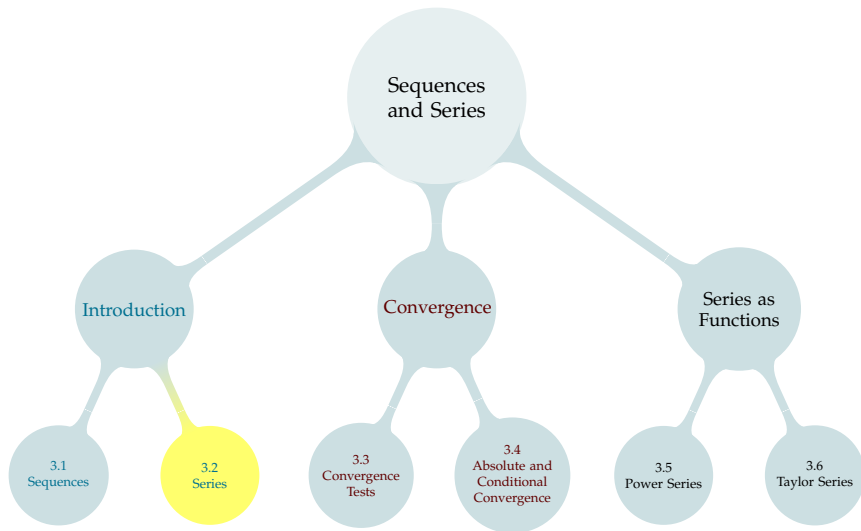


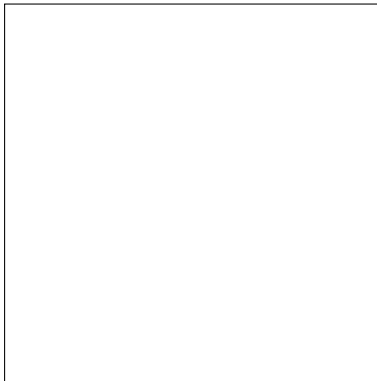
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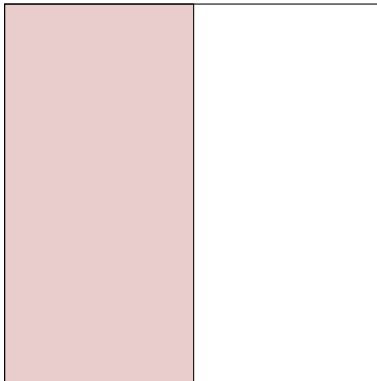
●○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○

SEQUENCES AND SERIES



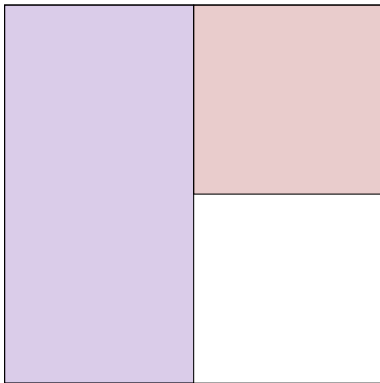
Square of Area 1

Size of Tiles: $\frac{1}{2}$



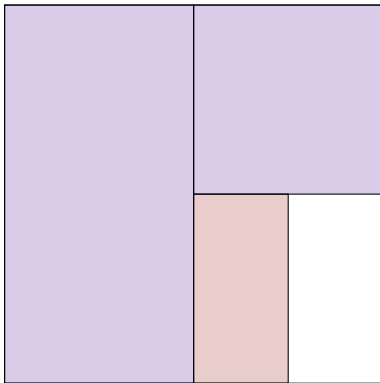
Covered Area: $\frac{1}{2}$

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}$



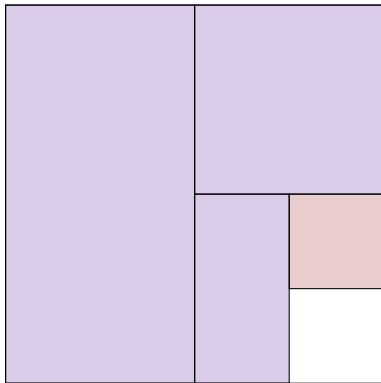
Covered Area: $\frac{1}{2} + \frac{1}{2^2}$

Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$



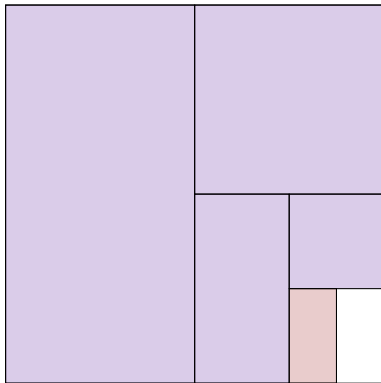
Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$

Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

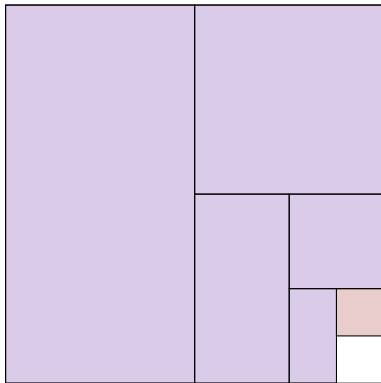
Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, $\frac{1}{2^5}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}$

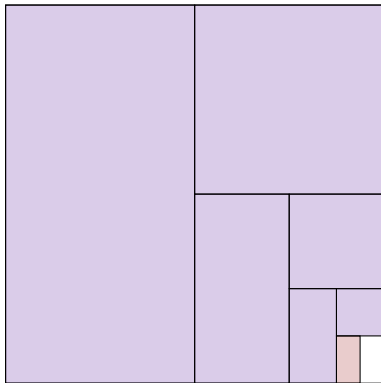
SEQUENCES AND SERIES

Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, $\frac{1}{2^5}$, $\frac{1}{2^6}$



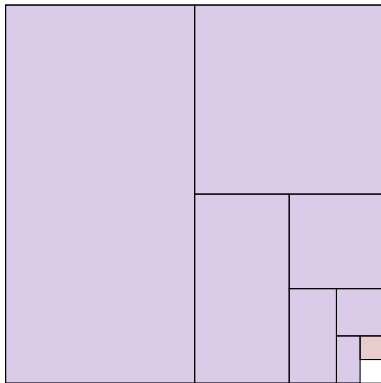
Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$

Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, $\frac{1}{2^5}$, $\frac{1}{2^6}$, $\frac{1}{2^7}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}$

Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, $\frac{1}{2^5}$, $\frac{1}{2^6}$, $\frac{1}{2^7}$, $\frac{1}{2^8}$



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8}$

Size of Tiles: $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, $\frac{1}{2^5}$, $\frac{1}{2^6}$, $\frac{1}{2^7}$, $\frac{1}{2^8}$, $\frac{1}{2^9}$, \dots

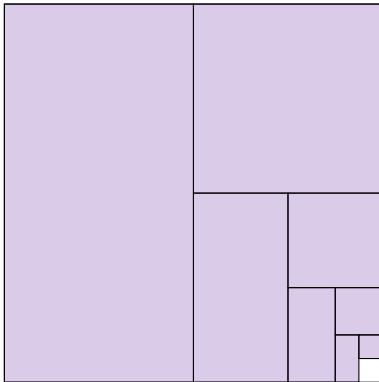
Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

Size of Tiles:

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$$

Sequence

List of numbers,
approaching



Covered Area:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$$

Size of Tiles:

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$$

The diagram illustrates a recursive partitioning of a square. The top-left quadrant is white. The top-right quadrant is further partitioned into four smaller squares, with the bottom-right one being white and the others light blue. The bottom-left and bottom-right quadrants are light blue.

**List of numbers,
approaching zero.**

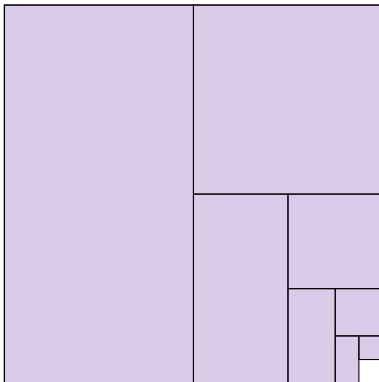
Covered Area:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$$

Size of Tiles: $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$

**List of numbers,
approaching zero.**

Series



Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$

SEQUENCES AND SERIES

Size of Tiles:

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$$

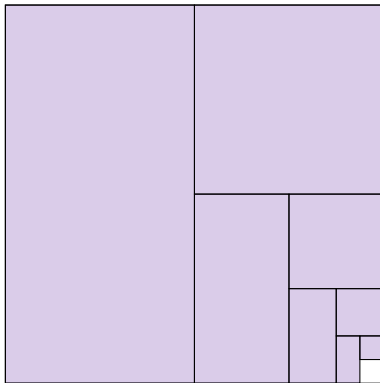
Sequence

**List of numbers,
approaching zero.**

Series

Sum of numbers,
approaching

Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$



SEQUENCES AND SERIES

Size of Tiles:

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \dots$$

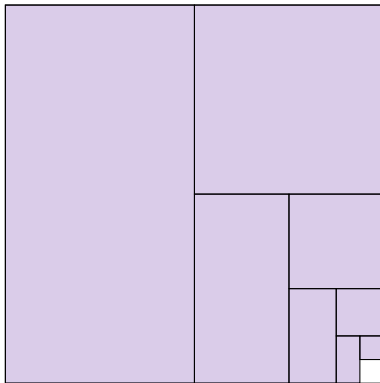
Sequence

**List of numbers,
approaching zero.**

Series

Sum of numbers,
approaching one.

Covered Area: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \cdots$



Recall:

$$\sum_{n=1}^5 \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

Recall:

$$\sum_{n=1}^5 \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =$$

QUICK REVIEW: SIGMA NOTATION

Recall:

$$\sum_{n=1}^5 \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

(a more rigorous definition will be discussed soon)

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (C \cdot a_n) =$$

$$\text{A. } \sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$$

$$\text{B. } \sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$$

$$\text{C. } C \sum_{n=1}^{\infty} a_n$$

$$\text{D. } a_n \sum_{n=1}^{\infty} C$$

E. none of the above



Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (C \cdot a_n) =$$

A. $\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$

C. $C \sum_{n=1}^{\infty} a_n$

D. $a_n \sum_{n=1}^{\infty} C$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n + b_n) =$$

$$\text{A. } \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$$

$$\text{B. } \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

C. $a_n + \sum_{n=1}^{\infty} b_n$

$$\text{D. } a_n \sum_{n=1}^{\infty} b_n$$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n + b_n) =$$

$$\text{A. } \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$$

B. $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $a_n + \sum_{n=1}^{\infty} b_n$

$$\text{D. } a_n \sum_{n=1}^{\infty} b_n$$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

$$\text{A. } \sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$$

B. $\left(\sum_{n=1}^{\infty} a_n\right)^C$

$$\text{C. } C^n \sum_{n=1}^{\infty} a_n$$

$$\text{D. } \sum_{n=1}^{\infty} C(a_n)^{C-1}$$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

$$\text{A. } \sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$$

B. $\left(\sum_{n=1}^{\infty} a_n\right)^C$

$$\text{C. } C^n \sum_{n=1}^{\infty} a_n$$

$$\text{D. } \sum_{n=1}^{\infty} C(a_n)^{C-1}$$

E. none of the above



What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

SERIES PHILOSOPHY

What does it really mean to add up infinitely many things?

$$\underbrace{1-1}_0 + \underbrace{1-1}_0 + \underbrace{1-1}_0 + \underbrace{1-1}_0 + \underbrace{1-1}_0 + \underbrace{1-1}_0 + \cdots$$

SERIES PHILOSOPHY

What does it really mean to add up infinitely many things?

$$\underbrace{1}_{1} - \underbrace{1+1}_{0} - \underbrace{1+1}_{0} - \underbrace{1+1}_{0} - \underbrace{1+1}_{0} - \underbrace{1+1}_{0} - \underbrace{1+\cdots}_{0}$$

SERIES PHILOSOPHY

What does it really mean to add up infinitely many things?

Diagram illustrating the construction of a sequence of 1s and -1s. The sequence is $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$. Below the sequence, there are three 1s. Arrows point from these 1s to the sequence: a purple arrow from the first 1 to the first 1, a blue arrow from the second 1 to the second -1, a blue arrow from the second 1 to the third 1, a purple arrow from the third 1 to the fourth -1, a purple arrow from the third 1 to the fifth 1, and a purple arrow from the third 1 to the sixth -1.

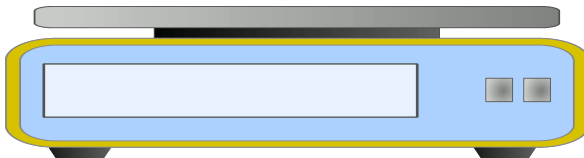
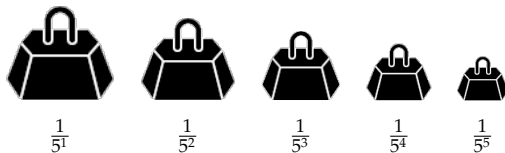
SERIES PHILOSOPHY

What does it really mean to add up infinitely many things?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

We need an unambiguous definition.

SEQUENCE OF PARTIAL SUMS



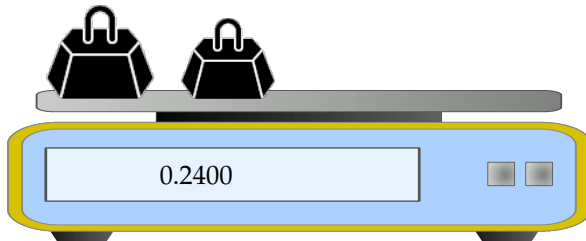
SEQUENCE OF PARTIAL SUMS

$\frac{1}{5^2}$
 $\frac{1}{5^3}$
 $\frac{1}{5^4}$
 $\frac{1}{5^5}$



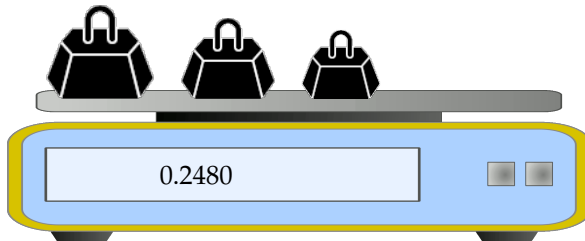
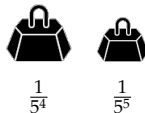
SEQUENCE OF PARTIAL SUMS

$$S_2 = 0.2400$$



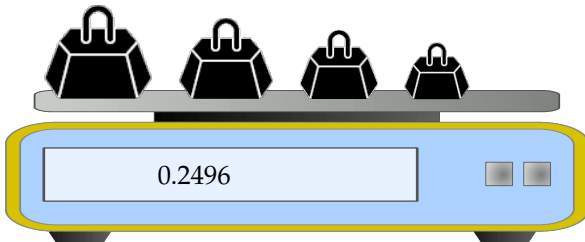
SEQUENCE OF PARTIAL SUMS

$$S_3 = 0.2480$$



SEQUENCE OF PARTIAL SUMS

$$S_4 = 0.2496$$



SEQUENCE OF PARTIAL SUMS

$$S_5 = 0.2499$$



Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2$$

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Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

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$$a_1 = \frac{1}{5} = 0.2 \quad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

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$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2$$

$$S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04$$

$$S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$S_4 = 0.2496$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2$$

$$S_1 = 0.2$$

$$a_2 = \frac{1}{5^2} = 0.04$$

$$S_2 = 0.24$$

$$a_3 = \frac{1}{5^3} = 0.008$$

$$S_3 = 0.248$$

$$a_4 = \frac{1}{5^4} = 0.0016$$

$$S_4 = 0.2496$$

$$a_5 = \frac{1}{5^5} = 0.00032$$

$$S_5 = 0.24992$$

We define $\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n = \lim_{N \rightarrow \infty} S_N$.

$a_1 = \frac{1}{5} = 0.2$	$S_1 = 0.2$	$a_5 = \frac{1}{5^5} = 0.00032$	$S_5 = 0.24992$
$a_2 = \frac{1}{5^2} = 0.04$	$S_2 = 0.24$	$a_6 = \frac{1}{5^6} = 0.000064$	$S_6 = 0.249984$
$a_3 = \frac{1}{5^3} = 0.008$	$S_3 = 0.248$	$a_7 = \frac{1}{5^7} = 0.0000128$	$S_7 = 0.2499968$
$a_4 = \frac{1}{5^4} = 0.0016$	$S_4 = 0.2496$	$a_8 = \frac{1}{5^8} = 0.00000256$	$S_8 = 0.24999936$

From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} = \lim_{N \rightarrow \infty} S_N =$$

We define $\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n = \lim_{N \rightarrow \infty} S_N$.

$a_1 = \frac{1}{5} = 0.2$	$S_1 = 0.2$	$a_5 = \frac{1}{5^5} = 0.00032$	$S_5 = 0.24992$
$a_2 = \frac{1}{5^2} = 0.04$	$S_2 = 0.24$	$a_6 = \frac{1}{5^6} = 0.000064$	$S_6 = 0.249984$
$a_3 = \frac{1}{5^3} = 0.008$	$S_3 = 0.248$	$a_7 = \frac{1}{5^7} = 0.0000128$	$S_7 = 0.2499968$
$a_4 = \frac{1}{5^4} = 0.0016$	$S_4 = 0.2496$	$a_8 = \frac{1}{5^8} = 0.00000256$	$S_8 = 0.24999936$

From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \lim_{N \rightarrow \infty} S_N = \frac{1}{4}$$

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NOTATION: $S_N = \sum_{n=1}^N a_n$



a_1



a_2



a_3



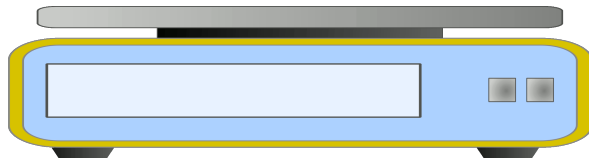
a_4



a_5



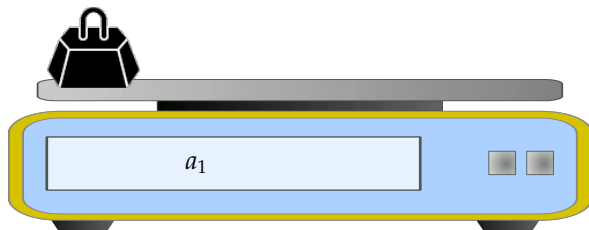
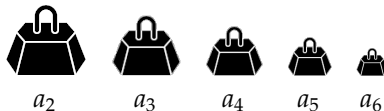
a_6



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NOTATION: $S_N = \sum_{n=1}^N a_n$

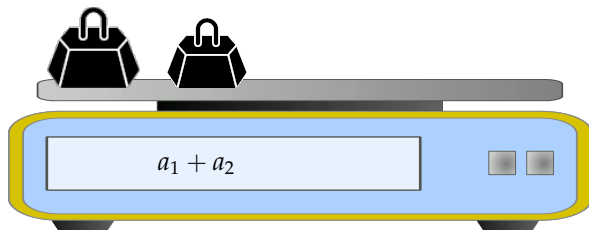
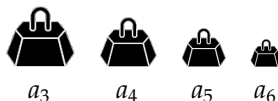
$$S_1 = a_1$$



