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Characterization and Efficient Computation of the Structured Singular Value

by

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Abstract

The concept of *structured singular value* was recently introduced by Doyle as a tool for the analysis and synthesis of feedback systems with structured uncertainties. It was found later to be a key to the design of control systems under joint robustness and performance specifications and to very nicely complement the H^{∞} approach to control system design. In this paper, it is shown that the structured singular value can be obtained as the solution of several *smooth* optimization problems. Properties of these optimization problems are exhibited, leading to fast algorithms for their solution.

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1. Introduction and preliminaries

The concept of *structured singular value* was recently introduced by Doyle [1] as a tool for the analysis and synthesis of feedback systems with structured uncertainties. It was found later to be a key to the design of control systems under joint robustness and performance specifications and to very nicely complement the H^∞ approach to control system design [2]. In this paper, it is shown that the structured singular value can be obtained as the solution of several *smooth* optimization problems. Properties of these optimization problems are exhibited, leading to fast algorithms for their solution.

Throughout the paper, given any square complex matrix M, we denote by $\rho(M)$ its spectrum radius, by $\overline{\sigma}(M)$ its largest singular value and by M^H its complex conjugate transpose. Given any complex vector x, x^H indicates its complex conjugate transpose and $||x||$ its Euclidean norm. For any positive integer k, we denote by O_k the $k \times k$ matrix whose entries are all zero and by I_k the $k \times k$ identity matrix. We also make use of the following notation and nomenclature, largely inspired from that used in [1]. We call *block-structure* of size m any m-tuple $\mathcal{k} = (k_1, \cdots, k_m)$ of positive integers. (Note that, in the terminology of [1], this corresponds to structures with *no repeated blocks*.) Given a block-structure $\mathcal{k}$ of size m, we make use of the projection matrices

$$P_i = \mathrm{diag}\,(O_{k_1}, \cdots, O_{k_{i-1}}, I_{k_i}, O_{k_{i+1}}, \cdots, O_{k_m}) \, ; \tag{1.1}$$

of the family of diagonal matrices

$$\mathcal{d} = \{\mathrm{diag}(d_1 I_{k_1}, \cdots, d_m I_{k_m}) \mid d_i \in (0,\infty)\} \; ; \tag{1.2}$$

of the family of block unitary matrices

$$\mathcal{U} = \{\mathrm{diag}(U_1, \cdots, U_m) \mid U_i \text{ is a } k_i \times k_i \text{ unitary matrix}\} \; ; \tag{1.3}$$

and, for any positive scalar δ (possibly ∞), of the family of block diagonal matrices

$$X_\delta = \{\mathrm{diag}(\Delta_1, \cdots, \Delta_m) \mid \Delta_i \text{ is a } k_i \times k_i \text{ complex matrix satisfying } \overline{\sigma}(\Delta_i) \le \delta\} \; . \tag{1.4}$$

All of the above have dimension $n \times n$, where

$$n = \sum_{j=1}^{m} k_j \; . \tag{1.5}$$

Definition 1.1.

The *structured singular value* $\mu(M)$ of a complex $n \times n$ matrix M *with respect to block-structure* $\mathcal{k}$ is the positive number μ having the property that

$$\det(I + M\Delta) \neq 0 \quad \text{for all } \Delta \in X_\delta \tag{1.6}$$

if, and only if,

$$\delta\mu < 1 \; . \tag{1.7}$$

In other words, $\mu(M)$ is 0 if there is no Δ in X_∞ such that $\det\,(I+M\Delta)=0$, and $\left(\min_{\Delta \in X_\infty} \{\, \overline{\sigma}(\Delta) \mid \det(I+M\Delta)=0 \,\}\right)^{-1}$ otherwise.

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It should be emphasized that P_i, $\mathcal{d}$, $\mathcal{U}$, X_δ, and $\mu(M)$ all depend on the underlying block-structure. In most instances however, we will not explicitly specify this block-structure.

We will make repeated use of the following easily derived fact [1].

Fact 1.0. For all $U \in \mathcal{U}$,

$$\mu(M) = \mu(MU) = \mu(UM)$$

and, for all $D \in \mathcal{d}$,

$$\mu(M) = \mu(DMD^{-1})$$

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In order to evaluate the structured singular value, more manageable expressions than those provided in Definition 1.1 are desirable. Such expressions are provided by the following facts [1].

Fact 1.1

$$\mu(M) = \max_{U \in \mathcal{U}} \rho(MU) = \max_{U \in \mathcal{U}} \rho(UM) \,, \tag{1.8}$$

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Fact 1.2. For block-structures of size less than 4,

$$\mu(M) = \inf_{D \in \mathcal{d}} \bar{\sigma}(DMD^{-1}) \,. \tag{1.9}$$

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In fact, in many (but not all) cases, (1.9) is correct with block-structures of larger size.

Using (1.8) to compute the structured singular value is discarded in [1] because the optimization problem it involves generally has non-global maxima. The function to be minimized in (1.9) is convex [3] and thus all stationary points are global minima. Algorithms are available for solving this problem and this yields a reliable way of computing μ for block-structures of size less than 4. However such algorithms are iterative in nature, thus requiring a large number of function evaluations, each of which involves a CPU demanding singular value decomposition. Finally, both (1.8) and (1.9) pertain to *non-differentiable* optimization, and such problems are inherently 'hard' to solve.

In view of the above, there is need for a *fast*, reliable method for computing μ for *any block-structure*. The results presented in this paper represent a significant step in that direction. In Section 2, we show that $\mu(M)$ can be obtained as the solution of several *smooth* optimization problem which do not involve any eigenvalue or singular value computation. These results are used in Section 3 to show that, for the purpose of computing μ, any matrix can be reduced, together with the given block-structure, in such a way that no block of the reduced structure has size larger than the rank of the original matrix. A particular simple expression is then obtained for the structured singular value of a matrix of rank one. In Section 4, the solutions of various optimization problems are related to each other, leading to a fast algorithm for computing μ. Numerical experiments are presented in Section 5. To avoid any loss of continuity, most proofs are given in appendix. Appendices A, B, C, and D contain proofs of results stated in Sections 2, 3, 4, and 5 respectively.

2. Several smooth optimization problems

Our first theorem gives several new expressions for the structured singular value.

Theorem 2.1. The following two relations hold (and, in both, the 'max' is achieved):

$$\mu(M) = \max_{x \in C^n} \{ \, ||Mx|| \mid ||P_i x|| \, ||Mx|| = ||P_i Mx||, \ i=1,\cdots,m \, \} \tag{2.1}$$

and

$$\mu(M) = \max_{x \in C^n} \{ \, ||Mx|| \mid ||P_i x|| \, ||Mx|| \leq ||P_i Mx||, \ i=1,\cdots,m \, \} \tag{2.2}$$

Furthermore, both equalities are preserved if the maximization is restricted to the unit sphere $\{x \in C^n \mid ||x|| = 1\}$ or to the unit ball $\{x \in C^n \mid ||x|| \leq 1\}$.

