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Relationship between the homotopy analysis method and harmonic balance method

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ABSTRACT

This paper presents a study of the relationship between the homotopy analysis method (HAM) and harmonic balance (HB) method. The HAM is employed to obtain periodic solutions of conservative oscillators and limit cycles of self-excited systems, respectively. Different from the usual procedures in the existing literature, the HAM is modified by retaining a given number of harmonics in higher-order approximations. It is proved that as long as the solution given by the modified HAM is convergent, it converges to one HB solution. The Duffing equation, the van der Pol equation and the flutter equation of a two-dimensional airfoil are taken as illustrations to validate the attained results.

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1. Introduction

Over the past decade, Liao described a nonlinear analytical technique which does not require small parameters and thus can be applied to solve nonlinear problems without small or large parameters [1–4]. This technique is based on homotopy theory, which is an important part of topology, thus called the homotopy analysis method (HAM). Its fundamental idea is to construct a class of homotopy in a rather general form by introducing an auxiliary parameter, through which nonlinear problems can be transformed into a series of linear sub-problems. The auxiliary parameter can provide us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. The systematical description of this method was given in Ref. [5]. Also in this paper, the author discussed the convergence of the solution series and showed that as long as the series given by HAM converges, it must converge to one solution of the nonlinear problem under consideration. In the rapid development of HAM, it has been widely used in various nonlinear problems [6–10].

Perturbation method [11] is one of the most widely applied analytic tools for nonlinear problems. Essentially, perturbation techniques are based on the existence of a small/large parameter or variable, which is often called perturbation quantity. The existence of perturbation quantities, however, is a cornerstone of these techniques. The dependence of perturbation techniques on small/large parameters might be avoided by introducing a so-called artificial small parameter, such as the Lyapunov artificial small parameter method [12], the δ-expansion method [13] and the Adomian's decomposition method [14]. Liao [15] proved that they are all special cases of the HAM, which implies the HAM is more generalized. Additionally,

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exploring the inner relationship between existing computational techniques is of fundamental interest to many researchers engaged in computing science. Thus, it is worth and interesting to investigate its relationship of the HAM to other methods for nonlinear systems.

The main aim of this paper is to study the relationship between the HAM and harmonic balance (HB) method. The basic procedure of the HB method is to transform the problem under consideration into a set of nonlinear algebraic equations by describing the possible periodic/limit cycle solution as truncated Fourier series [16]. That means the solutions given by the HB method possess a limited number of harmonics. However, the highest harmonic of the periodic solutions obtained by the HAM increases unboundedly [6]. For this issue, the HAM is slightly modified by retaining several lower-order harmonics to obtain solutions in the same form of HB ones. A major finding of this paper is that as long as the solution given by the modified HAM converges, it must converge to one HB solution. In order to validate it, proofs are given and three numerical examples are also presented.

2. Homotopy analysis method

Consider a nonlinear autonomous system described by

$$f(x,\dot{x},\ddot{x})=0 \tag{1}$$

where the superscript denotes the differentiation with respect to time t. In this study, system (1) may either be a conservative or self-excited system so that it possesses at least one periodic (or limit cycle) solution. Introducing a new time scale

$$\tau=\omega t \tag{2}$$

where ω is the angular frequency of the possible periodic solution, then (1) becomes

$$f(x,\omega x',\omega^2 x'')=0 \tag{3}$$

where the superscript denotes the differentiation with respect to τ.

2.1. Self-excited system

In general, limit cycles of self-excited oscillating systems contain two important physical parameters, i.e., the frequency ω and the amplitude a of oscillation. They are both independent upon the initial conditions. Without loss of generality, consider simple initial conditions

$$x(0)=a,\quad x'(0)=0. \tag{4}$$

For self-excited system, a is the amplitude of the limit cycle to be determined.

According to Eqs. (3) and (4), let

$$x_0(\tau)=a_0\cos(\tau) \tag{5}$$

be the initial guess of $x(\tau)$, where a_0 is the one of a. Likewise, let ω_0 be the initial approximation of ω. The HAM is based on such continuous variations $\phi(\tau,p)$, $\Omega(p)$ and $A(p)$ that, as the embedding parameter p increases from 0 to 1, $\phi(\tau,p)$ varies from the initial guess $x_0(\tau)=a\cos\tau$ to the exact solution $x(\tau)$, so do $\Omega(p)$ and $A(p)$ from ω_0 and a_0 to ω and a, respectively.

