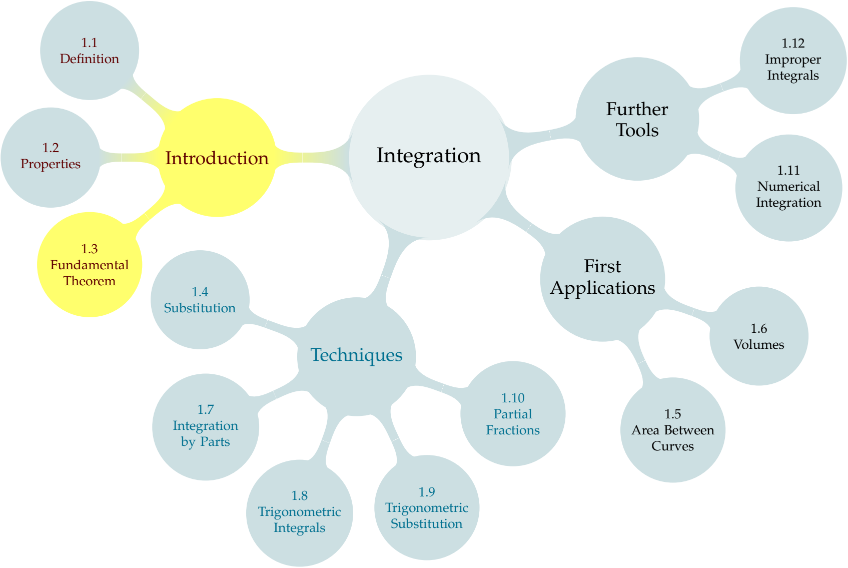
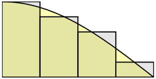
Table of Contents

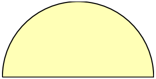


Review: Area under a Curve

Methods for finding the area under a curve.



- Limit of a Riemann Sum
  - Conceptually easy – cut into rectangles
  - Computationally rough $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$



- Use Geometry
  - Computationally nice when it's available! (Circles, triangles, symmetry, etc.)
  - Often not available – most functions don't make such nice shapes.

- Up next: Fundamental Theorem of Calculus
  - Conceptually less obvious – we'll spend about a day explaining why it works
  - Computationally generally nicer than Riemann sums
  - Doesn't work for every function

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

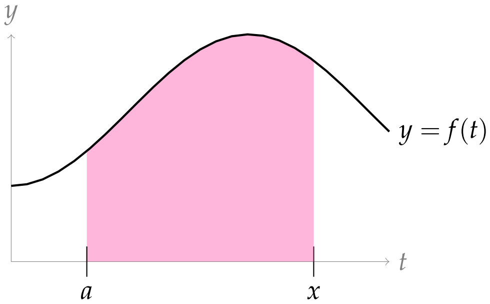
for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x) .$$

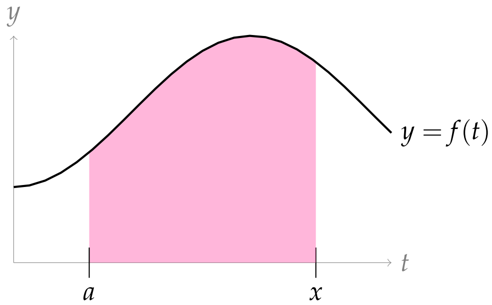
FTC(I) gives us the derivative of a very specific function (subject to some fine print).

It shows a close relationship between integrals and derivatives.

Area Function: $A(x) = \int_a^x f(t)\,\mathrm{d}t$ for $a \leq x \leq b$



Area Function: $A(x) = \int_a^x f(t)\mathrm{d}t$ for $a \le x \le b$



Notation: the function A depends on the variable x.

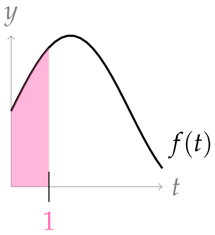
We need to know how the function f behaves on the whole interval $(0, x)$ to find $A(x)$. That's why we use $f(t)$, not $f(x)$.

