AIAA JOURNAL
Vol. 40, No. 9, September 2002


# Nonlinear Aeroelasticity and Unsteady Aerodynamics

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**In this von Kármán lecture, a subject is addressed whose foundations were significantly influenced by the work of Theodore von Kármán. A classic paper by von Kármán and Sears first considered the determination of aerodynamic forces on an airfoil undergoing general time-dependent motion. Also, early in his career, von Kármán investigated fundamental issues in structural mechanics and derived the celebrated von Kármán plate equations for determining the large (nonlinear) deflections of an elastic plate under a distributed force. Finally, he authored a widely cited paper on the importance of nonlinearities for engineers and engineering. In this lecture, these themes are recalled and the current state of the art in nonlinear aeroelasticity and unsteady aerodynamics is discussed. Several of the most significant nonlinearities arising in a structure or in an aerodynamic flow field are identified. Recent and relevant theoretical and experimental studies are reviewed and future developments are projected that are expected to have a significant impact on our ability to understand and beneficially use nonlinear dynamic aeroelastic behavior.**



## Nomenclature

| | | |
|---|---|---|
| $b$ | = | semichord of the airfoil section |
| $\bar{c}_l$ | = | first harmonic aerodynamic nondimensional lift |
| $\bar{c}_m$ | = | first harmonic aerodynamic nondimensional moment |
| $h, \bar{h}$ | = | plunge displacement and amplitude |
| $M$ | = | Mach number |
| $r_\alpha$ | = | nondimensional rotational inertia about the elastic axis |
| $t$ | = | time |
| $U_F$ | = | flutter velocity |
| $V$ | = | flow velocity |
| $x_\alpha$ | = | nondimensional distance between elastic axis and c.g. of the airfoil section |
| $\alpha, \bar{\alpha}$ | = | pitch angle and pitch amplitude |
| $\beta, \bar{\beta}$ | = | flap angle and amplitude |
| $\delta$ | = | freeplay region of control surface |
| $\mu$ | = | mass ratio |
| $\rho$ | = | air density |
| $\omega$ | = | frequency |
| $\omega_F$ | = | flutter oscillatory frequency |
| $\omega_h, \omega_\alpha$ | = | uncoupled plunge and torsional natural frequencies |
| $\omega_\beta$ | = | uncoupled flap natural frequency |

## Introduction

NONLINEAR phenomena in aeroelasticity have been known for many years. In the past decade or so, such effects have become of more serious concern to practitioners.[1–4] For that reason, and also because of the advance in theoretical and experimental methods, a more substantial and concentrated effort has been made by the research community to understand and pursue how unfavorable nonlinear aeroelastic effects may be diminished and favorable effects exploited.

Moreover, in the past few years, significant advances have been made in constructing reduced-order models for unsteady aerodynamic flows.[5] This key enabling methodology is also discussed here as it relates to nonlinear aeroelasticity.

Presented as Paper 2002-0003 at the AIAA 40th Aerospace Sciences Meeting, Reno, NV, 14–17 January 2002; received 5 February 2002; revision received 7 April 2002; accepted for publication 22 April 2002. 

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There are two possible principal consequences of any nonlinear effect. One is that the exponentially growing oscillations predicted by an unstable linear model are attenuated due to the nonlinear effects, and a finite amplitude, steady-state oscillation is obtained. Such limit-cycle oscillations (LCO) have been observed in operational aerospacecraft and in wind-tunnel models. LCO per se is, thus, benign in that the nonlinearity reduces the amplitude of the oscillations. Of course, structural integrity may still be at issue if the LCO amplitudes are too large. The second principal consequence is wholly detrimental. In this instance, a system that may be stable to a sufficiently small perturbation can become unstable due to a large disturbance.

The generic possibilities are illustrated in Fig. 1, where the LCO amplitude is plotted vs flight speed. In Fig. 1a, an aeroelastic system is depicted that is stable to small or large perturbations below the flutter (instability) boundary predicted by a linear dynamic model. Beyond the flutter boundary, LCO arise due to some nonlinear effect, and typically the amplitude of the LCO increases as the flight speed increases beyond the flutter speed. In Fig. 1b, the other generic possibility is shown. Whereas again LCO exist beyond the flutter boundary, now LCO may also exist below the flutter boundary, if the disturbances to the system are sufficiently large. Moreover both stable (solid line) and unstable (dotted line) LCO now are present. Stable LCO exist when, for any small disturbance, the motion returns to the same LCO at a large time. Unstable LCO are those for which any small perturbation will cause the motion to move from the unstable LCO to a stable LCO. Theoretically, in the absence of any disturbance, both stable and unstable LCO are possible dynamic, steady-state motions of the system. Information about the magnitude of the disturbance, required to move from one stable LCO to another stable LCO can also be obtained from data such as those shown in Fig. 1b. Note also that the hysteretic response as flight speed increases and then decreases.

The balance of this paper summarizes several distinct yet related research thrusts that have proven particularly fruitful. Although the authors draw largely on the experience of the Duke University aeroelasticity team, reference is made to the work of many other investigators for those readers who wish to pursue the ever-increasing literature on these topics. However, an in-depth summary of the total literature is not included, but rather a selective commentary is provided on the efforts of the broader community in the context of the work discussed here.

There are several physical sources of nonlinearity in either the aerodynamic flow or elastic structure. These are listed and briefly described as follows and will be further considered throughout the paper.

The physical sources of nonlinearity in elastic structure include 1) free play, or bilinear stiffness arising from loosely connected structural components; 2) strain displacement or geometric nonlinearity,

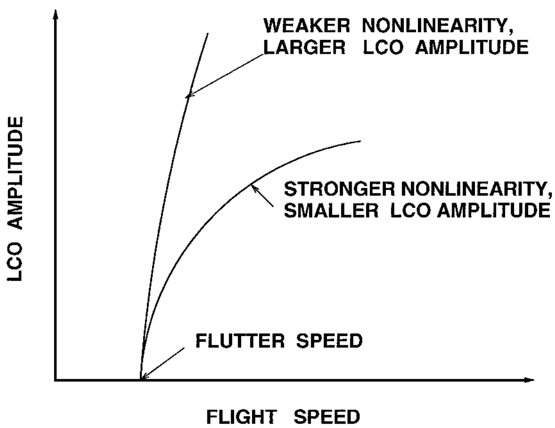



**Fig. 1a Schematic of LCO response for a benign nonlinearity leading to (only) stable LCO.**

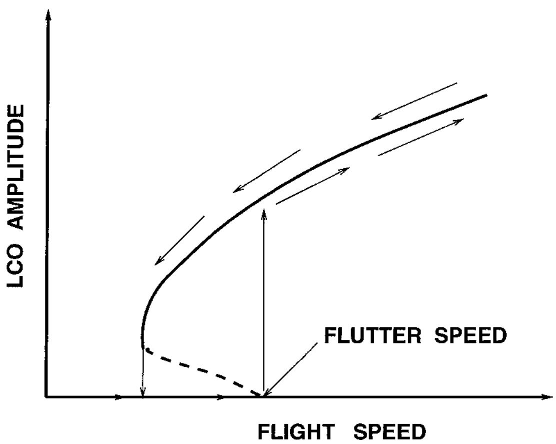



**Fig. 1b Schematic of LCO response for a detrimental nonlinearity leading to both stable (——) and unstable (- - -) LCO. Arrows denote path of system response when flight speed is increasing (—→ ) or decreasing (←— ).**



which comprise the nonlinear stiffness arising from large displacement gradients; and 3) dry friction, or nonlinear damping arising from structural components in sliding contact.

The physical sources of nonlinearity in aerodynamic flow (fluid sources) include 1) shock motions in transonic flow, which are particularly important for low, reduced frequencies and 2) separated flow, the most common in transonic flow (may be induced by shock) and/or at large angles of attack.

