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WARMUP

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}}\, dx$.

ans

WARMUP

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x$.

$$\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x = \int_3^7 \frac{1}{\sqrt{(x+1)^2}}\,\mathrm{d}x$$

$$= \int_3^7 \frac{1}{|x+1|}\,\mathrm{d}x$$

When $3 \le x \le 7$, we have $|x+1| = x+1$.

$$= \int_3^7 \frac{1}{x+1}\,\mathrm{d}x$$

$$= [\log|x+1|]_3^7$$

$$= \log 8 - \log 4 = \log 2$$

Idea: $\sqrt{(\text{something})^2} = |\text{something}|$. We cancelled off the square root.

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\, \mathrm{d}x$.

Evaluate $\int \dfrac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$.

We still want to cancel off the square root, but x^2+1 is not obviously of the form (something)^2.

Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$.

$$\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \int \frac{1}{\sqrt{\tan^2\theta+1}}\sec^2\theta\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \log|\sec\theta+\tan\theta| + C$$

We need to get these back in terms of x. From our substitution, we know $\tan\theta = x$. From simplifying our denominator, we also know $\sec\theta = \sqrt{x^2+1}$.

$$= \log\left|\sqrt{x^2+1} + x\right| + C$$

Same idea: $\sqrt{(\text{something})^2} = |\text{something}|$; cancel off the square root.

CHECK OUR WORK

Let's verify that $\displaystyle\int \frac{1}{\sqrt{x^2+1}} = \log\left|\sqrt{x^2+1}+x\right| + C.$
Seems improbable, right?

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Seems improbable, right?

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log\left|\sqrt{x^2+1}+x\right|+C\right] = \frac{1}{\sqrt{x^2+1}+x} \cdot \left(\frac{2x}{2\sqrt{x^2+1}}+1\right)$$

$$= \frac{x+\sqrt{x^2+1}}{(\sqrt{x^2+1}+x)\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

So, our answer works!

METHOD (ONE STANDARD CASE)

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  ► $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2 - 1}$ into $\sqrt{\tan^2\theta} = |\tan\theta|$

► After integrating, convert back to the original variable (possibly using a triangle–more details later)

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 - 1}$

- $\sqrt{x^2 + 1}$

- $\sqrt{1 - x^2}$

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$$1 - \sin^2 \theta = \cos^2 \theta \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 - 1}$
  Let $x = \sec \theta$, so $\sqrt{x^2 - 1}$ becomes $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2 \theta} = |\tan \theta|$

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$$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 - 1}$
  Let $x = \sec\theta$, so $\sqrt{x^2 - 1}$ becomes $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta|$

- $\sqrt{x^2 + 1}$
  Let $x = \tan\theta$, so $\sqrt{x^2 + 1}$ becomes $\sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = |\sec\theta|$

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$$1 - \sin^2 \theta = \cos^2 \theta \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

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▶ $\sqrt{x^2 - 1}$
Let $x = \sec \theta$, so $\sqrt{x^2 - 1}$ becomes $\sqrt{\sec^2 \theta - 1} = \sqrt{\tan^2 \theta} = |\tan \theta|$

▶ $\sqrt{x^2 + 1}$
Let $x = \tan \theta$, so $\sqrt{x^2 + 1}$ becomes $\sqrt{\tan^2 \theta + 1} = \sqrt{\sec^2 \theta} = |\sec \theta|$

▶ $\sqrt{1 - x^2}$
Let $x = \sin \theta$ so $\sqrt{1 - x^2}$ becomes $\sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta|$
(Alternately, $x = \cos \theta$ works as well)

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

▶ $\sqrt{x^2 + 7}$

▶ $\sqrt{3 - 2x^2}$

FOCUS ON THE ALGEBRA

$$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 + 7}$

  Adjust a given identity by multiplying both sides by 7:
  $7\tan^2\theta + 7 = 7\sec^2\theta$. Now we see we want $x^2 = 7\tan^2\theta$. That is, $x = \sqrt{7}\tan\theta$:
  $$\sqrt{x^2 + 7} = \sqrt{7\tan^2\theta + 7} = \sqrt{7(\sec^2\theta)} = \sqrt{7}\,|\sec\theta|$$

