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Numerical Stablity of Path Tracing in Polyhedral Homotopy Continuation Methods

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Abstract.
The reliability of polyhedral homotopy continuation methods for solving a polynomial system becomes increasingly important as the dimension of the polynomial system increases. High powers of the homotopy continuation parameter t and ill-conditioned Jacobian matrices encountered in tracing of homotopy paths affect the numerical stability. We present modified homotopy functions with a new homotopy continuation parameter s and various scaling strategies to enhance the numerical stability. Advantages of employing the new homotopy parameter s are discussed. Numerical results are included to illustrate the improved performance of the presented techniques.

AMS subject classification.
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Polynomial system, polyhedral homotopy continuation methods, path tracing, numerical stability.

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1 Introduction

We consider a polynomial system $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$ where $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_n(\boldsymbol{x})) \in \mathbb{C}$, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ a variable vector in the $\mathbb{C}$ is the n-dimensional complex space $\mathbb{C}^n$. An example of a polynomial system can be shown as the following cyclic-n polynomial system [4].

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0, \\ x_1x_2 + x_2x_3 + \cdots + x_nx_1 &= 0, \\ &\vdots \\ x_1x_2 \dots x_{n-1} + x_2x_3 \dots x_n + \cdots + x_nx_1 \dots x_{n-2} &= 0, \\ x_1x_2 \dots x_n - 1 &= 0. \end{aligned}$$

Given a system of n polynomials $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_n(\boldsymbol{x}))$, the basic approach of homotopy continuation methods for solving $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$ is to define a *homotopy* system

$$\boldsymbol{h}(\boldsymbol{x}, t) = (h_1(\boldsymbol{x}, t), h_2(\boldsymbol{x}, t), \dots, h_n(\boldsymbol{x}, t)) = \mathbf{0}$$

with a continuation parameter $t \in [0, 1]$ using the algebraic structure of the polynomial system. The homotopy system $\boldsymbol{h}(\boldsymbol{x}, t) : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ is constructed so that

(a) all solutions of *the starting polynomial system* $\boldsymbol{h}(\boldsymbol{x}, 0) = \mathbf{0}$ are easily attainable,

(b) *the target polynomial system* $\boldsymbol{h}(\boldsymbol{x}, 1) = \mathbf{0}$ coincides with $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$,

(c) for all t in $[0, 1)$, the system $\boldsymbol{h}(\boldsymbol{x}, t) = \mathbf{0}$ has only nonsingular solutions.

Then, trace a solution curve of $\boldsymbol{h}(\boldsymbol{x}, t) = \mathbf{0}$ numerically in the space $\mathbb{C}^n \times [0, 1]$ starting from a known solution $\boldsymbol{x}^0$ of $\boldsymbol{h}(\boldsymbol{x}, 0) = \mathbf{0}$ with *the homotopy parameter* $t = 0$.

Let us write each component $f_k(\boldsymbol{x})$ of $\boldsymbol{f}(\boldsymbol{x})$ as

$$f_k(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}_k} c_k(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}},$$

where $c_k(\boldsymbol{a})$ denotes a nonzero complex number, $\mathcal{A}_k \subset \mathbb{Z}^n_+$ the support of $f_k(\boldsymbol{x})$ and $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for every variable vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$ and every n-dimensional nonnegative integer vector $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$. In this paper, we employ the homotopy system $\boldsymbol{h}(\boldsymbol{x}, t) : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ whose kth component is of the form

$$h_k(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \mathcal{A}_k} \left((1 - t)\tilde{c}_k(\boldsymbol{a}) + t c_k(\boldsymbol{a})\right) \; \boldsymbol{x}^{\boldsymbol{a}} t^{\rho_k(\boldsymbol{a})}. \tag{1}$$

Here $\tilde{c}_k(\boldsymbol{a}) \in \mathbb{C}$ and $0 \le \rho_k(\boldsymbol{a}) \in \mathbb{R}$ are given numbers ($\boldsymbol{a} \in \mathcal{A}_k$). For the cyclic n-polynomial system, the first equation of (1) is of the form

$$h_1(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \mathcal{A}_1} \left((1 - t)\tilde{c}_1(\boldsymbol{a}) + t)\right) \; x^{\boldsymbol{a}} t^{\rho_1(\boldsymbol{a})}.$$

where $\mathcal{A}_1$ is the set of the vectors $\boldsymbol{e}_i$ of one in ith position and zeros in the other positions, and $\tilde{c}_1(\boldsymbol{a}) \in \mathbb{C}$. If $\rho_k(\boldsymbol{a})$ is 0 in (1), then the homotopy system (1) represents a linear homotopy [1, 7]. Otherwise, it is called as a polyhedral homotopy [10, 15] when $\tilde{c}_k(\boldsymbol{a}) = c_k(\boldsymbol{a})$, and a polyhedral-linear homotopy [15] when $\tilde{c}_k(\boldsymbol{a})$ and $c_k(\boldsymbol{a})$ are chosen differently.

The polyhedral homotopy based on Bernshtein theory [2, 10, 15, 20, 23], which bounds the number of the solutions of $\boldsymbol{f}(\boldsymbol{x}) = 0$ by the mixed volume, provides much fewer homotopy paths to trace than the classical linear homotopy continuation method [1, 7, 14]. The mixed volume is known to give a tighter bound than Bézout bound for the number of solutions in $(\mathbb{C}\backslash\{\mathbf{0}\})^n$. Therefore, the polyhedral homotopy method provides computational advantage over the linear homotopy continuation method.

Main parts in implementation of polyhedral homotopy continuation methods are computation of the fine mixed cells of a given polynomial system [8, 16, 21] from which we construct a family of polyhedral (or polyhedral-linear) homotopy functions, and tracing the solution paths of the homotopy systems [9, 12, 13, 15, 24]. After the initial stage of constructing mixed cells, finding all solutions of polynomial systems using polyhedral homotopy continuation methods depends largely on tracing the solution paths of polyhedral homotopy systems. Many numerical challenges arise while tracing paths. Main sources of the challenges are increasingly high power $\rho_k(\boldsymbol{a})$ of the homotopy parameter t in the homotopy system (1), increasingly large number of homotopy paths as larger polynomial systems are solved, and distinguishing the types of solutions such as nonsingular and singular solutions, which means that the computed Jacobian is nonsingular and singular at the solution, respectively. As a result, implementing a procedure for tracing paths often encounters situations like failing to find a point on a homotopy path, unexpected jumps to a different path, difficulty of determining divergent paths, and finding a singular solution from numerical computation.

The term 'numerical stability of path tracing' is used to indicate tracing homotopy paths correctly to desired solutions to the end without failure in the middle of tracing. Numerical stability is a critical issue for overall performance of polyhedral homotopy continuation methods in view that successful finding of all solutions of a polynomial system depends on how efficiently and stably numerical methods used in path tracing perform.

Three important factors to determine the numerical stability are magnitudes of powers of the homotopy continuation parameter t, solving linear systems in predictor-corrector procedures of tracing solution paths, and determining convergent and divergent paths in the final stage of path tracing. The focus of this paper is on developing stable numerical methods to deal with numerical challenges from large magnitudes of powers of t and ill-conditioned linear systems. For details of the last factor, see the paper [11].

High powers of the continuation parameter t in polyhedral homotopy continuation methods have been an important issue to achieve numerical stability as described in [8]. When polyhedral homotopy functions contain very high powers of the continuation parameter t, values of the functions change very rapidly during path tracing, especially near the end of $t = 1$. Then, it becomes necessary to take very small steps to trace, resulting in numerical inefficiency. One way to handle the difficulty caused by large magnitudes of powers is to balance the powers by computing new lifting values for the supports of a polynomial system, which leads to balanced powers of the continuation parameter t [8, 9]. We can reduce powers to some extent by this approach. However, as the dimension of a polynomial system grows, magnitudes of powers of the continuation parameter become large. We encounter a similar situation of large magnitudes of powers in larger dimensional problems again.

