# Application of a New Continuation Method to Flutter Equations *

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**Abstract**

Recent developments in the field of numerical analysis hold promise for a more robust and efficient method for the solution of flutter problems which include active controls, s-plane aerodynamics, and parameter variations. Over the past several years a considerable amount of work has been done on a class of methods known as continuation methods. This paper describes continuation methods and shows how the progress which has been made on these methods can be applied to flutter problems. Specifically, it is shown how a package of subroutines developed by Rheinboldt[1] is used at Boeing to solve a variety of flutter problems.

## 1 Nomenclature

| | |
|---|---|
| $\mathbf{A}$ | unsteady aerodynamic matrix |
| $a$ | sonic velocity |
| $\mathbf{C}$ | control system matrix |
| $\mathbf{D}$ | dynamic matrix |
| $\mathbf{E}$ | real representation of $\mathbf{D}$ |
| $\mathbf{G}$ | gyroscopic matrix |
| $g$ | added structural damping $\approx 2\sigma/\omega$ |
| $\mathbf{J}$ | Jacobian matrix $\partial\mathbf{y}/\partial\mathbf{x}$ |
| $\mathbf{K}$ | stiffness matrix |
| $\mathbf{K_r}$ | centrifugal stiffening matrix |
| $k$ | reduced frequency $\omega/V$ |
| $\mathbf{M}$ | mass matrix |
| $M$ | Mach number $V/a$ |
| $n$ | order of the dynamic matrix |
| $\mathbf{p}$ | vector of system parameters |
| $p$ | complex reduced frequency $s/V$ |
| $q$ | dynamic pressure |
| $s$ | characteristic exponent $\sigma + i\omega$ |

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| | |
|---|---|
| $t$ | artificial parameter for starting continuation process |
| $\mathbf{t}$ | tangent vector |
| $\mathbf{u}$ | generalized coordinates |
| $V$ | velocity (true airspeed) |
| $\mathbf{v}$ | real representation of $\mathbf{u}$ |
| $\mathbf{x}$ | independent variables |
| $\mathbf{y}$ | residual vector |
| $z$ | altitude |
| $\gamma$ | growth rate $2\sigma/\omega$ |
| $\rho$ | fluid density |
| $\sigma$ | real part of characteristic exponent |
| $\omega$ | oscillation frequency |
| $\Omega$ | rotation rate |

*Superscripts*

| | |
|---|---|
| $()^*$ | conjugate transpose |
| $()^T$ | transpose |

## 2 Introduction

Throughout the design of a modern aircraft a number of calculations are performed to ensure that the aircraft will be free of flutter. In the preliminary stages of design various combinations of design parameters are considered; as the design becomes fixed flutter calculations are required at various combinations of fuel, payload, altitude, and other parameters that define flight conditions. The flutter analyst is therefore concerned with two basic types of calculations: a search for the lowest (critical) flutter speed under a given set of conditions, and the variation of the critical flutter speed with certain system parameters. Flutter equations commonly encountered in modern structures contain terms which make them difficult or impossible to treat with the traditional $V-g$ approach. A partial list includes active controls, s-plane aerodynamics, gyroscopics, centrifugal stiffening, and Mach-dependent aerodynamics. This paper is an attempt to address both the flutter problem and the parameter-variation problem with a unified solution process based on recent research in continuation methods.

Continuation methods are a class of techniques for the solution of systems of nonlinear equations which are functions of a number of parameters, over a specified range of the parameters. Closely related to continuation methods, as they are known in numerical analysis, are the incremental methods of nonlinear structural analysis. Indeed, continuation and incremental methods have been developed simultaneously over the last thirty years, and have only recently been considered facets of the same problem[2]

## 3 Flutter Equations

The flutter equation for structures in inertial reference frames can be written as

$$\left[s^2\mathbf{M} + s\mathbf{G} + (1+ig)\mathbf{K} + i\mathbf{S} - q\mathbf{A}(p,M) + \mathbf{C}\right]\mathbf{u} = \mathbf{0} \tag{1}$$