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Problem (2.1) will be referred to below as $\mathcal{P}(M)$ and problem (2.2) as $\mathcal{P}'(M)$. The following corollary will lead, in Section 4, to a simple test of global optimality of a candidate solution for (2.1) or (2.2).

Corollary 2.1. If $D \in \mathcal{d}$ and if $x \in C^n$ satisfies the constraints in (2.2) then

$$||Mx|| \leq \mu(M) \leq \overline{\sigma}(DMD^{-1}) \tag{2.3}$$

In particular, if $||Mx|| = \overline{\sigma}(DMD^{-1})$, then both sides are equal to $\mu(M)$ and x solves (2.1) and (2.2).

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Remark 2.1. The maximizer in (2.1) and (2.2) is clearly non-unique, for if x^* is such a maximizer, θx^* is also one, for any complex number θ of unit magnitude.

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A major difference between expressions (1.8) and (1.9) on the one hand, and (2.1) and (2.2) on the other hand, is that the objective and constraints appearing in the latter are inexpensive to compute and that, after squaring all the norms, objective and constraints in the latter are smooth. However, it is not clear whether the global maximum in these optimization problems can easily be obtained. A first step towards answering this question is taken in the next section.

3. Block-structure reduction. Matrices of rank one

Given a complex matrix M and a block-structure $\mathcal{k}$, the algorithm below constructs a reduced matrix $\overline{M}$ of same rank as M, and a corresponding reduced block-structure $\overline{\mathcal{k}}$ in which no block size is larger than the rank of M. Proposition 3.1 then asserts that the structured singular value is invariant under such transformation.

Algorithm 3.1

Data.

$$M \in C^{n \times n}, \; \mathcal{k} = (k_1, \cdots, k_m)$$

Step 1.

Perform a singular value decomposition of M. Let $r = \text{rank}(M)$. For $i = 1, \cdots, m$, let U_i and V_i be $k_i \times r$ matrices such that

$$\begin{bmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_m \end{bmatrix}$$

consist respectively of the first r left and right singular vectors of M. Thus one has

$$M = \begin{bmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_m \end{bmatrix} \Sigma \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_m \end{bmatrix}^H$$

for some positive definite diagonal matrix Σ=diag$(\sigma_1, \cdots, \sigma_r)$.

Step 2.

Obtain W^U=diag$(W_1^U, \cdots, W_m^U)$ and W^V=diag$(W_1^V, \cdots, W_m^V)$, both in $\mathcal{U}$, such that, for $i=1, \cdots, m$, $W_i^U U_i$ and $W_i^V V_i$ have respectively their $(k_i - r_i^U)$ and $(k_i - r_i^V)$ last rows identically zero, where r_i^U and r_i^V are the ranks of U_i and V_i respectively.

Step 3.

For $i=1, \cdots, m$, let

$$\overline{k}_i = \max\,(r_i^U, r_i^V)$$

and let $\overline{U}_i$ and $\overline{V}_i$ be $\overline{k}_i \times r$ matrices consisting of the first $\overline{k}_i$ rows of $W_i^U U_i$ and $W_i^V V_i$ respectively. Let

$$\overline{M} = \begin{bmatrix} \overline{U}_1 \\ \cdot \\ \cdot \\ \cdot \\ \overline{U}_m \end{bmatrix} \Sigma \begin{bmatrix} \overline{V}_1 \\ \cdot \\ \cdot \\ \cdot \\ \overline{V}_m \end{bmatrix}^H$$

Let

$$\overline{k} = (\overline{k}_1, \cdots, \overline{k}_m)$$

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It should be noted that the W^U and W^V matrices are not unique. Thus we will refer to $\overline{M}$ merely as 'a' reduced matrix corresponding to M.

Proposition 3.1.

$$\mu(M) = \overline{\mu}(\overline{M})$$

where $\overline{\mu}$ indicates the structured singular value with respect to block-structure $\overline{k}$, and $\overline{M}$ and $\overline{k}$ are constructed by Algorithm 3.1.

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Proposition 3.2 and Theorem 3.1 give analytical expressions in the case of rank one matrices.

Proposition 3.2. Suppose that $M = uv^T$, where $u, v \in \mathbb{R}^n$ are in the nonnegative orthant and suppose that the given block-structure is $k = (1, \cdots, 1)$. Then $\mu(M) = u^T v$.

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Theorem 3.1. Suppose that matrix M has rank one, i.e., $M = uv^H$ for some $u, v \in C^n$. Let $\overline{u}, \overline{v} \in R^m$ be defined as

$$\overline{u} = [\ \overline{u}_1, \cdots, \overline{u}_m\ ]^T$$

$$\overline{v} = [\ \overline{v}_1, \cdots, \overline{v}_m\ ]^T$$

where

$$\overline{u}_i = ||P_i u|| \qquad i = 1, \cdots, m$$

$$\overline{v}_i = ||P_i v|| \qquad i=1,\cdots,m.$$

Then

$$\mu(M) = \overline{u}^T \overline{v}$$

Proof. It is easily checked that, for the given M, there is a reduced matrix $\overline{M}$, constructed by Algorithm 3.1, given by

$$\overline{M} = \overline{u}\ \overline{v}^T$$

and the reduced block-structure is given by

$$\overline{k} = (1,\cdots,1)\ .$$

The result then follows directly from Propositions 3.1 and 3.2.

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4. Properties of various optimization problems

Suppose that $x \in C^n$ is a global maximizer for (2.1). Then, from Theorem 2.1, it also solves (2.2). Consider problem (2.2) as an optimization problem in $\mathbb{R}^{2n}$. With $z = \begin{bmatrix} \mathrm{Re}x \\ \mathrm{Im}x \end{bmatrix} \in \mathbb{R}^{2n}$, the constraints on z in (2.2) can be written as

$$g_i(z) \triangleq ||P_i x||^2 ||Mx||^2 - ||P_i Mx||^2 \leq 0,\ i = 1,\cdots,m\ .$$

If the Kuhn-Tucker constraint qualification holds at z, the Kuhn-Tucker first order conditions of optimality for problem (2.2) must be satisfied. This leads to the following definition and proposition.

Definition 4.1. We call *regular point* for (2.1) and $x \in C^n$ for which the corresponding $z \in \mathbb{R}^{2n}$ is such that $\{\nabla g_i(z), i=1,\cdots,m\}$ forms a set of vectors that are linearly independent over the field of real numbers.

