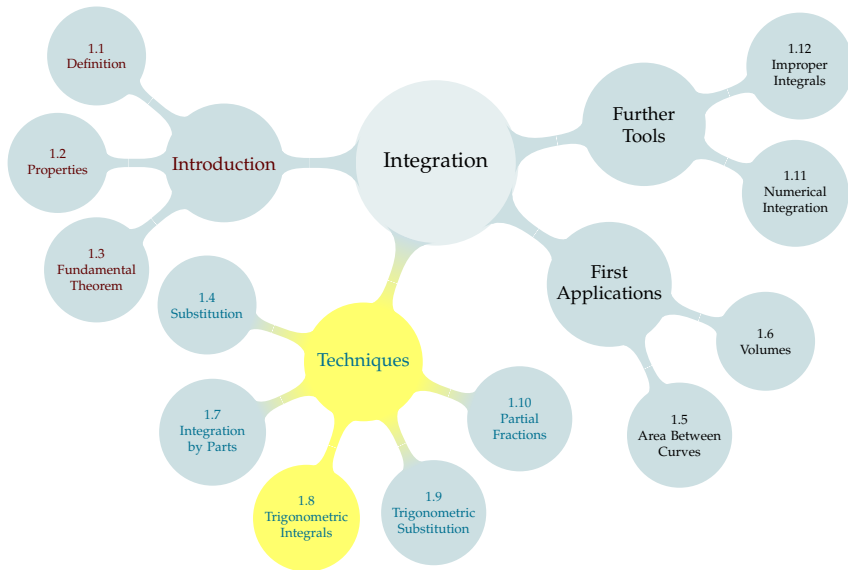


TABLE OF CONTENTS



1.8 TRIGONOMETRIC INTEGRALS

Recall:

- ▶ $\sin^2 x + \cos^2 x = 1$
- ▶ $\tan^2 x + 1 = \sec^2 x$
- ▶ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- ▶ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- ▶ $\sin(2x) = 2 \sin x \cos x$

INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin x \cos x dx =$$

INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin x \cos x dx =$$

$$\int \sin^{10} x \cos x dx =$$

CHECK OUR WORK

If we are correct that $\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C$, then it should be true that $\frac{d}{dx} \left\{ \frac{\sin^2 x}{2} + C \right\} = \sin x \cos x$.

CHECK OUR WORK

If we are correct that $\int \sin^{10} x \cos x dx = \frac{\sin^{11} x}{11} + C$, then it should be true that $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} + C \right\} = \sin^{10} x \cos x$.

INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x dx =$$

CHECK OUR WORK

If we are correct that $\int \sin^{\pi+1} x \cos x dx = \frac{\sin^{\pi+2} x}{\pi+2} + C$, then it should be true that $\frac{d}{dx} \left\{ \frac{\sin^{\pi+2} x}{\pi+2} + C \right\} = \sin^{\pi+1} x \cos x$.



INTEGRATING PRODUCTS OF SINE AND COSINE

Let $u = \sin x$, $du = \cos x dx$.

$$\int \sin^{10} x \cos^3 x dx =$$

INTEGRATING PRODUCTS OF SINE AND COSINE

Let $u = \sin x$, $du = \cos x dx$.

Use $\sin^2 x + \cos^2 x = 1$.

$$\int \sin^{10} x \cos^3 x dx =$$

CHECK OUR WORK

If we are correct that $\int \sin^{10} x \cos^3 x dx = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$, then it should be true that $\frac{d}{dx} \left\{ \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C \right\} = \sin^{10} x \cos^3 x$.

INTEGRATING PRODUCTS OF SINE AND COSINE

$$\int \sin^5 x \cos^4 x dx =$$

CHECK OUR WORK

If we are correct that

$$\int \sin^5 x \cos^4 x dx = -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C, \text{ then it should}$$

be true that $\frac{d}{dx} \left\{ -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + C \right\} = \sin^5 x \cos^4 x.$

GENERALIZE: $\int \sin^m x \cos^n bx dx$

To use the substitution $u = \sin x$, $du = \cos x dx$:

GENERALIZE: $\int \sin^m x \cos^n bx dx$

To use the substitution $u = \sin x$, $du = \cos x dx$:

- We need to reserve one $\cos x$ for the differential.

GENERALIZE: $\int \sin^m x \cos^n bx dx$

To use the substitution $u = \sin x$, $du = \cos x dx$:

- ▶ We need to **reserve** one **$\cos x$** for the differential.
- ▶ We need to **convert** the remaining **$\cos^{n-1} x$** to **$\sin x$** terms.

GENERALIZE: $\int \sin^m x \cos^n bx dx$

To use the substitution $u = \sin x$, $du = \cos x dx$:

- ▶ We need to **reserve** one **$\cos x$** for the differential.
- ▶ We need to **convert** the remaining **$\cos^{n-1} x$** to **$\sin x$** terms.
- ▶ We convert using **$\cos^2 x = 1 - \sin^2 x$** . To avoid square roots, that means $n - 1$ should be **even when we convert**.

