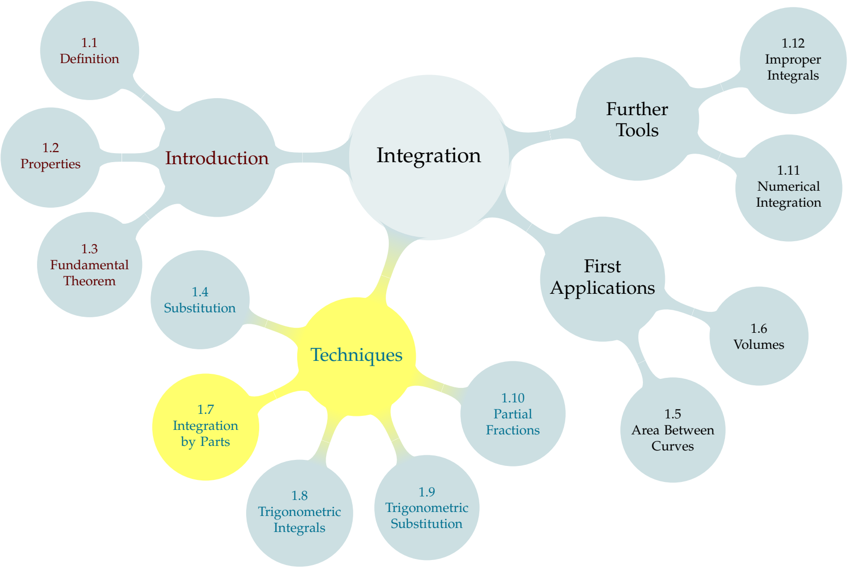
Table of Contents



Reverse the Product Rule

Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{u(x)\cdot v(x)\right\} = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

Related fact:

$$\int \Big[u'(x)\cdot v(x) + u(x)\cdot v'(x)\Big]\mathrm{d}x = u(x)\cdot v(x) + C$$

Rearrange:

$$\implies \int \Big[u'(x)v(x)\Big]\mathrm{d}x + \int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) + C$$

$$\implies \int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) - \int \Big[v(x)u'(x)\Big]\mathrm{d}x + C$$

Integration by Parts

$$\int \left[u(x)v'(x)\right] \mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right] \mathrm{d}x$$

Example: $\int xe^x \mathrm{d}x$

Check Our Work

In the previous slide, we evaluated

$$\int xe^x \mathrm{d}x = xe^x - e^x + C$$

for some constant C. We can check that this is correct by differentiating.

Integration by Parts (IBP): A Closer Look

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\underbrace{\int xe^x \mathrm{d}x}_{\text{How to integrate??}} = x(e^x) - \underbrace{1\int e^x \mathrm{d}x}_{\text{Easy to integrate!}} + C$$

We start and end with an integral. IBP is only useful if the new integral is somehow an improvement.

We differentiate the function we choose as $u(x)$, and antidifferentiate the function we choose as $v'(x)$

Choosing $u(x)$ and $v(x)$

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int \left[x\sin x\right]\mathrm{d}x =$$

Option A:

$u(x) = x$
$v'(x) = \sin x$

Option B:

$u(x) = \sin x$
$v'(x) = x$

Q

Check Our Work

To check our work, we can calculate $\frac{d}{dx}\left\{-x\cos x+\sin x+C\right\}$. It should work out to be $x\sin x$.

Choosing $u(x)$ and $v(x)$

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int \left[x^2 \log x\right]\mathrm{d}x =$$

Option A:	Option B:
$u(x) = x^2$	$u(x) = \log x$
$v'(x) = \log x$	$v'(x) = x^2$

Q

Check Our Work

To check our work, we can calculate $\frac{d}{dx}\left\{\frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C\right\}$. It should work out to be $x^2 \log x$.

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int \left[\tfrac{1}{2}xe^{6x}\right]\mathrm{d}x =$$

Option A:

$$u(x) = \tfrac{1}{2}x$$
$$v'(x) = e^{6x}$$

Option B:

$$u(x) = e^{6x}$$
$$v'(x) = \tfrac{1}{2}x$$

Q

Check Our Work

We check that $\displaystyle\int \left[\frac{1}{2}xe^{6x}\right] \mathrm{d}x = \frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C$ by differentiating.

