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Methods and advances in the study of aeroelasticity with uncertainties

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Abstract Uncertainties denote the operators which describe data error, numerical error and model error in the mathematical methods. The study of aeroelasticity with uncertainty embedded in the subsystems, such as the uncertainty in the modeling of structures and aerodynamics, has been a hot topic in the last decades. In this paper, advances of the analysis and design in aeroelasticity with uncertainty are summarized in detail. According to the non-probabilistic or probabilistic uncertainty, the developments of theories, methods and experiments with application to both robust and probabilistic aeroelasticity analysis are presented, respectively. In addition, the advances in aeroelastic design considering either probabilistic or non-probabilistic uncertainties are introduced along with aeroelastic analysis. This review focuses on the robust aeroelasticity study based on the structured singular value method, namely the μ method. It covers the numerical calculation algorithm of the structured singular value, uncertainty model construction, robust aeroelastic stability analysis algorithms, uncertainty level verification, and robust flutter boundary prediction in the flight test, etc. The key results and conclusions are explored. Finally, several promising problems on aeroelasticity with uncertainty are proposed for future investigation.

1. Fundamentals of aeroelasticity and uncertainty source

Aeroelasticity is the study of the stability and response of elastic structures undergoing aerodynamic forces. Typical aeroelasticity involves the coupling of elasticity, inertial force, and aerodynamics, which is indicated in Fig. 1.[1] This typical scheme only indicates the three forces' coupling in aeroelasticity. More general aeroelasticity, such as aeroservoelasticity or aerothermoelasticity, may include control systems or thermodynamics. Models of different fidelities can be employed to depict their individual effects on aeroelasticity. With the linear assumption, the equation of motion for typical aeroelasticity is written as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{M}_{\mathrm{c}}\ddot{\boldsymbol{\delta}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{F}_{\mathrm{q}} + \boldsymbol{F}_{\delta} + \boldsymbol{F}_{\mathrm{g}} \tag{1}$$

where $\boldsymbol{q}$ represents the coordinate vector, $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the total matrices for mass, damping and stiffness, respectively, $\boldsymbol{M}_{\mathrm{c}}$ is the coupling mass matrix of the control surface, $\boldsymbol{F}_{\mathrm{q}}$ and $\boldsymbol{F}_{\delta}$ are the aerodynamic forces due to the dynamic motion and the deflections of the control surface, respectively, $\boldsymbol{F}_{\mathrm{g}}$ is the external force, especially the aerodynamic force by the gust. The above aeroelastic equation can be represented in the modal basis

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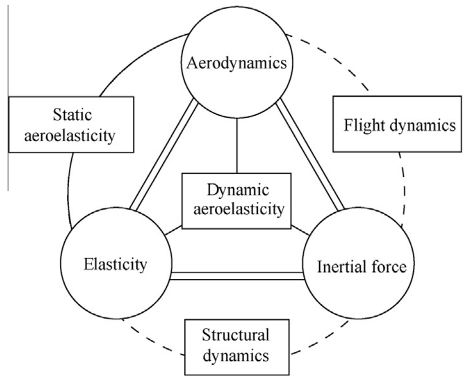

Fig. 1 Schematic of the study of aeroelasticity.

coordinates or in the physical coordinates represented by the degrees-of-freedom. The above dynamic equation contains the aerodynamic force, the elastic force and the inertial force. It is the fundamental equation for aeroelastic analysis and design. More forces' coupling, such as the control system or the thermo-stress, can be added or modified according to Eq. (1).

With the rapid development of the computers and mathematics, the aeroelastic models for modern aircraft and missiles are more accurate than before. Traditionally, aeroelasticity was modeled by definite models. However, due to material dispersivity, manufacturing tolerance, non-uniform airflow and random gust environment, we have to include uncertainty for input parameters, such as the aerodynamic forces, into the aeroelasticity model. Hence aeroelasticity models with uncertainty included have been developed. Uncertainties denote operators which describe data error, model error and numerical error in the mathematical methods. These uncertainties will result in differences between the numerical analysis and the real aeroelastic properties of aircraft. According to Fig. 1, an aeroelastic system can be constructed with individual subsystems. Though the complete aeroelastic system is complicated, the uncertainty source in the subsystem level or in the physical level is clear to be determined. It will be a good start to model the uncertainty in these levels.

The usual uncertainty source in the physical level is listed in Fig. 2. From the perspective of physical level, the uncertainties of data error, model error and numerical error can be determined and modeled. For example, in the subsystem of structural dynamics, the uncertainties may come from the material discrepancy, or the free-play nonlinearity, or the variation of calculated natural frequencies. Usually, the parametric variations of structural dynamics can be estimated by experiment data or designers' experience. For example, the variation of the general stiffness is less than 10%, and the module of composite materials is less than 8%, according to the material's standard. In aeroelasticity scope, how to quantify their influence on aeroelasticity is a challenge for engineers.

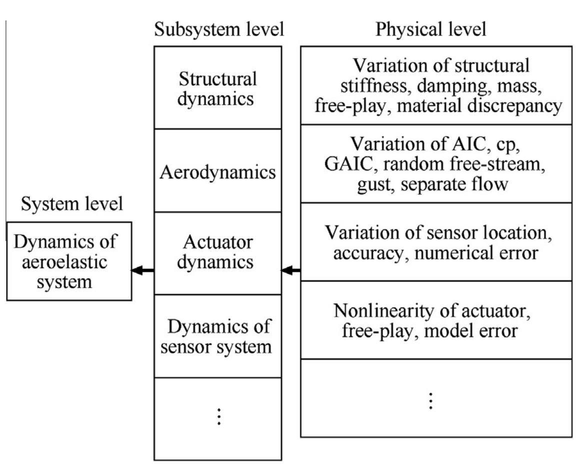

Fig. 2 Uncertainty sources for physical parameters of aeroelastic system.

The influence of the above uncertainty sources on aeroelasticity is different. In the process of aeroelastic analysis and design, we are frequently confronted with the following problems related with uncertainty:

(1) Composite materials are being increasingly used in aircraft structures. However, the variations in elastic modulus of composite materials are more obvious than those in metallic materials, which is one important uncertainty source indicated in Fig. 1. It will certainly affect an aircraft's structural dynamics and aeroelastic behavior. When the natural frequencies vary in a certain range, how do they affect the aeroelastic stability boundary? Will it be sensitive to these variations? How to design a robust structure that is not sensitive to the variations of structural parameters?
(2) The inertia of an aircraft may be changed due to fuel burn, cargo movement, weapon emitting, etc. The number of configurations in an aircraft series is large. It is a burden to analyze the aeroelastic behavior for all configurations. How to investigate the influence of successive inertia's variations on aeroelastic performance efficiently and precisely?
(3) When an aircraft structure is damaged, would the aircraft be still clear of aeroelastic instability? How to design an aircraft structure with a minimum weight, which is robustly safe with respect to uncertainties due to structural damage?
(4) When an aircraft is flying in a thermodynamic environment, the elastic modulus and other material properties will change. In addition, extra stress is produced in the structures. Hence, what is the influence of random thermodynamic flow on the aeroelastic stability and response?
(5) The modern aircraft is becoming more and more flexible. Nonlinearity is inevitable in many aircraft. It may lie in the structural dynamics or the unsteady aerodynamics, such as free-play, nonlinear damping or shocks. Must we employ the nonlinear structural and aerodynamic models to get an adequate aeroelastic behavior? Can we use an uncertainty description to take the place of some benign nonlinearity? Can this technique improve the efficiency of aeroelastic analysis and design? If we construct the nonlinear model directly, does uncertainty source still exist in the nonlinear part? How to analyze the influence of nonlinear uncertainty on aeroelastic stability and performance?
(6) Parametric uncertainties exist in the definite mathematical methods. On the other hand, the identification approach from the flight test may fail if the data is of poor quality. The results from the analytical methods may be very different from the flight test data. How to utilize both the theoretical model and the experimental

data for more precise results? When the aeroelastic phenomena observed from the wind tunnel test and the flight test are different from the predictions of an analytical model, how to find out the causes leading to the difference from the aspect of uncertainty quantification?

