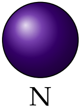
Table of Contents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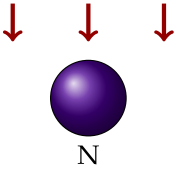
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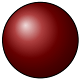

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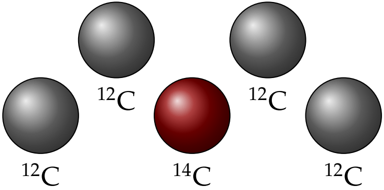
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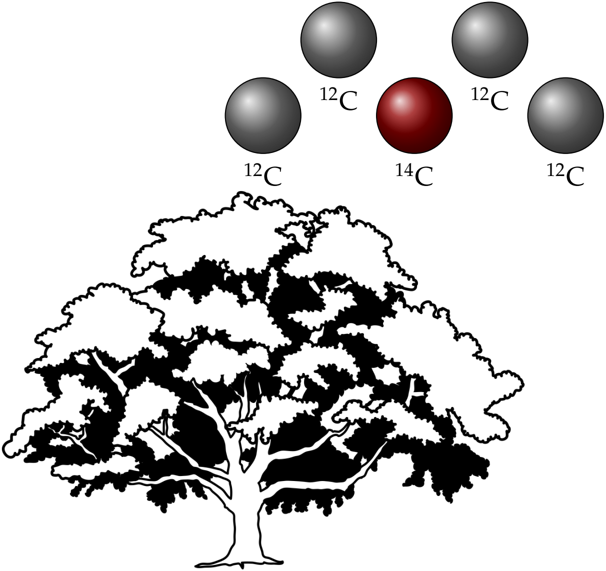

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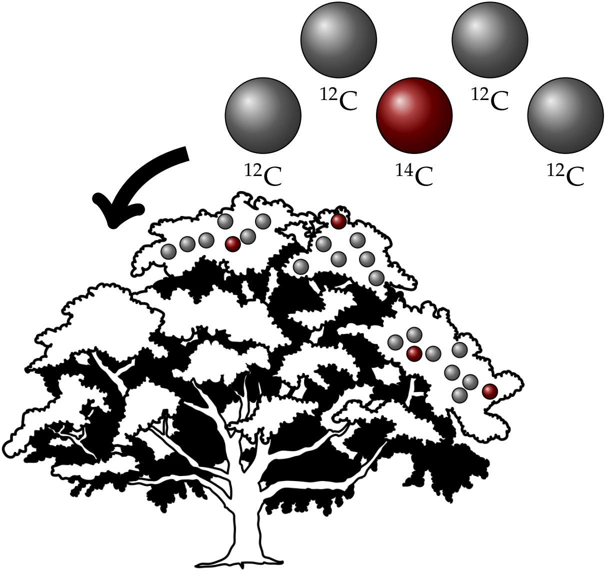
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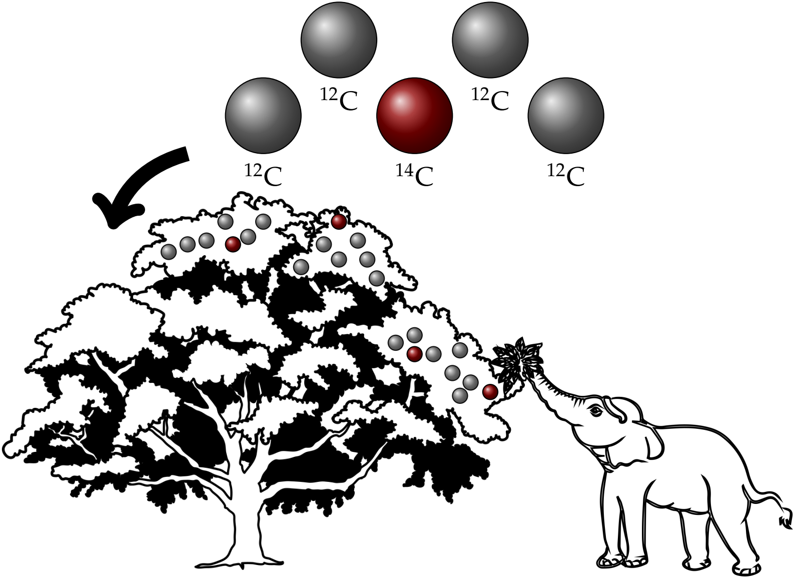

