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Inclusion Monotonic Property of Courant–Fischer Minimax Characterization on Interval Eigenproblems for Symmetric Interval Matrices

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Abstract

The classical Courant–Fischer minimax theorem for symmetric matrix is extended to the interval symmetric matrix. Inclusion monotonic property for interval version of Courant–Fischer theorem is discussed in this article and proved. By introducing the center–radius representation of the symmetric interval matrix and sign matrices with respect to the eigenvectors of the center matrix, the interval Courant–Fischer minimax equation is split into two ordinary minimax equations corresponding to the lower bound and upper bound eigenvalues. The two minimax equations are thus equivalent to two ordinary eigenvalue problems. The assumption that the signs of the elements of the eigenvectors are not changed for matrices ranging over the intervals is applied and is crucial on determining the sign matrices. A simple example demonstrate the algorithm presented in this article.

1. Introduction

The interval eigenvalue problem,

$$\mathbf{A}^I\mathbf{x} = \lambda^I\mathbf{x}, \tag{1}$$

for symmetric interval matrix $\mathbf{A}^I$ can be found in many fields. Specifically, when the parameters are uncertain or range over intervals, the interval analysis is a very important mathematical tool on dealing with such problems. The formal interval algebra was introduced by Sunaga[14], and by Moore[8] who introduced interval analysis and defined the interval vector and interval matrix. The further research on interval matrix was studied by Apostolatos and Kulisch[1]. The inverse interval matrix was discussed in Hansen[5]. Hansen and Smith[6] proposed a method using LU–decomposition and interval Gaussian elimination to find the determinant of an interval matrix. Eigenvalues of interval matrices were studied by Deif[3], Rohn[12], and Rohn and Deif[13]. The bounds of interval eigenvalues for symmetric interval matrix were investigated by Nickel[10] and Rohn[11]. Hertz[7] found that the exact minimum and maximum of all eigenvalues for all matrix in an symmetric interval matrix are the eigenvalues of some special vertex matrices.

The classical Courant–Fischer minimax theorem is a proper characterization for the eigenvalues of a symmetric matrix. In this article, we extend the classical minimax theorem to the symmetric interval matrix and prove that the inclusion monotonic property is still valid for interval version of minimax theorem. Based on Chen, Qiu and Song[2], an algorithm is developed to find the interval eigenvalues. We also discover that the associated eigenvectors can not be an interval in whole, whereas the elements in eigenvectors can be bounds for those of the eigenvectors of matrices in interval matrix.

2. Interval Arithmetic

2.1. Basic Definitions

The interval arithmetic is an extension of the ordinary arithmetic. We shall denote A^I by a closed interval of the form,

$$A^I = [\underline{a}, \overline{a}] = \{a \mid \underline{a} \le a \le \overline{a},\ \underline{a},\ \overline{a} \in \mathbb{R}\}, \tag{2}$$

and define the center and radius of A^I respectively as follows,

$$\text{center}: \ a^c = \frac{1}{2}(\underline{a} + \overline{a}), \tag{3}$$

$$\text{radius}: \ \Delta a = \frac{1}{2}(\overline{a} - \underline{a}), \tag{4}$$

thus,

$$A^I = [a^c - \Delta a,\ a^c + \Delta a] = a^c + \Delta a\,[-1, 1]. \tag{5}$$

The right side of Eq.(5) is so called center–radius representation of an interval. The absolute value is defined by the following equation and can be written in terms of center and radius.

$$\begin{aligned} |A^I| &\triangleq \max(|\underline{a}|, |\overline{a}|) \\ &= |a^c| + \Delta a. \end{aligned} \tag{6}$$

