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Motivation

How to integrate $\int \frac{x-2}{(x+1)(2x-1)}\,\mathrm{d}x$?

Useful fact: $\frac{x-2}{(x+1)(2x-1)} = \frac{1}{x+1} - \frac{1}{2x-1}$

So:

$$\begin{aligned}\int \frac{x-2}{(x+1)(2x-1)}\,\mathrm{d}x &= \int \frac{1}{x+1}\,\mathrm{d}x - \int \frac{1}{2x-1}\,\mathrm{d}x \\ &= \log|x+1| - \frac{1}{2}\log|2x-1| + C\end{aligned}$$

Method of Partial Fractions: Algebraic method to turn any rational function (i.e. ratio of two polynomials) into the sum of easier-to-integrate rational functions.

Distinct Linear Factors

The rational function

$$\frac{\text{numerator}}{K(x-a_1)(x-a_2)\cdots(x-a_j)}$$

can be written as

$$\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\cdots+\frac{A_j}{x-a_j}$$

for some constants $A_1, A_2, \ldots, A_j$, provided

(1) the linear roots $a_1, \cdots a_j$ are distinct, and

(2) the degree of the numerator is strictly less than the degree of the denominator.

Distinct Linear Factors

$$\frac{7x+13}{(2x+5)(x-2)} =$$

To find A and B, simplify the right-hand side by finding a common denominator.

$$\frac{7x+13}{2x^2+x-10} = \frac{A}{2x+5} + \frac{B}{x-2} = \frac{A(x-2)}{(2x+5)(x-2)} + \frac{B(2x+5)}{(2x+5)(x-2)}$$
$$= \frac{A(x-2)+B(2x+5)}{2x^2+x-10}$$

Cancel denominators

$$7x+13 = A(x-2) + B(2x+5)$$

Distinct Linear Factors

We found $7x + 13 = A(x - 2) + B(2x + 5)$ for some constants A and B. What are A and B?

Method 1: set x to convenient values.

When $x = 2$ (chosen to eliminate A from the right hand side), we have $14 + 13 = B \cdot 9$, so $B = 3$.
If $x = -\frac{5}{2}$ (chosen to eliminate B from the right hand side), then $-\frac{35}{2} + 13 = A\left(-\frac{5}{2} - 2\right)$, so $A = 1$.

Method 2: match coefficients of powers of x.

$7x + 13 = (A + 2B)x + (-2A + 5B)$, so $7 = A + 2B$ and $13 = -2A + 5B$.
Then $A = 7 - 2B$, so $13 = -2(7 - 2B) + 5B$.
Then $B = 3$ and $A = 1$.

Distinct Linear Factors

All together:

$$\frac{7x+13}{2x^2+x-10} = \frac{A}{2x+5} + \frac{B}{x-2}$$

$$A = 1, \qquad B = 3$$

$$\frac{7x+13}{2x^2+x-10} = \frac{1}{2x+5} + \frac{3}{x-2}$$

$$\int \frac{7x+13}{2x^2+x-10}\,\mathrm{d}x = \int \left(\frac{1}{2x+5} + \frac{3}{x-2}\right)\,\mathrm{d}x$$

$$= \frac{1}{2}\log|2x+5| + 3\log|x-2| + C$$

Check Our Work

We check that $\displaystyle\int \frac{7x+13}{2x^2+x-10} =$ by differentiating.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\log|2x+5| + 3\log|x-2| + C\right] &= \frac{1}{2}\cdot\frac{1}{2x+5}\cdot 2 + 3\cdot\frac{1}{x-2}\\ &= \frac{1}{2x+5}\left(\frac{x-2}{x-2}\right) + \frac{3}{x-2}\left(\frac{2x+5}{2x+5}\right)\\ &= \frac{(x-2)+(6x+15)}{(x-2)(2x+5)} = \frac{7x+13}{2x^2+x-10}\end{aligned}$$

So, our work checks out.

Distinct Linear Factors

$\dfrac{x^2+5}{2x(3x+1)(x+5)}$ is hard to antidifferentiate, but it can be written as $\dfrac{A}{2x}+\dfrac{B}{3x+1}+\dfrac{C}{x+5}$ for some constants A, B, and C.

