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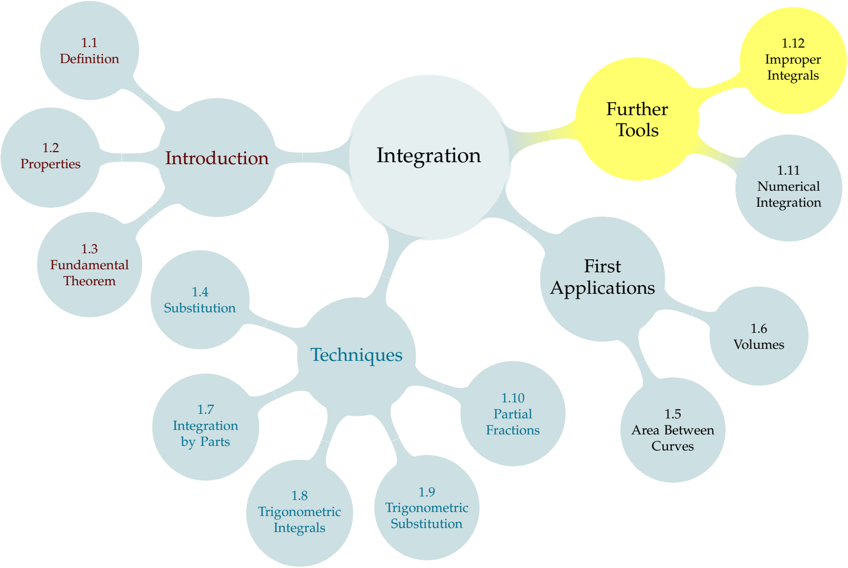
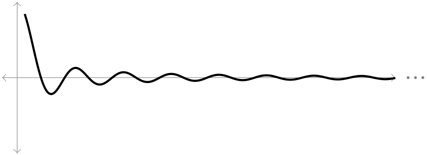


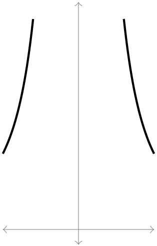
An integral is improper if one or both of the following happen:

- The region of integration is unbounded, e.g. $\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x$



$$\Delta x = \frac{b-a}{n} = \frac{\infty}{n} \text{ ???}$$

- The integrand is unbounded over the interval, e.g. $\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x$



$$f(0)\Delta x = \text{???}$$

Strategy

In both cases, we eliminate the offending parts of the integral using limits.

$$\int_1^\infty \frac{\sin x}{x}\,\mathrm{d}x = \lim_{b\to\infty}\left[\int_1^b \frac{\sin x}{x}\,\mathrm{d}x\right]$$

$$\int_0^3 \frac{1}{x}\,\mathrm{d}x = \lim_{a\to 0^+}\left[\int_a^3 \frac{1}{x}\,\mathrm{d}x\right]$$

If the limit doesn't exist, we say the integral diverges. Otherwise it converges.

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x =$$

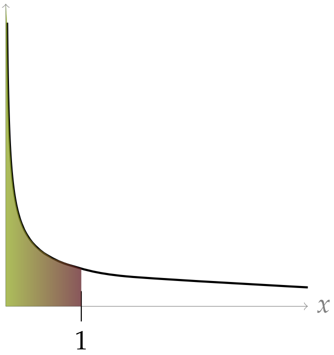
$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x =$$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$

When an integral has multiple sources of impropriety, we break it up into integrals that have only one source each. If all of them converge, the original integral converges. If any of them diverges, the original integral diverges as well.

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$$

Evaluate $\displaystyle\int_0^1 \frac{1}{2\sqrt{x}}\,\mathrm{d}x$



Same idea: we solve our problems by ignoring them (temporarily).
Eliminate the problematic part of the integral using a limit.

$$\int_0^1 \frac{1}{2\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{2\sqrt{x}}\,\mathrm{d}x\right] = \lim_{a\to 0^+}\left[1-\sqrt{a}\right] = 1$$

Evaluate $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\,\mathrm{d}x$

$$\int x^{-2}\,\mathrm{d}x = -x^{-1} + C = -\frac{1}{x} + C$$

$$\begin{aligned}\lim_{a\to 0^+} \int_a^1 \frac{1}{x^2}\,\mathrm{d}x &= \lim_{a\to 0^+} \left[-\frac{1}{x}\right]_a^1 \\ &= \lim_{a\to 0^+} \left[-1 + \frac{1}{a}\right] = \infty\end{aligned}$$

Once we see that one part of the improper integral diverges, we stop: the entire integral diverges, regardless of what happens to the left of the y-axis.

Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x$, or show that it diverges.

$$\begin{aligned}
u &= \sin x,\ \mathrm{d}u = \cos x\,\mathrm{d}x\\
u(0) &= 0\\
\lim_{b\to\infty}\left[\int_0^b \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x\right] &= \lim_{b\to\infty}\left[\int_0^{\sin b}\frac{1}{1+u^2}\,\mathrm{d}u\right]\\
&= \lim_{b\to\infty}\left[\arctan(\sin b)-\arctan(0)\right]\\
&= \lim_{b\to\infty}\left[\arctan(\sin b)\right]
\end{aligned}$$

As b goes to infinity, $\sin b$ oscillates between -1 and 1, so $\arctan(\sin b)$ oscillates between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. Since its limit does not exist, the integral diverges.

Warning: Sneaky Divergence

If you don't realize that an integral diverges, you can generate answers that look plausible but are secretly nonsense.

For example, attempting to use the Fundamental Theorem of Calculus in the example $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\mathrm{d}x$ gives $\displaystyle\left[-\frac{1}{x}\right]_{-2}^{1} = -\frac{3}{2}$: a poor approximation for positive infinity!

Evaluate $\displaystyle\int_0^1 \frac{1}{x^p}\mathrm{d}x$ and $\displaystyle\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ when p is constant.

$$\int \frac{1}{x^p}\,\mathrm{d}x = \int x^{-p}\,\mathrm{d}x = \begin{cases} \log|x| + C & \text{if } p = 1 \\ \frac{x^{1-p}}{1-p} + C & \text{if } p \neq 1 \end{cases}$$

$$\int_a^b \frac{1}{x^p}\,\mathrm{d}x = \begin{cases} \log|b| - \log|a| & \text{if } p = 1 \\ \frac{b^{1-p}-a^{1-p}}{1-p} & \text{if } p \neq 1 \end{cases} \quad \text{if } x = 0 \text{ is not in } [a,b]$$

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \begin{cases} \lim\limits_{b\to\infty} \log|b| & \text{if } p = 1 \\ \lim\limits_{b\to\infty} \left[\frac{b^{1-p}-1}{1-p}\right] & \text{if } p \neq 1 \end{cases} \quad : \quad \begin{cases} \text{divergent} & \text{if } p = 1 \\ \text{divergent} & \text{if } p < 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,\mathrm{d}x = \begin{cases} \lim\limits_{a\to 0^+} -\log|a| & \text{if } p = 1 \\ \lim\limits_{a\to 0^+} \left[\frac{1-a^{1-p}}{1-p}\right] & \text{if } p \neq 1 \end{cases} \quad : \quad \begin{cases} \text{divergent} & \text{if } p = 1 \\ \frac{1}{1-p} & \text{if } p < 1 \\ \text{divergent} & \text{if } p > 1 \end{cases}$$

p-test

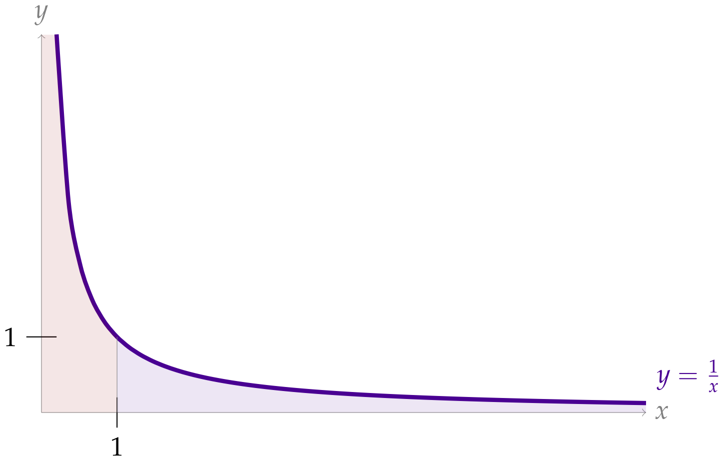
Let p be a constant.

