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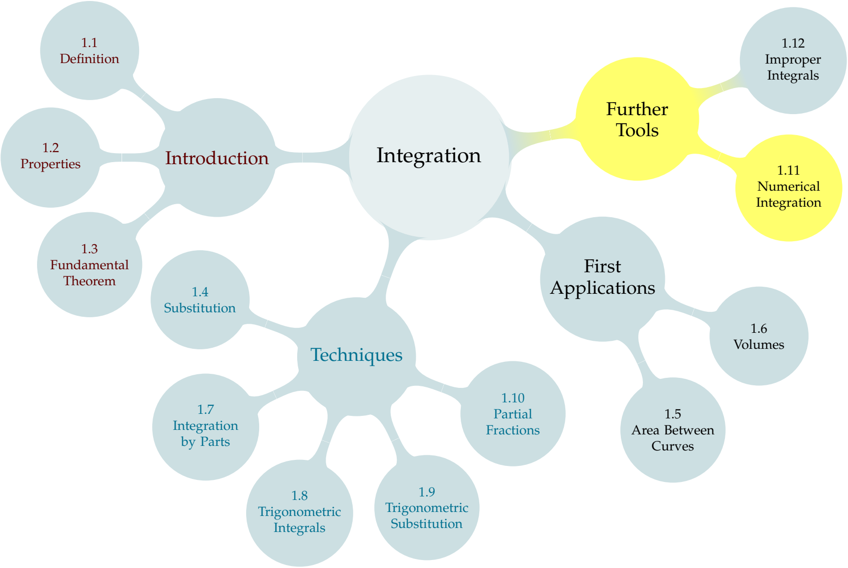

Sometimes, integrals can't be evaluated using the fundamental theorem of calculus:

$$\int_0^1 e^{x^2} \mathrm{d}x = ? \qquad \int_0^1 \sin(x^2) \mathrm{d}x = ?$$

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$$\int_0^3 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(3) = \ldots ?$$

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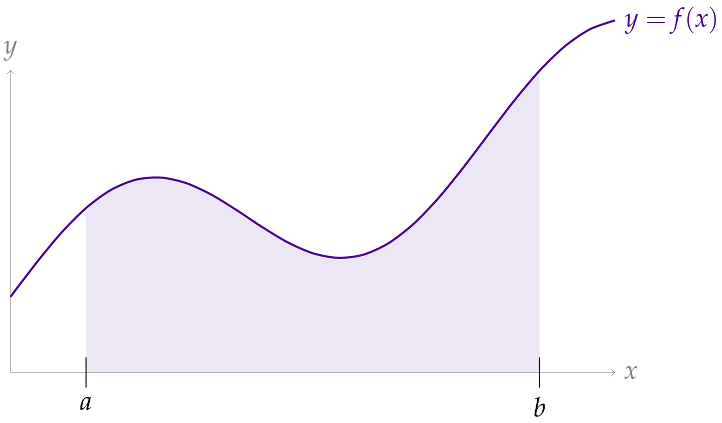
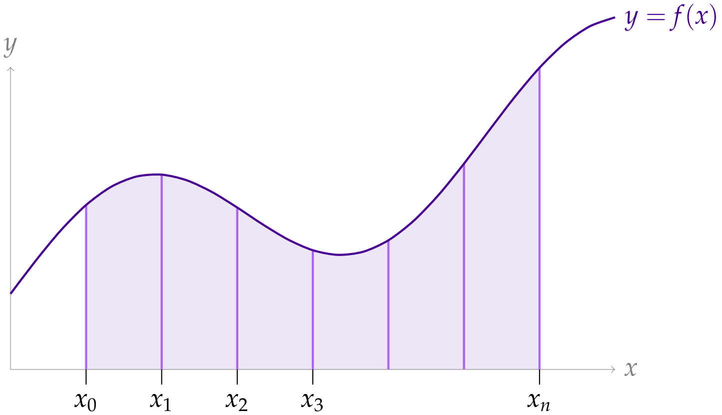
A numerical approximation will give us an approximate number for a definite integral.



We can approximate the area $\int_a^b f(x)\,\mathrm{d}x$ by cutting it into slices and approximating the area of those slices with a simple geometric figure, such as a rectangle, a trapezoid, or a parabola.



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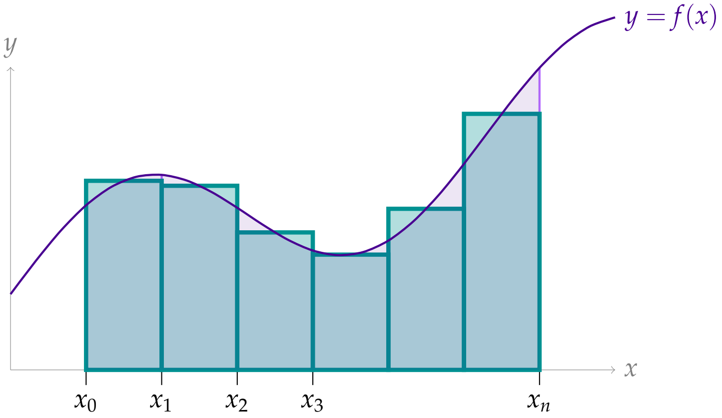

