An adapted equivalent linearization to predict higher harmonics of LCO in aeroelastic system with freeplay

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Abstract

Control surfaces nonlinearities can lead to aeroelastic systems with limit cycle oscillations (LCO). Several methods have been proposed to predict LCO, as Harmonic Balance-based methods (HB). Describing function (DF) is a particular case of HB with a single harmonic motion assumed, and the approach can be combined with a classical eigenvalue stability analysis via the Equivalent Linearization Technique (ELT). On the other hand, High-Order Harmonic Balance (HOHB) methods consider higher number of harmonics, but lead to a more complex nonlinear algebraic system of equations. This work investigates a new approach to adapt the ELT to consider both first and third harmonics in LCOs. Numerical simulations are performed by considering the aeroelastic typical section with control surface freeplay and both first and third harmonics. The results show that the new method provides good estimations for the third harmonic LCO amplitude, whereas the classical DF provides a more accurate estimation for the first harmonic amplitude.

Keywords: Describing Function, Equivalent Linearization Technique, Multiple Harmonics, Discrete Nonlinearities

1 Introduction

Aeroelastic systems typically have structural nonlinearities such as freeplay, which can lead the system to have limit cycle oscillations (LCO), as noted by [1]. LCO are usually characterized by periodic motions, which may be detrimental in terms of fatigue failures. An important task for aeronautical engineers is to predict the flight conditions (velocity and air density) in which these oscillatory phenomena takes place, including amplitude and frequency of motion in the flight envelope.

Several numerical methods have been employed to predict limit cycle oscillations in aeroelastic systems. Classic numerical time integrations are suitable, but because they are time consuming other techniques are often proposed. The Describing Function (DF) combined with the Equivalent Linearization Technique (ELT) is a typical method used instead, mainly due to the reduced computational time required to perform the analysis.

The ELT is related to the linear stability analysis, i.e., it is related to an eigenvalue problem. The approach associates a linear equivalent system for each nonlinear condition, such as each freeplay amplitude. In particular for freeplay, a mathematical expression for an equivalent stiffness is obtained via the balance between the harmonics of the Fourier series of the nonlinear element, from which arises the term Harmonic Balance (HB). The general procedure allows one to include more harmonics for both assumed limit cycle oscillation (i.e., the system response) and its associated effort. Particularly if only a single harmonic is considered to represent the LCO and to truncate the Fourier series, the term Describing Function (DF) is typically used in the literature, as shown in several works [2, 3, 4]. The authors show that the DF approach is suitable to predict LCO of systems presenting only the first harmonic. On the other hand, there are different works considering a larger number of harmonics to deal with systems containing LCOs with multiple harmonics. The general idea involves the High Order Harmonic Balance (HOHB) method, which replaces the assumed motion and the linear equivalent effort to define an algebraic system of equations with multiple variables [5]. Then, whilst the DF for a single harmonic comes to simplify the LCO prediction, the HOHB method usually involves more complex algebraic manipulations and requires the solution of a high order system of nonlinear equations.

In this context, the present article introduces a new approach adapting the ELT to predict limit cycle oscillations with both first and third harmonics. The proposal replaces the solution of a classical describing function by a nonlinear system of equations involving four unknown parameters. Numerical simulations are carried out by considering the three degree of freedom typical section airfoil with control surface freeplay. Time domain simulations previously presented in [6] are considered as reference results to demonstrate the approach. The results show that the classical DF provides slightly more accurate estimations for the first harmonic of LCO. However, the proposed approach also provides small errors for predicting these amplitudes. In addition, it provides interesting results for predicting the third harmonic of LCO mainly at airspeed V in the range $0.35 \leq V/V_f \leq 0.85$, where V_f is the flutter speed. The proposed approach requires further investigations, but the present results are encouraging.