NOTATION: $S_N = \sum_{n=1}^N a_n$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

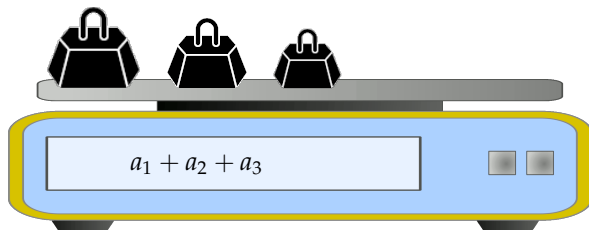


NOTATION: $S_N = \sum_{n=1}^N a_n$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$



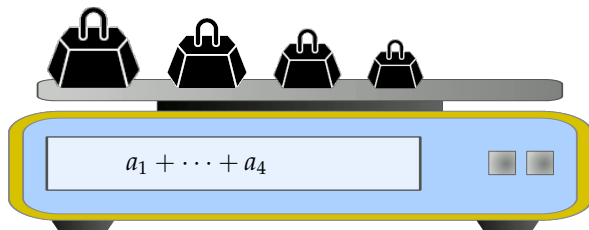
NOTATION: $S_N = \sum_{n=1}^N a_n$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$



NOTATION: $S_N = \sum_{n=1}^N a_n$

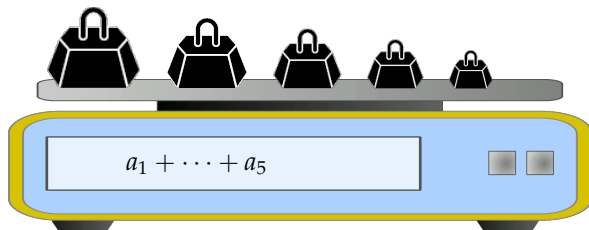
$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

$$S_5 = a_1 + \cdots + a_5$$


 a_6


NOTATION: $S_N = \sum_{n=1}^N a_n$

$$S_1 = a_1$$

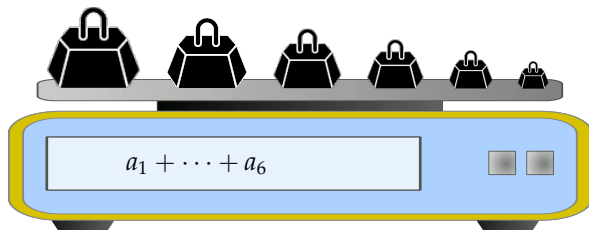
$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + \cdots + a_4$$

$$S_5 = a_1 + \cdots + a_5$$

$$S_6 = a_1 + \cdots + a_6$$



NOTATION PRACTICE

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$.

► Evaluate $\sum_{n=1}^{100} a_n$.

► Evaluate $\sum_{n=1}^{\infty} a_n$.

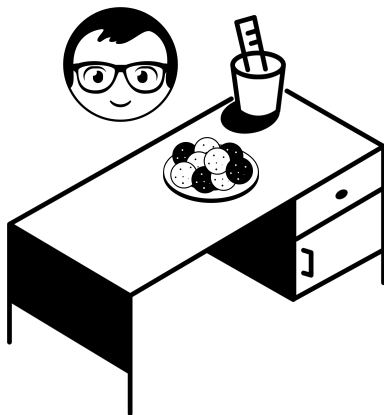
NOTATION PRACTICE

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$.

► Evaluate $\sum_{n=1}^{100} a_n$. $\sum_{n=1}^{100} a_n = S_{100} = \frac{100}{101}$

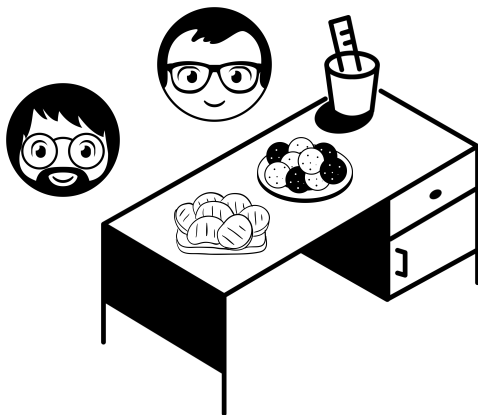
► Evaluate $\sum_{n=1}^{\infty} a_n$. $\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \frac{N}{N+1} = 1$

NOTATION PRACTICE



Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

NOTATION PRACTICE

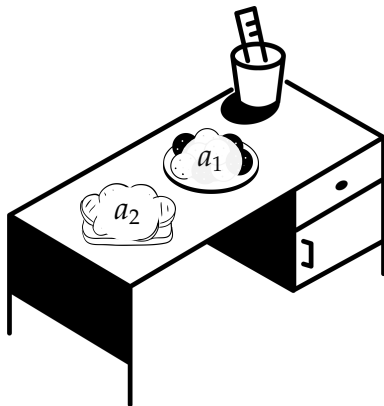


Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

NOTATION PRACTICE



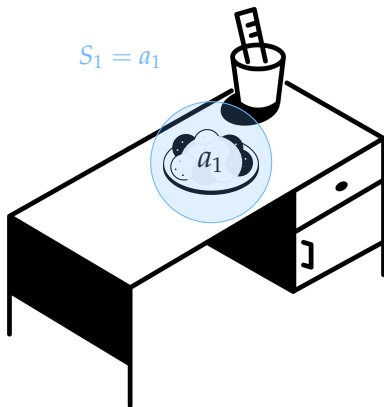
Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

Andrew brought 10, and Joel brought $19 - 10 = 9$.

NOTATION PRACTICE



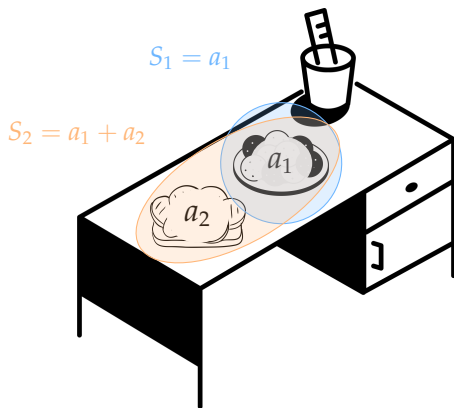
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NOTATION PRACTICE

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$.

- Find a_1 .
- Give an explicit expression for a_n , when $n > 1$.

NOTATION PRACTICE

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$.

- Find a_1 . $a_1 = \sum_{n=1}^1 a_n = S_1 = \frac{1}{2}$
- Give an explicit expression for a_n , when $n > 1$.

$$\begin{aligned} a_n &= \left(\sum_{k=1}^n a_k \right) - \left(\sum_{k=1}^{n-1} a_k \right) = S_n - S_{n-1} \\ &= \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)} \end{aligned}$$

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$$S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$$


 a_1

 a_2

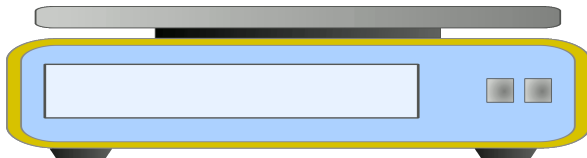
 a_3

 a_4

 a_5

 a_6

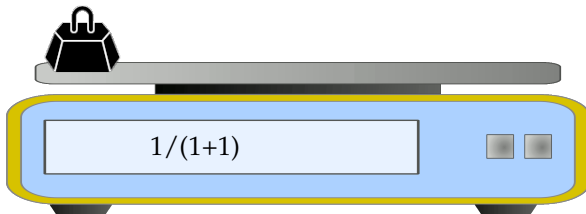
 a_7

 a_8


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$$S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$$

$$S_1 = 1/(1+1)$$

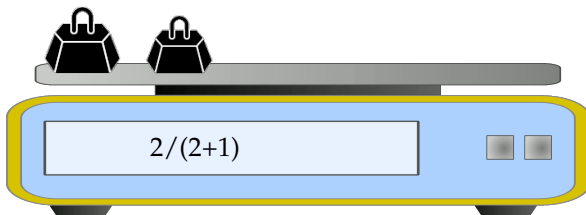


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$$S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$$

$$S_1 = 1/(1+1)$$

$$S_2 = 2/(2+1)$$



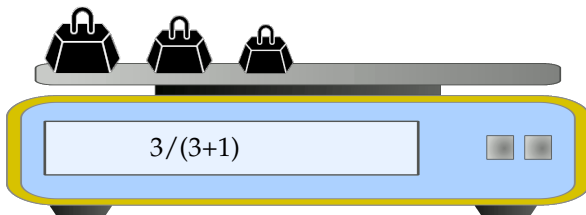
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$$S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$$

$$S_1 = 1/(1+1)$$

$$S_2 = 2/(2+1)$$

$$S_3 = 3/(3+1)$$



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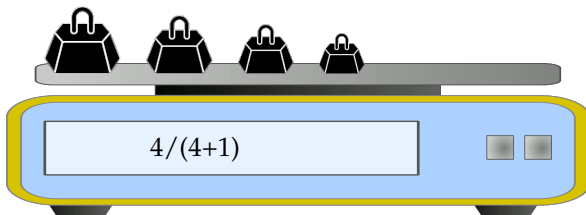
$$S_N = \sum_{n=1}^N a_n = \frac{N}{N+1}$$

$$S_1 = 1/(1+1)$$

$$S_2 = 2/(2+1)$$

$$S_3 = 3/(3+1)$$

$$S_4 = 4/(4+1)$$



Definition

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^N a_n.$$

The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \rightarrow S$ as $N \rightarrow \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Geometric Series

Let a and r be two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + ar^3 + \cdots$$

is called the **geometric series** with first term a and ratio r .

