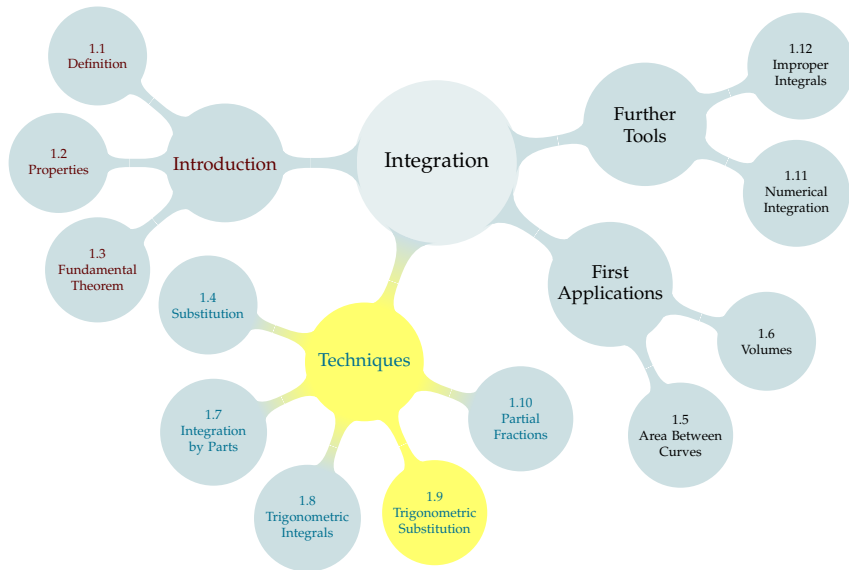


# TABLE OF CONTENTS



## WARMUP

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$$\begin{aligned}\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}} dx &= \int_3^7 \frac{1}{\sqrt{(x+1)^2}} dx \\ &= \int_3^7 \frac{1}{|x+1|} dx\end{aligned}$$

When  $3 \leq x \leq 7$ , we have  $|x+1| = x+1$ .

$$\begin{aligned}&= \int_3^7 \frac{1}{x+1} dx \\ &= [\log |x+1|]_3^7 \\ &= \log 8 - \log 4 = \log 2\end{aligned}$$

Idea:  $\sqrt{(\text{something})^2} = |\text{something}|$ . We cancelled off the square root.

Evaluate  $\int \frac{1}{\sqrt{x^2 + 1}} dx$ .

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We still want to cancel off the square root, but  $x^2 + 1$  is not obviously of the form (something)<sup>2</sup>.

Let  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ .

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 1}} dx &= \int \frac{1}{\sqrt{\tan^2 \theta + 1}} \sec^2 \theta d\theta = \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}} d\theta \\ &= \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + C \end{aligned}$$

We need to get these back in terms of  $x$ . From our substitution, we know  $\tan \theta = x$ . From simplifying our denominator, we also know  $\sec \theta = \sqrt{x^2 + 1}$ .

$$= \log \left| \sqrt{x^2 + 1} + x \right| + C$$

Same idea:  $\sqrt{(\text{something})^2} = |\text{something}|$ ; cancel off the square root.

# CHECK OUR WORK

Let's verify that  $\int \frac{1}{\sqrt{x^2 + 1}} = \log \left| \sqrt{x^2 + 1} + x \right| + C.$

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$$\begin{aligned} \frac{d}{dx} \left[ \log \left| \sqrt{x^2+1} + x \right| + C \right] &= \frac{1}{\sqrt{x^2+1} + x} \cdot \left( \frac{2x}{2\sqrt{x^2+1}} + 1 \right) \\ &= \frac{x + \sqrt{x^2+1}}{(\sqrt{x^2+1} + x)\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

So, our answer works!