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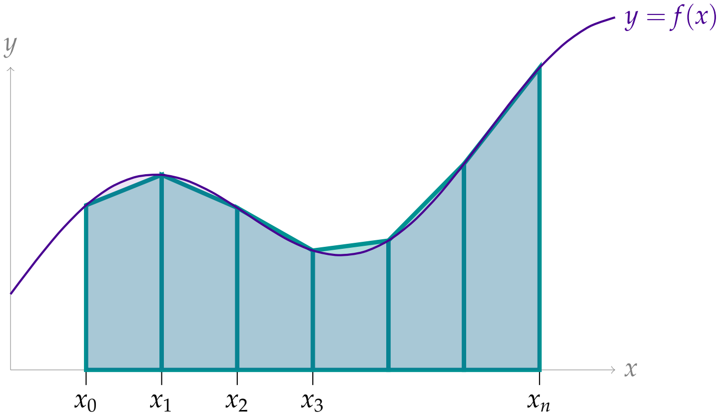

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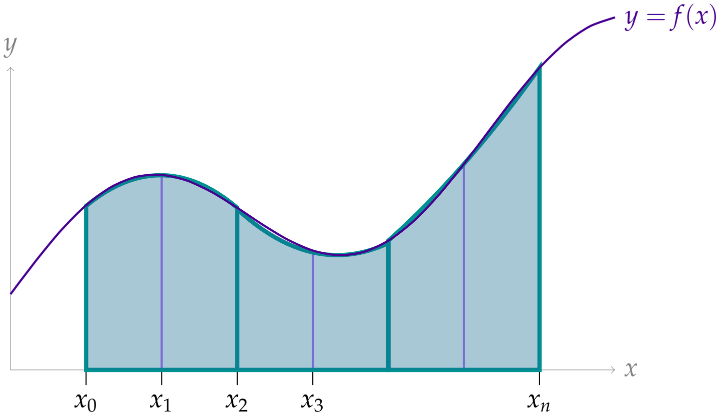
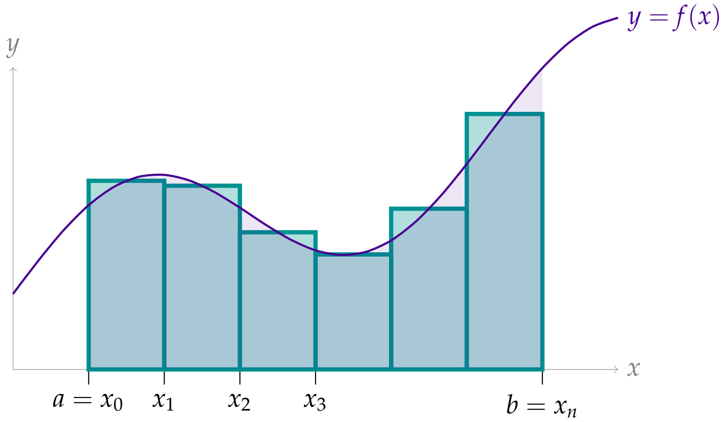


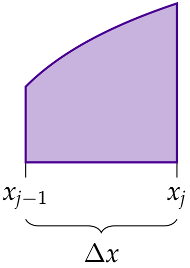
We can approximate the area $\int_a^b f(x)\,\mathrm{d}x$ by cutting it into slices and approximating the area of those slices with a simple geometric figure, such as a rectangle, a trapezoid, or a parabola.

The midpoint rule approximates $\int_a^b f(x)\,\mathrm{d}x$ as its midpoint Riemann sum with n intervals.



Approximate the area under the curve $y = f(x)$ from $x = x_{j-1}$ to $x = x_j$ with a rectangle.

To make our writing cleaner, let $\overline{x_j} = \frac{x_{j-1}+x_j}{2}$



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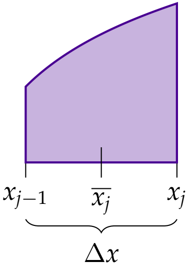

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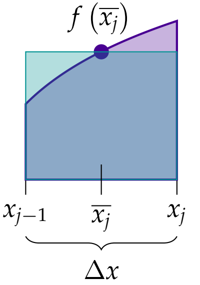

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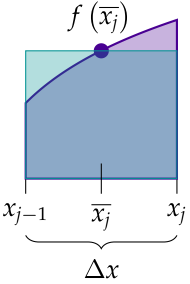


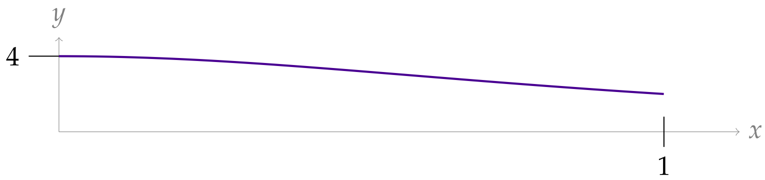
Midpoint Rule

The midpoint rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \left[f\left(\overline{x_1}\right) + f\left(\overline{x_2}\right) + \cdots + f\left(\overline{x_n}\right)\right]\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_j = a + j\Delta x$

Approximate $\displaystyle\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$ using the midpoint rule and $n = 4$ slices. Leave your answer in calculator-ready form.



