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Short communication

Equivalent linearization method for the flutter system of an airfoil with multiple nonlinearities

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ABSTRACT

The equivalent linearization method (ELM) was extended to analyze the flutter system of an airfoil with multiple nonlinearities. By replacing the cubic plunging and pitching stiffnesses by equivalent quantities, linearized equations for the nonlinear system were deduced. According to the linearized equations, approximate solutions for limit cycle oscillations (LCOs) were obtained in good agreement with numerical results. The influences of the linear and cubic stiffnesses on LCOs were analyzed in detail. Reducing linear pitching stiffness leads to decreasing of the critical flutter speed. For linear plunging stiffness, the opposite is true. Also, it reveals that the bifurcation could be supercritical or subcritical, which is related to the ratio between the coefficient of cubic pitching stiffness and that of plunging one.

1. Introduction

The nonlinear flutter system of an airfoil is a typical of self-excited vibration, exhibiting very rich the nonlinear dynamic behaviors [1]. A lot of contributions have been made to this problem [2–6].

Predicting the solutions of nonlinear flutter systems via numerical as well as analytical techniques has been an active field for many years. The techniques such as Runge–Kutta and finite-difference methods were generally adopted to numerically solve nonlinear flutter systems [7–9]. Well-known, the numerical techniques can only detect stable solutions [10]. Analytical techniques such as asymptotic approaches were widely applied to analyze the LCO characteristics [11,12]. The asymptotic methods have developed for more than half of one century, and many of them were applied to nonlinear flutter systems, for example the describing functions method, harmonic balance method, perturbation-incremental method [13], and homotopy analysis method [14], to mention a few. Among these techniques, the equivalent linearization method (ELM) was widely employed in investigating nonlinear flutter systems. The equivalent linear stiffness for nonlinearities is usually obtained by the average method, KBM and/or other methods. By this means, an equivalent system can be deduced hence the linear flutter analysis methods can be applied [15].

The ELM has stimulated the curiosity and interests of many researchers because of its simplicity and effectiveness. In addition, the approximate solution of the equivalent linear system has clear physical significance, which makes it possible to discuss the nonlinear aeroelastic behaviors. Liu and Zhao [16] used the average method to get the equivalent stiffness for the cubic pitching nonlinearity, and then studied bifurcation points of airfoil flutter. Shahrzad and Mahzoon [17] investigated three models of airfoil flutter by ELM. Lim and Wu [18] extended the ELM to strongly nonlinear vibration by combining linearization and the harmonic balance method. Chen and Liu [19] utilized the Lim's method to improve the accuracy of the equivalent stiffness, and analyzed the flutter system of an airfoil with both quadratic and cubic pitching stiffnesses.

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Note that, the above mentioned studies concentrated on flutter problems of airfoils with a single nonlinearity in the pitching direction. Well-known, multiple nonlinearities would be usually encountered in flutter analysis and design. For instance, Cai et al. [20] applied the incremental harmonic balance method to the flutter of an airfoil with cubic stiffnesses in both the pitching and plunging directions.

To the best of our knowledge, the ELM has not been successfully implemented in flutter systems with multiple nonlinearities. The major motivation of this study is to extend the ELM for flutter system of an airfoil subject to cubic stiffnesses in both the pitching and plunging degree-of-freedoms (DOFs).

2. Flutter model

The physical model given in Fig. 1 is a two-dimensional airfoil, oscillating in pitch and plunge, which has been employed by many authors. The pitch angle about the elastic axis is denoted by α, positive with the nose up; the plunge deflection is denoted by h, positive in the downward direction. The elastic axis is located at a distance ab from the mid-chord, while the mass center is located at a distance $x_\alpha b$ from the elastic axis. Both distances are positive when measured towards the trailing edge of the airfoil. Let $r_\alpha b$ be the radius of gyration of the airfoil with respect to the elastic axis, ω_h and ω_α be the eigenfrequencies of the constrained one-degree-of-freedom system associated with the linear plunging and the pitching springs, respectively. Introduce a non-scale wind speed as $Q = (V/b\omega_\alpha)^2$.

