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Hard to guess the antiderivative without seeing the derivative first!

Antiderivatives

Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sin\left(\underbrace{x^2+x}_{\text{inside function}}\right)\right\} =$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sin\left(\underbrace{x^2+x}_{\text{inside function}}\right)\right\} = \left(\underbrace{2x+1}_{\substack{\text{derivative of}\\ \text{inside function}}}\right)\cos\left(\underbrace{x^2+x}_{\text{inside function}}\right)$$

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Hallmark of the chain rule: an "inside" function, with that function's derivative multiplied.

Solve by Inspection

$$\int 2xe^{x^2+1}\,\mathrm{d}x$$

$$\int \frac{1}{x}\cos(\log x)\,\mathrm{d}x$$

$$\int 3(\sin x + 1)^2 \cos x\,\mathrm{d}x$$

(Look for an “inside” function, with its derivative multiplied.)

Undoing the Chain Rule

Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{f(u(x))\right\} = f'(u(x)) \cdot u'(x)$$

(Here, $u(x)$ is our "inside function")

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x)\ \mathrm{d}x = f(u(x)) + C$$

Undoing the Chain Rule

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Shorthand: call $u(x)$ simply u;
since $\frac{\mathrm{d}u}{\mathrm{d}x} = u'(x)$, call $u'(x)\, \mathrm{d}x$ simply $\mathrm{d}u$.

Undoing the Chain Rule

Antiderivative Fact:

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since $\frac{\mathrm{d}u}{\mathrm{d}x} = u'(x)$, call $u'(x)\, \mathrm{d}x$ simply $\mathrm{d}u$.

$$\int f'(u(x)) \cdot u'(x)\, \mathrm{d}x = \int f'(u)\, \mathrm{d}u \Big|_{u=u(x)} = f\big(u(x)\big) + C$$

This is the substitution rule.

We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1}\,\mathrm{d}x$$

$$\int \frac{1}{x}\cos(\log x)\,\mathrm{d}x$$

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$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x =$$

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"Inside" function: x^3+1.

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Shorthand: $x^3+1 \to u, \quad 3x^2\,\mathrm{d}x \to \mathrm{d}u$

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u\Big|_{u=x^3+1}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u, \quad 3x^2\,\mathrm{d}x \to \mathrm{d}u$

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = \left.\int \sin(u)\,\mathrm{d}u\right|_{u=x^3+1}$$
$$= -\cos(u) + C|_{u=x^3+1}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u, \quad 3x^2\,\mathrm{d}x \to \mathrm{d}u$

$$\begin{aligned}\int (3x^2)\sin(x^3+1)\,\mathrm{d}x &= \left.\int \sin(u)\,\mathrm{d}u\right|_{u=x^3+1}\\ &= -\cos(u)+C|_{u=x^3+1}\\ &= \cos(x^3+1)+C\end{aligned}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u$, $3x^2\,\mathrm{d}x \to \mathrm{d}u$

$$\begin{aligned}\int (3x^2)\sin(x^3+1)\,\mathrm{d}x &= \left.\int \sin(u)\,\mathrm{d}u\right|_{u=x^3+1}\\ &= -\cos(u)+C|_{u=x^3+1}\\ &= \cos(x^3+1)+C\end{aligned}$$

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Warning 1: We don't just change dx to du. We need to couple dx with the derivative of our inside function.
After all, we're undoing the chain rule! We need to have an "inside derivative."

$$
\begin{aligned}
\int (3x^2)\sin(x^3+1)\,\mathrm{d}x &= \left.\int \sin(u)\,\mathrm{d}u\right|_{u=x^3+1} \\
&= -\cos(u) + C|_{u=x^3+1} \\
&= \cos(x^3+1) + C
\end{aligned}
$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u$, $\quad 3x^2\,\mathrm{d}x \to \mathrm{d}u$

Warning 2: The final answer is a function of x.

We used the substitution rule to conclude

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = -\cos(x^3+1)+C$$

We can check that our antiderivative is correct by differentiating.

We saw:

$$\int 3x^2 \sin(x^3+1)\,\mathrm{d}x = -\cos(x^3+1) + C$$

So, we can evaluate:

$$\int_0^1 3x^2 \sin(x^3+1)\,\mathrm{d}x$$

Notation: limits of integration

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} \, \mathrm{d}x$$

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Let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$. Note the limits (or bounds) of integration $\pi/4$ and $\pi/2$ are values of x, not u: they follow the differential, unless otherwise specified.

