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1.8 Trigonometric Integrals

Recall:

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\, dx$

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Integrating Products of Sine and Cosine

Let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$

$$\int \sin x \cos x\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Let $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$

$$\int \sin^{10} x \cos x\,\mathrm{d}x = \int u^{10}\,\mathrm{d}u = \frac{1}{11}u^{11} + C = \frac{1}{11}\sin^{11} x + C$$

Check Our Work

If we are correct that $\displaystyle\int \sin x \cos x \,\mathrm{d}x = \qquad\qquad$, then it should be true that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\qquad\qquad\right\} = \sin x \cos x$.

We differentiate, using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin^2 x}{2} + C\right\} = \frac{2}{2}\sin x \cos x = \sin x \cos x$$

Our answer works.

Check Our Work

If we are correct that $\int \sin^{10} x \cos x \, dx = \qquad\qquad$, then it should be true that $\frac{d}{dx}\left\{ \qquad\qquad \right\} = \sin^{10} x \cos x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^{11} x}{11} + C\right\} = \frac{11}{11}\sin^{10} x \cos x = \sin^{10} x \cos x$$

Our answer works.

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\, dx$

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x\, dx = \int_{\sin(0)}^{\sin(\pi/2)} u^{\pi+1} du = \left. \frac{1}{\pi+2} u^{\pi+2} \right|_0^1$$
$$= \frac{1}{\pi+2}$$

Q

CHECK OUR WORK

If we are correct that $\int \sin^{\pi+1} x \cos x \, dx =$, then it should be true that $\frac{d}{dx}\left\{ \qquad \right\} = \sin^{\pi+1} x \cos x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^{\pi+2} x}{\pi+2} + C\right\} = \frac{\pi+2}{\pi+2}\sin^{\pi+1} x \cos x = \sin^{\pi+1} x \cos x$$

Our answer works.

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\,dx$.

$$\int \sin^{10} x \cos^3 x\,dx = \int \sin^{10} x \cos^2 x \cos x\,dx$$

$$= \int \sin^{10} x\,(1 - \sin^2 x) \cos x\,dx$$

$$= \int u^{10}(1 - u^2)\,du = \int (u^{10} - u^{12})\,du$$

$$= \frac{1}{11}u^{11} - \frac{1}{13}u^{13} + C = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$$

Check Our Work

If we are correct that $\int \sin^{10} x \cos^3 x\, dx =$, then it should be true that $\frac{d}{dx}\left\{ \quad \right\} = \sin^{10} x \cos^3 x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C\right\} = \frac{11}{11}\sin^{10} x \cos x - \frac{13}{13}\sin^{12} x \cos x$$
$$= \sin^{10} x\left(1 - \sin^2 x\right)\cos x = \sin^{10} x \cos^2 x \cos x$$
$$= \sin^{10} x \cos^3 x$$

Our answer works.

Integrating Products of Sine and Cosine

$u = \cos x$, $du = -\sin x \, dx$ $\qquad \sin^2 x + \cos^2 x = 1.$

$$\begin{aligned}
\int \sin^5 x \cos^4 x \, dx &= \int (\sin^2 x)^2 \cos^4 x \sin x \, dx \\
&= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx \\
&= - \int (1-u^2)^2 u^4 \, du = - \int (1 - 2u^2 + u^4) u^4 du \\
&= - \int (u^4 - 2u^6 + u^8) du = -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C \\
&= -\tfrac{1}{5}\cos^5 x + \tfrac{2}{7}\cos^7 x - \tfrac{1}{9}\cos^9 x + C
\end{aligned}$$

Check Our Work

If we are correct that $\displaystyle\int \sin^5 x \cos^4 x \,dx =$, then it should be true that $\frac{d}{dx}\{ \quad \} = \sin^5 x \cos^4 x$.

We differentiate, using the chain rule:

$$\begin{aligned}
&\frac{d}{dx}\left\{-\tfrac{1}{5}\cos^5 x + \tfrac{2}{7}\cos^7 x - \tfrac{1}{9}\cos^9 x + C\right\} \\
&= \frac{5}{5}\cos^4 x \sin x - \frac{2\cdot 7}{7}\cos^6 x \sin x + \frac{9}{9}\cos^8 x \sin x \\
&= \cos^4 x \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \\
&= \cos^4 x \sin x \left(1 - \cos^2 x\right)^2 = \cos^4 x \sin x \left(\sin^2 x\right)^2 \\
&= \sin^5 x \cos^4 x
\end{aligned}$$

Our answer works.