The transitive order relation $<$ of real numbers can be extended to the intervals as follows:

$$A^I < B^I \equiv \overline{a} < \underline{b}. \tag{7}$$

The other relations, $\leq, >, \geq$, are defined similarly. It happens that if $A^I \not< B^I$, then there are three possibilities (not as in real numbers): $A^I > B^I$, $A^I = B^I$ or $A^I \cap B^I \neq \varnothing$.
For the last case, we say that A^I and B^I are not comparable. However, the maximum and minimum can be defined on non-comparable intervals,

$$\begin{aligned} \max(A^I, B^I) &\triangleq \left[\max(\underline{a}, \underline{b}), \max(\overline{a}, \overline{b})\right], \\ \min(A^I, B^I) &\triangleq \left[\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})\right]. \end{aligned} \tag{8}$$

In general,

$$\begin{aligned} \max_i(A_i^I) &= \left[\max_i(\underline{a}_i), \max_i(\overline{a}_i)\right], \\ \min_i(A_i^I) &\doteq \left[\min_i(\underline{a}_i), \min_i(\overline{a}_i)\right]. \end{aligned} \tag{9}$$

Thus, the order relations of intervals can be defined by the min and max operators as follows:

$$A^I \leq B^I \Longleftrightarrow \max(A^I, B^I) = B^I. \tag{10}$$

Let A^I and B^I be two intervals, and $*$ be one of the binary operators $(+, -, \times, /)$. The interval arithmetic of two intervals is a set, and

$$A^I * B^I = \left\{a * b \,|\, a \in A^I; b \in B^I\right\}. \tag{11}$$

Note that B^I should not contain 0 if $* = /$. By the definitions of A^I and B^I, one can easily obtain that

$$A^I + B^I = (a^c + b^c) + (\Delta a + \Delta b)[-1, 1], \tag{12}$$

$$A^I - B^I = (a^c - b^c) + (\Delta a + \Delta b)[-1, 1], \tag{13}$$

$$A^I \times B^I = \left[\min(\underline{a}\,\underline{b}, \underline{a}\,\overline{b}, \overline{a}\,\underline{b}, \overline{a}\,\overline{b}), \max(\underline{a}\,\underline{b}, \underline{a}\,\overline{b}, \overline{a}\,\underline{b}, \overline{a}\,\overline{b})\right], \tag{14}$$

$$A^I / B^I \triangleq [\underline{a}, \overline{a}] \times \left[\frac{1}{\overline{b}}, \frac{1}{\underline{b}}\right] = \left[\min(\frac{\underline{a}}{\overline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\underline{b}}), \max(\frac{\underline{a}}{\overline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\underline{b}})\right], \text{ if } 0 \notin B^I. \tag{15}$$

There are no closed forms of center–radius representations for $\times$ and $/$, however, if $A^I = [-\triangle a, \triangle a]$ is so called symmetric interval, i.e. $a^c = 0$, and $B^I = [\underline{b}, \overline{b}] = b^c + \triangle b\,[-1,1]$ is any interval, then

$$\begin{aligned} A^I \times B^I &= \triangle a \left|B^I\right| [-1,1] \\ &= \triangle a \left(|b^c| + \triangle b\right)[-1,1]. \end{aligned} \tag{16}$$

We shall use large bold letter to stand for matrix and small bold letter for vector in this article, and add an ^{I} superscript for interval matrix or interval vector. Let $\mathbf{A}^I = [a^I_{ij}] = [\underline{\mathbf{A}}, \overline{\mathbf{A}}]$ be an $n \times n$ interval matrix, in which $\underline{\mathbf{A}} = [\underline{a}_{ij}]$ and $\overline{\mathbf{A}} = [\overline{a}_{ij}]$ are ordinary matrices and stand for lower bound and upper bound of $\mathbf{A}^I$. We define its center matrix and radius matrix as follows,

$$\text{center matrix}: \ \mathbf{A}^c \triangleq \frac{1}{2}\left(\underline{\mathbf{A}} + \overline{\mathbf{A}}\right), \tag{17}$$

$$\text{raidus matrix}: \ \triangle\,\mathbf{A} \triangleq \frac{1}{2}\left(\overline{\mathbf{A}} - \underline{\mathbf{A}}\right). \tag{18}$$

