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BIFURCATION ANALYSIS OF AIRFOILS IN INCOMPRESSIBLE FLOW

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Bifurcations of airfoils with non-linear pitching stiffness in incompressible flow are investigated. The harmonic balance method, which is more practical, is adopted for the analysis, and numerical integration results are used as standards of checking. Analytical solutions calculated by asymptotic expansion method and averaging method with the aid of computer algebra have confirmed the feasibility of the harmonic balance method.

1. INTRODUCTION

An airfoil in incompressible flow is a classical example of self-excited vibration. Introducing non-linear stiffness into this system will generally lead to limit cycle flutter phenomenon, for which Yang and Zhao [1] have made systematic investigations by the aid of harmonic balance method. Also, they [2] have analyzed and found complicated period doubling motions and chaotic motions by numerical integrations. In this paper, it is shown that the harmonic balance method can also provide sufficiently accurate information about the bifurcations of the airfoil with respect to the airspeed. Supercritical and subcritical Hopf bifurcations are found for two airfoils with the same cubic pitching stiffness and other parameters but different linear stiffness. As for most non-linear problems, numerical solutions play a role in obtaining exact results. Herein, the fourth order Runge–Kutta method is adopted to obtain limit cycles and phase portraits. Asymptotic expansion and the averaging method together with center manifold theory are also used to obtain the analytical solutions. For implementation of these, computer algebra [3] is employed; calculation by hand would be hardly possible. Numerical and analytical results have confirmed the feasibility of the harmonic balance method.

2. MATHEMATICAL MODEL

A sketch of the airfoil under study is shown in Figure 1: the meanings of all the parameters can be found in reference [2]. The governing equations are the same as equations (2) of reference [2]. Upon replacing the definite value 0·5 in $0{\cdot}5a$ by K_0 to allow for

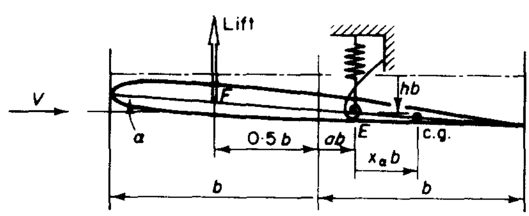

Figure 1. Sketch of the two-dimensional airfoil.

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a study of the effects of the linear stiffness on the bifurcations, the governing equations become

$$\ddot{h}+0\cdot25\ddot{\alpha}+0\cdot1\dot{h}+0\cdot2h+0\cdot1Q\alpha=0, \qquad 0\cdot25\ddot{h}+0\cdot5\ddot{\alpha}+0\cdot1\dot{\alpha}+K_0\alpha+e\alpha^3-0\cdot04Q\alpha=0. \tag{1}$$

3. HARMONIC BALANCE METHOD

According to the concept of first order harmonic balance, the stiffness term $(K_0\alpha+e\alpha^3)$ in equations (1) is replaced by an equivalent stiffness term $K_\alpha \alpha$, and the flutter determinant is

$$\begin{vmatrix} -\omega^2+0\cdot1\mathrm{i}\omega+0\cdot2 & -0\cdot25\omega^2+0\cdot1Q \\ -0\cdot25\omega^2 & -0\cdot5\omega^2+0\cdot1\mathrm{i}\omega+K_\alpha-0\cdot04Q \end{vmatrix}=0, \tag{2}$$

where Q and ω are the non-dimensional flutter speed squared and the flutter frequency respectively, and $\mathrm{i}=\sqrt{-1}$.

By solving the real and imaginary parts of equation (2) simultaneously, the following results are obtained:

$$0\cdot32Q^2-(12\cdot25K_\alpha+0\cdot11)Q+(106\cdot25K_\alpha^2-16K_\alpha+1\cdot55)=0, \tag{3}$$

$$\omega^2=(K_\alpha+0\cdot2-0\cdot04Q)/1\cdot5. \tag{4}$$

In our study, e is taken to be 20, and the equivalent linear stiffness term of $e\alpha^3$ is $(3/4)eA^2\alpha$: i.e., $15A^2\alpha$. As a result,

$$K_\alpha=K_0+15A^2 \qquad \text{or} \qquad A=\sqrt{(K_\alpha-K_0)/15}, \tag{5}$$

where A is the flutter amplitude of α.

