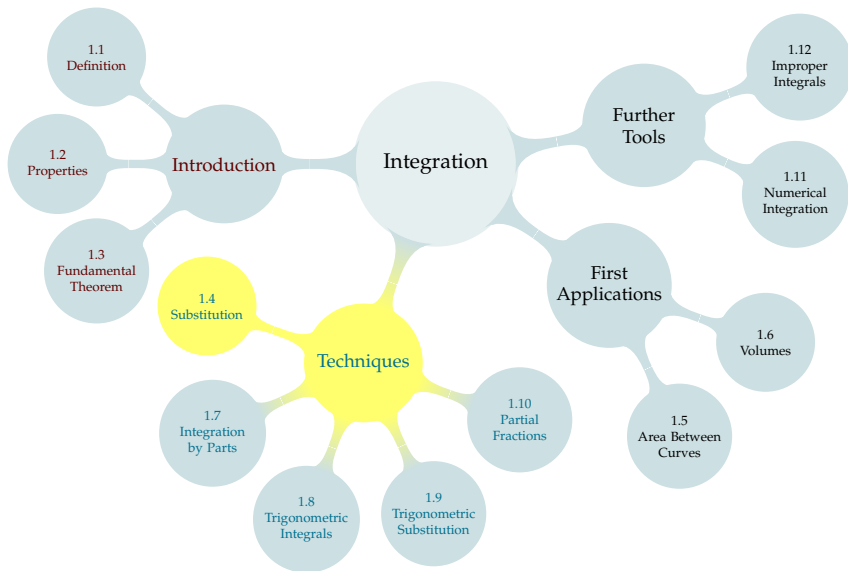


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Hard to guess the antiderivative without seeing the derivative first!

# ANTIDERIVATIVES

Chain Rule:

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Hallmark of the chain rule: an “inside” function, with that function’s derivative multiplied.

## SOLVE BY INSPECTION

$$\int 2xe^{x^2+1} dx$$

$$\int \frac{1}{x} \cos(\log x) dx$$

$$\int 3(\sin x + 1)^2 \cos x dx$$

(Look for an “inside” function, with its derivative multiplied.)

# UNDOING THE CHAIN RULE

Chain Rule:

$$\frac{d}{dx} \{f(u(x))\} = f'(u(x)) \cdot u'(x)$$

(Here,  $u(x)$  is our “inside function”)

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x) \, dx = f(u(x)) + C$$

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Shorthand: call  $u(x)$  simply  $u$ ;  
since  $\frac{du}{dx} = u'(x)$ , call  $u'(x) \, dx$  simply  $du$ .

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$$\int f'(u(x)) \cdot u'(x) \, dx = \int f'(u) \, du \Big|_{u=u(x)} = f(u(x)) + C$$

This is the **substitution rule**.

We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1} dx$$

$$\int \frac{1}{x} \cos(\log x) dx$$

$$\int 3(\sin x + 1)^2 \cos x dx$$



$$\int (3x^2) \sin(x^3 + 1) \, dx =$$



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“Inside” function:  $x^3 + 1$ .

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“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 \, dx \rightarrow du$

$$\int (3x^2) \sin(x^3 + 1) dx = \int \sin(u) du \Big|_{u=x^3+1}$$

“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 dx \rightarrow du$

$$\begin{aligned}\int (3x^2) \sin(x^3 + 1) \, dx &= \int \sin(u) \, du \Big|_{u=x^3+1} \\ &= -\cos(u) + C \Big|_{u=x^3+1}\end{aligned}$$

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 \int (3x^2) \sin(x^3 + 1) dx &= \int \sin(u) du \Big|_{u=x^3+1} \\
 &= -\cos(u) + C \Big|_{u=x^3+1} \\
 &= \cos(x^3 + 1) + C
 \end{aligned}$$

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“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 dx \rightarrow du$

Warning 1: We don't just change  $dx$  to  $du$ . We need to couple  $dx$  with the derivative of our inside function.

After all, we're undoing the chain rule! We need to have an “inside derivative.”

$$\begin{aligned}\int (3x^2) \sin(x^3 + 1) dx &= \int \sin(u) du \Big|_{u=x^3+1} \\ &= -\cos(u) + C \Big|_{u=x^3+1} \\ &= \cos(x^3 + 1) + C\end{aligned}$$

“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 dx \rightarrow du$

Warning 2: The final answer is a function of  $x$ .



