Flutter pseudospectra and pseudospectral continuation

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Nomenclature

χ	=	structural time-eigenvalue, rad/s
U	=	free-stream airspeed, m/s
Υ	=	free-stream airspeed per semichord, Hz
ζ	=	modal damping ratio
ξ	=	damping-frequency ratio
A	=	nonlinear problem matrix function
$\lVert\cdot\rVert$	=	two-norm
$\bar{x}$	=	complex conjugation
ι	=	imaginary unit

Subscripts

R	=	real part
I	=	imaginary part
ϵ	=	pseudospectral quantity

I. Introduction

Trefethen [1] introduced the concept of *pseudospectra*: the sets of points for which an eigenvalue problem is nearly satisfied. This is usually formulated in terms of an ϵ-pseudospectra: for a linear operator A, generating an eigenvalue problem $(\mathrm{A}-\lambda\mathrm{I})\mathbf{x}=0$, the ϵ-pseudospectra is the set of λ_ϵ such that there exists $\mathbf{x}_\epsilon$ such that $\lVert(\mathrm{A}-\lambda_\epsilon\mathrm{I})\mathbf{x}_\epsilon\rVert<\epsilon$. That is, the eigenvalue problem is solved to within a residual of ϵ. Pseudospectra have been similarly defined for matrix polynomials [2]. They have been applied to a wide range of physical problems, including control theory [3] and hydrodynamic stability [4]. They provide important and interesting qualitative information about system stability, and hence are of potential use in aeroelasticity.

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Pons and Gutschmidt [5,6] studied the application of multiparameter spectral theory to aeroelasticity, and showed how the flutter instability problem can be recast as a multiparameter eigenvalue problem (MPEVP). The pseudospectra of MPEVPs have been studied in brief [7], though as of yet there have been no applications. Yet aeroelasticity presents several problems that may be amenable to pseudospectral analysis: the analysis of subcritical and supercritical modal behaviour (rather than just of the flutter point); defining flight envelopes for a maximum acceptable modal damping; and the characterisation of flutter points based on the modal damping gradient (hard, soft, etc.). Multiparameter pseudospectral analysis has the potential to provide new methods for approaching these problems.

II. Pseudospectral Definitions

As a beginning, it is possible to apply ϵ-pseudospectra directly to multiparameter problems. For an n-parameter problem $\mathrm{A}(\boldsymbol{\lambda})\mathbf{x} = \mathbf{0}$, the ϵ-pseudospectrum is $\boldsymbol{\lambda}_\epsilon \subseteq \mathbb{C}^n$ such that there exists a normalized $\mathbf{x}_\epsilon$ such that $\|\mathrm{A}(\boldsymbol{\lambda}_\epsilon)\mathbf{x}_\epsilon\| < \epsilon$. Via the min-max theorem [8], these minimal $\|\mathrm{A}(\boldsymbol{\lambda})\mathbf{x}\|$ values are equivalent to the minimum singular values of $\mathrm{A}(\boldsymbol{\lambda})$. Thus, following Trefethen [9], we generate a field of minimal singular values over the space of $\boldsymbol{\lambda}$ and compute the pseudospectral contours. This method is by no means the most efficient method of computing pseudospectra; it is simply one that generalizes conveniently to the multiparameter nonlinear case.

The ϵ definition of the pseudospectrum is not, however, the most convenient for aeroelastic analysis. It is more obvious to measure distance from stability not in terms of the eigenproblem residual (ϵ) but in terms of a modal damping parameter. A simple way of doing this to introduce an imaginary component into the structural eigenvalue, which in the multiparameter analysis is normally assumed to be real – i.e. purely oscillatory; we are working with an eigenvalue, χ, defined such that $\hat{\mathbf{x}}(t) = \mathbf{x}\exp(\iota\chi t)$. That is, where ordinarily $\chi = \chi_R \in \mathbb{R}$, we take $\chi = \chi_R + \iota\chi_I,\ \chi_R, \chi_I \in \mathbb{R}$. For an arbitrary nonlinear problem $\mathrm{A}(\chi, U)\mathbf{x} = \mathbf{0}$, with airspeed parameter U, we have simply $\mathrm{A}(\chi_R + \iota\chi_I, U)\mathbf{x} = \mathbf{0}$.