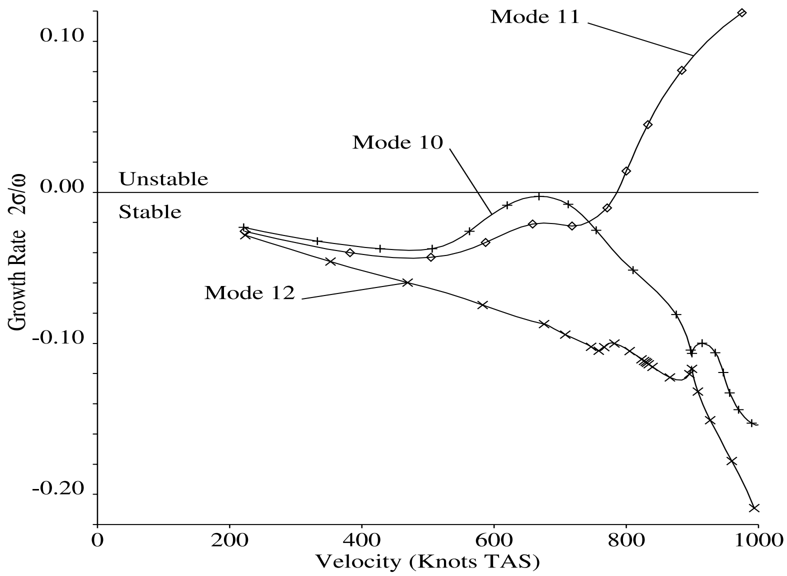


Figure 1: Typical Neutral-Stability Plot

or, in the case of rotating structures, such as turbofan blades,

$$\left[s^2\mathbf{M} + s\Omega\mathbf{G} + \Omega^2\mathbf{K}_r + (1+ig)\mathbf{K} + i\mathbf{S} - q_r\mathbf{A}(p_r, M_r) + \mathbf{C}\right]\mathbf{u} = \mathbf{0} \tag{2}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{G}$, and $\mathbf{A}$ are the (n by n) mass, stiffness, gyroscopic, and unsteady aerodynamic matrices, $\mathbf{K_r}$ is the centrifugal-stiffening matrix, $s = \sigma + i\omega$ is the complex characteristic exponent, $\mathbf{u}$ is a complex vector of generalized coordinates, $\mathbf{C}$ is a matrix of control-system coefficients, $q = \rho V^2/2$ (or $q_r = \rho\Omega^2/2$) is the dynamic pressure, $p = s/V$ (or $p_r = s/\Omega$) is the complex reduced frequency, $M = V/a$ is the free-stream Mach number, $a$ is the sonic velocity, and $M_r$ is the tip Mach number. The interest in these equations is in the behavior of the real part of the characteristic exponent ($s$) with velocity, Negative values indicate decaying oscillations, positive values indicate growing oscillations and zero indicates neutral stability. To avoid semantic difficulties with the traditional term *decay rate*[3] defined as $\sigma/\omega$, we prefer the term *growth rate*, $\gamma = 2\sigma/\omega$. From Eq. (1) or (2), $n$ physically meaningful *aeroelastic modes* can be calculated at each set of system parameters. Normally the flutter analyst is interested in only a few of these modes. As an example of the results from such an analysis, referred to here as a *neutral-stability* analysis, Figure 1 shows the growth rate plotted against velocity for three aeroelastic modes. Mode 11 is evidently neutrally-stable at about 800 knots, while modes 10 and 12 are stable throughout the velocity range. Once the lowest neutral-stability velocity (critical flutter speed) $V_f$ has been found, interest usually turns to studying the variation in critical flutter speed with various model parameters such as mass and stiffness distributions.

Parameter studies use a slightly different form of the flutter equations:

$$\left[\omega^2\mathbf{M}(\mathbf{p}) + i\omega\mathbf{G}(\mathbf{p}) + \mathbf{K}(\mathbf{p}) - q\mathbf{A}(k, M, \mathbf{p}) + \mathbf{C}(\omega, V, \mathbf{p}\right]\mathbf{u} = \mathbf{0} \tag{3}$$

where $\sigma$ is implicitly zero, $\mathbf{p}$ is a vector of the system parameters of interest, and the coefficient matrices are, in general functions of these parameters. Each of Eq. (1), (2), or (3) is a set of $n$ complex nonlinear algebraic equations which are linear and homogeneous in the generalized coordinates ($\mathbf{u}$), and nonlinear in $\sigma, \omega, \rho, V$ (or $\Omega$), $a$, and $\mathbf{p}$. By expanding the notion of system parameters to include velocity, frequency, growth rate, density, and sonic velocity, each of these equations may be expressed as a set of algebraic equations of the form

$$\mathbf{D}(\mathbf{x})\mathbf{u} = \mathbf{0} \tag{4}$$

where $\mathbf{D}$ is the ($n$ by $n$) complex dynamic matrix in (1), (2), or (3), and $\mathbf{x}$ is a vector of system parameters. Depending on the choice of variables included in $\mathbf{x}$ a variety of formulations are possible.