Once we find A, B, and C, integration is easy:

$$\begin{aligned}
&\int \frac{x^2-24x+5}{2x(3x+1)(x+5)}\,\mathrm{d}x \\
&\quad=\int\left(\frac{A}{2x}+\frac{B}{3x+1}+\frac{C}{x+5}\right)\mathrm{d}x \\
&\quad=\frac{A}{2}\log|x|+\frac{B}{3}\log|3x+1|+C\log|x+5|+D
\end{aligned}$$

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Distinct Linear Factors

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}$$

Find constants A, B, and C.
Start: make a common denominator

$$= \frac{A(3x+1)(x+5)}{2x(3x+1)(x+5)} + \frac{B(2x)(x+5)}{2x(3x+1)(x+5)} + \frac{C(2x)(3x+1)}{2x(3x+1)(x+5)}$$
$$= \frac{A(3x+1)(x+5)+B(2x)(x+5)+C(2x)(3x+1)}{2x(3x+1)(x+5)}$$

Cancel off denominator

$$x^2+5 = A(3x+1)(x+5)+B(2x)(x+5)+C(2x)(3x+1)$$

Check Our Work

Let's check that

$$\frac{x^2+5}{2x(3x+1)(x+5)} =$$

$$\begin{aligned}
&\frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5} \\
&= \frac{1(3x+1)(x+5)}{2x(3x+1)(x+5)} - \frac{23/14(2x)(x+5)}{(2x)(3x+1)(x+5)} + \frac{3/14(2x)(3x+1)}{(2x)(3x+1)(x+5)} \\
&= \frac{(3x^2+16x+5) - (\frac{23}{7}x^2 + \frac{115}{7}x) + (\frac{9}{7}x^2 + \frac{3}{7}x)}{2x(3x+1)(x+3)} \\
&= \frac{x^2+5}{2x(3x+1)(x+3)}
\end{aligned}$$

So, our algebra is good.

Distinct Linear Factors

All together:

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}$$

$$\int \frac{x^2-24x+5}{2x(3x+1)(x+5)}\,\mathrm{d}x = \int \left(\frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}\right)\,\mathrm{d}x$$

$$= \frac{1}{2}\log|x| - \frac{23}{42}\log|3x+1| + \frac{3}{14}\log|x+5| + C$$

Repeated Linear Factors

A rational function $\dfrac{P(x)}{(x-1)^4}$, where $P(x)$ is a polynomial of degree strictly less than 4, can be written as

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}$$

for some constants A, B, C, and D.

$$\frac{5x-11}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

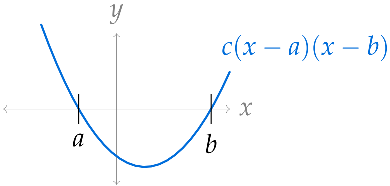
Set up the form of the partial fractions decomposition. (You do not have to solve for the parameters.)

$$\frac{3x+16}{(x+5)^3} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3}$$

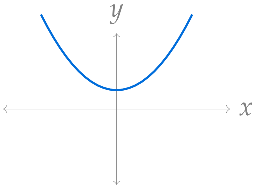
$$\frac{-2x-10}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

Irreducible Quadratic Factors

Sometimes it's not possible to factor our denominator into linear factors with real terms.



If a quadratic function has real roots a and b (possibly $a = b$, possibly $a \neq b$), then we can write it as $c(x-a)(x-b)$ for some constant c.



If a quadratic function has no real roots, then it can't be factored into (real) linear factors. It is irreducible.

IRREDUCIBLE QUADRATIC FACTORS

When the denominator has an irreducible quadratic factor x^2+bx+c, we add a term $\dfrac{Ax+B}{x^2+bx+c}$ to our composition. (The degree of the numerator must still be smaller than the degree of the denominator.) Write out the form of the partial fraction decomposition (but do not solve for the parameters):

- $$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

- $$\frac{3x^2-x+5}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

The purpose of the partial fraction decomposition is to end up with functions that we can integrate.

