Table of Contents

We defined the definite integral using a limit and a sum.

Definition

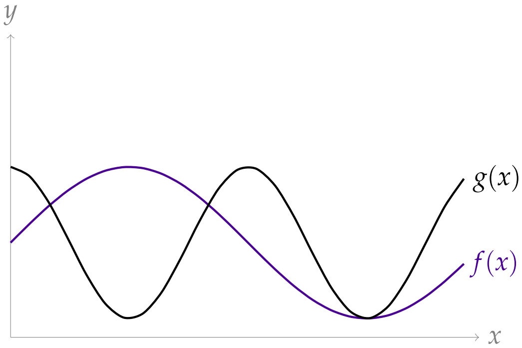
Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define $\Delta x = \frac{b-a}{N}$ and

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_{i,N}^*) \cdot \Delta x$$

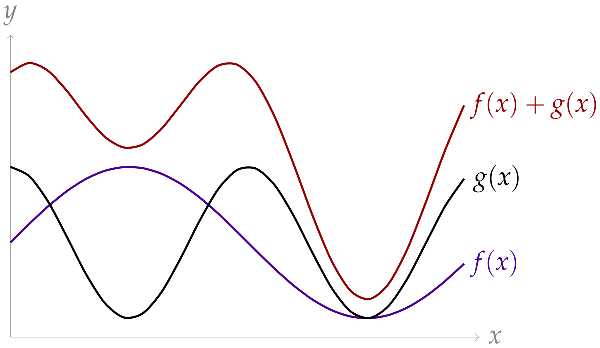
when the limit exists and when the choice of $x_{i,N}^*$ in the i^{th} interval doesn't matter.

Many of the operations that work nicely with sums and limits will also work nicely with integrals.

Adding (and Subtracting) Functions



Adding (and Subtracting) Functions



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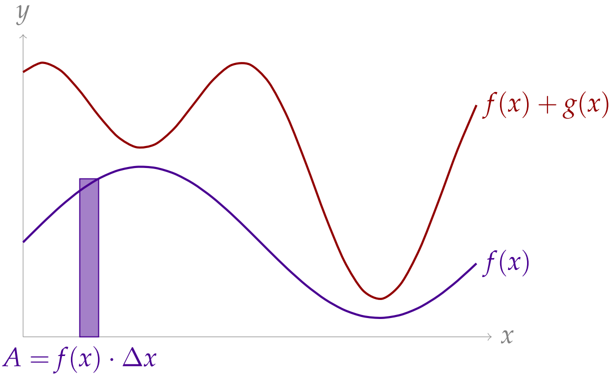

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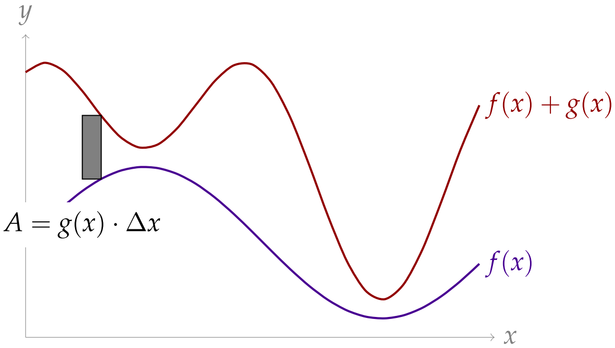

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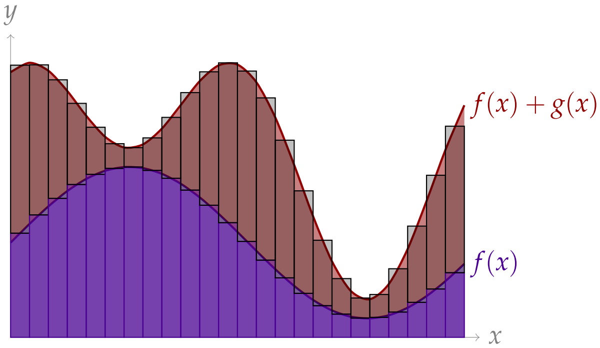
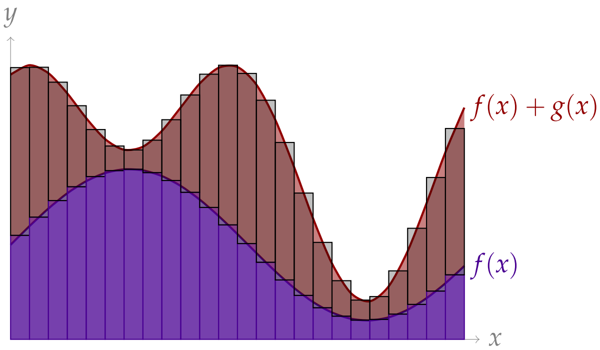


