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A family of multi-point iterative methods for solving systems of nonlinear equations☆

Gyurhan H. Nedzhibov

Faculty of Mathematics and Informatics, Shumen University, Shumen 9700, Bulgaria

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Abstract

We extend to n-dimensional case a known multi-point family of iterative methods for solving nonlinear equations. This family includes as particular cases some well known and also some new methods. The main advantage of these methods is they have order three or four and they do not require the evaluation of any second or higher order Fréchet derivatives. A local convergence analysis and numerical examples are provided.

MSC: 65H05

Keywords: Nonlinear equation; Iterative method; Order of convergence; Chebyshev–Halley family

1. Introduction

In many areas of the sciences and engineering, the classical problem of finding the zeros of a given nonlinear function F arises, i.e. that of finding the solutions of the equation

$$F(x) = 0, \tag{1}$$

where $F : V \subset \mathbb{R}^n \to \mathbb{R}^n$. The zeros of a nonlinear equation cannot in general be expressed in closed form; thus we have to use approximate (iterative) methods, where by starting from an initial approximation $x^{(0)}$, we compute the next approximations $x^{(i)}$, $i = 1, 2, \dots$, recurrently:

$$x^{(n+1)} = \varphi(x^{(n)}), \quad n = 0, 1, 2, \dots .$$

There are many known iterative methods for solving Eq. (1). One of the best-known families of iterative functions (for $F : V \subset \mathbb{R}^1 \to \mathbb{R}^1$) is the Chebyshev–Halley family:

$$\varphi(x) = x - \frac{F(x)}{F'(x)}\left(1 + \frac{1}{2}\frac{T_F(x)}{1 - \lambda T_F(x)}\right), \tag{2}$$

where $T_F(x) = \frac{F(x)F''(x)}{F'(x)^2}$ and λ is an arbitrary real parameter. These are order three methods; some particular cases

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E-mail address: gyurhan@shu-bg.net.

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are: Chebyshev's method ($\lambda = 0$), Halley's method ($\lambda = \frac{1}{2}$), and the so-called super Halley's method ($\lambda = 1$). When $\lambda \to \pm\infty$, we get Newton's method (of order two). This family was studied by Werner in 1980 (see [1]), and also can be found in [2,3]. For some interesting results on the methods of family (2), we refer the reader to [4,5]. In [6,7] a modification of family (2) for the multidimensional case is observed.

Recently, in [8] a semi-discrete analogue of The family (2), was presented:

$$\varphi(x) = x - u(x)\left(1 + \frac{1}{2}\frac{F'(x) - F'(x - \beta u(x))}{(\beta - 2)F'(x) + \lambda F'(x - \beta u(x))}\right), \tag{3}$$

where $u(x) = \frac{F(x)}{F'(x)}$, and λ and β are arbitrary real parameters. This modification is obtained from (2) by using the following approximation of the second derivative:

$$F''(x) \approx \frac{F'(x) - F'(x - \beta u(x))}{\beta u(x)}.$$

The following convergence result is proved.

Theorem 1. *Let $F : A \to R$ be a continuous and highly enough times differentiable function on the open interval A. If $F(x)$ has a simple root $\alpha \in A$, then for sufficiently close initial approximations to α, the family (3) has an order of convergence*

(i) 3, *for* $(\lambda, \beta) \neq (1, \frac{2}{3})$;
(ii) 4, *for* $\lambda = 1$ *and* $\beta = \frac{2}{3}$.

Some interesting particular cases of the family (3) are:

$$\varphi(x) = x - \frac{2F(x)}{F'(x) + F'(x - u(x))}, \quad \text{for } \lambda = \frac{1}{2} \text{ and } \beta = 1; \tag{4}$$

$$\varphi(x) = x - \frac{F(x)}{F'\left(x - \frac{u(x)}{2}\right)}, \quad \text{for } \lambda = \frac{1}{2} \text{ and } \beta = \frac{1}{2}; \tag{5}$$

$$\varphi(x) = x - \frac{F'\left(x + \frac{u(x)}{2}\right)}{F'(x)}u(x), \quad \text{for } \lambda = 0 \text{ and } \beta = -\frac{1}{2}; \tag{6}$$

$$\varphi(x) = x - \frac{u(x)}{2}\left(\frac{3F'(y) + F'(x)}{3F'(y) - F'(x)}\right), \quad \text{for } y = x - \frac{3}{2}u(x), \lambda = 1 \text{ and } \beta = \frac{3}{2}. \tag{7}$$

The order of convergence of iterations (4) and (5) is three (see [9]); iteration (7) is known as Jarratt's method and has order four (see [10,11]).

