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Recall the geometric series: for a constant r, with $|r| < 1$:

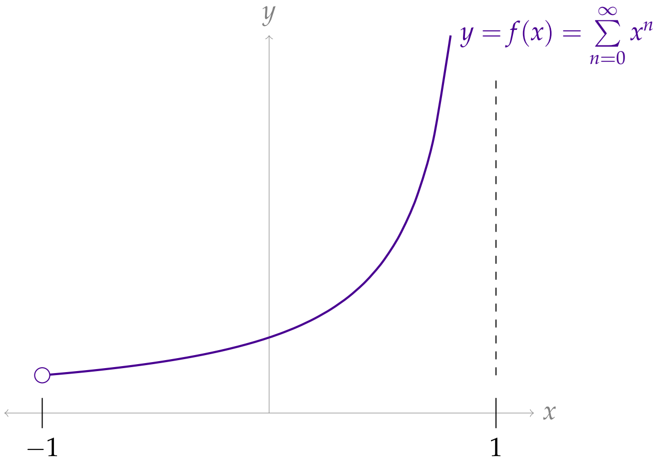
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

We can think of this as a function. If we set

$$f(x) = \sum_{n=0}^{\infty} x^n$$

and restrict our domain to $-1 < x < 1$, then

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$


y
$y = f(x) = \sum_{n=0}^{\infty} x^n$
x
-1
1

Why would we ever prefer to write $\sum_{n=0}^{\infty} x^n$ instead of $\frac{1}{1-x}$?

The function

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

isn't a polynomial, but in certain ways it behaves like one. For $|x| < 1$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\}\right) = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\int \frac{1}{1-x}\,\mathrm{d}x = \int\left(\sum_{n=0}^{\infty} x^n\right)\mathrm{d}x = \sum_{n=0}^{\infty}\left(\int x^n\,\mathrm{d}x\right) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

Definition

A series of the form

$$\sum_{n=0}^{\infty} A_n (x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

is called a *power series in* $(x-c)$ or a *power series centered on* c. The numbers A_n are called the coefficients of the power series.

One often considers power series centered on $c = 0$ and then the series reduces to

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots = \sum_{n=0}^{\infty} A_n x^n$$

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

In a power series, we think of the coefficients A_n as fixed constants, and we think of x as the variable of a function.

Evaluate the power series $\sum\limits_{n=0}^{\infty} A_n(x-c)^n$ when $x = c$:

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

$$\begin{aligned}\sum_{n=0}^{\infty} A_n(c-c)^n &= A_0 + A_1\underbrace{(c-c)}_{0} + A_2\underbrace{(c-c)^2}_{0} + A_3\underbrace{(c-c)^3}_{0} + \cdots \\ &= A_0 \qquad \text{(In particular, the series converges when } x = c\text{.)}\end{aligned}$$

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges.

This looks somewhat like a geometric series, but not exactly, so the ratio test is a good option.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| \left(\frac{n}{n+1} \right)$$

$$= \lim_{n\to\infty} |x| \left(\frac{n}{n+1} \right) = |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$, we have the harmonic series, which diverges. When $x = -1$, we have the alternating harmonic series, which converges.

So, all together, the series converges when $-1 \le x < 1$, and diverges

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Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots .$$

This still looks somewhat like a geometric series, so the ratio test is a still good option to start.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+1}}{2^n(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\right|\left(\frac{2^{n+1}}{2^n}\right)$$
$$= 2\,|x-1|$$

So we see that the series converges when $|x-1| < \frac{1}{2}$ and diverges when $|x-1| > \frac{1}{2}$.
When $x - 1 = -\frac{1}{2}$, i.e. $x = \frac{1}{2}$, our series is

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2} - 1\right)^n = \sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$$

When $x - 1 = \frac{1}{2}$, i.e. $x = \frac{3}{2}$, our series is

$$\sum^{\infty} 2^n \left(\frac{3}{2} - 1\right)^n = \sum^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum^{\infty} 1$$

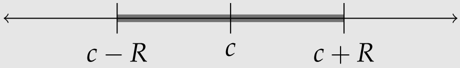
What happens if we apply the ratio test to a generic power series, $\sum_{n=0}^{\infty} A_n(x-c)^n$?

