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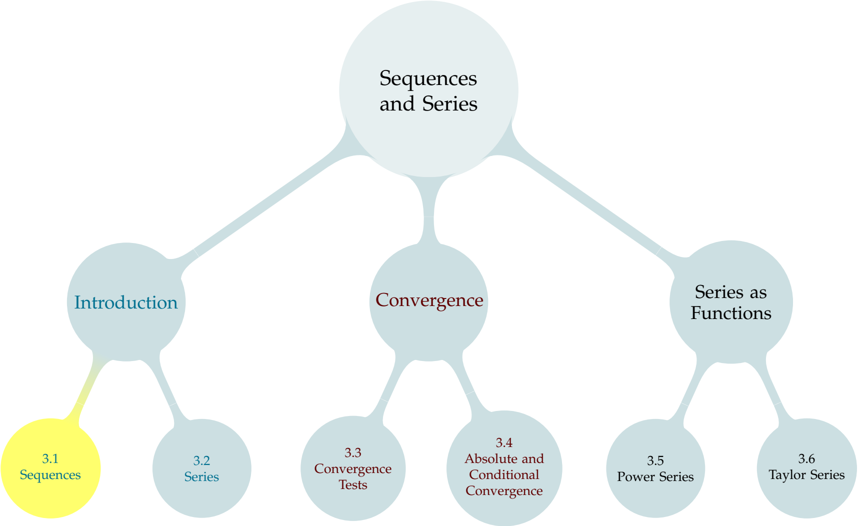

We can imagine the list of numbers below carrying on forever:

$$
\begin{aligned}
a_1 &= 0.1 \\
a_2 &= 0.01 \\
a_3 &= 0.001 \\
a_4 &= 0.0001 \\
a_5 &= 0.00001 \\
&\vdots
\end{aligned}
$$

A sequence is a list of infinitely many numbers with a specified order.
It is denoted $\{a_1,\ a_2,\ \cdots,a_n,\ \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.
Imagine *adding up* this sequence of numbers.
A series is a sum $a_1 + a_2 + \cdots + a_n + \cdots$ of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- What does it mean to add up infinitely many things?
- Should infinitely many things add up to an infinitely large number?
- Does the order in which the numbers are added matter?
- Can we add up infinitely many functions, instead of just infinitely many numbers?

Sequence

A sequence is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)

Sequence

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$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.
- ∞: there is no end to our sequence.
- $\frac{1}{n}$: this tells us the value of a_n.
- Often we omit the limits and even the brackets, writing $a_n = \frac{1}{n}$.

SEQUENCE NOTATION

For convenience, we write a_1 for the first term of a sequence, a_2 for the second term, etc.

In the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots,$
a_3 is another name for $\frac{1}{3}$.

Sometimes we can find a rule for a sequence.
In the above sequence, $a_n = \frac{1}{n}$ (whenever n is a whole number).

We can write $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

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- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
  This sequence diverges, growing without bound, not approaching a real number.
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
  This sequence diverges, since it bounces around, not approaching a real number.
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)
  This sequence diverges, since it bounces around, not approaching a real number.

Does the sequence $a_n = \dfrac{n}{2n+1}$ converge or diverge?

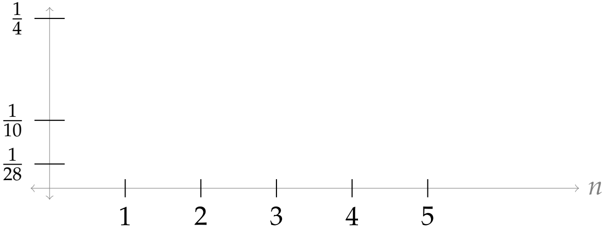
To study the behaviour of $\dfrac{n}{2n+1}$ as $n \to \infty$, it is a good idea to write it as:

$$\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

As $n \to \infty$, the $\frac{1}{n}$ in the denominator tends to zero, so that the denominator $2 + \frac{1}{n}$ tends to 2 and $\frac{1}{2+\frac{1}{n}}$ tends to $\frac{1}{2}$. So

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}$$

Consider the sequence $a_n = \dfrac{1}{3^n+1}$. $\quad \lim_{n\to\infty} a_n = 0$



Theorem 3.1.6

If

$$\lim_{x\to\infty} f(x) = L$$

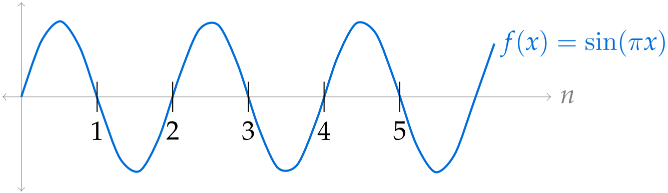
and if $a_n = f(n)$ for all positive integers n, then

$$\lim_{n\to\infty} a_n = L$$

Cautionary Tale

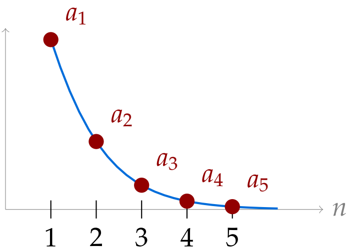
Consider the sequence $b_n = \sin(\pi n) = \{0, 0, 0, 0, 0, \ldots\}$

