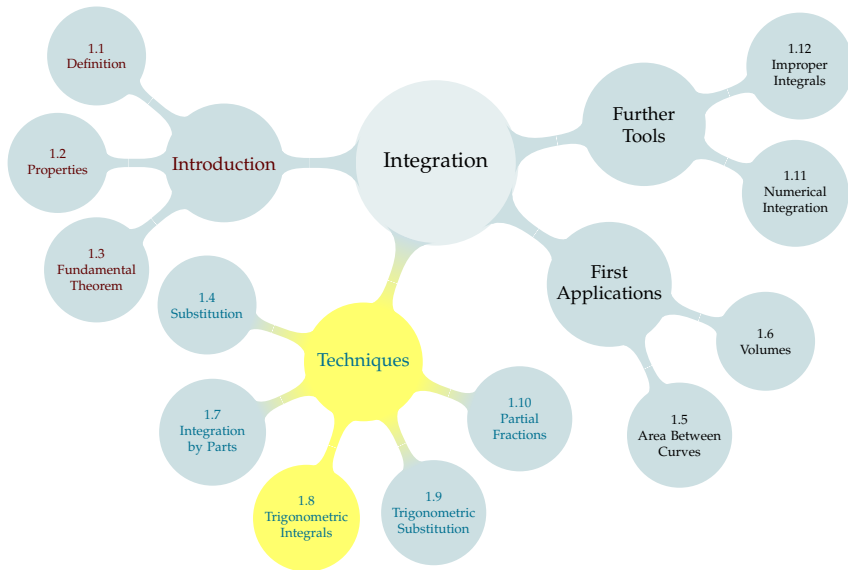


# TABLE OF CONTENTS



# 1.8 TRIGONOMETRIC INTEGRALS

Recall:

- ▶  $\sin^2 x + \cos^2 x = 1$
- ▶  $\tan^2 x + 1 = \sec^2 x$
- ▶  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- ▶  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- ▶  $\sin(2x) = 2 \sin x \cos x$

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin x \cos x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$

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$$\int \sin^{10} x \cos x dx =$$

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# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$

$$\int \sin x \cos x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2} \sin^2 x + C$$

Let  $u = \sin x$ ,  $du = \cos x dx$

$$\int \sin^{10} x \cos x dx = \int u^{10} du = \frac{1}{11}u^{11} + C = \frac{1}{11} \sin^{11} x + C$$



## CHECK OUR WORK

If we are correct that  $\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^2 x}{2} + C \right\} = \sin x \cos x$ .

# CHECK OUR WORK

If we are correct that  $\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^2 x}{2} + C \right\} = \sin x \cos x$ .

We differentiate, using the chain rule:

$$\frac{d}{dx} \left\{ \frac{\sin^2 x}{2} + C \right\} = \frac{2}{2} \sin x \cos x = \sin x \cos x$$

Our answer works.

# CHECK OUR WORK

If we are correct that  $\int \sin^{10} x \cos x dx = \frac{\sin^{11} x}{11} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} + C \right\} = \sin^{10} x \cos x$ .

# CHECK OUR WORK

If we are correct that  $\int \sin^{10} x \cos x dx = \frac{\sin^{11} x}{11} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} + C \right\} = \sin^{10} x \cos x$ .

We differentiate, using the chain rule:

$$\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} + C \right\} = \frac{11}{11} \sin^{10} x \cos x = \sin^{10} x \cos x$$

Our answer works.

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x dx &= \int_{\sin(0)}^{\sin(\pi/2)} u^{\pi+1} du = \frac{1}{\pi+2} u^{\pi+2} \Big|_0^1 \\ &= \frac{1}{\pi+2} \end{aligned}$$

# CHECK OUR WORK

If we are correct that  $\int \sin^{\pi+1} x \cos x dx = \frac{\sin^{\pi+2} x}{\pi+2} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{\pi+2} x}{\pi+2} + C \right\} = \sin^{\pi+1} x \cos x$ .





