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Finite Dimensional Approximation of Nonlinear Problems

Part I: Branches of Nonsingular Solutions

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Summary. We begin in this paper the study of a general method of approximation of solutions of nonlinear equations in a Banach space. We prove here an abstract result concerning the approximation of branches of nonsingular solutions. The general theory is then applied to the study of the convergence of two mixed finite element methods for the Navier-Stokes and the von Kármán equations.

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1. Introduction

This series of papers is devoted to the study of the numerical approximation of the solutions (λ, u) of nonlinear problems of the form

$$F(\lambda, u) = 0, \tag{1.1}$$

where $F: \Lambda \times V \to V$ for some interval $\Lambda \subset \mathbb{R}$ and some Banach space V.

In this paper, we shall be only concerned with the approximation of branches $\{(\lambda, u(\lambda));\ \lambda \in \Lambda\}$ of *nonsingular* solutions of (1.1), i.e. which satisfy the following properties:

$$\lambda \to u(\lambda) \quad \text{is a continuous function from } \Lambda \text{ into } V; \tag{1.2}$$

$$F(\lambda, u(\lambda)) = 0; \tag{1.3}$$

$$D_u F(\lambda, u(\lambda)) \quad \text{is an isomorphism of } V. \tag{1.4}$$

Our purpose is to study a fairly general method of approximation of problem (1.1) which includes nonstandard finite element methods such as mixed or hybrid methods. In this respect, our results appear to be a generalization of some of [4, Sect. 3] where conforming finite element methods are considered.

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In subsequent papers, we shall study the numerical approximation of singular solutions of (1.1) such as turning points and bifurcation points in the general framework introduced here.

An outline of the paper is as follows. Section 2 is devoted to the derivation of abstract global results of existence and approximation of implicit functions: this is obtained by first proving a generalization of the implicit function theorem and then using continuation arguments. These results, which may be viewed as a generalization of pointwise results of [9], will be constantly used in all the papers of this series as an essential technical tool.

In Sect. 3, we introduce a general method of approximation of problem (1.1) and, using the results of Sect. 2, we give natural sufficient conditions for obtaining a branch of approximate solutions which converges uniformly in λ to the branch of nonsingular solutions of (1.1).

In Sect. 4, we apply the abstract theory to the study of the convergence of a mixed finite element method for the Navier-Stokes equations, using a stream function-vorticity formulation. We then obtain an extension of the results of [6]. We refer to [5] for a mathematical discussion of some finite element methods for the Navier-Stokes equations and to [7] for a generalization of our results when using an upwind treatment of the convective terms.

Finally, Sect. 5 is devoted to the study of the "Hellan-Hermann-Johnson" mixed element method for the Von Kármán equations; we then improve the results of [1].

Some results of this paper have been announced in [12].

2. An Extension of the Implicit Function Theorem

We begin by stating a convenient form of the classical inverse function theorem. We are given two Banach spaces X, Y and a C^1 mapping f defined in a neighborhood of some point $x_0 \in X$ with range in Y. We set $y_0 = f(x_0)$ and we denote by $Df(x_0) \in \mathscr{L}(X, Y)$ the derivative of f at the point x_0. We denote by $\|.\|$ the various norms in $X, Y, \mathscr{L}(X, Y), \ldots$ and we introduce the closed balls

$$S(x_0; \delta) = \{x \in X;\ \|x - x_0\| \leqq \delta\}, \qquad S(y_0; \varepsilon) = \{y \in Y;\ \|y - y_0\| \leqq \varepsilon\}.$$

Lemma 1. *Assume that $Df(x_0)$ is an isomorphism of X onto Y with*

$$\|Df(x_0)^{-1}\| \leqq M. \tag{2.1}$$

If $\delta > 0$ is chosen in such a way that

$$\|x - x_0\| \leqq \delta \Rightarrow \|Df(x) - Df(x_0)\| \leqq \frac{1}{2M}, \tag{2.2}$$

then there exists a unique C^1 function g defined in $S\left(y_0; \frac{\delta}{2M}\right)$ with range in $S(x_0; \delta)$ such that $f_0 g = id_Y$. Moreover, we get for all $y \in S\left(y_0; \frac{\delta}{2M}\right)$

$$\|g(y) - g(y_0)\| \leqq 2M \|y - y_0\|. \tag{2.3}$$

Proof. Consider the mapping A_y: $X \to X$ defined by

$$A_y(x) = x_0 + Df(x_0)^{-1}(y - y_0 + f(x_0) + Df(x_0).(x - x_0) - f(x)).$$

Using (2.1) and (2.2), it is an easy and classical matter to check that, for $y \in S\left(y_0; \frac{\delta}{2M}\right)$, A_y is a strict contraction mapping of $S(x_0; \delta)$ into itself. Hence A_y has a unique fixed point $x \in S(x_0; \delta)$ or, equivalently, there exists a unique $x \in S(x_0; \delta)$ such that $y = f(x)$. This defines a function g: $y \in S\left(y_0; \frac{\delta}{2M}\right) \to x = g(y) \in S(x_0; \delta)$ which is easily seen to be a C^1 function.

Let us now prove the inequality (2.3). Let $x \in S(x_0; \delta)$; it follows from (2.1) and (2.2) that

$$\|Df(x_0)^{-1}(Df(x) - Df(x_0))\| \leqq \tfrac{1}{2}.$$

Hence

$$Df(x) = Df(x_0)(I + Df(x_0)^{-1}(Df(x) - Df(x_0)),$$

is invertible and we have

$$\|Df(x)^{-1}\| \leqq \|Df(x_0)^{-1}\| \, \|(I + Df(x_0)^{-1}(Df(x) - Df(x_0))^{-1}\|$$
$$\leqq \frac{\|Df(x_0)^{-1}\|}{1 - \|Df(x_0)^{-1}(Df(x) - Df(x_0))\|} \leqq 2M.$$

Thus, we get for all $y \in S\left(y_0; \frac{\delta}{2M}\right)$

$$\|Dg(y)\| \leqq 2M,$$

from which the inequality (2.3) follows immediately. ∎

Let us now establish a generalized form of the implicit function theorem. We introduce three Banach spaces X, Y, Z and a C^1 mapping f defined in a neighborhood of a point $(x_0, y_0) \in X \times Y$ with range in Z. We set $f_0 = f(x_0, y_0)$ and we denote by $Df_0 = Df(x_0, y_0) \in \mathscr{L}(X \times Y; Z)$ the derivative of f at the point (x_0, y_0) and by $D_x f_0 = D_x f(x_0, y_0) \in \mathscr{L}(X; Z)$, $D_y f_0 = D_y f(x_0, y_0) \in \mathscr{L}(Y; Z)$ the corresponding partial derivatives. Again, we denote by $\|.\|$ the various norms in $X, Y, Z, \ldots$. We provide the product space $X \times Y$ with the norm $\|(x, y)\| = \|x\| + \|y\|$ and we introduce the closed ball

$$S(x_0, y_0; \xi) = \{(x, y) \in X \times Y;\ \|(x, y) - (x_0, y_0)\| \leqq \xi\}.$$

Lemma 2. *Assume the following hypotheses:*

(i) *the mapping* $D_y f_0$ *is an isomorphism of* Y *onto* Z *with*

$$\|(D_y f_0)^{-1}\| \leqq c_0; \tag{2.4}$$

(ii) *we have*

$$\|D_x f_0\| \leqq c_1, \tag{2.5}$$

and there exists a monotonically increasing function L_1: $\mathbb{R}_+ \to \mathbb{R}_+$ *such that for all* $(x, y) \in S(x_0, y_0; \xi)$

$$\|Df(x, y) - Df_0\| \leqq L_1(\xi)(\|x - x_0\| + \|y - y_0\|). \tag{2.6}$$

Then, one can find three constants $\alpha, \beta, \gamma > 0$ depending only on c_0, c_1 and L_1 such that, under the condition

$$\|f_0\| \leqq \gamma, \tag{2.7}$$

there exists a unique C^1 function g defined in $S(x_0; \alpha)$ with range in $S(y_0; \beta)$ which satisfies

$$f(x, g(x)) = 0. \tag{2.8}$$

Moreover, we have for some constant $K_0 = K_0(c_0, c_1) > 0$ and for all $x \in S(x_0; \alpha)$

$$\|g(x) - y_0\| \leqq K_0 \{\|x - x_0\| + \|f_0\|\}. \tag{2.9}$$

Proof. Let us introduce the mapping F defined in a neighborhood of (x_0, y_0) with range in $X \times Z$ by

$$F(x, y) = (x, f(x, y)).$$

In order to apply Lemma 1 to the function F, we first check that the conditions (2.1) and (2.2) hold.

On the one hand, using (2.4), $DF_0 = DF(x_0, y_0) \in \mathscr{L}(X \times Y; X \times Z)$ is invertible and we have, for all $x \in X$ and all $z \in Z$

$$(DF_0)^{-1}(x, z) = (x, (D_y f_0)^{-1} \cdot (z - D_x f_0 \cdot x)).$$

Hence, it follows from (2.4) and (2.5) that

$$\|(DF_0)^{-1} \cdot (x, z)\| \leqq \|x\| + c_0 \|z - D_x f_0 \cdot x\| \leqq (1 + c_0 c_1) \|x\| + c_0 \|z\|$$

and therefore

$$\|(DF_0)^{-1}\| \leqq \max(c_0, 1 + c_0 c_1) = M.$$

On the other hand, we have for all $(x, y) \in X \times Y$

$$DF(x, y) - DF_0 = (0, Df(x, y) - Df_0),$$

and, by (2.6), we get if $(x, y) \in S(x_0, y_0; \xi)$

$$\|DF(x, y) - DF_0\| \leqq \xi L_1(\xi).$$

Now, if we choose the constant $\beta > 0$ such that $\beta L_1(\beta) \leqq \dfrac{1}{2M}$, we obtain for all $(x, y) \in S(x_0, y_0; \beta)$

$$\|DF(x, y) - DF_0\| \leqq \frac{1}{2M}.$$

It then follows from Lemma 1 that there exists a unique C^1 function G defined in the ball $S\left(x_0, f_0; \dfrac{\beta}{2M}\right)$ with range in the ball $S(x_0, y_0; \beta)$ such that $F \circ G = id_{X \times Z}$. Clearly, the function G is of the form

$$G(x, z) = (x, \varphi(x, z)) \qquad \text{with } f(x, \varphi(x, z)) = z.$$

As a consequence, choosing $z = 0$ and assuming that $\|f_0\| \leqq \gamma \leqq \dfrac{\beta}{4M}$ (say), we get the existence of a unique C^1 function g: $x \to g(x) = \varphi(x, 0)$ defined in $S(x_0; \alpha)$,

$\alpha = \frac{\beta}{4M}$, with range in $S(y_0; \beta)$ such that

$$f(x, g(x)) = 0.$$

Finally, by applying the inequality (2.3) to the function G and by noticing that $\varphi(x_0, f_0) = y_0$, we obtain for all $x \in S(x_0; \alpha)$

$$\|g(x) - y_0\| \leqq \|G(x, 0) - G(x_0, f_0)\| \leqq 2M(\|x - x_0\| + \|f_0\|),$$

which gives the desired estimate (2.9) with $K_0 = 2M$. ■

Remark 1. Let $\theta \in [0, 1]$; by replacing in the proof of Lemma 2 β by $\theta\beta$ and assuming that $\|f_0\| \leqq \frac{\theta\beta}{4M}$, we find that the function g maps the ball $Sx_0; \theta\alpha)$ into the ball $Sy_0; \theta\beta)$. ■

