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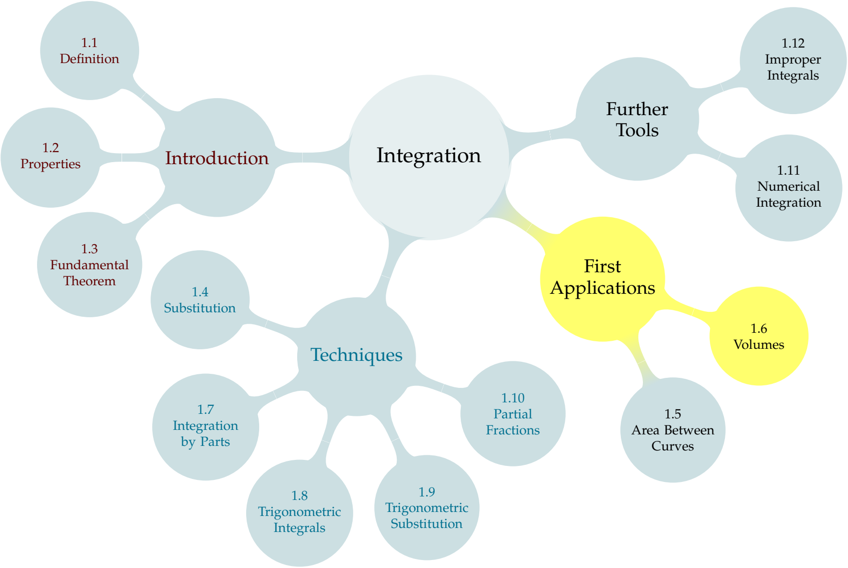
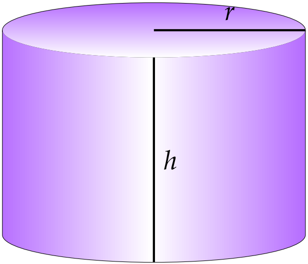
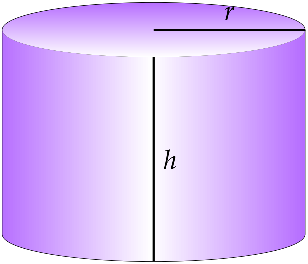


Quick Refresher: Volumes of Cylinders



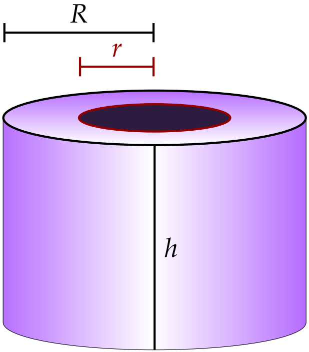
The volume of a cylinder with radius r and height h is:

Quick Refresher: Volumes of Cylinders



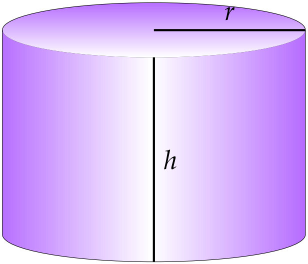
The volume of a cylinder with radius r and height h is:

$$\pi r^2 h$$



The volume of a washer, with outer radius R, inner radius r, and height h is:

Quick Refresher: Volumes of Cylinders



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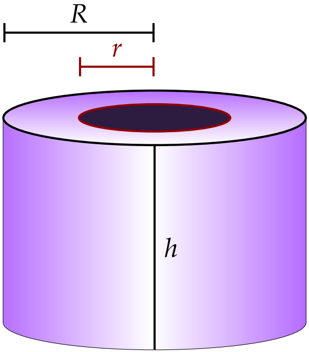
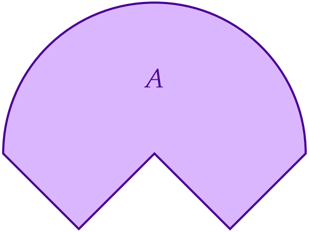


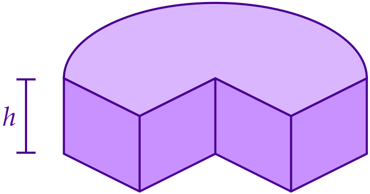
The volume of a washer, with outer radius R, inner radius r, and height h is:

$$\left(\pi R^2 h - \pi r^2 h\right) = \pi h \left(R^2 - r^2\right)$$

Quick Refresher: Volumes of Cylinders

More generally, if we have a shape of area A, and we extrude it into a solid of height h, the resulting solid has volume:





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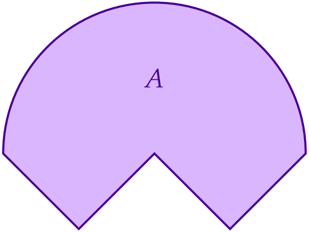
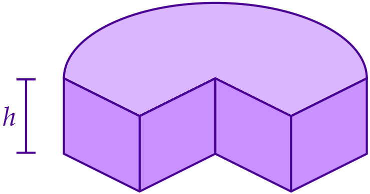
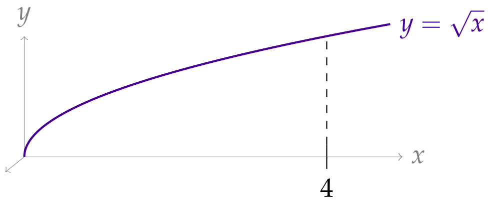
More generally, if we have a shape of area A, and we extrude it into a solid of height h, the resulting solid has volume: Ah





Consider the volume, V, enclosed by rotating the curve $y=\sqrt{x}$, from $x=0$ to $x=4$, around the x-axis.



