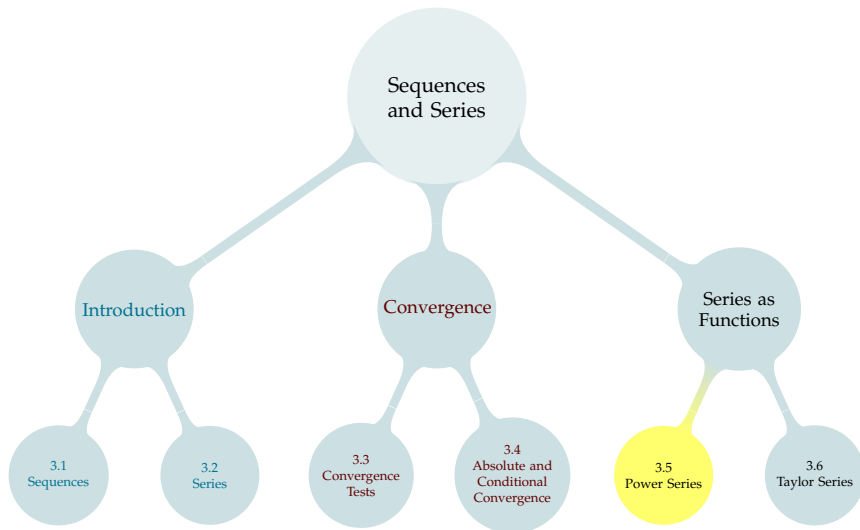


# TABLE OF CONTENTS



Recall the geometric series: for a constant  $r$ , with  $|r| < 1$ :

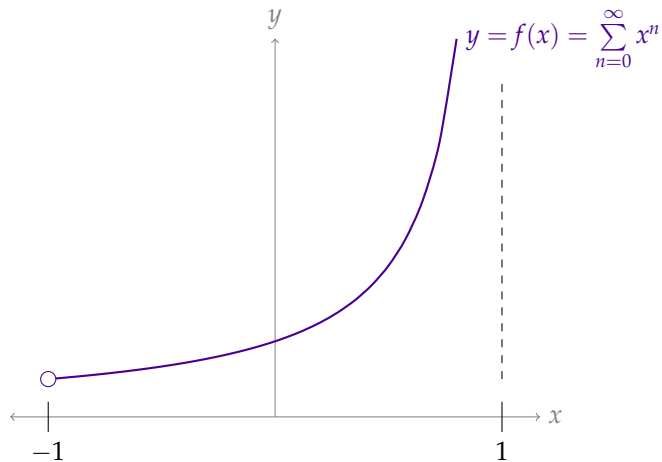
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

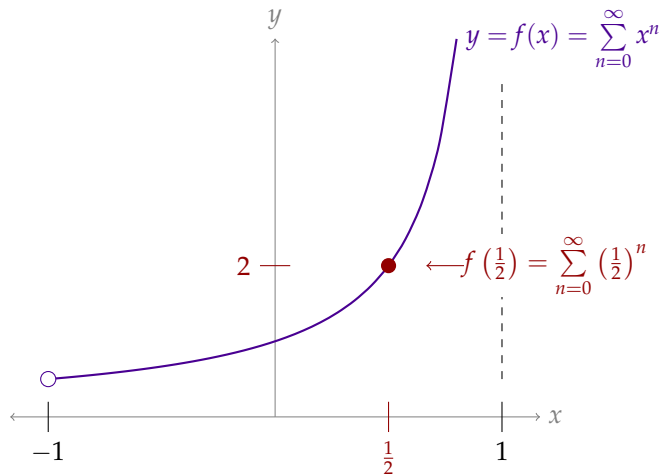
We can think of this as a function. If we set