$$\int_a^b \big(f(x)+g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

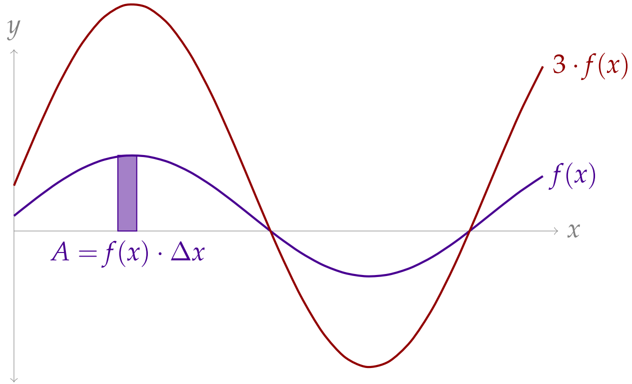
Adding (and Subtracting) Functions



$$\int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$$

Multiplying a Function by a Constant

Multiplying a Function by a Constant



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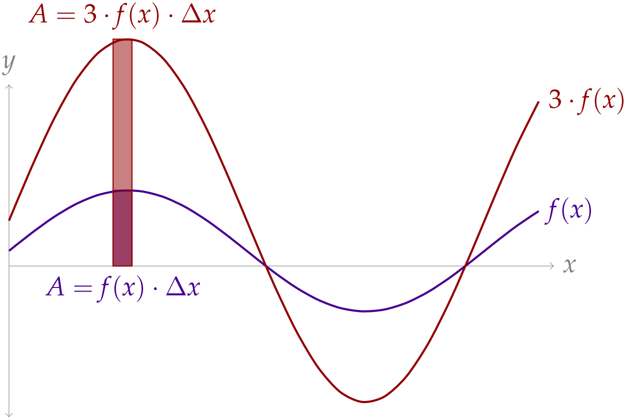


Multiplying a Function by a Constant

$$\int_a^b c \cdot f(x)\,\mathrm{d}x = c \int_a^b f(x)\,\mathrm{d}x$$

ARITHMETIC OF INTEGRATION

When a, b, and c are real numbers, and the functions $f(x)$ and $g(x)$ are integrable on an interval containing a and b:

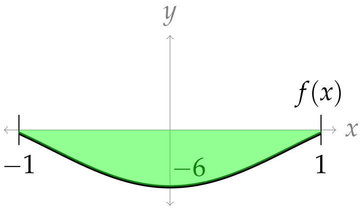
(a) $\displaystyle\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

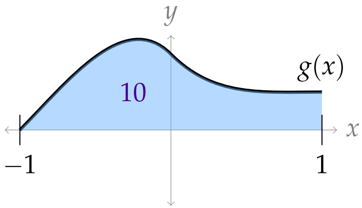
(b) $\displaystyle\int_a^b [f(x) - g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$

(c) $\displaystyle\int_a^b c \cdot f(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x \quad \text{when } c \text{ is constant}$

Arithmetic of Integration

Suppose $\int_{-1}^{1} f(x)\,\mathrm{d}x = -6$ and $\int_{-1}^{1} g(x)\,\mathrm{d}x = 10$.





$$\int_{-1}^{1} (2f(x) + g(x))\,\mathrm{d}x =$$

Q

ans

Arithmetic of Integration

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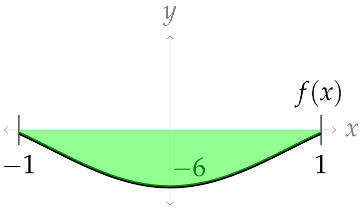
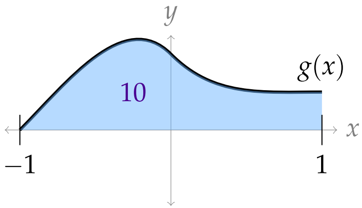
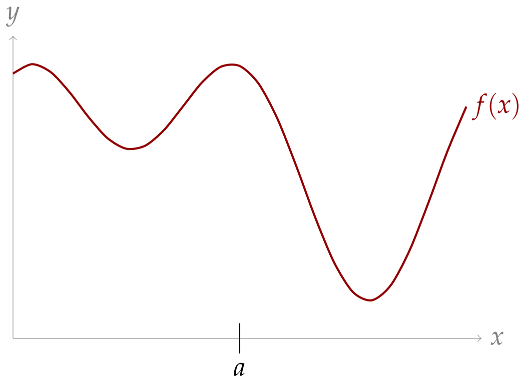




$$\int_{-1}^{1} (2f(x) + g(x))\,\mathrm{d}x = 2\int_{-1}^{1} f(x)\,\mathrm{d}x + \int_{-1}^{1} g(x)\,\mathrm{d}x = 2(-6) + 10 = -2$$

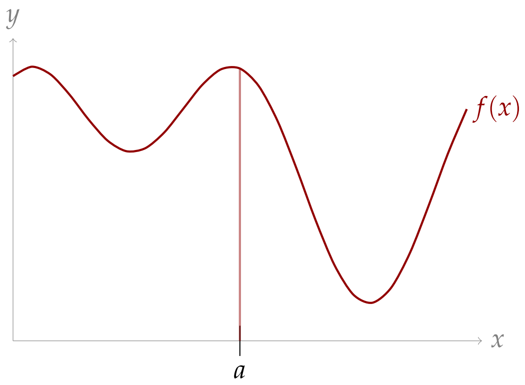
Interval of Integration



$$\int_a^a f(x)\,\mathrm{d}x =$$

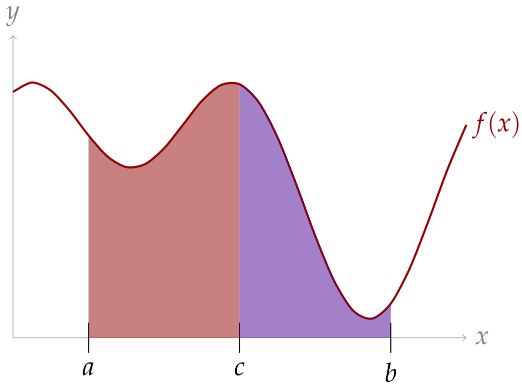
ans

Interval of Integration



$$\int_a^a f(x)\,\mathrm{d}x = 0$$

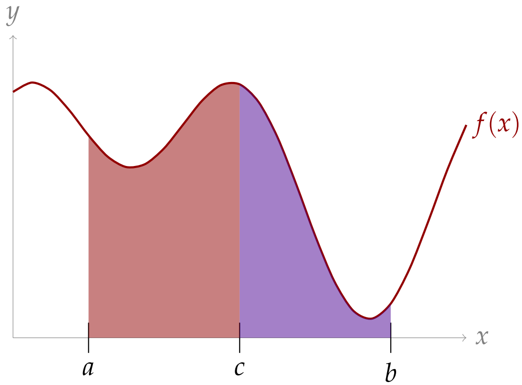
Interval of Integration



What rule do you think is being illustrated?