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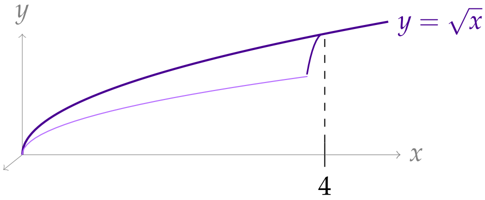

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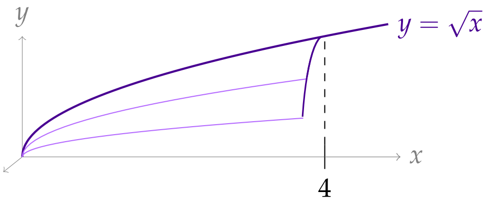

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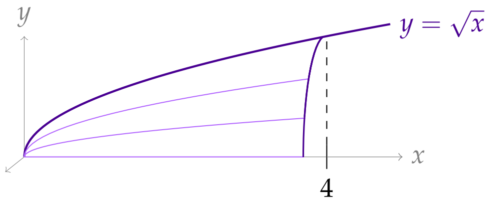

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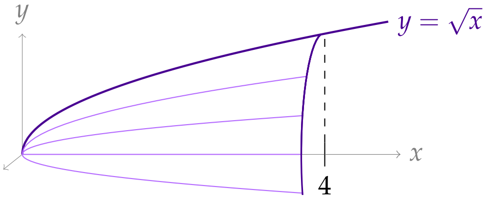

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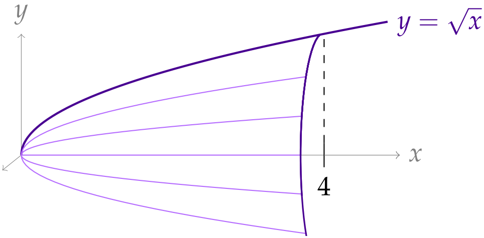

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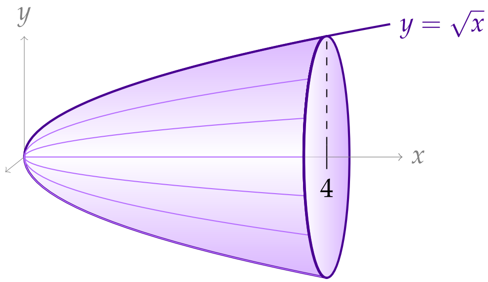

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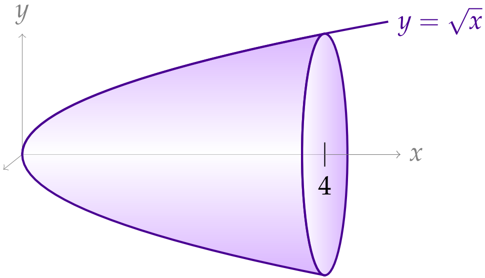

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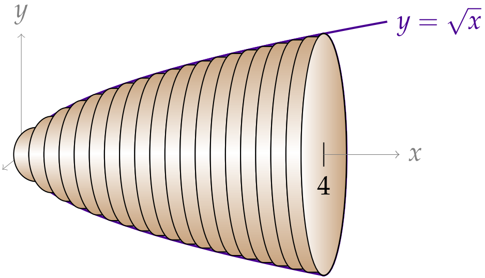
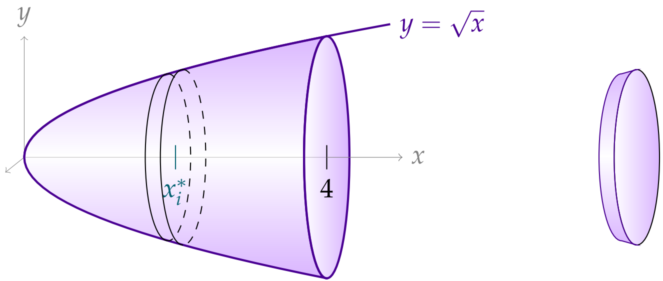


We cut the solid into slices, and approximate the volume of each slice.

Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.



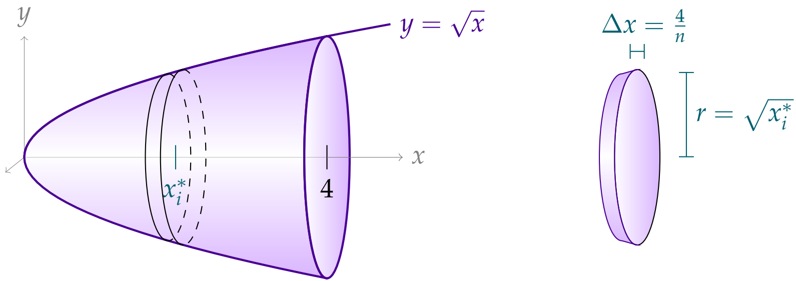
We cut the solid into slices, and approximate the volume of each slice. Each thin slice is *approximately* a cylinder.

