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Technical Notes

Continuation and Bifurcation in Linear Flutter Equations

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Nomenclature

$\boldsymbol{f}$	=	flutter equations cast into a set of m real equations
$\boldsymbol{h}^i$	=	ith corrector step
$\boldsymbol{J}$	=	$\boldsymbol{f}'(\boldsymbol{x})$, the Jacobian matrix of $\boldsymbol{f}$
$\mathbb{R}^n$, $\mathbb{R}^{m\times n}$	=	the real n vectors and $m \times n$ matrices
sgn z	=	-1, 0, or 1 if z is negative, zero, or positive, respectively
$\boldsymbol{t}_b$, $\boldsymbol{t}_c$	=	tangents to the intersecting and primary curves at a bifurcation
$\boldsymbol{t}_j$	=	tangent to the solution curve at the jth continuation step
$\boldsymbol{u}_j$	=	jth left singular vector
$\boldsymbol{v}_j$	=	jth right singular vector
$\dot{\boldsymbol{x}}$	=	derivative of $\boldsymbol{x}$ with respect to arclength
$\boldsymbol{x}_b$	=	independent variables at a bifurcation
$\boldsymbol{x}_j$	=	independent variables at the jth continuation step
η	=	factor (± 1) which corrects the tangent direction
μ	=	determinant of the Jacobian augmented with the tangent
σ_j	=	jth singular value of the Jacobian matrix
τ	=	arclength along the solution curve

I. Introduction

CONTINUATION methods are a class of methods for solving systems of nonlinear equations over a continuous range of parameters ([1] p. 4) and are ideally suited to solving linear frequency-domain flutter equations [2].

The term linear flutter equations refers to the fact that they are linear functions of the displacements; however, they are always nonlinear functions of various parameters such as velocity and frequency, and so when tracing flutter solutions through a range of parameters, it is possible to encounter bifurcation, where a curve splits into multiple branches. Including the effects of active controls, which tend to be highly nonlinear functions of at least frequency, increases the chances. Because bifurcation is not ordinarily associated with linear flutter solutions it is possible for a continuation process to pass unaware over a bifurcation point, possibly missing an important branch curve. Detecting bifurcation adds a trivial calculation to a continuation process, and the resulting branches can be traced in the same fashion as the original curve.

An extensive literature exists on bifurcation theory and bifurcation in continuation processes; Allgower and Georg [1], Keller [3], and Seydel [4] give numerous references, both theory and numerical techniques. Here, these techniques have been adapted for use with the minimum-norm arclength continuation process presented in [2]. An economical test for bifurcation uses the QR factorization ([5] p. 246) from the continuation process, and a singular value decomposition is shown to aid in the detection of bifurcations and branching from them. A simple contrived example shows the possibility of bifurcation in linear frequency-domain flutter analyses and illustrates the technique.

II. Continuation Method

A brief summary of a continuation method applied to the solution of flutter equations follows (for more details see [2]). It is assumed that the flutter equations have been cast into an underdetermined system of real equations

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0} \in \mathbb{R}^m \quad \boldsymbol{x} \in \mathbb{R}^{m+1} \tag{1}$$

where the number of equations m is one fewer than the number of independent variables, and curves are traced with a pseudoarclength continuation method. At step j, the next point is predicted using the tangent to the curve:

$$\boldsymbol{x}^0_{j+1} = \boldsymbol{x}_j + h\boldsymbol{t}_j \tag{2}$$

where $\boldsymbol{t}_j$ is the tangent and h is the step size. The predictor is followed by an iterative correction process

$$\boldsymbol{J}(\boldsymbol{x}^i)\boldsymbol{h}^i = -\boldsymbol{f}(\boldsymbol{x}^i) \quad \boldsymbol{x}^{i+1} = \boldsymbol{x}^i + \boldsymbol{h}^i \qquad i = 0, i_{\max} \tag{3}$$

where $\boldsymbol{J} = \boldsymbol{f}'(\boldsymbol{x})$ is the $(m, m+1)$ Jacobian matrix of derivatives of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$, and Eq. (3) is solved using a QR factorization of the transposed Jacobian