MNEMONIC

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u + C$$

We abbreviate:

- $u(x) \to u$
- $u'(x)\,\mathrm{d}x \to \mathrm{d}u$
- $v(x) \to v$
- $v'(x)\,\mathrm{d}x \to \mathrm{d}v$

Choosing u, dv in your head

Choose u and dv to evaluate the integral below:

$$\int (3t+5)\cos(t/4)\mathrm{d}t$$

Thoughts: $\displaystyle\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

u gets differentiated, and dv gets antidifferentiated.

Evaluate, using IBP or Substitution

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u + C$$

- $\displaystyle\int x e^{x^2}\,\mathrm{d}x$

- $\displaystyle\int x^2 e^x\,\mathrm{d}x$

- $\displaystyle\int e^{x+e^x}\,\mathrm{d}x$

Definite Integration by Parts

Method 1: Antidifferentiate first, then plug in limits of integration.

Method 2: Plug as you go.

Evaluate $\int_1^e \log^2 x \, \mathrm{d}x$

SPECIAL TECHNIQUE: $v'(x) = 1$

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u + C$$

Evaluate $\int \log x \, \mathrm{d}x$ using integration by parts.

Check Our Work

Let's check that $\int \log x \, \mathrm{d}x = x \log x - x + C$.

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u + C$$

Evaluate $\int \arctan x \, \mathrm{d}x$ using integration by parts.

Q

Check Our Work

Let's check that $\displaystyle\int \arctan x \, \mathrm{d}x = x \arctan x - \frac{1}{2} \log|1 + x^2| + C$.

Setting $\mathrm{d}v = 1\,\mathrm{d}x$ is a very specific technique. You'll probably only see it in situations integrating logarithms and inverse trigonometric functions.

$$\int \log x \,\mathrm{d}x, \qquad \int \arcsin x \,\mathrm{d}x, \qquad \int \arccos x \,\mathrm{d}x, \qquad \int \arctan x \,\mathrm{d}x, \qquad \text{etc.}$$

Evaluate $\displaystyle\int e^x \cos x \, \mathrm{d}x$ using integration by parts.

Let $u = e^x$ and $\mathrm{d}v = \cos x \, \mathrm{d}x$. Then $\mathrm{d}u = e^x \, \mathrm{d}x$ and $v = \sin x$:

$$\int e^x \cos x \, \mathrm{d}x = e^x \sin x - \int e^x \sin x \, \mathrm{d}x$$

Let $u = e^x$ and $\mathrm{d}v = \sin x \, \mathrm{d}x$. Then $\mathrm{d}u = e^x \, \mathrm{d}x$ and $v = -\cos x$:

$$= e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, \mathrm{d}x \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, \mathrm{d}x$$

$$\int e^x \cos x \, \mathrm{d}x = e^x \sin x + e^x \cos x - \int e^x \cos x \, \mathrm{d}x$$

$$2 \int e^x \cos x \, \mathrm{d}x = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, \mathrm{d}x = \frac{1}{2} \left(e^x \sin x + e^x \cos x \right) + C$$

Integrating around in a circle

We can use this technique to antidifferentiate products of two functions that almost, but don't quite, stay the same under (anti)differentiation.

Use integration by parts a number of times, ending up with an expression involving (a scalar multiple of) the original integral.

To do this, be consistent with your choice of u and $\mathrm{d}v$.

Evaluate $\displaystyle\int \cos(\log x)\,\mathrm{d}x$.

Q

Check Our Work

We check that $\displaystyle\int \cos(\log x)\mathrm{d}x = \frac{x}{2}\left[\,\cos(\log x) + \sin(\log x)\right] + C$ by differentiating.

Evaluate $\int e^{2x} \sin x \, \mathrm{d}x$ using integration by parts.

Q

Check Our Work

We can check our work by differentiating $\frac{1}{5}e^{2x}[2\sin x - \cos x] + C$. We should end up with $e^{2x}\sin x$.