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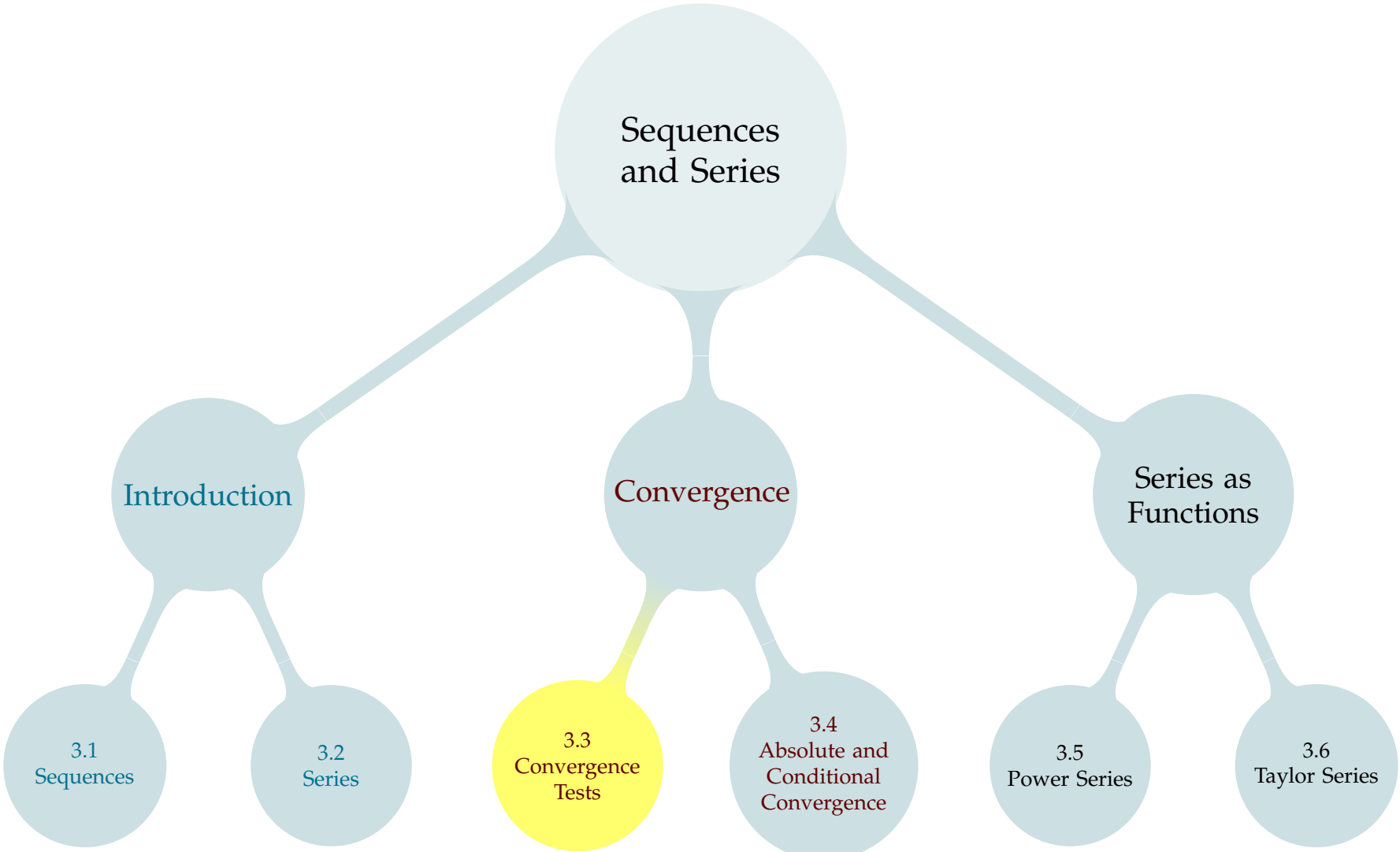

Review

Let $S_N = \sum_{n=1}^{N} a_n$.

Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

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Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N$$
$$S_{N-1} = a_1 + a_2 + a_3 + \cdots + a_{N-1}$$

Review

Let $S_N = \sum_{n=1}^{N} a_n$.

Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

$$\begin{aligned} S_N &= a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N \\ S_{N-1} &= a_1 + a_2 + a_3 + \cdots + a_{N-1} \\ S_N - S_{N-1} &= a_N \end{aligned}$$

Alternating Series

Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

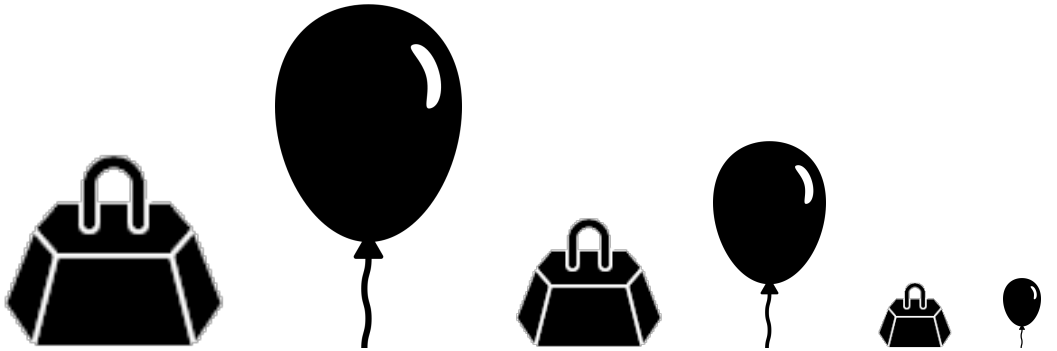
is alternating if every $A_n \geq 0$.

Alternating series:

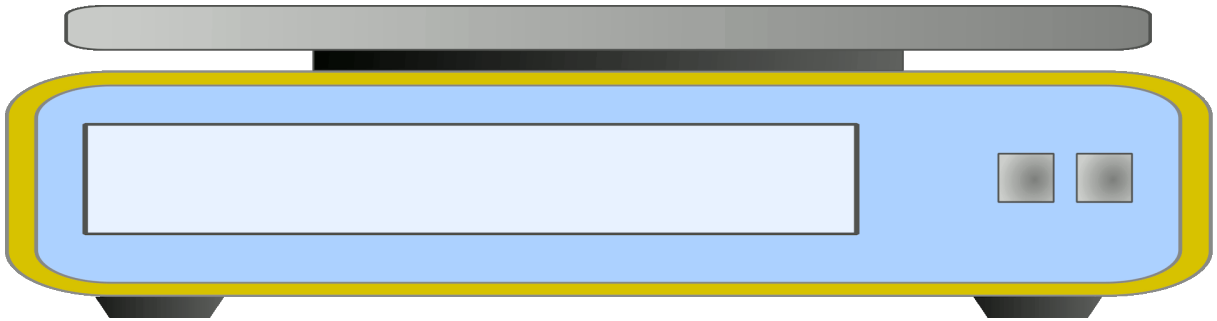
- $1-2+3-4+5-6+7-8+\cdots$
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

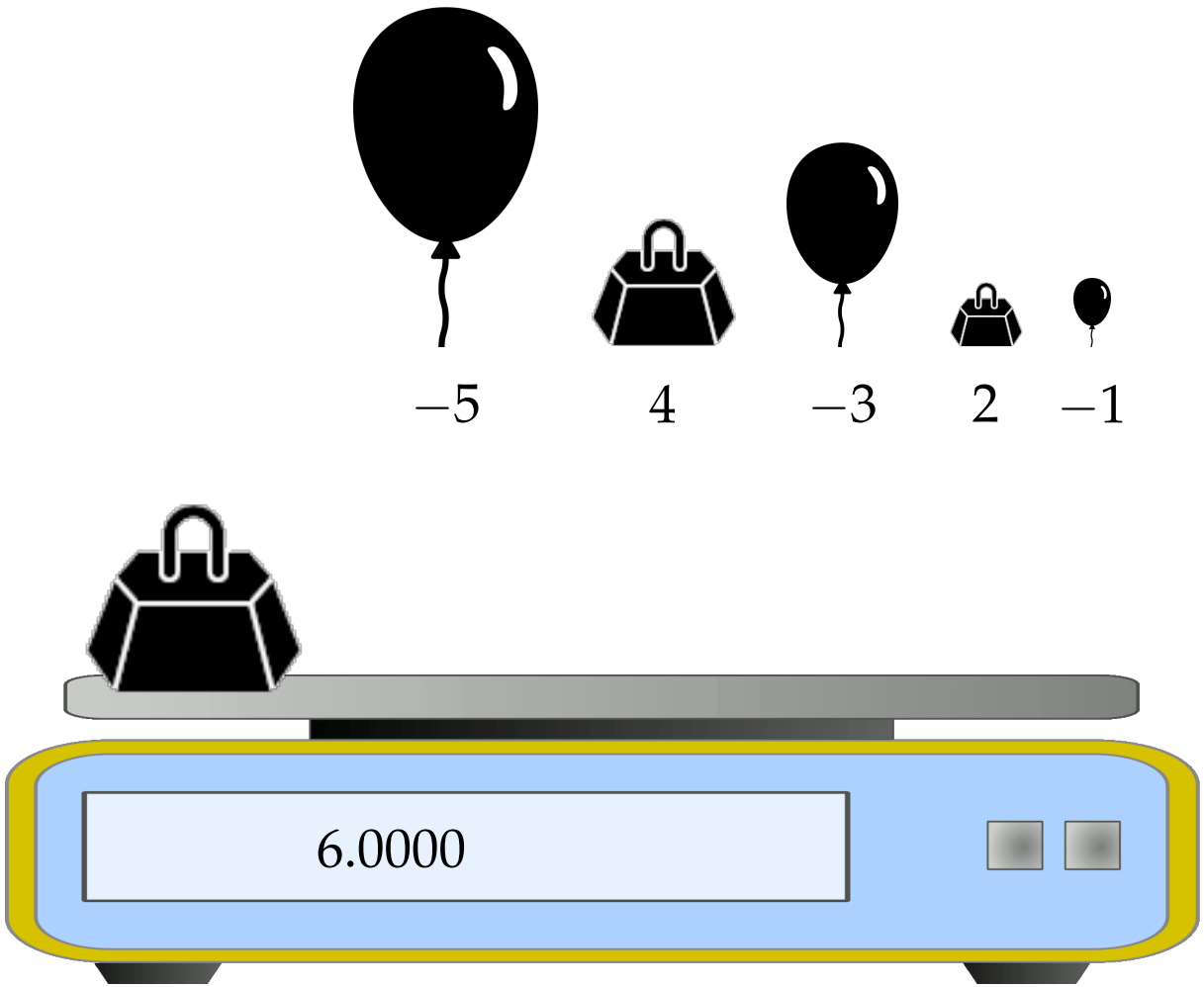
Not alternating:

- $\cos(1) + \cos(2) + \cos(3) + \cdots$
- $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$



Note: these terms alternate signs, and their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$





$$S_1 = 6.0000$$

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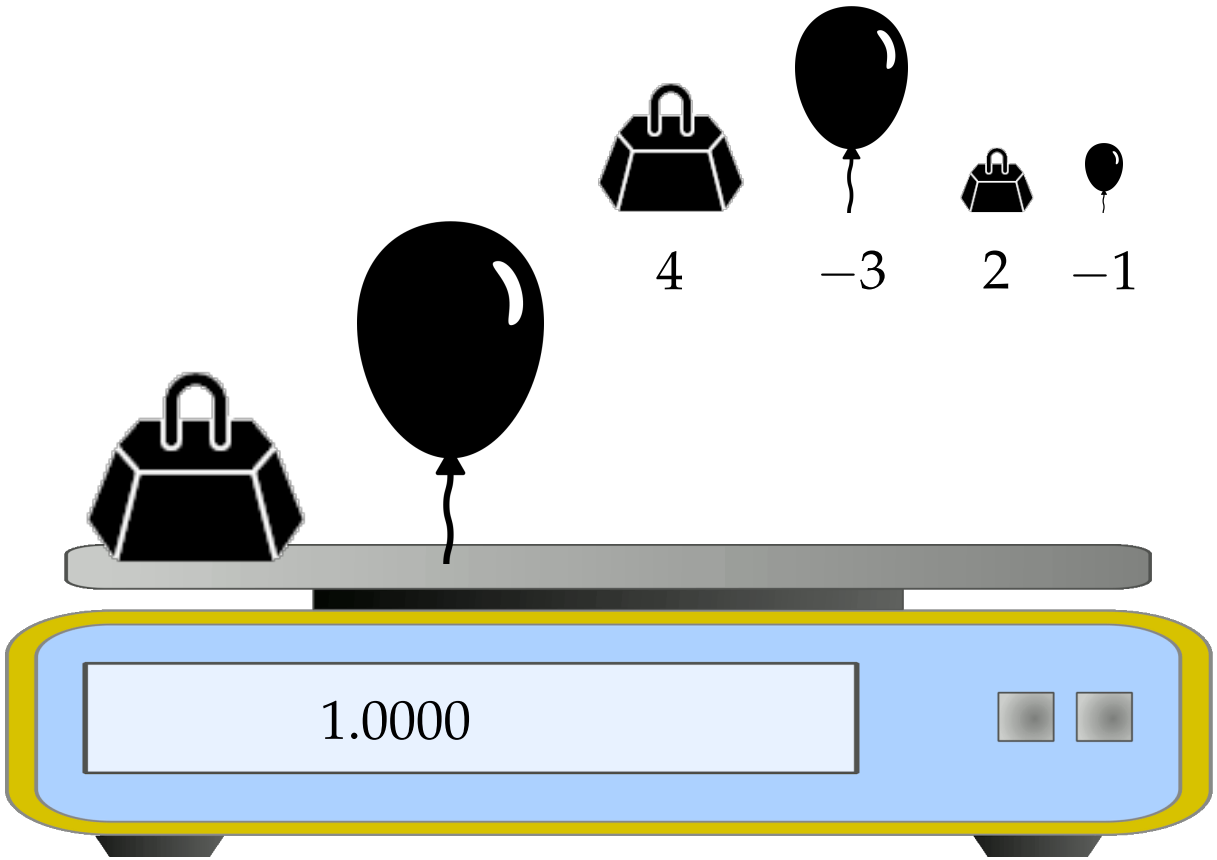
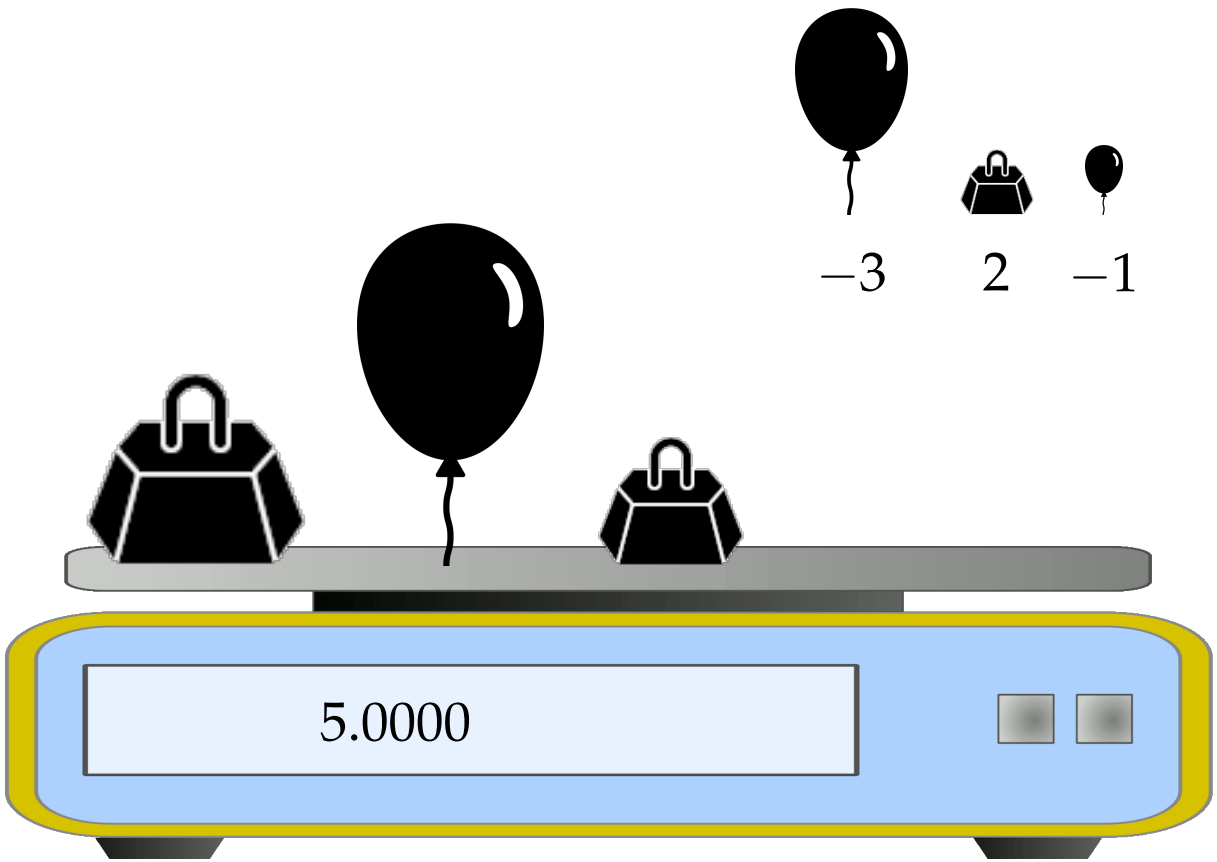
$S_2 = 1.0000$





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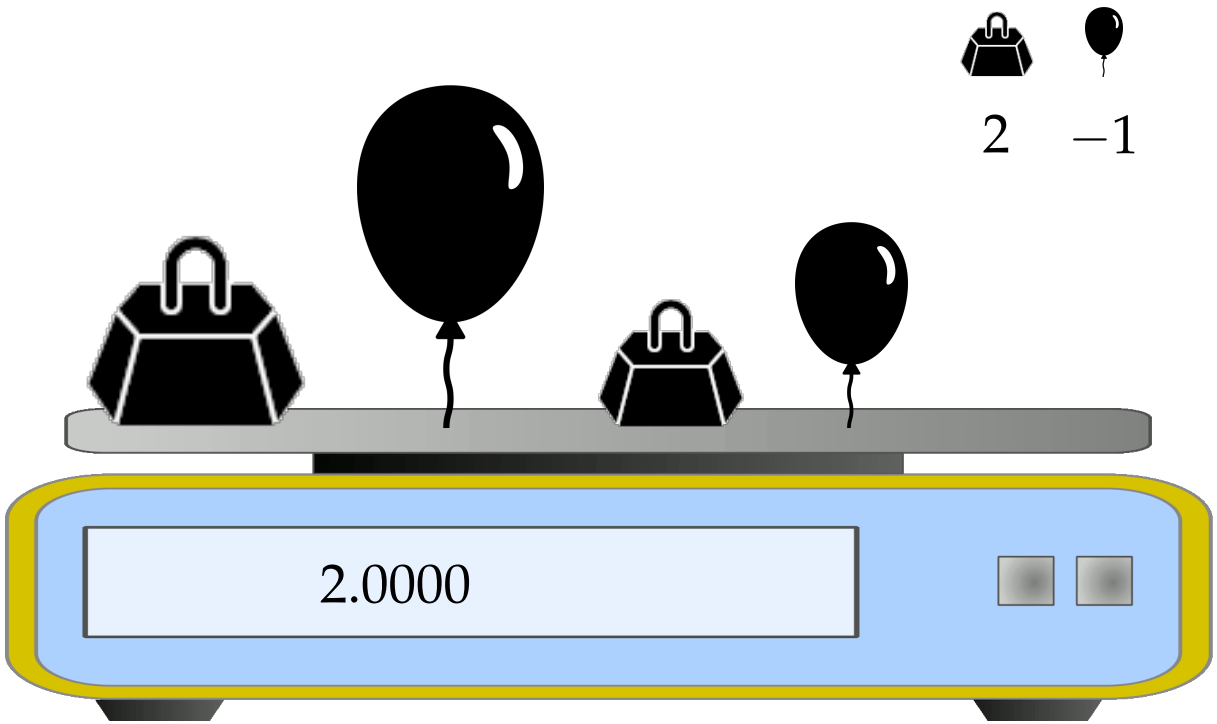

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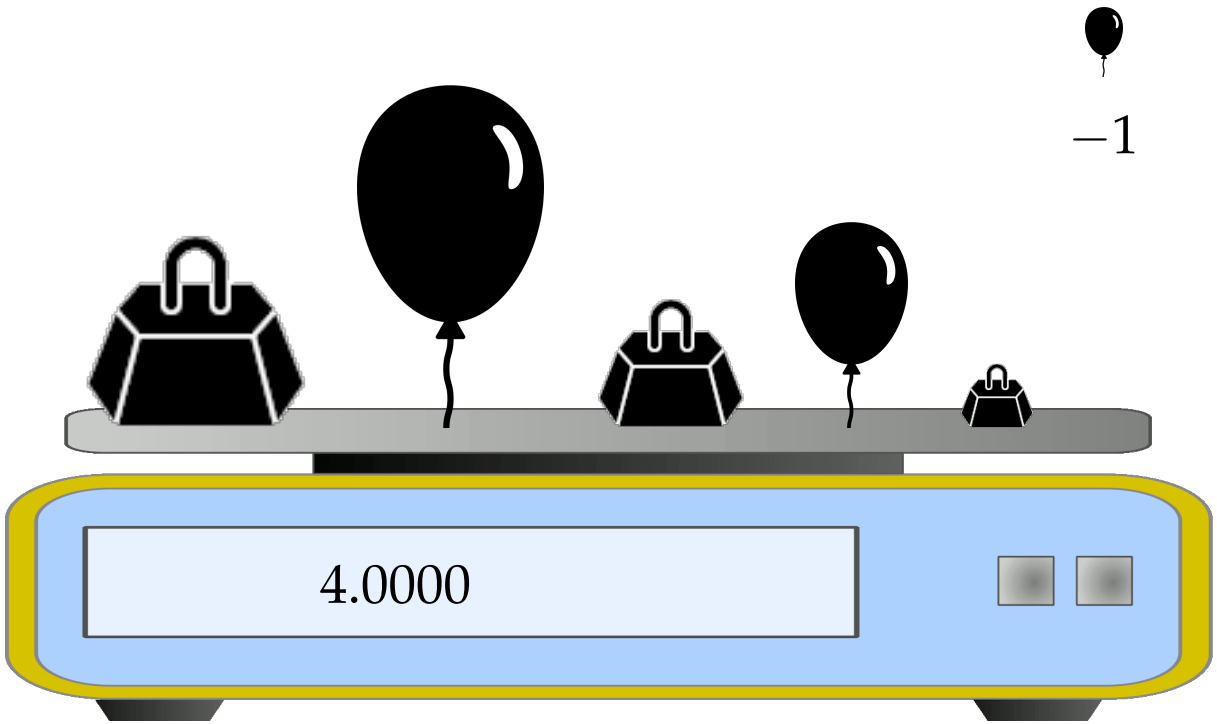

