Table of Contents

Warmup

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x$.

Warmup

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x$.

$$\begin{aligned}\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,\mathrm{d}x &= \int_3^7 \frac{1}{\sqrt{(x+1)^2}}\,\mathrm{d}x\\ &= \int_3^7 \frac{1}{|x+1|}\,\mathrm{d}x\end{aligned}$$

When $3 \le x \le 7$, we have $|x+1| = x+1$.

$$\begin{aligned}&= \int_3^7 \frac{1}{x+1}\,\mathrm{d}x\\ &= \left[\log|x+1|\right]_3^7\\ &= \log 8 - \log 4 = \log 2\end{aligned}$$

Idea: $\sqrt{(\text{something})^2} = |\text{something}|$. We cancelled off the square root.

Evaluate $\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$.

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$.

We still want to cancel off the square root, but x^2+1 is not obviously of the form $(\text{something})^2$.
Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$.

$$\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \int \frac{1}{\sqrt{\tan^2\theta+1}}\sec^2\theta\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\mathrm{d}\theta$$
$$= \int \frac{\sec^2\theta}{\sec\theta}\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \log|\sec\theta+\tan\theta| + C$$

We need to get these back in terms of x. From our substitution, we know $\tan\theta = x$. From simplifying our denominator, we also know $\sec\theta = \sqrt{x^2+1}$.

$$= \log\left|\sqrt{x^2+1}+x\right| + C$$

Same idea: $\sqrt{(\text{something})^2} = |\text{something}|$; cancel off the square root.

Check Our Work

Let's verify that $\int \frac{1}{\sqrt{x^2+1}} = \log\left|\sqrt{x^2+1}+x\right| + C.$

Seems improbable, right?

Check Our Work

Let's verify that $\displaystyle\int \frac{1}{\sqrt{x^2+1}} = \log\left|\sqrt{x^2+1}+x\right| + C.$
Seems improbable, right?

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\log\left|\sqrt{x^2+1}+x\right|+C\right] &= \frac{1}{\sqrt{x^2+1}+x}\cdot\left(\frac{2x}{2\sqrt{x^2+1}}+1\right)\\ &= \frac{x+\sqrt{x^2+1}}{(\sqrt{x^2+1}+x)\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

So, our answer works!

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.

- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.

- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:

  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$

  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.

- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:

  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into $\sqrt{\tan^2\theta} = |\tan\theta|$

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1 - \sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into $\sqrt{\cos^2\theta} = |\cos\theta|$
  - $x = \tan\theta$, $1 + \tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into $\sqrt{\sec^2\theta} = |\sec\theta|$
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into $\sqrt{\tan^2\theta} = |\tan\theta|$
- After integrating, convert back to the original variable (possibly using a triangle–more details later)

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 - 1}$
- $\sqrt{x^2 + 1}$
- $\sqrt{1 - x^2}$

ans

Focus on the Algebra

$1-\sin^2\theta=\cos^2\theta \qquad 1+\tan^2\theta=\sec^2\theta \qquad \sec^2\theta-1=\tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2-1}$
  Let $x=\sec\theta$, so $\sqrt{x^2-1}$ becomes $\sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=|\tan\theta|$
- $\sqrt{x^2+1}$
- $\sqrt{1-x^2}$

ans

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta$ $\qquad$ $1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $\sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2 - 1}$
  Let $x = \sec\theta$, so $\sqrt{x^2-1}$ becomes $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta|$
- $\sqrt{x^2 + 1}$
  Let $x = \tan\theta$, so $\sqrt{x^2+1}$ becomes $\sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = |\sec\theta|$
- $\sqrt{1 - x^2}$

ans

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $\sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2-1}$
  Let $x = \sec\theta$, so $\sqrt{x^2-1}$ becomes $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta|$
- $\sqrt{x^2+1}$
  Let $x = \tan\theta$, so $\sqrt{x^2+1}$ becomes $\sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta} = |\sec\theta|$
- $\sqrt{1-x^2}$
  Let $x = \sin\theta$ so $\sqrt{1-x^2}$ becomes $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$
  (Alternately, $x = \cos\theta$ works as well)