The rule of solution expression [6] states that $x(\tau)$ can be described as a set of base functions $\{\cos(k\tau),\sin(k\tau)|\ k=0,1,2,\ldots\}$, based on which we can choose such an auxiliary linear operator

$$L[\phi(\tau,p)]=\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+\phi(\tau,p) \tag{6}$$

so that

$$L[\cos\tau]=L[\sin\tau]=0. \tag{7}$$

Then according to Eq. (3), we define the following nonlinear operator:

$$\Psi[\phi(\tau,p),\Omega(p),A(p)]=f\left[\phi(\tau,p),\Omega(p)\frac{\partial\phi(\tau,p)}{\partial\tau},\Omega^2(p)\frac{\partial^2\phi(\tau,p)}{\partial^2\tau}\right] \tag{8}$$

where $p\in[0,1]$ is the embedding parameter. Letting h be a nonzero auxiliary parameter, we construct such a homotopy in a general form

$$H[\phi(\tau,p);h,p]=(1-p)L[\phi(\tau,p)-x_0(\tau)]-hp\Psi[\phi(\tau,p),\Omega(p),A(p)]. \tag{9}$$

Setting $H[\phi(\tau,p);h,p]=0$ yields a family of equations

$$(1-p)L[\phi(\tau,p)-x_0(\tau)]=hp\Psi[\phi(\tau,p),\Omega(p),A(p)] \tag{10}$$

subject to the initial conditions

$$\phi(0,p) = A(p), \quad \frac{\partial \phi(\tau,p)}{\partial \tau}|_{\tau=0} = 0 \tag{11}$$

which are called the zeroth-order deformation equations. Obviously, when $p = 0$, Eqs. (10) and (11) have the solution

$$\phi(\tau,0) = x_0(\tau) = a_0 \cos\tau \tag{12}$$

when $p = 1$, they are exactly the same as Eqs. (3) and (4) provided that

$$\phi(\tau,1) = x(\tau), \quad \Omega(1) = \omega, \quad A(1) = a. \tag{13}$$

Assuming that the so-called deformation derivatives

$$x_0^{(k)}(\tau) = \frac{\partial^k \phi(\tau,p)}{\partial p^k}|_{p=0}, \quad \omega_0^{(k)} = \frac{\partial^k \Omega(p)}{\partial p^k}|_{p=0}, \quad a_0^{(k)} = \frac{\partial^k A(p)}{\partial p^k}|_{p=0} \tag{14}$$

all exist, $\phi(\tau,p), \Omega(p)$ and $A(p)$ can be expanded in the series of p as follows:

$$\phi(\tau,p) = \sum_{k=0}^{+\infty} x_k(\tau)p^k, \quad \Omega(p) = \sum_{k=0}^{+\infty} \omega_k p^k, \quad A(p) = \sum_{k=0}^{+\infty} a_k p^k \tag{15}$$

where

$$x_k = \frac{x_0^{(k)}(\tau)}{k!}, \quad \omega_k = \frac{\omega_0^{(k)}}{k!}, \quad a_k = \frac{a_0^{(k)}}{k!}. \tag{16}$$

Note that series (15) contain the auxiliary parameter h, which determines their convergence regions. Choosing such a proper value of h that all of these series are convergent at $p = 1$, we can obtain the results to mth-order approximation as

$$x(\tau) = \sum_{k=0}^{m} x_k(\tau), \quad \omega = \sum_{k=0}^{m} \omega_k, \quad a = \sum_{k=0}^{m} a_k. \tag{17}$$

Differentiating k times Eqs. (10) and (11) with respect to p and then setting $p = 0$ and finally dividing them by $k!$, we can deduce the governing equations of $x_k(\tau)$ as

$$L\left[x_k(\tau) - \chi_k x_{k-1}\right] = hR_k(\tau) \tag{18}$$

subject to the initial conditions

$$x_k(0) = a_k, \quad x_k'(0) = 0 \tag{19}$$

where

$$R_k(\tau) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \Psi[\phi(\tau,p), \Omega(p), A(p)]}{\partial p^{k-1}}|_{p=0} \tag{20}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1 \\ 1, & k > 1 \end{cases} \tag{21}$$

Rewrite $R_k(\tau)$ as

$$R_k(\tau) = \sum_{n=1}^{\varphi(k)} \left[c_{k,n}\cos(n\tau) + s_{k,n}\sin(n\tau)\right] \tag{22}$$

where

$$c_{k,n} = \frac{2}{\pi}\int_0^\pi R_k(\tau)\cos(n\tau)\,d\tau, \quad s_{k,n} = \frac{2}{\pi}\int_0^\pi R_k(\tau)\sin(n\tau)\,d\tau$$

the integer $\varphi(k)$ is a function of k and the coefficients $c_{k,n}$ and $s_{k,n}$ become zeroes when $n > \varphi(k)$.