# METHOD (ONE STANDARD CASE)

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- ▶ After integrating, convert back to the original variable (possibly using a triangle—more details later)



## FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

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Let  $x = \sin \theta$  so  $\sqrt{1 - x^2}$  becomes  $\sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta|$   
(Alternately,  $x = \cos \theta$  works as well)

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►  $\sqrt{x^2 + 7}$

►  $\sqrt{3 - 2x^2}$

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Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

►  $\sqrt{x^2 + 7}$

Adjust a given identity by multiplying both sides by 7:

$7 \tan^2 \theta + 7 = 7 \sec^2 \theta$ . Now we see we want  $x^2 = 7 \tan^2 \theta$ . That is,

$x = \sqrt{7} \tan \theta$ :

$$\sqrt{x^2 + 7} = \sqrt{7 \tan^2 \theta + 7} = \sqrt{7(\sec^2 \theta)} = \sqrt{7} |\sec \theta|$$

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►  $\sqrt{3 - 2x^2}$

Adjust a given identity by multiplying both sides by 3:

$3 - 3 \sin^2 \theta = 3 \cos^2 \theta$ . Now we see we want  $2x^2 = 3 \sin^2 \theta$ , so

$x = \sqrt{\frac{3}{2}} \sin \theta$ :

$$\sqrt{3 - 2x^2} = \sqrt{3 - 2 \left( \frac{3}{2} \sin^2 \theta \right)} = \sqrt{3 - 3 \sin^2 \theta} = \sqrt{3 \cos^2 \theta} = \sqrt{3} |\cos \theta|$$

## CLOSER LOOK AT ABSOLUTE VALUES

▶ SKIP CLOSER LOOK

Consider the substitution  $x = \sin \theta$ ,  $dx = \cos \theta \, d\theta$  for the integral:

$$\int_0^1 \sqrt{1-x^2} \, dx$$

When  $x = 0$  (lower limit of integration), what is  $\theta$ ?

When  $x = 1$  (upper limit of integration), what is  $\theta$ ?



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If  $x = 0$ , then  $\sin \theta = 0$ , but there are infinitely many values of  $\theta$  that could make this true. To use the substitution  $x = \sin \theta$ , we need the function  $x = \sin \theta$  to be invertible. That way, we can unambiguously convert between  $x$  and  $\theta$ . With that in mind, we'll actually set  $\theta = \arcsin x$ . Now  $\theta$  is restricted to the interval  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

$$\begin{aligned} \int_0^1 \sqrt{1-x^2} \, dx &= \int_{\arcsin 0}^{\arcsin 1} \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 \theta} \cdot \cos \theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} |\cos \theta| \cdot \cos \theta \, d\theta \end{aligned}$$

For  $0 \leq \theta \leq \frac{\pi}{2}$ , we have  $\cos \theta \geq 0$ , so  $|\cos \theta| = \cos \theta$ .

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More generally, suppose  $a$  is a positive constant and we use the substitution  $x = a \sin \theta$  for the term  $\sqrt{a^2 - x^2}$ .

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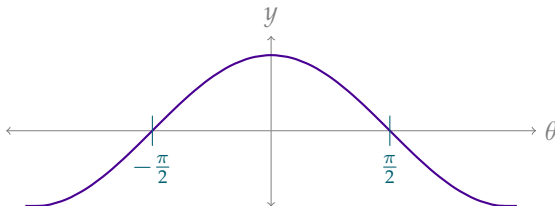
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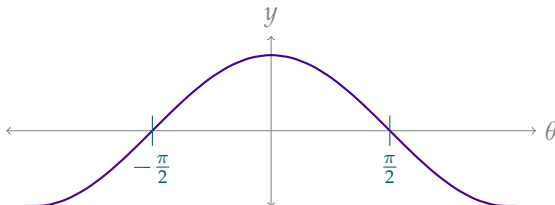


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Now, consider the substitution  $x = a \tan \theta$  for  $\sqrt{a^2 + x^2}$ , where  $a$  is a positive constant.