All of the above questions can be answered in the category of aeroelasticity with uncertainty. Some scholars name it "uncertainty quantification in aeroelasticity". It means quantifying the uncertainty of aeroelastic behavior due to the input parametric uncertainty in the mathematical methods. This research field is to investigate the aeroelastic stability, response and design considering the influence of all kinds of uncertainties. The uncertainty sources in the analytical methods can be reduced but not eliminated. Therefore, In order to guarantee aircraft safety and performance, aeroelastic uncertainty quantification should be investigated during the airframe certification. It has received more and more attention by academicians and engineers.

Engineers always take into account the effect of parametric uncertainty by enumeration, calculating aeroelastic behavior of all the possible uncertainty combinations. It is indeed an efficient approach. However, when the number of parametric uncertainties is large, there are too many combinations of uncertainty set to quantify all the possible uncertainties. Moreover, the enumeration method lacks sufficient mathematical foundation with discontinuous numerical simulation. Hence, it is necessary to develop a theoretical framework to quantify the influence of uncertainty on aeroelastic behavior. Uncertainties can be handled by several theories, depending on the basic assumptions for them. Assuming the parametric uncertainty to be a definite variable located in a certain range, the widely used robust aeroservoelastic analysis based on structured singular value (μ) theory was introduced by Lind in 1997. After decades of development, extensive activities by applying this theory to robust aeroelastic analysis, wind tunnel tests, and especially flight tests have been reported. In 2004, Pettit summarized the results and advances about uncertainty quantification in aeroelasticity, which included the probabilistic aeroelastic analysis, probabilistic aeroelastic design and flight test methods.[2] His main concern was the probabilistic uncertainty quantification method, originated from the reliability field and system engineering. After ten years' development, much work in the field of aeroelasticity with uncertainty has emerged, not only in the aeroelastic quantification with probabilistic uncertainty, but also the aeroelastic research for non-probabilistic uncertainty, especially the robust aeroelastic analysis and design. The research work of aeroelasticity with uncertainty is abundant. The authors have no intention to mention all the aeroelastic activities with uncertainty. Different from the point of Pettit's comprehensive review of probabilistic uncertainty quantification, the research advances and methods of μ-based robust aeroelasticity are emphasized in this review, with a brief mention of other methods which are merely applied.

From the definition of probabilistic aeroelasticity and robust aeroelasticity, their differences are distinct. Probabilistic aeroelastic analysis originates from structure reliability analysis. It tackles aleatory uncertainty in the physical level, such as variations of material properties, structural dimensions, boundary conditions and non-uniform free-stream flow. In this research scope, all the uncertainties are taken as probabilistic variables. However, the differences between aleatory and epistemic uncertainties are sidestepped in the non-probabilistic aeroelastic research. There is no need to know the probabilistic distribution of uncertain physical parameters. In addition, we are always focused on the extreme aeroelastic stability and response in all the aeroelastic uncertainty set, namely the "worst case" or "robust".

The outlines of the reviews are as follows: first, the fundamentals of aeroelasticity are introduced. Then, we review the modeling, analysis, experiment advances of robust aeroelasticity based mainly on the μ control theory. By comparison, the probabilistic aeroelastic analysis and design approaches were introduced in the last decades, mainly based on the widely used polynomial chaos expansion method. Finally, according to the newest advances of both robust and probabilistic aeroelasticity, several promising research topics on aeroelasticity with uncertainty are put forward.

2. Robust aeroelastic analysis and design

As we know, aeroelastic instability involves an aircraft's safety, which must be cleared during the design and flight test. When a parametric uncertainty is introduced to traditional definite aeroelastic analysis, it is of course always taken as non-probabilistic, and we are concerned with the "worst case" aeroelastic stability boundary and the worst performance with much care. The "worst case" indicates that in addition to the case that the aircraft is aeroelastically safe with the nominal model, it should also be safe with all the combinations in the uncertainty set. Otherwise, if only one combination of the uncertainties leads to the occurrence of flutter, the flutter boundary is regarded as "not robust". The worst case analysis infers a basic idea in the aircraft design process that we can never let the instability happen, though with uncertainty.

Regarding robust aeroelasticity, the interval theory, the perturbation method and the structured singular value (μ) methods[3,4] are the most widely used approaches to tackle with non-probabilistic uncertainty. Among them, based on sensitivity calculation, the perturbation theory can handle either the non-probabilistic uncertainty or the probabilistic one. The interval theory developed by Wang et al. is mostly applied to static and dynamic aeroelastic analysis, to obtain the intervals of aeroelastic behavior.[5,6] It is also based on the sensitivity calculation. μ theory is originated from the modern control theory. Hence it can be applied to not only dynamic aeroservoelastic analysis, but also the controller design and flutter boundary prediction. Notably, Wu introduced this theory to robust static divergence analysis,[7] which is a significant extension of this theory. In view of its good applicability to aeroelastic problems, the μ method has become the primary selection to consider non-probabilistic uncertainty in aeroelasticity. The related work in this area is also abundant, which includes the calculation of μ, uncertainty modeling, uncertainty level verification, algorithms for robust flutter analysis, robust flutter prediction, etc. These advances in the last decades will be summarized in detail in this section.

2.1. μ calculation

For a feedback control system, given a matrix and a block structure for uncertainty, the variable $1/\mu$ defines a distance to singularity of the closed system in an allowable uncertainty space. For the standard μ problem, the uncertainty space is measured using the infinitive norm, which is a hypercube space.

The definition of μ comes from the small gain theory in the control community. It is a tradeoff between fidelity and simplicity. It can improve the analysis accuracy by introducing the structure of uncertainty block. However, the computation of its accurate value is a bit complicated. Young proved that μ with complex-valued parametric uncertainty was continuous with the matrixes' variable. Consequently, the accurate value of μ can be calculated numerically.[8] However, the computation of real-valued μ (standard μ with real-valued elements of uncertainties) is non-convex and it is proved to be a NP-hard problem (non-deterministic polynomial-time hard).[9,10] In addition, the computational time increases exponentially. Several algorithms to compute the approximation of μ with complex or mixed valued uncertainty matrix were developed by Young.[8] Based on these algorithms, the MATLAB μ toolbox[11]was established by Balas et al., which was effective and efficient to calculate the approximation of μ. However, the mass variation and most of other aeroelastic parametric uncertainties are real valued variables. When this μ toolbox is applied to robust aeroelastic analysis with uncertainty, the gap between the upper and lower bounds of μ is too large in tolerance. For this sake, Dai presented an algorithm to calculate the exact value of μ of a non-repeated diagonal real-valued uncertainty matrix. In this algorithm, since the variables in the uncertainty matrix are totally real valued data, some fundamental algebraic transformations were applied to make use of this real nature. Finally, the singularity constraint in the definition of μ is transformed to an explicit expression of uncertain variables. Fig. 3 is the accurate value of μ with three real-valued parametric uncertainties by the algorithm in Ref.[12], compared with the results in MATLAB μ toolbox.

From the comparison of Fig. 3, some particular calculation problems for real μ exist as follows.

