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Robust flutter analysis of a nonlinear aeroelastic system with parametric uncertainties

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Abstract

This paper deals with the problem of robust stability of a 2-D nonlinear aeroelastic system with structural and aerodynamic uncertainties using μ-method and value set approach. The model includes a nonlinear spring in the pitch degree-of-freedom and a nonlinear aerodynamic model. Two types of uncertainty named vanishing and nonvanishing perturbation are investigated. For vanishing perturbations, μ-method can be used directly to examine the robust stability of the nonlinear aeroelastic system by analyzing its linearized system. For nonvanishing perturbations, the functional relationship between equilibrium point and uncertain parameters are expressed as Taylor series expansions in order to consider the uncertainty of the equilibrium point in μ-analysis framework. Furthermore, the value set approach is also used to examine the robust stability of the uncertain system. The value sets of the characteristic polynomials are computed and zero exclusion condition is applied to check the stability of the entire family of the characteristic polynomials. Numerical results are presented for a set of values of the flow velocities, and the lower bound and upper bound of robust flutter speeds are obtained from the V–μ graph and the motion of the value sets.

Keywords: Robust flutter analysis; μ-analysis method; Nonlinear aeroelastic system; Value set approach

1. Introduction

In recent years, robust flutter analysis has been an active field of research and gained much attention in the flight flutter test community. In contrast with traditional flutter solvers, this approach considers stability of a family of aeroelastic systems which are subjected to various uncertainties such as modeling errors, parametric uncertainties, disturbances and so on, and computes a robust flutter boundary which indicates the worst-case of the uncertain aeroelastic system. Structure singular value is used as the indicator of robust stability instead of the eigenvalues.

Several researchers have developed a lot of methods and tools to capture the stability characteristics of the uncertain aeroelastic systems with μ-method and obtained numerical and experimental results. A comprehensive and systematic presentation of robust flutter analysis can be found in [16,18] which present the entire procedures of robust flutter analysis including uncertainty description, μ-method analysis, model validation and flight flutter test. An online flight flutter test tool called flutterometer which uses both analytical model and flight data has been developed to predict the robust flutter boundary [17]. A real flight test of the aerostructure test wing was performed to evaluate the flutterometer and the results demonstrated the validity and safety of this approach [21]. To reduce the conservatism of the results generated by robust flutter margin prediction, wavelet analysis and Volterra modeling approach have been developed and more credible predictions were obtained [9,19,22]. In literature [15], a match point solution has been achieved by setting airspeed as the perturbation variable in stead of dynamic pressure such that the flight condition of the model depends on a single parameter. Mach number and flight altitude can also be used as the single parameter that describes the flight condition and generate match point solutions [5,26]. Borglund [6] has developed the μ–k method. The uncertainty of aerodynamic forces can be described in detail and robust flutter analysis is performed in frequency-domain. The dynamic pressure was considered as a flight parameter in stead of perturbation

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variable and consequently, match-point solutions were inherently generated. Moreover, both the lower and upper bound flutter speeds can be obtained which represent the worst case and best case, respectively [7]. In literature [8], a new formulation connecting μ–k approach and traditional frequency-domain flutter analysis was established to develop an efficient algorithm for computation of robust flutter boundaries, and an F-16 sample test case with uncertain external stores aerodynamics demonstrated the efficiency of this method. Researches have also explored the dynamic behavior of nonlinear aeroelastic system. The block-oriented identification technique and Volterra series representation were used to estimate nonlinearities by analyzing experimental data and the μ-method was used to predict robust stability margins which indicated both flutter and LCO instabilities [2,20].

Nonlinear systems exhibit a variety of complex phenomena and the characterization of the dynamic behavior of nonlinear system with uncertainty is difficult, but several theorems, analysis methods and stability criterions were established [10]. These theoretical results, although complicated, provide the general theories and methods to examine the robust stability of nonlinear uncertain systems and become the foundation of nonlinear robust control. Nevertheless, in flutter analysis, one has to predict the instability in order to obtain the critical stability conditions. These methods are unable to predict the instability because they can not provide a sufficient and necessary condition of stability and generate an instability criterion.

In this paper, the stability of a nonlinear 2-D aeroelastic system with structural and aerodynamic uncertainties is examined using μ-analysis method and value set approach. The model includes a nonlinear spring denoted by a fifth degree polynomial in pitch degree-of-freedom and a nonlinear quasi-steady aerodynamic model. Two types of perturbation are investigated. The vanishing perturbation has no effect on the location of equilibrium point, because the perturbation vanishes at the equilibrium point of the nominal system. Whereas, in the case of nonvanishing perturbation, the equilibrium point of the perturbed system may be different from that of the nominal system. For vanishing perturbation, the stability analysis of the nonlinear aeroelastic system can be performed by directly applying μ-analysis method to the reduced linear system, and a V–μ graph can display the lower bound and upper bound robust flutter speeds which imply that the true flutter speed lies within the predicted bounds if the model can be validated properly and the robust flutter analysis is performed successfully.

However, more difficulties will be encountered in the case of nonvanishing perturbation, because the perturbation terms do not vanish at the equilibrium point of the nominal system. In order to consider the uncertainty of the equilibrium point in μ-analysis framework, the equilibrium point is considered as a function of uncertainties and expressed as a Taylor series expansion. Thus, a LFT (Linear Fractional Transformation) representation is obtained and μ-method can be used to generate a V–μ graph. Also, the value set approach is presented to analyze robust stability of the uncertain aeroelastic system. The value sets of the characteristic polynomials with uncertain coefficients are evaluated for the intervals of the uncertain parameters, and then the zero exclusion condition is used to determine the stability of the entire family of the uncertain polynomials. The lower and upper bound flutter speeds are also obtained by analyzing the motion and distribution of the value sets on a complex plane. The numerical results demonstrate the validity of these two approaches.

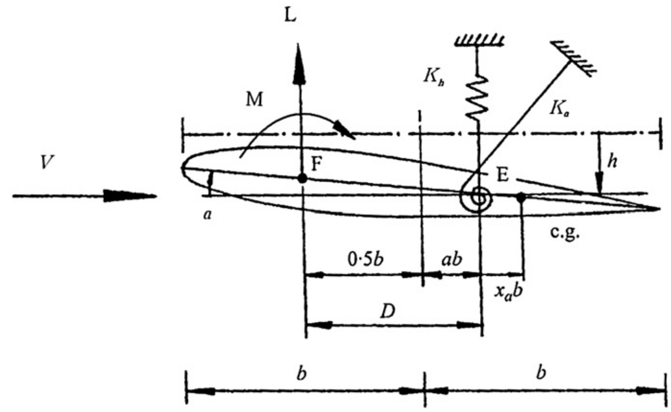

Fig. 1. Two-dimensional aeroelastic model.

