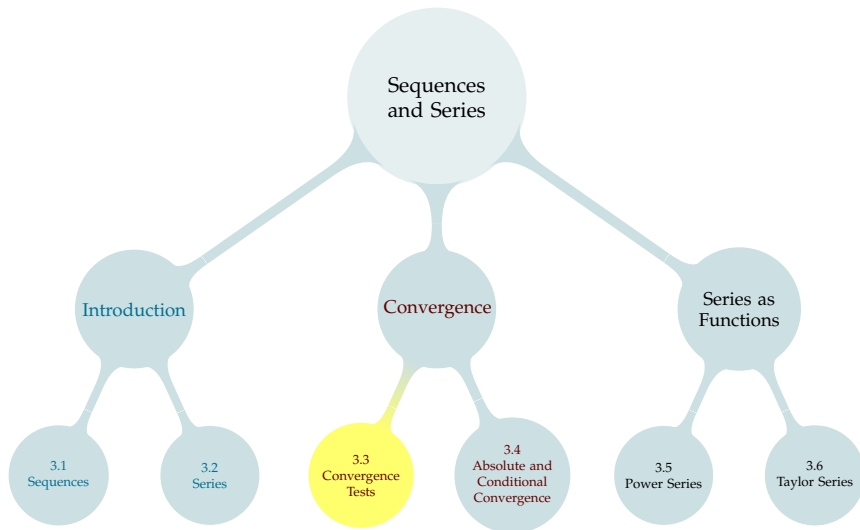


# TABLE OF CONTENTS



# REVIEW

$$\text{Let } S_N = \sum_{n=1}^N a_n.$$

Simplify:  $S_N - S_{N-1}$ .

(This will come in handy soon.)

# REVIEW

$$\text{Let } S_N = \sum_{n=1}^N a_n.$$

Simplify:  $S_N - S_{N-1}$ .

(This will come in handy soon.)

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N$$

$$S_{N-1} = a_1 + a_2 + a_3 + \cdots + a_{N-1}$$

# ALTERNATING SERIES

## Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

is alternating if every  $A_n \geq 0$ .

Alternating series:

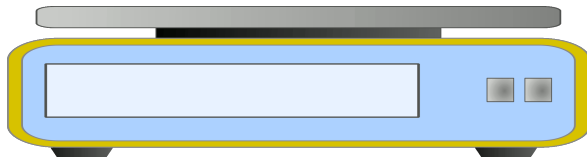
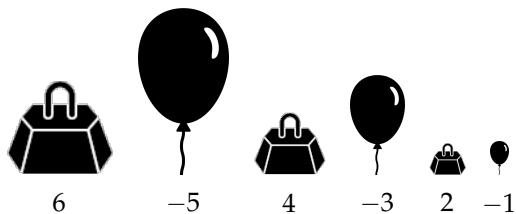
►  $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$

►  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

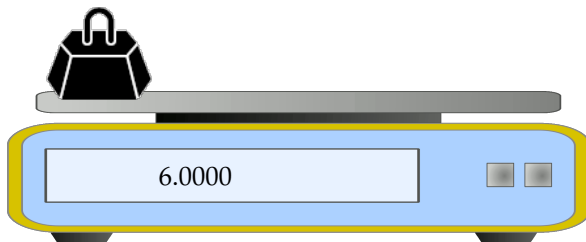
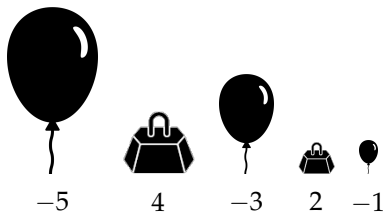
Not alternating:

►  $\cos(1) + \cos(2) + \cos(3) + \cdots$

►  $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$

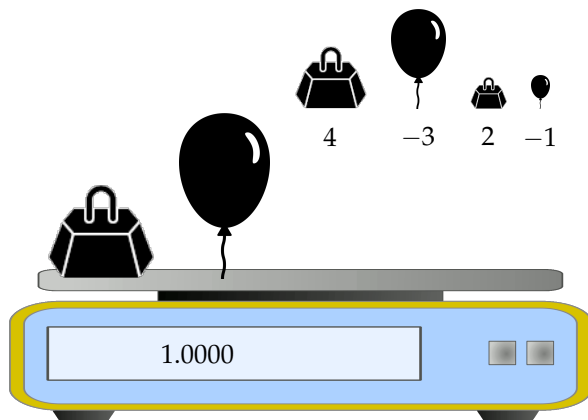


$$S_1 = 6.0000$$



$$S_1 = 6.0000$$

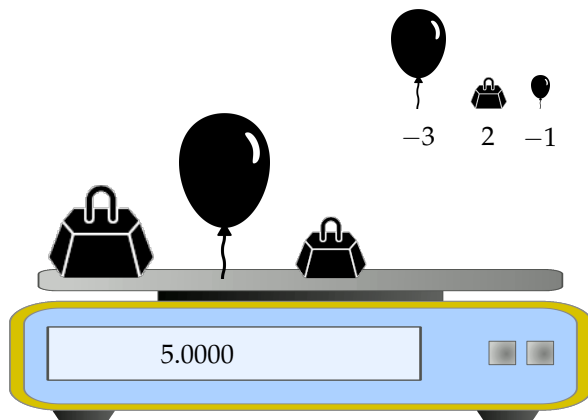
$$S_2 = 1.0000$$



$$S_1 = 6.0000$$

$$S_2 = 1.0000$$

$$S_3 = 5.0000$$



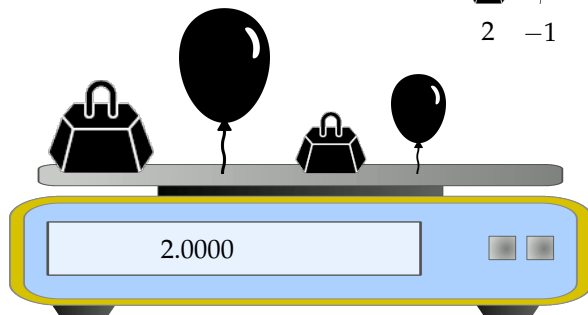


$$S_1 = 6.0000$$

$$S_2 = 1.0000$$

$$S_3 = 5.0000$$

$$S_4 = 2.0000$$



$$S_1 = 6.0000$$

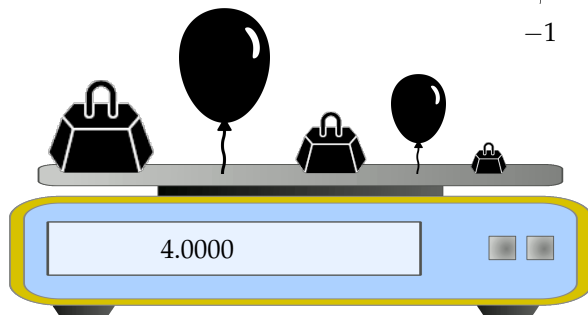
$$S_2 = 1.0000$$

$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

●  
-1



$$S_1 = 6.0000$$

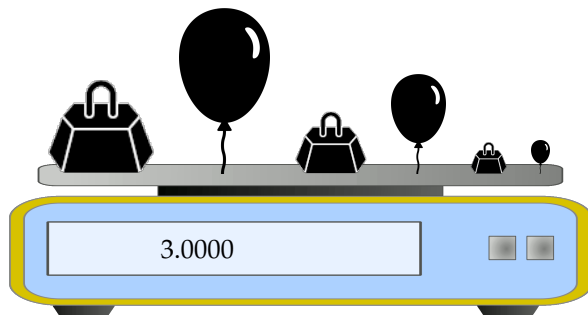
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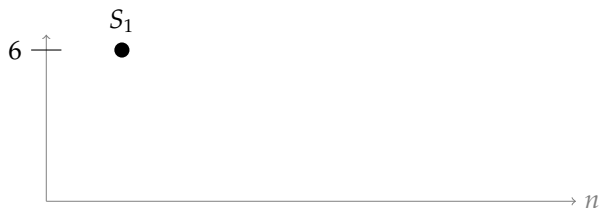
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