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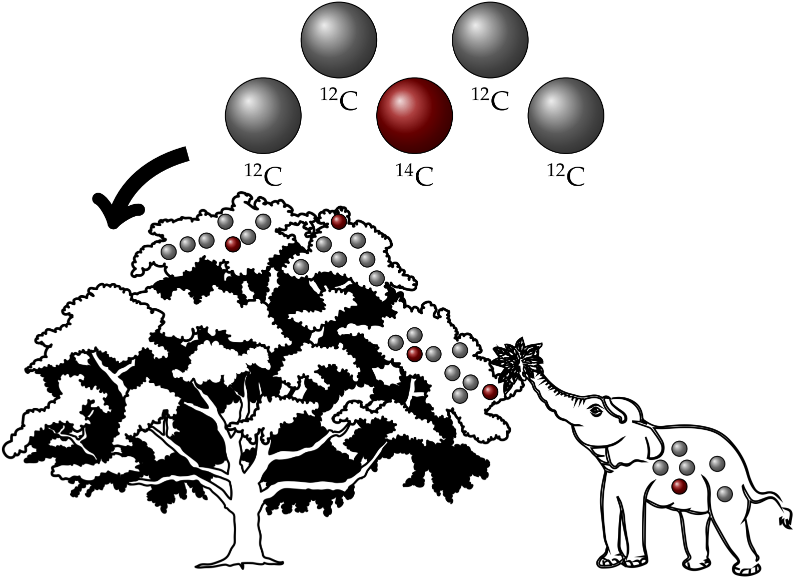

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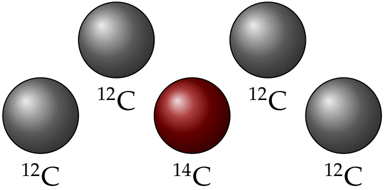
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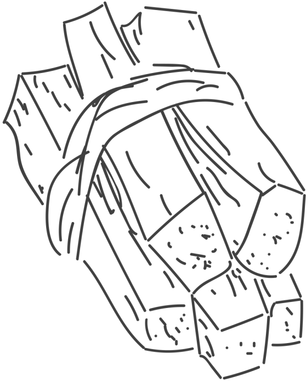

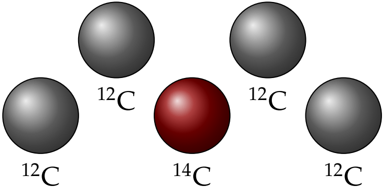
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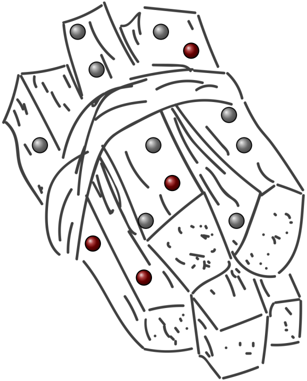

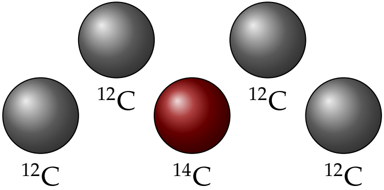
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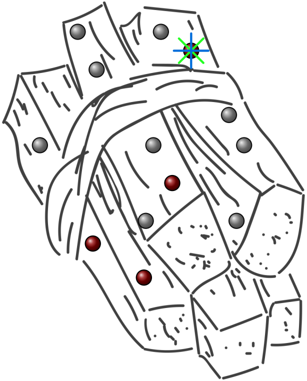

