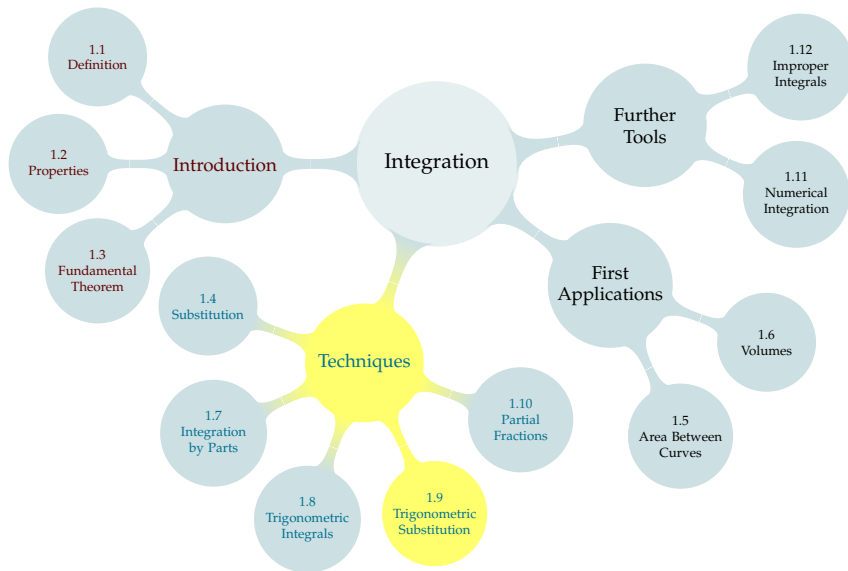


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WARMUP

Evaluate $\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}} dx$.

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$$\begin{aligned}\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}} dx &= \int_3^7 \frac{1}{\sqrt{(x+1)^2}} dx \\ &= \int_3^7 \frac{1}{|x+1|} dx\end{aligned}$$

When $3 \leq x \leq 7$, we have $|x+1| = x+1$.

$$\begin{aligned}&= \int_3^7 \frac{1}{x+1} dx \\ &= [\log |x+1|]_3^7 \\ &= \log 8 - \log 4 = \log 2\end{aligned}$$

Idea: $\sqrt{(\text{something})^2} = |\text{something}|$. We cancelled off the square root.

Evaluate $\int \frac{1}{\sqrt{x^2 + 1}} dx$.

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We still want to cancel off the square root, but $x^2 + 1$ is not obviously of the form (something)².

Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$.

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 1}} dx &= \int \frac{1}{\sqrt{\tan^2 \theta + 1}} \sec^2 \theta d\theta = \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}} d\theta \\ &= \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + C \end{aligned}$$

We need to get these back in terms of x . From our substitution, we know $\tan \theta = x$. From simplifying our denominator, we also know $\sec \theta = \sqrt{x^2 + 1}$.

$$= \log \left| \sqrt{x^2 + 1} + x \right| + C$$

Same idea: $\sqrt{(\text{something})^2} = |\text{something}|$; cancel off the square root.

CHECK OUR WORK

Let's verify that $\int \frac{1}{\sqrt{x^2+1}} = \log \left| \sqrt{x^2+1} + x \right| + C.$

Seems improbable, right?

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Seems improbable, right?

$$\begin{aligned} \frac{d}{dx} \left[\log \left| \sqrt{x^2+1} + x \right| + C \right] &= \frac{1}{\sqrt{x^2+1} + x} \cdot \left(\frac{2x}{2\sqrt{x^2+1}} + 1 \right) \\ &= \frac{x + \sqrt{x^2+1}}{(\sqrt{x^2+1} + x)\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

So, our answer works!

