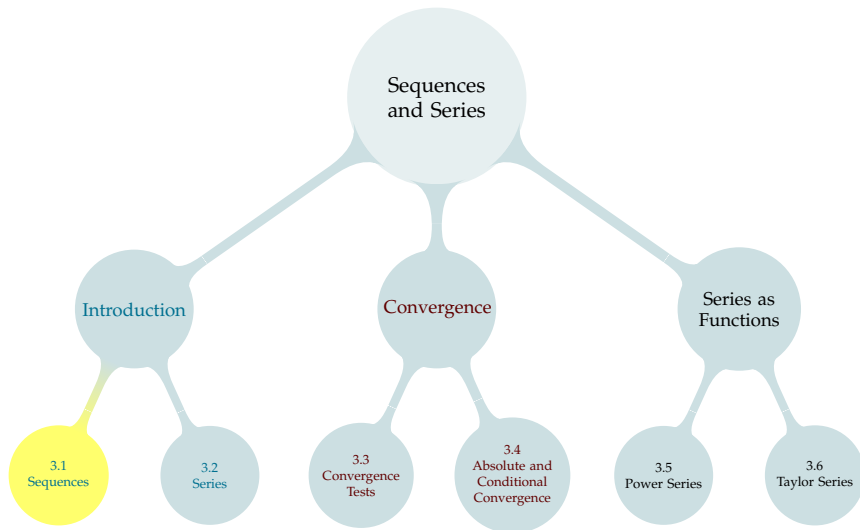


# TABLE OF CONTENTS



We can imagine the list of numbers below carrying on forever:

 $0.1$  $0.01$  $0.001$  $0.0001$  $0.00001$  $\vdots$ 

A **sequence** is a list of infinitely many numbers with a specified order.

We can imagine the list of numbers below carrying on forever:

$$a_1 = 0.1$$

$$a_2 = 0.01$$

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$$a_4 = 0.0001$$

$$a_5 = 0.00001$$

$$\vdots$$

A **sequence** is a list of infinitely many numbers with a specified order. It is denoted  $\{a_1, a_2, \dots, a_n, \dots\}$  or  $\{a_n\}_{n=1}^{\infty}$ , etc.

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Imagine *adding up* this sequence of numbers.

A **series** is a sum  $a_1 + a_2 + \dots + a_n + \dots$  of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- ▶ What does it mean to add up infinitely many things?
- ▶ Should infinitely many things add up to an infinitely large number?
- ▶ Does the order in which the numbers are added matter?
- ▶ Can we add up infinitely many functions, instead of just infinitely many numbers?

## Sequence

A **sequence** is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- ▶  $\{1, 2, 3, 4, 5, 6, 7, 8, \dots\}$  (natural numbers)
- ▶  $\{3, 1, 4, 1, 5, 9, 2, 6, \dots\}$  (digits of  $\pi$ )
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Sometimes it's 0, sometimes something else, for example a year.

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- ▶ Often we omit the limits and even the brackets, writing  $a_n = \frac{1}{n}$ .

# SEQUENCE NOTATION

For convenience, we write  $a_1$  for the first term of a sequence,  $a_2$  for the second term, etc.

In the sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ ,  
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Our primary concern with sequences will be the behaviour of  $a_n$  as  $n$  tends to infinity and, in particular, whether or not  $a_n$  “settles down” to some value as  $n$  tends to infinity.

## Convergence

A sequence  $\{a_n\}_{n=1}^{\infty}$  is said to **converge** to the limit  $A$  if  $a_n$  approaches  $A$  as  $n$  tends to infinity. If so, we write

$$\lim_{n \rightarrow \infty} a_n = A \quad \text{or} \quad a_n \rightarrow A \text{ as } n \rightarrow \infty$$

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To study the behaviour of  $\frac{n}{2n+1}$  as  $n \rightarrow \infty$ , it is a good idea to write it as:

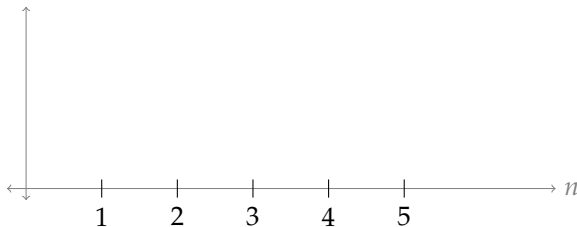
$$\frac{n}{2n+1} = \frac{1}{2 + \frac{1}{n}}$$

As  $n \rightarrow \infty$ , the  $\frac{1}{n}$  in the denominator tends to zero, so that the denominator  $2 + \frac{1}{n}$  tends to 2 and  $\frac{1}{2 + \frac{1}{n}}$  tends to  $\frac{1}{2}$ . So

$$\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}$$

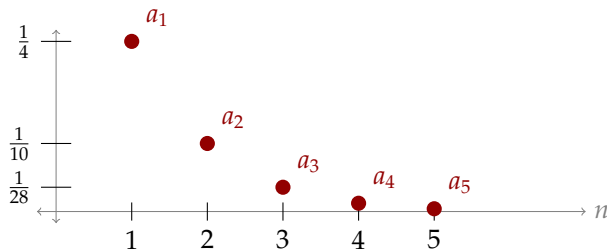
Consider the sequence  $a_n = \frac{1}{3^n + 1}$ .

$$\lim_{n \rightarrow \infty} a_n =$$

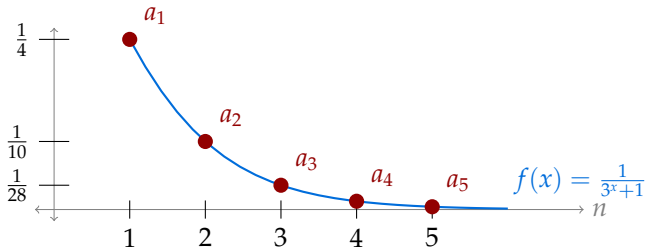


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### Theorem 3.1.6

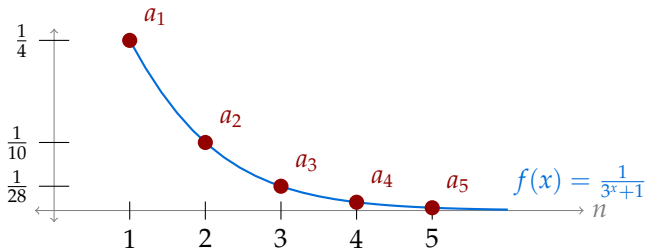
If  $\lim_{x \rightarrow \infty} f(x) = L$

and if  $a_n = f(n)$  for all positive integers  $n$ , then

$$\lim_{n \rightarrow \infty} a_n = L$$



Consider the sequence  $a_n = \frac{1}{3^n + 1}$ .  $\lim_{n \rightarrow \infty} a_n = 0$



### Theorem 3.1.6

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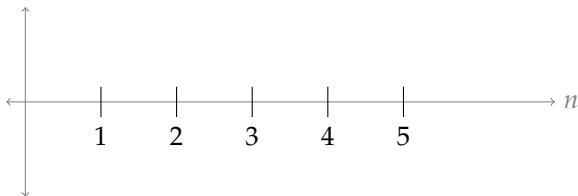
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# CAUTIONARY TALE