The organization of the paper is as follow: Section 2 presents the aeroelastic model. Uncertainty descriptions are discussed in Section 3. Robust flutter of nonlinear aeroelastic system with two types of perturbation is obtained in Section 4. The applications of value set approach to robust flutter analysis are presented in Section 5. Finally, Section 6 provides the numerical results.

2. Aeroelastic model

The 2-D aeroelastic wing section is shown in Fig. 1. The governing equations of motion are provided in [14,23,24] and is given by

$$\begin{bmatrix} m_t & m_w x_\alpha b \\ m_w x_\alpha b & I_\alpha \end{bmatrix}\begin{bmatrix} \ddot{h} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\alpha \end{bmatrix}\begin{bmatrix} \dot{h} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\alpha(\alpha) \end{bmatrix}\begin{bmatrix} h \\ \alpha \end{bmatrix} = \begin{bmatrix} -L \\ M \end{bmatrix}, \tag{1}$$

where h is the plunge displacement and α is the pitch angle. In Eq. (1), m_t is the total mass; m_w is the mass of the wing; b is the semichord of the wing; I_α is the moment of inertia; x_α is the nondimensionalized distance of the center of mass from the elastic axis; k_h is plunge stiffness coefficient; c_h and c_α are plunge and pitch damping coefficients, respectively; L and M are the aerodynamic lift and moment. The nonlinear quasi-steady aerodynamic model considering stall effect is as follow

$$L = \rho V^2 b c_{l\alpha}(\alpha_{\text{eff}} - c_3\alpha_{\text{eff}}^3), \tag{2}$$

$$M = \rho V^2 b^2 c_{m\alpha}(\alpha_{\text{eff}} - c_3\alpha_{\text{eff}}^3), \tag{3}$$

where ρ is the air density, V is the flow velocity, and $c_{l\alpha}$ and $c_{m\alpha}$ are the lift and moment coefficients. c_3 is a nonlinear parameter associated with stall model. α_{eff} is the effective angle of attack defined by

$$\alpha_{\text{eff}} = \left[\alpha + \left(\frac{\dot{h}}{V}\right) + \left(\frac{1}{2} - a\right)b\frac{\dot{\alpha}}{V}\right], \tag{4}$$

where a is the nondimensionalized distance from the midchord to the elastic axis. Parameter c_3 is defined as follow for NACA 0012 airfoil,

$$c_3 = 0.00034189(180/\pi)^3/c_{l\alpha}. \tag{5}$$

This aerodynamic model is valid for $\alpha_{\text{eff}} \in [-11,\ 11]$ degree. The function $k_\alpha(\alpha)$ is considered as a polynomial given by

$$\alpha k_\alpha(\alpha) = k_{\alpha 1}\alpha + k_{\alpha 2}\alpha^2 + k_{\alpha 3}\alpha^3 + k_{\alpha 4}\alpha^4 + k_{\alpha 5}\alpha^5, \tag{6}$$

where $k_{\alpha j},\ j = 1, 2, 3, 4, 5$, are constants.

Defining a state vector $x = (x_1\ x_2\ x_3\ x_4)^T = (h\ \alpha\ \dot{h}\ \dot{\alpha})$, a state-space representation can be obtained in the form

$$\dot{x} = f(x), \tag{7}$$

and the reduced linear system is

$$\dot{x} = Ax. \tag{8}$$

Matrix A is Jacobian matrix evaluated at an equilibrium point.

The stability of linear system (8) is determined by the eigenvalues of matrix A at different values of the flow velocities. The flutter speed is obtained when a critical eigenvalue crosses the stability boundary. In this analysis procedure, the process to find the equilibrium point, commonly, is to solve a set of nonlinear equations. But in the case of uncertain system, the existences of uncertain parameters in those equations may result in uncertainty of the solutions of equilibrium point. The question is how to deal with the uncertainty of equilibrium point within μ-analysis framework?

3. Uncertainty description

Uncertainties in aeroelastic systems derive from numerous sources such as damping, stiffness and unsteady aerodynamic forces modeling. In this paper, parametric uncertainties in structure and aerodynamic load are investigated. This is shown as follow

$$\begin{aligned}
&c_h = c_{h0} + Wc_h\delta c_h, \qquad c_\alpha = c_{\alpha 0} + Wc_\alpha\delta c_\alpha,\\
&k_h = k_{h0} + Wk_h\delta k_h,\\
&k_{\alpha j} = k_{\alpha j0} + Wk_{\alpha j}\delta k_{\alpha j}, \quad j = 1, 2, 3, 4, 5,\\
&c_3 = c_{30} + Wc_3\delta c_3,
\end{aligned} \tag{9}$$

where $c_{h0}, c_{\alpha 0}, k_{h0}$ and $k_{\alpha j0}$ are nominal values of damping and stiffness coefficients in plunge and pitch degree-of-freedom, respectively. c_{30} is the nominal value of the nonlinear aerodynamic parameter. $Wc_h, Wc_\alpha, Wk_h, Wk_{\alpha j}, Wc_3$ are weighting coefficients which determine the magnitude of the uncertainties. $\delta c_h, \delta c_\alpha, \delta k_h, \delta k_{\alpha j}, \delta c_3$ are the uncertainty operators and have a norm less than unity. For the various sources of uncertainty in aeroelastic systems, the uncertainty of unsteady aerodynamic forces may be the most important. So it is quite reasonable to consider the parameter c_3 as an uncertain parameter accounting for the uncertainty of aerodynamic loads. The uncertainty of pitch spring is described by setting each coefficient of $k_\alpha(\alpha)$ as an uncertain parameter. The uncertainty of mass is not included, because there is no fuel consuming and the mass of the aeroelastic section wing dose not vary in simulation or in wind tunnel test. The forms of uncertainty representation are additive and multiplicative uncertainty which depend on the choices of the weighting coefficients.

In fact, the numerous sources and various forms of uncertainty in aeroelastic system make it difficult to obtain a proper uncertainty model. Uncertainty modeling is problem-dependent and needs a profound understanding of aeroelasticity and abundance of engineering experience. In this paper, the detailed uncertainty modeling is not included and the weighting coefficients which should be estimated by model validation are obtained by assigning a proper value artificially. However, the purpose of this paper is to present a method for robust stability analysis of nonlinear aeroelastic system and the presented uncertainty model is enough to demonstrate the validity of the methods.

