On the Determination of the Number and Multiplicity of Zeros of a Function

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Abstract — Zusammenfassung

On the Determination of the Number and Multiplicity of Zeros of a Function. It is shown that certain simple integrals determine the number of zeros with a certain multiplicity of a function of one variable in an arbitrary interval. Several typical numerical examples are given.

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Zur Bestimmung der Anzahl und Vielfachheit der Nullstellen einer Funktion. Wir zeigen, daß man mit Hilfe gewisser Integrale die Anzahl der Nullstellen einer gegebenen Vielfachheit für eine Funktion einer Veränderlichen in einem beliebigen Intervall bestimmen kann. Das Vorgeben wird durch typische Beispiele erläutert.

1. Introduction

In previous papers, Hoenders, Slump [1], Slump, Hoenders [2], we showed that a certain simple integral, (1.1), determines the number of zeros N of a suitable function $f(x)$ in an interval $[a, b]$, ($x \in [a, b]$):

$$N = \oint_c d \arctan\left(\frac{zf'}{f}\right). \tag{1.1}$$

The prime denotes differentiation with respect to x, i.e. $f' = \dfrac{\partial f(x)}{\partial x}$, and c denotes a rectangle with vertices at $(a, \pm \varepsilon)$ and $(b, \pm \varepsilon)$ in the $x - z$-plane.[1] The theory of this so-called Kronecker-Picard (K.P.) integral (1.1) is closely connected with the analytical formulation of the solid angle and has been developed, Hoenders, Slump [1], Slump, Hoenders [2], for the general case of the number of zeros common to n functions in R_n.

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[1] We choose this particular contour because it turned out to be very handy for numerical calculations. However, any other contour c enclosing the interval $[a, b]$ and with $x \in [a, b]$ if $(x, z) \in c$ yields the same result.

In the previous papers we restricted ourselves to the case of simple zeros, and announced that the theory of multiple zeros would be analyzed in forthcoming papers.

It will be the aim of this paper to develop the theory for the case of multiple zeros of a function of *one* variable only, and to give some typical numerical examples.

The reason for restricting ourselves to the case of one variable is that the theory of multiple zeros common to n functions in R_n is quite different from the one dimensional case. The need for such a more dimensional theory can be deduced from statements made quite recently by Press et al. [3]. In 9.6, concerning the determination of the zeros of sets of functions they state: "It is not hard to see why (very likely) there never will be any good, general methods". The reason they give is that one never can be sure not to have missed a zero because of the lack of a prior knowledge of the zero curves, (surfaces) of the functions involved. Our theory then provides an exact answer to this problem of missing of zeros.

2. Theory

We first recall the integral I_n (eq. 3 of [1], eq. 13 of [2]):

$$I_n = \frac{1}{\Omega_n} \int\int_{S_n} \cdots \int \frac{\sum_{i=1}^{n} A_i \, dx_1 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_n}{(f_1^2 + f_2^2 + \cdots f_n^2)^{1/2n}}, \tag{2.1}$$

which is equal to the number of zeros common to the set of functions $\{f_n\}$ within a n-dimensional domain of the $x_1 \ldots x_n$ space, bounded by a closed surface S_n.

Furthermore,

$$A_i = \begin{vmatrix} f_1 \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{i-1}} & \dfrac{\partial f_1}{\partial x_{i+1}} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ f_2 \dfrac{\partial f_2}{\partial x_1} & \cdots & \dfrac{\partial f_2}{\partial x_{i-1}} & \dfrac{\partial f_2}{\partial x_{i+1}} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ f_n \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_{i-1}} & \dfrac{\partial f_n}{\partial x_{i+1}} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}, \tag{2.2}$$

if $n =$ even. A_i equals the determinant (2.2) multiplied by $(-1)^{i-1}$ if n is odd. The number Ω_n denotes the surface of a hypersphere with radius unity in R_n:

$$\Omega_n = \frac{2\pi^{(1/2)n}}{\Gamma(\frac{1}{2}n)}, \tag{2.3}$$

and S_n denotes a $(n-1)$-dimensional surface enclosing the set of zeros of the set of functions $\{f_n\}$ in a certain domain of the $\{x_1 \ldots x_n\}$-space.