2 Harmonic Balance Fundamentals

The aeroelastic system considered is the typical section of Theodorsen [7] with a control surface freeplay amplitude 2δ, as illustrated in Fig. 1. The nonlinear control surface restoring torque $T_\beta \equiv f_l^\delta$ due the freeplay is given by

$$f_l^\delta(t) = \begin{cases} k_{u_l}\left[u_l(t)+\delta\right] & , \quad u_l < -\delta \\ 0 & , \quad |u_l| < \delta \\ k_{u_l}\left[u_l(t)-\delta\right] & , \quad u_l > \delta \end{cases} \tag{1}$$

where the subscript l indicates the degree of freedom (DOF) associated to the freeplay, and in this case $u_l \equiv \beta$ and $k_{u_l} \equiv k_\beta$.

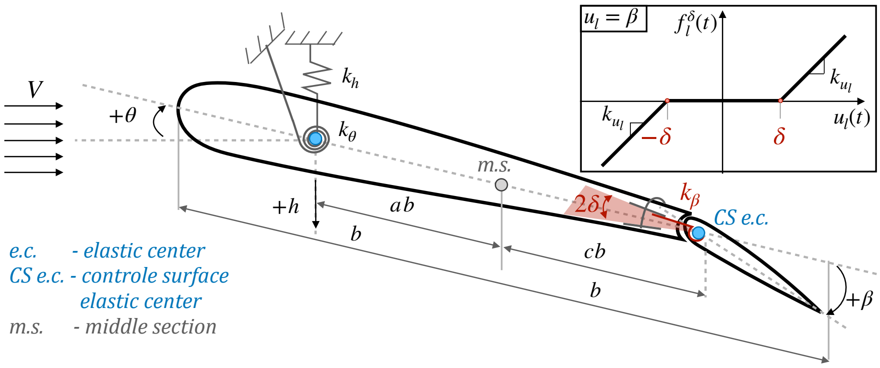

Figure 1: Three DOF typical section with control surface freeplay.

The main idea of the Harmonic Balance method (HB) is also employed when using both ELT and HOHB. The main difference between them is the procedure performed after the harmonic balance, as described in sections 2.1 and 2.2. For the freeplay case, the central premise assumed by the harmonic balance is to write a linear equivalent effort $\hat{f}_l^\delta = \hat{k}_{u_l}\hat{u}_l(t)$ to the describe the nonlinear, where $\hat{(.)}$ indicates the linear equivalent term associated to the exact $(.)$. Figure 2 shows the steps considered to address this premise: $i.$ choose a mathematical expression $\hat{u}_l$ composed by harmonic functions (sine and/or cosine) to properly describe the system response, i.e., the expected LCO (right-hand side of the upper equation in Fig. 2); $ii.$ choose which harmonics are used to compose $\hat{f}_l^\delta$ by using a Fourier series ($\frac{A_0}{2} + \sum A_n \cos n\theta + B_n \sin n\theta$) to express the nonlinear effort f_l^δ (left-hand side of the upper equation in Fig. 2).
Due to the discontinuous behavior of the freeplay representation, the integral limits need to be defined to compute the coefficients B_n of the Fourier series ($A_0 = A_n = 0$ for all $n \geq 1$, because this nonlinear effort is an odd function) - see the lower equation in Fig. 2. These limits are well-known in terms of u_l, as shown in Eq. (1) (i.e., $u_l = \pm\delta$). However, they must to be known in terms of the reduced time $\theta = \omega_{u_l} t$ over a complete period of oscillation, where ω_{u_l} is the LCO frequency of the first harmonic. Then, the unknown parameters θ_δ are introduced, and they represent the values of θ for which transition points among the locally linear models once this nonlinear effort is piecewise linear. In practice, these limits correspond to $\hat{u}_l = \delta$. Note that the number of unknown parameters θ_δ depends on the type of motion assumed by $\hat{u}_l$ once it is directly related to the number of transitions through the freeplay boundaries in a single cycle.

