2}$

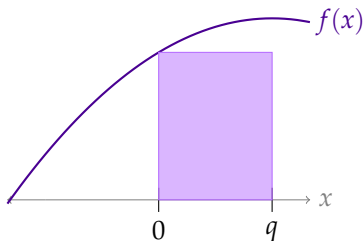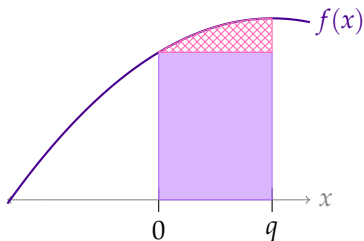$$\int_0^q f(x)\, \mathrm{d}x = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x)\, \mathrm{d}x$$
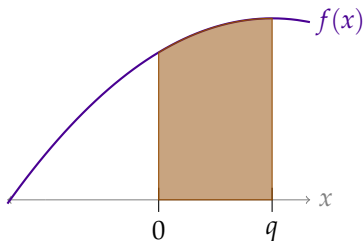
$$\int_0^q f(x)\,\mathrm{d}x \quad = \quad q\cdot f(0) \quad + \quad \frac{q^2}{2}\cdot f'(0) + \int_0^q \frac{(x-q)^2}{2}f''(x)\,\mathrm{d}x$$

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exact   approximate $\pm$ error

Repeat for the other half of the slice:

$$\int_{-q}^{0} \underbrace{f(x)}_{u} \underbrace{\mathrm{d}x}_{\mathrm{d}v} = \left[\underbrace{f(x)}_{u} \cdot \underbrace{(x+q)}_{v} \right]_{-q}^{0} - \int_{-q}^{0} \underbrace{(x+q)}_{v} \cdot \underbrace{f'(x)\,\mathrm{d}x}_{\mathrm{d}u}$$

$$= q \cdot f(0) - \int_{-q}^{0} \underbrace{f'(x)}_{\hat{u}} \cdot \underbrace{(x+q)\,\mathrm{d}x}_{\mathrm{d}\hat{v}}$$

$$= q \cdot f(0) - \left[\underbrace{f'(x)}_{\hat{u}} \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}} \right]_{-q}^{0} + \int_{-q}^{0} \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}} \underbrace{f''(x)\,\mathrm{d}x}_{\mathrm{d}\hat{u}}$$

$$= q \cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x$$
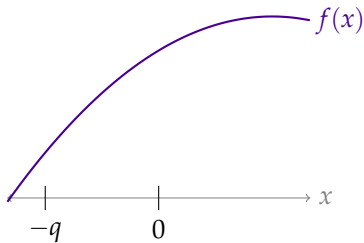
$$\int_{-q}^{0} f(x) \, dx \quad = \quad q \cdot f(0) \quad - \quad \frac{q^2}{2} \cdot f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x) \, dx$$
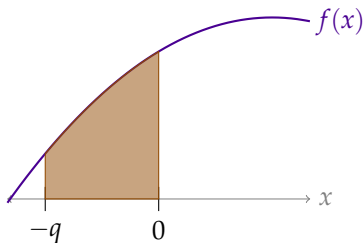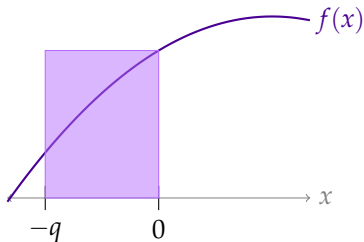
$$\int_{-q}^{0} f(x)\, \mathrm{d}x \quad = \quad q \cdot f(0) \quad - \quad \frac{q^2}{2} \cdot f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\, \mathrm{d}x$$

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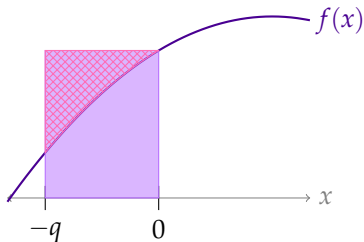
exact   approximate

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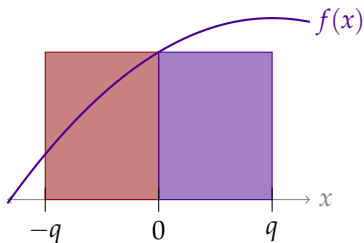
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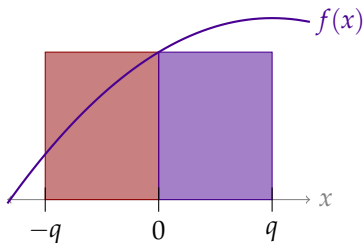
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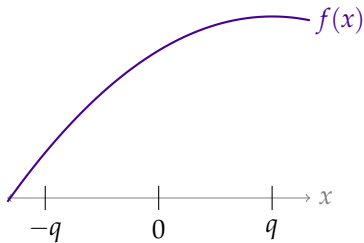
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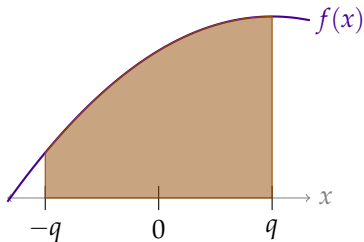
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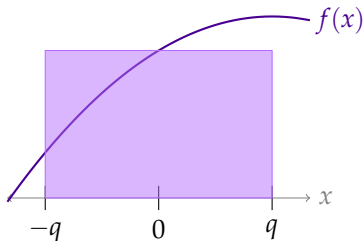
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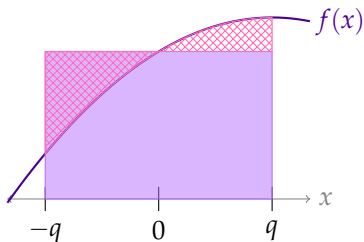
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exact   approximate $\pm$ error