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $\sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2+7}$
- $\sqrt{3-2x^2}$

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2+7}$
  Adjust a given identity by multiplying both sides by 7:
  $7\tan^2\theta + 7 = 7\sec^2\theta$. Now we see we want $x^2 = 7\tan^2\theta$. That is,
  $x = \sqrt{7}\tan\theta$:
  $\sqrt{x^2+7} = \sqrt{7\tan^2\theta + 7} = \sqrt{7(\sec^2\theta)} = \sqrt{7}\,|\sec\theta|$

- $\sqrt{3-2x^2}$

ans

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $\sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2+7}$
  Adjust a given identity by multiplying both sides by 7:
  $7\tan^2\theta + 7 = 7\sec^2\theta$. Now we see we want $x^2 = 7\tan^2\theta$. That is, $x = \sqrt{7}\tan\theta$:
  $\sqrt{x^2+7} = \sqrt{7\tan^2\theta + 7} = \sqrt{7(\sec^2\theta)} = \sqrt{7}\,|\sec\theta|$

- $\sqrt{3-2x^2}$
  Adjust a given identity by multiplying both sides by 3:
  $3 - 3\sin^2\theta = 3\cos^2\theta$. Now we see we want $2x^2 = 3\sin^2\theta$, so $x = \sqrt{\frac{3}{2}}\sin\theta$:
  $\sqrt{3-2x^2} = \sqrt{3 - 2\left(\frac{3}{2}\sin^2\theta\right)} = \sqrt{3 - 3\sin^2\theta} = \sqrt{3\cos^2\theta} =$
  $\sqrt{3}\,|\cos\theta|$

Closer look at absolute values

▸▸ SKIP CLOSER LOOK

Consider the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?

ans

Closer look at absolute values

SKIP CLOSER LOOK

Consider the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\, \mathrm{d}\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2}\, \mathrm{d}x$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?
If $x = 0$, then $\sin\theta = 0$, but there are infinitely many values of θ that could make this true. To use the substitution $x = \sin\theta$, we need the function $x = \sin\theta$ to be invertible. That way, we can unambiguously convert between x and θ. With that in mind, we'll actually set $\theta = \arcsin x$. Now θ is restricted to the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\, \mathrm{d}x = \int_{\arcsin 0}^{\arcsin 1} \sqrt{1-\sin^2\theta}\, \cos\theta\, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta} \cdot \cos\theta\, \mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{2}} |\cos\theta| \cdot \cos\theta\, \mathrm{d}\theta$$

For $0 \le \theta \le \frac{\pi}{2}$, we have $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

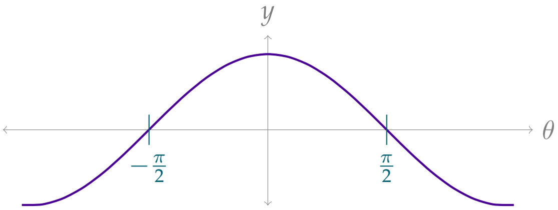
More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$

Closer look at absolute values

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

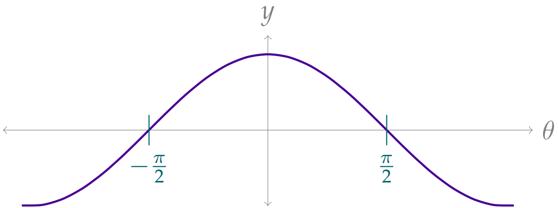
- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$
- On the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$



SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2 - x^2}$.

