# A Method for Efficient Flutter Analysis of Systems with Uncertain Modeling Parameters

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**Abstract**

A method for efficient analysis of variations in flutter speed predictions caused by parameter variations in the mass, stiffness and aerodynamic models of a wing structure is presented. The analysis method considers a linear uncertainty formulation and performs a perturbation analysis based on computation of eigenvalue differentials of a nominal system matrix. The method is applied in a test case study in which flutter speed variations caused by variations in mass and aerodynamic properties of a delta wing model is analyzed. The report is concluded with a discussion of the validity of the results and of how the applicability of the method is affected by the assumptions on which it builds.

# En Metod för Effektiv Uppskattning av Fladderhastigheten för System med Osäkra Parametrar

**Sammanfattning**

En metod för effektiv uppskattning av fladderhastigheten för system med osäkra parametrar i modellerna för massa, styvhet och aerodynamik presenteras. Metoden förutsätter en linjär osäkerhetsmodell och uppskattar den lägsta fladderhastighet som kan orsakas av parametervariationerna genom en analys som bygger på differentiering av egenvärden. Metoden appliceras i en studie av en delta vinge där skillnader i fladderhastighet orsakade av massvariationer och aerodynamiska osäkerheter analyseras. Rapporten avslutas med en diskussion om resultaten från teststudien och om hur applicerbarheten av metoden påverkas av de antaganden på vilka den bygger.

# Contents

# 1 Introduction

Aircraft flying at high speed are prone to encounter so called flutter. It is a dynamic instability which is caused by a complex interaction between aerodynamic, elastic and inertial forces. The interaction between the forces causes the aircraft structure to perform periodic, undamped oscillations, which may both cause structural damage and impose a problem for the control system. Since the damage caused by the oscillations may be costly, and even dangerous, it is highly desirable to be able to predict the flutter speed already during the design process. This so that one can choose a design for which the flutter speed is outside the speed envelop in which the aircraft is to be operated.

Throughout the years, a number of methods for flutter predictions have been developed, see for example [1],[2],[3]. Some of them build on linearized models of the aircraft dynamics and are usually able to give fairly accurate predictions of the flutter speed. However, due to various simplifying assumptions made during the modeling process, there will most likely be deviations between the modeled dynamics and the dynamics of the real aircraft. As a consequence, there will also be deviations between the predicted flutter speed and the actual flutter speed. The magnitude of the speed difference can be estimated by computing the flutter speed for a number of parameter perturbations, in search of the worst case perturbation giving the largest speed difference. However, the number of perturbation combinations to be evaluated can quickly grow large and it may thus be desirable to instead use some method which considers all possible combinations at the same time and computes a worst case flutter speed based on the perturbation shapes and boundaries.

Throughout the last decades an area called robust flutter analysis has been developed for this purpose. The robust analysis methods, see for example [4],[5] and [6],[7], consider a nominal model subjected to a number of uncertain parameters and aim at answering the question if these uncertainties could destabilize a nominally stable system. In theory, this sort of analysis could be very efficient for analyzing if, or proving that, the flight envelop is free of flutter instabilities. In practice, these methods are indeed very useful for analyzing how the system responds to various perturbations, but proving that a flutter instability could not occur in a certain speed regime may not be as simple.

One reason for this is that, for the robust stability results to be accurate, the uncertainty model must cover the true parameter variations both regarding magnitude and shape. When considering mass and stiffness uncertainties, this may not be a problem since one usually has a relatively good idea about the mass variations of various components and also relatively accurate material data and geometry information about the aircraft structure. But a non-negligible part of the modeling uncertainties usually come from the aerodynamic model, for which there may be little knowledge about both distribution and size of the modeling uncertainties. Thus, it may be difficult to know if the chosen uncertainty model covers the actual variations and, consequently, also difficult to know if the results from the robust analysis are accurate.

The example above aims at illustrating that robust stability may be difficult to guarantee in a practical application. The main advantage of the robust methods may thus be that they provide an efficient way of including several parameter variations at the same time when analyzing a system's response to modeling uncertainties. But it may then be an alternative to use some method for a more efficient perturbation analysis instead, if that could allow a wider variation of modeling uncertainties to be taken into account and perhaps also be more computationally efficient.

Such a method for perturbation analysis is presented in this study. The method exploits the diagonal structure of the modal flutter equation in order to perform a perturbation analysis based on computation of eigenvalues and eigenvalue differentials. The method allows a general linear uncertainty formulation and efficiently computes perturbation boundaries for the speed variations. The method is applied in a test case study of a delta wing model in which worst case flutter speeds predicted by the perturbation method, for parameter variations within certain bounds, are compared to worst case speed variations found from a number of nominal analyzes for various combinations of parameter variations within the same bounds. The study is concluded with a discussion about various advantages and limitations of the analysis method and with suggestions for further development.

## 2 Stability of the Nominal System

The nominal flutter equation can be written as

$$\left[p^2\overline{\boldsymbol{M}} + \left(\frac{b}{u}\right)^2 \overline{\boldsymbol{K}} - \frac{1}{2}\rho b^2 \overline{\boldsymbol{Q}}(p)\right]\boldsymbol{\eta} = 0 \tag{1}$$

where $\overline{\boldsymbol{M}}$, $\overline{\boldsymbol{K}}$ and $\overline{\boldsymbol{Q}}$ represent the mass, stiffness and aerodynamic matrices respectively, $p = g + ik$ is the reduced Laplace variable, $b$ is the aerodynamic reference length, $u$ is the airspeed and $\rho$ is the density of the air.