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Proposition 4.1. Suppose $x \in C^n$ is a regular point and a local maximizer for (2.1). Then there exists a unique m-tuple of *real* numbers $\lambda_1, \cdots, \lambda_m$, such that the following relations hold:

$$M^H Mx + \sum_{i=1}^{m} \lambda_i (M^H P_i Mx - ||P_i x||^2 M^H Mx - ||Mx||^2 P_i x) = 0 \tag{4.1}$$

$$\lambda_j \geq 0, \; j = 1, \cdots, m \tag{4.2}$$

$$||P_i Mx|| - ||P_i x|| \, ||Mx|| \geq 0, \; i = 1, \cdots, m \tag{4.3}$$

The λ_i's are called *multipliers* associated with x.

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We are now ready to state the main result of this section, exhibiting relationships between the arguments solving various optimization problems encountered so far for the computation of the structured singular value.

Theorem 4.1. Consider the following three optimization problems (Theorem 2.1, Fact 1.2, Fact 1.1):

$$\max_{x \in C^n} \{ ||Mx|| \mid ||P_i x|| \, ||Mx|| = ||P_i Mx||, \; i = 1, \cdots, m \} \tag{P1}$$

$$\inf_{D \in \mathcal{d}} \bar{\sigma}(DMD^{-1}) \tag{P2}$$

$$\max_{U \in \mathcal{U}} \rho(MU) \tag{P3}$$

and suppose that (i) x^* is such that the only global solutions to (P1) are of the form θx^*, with θ a complex number of unit magnitude, and x^* is a regular points for

(P1), (ii) there exists a $D \in \mathcal{d}$ such that $\mu(M) = \bar{\sigma}(DMD^{-1})$, i.e., the max in (P2) is achieved and (1.9) holds. Under these assumptions, the following results hold:

(a) Suppose that $D^* \in \mathcal{d}$ solves (P2), then there exists a right singular vector corresponding to the largest singular value of $D^*MD^{*^{-1}}$, say y^*, such that $x = (D^{*^{-1}}y^*)/||D^{*^{-1}}y^*||$ solves (P1).

(b) Let $\lambda^* = [\lambda_1^*, \cdots, \lambda_m^*]^T$ be the (unique) multiplier vector associated with x^*. Then $D \triangleq \mathrm{diag}\{\lambda_i^{*^{1/2}} I_{k_i}\}$ belongs to $\mathcal{d}$ and, up to scalar multiplication, it is the only solution to (P2).

(c) For $i=1, \cdots, m$, let Q_i be a $k_i \times n$ matrix consisting of all rows of P_i from $\sum_{j=1}^{i-1} k_j + 1$ to $\sum_{j=1}^{i} k_j$, i.e., Q_i is the ith block-row of P_i. Let $U = \mathrm{diag}(U_1, \cdots, U_m)$, where, for $i=1, \cdots, m$, U_i is a $k_i \times k_i$ matrix satisfying

$$U_i = V_i W_i^H$$

where V_i and W_i are both square and unitary and, if $Q_i x^* \neq 0$, have respectively $\dfrac{Q_i x}{||Q_i x||}$ and $\dfrac{Q_i Mx}{||Q_i Mx||}$ as their first column. Then U solves (P3).

(d) Suppose that $U^* \in \mathcal{U}$ solves (P3), then for any unit norm eigenvector z of MU^* corresponding to an eigenvalue of magnitude equal to the spectral radius, $.x = U^* z$ solves (P1).

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Suppose now that the assumptions of Theorem 4.1 are satisfied. Suppose one has obtained a candidate x^* (resp. U^*), possibly solving (P1) (resp. (P3)), but that one is not sure whether or not $\mu(M)$ has been achieved, i.e., whether or not x^* (resp. U^*) is a global maximizer. Then Theorem 4.1 (together with Corollary 2.1) provides (i) an infallible test determining whether or not $\mu(M)$ has been achieved and (ii) in case the answer is negative, an initial guess D for problem (P3). In case (1.9) (or any of the assumptions in Theorem 4.1) does not hold, this test still provides a *sufficient* condition and an initial guess D for (P3) in case of failure[1]. Thus a reasonable approach to computing $\mu(M)$ would be as follows. First solve (P1) using a local optimization algorithm. This will be very fast since problem (P1) is smooth and the objective and constraints inexpensive to evaluate. Second, check for global optimality as just suggested. It is hoped that, with an appropriately selected initial guess for (P1), the outcome of this test will in most cases be positive. Third, if necessary, solve problem (P2) starting from the initial guess provided by Theorem 4.1.

5. Complexity considerations and computational experiments

In view of the solution scheme just sketched out, the purpose of these experiments is twofold. First, we want to assess how often it may be necessary to resort to the third phase (solving (P2)). Obviously this test will be fair only if one makes sure that (1.9) always holds; this question is addressed in Algorithm 5.1 and Proposition 5.1 below. Second, in case of success in the first phase, we want to compare the computation time necessary for solving (P1) and (P2) respectively[2].

[1] However, unless (1.9) is known to hold, one cannot be sure that $\mu(M)$ has been computed correctly.

[2] (P3) was not tested since it does not seem to have any clear advantage over (P1) or (P2).

Experiments were performed on two types of problems. Examples 1 through 3 below have to do with computation of the structured singular value for a number of independently randomly generated matrices for which (1.9) holds. These matrices were generated using the following algorithm.

Algorithm 5.1

Step 1.

Arbitrarily generate $u_1, v_1 \in C^n$ such that, for $i=1,\cdots,m$, $||P_i u_1|| = ||P_i v_1||$ and $||u_1|| = ||v_1|| = 1$.

Step 2.

Arbitrarily generate U, V, unitary, with respectively u_1 and v_1 as first columns.

Step 3.

Arbitrarily generate $\Sigma = \mathrm{diag}(1, \sigma_2, \cdots, \sigma_m)$ where, for $i=2,\cdots,m$, $\sigma_i \in [0,1]$.

Step 4.

Arbitrarily generate $D \in d$. Let $M = D^{-1} U \Sigma V^H D$.

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Proposition 5.1. A matrix M has the property that, for some $D \in d$,

$$\mu(M) = \overline{\sigma}(DMD^{-1}) = 1$$

if, and only if, M can be generated by Algorithm 5.1

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Clearly there is no loss of generality in only considering matrices with structured singular value equal to unity. To best simulate the chronic ill-conditioning of matrices arising from real-world applications, entries of the various matrices used in

Algorithm 5.1 were generated according to a logarithmic distribution, with both real part and imaginary part lying between 10^{-5} and 10^{5} in absolute value. One hundred matrices were generated for each one of the first three examples. The dimension of these matrices (resp. 10, 10 and 20) and the size (number of blocks) of the block-structure (resp. 3, 5 and 2) were kept constant throughout each example, while the sizes of the various blocks were randomly generated.

In each one of examples 4 through 8, the structured singular value was computed over a range of frequencies for a fixed rational transfer function matrix. Coefficients in the entries of the latter matrices were also randomly generated. Each one of these examples correspond to one such transfer function matrix. All the matrices have dimension 6. Block-structures are (2,2,2) for examples 4 through 7 and (1,1,1,1,1,1) for example 8. Frequency ranges from 1 to 100, with 50 points per decade.