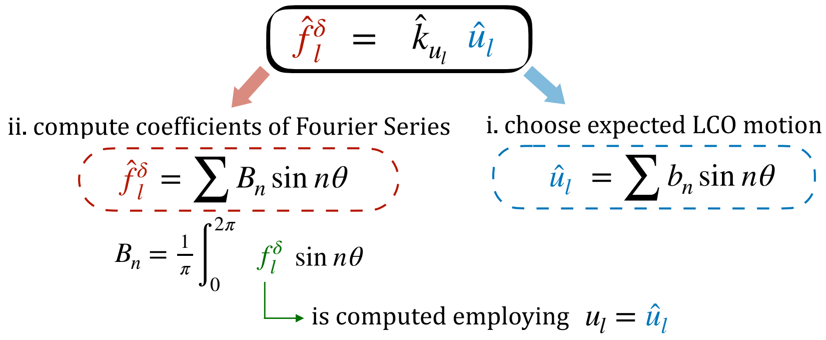

Figure 2: Central premise with the two steps required to derive the linear equivalent stiffness $\hat{k}_{u_l}$, for f_l^δ presented in Eq. (1).

2.1 Conventional Describing Function (DF) for freeplay

The conventional Describing Function (DF) is a particular case of the idea above, and it is derived by assuming a LCO with a single harmonic (i.e., $\hat{u}_l = b_1 \sin\theta$) and also truncating the Fourier series with only one harmonic (i.e., $\hat{f}_l^\delta = B_1 \sin\theta$). Applying the harmonic balance it is obtained that $\hat{k}_{u_l} = B_1/b_1$, where b_1 is the LCO amplitude. The final DF relates the linear equivalent stiffness, and it is given by $\hat{k}_{u_l} = \frac{k_{u_l}}{\pi}[\pi - 2\theta_\delta - \sin(2\theta_\delta)]$, where $\theta_\delta = \sin^{-1}(\delta/b_1)$. The special feature of this approach in comparison with a HOHB is that a classical stability analysis (i.e., eigenvalue extractions) is performed to identify each value of stiffness (in the range $0 < \hat{k}_{u_l} < k_{u_l}$) that implies to eigenvalues with null real parts. Each corresponding stiffness is associated with an equivalent linear system, and the possible LCO has its amplitude given by DF and frequency determined from the imaginary part of the eigenvalue. This procedure corresponds to the Equivalent Linearization Technique (ELT). The main advantages of this approach are to obtain the LCO frequency from a classical eigenvalue analysis and compute the LCO amplitude by using a describing function.

2.2 Considerations on the HOHB Methods in the Context of Describing Functions

HOHB methods allow one to define which harmonics are present on motion $\hat{u}_l$ considering LCO with multiple harmonics. They also provide the amplitude of motion for all degrees of freedom (DOF), instead of computing it only for the nonlinear DOF, because they consider harmonic motions in the form such as $\frac{c_0}{2} + \sum c_n \cos n\theta + d_n \sin n\theta$ (see [5] for details). On the other hand, the general procedure of HOHB methods is that all assumed displacements ($\hat{h}$, $\hat{\theta}$...) are directly included in the system of ordinary differential equations (ODEs). Similar procedure is used for the velocities, accelerations and the linear equivalent effort $\hat{f}_l^\delta$ computed as Fourier series. In particular for the typical section with control surface freeplay it is $S_\beta \ddot{\hat{h}} + [I_\beta + b(c-a)S_\beta]\ddot{\hat{\theta}} + I_\beta \ddot{\hat{\beta}} + \hat{f}_\beta^\delta = M_\beta$. Hence, a typically nonlinear algebraic system of equations is obtained with a large number of unknown parameters, i.e., the coefficients assumed for the motions of all DOFs, the unknown parameters θ_δ, and also the LCO frequency of the first harmonic of ω_{u_l}. Note that the main advantages of a HOHB method is to admit multiple harmonics, but they can involve much more complex solutions for the system of equations in comparison with a describing function.

3 Predict LCO with higher harmonics - an ELT-based approach

This section introduces an approach to consider the third harmonic in a LCO using an adapted Equivalent Linearization Technique. The central idea is to employ the eigenvalue analysis combined with a system of equations obtained after considering two terms to represent both the discrete nonlinearity by the Fourier serie and the assumed aeroelastic response.