(1) The gap between the upper and lower bounds of μ is sometimes too large to estimate its accurate value. Moreover, the lower bounds over the frequency range are always equal to zeros. Replacing the exact μ by its upper bound, the robust margin of flutter analysis may be overly conservative.

(2) The consequent parametric uncertainty of this value of μ cannot be obtained from the upper bound calculation. In this case, we can only calculate the uncertainty value from the lower bound algorithm. Hence, the worst-case perturbation from the lower bound is not reliable when the upper bound is not close to the lower one. Moreover, the un-symmetric uncertainty range cannot be modeled in the standard μ framework with the toolbox.

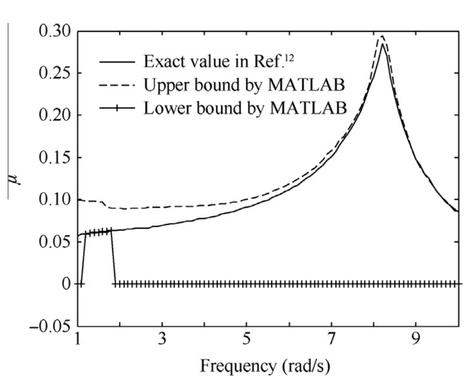

Fig. 3 Results of real μ calculated by MATLAB μ toolbox and Ref.[12].

Therefore, an appropriate complex-valued uncertainty has to be added when applying the MATLAB μ toolbox, or alternatively we use the upper bound to determine the robust stability. It should be noted that these two handling techniques will introduce extra conservatism of robust aeroelastic analysis.

Another development of the μ method is the extension of its definition, that is, spherical μ, by changing the uncertainty space from the infinitive norm to the Euclid norm. In 1998, Khatri and Parrilo developed an algorithm to calculate the upper bound of spherical μ by linear matrix inequality (LMI).[13] In 2001, Ishimoto and Terui provided a reasonable physical explanation of spherical μ.[14] They noted that when the element of uncertainty matrix was subject to a Gaussian distribution, robust analysis by special μ was less conservative than the one by standard μ above. An assumption of spherical μ is that flutter with little probability will not occur in the real world. Hence, it ignores the uncertainty combinations with very little probability density. In this sense, the results will be less conservative by spherical μ.

Similar to standard μ, it is also very difficult to calculate the accurate value of spherical μ. Therefore, Ref.[15] developed an algorithm to calculate the accurate value of spherical μ and robust flutter velocity by a genetic algorithm. The comparison of robust flutter velocities by the standard and spherical μ is demonstrated in Table 1,[15] in the table, V_{rob} is the robust flutter velocity, f_{rob} is the robust flutter frequency.

From Table 1, the robust flutter velocity predicted by spherical μ is higher than the one by standard μ. And the former is closer to the nominal flutter velocity. In fact, the robustness of spherical μ is only a relative estimate, since the worst case in the vicinity of the uncertainty range is out of our consideration. The spherical μ assumes that the worst case in the uncertainty corner with so small probability will never occur in reality. The implication by spherical μ is validated by Monte Carlo simulation (MCS). At the velocity of 86.4 m/s, there is no aeroelastic instability by the MCS, when the uncertain parametric uncertainty is subjected to a Gaussian distribution.

Lind first introduced the μ theory to aeroservoelastic stability analysis.[16,17] In addition, this method was successfully applied to the robust stability analysis of the F/A18 aircraft.[16] The experimental and analysis results indicated that the μ method can improve the accuracy of flutter boundary prediction. In recent years, abundant advances have emerged in China in the robust aeroelastic analysis with application to engineering,[4,7] such as the analysis of robust aeroservoelastic stability margin.[18]

Table 1 Robust flutter analysis by the standard μ and spherical μ.[15]

μ	V_{rob} (m/s)	f_{rob} (Hz)
Standard	82.6	12.5
Spherical	86.4	12.3

The related research about aeroelastic uncertainty quantification with μ method contains the following aspects: uncertainty modeling, algorithms for robust aeroelastic stability analysis, uncertainty level verification and flutter boundary prediction. These research contents are related with each other in sequence. That is: the aeroelastic model with uncertainty under the linear fractional transformation (LFT) framework should first be constructed. Then the uncertainty bound is estimated according to the experimental data. After these two steps, the robust stability analysis algorithm is developed to get the "worst case" flutter boundary. The final goal of all these is to predict a safe flutter boundary either in the wind tunnel test or in the flight test.

2.2. Uncertainty modeling

Most of the uncertainty sources in the aeroelastic system are indicated in Fig. 2. Because the robust aeroelastic analysis and design by the μ method are conducted in the LFT framework, the parametric uncertainty incorporating other aeroelastic parts should be transformed into this framework. Depending on the specific form of the nominal aeroelastic model, the uncertainty modeling can be divided into three levels: uncertainty model on state-space systematic level, on the subsystem level and on the physical parameter level.

When we are modeling the uncertainties in a systematic level, the parametric uncertainties of state-space parameters are considered by an unstructured multiplicative or additive uncertainty operator. When we are modeling the uncertainties in a subsystem level, the uncertainties embedded in the aerodynamic subsystems or the structural dynamic subsystems are considered, respectively. Then, they compose the whole aeroelastic system.

When we lack a priori information about uncertainty, the uncertainty modeling level of aeroservoelastic subsystem is applied. This expression is simple and easy to conduct. However, it lacks the explicit physical insight and the uncertainty bound cannot be estimated directly.

In the modeling process of physical parameter uncertainty, the aerodynamic calculation error, sensor uncertainties and dynamic parameter's variations due to manufacturing error, material dispersity, and environment change can be considered in depth. It is a straightforward choice for uncertainty sources. In this level, all the physical uncertainty sources are assumed as parametric uncertainties in a specific mathematical model.

It is physically clear to construct the uncertainty model under the level of physical parameters. Summarized by Wu's research group, the framework of uncertainty sources and uncertainty modeling under the physical parameter level is shown in Fig. 2. The general aeroelastic system can be divided into subsystems of structural dynamics, aerodynamics, actuators and sensors, etc. Then, the physical parameter uncertainty modeling can be conducted for each subsystem.

The developments of uncertainty modeling in the physical parameter level for each subsystem are illustrated in detail in the following.

2.2.1. Uncertainty model with unmodeled dynamics

Since the degrees-of-freedom of an aircraft are very large, in order to reduce computational time, flutter analysis is always conducted in the modal coordinate system, in which only several lowest degrees-of-freedom are used. Hence, this will introduce some modeling errors in the high frequency range, which will affect the accuracy of aeroelastic response and performance. In this case, an unstructured multiplicative or additive uncertainty block is usually employed. By this idea, Kapel constructed a reduced-order model of a complex aeroservoelastic system with unstructured uncertainty in 2002.[19] In his work, the influence of unmodeled uncertainty on robust stability was analyzed in detail. Results indicated that a low-order model together with unmodeled uncertainties can take the place of a full-order aeroservoelastic model for low-order controller design. By introducing the additive uncertainty, the order of the controller can be greatly reduced.

2.2.2. Uncertainty modeling for elastic structure

In the uncertainty modeling process for an elastic structure, the general uncertainties are structural mass, structural stiffness, total mass, center of gravity, inertia, natural frequencies and damping, etc. In order to utilize μ theory, the uncertainty sources have to be modeled in a LFT form. With the development of the finite element method, the uncertainty modeling for structures based on modal base assumption is attractive and convenient. In addition, most uncertainty bounds can be estimated through the ground vibration test.

