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Technical Notes

Overcoming Overestimation Characteristic to Classical Interval Analysis

I. Elishakoff* and K. Thakkar*
Florida Atlantic University, Boca Raton, Florida 33431-0991

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Nomenclature

k_1	=	spring constant of the spring attached at the left end of the bar
k_2	=	spring constant of the spring attached at in-between the bar
L	=	length of the bar
M	=	mass of the sphere attached to the bar
m	=	mass of the bar
R	=	discriminant
$\bar{x}$	=	upper-bound value of variable x
$\underline{x}$	=	lower-bound value of variable x
x_{avg}	=	average value of variable x
x_{dev}	=	deviation in the value of variable x
α, β	=	artificial parameters
ω	=	circular frequency of the bar
ω_1, ω_2	=	natural frequencies of the system

I. Introduction

INTERVAL analysis is a widespread tool of analyzing uncertainty in mechanical and aerospace systems. The method was developed by mathematicians since 1960 to get bounds on rounding error and measurement errors in mathematical evaluation of expressions to develop numerical methods with reliable, guaranteed methods. In aerospace applications, it was apparently introduced by few researchers, starting with the paper by Rao and Berke [1], and followed by those of Rao and Majumder [2–4] and Majumder and Rao [5]. Interesting papers by Guo and Du [6], Lew [7], Lew et al. [8], and Hua et al. [9] addressed the problems associated with the classical interval analysis; they note that there

> "are situations where the method has limitations for extended or complicated calculations because of the dependency problem, which is characterized by a cancellation of various sub-parts of the function that cannot be detected by direct use of interval methods. This effect often leads to pessimism and sometimes even drastic overestimation of range enclosure."

Likewise, Makino and Berz [10] stress that, unfortunately, it is not uncommon for the natural interval extension to provide a drastic overestimation for FR. This can be seen, for example, using the simple expression $f(x) = x/(x-1)$, and evaluating its interval extension for $x \in [2, 3]$. Computing the natural interval extension yields $F([2, 3]) = [2, 3]/([2, 3] - 1) = [2, 3]/[1, 2] = [1, 3]$, which is a considerable overestimate of the true range $FR([2, 3]) = [1.5, 2]$. In general, such overestimations may occur when an interval variable appears more than once in an expression. This so-called dependence problem occurs because interval arithmetic essentially treats all occurrences of a variable independently rather than recognizing their dependence. For example, if one rearranges $f(x) = x/(x-1)$, so that x appears only once, $f(x) = 1 + 1/(x-1)$, and then the natural interval extension yields $F([2, 3]) = 1 + 1/([2, 3] - 1) = 1 + 1/[1, 2] = 1 + [0.5, 1] = [1.5, 2]$, which is the same as FR. In fact, it can be shown that, if each variable in an expression appears only once, then the natural interval extension will always yield the true (although, in computational practice, outwardly rounded) range FR. Unfortunately, for most of the functions of interest in phase stability analysis, it is not possible to perform rearrangements that eliminate all but one occurrence of each variable. Elishakoff and Miglis [11] suggested a cure for the mentioned illness, namely, parameterization of intervals. In this Note, we employ their methodology to a simple vibration problem, and contrast it with classical analysis.

II. Classical Interval Analysis

A simple two-degree-of-freedom system is depicted in Fig. 1. The governing differential equations read

$$\frac{3}{4}ML\ddot{\theta}_1 + \left(\frac{3}{2}mL + M\right)\ddot{\theta}_2 - \frac{1}{4}L(3k_1 - k_2)\theta_1 + (k_1 + k_2)\theta_2 = 0 \tag{1}$$

$$\frac{9}{32}L^2(mL + 2M)\ddot{\theta}_1 + \frac{3}{4}ML\ddot{\theta}_2 + \frac{1}{16}L^2(9k_1 + k_2)\theta_1 - \frac{1}{4}L(3k_1 - k_2)\theta_2 = 0 \tag{2}$$

Now, considering harmonic vibrations, we set $\theta_1 = X_1 \sin(\omega t)$, $\theta_2 = X_2 \sin(\omega t)$, leading to the frequency equation

$$\frac{27}{64}L^4\omega^4m^2 + \frac{9}{8}\omega^4ML^3m - \frac{9}{8}k_1\omega^2mL^3 - \frac{3}{8}k_2\omega^2mL^3 - \frac{9}{4}L^2k_1\omega^2M + L^2k_1k_2 - \frac{1}{4}L^2\omega^2Mk_2 = 0 \tag{3}$$