The system (1) is formulated in the Laplace domain and can be seen as a non-linear eigenvalue problem. The Laplace variable $p$ represents an eigenvalue and $\boldsymbol{\eta}$ represents the corresponding eigenvector. The system is said to be stable if all eigenvalues that makes the flutter matrix singular have negative real part. This since these solutions correspond to a stable motion with decreasing amplitude in the time domain. If some eigenvalue has a real part $g \geq 0$ the system is said to be nominally unstable. This since those solutions correspond to an unstable motion with constant or increasing amplitude in the time domain.

### 2.1 Modal Projection

The mass and stiffness matrices are usually obtained from a finite element model of the aircraft structure and the dimension of the flutter problem (1) can thus be fairly large. In order to reduce the size of the flutter problem the solution vector $\boldsymbol{\eta}$ can be approximated using a so called modal basis. This does not only reduce the computational time, but it also gives the resulting modal mass and stiffness matrices a certain structure [8].

The modal basis consists of a number of eigenvectors to the structural eigenvalue problem

$$[-\omega^2\overline{\boldsymbol{M}} + \overline{\boldsymbol{K}}]\boldsymbol{z} = 0. \tag{2}$$

where $\boldsymbol{z}$ is an eigenvector and $\omega^2$ is an eigenvalue. If denoting the corresponding modal basis $\boldsymbol{Z} = [\boldsymbol{z}_1\boldsymbol{z}_2 ... \boldsymbol{z}_n]$ and letting $\boldsymbol{\eta} = \boldsymbol{Z}\boldsymbol{v}$, the resulting modal mass and stiffness matrices can be written as $\boldsymbol{Z}^T\overline{\boldsymbol{M}}\boldsymbol{Z} = \boldsymbol{I}$ and $\boldsymbol{Z}^T\overline{\boldsymbol{K}}\boldsymbol{Z} = \boldsymbol{\Omega}$ [8]. Here, $\boldsymbol{I}$ represents an identity matrix and $\boldsymbol{\Omega} = \mathrm{diag}(\omega_n^2)$. The aerodynamic modal matrix is denoted by $\boldsymbol{Z}^T\overline{\boldsymbol{Q}}\boldsymbol{Z} = \boldsymbol{Q}$. Thus, the modal flutter equation becomes

$$\left[p^2\boldsymbol{I} + \left(\frac{b}{u}\right)^2 \boldsymbol{\Omega} - \frac{1}{2}\rho b^2 \boldsymbol{Q}(p)\right]\boldsymbol{v} = 0. \tag{3}$$

### 2.2 Polar Formulation of the Modal Flutter Equation

In this section, the property $\boldsymbol{Z}^T\overline{\boldsymbol{M}}\boldsymbol{Z} = \boldsymbol{I}$ will be exposed to derive an alternative formulation of the modal flutter equation (3). Using this formulation, the stability of (3) can be analyzed by computation of eigenvalues of an ordinary linear system.

For flutter analysis, a usual approximation is to let $\boldsymbol{Q}(p) \approx \boldsymbol{Q}(k)$ close to the imaginary axis [8]. If making a change of variables so that $p = re^{i\theta}$ then close to the imaginary axis $k \approx r$ and so $\boldsymbol{Q}(p)$ can be approximated by $\boldsymbol{Q}(r)$ in this area. Thus, (3) can be written

$$\left[r^2 e^{i2\theta}\boldsymbol{I} + \left(\frac{b}{u}\right)^2 \boldsymbol{\Omega} - \frac{1}{2}\rho b^2 \boldsymbol{Q}(r)\right]\boldsymbol{v} = 0. \tag{4}$$

When $r \neq 0$, (4) can be reformulated as

$$\left[e^{i2\theta}\boldsymbol{I} + \frac{1}{r^2}\left(\frac{b}{u}\right)^2 (\boldsymbol{\Omega} - \frac{1}{2}\rho u^2 \boldsymbol{Q}(r))\right]\boldsymbol{v} = 0. \tag{5}$$

The first term in (5) is a function of $\theta$ only and, since it is the identity matrix multiplied by a complex number, any $\boldsymbol{v}$ that is multiplied by it will become $e^{i2\theta}\boldsymbol{v}$. Thus (5) is equal to

$$-e^{i2\theta}\boldsymbol{v} = \left[\frac{1}{r^2}\left(\frac{b}{u}\right)(\boldsymbol{\Omega} - \frac{1}{2}\rho u^2 \boldsymbol{Q}(r))\right]\boldsymbol{v}. \tag{6}$$

Letting $\boldsymbol{A}(u,r) = \left[\frac{1}{r^2}\left(\frac{b}{u}\right)^2(\boldsymbol{\Omega} - \frac{1}{2}\rho u^2 \boldsymbol{Q}(r))\right]$ (5) can be written as

$$-e^{i2\theta}\boldsymbol{v} = \boldsymbol{A}(u,r)\boldsymbol{v}. \tag{7}$$

Since $|-e^{i2\theta}| = 1$, this means that for (5) to have a solution, $\boldsymbol{A}$ has to have an eigenvalue $\lambda$ with magnitude equal to one. Since $\boldsymbol{A}$ is a function of $u, r$ and does not depend on $\theta$, this also means that if there exists a $\lambda_j$ with $|\lambda_j| = 1$ for some $u, r$, then it is possible to find a solution to (5). This since $-e^{i2\theta}$ can represent any complex number with magnitude equal to one and $\theta$ thus can be chosen so that $-e^{i2\theta} = \lambda_j$.