Due to the rule of solution expression and the definition of the linear operator L, the solutions of Eqs. (18) and (19) should not contain the so-called secular terms $\tau\sin\tau$ and $\tau\cos\tau$, resulted from $\cos\tau$ and $\sin\tau$ respectively. To ensure so, we can equate the coefficients of $\sin\tau$ and $\cos\tau$ to zeros

$$c_{k,1}(\omega_0, \omega_1, \ldots, \omega_{k-1}, a_0, a_1, \ldots, a_{k-1}) = 0 \tag{23}$$
$$s_{k,1}(\omega_0, \omega_1, \ldots, \omega_{k-1}, a_0, a_1, \ldots, a_{k-1}) = 0. \tag{24}$$

Then, ω_{k-1} and a_{k-1} $(k = 1, 2, 3, \ldots)$ can be determined by solving (23) and (24). The above two algebraic equations are usually nonlinear for ω_0 and a_0, but always linear when $k \geqslant 2$ [6].

2.2. Conservative system

When Eq. (1) denotes a conservative system, its solution is periodic for any given conditions. For brevity, adopting the simple initial conditions in the form of (4), the initial guess of the solution can then be described as

$$x_0(\tau) = a\cos(\tau) \tag{25}$$

where a is a given constant. Then, $A(p) = a$ need not be expanded. Accordingly, there is only one unknown (i.e., ω_{k-1}) to be determined by the kth-order deformation equation. Assuming that $R_k(\tau)$ does not contain sine harmonics (e.g., f is an odd function), using the above proposed procedures we can determine ω_{k-1} by solving

$$c_{k,1}(\omega_0, \omega_1, \cdots, \omega_{k-1}, a) = 0 \tag{26}$$

which is also a linear algebraic equation as $k \geqslant 2$.

3. A modified HAM

In general, the highest harmonic of the kth-order approximation $x_k(\tau)$ increases with k. For many periodic motions, the amplitudes of high harmonics are just small quantities compared with those of lower-order harmonics. Thus, it is enough to obtain high accurate solutions by retaining only several harmonics.

Neglect all the harmonics higher than Nth-order in $R_k(\tau)$ and rewrite $R_k(\tau)$ as

$$R_k^N(\tau) = \sum_{n=1}^{N}\left[c_{k,n}\cos(n\tau) + s_{k,n}\sin(n\tau)\right] \tag{27}$$

where $c_{k,n}$ and $s_{k,n}$ are as defined above. According to Eqs. (17) and (27), the solution to mth-order approximation contains N harmonics if m is large enough. In the following, we call the modified approach as HAM N for convenience.

4. Harmonic balance method

For self-excited vibrating system (3), the HB N solution can be described as

$$x_{\mathrm{HB}N}(\tau) = \sum_{n=1}^{N}[\alpha_n\cos(n\tau) + \beta_n\sin(n\tau)]. \tag{28}$$

Considering the initial conditions (4), we have

$$\sum_{n=1}^{N} n\beta_n = 0. \tag{29}$$

Substituting Eq. (28) into (3) and using the Garlerkin procedure result in the following harmonic balance equations:

$$\begin{aligned}
\Gamma_{n,c}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\omega) &= \frac{2}{\pi}\int_0^{\pi} f(x_{\mathrm{HBN}}, \omega x'_{\mathrm{HBN}}, \omega x''_{\mathrm{HBN}})\cos(n\tau)\,d\tau = 0\\
\Gamma_{n,s}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\omega) &= \frac{2}{\pi}\int_0^{\pi} f(x_{\mathrm{HBN}}, \omega x'_{\mathrm{HBN}}, \omega x''_{\mathrm{HBN}})\sin(n\tau)\,d\tau = 0
\end{aligned} \tag{30}$$

Then $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$ and ω can be obtained by solving Eqs. (29) and (30).