Solving the equation for angular frequency

$$\begin{aligned}\omega_{1,2}^2 =& 8\frac{k_1M}{(mL)(3mL+8M)} + 4\frac{k_1}{(3mL+8M)} + \frac{4}{3}\frac{k_2}{(3mL+8M)} \\ &+ \frac{8}{9}\frac{Mk_2}{(mL)(3mL+8M)} \mp \left(\frac{4}{9}\right)\left[324\frac{k_1^2M^2}{m^2L^2(3mL+8M)^2}\right. \\ &+ 324\frac{k_1^2M}{mL(3mL+8M)^2} - 144\frac{k_1Mk_2}{mL(3mL+8M)^2} \\ &+ 72\frac{k_1M^2k_2}{m^2L^2(3mL+8M)^2} + 81\frac{k_1^2}{(3mL+8M)^2} - 54\frac{k_1k_2}{(3mL+8M)^2} \\ &\left.+ 9\frac{k_2^2}{(3mL+8M)^2} + 12\frac{k_2^2M}{mL(3mL+8M)^2} + 4\frac{M^2k_2^2}{m^2L^2(3mL+8M)^2}\right]^{1/2}\end{aligned} \tag{4}$$

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*Department of Ocean and Mechanical Engineering.

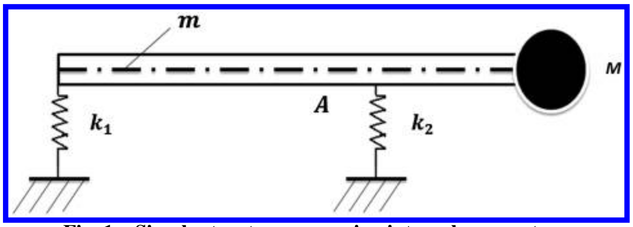

Fig. 1 Simple structure possessing interval parameters.

Each of the parameters in Eq. (3) represents an interval variable, namely

$$M=[\underline{M},\bar{M}],\quad k_1=[\underline{k}_1,\bar{k}_1],\quad k_2=[\underline{k}_2,\bar{k}_2],\quad m=[\underline{m},\bar{m}],\quad L=[\underline{L},\bar{L}] \tag{5}$$

in which the underbar designates a lower value of the variable, and the upper bar denotes the upper value of the variable. We are interested in determining the smallest possible value of the squared natural frequencies $\underline{\omega}_1^2$, $\underline{\omega}_2^2$, as well as the greatest possible values of the natural frequencies $\bar{\omega}_1^2, \bar{\omega}_2^2$. First of all, we determine the lower-bound and upper-bound values of expression $R2$, with R being the discriminant in Eq. (4):

$$R^2=\left[324\frac{k_1^2M^2}{m^2L^2(3mL+8M)^2}+324\frac{k_1^2M}{mL(3mL+8M)^2}-144\frac{k_1Mk_2}{mL(3mL+8M)^2}+72\frac{k_1M^2k_2}{m^2L^2(3mL+8M)^2}+81\frac{k_1^2}{(3mL+8M)^2}-54\frac{k_1k_2}{(3mL+8M)^2}+9\frac{k_2^2}{(3mL+8M)^2}+12\frac{k_2^2M}{mL(3mL+8M)^2}+4\frac{M^2k_2^2}{m^2L^2(3mL+8M)^2}\right] \tag{6}$$

For the external values of the interval

$$R=[\underline{R},\bar{R}] \tag{7}$$

we obtain

$$\bar{R}^2=\left[324\frac{\bar{k}_1^2\bar{M}^2}{\underline{m}^2\underline{L}^2(3\underline{m}\,\underline{L}+8\underline{M})^2}+324\frac{\bar{k}_1^2\bar{M}}{\underline{m}\,\underline{L}\,(3\underline{m}\,\underline{L}+8\underline{M})^2}-144\frac{\underline{k}_1\underline{M}\,\underline{k}_2}{\bar{m}\,\bar{L}\,(3\bar{m}\,\bar{L}+8\bar{M})^2}+72\frac{\bar{k}_1\bar{M}^2\bar{k}_2}{\underline{m}^2\underline{L}^2(3\underline{m}\,\underline{L}+8\underline{M})^2}+81\frac{\bar{k}_1^2}{(3\underline{m}\,\underline{L}+8\underline{M})^2}-54\frac{\underline{k}_1\underline{k}_2}{(3\bar{m}\,\bar{L}+8\bar{M})^2}+9\frac{\bar{k}_2^2}{(3\underline{m}\,\underline{L}+8\underline{M})^2}+12\frac{\bar{k}_2^2\bar{M}}{\underline{m}\,\underline{L}\,(3\underline{m}\,\underline{L}+8\underline{M})^2}+4\frac{\bar{M}^2\bar{k}_2^2}{\underline{m}^2\underline{L}^2(3\underline{m}\,\underline{L}+8\underline{M})^2}\right] \tag{8}$$