A stability criteria can be found by noting that any angle $\gamma$ in the complex plane can be written as $\gamma = \gamma_0 + 2\pi a$ where $0 < \gamma_0 \leq 2\pi$ and $a$ is an arbitrary, positive or negative, digit. One thus has $-1 = e^{i(\pi + b\cdot 2\pi)}$ and $e^{i\alpha} = e^{i(\alpha_0 + c\cdot 2\pi)}$. This gives the equation $e^{i(2\theta + pi + b\cdot 2\pi)} = e^{i(\alpha_0 + c\cdot 2\pi)}$ with solution $\theta = (\alpha_0 - \pi)/2 + (b + c)\pi$. Thus, for each $\alpha_0$, $\theta$ can take any value $\theta = \theta_0 + d\pi$, where $\theta_0 = (\alpha_0 - \pi)/2$ and $d = b + c$. Thus, if $d$ is an even number, $\theta$ and $\theta_0$ will represent the same angle, while if $d$ is an odd number the angle represented by $\theta$ and $\theta_0$ will differ by an angle $\pi$. However, only angles corresponding to a reduced frequency $k \geq 0$ make any physical sense for the current approximation of the aerodynamic model, so if some $\theta_a$ fulfills $k > 0$ the corresponding angle is the only physically feasible solution. If an angle $\theta_b$ corresponds to $k = 0$, then so does the angle $\pi$ radians away, and both are physically possible.

Based on these possibilities, solutions fulfilling $0 < \alpha_0 < \pi$ are concluded to be stable, since they give feasible solutions corresponding to $\pi/2 < \theta_0 < \pi$, i.e. solutions $p = g + ik$ with $g < 0$. Solutions with $\pi \leq \alpha_0 \leq 2\pi$ are concluded to be unstable, since they give feasible solutions corresponding to $0 \leq \theta_0 \leq \pi/2$, i.e. solutions $p = g + ik$ with $g \geq 0$.

### 2.3 Discretization into $r$-intervals

The elements of $\boldsymbol{A}(u,r)$ are assumed to be continuous functions $a_{ij}(u,r)$ and so the eigenvalues of the matrix $\lambda_j(u,r)$ will be continuous functions of the same variables [9]. However in most practical applications information about $a_{ij}(u,r)$ will only be available at a discrete set of points and, as a consequence, the eigenvalues will also only be known at a discrete set of points. But by assuming that the eigenvalues vary linearly in a neighborhood around each point, approximate eigenvalue paths can be obtained by computing the eigenvalues at the discrete points and differentiating them with respect to $u, r$. The eigenvalue derivatives are defined for simple eigenvalues but may not be available for multiple eigenvalues [10]. For this reason, the eigenvalues will be assumed to be simple in the following sections.

In this study, the stability analysis will be performed at a constant speed and altitude. Thus, the speed will be a constant $u_f$ and each element in $\boldsymbol{A}$ will be a function of $r$ only, i.e. $a_{ij}(u_f, r) = a_{ij}(r)$. In an interval close to the point $r_m$, $R_m = \{r | r_m \leq r < r_m + \Delta r_m\}$ a simple eigenvalue $\lambda_n$ can be approximated by

$$\lambda_j(r) = \lambda_j(r_m) + (\partial\lambda_j/\partial r)|_{r_m}(r - r_m). \tag{8}$$

Since $\lambda_j(r)$ in this approximation is a linear function of $r$ it can be investigated if an eigenvalue fulfilling $\lambda_j(r) = e^{i\alpha}$ is possible for some $r \in R_m$. If that is the case, the corresponding $\theta$ can be evaluated to see if this $\lambda_j(r)$ corresponds to a stable or unstable solution of (5).

If no eigenvalue $\lambda_j(r) = e^{i\alpha}$ is possible or if there exists a $\lambda_j(r) = e^{i\alpha}$ corresponding to a stable solution of (5), the eigenvalue is concluded to be stable at $u_f$ and for all $r \in R_m$. If there exists a $\lambda_j(r) = e^{i\alpha}$ corresponding to an unstable solution of (5), the eigenvalue is concluded to be unstable at $u_f$ for some $r \in R_m$.

Here it may be informative to note that the assumed eigenvalue variation in (8) does not necessarily imply that the eigenvalues are continuous in between the different $R_m$ intervals. However, for the stability conditions to hold it is only necessary that the eigenvalues are continuous functions in each $R_m$. This since, under the assumptions made, the stability is only evaluated in one $R_m$ interval at the time.