The shorthand notation of eq. (2.1) should be interpreted as follows: The surface differential $dx_1 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_n$ consists of $(n-1)$ differentials such that this element reads as $dx_2 \ldots dx_n$ if $i = 1$, and $dx_1 \ldots dx_{n-1}$ if $i = n$. In three dimensional

space eq. (2.1) would therefore read as:

$$I = \frac{1}{\Omega_n} \iint_{S_n} A \, dy \, dz + B \, dz \, dx + C \, dx \, dy, \tag{2.4}$$

if

$$A = \begin{vmatrix} f_1 & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\[2mm] f_2 & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\[2mm] f_3 & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}, \quad B = \begin{vmatrix} f_1 & \dfrac{\partial f_1}{\partial z} & \dfrac{\partial f_1}{\partial x} \\[2mm] f_2 & \dfrac{\partial f_2}{\partial z} & \dfrac{\partial f_2}{\partial x} \\[2mm] f_3 & \dfrac{\partial f_3}{\partial z} & \dfrac{\partial f_3}{\partial x} \end{vmatrix}, \quad \text{and} \quad C = \begin{vmatrix} f_1 & \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\[2mm] f_2 & \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \\[2mm] f_3 & \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} \end{vmatrix}. \tag{2.5}$$

For $n = 2$ the symbolic notation of eq. (2.1) becomes difficult to interpret. In this case the integral (2.1) reads as:

$$I_2 = \frac{1}{2\pi} \int_c \frac{f_1 \, df_2 - f_2 \, df_1}{(f_1^2 + f_2^2)} = \frac{1}{2\pi} \int_c d \arctan\left(\frac{f_1}{f_2}\right), \tag{2.6}$$

if the operations df_1, df_2 etc. denote differentiation along the contour.

A straightforward, but tedious calculation shows that

$$\nabla \cdot \{(f_1^2 + f_2^2 + \cdots f_n^2)^{-1/2n} \underline{A}\} = 0, \tag{2.7}$$

where $A = (A_1(\underline{x}), A_2(\underline{x}) \ldots A_n(\underline{x}))$. We can therefore, by Gauss's theorem, change the integral (2.1) into a sum of integrals taken over surfaces containing only one singularity of the integrand of (2.1), i.e. one zero common to the functions $f_1(\underline{x}) \ldots f_n(\underline{x})$. For further details and references concerning the theory of the K.P. integral we refer to [1] and [2].

We will from now on restrict ourselves to the case $n = 2$, appropriate for the calculation of the number of (multiple) zeros of a suitable function $f(x)$. We remark in passing that the integral (1.1) is equal to the integral (2.1), with $n = 2$. Some specific cases are considered below. Suppose the $f(x)$ is a k times piecewise continuously differentiable function in the interval $a \leq x \leq b$, and that $x = x_0$ is a zero with multiplicity m, $(k > m)$, viz.:

$$\begin{cases} f(x_0) = f'(x_0) = f''(x_0) = f^{(3)}(x_0) \cdots = f^{(m-1)}(x_0) = 0, \\ f^{(m)}(x_0) \neq 0. \end{cases} \tag{2.8}$$

Let us take

$$\begin{cases} f_1(x) = zf'(x), \\ f_2(x) = f(x). \end{cases} \tag{2.9}$$

The contribution of a zero point x_0 to the K.P. integral can be calculated choosing for the contour c around x_0 a small circle with radius ε, such that c encloses only one zero of $f(x)$, viz. $f(x) = x_0^m$, then

$$\begin{cases} z = \varepsilon \sin \phi, \\ x = \varepsilon \cos \phi. \end{cases} \tag{2.10}$$

(We remark again that his contribution is independent of the particular choice for the contour, as long as the contour encloses the zero point x_0).

Eqs. (2.6), (2.9) and (2.10) yield:

$$I_2 = \frac{1}{2\pi} \int_0^{2\pi} d \arctg \left(\frac{\varepsilon \sin \phi f'(x_0 + \varepsilon \cos \phi)}{f(x_0 + \varepsilon \cos \phi)} \right). \tag{2.11}$$

From Taylor's theorem and eq. (2.8) we derive that for small values of ε

$$f(x_0 + \varepsilon \cos \phi) = a_m (\varepsilon \cos \phi)^m (1 + O(\varepsilon)), \tag{2.12}$$

and

$$f'(x_0 + \varepsilon \cos \phi) = m a_m (\varepsilon \cos \phi)^{m-1} (1 + O(\varepsilon)), \tag{2.13}$$

because it is assumed that f is k times continuously differentiable at $x = x_0$, and $k > m$. The number a_m is the coefficient of the first term of the Taylor expansion involved.

