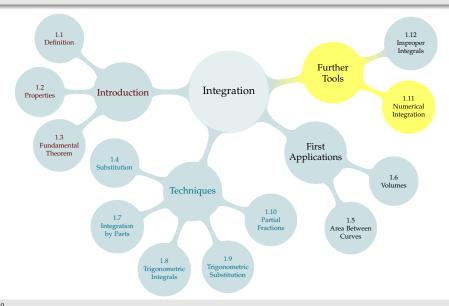
## TABLE OF CONTENTS



# Numerical integration errors

Assume that  $|f''(x)| \le M$  for all  $a \le x \le b$  and  $|f^{(4)}(x)| \le L$  for all  $a \le x \le b$ . Then

- ► the total error introduced by the midpoint rule is bounded by  $\frac{M}{24} \frac{(b-a)^3}{n^2},$
- ► the total error introduced by the trapezoidal rule is bounded by  $\frac{M}{12} \frac{(b-a)^3}{n^2}$ , and
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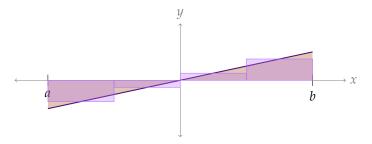
when approximating  $\int_a^b f(x) dx$ .

## WHY THE second DERIVATIVE?

The midpoint rule gives the exact area under the curve for

$$f(x) = ax + b$$

when a and b are any constants.



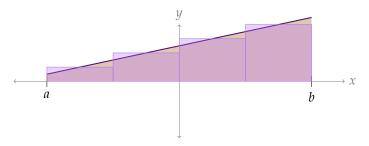
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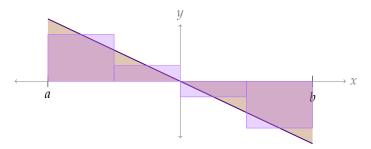
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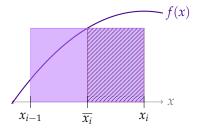
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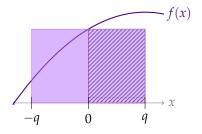
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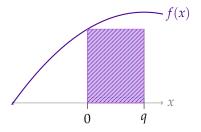
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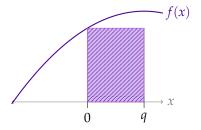
To avoid messiness, let's also consider a simplified location:



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We want to relate the actual area of this half-slice to its approximate area:

$$\int_0^q f(x) \, \mathrm{d}x \approx q \cdot f(0)$$

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If you squint just right, the right-hand side looks a bit like the " $u \cdot v$ " term from integration by parts, where u = f(x) and dv = dx.

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Set u = f(x) and dv = dx, so du = f'(x) dx. We choose v(x) = x - q, so that f(v(q)) = f(0).

$$\int_0^q f(x) \, dx = \left[ (x - q)f(x) \right]_0^q - \int_0^q (x - q)f'(x) \, dx$$
$$= q \cdot f(0) - \int_0^q (x - q)f'(x) \, dx$$

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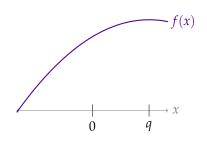
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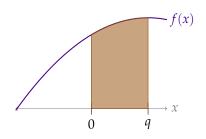
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▶ We know something about the second derivative, not the first, so repeat: set u = f'(x), dv = (x - q) dx; du = f''(x) dx,  $v = \frac{(x - q)^2}{2}$ 

$$\int_0^q f(x) \, \mathrm{d}x = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x) \, \mathrm{d}x$$

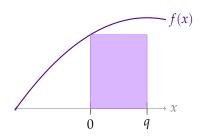


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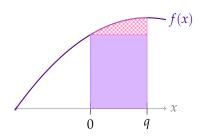


$$\int_0^q f(x) \, dx = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx$$
exact





$$\int_0^q f(x) dx = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x) dx$$
exact approximate



$$\int_0^q f(x) dx = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x) dx$$
exact approximate  $\pm \text{ error}$ 

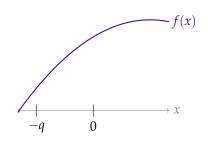
### Repeat for the other half of the slice:

$$\int_{-q}^{0} \underbrace{f(x)}_{u} \underbrace{dx}_{dv} = \left[ \underbrace{f(x)}_{u} \cdot \underbrace{(x+q)}_{v} \right]_{-q}^{0} - \int_{-q}^{0} \underbrace{(x+q)}_{v} \cdot \underbrace{f'(x)}_{du} dx$$

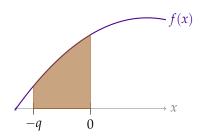
$$= q \cdot f(0) - \int_{-q}^{0} \underbrace{f'(x)}_{\hat{u}} \cdot \underbrace{(x+q)}_{d\hat{v}} dx$$

$$= q \cdot f(0) - \left[ \underbrace{f'(x)}_{\hat{u}} \underbrace{(x+q)^{2}}_{\hat{v}} \right]_{-q}^{0} + \int_{-q}^{0} \underbrace{(x+q)^{2}}_{\hat{v}} \underbrace{f''(x)}_{d\hat{u}} dx$$

$$= q \cdot f(0) - \frac{q^{2}}{2} f'(0) + \int_{-q}^{0} \underbrace{(x+q)^{2}}_{2} f''(x) dx$$