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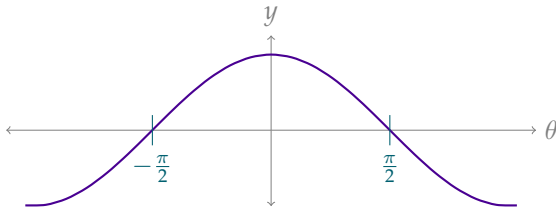
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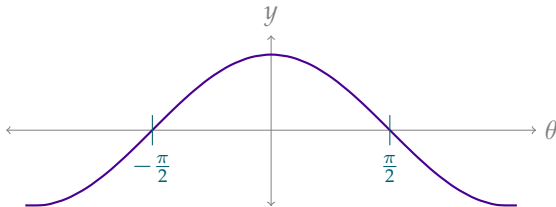


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Finally, consider the substitution  $x = a \sec \theta$  for  $\sqrt{x^2 - a^2}$ , where  $a$  is a positive constant.

▶  $\sec \theta = \frac{x}{a}$ , so  $\cos \theta = \frac{a}{x}$ , so  $\theta = \arccos\left(\frac{a}{x}\right)$ . Then  $0 \leq \theta \leq \pi$

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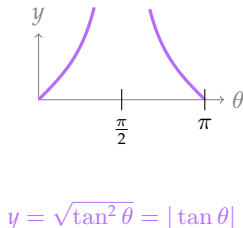
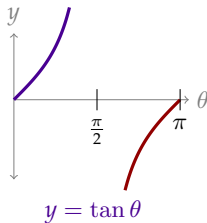
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  - ▶ When  $x \geq a$ , we have  $0 \leq \theta < \frac{\pi}{2}$ ,  $\tan \theta \geq 0$ , so  $|\tan \theta| = \tan \theta$ .

## CLOSER LOOK AT ABSOLUTE VALUES

▶ SKIP CLOSER LOOK

Finally, consider the substitution  $x = a \sec \theta$  for  $\sqrt{x^2 - a^2}$ , where  $a$  is a positive constant.

- ▶  $\sec \theta = \frac{x}{a}$ , so  $\cos \theta = \frac{a}{x}$ , so  $\theta = \arccos\left(\frac{a}{x}\right)$ . Then  $0 \leq \theta \leq \pi$
- ▶  $\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$
- ▶ Now this case gets slightly more complicated than the others:
  - ▶ For  $\sqrt{x^2 - a^2}$  to be defined, we need  $x^2 \geq a^2$ . I.e.  $x \geq a$  or  $x \leq -a$ .
  - ▶ When  $x \geq a$ , we have  $0 \leq \theta < \frac{\pi}{2}$ ,  $\tan \theta \geq 0$ , so  $|\tan \theta| = \tan \theta$ .
  - ▶ When  $x \leq -a$ , we have  $\frac{\pi}{2} < \theta \leq \pi$ ,  $\tan \theta < 0$ , so  $|\tan \theta| = -\tan \theta$ .



# ABSOLUTE VALUES

From now on, we will assume:

- ▶ With the substitution  $x = a \sin \theta$  for  $\sqrt{a^2 - x^2}$ ,  $|\cos \theta| = \cos \theta$
- ▶ With the substitution  $x = a \tan \theta$  for  $\sqrt{a^2 + x^2}$ ,  $|\sec \theta| = \sec \theta$

## Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate  $\int_0^1 (1 + x^2)^{-3/2} dx$

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Let  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ . When  $x = 0$ , then  $\theta = \arctan 0 = 0$ ;  
when  $x = 1$ , then  $\theta = \arctan 1 = \frac{\pi}{4}$ .

$$\begin{aligned} \int_0^1 (1+x^2)^{-3/2} dx &= \int_{\theta=0}^{\theta=\pi/4} \frac{1}{\sqrt{1+\tan^2 \theta}^3} \sec^2 \theta d\theta \\ &= \int_0^{\pi/4} \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}^3} d\theta = \int_0^{\pi/4} \frac{\sec^2 \theta}{|\sec \theta|^3} d\theta \\ &= \int_0^{\pi/4} \frac{1}{|\sec \theta|} d\theta = \int_0^{\pi/4} |\cos \theta| d\theta \end{aligned}$$

