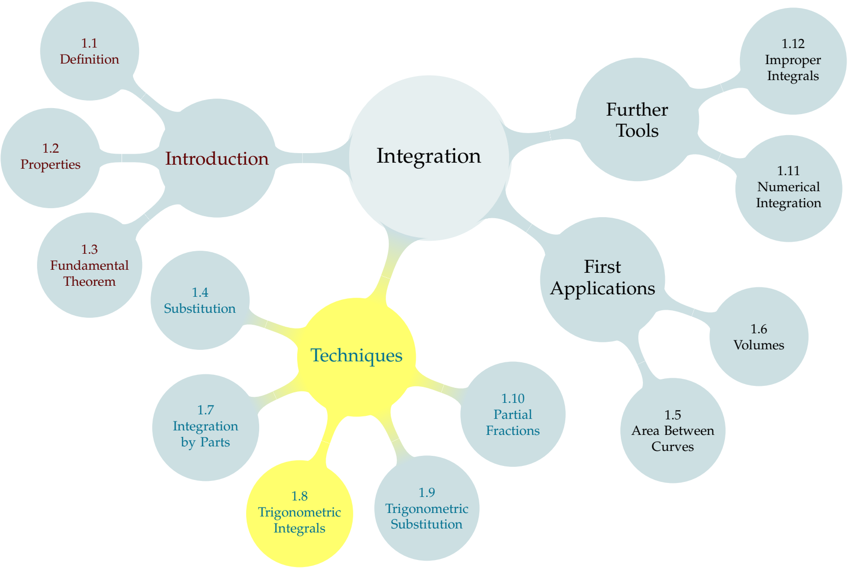
Table of Contents



Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd
- $u = \tan x$ if m is even and $n = 0$
  (after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

Remaining case: n odd and m is even.

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The general remaining case is known, but complicated. Instead of treating it exhaustively, we'll show examples of two methods.

$\int \sec x \, \mathrm{d}x$

We saw a way of integrating secant with the following trick:

$$\int \sec x \, \mathrm{d}x = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x$$
$$= \int \frac{1}{u} \mathrm{d}u \quad \text{with } u = \sec x + \tan x$$

Another trick: this time let $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

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$$\begin{aligned}\int \sec x \, \mathrm{d}x = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x \\ = \int \frac{1}{u} \mathrm{d}u \quad \text{with } u = \sec x + \tan x\end{aligned}$$

Another trick: this time let $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

$$\begin{aligned}\int \sec x \, \mathrm{d}x &= \int \frac{1}{\cos x} \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x \\ &= \int \frac{1}{1 - \sin^2 x} \cos x \, \mathrm{d}x = \int \frac{1}{1 - u^2} \mathrm{d}u\end{aligned}$$

The integrand $\frac{1}{1-u^2}$ is a rational function of u (i.e. a ratio of two polynomials). There is a procedure, called Partial Fractions, that can be used to evaluate all integrals of rational functions. We will learn it in Section 1.10.

$\int \sec^3 x \, \mathrm{d}x$

We can integrate around in a circle (with integration by parts) to evaluate $\int \sec^3 x \, \mathrm{d}x$. Let $u = \sec x$, $\mathrm{d}v = \sec^2 x \, \mathrm{d}x$. Then $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$ and $v = \tan x$.

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$$\begin{aligned}
\int \sec^3 x \, \mathrm{d}x &= \sec x \tan x - \int \sec x \tan^2 x \, \mathrm{d}x \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) \mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \int \sec x \, \mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \log|\sec x + \tan x| + C' \\
2\int \sec^3 x \, \mathrm{d}x &= \sec x \tan x + \log|\sec x + \tan x| + C' \\
\int \sec^3 x \, \mathrm{d}x &= \frac{1}{2}\left(\sec x \tan x + \log|\sec x + \tan x|\right) + C
\end{aligned}$$

with $C = C'/2$.