When Eq. (3) is conservative and f is an odd function, the solution contains no sine harmonics. Then, the HB N solution can be given as

$$x_{\mathrm{HB}N}(\tau) = \sum_{n=1}^{N}\alpha_n\cos(n\tau). \tag{31}$$

Considering the initial conditions (4), we introduce an additional equation

$$\sum_{n=1}^{N}\alpha_n = a \tag{32}$$

where a is a constant. The harmonic balance equations are as follows:

$$\Gamma_{n,c}(\alpha_1,\ldots,\alpha_n,\omega) = \frac{2}{\pi}\int_0^{\pi} f(x_{\mathrm{HBN}}, \omega x'_{\mathrm{HBN}}, \omega x''_{\mathrm{HBN}})\cos(n\tau)\,d\tau = 0. \tag{33}$$

Then, $\alpha_1, \alpha_2, \ldots, \alpha_n$ and ω can be obtained by Eqs. (32) and (33).

5. Relationship between HAMN and HBN

In this section, we will prove that as long as the HAM N solution is convergent, it must converge to one HB N solution.

Without loss of generality, we only consider the case when system (1) is self-excited. If the auxiliary parameter h is properly chosen so that series (15) converge at $p = 1$, it holds that

$$\lim_{k\to\infty} x_k(\tau) = 0, \tag{34}$$

According to Eq. (28), we have

$$\begin{aligned} &L[x_1(\tau)] = hR_1^N(\tau) \\ &L[x_2(\tau) - x_1(\tau)] = hR_2^N(\tau) \\ &\cdots \\ &L[x_k(\tau) - x_{k-1}(\tau)] = hR_k^N(\tau) \end{aligned} \tag{35}$$

Summarizing Eq. (35) results in

$$L[x_k(\tau)] = h\sum_{i=1}^{k} R_i^N(\tau) \tag{36}$$

Due to the definition of the linear operator L and Eqs. (34) and (36), and considering that h is a nonzero constant, we know

$$\lim_{k\to\infty} \sum_{i=1}^{k} R_i^N(\tau) = 0 \tag{37}$$

which implies for $1 \leqslant n \leqslant N$ that

$$C_n = \lim_{k\to\infty} \sum_{i=1}^{k} c_{i,n} = 0, S_n = \lim_{k\to\infty} \sum_{i=1}^{k} s_{i,n} = 0 \tag{38}$$

According to the definitions of $R_i(\tau)$ and of the nonlinear operator, we have

$$\Psi[\phi(\tau,p),\Omega(p),A(p)] = \sum_{i=1}^{\infty} R_i(\tau)p^{i-1} = \sum_{n=1}^{\infty}\left[(\lim_{k\to\infty}\sum_{i=1}^{k} c_{i,n}p^{i-1})\cos(n\tau) + (\lim_{k\to\infty}\sum_{i=1}^{k} s_{i,n}p^{i-1})\sin(n\tau)\right] \tag{39}$$

thus

$$\begin{aligned} \Psi[\phi(\tau,1),\Omega(1),A(1)] &= f\left[\phi(\tau,1),\Omega(1)\frac{\partial\phi(\tau,1)}{\partial\tau},\Omega^2(1)\frac{\partial^2\phi(\tau,1)}{\partial\tau^2}\right] = f\left[x_{\mathrm{HAM}N}(\tau),\omega x'_{\mathrm{HAM}N}(\tau),\omega^2 x''_{\mathrm{HAM}N}(\tau)\right] \\ &= \sum_{n=1}^{\infty}\left[(\lim_{k\to\infty}\sum_{i=1}^{k} c_{i,n})\cos(n\tau) + (\lim_{k\to\infty}\sum_{i=1}^{k} s_{i,n})\sin(n\tau)\right] \end{aligned} \tag{40}$$

where ω is given by the HAMN. Substituting Eq. (38) into (40), the first to Nth-order harmonics in the right hand side term of (40) vanish, which implies that the solution $x_{\mathrm{HAM}N}$ complies with the principle of harmonic balance. In other words, the HAM N solution must converge to an HB N solution.

6. Numerical examples

For the Duffing equation, $f(x,\dot{x},\ddot{x})$ are described as

$$f(x,\dot{x},\ddot{x}) = \ddot{x} + x + k_3x^3 \tag{41}$$

subject to the initial conditions described as Eq. (4) with a as a given constant. The HAM5 and HB5, HAM7 and HB7 solutions are shown in Tables 1 and 2, respectively. As m increases, the convergence of HAM5 to HB5 and of HAM7 to HB7 solutions can be observed.