Given our previous investigation,

$$\begin{aligned} &= \int_0^{\pi/4} \cos \theta d\theta = [\sin \theta]_0^{\pi/4} \\ &= \sin \frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}} \end{aligned}$$

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Evaluate  $\int \sqrt{1 - 4x^2} \, dx$

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Under the square root, we have “one minus a term with a variable,” which matches the identity  $1 - \sin^2 \theta$ . So, we want  $4x^2$  to become  $\sin^2 \theta$ . That is,  $x = \frac{1}{2} \sin \theta$ . Then  $dx = \frac{1}{2} \cos \theta \, d\theta$ .

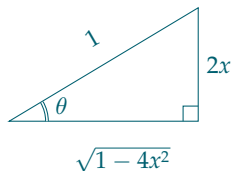
$$\begin{aligned} \int \sqrt{1 - 4x^2} \, dx &= \int \sqrt{1 - 4 \left( \frac{1}{2} \sin \theta \right)^2} \cdot \frac{1}{2} \cos \theta \, d\theta \\ &= \frac{1}{2} \int \sqrt{1 - \sin^2 \theta} \cdot \cos \theta \, d\theta = \frac{1}{2} \int \sqrt{\cos^2 \theta} \cdot \cos \theta \, d\theta \\ &= \frac{1}{2} \int |\cos \theta| \cdot \cos \theta \, d\theta = \frac{1}{2} \int \cos^2 \theta \, d\theta \\ &= \frac{1}{2} \int \left( \frac{1 + \cos(2\theta)}{2} \right) d\theta = \frac{1}{4} \int (1 + \cos(2\theta)) d\theta \\ &= \frac{1}{4} \left[ \theta + \frac{1}{2} \sin(2\theta) \right] + C = \frac{1}{4} [\theta + \sin \theta \cos \theta] + C \end{aligned}$$

It remains to convert  $\theta$  back into  $x$ .

Evaluate  $\int \sqrt{1-4x^2} \, dx$

The substitution  $x = \frac{1}{2} \sin \theta$  tells us  $\sin \theta = 2x$ . This in turn gives us  $\theta = \arcsin(2x)$ . We should still convert  $\cos \theta$  back into terms of  $x$ . You might notice in the calculation we did that  $\sqrt{1-4x^2}$  turned into  $\cos \theta$ , so  $\cos \theta = \sqrt{1-4x^2}$ .

Alternately, to find  $\cos \theta$  in terms of  $x$ , we can use a triangle. From  $\sin \theta = 2x$ , and the understanding that  $\sin \theta$  is the ratio  $\frac{\text{opposite}}{\text{hypotenuse}}$  for a right triangle with angle  $\theta$ , we can set up a triangle whose opposite side has length  $2x$ , and hypotenuse has length 1.



The Pythagorean theorem tells us the side adjacent to  $\theta$  has length

$$\sqrt{1-4x^2}. \text{ So}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{1-4x^2}.$$

$$\int \sqrt{1-4x^2} \, dx = \frac{1}{4} \left( \underbrace{\arcsin(2x)}_{\theta} + \underbrace{2x\sqrt{1-4x^2}}_{\sin \theta \cos \theta} \right) + C$$

# CHECK OUR WORK

In the last example, we computed

$$\int \sqrt{1-4x^2} \, dx = \frac{1}{4} (\arcsin(2x) + 2x\sqrt{1-4x^2}) + C.$$

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$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{1}{4} (\arcsin(2x) + 2x\sqrt{1-4x^2}) + C \right\} \\ &= \frac{1}{4} \left( \frac{2}{\sqrt{1-(2x)^2}} + 2x \frac{-8x}{2\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} \right) \\ &= \frac{1}{4} \left( \frac{2}{\sqrt{1-4x^2}} - \frac{8x^2}{\sqrt{1-4x^2}} + \frac{2(1-4x^2)}{\sqrt{1-4x^2}} \right) \\ &= \frac{1}{4} \left( \frac{2-8x^2+2-8x^2}{\sqrt{1-4x^2}} \right) = \frac{1-4x^2}{\sqrt{1-4x^2}} = \sqrt{1-4x^2} \quad \checkmark \end{aligned}$$

## Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

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Evaluate  $\int \frac{1}{\sqrt{x^2 - 1}} dx$

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We use the substitution  $x = \sec \theta$ ,  $dx = \sec \theta \tan \theta d\theta$ .