Another uncertainty description is the interval model. The uncertain parameters can be modeled as follow

$$\begin{aligned}
&c_h^- \leqslant c_h \leqslant c_h^+, \qquad c_\alpha^- \leqslant c_\alpha \leqslant c_\alpha^+, \qquad k_h^- \leqslant k_h \leqslant k_h^+,\\
&k_{\alpha j}^- \leqslant k_{\alpha j} \leqslant k_{\alpha j}^+, \quad j = 1, 2, 3, 4, 5,\\
&c_3^- \leqslant c_3 \leqslant c_3^+,
\end{aligned} \tag{10}$$

where $(\cdot)^-$ and $(\cdot)^+$ are the lower and upper bounds of the parameters, respectively. Eq. (10) can also be write as

$$\begin{aligned}
&c_h \in c_h^I = [c_h^-, c_h^+], \qquad c_\alpha \in c_\alpha^I = [c_\alpha^-, c_\alpha^+],\\
&k_h \in k_h^I = [k_h^-, k_h^+],\\
&k_{\alpha j} \in k_{\alpha j}^I = [k_{\alpha j}^-, k_{\alpha j}^+], \quad j = 1, 2, 3, 4, 5,\\
&c_3 \in c_3^I = [c_3^-, c_3^+],
\end{aligned} \tag{11}$$

in which $c_h^I, c_\alpha^I, k_h^I, k_{\alpha j}^I, c_3^I$ are interval parameters, and the middle point and radius of the intervals can be expressed as

$$\begin{aligned}
&c_h^m = (c_h^- + c_h^+)/2, \qquad c_\alpha^m = (c_\alpha^- + c_\alpha^+)/2,\\
&k_h^m = (k_h^- + k_h^+)/2,\\
&k_{\alpha j}^m = (k_{\alpha j}^- + k_{\alpha j}^+)/2, \quad j = 1, 2, 3, 4, 5,\\
&c_3^m = (c_3^- + c_3^+)/2,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\Delta c_h = (c_h^+ - c_h^-)/2, \qquad \Delta c_\alpha = (c_\alpha^+ - c_\alpha^-)/2,\\
&\Delta k_h = (k_h^+ - k_h^-)/2,\\
&\Delta k_{\alpha j} = (k_{\alpha j}^+ - k_{\alpha j}^-)/2, \quad j = 1, 2, 3, 4, 5,\\
&\Delta c_3 = (c_3^+ - c_3^-)/2.
\end{aligned} \tag{13}$$

A center interval representation can be produced in term of the middle points and radius of the interval parameters and is given by

$$\begin{aligned}
&c_h^I = c_h^m + \Delta c_h e_\Delta, \qquad c_\alpha^I = c_\alpha^m + \Delta c_\alpha e_\Delta,\\
&k_h^I = k_h^m + \Delta k_h e_\Delta,\\
&k_{\alpha j}^I = k_{\alpha j}^m + \Delta k_{\alpha j} e_\Delta, \quad j = 1, 2, 3, 4, 5,\\
&c_3^I = c_3^m + \Delta c_3 e_\Delta,
\end{aligned} \tag{14}$$

where $e_\Delta = [-1,\ 1]$, and can vary independently in each interval parameter of Eq. (14).

Although the two uncertainty expressions are different, both of them can represent the characteristics of the uncertain parameters, such as middle points/nominal values and bounds/weighting coefficients. The expression of Eq. (9) is suitable for μ-method analysis, while another model is suitable for the value set approach.

The presented uncertainties can be sorted as vanishing and nonvanishing perturbations in term of the values of the perturbation terms at the nominal equilibrium point. The uncertain system can be considered as follow

$$\dot{x} = f(x) + \Delta f(x), \tag{15}$$

where $\dot{x} = f(x)$ is the nominal system and $\Delta f(x)$ is the unknown perturbation. Suppose x_e is an equilibrium point of nominal system $\dot{x} = f(x)$. In order to analyze the stability of perturbed system, one has to determine whether the perturbation term vanishes at the point $x = x_e$. If $\Delta f(x_e) = 0$, the perturbation term $\Delta f(x)$ is called vanishing perturbation and x_e will be an equilibrium point of the perturbed system. If $\Delta f(x_e) \neq 0$, the perturbation term is called nonvanishing perturbation and x_e will not be an equilibrium point of the perturbed system. Thus the influence of nonvanishing perturbation on the equilibrium point should be taken into account in stability analysis of the perturbed system.

For the 2-D nonlinear aeroelastic system studied in this paper, the equilibrium point of Eq. (1) is $x_e = (x_{1e}, x_{2e}, x_{3e}, x_{4e})^T = (h_e, \alpha_e, \dot{h}_e, \dot{\alpha}_e)^T = (h_e, \alpha_e, 0, 0)^T$. According to the values of the perturbation terms at x_e, the uncertainty parameters c_h and c_α which are associated with state variables $\dot{h}$ and $\dot{\alpha}$ generate vanishing perturbations since the perturbation terms $\delta c_h \dot{h}$ and $\delta c_\alpha \dot{\alpha}$ always vanish at the equilibrium point x_e. While k_h, $k_{\alpha j}$ and c_3 produce nonvanishing perturbations. The perturbation terms $\delta k_h h$, $\delta k_{\alpha j} \alpha^j$ and $\delta c_3 \alpha^3$ do not vanish at x_e and affect the location of equilibrium point of the perturbed system.

4. Robust flutter analysis

Case I: Robust flutter analysis for aeroelastic system with only vanishing perturbations

Usually in robust flutter analysis, the family of plants is defined by a nominal system and an uncertainty model which describes the sources, forms and magnitudes of the uncertainties. In this specific system family, the flutter speed of each system is different and determined by the magnitudes and function of the uncertainties which could improve or deteriorate the perturbed system and correspondingly, increase or decrease the flutter speed. As it is expected, there must be a system of which the flutter speed is the lowest in the system family. This system is called the worst-case system and the corresponding flutter speed is robust flutter speed. The perturbation that represents the error between the nominal system and the worst-case system is defined as the worst-case perturbation. The worst-case perturbation is a combination of the various factors, including structure parameters and aerodynamic modeling which are discussed before, such that the flutter speed is shifted to the lowest. The Structured Singular Value theory is used to compute the flutter speed of the worst-case system using $\mu = 1$ as the critical condition and the method is shown as follow.

The linearized system of Eq. (15) is shown as

$$\dot{x} = A_\Delta x, \tag{16}$$

where matrix A_Δ is the Jacobian matrix evaluated at the equilibrium point x_e and defined as

$$A_\Delta = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ M_1 & M_2 \end{bmatrix}. \tag{17}$$

In Eq. (17), 0 and I are null and identity matrices, the matrices M_1 and M_2 can be obtained from Eq. (1) easily.

If only vanishing perturbations exist, the stability behavior of nominal equilibrium point x_e can be analyzed as an equilibrium point of the perturbed system. Since the system is subjected to the uncertain parameters c_h and c_α, Eqs. (15) and (16) define a family of systems and the corresponding linearized systems family, respectively. It turns out that investigating the stability behavior of the nonlinear perturbed system (15) is equivalent to examining the stability of the entire family of plants (16). That is to say, the stability characteristics of the perturbed system (15) can be obtained by analyzing the robust stability of the linear uncertain system (16). Although the eigenvalues of matrix A_Δ can not be obtained directly, the robust stability of the uncertain system (16) can be investigated by μ-analysis method. Without computing the eigenvalues of matrix A_Δ, μ method computes a minimum singular value of the uncertainty operator that destabilizes the nominal system.