Lind illustrated the uncertainty modeling for frequency and damping systematically. Moreover, he estimated their uncertainty bounds, using the experimental data of ground vibration test. From this aspect, the uncertainty model was validated by experiments. Therefore, the corresponding robust flutter velocity was reasonable.[20] From his pioneering work, a key feature was the ability to combine the theoretical uncertainty model with the experimental data. Moreover, the theoretical model can be validated through the experimental data. This uncertainty modeling gave us a feasible way to tackle with uncertainty for engineering application.

Tip-mass is commonly seen as a military aircraft. In addition, the tip-mass will be changed through the flight history. Therefore, attention was paid to its uncertainty model by many researchers. Moulin et al.[21] constructed the weight uncertainty of the tip-mass together with its location variation in a state-space form. It is advantageous that the uncertainty model was constructed in the time domain. Hence, it was convenient to apply the modern control theory for controller design and aeroservoelastic analysis.

In China, Wu and Yang first applied the μ theory to robust aeroservoelastic research.[4] Many kinds of structural parametric uncertainties were modeled by him, such as the uncertainty model for natural frequency, damping, modal shapes, the zeros and poles of state-space aeroservoelastic equation. Notably, the uncertainty models for fuel burn and structural damages are represented as the mass and stiffness variations in the finite element model.[22] This modeling can somewhat answer the second and third questions in the first section. Similarly, Heinze also constructed an uncertainty model for fuel burn of a rectangular tank.[23] It was more detailed than the model constructed by Wu. Other structural uncertainties were also studied in the last decades. For example, Danowsky developed the uncertainty models for structural elastic modulus and density.[24]

In the above structural uncertainty models, the modal basis is assumed to be fixed as constant, regardless of the parametric variable changes in the uncertainty space. In fact, because of the orthogonality property of the modal shapes, the modal basis will certainly alter with structural parameters. Consequently, the general aerodynamics together with the robust flutter velocity will also be affected. The influence of modal shapes on robust flutter boundary was studied by Danowsky et al., based on the method of first-order perturbation for structural frequency.[25] Several approaches to consider the uncertainty for modal shapes were studied by Heinze and Borglund, which cover the fixed basis method, Taylor expansion and iteration algorithm for modal shapes.[26] His results indicated that the fixed basis method is applicable when the variation of mass was small. However, this conclusion was not correct when the mass variation was significant. The iteration algorithm for modal shape was also studied by Dai et al.[27] for another numerical example. Results indicated that the modal shape uncertainties can be ignored by increasing the number of fixed modal shapes.

Generally speaking, the uncertainty modeling for elastic structural parameters is well developed, compared with other uncertainty modeling, such as aerodynamics or nonlinearity. These uncertainty models are summarized in the following sections.

2.2.3. Uncertainty modeling for aerodynamics

The unsteady modeling of aerodynamics is very important to aeroelastic behavior. Moreover, its calculation is very complicated. Therefore, the uncertainty modeling for unsteady aerodynamics is more difficult than the uncertainty modeling for structural parameters.

Generally, the aerodynamics in the frequency domain is employed for open-loop aeroelastic analysis. Hence, the uncertainty model for unsteady aerodynamics is always constructed in the frequency domain. However, for the aeroservoelastic analysis, we always employ a state-space model in the time domain. In this case, the aerodynamics should be transformed from the frequency domain to the time domain by the algorithm of rational function approximation. Some studies are focused on modeling uncertainties of lag roots resulting from approximation errors. However, this is not the main source of uncertainty. According to our experience, the numerical error in calculating the aerodynamic influence coefficients may be the leading uncertainty source. Therefore, in this summary, without special explanation, the aerodynamic uncertainty model means the numerical errors in the frequency domain.

In aeroelasticity, the generally used aerodynamics is as follows: Theodorson theory and strip theory for two-dimensional wing-panel configurations, lift-line theory for large aspect-ratio wing-panel configurations, doublet lattice method (DLM) based on the potential theory and the computational aerodynamics based on the Navier–Stokes theory. All the aerodynamic uncertainty models should be constructed based on a specific nominal model by the above aerodynamic calculation methods.

Lind constructed a simple uncertainty model for the derivative of aerodynamic forces in the time domain.[28] An uncertainty model based on the lift-line theory was developed by Borglund in 2002.[29] It was applicable for a large aspect ratio wing or civil aircraft. A good conclusion was made by Borglund that we only need to construct the uncertainty model of aerodynamics associated with flutter. This will greatly reduce the uncertainty number and computational time. Based on the Theodorson theory, the parametric uncertainties of Theodorson function, air density and flight velocity were constructed by Chun et al.[30–32] This method was limited to the application of the configurations of two degrees-of-freedom, not applicable for most complicated aircraft configurations.

The widely used aerodynamic model is based on the DLM method. Hence, it is a good demonstration for this aerodynamic uncertainty modeling. Moulin first constructed the uncertainty model for aerodynamic influence coefficients (AIC).[33] However, AIC has no explicit physical meaning and its uncertainty bound is difficult to estimate. There is still a long way to go to apply this uncertainty model to tackle engineering problems.

Based on the nominal DLM, an alternative and attractive method is to model the uncertainty of unsteady aerodynamic pressure. It was first developed by Borglund and Ringertz[34,35] notably. They applied the frequency–response-function validation method[36] to estimate its uncertainty bound. It was an improvement of the AIC uncertainty model. First, it provides us with a physical insight of aerodynamic calculation errors. In addition, the uncertainty model can be validated by experimental data. Martin also constructed an uncertainty model for aerodynamic pressure[37] to conduct the robust aeroelastic stability analysis.

Based on the DLM aerodynamic model, Dai et al. constructed the aerodynamic uncertainty model in a LFT form in detail. From the derivation process of general AIC, three uncertainty models in different levels were given systematically.[38] They were the uncertainty model for AIC, uncertainty model for pressure and uncertainty model for general AIC. In engineering, the uncertainty formula for aerodynamic pressure coefficients is more convenient, which is written as:

$$\boldsymbol{C}_{\mathrm{p}} = \boldsymbol{C}_{\mathrm{p0}} + \boldsymbol{W}_{\mathrm{cpl}} \times \boldsymbol{\Delta}_{\mathrm{cp}} \times \boldsymbol{W}_{\mathrm{cpr}} \tag{2}$$

In this model, the uncertainty bound can be estimated either by the experimental aerodynamic data or the frequency response data. From the example of AGARD445.6, the authors found that not only the uncertainty bound of pressure but also the uncertainty bound for general AIC can be estimated. For more details of the uncertainty modeling, one can see Ref.[38].

2.2.4. Uncertainty modeling for nonlinearity

As we know, the basis of the μ method is LFT. It is applicable for linear systems but not for nonlinear ones. Under the LFT framework, it is indeed very difficult to model the nonlinearities associated with uncertainties. Some clever approaches were employed to sidestep the contradiction. Generally, the nonlinearity induced by the structures was studied with more attention, since it was less difficult than the aerodynamic nonlinearity.

Lind et al. pointed out that, the nonlinearity can be replaced approximately by a linear uncertainty operator when the nonlinearity effect is benign.[28] This was a typical approach to avoid nonlinear uncertainties. However, the result may be overly conservative by this representation. Usually, when the uncertainty bound for nonlinearity is more than 20%, the quantitative results may be conservative. By the identification of Volterra kernel, the uncertainty for nonlinearity was estimated. After the validated uncertainty model, robust aeroelastic stability for a nonlinear system can be predicted correctly.[39,40]

Another generally used approach to tackle both nonlinearity and uncertainty was to separate the linear and nonlinear parts by mathematical blocks, such as the nonlinear Hammerstein model or nonlinear Winner model.[41] Based on these block-oriented models, a general model including both nonlinearity and uncertainty can be constructed. In addition, by applying the classical describing function, Baldelli predicted the aeroelastic stability boundary[42,43] of an aircraft with complicated configuration. It was a tradeoff method to consider both the uncertainty and nonlinearity in a linear LFT framework. However, how to investigate the uncertainty sources for nonlinearity has not been solved yet. It is a great limitation due to the linearity nature of the LFT method.