Problem (P2) was solved using an algorithm due to one of the authors [4], to our knowledge the best algorithm currently available for this problem. Locally around the solution, the algorithm performs Newton iterations. Problem (P1) was solved using an algorithm combining Harwell routine VF02AD [5], based on a successive-quadratic-programming type algorithm due to Powell [6], and a Newton iteration on both primal and dual variables. Again the latter was used in the neighborhood of a solution.

When solving (P2), computation was terminated when the objective value was within 10^{-6} (relative distance) of the exact structured singular value (which is 1 for examples 1 through 3 and was precomputed for examples 4 through 8). In solving

(P1), it was further required that constraint violations be less than 10^{-5}. As initial guess for problem (P1) we used an unconstrained maximizer, i.e., a right singular vector corresponding to the largest singular value. The corresponding value of D, as provided by Theorem 4.1, was used as initial guess for (P2). In examples 4 through 8, this initial guess was used only for the first frequency in the given frequency range. The solution obtained at any given frequency was then used as initial guess at the next frequency. All the experiments were performed on a Pyramid 90X machine, running UNIX 4.2 bsd, using double precision complex arithmetic.

The results are shown in Tables 5.1 through 5.4. In all cases, T_1 indicates the central processor time used in solving (P1), and similarly with T_2 for (P2). Dashes in the T_1 column indicate that the global maximizer was not achieved. In each of the first three examples (Tables 5.1 through 5.3), the number of such 'failures' is of the order of 10%. This is relatively low in view of the fact that, due to the logarithmic distribution of entries, some matrices are severely ill-conditioned, making it likely that our initial guess be away from the global maximizer. The damage caused by such failures is in any case limited since they can be detected and since one can then switch to phase 3 of our scheme (see end of Section 4), i.e., solve (P2) with an 'educated' initial guess. Concerning computational speed, Tables 5.1-5.3 show that (P1) often has a dramatic advantage over (P2), while (P2) is slightly faster than (P1) in about 10% of the cases. On the average, the superiority of (P1) is significant. Examples 4 through 8, which are representative of problems arising in the context of control system design, demonstrate the full power of the new approach. There, typical speedup of (P1) compared to (P2) is of one full order of magnitude.

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Appendix A.

Proof of Theorem 2.1. Let

$$\hat{\mu}(M) \triangleq \sup_{x \in C^n} \{ \, ||Mx|| \mid ||P_i x|| \, ||Mx|| = ||P_i Mx||, \ i=1, \cdots, m \, \} \tag{A.1}$$

If x is feasible for (A.1), then

$$||x||^2 ||Mx||^2 = \sum_{i=1}^{m} ||P_i x||^2 ||Mx||^2 = \sum_{i=1}^{m} ||P_i Mx||^2 = ||Mx||^2 \tag{A.2}$$

i.e., $||x||=1$ or $||Mx||=0$. Clearly, the optimal objective value will not change if one restricts x to lie on the unit sphere. Since the resulting constraint set is compact, (A.1) can be written as

$$\hat{\mu}(M) = \max_{x \in C^n} \{ \, ||Mx|| \mid ||P_i x|| \, ||Mx|| = ||P_i Mx||, \ i=1, \cdots, m \, \} \tag{A.3}$$

or, equivalently, as

$$\hat{\mu}(M) = \max_{x \in C^n} \{ \, ||Mx|| \mid ||P_i x|| \, ||Mx|| = ||P_i Mx||, \ i=1, \cdots, m, \ ||x||=1 \, \} \tag{A.4}$$

Similarly, $\tilde{\mu}(M)$ defined as

$$\tilde{\mu}(M) \triangleq \sup_{x \in C^n} \{ \, ||Mx|| \mid ||P_i x|| \, ||Mx|| \le ||P_i Mx||, \ i=1, \cdots, m \, \} \tag{A.5}$$

can be written as

$$\tilde{\mu}(M) = \max_{x \in C^n} \{ ||Mx|| \mid ||P_i x|| ||Mx|| \leq ||P_i Mx||, i=1, \cdots, m \} \quad (A.6)$$

or as

$$\hat{\mu}(M) = \max_{x \in C^n} \{ ||Mx|| \mid ||P_i x|| ||Mx|| \leq ||P_i Mx||, i=1, \cdots, m, ||x|| \leq 1 \} \quad (A.7)$$

Obviously $\tilde{\mu}(M) \geq \hat{\mu}(M)$. To complete the proof of Theorem 2.1, we now show that

$$\hat{\mu}(M) \geq \mu(M) \geq \tilde{\mu}(M) . \quad (A.8)$$

First, since every eigenvector of M with unit norm satisfies the constraints in (2.1), we have

$$\hat{\mu}(M) \geq \rho(M) \quad . \quad (A.9)$$

Second, for any $U \in \mathcal{U}$, $P_i U = UP_i$ and if $v \in C^n$, $||Uv|| = ||v||$. Thus

$$\hat{\mu}(M) = \max_{x \in C^n} \{ ||UMx|| \mid ||P_i x|| ||UMx|| = ||P_i UMx||, i=1, \cdots, m \}$$
$$= \hat{\mu}(UM) . \quad (A.10)$$

From (A.9) and (A.10) we conclude that, for any $U \in \mathcal{U}$, $\hat{\mu}(M) = \hat{\mu}(UM) \geq \rho(UM)$. In view of Fact 1.1, this implies

$$\hat{\mu}(M) \geq \max_{U \in \mathcal{U}} \rho(UM) = \mu(M) \quad (A.11)$$

and the first inequality in (A.8) is proven. In order to prove the second inequality in (A.8), suppose that x^* solve (A.6), thus $\tilde{\mu}(M) = ||Mx^*||$. If $\tilde{\mu}(M) = 0$, the claim obviously holds. Thus suppose that $\tilde{\mu}(M) > 0$. Since x^* is feasible for (A.6), for $i=1, \cdots, m$, there exists a $k_i \times k_i$ matrix R_i, with $||R_i|| \leq 1$, such that

$$R_i P_i Mx^* = -\tilde{\mu}(M) P_i x^* \quad (A.12)$$

Hence $R = \text{diag}(R_i) \in X_1$ satisfies

$$RMx^* = -\tilde{\mu}(M)x^* \tag{A.13}$$

or

$$(I+\tilde{\mu}(M)^{-1}RM)x^* = 0 \tag{A.14}$$

which implies

$$\det(I+M\Delta) = \det(I+\Delta M) = 0 \tag{A.15}$$

with

$$\Delta = \tilde{\mu}(M)^{-1}R \;\in\; X_{\tilde{\mu}(M)^{-1}} \tag{A.16}$$

which, from Definition 1, implies

$$\tilde{\mu}(M) \leq \mu(M) \tag{A.17}$$

which proves the second inequality in (A.8). This completes the proof of Theorem 2.1.