Note that $\triangle\mathbf{A}$ is a non–negative real matrix. Therefore, the center–radius representation of an interval matrix $\mathbf{A}^I$ is

$$\mathbf{A}^I = \mathbf{A}^c + \triangle\mathbf{A}\,[-1,1]. \tag{19}$$

Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be a real vector, then

$$\mathbf{A}^I\mathbf{x} = \mathbf{A}^c\mathbf{x} + \triangle\mathbf{A}\,|\mathbf{x}|\,[-1,1], \tag{20}$$

where $|\mathbf{x}| = (|x_1|, |x_2|, \cdots, |x_n|)^T$, furthermore,

$$\mathbf{x}^T\mathbf{A}^I\mathbf{x} = \mathbf{x}^T\mathbf{A}^c\mathbf{x} + |\mathbf{x}|^T \triangle\,\mathbf{A}\,|\mathbf{x}|\,[-1,1]. \tag{21}$$

2.2. Some Properties of Interval Mathematics

Since the results from the interval algebra are sets, some classical algebra properties may not hold for interval arithmetic.

2.2.1. Subdistributivity

Not all the algebraic properties for real numbers hold for intervals, especially the distributive law. For example, $[-1,1][1,2] + [-1,1][-3,-2] = [-2,2] + [-3,3] = [-5,5]$, but $[-1,1]([1,2] + [-3,-2]) = [-1,1][-2,0] = [-2,2] \subseteq [-5,5]$. We have the following subdistributivity or inclusion theorem.

Theorem 1 (Subdistributivity) .*Let A^I, B^I and C^I be three intervals, then, in general,*

$$A^I(B^I + C^I) \subseteq A^I B^I + A^I C^I. \tag{22}$$

Proof. Let $A^I = [\underline{a}, \overline{a}]$, $B^I = [\underline{b}, \overline{b}]$ and $C^I = [\underline{c}, \overline{c}]$.
$\forall x \in A^I(B^I + C^I) \Rightarrow \exists\, y \in A^I$, $z \in B^I + C^I$ such that $x = yz$, and

$$
\begin{aligned}
&\underline{a} \leq y \leq \overline{a} \text{ and } \underline{b} + \underline{c} \leq z \leq \overline{b} + \overline{c} \\
\Rightarrow\ & \underline{ab} + \underline{ac} \leq yz \leq \overline{a}\overline{b} + \overline{ac} \\
&\text{or } \underline{a}\overline{b} + \underline{a}\overline{c} \leq yz \leq \overline{a}\underline{b} + \overline{a}\underline{c} \\
&\text{or } \overline{a}\underline{b} + \overline{a}\underline{c} \leq yz \leq \underline{a}\overline{b} + \underline{a}\overline{c} \\
&\text{or } \overline{a}\overline{b} + \overline{ac} \leq yz \leq \underline{ab} + \underline{ac} \\
\Rightarrow\ & \min(\underline{ab} + \underline{ac}, \underline{a}\overline{b} + \underline{a}\overline{c}, \overline{a}\underline{b} + \overline{a}\underline{c}, \overline{a}\underline{b} + \overline{ac}) \leq x \leq \max(\underline{ab} + \underline{ac}, \underline{a}\overline{b} + \underline{a}\overline{c}, \overline{a}\underline{b} + \overline{a}\underline{c}, \overline{a}\underline{b} + \overline{ac}) \\
\Rightarrow\ & \min(\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\underline{b}) + \min(\underline{ac}, \underline{a}\overline{c}, \overline{a}\underline{c}, \overline{ac}) \leq x \leq \max(\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\underline{b}) + \max(\underline{ac}, \underline{a}\overline{c}, \overline{a}\underline{c}, \overline{ac}) \\
\Rightarrow\ & x \in A^I B^I + A^I C^I.
\end{aligned}
$$

However, if $B^I C^I > \dot{0}$, then $A^I(B^I + C^I) = A^I B^I + A^I C^I$.

