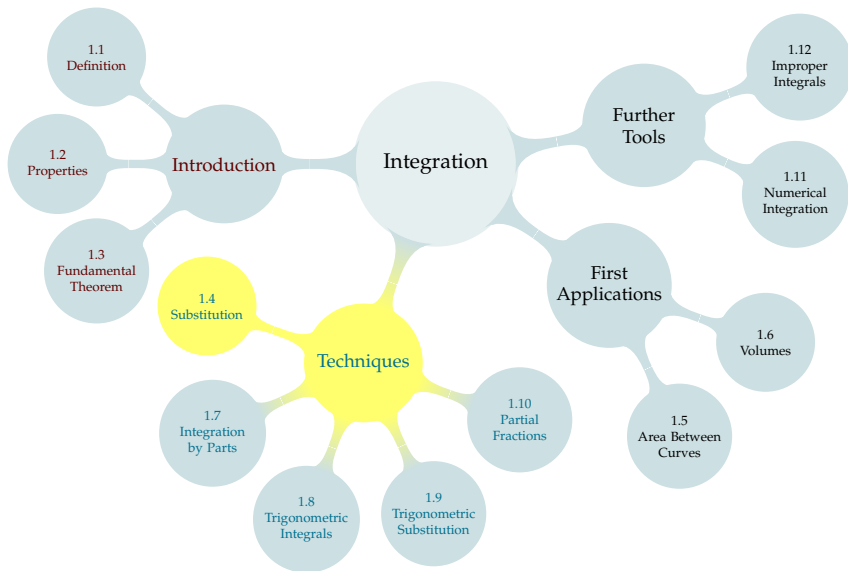


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# ANTIDERIVATIVES

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Hard to guess the antiderivative without seeing the derivative first!

# ANTIDERIVATIVES

Chain Rule:

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Hallmark of the chain rule: an “inside” function, with that function’s derivative multiplied.

## SOLVE BY INSPECTION

$$\int 2xe^{x^2+1} dx$$

$$\int \frac{1}{x} \cos(\log x) dx$$

$$\int 3(\sin x + 1)^2 \cos x dx$$

(Look for an “inside” function, with its derivative multiplied.)

## SOLVE BY INSPECTION

$$\int 2xe^{x^2+1} dx = e^{x^2+1} + C$$

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(Look for an “inside” function, with its derivative multiplied.)

# UNDOING THE CHAIN RULE

Chain Rule:

$$\frac{d}{dx} \{f(u(x))\} = f'(u(x)) \cdot u'(x)$$

(Here,  $u(x)$  is our “inside function”)

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x) \, dx = f(u(x)) + C$$

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Shorthand: call  $u(x)$  simply  $u$ ;  
since  $\frac{du}{dx} = u'(x)$ , call  $u'(x) \, dx$  simply  $du$ .

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$$\int f'(u(x)) \cdot u'(x) \, dx = \int f'(u) \, du \Big|_{u=u(x)} = f(u(x)) + C$$

This is the **substitution rule**.

We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1} dx$$

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We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1} dx$$

Using  $u$  as shorthand for  $x^2 + 1$ , and  $du$  as shorthand for  $2x dx$ :

$$\int 2xe^{x^2+1} dx = \int e^u du = e^u + C = e^{x^2+1} + C$$

$$\int \frac{1}{x} \cos(\log x) dx$$

Using  $u$  as shorthand for  $\log x$ , and  $du$  as shorthand for  $\frac{1}{x} dx$ :

$$\int \frac{1}{x} \cos(\log x) dx = \int \cos(u) du = \sin(u) + C = \sin(\log x) + C$$

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Using  $u$  as shorthand for  $\sin x + 1$ , and  $du$  as shorthand for  $\cos x dx$ :

$$\int 3(\sin x + 1)^2 \cos x dx = \int 3u^2 du = u^3 + C = (\sin x + 1)^3 + C$$

$$\int (3x^2) \sin(x^3 + 1) \, dx =$$

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Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 \, dx \rightarrow du$

$$\int (3x^2) \sin(x^3 + 1) dx = \int \sin(u) du \Big|_{u=x^3+1}$$

“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 dx \rightarrow du$

$$\begin{aligned}\int (3x^2) \sin(x^3 + 1) \, dx &= \int \sin(u) \, du \Big|_{u=x^3+1} \\ &= -\cos(u) + C \Big|_{u=x^3+1}\end{aligned}$$