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Appendix B.

Proof of Proposition 3.1

By Fact 1.0, it is clear that

$$\mu(M) = \mu(M_1)$$

Suppose x is feasible for $\mathcal{P}'(M_1)$, and without loss of generality, suppose that $||x|| = 1$. Define

$$\overline{P}_i = \text{diag}\,(O_{\overline{k}_1}, \cdots, O_{\overline{k}_{i-1}}, I_{\overline{k}_i}, O_{\overline{k}_{i+1}}, \cdots, O_{\overline{k}_m});$$

$$\overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \cdot \\ \overline{x}_m \end{bmatrix}$$

where, for $i=1, \cdots, m$, $\overline{x}_i$ consists of the entries $\sum_{j=1}^{i-1} k_j + 1$ to $\sum_{j=1}^{i-1} k_j + \overline{k}_i$ of x.

Then,

$$||M_1 x|| = ||\overline{M}\overline{x}|| \tag{B.1}$$

and, for $i=1, \cdots, m$,

$$||P_i M_1 x|| = ||\overline{P}_i \overline{M}\overline{x}||$$

and

$$||P_i x|| \geq ||\overline{P}_i \overline{x}||$$

Therefore,

$$\begin{aligned} ||\overline{P}_i \overline{x}|| \; ||\overline{M}\overline{x}|| &\leq ||P_i x|| \; ||M_1 x|| \\ &\leq ||P_i M_1 x|| \\ &= ||\overline{P}_i \overline{M}\overline{x}|| \end{aligned}$$

Hence $\overline{x}$ is feasible for $\mathcal{P}'(\overline{M})$ (see Theorem 2.1) and in view of (B.1),

$$\overline{\mu}(\overline{M}) \geq \mu(M_1)$$

where $\overline{\mu}$ indicates the structured singular value with respect to block-structure

$$\overline{\mathcal{K}} = (\overline{k}_1, \cdots, \overline{k}_m).$$

To prove $\mu(M_1) \geq \overline{\mu}(\overline{M})$, suppose $\overline{x}$ is feasible for $\mathcal{P}'(\overline{M})$. Again, without loss of generality, suppose that $||\overline{x}||=1$. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \end{bmatrix}$$

where x_i is a k_i-vector given by

$$x_i = \begin{bmatrix} \overline{x}_i \\ 0 \\ \cdot \\ 0 \end{bmatrix}$$

Then

$$||\overline{M}\overline{x}\,|| = ||M_1 x\,|| \tag{B.2}$$

and, for $i=1, \cdots, m$,

$$||\overline{P}_i\overline{M}\overline{x}\,|| = ||P_i M_1 x\,||$$

and

$$||\overline{P}_i\,\overline{x}\,|| = ||P_i\,x\,||$$

Therefore,

$$||P_i\,x\,||\;||M_1 x\,|| = ||P_i M_1 x\,||$$

So, x is feasible for $\mathcal{P}'\,(M_1)$ and in view of (B.2),

$$\mu(M_1) \geq \overline{\mu}(\overline{M}\,)$$

which completes the proof.

#

Proof of Proposition 3.2

Let $\beta \triangleq u^T v$. From Definition 1.1, it is clear that $\mu(M) \geq \beta$ since, if $\beta \neq 0$ and $\Delta = \frac{-1}{\beta} I$,

$$\det(I + M\Delta) = \det(I - u\ v^T \frac{1}{\beta} I) = 1 - \frac{v^T u}{\beta} = 0,$$

and $\Delta \in X_{\frac{1}{\beta}}$. On the other hand, according to (1.8), there is a $U \triangleq \mathrm{diag}\{\ u_i \mid u_i \in C, |\ u_i\ | = 1,\ i = 1, \cdots, m\ \}$ such that

$$\mu(M) = \rho(MU)$$

Therefore there exists a complex number θ with unit magnitude such that

$$\det(I + \frac{\theta}{\mu(M)} M\ U) = 0$$

Since

$$\det(I + \frac{\theta}{\mu(M)} MU) = \det(1 + \frac{\theta}{\mu(M)} v^T\, Uu),$$

this implies that

$$|\ v^T\, Uu\ | = \mu(M)\ .$$

Since, in view of the structure of U,

$$\beta \geq |\ v^T\, Uu\ |\ ,$$

the proof is complete.

#

Appendix C.

Proof of Proposition 4.1

Define

$$f \triangleq ||Mx||^2$$

and, for $i=1,\cdots,m$,

$$g_i \triangleq ||P_i x||^2||Mx||^2 - ||P_i Mx||^2$$

Let us write M and x in terms of their real and imaginary parts, respectively as $M = M_r + jM_i$ and $x = x_r + jx_i$. Define

$$z = \begin{bmatrix} x_r \\ x_i \end{bmatrix} \tag{C.1}$$

Suppose that x is a local maximizer for problem (2.2), or equivalently that z, defined in (C.1), is a local maximizer for problem

$$\max_{z \in \mathbb{R}^{2n}} f \tag{C.2}$$

subject to

$$g_i \leq 0, \quad i=1,\cdots,m$$

where f and the g_i's are now viewed as defined on $\mathbb{R}^{2n}$. Since x is a regular point for (2.1), there is a unique m-tuple of real numbers $\lambda_1,\cdots,\lambda_m$, such that

$$\nabla_z f(z) - \sum_{i=1}^{m} \lambda_i \nabla_z g_i(z) = 0, \tag{C.3}$$

$$\lambda_i \geq 0, \quad i=1,\cdots,m,$$

$$g_i(z) \leq 0, \quad i=1,\cdots,m.$$

Since

$$
\begin{aligned}
f &= ||Mx||^2 = ||(M_r + jM_i)(x_r + jx_i)||^2 \\
&= ||M_r x_r - M_i x_i||^2 + ||M_i x_r - M_r x_i||^2 \\
&= ||\begin{matrix} M_r x_r - M_i x_i \\ M_i x_r - M_r x_i \end{matrix}||^2 \\
&= ||\begin{bmatrix} M_r & -M_i \\ M_i & M_r \end{bmatrix} z ||^2 ,
\end{aligned}
$$

we have

$$
\begin{aligned}
\nabla_z f &= 2 \begin{bmatrix} M_r & -M_i \\ M_i & M_r \end{bmatrix}^T \begin{bmatrix} M_r & -M_i \\ M_i & M_r \end{bmatrix} z \\
&= 2 \begin{bmatrix} \mathrm{Re}(M^H Mx) \\ \mathrm{Im}(M^H Mx) \end{bmatrix} .
\end{aligned}
$$

A similar manipulation yields

$$
\nabla_z g_i = 2 \begin{bmatrix} \mathrm{Re}(||P_i x||^2 M^H Mx + ||Mx||^2 P_i x - M^H P_i Mx) \\ \mathrm{Im}(||P_i x||^2 M^H Mx + ||Mx||^2 P_i x - M^H P_i Mx) \end{bmatrix} , i = 1, \cdots, m .
$$

Therefore, (C.3) is equivalent to (4.1).

