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So:

$$\begin{aligned}\int \frac{x-2}{(x+1)(2x-1)}\,\mathrm{d}x &= \int \frac{1}{x+1}\,\mathrm{d}x - \int \frac{1}{2x-1}\,\mathrm{d}x \\ &= \log|x+1| - \frac{1}{2}\log|2x-1| + C\end{aligned}$$

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Method of Partial Fractions: Algebraic method to turn any rational function (i.e. ratio of two polynomials) into the sum of easier-to-integrate rational functions.

Distinct Linear Factors

The rational function

$$\frac{\text{numerator}}{K(x-a_1)(x-a_2)\cdots(x-a_j)}$$

can be written as

$$\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\cdots+\frac{A_j}{x-a_j}$$

for some constants $A_1, A_2, \ldots, A_j$, provided

(1) the linear roots $a_1, \cdots a_j$ are distinct, and

(2) the degree of the numerator is strictly less than the degree of the denominator.

Distinct Linear Factors

$$\frac{7x+13}{(2x+5)(x-2)} =$$

Distinct Linear Factors

$$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$

Distinct Linear Factors

We found $7x + 13 = A(x-2) + B(2x+5)$ for some constants A and B. What are A and B?

Method 1: set x to convenient values.

Method 2: match coefficients of powers of x.

Distinct Linear Factors

All together:

$$\frac{7x+13}{2x^2+x-10} = \frac{A}{2x+5} + \frac{B}{x-2}$$
$$A = 1, \qquad B = 3$$

Check Our Work

We check that $\int \frac{7x+13}{2x^2+x-10} = \frac{1}{2}\log|2x+5| + 3\log|x-2| + C$ by differentiating.

Distinct Linear Factors

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Once we find A, B, and C, integration is easy:

$$\begin{aligned}
&\int \frac{x^2-24x+5}{2x(3x+1)(x+5)}\,\mathrm{d}x \\
&\quad = \int \left(\frac{A}{2x}+\frac{B}{3x+1}+\frac{C}{x+5}\right)\,\mathrm{d}x \\
&\quad = \frac{A}{2}\log|x| + \frac{B}{3}\log|3x+1| + C\log|x+5| + D
\end{aligned}$$

Distinct Linear Factors

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}$$

Find constants A, B, and C.
Start: make a common denominator

Distinct Linear Factors

$$x^2 + 5 = A(3x+1)(x+5) + B(2x)(x+5) + C(2x)(3x+1)$$

Check Our Work

Let's check that

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}.$$

Distinct Linear Factors

All together:

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}$$

Repeated Linear Factors

A rational function $\frac{P(x)}{(x-1)^4}$, where $P(x)$ is a polynomial of degree strictly less than 4, can be written as

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}$$

for some constants A, B, C, and D.

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for some constants A, B, C, and D.

$$\frac{5x-11}{(x-1)^2} =$$

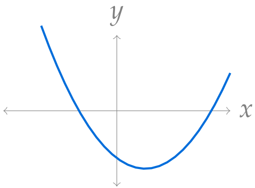
Set up the form of the partial fractions decomposition. (You do not have to solve for the parameters.)

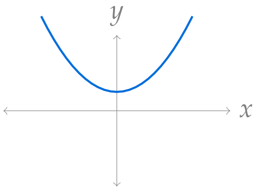
$$\frac{3x+16}{(x+5)^3} =$$

$$\frac{-2x-10}{(x+1)^2(x-1)} =$$

Irreducible Quadratic Factors

Sometimes it's not possible to factor our denominator into linear factors with real terms.





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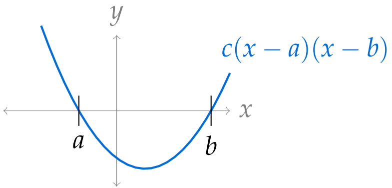
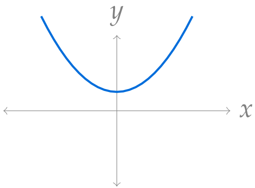


If a quadratic function has real roots a and b (possibly $a = b$, possibly $a \neq b$), then we can write it as $c(x-a)(x-b)$ for some constant c.



If a quadratic function has no real roots, then it can't be factored into (real) linear factors. It is irreducible.

Irreducible Quadratic Factors

When the denominator has an irreducible quadratic factor $x^2 + bx + c$, we add a term $\dfrac{Ax + B}{x^2 + bx + c}$ to our composition. (The degree of the numerator must still be smaller than the degree of the denominator.) Write out the form of the partial fraction decomposition (but do not solve for the parameters):

- $\dfrac{1}{(x+1)(x^2+1)} =$

Irreducible Quadratic Factors

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- $\dfrac{1}{(x+1)(x^2+1)} =$

- $\dfrac{3x^2 - x + 5}{(x^2+1)(x^2+2)} =$

The purpose of the partial fraction decomposition is to end up with functions that we can integrate.

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- Recall: $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + C$.

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- Recall: $\displaystyle\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan x + C.$

- Evaluate: $\displaystyle\int \frac{1}{(x+1)^2+1}\,\mathrm{d}x$

Evaluate $\displaystyle\int \frac{4}{(3x+8)^2+9}\,\mathrm{d}x$

Check Our Work

We found $\displaystyle\int \frac{4}{(3x+8)^2+9}\,\mathrm{d}x = \frac{4}{9}\arctan\left(x+\frac{8}{3}\right)+C.$

Evaluate $\int \frac{x+1}{x^2+2x+2}\mathrm{d}x.$

(Hint: start by completing the square.)

Q

Check Our Work

We found $\displaystyle\int \frac{x+1}{x^2+2x+2}\,\mathrm{d}x = \frac{1}{2}\log\left|(x+1)^2+1\right| + C.$

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$$\int \frac{x^3}{(x-2)^2(x-3)(x-4)^2}\,\mathrm{d}x \qquad \int \frac{x^5}{(x-2)^2(x-3)(x-4)^2}\,\mathrm{d}x$$

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If the degree of the numerator is too large, we use polynomial long division.

Evaluate $\displaystyle\int \frac{8x^2 + 22x + 23}{2x+3}\,\mathrm{d}x.$

Check Our Work

We computed

$$\int \frac{8x^2 + 22x + 23}{2x + 3}\, \mathrm{d}x = 2x^2 + 5x + 4 \log |2x + 3| + C.$$

Evaluate $\displaystyle\int \frac{3x^3+x+3}{x-2}\,\mathrm{d}x$.

Check Our Work

We found

$$\int \frac{3x^3+x+3}{x-2}\,\mathrm{d}x = x^3+3x^2+13x+29\log|x-2|+C.$$

Evaluate $\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x$.

Check Our Work

We found $\displaystyle\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x = 3x + \frac{1}{5}\log|x| - \frac{76}{5}\log|x+5| + C.$

Factoring

$$P(x) = x^3 + 2x^2 - 5x - 6$$

FACTORING

$$P(x) = 2x^3 - 3x^2 + 4x - 6$$

Included Work