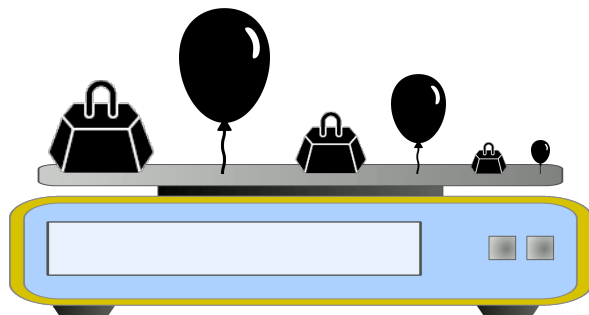
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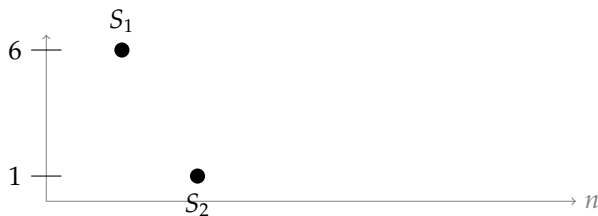
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$$S_4 = 2.0000$$

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$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

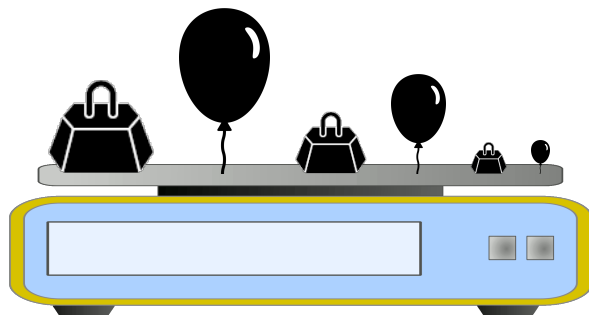
$$S_2 = 1.0000$$

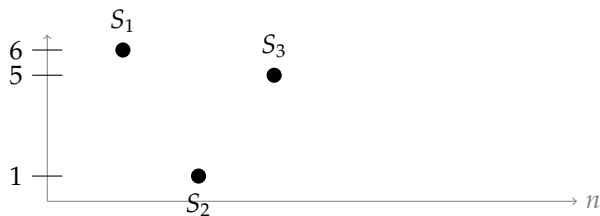
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$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

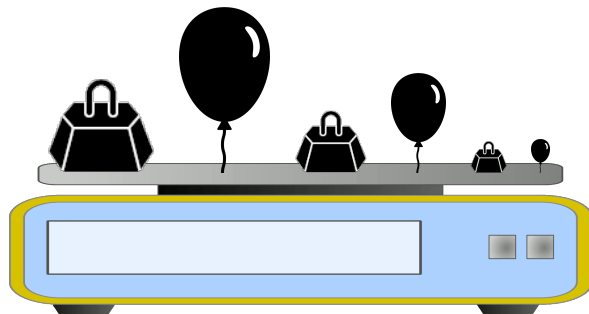
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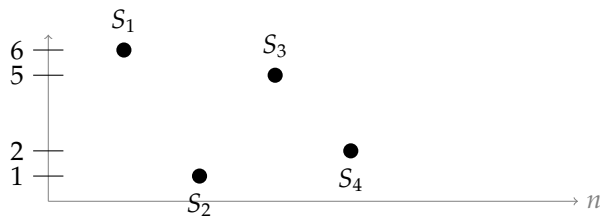
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$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

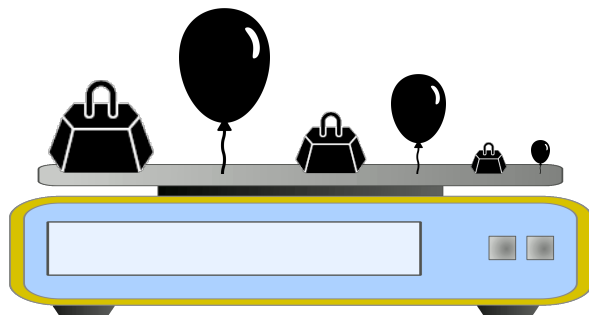
$$S_2 = 1.0000$$

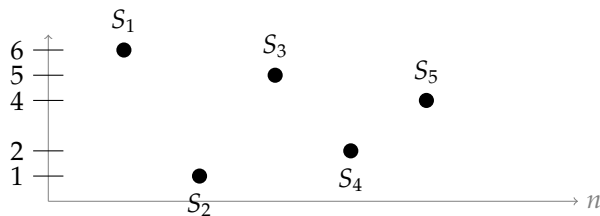
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

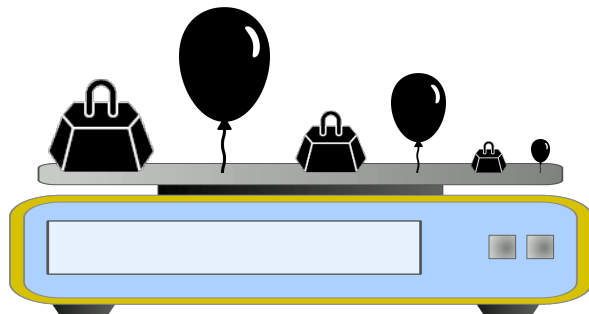
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$$S_3 = 5.0000$$

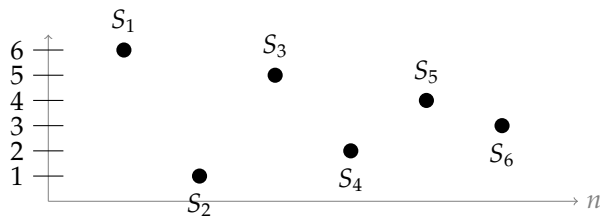
$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$







$$S_1 = 6.0000$$

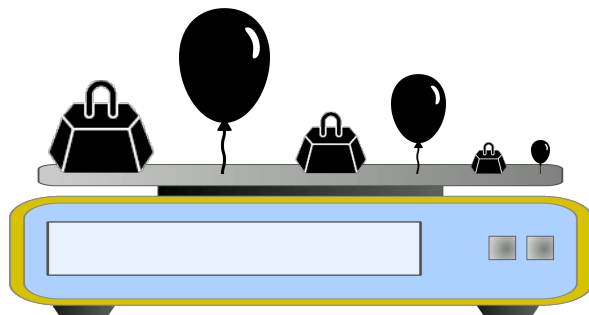
$$S_2 = 1.0000$$

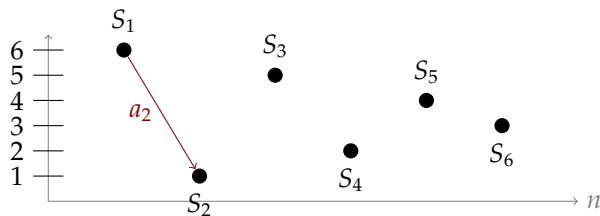
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

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$$S_1 = 6.0000$$

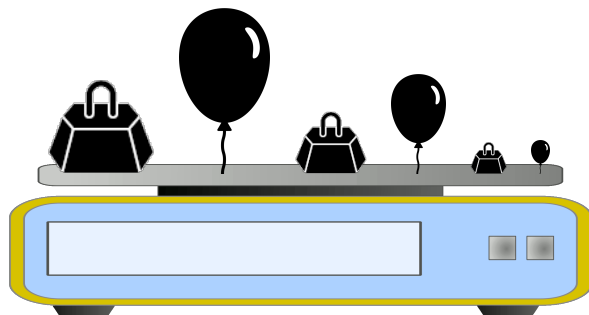
$$S_2 = 1.0000$$

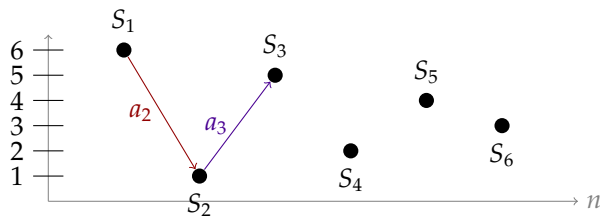
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

