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We can imagine the list of numbers below carrying on forever:

$$
\begin{aligned}
a_1 &= 0.1 \\
a_2 &= 0.01 \\
a_3 &= 0.001 \\
a_4 &= 0.0001 \\
a_5 &= 0.00001 \\
&\vdots
\end{aligned}
$$

A sequence is a list of infinitely many numbers with a specified order.
It is denoted $\{a_1, a_2, \cdots, a_n, \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.
Imagine *adding up* this sequence of numbers.
A series is a sum $a_1 + a_2 + \cdots + a_n + \cdots$ of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- What does it mean to add up infinitely many things?
- Should infinitely many things add up to an infinitely large number?
- Does the order in which the numbers are added matter?
- Can we add up infinitely many functions, instead of just infinitely many numbers?

Sequence

A sequence is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)

Sequence

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$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.
- ∞: there is no end to our sequence.
- $\frac{1}{n}$: this tells us the value of a_n.
- Often we omit the limits and even the brackets, writing $a_n = \frac{1}{n}$.

Sequence Notation

For convenience, we write a_1 for the first term of a sequence, a_2 for the second term, etc.

In the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots,$
a_3 is another name for

Sometimes we can find a rule for a sequence.
In the above sequence, $a_n =$

We can write $\{a_n\}_{n=1}^{\infty} =$

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

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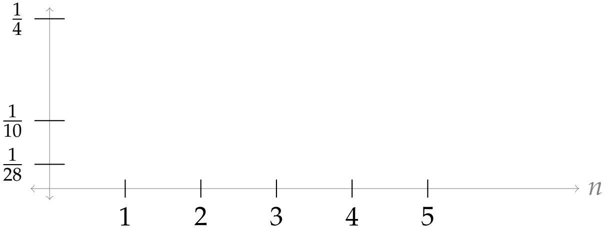
  This sequence
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of -1 : $(-1)^0$, $(-1)^1$, $(-1)^2$, etc.)
  This sequence

Does the sequence $a_n = \dfrac{n}{2n+1}$ converge or diverge?

Consider the sequence $a_n = \dfrac{1}{3^n + 1}$.

$$\lim_{n \to \infty} a_n =$$



Theorem 3.1.6

If

$$\lim_{x \to \infty} f(x) = L$$

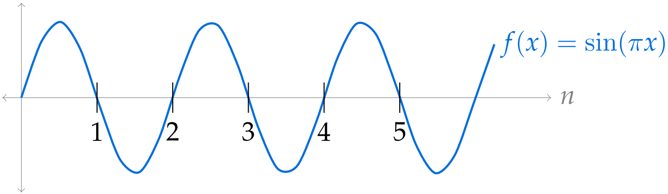
and if $a_n = f(n)$ for all positive integers n, then

$$\lim_{n \to \infty} a_n = L$$

Cautionary Tale

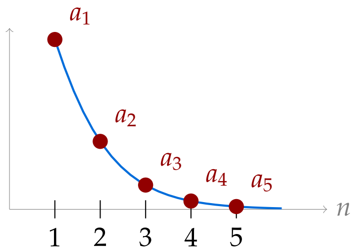
Consider the sequence $b_n = \sin(\pi n) =$

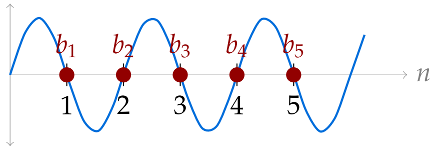
$\lim_{n\to\infty} b_n =$ $\lim_{x\to\infty} f(x)$



Theorem

If $\lim_{x\to\infty} f(x) = L$ and if $a_n = f(n)$ for all natural n, then $\lim_{n\to\infty} a_n = L$.





Arithmetic of Limits

Let A, B and C be real numbers and let the two sequences $\left\{a_n\right\}_{n=1}^{\infty}$ and $\left\{b_n\right\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that

$$\lim_{n\to\infty} a_n = A \qquad\qquad \lim_{n\to\infty} b_n = B$$

Then the following limits hold.

(a) $\displaystyle\lim_{n\to\infty}\left[a_n + b_n\right] = A + B$

(b) $\displaystyle\lim_{n\to\infty}\left[a_n - b_n\right] = A - B$

(c) $\displaystyle\lim_{n\to\infty} Ca_n = CA.$

(d) $\displaystyle\lim_{n\to\infty} a_n\, b_n = A\, B$

(e) If $B \neq 0$, then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}$

Evaluate the following limits:

- $\lim_{n\to\infty} e^{-n} =$
- $\lim_{n\to\infty} \frac{1+n}{n} =$
- $\lim_{n\to\infty} \frac{1}{n^2} =$
- $\lim_{n\to\infty} 2n^2 =$
- $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)\left(2n^2\right) =$

Continuous functions of limits

If $\lim_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Evaluate $\lim_{n\to\infty} \left[\sin\left(\frac{\pi n}{2n+1}\right)\right]$

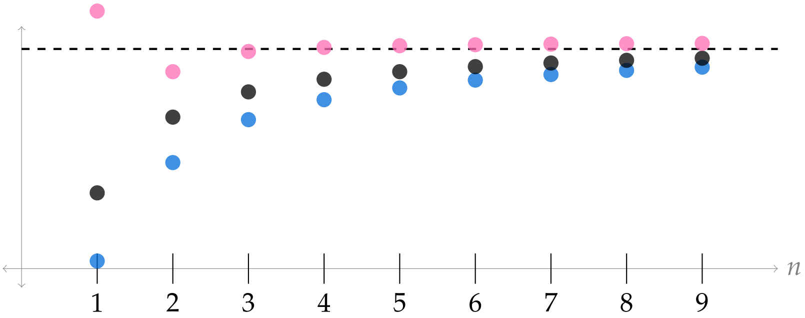
Squeeze Theorem

If $a_n \leq c_n \leq b_n$ for all sufficiently large natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$



Evaluate

$$\lim_{n \to \infty} \left(\frac{2n + \cos n}{n + 1} \right)$$

Let $a_n = (-n)^{-n}$. Evaluate $\lim_{n\to\infty} a_n$.