The classical linear stability analysis involving an eigenvalue problem consists in extracting the eigenvalues λ of a dynamic matrix $\mathbf{A}$ for each i-th flight condition ($V_i = V_0\sqrt{\rho_i/\rho_0}$, $i = 1, ...$) and evaluate the sign of their real part $\Re(\lambda) \equiv g$, where (V_0, ρ_0) are the airspeed and air density at sea level. If the ELT considers the fundamental harmonic motion (i.e., the first one), DF provides the coefficient b_1 from a scalar equation. On the other hand, if the first and third harmonics are assumed, and defining oscillations such as illustrated in Fig. 3, the vector of unknown parameters is given by $\mathbf{y} = \{b_1 \;\; b_3 \;\; \theta_{\delta 1} \;\; \theta_{\delta 2}\}^T$, which allows one to write a nonlinear system of equation $\mathbf{f}(\mathbf{y}) = \mathbf{0}$, such that

$$\mathbf{f}(\mathbf{y}) = \left\{ \begin{array}{l} b_1 \hat{k}_{u_l} - B_1 \\ b_3 \hat{k}_{u_l} - B_3 \\ -\delta + b_1 \sin\theta_{\delta_1} + b_3 \sin 3\theta_{\delta_1} \\ -\delta + b_1 \sin\theta_{\delta_2} + b_3 \sin 3\theta_{\delta_2} \end{array} \right\} \tag{2}$$

Note that the Fourier series coefficients B_1 and B_3 depend on the parameters b_1, b_3, $\theta_{\delta 1}$ and $\theta_{\delta 2}$.

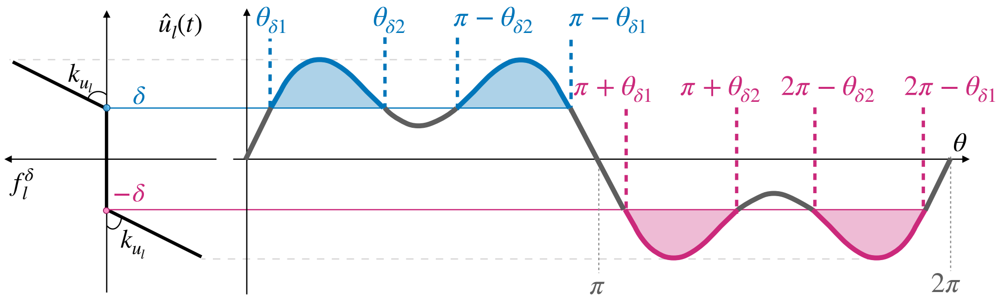

Figure 3: Schematic illustration of one cycle of LCO assuming $\hat{u}_l = b_1 \sin\theta + b_3 \sin 3\theta$ for freeplay nonlinearity.

The linear equivalent stiffness $\hat{k}_{u_l}$ is obtained from the ELT by extracting the eigenvalues, as performed when using the classical DF. However, it is assumed that $\hat{u}_l = b_1 \sin\theta + b_3 \sin 3\theta \equiv \beta$, instead of $b_1 \sin\theta$. Hence, it is convenient to assume only two terms to represent f_l^δ by the Fourier series, otherwise, it is not possible to compare directly the coefficients of the corresponding harmonics to define the algebraic system of equations. On the other hand, this consideration imposes the spectral content of the nonlinear restoring torque to be equivalent to that of the LCO, which may not be necessarily physically consistent for all practical applications. However, this approach provides good estimations of the third harmonic LCO amplitude without requiring more a complex solution typically involved in HOHB methods.

4 Results and Discussions

The three DOF typical section is considered with freeplay acting on the control surface rotation $\beta \equiv u_l$, and its parameters are shown in Tab. 1. Numerical simulations for this system are presented in the time domain by [6]. The eigenvalue analysis of the nominal linear system (i.e., without freeplay) reveals that flutter occurs at velocity and frequency $V_f = 18.70$ m/s and $f_f = 4.99$ Hz, respectively. Including freeplay $\delta = 0.5^o$ at $V = 0.51V_f$, both spectral displacement and restoring torque obtained by time integration are shown in Fig. 4. Note that for this case these spectrum are different from each other. In addition, the ratio between amplitude of the force in relation to the amplitude of displacement (i.e., B_j/b_j) is not a constant value of equivalent stiffness $\hat{k}_\beta$, such as previously assumed in Eq. (2), if the fifth harmonic is compared with the third harmonic.

