Verification of zeros in underdetermined systems.

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Introduction

Let $B \subseteq \mathbb{R}^m$ be a box (product of closed intervals). While the nonexistence of a zero of a continuous functions $f\colon B \to \mathbb{R}^n$ can often be verified by interval arithmetic, existence verification requires additional ingredients. In case of a continuous function $f\colon [-1,1]^n \to \mathbb{R}^n$, such ingredients include Brouwer's fixed point theorem, Miranda's theorem, or, more generally, topological degree.

For underdetermined systems—that is, $\dim B > n$—the problem of zero verification may be surprisingly complex, with connections to the field of computational topology.

The section method and its incompleteness

One way to reduce underdetermined systems to "square" systems is to fix some coordinates and verify the existence of a zero of $f(\cdot, y_0)$ where y_0 represents some fixed $\dim B - n$ coordinates. If the Jacobian f' has maximal rank n in every $x \in f^{-1}(0)$, then the zero set has dimension $\dim B - n$ and—possibly after a rotation—we can eventually find y_0 so that $f(\cdot, y_0)$ has a zero and verify it. We call this "section method" due to the fact that we analyze the restriction of f to a section $B' \times \{y_0\}$.

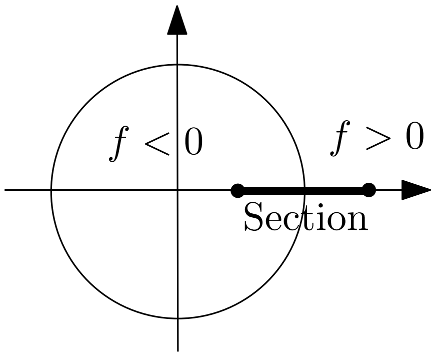

Figure 1: Zero detection for $f(x, y) = x^2 + y^2 - 1$.

There is a potential problem with this approach: if 0 is a singular value of f, then the zero set may have lower dimension and resist detection by the section method. This is illustrated by the quadratic function $f\colon [-1, 1]^4 \to \mathbb{R}^3$ defined component-wise by $(x_1x_3 + x_2x_4, x_2x_3 - x_1x_4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$ which has a single singular zero at the origin (and also every "close enough" perturbation of f has a zero). The restriction of f to the unit sphere $S^3 = \{x : |x| = 1\}$ is the Hopf map $S^3 \to S^2$. For any rotation R of $\mathbb{R}^4$, once we use the 3-dimensional section $[-1, 1]^3 \times \{0\}$ containing the origin, the "section method" for zero verification of $(f \circ R)|_{[-1,1]^3 \times \{0\}}$ fails. Arbitrarily small perturbation of $f \circ R$ have no zero in $[-1, 1]^3 \times \{0\}$, thus the zero cannot be verified by common methods such as Brouwer or Miranda's theorem (as these methods are stable with respect to small perturbations). However, the zero of f cannot be removed by perturbations: in particular, any continuous g such that $\|g - f\| \leq 1$ has a nonempty zero set. Therefore, it is natural to expect that more sophisticated verification methods should detect the zero of f.

The key is to analyze the function on a subdomain A where it is further away from zero, that is, $A := \{x\colon |f(x)| \geq r\}$ for some $r > 0$. If $f|_A : A \to \mathbb{R}^n \setminus \{0\}$ *cannot* be extended to a map $B \to \mathbb{R}^n \setminus \{0\}$ then f has a zero in B that is r-robust in the sense explained below. We summarize the up-to-date results about decidability of the extension problem and its implications for the zero verification problem.

Zero verification with incomplete information

The above mentioned method using non-extendability can not only verify a zero of f, but also a zero of any continuous r-perturbation g of f, that is, $g\colon B \to \mathbb{R}^n$ s.t. $\|g - f\|_\infty \leq r$. The non-extendability criterion is complete in the sense that once it fails to verify a zero, there has to be a continuous r-perturbation g of f without a zero.

Thus the method is useful in situations where we don't know the exact values of the function f and only deal with its approximation. For example, the function may be inferred from measurements or come from a scientific model that only approximates the reality. An important instance is the case when the function f is given just by its values on the vertices of a cubical grid and a Lipschitz constant, that is, as a higher dimensional bitmap.

References

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