We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

Geometric Series

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We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

Another useful way of identifying geometric series is to determine whether all pairs of consecutive terms have the same ratio.

- ▶ Geometric: $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \dots$
- ▶ Geometric: $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- ▶ Not geometric: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

$$rS_N =$$

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$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

$$rS_N = \quad ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1}$$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

$$rS_N = ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1}$$

$$rS_N - S_N =$$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

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$$rS_N - S_N = -a + ar^{N+1}$$

Consider the partial sum S_N of a geometric series:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

$$rS_N = ar + ar^2 + ar^3 + \cdots + ar^N + ar^{N+1}$$

$$rS_N - S_N = -a + ar^{N+1}$$

$$S_N(r - 1) = ar^{N+1} - a$$

If $r \neq 1$, then

$$S_N = \frac{ar^{N+1} - a}{r - 1} = a \frac{r^{N+1} - 1}{r - 1}$$

Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

▶ If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a \frac{r^{N+1} - 1}{r - 1}.$

▶ If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N =$

▶ If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n =$

▶ If $r = 1$, then $\sum_{n=0}^{\infty} ar^n$

▶ If $r = -1$, then $\sum_{n=0}^{\infty} ar^n$

▶ If $|r| > 1$, then $\sum_{n=0}^{\infty} ar^n$

Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

▶ If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a \frac{r^{N+1} - 1}{r - 1}$.

▶ If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = (N + 1)a$.

▶ If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \lim_{N \rightarrow \infty} a \frac{r^{N+1} - 1}{r - 1} = a \frac{1}{1 - r}$

▶ If $r = 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges

▶ If $r = -1$, then $\sum_{n=0}^{\infty} ar^n$ diverges

▶ If $|r| > 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges

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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$



a



a



a



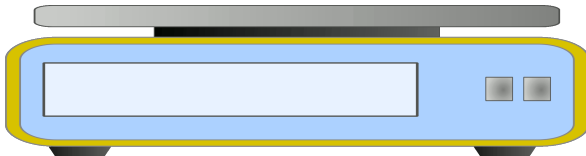
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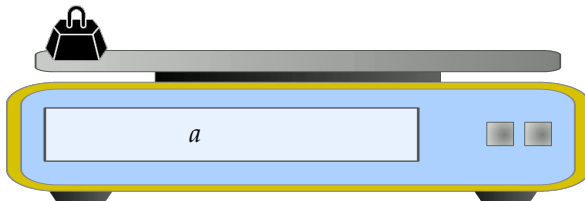
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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

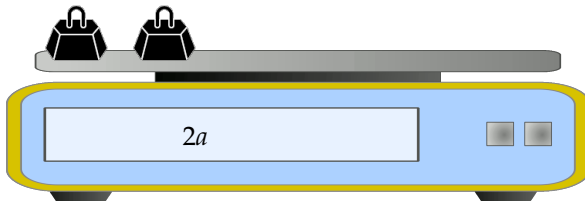
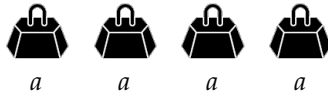


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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 2a$$



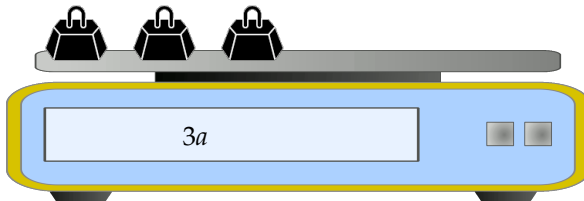
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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 2a$$

$$S_2 = 3a$$



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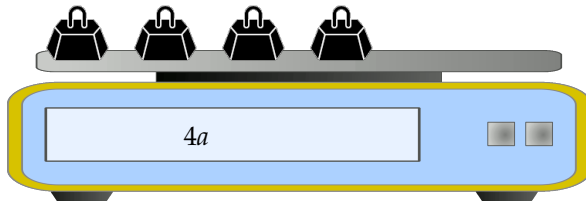
$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$



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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 2a$$

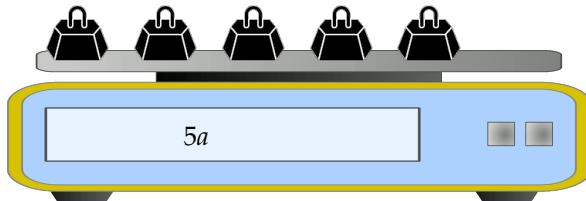
$$S_2 = 3a$$

$$S_3 = 4a$$

$$S_4 = 5a$$



a



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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$

$$S_0 = a$$

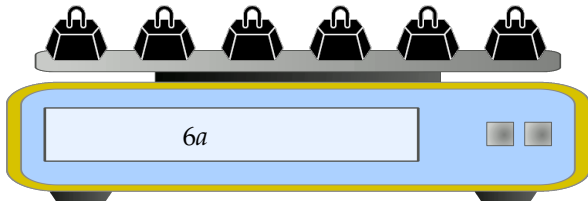
$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$

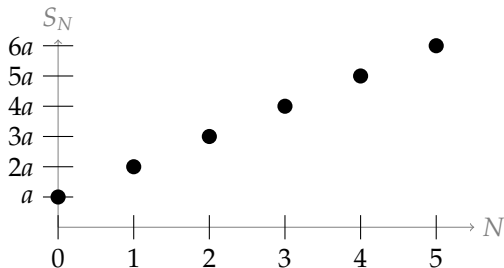
$$S_4 = 5a$$

$$S_5 = 6a$$



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$$\sum_{n=0}^{\infty} ar^n, r = 1, a \neq 0$$



$$S_0 = a$$

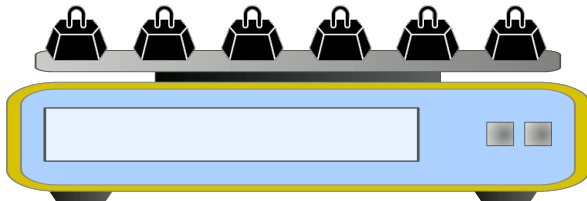
$$S_1 = 2a$$

$$S_2 = 3a$$

$$S_3 = 4a$$

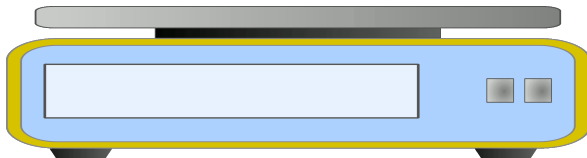
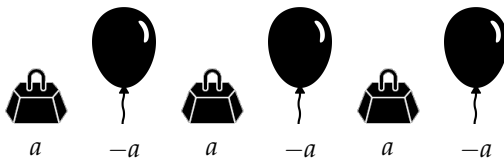
$$S_4 = 5a$$

$$S_5 = 6a$$



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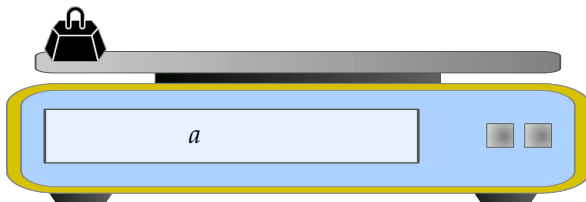
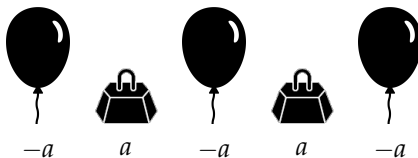
$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



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$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

$$S_0 = a$$

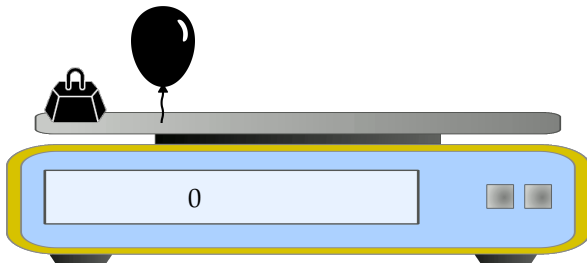
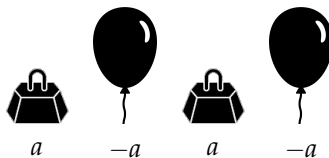


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$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 0$$



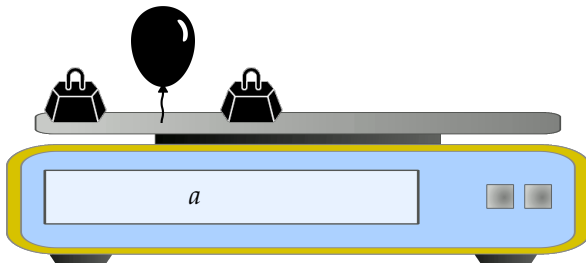
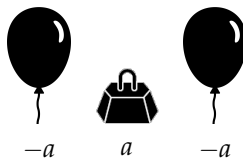
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$$S_0 = a$$

$$S_1 = 0$$

$$S_2 = a$$



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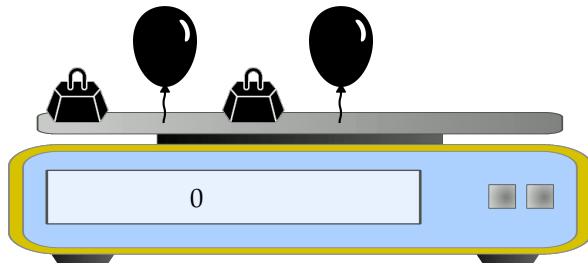
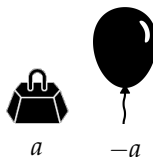
$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

$$S_0 = a$$

$$S_1 = 0$$

$$S_2 = a$$

$$S_3 = 0$$



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$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