Tracing paths in polyhedral homotopy continuation methods is implemented using predictor-corrector procedures, which involve solving linear systems with Jacobian matrices of a polyhedral homotopy system to obtain points in a solution path. Solving the linear systems accurately is central to achieve numerical efficiency and stability for path tracing. The accuracy of solutions of the linear systems may deteriorate by nearly singular Jacobian matrices. While tracing solutions paths of polynomial systems, ill-conditioned linear systems are often inevitable, for instance, when two solution paths come very closely or magnitudes of some coordinates of a point on a solution path is extremely large. As a result, we obtain inaccurate solution points and tracing the path becomes unsuccessful; it is impossible to reach the end of the path with $t = 1$.

The purpose of this paper is to provide stable numerical algorithms to trace solution paths in polyhedral homotopy methods. To resolve the issue arising from large magnitudes of powers of the continuation parameter t, we introduce a modified homotopy with a new continuation parameter s using a change of the parameter t, $s = \log t$. This modified homotopy enables us to trace solution paths more accurately within available machine precision. In particular, it provides a longer distance to travel than the one from $t = 0$ to $t = 1$. Therefore, we can trace solution paths more carefully in the sense that sudden changes in values of homotopy functions can be reduced by taking smaller step lengths and accidental jumps from one solution path to another can be prevented. Various scaling techniques to increase the accuracy of solutions of linear systems in predictor-corrector procedures are also presented.

Currently available software packages based on polyhedral homotopy continuation methods are PHCpack [25], CMPSm [12] and PHoM [9]. PHCpack has been one of the most successful polynomial system solvers by polyhedral homotopy continuation written in Ada language. The package offers various methods for computing fine mixed cells and several modes to operate. PHoM is a software package in C++ implementing polyhedral homotopy continuation methods from constructing a family of polyhedral-linear homotopy functions to tracing solution paths. CMPSm is a MATLAB code and CMPSc [13] a C++ program for tracing solution paths. CMPSm served as a prototype for CMPSc, which is integrated into PHoM. It is shown that PHoM can handle larger dimensional polynomial systems than PHCpack [9]. Numerical experiments in this paper were done using a revised version of CMPSm, which included the modified homotopy with the new continuation parameter s and scaling techniques discussed in the succeeding sections.

This paper is organized as follows: After discussing basic polyhedral-linear homotopy systems, we address numerical difficulties arising from tracing paths in implementation of polyhedral-linear homotopy methods in Section 2. These include some effects of high powers of the continuation parameter t and ill-conditioned Jacobian matrices during path tracing. In Section 3, we propose a modified polyhedral-linear homotopy using a nonlinear scaling $s = \log t$ of the continuation parameter t. The modified homotopy provides computational advantages for a predictor step length control in tracing paths. Section 4 contains scaling techniques for linear systems such as scalings based on function values, magnitudes of variables, and Jacobian matrices. Next, we present numerical results obtained from the modified polyhedral-linear homotopy and the scaling techniques in Section 5, and show some effectiveness of the new continuation parameter s and the scaling strategies. Finally, Section 6 is devoted to concluding discussions.

We introduce notation and symbols for the following discussions. Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_+$ denote the set of real numbers, the set of complex numbers and the set of nonnegative integers, respectively. For every variable vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ and every $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_+$, we use the notation $\boldsymbol{x}^{\boldsymbol{a}}$ for the term $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then we can write any polynomial $\phi(\boldsymbol{x})$ in the variable vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ as $\phi(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}} c(\boldsymbol{a}) \boldsymbol{x}^a$ for some finite subset $\mathcal{A}$ of $\mathbb{Z}^n_+$ and some $c(\boldsymbol{a}) \in \mathbb{C}$ ($\boldsymbol{a} \in \mathcal{A}$). We call $\mathcal{A}$ the *support* of the polynomial $\phi(\boldsymbol{x})$.

2 Some difficulties in polyhedral homotopy continuation methods

2.1 A polyhedral-linear homotopy system

We compute solutions of a system of n polynomial equations

$$\boldsymbol{f}(\boldsymbol{x}) \equiv (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))^T = \boldsymbol{0} \tag{2}$$

in an n-dimensional complex vector variable $\boldsymbol{x} \equiv (x_1, \ldots, x_n) \in \mathbb{C}^n$. Throughout the paper, we assume that each component $f_j(\boldsymbol{x})$ of $\boldsymbol{f}(\boldsymbol{x})$ is of the form

$$f_j(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \mathcal{A}_j} c_j(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}},$$

where $c_j(\boldsymbol{a})$ denotes a nonzero complex number and $\mathcal{A}_j \subset \mathbb{Z}^n_+$ the support of $f_j(\boldsymbol{x})$.

In homotopy continuation methods, we first define a smooth *homotopy* system with a *continuation parameter* $t \in [0, 1]$

$$\boldsymbol{h}(\boldsymbol{x}, t) \equiv (h_1(\boldsymbol{x}, t), h_2(\boldsymbol{x}, t), \ldots, h_n(\boldsymbol{x}, t))^T = \boldsymbol{0}$$

using the algebraic structure of the polynomial system, where

$$h_j(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \mathcal{A}_j} \left((1-t)\tilde{c}_j(\boldsymbol{a}) + t c_j(\boldsymbol{a})\right) \ \boldsymbol{x}^{\boldsymbol{a}} t^{\rho_j(\boldsymbol{a})} \quad (j = 1, \ldots, n).$$

Depending on the choices of $\rho_j(\boldsymbol{a})$ and $\tilde{c}_j(\boldsymbol{a})$, several variants of homotopy continuation methods can be derived as mentioned in Section 1. It has been shown that polyhedral-linear homotopy functions, which were proposed by [15] as the name of cheater's homotopy, are regarded as an efficient way to implement polyhedral homotopy continuation methods. For details, see [15].

The software packages CMPSm [12] and CMPSc [13] employ a family of polyhedral-linear homotopy functions $\underline{\boldsymbol{h}}^p : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ ($p = 1, 2, \ldots, \ell$). The number ℓ of the polyhedral-linear homotopy functions in the family corresponds to the number of fine mixed cells, which should be computed to construct the family of polyhedral-linear homotopy functions. The family satisfy the property that for each isolated solution $\boldsymbol{x}^1$ of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, there exist an index p and a solution $\boldsymbol{x}^0$ of $\underline{\boldsymbol{h}}^p(\boldsymbol{x}, 0) = \boldsymbol{0}$ such that $(\boldsymbol{x}^0, 0)$ is connected to $(\boldsymbol{x}^1, 1)$ through a homotopy path, a solution path $\{(\boldsymbol{\xi}(t), t) : t \in [0, 1)\}$ of $\underline{\boldsymbol{h}}^p(\boldsymbol{x}, t) = \boldsymbol{0}$ in the space $\mathbb{C}^n \times \mathbb{R}$. This property is essential to compute all isolated solutions of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$.

Each component $\underline{h}^p_j(\boldsymbol{x},t)$ of $\underline{\boldsymbol{h}}^p(\boldsymbol{x},t)$ is of the form

$$\begin{aligned}
\underline{h}^p_j(\boldsymbol{x},t) &= \sum_{\boldsymbol{a}\in\mathcal{A}_j} \left((1-t^{\beta/\gamma_p})\tilde{c}_j(\boldsymbol{a}) + t^{\beta/\gamma_p}c_j(\boldsymbol{a})\right)\boldsymbol{x}^{\boldsymbol{a}} t^{\rho^p_j(\boldsymbol{a})/\gamma_p} \\
&= \sum_{\boldsymbol{a}\in\mathcal{A}_j} \left(\tilde{c}_j(\boldsymbol{a})t^{\rho^p_j(\boldsymbol{a})/\gamma_p} + (c_j(\boldsymbol{a})-\tilde{c}_j(\boldsymbol{a}))t^{(\rho^p_j(\boldsymbol{a})+\beta)/\gamma_p}\right)\boldsymbol{x}^{\boldsymbol{a}}. \qquad (3)
\end{aligned}$$

$(p = 1,2,\ldots,\ell)$. Here $0^0 = 1$, and β and γ_p $(p = 1,2,\ldots,\ell)$ are positive parameters which we describe below; $\rho^p_j(\boldsymbol{a})$ $(\boldsymbol{a}\in\mathcal{A}_j,\ j=1,2,\ldots,n,\ p=1,2,\ldots,\ell)$ are nonnegative numbers obtained through computation of the fine mixed cells, and for each p and j, exactly two $\rho^p_j(\boldsymbol{a})$'s among $\rho^p_j(\boldsymbol{a})$ $(\boldsymbol{a}\in\mathcal{A}_j)$ are zero and all others are positive. Hence, for each $p=1,2,\ldots,\ell$, the initial system from which homotopy solution paths starts is a system of binomial equations

$$\sum_{\boldsymbol{a}\in\mathcal{A}^0_{pj}} \tilde{c}_j(\boldsymbol{a})\boldsymbol{x}^{\boldsymbol{a}} = 0 \ (j=1,2,\ldots,n),$$

where $\mathcal{A}^0_{pj} = \left\{\boldsymbol{a}\in\mathcal{A}_j : \rho^p_j(\boldsymbol{a}) = 0\right\}$, so that the starting points of the homotopy solution paths can be easily computed. We can also verify that $\underline{\boldsymbol{h}}^p(\boldsymbol{x},1) = \boldsymbol{f}(\boldsymbol{x})$ for all $\boldsymbol{x}\in\mathbb{C}^n$.