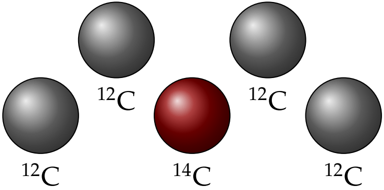
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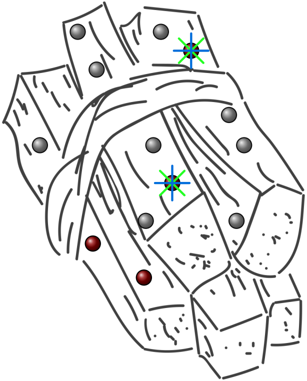

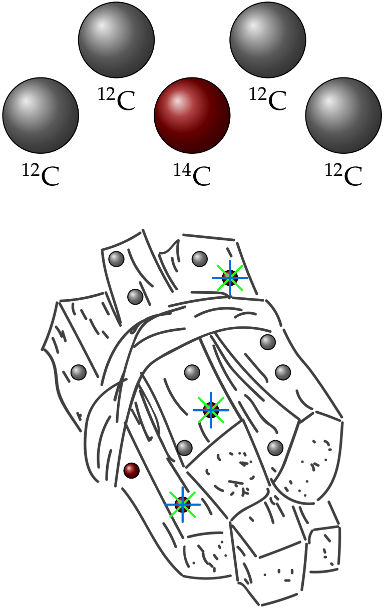
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Radioactive Decay

One model for radioactive decay says that the rate at which an isotope decays is proportional to the amount present. So if $Q(t)$ is the amount of a radioactive substance, then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ(t)$$

for some constant[1] k.

This is a first-order linear differential equation. Its explicit solutions have the form:

[1]By including the negative sign, we ensure k will be positive, but of course we could also write "$\frac{\mathrm{d}Q}{\mathrm{d}t} = KQ(t)$ for some [negative] constant K".

Half-life

The half-life of an isotope is the time required for half of that isotope to decay. If we know the half-life of a substance is $t_{1/2}$, and its quantity at time t is given by $Q(0)e^{-kt}$ we can find the constant k:

Radioactive Decay

The function $Q(t)$ satisfies the equation $\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ(t)$ if and only if

$$Q(t) = Q(0)\, e^{-kt}$$

The half–life is defined to be the time $t_{1/2}$ which obeys $Q\left(t_{1/2}\right) = \frac{1}{2}\, Q(0)$. The half–life is related to the constant k by $t_{1/2} = \frac{\log 2}{k}$. Then

$$Q(t) = Q(0)\, e^{-\frac{\log 2}{t_{1/2}} t} = Q(0) \cdot \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

If the half-life of $^{14}\mathrm{C}$ is $t_{1/2} = 5730$ years, then the quantity of carbon-14 present in a sample after t years is:

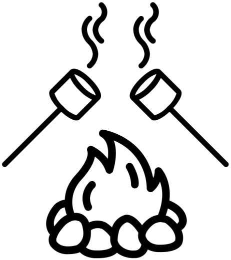
A particular piece of flax parchment contains about 64% as much ^{14}C as flax plants do today. We will estimate the age of the parchment, using 5730 years as the half-life of ^{14}C.

First, a rough estimate: is the parchment older or younger than 5730 years?

Newton's law of cooling

The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings.

The temperature of the surroundings is sometimes called the ambient temperature.



$$\frac{\mathrm{d}T}{\mathrm{d}t} = K\left(\qquad\qquad \right)$$

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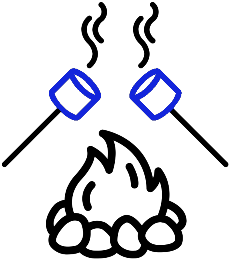

$$\frac{\mathrm{d}T}{\mathrm{d}t} = K\left(T(t) \qquad\quad \right)$$

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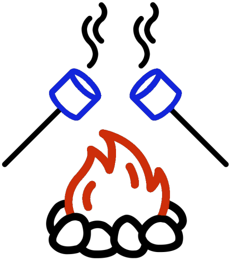

$$\frac{\mathrm{d}T}{\mathrm{d}t} = K\left(T(t) - A\right)$$

Linear First-Order Differential Equations

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y - b)$$

if and only if

$$y(x) = \{y(0) - b\}\, e^{ax} + b$$

Find an explicit formula for functions $T(t)$ solving the differential equation $\frac{\mathrm{d}T}{\mathrm{d}t} = K(T(t) - A)$ for some constants K and A.