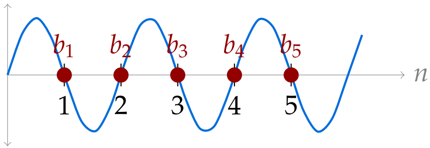
$\lim_{n\to\infty} b_n = 0$ $\lim_{x\to\infty} f(x)$ DNE



Theorem

If $\lim_{x\to\infty} f(x) = L$ and if $a_n = f(n)$ for all natural n, then $\lim_{n\to\infty} a_n = L$.





Arithmetic of Limits

Let A, B and C be real numbers and let the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that

$$\lim_{n\to\infty} a_n = A \qquad\qquad \lim_{n\to\infty} b_n = B$$

Then the following limits hold.

(a) $\displaystyle\lim_{n\to\infty} \big[a_n + b_n\big] = A + B$

(b) $\displaystyle\lim_{n\to\infty} \big[a_n - b_n\big] = A - B$

(c) $\displaystyle\lim_{n\to\infty} Ca_n = CA.$

(d) $\displaystyle\lim_{n\to\infty} a_n\, b_n = A\, B$

(e) If $B \neq 0$, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

Evaluate the following limits:

- $\lim\limits_{n\to\infty} e^{-n} = 0$
- $\lim\limits_{n\to\infty} \frac{1+n}{n} = 1$
- $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$
- $\lim\limits_{n\to\infty} 2n^2 = \infty$
- $\lim\limits_{n\to\infty} \left(\frac{1}{n^2}\right)\left(2n^2\right) = 2$

(As you might guess, the expression "$\lim\limits_{n\to\infty} a_n = \infty$" means that a_n grows without bound as $n \to \infty$.)

Continuous functions of limits

If $\lim\limits_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Evaluate $\lim\limits_{n\to\infty} \left[\sin\left(\frac{\pi n}{2n+1}\right)\right]$

$$\lim_{n\to\infty} \left[\frac{\pi n}{2n+1}\right] = \lim_{n\to\infty} \left[\frac{\pi}{2+\frac{1}{n}}\right] = \frac{\pi}{2}$$

$$\lim_{n\to\infty} \left[\sin\left(\frac{\pi n}{2n+1}\right)\right] = \sin\left(\frac{\pi}{2}\right) = 1$$

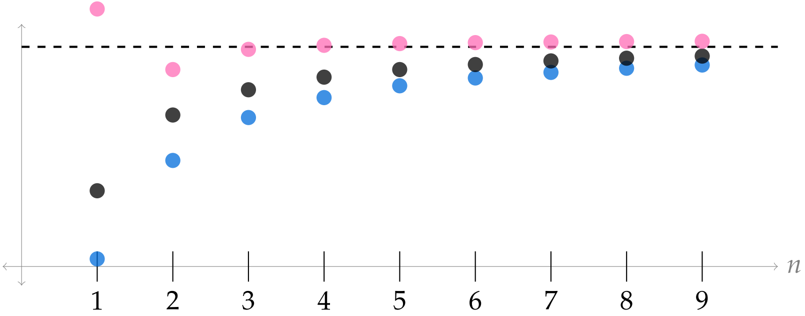
Squeeze Theorem

If $a_n \leq c_n \leq b_n$ for all sufficiently large natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$



Evaluate

$$\lim_{n\to\infty}\left(\frac{2n+\cos n}{n+1}\right)$$

Use squeeze theorem:

$$\begin{aligned}
-1 \le \cos n \le 1 \\
2n-1 \le 2n+\cos n \le 2n+1 \\
\frac{2n-1}{n+1} \le \frac{2n+\cos n}{n+1} \le \frac{2n+1}{n+1} \\
\lim_{n\to\infty}\frac{2n-1}{n+1} = \lim_{n\to\infty}\frac{2n+1}{n+1} = 2 \\
2 = \lim_{n\to\infty}\frac{2n+\cos n}{n+1}
\end{aligned}$$

Let $a_n = (-n)^{-n}$. Evaluate $\lim_{n\to\infty} a_n$.

First, we note $a_n = (-1)^{-n} \cdot (n^{-n}) = \frac{(-1)^n}{n^n}$ because $(-1)^{-n} = \left((-1)^{-1}\right)^n = (-1)^n$.

This sequence alternates between positive and negative terms. We can show that the positive terms tend to zero and the negative terms tend to zero. So, we can apply the squeeze theorem.

$$\text{Set } b_n = \frac{-1}{n^n} \text{ and } c_n = \frac{1}{n^n}$$

$$\text{Then, } b_n < a_n < c_n \text{ for all natural } n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$$

$$\text{So, } \lim_{n\to\infty} a_n = 0$$