- $\theta = \arcsin\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$
- On the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$



- So, in general, when we use the substitution $x = \sin\theta$ with trigonometric substitution, we can expect $|\cos\theta| = \cos\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

Closer look at absolute values

▸▸ SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Closer look at absolute values

SKIP CLOSER LOOK

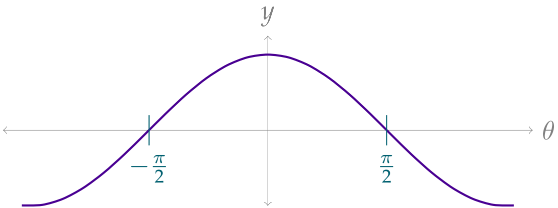
Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = \frac{a}{|\cos\theta|}$

Closer look at absolute values

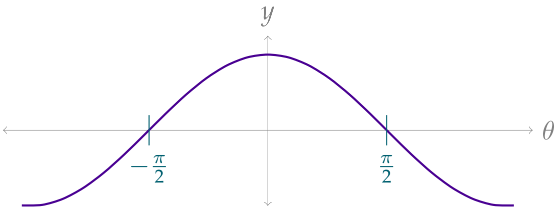
Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = \frac{a}{|\cos\theta|}$
- On the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$ and $|\sec\theta| = \sec\theta$.



Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

- $\theta = \arctan\left(\frac{x}{a}\right)$, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = \frac{a}{|\cos\theta|}$
- On the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$ and $|\sec\theta| = \sec\theta$.



- So, in general, when we use the substitution $x = \tan\theta$ with trigonometric substitution, we can expect $|\sec\theta| = \sec\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

Closer look at absolute values

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$

Closer look at absolute values

▸▸ SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:
  - For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \geq a^2$. I.e. $x \geq a$ or $x \leq -a$.

Closer look at absolute values

▶▶ SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

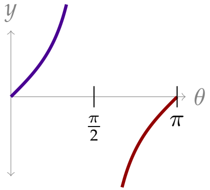
- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \leq \theta \leq \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:
  - For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \geq a^2$. I.e. $x \geq a$ or $x \leq -a$.
  - When $x \geq a$, we have $0 \leq \theta < \frac{\pi}{2}$, $\tan\theta \geq 0$, so $|\tan\theta| = \tan\theta$.

Closer look at absolute values

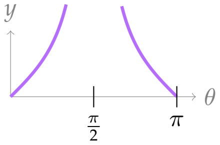
SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

- $\sec\theta = \frac{x}{a}$, so $\cos\theta = \frac{a}{x}$, so $\theta = \arccos\left(\frac{a}{x}\right)$. Then $0 \le \theta \le \pi$
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$
- Now this case gets slightly more complicated than the others:
  - For $\sqrt{x^2 - a^2}$ to be defined, we need $x^2 \ge a^2$. I.e. $x \ge a$ or $x \le -a$.
  - When $x \ge a$, we have $0 \le \theta < \frac{\pi}{2}$, $\tan\theta \ge 0$, so $|\tan\theta| = \tan\theta$.
  - When $x \le -a$, we have $\frac{\pi}{2} < \theta \le \pi$, $\tan\theta < 0$, so $|\tan\theta| = -\tan\theta$.



$y = \tan\theta$



$y = \sqrt{\tan^2\theta} = |\tan\theta|$

Absolute Values

From now on, we will assume:

- With the substitution $x = a\sin\theta$ for $\sqrt{a^2 - x^2}$, $|\cos\theta| = \cos\theta$
- With the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, $|\sec\theta| = \sec\theta$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

Evaluate $\displaystyle\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

ans

Evaluate $\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x$

Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. When $x = 0$, then $\theta = \arctan 0 = 0$; when $x = 1$, then $\theta = \arctan 1 = \frac{\pi}{4}$.

$$\int_0^1 (1+x^2)^{-3/2}\,\mathrm{d}x = \int_{\theta=0}^{\theta=\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}^3} \sec^2\theta\,\mathrm{d}\theta$$
$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}^3}\mathrm{d}\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{|\sec\theta|^3}\mathrm{d}\theta$$
$$= \int_0^{\pi/4} \frac{1}{|\sec\theta|}\,\mathrm{d}\theta = \int_0^{\pi/4} |\cos\theta|\,\mathrm{d}\theta$$