Let

$$\begin{aligned}
\rho^p_{\max} &= \max\left\{\rho^p_j(\boldsymbol{a}) : \boldsymbol{a}\in\mathcal{A}_j,\ j=1,2,\ldots,n\right\}\ (p=1,2,\ldots,\ell),\\
\rho_{\max} &= \max\left\{\rho^p_{\max} : p=1,2,\ldots,\ell\right\},\\
\rho^p_{\min} &= \min\left\{\rho^p_j(\boldsymbol{a}) : \boldsymbol{a}\in\mathcal{A}_j\backslash\mathcal{A}^0_{pj},\ j=1,2,\ldots,n\right\}\ (p=1,2,\ldots,\ell).
\end{aligned}$$

For computational purposes, we want to choose $\beta\geq 1$ and $\gamma_p\geq 1$ $(p=1,2,\ldots,\ell)$ such that, for each p, the minimum of positive powers

$$\rho^p_j(\boldsymbol{a})/\gamma_p \ \ (\boldsymbol{a}\in\mathcal{A}_j\backslash\mathcal{A}^0_{pj},\ j=1,2,\ldots,n),\ (\rho^p_j(\boldsymbol{a})+\beta)/\gamma_p\ (\boldsymbol{a}\in\mathcal{A}_j,\ j=1,2,\ldots,n)$$

of t of the polyhedral-linear homotopy function $\underline{\boldsymbol{h}}^p(\boldsymbol{x},0)=\boldsymbol{0}$ is normalized to 1, and that all the positive powers

$$\left.\begin{array}{l}
\rho^p_j(\boldsymbol{a})/\gamma_p \ \ (\boldsymbol{a}\in\mathcal{A}_j\backslash\mathcal{A}^0_{pj},\ j=1,2,\ldots,n,\ p=1,2,\ldots,\ell),\\
(\rho^p_j(\boldsymbol{a})+\beta)/\gamma_p\ (\boldsymbol{a}\in\mathcal{A}_j,\ j=1,2,\ldots,n,\ p=1,2,\ldots,\ell)
\end{array}\right\} \qquad (4)$$

become smaller. Given a $\beta>0$, we choose $\gamma_p=\min\{\rho^p_{\min},\ \beta\}$ to meet the first requirement. Through numerical experiments, we observed that the choice $\beta\in[0.01,\ 0.05]\times\rho_{\max}$ reduces the positive powers (4) considerably.

On the other hand, we can take a very large β such as $\beta=10^{10}$ or 10^{20} to construct polyhedral-linear homotopies with very large powers (4). We utilize such artificially highly nonlinear homotopies to test effectiveness and robustness of the proposed techniques.

2.2 Powers of the continuation parameter t

Each term in $\underline{h}^p(\boldsymbol{x},t)$ has a coefficient ct^ρ for some complex number c and nonnegative number ρ. The power ρ of each term is a function of $\rho^p_j(\boldsymbol{a})$ $(\boldsymbol{a}\in\mathcal{A}_j,\ j=1,2,\ldots,n,\ p=$

$1, 2, \ldots, \ell$) determined by the fine mixed cells and $\beta > 0$. Its magnitude can be very large when $\rho^p_{\min}$ is extremely small and/or $\rho^p_{\max}$ is extremely large; for example, when $\rho^p_{\min} = 0.01$ and $\rho^p_{\max} = 1,000$, $\rho = (\rho^p_{\max} + \beta)/\gamma_p \geq 100,000$ no matter how we choose $\beta > 0$ (recall that $\gamma_p = \min\{\rho^p_{\min},\ \beta\}$). Such huge powers may cause numerical inefficiency and affects stability of tracing homotopy paths of $\boldsymbol{h}^p(\boldsymbol{x}, t) = \boldsymbol{0}$. As a way to overcome the difficulty, balancing the quantities $\rho^p_j(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n,\ p = 1, 2, \ldots, \ell$) was proposed in [8]. This approach is based on computing new lifting values, which maintain the fine mixed cells, for the supports of the polynomial system. To balance the quantities $\rho^p_j(\boldsymbol{a})$ ($\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, \ldots, n,\ p = 1, \ldots, \ell$), we decrease the ratio of $\rho^p_{\max}$ and $\rho^p_{\min}$. However, the numerical difficulty remains for larger dimensional polynomials, even after balancing is performed.

2.3 Ill-conditioned Jacobian matrices

Another obstacle to have stable numerical algorithms for polyhedral homotopy continuation occurs when we solve a linear system

$$\boldsymbol{D}_{\boldsymbol{x}} \underline{\boldsymbol{h}}^p(\boldsymbol{x}, t) \boldsymbol{dx} = -\boldsymbol{D}_t \underline{\boldsymbol{h}}^p(\boldsymbol{x}, t) \quad \text{or} \quad -\underline{\boldsymbol{h}}^p(\boldsymbol{x}, t) \tag{5}$$

for $\boldsymbol{dx} \in \mathbb{C}^n$. Solving the linear system (5) is necessary when computing a predictor direction or a corrector direction, respectively. The Jacobian matrix $\boldsymbol{D}_{\boldsymbol{x}} \underline{\boldsymbol{h}}^p(\boldsymbol{x}, t)$ is likely to become more ill-conditioned if the current point $(\boldsymbol{x}, t)$ gets closer to two different homotopy paths or the magnitude of $\boldsymbol{x}$ grows larger. Very ill-conditioned Jacobian matrices appear in some of polynomial systems. For instance, tracing a homotopy path of the economic-14 polynomial system yielded the condition number of the Jacobian matrix larger than 10^{20}. In that case, a solution of the linear system (5) is not accurate and tracing solution path fails.

To improve accuracy of solutions obtained from (5), we utilize the singular value decomposition. Let $\boldsymbol{A}\boldsymbol{dy} = \boldsymbol{b}$ denote a target linear system to be solved, where $\boldsymbol{A}$ is an $n \times n$ complex matrix and $\boldsymbol{b} \in \mathbb{C}^n$. We obtain the singular value decomposition of $\boldsymbol{A} = \boldsymbol{USV}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are $n \times n$ orthogonal matrices, and $\boldsymbol{S}$ a diagonal matrices with nonnegative real diagonal entries. When the ratio of the minimum of nonnegative diagonal entries of $\boldsymbol{S}$ to the maximum is nearly zero (or zero), the matrix $\boldsymbol{A}$ is ill-conditioned (or singular). To cope with such a case, the diagonal matrix $\boldsymbol{S}$ is perturbed as

$$\boldsymbol{S} = \mathrm{diag}\ \left(S_{11}, \max\{S_{22}, \epsilon S_{11}\}, \ldots, \max\{S_{nn},\ \epsilon S_{11}\}\right).$$

Here ϵ denotes an extremely small positive number, say $\epsilon = 10^{-20}$, and $S_{ii} (i = 2, \ldots, n)$ denote the diagonal entries of $\boldsymbol{S}$ such that $S_{11} \geq S_{22} \geq \cdots \geq S_{nn} \geq 0$. Then, we compute the solution of $\boldsymbol{USV}\boldsymbol{dy} = \boldsymbol{b}$: $\boldsymbol{dy} = \boldsymbol{V}^* \boldsymbol{S}^{-1} \boldsymbol{U}^* \boldsymbol{b}$.