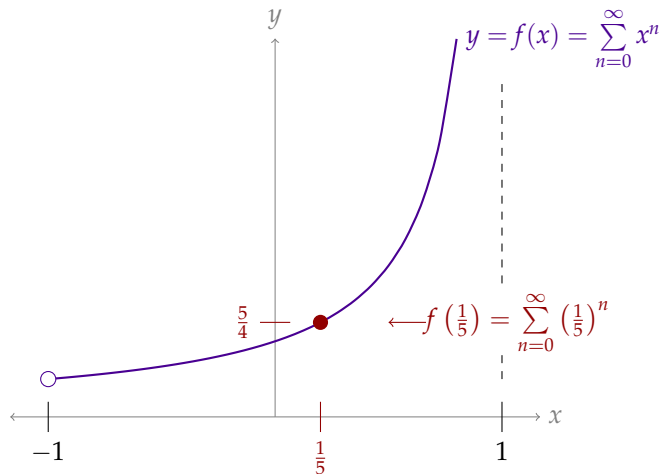
$$f(x) = \sum_{n=0}^{\infty} x^n$$

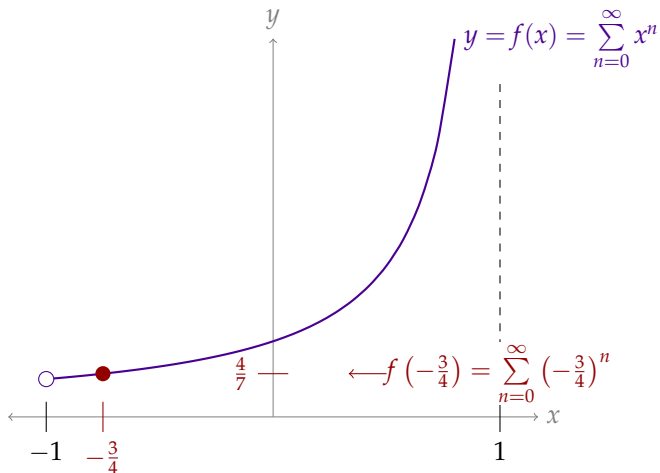
and restrict our domain to  $-1 < x < 1$ , then

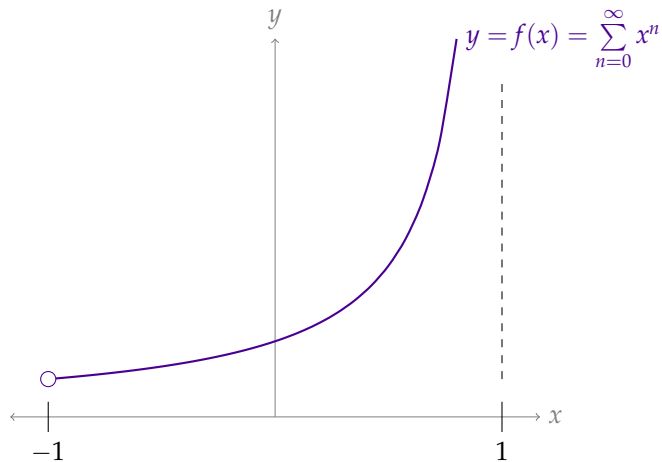
$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$











Why would we ever prefer to write  $\sum_{n=0}^{\infty} x^n$  instead of  $\frac{1}{1-x}$ ?



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The function

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isn't a polynomial, but in certain ways it behaves like one. For  $|x| < 1$ :

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$$\int \frac{1}{1-x} dx = \int \left( \sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=0}^{\infty} \left( \int x^n dx \right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

## Definition

A series of the form

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

is called a *power series in  $(x-c)$*  or a *power series centered on  $c$* . The numbers  $A_n$  are called the coefficients of the power series.

One often considers power series centered on  $c = 0$  and then the series reduces to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots = \sum_{n=0}^{\infty} A_nx^n$$

$$\sum_{n=0}^{\infty} A_n (x - c)^n = A_0 + A_1(x - c) + A_2(x - c)^2 + A_3(x - c)^3 + \cdots$$

In a power series, we think of the coefficients  $A_n$  as fixed constants, and we think of  $x$  as the variable of a function.

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Evaluate the power series  $\sum_{n=0}^{\infty} A_n(x-c)^n$  when  $x = c$  :





A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of  $x$  for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

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## Definition

Consider the power series

$$\sum_{n=0}^{\infty} A_n(x - c)^n.$$

The set of real  $x$ -values for which it converges is called the interval of convergence of the series.

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots .$$

What happens if we apply the ratio test to a generic power series,

$$\sum_{n=0}^{\infty} A_n(x - c)^n?$$

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$$\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}(x - c)^{n+1}}{A_n(x - c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} (x - c) \right| = |x - c| \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right|$$

- ▶ If  $\left| \frac{A_{n+1}}{A_n} \right|$  does not approach a limit as  $n \rightarrow \infty$ , the ratio test tells us nothing. (We should try other tests.)
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = 0$ , then
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \infty$ , then
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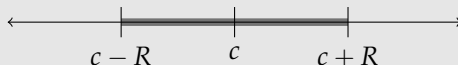
$$\sum_{n=0}^{\infty} A_n(x - c)^n?$$

$$\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}(x - c)^{n+1}}{A_n(x - c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} (x - c) \right| = |x - c| \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right|$$

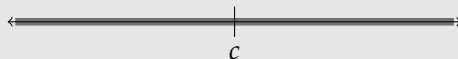
- ▶ If  $\left| \frac{A_{n+1}}{A_n} \right|$  does not approach a limit as  $n \rightarrow \infty$ , the ratio test tells us nothing. (We should try other tests.)
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = 0$ , then the series converges for all  $x$ .
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \infty$ , then the series converges when  $x = c$ , and diverges otherwise.
- ▶ If  $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = A$  for some real number  $A$ , then the series converges when  $|x - c| < \frac{1}{A}$ , and diverges for  $|x - c| > \frac{1}{A}$ . The cases  $|x - c| = \frac{1}{A}$  need further inspection.