In China, Gu and Yang constructed an uncertainty model for free-play nonlinearity.[44] The robust flutter stability of a nonlinear wing section with parameter uncertainty was analyzed by Yun and Han by the μ method and characteristic polynomials.[45]

The advances of robust flutter and limit cycle oscillation prediction were greatly promoted by the developments of nonlinear uncertainty identification and modeling. However, there is still a long way to go to model the nonlinear uncertainty and to investigate its influence on aeroelastic stability.

2.3. Algorithm for robust aeroelastic and aeroservoelastic stability analysis

There are three types of aeroelastic stability problems. They include static aeroelastic stability (divergence), dynamic aeroelastic stability (flutter) and aeroservoelastic stability. In fact, μ is a definition originated in the control field. In the traditional aeroelastic stability, we are usually focused on the critical stability index, such as flutter velocity in the flutter analysis and stability margin in the aeroservoelastic analysis. In order to combine μ with the traditional aeroelastic equation and to obtain a compatible stability index, a specific algorithm should be developed, with both the μ theory and traditional aeroelastic stability analysis method. Many algorithms were developed to solve the robust aeroelastic/aeroservoelastic stability problems, either in the time domain or in the frequency domain, depending on which form of the aeroelastic equation we use. All of these algorithms for aeroelastic stability problems are presented in the following.

2.3.1. Algorithm in the time domain

Lind and Brenner proposed a robust flutter analysis algorithm based on the state-space equation in the time domain.[16] Based on this algorithm, a match-point robust flutter boundary solution was made by them.[46] It makes the μ method to be applicable to aeroelastic solution. Kou and Qiu developed a flutter prediction approach according to this robust match-point algorithm.[47] The main problem for the algorithm in the time domain is that the rational function approximation for aerodynamics can be unavoidable, which will definitely bring extra uncertainties for aerodynamics.

2.3.2. μ-k algorithm

Borglund proposed a μ-k algorithm to conduct robust flutter analysis in the frequency domain.[34] In this algorithm, the Mach number Ma was fixed as a constant value, while the reduced frequency k and flight velocity were represented by the flight height h. Consequently, we were able to solve the robust flight height at a fixed Mach number by the μ theory. The advantage was that the match-point robust safe boundary could be represented by h and Ma. Considering the perturbation of height, Yun and Han also developed a match-point robust aeroelastic solution of height.[48,49]

2.3.3. V-μ algorithm

Wu and Yang proposed the V-μ robust flutter analysis algorithm in the frequency domain.[22] The details of this method were summarized in Fig. 4.[22] Notably, the influence of flight height was ignored for this method. It was not a match-point solution, which was applicable for aeroelastic tests in the wind tunnel. By this method, we do not need to worry about the "mode jump" problems for different modal shapes.

2.3.4. μ-p algorithm

Based on the traditional flutter analysis approach, noted as p-k method, Borglund proposed a μ-p method.[50] Similar to the p-k method, the stable range of eigenvalues was concerned in the μ-p algorithm. Consequently, for a fixed dynamic pressure, the maximum and minimum eigenvalues can be calculated. In the uncertain eigenvalue set, the minimum one, noted as the worst-case eigenvalue, was calculated with more interest. Its attractive advantage was that the result was compatible with traditional p-k method. Hence, the uncertainty model could be validated through experimental on-line frequencies and dampings. However, the drawback was that the on-line frequencies of different modes might "jump", similar to the p-k method.

Note that the μ-k algorithm was a special case of μ-p. For the μ-k algorithm, the eigenvalue was located in the imaginary axis, not in the whole complex plane as μ-p. These two algorithms were both compatible with the traditional flutter solution, and they were practical in uncertain aeroelastic engineering applications, either in the wind tunnel test or in the flight flutter test.

2.3.5. μ-ω algorithm

Based on the algorithm in the time domain, Gu developed the μ-ω algorithm considering the perturbation of dynamic pressure. In this algorithm, because the uncertainty of dynamic pressure was real valued, the μ of real uncertainty must be treated carefully.[51,52] This algorithm can not only be applied to robust flutter prediction, but also to nominal flutter prediction, when there is only uncertainty for dynamic pressure. For this case, recalling Section 2.1, the μ subject to pure real values should be calculated with much care, and not only by employing the MATLAB μ toolbox. The method was extended to calculate the aeroservoelastic critical stability point by introducing a complex uncertainty of dynamic pressure.[53]

2.3.6. Algorithm for robust aeroservoelastic stability margin

For aeroservoelastic stability problems, we are not only concerned with whether the system is stable, but also with

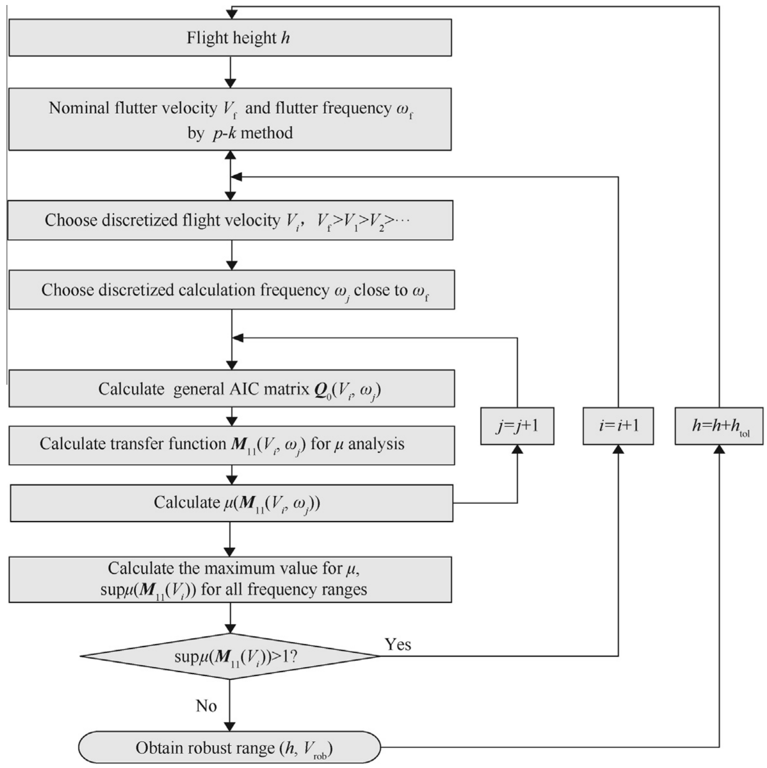

Fig. 4 Flow chart for V-μ algorithm.[22]

how far the aeroservoelastic system is from the critical stability point. Therefore, the traditional indices in the control theory, such as gain margin, phase margin, are of significance. The Bode diagram is appropriate to express the distance of the present pressure to instability point. Motivated by this idea, a framework based on μ analysis incorporated with gain margin was introduced to evaluate the robust stability margin of an SISO aeroservoelastic system by Dai et al.[18] The essence of the proposed method was to extend the nominal gain margin concept to robust stability margin by introducing an extra uncertainty. It was found sometimes the usually used gain margin of 6 dB may not be safe enough for the nominal aeroelastic system to resist the structural parameters' uncertainties, as indicated in Fig. 5.[18] This algorithm is applicable to most aeroservoelastic problems considering uncertainties.

