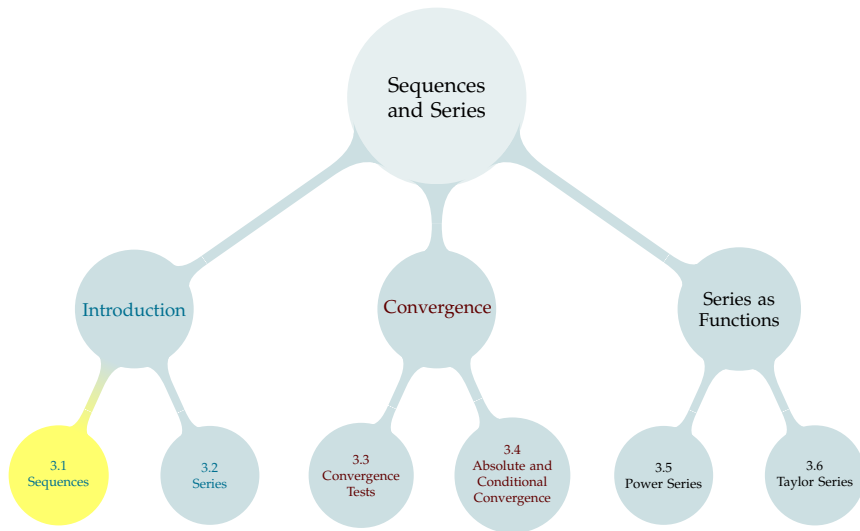


# TABLE OF CONTENTS



We can imagine the list of numbers below carrying on forever:

$$a_1 = 0.1$$

$$a_2 = 0.01$$

$$a_3 = 0.001$$

$$a_4 = 0.0001$$

$$a_5 = 0.00001$$

$$\vdots$$

A **sequence** is a list of infinitely many numbers with a specified order.

It is denoted  $\{a_1, a_2, \dots, a_n, \dots\}$  or  $\{a_n\}_{n=1}^{\infty}$ , etc.

Imagine *adding up* this sequence of numbers.

A **series** is a sum  $a_1 + a_2 + \dots + a_n + \dots$  of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- ▶ What does it mean to add up infinitely many things?
- ▶ Should infinitely many things add up to an infinitely large number?
- ▶ Does the order in which the numbers are added matter?
- ▶ Can we add up infinitely many functions, instead of just infinitely many numbers?

## Sequence

A **sequence** is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- ▶  $\{1, 2, 3, 4, 5, 6, 7, 8, \dots\}$  (natural numbers)
- ▶  $\{3, 1, 4, 1, 5, 9, 2, 6, \dots\}$  (digits of  $\pi$ )
- ▶  $\{1, -1, 1, -1, 1, \dots\}$  (powers of  $-1$  :  $(-1)^0, (-1)^1, (-1)^2$ , etc.)

## Sequence

A **sequence** is a list of infinitely many numbers with a specified order. It is denoted  $\{a_1, a_2, a_3, \dots, a_n, \dots\}$  or  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$ , etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

- ▶  $n = 1$ : this is the index of the first term of our sequence.  
Sometimes it's 0, sometimes something else, for example a year.
- ▶  $\infty$ : there is no end to our sequence.
- ▶  $\frac{1}{n}$ : this tells us the value of  $a_n$ .
- ▶ Often we omit the limits and even the brackets, writing  $a_n = \frac{1}{n}$ .

# SEQUENCE NOTATION

For convenience, we write  $a_1$  for the first term of a sequence,  $a_2$  for the second term, etc.

In the sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ ,  
 $a_3$  is another name for

Sometimes we can find a rule for a sequence.  
In the above sequence,  $a_n =$

We can write  $\{a_n\}_{n=1}^{\infty} =$

Our primary concern with sequences will be the behaviour of  $a_n$  as  $n$  tends to infinity and, in particular, whether or not  $a_n$  “settles down” to some value as  $n$  tends to infinity.

## Convergence

A sequence  $\{a_n\}_{n=1}^{\infty}$  is said to **converge** to the limit  $A$  if  $a_n$  approaches  $A$  as  $n$  tends to infinity. If so, we write

$$\lim_{n \rightarrow \infty} a_n = A \quad \text{or} \quad a_n \rightarrow A \text{ as } n \rightarrow \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

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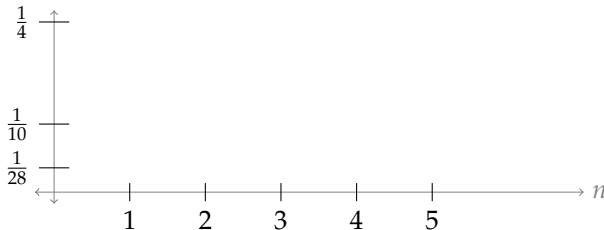
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- ▶  $\{1, 2, 3, 4, 5, 6, 7, 8, \dots\}$  (natural numbers)  
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This sequence



Does the sequence  $a_n = \frac{n}{2n+1}$  converge or diverge?