### 3.1 Solution Methods

Techniques for solving (4) are necessarily iterative in nature. Solutions procedures are usually based on one of two views of the equations. In the first approach, it is recognized that a solution requires that the determinant of $\mathbf{D}$ be zero, resulting in a complex scalar nonlinear equation at each set of system variables. All system variables except the characteristic exponent $s$ are fixed, resulting in a single complex nonlinear equation. Hassig[3] proposed solving the determinant equation using the Regula Falsi method. Stark[4] uses a Newton method with finite difference derivative approximations (also known as a *secant* method). The basic drawback to solving the determinant equation is that since it is impractical to compute derivatives of the determinant with respect to system parameters, only first-order convergent methods such as Regula Falsi or a secant method can be used, instead of more rapidly-converging methods such as Newton's method. Moreover, the lack of derivative information means less accuracy in the predictor phase; hence smaller steps must be taken not only to ensure convergence in the corrector but also to ensure that the corrector converges to the correct root. Convergence to the wrong root, known as *mode switching*, can occur when two modes are nearly equal. The danger with mode switching is that a mode may be left unaccounted for. For example, Fig. 1 shows two modes, 10 and 12, that have nearly the same growth rate at about 900 knots; indeed the discontinuity in both modes appears to be caused by both modes switching. That this is not the case will be demonstrated below.

A second approach, and the one adopted here, is to consider (4) as a system of $2n$ real nonlinear equations in the variables $\mathbf{x}$. Equation (4) is converted to a set of real equations by replacing matrix elements by their 2 by 2 equivalents[5]:

$$d_{jk} = a_{jk} + ib_{jk} \leftrightarrow \begin{bmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk} \end{bmatrix}$$

and vector elements with their 2 by 1 real equivalents:

$$u_j = v_j + iw_j \leftrightarrow \left\{ \begin{array}{c} v_j \\ w_j \end{array} \right\}$$

An equivalent real expression for the dynamic equation (4) is

$$\mathbf{D}(\mathbf{x})\mathbf{u} = \mathbf{0} \leftrightarrow \mathbf{E}(\mathbf{x})\mathbf{v} = \mathbf{0} \tag{5}$$

Since (4) is linear and homogeneous in $\mathbf{u}$ it is necessary to add a normalization condition; here we chose the conditions

$$\begin{aligned} \mathbf{u}^*\mathbf{u} &= 1 \\ Im(u_m) &= 0 \end{aligned} \tag{6}$$

where $m$ is the index of the largest component of $\mathbf{u}$ at the start of the solution process, and is fixed throughout, and $()^*$ indicates the conjugate transpose. In terms of real equivalents,

$$\begin{aligned} \mathbf{v}^{\mathbf{T}}\mathbf{v} &= 1 \\ v_{2m} &= 0 \end{aligned} \tag{7}$$

Combining (4) with the two real equations (7) we have the set of $2n+2$ real nonlinear equations

$$\mathbf{y}(\mathbf{x}) = \left\{ \begin{array}{c} \mathbf{E}\mathbf{v} \\ \mathbf{v}^{\mathbf{T}}\mathbf{v} - 1 \\ v_{2m} \end{array} \right\} = \mathbf{0} \tag{8}$$

Mantegazza[6] first proposed solving flutter equations with continuation methods, using velocity as the continuation parameter. The method presented in Ref. [6] uses a Huen predictor and a Newton corrector.

Historically, continuation methods have had two major problems: step-size control and parameter limit points. Step-size control is important for reasons of economy - it is desirable to pick the largest possible predictor step-size while maintaining convergence in the corrector phase.

Limit points can cause numerical problems if the continuation parameter is not chosen properly. A solution curve is said to have a limit point in a parameter when that parameter's component of the tangent to the curve is zero. Continuation methods typically solve sets of equations with one more unknown than equations. One unknown is chosen as the *continuation parameter*, the parameter which is fixed during the corrector phase in order to give the same number of equations as unknowns. If the continuation parameter has a limit point on the curve the corrector cannot converge because all iterates lie on the tangent to the curve. This condition also gives rise to a singular Jacobian matrix. Although in some problems the choice of a continuation parameter seems obvious, others have no single parameter which is best throughout the range of interest, and a poor choice can lead to severe numerical difficulties. For example, the solution curve in Fig. 2 has a limit point in parameter $p$ (a nacelle bending frequency) *and* a limit point in the velocity $V_f$. The technique in Ref. [6] would break down at one of these points since it uses the same continuation parameter throughout the solution. Numerical difficulties might also arise with the technique in Ref. [6] at a point such as the discontinuity at 900 knots shown on Figure 1, if velocity were the continuation parameter. The continuation method discussed below avoids limit points by choosing the continuation parameter at each step based on information from the tangent vector, hence the name *locally parameterized continuation process.*

