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Flutter analysis including structural uncertainties

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Abstract The common practice in industry is to perform flutter analyses considering the generalized stiffness and mass matrices obtained from finite element method (FEM) and aerodynamic generalized force matrices obtained from a panel method, as the doublet lattice method. These analyses are often re-performed if significant differences are found in structural frequencies and damping ratios determined from ground vibration tests compared to FEM. This unavoidable rework can result in a lengthy and costly process of analysis during the aircraft development. In this context, this paper presents an approach to perform flutter analysis including uncertainties in natural frequencies and damping ratios. The main goal is to assure the nominal system's stability considering these modal parameters varying in a limited range. The aeroelastic system is written as an affine parameter model and the robust stability is verified solving a Lyapunov function through linear matrix inequalities and convex optimization.

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1 Introduction

Fluid and structure interaction problems have been studied by many researchers especially since the work of Theodersen [1] explained the mechanism of aeroelastic instability. Different approaches have been proposed to study linear and nonlinear problems and, because its catastrophic nature, flutter probably is the most important topic involving structural and aerodynamic forces [2–6].

Different problems of engineering have required static and dynamic analysis involving structural uncertainties [7]. In a recent past, authors also included structural and aerodynamic uncertainties on the flutter problem. In 1999 Lind and Brenner [9] introduced structured singular value μ into the aeroelastic field and developed a μ-method which the aeroelastic system is parameterized with dynamic pressure perturbations. Chung et al. [10] included aerodynamic uncertainties by considering variation of Mach number, which was represented by the boundary of Theodorsen's lift deficiency function [10]. Chung and Shin [11] applied the μ-method to identify the flutter boundary considering the structural variation due to changing natural frequency and aerodynamic variation. The approach was applied only for simple systems and, according to the authors, there are some limitations regarding its accuracy because it does not consider an updated relationship between each uncertainty variable [11].

For real problems of flutter analysis, the number of degrees of freedom to describe an aircraft can be very large (tens of thousands). In these cases, the numerical model which represents those aircraft can involve many structural elements and, as a consequence, to include uncertainties on the physical properties of those all elements is truly impractical. For this reason, the common practice in aeronautical industry is to perform flutter analyses considering a nominal aeroelastic model written in generalized coordinate system (GCS). This kind of model, which is truncated with few modes to reduce the size of the matrices, includes the generalized stiffness and mass matrices obtained from FEM and aerodynamic generalized force matrices computed using a panel method, as for instance, the DLM [8]. To consider some structural uncertainty, a very limited number of physical parameters are varied to create parametric (non-nominal) models in GCS.

During an aircraft development, flutter analyses are often re-performed if significant differences are found in structural frequencies and damping ratios determined from ground vibration tests (GVT) in comparison with their values previously computed by the numerical model. Based on the process described above, this is an unavoidable rework that result in a lengthy and costly process of re-analysis. In addition, there are serious difficulties to apply this procedure due to the limited availability of the aircraft and the fact that multiple configurations need to be tested, as discussed in reference [12]. Also, GVTs of aircraft are typically performed very late in the development process [12]. In fact, if flutter analysis must be re-performed an extreme time pressure can exist to get the final results.

In this context, this paper presents an approach to perform flutter analysis including uncertainties in structural frequencies and damping ratios for aeroelastic models written in generalized coordinate system. The main goal is to assure the nominal system stability considering these modal parameters varying in a limited range previously assumed by experience. The aeroelastic system is written as an affine parameter model and the robust stability is verified solving a Lyapunov function through LMI. The method is written in time domain using a rational function approximation for the aerodynamic forces. It is shown that the introduced methodology offers promise for robust flutter analysis using convex optimization.

2 Affine parameter model

The method described in this section considers the aeroelastic model written as a second order model in generalized coordinates. This is a fundamental approach mainly to analyze large and complex structures. The system of equations is projected on the structural eigenvector obtained without structural damping. The matrix $\mathbf{M}_m$ is an identity matrix (using the eigenvector normalized by the modal mass) and the stiffness and damping matrices are both diagonal matrices respectively given by

$$\begin{aligned} \mathbf{K}_m(x,x) &= \lambda_x = \omega_x^2 \\ \mathbf{D}_m(x,x) &= 2\xi_x\omega_x, \quad x = 1,\ldots,m \end{aligned} \tag{1}$$

The proposed approach is formulated for system at which variations on frequencies and damping ratios do not imply to significant changes on their correspondent structural modes. This assumption is commonly used by researchers in different methods [9, 13, 14] and [15]. It was considered valid for the example shown in this work and, mainly for complex structures, it can be previously confirmed through numerical tests, as for instance, by computing the modal assurance criterion [16, 17].