Let us next give bounds for the derivatives of the function g. We begin with the first derivative Dg and we introduce the function $f^{(1)}$: $(x, y, y^{(1)}) \in X \times Y \times \mathscr{L}(X; Y) \to f^{(1)}(x, y, y^{(1)}) \in \mathscr{L}(X; Z)$ defined by

$$f^{(1)}(x, y, y^{(1)}) = D_x f(x, y) + D_y f(x, y) \cdot y^{(1)}. \tag{2.10}$$

Note that

$$f^{(1)}(x, g(x), Dg(x)) = 0. \tag{2.11}$$

Lemma 3. *Assume the hypotheses of Lemma* 2 *and in addition*

$$\|Df_0\| \leqq c_1. \tag{2.12}$$

Then, there exists a continuous function K_1: $\mathbb{R}_+ \to \mathbb{R}_+$ *which depends only on* c_0, c_1 and L_1, *such that for all* $x \in S(x_0; \alpha)$ *and all* $y^{(1)} \in \mathscr{L}(X; Y)$

$$\|Dg(x) - y^{(1)}\| \leqq K_1(\|y^{(1)}\|)\{\|x - x_0\| + \|f_0\| + \|f^{(1)}(x_0, y_0, y^{(1)})\|\}. \tag{2.13}$$

Proof. Let $x \in S(x_0; \alpha)$; using (2.6), (2.7) and the bound (2.9), we obtain

$$\begin{aligned}\|D_y f(x, g(x)) - D_y f_0\| &\leqq L_1(\alpha + \beta)(\|x - x_0\| + \|g(x) - y_0\|)\\ &\leqq (1 + K_0) L_1(\alpha + \beta)(2\alpha + \gamma)\end{aligned}$$

so that we get by choosing α and γ small enough

$$\|D_y f(x, g(x)) - D_y f_0\| \leqq \frac{1}{2c_0}.$$

Now, it follows from (2.4) that

$$D_y f(x, g(x)) = D_y f_0 (I + (D_y f_0)^{-1} (D_y f(x, g(x)) - D_y f_0)),$$

is an isomorphism of Y onto Z with

$$\|D_y f(x, g(x))^{-1}\| \leqq 2c_0. \tag{2.14}$$

On the other hand, it follows from (2.11) that we have

$$Dg(x) = -(D_y f(x, g(x))^{-1} D_x f(x, g(x)),$$

so that we may write for all $y^{(1)} \in \mathscr{L}(X;Y)$

$$Dg(x)-y^{(1)} = -(D_y f(x,g(x)))^{-1}\{f^{(1)}(x_0,y_0,y^{(1)})+(D_x f(x,g(x))-D_x f_0) \\ +(D_y f(x,g(x))-D_y f_0)\cdot y^{(1)}\}.$$

Applying again (2.6), (2.7), (2.9) together with (2.14) gives

$$\|Dg(x)-y^{(1)}\| \leqq c\{(1+\|y^{(1)}\|)(\|x-x_0\|+\|f_0\|)+f^{(1)}(x_0,y_0,y^{(1)})\|\},$$

for some constant $c=c(c_0,c_1,L_1)>0$. Hence the inequality (2.13) holds with $K_1(\xi)=c(1+\xi)$. ∎

The previous result can now be extended so as to cover the case of higher derivatives of the function g. If f is a C^m function, then g is also a C^m function and we denote by $D^m f(x,y) \in \mathscr{L}_m(X\times Y;Z)$ and $D^m g(x) \in \mathscr{L}_m(X;Y)$ the corresponding m-th derivatives, where $\mathscr{L}_m(X;Y)$ is the space of all continuous m-linear mappings of X^m into Y [1]. Let us next define by induction the functions $f^{(l)}$: $(x,y,y^{(1)},\dots,y^{(l)}) \in X\times Y\times\mathscr{L}(X;Y)\times\dots\times\mathscr{L}_l(X;Y) \to f^{(l)}(x,y,y^{(1)},\dots,y^{(l)}) \in \mathscr{L}_l(X;Z)$, $2\leqq l\leqq m$, by

$$f^{(l+1)}(x,y,y^{(1)},\dots,y^{(l+1)}) = D_x f^{(l)}(x,y,y^{(1)},\dots,y^{(l)}) \\ +D_y f^{(l)}(x,y,\dots,y^{(l)})\cdot y^{(1)}+\sum_{i=1}^{l} D_{y^{(i)}}\, f^{(l)}(x,y,\dots,y^{(l)})\cdot y^{(i+1)}. \tag{2.15}$$

We have

$$f^{(l)}(x,g(x),Dg(x),\dots,D^l g(x))=0. \tag{2.16}$$

Lemma 4. *Assume the hypotheses of Lemma 3. Assume in addition that the mapping f is of class C^m, $m\geqq 2$, and there exist monotonically increasing functions L_l: $\mathbb{R}_+ \to \mathbb{R}_+$, $2\leqq l\leqq m$, such that for all $(x,y)\in S(x_0,y_0;\xi)$*

$$\|D^l f(x,y)-D^l f_0\| \leqq L_l(\xi)(\|x-x_0\|+\|y-y_0\|), \quad 2\leqq l\leqq m. \tag{2.17}$$

Then, there exists a continuous function K_m: $\mathbb{R}_+^m \to \mathbb{R}_+$ which depends only on $c_0,c_1,L_1,\dots,L_m$ such that for all $x\in S(x_0;\alpha)$ and all $y^{(l)}\in\mathscr{L}_l(X;Y)$, $1\leqq l\leqq m$

$$\|D^m g(x)-y^{(m)}\| \leqq K_m(\|y^{(1)}\|,\dots,\|y^{(m)}\|)\left\{\|x-x_0\|+\|f_0\| \right. \\ \left. +\sum_{l=1}^{m}\|f^{(l)}(x_0,y_0,y^{(1)},\dots,y^{(l)})\|\right\}. \tag{2.18}$$

Proof. For simplicity, we only sketch the proof in the case $m=2$. Denote by c various positive constants which depend only on c_0,c_1,L_1 and L_2. Let $x\in S(x_0;\alpha)$; using (2.13) with $y^{(1)}=0$, we have

$$\|Dg(x)\| \leqq K_1(0)\{\|x-x_0\|+\|f_0\|+\|D_x f_0\|\}$$

so that by (2.5) and (2.7)

$$\|Dg(x)\| \leqq c. \tag{2.19}$$

[1] More generally, $\mathscr{L}_m(X_1,\dots,X_m;Y)$ is the space of all m-linear continuous mappings of $\prod_{i=1}^{m} X_i$ into Y and we set $\mathscr{L}_m(X;Y)=\mathscr{L}_m(\underbrace{X,\dots,X}_{m\text{ times}};Y)$

Next, using (2.16) with $l=2$, we get for all $y^{(k)} \in \mathscr{L}_k(X;Y)$, $k=1,2$

$$\begin{aligned} D_y f(x, g(x)) \cdot (y^{(2)} - D^2 g(x)) &= f^2(x, g(x), Dg(x), y^{(2)}) \\ &= f^{(2)}(x_0, y_0, y^{(1)}, y^{(2)}) + (f^{(2)}(x, g(x), Dg(x), y^{(2)}) \\ &\quad - f^{(2)}(x_0, y_0, y^{(1)}, y^{(2)})). \end{aligned} \tag{2.20}$$

Let us evaluate

$$\begin{aligned} & f^{(2)}(x, g(x), Dg(x), y^{(2)}) - f^{(2)}(x_0, y_0, y^{(1)}, y^{(2)}) \\ &= (D^2_{xx} f(x, g(x)) - D^2_{xx} f_0) + 2(D^2_{xy} f(x, g(x)) \cdot Dg(x) - D^2_{xy} f_0 \cdot y^{(1)}) \\ &\quad + (D^2_{yy} f(x, g(x)) \cdot (Dg(x), Dg(x)) - D^2_{yy} f_0 \cdot (y^{(1)}, y^{(1)}) \\ &\quad + (D_y f(x, g(x)) - D_y f_0) \cdot y^{(2)}. \end{aligned}$$

Applying (2.9) and (2.17) with $l=2$, we obtain

$$\|D^2_{xx} f(x, g(x)) - D^2_{xx} f_0\| \leqq c(\|x - x_0\| + \|f_0\|). \tag{2.21}$$

Next, we write

$$\begin{aligned} D^2_{xy} f(x, g(x)) \cdot Dg(x) - D^2_{xy} f_0 \cdot y^{(1)} &= (D^2_{xy} f(x, g(x)) - D^2_{xy} f_0) \cdot Dg(x) \\ &\quad + D^2_{xy} f_0 \cdot (Dg(x) - y^{(1)}). \end{aligned}$$

Using again (2.9) and (2.17) together with (2.19), we have

$$\|(D^2_{xy} f(x, g(x)) - D^2_{xy} f_0) \cdot Dg(x)\| \leqq c(\|x - x_0\| + \|f_0\|).$$

On the other hand, it follows from (2.6) that

$$\|D^2 f_0\| \leqq c$$

and by (2.13)

$$\|D^2_{xy} f_0 \cdot (Dg(x) - y^{(1)})\| \leqq c K_1(\|y^{(1)}\|) \{\|x - x_0\| + \|f_0\| + \|f^{(1)}(x_0, y_0, y^{(1)})\|\}.$$

Hence we get

$$\begin{aligned} & \|D^2_{xy} f(x, g(x)) \cdot Dg(x) - D^2_{xy} f_0 \cdot y^{(1)}\| \\ & \quad \leqq c(1 + K_1(\|y^{(1)}\|)) \{\|x - x_0\| + \|f_0\| + \|f^{(1)}(x_0, y_0, y^{(1)})\|\}. \end{aligned} \tag{2.22}$$

Similarly, we obtain

$$\begin{aligned} & \|D^2_{yy} f(x, g(x)) \cdot (Dg(x), Dg(x)) - D^2_{yy} f_0 \cdot (y^{(1)}, y^{(1)})\| \\ & \quad \leqq c(1 + K_1(\|y^{(1)}\|)(1 + \|y^{(1)}\|) \{\|x - x_0\| + \|f_0\| + \|f^{(1)}(x_0, y_0, y^{(1)})\|\} \end{aligned} \tag{2.23}$$

and

$$\|(D_y f(x, g(x)) - D_y f_0) \cdot y^{(2)}\| \leqq c\, \|y^{(2)}\| \,(\|x - x_0\| + \|f_0\|). \tag{2.24}$$

Combining (2.21)–(2.24), we find

$$\begin{aligned} & \|f^{(2)}(x, g(x), Dg(x), y^{(2)}) - f^{(2)}(x_0, y_0, y^{(1)}, y^{(2)})\| \\ & \quad \leqq c\,\{(1 + K_1(\|y^{(1)}\|))(1 + \|y^{(1)}\|) + \|y^{(2)}\|\} \\ & \qquad \cdot \{\|x - x_0\| + \|f_0\| + \|f^{(1)}(x_0, y_0, y^{(1)})\|\} \end{aligned} \tag{2.25}$$

For $m=2$, the desired inequality (2.18) follows from (2.20) and the bounds (2.14) and (2.25). ■

So far, we have only obtained local results for the implicit function g. We now want to derive global properties. To this end, we assume that there exists a function $x \to y(x)$ defined in a subset B of X with range in Y which satisfies the uniform Lipschitz condition

$$\|y(x)-y(x^*)\| \leqq c_2 \|x-x^*\| \qquad \text{for all } x, x^* \in B. \tag{2.26}$$

Also, we assume that the C^1 mapping f is defined in a neighborhood of $B \times y(B)$.