2. The main results

Our purpose in this study is to extend the family (3) to the multidimensional case. We introduce the following modification:

$$\varphi(x) = x - \left(I + \frac{1}{2\beta}\left(I - \frac{\lambda}{\beta}G(x)\right)^{-1}G(x)\right)h(x), \tag{8}$$

where

$$G(x) = I - H(x)F'(y), \quad y = x - \beta h(x), \quad h(x) = H(x)F(x) \quad \text{and} \quad H(x) = F'(x)^{-1}.$$

In [12], Ezquerro et al. present the following biparametric family:

$$\varphi(x) = x - F'(x)^{-1}F'(x) + \frac{1}{2}\theta(x, y)(I + \lambda\theta(x, y))h(x), \tag{9}$$

$$\theta(x, y) = \frac{1}{\beta}H(x)(F'(y) - F'(x)), \quad y = x - \beta h(x)$$

For both the families (8) and (9) in particular, when $\lambda = 0$, we obtain the same one-parametric family:

$$\varphi(x) = x - \left(I + \frac{1}{2\beta} H(x)(F'(x) - F'(z))\right) h(x).$$

When $\lambda \neq 0$, the two families (8) and (9) produce different iterative formulas. For example, if we choose $\lambda = 1$ and $\beta = \frac{2}{3}$, from (8) we obtain a modification of Jarrat's method (7) — see formula (15); whereas for the same values of λ and β, from (9) we obtain a modification of the Inverse-free Jarrat's method:

$$\varphi(x) = x - u(x) + \frac{3}{4} u(x)\theta(x) \left(1 - \frac{3}{2}\theta(x)\right),$$

where

$$\theta(x) = \frac{f'\left(x - \frac{2}{3}u(x) - f'(x)\right)}{f'(x)} \quad \text{and} \quad u(x) = \frac{f(x)}{f'(x)}.$$

Before explore the local convergence properties of the family (8), we will state the following result on Taylor's expression on vector functions (see [13]).

Lemma 2. *Let $F : V \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^p function defined on $V = \{x : \|x - a\| < r\}$; then for any $\|v\| \le r$, the following expression holds:*

$$F(a + v) = F(a) + F'(a)v + \frac{1}{2} F''(a)vv + \cdots + \frac{1}{p!} F^{(p)}(a)v \ldots v + R_p,$$

where

$$\|R_p\| \le \sup_{x \in V} \frac{\|v\|^p}{p!} \|F^{(p)}(x) - F^{(p)}(a)\|.$$

We at first will prove the following lemma.

Lemma 3. *Assume that $F : V \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^4 function, and that it has a locally unique root $\alpha \in V$. Further, assume the Jacobian $F'(x)$ is invertible in a neighbourhood of α; then the following expressions are satisfied:*

$$h''(\alpha) = -H(\alpha)F''(\alpha)$$
$$h'''(\alpha) = 2H''(\alpha)F'(\alpha) + H'(\alpha)F''(\alpha)$$
$$B''(\alpha) = H''(\alpha)F'(\alpha) + 2(1 - \beta)H'(\alpha)F''(\alpha) + (1 - \beta)^2 H(\alpha)F'''(\alpha) + \beta \left(H(\alpha)F''(\alpha)\right)^2,$$

where

$$B(x) = H(x)F'(y), \quad y = y(x) = x - \beta h(x), \quad h(x) = H(x)F(x) \quad \text{and} \quad H(x) = F'(x)^{-1}.$$

Proof. Obviously $h(\alpha) = 0$. Let us write $H(x) = F'(x)^{-1}$ and differentiate the $h(x) = H(x)F(x)$; consequently:

$$h'(x) = H'(x)F(x) + H(x)F'(x) = H'(x)F(x) + I \Rightarrow h'(\alpha) = I;$$
$$h''(x) = H''(x)F(x) + H'(x)F'(x) \Rightarrow h''(\alpha) = H'(\alpha)F'(\alpha) = -H(\alpha)F''(\alpha);$$
$$\begin{aligned} h'''(x) &= H'''(x)F(x) + 2H''(x)F'(x) + H'(x)F''(x) \\ &\Rightarrow h'''(\alpha) = 2H''(\alpha)F'(\alpha) + H'(\alpha)F''(\alpha). \end{aligned}$$