2.2.2. Inclusion Monotonicity

Because of the set theoretic definitions of interval mathematics, the interval arithmetic has the monotonic property in interval functions. First, let us define two extensions from ordinary functions to interval functions. Let $\mathcal{P}(A)$ stand for the power set of A.

Definition 2 .*Let $f : A_1 \times A_2 \times \cdots A_n \longrightarrow \mathbb{R}$ be a real valued function. The united extension $\overline{f} : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \mathcal{P}(A_n) \longrightarrow \mathcal{P}(\mathbb{R})$ is defined by,*

$$\overline{f}(X_1, X_2, \ldots, X_n) = \{f(x_1, x_2, \ldots, x_n) \,|\, x_i \in X_i,\ X_i \in \mathcal{P}(A_i), i = 1, 2, \cdots n\}. \tag{23}$$

One can see that this is well–defined, and it has the *subset* property that

$$
\begin{aligned}
& X_1 \subseteq Y_1, X_2 \subseteq Y_2, \cdots, X_n \subseteq Y_n \\
\Rightarrow\ & \overline{f}(X_1, X_2, \ldots, X_n) \subseteq \overline{f}(Y_1, Y_2, \ldots, Y_n).
\end{aligned} \tag{24}
$$

This property is referred to the *inclusion monotonicity*. The interval arithmetic functions, Eqs.(12), (13), (14) and (15 are united extensions of ordinary arithmetic operations, so they are of course inclusion monotonic. Furthermore, all the rational interval functions are inclusion monotonic, since they are natural united extensions of the real–valued rational functions.

Let $\mathcal{I}(A)$ stand for the set of closed intervals of set A. We have the following definition for interval extension.

Definition 3 .*Let $f : A_1 \times A_2 \times \cdots A_n \longrightarrow \mathbb{R}$ be a real valued function of n variables $x_1, x_2, \ldots x_n$, and $F : \mathcal{I}(A_1) \times \mathcal{I}(A_2) \times \cdots \mathcal{I}(A_n) \longrightarrow \mathcal{I}(\mathbb{R})$ be an interval valued function of n interval variables $X_1^I, X_2^I, \cdots X_n^I$. If*

$$F(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n), \quad \text{for } x_i \in X_i^I, \tag{25}$$

then F is called the interval extension of f.

In general, the inclusion monotonic property do not hold for the interval extension. However, if F is an inclusion monotonic interval extension, then $\overline{f}(X_1, X_2, \cdots, X_n) \subseteq F(X_1, X_2, \cdots, X_n)$([9], p. 21). The interval extension of a function is also referred to the *interval evaluation.* By the interval evaluation of a rational function f, we mean that each variable in f is replaced by a corresponding interval variable and the arithmetic operations are replaced respectively by the corresponding interval arithmetic operations. Moreover, *the rational interval extensions shall be inclusion monotonic.* Unfortunately, since the distributive law does not hold for interval arithmetic, two interval extensions from two equivalent rational functions may not be equivalent. For example, $f(x) = x^3 - x$ and $g(x) = x(x-1)(x+1)$ are equivalent, but $F(X) = X \cdot X \cdot X - X$ and $G(X) = X \cdot (X-1) \cdot (X+1)$ are not equivalent, since $F([0,1]) = [-1,1]$ and $G([0,1]) = [-2,0]$. In fact, the exact range for $f(x)$ when $x \in [0,1]$ is $\left[-2/(3\sqrt{3}), 0\right]$. This irrational greatest lower bound can not be obtained by any rational interval extension of any equivalent form of $f(x)$. The interval extension can give only the bounds for the values of a rational function when its variable lies in an interval. For many applications, it is necessary that the exact range , $\left[\min_{x \in X} f(x), \max_{x \in X} f(x)\right]$ be given. Nevertheless, the exact range could be determined by the interval extensions of some kinds of rational functions. Let us quote this from Moore ([9] p. 23):