## Definition: Radius of Convergence

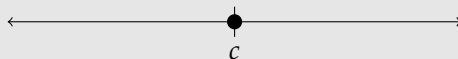
- (a) Let  $0 < R < \infty$ . If  $\sum_{n=0}^{\infty} A_n(x - c)^n$  converges for  $|x - c| < R$ , and diverges for  $|x - c| > R$ , then we say that the series has radius of convergence  $R$ .



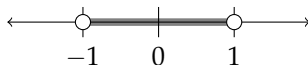
- (b) If  $\sum_{n=0}^{\infty} A_n(x - c)^n$  converges for every number  $x$ , we say that the series has an infinite radius of convergence.



- (c) If  $\sum_{n=0}^{\infty} A_n(x - c)^n$  diverges for every  $x \neq c$ , we say that the series has radius of convergence zero.

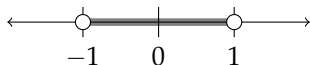


- We saw that  $\sum_{n=0}^{\infty} x^n$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series has radius of convergence  $R =$

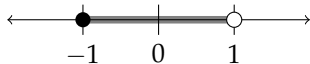




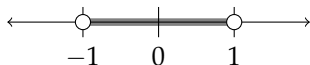
- We saw that  $\sum_{n=0}^{\infty} x^n$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series has radius of convergence  $R = 1$ .



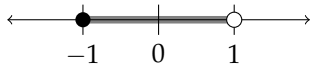
- We saw that  $\sum_{n=1}^{\infty} \frac{x^n}{n}$  converges when  $|x| < 1$  and diverges when  $|x| > 1$ , so this series also has radius of convergence  $R =$



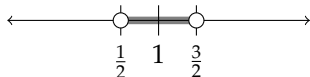
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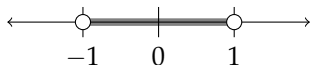
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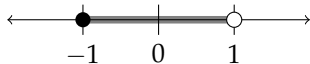
- We saw that  $\sum_{n=1}^{\infty} 2^n(x-1)^n$  converges when  $|x-1| < \frac{1}{2}$  and diverges when  $|x-1| > \frac{1}{2}$ , so this series has radius of convergence  $R =$



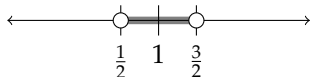
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What is the radius of convergence for the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  ?

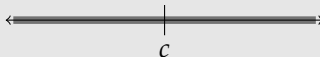
*Recall:*  $n! = (n)(n-1)(n-2) \cdots (2)(1)$ .

What is the radius of convergence for the series  $\sum_{n=0}^{\infty} n! \cdot (x - 3)^n$  ?

## Theorem

Given a power series (say with centre  $c$ ), one of the following holds.

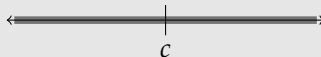
- (a) The power series converges for every number  $x$ . In this case we say that the radius of convergence is  $\infty$ .



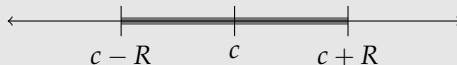
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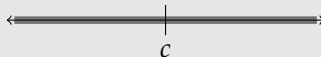
- (b) There is a number  $0 < R < \infty$  such that the series converges for  $|x - c| < R$  and diverges for  $|x - c| > R$ . Then  $R$  is called the radius of convergence.



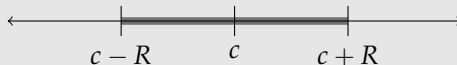
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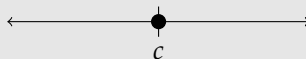
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- (c) The series converges for  $x = c$  and diverges for all  $x \neq c$ . In this case, we say that the radius of convergence is 0.

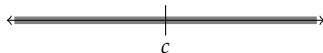




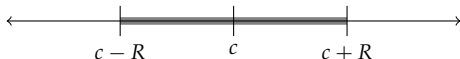
We are told that a certain power series with centre  $c = 3$  converges at  $x = 4$  and diverges at  $x = 1$ . What else can we say about the convergence or divergence of the series for other values of  $x$ ?