Let $K_\alpha=K_0$. It is not difficult to find the bifurcation points from equation (3). Our discussion is limited to airspeeds lower than the static divergent speed Q_D, which can be found from equations (1) to be $K_0/0\cdot04$.

First case: $K_0=0\cdot5$

For this case, $Q_D=12\cdot5$, and hence only the motion for $Q<12\cdot5$ is to be considered. The Q versus K_α and K_α versus A curves are obtained from equations (3) and (5), respectively. The bifurcation plot A versus Q is plotted by combining these two curves. These three diagrams are all depicted in Figure 2. Supercritical bifurcation occurs at $Q_B=4\cdot0801512$.

Second case: $K_0=0\cdot0816$

For this case, $Q_D=2\cdot04$, and the Q versus K_α, K_α versus A and A versus Q plots are all shown in Figure 3.

When $K_\alpha=0\cdot1267396$, corresponding to the minimum speed $Q_C=0\cdot8924225$, a global bifurcation arises. At $Q_{B1}=1\cdot5568623$ and $Q_{B2}=1\cdot9106378$, subcritical and supercritical Hopf bifurcations occur respectively.

Some results are listed in Tables 1 and 2.

4. NUMERICAL INTEGRATION

The equations (1) are first transformed into state space,

$$\dot{X}=AX+f(X), \tag{6}$$

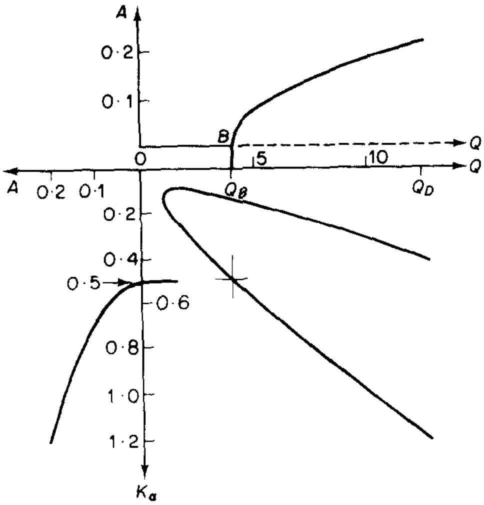

Figure 2. Case $K_0=0{\cdot}5$: Q versus K_α and K_α versus A curves, and A versus Q bifurcation diagram. ——, Stable state; – – –, unstable state.

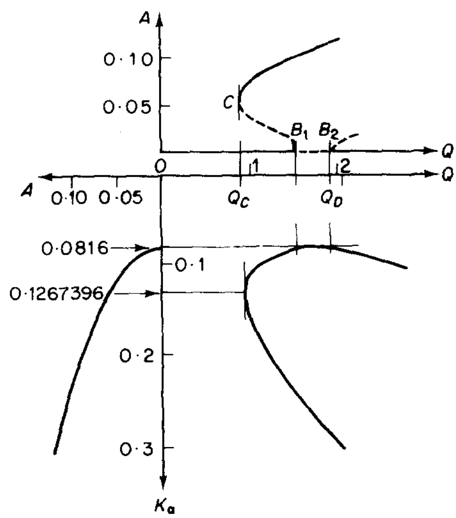

Figure 3. Case $K_0=0{\cdot}0816$: Q versus K_α and K_α versus A curves, and A versus Q bifurcation diagram.