Unfortunately, this approach does not solve the ill-conditioning of the linear system (5) completely. We need a method to reduce the condition number of the Jacobian matrix.

3 A change of the continuation parameter t

3.1 A modified polyhedral-linear homotopy

We introduce a modified polyhedral-linear homotopy in this section using a change of the continuation parameter $t \in [0, 1]$ as $s = \log t$. The purpose of the modified polyhedral-linear homotopy is to resolve numerical difficulty originated from high powers of the continuation parameter t in the homotopy functions.

Let $p \in \{1, 2, \ldots, \ell\}$ be fixed. For simplicity of notation, we write the polyhedral-linear homotopy function $\underline{\boldsymbol{h}}^p(\boldsymbol{x}, t)$ in (3) as $\underline{\boldsymbol{h}}(\boldsymbol{x}, t)$ with no superscript p, and its component $\underline{h}_j^p(\boldsymbol{x}, t)$ as $\underline{h}_j(\boldsymbol{x}, t)$. Let

$$\varphi_j(\boldsymbol{a}) = \rho_j^p(\boldsymbol{a})/\gamma_p,\ \psi_j(\boldsymbol{a}) = (\rho_j^p(\boldsymbol{a}) + \beta)/\gamma_p \ \text{ and } \hat{c}_j(\boldsymbol{a}) = c_j(\boldsymbol{a}) - \tilde{c}_j(\boldsymbol{a})$$

$(\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n)$. Then, we have

$$\underline{h}_j(\boldsymbol{x}, t) = \sum_{\boldsymbol{a} \in \mathcal{A}_j} \left(\tilde{c}_j(\boldsymbol{a}) t^{\varphi_j(\boldsymbol{a})} + \hat{c}_j(\boldsymbol{a}) t^{\psi_j(\boldsymbol{a})}\right) \boldsymbol{x}^{\boldsymbol{a}}. \tag{6}$$

The homotopy (6) is general in the sense that it covers the classical linear [6, 7], polyhedral [10, 15, 23] and polyhedral-linear homotopies (cheater's homotopies) [15] as special cases. If we take $\varphi_j(\boldsymbol{a}) = 0$ and $\psi_j(\boldsymbol{a}) = 1$, we have a classical linear homotopy. With the choices of $\beta = 1$ and $\gamma_p = 1$, it represents a polyhedral-linear homotopy. And, $\hat{c}_j(\boldsymbol{a}) = 0$ in (6) results in a polyhedral homotopy. For more general homotopies with coefficient-parameter continuation, see [18].

We use a change of the parameter t in the interval $[0, 1]$ such that $s = \log t$ or $t = \exp(s)$ in $\underline{h}_j(\boldsymbol{x}, t)$. Then each term $t^{\varphi_j(\boldsymbol{a})}$ and $t^{\psi_j(\boldsymbol{a})}$ appeared in (6) are changed to $\exp(\varphi_j(\boldsymbol{a})s)$ and $\exp(\psi_j(\boldsymbol{a})s)$, respectively. The modified homotopy becomes

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{x}, s) &= (h_1(\boldsymbol{x}, s), h_2(\boldsymbol{x}, s), \ldots, h_n(\boldsymbol{x}, s))^T, \\ h_j(\boldsymbol{x}, s) &= \sum_{\boldsymbol{a} \in \mathcal{A}_j} (\tilde{c}_j(\boldsymbol{a}) \exp(\varphi_j(\boldsymbol{a})s) + \hat{c}_j(\boldsymbol{a}) \exp(\psi_j(\boldsymbol{a})s))\, \boldsymbol{x}^{\boldsymbol{a}}\ (j = 1, 2, \ldots, n). \end{aligned}$$

The initial parameter value 0 in the original continuation parameter t corresponds to $-\infty$ in the modified continuation parameter s in theory. We can take a sufficiently small negative number s^0 in practice such that

$$\exp(\varphi_j(\boldsymbol{a})s^0)\ (\boldsymbol{a} \in \mathcal{A}_j \backslash \mathcal{A}_j^0,\ j = 1, 2, \ldots, n) \ \text{ and } \ \exp(\psi_j(\boldsymbol{a})s^0)\ (\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n)$$

are all negligibly small positive numbers, and that we can employ

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{x}, s^0) &= (h_1(\boldsymbol{x}, s^0), h_2(\boldsymbol{x}, s^0), \ldots, h_n(\boldsymbol{x}, s^0))^T, \\ h_j(\boldsymbol{x}, s^0) &= \sum_{\boldsymbol{a} \in \mathcal{A}_j^0} \tilde{c}_j(\boldsymbol{a}) \boldsymbol{x}^{\boldsymbol{a}} \exp(\varphi_j(\boldsymbol{a})s^0)\ (j = 1, 2, \ldots, n) \end{aligned}$$

as a starting system of binomial equations; for example, take $s^0 = -20$ or -50 in the double precision arithmetic.

3.2 An advantage of the modified polyhedral-linear homotopy

We trace a solution path of the homotopy system of polynomial equations with the modified polyhedral-linear homotopy,

$$h_j(\boldsymbol{x}, s) \equiv \sum_{\boldsymbol{a} \in \mathcal{A}_j} (\tilde{c}_j(\boldsymbol{a}) \exp(\varphi_j(\boldsymbol{a})s) + \hat{c}_j(\boldsymbol{a}) \exp(\psi_j(\boldsymbol{a})s))\, \boldsymbol{x}^{\boldsymbol{a}} = 0\ (j = 1, 2, \ldots, n) \tag{7}$$

by applying a predictor-corrector procedure. Now, the continuation parameter s starts from $s = s^0 < 0$ and terminates at $s = 0$. As the continuation parameter $s < 0$ approaches 0, a smaller step length $ds > 0$ satisfying $s + ds \le 0$ is required. When a fixed finite precision arithmetic for numerical computation is used, the new parameterization $s \in (-\infty, 0]$ works effectively near 0. Namely, we can use an $s < 0$ very close to 0; for example, $s = -1.0 \times 10^{-200}$ in the double precision arithmetic. Furthermore, we can take a smaller positive step length than the magnitude of $s < 0$; when $s = -1.0 \times 10^{-200}$, ds can be 1.23×10^{-203} so that $s + ds = 0.99877 \times 10^{-200}$ is numerically meaningful in the double precision arithmetic.

For most of polynomial systems, such an extremely small step length ds may not be necessary. However, there exist some homotopy systems whose $\varphi_j(\boldsymbol{a})$ and $\psi_j(\boldsymbol{a})$ can be extremely large positive numbers. For instance, suppose that $\psi_j(\boldsymbol{a}) = 1.0 \times 10^{10}$. If we take $s = -1.0 \times 10^{-20}$ and $ds = 1.0 \times 10^{-22}$, then $\psi_j(\boldsymbol{a})s = -1.0 \times 10^{-10}$ and the values $t^{\psi_j(\boldsymbol{a})} = \exp(\psi_j(\boldsymbol{a})s)$ and $t_+^{\psi_j(\boldsymbol{a})} = \exp(\psi_j(\boldsymbol{a})(s + ds))$ in the t-space corresponding to s and $s_+ = s + ds$ in the s-space become $9.999999999000000 \times 10^{-1}$ and $9.999999998990000 \times 10^{-1}$, respectively. Thus, taking such small s and ds are necessary and effective in this case. It should be noted that neither t nor t_+ can not be numerically distinguishable from 1 in the double precision arithmetic. Hence the original polyhedral-linear homotopy system with the continuation parameter $t \in [0, 1]$ would encounter a difficulty in dealing with such a case effectively.