- Recall: $\displaystyle\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan x + C.$

- Evaluate: $\displaystyle\int \frac{1}{(x+1)^2+1}\,\mathrm{d}x$

$u = x+1$, $\mathrm{d}u = \mathrm{d}x$:

$$\int \frac{1}{u^2+1}\,\mathrm{d}u = \arctan u + C = \arctan(x+1) + C$$

Evaluate $\displaystyle\int \frac{4}{(3x+8)^2+9}\,\mathrm{d}x$

$$= \int \frac{4}{9\left(\frac{(3x+8)^2}{9}+1\right)}\mathrm{d}x$$

$$= \frac{4}{9}\int \frac{1}{\left(\frac{3x+8}{3}\right)^2+1}\,\mathrm{d}x$$

$$= \frac{4}{9}\int \frac{1}{\left(x+\frac{8}{3}\right)^2+1}\,\mathrm{d}x$$

$$u = x+\frac{8}{3}, \quad \mathrm{d}u = \mathrm{d}x \qquad = \frac{4}{9}\int \frac{1}{u^2+1}\,\mathrm{d}u$$

$$= \frac{4}{9}\arctan u + C$$

$$= \frac{4}{9}\arctan\left(x+\frac{8}{3}\right)+C$$

Check Our Work

We found $\displaystyle\int \frac{4}{(3x+8)^2+9}\,\mathrm{d}x =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{4}{9}\arctan\left(x+\frac{8}{3}\right)+C\right\} = \frac{4}{9}\cdot\frac{1}{\left(x+\frac{8}{3}\right)^2+1}$$

$$= \frac{4}{9\left(\left(x+\frac{8}{3}\right)^2+1\right)}$$

$$= \frac{4}{3^2\left(x+\frac{8}{3}\right)^2+9}$$

$$= \frac{4}{(3x+8)^2+9}$$

So, our answer works.

Evaluate $\displaystyle\int \frac{x+1}{x^2+2x+2}\,\mathrm{d}x$.

(Hint: start by completing the square.)

$$= \int \frac{x+1}{(x+1)^2+1}\,\mathrm{d}x$$

Let $y = x+1$, $\mathrm{d}y = \mathrm{d}x$:
$$= \int \frac{y}{y^2+1}\,\mathrm{d}y$$

Let $u = y^2+1$, $\mathrm{d}u = 2y\,\mathrm{d}y$:
$$= \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u$$

$$= \frac{1}{2}\log|u| + C$$

$$= \frac{1}{2}\log\left|y^2+1\right| + C$$

$$= \frac{1}{2}\log\left|(x+1)^2+1\right| + C$$

Check Our Work

We found $\displaystyle\int \frac{x+1}{x^2+2x+2}\,\mathrm{d}x =$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{2}\log\left|(x+1)^2+1\right|+C\right\} &= \frac{1}{2}\cdot\frac{2(x+1)}{(x+1)^2+1} \\
&= \frac{x+1}{(x+1)^2+1} \\
&= \frac{x+1}{x^2+2x+2}
\end{aligned}$$

So, our answer works.

These rules work only when the degree of the numerator is less than the degree of the denominator.

$$\int \frac{x^3}{(x-2)^2(x-3)(x-4)^2}\,\mathrm{d}x \qquad \int \frac{x^5}{(x-2)^2(x-3)(x-4)^2}\,\mathrm{d}x$$

If the degree of the numerator is too large, we use polynomial long division.

Evaluate $\int \frac{8x^2+22x+23}{2x+3}\,\mathrm{d}x$.

$$\begin{array}{r@{}l} & \;4x\quad +5 \\ \cline{2-2} 2x+3 & \big)\; 8x^2+22x+23 \\ & \;-8x^2-12x \\ \cline{2-2} & \qquad\quad 10x+23 \\ & \qquad\; -10x-15 \\ \cline{2-2} & \qquad\qquad\qquad 8 \end{array}$$

So,

$$\frac{8x^2+22x+23}{2x+3} = 4x+5+\frac{8}{2x+3}$$

$$\int \frac{8x^2+22x+23}{2x+3}\,\mathrm{d}x = 2x^2+5x+4\log|2x+3|+C$$

Check Our Work

We computed

$$\int \frac{8x^2+22x+23}{2x+3}\,\mathrm{d}x =$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}&\left\{2x^2+5x+4\log|2x+3|+C\right\}\\
&=4x+5+\frac{8}{2x+3}\\
&=\frac{4x(2x+3)+5(2x+3)+8}{2x+3}\\
&=\frac{8x^2+12x+10x+15+8}{2x+3}\\
&=\frac{8x^2+22x+23}{2x+3}
\end{aligned}$$

So, our solution works.

