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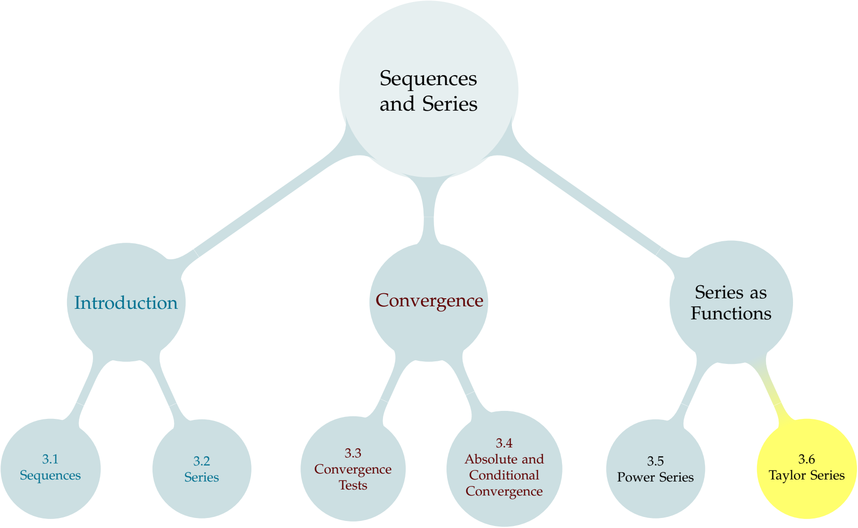


Taylor polynomial

Let a be a constant and let n be a non-negative integer. The n^{th} order Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \cdot (x-a)^k.$$

Taylor series

The Taylor series for the function $f(x)$ expanded around a is the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n.$$

When $a = 0$ it is also called the Maclaurin series of $f(x)$.

Let’s compute some Taylor series, using the definition.

The method is nearly identical to finding Taylor *polynomials*, which is covered in CLP–1.

Find the Maclaurin series for $f(x) = \sin x$.

Taylor series

The Taylor series for the function $f(x)$ expanded around a is the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n.$$

When $a = 0$ it is also called the Maclaurin series of $f(x)$.

$$\begin{array}{rclcrcl} f(x) & = & \sin x & \quad & f(0) & = & 0 \\ f'(x) & = & \cos x & & f'(0) & = & 1 \\ f''(x) & = & -\sin x & & f''(0) & = & 0 \\ f'''(x) & = & -\cos x & & f'''(0) & = & -1 \end{array}$$

The derivatives then repeat. Notice we only have non-zero derivatives for odd orders, and these alternate in sign.
We can write the Maclaurin series as follows:

Find the Maclaurin series for $f(x) = \cos x$.

$$\begin{array}{rclrcl} f(x) & = & \cos x & f(0) & = & 1 \\ f'(x) & = & -\sin x & f'(0) & = & 0 \\ f''(x) & = & -\cos x & f''(0) & = & -1 \\ f'''(x) & = & \sin x & f'''(0) & = & 0 \end{array}$$

The derivatives then repeat. Notice we only have non-zero derivatives for even orders, and these alternate in sign. We can write the Maclaurin series as follows:

$$\begin{aligned} \cos x &\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

The Maclaurin series for $f(x) = e^x$ is: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Every derivative of e^x is e^x, so all coefficients $f^{(n)}(0)$ are e^0, i.e. 1.

$$\begin{aligned} e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

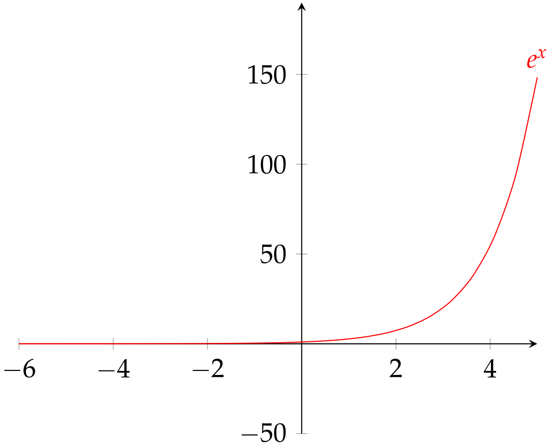
Q

Let $T_n(x)$ be the n-th order Taylor polynomial of the function $f(x)$, centred at a.

When we introduced Taylor polynomials in CLP–1, we framed $T_n(x)$ as an approximation of $f(x)$.

Let's see how those approximations look in two cases:

Taylor Polynomials for e^x



It seems like high-order Taylor polynomials do a pretty good job of approximating the function e^x, at least when x is near enough to 0.