ans

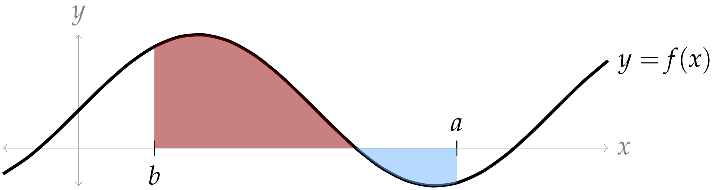
Interval of Integration



What rule do you think is being illustrated?

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

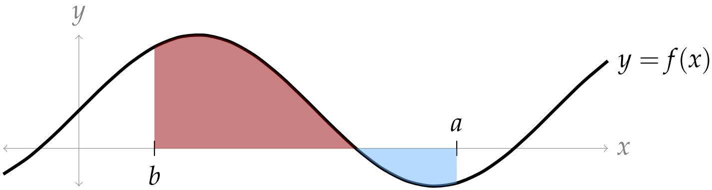
What happens in $\int_a^b f(x)\,\mathrm{d}x$ when $b < a$?



$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

This is the definition of a definite integral *whether or not* $a < b$.

What happens in $\int_a^b f(x)\,\mathrm{d}x$ when $b < a$?



Choose a number of intervals, n.

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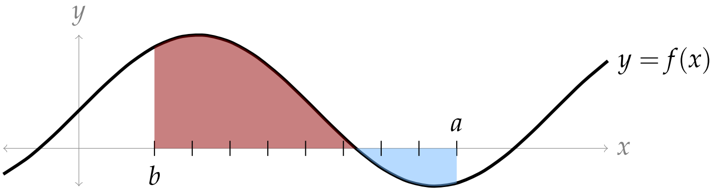

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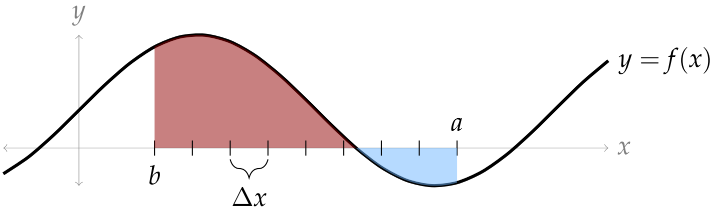

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The (signed) width of each interval is $\Delta x = \frac{b-a}{n}$, which is negative

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

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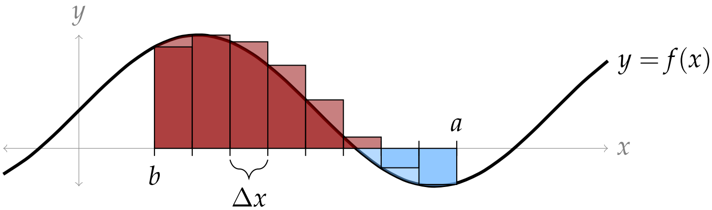


Choose a number of intervals, n.
The (signed) width of each interval is $\Delta x = \frac{b-a}{n}$, which is negative

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \left(-\frac{a-b}{n}\right) = -\int_b^a f(x)\,\mathrm{d}x$$

Property of Definite Integrals

$$\int_{a}^{b} f(x)\,\mathrm{d}x = -\int_{b}^{a} f(x)\,\mathrm{d}x$$

As strictly a measure of area, not usually a super useful fact – but helps later when we do arithmetic with integrals.

Property of Definite Integrals

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

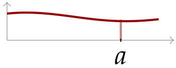
As strictly a measure of area, not usually a super useful fact – but helps later when we do arithmetic with integrals.

It's also useful that the definition works without having to worry about which limit of integration (a or b) is larger.

Arithmetic for Domain of Integration

When a, b, and c are constants, and $f(x)$ is integrable over a domain containing all three:

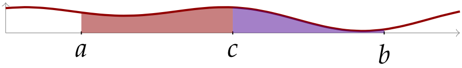
(a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$



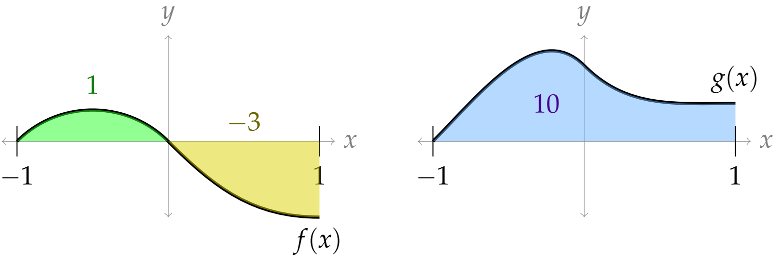
(b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

$$\Delta x = \frac{b-a}{n} = -\frac{a-b}{n}$$

(c) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ for constant c



Suppose $\int_{-1}^{0} f(x)\,\mathrm{d}x = 1$, $\int_{0}^{1} f(x)\,\mathrm{d}x = -3$, and $\int_{-1}^{1} g(x)\,\mathrm{d}x = 10$.