Consider the sequence  $a_n = \frac{1}{3^n + 1}$ .  $\lim_{n \rightarrow \infty} a_n =$



### Theorem 3.1.6

If  $\lim_{x \rightarrow \infty} f(x) = L$

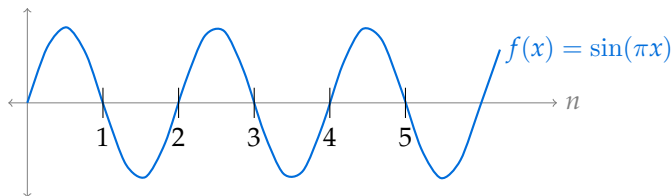
and if  $a_n = f(n)$  for all positive integers  $n$ , then

$$\lim_{n \rightarrow \infty} a_n = L$$

# CAUTIONARY TALE

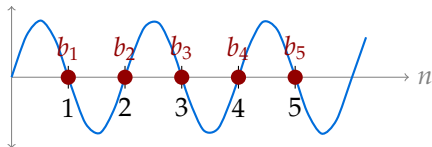
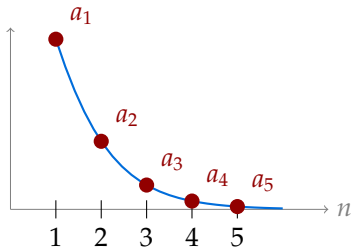
Consider the sequence  $b_n = \sin(\pi n) =$

$$\lim_{n \rightarrow \infty} b_n = \lim_{x \rightarrow \infty} f(x)$$



## Theorem

If  $\lim_{x \rightarrow \infty} f(x) = L$  and if  $a_n = f(n)$  for all natural  $n$ , then  $\lim_{n \rightarrow \infty} a_n = L$ .



## Arithmetic of Limits

Let  $A$ ,  $B$  and  $C$  be real numbers and let the two sequences  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  converge to  $A$  and  $B$  respectively. That is, assume that

$$\lim_{n \rightarrow \infty} a_n = A$$

$$\lim_{n \rightarrow \infty} b_n = B$$

Then the following limits hold.

(a)  $\lim_{n \rightarrow \infty} [a_n + b_n] = A + B$

(b)  $\lim_{n \rightarrow \infty} [a_n - b_n] = A - B$

(c)  $\lim_{n \rightarrow \infty} Ca_n = CA.$

(d)  $\lim_{n \rightarrow \infty} a_n b_n = AB$

(e) If  $B \neq 0$ , then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$

Evaluate the following limits:

$$\blacktriangleright \lim_{n \rightarrow \infty} e^{-n} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1+n}{n} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \frac{1}{n^2} =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} 2n^2 =$$

$$\blacktriangleright \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) (2n^2) =$$

## Continuous functions of limits

If  $\lim_{n \rightarrow \infty} a_n = L$  and if the function  $g(x)$  is continuous at  $L$ , then

$$\lim_{n \rightarrow \infty} g(a_n) = g(L)$$

Evaluate  $\lim_{n \rightarrow \infty} \left[ \sin \left( \frac{\pi n}{2n+1} \right) \right]$

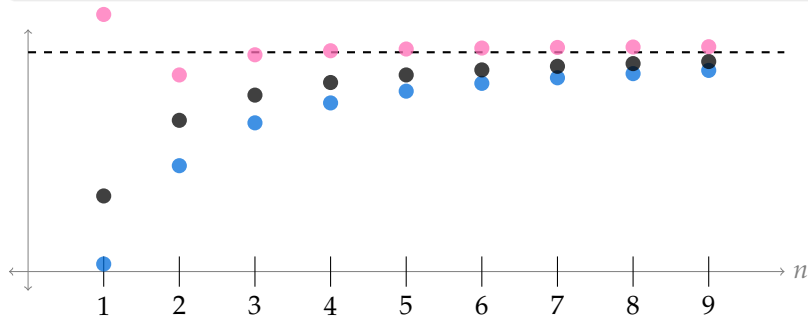
## Squeeze Theorem

If  $a_n \leq c_n \leq b_n$  for all sufficiently large natural numbers  $n$ , and if

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$$

then

$$\lim_{n \rightarrow \infty} c_n = L$$





Evaluate

$$\lim_{n \rightarrow \infty} \left( \frac{2n + \cos n}{n + 1} \right)$$

Let  $a_n = (-n)^{-n}$ . Evaluate  $\lim_{n \rightarrow \infty} a_n$ .