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 \int (3x^2) \sin(x^3 + 1) dx &= \int \sin(u) du \Big|_{u=x^3+1} \\
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 &= \cos(x^3 + 1) + C
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“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 dx \rightarrow du$

Warning 1: We don't just change  $dx$  to  $du$ . We need to couple  $dx$  with the derivative of our inside function.

After all, we're undoing the chain rule! We need to have an “inside derivative.”

$$\begin{aligned}\int (3x^2) \sin(x^3 + 1) dx &= \int \sin(u) du \Big|_{u=x^3+1} \\ &= -\cos(u) + C \Big|_{u=x^3+1} \\ &= \cos(x^3 + 1) + C\end{aligned}$$

“Inside” function:  $x^3 + 1$ . Its derivative:  $3x^2$

Shorthand:  $x^3 + 1 \rightarrow u$ ,  $3x^2 dx \rightarrow du$

Warning 2: The final answer is a function of  $x$ .

We used the substitution rule to conclude

$$\int (3x^2) \sin(x^3 + 1) \, dx = -\cos(x^3 + 1) + C$$

We can check that our antiderivative is correct by differentiating.

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We can check that our antiderivative is correct by differentiating.

$$\frac{d}{dx} \{-\cos(x^3 + 1) + C\} = \sin(x^3 + 1)(3x^2)$$

We saw:

$$\int 3x^2 \sin(x^3 + 1) \, dx = -\cos(x^3 + 1) + C$$

So, we can evaluate:

$$\int_0^1 3x^2 \sin(x^3 + 1) \, dx$$

We saw:

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So, we can evaluate:

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Alternately, we can put in the limits of integration as we substitute. The bounds are originally given as values of  $x$ ; we simply change them to values of  $u$ .

If  $u(x) = x^3 + 1$ , then  $u(0) = 1$  and  $u(1) = 2$ .

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If  $u(x) = x^3 + 1$ , then  $u(0) = 1$  and  $u(1) = 2$ .

$$\underbrace{\int_0^1}_{x\text{-values}} 3x^2 \sin(x^3 + 1) \, dx = \underbrace{\int_1^2}_{u\text{-values}} \sin(u) \, du = -\cos(2) + \cos(1)$$



# NOTATION: LIMITS OF INTEGRATION

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

## NOTATION: LIMITS OF INTEGRATION

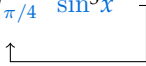
$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

Let  $u = \sin x$ ,  $du = \cos x dx$ . Note the limits (or bounds) of integration  $\pi/4$  and  $\pi/2$  are values of  $x$ , not  $u$ : they follow the differential, unless otherwise specified.

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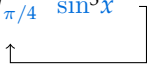
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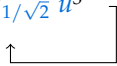
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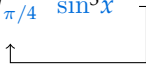
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$$\int_{1/\sqrt{2}}^1 \frac{1}{u^3} du$$

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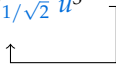
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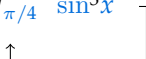
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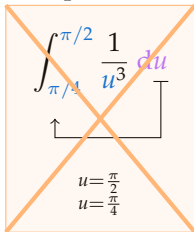
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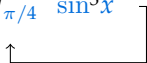
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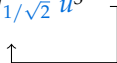
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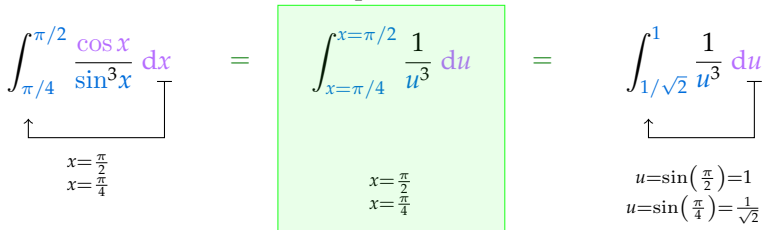
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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx = \int_{x=\pi/4}^{x=\pi/2} \frac{1}{u^3} du = \int_{u=\sin(\pi/4)=\frac{1}{\sqrt{2}}}^{u=\sin(\pi/2)=1} \frac{1}{u^3} du$$


not standard, but OK

# NOTATION: LIMITS OF INTEGRATION

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$x = \frac{\pi}{2}$   
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$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx$$

Let  $u = \sin x$ ,  $du = \cos x dx$ .