If $p < 1$, then $\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ converges

If $p \geq 1$, then $\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ diverges

If $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converges

If $p \leq 1$, then $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ diverges



$\int_0^1 \frac{1}{x}\,\mathrm{d}x$ diverges

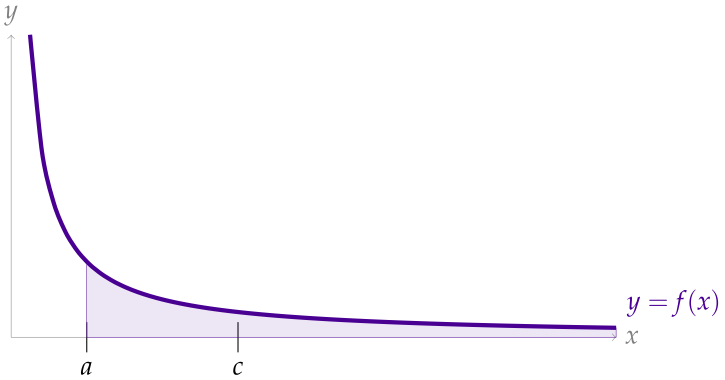
$\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ diverges

Decide whether each integral converges or diverges.

- $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,\mathrm{d}x$ converges
- $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges
- $\displaystyle\int_0^1 \frac{1}{x}\,\mathrm{d}x$ diverges
- $\displaystyle\int_0^1 \frac{1}{x^{1.5}}\,\mathrm{d}x$ diverges
- $\displaystyle\int_1^\infty \frac{1}{x^{1/3}}\,\mathrm{d}x$ diverges
- $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$ diverges
- $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ diverges
- $\displaystyle\int_1^\infty \frac{1}{x^{1.5}}\,\mathrm{d}x$ converges

Theorem 1.12.20

Let a and c be real numbers with $a < c$ and let the function $f(x)$ be continuous for all $x \geq a$. Then the improper integral $\int_a^\infty f(x)\,\mathrm{d}x$ converges if and only if the improper integral $\int_c^\infty f(x)\,\mathrm{d}x$ converges.



Decide whether each integral converges or diverges.

- $\int_0^9 \frac{1}{x^{0.3}}\,\mathrm{d}x$ converges
- $\int_0^{81} \frac{1}{x^2}\,\mathrm{d}x$ diverges
- $\int_0^{\frac{1}{2}} \frac{1}{x^3}\,\mathrm{d}x$ diverges
- $\int_{15}^{\infty} \frac{1}{x^{0.3}}\,\mathrm{d}x$ diverges
- $\int_{0.4}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges
- $\int_{\frac{1}{2}}^{\infty} \frac{1}{x^3}\,\mathrm{d}x$ converges

It is very common to encounter integrals that are too complicated to evaluate explicitly. Numerical approximation schemes, evaluated by computer, are often used instead. You want to be sure that at least the integral converges before feeding it into a computer.

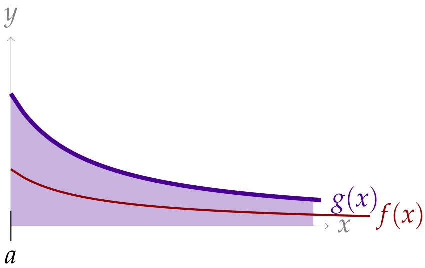
Fortunately it is usually possible to determine whether or not an improper integral converges even when you cannot evaluate it explicitly.

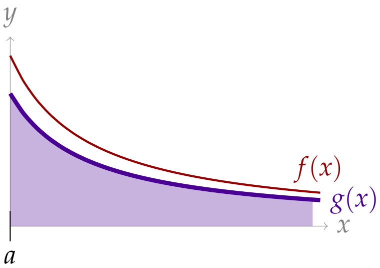
Comparison

Let a be a real number. Let f and g be functions that are defined and continuous for all $x \geq a$ and assume that $g(x) \geq 0$ for all $x \geq a$.