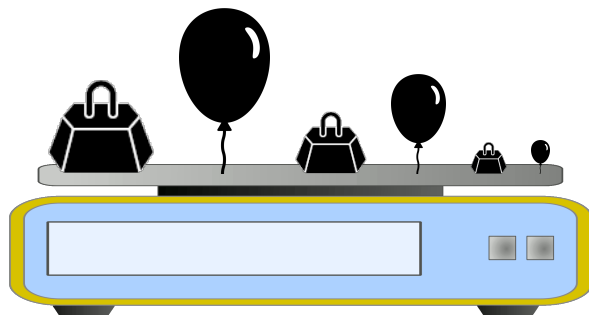
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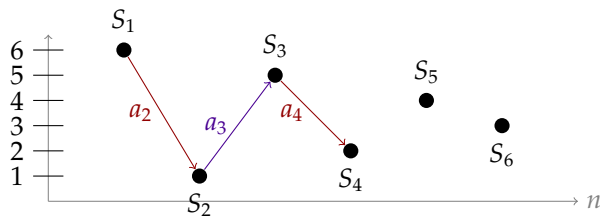
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

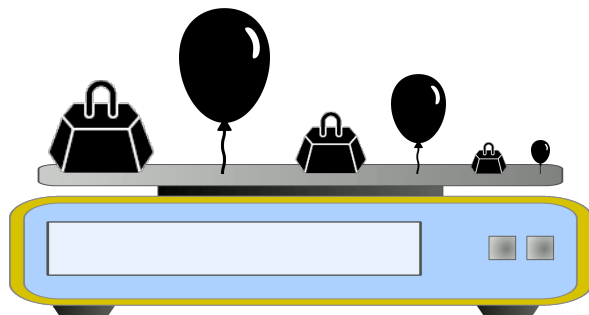
$$S_2 = 1.0000$$

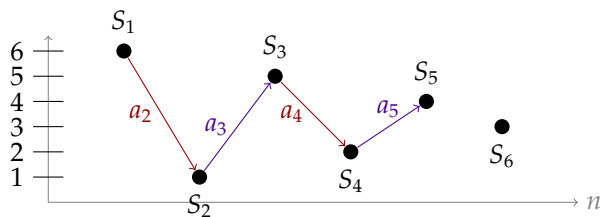
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

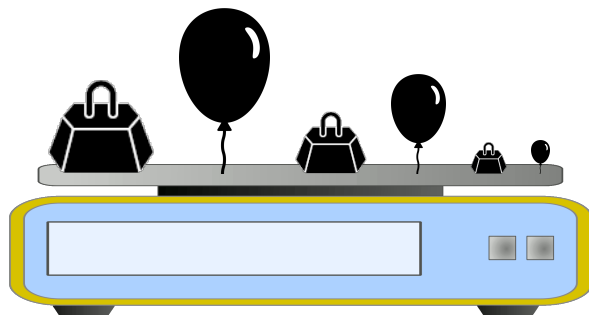
$$S_2 = 1.0000$$

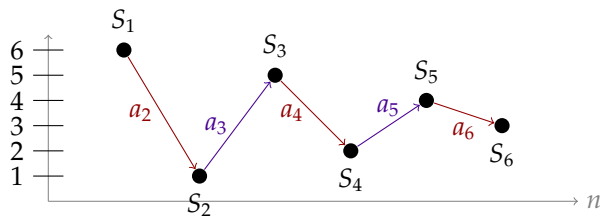
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$





$$S_1 = 6.0000$$

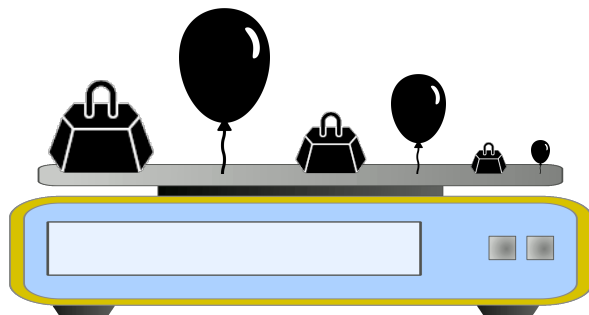
$$S_2 = 1.0000$$

$$S_3 = 5.0000$$

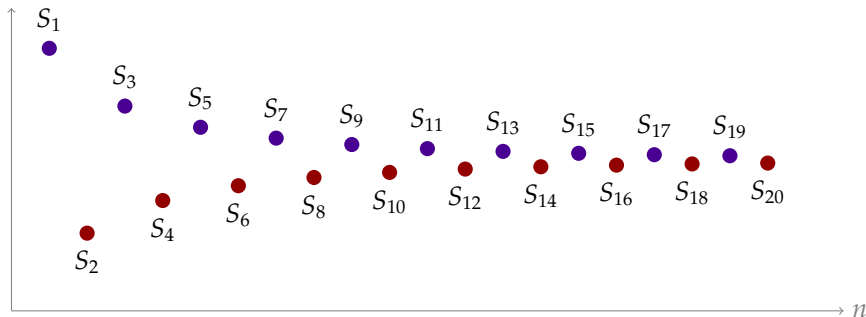
$$S_4 = 2.0000$$

$$S_5 = 4.0000$$

$$S_6 = 3.0000$$

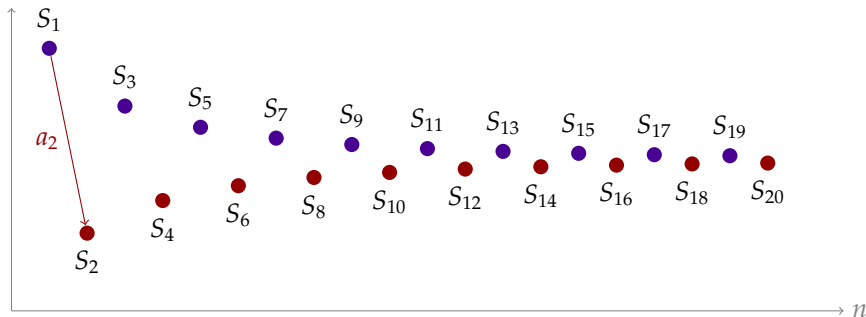


Consider an alternating series  $a_1 - a_2 + a_3 - a_4 + \cdots$ , where  $\{a_n\}$  is a sequence with positive, **decreasing** terms and with  $\lim_{n \rightarrow \infty} a_n = 0$ .



Since  $a_2 > a_3$ , we have  $a_1 - (a_2 - a_3) < a_1$ , so  $S_3 < S_1$ .

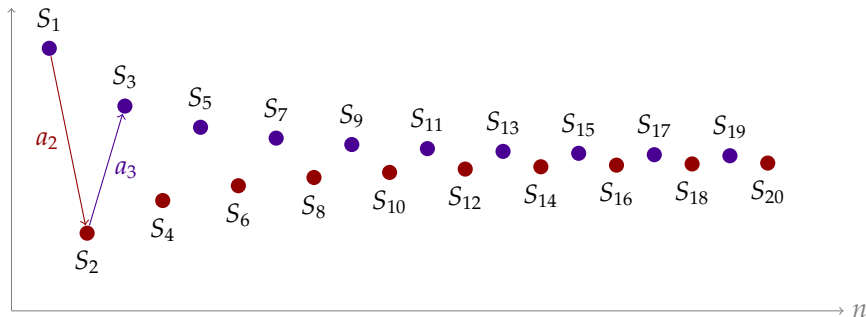
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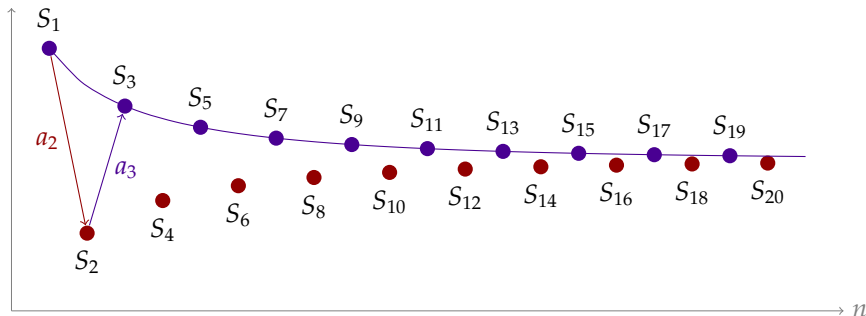


Consider an alternating series  $a_1 - a_2 + a_3 - a_4 + \cdots$ , where  $\{a_n\}$  is a sequence with positive, **decreasing** terms and with  $\lim_{n \rightarrow \infty} a_n = 0$ .



