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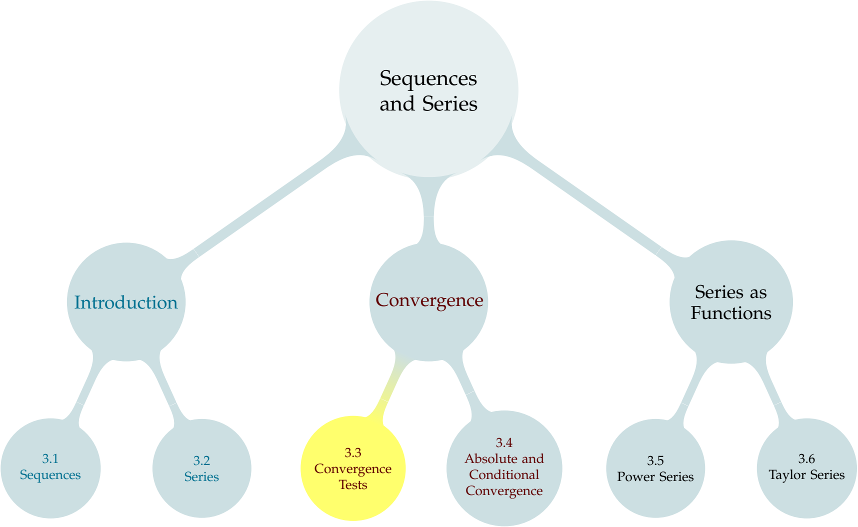

Review

Let $S_N = \sum_{n=1}^{N} a_n$.

Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

REVIEW

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$$\begin{aligned} S_N &= a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N \\ S_{N-1} &= a_1 + a_2 + a_3 + \cdots + a_{N-1} \end{aligned}$$

Review

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Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

$$\begin{aligned} S_N &= a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N \\ S_{N-1} &= a_1 + a_2 + a_3 + \cdots + a_{N-1} \\ S_N - S_{N-1} &= a_N \end{aligned}$$

Alternating Series

Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

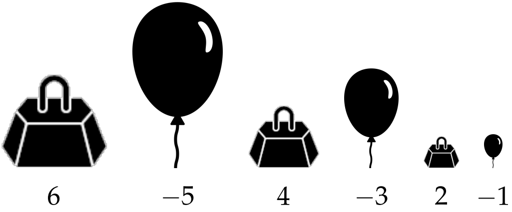
is alternating if every $A_n \geq 0$.

Alternating series:

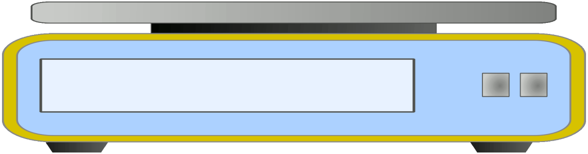
- $1-2+3-4+5-6+7-8+\cdots$
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

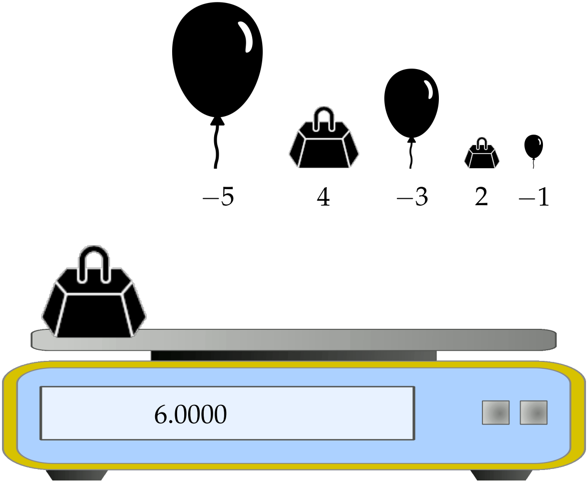
Not alternating:

- $\cos(1) + \cos(2) + \cos(3) + \cdots$
- $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$

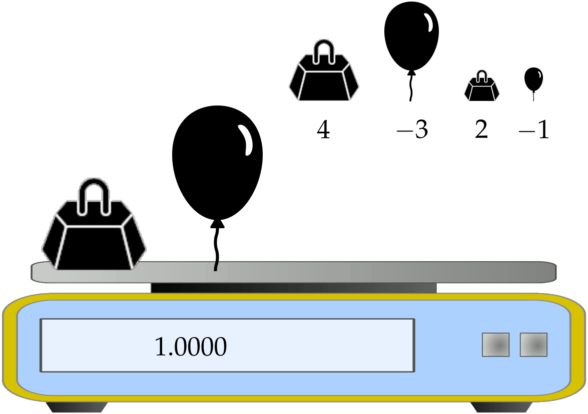


Note: these terms alternate signs, and their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$





$S_1 = 6.0000$



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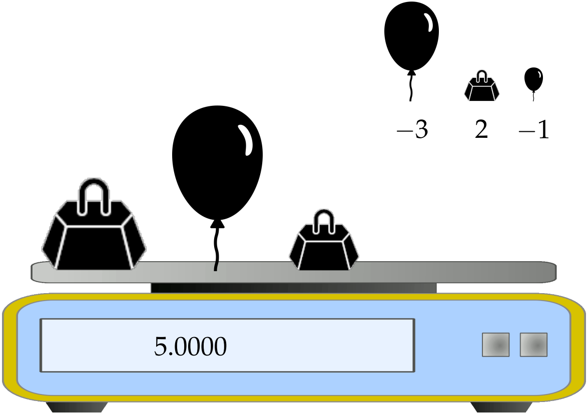

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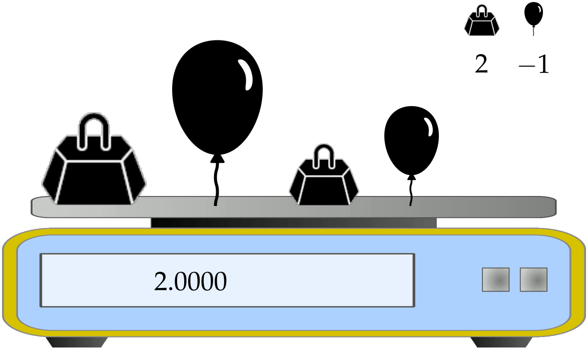

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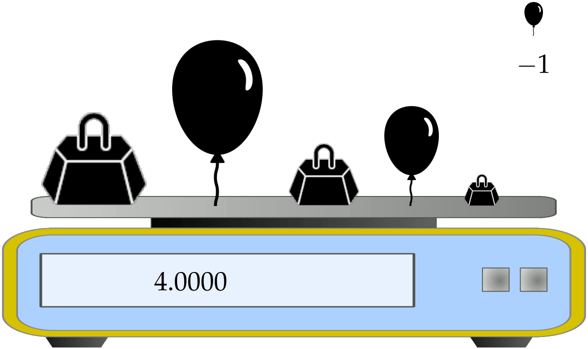
$$S_4 = 2.0000$$



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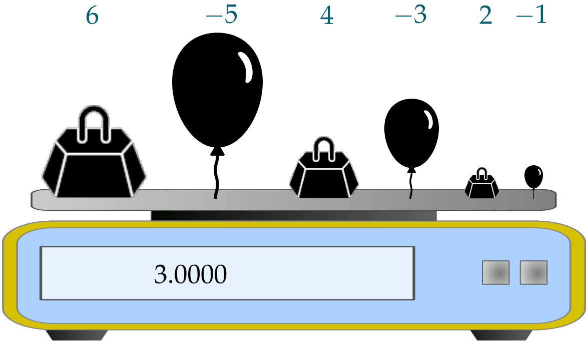

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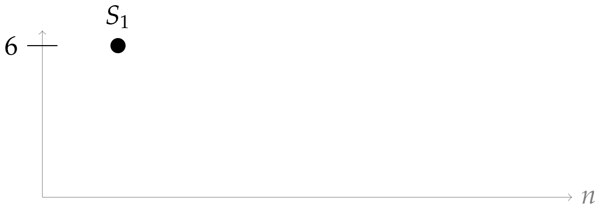

