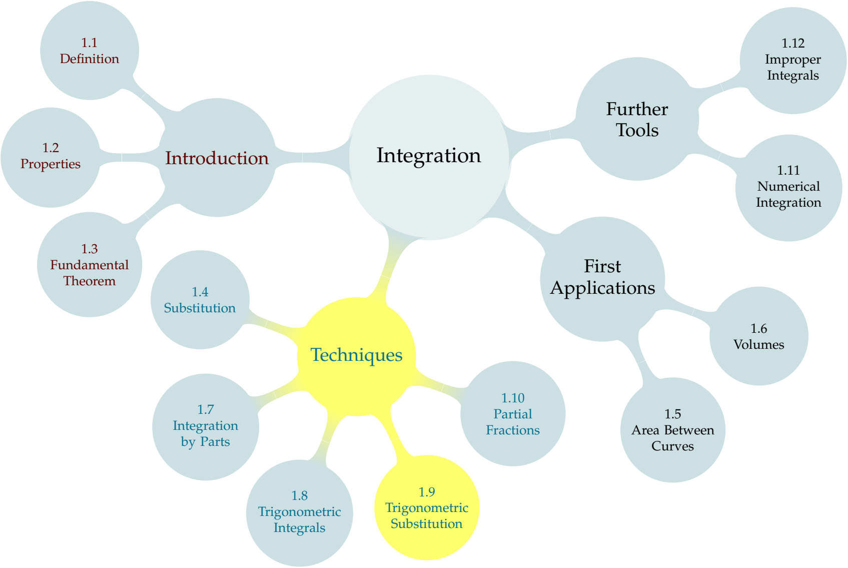
Table of Contents



Warmup

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x.$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$.

Check Our Work

Let's verify that $\displaystyle\int \frac{1}{\sqrt{x^2+1}} = \log\left|\sqrt{x^2+1}+x\right| + C.$

Seems improbable, right?

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.

METHOD (ONE STANDARD CASE)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:

METHOD (ONE STANDARD CASE)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1 - x^2}$ into

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$

METHOD (ONE STANDARD CASE)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into

METHOD (ONE STANDARD CASE)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$

METHOD (ONE STANDARD CASE)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1-\sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1+\tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into $\sqrt{\tan^2\theta} = |\tan\theta|$

METHOD (ONE STANDARD CASE)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into $\sqrt{\tan^2\theta} = |\tan\theta|$
- After integrating, convert back to the original variable (possibly using a triangle–more details later)

Focus on the Algebra

$$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 - 1}$
- $\sqrt{x^2 + 1}$
- $\sqrt{1 - x^2}$

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2+7}$

- $\sqrt{3-2x^2}$

Closer look at absolute values

SKIP CLOSER LOOK

Consider the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

Closer look at absolute values

SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Closer look at absolute values

Skip closer look

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

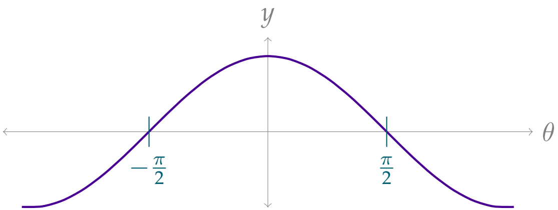
- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$

Closer look at absolute values

SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$
- On the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$, so $|\cos\theta| = \cos\theta$

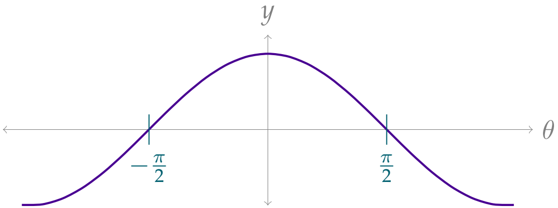


Closer look at absolute values

SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$
- On the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$



- So, in general, when we use the substitution $x = \sin\theta$ with trigonometric substitution, we can expect $|\cos\theta| = \cos\theta$.

CLOSER LOOK AT ABSOLUTE VALUES

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

Closer look at absolute values

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Closer look at absolute values

Skip closer look

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

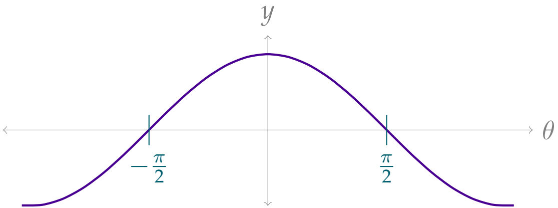
- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = \frac{a}{|\cos\theta|}$

Closer look at absolute values

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = \frac{a}{|\cos\theta|}$
- On the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$, so $|\cos\theta| = \cos\theta$ and $|\sec\theta| = \sec\theta$.