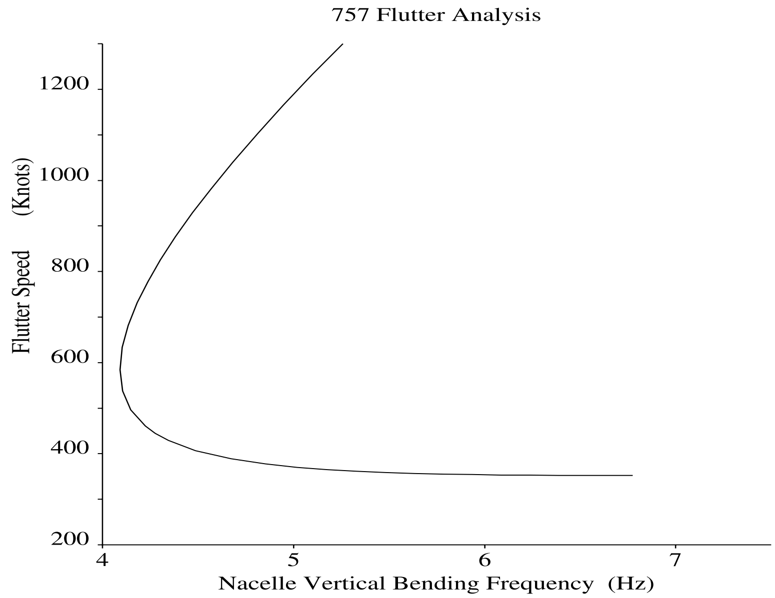


Figure 2: Parameter Variation with Limit Points

### 3.2 Comparison of the two Methods

Although solving a system of $2n+3$ nonlinear equations may appear to involve much more work than solving the single scalar determinant equation, this is not actually the case. Computation of the determinant of an $n$ by $n$ complex matrix requires approximately $2/3n^3$ complex multiplications or $8/3n^3$ real multiplications. Each iteration of Newton's method on the other hand requires the decomposition of a $2n$ by $2n$ real matrix, which takes approximately $16/3n^3$ multiplications, or twice as much as the determinant method. However, if a modified Newton's method is used whereby the Jacobian is computed and decomposed only once at the begining of the iteration process, this method would actually be cheaper than the determinant method if more than two iterations are required. In light of the comments above regarding the lack of derivative information in the determinant method, continuation methods should require fewer steps and fewer corrector iterations.

A more important consideration however, is mode switching. Because more information regarding the mode being tracked (namely the generalized coordinates) is carried along during the nonlinear system solution process than in the determinant approach, mode switching is expected to be far less likely. In the example discussed above it was pointed out that modes 10 and 12 in Fig. 1 appear to have switched; Figure 3 shows the corresponding plot of frequency vs. velocity. From these two figures it is evident that both the real *and* imaginary parts of the characteristic exponent are nearly equal at about 900 knots. In such circumstances the determinant method could not be depended upon to distinguish between the two modes. The fact that these two modes have not in fact switched is evidenced by Figure 4 which shows, in an expanded scale, the variation of generalized coordinate 12 with velocity in the vicinity of the apparent discontinuity (898 knots). Here aeroelastic modes 10 and 12 have been normalized according to Eq.(6) with $m = 16$ for purposes of comparison. Continuity clearly has been maintained in the two modes through the problem area.