> $\cdots$ *any natural interval extension of a rational function in which each variable occurs only once (if at all) and to the first power only will compute the exact range of values* $\cdots$

3. Courant–Fischer Minimax Theorem

By a symmetric interval matrix $\mathbf{A}^I$, we mean that if $\mathbf{A} \in \mathbf{A}^I$, then $\mathbf{A}$ is symmetric. The interval extension of Rayleigh's quotient of a symmetric $n \times n$ matrix $\mathbf{A}$, $f(\mathbf{x};\mathbf{A}) = \dfrac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is

$$F(\mathbf{x};\mathbf{A}^I) = \frac{\mathbf{x}^T\mathbf{A}^I\mathbf{x}}{\mathbf{x}^T\mathbf{x}}. \tag{26}$$

Note that the vector variable $\mathbf{x}$ is not extended. The inclusion monotonic property holds for Eq.(26). Let $\mathbf{A}^I$ and $\mathbf{B}^I$ be two symmetric interval matrices, then

$$\mathbf{A}^I \subseteq \mathbf{B}^I \Rightarrow \frac{\mathbf{x}^T\mathbf{A}^I\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \subseteq \frac{\mathbf{x}^T\mathbf{B}^I\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Solveing the eigenproblem of a symmetric matrix is equivalent to finding the stationary value of its Rayleigh's quotient.

Theorem 4 (Courant–Fischer Minimax Theorem) *.If* $\mathbf{A} \subseteq \mathbb{R}^{n\times n}$ *is symmetric and* λ_i *are its eigenvalues ordered as* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$*, then*

$$\lambda_i = \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \left(\mathbf{x}^T\mathbf{A}\mathbf{x}\right), \tag{27}$$

for $i = 1, 2, \ldots n$.

As Eq.(26), the interval version of Courant–Fischer Minimax Theorem is obtained by the interval extension of Eq.(27), i.e.

$$\lambda_i^I(\mathbf{A}^I) = [\underline{\lambda}_i(\mathbf{A}^I), \overline{\lambda}_i(\mathbf{A}^I)] = \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \left(\mathbf{x}^T\mathbf{A}^I\mathbf{x}\right), \text{ for } i = 1, 2, \ldots, n, \tag{28}$$

where $\mathbf{A}^I$ is a symmetric interval matrix. The min or max operation in Eq.(28) is for intervals, and can be defined according to Eq.(8). Hertz ([7]) found that the $\min_i(\underline{\lambda}_i\,(A^I))$ and $\max_i(\overline{\lambda}_i\,(A^I))$ can be computed from some 2^{n-1} vertex matrices of $\mathbf{A}^I$. We mean the vertex matrix of $\mathbf{A}^I$ by the matrix $\mathbf{C} = [c_{ij}]$ in which $c_{ij} = \underline{a}_{ij}$ or $\overline{a}_{ij}$, for all i, j.

Based on the fact from Moore and inclusion monotonic property, we have

Theorem 5 *.If* A^I *and* B^I *are symmetric interval matrices, then*

$$\begin{aligned} \mathbf{A}^I \subseteq \mathbf{B}^I &\Rightarrow \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \left(\mathbf{x}^T\mathbf{A}^I\mathbf{x}\right) \subseteq \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \left(\mathbf{x}^T\mathbf{B}^I\mathbf{x}\right) \\ &\Rightarrow \lambda_i^I(\mathbf{A}^I) \subseteq \lambda_i^I(\mathbf{B}^I), \end{aligned} \tag{29}$$

for $i = 1, 2, \cdots n$*, where* $\lambda_i^I(\cdot)$ *is the ith interval eigenvalue of interval matrix* $\cdot$.