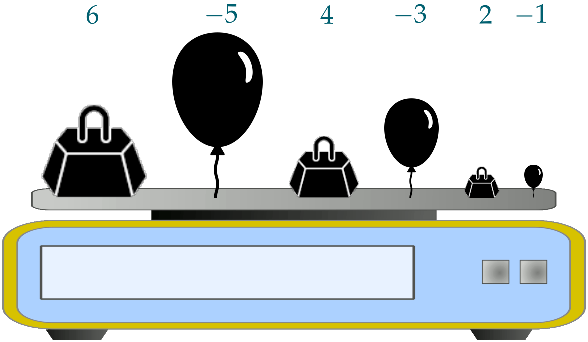

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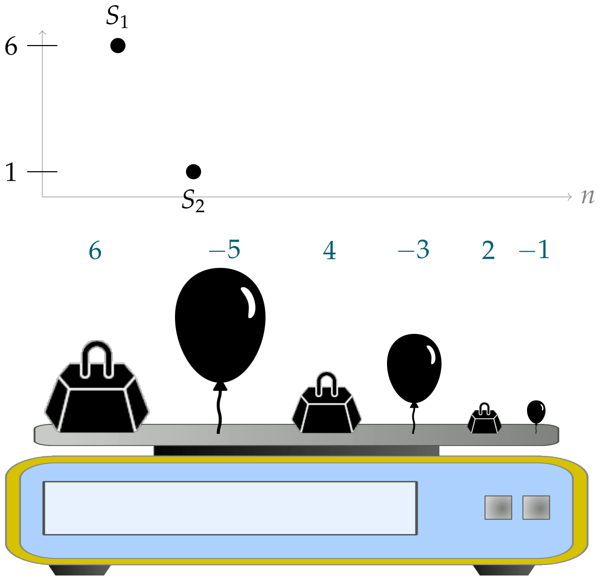

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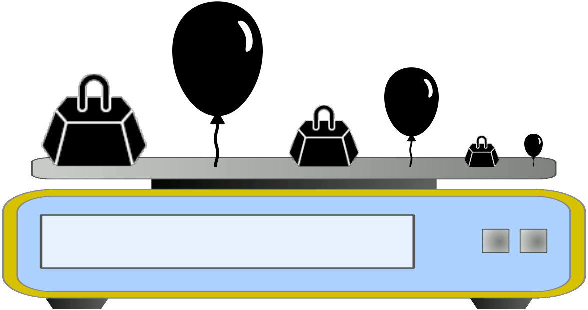

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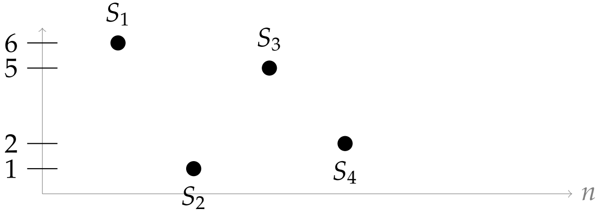

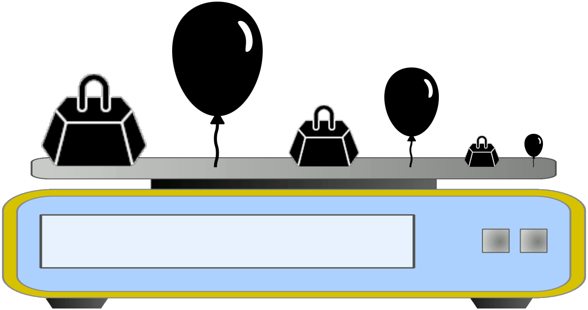

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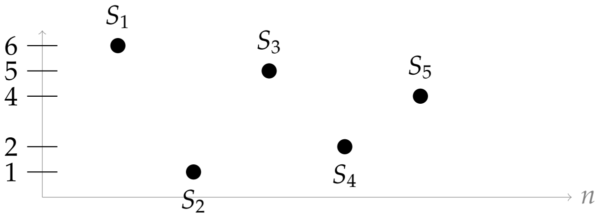

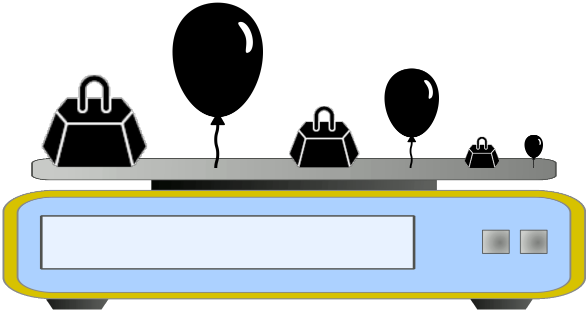

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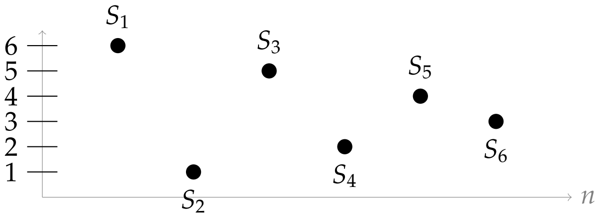

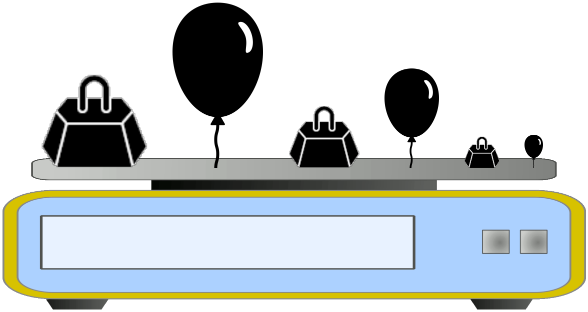

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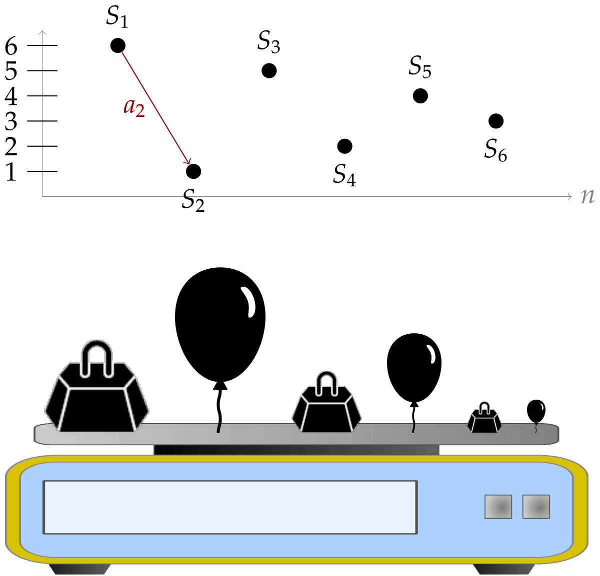

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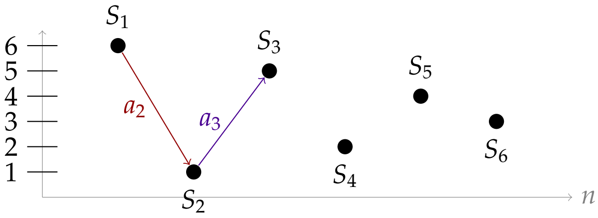

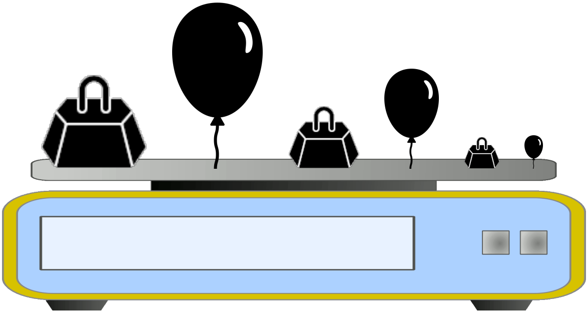

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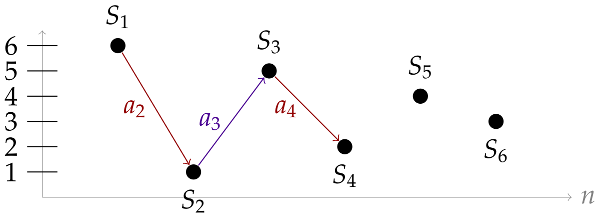

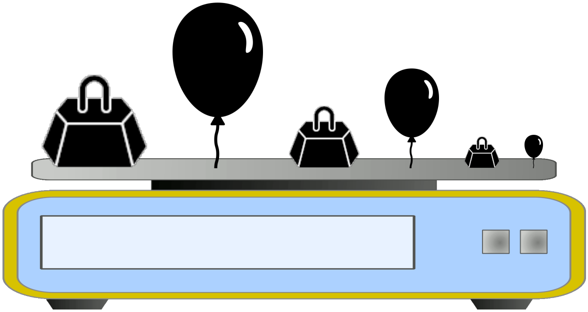

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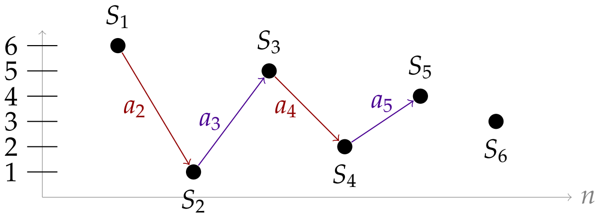