## Scope of Paper

Four specific aeroelastic models are considered in this paper. The first is an airfoil with a control surface that has freeplay in its attachment to the airfoil. Such a configuration exhibits LCO due to freeplay well below the classical linear flutter speed (LFS). The second is a wing with a platelike structure that undergoes LCO once the LFS is exceeded but not usually below the LFS. Here, the nonlinearity is a result of the tension induced in the plane of the plate wing when the wing bending is on the order the wing thickness or greater. The third is a very high-aspect-ratio wing that exhibits LCO above the LFS but that also exhibits a sensitivity to initial disturbances below the LFS that may lead to LCO there as well. Here, the nonlinearity is due to the coupling among flapwise bending, chordwise (or lag) bending, and torsion of the wing structure. This coupling has long been known to be important for rotorcraft blades that are cantilevered at the rotor hub (sometimes called hingeless blades). Also modeled semi-empirically are the effects of aerodynamic stall. More recently, very high-aspect-ratio wings have been identified as of importance for high-altitude long-endurance (HALE) uninhabited air vehicles (UAV). Fourth and finally, the aeroelastic responses due to nonlinear aerodynamic forces arising from large shock motions in the transonic flow range are considered.

For each of the first three models, experiments (conducted in a low-speed wind tunnel) are also discussed and correlated with theory. For the fourth model, the NASA Langley Research Center aeroelasticity team has provided valuable benchmark experiments that are directed toward aerodynamic nonlinearities in the transonic flow regime.[6–9] Some encouraging correlations with available theory have been made by the NASA Langley Research Center team as well. It can be expected that our ability to pursue more such correlations will continue to advance due to the work of the NASA Langley Research Center team and the efforts of the aeroelastic community at large. Also noteworthy, the aeroelasticity group at DLR, Gottingen has conducted valuable two-dimensional oscillating airfoil flutter and LCO experimental/theoretical studies as reported in Refs. 10–12.

Cunningham[13] provides an insightful discussion of certain nonlinear aerodynamics issues based on his extensive experience with operational aircraft and wind-tunnel models. Friedmann[14] has touched on particular nonlinear aeroelasticity issues in his broad-ranging survey of aeroelasticity as well. Both are recommended to the reader who wishes further background and a broader context for the present discussion.

### Airfoil Plus a Control Surface with Freeplay

Many investigators over the past 50 years have considered the effects of a structural stiffness nonlinearity on an airfoil with or without a control surface.[15–43] Broadly speaking, the literature is characterized by the type of nonlinearity, that is, continuous or discontinuous (freeplay), whether the nonlinear spring stiffness is for the airfoil per se or the control surface, and, finally, the nature of the aerodynamic model. Most of the analysis has been done at low Mach number using classical Théodorsen aerodynamic theory or approximations thereto, or at very high Mach number where piston theory aerodynamics can be applied. Also, most of the studies have been theoretical/numerical, but some interesting experimental work has been done as well. For a more thorough review of the literature, particularly as regards freeplay nonlinearities, see Connor et al.[36] For continuous spring nonlinearities, the recent publication by Liu et al.[42] has a nice summary.

Here, we focus on theoretical/experimental correlation as achieved at low Mach number, the theoretical effects of transonic Mach number, and the physical and fundamental insights that have been gained over the years. For another recent study of freeplay for an airfoil in the transonic, low supersonic range, see Kim and Lee.[41] Their results appear comparable to those discussed here, although they use different solution methods and do not consider a control surface per se.

The configuration considered is the simplest that models the fundamental physical phenomena. See Fig. 2, which depicts an airfoil with a control surface in two-dimensional flow. The airfoil structure has three degrees of freedom, plunge (vertical translation), pitch (rotation about the spring or elastic axis), and flap rotation (rotation of the control surface relative to the airfoil per se). The nonlinearity modeled is the bilinear torsional stiffness of the attachment of the flap to the airfoil (Fig. 3). For small flap rotation, the bilinear torsional stiffness is very small (here set to zero and, hence, the term free play), whereas for large flap rotation the torsional spring

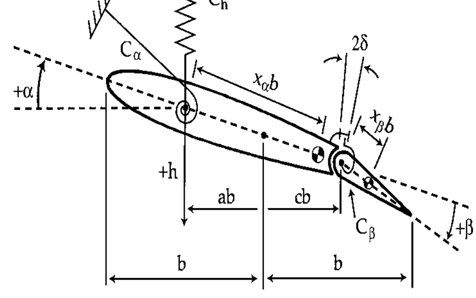



**Fig. 2 Aeroelastic typical section with control surface.**

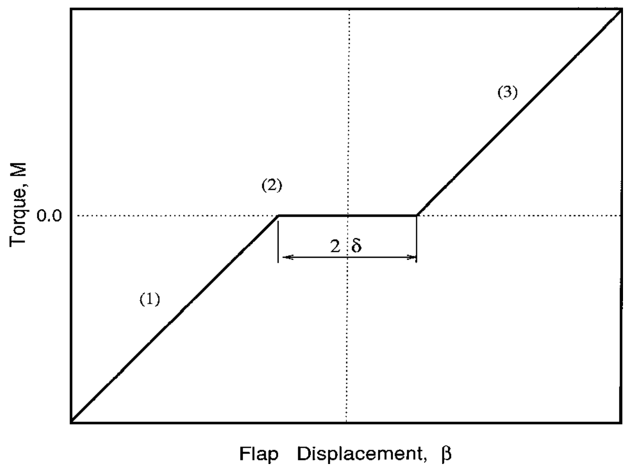



**Fig. 3 Elastic restoring moment or torque with a symmetric freeplay region about $\beta = 0$.**

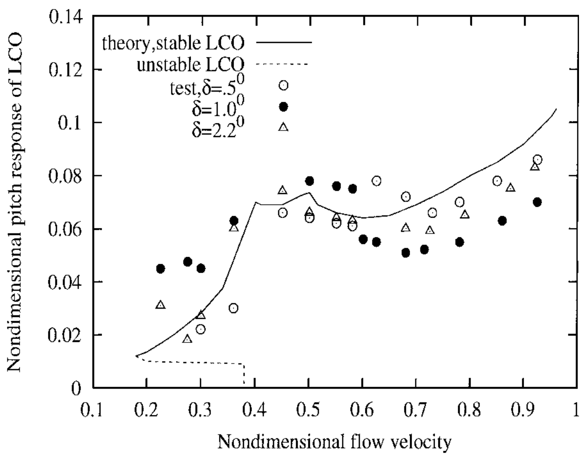



**Fig. 4 Numerical and experimental normalized steady-state rms amplitude for pitch vs normalized flow velocity.**

stiffness approaches a nominal fixed value. By the use of Lagrange's equations for the structure, including the bilinear free-play nonlinearity and an appropriate aerodynamic model, one may determine the flutter boundary and the LCO. For the details of the theoretical model and the wind-tunnel experimental model, see Refs. 36, 38, and 43. Theodorsen aerodynamics or the corresponding theory of von Kármán and Sears[44] is used for the low-Mach-number analyses and results.

In Fig. 4, an LCO amplitude (in pitch) is plotted vs flow speed (velocity) where the latter is normalized by the flutter speed in the absence of free play. Beyond the flutter speed without free play (here normalized to unity), the LCO amplitude becomes very large (infinite in the present theoretical model), and no experimental testing was done for this range of flow speed due to safety concerns. Note that the effect of free play is detrimental in that LCO is induced by free play at flow speeds substantially below the nominal flutter speed without free play. No LCO exists above the flutter boundary predicted by linear theory for this configuration, but rather exponentially growing oscillations (diverging flutter) are predicted by the theoretical model.

Now consider the LCO amplitude below the nominal linear flutter boundary. As shown in Fig. 4, the LCO amplitude in pitch response $\alpha$ is normalized by the free-play angle $\delta$. See Fig. 2 for the definition of $\delta$. The theoretical model predicts that by using this normalization the normalized LCO amplitude results for any $\delta$ will lie on a single, universal curve (Fig. 4). The results from numerical simulation confirm this, and the results from experiment for several $\delta$ over a range from 0.5 to 2.2 deg are also in close correspondence with each other and the theoretical prediction.

Note that there are both stable and unstable LCO predicted by theory. The unstable LCO is indicated by the dashed line in Fig. 4. This has a simple and important physical interpretation, as follows. In addition to the nominal flutter speed without free play, there is another and different flutter speed predicted by the linear theory when there is only free play (formally, when $\delta \longrightarrow \infty$). For the combination of flow and structural parameters considered in the present example[36,38,43] this normalized flow speed is about 0.37 (Fig. 4). Note, however, that both a stable and an unstable LCO exist at yet lower flow speeds. The lowest flow speed for an LCO to exist (where the stable and unstable LCO meet) corresponds to the minimum flow speed for flutter to occur for any control surface torsional stiffness between zero (oscillations entirely in the free-play range) and the nominal stiffness in the absence of free play. In the present example, this normalized flow velocity is about 0.18. See Ref. 38 for a more detailed discussion.