GENERALIZE: $\int \sin^m x \cos^n bx \, dx$

To use the substitution $u = \sin x$, $du = \cos x \, dx$:

- We need to reserve one $\cos x$ for the differential.
- We need to convert the remaining $\cos^{n-1} x$ to $\sin x$ terms.
- We convert using $\cos^2 x = 1 - \sin^2 x$. To avoid square roots, that means $n - 1$ should be even when we convert.
- So, we can use this substitution when the original power of cosine, n, is ODD: one cosine goes to the differential, the rest are converted to sines.

GENERALIZE: $\int \sin^m x \cos^n x \, dx$

To use the substitution $u = \cos x$, $du = -\sin x \, dx$:

- We need to reserve one $\sin x$ for the differential.
- We need to convert the remaining $\sin^{m-1} x$ to $\cos x$ terms.
- We convert using $\sin^2 x = 1 - \cos^2 x$. To avoid square roots, that means $m - 1$ should be even when we convert.
- So, we can use this substitution when the original power of sine, m, is ODD: one sine goes to the differential, the rest are converted to cosines.

Mnemonic: "Odd One Out"

$$\text{Integrating } \int \sin^m x \cos^n x \, dx$$

If you want to use $u = \sin x$, there should be an odd power of cosine.

If you want to use $u = \cos x$, there should be an odd power of sine.

Carry out a suitable substitution (but do not evaluate the resulting integral):

- $\displaystyle\int \sin^4 x \cos^7 x \,dx$
- $\displaystyle\int \sin^7 x \cos^4 x \,dx$
- $\displaystyle\int \sin^7 x \cos^7 x \,dx$

$$\int \sin^4 x \cos^7 x \,dx$$

The power of cosine is odd, so it becomes our differential. That is, we use $u = \sin x$, $du = \cos x\,dx$.

$$\begin{aligned}
&\int \sin^4 x \cos^7 x \,dx \\
&= \int \sin^4 x\,(\cos^2 x)^3 \cos x\,dx \\
&= \int \sin^4 x\,(1-\sin^2 x)^3 \cos x\,dx \\
&= \int u^4 (1-u^2)^3\,du
\end{aligned}$$

To evaluate $\int \sin^m x \cos^n x \, dx$, we use:

- $u = \sin x$ if n is odd, and/or
- $u = \cos x$ if m is odd

What if n and m are both even?

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\begin{aligned}\int \sin^2 x \,\mathrm{d}x &= \int \frac{1-\cos 2x}{2}\,\mathrm{d}x \\ &= \frac{1}{2}\int (1-\cos 2x)\,\mathrm{d}x \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C\end{aligned}$$

Check Our Work

We check that $\int \sin^2 x \, dx =$ by differentiating:

$$\frac{d}{dx}\left\{\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+C\right\}=\frac{1}{2}\left(1-\frac{1}{2}(\cos 2x)(2)\right)$$
$$=\frac{1-\cos 2x}{2}=\sin^2 x$$

So, our answer works.

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

Evaluate $\displaystyle\int \sin^4 x \, dx$.

$$\begin{aligned}
\int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \, dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x) \, dx + \frac{1}{4}\int \cos^2(2x) \, dx \\
&= \frac{1}{4}(x - \sin 2x) + \frac{1}{4}\int \left(\frac{1+\cos(4x)}{2}\right) dx \\
&= \frac{1}{4}(x - \sin 2x) + \frac{1}{8}\left(x + \frac{1}{4}\sin(4x)\right) + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

Q

Check Our Work

We want to check that $\int \sin^4 x \, dx =$

Note $\sin^2 x = \frac{1-\cos(2x)}{2}$, so $\cos(2x) = 1 - 2\sin^2 x$. Also remember $\frac{1}{2}\sin(2x) = \sin x \cos x$.