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\, \mathrm{d}x$$

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\, \mathrm{d}x$$

$$\int_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}}$$

NOTATION: LIMITS OF INTEGRATION

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\, \mathrm{d}x$$

$$x=\frac{\pi}{2}$$
$$x=\frac{\pi}{4}$$

$$\int_{1/\sqrt{2}}^{1} \frac{1}{u^3}\, \mathrm{d}u$$

$$u=\sin\left(\frac{\pi}{2}\right)=1$$
$$u=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\, \mathrm{d}x \qquad \begin{matrix} x=\frac{\pi}{2} \\ x=\frac{\pi}{4} \end{matrix}$$

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x$$

$x = \frac{\pi}{2}$
$x = \frac{\pi}{4}$

$$\cancel{\int_{\pi/4}^{\pi/2} \frac{1}{u^3}\,\mathrm{d}u}$$

$\cancel{u = \frac{\pi}{2}}$
$\cancel{u = \frac{\pi}{4}}$

different

$$\int_{1/\sqrt{2}}^{1} \frac{1}{u^3}\,\mathrm{d}u$$

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x \qquad \int_{x=\pi/4}^{x=\pi/2} \frac{1}{u^3}\,\mathrm{d}u \qquad \int_{1/\sqrt{2}}^{1} \frac{1}{u^3}\,\mathrm{d}u$$

$x=\frac{\pi}{2}$
$x=\frac{\pi}{4}$

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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x \quad = \quad \boxed{\int_{x=\pi/4}^{x=\pi/2} \frac{1}{u^3}\,\mathrm{d}u} \quad = \quad \int_{1/\sqrt{2}}^{1} \frac{1}{u^3}\,\mathrm{d}u$$

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not standard, but OK

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$$\begin{matrix} x=\frac{\pi}{2} \\ x=\frac{\pi}{4} \end{matrix} \qquad\qquad \begin{matrix} x=\frac{\pi}{2} \\ x=\frac{\pi}{4} \end{matrix} \qquad\qquad \begin{matrix} u=\sin\left(\frac{\pi}{2}\right)=1 \\ u=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \end{matrix}$$

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x$$

Let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$.

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,\mathrm{d}x = \int_{1/\sqrt{2}}^{1} \frac{1}{u^3}\,\mathrm{d}u$$

True or False?

1. Using $u = x^2$,

$$\int e^{x^2}\, \mathrm{d}x = \int e^u\, \mathrm{d}u$$

2. Using $u = x^2 + 1$,

$$\int_0^1 x \sin(x^2+1)\, \mathrm{d}x = \int_0^1 \frac{1}{2} \sin u \;\, \mathrm{d}u$$

Evaluate $\int_0^1 x^7 \left(x^4+1\right)^5 \, \mathrm{d}x$.

Time permitting, more examples using the substitution rule

Evaluate $\displaystyle\int \sin x \cos x \, \mathrm{d}x.$

Check Our Work

We can check that $\int \sin x \cos x \, \mathrm{d}x = \frac{1}{2} \sin^2 x + C$ by differentiating.

We can check that $\int \sin x \cos x \, \mathrm{d}x = -\frac{1}{2} \cos^2 x + C$ by differentiating.

Evaluate $\displaystyle\int \frac{\log x}{3x}\,\mathrm{d}x.$

Check Our Work

We can check that $\displaystyle\int \frac{\log x}{3x}\,\mathrm{d}x = \frac{1}{6}\log^2 x + C$ by differentiating.

Evaluate $\int \frac{e^x}{e^x + 15}\,\mathrm{d}x.$

Evaluate $\int x^4 (x^5 + 1)^8\,\mathrm{d}x.$

Check Our Work

We can check that $\displaystyle\int \frac{e^x}{e^x+15}\,\mathrm{d}x = \log|e^x+15| + C$ by differentiating.

We can check that $\displaystyle\int x^4(x^5+1)^8\,\mathrm{d}x = \frac{1}{45}(x^5+1)^9 + C$ by differentiating.

Evaluate $\displaystyle\int_4^8 \frac{s}{s-3}\,\mathrm{d}s$. Be careful to use correct notation.

Evaluate $\displaystyle\int x^9(x^5+1)^8\,\mathrm{d}x.$

Q

Check Our Work

We can check that $\int x^9(x^5+1)^8\,\mathrm{d}x = \frac{1}{5}\left[\frac{(x^5+1)^{10}}{10} - \frac{(x^5+1)^9}{9}\right] + C$ by differentiating.

Particularly Tricky Substitution

$$\text{Evaluate } \int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x.$$

Q

Check Our Work

We can check that $\int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x = \arctan(e^x) + C$ by differentiating.

Included Work

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