An affine parameter model is a special state space equation in which some constant uncertain parameter have a fixed value that is known only approximately. Gahinet et al. [18] discuss details and advantages to use this approach and, in practice, the equation of motion is written as

$$\mathbf{E}(\bar{\theta})\dot{\mathbf{x}}(t) = \mathbf{A}_E(\bar{\theta})\mathbf{x}(t) \tag{2}$$

where $\mathbf{E}(\bar{\theta})$ and $\mathbf{A}(\bar{\theta})$ are known matrices written as functions of the vector $\bar{\theta} = (\bar{\theta}_1,\ldots,\bar{\theta}_{n_{\bar{\theta}}}) \in \mathbf{R}^{n_{\bar{\theta}}}$ of real uncertain parameters $\bar{\theta}_x \in [\bar{\theta}_x^{min}, \bar{\theta}_x^{max}], x = 1,\ldots,n_{\bar{\theta}}$; and

$$\mathbf{A}_E = \mathbf{A}_0 + \bar{\theta}_1\mathbf{A}_{\bar{\theta}_1} + \cdots + \bar{\theta}_{n_{\bar{\theta}}}\mathbf{A}_{\bar{\theta}_{n_{\bar{\theta}}}} \tag{3}$$

In this work the robust aeroelastic quadratic stability is verified considering that $\bar{m}$ structural frequencies and damping ratios are uncertain and can vary in a limited range, as shown in the following equations

$$
\begin{aligned}
\omega_x^{unc} &= \omega_x + \Delta\omega_x \\
\xi_x^{unc} &= \xi_x + \Delta\xi_x \\
x &= 1,\ldots,\bar{m} \le m
\end{aligned}
\tag{4}
$$

where ω_x is a nominal value computed by FEM. The values of ξ_x are assumed proportional to the modal stiffness and mass. Using this approach three analysis can be performed: case (1) some structural frequencies are uncertain (the structural damping is neglected); case (2) some damping ratios are uncertain; and case (3) both structural frequencies and damping ratios are uncertain (see an illustrative scheme in Fig. 1). Note that if the structural damping increases, the aeroelastic system becomes safer in terms of stability.

3 Robust stability analysis

Linear matrix inequalities (LMI) have been extensively applied in modern control theory [19]. LMI contributed to overcome many difficulties in control design. In the last decade, LMIs have been used to solve many problems that until then were unfeasible through others methodologies, due mainly to the emerging of powerful algorithms to solve convex optimization problems, as for instance, the interior point method [18–20]. There are few studies involving LMIs to solve problem in aeroelastic fields and this work introduces an extension of the ideas previously discussed in references [21] and [22].

Based on LMI, a sufficient condition for the quadratic stability of affine system represented by

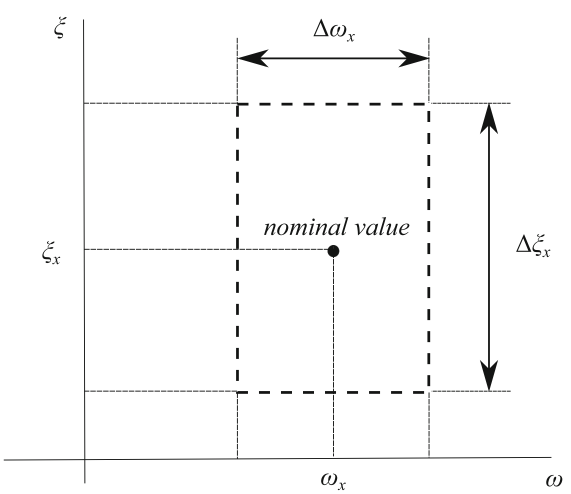

Fig. 1 A box limiting the xth frequency and damping ratio variation

$\dot{\mathbf{x}}(t) = \mathbf{E}_0^{-1}\mathbf{A}(\bar{\theta})$ is the existence of $\bar{m}+1$ symmetric matrices $\mathbf{P}_i$ such that [18]

$$
\begin{aligned}
&\mathbf{A}(\bar{\theta})^T\mathbf{P}(\bar{\theta}) + \mathbf{P}(\bar{\theta})\mathbf{A}(\bar{\theta}) < \mathbf{0}, && \forall\, \bar{\theta} \in v \\
&\mathbf{P}(\bar{\theta}) > \mathbf{I}, && \forall\, \bar{\theta} \in v \\
&\mathbf{A}_i^T\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i \ge \mathbf{0} && \forall\, i = 1,\ldots,\bar{m} \in v
\end{aligned}
\tag{5}
$$

where v denotes a set of vertices of a hyperrectangle and $\mathbf{P}(\bar{\theta}) := \mathbf{P}_0 + \bar{\theta}_1\mathbf{P}_1 + \cdots + \bar{\theta}_m\mathbf{P}_m$. Gahinet et al. [18] show that if this LMI system is feasible, the quadratic stability is assured for all values of $\bar{\theta}_x$ in $[\bar{\theta}_x^{min}, \bar{\theta}_x^{max}]$, where $\bar{\theta}_x^{min} = \bar{\theta}_x\left(1 - \Theta\delta_{\bar{\theta}_x}\right)$ and $\bar{\theta}_x^{max} = \bar{\theta}_x\left(1 + \Theta\delta_{\bar{\theta}_x}\right), \forall\, x = 1,\ldots,\bar{m}$. Note that $\mathbf{E}_0$ is a non-singular matrix; otherwise, see reference [23].

