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graph LR; Integration((Integration)) --- Introduction((Introduction)); Integration --- Techniques((Techniques)); Integration --- FirstApplications((First Applications)); Integration --- FurtherTools((Further Tools)); Introduction --- 1.1((1.1 Definition)); Introduction --- 1.2((1.2 Properties)); Introduction --- 1.3((1.3 Fundamental Theorem)); Techniques --- 1.4((1.4 Substitution)); Techniques --- 1.7((1.7 Integration by Parts)); Techniques --- 1.8((1.8 Trigonometric Integrals)); Techniques --- 1.9((1.9 Trigonometric Substitution)); Techniques --- 1.10((1.10 Partial Fractions)); FirstApplications --- 1.5((1.5 Area Between Curves)); FirstApplications --- 1.6((1.6 Volumes)); FurtherTools --- 1.11((1.11 Numerical Integration)); FurtherTools --- 1.12((1.12 Improper Integrals));
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The mind map is structured as follows:

- Integration**
 - Introduction**
 - 1.1 Definition
 - 1.2 Properties
 - 1.3 Fundamental Theorem
 - Techniques**
 - 1.4 Substitution
 - 1.7 Integration by Parts
 - 1.8 Trigonometric Integrals
 - 1.9 Trigonometric Substitution
 - 1.10 Partial Fractions
 - First Applications**
 - 1.5 Area Between Curves
 - 1.6 Volumes
 - Further Tools**
 - 1.11 Numerical Integration
 - 1.12 Improper Integrals

Methods for finding the area under a curve.

- 2/123

Methods for finding the area under a curve.

- ▶ Limit of a Riemann Sum
 - ▶ Conceptually easy – cut into rectangles



Methods for finding the area under a curve.

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$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x; \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

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- Up next: Fundamental Theorem of Calculus

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-

REVIEW: AREA UNDER A CURVE

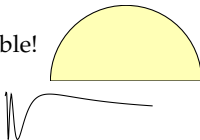
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- Use Geometry
 - Computationally nice when it's available! (Circles, triangles, symmetry, etc.)
 - Often not available – most functions don't make such nice shapes.



- ▶ Up next: Fundamental Theorem of Calculus
 - ▶ **Conceptually** less obvious – we'll spend about a day explaining why it works
 - ▶ **Computationally** generally nicer than Riemann sums

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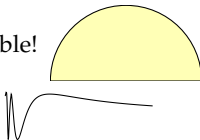
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Fundamental Theorem of Calculus, Part 1

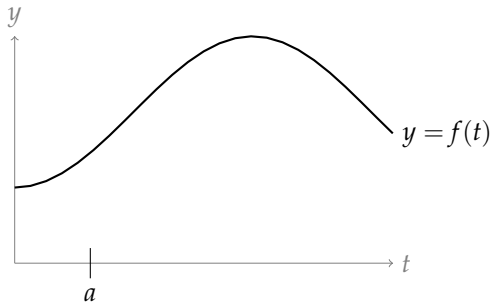
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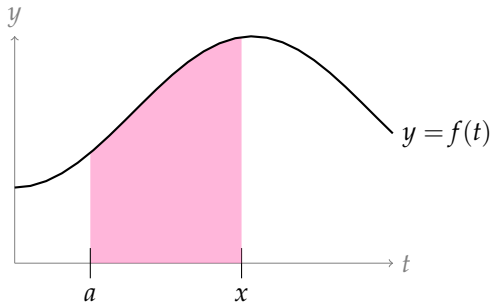
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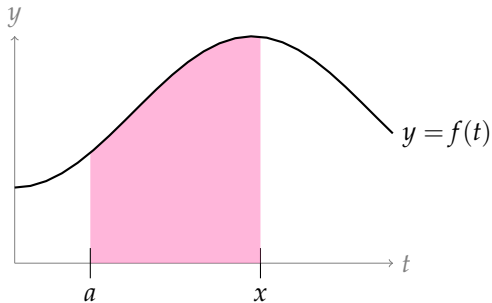
AREA FUNCTION: $A(x) = \int_a^x f(t)dt$ FOR $a \leq x \leq b$