$$\boldsymbol{J}^T = \boldsymbol{Q}\boldsymbol{R} = [\boldsymbol{Q}_1 \quad \boldsymbol{Q}_2]\begin{bmatrix}\boldsymbol{R}\\ 0\end{bmatrix} = \boldsymbol{Q}_1\boldsymbol{R} \quad \boldsymbol{R} \in \mathbb{R}^{m\times m} \tag{4}$$

resulting in a minimum-norm solution for $\boldsymbol{h}^i$ ([5] p. 300),

$$\boldsymbol{h}^i = -\boldsymbol{Q}_1\boldsymbol{R}^{-T}\boldsymbol{f}(\boldsymbol{x}^i) \tag{5}$$

The tangent vector at step $j+1$ is

$$\boldsymbol{t}_{j+1} = \eta\boldsymbol{Q}_2 \tag{6}$$

$$\eta = \mathrm{sgn}(\boldsymbol{Q}_2^T\boldsymbol{t}_j) \tag{7}$$

where sgn z is -1, 0, or $+1$ if z is negative, zero, or positive, respectively. A characterization of the tangent vector, useful in the sequel, is the rate of change of the independent variables with respect to arclength τ along the curve, $\boldsymbol{t} = \dot{\boldsymbol{x}}(\tau)$. Differentiating Eq. (1) with respect to τ

$$\boldsymbol{f}'(\boldsymbol{x})\dot{\boldsymbol{x}}(\tau) = \boldsymbol{J}\boldsymbol{t} = 0 \in \mathbb{R}^m \tag{8}$$

which shows that the tangent is a null vector of the Jacobian. Also, because

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$$t = \dot{x} = \frac{dx}{d\tau} = \frac{dx}{\sqrt{dx^T dx}} \tag{9}$$

the tangent has unit length $t^T t = 1$.

III. Detecting Bifurcations

A straightforward and easily computed method for detecting bifurcations is to monitor the determinant of the Jacobian matrix augmented with the tangent vector:

$$\mu(x) = \det\begin{bmatrix} J \\ t^T \end{bmatrix} = \det[\, J^T \quad t\,] \tag{10}$$

A change in sign indicates the orientation has changed and a bifurcation has been passed ([1] p. 81).

Evaluating the determinant is simplified by recognizing that the tangent came from the QR factorization, that the determinant of an orthogonal matrix is ± 1, and the determinant of a triangular matrix is the product of the diagonals. Furthermore, the tangent and the QR factorization of the Jacobian it was computed from are available from the last corrector iteration, and so the desired determinant is simply

$$\det[\, J^T \quad t\,] = \det[\, QR \quad t\,] = \det[\, Q_1 \quad Q_2\,]\begin{bmatrix} R & 0 \\ 0 & \eta \end{bmatrix}$$
$$= \eta \det Q \det R = (-1)^k \eta \prod_{i=1}^{m} R_{ii} \tag{11}$$

where k is the number of Householder reflections for Householder-based QR factorizations; for example, it is the number of nonzero elements of the TAU array in LAPACK QR routines [6].

To locate the bifurcation point more precisely, the continuation process could proceed with a smaller step size from the point before the sign change; however, the Newton corrector will encounter numerical problems close to the bifurcation as the R matrix approaches singularity and a final approximation x_b must be obtained by interpolation:

$$x_b \approx x_j - \frac{\mu_j}{\mu_{j+1} - \mu_j}(x_{j+1} - x_j) \tag{12}$$

in which $\mu_j = \mu(x_j)$ and $\mu_j \mu_{j+1} < 0$. Repeated application of Eq. (12) can be used to improve the approximation in a secant-method fashion [3], however, it is only necessary to approximate the bifurcation point closely enough that the first points computed on the branches converge.

If, in addition to a change in orientation, the Jacobian matrix has one rank deficiency at x_b, that is, its rank is $m-1$, then it is a simple bifurcation ([1] p. 76). Instead of passing over a bifurcation and detecting it by a determinant sign change, it is possible the continuation process might come close enough that the Jacobian has a rank deficiency and the point may be considered a possible bifurcation without a sign change.

A singular value decomposition (SVD) ([5] p. 76) of the Jacobian at x_b

$$J = U\Sigma V^T, \qquad U \in \mathbb{R}^{m\times m}, \qquad V \in \mathbb{R}^{m+1\times m+1} \tag{13}$$

reveals the rank and is necessary for branching from the bifurcation. U and V are the left and right orthogonal singular vectors, respectively, and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^{m\times m+1}$ is a diagonal matrix with the singular values on the first m diagonals with $\sigma_{i+1} \le \sigma_i$ and the last $(m+1)$ column zero. At regular points on the curve, the rank of J is m and all singular values are positive. With one rank deficiency, the last singular value is zero. In practice, the smallest singular value σ_m will not be exactly zero because the bifurcation point is only an approximation, but it should be small relative to the largest:

$$\sigma_m \le \epsilon_r \sigma_1 \tag{14}$$

for some small ϵ_r, for example, $\sqrt{\epsilon_m}$, where ϵ_m is the machine precision ([5] p. 95).

IV. Branching from Bifurcations

At a regular point on the curve, the last column of V, v_{m+1}, is a Jacobian null vector and (within a sign) is the tangent vector. At a bifurcation point, the last column of U, u_m, and the last two columns of v are left and right null vectors, respectively:

$$u_m^T J = 0 \quad J v_m = J v_{m+1} = 0 \tag{15}$$

and there are two tangents, combinations of the right null vectors:

$$t_k = \alpha_k v_m + \beta_k v_{m+1}, \qquad k = 1, 2 \tag{16}$$

The task is to find the scalars $(\alpha_k, \beta_k, k = 1, 2)$.

Equation (8), the derivative of Eq. (1) with respect to arclength τ, defines the tangent to the curve for a regular point. With two tangents at a simple bifurcation, another procedure must be used, starting with the second derivative of Eq. (1) with respect to arclength:

$$f''\dot{x}\dot{x} + f'\ddot{x} = 0 \tag{17}$$

Multiplying by the left null vector eliminates the second term, resulting in a scalar equation both tangents must satisfy:

$$u_m^T f''\dot{x}\dot{x} = u_m^T f'' t t = 0 \tag{18}$$

Substituting Eq. (16),

$$\begin{aligned} &u_m^T f''(\alpha_k v_m + \beta_k v_{m+1})(\alpha_k v_m + \beta_k v_{m+1}) \\ &\quad = u_m^T f'' v_m v_m \alpha_k^2 + 2 u_m^T f'' v_m v_{m+1} \alpha_k \beta_k + u_m^T f'' v_{m+1} v_{m+1} \beta_k^2 \\ &\quad = a_{11}\alpha_k^2 + 2a_{12}\alpha_k\beta_k + a_{22}\beta_k^2 = 0 \end{aligned} \tag{19}$$

Solving for α_k in terms of β_k if $a_{11} \ne 0$, or β_k in terms of α_k,

$$\alpha_k = \frac{-a_{12} + (-1)^k\sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}\beta_k = \frac{c_k}{a_{11}}\beta_k \qquad \beta_k = \frac{c_k}{a_{22}}\alpha_k$$
$$c_k = -a_{12} + (-1)^k\sqrt{a_{12}^2 - a_{11}a_{22}} \tag{20}$$

A necessary condition for a simple bifurcation point is that the discriminant be positive ([4] p. 165)

$$a_{12}^2 - a_{11}a_{22} > 0 \tag{21}$$

To complete the determination of α_k and β_k, another relationship is necessary. Tangent vectors must have unit length, and the singular vectors are orthonormal, leading to

$$\begin{aligned} t_k^T t_k &= \alpha_k^2 v_m^T v_m + 2\alpha_k\beta_k v_m^T v_{m+1} + \beta_k^2 v_{m+1}^T v_{m+1} \\ &= \alpha_k^2 + \beta_k^2 = 1 \end{aligned} \tag{22}$$

Substituting Eq. (20),

$$\alpha_k = \frac{c_k}{\sqrt{c_k^2 + a_{11}^2}} = \frac{a_{22}}{\sqrt{c_k^2 + a_{22}^2}} \qquad \beta_k = \frac{c_k}{\sqrt{c_k^2 + a_{22}^2}} = \frac{a_{11}}{\sqrt{c_k^2 + a_{11}^2}} \tag{23}$$