Consider the sequence  $b_n = \sin(\pi n) =$

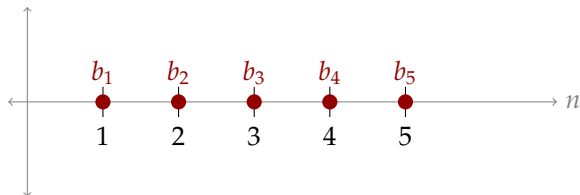
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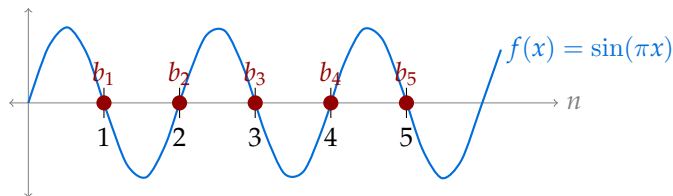
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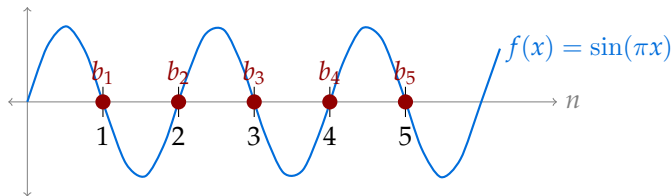


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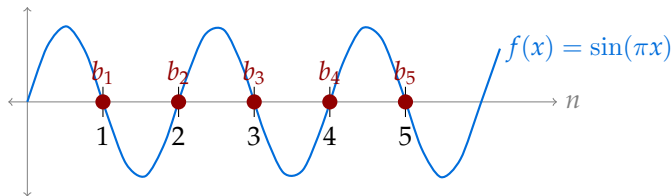
$$\lim_{x \rightarrow \infty} f(x)$$



# CAUTIONARY TALE

Consider the sequence  $b_n = \sin(\pi n) = \{0, 0, 0, 0, 0, \dots\}$

$$\lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} f(x)$$

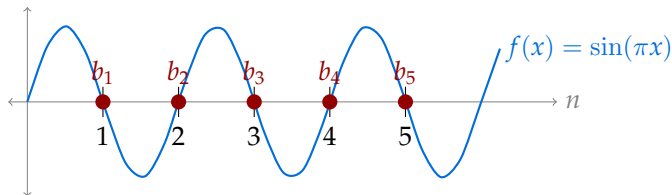


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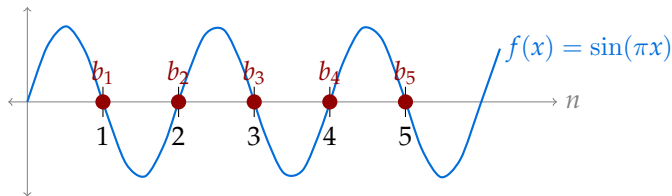


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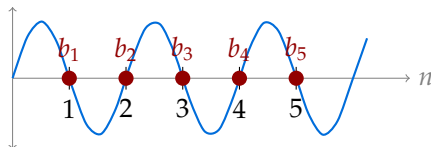
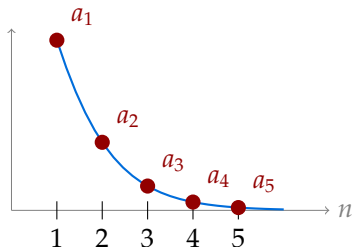
$$\lim_{n \rightarrow \infty} b_n = 0$$

$$\lim_{x \rightarrow \infty} f(x) \text{ DNE}$$



## Theorem

If  $\lim_{x \rightarrow \infty} f(x) = L$  and if  $a_n = f(n)$  for all natural  $n$ , then  $\lim_{n \rightarrow \infty} a_n = L$ .



## Arithmetic of Limits

Let  $A$ ,  $B$  and  $C$  be real numbers and let the two sequences  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  converge to  $A$  and  $B$  respectively. That is, assume that

$$\lim_{n \rightarrow \infty} a_n = A$$

$$\lim_{n \rightarrow \infty} b_n = B$$

Then the following limits hold.