Theorem 1. *Suppose that the function $x \to y(x)$ satisfies* (2.26). *Suppose in addition that the following hypotheses hold:*

(i) *for all $x_0 \in B$, $D_y f(x_0, y(x_0))$ is an isomorphism of Y onto Z with*

$$\sup_{x_0 \in B} \|(D_y f(x_0, y(x_0)))^{-1}\| \leqq c_0; \tag{2.27}$$

(ii) *we have*

$$\sup_{x_0 \in B} \|(D_x f(x_0, y(x_0)))\| \leqq c_1 \tag{2.28}$$

and there exists a monotonically increasing function L_1: $\mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x_0 \in B$ and all $(x, y) \in S(x_0 y(x_0); \xi)$

$$\|Df(x,y) - Df(x_0, y(x_0))\| \leqq L_1(\xi)(\|x-x_0\| + \|y-y(x_0)\|). \tag{2.29}$$

Then, one can find three constants $a, b, d > 0$ depending only on c_0, c_1, c_2 and L_1 such that, under the condition

$$\sup_{x_0 \in B} \|f(x_0, y(x_0))\| \leqq d, \tag{2.30}$$

there exists a unique C^1 function g: $\bigcup_{x_0 \in B} S(x_0; a) \to Y$ which satisfies

$$f(x, g(x)) = 0, \tag{2.31}$$

and maps $S(x_0; a)$ into $S(y(x_0); b)$, $x_0 \in B$. Moreover, we have for all $x_0 \in B$ and all $x \in S(x_0; a)$

$$\|g(x) - y(x_0)\| \leqq K_0(\|x-x_0\| + \|f(x_0, y(x_0))\|), \tag{2.32}$$

where the constant $K_0 > 0$ depends only on c_0, c_1.

Proof. We may apply Lemma 2 at each point $(x_0, y(x_0))$, $x_0 \in B$. Now, using the above hypotheses, one can easily check that the constants α, β, γ and K_0 appearing in Lemma 2 an be chosen independently of $x_0 \in B$. Hence, under the condition

$$\sup_{x_0 \in B} \|f(x_0, y(x_0))\| \leqq \gamma,$$

and for each $x_0 \in B$, there exists a unique C^1 function g_0 defined in $S(x_0; \alpha)$ with range in $S(y(x_0); \beta)$ and satisfying (2.31). We have to prove that these functions match together.

Let $\delta=\min\left(\frac{\alpha}{2},\frac{\beta}{2c_2}\right)$ and let $x_0, x_1 \in B$ with $\|x_0-x_1\|<\delta$; we want to show that the corresponding implicit functions g_0 and g_1 coincide in the intersection of the two open balls $\mathring{S}(x_0;\delta)$ and $\mathring{S}(x_1;\delta)$. In fact, let x belong to this intersection; we write

$$\|g_1(x)-y(x_0)\| \leqq \|g_1(x)-y(x_1)\| + \|y(x_1)-y(x_0)\|.$$

Using Remark 1, we have for d small enough $\left(d \leqq \frac{\beta}{8M}\right)$

$$\|g_1(x)-y(x_1)\| \leqq \frac{\beta}{2}.$$

On the other hand, it follows from (2.26) that

$$\|y(x_1)-y(x_0)\| \leqq \frac{\beta}{2}.$$

Hence $g_1(x) \in S(y(x_0);\beta)$. By the uniqueness of the implicit function g_0: $S(x_0;\alpha) \to S(y(x_0);\beta)$, we conclude that

$$g_1(x)=g_0(x) \qquad \text{for all } x \in \mathring{S}(x_0;\delta) \cap \mathring{S}(x_1;\delta).$$

By setting

$$g(x)=g_0(x) \qquad \text{in } \mathring{S}(x_0;\delta),\ x_0 \in B,$$

we thus define a unique C^1 function $g\colon \bigcup_{x_0 \in B} \mathring{S}(x_0;\delta) \to Y$ which satisfies (2.31) and maps $\mathring{S}(x_0;\delta)$ into $S\left(y(x_0);\frac{\beta}{2}\right)$ for all $x_0 \in B$. Therefore, the first part of the theorem holds with $a<\delta$ and $b=\frac{\beta}{2}$.

Finally, the estimate (2.32) is a consequence of (2.9) used with $y_0=y(x_0)$. ■

Let us next state global results for the derivatives of the function g. For every integer $m \geqq 1$, we introduce a function $y^{(m)}\colon B \to \mathscr{L}_m(X;Y)$. Then applying Lemma 4 gives

Theorem 2. *Assume the hypotheses of Theorem* 1. *Assume in addition that the mapping f is of class C^m, $m \geqq 1$, and satisfies:*

(i) $\sup_{x_0 \in B} \|Df(x_0, y(x_0))\| \leqq c_1;$ (2.33)

(ii) *there exists monotonically increasing functions $L_l\colon \mathbb{R}_+ \to \mathbb{R}_+$, $2 \leqq l \leqq m$, such that for all $(x,y) \in S(x_0, y(x_0);\xi)$*

$$\|D^l f(x,y) - D^l f(x_0, y(x_0))\| \leqq L_l(\xi)(\|x-x_0\| + \|y-y(x_0)\|), \qquad 2 \leqq l \leqq m. \tag{2.34}$$

Then, there exists a continuous function K_m: $\mathbb{R}_+^m \to \mathbb{R}_+$ *which depends only on* $c_0, c_1, L_1, \ldots, L_m$ *such that for all* $x_0 \in B$ *and all* $x \in S(x_0; \alpha)$

$$\begin{aligned} \|D^m g(x) - y^{(m)}(x_0)\| \leqq K_m(\|y^{(1)}(x_0)\|, \ldots, \|y^{(m)}(x_0)\|) \\ \times \left\{ \|x - x_0\| + \|f(x_0, y(x_0))\| + \sum_{l=1}^{m} \|f^{(l)}(x_0, y(x_0), \ldots, y^{(l)}(x_0))\| \right\}. \end{aligned} \tag{2.35}$$

3. Approximation of Branches of Nonsingular Solutions

Let V and W be two Banach spaces, Λ be a compact interval of the real line $\mathbb{R}$. We introduce a C^1 mapping G: $\Lambda \times V \to W$ and a linear continuous mapping $T \in \mathscr{L}(W; V)$. We set:

$$F(\lambda, u) = u + TG(\lambda, u). \tag{3.1}$$

We want to find a pair $(\lambda, u) \in \Lambda \times V$ solution of the equation

$$F(\lambda, u) = 0. \tag{3.2}$$

We shall assume in all the sequel that

(i) for all $(\lambda, u) \in$, $\Lambda \times V$, the operator $TD_u G(\lambda, u) \in \mathscr{L}(V; V)$ is *compact;*

(ii) there exists a branch $\{(\lambda, u(\lambda));\ \lambda \in \Lambda\}$ of nonsingular solutions of the equation (3.2).

Note that by the implicit function theorem $\lambda \to u(\lambda)$ is a C^1 function from Λ into V.

The purpose of this section is to study the approximation of such a branch of nonsingular solutions. To do this, we introduce three Banach spaces $\tilde{V}$, $\tilde{W}$ and Z such that

$$Z \subset V \subset \tilde{V}, \quad \tilde{W} \subset W \text{ with continuous imbeddings.}$$

We assume that, for $u \in Z$, $DG(\lambda, u)$ may be extended as an operator of $\mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{W})$ with the following properties:

(i) for $(\lambda, u) \in \Lambda \times Z$, the operator $TD_u G(\lambda, u) \in \mathscr{L}(\tilde{V}; \tilde{V})$ is *compact*; (3.3)

(ii) the mapping $(\lambda, u) \in \Lambda \times Z \to DG(\lambda, u) \in \mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{W})$ is Lipschitz continuous on the bounded subsets of $\Lambda \times Z$, i.e. there exists a function $L: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ monotonically increasing with respect to each variable such that for all $\lambda, \lambda^* \in$, and all $u, u^* \in Z$

$$\begin{aligned} &\|DG(\lambda^*, u^*) - DG(\lambda, u)\|_{\mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{W})} \\ &\leqq L(|\lambda^*| + \|u^*\|_Z, |\lambda| + \|u\|_Z)(|\lambda^* - \lambda| + \|u^* - u\|_Z). \end{aligned} \tag{3.4}$$

Moreover, we suppose that for any $\lambda \in \Lambda$, $u(\lambda)$ belongs to Z and

$$\text{the function } \lambda \to u(\lambda) \text{ is continuous from } \Lambda \text{ into } Z. \tag{3.5}$$

Next, for each value of a real parameter $h > 0$ which will tend to zero, we are given a finite-dimensional subspace V_h of the space Z and an operator $T_h \in \mathscr{L}(W; V_h)$. We set:

$$F_h(\lambda, u_h) = u_h + T_h G(\lambda, u_h), \quad \lambda \in \Lambda,\ u_h \in V_h. \tag{3.6}$$

The approximate problem consists in finding a pair $(\lambda, u_h) \in \Lambda \times V_h$ solution of the equation

$$F_h(\lambda, u_h) = 0. \tag{3.7}$$

We want to establish the existence of a branch $\{(\lambda, u_h(\lambda));\ \lambda \in \Lambda\}$ of solutions of the Eq. (3.7) which approximates the branch of nonsingular solutions of (3.2).

In all the sequel, we shall denote by C a generic constant >0 independent of h and λ. Then, we assume that the following inverse inequality holds

$$\|v_h\|_Z \leqq C h^{-r} \|v_h\|_{\tilde{V}}, \tag{3.8}$$

for some $r \geqq 0$ and for all $v_h \in V_h$. Moreover, we suppose that there exists a function $\Pi_h u\colon \lambda \in \Lambda \to \Pi_h u(\lambda) \in V_h$ such that

$$\|\Pi_h u(\lambda^*) - \Pi_h u(\lambda)\|_{\tilde{V}} \leqq C |\lambda^* - \lambda| \qquad \text{for all } \lambda, \lambda^* \in \Lambda \tag{3.9}$$

and

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|u(\lambda) - \Pi_h u(\lambda)\|_Z = 0. \tag{3.10}$$

Theorem 3. *Assume the hypotheses* (3.3), (3.4), (3.5), (3.8), (3.9) *and* (3.10). *Assume in addition that:*

$$\lim_{h \to 0} \|T - T_h\|_{\mathscr{L}(\tilde{W}; \tilde{V})} = 0. \tag{3.11}$$

Then, under the condition:

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} h^{-r} \|F_h(\lambda, \Pi_h u(\lambda))\|_{\tilde{V}} = 0, \tag{3.12}$$

and for $h \leqq h_0$ small enough, there exists a constant $b > 0$ independent of h and a unique C^1 mapping $\lambda \in \Lambda \to u_h(\lambda) \in V_h$ such that for all $\lambda \in \Lambda$

$$\begin{gathered} F_h(\lambda, u_h(\lambda)) = 0, \\ \|u_h(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} \leqq b h^r \end{gathered} \tag{3.13}$$

Moreover, we have for some constant $K_0 > 0$ independent of h and λ:

$$\|u_h(\lambda) - u(\lambda)\|_{\tilde{V}} \leqq K_0 \{\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T_h - T)\, G(\lambda, u(\lambda))\|_{\tilde{V}}\}. \tag{3.14}$$

Proof. In order to apply Theorem 1, we first check that there exists a constant $c_0 > 0$ (independent of h and λ) with

$$\|D_u F_h(\lambda, \Pi_h u(\lambda)) \cdot v_h\|_{\tilde{V}} \geqq c_0^{-1} \|v_h\|_{\tilde{V}} \qquad \text{for all } v_h \in V_h. \tag{3.15}$$

Set $J = D_u G(\lambda, u(\lambda))$. Since $u(\lambda)$ is a nonsingular solution of (3.2), the operator $D_u F(\lambda, u(\lambda)) = I + TJ$ is an isomorphism of V. Moreover, since $u(\lambda)$ is assumed to belong to the space Z, it follows from (3.3) that the operator $TJ \in \mathscr{L}(\tilde{V}; \tilde{V})$ is compact. Let $v \in \tilde{V}$ satisfy $(I + TJ)v = 0$; we have $-v = TJv \in V$ so that $v = 0$. Hence it follows from the compactness of TJ and the Fredholm alternative that $D_u F(\lambda, u)$ is also an isomorphism of $\tilde{V}$.