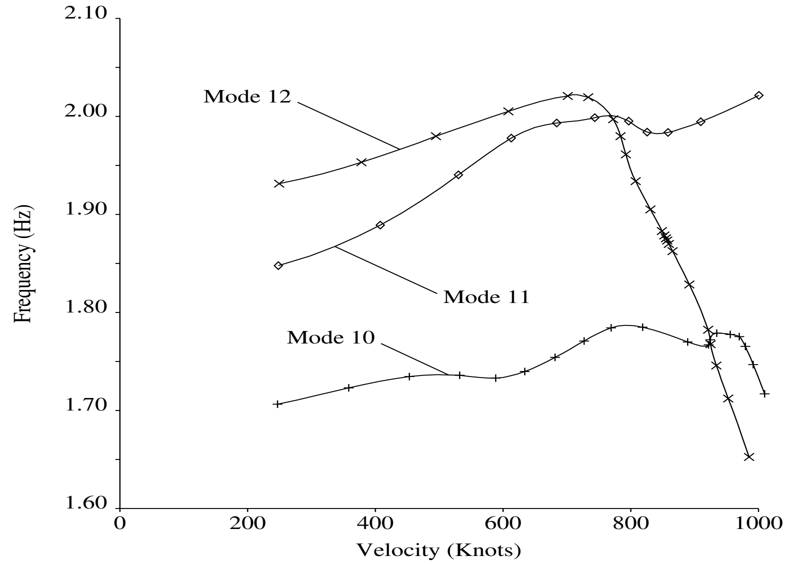


Figure 3: Typical Neutral-Stability Velocity-Frequency Plot

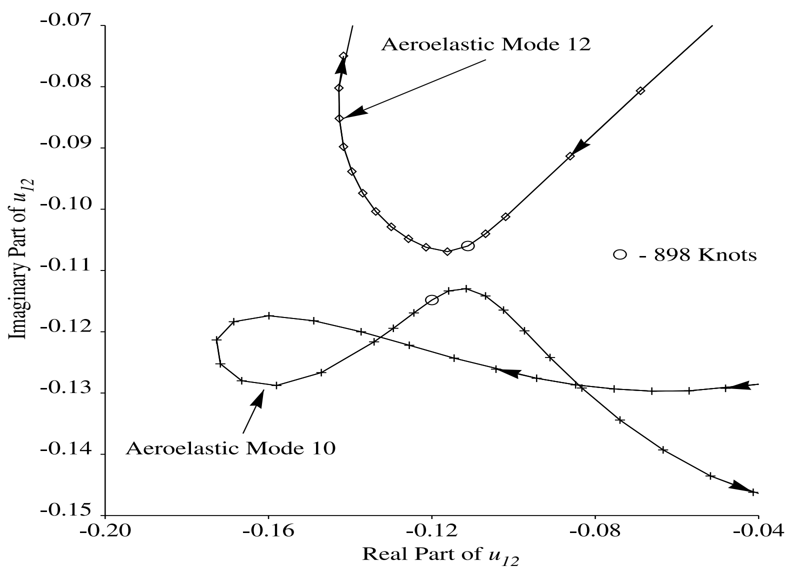


Figure 4: Generalized-Coordinate 12 Near the Discontinuity

# 4 Continuation Methods

## 4.1 Background

Common to all continuation methods is a predictor phase followed by a corrector phase. One of the system variables is chosen as the continuation parameter, then starting from a known solution, the predictor phase computes a starting guess which is then used by the corrector to obtain a solution at the new value of the continuation parameter. Reminicent of ordinary differential equation solvers, the primary difference with contination methods is that the accuracy of the solution at any point is not dependent on previous solutions.

Many continuation methods use the tangent vector in the predictor phase, resulting in an implicit parameterization of the solution curve by *arc-length.* The tangent vector is the vector $t$ with the properties

$$\begin{aligned} \mathbf{Jt} &= \mathbf{0} \\ \mathbf{t}^T\mathbf{t} &= 1, \end{aligned}$$

where $\mathbf{J} = \partial\mathbf{y}/\partial\mathbf{x}$ is the Jacobian matrix.

## 4.2 The PITCON Package

Over the past several years a considerable amount of work has been done on a continuation method by Rheinboldt and others[1],[2],[7],[8] resulting in a highly refined package of Fortran subroutines known as the PITCON package. A great deal of emphasis was placed on solving some of the traditional problems associated with continuation problems such as step-size control and the identification of the optimal continuation parameter.

PITCON was designed to be a general-purpose package for the solution of underdetermined systems of nonlinear equations in which the number of equations is one less than the number of unknowns. It requires the application program to supply, in the form of a Fortran subroutine the Jacobian matrix of partial derivatives $\mathbf{J} = \partial\mathbf{y}/\partial\mathbf{x}$. The Jacobian is used internally to compute the tangent to the solution curve at the known solution point, in addition to the Newton corrector iterates. From convergence characteristics of previous solutions, the step size is adjusted so that the predicted point lies within the estimated radius of convergence of the corrector. This new step size is then used with the tangent vector to predict the solution at the next point on the curve. Based on the largest component of the tangent vector and estimates of the local curvature of the solution curve, one of the independent variables is chosen as the local continuation parameter to be fixed throughout the corrector phase. Fixing the continuation parameter is necessary to obtain an exactly determined set of equations from the underdetermined set. Finally, the corrector process is applied until convergence is obtained and the package returns the new solution point. The corrector process can, at the user's option, be either Newton's method or a modified Newton's method where the Jacobian at the predicted point is used throughout the corrector process.