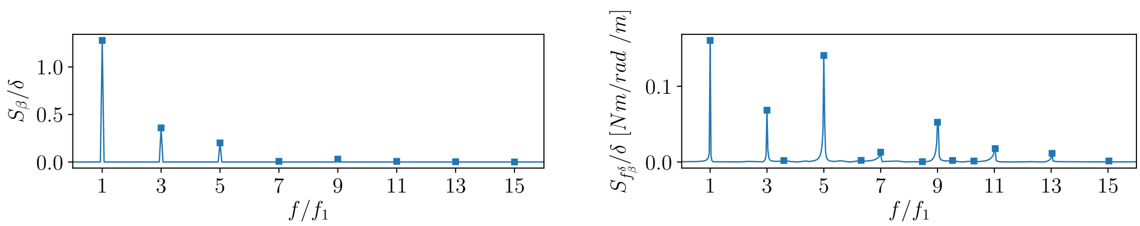

Figure 4: Nonlinear displacement (S_β) and nonlinear force ($S_{f_\beta^\delta}$) in the frequency domain computed at airspeed $V/V_f = 0.51$ and for an equivalent stiffness $\hat{k}_\beta = 0.11787$ Nm/rad /m ($f_1 = 3.63$ Hz).

Figure 5 presents the LCO spectrum computed by Fourier transform of the time domain response. The LCO amplitude (b_1) computed by using the classical DF is highlighted by the circle symbol, and it corresponds to the value computed from the time domain solution. This figure also shows both first and third harmonic amplitudes computed by the proposed approach. Note that the first harmonic is (slightly) better predicted by DF. On the other hand, the prediction of the third harmonic amplitude by this new approach can provide interesting additional results for this type of motion.

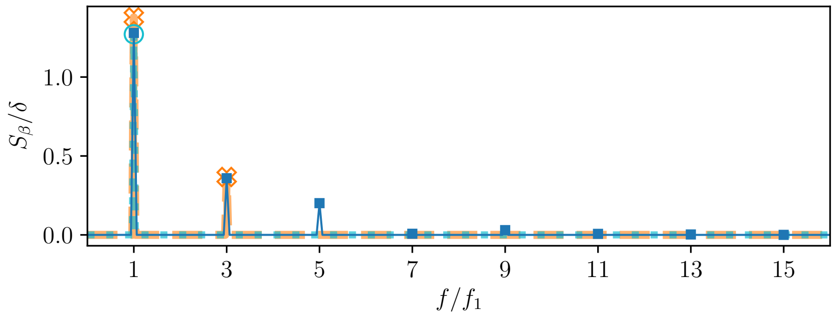

Figure 5: LCO in the time domain: computed from the time integration (solid line and square ■); proposed approach (dashed line and ⯐); $\hat{\beta} = b_1 \sin\theta$, DF (dotted line and circle ○). The airspeed $V/V_f = 0.51$ and equivalent stiffness $\hat{k}_\beta = 0.11787$ Nm/rad /m ($f_1 = 3.63$ Hz).

Figure 6 shows both amplitude (left-hand side) and frequency (right-hand side) predictions by this proposed approach. Square symbols indicate the results from the time domain simulations, whereas circle symbols show amplitude and frequency for the first harmonic predicted by the classical DF. On the other hand, cross symbols (⯐) show that the amplitudes for the third harmonic (b_3) are correctly predicted for a wide range of airspeed ($0.35 \leq V/V_f \leq 0.85$). Note however that at low airspeed (i.e., $V/V_f < 0.35$) both DF and this new approach predict LCOs that are not confirmed by time domain simulations. This characteristic has already been reported in the literature for DF, as shown by [8] and [9] for example. Also for $V/V_f > 0.9$ this proposed approach exhibits less accurate LCO amplitudes for the third harmonic, and in general they are larger than their values computed from the time domain simulations.