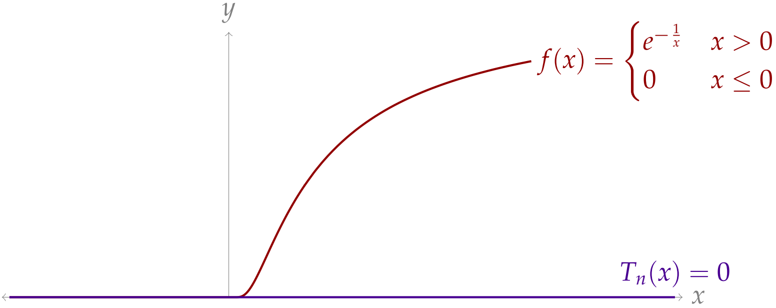
Taylor Polynomials for a different function

But that is not the case for all functions. Define

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Using the definition of the derivative and l'Hôpital's rule, one can show that $f^{(n)}(0) = 0$ for all natural numbers n.

Taylor Polynomials for a different function



Taylor polynomial approximations don't always get better as their orders increase – it depends on the function being approximated.

Investigation

- We found the Maclaurin series for $f(x) = e^x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- But, it's not immediately clear whether $e^x \stackrel{?}{=} \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- We're going to demonstrate that e^x is in fact equal to $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. The proof involves a particular limit: $\displaystyle\lim_{n\to\infty} \frac{|x|^n}{n!}$. We'll talk about that limit first, so that it doesn't distract us later.

Intermediate result: $\lim\limits_{n\to\infty} \frac{|x|^n}{n!}$, when x is some fixed number.

For large n, we can think of $\frac{|x|^n}{n!}$ as a long multiplication, with decreasing terms. At some point, those terms are all decreasing *and less than 1*.

$$\frac{|x|^n}{n!} = \frac{|x| \cdot |x| \cdot |x| \cdot |x| \cdot |x| \cdot |x| \cdot \ldots \cdot |x|}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \ldots \cdot n}$$

$$=$$

 Convenient notation: $\lceil x \rceil$ is the number you get when you round x up to the nearest whole number.

Intermediate result: $\lim\limits_{n\to\infty} \frac{|x|^n}{n!}$, when x is some fixed number.
We're multiplying terms that are closer and closer to 0, so it seems quite reasonable that this sequence should converge to 0.

For a more formal proof, we can use the squeeze theorem to compare this sequence to a geometric sequence.

Convenient notation: $\lceil x \rceil$ is the number you get when you round x up to the nearest whole number.

INVESTIGATION

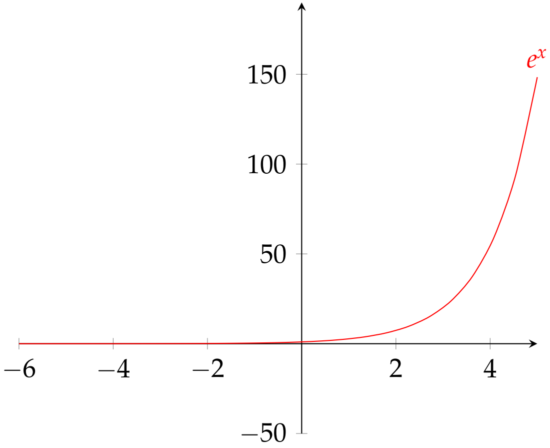
- We found the Maclaurin series for $f(x) = e^x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- But, it's not immediately clear whether $e^x \stackrel{?}{=} \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. How could we determine this?
-

$$
\begin{aligned}
& e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\iff & 0 = e^x - \sum_{n=0}^{\infty} \frac{x^n}{n} = e^x - \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n} \frac{x^k}{k!}}_{T_n(x)} = \lim_{n\to\infty} \underbrace{\left[e^x - T_n(x)\right]}_{E_n(x)} \\
\iff & 0 = \lim_{n\to\infty} E_n(x) \qquad \text{(for all } x\text{)}
\end{aligned}
$$

Taylor Polynomial Error for $f(x) = e^x$

If $\lim_{n\to\infty} E_n(x) = 0$ for all x, then $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

It *looks* plausible, especially when x is close to 0. Let's try to prove it.