TABLE 1

α-amplitudes of limit cycles at different airspeeds for $K_0=0{\cdot}5$; LC means limit cycle

Q	Harmonic balance (stable LC)	Numerical Integration (stable LC)
4·25	0·03186	0·03205
4·5	0·04998	0·05012
5	0·07365	0·07393
7	0·12948	0·13156
9	0·16657	0·17035
11	0·19625	0·20217

TABLE 2

As Table 1 but for $K_0 = 0{\cdot}0816$

Q	Harmonic balance		Numerical integration	
0·8924225	0·05486, Semistable LC		0·05582, Semistable LC	
	Stable LC	Unstable LC	Stable LC	Unstable LC
1·0	0·07512	0·03468	0·07605	0·03557
1·25	0·09190	0·01792	0·09251	0·01922
1·75	0·11244	No	0·11571	No
1·94	0·11857	0·00327	0·12214	0·00322
1·98	0·11979	0·00523	0·12371	0·00519

where

$$X = (\alpha, \dot{\alpha}, h, \dot{h})^{\mathrm{T}} = (x_1, x_2, x_3, x_4)^{\mathrm{T}},$$
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4/7(0{\cdot}26Q - 4K_0) & -1{\cdot}6/7 & 0{\cdot}8/7 & 0{\cdot}4/7 \\ 0 & 0 & 0 & 1 \\ 4/7(K_0 - 0{\cdot}24Q) & 0{\cdot}4/7 & -1{\cdot}6/7 & -0{\cdot}8/7 \end{bmatrix}, \tag{7}$$

$$f(X) = (0, -16/7, 0, 4/7)^{\mathrm{T}} e x_1^3. \tag{8}$$

The fourth order Runge–Kutta method is used to integrate equation (6) step by step with given initial conditions. For an unstable or a semi-stable limit cycle, only its approximate amplitude can be obtained by a trial and error searching process. Some results obtained by this method are also listed in Tables 1 and 2.

5. ASYMPTOTIC EXPANSION METHOD

If a system $\dot{X} = f(X; u)$ satisfies the conditions of Hopf bifurcation theory, there exists an $\varepsilon_0 > 0$ and $u(\varepsilon) = \sum_{j=2}^{\infty} u_j \varepsilon^j$, such that the system has a periodic solution, the period of which is $T(\varepsilon) = 2\pi(1 + \sum_{j=2}^{\infty} \tau_j \varepsilon^j)/\omega_0$, and the non-zero Floquet exponent of which is $\beta(\varepsilon) = \sum_{j=2}^{\infty} \beta_j \varepsilon^j$ for $0 < \varepsilon < \varepsilon_0$, with $\omega_0 = \mathrm{Re}\ \lambda_1(0)$ and $\lambda_1(0)$ the eigenvalue of largest real part of the linearized matrix of $f(X; u)$. The periodic solution is stable if $\beta(\varepsilon) < 0$, but is

TABLE 3

Results of the asymptotic expansion method

Bifurcation point	u_2	τ_2	β_2	α-amplitude
4·0801512	166·726 (>0, supercritical)	−7·754	−44·474 (<0, stable)	$\sqrt{\dfrac{Q - 4{\cdot}0801512}{166{\cdot}726}}$
1·5568623	−2343·857 (<0, subcritical)	−247·09	37·746 (>0, unstable)	$\sqrt{\dfrac{1{\cdot}5568623 - Q}{2343{\cdot}857}}$
1·9106378	2921·166 (>0, supercritical)	248·78	39·975 (>0, unstable)	$\sqrt{\dfrac{Q - 1{\cdot}9106378}{2921{\cdot}166}}$

unstable if $\beta(\varepsilon)>0$. Hassard and Wan [4] have developed a computational procedure for determining u_j, τ_j and β_j. With j taken to be 2, the following formulae are obtained:

$$u_2=-\mathrm{Re}\,C_1(0)/\alpha'(0), \qquad \tau_2=-[\mathrm{Im}\,C_1(0)+u_2\omega'(0)]/\omega_0, \qquad \beta_2=2\,\mathrm{Re}\,C_1(0). \tag{9–11}$$

Here $\alpha'(0)=\mathrm{Re}\,\lambda_1'(0)$, $\omega'(0)=\mathrm{Im}\,\lambda_1'(0)$ and the meaning of $C_1(0)$ can be found in reference [4].

As obtained by adopting Hassard and Wan algorithm and using computer algebra, the results calculated are listed in Table 3.