Proof. Let $\mathcal{S} \subseteq \mathbb{R}^n$, since $\mathbf{A}^I \subseteq \mathbf{B}^I$, then for each $\mathbf{x} \in \mathcal{S}$ with $\|\mathbf{x}\| = 1$, we have

$$\mathbf{x}^T\mathbf{A}^I\mathbf{x} \subseteq \mathbf{x}^T\mathbf{B}^I\mathbf{x}. \tag{30}$$

We claim that for some $\mathcal{S} \subseteq \mathbb{R}^n$

$$\max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|=1}}(\mathbf{x}^T\mathbf{A}^I\mathbf{x}) \subseteq \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|=1}}\left(\mathbf{x}^T\mathbf{B}^I\mathbf{x}\right). \tag{31}$$

Since if

$$\mathbf{x}_k^T\mathbf{A}^I\mathbf{x}_k = \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|=1}}(\mathbf{x}^T\mathbf{A}^I\mathbf{x}) \supset \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|=1}}(\mathbf{x}^T\mathbf{B}^I\mathbf{x}) \tag{32}$$

for some unit vector $\mathbf{x}_k \in \mathcal{S}$, then from Eq.(30) and Eq.(32),

$$\begin{aligned}
&\mathbf{x}_k^T\mathbf{A}^I\mathbf{x}_k \subseteq \mathbf{x}_k^T\mathbf{B}^I\mathbf{x}_k \\
\Rightarrow &\max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|=1}}(\mathbf{x}^T\mathbf{B}^I\mathbf{x}) \subset \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|=1}}(\mathbf{x}^T\mathbf{A}^I\mathbf{x}) = \mathbf{x}_k^T\mathbf{A}^I\mathbf{x}_k \subseteq \mathbf{x}_k^T\mathbf{B}^I\mathbf{x}_k .
\end{aligned}$$

This comes up a contradiction. Hence,

$$\min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}}\ \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}}\left(\mathbf{x}^T\mathbf{A}^I\mathbf{x}\right) \subseteq \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}}\ \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}}\left(\mathbf{x}^T\mathbf{B}^I\mathbf{x}\right). \tag{33}$$

From above theorem, we know that the Courant–Fischer minimax characterization for eigenvalues can be extended for interval symmetric matrix and still have the interval monotonic property.

4. Interval Eigenvalue Problems

In the section, we are to find the interval eigenvalues by spliting Eq. (28) into two ordinary eigenvalue problems for both lower and upper bounds of the interval eigenvalues.

Substitute the center–radius representation, $\mathbf{A}^I = \mathbf{A}^c + \triangle\mathbf{A}\,[-1,1]$, into Eq.(28), and from Eq.(??), then

$$\begin{aligned}
\lambda_i^I &= \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}}\ \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}}\left(\mathbf{x}^T\mathbf{A}^c\mathbf{x} + |\mathbf{x}|^T \triangle\,\mathbf{A}\,|\mathbf{x}|\,[-1,1]\right) \\
&= \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}}\ \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}}\left[\mathbf{x}^T\mathbf{A}^c\mathbf{x} - |\mathbf{x}|^T\triangle\mathbf{A}\,|\mathbf{x}|\,,\ \mathbf{x}^T\mathbf{A}^c\mathbf{x} + |\mathbf{x}|^T\triangle\mathbf{A}\,|\mathbf{x}|\right].
\end{aligned}$$

Let $|\mathbf{x}| = S^{\mathbf{x}}\mathbf{x}$, where $S^{\mathbf{x}}$ =diag([sign(x_1),sign(x_2),... ,sign(x_n)]) is a sign diagonal matrix of $\mathbf{x}$. Thus,

$$\lambda_i^I = \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \left[\mathbf{x}^T\left(\mathbf{A}^c - S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x},\, \mathbf{x}^T\left(\mathbf{A}^c + S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x}\right]$$

$$= \left[\min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \mathbf{x}^T\left(\mathbf{A}^c - S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x},\; \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \mathbf{x}^T\left(\mathbf{A}^c + S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x}\right].$$