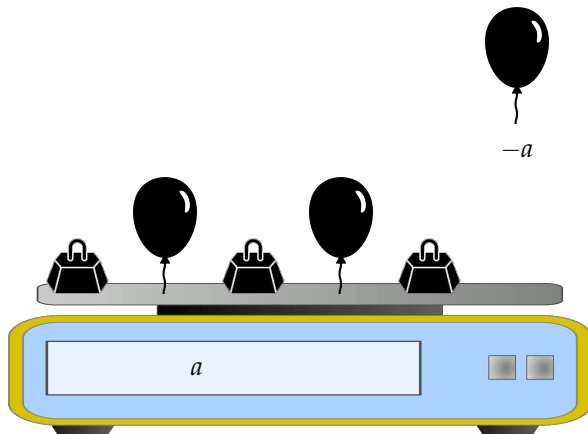
$$S_0 = a$$

$$S_1 = 0$$

$$S_2 = a$$

$$S_3 = 0$$

$$S_4 = a$$



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$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$

$$S_0 = a$$

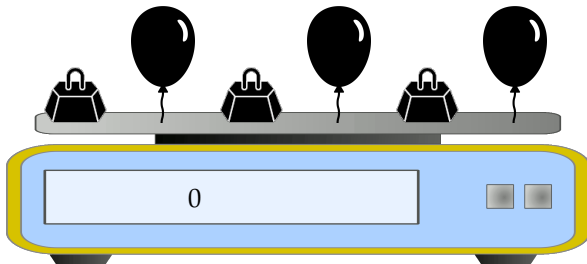
$$S_1 = 0$$

$$S_2 = a$$

$$S_3 = 0$$

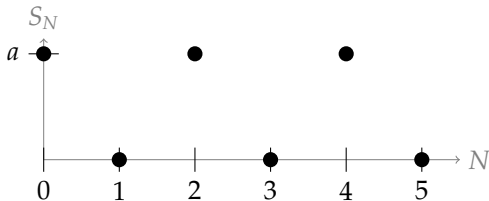
$$S_4 = a$$

$$S_5 = 0$$



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$$\sum_{n=0}^{\infty} ar^n, r = -1, a \neq 0$$



$$S_0 = a$$

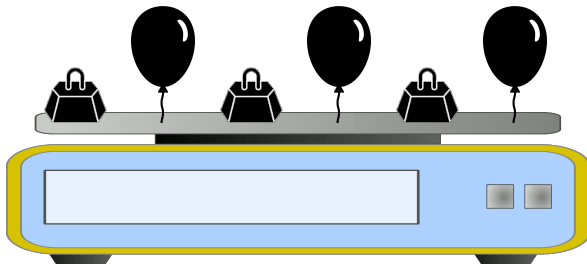
$$S_1 = 0$$

$$S_2 = a$$

$$S_3 = 0$$

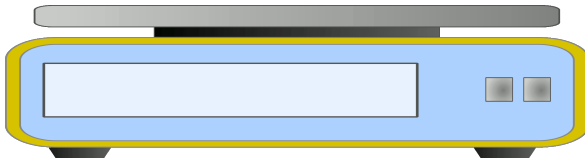
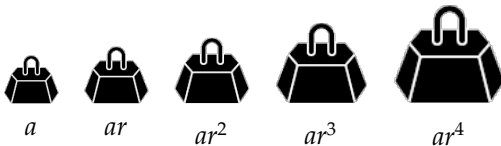
$$S_4 = a$$

$$S_5 = 0$$



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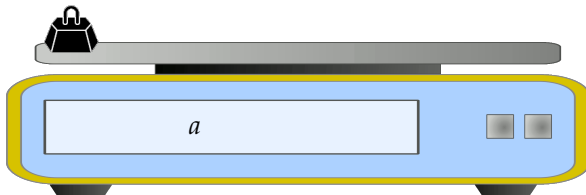
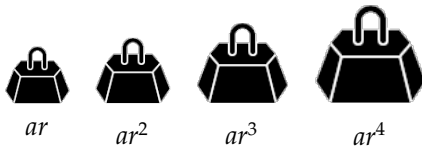
$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$



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$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

$$S_0 = a$$

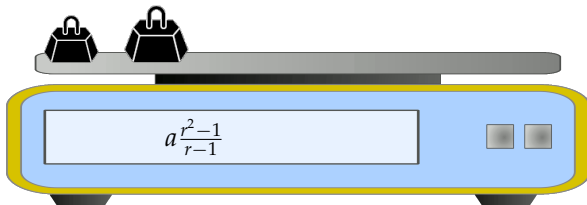
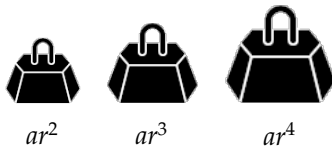


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$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

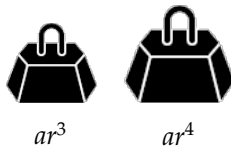
$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$



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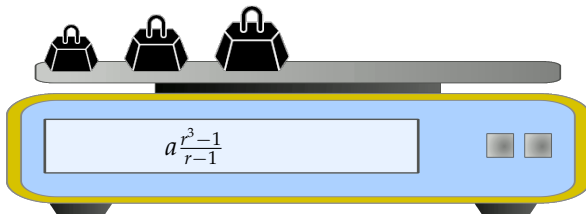
$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$



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$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

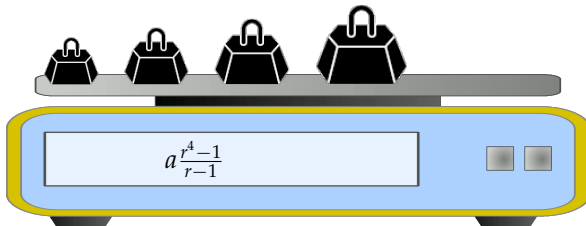

 ar^4

$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$



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$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$

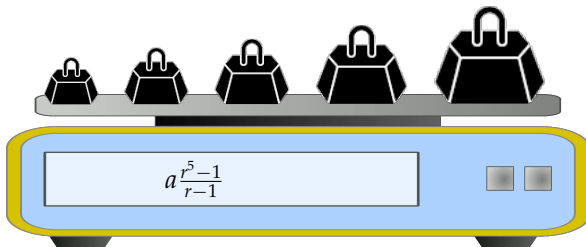
$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

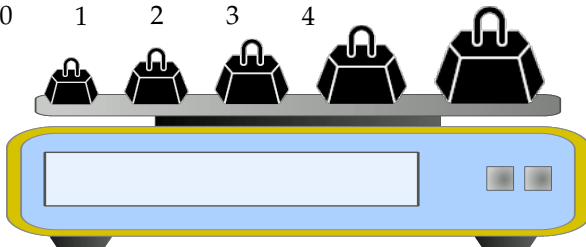
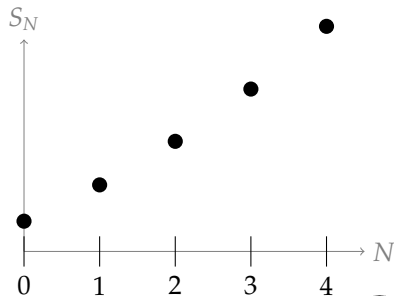
$$S_3 = a \frac{r^4 - 1}{r - 1}$$

$$S_4 = a \frac{r^5 - 1}{r - 1}$$



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$$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

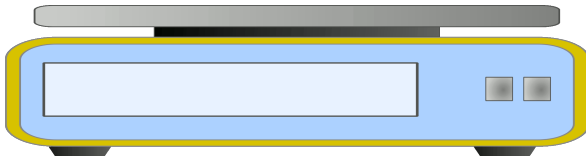
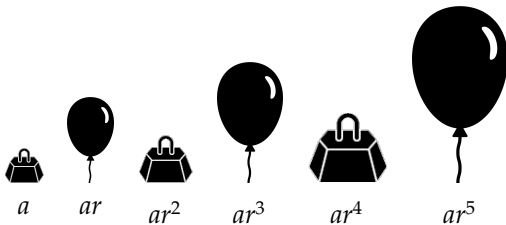
$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$

$$S_4 = a \frac{r^5 - 1}{r - 1}$$

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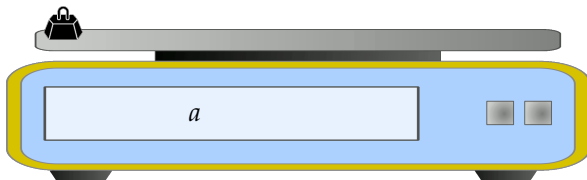
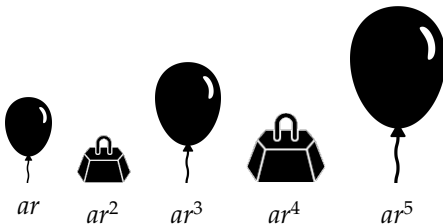
$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



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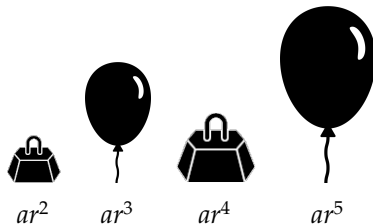
$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$

$$S_0 = a$$



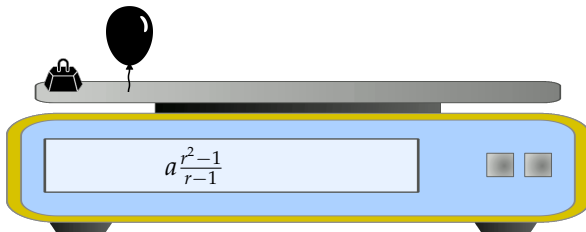
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$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



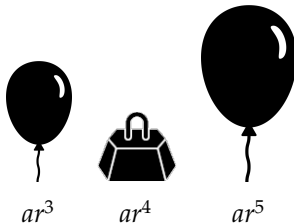
$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$