Now, the hypotheses (3.4) and (3.5) imply that the mapping $\lambda \to D_u F(\lambda, u(\lambda))$ is continuous from Λ into $\mathscr{L}(\tilde{V}; \tilde{V})$ and the same is indeed true of the mapping

$\lambda \to (D_u F(\lambda, u(\lambda))^{-1}$. We then set:

$$\alpha = \max_{\lambda \in \Lambda} \|(D_u F(\lambda, u(\lambda)))^{-1}\|_{\mathscr{L}(\tilde{V}; \tilde{V})}.$$

Next, we write

$$D_u F_h(\lambda, \Pi_h u(\lambda)) = D_u F(\lambda, u(\lambda)) + (T_h - T) D_u G(\lambda, u(\lambda)) \\ + T_h(D_u G(\lambda, \Pi_h u(\lambda)) - D_u G(\lambda, u(\lambda))),$$

so that we obtain for all $v_h \in V_h$

$$\|D_u F_h(\lambda, \Pi_h u(\lambda)) \cdot v_h\|_{\tilde{V}} \geqq (\alpha^{-1} - \|T_h - T\|_{\mathscr{L}(\tilde{W}; \tilde{V})} \|D_u G(\lambda, u(\lambda))\|_{\mathscr{L}(\tilde{V}; \tilde{W})} \\ - \|T_h\|_{\mathscr{L}(\tilde{W}; \tilde{V})} \|D_u G(\lambda, \Pi_h u(\lambda)) - D_u G(\lambda, u(\lambda))\|_{\mathscr{L}(\tilde{V}; \tilde{W})}) \|v_h\|_{\tilde{V}}.$$

Using the hypotheses (3.4), (3.5), (3.10) and (3.11), we get for all $\lambda \in \Lambda$

$$\|D_u F_h(\lambda, \Pi_h u(\lambda)) \cdot v_h\|_{\tilde{V}} \geqq (\alpha^{-1} - \varepsilon(h)) \|v_h\|_{\tilde{V}},$$

with $\lim_{h \to 0} \varepsilon(h) = 0$. Hence, (3.15) follows for h small enough. Consider now

$$D_\lambda F_h(\lambda, \Pi_h u(\lambda)) = T_h D_\lambda G(\lambda, \Pi_h u(\lambda)).$$

Using again the hypotheses (3.4), (3.5), (3.10) and (3.11), we have for some constant $c_1 > 0$ (independent of h and λ)

$$\|D_\lambda F_h(\lambda, \Pi_h(\lambda))\|_{\tilde{V}} \leqq c_1. \tag{3.16}$$

Finally, using (3.4) and (3.11), we obtain for all $\lambda, \lambda^* \in \Lambda$ and all $u_h^* \in V_h$

$$\|DF_h(\lambda^*, u_h^*) - DF_h(\lambda, \Pi_h u(\lambda))\|_{\mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{V})} \\ \leqq CL(|\lambda^*| + \|u_h^*\|_Z, |\lambda| + \|\Pi_h u(\lambda)\|_Z)(|\lambda^* - \lambda| + \|u_h^* - \Pi_h u(\lambda)\|_Z).$$

Now, applying the inverse inequality (3.8) and the condition (3.10), there exists a monotonically increasing function $L_1\colon \mathbb{R}_+ \to \mathbb{R}_+$ independent of h such that we have for all $\lambda, \lambda^* \in \Lambda$ and all $u_h^* \in V_h$ with $h^{-r}(|\lambda^* - \lambda| + \|u_h^* - \Pi_h u(\lambda)\|_{\tilde{V}}) \leqq \xi$

$$\|DF_h(\lambda^*, u_h^*) - DF_h(\lambda, \Pi_h u(\lambda))\|_{\mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{W})} \\ \leqq L_1(\xi) h^{-r}(|\lambda^* - \lambda| + \|u_h^* - \Pi_h u(\lambda)\|_{\tilde{V}}). \tag{3.17}$$

Thus, it follows from (3.9), (3.15), (3.16) and (3.17) that we may apply Theorem 1 to the mapping F_h in the following situation:

$$\begin{aligned} X &= \mathbb{R} && \text{provided with the norm } h^{-r}|\lambda|, \\ Y = Z &= V_h && \text{provided with the norm } h^{-r}\|v_h\|_{\tilde{V}}, \\ y(\lambda) &= \Pi_h u(\lambda). \end{aligned} \tag{3.18}$$

Since

$$\|A_h\|_{\mathscr{L}(X \times Y; Z)} = \|A_h\|_{\mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{V})} \quad \text{for all } A_h \in \mathscr{L}(\mathbb{R} \times V_h; V_h)$$

and under the condition (3.12), there exists, for all $h \leqq h_0$ small enough, a constant $b>0$ independent of h and a unique C^1 function $\lambda \in \Lambda \to u_h(\lambda) \in V_h$ such that (3.13) and the inequality

$$\|u_h(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} \leqq C \|F_h(\lambda, \Pi_h u(\lambda))\|_{\tilde{V}},$$

hold for all $\lambda \in \Lambda$. For getting the desired estimate (3.14), it remains only to check the inequality

$$\|F_h(\lambda, \Pi_h u(\lambda))\|_{\tilde{V}} \leqq C\{\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T_h - T)\, G(\lambda, u(\lambda))\|_{\tilde{V}}\}. \tag{3.19}$$

In fact, we have

$$\begin{aligned} F_h(\lambda, \Pi_h u(\lambda)) = F_h(\lambda, \Pi_h u(\lambda)) - F(\lambda, u(\lambda)) = \Pi_h u(\lambda) - u(\lambda) \\ + (T_h - T)\, G(\lambda, u(\lambda)) + T_h(G(\lambda, \Pi_h u(\lambda)) - G(\lambda, u(\lambda))). \end{aligned}$$

The bound (3.19) will be proved if we show that

$$\|T_h(G(\lambda, \Pi_h u(\lambda)) - G(\lambda, u(\lambda)))\|_{\tilde{V}} \leqq C \|\Pi_h u(\lambda) - u(\lambda)\|_{\tilde{V}}.$$

But again, this is a simple consequence of the hypotheses (3.4), (3.5), (3.10), and (3.11). ∎

Remark 2. By using the inequality (3.19), we may as well suppose that in Theorem 3 the condition (3.12) is replaced by

$$\begin{aligned} &\lim_{h \to 0} \sup_{\lambda \in \Lambda} h^{-r} \|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} = 0, \\ &\lim_{h \to 0} \sup_{\lambda \in \Lambda} h^{-r} \|(T_h - T)\, G(\lambda, u(\lambda))\|_{\tilde{V}} = 0. \quad ∎ \end{aligned} \tag{3.20}$$

As an obvious corollary of Theorem 3, we get

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|u_h(\lambda) - u(\lambda)\|_{\tilde{V}} = 0, \tag{3.21}$$

i.e. the branch $\{(\lambda, u_h(\lambda)); \lambda \in \Lambda\}$ of approximate solutions converges to the branch $\{(\lambda, u(\lambda)); \lambda \in \Lambda\}$ of nonsingular solutions of (3.2) in the space $\tilde{V}$ and uniformly in $\lambda \in \Lambda$.

We now turn to the approximation of the first derivative $u'(\lambda)$ of the function $u(\lambda)$. We define as in (2.10) the mappings $F^{(1)}: \Lambda \times V^2 \to V$, $G^{(1)}: \Lambda \times V^2 \to W$ and $F_h^{(1)}: \Lambda \times V_h^2 \to V_h$. We introduce a function $\Pi_h u': \lambda \in \Lambda \to \Pi_h u'(\lambda) \in V_h$ and we suppose that

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|u'(\lambda) - \Pi_h u'(\lambda)\|_{\tilde{V}} = 0. \tag{3.22}$$

We obtain

Theorem 4. *Assume the hypotheses of Theorem 3 together with* (3.22). *Then, we have for some constant K_1 independent of h and λ*

$$\begin{aligned} \|u_h'(\lambda) - u'(\lambda)\|_{\tilde{V}} \leqq K_1 \{h^{-r}[\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T_h - T)\, G(\lambda, u(\lambda))\|_{\tilde{V}}] \\ + \|u(\lambda) - \Pi_h u(\lambda)\|_Z + \|u'(\lambda) - \Pi_h u'(\lambda)\|_{\tilde{V}} + \|(T_h - T)\, G^{(1)}(\lambda, u(\lambda), u'(\lambda))\|_{\tilde{V}}\}. \end{aligned} \tag{3.23}$$

Proof. Just as in the proof of Theorem 3, we check that

$$\|DF_h(\lambda, \Pi_h u(\lambda))\|_{\mathscr{L}(\mathbb{R}\times\tilde{V};\tilde{V})} \leqq C.$$

Thus, we may apply Theorem 2 to the mapping F_h in the case $m=1$ and in the situation (3.18) with in addition $y^{(1)}(\lambda)=\Pi_h u'(\lambda)$. Since by (3.22)

$$\|\Pi_h u'(\lambda)\|_{\tilde{V}} \leqq C,$$

and

$$\|\cdot\|_{\mathscr{L}(X;Y)} = \|\cdot\|_{\mathscr{L}(X;Z)} = \|\cdot\|_{\tilde{V}}$$

we obtain

$$\begin{aligned}&\|u_h'(\lambda)-\Pi_h u'(\lambda)\|_{\tilde{V}}\\ &\quad\leqq C\{h^{-r}\|F_h(\lambda, \Pi_h u(\lambda))\|_{\tilde{V}} + \|F_h^{(1)}(\lambda, \Pi_h u(\lambda), \Pi_h u'(\lambda))\|_{\tilde{V}}\}.\end{aligned}$$

It remains to check that this implies (3.23).