$$\int_{-1}^{1} \left(2f(x) + g(x)\right) \mathrm{d}x =$$

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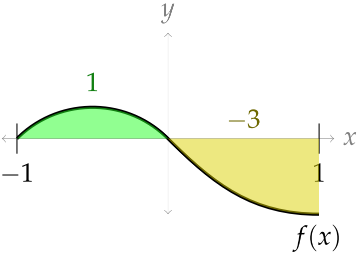
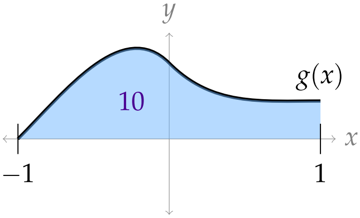
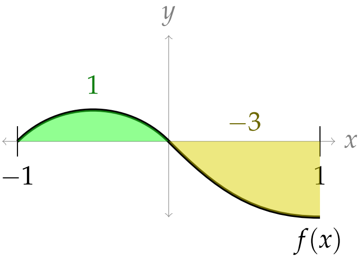




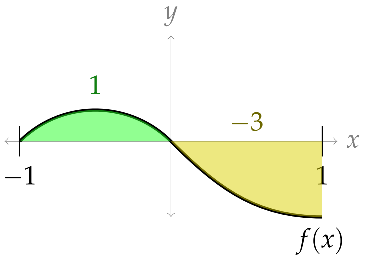
$$\int_{-1}^{1} (2f(x) + g(x))\,\mathrm{d}x = 2\left[\int_{-1}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x\right] + \int_{-1}^{1} g(x)\,\mathrm{d}x$$
$$= 2\,[1-3] + 10 = 6$$

Suppose $\int_{-1}^{0} f(x)\,\mathrm{d}x = 1$ and $\int_{0}^{1} f(x)\,\mathrm{d}x = -3$.



$$\int_{-1}^{3} f(x)\,\mathrm{d}x + \int_{3}^{0} f(x)\,\mathrm{d}x =$$

Suppose $\int_{-1}^{0} f(x)\,\mathrm{d}x = 1$ and $\int_{0}^{1} f(x)\,\mathrm{d}x = -3$.

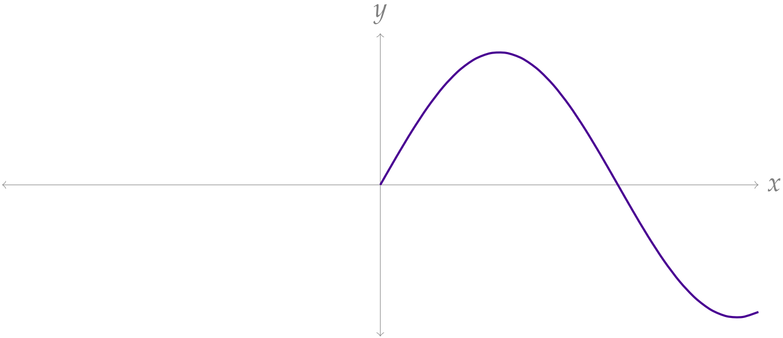


$$\int_{-1}^{3} f(x)\,\mathrm{d}x + \int_{3}^{0} f(x)\,\mathrm{d}x = \int_{-1}^{0} f(x)\,\mathrm{d}x = 1$$

Even and Odd Functions

Let $f(x)$ be a function.

- We say $f(x)$ is even when $f(x) = f(-x)$ for all x, and
- we say $f(x)$ is odd when $f(x) = -f(-x)$ for all x.



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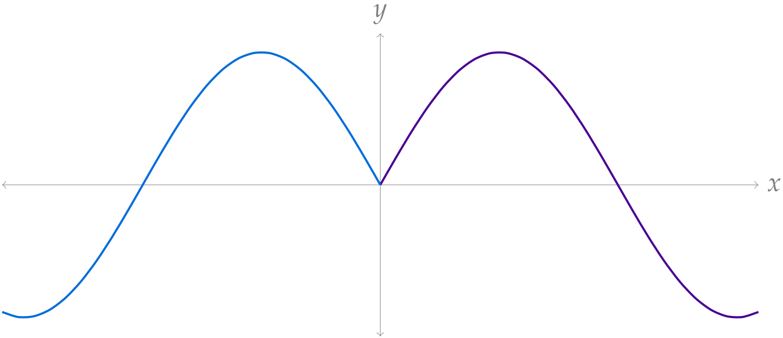

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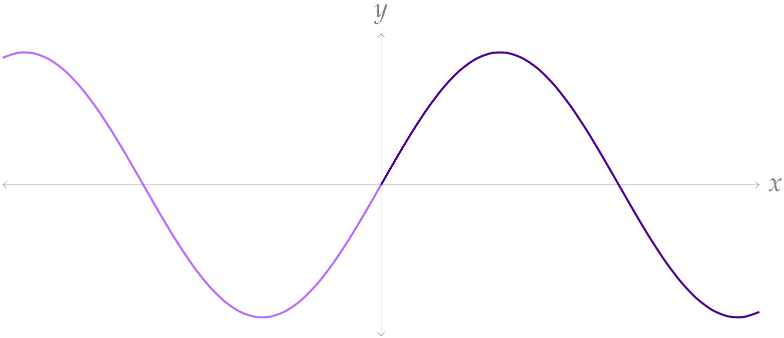
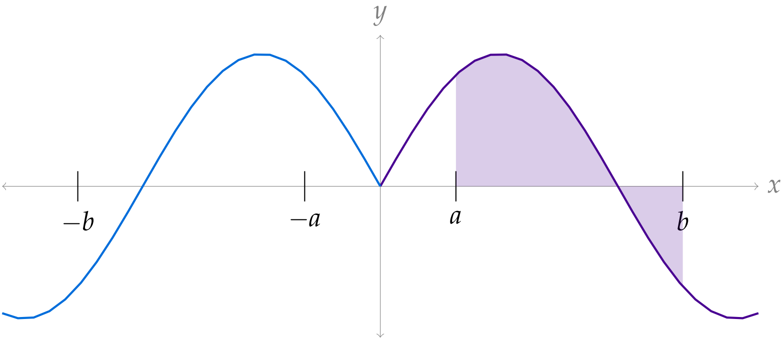