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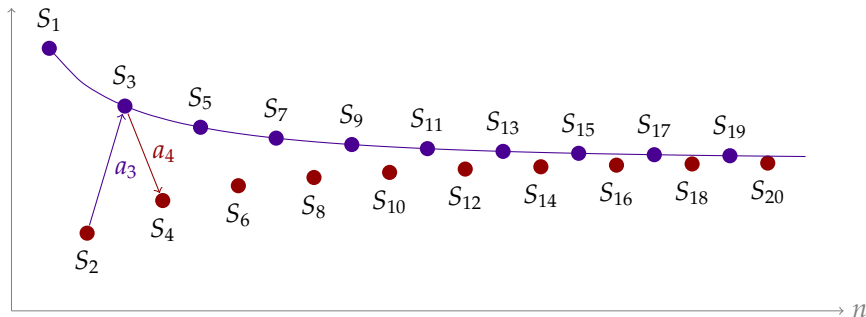
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Since  $a_2 > a_3$ , we have  $a_1 - (a_2 - a_3) < a_1$ , so  $S_3 < S_1$ .

Odd-indexed partial sums are decreasing.

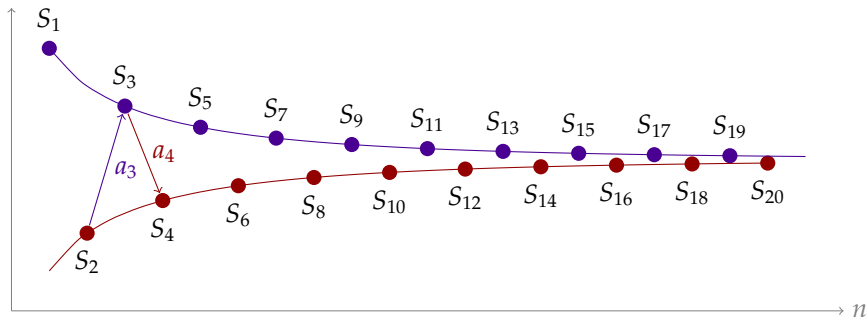
Consider an alternating series  $a_1 - a_2 + a_3 - a_4 + \cdots$ , where  $\{a_n\}$  is a sequence with positive, **decreasing** terms and with  $\lim_{n \rightarrow \infty} a_n = 0$ .



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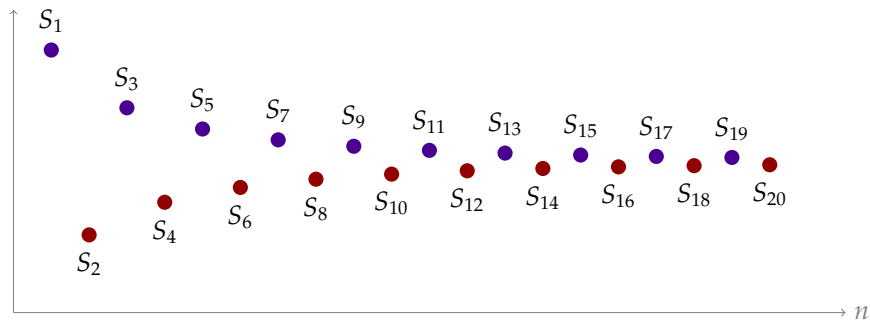


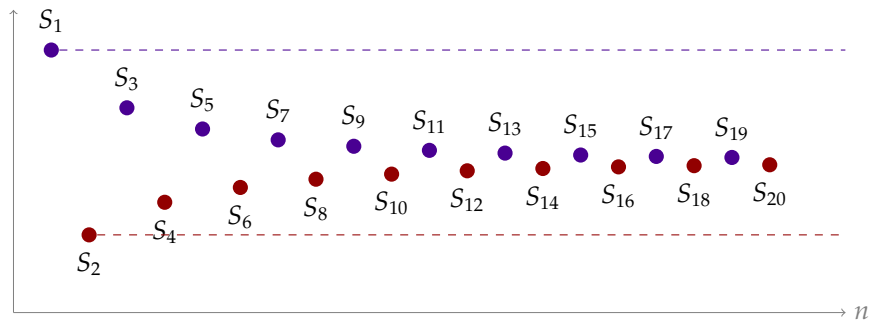
Since  $a_2 > a_3$ , we have  $a_1 - (a_2 - a_3) < a_1$ , so  $S_3 < S_1$ .

Odd-indexed partial sums are decreasing.

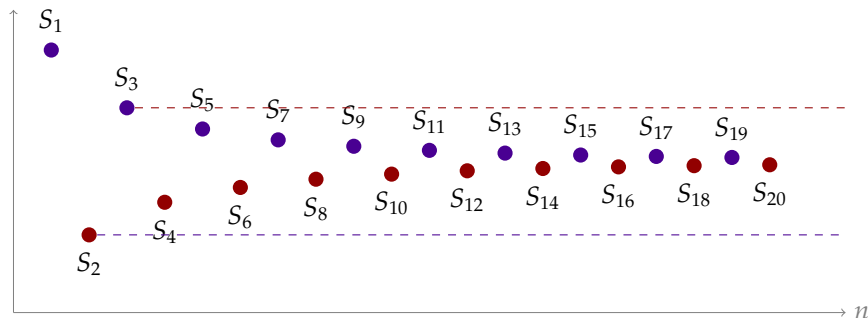
Since  $a_3 > a_4$ , we have  $a_1 - a_2 + (a_3 - a_4) > a_1 - a_2$ , so  $S_4 > S_2$ .

Even-indexed partial sums are increasing.

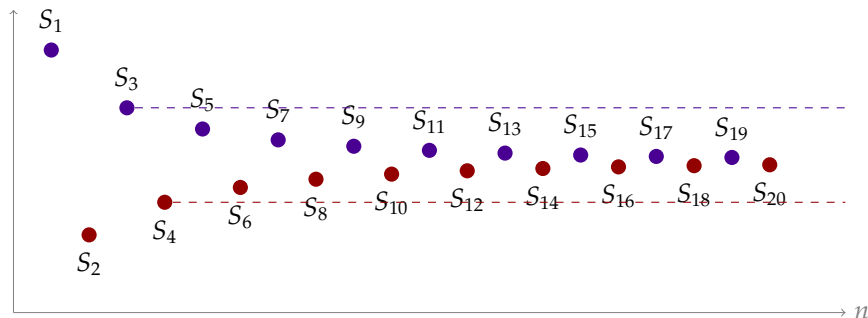




► For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .

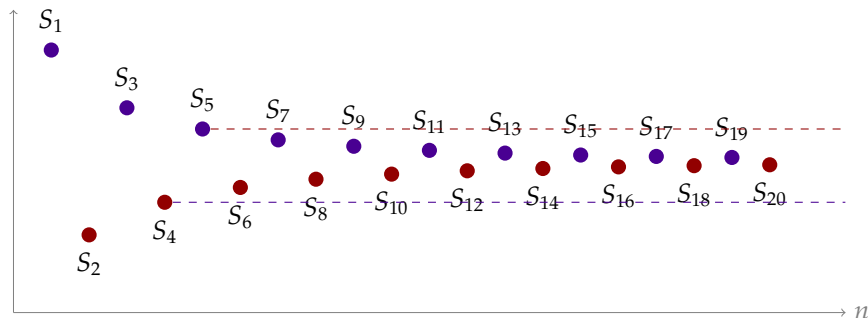


- For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .
- For all  $n \geq 3$ ,  $S_n$  lies between  $S_2$  and  $S_3$ .