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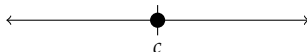
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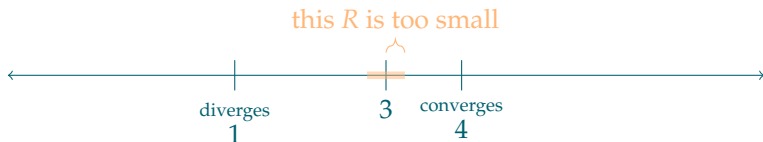
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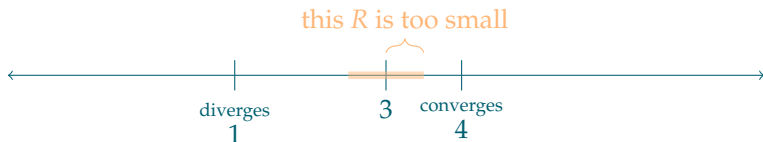
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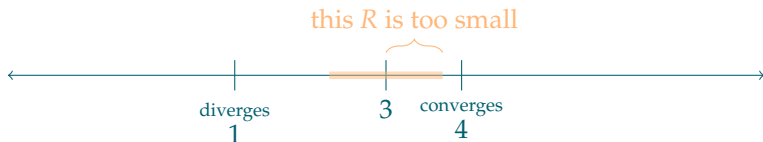
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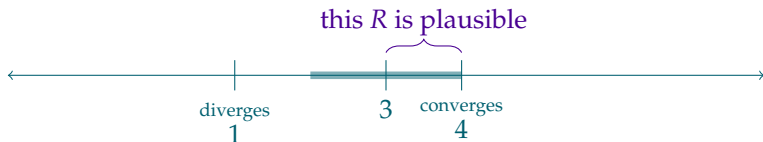
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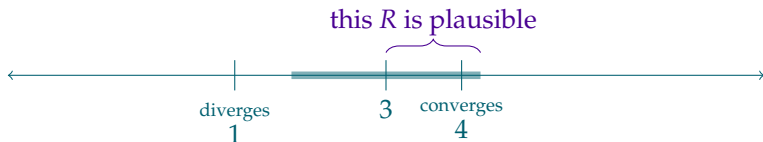
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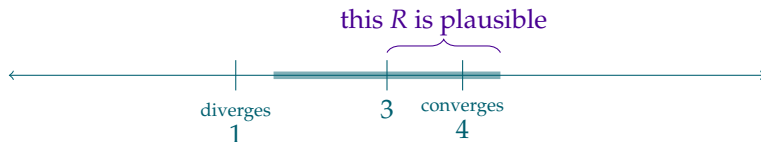
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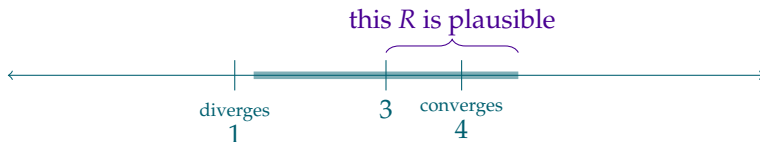
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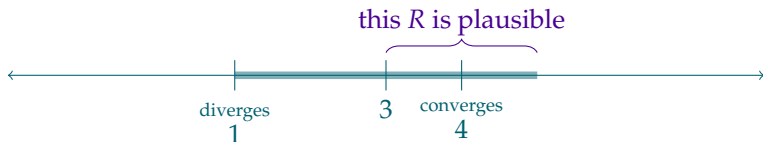
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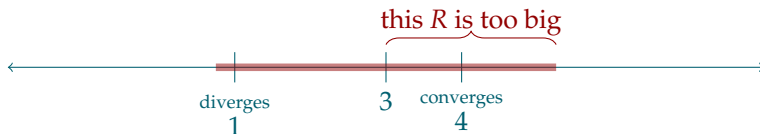
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From the theorem, we know that there is some real number  $R$  such that the series converges when  $|x - 3| < R$  and diverges when  $|x - 3| > R$ .