$$\underline{R}^2=\left[324\frac{\underline{k}_1^2\underline{M}^2}{\bar{m}^2\bar{L}^2(3\bar{m}\,\bar{L}+8\bar{M})^2}+324\frac{\underline{k}_1^2\underline{M}}{\bar{m}\,\bar{L}\,(3\bar{m}\,\bar{L}+8\bar{M})^2}-144\frac{\bar{k}_1\bar{M}\bar{k}_2}{\underline{m}\,\underline{L}\,(3\underline{m}\,\underline{L}+8\underline{M})^2}+72\frac{\underline{k}_1\underline{M}^2\underline{k}_2}{\bar{m}^2\bar{L}^2(3\bar{m}\,\bar{L}+8\bar{M})^2}+81\frac{\underline{k}_1^2}{(3\bar{m}\,\bar{L}+8\bar{M})^2}-54\frac{\bar{k}_1\bar{k}_2}{(3\underline{m}\,\underline{L}+8\underline{M})^2}+9\frac{\underline{k}_2^2}{(3\bar{m}\,\bar{L}+8\bar{M})^2}+12\frac{\underline{k}_2^2\underline{M}}{\bar{m}\,\bar{L}\,(3\bar{m}\,\bar{L}+8\bar{M})^2}+4\frac{\underline{M}^2\underline{k}_2^2}{\bar{m}^2\bar{L}^2(3\bar{m}\,\bar{L}+8\bar{M})^2}\right] \tag{9}$$

We get the lower and upper bounds of the angular frequencies as follows:

$$[\underline{\omega}_1,\bar{\omega}_1]=\left\{\left(\frac{4}{9}\right)\left[\frac{18\underline{k}_1\underline{M}+9\underline{k}_1\underline{m}\,\underline{L}+3\underline{k}_2\underline{m}\,\underline{L}+2\underline{M}\,\underline{k}_2}{\bar{m}\,\bar{L}(3\bar{m}\,\bar{L}+8\bar{M})}-\bar{R}\right]\times\left(\frac{4}{9}\right)\left[\frac{18\bar{k}_1\bar{M}+9\bar{k}_1\bar{m}\,\bar{L}+3\bar{k}_2\bar{m}\,\bar{L}+2M\bar{k}_2}{\underline{m}\,\underline{L}(3\underline{m}\,\underline{L}+8\underline{M})}-\underline{R}\right]\right\} \tag{10}$$

$$[\underline{\omega}_2,\bar{\omega}_2]=\left\{\left(\frac{4}{9}\right)\left[\frac{18\underline{k}_1\underline{M}+9\underline{k}_1\underline{m}\,\underline{L}+3\underline{k}_2\underline{m}\,\underline{L}+2\underline{M}\,\underline{k}_2}{\bar{m}\,\bar{L}(3\bar{m}\,\bar{L}+8\bar{M})}+\bar{R}\right]\times\left(\frac{4}{9}\right)\left[\frac{18\underline{k}_1\underline{M}+9\underline{k}_1\underline{m}\,\underline{L}+3\underline{k}_2\underline{m}\,\underline{L}+2\underline{M}\,\underline{k}_2}{\bar{m}\,\bar{L}(3\bar{m}\,\bar{L}+8\bar{M})}-\underline{R}\right]\right\} \tag{11}$$