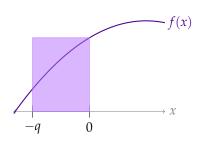


$$\int_{-q}^{0} f(x) \, dx = q \cdot f(0) - \frac{q^{2}}{2} \cdot f'(0) + \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx$$

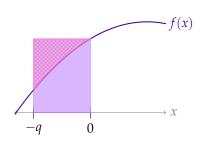


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exact

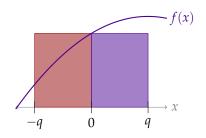




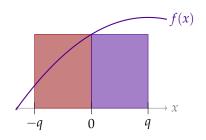
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exact approximate



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exact approximate  $\pm \text{ error}$ 



$$\int_{-q}^{0} f(x) \, dx = q \cdot f(0) - \frac{q^{2}}{2} f'(0) + \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx$$
$$\int_{0}^{q} f(x) \, dx = q \cdot f(0) + \frac{q^{2}}{2} \cdot f'(0) + \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx$$

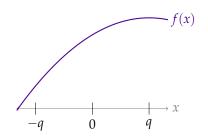


$$\int_{-q}^{0} f(x) \, dx = q \cdot f(0) - \frac{q^{2}}{2} f'(0) + \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx$$

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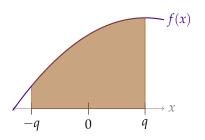
$$\int_{-q}^{q} f(x) \, dx = 2q \cdot f(0) + \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx + \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx$$

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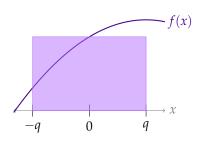
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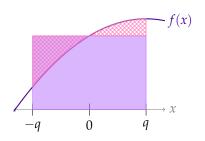
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$$\int_{-q}^{q} f(x) dx = 2q \cdot f(0) + \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) dx + \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) dx$$
exact
approximate





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exact approximate  $\pm \text{ error}$ 

We re-arrange to write the error as the difference between the actual area of one slice and its rectangular approximation.

$$\int_{-q}^{q} f(x) \, dx - 2q \cdot f(0) = \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx + \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx$$

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$$\text{error} = \left| \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx + \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx \right|$$

$$\leq \left| \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx \right| + \left| \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx \right|$$

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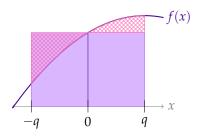
$$error = \left| \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx + \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx \right|$$

$$\leq \left| \int_{-q}^{0} \frac{(x+q)^{2}}{2} f''(x) \, dx \right| + \left| \int_{0}^{q} \frac{(x-q)^{2}}{2} f''(x) \, dx \right|$$

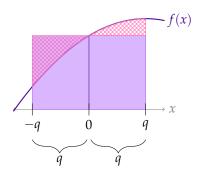
$$\leq \int_{-q}^{0} \frac{(x+q)^{2}}{2} M \, dx + \int_{0}^{q} \frac{(x-q)^{2}}{2} M \, dx$$

$$= M \left[ \frac{(x+q)^{3}}{6} \right]_{-q}^{0} + M \left[ \frac{(x-q)^{3}}{6} \right]_{0}^{q}$$

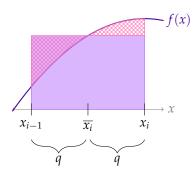
$$= \frac{M \cdot q^{3}}{3}$$



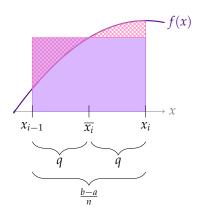
$$\left| \int_{-q}^{q} f(x) \, \mathrm{d}x - 2q \cdot f(0) \right| \le \frac{M}{3} \cdot q^{3}$$



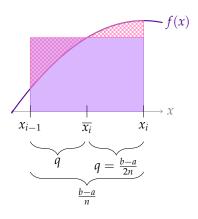
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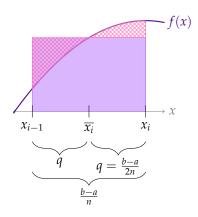
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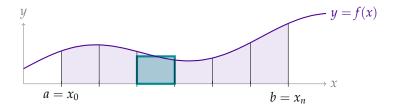


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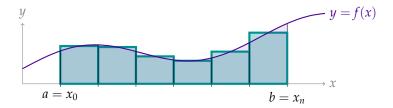
$$\left| \int_{x_{i-1}}^{x_i} f(x) \, \mathrm{d}x - \frac{b-a}{n} \cdot f(\overline{x_i}) \right| \le \frac{M}{3} \left( \frac{b-a}{2n} \right)^3 = \frac{M}{24} \frac{(b-a)^3}{n^3}$$



lacktriangle The error in each slice is at most

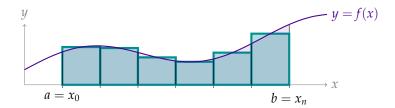
$$\frac{M}{24} \frac{(b-a)^3}{n^3}$$





- ► The error in each slice is at most
- ightharpoonup There are n slices

 $\frac{M}{24} \frac{(b-a)^3}{n^3}$ 



- ► The error in each slice is at most
- ightharpoonup There are n slices
- ► The overall error is at most  $n \cdot \frac{M}{24} \frac{(b-a)^3}{n^3} = \frac{M}{24} \frac{(b-a)^3}{n^2}$

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