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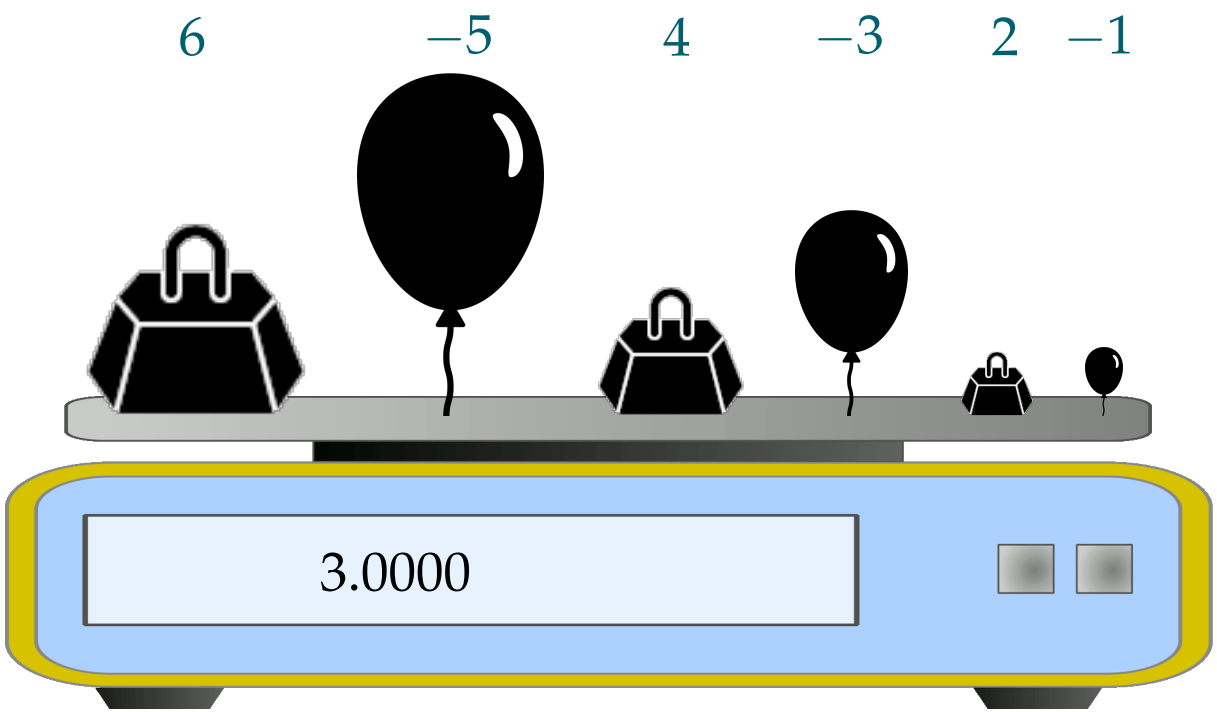

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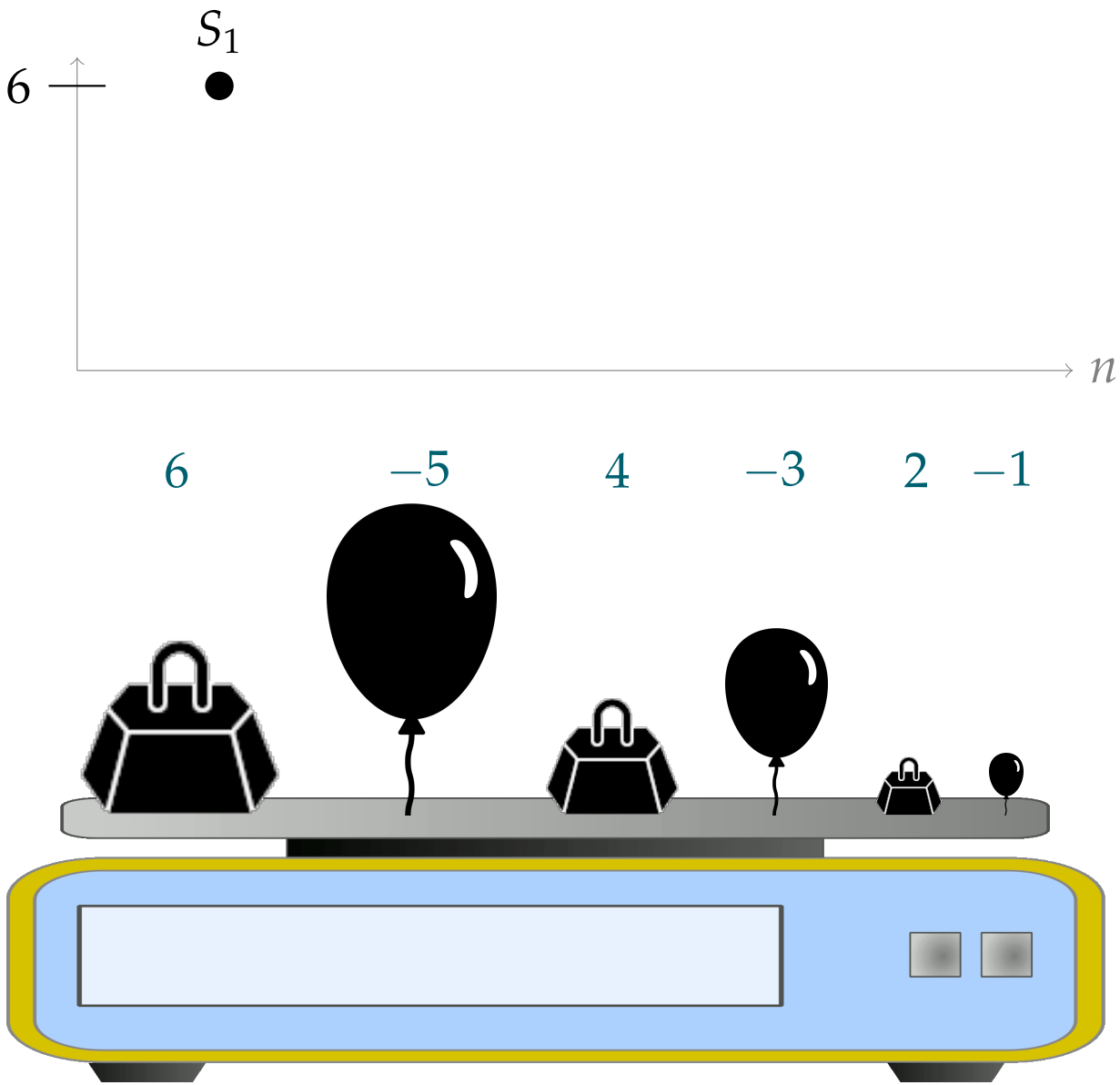

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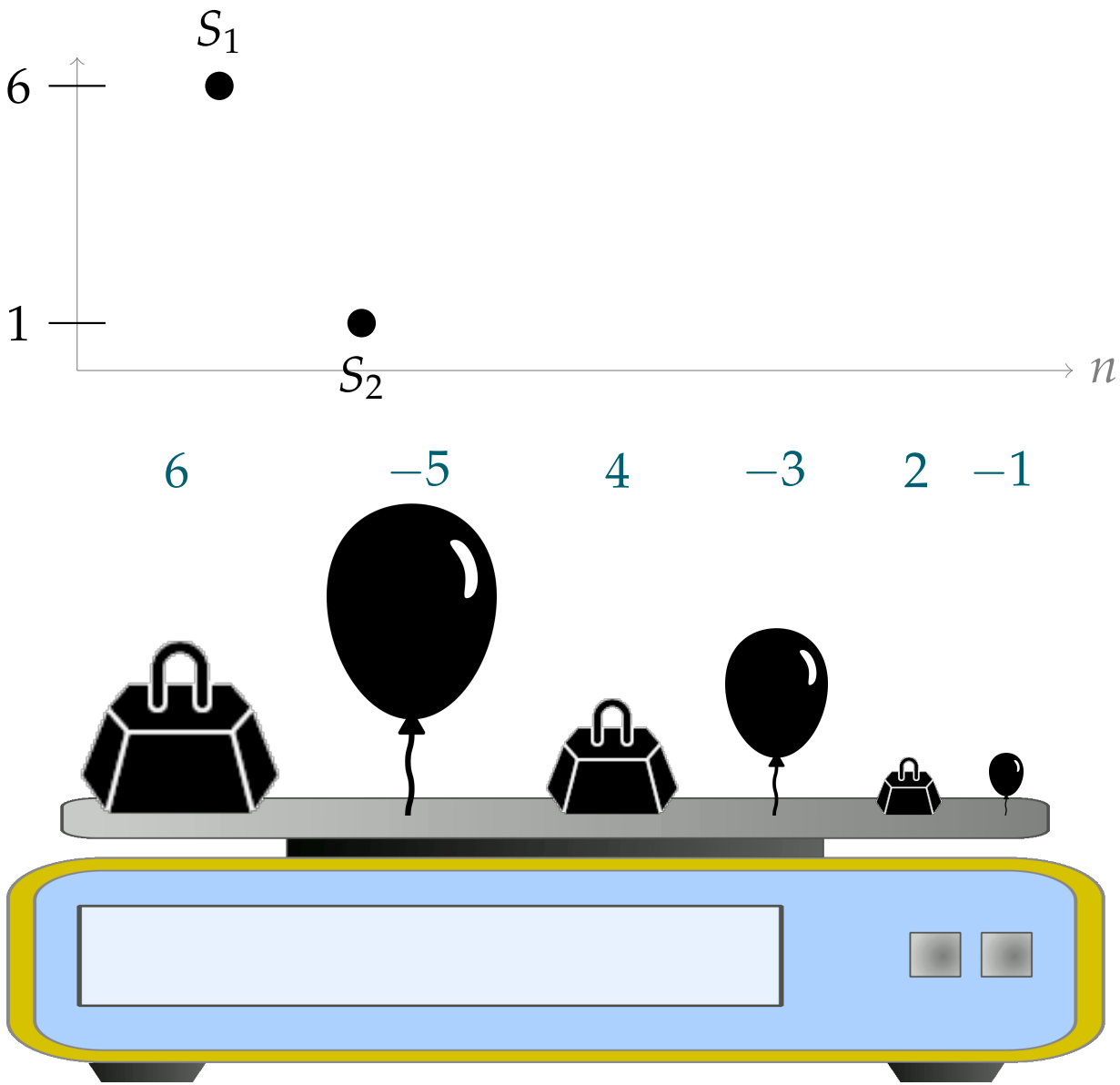

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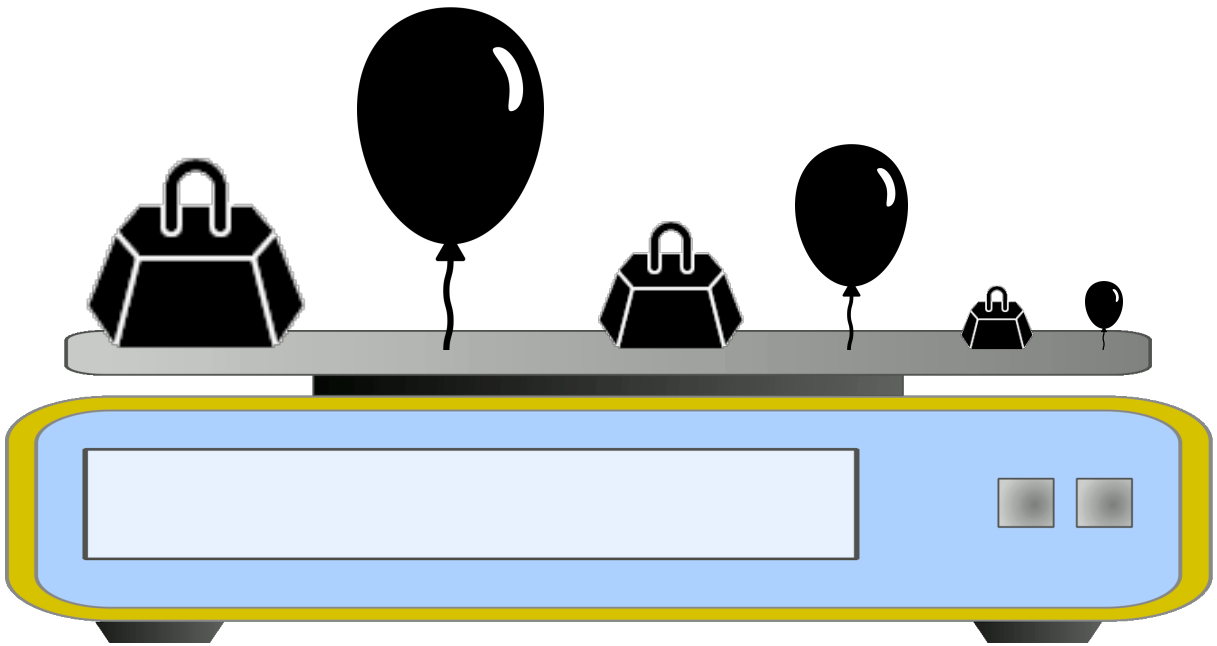

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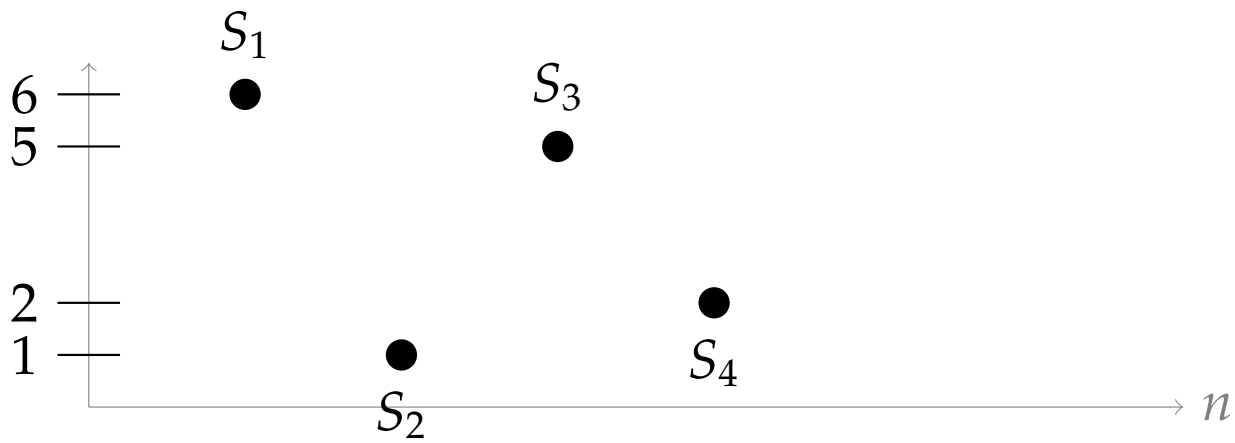

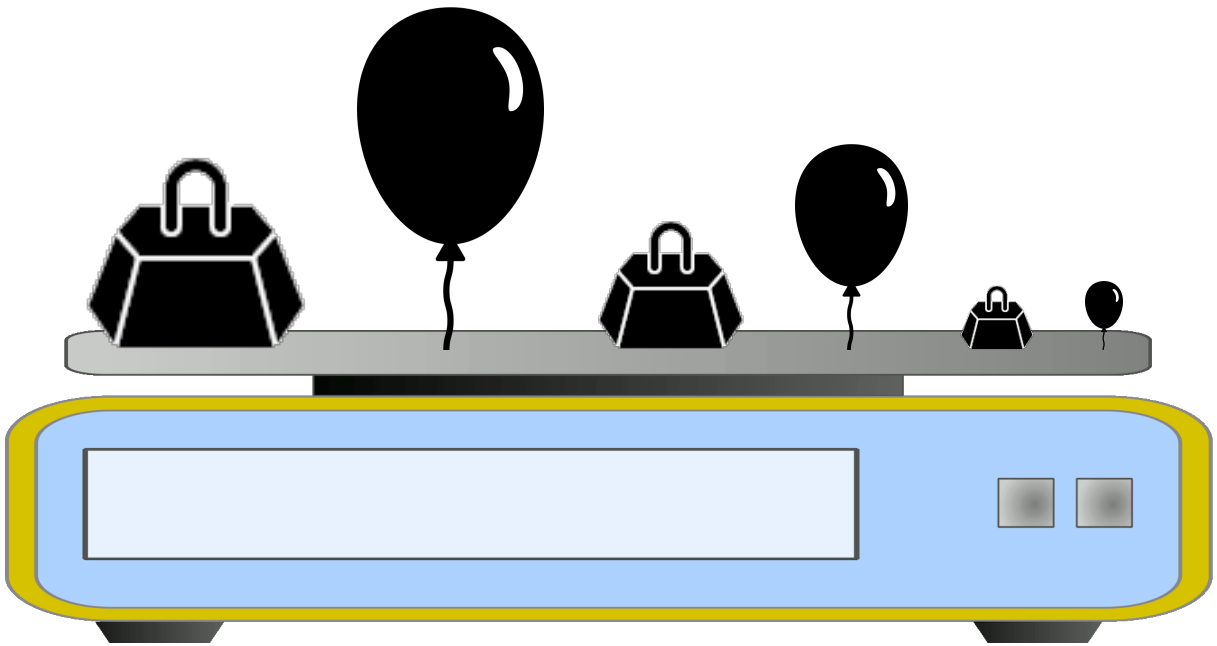

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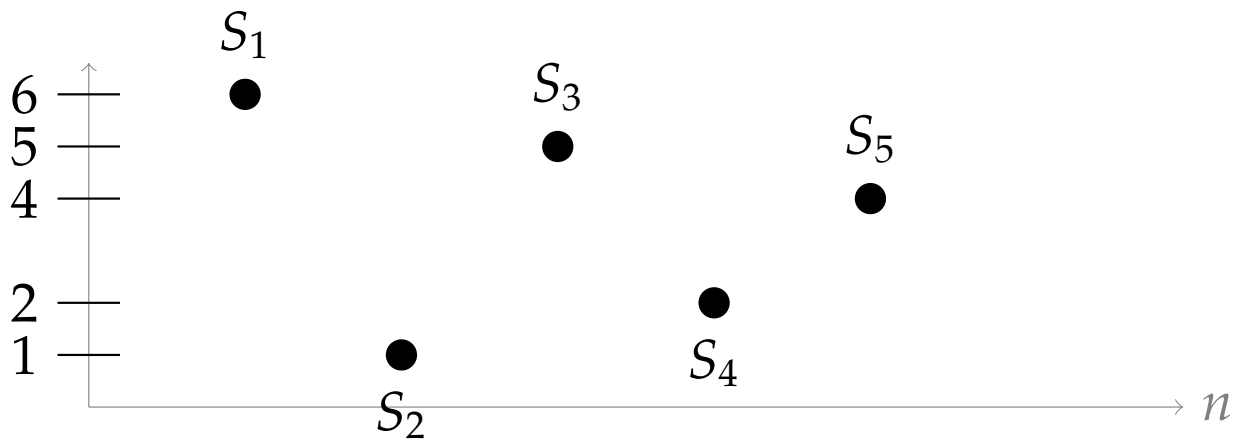

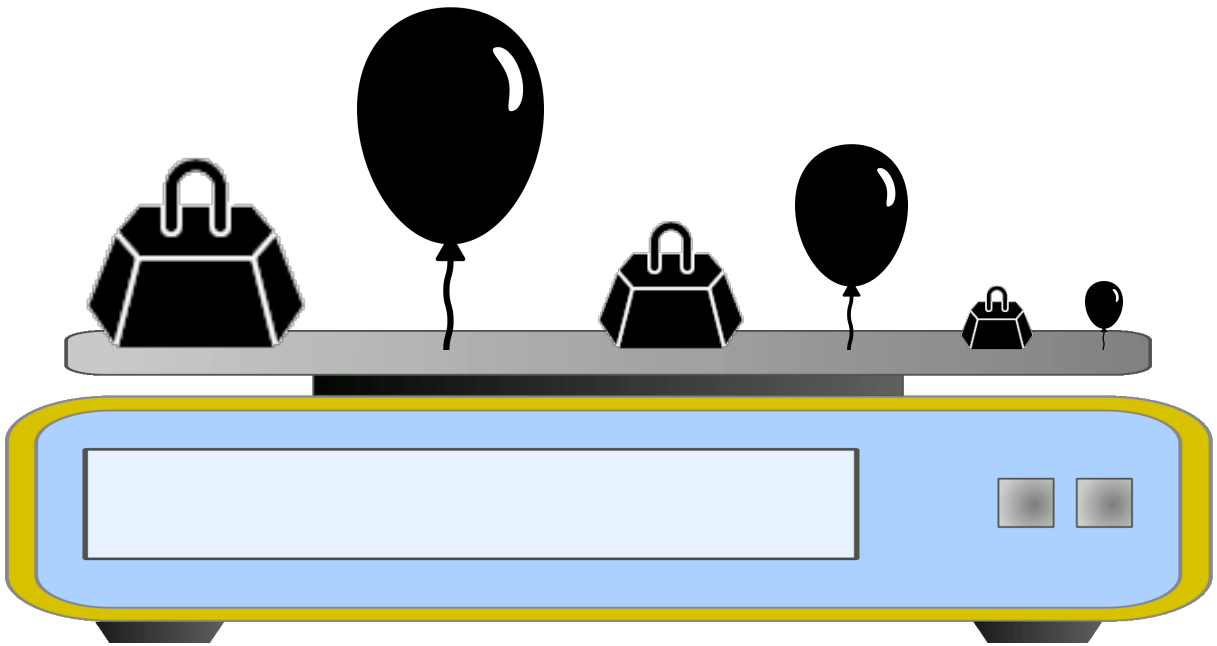

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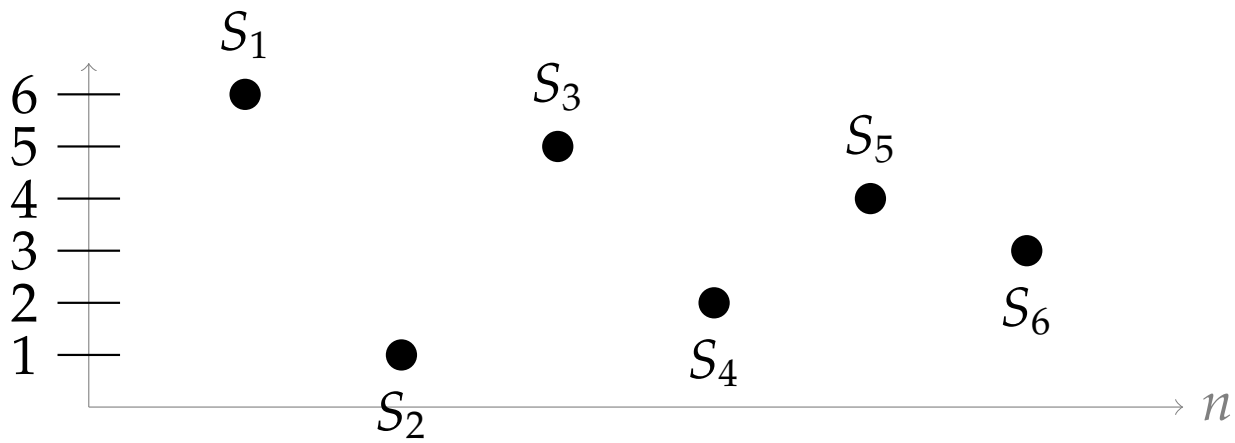