We re-arrange to write the error as the difference between the actual area of one slice and its rectangular approximation.

$$\int_{-q}^{q} f(x) \, dx - 2q \cdot f(0) = \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x) \, dx + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x) \, dx$$
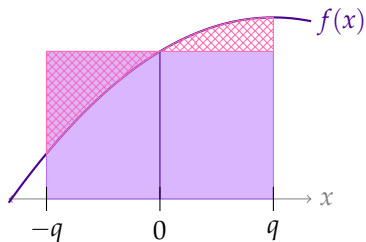
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$$\text{error} = \left| \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\, dx + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\, dx \right|$$

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$$\leq \int_{-q}^{0} \frac{(x+q)^2}{2} M \, dx + \int_{0}^{q} \frac{(x-q)^2}{2} M \, dx$$

$$= M \left[\frac{(x+q)^3}{6} \right]_{-q}^{0} + M \left[\frac{(x-q)^3}{6} \right]_{0}^{q}$$

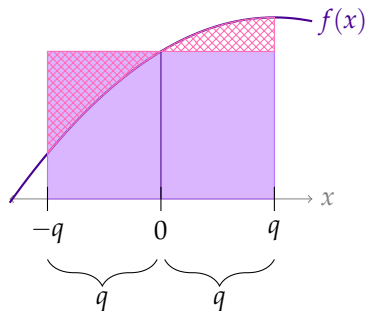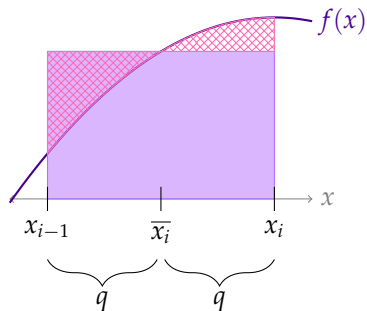$$= \frac{M \cdot q^3}{3}$$

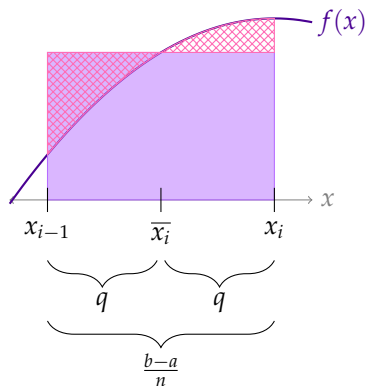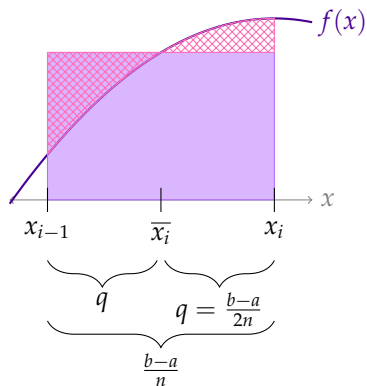Now we can bound the error of a single slice:

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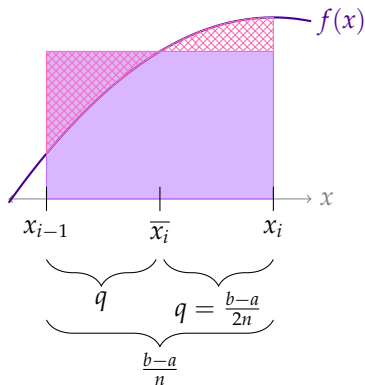Now we can bound the error of a single slice:

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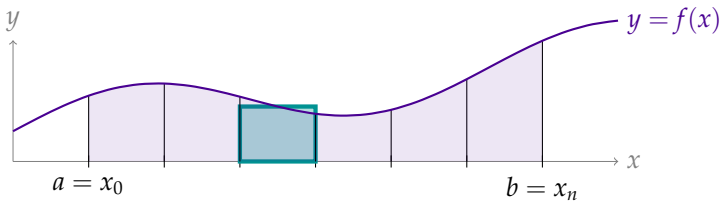Now we can bound the error of a single slice:

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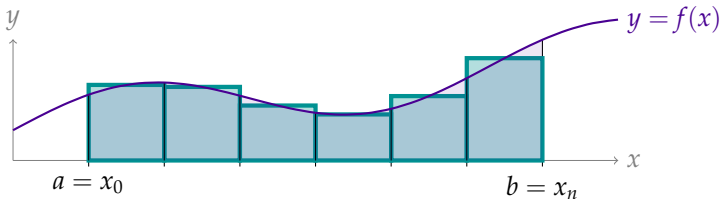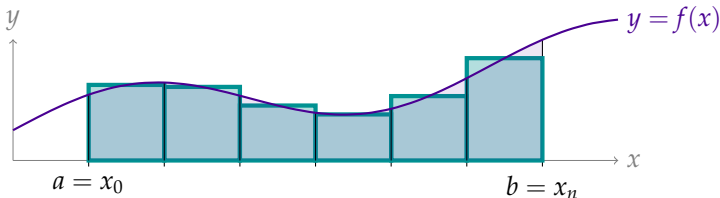
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$$\left| \int_{x_{i-1}}^{x_i} f(x) \, \mathrm{d}x - \frac{b-a}{n} \cdot f(\overline{x_i}) \right| \leq \frac{M}{3} \left(\frac{b-a}{2n} \right)^3 = \frac{M}{24} \frac{(b-a)^3}{n^3}$$

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