Evaluate $\int \frac{3x^3+x+3}{x-2}\,\mathrm{d}x$.

$$
\begin{array}{r@{}l}
& \;3x^2 + 6x + 13 \\
\cline{2-2}
x-2 \big) & \;3x^3 \qquad\qquad + x + 3 \\
& -3x^3 + 6x^2 \\
\cline{2-2}
& \qquad\quad 6x^2 + x \\
& \qquad -6x^2 + 12x \\
\cline{2-2}
& \qquad\qquad\quad 13x + 3 \\
& \qquad\qquad -13x + 26 \\
\cline{2-2}
& \qquad\qquad\qquad\quad 29
\end{array}
$$

So,

$$
\begin{aligned}
\int \frac{3x^3+x+3}{x-2}\,\mathrm{d}x &= \int \left(3x^2+6x+13+\frac{29}{x-2}\right)\,\mathrm{d}x \\
&= x^3+3x^2+13x+29\log|x-2|+C
\end{aligned}
$$

Check Our Work

We found

$$\int \frac{3x^3+x+3}{x-2}\,\mathrm{d}x =$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^3+3x^2+13x+29\log|x-2|+C\right\}\\
&\quad= 3x^2+6x+13+\frac{29}{x-2}\\
&\quad= \frac{3x^2(x-2)+6x(x-2)+13(x-2)+29}{x-2}\\
&\quad= \frac{3x^3-6x^2+6x^2-12x+13x-26+29}{x-2}\\
&\quad= \frac{3x^3+x+3}{x-2}
\end{aligned}$$

Evaluate $\displaystyle\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x$.

$$\begin{array}{r|l} & \quad 3 \\ \hline x^2+5x & 3x^2 \qquad\qquad +1 \\ & -3x^2 - 15x \\ \hline & \qquad -15x + 1 \end{array}$$

So, $\displaystyle\frac{3x^2+1}{x^2+5x} = 3 + \frac{-15x+1}{x^2+5x}$

Now, we can use partial fraction decomposition.

$$\frac{-15x+1}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5} = \frac{(A+B)x+5A}{x(x+5)}$$

$$A = \frac{1}{5}, \quad B = -15 - \frac{1}{5} = -\frac{76}{5}$$

$$\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x = \int \left(3 + \frac{1/5}{x} - \frac{76/5}{x+5}\right)\,\mathrm{d}x$$

$$= 3x + \frac{1}{5}\log|x| - \frac{76}{5}\log|x+5| + C$$

Check Our Work

We found $\displaystyle\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{3x + \frac{1}{5}\log|x| - \frac{76}{5}\log|x+5| + C\right\}$$
$$= 3 + \frac{1}{5x} - \frac{76}{5(x+5)}$$
$$= 3\left(\frac{5x(x+5)}{5x(x+5)}\right) + \frac{1}{5x}\left(\frac{x+5}{x+5}\right) - \frac{76}{5(x+5)}\left(\frac{x}{x}\right)$$
$$= \frac{(15x^2+75x)+(x+5)-(76x)}{5x(x+5)}$$
$$= \frac{15x^2+5}{5x(x+5)} = \frac{3x^2+1}{x^2+5x}$$

So, our solution works.

Factoring

$$P(x) = x^3 + 2x^2 - 5x - 6$$

- To start, let's guess a root.
  - Since $P(x)$ has integer coefficients, any integer root must divide 6 exactly.
  - So the only possible integer roots are ±1, ±2, ±3, and ±6. We'll try each until one works.
    - $P(1) = -8 \neq 0 \implies 1$ is not a root
    - $P(-1) = 0 \implies$ -1 is a root. Therefore, $(x+1)$ is a factor.
- Long division gives the rest:

$$\begin{array}{r|l}
 & x^2 + x - 6 \\ \hline
x+1 & x^3 + 2x^2 - 5x - 6 \\
 & -x^3 - x^2 \\ \hline
 & x^2 - 5x \\
 & -x^2 - x \\ \hline
 & -6x - 6 \\
 & 6x + 6 \\ \hline
 & 0
\end{array}$$

$$P(x) = (x+1)(x^2 + x - 6) = (x+1)(x-2)(x+3)$$

Factoring

$$P(x) = 2x^3 - 3x^2 + 4x - 6$$

Notice that the first two terms and the last two terms have the same ratios: $\frac{2x^3}{-3x^2} = \frac{2x}{-3} = \frac{4x}{-6}$. So, we can factor $2x - 3$ out of both pairs.

$$\begin{aligned} P(x) &= 2x^3 - 3x^2 \qquad && +4x - 6 \\ &= (2x-3)(x^2) && +(2x-3)(2) \\ &= (2x-3)(x^2+2) \end{aligned}$$