If $f(x,\dot{x},\ddot{x})$ is given as

$$f(x,\dot{x},\ddot{x}) = \ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} \tag{42}$$

where ε is a constant, Eq. (1) is called as the van der Pol equation. This equation possesses a limit cycle, which is investigated by many authors [6,10,17]. The HAM3 and HB3 solutions are given in Table 3, which shows that the HAM3 solutions converge to the HB3 results.

Since it is rather difficult to obtain the higher HB solutions, alternatively we check whether the solutions provided by the modified HAM satisfy the harmonic balance equation, i.e., Eq. (30). Table 4 shows the amplitudes of the harmonics in the

Table 1
Comparison of the HAM5 with HB5 solutions of the Duffing equation with $k_3 = 1$ and $a = 10$.

HAM $h = -0.5$	C_1	C_3	C_5	ω
$m = 0$	10	0	0	8.71779788708135
$m = 25$	9.55749682827188	0.42445891229652	0.01804425943159	8.53429908176645
$m = 50$	9.55747352937422	0.42447550988862	0.01805096073715	8.53429357009445
$m = 150$	9.55747352935610	0.42447550989885	0.01805096074504	8.53429357009400
HB5	α_1	α_3	α_5	ω
	9.55747352935600	0.42447550989884	0.01805096074504	8.53429357009391

Table 2
Comparison of the HAM7 with HB7 solutions of the Duffing equation with $k_3 = 10$ and $a = 10$.

HAM $h = -0.5$	C_1	C_3	C_5	C_7	ω
$m = 0$	10	0	0	0	27.40437921208944
$m = 25$	9.55076735957974	0.42988225642031	0.01855081948431	0.00079956451564	26.81084017515133
$m = 50$	9.55076438499581	0.42988416387164	0.01855167026962	0.00079978086293	26.81083764285547
$m = 150$	9.55076438494495	0.42988416390005	0.01855167028649	0.00079978086851	26.81083764282150
HB7	a_1	a_3	a_5	a_7	ω
	9.55076438494454	0.42988416390003	0.01855167028649	0.00079978086851	26.81083764282035

Table 3
Comparison of the HAM3 with HB3 solutions of the van der Pol equation with $\varepsilon = 1$.

HAM3 $h = -0.5$	C_1	C_3	S_3	ω
$m = 0$	2	0	0	1
$m = 25$	1.91261513245611	0.13757329517960	−0.21428419027216	0.94264330571373
$m = 50$	1.91262564245630	0.13755333566515	−0.21426886159367	0.94264340010520
$m = 100$	1.91262563847376	0.13755332871588	−0.21426886445153	0.94264341005767
HB3	α_1	α_3	β_3	ω
	1.91262563847380	0.13755332871578	−0.21426886445147	0.94264341005769

Table 4
Amplitudes of the harmonics in the residual of the van der Pol equation for HAM5 solution.

HAM5 $h = -0.3$	$\Gamma_{1,c}$	$\Gamma_{1,s}$	$\Gamma_{3,c}$	$\Gamma_{3,s}$	$\Gamma_{5,c}$	$\Gamma_{5,s}$	$\Gamma_{7,c}$	$\Gamma_{7,s}$	$\Gamma_{9,c}$	$\Gamma_{9,s}$
$m = 0$	0	0	0	−2	0	0	0	0	0	0
$m = 25$	−1.2E−4	1.0E−3	−3.8E−3	−1.0E−2	−1.1E−2	1.1E−2	−0.4909	−0.1119	−0.0689	0.0844
$m = 50$	−2.3E−5	4.3E−5	−2.8E−4	−4.0E−4	−3.4E−4	7.2E−4	−0.4916	−0.1058	−0.0673	0.0855
$m = 120$	−1.1E−8	−6.2E−8	−1.4E−9	5.2E−7	8.0E−7	−2.9E−7	−0.4916	−0.1055	−0.0671	0.0855

residuals of Eq. (30) with x being given by HAM5. Both $\Gamma_{i,c}$ and $\Gamma_{i,s}$ $(i = 1, 3, 5)$ converge to 0 when m increasing, however the coefficients of higher-order harmonics, i.e., $\Gamma_{i,c}$ and $\Gamma_{i,s}$ $(i = 7, 9)$, converge to constants. That means the HAM5 solution satisfies (30) when $N = 5$.

