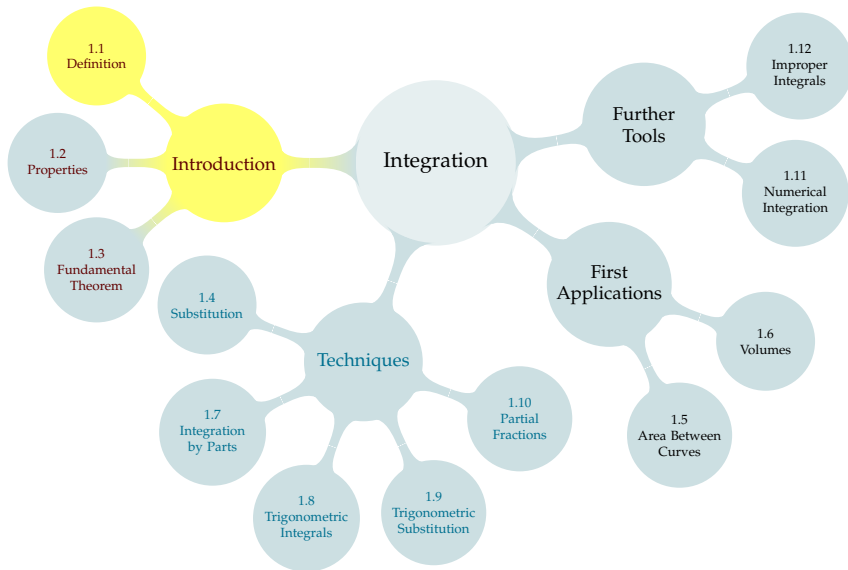


TABLE OF CONTENTS



Calculus is build on two operations: **differentiation** and **integration**.

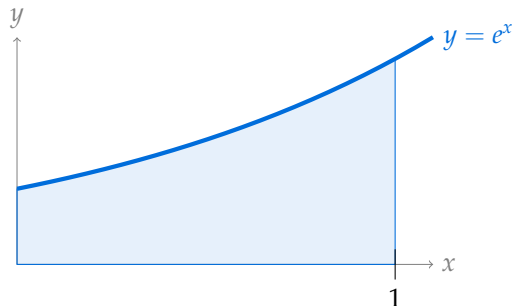
Differentiation

- ▶ Slope of a line
- ▶ Rate of change
- ▶ Optimization
- ▶ Numerical Approximations

Integration

- ▶ Area under a curve
- ▶ “Reverse” of differentiation
- ▶ Solving differential equations
- ▶ Calculate net change from rate of change
- ▶ Volume of solids
- ▶ Work (in the physics sense)

Approximate the area of the shaded region using rectangles.



We're going to be doing a lot of adding.

SUMMATION (SIGMA) NOTATION

$$\sum_{i=a}^b f(i)$$

- ▶ a, b (integers with $a \leq b$) “bounds”
- ▶ i “index:” integer which runs from a to b
- ▶ $f(i)$ “summands:” compute for every i , add

$$\sum_{i=a}^b f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

SIGMA NOTATION

Expand $\sum_{i=2}^4 (2i + 5).$

SIGMA NOTATION

Expand $\sum_{i=1}^4 (i + (i - 1)^2).$

Write the following expressions in sigma notation:

► $3 + 4 + 5 + 6 + 7$

► $8 + 8 + 8 + 8 + 8$

► $1 + (-2) + 4 + (-8) + 16$

ARITHMETIC OF SUMMATION NOTATION

Let c be a constant.

► Adding constants: $\sum_{i=1}^{10} c =$

► Factoring constants: $\sum_{i=1}^{10} 5(i^2) =$

► Addition is Commutative: $\sum_{i=1}^{10} (i + i^2) =$

ARITHMETIC OF SUMMATION NOTATION

Let c be a constant.