A V–μ graph can be obtained by performing robust flutter analysis for a set of values of flow velocities. This approach uses flow velocity as a flight condition parameter in stead of perturbation variable and consequently generates match-point solutions. This treatment of the flight condition in robust flutter analysis was developed by Borglund and illustrated in Literature [6].

Case II: Robust flutter analysis for aeroelastic system with vanishing and nonvanishing perturbations

In the case of nonvanishing perturbations, the equilibrium point of perturbed system differs from that of nominal system. The influences of nonvanishing perturbations on equilibrium point have to be taken into consideration in the procedure of robust flutter analysis. If the uncertainty of equilibrium point is formulated as an independent uncertainty, the stability of system (16) is influenced by the parametric uncertainties and uncertainty of equilibrium point simultaneously. Under this assumption, the worst-case perturbation is composed of two parts: the perturbations of the physical parameters and the perturbation of equilibrium point which is defined as the worst-case equilibrium point. Otherwise, if the dependency of the equilibrium point on the uncertain parameters is admitted, the Jacobian matrix A_Δ is entirely determined by the uncertain parameters. In this case, the worst-case perturbation is a combination of the physical parameters which deteriorates the stability boundary worst and the corresponding equilibrium point determined by these specific parameters is the worst-case equilibrium point.

The equations which are solved to obtain the equilibrium point of the perturbed system (15) can be written as

$$\dot{x} = f(x) + \Delta f(x) = 0, \tag{18}$$

and expressed in an expanded form

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ G_1(q, x_1, x_2) \\ G_2(q, x_1, x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_3(q)x_2^3 + a_2(q)x_2^2 + a_1(q)x_2 + a_h(q)x_1 + a_0(q) \\ b_5(q)x_2^5 + b_4(q)x_2^4 + b_3(q)x_2^3 + b_2(q)x_2^2 + b_1(q)x_2 + b_0(q) \end{bmatrix}, \tag{19}$$

where q is the uncertainty vector denoted by $q = (k_h, k_{\alpha j}, c_3)$, $j = 1, 2, 3, 4, 5$, and $a_m(q)$, $m = 0, 1, 2, 3$, $b_n(q)$, $n = 0, 1, 2, 3, 4, 5$, and $a_h(q)$ are coefficients which are assumed to be continuous functions of uncertainty vector q. It is also assumed that the solution of equations $G_1(q, x_1, x_2) = 0$ and $G_2(q, x_1, x_2) = 0$ are continuous functions of the coefficients, and consequently, are determined by continuous functions of uncertain parameters k_h, $k_{\alpha j}$ and c_3. Here, sufficient conditions of the implicit function existence theorem are checked and satisfied and thus x_1 and x_2 can be written as

$$\begin{aligned} x_1 &= F_1(k_h, k_{\alpha 1}, \ldots, k_{\alpha 5}, c_3), \\ x_2 &= F_2(k_h, k_{\alpha 1}, \ldots, k_{\alpha 5}, c_3). \end{aligned} \tag{20}$$

Although the explicit expressions of Eq. (20) are difficult to be obtained, F_1 and F_2 can be expanded in Taylor series at the nominal values of the uncertain parameters to formulate the approximations of F_1 and F_2. Neglecting second and higher order terms gives

$$\begin{aligned} x_2 = {} & F_2(k_{h0}, k_{\alpha 10}, \ldots, k_{\alpha 50}, c_{30}) \\ & + \frac{\partial F_2(k_{h0}, k_{\alpha 10}, \ldots, k_{\alpha 50}, c_{30})}{\partial k_h}(k_h - k_{h0}) \\ & + \sum_{j=1}^{5} \frac{\partial F_2(k_{h0}, k_{\alpha 10}, \ldots, k_{\alpha 50}, c_{30})}{\partial k_{\alpha j}}(k_{\alpha j} - k_{\alpha j0}) \\ & + \frac{\partial F_2(k_{h0}, k_{\alpha 10}, \ldots, k_{\alpha 50}, c_{30})}{\partial c_3}(c_3 - c_{30}). \end{aligned} \tag{21}$$

In Eq. (21), $F_2(k_{h0}, k_{\alpha 10}, \ldots, k_{\alpha 50}, c_{30})$ generates the nominal value of x_2 and the partial derivatives of F_2 can be obtained by using the derivative law of the implicit function which are given by

$$\begin{aligned} \frac{\partial F_2}{\partial k_h} &= \frac{\partial G_2}{\partial k_h} \Big/ \frac{\partial G_2}{\partial x_2}, \\ \frac{\partial F_2}{\partial k_{\alpha j}} &= \frac{\partial G_2}{\partial k_{\alpha j}} \Big/ \frac{\partial G_2}{\partial x_2}, \quad j = 1, 2, 3, 4, 5, \\ \frac{\partial F_2}{\partial c_3} &= \frac{\partial G_2}{\partial c_3} \Big/ \frac{\partial G_2}{\partial x_2}. \end{aligned} \tag{22}$$

Thus, the explicit expression of x_2 is obtained and the expression of x_1 can also be obtained by substituting Eq. (21) in G_1.

The robust stability of the perturbed system (15) are determined by the uncertain parameters $c_h, c_\alpha, k_h, k_{\alpha j}$ and c_3 and the uncertain equilibrium point x_{1e} and x_{2e} which are influenced by the nonvanishing perturbations. In term of Eq. (21), which simply represents the relation between uncertain parameters and equilibrium point as a polynomial, matrix A_Δ can be viewed as an uncertain matrix subjected to these uncertain parameters. Although the entrance of Eq. (21) into matrix A_Δ always produces nonlinearities of perturbations, polynomials in the perturbations can be removed by introducing fictitious input and output feedback signals that relate the nominal plant to the perturbations. This method was developed by Lind [15] and can express a system containing polynomial perturbation as an LFT form, which separates the uncertainty operator from the nominal plant. Then the robust flutter analysis can be performed by μ-analysis method.

5. Value set approach

The Taylor series expansion of Eq. (20), although produce an explicit expression, can not analytically reflect the functional relationship between equilibrium point and uncertain parameter. The second order Taylor series expansion produces more accurate results, but the second order as well as higher order Taylor series generate arduous algebraic manipulation when it is introduced to matrix A_Δ to replace x_{1e} and x_{2e}.