From the expression $y(x) = x - \beta h(x)$, it follows that $y(\alpha) = \alpha$. Hence, we get

$$y'(x) = I - \beta h'(x) \Rightarrow y'(\alpha) = (1 - \beta)I;$$
$$y''(x) = -\beta h''(x) \Rightarrow y''(\alpha) = -\beta h''(\alpha) = \beta H(\alpha)F''(\alpha).$$

From $B(x) = F'(x)^{-1}F'(y)$, it follows $B(\alpha) = I$. Furthermore, we can obtain in turn

$$B'(x) = H'(x)F'(y) + H(x)F''(y)y'(x) \Rightarrow B'(\alpha) = -\beta H(\alpha)F''(\alpha);$$

$$B''(x) = H''(x)F'(y) + 2H'(x)F''(y)y'(x) + H(x)F'''(y)y'(x)^2 + H(x)F''(y)y''(x)$$

$$\Rightarrow B''(\alpha) = H''(\alpha)F'(\alpha) + 2(1-\beta)H'(\alpha)F''(\alpha) + (1-\beta)^2 H(\alpha)F'''(\alpha) + \beta\left(H(\alpha)F''(\alpha)\right)^2.$$

The proof is complete. □

We can now state and prove the main result.

Theorem 4. *Let $F : V \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^4 function in an open convex set $V \subset \mathbb{R}^n$. Assume that there exists an $\alpha \in V$ such that $F(\alpha) = 0$ and $F'(\alpha)^{-1}$ exists. Then there exists an $\varepsilon > 0$ such that for every $x^{(0)} \in U(\alpha, \varepsilon)$, the sequence $x^{(n+1)} = \varphi(x^{(n)})$ generated by (8) is well defined, converges to α, and the process has order*

(i) 3, *for* $(\lambda, \beta) \neq (1, \frac{2}{3})$;
(ii) 4, *for* $\lambda = 1$ *and* $\beta = \frac{2}{3}$.

Proof. From Eq. (8), we get

$$\varphi(x) - \alpha = x - \alpha - \left(I + \frac{1}{2\beta}\left(I - \frac{\lambda}{\beta}G(x)\right)^{-1} G(x)\right)h(x),$$

where

$$G(x) = I - F'(x)^{-1}F'(y) = I - B(x), \quad y = x - \beta h(x), \quad \text{and} \quad h(x) = F'(x)^{-1}F(x).$$

We introduce the notations $L(x) = I - \frac{\lambda}{\beta}G(x)$, $\delta = \varphi(x) - \alpha$ and $\varepsilon = x - \alpha$; then we obtain

$$\delta = L(x)^{-1}\left(L(x)\varepsilon - \left(L(x) + \frac{1}{2\beta}G(x)\right)h(x)\right),$$

and hence

$$\delta = L(x)^{-1}\left(L(x)(\varepsilon - h(x)) - \frac{1}{2\beta}G(x)h(x)\right).$$

Using Lemma 2, we present $h(x)$ by using Taylor series:

$$\begin{aligned} h(x) &= h(\alpha) + h'(\alpha)\varepsilon + \frac{1}{2}h''(\alpha)\varepsilon\varepsilon + \frac{1}{6}h'''(\alpha)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4) \\ &= \varepsilon + \frac{1}{2}h''(\alpha)\varepsilon\varepsilon + \frac{1}{6}h'''(\alpha)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4). \end{aligned} \tag{10}$$

Similarly, we express:

$$B(x) = B(\alpha) + B'(\alpha)\varepsilon + \frac{1}{2}B''(\alpha)\varepsilon\varepsilon + O(\|\varepsilon\|^4). \tag{11}$$