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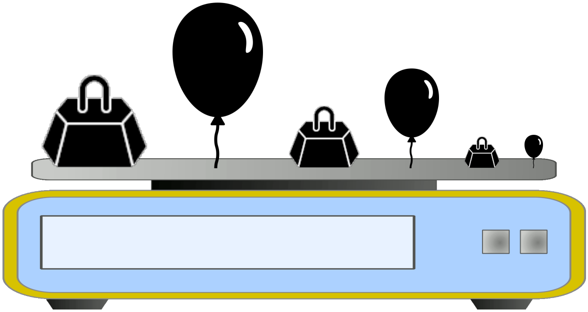

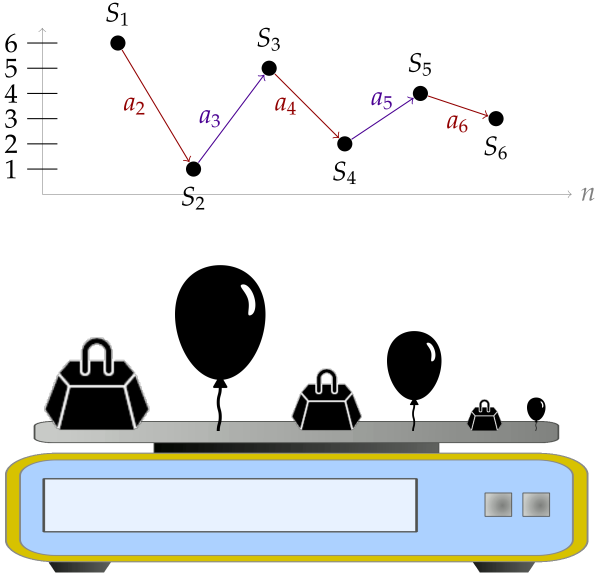

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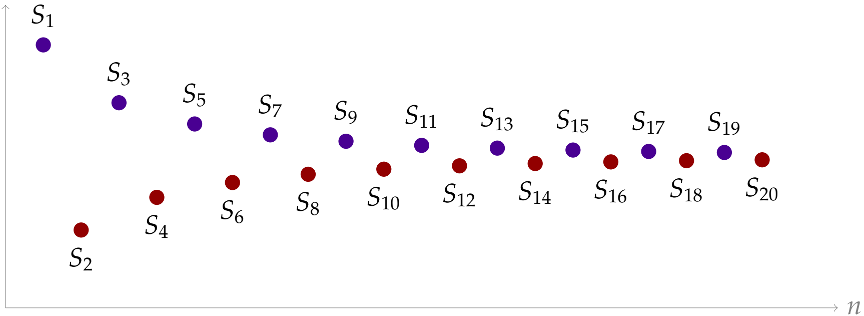
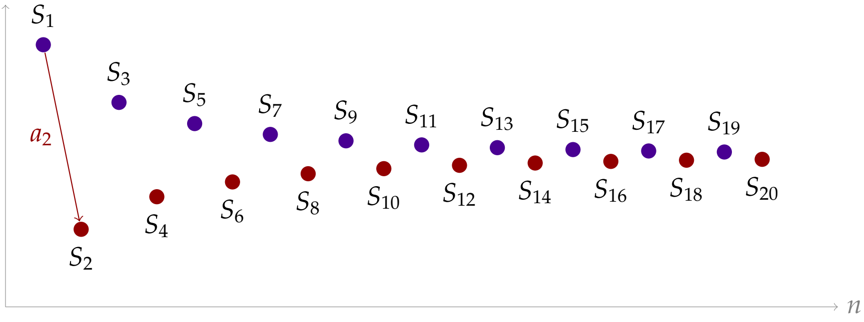
Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.

Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



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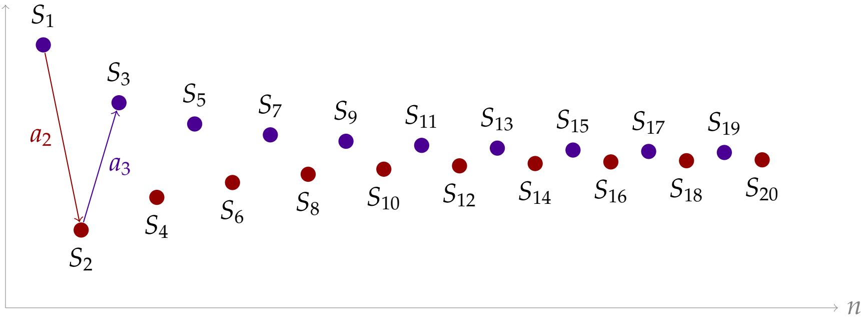

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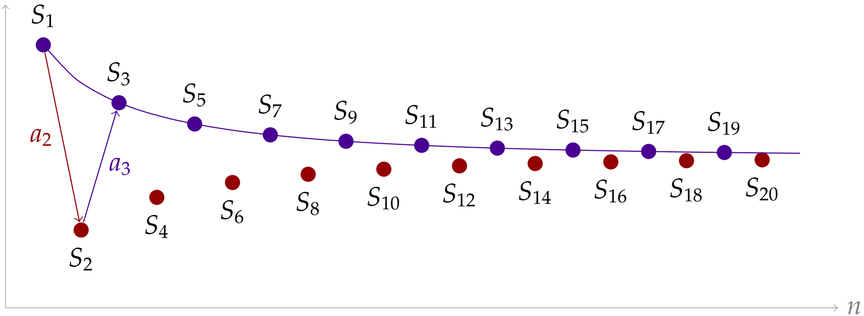
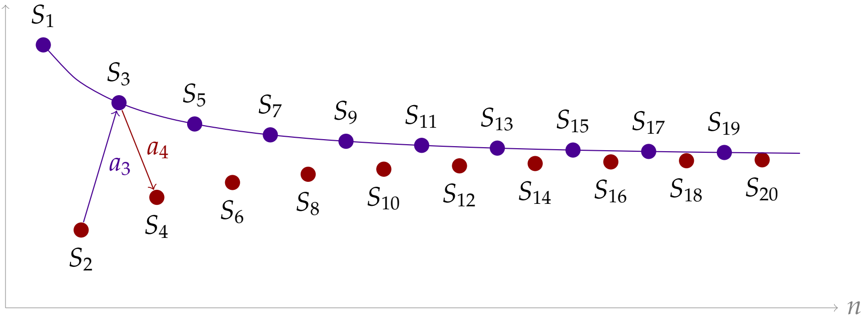


Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.
Odd-indexed partial sums are decreasing.

Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



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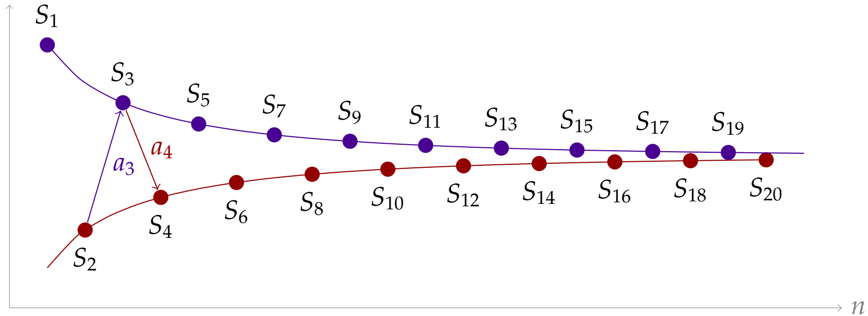
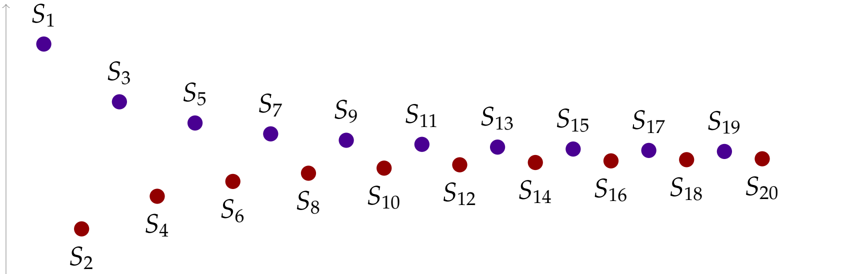


Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.
Odd-indexed partial sums are decreasing.

Since $a_3 > a_4$, we have $a_1 - a_2 + (a_3 - a_4) > a_1 - a_2$, so $S_4 > S_2$.
Even-indexed partial sums are increasing.