- $\sqrt{3 - 2x^2}$

ans

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

▶ $\sqrt{x^2 + 7}$
   Adjust a given identity by multiplying both sides by 7:
   $7 \tan^2 \theta + 7 = 7 \sec^2 \theta$. Now we see we want $x^2 = 7 \tan^2 \theta$. That is,
   $x = \sqrt{7} \tan \theta$:
   $\sqrt{x^2 + 7} = \sqrt{7 \tan^2 \theta + 7} = \sqrt{7(\sec^2 \theta)} = \sqrt{7} \, |\sec \theta|$

▶ $\sqrt{3 - 2x^2}$
   Adjust a given identity by multiplying both sides by 3:
   $3 - 3 \sin^2 \theta = 3 \cos^2 \theta$. Now we see we want $2x^2 = 3 \sin^2 \theta$, so
   $x = \sqrt{\frac{3}{2}} \sin \theta$:
   $\sqrt{3 - 2x^2} = \sqrt{3 - 2 \left(\frac{3}{2} \sin^2 \theta\right)} = \sqrt{3 - 3 \sin^2 \theta} = \sqrt{3 \cos^2 \theta} = \sqrt{3} \, |\cos \theta|$

CLOSER LOOK AT ABSOLUTE VALUES

Consider the substitution $x = \sin\theta$, $dx = \cos\theta \, d\theta$ for the integral:

$$\int_0^1 \sqrt{1 - x^2} \, dx$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?

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Consider the substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2}\,dx$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?
If $x = 0$, then $\sin\theta = 0$, but there are infinitely many values of θ that could make this true. To use the substitution $x = \sin\theta$, we need the function $x = \sin\theta$ to be invertible. That way, we can unambiguously convert between x and θ. With that in mind, we'll actually set $\theta = \arcsin x$. Now θ is restricted to the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\arcsin 0}^{\arcsin 1} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} |\cos\theta|\cdot\cos\theta\,d\theta$$

For $0 \leq \theta \leq \frac{\pi}{2}$, we have $\cos\theta \geq 0$, so $|\cos\theta| = \cos\theta$.

CLOSER LOOK AT ABSOLUTE VALUES

More generally, suppose a is a positive constant and we use the substitution $x = a \sin \theta$ for the term $\sqrt{a^2 - x^2}$.

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▶ $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

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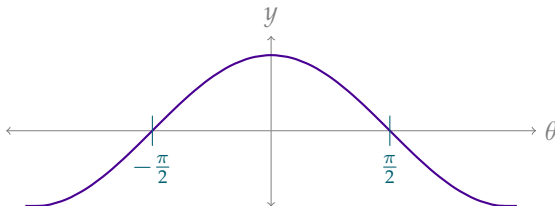
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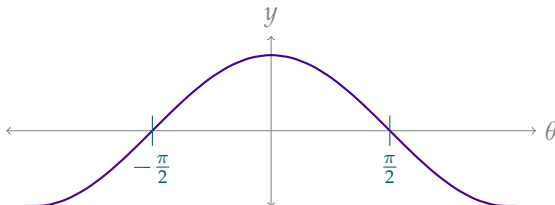
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More generally, suppose a is a positive constant and we use the substitution $x = a \sin \theta$ for the term $\sqrt{a^2 - x^2}$.

▶ $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

▶ $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2 \cos^2 \theta} = a|\cos \theta|$

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▶ So, in general, when we use the substitution $x = \sin \theta$ with trigonometric substitution, we can expect $|\cos \theta| = \cos \theta$.