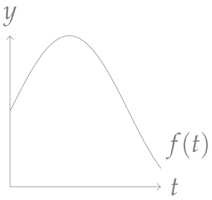
Area function notation

It might look strange at first to see two different variables. Let's consider the alternatives:

$$A(x) = \int_0^x f(t)\,\mathrm{d}t \qquad B(x) = \int_0^x f(x)\,\mathrm{d}t \qquad C(x) = \int_0^x f(x)\,\mathrm{d}x$$

$$A(1) = \int_0^1 f(t)\,\mathrm{d}t \qquad B(1) = \int_0^1 f(1)\,\mathrm{d}t \qquad C(1) = \int_0^1 f(1)\,\underbrace{\mathrm{d}1}_{??}$$





Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

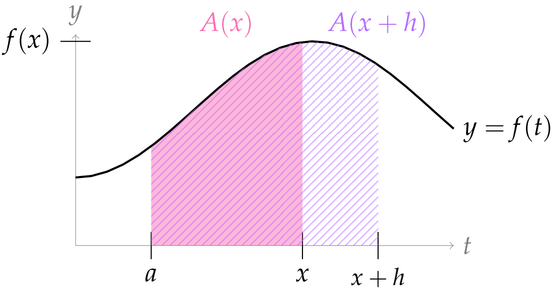
$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x)\ .$$

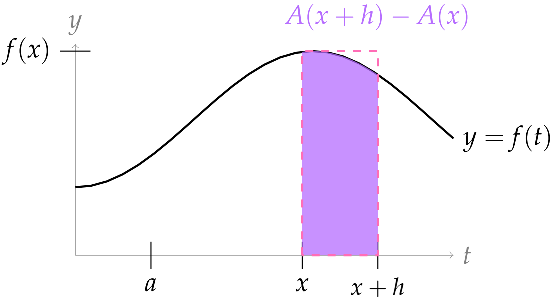
Question: Why is it true?

Derivative of Area Function, $A(x) = \int_a^x f(t)\mathrm{d}t$



$$A'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{hf(x)}{h} = f(x)$$

Derivative of Area Function, $A(x) = \int_a^x f(t)\mathrm{d}t$



$$A'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{hf(x)}{h} = f(x)$$

When h is very small, the purple area looks like a rectangle with base h and height $f(x)$, so $A(x+h) - A(x) \approx hf(x)$ and $\frac{A(x+h)-A(x)}{h} \approx f(x)$. As h tends to zero, the error in this approximation approaches 0.

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x)\ .$$

Suppose $A(x) = \int_2^x \sin t\,\mathrm{d}t$. What is $A'(x)$?

$A'(x) = \sin x$

Suppose $B(x) = \int_x^2 \sin t\,\mathrm{d}t$. What is $B'(x)$?

$B'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{-\int_2^x f(t)\,\mathrm{d}t\right\} = -\sin x$

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x) .$$

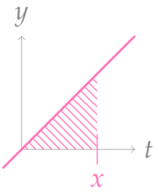
Suppose $C(x) = \int_2^{e^x} \sin t\,\mathrm{d}t$. What is $C'(x)$?

$C'(x) = e^x \sin(e^x)$: if we set $a = 2$, then

$$\begin{aligned} C(x) &= A(e^x) \\ \implies C'(x) &= A'(e^x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} = \sin(e^x) \cdot e^x \end{aligned}$$

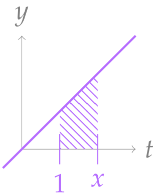
It's possible to have two different functions with the same derivative.

$A(x) = \int_0^x 2t\,\mathrm{d}t = x^2$



$A'(x) = 2x$

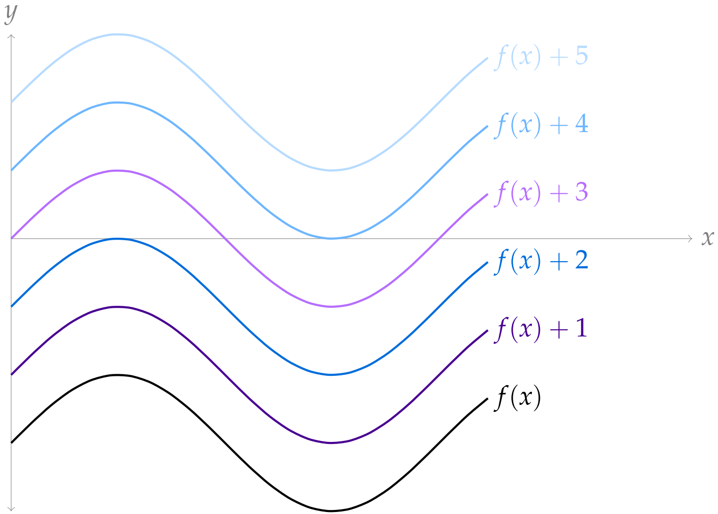
$B(x) = \int_1^x 2t\,\mathrm{d}t = x^2 - 1$



$B'(x) = 2x$

When two functions have the same derivative, they differ only by a constant.

