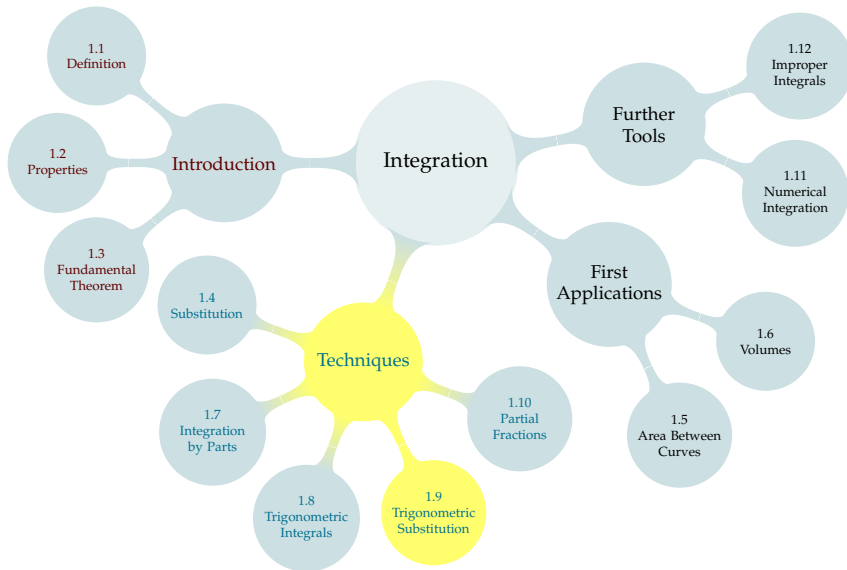


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WARMUP

Evaluate $\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}} dx$.

Evaluate $\int \frac{1}{\sqrt{x^2 + 1}} dx$.

CHECK OUR WORK

Let's verify that $\int \frac{1}{\sqrt{x^2 + 1}} =$

Seems improbable, right?

METHOD (ONE STANDARD CASE)

- ▶ An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- ▶ So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
 - ▶ $x = \sin \theta$, $1 - \sin^2 \theta = \cos^2 \theta$ changes $\sqrt{1 - x^2}$ into
 - ▶ $x = \tan \theta$, $1 + \tan^2 \theta = \sec^2 \theta$ changes $\sqrt{1 + x^2}$ into
 - ▶ $x = \sec \theta$, $\sec^2 \theta - 1 = \tan^2 \theta$ changes $\sqrt{x^2 - 1}$ into
- ▶ After integrating, convert back to the original variable (possibly using a triangle—more details later)

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

▶ $\sqrt{x^2 - 1}$

▶ $\sqrt{x^2 + 1}$

▶ $\sqrt{1 - x^2}$

FOCUS ON THE ALGEBRA

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

► $\sqrt{x^2 + 7}$

► $\sqrt{3 - 2x^2}$

CLOSER LOOK AT ABSOLUTE VALUES

▶ SKIP CLOSER LOOK

Consider the substitution $x = \sin \theta$, $dx = \cos \theta \, d\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2} \, dx$$

When $x = 0$ (lower limit of integration), what is θ ?

When $x = 1$ (upper limit of integration), what is θ ?

CLOSER LOOK AT ABSOLUTE VALUES

[▶ SKIP CLOSER LOOK](#)

More generally, suppose a is a positive constant and we use the substitution $x = a \sin \theta$ for the term $\sqrt{a^2 - x^2}$.

CLOSER LOOK AT ABSOLUTE VALUES

[▶ SKIP CLOSER LOOK](#)

Now, consider the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

CLOSER LOOK AT ABSOLUTE VALUES

[▶ SKIP CLOSER LOOK](#)

Finally, consider the substitution $x = a \sec \theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

ABSOLUTE VALUES

From now on, we will assume:

- ▶ With the substitution $x = a \sin \theta$ for $\sqrt{a^2 - x^2}$, $|\cos \theta| = \cos \theta$
- ▶ With the substitution $x = a \tan \theta$ for $\sqrt{a^2 + x^2}$, $|\sec \theta| = \sec \theta$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\int_0^1 (1 + x^2)^{-3/2} dx$

Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\int \sqrt{1 - 4x^2} \, dx$

CHECK OUR WORK

In the last example, we computed

$$\int \sqrt{1 - 4x^2} \, dx =$$

To check, we differentiate.

Identities

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\int \frac{1}{\sqrt{x^2 - 1}} dx$

CHECK OUR WORK

Let's check our result, $\int \frac{1}{\sqrt{x^2 - 1}} dx =$

COMPLETING THE SQUARE

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two “parts” that turn into one part:

▶ $1 - \sin^2 \theta = \cos^2 \theta$

▶ $1 + \tan^2 \theta = \sec^2 \theta$

▶ $\sec^2 \theta - 1 = \tan^2 \theta$

But our quadratic expression has *three* parts.

Fact: $3 - x^2 + 2x = 4 - (x - 1)^2$

COMPLETING THE SQUARE

$$(x + b)^2 = x^2 + 2bx + b^2$$

$$c - (x + b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x + b)^2$ for constants b, c .

1. Find b :
2. Solve for c :
3. All together:

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx$.

Identities have two “parts” that turn into one part:

- ▶ $1 - \sin^2 \theta = \cos^2 \theta$
- ▶ $1 + \tan^2 \theta = \sec^2 \theta$
- ▶ $\sec^2 \theta - 1 = \tan^2 \theta$

One of those parts is a constant, and one is squared.

Evaluate $\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} dx = \int \frac{(x - 3)^2}{\sqrt{9 - (x - 3)^2}} dx.$

CHECK OUR WORK

Let's verify that

$$\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}} =$$