This physical phenomenon persists into the transonic flow range, although with some added features. Turning to transonic flow, consider first the flutter boundary in the absence of free play. These results were obtained using a reduced-order aerodynamic model based on proper orthogonal decomposition for the time-linearized Euler equations of fluid mechanics.[43] The flutter boundary is shown in Fig. 5 in terms of a nondimensional flutter speed (flutter index) vs Mach number. At low Mach number the results are similar to those shown before for $M = 0$. However, for a high subsonic/transonic Mach number, there are important differences. Note that the flutter speed tends to a minimum in the transonic Mach number regime; however, also note that the flutter speed rises sharply with Mach number after the flutter minimum occurs. This is often called the transonic bucket.

The corresponding flutter frequency is shown in Fig. 6. This illustrates another important characteristic of transonic flow, that is, the change in flutter mode as the transonic Mach number range is traversed. For low Mach numbers, it is the plunge mode that dominates the flutter motion, although all modes participate to some

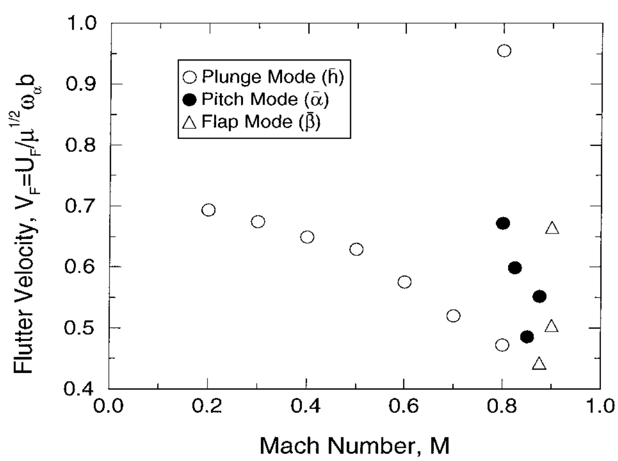



**Fig. 5 Flutter velocity vs Mach number.**

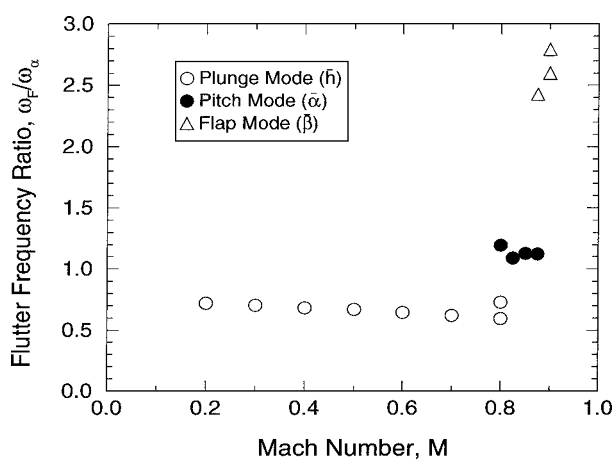
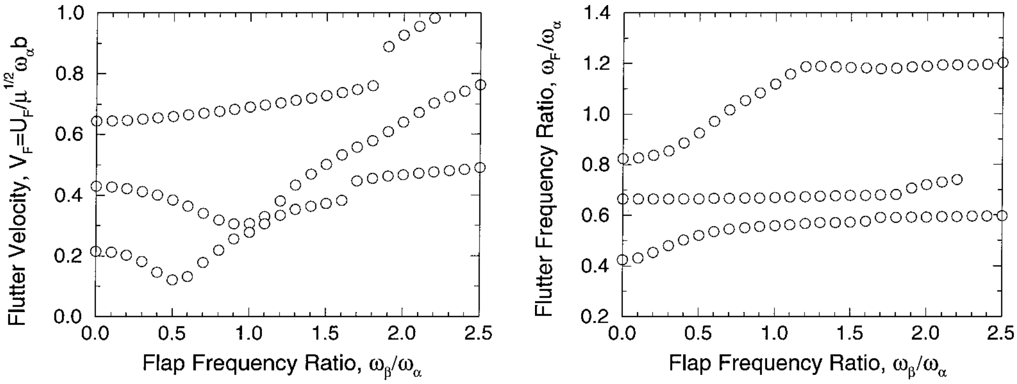



**Fig. 6 Flutter frequency vs Mach number.**




**Flutter velocity** **Flutter frequency ratio**

**Fig. 7 Flutter velocity and frequency vs flap frequency ratio; $M = 0.8$.**

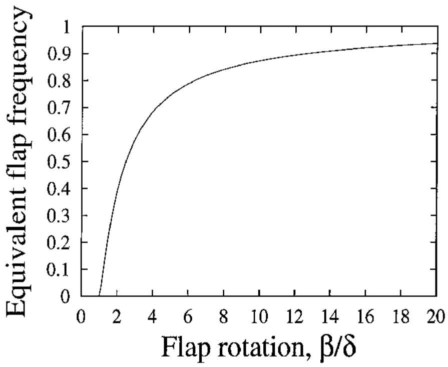



**Fig. 8 Nondimensional equivalent flap frequency, $\omega_\beta/\omega_{\beta,\mathrm{nominal}}$, vs normalized flap rotation $\beta/\delta$.**



degree. Above $M = 0.8$, the pitch mode becomes dominant, and for yet higher Mach numbers, the flap mode dominates. To detect these rapid changes in flutter mode, many flutter calculations must be performed for a range of Mach numbers and other parameters. Reduced-order aerodynamic modeling is a key enabling methodology for these calculations.[43]

Once the flutter boundary and flutter mode are determined, the harmonic balance method is a very useful and computationally effective method to compute LCO. The present configuration provides a good example of how the LCO may be determined using a harmonic balance approach.

Consider a fixed Mach number and all other parameters specified except the flap torsional stiffness (or equivalently, natural frequency). Treating the flap stiffness or flap natural frequency as a free parameter, one may determine the flutter speed as a function of, for example, flap natural frequency using a linear aeroelastic model (Fig. 7). However, in reality, because of free play, the flap natural frequency is a function of flap rotation amplitude (Fig. 8). For an assumed harmonic motion of the flap, the equivalent stiffness or natural frequency of the flap may be determined as a function of flap amplitude from a (nonlinear) Fourier analysis of the bilinear torque–flap rotation characteristic, that is, by a harmonic balance method (recall Fig. 3), and the result is shown in Fig. 8. Cross plotting Figs. 7 and 8, one may determine the LCO amplitude vs flow speed or velocity relationship. In a sense, a nonlinear eigenvalue problem has been solved, that is, the amplitude (at a given flow speed) for which neutrally stable oscillations (LCO) may exist has been determined. By considering perturbations in amplitude or flow speed about these LCO, one may determine whether the LCO is stable or unstable. The result for LCO amplitude vs flow speed is shown in Fig. 9. Such results can be and have been obtained over a broad range of Mach numbers.[43]

