1.8.3 Optional – integrating $\sec x$, $\csc x$, $\sec^3 x$, and $\csc^3 x$.
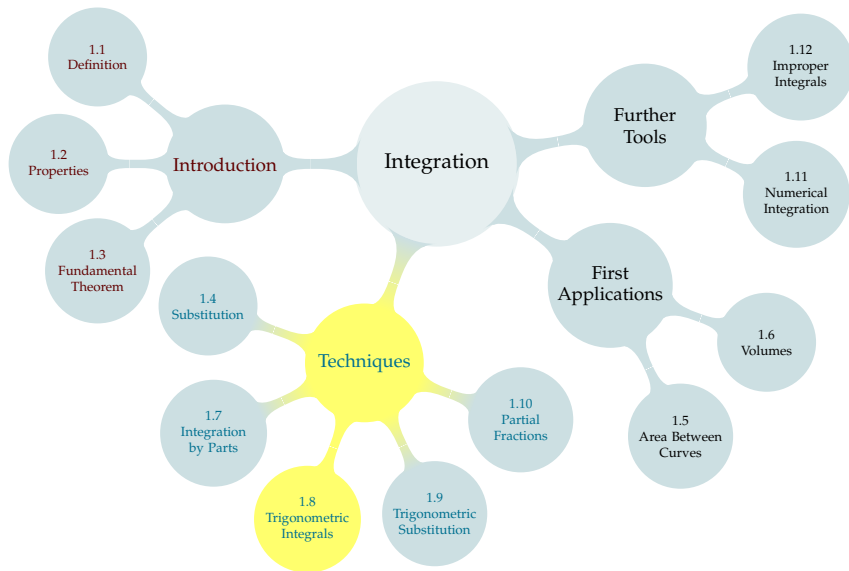
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TABLE OF CONTENTS

Evaluating $\int \tan^m x \sec^n \, \mathrm{d}x$

To evaluate $\int \tan^m x \sec^n \, \mathrm{d}x$, we can use:

▶ $u = \sec x$ if m is odd and $n \geq 1$

▶ $u = \tan x$ if n is even and $n \geq 2$

▶ $u = \cos x$ if m is odd

▶ $u = \tan x$ if m is even and $n = 0$
(after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

Remaining case: n odd and m is even.

The general remaining case is known, but complicated. Instead of treating it exhaustively, we'll show examples of two methods.

$\int \sec x \, \mathrm{d}x$

We saw a way of integrating secant with the following trick:

$$\int \sec x \, \mathrm{d}x = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x$$

$$= \int \frac{1}{u} \mathrm{d}u \quad \text{with } u = \sec x + \tan x$$

Another trick: this time let $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

$$\int \sec x \, \mathrm{d}x = \int \frac{1}{\cos x} \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x$$

$$= \int \frac{1}{1 - \sin^2 x} \cos x \, \mathrm{d}x = \int \frac{1}{1 - u^2} \mathrm{d}u$$

The integrand $\frac{1}{1-u^2}$ is a rational function of u (i.e. a ratio of two polynomials). There is a procedure, called Partial Fractions, that can be used to evaluate all integrals of rational functions. We will learn it in Section 1.10.

1.8.3 Optional – integrating sec x, csc x, sec^3 x, and csc^3 x.

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$\int \sec^3 x \, dx$

We can integrate around in a circle (with integration by parts) to evaluate $\int \sec^3 x \, dx$. Let $u = \sec x$, $dv = \sec^2 x \, dx$. Then $du = \sec x \tan x \, dx$ and $v = \tan x$.

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \log|\sec x + \tan x| + C'$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \log|\sec x + \tan x| + C'$$

$$\int \sec^3 x \, dx = \frac{1}{2} \left(\sec x \tan x + \log|\sec x + \tan x|\right) + C$$

with $C = C'/2$.