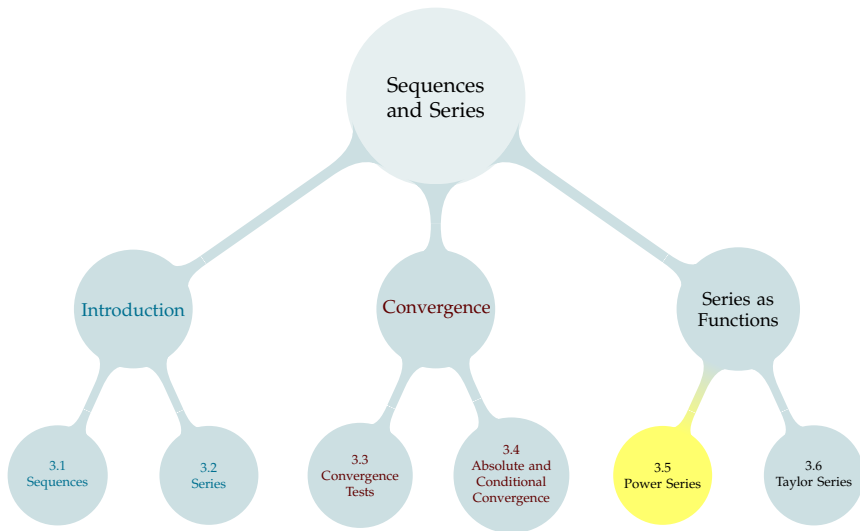


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Recall the geometric series: for a constant r , with $|r| < 1$:

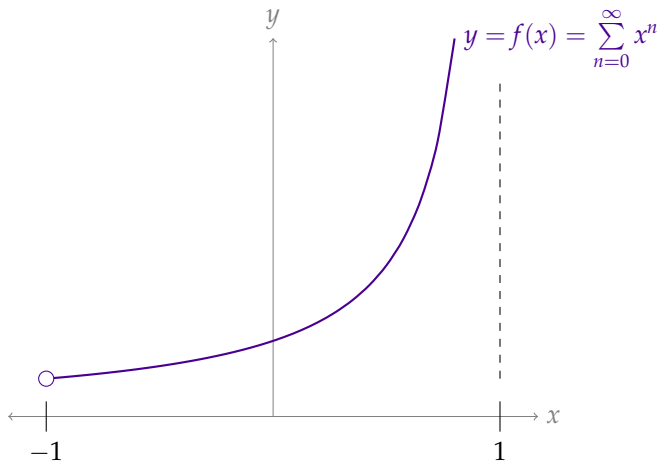
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

We can think of this as a function. If we set

$$f(x) = \sum_{n=0}^{\infty} x^n$$

and restrict our domain to $-1 < x < 1$, then

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$



Why would we ever prefer to write $\sum_{n=0}^{\infty} x^n$ instead of $\frac{1}{1-x}$?

The function

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

isn't a polynomial, but in certain ways it behaves like one. For $|x| < 1$:

$$\frac{d}{dx} \left\{ \frac{1}{1-x} \right\} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\frac{d}{dx} \{x^n\} \right) = \sum_{n=0}^{\infty} nx^{n-1}$$

$$\int \frac{1}{1-x} dx = \int \left(\sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=0}^{\infty} \left(\int x^n dx \right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Definition

A series of the form

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

is called a *power series in $(x-c)$* or a *power series centered on c* . The numbers A_n are called the coefficients of the power series.

One often considers power series centered on $c = 0$ and then the series reduces to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots = \sum_{n=0}^{\infty} A_nx^n$$

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

In a power series, we think of the coefficients A_n as fixed constants, and we think of x as the variable of a function.

Evaluate the power series $\sum_{n=0}^{\infty} A_n(x-c)^n$ when $x = c$:

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges.

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges.

Definition

Consider the power series

$$\sum_{n=0}^{\infty} A_n(x - c)^n.$$

The set of real x -values for which it converges is called the interval of convergence of the series.

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots .$$

What happens if we apply the ratio test to a generic power series,

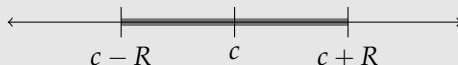
$$\sum_{n=0}^{\infty} A_n(x-c)^n?$$

$$\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n}(x-c) \right| = |x-c| \lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right|$$

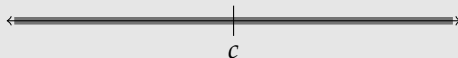
- ▶ If $\left| \frac{A_{n+1}}{A_n} \right|$ does not approach a limit as $n \rightarrow \infty$, the ratio test tells us nothing. (We should try other tests.)
- ▶ If $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = 0$, then
- ▶ If $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = \infty$, then
- ▶ If $\lim_{n \rightarrow \infty} \left| \frac{A_{n+1}}{A_n} \right| = A$ for some real number A , then

Definition: Radius of Convergence

- (a) Let $0 < R < \infty$. If $\sum_{n=0}^{\infty} A_n(x - c)^n$ converges for $|x - c| < R$, and diverges for $|x - c| > R$, then we say that the series has radius of convergence R .



