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1.8 Trigonometric Integrals

Recall:

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$

Integrating Products of Sine and Cosine

$$\int \sin x \cos x \, \mathrm{d}x =$$

Integrating Products of Sine and Cosine

$$\int \sin x \cos x \, \mathrm{d}x =$$

$$\int \sin^{10} x \ \cos x \, \mathrm{d}x =$$

Q

Check Our Work

If we are correct that $\displaystyle\int \sin x \cos x \, \mathrm{d}x = \frac{\sin^2 x}{2} + C$, then it should be true that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin^2 x}{2} + C\right\} = \sin x \cos x$.

Check Our Work

If we are correct that $\int \sin^{10} x \cos x \, \mathrm{d}x = \frac{\sin^{11} x}{11} + C$, then it should be true that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin^{11} x}{2} + C\right\} = \sin^{10} x \cos x$.

Integrating Products of Sine and Cosine

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \, \cos x \, \mathrm{d}x =$$

Q

Check Our Work

If we are correct that $\int \sin^{\pi+1} x \cos x \, \mathrm{d}x = \dfrac{\sin^{\pi+2} x}{\pi+2} + C$, then it should be true that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin^{\pi+2} x}{\pi+2} + C\right\} = \sin^{\pi+1} x \cos x$.

Let $u = \sin x$, $\mathrm{d}u = \cos x \,\mathrm{d}x$.

$$\int \sin^{10} x \cos^3 x \,\mathrm{d}x =$$

Integrating Products of Sine and Cosine

Let $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$.
Use $\sin^2 x + \cos^2 x = 1$.

$$\int \sin^{10} x \cos^3 x \, \mathrm{d}x =$$

Q

Check Our Work

If we are correct that $\int \sin^{10} x \cos^3 x \, \mathrm{d}x = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$, then it should be true that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C\right\} = \sin^{10} x \cos^3 x$.

Integrating Products of Sine and Cosine

$$\int \sin^5 x \cos^4 x \, \mathrm{d}x =$$

Check Our Work

If we are correct that $\displaystyle\int \sin^5 x \cos^4 x \, \mathrm{d}x = -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C$, then it should be true that $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left\{-\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C\right\} = \sin^5 x \cos^4 x$.

GENERALIZE: $\int \sin^m x \cos^n bx \, \mathrm{d}x$

To use the substitution $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

Generalize: $\int \sin^m x \cos^n bx \, \mathrm{d}x$

To use the substitution $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

- We need to reserve one $\cos x$ for the differential.

GENERALIZE: $\int \sin^m x \cos^n bx \, dx$

To use the substitution $u = \sin x$, $du = \cos x \, dx$:

- We need to reserve one $\cos x$ for the differential.
- We need to convert the remaining $\cos^{n-1} x$ to $\sin x$ terms.

Generalize: $\int \sin^m x \cos^n bx \, \mathrm{d}x$

To use the substitution $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

- We need to reserve one $\cos x$ for the differential.
- We need to convert the remaining $\cos^{n-1} x$ to $\sin x$ terms.
- We convert using $\cos^2 x = 1 - \sin^2 x$. To avoid square roots, that means $n - 1$ should be even when we convert.

Generalize: $\int \sin^m x \cos^n bx \, \mathrm{d}x$

To use the substitution $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$:

- We need to reserve one $\cos x$ for the differential.
- We need to convert the remaining $\cos^{n-1} x$ to $\sin x$ terms.
- We convert using $\cos^2 x = 1 - \sin^2 x$. To avoid square roots, that means $n - 1$ should be even when we convert.
- So, we can use this substitution when the original power of cosine, n, is ODD: one cosine goes to the differential, the rest are converted to sines.

GENERALIZE: $\int \sin^m x \cos^n x \, \mathrm{d}x$

To use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x \, \mathrm{d}x$:

GENERALIZE: $\int \sin^m x \cos^n x \, dx$

To use the substitution $u = \cos x$, $du = -\sin x \, dx$:

- We need to reserve one $\sin x$ for the differential.

GENERALIZE: $\int \sin^m x \cos^n x \, dx$

To use the substitution $u = \cos x$, $du = -\sin x \, dx$:

- We need to reserve one $\sin x$ for the differential.
- We need to convert the remaining $\sin^{m-1} x$ to $\cos x$ terms.