METHOD (ONE STANDARD CASE)

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 - ▶ $x = \sec \theta$, $\sec^2 \theta - 1 = \tan^2 \theta$ changes $\sqrt{x^2 - 1}$ into $\sqrt{\tan^2 \theta} = |\tan \theta|$
- ▶ After integrating, convert back to the original variable (possibly using a triangle—more details later)

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

▶ $\sqrt{x^2 - 1}$

▶ $\sqrt{x^2 + 1}$

▶ $\sqrt{1 - x^2}$

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Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

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Let $x = \tan \theta$, so $\sqrt{x^2 + 1}$ becomes $\sqrt{\tan^2 \theta + 1} = \sqrt{\sec^2 \theta} = |\sec \theta|$

► $\sqrt{1 - x^2}$

Let $x = \sin \theta$ so $\sqrt{1 - x^2}$ becomes $\sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta|$
(Alternately, $x = \cos \theta$ works as well)

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

► $\sqrt{x^2 + 7}$

► $\sqrt{3 - 2x^2}$

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

► $\sqrt{x^2 + 7}$

Adjust a given identity by multiplying both sides by 7:

$7 \tan^2 \theta + 7 = 7 \sec^2 \theta$. Now we see we want $x^2 = 7 \tan^2 \theta$. That is,

$x = \sqrt{7} \tan \theta$:

$$\sqrt{x^2 + 7} = \sqrt{7 \tan^2 \theta + 7} = \sqrt{7(\sec^2 \theta)} = \sqrt{7} |\sec \theta|$$

► $\sqrt{3 - 2x^2}$

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

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$$\sqrt{x^2 + 7} = \sqrt{7 \tan^2 \theta + 7} = \sqrt{7(\sec^2 \theta)} = \sqrt{7} |\sec \theta|$$

► $\sqrt{3 - 2x^2}$

Adjust a given identity by multiplying both sides by 3:

$3 - 3 \sin^2 \theta = 3 \cos^2 \theta$. Now we see we want $2x^2 = 3 \sin^2 \theta$, so

$x = \sqrt{\frac{3}{2}} \sin \theta$:

$$\sqrt{3 - 2x^2} = \sqrt{3 - 2 \left(\frac{3}{2} \sin^2 \theta \right)} = \sqrt{3 - 3 \sin^2 \theta} = \sqrt{3 \cos^2 \theta} = \sqrt{3} |\cos \theta|$$

CLOSER LOOK AT ABSOLUTE VALUES

▶ SKIP CLOSER LOOK

Consider the substitution $x = \sin \theta$, $dx = \cos \theta \, d\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2} \, dx$$

When $x = 0$ (lower limit of integration), what is θ ?

When $x = 1$ (upper limit of integration), what is θ ?

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When $x = 0$ (lower limit of integration), what is θ ?

When $x = 1$ (upper limit of integration), what is θ ?

If $x = 0$, then $\sin \theta = 0$, but there are infinitely many values of θ that could make this true. To use the substitution $x = \sin \theta$, we need the function $x = \sin \theta$ to be invertible. That way, we can unambiguously convert between x and θ . With that in mind, we'll actually set $\theta = \arcsin x$. Now θ is restricted to the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} \int_0^1 \sqrt{1-x^2} \, dx &= \int_{\arcsin 0}^{\arcsin 1} \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 \theta} \cdot \cos \theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} |\cos \theta| \cdot \cos \theta \, d\theta \end{aligned}$$

For $0 \leq \theta \leq \frac{\pi}{2}$, we have $\cos \theta \geq 0$, so $|\cos \theta| = \cos \theta$.

CLOSER LOOK AT ABSOLUTE VALUES

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More generally, suppose a is a positive constant and we use the substitution $x = a \sin \theta$ for the term $\sqrt{a^2 - x^2}$.

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More generally, suppose a is a positive constant and we use the substitution $x = a \sin \theta$ for the term $\sqrt{a^2 - x^2}$.

► $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

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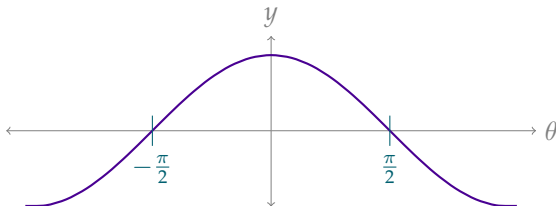
- ▶ $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
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- ▶ On the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos \theta \geq 0$, so $|\cos \theta| = \cos \theta$

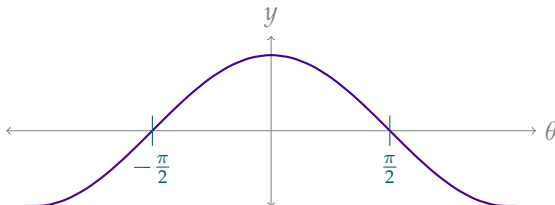


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- ▶ So, in general, when we use the substitution $x = \sin \theta$ with trigonometric substitution, we can expect $|\cos \theta| = \cos \theta$.