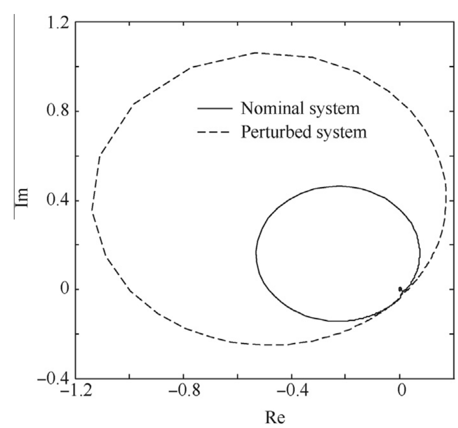

Fig. 5 Nyquist diagrams of the nominal system and perturbed system at the velocity of 35.5 m/s (with 0.9% relative error for frequency).

2.3.7. Algorithm for robust static aeroelastic stability analysis

Wu and Yang extended the robust flutter analysis algorithm to calculate the static aeroelastic stability.[7] He pointed out the static aeroelastic stability boundary could be calculated by a method similar to V-μ. The key distinction was that the critical stability frequency for divergence was nearly zero. Motivated by this, the same algorithm to V-μ can be utilized for robustness analysis of static aeroelastic stability and performance. In this case, the critical frequency is zero.

2.4. Uncertainty level verification

The uncertainty for robust analysis is a non-probabilistic one, not probabilistic. Therefore, in this case, it is only necessary to estimate the upper and lower bounds for each parametric uncertainty variable. Uncertainty level verification is an association of analytical methods and experimental data. In order to validate the uncertainty model indicated in Section 2.2, the uncertainty level verification is a necessary stage. The criterion is that the experimental parameters or outputs should be included through the verified uncertain model set.

The methods for uncertainty level verification can be classified into two types: the forward uncertainty bound estimation and the backward uncertainty bound identification, illustrated in Fig. 6. For the former one, we have to construct the mathematical model according to physical laws first. Then the

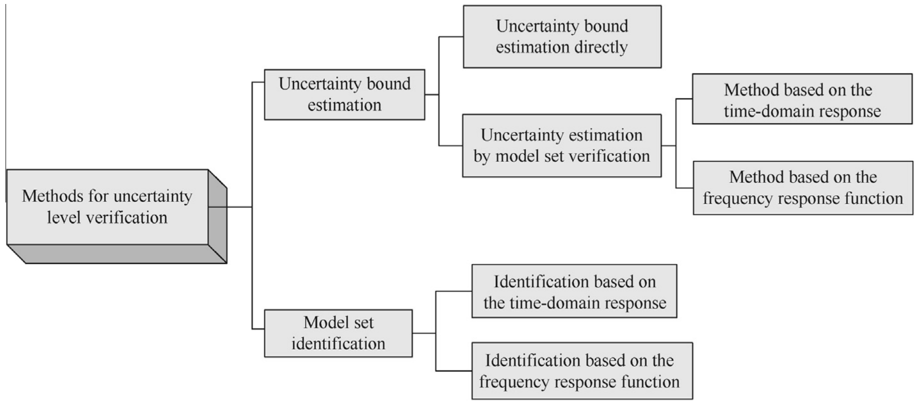

Fig. 6 Methods for uncertainty level verification.

uncertain aeroelastic model should be verified as whether invalid. When the uncertainty can be measured directly, such as the variations of natural frequency and mass distribution, we can calculate the uncertainty bounds directly. Otherwise, when the parametric uncertainty bound cannot be measured directly or explicitly, a model verification method should be employed to validate the uncertain model.[38] From the validated uncertain model, we can obtain the required parametric uncertainty bound. For example, we cannot directly measure the unsteady pressure for each dynamic modal motion, so its uncertainty bound is estimated by other indirect experimental data, by a model validation method.

Kumar and Balas proposed a μ validation test, based on the small gain theory, to estimate the amount of uncertainty underlying in the uncertainty model set.[54] Borglund applied it to estimate the uncertainty bound of aerodynamic pressure in the frequency domain.[29] The criterion was that the uncertain model could match the experimental frequency–response identically.

The above verification method was only applicable for a single-input–single-output system. For a multi-input–multi-output (MIMO) system, Newlin proposed an uncertainty level verification method based on the generalized μ theory.[55] The famous Carathe-dory–Fejer and Nevanlinna–Pick interpolation theory in the control field was brought to estimate the amount of aerodynamic uncertainty by Huang et al.[56] They were good choices for uncertain model validation of MIMO aeroelastic systems.

Wu et al. investigated the uncertainty modeling and validation for aerodynamics for an aircraft by a wind tunnel test,[57] shown in Fig. 7.

The uncertainty model was expressed as the dynamic pressure variation on the wing surface behind the nacelle. Either by the online frequency response data or the online time-history data, the amount of aerodynamic uncertainty can be estimated. After the model validation, we can get a reasonable aerodynamic uncertainty to match the experimental data. Consequently, the result of robust flutter analysis was reliable, as shown in Table 2.[57]

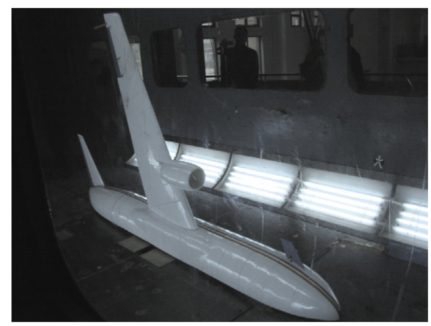

Fig. 7 Aircraft model mounted in a wind tunnel.[57]

From the experiment study, Wu found that for robust stability analysis, the online response in the time domain was better than the FRF to estimate the uncertainty bound of dynamic pressure, since the error of dynamic pressure due to aileron's motion did not need to be considered in this case.

It should be noted that, because the uncertainty for the distributed unsteady dynamic pressure was difficult to measure, we had to apply the model validation method to estimate its uncertainty bound indirectly. Otherwise, the aerodynamic uncertainty bound could also be estimated directly, as the structural parameter's uncertainty.

When a motion equation for an aeroelastic system was given already, the model verification method would be helpful to estimate the uncertainty bound. However, sometimes we lacked the physical laws of motion for a complicated system a priori or there were some unknown nonlinearities in the model. In this situation, no existing physical model could be employed to describe the real aircraft plant. Hence, we can

Table 2 Robust flutter analysis for an aircraft model.[57]

Parameter	Test result	*p-k* analysis	Robust analysis with uncertainty bound verified by FRF	Robust analysis with uncertainty verified by time-domain response
V_f (m/s)	$30.5_{0}^{0.5}$	31.5	27.5	28.5
ω_f (Hz)	6.3	6.4	6.3	6.2

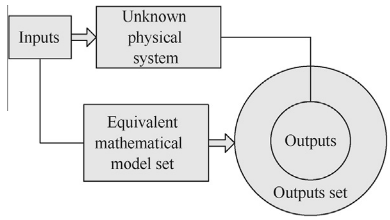

Fig. 8 Schematic diagram of model set identification.

try to introduce a mathematical block structure of the system. After the block structure is given, the remaining is to identify parameters of the mathematical model according to experimental data, as illustrated in Fig. 8. The criterion for model set identification is that the nominal model together with its uncertainty bound can match the experimental data. A significant advantage of model set identification is that we do not need to establish the motion equation according to the physical laws a priori. Hence, it is useful for nonlinear and complicated aeroelastic systems.[58]

By the data-based method, Dai et al. transformed the nonlinear aeroelastic system with uncertainty to an identical mathematical model set,[59] whose upper and lower bounds of parameters in the model set were identified both in the time domain and in the frequency domain. For the method of data-based model set identification, there is no need to establish its physical model ahead. It is useful in the wind tunnel test or flight test, without much knowledge of theoretical aeroelastic model parameters. Compared with nominal model identification, the model set with parameter's uncertainty is more robust to match time-varying experiment data.