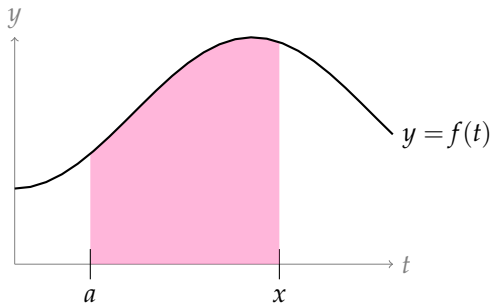
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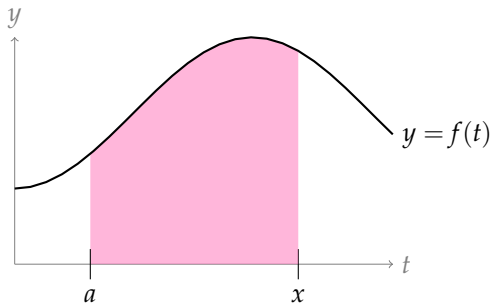
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Notation: the function A depends on the variable x .

We need to know how the function f behaves on the whole interval $(0, x)$ to find $A(x)$. That's why we use $f(t)$, not $f(x)$.

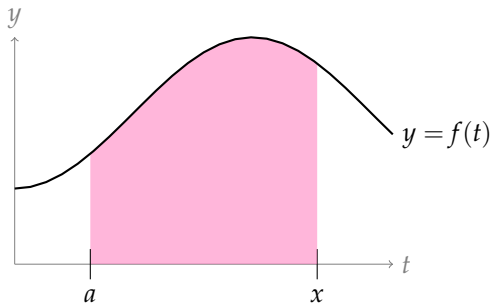
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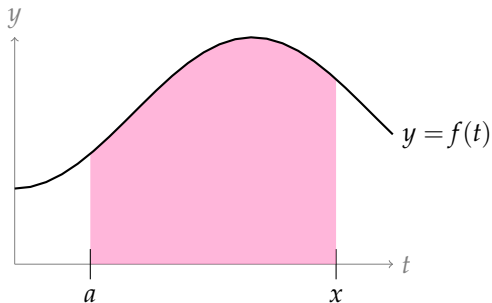
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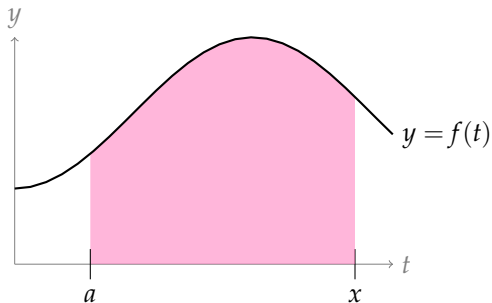
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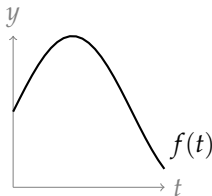
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It might look strange at first to see two different variables. Let's consider the alternatives:

$$A(x) = \int_0^x f(t) \, dt$$

$$B(x) = \int_0^x f(x) \, dt$$

$$C(x) = \int_0^x f(x) \, dx$$



AREA FUNCTION NOTATION

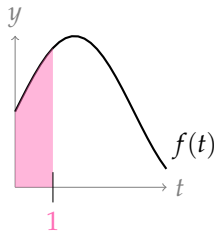
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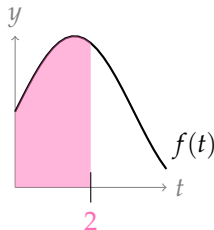
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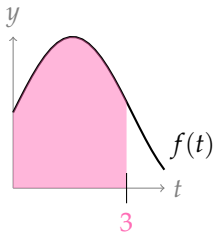
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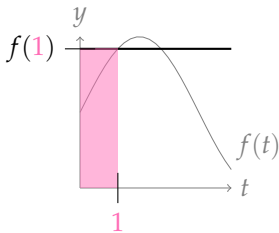
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It might look strange at first to see two different variables. Let's consider the alternatives:

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AREA FUNCTION NOTATION

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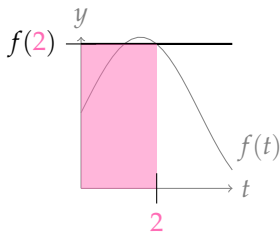
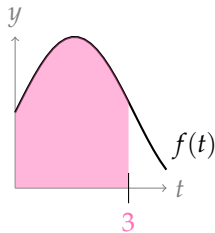
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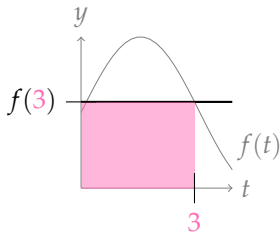
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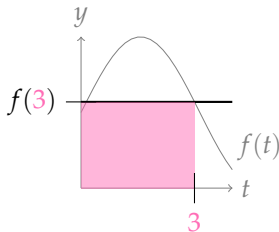
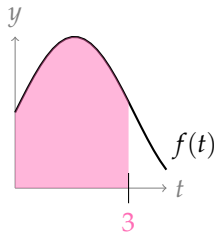
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Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

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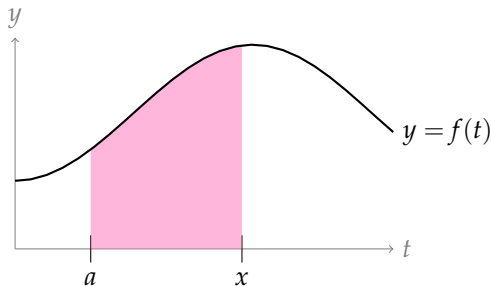
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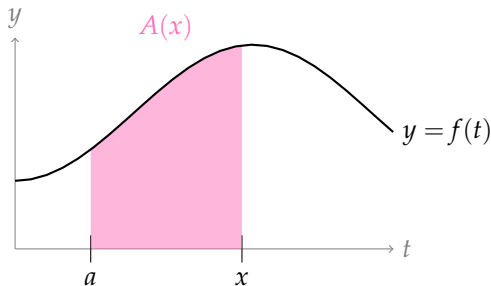
Question: Why is it true?

DERIVATIVE OF AREA FUNCTION, $A(x) = \int_a^x f(t) dt$



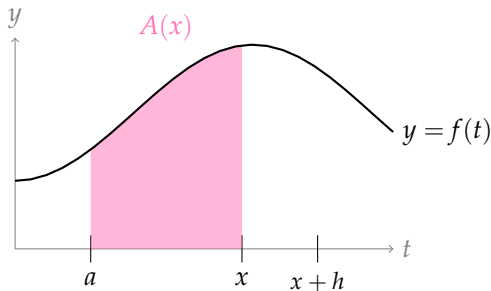
$$A'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h}$$