Inserting (2.12), (2.13) into (2.11) yields

$$I_2 = \frac{1}{2\pi} \int_0^{2\pi} d \left(\arctg \frac{m \sin \phi + O(\varepsilon)}{\cos \phi + O(\varepsilon)} \right) \quad (a)$$

$$= 1. \quad (b) \tag{2.14}$$

This value for the integral (2.14a) is immediately derived from the observation that the integrand of (2.14a) is a total differential. The integration can therefore be performed analytically, leading to

$$I_2 = \frac{1}{2\pi} \left\{ \arctg \left(\frac{m \sin \phi + O(\varepsilon)}{\cos \phi + O(\varepsilon)} \right) \Big| \begin{array}{c} \phi_1 \\ 0 \end{array} \Big| \begin{array}{c} \phi_2 \\ \phi_1 \end{array} \Big| \begin{array}{c} 2\pi \\ \phi_2 \end{array} \right\}, \tag{2.15}$$

if ϕ_1 and ϕ_2 denote the zeros of the numerator of the argument of the arctg. The integration over $0 \le \phi \le \phi_1$ then yields the contribution $(2\pi)^{-1} \pi/2$ to the integral, and the integrations over $\phi_1 \le \phi < \phi_2$ and $\phi_2 \le \phi \le 2\pi$ resp. yield the contributions $(2\pi)^{-1} \pi$, and $(2\pi)^{-1} \pi/2$, resp. to the integral.

Next consider the value of the K.P. integral if we choose

$$f_1 = z f_2' = z(2ff' + 2f'f''),$$
$$f_2 = f^2 + f'^2. \tag{2.16}$$

The contribution of the point $x = x_0$ to the K.P. integral for the case (2.16) will be zero if x_0 is a simple zero of $f(x)$, because $f_2 \neq 0$ in an interval around x_0. The integrand of (2.6) is therefore not singular within the contour c enclosing x_0, and the K.P. integral hence equals zero by virtue of (2.4) and Gauss's theorem. If x_0 is a zero of $f(x)$ with multiplicity $m \ge 2$, we derive from (2.6) and (2.16)

$$I_2 = \frac{1}{2\pi} \int_0^{2\pi} d \arctg \left(\frac{(\sin \phi + O(\varepsilon))(2m - 2)}{\cos \phi + O(\varepsilon)} \right) \quad (a)$$

$$= 1 \quad (b). \tag{2.17}$$

Similarly, we derive the result that the contribution of a zero of $f(x)$ at $x = x_0$ to the K.P. integral with

$$f_1 = zf_2',$$
$$f_2 = f^2 + f'^2 + f''^2, \qquad (2.18)$$

equals zero if its multiplicity is 2, and that its contribution to the K.P. integral equals 1 if the multiplicity of the zero is larger than, or equal to three. We can therefore derive the following theorem:

Theorem 1: *Let $f(x)$ be a k times piecewise continuously differentiable function of x in the interval $[a, b]$. Suppose that $f(x)$ has a zero with multiplicity m at the point $x_0 \in (a, b)$.*

Then if $f(x)$ has no other zeros in the interval (a, b), the K.P. integral I_2,

$$I_2 = \frac{1}{2\pi} \oint_c d \arctg \left(\frac{f_1}{f_2} \right), \qquad (2.19)$$

with

$$f_1 = zf_2', \qquad (2.20)$$

$$f_2 = f^2 + \sum_{l=0}^{j} \left(\frac{\partial^l f}{\partial x^l} \right)^2, \qquad (2.21)$$

equals 1 if $j < m$ and 0 if $m \le j \le k$. The curve c denotes a contour in the $x - z$-plane enclosing x_0, such that if $(x, z) \in c$, $x \in [a, b]$.

Proof: We first consider the case $m \le j \le k$. If the point $x = x_0$ is a zero of multiplicity m of $f(x)$ we have:

$$\frac{\partial f^m(x_0)}{\partial x_0^m} \ne 0, \qquad (2.22)$$

The function f_2, viz. eq. (2.21) is therefore by virtue of eq. (2.22) never zero in the interval $[a, b]$. The integral (2.19) is equal to the general integral (2.1) with $n = 2$, and is therefore zero if f_1 and f_2 have no common zeros. Because f_2 is never zero, as we showed above, we conclude that the integral (2.19) is equal to zero if $m \le j \le k$.