Since $F^{(1)}(\lambda, u(\lambda), u'(\lambda))=0$, we get

$$\begin{aligned}F_h^{(1)}(\lambda, \Pi_h u(\lambda), \Pi_h u'(\lambda)) &= F_h^{(1)}(\lambda, \Pi_h u(\lambda), \Pi_h u'(\lambda)) - F^{(1)}(\lambda, u(\lambda), u'(\lambda))\\ &= \Pi_h u'(\lambda) - u'(\lambda) + (T_h - T)\,G^{(1)}(\lambda, u(\lambda), u'(\lambda))\\ &\quad + T_h(G^{(1)}(\lambda, \Pi_h u(\lambda), \Pi_h u'(\lambda)) - G^{(1)}(\lambda, u(\lambda), u'(\lambda))).\end{aligned}$$

Let us consider the expression

$$\begin{aligned}&G^{(1)}(\lambda, \Pi_h u(\lambda), \Pi_h u'(\lambda)) - G^{(1)}(\lambda, u(\lambda), u'(\lambda))\\ &\quad = D_\lambda G(\lambda, \Pi_h u(\lambda)) - D_\lambda G(\lambda, u(\lambda)) + (D_u G(\lambda, \Pi_h u(\lambda)) - D_u G(\lambda, u(\lambda)))\cdot u'(\lambda)\\ &\qquad + D_u G(\lambda, \Pi_h u(\lambda))\cdot(\Pi_h u'(\lambda) - u'(\lambda)).\end{aligned}$$

Using (3.4) and (3.10), we have

$$\begin{aligned}&\|D_\lambda G(\lambda, \Pi_h u(\lambda)) - D_\lambda G(\lambda, u(\lambda))\|_{\tilde{W}} \leqq C\|u(\lambda) - \Pi_h u(\lambda)\|_Z,\\ &\|(D_u G(\lambda, \Pi_h u(\lambda)) - D_u G(\lambda, u(\lambda)))\cdot u'(\lambda)\|_{\tilde{W}} \leqq C\,\|u(\lambda) - \Pi_h u(\lambda)\|_Z,\end{aligned}$$

so that

$$\begin{aligned}&\|G^{(1)}(\lambda, \Pi_h u(\lambda), \Pi_h u'(\lambda)) - G^{(1)}(\lambda, u(\lambda), u'(\lambda))\|_{\tilde{W}}\\ &\quad\leqq C\{\|u(\lambda) - \Pi_h u(\lambda)\|_Z + \|u'(\lambda) - \Pi_h u'(\lambda)\|_{\tilde{V}}\}.\end{aligned}$$

Hence, we get the estimate

$$\begin{aligned}\|u_h'(\lambda) - u'(\lambda)\|_{\tilde{V}} &\leqq C\{h^{-r}\|F_h(\lambda, \Pi_h u(\lambda))\|_{\tilde{V}} + \|u(\lambda) - \Pi_h u(\lambda)\|_Z\\ &\quad + \|u'(\lambda) - \Pi_h u'(\lambda)\|_{\tilde{V}} + \|(T_h - T)\,G^{(1)}(\lambda, u(\lambda), u'(\lambda))\|_{\tilde{V}}\}.\end{aligned}$$

Together with (3.19), this implies the desired inequality (3.23). ∎

Remark 3. Under the additional hypothesis (3.20) and using (3.10) and (3.22), we obtain as a consequence of Theorem 4

$$\lim_{h\to 0}\sup_{\lambda\in\Lambda}\|u_h'(\lambda) - u'(\lambda)\|_{\tilde{V}} = 0. \quad\blacksquare \tag{3.24}$$

Let us now assume that G is a C^m mapping, $m \geqq 2$, from $\Lambda \times V$ into W so that $\lambda \to u(\lambda)$ is a C^m function from Λ into V. We turn to the approximation of the m-th derivative $u^{(m)}(\lambda)$ of $u(\lambda)$. We suppose that, for $u \in Z$, the l-th derivative $D^l G(\lambda, u) \in \mathscr{L}_l(\mathbb{R} \times V; W)$ may be extended as a l-linear continuous operator of $(\mathbb{R} \times Z)^{l-1} \times (\mathbb{R} \times \tilde{V})$ into $\tilde{W}$, $2 \leqq l \leqq m$, such that for all $\lambda, \lambda^* \in \Lambda$ and all $u, u^* \in Z$

$$\begin{aligned}&\|D^l G(\lambda^*, u^*) - D^l G(\lambda, u)\|_{\mathscr{L}_l(\mathbb{R} \times Z, \ldots, \mathbb{R} \times Z, \mathbb{R} \times \tilde{V}; \tilde{W})} \\ &\leqq L(|\lambda^*| + \|u^*\|_Z, |\lambda| + \|u\|_Z)(|\lambda^* - \lambda| + \|u^* - u\|_Z), \quad 2 \leqq l \leqq m.\end{aligned} \tag{3.25}$$

Moreover, we suppose that

$$\lambda \to u(\lambda) \quad \text{is a } C^{m-1} \text{ function from } \Lambda \text{ into } Z. \tag{3.26}$$

As in (2.15), we define by induction the mappings $F^{(l)}: \Lambda \times V^{l+1} \to V$, $G^{(l)}: \Lambda \times V^{l+1} \to W$ and $F_h^{(l)}: \Lambda \times V_h^{l+1} \to V_h$, $2 \leqq l \leqq m$.

We introduce functions $\Pi_h u^{(l)}: \lambda \to \Pi_h u^{(l)}(\lambda) \in V_h$, $2 \leqq l \leqq m$, and we assume that

$$\begin{aligned}&\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|u^{(l)}(\lambda) - \Pi_h u^{(l)}(\lambda)\|_Z = 0, \quad 1 \leqq l \leqq m-1, \\ &\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|u^{(m)}(\lambda) - \Pi_h u^{(m)}(\lambda)\|_{\tilde{V}} = 0.\end{aligned} \tag{3.27}$$

We can now state:

Theorem 5. *Assume the hypotheses of Theorem 4. Assume in addition that the hypotheses* (3.25), (3.26), *and* (3.27) *hold. Then,* $\lambda \to u_h(\lambda)$ *is a* C^m *function from* Λ *into* V_h *and we have for some constant* $K_m > 0$ *independent of* h *and* λ

$$\begin{aligned}&\|u_h^{(m)}(\lambda) - u^{(m)}(\lambda)\|_{\tilde{V}} \\ &\quad \leqq K_m \{h^{-mr}[\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T_h - T) G(\lambda, u(\lambda))\|_{\tilde{V}}] \\ &\quad + h^{(1-m)r} \|u(\lambda) - \Pi_h u(\lambda)\|_Z \\ &\quad + \sum_{l=1}^{m} h^{(l-m)r}(\|u^{(l)}(\lambda) - \Pi_h u^{(l)}(\lambda)\|_{\tilde{V}} + \|(T_h - T) G^{(l)}(\lambda, u(\lambda), \ldots, u^{(l)}(\lambda))\|_{\tilde{V}})\}.\end{aligned} \tag{3.28}$$

Proof. It follows from (3.8), (3.10), (3.11), and (3.25), that there exists monotonically increasing functions $L_l: \mathbb{R}_+ \to \mathbb{R}_+$, $2 \leqq l \leqq m$, independent of h such that for all $\lambda, \lambda^* \in \Lambda$ and all $u_h^* \in V_h$ with $h^{-r}(|\lambda^* - \lambda| + \|u_h^* - \Pi_h u(\lambda)\|_{\tilde{V}}) \leqq \xi$

$$\begin{aligned}&\|D^l F_h(\lambda^*, u_h^*) - D^l F_h(\lambda, \Pi_h u(\lambda))\|_{\mathscr{L}_l(\mathbb{R} \times Z, \ldots, \mathbb{R} \times Z, \mathbb{R} \times \tilde{V}; \tilde{V})} \\ &\leqq L_l(\xi) h^{-r}(|\lambda^* - \lambda| + \|u_h^* - \Pi_h u(\lambda)\|_{\tilde{V}}).\end{aligned}$$

Then, we apply Theorem 2 to the mapping F_h in the situation (3.18) with in addition $y^{(l)}(\lambda) = \Pi_h u^{(l)}(\lambda)$, $1 \leqq l \leqq m$. In fact, by the inverse inequality (3.8), we have for all $A_h \in \mathscr{L}_l(\mathbb{R} \times V_h; V_h)$

$$\|A_h\|_{\mathscr{L}_l(X \times Y; Z)} \leqq C \|A_h\|_{\mathscr{L}_l(\mathbb{R} \times Z, \ldots, \mathbb{R} \times Z, \mathbb{R} \times \tilde{V}; \tilde{V})}.$$

Moreover, we may write for all $v_h \in V_h$

$$\|v_h\|_{\mathscr{L}_l(X; Y)} = \|v_h\|_{\mathscr{L}_l(X; Z)} = h^{(l-1)r} \|v_h\|_{\tilde{V}}.$$

Hence, using (3.27), it follows from the estimate (2.35) that

$$\|u_h^{(m)}(\lambda)-\Pi_h u^{(m)}(\lambda)\|_{\tilde V}\leqq C\Big\{h^{-mr}\|F_h(\lambda,\Pi_h u(\lambda))\|_{\tilde V}$$
$$+\sum_{l=1}^{m} h^{(l-m)r}\|F_h^{(l)}(\lambda,\Pi_h u(\lambda),\ldots,\Pi_h u^{(l)}(\lambda))\|_{\tilde V}\Big\}.$$

The estimate (3.28) will be proved if we check that

$$\|F_h^{(l)}(\lambda,\Pi_h u(\lambda),\ldots,\Pi_h u^{(l)}(\lambda))\|_{\tilde V}$$
$$\leqq C\Big\{\|u(\lambda)-\Pi_h u(\lambda)\|_Z+\sum_{k=1}^{l}\|u^{(k)}(\lambda)-\Pi_h u^{(k)}(\lambda)\|_{\tilde V}$$
$$+\|(T_h-T)\,G^{(l)}(\lambda,u(\lambda),\ldots,u^{(l)}(\lambda))\|_{\tilde V}\Big\}.$$

But, by working along the same lines as in the proof of Theorem 4, this bound is obtained as a consequence of (3.25), (3.26) and (3.27). ■

Finally, let us consider the simple case where $Z=\tilde V=V$ and $\tilde W=W$. We introduce a linear operator $\Pi_h\in\mathscr{L}(V;V_h)$ such that

$$\lim_{h\to 0}\|v-\Pi_h v\|_V=0 \qquad \text{for all } v\in V. \tag{3.29}$$

We get the following natural result

Theorem 6. *Assume that G is a C^{m+1} mapping from $\Lambda\times V$ into W for some integer $m\geqq 1$ and the mapping $D^{m+1}G$ is bounded on all bounded subsets of $\Lambda\times V$. Assume in addition that the conditions* (3.29) *and*

$$\lim_{h\to 0}\|T_h-T\|_{\mathscr{L}(W;V)}=0. \tag{3.30}$$

hold. Then, there exists a neighborhood $\mathcal{O}$ of the origin in V and, for $h\leqq h_0$ small enough, a unique C^{m+1} function $\lambda\in\Lambda\to u_h(\lambda)\in V_h$ such that for all $\lambda\in\Lambda$

$$F_h(\lambda,u_h(\lambda))=0, \qquad u_h(\lambda)-u(\lambda)\in\mathcal{O}. \tag{3.31}$$

Furthermore, we have for some constants K_l, $0\leqq l\leqq m$, independent of h and λ

$$\|u_h^{(l)}(\lambda)-u^{(l)}(\lambda)\|_V\leqq K_l\sum_{k=0}^{l}\{\|u^{(k)}(\lambda)-\Pi_h u^{(k)}(\lambda)\|_V$$
$$+\|(T_h-T)\,G^{(k)}(\lambda,u(\lambda),\ldots,u^{(k)}(\lambda))\|_V\}. \tag{3.32}$$

In (3.32), $u^{(0)}$, $u_h^{(0)}$, $\Pi_h u^{(0)}$, $G^{(0)}$ stand for u, u_h, $\Pi_h u$ and G respectively.