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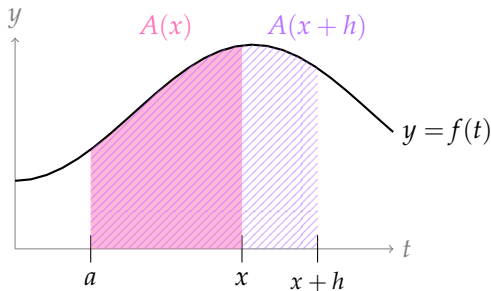
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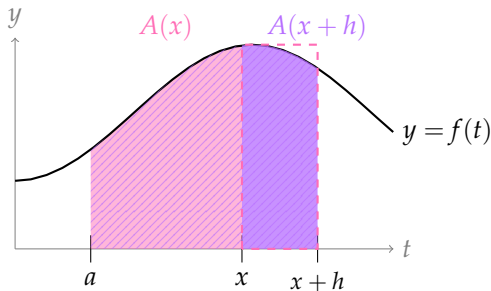
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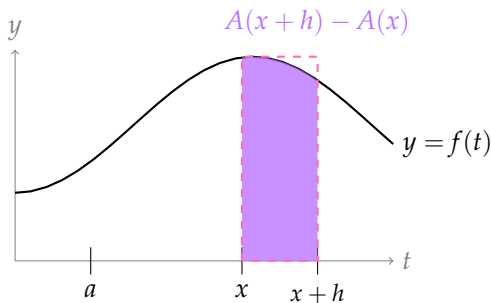
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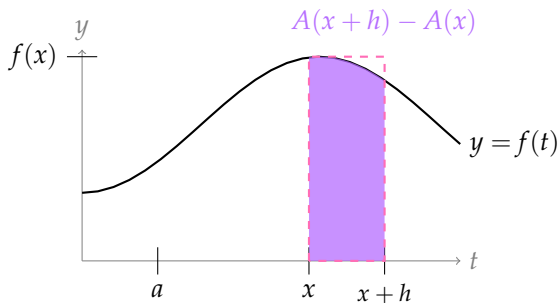
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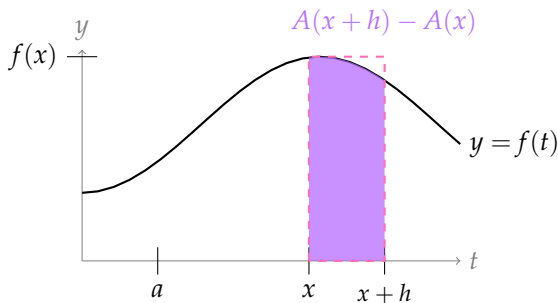
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When h is very small, the purple area looks like a rectangle with base h and height $f(x)$, so $A(x+h) - A(x) \approx hf(x)$ and $\frac{A(x+h) - A(x)}{h} \approx f(x)$. As h tends to zero, the error in this approximation approaches 0.

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

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Suppose $A(x) = \int_2^x \sin t \, dt$. What is $A'(x)$?

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$$B'(x) = \frac{d}{dx} \left\{ - \int_2^x f(t) \, dt \right\} = - \sin x$$

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$C'(x) = e^x \sin(e^x)$: if we set $a = 2$, then

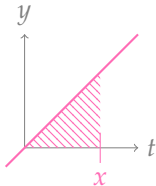
$$C(x) = A(e^x)$$

$$\implies C'(x) = A'(e^x) \cdot \frac{d}{dx}\{e^x\} = \sin(e^x) \cdot e^x$$

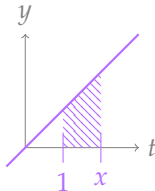


It's possible to have two different functions with the same derivative.

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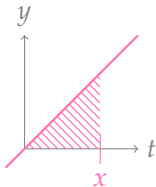


$$B(x) = \int_1^x 2t \, dt$$



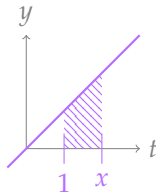
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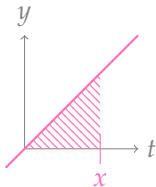
$$A'(x) = 2x$$

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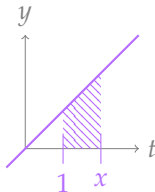
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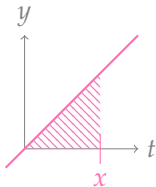
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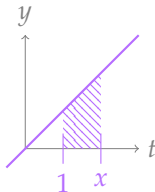
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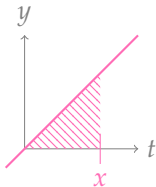
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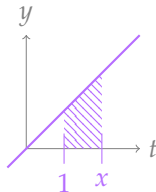
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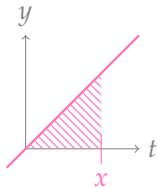
$$B(x) = \int_1^x 2t \, dt = x^2 - 1$$



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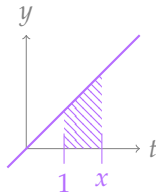
It's possible to have two different functions with the same derivative.