S_1
S_2
S_3
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S_5
S_6
S_7
S_8
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S_{10}
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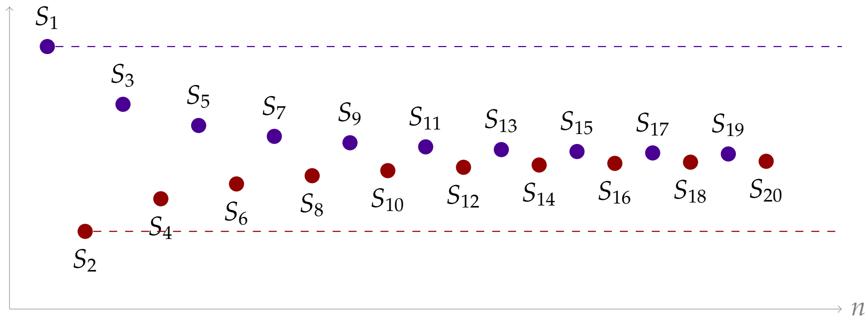
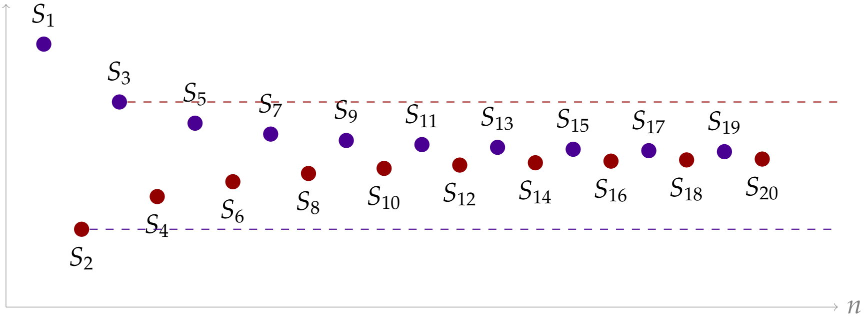
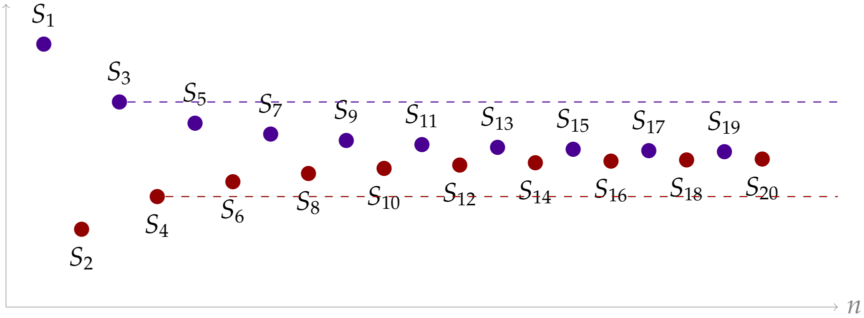


- For all $n \geq 2$, S_n lies between S_1 and S_2.



- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.



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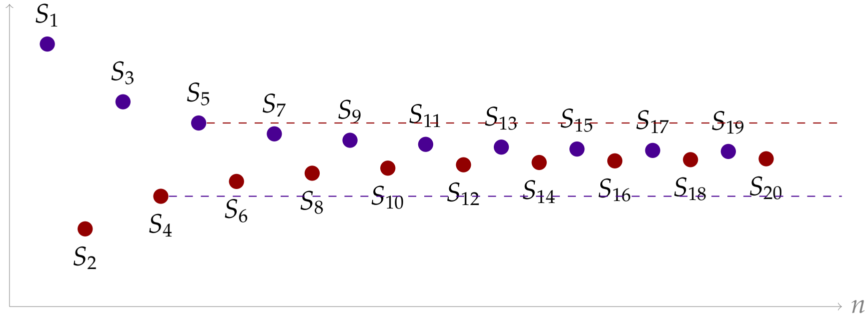

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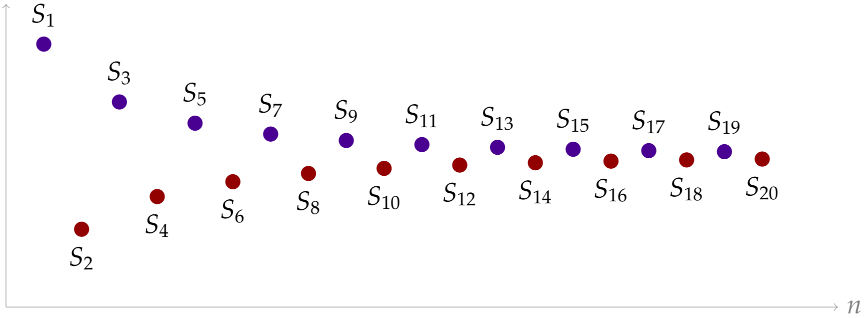

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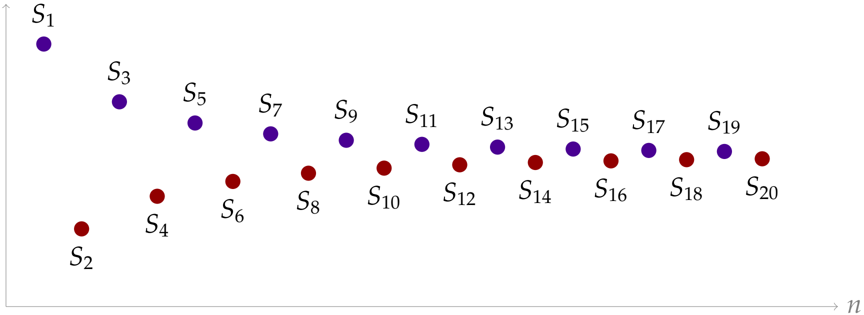
The difference between consecutive sums S_n and S_{n-1} is:



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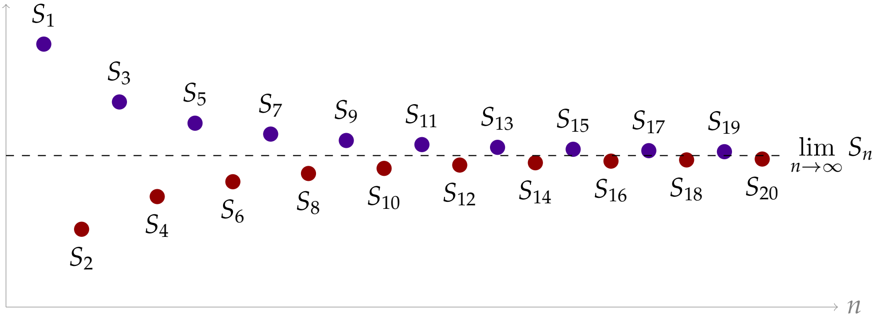
The difference between consecutive sums S_n and S_{n-1} is: $|a_n|$, which approaches 0.



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The difference between consecutive sums S_n and S_{n-1} is: $|a_n|$, which approaches 0.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \ge 0$ for all $n \ge 1$;

(ii) $a_{n+1} \le a_n$ for all $n \ge 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim\limits_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum\limits_{n=1}^{N} (-1)^{n-1} a_n$.

Alternating Series Test (abridged)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

- True or false: the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges.
- True or false: the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Let $a_n = \frac{1}{n}$.

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(i) $a_n \geq 0$

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(i) $a_n \geq 0$

(ii) $a_{n+1} \leq a_n$

(iii) $\lim\limits_{n\to\infty} a_n = 0$

- We've already seen that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges. That is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

Divergence Test + Alternating Series Test

$$\lim_{n\to\infty} a_n = ?$$

Divergence Test + Alternating Series Test

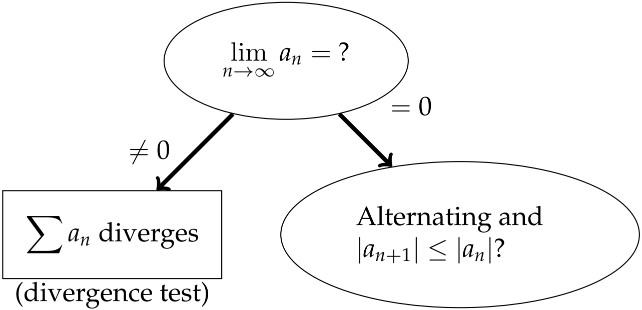
$$\lim_{n\to\infty} a_n = ?$$

$$\big\downarrow{\scriptstyle \neq 0}$$

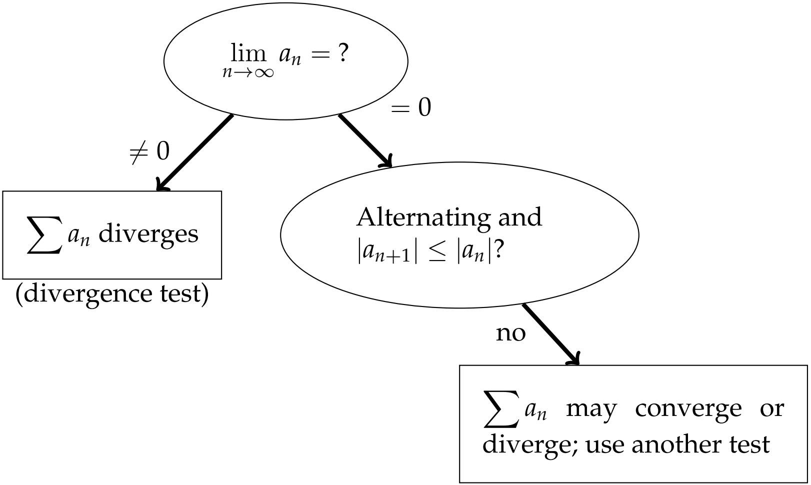
$$\boxed{\sum a_n \text{ diverges}}$$

(divergence test)

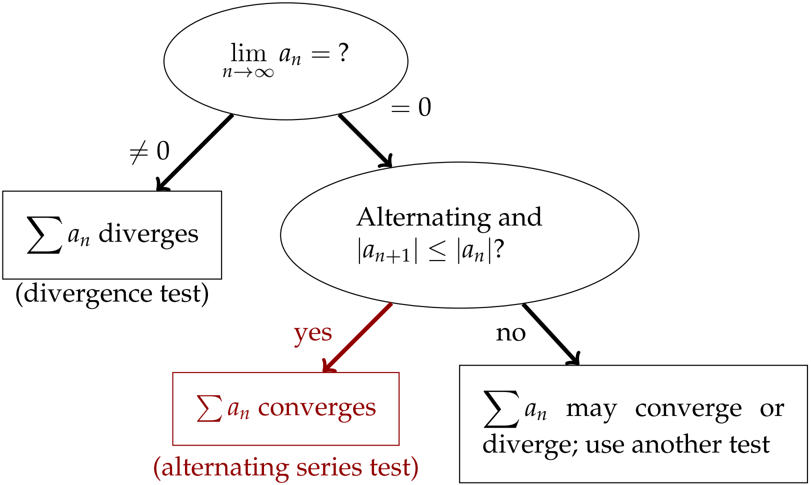
Divergence Test + Alternating Series Test



Divergence Test + Alternating Series Test



Divergence Test + Alternating Series Test



Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\displaystyle\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698$.

How close is that to the value $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

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How close is that to the value $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

$$\frac{-1}{100} = \frac{(-1)^{100-1}}{100} \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{99} \frac{(-1)^n}{n} \leq 0.$$

That is, the actual series has a sum in the interval $[0.688, 0.698]$.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\displaystyle\sum_{n=1}^{19}(-1)^{n-1}\frac{n^2}{n^2+1} \approx 0.6347$.

How close is that to the value $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$?

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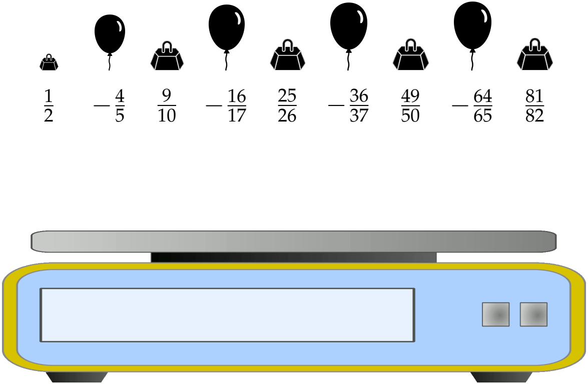
How close is that to the value $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$?

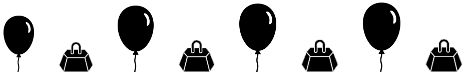
Not close at all: the series is divergent (which we can see by the divergence test).

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

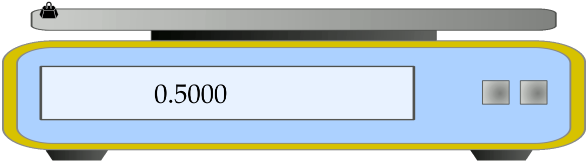


$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$S_1 = 0.5000$



$-\frac{4}{5}$ $\frac{9}{10}$ $-\frac{16}{17}$ $\frac{25}{26}$ $-\frac{36}{37}$ $\frac{49}{50}$ $-\frac{64}{65}$ $\frac{81}{82}$

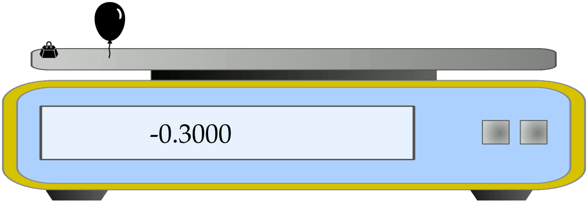


$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$\frac{9}{10} \quad -\frac{16}{17} \quad \frac{25}{26} \quad -\frac{36}{37} \quad \frac{49}{50} \quad -\frac{64}{65} \quad \frac{81}{82}$$

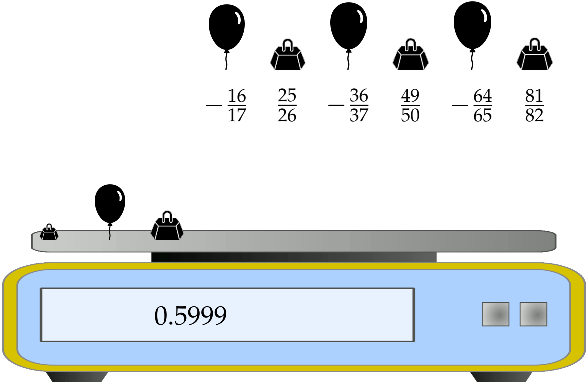


$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

$$S_3 = 0.5999$$



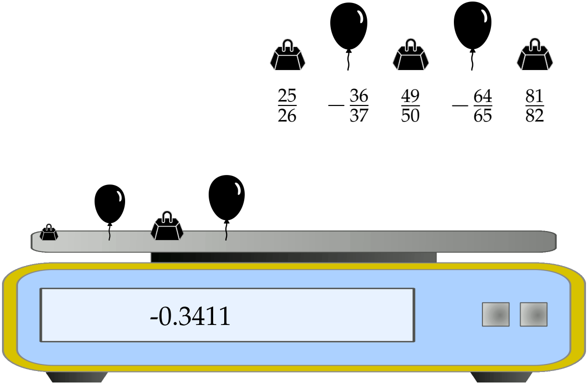
$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES

$$S_1 = 0.5000$$

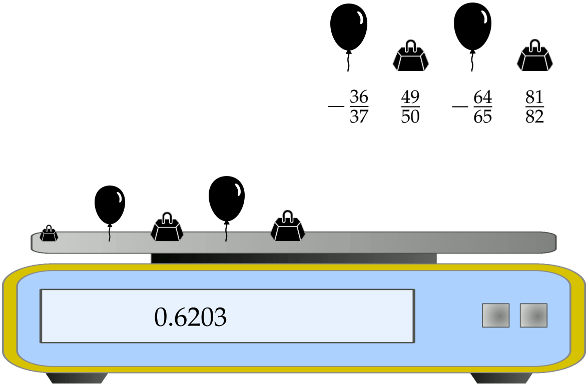
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$



$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$$S_1 = 0.5000$$

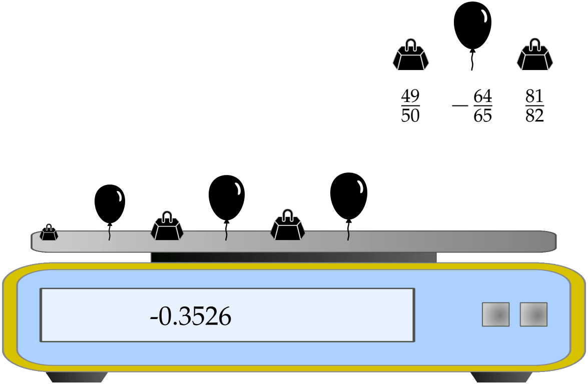
$$S_2 = -0.3000$$

$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$$S_1 = 0.5000$$

$$S_2 = -0.3000$$

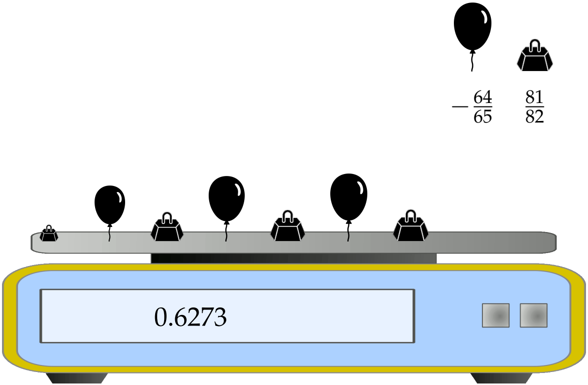
$$S_3 = 0.5999$$

$$S_4 = -0.3411$$

$$S_5 = 0.6203$$

$$S_6 = -0.3526$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



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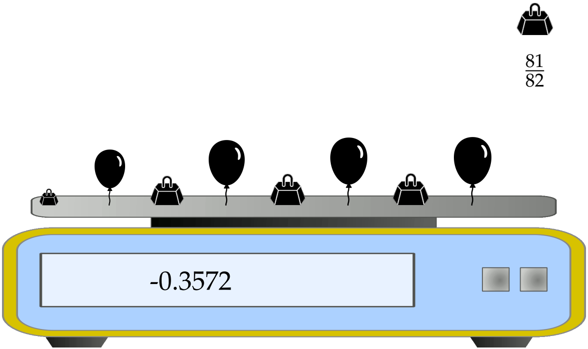
$$S_7 = 0.6273$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$S_1 = 0.5000$

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$S_3 = 0.5999$

$S_4 = -0.3411$

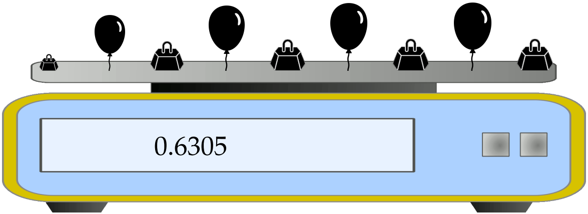
$S_5 = 0.6203$

$S_6 = -0.3526$

$S_7 = 0.6273$

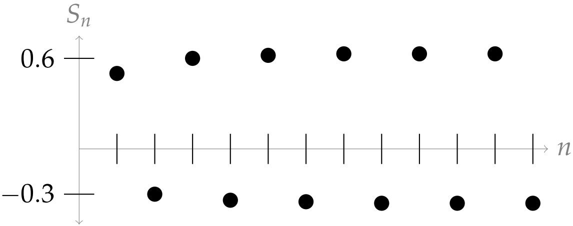
$S_8 = -0.3572$

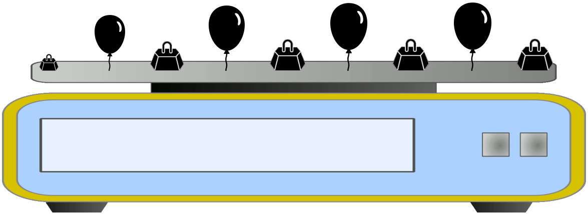
$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES



$$S_1 = 0.5000$$
$$S_2 = -0.3000$$
$$S_3 = 0.5999$$
$$S_4 = -0.3411$$
$$S_5 = 0.6203$$
$$S_6 = -0.3526$$
$$S_7 = 0.6273$$
$$S_8 = -0.3572$$
$$S_9 = 0.6305$$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$ DIVERGES





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$S_5 = 0.6203$

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$S_8 = -0.3572$

$S_9 = 0.6305$

Recall for a geometric series, the ratios of consecutive terms is constant.

$$\overset{\times \frac{1}{2}}{\curvearrowright}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

If that ratio has magnitude less then one, then the series converges.
If the ratio has magnitude greater than one, the series diverges.

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$$\frac{1}{2} \overset{\times\frac{1}{2}}{\curvearrowright} \frac{1}{4} \overset{\times\frac{1}{2}}{\curvearrowright} \frac{1}{8}$$

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$$\underbrace{\frac{1}{2} \xrightarrow{\times\frac{1}{2}} \frac{1}{4}}_{\frac{1/4}{1/2} =} \xrightarrow{\times\frac{1}{2}} \frac{1}{8} \xrightarrow{\times\frac{1}{2}} \frac{1}{16} \xrightarrow{\times\frac{1}{2}} \frac{1}{32}$$

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$$\underbrace{\frac{1}{2} + \frac{1}{4}}_{\frac{1/4}{1/2}\,=} + \underbrace{\frac{1}{8}}_{\frac{1/8}{1/4}\,=} + \frac{1}{16} + \frac{1}{32} \cdots$$

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$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

$$\frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} =$$

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$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

$$\frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} = \frac{1/32}{1/16} = \frac{1}{2}$$

If that ratio has magnitude less then one, then the series converges. If the ratio has magnitude greater than one, the series diverges.

For series convergence, we are concerned with what happens to terms a_n when n is sufficiently large.
Suppose for a sequence a_n, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ for some constant L.

$$\underbrace{a_n \quad + \quad a_{n+1}}_{\frac{a_{n+1}}{a_n} \approx} \quad + \quad a_{n+2} \quad + \quad a_{n+3} \quad + \quad a_{n+4} \quad + \cdots$$

Like in a geometric series:

If L has magnitude less then one, then the series converges.
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$$\underbrace{a_n \quad + \quad a_{n+1}}_{\frac{a_{n+1}}{a_n}} \underbrace{\quad + \quad a_{n+2}}_{\approx \frac{a_{n+2}}{a_{n+1}}} \underbrace{\quad + \quad a_{n+3}}_{\approx \frac{a_{n+3}}{a_{n+2}}} \underbrace{\quad + \quad a_{n+4}}_{\approx \frac{a_{n+4}}{a_{n+3}}} \underbrace{\quad + \quad \cdots}_{\approx \frac{a_{n+5}}{a_{n+4}} \approx}$$

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Like in a geometric series:

If L has magnitude less then one, then the series converges.
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Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

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Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges or diverges.

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Q

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converges or diverges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{3}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, by the ratio test, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$ coverges.

Remark

The series we just considered, $\sum_{n=1}^{\infty} \frac{n}{3^n}$, looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

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We could have used other tests, but ratio was probably the easiest.

- Integral test:

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- Integral test: $\int \frac{x}{3^x}\,\mathrm{d}x$ can be evaluated using integration by parts.
- Comparison test:

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  - $\sum \frac{1}{3^n}$ is not a valid comparison series, nor is $\sum n$.

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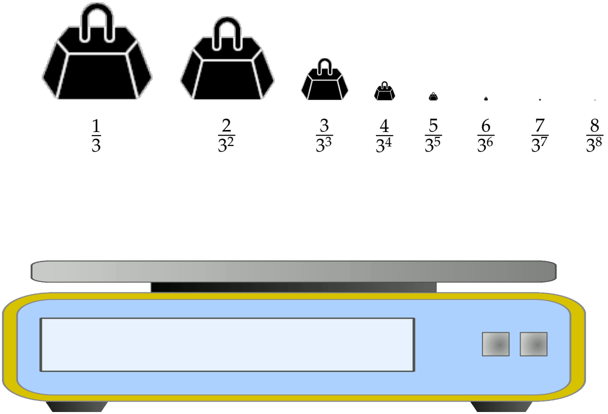
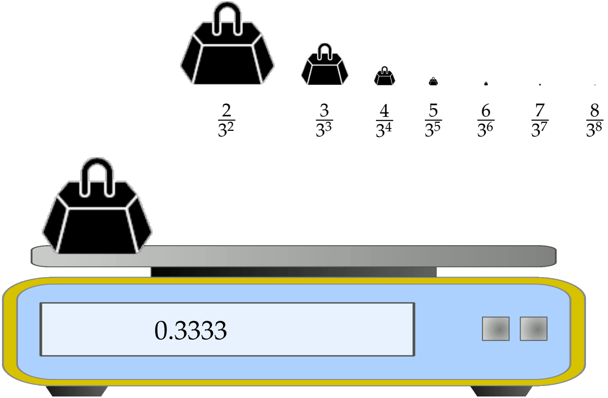
  - Because $n < 2^n$ for all $n \geq 1$, the series $\sum \left(\frac{2}{3}\right)^n$ will work.
- The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

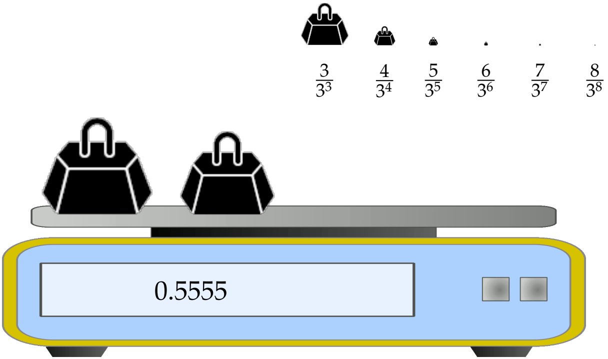
$$S_1 = 0.3333$$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

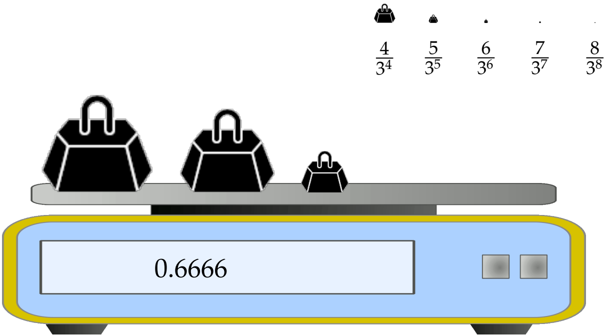


$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$S_1 = 0.3333$

$S_2 = 0.5555$

$S_3 = 0.6666$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

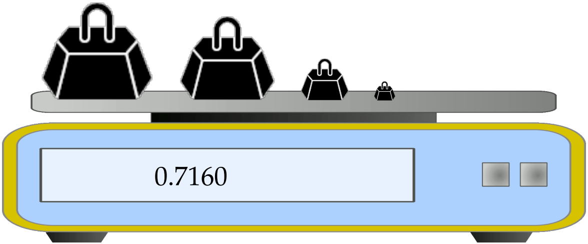
$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

$$S_3 = 0.6666$$

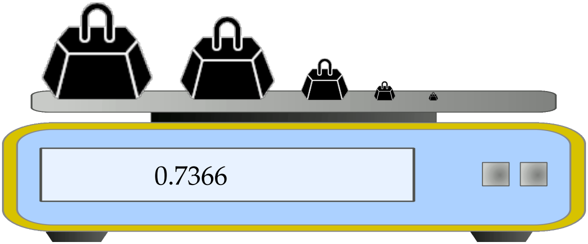
$$S_4 = 0.7160$$

$\frac{5}{3^5}$ $\frac{6}{3^6}$ $\frac{7}{3^7}$ $\frac{8}{3^8}$



$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$\frac{6}{3^6}$ $\frac{7}{3^7}$ $\frac{8}{3^8}$



$S_1 = 0.3333$

$S_2 = 0.5555$

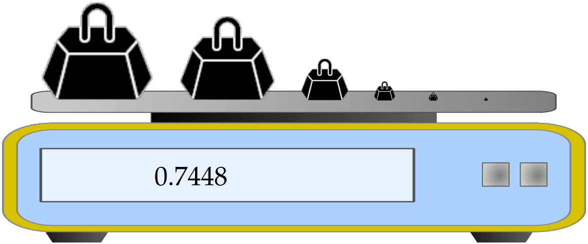
$S_3 = 0.6666$

$S_4 = 0.7160$

$S_5 = 0.7366$

$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES

$\frac{7}{3^7}$ $\frac{8}{3^8}$



$$S_1 = 0.3333$$

$$S_2 = 0.5555$$

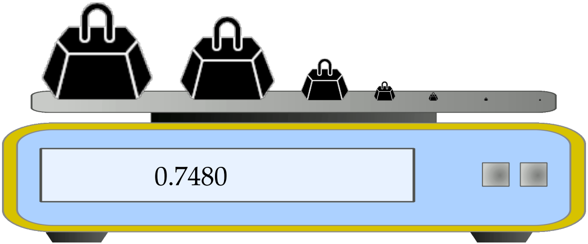
$$S_3 = 0.6666$$

$$S_4 = 0.7160$$

$$S_5 = 0.7366$$

$$S_6 = 0.7448$$

$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES



$\frac{8}{3^8}$

$S_1 = 0.3333$

$S_2 = 0.5555$

$S_3 = 0.6666$

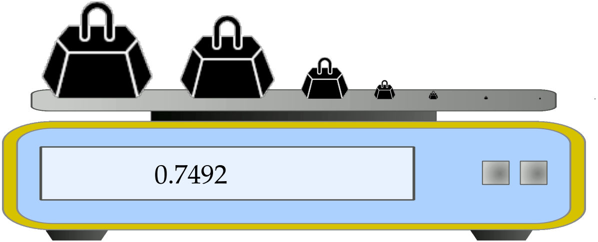
$S_4 = 0.7160$

$S_5 = 0.7366$

$S_6 = 0.7448$

$S_7 = 0.7480$

$\sum_{n=1}^{\infty} \frac{n}{3^n}$ CONVERGES



$S_1 = 0.3333$

$S_2 = 0.5555$

$S_3 = 0.6666$

$S_4 = 0.7160$

$S_5 = 0.7366$

$S_6 = 0.7448$

$S_7 = 0.7480$

$S_8 = 0.7492$

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Let a and x be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of a and x.)

$$\sum_{n=1}^{\infty} anx^{n-1}$$

$$\sum_{n=1}^{\infty} anx^{n-1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a(n+1)x^n}{anx^{n-1}}\right| = \left|\left(\frac{n+1}{n}\right)x\right| = \left(1+\frac{1}{n}\right)|x|$$
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. For the cases $x = \pm 1$, the ratio test is inconclusive, so we'll need another test. Fortunately, the divergence test makes things quick.

$$\text{For } x = 1: \qquad \lim_{n\to\infty} an(1)^{n-1} = \lim_{n\to\infty} an \neq 0$$
$$\text{For } x = -1: \qquad \lim_{n\to\infty} an(-1)^{n-1} \neq 0$$

All together, for any nonzero a, the series diverges when $|x| \geq 1$ and converges when $|x| < 1$.

A

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x.)

Q

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n\sqrt{n+1}}{2n+3}x^n$$

converges or diverges. (This may depend on the value of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}\sqrt{n+2}}{2(n+1)+3}x^{n+1}}{\frac{(-3)^n\sqrt{n+1}}{2n+3}x^n}\right| = \left|\frac{(-3)^{n+1}}{(-3)^n}\cdot\frac{\sqrt{n+2}}{\sqrt{n+1}}\cdot\frac{2n+3}{2n+5}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$= 3\cdot\sqrt{\frac{n+2}{n+1}}\cdot\left(\frac{2n+3}{2n+5}\right)\cdot|x| = 3\sqrt{\frac{1+2/n}{1+1/n}}\cdot\left(\frac{2+3/n}{2+5/n}\right)\cdot|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3\sqrt{\frac{1}{1}}\left(\frac{2}{2}\right)|x| = 3|x|$$

So the series converges when $3|x| < 1$ and diverges when $3|x| > 1$.
So for $|x| < \frac{1}{3}$, the series converges, and for $|x| > \frac{1}{3}$, it diverges.

Consider $x = \frac{1}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}}{2n+3}$$

This is an alternating series. Let's use the alternating series test.

(i) $a_n = \frac{\sqrt{n+1}}{2n+3} \geq 0$ for all $n \geq 1$,

(ii) To show that a_n is monotonically decreasing, consider the derivative of $f(t) = \frac{\sqrt{t+1}}{2t+3}$:

$$f'(t) = \frac{(2t+3)\frac{1}{2\sqrt{t+1}} - \sqrt{t+1}(2)}{(2t+3)^2} \left(\frac{\sqrt{t+1}}{\sqrt{t+1}} \right)$$

$$= \frac{\left(t + \frac{3}{2}\right) - (t+1)(2)}{(2t+3)^2\sqrt{t+1}} = \frac{-t - \frac{1}{2}}{(2t+3)^2\sqrt{t+1}}$$

Since $f'(t) < 0$ for all $t > 0$, we see it is a decreasing function on that domain, so $a_{n+1} < a_n$ for all $n \geq 1$.

(iii) $\lim_{n \to \infty} a_n = 0$

So, our series converges by the alternating series test when $x = \frac{1}{3}$.

Finally, consider $x = -\frac{1}{3}$.

$$\sum_{n=1}^{\infty} \frac{(-3)^n\sqrt{n+1}}{2n+3}x^n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3}\frac{(-3)^n}{(-3)^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3}$$

We will use the limit comparison test, with comparison series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{2n+3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n+1}\sqrt{n}}{2n+3} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}}{2n+3}\left(\frac{1/n}{1/n}\right)$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n}}{2+3/n} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2}$$

Since $\frac{1}{2}$ is a nonzero constant, and since $\sum \frac{1}{\sqrt{n}}$ diverges (by the p-test), our series diverges as well.