It remains to compute the scalars

$$\begin{aligned} &a_{11} = u_m^T f'' v_m v_m, \qquad a_{12} = u_m^T f'' v_m v_{m+1}, \\ &a_{22} = u_m^T f'' v_{m+1} v_{m+1} \end{aligned} \tag{24}$$

containing second derivatives of $\boldsymbol{f}(\boldsymbol{x})$ which may not be readily available and must be approximated numerically. To this end let

$$g(\xi_1, \xi_2) = \boldsymbol{u}_m^T \boldsymbol{f}(\boldsymbol{x}_b + \xi_1 \boldsymbol{v}_m + \xi_2 \boldsymbol{v}_{m+1}) \tag{25}$$

Second-order central difference approximations of the second derivatives with respect to arclength τ in the directions $\boldsymbol{v}_m$ and $\boldsymbol{v}_{m+1}$ are

$$\begin{aligned} a_{11} &= \frac{\partial^2 g}{\partial \xi_1^2} \approx \frac{1}{\epsilon^2}[g(\epsilon, 0) - 2g(0,0) + g(-\epsilon, 0)] \\ a_{12} &= \frac{\partial^2 g}{\partial \xi_1 \partial \xi_2} \approx \frac{1}{4\epsilon^2}[g(\epsilon, \epsilon) + g(-\epsilon, -\epsilon) - g(\epsilon, -\epsilon) - g(-\epsilon, \epsilon] \\ a_{22} &= \frac{\partial^2 g}{\partial \xi_2^2} \approx \frac{1}{\epsilon^2}[g(0, \epsilon) - 2g(0,0) + g(0, -\epsilon)] \end{aligned} \tag{26}$$

with ϵ a small number, for example $\sqrt[3]{\epsilon_m}$ ([1] p. 90). As mentioned earlier, approximations in computing the tangents are acceptable, provided the first continuation step taken with these tangents converges.

One of the two tangents, the primary tangent $\boldsymbol{t}_c$, is tangent to the curve being traced, the other $\boldsymbol{t}_b$ is tangent to the intersecting curve. The intersecting tangent is used to trace the two branches, first using it as calculated to start a continuation process, then multiplied by -1 to start another. The primary tangent is only needed to continue tracing in the same direction, and so it is necessary to determine which is which by comparing with the previous tangent:

$$\begin{gathered} t_1 = |\boldsymbol{t}_j^T \boldsymbol{t}_1|, \qquad t_2 = |\boldsymbol{t}_j^T \boldsymbol{t}_2| \\ \boldsymbol{t}_c, \boldsymbol{t}_b = \begin{cases} \boldsymbol{t}_1, \boldsymbol{t}_2 & \text{if } \ t_1 > t_2 \\ \boldsymbol{t}_2, \boldsymbol{t}_1 & \text{otherwise} \end{cases} \end{gathered} \tag{27}$$

and ensure the primary tangent points in the right direction,

$$\text{if } \boldsymbol{t}_j^T \boldsymbol{t}_c < 0 \text{ then } \boldsymbol{t}_c = -\boldsymbol{t}_c \tag{28}$$

V. Summary

To summarize the conditions necessary for a simple bifurcation: 1) change in orientation of the curve [change in sign of μ, Eq. (10)]; 2) one rank deficiency in the Jacobian [Eq. (14)]; 3) positive discriminant [Eq. (21)].

The steps involved in tracing the branches can be summarized as follows:

1) Interpolate to get an approximation to the bifurcation point $\boldsymbol{x}_b$ [Eq. (12)].

2) Compute the SVD of the Jacobian evaluated at $\boldsymbol{x}_b$.

3) Compute the scalars a_{11}, a_{12}, and a_{22} [Eq. (26)].

4) Compute the scalars (α_k, β_k) [Eqs. (20) and (23)].

5) Compute $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, the tangents emanating from the bifurcation point [Eq. (16)].

6) Associate $\boldsymbol{t}_b$ and $\boldsymbol{t}_c$, the intersecting and primary tangents, with $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ [Eqs. (27) and (28)].

7) Using $\boldsymbol{x}_b$ and $\boldsymbol{t}_b$ as start points, trace the bifurcation branch, then in the opposite direction with $-\boldsymbol{t}_b$.