$\Longrightarrow$

$$\underline{\lambda}_i = \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \mathbf{x}^T\left(\mathbf{A}^c - S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x}, \tag{34}$$

$$\overline{\lambda}_i = \min_{\substack{\dim(\mathcal{S})=i,\\ \mathcal{S}\subseteq\mathbb{R}^n}} \max_{\substack{\mathbf{x}\in\mathcal{S},\\ \|\mathbf{x}\|_2=1}} \mathbf{x}^T\left(\mathbf{A}^c + S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x}. \tag{35}$$

The above two equations are the Courant–Fischer minimax characterization for the eigenvalues, $\underline{\lambda}_i$ and $\overline{\lambda}_i$ of matrices, $\mathbf{A}^c - S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}$ and $\mathbf{A}^c + S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}$, respectively. In other words, the minimax value of the quadratic function $\mathbf{x}^T\left(\mathbf{A}^c - S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x}$ is the ith eigenvalue $\underline{\lambda}_i$ associated with the eigenvector $\underline{\mathbf{x}}_i$, and the minimax value of the quadratic function $\mathbf{x}^T\left(\mathbf{A}^c + S^{\mathbf{x}}\Delta\mathbf{A}S^{\mathbf{x}}\right)\mathbf{x}$ is the ith eigenvalue $\overline{\lambda}_i$ associated with the eigenvector $\overline{\mathbf{x}}_i$. Correspondingly, Eqs. (34) and (35) are equivalent to the following two ordinary eigenvalue problems,

$$\left(\mathbf{A}^c - S^{\mathbf{x}_i}\Delta\mathbf{A}S^{\mathbf{x}_i}\right)\mathbf{x}_i = \underline{\lambda}_i\mathbf{x}_i, \tag{36}$$

$$\left(\mathbf{A}^c + S^{\mathbf{x}_i}\Delta\mathbf{A}S^{\mathbf{x}_i}\right)\mathbf{x}_i = \overline{\lambda}_i\mathbf{x}_i. \tag{37}$$

In order to solve the above eigenvalue problems, we put a restriction on the sign matrix S based on the assumption from Deif[3] and Rohn[12] that *the sign of the elements of eigenvectors do not change within the intervals of the eigenvalues.* Under this restriction, the eigenvalues found from Eqs.(36), (37) are exact bounds for $\lambda_i^I(\mathbf{A}^I)$. Although this restriction is difficult to verify[13], it wouldn't be a dangerous one[4]. Since the validity of the assumption can be examined *a posteriori.* One may find the ith eigenvector from $\mathbf{A}^c\mathbf{x}_i = \lambda_i\mathbf{x}_i$ and construct S from $\mathbf{x}_i$ then solve $\underline{\lambda}_i$ and $\overline{\lambda}_i$ from Eqs.(36), (37), respectively, for $i = 1, 2, \ldots n$. Note that the associated eigenvectors shall not be intervals in general. Since the dimension of eigenspace is greater than or equal to 1, say k, then there are also k elements of the eigenvector left to be arbitrarily determined in the sense of linear independence. Therefore, it is not necessary that each element of the eigenvector with lower bound eigenvalue, $\underline{\lambda}_i$ be less than or equal to corresponding element of the eigenvector with upper bound eigenvalue, $\overline{\lambda}_i$. We summarize the above discussion for calculating the interval eigenvalues of a symmetric interval matrix as follows:

1. Determine: $\mathbf{A}^c = \frac{1}{2}\left(\underline{\mathbf{A}} + \overline{\mathbf{A}}\right)$ and $\triangle\mathbf{A} = \frac{1}{2}\left(\overline{\mathbf{A}} - \underline{\mathbf{A}}\right)$
2. Solve: $\mathbf{A}^c\mathbf{x}_i = \lambda_i\mathbf{x}_i$
3. Construct S^i with respect to each eigenvector $\mathbf{x}_i$ from step 2 above
4. Solve: $\left(\mathbf{A}^c - S^i\triangle\mathbf{A}S^i\right)\mathbf{x}_i = \underline{\lambda}_i\mathbf{x}_i$ and $\left(\mathbf{A}^c + S^i\triangle\mathbf{A}S^i\right)\mathbf{x}_i = \overline{\lambda}_i\mathbf{x}_i$ for each i