Note that the model set identification is an alternative method when there is a lack of investigation of the physical aeroelastic plant, such as complicated nonlinearities. It is not helpful for investigating the mechanism of aeroelastic stability. Especially, it may fail when the experimental data is of poor quality. More studies can be concentrated on the uncertainty identification both with the theoretical physical model and with experimental data.

2.5. Flutter boundary prediction in the flight test

The final purpose of uncertainty modeling and model validation is to predict flutter boundary fast and precisely. Since flutter flight test is one of the most expensive parts in the process of aircraft certification, any method that can accelerate the process may be beneficial to the aeronautical industry, and will motivate the development of flutter boundary prediction methods.

Traditionally, the flutter criterion is based on modal damping extrapolation.[60,61] The key of this method is to identify the on-line frequency and damping accurately and quickly. Recently, the wavelet transformation[62] and Hilbert-Huang transformation[63] have been applied to identify damping and to predict the flutter speed. However, it is difficult to extrapolate an accurate flutter speed because the damping may decrease suddenly just before flutter occurrence. An alternative method, named Zimmermann and Weissenburger, introduces flutter margins to predict flutter speed. It is based on Routh's stability criteria and applied both to the modal frequency and damping. It can be applied to a two-degree-of-freedom flutter system, and it is later extended to a three-degree-of-freedom system. System identification, called auto regressive moving average (ARMA), is also a good data-based flutter prediction approach.[64,65] The above flutter prediction methods may be applicable at high speeds, but not accurate at low speeds. Moreover, when the structural modes are close to each other, all these methods may fail to predict the flutter boundary.

Therefore, based on the μ theory, Lind and Brenner have introduced the "flutterometer".[66] It is quite different from the above approaches. Notably, it is a model-based approach. The significant charm is that it combines a theoretical aeroelastic equation with test data, and it is able to predict a reliable flutter speed using low-speed test points. This will be safe for flight test. Moreover, the uncertainties can be considered in the theoretical model. In this case, a worst-case critical speed will be predicted and it infers a more safe critical flight boundary. This method has been applied to many practical aircraft.[61,67]

An alternative flutter prediction method by the μ theory is a data-based method,[68] similar to the model set identification method in Section 2.4. There is no need to construct a theoretical model for the data-based μ prediction method, which is applicable to the nonlinear and complex configurations.

How to apply both the theoretical model and flight test data to predict and to update the flutter boundary is a good attempt under the μ framework.

2.6. Robust aeroelastic design

There are two types of aeroelastic designs with non-probabilistic uncertainty. Some researchers are concentrated on designing the structural parameters with uncertainties to get a lighter and robust aircraft structure. The static aeroelastic performance and stress limits are always the constraints. Others are devoted to designing a robust control law with uncertainty in the aeroelastic plant. The dynamic performance and stability are usually concerned. The purpose of robust aeroelastic design is to obtain a lighter weight and to satisfy the performance and stability with whatever uncertainty variations in their ranges.

2.6.1. Robust structural design

The goal of robust structural design is to obtain an optimized structure, whose weight and performance are not sensitive to the uncertain design variables. Hence, though there is uncertainty underlying the structure or aerodynamic load, the static performance and stability can also be satisfied.

Odaka and Furuya considered the bifurcation of higher mode to design a plate wing, in order to improve the critical flutter speed.[69] The aeroelastic tailing with the uncertainty of composite materials was conducted by Kuttenkeuler and Ringertz.[70] With aeroelastic static and flutter constraints, Wan et al. developed a robust structural design methodology considering the uncertainty of thickness for a composite wing by the sensitivity analysis and genetic algorithm.[71] This method was applicable for designing a complex-configuration aircraft. Based on the same design methodology, both the uncertainties for the gear-ratio and skin thickness were considered to design a robust structure with four control surfaces.[72,73]

2.6.2. Robust controller design

The purpose of robust controller design is to design a control law that can resist parameter variations in the aeroelastic plant. In the process of robust controller design, the parameter uncertainties underlying the structures or actuators are considered. However, the parameters in the control law are assumed to be constants without uncertainty. In this case, the closed-loop aeroservoelastic system can still satisfy the performance and stability when the parameters in the plant vary.

The usual methods for aeroelastic control design are the PID controller and linear quadratic Gauss (LQG). The aeroservoelastic system is not robust to resist parametric uncertainty with these algorithms. H_∞ and μ synthesis are regarded as robust controller design methods. When parametric uncertainties lie in the mathematical plant, μ along with order-reducing algorithm works well for the aeroelastic design. A problem appears as to how to incorporate the robust aeroelastic design and the robust control design in a general frame. From this aspect, Kapel has designed an optimal structure with ASTROS code and a robust controller in a general framework.[19] Inspired by this idea, Dai et al. has developed a framework for both robust structure design and robust controller design, based on the genetic algorithm and μ theory.[74] Results indicate that by the synthesis, not only can the weight of the aircraft structures be reduced, but the closed-loop performance is also enhanced before and after the perturbation of the structure's parameters. For more results, please see Ref.[74].

3. Probabilistic aeroelastic analysis and design

3.1. Aeroelastic analysis with probabilistic uncertainties

In robust aeroelastic analysis, we only need to estimate the uncertainty bound, not the probability density. However, we cannot get the probability for aeroelastic behavior either. Sometimes, the robust flutter boundary may be overly conservative. In particular, robust aeroelastic analysis may not provide engineers with rational decision when there is ''bad experimental data''. This is why aeroelastic analysis with probabilistic uncertainty comes into our mind: considering the probabilistic uncertainty in the parameters or input, to estimate the probability of aeroelastic behavior or to design an upgraded aircraft. This idea comes from the field of reliability. It is regarded as a powerful tool for aircraft design.

Since flutter will result in highly dangerous catastrophe, little attention was paid to the risk-based aeroelasticity by engineers for a long time. Driven by the growing need of aircraft with high performance and light weight, the concepts of risk and reliability were accepted by aeroelastic engineers gradually.

Compared with robust flutter analysis, the modeling for probabilistic uncertainty is less difficult. So academicians are focused on the latter two aspects for aeroelastic analysis with probabilistic uncertainty.[75] One issue is to estimate the probabilistic distribution for parametric uncertainty according to experimental data.[76] The other is to investigate the influence of the parameter's uncertainty on aeroelastic behavior. In the former aspect, Mehrez et al. estimates the uncertainty probability by the Karhunen–Loeve expansion and polynomial chaos expansion (PCE).[77] It is originated in the structural dynamics and applicable for most aeroelastic problems with stochastic structural uncertainty.

Researchers are focused on the latter research aspect. That is, analyzing the influence of parameter uncertainty on aeroelastic stability. The methods are mostly based on MCS and PCE methods. MCS is regarded as a straightforward choice for uncertainty quantification of probabilistic aeroelasticity. The influence of uncertainty of structural parameters on flutter boundary was analyzed by Pitt et al.[78] Notably, he strongly recommended that uncertainty margin should be introduced to aircraft certification process. When we are applying computational fluid dynamics to calculate the unsteady aerodynamics, MCS may increase the computational time rapidly.[75] Hence, the potential of PCE method is tremendous in the field of computational aeroelasticity. For this method, the uncertainties of inputs and outputs of an aeroelastic system are projected into the stochastic space. Beran applied the PCE method to aeroelastic limit cycle oscillation.[79] The stochastic basis construction of the PCE method was an emphasis in his research. The advantage and drawback of MCS and PCE were compared by Badcock et al.[80] The influence of uncertainty of flow velocity on flutter probability was analyzed by Bruno et al. with MCS and generalized PCE method.[81] In that paper, the uncertainty model of Reynolds number was constructed. It was a novel consideration in probabilistic aeroelasticity.