$$A(x) = \int_0^x 2t \, dt = x^2$$



$$A'(x) = 2x$$

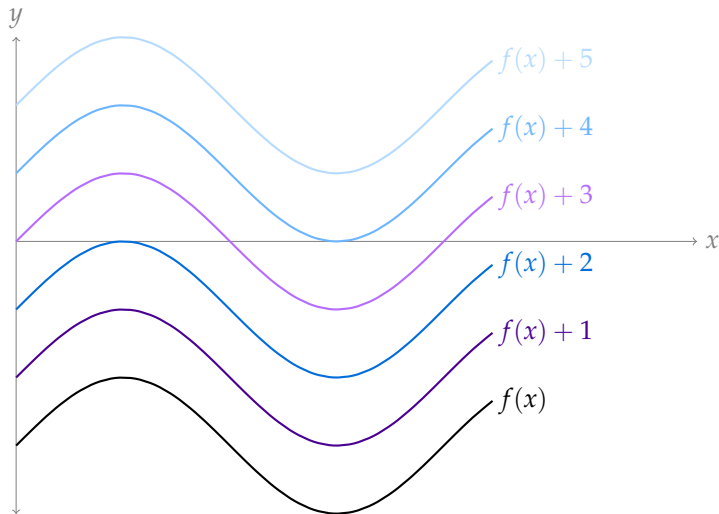
$$B(x) = \int_1^x 2t \, dt = x^2 - 1$$



$$B'(x) = 2x$$

When two functions have the same derivative, **they differ only by a constant.**

In this example: $B(x) = A(x) - 1$



If two continuous functions have the same derivative, then one is a constant plus the other.

Two clues for finding $A(x) = \int_a^x f(t) \, dt$:

- ▶ If $A(x) = \int_a^x f(t) \, dt$, then¹ $A'(x) = f(x)$
- ▶ If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C .

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- ▶ $A'(x) = e^x$.
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- ▶ $A'(x) = \cos x$.
- ▶ Guess a function with derivative $\cos x$: $F(x) = \sin x$.
- ▶ Then $A(x) = \sin x + C$ for some constant C .

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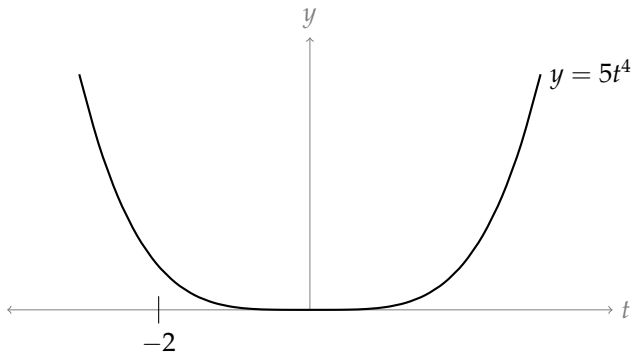
- ▶ $A'(x) = 5x^4$.
- ▶ Guess a function with derivative $5x^4$: $F(x) = x^5$.
- ▶ Then $A(x) = x^5 + C$ for some constant C .
- ▶ We ALSO know $A(-2) = \int_{-2}^{-2} 5t^4 dt = 0$, so we can find C :

$$0 = A(-2) = (-2)^5 + C \implies C = 32$$

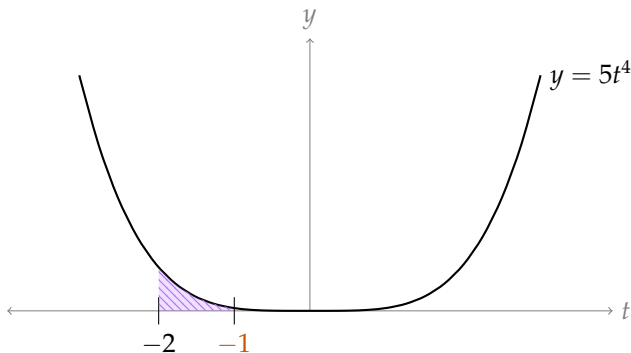
- ▶ So, $A(x) = x^5 + 32$

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$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$