Given our previous investigation,

$$= \int_0^{\pi/4} \cos\theta\,\mathrm{d}\theta = \left[\sin\theta\right]_0^{\pi/4}$$
$$= \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}}$$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\int \sqrt{1 - 4x^2}\,\mathrm{d}x$

Evaluate $\int \sqrt{1-4x^2}\,\mathrm{d}x$

ans

Evaluate $\int \sqrt{1-4x^2}\,\mathrm{d}x$

Under the square root, we have "one minus a term with a variable," which matches the identity $1-\sin^2\theta$. So, we want $4x^2$ to become $\sin^2\theta$. That is, $x = \frac{1}{2}\sin\theta$. Then $\mathrm{d}x = \frac{1}{2}\cos\theta\,\mathrm{d}\theta$.

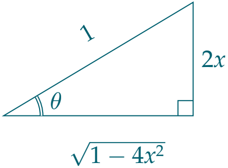
$$\begin{aligned}
\int \sqrt{1-4x^2}\,\mathrm{d}x &= \int \sqrt{1-4\left(\frac{1}{2}\sin\theta\right)^2}\cdot\frac{1}{2}\cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int \sqrt{1-\sin^2\theta}\cdot\cos\theta\mathrm{d}\theta = \frac{1}{2}\int\sqrt{\cos^2\theta}\cdot\cos\theta\mathrm{d}\theta \\
&= \frac{1}{2}\int |\cos\theta|\cdot\cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int\cos^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int\left(\frac{1+\cos(2\theta)}{2}\right)\mathrm{d}\theta = \frac{1}{4}\int(1+\cos(2\theta))\mathrm{d}\theta \\
&= \frac{1}{4}\left[\theta+\frac{1}{2}\sin(2\theta)\right]+C = \frac{1}{4}\left[\theta+\sin\theta\cos\theta\right]+C
\end{aligned}$$

It remains to convert θ back into x.

Evaluate $\int \sqrt{1-4x^2}\,\mathrm{d}x$

The substitution $x = \frac{1}{2}\sin\theta$ tells us $\sin\theta = 2x$. This in turn gives us $\theta = \arcsin(2x)$. We should still convert $\cos\theta$ back into terms of x. You might notice in the calculation we did that $\sqrt{1-4x^2}$ turned into $\cos\theta$, so $\cos\theta = \sqrt{1-4x^2}$.

Alternately, to find $\cos\theta$ in terms of x, we can use a triangle. From $\sin\theta = 2x$, and the understanding that $\sin\theta$ is the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ for a right triangle with angle θ, we can set up a triangle whose opposite side has length $2x$, and hypotenuse has length 1.



The Pythagorean theorem tells us the side adjacent to θ has length $\sqrt{1-4x^2}$. So $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{1-4x^2}$.

$$\int \sqrt{1-4x^2}\,\mathrm{d}x = \frac{1}{4}\left(\underbrace{\arcsin(2x)}_{\theta} + \underbrace{2x\sqrt{1-4x^2}}_{\sin\theta\cos\theta}\right) + C$$

Check Our Work

In the last example, we computed

$$\int \sqrt{1-4x^2}\,\mathrm{d}x = \frac{1}{4}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right) + C.$$

To check, we differentiate.