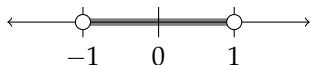
- (b) If $\sum_{n=0}^{\infty} A_n(x - c)^n$ converges for every number x , we say that the series has an infinite radius of convergence.



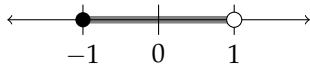
- (c) If $\sum_{n=0}^{\infty} A_n(x - c)^n$ diverges for every $x \neq c$, we say that the series has radius of convergence zero.



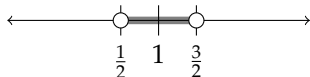
- We saw that $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$ and diverges when $|x| > 1$, so this series has radius of convergence $R =$



- We saw that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges when $|x| < 1$ and diverges when $|x| > 1$, so this series also has radius of convergence $R =$



- We saw that $\sum_{n=1}^{\infty} 2^n(x-1)^n$ converges when $|x-1| < \frac{1}{2}$ and diverges when $|x-1| > \frac{1}{2}$, so this series has radius of convergence $R =$



What is the radius of convergence for the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

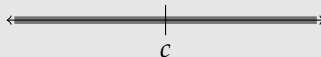
Recall: $n! = (n)(n-1)(n-2) \cdots (2)(1)$.

What is the radius of convergence for the series $\sum_{n=0}^{\infty} n! \cdot (x - 3)^n$?

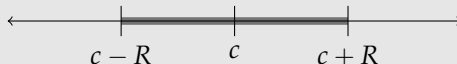
Theorem

Given a power series (say with centre c), one of the following holds.

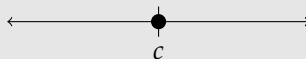
- (a) The power series converges for every number x . In this case we say that the radius of convergence is ∞ .



- (b) There is a number $0 < R < \infty$ such that the series converges for $|x - c| < R$ and diverges for $|x - c| > R$. Then R is called the radius of convergence.



- (c) The series converges for $x = c$ and diverges for all $x \neq c$. In this case, we say that the radius of convergence is 0.



We are told that a certain power series with centre $c = 3$ converges at $x = 4$ and diverges at $x = 1$. What else can we say about the convergence or divergence of the series for other values of x ?

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x - c)^n$$

for all x obeying $|x - c| < R$. Let K be a constant. Then:

$$f(x) + g(x) = \sum_{n=0}^{\infty} [A_n + B_n] (x - c)^n$$

$$Kf(x) = \sum_{n=0}^{\infty} K A_n (x - c)^n$$

for all x obeying $|x - c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$\begin{aligned} (x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n (x-c)^{n+N} \quad \text{for any integer } N \geq 1 \\ &= \sum_{k=N}^{\infty} A_{k-N} (x-c)^k \quad \text{where } k = n+N \end{aligned}$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x - c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n (x - c)^n$$

for all x obeying $|x - c| < R$. Let K be a constant. Then:

$$f'(x) = \sum_{n=0}^{\infty} A_n n (x - c)^{n-1} = \sum_{n=1}^{\infty} A_n n (x - c)^{n-1}$$

$$\int_c^x f(t) \, dt = \sum_{n=0}^{\infty} A_n \frac{(x - c)^{n+1}}{n + 1}$$

$$\int f(x) \, dx = \left[\sum_{n=0}^{\infty} A_n \frac{(x - c)^{n+1}}{n + 1} \right] + C \quad \text{with } C \text{ an arbitrary constant}$$

for all x obeying $|x - c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n \quad g(x) = \sum_{n=0}^{\infty} B_n(x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

for all x obeying $|x-c| < R$.

Differentiating, antidifferentiating, multiplying by a nonzero constant, and multiplying by a positive power of $(x-c)$ do not change the radius of convergence of $f(x)$ (although they may change the interval of convergence).

Given that $\frac{d}{dx} \left\{ \frac{1}{1-x} \right\} = \frac{1}{(1-x)^2}$, find a power series representation for $\frac{1}{(1-x)^2}$ when $|x| < 1$.

Find a power series representation for $\log(1 + x)$ when $|x| < 1$.



Find a power series representation for $\arctan(x)$ when $|x| < 1$.



Substituting in a Power Series

Assume that the function $f(x)$ is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all x in the interval I . Also let K and k be real constants. Then

$$f(Kx^k) = \sum_{n=0}^{\infty} A_n K^n x^{kn}$$

whenever Kx^k is in I . In particular, if $\sum_{n=0}^{\infty} A_n x^n$ has radius of convergence R , K is nonzero and k is a natural number, then $\sum_{n=0}^{\infty} A_n K^n x^{kn}$ has radius of convergence $\sqrt[k]{R/|K|}$.

Find a power series representation for $\frac{1}{5-x}$ with centre 3.