Proof. It follows from (3.29) and the uniform boundedness theorem that $\|\Pi_h\|_{\mathscr{L}(V;V)}\leqq C$. Thus (3.9) holds. Moreover, using the continuity of each function $\lambda\to u^{(l)}(\lambda)$, it is an easy matter to show that

$$\lim_{h\to 0}\sup_{\lambda\in\Lambda}\|(I-\Pi_h)\,u^{(l)}(\lambda)\|_V=0, \qquad 0\leqq l\leqq m.$$

Finally, since $D^{m+1}G$ is bounded on all bounded subsets of $\Lambda \times V$, the same property is true for $D^l G$, $0 \leqq l \leqq m+1$. Hence, using the mean value theorem, we get for all $1 \leqq l \leqq m$

$$\begin{aligned}&\|D^l G(\lambda^*, u^*) - D^l G(\lambda, u)\|_{\mathscr{L}_l(\mathbb{R}\times V;\, W)}\\ &\qquad \leqq L(|\lambda^*| + \|u^*\|_V, |\lambda| + \|u\|_V)(|\lambda^* - \lambda| + \|u^* - u\|_V),\end{aligned}$$

for some function L: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ monotonically increasing with respect to each variable. Hence, all the hypotheses of Theorems 3, 4 and 5 hold with $r=0$, $\tilde{V}=Z=V$, $\tilde{W}=W$. Applying these theorems in that case yields the desired result. ■

4. Application I: A Mixed Finite Element Approximation of the Navier-Stokes Equations

Let Ω be a bounded simply connected plane domain with boundary Γ; we consider the Navier-Stokes equations for an incompressible viscous fluid confined in Ω in the stream function formulation

$$\begin{aligned}&\nu\Delta^2\psi + \frac{\partial}{\partial x_1}\left((-\Delta\psi)\frac{\partial\psi}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left((-\Delta\psi)\frac{\partial\psi}{\partial x_1}\right) = f \quad \text{in } \Omega,\\ &\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma,\end{aligned} \tag{4.1}$$

where f is given in $H^{-1}(\Omega)$ (say), $\nu > 0$ is the viscosity coefficient and $\dfrac{\partial}{\partial n}$ denotes the normal derivative along Γ. As it is well known (cf. [11] for instance), problem (4.1) has at least one solution $\psi \in H_0^2(\Omega)$.

Let us put the above problem into the framework of Sect. 3. We are now looking for a pair $u = (\psi, \omega)$ where $\omega = -\Delta\psi$ is the vorticity. Then, we set with standard notations for the Sobolev spaces

$$V = W_0^{1,4}(\Omega) \times L^2(\Omega), \qquad W = W^{-1,\frac{4}{3}}(\Omega). \tag{4.2}$$

We introduce the linear operator T: $g \in H^{-2}(\Omega) \to u = (\psi, \omega) = Tg \in H_0^2(\Omega) \times L^2(\Omega)$ defined by

$$\begin{aligned}&\Delta^2\psi = g \quad \text{in } \Omega\\ &\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma\end{aligned}, \qquad \omega = -\Delta\psi. \tag{4.3}$$

By the Sobolev imbedding theorem, we have $H_0^2(\Omega) \subset W_0^{1,4}(\Omega)$ and $W^{-1,\frac{4}{3}}(\Omega) \subset H^{-2}(\Omega)$ with continuous imbeddings so that the operator T also belongs to $\mathscr{L}(W; V)$. We next define the C^∞ mapping G: $(\lambda, u = (\psi, \omega)) \in \mathbb{R}_+ \times V \to G(\lambda, u) \in W$ by

$$G(\lambda, u) = \lambda\left\{\frac{\partial}{\partial x_1}\left(\omega\frac{\partial\psi}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\omega\frac{\partial\psi}{\partial x_1}\right) - f\right\}. \tag{4.4}$$

Clearly, a pair $u=(\psi, \omega)\in V$ satisfies the equation

$$F(\lambda, u)\equiv u+TG(\lambda, u)=0$$

if and only if the function ψ is a solution of the Navier-Stokes problem (4.1) corresponding to $\nu=\dfrac{1}{\lambda}$ and $\omega=-\Delta\psi$.

Now, assuming very weak regularity hypotheses on the domain Ω, T is a linear continuous operator from $W^{-1,\frac{4}{3}}(\Omega)$ into $H^{2+\delta}(\Omega)\times H^{\delta}(\Omega)$ for some $\delta>0$ and therefore a compact operator from W into V. Hence, $TD_uG(\lambda, u)$ is a compact operator of $\mathscr{L}(V; V)$.

Finally, let $\psi\in H_0^2(\Omega)$ be a solution of the Navier-Stokes problem (4.1) which is nonsingular in the sense that the linearized Navier-Stokes operator

$$X\to\nu\Delta^2X+\frac{\partial}{\partial x_1}\left((-\Delta\psi)\frac{\partial X}{\partial x_2}+(-\Delta X)\frac{\partial\psi}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left((-\Delta\psi)\frac{\partial X}{\partial x_1}+(-\Delta X)\frac{\partial\psi}{\partial x_1}\right)$$

is an isomorphism from $H_0^2(\Omega)$ onto $H^{-2}(\Omega)$.

Lemma 5. *A solution $\psi\in H_0^2(\Omega)$ of* (4.1) *is nonsingular if and only if the operator $D_uF(\lambda, u)$ is an isomorphism of V, where $\lambda=\dfrac{1}{\nu}$ and $u=(\psi, \omega=-\Delta\psi)$.*

Proof. Assume that ψ is a nonsingular solution of (4.1). This means that $D_uF(\lambda, u)=I+TD_uG(\lambda, u)$ is an isomorphism of U where

$$U=\{v=(\varphi, \theta);\ \varphi\in H_0^2(\Omega),\ \theta=-\Delta\varphi\}.$$

Now, since the operator $TD_uG(\lambda, u)\in\mathscr{L}(V; V)$ is compact, it follows from the Fredholm alternative that $D_uF(\lambda, u)$ is also an isomorphism of V. Conversely, it is an easy matter to check that, if $D_uF(\lambda, u)$ is an isomorphism of V, it is also an isomorphism of U.

Let us introduce a mixed finite element method for approximating the nonsingular solutions of (4.1). We assume, for simplicity, that Ω is a polygonal domain. If, in addition, we suppose that Ω is convex, we have that the operator T is continuous from $W^{-1,\frac{4}{3}}(\Omega)$ into $W^{3,\frac{4}{3}}(\Omega)\times W^{1,\frac{4}{3}}(\Omega)$ (cf. [10], [8]).

We introduce a family $(\mathscr{T}_h)$ of triangulations of $\bar{\Omega}$ made with triangles K whose diameters are $\leqq h$. We assume that the familly $(\mathscr{T}_h)$ is uniformly regular in the sense that there exists two constants $\sigma, \tau>0$ independent of h such that

$$h_K\leqq\sigma\rho_K, \qquad \tau h\leqq h_K\leqq h, \tag{4.5}$$

where h_K is the diameter of K and ρ_K is the diameter of the inscribed circle in K. Then, we define for each integer $l\geqq 1$ the finite dimensional spaces

$$\Theta_h=\Theta_h^{(l)}=\{\theta\in C^0(\bar{\Omega});\ \theta|_K\in P_l \text{ for all } K\in\mathscr{T}_h\}, \tag{4.6}$$

$$\Phi_h=\Phi_h^{(l)}=\{\varphi\in\Theta_h;\ \varphi=0 \text{ on } \Gamma\}, \tag{4.7}$$

and

$$V_h=V_h^{(l)}=\Phi_h\times\Theta_h\subset V, \tag{4.8}$$

where P_l denotes the space of all polynomials of degree $\leq l$ in the two variables x_1, x_2.

A mixed finite element approximation of the Navier-Stokes problem consists in finding a pair $u_h = (\psi_h, \omega_h) \in V_h$ solution of the equations

$$\begin{aligned} &\int_\Omega \left\{ \nu \frac{\partial \omega_h}{\partial x_i} \frac{\partial \varphi_h}{\partial x_i} + \omega_h \left(\frac{\partial \psi_h}{\partial x_1} \frac{\partial \varphi_h}{\partial x_2} - \frac{\partial \psi_h}{\partial x_2} \frac{\partial \varphi_h}{\partial x_1} \right) \right\} dx = \langle f, \varphi_h \rangle \quad \text{for all } \varphi_h \in \Phi_h, \\ &\int_\Omega \left\{ \omega_h \theta_h - \frac{\partial \psi_h}{\partial x_i} \frac{\partial \theta_h}{\partial x_i} \right\} dx = 0 \qquad \text{for all } \theta_h \in \Theta_h, \end{aligned} \tag{4.9}$$

where $\langle .,. \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ or between $W^{-1,\frac{4}{3}}(\Omega)$ and $W_0^{1,4}(\Omega)$. In (4.9), we have used the classical summation convention.

Let us define the linear operator T_h: $g \in W \to u_h = (\psi_h, \omega_h) = T_h\, g \in V_h$ by

$$\begin{aligned} &\int_\Omega \frac{\partial \omega_h}{\partial x_i} \frac{\partial \varphi_h}{\partial x_i} dx = \langle g, \varphi_h \rangle \qquad \text{for all } \varphi_h \in \Phi_h, \\ &\int_\Omega \left\{ \omega_h \theta_h - \frac{\partial \psi_h}{\partial x_i} \frac{\partial \theta_h}{\partial x_i} \right\} dx = 0 \qquad \text{for all } \theta_h \in \Theta_h. \end{aligned} \tag{4.10}$$

Thus, an equivalent form of problem (4.9) consists in finding $u_h \in V_h$ solution of

$$F_h(\lambda, u_h) \equiv u_h + T_h G(\lambda, u_h) = 0, \qquad \lambda = \frac{1}{\nu}.$$

Now, for applying Theorem 6, we need to state the convergence properties of the mixed finite element approximation (4.10) of the biharmonic problem (4.3).

Lemma 6. *Assume that the polygonal domain Ω is convex. Then, for all $g \in W$ we have the estimate*

$$\|(T - T_h) g\|_V \leq \begin{cases} C h^{\frac{1}{2}} \|g\|_W & \text{if } l \geq 2, \\ C h^{\frac{1}{4}} |\ln h|^{\frac{1}{2}} \|g\|_V & \text{if } l = 1. \end{cases} \tag{4.11}$$

If g is chosen in such a way that $u = (\psi, \omega) = Tg$ satisfies the smoothness property $\psi \in H^{k+\frac{3}{2}}(\Omega) \cap W^{k+1,\infty}(\Omega)$ for some $k \in \mathbb{R}$ with $1 \leq k \leq l$, we have

$$\|(T - T_h) g\|_V \leq C h^{k-\frac{1}{2}} |\ln h|^\beta (\|\psi\|_{H^{k+\frac{3}{2}}(\Omega)} + \|\psi\|_{W^{k+1,\infty}(\Omega)}), \tag{4.12}$$

where $\beta = 0$ if $l \geq 2$ and $\beta = 1$ if $l = 1$.