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Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

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Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

▶ $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

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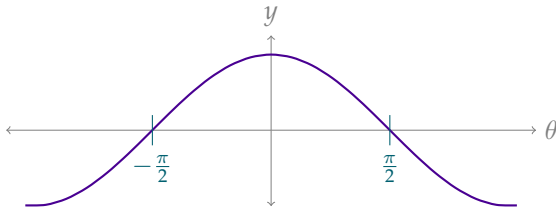
▶ $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = \sqrt{a^2 \sec^2 \theta} = \frac{a}{|\cos \theta|}$

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Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

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- ▶ On the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos \theta \geq 0$, so $|\cos \theta| = \cos \theta$ and $|\sec \theta| = \sec \theta$.

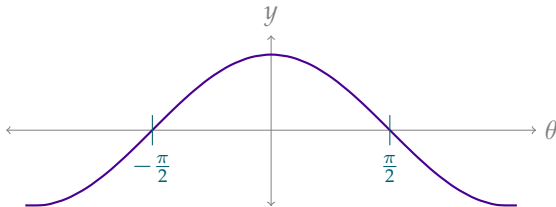


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Finally, consider the substitution $x = a \sec \theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

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Finally, consider the substitution $x = a \sec \theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

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- ▶ $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$

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- ▶ $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$
- ▶ Now this case gets slightly more complicated than the others:

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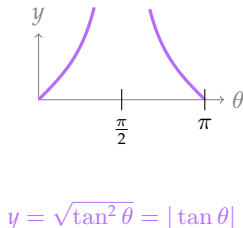
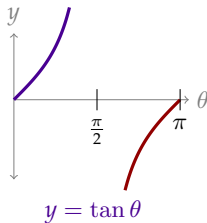
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- ▶ $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$
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 - ▶ For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \geq a^2$. I.e. $x \geq a$ or $x \leq -a$.
 - ▶ When $x \geq a$, we have $0 \leq \theta < \frac{\pi}{2}$, $\tan \theta \geq 0$, so $|\tan \theta| = \tan \theta$.
 - ▶ When $x \leq -a$, we have $\frac{\pi}{2} < \theta \leq \pi$, $\tan \theta < 0$, so $|\tan \theta| = -\tan \theta$.



ABSOLUTE VALUES

From now on, we will assume:

- ▶ With the substitution $x = a \sin \theta$ for $\sqrt{a^2 - x^2}$, $|\cos \theta| = \cos \theta$
- ▶ With the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, $|\sec \theta| = \sec \theta$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\int_0^1 (1 + x^2)^{-3/2} dx$

Evaluate $\int_0^1 (1+x^2)^{-3/2} dx$

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Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$. When $x = 0$, then $\theta = \arctan 0 = 0$;
when $x = 1$, then $\theta = \arctan 1 = \frac{\pi}{4}$.

$$\begin{aligned} \int_0^1 (1+x^2)^{-3/2} dx &= \int_{\theta=0}^{\theta=\pi/4} \frac{1}{\sqrt{1+\tan^2 \theta}^3} \sec^2 \theta d\theta \\ &= \int_0^{\pi/4} \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}^3} d\theta = \int_0^{\pi/4} \frac{\sec^2 \theta}{|\sec \theta|^3} d\theta \\ &= \int_0^{\pi/4} \frac{1}{|\sec \theta|} d\theta = \int_0^{\pi/4} |\cos \theta| d\theta \end{aligned}$$

Given our previous investigation,

$$\begin{aligned} &= \int_0^{\pi/4} \cos \theta d\theta = [\sin \theta]_0^{\pi/4} \\ &= \sin \frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}} \end{aligned}$$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\int \sqrt{1 - 4x^2} \, dx$

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Under the square root, we have “one minus a term with a variable,” which matches the identity $1 - \sin^2 \theta$. So, we want $4x^2$ to become $\sin^2 \theta$. That is, $x = \frac{1}{2} \sin \theta$. Then $dx = \frac{1}{2} \cos \theta \, d\theta$.