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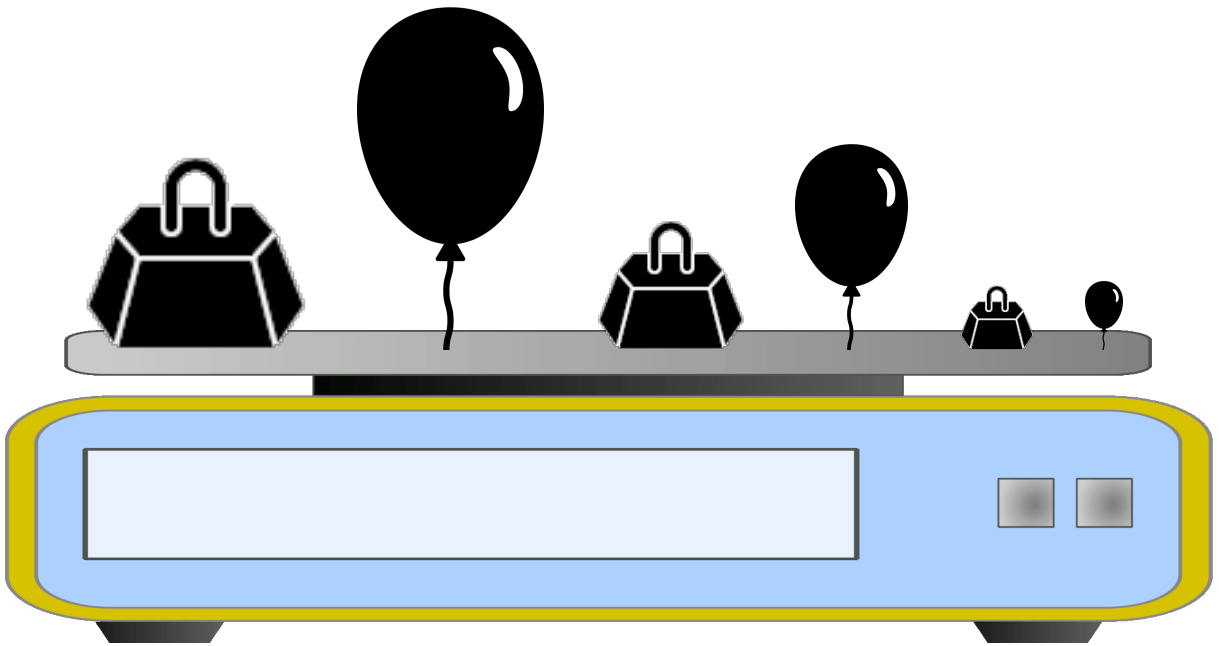

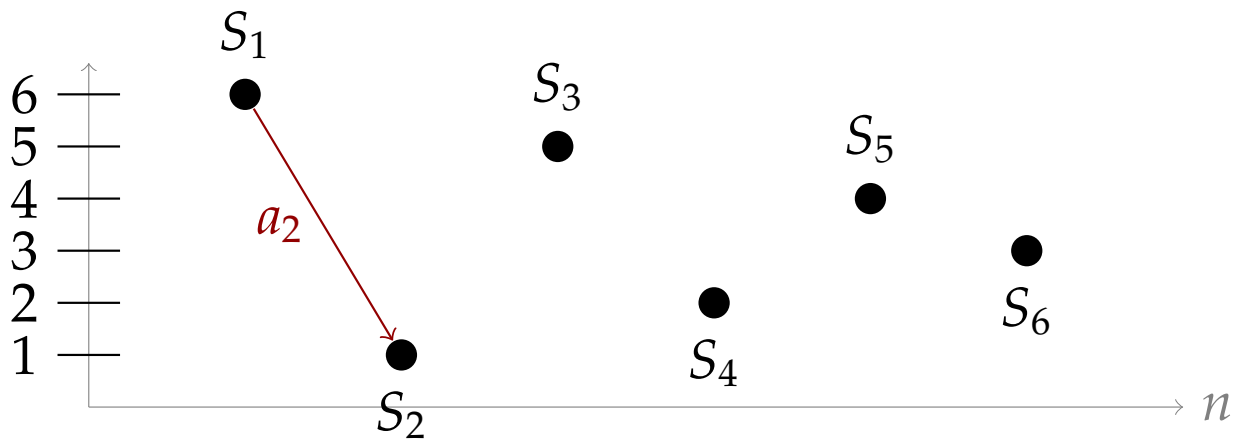

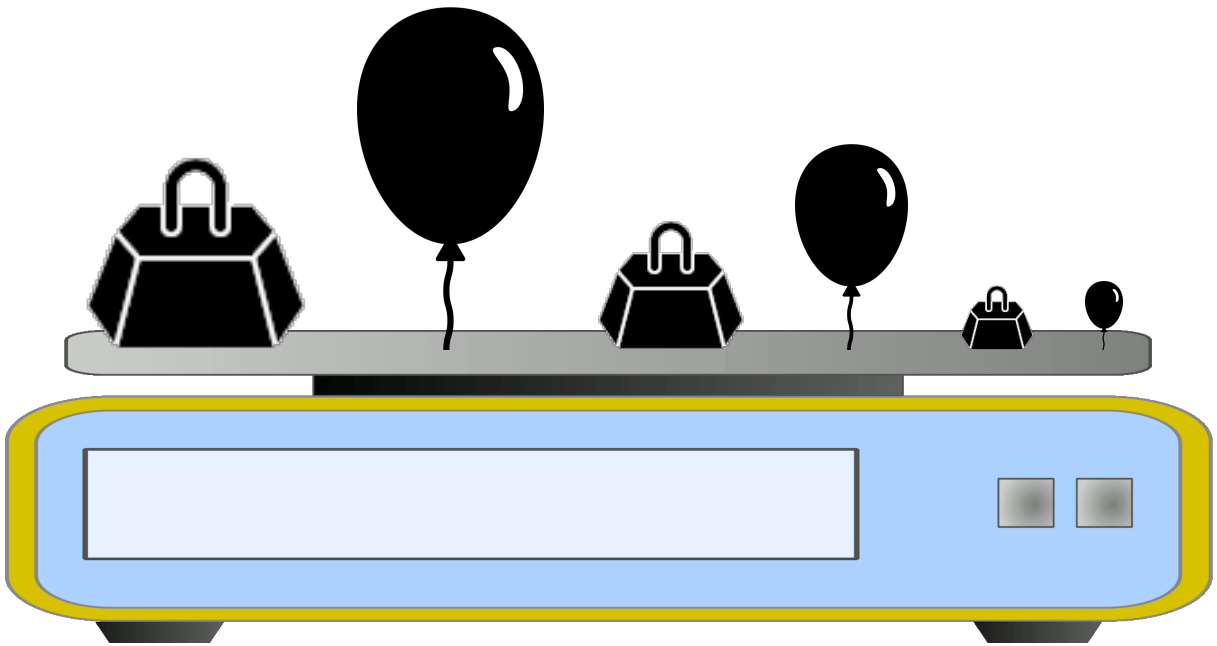

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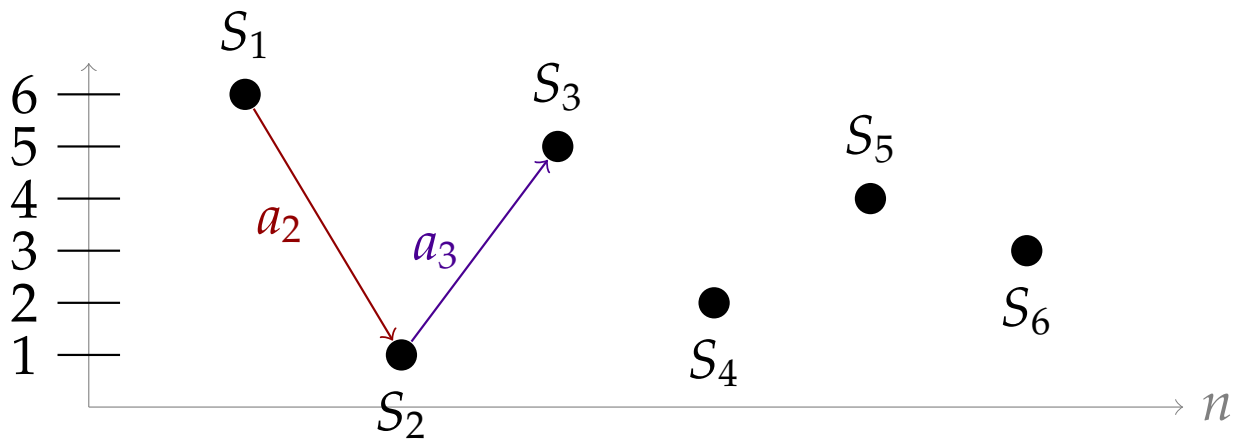

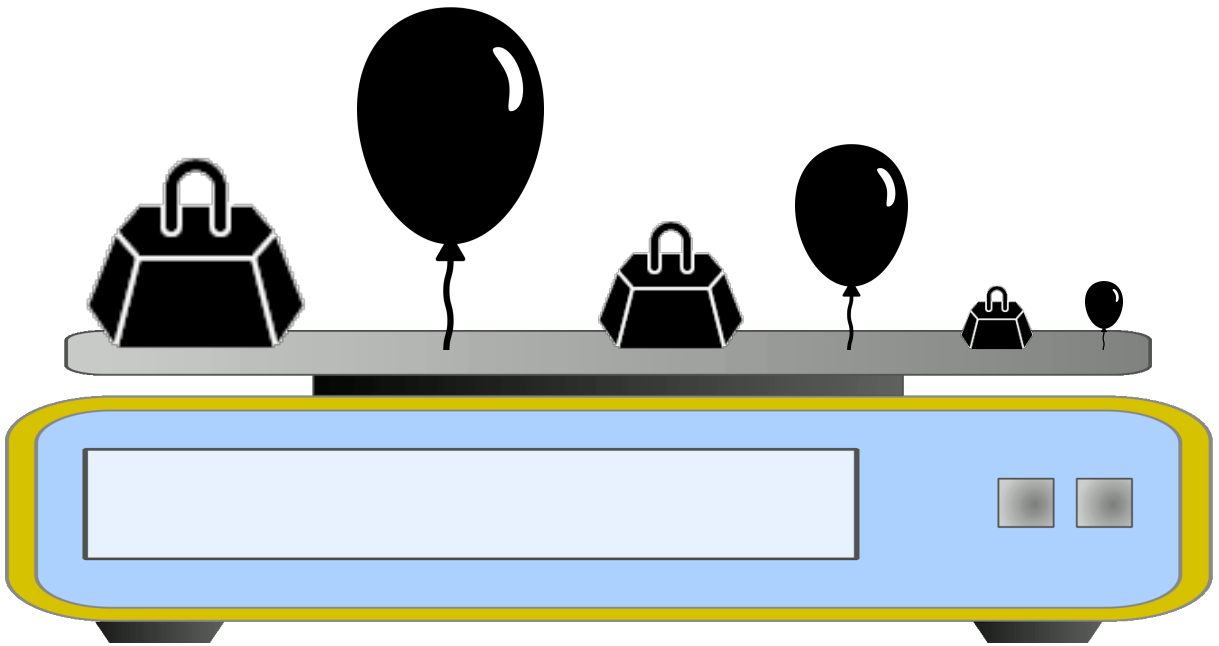

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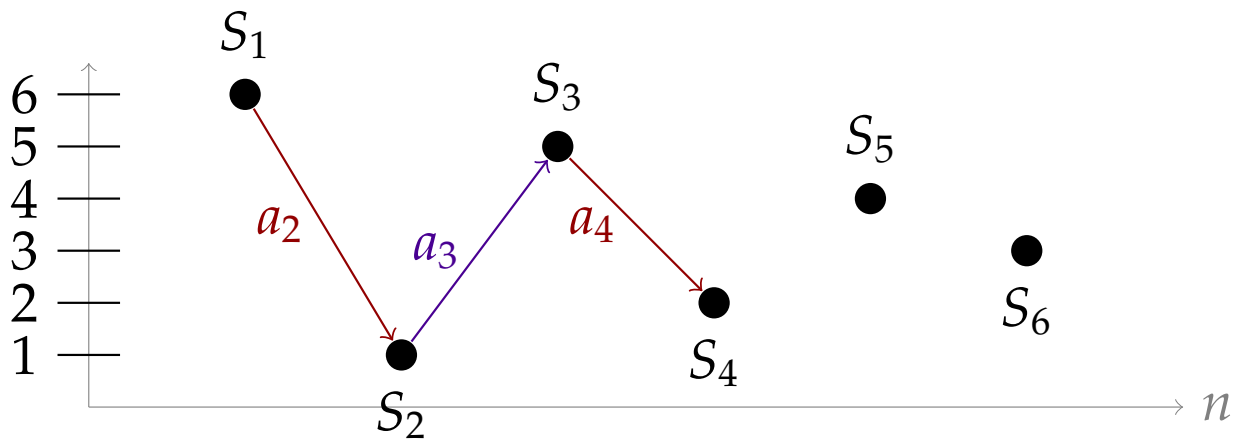

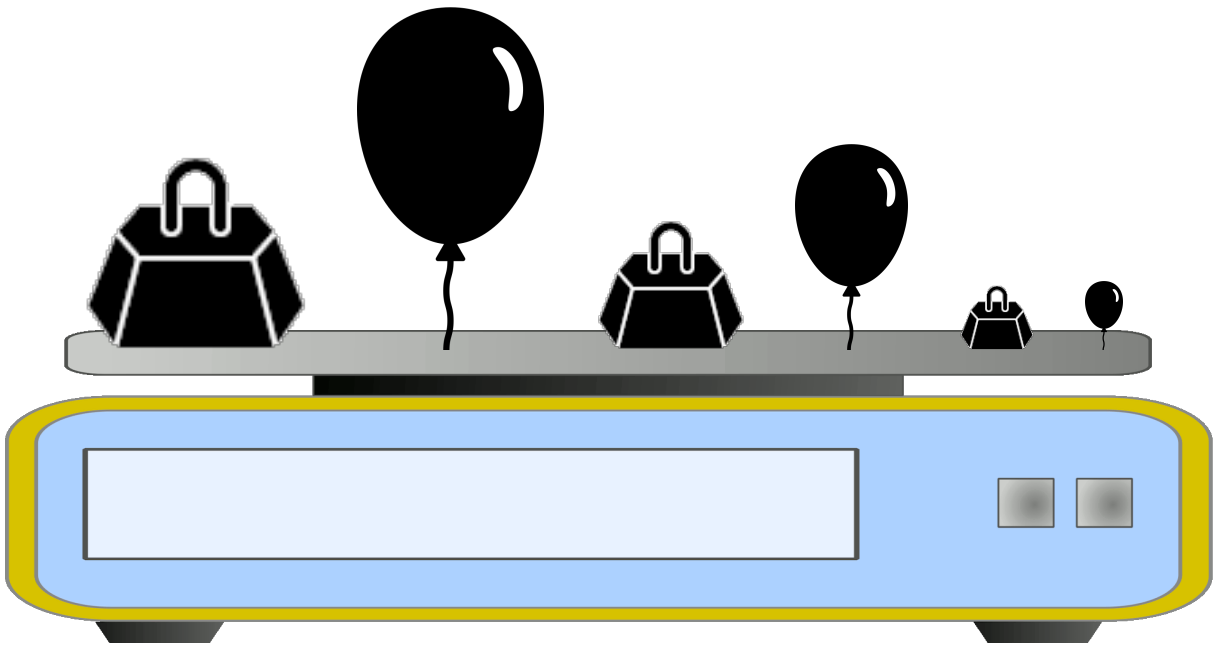

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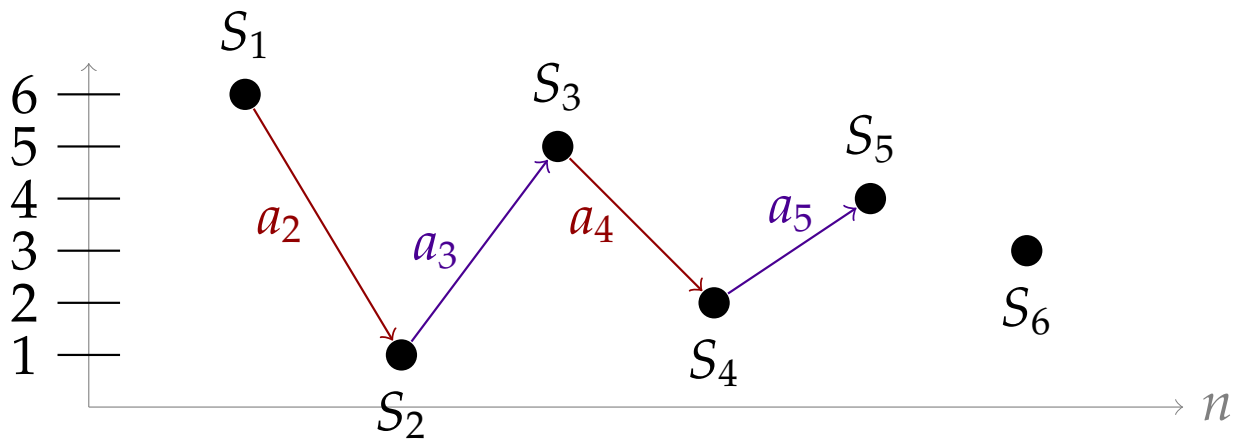

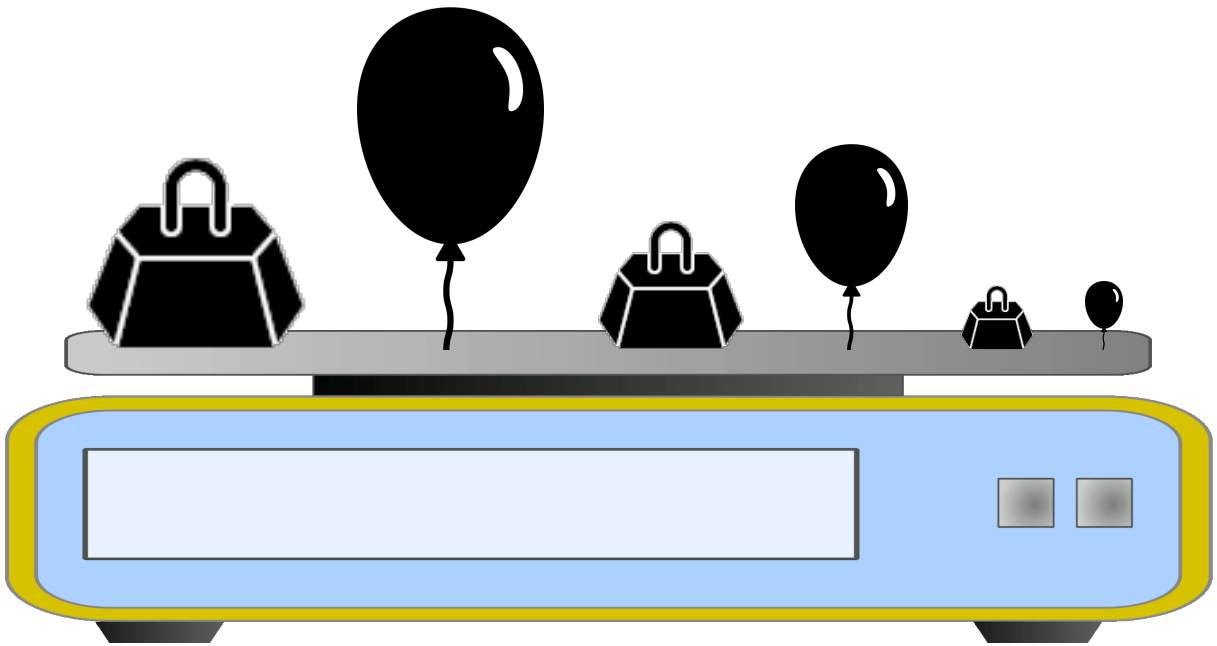

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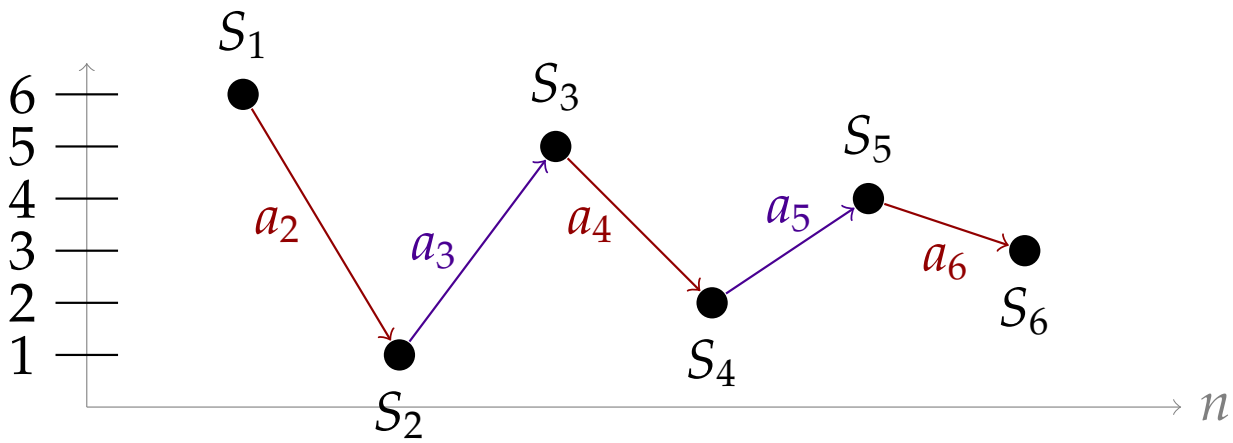