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x} dx &= \int_{1/\sqrt{2}}^1 \frac{1}{u^3} du \\ &= \int_{1/\sqrt{2}}^1 u^{-3} du \\ &= \left[ \frac{1}{-2u^2} \right]_{1/\sqrt{2}}^1 \\ &= -\frac{1}{2} - (-1) = \frac{1}{2} \end{aligned}$$

## TRUE OR FALSE?

1. Using  $u = x^2$ ,

$$\int e^{x^2} dx = \int e^u du$$

2. Using  $u = x^2 + 1$ ,

$$\int_0^1 x \sin(x^2 + 1) dx = \int_0^1 \frac{1}{2} \sin u du$$

## TRUE OR FALSE?

1. Using  $u = x^2$ ,

$$\int e^{x^2} dx = \int e^u du$$

False: missing  $u'(x)$ .

$$du = (2x dx) \neq dx$$

2. Using  $u = x^2 + 1$ ,

$$\int_0^1 x \sin(x^2 + 1) dx = \int_0^1 \frac{1}{2} \sin u du$$

False: limits of integration didn't translate.

When  $x = 0$ ,  $u = 0^2 + 1 = 1$ , and when  $x = 1$ ,  $u = 1^2 + 1 = 2$ .

Evaluate  $\int_0^1 x^7 (x^4 + 1)^5 \, dx$ .



Evaluate  $\int_0^1 x^7 (x^4 + 1)^5 dx$ .

$$u = x^4 + 1, \quad du = 4x^3 dx$$

$$u(0) = 1, \quad u(1) = 2$$

$$x^4 = u - 1, \quad x^3 dx = \frac{1}{4} du$$

$$\int_0^1 x^7 (x^4 + 1)^5 dx = \int_0^1 (x^4) \cdot (x^4 + 1)^5 \cdot x^3 dx$$

$$= \int_1^2 (u - 1) \cdot u^5 \cdot \frac{1}{4} du$$

$$= \frac{1}{4} \int_1^2 (u^6 - u^5) du$$

$$= \frac{1}{4} \left[ \frac{1}{7} u^7 - \frac{1}{6} u^6 \right]_1^2$$

$$= \frac{1}{4} \left[ \frac{2^7}{7} - \frac{2^6}{6} - \frac{1}{7} + \frac{1}{6} \right]$$

Time permitting, more examples using the substitution rule

Evaluate  $\int \sin x \cos x \, dx$ .

Evaluate  $\int \sin x \cos x \, dx$ .

Let  $u = \sin x$ ,  $du = \cos x \, dx$ :

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Or, let  $u = \cos x$ ,  $du = -\sin x \, dx$ :

$$\int \cos x \sin x \, dx = -\int u \, du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$$

Recall  $\sin^2 x + \cos^2 x = 1$  for all  $x$ , so  $\frac{1}{2}\sin^2 x = -\frac{1}{2}\cos^2 x + \frac{1}{2}$ . The two answers look different, but they only differ by a constant, which can be absorbed in the arbitrary constant  $C$ . If we rename the second  $C$  to  $C'$  so that the second answer is  $-\frac{1}{2}\cos^2 x + C'$ , then  $C' = C + \frac{1}{2}$ .

## CHECK OUR WORK

We can check that  $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$  by differentiating.

We can check that  $\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + C$  by differentiating.

## CHECK OUR WORK

We can check that  $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$  by differentiating.

$$\frac{d}{dx} \left\{ \frac{1}{2} \sin^2 x + C \right\} = \frac{2}{2} \sin x \cdot \cos x = \sin x \cos x$$

Our answer works.