(a)  $\lim_{n \rightarrow \infty} [a_n + b_n] = A + B$

(b)  $\lim_{n \rightarrow \infty} [a_n - b_n] = A - B$

(c)  $\lim_{n \rightarrow \infty} Ca_n = CA.$

(d)  $\lim_{n \rightarrow \infty} a_n b_n = AB$

(e) If  $B \neq 0$ , then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$

Evaluate the following limits:

$$\blacktriangleright \lim_{n \rightarrow \infty} e^{-n} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1+n}{n} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1}{n^2} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} 2n^2 =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) (2n^2) =$$

Evaluate the following limits:

$$\blacktriangleright \lim_{n \rightarrow \infty} e^{-n} = 0$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1+n}{n} = 1$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

$$\blacktriangleright \lim_{n \rightarrow \infty} 2n^2 = \infty$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) (2n^2) = 2$$

(As you might guess, the expression “ $\lim_{n \rightarrow \infty} a_n = \infty$ ” means that  $a_n$  grows without bound as  $n \rightarrow \infty$ .)

## Continuous functions of limits

If  $\lim_{n \rightarrow \infty} a_n = L$  and if the function  $g(x)$  is continuous at  $L$ , then

$$\lim_{n \rightarrow \infty} g(a_n) = g(L)$$

Evaluate  $\lim_{n \rightarrow \infty} \left[ \sin \left( \frac{\pi n}{2n+1} \right) \right]$

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$$\lim_{n \rightarrow \infty} \left[ \frac{\pi n}{2n+1} \right] = \lim_{n \rightarrow \infty} \left[ \frac{\pi}{2 + \frac{1}{n}} \right] = \frac{\pi}{2}$$

$$\lim_{n \rightarrow \infty} \left[ \sin \left( \frac{\pi n}{2n+1} \right) \right] = \sin \left( \frac{\pi}{2} \right) = 1$$

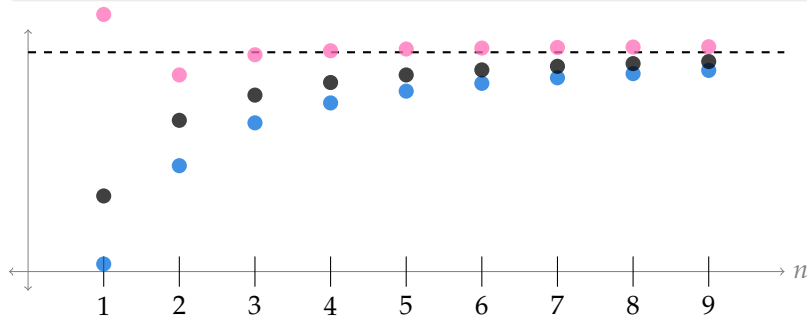
## Squeeze Theorem

If  $a_n \leq c_n \leq b_n$  for all sufficiently large natural numbers  $n$ , and if

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$$

then

$$\lim_{n \rightarrow \infty} c_n = L$$





Evaluate

$$\lim_{n \rightarrow \infty} \left( \frac{2n + \cos n}{n + 1} \right)$$

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Use squeeze theorem:

$$-1 \leq \cos n \leq 1$$

$$2n - 1 \leq 2n + \cos n \leq 2n + 1$$

$$\frac{2n - 1}{n + 1} \leq \frac{2n + \cos n}{n + 1} \leq \frac{2n + 1}{n + 1}$$

$$\lim_{n \rightarrow \infty} \frac{2n - 1}{n + 1} = \lim_{n \rightarrow \infty} \frac{2n + 1}{n + 1} = 2$$

$$2 = \lim_{n \rightarrow \infty} \frac{2n + \cos n}{n + 1}$$

Let  $a_n = (-n)^{-n}$ . Evaluate  $\lim_{n \rightarrow \infty} a_n$ .

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First, we note  $a_n = (-1)^{-n} \cdot (n^{-n}) = \frac{(-1)^n}{n^n}$  because  $(-1)^{-n} = ((-1)^{-1})^n = (-1)^n$ .

This sequence alternates between positive and negative terms. We can show that the positive terms tend to zero and the negative terms tend to zero. So, we can apply the squeeze theorem.

$$\text{Set } b_n = \frac{-1}{n^n} \text{ and } c_n = \frac{1}{n^n}$$

Then,  $b_n < a_n < c_n$  for all natural  $n$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = 0$$

$$\text{So, } \lim_{n \rightarrow \infty} a_n = 0$$

## Included Work



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