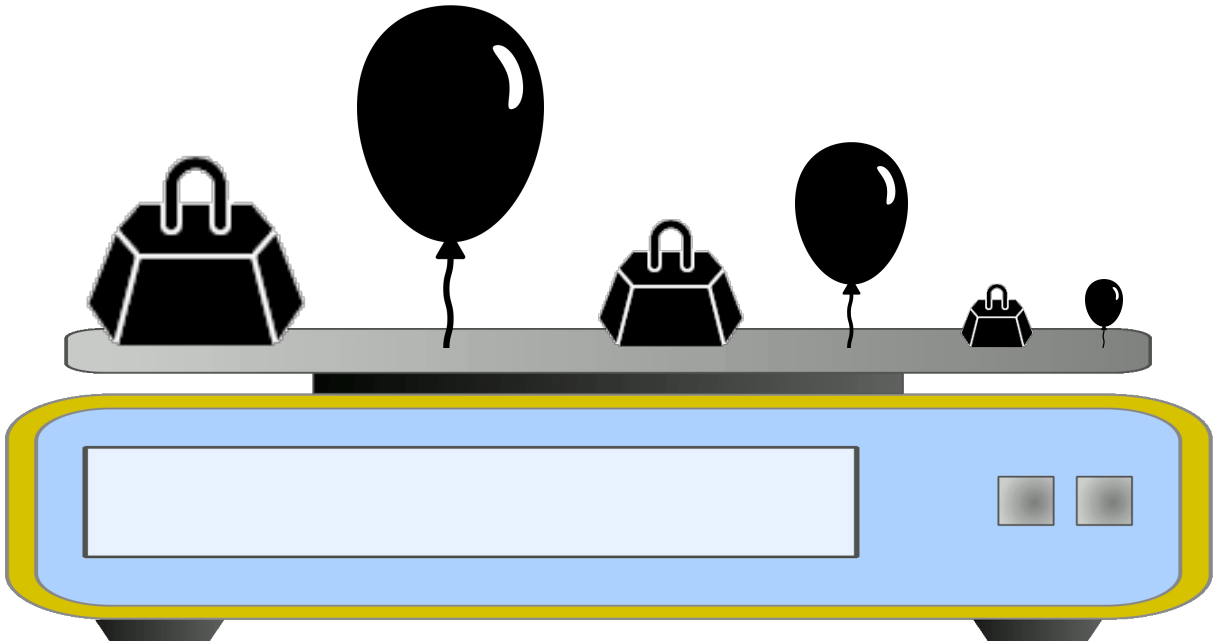

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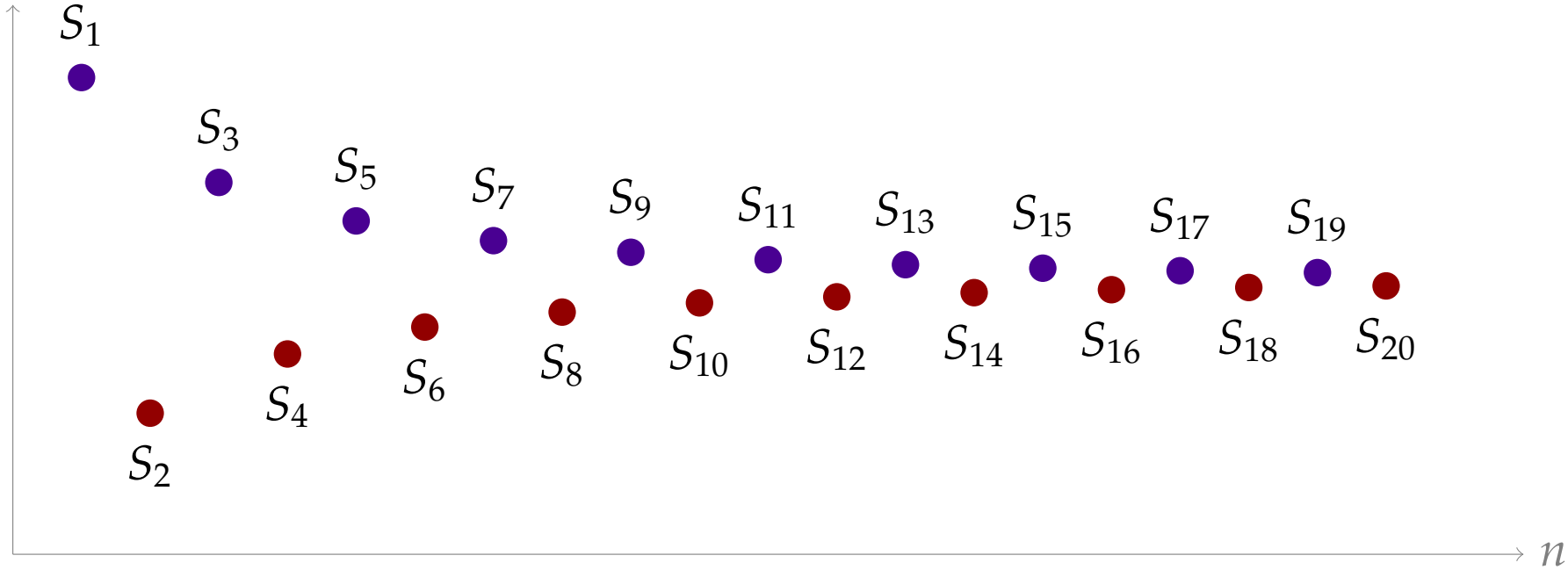
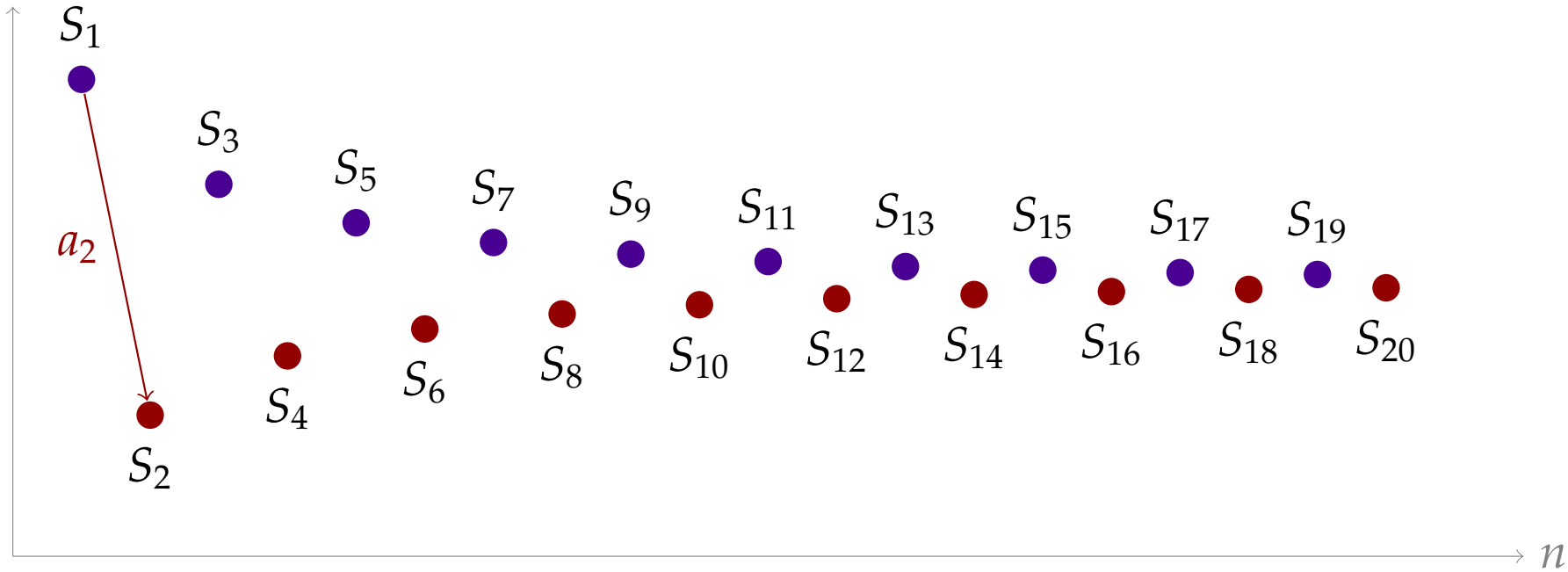
Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.

Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



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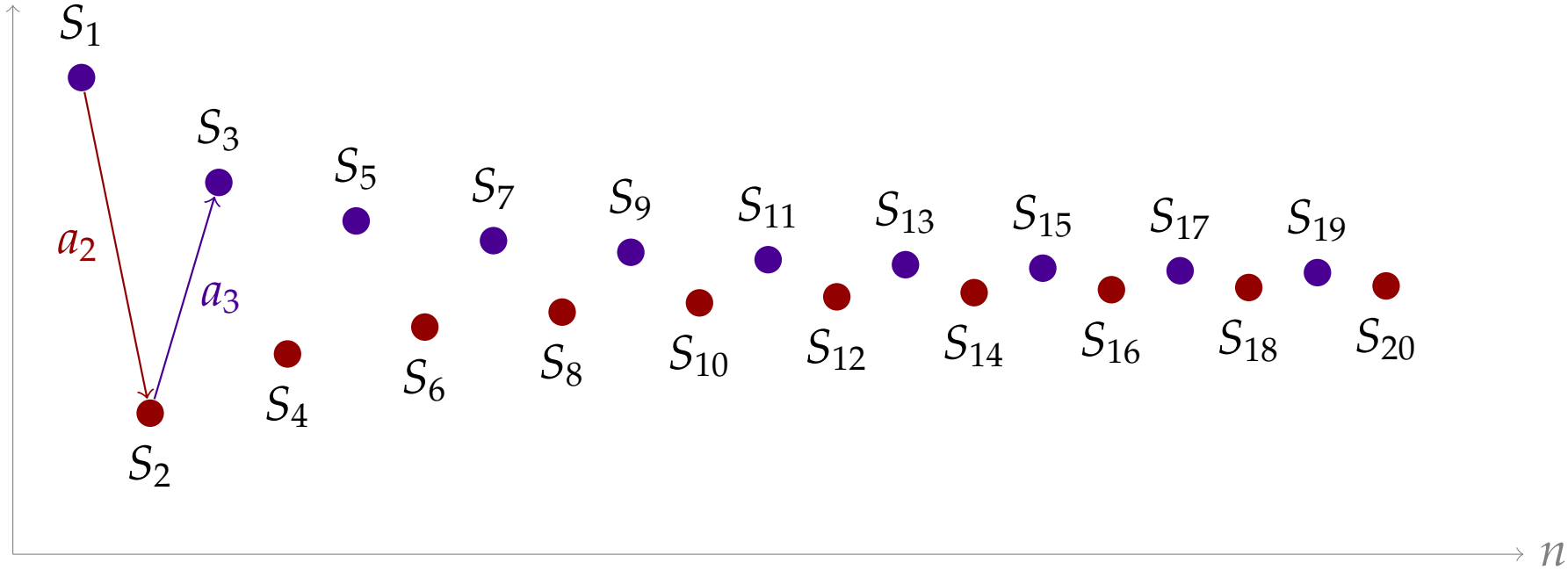

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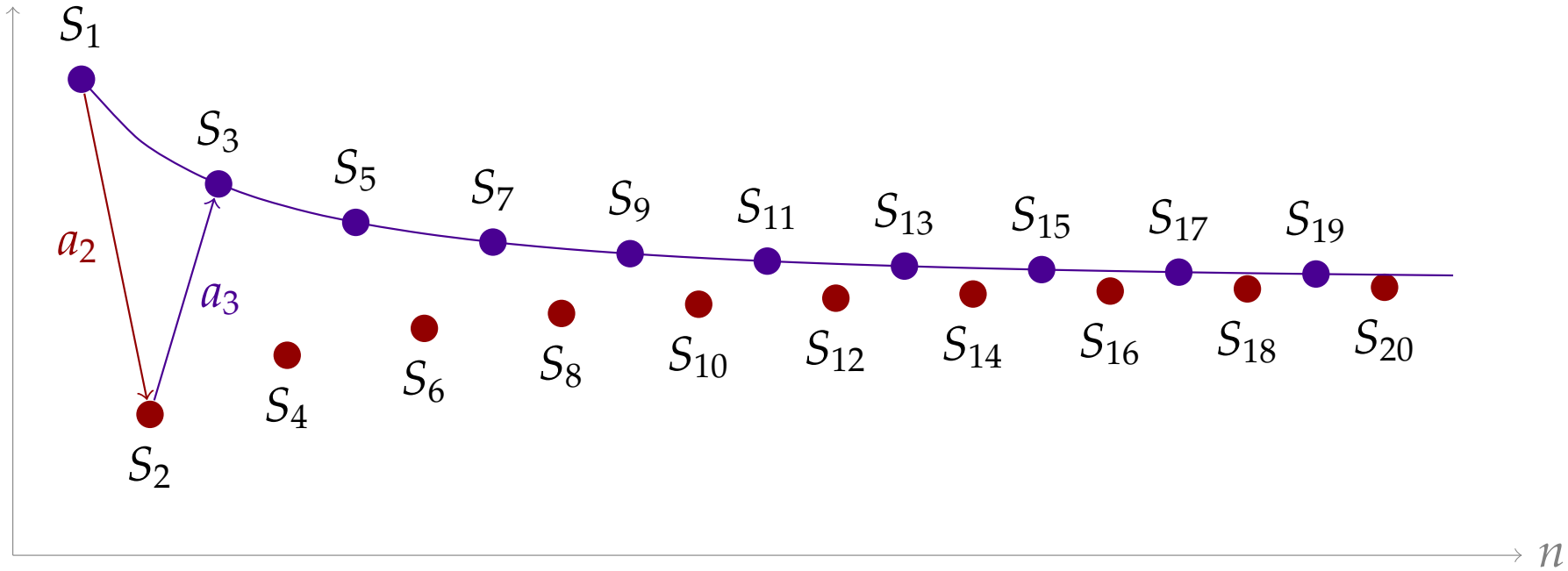
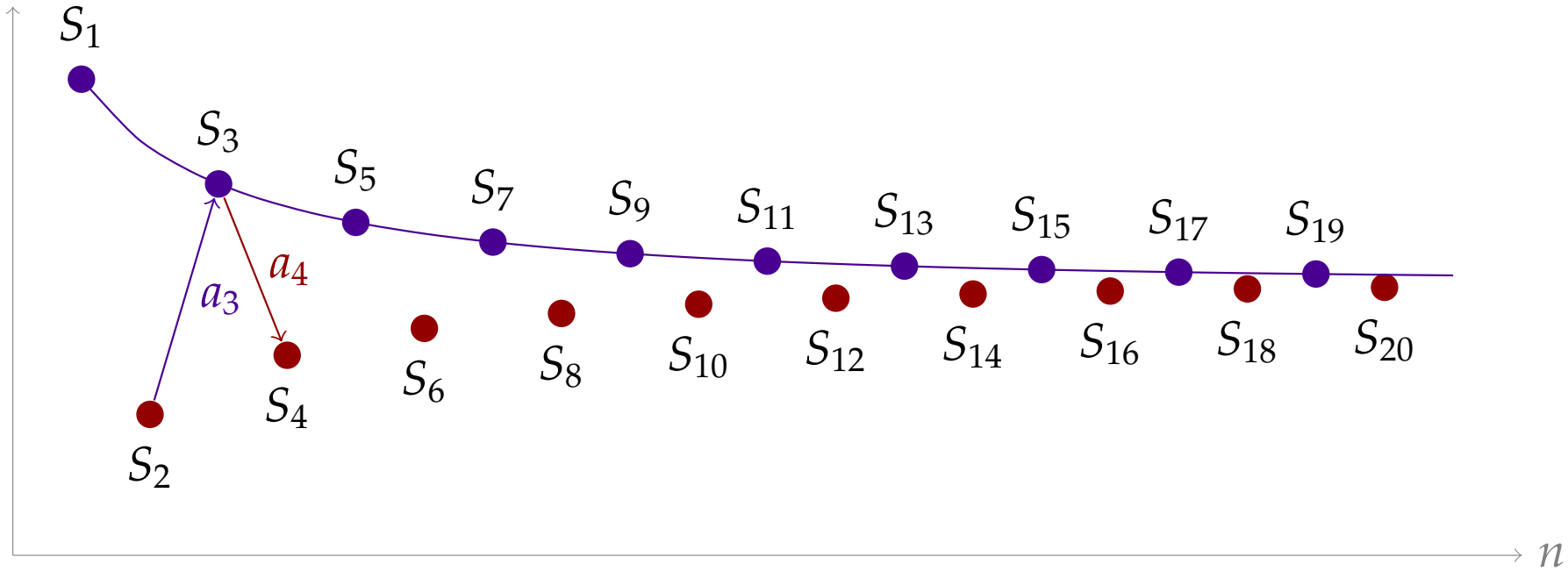


Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.
Odd-indexed partial sums are decreasing.

Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



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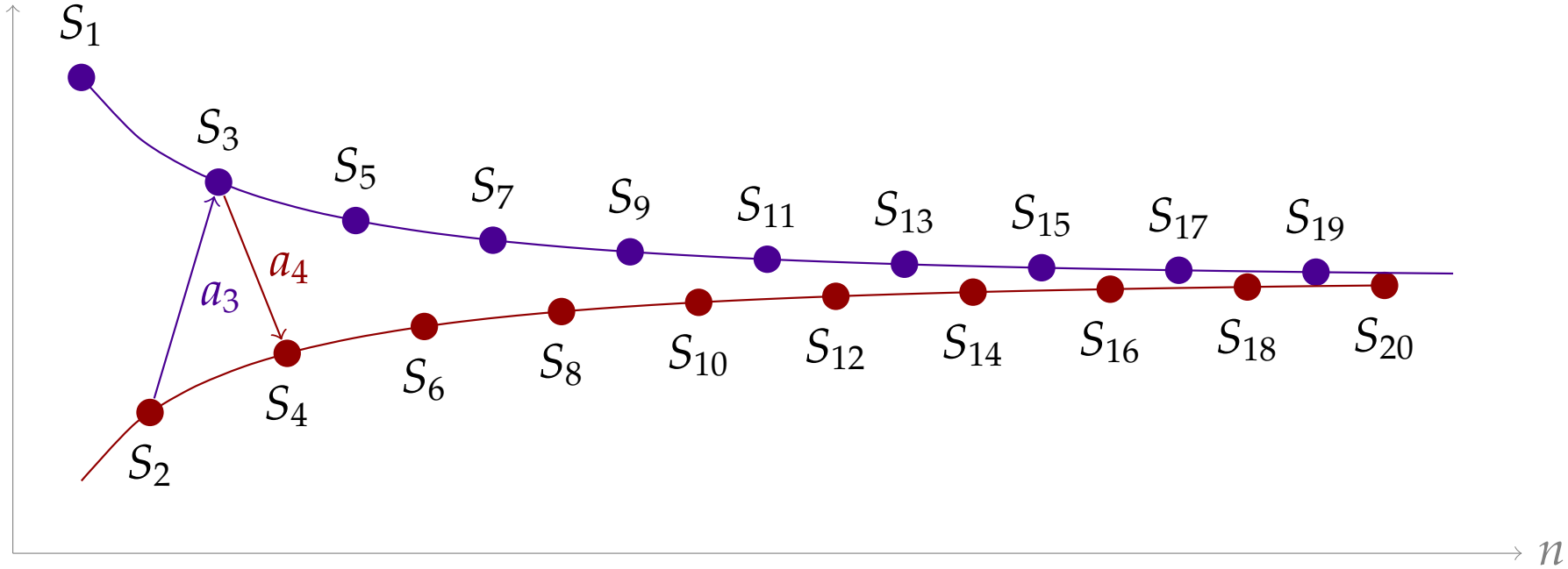
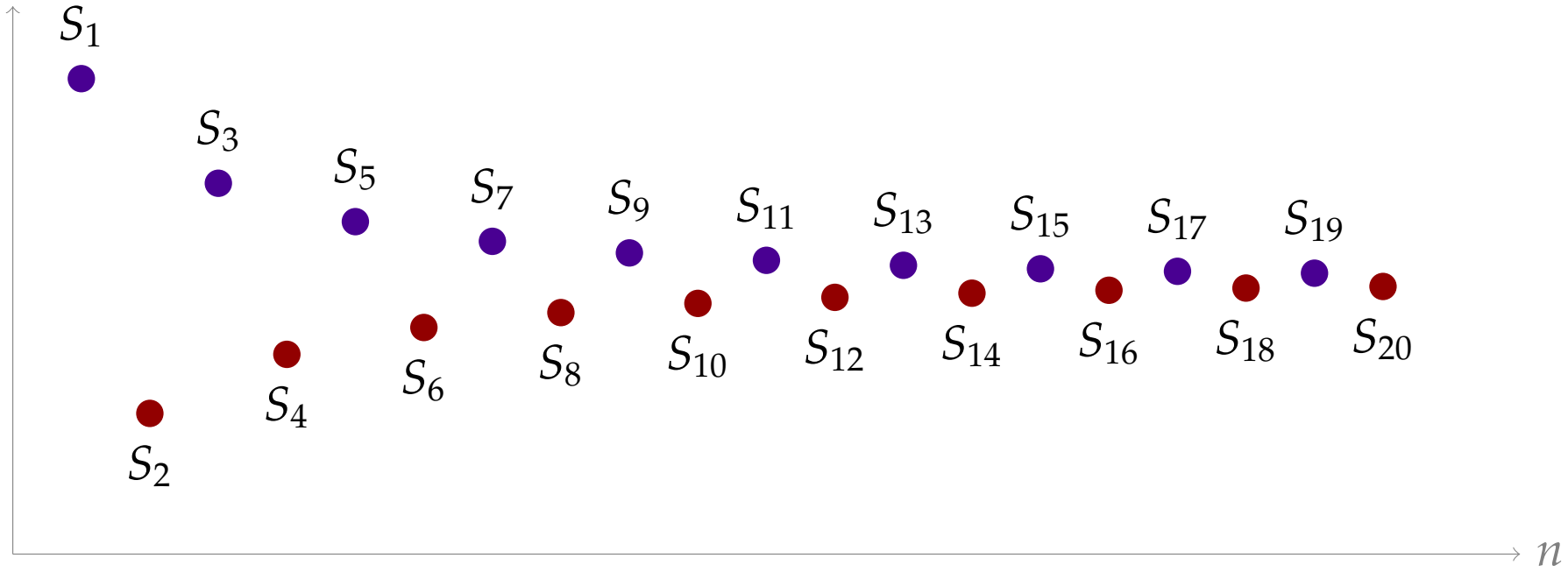


Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.
Odd-indexed partial sums are decreasing.

Since $a_3 > a_4$, we have $a_1 - a_2 + (a_3 - a_4) > a_1 - a_2$, so $S_4 > S_2$.
Even-indexed partial sums are increasing.