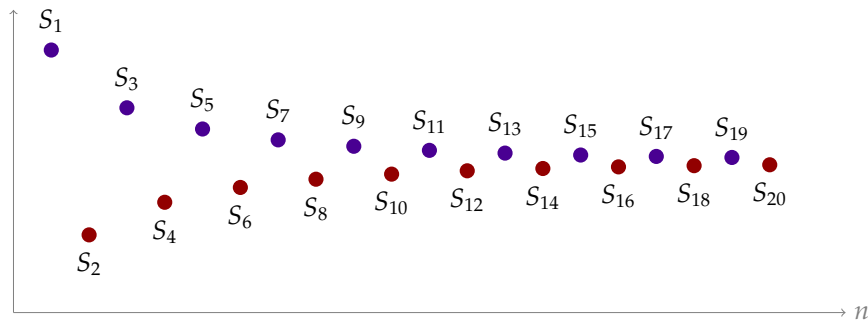


- ▶ For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .
- ▶ For all  $n \geq 3$ ,  $S_n$  lies between  $S_2$  and  $S_3$ .
- ▶ For all  $n \geq 4$ ,  $S_n$  lies between  $S_3$  and  $S_4$ .



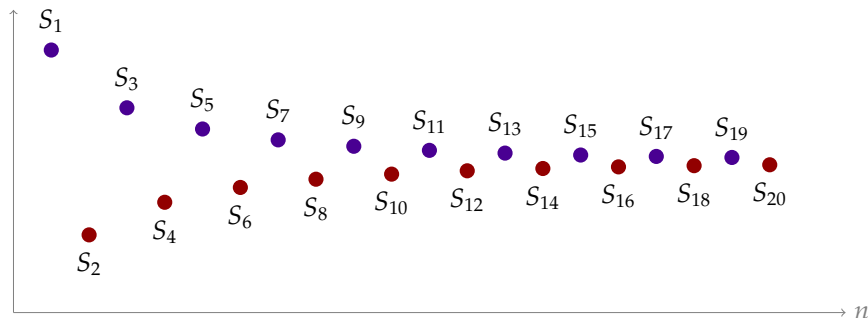


- ▶ For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .
- ▶ For all  $n \geq 3$ ,  $S_n$  lies between  $S_2$  and  $S_3$ .
- ▶ For all  $n \geq 4$ ,  $S_n$  lies between  $S_3$  and  $S_4$ .
- ▶ For all  $n \geq 5$ ,  $S_n$  lies between  $S_4$  and  $S_5$ .



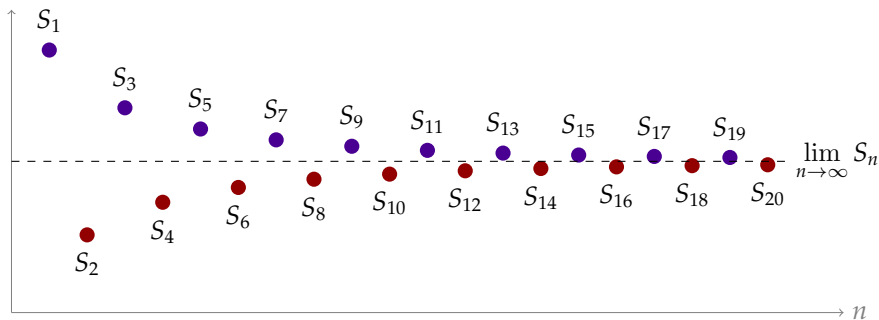
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- ▶ For all  $n \geq 5$ ,  $S_n$  lies between  $S_4$  and  $S_5$ .

The difference between consecutive sums  $S_n$  and  $S_{n-1}$  is:



- ▶ For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .
- ▶ For all  $n \geq 3$ ,  $S_n$  lies between  $S_2$  and  $S_3$ .
- ▶ For all  $n \geq 4$ ,  $S_n$  lies between  $S_3$  and  $S_4$ .
- ▶ For all  $n \geq 5$ ,  $S_n$  lies between  $S_4$  and  $S_5$ .

The difference between consecutive sums  $S_n$  and  $S_{n-1}$  is:  
 $|a_n|$ , which approaches 0.



- For all  $n \geq 2$ ,  $S_n$  lies between  $S_1$  and  $S_2$ .
- For all  $n \geq 3$ ,  $S_n$  lies between  $S_2$  and  $S_3$ .
- For all  $n \geq 4$ ,  $S_n$  lies between  $S_3$  and  $S_4$ .
- For all  $n \geq 5$ ,  $S_n$  lies between  $S_4$  and  $S_5$ .

The difference between consecutive sums  $S_n$  and  $S_{n-1}$  is:  
 $|a_n|$ , which approaches 0.

## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys

- (i)  $a_n \geq 0$  for all  $n \geq 1$ ;
- (ii)  $a_{n+1} \leq a_n$  for all  $n \geq 1$  (i.e. the sequence is monotone decreasing);
- (iii) and  $\lim_{n \rightarrow \infty} a_n = 0$ .

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number  $N$ ,  $S - S_N$  is between 0 and (the first dropped term)  $(-1)^N a_{N+1}$ . Here  $S_N$  is, as previously, the  $N^{\text{th}}$  partial sum  $\sum_{n=1}^N (-1)^{n-1} a_n$ .

## Alternating Series Test (abridged)

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys

- (i)  $a_n \geq 0$  for all  $n \geq 1$ ;
- (ii)  $a_{n+1} \leq a_n$  for all  $n \geq 1$  (i.e. the sequence is monotone decreasing);
- (iii) and  $\lim_{n \rightarrow \infty} a_n = 0$ .

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

► True or false: the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  converges.

► True or false: the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges.

Let  $a_n = \frac{1}{n}$ .

Let  $a_n = \frac{1}{n}$ .

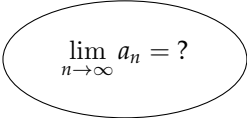
(i)  $a_n \geq 0$

(ii)  $a_{n+1} \leq a_n$

(iii)  $\lim_{n \rightarrow \infty} a_n = 0$



# DIVERGENCE TEST + ALTERNATING SERIES TEST


$$\lim_{n \rightarrow \infty} a_n = ?$$

# DIVERGENCE TEST + ALTERNATING SERIES TEST

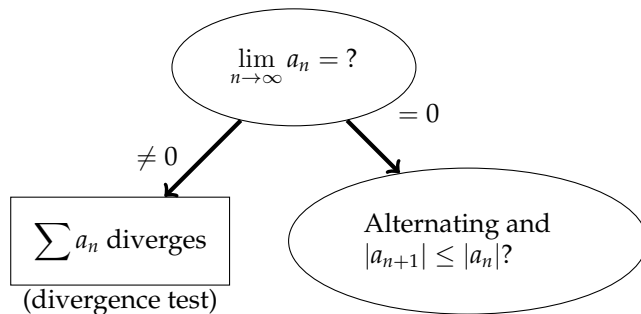
$$\lim_{n \rightarrow \infty} a_n = ?$$

$\neq 0$

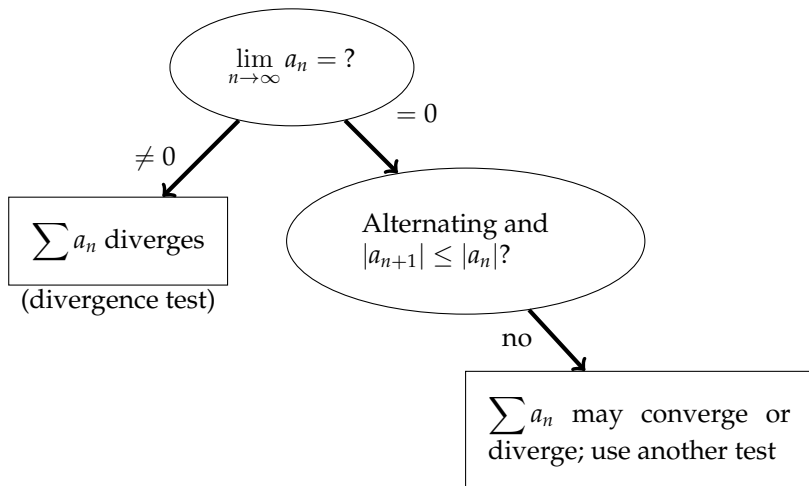
$$\sum a_n \text{ diverges}$$

(divergence test)