Proof. Let us introduce the following subspace U_h of the space V_h:

$$U_h = \left\{ v_h = (\varphi_h, \theta) \in V_h;\ \int_\Omega \left(\frac{\partial \varphi_h}{\partial x_i} \frac{\partial \mu_h}{\partial x_i} - \theta_h \mu_h \right) dx = 0 \quad \text{for all } \mu_h \in \Theta_h \right\}$$

We consider the following problem: Find $u_h = (\psi_h, \omega_h) \in U_h$ such that

$$\int_\Omega \omega_h \theta_h dx = \langle g, \varphi_h \rangle \qquad \text{for all } v_h = (\varphi_h, \theta_h) \in U_h. \tag{4.13}$$

Problems (4.10) and (4.13) are equivalent. In fact, if $u_h \in U_h$ is a solution of (4.13), the 2nd equation (4.10) clearly holds. Moreover, let φ_h be in Φ_h and let θ_h be the unique element of Θ_h such that $v_h = (\varphi_h, \theta_h)$ belongs to the space U_h. Then

$$\int_\Omega \omega_h \theta_h dx = \int_\Omega \frac{\partial \omega_h}{\partial x_i} \frac{\partial \varphi_h}{\partial x_i} dx$$

so that the 1st equation (4.10) holds. Conversely, any solution of (4.10) is trivially a solution of (4.13).

Now, we may apply the results of [6] concerning problem (4.13). The estimate (4.11) follows from [6, Theorem 4.1] used with $r = \frac{4}{3}$, $s = 4$, while (4.12) follows from [6, Remark 6.1]. ■

Thus, we are able to derive

Theorem 7. *Assume that Λ is a compact interval of $\mathbb{R}_+$ and there exists a branch $\{(\lambda, \psi(\lambda)); \lambda \in \Lambda\}$ of non singular solutions of the Navier-Stokes problem* (4.1). *Then, there exists a neighborhood $\mathcal{O}$ of the origin in $W_0^{1,4}(\Omega) \times L^2(\Omega)$ and, for $h \leq h_0$ small enough, a unique branch $\{(\lambda, u_h(\lambda) = (\psi_h(\lambda), \omega_h(\lambda))); \lambda \in \Lambda\}$ of solutions of* (4.9) *such that $u_h(\lambda) - u(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$ with $u(\lambda) = (\psi(\lambda), \omega(\lambda) = -\Delta \psi(\lambda))$. Moreover, we have:*

$$\lambda \to u_h(\lambda) \text{ is a } C^\infty \text{ function from } \Lambda \text{ into } V_h; \tag{4.14}$$

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \{\|\psi_h^{(m)}(\lambda) - \psi^{(m)}(\lambda)\|_{W_0^{1,4}(\Omega)} + \|\omega_h^{(m)}(\lambda) - \omega^{(m)}(\lambda)\|_{L^2(\Omega)}\} = 0, \tag{4.15}$$

for all $m \geq 0$. If, in addition, $\lambda \to \psi(\lambda)$ is a C^m function from Λ into $H^{k+\frac{3}{2}}(\Omega) \cap W^{k+1,\infty}(\Omega)$ for some $k \in \mathbb{R}$ with $1 \leq k \leq l$, we get the error estimate

$$\|\psi_h^{(m)}(\lambda) - \psi^{(m)}(\lambda)\|_{W_0^{1,4}(\Omega)} + \|\omega_h^{(m)}(\lambda) - \omega^{(m)}(\lambda)\|_{L^2(\Omega)} \leq C h^{k-\frac{1}{2}} |\ln h|^\beta \tag{4.16}$$

where $\beta = 0$ if $l \geq 2$ and $\beta = 1$ if $l = 1$ and C is a constant independent of h and λ.

Proof. Let us check the hypotheses of Theorem 6. Clearly, we may write $G(\lambda, u) = \lambda H(u)$ where H is a C^∞ *quadratic* mapping from V into W and $D^2 H$ is bounded on all bounded subsets of V. Next, (3.30) is a consequence of the estimate (4.11). Finally, we introduce the operator

$$\Pi_h \in \mathcal{L}(V; V_h): v = (\varphi, \theta) \in V \to \Pi_h v = (\varphi_h, \theta_h) \in V_h$$

defined as follows:

(i) φ_h is the function of Φ_h which interpolates the function $\varphi \in W_0^{1,4}(\Omega)$ at the usual finite element nodes;

(ii) θ_h is the orthogonal projection in $L^2(\Omega)$ of θ onto the space Θ_h.

Then, using standard approximation results in finite element theory, we get (3.29). Moreover, if $\varphi \in W^{k+1,4}(\Omega)$ (say) and $\theta \in H^{k-\frac{1}{2}}(\Omega)$ for some $k \in \mathbb{R}$ with $1 \leq k \leq l$, we obtain

$$\|v - \Pi_h v\|_V \leq C h^{k-\frac{1}{2}} (\|\varphi\|_{W^{k+1,4}(\Omega)} + \|\theta\|_{H^{k-\frac{1}{2}}(\Omega)}). \tag{4.17}$$

Now, we may apply Theorem 6: there exists a neighborhood $\mathcal{O}$ of the origin in V and, for $h \leq h_0$ small enough, a unique C^∞ function $\lambda \in \Lambda \to u_h(\lambda)$

$=(\psi_h(\lambda), \omega_h(\lambda)) \in V_h$ such that, for all $\lambda \in \Lambda$, $u_h(\lambda)$ is a solution of (4.9) and $u_h(\lambda) - u(\lambda) \in \mathcal{O}$. Furthermore, we have for all integer $m \geq 0$

$$\|u_h^{(m)}(\lambda) - u^{(m)}(\lambda)\|_V \leqq K_m \sum_{i=0}^{m} \{\|u^{(i)}(\lambda) - \Pi_h u^{(i)}(\lambda)\|_V + \|(T - T_h) G^{(i)}(\lambda, u(\lambda), u'(\lambda), \ldots, u^{(i)}(\lambda))\|_V\},$$

where $u(\lambda) = (\psi(\lambda), \omega(\lambda) = -\Delta \psi(\lambda))$. This implies (4.15).

Now, assume that $\lambda \to \psi(\lambda)$ is a C^m function from Λ into $H^{k+\frac{3}{2}}(\Omega) \cap W^{k+1,\infty}(\Omega)$ for some k with $1 \leqq k \leqq l$. Since

$$T G^{(i)}(\lambda, u(\lambda), u'(\lambda), \ldots, u^{(i)}(\lambda)) = -u^{(i)}(\lambda),$$

it follows from (4.12) that

$$\sup_{\lambda \in \Lambda} \|(T - T_h) G^{(i)}(\lambda, u(\lambda), u'(\lambda), \ldots, u^{(i)}(\lambda))\|_V \leqq C h^{k-\frac{1}{2}} |\ln h|^\beta$$

for $0 \leqq i \leqq m$, where $\beta = 1$ for $l = 1$ and $\beta = 0$ for $l \geqq 2$. Moreover, $\lambda \to u(\lambda)$ is a C^m function from Λ into $W^{k+1,4}(\Omega) \times H^{k-\frac{1}{2}}(\Omega)$ so that we have as a consequence of (4.17)

$$\sup_{\lambda \in \Lambda} \|u^{(i)}(\lambda) - \Pi_h u^{(i)}(\lambda)\|_V \leqq C h^{k-\frac{1}{2}}, \qquad 0 \leqq i \leqq m.$$

This proves the estimate (4.16). ∎

5. Application II: A Mixed Method for the Von Kármán Equations

Consider now the Von Kármán equations for a clamped plate

$$\begin{aligned} \Delta^2 \psi^1 &= -\tfrac{1}{2}[\psi^2, \psi^2] && \text{in } \Omega, \\ \Delta^2 \psi^2 &= [\psi^1, \psi^2] + \lambda f && \text{in } \Omega, \\ \psi^1 = \frac{\partial \psi^1}{\partial n} &= \psi^2 = \frac{\partial \psi^2}{\partial n} = 0 && \text{on } \Gamma, \end{aligned} \tag{5.1}$$

where

$$[\varphi, \psi] = \frac{\partial^2 \varphi}{\partial x_1^2} \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_2^2} \frac{\partial^2 \psi}{\partial x_1^2} - 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2}. \tag{5.2}$$

In (5.1) ψ^1 is the Airy stress function, ψ^2 is the vertical deflection of the plate, λf is an external vertical load on the plate which depends on the loading parameter λ. We assume that f belongs to the space $H^{-1}(\Omega)$. Then, problem (5.1) has at least one solution $\psi = (\psi^1, \psi^2) \in H_0^2(\Omega)^2$ (cf. [11] for instance).

Again, let us put the above problem into the framework of Sect. 3. We begin by introducing a new dependent variable $\sigma = (\sigma^1, \sigma^2)$ where

$$\sigma^k = (\sigma_{ij}^k)_{1 \leqq i, j \leqq 2}, \qquad \sigma_{ij}^k = \frac{\partial^2 \psi^k}{\partial x_i \partial x_j}, \qquad i, j, k = 1, 2, \tag{5.3}$$

and we are now looking for a pair $u=(\boldsymbol{\psi},\boldsymbol{\sigma})$. Next, given an arbitrarily small $\varepsilon>0$, we set $p=\dfrac{2}{1-\varepsilon}$ and we define the spaces

$$V=W_0^{1,p}(\Omega)^2\times(L^2(\Omega)_s^4)^2,\qquad W=W^{-1,q}(\Omega)^2,\quad \frac{1}{p}+\frac{1}{q}=1,\tag{5.4}$$

where

$$L^2(\Omega)_s^4=\{\tau=(\tau_{ij})_{1\leq i,j\leq 2};\ \tau_{ij}\in L^2(\Omega),\ \tau_{12}=\tau_{21}\}.$$

Let T: $\mathbf{g}=(g^1,g^2)\in H^{-2}(\Omega)^2\to u=(\boldsymbol{\psi},\boldsymbol{\sigma})=T\mathbf{g}\in H_0^2(\Omega)^2\times(L^2(\Omega)_s^4)^2$ be the linear continuous operator defined by

$$\left.\begin{aligned}\Delta^2\psi^k&=g^k &&\text{in } \Omega\\ \psi^k&=\frac{\partial\psi^k}{\partial n}=0 &&\text{on } \Gamma\end{aligned}\right\},\qquad \sigma_{ij}^k=\frac{\partial^2\psi^k}{\partial x_i\partial x_j},\qquad k=1,2.\tag{5.5}$$

Using the Sobolev imbedding theorem, we find that the operator T belongs also to the space $\mathscr{L}(W;V)$.

Finally, we introduce the C^∞ mapping G: $(\lambda,u=(\boldsymbol{\psi},\boldsymbol{\sigma}))\in\mathbb{R}\times V\to G(\lambda,u)\in W$ defined by

$$G(\lambda,u)=(\tfrac{1}{2}[\sigma^2,\sigma^2],\ -[\sigma^1,\sigma^2]-\lambda f),\tag{5.6}$$

where

$$[\sigma,\tau]=\sigma_{11}\tau_{22}+\sigma_{22}\tau_{11}-2\sigma_{12}\tau_{12},\qquad \sigma,\tau\in L^2(\Omega)_s^4.\tag{5.7}$$

Now, problem (5.1) amounts to find a pair $u=(\boldsymbol{\psi},\boldsymbol{\sigma})\in V$ solution of the equation

$$F(\lambda,u)\equiv u+TG(\lambda,u)=0.$$

Again, $TD_uG(\lambda,u)$ is a compact operator of $\mathscr{L}(V;V)$ and the analogue of Lemma 5 clearly holds.