To make the substitution work, we're actually taking  $\theta = \arccos\left(\frac{1}{x}\right)$ , and so  $0 \leq \theta \leq \pi$ .

Note that the integrand exists on the intervals  $x < -1$  and  $x > 1$ .

- ▶ When  $x > 1$ , then  $0 < \frac{1}{x} < 1$ , so  $0 < \arccos\left(\frac{1}{x}\right) < \frac{\pi}{2}$ .  
That is,  $0 < \theta < \frac{\pi}{2}$ , so  $|\tan \theta| = \tan \theta$ .
- ▶ When  $x < -1$ , then  $-1 < \frac{1}{x} < 0$ , so  $\frac{\pi}{2} < \arccos\left(\frac{1}{x}\right) < \pi$ .  
That is,  $\frac{\pi}{2} < \theta < \pi$ , so  $|\tan \theta| = -\tan \theta$ .

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 1}} dx &= \int \frac{1}{\sqrt{\sec^2 \theta - 1}} \cdot \sec \theta \tan \theta d\theta = \int \frac{\sec \theta \tan \theta}{\sqrt{\tan^2 \theta}} d\theta \\ &= \int \sec \theta \left( \frac{\tan \theta}{|\tan \theta|} \right) d\theta = \begin{cases} \int \sec \theta d\theta & 0 < \theta < \frac{\pi}{2} \\ -\int \sec \theta d\theta & \frac{\pi}{2} < \theta < \pi \end{cases} \\ &= \begin{cases} \log |\sec \theta + \tan \theta| + C & 0 < \theta < \frac{\pi}{2} \\ -\log |\sec \theta + \tan \theta| + C & \frac{\pi}{2} < \theta < \pi \end{cases} \end{aligned}$$

Our substitution tells us  $\sec \theta = x$ . We saw from the denominator of our integrand that  $\sqrt{x^2 - 1} = |\tan \theta|$ .

- ▶ When  $0 < \theta < \frac{\pi}{2}$ ,  $\tan \theta = |\tan \theta| = \sqrt{x^2 - 1}$
- ▶ When  $\frac{\pi}{2} < \theta < \pi$ ,  $\tan \theta = -|\tan \theta| = -\sqrt{x^2 - 1}$

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \begin{cases} \log |x + \sqrt{x^2 - 1}| + C & x > 1 \\ -\log |x - \sqrt{x^2 - 1}| + C & x < -1 \end{cases}$$

Although the two branches look different, they are actually equivalent. Remember  $-\log(A) = \log(A^{-1}) = \log\left(\frac{1}{A}\right)$ :

$$\begin{aligned} -\log |x - \sqrt{x^2 - 1}| &= \log \left| \frac{1}{x - \sqrt{x^2 - 1}} \right| = \log \left| \frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} \right| \\ &= \log \left| \frac{x + \sqrt{x^2 - 1}}{x^2 - x^2 + 1} \right| = \log |x + \sqrt{x^2 - 1}| \end{aligned}$$

So,

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \log |x + \sqrt{x^2 - 1}| + C$$



# CHECK OUR WORK

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$$\begin{aligned} \frac{d}{dx} \left\{ \log \left| x + \sqrt{x^2-1} \right| + C \right\} &= \frac{1 + \frac{2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} \\ &= \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} \left( \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} \right) = \frac{(\sqrt{x^2-1} + x)}{(x + \sqrt{x^2-1}) \sqrt{x^2-1}} \\ &= \frac{1}{\sqrt{x^2-1}} \end{aligned}$$

So, our answer works.