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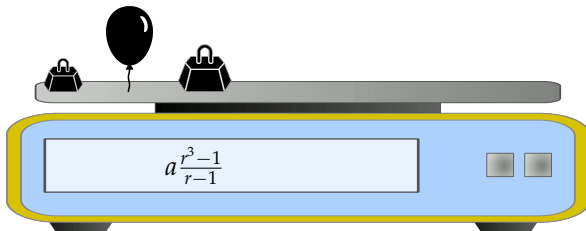
$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

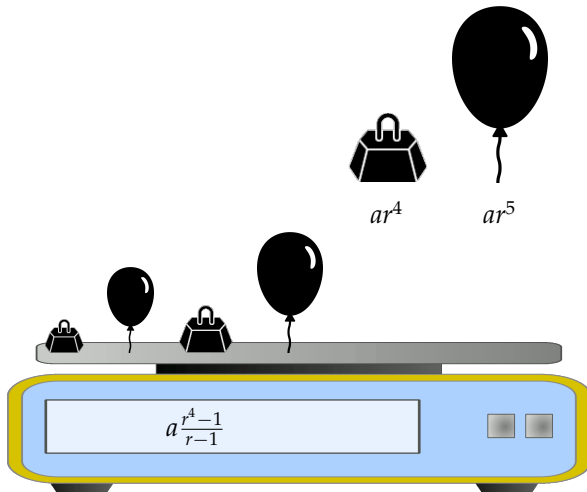
$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$



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$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

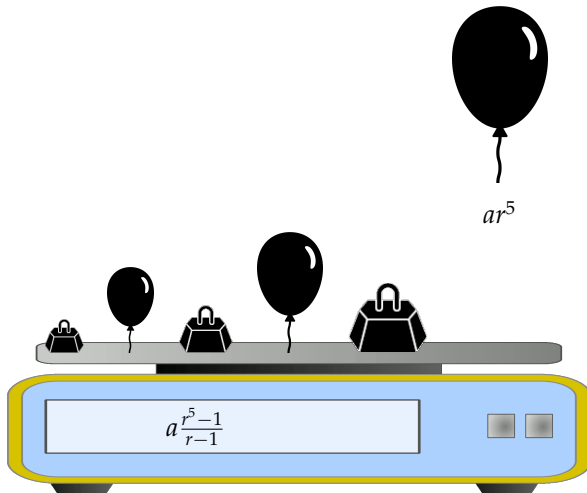
$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$

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$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

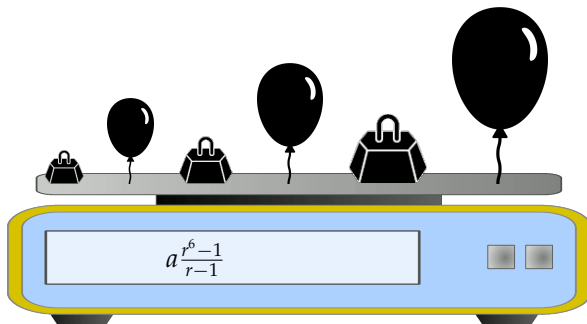
$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$

$$S_4 = a \frac{r^5 - 1}{r - 1}$$

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$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

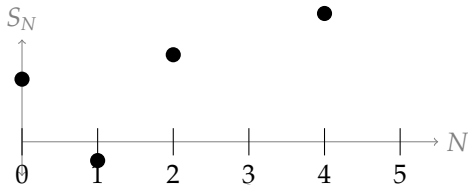
$$S_3 = a \frac{r^4 - 1}{r - 1}$$

$$S_4 = a \frac{r^5 - 1}{r - 1}$$

$$S_5 = a \frac{r^6 - 1}{r - 1}$$

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$$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$$



$$S_0 = a$$

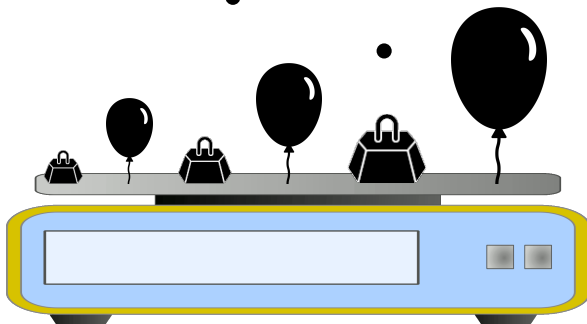
$$S_1 = a \frac{r^2 - 1}{r - 1}$$

$$S_2 = a \frac{r^3 - 1}{r - 1}$$

$$S_3 = a \frac{r^4 - 1}{r - 1}$$

$$S_4 = a \frac{r^5 - 1}{r - 1}$$

$$S_5 = a \frac{r^6 - 1}{r - 1}$$



GEOMETRIC SERIES

New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half.

- ▶ Each of the first 210,000 solutions can produce 50 bitcoins.
- ▶ Each of the next 210,000 solutions can produce $\frac{50}{2}$ bitcoins.
- ▶ Each of the next 210,000 solutions can produce $\frac{50}{2^2}$ bitcoins.
- ▶ Each of the next 210,000 solutions can produce $\frac{50}{2^3}$ bitcoins.

Assume that this continues forever, and that bitcoins are infinitely divisible.¹ How many bitcoins can possibly be produced?

¹Actually the smallest allowed division of a bitcoin is 10^{-8} .

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Assume that this continues forever, and that bitcoins are infinitely divisible.¹ How many bitcoins can possibly be produced?

We start by writing the total number of bitcoin produced as a series. Since we want to know an upper bound, we'll assume that infinite solutions can be found and used to make bitcoin.

$$210\,000(50) + 210\,000\left(\frac{50}{2}\right) + 210\,000\left(\frac{50}{2^2}\right) + \cdots = \sum_{n=0}^{\infty} (210\,000) \left(\frac{50}{2^n}\right)$$

¹Actually the smallest allowed division of a bitcoin is 10^{-8} .

GEOMETRIC SERIES

$$\sum_{n=0}^{\infty} (210\,000) \left(\frac{50}{2^n} \right) =$$

GEOMETRIC SERIES

$$\begin{aligned}
 \sum_{n=0}^{\infty} (210\,000) \left(\frac{50}{2^n} \right) &= \sum_{n=0}^{\infty} (210\,000 \cdot 50) \left(\frac{1}{2} \right)^n \\
 &= (210\,000 \cdot 50) \frac{1}{1 - \frac{1}{2}} \\
 &= (210\,000 \cdot 50)(2) \\
 &= 21\,000\,000
 \end{aligned}$$

So there will never be more than 21,000,000 bitcoins produced this way.

$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$



10 500 000



5 250 000



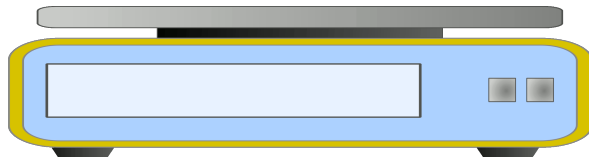
2 625 000



1 312 500



656 250



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$



5 250 000



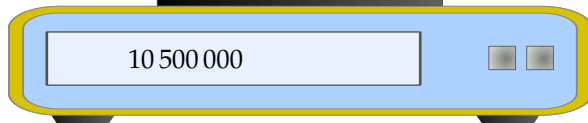
2 625 000



1 312 500



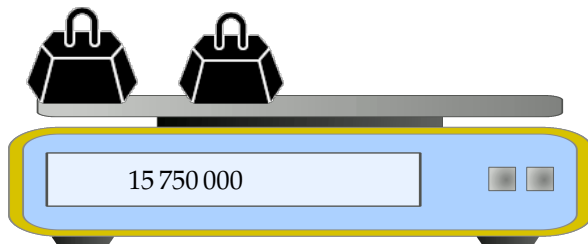
656 250



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

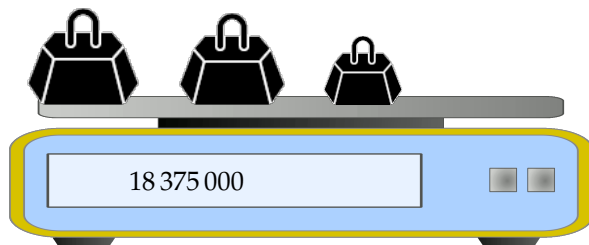


$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$

$$S_0 = 10\,500\,000$$

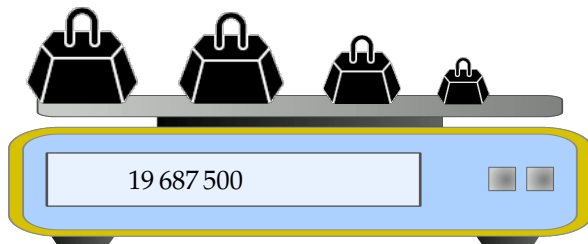
$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$