$$\lim_{n\to\infty}\left|\frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}(x-c)\right| = |x-c|\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$$

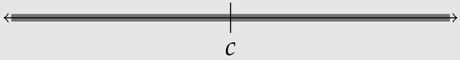
- If $\left|\frac{A_{n+1}}{A_n}\right|$ does not approach a limit as $n \to \infty$, the ratio test tells us nothing. (We should try other tests.)
- If $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = 0$, then
- If $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \infty$, then
- If $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = A$ for some real number A, then

Definition: Radius of Convergence

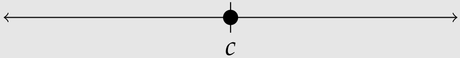
(a) Let $0 < R < \infty$. If $\sum\limits_{n=0}^{\infty} A_n(x-c)^n$ converges for $|x-c| < R$, and diverges for $|x-c| > R$, then we say that the series has radius of convergence R.



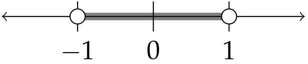
(b) If $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for every number x, we say that the series has an infinite radius of convergence.



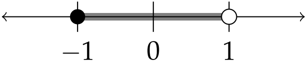
(c) If $\sum_{n=0}^{\infty} A_n(x-c)^n$ diverges for every $x \neq c$, we say that the series has radius of convergence zero.



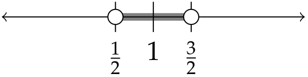
- We saw that $\sum\limits_{n=0}^{\infty} x^n$ converges when $|x| < 1$ and diverges when $|x| > 1$, so this series has radius of convergence $R =$



- We saw that $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}$ converges when $|x| < 1$ and diverges when $|x| > 1$, so this series also has radius of convergence $R =$



- We saw that $\sum\limits_{n=1}^{\infty} 2^n(x-1)^n$ converges when $|x-1| < \frac{1}{2}$ and diverges when $|x-1| > \frac{1}{2}$, so this series has radius of convergence $R =$



What is the radius of convergence for the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

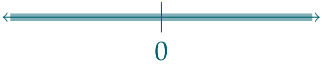
Recall: $n! = (n)(n-1)(n-2)\cdots(2)(1)$.

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty}|x|\frac{(n)(n-1)(n-2)\cdots(2)(1)}{(n+1)(n)(n-1)(n-2)\cdots(2)(1)}$$

$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

For every real x, the limit is less than one, so the series converges. That is, its radius of convergence is ∞.



What is the radius of convergence for the series $\sum_{n=0}^{\infty} n! \cdot (x-3)^n$?

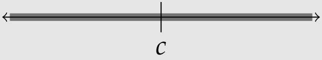
$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(n+1)!(x-3)^{n+1}}{(n!)(x-3)^n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{n!}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)(n)(n-1)(n-2)\cdots(2)(1)}{(n)(n-1)(n-2)\cdots(2)(1)}|x-3| \\
&= \lim_{n\to\infty}(n+1)|x-3|
\end{aligned}$$

For every real x except $x=3$, the limit is greater than one, so the series diverges. The series only converges at $x=3$. That is, its radius of convergence is 0.

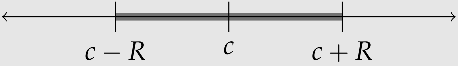
Theorem

Given a power series (say with centre c), one of the following holds.

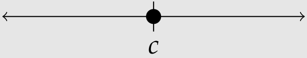
(a) The power series converges for every number x. In this case we say that the radius of convergence is ∞.



(b) There is a number $0 < R < \infty$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. Then R is called the radius of convergence.



(c) The series converges for $x = c$ and diverges for all $x \neq c$. In this case, we say that the radius of convergence is 0.



We are told that a certain power series with centre $c = 3$ converges at $x = 4$ and diverges at $x = 1$. What else can we say about the convergence or divergence of the series for other values of x?