In this example: $B(x) = A(x) - 1$



If two continuous functions have the same derivative, then one is a constant plus the other.

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

- If $A(x) = \displaystyle\int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_a^x e^t\,dt$. What functions could $A(x)$ be?

- $A'(x) = e^x$.
- Guess a function with derivative e^x: $F(x) = e^x$.
- Then $A(x) = e^x + C$ for some constant C.

[1](as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

- If $A(x) = \displaystyle\int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_a^x \cos t\,dt$. What functions could $A(x)$ be?

- $A'(x) = \cos x$.
- Guess a function with derivative $\cos x$: $F(x) = \sin x$.
- Then $A(x) = \sin x + C$ for some constant C.

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Two clues for finding $A(x) = \int_a^x f(t)\,\mathrm{d}t$:

- If $A(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_{-2}^x 5t^4\,\mathrm{d}t$. What functions could $A(x)$ be?

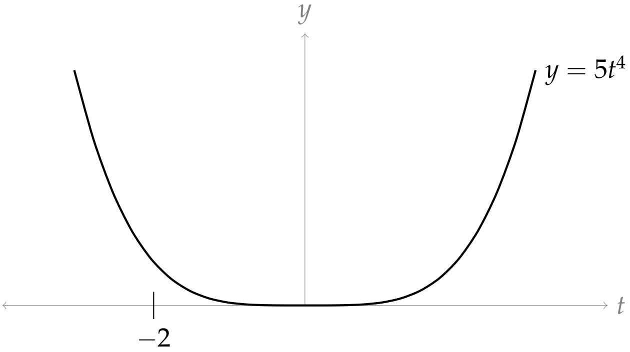
- $A'(x) = 5x^4$.
- Guess a function with derivative $5x^4$: $F(x) = x^5$.
- Then $A(x) = x^5 + C$ for some constant C.
- We ALSO know $A(-2) = \int_{-2}^{-2} 5t^4\,\mathrm{d}t = 0$, so we can find C:

$$0 = A(-2) = (-2)^5 + C \implies C = 32$$

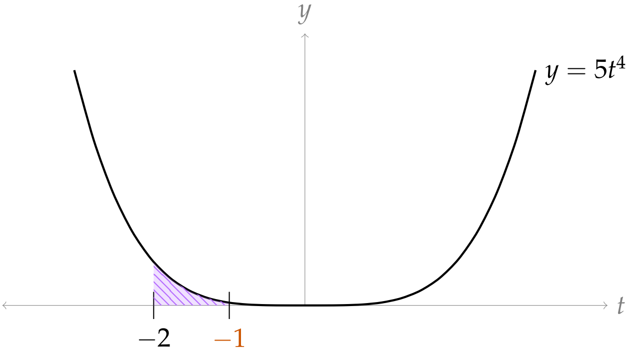
- So, $A(x) = x^5 + 32$

[1](as long as $f(t)$ is continuous on $[a, x]$)

$$A(x) = \int_{-2}^{x} 5t^4 \,\mathrm{d}t = x^5 + 32$$

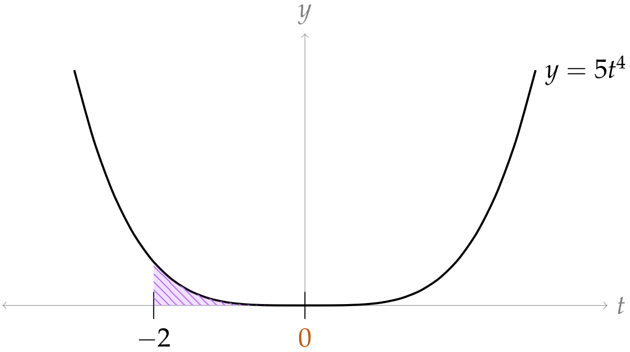


$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



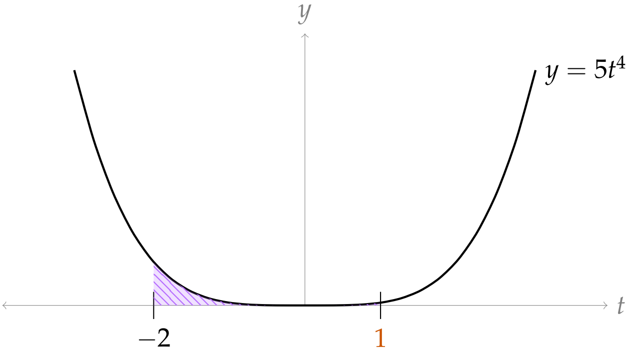
$$A(-1) = \int_{-2}^{-1} 5t^4 \, \mathrm{d}t = (-1)^5 + 32 = 31$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



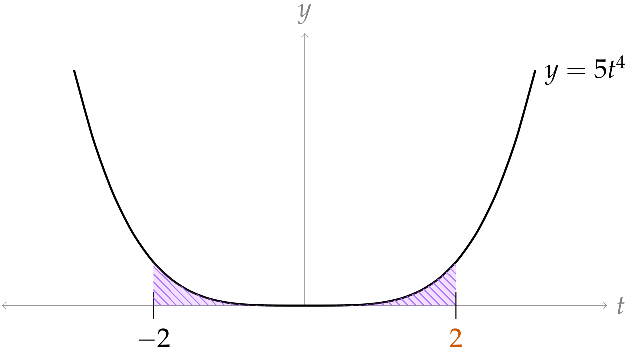
$$A(0) = \int_{-2}^{0} 5t^4 \, \mathrm{d}t = (0)^5 + 32 = 32$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



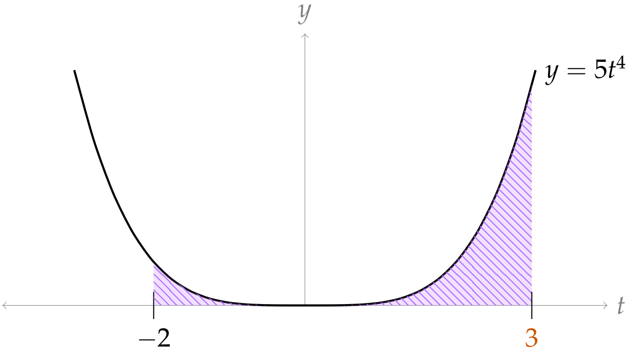
$$A(1) = \int_{-2}^{1} 5t^4 \, \mathrm{d}t = (1)^5 + 32 = 33$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



$$A(2) = \int_{-2}^{2} 5t^4 \, \mathrm{d}t = (2)^5 + 32 = 64$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



$$A(3) = \int_{-2}^{3} 5t^4 \, \mathrm{d}t = (3)^5 + 32 = 275$$

Two clues for finding $A(x) = \int_a^x f(t)\,\mathrm{d}t$:

- If $A(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$. What functions could $A(x)$ be?

[1](as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

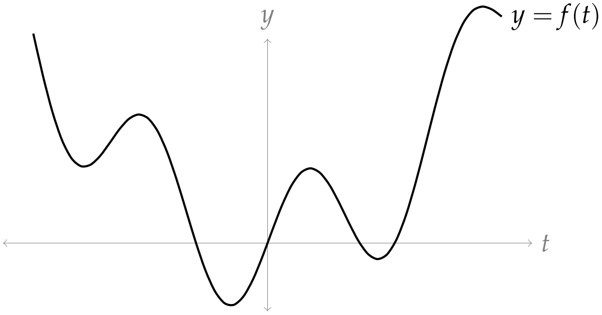
- If $A(x) = \displaystyle\int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(b) = \displaystyle\int_a^b f(t)\,dt$. What functions could $A(b)$ be?