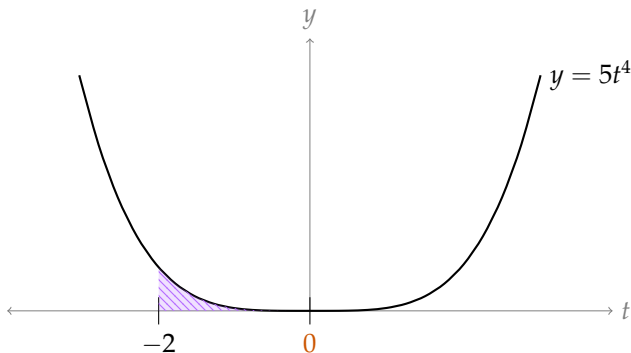


$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$



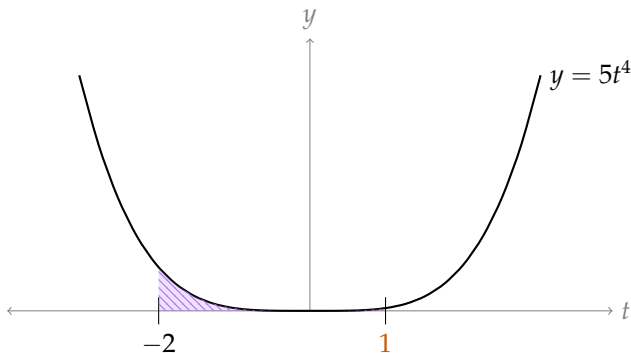
$$A(-1) = \int_{-2}^{-1} 5t^4 dt = (-1)^5 + 32 = 31$$

$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$



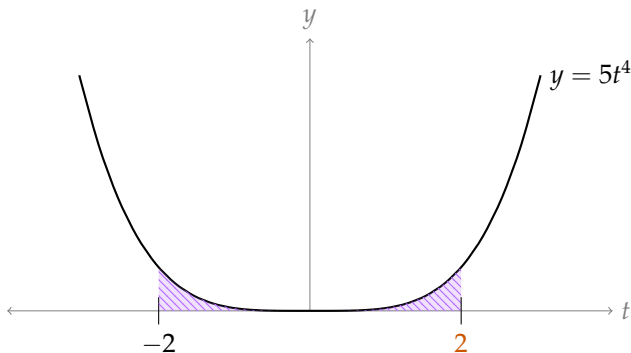
$$A(0) = \int_{-2}^0 5t^4 dt = (0)^5 + 32 = 32$$

$$A(x) = \int_{-2}^x 5t^4 \, dt = x^5 + 32$$



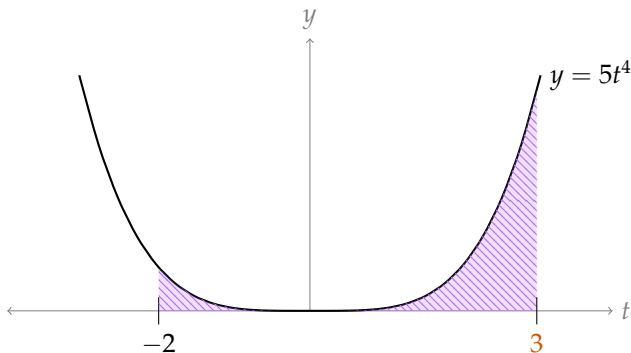
$$A(\textcolor{brown}{1}) = \int_{-2}^{\textcolor{brown}{1}} 5t^4 \, dt = (\textcolor{brown}{1})^5 + 32 = 33$$