We can check that  $\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + C$  by differentiating.

$$\frac{d}{dx} \left\{ -\frac{1}{2} \cos^2 x + C \right\} = -\frac{2}{2} \cos x \cdot (-\sin x) = \sin x \cos x$$

This answer works too.

Evaluate  $\int \frac{\log x}{3x} dx$ .

Evaluate  $\int \frac{\log x}{3x} dx$ .

Let  $u = \log x$ ,  $du = \frac{1}{x} dx$ :

$$\begin{aligned}\int \frac{\log x}{3} \cdot \frac{1}{x} dx &= \frac{1}{3} \int u du \\ &= \frac{1}{6} u^2 + C \\ &= \frac{1}{6} \log^2 x + C\end{aligned}$$



## CHECK OUR WORK

We can check that  $\int \frac{\log x}{3x} dx = \frac{1}{6} \log^2 x + C$  by differentiating.

## CHECK OUR WORK

We can check that  $\int \frac{\log x}{3x} dx = \frac{1}{6} \log^2 x + C$  by differentiating.

$$\frac{d}{dx} \left\{ \frac{1}{6} \log^2 x + C \right\} = \frac{2}{6} \log x \cdot \frac{1}{x} = \frac{\log x}{3x}$$

Our answer works.

Evaluate  $\int \frac{e^x}{e^x + 15} dx$ .

Evaluate  $\int x^4(x^5 + 1)^8 dx$ .

Evaluate  $\int \frac{e^x}{e^x + 15} dx$ .

Let  $u = e^x + 15$ ,  $du = e^x dx$

$$\int \frac{e^x}{e^x + 15} dx = \int \frac{1}{u} du = \log |u| + C = \log |e^x + 15| + C$$

In this case, since  $e^x + 15 > 0$ , the absolute values on  $|e^x + 15|$  are optional.

Evaluate  $\int x^4(x^5 + 1)^8 dx$ .



Evaluate  $\int \frac{e^x}{e^x + 15} dx$ .

Evaluate  $\int x^4(x^5 + 1)^8 dx$ .

Let  $u = x^5 + 1$ ,  $du = 5x^4 dx$ . Then,  $x^4 dx = \frac{1}{5} du$ .

$$\begin{aligned}\int x^4(x^5 + 1)^8 dx &= \int \frac{1}{5}(u)^8 du \\ &= \frac{1}{5} \cdot \frac{1}{9} u^9 + C = \frac{1}{45}(x^5 + 1)^9 + C\end{aligned}$$



## CHECK OUR WORK

We can check that  $\int \frac{e^x}{e^x + 15} dx = \log |e^x + 15| + C$  by differentiating.

We can check that  $\int x^4(x^5 + 1)^8 dx = \frac{1}{45}(x^5 + 1)^9 + C$  by differentiating.



## CHECK OUR WORK

We can check that  $\int \frac{e^x}{e^x + 15} dx = \log |e^x + 15| + C$  by differentiating.

$$\frac{d}{dx} \{\log |e^x + 15| + C\} = \frac{1}{e^x + 15} \cdot e^x = \frac{e^x}{e^x + 15}$$

Our answer works.

We can check that  $\int x^4(x^5 + 1)^8 dx = \frac{1}{45}(x^5 + 1)^9 + C$  by differentiating.

$$\frac{d}{dx} \left\{ \frac{1}{45}(x^5 + 1)^9 + C \right\} = \frac{9}{45}(x^5 + 1)^8 \cdot 5x^4 = (x^5 + 1)^8 x^4$$

Our answer works.



Evaluate  $\int_4^8 \frac{s}{s-3} ds$ . Be careful to use correct notation.





Evaluate  $\int_4^8 \frac{s}{s-3} ds$ . Be careful to use correct notation.

Let  $u = s - 3$ ,  $du = ds$ .