Using (10) and (11), we obtain in turn

$$L(x) = I - \frac{\lambda}{\beta}G(x) = \cdots = I + \lambda h''(\alpha)\varepsilon + \frac{\lambda}{2\beta}B''(\alpha)\varepsilon\varepsilon + O(\|\varepsilon\|^3)$$

and

$$\begin{aligned} \delta &= L(x)^{-1}\Bigg[\left(I + \lambda h''(\alpha)\varepsilon + \frac{\lambda}{2\beta}B''(\alpha)\varepsilon\varepsilon + O(\|\varepsilon\|^3)\right)\left(-\frac{1}{2}h''(\alpha)\varepsilon\varepsilon - \frac{1}{6}h'''(\alpha)\varepsilon\varepsilon\varepsilon + \cdots\right) \\ &\quad - \frac{1}{2\beta}\left(-\beta h''(\alpha)\varepsilon - \frac{1}{2}B''(\alpha)\varepsilon\varepsilon + O(\|\varepsilon\|^3)\right)\left(\varepsilon + \frac{1}{2}h''(\alpha)\varepsilon\varepsilon + O(\|\varepsilon\|^3)\right)\Bigg] \\ &= L(x)^{-1}\Bigg[-\frac{1}{2}h''(\alpha)\varepsilon\varepsilon - \left(\frac{1}{6}h'''(\alpha) + \frac{\lambda}{2}h''(\alpha)^2\right)\varepsilon\varepsilon\varepsilon \\ &\quad + \frac{1}{2}h''(\alpha)\varepsilon\varepsilon + \frac{1}{4\beta}B''(\alpha)\varepsilon\varepsilon\varepsilon + \frac{1}{4}h''(\alpha)^2\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4)\Bigg]. \end{aligned}$$

Further, we obtain

$$\delta = L(x)^{-1}\left[\left(\frac{1-2\lambda}{4}h''(\alpha)^2 - \frac{1}{6}h'''(\alpha) + \frac{1}{4\beta}B''(\alpha)\right)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4)\right].$$

According to statement of Lemma 3, substituting the expressions of h'', h''' and B'', we get

$$\begin{aligned}\delta &= L(x)^{-1}\left[\left(\frac{1-\lambda}{2}\left(H(\alpha)F''(\alpha)\right)^2 + \frac{3-4\beta}{12\beta}H''(\alpha)F'(\alpha) + \frac{3-4\beta}{6\beta}H'(\alpha)F''(\alpha)\right.\right.\\ &\quad\left.\left.+\ \frac{(1-\beta)^2}{4\beta}H(\alpha)F'''(\alpha)\right)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4)\right]\\ &= L(x)^{-1}\left[\left(\frac{1-\lambda}{2}\left(H(\alpha)F''(\alpha)\right)^2 + \frac{3-4\beta}{12\beta}\left(H''(\alpha)F'(\alpha) + 2H'(\alpha)F''(\alpha) + H(\alpha)F'''(\alpha)\right)\right.\right.\\ &\quad\left.\left.+\frac{3\beta^2-2\beta}{12\beta}H(\alpha)F'''(\alpha)\right)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4)\right].\end{aligned}$$

Let us differentiate twice the equation $H(x)F'(x) = I$; we thereby obtain

$$H''(x)F'(x) + 2H'(x)F''(x) + H(x)F'''(x) = 0.$$

Finally, we get

$$\delta = L(x)^{-1}\left[\left(\frac{1-\lambda}{2}\left(H(\alpha)F''(\alpha)\right)^2 + \frac{3\beta-2}{12}H(\alpha)F'''(\alpha)\right)\varepsilon\varepsilon\varepsilon + O(\|\varepsilon\|^4)\right].$$

The latter implies the assertions (i) and (ii) in the statement of Theorem, and this completes the proof. □

2.1. *Some of particular cases of the family (8)*

Corresponding to (4)–(7) are the particular cases:

$$\varphi(x) = x - \left(F'(x) + F'(x-h(x))\right)^{-1}F(x), \quad \text{for } \lambda = \frac{1}{2} \text{and } \beta = 1; \tag{12}$$

$$\varphi(x) = x - F'\left(x - \frac{1}{2}h(x)\right)^{-1}F(x), \quad \text{for } \lambda = \frac{1}{2} \text{ and } \beta = \frac{1}{2}; \tag{13}$$

$$\varphi(x) = x - F'(x)^{-1}F'\left(x + \frac{1}{2}h(x)\right)h(x) \quad \text{for } \lambda = 0 \text{ and } \beta = -\frac{1}{2}; \tag{14}$$

$$\varphi(x) = x - \frac{1}{2}\left(3F'(y) - F'(x)\right)^{-1}\left(3F'(y) + F'(x)\right)h(x), \quad \text{for } \lambda = 1 \text{ and } \beta = \frac{2}{3}; \tag{15}$$

where $y = y(x) = x - \frac{2}{3}h(x)$.

The first three methods (12)–(14) have Order three, while method (15) is of the fourth order.

3. Numerical experiments

In order to demonstrate the performance of the introduced iterative methods of family (8), we have tested them on two examples. We present the results of our comparison of the methods (12), (13) and (15) with the classical Newton's method (NM):

$$\varphi(x) = x - H(x)F(x), \quad \text{where } H(x) = F'(x)^{-1}.$$

The calculations were done using MATLAB. We compare the iterative methods obtained on the basis of following criteria: *number of iterations*, and *computational order of convergence*. We use the following stopping test for our computer programs:

$$\|x^{(k+1)} - x^{(k)}\| < \varepsilon, \quad (\text{where } \varepsilon = 2.22e-16 \text{ is a MATLAB constant}).$$

Table 1
Number of iterations and computational order of convergence for Example 1

Iterative function	$x^0 = (-1, 0.4)$		$x^0 = (-2, 4)$		$x^0 = (-2.5, 3.5)$	
	iter	$\rho_{\text{avg}}(3)$	iter	$\rho_{\text{avg}}(3)$	iter	$\rho_{\text{avg}}(3)$
NM	8	1.80	8	1.80	8	1.83
(12)	5	2.75	5	2.89	5	2.91
(13)	5	2.75	5	2.89	5	2.91
(15)	5	3.91	5	3.92	5	3.95

Table 2
Number of iterations and computational order of convergence for Example 2

Iterative function	$x^0 = (-0.1, 0.4)$		$x^0 = (-0.1, 0.6)$		$x^0 = (-1, 2)$	
	iter	$\rho_{\text{avg}}(3)$	iter	$\rho_{\text{avg}}(3)$	iter	$\rho_{\text{avg}}(3)$
NM	9	1.99	7	2.02	8	1.87
(12)	6	3.03	5	2.87	5	2.78
(13)	6	3.62	5	2.36	5	2.98
(15)	5	3.75	3	3.47	3	3.83

The computational order of convergence $\rho(i, k)$, for series $\{x_i^{(k)}\}$, is computed by [14]:

$$\rho(i, k+1) = \frac{\ln \left| (x_i^{(k+1)} - \alpha_i)/(x_i^{(k)} - \alpha_i) \right|}{\ln \left| (x_i^{(k)} - \alpha_i)/(x_i^{(k-1)} - \alpha_i) \right|}, \quad i = 1, \dots, n, \ k >= 2.$$

We introduce the notation: $x^{(0)}$ — an initial approximation; iter — number of iterations; coc — computational order of convergence computed by $\rho_{\text{avg}}(k) = \text{average}(\rho(1, k), \dots, \rho(n, k))$.

Example 1. Consider the system:

$$\left| \begin{array}{l} x_1^2 - 4x_1 + x_2^2 = 0 \\ 2x_1 + x_2^2 - 2 = 0, \end{array} \right.$$

This has a zero close to $\alpha \approx (0.3542486889, 1.136442969)^{\mathrm{T}}$—see the results in Table 1.

Example 2. Consider the system:

$$\left| \begin{array}{l} x_1^3 - 3x_1 x_2^2 - 1 = 0 \\ 3x_1^2 x_2 - x_2^3 + 1 = 0, \end{array} \right.$$

This has a zero close to $\alpha \approx (-0.290514555, 1.084215081)^{\mathrm{T}}$—see the results in Table 2.

4. Conclusion

The study presents a family of multi-point iterative methods for solving nonlinear equations, in the multivariate case. This family is a generalization of a known univariate family and depends on two real parameters. As particular cases, we obtain new methods, including some existing well-known methods. These methods have order three or four, and they do not require the execution of second or higher order Fréchet derivatives of the function F. Per iteration, each one of the presented methods needs one evaluation of the function vector F and the solving of two linear systems, where the Jacobian F' has to be evaluated twice.

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