8) Using $\boldsymbol{x}_b$ and $\boldsymbol{t}_c$ as start points, continue tracing the primary curve.

Bifurcations are rare in flutter solutions, but they must be treated properly or unstable flutter modes could be missed, as the following example shows.

VI. Example

Example 9-1 in [7] is often used to demonstrate flutter formulations and solution techniques; in MSC-NASTRAN [8], it is example HA145b. Here, it is built with a generalized coordinate basis of 10 free-vibration modes and only one aeroelastic mode traced starting at zero dynamic pressure and 11.6 Hz, referred to as mode 4.

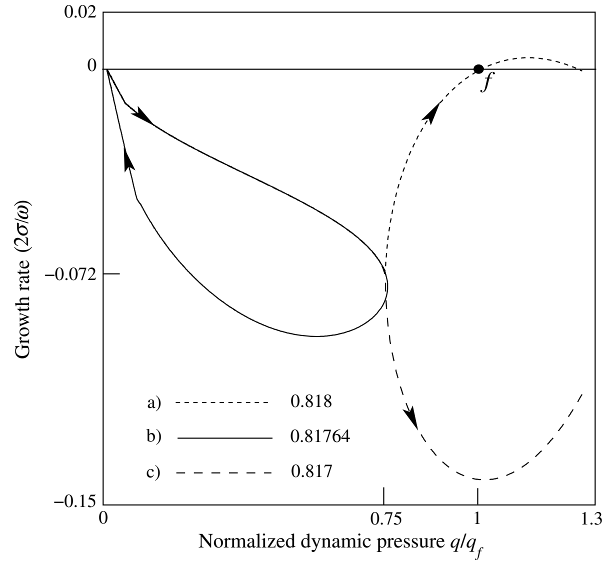

Fig. 1 Aeroelastic mode 4 with scaled stiffness.

This mode exhibits bifurcation when the stiffness matrix is modified slightly and the flutter equation

$$[s^2 \boldsymbol{M} + \boldsymbol{K} - q\boldsymbol{U}(\kappa)]\boldsymbol{q} = 0 \tag{29}$$

is analyzed with independent variables dynamic pressure $(q = \rho V^2/2)$ and characteristic exponent $s = \sigma + i\omega$, where ω is the frequency and the real part σ determines the stability of trivial $(\boldsymbol{q} = \boldsymbol{0})$ solutions. $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{U}$ are the mass, stiffness, and unsteady aerodynamics matrices, respectively. The unsteady aerodynamics matrix is a function of reduced frequency $\kappa = \omega/V$.

Scaling the fourth diagonal of the stiffness by three factors, a) 0.818, b) 0.81764, and c) 0.817, results in three very different curves, shown in Fig. 1. Scale factor a results in a mode that goes unstable at q_f, scale factor c causes the mode to remain stable, and scale factor b causes the mode to reverse direction, ending at zero. Figure 2 shows this behavior more clearly. Curves a and c appear normal, but curve b is suspicious, and indeed the sign of the determinant [Eq. (10)] changes close to where the curves diverge, as shown in Fig. 3, with arrows indicating the orientation of the curve. Evidently, curve b passes a bifurcation, located more precisely by

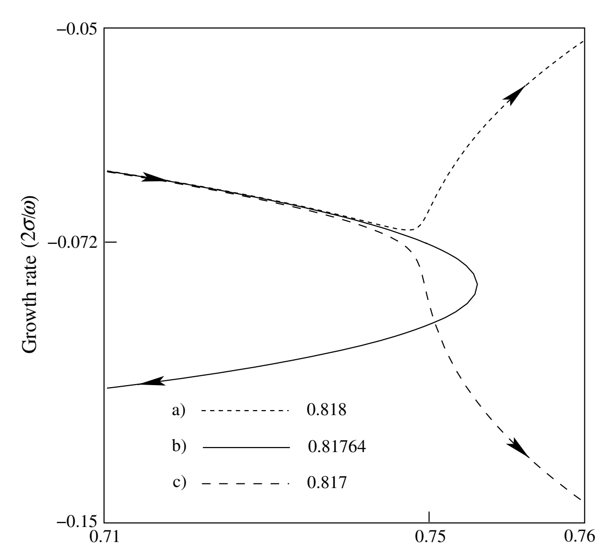

Fig. 2 Transition region.