5. Example

Let $\mathbf{A}^I = \begin{bmatrix} [9,15] & [-4,-2] & 0 \\ [-4,-2] & [10,16] & [-8,-4] \\ 0 & [-8,-4] & [10,14] \end{bmatrix}$, then $\mathbf{A}^c = \begin{bmatrix} 12 & -3 & 0 \\ -3 & 13 & -6 \\ 0 & -6 & 12 \end{bmatrix}$ and $\triangle\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}$, the eigenpais of $\mathbf{A}^c$are $\mathbf{x}_1 = \begin{Bmatrix} 1.0000 \\ 2.0756 \\ 2.0000 \end{Bmatrix}$ for $\lambda_1 = 5.7732$, and $\mathbf{x}_2 = \begin{Bmatrix} 1.0000 \\ 0.0000 \\ -0.5000 \end{Bmatrix}$ for $\lambda_2 = 12$ and $\mathbf{x}_3 = \begin{Bmatrix} 1.0000 \\ -2.4089 \\ 2.0000 \end{Bmatrix}$ for $\lambda_3 = 19.2268$. Thus, S^1 =diag([1, 1, 1]), S^2 =diag([1, 0, −1]), S^3 =diag([1, −1, 1]). Thus, we have the following results:

1. $\lambda_1^I = \left[\underline{\lambda}_1, \overline{\lambda}_1\right] = [0.9454, 10.5158]$, $\underline{\mathbf{x}}_1 = \begin{Bmatrix} 1.0000 \\ 2.0137 \\ 1.7791 \end{Bmatrix}$, $\overline{\mathbf{x}}_1 = \begin{Bmatrix} 1.0000 \\ 2.2421 \\ 2.5740 \end{Bmatrix}$
2. $\lambda_2^I = \left[\underline{\lambda}_2, \overline{\lambda}_2\right] = [9.2142, 14.7935]$, $\underline{\mathbf{x}}_2 = \begin{Bmatrix} 1.0000 \\ -0.0714 \\ -0.5450 \end{Bmatrix}$, $\overline{\mathbf{x}}_2 = \begin{Bmatrix} 1.0000 \\ 0.0688 \\ -0.5206 \end{Bmatrix}$
3. $\lambda_3^I = \left[\underline{\lambda}_3, \overline{\lambda}_3\right] = [14.3884, 24.0972]$, $\underline{\mathbf{x}}_3 = \begin{Bmatrix} 1.0000 \\ -2.6942 \\ 2.4558 \end{Bmatrix}$, $\overline{\mathbf{x}}_3 = \begin{Bmatrix} 1.0000 \\ -2.2743 \\ 1.8019 \end{Bmatrix}$

Note that the lower and upper eigenvectors may not form an interval in general. One can take a look at the 3rd elements of the 3rd eigenvectors for example. In other words, the form $[\underline{\mathbf{x}}_i, \overline{\mathbf{x}}_i]$ is meaningless.

6. Conclusion

The center–radius representation of an interval is a tool on simplifying the interval arithmetic. Some classical problems can then be handled when those parameters are replaced with intervals by interval extension or evaluation. However, the inclusion monotonic property shall not pertain to all interval extensions. Fortunately, it holds for the interval Courant–Fischer minimax equation. This guarantee that the eigenvalues (and eigenvectors) found from the vertex matrices contain all the eigenvalues of any matrix in interval. Even though the procedure needs the assumption of unchanged sign eigenvectors, it can be easily verified *a posteriori* that it is true. The example shows that the eigenvectors can not form an interval in whole, whereas both corresponding elements of the eigenvectors from the two eigenproblems are the bounds of those of other matrices inside interval when all the eigenvectors are normalized in the same manner.

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