We used the substitution rule to conclude

$$\int (3x^2) \sin(x^3 + 1) \, dx = -\cos(x^3 + 1) + C$$

We can check that our antiderivative is correct by differentiating.

We saw:

$$\int 3x^2 \sin(x^3 + 1) \, dx = -\cos(x^3 + 1) + C$$

So, we can evaluate:

$$\int_0^1 3x^2 \sin(x^3 + 1) \, dx$$

# NOTATION: LIMITS OF INTEGRATION

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

## NOTATION: LIMITS OF INTEGRATION

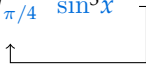
$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

Let  $u = \sin x$ ,  $du = \cos x dx$ . Note the limits (or bounds) of integration  $\pi/4$  and  $\pi/2$  are values of  $x$ , not  $u$ : they follow the differential, unless otherwise specified.

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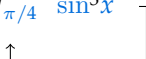
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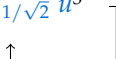
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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

$$\begin{aligned} x &= \frac{\pi}{2} \\ x &= \frac{\pi}{4} \end{aligned}$$

$$\int_{1/\sqrt{2}}^1 \frac{1}{u^3} du$$

$$\begin{aligned} u &= \sin\left(\frac{\pi}{2}\right) = 1 \\ u &= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \end{aligned}$$


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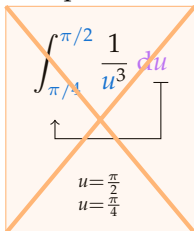
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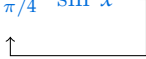
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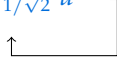
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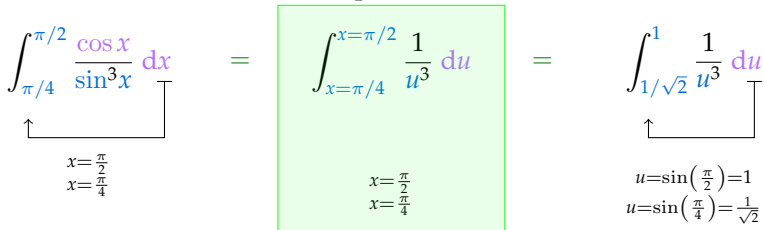
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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx = \int_{x=\pi/4}^{x=\pi/2} \frac{1}{u^3} du = \int_{u=\sin(\pi/4)=\frac{1}{\sqrt{2}}}^{u=\sin(\pi/2)=1} \frac{1}{u^3} du$$


not standard, but OK

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## TRUE OR FALSE?

1. Using  $u = x^2$ ,

$$\int e^{x^2} dx = \int e^u du$$

2. Using  $u = x^2 + 1$ ,

$$\int_0^1 x \sin(x^2 + 1) dx = \int_0^1 \frac{1}{2} \sin u du$$

Evaluate  $\int_0^1 x^7 (x^4 + 1)^5 \, dx$ .



Time permitting, more examples using the substitution rule

Evaluate  $\int \sin x \cos x \, dx$ .

## CHECK OUR WORK

We can check that  $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$  by differentiating.

We can check that  $\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + C$  by differentiating.

Evaluate  $\int \frac{\log x}{3x} dx$ .

## CHECK OUR WORK

We can check that  $\int \frac{\log x}{3x} dx = \frac{1}{6} \log^2 x + C$  by differentiating.

Evaluate  $\int \frac{e^x}{e^x + 15} dx$ .

Evaluate  $\int x^4(x^5 + 1)^8 dx$ .



## CHECK OUR WORK

We can check that  $\int \frac{e^x}{e^x + 15} dx = \log |e^x + 15| + C$  by differentiating.

We can check that  $\int x^4(x^5 + 1)^8 dx = \frac{1}{45}(x^5 + 1)^9 + C$  by differentiating.

Evaluate  $\int_4^8 \frac{s}{s-3} ds$ . Be careful to use correct notation.





Evaluate  $\int x^9(x^5 + 1)^8 dx$ .

## CHECK OUR WORK

We can check that  $\int x^9(x^5 + 1)^8 \, dx = \frac{1}{5} \left[ \frac{(x^5 + 1)^{10}}{10} - \frac{(x^5 + 1)^9}{9} \right] + C$  by differentiating.

# PARTICULARLY TRICKY SUBSTITUTION

Evaluate  $\int \frac{1}{e^x + e^{-x}} dx$ .



## CHECK OUR WORK

We can check that  $\int \frac{1}{e^x + e^{-x}} dx = \arctan(e^x) + C$  by differentiating.

## Included Work



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