The non-dimensional time $t = \omega_\alpha t_1$ (t_1 is the real time), and non-dimensional plunge displacement $h_1 = \frac{h}{b}$, which denoted as h again for facilitate expression. Then, the coupled equations modeling the motions of the airfoil subject to an incompressible flow can be written as [1]

$$\begin{cases} \mu\ddot{h} + \mu x_\alpha\ddot{\alpha} + \mu(\frac{\omega_h}{\omega_\alpha})^2 h = -2Q\alpha \\ \mu x_\alpha \ddot{h} + \mu r_\alpha^2\ddot{\alpha} + \mu r_\alpha^2 \alpha = (1+2a)Q\alpha \end{cases} \tag{1}$$

The parameters are given as [16]: $\mu = 20$, $a = -0.1$, $b = 1$ m, $x_\alpha = 0.25$, $r_\alpha^2 = 0.5$, $(\omega_h/\omega_\alpha)^2 = 0.2$, $\omega_\alpha = 62.8Hz$. Introducing viscous dampers and cubic stiffnesses in the plunging and pitching DOFs, respectively, then Eq. (1) can be rewritten as

$$\begin{cases} \ddot{h} + 0.25\ddot{\alpha} + 0.1\dot{h} + k_h h + e_h h^3 + 0.1Q\alpha = 0 \\ 0.25\ddot{h} + 0.5\ddot{\alpha} + 0.1\dot{\alpha} + k_\alpha\alpha + e_\alpha\alpha^3 - 0.04Q\alpha = 0 \end{cases} \tag{2}$$

3. Equivalent linearization method

In the ELM, firstly, the stiffnesses $k_h h + e_h h^3$ and $k_\alpha\alpha + e_\alpha\alpha^3$ are replaced by equivalent linear quantities $k_{heq}h$ and $k_{\alpha eq}\alpha$, respectively, such as [15,16]

$$k_{heq} = k_h + \frac{3}{4}e_h H^2, k_{\alpha eq} = k_\alpha + \frac{3}{4}e_\alpha A^2 \tag{3}$$

where H and A are amplitudes of the limit cycle oscillation (LCO). Substitution of Eq. (3) into Eq. (2) yields

$$\begin{cases} \ddot{h} + 0.25\ddot{\alpha} + 0.1\dot{h} + k_{heq}h + 0.1Q\alpha = 0 \\ 0.25\ddot{h} + 0.5\ddot{\alpha} + 0.1\dot{\alpha} + k_{\alpha eq}\alpha - 0.04Q\alpha = 0 \end{cases} \tag{4}$$

The LCO can be expressed by

$$h = He^{i\omega t}, \alpha = Ae^{i\omega t} \tag{5}$$

Substituting Eq. (5) into Eq. (4), one can obtain the equivalent linear system

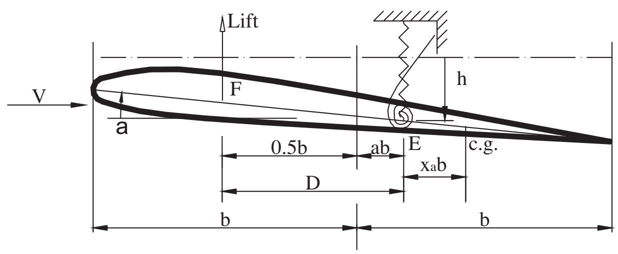

Fig. 1. Two-degree-of-freedom airfoil motion.

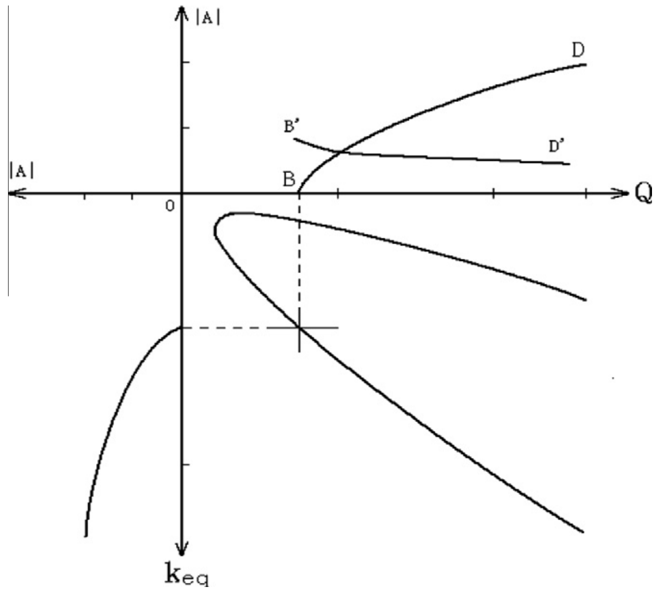

Fig. 2. The curve $Q \sim k_{\alpha eq}$ is plotted according to Eq. (11), and $A \sim k_{\alpha eq}$ according to Eq. (3). *BD* is drawn by combining $Q \sim k_{\alpha eq}$ and $A \sim k_{\alpha eq}$, and *B'D'* is based on Eq. (6).