When evaluating the stability in a certain $R_m$ the method says nothing about the stability in the rest of $R$. However, by dividing $R$ into a number of connected intervals $R_m$ and evaluating the stability in each $R_m$ it is possible to investigate the stability in all of $R$. This since the different $R_m$ together cover all of $R$ and an instability thus cannot be present in $R$ without being present in some $R_m$. So by evaluating the stability in all different $R_m$ one also evaluates the stability in $R$.

# 3 Stability of the Perturbed System

The system (1) is nominally stable in a certain flight state if all solutions $p$ have a negative real part. If introducing uncertainties to (1), the perturbed system is stable if all possible solutions still fulfill the stability criteria for the uncertainties under consideration. The critical flutter speed of the perturbed system is then the lowest speed for which the perturbed system has an unstable solution.

## 3.1 Aerodynamic uncertainties

In this part it will be explained how to formulate the aerodynamic uncertainties as variations in pressure coefficients at a number of aerodynamic nodes. This method is suitable for aerodynamic matrices obtained from Doublet-Lattice Theory [7]. It may be useful to note that the analysis procedure which will be employed for these uncertainties is not restricted to this exact uncertainty formulation, but can be applied to any uncertainty formulation that can be written as a linear combination of uncertainty matrices.

In order to be able to formulate the aerodynamic uncertainties as variations in pressure coefficients it is assumed that the aerodynamic matrix can be divided into left and right partitions $\boldsymbol{Q}_L, \boldsymbol{Q}_R$ that relates displacements and forces through pressure coefficients in different boxes [7], [11]. Letting $\boldsymbol{C}_{p0}$ =diag($c_{p0_u}$), where $c_{p0_u}$ denotes the pressure coefficient in box $u$, the aerodynamic matrix can be written as $\boldsymbol{Q}(r) = \boldsymbol{Q}_L \boldsymbol{C}_{p0} \boldsymbol{Q}_R$. Introducing uncertainties in the pressure coefficients as $c_{p_u} = c_{p0_u}(1+x_{a_u}e^{i\beta_u})$, where $x_{a_u} \in \mathbb{R}$ denotes the magnitude of uncertainty $u$ and $e^{i\beta_u}$ is a complex number fulfilling $|e^{i\beta_u}| = 1$, and letting $d\boldsymbol{C}_p(r, \boldsymbol{x}_a)$ =diag($c_{p0_u} x_{a_u} e^{i\beta_u}$) the aerodynamic uncertainty matrix can be written as

$$d\boldsymbol{Q}(r, \boldsymbol{x}_a) = \boldsymbol{Q}_L d\boldsymbol{C}_p \boldsymbol{Q}_R. \tag{9}$$

It can be seen that $d\boldsymbol{C}_p$ can be represented by a matrix sum

$$d\boldsymbol{C}_p = \sum_u c_{p0_u} x_{a_u} e^{i\beta_u} d\boldsymbol{C}_u, \tag{10}$$

where $\boldsymbol{D}_u$ is a matrix with element $(u, u)$ equal to one and all other elements equal to zero. Thus, by the distributive law of matrix algebra, (9) can be rewritten as

$$d\boldsymbol{Q}(r, \boldsymbol{x}_a) = \sum_u w_u e^{i\beta_u} \boldsymbol{Q}_L d\boldsymbol{C}_u \boldsymbol{Q}_R. \tag{11}$$

This gives the $\boldsymbol{A}$-matrix

$$\boldsymbol{A}(r, \boldsymbol{x}_a) = \left[ \frac{1}{r^2} \left( \frac{b}{u_f} \right)^2 (\boldsymbol{\Omega} - \frac{1}{2}\rho u_f^2 (\boldsymbol{Q}(r) + \sum_u x_{a_u} e^{i\beta_u} \boldsymbol{Q}_L d\boldsymbol{C}_u \boldsymbol{Q}_R) \right] \tag{12}$$

and so the uncertain flutter equation can be written as

$$[e^{i2\theta}\boldsymbol{I} + \boldsymbol{A}(r, \boldsymbol{x}_a)]\boldsymbol{v} = 0. \tag{13}$$

Comparing (13) to (7) it can be seen that (7) is a special case of (13) obtained by letting $\boldsymbol{x}_a = \boldsymbol{0}$. Thus, based on the same reasoning as for (7), it can be seen that the system is stable if and only if $\boldsymbol{A}(r, \boldsymbol{x}_a)$ has no eigenvalue on the lower half of the unit circle.

In order to be able to evaluate if the stability criteria is fulfilled for (13) it is assumed that each $\lambda_j$ varies linearly with $\boldsymbol{x}_a$. This can be seen as a reasonable assumption for small $x_{a_u}$ and the variation can be found by computation of eigenvalue derivatives. These derivatives are available for simple eigenvalues and can be found from

$$\frac{\partial \lambda_j}{\partial x_{a_u}} = \boldsymbol{v}_{l_j}^* \frac{\partial \boldsymbol{A}(q, r, \boldsymbol{x}_a)}{\partial x_{a_u}} \boldsymbol{v}_{r_j}, \tag{14}$$

where $\boldsymbol{v}_{l_j}$ and $\boldsymbol{v}_{r_j}$ are the left and right eigenvectors to eigenvalue $j$ [10]. Using (9) one obtains

$$\frac{\partial \boldsymbol{A}(r, \boldsymbol{x}_a)}{\partial x_{a_u}} = e^{i\beta_u} \boldsymbol{v}_{l_j}^* \boldsymbol{Q}_L d\boldsymbol{D}_u \boldsymbol{Q}_R \boldsymbol{v}_{r_j}. \tag{15}$$

It can be seen that, since $e^{i\beta_u}$ represents an arbitrary complex number with magnitude equal to one, each of these terms describes a circle in the complex plane. Thus, the sum of the different eigenvalues differentials can be bounded within a distance $d_{a_{qm}}$ from the nominal eigenvalue path.