6. AVERAGING METHOD

For the system (6) with $K_0=0{\cdot}5$ at $Q=Q_B=4{\cdot}0801512$, let $u=Q-Q_B$ be the bifurcation parameter. Then, at $u=0$, the four eigenvalues of the linearized matrix (7) are

$$\lambda_1=\bar{\lambda}_2=0{\cdot}5982162\mathrm{i}, \qquad \lambda_3=\bar{\lambda}_4=-0{\cdot}1714286+0{\cdot}6331204\mathrm{i}. \tag{12}$$

Upon introducing the transformation

$$X=PY, \tag{13}$$

where $Y=(y_1, y_2, y_3, y_4)^{\mathrm{T}}$, and P is constructed from the real and imaginary parts of the eigenvectors corresponding to λ_2 and λ_4, equation (6) is reduced to

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix}=\begin{bmatrix}0 & -0{\cdot}5982162 & & \\ 0{\cdot}5982162 & 0 & & \\ & & -0{\cdot}1714286 & -0{\cdot}6331204\\ & & 0{\cdot}6331204 & -0{\cdot}1714286\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix}$$

$$+u\begin{bmatrix}0{\cdot}2668131 & 0 & 0{\cdot}2668131 & 0\\ 0{\cdot}0495631 & 0 & 0{\cdot}0495631 & 0\\ -0{\cdot}2668131 & 0 & -0{\cdot}2668131 & 0\\ -0{\cdot}2092517 & 0 & -0{\cdot}2092517 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix}+20\begin{bmatrix}-2{\cdot}964942\\ 0{\cdot}068299\\ 2{\cdot}964942\\ 2{\cdot}742892\end{bmatrix}(y_1+y_3)^3. \tag{14}$$

According to the center manifold theorem [5], the system (14) supplemented by a trivial equation $\dot{u}=0$ has a center manifold $y_3=h_1(y_1, y_2, u)$, $y_4=h_2(y_1, y_2, u)$. With the aid of computer algebra, the approximate form of the center manifold is obtained as

$$h_1=-0{\cdot}5593513y_1u+0{\cdot}6202773y_2u, \qquad h_2=-0{\cdot}8560522y_1u-0{\cdot}6964650y_2u. \tag{15}$$

The flow on the center manifold is

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}0 & -0{\cdot}5982162\\ 0{\cdot}5982162 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+u\begin{bmatrix}0{\cdot}2668131\\ 0{\cdot}0495631\end{bmatrix}(y_1+h_1)$$

$$+20\begin{bmatrix}-2{\cdot}964942\\ 0{\cdot}068299\end{bmatrix}(y_1+h_1)^3. \tag{16}$$

Let

$$y_1=r\cos\varphi, \qquad y_2=r\sin\varphi. \tag{17}$$

Then equation (16) is transformed to

$$\dot{r}=f_1(r,\varphi), \qquad \dot{\varphi}=g_1(r,\varphi). \tag{18}$$

By employing the averaging method, the first of equations (18) can be reduced to

$$\dot{r}=F_1(r)=r^3(8\cdot8985678u^3-29\cdot7832135u^2+37\cdot6327256u-22\cdot237065)$$
$$+r(-0\cdot05924969u^2+0\cdot1334065u). \tag{19}$$

The non-zero equilibrium point r_1 of equation (19) is the amplitude of the limit cycle of the system (17): i.e., of system (16). The sign of $F_1'(r_1)$ determines the stability of the limit cycle: $F_1'(r_1)<0$ and $F_1'(r_1)>0$ correspond to the stable limit cycle and the unstable limit cycle, respectively.