If we use n slices, the width of each is:

The radius of the slice at $x = x_i^*$ is:

ans

Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.

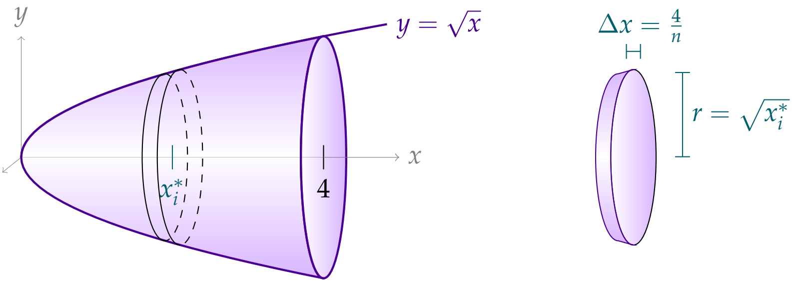


We cut the solid into slices, and approximate the volume of each slice. Each thin slice is *approximately* a cylinder.

If we use n slices, the width of each is: $\frac{4}{n}$.

The radius of the slice at $x = x_i^*$ is: $\sqrt{x_i^*}$.

Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.



$$V \approx \sum_{i=1}^{n} (\text{volume of each slice})$$

ans

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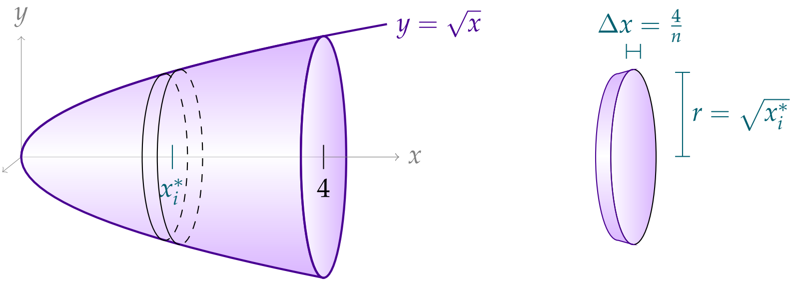
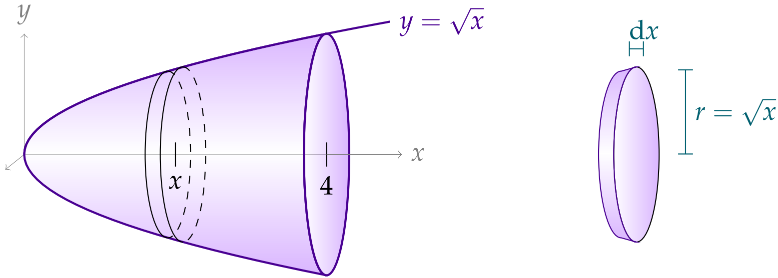


$$V \approx \sum_{i=1}^{n} (\text{volume of each slice}) = \sum_{i=1}^{n} \pi \left(\sqrt{x_i^*}\right)^2 \frac{4}{n} = \sum_{i=1}^{n} \underbrace{\pi x_i^*}_{f(x_i^*)} \underbrace{\frac{4}{n}}_{\Delta x}$$

This is a Riemann sum for $\displaystyle\int_0^4 \pi x \, dx$.

Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.



Informally, we think of one slice, at position x, as having thickness dx. So, we can write the volume of this slice as:

Summing up the volumes of slices from $x = 0$ to $x = 4$, our total volume is:

ans

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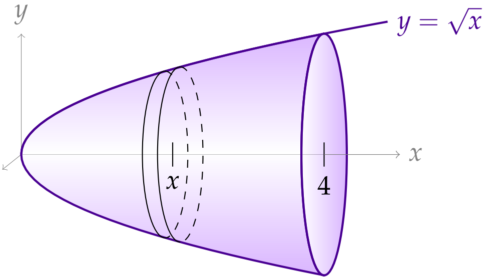
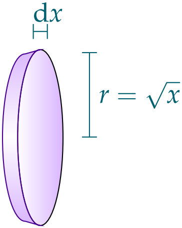
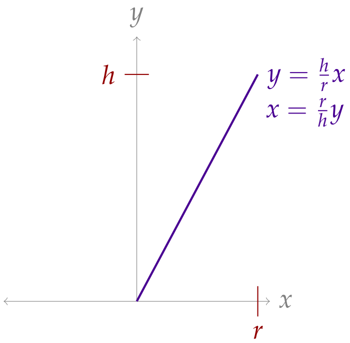




Informally, we think of one slice, at position x, as having thickness $\mathrm{d}x$. So, we can write the volume of this slice as: $\pi x \, \mathrm{d}x$.