Equation 3.6.1-b

Let $T_n(x)$ be the n-th order Taylor approximation of a function $f(x)$, centred at a. Then $E_n(x) = f(x) - T_n(x)$ is the error in the n-th order Taylor approximation.
For some c strictly between x and a,

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \cdot (x-a)^{n+1}$$

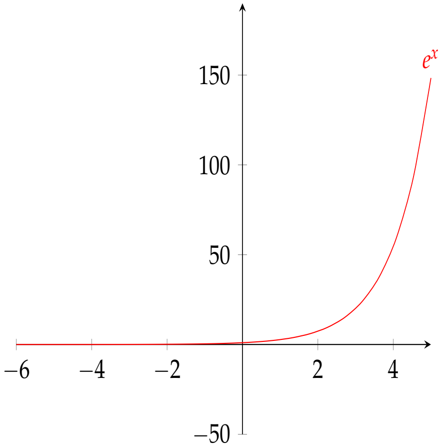
When $f(x) = e^x$,

$$E_n(x) = e^c \frac{x^{n+1}}{(n+1)!}$$

for some c between 0 and x.

$$
\begin{aligned}
E_n(x) &= e^x - T_n(x) \\
&= e^c \frac{x^{n+1}}{(n+1)!} && \text{for some } c \text{ between 0 and } x \\
0 \le |E_n(x)| &< \left| e^c \frac{x^{n+1}}{(n+1)!} \right| \\
&\le e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \\
0 &= \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} && \text{by our previous result} \\
\implies 0 &= \lim_{n\to\infty} |E_n(x)| && \text{by the squeeze theorem}
\end{aligned}
$$

We found $0 \le |E_n(x)| < e^{|x|}\frac{|x|^{n+1}}{(n+1)!}$ for large n, hence $\lim\limits_{n\to\infty} |E_n(x)| = 0$.



For a particular value of x:

We saw $\quad 0 = \lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$

so $\quad 0 = \lim\limits_{n\to\infty} E_n(x)$

That is, $\quad 0 = \lim\limits_{n\to\infty} \left[e^x - T_n(x)\right]$

So, $e^x = \lim\limits_{n\to\infty} T_n(x)$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

TAYLOR POLYNOMIAL ERROR FOR SINE AND COSINE

Equation 3.6.1-b

Let $T_n(x)$ be the n-th order Taylor approximation of a function $f(x)$, centred at a. Then $E_n(x) = f(x) - T_n(x)$ is the error in the n-th order Taylor approximation.

For some c strictly between x and a,

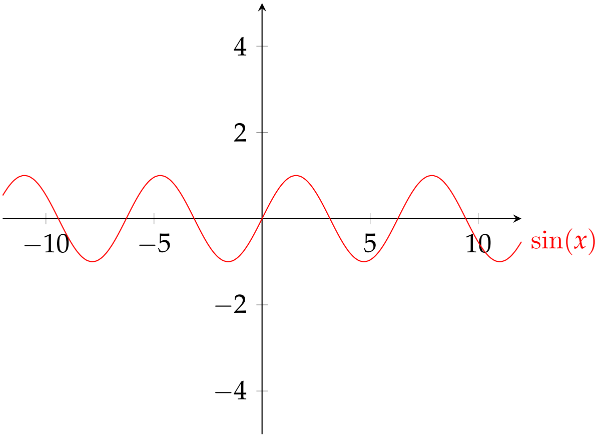
$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \cdot (x-a)^{n+1}$$

Suppose $f(x)$ is either $\sin x$ or $\cos x$. Is $f(x)$ equal to its Maclaurin series? In either case, $|f^{(n+1)}(c)|$ is either $|\sin c|$ or $|\cos c|$, so it's between 0 and 1.

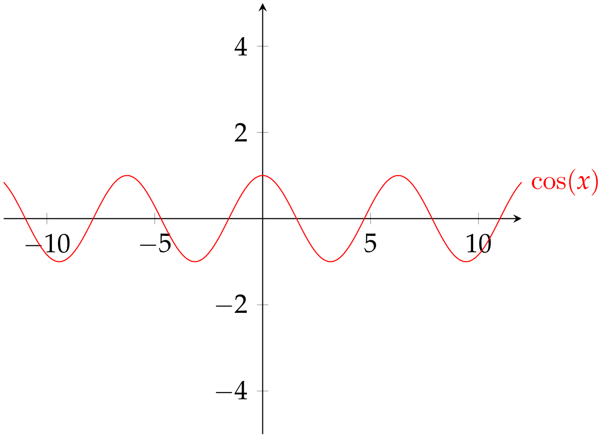
$$|E_n(x)| = \frac{1}{(n+1)!} \left|f^{(n+1)}(c)\right| |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\implies 0 \leq |E_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Taylor Polynomials for $\sin(x)$



Taylor Polynomials for $\cos(x)$



Selected Taylor series that equal their functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } -\infty < x < \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \qquad \text{for all } -\infty < x < \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} \qquad \text{for all } -\infty < x < \infty$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for all } -1 < x < 1$$

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \text{for all } -1 < x \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all } -1 \le x \le 1$$

COMPUTING π

Use the fact that $\arctan 1 = \frac{\pi}{4}$ to find a series converging to π whose terms are rational numbers.

