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A new structural reliability index based on uncertainty theory

Pidong WANG, Jianguo ZHANG *, Hao ZHAI, Jiwei QIU

School of Reliability and Systems Engineering, Beihang University, Beijing 100083, China
Science and Technology on Reliability and Environment Engineering Laboratory, Beihang University, Beijing 100083, China

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Abstract The classical probabilistic reliability theory and fuzzy reliability theory cannot directly measure the uncertainty of structural reliability with uncertain variables, i.e., subjective random and fuzzy variables. In order to simultaneously satisfy the duality of randomness and subadditivity of fuzziness in the reliability problem, a new quantification method for the reliability of structures is presented based on uncertainty theory, and an uncertainty-theory-based perspective of classical Cornell reliability index is explored. In this paper, by introducing the uncertainty theory, we adopt the uncertain measure to quantify the reliability of structures for the subjective probability or fuzzy variables, instead of probabilistic and possibilistic measures. We utilize uncertain variables to uniformly represent the subjective random and fuzzy parameters, based on which we derive solutions to analyze the uncertainty reliability of structures with uncertainty distributions. Moreover, we propose the Cornell uncertainty reliability index based on the uncertain expected value and variance. Experimental results on three numerical applications demonstrate the validity of the proposed method.

1. Introduction

Reliability analysis has been a hot research topic in recent years, as the influences of uncertainty arising on loads, material properties, dimensions, and geometries become more and more profound.[1] This topic has a wide range of applications in the design and analysis of structural systems for aerospace vehicles, civil engineering, and manufacturing industry.[2]

Probability theory is one of the most classical and prevailing tools in dealing with uncertain variables, which has been widely used to estimate structural reliability and safety by calculating the probability of failure and the reliability index β,[2–5] and a lot of probabilistic reliability methods have been proposed in literature such as the first-order method,[2,5,6] the second-order method,[2,6] the response surface method,[7,8] and the numerical sampling simulation method.[7,9]

* Corresponding author.
E-mail addresses: wpd-pizi@163.com (P. WANG), zjg@buaa.edu.cn (J. ZHANG).

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However, in practical structural engineering, beside randomness which can be modelled by probabilistic theory with Probability Distribution Functions (PDFs)[3,9,10] we encounter epistemic uncertainty,[3,10] caused by things such as loss of information, limited knowledge, and inevitable man-made mistakes.[3,11,12] It cannot be well explained by randomness and probabilistic models. Although, for uncertain problems in structural engineering, a random variable is always employed to represent a kind of subject probability, which is conducted by experts' judgments (subjective interpretation),[10] the uncertainty of this variable is actually fuzziness from experts' judgements.[3] In this sense, that results in a juxtaposition with randomness.[3,11] As a consequence, fuzzy variables and credibility theory are introduced to describe fuzziness with Membership Functions (MFs).[3,9,13] A new definition for structural reliability index based on credibility theory is explored, which is named "Cornell fuzzy reliability index".[3] This index and the credibility of failure are calculated based on the possibilistic principle instead of the probabilistic one.[3,9,14,15] Nevertheless, in most practical situations, some input parameters of structures might be represented with PDFs and some with MFs.[9] For completeness, different knowledge conditions for each uncertainty parameter derive "hybrid" uncertain variables in structural reliability analysis.[3] Therefore, in order to comprehensively analyze the reliability of structures correctly, subjective randomness and fuzziness should be jointly considered, resulting in a hybrid model with random and fuzzy variables.[3,9,15,16]

Many approaches have been proposed to solve the aforementioned hybrid reliability problems in structures. Most of them separate random and fuzzy parameters based on a double-sampling framework,[9] such as the Monte Carlo (MC) method,[17] improved sampling methods,[18] transformation-based methods,[19,20] and the iteration method.[21] To avoid the deterioration of efficiency and accuracy, several works have attempted to combine stochastic expansions with traditional optimization methods.[9,22–26] They are mainly concentrated on explaining fuzzy variables by adopting the probability theory,[13] and calculating the probability of failure and the structural reliability index β based on a probability measure.[2,5] However, the probability measure with additivity used by these methods fails to satisfy the subadditivity axiom of fuzziness, and the possibilistic measure cannot satisfy the duality axiom of randomness.[10,15] To that end, it is suggested that a reliability quantification model based on probability theory and one based on credibility theory frequently yield infeasible solutions with large differences and paradoxical results.[1,5,10] In other words, neither probability theory nor credibility theory can deal with structural reliability problems under epistemic uncertainty with hybrid subjective random and fuzzy variables,[1,3,15,27,28] because neither measure of the two theories can satisfy duality and subadditivity simultaneously.[10,15,27,28]