Quite a number of relevant work of uncertainty quantification in structural dynamics and aeroelasticity emerged in China. Li and Yang estimated the probability density of kernel function and applied MCS to determine the probability of flutter occurrence.[82] It was an efficient method to consider a structural uncertainty. The line sampling method was employed by Song et al., it introduced ''flutter reliability'' in the field of aeroelasticity.[83] This method was in fact a modification of MCS, which could estimate the sensitivity of reliability efficiently. By the fast Fourier transformation method, the sensitivity of flutter to natural frequencies and to the center of mass was analyzed in detail.[84] It was much more efficient than the standard MCS method. The above studies were focused on structural uncertainty. Therefore, how to evaluate the effect of aerodynamic uncertainty on aeroelastic stability is an urged problem to be solved.

Sometimes, too much robustness may lead to overly conservativeness. On the contrary, a large amount of instability probability cannot be accepted by the aeronautical engineers. Motivated by this contradiction, another development is to balance the probability and robustness of an aeroelastic system. The tradeoff definition is ''probabilistic robustness''. It allows a very small probability of risk, while the system is still relatively robust to some extent. The pioneer theoretical work was developed by Zhu[85] and Chen et al.,[86] who defined the probabilistic robustness and gave a rough calculation algorithm. Dai et al. applied this definition and algorithm to calculate the aeroelastic instability risk.[87] It is found that if we accept a little amount of probability of instability risk, the critical flutter boundary can be increased significantly. Problems appear as to how to employ the conclusion for decision making in aircraft design. This idea requires a great amount of efforts to become applicable.

3.2. Aeroelastic design with probabilistic uncertainties

The properties of materials in real aircraft may not be the same as the ones in the design status. In this case, the aircraft's performance is different from analytical prediction. In the design perspective, if we use a definite performance or stability margin as our optimized objective or constraints, the resulting aircraft may not always meet our ideal design goal, due to the numerical error, data error or the model error. Usually at the beginning of aircraft structure design, most of the parameters are unknown, let alone their uncertainty level. Hence, structures are optimized in the aeroelastic design considering the variables' probabilistic uncertainty. Its difference from a robust design is that the design variables are not only located in a certain range, but also atisfy some probability distribution. Therefore, the aeroelastic constraints also have to be met probabilistically, not definitely. This concept is familiar to structural designers. Some definite methods in probabilistic aeroelastic design originated from the structural research area. However, for aeroelastic designers, to do so is more difficult than in pure structural design. They have to add the constraints of static aeroelastic stability, performance and dynamic aeroelastic stability to the existing structural constraints, which involves complicated calculation of unsteady aerodynamics.

Pettit and Grandhi developed a framework for aeroelastic design based on uncertainty,[88] in which the static deformation and gust response were constrained with a probabilistic distribution. Driven by the application of composite materials, the stochastic variations of materials were considered by Manan and Cooper. He designed a composite wing based on the PCE method.[89] In the abundant related references, the sampling, PCE and reliability methods were generally applied for probabilistic aeroelastic design.[90,91] In China, Zhang et al. developed an aeroelastic design method for a wing considering stochastic structural uncertainty.[92] In his work, the flutter constraint was introduced to consider aeroelastic constraints.

A still declined concept in aircraft industry is the "flutter probability". Engineers never accept flutter occurrence in any case for the sake of safety. This slows down the development of aeroelastic design with probabilistic uncertainty.

4. Conclusions and future work

This paper summarizes the work in aeroelasticity research with uncertainty in the last few decades, especially the advances in the fields of PCE-based probabilistic aeroelasticity and μ-based robust aeroelasticity, including both aeroelastic analysis and design. Firstly, several aeroelastic problems with uncertainties were put forth. Then the methods and concepts to solve these problems were reviewed and classified. According to the development of robust aeroelasticity in the last decades, the μ theory with its application to aeroelasticity was illustrated in detail. Finally, the activities undertaken in stochastic aeroelastic analysis and design were summarized.

From the advances abroad and in China, the μ theory applied to aeroelasticity is well developed. It provides a good tool to consider non-probabilistic parametric uncertainty in the aeroelastic analysis and design. In the research field of robust aeroelasticity based on μ theory, there are still some problems that need to be solved.

(1) Uncertainty modeling for nonlinearity. It is the most difficult part in aeroelastic problems with uncertainty. It has been pointed out that we can apply the block-oriented model incorporating experimental data to represent uncertainty with nonlinear parts. However, it still lacks physical insight in nonlinear mechanism. An effort can be made to model a nonlinear uncertainty from the real physical prospect. A related activity is to validate and verify the modeled nonlinear uncertainty. These would be a helpful supplement to the existing data-based identification method.
(2) Application to flutter boundary prediction with efficiency. As is known, flutter test is dangerous and time-consuming. Hence, high efficiency is a basic requirement for this test. In order to utilize the flight test data and the theoretical model sufficiently, the flutterometer-based methods should be developed from the data-based-model-based aspect to reduce the prediction time. It cannot be applied to real flight test as a mature technology unless the accuracy and efficiency requirements can be met.

Aeroelasticity research considering stochastic parametric uncertainty is developed from another prospect. It is mainly the focus of a theoretical community. Nevertheless, in the future, it is a trend to introduce "risk" and "reliability" to engineering industry in the aeroelasticity field, not only in the field of structural dynamics. Several promising research topics are proposed in the following:

(1) The stochastic uncertainty quantification of aeroelasticity is focused on the structural dynamics. Few published work focuses on the uncertainty modeling and validation of unsteady aerodynamics. How to apply an adequate aerodynamic model and how to estimate the effect of aerodynamic uncertainty on aeroelastic flutter and LCO in a probabilistic context are subjects that have to be undertaken. Much attention should be paid to both the theoretical methods and engineering applications with aerodynamic uncertainty.
(2) Most of the uncertainties nowadays in the aeroelastic area are assumed to be a specific distribution, either the Gaussian one or the Uniform one. However, the distribution of aerodynamic uncertainty is very complicated. It may not be the typical and known probability function. There is little research in aeroelasticity to consider uncertainty with a generalized distribution. It limits the development of probabilistic aeroelasticity.
(3) In order to validate the uncertainty model, an urgent need is to estimate the uncertainty distribution with experimental data, especially for the aerodynamic uncertainties. How to estimate aerodynamic uncertainty and how to validate the stochastic uncertainty model is a worthy topic in aeroelasticity. It needs both the developments of theoretical methods and experimental validation.
(4) When the aircraft is flying at a hypersonic velocity, the effect of thermodynamics on aerodynamics and structure is significant. Consequently, the variations of material properties, structural dynamics and the aerodynamics due to heat transfer cannot be neglected in a hypersonic environment.

Based on the aeroelastic analysis with probabilistic uncertainty, it is promising to develop aeroelastic design algorithms with reasonable probabilistic performance and stability constraints. This may play an important role in the future aircraft design with composite and other active materials. Though there may be some obstacles, this topic requires much attention. Another useful activity is needed to develop the existing analysis methods to aircraft industry.

Acknowledgements

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