The equations of motions of a two-dimensional airfoil with cubic pitching stiffness in incompressible flow can be described as [18]

$$\mathbf{M}\ddot{\mathbf{x}} + \mu\dot{\mathbf{x}} + \mathbf{K_Q}\mathbf{x} + \mathbf{g}(\mathbf{x}) = 0 \tag{43}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$\mathbf{K}_Q = \begin{bmatrix} 0.2 & 0.1Q \\ 0 & 0.5 - 0.04Q \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ k_3x_2^3 \end{bmatrix},$$

and Q is the generalized flow speed, k_3 is a positive constant. For more details, please refers to [18]. Eq. (43) possesses one stable limit cycle when Q is larger than the linear flutter boundary $Q = 4.08$. The convergence of HAM3 and HAM5 solutions are validated by checking the residuals of Eq. (43), as shown in Tables 5 and 6 respectively. Obviously, both $\Gamma_{i,c}$ and $\Gamma_{i,s}$ ($i = 1, 3$ for HAM3 and $i = 1, 3, 5$ for HAM5 solution) converge to 0 as m increases, however, the coefficients of higher-order harmonics, e.g., $\Gamma_{i,c}$ and $\Gamma_{i,s}$ ($i = 5, 7, 9$ for HAM3, $i = 7, 9$ for HAM5 solution) converge to constants. Notice that the residuals of the higher-order harmonics of the first equation in Eq. (43) always remain as 0 because this equation is linear.

Let E_i ($i = 1, 3, \ldots$) be the amplitudes of the ith-order harmonics of the residual of Eq. (43), that

$$E_i = \sqrt{\Gamma_{i,c}^2 + \Gamma_{i,s}^2} \tag{44}$$

Figs. 1 and 2 show the amplitudes of the harmonics of the residuals for HAM3 solution versus m, where the second subscript "1" or "2" denotes the first or second equation in (43), respectively. As m increases, the amplitudes of lower-order harmonics

Table 5
Amplitudes of the harmonics in the residual of the flutter equation for HAM3 solution.

HAM3 $h = -3$	$\Gamma_{1,c}$	$\Gamma_{1,s}$	$\Gamma_{3,c}$	$\Gamma_{3,s}$	$\Gamma_{5,c}$	$\Gamma_{5,s}$	$\Gamma_{7,c}$	$\Gamma_{7,s}$	$\Gamma_{9,c}$	$\Gamma_{9,s}$
$m = 0$	−2.4E−17	2.3E−18	0	0	0	0	0	0	0	0
	−1.6E−18	−1.0E−18	1.7E−2	0	0	0	0	0	0	0
$m = 25$	3.5E−8	1.1E−8	2.3E−8	−8.6E−9	0	0	0	0	0	0
	1.3E−8	−1.6E−9	8.2E−8	−3.0E−8	3.8E−3	−2.4E−4	2.8E−4	−4.8E−5	7.5E−6	−2.2E−6
$m = 50$	1.9E−12	5.9E−13	1.2E−12	−2.9E−13	0	0	0	0	0	0
	7.1E−13	−3.9E−14	3.8E−12	−1.3E−12	3.8E−3	−2.4E−4	2.8E−4	−4.8E−5	7.4E−6	−2.2E−6
$m = 100$	−7.2E−17	3.0E−17	−7.4E−18	9.8E−19	0	0	0	0	0	0
	7.7E−17	−5.4E−18	4.6E−17	4.0E−18	3.8E−3	−2.4E−4	2.8E−4	−4.8E−5	7.4E−6	−2.2E−6

Table 6
Amplitudes of the harmonics in the residual of the flutter equation for HAM5 solution.