Since the quadratic stability is a sufficient condition, this approach computes the range which the parameters can vary keeping the system stable. However, it does not necessarily compute the largest variation that is possible.

3.1 Case 1: uncertain structural frequencies

Consider an undamped system ($\mathbf{D}_m$ is a null matrix) with uncertainties in its $\bar{m}$ elastic frequencies ω_x, $x = 1,..,\bar{m}$. Since each xth frequency is written as its square into the generalized stiffness matrix, the approach is mathematically formulated considering the parameter $\bar{\theta}_x^{unc} = (\omega_x^2)^{unc}$. This notation allows to write the aeroelastic matrix as shown bellow

$$
\mathbf{A}_E = \mathbf{A}_0 + (\omega_1^2)^{unc}\mathbf{A}_{\omega_1} + \cdots + (\omega_{\bar{m}}^2)^{unc}\mathbf{A}_{\omega_{\bar{m}}}
\tag{6}
$$

and the system described by Eq. (2) is written such that

$$
\mathbf{E}_0 = \begin{bmatrix} \mathbf{M}_{am} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots \\ \mathbf{0} & \cdots & \ddots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{bmatrix}
\tag{7}
$$

$$
\mathbf{M}_{am} = \mathbf{M}_m - q\left(\frac{b}{V}\right)^2\mathbf{Q}_{m2}
\tag{8}
$$

$$
\mathbf{A}_0 = \begin{bmatrix} q\dfrac{b}{V}\mathbf{Q}_{m1} & -(\bar{\mathbf{K}}_m - q\mathbf{Q}_{m0}) & \bar{\mathbf{A}}_3 \\ \bar{\mathbf{A}}_4 & \bar{\mathbf{A}}_5 & \bar{\mathbf{A}}_6 \end{bmatrix}
\tag{9}
$$

where the submatrices $\bar{\mathbf{A}}_j$ ($j = 3, 4, 5, 6$) and $\mathbf{Q}_{m(j)}$ are defined in the Appendix, $\bar{\mathbf{K}}_m$ is the modal stiffness matrix setting zero for each xth frequency considered uncertain. For each matrix $\mathbf{A}_{\omega_x} \in m(2+n_{lag}) \times m(2+n_{lag})$ in Eq. (3), $a_{(x,x+m)} = -1$ and the other elements are zero. Note that each of these matrices have only one non-zero element and each $\mathbf{E}_{\omega_x}$ are null matrices. The uncertain parameter $(\omega_x^{unc})^2$ is defined in a continuous range through its minimum and maximum values according to the following equation

$$\begin{aligned} (\omega_x^{unc})^2 &= \omega_x^2(1 \pm \delta_{\omega_x}) \quad \text{such that,} \\ (\omega_x^{unc})^2_{min} &= \omega_x^2(1 - \delta_{\omega_x}) \\ (\omega_x^{unc})^2_{max} &= \omega_x^2(1 + \delta_{\omega_x}) \\ x &= 1, \ldots, \bar{m} \end{aligned} \tag{10}$$

where δ_{ω_x} is the percentage of uncertainty in the xth eigenvalue λ_x (or ω_x^2). It is possible to rewrite Eq. (10) introducing the parameter $\Theta > 0$ for computing the largest portion of the specified parameter range $\bar{\theta}_x \in [\bar{\theta}_x^{min}, \bar{\theta}_x^{max}]$ where quadratic stability is assured, as shown bellow

$$\begin{aligned} (\omega_x^{unc})^2_{min} &= \omega_x^2(1 - \Theta\delta_x) \\ (\omega_x^{unc})^2_{max} &= \omega_x^2(1 + \Theta\delta_x) \end{aligned} \tag{11}$$

In this case, the aeroelastic quadratic stability is assured considering that each xth structural frequency can vary τ_{ω_x} percent with respect to its nominal value, where

$$\tau_{\omega_x} = 10^2\left[(1 \pm \Theta\delta_{\omega_x})^{1/2} - 1\right] \tag{12}$$

Note that $\omega_x^{unc} = \omega_x[1 \pm 10^{-2}\tau_{\omega_x}]$; and δ_{ω_x} is previously defined to solve an LMI system and Θ is an output.