# 5 Application to Flutter Equations

The PITCON package solves systems of nonlinear equations in which the number of independent variables is one larger than the number of equations. In our case we have $2n+2$ equations, so we must choose $2n+3$ system parameters as the independent variables. $2n$ of these variables are the real and imaginary parts of the generalized coordinates. Thus $\mathbf{x}$ has the form

$$\mathbf{x} = \lfloor\ x_1 \quad x_2 \quad x_3 \quad \mathbf{v}^T\ \rfloor^T$$

Various flutter problems can be solved with PITCON, Depending on the choice of the three variables $x_1, x_2$, and $x_3$. Four examples follow while others are easily imagined.

Input to the PITCON package comprises the names of two subroutines supplied by the application program; one computes the value of $\mathbf{y}$ at a given $\mathbf{x}$, and the other computes the $2n+2$ by $2n+3$ Jacobian matrix $\partial\mathbf{y}/\partial\mathbf{x}$. The Jacobian matrix for (8) has the form

$$\partial\mathbf{y}/\partial\mathbf{x} = \begin{bmatrix} \partial\mathbf{E}/\partial x_1\mathbf{v} & \partial\mathbf{E}/\partial x_2\mathbf{v} & \partial\mathbf{E}/\partial x_3\mathbf{v} & \mathbf{E} \\ 0 & 0 & 0 & 2\mathbf{v}^T \\ 0 & 0 & 0 & \mathbf{e}_{2m}^T \end{bmatrix} \tag{9}$$

for all choices of independent variables, where $\mathbf{e}_{2m}$ is column $2m$ of the identity matrix.

## 5.1 Examples

I) *Neutral-Stability.* In general, neutral-stability solutions use

$$\mathbf{x} = \lfloor\ V \quad \sigma \quad \omega \quad \mathbf{v}^T\ \rfloor^T$$

and the terms of the Jacobian are

$$\partial\mathbf{E}/\partial x_1 = \partial\mathbf{E}/V = (\partial q/\partial V)\mathbf{A} + q\partial\mathbf{A}/\partial V + \partial\mathbf{C}/\partial V$$

$$\partial\mathbf{E}/\partial x_2 = \partial\mathbf{E}/\sigma = 2s\mathbf{M} + \mathbf{G} - q\partial\mathbf{A}/\partial p + \partial\mathbf{C}/\partial s$$

$$\partial\mathbf{E}/\partial x_3 = \partial\mathbf{E}/\omega = 2is\mathbf{M} + i\mathbf{G} - q(\partial p/\partial\omega)\partial\mathbf{A}/\partial p + i\partial\mathbf{C}/\partial s$$

where

$$\partial\mathbf{A}/\partial V = -(s/V^2)\partial\mathbf{A}/\partial p + (\partial M/\partial V)\partial\mathbf{A}/\partial M.$$

Some variations of this general form are:

a) Fixing $\rho$, $a$ and $M$ results in a neutral-stability form of the equations in which the Mach number is not consistent with the velocity and speed of sound. In many cases this is an acceptable approximation; if it is not, a form like example 2 might be more appropriate. For this constant-Mach constant-density form of the equations $\partial q/\partial V = -\rho V$ and $\partial\mathbf{A}/\partial V = (s/V^2)\partial\mathbf{A}/\partial p$

b) Often the aerodynamic matrix is only a function of (real) reduced frequency $k$, in which case

$$\partial \mathbf{A}/\partial V = -(\omega/V^2)\partial \mathbf{A}/\partial k$$

c) If the unsteady aerodynamic matrix is a function of (real) reduced frequency and Mach number it is possible to formulate a neutral-stability problem where Mach number, speed of sound and velocity are consistent. It is necessary to specify a relationship between velocity, speed of sound, and density. One possibility is a constant-altitude form where density and speed of sound are constant and

$$\partial \mathbf{A}/\partial V = (\omega/V^2)\partial \mathbf{A}/\partial k + (1/a)\partial \mathbf{A}/\partial M$$

Wind tunnel conditions can be simulated by specifying the relationship between velocity, speed of sound, temperature, density, and static pressure, and specifying $\partial M/\partial V$ and $\partial q/\partial V$.

II) *Constant-Mach-Variable-Density.* If $z$ is altitude in the standard atmosphere and the aerodynamic matrix is valid for one particular Mach number $M = M_0$ the choice

$$\mathbf{x} = \lfloor \; z \quad \sigma \quad \omega \quad \mathbf{v}^T \; \rfloor^T \tag{10}$$

results in a constant-Mach-variable-density formulation in which reduced frequency, velocity, and Mach number are consistent. The reduced frequency is computed for a particular $x$ by determining the standard atmosphere density and speed of sound $a$ at $z$ from standard atmosphere data[9] computing the true airspeed $V = M_0/a$, and applying the definition $k = \omega/V$ (or $p = s/V$).