► Adding constants: $\sum_{i=1}^{10} c = 10c$

► Factoring constants: $\sum_{i=1}^{10} 5(i^2) = 5 \sum_{i=1}^{10} (i^2)$

► Addition is Commutative: $\sum_{i=1}^{10} (i + i^2) = \left(\sum_{i=1}^{10} i \right) + \left(\sum_{i=1}^{10} i^2 \right)$

COMMON SUMS

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^n ar^i = a + ar + ar^2 + \cdots + ar^n = a \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^n ar^i = a + ar + ar^2 + \cdots + ar^n = a \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\sum_{i=1}^{13} (i^2 + i^3)$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^n ar^i = a + ar + ar^2 + \cdots + ar^n = a \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

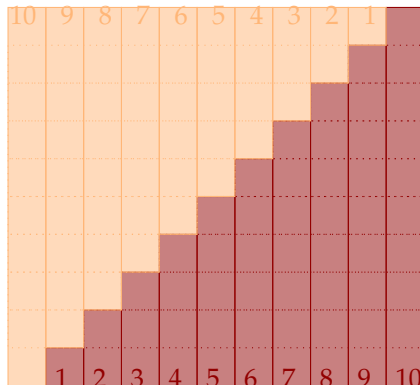
Simplify: $\sum_{i=1}^{50} (1 - i^2)$

(OPTIONAL) PROOF OF A COMMON SUM

Here is a derivation of $\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}$:

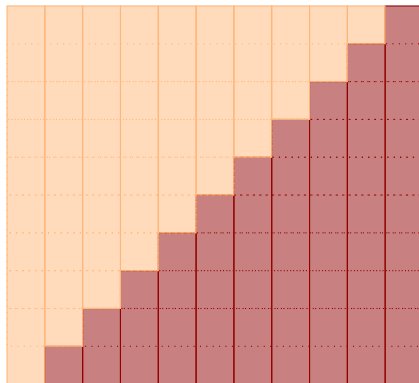
(OPTIONAL) PROOF OF ANOTHER COMMON SUM

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 =$$



(OPTIONAL) PROOF OF A COMMON SUM

$$\sum_{i=1}^n i = 1 + 2 + 3 + \cdots + n =$$



The purpose of these sums is to describe areas.

Notation

The symbol

$$\int_a^b f(x) \, dx$$

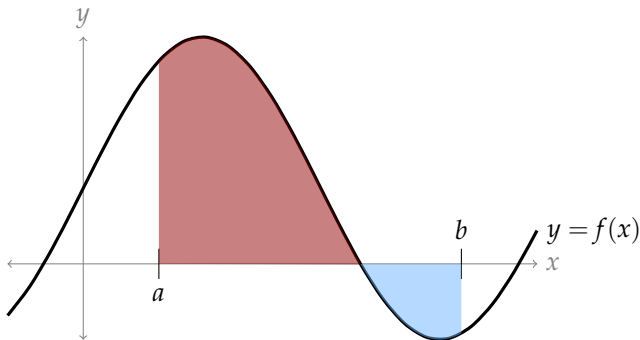
is read “the definite integral of the function $f(x)$ from a to b .”

- ▶ $f(x)$: integrand
- ▶ a and b : limits of integration
- ▶ dx : differential

If $f(x) \geq 0$ and $a \leq b$, one interpretation of

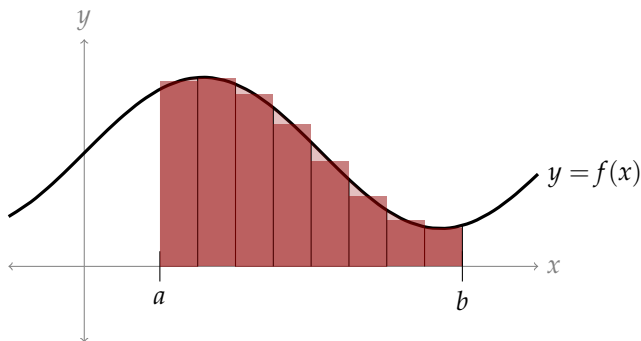
$$\int_a^b f(x) \, dx$$

is the **signed** area of the region between $y = f(x)$ and $y = 0$, from $x = a$ to $x = b$. Area **above** the axis is **positive**, and area **below** it is **negative**.



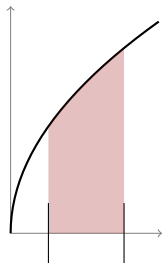
RIEMANN SUMS

A **Riemann sum** approximates the area under a curve by cutting it into equal-width segments, and approximating each segment as a rectangle.