It follows from Theorem 4.2 of van Dorsselaer et al. [10] that, for the linear problem, the χ_I-pseudospectrum is equivalent to the ϵ-pseudospectrum: however, this equivalence is in doubt for nonlinear problems. Moreover, from this modal damping metric we can define more useful dimensionless parameters, such as the modal damping ratio: $\zeta = \chi_I / \|\chi\|$ (note χ_R represents the damped (*not* undamped) natural frequency). ζ can be substituted directly into a problem, resulting in a system of the form $\mathrm{A}(\chi_R, \zeta, U)\mathbf{x} = \mathbf{0}$, but this may introduce nonlinearity into an otherwise linear MPEVP: something which is not significant if ζ is assumed *a priori*, but which will make solving for ζ more difficult. As a *via media*, we can define the damping parameter $\chi = \chi_R(1 \pm \xi\iota)$, which is both linear and dimensionless; scaling well with the modal frequency (χ_R). Table 1 presents an overview of these three physical pseudospectral parameters. This concept of a flutter pseudospectra is easily generalized: for example, a k- or p-k method formulation yields naturally the a pseudospectrum in g; the artificial structural damping term introduced into the k- and p-k method systems.

For an aeroelastic system in the pseudospectral form, e.g. $\mathrm{A}(\chi_R, \zeta, U)\mathbf{x} = \mathbf{0}$, substituting known values of the pseudospectral parameter allows us to map out the corresponding pseudospectrum. This can be used to map out the flight envelope corresponding to a maximum acceptable modal damping. Moreover, in the near vicinity of a destabilising flutter point, it may be assumed that the modal damping is monotonically decreasing, and hence the minimum acceptable damping value will provide the safe bounds around the flutter point. At the flutter point itself we can define the derivative of modal damping with respect to the airspeed parameter as a finite difference

$$\frac{\partial \xi}{\partial \Upsilon} = \frac{\xi(\Upsilon + \Delta\Upsilon)}{\Delta\Upsilon} \tag{1}$$

where $\xi(\Upsilon)$ denotes the ξ-value at Υ. This allows a characterisation of the flutter point as hard or soft [11] (a large or small derivative respectively); though some situation-specific quantification of *large* and *small* is needed. It also allows an easy quantification of whether the flutter point is a destabilising or restabilising (negative and positive derivative, respectively).

III. Pseudospectral continuation

The procedure we have just devised provides a method of generating the modal damping via numerical continuation along the pseudospectral parameter. At the most basic level this involves specifying a grid of modal damping values and then solving the pseudospectral problem at each of these points. If the system is linear or polynomial this can be done directly [5]; if the system is nonlinear then the previous point can supply an initial guess for the iterative solution of the next point. Continuation of this form has been applied to flutter problems before [12–14]: indeed, the traditional modal damping plot corresponds to a form of natural parameter continuation, with the continuation parameter being U or another airspeed parameter, and the pseudospectral parameter (modal damping) being solved for in the eigenvalue problem. However, rather than starting the continuation at zero airspeed and then marching to a flutter point, we can simply compute the flutter points initially via the standard multiparameter approach (cf. [5,6]), and then expand the modal damping paths out from these points. This produces an accurate picture of the system behaviour around the flutter point, without wasting computational effort on areas which are of no interest. Continuation with a pseudospectral parameter (e.g. the modal damping ratio) is even more useful as a form of local analysis, as it allows us to specify a modal damping range of interest directly and thus efficiently compute damping-related flight envelopes. Unfortunately it also restricts the continuation to areas of monotonic damping behaviour: the method will not pass a modal damping turning point.