Thus, if the distance between $\lambda_j(r, \boldsymbol{x}_a)$ and the lower half of the unit circle is larger than $d_{a_{qm}}$, the aerodynamic uncertainties cannot destabilize the system in that $R_m$. If on the other hand the distance is smaller than, or equal to, $d_{a_{qm}}$, the uncertainties could destabilize the system.

## 3.2 Mass and stiffness uncertainties

The eigenvalue method described above exposes the property $\boldsymbol{Z}^T\overline{\boldsymbol{M}}\boldsymbol{Z} = \boldsymbol{I}$ when analyzing the stability of the system. To be able to use the same eigenvalue approach when mass uncertainties are present, the uncertainties have to be expressed in such a way that this property is not lost. One way to achieve this is to express both mass and stiffness uncertainties as variations in structural eigenvalues and eigenvectors. The required eigenvalue and eigenvector differentials are available for simple eigenvalues but may not be available for multiple eigenvalues [12]. In the following sections, the eigenvalues will therefore be assumed to be simple. Using this approach, any uncertainty formulation that can be written as a linear combination of uncertainty matrices can be allowed for both mass and stiffness uncertainties.

Denoting the nominal structural eigenvalue problem

$$[-\omega^2\overline{\boldsymbol{M}} + \overline{\boldsymbol{K}}]\boldsymbol{z} = 0 \tag{16}$$

one has $\boldsymbol{Z}^T\overline{\boldsymbol{M}}\boldsymbol{Z} = \boldsymbol{I}$ and $\boldsymbol{Z}^T\overline{\boldsymbol{K}}\boldsymbol{Z} = \boldsymbol{\Omega}$. Letting $\overline{\boldsymbol{M}}_{x_{ms}} = \overline{\boldsymbol{M}} + \sum\limits_v \frac{\partial \overline{\boldsymbol{M}}}{\partial x_{m_v}} x_{m_v}$ and $\overline{\boldsymbol{K}}_{x_{ms}} = \overline{\boldsymbol{K}} + \sum\limits_w \frac{\partial \overline{\boldsymbol{K}}}{\partial x_{s_w}} x_{s_w}$, where $x_{m_v}$ and $x_{s_w}$ denote the mass and stiffness uncertainties, one obtains the uncertain structural eigenvalue problem

$$[-\omega^2\overline{\boldsymbol{M}}_{x_{ms}} + \overline{\boldsymbol{K}}_{x_{ms}}]\boldsymbol{z}_{x_{ms}} = 0. \tag{17}$$

The maximum magnitudes of $x_{m_v}$ and $x_{s_w}$ are denoted by $\hat{x}_{m_v}$ and $\hat{x}_{s_w}$ respectively. If they are small, it can be assumed that the eigenvalues and eigenvectors vary according to $\omega^2_{x_{ms}} = \omega^2 + \sum\limits_v \frac{\partial \omega^2}{\partial x_{m_v}} x_{m_v} + \sum\limits_w \frac{\partial \omega^2}{\partial x_{s_w}} x_{s_w}$ and $\boldsymbol{z}_{x_{ms}} = \boldsymbol{z} + \sum\limits_v \frac{\partial \boldsymbol{z}}{\partial x_{m_v}} x_{m_v} + \sum\limits_v \frac{\partial \boldsymbol{z}}{\partial x_{s_w}} x_{s_w}$ respectively. Denoting the corresponding modal basis $\boldsymbol{Z}_{x_{ms}}$ one has $\boldsymbol{Z}^T_{x_{ms}}\overline{\boldsymbol{M}}_{x_{ms}}\boldsymbol{Z}_{x_{ms}} = \boldsymbol{I}$ and $\boldsymbol{Z}^T_{x_{ms}}\overline{\boldsymbol{K}}_{x_{ms}}\boldsymbol{Z}_{x_{ms}} = \boldsymbol{\Omega}_{x_{ms}}$, which gives the uncertain flutter equation

$$\left[e^{i2\theta}\boldsymbol{I} + \frac{1}{r^2}\left(\frac{b}{u_f}\right)^2 (\boldsymbol{\Omega}_{x_{ms}} - \frac{1}{2}\rho u_f^2 \boldsymbol{Z}^T_{x_{ms}}\boldsymbol{Q}(r)\boldsymbol{Z}_{x_{ms}})\right]\boldsymbol{v} = 0. \tag{18}$$