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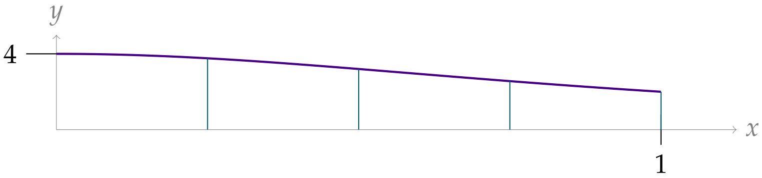

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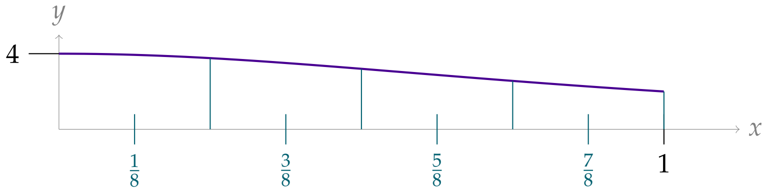

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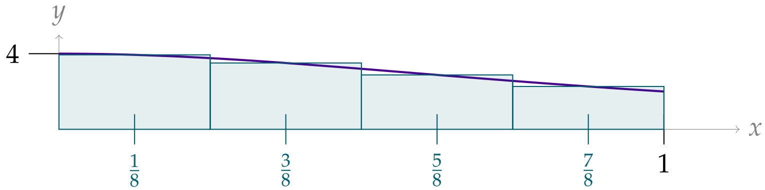
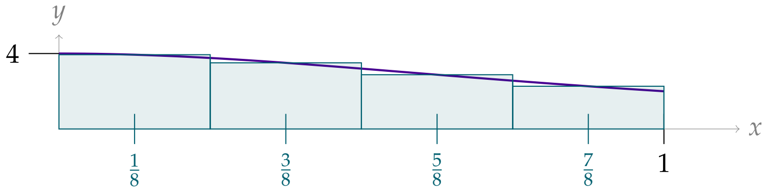


$$\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$

Approximate $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$ using the midpoint rule and $n = 4$ slices. Leave your answer in calculator-ready form.



$$\int_0^1 \frac{4}{1+x^2}\,dx \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$

$$\int_0^1 \frac{4}{1+x^2}\,dx = 4\arctan(1) = 4 \cdot \frac{\pi}{4} = \pi$$

ERROR

$$\pi = \int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$

$$\approx 3.14680$$

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$$|\text{exact} - \text{approximate}| \approx |3.14159 - 3.14680| = 0.00521$$

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Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right|$$

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Relative error:

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Percent error:

$$100\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right|$$

Error

$$\pi = \int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right] \cdot \frac{1}{4}$$
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Error:

$$|\text{exact} - \text{approximate}| \approx |3.14159 - 3.14680| = 0.00521$$

Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right| \approx \frac{0.00521}{3.14159} \approx 0.001658$$

Percent error:

$$100\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right| \approx 100(0.001658) = 0.1658\%$$

Error

A numerical approximation will give us an approximate value for a definite integral.
This is most useful if we know something about its accuracy.

A: approximation E: exact number

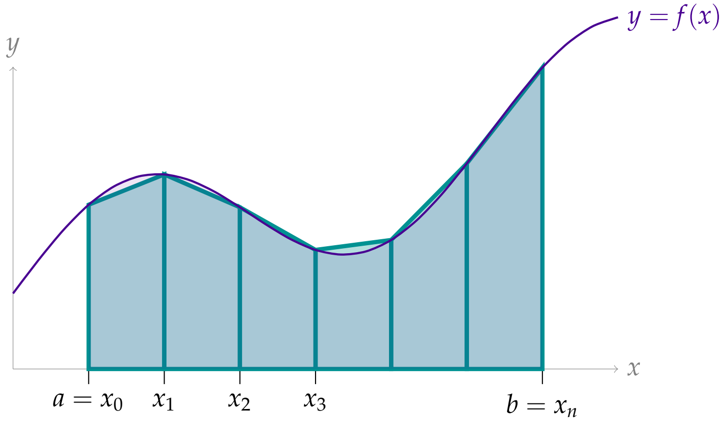
Error: $|A - E|$

Relative Error: $\left|\dfrac{A - E}{E}\right|$

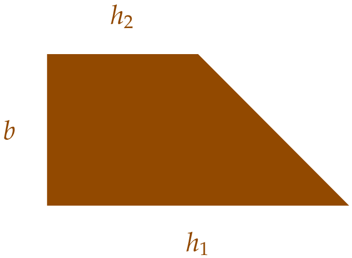
Percent Error: $100\left|\dfrac{A - E}{E}\right|$

We will discuss error more after we've learned the three approximation rules. For now, we're using error to illustrate that our methods have the potential to produce reasonable approximations without too much work.

The trapezoidal rule approximates each slice of $\int_a^b f(x)\,\mathrm{d}x$ with a trapezoid.



Recall the area of a right trapezoid with base b and heights h_1 and h_2:



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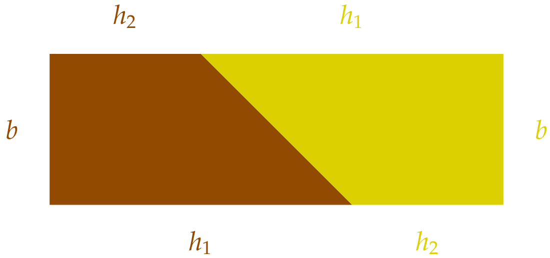