656 250



$$\sum_{n=0}^{\infty} 210\,000 \left(\frac{50}{2^n} \right) = 21\,000\,000$$

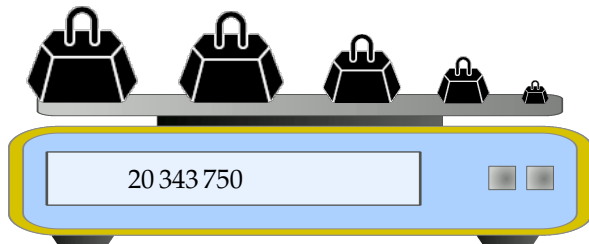
$$S_0 = 10\,500\,000$$

$$S_1 = 15\,750\,000$$

$$S_2 = 18\,375\,000$$

$$S_3 = 19\,687\,500$$

$$S_4 = 20\,343\,750$$



Arithmetic of Series

Let S , T , and C be real numbers. Let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T respectively. Then

$$\sum_{n=1}^{\infty} [a_n + b_n] = S + T$$

$$\sum_{n=1}^{\infty} [a_n - b_n] = S - T$$

$$\sum_{n=1}^{\infty} [Ca_n] = CS$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^N ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right)$

$$\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right) =$$

$$\begin{aligned}
 \sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{4}{5^n} \right) &= \sum_{n=0}^{\infty} \frac{2}{3^n} + \sum_{n=0}^{\infty} \frac{4}{5^n} \\
 &= \sum_{n=0}^{\infty} 2 \left(\frac{1}{3} \right)^n + \sum_{n=0}^{\infty} 4 \left(\frac{1}{5} \right)^n \\
 &= \frac{2}{1 - \frac{1}{3}} + \frac{4}{1 - \frac{1}{5}} \\
 &= \frac{2}{2/3} + \frac{4}{4/5} \\
 &= 3 + 5 = 8
 \end{aligned}$$

Geometric Series and Partial Sums

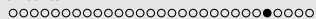
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$$\sum_{n=0}^N ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right)$



$$\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right) =$$



$$\sum_{n=6}^{\infty} \left(\frac{3^{n-1}}{5^{2n}} \right) = \sum_{n=6}^{\infty} \frac{1}{3} \left(\frac{3^n}{5^{2n}} \right) = \sum_{n=6}^{\infty} \frac{1}{3} \left(\frac{3^n}{25^n} \right) = \sum_{n=6}^{\infty} \frac{1}{3} \left(\frac{3}{25} \right)^n$$

Set $k = n - 6$. Then $n = k + 6$ and when $n = 6$, $k = 0$.

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{1}{3} \left(\frac{3}{25} \right)^{k+6} = \sum_{k=1}^{\infty} \frac{1}{3} \left(\frac{3}{25} \right)^6 \left(\frac{3}{25} \right)^k \\ &= \frac{1}{3} \left(\frac{3}{25} \right)^6 \cdot \frac{1}{1 - 3/25} = \frac{3^5}{25^6} \cdot \frac{25}{22} = \frac{3^5}{25^5 \cdot 22} \end{aligned}$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^N ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right)$

$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) =$$



$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) &= \sum_{n=0}^{\infty} \left(\frac{4^n}{3^n} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{4}{3} \right)^n\end{aligned}$$

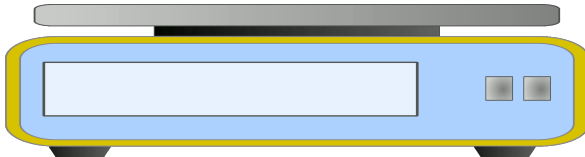
Since $\left| \frac{4}{3} \right| > 1$, the series diverges.



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

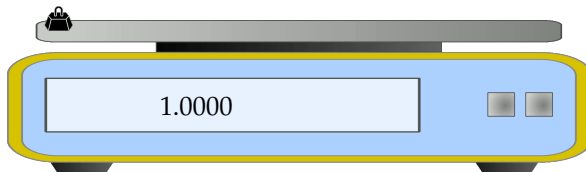
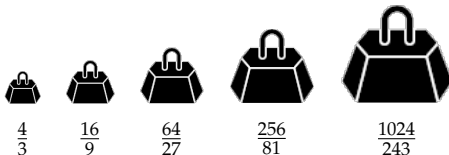


1

 $\frac{4}{3}$  $\frac{16}{9}$  $\frac{64}{27}$  $\frac{256}{81}$  $\frac{1024}{243}$ 

$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

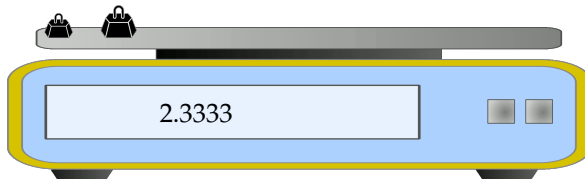
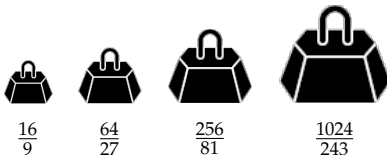
$$S_0 = 1.0000$$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$$S_0 = 1.0000$$

$$S_1 = 2.3333$$

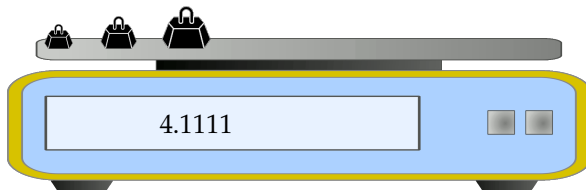
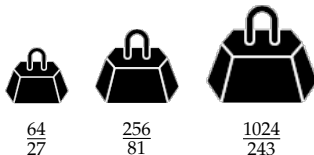


$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$$S_0 = 1.0000$$

$$S_1 = 2.3333$$

$$S_2 = 4.1111$$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$



$$\frac{256}{81}$$

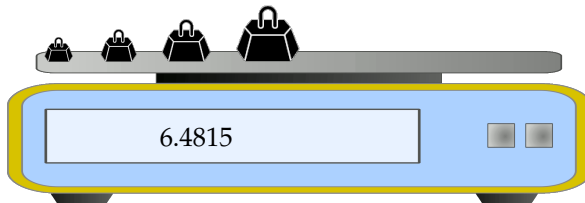
$$\frac{1024}{243}$$

$$S_0 = 1.0000$$

$$S_1 = 2.3333$$

$$S_2 = 4.1111$$

$$S_3 = 6.4815$$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$



$$\frac{1024}{243}$$

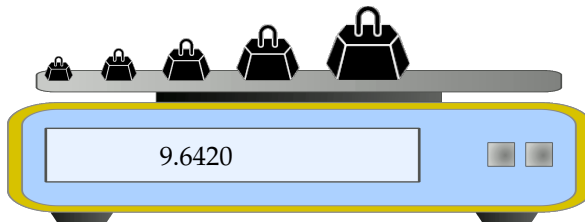
$$S_0 = 1.0000$$

$$S_1 = 2.3333$$

$$S_2 = 4.1111$$

$$S_3 = 6.4815$$

$$S_4 = 9.6420$$



$$\sum_{n=0}^{\infty} \left(\frac{2^{2n}}{3^n} \right) \text{ diverges}$$

$$S_0 = 1.0000$$

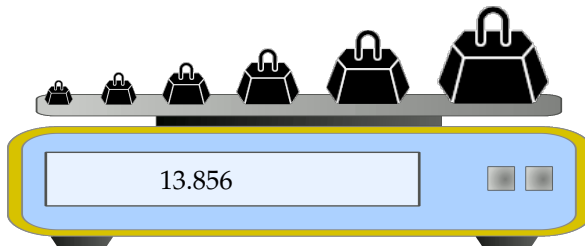
$$S_1 = 2.3333$$

$$S_2 = 4.1111$$

$$S_3 = 6.4815$$

$$S_4 = 9.6420$$

$$S_5 = 13.856$$



TELESCOPING SUMS

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right).$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$

TELESCOPING SUMS

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{cccc} a_1 : & \frac{1}{1} & - & \frac{1}{2} \\ a_2 : & \frac{1}{2} & - & \frac{1}{3} \\ a_3 : & \frac{1}{3} & - & \frac{1}{4} \\ a_4 : & \frac{1}{4} & - & \frac{1}{5} \\ \vdots & & & \\ a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\ a_N : & \frac{1}{N} & - & \frac{1}{N+1} \end{array}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$S_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

•

•

•

$$S_N =$$

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{llll} a_1 : & \frac{1}{1} & - & \frac{1}{2} \\ a_2 : & \frac{1}{2} & - & \frac{1}{3} \\ a_3 : & \frac{1}{3} & - & \frac{1}{4} \\ a_4 : & \frac{1}{4} & - & \frac{1}{5} \\ \vdots & & & \\ a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\ a_N : & \frac{1}{N} & - & \frac{1}{N+1} \end{array}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rcl} S_1 = & \frac{1}{1} & - \quad \frac{1}{2} \\ S_2 = & \frac{1}{1} & - \quad \frac{1}{3} \\ S_3 = & & \\ S_4 = & & \\ \vdots & & \\ S_N = & & \end{array}$$