$$\begin{aligned}
\frac{d}{dx}\left\{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\right\} &= \frac{3}{8} - \frac{2}{4}\cos(2x) + \frac{4}{32}\cos(4x)\\
&= \frac{3}{8} - \frac{1}{2}\left(1 - 2\sin^2 x\right) + \frac{1}{8}\left(1 - 2\sin^2(2x)\right)\\
&= \frac{3}{8} - \frac{1}{2} + \sin^2 x + \frac{1}{8} - \frac{1}{4}\sin^2(2x)\\
&= \sin^2 x - \left(\frac{1}{2}\sin 2x\right)^2 = \sin^2 x - \sin^2 x \cos^2 x\\
&= \sin^2 x(1 - \cos^2 x) = \sin^2 x(\sin^2 x) = \sin^4 x
\end{aligned}$$

So, our answer works.

Recall:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{\tan x\} = \sec^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{\sec x\} = \sec x \ \tan x$
- $\tan^2 x + 1 = \sec^2 x$

$$\begin{aligned}\int \tan x\,\mathrm{d}x &= \int \frac{\sin x}{\cos x}\,\mathrm{d}x \qquad u = \cos x \quad \mathrm{d}u = -\sin x\,\mathrm{d}x\\ &= -\int \frac{1}{u}\,\mathrm{d}u = -\log|u| + C\\ &= \log|u^{-1}| + C = \log\left|\frac{1}{\cos x}\right| + C\\ &= \log|\sec x| + C\end{aligned}$$

Check Our Work

Let's check that $\int \tan x dx =$ by differentiating.

$$\frac{d}{dx}\left\{\log|\sec x| + C\right\} = \frac{\sec x \tan x}{\sec x} = \tan x$$

So, our answer works.

Optional: A nifty trick – you won't be expected to come up with it. There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx$$

set $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\, dx$

$$= \int \frac{1}{u} \, du = \log |u| + C$$

$$= \log |\sec x + \tan x| + C$$

Useful integrals:

- $\displaystyle\int \tan x \,\mathrm{d}x = \log|\sec x| + C$
- $\displaystyle\int \sec x \,\mathrm{d}x = \log|\sec x + \tan x| + C$

1. $\displaystyle\int \sec x \tan x \, \mathrm{d}x = \sec x + C$

2. $\displaystyle\int \sec^2 x \, \mathrm{d}x = \tan x + C$

3. $\displaystyle\int \tan x \, \mathrm{d}x = \log|\sec x| + C$

4. $\displaystyle\int \sec x \, \mathrm{d}x = \log|\sec x + \tan x| + C$

Evaluate using the substitution rule:

$u = \tan x,\ \mathrm{d}u = \sec^2 x\,\mathrm{d}x$

$$\int \tan^5 x \sec^2 x\,\mathrm{d}x = \int u^5 \mathrm{d}u = \frac{1}{6}u^6 + C = \frac{1}{6}\tan^6 x + C$$

$u = \sec x,\ \mathrm{d}u = \sec x \tan x\,\mathrm{d}x$

$$\int \sec^4 x\,(\sec x \tan x)\,\mathrm{d}x = \int u^4 \mathrm{d}u = \frac{1}{5}u^5 + C = \frac{1}{5}\sec^5 x + C$$

Q

Check Our Work

Let's check that $\int \tan^5 x \sec^2 x \, dx =$ by differentiating.

$$\frac{d}{dx}\left\{\frac{1}{6}\tan^6 x + C\right\} = \frac{6}{6}\tan^5 x \sec^2 x = \tan^5 x \sec^2 x$$

So, our answer works.

Evaluate using the identity $\sec^2 x = 1 + \tan^2 x$

$$\int \tan^4 x \sec^6 x \,\mathrm{d}x =$$

$$\int \tan^3 x \sec^5 x \,\mathrm{d}x =$$

CHECK OUR WORK

Let's check that $\displaystyle\int \tan^4 x \sec^6 x \, dx =$

$$\begin{aligned}
&\frac{d}{dx}\left\{\frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C\right\} \\
&\qquad = \tan^4 x \sec^2 x + 2\tan^6 x \sec^2 x + \tan^8 x \sec^2 x \\
&\qquad = \tan^4 x \sec^2 x(1 + 2\tan^2 x + \tan^4 x) = \tan^4 x \sec^2 x(1 + \tan^2 x)^2 \\
&\qquad = \tan^4 x \sec^2 x(\sec^2 x)^2 = \tan^4 x \sec^6 x
\end{aligned}$$

So, our answer works.