# CHECK OUR WORK

If we are correct that  $\int \sin^{\pi+1} x \cos x dx = \frac{\sin^{\pi+2} x}{\pi+2} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{\pi+2} x}{\pi+2} + C \right\} = \sin^{\pi+1} x \cos x$ .

We differentiate, using the chain rule:

$$\frac{d}{dx} \left\{ \frac{\sin^{\pi+2} x}{\pi+2} + C \right\} = \frac{\pi+2}{\pi+2} \sin^{\pi+1} x \cos x = \sin^{\pi+1} x \cos x$$

Our answer works.



# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$ .

$$\int \sin^{10} x \cos^3 x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$ .

Use  $\sin^2 x + \cos^2 x = 1$ .

$$\int \sin^{10} x \cos^3 x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

Let  $u = \sin x$ ,  $du = \cos x dx$ .

Use  $\sin^2 x + \cos^2 x = 1$ .

$$\begin{aligned}
 \int \sin^{10} x \cos^3 x dx &= \int \sin^{10} x \cos^2 x \cos x dx \\
 &= \int \sin^{10} x (1 - \sin^2 x) \cos x dx \\
 &= \int u^{10} (1 - u^2) du = \int (u^{10} - u^{12}) du \\
 &= \frac{1}{11} u^{11} - \frac{1}{13} u^{13} + C = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C
 \end{aligned}$$

## CHECK OUR WORK

If we are correct that  $\int \sin^{10} x \cos^3 x dx = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C \right\} = \sin^{10} x \cos^3 x$ .

# CHECK OUR WORK

If we are correct that  $\int \sin^{10} x \cos^3 x dx = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$ , then it should be true that  $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C \right\} = \sin^{10} x \cos^3 x$ .

We differentiate, using the chain rule:

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C \right\} &= \frac{11}{11} \sin^{10} x \cos x - \frac{13}{13} \sin^{12} x \cos x \\ &= \sin^{10} x (1 - \sin^2 x) \cos x = \sin^{10} x \cos^2 x \cos x \\ &= \sin^{10} x \cos^3 x \end{aligned}$$

Our answer works.

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin^5 x \cos^4 x dx =$$

# INTEGRATING PRODUCTS OF SINE AND COSINE

$$u = \cos x, du = -\sin x dx$$

$$\sin^2 x + \cos^2 x = 1.$$

$$\int \sin^5 x \cos^4 x dx =$$



# INTEGRATING PRODUCTS OF SINE AND COSINE

$$u = \cos x, du = -\sin x dx$$

$$\sin^2 x + \cos^2 x = 1.$$

$$\begin{aligned} \int \sin^5 x \cos^4 x dx &= \int (\sin^2 x)^2 \cos^4 x \sin x dx \\ &= \int (1 - \cos^2 x)^2 \cos^4 x \sin x dx \\ &= - \int (1 - u^2)^2 u^4 du = - \int (1 - 2u^2 + u^4) u^4 du \\ &= - \int (u^4 - 2u^6 + u^8) du = -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C \\ &= -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C \end{aligned}$$

# CHECK OUR WORK

If we are correct that

$$\int \sin^5 x \cos^4 x dx = -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C, \text{ then it should}$$

be true that  $\frac{d}{dx} \left\{ -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C \right\} = \sin^5 x \cos^4 x.$

# CHECK OUR WORK

If we are correct that

$$\int \sin^5 x \cos^4 x dx = -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C, \text{ then it should}$$

$$\text{be true that } \frac{d}{dx} \left\{ -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C \right\} = \sin^5 x \cos^4 x.$$

We differentiate, using the chain rule:

$$\begin{aligned} & \frac{d}{dx} \left\{ -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C \right\} \\ &= \frac{5}{5} \cos^4 x \sin x - \frac{2 \cdot 7}{7} \cos^6 x \sin x + \frac{9}{9} \cos^8 x \sin x \\ &= \cos^4 x \sin x (1 - 2 \cos^2 x + \cos^4 x) \\ &= \cos^4 x \sin x (1 - \cos^2 x)^2 = \cos^4 x \sin x (\sin^2 x)^2 \\ &= \sin^5 x \cos^4 x \end{aligned}$$

Our answer works.



GENERALIZE:  $\int \sin^m x \cos^n bx dx$

To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

GENERALIZE:  $\int \sin^m x \cos^n x dx$

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- We need to reserve one  $\cos x$  for the differential.

GENERALIZE:  $\int \sin^m x \cos^n bx dx$

To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

- We need to **reserve** one  **$\cos x$**  for the differential.
- We need to **convert** the remaining  **$\cos^{n-1} x$**  to  **$\sin x$**  terms.

GENERALIZE:  $\int \sin^m x \cos^n bx dx$

To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

- ▶ We need to **reserve** one  **$\cos x$**  for the differential.
- ▶ We need to **convert** the remaining  **$\cos^{n-1} x$**  to  **$\sin x$**  terms.
- ▶ We convert using  **$\cos^2 x = 1 - \sin^2 x$** . To avoid square roots, that means  $n - 1$  should be **even when we convert**.

GENERALIZE:  $\int \sin^m x \cos^n bx dx$

To use the substitution  $u = \sin x$ ,  $du = \cos x dx$ :

- ▶ We need to **reserve** one  **$\cos x$**  for the differential.
- ▶ We need to **convert** the remaining  **$\cos^{n-1} x$**  to  **$\sin x$**  terms.
- ▶ We convert using  **$\cos^2 x = 1 - \sin^2 x$** . To avoid square roots, that means  $n - 1$  should be **even when we convert**.
- ▶ So, we can use this substitution when the original power of cosine,  $n$ , is ODD: one cosine goes to the differential, the rest are converted to sines.



GENERALIZE:  $\int \sin^m x \cos^n x dx$

To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

GENERALIZE:  $\int \sin^m x \cos^n x dx$

To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

- We need to **reserve** one  **$\sin x$**  for the differential.

GENERALIZE:  $\int \sin^m x \cos^n x dx$

To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

- ▶ We need to **reserve** one  **$\sin x$**  for the differential.
- ▶ We need to **convert** the remaining  **$\sin^{m-1} x$**  to  **$\cos x$**  terms.

GENERALIZE:  $\int \sin^m x \cos^n x dx$

To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

- ▶ We need to **reserve** one  $\sin x$  for the differential.
- ▶ We need to **convert** the remaining  $\sin^{m-1} x$  to  $\cos x$  terms.
- ▶ We convert using  $\sin^2 x = 1 - \cos^2 x$ . To avoid square roots, that means  $m - 1$  should be **even when we convert**.

GENERALIZE:  $\int \sin^m x \cos^n x dx$

To use the substitution  $u = \cos x$ ,  $du = -\sin x dx$ :

- ▶ We need to **reserve** one  **$\sin x$**  for the differential.
- ▶ We need to **convert** the remaining  **$\sin^{m-1} x$**  to  **$\cos x$**  terms.
- ▶ We convert using  **$\sin^2 x = 1 - \cos^2 x$** . To avoid square roots, that means  $m - 1$  should be **even when we convert**.
- ▶ **So, we can use this substitution when the original power of sine,  $m$ , is ODD: one sine goes to the differential, the rest are converted to cosines.**

# MNEMONIC: "ODD ONE OUT"

$$\text{Integrating } \int \sin^m x \cos^n x dx$$

If you want to use  $u = \sin x$ , there should be an odd power of **cosine**.

If you want to use  $u = \cos x$ , there should be an odd power of **sine**.

Carry out a suitable substitution (but do not evaluate the resulting integral):

►  $\int \sin^4 x \cos^7 x dx$

►  $\int \sin^7 x \cos^4 x dx$

►  $\int \sin^7 x \cos^7 x dx$

$$\int \sin^4 x \cos^7 x dx$$

The power of **cosine** is odd, so it becomes our differential. That is, we use  $u = \sin x$ ,  **$du = \cos x dx$** .

$$\begin{aligned} & \int \sin^4 x \cos^7 x dx \\ &= \int \sin^4 x (\cos^2 x)^3 \mathbf{\cos x dx} \\ &= \int \sin^4 x (1 - \sin^2 x)^3 \mathbf{\cos x dx} \\ &= \int u^4 (1 - u^2)^3 \mathbf{du} \end{aligned}$$



$$\int \sin^7 x \cos^4 x dx$$

The power of **sine** is odd, so it becomes our differential. That is, we use  $u = \cos x$ ,  **$du = -\sin x dx$** .

$$\begin{aligned} & \int \sin^7 x \cos^4 x dx \\ &= \int (\sin^2 x)^3 \cos^4 x \sin x dx \\ &= \int (1 - \cos^2 x)^3 \cos^4 x \sin x dx \\ &= - \int (1 - u^2)^3 u^4 du \end{aligned}$$

$$\int \sin^7 x \cos^7 x dx$$

The powers of sine and cosine are both odd, so we can use either as our differential.

Solution 1:

$$u = \sin x, du = \cos x$$

$$\begin{aligned} & \int \sin^7 x \cos^7 x dx \\ &= \int \sin^7 x (\cos^2 x)^3 \cos x dx \\ &= \int \sin^7 x (1 - \sin^2 x)^3 \cos x dx \\ &= \int u^7 (1 - u^2)^3 du \Big|_{u=\sin x} \end{aligned}$$

Solution 2:

$$u = \cos x, du = -\sin x dx$$

$$\begin{aligned} & \int \sin^7 x \cos^7 x dx \\ &= \int (\sin^2 x)^3 \cos^7 x \sin x dx \\ &= \int (1 - \cos^2 x)^3 \cos^7 x \sin x dx \\ &= - \int (1 - u^2)^3 u^7 du \Big|_{u=\cos x} \end{aligned}$$

To evaluate  $\int \sin^m x \cos^n x dx$ , we use:

- ▶  $u = \sin x$  if  $n$  is odd, and/or
- ▶  $u = \cos x$  if  $m$  is odd

What if  $n$  and  $m$  are both even?

To evaluate  $\int \sin^m x \cos^n x dx$ , we use:

- ▶  $u = \sin x$  if  $n$  is odd, and/or
- ▶  $u = \cos x$  if  $m$  is odd

What if  $n$  and  $m$  are both even?

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \sin^2 x dx =$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\begin{aligned} \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C \end{aligned}$$

# CHECK OUR WORK

We check that  $\int \sin^2 x dx = \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C$  by differentiating:

# CHECK OUR WORK

We check that  $\int \sin^2 x dx = \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C$  by differentiating:

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C \right\} &= \frac{1}{2} \left( 1 - \frac{1}{2} (\cos 2x)(2) \right) \\ &= \frac{1 - \cos 2x}{2} = \sin^2 x \end{aligned}$$

So, our answer works.



$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Evaluate  $\int \sin^4 x \, dx$ .

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

Evaluate  $\int \sin^4 x dx$ .

$$\begin{aligned} \int \sin^4 x dx &= \int (\sin^2 x)^2 dx = \int \left( \frac{1 - \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int (1 - 2 \cos 2x) dx + \frac{1}{4} \int \cos^2(2x) dx \\ &= \frac{1}{4} (x - \sin 2x) + \frac{1}{4} \int \left( \frac{1 + \cos(4x)}{2} \right) dx \\ &= \frac{1}{4} (x - \sin 2x) + \frac{1}{8} \left( x + \frac{1}{4} \sin(4x) \right) + C \\ &= \frac{3}{8} x - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C \end{aligned}$$

# CHECK OUR WORK

We want to check that  $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$

## CHECK OUR WORK

We want to check that  $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$ .

Note  $\sin^2 x = \frac{1 - \cos(2x)}{2}$ , so  $\cos(2x) = 1 - 2\sin^2 x$ . Also remember  $\frac{1}{2}\sin(2x) = \sin x \cos x$ .

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C \right\} &= \frac{3}{8} - \frac{2}{4}\cos(2x) + \frac{4}{32}\cos(4x) \\ &= \frac{3}{8} - \frac{1}{2}(1 - 2\sin^2 x) + \frac{1}{8}(1 - 2\sin^2(2x)) \\ &= \frac{3}{8} - \frac{1}{2} + \sin^2 x + \frac{1}{8} - \frac{1}{4}\sin^2(2x) \\ &= \sin^2 x - \left( \frac{1}{2}\sin 2x \right)^2 = \sin^2 x - \sin^2 x \cos^2 x \\ &= \sin^2 x(1 - \cos^2 x) = \sin^2 x(\sin^2 x) = \sin^4 x \end{aligned}$$

So, our answer works.



Recall:

- ▶  $\frac{d}{dx} \{\tan x\} = \sec^2 x$
- ▶  $\frac{d}{dx} \{\sec x\} = \sec x \tan x$
- ▶  $\tan^2 x + 1 = \sec^2 x$

$$\int \tan x \, dx =$$

$$\begin{aligned}
 \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx & u &= \cos x & du &= -\sin x \, dx \\
 &= - \int \frac{1}{u} \, du = -\log |u| + C \\
 &= \log |u^{-1}| + C = \log \left| \frac{1}{\cos x} \right| + C \\
 &= \log |\sec x| + C
 \end{aligned}$$

# CHECK OUR WORK

Let's check that  $\int \tan x dx = \log |\sec x| + C$  by differentiating.



# CHECK OUR WORK

Let's check that  $\int \tan x dx = \log |\sec x| + C$  by differentiating.

$$\frac{d}{dx} \{\log |\sec x| + C\} = \frac{\sec x \tan x}{\sec x} = \tan x$$

So, our answer works.

Optional: A nifty trick – you won't be expected to come up with it.  
There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx =$$

Optional: A nifty trick – you won't be expected to come up with it.  
There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx = \int \sec x \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

Optional: A nifty trick – you won't be expected to come up with it. There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\begin{aligned}
 \int \sec x \, dx &= \int \sec x \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\
 &= \int \left( \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx \\
 &\quad \text{set } u = \sec x + \tan x, \, du = (\sec x \tan x + \sec^2 x) dx \\
 &= \int \frac{1}{u} du = \log |u| + C \\
 &= \log |\sec x + \tan x| + C
 \end{aligned}$$

Useful integrals:

$$\blacktriangleright \int \tan x \, dx = \log |\sec x| + C$$

$$\blacktriangleright \int \sec x \, dx = \log |\sec x + \tan x| + C$$

1.  $\int \sec x \tan x dx =$

2.  $\int \sec^2 x dx =$

3.  $\int \tan x dx =$

4.  $\int \sec x dx =$

$$1. \int \sec x \tan x dx = \sec x + C$$

$$2. \int \sec^2 x dx = \tan x + C$$

$$3. \int \tan x dx = \log |\sec x| + C$$

$$4. \int \sec x dx = \log |\sec x + \tan x| + C$$

Evaluate using the substitution rule:

$$\int \tan^5 x \sec^2 x \, dx =$$

$$\int \sec^4 x (\sec x \tan x) \, dx =$$



Evaluate using the substitution rule:

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$\int \tan^5 x \sec^2 x \, dx =$$

$$\int \sec^4 x (\sec x \tan x) \, dx =$$

Evaluate using the substitution rule:

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$\int \tan^5 x \sec^2 x \, dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}\tan^6 x + C$$

$$\int \sec^4 x (\sec x \tan x) \, dx =$$

Evaluate using the substitution rule:

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$\int \tan^5 x \sec^2 x \, dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}\tan^6 x + C$$

$$u = \sec x, \quad du = \sec x \tan x \, dx$$

$$\int \sec^4 x (\sec x \tan x) \, dx =$$

Evaluate using the substitution rule:

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$\int \tan^5 x \sec^2 x \, dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}\tan^6 x + C$$

$$u = \sec x, \quad du = \sec x \tan x \, dx$$

$$\int \sec^4 x (\sec x \tan x) \, dx = \int u^4 du = \frac{1}{5}u^5 + C = \frac{1}{5}\sec^5 x + C$$

# CHECK OUR WORK

Let's check that  $\int \tan^5 x \sec^2 x dx = \frac{1}{6} \tan^6 x + C$  by differentiating.