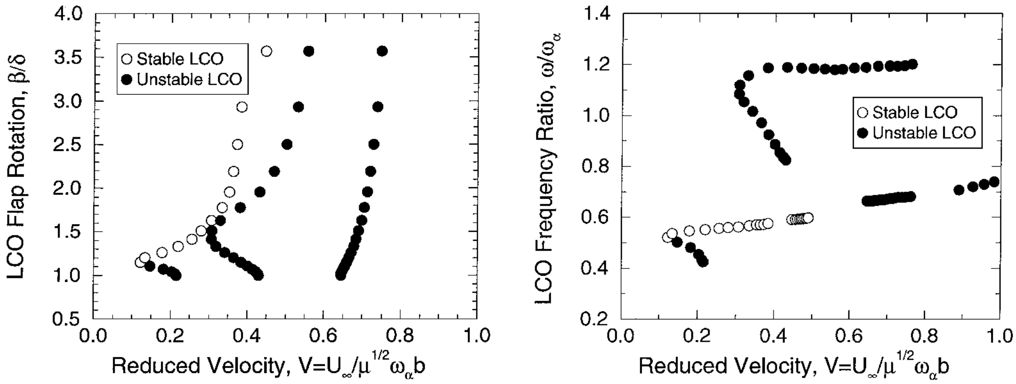
## Low-Aspect-Ratio, Platelike Wing

From one perspective, the literature for this configuration in the context of nonlinear aeroelasticity is very recent and relatively small.[45–51] However, if one takes a broader view and notes the basic physical mechanism for the nonlinear effect (that a tension force is induced in the midplane of the plate wing by the out-of-plane bending when the latter deflection is of the order of the plate thickness), then we recall that this is indeed the same physical mechanism that leads to nonlinear effects in general, and LCO in particular, for panel flutter, or the flutter of plates and shells. A panel is a local portion of a wing between pairs of spars and stringers. See the monograph by Dowell,[45] which gives an early account of the fundamentals of the phenomena, and the recent review by Mei et al.[46] that summarizes the recent literature on panel flutter. Note that in much of the recent (and not so recent) panel flutter research, the simple piston theory aerodynamic model is used that is valid only at sufficiently high supersonic Mach number. However, it is the subsonic, transonic, and low supersonic flow regimes that are often most important for applications. Fortunately, recent theoretical advances make calculations in these Mach number regimes more feasible and attractive. When such calculations were first done 25 years ago, they were a feat.[45] Today, although they do require an understanding of the more sophisticated aerodynamic models, the calculations themselves are no longer extraordinary in their demand on computer resources.[5,47,48] The most recent work for a platelike wing per se is discussed in Refs. 49–51.

For a low-aspect-ratio wing structure that has significant bending in both the chordwise and spanwise direction, it may, within a certain approximation, be modeled as a plate (as distinct from a bending beam/torsional rod model that is often used for high-aspect-ratio wing structures). It was shown by von Kármán[52] that, if the bending deflections of a plate are comparable to the plate thickness, then there is a tension induced into the plate by midplane stretching that varies as the square of the plate deflection (more precisely as the square of the local slopes). This tension, when appropriately multiplied by the plate curvature to give the relevant transverse force, gives rise to a cubic stiffness nonlinearity in the bending plate deflection.

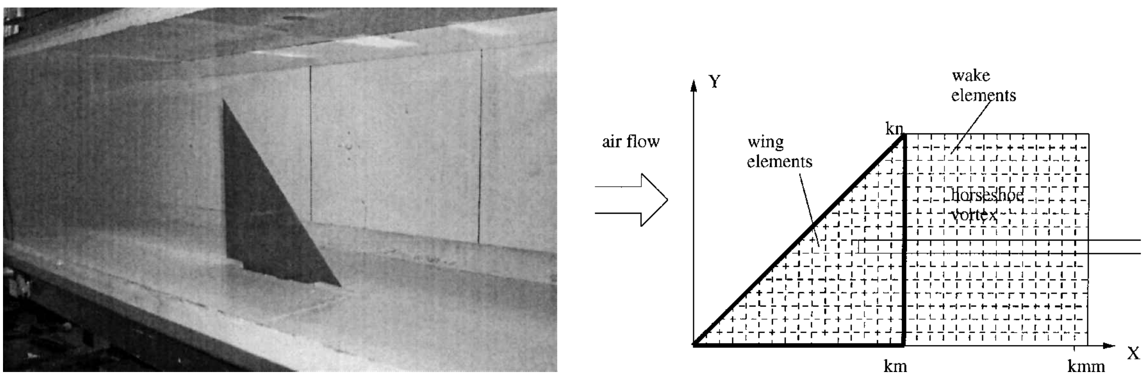
Von Kármán[52] and many subsequent investigators were concerned with plate postbuckling. In the early days of aeronautics and continuing to the present, plate skins on aircraft between spars and stringers are allowed to buckle. Yet, because of the stiffness nonlinearity, they retain some useful stiffness even when buckled. Many years after von Kármán's original studies, his theoretical model was a key to our understanding of the flutter of aerospacecraft skin panels. This is because the static pressure and thermal stress loading of thin-skin aircraft panels can often deform them into the nonlinear regime. Thus, to predict the flutter boundary of such panels, not to mention LCO, requires the nonlinear plate theory of von Kármán. Indeed, this theory may be used to determine the LCO as well. As expected, the LCO is almost invariably benign, and the LCO amplitude is typically of the order of the plate thickness. For a further discussion of these matters, see Refs. 45 and 46.

Now, at least in retrospect, none of the preceding is surprising perhaps, although for several years there was distressing disagreement between theory and experiment for panel flutter until the von Kármán[52] nonlinear model was adopted. After all, when an elastic plate or beam is fixed at its edges, it will stretch when it bends, and a




**LCO flap rotation** **LCO frequency ratio**

**Fig. 9 Normalized LCO flap rotation $\beta/\delta$ and LCO frequency ratio $\omega/\omega_\alpha$ vs reduced velocity; $M = 0.8$.**




**Photograph** **Physical representation**

**Fig. 10 Photograph of experimental model for a delta wing in a wind tunnel and physical representation of the experimental model; both a finite element grid for the wing structure and a boundary element, horseshoe vortex grid for the three-dimensional vortex lattice model (linear) of the unsteady flow are shown.**

nonlinear tension force will be induced. However, what may be less expected, but nevertheless also true, is that when a plate is only restrained on a single edge (e.g., a low-aspect-ratio wing cantilevered to a wind-tunnel wall or an aerospacecraft fuselage), a significant tension may still be induced as predicted by the von Kármán theory. Typically, however, the deflections must be larger for a cantilevered plate than for a plate fixed on opposite edges for the nonlinear effect to be important. The LCO amplitude of a cantilevered plate may be as large as several plate thicknesses, whereas that for a plate clamped on all edges is usually less than one plate thickness.

In Fig. 10, a low-aspect-ratio delta wing model cantilevered to a wind-tunnel wall is shown. Von Kármán's[52] plate theory and an appropriate reduced-order aerodynamic model are used to determine the LCO amplitude as a function of flow speed. A representative result is shown in Fig. 11 (Refs. 49–51). Note that the LCO is entirely benign. No LCO occurs below the flutter boundary predicted by linear theory. The small oscillations seen in the experiment at flow speeds below the flutter speed are a result of small turbulent flow fluctuations in the wind tunnel not modeled in the present theory. Theory and experiment are in essential agreement for both the onset of flutter and the LCO beyond the flutter boundary.