We now consider the case $j < m$. From the general theory of the integral (2.1) we know that it is possible to change the contour c into a circle with radius ε and centre at $x = x_0$ in the $x - z$-plane without altering the value of the integral (2.19). (see eqs. (2.1) and (2.4) with $n = 2$).

Inserting the Taylor series expansion of f_1 and f_2 into (2.19) yields for sufficiently small values of ε an integral equal to the r.h.s. of eq. (2.17a). We therefore conclude that the integral (2.19) equals 1 if $j < m$. ∎

From Theorem 1 we immediately derive theorem 2:

Theorem 2: *Let $f(x)$ be a k times piecewise continuously differentiable function of x in the interval $[a, b]$. Suppose that $f(x)$ has P zeros at the points $x = x_p$ with multiplicity m_p such that*

$$m_p \leq k, \qquad p = 1, 2, \ldots, P, \tag{2.23}$$

The integral

$$I_2 = \frac{1}{2\pi} \oint_c d\arctg\left(\frac{f_1}{f_2}\right), \tag{2.24}$$

with f_1 and f_2 defined by (2.20), (2.21) respectively, then equals

$$I_2 = P - \sum_{p=1}^{P} U(j - m_p), \tag{2.25}$$

where $U(x)$ denotes the Heaviside unit step function:

$$\begin{aligned} U(x) &= 0, \qquad x < 0, \\ U(x) &= 1, \qquad x \geq 0. \end{aligned} \tag{2.26}$$

Proof: From eq. (2.4) and Gauss's theorem we observe that the value of the K.P. integral (1.1) of the system is equal to the sum of the contributions of P K.P. integrals taken over sufficiently small contours containing only one zero of $f(x)$. The values of each of these integrals has been calculated in Theorem 1, so that the desired result follows from the addition of each of these values. ∎

We will now formulate the key result:

Theorem 3: *Let $f(x)$ satisfy the conditions of theorem 2, and suppose that we calculate the values of the $(k + 1)$ K.P. integrals (2.18) with eqs. (2.20) and (2.21), and $0 \leq j \leq k$. These $(k + 1)$ values then determine the number of zeros of $f(x)$ in the interval $[a, b]$ with multiplicity k.*

Proof: The eq. (2.25) shows that $I_2(j + 1) - I_2(j)$ is equal to the number of zeros of $f(x)$ with multiplicity j in the interval $[a, b]$.

3. Numerical Examples

The theory of the preceding sections will be illustrated by the following three examples:

1. The first example consists of the following ten functions:

$$\begin{aligned} &1.\ f_l(x) = x^l; \qquad l = 1, \ldots, 5, \\ &6.\ f_6(x) = 1 - \cos 2\pi x, \\ &7.\ f_7(x) = (x^2 - 1)(x - 2), \\ &8.\ f_8(x) = (x^2 - 2)^2(x - 1), \\ &9.\ f_9(x) = x(x^2 - 1)^3, \\ &10.\ f_{10}(x) = x(x^2 - 4)^4. \end{aligned} \tag{3.1}$$

For each of these functions we determined the number of zeros together with their

multiplicity, using our theory by calculating the following integrals

$$I = \frac{1}{2\pi} \oint_c d \arctg\left(\frac{f_1}{f_2}\right),$$ (3.2)

with

$$f_1 = zf_2'$$ (3.3)

$$f_2 = f^2 + \sum_{l=0}^{m} \left(\frac{\partial^l f}{\partial x^l}\right)^2$$ (3.4)

The contour c consists of a rectangle in the $x - z$-plane with vertices $(a, \pm\varepsilon), (b, \pm\varepsilon)$. The value chosen for ε was 1, and ε denotes the distance along the z-axis. The values along the x-axis, (a, b), of the vertices are indicated in the Tables 1, 2 and 3.