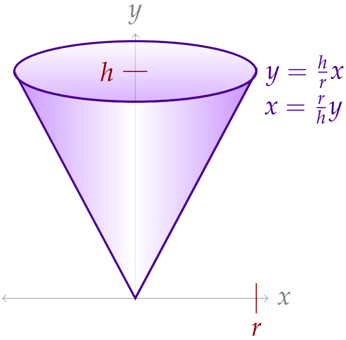
Summing up the volumes of slices from $x = 0$ to $x = 4$, our total volume is:

$$\int_0^4 \pi x \, \mathrm{d}x = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$$



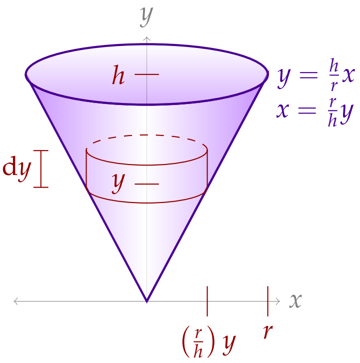
Let h and r be positive constants.

1. What familiar solid results from rotating the line segment from $(0,0)$ to (r,h) around the y-axis?



Let h and r be positive constants.

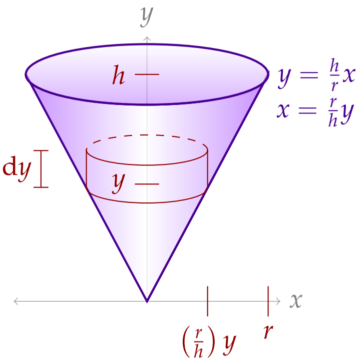
1. What familiar solid results from rotating the line segment from $(0,0)$ to (r,h) around the y-axis?
2. In the informal manner of the last example, describe the volume of a horizontal slice of the cone taken at height y.



Slice volume: $\pi\left(\frac{r}{h}y\right)^2 \mathrm{d}y$

Let h and r be positive constants.

1. What familiar solid results from rotating the line segment from $(0,0)$ to (r,h) around the y-axis?
2. In the informal manner of the last example, describe the volume of a horizontal slice of the cone taken at height y.
3. What is the volume of the entire cone?



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3. What is the volume of the entire cone?

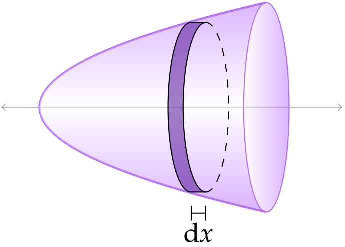
Slice volume: $\pi \left(\frac{r}{h}y\right)^2 dy$

Cone volume: $\displaystyle\int_0^h \pi \left(\frac{r}{h}y\right)^2 dy = \left[\frac{\pi r^2}{3h^2}y^3\right]_{y=0}^{y=h} = \frac{\pi r^2}{3h^2}(h^3 - 0) = \frac{\pi}{3}r^2 h$

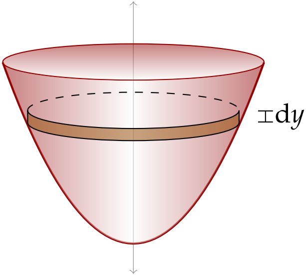
Observation

When we rotated around the horizontal axis, the width of our cylindrical slices was dx, and our integrand was written in terms of x.

When we rotated around the vertical axis, the width of our cylindrical slices was dy, and we integrated in terms of y.

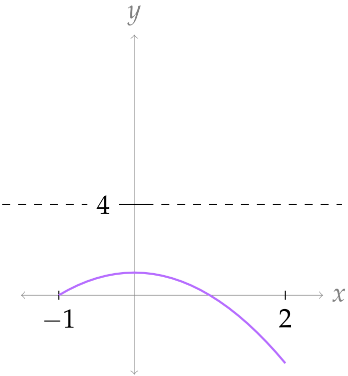


Vertical slices are approximately cylinders



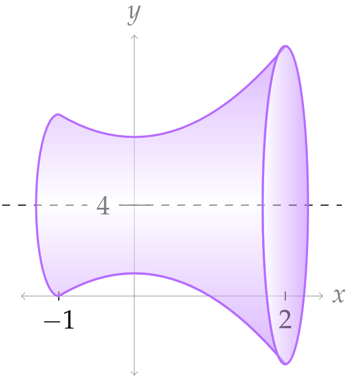
Horizontal slices are approximately cylinders

In this question, we will find the volume enclosed by rotating the curve $y = 1 - x^2$, from $x = -1$ to $x = 2$, about the line $y = 4$.



1. Sketch the surface traced out by the rotating curve.
2. Sketch a cylindrical slice. (Consider: will it be horizontal or vertical?)
3. Give the volume of your slice. Use dx or dy for the width, as appropriate.
4. Integrate (with the appropriate limits of integration) to find the volume of the solid.

In this question, we will find the volume enclosed by rotating the curve $y = 1 - x^2$, from $x = -1$ to $x = 2$, about the line $y = 4$.