A value set approach is presented to perform robust stability analysis of the reduced system (16). The characteristic polynomial of matrix A_Δ is given by

$$P(p, s) = s^4 + r_3(p)s^3 + r_2(p)s^2 + r_1(p)s + r_0(p), \tag{23}$$

where p is the vector of uncertain parameter defined by $p = (c_h, c_\alpha, k_h, k_{\alpha j}, c_3)$, $j = 1, 2, 3, 4, 5$, and r_i, $i = 0, 1, 2, 3$ are the functions of the uncertain parameters. In terms of the interval descriptions of these uncertain parameters (10) and (11), p can be viewed as an interval vector and expressed as

$$[\mathbf{P}] = \begin{pmatrix} [c_h^-, c_h^+] \\ [c_\alpha^-, c_\alpha^+] \\ [k_h^-, k_h^+] \\ [k_{\alpha 1}^-, k_{\alpha 1}^+] \\ \vdots \\ [k_{\alpha 5}^-, k_{\alpha 5}^+] \\ [c_3^-, c_3^+] \end{pmatrix}, \quad p \in [\mathbf{P}]. \tag{24}$$

The question is whether the characteristic polynomial $P(p, s)$ is robustly stable in $[\mathbf{P}]$? Refs. [3,13] provide a stability theorem to analyze the stability characteristics of an interval polynomial with real or complex coefficients. However, this approach is only valid for polynomials of which the coefficients are independent interval variables. If $P(p, s)$ is considered as an interval polynomial in spite of the functional relationship between coefficients and uncertain parameters, the stability analysis of $P(p, s)$ will produce a conservative result by using Kharitonov's theorem, because $P(p, s)$ is a subset of that assumed interval polynomial. The edge theorem [4] has also been developed to deal with the stability of uncertain polynomial of which the coefficients depend affinely on the uncertain parameters. But the coefficients of polynomial $P(p,s)$ are nonlinear functions of the uncertain parameters.

A computational approach based on interval analysis [11] has been developed to analyze the stability of a generic uncertain polynomial and is shown as follow.

Theorem 1. *If*

1. *the coefficients of $P(p,s)$ are continuous functions of p,*
2. *the leading coefficient never vanishes over* $[\mathbf{P}]$,
3. *there exists p_0 in $[\mathbf{P}]$ such that $P(p_0,s)$ is stable,*

then $P([\mathbf{P}],s)$ is robust stable if and only if $P(p,j\omega)=0$, $p\in[\mathbf{P}]$, $\omega\in R$ has no solution.

Assume that the coefficients of $P(p,s)$ are continuous functions of p and the roots of the characteristic polynomial are continuous with respect to its coefficients. It can be concluded that the roots of $P(p,s)$ are continuous with respect to uncertain vector p. Assume also that $[\mathbf{P}]$ contains two vectors p_0 and p_1 such that $P(p_0,s)$ is stable and $P(p_1,s)$ is unstable. All the roots associated with p_0 have negative real part, while one of the roots of $P(p_1,s)$ has a positive real part at least. According to the continuity of the roots of the characteristic polynomial with respect to p, there exists a vector p_2 such that one of the roots of $P(p_2,s)$ is pure imaginary. The inexistence of $p\in[\mathbf{P}]$ and $\omega\in R$ satisfying $P(p,j\omega)=0$ leads to the robust stability of the characteristic polynomial. That is to say the value set of $P(p,j\omega)$ on complex plane does not contain the origin for $p\in[\mathbf{P}]$ and $\omega\in R$. Another formulation of Theorem 1 is the so-called zero-exclusion condition stated as below.

Theorem 2. *If the coefficients of the polynomial $P(p,s)$ are continuous functions of p and if there exists p_0 in $[\mathbf{P}]$ such that $P(p_0,s)$ is stable, then $P([\mathbf{P}],s)$ is robustly stable if and only if for any $\omega\geqslant 0$, $0\notin P([\mathbf{P}],j\omega)$.*

The stability analysis of the uncertain characteristic polynomial can then be performed by computing its value set for a set of values of ω. The domain of ω can be restricted to $\omega\in[0,+\infty]$, because ω and $-\omega$ produce conjugated value sets. Furthermore, ω can be restricted to an upper bound ω_c, such that the value set of $P(p,j\omega)$ for $p\in[\mathbf{P}]$ will never contain origin if ω is beyond it. The Cauchy theorem is presented to compute an upper bound of ω [11].

Theorem 3 *(Cauchy theorem). All the roots of the polynomial $P(s)=a_ns^n+\cdots+a_1s+a_0$, with $a_n\neq 0$ satisfy*

$$|s|<1+\frac{A}{|a_n|},\tag{25}$$

where A is defined as $A=\max\{|a_{n-1}|,|a_{n-2}|,\ldots,|a_0|\}$.

It can be seen in Ref. [11] that the Cauchy theorem is also used for polynomials with uncertain coefficients. Thus ω can be considered as an interval variable restricted in the interval $[0,\omega_c]$. Then $\omega\in[0,\omega_c]$ as well as the parameters restricted in $[\mathbf{P}]$ are used to compute the value sets of the characteristic polynomial for a set of values of the flow velocities.

For a certain flow velocity, if there exists p_0 in $[\mathbf{P}]$ such that the polynomial is stable/unstable and the origin is excluded from the value set for all frequencies $\omega\geqslant 0$, the polynomial is robustly stable/unstable with parameter p in $[\mathbf{P}]$ for this flow velocity. Thus by analyzing the motion and distribution of the value sets on a complex plane, the lower bound and upper bound of robust flutter speed can also be obtained. This is demonstrated with numerical results and discussed in next section.

6. Numerical results

This section presents the numerical results. The model parameters are given in Appendix A and $c_h, c_\alpha, k_h, k_{\alpha 1}, c_3$ are chosen to be the uncertain parameters. The initial angle of attack is set to 5 degrees and nominal flutter speed $V=13.92$ m/s is obtained. The computation of μ value is performed by Matlab μ-Toolbox [1].

Fig. 2 displays the μ values of the uncertain aeroelastic system with only vanishing perturbations for a set of values of flow velocities. V_{lower} and V_{upper} are used to denote the flutter speed of worst-case and best-case, respectively, and are obtained when the curve of μ value crosses the critical condition $\mu=1$ which is denoted by the dashed line. The flutter speeds of the entire family of plants (15) are bounded between V_{lower} and V_{upper}, so the real or experimental flutter speed lies within that bound if the model validation can be performed effectively. The system is robust stable when flow velocity V is less than V_{lower} where $\mu<1$ is achieved. While the system is robustly unstable when V is larger than V_{upper}. In this case, $\mu<1$ means that there is no uncertainty operator satisfying $\|\Delta\|_\infty\leqslant 1$ can shift any eigenvalue of the nominal system to the critical stability condition. When the flow velocity is larger than V_{lower} and less than V_{upper}, μ value is larger than 1 which means that there exist uncertainty operators that can destabilize the nominal system. Note that the μ value tends to infinity at nominal flutter speed. The nominal system needs no perturbation to reach the critical stability condition at nominal flutter speed, so the minimum singular value of the uncertainty operator that destabilize the nominal system is zero and thus a infinite μ value is produced. Ref. [7] discussed this topic in detail.