GENERALIZE: $\int \sin^m x \cos^n bx dx$

To use the substitution $u = \sin x$, $du = \cos x dx$:

- ▶ We need to **reserve** one **$\cos x$** for the differential.
- ▶ We need to **convert** the remaining **$\cos^{n-1} x$** to **$\sin x$** terms.
- ▶ We convert using **$\cos^2 x = 1 - \sin^2 x$** . To avoid square roots, that means $n - 1$ should be **even when we convert**.
- ▶ **So, we can use this substitution when the original power of cosine, n , is ODD: one cosine goes to the differential, the rest are converted to sines.**

GENERALIZE: $\int \sin^m x \cos^n x dx$

To use the substitution $u = \cos x$, $du = -\sin x dx$:

GENERALIZE: $\int \sin^m x \cos^n x dx$

To use the substitution $u = \cos x$, $du = -\sin x dx$:

- We need to **reserve** one **$\sin x$** for the differential.

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- ▶ We need to **reserve** one $\sin x$ for the differential.
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GENERALIZE: $\int \sin^m x \cos^n x dx$

To use the substitution $u = \cos x$, $du = -\sin x dx$:

- ▶ We need to **reserve** one $\sin x$ for the differential.
- ▶ We need to **convert** the remaining $\sin^{m-1} x$ to $\cos x$ terms.
- ▶ We convert using $\sin^2 x = 1 - \cos^2 x$. To avoid square roots, that means $m - 1$ should be **even when we convert**.

GENERALIZE: $\int \sin^m x \cos^n x dx$

To use the substitution $u = \cos x$, $du = -\sin x dx$:

- ▶ We need to **reserve** one **$\sin x$** for the differential.
- ▶ We need to **convert** the remaining **$\sin^{m-1} x$** to **$\cos x$** terms.
- ▶ We convert using **$\sin^2 x = 1 - \cos^2 x$** . To avoid square roots, that means $m - 1$ should be **even when we convert**.
- ▶ **So, we can use this substitution when the original power of sine, m , is ODD: one sine goes to the differential, the rest are converted to cosines.**

MNEMONIC: "ODD ONE OUT"

$$\text{Integrating } \int \sin^m x \cos^n x dx$$

If you want to use $u = \sin x$, there should be an odd power of **cosine**.

If you want to use $u = \cos x$, there should be an odd power of **sine**.

Carry out a suitable substitution (but do not evaluate the resulting integral):

► $\int \sin^4 x \cos^7 x dx$

► $\int \sin^7 x \cos^4 x dx$

► $\int \sin^7 x \cos^7 x dx$



To evaluate $\int \sin^m x \cos^n x dx$, we use:

- ▶ $u = \sin x$ if n is odd, and/or
- ▶ $u = \cos x$ if m is odd

What if n and m are both even?

To evaluate $\int \sin^m x \cos^n x dx$, we use:

- ▶ $u = \sin x$ if n is odd, and/or
- ▶ $u = \cos x$ if m is odd

What if n and m are both even?

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \sin^2 x dx =$$

CHECK OUR WORK

We check that $\int \sin^2 x dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C$ by differentiating:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Evaluate $\int \sin^4 x dx$.

CHECK OUR WORK

We want to check that $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$

Recall:

- ▶ $\frac{d}{dx} \{\tan x\} = \sec^2 x$
- ▶ $\frac{d}{dx} \{\sec x\} = \sec x \tan x$
- ▶ $\tan^2 x + 1 = \sec^2 x$

$$\int \tan x \, dx =$$

CHECK OUR WORK

Let's check that $\int \tan x dx = \log |\sec x| + C$ by differentiating.

Optional: A nifty trick – you won't be expected to come up with it.
There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx =$$

Optional: A nifty trick – you won't be expected to come up with it.
 There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

Useful integrals:

$$\blacktriangleright \int \tan x \, dx = \log |\sec x| + C$$

$$\blacktriangleright \int \sec x \, dx = \log |\sec x + \tan x| + C$$

$$1. \int \sec x \tan x \, dx =$$

$$2. \int \sec^2 x \, dx =$$

$$3. \int \tan x \, dx =$$

$$4. \int \sec x \, dx =$$

Evaluate using the substitution rule:

$$\int \tan^5 x \sec^2 x dx =$$

$$\int \sec^4 x (\sec x \tan x) dx =$$

CHECK OUR WORK

Let's check that $\int \tan^5 x \sec^2 x dx = \frac{1}{6} \tan^6 x + C$ by differentiating.

CHECK OUR WORK

Let's check that $\int \sec^4 x (\sec x \tan x) dx = \frac{1}{5} \sec^5 x + C$ by differentiating.

Evaluate using the identity $\sec^2 x = 1 + \tan^2 x$

$$\int \tan^4 x \sec^6 x dx =$$

$$\int \tan^3 x \sec^5 x dx =$$

CHECK OUR WORK

Let's check that $\int \tan^4 x \sec^6 x dx = \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.$

CHECK OUR WORK

Let's check that $\int \tan^3 x \sec^5 x dx = \frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C.$

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \sec x$, $du = \sec x \tan x dx$:

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \sec x$, $du = \sec x \tan x dx$:

- Reserve $\sec x \tan x$ for the differential.