3.2 Case 2: uncertain damping ratio

Consider that each xth uncertain modal damping ratio ξ_x^{unc} can vary $\pm\delta_{\xi_x}$ percent. It is possible to write an affine parameter system for which the matrix $\mathbf{E}_0$ is given by Eq. (7) and the matrix $\mathbf{A}_E(\bar{\theta})$ is written as

$$\mathbf{A}_E = \mathbf{A}_0 + \xi_1^{unc}\mathbf{A}_{\xi_1} + \cdots + \xi_{\bar{m}}^{unc}\mathbf{A}_{\xi_{\bar{m}}} \tag{13}$$

such that,

$$\mathbf{A}_0 = \begin{bmatrix} -\left(\bar{\mathbf{D}}_m - q\dfrac{b}{V}\mathbf{Q}_{m1}\right) & -(\mathbf{K}_m - q\mathbf{Q}_{m0}) & \bar{\mathbf{A}}_3 \\ \bar{\mathbf{A}}_4 & \bar{\mathbf{A}}_5 & \bar{\mathbf{A}}_6 \end{bmatrix} \tag{14}$$

and $\bar{\mathbf{D}}_m$ is a diagonal matrix such that its element $\bar{D}_{xx}$ is zero if the xth damping ratio is considered uncertain; otherwise $\bar{D}_{xx} = 2\xi_x\omega_x$. Similarly, each matrix $\mathbf{A}_{\xi_x}$ has the element $a_{(x,x)} = -2\omega_x$; and the other ones equal zero. The uncertain damping ratio is $\xi_x^{unc} \in [\xi_x - \Theta\delta_{\xi_x}, \xi_x + \Theta\delta_{\xi_x}] = [\xi_x^{min}, \xi_x^{max}]$. In this case, the aeroelastic quadratic stability is assured if the percentage of uncertainty in the xth damping ratio is given by

$$\tau_{\xi_x} = \pm 10^2\Theta\delta_{\xi_x} \tag{15}$$

where δ_{ξ_x} is the input and Θ is the output. The xth uncertain damping ratio is given by $\xi_x^{unc} = \xi_x(1 \pm \Theta\delta_{\xi_x})$.

3.3 Case 3: uncertain frequencies and damping ratios

Consider now that $\bar{m}$ structural frequencies and damping ratios can vary simultaneously. The system of equations is represented through an affine parameter state space model for which the matrix $\mathbf{E}_0$ is shown in Eq. (7). In this case, the matrix $\mathbf{A}_E$ is given by

$$\mathbf{A}_E = \mathbf{A}_0 + \sum_{j=1}^{\bar{m}}(\omega_j^2)^{unc}\mathbf{A}_{\omega_j} + \sum_{j=1}^{\bar{m}}(\xi_j\omega_j)^{unc}\mathbf{A}_{(\xi\omega)_j} \tag{16}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} -\left(\bar{\mathbf{D}}_m - q\dfrac{b}{V}\mathbf{Q}_{m1}\right) & -(\bar{\mathbf{K}}_m - q\mathbf{Q}_{m0}) & \bar{\mathbf{A}}_3 \\ \bar{\mathbf{A}}_4 & \bar{\mathbf{A}}_5 & \bar{\mathbf{A}}_6 \end{bmatrix} \tag{17}$$

and $(\xi_x\omega_x)^{unc}$ is written as $(\xi_x\omega_x)[1 \pm \Theta\delta_{(\xi\omega)_x}]$ and the matrices $\mathbf{A}_{\omega_x}$ are defined in Sect. (3.1). Each matrix $\mathbf{A}_{(\xi\omega)_x}$ is written such that its element $\mathrm{a}_{i,j} = -2$ if $i = j = x$; otherwise $\mathrm{a}_{i,j}$.

It is considered that damping ratios and structural frequencies have independent uncertainties. So, the parameter $\bar{\theta}_x^{unc} = (\xi_x\omega_x)^{unc} = (\xi_x\omega_x)[1 \pm \Theta\delta_{(\xi\omega)_x}]$ is rewritten as $(\xi_x\omega_x)^{unc} = \xi_x^{unc}\omega_x^{unc}$. In this case,

$$\begin{aligned} \xi_x^{unc} &= \frac{(\xi_x\omega_x)}{\omega_x^{unc}}\left[1 \pm \Theta\delta_{(\xi\omega)_x}\right] \\ &= \xi_x\frac{1}{(1 \pm 10^{-2}\tau_{\omega_x})}\left[1 \pm \Theta\delta_{(\xi\omega)_x}\right] \end{aligned} \tag{18}$$

or,

$$\begin{aligned}\xi_x^{min} &= \xi_x \frac{\left[1 - \Theta\delta_{(\xi\omega)_x}\right]}{\left[1 + 10^{-2}\tau_{\omega_x}\right]}\\ \xi_x^{unc} &\in [\xi_x^{min}, \xi_x^{max}] \Leftrightarrow\\ \xi_x^{max} &= \xi_x \frac{\left[1 + \Theta\delta_{(\xi\omega)_x}\right]}{\left[1 - 10^{-2}\tau_{\omega_x}\right]}\end{aligned} \tag{19}$$