GENERALIZE: $\int \sin^m x \cos^n x \,\mathrm{d}x$

To use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x \,\mathrm{d}x$:

- We need to reserve one $\sin x$ for the differential.
- We need to convert the remaining $\sin^{m-1} x$ to $\cos x$ terms.
- We convert using $\sin^2 x = 1 - \cos^2 x$. To avoid square roots, that means $m - 1$ should be even when we convert.

GENERALIZE: $\int \sin^m x \cos^n x \, dx$

To use the substitution $u = \cos x$, $du = -\sin x \, dx$:

- We need to reserve one $\sin x$ for the differential.
- We need to convert the remaining $\sin^{m-1} x$ to $\cos x$ terms.
- We convert using $\sin^2 x = 1 - \cos^2 x$. To avoid square roots, that means $m - 1$ should be even when we convert.
- So, we can use this substitution when the original power of sine, m, is ODD: one sine goes to the differential, the rest are converted to cosines.

MNEMONIC: "ODD ONE OUT"

Integrating $\int \sin^m x \cos^n x \, dx$

If you want to use $u = \sin x$, there should be an odd power of cosine.

If you want to use $u = \cos x$, there should be an odd power of sine.

Carry out a suitable substitution (but do not evaluate the resulting integral):

- $\int \sin^4 x \cos^7 x \, \mathrm{d}x$
- $\int \sin^7 x \cos^4 x \, \mathrm{d}x$
- $\int \sin^7 x \cos^7 x \, \mathrm{d}x$

To evaluate $\int \sin^m x \cos^n x \, \mathrm{d}x$, we use:

- $u = \sin x$ if n is odd, and/or
- $u = \cos x$ if m is odd

What if n and m are both even?

To evaluate $\int \sin^m x \cos^n x \, \mathrm{d}x$, we use:

- $u = \sin x$ if n is odd, and/or
- $u = \cos x$ if m is odd

What if n and m are both even?

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int \sin^2 x \, \mathrm{d}x =$$

Check Our Work

We check that $\int \sin^2 x \, \mathrm{d}x = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$ by differentiating:

A

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

Evaluate $\displaystyle\int \sin^4 x \, \mathrm{d}x$.

Q

Check Our Work

We want to check that $\int \sin^4 x \, \mathrm{d}x = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$

Recall:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{\tan x\} = \sec^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{\sec x\} = \sec x \ \tan x$
- $\tan^2 x + 1 = \sec^2 x$

$$\int \tan x \, \mathrm{d}x =$$

Check Our Work

Let's check that $\displaystyle\int \tan x \mathrm{d}x = \log|\sec x| + C$ by differentiating.

Optional: A nifty trick – you won't be expected to come up with it. There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, \mathrm{d}x =$$

Optional: A nifty trick – you won't be expected to come up with it. There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx$$

Useful integrals:

- $\displaystyle\int \tan x \, \mathrm{d}x = \log|\sec x| + C$
- $\displaystyle\int \sec x \, \mathrm{d}x = \log|\sec x + \tan x| + C$

1. $\displaystyle\int \sec x \tan x \, \mathrm{d}x =$

2. $\displaystyle\int \sec^2 x \, \mathrm{d}x =$

3. $\displaystyle\int \tan x \, \mathrm{d}x =$

4. $\displaystyle\int \sec x \, \mathrm{d}x =$

Evaluate using the substitution rule:

$$\int \tan^5 x \ \sec^2 x \, \mathrm{d}x =$$

$$\int \sec^4 x \, (\sec x \tan x) \, \mathrm{d}x =$$

Check Our Work

Let's check that $\int \tan^5 x \sec^2 x \, \mathrm{d}x = \frac{1}{6}\tan^6 x + C$ by differentiating.

Check Our Work

Let's check that $\displaystyle\int \sec^4 x(\sec x \tan x)\, \mathrm{d}x = \frac{1}{5}\sec^5 x + C$ by differentiating.

Evaluate using the identity $\sec^2 x = 1 + \tan^2 x$

$$\int \tan^4 x \sec^6 x \, \mathrm{d}x =$$

$$\int \tan^3 x \sec^5 x \, \mathrm{d}x =$$

Check Our Work

Let's check that $\int \tan^4 x \sec^6 x \, \mathrm{d}x = \frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C.$

Check Our Work

Let's check that $\int \tan^3 x \sec^5 x \, \mathrm{d}x = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.$

Choosing a Substitution: $\int \tan^m x \sec^n x \, \mathrm{d}x$

Using $u = \sec x$, $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$:

CHOOSING A SUBSTITUTION: $\int \tan^m x \sec^n x \,\mathrm{d}x$

Using $u = \sec x$, $\mathrm{d}u = \sec x \tan x \,\mathrm{d}x$:

- Reserve $\sec x \tan x$ for the differential.