To solve this problem higher-order continuation methods are available, such as as pseudo-arclength (or Riks') continuation [12,13,15]. Pseudo-arclength continuation is a predictor-corrector procedure; the predictor step involves incrementing $[U, \chi_R, \chi_I]$ along a specified arclength Δs, according to

$$\begin{bmatrix} U \\ \chi_R \\ \chi_I \end{bmatrix} = \begin{bmatrix} U_0 \\ \chi_{R,0} \\ \chi_{I,0} \end{bmatrix} + \Delta s \begin{bmatrix} \dot{U}_0 \\ \dot{\chi}_{R,0} \\ \dot{\chi}_{I,0} \end{bmatrix}; \tag{2}$$

using an initial guess $\left[U_0, \chi_{R,0}, \chi_{I,0}\right]$ and the local tangent $\left[\dot{U}_0, \dot{\chi}_{R,0}, \dot{\chi}_{I,0}\right]$. We compute this tangent using a finite-difference approximation based on the previous point computed on the continuation curve. The corrector step involves solving for the system's local modal properties subject to a pseudo-arclength constraint. However, instead of the large system of nonlinear equations employed Meyer [12,13], we can use the pseudospectral approach to formulate the corrector step as three-parameter eigenvalue problem:

$$\begin{gathered} \mathrm{A}(\chi_R, \chi_I, U)\mathbf{x} = \mathbf{0}. \\ \overline{\mathrm{A}}(\chi_R, \chi_I, U)\bar{\mathbf{x}} = \mathbf{0}. \\ (\,\dot{U}_0(U - U_0) + \dot{\chi}_{R,0}(\chi_R - \chi_{R,0}) + \dot{\chi}_{I,0}(\chi_I - \chi_{I,0}) - \Delta s)y = 0. \end{gathered} \tag{3}$$

where the conjugate equation enforces $\chi_R, \chi_I \in \mathbb{R}$, as per [5,6], and the pseudo-arclength constraint [15] has been recast as an MPEVP using an arbitrary scalar eigenvalue y. This is a three-parameter nonlinear eigenvalue problem which we solve with the method of successive linear problems [5]. We note in passing that, for the SLP algorithm, a significantly simpler form of the constraint in Eq. 3 is:

$$(\dot{U}_0 \Delta U + \dot{\chi}_{R,0} \Delta \chi_R + \dot{\chi}_{I,0} \Delta \chi_I) y = 0. \quad (3)$$

with $\Delta x = x_{k+1} - x_k$ as per [5]. This continuation approach can easily be generalized to other sets of parameters; e.g. the computation of flight envelopes at a specified damping as a function of a model parameter.

IV. Application

As a demonstration of our method, we analyse the well-established Goland wing test case. Model parameter values are taken from [16]. However, we reduce the Goland wing partial-differential model to an algebraic MPEVP not via discretisation but via the Generalised Laplace Transform Method [16]: the result is an unstructured nonlinear MPEVP in the modal frequency (χ) and airspeed (U). The problem is then restricted via the restriction method of Pons [5] to simplify the system behaviour at nonphysical parameter values ($\chi < 0$ and $U < 0$). We compute the first flutter point of this system using the iterated contour plot algorithm presented in Pons [5], and then, using the ϵ-pseudospectrum, compute the set of points with $\epsilon = 0.04$ and $\epsilon = 0.08$. Figure 1 shows these results, visualised over a contour plot of the system [5,16]. The intersections of real and imaginary contours represent flutter points. One particularly interesting feature is observed: a pseudospectral area (i.e. a mode near the stability boundary) at $U = 600$ m/s, not associated with any flutter point. However with ϵ-pseudospectrum it is not possible to tell whether this pseudospectral area represents a stable mode near instability or vice versa: for this we need modal pseudospectra.