Substituting $\tau_{\omega_x} = 10^2\left[(1 \pm \Theta\delta_{\omega_x})^{1/2} - 1\right]$, the xth uncertain structural frequency and the damping ratio are respectively given by

$$\begin{aligned}\omega_x^{unc} &= \omega_x(1 \pm \Theta\delta_{\omega_x})^{1/2}\\ \xi_x^{min} &= \xi_x \frac{\left[1 - \Theta\delta_{(\xi\omega)_x}\right]}{\left[1 + \Theta\delta_{\omega_x}\right]^{1/2}}\\ \xi_x^{unc} &\in [\xi_x^{min}, \xi_x^{max}] \Leftrightarrow\\ \xi_x^{max} &= \xi_x \frac{\left[1 + \Theta\delta_{(\xi\omega)_x}\right]}{2 - (1 + \Theta\delta_{\omega_x})^{1/2}}\end{aligned} \tag{20}$$

or, if $\xi_x^{min} = \xi_x(1 + \tau_{\xi_x}^{min})$ and $\xi_x^{max} = \xi_x(1 + \tau_{\xi_x}^{max})$, in percentage,

$$\begin{aligned}\tau_{\xi_x}^{min} &= -10^2\left\{1 - \frac{\left[1 - \Theta\delta_{(\xi\omega)_x}\right]}{\left[1 + \Theta\delta_{\omega_x}\right]^{1/2}}\right\}\\ \tau_{\xi_x}^{max} &= 10^2\left\{\frac{\left[1 + \Theta\delta_{(\xi\omega)_x}\right]}{2 - (1 + \Theta\delta_{\omega_x})^{1/2}} - 1\right\}\end{aligned} \tag{21}$$

that indicate the minimum and maximum percentages which the xth nominal damping ratio can vary keeping the system stable.

4 Numerical application

To illustrate the effectiveness of the method, numerical simulations were developed on a benchmark wing structure AGARD 445.6 shown in references [24] and [25]. The linear structural model for the AGARD 445.6 wing was created using the MSC/NASTRAN program. The wing is modelled with plate elements as a single layer orthotropic material consisting of 231 nodes and 200 elements. The thickness distribution was governed by the airfoil shape. The material properties used are $E_1 = 3.1511$ GPa, $E_2 = 0.4162$ GPa, $v = 0.31$, $G = 0.4392$ GPa and $\rho_{mat} = 381.98$ kg/m^3, where E_1 and E_2 are the moduli of elasticity in the longitudinal and lateral directions respectively, the v is Poisson's ratio, G is the shear modulus in each plane and ρ_{mat} is the mass density.

Aerodynamic and structural matrices are obtained by MSC/NASTRAN program (from solution 145) for Mach number 0.50 and $\rho_{REF} = 1.225$ kg/m^3 (air density). Figure 2 shows the flight envelope considered to illustrate this application. The values of reduced frequencies are 10^{-3}, 2.10^{-3}, 5.10^{-3}, 10^{-2}, 5.10^{-2}, 0.1, 0.2, 0.3, 0.5, 0.6, 0.8, 1.0, 1.5, 2.0, 3.0 and 4.0. The model has a length of reference $2b = 0.5578$ m, a sweep angle is 45° at the quarter chord line, a semi-span of 0.762 m and a taper ratio of 0.66. The flutter boundary is investigated using the first four fundamental structural modes ($m = 4$). Their natural frequencies $v = 0$ are $\omega_1 = 9.45$ Hz, $\omega_2 = 39.69$ Hz, $\omega_3 = 49.45$ Hz and $\omega_4 = 95.10$ Hz. A state space model was obtained using four parameters of lag $\beta_1 = 0.55$, $\beta_2 = 1.40$, $\beta_3 = 1.90$ and $\beta_4 = 2.90$.

To apply the approach is verified that the structural modes are not significantly affected by the uncertainties considered during the analysis. This is an assumption to assure the results specially for systems represented by equation of motion written in the generalized coordinate system with modal truncation, however, it is not required for small systems written in the physical coordinate system. For the system presented in this work that behaviour was verified by computing the modal assurance criterion (MAC) using the structural eigenvectors obtained from different values of longitudinal modulus of elasticity. The particular example shown in Fig. 3 demonstrates that

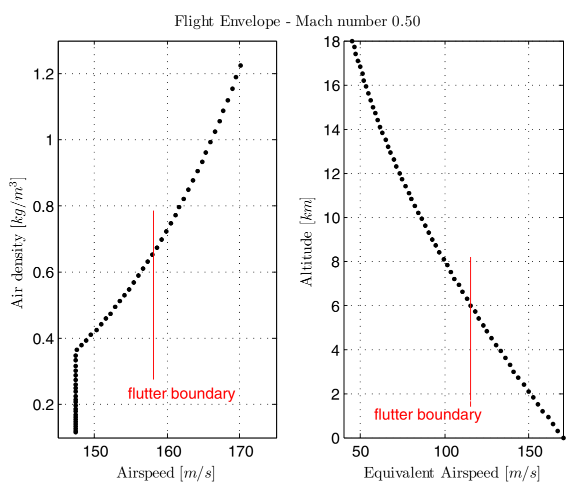

Fig. 2 Flight envelope corresponding to the Mach number 0.50

two sets of eigenvectors are consistent when the structural frequencies are not their nominal values (see Table 1).