Integrals of Even Functions

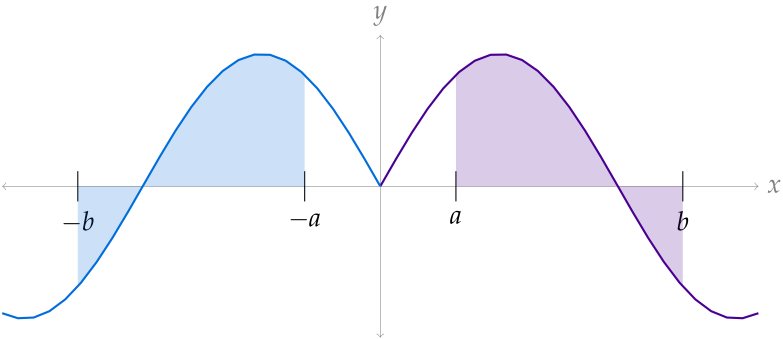


Suppose $f(x)$ is even. Then

$$\int_a^b f(x)\,\mathrm{d}x =$$

ans

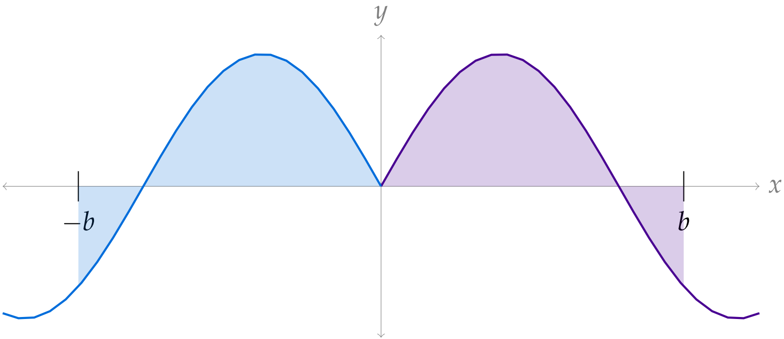
Integrals of Even Functions



Suppose $f(x)$ is even. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(x)\,\mathrm{d}x$$

Integrals of Even Functions

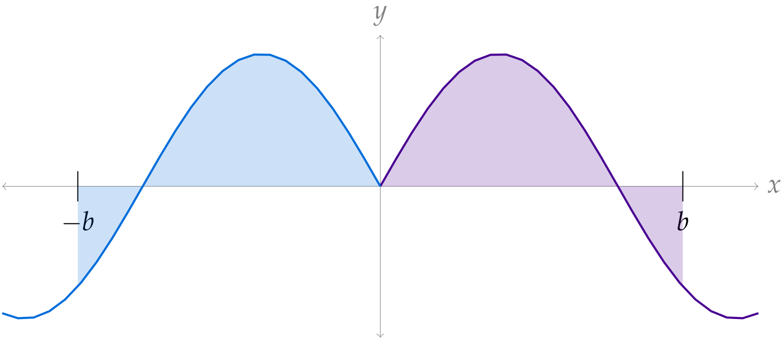


Suppose $f(x)$ is even. Then

$$\int_{-b}^{b} f(x)\,dx =$$

ans

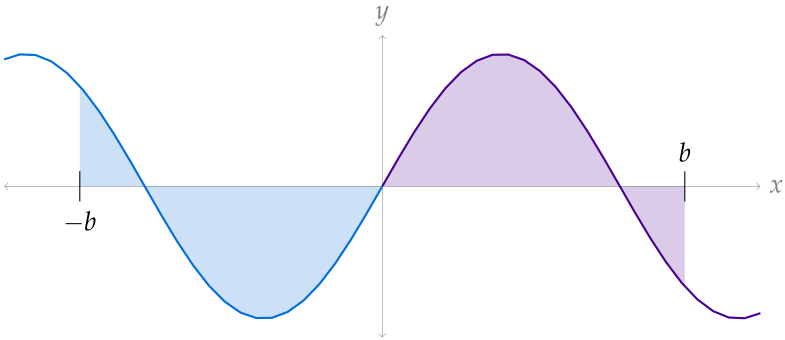
Integrals of Even Functions



Suppose $f(x)$ is even. Then

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 2\int_{0}^{b} f(x)\,\mathrm{d}x$$