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$f(x) + g(x) = \sum_{n=0}^{\infty} [A_n + B_n] (x-c)^n$$

$$Kf(x) = \sum_{n=0}^{\infty} K A_n (x-c)^n$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$\begin{aligned}(x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n \, (x-c)^{n+N} \quad \text{for any integer } N \geq 1 \\ &= \sum_{k=N}^{\infty} A_{k-N} \, (x-c)^k \quad \text{where } k = n+N\end{aligned}$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$f'(x) = \sum_{n=0}^{\infty} A_n\, n\, (x-c)^{n-1} = \sum_{n=1}^{\infty} A_n\, n\, (x-c)^{n-1}$$

$$\int_c^x f(t)\, \mathrm{d}t = \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}$$

$$\int f(x)\, \mathrm{d}x = \left[\sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}\right] + C \quad \text{with } C \text{ an arbitrary constant}$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

for all x obeying $|x-c| < R$.

Differentiating, antidifferentiating, multiplying by a nonzero constant, and multiplying by a positive power of $(x-c)$ do not change the radius of convergence of $f(x)$ (although they may change the interval of convergence).

Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{1-x}\right\} = \dfrac{1}{(1-x)^2}$, find a power series representation for $\dfrac{1}{(1-x)^2}$ when $|x| < 1$. For $|x| < 1$:

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\} \\
&= \sum_{n=0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\}\right) \\
&= \sum_{n=0}^{\infty} nx^{n-1} \\
&= \sum_{n=1}^{\infty} nx^{n-1}
\end{aligned}$$

Q

Find a power series representation for $\log(1+x)$ when $|x|<1$.

First, note $\frac{\mathrm{d}}{\mathrm{d}x}\{\log(1+x)\} = \frac{1}{1+x}$. Our plan is to antidifferentiate a power series representation of $\frac{1}{1+x}$. For $|x|<1$:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\int \frac{1}{1+x}\,\mathrm{d}x = \int \left(\sum_{n=0}^{\infty}(-1)^n x^n\right)\mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\left(\int (-1)^n x^n\,\mathrm{d}x\right)$$

So, for some constant C,

$$\log(1+x) = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

To find C, let's plug in a value for x where both sides of the equation are easy to evaluate: $x=0$.

Find a power series representation for $\arctan(x)$ when $|x| < 1$.

First, note $\frac{\mathrm{d}}{\mathrm{d}x}\{\arctan x\} = \frac{1}{1+x^2}$. To obtain a power series representation of $\frac{1}{1+x^2}$, we'll substitute into the geometric series. Let $y = -x^2$ with $|y| < 1$. Then:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

$$\implies \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\implies \int \frac{1}{1+x^2}\,\mathrm{d}x = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right)\mathrm{d}x = \sum_{n=0}^{\infty}\left(\int (-1)^n x^{2n}\,\mathrm{d}x\right)$$

$$\implies \arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for some constant C. To find C, we'll plug in $x = 0$, which makes both sides of the last equation easy to evaluate.

$$\arctan 0 = C + \sum^{\infty} (-1)^n \frac{x^{2n+1}}{}$$

Substituting in a Power Series

Assume that the function $f(x)$ is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all x in the interval I. Also let K and k be real constants. Then

$$f\left(Kx^k\right) = \sum_{n=0}^{\infty} A_n K^n \, x^{kn}$$

whenever Kx^k is in I. In particular, if $\sum_{n=0}^{\infty} A_n x^n$ has radius of convergence R, K is nonzero and k is a natural number, then $\sum_{n=0}^{\infty} A_n K^n \, x^{kn}$ has radius of convergence $\sqrt[k]{R/|K|}$.

Find a power series representation for $\dfrac{1}{5-x}$ with centre 3.

We know that $\frac{1}{1-(x-3)} = \sum_{n=0}^{\infty}(x-3)^n$ when $|x-3|<1$. To take advantage of our ability to substitute into power functions, we'd like to write $\frac{1}{5-x}$ in the form $\frac{1}{1-K(x-3)^k}$ for some constant K and some whole number k.

$$\frac{1}{5-x} = \frac{1}{2-(x-3)} = \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x-3}{2}\right)}$$

Set $y = \frac{x-3}{2}$. When $|y|<1$:

$$\begin{aligned} \frac{1}{2}\cdot\frac{1}{1-y} &= \frac{1}{2}\sum_{n=0}^{\infty} y^n \\ \implies \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x-3}{2}\right)} &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n \\ \implies \frac{1}{5-x} &= \sum_{n=0}^{\infty}\frac{(x-3)^n}{2^{n+1}}. \end{aligned}$$

The series converges when: