Computing the Topological Degree of a Mapping in Rn .

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Computing the Topological Degree of a Mapping in R^n *

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Summary. Let P be a connected n-dimensional polyhedron, and let

$$(1) \qquad b(P) = \sum_{j=1}^{m} t_j [Y_1^{(j)} \dots Y_n^{(j)}]$$

be the oriented boundary of P in terms of oriented $n-1$ simplexes $t_j[Y_1^{(j)} \dots Y_n^{(j)}]$, where $Y_i^{(j)}$ is a vertex of a simplex and $t_j = \pm 1$. Let $F = (f^1, \dots, f^n)$ be a vector of real, continuous functions defined on P, and let $F \neq \theta \equiv (0, \dots, 0)$ on $b(P)$. Assume that for $1 < \mu \leqq n$, and $\Phi_\mu = (\varphi^1, \dots, \varphi^\mu)$ where $\varphi^i = f^{j_i}$, $j_k \neq j_l$ if $k \neq l$, the sets $S(A_\mu) = \{X \in b(P)\colon \Phi_\mu(X)/|\Phi_\mu(X)| = A_\mu\} \cap H_\mu$ and $b(P) - S(A_\mu)$ consist of a finite number of connected subsets of $b(P)$, for all vectors $A_\mu = (\pm 1, 0, \dots, 0)$, $(0, \pm 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, \pm 1)$ and for all $\mu - 1$ dimensional simplexes H_μ on $b(P)$. It is shown that if m is sufficiently large, and $\max_{(j;\, 1 \leqq k < l \leqq n)} |Y_k^{(j)} - Y_l^{(j)}|$ sufficiently small, then $d(F, P, \theta)$, the topological degree of F at θ relative to P, is given by

$$(2) \qquad d(F, P, \theta) = \frac{1}{2^n n!} \sum_{j=1}^{m} t_j \Delta(\operatorname{sgn} F(Y_1^{(j)}), \dots, \operatorname{sgn} F(Y_n^{(j)}))$$

where the t_j and $Y_i^{(j)}$ are the same as those in (1), where $\operatorname{sgn} F = (\operatorname{sgn} f^1, \dots, \operatorname{sgn} f^n)$, where for a real, $\operatorname{sgn} a = 1, 0$ or -1 if $a > 0$, $= 0$ or < 0 respectively, and where $\Delta(B_1, \dots, B_n)$ denotes the determinant of the $n \times n$ matrix with i'th row B_i. An algorithm is given for computing $d(F, P, \theta)$ using (2), and the use of (2) is illustrated in examples.

1. Introduction and Summary

Let $\mathscr{D}$ be an n-dimensional region in R^n and let $F = (f^1, \dots, f^n)$ be a vector of real, differentiable functions defined on $\bar{\mathscr{D}}$, the closure of $\mathscr{D}$. A point in R^n will be denoted by $X = (x^1, \dots, x^n)$, and we shall write $F(X) = (f^1(X), \dots, f^n(X))$.

If $B_i = (b_{i1}, \dots, b_{in})$ are q vectors, we shall use the convenient notation

$$(1.1) \qquad \Delta_q(B_1, \dots, B_q) = \begin{vmatrix} b_{11} \dots b_{1q} \\ b_{q1} \dots b_{qq} \end{vmatrix}$$

for the determinant of the matrix $[b_{ij}]$. The Jacobian $j(F)$ of F is thus given by

$$(1.2) \qquad j(F) = \Delta_n\left(\frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^n}\right).$$

Let us assume that there are no points $X \in b(\mathscr{D})$ ($b(\mathscr{D})$ = the boundary of $\mathscr{D}$) such that

$$(1.3) \qquad F(X) = \theta \equiv (0, \dots, 0).$$

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The topological degree of F at θ relative to $\mathscr{D}$, which we denote by $d(F, \mathscr{D}, \theta)$, is often defined as follows. Let us assume that if X is a point in $\mathscr{D}$ such that $F(X) = \theta$, then $j(F)(X) \neq 0$. Let N_+ (N_-) denote the number of solutions in $\mathscr{D}$ of (1.3) such that $j(F) > 0 \,(< 0)$.

Then $d(F, \mathscr{D}, \theta) = N_+ - N_-$. Kronecker (see Alexandroff-Hopf [1, pp. 465–467]) gave a more general definition of $d(F, \mathscr{D}, \theta)$ for $n > 1$, namely

$$(1.4) \qquad d(F, \mathscr{D}, \theta) = \frac{1}{\Omega_{n-1}} \int\limits_{X(U) \in b(\mathscr{D})} \frac{1}{|F|^n} \Delta_n\left(F, \frac{\partial F}{\partial u^1}, \ldots, \frac{\partial F}{\partial u^{n-1}}\right) du^1 \ldots du^{n-1},$$

where

$$|F| = [(f^1)^2 + \cdots + (f^n)^2]^{\frac{1}{2}}$$

$$(1.5) \qquad X(U) = X(u^1, \ldots, u^{n-1})$$

$$\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

For the case $n = 1$, we adopt the definition

$$(1.4)' \qquad d(F, \mathscr{D}, \theta) = d(f, \mathscr{D}_1, 0) = \frac{1}{2}\left\{\frac{f(B)}{|f(B)|} - \frac{f(A)}{|f(A)|}\right\}.$$

In (1.4)′ $\mathscr{D}_1$ may be an arc C embedded in R^p, $p \geqq 1$, where C begins at A and ends at B. In (1.4) $\mathscr{D}$ may be an n-dimensional region embedded in R^p, $p \geqq n$ (see the first paragraph of Sect. 4.2). In (1.4) and (1.5) the vector $X = X(U)$ is a one-one parametrization of $U = (u^1, \ldots, u^{n-1})$, such that if i_j denotes the unit vector in the x^j direction in R^n, and we define a vector A by

$$(1.6) \qquad A = \begin{vmatrix} i_1 & i_2 & \ldots & i_n \\ \frac{\partial f^1}{\partial u^1} & \frac{\partial f^2}{\partial u^1} & \ldots & \frac{\partial f^n}{\partial u^1} \\ & \ldots & & \\ \frac{\partial f^1}{\partial u^{n-1}} & \frac{\partial f^2}{\partial u^{n-1}} & \ldots & \frac{\partial f^n}{\partial u^{n-1}} \end{vmatrix}.$$

Then A is in the direction of the outward normal to $F(b(\mathscr{D}))$ at $F(X(U))$. For example, the n-1-dimensional volume of $F(b(\mathscr{D}))$ is given by

$$(1.7) \qquad V(F(b(\mathscr{D}))) = \int\limits_{X(U) \in b(\mathscr{D})} |A| \, du^1 \ldots du^{n-1}.$$

The definitions (1.4)′–(1.4) allow the computation of $d(F, \mathscr{D}, \theta)$ in cases where $j(F) = 0$ at points in $\mathscr{D}$ at which $F(X) = \theta$, such as, for example, when F vanishes identically on closed sets of positive n-dimensional Lebesgue measure in the interior of $\mathscr{D}$, so long as F is differentiable on the boundary of $\mathscr{D}$, continuous in $\overline{\mathscr{D}}$, and $F \neq \theta$ on the boundary of $\mathscr{D}$. The right hand side of (1.4)′ or (1.4) is then always as integer. If F is merely continuous and not of class C^1 on $\overline{\mathscr{D}}$, then we define $d(F, \mathscr{D}, \theta)$ by $d(F, \mathscr{D}, \theta) = \lim\limits_{(\nu \to \infty)} d(F^{(\nu)}, \mathscr{D}, \theta)$ where $F^{(\nu)}$ is of class C^1 on $\overline{\mathscr{D}}$ for $\nu = 1, 2, \ldots$, $\sup\limits_{(X \in \mathscr{D})} |F(X) - F^{(\nu)}(X)| \to 0$ as $\nu \to \infty$, and $d(F^{(\nu)}, \mathscr{D}, \theta)$ is defined by means of (1.4)′ or (1.4).

Another analytical expression for $d(F, \mathscr{D}, \theta)$ was given by Heinz [3] in the form

$$(1.8)\qquad d(F, \mathscr{D}, \theta) = \int_{\mathscr{D}} \varphi(|F|)\, j(F)\, dx^1 \ldots dx^n.$$

This is applicable if F is of class C^1 in $\mathscr{D}$, $F \neq \theta$ on $b(\mathscr{D})$, and $j(F) \neq 0$ at each point $X \in \mathscr{D}$ such that $F(X) = \theta$. In (1.8) φ is a function of $|X|$, with support on $\{X\colon 0 < r_1 \leqq |X| \leqq r_2 < \infty\}$, where r_1 and $r_2 > r_1$ are sufficiently small (depending on $\mathscr{D}$ and F), and such that

$$(1.9)\qquad \int_{R^n} \varphi(|X|)\, dx^1 \ldots dx^n = 1.$$

Various other properties of $d(F, \mathscr{D}, \theta)$ are described in [1, 2, 4, 5 and 8].

Neither (1.4) nor (1.8) make it possible to explicitly evaluate $d(F, \mathscr{D}, \theta)$ in general. In this paper we shall describe a simple formula for computing $d(F, P, \theta)$, which depends only on the sign of the coordinates f^i of F at a finite number of points of $b(P)$, where P is a polyhedron in $\overline{\mathscr{D}}$, and where $b(P)$ denotes the boundary of P. The polyhedron P and its boundary $b(P)$ are represented as "sums" of "oriented" simplexes in the form

$$(1.10)\qquad P = \sum_{j=1}^{m'} [X_0^{(j)} \ldots X_n^{(j)}],$$

$$(1.11)\qquad b(P) = \sum_{j=1}^{m} t_j [Y_1^{(j)} \ldots Y^{(j)}]$$

where the $X_i^{(j)}$, $i = 0, 1, \ldots, n$ $(Y_i^{(j)}, i = 1, 2, \ldots, n)$ are the vertices of the oriented n-simplex (n-1-simplex) $[X_0^{(j)} \ldots X_n^{(j)}]$ $([Y_1^{(j)} \ldots Y_n^{(j)}])$, $j = 1, 2, \ldots, m'$ $(j = 1, 2, \ldots, m)$, and where $t_j = \pm 1$.

We assume that $F \neq \theta$ on $b(P)$, and that the components f^i of F are real and continuous on P. Moreover, if $n > 1$, we assume that for $1 < \mu \leqq n$ and $\Phi_\mu \equiv (\varphi^1, \ldots, \varphi^\mu)$, where $\varphi^i = f^{j_i}$, $j_k \neq j_l$ if $k \neq l$, the sets $S(A_\mu) \equiv \{X \in b(P)\colon \Phi_\mu(X) / |\Phi_\mu(X)| = A_\mu\} \cap H_\mu$ and $b(P) - S(A_\mu)$ consist of a finite number of connected subsets of $b(P)$, for all vectors $A_\mu = (\pm 1, 0, \ldots, 0)$, $(0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)$ and for all $\mu - 1$-dimensional simplexes H_μ on $b(P)$. This assumption enables us to eliminate "wild" cases of F such as when n-1 of the components of F have an infinite number of isolated zeros on $b(P)$. Under this assumption, if m is sufficiently large, and $\max_{(j=1,2,\ldots,m;\, 1 \leqq k < l \leqq n)} |Y_k^{(j)} - Y_l^{(j)}|$ is sufficiently small, we show that

$$(1.12)\qquad d(F, \mathscr{D}, \theta) = \frac{1}{2^n n!} \sum_{j=1}^{m} t_j \Delta_n(\operatorname{sgn} F(Y_1^{(j)}), \ldots, \operatorname{sgn} F(Y_n^{(j)}))$$

where the t_j and $Y_1^{(j)}$ are the same as those in (1.11), where $\Delta_n(\cdot)$ is defined as in (1.1), and where $\operatorname{sgn} F = (\operatorname{sgn} f^1, \ldots, \operatorname{sgn} f^n)$. We also describe an algorithm for computing $d(F, P, \theta)$ using the formula (1.12).

Under the assumptions on F in the above paragraph, it is possible to subdivide the boundary $b(P)$ into a finite number of closed, connected region β_{n-1}^i, $i = 1, 2, \ldots, \varkappa$, such that each region β_{n-1}^i is the "sum" of simplexes belonging to the representation (1.11), such that $f^{j_i} \neq 0$ or β_{n-1}^i, and such that $F_{n-1}^{j_i} \equiv (f^1, \ldots, f^{j_i - 1},$

$f^{j_i+1}, \ldots, f^n) \neq \theta_{n-1} \equiv (0, \ldots, 0)$ on the boundary $b(\beta^i_{n-1})$ of β^i_{n-1}. We, show that

$$(1.13)\qquad d(F, P, \theta) = \frac{1}{2n} \sum_{i=1}^{\varkappa} (-1)^{j_i-1}\, d(F^{j_i}_{n-1}, \beta^i_{n-1}, \theta_{n-1}) \operatorname{sgn} f^{j_i}(\beta^i_{n-1}).$$

The inductive degree relation (1.13) plays a key role in our inductive proof of (1.12).

In Section 2 of this paper we set up the oriented regions P and $b(P)$, in Section 3, we state a theorem on the convergence of an algorithm for computing $d(F, P, \theta)$ using (1.12), and in Section 4 we derive (1.13) using (1.4)′ and (1.4) and we use (1.13) to give an inductive proof of the theorem in Section 3. Finally, in Section 5, we illustrate applications of the previously derived results in two-dimensional examples, and we give a novel proof of the Miranda fixed point theorem [7] using (1.13).

By Kronecker's theorem (see e.g. [8]), if $F \neq \theta$ on $b(P)$, and if $d(F, P, \theta) \neq 0$, then the Eq. (1.3) has at least one solution in the interior of P. In a future paper we shall study the approximate solution of the problem (1.3) by starting with a simplex P_0, computing $d(F, P_0, \theta)$ by use of (a variant of) the Algorithm in Section 3, and if $d(F, P_0, \theta) \neq 0$, bisecting P_0 into two simplexes P_{1i} by bisecting the longest edge of P_0, next bisecting P_{1i} into P_{2j}, $j = 1, 2$, if $d(F, P_{1i}, \theta) \neq 0$, etc.

2. The Oriented Region and Boundary

Let $X_0, \ldots, X_q$ denote $q+1$ points in R^n such that the vectors $X_i - X_0$ $(i = 1, 2, \ldots, q)$ are linearly independent. A q-simplex $S^q(X_0, \ldots, X_q)$ in R^n is defined by

$$(2.1)\qquad S^q(X_0, \ldots, X_q) = \{X \in R^n : X = \sum_{i=0}^{q} \lambda_i X_i,\ \lambda_i \geqq 0,\ \sum_{i=0}^{q} \lambda_i = 1\}.$$

The points X_i, $i = 0, 1, \ldots, q$ are called the extreme points of $S^q(X_0, \ldots, X_q)$.

Let us associate an orientation with the n-simplex $S^n(X_0, \ldots, X_n)$ in R^n. We shall say that the n-simplex $S^n(X_0, \ldots, X_n)$ is positively (negatively) oriented in R^n if the determinant

$$(2.2)\qquad \Delta_{n+1}(Z_0, \ldots, Z_n) > 0 \; (< 0)$$

where $Z_i = (1, X_i)$, $i = 0, 1, \ldots, n$, and $X_i = (x_i^1, x_i^2, \ldots, x_i^n)$. Thus, whereas $S^n(X_0, \ldots, X_n)$ defined in (2.1) represents a set of points in R^n, we shall write $[X_0, \ldots, X_n]$ for the oriented simplex. In general, an odd permutation of the points $X_0, \ldots, X_n$ changes the sign of the orientation of $[X_0, \ldots, X_n]$, while an even permutation leaves the sign of the orientation unchanged. Thus if $[Y_0 \ldots Y_n]$ is as oriented simplex for which the points $Y_0, \ldots, Y_n$ are a permutation of the points $X_0, \ldots, X_n$, we shall write

$$(2.3)\qquad [Y_0 \ldots Y_n] = [X_0 \ldots X_n] \quad ([Y_0 \ldots Y_n] = -[X_0 \ldots X_n])$$

if the permutation is even (odd).

As in Cronin [3 p. 6] we define the oriented boundary $b[X_0 \ldots X_n]$ of the oriented simplex $[X_0 \ldots X_n]$ by

$$(2.4)\qquad b[X_0 \ldots X_n] = \sum_{i=0}^{n} (-1)^i [X_0 \ldots X_{i-1} X_{i+1} \ldots X_n],$$

where $[X_0 \ldots X_{i-1}X_{i+1} \ldots X_n]$ in an oriented n-1 simplex in R^n. More generally, if $[X_0 \ldots X_q]$ is an oriented q-simplex embedded in R^n, its oriented boundary is defined by

$$b[X_0 \ldots X_q] = \sum_{i=0}^{q} (-1)^i [X_0 \ldots X_{i-1}X_{i+1} \ldots X_q], \tag{2.5}$$

where we write $[Y_1 \ldots Y_q] = [Z_1 \ldots Z_q]$ ($[Y_1 \ldots Y_q] = -[Z_1 \ldots Z_q]$) if the points $Y_1, \ldots, Y_q$ are an even (odd) permutation of the points $Z_1, \ldots Z_q$.

Lemma 2.1. *Let $[X_0 \ldots X_q]$ be an oriented q-simplex in R_q, with oriented boundary*

$$b[X_0 \ldots X_q] = \sum_{i=0}^{q} (-1)^i [X_0 \ldots X_{i-1}X_{i+1} \ldots X_q]. \tag{2.6}$$

Then

$$\Delta_{q+1}(Z_0, \ldots, Z_q) = \sum_{i=0}^{q} (-1)^i \Delta_q(X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_q). \tag{2.7}$$

where $Z_i = (1, X_i)$, $i = 0, 1, \ldots, q$.

Proof. The proof follows by expansion of the determinant on the left hand side of (2.7) along the first column.

Let P be a polyhedron in R^n defined by the union of m' distinct positively oriented n-simplexes $[X_0^{(1)} \ldots X_n^{(j)}]$, $j = 1, 2, \ldots, m'$, such that:

(a) The n-dimensional volume of the intersection of any two of the n-simplexes is zero;

(b) The interior of P is connected.

We set

$$P = \sum_{j=1}^{m'} [X_0^{(j)} \ldots X_n^{(j)}]. \tag{2.8}$$

The representation (2.5) enables us to represent the boundary of P as

$$\begin{aligned} b(P) &= \sum_{j=1}^{m'} b[X_0^{(j)} \ldots X_n^{(j)}] \\ &= \sum_{j=1}^{m'} \sum_{i=0}^{n} (-1)^i [X_0^{(j)} \ldots X_{i-1}^{(i)} X_{i+1}^{(j)} \ldots X_n^{(j)}] \end{aligned} \tag{2.9}$$

where some cancellation may occur, due to the appearance of both an oriented simplex $[Y_1 \ldots Y_n]$ and its negative, $-[Y_1 \ldots Y_n]$. Due to possible cancellation, the expression for $b(P)$ thus reduces to the form

$$b(P) = \sum_{j=1}^{m} t_j [Y_1^{(j)} \ldots Y_n^{(j)}] \tag{2.10}$$

where $t_j = \pm 1$.

More generally, let

$$P_\mu = \sum_{j=1}^{m'_\mu} [X_0^{(j)} \ldots X_\mu^{(j)}] \tag{2.11}$$

be a connected region represented as a "sum" of oriented μ-simplexes embedded in R^n, such that the intersection of any two of the simplexes has zero μ-dimensional volume, and let

$$(2.12) \qquad b(P_\mu) = \sum_{j=1}^{m_\mu} t_j [Y_1^{(j)} \ldots Y_\mu^{(j)}]$$

denote the oriented boundary of P_μ. Lemma 2.1 and the definition of $b(P_\mu)$ in terms of that of P_μ then yield

Lemma 2.2. *Let G be an arbitrary mapping of R^n into R^μ and set $H \equiv (1, G)$. Then*

$$(2.13) \qquad \sum_{j=1}^{m'_\mu} \Delta_{\mu+1}(H(X_0^{(j)}), \ldots, H(X_\mu^{(j)})) = \sum_{j=1}^{m_\mu} t_j \Delta_\mu(G(Y_1^{(j)}), \ldots, G(Y_\mu^{(j)}))$$

where the $X_i^{(j)}$ are those in (2.11) *and the t_j and $Y_i^{(j)}$ are those in* (2.12).

3. Formula for the Topological Degree

In this section we shall describe a formula for evaluating

$$(3.1) \qquad \begin{aligned} d(f, P, 0) &= \frac{1}{2}\left\{\frac{f(B)}{|f(B)|} - \frac{f(A)}{|f(A)|}\right\}, \quad n = 1, \\ d(F, P, \theta) &= \frac{1}{\Omega_{n-1}} \int_{X(u)\in b(P)} \frac{1}{|F|^n} \Delta_n\left(F, \frac{\partial F}{\partial u^1}, \ldots, \frac{\partial F}{\partial u^{n-1}}\right) du^1 \ldots du^{n-1}, \quad n > 1 \end{aligned}$$

where P and $b(P)$ are defined as in Section 2, and $X(U)$ and $\Delta_n(\cdot)$ as in Eq. (1.4).

Let a be a real number, and let us set

$$(3.2) \qquad \operatorname{sgn} a = \begin{cases} 1 & \text{if} \quad a > 0 \\ 0 & \text{if} \quad a = 0 \\ -1 & \text{if} \quad a < 0. \end{cases}$$

Let us define sgn F by

$$(3.3) \qquad \operatorname{sgn} F = (\operatorname{sgn} f^1, \ldots, \operatorname{sgn} f^n)$$

where $F = (f^1, \ldots, f^n)$ and where the f^i are real. Let $b(P)$ be defined as in Eq. (2.10), and let us set

$$(3.4) \qquad \delta_m(F, P, \theta) = \frac{1}{2^n n!} \sum_{j=1}^{m} t_j \Delta_n(\operatorname{sgn} F(Y_1^{(j)}), \ldots, \operatorname{sgn} F(Y_n^{(j)}))$$

where the t_j and $Y_i^{(j)}$ are the same as those in (2.10).

Let the following assumption be satisfied.

Assumption 3.1. Let $F = (f^1, \ldots, f^n)$ be continuous and real on P, where P is defined as in Eq. (2.8). *Let $b(P)$ be defined as in Eq.* (2.10), *and let $F \neq \theta$ on $b(P)$. Let us assume that for all Φ_μ of the form $\Phi_\mu = (\varphi^1, \ldots, \varphi^\mu)$ where $1 < \mu \leqq n$, $\varphi^i = f^{j_i}$, $j_k \neq j_l$ if $k \neq l$, the sets $S(A_\mu) = \{X \in b(P) \colon \Phi_\mu(X)/|\Phi_\mu(X)| = A_\mu\} \cap H_\mu$ and $b(P) - S(A_\mu)$ consist of a finite number of connected subsets of $b(P)$, for all vectors $A_\mu = (\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)$ and for all $\mu - 1$-dimensional simplexes H_μ on $b(P)$.*

This assumption eliminates "wild" cases of vectors Φ_μ, such as, for example, when n-1 of the components of F vanish simultaneously as an infinite number of isolated points on $b(P)$. More importantly, it insures that $d(F, P, \theta)$ can be computed by means of the following algorithm.

Algorithm 3.2.

1. *Let p be a fixed positive integer.*

2. *Set $\delta = \delta_m(F, P, \theta)$ as defined in* (3.4).

3. *Revise the definition* (2.10) *of of $b(P)$ as follows: For $j = 1, 2, \ldots, m$,*

(a) *locate the longest segment $\overline{Y_k^{(j)} Y_l^{(j)}}$ $(k < l)$ of the oriented simplex $t_j [Y_1^{(j)} \ldots Y_n^{(j)}]$ in* (2.10) *and set $A = (Y_k^{(j)} + Y_l^{(j)})/2$;*

(b) *Replace*[1] *$t_j [Y_1^{(j)} \ldots Y_n^{(j)}]$ and define a new oriented simplex $t_{j+m} [Y_1^{(j+m)} \ldots Y_n^{(j+m)}]$ according to:*

$$(3.5)\qquad \begin{array}{l} t_j [Y_1^{(j)} \ldots Y_k^{(j)} \ldots Y_l^{(j)} \ldots Y_n^{(j)}] \leftarrow t_j [Y_1^{(j)} \ldots A \ldots Y_l^{(j)} \ldots Y_n^{(j)}]. \\ t_{j+m} [Y_1^{(j+m)} \ldots Y_k^{(j+m)} \ldots Y_l^{(j+m)} \ldots Y_n^{(j+m)}] \leftarrow t_j [Y_1^{(j)} \ldots Y_k^{(j)} \ldots A \ldots Y_n^{(j)}]. \end{array}$$

4. *Replace m by $2m$ to get a new decomposition of $b(P)$ of the form* (2.10) *in terms of (twice as many) oriented simplexes.*

5. *Set $e = \delta_m(F, P, \theta)$ as defined in* (3.4) *with the new $b(P)$.*

6. *If $\delta = e =$ integer, go to Step. 7. Otherwise set $\delta = e$ and return to Step 3.*

7. *Replace p by p-1. If the resulting p is positive, return to Step 3. Otherwise print out m, δ.*

It is readily seen that in carrying out Step 3 of the above algorithm we merely replace each n-1-simplex in (2.10) by two new ones whose union is the original n-1-simplex and whose intersection has zero n-1 dimensional volume. Considered as a set of points embedded in R^n, $b(P)$ is therefore left unchanged. Also, it is readily verified that the new simplexes produced by (3.5) have the correct orientations of $b(P)$ with reference to Lemma 2.2, i.e., the same representations of $b(P)$ could have been obtained if we had started with a more refined representation (2.8) of P in terms of suitable positively oriented n-simplexes.

Theorem 3.3. *If Assumption* 3.1 *is satisfied, and if the integer p in Algorithm* 3.2 *is chosen sufficiently large, then Algorithm* 3.2 *prints out finite integers m and δ, where $\delta = d(F, P, \theta)$, and where P is defined as in* (2.8).

The proof of this theorem will be carried out in Section 4.

4. Proof of Theorem 3.3

We divide the proof of Theorem 3.3 into three parts.

The first part, in Section 4.1, describes an inductive definition of "sufficient refinement of $b(P_\mu)$ relative to sgn Φ_μ", and it is shown that this condition is met after Step 3 of Algorithm 3.2 is traversed a sufficient number of times.

Next, in Sec. 4.2, an inductive relationship is derived, involving the degree $d(\Phi_\mu, P_\mu, \theta_\mu)$ and a family of degrees $d(\Phi_{\mu-1}^{ji}, \beta_{\mu-1}^i, \theta_{\mu-1})$ where $\theta_\mu(\theta_{\mu-1})$ is the zero vector in $R^\mu(R^{\mu-1})$, $\Phi_\mu = (\varphi^1, \ldots, \varphi^\mu)$, $\Phi_{\mu-1}^{ji} = (\varphi^1, \ldots, \varphi^{ji-1}, \varphi^{ji+1}, \ldots, \varphi^\mu)$, and the $\beta_{\mu-1}^i$ are suitable connected subsets of $b(P_\mu)$ whose union is $b(P_\mu)$.

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1 In (3.5) the symbol "$\leftarrow$" reads "is replaced by".

In Section 4.3 we use the results of the previous sections to prove by induction that if $b(P_\mu)$ is sufficiently refined relative to sgn F, then the sum on the right hand side of (3.4) is $d(F, P, \theta)$, i.e., we complete the proof of Theorem 3.3.

4.1. *The Sufficient Refinement of* $b(P_\mu)$

Let $\Phi_\mu = (\varphi^1, \dots, \varphi^\mu)$, $1 \leqq \mu \leqq n$ be a vector of real continuous functions defined in the region $\mathscr{D}$ in R^n, and let us set

$$\operatorname{sgn} \Phi_\mu = (\operatorname{sgn} \varphi^1, \dots, \operatorname{sgn} \varphi^\mu). \tag{4.1}$$

Let P_μ be a connected μ-dimensional oriented polyhedron in $\mathscr{D}$. We assume that $b(P_\mu)$ may be represented as a "sum" of oriented μ-simplexes in the form

$$P_\mu = \sum_{j=1}^{m_\mu} [X_0^{(j)} \dots X_\mu^{(j)}] \tag{4.2}$$

where the intersection of any two of the simplexes in the representation (4.2) has zero μ-dimensional volume. We may represent the boundary of P_μ, by means of (2.5), in the form

$$b(P_\mu) = \sum_{j=1}^{m_\mu} t_j [Y_1^{(j)} \dots Y_\mu^{(j)}]. \tag{4.3}$$

For example, when $\mu = 1$,

$$P_\mu = P_1 = \sum_{i=0}^{m-1} [X_i X_{i+1}] \tag{4.4}$$

so that

$$b(P_1) = \sum_{i=0}^{m-1} \{[X_{i+1}] - [X_i]\} = [X_m] - [X_0]. \tag{4.5}$$

Assumption 4.1. On $b(P_\mu)$

$$\Phi_\mu \neq \theta_\mu \equiv (0, \dots, 0). \tag{4.6}$$

Definition 4.2. *If* $\mu = 1$, $b(P_\mu) = b(P_1)$ (Eq. (4.5)) *is said to be sufficiently refined relative to* sgn $\Phi_1 = \operatorname{sgn} \varphi^1$ *if* $\varphi^1(X_m)\varphi^1(X_0) \neq 0$.

Definition 4.3. *Let* $\mu > 1$. *A* Q_μ — *set is a connected subset of* $b(P_\mu)$ *consisting of the set of all points* $Q \in b(P_\mu)$ *such that*

$$\Phi_\mu(Q)/|\Phi_\mu(Q)| = \pm Z_i^\mu \tag{4.7}$$

for some $i = 1, 2, \dots, \mu$, *where*

$$Z_i^\mu = (\delta_{i1}, \dots, \delta_{i\mu}), \tag{4.8}$$

where $\delta_{ij} = 1$ *if* $i = j$ *and* 0 *if* $i \neq j$, *and where* $|\Phi_\mu| = [(\varphi^1)^2 + \dots + (\varphi^\mu)^2]^{\frac{1}{2}}$.

Definition 4.4. *If* $\mu > 1$, $b(P_\mu)$ *is said to be sufficiently refined relative to* sgn Φ_μ *if* $b(P_\mu)$ *has been subdivided into a finite number of regions*

$$\beta_{\mu-1}^1, \dots, \beta_{\mu-1}^{\varkappa_\mu}$$

consisting of the union of oriented simplexes of $b(P_\mu)$, *such that*:

(a) *The interiors of the regions* $\beta^s_{\mu-1}(s=1,2,\ldots,\varkappa_\mu)$ *are disjoint and each region* $\beta^s_{\mu-1}$ *is connected;*

(b) *At least one of the functions* $\varphi^1,\ldots,\varphi^\mu$, *say* $\varphi^{j_r}\neq 0$ *on each region* $\beta^r_{\mu-1}$;

(c) *Unless* $b(P_\mu)$ *is itself a* Q^μ*-set, each* Q^μ*-set lies in the interior of a region* $\beta^r_{\mu-1}$, *and each region* $\beta^r_{\mu-1}$ *contains at most one* Q^μ*-set;*

(d) *If* $\varphi^{j_r}\neq 0$ *on* $\beta^r_{\mu-1}$, *then* $b(\beta^r_{\mu-1})$ *the boundary of* $\beta^r_{\mu-1}$, *is sufficiently refined relative to* sgn $\Phi^{j_r}_{\mu-1}$, *where* $\Phi^{j_r}_{\mu-1}=(\varphi^1,\ldots,\varphi^{j_r-1},\varphi^{j_r-1},\ldots,\varphi^\mu)$.

Theorem 4.5. *Let* Assumption 3.1 *be satisfied. There exists an integer* $\nu\geqq 0$ *depending only on F and the original definition* (2.10) *of* $b(P)$, *such that if* Step 3 *of* Algorithm 3.2 *is carried out* ν *times, the resulting representation of* $b(P)$ *becomes sufficiently refined relative to* sgn F.

Proof. There is nothing to prove for the case $\mu=1$, since by Assumption 4.1, $\varphi^1(X_0)\,\varphi^1(X_m)\neq 0$.

Next, let $1<\mu\leqq n$, let $\Phi_\mu=(\varphi^1,\ldots,\varphi^\mu)$ where $\varphi^i=f^{j_i}$, $i=1,2,\ldots,\mu$, $j_k\neq j_l$ if $k\neq l$, and let $\Phi_\mu\neq\theta_\mu$ on $b(P_\mu)$ where $b(P_\mu)$ is defined as in (4.3). Let us make the following induction hypothesis for $r=1,2,\ldots,\mu-1$:

> H: $b(P_r)$ defined as in (4.3) becomes sufficiently refined relative to sgn Φ_r after Step 3 of Algorithm 3.2 is carried out ν times, where $\Phi_r=(\varphi^1,\ldots,\varphi^r)$, $\varphi^i=f^{j_i}$, $i=1,2,\ldots,r$, $j_0\neq j_l$ if $k\neq l$ and where $\Phi_r\neq\theta_r$ on $b(P_r)$.

We proceed to the case $r=\mu$ under the hypothesis H. Let P_μ and $b(P_\mu)$ be defined as in (4.2) and (4.3) respectively. Consider a Q^μ-set which is a proper subset of $b(P_\mu)$. Since $\mu-1$ of the functions $\varphi^1,\ldots,\varphi^\mu$ are zero on this Q^μ-set, one of these, say φ^{j_ρ}, is not zero there. Hence there exist two closed, connected, $\mu-1$ dimensional regions S_1 and S_2 on $b(P_\mu)$ such that the Q^μ-set is interior to S_1 and S_1 is interior to S_2, such that $\varphi^{j_\rho}\neq 0$ on S_2, and such that S_2 does not contain any other Q^μ-set. Let $\eta>0$ be the shortest distance between the boundaries of S_1 and S_2. Here we may assume that $\eta>0$ since by Assumption 3.1 the complement in $b(P_\mu)$ of any Q_μ-set consists of at most a finite number of connected subsets of $b(P_\mu)$. Assume that Step 3 of Algorithm 3.2 is traversed a sufficient number of times so that the diameter (i.e. the longest line segment connecting any two points of a simplex) of all the simplexes is at most $\eta/3$. We can then choose a connected subset of $\beta^\varrho_{\mu-1}$ of $b(P_\mu)$ consisting of a finite number of simplexes such that $S_1\subset\beta^\varrho_{\mu-1}\subset S_2$. By continuing the process of refinement as described in Step 3 of Algorithm 3.2 if necessary, it is possible, by the induction hypothesis H, to make $b(\beta^\varrho_{\mu-1})$ sufficiently refined relative to sgn $\Phi^{j_\rho}_{\mu-1}$, where $\Phi^{j_\rho}_{\mu-1}=(\varphi^1,\ldots,\varphi^{j_\rho-1},\varphi^{j_\rho+1},\ldots,\varphi^\mu)$.

By Assumption 3.1, there are at most a finite number of Q^μ-sets an $b(P_\mu)$. Thus the above process can clearly be carried out for every Q^μ-set on $b(P_\mu)$: corresponding to each Q^μ-set on $b(P_\mu)$ there exists a connected subset $\beta^\varrho_{\mu-1}$ constructed as above, on which $\varphi^{j_\rho}\neq 0$, such that $b(\beta^\varrho_{\mu-1})$ is sufficiently refined relative to sgn $\Phi^{j_\rho}_{\mu-1}$, $\varrho=1,2,\ldots,\varkappa'_\mu$, and such-that the interiors of the sets sets $\beta^\varrho_{\mu-1}$ are disjoint subsets of $b(P_\mu)$.

Consider now an arbitrary $\mu-1$-simplex $[Z^{\mu-1}]$ on $b(P_\mu)$ which is not in any of the sets $\beta^\varrho_{\mu-1}$, $\varrho=1,2,\ldots,\varkappa'_\mu$. After Step 3 of Algorithm 3.2 is traversed sufficiently often, at least one of the components φ^{j_z} of Φ_μ must be non-zero on every such $[Z^{\mu-1}]$, for otherwise, as a consequence of the diameter of the simplexes going to zero under sufficient refinement, there would exist a point X on $b(P_\mu)$ such that $\Phi_\mu(X) = \theta_\mu$, which is not possible by Assumption 4.1. Furthermore, none of the simplexes $[Z^{\mu-1}]$ can contain a Q^μ-set. Hence after Step 3 of Algorithm 3.2 is traversed sufficiently often, the boundary, $b[Z^{\mu-1}]$ of each oriented simplex $[Z^{\mu-1}]$ becomes sufficiently refined relative to sgn $(\varphi^1, \ldots, \varphi^{j_s-1}, \varphi^{j_s+1}, \ldots, \varphi^\mu)$, where $\varphi^{j_s} \neq 0$ on $[Z^{\mu-1}]$.

The statement of Theorem 4.5 thus follows by induction.

Remark 4.6. It follows at once for the case $n=2$, that if P_2 is simply connected, then

$$b(P_2) = \sum_{j=0}^{m-1} [X_j X_{j+1}], \quad \text{where } X_m = X_0,$$

is sufficiently refined relative to $\Phi_2 = (\varphi^1, \varphi^2)$ if at most one of φ^1 and φ^2 changes sign at most once on each line segment $\overline{X_j X_{j+1}}$.

4.2. *An Inductive Degree Relation*

Let us consider a μ-dimensional connected, oriented region $\mathscr{D}_\mu^n$ embedded in R^n, where $1 < \mu \leqq n$. Let $\Phi_\mu = (\varphi^1, \ldots, \varphi^\mu)$ where $\varphi^i = \varphi^i(x^1, \ldots, x^n)$ is a vector of real continuous functions defined in an n-dimensional region $\mathscr{D}_n^n \supset \mathscr{D}_\mu^n$.

Assumption 4.7. *Let Φ_μ and $\mathscr{D}_\mu^n$ be defined as above, let $\Phi_\mu \neq \theta_\mu$ on the oriented boundary $b(\mathscr{D}_\mu^n)$ of $\mathscr{D}_\mu^n$, and let $b(\mathscr{D}_\mu^n)$ be subdivided into a finite number of closed, connected $\mu-1$-dimensional oriented subsets $\beta_{\mu-1}^i$, $i=1, 2, \ldots, \varkappa_\mu$,* i.e., $b(\mathscr{D}_\mu^n) = \sum_{i=1}^{\varkappa_\mu} \beta_{\mu-1}^i$, *such that:*

(i) *at least one of the functions $\varphi^1, \ldots, \varphi^\mu$, say φ^{j_i}, $\neq 0$ on $\beta_{\mu-1}^i$;*

(ii) $\Phi_{\mu-1}^{j_i} \equiv (\varphi^1, \ldots, \varphi^{j_i-1}, \varphi^{j_i+1}, \ldots, \varphi^\mu) \neq \theta_{\mu-1}$.

on the oriented boundary $b(\beta_{\mu-1}^i)$ of $\beta_{\mu-1}^i$.

We shall prove

Theorem 4.8. *If* Assumption 4.7 *is satisfied, then*

$$d(\Phi_\mu, \mathscr{D}_\mu^n, \theta_\mu) = \frac{1}{2\mu} \sum_{i=1}^{\varkappa_\mu} (-1)^{j_i-1} d(\Phi_{\mu-1}^{j_i}, \beta_{\mu-1}^i, \theta_{\mu-1}). \tag{4.9}$$

Proof. Let us form a homeomorphic parametrization $X = (x^1, \ldots, x^n)$ as a function of $U_{\mu-1} = (u^1, \ldots, u^{\mu-1})$ such that X: $\mathscr{D}_{\mu-1}^\mu \to b(\mathscr{D}_\mu^n)$ where $\mathscr{D}_{\mu-1}^\mu$ is the boundary of a region $\mathscr{D}_\mu^\mu$ in R^μ, and such that the function Ψ_μ which is continuous on $\mathscr{D}_\mu^\mu$ and defined on the boundary $\mathscr{D}_{\mu-1}^\mu$ of $\mathscr{D}_\mu^\mu$ by $\Psi_\mu(U_{\mu-1}) = \Phi_\mu(X(U_{\mu-1}))$ satisfies $d(\Psi_\mu, \mathscr{D}_\mu^\mu, \theta_\mu) = d(\Phi_\mu, \mathscr{D}_\mu^n, \theta_\mu)$. That this is possible follows from [1, p. 475]. Consider a subset $\beta_{\mu-1}^i$ of $b(\mathscr{D}_\mu^n)$ on which $\varphi^{j_i} \neq 0$ and such that $d(\Phi_{\mu-1}^{j_i}, \beta_{\mu-1}^i, \theta_{\mu-1}) \neq 0$, where $\Phi_{\mu-1}^{j_i}$ is defined as in Assumption 4.7. We shall assume without loss of generality that the points W on $\beta_{\mu-1}$ at which $\Phi_{\mu-1}^{j_i} = \theta_{\mu-1}$ are isolated and such that

$$\Delta_{\mu-1}\left(\frac{\partial \Phi_{\mu-1}^{j_i}}{\partial u^1}, \ldots, \frac{\partial \Phi_{\mu-1}^{j_i}}{\partial u^{\mu-1}}\right) \neq 0 \tag{4.10}$$

at these points. For if this is not the case, we can replace the φ^i by suitable C^1 approximation φ^i_* which leave $d(\Phi_\mu, \mathcal{D}^n_\mu, \theta_\mu)$ and $d(\Phi^{ji}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$, as well as the conditions of Assumption 4.7 unchanged, and for which (4.10) is satisfied (see [2, p. 23]). (In (4.10) and in what follows, $\Phi_\mu = \Phi_\mu(X(U_{\mu-1}))$, i.e., for simplicity, we write Φ in place of Ψ.) At these points W,

$$(4.11) \qquad \frac{\Phi_\mu}{|\Phi_\mu|} = Z^\mu_{ji} \operatorname{sgn} \varphi^{ji},$$

where Z^μ_{ji} is defined as in (4.8). Furthermore, at these points W,

$$(4.12) \quad \Delta_\mu\left(\Phi_\mu, \frac{\partial \Phi_\mu}{\partial u^1}, \ldots, \frac{\partial \Phi_\mu}{\partial u^{\mu-1}}\right) = (-1)^{j_i-1} \Delta_{\mu-1}\left(\frac{\partial \Phi_\mu}{\partial u^1}, \ldots, \frac{\partial \Phi_\mu}{\partial u^{\mu-1}}\right) \varphi^{ji}.$$

where (4.12) follows in view of (4.11), by expanding the determinant in (4.12) about the first row.

In view of the representation (3.1) and Krasnosel'skiĭ [5, p. 79], if ν_+ (ν_-) denotes the number of points W or $b(\mathcal{D}^n_\mu)$ at which $\Phi^{ji}_{\mu-1} = \theta_{\mu-1}$, at which sgn $\varphi^{ji} = t$, and at which

$$(4.13) \qquad \Delta_\mu\left(\Phi_\mu, \frac{\partial \Phi_\mu}{\partial u^1}, \ldots, \frac{\partial \Phi_\mu}{\partial u^{\mu-1}}\right) > 0 \quad (<0),$$

where j is fixed in the range $1 \leqq j_i \leqq \mu$ and t is either $+1$ or -1, then

$$(4.14) \qquad d(\Phi_\mu, \mathcal{D}^n_\mu, \theta_\mu) = \nu_+ - \nu_- .$$

Corresponding to an integer s in the range $1 \leqq s \leqq \mu$, let $J^+(s)$ denote the subset of the integers $\{1, 2, \ldots, \varkappa_\mu\}$ such that if $i \in J^+(s)$ then $\varphi^s \neq 0$ and sgn $\varphi^s = 1$ on $\beta^i_{\mu-1}$. Then it follows from our remarks of (4.12), (4.13) and (4.14), that

$$(4.15) \qquad d(\Phi_\mu, \mathcal{D}^n_\mu, \theta_\mu) = (-1)^{\;-1} \sum_{i \in J^+(s)} d(\Phi^s_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$$

where per usual, the right hand side of (4.15) is zero if $J^+(s)$ is the null set. Similarly, if $J^-(s)$ denotes the subset of integers $\{1, 2, \ldots, \varkappa_\mu\}$ such that if $i \in J^-(s)$ then $\varphi^s \neq 0$ and sgn $\varphi^s = -1$ on $\beta^i_{\mu-1}$, then

$$(4.16) \qquad d(\Phi_\mu, \mathcal{D}^n_\mu, \theta_\mu) = -(-1)^{s-1} \sum_{i \in J^-(s)} d(\Phi^s_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}).$$

Hence if we set $J(s) = J^+(s) \cup J^-(s)$, then

$$(4.17) \quad 2d(\Phi_\mu, \mathcal{D}^n_\mu, \theta_\mu) = \sum_{i \in J(s)} (-1)^{s-1} d(\Phi^s_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) \operatorname{sgn} \varphi^s(\beta^i_{\mu-1}).$$

Summing (4.17) over $s = 1, 2, \ldots, \mu$, we get

$$(4.18) \quad 2\mu\, d(\Phi_\mu, \mathcal{D}^\nu_\mu, \theta_\mu) = \sum_{s=1}^{\mu} \sum_{i \in J(s)} (-1)^{s-1} d(\Phi^s_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) \operatorname{sgn} \varphi^s(\beta^i_{\mu-1}).$$

In (4.18) we note that if i is in two (or more) sets $J(s_1)$ and $J(s_2)$, where s_1 and s_2 are integers such that $1 \leqq s_1 < s_2 \leqq \mu$, then two distinct functions φ^{s_1} and φ^{s_2} are non-zero on $\beta^i_{\mu-1}$, and hence $d(\Phi^s_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) = 0$ for $s = s_1$ and also for $s = s_2$. We may therefore delete that particular i in the summation on the right hand side of (4.18), or include it only once. Hence in the notation of Assumption 4.7;

$$(4.19) \qquad 2\mu\, d(\Phi_\mu, \mathcal{D}^n_\mu, \theta_\mu) = \sum_{i=1}^{\varkappa_\mu} (-1)^{j_i-1} d(\Phi^{ji}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}).$$

This completes the proof of Theorem 4.8.

4.3. Completion of Proof of Theorem 3.3

We shall prove by induction that if $b(P)$ is sufficiently refined relative to sgn F, then $\delta_m(F, P, \theta) = d(F, P, \theta)$, where δ_m is given in (3.4).

On comparing the first of (3.1) with (3.4), we conclude, under the assumption that $F \neq \theta$ on $b(P)$, that $\delta_m(F, P, \theta) = d(F, P, \theta)$ for $n = 1$.

Let us now assume that $\delta_m(F, P, \theta) = d(F, P, \theta)$ whenever $b(P)$ is sufficiently refined relative to sgn F, for $n = 1, 2, \ldots, \mu - 1$, where $\mu > 1$, and let us prove it true for $n = \mu$.

Let us assume that the sets $\beta^i_{\mu-1}$ are defined as in Def. 4.4, and let us consider the sum

$$\sigma_i \equiv \frac{1}{2^\mu \mu!} \sum_{j \in J_i} t_j \Delta_\mu (\operatorname{sgn} \Phi_\mu(Y_1^{(j)}), \ldots, \operatorname{sgn} \Phi_\mu(Y_\mu^{(j)})) \tag{4.20}$$

where the sets J_i are defined such that

$$\beta^i_{\mu-1} = \sum_{j \in J_i} t_j [Y_1^{(j)} \ldots Y_\mu^{(j)}]. \tag{4.21}$$

By Assumption 4.7, $\varphi^{ji} \neq 0$ on $\beta^i_{\mu-1}$; hence defining H by $H \equiv (1, \Phi^{ji}_{\mu-1})$, where $\Phi^{ji}_{\mu-1}$ is defined as in Assumption 4.7, and interchanging the jth and first column of each determinant on the right hand side of (4.20), we get

$$\sigma_i = \frac{(-1)^{ji-1} \operatorname{sgn} \varphi^{ji}(\beta^i_{\mu-1})}{2^\mu \mu!} \sum_{j \in J_i} t_j \Delta_\mu (\operatorname{sgn} H(Y_1^{(j)}), \ldots, \operatorname{sgn} H(Y_\mu^{(j)})). \tag{4.22}$$

Let us now recall Lemma 2.2 in order to replace the sum in (4.22) by one involving determinants of order $\mu - 1$. That is, if we define the oriented boundary

$$b(\beta^i_{\mu-1}) \equiv \sum_{j \in K_i} \tau_j [Z_1^{(j)} \ldots Z_{\mu-1}^{(j)}] \tag{4.23}$$

by the procedure of Section 2, then by Lemma 2.2, (4.22) becomes

$$\sigma_i = \frac{(-1)^{ji} \operatorname{sgn} \varphi^{ji}(\beta^i_{\mu-1})}{2^\mu \mu!} \sum_{j \in K_i} \tau_j \Delta_{\mu-1} (\operatorname{sgn} \Phi^{ji}_{\mu-1}(Z_1^{(j)}), \ldots, \operatorname{sgn} \Phi^{ji}_{\mu-1}(Z_{\mu-1}^{(j)})). \tag{4.24}$$

By the induction hypothesis, $d(\Phi^{ji}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$ can be computed by means of formula (3.4) under the assumption that $b(\beta^i_{\mu-1})$ is sufficiently refined relative to sgn $\Phi^{ji}_{\mu-1}$. By Theorem 4.5, we may assume that $b(\beta^i_{\mu-1})$ is sufficiently refined relative to sgn $\Phi^{ji}_{\mu-1}$. We find that by comparing (3.4) with $n = \mu - 1$ with (4.24) that

$$\begin{aligned}\sigma_i &= \frac{(-1)^{ji-1} \operatorname{sgn} \varphi^{ji}(\beta^i_{\mu-1})\, 2^{\mu-1}(\mu-1)!}{2^\mu \mu!} d(\Phi^{ji}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) \\ &= \frac{(-1)^{ji-1} \operatorname{sgn} \varphi^{ji}(\beta^i_{\mu-1})}{2\mu} d(\Phi^{ji}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}).\end{aligned} \tag{4.25}$$

Summing over $i = 1, 2, \ldots, \varkappa_\mu$, and using Theorem 4.8., we find that if $b(P_\mu)$ is sufficiently refined relative to sgn Φ_μ, then

$$\begin{aligned}d(\Phi_\mu, P_\mu, \theta_\mu) &= \sum_{i=1}^{\varkappa_\mu} \sigma_i \\ &= \frac{1}{2^\mu \mu!} \sum_{j=1}^{m_\mu} t_j \Delta_\mu (\operatorname{sgn} \Phi_\mu(Y_1^{(j)}), \ldots, \operatorname{sgn} \Phi_\mu(Y_\mu^{(j)})).\end{aligned} \tag{4.26}$$

where the sum on the extreme right of (4.26) extends over all the oriented simplexes $t_j[Y_1^{(j)} \ldots Y_\mu^{(j)}]$ $(j=1, 2, \ldots, m_\mu)$ of $b(P_\mu)$.

We conclude, therefore, that (3.4) yields $d(F, P, \theta)$ whenever $b(P)$ is sufficiently refined relative to sgn F.

This completes the proof of Theorem 3.3.

5. Examples

5.1. *The Case* $n=2$

For this case, let $P_n = P_2$ take the form

$$P_2 = \sum_{i=1}^{m} [X_i Y_i Z_i]. \tag{5.1}$$

It is assumed that if the points X_i, Y_i and Z_i all lie in a plane where they take the form $X_i=(x_i^1, x_i^2)$, $Y_i=(y_i^1, y_i^2)$, $Z_i=(z_i^1, z_i^2)$, then the determinants

$$\begin{vmatrix} 1 & x_i^1 & x_i^2 \\ 1 & y_i^1 & y_i^2 \\ 1 & z_i^1 & z_i^2 \end{vmatrix} > 0 \quad \text{for} \quad i=1, 2, \ldots, m'.$$

In general, let P_2 be simply-connected and suitably oriented, so that $b(P_2)$ can be expressed in the form

$$b(P_2) = \sum_{i=1}^{m-1} [Y_i Y_{i+1}] \tag{5.2}$$

where $Y_m = Y_1$. Let $\Phi_2 = (\varphi^1, \varphi^2)$ be continuous on P_2, let $\Phi_2 \neq \theta_2$ on $b(P_2)$, and let the product $\varphi^1 \varphi^2$ change sign at most once on each segment $[Y^i Y^{i+1}]$ of $b(P_2)$. In this case it follows by Remark 4.6, Theorem 3.3 and Eq. (3.4) that

$$d(\Phi_2, P_2, \theta_2) = \frac{1}{8} \sum_{i=1}^{m-1} \begin{vmatrix} \operatorname{sgn} \varphi^1(Y_i) & \operatorname{sgn} \varphi^2(Y_i) \\ \operatorname{sgn} \varphi^1(Y_{i+1}) & \operatorname{sgn} \varphi^2(Y_{i+1}) \end{vmatrix} \tag{5.3}$$

We shall illustrate the application of this formula in greater detail is the following example.

5.2. *An Explicit Example for the Case* $n=2$

Let us apply the formula (3.4) to show that the system of equations

$$f(x, y) \equiv x^2 - 4y = 0, \quad g(x, y) \equiv y^2 - 2x + 4y = 0 \tag{5.4}$$

has at least one solution in the domain

$$P_2 = \{x, y): |x| \leqq 2, \quad |y| \leqq 1/4\} \tag{5.5}$$

notice that the system (5.7) has the solution $(x, y) = (0, 0)$ in P_2.

Solution. Let us set (see Fig. 5.1)

$$P_2 = [X_1 X_2 X_3] + [X_1 X_3 X_4] + [X_1 X_4 X_7] + [X_4 X_5 X_7] + [X_5 X_6 X_7], \tag{5.6}$$

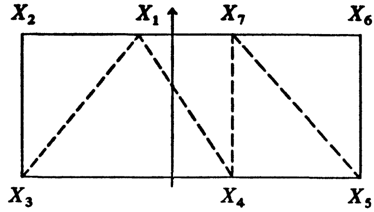

Fig. 5.1. Triangulation of the region P_2

where the $X_i = (x_i, y_i)$ are given in Table 5.2. Using the formula (2.4), we get

$$
\begin{aligned}
b(P_2) = {} & [X_2X_3] - [X_1X_3] + [X_1X_2] \\
& + [X_3X_4] - [X_1X_4] + [X_1X_3] \\
& + [X_4X_7] - [X_1X_7] + [X_1X_4] \\
& + [X_5X_7] - [X_4X_7] + [X_4X_5] \\
& + [X_6X_7] - [X_5X_7] + [X_5X_6]
\end{aligned}
\tag{5.7}
$$

which, due to cancellation, becomes

$$
\begin{aligned}
b(P_2) = {} & [X_1X_2] + [X_2X_3] + [X_3X_4] + [X_4X_5] + [X_5X_6] \\
& + [X_6X_7] + [X_7X_1].
\end{aligned}
\tag{5.8}
$$

Using (5.3), we set

$$\delta_7(F_2, P_2, \theta_2) = \frac{1}{8} \sum_{j=1}^{7} b_j, \tag{5.9}$$

where $F_2 = (f, g)$, and where

$$b_j = \begin{vmatrix} \operatorname{sgn} f(X_j) & \operatorname{sgn} g(X_j) \\ \operatorname{sgn} f(X_{j+1}) & \operatorname{sgn} g(X_{j+1}) \end{vmatrix}. \tag{5.10}$$

The results $f_j = f(X_j)$, $g_j = g(X_j)$, sgn f_j, sgn g_i and b_j are given in Table 5.2, so that $\sum_{j=1}^{7} b_j = -8$, and $\delta_7(F_2, P_2, \theta_2) = -1$. It may be shown that even if we were to add more points to $b(P_2)$ in order to get a more refined representation, we would get the same result, $\delta_m(F_2, P_2, \theta_2) = -1$. Hence $d(F_2, P_2, \theta_2) = -1$, and by Kronecker's theorem [8, p. 161] there exists at least one point X in interior of P_2 such that $F_2(X) = \theta_2$.

Table 5.2. Zeros in P_2 of $(f, g) = (0, 0)$

j	x_j	y_j	f_j	g_j	sgn f_j	sgn g_j	b_j
1	−0.5	0.25	−0.75	2.0625	−1	1	−2
2	−2.0	0.25	3.0	5.0625	1	1	0
3	−2.0	−0.25	5.0	3.0625	1	1	−2
4	0.75	−0.25	1.5625	−2.4375	1	−1	0
5	2.0	−0.25	5.0	−4.9375	1	−1	0
6	2.0	0.25	3.0	−2.9375	1	−1	−2
7	0.75	0.25	−0.4375	−0.4375	−1	−1	−2

Notice that it does not suffice to take only the points X^2, X^3, X^5 and X^6, i.e., the corner points of the rectangular region P_2, although the points X^3, X^4 and X^5 could, for example, have been dropped.

5.3. *An Alternative Proof of the Miranda Fixed Point Theorem*

Let us apply Theorem 4.8 to obtain a novel proof of the Miranda fixed point theorem. We shall prove

Theorem 5.1. *Let $F_n=(f^1, \dots, f^n)$ be a vector function which is real and continuous on $P_n=\{X_n: (x, \dots, x^n): -1 \leqq x^i \leqq 1, i=1, 2, \dots, n\}$, and such that*

$$(5.11)\qquad \begin{aligned} &f^1(-1, x^2, \dots, x^n)<0<f(1, x^2, \dots, x^n)\\ &f^2(x^1, -1, x^3, \dots, x^n)<0<f^2(x^1, 1, x^3, \dots, x^n)\\ &\vdots\\ &f^n(x^1, \dots, x^{n-1}, -1)<0<f^n(x^1, \dots, x^{n-1}, 1), \end{aligned}$$

where the inequalities involving f^i in (5.11) are valid for $-1 \leqq x^j \leqq 1, j \neq i$. Then there exists a point Y interior to P_n such that $F_n(Y)=\theta_n$.

Proof. We define the sets $\beta^i_{\mu-1}$ of Theorem 4.8 by

$$(5.12)\qquad \begin{aligned} &\beta^i_{\mu-1}=\{X_\mu: x^i=1, -1 \leqq x^j \leqq 1, j \neq i\}, i=1, 2, \dots, \mu\\ &\beta^i_{\mu-1}=\{X_\mu: x^{i-\mu}=-1, -1 \leqq x^j \leqq 1, j \neq i-\mu\}, i=\mu+1, \mu+2, \dots, 2\mu \end{aligned}$$

where $X_\mu=(x^1, \dots, x^\mu)$. Then

$$(5.13)\qquad b(P_\mu)=\sum_{i=1}^{2\mu} \beta^i_{\mu-1}.$$

We shall prove that

$$(5.14)\qquad d(F_n, P_n, \theta_n)=d(I_n, P_n, \theta_n)=1, \qquad n=1, 2, \dots.$$

where "I_n" in $d(I_n, P_n, \theta_n)$ stands for the identity mapping of p_n. Under the assumptions (5.11), the assertion (5.14) is clearly satisfied for the case $n=1$. Let us therefore assume that it is satisfied for the case (s) $n=1, 2, \dots, \mu-1$, where $\mu>1$, and prove that under the assumptions of Theorem 5.1, Eq. (5.14) also follows for $n=\mu$. To this end we define $X^i_{\mu-1}$, the identity mapping $I^i_{\mu-1}$ and the mapping $F^i_{\mu-1}$ by

$$(5.15)\qquad \left.\begin{aligned} &X^i_{\mu-1}=(x^1, \dots, x^{i+1}, x^{i+1}, \dots, x^\mu)\\ &I^i_{\mu-1}(X^i_{\mu-1})=X^i_{\mu-1}\\ &F^i_{\mu-1}=(f^1, \dots, f^{i-1}, f^{i+1}, \dots, f^\mu) \end{aligned}\right\} \quad i=1, 2, \dots, \mu$$

Clearly $I^i_{\mu-1}\neq\theta_{\mu-1}$, $F^i_{\mu-1}\neq\theta_{\mu-1}$ on $b(\beta^i_{\mu-1})$ for $i=1, 2, \dots, \mu$, and $I^{\mu-i}_{\mu-1}\neq\theta_{\mu-1}$, $F^{\mu-i}_{\mu-1}\neq\theta_{\mu-1}$ on $b(\beta^i_{\mu-1}$ for $i=\mu+1, \dots, 2\mu$. Thus the degrees $d(I^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$, $d(F^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$, $i=1, 2, \dots, \mu$ and $d(I^{i-\mu}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$, $d(F^{i-\mu}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})$, $i=\mu+1, \dots, 2\mu$ are well defined. The relations

$$(5.16)\qquad \begin{aligned} &d(I^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})=(-1)^{i-1}, i=1, 2, \dots, \mu\\ &d(I^{i-\mu}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})=-(-1)^{i-1}, i=\mu+1, \dots, 2\mu \end{aligned}$$

are a consequence of Theorem 4.8, since

$$
\begin{aligned}
d(I_\mu, P_\mu, \theta_\mu) = \frac{1}{2\mu}\Bigg\{&\sum_{i=1}^{\mu} (-1)^{i-1} \operatorname{sgn}(1)\, d(I^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) \\
&+ \sum_{i=\mu+1}^{2\mu} (-1)^{i-1} \operatorname{sgn}(-1)\, d(I^{i-\mu}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1})\Bigg\} = 1.
\end{aligned} \tag{5.17}
$$

However, by the induction hypothesis,

$$
\begin{aligned}
&d(F^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) = d(I^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}), \qquad i = 1, 2, \ldots, \mu \\
&d(F^{i-\mu}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) = d(I^{i-\mu}_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}), i = \mu+1, \ldots, 2\mu.
\end{aligned} \tag{5.18}
$$

Since moreover $\operatorname{sgn} f^i(X) = 1$ for $X \in \beta^i_{\mu-1}$, $i = 1, 2, \ldots, \mu$, and $\operatorname{sgn} f^{i-\mu}(X) = -1$ for $X \in \beta^i_{\mu-2}$, $i = \mu+1, \ldots, 2\mu$, another application of Theorem 4.8 yields

$$
\begin{aligned}
d(F_\mu, P_\mu, \theta_\mu) &= \frac{1}{2\mu} \sum_{i=1}^{\mu} (-1)^{i-1} \{d(F^i_{\mu-1}, \beta^i_{\mu-1}, \theta_{\mu-1}) - d(F^i_{\mu-1}, \beta^{\mu+i}_{\mu-1}, \theta_{\mu-1})\} \\
&= \frac{1}{2\mu} \sum_{i=1}^{\mu} 2 = 1
\end{aligned} \tag{5.19}
$$

We have thus proved the assertion (5.14) for $n = 1, 2, \ldots$. Since $d(F_n, P_n, \theta_n) = 1 \neq 0$, it follows by Kronecker's theorem [8, p. 161] that the system $F_n = \theta_n$ has at least one solution in the interior of P_n.

This completes the proof of Theorem 5.1.

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