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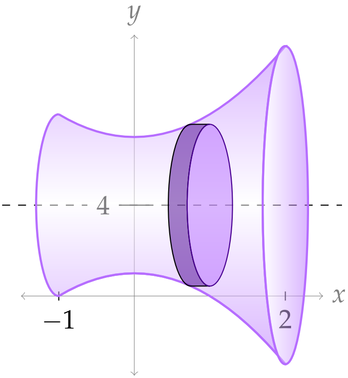

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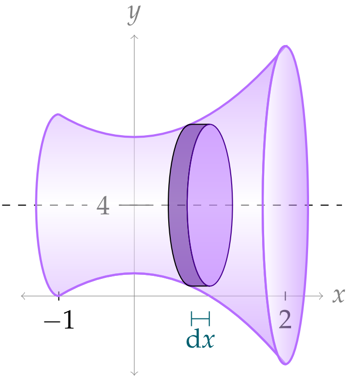


1. Sketch the surface traced out by the rotating curve.
2. Sketch a cylindrical slice. (Consider: will it be horizontal or vertical?)
3. Give the volume of your slice. Use $\mathrm{d}x$ or $\mathrm{d}y$ for the width, as appropriate.
4. Integrate (with the appropriate limits of integration) to find the volume of the solid.

Slice volume: $\pi \underbrace{\left(4-(1-x^2)\right)^2}_{\text{radius}^2} \mathrm{d}x = \pi(3+x^2)^2 \mathrm{d}x$

To find the volume of the entire object, we “add up” the slices from $x = -1$ to $x = 2$ by integrating.

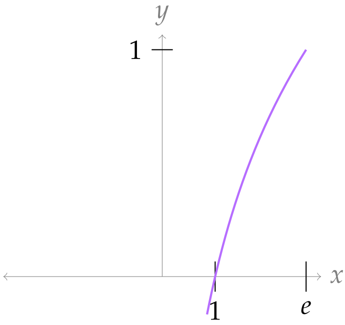
$$\int_{-1}^{2} \pi(3+x^2)^2 \mathrm{d}x =$$

ans

To find the volume of the entire object, we "add up" the slices from $x = -1$ to $x = 2$ by integrating.

$$\begin{aligned}
\int_{-1}^{2} \pi(3+x^2)^2 \mathrm{d}x &= \pi \int_{-1}^{2} \left(9 + 6x^2 + x^4\right) \mathrm{d}x \\
&= \pi \left[9x + 2x^3 + \frac{1}{5}x^5\right]_{-1}^{2} \\
&= \pi \left[\left(18 + 16 + \frac{32}{5}\right) - \left(-9 - 2 - \frac{1}{5}\right)\right] \\
&= \pi \left[\left(40 + \frac{2}{5}\right) + \left(11 + \frac{1}{5}\right)\right] \\
&= 51.6\pi
\end{aligned}$$

Let A be the area between the curve $y = \log x$ and the x-axis, from $(1, 0)$ to $(e, 1)$. In this question, we will consider the volume of the solid formed by rotating A about the y-axis.



1. Sketch A.
2. Sketch a washer-shaped slice of the solid. (Should it be horizontal or vertical?)
3. Give the volume of your slice. Use dx or dy for the width, as appropriate.
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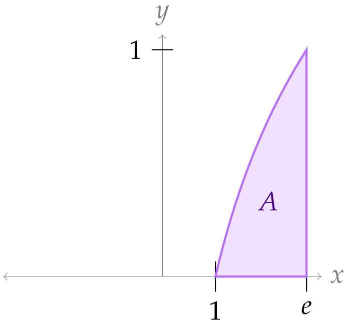

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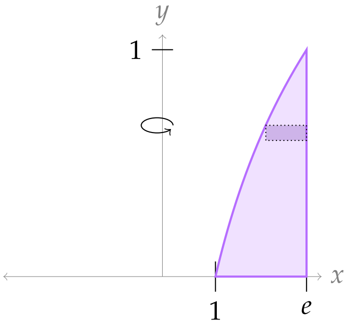

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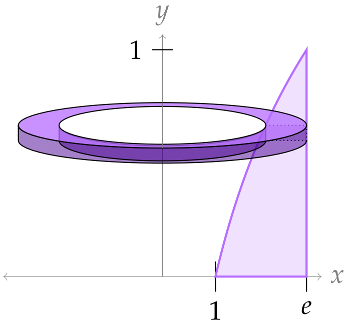

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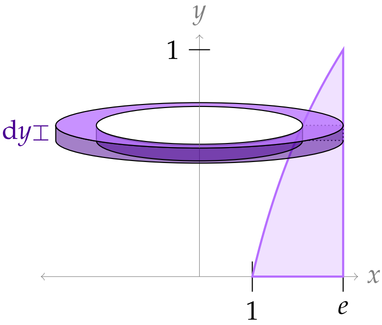

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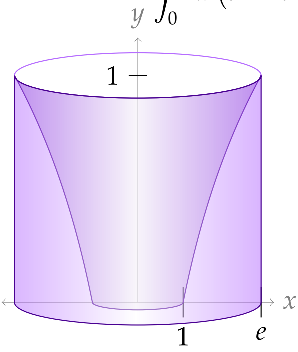
3. Give the volume of your slice. Use $\mathrm{d}x$ or $\mathrm{d}y$ for the width, as appropriate.
4. Integrate to find the volume of the entire solid.