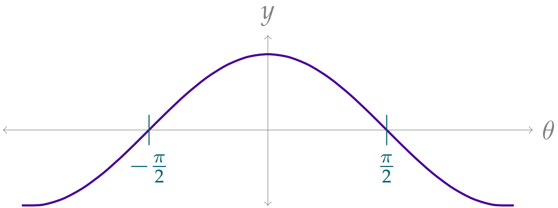


Closer look at absolute values

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = \frac{a}{|\cos\theta|}$
- On the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$, so $|\cos\theta| = \cos\theta$ and $|\sec\theta| = \sec\theta$.



- So, in general, when we use the substitution $x = \tan\theta$ with trigonometric substitution, we can expect $|\sec\theta| = \sec\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2-a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \le \theta \le \pi$
- $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2-a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \le \theta \le \pi$
- $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:
  - For $\sqrt{x^2-a^2}$ to be defined, we need $x^2 \ge a^2$. I.e. $x \ge a$ or $x \le -a$.

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

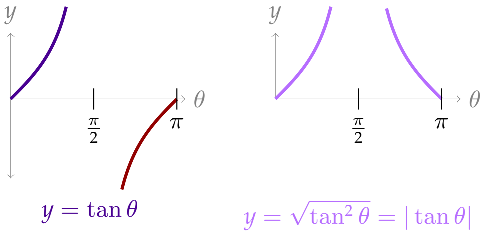
- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:
  - For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \geq a^2$. I.e. $x \geq a$ or $x \leq -a$.
  - When $x \geq a$, we have $0 \leq \theta < \frac{\pi}{2}$, $\tan\theta \geq 0$, so $|\tan\theta| = \tan\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2-a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \le \theta \le \pi$
- $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:
  - For $\sqrt{x^2-a^2}$ to be defined, we need $x^2 \ge a^2$. I.e. $x \ge a$ or $x \le -a$.
  - When $x \ge a$, we have $0 \le \theta < \frac{\pi}{2}$, $\tan\theta \ge 0$, so $|\tan\theta| = \tan\theta$.
  - When $x \le -a$, we have $\frac{\pi}{2} < \theta \le \pi$, $\tan\theta < 0$, so $|\tan\theta| = -\tan\theta$.



Absolute Values

From now on, we will assume:

- With the substitution $x = a\sin\theta$ for $\sqrt{a^2 - x^2}$, $|\cos\theta| = \cos\theta$
- With the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, $|\sec\theta| = \sec\theta$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

Evaluate $\displaystyle\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \sqrt{1-4x^2}\,\mathrm{d}x$

Evaluate $\int \sqrt{1-4x^2}\,\mathrm{d}x$

CHECK OUR WORK

In the last example, we computed

$$\int \sqrt{1-4x^2}\,\mathrm{d}x = \frac{1}{4}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right) + C.$$

To check, we differentiate.

Identities

$$1-\sin^2\theta=\cos^2\theta \qquad \sin(2\theta)=\cos\theta$$

$$1+\tan^2\theta=\sec^2\theta \qquad \sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta-1=\tan^2\theta \qquad \cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}}\,\mathrm{d}x$

Check Our Work

Let's check our result, $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \log\left|x+\sqrt{x^2-1}\right| + C.$

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3-x^2+2x}$.

Identities have two “parts” that turn into one part:

- $1-\sin^2\theta=\cos^2\theta$
- $1+\tan^2\theta=\sec^2\theta$
- $\sec^2\theta-1=\tan^2\theta$

But our quadratic expression has *three* parts.
Fact: $3-x^2+2x=4-(x-1)^2$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b:

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c:

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c: $3 = c - b^2 = c - 1$, so $c = 4$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c: $3 = c - b^2 = c - 1$, so $c = 4$
3. All together:

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c: $3 = c - b^2 = c - 1$, so $c = 4$
3. All together: $3 - x^2 + 2x = 4 - (x-1)^2$

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\, \mathrm{d}x$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

One of those parts is a constant, and one is squared.

Evaluate $\displaystyle\int \frac{x^2-6x+9}{\sqrt{6x-x^2}}\,\mathrm{d}x = \int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x.$

Check Our Work

Let's verify that

$$\int \frac{x^2-6x+9}{\sqrt{6x-x^2}} = \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C:$$

Included Work

 'Notebook' by Iconic is licensed under CC BY 3.0 (accessed 9 June 2021, modified), 14–16

 'Notebook' by Iconic is licensed under CC BY 3.0 (accessed 9 June 2021), 35, 37, 40, 52