#

In order to prove Theorem 4.1, we need the following lemma.

Lemma C.1. Suppose x and y, both of unit norm, are related by

$$
y = Dx
$$

for some $D \in \mathcal{d}$. Then x is feasible for $\mathcal{p}(M)$ if and only if y is feasible for $\mathcal{p}(DMD^{-1})$, in which case

$$
||Mx|| = ||DMD^{-1}y|| .
$$

Furthermore, x solves $\mathcal{p}(M)$ if, and only if, y solves $\mathcal{p}(DMD^{-1})$.

Proof. Suppose x is feasible for $\mathcal{P}(M)$ and let $D = \text{diag}(d_i I_{k_i}) \in \mathcal{d}$. Since $P_i D = d_i P_i$, feasibility of x for $\mathcal{P}(M)$ implies

$$||P_i Dx|| \, ||Mx|| = ||P_i DMx|| \quad i = 1, \cdots, m \tag{C.4}$$

which implies, since $||Dx|| = 1$,

$$||Mx|| = ||DMx|| \, . \tag{C.5}$$

(C.4) and (C.5) then yield

$$||P_i Dx|| \, ||DMx|| = ||P_i DMx|| \quad i = 1, \cdots, m$$

or, since $y = Dx$,

$$||P_i y|| \, ||DMD^{-1}y|| = ||P_i DMD^{-1}y|| \quad i = 1, \cdots, m \; .$$

Thus y is feasible for $\mathcal{P}(DMD^{-1})$. Since $x = D^{-1}y$, with $D^{-1} \in \mathcal{d}$, the argument is reversible. Thus the first claim is proven. The second claim follows from the fact that (C.5) implies

$$||Mx|| = ||DMD^{-1}y||$$

This completes the proof of Lemma C.1.

#

Proof of Theorem 4.1. Suppose that $D^* = \text{diag}\{d_i^* I_{k_i}\} \in \mathcal{d}$ solves (P2), i.e., from Fact 1.0,

$$\mu(M) = \mu(D^* M D^{*^{-1}}) = \overline{\sigma}(D^* M D^{*^{-1}}) \tag{C.6}$$

Then, clearly, there exists a top right singular vector of $D^* M D^{*^{-1}}$, say y^*, such that y^* solves $\mathcal{P}(D^* M D^{*^{-1}})$. Lemma C.1 then implies that $x = (D^{*^{-1}} y^*)/||D^{*^{-1}} y^*||$ solves (P1). Thus (a) holds. Moreover, from our uniqueness assumption, $x = \theta x^*$ for some complex θ of unit magnitude. Furthermore, from (C.6), such y^* must satisfy

$$D^{*-1}M^H D^* D^* M D^{*-1} y^* = \bar{\sigma}(D^* M D^{*^{-1}}) y^* = \mu(M)^2 y^* = ||M(x)||^2 y^*$$

Since, for some scalar s, $y^* = sD^* x$, this implies

$$M^H D^{*^2} Mx = ||Mx||^2 D^{*^2} x \tag{C.7}$$

Define

$$\beta_i = \alpha d_i^{*2} \quad i=1, \cdots, m \tag{C.8}$$

where α is chosen such that $\sum_{i=1}^{m} \beta_i ||P_i x||^2 = 1$. Since $\alpha D^{*2} = \sum_{i=1}^{m} \beta_i P_i$, (C.7) becomes

$$\sum_{i=1}^{m} \beta_i (M^H P_i Mx - ||Mx||^2 P_i x) = 0$$

or equivalently,

$$M^H Mx + \sum_{i=1}^{m} \beta_i (M^H P_i Mx - ||Mx||^2 P_i x - ||P_i x||^2 M^H Mx) = 0$$

and since $x = \theta x^*$ with $|\theta| = 1$,

$$M^H Mx^* + \sum_{i=1}^{m} \beta_i (M^H P_i Mx^* - ||Mx^*||^2 P_i x^* - ||P_i x^*||^2 M^H Mx^*) = 0$$

From Proposition 4.1, $\beta_i = \lambda_i$ for $i=1, \cdots, m$ and together with (C.8) this implies that (b) holds true. To prove (c), first note that for any $v \in C^n$ and $i=1, \cdots, m$, $||P_i v|| = ||Q_i v||$. Also note that assumption (i) implies that $||Mx^*|| \neq 0$. Since

$$||P_i Mx^*|| = ||Mx^*|| \; ||P_i x^*|| \quad i=1, \cdots, m$$

$||P_i Mx^*||$ is nonzero whenever $||P_i x^*||$ is. Thus for any i such that $||P_i x|| \neq 0$, we can write, since $||Mx^*|| = \mu(M)$,

$$U_i Q_i M x^* = \frac{||P_i M x^*||}{||P_i x^*||} Q_i x^*$$

$$= \mu(M) Q_i x^*$$

When $||P_i x||=0$, the last relationship holds trivially. Filling identically zero rows, one gets

$$U_i P_i M x^* = \mu(M) P_i x^* \ . \tag{C.9}$$

Since U_i and P_i commute, one obtains, after summing (C.9) over all values of i,

$$U M x^* = \mu(M) x^* \ .$$

Thus

$$\rho(UM) \geq \mu(M)$$

which implies that U solves (P3) and this proves (c). Suppose now that $U^* \in \mathcal{U}$ solves (P3). Let z^* be any unit norm eigenvector of MU^* corresponding to an eigenvalue of magnitude equal to the spectral radius, i.e.,

$$M U^* z^* = \theta \mu(M) z^*$$

where θ is some complex number such that $|\theta|=1$. Letting $x = U^* z^*$, we get

$$M x = \theta \mu(M) U^{*^H} x$$

which implies

$$||P_i M x|| = \mu(M) ||P_i x|| \quad i=1, \cdots, m$$

whence

$$||Mx|| = \left[\sum_{i=1}^{m} ||P_i M x||^2 \right]^{1/2}$$

$$= \left[\sum_{i=1}^{m} \mu^2(M) ||P_i x||^2 \right]^{1/2}$$

$$= \mu(M)$$

i.e. x solves (P1). Thus (d) holds. The proof of Theorem 4.1 is now complete.

#

Appendix D

Consider the following algorithm, to be compared with Algorithm 5.1 (steps have been numbered correspondingly).

Algorithm D.1

Step 1.a.