The outer radius is e, while the inner radius at height y is $x = e^y$.

Slice volume at height y: $\quad \pi\left(e^2 - \left(e^y\right)^2\right)\mathrm{d}y = \pi\left(e^2 - e^{2y}\right)\mathrm{d}y$

To find the volume of the entire object, we "add up" the slices from $y = 0$ to $y = 1$ by integrating.

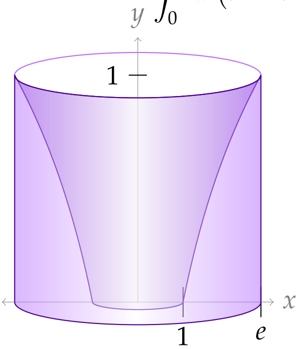
$$\int_0^1 \pi\left(e^2 - e^{2y}\right)\,\mathrm{d}y =$$



ans

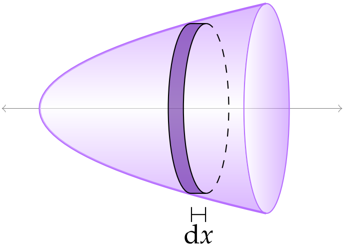
To find the volume of the entire object, we “add up” the slices from $y = 0$ to $y = 1$ by integrating.

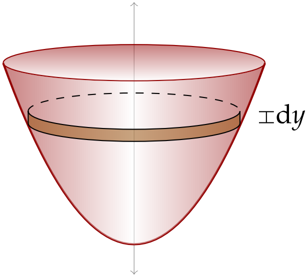
Below we use the substitution rule with $u = 2y$ and $\mathrm{d}u = 2\mathrm{d}y$. With practice, you'll probably be able to do this substitution in your head, but we have written it out for clarity



$$\begin{aligned}\int_0^1 \pi\left(e^2 - e^{2y}\right)\mathrm{d}y &= \pi\int_{u(0)}^{u(1)}\left(e^2 - e^u\right)\cdot\frac{1}{2}\mathrm{d}u\\ &= \frac{\pi}{2}\int_0^2\left(e^2 - e^u\right)\mathrm{d}u\\ &= \frac{\pi}{2}\left[e^2u - e^u\right]_0^2\\ &= \frac{\pi}{2}\left[\left(2e^2 - e^2\right) - (0 - 1)\right]\\ &= \frac{\pi}{2}\left[e^2 + 1\right]\end{aligned}$$

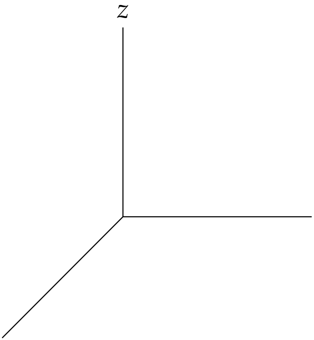
So far, we've found the volume of solids formed by rotating a curve. When a point rotates about a fixed centre, the result is a circle, so we could slice those solids up into pieces that are approximately cylinders.





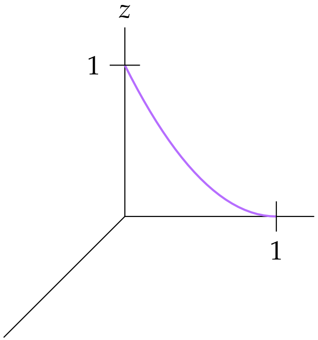
We can find the volumes of other shapes, as long as we can find the areas of their cross-sections.

The corner of a room is sealed off as follows:



The corner of a room is sealed off as follows:

On both walls, a parabola of the form $z = (x-1)^2$ is drawn, where z is the vertical axis and x is the horizontal. They start one metre above the corner, and end one metre to the side of the corner.



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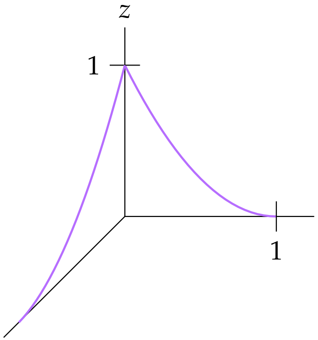

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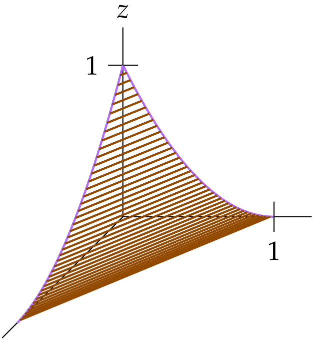

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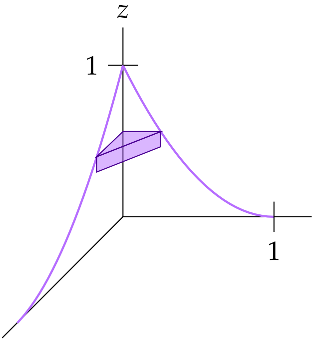

