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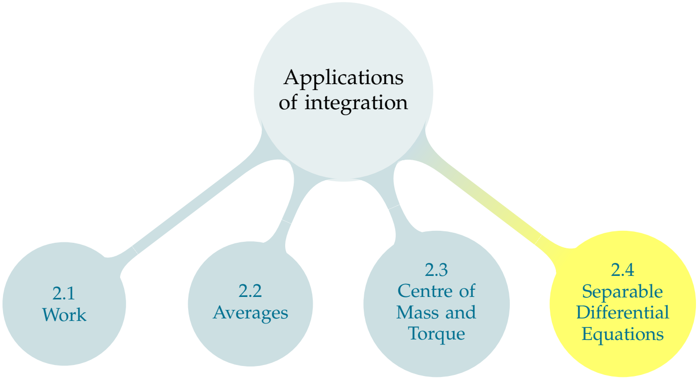

Differential Equation

A differential equation is an equation for an unknown function that involves the derivative of the unknown function.

Differential equations play a central role in modelling a huge number of different phenomena. Here is a table giving a bunch of named differential equations and what they are used for. It is far from complete.

Newton's Law of Motion	describes motion of particles
Maxwell's equations	describes electromagnetic radiation
Navier–Stokes equations	describes fluid motion
Heat equation	describes heat flow
Wave equation	describes wave motion
Schrödinger equation	describes atoms, molecules and crystals
Stress-strain equations	describes elastic materials
Black–Scholes models	used for pricing financial options
Predator–prey equations	describes ecosystem populations
Einstein's equations	connects gravity and geometry
Ludwig–Jones–Holling's equation	models spruce budworm/Balsam fir ecosystem
Zeeman's model	models heart beats and nerve impulses
Sherman–Rinzel–Keizer model	for electrical activity in Pancreatic β–cells
Hodgkin–Huxley equations	models nerve action potentials

Disclaimer:
We are dipping our toes into a vast topic. Most universities offer half a dozen different undergraduate courses on various aspects of differential equations. We will just look at one special, but important, type of equation.

- We will first learn to verify solutions without finding them. (If you learned about differential equations last semester, this will be review.)
- Then, we will learn to solve one particular type of differential equation.

Differential Equations

Definition

A **differential equation** is an equation involving the derivative of an unknown function.

Examples: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$; $\qquad x\frac{\mathrm{d}y}{\mathrm{d}x} = 7xy + y$

Definition

If a function makes a differential equation true, we say it **satisfies** the differential equation, or is a solution to the differential equation.

Example: $y = x^2$ and $y = x^2 + 1$ both satisfy the first differential equation

Verifying Solutions

Consider the equation

$$x + 2 = x^3 - x^2$$

How would you verify whether $x = 1$ satisfies the equation?
How would you verify whether $x = 2$ satisfies the equation?
Plug x into the equation, check whether the left-hand side and the right-hand side are the same **number**.

Verifying Solutions

Consider the differential equation

$$\frac{dy}{dx} = 2y + 4x$$

How would you verify whether $y = e^{2x} - 2x$ satisfies the equation?
How would you verify whether $y = e^{2x} - 2x - 1$ satisfies the equation?

Replace y and $\frac{dy}{dx}$ in the equation, check whether the left-hand side and the right-hand side are the same **function**.

- If $y = e^{2x} - 2x$, then $\frac{dy}{dx} = 2e^{2x} - 2$. Plug these into both sides of the differential equation, replacing anything depending on y:

$$\frac{dy}{dx} = 2y + 4x$$

$$2e^{2x} - 2 \stackrel{?}{=} 2(e^{2x} - 2x) + 4x$$

$$2e^{2x} - 2 \stackrel{?}{=} 2e^{2x}$$

Since the functions on the left and right are not the same

Differential equation:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 7xy + y$$

Interpretation:

There is a function $y(x)$ that makes the left-hand side and the right-hand side into the same function.

To check whether a given function satisfies the differential equation, plug it in for everything with a "y": y itself and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Is $y = xe^{7x+9}$ a solution to the differential equation?

Which of the following solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$?

A. $y = -x$ B. $y = x + 5$ C. $y = \sqrt{x^2+5}$

- If $y = -x$, then $\frac{dy}{dx} = -1$. Plugging into the differential equation yields: $-1 \stackrel{?}{=} \frac{x}{-x}$. Since the left and right are the same function (except for the single point when $x = 0$), we say $y = -x$ solves the differential equation.
- If $y = x + 5$, then $\frac{dy}{dx} = 1$. Plugging into the differential equation yields: $1 \stackrel{?}{=} \frac{x}{x+5}$. Since the left and right are not the same function, $y = x + 5$ does not solve the differential equation.
- If $y = \sqrt{x^2+5}$, then $\frac{dy}{dx} = \frac{2x}{2\sqrt{x^2+5}} = \frac{x}{\sqrt{x^2+5}}$. Plugging into the differential equation yields: $\frac{x}{\sqrt{x^2+5}} \stackrel{?}{=} \frac{x}{\sqrt{x^2+5}}$. Since the left and right are the same function, we say $y = \sqrt{x^2+5}$ solves the differential equation.

First Example of a Separable DE

Definition

A separable differential equation is an equation for a function y(x) that can be written in the form

$$g(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

(It may take some rearranging to get the equation into this form.)

For example:

$$y^2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 4x$$

$$\int \left(y^2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x = \int 4x \, \mathrm{d}x$$

$$\int y^2 \, \mathrm{d}y = 2x^2 + C$$

General Method for Solving Separable DEs

$$g(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$g(y(x)) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$\int \left(g(y(x)) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x = \int f(x)\, \mathrm{d}x$$

y-substitution:

$$\int g(y)\, \mathrm{d}y = \int f(x)\, \mathrm{d}x$$

General Method for Solving Separable DEs

$$g(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$g(y(x)) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$\int \left(g(y(x)) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x = \int f(x)\,\mathrm{d}x$$

y-substitution:

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

Shorthand:

$$g(y) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x$$

$$\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 x$$

1. "Separate" y's from x's.
2. Integrate.
3. Solve explicitly for y.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (xy)^4, \qquad y(0) = \frac{1}{2}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x^4 y^4 \\
y^{-4}\,\mathrm{d}y &= x^4 \mathrm{d}x \\
\int y^{-4} \mathrm{d}y &= \int x^4 \mathrm{d}x \\
\frac{1}{-3} y^{-3} &= \frac{1}{5} x^5 + C \\
\frac{1}{y^3} &= -3\left(\frac{1}{5}x^5 + C\right) \\
y &= \frac{1}{-\sqrt[3]{3\left(\frac{1}{5}x^5 + C\right)}}
\end{aligned}$$

$$\begin{aligned}
y(0) &= -\sqrt[3]{\frac{1}{3\left(\frac{1}{5}x^5 + C\right)}}\Bigg|_{x=0} \\
\frac{1}{2} &= -\sqrt[3]{\frac{1}{3C}} \\
2 &= -\sqrt[3]{3C} \\
3C &= -8 \\
y(x) &= -\sqrt[3]{\frac{1}{\frac{3}{5}x^5 - 8}} \\
&= \sqrt[3]{\frac{1}{8 - \frac{3}{5}x^5}}
\end{aligned}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(4x^3 - 1) \qquad y(0) = -2$$

$$\frac{1}{y}\,\mathrm{d}y = \left(4x^3 - 1\right)\mathrm{d}x$$

$$\int \frac{1}{y}\,\mathrm{d}y = \int \left(4x^3 - 1\right)\mathrm{d}x$$

$$\log|y| = x^4 - x + C$$

When $x = 0$, $\log|-2| = 0^4 - 0 + C$

$$C = \log 2$$

$$|y(x)| = e^{x^4 - x + \log 2}$$

$$y(x) = e^{x^4 - x + \log 2} \quad \text{or} \quad y(x) = -e^{x^4 - x + \log 2}$$

$$y(x) = -e^{x^4 - x + \log 2} = -2e^{x^4 - x} \quad \text{to make } y(0) = -2$$

Q

Let a and b be any two constants. We'll now solve the family of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

using our mnemonic device.

$$\begin{aligned}
\frac{\mathrm{d}y}{y-b} &= a\,\mathrm{d}x \\
\int \frac{\mathrm{d}y}{y-b} &= \int a\,\mathrm{d}x \\
\log|y-b| &= ax + c \\
|y-b| &= e^{ax+c} = e^c e^{ax} \\
y - b &= \pm e^c e^{ax} = Ce^{ax}
\end{aligned}$$

where the constant C can be any real number. (Even $C = 0$ works, i.e. $y(x) = b$ solves $\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$.) Note that when $y(x) = Ce^{ax} + b$ we have $y(0) = C + b$. So $C = y(0) - b$ and the general solution is

$$y(x) = \{y(0) - b\}\, e^{ax} + b$$

Linear First-Order Differential Equations

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y - b)$$

if and only if

$$y(x) = \{y(0) - b\}\, e^{ax} + b$$

Find a function $y(x)$ with $y' = 3y + 7$ and $y(2) = 5$.

To avoid re-inventing the wheel, we'll use our equation. But first, we should re-write our differential equation so the formatting matches.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\left(y + \frac{7}{3}\right)$$

Since we aren't given $y(0)$, we can't use the theorem as a shortcut to find C. We'll do it the old-fashioned way.

$$5 = y(2) = Ce^{3(2)} - \frac{7}{3}$$

$$\frac{22}{3} = Ce^{6}$$

The rate at which a medicine is metabolized (broken down) in the body depends on how much of it is in the bloodstream. Suppose a certain medicine is metabolized at a rate of $\frac{1}{10}A$ μg/hr, where A is the amount of medicine in the patient. The medicine is being administered to the patient at a constant rate of 2 μg/hr.
If the patient starts with no medicine in their blood at $t = 0$, give the formula for the amount of medicine in the patient at time t. What happens to the amount over time?

The rate of change of the amount of medicine in the patient is given by how quickly the medicine is being administered, minus how quickly it is metabolized:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2 - \frac{1}{10}A$$

Q

Linear First-Order Differential Equations

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

if and only if

$$y(x) = \{y(0)-b\}\, e^{ax} + b$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2 - \frac{1}{10}A = -\frac{1}{10}(A-20) \qquad A(0)=0$$

$$a = -\frac{1}{10}, \quad b = 20$$

$$A(t) = (A(0)-20)e^{-t/10} + 20$$

$$A(t) = -20e^{-t/10} + 20$$

This is an increasing function, with $\lim_{t\to\infty} A(t) = 20$. So the amount of medicine initially increases, but eventually almost holds steady at 20 μg.