Fig. 3 exhibits the μ value of the uncertain aeroelastic system subjected to vanishing and nonvanishing perturbations at a set of values of airspeeds. In robust flutter analysis of case II, the existences of the nonvanishing perturbations as well as the uncertainty of equilibrium point make the lower bound V_{lower} more conservative, and correspondingly, the upper bound V_{upper} more optimistic. It can be seen that to develop an accurate uncertainty description is vital for robust flutter analysis. An unreasonable choice of uncertainty sources or an overly conservative uncertainty description which is excessive from the true model errors could result in failure of the prediction. Although the true flutter speed can be guaranteed to lie within the interval, it is difficult to localize it with enough accuracy. On the contrary, an overly optimistic uncertainty description which

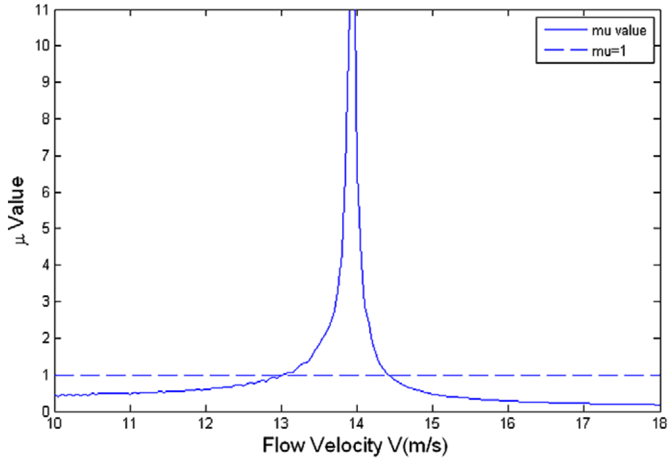

Fig. 2. V–μ graph of the uncertain aeroelastic system with only vanishing perturbations.

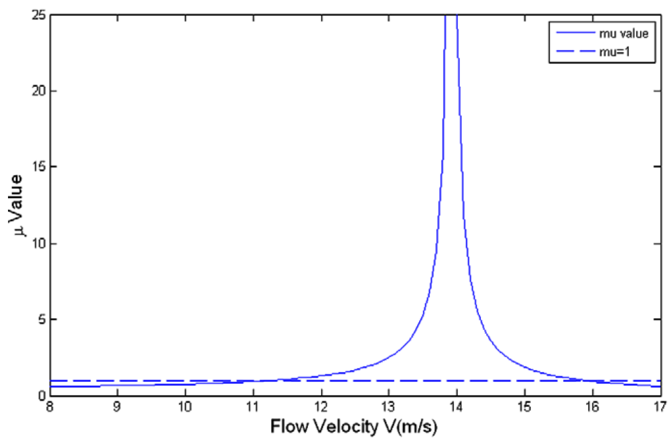

Fig. 3. V–μ graph of the uncertain aeroelastic system with vanishing and nonvanishing perturbations.

is deficient to account for the model errors makes the results unreliable, because the interval $[V_{\text{lower}}, V_{\text{upper}}]$ could exclude the true flutter speed. The results are highly dependent on uncertainty modeling and the robust flutter boundary interval should be produced as narrow as possible in order to localize the true flutter speed accurately.

As for the stability problem of the characteristic polynomial, several methods have been developed to compute the value sets. A straightforward method is to intensively discretize the parameter interval vector and compute the polynomial values of each grid point. In Ref. [11], a procedure called IMAGESp is used to compute the value set by interval computation. The interval vector $[\mathbf{P}]$ is discretized by subpaving into small boxes and the corresponding value boxes are computed and regularized to approximate the real value set. The two methods are computational expensive, but there is no easy trade-off between computational complexity and conservatism. Some methods use another strategy to solve this problem. Ref. [11] gives a method to determine the robust stability by estimating the existence of the solution of $P(j\omega, p) = 0$ in the interval vector $[\mathbf{P}]$. This is a constrained satisfactory problem (CSP) and can be solved efficiently using the interval analysis techniques. Moreover, it is observed that the coefficients of the characteristic polynomials are polynomic functions of the uncertain parameters and the feasible set $[\mathbf{P}]$ is a box. In this case, an approach based on Hurwitz stability criterion and signomial geometric programming has been developed to solve this specific problem [25]. But these two methods are not very visual and they do not compute the value sets directly. In this paper, the value sets are computed by intensive parameter griding.

Fig. 4 displays the value sets of the uncertain characteristic polynomial for a set of values of the flow velocities and discrete ω varying from 0 to 15. The value sets evaluated at $V = 10.50$ m/s are shown in Fig. 4(a-1). It is clear that the origin is excluded from the value set and a p_0 in $[\mathbf{P}]$ is easy to find such that $P(p_0, s)$ is stable. Furthermore, Mikhailov's criterion [12] can be simply used to examine the stability of a polynomial by analyzing the motion of the value sets on the complex plane. It states that the real polynomial $P(p, s)$ of degree n is Hurwitz if and only if the plot of $P(p, j\omega)$ with ω increasing from 0 to $+\infty$ starts on the positive real semi-axis and turns strictly counterclockwise around the origin through n quadrants. It is seen from Fig. 4(a-1) that, the value set starts on the positive real semi-axis and encircles the origin in a counterclockwise direction through an angle $n\pi/2$. The motion of the value sets demonstrates that the uncertain polynomial is robustly stable for p in $[\mathbf{P}]$. This can be seen more clearly in Fig. 4(a-2). The value sets for $V = 11.35$ m/s is shown in Fig. 4(b-1) and