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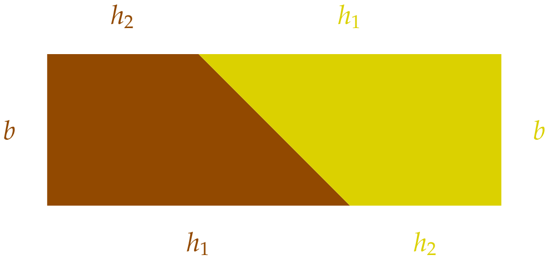
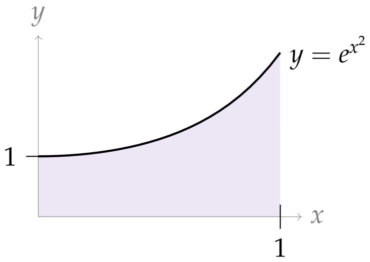
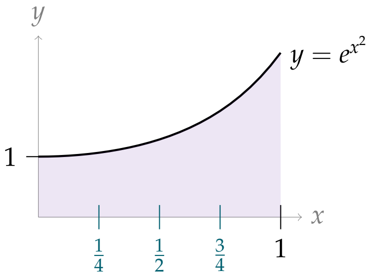


Rectangle area: $b(h_1 + h_2)$
Trapezoid area: $\frac{b}{2}(h_1 + h_2)$



Trapezoid area: $\frac{\text{base}}{2}(h_1 + h_2)$

Approximate $\displaystyle\int_0^1 e^{x^2}\,\mathrm{d}x$ using $n = 4$ trapezoids.
Leave your answer in calculator-ready form.



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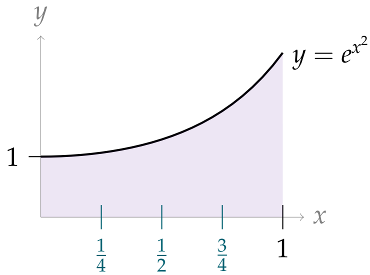

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Approximate $\int_0^1 e^{x^2}\,\mathrm{d}x$ using $n = 4$ trapezoids.
Leave your answer in calculator-ready form.

$$\int_0^1 e^{x^2}\,\mathrm{d}x \approx \frac{1/4}{2}\left(e^0 + e^{\frac{1}{16}} + e^{\frac{1}{16}} + e^{\frac{1}{4}} + e^{\frac{1}{4}} + e^{\frac{9}{16}} + e^{\frac{9}{16}} + e\right)$$
$$= \frac{1/4}{2}\left(e^0 + 2e^{1/16} + 2e^{1/4} + 2e^{9/16} + e\right)$$

Trapezoidal Rule

The trapezoidal rule approximation is

$$\int_a^b f(x)\mathrm{d}x \approx \Delta x \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

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Using $n = 3$ trapezoids, approximate $\displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x$.

Trapezoidal Rule

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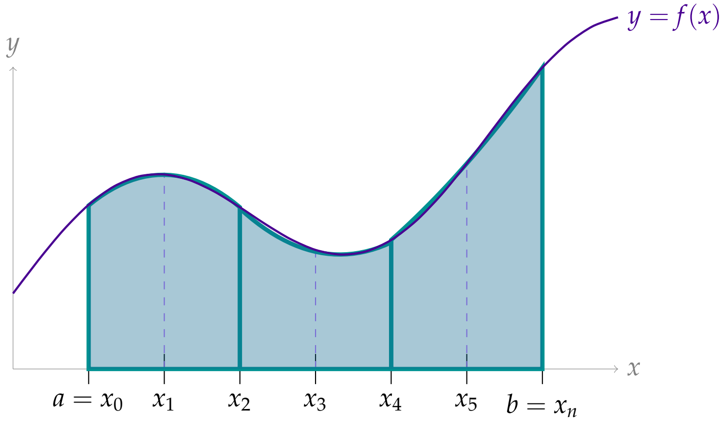
where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Using $n = 3$ trapezoids, approximate $\displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x$.

$$\Delta x = \frac{10-1}{3} = 3 \quad x_0 = 1 \quad x_1 = 4 \quad x_2 = 7 \quad x_3 = 10$$

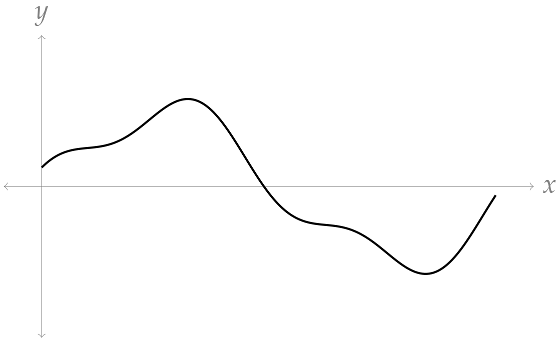
$$\int_1^{10} \frac{1}{x}\,\mathrm{d}x \approx 3\left[\frac{1}{2}(1) + \frac{1}{4} + \frac{1}{7} + \frac{1}{2}\left(\frac{1}{10}\right)\right] = \frac{99}{35}$$

Simpson's rule approximates each *pair of slices* of $\int_a^b f(x)\,\mathrm{d}x$ with a parabola.



Simpson's Rule

Add up parabolas.