Check Our Work

In the last example, we computed

$$\int \sqrt{1-4x^2}\,\mathrm{d}x = \frac{1}{4}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right) + C.$$

To check, we differentiate.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{4}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right) + C\right\} \\
&= \frac{1}{4}\left(\frac{2}{\sqrt{1-(2x)^2}} + 2x\frac{-8x}{2\sqrt{1-4x^2}} + 2\sqrt{1-4x^2}\right) \\
&= \frac{1}{4}\left(\frac{2}{\sqrt{1-4x^2}} - \frac{8x^2}{\sqrt{1-4x^2}} + \frac{2(1-4x^2)}{\sqrt{1-4x^2}}\right) \\
&= \frac{1}{4}\left(\frac{2-8x^2+2-8x^2}{\sqrt{1-4x^2}}\right) = \frac{1-4x^2}{\sqrt{1-4x^2}} = \sqrt{1-4x^2} \qquad \checkmark
\end{aligned}$$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$

Evaluate $\int \frac{1}{\sqrt{x^2 - 1}}\, \mathrm{d}x$

ans

We use the substitution $x = \sec\theta$, $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$.
To make the substitution work, we're actually taking $\theta = \arccos\left(\frac{1}{x}\right)$, and so $0 \le \theta \le \pi$.
Note that the integrand exists on the intervals $x < -1$ and $x > 1$.

- When $x > 1$, then $0 < \frac{1}{x} < 1$, so $0 < \arccos\left(\frac{1}{x}\right) < \frac{\pi}{2}$.
  That is, $0 < \theta < \frac{\pi}{2}$, so $|\tan\theta| = \tan\theta$.
- When $x < -1$, then $-1 < \frac{1}{x} < 0$, so $\frac{\pi}{2} < \arccos\left(\frac{1}{x}\right) < \pi$.
  That is, $\frac{\pi}{2} < \theta < \pi$, so $|\tan\theta| = -\tan\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}}\mathrm{d}x &= \int \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot \sec\theta\tan\theta\,\mathrm{d}\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\mathrm{d}\theta \\
&= \int \sec\theta\left(\frac{\tan\theta}{|\tan\theta|}\right)\mathrm{d}\theta = \begin{cases} \int \sec\theta\,\mathrm{d}\theta & 0 < \theta < \frac{\pi}{2} \\ -\int \sec\theta\,\mathrm{d}\theta & \frac{\pi}{2} < \theta < \pi \end{cases} \\
&= \begin{cases} \log|\sec\theta + \tan\theta| + C & 0 < \theta < \frac{\pi}{2} \\ -\log|\sec\theta + \tan\theta| + C & \frac{\pi}{2} < \theta < \pi \end{cases}
\end{aligned}$$

Our substitution tells us $\sec\theta = x$. We saw from the denominator of our integrand that $\sqrt{x^2-1} = |\tan\theta|$.

- When $0 < \theta < \frac{\pi}{2}$, $\tan\theta = |\tan\theta| = \sqrt{x^2-1}$
- When $\frac{\pi}{2} < \theta < \pi$, $\tan\theta = -|\tan\theta| = -\sqrt{x^2-1}$

$$\int \frac{1}{\sqrt{x^2-1}}\mathrm{d}x = \begin{cases} \log|x+\sqrt{x^2-1}| + C & x > 1 \\ -\log|x-\sqrt{x^2-1}| + C & x < -1 \end{cases}$$

Although the two branches look different, they are actually equivalent. Remember $-\log(A) = \log\left(A^{-1}\right) = \log\left(\frac{1}{A}\right)$:

$$\begin{aligned} -\log|x-\sqrt{x^2-1}| &= \log\left|\frac{1}{x-\sqrt{x^2-1}}\right| = \log\left|\frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right| \\ &= \log\left|\frac{x+\sqrt{x^2-1}}{x^2-x^2+1}\right| = \log\left|x+\sqrt{x^2-1}\right| \end{aligned}$$

So,

$$\int \frac{1}{\sqrt{x^2-1}}\mathrm{d}x = \log\left|x+\sqrt{x^2-1}\right| + C$$

Check Our Work

Let's check our result, $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \log\left|x+\sqrt{x^2-1}\right| + C.$

Check Our Work

Let's check our result, $\int \frac{1}{\sqrt{x^2-1}}\mathrm{d}x = \log\left|x+\sqrt{x^2-1}\right| + C.$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log\left|x+\sqrt{x^2-1}\right| + C\right\} = \frac{1+\frac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$

$$= \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}\right) = \frac{(\sqrt{x^2-1}+x)}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

So, our answer works.