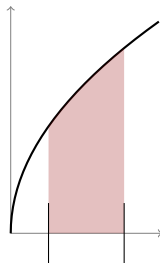


There are different ways to choose the height of each rectangle.

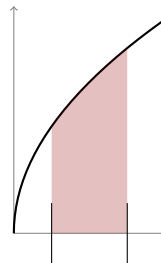
TYPES OF RIEMANN SUMS (RS)



Left RS

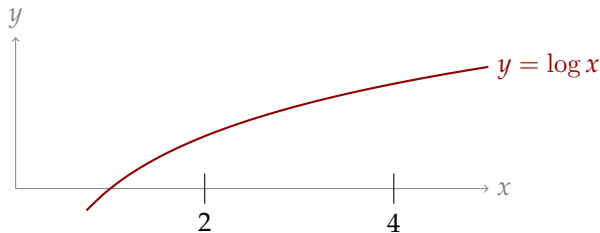


Right RS

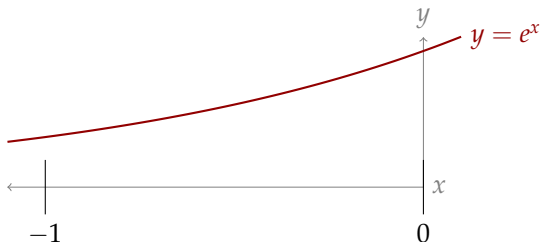


Midpoint RS

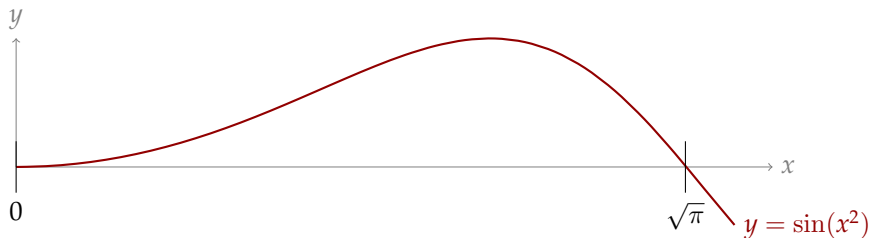
Approximate $\int_2^4 \log(x) \, dx$ using a **right Riemann sum** with $n = 4$ rectangles. For now, do not use sigma notation.



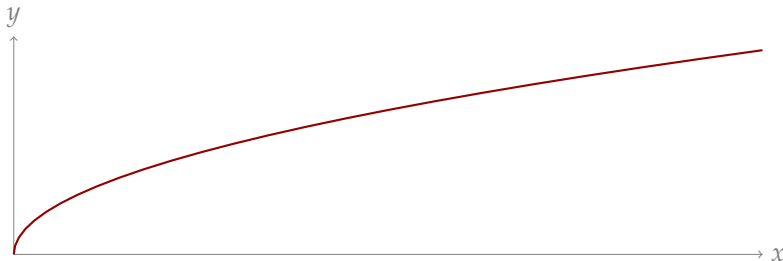
Approximate $\int_{-1}^0 e^x dx$ using a **left Riemann sum** with $n = 3$ rectangles. For now, do not use sigma notation.



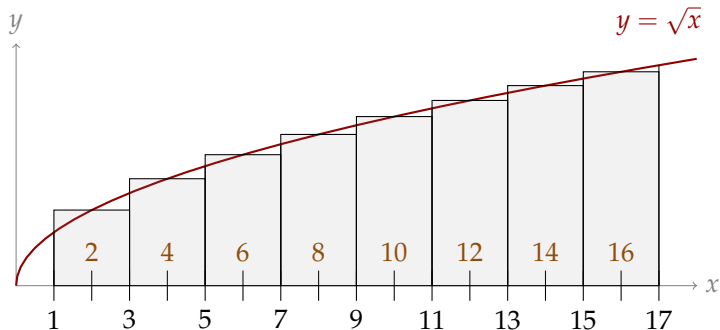
Approximate $\int_0^{\sqrt{\pi}} \sin(x^2) dx$ using a **midpoint Riemann sum** with $n = 5$ rectangles. For now, do not use sigma notation.