# COMPLETING THE SQUARE

Choose a trigonometric substitution to simplify  $\sqrt{3 - x^2 + 2x}$ .

Identities have two “parts” that turn into one part:

▶  $1 - \sin^2 \theta = \cos^2 \theta$

▶  $1 + \tan^2 \theta = \sec^2 \theta$

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But our quadratic expression has *three* parts.

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$$\text{Fact: } 3 - x^2 + 2x = 4 - (x - 1)^2$$

$$\sqrt{3 - x^2 + 2x} = \sqrt{4 - (x - 1)^2}$$

We want  $(x - 1)^2 = 4 \sin^2 \theta$ , so let  $(x - 1) = 2 \sin \theta$

$$= \sqrt{4 - 4 \sin^2 \theta} = \sqrt{4 \cos^2 \theta} = 2 \cos \theta$$

# COMPLETING THE SQUARE

$$(x + b)^2 = x^2 + 2bx + b^2$$

$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

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3. All together:  $3 - x^2 + 2x = 4 - (x - 1)^2$

Evaluate  $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx$ .

Identities have two “parts” that turn into one part:

- ▶  $1 - \sin^2 \theta = \cos^2 \theta$
- ▶  $1 + \tan^2 \theta = \sec^2 \theta$
- ▶  $\sec^2 \theta - 1 = \tan^2 \theta$

One of those parts is a constant, and one is squared.

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One of those parts is a constant, and one is squared.

Write  $6x - x^2$  as  $c - (x + b)^2$ .

$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

$$6x = -2bx \implies b = -3$$

$$0 = c - b^2 = c - 9 \implies c = 9$$

$$6x - x^2 = 9 - (x - 3)^2$$

Evaluate  $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx = \int \frac{(x - 3)^2}{\sqrt{9 - (x - 3)^2}} dx.$



Evaluate  $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx = \int \frac{(x-3)^2}{\sqrt{9 - (x-3)^2}} dx.$

We use the identity  $9 - 9\sin^2 \theta = 9\cos^2 \theta$ .

We want  $(x-3)^2 = 9\sin^2 \theta$ , so take  $(x-3) = 3\sin \theta$ ,  $dx = 3\cos \theta d\theta$ .

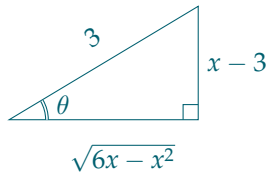
$$\int \frac{(x-3)^2}{\sqrt{9 - (x-3)^2}} dx = \int \frac{9\sin^2 \theta}{\sqrt{9 - 9\sin^2 \theta}} 3\cos \theta d\theta$$

$$= \int \frac{9\sin^2 \theta}{\sqrt{9\cos^2 \theta}} 3\cos \theta d\theta = \int 9\sin^2 \theta d\theta$$

$$= \frac{9}{2} \int (1 - \cos 2\theta) d\theta = \frac{9}{2} \left( \theta - \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C$$

$$= \frac{9}{2} \left( \arcsin \left( \frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C$$



# CHECK OUR WORK

Let's verify that

$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} = \frac{9}{2} \left( \arcsin \left( \frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C :$$

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$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{9}{2} \left( \arcsin \left( \frac{x-3}{3} \right) - \frac{x-3}{3} \cdot \frac{\sqrt{6x-x^2}}{3} \right) + C \right\} \\ &= \frac{9}{2} \left( \frac{1/3}{\sqrt{1 - \left(\frac{x-3}{3}\right)^2}} - \frac{x-3}{3} \cdot \frac{3-x}{3\sqrt{6x-x^2}} - \frac{1}{9} \sqrt{6x-x^2} \right) \\ &= \frac{9}{2} \left( \frac{9}{9\sqrt{6x-x^2}} - \frac{6x-x^2-9}{9\sqrt{6x-x^2}} - \frac{6x-x^2}{9\sqrt{6x-x^2}} \right) \\ &= \frac{9-6x+x^2}{\sqrt{6x-x^2}} \end{aligned}$$

So, our answer works.

## Included Work

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