In order to achieve a reasonable solution to these structural reliability problems, solving the limitations of the two measures, we introduce the uncertainty theory and an uncertain measure proposed by Liu[15,27,28] in structural reliability, including the normality, duality, subadditivity, and product axioms. This theory relies on the uncertain measure to describe the belief degrees of events affected by epistemic uncertainty.[15,27] Liu firstly employed it in system reliability and defined reliability index as a measure of systems' reliability.[10,29] This theory provides a concrete mathematical description on different types of uncertain parameters in the uncertainty space.[15] The "belief degree" in the range of [0, 1] is adopted to represent the level of confidence about the occurrence of a particular event[5] in engineering structures. Belief reliability was defined by Zeng et al. as the uncertainty measure of a system to perform specified functions within given time under given operating conditions.[10,30] Based on the new theory, this paper will explore a new reliability index and quantification model, and discuss how to apply them to quantify the reliability of structures with the hybrid uncertainty problem. The main contributions of this paper lie in the following two-folds:

(1) In the framework of the uncertainty theory, the new reliability index and quantification model are based on the uncertain measure instead of a probabilistic or possibilistic one.
(2) We uniformly treat subjective random and fuzzy parameters as uncertain variables in the uncertainty space. Based on the new theory and variables, a new definition and formulations are proposed to deal with epistemic uncertainty, especially for the two parameters mixed, simultaneously in structural reliability problems. With the proposed formulations, not only the duality of randomness can be described, but also the subadditivity of fuzziness can be explained.

The remainder of this paper is organized as follows. In Section 2, some useful concepts in the uncertainty theory are illustrated in Section 2.1 such as uncertain measure, uncertain variable, and uncertainty distribution. Subsequently, in Section 2.2, based on the uncertain measure and uncertainty distributions, we prove two important theorems (Theorems 2 and 3) and derive formulas to quantify the uncertainty reliability and the uncertainty of failure with the limit state function of structures. According to the similarity between formulations based on the uncertain measure and the presented probabilistic one, the "Cornell uncertain reliability index" is defined by the uncertain expected value and variance in Section 3. Finally, three numerical applications to demonstrate its rationality and practicability are presented. Firstly, reliability analysis for strengths of materials is used to prove that these methods are effective and accurate, and to analyze the relationship between the Cornell uncertainty reliability index and uncertainty reliability, due to low analytical complexity of the linear limit state function. Secondly, the methods are applied to deal with the reliability problem in a real structure, in this specific case a beam structure.

2. Uncertainty quantification for structural reliability

2.1. Overview of uncertainty theory

Definition 1 (*Uncertain measure*[15]). Let $(\Gamma, \mathcal{L})$ be a measurable space. For $\forall \Lambda \in \mathcal{L}$, the element Λ is called a measurable set. Then, the set function M is called an uncertain measure if it satisfies the following four axioms:

Axioms 1 (*Normality axiom*). For the universal set Γ, $M\{\Gamma\} = 1$.

Axioms 2 (*Duality axiom*). For $\forall \Lambda$, $M\{\Lambda\} + M\{\Lambda^C\} = 1$, where Λ^C is the complement set of Λ.

Axioms 3 (*Subadditivity axiom*). For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leqslant \sum_{i=1}^{\infty}M\{\Lambda_i\} \tag{1}$$

Axiom 4 (*Product axiom*). Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces and Λ_k be events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively. The product uncertain measure M is an uncertain measure if

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\} \tag{2}$$

Definition 2 (*Uncertainty space*[15]). Assume that Γ is a nonempty set, $\mathcal{L}$ is a $\sigma -$ algebra over Γ, and M is an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, M)$ forms an uncertainty space. This space is called complete, if the following conditions hold:

(1) $M\{\Lambda_1\} = M\{\Lambda_2\}$ for $\forall \Lambda_1, \Lambda_2 \in \mathcal{L}$.
(2) For any subset A, if $\Lambda_1 \subset A \subset \Lambda_2$, then $A \subset \Omega$, and $M\{\Lambda_1\} = M\{A\} = M\{\Lambda_2\}$.