4.1 Case 1: first structural frequency uncertain

Let the first structural frequency uncertain ($\bar{m} = 1$). Using the pk-method the flutter speed was computed considering the nominal value of ω_1 (see the *V-g* plot in Fig. 4). The LMI system shown in Eq. (5) was solved considering $\delta_1 = 0.2$, which approximately represents[1] a variation from 89.4 to 109.5 % of the nominal value of ω_1. According to the proposed approach, Fig. 5 shows the range which the first structural frequency can vary keeping the system stable. The stability was confirmed by performing the pk-method in the limits of the range using $\lambda_1^{unc} = (1+\delta_1)\omega_1^2$ and $(1-\delta_1)\omega_1^2$ into the stiffness matrix.

4.2 Case 1: first and second frequencies uncertain

Let both first and second structural frequencies uncertain ($\bar{m} = 2$). The LMI system shown in Eq. (5) was solved considering $\delta_1 = 0.20$ and $\delta_2 = 0.15$. Figures 6 and 7 show the range which first and second structural frequencies can vary keeping the system stable. It is possible to note the region where the LMIs are feasible in comparison with the region used as the convex domain obtained setting $\Theta = 1$ in Eq. (12).

4.3 Case 2: damping ratios uncertain

Commonly a conservative flutter analysis does not consider structural damping. However, if necessary, it is taken into account to assure aeroelastic stability into the flight envelope. Table 2 shows the damping for the first aeroelastic mode in three different points into the considered flight envelope. These aeroelastic damping ratios were obtained using the classical pk-method considering different values of structural damping $\xi_x, \forall x = 1, \ldots, 4$. After performing classical flutter analysis, the proposed approach was used to verify if the system keeps its stability when damping ratios vary ±50 %. Figure 8 shows the range which ξ_1 can vary keeping the system stable. The results for the modes two through four are similar because it was considered ξ_x and $\delta_x \pm 50\%, \forall x = 1, \ldots, 4$.

4.4 Case 3: structural frequencies and damping ratios uncertain

Consider simultaneously the four first structural frequencies and structural damping ratios uncertain ($\bar{m} = m = 4$). The LMI system was solved considering $\delta_{\omega_x} = \delta_{(\xi\omega)_x} = 0.05$ if $x = 1, 2$; and $\delta_{\omega_x} = \delta_{(\xi\omega)_x} = 0.08$ if $x = 3, 4$. Figures 9 and 10 show respectively the range which ω_x^{unc} and ξ_x^{unc} can vary keeping the system stability.

5 Final remarks

This paper presented the aeroelastic stability analysis using affine parameter state space model and linear matrix inequalities. It was used the quadratic criterion as a sufficient condition to assure that the system keeps its nominal stability when its modal parameters are uncertain. It was proposed three cases of analysis assuming that structural frequencies and damping ratios can vary in a limited range previously defined by experience. The main idea is to avoid the need to re-perform flutter analysis when the difference between structural frequencies and damping ratios obtained from GVT and from FEM are smaller or equal those assumed in that robust analysis. The benchmark AGARD 445.6 wing was used to demonstrate the approach through numerical simulations. It was demonstrated that the proposed formulation is appropriate also for large aeroelastic systems for which the generalized coordinate system is used to obtain the numerical models.

The proposed approach can also assure the robust aeroelastic stability when natural frequencies and damping ratios change during operation. This is an important aspect of this method because aircraft are systems with time-varying parameters during the flight. Note that the frequency of the bending mode changes slowly due to the consume of fuel inside the wing; the control surface natural rotation can change due to the actuator stiffness variations in failure conditions; also due to the influence of a large range of

[1] The range of variation of each xth frequency can be computed by $(1-\delta_x)^{1/2}$ and $(1+\delta_x)^{1/2}$, where δ_x is the associated parameter.

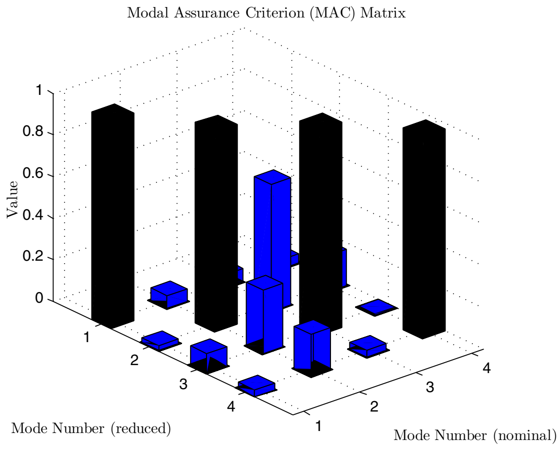

Fig. 3 A three-dimensional presentation of MAC values

Table 1 Structural frequencies for two different longitudinal moduli of elasticity