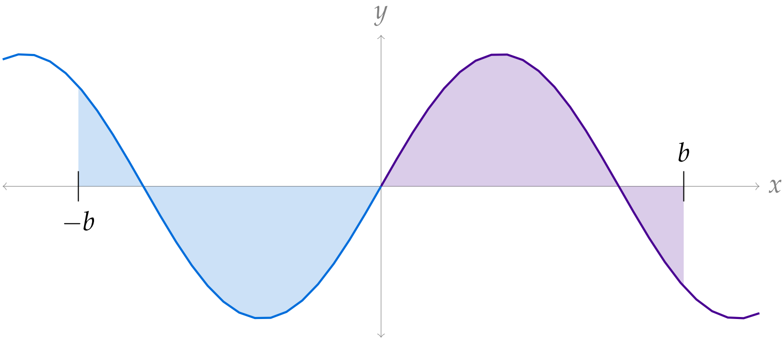
Integrals of Odd Functions



Suppose $f(x)$ is odd. Then

$$\int_{-b}^{b} f(x)\,\mathrm{d}x =$$

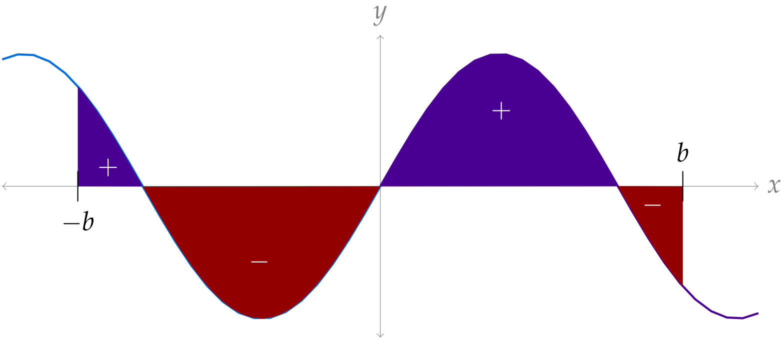
Integrals of Odd Functions



Suppose $f(x)$ is odd. Then

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 0$$

Integrals of Odd Functions



Suppose $f(x)$ is odd. Then

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 0$$

Theorem 1.2.11 (Even and Odd)

Let $a > 0$.

(a) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

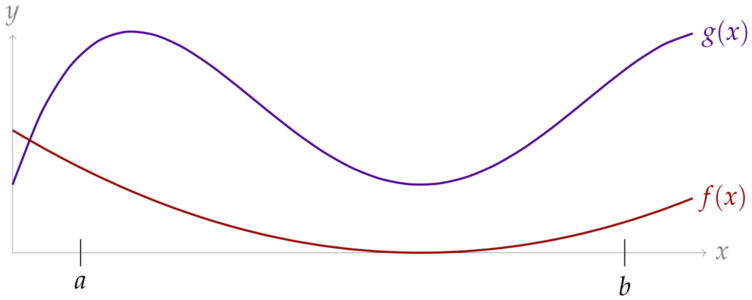
(b) If $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

Integral Inequality

Let $a \leq b$ be real numbers and let the functions $f(x)$ and $g(x)$ be integrable on the interval $a \leq x \leq b$.
If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$



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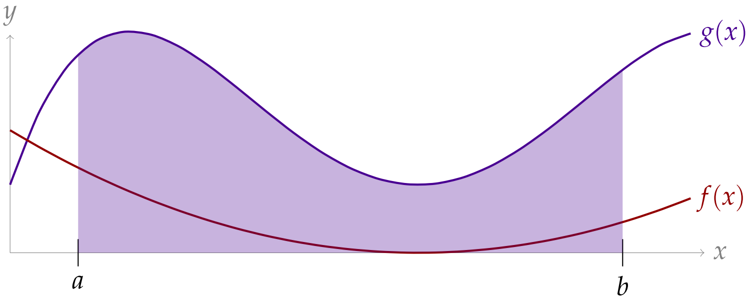

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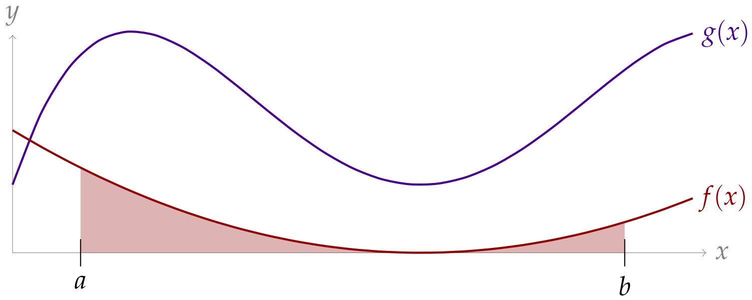
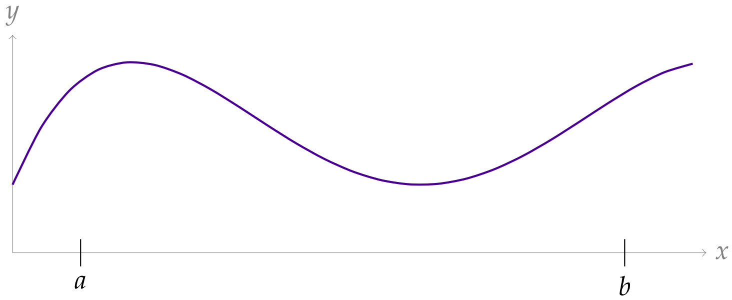


Integral Inequality

Let $a \leq b$ and $m \leq M$ be real numbers and let the function $f(x)$ be integrable on the interval $a \leq x \leq b$.
If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$



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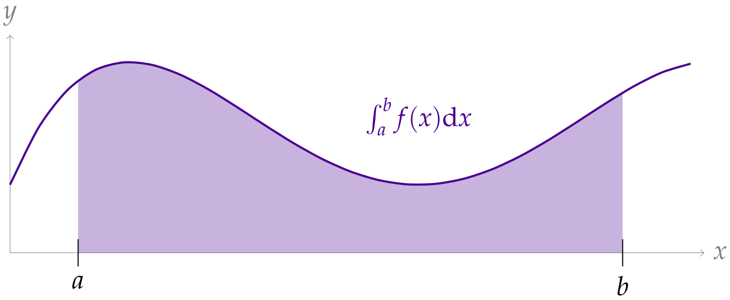