$$\begin{cases} (-\omega^2 + 0.1i\omega + k_{heq})H + (-0.25\omega^2 + 0.1Q)A = 0 \\ (-0.25\omega^2)H + (-0.5\omega^2 + 0.1i\omega + k_{\alpha eq} - 0.04Q)A = 0 \end{cases} \tag{6}$$

That Eq. (6) has non-zero solution implies the flutter determinant must be 0, i.e.,

$$\begin{vmatrix} -\omega^2 + 0.1i\omega + k_{heq} & -0.25\omega^2 + 0.1Q \\ -0.25\omega^2 & -0.5\omega^2 + 0.1i\omega + k_{\alpha eq} - 0.04Q \end{vmatrix} = 0 \tag{7}$$

Separating the imaginary and real parts of Eq. (7), one obtains

$$0.4375\omega^4 + (-0.01 + 0.065Q - k_{\alpha eq} - 0.5k_{heq})\omega^2 + k_{heq}(k_{\alpha eq} - 0.04Q) = 0 \tag{8}$$

$$-0.15\omega^3 + (0.1k_{\alpha eq} - 0.004Q + 0.1k_{heq})\omega = 0 \tag{9}$$

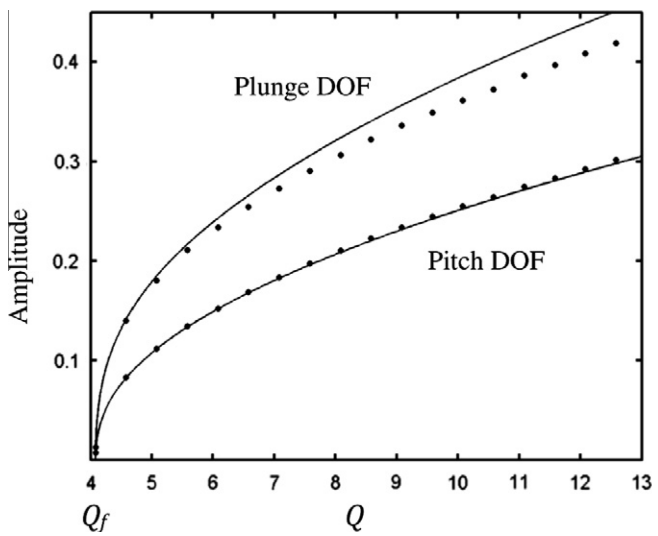

Fig. 3. Comparisons of LCO amplitudes for system (2) with k_h = 0.2, e_h = 5, k_α = 0.5, e_α = 20; solid lines denote the ELM solutions, and dots denote numerical ones.

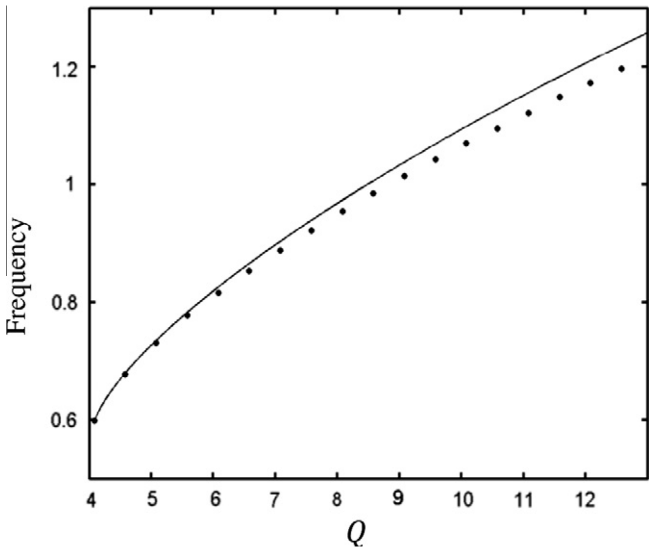

Fig. 4. Comparisons of LCO frequencies for system (2) with $k_h = 0.2$, $e_h = 5$, $k_\alpha = 0.5$, $e_\alpha = 20$; solid lines denote the ELM solutions, and dots denote numerical ones.

For $\omega \neq 0$, the solution of Eq. (9) is

$$\omega^2 = \frac{k_{\alpha eq} - 0.04Q + k_{heq}}{1.5} \tag{10}$$

Substituting Eq. (10) into Eq. (8), then Eq. (8) can be expressed as

$$0.32Q^2 - (12.25k_{\alpha eq} + 0.25k_{heq} + 0.06)Q + (106.25k_{\alpha eq}^2 - 87.5k_{\alpha eq}k_{heq} + 31.25k_{heq}^2 + 1.5k_{\alpha eq} + 1.5k_{heq}) = 0 \tag{11}$$

The bifurcation charts are drawn as follows:

(1) Assume H as a positive constant (e.g., change H from 0.01 to 0.5 step by step), then set Q and A as variables. Draw the curve of $Q \sim k_{\alpha eq}$, according to Eq. (11), shown as Fig. 2.
(2) Based on Eq. (3), draw the curve of $A \sim k_{\alpha eq}$.
(3) Combined the two curves to plot the curve for $Q \sim A$, shown as BD.