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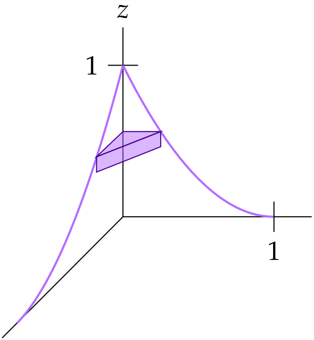

Q
ans

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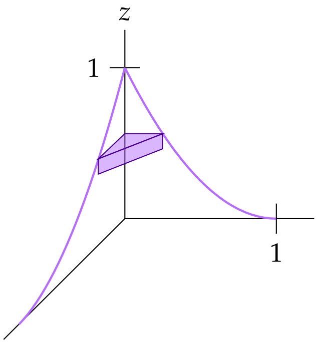
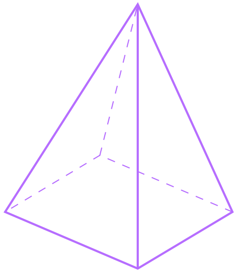


At height z, the cross-section is a right triangle. Its side length is the x-value on the parabola.
Solving $z = (x-1)^2$ for x, we find $x = \sqrt{z} + 1$.
So, the area of a cross-section at height z is $\frac{1}{2}\left(\sqrt{z}+1\right)^2$. We call its width $\mathrm{d}z$.
All together, the enclosed volume is $\int_0^1 \frac{1}{2}\left(z + 2\sqrt{z} + 1\right)\,\mathrm{d}z = \frac{17}{12}$ cubic metres.

A pyramid with height h metres has a square base with side-length b metres. At an elevation of y metres above the base, $0 \leq y \leq h$, the cross-section of the pyramid is a square with side-length $\frac{b}{h}(h-y)$. What is the volume of the pyramid?



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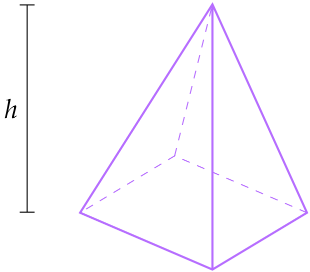

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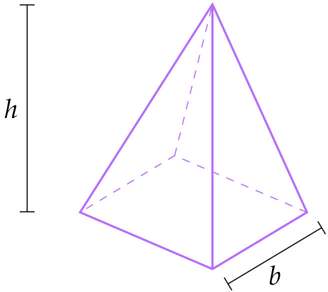

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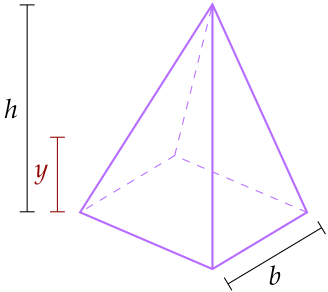

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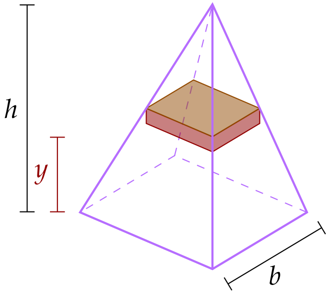

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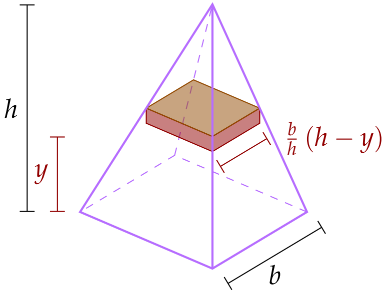

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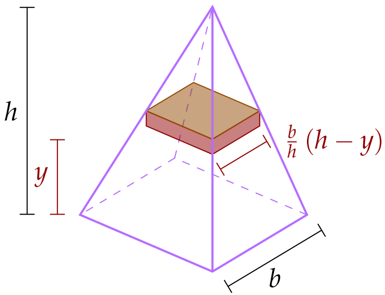


The area of the square cross-section at height y is $\left[\frac{b}{h}(h-y)\right]^2 = \frac{b^2}{h^2}\left(h^2 - 2hy + y^2\right)$.

If we give a horizontal slice width $\mathrm{d}y$, then the slice volume is $\frac{b^2}{h^2}\left(h^2-2hy+y^2\right)\mathrm{d}y$. So, the total volume of the pyramid is

$$\int_0^h \frac{b^2}{h^2}\left(h^2-2hy+y^2\right)\mathrm{d}y$$

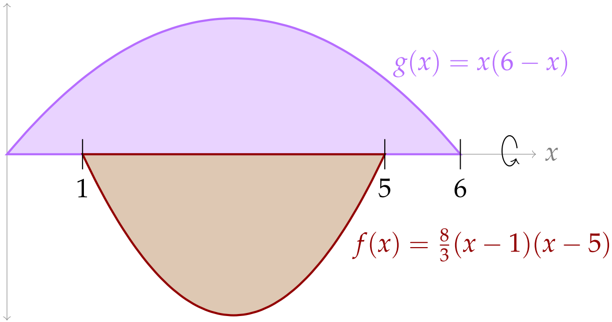
$$=\frac{b^2}{h^2}\left[h^2y - hy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=h}$$