For all $-1 \leq x \leq 1$:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$4 \arctan x = 4 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\pi = 4 \arctan 1 = 4 \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

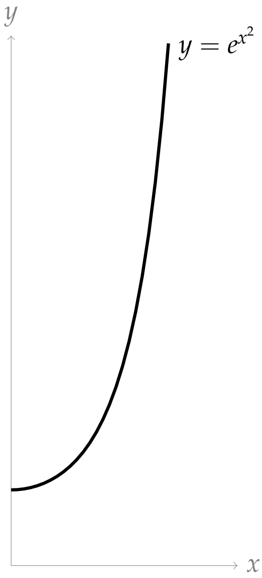
$$= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

Error function

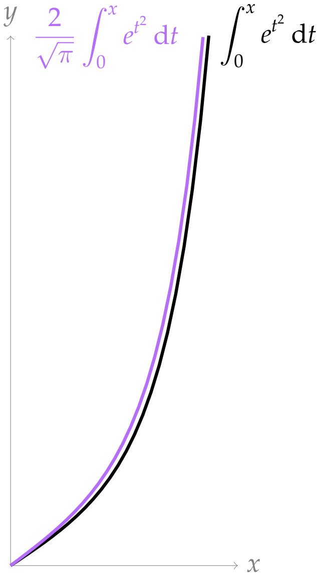
The *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t$$

is used in computing "bell curve" probabilities.



y
$y = e^{x^2}$
x



y
$\frac{2}{\sqrt{\pi}} \int_0^x e^{t^2}\,\mathrm{d}t$
$\int_0^x e^{t^2}\,\mathrm{d}t$
x

Error function

The *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t$$

is used in computing "bell curve" probabilities.

The indefinite integral of the integrand e^{-t^2} cannot be expressed in terms of standard functions. But we can still evaluate the integral to within any desired degree of accuracy by using the Taylor expansion of the exponential.

For example, evaluate $\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$.

$$\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\Bigg|_{x=-t^2} \mathrm{d}t$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right) \mathrm{d}t = \frac{2}{\sqrt{\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)n!}\right]_0^{\frac{1}{\sqrt{2}}}$$

Q

Evaluating a convergent series

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$

$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	for all $-\infty < x < \infty$
$\sin(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$	for all $-\infty < x < \infty$
$\cos(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$	for all $-\infty < x < \infty$
$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$	for all $-1 < x < 1$
$\log(1+x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	for all $-1 < x \leq 1$
$\arctan x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	for all $-1 \leq x \leq 1$

The series most closely resembles the Taylor series $\log(1+x) = \displaystyle\sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1}$. To make that relation clearer, set

Q

Finding a high-order derivative

Let $f(x) = \sin(2x^3)$. Find $f^{(15)}(0)$, the fifteenth derivative of f at $x = 0$.

Differentiating directly gets messy quickly. Instead, let's find the Taylor series. Let $y = 2x^3$:

$$\sin(y) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} y^{2n+1}$$

$$\implies \quad f(x) = \sin(2x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(2x^3\right)^{2n+1}$$

$$\implies \quad f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{6n+3}$$

The coefficients of x^{15} on the left and right series must match for the series to be equal.

When $m = 15$ on the left-hand side, we get the term $\frac{f^{(15)}(0)}{15!} x^{15}$. The right-hand side term corresponding to x^{15} occurs when $6n + 3 = 15$, i.e. when $n = 2$.

Given that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have a new way of evaluating the familiar limit

$$\lim_{x\to 0} \frac{\sin x}{x}:$$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} \\
&= \lim_{x\to 0} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] \\
&= 0
\end{aligned}$$

This technique is sometimes faster than l'Hôpital's rule.

Evaluate $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{\sin x - x}$.

$$\begin{aligned}\arctan x - x &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) - x\\ &= -\frac{x^3}{3} + \frac{x^5}{5} - \cdots\end{aligned}$$

$$\begin{aligned}\sin x - x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x\\ &= -\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0} \frac{\arctan x - x}{\sin x - x} &= \lim_{x\to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)\\ &= \lim_{x\to 0} \frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{x^2}{5!} - \cdots} = \frac{-\frac{1}{3}}{-\frac{1}{6}} = 2\end{aligned}$$

Q