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Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

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CLOSER LOOK AT ABSOLUTE VALUES $\quad$ ⏭ SKIP CLOSER LOOK

Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

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▶ $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = \sqrt{a^2 \sec^2 \theta} = \frac{a}{|\cos \theta|}$

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Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

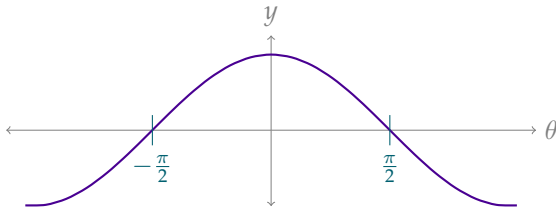▶ $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

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▶ On the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos \theta \geq 0$, so $|\cos \theta| = \cos \theta$ and $|\sec \theta| = \sec \theta$.

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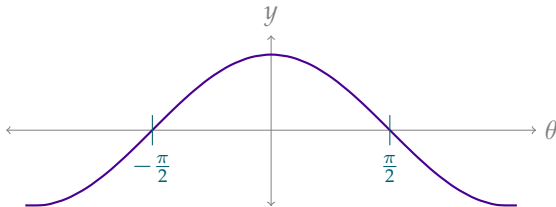

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▶ $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$

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▶ $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$

▶ Now this case gets slightly more complicated than the others:

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  ▶ For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \geq a^2$. I.e. $x \geq a$ or $x \leq -a$.

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   ▶ For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \geq a^2$. I.e. $x \geq a$ or $x \leq -a$.
   ▶ When $x \geq a$, we have $0 \leq \theta < \frac{\pi}{2}$, $\tan \theta \geq 0$, so $|\tan \theta| = \tan \theta$.

CLOSER LOOK AT ABSOLUTE VALUES

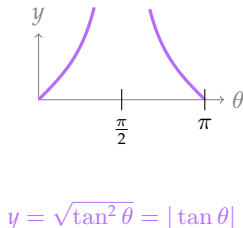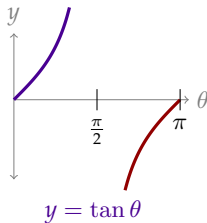Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

► $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \le \theta \le \pi$

► $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$

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  ► For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \ge a^2$. I.e. $x \ge a$ or $x \le -a$.
  ► When $x \ge a$, we have $0 \le \theta < \frac{\pi}{2}$, $\tan\theta \ge 0$, so $|\tan\theta| = \tan\theta$.
  ► When $x \le -a$, we have $\frac{\pi}{2} < \theta \le \pi$, $\tan\theta < 0$, so $|\tan\theta| = -\tan\theta$.

$y = \tan\theta$

$y = \sqrt{\tan^2\theta} = |\tan\theta|$

ABSOLUTE VALUES

From now on, we will assume:

▶ With the substitution $x = a\sin\theta$ for $\sqrt{a^2 - x^2}$, $|\cos\theta| = \cos\theta$
▶ With the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, $|\sec\theta| = \sec\theta$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad\qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad\qquad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad\qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\displaystyle\int_0^1 (1 + x^2)^{-3/2}\,\mathrm{d}x$

Evaluate $\displaystyle\int_0^1 (1 + x^2)^{-3/2} \, \mathrm{d}x$

ans

Evaluate $\displaystyle\int_0^1 (1 + x^2)^{-3/2}\, \mathrm{d}x$

Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. When $x = 0$, then $\theta = \arctan 0 = 0$; when $x = 1$, then $\theta = \arctan 1 = \frac{\pi}{4}$.

$$
\begin{aligned}
\int_0^1 (1 + x^2)^{-3/2}\, \mathrm{d}x &= \int_{\theta=0}^{\theta=\pi/4} \frac{1}{\sqrt{1 + \tan^2\theta}^3} \sec^2\theta\, \mathrm{d}\theta \\
&= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}^3}\, \mathrm{d}\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{|\sec\theta|^3}\, \mathrm{d}\theta \\
&= \int_0^{\pi/4} \frac{1}{|\sec\theta|}\, \mathrm{d}\theta = \int_0^{\pi/4} |\cos\theta|\, \mathrm{d}\theta
\end{aligned}
$$