3.3 Bounding changes of the coefficients depending on the continuation parameter in predictor iterations

An additional important feature of the new parameterization using $s \in (-\infty, 0]$ is that it can provide an effective technique to bound changes of the numbers $\exp(\varphi_j(\boldsymbol{a})s)$ and $\exp(\psi_j(\boldsymbol{a})s)$ ($\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n$) in the coefficients, depending on the continuation parameter s in predictor iterations. Suppose that $(\bar{\boldsymbol{x}}, \bar{s}) \in \mathbb{C}^n \times (-\infty, 0)$ be the current iterate that lies approximately on a homotopy solution path;

$$h_j(\bar{\boldsymbol{x}}, \bar{s}) \ \equiv \ \sum_{\boldsymbol{a} \in \mathcal{A}_j} \left(\tilde{c}_j(\boldsymbol{a}) \exp(\varphi_j(\boldsymbol{a})\bar{s}) + \hat{c}_j(\boldsymbol{a}) \exp(\psi_j(\boldsymbol{a})\bar{s})\right) \bar{\boldsymbol{x}}^{\boldsymbol{a}} \approx 0 \ (j = 1, 2, \ldots, n).$$

Then, we compute a direction $(\boldsymbol{dx}, 1)$ by solving the linear system of equations

$$\boldsymbol{D_x h}(\bar{\boldsymbol{x}}, \bar{s})\boldsymbol{dx} + \boldsymbol{D}_s \boldsymbol{h}(\bar{\boldsymbol{x}}, \bar{s}) = \boldsymbol{0}.$$

Now we want to choose a step length $ds > 0$ that determines a predicted point $(\boldsymbol{x}_+, s_+) = (\bar{\boldsymbol{x}}, \bar{s}) + ds(\boldsymbol{dx}, 1)$. Note that s_+ needs to remain nonpositive. ¿From $(\boldsymbol{x}_+, s_+)$, we apply the corrector iteration to find a point on the homotopy solution path. In general, the predicted point $(\boldsymbol{x}_+, s_+)$ deviates more from the homotopy solution path as we take a larger $ds = s_+ - \bar{s}$. Although we want to take a larger step length $ds = s_+ - \bar{s}$ to reduce the number of predictor iterations, we need to choose a step length $ds = s_+ - \bar{s}$ so that the corrector procedure from $(\boldsymbol{x}_+, s_+)$ securely generates a sequence of $\{(\boldsymbol{x}^k, \bar{s})\}$ converging to a point on the same homotopy solution path. For this purpose, we derive below an upper bound for ds so that the changes

$$\exp(\varphi_j(\boldsymbol{a})s_+) - \exp(\varphi_j(\boldsymbol{a})\bar{s}),\ \exp(\psi_j(\boldsymbol{a})s_+) - \exp(\psi_j(\boldsymbol{a})\bar{s}) \ (\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n) \quad (8)$$

of the coefficients

$$\exp(\varphi_j(\boldsymbol{a})s),\ \exp(\psi_j(\boldsymbol{a})s)\ (\boldsymbol{a} \in \mathcal{A}_j,\ j = 1, 2, \ldots, n),$$

do not exceed a given fixed positive number κ. Notice that all the changes in (8) are nonnegative whenever $ds \geq 0$.

Let us consider the term $\exp(\rho s)$ and investigate how it changes from $s = \bar{s}$ to $s = \bar{s} + \lambda(-\bar{s}) = (1-\lambda)\bar{s}$, where $\rho \geq 1$ and $\lambda \in [0, 1)$. Then we observe that

$$0 \ \leq \ \exp(\rho\bar{s}(1-\lambda)) - \exp(\rho\bar{s}) \leq -\rho\bar{s}\lambda \exp(\rho\bar{s}(1-\lambda)).$$

We now regard $\zeta(\rho) \equiv -\rho\bar{s}\lambda \exp(\rho\bar{s}(1-\lambda))$ as a function of ρ, then

$$\begin{aligned}
\zeta'(\rho) &= -\bar{s}\lambda\left(1 + \rho\bar{s}(1-\lambda)\right)\exp\left(\rho\bar{s}(1-\lambda)\right), \\
\zeta'(\rho) &> 0 \text{ if } \rho < 1/\left(-\bar{s}(1-\lambda)\right), \\
\zeta'(\rho) &= 0 \text{ if } \rho = 1/\left(-\bar{s}(1-\lambda)\right), \\
\zeta'(\rho) &< 0 \text{ if } \rho > 1/\left(-\bar{s}(1-\lambda)\right).
\end{aligned}$$

Hence, $\zeta(\rho)$ attains the maximum $\lambda/\left((1-\lambda)\exp(1)\right)$ at the solution $\rho = 1/\left(-\bar{s}(1-\lambda)\right)$ over the set of nonnegative numbers, and for every $\rho \in [1, \rho_{\max}]$ and every $\lambda \in [0, 1)$,

$$\zeta(\rho) \ \leq \ \begin{cases} \lambda/\left((1-\lambda)\exp(1)\right) & \text{if } 1/(-\bar{s}) \leq \rho_{\max}, \\ -\rho_{\max}\bar{s}\lambda\exp(\rho_{\max}\bar{s}(1-\lambda)) \leq -\rho_{\max}\bar{s}\lambda & \text{if } \rho_{\max} \leq 1/(-\bar{s}) \end{cases}$$

Therefore, if $\bar{s} < 0$ and $\lambda \in [0, 1)$ satisfy the set of inequalities

$$1 \leq -\rho_{\max}\bar{s} \ \text{ and } \lambda/\left((1-\lambda)\exp(1)\right) \leq \kappa \ \left(\textit{i.e. } \lambda \leq \kappa\exp(1)/\left(1 + \kappa\exp(1)\right)\right)$$

or the set of inequalities

$$-\rho_{\max}\bar{s} \leq 1 \ \text{ and } \lambda \leq \kappa/(-\rho_{\max}\bar{s}),$$

then

$$0 \ \leq \ \exp(\rho\bar{s}(1-\lambda)) - \exp(\rho\bar{s}) \leq \zeta(\rho) \leq \kappa$$

holds independent of $\rho \geq 1$.

4 Scaling strategies for linear systems

In this section, we address numerical aspects of solving linear systems. As discussed in Section 2, we encounter ill-conditioned linear systems frequently in predictor-corrector procedures. To improve the numerical stability of path tracing, we present three scaling techniques; a scaling based on function values, a scaling based on magnitudes of variables and a scaling based on Jacobian matrices.

Two types of scalings called variable scaling and equation scaling were described in [17]. In essence, they scale using the coefficients and powers of a given polynomial system before starting to solve the polynomial system. As a result, they would not be dependent on homotopy functions nor homotopy paths if they were used in the implementation of polyhedral homotopy continuation methods.

The aim of scalings presented here is to minimize numerical instability that might occur while tracing homotopy functions. As we have seen in many numerical experiments, homotopy paths can vary very much. One homotopy path needs scalings at some points, even if the path can be traced stably at other points without any scaling. This is why we need scaling techniques dependent on homotopy functions and each point of paths. In view of varying nature of the scalings on points from $s = -\infty$ to $s = 0$, we call the scalings in this paper dynamic scalings, as opposed to static nature of the scalings discussed in [17]. These dynamic scalings are used to improve ill-conditioned linear systems. The first scaling, a scaling based on function values, is also used in a stopping criterion for corrector iterations. The effectiveness of these scalings are shown with numerical results in Section 5. Throughout this section, let $(\tilde{\boldsymbol{x}}, \tilde{s}) \in \mathbb{C}^n \times (-s^0, 0]$ be a fixed point at which we perform the scalings.