HAM5 $h = -3$	$\Gamma_{1,c}$	$\Gamma_{1,s}$	$\Gamma_{3,c}$	$\Gamma_{3,s}$	$\Gamma_{5,c}$	$\Gamma_{5,s}$	$\Gamma_{7,c}$	$\Gamma_{7,s}$	$\Gamma_{9,c}$	$\Gamma_{9,s}$
$m = 0$	−2.4E−17	2.3E−18	0	0	0	0	0	0	0	0
	−1.6E−18	−1.0E−18	1.7E−2	0	0	0	0	0	0	0
$m = 25$	9.1E−8	3.6E−8	5.9E−8	−1.4E−8	−2.7E−8	9.5E−9	0	0	0	0
	4.6E−8	−7.9E−9	2.1E−7	−6.4E−8	2.1E−7	−2.2E−8	6.0E−4	−7.7E−5	5.2E−5	−1.1E−5
$m = 50$	1.6E−11	5.7E−12	9.4E−12	−1.9E−12	−3.6E−12	1.1E−12	0	0	0	0
	7.4E−12	−8.4E−13	2.8E−11	−9.1E−12	2.9E−11	−2.9E−12	6.0E−4	−7.7E−5	5.2E−5	−1.1E−5
$m = 100$	1.7E−16	−7.0E−18	4.1E−18	−6.9E−19	1.3E−18	−7.3E−20	0	0	0	0
	1.1E−16	−7.3E−18	2.8E−17	−1.8E−18	1.1E−17	−5.5E−19	6.0E−4	−7.7E−5	5.2E−5	−1.1E−5

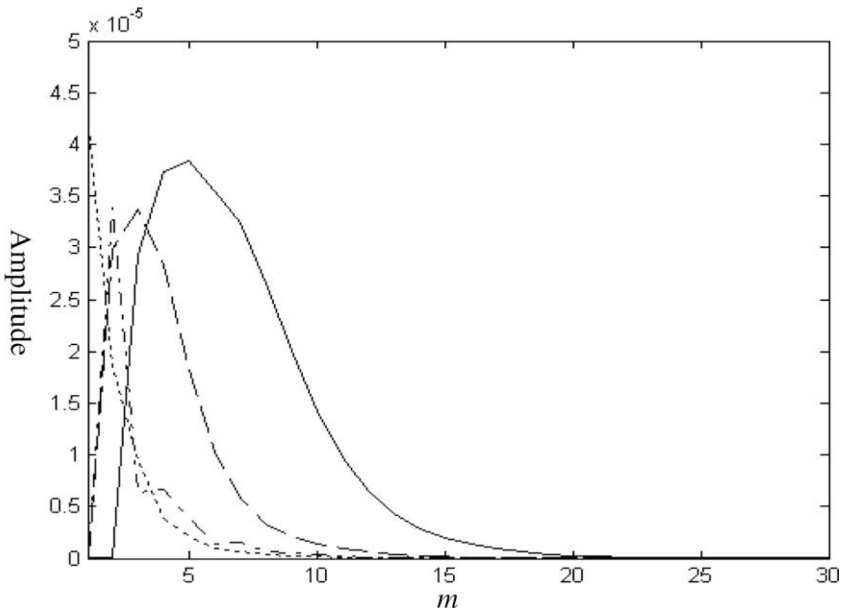

Fig. 1. Amplitudes of the harmonics in the residuals of Eq. (43) for HAM3 solutions with $Q = 8$ and $h = -3$. Real line denotes $E_{1,1}$, dashed line for $E_{1,2}/5$, dash-dot line for $E_{3,1}/50$, and dot line for $E_{3,2}/400$.