Given our previous investigation,

$$
\begin{aligned}
&= \int_0^{\pi/4} \cos\theta\, \mathrm{d}\theta = \big[\sin\theta\big]_0^{\pi/4} \\
&= \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta \qquad\qquad \sin(2\theta) = \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta \qquad\qquad \sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2 \theta - 1 = \tan^2 \theta \qquad\qquad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \sqrt{1 - 4x^2} \, \mathrm{d}x$

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Under the square root, we have "one minus a term with a variable," which matches the identity $1 - \sin^2\theta$. So, we want $4x^2$ to become $\sin^2\theta$. That is, $x = \frac{1}{2}\sin\theta$. Then $\mathrm{d}x = \frac{1}{2}\cos\theta\,\mathrm{d}\theta$.

$$\int \sqrt{1 - 4x^2}\, \mathrm{d}x = \int \sqrt{1 - 4\left(\frac{1}{2}\sin\theta\right)^2} \cdot \frac{1}{2}\cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int \sqrt{1 - \sin^2\theta} \cdot \cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int \sqrt{\cos^2\theta} \cdot \cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int |\cos\theta| \cdot \cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int \cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int \left(\frac{1 + \cos(2\theta)}{2}\right)\mathrm{d}\theta = \frac{1}{4}\int (1 + \cos(2\theta))\mathrm{d}\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C = \frac{1}{4}\left[\theta + \sin\theta\cos\theta\right] + C$$

It remains to convert θ back into x.

Evaluate $\int \sqrt{1 - 4x^2}\, dx$

The substitution $x = \frac{1}{2}\sin\theta$ tells us $\sin\theta = 2x$. This in turn gives us $\theta = \arcsin(2x)$. We should still convert $\cos\theta$ back into terms of x. You might notice in the calculation we did that $\sqrt{1 - 4x^2}$ turned into $\cos\theta$, so $\cos\theta = \sqrt{1 - 4x^2}$.

Alternately, to find $\cos\theta$ in terms of x, we can use a triangle. From $\sin\theta = 2x$, and the understanding that $\sin\theta$ is the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ for a right triangle with angle θ, we can set up a triangle whose opposite side has length $2x$, and hypotenuse has length 1.
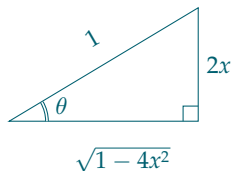
The Pythagorean theorem tells us the side adjacent to θ has length $\sqrt{1 - 4x^2}$. So $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{1 - 4x^2}$.

$\int \sqrt{1 - 4x^2}\, dx = \frac{1}{4}\left(\underbrace{\arcsin(2x)}_{\theta} + \underbrace{2x\sqrt{1 - 4x^2}}_{\sin\theta\,\cos\theta} \right) + C$

CHECK OUR WORK

In the last example, we computed

$$\int \sqrt{1 - 4x^2}\, \mathrm{d}x = \frac{1}{4}\left(\arcsin(2x) + 2x\sqrt{1 - 4x^2}\right) + C.$$

To check, we differentiate.

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In the last example, we computed

$$\int \sqrt{1 - 4x^2}\, dx = \frac{1}{4}\big(\arcsin(2x) + 2x\sqrt{1 - 4x^2}\big) + C.$$

To check, we differentiate.