4.1 A scaling based on function values

We investigate how much accuracy we require to stop iterations in our corrector procedure, and derive a way to scale the homotopy function in this subsection. Recall that each component of the value of the homotopy function $\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})$ is of the form

$$h_j(\tilde{\boldsymbol{x}}, \tilde{s}) \equiv \sum_{\boldsymbol{a} \in \mathcal{A}_j} \left(\tilde{c}_j(\boldsymbol{a}) \exp(\varphi_j(\boldsymbol{a})\tilde{s}) + \hat{c}_j(\boldsymbol{a}) \exp(\psi_j(\boldsymbol{a})\tilde{s})\right) \tilde{\boldsymbol{x}}^{\boldsymbol{a}} \ (j = 1, 2, \ldots, n).$$

For every j, let σ_j^f denote the maximum of the absolute values of all the terms appeared in the sum:

$$\sigma_j^f = \max_{\boldsymbol{a} \in \mathcal{A}_j} \left\{ \left| \left(\tilde{c}_j(\boldsymbol{a}) \exp(\varphi_j(\boldsymbol{a})\tilde{s}) + \hat{c}_j(\boldsymbol{a}) \exp(\psi_j(\boldsymbol{a})\tilde{s})\right) \tilde{\boldsymbol{x}}^{\boldsymbol{a}} \right| \right\}.$$

Roundoff errors in the evaluation of $h_j(\tilde{\boldsymbol{x}}, \tilde{s})$ are expected to be proportional to σ_j^f. Given a sufficiently small positive ϵ, it is reasonable to require for an approximate solution $(\tilde{\boldsymbol{x}}, \tilde{s})$ of $\boldsymbol{h}(\boldsymbol{x}, s) = \boldsymbol{0}$ to satisfy

$$h_j(\tilde{\boldsymbol{x}}, \tilde{s}) / \max\left\{\sigma_j^f,\ 1\right\} \leq \epsilon \ (j = 1, 2, \ldots, n) \quad \text{or} \quad \left\| \boldsymbol{\Sigma}^f \boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s}) \right\|_\infty \leq \epsilon,$$

where

$$\boldsymbol{\Sigma}^f = \mathrm{diag}\ \left(1/\max\{\sigma_1^f,\ 1\}, 1/\max\{\sigma_2^f,\ 1\}, \ldots, 1/\max\{\sigma_n^f,\ 1\}\right).$$

We use the diagonal matrix $\boldsymbol{\Sigma}^f$ when evaluating the homotopy function $\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})$.

4.2 A scaling based on magnitudes of variables

Define the diagonal matrix

$$\boldsymbol{\Sigma}^v = \mathrm{diag}\ \left(\max\left\{|\tilde{x}_1|,\ 1\right\}, \max\left\{|\tilde{x}_2|,\ 1\right\}, \ldots, \max\left\{|\tilde{x}_n|,\ 1\right\}\right),$$

and consider the linear scaling $\boldsymbol{x} \in \mathbb{C}^n \to \boldsymbol{y} \in \mathbb{C}^n$ such that

$$\boldsymbol{x} = \boldsymbol{\Sigma}^v \boldsymbol{y} \ \text{ or } \boldsymbol{y} = (\boldsymbol{\Sigma}^v)^{-1}\boldsymbol{x}.$$

Then,

$$\boldsymbol{D}_{\boldsymbol{y}}\boldsymbol{h}(\boldsymbol{\Sigma}^v\boldsymbol{y}, \tilde{s}) \;=\; \boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\boldsymbol{\Sigma}^v\boldsymbol{y}, \tilde{s})\boldsymbol{\Sigma}^v = \boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\boldsymbol{x}, \tilde{s})\boldsymbol{\Sigma}^v.$$

Thus the linear scaling $\boldsymbol{x} \in \mathbb{C}^n \to (\boldsymbol{\Sigma}^v)^{-1}\boldsymbol{y} \in \mathbb{C}^n$ induces a column scaling with the diagonal matrix $\boldsymbol{\Sigma}^v$ for the Jacobian matrix of $\boldsymbol{h}(\boldsymbol{x}, \tilde{s})$ with respect to $\boldsymbol{x}$.

4.3 Scaling based on Jacobian matrices

The Newton system for a corrector direction $\boldsymbol{dx}$ is of the form

$$\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\boldsymbol{dx} = -\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s}). \tag{9}$$

We first apply the row scaling $\boldsymbol{\Sigma}^f$ and the column scaling $\boldsymbol{\Sigma}^v$ to the system as follows:

$$\boldsymbol{\Sigma}^f\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\boldsymbol{\Sigma}^v\boldsymbol{dy} = -\boldsymbol{\Sigma}^f\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s}),\ \boldsymbol{dx} = \boldsymbol{\Sigma}^v\boldsymbol{dy}. \tag{10}$$

For each $j = 1, 2, \ldots, n$, let

$$\begin{aligned}\sigma_j^r &= \text{the maximum of the absolute values of all the } n \text{ terms in the } j\text{th row of} \\ &\quad\ \text{the coefficient matrix } \boldsymbol{\Sigma}^f\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\boldsymbol{\Sigma}^v, \\ \sigma_j^r &= \max\{\sigma_j^r\ 1\}.\end{aligned}$$

We further apply another row scaling matrix

$$\boldsymbol{\Sigma}^r \;=\; \text{diag } \left(1/\sigma_1^r, 1/\sigma_2^r, \ldots, 1/\sigma_n^r\right).$$

to (10). Finally, we obtain the scaled Newton system

$$\boldsymbol{\Sigma}^r\boldsymbol{\Sigma}^f\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\boldsymbol{\Sigma}^v\boldsymbol{dy} = -\boldsymbol{\Sigma}^r\boldsymbol{\Sigma}^f\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s}),\ \boldsymbol{dx} = \boldsymbol{\Sigma}^v\boldsymbol{dy}. \tag{11}$$

As a stopping criterion for corrector iterations, the following is used.

$$\left\|\boldsymbol{\Sigma}^f\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\right\|_\infty \le \epsilon^h \ \text{ and } \|\boldsymbol{dx}\| \le \epsilon^x, \tag{12}$$

where $\epsilon^h > 0$ and $\epsilon^x > 0$.

Now, we focus our attention to the Newton system for the predictor direction $(\boldsymbol{dx}, 1)$

$$\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\boldsymbol{dx} = -\boldsymbol{D}_s\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s}). \tag{13}$$

We apply the same scalings as the ones in the predictor procedure:

$$\boldsymbol{\Sigma}^r\boldsymbol{\Sigma}^f\boldsymbol{D}_{\boldsymbol{x}}\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s})\boldsymbol{\Sigma}^v\boldsymbol{dy} = -\boldsymbol{\Sigma}^r\boldsymbol{\Sigma}^f\boldsymbol{D}_s\boldsymbol{h}(\tilde{\boldsymbol{x}}, \tilde{s}),\ \boldsymbol{dx} = \boldsymbol{\Sigma}^v\boldsymbol{dy}. \tag{14}$$

5 Numerical experiments

We show numerical results obtained from numerical experiments focused on high powers of the continuation parameter t and ill-conditioned Jacobian matrices. All the tests were done using a revised version of CMPSm [12], which is a MATLAB program for tracing solution paths of the polyhedral-linear homotopies $\underline{\boldsymbol{h}}^p(\boldsymbol{x}, t)$ given in (3). The original CMPSm served as a prototype for CMPSc [12], a C++ program for tracing solution paths of the same homotopies, and PHoM [9], a C++ program package, including CMPSc, for solving polynomial systems.

For numerical tests on very high powers of the continuation parameter t, we used problems as reimer-4 [22], noon-5 [19], katsura-7 [3] and economic-14 [17] polynomials. The original polyhedral-linear homotopies (3) with $\beta = 1$ and $\gamma_p = 1$ for these polynomials contain some powers of t ranging up to the order of magnitude 10^5, which are large enough to cause possible numerical difficulties when implementing polyhedral-linear homotopy continuation method in a naive way. We expect to have much larger magnitudes of powers of t, as the dimension n of the polynomials increase. To test the effects of extremely high powers of t, we create highly nonlinear homotopy using (3). Specifically, huge powers are created artificially without applying any balancing technique and/or by choosing a large β such as $\beta = 10^{10},\ 10^{20}$ in the modified homotopy (3); the original CMPSm can not correctly work on those resulting homotopies.

We have noticed from various numerical experiments that very ill-conditioned Jacobian matrices occur at a point $\boldsymbol{x}$ with a huge magnitude, *e.g.*, $\|\boldsymbol{x}\| > 10^{10}$. To observe the effectiveness of the scaling techniques, we similarly made the magnitude of powers of t very large by changing the values of β in (3).

In the following tables, we use the notation in Table 1. We compute the average and maximum of several quantities obtained while tracing all the paths used for tests. More precisely, tracing one path provides the information such as the maximum power of t, the number of predictor iterations, the last value of the continuation parameter s used before reaching $s = 0$, $\|\boldsymbol{f}(\hat{\boldsymbol{x}})\|_\infty$ at a solution $\hat{\boldsymbol{x}}$, and the minimum singular value of $\boldsymbol{Df}(\hat{\boldsymbol{x}})$ at $\hat{\boldsymbol{x}}$. After tracing all the paths, we find the average and the maximum of these values of all the paths.