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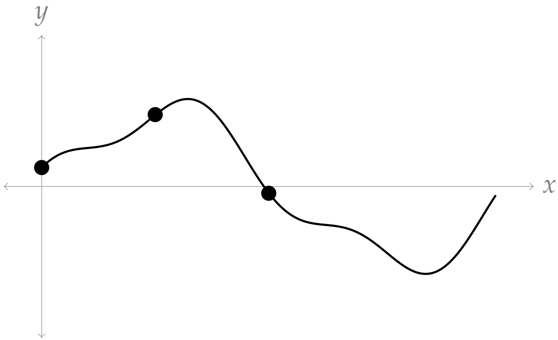

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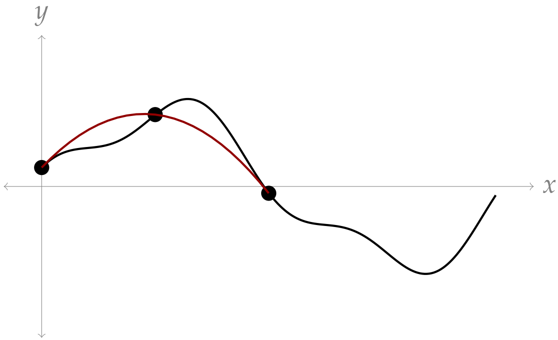

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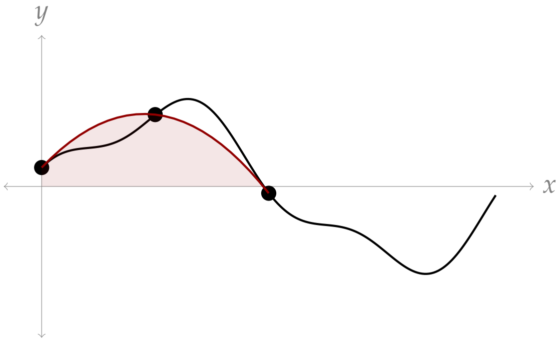

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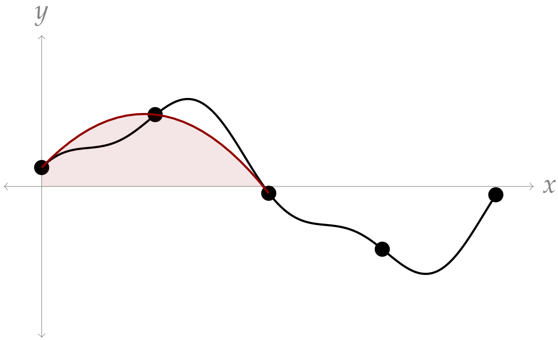

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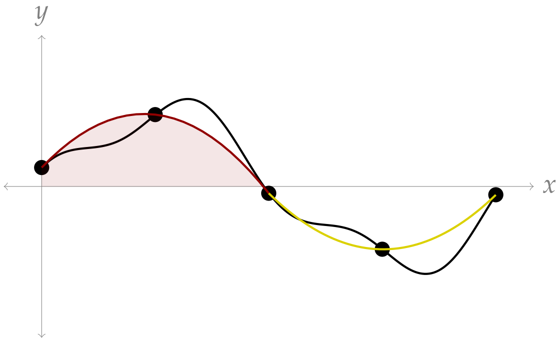

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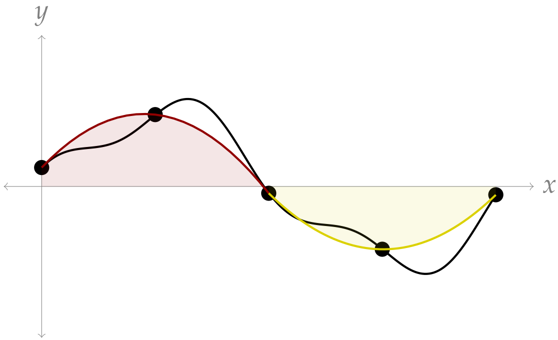
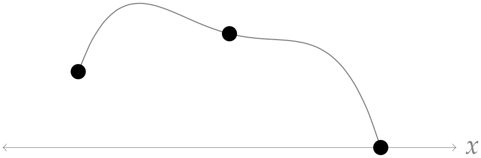


Simpson's Rule Derivation

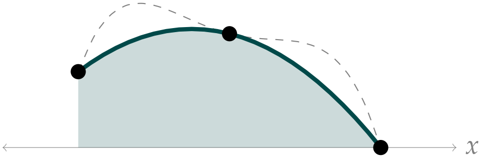
Skip derivation of Simpson's rule



What is the area under the parabola passing through three specified points?

Simpson's Rule Derivation

Skip derivation of Simpson's rule



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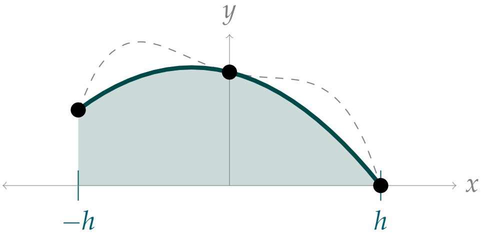
Simpson's Rule Derivation

SKIP DERIVATION OF SIMPSON'S RULE



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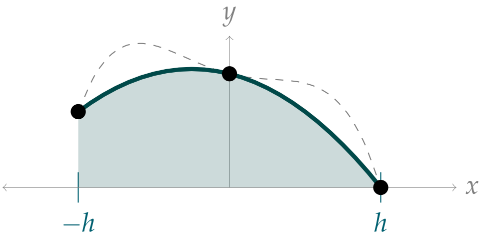
SIMPSON'S RULE DERIVATION

SKIP DERIVATION OF SIMPSON'S RULE

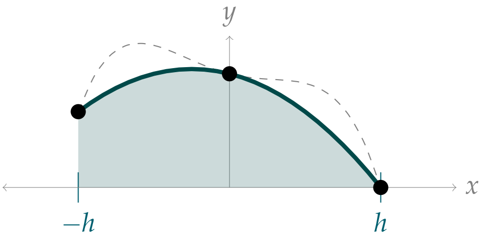


What is the area under the parabola passing through three specified points?