To get an insight, we fix variables at the following values:

$$k_1=[\underline{k}_1,\bar{k}_1]=[50,55]\left(\frac{\mathrm{N}}{\mathrm{m}}\right) \tag{12}$$

$$k_2=[\underline{k}_2,\bar{k}_2]=[100,105]\left(\frac{\mathrm{N}}{\mathrm{m}}\right) \tag{13}$$

$$m=[\underline{m},\bar{m}]=[4,4.125]\left(\frac{\mathrm{kg}}{\mathrm{m}}\right) \tag{14}$$

$$M=[\underline{M},\bar{M}]=[8,8.125](\mathrm{kg}) \tag{15}$$

$$L=[\underline{L},\bar{L}]=[10,10.125](\mathrm{m}) \tag{16}$$

Now, solving the frequency values through the classical interval analysis by using the given parameter values, we will get the following results:

$$\omega_1=[\underline{\omega}_1,\bar{\omega}_1]=[1.040233521,21.33479640] \tag{17}$$

$$\omega_2=[\underline{\omega}_2,\bar{\omega}_2]=[2.309537008,22.60409989] \tag{18}$$

The parameterization method was suggested by Elishakoff and Miglis [11]. According to this method, each interval encountered in a problem at hand is represented via an artificial parameter α. For example, intervals $A=[\underline{a},\bar{a}]$, $B=[\underline{b},\bar{b}]$, $C=[\underline{c},\bar{c}]$ are represented, respectively, as $A=A_{\mathrm{avg}}+A_{\mathrm{dev}}\cdot\sin(\alpha_1)$, $B=B_{\mathrm{avg}}+B_{\mathrm{dev}}\cdot\sin(\alpha_2)$, $C=C_{\mathrm{avg}}+C_{\mathrm{dev}}\cdot\sin(\alpha_3)$, in which A_{avg}, B_{avg}, C_{avg} represent the average bounds, whereas A_{dev}, B_{dev}, C_{dev} represent the respective deviations from the average values. The angles α_1, α_2, α_3 vary between $-\pi/2$ to $\pi/2$.

Likewise, simple representations $A=A_{\mathrm{avg}}+A_{\mathrm{dev}}\cdot t_1$, $B=B_{\mathrm{avg}}+B_{\mathrm{dev}}\cdot t_2$, and $C=C_{\mathrm{avg}}+C_{\mathrm{dev}}\cdot t_3$ are introduced by Elishakoff and Ducreux [12], in which t_1, t_2, t_3 vary between -1 to 1.

Then, the obtained expressions are subjected to extrema determination either analytically or numerically. It should be stressed that, if two intervals, say A and B, are dependent, we set $\alpha_1=\alpha_2$. Likewise, if all three intervals A, B, and C are dependent, we set $\alpha_1=\alpha_2=\alpha_3$ or $t_1=t_2=t_3$. In case that intervals are independent, we are adopting noncoinciding artificial parameters.

Now, by the parameterization method, we first determine the average and deviation values for each variable:

$$k_{1_{\mathrm{avg}}}=\frac{50+55}{2}=52.5,\qquad k_{2_{\mathrm{avg}}}=\frac{100+105}{2}=102.5 \tag{19}$$

$$m_{\text{avg}} = \frac{4 + 4.125}{2} = 4.0625, \qquad M_{\text{avg}} = \frac{8 + 8.125}{2} = 8.0625 \tag{20}$$

$$L_{\text{avg}} = \frac{10 + 10.125}{2} = 10.0625, \quad L_{\text{dev}} = \frac{10.125 - 10}{2} = 0.0625 \tag{21}$$

$$k_{1_{\text{dev}}} = \frac{55 - 50}{2} = 2.5, \qquad k_{2_{\text{dev}}} = \frac{105 - 100}{2} = 2.5 \tag{22}$$

$$k_{1_{\text{dev}}} = \frac{55 - 50}{2} = 2.5, \qquad k_{2_{\text{dev}}} = \frac{105 - 100}{2} = 2.5 \tag{23}$$

$$m_{\text{dev}} = \frac{4.125 - 4}{2} = 0.0625, \qquad M_{\text{dev}} = \frac{8.125 - 8}{2} = 0.0625 \tag{24}$$