Approximate $\int_1^{17} \sqrt{x} \, dx$ using a **midpoint Riemann sum** with 8 rectangles. Write the result in sigma notation.



$$\sum_{i=1}^8 2\sqrt{2i} = \underbrace{2\sqrt{2}}_{i=1} + \underbrace{2\sqrt{4}}_{i=2} + \underbrace{2\sqrt{6}}_{i=3} + \underbrace{2\sqrt{8}}_{i=4} + \underbrace{2\sqrt{10}}_{i=5} + \underbrace{2\sqrt{12}}_{i=6} + \underbrace{2\sqrt{14}}_{i=7} + \underbrace{2\sqrt{16}}_{i=8}$$

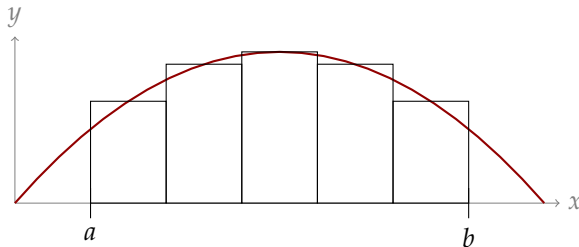


Riemann sum with n rectangles

$$\int_a^b f(x) \, dx \approx \sum_{i=1}^n \Delta x \cdot f(x_{i,n}^*)$$

where $\Delta x = \frac{b-a}{n}$ and $x_{i,n}^*$ is an x -value in the i th rectangle.

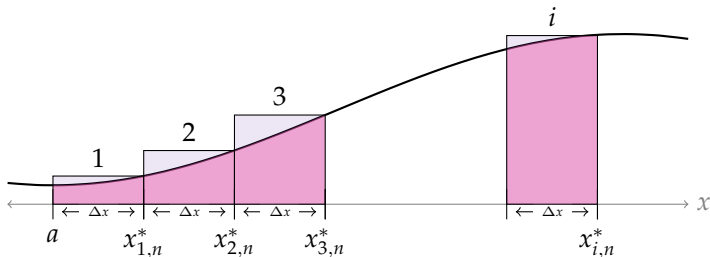
$$\sum_{i=1}^n \Delta x \cdot f(x_{i,n}^*) = \Delta x \cdot f(x_{1,n}^*) + \Delta x \cdot f(x_{2,n}^*) + \Delta x \cdot f(x_{3,n}^*) + \cdots + \Delta x \cdot f(x_{n,n}^*)$$



Right Riemann sum with n rectangles

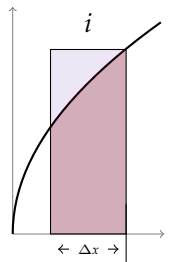
$$\int_a^b f(x) dx \approx \sum_{i=1}^n \Delta x \cdot f(\quad)$$

where $\Delta x = \frac{b-a}{n}$ and $x_{i,n}^* =$



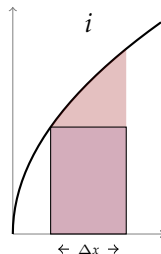
TYPES OF RIEMANN SUMS (RS)

What height would you choose for the i th rectangle?

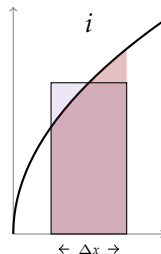


$$a + i\Delta x$$

Right RS



Left RS



Midpoint RS

Riemann sums with n rectangles. Let $\Delta x = \frac{b-a}{n}$

The **right** Riemann sum approximation of $\int_a^b f(x) \, dx$ is:

$$\sum_{i=1}^n \Delta x \cdot f(a + i\Delta x)$$

The **left** Riemann sum approximation of $\int_a^b f(x) \, dx$ is:

$$\sum_{i=1}^n \Delta x \cdot f(a + (i-1)\Delta x)$$

The **midpoint** Riemann sum approximation of $\int_a^b f(x) \, dx$ is:

$$\sum_{i=1}^n \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right) \Delta x\right)$$

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The **right** Riemann sum approximation of $\int_a^b f(x) \, dx$ is:

$$\sum_{i=1}^n \Delta x \cdot f(a + i\Delta x)$$

Give a right Riemann Sum for the area under the curve $y = x^2 - x$ from $a = 1$ to $b = 6$ using $n = 1000$ intervals.

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The **midpoint** Riemann sum approximation of $\int_a^b f(x) \, dx$ is:

$$\sum_{i=1}^n \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right) \Delta x\right)$$

Give a midpoint Riemann Sum for the area under the curve $y = 5x - x^2$ from $a = 6$ to $b = 9$ using $n = 1000$ intervals.

EVALUATING RIEMANN SUMS

[▶ SKIP RIEMANN EVALUATIONS](#)

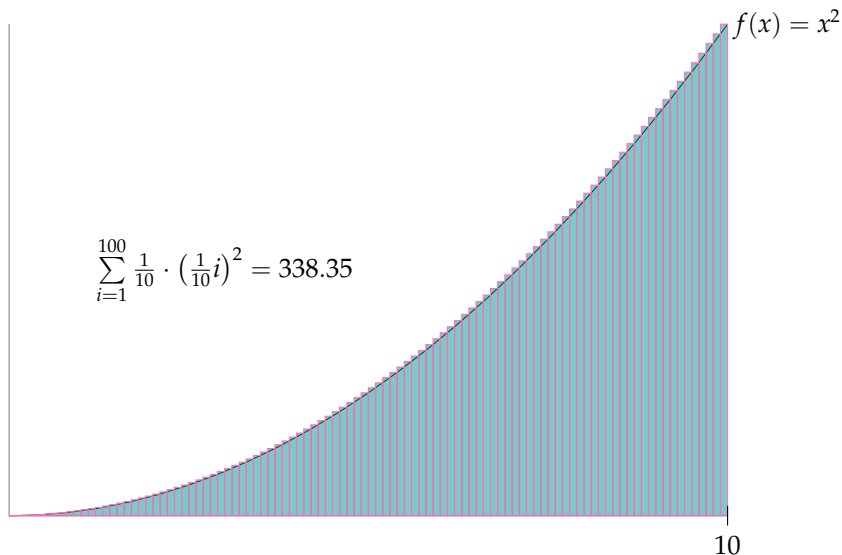
$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^2$ from $a = 0$ to $b = 10$, $n = 100$:

$$\sum_{i=1}^n \Delta x \cdot f(a + i\Delta x) =$$



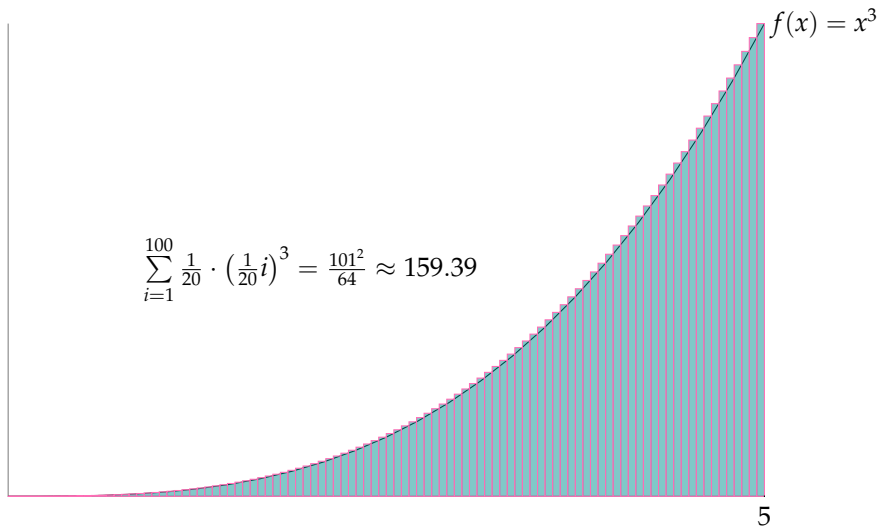
EVALUATING RIEMANN SUMS IN SIGMA NOTATION

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^3$ from $a = 0$ to $b = 5$, $n = 100$:



Definition

Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b . Then we define $\Delta x = \frac{b-a}{N}$ and

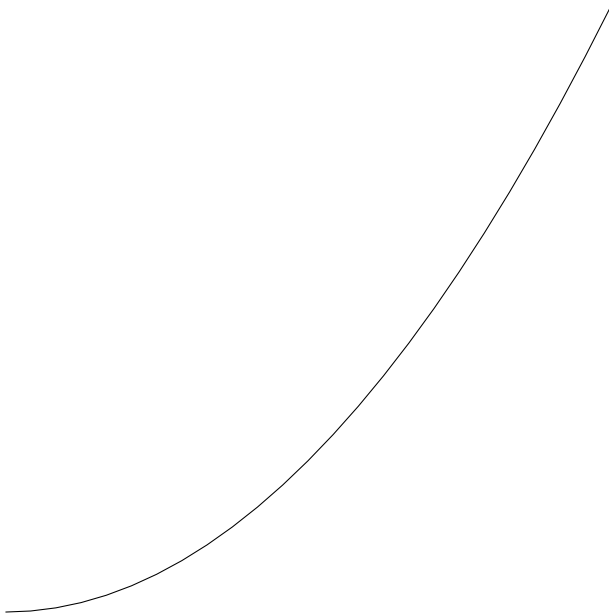
$$\int_a^b f(x) \, dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_{i,N}^*) \cdot \Delta x$$

when the limit exists and when the choice of $x_{i,N}^*$ in the i^{th} interval doesn't matter.

\sum, \int both stand for “sum”

$\Delta x, dx$ are tiny pieces of the x -axis, the bases of our very skinny rectangles

It's understood we're taking a limit as N goes to infinity, so we don't bother specifying N (or each location where we find our height) in the second notation.



$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

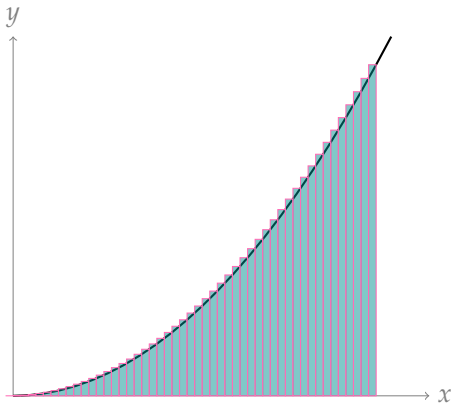
$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ with n slices, and simplify:

We found the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ using n slices:

$$\frac{125}{6} \cdot \frac{2n^2 + 3n + 1}{n^2}$$

Use it to find the exact area under the curve.



REFRESHER: LIMITS OF RATIONAL FUNCTIONS

$$\lim_{n \rightarrow \infty} \frac{n^2 + 2n + 15}{3n^2 - 9n + 5} =$$

When the degree of the top and bottom are the same, the limit as n goes to infinity is the ratio of the leading coefficients.

$$\lim_{n \rightarrow \infty} \frac{n^2 + 2n + 15}{3n^3 - 9n + 5} =$$

When the degree of the top is smaller than the degree of the bottom, the limit as n goes to infinity is 0.

$$\lim_{n \rightarrow \infty} \frac{n^3 + 2n + 15}{3n^2 - 9n + 5} =$$

When the degree of the top is larger than the degree of the bottom, the limit as n goes to infinity is positive or negative infinity.

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

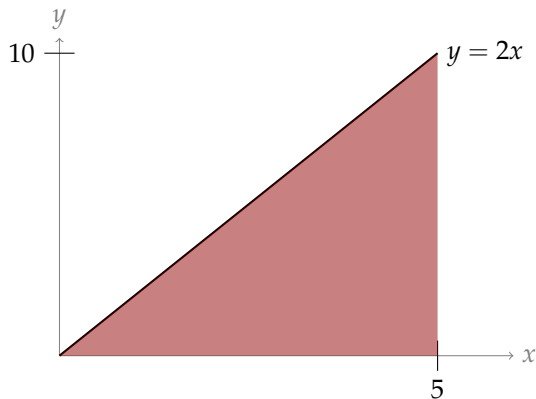
$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