TELESCOPING SUMS

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rclcl} a_1 : & \frac{1}{1} & - & \frac{1}{2} \\ a_2 : & \frac{1}{2} & - & \frac{1}{3} \\ a_3 : & \frac{1}{3} & - & \frac{1}{4} \\ a_4 : & \frac{1}{4} & - & \frac{1}{5} \\ & \vdots & & \\ a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\ a_N : & \frac{1}{N} & - & \frac{1}{N+1} \end{array}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rcl} S_1 = & \frac{1}{1} & - & \frac{1}{2} \\ S_2 = & \frac{1}{1} & - & \frac{1}{3} \\ S_3 = & \frac{1}{1} & - & \frac{1}{4} \\ S_4 = & & & \\ \vdots & & & \end{array}$$

$$S_N =$$

TELESCOPING SUMS

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rclcl} a_1 : & \frac{1}{1} & - & \frac{1}{2} \\ a_2 : & \frac{1}{2} & - & \frac{1}{3} \\ a_3 : & \frac{1}{3} & - & \frac{1}{4} \\ a_4 : & \frac{1}{4} & - & \frac{1}{5} \\ & \vdots & & \\ a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\ a_N : & \frac{1}{N} & - & \frac{1}{N+1} \end{array}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rclcl} S_1 & = & \frac{1}{1} & - & \frac{1}{2} \\ S_2 & = & \frac{1}{1} & - & \frac{1}{3} \\ S_3 & = & \frac{1}{1} & - & \frac{1}{4} \\ S_4 & = & \frac{1}{1} & - & \frac{1}{5} \\ & \vdots & & & \end{array}$$

$$S_N =$$

TELESCOPING SUMS

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rclcl} a_1 : & \frac{1}{1} & - & \frac{1}{2} \\ a_2 : & \frac{1}{2} & - & \frac{1}{3} \\ a_3 : & \frac{1}{3} & - & \frac{1}{4} \\ a_4 : & \frac{1}{4} & - & \frac{1}{5} \\ & \vdots & & \\ a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\ a_N : & \frac{1}{N} & - & \frac{1}{N+1} \end{array}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rcl} S_1 = & \frac{1}{1} & - \frac{1}{2} \\ S_2 = & \frac{1}{1} & - \frac{1}{3} \\ S_3 = & \frac{1}{1} & - \frac{1}{4} \\ S_4 = & \frac{1}{1} & - \frac{1}{5} \\ & \vdots & \end{array}$$

$$S_N =$$

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rclcl} a_1 : & \frac{1}{1} & - & \frac{1}{2} \\ a_2 : & \frac{1}{2} & - & \frac{1}{3} \\ a_3 : & \frac{1}{3} & - & \frac{1}{4} \\ a_4 : & \frac{1}{4} & - & \frac{1}{5} \\ \vdots & & & \\ a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\ a_N : & \frac{1}{N} & - & \frac{1}{N+1} \end{array}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

$$\begin{array}{rclcl} S_1 & = & \frac{1}{1} & - & \frac{1}{2} \\ S_2 & = & \frac{1}{1} & - & \frac{1}{3} \\ S_3 & = & \frac{1}{1} & - & \frac{1}{4} \\ S_4 & = & \frac{1}{1} & - & \frac{1}{5} \\ & \vdots & & & \\ S_N & = & \frac{1}{1} & - & \frac{1}{N+1} = \frac{N}{N+1} \end{array}$$

TELESCOPING SUMS

Evaluate $\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right).$

$$\begin{array}{rclcl}
 a_1 : & \frac{1}{1} & - & \frac{1}{2} \\
 a_2 : & \frac{1}{2} & - & \frac{1}{3} \\
 a_3 : & \frac{1}{3} & - & \frac{1}{4} \\
 a_4 : & \frac{1}{4} & - & \frac{1}{5} \\
 & \vdots & & \\
 a_{N-1} : & \frac{1}{N-1} & - & \frac{1}{N} \\
 a_N : & \frac{1}{N} & - & \frac{1}{N+1}
 \end{array}$$

$$\sum_{n=1}^{800} \left(\frac{1}{n} - \frac{1}{n+1} \right) = S_{800} = \frac{800}{801}$$

Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$

$$\begin{array}{rclcl}
 S_1 = & \frac{1}{1} & - & \frac{1}{2} \\
 S_2 = & \frac{1}{1} & - & \frac{1}{3} \\
 S_3 = & \frac{1}{1} & - & \frac{1}{4} \\
 S_4 = & \frac{1}{1} & - & \frac{1}{5} \\
 & \vdots & & \\
 S_N = & \frac{1}{1} & - & \frac{1}{N+1} = \frac{N}{N+1}
 \end{array}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{N \rightarrow \infty} S_N = 1$$

Evaluate $\sum_{n=1}^{1000} \log \left(\frac{n+1}{n} \right).$

Evaluate $\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right).$

Evaluate $\sum_{n=1}^{1000} \log \left(\frac{n+1}{n} \right).$

$$a_1 : \log(2) - \log(1)$$

$$a_2 : \log(3) - \log(2)$$

$$a_3 : \log(4) - \log(3)$$

$$\vdots$$

$$a_{n-1} : \log(n) - \log(n-1)$$

$$a_n : \log(n+1) - \log(n)$$

Evaluate $\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right).$

$$S_1 = \log(2)$$

$$S_2 = \log(3)$$

$$S_3 = \log(4)$$

$$S_n = \log(n+1)$$

So, $\sum_{n=1}^{1000} \log \left(\frac{n+1}{n} \right) = S_{1000} = \log(1001)$ and

$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) = \lim_{n \rightarrow \infty} \log(n+1) = \infty$$

$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$


 $\log 2$

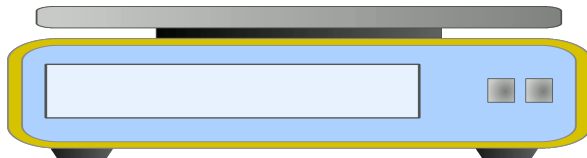
 $\log \frac{3}{2}$

 $\log \frac{4}{3}$

 $\log \frac{5}{4}$

 $\log \frac{6}{5}$

 $\log \frac{7}{6}$

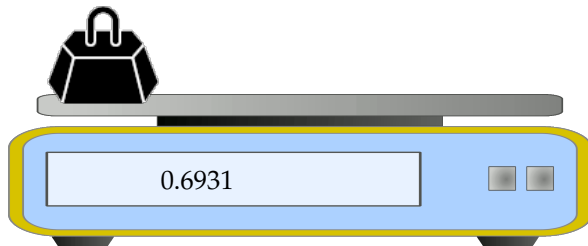
 $\log \frac{8}{7}$


$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

$$S_1 = 0.6931$$




$$\log \frac{3}{2} \quad \log \frac{4}{3} \quad \log \frac{5}{4} \quad \log \frac{6}{5} \quad \log \frac{7}{6} \quad \log \frac{8}{7}$$



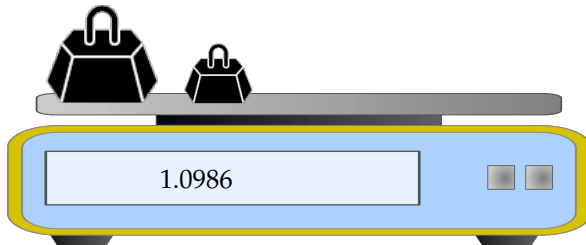
$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$



$$\log \frac{4}{3} \quad \log \frac{5}{4} \quad \log \frac{6}{5} \quad \log \frac{7}{6} \quad \log \frac{8}{7}$$



$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

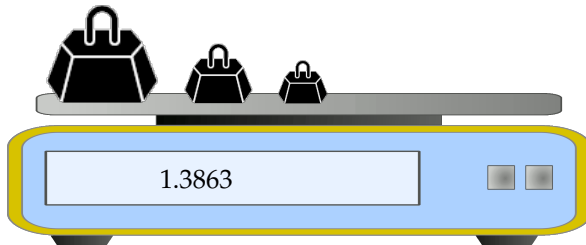
$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

$$S_3 = 1.3863$$



$$\log \frac{5}{4} \quad \log \frac{6}{5} \quad \log \frac{7}{6} \quad \log \frac{8}{7}$$



$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

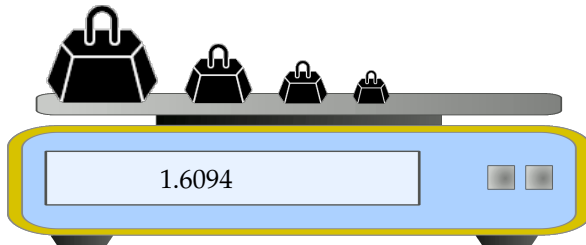
$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

$$S_4 = 1.6094$$

$$\begin{array}{ccc} \text{[Weight Icon]} & \text{[Weight Icon]} & \text{[Weight Icon]} \\ \log \frac{6}{5} & \log \frac{7}{6} & \log \frac{8}{7} \end{array}$$



$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

$$S_1 = 0.6931$$

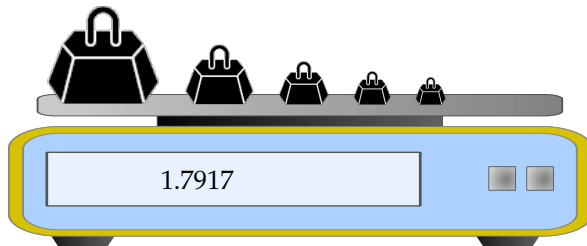
$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

$$S_4 = 1.6094$$

$$S_5 = 1.7917$$

$$\log \frac{7}{6} \quad \log \frac{8}{7}$$



$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

$$S_3 = 1.3863$$

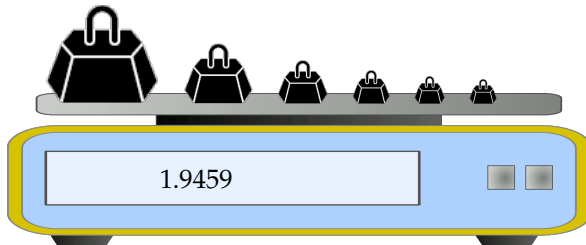
$$S_4 = 1.6094$$

$$S_5 = 1.7917$$

$$S_6 = 1.9459$$



$$\log \frac{8}{7}$$



$$\sum_{n=1}^{\infty} \log \left(\frac{n+1}{n} \right) \text{ diverges}$$

$$S_1 = 0.6931$$

$$S_2 = 1.0986$$

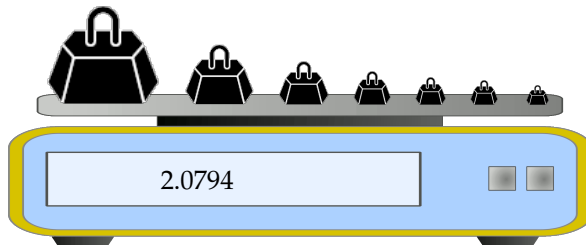
$$S_3 = 1.3863$$

$$S_4 = 1.6094$$


$$S_5 = 1.7917$$


$$S_6 = 1.9459$$


$$S_7 = 2.0794$$




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