Arbitrarily generate M_1, a complex $n \times n$ matrix. Obtain its singular value decomposition

$$M_1 = U \Sigma \tilde{V}^H = [u_1 \cdots u_n] \mathrm{diag}(\sigma_1, \cdots, \sigma_n) [\tilde{v}_1 \cdots \tilde{v}_n]^H$$

If there are several identical largest singular values, arbitrarily pick one of the corresponding right singular vector as $\tilde{v}_1$ (and similarly for u_1 in a consistent manner). If there exists a j such that $||P_j \tilde{v}_1||=0$ but $||P_j u_1|| \neq 0$, discard M_1 and restart Step 1.a. [This gives an equivalent singular value decomposition of M_1.]

Step 1.b.

Let

$$v_1 = \sum_{i=1}^{m} \alpha_i P_i \tilde{v}_1 ,$$

where

$$\alpha_i = \frac{||P_i \tilde{u}_1||}{||P_i v_1||} \quad \text{if } ||P_i \tilde{v}_1|| \neq 0$$

$$= 1 \qquad \text{otherwise}$$

[Thus $||u_1|| = ||v_1|| = 1$ and, for $i = 1, \cdots, m$, $||P_i u_1|| = ||P_i v_1||$.]

Step 2.

Arbitrarily generate a unitary matrix V with v_1 as first column.

Step 3.

Set

$$\Sigma = \text{diag}(\frac{\sigma_i}{\sigma_1}) \quad \text{if } \sigma_1 \neq 0$$

$$O_n \qquad \text{otherwise}$$

Step 4.

Arbitrarily generate $D \in \mathcal{d}$. Let

$$M = D^{-1} U \Sigma V^H D$$

#

Proof of Proposition 5.1. First, it is clear that any matrix generated by Algorithm D.1 can be generated by Algorithm 5.1. We now show that (i) any matrix M generated by Algorithm 5.1 satisfies condition (1.9) and (ii) any matrix $\hat{M}$ satisfying condition (1.9) can be generated by Algorithm D.1. Since any matrix M generated by Algorithm D.1 can be generated by Algorithm 5.1, this will prove the proposition. Suppose that M is generated by Algorithm 5.1. Then

$$\mu(M) \leq \bar{\sigma}(DMD^{-1}) = \bar{\sigma}(U\Sigma V^H) = 1$$

Now, since

$$||U\Sigma V^H v^1|| = ||u_1||=1 \ ,$$

we have, for $i=1, \cdots, m$,

$$||P_i U\Sigma V^H v_1|| = ||P_i u_1|| = ||P_i v_1|| = ||U\Sigma V^H v_1||\cdot||P_i v_1|| \ .$$

Thus v_1 is feasible for $\mathcal{P}(U\Sigma V^H)$ and hence

$$\mu(U\Sigma V^H) \geq ||U\Sigma V^H v_1|| = 1 \ .$$

Since $U\Sigma V^H = DMD^{-1}$ with $D \in \mathcal{d}$,

$$\mu(M) = \mu(U\Sigma V^H)$$

This proves (i). Suppose now that $\mu(\hat{M})=\bar{\sigma}(\hat{D}\hat{M}\hat{D}^{-1})=1$, for some $\hat{D} \in \mathcal{d}$. let $\hat{M}_1=\hat{D}\hat{M}\hat{D}^{-1}$. Then $\mu(\hat{M}_1)=\bar{\sigma}(\hat{M}_1)=1$. Suppose that $\hat{M}_1$ is generated in Step 1.a of Algorithm D.1. Since $\mu(\hat{M}_1)=\bar{\sigma}(\hat{M}_1)$, there is a right singular vector $\hat{v}_1$ corresponding to the singular value 1 such that

$$||P_i u_1|| = ||P_i \hat{M}_1 \hat{v}_1|| = ||P_i \hat{v}_1||\cdot||\hat{M}_1 \hat{v}_1|| = ||P_i \hat{v}_1|| \ .$$

Suppose that $\hat{v}_1 = \tilde{v}_1$. Then $\hat{M}_1$ will not be discarded in Step 1.a. Suppose that the columns of U and V are respectively left and right singular vectors of $\hat{M}_1$. Then, if Step 4 generates $\hat{D}$, the resulting matrix M will be, since $\sigma_1=1$,

$$M = \hat{D}^{-1}U\Sigma V^H\hat{D} = \hat{D}^{-1}\hat{M}_1\hat{D} = \hat{M} \ .$$

This proves (ii). The proof of Proposition 5.1 is now complete.

#

References

[1] J.C. Doyle, "Analysis of Feedback Systems with Structured Uncertainties," *Proceedings of IEE* vol. 129, no. 6, pp. 242-250 (November 1982).

[2] J.C. Doyle, J.E. Wall, and G. Stein, "Performance and Robustness Analysis for Structured Uncertainty," *Proc. 21st IEEE Conf. on Decision and Control,* pp. 629-636 (December 1982).

[3] M.G. Safonov and J.C. Doyle, *Minimizing Conservativeness of Robustness Singular Values.*

[4] M.K.H. Fan, "An Algorithm to Compute the Structured Singular Value," unpublished notes, Electrical Engineering Department, University of Maryland, College Park (1984).

[5] "Harwell Subroutine Library," Library Reference Manual, Harwell, England (1982).

[6] M. J. D. Powell, "A Fast Algorithm for Nonlinearly Constrained Optimization Calculations," pp. 144-157 in *Numerical Analysis, Dundee, 1977, Lecture Notes in Mathematics 630,* ed. G.A. Watson, Springer-Verlag (1977).

Example 1														
No.	T1	T2	No.	T1	T2	No.	T1	T2	No.	T1	T2	No.	T1	T2
1	32.65	76.93	21	33.85	28.01	41	26.05	25.10	61	42.35	190.5	81	33.23	35.68
2	25.60	849.0	22	22.45	750.4	42	4.600	829.4	62	32.13	186.7	82	24.66	608.1
3	24.63	297.9	23	----	71.55	43	21.31	31.23	63	----	462.1	83	69.18	26.53
4	48.01	29.88	24	33.76	121.1	44	44.10	110.3	64	6.333	816.0	84	28.56	595.5
5	----	22.55	25	32.71	109.1	45	23.38	841.7	65	----	110.9	85	27.03	723.9
6	5.950	862.3	26	----	713.8	46	5.433	19.56	66	29.53	22.33	86	30.45	481.4
7	30.85	427.6	27	20.43	95.45	47	5.350	655.6	67	39.31	169.0	87	25.21	27.63
8	5.516	18.83	28	4.816	790.0	48	5.466	177.6	68	31.31	23.38	88	32.98	66.73
9	18.26	31.48	29	40.80	27.58	49	22.96	571.6	69	22.05	178.3	89	26.95	21.76
10	39.10	23.80	30	23.26	746.7	50	----	108.1	70	35.33	64.78	90	35.16	312.4
11	26.76	163.4	31	6.066	998.4	51	45.13	121.9	71	----	520.9	91	34.13	111.7
12	----	212.5	32	5.900	464.0	52	19.90	255.6	72	----	35.13	92	28.48	31.05
13	21.63	134.3	33	4.800	317.3	53	29.76	20.01	73	5.400	139.4	93	21.81	362.4
14	----	40.51	34	5.633	89.18	54	22.51	870.2	74	4.183	129.6	94	30.20	106.1
15	5.383	88.93	35	28.26	463.1	55	25.15	881.6	75	29.98	174.0	95	32.16	21.91
16	24.61	782.7	36	26.43	394.8	56	27.76	790.4	76	5.333	24.01	96	23.18	673.6
17	----	25.83	37	86.06	88.95	57	75.78	38.41	77	----	605.4	97	24.61	863.9
18	----	19.80	38	25.70	478.4	58	43.20	53.25	78	29.16	655.0	98	25.16	181.0
19	24.55	536.5	39	----	190.7	59	6.283	786.3	79	26.30	799.6	99	32.35	75.50
20	31.85	33.51	40	20.96	359.9	60	5.616	759.9	80	6.033	679.4	100	----	127.6