$$\frac{d}{dx}\left\{\frac{1}{4}\big(\arcsin(2x) + 2x\sqrt{1 - 4x^2}\big) + C\right\}$$

$$= \frac{1}{4}\left(\frac{2}{\sqrt{1 - (2x)^2}} + 2x\frac{-8x}{2\sqrt{1 - 4x^2}} + 2\sqrt{1 - 4x^2}\right)$$

$$= \frac{1}{4}\left(\frac{2}{\sqrt{1 - 4x^2}} - \frac{8x^2}{\sqrt{1 - 4x^2}} + \frac{2(1 - 4x^2)}{\sqrt{1 - 4x^2}}\right)$$

$$= \frac{1}{4}\left(\frac{2 - 8x^2 + 2 - 8x^2}{\sqrt{1 - 4x^2}}\right) = \frac{1 - 4x^2}{\sqrt{1 - 4x^2}} = \sqrt{1 - 4x^2} \qquad \checkmark$$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta \qquad\qquad \sin(2\theta) = \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta \qquad\qquad \sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

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Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}} \, dx$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}} \, \mathrm{d}x$

We use the substitution $x = \sec\theta$, $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$.

To make the substitution work, we're actually taking $\theta = \arccos\left(\frac{1}{x}\right)$, and so $0 \le \theta \le \pi$.

Note that the integrand exists on the intervals $x < -1$ and $x > 1$.

▶ When $x > 1$, then $0 < \frac{1}{x} < 1$, so $0 < \arccos\left(\frac{1}{x}\right) < \frac{\pi}{2}$.
  That is, $0 < \theta < \frac{\pi}{2}$, so $|\tan\theta| = \tan\theta$.

▶ When $x < -1$, then $-1 < \frac{1}{x} < 0$, so $\frac{\pi}{2} < \arccos\left(\frac{1}{x}\right) < \pi$.
  That is, $\frac{\pi}{2} < \theta < \pi$, so $|\tan\theta| = -\tan\theta$.

$$\int \frac{1}{\sqrt{x^2 - 1}}\mathrm{d}x = \int \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot \sec\theta\tan\theta\,\mathrm{d}\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\mathrm{d}\theta$$

$$= \int \sec\theta\left(\frac{\tan\theta}{|\tan\theta|}\right)\mathrm{d}\theta = \begin{cases} \int \sec\theta\,\mathrm{d}\theta & 0 < \theta < \frac{\pi}{2} \\ -\int \sec\theta\,\mathrm{d}\theta & \frac{\pi}{2} < \theta < \pi \end{cases}$$

$$= \begin{cases} \log|\sec\theta + \tan\theta| + C & 0 < \theta < \frac{\pi}{2} \\ -\log|\sec\theta + \tan\theta| + C & \frac{\pi}{2} < \theta < \pi \end{cases}$$

Our substitution tells us $\sec\theta = x$. We saw from the denominator of our integrand that $\sqrt{x^2 - 1} = |\tan\theta|$.

▶ When $0 < \theta < \frac{\pi}{2}$, $\tan\theta = |\tan\theta| = \sqrt{x^2 - 1}$

▶ When $\frac{\pi}{2} < \theta < \pi$, $\tan\theta = -|\tan\theta| = -\sqrt{x^2 - 1}$

$$\int \frac{1}{\sqrt{x^2 - 1}}\mathrm{d}x = \begin{cases} \log|x + \sqrt{x^2 - 1}| + C & x > 1 \\ -\log|x - \sqrt{x^2 - 1}| + C & x < -1 \end{cases}$$

Although the two branches look different, they are actually equivalent. Remember $-\log(A) = \log\left(A^{-1}\right) = \log\left(\frac{1}{A}\right)$:

$$-\log|x - \sqrt{x^2 - 1}| = \log\left|\frac{1}{x - \sqrt{x^2 - 1}}\right| = \log\left|\frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\right|$$

$$= \log\left|\frac{x + \sqrt{x^2 - 1}}{x^2 - x^2 + 1}\right| = \log\left|x + \sqrt{x^2 - 1}\right|$$

So,

$$\int \frac{1}{\sqrt{x^2 - 1}}\mathrm{d}x = \log\left|x + \sqrt{x^2 - 1}\right| + C$$

CHECK OUR WORK

Let's check our result, $\displaystyle\int \frac{1}{\sqrt{x^2 - 1}}\,dx = \log\left|x + \sqrt{x^2 - 1}\right| + C.$

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$$\frac{d}{dx}\left\{\log\left|x + \sqrt{x^2-1}\right| + C\right\} = \frac{1 + \frac{2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}}$$

$$= \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}\right) = \frac{(\sqrt{x^2-1} + x)}{\left(x + \sqrt{x^2-1}\right)\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

So, our answer works.