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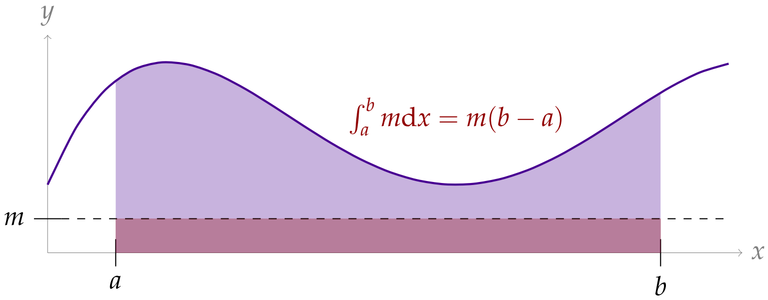
If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

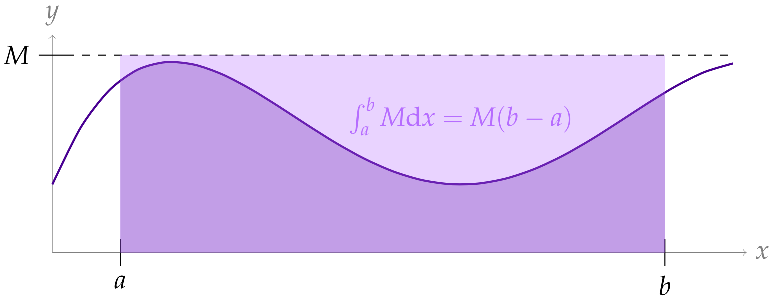
$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$



Integral Inequality

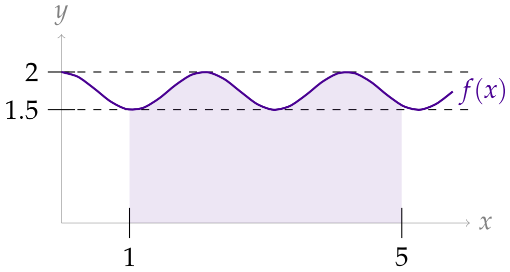
Let $a \leq b$ and $m \leq M$ be real numbers and let the function $f(x)$ be integrable on the interval $a \leq x \leq b$.
If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$



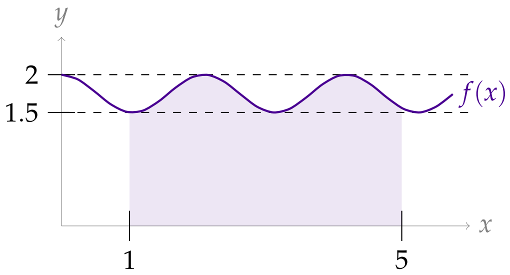
Find a lower bound c and an upper bound d such that

$$c \leq \int_1^5 f(x)\,\mathrm{d}x \leq d$$



Find a lower bound c and an upper bound d such that

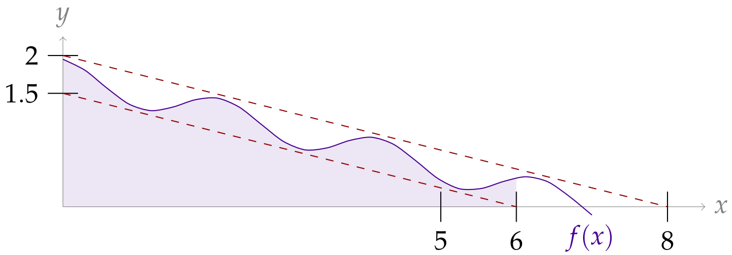
$$c \leq \int_1^5 f(x)\,\mathrm{d}x \leq d$$



$$1.5 \leq f(x) \leq 2 \implies \overbrace{1.5(5-1)}^{6} \leq \int_1^5 f(x)\,\mathrm{d}x \leq \overbrace{2(5-1)}^{8}$$

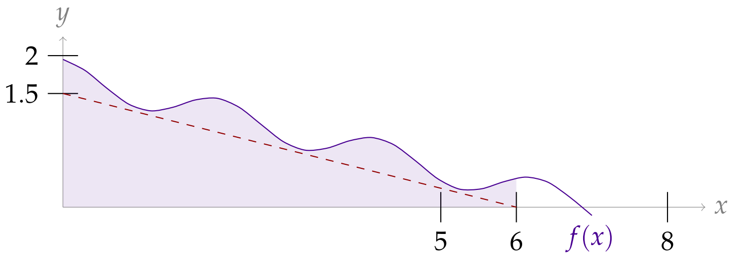
Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$



Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

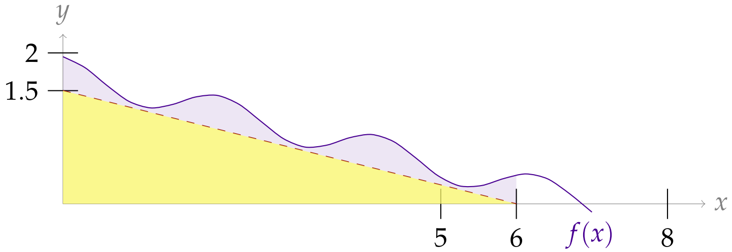
$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$



ans

Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$

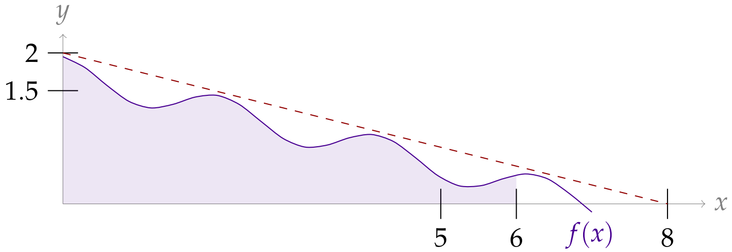


The area under the curve is no smaller than the area of the highlighted triangle.