The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°. Assume the temperature of the tea as it cools follows Newton's law of cooling,

$$T(t) = (T(0) - A)\, e^{Kt} + A$$

(a) Determine the temperature as a function of time.

(b) When the tea will reach a temperature of 14° ?

A glass of room-temperature water is carried out onto a balcony from an apartment where the temperature is 22°C. After one minute the water has temperature 26°C and after two minutes it has temperature 28°C. Assuming the water warms according to Newton's law of cooling, what is the outdoor temperature?
Assume that the temperature of the water obeys Newton's law of cooling.

Let P the the size of a population, and let K be the carrying capacity of its environment (i.e. the population size that can be sustainably supported).

When P is much less than K, our population has...

A. not enough resources
B. just enough resources
C. extra resources

So when the P is much less than K, we expect the population to...

A. shrink
B. stay the same
C. grow

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Malthusian growth

The Malthusian growth model relates population growth to population size:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = bP(t)$$

where b is a constant representing net birthrate per member of the population.

Let P the the size of a population, and let K be the carrying capacity of its environment (i.e. the population size that can be sustainably supported).

When P is greater than K, our population has...

A. not enough resources
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So when the P is greater than K, we expect the population to...

A. shrink
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Logistic growth models population growth as:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = b_0 \left(1 - \frac{P(t)}{K}\right) P(t)$$

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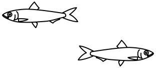
- If $P << K$, then $\frac{\mathrm{d}P}{\mathrm{d}t} \approx$



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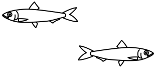

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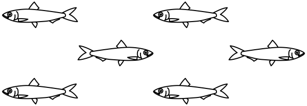

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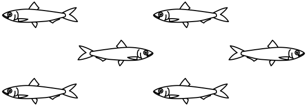

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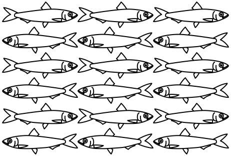
- If $P > K$, then $\frac{\mathrm{d}P}{\mathrm{d}t}$



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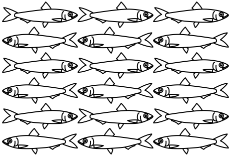
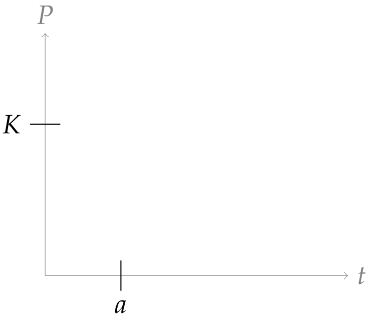


Before we solve explicitly, let's sketch some solutions to

$$\frac{\mathrm{d}P}{\mathrm{d}t} = b_0 \left(1 - \frac{P(t)}{K}\right) P(t)$$



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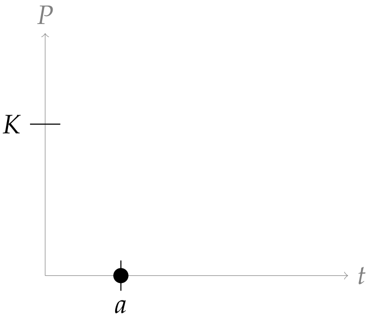
- If $P(a) = 0$:



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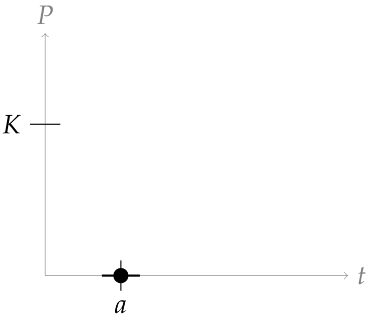

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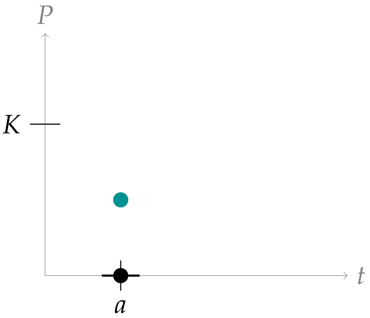
- If $0 < P(a) < K$:



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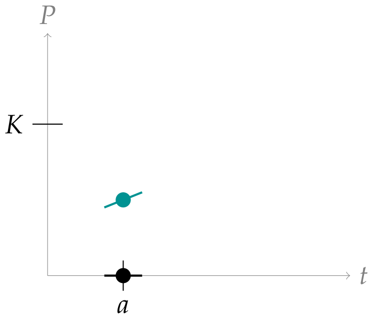
- If $0 < P(a) < K$: $\frac{\mathrm{d}P}{\mathrm{d}t}(a) > 0$



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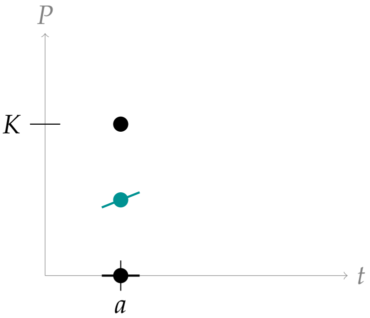

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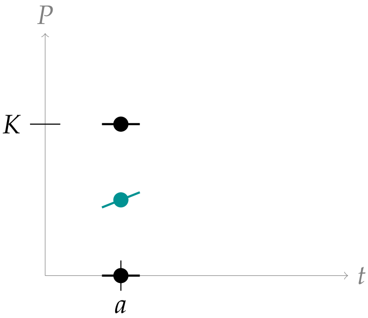

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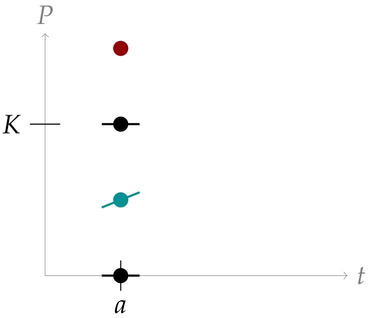
- If $K < P(0)$:



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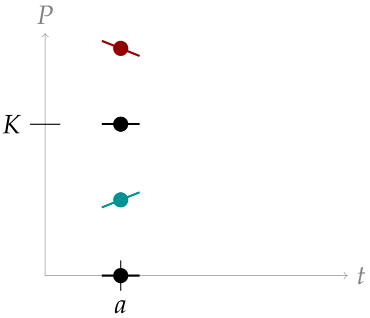

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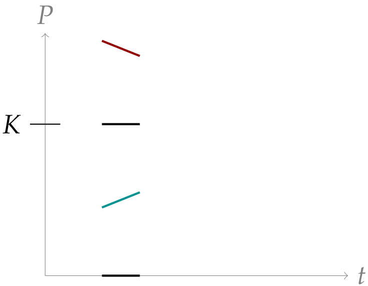

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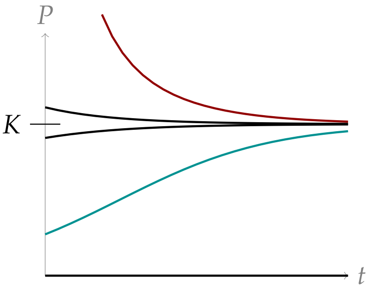


Find the explicit solutions to

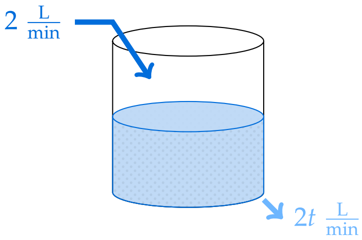
$$\frac{\mathrm{d}P}{\mathrm{d}t} = b\left(1 - \frac{P(t)}{K}\right)P(t)$$

when b and K are constants.

At time $t = 0$, where t is measured in minutes, a large tank contains 3 litres of water in which 1 kg of salt is dissolved. Fresh water enters the tank at a rate of 2 litres per minute and the fully mixed solution leaks out of the tank at the varying rate of $2t$ litres per minute.