We then compute the mode paths of the system via pseudo-arclength continuation, starting from two areas of interest: the first flutter point and the pseudospectral area noted in Figure 1. Short continuation paths are extended from these points: these are shown in Figure 2. Note that the pseudo-arclength method operates in the χ_I pseudospectral definition: other variables produce poorly-scaled arclength definitions due to the difference in magnitude e.g. between ζ and χ_R. The χ_I results are postprocessed into ζ for Figure 1. Using only one continuation point we can both compute a flight envelope around the flutter point (e.g. for $\zeta > 0.05$, $U < 125.6$ m/s), and estimate a flutter point damping ratio / airspeed gradient of $\partial\zeta/\partial U = -0.004$, which we would interpret as a hard flutter event, relative to the gradients observed in the rest of the plot. Further continuation provides information on the wider modal trends: e.g. that the pseudospectral area noted in Figure 1 represents a near-restabilisation, reaching $\zeta = -0.08$ maximum. Extending the mode further, or analysing the modeshape, we observe that the restabilising mode the one destabilised at the first flutter point (not, e.g., the divergence point).

V. Conclusion

In this work we have presented a wholly novel application of pseudospectral analysis to the study of aeroelastic flutter points, and introduced the concept of pseudospectral continuation. The former broadens the multiparameter approach to flutter beyond the simple computation of flutter points; to the analysis of local modal behaviour in the vicinity of a flutter point (or a near-flutter point). The latter extends this analysis out into the wider local model context of a flutter or near-flutter point. The pseudospectral approach is particularly useful in the computation of advanced secondary flutter results, such as aeroelastic flight envelopes. The Goland wing benchmark system has been used as an example application of these methods and analyses.

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Tables

Table 1: Modal pseudospectral parameters

Parameter	Definition	Comments
χ_I	$\chi = \chi_R + \iota\chi_I$	Simple but dimensional, equivalent to ϵ parameter in at least some cases.
ζ	$\chi = \chi_R \left(\iota + \frac{\zeta}{\sqrt{1-\zeta^2}}\right)$.	Dimensionless and most physically relevant, but introduces strong nonlinearity into the system.
ξ	$\chi_R(1 + \xi\iota)$	Simple and dimensionless.