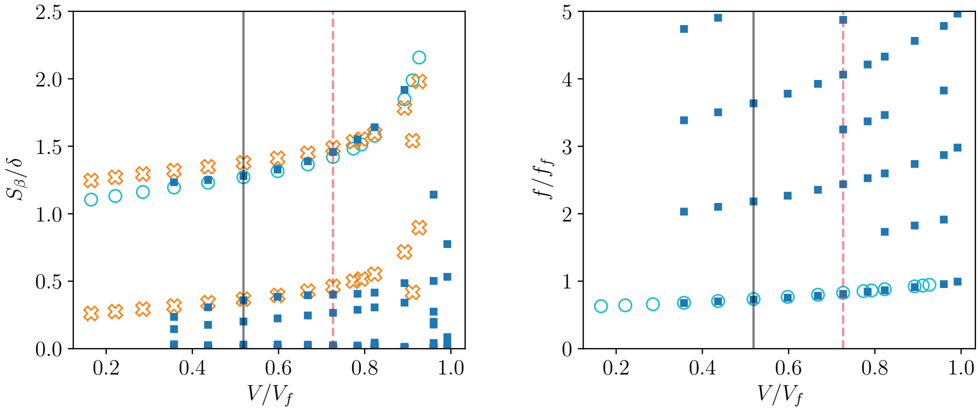

Figure 6: LCO amplitudes (left-hand side) for the first (b_1) and third (b_3: results from the time domain simulations (square ■); DF (circle ○); proposed approach (cross symbol ⯐). The solid vertical line indicates $V/V_f = 0.51$.

5 Final Remarks

This article has presented a new approach to predict the third harmonic in limit cycle oscillations due to freeplay. The approach combines the Equivalent Linearization Technique with a nonlinear system of equations obtained by the concept of describing function. The method employs a harmonic balance, but it does not require the solution of more complex nonlinear system of equations such as high-order Harmonic Balance methods.
Based on the results it is noted that the classical describing function provides slightly more accurate predictions for the first harmonic of the limit cycle oscillations, in comparison with this proposed approach. However, small differences are observed when comparing both approaches with the results from the time domain simulations. In particular for the third harmonic, the proposed approach predicts the LCO amplitudes for the range $0.35 \leq V/V_f \leq 0.85$, and because of this it is an interesting strategy to compute LCOs.

6 Acknowledgements

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A The 3-DOF Typical Section

This appendix shows the dynamic matrix (Eq. 3) for the system without freeplay, and used to perform the ELT. The Roger's approximation is used to write the unsteady aerodynamic forces in the time domain [10].

$$\mathbf{A} = \begin{bmatrix} [\bar{\mathbf{A}}]_{2n\times 2n} & [\bar{\mathbf{Q}}]_{2n\times n_{lag}n} \\ [\bar{\mathbf{I0}}]_{n_{lag}n\times 2n} & [\bar{\mathbf{\Gamma}}]_{n_{lag}n\times n_{lag}n} \end{bmatrix}_{N\times N} \tag{3}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} -\mathbf{M}_a^{-1}\mathbf{D}_a & -\mathbf{M}_a^{-1}\mathbf{K}_a \\ \mathbf{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \quad \bar{\mathbf{Q}} = \begin{bmatrix} q\mathbf{M}_a^{-1}\mathbf{Q}_{1+2} & \dots & q\mathbf{M}_a^{-1}\mathbf{Q}_{(n_{lag}+2)} \\ \mathbf{0}_{n\times n} & \dots & \mathbf{0}_{n\times n} \end{bmatrix}$$

$$\bar{\mathbf{I0}} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times n} \\ \vdots & \vdots \\ \mathbf{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \qquad \mathbf{\Gamma} = -\frac{V}{b}\begin{bmatrix} \gamma_1\mathbf{I}_n & \dots & \mathbf{0}_{n\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n\times n} & \dots & \gamma_{n_{lag}}\mathbf{I}_n \end{bmatrix} \tag{4}$$

and, also,

$$\mathbf{M}_a = \mathbf{M} - q\left(\frac{b}{V}\right)^2\mathbf{Q}_2 \quad ; \quad \mathbf{D}_a = \mathbf{D} - q\left(\frac{b}{V}\right)\mathbf{Q}_1 \quad ; \quad \mathbf{K}_a = \mathbf{K} - q\mathbf{Q}_0 \tag{5}$$