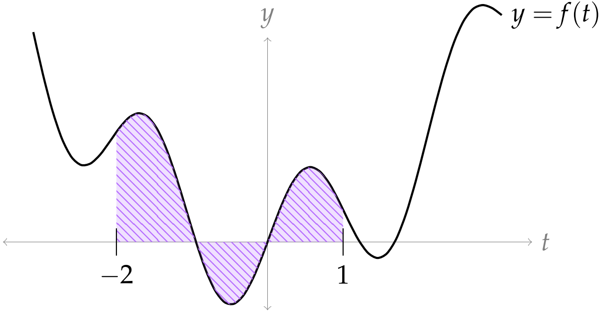
- $A'(x) = f(x)$.
- Guess a function with derivative $f(x)$: $F(x)$.
- Then $A(x) = F(x) + C$ for some constant C.
- Also $A(a) = 0$, so $0 = F(a) + C$, so $C = -F(a)$
- So, $A(b) = F(b) - F(a)$

[1](as long as $f(t)$ is continuous on $[a, x]$)

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$

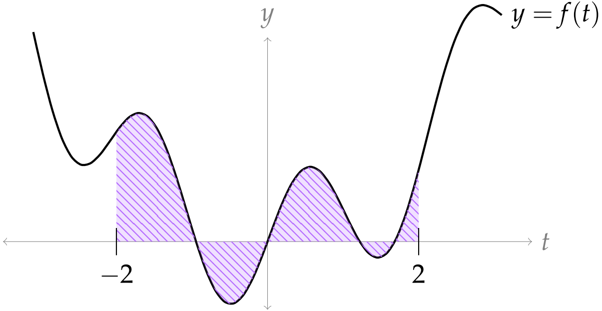


$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



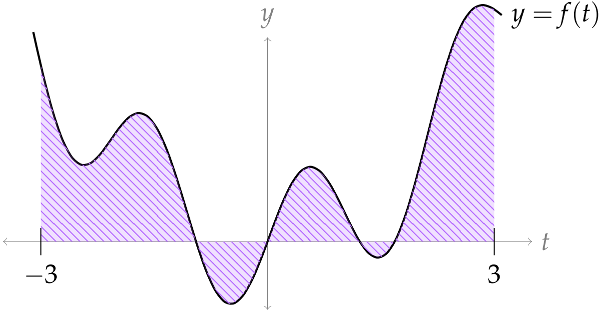
$$\int_{-2}^1 f(t)\,\mathrm{d}t = F(1) - F(-2)$$

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-2}^{2} f(t)\,\mathrm{d}t = F(2) - F(-2)$$

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-3}^{3} f(t)\,\mathrm{d}t = F(3) - F(-3)$$

Fundamental Theorem of Calculus, Part 2

Let $F(x)$ be differentiable, defined, and continuous on the interval $[a, b]$ with $F'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Examples:

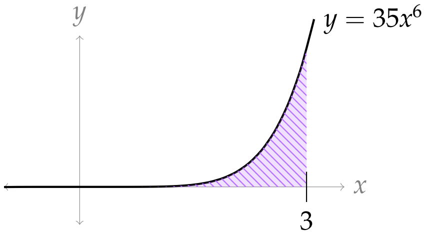
$\frac{\mathrm{d}}{\mathrm{d}x}\left\{5x^7\right\} = 35x^6$, so

$$\int_0^3 35x^6\,\mathrm{d}x = 5x^7\Big|_{x=3} - 5x^7\Big|_{x=0} = 5(3^7) - 5(0^7) = 5 \cdot 3^7$$

$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan x\} = \sec^2 x$, so

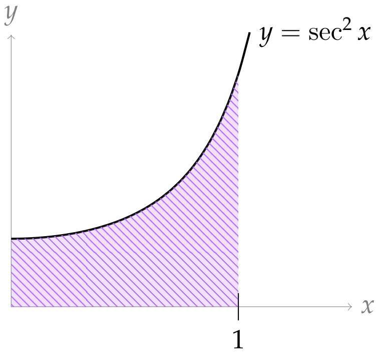
$$\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x = \tan x\Big|_{x=\frac{\pi}{4}} - \tan x\Big|_{x=0} = \tan(\pi/4) - \tan 0 = 1$$

$$\int_0^3 35x^6\,\mathrm{d}x = F(b) - F(a) \quad \text{where} \quad F(x) = 5x^7$$



$$\int_0^3 35x^6\,\mathrm{d}x = 5(3)^7 - 5(0)^7$$

$$\int_0^{\pi/4} \sec^2 x \, dx = F(b) - F(a) \quad \text{where} \quad F(x) = \tan x$$



$$\int_0^{\pi/4} \sec^2 x \, dx = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1$$

Relevant Vocabulary

Definition

If $F(x)$ is a function whose derivative is $f(x)$, we call $F(x)$ an **antiderivative** of $f(x)$.