Let us consider the mixed "Hellan-Hermann-Johnson" finite element scheme for approximating the branches of solutions of (5.1). Assume again that Ω is a *convex* polygonal domain so that the operator T is continuous from $H^{-1}(\Omega)^2$ into $H^3(\Omega)^2\times(H^1(\Omega)^4)^2$. We introduce a family $(\mathscr{T}_h)$ of uniformly regular triangulations of $\bar{\Omega}$ and, for each integer $l\geq 1$, the finite-dimensional spaces $\Phi_h=\Phi_h^{(l)}$ defined as in (4.7) and

$$\begin{aligned}\Sigma_h=\Sigma_h^{(l)}=\{\tau\in L^2(\Omega)_s^4;\ \tau_{ij|K}\in P_{l-1}\quad &\text{for all } K\in\mathscr{T}_h\\ \text{and } M_n(\tau) \text{ is continuous at the interelement boundaries}\},&\end{aligned}\tag{5.8}$$

where $M_n(\tau)=\tau_{ij}n_jn_i$ and $n=(n_1,n_2)$ is the unit outward normal along an element boundary ∂K. We set

$$V_h=\Phi_h^2\times\Sigma_h^2.\tag{5.9}$$

We next introduce the bilinear forms

$$a(\boldsymbol{\sigma},\boldsymbol{\tau})=\int_\Omega\sigma_{ij}^k\tau_{ij}^k\,dx,\qquad \boldsymbol{\sigma},\boldsymbol{\tau}\in\Sigma_h^2\tag{5.10}$$

$$\mathrm{b}(\tau,\varphi)=\sum_{K\in\mathscr{T}_h}\left\{\int_K \frac{\partial\tau_{ij}^k}{\partial x_j}\frac{\partial\varphi^k}{\partial x_i}\,dx-\int_{\partial K}\frac{\partial\varphi^k}{\partial t}M_{nt}(\tau^k)\,ds\right\},\qquad \varphi\in\Phi_h^2,\ \tau\in\Sigma_h^2, \tag{5.11}$$

where $t=(-n_2,n_1)$ is the unit tangent along ∂K and $M_{nt}=\tau_{ij}n_j t_i$.

Then, a mixed finite element approximation of the Von Kármán equations consists in finding $u_h=(\psi_h,\sigma_h)\in V_h$ solution of the equations

$$\begin{aligned}&a(\sigma_h,\tau_h)+b(\tau_h,\psi_h)=0 \qquad \text{for all } \tau_h\in\Sigma_h^2,\\ &b(\sigma_h,\varphi_h)=\int_\Omega \{\tfrac{1}{2}[\sigma_h^2,\sigma_h^1]\,\varphi_h^1-([\sigma_h^1,\sigma_h^2]+\lambda f)\,\varphi_h^2\}\,dx \qquad \text{for all } \varphi_h\in\Phi_h^2.\end{aligned} \tag{5.12}$$

Let us define the linear continuous operator T_h: $\mathbf{g}\in H^{-1}(\Omega)^2\to u_h=(\psi_h,\sigma_h)=T_h\mathbf{g}\in V_h$ by

$$\begin{aligned}&a(\sigma_h,\tau_h)+b(\tau_h,\psi_h)=0 \qquad \text{for all } \tau_h\in\Sigma_h^2,\\ &b(\sigma_h,\varphi_h)=\langle \mathbf{g},\varphi_h\rangle \qquad \text{for all } \varphi_h\in\Phi_h^2,\end{aligned} \tag{5.13}$$

where $\langle .,.\rangle$ is the duality pairing between $H^{-1}(\Omega)^2$ and $H_0^1(\Omega)^2$. Since G maps $\mathbb{R}\times V_h$ into $H^{-1}(\Omega)^2$, an equivalent form of problem (5.12) consists in finding $u_h\in V_h$ solution of

$$F_h(\lambda,u_h)\equiv u_h+T_hG(\lambda,u_h)=0.$$

Recall some approximation properties of the operator T_h. We set:

$$\tilde V=H_0^1(\Omega)^2\times(L^2(\Omega)_s^4)^2,\qquad \tilde W=H^{-1}(\Omega)^2. \tag{5.14}$$

We have [2] (cf. also [3]).

Lemma 7. *Assume that the polygonal domain Ω is convex. Then, for all $g\in\tilde W$, we have the estimate*

$$\|(T-T_h)\mathbf{g}\|_{\tilde V}\leqq Ch\,\|\mathbf{g}\|_{\tilde W}. \tag{5.15}$$

Moreover, if $u=(\psi,\sigma)=T\mathbf{g}$ satisfies $\psi\in H^{k+2}(\Omega)^2$ for some $k\in\mathbb{R}$ with $1\leqq k\leqq l$, we get

$$\|(T-T_h)\mathbf{g}\|_{\tilde V}\leqq Ch^k\,\|\psi\|_{H^{k+2}(\Omega)}. \tag{5.16}$$

We are now able to prove

Theorem 8. *Assume that Λ is a compact interval of $\mathbb{R}$ and $\{(\lambda,\psi(\lambda));\lambda\in\Lambda\}$ is a branch of nonsingular solutions of the Von Kármán equations* (5.1). *Then, for all $\varepsilon>0$ arbitrarily small, there exists a constant $b_\varepsilon>0$ independent of h and, for $h\leqq h_0$ small enough, a unique branch $\{(\lambda,u_h(\lambda)=(\psi_h(\lambda),\sigma_h(\lambda));\ \lambda\in\Lambda\}$ of solutions of* (5.12) *such that*

$$\sup_{\lambda\in\Lambda}\|u_h(\lambda)-u(\lambda)\|_{\tilde V}\leqq b_\varepsilon h^\varepsilon. \tag{5.17}$$

Moreover, $\lambda\to u_h(\lambda)$ is a C^∞ function from Λ into V_h. If, in addition, $\lambda\to\psi(\lambda)$ is a C^m function from Λ into $H^{k+2}(\Omega)^2$ for some $k\in\mathbb{R}$ with $1\leqq k\leqq l$, we have the error bound

$$\|\psi_h^{(m)}(\lambda)-\psi^{(m)}(\lambda)\|_{H_0^1(\Omega)}+\|\sigma_h^{(m)}(\lambda)-\sigma^{(m)}(\lambda)\|_{L^2(\Omega)_s^4}\leqq c_\varepsilon h^{k-m\varepsilon}, \tag{5.18}$$

where c_ε is a constant >0 independent of h and λ.

Proof. Let us check the hypotheses of Theorems 3, 4 and 5. For any $\varepsilon>0$ arbitrarily small, we set $p=\frac{2}{1-\varepsilon}$ and we introduce the space

$$Z=W_0^{1,p}(\Omega)^2\times(L^p(\Omega)_s^4)^2.$$

Next, for $u=(\psi,\sigma)$ and $v=(\varphi,\tau)$, we may write

$$D_\lambda G(\lambda,u)=(0,f)\in H^{-1}(\Omega)^2,$$
$$D_u G(\lambda,u)\cdot v=([\sigma^2,\tau^2],\ -[\tau^1,\sigma^2]-[\sigma^1,\tau^2]).$$

If $u\in Z$, $v\in\tilde V$, we have $D_u G(\lambda,u)\cdot v\in L^s(\Omega)^2$, $\frac{1}{s}=\frac{1}{2}-\frac{1}{p}$, so that $D_u G(\lambda,u)\cdot v\in H^{-1}(\Omega)^2$ by the Sobolev imbedding theorem. Hence, for $u\in Z$, $DG(\lambda,u)\in\mathscr{L}(\mathbb{R}\times\tilde V;\tilde W)$ and

$$\|D_u G(\lambda,u)\|_{\mathscr{L}(\tilde V;\tilde W)}\leqq C\,\|u\|_Z.$$

Now, the operator T being compact from $\tilde W$ into $\tilde V$, the condition (3.3) is satisfied. On the other hand, by the linearity of the mapping $u\to D_u G(\lambda,u)$, we get for all $\lambda,\lambda^*\in\Lambda$, and $u,u^*\in Z$

$$\|DG(\lambda^*,u^*)-DG(\lambda,u)\|_{\mathscr{L}(\mathbb{R}\times\tilde V;\tilde W)}\leqq C\,\|u^*-u\|_Z,$$

so that (3.4) holds. Moreover, since $D^2G(\lambda,u)$ does not depend on λ and u, the hypothesis (3.25) is trivially satisfied for all $l\geqq 2$. Finally, the family $(\mathscr{T}_h)$ being uniformly regular, the following inverse inequality is valid

$$\|v_h\|_Z\leqq c h^{\frac{2}{p}-1}\|v_h\|_{\tilde V}=c h^{-\varepsilon}\|v_h\|_{\tilde V}\qquad\text{for all } v_h\in V_h,$$

and (3.8) holds with $r=\varepsilon$.

Next, the operator T being continuous from $H^{-1}(\Omega)^2$ into $H^3(\Omega)^2\times(H^1(\Omega)^4)^2$, one can easily check that $\lambda\to\psi(\lambda)$ is a C^∞ mapping from Λ into $H^3(\Omega)^2$. Let us then define a linear continuous operator

$$\Pi_h\colon v=(\varphi,\tau)\in(H^2(\Omega)\cap H_0^1(\Omega))^2\times(H^1(\Omega)_s^4)^2\to\Pi_h v=(\varphi_h,\tau_h)\in V_h$$

in the following way

(i) φ_h is the function of Φ_h^2 which interpolates φ at the usual finite element nodes;

(ii) τ_h is the orthogonal projection in $L^2(\Omega)^4$ of τ onto the space Σ_h^2.

It follows from [I, Lemma 4] and classical results in finite element theory that we have if $\varphi\in H^{k+1}(\Omega)^2$ and $\tau\in(H^k(\Omega)^4)^2$ with $1\leqq k\leqq l$:

$$\|v-\Pi_h v\|_{\tilde V}+h^\varepsilon\|v-\Pi_h v\|_Z\leqq ch^k(\|\varphi\|_{H^{k+1}(\Omega)}+\|\tau\|_{K^k(\Omega)}).\tag{5.19}$$

Using this inequality with $k=1$, we find that the hypothesis (3.9) holds and

$$\sup_{\lambda\in\Lambda}\|u^{(i)}(\lambda)-\Pi_h u^{(i)}(\lambda)\|_{\tilde V}+h^\varepsilon\|u^{(i)}(\lambda)-\Pi_h u^{(i)}(\lambda)\|_Z\leqq ch,\qquad i\geqq 0.$$

Hence, it follows from (5.15) that we may apply Theorems 3–5 with $r=\varepsilon$: there exists a constant $b_\varepsilon>0$ independent of h and, for $h\leqq h_0(\varepsilon)$ small enough, a unique C^∞ function $\lambda\in\Lambda\to u_h(\lambda)=(\psi_h(\lambda),\sigma_h(\lambda))\in V_h$ such that $u_h(\lambda)$ is a solution of (5.12) for all $\lambda\in\Lambda$ and (5.17) holds. Moreover, we have for all $m\geq 0$

$$\begin{aligned}\|u_h^{(m)}(\lambda)-u^{(m)}(\lambda)\|_{\tilde V}\leqq K_m\sum_{i=0}^{m} h^{(i-m)\varepsilon}\{&\|u^{(i)}(\lambda)-\Pi_h u^{(i)}(\lambda)\|_{\tilde V}\\ &+\|(T_h-T)\,G^{(i)}(\lambda,u(\lambda),\dots,u^{(i)}(\lambda)\|_{\tilde V}\}.\end{aligned}\tag{5.20}$$

If we assume that $\lambda\to\psi(\lambda)$ is a C^m function from Λ into $H^{k+2}(\Omega)^2$ for some k with $1\leqq k\leqq l$, then the desired bound (5.18) is a consequence of (5.16), (5.19) and (5.20). ■

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