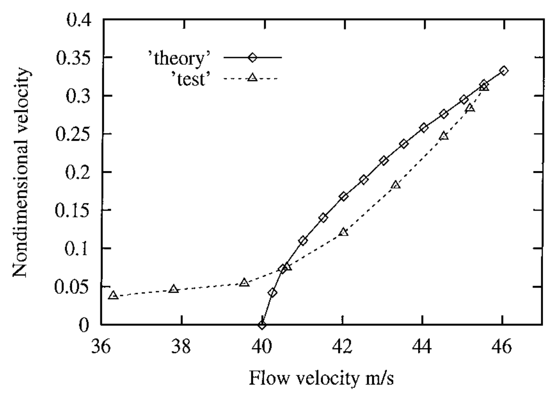


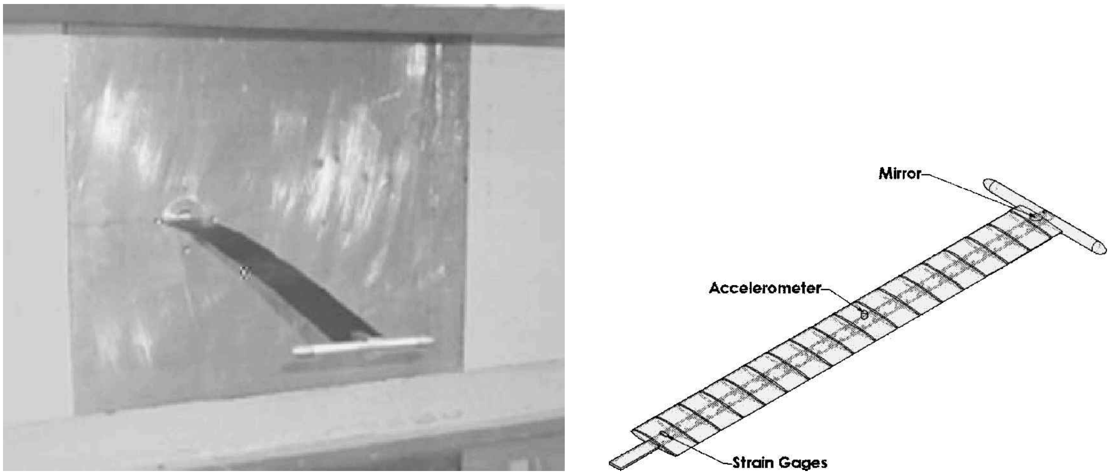

**Fig. 11 Theoretical and experimental nondimensional transverse velocity of LCO vs flow velocity.**

### High-Aspect-Ratio, Beamlike Wing

For very high-aspect-ratio beams that may bend and twist, it has been known for many years that the flapwise bending, chordwise bending (lag), and torsional deformation (twist) may couple among themselves to produce a significant structural nonlinearity.[53] Also, nonlinear aerodynamic stall effects are known to be important and have been modeled successfully, though semi-empirically.[54] Such issues were first pursued in the context of rotor blades that are often, of course, long and slender. They have not been important for fixed-wing aircraft for the most part. However, recently, and particularly in the context of some UAVs, very high-aspect-ratio fixed-wing configurations are of interest. Thus, researchers have pursued aeroelastic studies of this configuration. Notable work has been done by Patil et al.[55,56] and Patil and Hodges.[57] Their pioneering theoretical studies have shown a number of interesting nonlinear effects, including the presence of LCO, the sensitivity of the onset of classical flutter, as well as LCO to temporal disturbances including initial conditions. Their work and that of the Duke University research team[58] are discussed here. The latter includes both theoretical and experimental studies.

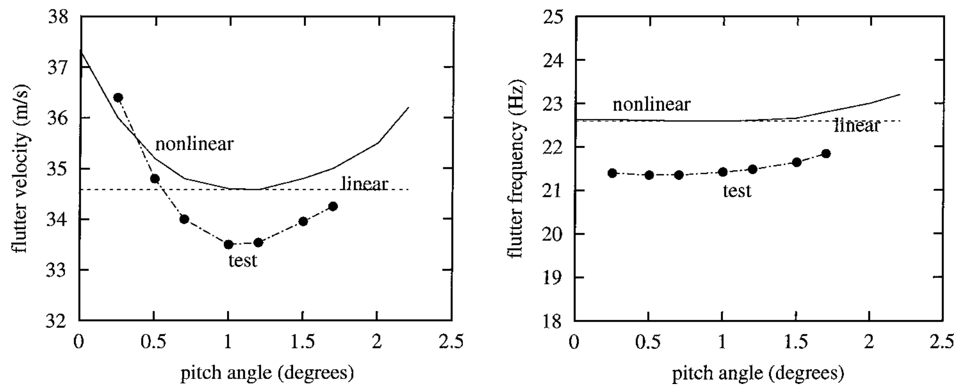
For a very high-aspect-ratio wing, the structural nonlinearity is of a kind very different from that for a low-aspect-ratio platelike wing. However, it can still be attributed to a nonlinear relationship between strain and displacement (gradient). Such a nonlinearity was first investigated in the context of helicopter rotor blades (which are very high-aspect wings) dating from the paper by Hodges and Dowell.[53] Subsequently, Hodges,[59] Friedmann,[14] and many others have improved on this original work.

The key physical aspect of the structural nonlinearity arises from a subtle, mutual coupling among the chordwise bending, as well as




**Photograph** **Physical representation**

**Fig. 12 Photograph of experimental model for a high-aspect-ratio wing in a wind tunnel and physical representation of the experimental model: NACA 0012 airfoil, span $L$ = 18 in., chord $c$ = 2 in., and bending stiffness ratio, $EI_2/EI_1$ = 44.**




**Flutter velocity** **Flutter frequency**

**Fig. 13 Perturbation stability solution of the nonlinear aeroelastic system vs pitch angle.**

transverse bending of the wing in conjunction with twisting about the beam–rod elastic axis of the very high aspect ratio wing. As in classical high-aspect-wing structural theory, the chordwise deformation is treated as a combination of rigid-body plunge (translation) and pitch (rotation), with all elastic deformation occurring along the (one-dimensional) axis of the beam–rod (Fig. 12). There can be both static, as well as dynamic, effects of the structural nonlinearity. For example, static gravity loading or static aerodynamic loading due to a static angle of attack may change the structural natural frequencies and, hence, the flutter speed. The effect of a static angle of attack on flutter speed is shown in Fig. 13. The LCO amplitude vs flow speed is shown in Fig. 14 for a specified angle of attack.

From Fig. 13, which shows both flutter velocity and flutter frequency vs static angle of attack or pitch angle, it is seen that the theoretical model captures well the results observed experimentally. The LCO results in Fig. 14 show hysteretic response in both the theory and experiment with increasing and decreasing flow velocity. When the nonlinear structural effects are studied with and without flow separation in the theoretical model, it is found that the hysteretic behavior is due to flow separation. Flow separation is accounted for theoretically using the ONERA aerodynamic model.[54]

The agreement between theory and experiment is encouragingly good for what is the rather complex behavior of a multidimensional nonlinear system. The flutter mode is dominated by the second spanwise transverse bending and first torsion structural natural modes. The first chordwise bending mode is also an essential contributor to the LCO, however. For further discussion of this case, see Ref. 58.

**Nonlinear Inviscid Aerodynamic Effects on Transonic Divergence Flutter and LCO**

The presence of a shock wave per se may give rise to two significant effects on aeroelastic response. First, if the shock wave motion is small enough, then the aerodynamic force will be linearly proportional to the structural motion. However, the linear relationship will be quantitatively different with a shock wave present than without. Hence, the flutter boundary will be changed by the shock even for small (infinitesimal) shock motions. However, in addition, to predict LCO, the nonlinear relationship between the aerodynamic forces and the structural motion that arise from large shock motions must be taken into account. This is the primary subject here. However, as will be seen, determining the flutter condition is a key step in determining the LCO. A representative sample of the relevant literature is contained in Refs. 60–64.

It has been found that both benign and detrimental LCO may result from aerodynamic nonlinearities due to large shock wave motion.

To solve the Euler equations of fluid dynamics for large motions, a novel form of the harmonic balance (HB) method has been derived by Hall et al.[62] and Thomas et al.[64] In this method, the aerodynamic forces are determined as a Fourier series in time for a prescribed structural motion that is a harmonic in time. The Fourier coefficients for the aerodynamic forces are, in general, nonlinear functions of the airfoil (or wing) motion amplitude. It is found that a single harmonic approximation for the structural motion is sufficient, but typically two or three harmonics are needed in the aerodynamic model to obtain a good first harmonic representation of the aerodynamic forces required for the structural equations of motion.

Now the keys to the efficient calculation of LCO are twofold. First, if the structural motion is known, then the HB method will determine the aerodynamic forces very efficiently relative to a time-marching code. Note that the results from a time-linearized model in the frequency domain for small airfoil or wing motions may be used to construct an initial guess for an iteration method to determine the HB aerodynamic solution for larger airfoil and shock wave motions.