$$\int_0^6 (\text{dashed line})\,\mathrm{d}x = \frac{1}{2} \cdot \frac{3}{2} \cdot 6 = \frac{9}{2} \leq \int_0^6 f(x)\,\mathrm{d}x$$

Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

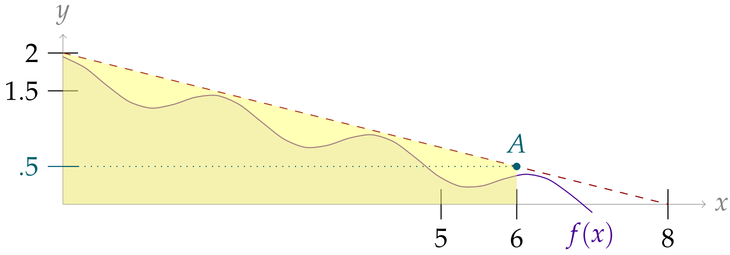
$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$



ans

Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

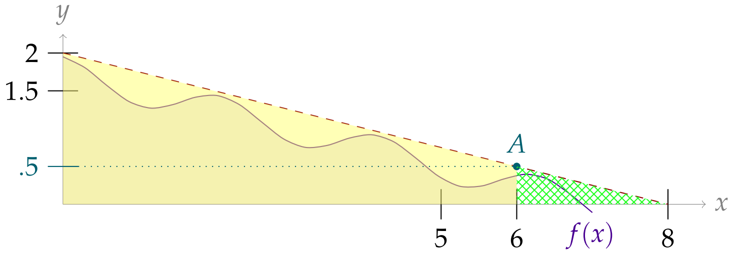
$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$



The area under the curve is not greater than the area under the solid yellow trapezoid. Because the dashed line has slope $-\frac{1}{4}$, the y–coordinate of point A is $\frac{1}{2}$.

Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$



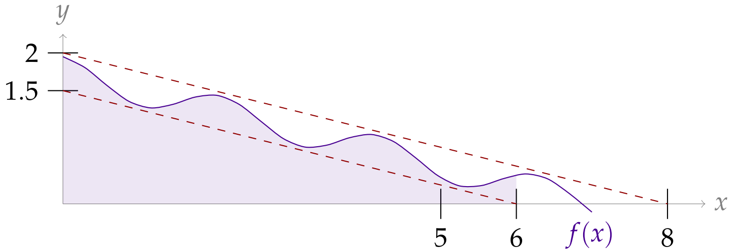
We can compute the area of the trapezoid as the difference in the area of the triangle under the dotted line, and the green cross-hatched triangle.

$$\int_0^6 f(x)\,\mathrm{d}x \leq \int_0^6 (\text{dashed line})\,\mathrm{d}x = \frac{1}{2}(8)(2) - \frac{1}{2}(2)\frac{1}{2} = \frac{15}{2}$$

ans

Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

$$c \leq \int_0^6 f(x)\,\mathrm{d}x \leq d$$

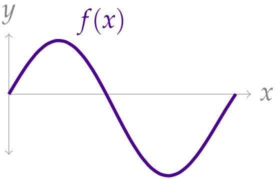


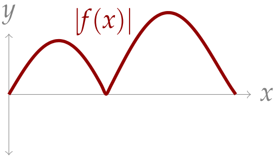
$$\frac{9}{2} \leq \int_0^6 f(x)\,\mathrm{d}x \leq \frac{15}{2}$$

Note $\frac{15}{2} - \frac{9}{2} = 3$, as required.
(Many bounds of the integral are possible, but looser bounds won't satisfy $d - c = 3$.)

Absolute Values

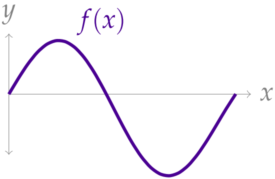
$f(x) \leq |f(x)|$ for any $f(x)$

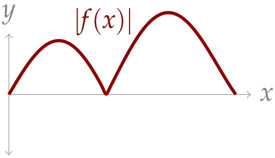




Absolute Values

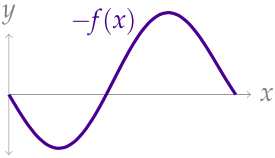
$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$

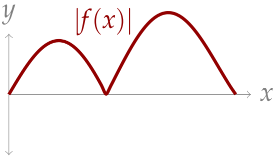




ABSOLUTE VALUES

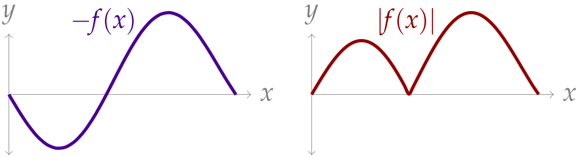
$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$





ABSOLUTE VALUES

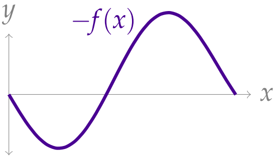
$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$

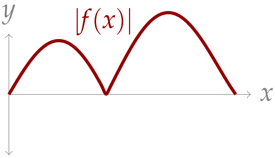


$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x \quad \text{and} \quad \int_a^b -f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$$

Absolute Values

$$f(x) \leq |f(x)| \text{ for any } f(x)$$
$$-f(x) \leq |f(x)| \text{ for any } f(x)$$





$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x \quad \text{and} \quad \int_a^b -f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$$

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f(x)|\mathrm{d}x$$

because $\left|\int_a^b f(x)\,\mathrm{d}x\right|$ is either $\int_a^b f(x)\,\mathrm{d}x$ or $-\int_a^b f(x)\,\mathrm{d}x$.

Included Work

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