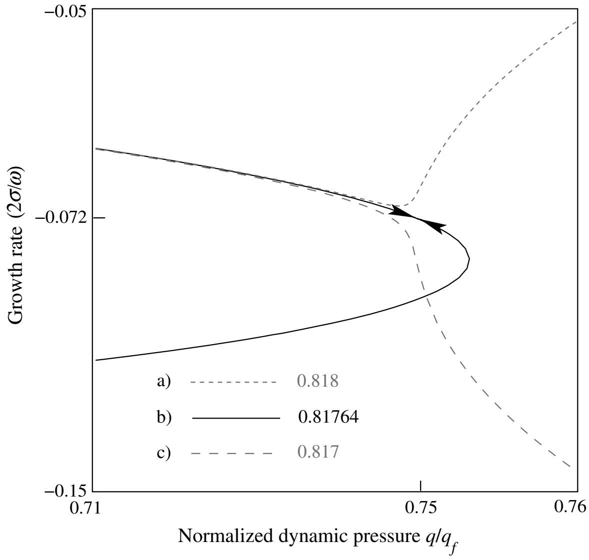

Fig. 3 Orientation change.

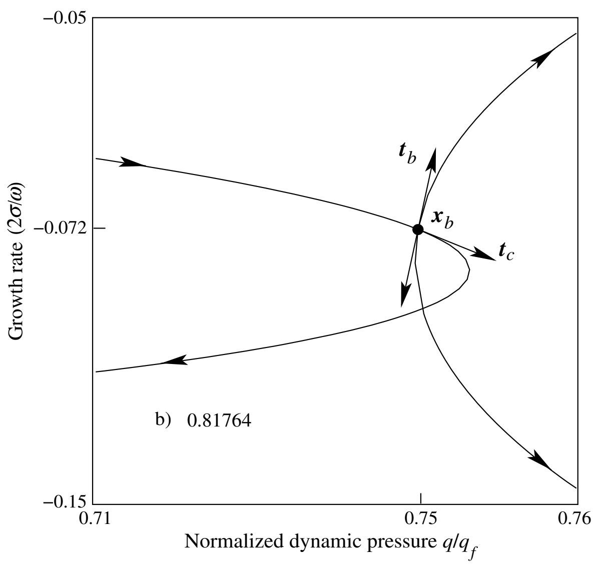

Fig. 4 Bifurcation branches.

reducing the continuation step size and continuing from the last point before the determinant changed signs, and finally using interpolation [Eq. (12)] to get an accurate estimate of the bifurcation point $\boldsymbol{x}_b$.

A singular value decomposition of the Jacobian at the bifurcation reveals it has one rank deficiency. Furthermore, computation of the discriminant [Eq. (21)] confirms this is a simple bifurcation. Equations (27) and (28) are then used to compute the tangents $\boldsymbol{t}_b$ and $\boldsymbol{t}_c$, with the aid of Eqs. (16), (20), (23), and (26). Three continuation processes starting from $\boldsymbol{x}_b$ with $\boldsymbol{t}_c$, $\boldsymbol{t}_b$, and $-\boldsymbol{t}_b$ produce the branches shown in Fig. 4. Away from the bifurcation, these curves are indistinguishable from the curves in Fig. 1.

Had the process continued without detecting and branching from the bifurcation, the curve would have reversed direction, missing the important branch that goes unstable. This example was contrived to demonstrate the possibility of encountering bifurcation in linear flutter continuation, in addition to the process of detecting and tracing branches.

VII. Conclusions

Bifurcations encountered in linear frequency-domain flutter equations when tracing aeroelastic modes with continuation methods are rare, although experience shows they are more common when active controls equations are included. Small perturbations in model parameters can often eliminate them. When they do appear, a continuation method can pass them without realizing it; the danger is that one of the two curves emanating from a bifurcation will be missed and may be an important curve. Detecting when a bifurcation has been passed and tracing the branches is an easy, inexpensive calculation and should be a routine part of continuation methods.

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