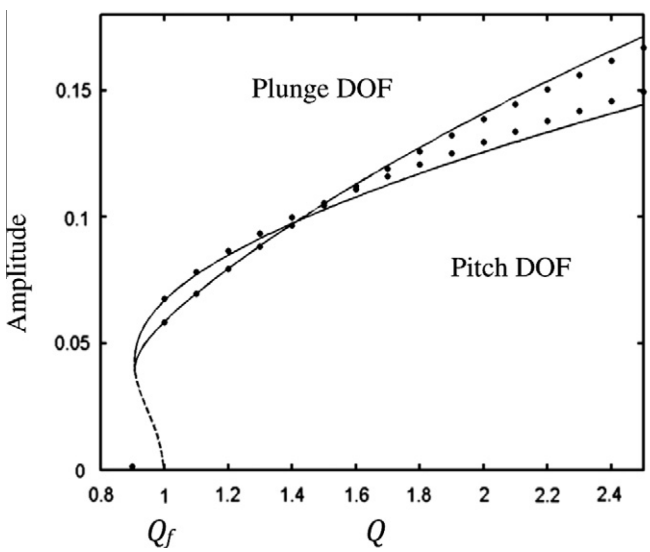

Fig. 5. Comparisons of LCO amplitudes for system (2) with $k_h = 0.2$, $e_h = 5$, $k_\alpha = 0.1$, $e_\alpha = 20$; solid lines denote the ELM solutions, and dots denote numerical ones.

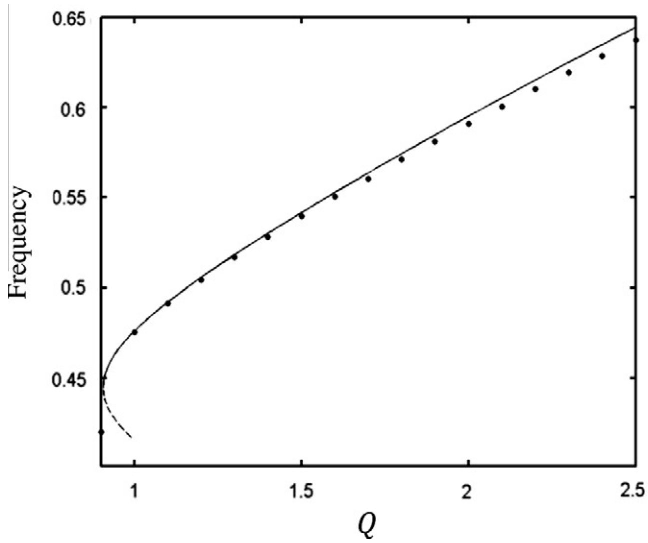

Fig. 6. Comparisons of LCO frequency for system (2) with $k_h = 0.2$, $e_h = 5$, $k_\alpha = 0.1$, $e_\alpha = 20$; solid lines denote the ELM solutions, and dots denote numerical ones.

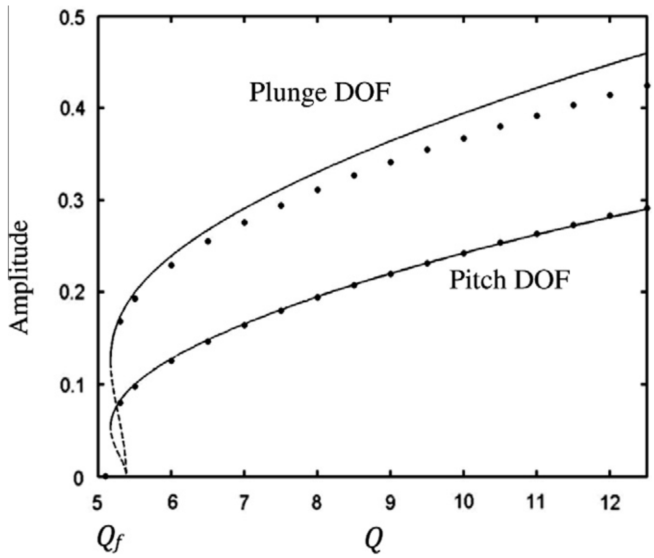

Fig. 7. Comparisons of LCO amplitudes for system (2) with $k_h = 0.08$, $e_h = 5$, $k_\alpha = 0.5$, $e_\alpha = 20$; solid lines denote the ELM solutions, and dots denote numerical ones.