III) *Parameter Variations.* The variation of critical flutter speed can be investigated using

$$\mathbf{x} = \lfloor \; V_f \quad p_j \quad \omega \quad \mathbf{v}^T \; \rfloor^T \tag{11}$$

where $p_j$ is a system parameter and the growth rate is implicitly zero. For example, Fig. 2 shows the variation of critical flutter speed with the first bending frequency of a nacelle.

IV) *Two-Parameter Variations.* If the flutter velocity is held constant a contour plot of this flutter speed can be obtained on a two-parameter plane. The vector of independent variables in this case is

$$\mathbf{x} = \lfloor \; p_1 \quad p_2 \quad \omega \quad \mathbf{v}^T \; \rfloor^T \tag{12}$$

where $p_1$ and $p_2$ are parameters, growth rate is implicitly zero, and $V_f$ is implicitly constant.

## 5.2 Starting Points

Parameter variation flutter problems have natural starting points - the neutral-stability solution. Neutral-stability problems themselves do not have readily available starting points, particularly when control-system, gyroscopic or viscous damping terms are included. One possibility is to use free-vibration modes and frequencies as starting points[6] Unfortunately there is no guarantee that the free-vibration solution will be close enough to an aeroelastic mode for the corrector to converge. Moreover, even if it does converge it may not be to the mode of interest, or worse, two starting points may converge to the same aeroelastic mode leaving a mode unaccounted for. A safer procedure is to use the PITCON package in a modified continuation process to obtain the starting points.

One of the classical uses for continuation methods is to provide a solution method which is (almost) guaranteed to converge even in the absence of a good initial guess[10]. What follows is a method for obtaining starting points for the neutral-stability example (I) given above; extension to other formulations is straightforward. The idea is to introduce an artificial parameter $t$ and define a set of equations with two properties: 1) when $t = 0$ we have a known solution, and 2) at $t = 1$ the desired starting point is obtained. To this end let

$$\bar{\mathbf{x}}(t) = \lfloor \ \bar{V}(t) \quad \sigma(t) \quad \omega(t) \quad \mathbf{v}^T(t) \ \rfloor^T \tag{13}$$

where $\bar{V}(t) = \omega(t)/k_{max}$ is the lowest velocity for this mode consistent with the largest reduced frequency $k_{max}$; $\bar{\mathbf{x}}$ differs from $\mathbf{x}$ in that the velocity is constrained according to the frequency and highest reduced frequency, and each element is implicitly a function of the parameter $t$. Let

$$\hat{\mathbf{x}}(t) = \mathbf{f}\left[\bar{\mathbf{x}}(t)\right] = \lfloor \ t \quad \sigma(t) \quad \omega(t) \quad \mathbf{v}^T(t) \ \rfloor^T \tag{14}$$

where $f$ simply expresses the fact that $\hat{\mathbf{x}}$ is obtained from $\bar{\mathbf{x}}$ by replacing the first element with $t$. The modified continuation equation is

$$\hat{\mathbf{y}}(\hat{\mathbf{x}} = (1-t)\mathbf{y}_0 + \mathbf{y}\left[\bar{\mathbf{x}}(t)\right] = 0 \tag{15}$$

where $\mathbf{y}_0 = \mathbf{y}\left[\hat{\mathbf{x}}\right]$. Equation (15) has the property that any choice of $\bar{\mathbf{x}}(0)$ in (14) results in $\hat{\mathbf{x}}\left[0, \bar{\mathbf{x}}\right]$ which is a solution to (15) at $t = 0$. Hence we are free to choose $\bar{\mathbf{x}}(0)$. Furthermore, continuation from $t = 0$ to $t = 1$ results in a solution $\mathbf{x}_0 = \bar{\mathbf{x}}(1)$ to (8) at the largest reduced frequency, suitable for a starting point for a neutral-stability continuation solution. In some sense $\mathbf{x}_0$ is the solution to (8) which is closest to $\hat{\mathbf{x}}(0)$; hence $\hat{\mathbf{x}}(0)$ should be chosen to have some physical significance. Here we choose $\hat{\mathbf{x}}(0)$ to be an aeroelastic mode from a $V-g$ formulation of the flutter equations. The $V-g$ form of the flutter equations is the eigenvalue problem[3]