S_1
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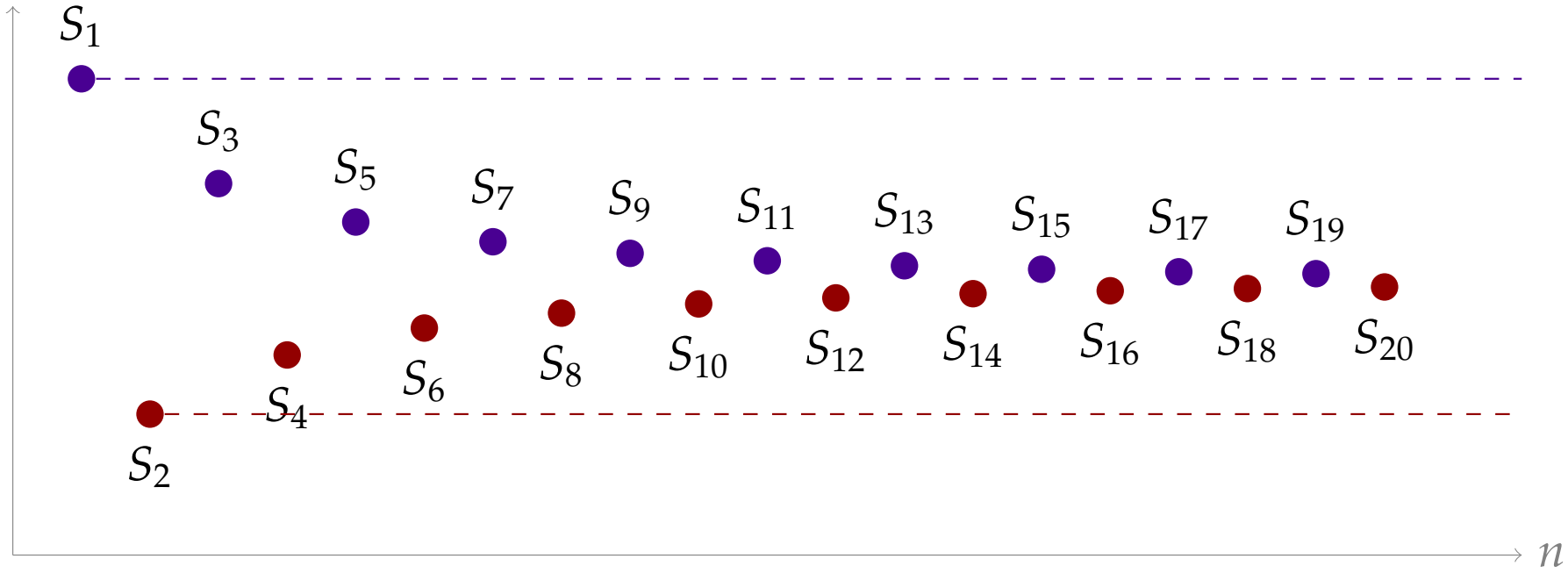
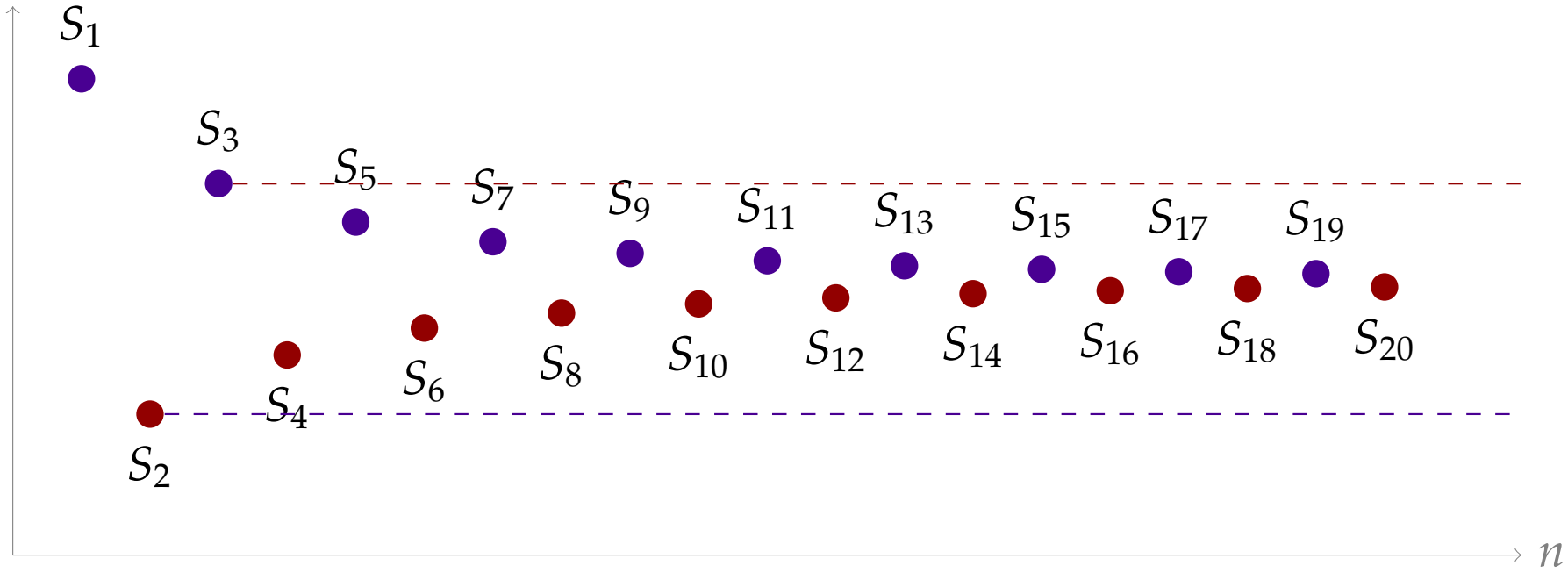
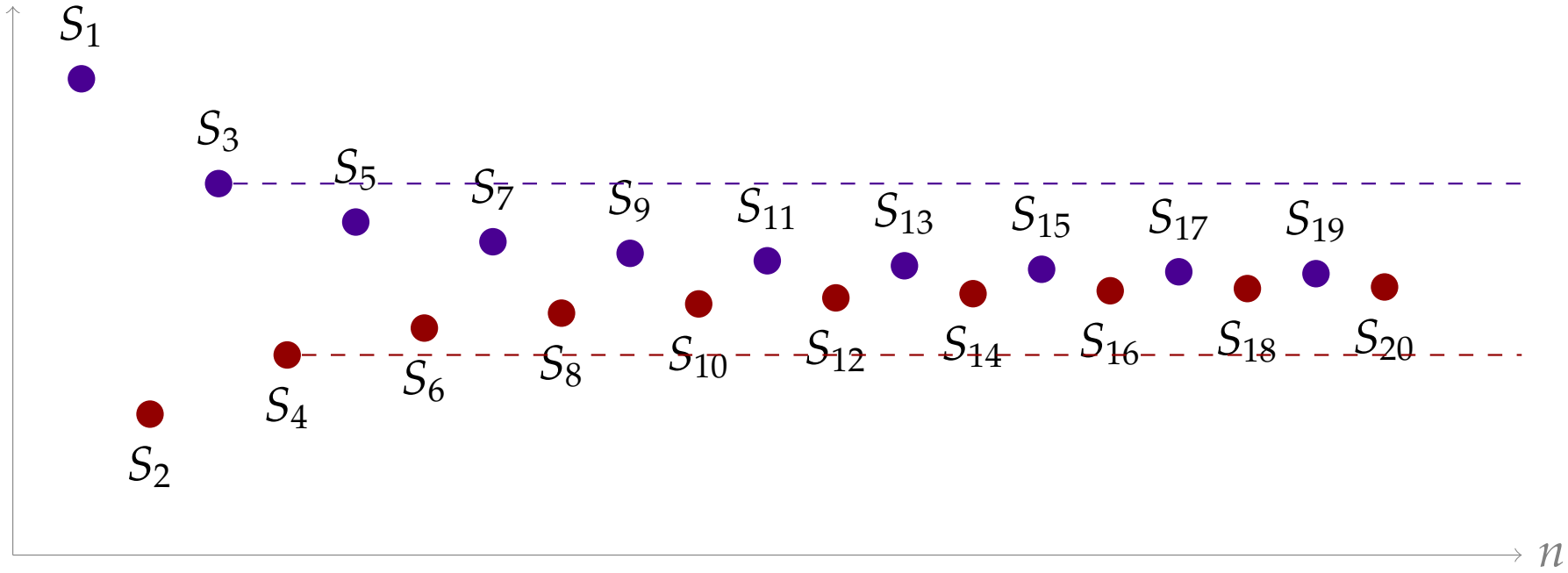
S_{18}
S_{20}
n



- For all $n \geq 2$, S_n lies between S_1 and S_2.



- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.



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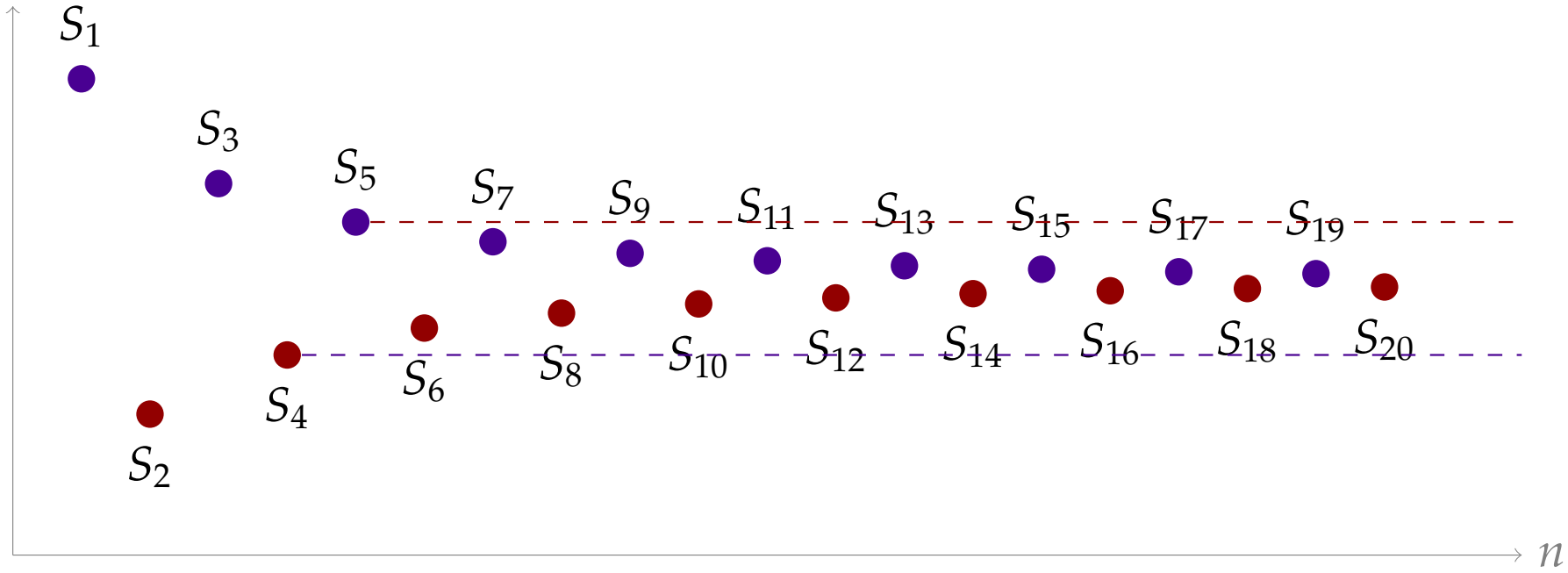
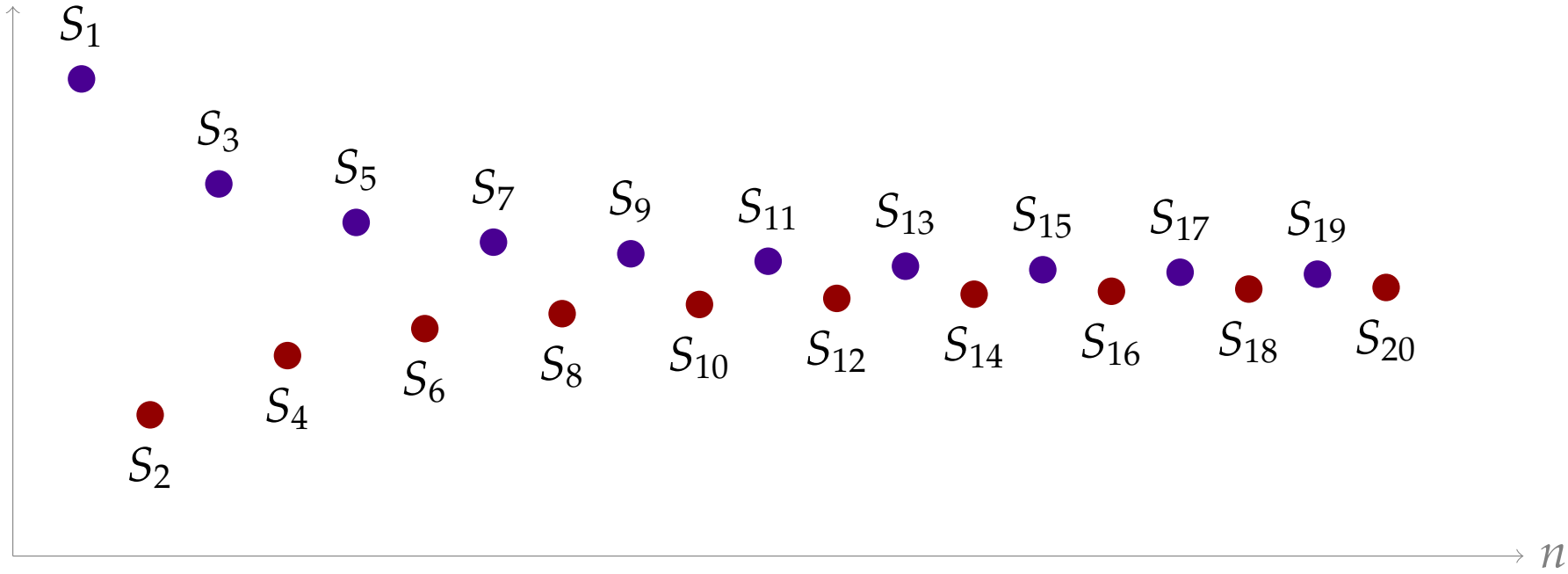
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.

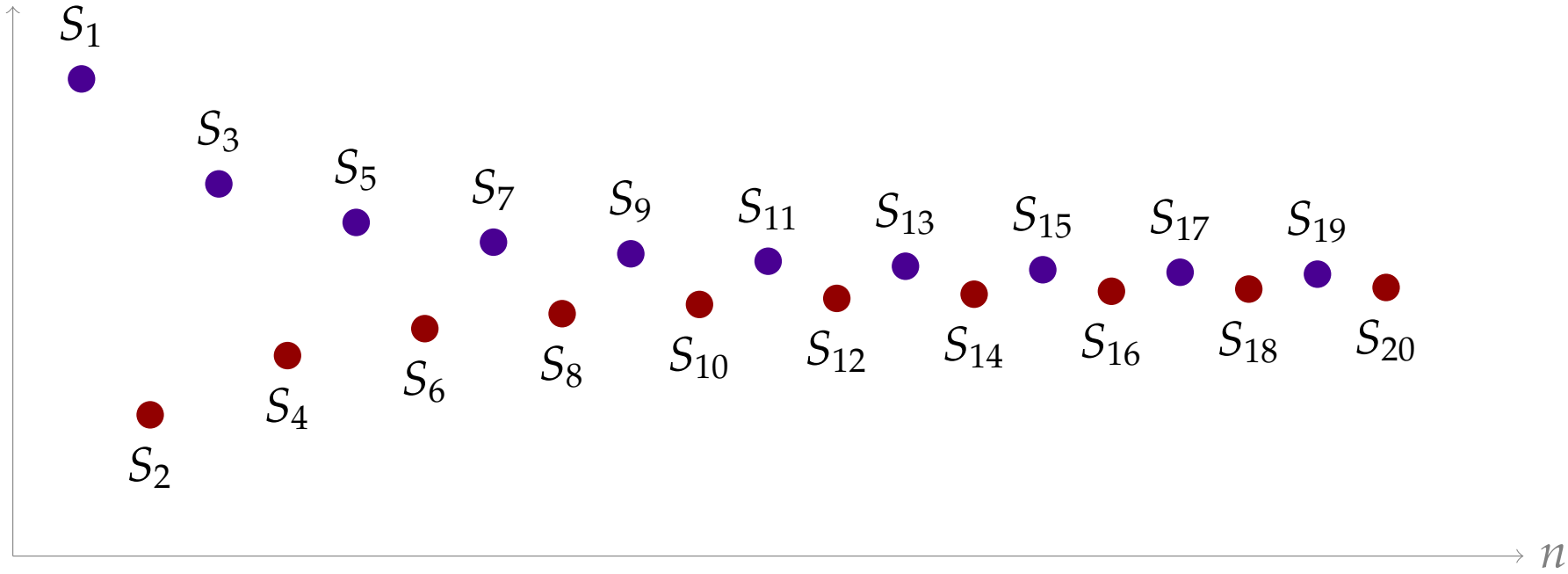


- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.
- For all $n \geq 5$, S_n lies between S_4 and S_5.



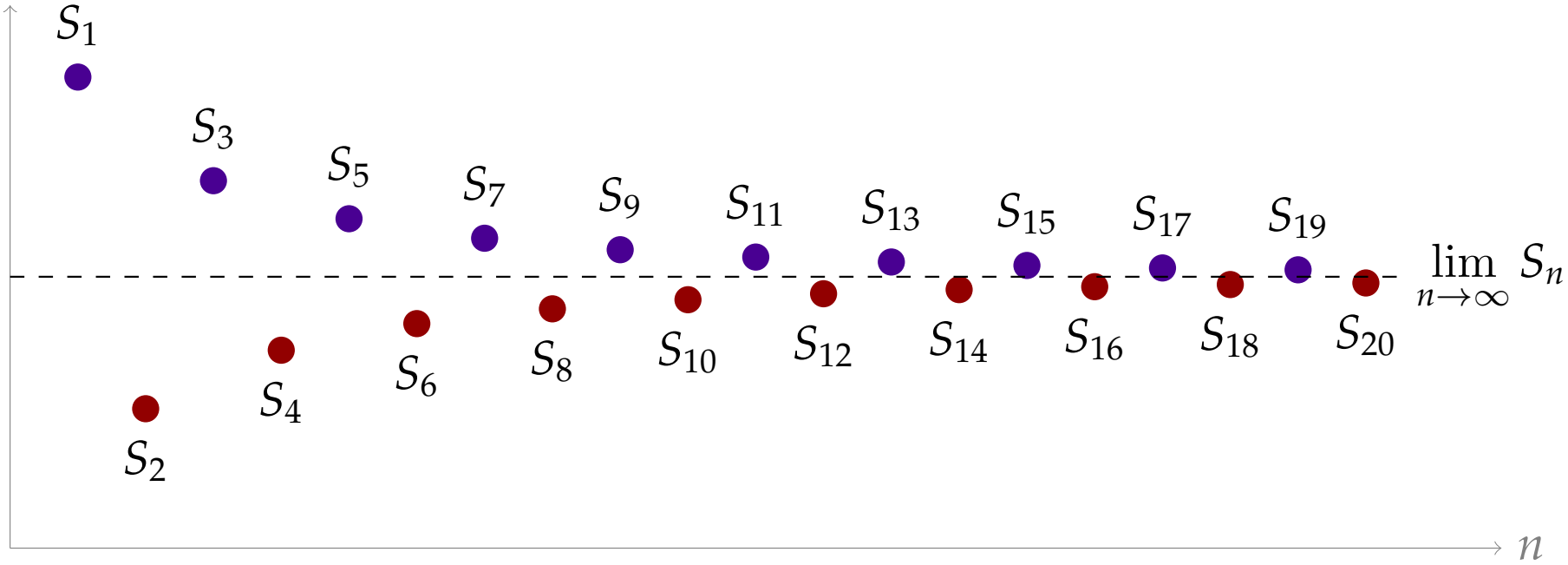
- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.
- For all $n \geq 5$, S_n lies between S_4 and S_5.

The difference between consecutive sums S_n and S_{n-1} is:



- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.
- For all $n \geq 5$, S_n lies between S_4 and S_5.

The difference between consecutive sums S_n and S_{n-1} is: $|a_n|$, which approaches 0.



- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.
- For all $n \geq 5$, S_n lies between S_4 and S_5.

The difference between consecutive sums S_n and S_{n-1} is: $|a_n|$, which approaches 0.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the Nth partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

Alternating Series Test (abridged)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

- True or false: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.
- True or false: the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Let $a_n = \frac{1}{n}$.

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(i) $a_n \geq 0$

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(iii) $\lim_{n\to\infty} a_n = 0$

Let $a_n = \frac{1}{n}$.

(i) $a_n \geq 0$

(ii) $a_{n+1} \leq a_n$

(iii) $\lim_{n\to\infty} a_n = 0$

- We've already seen that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- By the Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges. That is,

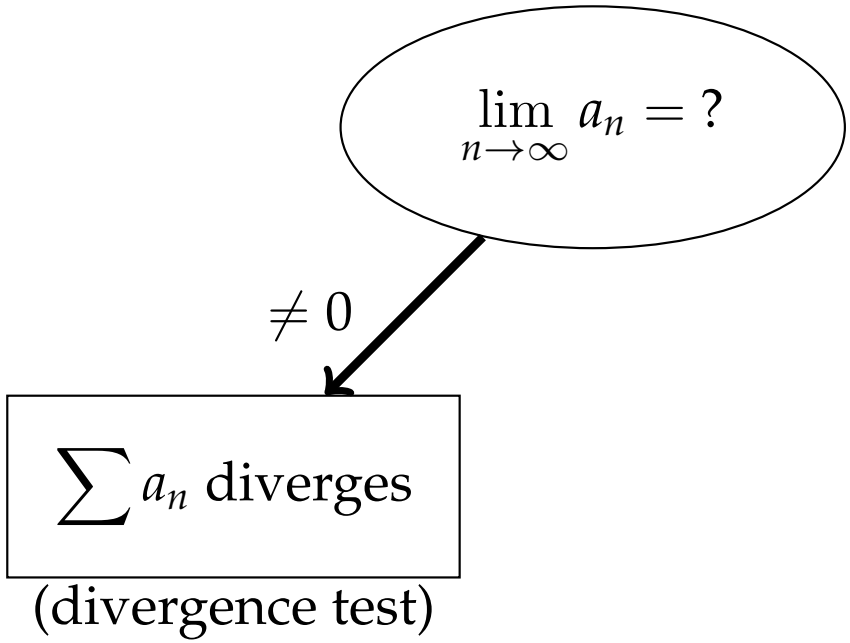
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

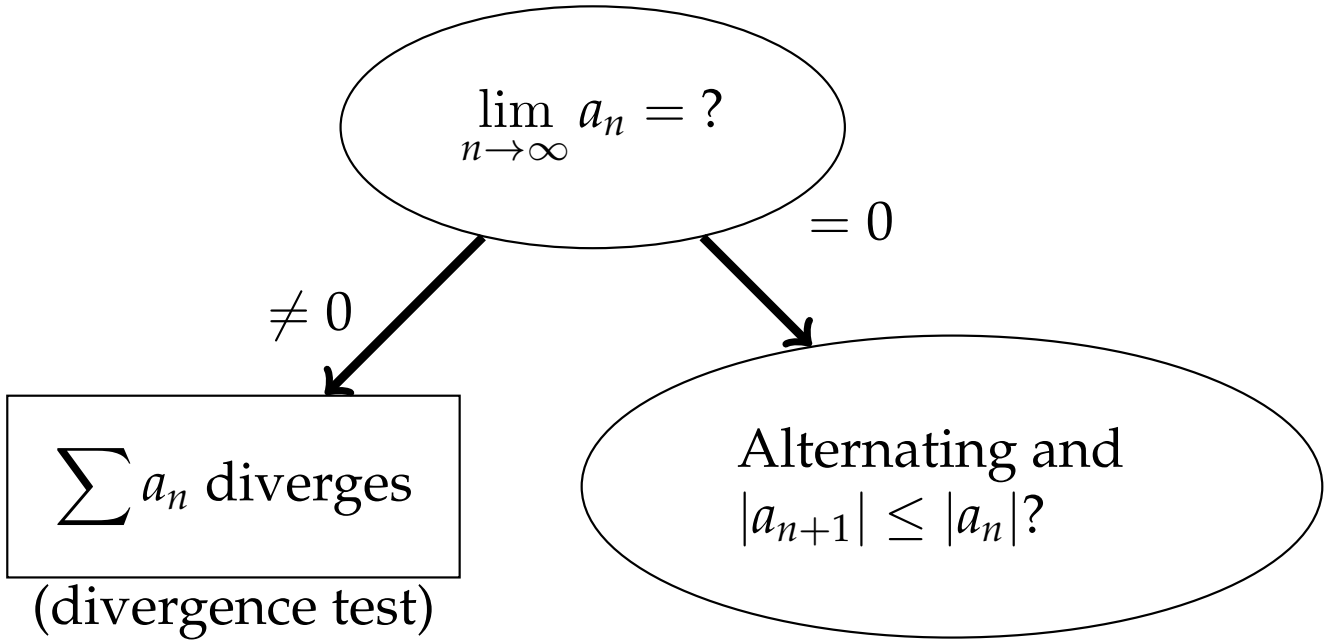
Divergence Test + Alternating Series Test

$$\lim_{n\to\infty} a_n = ?$$

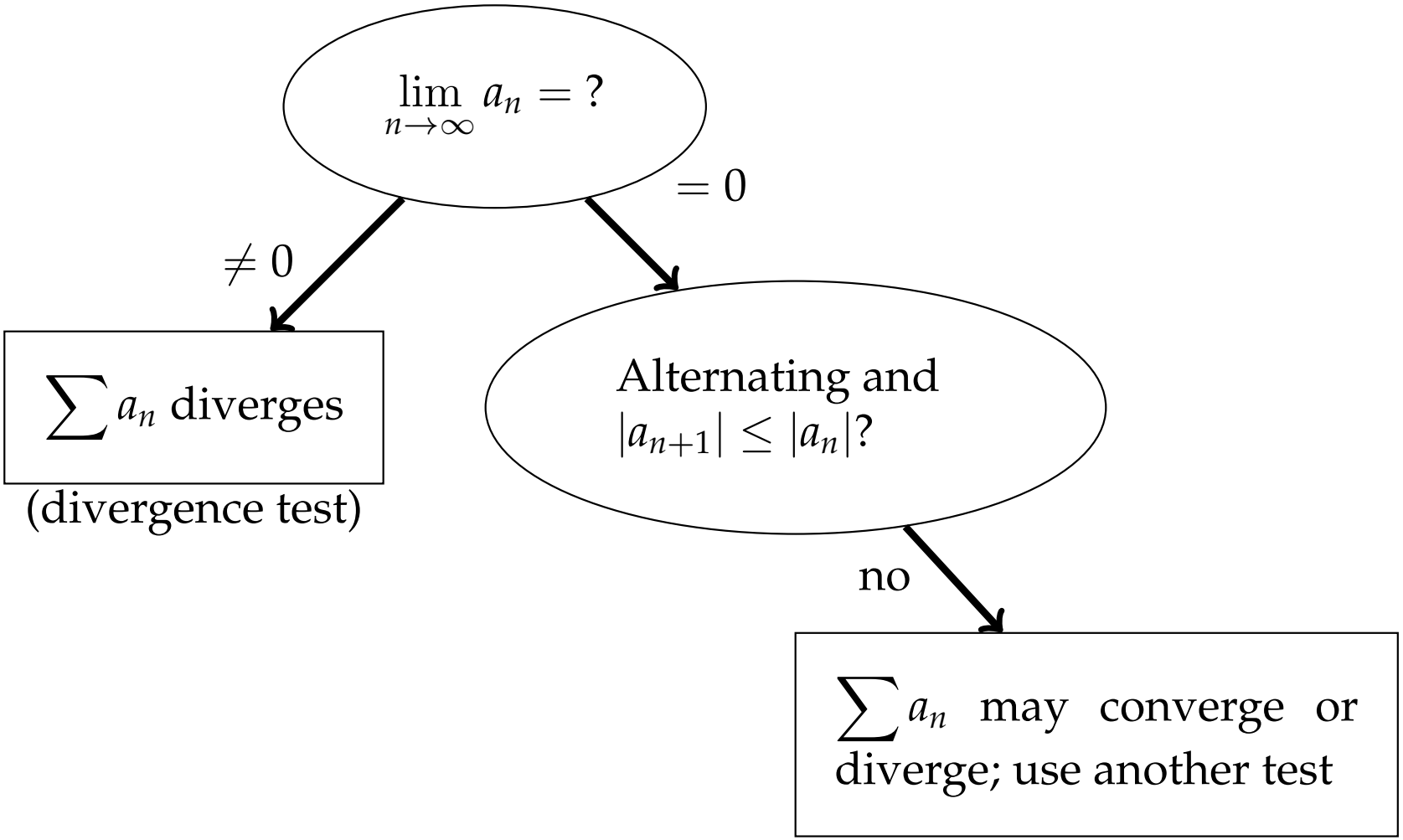
Divergence Test + Alternating Series Test