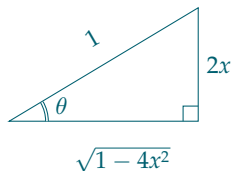
$$\begin{aligned} \int \sqrt{1 - 4x^2} \, dx &= \int \sqrt{1 - 4 \left(\frac{1}{2} \sin \theta \right)^2} \cdot \frac{1}{2} \cos \theta \, d\theta \\ &= \frac{1}{2} \int \sqrt{1 - \sin^2 \theta} \cdot \cos \theta \, d\theta = \frac{1}{2} \int \sqrt{\cos^2 \theta} \cdot \cos \theta \, d\theta \\ &= \frac{1}{2} \int |\cos \theta| \cdot \cos \theta \, d\theta = \frac{1}{2} \int \cos^2 \theta \, d\theta \\ &= \frac{1}{2} \int \left(\frac{1 + \cos(2\theta)}{2} \right) d\theta = \frac{1}{4} \int (1 + \cos(2\theta)) d\theta \\ &= \frac{1}{4} \left[\theta + \frac{1}{2} \sin(2\theta) \right] + C = \frac{1}{4} [\theta + \sin \theta \cos \theta] + C \end{aligned}$$

It remains to convert θ back into x .

Evaluate $\int \sqrt{1-4x^2} \, dx$

The substitution $x = \frac{1}{2} \sin \theta$ tells us $\sin \theta = 2x$. This in turn gives us $\theta = \arcsin(2x)$. We should still convert $\cos \theta$ back into terms of x . You might notice in the calculation we did that $\sqrt{1-4x^2}$ turned into $\cos \theta$, so $\cos \theta = \sqrt{1-4x^2}$.

Alternately, to find $\cos \theta$ in terms of x , we can use a triangle. From $\sin \theta = 2x$, and the understanding that $\sin \theta$ is the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ for a right triangle with angle θ , we can set up a triangle whose opposite side has length $2x$, and hypotenuse has length 1.



The Pythagorean theorem tells us the side adjacent to θ has length

$$\sqrt{1-4x^2}. \text{ So } \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{1-4x^2}.$$

$$\int \sqrt{1-4x^2} \, dx = \frac{1}{4} \left(\underbrace{\arcsin(2x)}_{\theta} + \underbrace{2x\sqrt{1-4x^2}}_{\sin \theta \cos \theta} \right) + C$$

CHECK OUR WORK

In the last example, we computed

$$\int \sqrt{1-4x^2} \, dx = \frac{1}{4} (\arcsin(2x) + 2x\sqrt{1-4x^2}) + C.$$

To check, we differentiate.

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$$\int \sqrt{1-4x^2} \, dx = \frac{1}{4} (\arcsin(2x) + 2x\sqrt{1-4x^2}) + C.$$

To check, we differentiate.

$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{1}{4} (\arcsin(2x) + 2x\sqrt{1-4x^2}) + C \right\} \\ &= \frac{1}{4} \left(\frac{2}{\sqrt{1-(2x)^2}} + 2x \frac{-8x}{2\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} \right) \\ &= \frac{1}{4} \left(\frac{2}{\sqrt{1-4x^2}} - \frac{8x^2}{\sqrt{1-4x^2}} + \frac{2(1-4x^2)}{\sqrt{1-4x^2}} \right) \\ &= \frac{1}{4} \left(\frac{2-8x^2+2-8x^2}{\sqrt{1-4x^2}} \right) = \frac{1-4x^2}{\sqrt{1-4x^2}} = \sqrt{1-4x^2} \quad \checkmark \end{aligned}$$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

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Evaluate $\int \frac{1}{\sqrt{x^2 - 1}} dx$

Evaluate $\int \frac{1}{\sqrt{x^2 - 1}} dx$

We use the substitution $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$.