COMPLETING THE SQUARE

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.

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$$\sqrt{3 - x^2 + 2x} = \sqrt{4 - (x - 1)^2}$$

We want $(x - 1)^2 = 4\sin^2 \theta$, so let $(x - 1) = 2\sin \theta$

$$= \sqrt{4 - 4\sin^2 \theta} = \sqrt{4\cos^2 \theta} = 2\cos \theta$$

COMPLETING THE SQUARE

$$(x + b)^2 = x^2 + 2bx + b^2$$
$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x + b)^2$ for constants b, c.

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1. Find b:

ans

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1. Find b: $-2bx = 2x$, so $b = -1$

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2. Solve for c:

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3. All together: $3 - x^2 + 2x = 4 - (x - 1)^2$

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\, dx$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
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One of those parts is a constant, and one is squared.

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Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\,\mathrm{d}x$.

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One of those parts is a constant, and one is squared.
Write $6x - x^2$ as $c - (x + b)^2$.

$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$
$$6x = -2bx \implies b = -3$$
$$0 = c - b^2 = c - 9 \implies c = 9$$
$$6x - x^2 = 9 - (x - 3)^2$$

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\ \mathrm{d}x\ =\ \int \frac{(x - 3)^2}{\sqrt{9 - (x - 3)^2}}\mathrm{d}x.$

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\,\mathrm{d}x \;=\; \int \frac{(x-3)^2}{\sqrt{9 - (x-3)^2}}\mathrm{d}x.$

We use the identity $9 - 9\sin^2\theta = 9\cos^2\theta$.
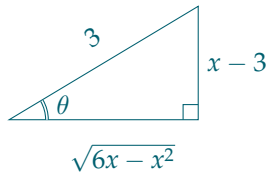We want $(x-3)^2 = 9\sin^2\theta$, so take $(x-3) = 3\sin\theta$, $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$.

$$\int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}}\,3\cos\theta\,\mathrm{d}\theta = \int 9\sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{9}{2}\int (1 - \cos 2\theta)\,\mathrm{d}\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C$$

Right triangle with hypotenuse 3, vertical side $x-3$, horizontal side $\sqrt{6x-x^2}$, and angle θ.

CHECK OUR WORK

Let's verify that
$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} = \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3}\right) + C:$$

CHECK OUR WORK

Let's verify that
$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} = \frac{9}{2} \left(\arcsin\left(\frac{x - 3}{3}\right) - \frac{x - 3}{3} \cdot \frac{\sqrt{6x - x^2}}{3} \right) + C :$$

$$\frac{d}{dx} \left\{ \frac{9}{2} \left(\arcsin\left(\frac{x - 3}{3}\right) - \frac{x - 3}{3} \cdot \frac{\sqrt{6x - x^2}}{3} \right) + C \right\}$$

$$= \frac{9}{2} \left(\frac{1/3}{\sqrt{1 - \left(\frac{x-3}{3}\right)^2}} - \frac{x - 3}{3} \cdot \frac{3 - x}{3\sqrt{6x - x^2}} - \frac{1}{9}\sqrt{6x - x^2} \right)$$

$$= \frac{9}{2} \left(\frac{9}{9\sqrt{6x - x^2}} - \frac{6x - x^2 - 9}{9\sqrt{6x - x^2}} - \frac{6x - x^2}{9\sqrt{6x - x^2}} \right)$$

$$= \frac{9 - 6x + x^2}{\sqrt{6x - x^2}}$$

So, our answer works.

Included Work