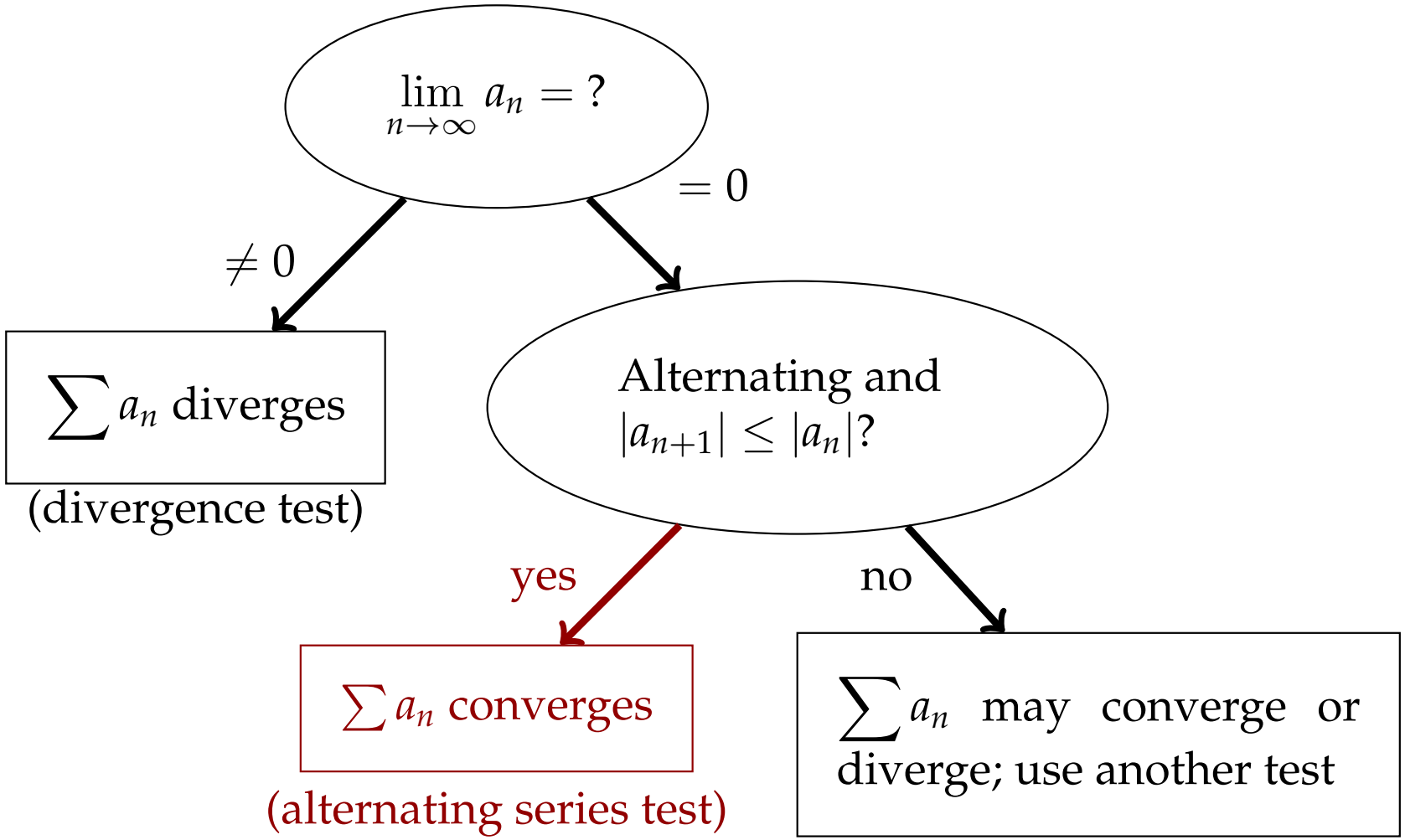
Divergence Test + Alternating Series Test



Divergence Test + Alternating Series Test



Divergence Test + Alternating Series Test



Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{99}\frac{(-1)^{n-1}}{n} \approx 0.698$.

How close is that to the value $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$?

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{99}\frac{(-1)^{n-1}}{n} \approx 0.698$.

How close is that to the value $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$?

$$\frac{-1}{100} = \frac{(-1)^{100-1}}{100} \leq \sum_{n=1}^{\infty}\frac{(-1)^n}{n} - \sum_{n=1}^{99}\frac{(-1)^n}{n} \leq 0.$$

That is, the actual series has a sum in the interval $[0.688, 0.698]$.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{19}(-1)^{n-1}\frac{n^2}{n^2+1} \approx 0.6347$.

How close is that to the value $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$?

Alternating Series Test

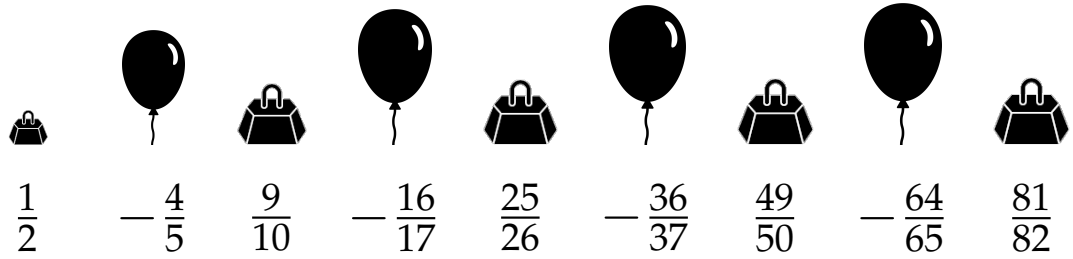
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{19}(-1)^{n-1}\frac{n^2}{n^2+1} \approx 0.6347$.

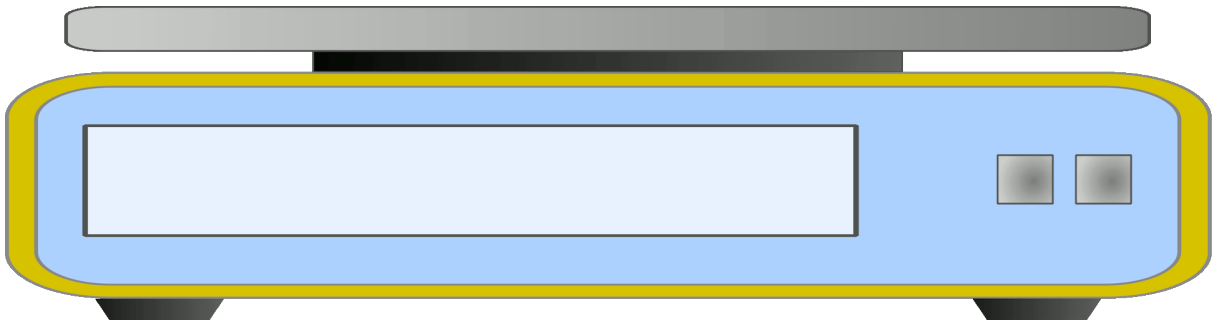
How close is that to the value $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$?

Not close at all: the series is divergent (which we can see by the divergence test).

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

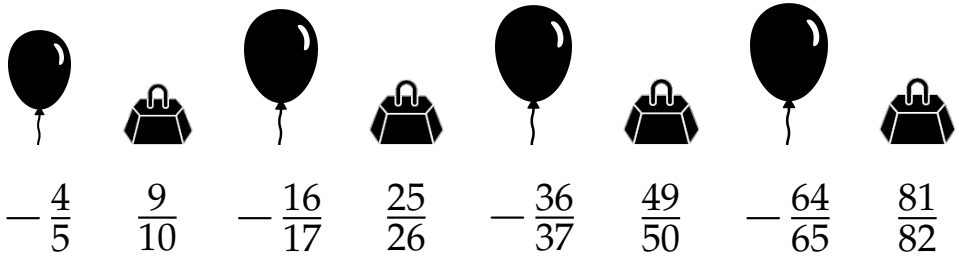


$\frac{1}{2}$ $-\frac{4}{5}$ $\frac{9}{10}$ $-\frac{16}{17}$ $\frac{25}{26}$ $-\frac{36}{37}$ $\frac{49}{50}$ $-\frac{64}{65}$ $\frac{81}{82}$

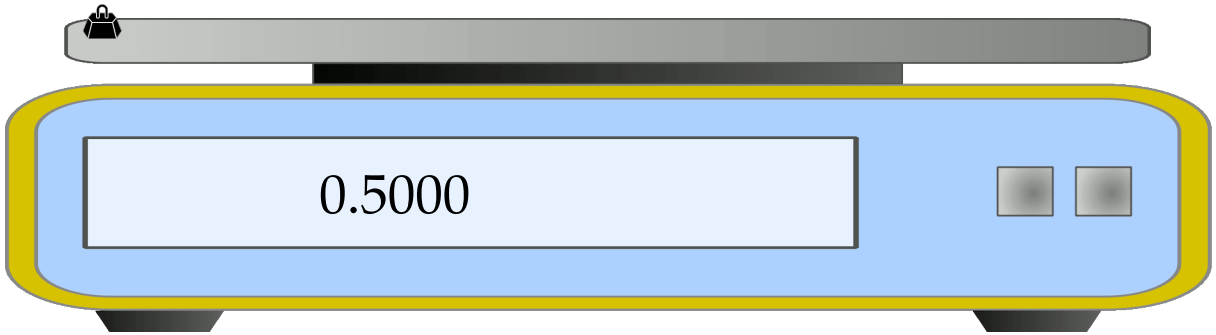


$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$S_1 = 0.5000$



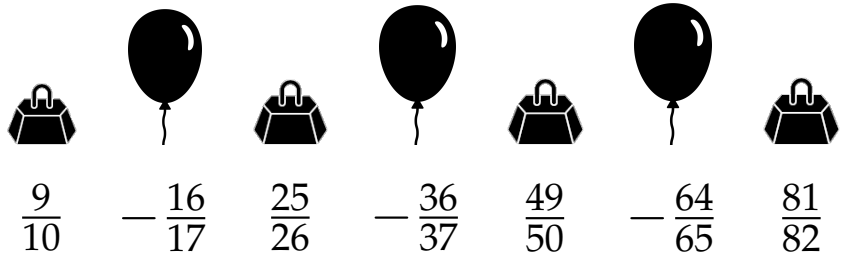
$$-\frac{4}{5} \quad \frac{9}{10} \quad -\frac{16}{17} \quad \frac{25}{26} \quad -\frac{36}{37} \quad \frac{49}{50} \quad -\frac{64}{65} \quad \frac{81}{82}$$



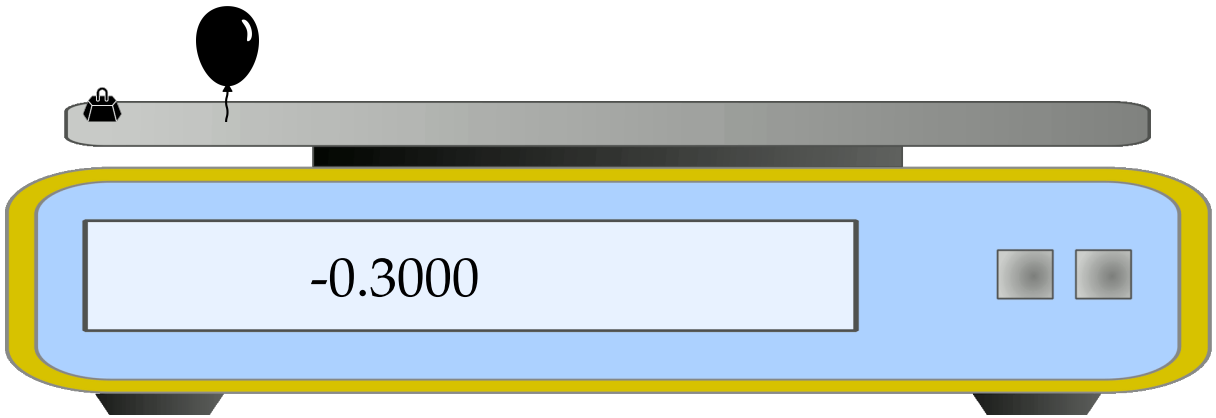
$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$



$$\frac{9}{10} \quad -\frac{16}{17} \quad \frac{25}{26} \quad -\frac{36}{37} \quad \frac{49}{50} \quad -\frac{64}{65} \quad \frac{81}{82}$$

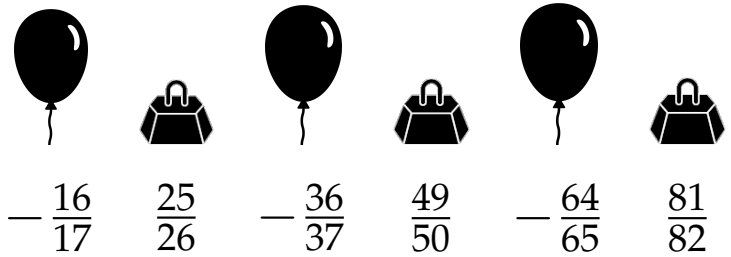


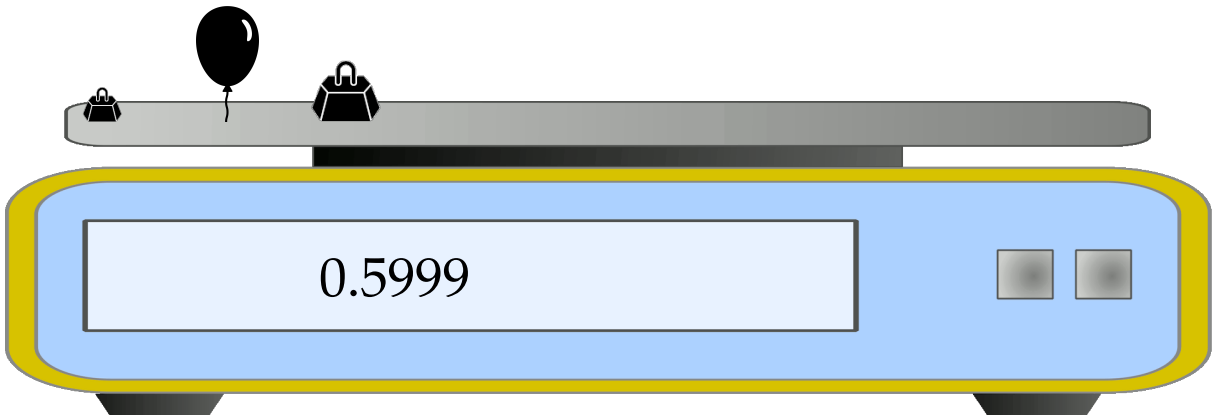
$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$$S_1 = 0.5000$$

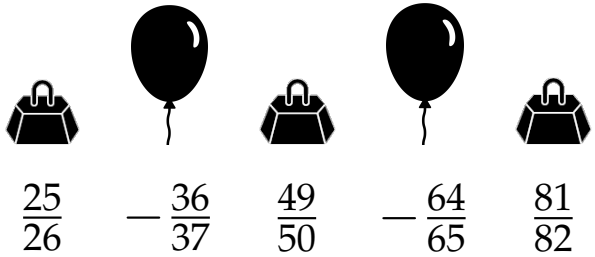
$$S_2 = -0.3000$$

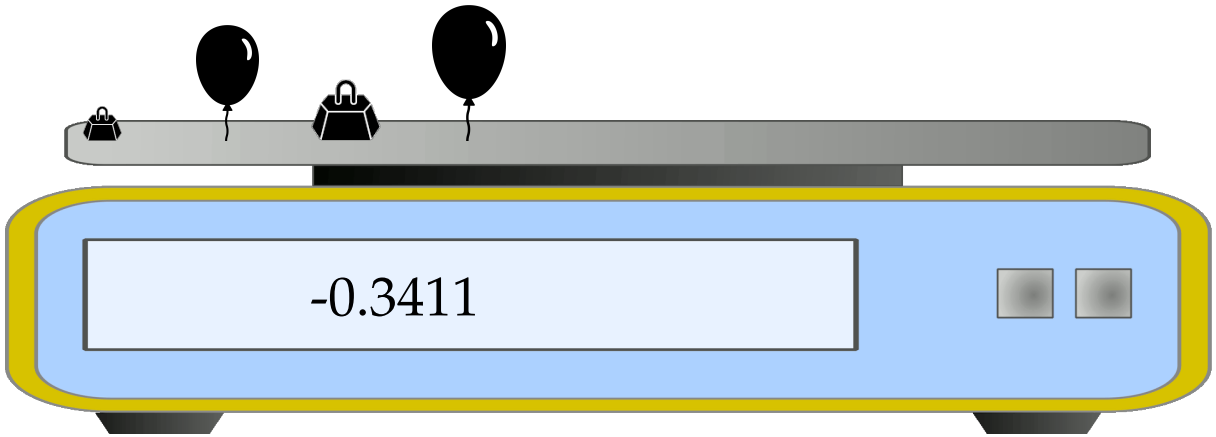
$$S_3 = 0.5999$$





$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES





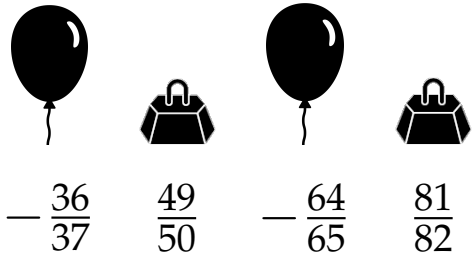
$$S_1 = 0.5000$$

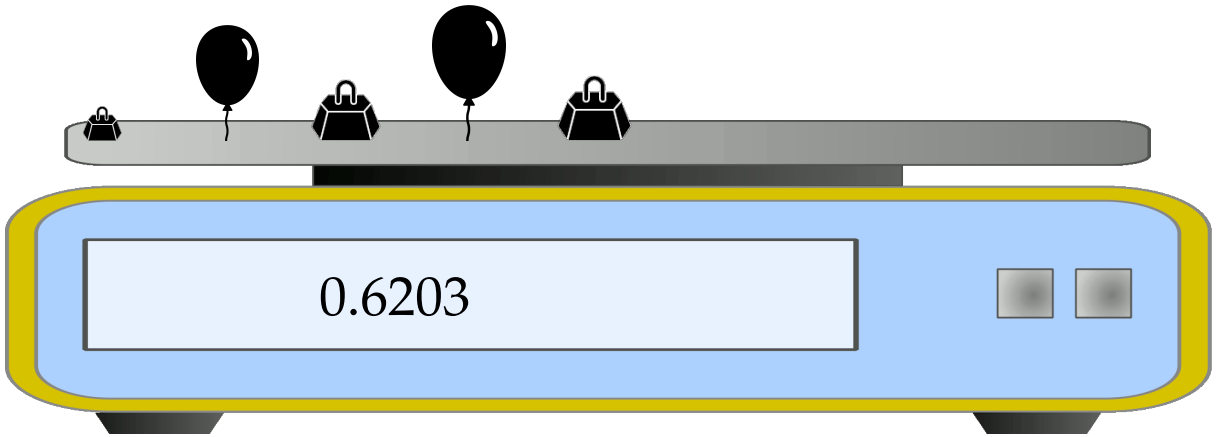
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES





$$S_1 = 0.5000$$

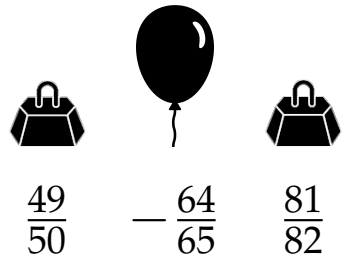
$$S_2 = -0.3000$$

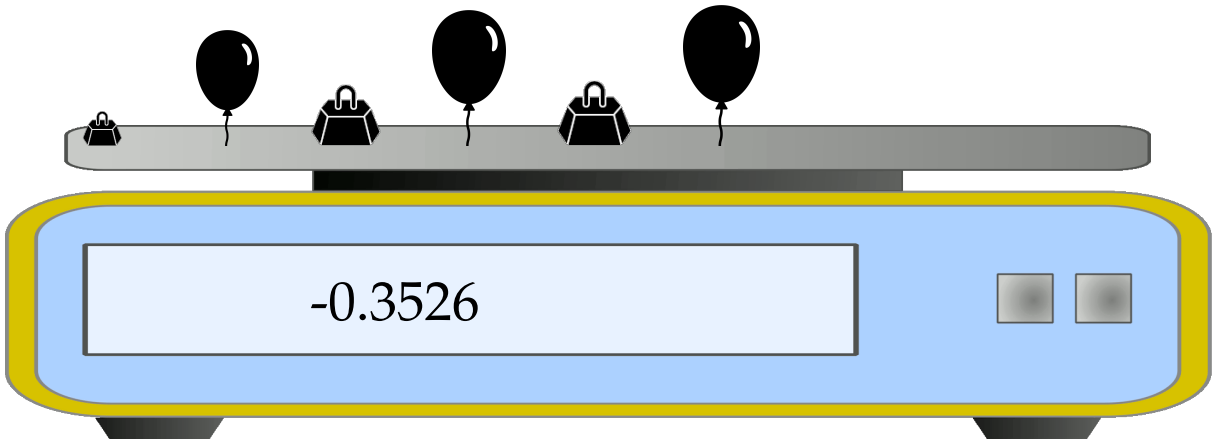
$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES





$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

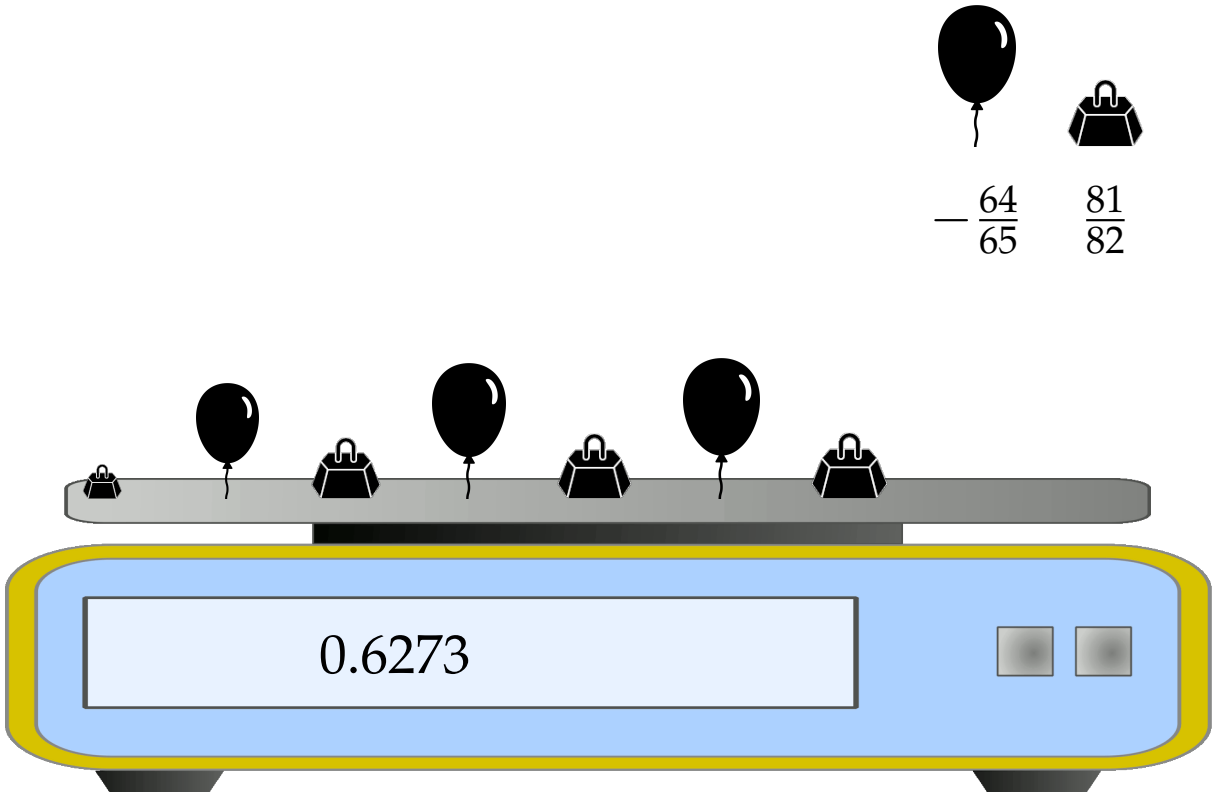
$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

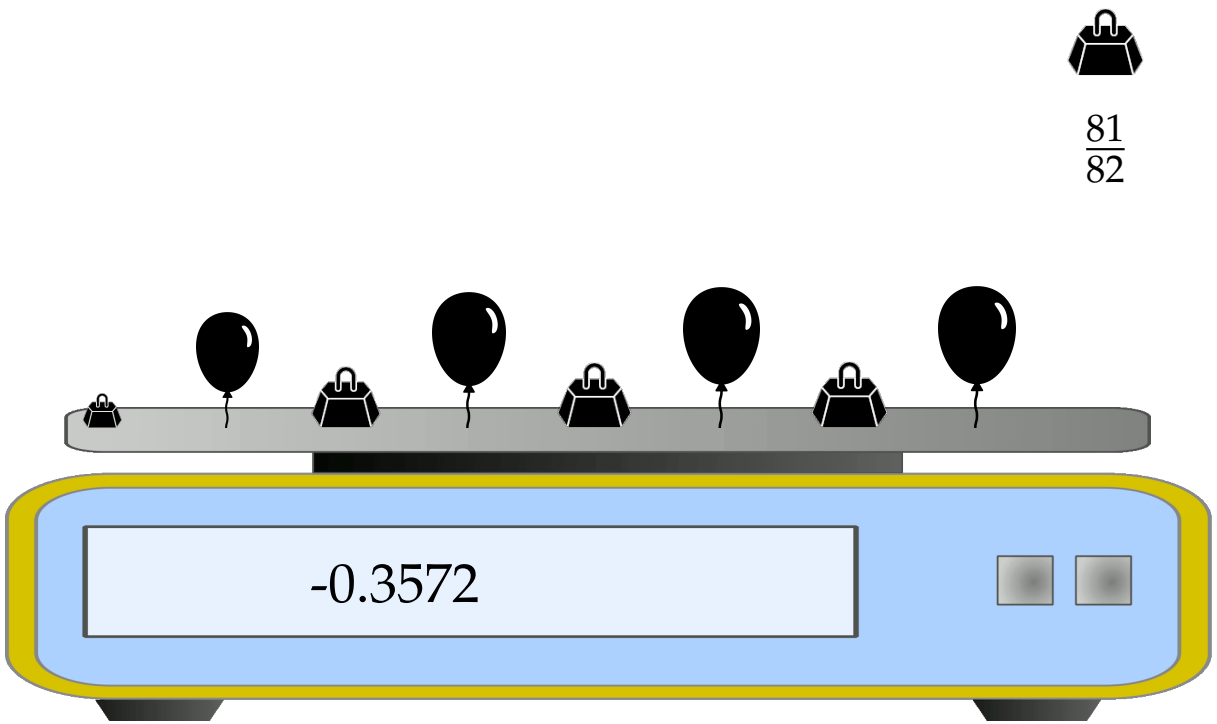
$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

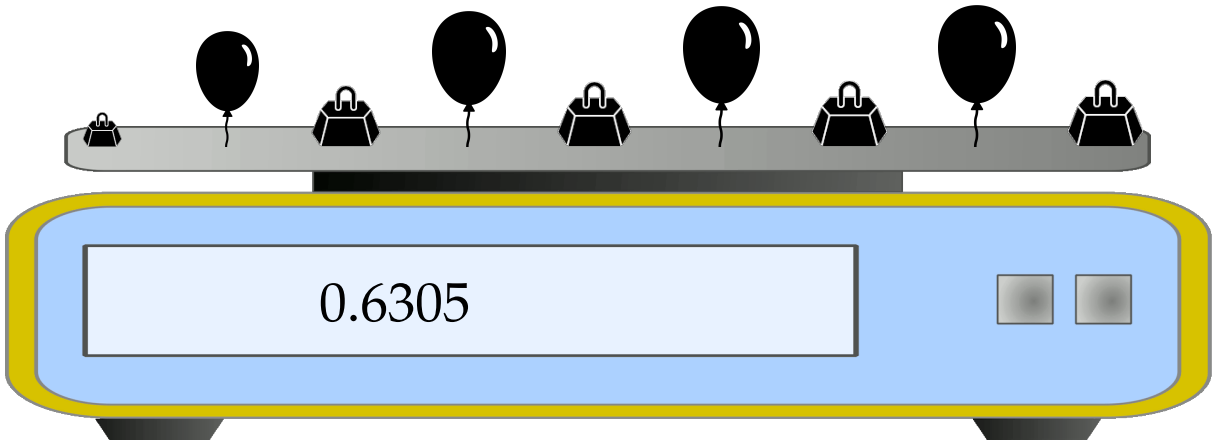
$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$$S_7 = 0.6273$$

$$S_8 = -0.3572$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

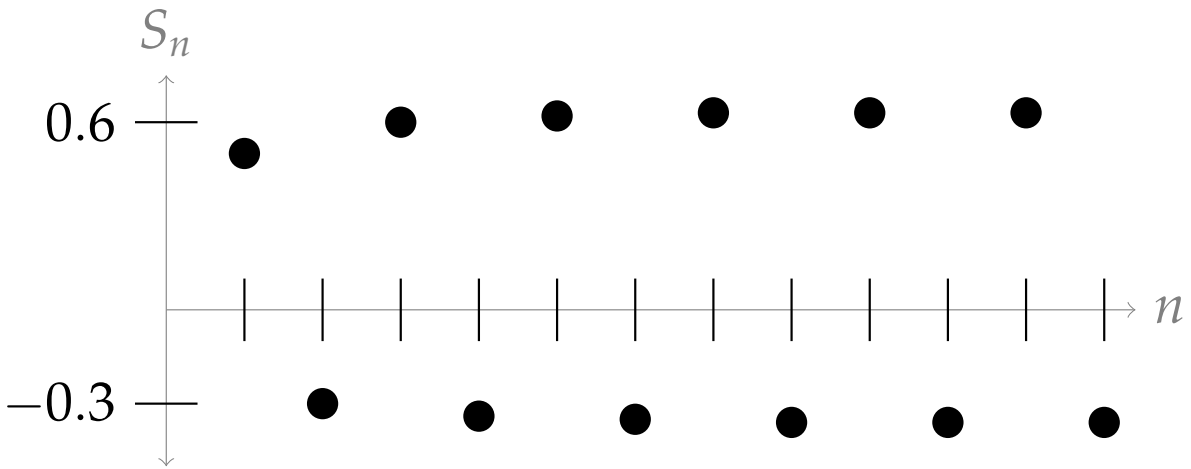
$$S_6 = -0.3526$$

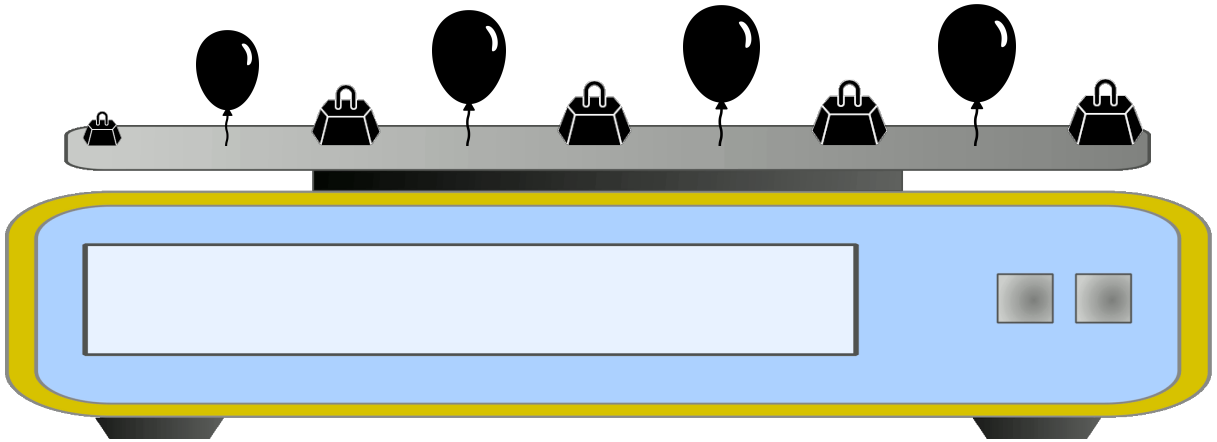
$$S_7 = 0.6273$$

$$S_8 = -0.3572$$

$$S_9 = 0.6305$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES





$S_1 = 0.5000$

$S_2 = -0.3000$

$S_3 = 0.5999$

$S_4 = -0.3411$

$S_5 = 0.6203$

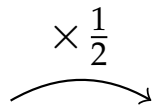
$S_6 = -0.3526$

$S_7 = 0.6273$

$S_8 = -0.3572$

$S_9 = 0.6305$

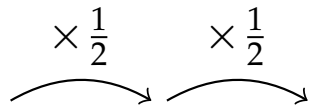
Recall for a geometric series, the ratios of consecutive terms is constant.



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

If that ratio has magnitude less then one, then the series converges.
If the ratio has magnitude greater than one, the series diverges.

Recall for a geometric series, the ratios of consecutive terms is constant.



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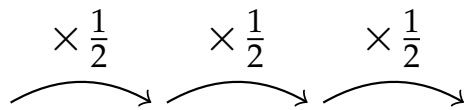

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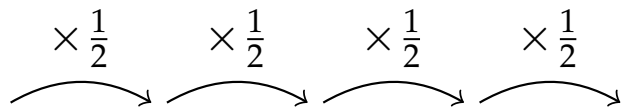

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Recall for a geometric series, the ratios of consecutive terms is constant.

$$\overset{\times\frac{1}{2}}{\frac{1}{2}} + \overset{\times\frac{1}{2}}{\frac{1}{4}} + \overset{\times\frac{1}{2}}{\frac{1}{8}} + \overset{\times\frac{1}{2}}{\frac{1}{16}} + \overset{\times\frac{1}{2}}{\frac{1}{32}} \cdots$$

$$\underbrace{\frac{1/4}{1/2}} = $$

If that ratio has magnitude less then one, then the series converges.
If the ratio has magnitude greater than one, the series diverges.

Recall for a geometric series, the ratios of consecutive terms is constant.

$$\overset{\times\frac{1}{2}}{\frown}\quad\overset{\times\frac{1}{2}}{\frown}\quad\overset{\times\frac{1}{2}}{\frown}\quad\overset{\times\frac{1}{2}}{\frown}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

$$\underbrace{\quad}_{\frac{1/4}{1/2} =}\ \underbrace{\quad}_{\frac{1/8}{1/4} =}$$

If that ratio has magnitude less then one, then the series converges.
If the ratio has magnitude greater than one, the series diverges.

Recall for a geometric series, the ratios of consecutive terms is constant.

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$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\cdots$$

$$\underbrace{\quad}_{\frac{1/4}{1/2}=\frac{1}{2}}\quad\underbrace{\quad}_{\frac{1/8}{1/4}=\frac{1}{2}}\quad\underbrace{\quad}_{\frac{1/16}{1/8}=\frac{1}{2}}\quad\underbrace{\quad}_{\frac{1/32}{1/16}=\frac{1}{2}}$$

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If the ratio has magnitude greater than one, the series diverges.

For series convergence, we are concerned with what happens to terms a_n when n is sufficiently large.
Suppose for a sequence a_n, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ for some constant L.

$$\underbrace{a_n + a_{n+1}}_{\frac{a_{n+1}}{a_n} \approx} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots$$

Like in a geometric series:

If L has magnitude less then one, then the series converges.
If L has magnitude greater than one, the series diverges.

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Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

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Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

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$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{3}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, by the ratio test, $\sum_{n=1}^{\infty} \frac{n}{3^n}$ coverges.

Remark

The series we just considered, $\sum_{n=1}^{\infty} \frac{n}{3^n}$, looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

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  - $\sum \frac{1}{3^n}$ is not a valid comparison series, nor is $\sum n$.

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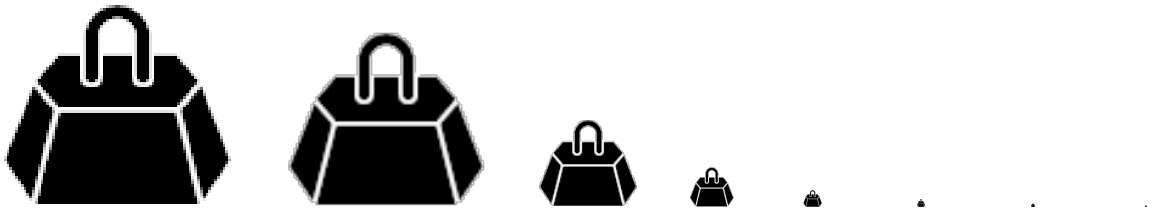
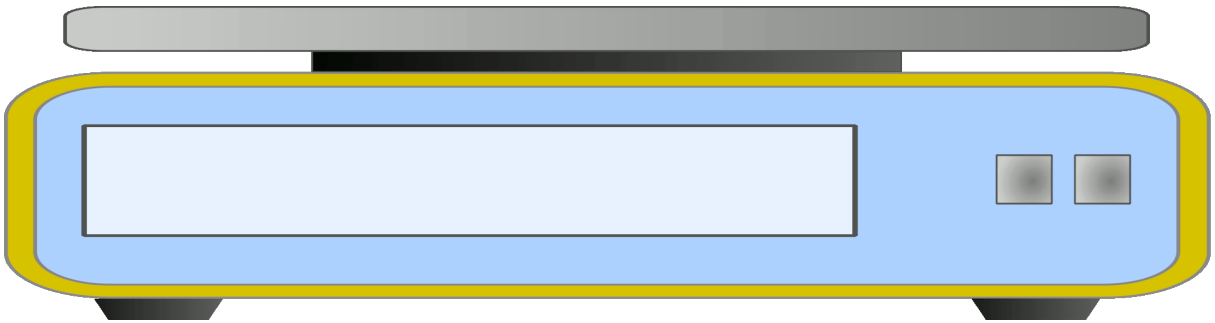
  - Because $n < 2^n$ for all $n \geq 1$, the series $\sum \left(\frac{2}{3}\right)^n$ will work.
- The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

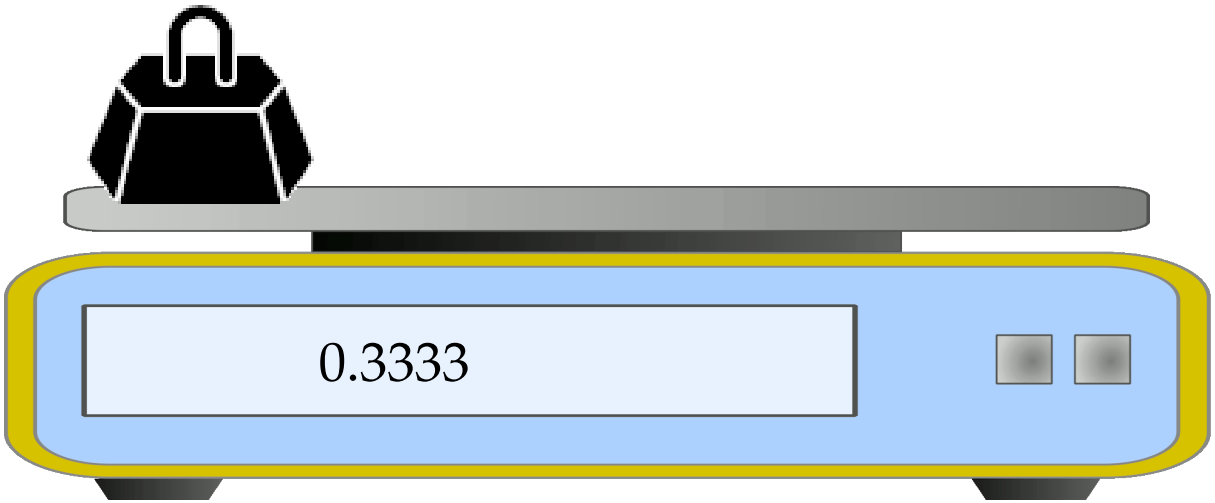


$\frac{1}{3}$ $\frac{2}{3^2}$ $\frac{3}{3^3}$ $\frac{4}{3^4}$ $\frac{5}{3^5}$ $\frac{6}{3^6}$ $\frac{7}{3^7}$ $\frac{8}{3^8}$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

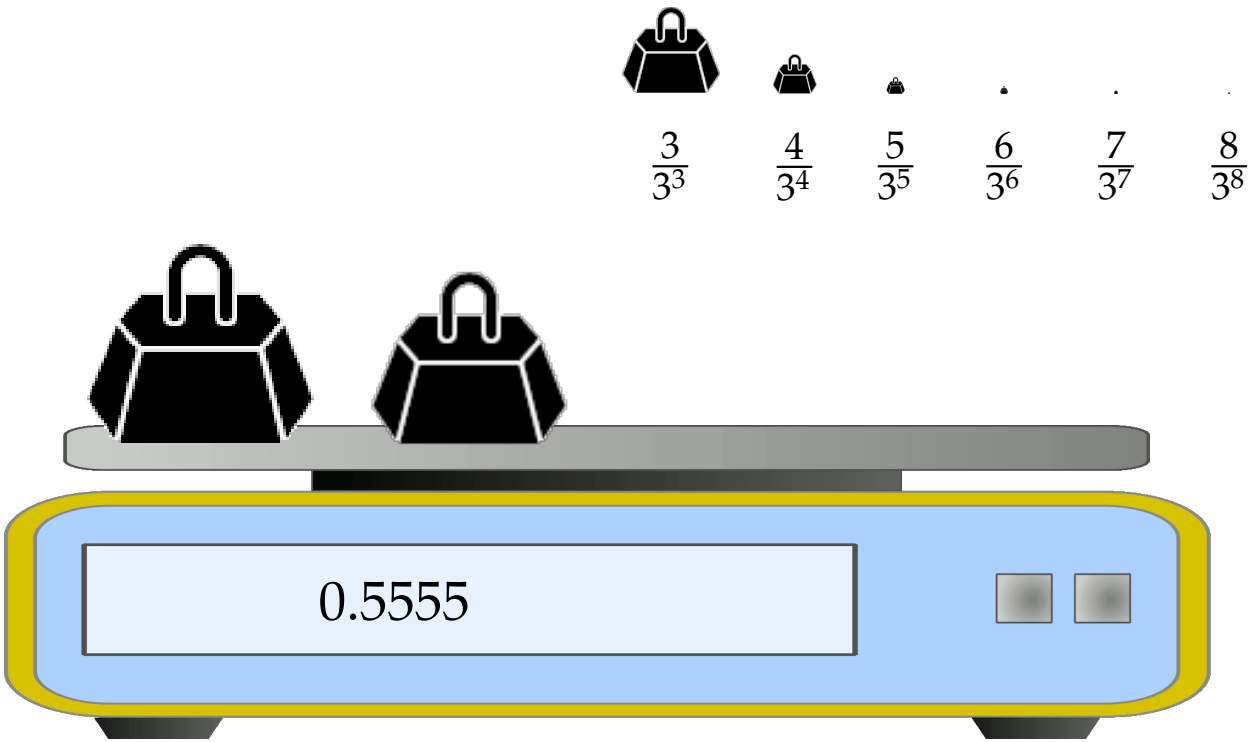
$$S_1 = 0.3333$$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$S_1 = 0.3333$

$S_2 = 0.5555$

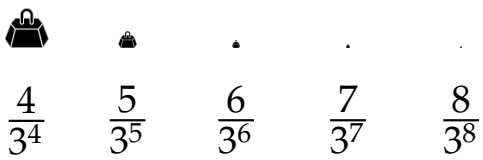


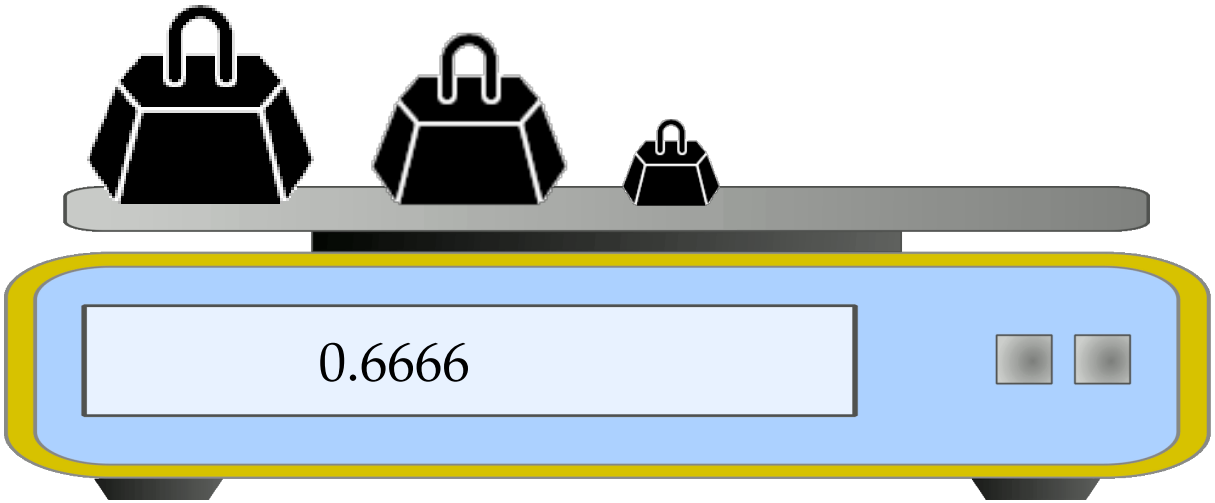
$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$S_1 = 0.3333$