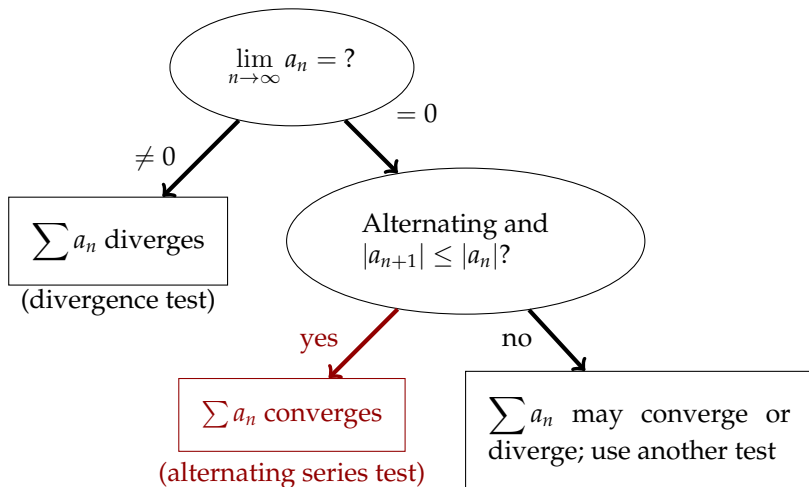
# DIVERGENCE TEST + ALTERNATING SERIES TEST



# DIVERGENCE TEST + ALTERNATING SERIES TEST



# DIVERGENCE TEST + ALTERNATING SERIES TEST



## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys  $a_n \geq 0$  for all  $n \geq 1$ ;  $a_{n+1} \leq a_n$  for all  $n \geq 1$ ; and  $\lim_{n \rightarrow \infty} a_n = 0$ . Then  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$  converges and  $S - S_N$  is between 0 and  $(-1)^N a_{N+1}$ .

Using a computer, you find  $\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698$ .

How close is that to the value  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ ?



## Alternating Series Test

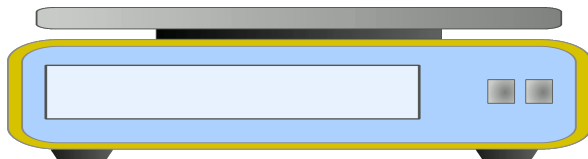
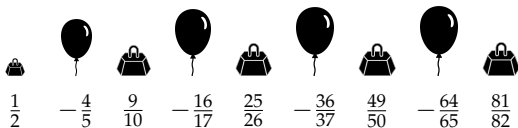
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Using a computer, you find  $\sum_{n=1}^{19} (-1)^{n-1} \frac{n^2}{n^2 + 1} \approx 0.6347$ .

How close is that to the value  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2 + 1}$ ?











$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

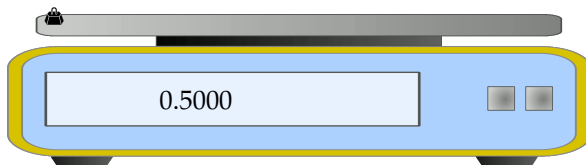




$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

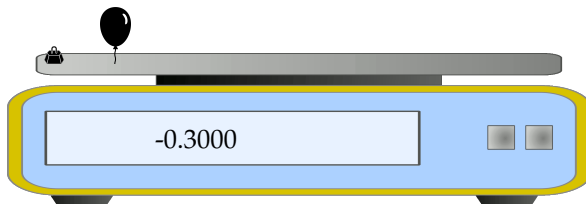
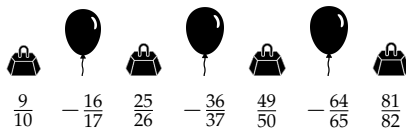
							
$-\frac{4}{5}$	$\frac{9}{10}$	$-\frac{16}{17}$	$\frac{25}{26}$	$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$









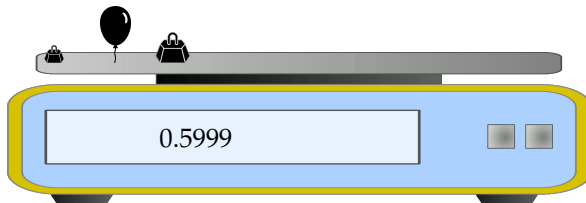
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

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$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

					
$-\frac{16}{17}$	$\frac{25}{26}$	$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$



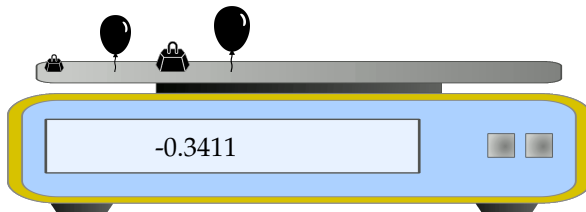
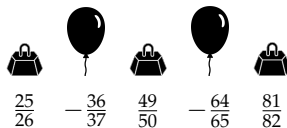
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

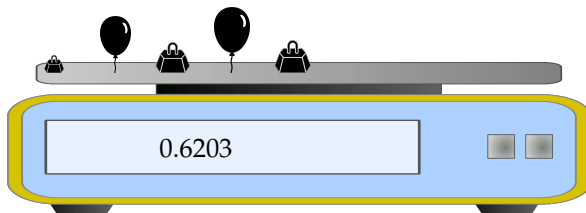
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

●	●	●	●
$-\frac{36}{37}$	$\frac{49}{50}$	$-\frac{64}{65}$	$\frac{81}{82}$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

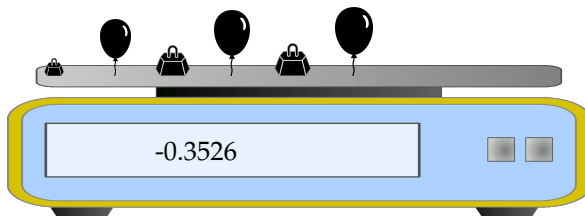
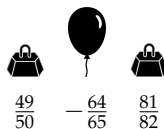
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

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$$S_6 = -0.3526$$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

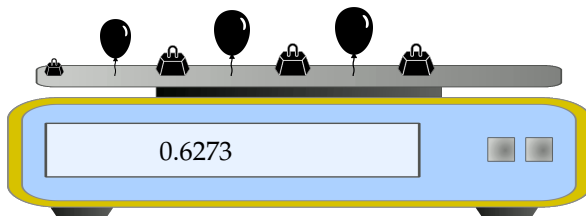
$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$$-\frac{64}{65} \quad \frac{81}{82}$$



$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

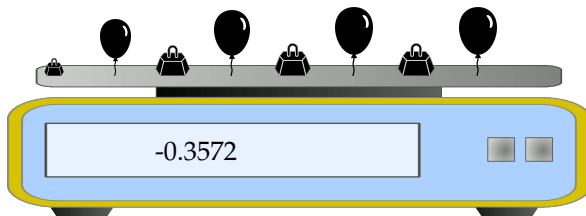
$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

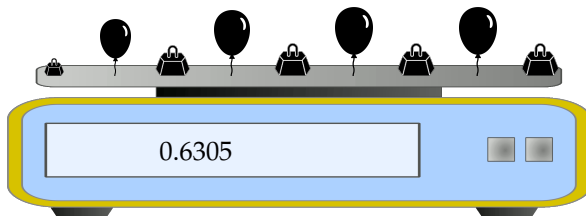
$$S_8 = -0.3572$$



$$\frac{81}{82}$$




$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$



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$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

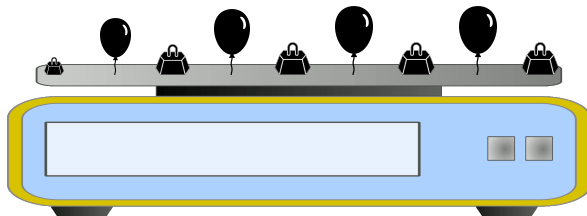
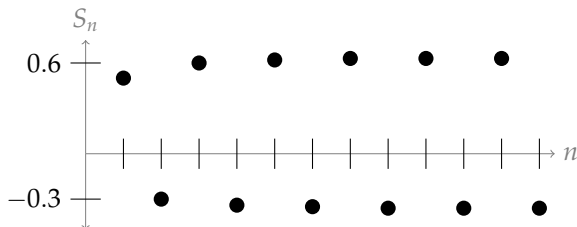
$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$$S_8 = -0.3572$$

$$S_9 = 0.6305$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+1} \text{ DIVERGES}$$



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$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$$S_8 = -0.3572$$

$$S_9 = 0.6305$$

Recall for a geometric series, the **ratios of consecutive terms** is constant.

$$\times \frac{1}{2}$$


$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \dots$$

If that ratio has magnitude **less than one**, then the series converges.  
If the ratio has magnitude **greater than one**, the series diverges.