(a) Determine the volume of solution $V(t)$ in the tank at time t.

(b) Determine the amount of salt $Q(t)$ in solution when the amount of water in the tank is at maximum.

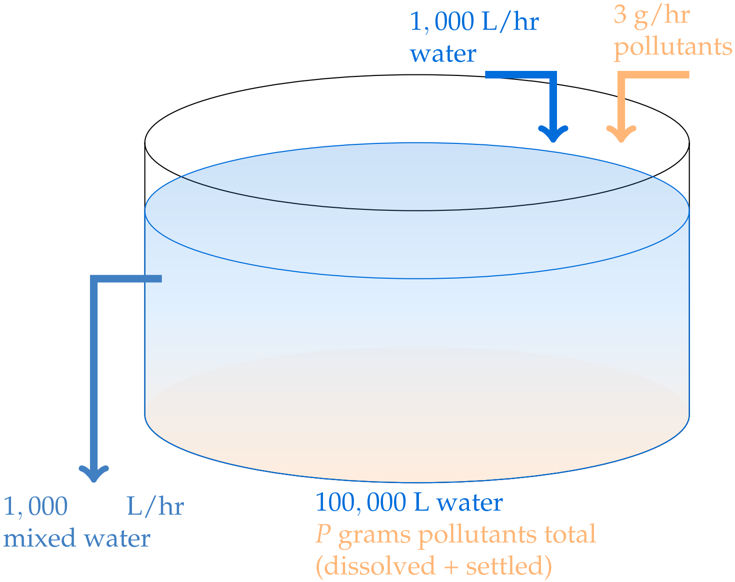


Settling Tank

A settling tank is filled with 100,000 litres of pure water. Every hour, 1,000 litres of water, containing 3 grams of pollutants, enters the tank.

90% of the pollutants in the settling tank sink to the bottom, with the remaining 10% well-mixed into the water. The tank drains 1,000 litres of this mixed water into the sewer every hour.

In order to drain the water into the local sewer, the concentration of pollutants cannot be more than 1 gram per 1,000 litres. How long can the settling tank take dirty water until the process must be stopped?


1, 000 L/hr
water
3 g/hr
pollutants
1, 000 L/hr
mixed water
100, 000 L water
P grams pollutants total
(dissolved + settled)

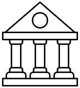
Let $P(t)$ be the total amount (in grams) of pollutants in the tank. Pollutants are entering at a rate of 3 grams per hour. How fast are they leaving?

So, the quantity of pollutants in the tank satisfies the differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 3 - \frac{1}{1000}P$$

is a linear first-order differential equation, so we know its solution:

You deposit \$$P$ in a bank account at time $t = 0$, and the account pays $r\%$ interest per year, compounded n times per year. Your balance at time t is $B(t)$.



You deposit $\$P$ in a bank account at time $t = 0$, and the account pays $r\%$ interest per year, compounded n times per year. Your balance at time t is $B(t)$.

If one interest payment comes at time t, then the next interest payment will be made at time $t + \frac{1}{n}$ and will be:

$$\frac{1}{n} \times \frac{r}{100} \times B(t) = \frac{r}{100n} B(t)$$

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So, calling $\frac{1}{n} = h$,

$$B(t+h) = B(t) + \frac{r}{100} B(t) h \qquad \text{or} \qquad \frac{B(t+h) - B(t)}{h} = \frac{r}{100} B(t)$$

If the interest is compounded continuously,

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}t}(t) &= \lim_{h \to 0} \frac{B(t+h) - B(t)}{h} = \frac{r}{100} B(t) \\ \implies B(t) &= B(0) \cdot e^{rt/100} = P \cdot e^{rt/100} \end{aligned}$$

Continuously compounding interest

If an account with balance $B(t)$ pays a continuously compounding rate of r% per year, then:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{r}{100}B$$
$$B(t) = B(0) \cdot e^{rt/100}$$

You invest \$200 000 into an account with continuously compounded interest of 5% annually. You want to withdraw from the account continuously at a rate of \$$W$ per year, for the next 20 years. How big can W be?

Included Work

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