Parabola: $Ax^2 + Bx + C$

Simpson's Rule Derivation

SKIP DERIVATION OF SIMPSON'S RULE



What is the area under the parabola passing through three specified points?

Parabola: $Ax^2 + Bx + C$
Area: $\int_{-h}^{h}(Ax^2 + Bx + C)\mathrm{d}x = \frac{h}{3}\left(2Ah^2 + 6C\right)$

Find

$$\frac{h}{3}\left(2Ah^2 + 6C\right)$$

for A, B, and C such that

$$Ah^2 - Bh + C = f(-h) \tag{E1}$$

$$C = f(0) \tag{E2}$$

$$Ah^2 + Bh + C = f(h) \tag{E3}$$

Try $(E1) + 4(E2) + (E3)$:

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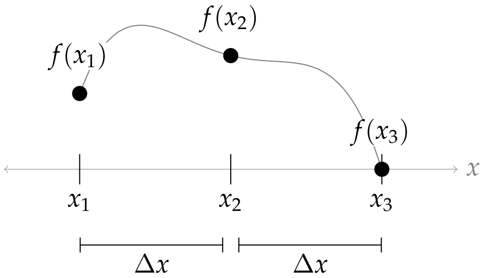
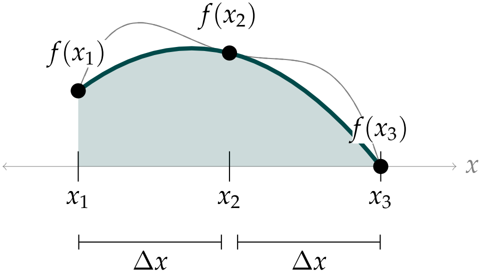
$$\begin{aligned} Ah^2 - Bh + C &= f(-h) && \text{(E1)} \\ C &= f(0) && \text{(E2)} \\ Ah^2 + Bh + C &= f(h) && \text{(E3)} \end{aligned}$$

Try $(E1) + 4(E2) + (E3)$:

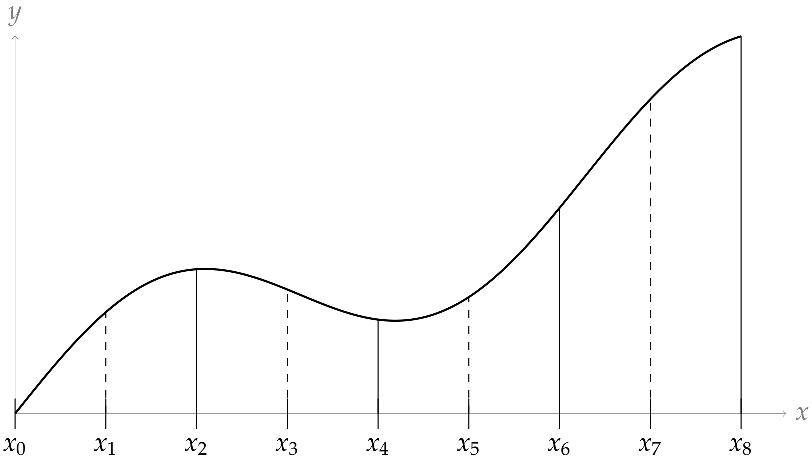
$$\begin{aligned} 2Ah^2 + 6C &= f(-h) + 4f(0) + f(h) \\ \text{Area} = \frac{h}{3}\left(2Ah^2 + 6C\right) &= \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right) \end{aligned}$$


$f(x_2)$
$f(x_1)$
$f(x_3)$
x
x_1
x_2
x_3
Δx
Δx



Area under parabola:

$$\frac{\Delta x}{3}\Big(f(x_1) + 4f(x_2) + f(x_3)\Big)$$


y
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x_0
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x_6
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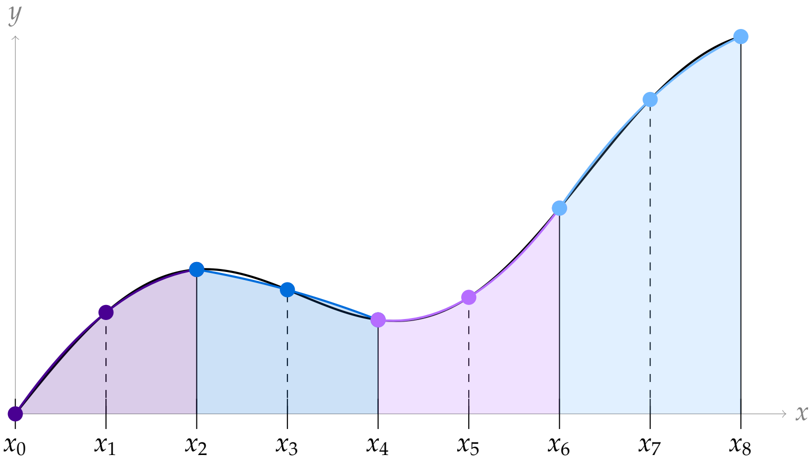
y
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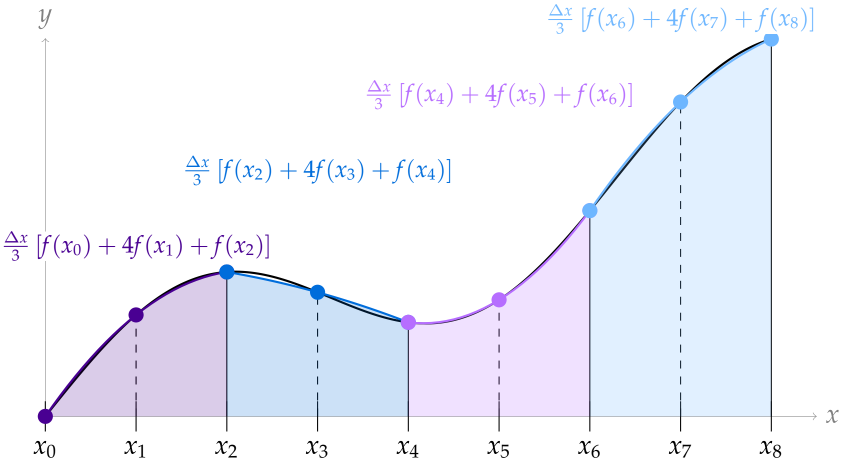
$\frac{\Delta x}{3}\left[f(x_6)+4f(x_7)+f(x_8)\right]$
$\frac{\Delta x}{3}\left[f(x_4)+4f(x_5)+f(x_6)\right]$
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y
x
x_0
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x_7
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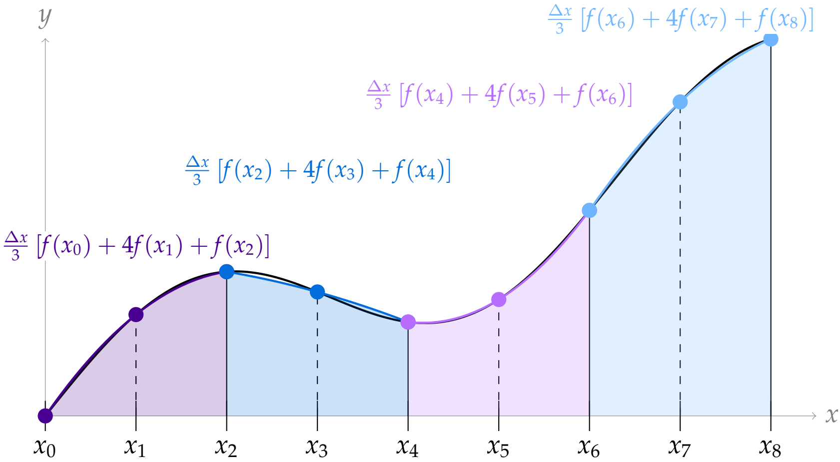

$$\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+2f(x_6)+4f(x_7)+f(x_8)\right]$$

Simpson's Rule

The Simpson's rule approximation is $\int_a^b f(x)\mathrm{d}x \approx$

$$\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$

(We'll call n the number of slices; some people call $n/2$ the number of slices, because that's the number of approximating parabolas.)

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Using Simpson's rule and $n = 8$ (i.e. 4 parabolas), approximate $\displaystyle\int_1^{17} \frac{1}{x}\,\mathrm{d}x$. Leave your answer in calculator-ready form.

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Using Simpson's rule and $n = 8$ (i.e. 4 parabolas), approximate $\displaystyle\int_1^{17} \frac{1}{x}\,\mathrm{d}x$. Leave your answer in calculator-ready form.

$\approx \frac{2}{3}\left[\frac{1}{1} + 4\cdot\frac{1}{3} + 2\cdot\frac{1}{5} + 4\cdot\frac{1}{7} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{11} + 2\cdot\frac{1}{13} + 4\cdot\frac{1}{15} + \frac{1}{17}\right]$

The instantaneous electricity use rate (kW/hr) of a factory is measured throughout the day.

time	12:00	1:00	2:00	3:00	4:00	5:00	6:00	7:00	8:00
rate	100	200	150	400	300	300	200	100	150

Use Simpson's Rule to approximate the total amount of electricity you used from noon to 8:00.

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Use Simpson's Rule to approximate the total amount of electricity you used from noon to 8:00.

We use $n = 8$, with $\Delta x = 1$ hour. Let's re-label the times as $x = 0$ as noon, $x = 1$ as 1 o'clock, etc.

$$\frac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + 2f(6) + 4f(7) + f(8)\right]$$

$$= \frac{1}{3}\left[100 + 800 + 300 + 1600 + 600 + 1200 + 400 + 400 + 150\right]$$

$$= 1850 \text{ kW}$$

Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

- the total error introduced by the midpoint rule is bounded by $\frac{M}{24}\frac{(b-a)^3}{n^2}$,
- the total error introduced by the trapezoidal rule is bounded by $\frac{M}{12}\frac{(b-a)^3}{n^2}$, and
- the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$

when approximating $\int_a^b f(x)\,\mathrm{d}x$.

Numerical integration errors

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Suppose we approximate $\int_0^3 \sin(x)\, \mathrm{d}x$ using the midpoint rule and $n = 6$ intervals. Give an upper bound of the resulting error.

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Suppose we approximate $\int_0^3 \sin(x)\,\mathrm{d}x$ using the midpoint rule and $n = 6$ intervals. Give an upper bound of the resulting error.

If $f(x) = \sin x$, then $f''(x) = -\sin x$. For $0 \leq x \leq 3$ (indeed, for any x), $|f''(x)| = |-\sin x| \leq 1$, so we take $M = 1$.

$$|\text{error}| \leq \frac{1}{24}\frac{(3-0)^3}{6^2} = \frac{1}{32}$$

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,\mathrm{d}x$.