Then  $s = u + 3$ ,  $u(4) = 1$  and  $u(8) = 5$ .

$$\begin{aligned}\int_4^8 \frac{s}{s-3} ds &= \int_1^5 \frac{u+3}{u} du \\&= \int_1^5 \left(1 + \frac{3}{u}\right) du \\&= [u + 3 \log |u|]_1^5 \\&= [5 + 3 \log 5] - [1 + 3 \log 1] \\&= 4 + 3 \log 5\end{aligned}$$



Evaluate  $\int x^9(x^5 + 1)^8 dx$ .



Evaluate  $\int x^9(x^5 + 1)^8 dx$ .

Let  $u = x^5 + 1$ ,  $du = 5x^4 dx$ .

Then  $x^4 dx = \frac{1}{5} du$ ,  $x^5 = u - 1$ .

$$\begin{aligned}\int x^9(x^5 + 1)^8 dx &= \int (x^4) \cdot (x^5) \cdot (x^5 + 1)^8 dx \\&= \int \left(\frac{1}{5}\right) \cdot (u - 1) \cdot u^8 du = \frac{1}{5} \int (u^9 - u^8) du \\&= \frac{1}{5} \left[ \frac{1}{10} u^{10} - \frac{1}{9} u^9 \right] + C \\&= \frac{1}{5} \left[ \frac{(x^5 + 1)^{10}}{10} - \frac{(x^5 + 1)^9}{9} \right] + C\end{aligned}$$



## CHECK OUR WORK

We can check that  $\int x^9(x^5 + 1)^8 \, dx = \frac{1}{5} \left[ \frac{(x^5 + 1)^{10}}{10} - \frac{(x^5 + 1)^9}{9} \right] + C$  by differentiating.



## CHECK OUR WORK

We can check that  $\int x^9(x^5 + 1)^8 \, dx = \frac{1}{5} \left[ \frac{(x^5 + 1)^{10}}{10} - \frac{(x^5 + 1)^9}{9} \right] + C$  by differentiating.

$$\begin{aligned} & \frac{d}{dx} \left\{ \frac{1}{5} \left[ \frac{(x^5 + 1)^{10}}{10} - \frac{(x^5 + 1)^9}{9} \right] + C \right\} \\ &= \frac{1}{5} [(x^5 + 1)^9 \cdot 5x^4 - (x^5 + 1)^8 \cdot 5x^4] \\ &= x^4(x^5 + 1)^9 - x^4(x^5 + 1)^8 \\ &= x^4(x^5 + 1)^8[(x^5 + 1) - 1] \\ &= x^4(x^5 + 1)^8[x^5] \\ &= x^9(x^5 + 1)^8 \end{aligned}$$

Our answer works.



# PARTICULARLY TRICKY SUBSTITUTION

Evaluate  $\int \frac{1}{e^x + e^{-x}} dx$ .



Q

# PARTICULARLY TRICKY SUBSTITUTION

Evaluate  $\int \frac{1}{e^x + e^{-x}} dx$ .

Let  $u = e^x$ ,  $du = e^x dx$ . Then  $dx = \frac{du}{e^x} = \frac{du}{u}$ .

$$\begin{aligned}\int \frac{1}{e^x + e^{-x}} dx &= \int \frac{1}{u + \frac{1}{u}} \frac{du}{u} \\&= \int \frac{1}{u^2 + 1} du \\&= \arctan(u) + C \\&= \arctan(e^x) + C\end{aligned}$$



## CHECK OUR WORK

We can check that  $\int \frac{1}{e^x + e^{-x}} dx = \arctan(e^x) + C$  by differentiating.





## CHECK OUR WORK

We can check that  $\int \frac{1}{e^x + e^{-x}} dx = \arctan(e^x) + C$  by differentiating.

$$\begin{aligned}\frac{d}{dx} \{\arctan(e^x) + C\} &= \frac{1}{(e^x)^2 + 1} \cdot e^x \\ &= \frac{e^x}{(e^x)^2 + 1} \\ &= \frac{e^x}{(e^x)^2 + 1} \cdot \frac{e^{-x}}{e^{-x}} \\ &= \frac{1}{e^x + e^{-x}}\end{aligned}$$

Our answer works.



## Included Work



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