The integral (3.2) then reduces to:

$$I = \frac{1}{2\pi} \int_a^b \frac{zf_2''f_2 - zf_2'^2}{f_2^2 + z^2f_2'^2} dx\bigg|_{z=-1} + \frac{1}{2\pi} \int_b^a \frac{zf_2''^2 f_2 - zf_2'^2}{f_2^2 + z^2f_2'^2} dx\bigg|_{z=1}$$

$$+ \frac{1}{2\pi} \arctg\left(\frac{zf_2'(b)}{f_2(b)}\right)\bigg|_{z=1} - \frac{1}{2\pi} \arctg\left(\frac{zf_2'(b)}{f_2(b)}\right)\bigg|_{z=-1}$$

$$+ \frac{1}{2\pi} \arctg\left(\frac{zf_2'(a)}{f_2(a)}\right)\bigg|_{z=-1} - \frac{1}{2\pi} \arctg\left(\frac{zf_2'(a)}{f_2(a)}\right)\bigg|_{z=+1}$$ (3.5)

or

$$I = \frac{1}{\pi} \int_a^b \frac{f_2''f_2 - f_2'^2}{f_2^2 + f_2'^2} + \frac{1}{\pi} \arctg\left(\frac{f_2'(b)}{f_2(b)}\right) - \frac{1}{\pi} \arctg\left(\frac{f_2'(a)}{f_2(a)}\right).$$ (3.6)

For the values of the arctg functions, occurring at the r.h.s. of eq. (3.6), we have to take the principal value. This stems from the observation that the functions $\arctg\left(\frac{\pm f_2'(b)}{f_2(b)}\right)$ have values at the same branch of the multiple valued arctg function, viz.

$$\arctg\left(\frac{\pm f_2'(b)}{f_2(b)}\right) = l\frac{\pi}{2} \pm \text{principal value},$$ (3.7)

where l denotes an arbitrary integer. The same argument applies to the terms containing $f(a)$ and $f'(a)$.

The equation (3.5) is implemented in an experimental FORTRAN program on a VAX 11/750 working under VMS operating system. The required integrations are performed by means of an elegant general purpose integration routine in double precision from the NAG library, i.e. the routine D01AJF.

The required absolute and relative accuracy of the result of the integrations procedure are both set to 10^{-3}. Typical calculations time is 10 s for the 10 integrals of the simple functions f_1-f_{10}. The other two examples each take about 15 s computation time.

Again we observe a switching from 2 to 1 for the case of the K.P.-integral ($j = 0$), however, for such smaller values of δ the behaviour of the double zero integral is not so much affected as we see comparing table 4 and table 5. The explanation for the observed phenomena is in the fact that the integrand has a very narrow peak positioned at the point $x = \frac{1}{2}\delta$ where $f'(x)$ equals zero. Viz. the case ($j = 0$) (i.e. the K.P.-integral) the value of the integrand for $x = 0$ and $x = \delta$ are qual to -1 and for $x = \dfrac{\delta}{2}$ the value of the integrand is equal to $\dfrac{-8}{\delta^2}$. In appendix A this behaviour of the integrand is analyzed in detail and it is proven to correspond with a δ-function for small values of δ. Depending on the quadrature algorithm and the requested accuracy, this narrow δ-function is missed in the calcluation which causes the observed switch of 1 unit in the results.

For the two-dimensional case one could ask the same question: Which are the numerical conditions such that the proposed method discriminates between two curves that touch each other and two curves laying closely together. Model studies show that the difference between two such situations show up as a δ-type peak behaviour, discussed in Appendix A, of the integrand of the integrals involved in our theory.

The stability problem, as well as the non-trivial extension of the theory to higher order dimensions will be discussed at full length in forthcoming papers. In the two-dimensional case the issue is about intersecting curves. The type of zero can be analyzed again by a series of integrals, to be calculated over a closed two-dimensional surface of a two-dimensional integrand. Here again we use the NAG library as it also contains integration routines in two or higher dimensions.

A. Appendix

We consider the Picard integral (2.6) with (2.9) and choose for the function f:

$$f(x) = x(x - \delta), \tag{A.1}$$

if δ denotes a positive real number, which can take arbitrarily small values. It has been explained in this paper that the integral (2.6) for values of $\delta \neq 0$ is equal to 2, whereas its value is equal to 1 if $\delta = 0$, provided the integration interval (a, b) includes δ. The integral (2.9) hence shows a discontinuous behaviour *qua* function of δ.