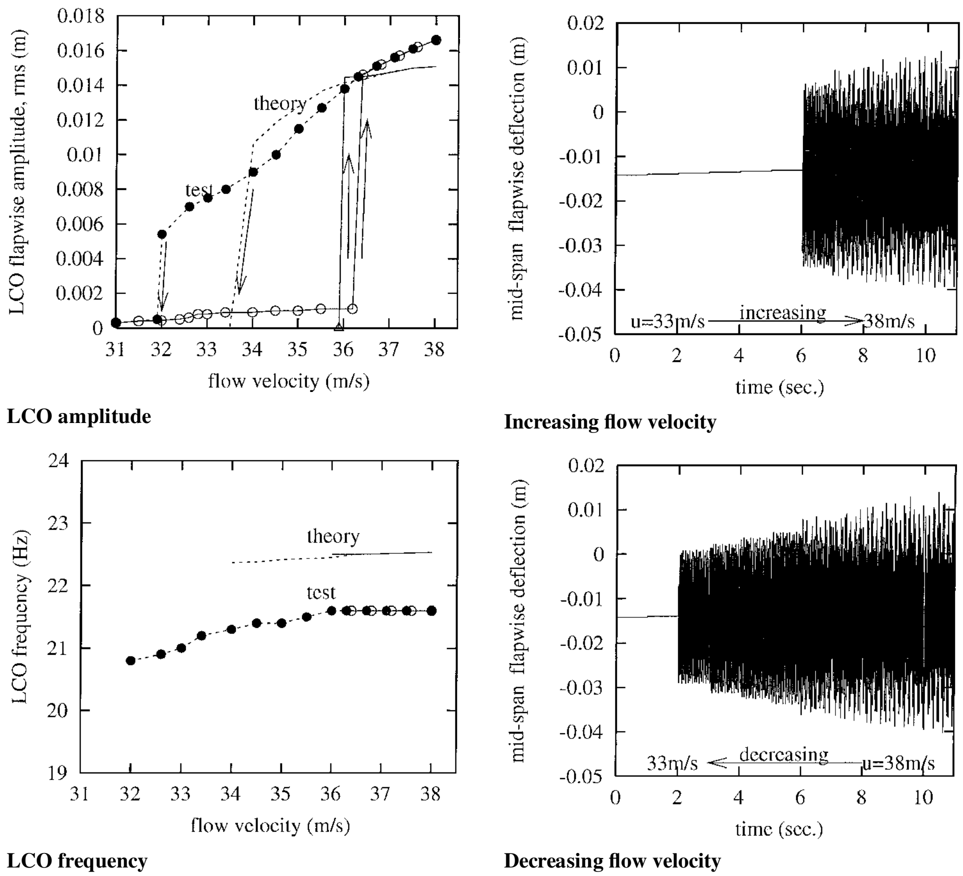



**Fig. 14 Midspan LCO amplitude, LCO frequency vs flow velocity, and LCO time history at midspan for increasing and decreasing flow velocities for $\theta_0 = 0.25$ deg.**

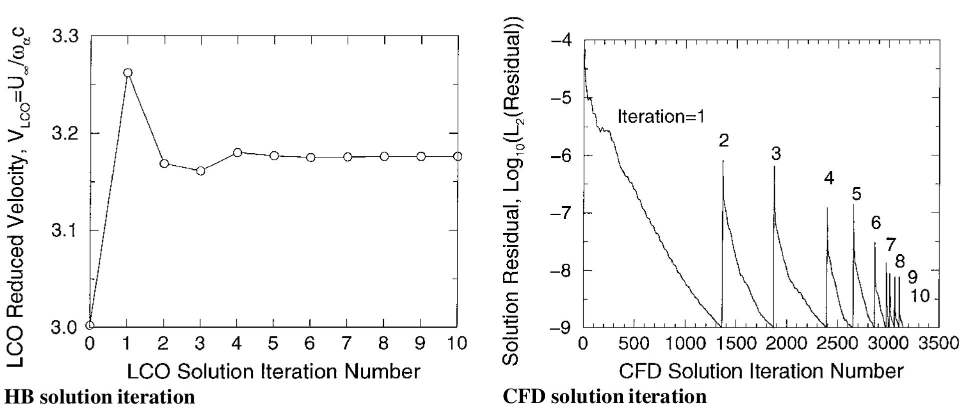



**Fig. 15 Iterative solution technique for LCO variables; structural and aerodynamic parameters are NACA 64A010A, $M = 0.8$, $\alpha_0 = 0$ deg, $a = -0.6$, $\bar{\alpha} = 2$ deg, $N = 2$, $x_\alpha = 0.25$, $r_\alpha^2 = 0.75$, $\omega_h/\omega_\alpha = 0.5$, and $\mu = 75$.**

Second, when dealing with the nonlinear aeroelastic equations, another rapid iteration method may be used to determine the LCO. One such method follows.

For definiteness and clarity, consider an airfoil with plunge $h$ and pitch $\alpha$ degrees of freedom. In a standard nondimensional notation, the structural/aeroelastic equations of motion for the first harmonic in time are

$$[-\bar{\omega}^2 M + (1/V^2)K]\{u\} = 4/\pi\mu\{f\} \tag{1}$$

where $h = \bar{h}e^{i\omega t}$, $\alpha = \bar{\alpha}e^{i\omega t}$, and

$$M = \begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix}, \qquad K = \begin{bmatrix} (\omega_h/\omega_\alpha)^2 & 0 \\ 0 & r_\alpha^2 \end{bmatrix} \tag{2}$$

$$\{u\} = \begin{Bmatrix} \bar{h}/b \\ \bar{\alpha} \end{Bmatrix}, \qquad \{f\} = \begin{Bmatrix} -\bar{c}_l \\ 2\bar{c}_m \end{Bmatrix} \tag{3}$$

where $\bar{c}_l$ and $\bar{c}_m$ are the first harmonic aerodynamic nondimensional lift and moment. In general, $\bar{c}_l$ and $\bar{c}_m$ are nonlinear functions of $\bar{h}$ and $\bar{\alpha}$. However, $\bar{h}$ and $\bar{\alpha}$ are to be determined as part of the solution.

One can construct a very efficient iteration scheme to solve these equations, as follows. First, select the amplitude (of the first harmonic) of $\bar{\alpha}$. Then, determine the corresponding amplitude (first harmonic including both in-phase and out-of-phase contributions) of $\bar{h}$ from the aeroelastic eigenvector calculated from a time-linearized model. Also choose the (reduced) frequency to be that from a time-linearized aeroelastic flutter analysis. Now $\bar{h}$ and $\bar{\alpha}$ are known to some approximation and, thus, $\bar{c}_l$ and $\bar{c}_m$ can be determined from a nonlinear HB aerodynamic analysis. With $\bar{c}_l$ and $\bar{c}_m$ known to a first approximation, Eq. (1) may be solved for new estimates of reduced frequency, reduced velocity (the first approximation is the reduced velocity at flutter), and the ratio of the in-phase and out-of-phase plunge amplitude to the chosen pitch amplitude. With these new estimates, $\bar{c}_l$ and $\bar{c}_m$ may be reevaluated, and the iteration continues. A typical result is shown in Fig. 15 for the LCO reduced velocity vs iteration number $N$ for a prescribed pitch amplitude of 2 deg. For $N = 0$, the reduced velocity is that given at the flutter point. Note that $N = 2$ or 3 gives a good approximation. The convergence of the HB computational fluid dynamic (CFD) solution is also shown in Fig. 15. Note that, as $N$ increases, the CFD solution converges more rapidly from one iteration to the next. This attractive result is because the difference between the aerodynamic solution for $N$ and that for $N + 1$ decreases as $N$ increases. In more recent work (as yet unpublished), it has been found that sometimes the preceding iteration method fails to converge. However, in those cases, a standard Newton–Raphson method has been used successfully to solve Eq. (1) with even more rapid convergence.

Repeating this calculation for different selected pitch amplitudes provides results for LCO pitch amplitude vs reduced velocity (Fig. 16). This is an example of a relatively strong transonic nonlinearity leading to a benign LCO. If the Mach number is reduced so that no shock is present, the LCO amplitude vs reduced velocity curve would be a near vertical line, indicating a weaker nonlinearity.

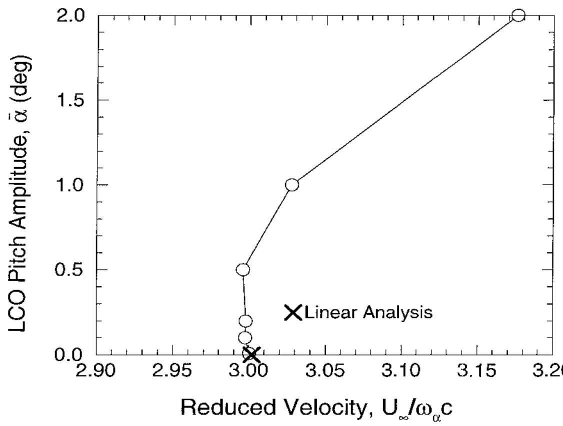



**Fig. 16 Computed LCO behavior; structural and aerodynamic parameters are NACA 64A010A, $M = 0.8$, $\alpha_0 = 0$ deg, $a = -0.6$, $\bar{\alpha} = 2$ deg, $N = 2$, $x_\alpha = 0.25$, $r_\alpha^2 = 0.75$, $\omega_h/\omega_\alpha = 0.5$, and $\mu = 75$.**

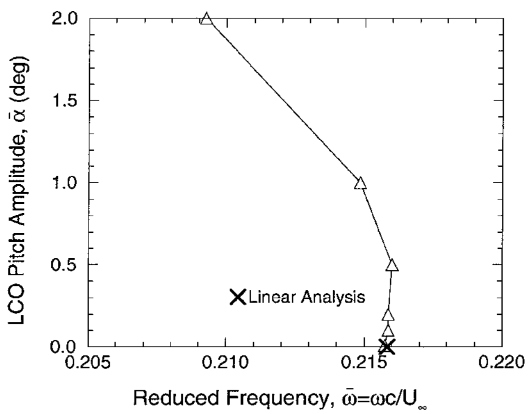
In addition to the result shown in Fig. 16, one may also calculate the reduced frequency, the ratio of LCO frequency $\omega$ to the pitch natural frequency $\omega_\alpha$, and the real and imaginary (in-phase and out-of-phase) components of $h$ normalized by $\alpha$ (Fig. 17). As expected, all of these quantities are weak functions of LCO pitch amplitude. Using Figs. 16 and 17, one can also construct plots of these quantities as a function of reduced velocity.