The eigenvalue variations caused by these uncertainties can be found from (14) and each eigenvalue variation can be bounded within an uncertainty distance $d_{ms}(r, \boldsymbol{x}_m, \boldsymbol{x}_s)$. The resulting area, within which an eigenvalue $\lambda_j(r, \boldsymbol{x}_m, \boldsymbol{x}_s)$ is allowed to vary, could then be assumed to include all points within the distance $d_{ms}(r, \boldsymbol{x}_m, \boldsymbol{x}_s)$ from the nominal eigenvalue $\lambda_j(r, \boldsymbol{0})$. However, since the eigenvalue variations in this case have certain directions, and cannot reach all points within the uncertainty distance, this may induce a certain amount of conservatism in the results. This can be avoided by bounding the uncertainties within their convex hull instead, which can be found by use of Minkowski sums [13]. The perturbed eigenvalue is then considered to be stable if none of the convex hull intersects with the lower half of the unit circle. If on the other hand some point intersects with the lower half of the unit circle, it is concluded that some uncertainty could destabilize the eigenvalue.

### 3.3 Combined uncertainties

Combining the uncertainties from (13) and (18), one obtains the uncertain flutter equation

$$\left[e^{i2\theta}\boldsymbol{I} + \frac{1}{r^2}\left(\frac{b}{u_f}\right)^2 (\boldsymbol{\Omega}_{x_{ms}} - \frac{1}{2}\rho u_f^2 \boldsymbol{Z}_{x_{ms}}^T \boldsymbol{Q}(r, \boldsymbol{x}_a)\boldsymbol{Z}_{x_{ms}})\right]\boldsymbol{v} = 0. \tag{19}$$

For an analysis of how the combination of aerodynamic, mass and stiffness uncertainties affect the stability of the system, one has to analyze if the uncertainty area caused by the combined uncertainties could have an intersection with the lower half of the unit circle.

The perhaps simplest way is to use the circular bound for both aerodynamic and mass and stiffness uncertainties. This would give rise to a combined uncertainty distance, within which the eigenvalues were allowed to vary. It would have the advantage of being simple, but could give unnecessarily conservative results as discussed in Section 3.2.

Another possibility is to bound the variations within the convex hull of the combination of the line segments from the mass and stiffness uncertainties and the uncertainty circles from the aerodynamic uncertainties. This method may be a bit more computationally demanding, but has the advantage that it avoids unnecessarily conservative results.

## 4 Flutter analysis of a delta wing model

To evaluate the practical usefulness of the perturbation analysis method a case study was performed on a delta wing model. The case study was aimed at analyzing if the flutter speed bounds predicted by the perturbation analysis method, for parameter variations within certain bounds, were similar to the worst case flutter speeds found when performing a number of nominal analyzes for different combinations of parameter variations, within the same bounds. For this analysis it was considered important to have well defined parameter variations which made physical sense. It was therefore decided to focus on mass uncertainties caused by fuel level variations and aerodynamic uncertainties caused by uncertain aerodynamics around the leading and trailing edges of the wing.

A relatively simple delta wing model, see Figure 1, was used. The model had a semi span of 0.86 m and a thickness of 4 mm. It was made of a glass fiber composite material with an elastic modulus of 28.5 GPa, a shear modulus of 6.0 GPa, a Poisson ratio of 0.1606 and a density of 2036 kg/m$^3$. The mass and stiffness matrices needed for the numerical analysis were obtained from a finite element model and the aerodynamic matrix was found using Doublet-Lattice aerodynamics [8].

The aerodynamic model was only considered to be accurate for reduced frequencies $k$

up to 1.5 and since the equations needed for the analysis are undefined in the origin only $r$ values between 0.1 and 1.5 were considered in this study. If the wing model has flutter instabilities for reduced frequencies outside this interval, they will thus not be found in the analysis. But since experimental flutter testing of wing models similar to the one in this study have shown to suffer from flutter instabilities at reduced frequencies close to the center of this interval, it is credible that also this wing model will have reduced frequencies within the interval and that the flutter speeds found in this study actually are the critical flutter speeds for the wing model in question.

The numerical model predicted the wing to have a flutter speed of 48.7 m/s and a flutter frequency of 7.4 Hz. An experimental flutter test of a physical model for which the geometric and material properties were assumed to be the same gave a flutter speed of 48.5 m/s and a flutter frequency of 8.4 Hz. The numerical model thus predicted a flutter speed similar to the one of the physical model but a slightly lower flutter frequency. For a flutter analysis of a real aircraft the numerical model would possibly need to be tuned to obtain a better prediction of the flutter frequency of the physical model. But in this study the main goal was not to analyze the exact flutter behavior of the physical wing model in question, but to investigate if the perturbation analysis method was able to predict the speed variations caused by mass variations for an arbitrary wing model. It was thus deemed sufficient that the numerical model gave results which made physical sense and for that purpose the match between the numerical and experimental analysis was deemed good enough.

The mass uncertainties were chosen to be variations in the fuel levels in nine fuel tanks at different span and chord wise positions of the wing, see Figure 2. The fuel mass was modeled as an increased density of the material, from 2036 kg/m$^3$ to 4000 kg/m$^3$, in the corresponding parts of the wing and the flutter speed variation for density variations within 10% of 4000 kg/m$^3$ was analyzed. Nominal analyzes for a number of combinations of density variations within different bounds were made. Worst case flutter speeds for the different uncertainty bounds were found as the lowest nominal flutter speed computed for some parameter combination within the bounds. These results were compared to the worst case flutter speeds predicted when using the same bounds for the density variation as input to the perturbation analysis method. The predicted worst case flutter speeds for the different uncertainty bounds can be seen in Figure 4.