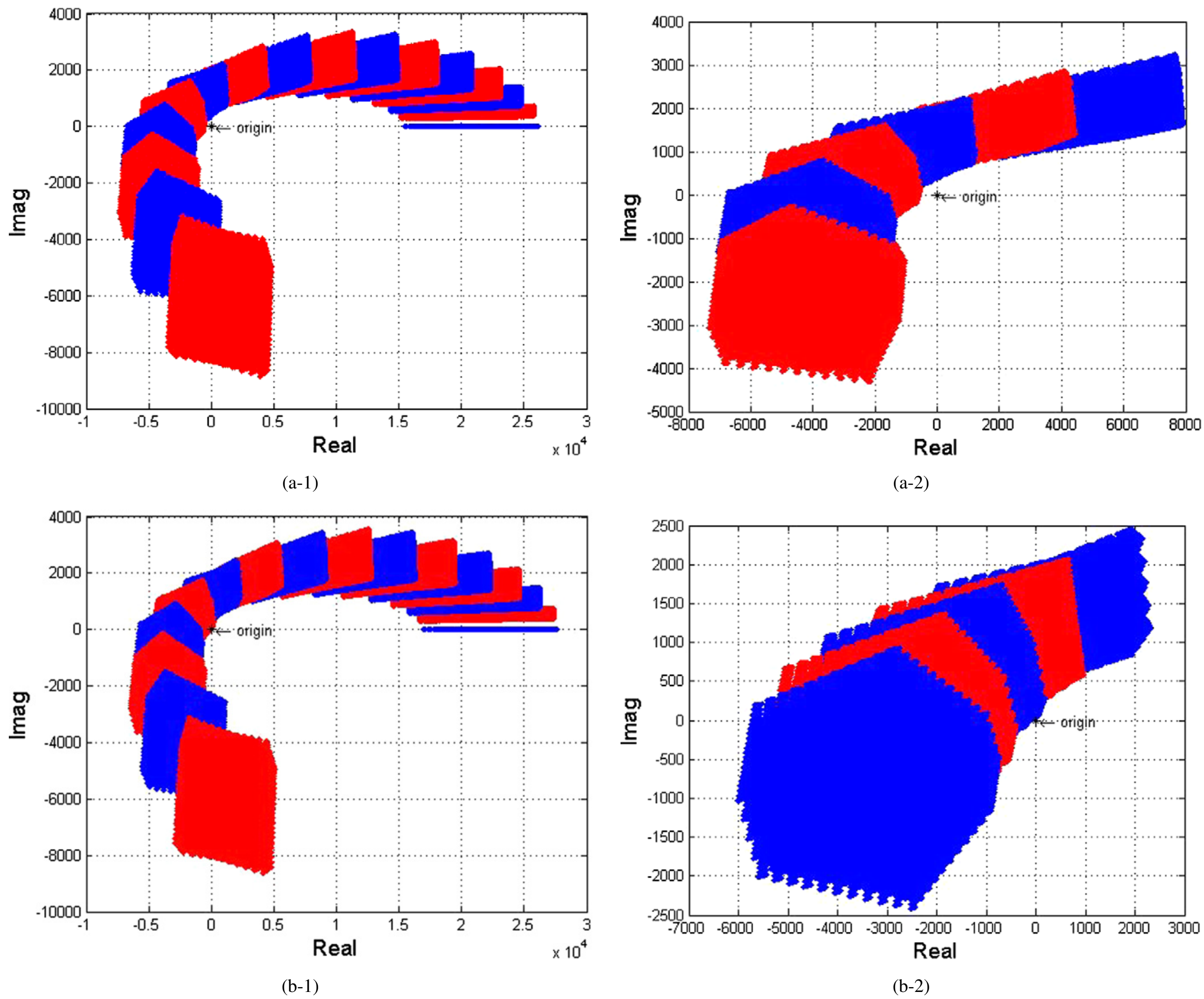

Fig. 4. Value sets of the uncertain characteristic polynomial. (a-1) $V = 10.50$ m/s, $\omega = 0, 1, \ldots, 15$. (a-2) $V = 10.50$ m/s, $\omega = 8, 9, \ldots, 13$. (b-1) $V = 11.35$ m/s, $\omega = 0, 1, \ldots, 15$. (b-2) $V = 11.35$ m/s, $\omega = 10, 10.5, 11, 11.5, 12$. (c) $V = 13.90$ m/s, $\omega = 0, 1, \ldots, 15$. (d-1) $V = 15.80$ m/s. $\omega = 0, 1, \ldots, 15$. (d-2) $V = 15.80$ m/s, $\omega = 12, 12.5, 13, 13.5, 14$. (e-1) $V = 16.50$ m/s, $\omega = 0, 1, \ldots, 15$. (e-2) $V = 16.50$ m/s, $\omega = 12, 12.5, 13, 13.5, 14$.

Fig. 4(b-2). The origin lies adjacent to the boundary of the value sets, so $V = 11.35$ m/s is the lower bound of the robust flutter speed. When the flow velocity increases to $V = 13.90$ m/s, the origin is included in the value sets which are exhibited in Fig. 4(c). In this case, the inclusion of the origin indicates that the polynomial is not robustly stable. Fig. 4(d-1) and Fig. 4(d-2) reveal the upper bound of the robust flutter speed. The value sets just border on the origin at $V = 15.80$ m/s. If the flow velocity continues to increase, the uncertain polynomial will be robustly unstable. As shown in Fig. 4(e-1) and Fig. 4(e-2), the origin is not contained in the value sets, but the motion of the value sets does not satisfy the Mikhailov's criterion. From another viewpoint, it is easy to find p_1 (e.g. the nominal value of p) such that $P(p_1, s)$ is unstable. So it can be concluded that the entire family of the characteristic polynomials is unstable.

Value set approach is also used for vanishing perturbation Case I and the results produced by the two methods are shown in Table 1. The first row shows the results computed by μ analysis method which are also indicated in Fig. 2 and Fig. 3. The curve of μ value crosses the critical condition $\mu = 1$ at two different flow velocities which indicate the two critical stability boundaries: the worst-case and the best-case flutter speed. For Case I, the robust flutter speed $V_{\text{lower}} = 13.02$ m/s corresponds to the first cross in Fig. 2, and with the airspeed increasing, the reduction of μ value induces the second cross which implies the upper bound of the stability boundary $V_{\text{upper}} = 14.41$ m/s. For Case II, the two critical speeds are also given. As it is expected, the worst-case flutter speed is depressed and the best-case is enhanced. The results obtained by value set approach are also shown in Table 1. The robust flutter speed $V_{\text{lower}} = 11.35$ m/s is determined when the origin is on the boundary of the value sets, which is shown in Fig. 4(b-1) and Fig. 4(b-2). If the airspeed continues to increase, $V_{\text{upper}} = 15.80$ m/s is obtained when the origin is excluded from the value sets again, which is shown in