Detrimental LCO due to shock-related nonlinearities may also occur below the nominal (linear) flutter-boundary, as reported in Ref. 64. Finally, this methodology can be extended to three-dimensional flows and to include viscous flow effects and structural models with several degrees of freedom. Such work is currently underway.

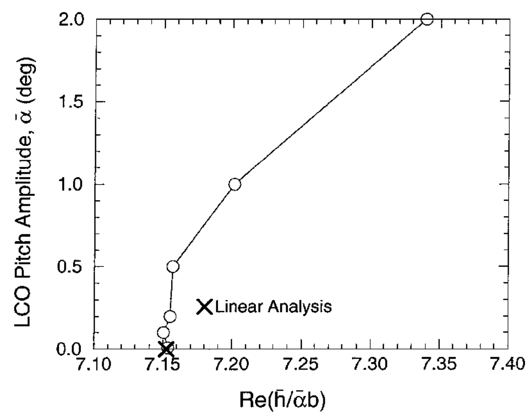
## Future Work

NASA Langley Research Center has developed, over the past decade, a substantial body of experimental data for transonic flutter and LCO that can be compared to the results from newly emerging computational models and methods. Also, the Institute for Aeroelasticity at DLR has provided some interesting experimental results. In addition, time-marching CFD codes have provided some benchmark theoretical results that may be used for comparisons with the newly emerging methods. Further insights into LCO will undoubtedly be developed as a result of such correlation studies.

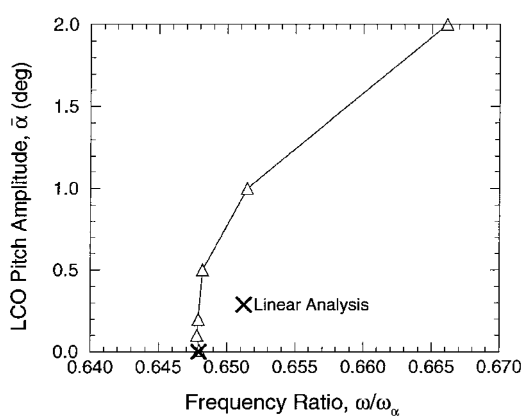
The inclusion of viscosity in nonlinear aerodynamic models remains an open challenge. There are global, semi-empirical models such as that developed by Tran and Petot[54] at ONERA. These use experimental data to represent aerodynamic forces directly, based on equation forms suggested by inviscid theory. On the other hand, there is extensive literature on semi-empirical models for the local flow (so-called turbulence models). Incorporation of these in emerging, unsteady aerodynamic models based on the concept of



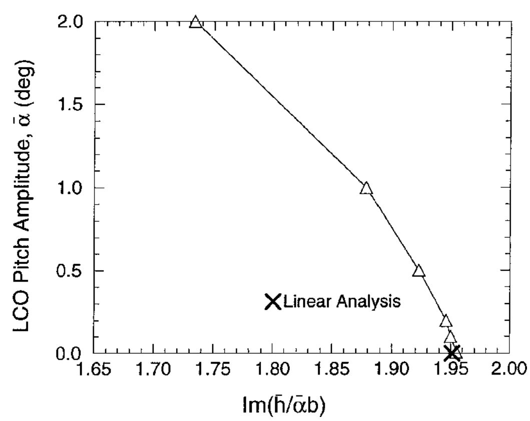

**LCO reduced frequency**




**Real part**




**LCO frequency ratio**




**Imaginary part**

**Fig. 17 Effect of pitch amplitude on LCO reduced frequency and frequency ratio and LCO structural mode shape for Mach number $M = 0.8$.**

aerodynamic modes is within reach, and results may be expected in the near term. However, the adequacy of semi-empirical turbulence models for the flows of interest in aeroelasticity remain to be assessed by systematic studies using reduced-order aerodynamic models. A far deeper question is whether the concept of aerodynamic modes will prove useful for developing solutions to the Navier–Stokes equations from first principles, without the necessity of empirical turbulence modeling. This remains very much an open question, albeit an intriguing one. However, it appears that representing the fluid flowfield in terms of aerodynamic modes is one of the few prospective and promising methods for solving the Navier–Stokes equations de novo at the Reynolds numbers of interest for aerospace applications.

Finally, and in summary, there is real cause for optimism based on the progress to date in developing a more fundamental understanding of LCO and other nonlinear aeroelastic phenomena, as a result of the recent emergence of computationally efficient models and methods. These are providing new insights into the fascinating phenomena of nonlinear aeroelasticity and unsteady aerodynamics, as well as providing a firmer basis for more rational designs and improved performance. One may hope that Theodore von Kármán would be pleased that today's aerospace engineers are grappling more successfully with nonlinearities than ever before.[65]

## Summary

Nonlinear effects in aeroelastic systems may be either favorable, unfavorable, or a combination of both. For example, when a structural stiffness nonlinearity is equivalent to a hardening spring, as in the von Kármán model of a plate, no LCOs will exist below the flutter boundary determined in the absence of nonlinearity. Above the flutter velocity, the nonlinearity will limit the response, a clearly favorable outcome. However, for other nonlinearities, such as structural free play or aerodynamic nonlinearities due to flow separation or large shock motion, the effect of the nonlinearity may be to induce LCO below the nominal flutter velocity but to still limit the LCO response to a finite amplitude both below and above the nominal flutter velocity. Whether such nonlinear effects are favorable or not will depend very much on the particular circumstances and parameters involved. Nonetheless, it is clear that nonlinear effects often lead to LCO, and, in their absence, the alternative would be catastrophic flutter leading to structural failure. Hence, nonlinearities in aeroelastic systems provide an opportunity for improved safety and performance of modern aerospacecraft if reliable and computationally effective analysis and design methods can be developed.

Recent advances in computational models and solution techniques now permit efficient nonlinear aeroelastic analysis, including the determination of LCOs for a number of nonlinearities as described in this paper. The most promising methods are based on two fundamental ideas. First, time-linearized models of the fluid (and the structure) may be used to construct highly compact, reduced-order aerodynamic (and structural) models. Second, with the flutter velocity, frequency, and eigenmode (the aeroelastic motion) determined from such an analysis, the results may be used as the first step in a rapidly converging iteration process to solve for the nonlinear LCO using a novel form of the harmonic balance method for determining the nonlinear aerodynamic forces and the structural deformation.

Nonlinearities that have been successfully modeled theoretically (and the results confirmed by experiment) include 1) structural free play, 2) structural geometric (strain-displacement) nonlinearities for low- and high-aspect wings of relatively simple construction, and 3) separated flow nonlinearities [at low Mach number using a semi-empirical (ONERA) aerodynamic model].

In addition, theoretical fluid models for inviscid, large shock motion have now been developed that permit computationally efficient flutter and LCO analysis arising from such aerodynamic nonlinearities. Correlation with experiment remains an open challenge.

## Appendix: Modeling of Fluid–Structural (Aeroelastic) Interaction

There are three aspects to be considered in the modeling of fluid–structural (aeroelastic) interactions: 1) the modeling of the time-dependent aerodynamic forces on a deforming structure, 2) the modeling of the deforming structure under the action of a distribution of forces, and 3) the determination of the aerodynamic forces and structural deformations simultaneously in time for purposes of a) assessing stability or instability (flutter) due to small perturbations and also b) the LCOs and sensitivity to finite disturbances arising from nonlinear effects. The nonlinearities may originate 1) in the fluid, 2) in the structure, or 3) in both.