Table 5.1

$n=10$; $m=3$;

$k=(k_1,k_2,k_3)$; k_1, k_2, and k_3 randomly generated.

For all matrices M, $\mu(M)=1$.

Example 2														
No.	T1	T2	No.	T1	T2	No.	T1	T2	No.	T1	T2	No.	T1	T2
1	40.55	41.73	21	40.26	380.8	41	23.90	870.0	61	81.56	100.2	81	----	524.5
2	32.83	153.8	22	85.20	219.4	42	----	103.3	62	42.51	200.1	82	30.16	592.6
3	27.60	50.86	23	28.70	910.1	43	----	23.01	63	----	42.96	83	116.8	53.36
4	48.38	512.6	24	35.40	108.4	44	----	158.3	64	74.25	32.98	84	24.25	24.03
5	29.60	286.1	25	37.38	271.0	45	----	48.31	65	----	50.75	85	36.06	477.6
6	70.06	94.00	26	36.38	1143.	46	8.250	995.6	66	----	978.9	86	33.25	379.1
7	53.66	217.8	27	25.80	281.0	47	31.43	242.7	67	32.60	959.0	87	31.06	746.9
8	----	36.96	28	42.45	33.98	48	36.58	152.2	68	44.28	943.1	88	36.01	24.21
9	33.56	98.11	29	41.35	255.4	49	33.48	259.2	69	43.46	235.3	89	59.25	445.8
10	7.916	40.53	30	----	370.7	50	34.40	553.6	70	23.18	219.3	90	31.03	158.0
11	24.33	584.2	31	23.51	79.01	51	42.03	213.0	71	31.55	198.4	91	33.93	187.4
12	43.28	173.5	32	61.66	362.5	52	62.13	43.60	72	56.01	171.5	92	67.93	255.0
13	32.46	150.8	33	52.10	146.4	53	34.65	468.9	73	45.40	38.93	93	39.31	125.2
14	30.45	668.3	34	9.266	896.4	54	29.66	254.8	74	33.81	1043.	94	8.783	671.2
15	41.68	420.9	35	59.05	596.9	55	76.96	40.70	75	39.86	216.3	95	67.00	870.5
16	29.88	582.9	36	30.43	602.3	56	33.20	695.1	76	40.25	25.71	96	28.50	521.9
17	39.68	29.70	37	----	537.4	57	28.46	29.25	77	90.83	157.4	97	45.83	43.35
18	28.10	244.4	38	113.4	637.9	58	24.95	1154.	78	----	545.2	98	7.966	1017.
19	50.08	26.73	39	27.95	130.6	59	28.25	177.0	79	24.26	30.73	99	24.50	903.5
20	31.35	653.5	40	35.48	92.11	60	70.13	57.13	80	27.83	32.66	100	35.33	314.3

Table 5.2

$n = 10$; $m = 5$;

$k = (k_1, k_2, k_3, k_4, k_5)$; k_1, k_2, k_3, k_4, and k_5 randomly generated.

For all matrices M, $\mu(M) = 1$.

Example 3														
No.	T1	T2	No.	T1	T2	No.	T1	T2	No.	T1	T2	No.	T1	T2
1	192.8	289.6	21	186.6	161.6	41	177.5	221.1	61	169.3	211.1	81	110.7	180.2
2	29.40	123.3	22	142.8	109.6	42	20.80	108.9	62	123.1	1093.	82	215.4	170.1
3	132.6	196.6	23	139.7	163.6	43	269.5	134.8	63	----	251.3	83	----	109.0
4	250.0	111.1	24	----	226.6	44	152.5	177.1	64	----	145.1	84	129.3	175.8
5	253.9	186.7	25	121.9	175.4	45	205.2	162.6	65	125.7	71.23	85	132.2	162.0
6	270.1	269.0	26	125.7	161.6	46	254.7	246.2	66	175.0	174.8	86	243.8	153.7
7	195.2	195.0	27	148.1	207.4	47	131.5	141.7	67	166.5	204.5	87	20.03	154.2
8	16.71	134.5	28	235.0	177.6	48	23.51	182.2	68	117.5	135.8	88	22.33	152.2
9	94.68	126.2	29	18.31	131.9	49	215.7	140.9	69	103.5	83.40	89	194.2	350.1
10	17.06	81.61	30	152.6	184.7	50	99.95	139.6	70	187.4	164.1	90	----	128.2
11	130.1	158.9	31	22.61	98.98	51	135.2	145.5	71	135.3	260.2	91	175.4	170.2
12	95.98	132.9	32	148.2	215.2	52	154.5	489.8	72	164.8	120.2	92	128.1	158.7
13	121.6	189.3	33	168.2	146.1	53	124.5	140.2	73	125.5	171.9	93	----	185.5
14	----	178.1	34	----	179.8	54	19.05	111.9	74	107.8	174.3	94	23.33	172.4
15	130.3	122.4	35	98.21	1004.	55	18.55	116.5	75	153.9	233.0	95	----	101.6
16	149.8	283.6	36	112.8	1268.	56	19.26	138.2	76	134.3	196.0	96	16.33	112.2
17	18.15	130.4	37	124.6	124.9	57	214.0	349.4	77	126.0	198.2	97	16.53	153.4
18	251.0	224.2	38	214.2	310.9	58	173.3	187.3	78	18.41	105.0	98	20.35	141.7
19	----	150.7	39	186.1	208.6	59	185.5	275.1	79	94.98	320.2	99	133.4	182.1
20	134.9	168.5	40	117.8	154.5	60	135.8	119.4	80	273.0	217.7	100	135.7	162.0

Table 5.3

$n=20$; $m=2$;

$k=(k_1,k_2)$; k_1 and k_2 randomly generated.
For all matrices M, $\mu(M)=1$.