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$$\begin{array}{c} \times \frac{1}{2} \quad \times \frac{1}{2} \\ \curvearrowright \quad \curvearrowright \end{array}$$

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 \times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2} \\
 \frown \quad \frown \quad \frown \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots
 \end{array}$$

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$$\begin{array}{ccccccc}
 & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} \\
 & \frown & & \frown & & \frown & & \frown \\
 & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
 \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} \cdots
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$$\begin{array}{c}
 \begin{array}{ccccccc}
 & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} \\
 \frown & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow \\
 & & & & & & & 
 \end{array} \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \underbrace{\hspace{1.5cm}} \\
 \frac{1/4}{1/2} =
 \end{array}$$

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 \times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2} \\
 \curvearrowright \quad \curvearrowright \quad \curvearrowright \quad \curvearrowright \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \\
 \frac{1/4}{1/2} = \frac{1/8}{1/4} =
 \end{array}$$

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$$\begin{array}{c}
 \begin{array}{ccccccc}
 & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} \\
 \frown & \rightarrow & \frown & \rightarrow & \frown & \rightarrow & \frown & \rightarrow
 \end{array} \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \begin{array}{c}
 \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \\
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$$\begin{array}{c}
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 \\
 \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots \\
 \\
 \underbrace{\quad} \quad \underbrace{\quad} \quad \underbrace{\quad} \quad \underbrace{\quad} \\
 \frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} = \frac{1/32}{1/16} = \frac{1}{2}
 \end{array}$$

If that ratio has magnitude **less than one**, then the series converges.  
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For series convergence, we are concerned with what happens to terms  $a_n$  when  $n$  is sufficiently large.

Suppose for a sequence  $a_n$ ,  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$  for some constant  $L$ .

$$\underbrace{a_n + a_{n+1}}_{\frac{a_{n+1}}{a_n} \approx} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots$$

Like in a geometric series:

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$$\begin{array}{ccccccc}
 a_n & + & a_{n+1} & + & a_{n+2} & + & a_{n+3} & + & a_{n+4} & + & \cdots \\
 \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & & & & & & & \\
 \frac{a_{n+1}}{a_n} & \approx & \frac{a_{n+2}}{a_{n+1}} & \approx & & & & & & & 
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 \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & & & & & \\
 \frac{a_{n+1}}{a_n} & \approx & \frac{a_{n+2}}{a_{n+1}} & \approx & \frac{a_{n+3}}{a_{n+2}} & \approx & & & & & 
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$$\begin{array}{ccccccc} a_n & + & a_{n+1} & + & a_{n+2} & + & a_{n+3} & + & a_{n+4} & + & \cdots \\ \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \\ \frac{a_{n+1}}{a_n} & \approx & \frac{a_{n+2}}{a_{n+1}} & \approx & \frac{a_{n+3}}{a_{n+2}} & \approx & \frac{a_{n+4}}{a_{n+3}} & \approx & \frac{a_{n+5}}{a_{n+4}} & \approx & L \end{array}$$

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## Ratio Test

Let  $N$  be any positive integer and assume that  $a_n \neq 0$  for all  $n \geq N$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

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Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

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The series we just considered,  $\sum_{n=1}^{\infty} \frac{n}{3^n}$ , looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

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► Integral test:

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- ▶ Integral test:  $\int \frac{x}{3^x} dx$  can be evaluated using integration by parts.
- ▶ Comparison test:

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- ▶ Integral test:  $\int \frac{x}{3^x} dx$  can be evaluated using integration by parts.
- ▶ Comparison test:
  - ▶  $\sum \frac{1}{3^n}$  is not a valid comparison series, nor is  $\sum n$ .



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  - ▶ Because  $n < 2^n$  for all  $n \geq 1$ , the series  $\sum \left(\frac{2}{3}\right)^n$  will work.
- ▶ The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$



$$\frac{1}{3}$$



$$\frac{2}{3^2}$$



$$\frac{3}{3^3}$$



$$\frac{4}{3^4}$$



$$\frac{5}{3^5}$$



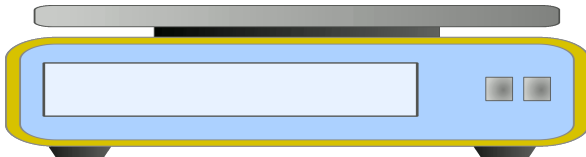
$$\frac{6}{3^6}$$



$$\frac{7}{3^7}$$

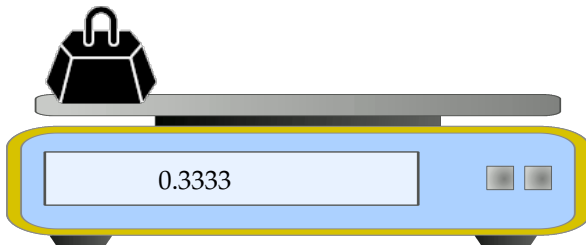


$$\frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

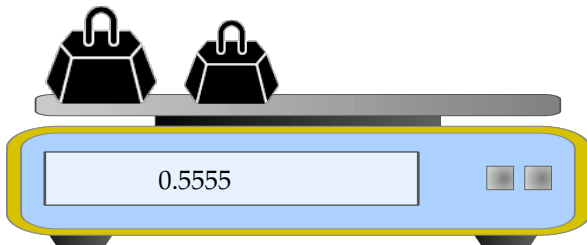
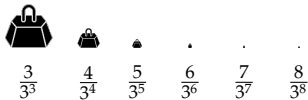
$$S_1 = 0.3333$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$








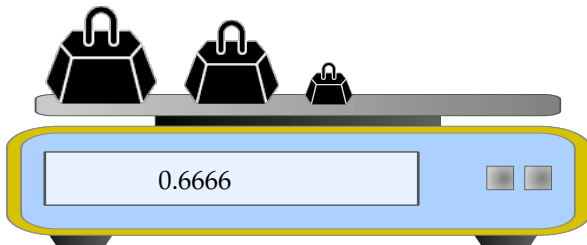
$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

				
$\frac{4}{3^4}$	$\frac{5}{3^5}$	$\frac{6}{3^6}$	$\frac{7}{3^7}$	$\frac{8}{3^8}$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

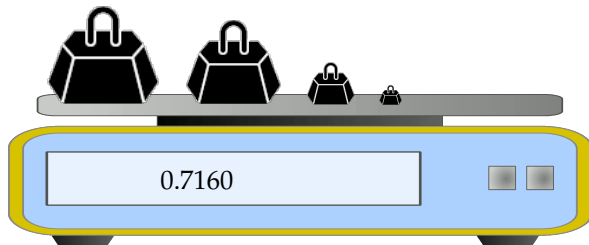
$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$\frac{5}{3^5} \quad \frac{6}{3^6} \quad \frac{7}{3^7} \quad \frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

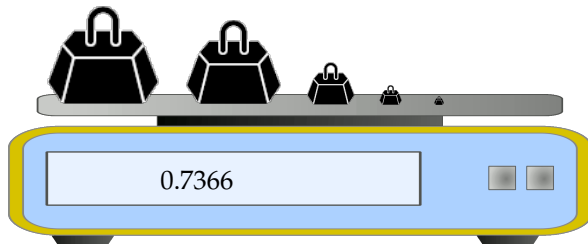
$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \frac{6}{3^6} & \frac{7}{3^7} & \frac{8}{3^8} \end{array}$$