5.1 Tracing with increased powers of t

We summarize numerical results for increased powers of t in Tables 2, 3, 4 and 5. Each table contains increased powers of t up to 10^{20}. The first rows of Table 2 to 5 show the results from paths without artificial increase of powers of t and from the homotopies with the variable change to s. We mention that all the paths were traced correctly to the same solutions for the three different cases. In particular, we could trace the paths in the row Max.power $= 10^{20}$ of all the tables, and arrived safely at the end of path tracing. Even with the increased powers of t, we notice that the growth in the numbers of predictor and corrector iterations was not very large as indicated in the second and third columns of the tables. The fourth column of the tables shows the last point of the new continuation parameter s that was used in the last corrector iteration before reaching $s = 0$. The average and the maximum values of s for increased powers of t shown in the second and third rows are very near to 0, (*e.g.*, -8.72×10^{-30} in the last row of Table 5) which means that tracing

Av.max.power	the average of maximum powers of t of each path.
Max.max.power	the maximum of powers of t of each path.
Conv.div	convergence to nonsingular or singular solutions, or divergence.
Av.pred.it	the average number of predictor iterations per path.
Av.cor.it	the average number of corrector iterations per path.
Max.pred.it	the maximum number of predictor iterations per path.
Max.cor.it	the maximum number of corrector iterations per path.
Av.last.s	the average of the last points of $s < 0$ of all the paths.
Max.last.s	the maximum of the last points of $s < 0$ of all the paths.
Av.$\|\boldsymbol{f}(\hat{\boldsymbol{x}})\|_\infty$	the average of $\|\boldsymbol{f}(\hat{\boldsymbol{x}})\|_\infty$'s at solution $\hat{\boldsymbol{x}}$'s.
Max.$\|\boldsymbol{f}(\hat{\boldsymbol{x}})\|_\infty$	the maximum of $\|\boldsymbol{f}(\hat{\boldsymbol{x}})\|_\infty$'s at solution $\hat{\boldsymbol{x}}$'s.
Av.min.σ	the average of the minimum singular values of $D\boldsymbol{f}(\hat{\boldsymbol{x}})$'s
Max.min.σ	the maximum of the minimum singular values of $D\boldsymbol{f}(\hat{\boldsymbol{x}})$'s
Av.max.$K(D\boldsymbol{f})$	the average of the maximum condition numbers of $D\boldsymbol{f}(\hat{\boldsymbol{x}})$'s
Max.max.$K(D\boldsymbol{f})$	the maximum of the maximum condition numbers of $D\boldsymbol{f}(\hat{\boldsymbol{x}})$'s

Table 1: Notation

Av.power (Max.power)	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)
2.22e1 (2.22e1)	77.2 (126)	159.5 (276)	-6.92e-7 (-9.57e-16)	1.16e-15 (2.12e-15)
1.00e10 (1.00e10)	127.2 (183)	203.6 (344)	-1.10e-15 (-2.32e-24)	2.33e-15 (1.79e-14)
1.00e20 (1.00e20)	158.5 (211)	200.3 (321)	-1.12e-25 (-2.04e-34)	1.23e-15 (3.20e-15)

Table 2: The reimer-4 polynomial: 36 paths among 120 paths converged to nonsingular solutions.

path can continue to very close to the terminal value 0 in the continuation parameter s. This is an advantage of tracing with the parameter s because we have not been able to trace with the original parameter t such close to the terminal value 1.

5.2 Obtaining singular solutions

We tested whether the modified polyhedral-linear homotopy with the continuation parameter s would be effective to find singular solutions. The test problems included cyclic-8,

Av.power (Max.power)	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)
5.53e2 (1.36e3)	85.6 (154)	168.3 (329)	-1.30e-8 (-3.91e-10)	1.88e-15 (3.97e-15)
2.01e9 (1.00e10)	121.6 (208)	205.5 (417)	-1.13e-15 (-3.60e-18)	1.89e-15 (4.37e-15)
2.01e19 (1.00e20)	154.1 (237)	200.9 (408)	-1.19e-25 (-3.54e-28)	1.81e-15 (5.18e-15)

Table 3: The noon-5 polynomial: 233 paths converged to nonsingular solutions.

Av.power (Max.power)	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)
1.03e2 (1.15e2)	81.9 (142)	168.3 (295)	-2.65e-8 (-4.13e-9)	5.39e-16 (1.05e-15)
1.51e9 (1.00e10)	161.6 (256)	297.0 (510)	-1.11e-16 (-3.26e-18)	9.25e-16 (1.88e-15)
1.51e19 (1.00e20)	194.7 (291)	289.8 (496)	-1.20e-26 (-3.86e-28)	9.31e-16 (2.05e-15)

Table 4: The katsura-7 polynomial: 128 paths converged to nonsingular solutions.

Av.power (Max.power)	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)
3.99e4 (2.18e5)	155.8 (363)	328.5 (837)	-4.10e-11 (-9.68e-17)	2.73e-11 (1.03e-8)
2.66e9 (1.00e10)	216.1 (406)	431.1 (859)	-1.34e-16 (-8.13e-21)	2.47e-11 (6.49e-9)
1.91e18 (1.00e20)	244.7 (424)	425.6 (861)	-3.07e-25 (-8.72e-30)	2.04e-11 (4.12e-9)

Table 5: The economic-14 polynomial: 4096 paths converged to nonsingular solutions.

cyclic-9 and cyclic-13 polynomials that are known to have singular solutions. Tables 6, 7 and 8 show summarized results for cyclic-8, cyclic-9 and cyclic-13, respectively. The rows of the tables contain information for nonsingular, singular solutions and divergence.

The mixed volumes of them are 2,560, 11,016 and 2,704,156, which amount to the total numbers of homotopy paths, respectively. Using a symmetric structure [5] of the cyclic polynomials, we only have to trace $320 = 2560/8$, $1,224 = 11016/9$ and $208,012 = 2,704,156/13$ homotopy paths, respectively, to approximate all of their isolated solutions.

It is known that the cyclic-8 polynomial has 1152 isolated nonsingular solutions. Hence, taking account of the symmetry, $144 = 1,152/8$ homotopy paths among $320 = 2560/8$ converge to isolated nonsingular solutions, which coincides with the numerical experiments shown in Table 6. The cyclic-8 problem also has solution components with a positive dimension. The singular solutions obtained in Table 6 are points on those solution components.

The cyclic-9 polynomial has $5,994$ isolated nonsingular solutions, and 162 isolated singular solutions with multiplicity 4. In Table 7, $666 = 5,994/9$ paths correctly reach to nonsingular solutions, and $72 = (162/9) \times 4$ paths correctly attain isolated singular solutions with multiplicity 4.

Even if we utilize the symmetric structure, the number of homotopy paths to be traced for computing all isolated solutions of the cyclic-13 polynomial is $208,012$. This is too many for CMPSm to handle on a single machine, so that we chose 1618 paths from 1000 mixed cells using the symmetric structure for the numerical experiments shown in Table 8.

As shown in the last columns of Tables 6, 7 and 8, singular solutions have much smaller minimum singular values of Jacobian matrices than nonsingular solutions. (*e.g.*, 7.68×10^{-1} and 1.55×10^{-7}, respectively in Table 6). In addition, the values of the continuation parameter s from which s proceeded to its terminal value $s = 0$ in the fourth columns of the tables are very small in the cases of singular solutions, compared with the cases of nonsingular solutions and divergent paths. We note that singular solutions achieve a relatively good accuracy in terms of function values, indicated under the column of $\|f(\hat{\boldsymbol{x}})\|_\infty$. We observe that improved numerical stability through the new continuation parameter s

Conv.div	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)	Av.min.σ (Max.min.σ)
Nonsingular solutions	104.6 (170)	227.2 (370)	-3.60e-8 (-5.06e-9)	8.52e-15 (8.47e-14)	7.68e-1 (2.19e)
Singular solutions	117.1 (172)	296.5 (452)	-4.61e-16 (-1.27e-16)	4.52e-10 (5.04e-8)	1.55e-7 (5.10e-7)
Divergent paths	125.0 (199)	281.0 (437)	-1.36e-6 (-6.07e-7)	- -	- -

Table 6: The cyclic-8 polynomial: Total 320 paths traced with the average and the maximum powers of t 1.20e2 and 3.32e2.