$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$



$$A(2) = \int_{-2}^2 5t^4 dt = (2)^5 + 32 = 64$$

$$A(x) = \int_{-2}^x 5t^4 \, dt = x^5 + 32$$



$$A(3) = \int_{-2}^3 5t^4 \, dt = (3)^5 + 32 = 275$$

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- ▶ Also $A(a) = 0$, so $0 = F(a) + C$, so $C = -F(a)$
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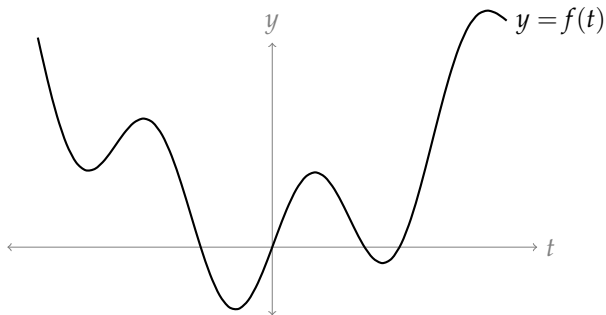
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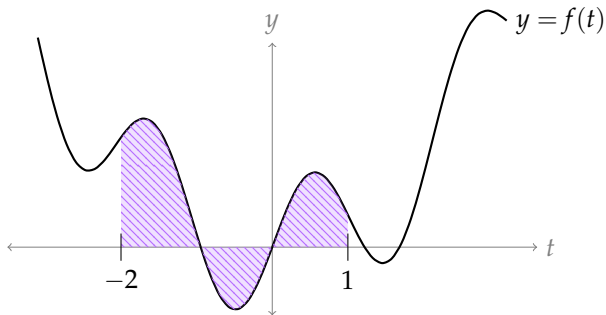
- ▶ $A'(x) = f(x)$.
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$$\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$

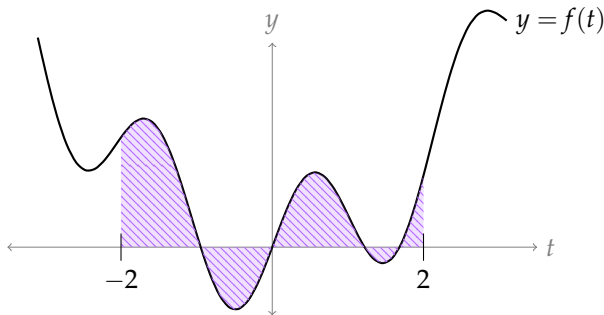


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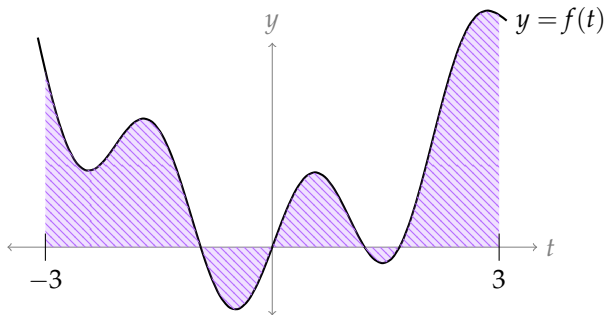
$$\int_{-2}^1 f(t) \, dt = F(1) - F(-2)$$

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$$\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-3}^3 f(t) \, dt = F(3) - F(-3)$$

Fundamental Theorem of Calculus, Part 2

Let $F(x)$ be differentiable, defined, and continuous on the interval $[a, b]$ with $F'(x) = f(x)$ for all $a < x < b$. Then

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Examples:

$$\frac{d}{dx} \{5x^7\} = 35x^6, \text{ so}$$

$$\int_0^3 35x^6 \, dx =$$

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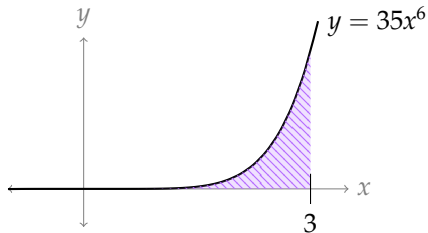
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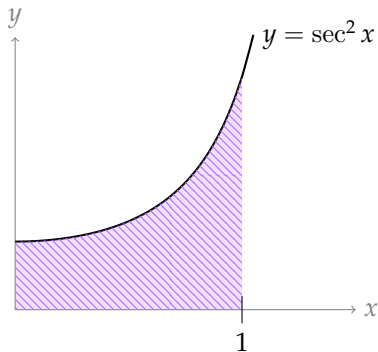
$$\int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big|_{x=\pi/4} - \tan x \Big|_{x=0} = \tan(\pi/4) - \tan 0 = 1$$

$$\int_0^3 35x^6 \, dx = F(b) - F(a) \quad \text{where} \quad F(x) = 5x^7$$



$$\int_0^3 35x^6 \, dx = 5(3)^7 - 5(0)^7$$

$$\int_0^{\pi/4} \sec^2 x \, dx = F(b) - F(a) \quad \text{where} \quad F(x) = \tan x$$



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An antiderivative of $\sin x$ is $-\cos x$, because $\frac{d}{dx} \{-\cos x\} = \sin x$.