For a more in-depth discussion, the interested reader is referred to Refs. 5 and 66. Here an overview is presented that may serve as an introduction for those to whom the subject is relatively new and as a useful summary for the expert. We begin with a discussion of linear models as a point of reference.

### Modeling of the Structure

Most structures of interest may be modeled as being constructed of linearly elastic materials, that is, it is assumed that there is a linear relationship between stress and strain with known material constants. If the strain–displacement (gradient) relationships are also linear (as they are for sufficiently small structural deformation) then the structural model will be entirely linear, that is, deformations will be proportional to the aerodynamic forces acting on the structure. Then, for the small damping typical of an elastic structure, the structure will possess a discrete set of resonant eigenmodes. For aeroelastic analysis (linear or nonlinear) the structural model may be most compactly (and usually most conveniently) expressed in terms of these eigenmodes and their modal amplitudes. A typical finite element structural model may have $10^3$–$10^4$ degrees of freedom (unknowns), whereas a typical eigenmode model of comparable accuracy for aeroelastic analysis will have $10$–$10^2$ degrees of freedom.

There are two further points to be made, however. One is that even for a linear structural model, if the structural configuration is at all complicated, then a finite element analysis of the structure will be required to determine the structural eigenmodes. If the structural configuration is then changed, one may need to do the finite element analysis again to determine the new eigenmodes. However, if the configuration changes are small enough, then the original eigenmodes may still be a sufficiently good basis for representing the modified structural model.

The second point concerns the impact of nonlinearity on the structural model. If, as is usually the case, one can express the kinetic energy and strain energy of the structure in terms of the structural deformations, then one can still expand the latter into a series of the eigenmodes. However, now the structural modal amplitude equations of motion, as derived from Lagrange's equations, for example, will be nonlinear, and, indeed, the nonlinear terms will couple the (modal) equations of motion even if one uses an eigenmode expansion.

### Modeling of the Fluid

Readers new to the field of aeroelasticity may not be surprised if they are told that one can also compactly model the aerodynamic flow in terms of fluid eigenmodes. However, the expert will recognize this approach for representing the aerodynamic flow is quite a recent development. This is the case for two reasons. First, it is substantially more difficult to compute the eigenmodes of an aerodynamic flow compared to the structural eigenmodes. Second, strictly speaking, the aerodynamic eigenmodes are a result of the discretization and finiteness of the computational domain in a typical CFD code. Indeed, in the classical Theodorsen/von Kármán and Sears[44] theory for incompressible flow over an oscillating airfoil in an infinite domain, only a branch cut appears in the exact solution, whereas poles or eigenvalues appear in the corresponding CFD or vortex lattice model[5] due to the approximation made in the latter models by spatial discretization and a finite computational domain.

Even so, because of recent advances, it is now recognized that the eigenmodes of CFD aerodynamic flow models are very useful for deriving a compact unsteady aerodynamic model. However, note that determining the fluid eigenmodes directly from a CFD model is quite difficult, and, indeed, an alternative set of aerodynamic modes or basis functions derived from a proper orthogonal decomposition

(POD) is a more attractive approach for constructing a compact reduced-order aerodynamic modal model. A typical CFD model in three-dimensional flow will have $10^5$–$10^6$ degrees of freedom, whereas an aerodynamic reduced-order modal model will usually have fewer than $10^2$.

In principle, the eigenmode or POD approach may be used to construct a time-linearized or a fully dynamically nonlinear reduced-order aerodynamic modal model. However, the nonlinear models still require further development, as of this writing, before they will be available for practical use. Time-linearized, three-dimensional inviscid flow, reduced-order aerodynamic modal models are now available.

For now, and perhaps for some time to come, the HB method developed for CFD models by Hall et al.[62] and Thomas et al.[64] is a very attractive approach to the fully nonlinear aerodynamic modeling challenge for reasons to be described hereafter.

### Solving the Aeroelastic Equations

First consider a dynamically linear aeroelastic model that is sufficient to establish the onset of flutter (instability with respect to small or strictly infinitesimal perturbations) and often the onset of LCOs. For the strictly linear case, the dynamic structural deformation will be about some trivial (zero deformation) static equilibrium, and the steady (static) flow solution will be one of uniform velocity everywhere in the flowfield before the dynamic perturbation. In general, the static equilibrium per se may be a nonlinear state, however.

By the use of dynamically linear structural and aerodynamic reduced-order modal models, the equations governing the dynamic perturbations about any (linear or nonlinear) static aeroelastic equilibrium state will themselves be in state-space form, that is, a linear system of ordinary differential equations (ODEs). The number of structural modes (states) will typically be of order 10 and the number of aerodynamic modes (states) less than 100. Thus, one may follow the migration of the aeroelastic poles or eigenvalues as some parameter, for example, flow velocity, density, or dynamic pressure, is varied. That is, one may construct a root locus. When the real part of any aeroelastic eigenvalue becomes positive corresponding to exponentially growing oscillations with time, the aeroelastic system is unstable with respect to small perturbations about the static equilibrium condition.

In transonic flow, the steady (or static) flow condition will be nontrivial and must be first determined by a nonlinear static or time-independent analysis before a small dynamic perturbation (flutter) analysis may be carried out. Similarly, static forces on the structure (due to steady aerodynamic flow or gravity, for example) may be sufficient to induce a nontrivial nonlinear static equilibrium condition for the structure. When this is the case, a nonlinear static aeroelastic analysis must be performed before the (linear) small dynamic perturbation analysis.

Finally, to determine LCOs or the sensitivity of any static or dynamic state to large or finite perturbations, a full nonlinear dynamic finite perturbation analysis must be performed. There are several possibilities for how this may be done.

The simplest conceptually, but the most expensive computationally, is to time march the nonlinear ODEs governing the structure (either a modal model or a finite element model) together with the nonlinear ODEs governing the aerodynamic flow CFD model. This has been done by several investigators, but the results are typically for a small parameter range because of the large computational expense.

The computational expense can be greatly reduced by using modal models for the structure and aerodynamic flow. Such structural modes for a variety of nonlinearities have been constructed and used effectively. However, the construction of such dynamically nonlinear aerodynamic flow models is still a subject for ongoing research. Although certainly possible in principle, the size and complex form of typical CFD models renders a practical modal reduction for fully dynamically nonlinear aerodynamic flows a more difficult challenge than, for example, for typical nonlinear structural models. Even so, one may expect useful techniques for such nonlinear aerodynamic modal reduction to be developed, and certainly efforts are underway to do so by the research community.

Another effective approach has been developed for determining LCOs and other nonlinear dynamic responses, however, that is very attractive. This is the well-known HB method, modified and extended for aeroelastic analysis by Hall et al.[62] and Thomas et al.[64] One reason this approach is computationally efficient for determining LCO is that the solution of the nonlinear equations for LCO is reduced to finding the solutions of a set of nonlinear, algebraic equations for the structural modal amplitudes and the corresponding aerodynamic generalized forces. This requires an iteration solution method, for example, Newton–Raphson, but the iteration converges rapidly because a flutter solution (based on a linear dynamic analysis) gives an excellent starting point for the iteration to determine LCO. Note that, in particular, the flutter solution provides the aeroelastic eigenvector for the flutter condition that gives the relative proportions of each structural modal amplitude or generalized coordinate (and also each aerodynamic flowfield variable). Thus, the iteration process for determining the LCO from the solution of the nonlinear, algebraic HB equation can begin with an excellent estimate of the spatial (or equivalently modal) distribution of the structural and flowfield dynamic amplitudes from the flutter solution. Hence, the iteration converges very rapidly, typically in one or two iteration cycles.

## Acknowledgments

This work was supported by the U.S. Air Force Office of Scientific Research. Dan Segalman is the Program Manager. The authors have greatly benefitted from the insights gained from discussing and reading the work of many colleagues in the aeroelastic community over the years. A partial listing of this work is contained in the references to this paper. In addition, the authors would especially like to acknowledge their colleagues at Duke University (Kenneth Hall, Jeffrey Thomas, Lawrence Virgin, and Robert Clark), the members of the Aeroelasticity Branch at NASA Langley Research Center, the aeroelasticity group at the Air Force Flight Dynamic Laboratory, and the industry-led Aerospace Flutter and Dynamics Council. Furthermore, the first author would like to acknowledge gratefully John Dugundji, Holt Ashley, Marten Landahl, Herbert Voss, and Jonathan Turner. Lest we forget.

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