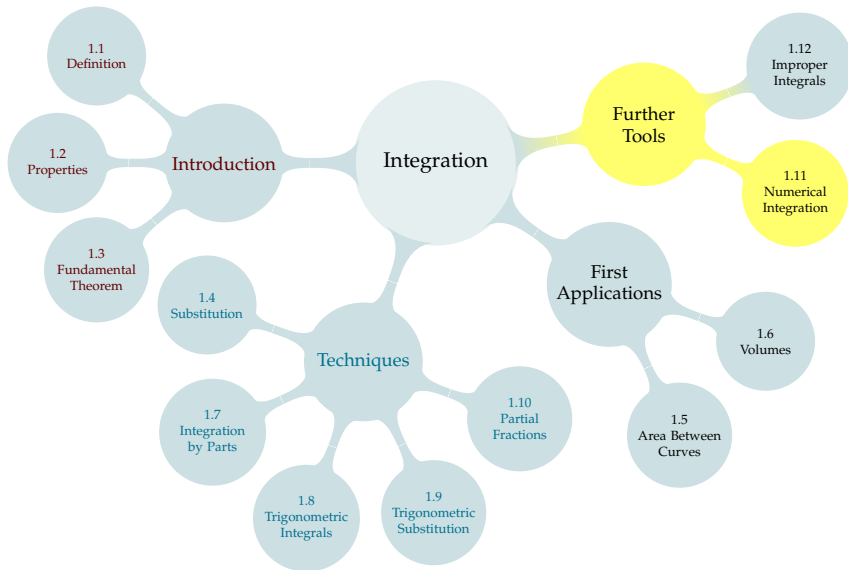


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Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

- ▶ the total error introduced by the midpoint rule is bounded by $\frac{M}{24} \frac{(b-a)^3}{n^2}$,
- ▶ the total error introduced by the trapezoidal rule is bounded by $\frac{M}{12} \frac{(b-a)^3}{n^2}$, and
- ▶ the total error introduced by Simpson's rule is bounded by $\frac{L}{180} \frac{(b-a)^5}{n^4}$

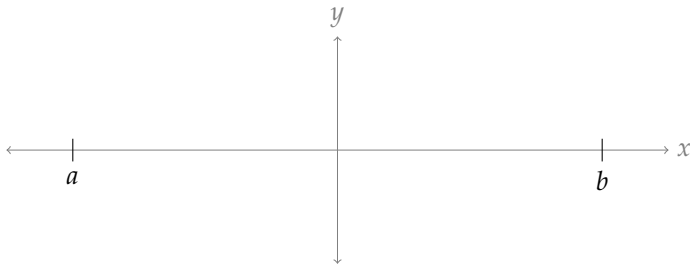
when approximating $\int_a^b f(x) \, dx$.

WHY THE *second* DERIVATIVE?

The midpoint rule gives the exact area under the curve for

$$f(x) = ax + b$$

when a and b are any constants.



The first derivative can be large without causing a large error.

Numerical integration errors

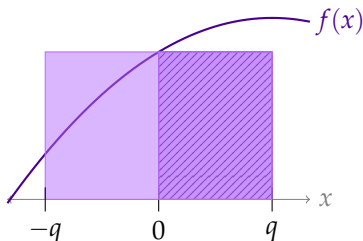
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when approximating $\int_a^b f(x) \, dx$.

We'll start small: let's consider one-half of a single interval being approximated using the midpoint rule.

To avoid messiness, let's also consider a simplified location:



We want to relate the actual area of this half-slice to its approximate area:

$$\int_0^q f(x) \, dx \approx q \cdot f(0)$$

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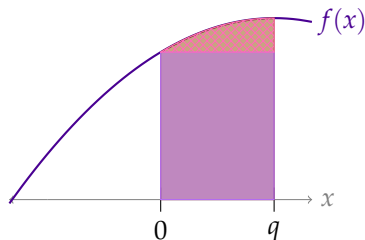
If you squint just right, the right-hand side looks a bit like the “ $u \cdot v$ ” term from integration by parts, where $u = f(x)$ and $dv = dx$.

- Set $u = f(x)$ and $dv = dx$, so $du = f'(x) \, dx$.
We choose $v(x) = x - q$, so that $f(v(q)) = f(0)$.

$$\begin{aligned} \int_0^q f(x) \, dx &= [(x - q)f(x)]_0^q - \int_0^q (x - q)f'(x) \, dx \\ &= q \cdot f(0) - \int_0^q (x - q)f'(x) \, dx \end{aligned}$$

- We know something about the second derivative, not the first, so repeat: set $u = f'(x)$, $dv = (x - q) \, dx$; $du = f''(x) \, dx$, $v = \frac{(x - q)^2}{2}$

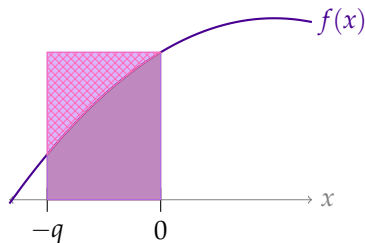
$$\int_0^q f(x) \, dx = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x - q)^2}{2} f''(x) \, dx$$