By transforming back to the physical co-ordinates, α is obtained as

$$\alpha=x_1=y_1+h_1=r_1[(1-0\cdot5593513u)\cos\varphi+0\cdot6202773u\sin\varphi]. \tag{20}$$

Hence the amplitude of α in the limit cycle is

$$|\alpha|=r_1\sqrt{(1-0\cdot5593513u)^2+(0\cdot6202773u)^2}. \tag{21}$$

The second case, $K_0=0\cdot0816$, can be analyzed in the same way as mentioned above, and results analogous to (19) and (21) are as follows:

for $Q_{B1}=1\cdot5568623$,

$$\dot{r}=F_2(r)=r^3(-3\cdot9255007u^3+16\cdot210584u^2-23\cdot6666365u+18\cdot86805)$$
$$+r(-0\cdot02663722u^2+0\cdot00803104u), \tag{22}$$

$$|\alpha|=r_2\sqrt{(1-0\cdot4255526u)^2+(0\cdot5786874u)^2}; \tag{23}$$

for $Q_{B2}=1\cdot9106378$,

$$\dot{r}=F_3(r)=r^3(-6\cdot7101201u^3+22\cdot488244u^2-30\cdot6769287u+18\cdot9875775)$$
$$+r(-0\cdot01371466u^2-0\cdot00649439u), \tag{24}$$

$$|\alpha|=r_3\sqrt{(1-0\cdot4951198u)^2+(0\cdot5656275u)^2}. \tag{25}$$

Here r_2 and r_3 are the non-zero fixed points of equations (22) and (24), respectively.

Some of the results are listed in Table 4.

TABLE 4

Results of the averaging method

i	Bifurcation point	Q	$F_i(r)$	r_i	$F_i'(r_i)$	α-amplitude
1	4·0801512	4·4	$-12\cdot956r^3+0\cdot0366r$	0·05315	<0	0·045
		4·642	$-8\cdot91683r^3+0\cdot05625r$	0·079425	Stable super-critical	0·061
2	1·5568623	1·25	$28\cdot1022r^3-0\cdot005r$	0·0133	>0	0·016
		1·35	$24\cdot46r^3-0\cdot0028r$	0·0107	Unstable sub-critical	0·012
3	1·9106378	1·947	$17\cdot91r^3-0\cdot0002522r$	0·003752	>0	0·00368
		2·007	$16\cdot23r^3-0\cdot006755r$	0·00682	Unstable super-critical	0·0065

7. DISCUSSION AND CONCLUSIONS

(1) An airfoil with linear plus cubic pitching stiffness in incompressible flow undergoes bifurcations for certain values of the linear stiffness. The bifurcation may be stable and unstable supercritical as well as subcritical Hopf bifurcation.

TABLE 5

Comparison of α-amplitudes obtained by the four different methods

K_0	Q	Stability of limit cycle	Harmonic balance method	Numerical integration method	Asymptotic expansion method	Averaging method
0·5	4·25	Stable	0·03186	0·03205	0·03192	0·0323
	4·5	Stable	0·04998	0·05012	0·0502	0·0519
	7	Stable	0·12948	0·13156		
0·0816	1·35	Stable	0·0968	0·09811		
		Unstable	0·01287	0·01373	0·0094	0·012
	1·5	Stable	0·10324	0·10791		
		Unstable	0·00541	0·00532	0·005	0·0053
	1·75	Stable	0·11244	0·11571		
	1·94	Stable	0·11857	0·12214		
		Unstable	0·00327	0·00322	0·00317	0·00329
	1·98	Stable	0·11979	0·12371		
		Unstable	0·00523	0·00519	0·00487	0·00538

(2) By comparison of the data in Tables 1–5, it is shown that the conclusions about stabilities reached via the four methods are the same. The amplitudes of limit cycles are also nearly equal to each other. Hence, for the title problem, the harmonic balance method can provide correct information about the steady periodic state of the system, and the analysis can be extended further from the critical point and even to global bifurcation cases.

(3) The asymptotic expansion method and averaging method both belong to analytic methods having firm mathematical foundations. The center manifold theory is used in both methods to reduce the dimensions of the system: hence they can give satisfactory results only in the neighbourhood of the critical point.

(4) If the airfoil studied has non-linearities in both the α and h degrees of freedom, employing the harmonic balance method inevitably involves some iterative processes [6] and loses its simplicity. With the aid of computer algebra, however, the analytic methods can easily be extended to dealing with this case without much increase in the labour.

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