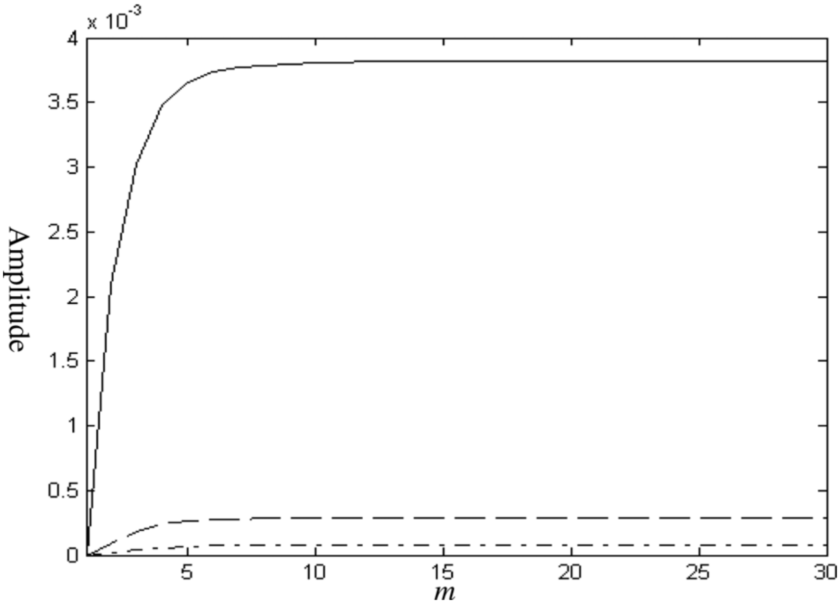

Fig. 2. Amplitudes of the harmonics in the residuals of Eq. (43) for HAM3 solutions with $Q = 8$ and $h = -3$. Real line for $E_{5,2}$, dashed line $E_{7,2}$, and dash-dot line for $10E_{9,2}$.

converge to zeros (see Fig. 1), while those of higher-order harmonics converge to constants (see Fig. 2). That means the HAM3 solution indeed complies with the principle of harmonic balance.

7. Conclusions and remarks

In this paper, the HAM is employed to obtain periodic solutions of both conservative and self-excited systems. The HAM solutions with a given number of harmonics are obtained by a modified scheme. This makes it convenient to study its relationship to the well-known harmonic balance method. Interestingly, it reveals that the solution with limited harmonics ob-

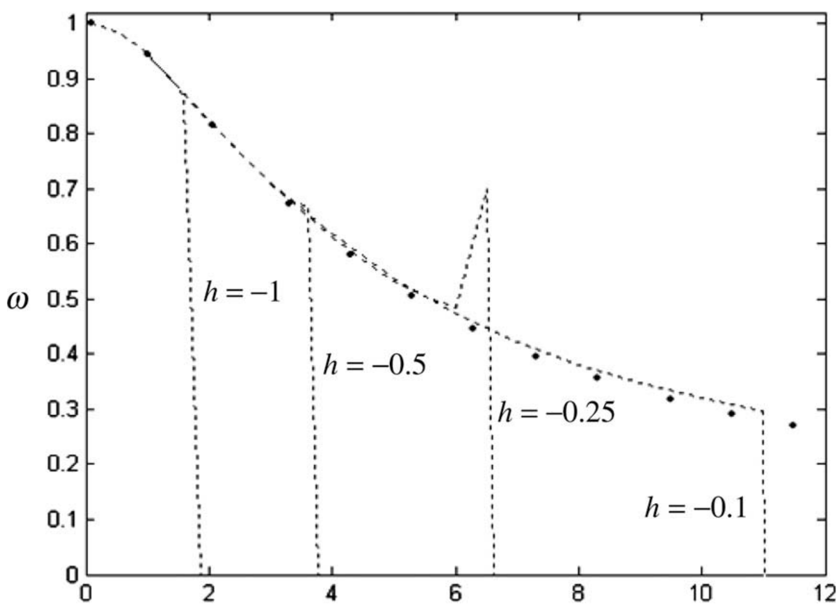

Fig. 3. Frequency of the limit cycle of van der Pol equation given by the 50th-order HAM solution with $N = 20$, where the heavy dots denote numerical solutions.

tained by the modified HAM converges to one of HB solutions, which implies the proposed theory can give some new insight into both the HAM and HB method.

Note that the convergence of the HAM's series depends on the auxiliary parameter h. Fig. 3 presents the curves of HAM solutions with 20 harmonics versus the nonlinear coefficient ε, where the longitudinal coordinate denotes the frequency of the limit cycle of van der Pol equation. When ε is relatively small, the HAM solutions are in good agreement with numerical results. As ε increases, however, the HAM's series become divergent and hence the results cease to be valid. That is because ε increases beyond the convergence region of the HAM's series. It also shows that the smaller h is, the larger the convergence radius is.

Therefore, the auxiliary parameter ensures it convenient to adjust and control the convergence of HAM's series. Due to this convenience, the HAM can be valid for both weakly and strongly nonlinear oscillators, as is shown in Fig. 3. On the other hand, the convergence region of the numerical solution process of HB method is uniquely determined and in many cases is very small. Well known, in the incremental HB method a major difficulty is to search a proper but usually very restricted initial guess. In addition, as stated above, the high-order HAM approximations can be obtained just by solving linear algebraic equations. While in the HB, usually, very complicated nonlinear algebraic equations have to be numerically solved, especially when many harmonics are taken into account. In summary, the HAM is in a sense more general, and for some cases is much easier to be implemented than is the HB.

Acknowledgments

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