$$\mathbf{BU} = \mathbf{K\Lambda U} \tag{16}$$

where

$$\mathbf{B} = \mathbf{M} + \alpha\mathbf{A}(k), \alpha = \rho/2k^2,$$

$\mathbf{U}$ is the $n$ by $n$ complex matrix of eigenvectors, and $\mathbf{\Lambda}$ is an $n$ by $n$ diagonal eigenvalue matrix with diagonal elements

$$\lambda_j = \omega_j^2/(1 + ig_j)$$

and $g_j$ is the added structural damping necessary for harmonic motion. Solving (16) at $k = k_max$ gives $V-g$ aeroelastic modes at the lowest velocity consistent with the aerodynamic matrix. Since for small values of $g$,

$$g \approx 2\sigma/\omega$$

we set

$$s_j = g_j\omega_j/2 + i\omega_j = \sigma_j + i\omega_j$$

and let $\mathbf{v}_j$ be the real representation of the $j^{th}$ eigenvector $\mathbf{u}_j$. Then defining

$$\bar{\mathbf{x}}(0) = \lfloor\ \omega_j/k_{max} \quad \sigma_j \quad \omega_j \quad \mathbf{v}^T\ \rfloor^T$$

and using

$$\hat{\mathbf{x}}(0) = \lfloor\ 0 \quad \sigma_j \quad \omega_j \quad \mathbf{v}^T\ \rfloor^T$$

to start the modified continuation process (15) results in a starting point for the neutral-stability continuation process which is closest to the $j^{th}$ aeroelastic mode from a $V-g$ solution. The modified continuation equation (15) is solved using PITCON from $t = 0$ to $t = 1$; in the PITCON terminology $t = 1$ is a *target* point. From $\hat{\mathbf{x}}(1)$ the starting point $\mathbf{x}_0 = \bar{\mathbf{x}}(1)$ is obtained by replacing the first element with $\bar{V}(1)$. This process is repeated for each aeroelastic mode of interest as identified by their $V-g$ frequency $\omega_f$

### 5.3 Scaling

PITCON uses the tangent vector to determine the optimum continuation parameter at each solution point. Because the tangent vector is effected by scaling of the independent variables, the PITCON package performs best when the variables are scaled so that their variations during the solution process are approximately equal. Here we scale the first three variables to have variations of approximately one; the generalized coordinates are not scaled. These scalings are necessarily based on very rough estimates of the range of the variables. It is usually not possible to predict, for example, the variation in growth rate or frequency. Nevertheless, even crude estimates seem to suffice.

## 6 Concluding Remarks

A procedure for solving various flutter equations that cannot be solved using the traditional $V-g$ method has been developed. The procedure uses a general-purpose library of Fortran subroutines developed by Rheinbold[1] for the solution of continuation problems. Application of this package to the solution of several example flutter problems has been shown to be straightforward. A new method for the computation of starting points, necessary to begin neutral-stability problems, has been developed. In practice the new starting procedure has proven to be safer and more reliable than traditional methods.

## References


[1] W.C. Rheinboldt. A locally parameterized continuation process. *ACM Trans. Math. Software*, 9(2):215–235, 1983.

[2] W.C. Rheinboldt. Numerical analysis of continuation methods for nonlinear structural problems. *Computers and Structures*, 13:103–113, 1981.

[3] H.J. Hassig. An approximate true damping solution of the flutter equation by determinant iteration. *AIAA Journal of Aircraft*, 8(11):885–889, Nov 1971.

[4] V.J.E. Stark. A flutter eigenvalue program based on the newton-raphson method. *AIAA Journal*, 22(7):993–995, July 1984.

[5] M.W. Hirsch and S. Smale. *Differential Equations, Dynamic Systems, and Linear Algebra*. Academic Press, New York, 1974.

[6] C.C. Mantegazza. Continuation and direct solution of the flutter equation. *Computers and Structures*, 8:185–192, 1978.

[7] W.C. Rheinboldt. Solution fields of nonlinear equations and continuation methods. *SIAM J. Numerical Analysis*, 17:221–237, 1980.

[8] C. Den Heijer and W.C. Rheinboldt. On steplength algorithms for a class of continuation methods. *SIAM J. Numerical Analysis*, 18:925–947, 1981.

[9] K.S.W. Champion, W.J. O'Sullivan, and S. Teweles. *U.S. Standard Atmosphere*. U.S. Govt. Printing Office,, Washington, D.C., 1962.

[10] H.J.(Ed.) Wacker. *Continuation Methods*. Academic Press, New York, 1978.