Suppose we approximate $\int_2^3 \frac{1}{x}\mathrm{d}x$ using Simpson's rule with $n = 10$ slices (5 parabolas). Give an upper bound of the resulting error.

Q

ans

Numerical integration errors

Assume that $|f^{(4)}(x)| \le L$ for all $a \le x \le b$. Then the total error introduced by Simpson's rule is bounded by $\dfrac{L}{180}\dfrac{(b-a)^5}{n^4}$ when approximating $\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Suppose we approximate $\displaystyle\int_2^3 \frac{1}{x}\mathrm{d}x$ using Simpson's rule with $n = 10$ slices (5 parabolas). Give an upper bound of the resulting error.

If $f(x) = \frac{1}{x}$, then $f^{(4)}(x) = \frac{24}{x^5}$. This is a positive, decreasing function for positive values of x, so its maximum value on the interval $[2, 3]$ is $f^{(4)}(2) = \frac{24}{2^5} = \frac{3}{4}$. So, we take $L = \frac{3}{4}$. Then the error is bounded by

$$\frac{3/4}{180}\frac{1^5}{10^4} = \frac{1}{240 \times 10^4} = \frac{1}{2\,400\,000}$$

Numerical integration errors

Assume that $|f^{(4)}(x)| \le L$ for all $a \le x \le b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,\mathrm{d}x$.

We will approximate $\int_0^{1/2} e^{x^2}\,\mathrm{d}x$ using Simpson's rule, and we need our error to be no more than $\frac{1}{10\,000}$. How many intervals will suffice?

You may use, without proof:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{e^{x^2}\right\} = 4e^{x^2}\left(4x^4 + 12x^2 + 3\right) \qquad \frac{25\sqrt[4]{e}}{180 \cdot 2^5} < \frac{1}{3^4}$$

Q

ans

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Since the fourth derivative is positive and increasing for positive x, its maximum value on the interval $\left[0,\frac{1}{2}\right]$ occurs when $x=\frac{1}{2}$

$$L = 4e^{(1/2)^2}\left(\frac{4}{2^4}+\frac{12}{2^2}+3\right) = 4\sqrt[4]{e}\left(\frac{1}{4}+3+3\right) = 25\sqrt[4]{e}$$

$$|\text{error}| \le \frac{25\sqrt[4]{e}}{180}\frac{(1/2-0)^5}{n^4} = \frac{25\sqrt[4]{e}}{180\cdot 2^5}\frac{1}{n^4} < \frac{1}{3^4 n^4}$$

$$\frac{1}{3^4 n^4} < \frac{1}{10\,000} = \frac{1}{10^4}$$

$$n^4 > \frac{10^4}{3^4} \implies n > \frac{10}{3} = 3.\overline{33}$$

So $n=4$ is a large enough number of intervals to guarantee that our error is no more than $\frac{1}{10\,000}$.

$$\int_0^{1/2} e^{x^2}\,\mathrm{d}x \qquad \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{e^{x^2}\right\} = 4e^{x^2}\left(4x^4 + 12x^2 + 3\right) \qquad \frac{25\sqrt[4]{e}}{180\cdot 2^5} < \frac{1}{3^4}$$

Remark: It's also true that $n = 6$ is good enough, as is $n = 1,000,000$. If you take some shortcuts and end up with $n = 6$ instead of $n = 4$, then the difference in the difficulty of your final calculation is not so much, so the shortcuts were probably OK. But if you guess a huge number like $n = 1,000,000$, this is no longer reasonable: computing Simpson's rule with this many intervals is extremely difficult.

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,\mathrm{d}x$.

It can be shown that the fourth derivative of $\frac{1}{x^2+1}$ has absolute value at most 24 for all real numbers x. Using this information, find a rational number approximating $\arctan(2)$ with an error of no more than $\frac{2^6}{3 \cdot 5^5} \approx 0.007$.

Q

ans

First, we'll set up our integral:

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$$\int_0^2 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(2) - \arctan(0) = \arctan 2$$

From the given information, we'll use $L = 24$.

$$\begin{aligned} |\text{error}| &\le \frac{L}{180}\frac{(2-0)^5}{n^4} \\ &= \frac{24 \cdot 2^5}{180n^4} = \frac{2^6}{15n^4} \\ \frac{2^6}{15n^4} &\le \frac{2^6}{15 \cdot 5^4} \\ \frac{1}{n^4} &\le \frac{1}{5^4} \\ n &\ge 5 \end{aligned}$$

Since n must be even, we'll use $n = 6$. Now, we can give the approximation.

A

ans

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$$\arctan(2) = \int_0^2 \frac{1}{1+x^2}\mathrm{d}x, \qquad n = 6, \qquad \Delta x = \frac{2-0}{6} = \frac{1}{3}$$

$$\approx \frac{1/3}{3}\left[f(0) + 4f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 4f(1) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{5}{3}\right) + f(2)\right]$$

$$= \frac{1}{9}\left[\frac{1}{1+0} + \frac{4}{1+1/9} + \frac{2}{1+4/9} + \frac{4}{1+1} + \frac{2}{1+16/9} + \frac{4}{1+25/9} + \frac{1}{1+4}\right]$$

Remark: Calculators and computers are pretty good at adding, subtracting, multiplying, and dividing integers. If we can use these operations to approximate an integral, then we can program a computer to evaluate the result. So, an approximation like the one we just did is a reasonable start to approximating $\arctan(2)$ as a decimal.

Included Work

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