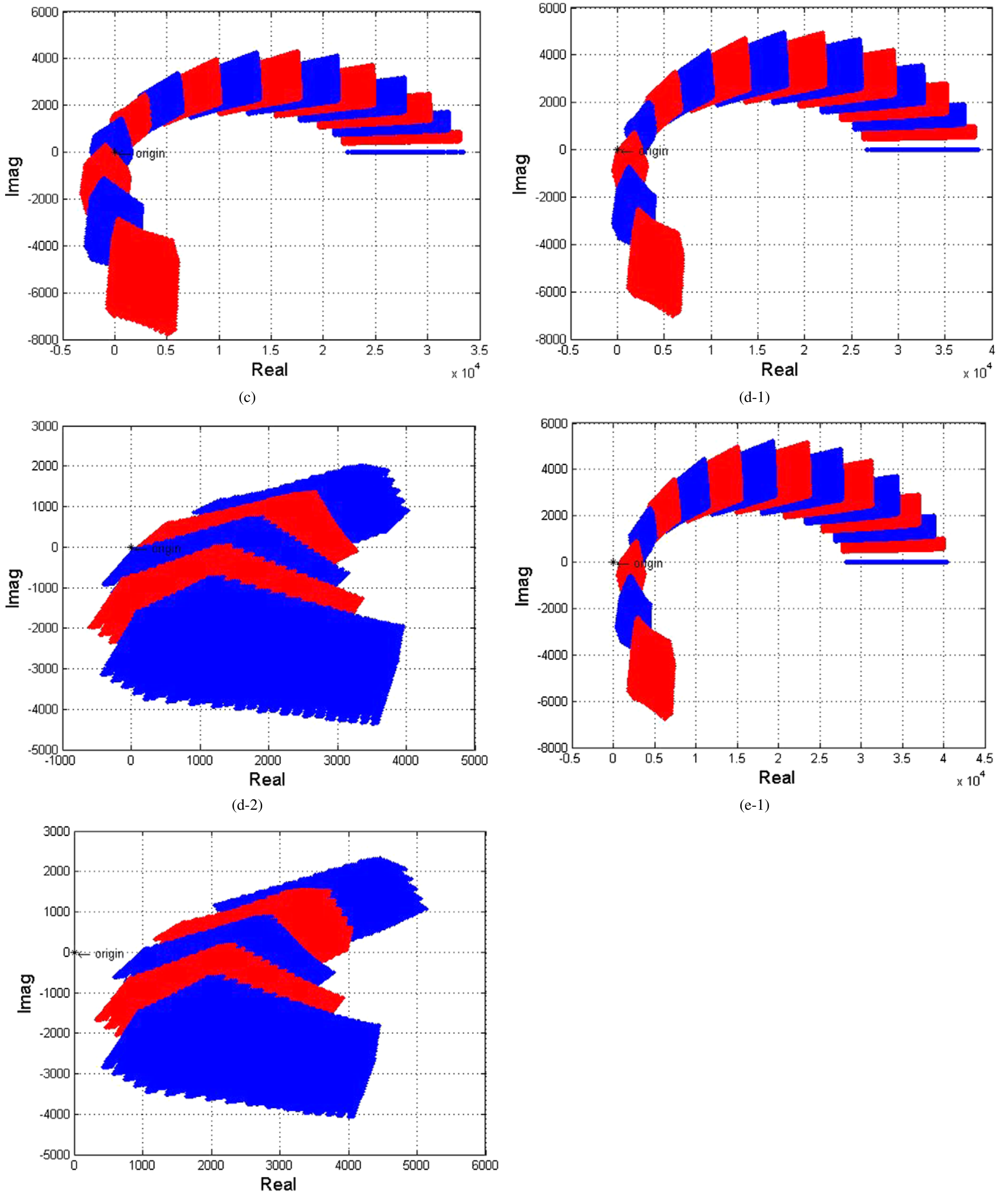

Fig. 4. (*continued*)

Table 1
Results of robust flutter analysis with μ-analysis method and value set approach

	Case I		Case II	
μ-analysis method	$V_{\text{lower}} = 13.02$ m/s	$V_{\text{upper}} = 14.41$ m/s	$V_{\text{lower}} = 11.27$ m/s	$V_{\text{upper}} = 15.85$ m/s
Value set approach	$V_{\text{lower}} = 13.02$ m/s	$V_{\text{upper}} = 14.41$ m/s	$V_{\text{lower}} = 11.35$ m/s	$V_{\text{upper}} = 15.80$ m/s
Nominal flutter speed	$V_{\text{flutter}} = 13.92$ m/s		$V_{\text{flutter}} = 13.92$ m/s	

Fig. 4(d-1) and Fig. 4(d-2). The nominal flutter speed is obtained by solving the eigenvalue problem of the state space equations. As it is shown in Fig. 2 and Fig. 3, the μ value is infinite under this flight condition.

From Table 1, it can be seen that value set approach produces a less conservative result for Case II than the μ-analysis method, because the equilibrium points are computed as accurate values in stead of Taylor series representation. As for Case I, the two methods generate the same results because the uncertainty of equilibrium point dose not exist. The results of value set approach are consistent with those produced by μ-analysis method, thus it is believed that value set approach is feasible to perform robust flutter analysis for the nonlinear aeroelastic system presented in this paper.

7. Conclusion

This paper deals with the robust flutter analysis of a nonlinear 2-D wing section with structural and aerodynamic uncertainty. The model presented contains nonlinear spring in pitch degree-of-freedom and the quasi-steady aerodynamics incorporated with a stall model. The parametric uncertainty is adopted to describe the uncertainties in structure and aerodynamics. Vanishing and nonvanishing perturbation are investigated respectively. For the system only with vanishing perturbations, the nonlinear system is linearized at equilibrium point and μ analysis is performed for a set of values of flow velocities to generate the lower and upper bounds of robust flutter speed which are associated with worst-case and best-case respectively. For the system with nonvanishing perturbations, the equilibrium point is viewed as a function of uncertain parameters and expanded in Taylor series. The influences of uncertain parameters on the location of equilibrium point are taken into consideration in μ-analysis framework. The value set approach is used to analyze the robust stability of a generic characteristic polynomial through its value sets evaluated with respect to $j\omega$ for different airspeeds. Stability characteristics of the uncertain system can be obtained from examining the motion and distribution of the value sets on complex plane. The numerical results demonstrate the validity of the developed methods.

The computation of the value sets costs a lot of time, therefore reducing the computation time is necessary for practical engineering application. The methods and algorithms presented in Refs. [11,25] are attractive and might be useful to improve the computation efficiency. The results are highly dependent on the uncertainty descriptions and especially sensitive to the magnitudes which are determined by model validation. Developing model validation methods for robust flutter analysis should be considered in future study and applications.

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Appendix A

These parameters are taken from [14,23,24] and the nominal values of the uncertain parameters are chosen to be the accurate values of the corresponding parameters in Ref. [24]. The weighting coefficients are determined artificially.

System parameters

$$m_t = 12.387 \text{ kg}, \qquad m_w = 2.049 \text{ kg}, \qquad b = 0.135 \text{ m},$$
$$x_\alpha = 0.2467, \qquad I_\alpha = 0.0540,$$
$$k_\alpha(\alpha) = k_{\alpha 1} + 3.272873\alpha + 276.612597\alpha^2 + 27.221849\alpha^3 - 2082.154122\alpha^4 \text{ N m/rad},$$
$$\rho = 1.225 \text{ kg/m}^3, \qquad a = -0.6,$$
$$c_{l\alpha} = 6.28, \qquad c_{m\alpha} = -0.628.$$

Uncertainty model

$$c_{h0} = 27.43 \text{ Ns/m}, \qquad Wc_h = 10.0;$$
$$c_{\alpha 0} = 0.036 \text{ Ns}, \qquad Wc_\alpha = 0.02;$$
$$k_{h0} = 2844.4 \text{ N/m}, \qquad Wk_h = 300.0;$$
$$k_{\alpha 10} = 2.861422 \text{ N m/rad}, \qquad Wk_{\alpha 1} = 1.0;$$
$$c_{30} = 10.255464, \qquad Wc_3 = 0.2c_{30}.$$

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