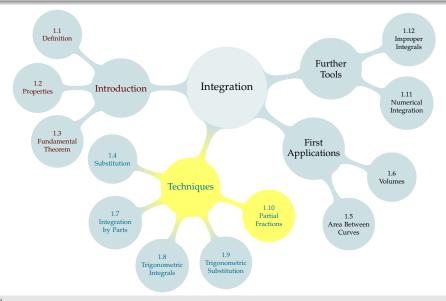
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#### **MOTIVATION**

How to integrate  $\int \frac{x-2}{(x+1)(2x-1)} dx$ ?

Useful fact: 
$$\frac{x-2}{(x+1)(2x-1)} = \frac{1}{x+1} - \frac{1}{2x-1}$$

So:

$$\int \frac{x-2}{(x+1)(2x-1)} dx = \int \frac{1}{x+1} dx - \int \frac{1}{2x-1} dx$$
$$= \log|x+1| - \frac{1}{2}\log|2x-1| + C$$

Method of Partial Fractions: Algebraic method to turn any rational function (i.e. ratio of two polynomials) into the sum of easier-to-integrate rational functions.

The rational function

$$\frac{\text{numerator}}{K(x-a_1)(x-a_2)\cdots(x-a_j)}$$

can be written as

$$\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_j}{x-a_j}$$

for some constants  $A_1, A_2, \dots, A_i$ , provided

- (1) the linear roots  $a_1, \dots a_i$  are distinct, and
- (2) the degree of the numerator is strictly less than the degree of the denominator.

$$\frac{7x + 13}{(2x + 5)(x - 2)} =$$

We found 7x + 13 = A(x - 2) + B(2x + 5) for some constants A and B. What are A and B?

Method 1: set *x* to convenient values.

Method 2: match coefficients of powers of *x*.

## All together:

$$\frac{7x+13}{2x^2+x-10} = \frac{A}{2x+5} + \frac{B}{x-2}$$

$$A = 1, \quad B = 3$$

We check that 
$$\int \frac{7x + 13}{2x^2 + x - 10} =$$
differentiating.

by

$$\frac{x^2 + 5}{2x(3x+1)(x+5)}$$
 is hard to antidifferentiate, but it can be written as 
$$\frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}$$
 for some constants  $A$ ,  $B$ , and  $C$ .

Once we find *A*, *B*, and *C*, integration is easy:

$$\int \frac{x^2 - 24x + 5}{2x(3x+1)(x+5)} dx$$

$$= \int \left(\frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}\right) dx$$

$$= \frac{A}{2} \log|x| + \frac{B}{3} \log|3x+1| + C \log|x+5| + D$$

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}$$

Find constants *A*, *B*, and *C*. Start: make a common denominator

#### Let's check that

$$\frac{x^2 + 5}{2x(3x+1)(x+5)} =$$

#### All together:

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}$$

## Repeated Linear Factors

A rational function  $\frac{P(x)}{(x-1)^4}$ , where P(x) is a polynomial of degree strictly less than 4, can be written as

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}$$

for some constants *A*, *B*, *C*, and *D*.

$$\frac{5x - 11}{(x - 1)^2} =$$

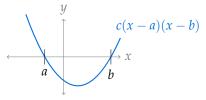
Set up the form of the partial fractions decomposition. (You do not have to solve for the parameters.)

$$\frac{3x + 16}{(x+5)^3} =$$

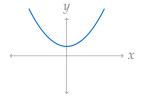
$$\frac{-2x-10}{(x+1)^2(x-1)} =$$

## IRREDUCIBLE QUADRATIC FACTORS

Sometimes it's not possible to factor our denominator into linear factors with real terms.



If a quadratic function has real roots a and b (possibly a = b, possibly  $a \neq b$ ), then we can write it as c(x - a)(x - b) for some constant c.



If a quadratic function has no real roots, then it can't be factored into (real) linear factors. It is irreducible.

# IRREDUCIBLE QUADRATIC FACTORS

When the denominator has an irreducible quadratic factor  $x^2 + bx + c$ , we add a term  $\frac{Ax + B}{x^2 + bx + c}$  to our composition. (The degree of the numerator must still be smaller than the degree of the denominator.) Write out the form of the partial fraction decomposition (but do not solve for the parameters):



The purpose of the partial fraction decomposition is to end up with functions that we can integrate.

$$\blacktriangleright \text{ Evaluate: } \int \frac{1}{(x+1)^2 + 1} \mathrm{d}x$$

Evaluate  $\int \frac{4}{(3x+8)^2+9} dx$ 

We found 
$$\int \frac{4}{(3x+8)^2+9} dx =$$

Evaluate 
$$\int \frac{x+1}{x^2+2x+2} dx.$$

(Hint: start by completing the square.)

We found 
$$\int \frac{x+1}{x^2+2x+2} dx =$$

These rules work only when the degree of the numerator is less than the degree of the denominator.

$$\int \frac{x^3}{(x-2)^2(x-3)(x-4)^2} dx \qquad \int \frac{x^5}{(x-2)^2(x-3)(x-4)^2} dx$$

If the degree of the numerator is too large, we use polynomial long division.

Evaluate  $\int \frac{8x^2 + 22x + 23}{2x + 3} dx.$ 

## We computed

$$\int \frac{8x^2 + 22x + 23}{2x + 3} \, \mathrm{d}x =$$

Evaluate  $\int \frac{3x^3 + x + 3}{x - 2} \, \mathrm{d}x.$ 



#### We found

$$\int \frac{3x^3 + x + 3}{x - 2} \, \mathrm{d}x =$$

Evaluate  $\int \frac{3x^2 + 1}{x^2 + 5x} \, \mathrm{d}x.$ 

We found 
$$\int \frac{3x^2 + 1}{x^2 + 5x} \, \mathrm{d}x =$$

### **FACTORING**

$$P(x) = x^3 + 2x^2 - 5x - 6$$

### **FACTORING**

$$P(x) = 2x^3 - 3x^2 + 4x - 6$$