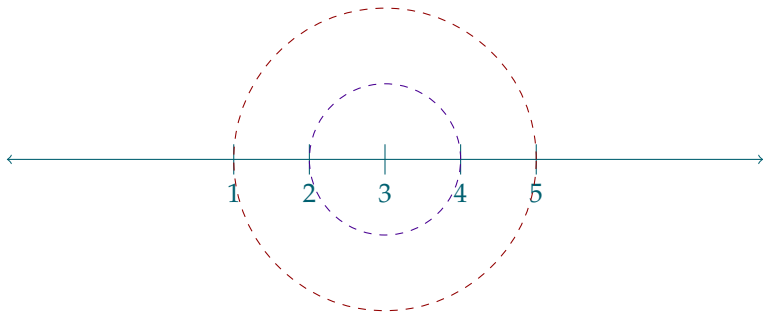


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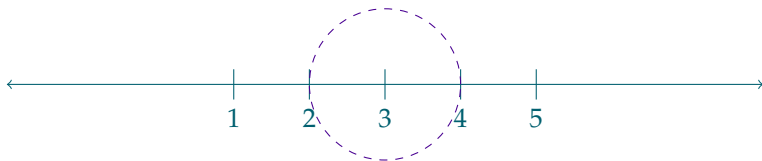


We are told that a certain power series with centre  $c = 3$  converges at  $x = 4$  and diverges at  $x = 1$ . What else can we say about the convergence or divergence of the series for other values of  $x$ ?



- ▶ From  $R \geq 1$ , we know
- ▶ From  $R \leq 2$ , we know
- ▶ We do not know whether the series converges for other  $x$ .

We are told that a certain power series with centre  $c = 3$  converges at  $x = 4$  and diverges at  $x = 1$ . What else can we say about the convergence or divergence of the series for other values of  $x$ ?



## Operations on Power Series

Assume that the functions  $f(x)$  and  $g(x)$  are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x - c)^n$$

for all  $x$  obeying  $|x - c| < R$ . Let  $K$  be a constant. Then:

$$f(x) + g(x) = \sum_{n=0}^{\infty} [A_n + B_n] (x - c)^n$$

$$Kf(x) = \sum_{n=0}^{\infty} K A_n (x - c)^n$$

for all  $x$  obeying  $|x - c| < R$ .

## Operations on Power Series

Assume that the functions  $f(x)$  and  $g(x)$  are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x - c)^n$$

for all  $x$  obeying  $|x - c| < R$ . Then:

$$\begin{aligned} (x - c)^N f(x) &= \sum_{n=0}^{\infty} A_n (x - c)^{n+N} \quad \text{for any integer } N \geq 1 \\ &= \sum_{k=N}^{\infty} A_{k-N} (x - c)^k \quad \text{where } k = n + N \end{aligned}$$

for all  $x$  obeying  $|x - c| < R$ .



## Operations on Power Series

Assume that the functions  $f(x)$  and  $g(x)$  are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x - c)^n$$

for all  $x$  obeying  $|x - c| < R$ . Then:

$$f'(x) = \sum_{n=0}^{\infty} A_n n (x - c)^{n-1} = \sum_{n=1}^{\infty} A_n n (x - c)^{n-1}$$

$$\int_c^x f(t) \, dt = \sum_{n=0}^{\infty} A_n \frac{(x - c)^{n+1}}{n + 1}$$

$$\int f(x) \, dx = \left[ \sum_{n=0}^{\infty} A_n \frac{(x - c)^{n+1}}{n + 1} \right] + C \quad \text{with } C \text{ an arbitrary constant}$$

for all  $x$  obeying  $|x - c| < R$ .

## Operations on Power Series

Assume that the functions  $f(x)$  and  $g(x)$  are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n(x-c)^n$$

for all  $x$  obeying  $|x-c| < R$ .

Differentiating, antidifferentiating, multiplying by a nonzero constant, and multiplying by a positive power of  $(x-c)$  do not change the radius of convergence of  $f(x)$  (although they may change the interval of convergence).

Given that  $\frac{d}{dx} \left\{ \frac{1}{1-x} \right\} = \frac{1}{(1-x)^2}$ , find a power series representation for  $\frac{1}{(1-x)^2}$  when  $|x| < 1$ .

Find a power series representation for  $\log(1 + x)$  when  $|x| < 1$ .



Find a power series representation for  $\arctan(x)$  when  $|x| < 1$ .



## Substituting in a Power Series

Assume that the function  $f(x)$  is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all  $x$  in the interval  $I$ . Also let  $K$  and  $k$  be real constants. Then

$$f(Kx^k) = \sum_{n=0}^{\infty} A_n K^n x^{kn}$$

whenever  $Kx^k$  is in  $I$ . In particular, if  $\sum_{n=0}^{\infty} A_n x^n$  has radius of convergence  $R$ ,  $K$  is nonzero and  $k$  is a natural number, then  $\sum_{n=0}^{\infty} A_n K^n x^{kn}$  has radius of convergence  $\sqrt[k]{R/|K|}$ .

Find a power series representation for  $\frac{1}{5-x}$  with centre 3.



## Included Work



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