(a) If $|f(x)| \leq g(x)$ for all $x \geq a$ and if $\int_a^\infty g(x)\,\mathrm{d}x$ converges, then $\int_a^\infty f(x)\,\mathrm{d}x$ converges.

(b) If $f(x) \geq g(x)$ for all $x \geq a$ and if $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.





Does the integral $\int_1^\infty e^{-x^2}$ converge or diverge?

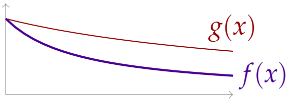
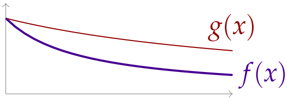
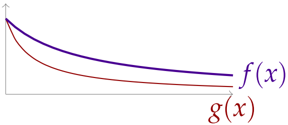
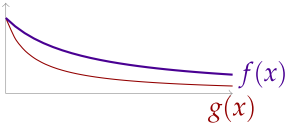
We know from previous examples that we can't evaluate $\int e^{-x^2}\,\mathrm{d}x$ directly. For $x \geq 1$:

$$x^2 > x \implies -x^2 < -x \implies e^{-x^2} < e^{-x}$$

$$\begin{aligned}
\int_1^\infty e^{-x}\,\mathrm{d}x &= \lim_{b\to\infty} \int_1^b e^{-x}\,\mathrm{d}x \\
&= \lim_{b\to\infty} \left[-e^{-x}\right]_1^b \\
&= \lim_{b\to\infty} \left[e^{-b} - e^{-1}\right] \\
&= e^{-1} = \frac{1}{e}
\end{aligned}$$

Since $0 \leq e^{-x^2} \leq e^{-x}$ for $x \geq 1$, and since $\int_1^\infty e^{-x}\,\mathrm{d}x$ converges, by the comparison test we conclude that $\int e^{-x^2}\mathrm{d}x$ converges, as well.

Let functions $f(x)$ and $g(x)$ be positive and continuous for all $x \geq a$.

	$\int_a^\infty g(x)\, dx$ converges	$\int_a^\infty g(x)\, dx$ diverges
$f(x) \leq g(x)$ for all $x \geq a$	 $\int_a^\infty f(x)$ converges	 inconclusive
$f(x) \geq g(x)$ for all $x \geq a$	 inconclusive	 $\int_a^\infty f(x)$ diverges

For each example below, decide whether the statement is a valid use of the comparison theorem.

- $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges and $0 \le \frac{1}{x^2} \le \frac{2+\sin x}{x^2}$ for $x \ge 1$. So by the comparison test, $\displaystyle\int_1^\infty \frac{2+\sin x}{x^2}\,\mathrm{d}x$ converges as well.

- $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges and $0 \le \frac{e^{-x}}{x^2} \le \frac{1}{x^2}$ for $x \ge 1$. So by the comparison test, $\displaystyle\int_1^\infty \frac{e^{-x}}{x^2}\,\mathrm{d}x$ converges as well.

- $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges and $-\frac{1}{x} \le \frac{1}{x^2}$ for $x \ge 1$. So by the comparison test, $\displaystyle\int_1^\infty \frac{-1}{x}\,\mathrm{d}x$ converges as well.

Limiting comparison

Let $-\infty < a < \infty$. Let f and g be functions that are defined and continuous for all $x \geq a$ and assume that $g(x) \geq 0$ for all $x \geq a$.
If the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and is nonzero, then either $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ both converge, or they both diverge.

Use limiting comparison to determine whether $\displaystyle\int_1^\infty \frac{1}{x+10}\,\mathrm{d}x$ converges or diverges.

An integrand that looks similar and simpler is $\frac{1}{x}$. Since $\frac{1}{x+10} < \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ diverges, we can't directly compare the two series. So, let's use limiting comparison. Set $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x+10}$. Then:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x}{1/(x+10)} = \lim_{x\to\infty} \frac{x+10}{x} = 1$$