Incorporating each variable's deviation and average values in the variable itself, we get

$$k_1 = k_{1_{\text{avg}}} + k_{1_{\text{dev}}} \sin(\alpha) = 52.5 + 2.5 \sin(\alpha) \tag{25}$$

$$k_2 = k_{2_{\text{avg}}} + k_{2_{\text{dev}}} \sin(\alpha) = 102.5 + 2.5 \sin(\alpha) \tag{26}$$

$$m = m_{\text{avg}} + m_{\text{dev}} \sin(\alpha) = 4.0625 + 0.0625 \sin(\alpha) \tag{27}$$

$$M = M_{\text{avg}} + M_{\text{dev}} \sin(\alpha) = 8.0625 + 0.0625 \sin(\alpha) \tag{28}$$

$$L = L_{\text{avg}} + L_{\text{dev}} \sin(\alpha) = 10.0625 + 10.0625 \sin(\alpha) \tag{29}$$

in which the angle α varies between $-\pi/2$ to $\pi/2$. It should be stressed that, in Eqs. (25–30), the same artificial parameter is used for all quantities. One may wonder why the inertial properties and stiffness properties would be interrelated. The answer to this interesting inquiry is as follows. Recently, there is a huge experimental, analytical, and numerical development for functionally graded materials [13,14]. In [15], for example, Adachi and Higuchi consider the case when the material density and the elastic modulus are functionally related.

Solving for the angular frequencies, we get via the Maple commands

$$\text{Maximize}\left[\left(\frac{4}{9}\right)\left(\frac{18k_1M + 9k_1mL + 3k_2mL + 2Mk_2}{mL(3mL + 8M)} - R\right), \; \alpha = -\frac{\pi}{2}\dots\frac{\pi}{2}\right] \tag{30}$$

$$\bar{\omega}_1 = 1.55838376738268, \quad \text{at } \alpha = 1.57079615746863 \tag{31}$$

$$\text{Minimize}\left\{\left(\frac{4}{9}\right)\left[\frac{18k_1M + 9k_1mL + 3k_2mL + 2Mk_2}{mL(3mL + 8M)} - R\right], \; \alpha = -\frac{\pi}{2}\dots\frac{\pi}{2}\right\} \tag{32}$$

$$\underline{\omega}_1 = 1.53138658857916, \quad \text{at } \alpha = -1.57079610937276 \tag{33}$$

$$\text{Maximize}\left\{\left(\frac{4}{9}\right)\left[\frac{18k_1M + 9k_1mL + 3k_2mL + 2Mk_2}{mL(3mL + 8M)} + R\right], \; \alpha = -\frac{\pi}{2}\dots\frac{\pi}{2}\right\} \tag{34}$$

$$\bar{\omega}_2 = 3.31561729264594, \quad \text{at } \alpha = 1.57079624514848 \tag{35}$$

$$\text{Minimize}\left\{\left(\frac{4}{9}\right)\left[\frac{18k_1M + 9k_1mL + 3k_2mL + 2Mk_2}{mL(3mL + 8M)} + R\right], \; \alpha = -\frac{\pi}{2}\dots\frac{\pi}{2}\right\} \tag{36}$$

$$\underline{\omega}_2 = 3.15460374958509, \quad \text{at } \alpha = -1.57079630682627 \tag{37}$$

Table 1 provides a comparison for the angular-frequency values with both approaches.

As is seen, the ratio of the upper bound to the lower bound is in excess of 20 within the classical interval analysis, whereas it equals only 1.018 for the parameterized solution, illustrating supremacy of the parameterization methodology. For the second frequency, the analogous overshooting factor equals 9.79, whereas its counterpart derived via parameterization analysis turns out to be much less than the preceding value, namely, it equals 1.051. Note that, by using the same single artificial parameter in analysis, we used the assumption that we deal with the analog of the functionally graded material [13,14], as in the paper by Adachi and Higuchi [15], in which the assumption was made for functionally graded foam materials about the functional dependence between the material density and elastic modulus.

Let us now consider a more common case when we do not deal with functionally graded material and, in which, therefore, inertial quantities are unrelated with the stiffness characteristics. We treat the case when m and M are totally dependent of each other, but otherwise independent of other parameters. We introduce the artificial parameter β to describe m and M:

$$m = m_{\text{avg}} + m_{\text{dev}} \sin(\beta) = 4.0625 + 0.0625 \sin(\beta) \tag{38}$$

$$M = M_{\text{avg}} + M_{\text{dev}} \sin(\beta) = 8.0625 + 0.0625 \sin(\beta) \tag{39}$$

Performing parameterization and solving for the angular frequencies, via the search of extreme for region $-\pi/2 \le \alpha, \beta \le \pi/2$, expectedly, we obtain broader interval bounds than in the fully dependent as follows:

Table 1 Comparison of the classical interval analysis and the parameterization method