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.
Fact: $3 - x^2 + 2x = 4 - (x-1)^2$

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3 - x^2 + 2x}$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$ $\qquad$ $4 - 4\sin^2\theta = 4\cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.
Fact: $\quad 3 - x^2 + 2x = 4 - (x-1)^2$

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3-x^2+2x}$.

Identities have two "parts" that turn into one part:

- $1-\sin^2\theta = \cos^2\theta$ $\qquad 4-4\sin^2\theta = 4\cos^2\theta$
- $1+\tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

But our quadratic expression has *three* parts.
Fact: $3-x^2+2x = 4-(x-1)^2$

$$\sqrt{3-x^2+2x} = \sqrt{4-(x-1)^2}$$

We want $(x-1)^2 = 4\sin^2\theta$, so let $(x-1) = 2\sin\theta$

$$= \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b:

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$

2. Solve for c:

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c: $3 = c - b^2 = c - 1$, so $c = 4$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c: $3 = c - b^2 = c - 1$, so $c = 4$
3. All together:

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b: $-2bx = 2x$, so $b = -1$
2. Solve for c: $3 = c - b^2 = c - 1$, so $c = 4$
3. All together: $3 - x^2 + 2x = 4 - (x-1)^2$

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\,\mathrm{d}x$.

Identities have two “parts” that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

One of those parts is a constant, and one is squared.

ans

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\,\mathrm{d}x$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

One of those parts is a constant, and one is squared.
Write $6x - x^2$ as $c - (x+b)^2$.

$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$
$$6x = -2bx \implies b = -3$$
$$0 = c - b^2 = c - 9 \implies c = 9$$
$$6x - x^2 = 9 - (x-3)^2$$

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\,\mathrm{d}x = \int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x.$

ans

Evaluate $\displaystyle\int \frac{x^2 - 6x + 9}{\sqrt{6x - x^2}}\,\mathrm{d}x = \int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x.$

We use the identity $9 - 9\sin^2\theta = 9\cos^2\theta$.
We want $(x-3)^2 = 9\sin^2\theta$, so take $(x-3) = 3\sin\theta$, $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$.

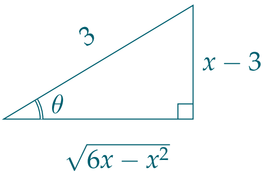
$$\int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}\mathrm{d}x = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}}\,3\cos\theta\,\mathrm{d}\theta = \int 9\sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{9}{2}\int (1-\cos 2\theta)\,\mathrm{d}\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C$$



Check Our Work

Let's verify that

$$\int \frac{x^2-6x+9}{\sqrt{6x-x^2}} = \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C:$$

Check Our Work

Let's verify that

$$\int \frac{x^2-6x+9}{\sqrt{6x-x^2}} = \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C:$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C\right\}\\
&= \frac{9}{2}\left(\frac{1/3}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}} - \frac{x-3}{3}\cdot\frac{3-x}{3\sqrt{6x-x^2}} - \frac{1}{9}\sqrt{6x-x^2}\right)\\
&= \frac{9}{2}\left(\frac{9}{9\sqrt{6x-x^2}} - \frac{6x-x^2-9}{9\sqrt{6x-x^2}} - \frac{6x-x^2}{9\sqrt{6x-x^2}}\right)\\
&= \frac{9-6x+x^2}{\sqrt{6x-x^2}}
\end{aligned}$$

So, our answer works.

Included Work

 'Notebook' by Iconic is licensed under CC BY 3.0 (accessed 9 June 2021, modified), 21, 22, 24

 'Notebook' by Iconic is licensed under CC BY 3.0 (accessed 9 June 2021), 44, 47, 53, 71