# CHECK OUR WORK

Let's check that  $\int \tan^5 x \sec^2 x dx = \frac{1}{6} \tan^6 x + C$  by differentiating.

$$\frac{d}{dx} \left\{ \frac{1}{6} \tan^6 x + C \right\} = \frac{6}{6} \tan^5 x \sec^2 x = \tan^5 x \sec^2 x$$

So, our answer works.

# CHECK OUR WORK

Let's check that  $\int \sec^4 x (\sec x \tan x) dx = \frac{1}{5} \sec^5 x + C$  by differentiating.

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$$\frac{d}{dx} \left\{ \frac{1}{5} \sec^5 x + C \right\} = \frac{5}{5} \sec^4 x (\sec x \tan x) = \sec^4 x (\sec x \tan x)$$

So, our answer works.



Evaluate using the identity  $\sec^2 x = 1 + \tan^2 x$

$$\int \tan^4 x \sec^6 x dx =$$

$$\int \tan^3 x \sec^5 x dx =$$

$$u = \tan x, du = \sec^2 x dx$$

Reserve  $\sec^2 x$ , change the rest of the secants to tangents.

$$\begin{aligned} \int \tan^4 x \sec^6 x dx &= \int \tan^4 x (\sec^2 x)^2 \sec^2 x dx \\ &= \int \tan^4 x (1 + \tan^2 x)^2 \sec^2 x dx = \int u^4 (1 + u^2)^2 du \\ &= \int (u^4 + 2u^6 + u^8) du \\ &= \frac{1}{5}u^5 + \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\ &= \frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C \end{aligned}$$

# CHECK OUR WORK

Let's check that  $\int \tan^4 x \sec^6 x dx = \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$

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Let's check that  $\int \tan^4 x \sec^6 x dx = \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C \right\} \\ &= \tan^4 x \sec^2 x + 2 \tan^6 x \sec^2 x + \tan^8 x \sec^2 x \\ &= \tan^4 x \sec^2 x (1 + 2 \tan^2 x + \tan^4 x) = \tan^4 x \sec^2 x (1 + \tan^2 x)^2 \\ &= \tan^4 x \sec^2 x (\sec^2 x)^2 = \tan^4 x \sec^6 x \end{aligned}$$

So, our answer works.

$$u = \sec x, du = \sec x \tan x dx$$

Reserve  $\sec x \tan x$ ,

change the rest of the tangents to secants.

$$\begin{aligned} \int \tan^3 x \sec^5 x dx &= \int \tan^2 x \sec^4 x (\sec x \tan x) dx \\ &= \int (\sec^2 x - 1) \sec^4 x (\sec x \tan x) dx \\ &= \int (u^2 - 1) u^4 du \\ &= \int (u^6 - u^4) du \\ &= \frac{1}{7} u^7 - \frac{1}{5} u^5 + C \\ &= \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C \end{aligned}$$

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Let's check that  $\int \tan^3 x \sec^5 x dx = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C.$

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Let's check that  $\int \tan^3 x \sec^5 x dx = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C$ .

$$\begin{aligned} \frac{d}{dx} \left\{ \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C \right\} &= \sec^6 x \sec x \tan x - \sec^4 x \sec x \tan x \\ &= \sec^5 x \tan x (\sec^2 x - 1) = \sec^5 x \tan x (\tan^2 x) = \tan^3 x \sec^5 x \end{aligned}$$

So, our answer works.

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :



# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

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- Reserve  $\sec x \tan x$  for the differential.

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- ▶ Reserve  $\sec x \tan x$  for the differential.
- ▶ From the remaining  $\tan^{m-1} x \sec^{n-1} x$ , convert all tangents to secants using  $\tan^2 x + 1 = \sec^2 x$ .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

- ▶ Reserve  $\sec x \tan x$  for the differential.  
( $m, n$  should each be at least 1)
- ▶ From the remaining  $\tan^{m-1} x \sec^{n-1} x$ , convert all tangents to secants using  $\tan^2 x + 1 = \sec^2 x$ .

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To use the substitution  $u = \sec x$ ,  $du = \sec x \tan x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be , and  $m$  should be .

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To use the substitution  $u = \sec x$ ,  $du = \sec x \tan x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be at least one, and  $m$  should be odd.

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  for the differential.
- ▶ From the remaining terms, convert all  to  using  $\tan^2 x + 1 = \sec^2 x$ .

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate  $\int \tan^m x \sec^n x dx$ ,  $n$  should be .



# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  $\sec^2 x$  for the differential.  
 $(n \geq 2)$
- ▶ From the remaining terms, convert all  to   
 using  $\tan^2 x + 1 = \sec^2 x$ .

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate  
 $\int \tan^m x \sec^n x dx$ ,  $n$  should be .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  $\sec^2 x$  for the differential.  
 $(n \geq 2)$
- ▶ From the remaining terms, convert all  $\secants$  to   
 using  $\tan^2 x + 1 = \sec^2 x$ .

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  $\sec^2 x$  for the differential.  
 $(n \geq 2)$
- ▶ From the remaining terms, convert all  $\secants$  to  $tangents$   
 using  $\tan^2 x + 1 = \sec^2 x$ .

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  $\sec^2 x$  for the differential.  
 $(n \geq 2)$
- ▶ From the remaining terms, convert all  $\secants$  to  $tangents$   
 using  $\tan^2 x + 1 = \sec^2 x$ .  
 $(n - 2 \text{ should be even, to avoid square roots})$

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be .

# CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using  $u = \tan x$ ,  $du = \sec^2 x dx$ :

- ▶ Reserve  $\sec^2 x$  for the differential.  
 $(n \geq 2)$
- ▶ From the remaining terms, convert all  $\secants$  to  $tangents$   
 using  $\tan^2 x + 1 = \sec^2 x$ .  
 $(n - 2 \text{ should be even, to avoid square roots})$

To use the substitution  $u = \tan x$ ,  $du = \sec^2 x dx$  to evaluate

$\int \tan^m x \sec^n x dx$ ,  $n$  should be  $\text{even (and at least 2)}$ .

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$

Choose a substitution for the integrals below.

▶  $\int \sec^2 x \tan^3 x dx$

▶  $\int \sec^2 x \tan^2 x dx$

▶  $\int \sec^3 x \tan^3 x dx$

$$\int \sec^2 x \tan^3 x dx$$

$$\int \sec^2 x \tan^3 x dx$$

Solution 1:  $u = \tan x$ ,  $du = \sec^2 x dx$ :

$$\int \sec^2 x \tan^3 x dx = \int u^3 du \Big|_{u=\tan x}$$

Solution 2:  $u = \sec x$ ,  $du = \sec x \tan x dx$ :

$$\begin{aligned} \int \sec^2 x \tan^3 x dx &= \int \tan^2 x \sec x (\sec x \tan x) dx \\ &= \int (\sec^2 x - 1) \sec x (\sec x \tan x) dx \\ &= \int (u^2 - 1) u du \Big|_{u=\sec x} \end{aligned}$$

(the rest you can do)



$$\int \sec^2 x \tan^2 x dx$$

$$\int \sec^2 x \tan^2 x dx$$

Let  $u = \tan x$  and  $du = \sec^2 x dx$ .

$$\int \sec^2 x \tan^2 x dx = \int u^2 du$$

(the rest you can do)

$$\int \sec^3 x \tan^3 x \, dx$$

$$\int \sec^3 x \tan^3 x dx$$

Let  $u = \sec x$  and  $du = \sec x \tan x dx$ .

$$\begin{aligned} \int \sec^3 x \tan^3 x dx &= \int \sec^2 x \tan^2 x (\sec x \tan x) dx \\ &= \int \sec^2 x (\sec^2 x - 1) (\sec x \tan x) dx \\ &= \int u^2 (u^2 - 1) du \end{aligned}$$

(the rest you can do)

Evaluate  $\int \tan^3 x dx$

Evaluate  $\int \tan^3 x dx = \int \frac{\sin^3 x}{\cos^3 x} dx$

$$\text{Evaluate } \int \tan^3 x \, dx = \int \frac{\sin^3 x}{\cos^3 x} \, dx$$

Let  $u = \cos x$ ,  $du = -\sin x \, dx$ .

$$\begin{aligned} &= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, dx = \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x \, dx \\ &= - \int \frac{1 - u^2}{u^3} du \\ &= \int \left( \frac{1}{u} - u^{-3} \right) du \\ &= \log |u| + \frac{1}{2} u^{-2} + C \\ &= \log |\cos x| + \frac{1}{2} \sec^2 x + C \end{aligned}$$

# CHECK OUR WORK

Let's check that  $\int \tan^3 x dx = \log |\cos x| + \frac{1}{2} \sec^2 x + C$  by differentiating.



# CHECK OUR WORK

Let's check that  $\int \tan^3 x dx = \log |\cos x| + \frac{1}{2} \sec^2 x + C$  by differentiating.

$$\begin{aligned}\frac{d}{dx} \left\{ \log |\cos x| + \frac{1}{2} \sec^2 x + C \right\} &= \frac{-\sin x}{\cos x} + \frac{1}{2} (2 \sec x) \sec x \tan x \\ &= -\tan x + \sec^2 x \tan x \\ &= -\tan x + (\tan^2 x + 1) \tan x \\ &= -\tan x + \tan^3 x + \tan x \\ &= \tan^3 x\end{aligned}$$

So, indeed,  $\int \tan^3 x dx = \log |\cos x| + \frac{1}{2} \sec^2 x + C$ .

Generalizing the last example:

$$\int \tan^m x \sec^n x dx =$$

Generalizing the last example:

$$\begin{aligned}
 \int \tan^m x \sec^n x dx &= \int \left( \frac{\sin x}{\cos x} \right)^m \left( \frac{1}{\cos x} \right)^n dx \\
 &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\
 &= \int \left( \frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx
 \end{aligned}$$

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \left( \frac{\sin x}{\cos x} \right)^m \left( \frac{1}{\cos x} \right)^n dx \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\ &= \int \left( \frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx\end{aligned}$$

To use  $u = \cos x$ ,  $du = -\sin x dx$ :

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \left( \frac{\sin x}{\cos x} \right)^m \left( \frac{1}{\cos x} \right)^n dx \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\ &= \int \left( \frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx\end{aligned}$$

To use  $u = \cos x$ ,  $du = -\sin x dx$ : we will convert  $\sin^{m-1}(x)$  into cosines, so  $m - 1$  must be even, so  $m$  must be odd.

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd

Evaluate  $\int \tan^2 x dx$

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd

Evaluate  $\int \tan^2 x dx$

## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:

- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd

Evaluate  $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$



## Evaluating $\int \tan^m x \sec^n x dx$

To evaluate  $\int \tan^m x \sec^n x dx$ , we can use:


- ▶  $u = \sec x$  if  $m$  is odd and  $n \geq 1$
- ▶  $u = \tan x$  if  $n$  is even and  $n \geq 2$
- ▶  $u = \cos x$  if  $m$  is odd
- ▶  $u = \tan x$  if  $m$  is even and  $n = 0$   
(after using  $\tan^2 x = \sec^2 x - 1$ , maybe several times)

Evaluate  $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

## Included Work

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