We will show, as is to be expected, that this discontinuous character can be explained in terms of δ-functions: the integrand of the integral (2.6) develops a δ-function around $x = \dfrac{\delta}{2}$ for values of δ sufficiently close to zero. We choose for the contour c in the $x - z$-plane a rectangle with vertices at $(a, \pm \varepsilon), (b, \pm \varepsilon)$. The value chosen for ε was 1, and ε denotes the distance along the z-axis. Then, as shown by the eqs. (2.9), (3.5), (3.6) and (A1) we end up with the integral:

$$\int_a^b \frac{2x(x - \delta) - (2x - \delta)^2}{x^2(x - \delta)^2 + (2x - \delta)^2} \, dx. \tag{A.2}$$

Let us take $a = 0$ and $b = \eta$, if η denotes an arbitrarily small fixed positive real number. We will now indicate that the integral (A2) tends to a finite constant, independent of η, if δ tends to zero. To this end we make the substitution $x = \delta p$ in the integral (A2), leading to:

$$\delta \int_0^{\eta\delta^{-1}} \frac{-2p^2 + 2p - 1}{(2p-1)^2 + \delta^2 p^2 (p-1)^2} \, dp. \tag{A.3}$$

We observe that the integrand of the integral (A3) tends to zero if δ tends to zero except for a small neighbourhood around $p = \frac{1}{2}$. We therefore apply another linear transformation to the integral (A3), leading very quickly to the desired result. Let

$$y = 2p - 1. \tag{A.4}$$

Then, we derive from eq. (A3):

$$I = \delta \int_{-1}^{(2\eta\delta^{-1}-1)} \frac{y - \frac{1}{2}(1+y)^2}{y^2 + \frac{\delta^2}{8}(1-y^2)^2} \, dy. \tag{A.5}$$

We divide the integration interval of the integral (A5) into three parts: $-1 \le y \le -\sigma$, $-\sigma \le y \le +\sigma$, $+\sigma \le y \le 2\eta\delta_{-1}^{-1}$, where σ denotes a small, fixed, positive number, leading to

$$I = I_1 + I_2 + I_3 = \int_{-1}^{-\sigma} (\cdots) \, dy + \int_{-\sigma}^{+\sigma} (\cdots) \, dy + \int_{+\sigma}^{2\eta\delta^{-1}-1} (\cdots) \, dy. \tag{A.6}$$

We observe that

$$\lim_{\delta \to 0} \frac{y - \frac{1}{2}(1+y)^2}{y^2 + \frac{\delta^2}{8}(1-y^2)^2} = 0. \tag{A.7}$$

This limit is uniform for values of the variable y in the interval $-1 \le y \le -\sigma$. We can therefore change the order of taking the $\lim_{\delta \to 0}$ and the integration in the integral I_1, showing that

$$\lim_{\delta \to 0} I_1 = 0. \tag{A.8}$$

We next observe from the eqs. (A5) and (A6) that the integral I_3, taking ∞ for the upper boundary, converges absolutely. So again, by a theorem concerning Tannery integrals, (see Bromwich [5]), we are allowed to change the $\lim_{\delta \to 0}$ and the integration in the integral I_3, leading to the result that

$$\lim_{\delta \to 0} I_3 = 0. \tag{A.9}$$

Combination of eqs. (A6), (A8), and (A9) shows that for sufficiently small values of δ the value of the integral I is determined only by the integral I_2. As σ denotes an *arbitrarily* small positive number we conclude that if δ tends to zero only values of the integrand near $y = 0$ contribute to the integral. If the value of this integral $\neq 0$ we have proven that the δ-type behaviour of the integrand of the integral I if δ tends to zero. The integral I_3 equals for sufficiently small values of δ and σ to:

$$I_3 = \int_{-\sigma}^{+\sigma} \frac{\dfrac{-1}{2\delta}}{\left(y^2 + \dfrac{\delta^2}{8}\right)} \, dy + 0(\delta, \sigma). \tag{A.10}$$

The integral (A10) can be integrated exactly, yielding:

$$I_3 = -\sqrt{2} \arctan(\sqrt{8}p)\big|_{-\sqrt{8}\sigma\delta^{-1}}^{+\sqrt{8}\sigma\delta^{-1}}. \tag{A.11}$$

The integral (A11) tends to $-\sqrt{2}\pi$ for every value of $\sigma \neq 0$ if δ tends to zero. Eqs. (A6), (A8), ... (A11) therefore show, as we have explained above, that the integrand of the integral I_3 develops a δ-function near $y = 0$ if $\delta \to 0$.

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