This present study considers the 3 degree-of-freedom airfoil for which the matrix of mass is given by

$$\mathbf{M} = \begin{bmatrix} m & S_\theta & S_\beta \\ S_\theta & I_\theta & I_\beta + b(c-a)S_\beta \\ S_\beta & I_\beta + b(c-a)S_\beta & I_\beta \end{bmatrix}; \qquad \mathbf{K} = \mathrm{diag}(k_h,\ k_\theta,\ k_\beta) \tag{6}$$

where m is the mass, b is the aerodynamic semichord, I_θ and S_θ are the inertia and static moments of mass of wing-aileron around elastic center of wing ($e.c.$); I_β and S_β are the inertia and static moments of mass of aileron around elastic center of aileron ($e.c.^\beta$). $\mathbf{K}$ is the stiffness matrix of the overlying system, with the nominal stiffness k_β placed on the 3rd row and 3rd column. The linear equivalent matrix $\hat{\mathbf{K}} = \mathrm{diag}(k_h, k_\theta, \hat{k}_\beta)$ is employed in Eq. 5 instead of $\mathbf{K}$ to compute the linear equivalent aeroelastic matrix $\hat{\mathbf{A}}$ (instead of $\mathbf{A}$), used to the equivalent linear stability analysis on the Equivalent Linearization Technique (ELT). The structural damping matrix $\mathbf{D}$ is assumed to be zero. The displacement vector is $\mathbf{u}(t) = \{h(t)\ \theta(t)\ \beta(t)\}^T$ (i.e., the plunge, pitch and control surface rotation, respectively). The unsteady aerodynamics of [7] are used with $n_{lag} = 7$ lag states in a rational function approximation ($\gamma_1 = 0.05$, $\gamma_2 = 0.21$, $\gamma_3 = 0.48$, $\gamma_4 = 0.85$, $\gamma_5 = 1.33$, $\gamma_6 = 1.91$ and $\gamma_7 = 2.60$). Note that $u_l = \beta$ and $\dot{u}_l = \dot{\beta}$. The air density is 1.225 kg/m^3 and the other parameters are shown in Table 1.

Table 1: Physical and geometric parameters for the three DOF typical section.

Description of the Meaning	**Value**	**Unity (S.I.)**
Span	$s = 0.3$	m
Semi-chord	$b = 0.115$	m
Total mass (p.u.s)**	$m = 2.40585$	kg/m
Nondimensional* w.r.t b from $m.s.$ to $e.c.$	$a = -0.42609$	
Nondimensional w.r.t b from $m.s.$ to $e.c.^{\beta}$	$c = 0.64783$	
Inertia moment (p.u.s) of wing-aileron around $e.c.$	$I_\theta = 1.38524\ 10^{-2}$	kgm^2 /m
Inertia moment (p.u.s) of aileron around $e.c.^{\beta}$	$I_\beta = 8.06206\ 10^{-5}$	kgm^2 /m
Static moment (p.u.s) of wing-aileron around $e.c.$	$S_\theta = 0.114219$	kgm/m
Static moment (p.u.s) of aileron around $e.c.^{\beta}$	$S_\beta = 0.003264$	kgm/m
Stiffness in plunge h (p.u.s)	$k_h = 854.81$	(N/m)/m
Stiffness in pitch θ (p.u.s)	$k_\theta = 26.80$	(N/rad)/m
Stiffness in β (p.u.s)	$k_\beta = 1.0312$	(N/rad)/m

*Positive value to the right-hand side of middle section ($m.s.$) and negative to the left-hand side of $m.s.$.
** Mass, inertia, static and stiffness are given per unit of span (p.u.s).

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