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \sec x$, $du = \sec x \tan x dx$:

- ▶ Reserve $\sec x \tan x$ for the differential.
- ▶ From the remaining $\tan^{m-1} x \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \sec x$, $du = \sec x \tan x dx$:

- ▶ Reserve $\sec x \tan x$ for the differential.
(m, n should each be at least 1)
- ▶ From the remaining $\tan^{m-1} x \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \sec x$, $du = \sec x \tan x dx$:

- ▶ Reserve $\sec x \tan x$ for the differential.
(m, n should each be at least 1)
- ▶ From the remaining $\tan^{m-1} x \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.
($m - 1$ should be even, to avoid square roots)

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \sec x$, $du = \sec x \tan x dx$:

- ▶ Reserve $\sec x \tan x$ for the differential.
 (m, n should each be at least 1)
- ▶ From the remaining $\tan^{m-1} x \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.
 ($m - 1$ should be even, to avoid square roots)

To use the substitution $u = \sec x$, $du = \sec x \tan x dx$ to evaluate

$\int \tan^m x \sec^n x dx$, n should be , and m should be .

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x dx$

Using $u = \tan x$, $du = \sec^2 x dx$:

- ▶ Reserve for the differential.
- ▶ From the remaining terms, convert all to using $\tan^2 x + 1 = \sec^2 x$.

To use the substitution $u = \tan x$, $du = \sec^2 x dx$ to evaluate $\int \tan^m x \sec^n x dx$, n should be .

Evaluating $\int \tan^m x \sec^n x dx$

To evaluate $\int \tan^m x \sec^n x dx$, we can use:

- ▶ $u = \sec x$ if m is odd and $n \geq 1$
- ▶ $u = \tan x$ if n is even and $n \geq 2$

Choose a substitution for the integrals below.

▶ $\int \sec^2 x \tan^3 x dx$

▶ $\int \sec^2 x \tan^2 x dx$

▶ $\int \sec^3 x \tan^3 x dx$

$$\int \sec^2 x \tan^3 x dx$$

$$\int \sec^2 x \tan^2 x dx$$

$$\int \sec^3 x \tan^3 x dx$$

Evaluate $\int \tan^3 x dx$

Evaluate $\int \tan^3 x dx = \int \frac{\sin^3 x}{\cos^3 x} dx$

CHECK OUR WORK

Let's check that $\int \tan^3 x dx = \log |\cos x| + \frac{1}{2} \sec^2 x + C$ by differentiating.

Generalizing the last example:

$$\int \tan^m x \sec^n x dx =$$

Generalizing the last example:

$$\begin{aligned}
 \int \tan^m x \sec^n x dx &= \int \left(\frac{\sin x}{\cos x} \right)^m \left(\frac{1}{\cos x} \right)^n dx \\
 &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\
 &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx
 \end{aligned}$$

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \left(\frac{\sin x}{\cos x} \right)^m \left(\frac{1}{\cos x} \right)^n dx \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\ &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx\end{aligned}$$

To use $u = \cos x$, $du = -\sin x dx$:

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \left(\frac{\sin x}{\cos x} \right)^m \left(\frac{1}{\cos x} \right)^n dx \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} dx \\ &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x} \right) \sin x dx\end{aligned}$$

To use $u = \cos x$, $du = \sin x dx$: we will convert $\sin^{m-1}(x)$ into cosines, so $m - 1$ must be even, so m must be odd.

Evaluating $\int \tan^m x \sec^n x dx$

To evaluate $\int \tan^m x \sec^n x dx$, we can use:

- ▶ $u = \sec x$ if m is odd and $n \geq 1$
- ▶ $u = \tan x$ if n is even and $n \geq 2$
- ▶ $u = \cos x$ if m is odd

Evaluate $\int \tan^2 x dx$

Evaluating $\int \tan^m x \sec^n x dx$

To evaluate $\int \tan^m x \sec^n x dx$, we can use:

- ▶ $u = \sec x$ if m is odd and $n \geq 1$
- ▶ $u = \tan x$ if n is even and $n \geq 2$
- ▶ $u = \cos x$ if m is odd

Evaluate $\int \tan^2 x dx$

Evaluating $\int \tan^m x \sec^n x dx$

To evaluate $\int \tan^m x \sec^n x dx$, we can use:

- ▶ $u = \sec x$ if m is odd and $n \geq 1$
- ▶ $u = \tan x$ if n is even and $n \geq 2$
- ▶ $u = \cos x$ if m is odd

Evaluate $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

Evaluating $\int \tan^m x \sec^n x dx$

To evaluate $\int \tan^m x \sec^n x dx$, we can use:


- ▶ $u = \sec x$ if m is odd and $n \geq 1$
- ▶ $u = \tan x$ if n is even and $n \geq 2$
- ▶ $u = \cos x$ if m is odd
- ▶ $u = \tan x$ if m is even and $n = 0$
(after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

Evaluate $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

Included Work

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