All together, the original series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, and diverges otherwise.

A

Fill in in the blanks

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ $\boxed{}$

then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \boxed{}$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \boxed{}$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Fill in in the blanks

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ fails to converge to zero as $n \to \infty$, then the series $\sum_{n=c}^{\infty} a_n$ diverges.

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FILL IN IN THE BLANKS

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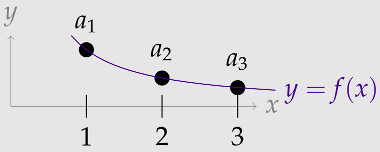
Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

(i) $\boxed{}$ and

(ii) $\boxed{}$ and

(iii) $f(n) = a_n$ for all $n \geq N_0$.



Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\, \mathrm{d}x \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_{N}^{\infty} f(x)\, \mathrm{d}x \qquad \text{for all } N \geq N_0$$

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Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

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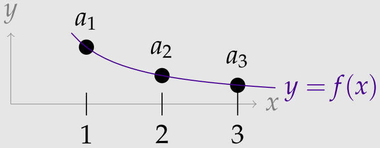

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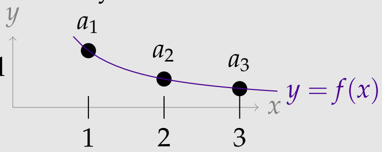

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Fill in in the blanks

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \boxed{} Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \boxed{} Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

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Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $\boxed{}$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

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Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $\boxed{}$

(ii) $a_{n+1} \le a_n$ for all $n \ge 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\boxed{}$

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

ans

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

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List of convergence tests

Divergence Test

When the n^{th} term in the series *fails* to converge to zero as n tends to infinity.

This is a good first thing to check: if it works, it's quick, but it doesn't always work.

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- don't forget to check that successive terms decrease in magnitude and tend to zero as n tends to infinity

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Integral Test

- works well when, if you substitute x for n in the n^{th} term you get a function, $f(x)$, that you can easily integrate
- don't forget to check that $f(x) \geq 0$ and that $f(x)$ decreases as x increases

List of convergence tests

Ratio Test

LIST OF CONVERGENCE TESTS

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- works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$
- this often happens when a_n contains powers, like 7^n, or factorials, like $n!$
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Comparison Test and Limit Comparison Test

- Comparison test lets you ignore pieces of a function that feel extraneous (like replacing $n^2 + 1$ with n^2) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- Limit comparison works well when, for very large n, the n^{th} term a_n is approximately the same as a simpler, nonnegative term b_n

- The integral test gave us the p-test. When you're looking for comparison series, p-series $\sum \frac{1}{n^p}$ are often good choices, because their convergence or divergence is so easy to ascertain.

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- Geometric series have the form $\sum a \cdot r^n$ for some nonzero constants a and r. The magnitude of r is all you need to know to deicide whether they converge or diverge, so these are also common comparison series.

- Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

The **divergence test** is inconclusive, because $\lim_{n\to\infty} \frac{\cos n}{2^n} = 0$ (which you can show with the squeeze theorem).

The **integral test** doesn't apply, because $f(x) = \frac{\cos x}{2^x}$ is not always positive (and not decreasing).

The **alternating series test** doesn't apply because the signs of the series do not strictly alternate every term.

The **ratio test** does not apply, because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

Comparison test: Let $a_n = \frac{\cos n}{2^n}$. Note $|a_n| \leq \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a geometric sum with ratio of consecutive terms $\frac{1}{2}$).

So by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges.

Limit comparison: Set $a_n = \frac{\cos n}{2^n}$ and $b_n = \left(\frac{2}{3}\right)^n$. Then

$$\frac{a_n}{b_n} = \frac{\frac{\cos n}{2^n}}{\frac{2^n}{3^n}} = \left(\frac{3}{4}\right)^n \cos n$$

$$-\left(\tfrac{3}{4}\right)^n \leq \left(\tfrac{3}{4}\right)^n \cos n \leq \left(\tfrac{3}{4}\right)^n, \text{ and } \lim_{n\to\infty} -\left(\tfrac{3}{4}\right)^n = \lim_{n\to\infty} \left(\tfrac{3}{4}\right)^n = 0$$

So, by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

Since $\sum_{n=1}^{\infty} b_n$ converges, by the limit comparison theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges as well.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Q

ans

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

The **alternating series test** doesn't apply because the signs of the series do not alternate.

The **integral test** doesn't apply $f(x) = \frac{2^x \cdot x^2}{(x+5)^5}$ is not a decreasing function.

Divergence test: $\lim_{n\to\infty} \frac{2^n \cdot n^2}{(n+5)^5} = \infty$ (which you can see because the numerator is larger than a power function; the denominator is a polynomial; and power functions grow faster than polynomials), so the series diverges by the divergence test.

This is the fastest option, but not the only one.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Ratio test:

$$\frac{a_n}{b_n} = \frac{\frac{2^{n+1}\cdot(n+1)^2}{(n+1+5)^5}}{\frac{2^n \cdot n^2}{(n+5)^5}} = \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{(n+5)^5}{(n+6)^5}$$

$$= 2\left(1+\frac{1}{n}\right)^2\left(1-\frac{1}{n+6}\right)^5$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 2(1)^2(1)^5 = 2$$

So, the limit of the ratio of consecutive terms is greater than 1.

Therefore $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges by the ratio test.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Comparison test: Since power functions grow faster than polynomials, for large values of n, $2^n > (n+5)^5$, so $\frac{2^n}{(n+5)^5} > 1$. Then, for large enough n,

$$\frac{2^n \cdot n^2}{(n+5)^5} > n^2 .$$

By the divergence test, $\sum_{n=1}^{\infty} n^2$ diverges. So by the comparison test, $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges as well.

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Limit comparison: Set $a_n = \frac{2^n \cdot n^2}{(n+5)^5}$ and $b_n = \frac{2^n}{n^3}$.
Then

$$\frac{a_n}{b_n} = \frac{\frac{2^n \cdot n^2}{(n+5)^5}}{\frac{2^n}{n^3}} = \frac{n^5}{(n+5)^5} = \left(1 - \frac{5}{n+5}\right)^5$$

So, $\lim_{n\to\infty} \frac{a_n}{b_n} = 1^5 = 1$

Note that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges. (You can show this using the tests we've already used on the original series, so this method isn't really an improvement.) Since $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is nonzero, by the limit comparison theorem, $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ diverges.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

Q

ans

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

The **divergence test** is inconclusive because $\displaystyle\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n} = 0$.

The **alternating series test** does not apply because we are not considering an alternating series.

The **integral test** won't work for us because $\int_1^\infty \frac{1}{x}\sin\left(\frac{1}{x}\right)\, dx$ cannot be evaluated with techniques we've learned in class so far.

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

The **ratio test** is inconclusive because $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+1}\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}\sin\left(\frac{1}{n}\right)} = 1$:

Set $x = \frac{1}{n+1}$. Then $\frac{1}{n} = \frac{x}{1-x}$:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}} = \lim_{x\to 0^+} \frac{\sin x}{\frac{x}{1-x}} = \lim_{x\to 0^+} (1-x)\frac{\sin x}{x} = 1 \cdot 1 = 1$$

Set $y = \frac{1}{n}$. Then $\frac{1}{n+1} = \frac{y}{1+y}$:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n+1}} = \lim_{y\to 0^+} \frac{\sin y}{\frac{y}{1+y}} = \lim_{y\to 0^+} (1+y)\frac{\sin y}{y} = 1 \cdot 1 = 1$$

Therefore,

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}\sin\left(\frac{1}{n}\right)} = 1$$

A

ans

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

Comparison test: For $n \geq 1$, $\frac{1}{n} > 0$. Then setting $\theta = \frac{1}{n}$ in the hint, $\sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$. Furthermore, $0 < \frac{1}{n} < \pi$, so $\sin\left(\frac{1}{n}\right) > 0$.

$$0 < \frac{1}{n} \sin\left(\frac{1}{n}\right) \leq \frac{1}{n}\left(\frac{1}{n}\right) = \frac{1}{n^2}$$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges as well.

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

Limit comparison: Set $a_n = \frac{1}{n}\sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}\sin\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

Setting $x = \frac{1}{n}$,

$$= \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{1}{n}\right)$ converges as well.

Included Work

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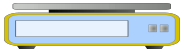 ‘Waage/Libra’ by B. Lachner is in the public domain (accessed April 2021, edited), 6–23, 52–62, 88–96

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