(4) Substituting H and Eq. (10) into the first equation of (6), another curve for $Q \sim A$ is drawn as $B'D'$.

The intersection point of BD and $B'D'$ provides the real amplitude A corresponding to wind speed Q.

4. Numerical examples

Using the Runge–Kutta method, the numerical solutions can be obtained, which will be used to validate the ELM results. The comparisons of the LCO amplitudes in the pitch and plunge DOFs by the ELM with the numerical solutions are shown in Fig. 3. Also what plotted in Fig. 4 is the LCO frequency. As one can see, the frequency monotonously increases with Q increasing. From these figures, we can see the ELM solutions agree well with the numerical ones, especially at the low air speed. In addition, a supercritical Hopf bifurcation is detected at the critical flutter speed ($Q_f = 4.08$) in Fig. 3. Intuitively, a LC solution that can be tracked by time marching integration is considered as stable. It can be perceived that the periodic solutions are stable.

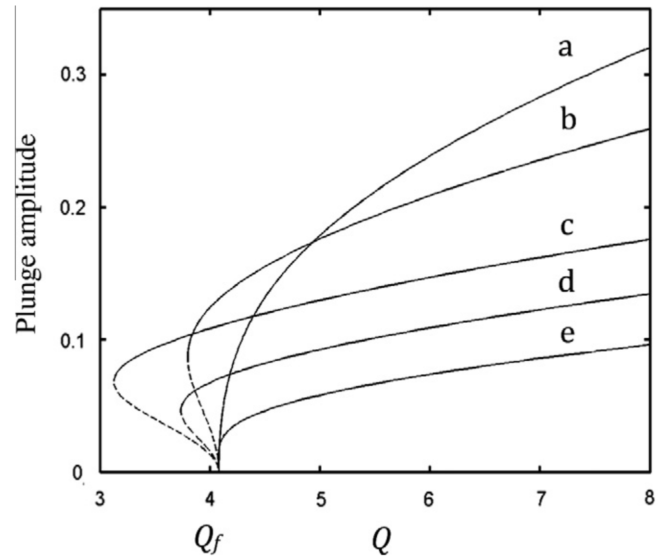

Fig. 8. Comparison of the plunge amplitudes for different nonlinear coefficients.

Since the LCOs can be obtained efficiently by the ELM, the influences of certain system parameters on LCOs are discussed in the following. As the linear stiffness in pitching DOF is given as $k_\alpha = 0.1$, the bifurcation charts (Figs. 5 and 6) demonstrate that the critical flutter speed decreases. Moreover, as it shown in Fig. 5, two LCOs could arise prior to Q_f. The lower LCO is stable and the upper one unstable. In this case, a subcritical Hopf bifurcation occurs at Q_f.

As for the linear plunging stiffness, the bifurcation chart is drawn in Fig. 7. As one observes, different from the case of k_α varying, the decreasing of k_h can enhance the critical flutter speed. Also, the bifurcation is subcritical, which means LCOs can occur prior to Q_f.

In order to analyze the influences of cubic stiffnesses, we give five pairs of nonlinear coefficients as (e_h, e_α) =a(5, 20); b(20, 20); c(80, 20); d(80, 70); e(80, 200). The bifurcations for different parameters are shown in Fig. 8. The bifurcation value remains the same as e_h and e_α varying. It implies the bifurcation value is only dependent on linear parameters of system (2). Generally, the amplitude decreases with the nonlinear coefficient increasing. Another interesting phenomenon is that LCO can arise prior to critical flutter speed as the ratio e_h/e_α varies. As e_h/e_α increases, it is more possible for system (2) to exhibit a subcritical bifurcation. The larger e_h/e_α is, the earlier a LCO could arise. For instance, a LCO can arise at about $Q = 3.7$ for line d with $e_h/e_\alpha = 8/7$. When e_h/e_α increases to 4, as line c shows, a LCO can occur at $Q = 3.2$.

5. Conclusion

The equivalent linearization method has been extended to analyze the bifurcation of the flutter system of an airfoil with cubic stiffnesses in both plunge and pitch DOFs. The influences of the stiffnesses on limit cycle oscillations and bifurcation are analyzed in detail by the proposed method. Numerical examples validate the feasibility and effectiveness of the presented method, which implies that it could be applicable in more dynamical systems, especially those with multiple nonlinearities.

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