Mode no.	Freq. (Hz) E_1 (nominal)	Freq. (Hz) E_1=2.50 Gpa	Variation (%)
1	9.45	8.49	−10.19
2	39.69	37.15	−6.41
3	19.45	47.04	−4.87
4	95.10	89.33	−6.06

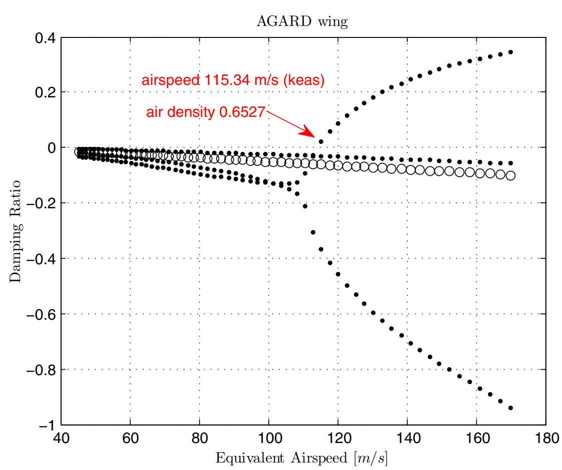

Fig. 4 Nominal *V-g* plot obtained from pk-method

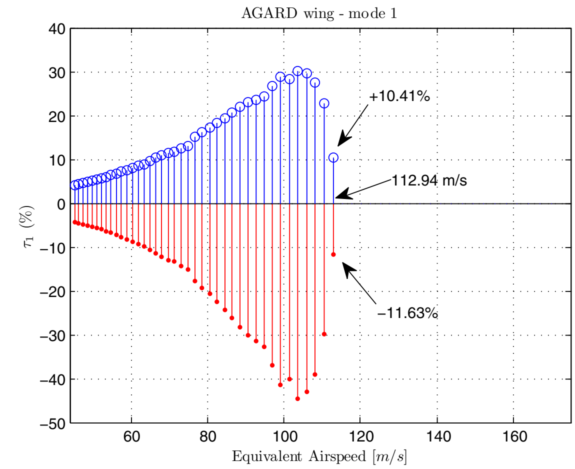

Fig. 5 Index τ for the first structural frequency uncertain (case 1)

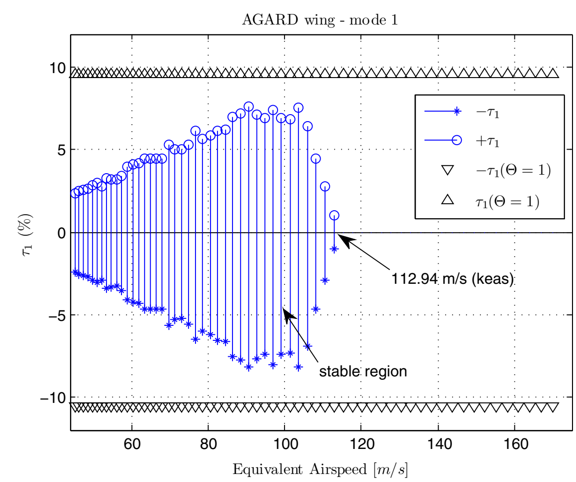

Fig. 6 Index τ for the first structural frequency uncertain (case 1)

external temperature on the damping ratios; and others. In this sense, this method offers promising for aeroelastic stability including structural uncertainties.

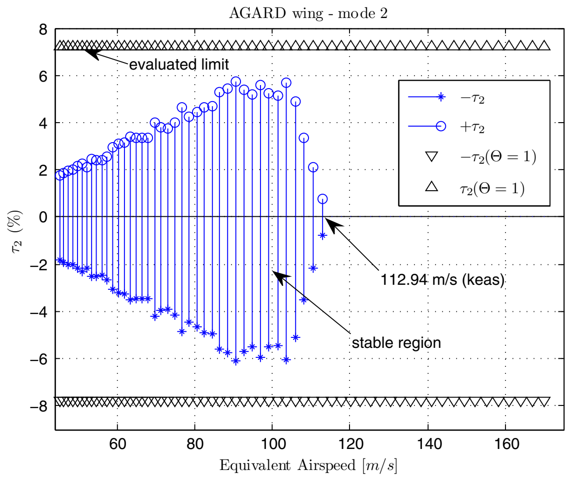

Fig. 7 Index τ for the second structural frequency uncertain (case 1)

Table 2 Aeroelastic damping for different structural damping ratios

Eq. Airspeed (m/s)	$\xi(0)$	$\xi(0.5\,\%)$	Condition
112.94	−0.1875	−0.06056	Stable
115.34	0.03902	0.03441	Unstable
117.76	0.1785	0.1092	Unstable

Eq. Airspeed (m/s)	$\xi(1.0\,\%)$	$\xi(3.0\,\%)$	Condition
112.94	−0.06202	−0.0672	Stable
115.34	0.03001	0.01408	Unstable
117.76	0.1037	0.08329	Unstable