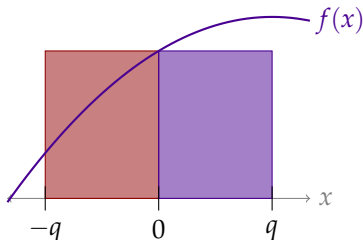
$$\begin{array}{ccccccc}
 \int_0^q f(x) \, dx & = & q \cdot f(0) & + & \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx \\
 \text{exact} & & \text{approximate} & & \pm \text{error}
 \end{array}$$

Repeat for the other half of the slice:

$$\begin{aligned}
 \int_{-q}^0 \underbrace{f(x)}_u \underbrace{dx}_{dv} &= \left[\underbrace{f(x)}_u \cdot \underbrace{(x+q)}_v \right]_{-q}^0 - \int_{-q}^0 \underbrace{(x+q)}_v \cdot \underbrace{f'(x)}_{du} dx \\
 &= q \cdot f(0) - \int_{-q}^0 \underbrace{f'(x)}_{\hat{u}} \cdot \underbrace{(x+q)}_{d\hat{v}} dx \\
 &= q \cdot f(0) - \left[\underbrace{f'(x)}_{\hat{u}} \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}} \right]_{-q}^0 + \int_{-q}^0 \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}} \underbrace{f''(x)}_{d\hat{u}} dx \\
 &= q \cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) dx
 \end{aligned}$$



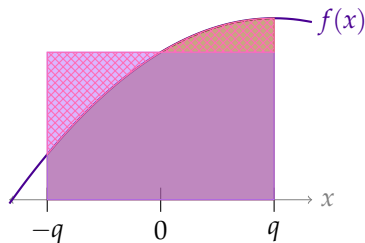
$$\begin{array}{ccccccc}
 \int_{-q}^0 f(x) \, dx & = & q \cdot f(0) & - & \frac{q^2}{2} \cdot f'(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx \\
 \text{exact} & & \text{approximate} & & \pm \text{error}
 \end{array}$$



$$\int_{-q}^0 f(x) \, dx = q \cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx$$

$$\int_0^q f(x) \, dx = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx$$

$$\int_{-q}^q f(x) \, dx = 2q \cdot f(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx$$

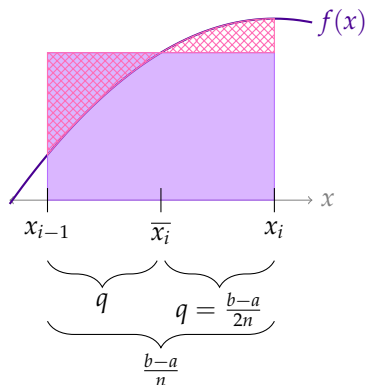


$$\begin{array}{rcl}
 \int_{-q}^q f(x) \, dx & = & 2q \cdot f(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx \\
 \text{exact} & \quad \quad \quad \text{approximate} & \quad \quad \quad \pm \text{error}
 \end{array}$$

We re-arrange to write the **error** as the difference between the **actual** area of one slice and its rectangular **approximation**.

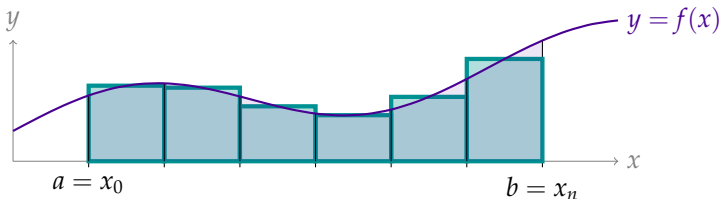
$$\begin{aligned}
 \int_{-q}^q f(x) \, dx - 2q \cdot f(0) &= \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx \\
 \text{error} &= \left| \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx + \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx \right| \\
 &\leq \left| \int_{-q}^0 \frac{(x+q)^2}{2} f''(x) \, dx \right| + \left| \int_0^q \frac{(x-q)^2}{2} f''(x) \, dx \right| \\
 &\leq \int_{-q}^0 \frac{(x+q)^2}{2} M \, dx + \int_0^q \frac{(x-q)^2}{2} M \, dx \\
 &= M \left[\frac{(x+q)^3}{6} \right]_{-q}^0 + M \left[\frac{(x-q)^3}{6} \right]_0^q \\
 &= \frac{M \cdot q^3}{3}
 \end{aligned}$$

Now we can bound the error of a single slice:



$$\left| \int_{-q}^q f(x) \, dx - 2q \cdot f(0) \right| \leq \frac{M}{3} \cdot q^3$$

$$\left| \int_{x_{i-1}}^{x_i} f(x) \, dx - \frac{b-a}{n} \cdot f(\bar{x}_i) \right| \leq \frac{M}{3} \left(\frac{b-a}{2n} \right)^3 = \frac{M}{24} \frac{(b-a)^3}{n^3}$$



- ▶ The error in each slice is at most $\frac{M}{24} \frac{(b-a)^3}{n^3}$
- ▶ There are n slices
- ▶ The overall error is at most $n \cdot \frac{M}{24} \frac{(b-a)^3}{n^3} = \frac{M}{24} \frac{(b-a)^3}{n^2}$