Examples:
The derivative of x^2 is $2x$, so:
x^2 is an **antiderivative** of $2x$.

When $x > 0$, the derivative of $\log x$ is $\frac{1}{x}$, so:
$\frac{1}{x}$ is an **antiderivative** of $\log x$.

For all x, the derivative of $\log|x|$ is $\frac{1}{x}$, so:
$\frac{1}{x}$ is an **antiderivative** of $\log|x|$.

An antiderivative of $\sin x$ is $-\cos x$, because $\frac{\mathrm{d}}{\mathrm{d}x}\{-\cos x\} = \sin x$.

Convenient Notation

Definition

$$f(x)\Big|_a^b = f(b) - f(a)$$

The function $f(x)$ evaluated from a to b

Examples:

$(5x + x^2)\Big|_1^2 = (10 + 4) - (5 + 1)$

$\frac{x^2}{x+2}\Big|_5^{-1} = \frac{1}{1} - \frac{25}{7}$

FTC Part 2, Abridged

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b$$

where $F(x)$ is an antiderivative of $f(x)$

Definition

The **indefinite integral** of a function $f(x)$:

$$\int f(x)\,\mathrm{d}x$$

means the *most general* antiderivative of $f(x)$.

Examples:

$$\int 2x\,\mathrm{d}x = x^2 + C,$$ C "arbitrary constant."

$$\int \frac{1}{x}\,\mathrm{d}x = \log|x| + C$$

Remember: two functions with the same derivative differ by a constant, so we include the "$+C$" for indefinite integrals.

Definite vs Indefinite Integrals

For each pair of properties below, decide which applies to **definite** integrals, and which to **indefinite** integrals.

No limits (or bounds) of integration, $\int f(x)\,\mathrm{d}x$	indefinite
Limits (or bounds) of integration, $\int_a^b f(x)\,\mathrm{d}x$	definite
Area under a curve	definite
Antiderivative	indefinite
Number	definite
Function	indefinite

Antidifferentiation by Inspection

1. $\displaystyle\int e^x \, \mathrm{d}x = e^x + C$
2. $\displaystyle\int \cos x \, \mathrm{d}x = \sin x + C$
3. $\displaystyle\int -\sin x \, \mathrm{d}x = \cos x + C$
4. $\displaystyle\int \frac{1}{x} \, \mathrm{d}x = \log|x| + C$
5. $\displaystyle\int 1 \, \mathrm{d}x = x + C$
6. $\displaystyle\int 2x \, \mathrm{d}x = x^2 + C$
7. $\displaystyle\int nx^{n-1} \, \mathrm{d}x = x^n + C \qquad (n \neq 0\text{, constant})$
8. $\displaystyle\int x^n \, \mathrm{d}x = \tfrac{1}{n+1}x^{n+1} + C \qquad (\, n \neq -1\text{, constant})$

Power Rule for Antidifferentiation

$$\int x^n \, \mathrm{d}x = \frac{1}{n+1} x^{n+1} + C$$

if $n \neq -1$ is a constant

Example:

$$\int \left(5x^2 - 15x + 3\right) \, \mathrm{d}x = \frac{5}{3}x^3 - \frac{15}{2}x^2 + 3x + C$$

ANTIDERIVATIVES TO RECOGNIZE

- $\int x^n \, \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$
- $\int a \, \mathrm{d}x = ax + C$
- $\int e^x \, \mathrm{d}x = e^x + C$
- $\int \frac{1}{x} \, \mathrm{d}x = \log|x| + C$
- $\int \sin x \, \mathrm{d}x = -\cos x + C$
- $\int \cos x \, \mathrm{d}x = \sin x + C$
- $\int \sec^2 x \, \mathrm{d}x = \tan x + C$
- $\int \sec x \tan x \, \mathrm{d}x = \sec x + C$
- $\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$
- $\int \csc^2 x \, \mathrm{d}x = -\cot x + C$
- $\int \frac{1}{1+x^2} \, \mathrm{d}x = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + C$