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FTC Part 2, Abridged

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b$$

where $F(x)$ is an antiderivative of $f(x)$

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The **indefinite integral** of a function $f(x)$:

$$\int f(x) \, dx$$

means the *most general* antiderivative of $f(x)$.

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$$\int 2x \, dx = x^2 + C, \quad C \text{ "arbitrary constant."}$$

$$\int \frac{1}{x} \, dx = \log |x| + C$$

Remember: two functions with the same derivative differ by a constant, so we include the “ $+C$ ” for indefinite integrals.

DEFINITE VS INDEFINITE INTEGRALS

For each pair of properties below, decide which applies to **definite** integrals, and which to **indefinite** integrals.

No limits (or bounds) of integration, $\int f(x) dx$	
Limits (or bounds) of integration, $\int_a^b f(x) dx$	
Area under a curve	
Antiderivative	
Number	
Function	

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Function	indefinite

ANTIDIFFERENTIATION BY INSPECTION

1. $\int e^x dx$

ANTIDIFFERENTIATION BY INSPECTION

$$1. \int e^x dx = e^x + C$$

ANTIDIFFERENTIATION BY INSPECTION

1. $\int e^x \, dx = e^x + C$

2. $\int \cos x \, dx$

ANTIDIFFERENTIATION BY INSPECTION

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$$3. \int -\sin x dx$$

ANTIDIFFERENTIATION BY INSPECTION

$$1. \int e^x dx = e^x + C$$

$$2. \int \cos x dx = \sin x + C$$

$$3. \int -\sin x dx = \cos x + C$$



ANTIDIFFERENTIATION BY INSPECTION

$$1. \int e^x \, dx = e^x + C$$

$$2. \int \cos x \, dx = \sin x + C$$

$$3. \int -\sin x \, dx = \cos x + C$$

$$4. \int \frac{1}{x} \, dx$$

ANTIDIFFERENTIATION BY INSPECTION

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$$2. \int \cos x dx = \sin x + C$$

$$3. \int -\sin x dx = \cos x + C$$

$$4. \int \frac{1}{x} dx = \log |x| + C$$

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$$5. \int 1 \, dx$$

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$$5. \int 1 \, dx = x + C$$



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$$6. \int 2x \, dx$$

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$$7. \int nx^{n-1} dx \quad (n \neq 0, \text{ constant})$$

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$$7. \int nx^{n-1} dx = x^n + C \quad (n \neq 0, \text{ constant})$$

$$8. \int x^n dx \quad (n \neq -1, \text{ constant})$$



ANTIDIFFERENTIATION BY INSPECTION

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$$5. \int 1 dx = x + C$$

$$6. \int 2x dx = x^2 + C$$

$$7. \int nx^{n-1} dx = x^n + C \quad (n \neq 0, \text{ constant})$$

$$8. \int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1, \text{ constant})$$



Power Rule for Antidifferentiation

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

if $n \neq -1$ is a constant

Example:

$$\int (5x^2 - 15x + 3) dx =$$

Power Rule for Antidifferentiation

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

if $n \neq -1$ is a constant

Example:

$$\int (5x^2 - 15x + 3) dx = \frac{5}{3}x^3 - \frac{15}{2}x^2 + 3x + C$$



ANTIDERIVATIVES TO RECOGNIZE

- ▶ $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
- ▶ $\int a dx = ax + C$
- ▶ $\int e^x dx = e^x + C$
- ▶ $\int \frac{1}{x} dx = \log |x| + C$
- ▶ $\int \sin x dx = -\cos x + C$
- ▶ $\int \cos x dx = \sin x + C$
- ▶ $\int \sec^2 x dx = \tan x + C$
- ▶ $\int \sec x \tan x dx = \sec x + C$
- ▶ $\int \csc x \cot x dx = -\csc x + C$
- ▶ $\int \csc^2 x dx = -\cot x + C$
- ▶ $\int \frac{1}{1+x^2} dx = \arctan x + C$
- ▶ $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

Included Work



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