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \sec x$, $du = \sec x \tan x \, dx$:

- Reserve $\sec x \tan x$ for the differential.
- From the remaining $\tan^{m-1} x \ \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \sec x$, $du = \sec x \tan x \, dx$:

- Reserve $\sec x \tan x$ for the differential. (m, n should each be at least 1)
- From the remaining $\tan^{m-1} x \ \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.

Choosing a Substitution: $\int \tan^m x \sec^n x \, \mathrm{d}x$

Using $u = \sec x$, $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$:

- Reserve $\sec x \tan x$ for the differential.
  (m, n should each be at least 1)
- From the remaining $\tan^{m-1} x \ \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.
  ($m - 1$ should be even, to avoid square roots)

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \sec x$, $du = \sec x \tan x \, dx$:

- Reserve $\sec x \tan x$ for the differential.
  (m, n should each be at least 1)
- From the remaining $\tan^{m-1} x \ \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$.
  ($m - 1$ should be even, to avoid square roots)

To use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$ to evaluate $\displaystyle\int \tan^m x \sec^n x \, dx$, n should be $\boxed{}$, and m should be $\boxed{}$.

Choosing a Substitution: $\int \tan^m x \sec^n x \, \mathrm{d}x$

Using $u = \tan x$, $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$:

- Reserve $\boxed{}$ for the differential.
- From the remaining terms, convert all $\boxed{}$ to $\boxed{}$ using $\tan^2 x + 1 = \sec^2 x$.

To use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$ to evaluate $\displaystyle\int \tan^m x \sec^n \, \mathrm{d}x$, n should be $\boxed{}$.

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$

Choose a substitution for the integrals below.

- $\displaystyle\int \sec^2 x \tan^3 x \, dx$
- $\displaystyle\int \sec^2 x \tan^2 x \, dx$
- $\displaystyle\int \sec^3 x \tan^3 x \, dx$

$$\int \sec^2 x \tan^3 x \, \mathrm{d}x$$

$$\int \sec^2 x \tan^2 x \, \mathrm{d}x$$

Q

$$\int \sec^3 x \tan^3 x \, \mathrm{d}x$$

Q

Evaluate $\int \tan^3 x \, \mathrm{d}x$

Evaluate $\int \tan^3 x \, \mathrm{d}x = \int \frac{\sin^3 x}{\cos^3 x} \, \mathrm{d}x$

CHECK OUR WORK

Let's check that $\displaystyle\int \tan^3 x \, \mathrm{d}x = \log|\cos x| + \frac{1}{2}\sec^2 x + C$. by differentiating.

Generalizing the last example:

$$\int \tan^m x \sec^n x \, dx =$$

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x \, \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n \mathrm{d}x \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} \mathrm{d}x \\ &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x}\right) \sin x \, \mathrm{d}x\end{aligned}$$

Generalizing the last example:

$$\begin{aligned}
\int \tan^m x \sec^n x \, \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n \mathrm{d}x \\
&= \int \frac{\sin^m x}{\cos^{m+n} x} \mathrm{d}x \\
&= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x}\right) \sin x \, \mathrm{d}x
\end{aligned}$$

To use $u = \cos x$, $\mathrm{d}u = \sin x \, \mathrm{d}x$:

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x \, \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n \mathrm{d}x \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} \mathrm{d}x \\ &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x}\right) \sin x \, \mathrm{d}x\end{aligned}$$

To use $u = \cos x$, $\mathrm{d}u = \sin x \, \mathrm{d}x$: we will convert $\sin^{m-1}(x)$ into cosines, so $m - 1$ must be even, so m must be odd.

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd

$$\text{Evaluate } \int \tan^2 x \, dx$$

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd

$$\text{Evaluate } \int \tan^2 x \, dx$$

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd

Evaluate $\displaystyle\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd
- $u = \tan x$ if m is even and $n = 0$
  (after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

Evaluate $\displaystyle\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x + x + C$$

Included Work

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