$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

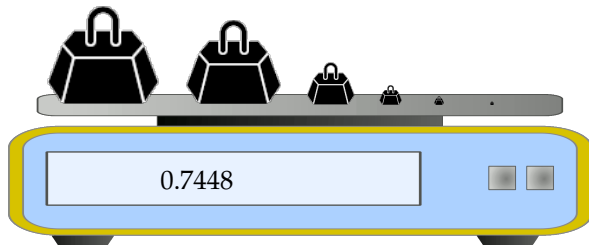
$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$\frac{7}{3^7} \quad \frac{8}{3^8}$$



$$\sum_{n=1}^{\infty} \frac{n}{3^n} \text{ CONVERGES}$$

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

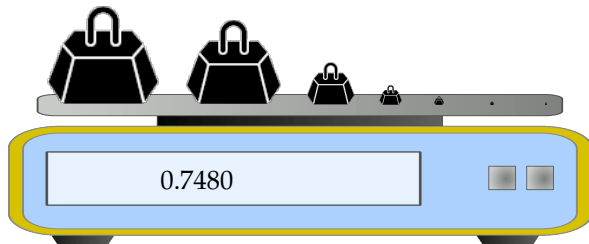
$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$S_7 = 0.7480$$

$$\frac{8}{3^8}$$



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$$S_1 = 0.3333$$

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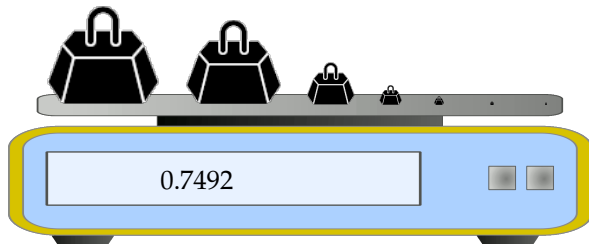
$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$S_7 = 0.7480$$

$$S_8 = 0.7492$$



## Ratio Test

Let  $N$  be any positive integer and assume that  $a_n \neq 0$  for all  $n \geq N$ .

(a) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

(b) If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

Let  $a$  and  $x$  be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of  $a$  and  $x$ .)

$$\sum_{n=1}^{\infty} anx^{n-1}$$

Let  $x$  be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of  $x$ .)

# FILL IN THE BLANKS

## Divergence Test

If the sequence  $\{a_n\}_{n=c}^{\infty}$    
 then the series  $\sum_{n=c}^{\infty} a_n$  diverges.

## Ratio Test

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If the sequence  $\{a_n\}_{n=c}^{\infty}$  fails to converge to zero as  $n \rightarrow \infty$ ,  
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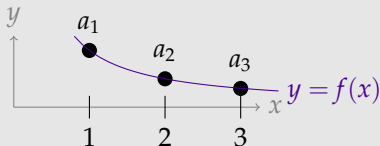
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## Integral Test

Let  $N_0$  be any natural number. If  $f(x)$  is a function which is defined and continuous for all  $x \geq N_0$  and which obeys

- (i)  and  
 (ii)  and  
 (iii)  $f(n) = a_n$  for all  $n \geq N_0$ .

Then



$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x) \, dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

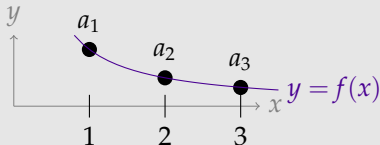
$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n \leq \int_N^{\infty} f(x) \, dx \quad \text{for all } N \geq N_0$$

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Let  $N_0$  be any natural number. If  $f(x)$  is a function which is defined and continuous for all  $x \geq N_0$  and which obeys

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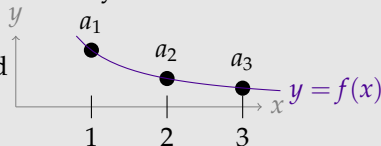
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- (i)  $f(x) \geq 0$  for all  $x \geq N_0$  and
- (ii)  $f(x)$  decreases as  $x$  increases and
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Then



$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x) \, dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

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# FILL IN THE BLANKS

## The Comparison Test

Let  $N_0$  be a natural number and let  $K > 0$ .

(a) If  $|a_n| \square Kc_n$  for all  $n \geq N_0$  and  $\sum_{n=0}^{\infty} c_n$  converges, then  $\sum_{n=0}^{\infty} a_n$  converges.

(b) If  $a_n \square Kd_n \geq 0$  for all  $n \geq N_0$  and  $\sum_{n=0}^{\infty} d_n$  diverges, then  $\sum_{n=0}^{\infty} a_n$  diverges.

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# FILL IN THE BLANKS

## Limit Comparison Theorem

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two series with  $b_n > 0$  for all  $n$ . Assume that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

exists.

- (a) If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges too.
- (b) If  $L \neq 0$  and  $\sum_{n=1}^{\infty} b_n$  diverges, then  $\sum_{n=1}^{\infty} a_n$  diverges too.

In particular, if , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges.

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In particular, if  $L \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges.

## Alternating Series Test

Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers that obeys

(i)

(ii)  $a_{n+1} \leq a_n$  for all  $n \geq 1$  (i.e. the sequence is monotone decreasing);

(iii) and

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number  $N$ ,  $S - S_N$  is between 0 and (the first dropped term)  $(-1)^N a_{N+1}$ . Here  $S_N$  is, as previously, the  $N^{\text{th}}$  partial sum  $\sum_{n=1}^N (-1)^{n-1} a_n$ .

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- (i)  $a_n \geq 0$  for all  $n \geq 1$ ;
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- (iii) and  $\lim_{n \rightarrow \infty} a_n = 0$ .

Then

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# LIST OF CONVERGENCE TESTS

## Divergence Test

When the  $n^{\text{th}}$  term in the series *fails* to converge to zero as  $n$  tends to infinity.

This is a good first thing to check: if it works, it's quick, but it doesn't always work.

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- ▶ successive terms in the series alternate in sign
- ▶ don't forget to check that successive terms decrease in magnitude and tend to zero as  $n$  tends to infinity



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## Alternating Series Test

- ▶ successive terms in the series alternate in sign
- ▶ don't forget to check that successive terms decrease in magnitude and tend to zero as  $n$  tends to infinity

## Integral Test

- ▶ works well when, if you substitute  $x$  for  $n$  in the  $n^{\text{th}}$  term you get a function,  $f(x)$ , that you can easily integrate
- ▶ don't forget to check that  $f(x) \geq 0$  and that  $f(x)$  decreases as  $x$  increases

# LIST OF CONVERGENCE TESTS

## Ratio Test

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## Ratio Test

- ▶ works well when  $\frac{a_{n+1}}{a_n}$  simplifies enough that you can easily compute  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$
- ▶ this often happens when  $a_n$  contains powers, like  $7^n$ , or factorials, like  $n!$
- ▶ don't forget that  $L = 1$  tells you nothing about the convergence/divergence of the series

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## Comparison Test and Limit Comparison Test

- ▶ Comparison test lets you ignore pieces of a function that feel extraneous (like replacing  $n^2 + 1$  with  $n^2$ ) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- ▶ Limit comparison works well when, for very large  $n$ , the  $n^{\text{th}}$  term  $a_n$  is approximately the same as a simpler, nonnegative term  $b_n$

- The integral test gave us the  $p$ -test. When you're looking for comparison series,  $p$ -series  $\sum \frac{1}{n^p}$  are often good choices, because their convergence or divergence is so easy to ascertain.

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- ▶ Geometric series have the form  $\sum a \cdot r^n$  for some nonzero constants  $a$  and  $r$ . The magnitude of  $r$  is all you need to know to decide whether they converge or diverge, so these are also common comparison series.
- ▶ Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

## Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series  $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$  converges or diverges.

## Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series  $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$  converges or diverges.


## Test List


- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison


Determine whether the series  $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$  converges or diverges.

*Hint:* If  $\theta \geq 0$  then  $\sin \theta \leq \theta$ .

## Included Work

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