The aerodynamic uncertainties were modeled as variations in the pressure coefficients in six patches along the leading and trailing edges of the wing, see Figure 3. Nominal analyzes for a number of combinations of pressure variations within different bounds were made. Worst case flutter speeds for the different uncertainty bounds were found as the lowest nominal flutter speed computed for some parameter combination within the bounds. These results were compared to the worst case flutter speeds predicted when using the same bounds for the pressure variation as input to the perturbation analysis method. The predicted worst case flutter speeds for the different uncertainty bounds can be seen in Figure 5.

An analysis of the flutter speed variations caused by the combination of mass and aerodynamic uncertainties was then made. Also this time, a number of nominal analyzes were made for various parameter combinations within certain bounds and the worst case

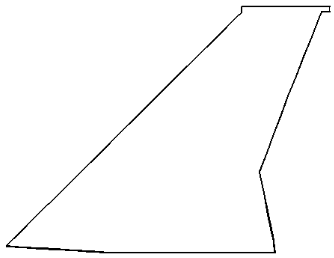

Figure 1: Wing model used for the nominal flutter analysis.

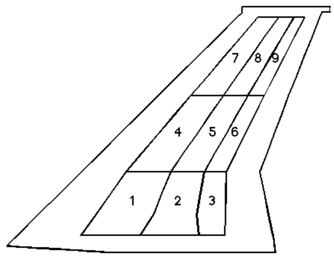


Figure 2: Numbering of the fuel tanks used for the mass uncertainties.

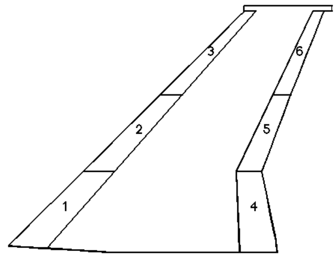


Figure 3: Numbering of the patches used for the aerodynamic uncertainties.

flutter speeds found from these analyzes were compared to the worst case flutter speeds found when using the same bounds in the perturbation analysis method. The results can be seen in Figure 6.

As can be seen in Figures 4,5, the worst case flutter speeds predicted by the perturbation analysis method show close resemblance to the worst case flutter speeds found from the nominal tests of different combinations of mass uncertainties and aerodynamic uncertainties. For the combined uncertainty models the flutter speeds in Figure 6 show that the predictions from the perturbation analysis method are slightly conservative as compared to the worst case flutter speeds found from the nominal analyzes.This could indicate that the perturbation analysis method in this case over predicted the decrease in flutter speed caused by the combined uncertainties. However, it could also mean that there are worse combinations of parameter variations than the ones investigated in the nominal analyzes and that the speed bounds predicted by the perturbation method are more accurate than the ones found from the nominal analyzes.

Another aspect to note about the results is that the worst nominal flutter speed found for a 10 % variation in the combination of mass and aerodynamic uncertainties is 22 % below the flutter speed for the nominal model without any uncertain parameters. One could thus perhaps question if this large uncertainties should be analyzed by a perturbation analysis method, or if one should try to improve the nominal model to reduce the magnitude of the uncertainties instead. If only analyzing the results for flutter speeds above 42.7 m/s, which corresponds to a 10% decrease from the nominal flutter speed, it can be noted that the speeds predicted by the perturbation analysis and the nominal analyzes are fairly close.

Since there are infinitely many possible combinations of parameter variations within the bounds, it is possible that there could exist some worse combination of parameter variations which was not found when testing different combinations and that the worst case flutter speed that could be caused by the parameter variations actually is below the one predicted by the perturbation analysis method. However, since the parameter variations tested in the nominal analyzes were chosen as combinations which were expected to give a noticeable decrease in flutter speed, it is credible that a combination close to the worst

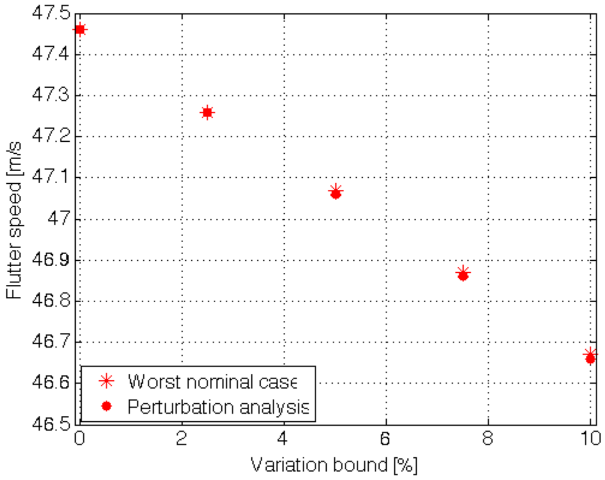


Figure 4: Worst case flutter speeds for mass uncertainties.