Figures

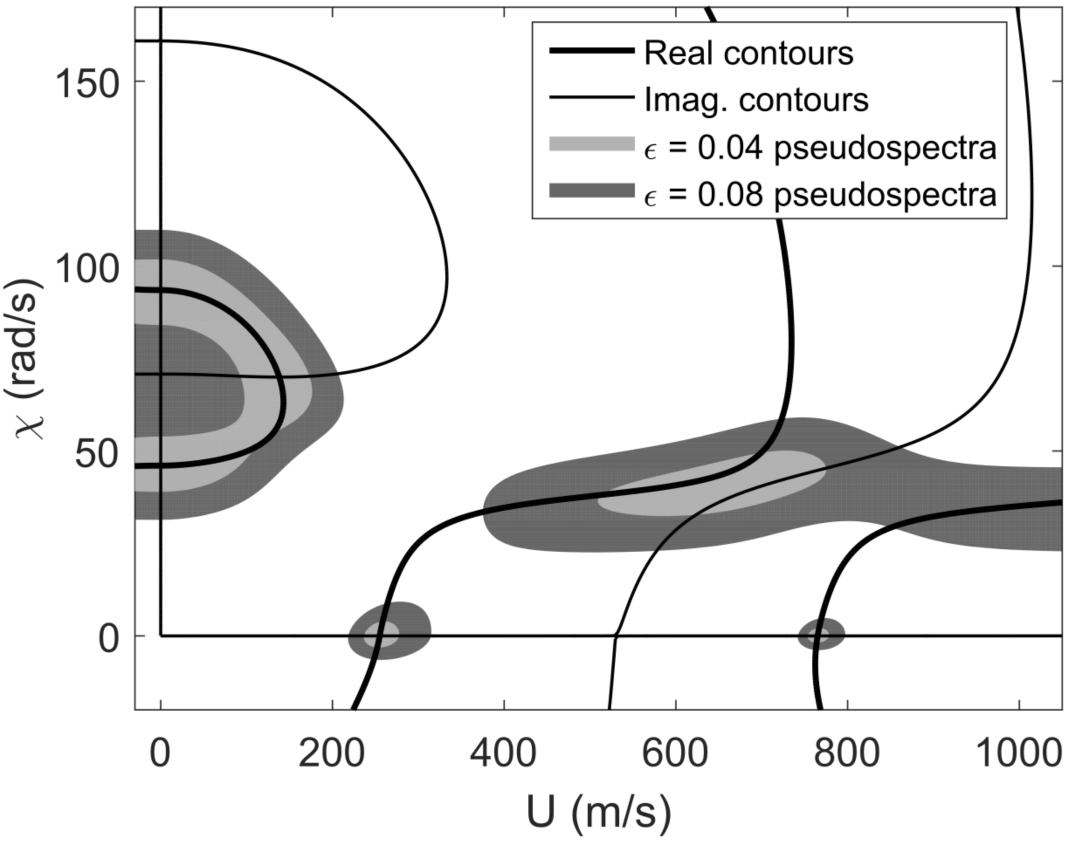

Figure 1: ϵ-pseudospectra of the Goland wing model

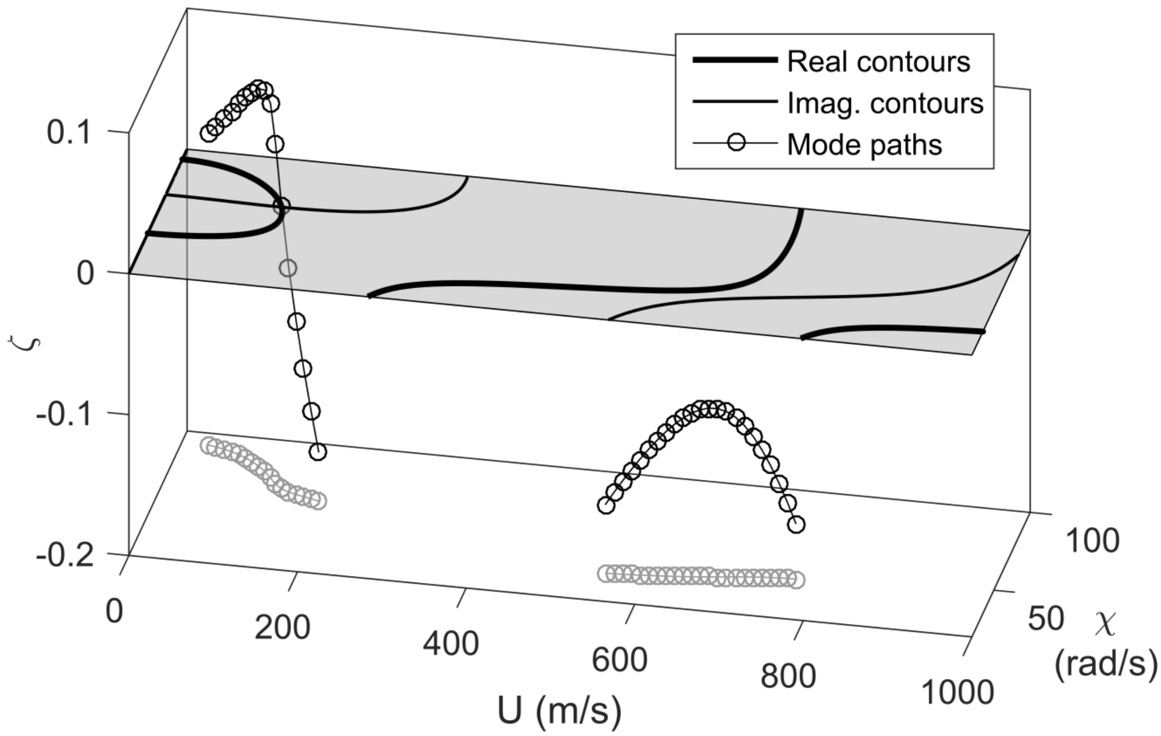

Figure 2: Modal paths of the Goland wing model in areas of interest, generated via pseudo-arclength continuation. The path projections onto the U-χ plane are in light grey.