To make the substitution work, we're actually taking $\theta = \arccos\left(\frac{1}{x}\right)$, and so $0 \leq \theta \leq \pi$.

Note that the integrand exists on the intervals $x < -1$ and $x > 1$.

- ▶ When $x > 1$, then $0 < \frac{1}{x} < 1$, so $0 < \arccos\left(\frac{1}{x}\right) < \frac{\pi}{2}$.
That is, $0 < \theta < \frac{\pi}{2}$, so $|\tan \theta| = \tan \theta$.
- ▶ When $x < -1$, then $-1 < \frac{1}{x} < 0$, so $\frac{\pi}{2} < \arccos\left(\frac{1}{x}\right) < \pi$.
That is, $\frac{\pi}{2} < \theta < \pi$, so $|\tan \theta| = -\tan \theta$.

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 1}} dx &= \int \frac{1}{\sqrt{\sec^2 \theta - 1}} \cdot \sec \theta \tan \theta d\theta = \int \frac{\sec \theta \tan \theta}{\sqrt{\tan^2 \theta}} d\theta \\ &= \int \sec \theta \left(\frac{\tan \theta}{|\tan \theta|} \right) d\theta = \begin{cases} \int \sec \theta d\theta & 0 < \theta < \frac{\pi}{2} \\ -\int \sec \theta d\theta & \frac{\pi}{2} < \theta < \pi \end{cases} \\ &= \begin{cases} \log |\sec \theta + \tan \theta| + C & 0 < \theta < \frac{\pi}{2} \\ -\log |\sec \theta + \tan \theta| + C & \frac{\pi}{2} < \theta < \pi \end{cases} \end{aligned}$$

Our substitution tells us $\sec \theta = x$. We saw from the denominator of our integrand that $\sqrt{x^2 - 1} = |\tan \theta|$.

- ▶ When $0 < \theta < \frac{\pi}{2}$, $\tan \theta = |\tan \theta| = \sqrt{x^2 - 1}$
- ▶ When $\frac{\pi}{2} < \theta < \pi$, $\tan \theta = -|\tan \theta| = -\sqrt{x^2 - 1}$

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \begin{cases} \log |x + \sqrt{x^2 - 1}| + C & x > 1 \\ -\log |x - \sqrt{x^2 - 1}| + C & x < -1 \end{cases}$$

Although the two branches look different, they are actually equivalent. Remember $-\log(A) = \log(A^{-1}) = \log\left(\frac{1}{A}\right)$:

$$\begin{aligned} -\log |x - \sqrt{x^2 - 1}| &= \log \left| \frac{1}{x - \sqrt{x^2 - 1}} \right| = \log \left| \frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} \right| \\ &= \log \left| \frac{x + \sqrt{x^2 - 1}}{x^2 - x^2 + 1} \right| = \log |x + \sqrt{x^2 - 1}| \end{aligned}$$

So,

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \log |x + \sqrt{x^2 - 1}| + C$$

CHECK OUR WORK

Let's check our result, $\int \frac{1}{\sqrt{x^2 - 1}} dx = \log \left| x + \sqrt{x^2 - 1} \right| + C.$

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Let's check our result, $\int \frac{1}{\sqrt{x^2-1}} dx = \log \left| x + \sqrt{x^2-1} \right| + C.$

$$\begin{aligned} \frac{d}{dx} \left\{ \log \left| x + \sqrt{x^2-1} \right| + C \right\} &= \frac{1 + \frac{2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} \\ &= \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} \left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} \right) = \frac{(\sqrt{x^2-1} + x)}{(x + \sqrt{x^2-1}) \sqrt{x^2-1}} \\ &= \frac{1}{\sqrt{x^2-1}} \end{aligned}$$

So, our answer works.

COMPLETING THE SQUARE

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two “parts” that turn into one part:

▶ $1 - \sin^2 \theta = \cos^2 \theta$

▶ $1 + \tan^2 \theta = \sec^2 \theta$

▶ $\sec^2 \theta - 1 = \tan^2 \theta$

But our quadratic expression has *three* parts.