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \sec x$, $du = \sec x \tan x \, dx$:

- Reserve $\sec x \tan x$ for the differential.
  (m, n should each be at least 1)
- From the remaining $\tan^{m-1} x \ \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.
  ($m - 1$ should be even, to avoid square roots)

To use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$ to evaluate $\int \tan^m x \sec^n x \, dx$, n should be $\boxed{\text{at least one}}$, and m should be $\boxed{\text{odd}}$.

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \tan x$, $du = \sec^2 x \, dx$:

- Reserve $\boxed{\sec^2 x}$ for the differential.
  ($n \geq 2$)
- From the remaining terms, convert all $\boxed{\text{secants}}$ to $\boxed{\text{tangents}}$ using $\tan^2 x + 1 = \sec^2 x$.
  ($n - 2$ should be even, to avoid square roots)

To use the substitution $u = \tan x$, $du = \sec^2 x \, dx$ to evaluate $\int \tan^m x \sec^n \, dx$, n should be $\boxed{\text{even (and at least 2)}}$.

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$

Choose a substitution for the integrals below.

- $$\int \sec^2 x \tan^3 x \, dx$$
- $$\int \sec^2 x \tan^2 x \, dx$$
- $$\int \sec^3 x \tan^3 x \, dx$$

$$\int \sec^2 x \tan^2 x \, \mathrm{d}x$$

Let $u = \tan x$ and $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$.

$$\int \sec^2 x \tan^2 x \, \mathrm{d}x = \int u^2 \, \mathrm{d}u$$

(the rest you can do)

$$\int \sec^3 x \tan^3 x \,\mathrm{d}x$$

Let $u = \sec x$ and $\mathrm{d}u = \sec x \tan x \,\mathrm{d}x$.

$$\begin{aligned}
\int \sec^3 x \tan^3 x \,\mathrm{d}x &= \int \sec^2 x \tan^2 x \,(\sec x \tan x)\,\mathrm{d}x \\
&= \int \sec^2 x(\sec^2 x - 1)(\sec x \tan x)\,\mathrm{d}x \\
&= \int u^2(u^2 - 1)\,\mathrm{d}u
\end{aligned}$$

(the rest you can do)

$$\text{Evaluate} \int \tan^3 x \, \mathrm{d}x = \int \frac{\sin^3 x}{\cos^3 x} \, \mathrm{d}x$$

Let $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$.

$$\begin{aligned}
&= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, \mathrm{d}x = \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x \, \mathrm{d}x \\
&= - \int \frac{1 - u^2}{u^3} \mathrm{d}u \\
&= \int \left(\frac{1}{u} - u^{-3} \right) \mathrm{d}u \\
&= \log|u| + \frac{1}{2} u^{-2} + C \\
&= \log|\cos x| + \frac{1}{2} \sec^2 x + C
\end{aligned}$$

Check Our Work

Let's check that $\displaystyle\int \tan^3 x\,dx =$ by differentiating.

$$\begin{aligned}
\frac{d}{dx}\left\{\log|\cos x| + \frac{1}{2}\sec^2 x + C\right\} &= \frac{-\sin x}{\cos x} + \frac{1}{2}(2\sec x)\sec x \tan x \\
&= -\tan x + \sec^2 x \tan x \\
&= -\tan x + (\tan^2 x + 1)\tan x \\
&= -\tan x + \tan^3 x + \tan x \\
&= \tan^3 x
\end{aligned}$$

So, indeed, $\displaystyle\int \tan^3 x\,dx = \log|\cos x| + \frac{1}{2}\sec^2 x + C.$

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x \,\mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n \mathrm{d}x \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} \mathrm{d}x \\ &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x}\right) \sin x \,\mathrm{d}x\end{aligned}$$

To use $u = \cos x$, $\mathrm{d}u = \sin x \,\mathrm{d}x$: we will convert $\sin^{m-1}(x)$ into cosines, so $m-1$ must be even, so m must be odd.

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd
- $u = \tan x$ if m is even and $n = 0$
  (after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

$$\text{Evaluate } \int \tan^2 x \, dx$$