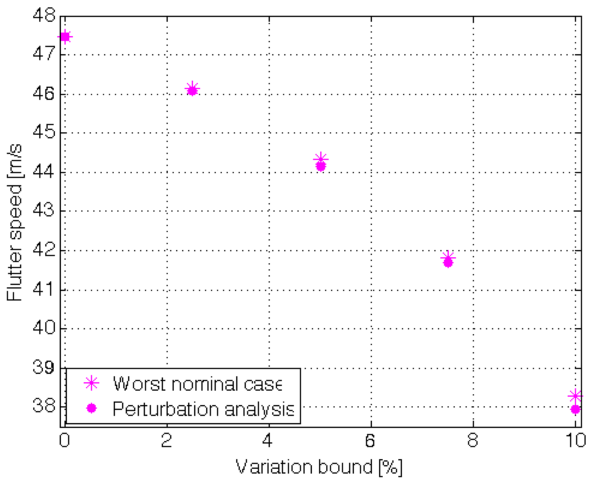


Figure 5: Worst case flutter speeds for aerodynamic uncertainties.

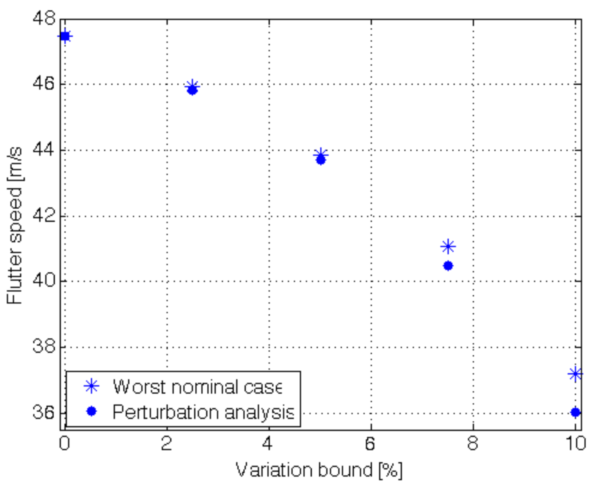


Figure 6: Worst case flutter speeds for combination of mass and aerodynamic uncertainties.

combination was found. This would indicate that the predicted worst case flutter speeds are fairly close to the actual ones and that the perturbation analysis method was successful in estimating these speed boundaries.

# 5 Discussion

As shown by the test case study in the previous section, the perturbation analysis method seems to be able to predict fairly accurate bounds for the speed variations caused by various parameter variations. If the assumptions about linear variation of eigenvalues and eigenvectors hold, this should also be the case. For most dynamical systems these assumptions are reasonable if the parameter variations causing them are small enough. It is thus reasonable to expect the speed variations predicted by the method to be accurate for small variations in mass, stiffness and aerodynamic properties. However, lots of analysis remains to be done to be able to tell how small the variations need to be to be considered as small enough.

Another assumption that needs discussing is the one about continuously varying eigenvalues and eigenvectors. As mentioned in Section 3.1, these derivatives may be undefined for multiple eigenvalues and for this reason the eigenvalues are, in this study, assumed to be simple. If multiple eigenvalues appear, it is difficult to know how to interpret the results obtained from the perturbation analysis. So far, undefined derivatives have not shown to be a problem but it may be recommendable to keep in mind that problems could appear. It could perhaps also be useful to investigate how to handle the problems in case of them appearing.

The fact that the governing equations are undefined in the origin may also need discussing. For aircraft with a high wing stiffness flutter instabilities will likely appear at fairly high values of the reduced frequency and it may thus not be a problem that the analysis cannot be performed close to the origin. For aircraft with a low wing stiffness however, flutter could appear at fairly low reduced frequencies. Since it may be difficult to analyze the system's behavior at low reduced frequencies with the above described method, the method may not be suitable for analysis of aircraft for which one suspects that flutter instabilities could appear at very low reduced frequencies.

Further, the perturbation method builds the analysis on the assumption that, close to the imaginary axis, the reduced frequency is approximately equal to the polar radius. A question that naturally arises is for what polar angles this is a reasonable assumption. To answer the question one needs to analyze for what angles the aerodynamic model is able to accurately represent the aerodynamic forces. No such analysis has been made in this study but, based on arguments about small angles, it is assumed to be a reasonable assumption for a few degrees variation about the imaginary axis. This means that the analysis method cannot be expected to give accurate flutter predictions outside this interval and that the analysis thus is restricted to predictions within this area. But the commonly used approximation of letting the aerodynamic matrix be a function of the reduced frequency, and thereby neglecting the effect of the damping of the motion, is also restricted to an area close to the imaginary axis. Thus, the effect of approximating the reduced frequency with the polar radius will hopefully not have a severe effect on the validity of the aerodynamic model.

# 6 Conclusions

The study has shown that the suggested method for perturbation analysis seems to be able to predict accurate bounds for flutter speed variations for small variations in mass and aerodynamic properties. The linear assumptions on which the method builds can likely be seen as reasonable for small parameter variations and the method can thus be expected to give accurate flutter predictions for small variations in modeling parameters. However, a lot of analysis remains to be done to be able to tell how small the variations need to be for them to be considered as small enough. Since important equations and derivatives are undefined in certain points, problems could appear if the analysis needs to be performed close to those points. Hopefully however, such cases will not be very common in practical applications.

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