Conv.div	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)	Av.min.σ (Max.min.σ)
Nonsingular solutions	125.1 (655)	263.4 (1189)	-9.42e-11 (-1.25e-11)	1.60e-14 (1.68e-13)	8.62e-1 (1.56e0)
Singular solutions	122.3 (199)	353.9 (500)	-1.11e-21 (-2.45e-22)	3.37e-13 (8.40e-13)	4.53e-7 (7.81e-7)
Divergent paths	187.5 (787)	448.6 (1503)	-2.18e-9 (-2.07e-14)	- -	- -

Table 7: The cyclic-9 polynomial: total 1224 paths traced with the average of the maximum powers (the maximum of the maximum powers) $4.86e4$ $(1.30e5)$ of all paths.

makes tracing path possible to extremely small values of s and enables us to compute singular solutions with higher accuracy.

5.3 Computing with ill-conditioned Jacobian matrices

Effectiveness of the scaling techniques described in Section 4 were tested numerically. The cyclic-13 and economic-14 polynomials were chosen as test problems because they often yield ill-conditioned Jacobian matrices while tracing homotopy paths. For each numerically traced homotopy path $\boldsymbol{\Xi}$ in Table 5 for the economic-14 polynomial and Table 8 for the

Conv.div	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)	Av.min.σ (Max.min.σ)
Nonsingular solutions	222.0 (1516)	427.8 (2997)	-1.51e-15 (-3.09e-16)	3.90e-14 (1.83e-12)	5.64e-1 (1.87e0)
Singular solutions	258.4 (621)	678.8 (1411)	-1.39e-21 (-9.62e-22)	2.12e-13 (2.60e-13)	1.92e-8 (2.57e-8)

Table 8: The cyclic-13 polynomial: total 1618 paths traced with the average of the maximum powers (the maximum of the maxium powers) $5.32e4$ $(1.85e5)$ of all paths.

cyclic-13 polynomial, we computed

$$\text{max.norm}(\Xi) = \max\{\|\boldsymbol{x}\| : (\boldsymbol{x}, s) \in \Xi\}.$$

We selected 20 homotopy paths Ξ with $\text{max.norm}(\Xi) \geq 10^{12}$ from Table 5, and 20 homotopy paths Ξ with $\text{max.norm}(\Xi) \geq 10^{6}$ from Table 8. The averages (the maximum) of $\text{max.norm}(\Xi)$ over those 20 paths are 7.60e16 (8.08e17) and 3.47e6 (3.66e7), respectively. Tables 9 and 10 show the effectiveness of the three scalings presented in Section 4 when they are applied to those 20 paths, respectively. Each row in Tables 9 and 10 indicates (a) no scaling, (b) the scaling based on function values described in subsection 4.1, (c) the two scalings based on function values and magnitudes of variables, in subsections 4.1 and 4.2, respectively, and (d) the three scalings based on function values, magnitudes of variables and Jacobian matrices in subsections 4.1, 4.2 and 4.3, respectively.

Table 9 shows that (a) no scaling resulted in failure to obtain solutions. However, we could find nonsingular solutions successfully in cases (b), (c) and (d). As we applied more scaling techniques from (a) to (d), the numbers of predictor and corrector iterations were reduced, and/or the maximum of the condition numbers of the Jacobian matrices decreased. The last points of $s < 0$ were much closer to $s = 0$ for (b), (c) and (d) than for (a). Consequently, we can say that the scalings helped to stabilize the numerical algorithms and increase the efficiency for the problems.

In Table 10, with (a) no scaling and (b) the scaling based on function values, only 6 paths out of the 20 paths converged to nonsingular solutions and tracing the rest 14 paths ended in the middle, resulting too small predictor step lengths to continue in the adaptive path tracing. When we used the two scalings based on function values and magnitudes of variables as in (c) and (d), we succeeded to find all twenty nonsingular solutions. The condition numbers of the Jacobian matrices for (c) and (d) were smaller than those of (a) and (b), and the values of $s < 0$ were closer to $s = 0$ for (c) and (d). The two scalings used in (c) and (d) provided stability in tracing. The effectiveness of the scaling based on Jacobian matrices is not clear in Table 10 except that it slightly improves the condition numbers of Jacobian matrices.

In the case (a) and/or (b) in Tables 9 and 10, tracing all or some paths were failed. We can conclude from the numerical results that the two scalings based on function values and magnitudes of variables work effectively, and that the scaling based on Jacobian matrices improves condition numbers of the Jacobian matrix.

6 Concluding discussions

Polyhedral homotopy continuation methods have been known as an efficient and reliable way to compute all isolated solutions of a polynomial system of equations. However, an implementation without a proper handling of large magnitudes of powers of t limits the capability to solve various polynomial systems. The highest power of t that we were able to deal with so far is about 10^5 in the cyclic-13 polynomial. We have presented a modified polyhedral-linear homotopy function with the continuation parameter s using a change of the continuation parameter t. Although the polyhedral-linear homotopy function obtained from this scaling is equivalent to the original one mathematically, it provides a convenient tool to improve the numerical stability in path tracing with extremely high powers of t.

Scalings	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)	Av.max.$K(D\boldsymbol{f})$ (Max.max.$K(D\boldsymbol{f})$)
(a)	271.8 (360)	866.4 (1233)	-6.09e-5 (-6.80e-11)	- -	- -
(b)	303.9 (390)	710.0 (869)	-6.80e-18 (-1.46e-19)	9.90e-11 (7.54e-10)	5.83e28 (9.66e29)
(c)	303.4 (390)	699.0 (859)	-6.80e-18 (-1.25e-19)	8.95e-11 (8.43e-10)	2.65e11 (4.12e12)
(d)	303.4 (390)	699.0 (859)	-6.80e-18 (-1.25e-19)	7.20e-11 (5.87e-10)	1.45e8 (1.40e9)

Table 9: 20 homotopy paths from the economic-14 polynomial.

Scalings	Av.pred.it (Max.pred.it)	Av.cor.it (Max.cor.it)	Av.last.s (Max.last.s)	Av.$\|f(\hat{\boldsymbol{x}})\|_\infty$ (Max.$\|f(\hat{\boldsymbol{x}})\|_\infty$)	Av.max.$K(D\boldsymbol{f})$ (Max.max.$K(D\boldsymbol{f})$)
(a)	625.3 (1001)	2099.9 (4167)	-4.38e-5 (-1.20e-11)	1.56e-14 (9.87e-14)	7.74e17 (6.55e18)
(b)	820.8 (1861)	1253.7 (2206)	-5.60e-5 (-3.21e-12)	4.88e-14 (1.05e-13)	2.04e18 (1.31e19)
(c)	652.6 (1185)	1330.9 (2206)	-5.31e-11 (-1.93e-18)	5.81e-14 (2.67e-13)	5.57e12 (5.13e13)
(d)	652.2 (1185)	1330.0 (2206)	-5.76e-11 (-1.93e-18)	6.56e-14 (2.57e-13)	3.40e12 (3.86e13)

Table 10: 20 homotopy paths from the cyclic-13 polynomial.

The proposed polyhedral-linear homotopy function has shown to be successful for tracing paths with powers of the original parameter t up to 10^{20}.

Therefore, we can challenge polynomials with larger dimensions (*e.g.* the cyclic-14 and economic-15 polynomials) employing the modified homotopy function from the viewpoint of powers of t.

As shown in Section 3, we can use extremely small step lengths at the end of tracing near $s = 0$ with the new continuation parameter s, and singular solutions are obtained with high accuracy.

The scaling techniques in Section 4 have been effective to resolve numerical difficulties in solving ill-conditioned linear systems associated with predictor-corrector procedures. For some cases, the scaling based on Jacobian matrices does not always decrease condition numbers of Jacobian matrices greatly as we have seen in Table 10. We may need to develop more effective techniques to improve ill-conditioning of linear systems.

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