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$$\blacktriangleright 1 - \sin^2 \theta = \cos^2 \theta \qquad 4 - 4 \sin^2 \theta = 4 \cos^2 \theta$$

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$$\text{Fact: } 3 - x^2 + 2x = 4 - (x - 1)^2$$

$$\sqrt{3 - x^2 + 2x} = \sqrt{4 - (x - 1)^2}$$

We want $(x - 1)^2 = 4 \sin^2 \theta$, so let $(x - 1) = 2 \sin \theta$

$$= \sqrt{4 - 4 \sin^2 \theta} = \sqrt{4 \cos^2 \theta} = 2 \cos \theta$$

COMPLETING THE SQUARE

$$(x + b)^2 = x^2 + 2bx + b^2$$

$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x + b)^2$ for constants b, c .

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1. Find b :

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$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x + b)^2$ for constants b, c .

1. Find b : $-2bx = 2x$, so $b = -1$

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2. Solve for c :

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2. Solve for c : $3 = c - b^2 = c - 1$, so $c = 4$

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Write $3 - x^2 + 2x$ in the form $c - (x + b)^2$ for constants b, c .

1. Find b : $-2bx = 2x$, so $b = -1$

2. Solve for c : $3 = c - b^2 = c - 1$, so $c = 4$

3. All together: $3 - x^2 + 2x = 4 - (x - 1)^2$

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx$.

Identities have two “parts” that turn into one part:

- ▶ $1 - \sin^2 \theta = \cos^2 \theta$
- ▶ $1 + \tan^2 \theta = \sec^2 \theta$
- ▶ $\sec^2 \theta - 1 = \tan^2 \theta$

One of those parts is a constant, and one is squared.

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx$.

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One of those parts is a constant, and one is squared.

Write $6x - x^2$ as $c - (x + b)^2$.

$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

$$6x = -2bx \implies b = -3$$

$$0 = c - b^2 = c - 9 \implies c = 9$$

$$6x - x^2 = 9 - (x - 3)^2$$

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx = \int \frac{(x - 3)^2}{\sqrt{9 - (x - 3)^2}} dx.$

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx = \int \frac{(x-3)^2}{\sqrt{9 - (x-3)^2}} dx.$

We use the identity $9 - 9\sin^2 \theta = 9\cos^2 \theta.$

We want $(x-3)^2 = 9\sin^2 \theta$, so take $(x-3) = 3\sin \theta$, $dx = 3\cos \theta d\theta.$

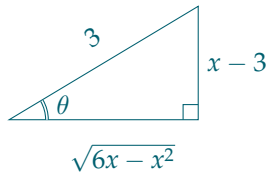
$$\int \frac{(x-3)^2}{\sqrt{9 - (x-3)^2}} dx = \int \frac{9\sin^2 \theta}{\sqrt{9 - 9\sin^2 \theta}} 3\cos \theta d\theta$$

$$= \int \frac{9\sin^2 \theta}{\sqrt{9\cos^2 \theta}} 3\cos \theta d\theta = \int 9\sin^2 \theta d\theta$$

$$= \frac{9}{2} \int (1 - \cos 2\theta) d\theta = \frac{9}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C$$

$$= \frac{9}{2} \left(\arcsin \left(\frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C$$



CHECK OUR WORK

Let's verify that

$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} = \frac{9}{2} \left(\arcsin \left(\frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C :$$

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$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} = \frac{9}{2} \left(\arcsin \left(\frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C :$$

$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{9}{2} \left(\arcsin \left(\frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C \right\} \\ &= \frac{9}{2} \left(\frac{1/3}{\sqrt{1 - \left(\frac{x-3}{3}\right)^2}} - \frac{x-3}{3} \cdot \frac{3-x}{3\sqrt{6x-x^2}} - \frac{1}{9} \sqrt{6x-x^2} \right) \\ &= \frac{9}{2} \left(\frac{9}{9\sqrt{6x-x^2}} - \frac{6x-x^2-9}{9\sqrt{6x-x^2}} - \frac{6x-x^2}{9\sqrt{6x-x^2}} \right) \\ &= \frac{9-6x+x^2}{\sqrt{6x-x^2}} \end{aligned}$$

So, our answer works.

Included Work

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