$$=\frac{b^2}{h^2}\left[h^3 - h^3 + \frac{1}{3}h^3\right] = \frac{1}{3}b^2h$$

Optional: Challenge Question

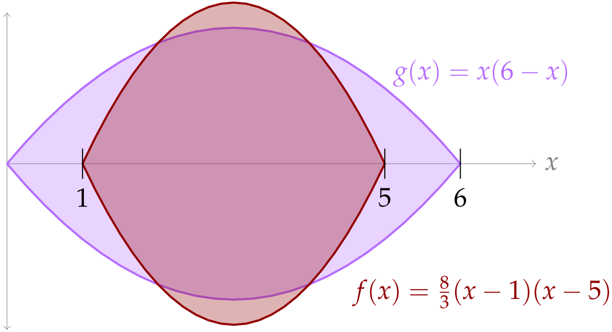
A paddle fixed to the x-axis has two flat blades. One blade is in the shape of $f(x) = \frac{8}{3}(x-1)(x-5)$, from $x = 1$ to $x = 5$. The other blade is in the shape of $g(x) = x(6-x)$, $0 \leq x \leq 6$. The paddle turns through a gelatinous fluid, scraping out a hollow cavity as it turns. What is the volume of this cavity?

You may leave your answer as an integral, or sum of integrals.



ans

The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.



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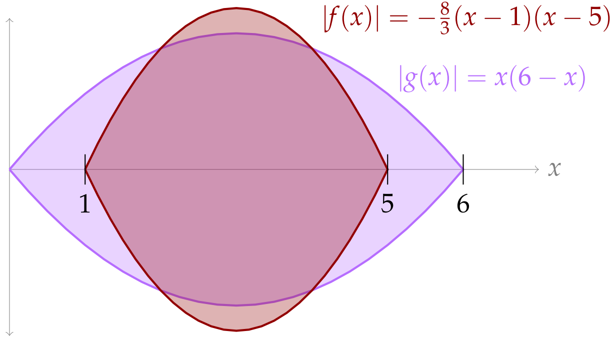


The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.

Let's find where $|f(x)| = |g(x)|$:

$$x(6-x) = -\frac{8}{3}(x-1)(x-5)$$

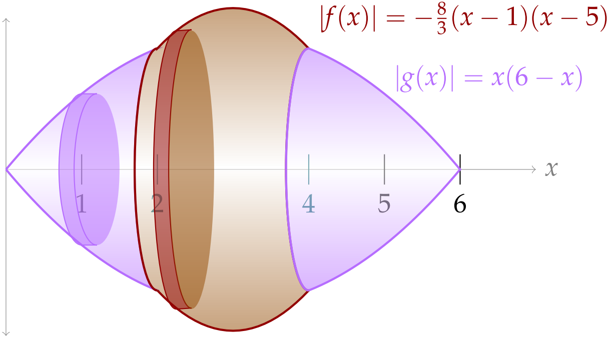
ans

The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.

Let's find where $|f(x)| = |g(x)|$:

$$x(6-x) = -\frac{8}{3}(x-1)(x-5)$$
$$6x - x^2 = -\frac{8}{3}\left(x^2 - 6x + 5\right) = -\frac{8}{3}x^2 + 16x - \frac{40}{3}$$
$$\frac{5}{3}x^2 - 10x + \frac{40}{3} = 0$$
$$x^2 - 6x + 8 = 0$$
$$(x-2)(x-4) = 0$$
$$x = 2,\ x = 4$$

The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.



The radius of a cylindrical slice is $|g(x)| = x(6-x)$ when $0 < x < 2$ and $4 < x < 6$, and the radius is $|f(x)| = -\frac{8}{3}(x-1)(x-5)$ when $2 < x < 4$.

The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.

The radius of a cylindrical slice is $|g(x)| = x(6-x)$ when $0 < x < 2$ and $4 < x < 6$, and the radius is $|f(x)| = -\frac{8}{3}(x-1)(x-5)$ when $2 < x < 4$.

The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.

The radius of a cylindrical slice is $|g(x)| = x(6-x)$ when $0 < x < 2$ and $4 < x < 6$, and the radius is $|f(x)| = -\frac{8}{3}(x-1)(x-5)$ when $2 < x < 4$.

$|f(x)|^2 = [f(x)]^2$, so we can drop our absolute values in this step.

$$\begin{aligned} \text{Volume} = &\int_0^2 \pi \left(6x - x^2\right)^2 \mathrm{d}x + \int_2^4 \pi \left(\frac{8}{3}\left(x^2 - 6x + 5\right)\right)^2 \mathrm{d}x \\ &+ \int_4^6 \pi \left(6x - x^2\right)^2 \mathrm{d}x \end{aligned}$$

We could make our calculation slightly shorter by noting that the shape is symmetric to the left and right of $x = 3$.

$$= 2 \underbrace{\left[\int_0^2 \pi \left(6x - x^2\right)^2 \mathrm{d}x + \int_2^3 \pi \left(\frac{8}{3}\left(x^2 - 6x + 5\right)\right)^2 \mathrm{d}x\right]}_{\text{Volume of half the object, } 0 \le x \le 3}$$

Included Work

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