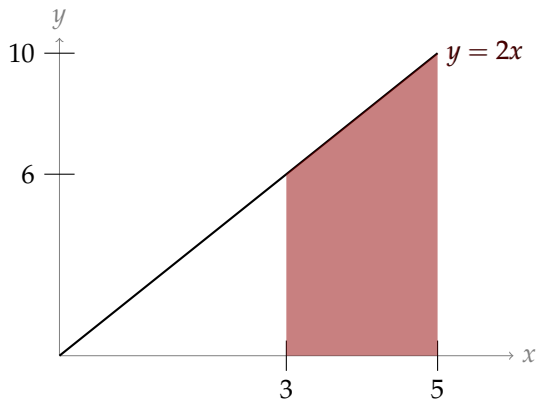
Evaluate $\int_0^1 x^2 dx$ exactly using midpoint Riemann sums.

Let's see some special cases where we can use geometry to evaluate integrals without Riemann sums.

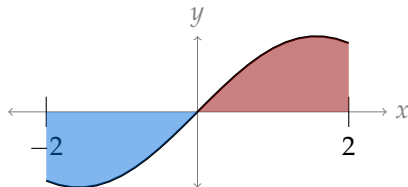
$$\int_0^5 2x \, dx$$



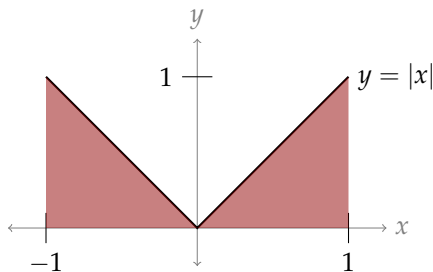
$$\int_3^5 2x \, dx$$



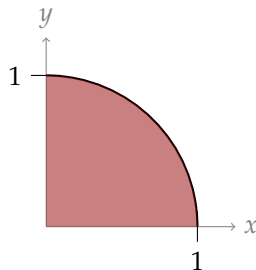
$$\int_{-2}^2 \sin x \, dx$$



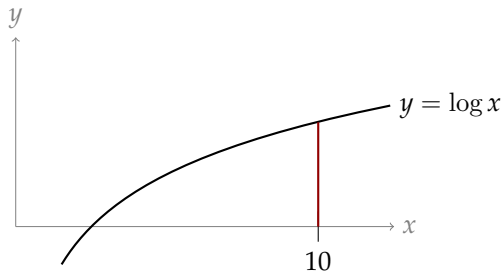
$$\int_{-1}^1 |x| \, dx$$



$$\int_0^1 \sqrt{1-x^2} \, dx$$



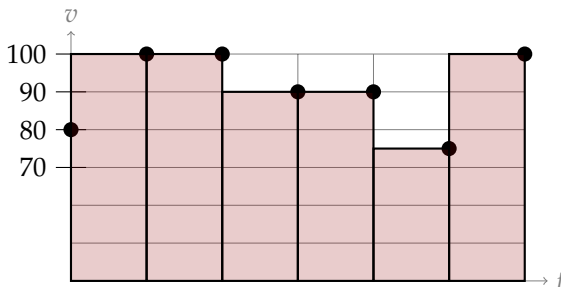
$$\int_{10}^{10} \log x \, dx$$



A car travelling down a straight highway records the following measurements:

Time	12:00	12:10	12:20	12:30	12:40	12:50	1:00
Speed (kph)	80	100	100	90	90	75	100

Approximately how far did the car travel from 12:00 to 1:00?



The computation

$$\text{distance} = \text{rate} \times \text{time}$$

looks a lot like the computation

$$\text{area} = \text{base} \times \text{height}$$

for a rectangle. This gives us another interpretation for an integral.

ANOTHER INTERPRETATION OF THE INTEGRAL

Let $x(t)$ be the position of an object moving along the x -axis at time t , and let $v(t) = x'(t)$ be its velocity. Then for all $b > a$,

$$x(b) - x(a) = \int_a^b v(t) \, dt$$

That is, $\int_a^b v(t) \, dt$ gives the *net distance* moved by the object from time a to time b .