Frequency domain	Classical interval analysis	Parameterization method	Angular frequency
Lower bound	1.0402	1.5314	ω_1
Upper bound	21.3348	1.5584	
Lower bound	2.3095	3.1546	ω_2
Upper bound	22.6041	3.3156	

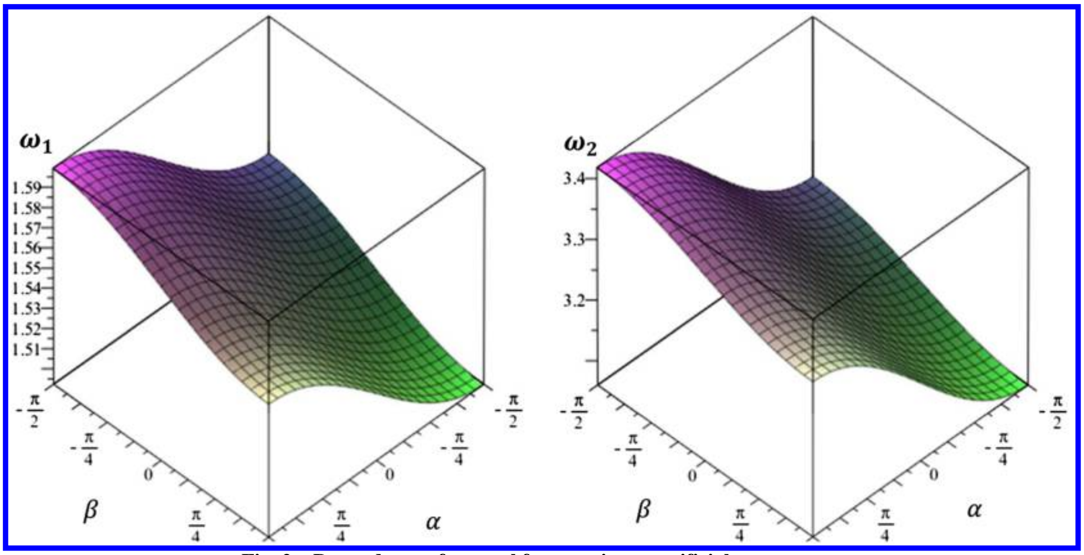

Fig. 2 Dependence of natural frequencies vs artificial parameters.

Table 2 Comparison of natural frequencies driven in various dependency conditions

Frequency domain	Classical interval analysis	Parameterization method ($\alpha = \beta$)	Parameterization method ($\alpha \neq \beta$)	Angular frequency
Lower bound	1.0402	1.5314	1.4924	ω_1
Upper bound	21.3348	1.5584	1.5993	
Lower bound	2.3095	3.1546	3.0600	ω_2
Upper bound	22.6041	3.3156	3.4180	

$$\omega_1 = [\underline{\omega}_1, \bar{\omega}_1] = [1.49236365357720846, 1.59926587853381562]$$

$$\omega_2 = [\underline{\omega}_2, \bar{\omega}_2] = [3.06000328289568912, 3.41798257686977624]$$

The dependence of natural frequencies ω_1 and ω_2 vs α and β is portrayed in Fig. 2. The final comparison of all three approaches is summarized in Table 2. The overshooting ratios turn out to be 1.07 and 1.117, respectively. We conclude that the smallest overshooting ratios are recorded for functionally graded materials.

III. Conclusions

In this Note, we compare the two versions of interval analysis. It turns out that the parameterized version is superior to the classical interval analysis. Two cases were considered for the parameterized analysis: 1) fully dependent intervals, mimicking the case of special, functionally graded materials; and 2) intervals describing material distribution being independent of the rest of the parameters, but fully dependent of each other.

It turns out that, in the latter case, the derived bounds are broader than in the former case. Both cases of parameterized treatment are associated with higher efficacy than the classical version of interval analysis. It should be noted that some other methods have been suggested in the literature to overcome the phenomenon of overestimation. For example, if the mathematical expression has multiple appearance of the same interval, it is advisable, if possible, to rewrite the expression in an equivalent form, in which the interval appears only once. It appears remarkable that the parameterized analysis in this case coincides with classical interval analysis. This implies that the following assertion could be made: the parameterized interval analysis performs as the classical interval analysis when the latter furnishes exact results, or better, when the classical interval analysis yields an overestimation.

It appears that further research ought to be recommended to exploit the advantages of the parameterized technique.

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R. Ghanem
Associate Editor