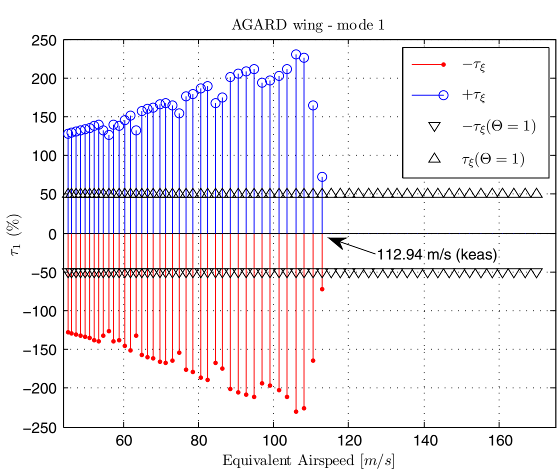

Fig. 8 Index τ computed from damping ratios uncertain (case 2)

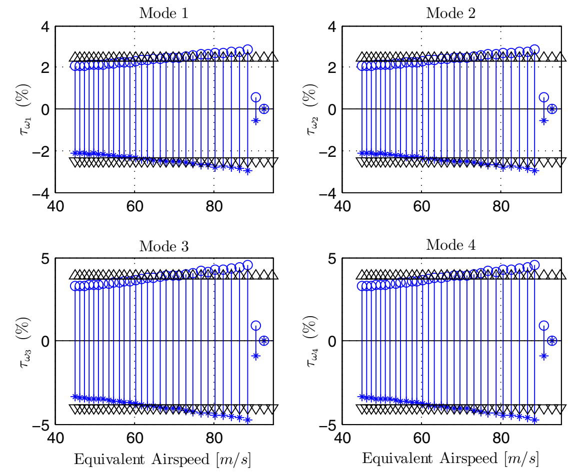

Fig. 9 Index τ for structural frequencies uncertain (case 3)

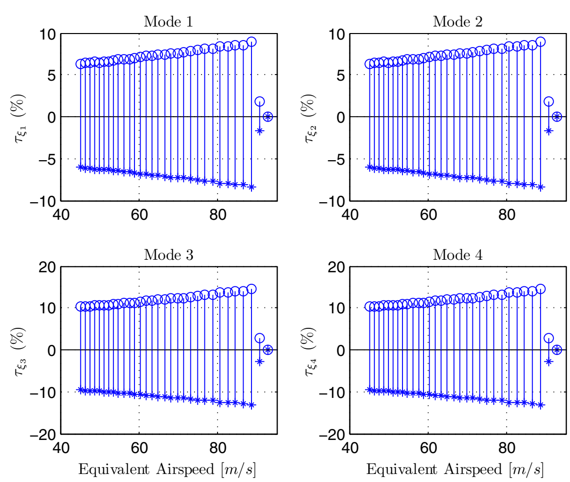

Fig. 10 Index τ for damping ratios uncertain (case 3)

Appendix: State space submatrices

This appendix presents the submatrices $\bar{\mathbf{A}}_j$ previously shown in Eq. (9).

$$\bar{\mathbf{A}}_3 = \left[q\mathbf{Q}_{m3} \;\cdots\; q\mathbf{Q}_{m(2+n_{lag})} \right] \tag{22}$$

$$\bar{\mathbf{A}}_4 = [\mathbf{I}\cdots\mathbf{I}\cdots\mathbf{I}]^T, \quad (n_{lag}+1)m \times m \quad | \quad \mathbf{I}, \quad m \times m \tag{23}$$

$$\bar{\mathbf{A}}_5 = [0]^{(n_{lag}+1)m\times m} \tag{24}$$

$$\bar{\mathbf{A}}_6 = V\left(-\frac{1}{b}\right)\begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \beta_1 \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \beta_{n_{lag}} \mathbf{I} \end{bmatrix}, \quad (25)$$

$(n_{lag}+1)m \times n_{lag}m$

Aerodynamic forces in time domain

The state space model was obtained considering the unsteady aerodynamic forces written in time domain through the Roger's rational function approximation shown in Eq. (26). Additional details can be found in reference [26].

$$\mathbf{Q}_m(s) = \left[\sum_{j=0}^{2} \mathbf{Q}_{mj} s^j \left(\frac{b}{V}\right)^j + \sum_{j=1}^{n_{lag}} \mathbf{Q}_{m(j+2)} \left(\frac{s}{s+\frac{b}{V}\beta_j}\right)\right] \mathbf{u}_m(s) \quad (26)$$

for which the parameters β_j were previously chosen to keep the aeroelastic frequency and damping ratios computed by the pk-method for the nominal system. The generalized displacement vector $\mathbf{u}_m$ is used to define the state space vector $\mathbf{x}_m = \left\{\dot{\mathbf{u}}_m \ \ \mathbf{u}_m \ \ \mathbf{u}_{am(1)} \cdots \mathbf{u}_{am(n_{lag})}\right\}^T$. The Laplace variable is s and $\mathbf{u}_{am(j)}$ is the jth vector of state of lags.

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