$S_2 = 0.5555$

$S_3 = 0.6666$





$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

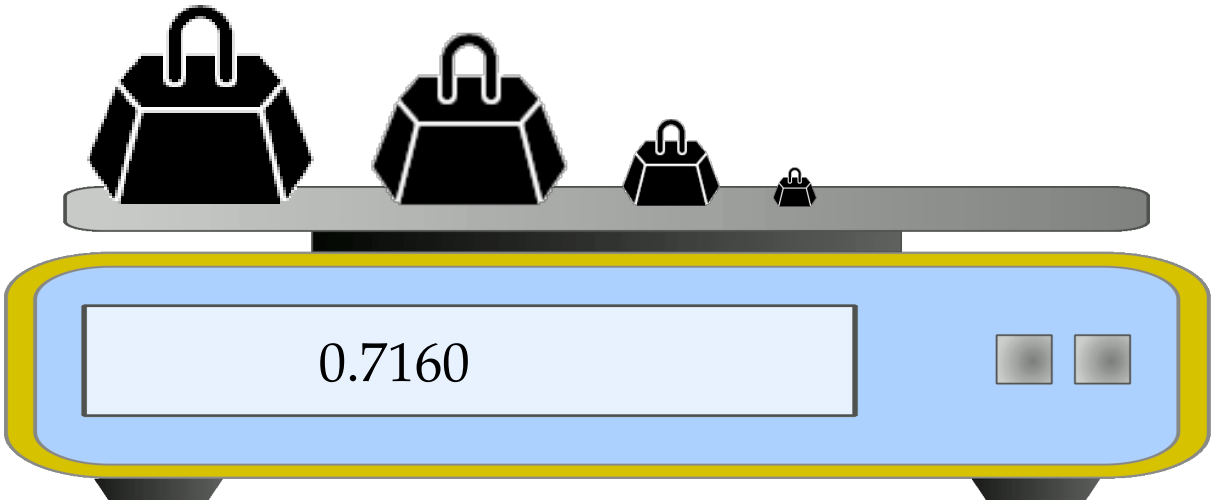
$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$\frac{5}{3^5}$ $\frac{6}{3^6}$ $\frac{7}{3^7}$ $\frac{8}{3^8}$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$$S_1 = 0.3333$$

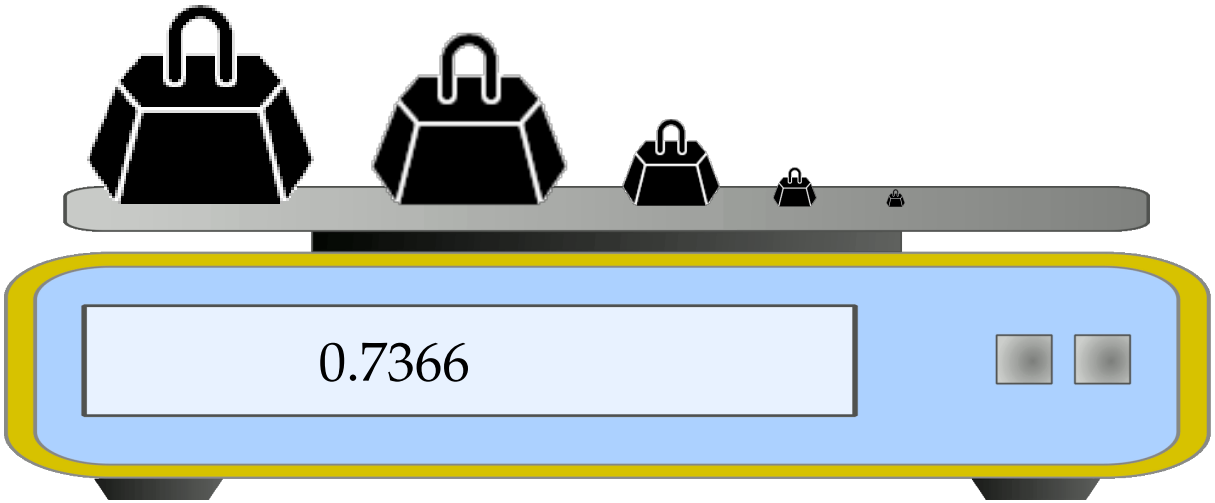
$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$\frac{6}{3^6}$ $\frac{7}{3^7}$ $\frac{8}{3^8}$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

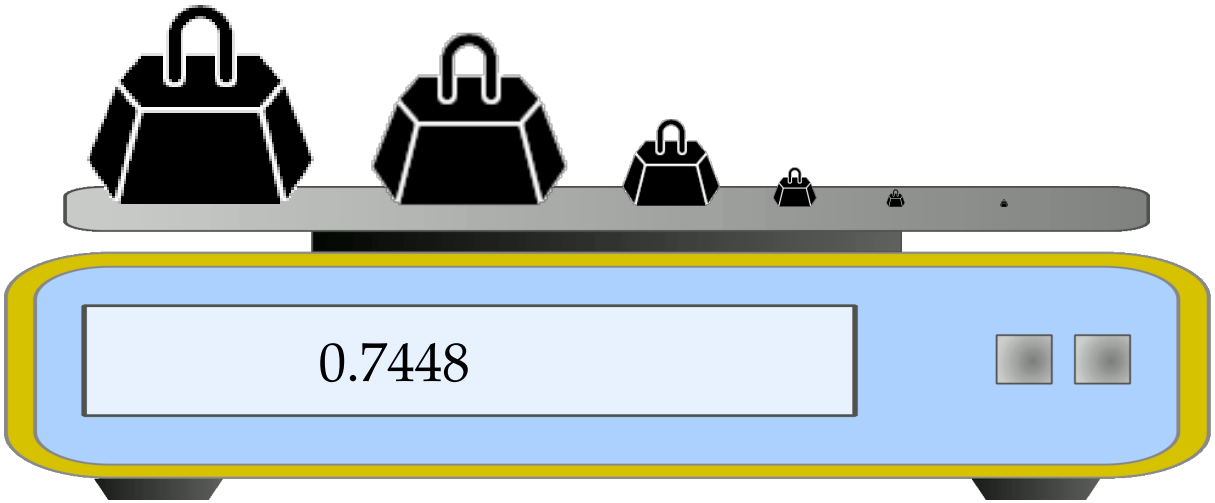
$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$\frac{7}{3^7}$ $\frac{8}{3^8}$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

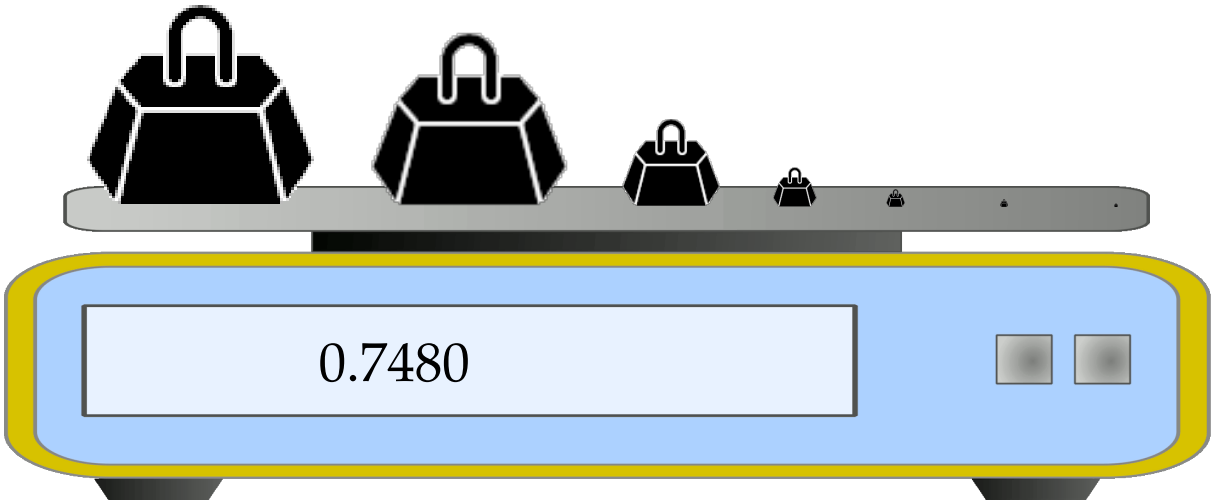
$$S_4 = 0.7160$$

$$\frac{8}{3^8}$$

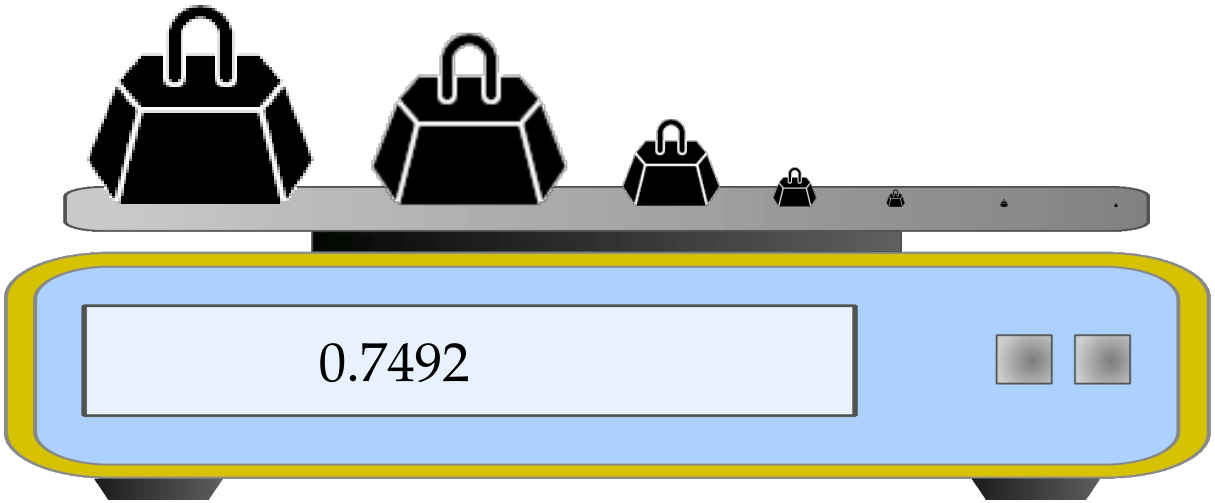
$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$$S_7 = 0.7480$$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES



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$$S_8 = 0.7492$$

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Let a and x be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of a and x.)

$$\sum_{n=1}^{\infty} anx^{n-1}$$

$$\sum_{n=1}^{\infty} anx^{n-1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a(n+1)x^n}{anx^{n-1}}\right| = \left|\left(\frac{n+1}{n}\right)x\right| = \left(1+\frac{1}{n}\right)|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. For the cases $x = \pm 1$, the ratio test is inconclusive, so we'll need another test. Fortunately, the divergence test makes things quick.

For $x = 1$: $\lim_{n\to\infty} an(1)^{n-1} = \lim_{n\to\infty} an \neq 0$

For $x = -1$: $\lim_{n\to\infty} an(-1)^{n-1} \neq 0$

All together, for any nonzero a, the series diverges when $|x| \geq 1$ and converges when $|x| < 1$.

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x.)

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}\sqrt{n+2}}{2(n+1)+3} x^{n+1}}{\frac{(-3)^n \sqrt{n+1}}{2n+3} x^n}\right| = \left|\frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{2n+3}{2n+5} \cdot \frac{x^{n+1}}{x^n}\right|$$

$$= 3 \cdot \sqrt{\frac{n+2}{n+1}} \cdot \left(\frac{2n+3}{2n+5}\right) \cdot |x| = 3\sqrt{\frac{1+2/n}{1+1/n}} \cdot \left(\frac{2+3/n}{2+5/n}\right) \cdot |x|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3\sqrt{\frac{1}{1}} \left(\frac{2}{2}\right) |x| = 3|x|$$

So the series converges when $3|x| < 1$ and diverges when $3|x| > 1$.
So for $|x| < \frac{1}{3}$, the series converges, and for $|x| > \frac{1}{3}$, it diverges.

A

Consider $x = \frac{1}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}}{2n+3}$$

This is an alternating series. Let's use the alternating series test.

(i) $a_n = \frac{\sqrt{n+1}}{2n+3} \geq 0$ for all $n \geq 1$,

(ii) To show that a_n is monotonically decreasing, consider the derivative of $f(t) = \frac{\sqrt{t+1}}{2t+3}$:

$$f'(t) = \frac{(2t+3)\frac{1}{2\sqrt{t+1}} - \sqrt{t+1}(2)}{(2t+3)^2} \left(\frac{\sqrt{t+1}}{\sqrt{t+1}} \right)$$

$$= \frac{\left(t+\frac{3}{2}\right) - (t+1)(2)}{(2t+3)^2\sqrt{t+1}} = \frac{-t-\frac{1}{2}}{(2t+3)^2\sqrt{t+1}}$$

Since $f'(t) < 0$ for all $t > 0$, we see it is a decreasing function on that domain, so $a_{n+1} < a_n$ for all $n \geq 1$.

(iii) $\lim_{n\to\infty} a_n = 0$

So, our series converges by the alternating series test when $x = \frac{1}{3}$.

Finally, consider $x = -\frac{1}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3} \frac{(-3)^n}{(-3)^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3}$$

We will use the limit comparison test, with comparison series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{2n+3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n+1}\sqrt{n}}{2n+3} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}}{2n+3} \left(\frac{1/n}{1/n}\right)$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n}}{2+3/n} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2}$$

Since $\frac{1}{2}$ is a nonzero constant, and since $\sum \frac{1}{\sqrt{n}}$ diverges (by the p-test), our series diverges as well.
All together, the original series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, and diverges otherwise.

FILL IN IN THE BLANKS

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ $\square$

then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ $\square$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ $\square$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

FILL IN IN THE BLANKS

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ fails to converge to zero as $n \to \infty$,

then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \square$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \square$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Fill in in the blanks

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then the series $\sum_{n=c}^{\infty} a_n$ diverges.

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Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \boxed{}$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Fill in in the blanks

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ fails to converge to zero as $n \to \infty$,

then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Integral Test

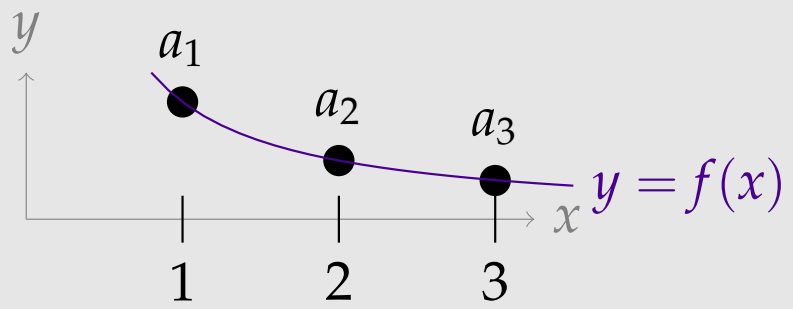
Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

(i) $\square$ and

(ii) $\square$ and

(iii) $f(n) = a_n$ for all $n \geq N_0$.

Then



$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_{N}^{\infty} f(x)\,\mathrm{d}x \qquad \text{for all } N \geq N_0$$

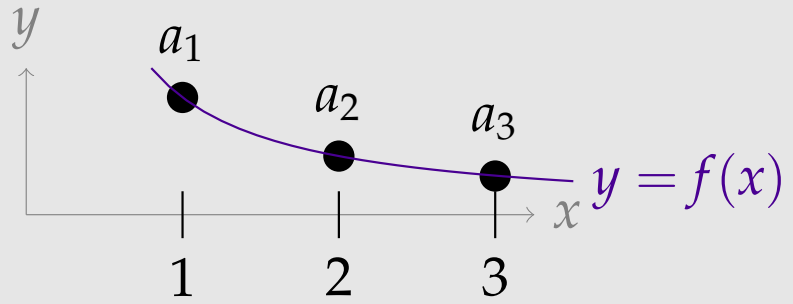
Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

(i) $f(x) \geq 0$ for all $x \geq N_0$ and

(ii) $\boxed{}$ and

(iii) $f(n) = a_n$ for all $n \geq N_0$.



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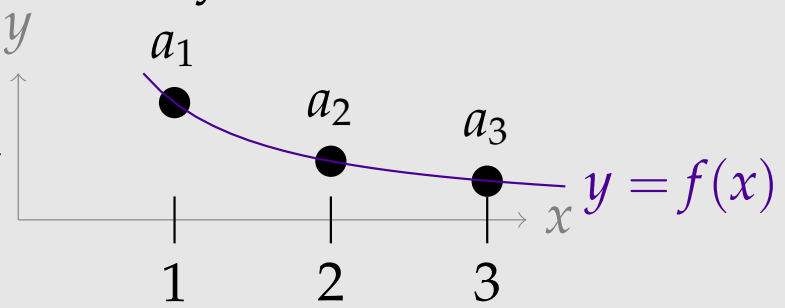

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FILL IN IN THE BLANKS

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \square Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \square Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

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FILL IN IN THE BLANKS

Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $\boxed{}$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

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Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) [ ]

(ii) $a_{n+1} \le a_n$ for all $n \ge 1$ (i.e. the sequence is monotone decreasing);

(iii) and [ ]

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the Nth partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

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List of convergence tests

Divergence Test

When the nth term in the series *fails* to converge to zero as n tends to infinity.

This is a good first thing to check: if it works, it's quick, but it doesn't always work.

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- successive terms in the series alternate in sign
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Integral Test

- works well when, if you substitute x for n in the nth term you get a function, $f(x)$, that you can easily integrate
- don't forget to check that $f(x) \geq 0$ and that $f(x)$ decreases as x increases

List of convergence tests

Ratio Test

List of convergence tests

Ratio Test

- works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$
- this often happens when a_n contains powers, like 7^n, or factorials, like $n!$
- don't forget that $L = 1$ tells you nothing about the convergence/divergence of the series

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Comparison Test and Limit Comparison Test

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Comparison Test and Limit Comparison Test

- Comparison test lets you ignore pieces of a function that feel extraneous (like replacing $n^2 + 1$ with n^2) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- Limit comparison works well when, for very large n, the nth term a_n is approximately the same as a simpler, nonnegative term b_n

- The integral test gave us the p-test. When you're looking for comparison series, p-series $\sum \frac{1}{n^p}$ are often good choices, because their convergence or divergence is so easy to ascertain.

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- Geometric series have the form $\sum a \cdot r^n$ for some nonzero constants a and r. The magnitude of r is all you need to know to deicide whether they converge or diverge, so these are also common comparison series.

- Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

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The **divergence test** is inconclusive, because $\lim_{n\to\infty} \frac{\cos n}{2^n} = 0$ (which you can show with the squeeze theorem).

The **integral test** doesn't apply, because $f(x) = \frac{\cos x}{2^x}$ is not always positive (and not decreasing).

The **alternating series test** doesn't apply because the signs of the series do not strictly alternate every term.

The **ratio test** does not apply, because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

Comparison test: Let $a_n = \frac{\cos n}{2^n}$. Note $|a_n| \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a geometric sum with ratio of consecutive terms $\frac{1}{2}$).

So by the comparison test, $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges.

Limit comparison: Set $a_n = \frac{\cos n}{2^n}$ and $b_n = \left(\frac{2}{3}\right)^n$. Then

$$\frac{a_n}{b_n} = \frac{\frac{\cos n}{2^n}}{\frac{2^n}{3^n}} = \left(\frac{3}{4}\right)^n \cos n$$

$$-\left(\tfrac{3}{4}\right)^n \leq \left(\tfrac{3}{4}\right)^n \cos n \leq \left(\tfrac{3}{4}\right)^n, \text{ and } \lim_{n\to\infty} -\left(\tfrac{3}{4}\right)^n = \lim_{n\to\infty} \left(\tfrac{3}{4}\right)^n = 0$$

So, by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

Since $\sum_{n=1}^{\infty} b_n$ converges, by the limit comparison theorem, $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges as well.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Q

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

The **alternating series test** doesn't apply because the signs of the series do not alternate.

The **integral test** doesn't apply $f(x) = \frac{2^x \cdot x^2}{(x+5)^5}$ is not a decreasing function.

Divergence test: $\lim_{n\to\infty} \frac{2^n \cdot n^2}{(n+5)^5} = \infty$ (which you can see because the numerator is larger than a power function; the denominator is a polynomial; and power functions grow faster than polynomials), so the series diverges by the divergence test.

This is the fastest option, but not the only one.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Ratio test:

$$\frac{a_n}{b_n} = \frac{\frac{2^{n+1}\cdot(n+1)^2}{(n+1+5)^5}}{\frac{2^n \cdot n^2}{(n+5)^5}} = \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{(n+5)^5}{(n+6)^5}$$

$$= 2\left(1+\frac{1}{n}\right)^2\left(1-\frac{1}{n+6}\right)^5$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 2(1)^2(1)^5 = 2$$

So, the limit of the ratio of consecutive terms is greater than 1.

Therefore $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges by the ratio test.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Comparison test: Since power functions grow faster than polynomials, for large values of n, $2^n > (n+5)^5$, so $\frac{2^n}{(n+5)^5} > 1$. Then, for large enough n,

$$\frac{2^n \cdot n^2}{(n+5)^5} > n^2 \, .$$

By the divergence test, $\sum_{n=1}^{\infty} n^2$ diverges. So by the comparison test, $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges as well.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Limit comparison: Set $a_n = \frac{2^n \cdot n^2}{(n+5)^5}$ and $b_n = \frac{2^n}{n^3}$.
Then

$$\frac{a_n}{b_n} = \frac{\frac{2^n \cdot n^2}{(n+5)^5}}{\frac{2^n}{n^3}} = \frac{n^5}{(n+5)^5} = \left(1 - \frac{5}{n+5}\right)^5$$

$$\text{So, } \lim_{n\to\infty} \frac{a_n}{b_n} = 1^5 = 1$$

Note that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges. (You can show this using the tests we've already used on the original series, so this method isn't really an improvement.) Since $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is nonzero, by the limit comparison theorem, $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

The **divergence test** is inconclusive because $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n} = 0$.

The **alternating series test** does not apply because we are not considering an alternating series.

The **integral test** won't work for us because $\int_1^{\infty} \frac{1}{x} \sin\left(\frac{1}{x}\right)\, dx$ cannot be evaluated with techniques we've learned in class so far.

A

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

The **ratio test** is inconclusive because $\lim_{n\to\infty} \frac{\frac{1}{n+1}\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}\sin\left(\frac{1}{n}\right)} = 1$:

Set $x = \frac{1}{n+1}$. Then $\frac{1}{n} = \frac{x}{1-x}$:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}} = \lim_{x\to 0^+} \frac{\sin x}{\frac{x}{1-x}} = \lim_{x\to 0^+} (1-x)\frac{\sin x}{x} = 1\cdot 1 = 1$$

Set $y = \frac{1}{n}$. Then $\frac{1}{n+1} = \frac{y}{1+y}$:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n+1}} = \lim_{y\to 0^+} \frac{\sin y}{\frac{y}{1+y}} = \lim_{y\to 0^+} (1+y)\frac{\sin y}{y} = 1\cdot 1 = 1$$

Therefore,

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}\sin\left(\frac{1}{n}\right)} = 1$$

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

Comparison test: For $n \geq 1$, $\frac{1}{n} > 0$. Then setting $\theta = \frac{1}{n}$ in the hint, $\sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$. Furthermore, $0 < \frac{1}{n} < \pi$, so $\sin\left(\frac{1}{n}\right) > 0$.

$$0 < \frac{1}{n} \sin\left(\frac{1}{n}\right) \leq \frac{1}{n}\left(\frac{1}{n}\right) = \frac{1}{n^2}$$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges as well.

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

Limit comparison: Set $a_n = \frac{1}{n}\sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}\sin\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

Setting $x = \frac{1}{n}$,

$$= \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{1}{n}\right)$ converges as well.

Included Work

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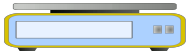 ‘Waage/Libra’ by B. Lachner is in the public domain (accessed April 2021, edited), 6–23, 52–62, 88–96

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