Definition 3 (*Uncertain variable*[15]). Let ξ be a measurable mapping function from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers. Then, ξ is called an uncertain variable. If $\xi_1, \xi_2, \ldots, \xi_n$ are uncertain variables, and f is a real valued measurable function, then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is also an uncertain variable, and can be written as

$$\xi(\tau) = f(\xi_1(\tau), \xi_2(\tau), \ldots, \xi_n(\tau)) \quad \forall \tau \in \Gamma \tag{3}$$

Then $\xi_1, \xi_2, \ldots, \xi_n$ are independent if and only if $M\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\} = \bigvee_{i=1}^{n} M\{\xi_i \in B_i\}$, for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.[15,31] For a set of independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$, if $f_1, f_2, \ldots, f_n$ are measurable functions, then $f_1(\xi_1), f_2(\xi_2), \ldots, f_n(\xi_n)$ are independent.[28]

Actually, an uncertainty distribution is commonly used to describe an uncertain variable, defined by Liu[15] as $\Phi(x) = M\{\xi \leqslant x\}$ for any real number x. Moreover, $\Phi(x)$ is regular if it is a continuous and strictly increasing function for x with $0 < \Phi(x) < 1$, and $\lim_{x\to-\infty}\Phi(x) = 0$, $\lim_{x\to\infty}\Phi(x) = 1$.[28] In this case, the inverse function $\Phi^{-1}(\alpha)$ is defined as the inverse uncertainty distribution of ξ.[28] It is suggested that the distributions are a carrier of incomplete information of uncertain variables.[15] For both subjective random variables and fuzzy ones, we can utilize uncertain variables and uncertainty distributions to describe them in the uncertainty space.[15,27,28,31]

Definition 4 (*Uncertain expected value*[15]). Let ξ be an uncertain variable. Then the expected value of ξ is defined as

$$E(\xi) = \int_0^{+\infty} M\{\xi \geqslant x\}\mathrm{d}x - \int_{-\infty}^{0} M\{\xi \leqslant x\}\mathrm{d}x \tag{4}$$

where $E(\bullet)$ is the uncertain expected value operator.

In addition, Liu[28] has proven the linearity of the expected value operator, and proposed that if there is an uncertainty distribution $\Phi(x)$ with ξ, then Eq. (4) turns into the following:

$$E(\xi) = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x) \tag{5}$$

Moreover, if $\Phi(x)$ is regular, then Eq. (4) will be rewritten[28] as

$$E(\xi) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha \tag{6}$$

Definition 5 (*Uncertain variance*[15]). Let ξ be an uncertain variable with a finite expected value e. Then the variance of ξ is

$$V(\xi) = E(\xi - e) = \int_0^{+\infty} \frac{1}{2}\Big\{M\{(\xi - e)^2 \geqslant x\} + 1 - M\{(\xi - e)^2 < x\}\Big\}\mathrm{d}x \tag{7}$$

where $V(\bullet)$ is the uncertain variance operator.

Similarly, if there is an uncertainty distribution $\Phi(x)$ with regard to ξ, then Eq. (7) turns into[28]

$$V(\xi) = \int_{-\infty}^{+\infty} (x - e)^2\mathrm{d}\Phi(x) \tag{8}$$

Moreover, if $\Phi(x)$ is regular, then the following equation holds[32]:

$$V(\xi) = \int_0^1 \left(\Phi^{-1}(\alpha) - e\right)^2\mathrm{d}\alpha \tag{9}$$

Theorem 1 [15]. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing for $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing for $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has the following uncertainty distribution:*

$$\Phi(x) = \sup_{f(x_1,x_2,\ldots,x_n)=x}\left(\min_{0\leqslant i\leqslant k}\Phi_i(x_i) \wedge \min_{k+1\leqslant i\leqslant n}(1 - \Phi_i(x_i))\right) \tag{10}$$

Moreover, if $\Phi_1, \Phi_2, \ldots, \Phi_n$ are regular uncertainty distributions, then

$$M\{f(\xi_1, \xi_2, \ldots, \xi_n) \leqslant 0\} = \lambda \tag{11}$$

where λ satisfies the following equation[29]:

$$f(\Phi_1^{-1}(\lambda), \Phi_2^{-1}(\lambda), \ldots, \Phi_k^{-1}(\lambda), \Phi_{k+1}^{-1}(1 - \lambda), \ldots, \Phi_n^{-1}(1 - \lambda)) = 0 \tag{12}$$

Considering the monotone function $f(\xi_1, \xi_2, \ldots, \xi_n)$ of the independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, which is strictly increasing for $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing for $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$, the uncertain expected value of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ can be obtained as follows[33]:

$$E(\xi) = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1 - \alpha), \ldots, \Phi_n^{-1}(1 - \alpha))\mathrm{d}\alpha \tag{13}$$

The uncertain variance of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ with the uncertain expected value e can be calculated by[32]

$$V(\xi) = \int_0^1 \Big[f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) - \theta\Big]^2 \mathrm{d}\alpha \tag{14}$$

2.2. Uncertainty reliability and uncertainty of failure

With the limit state function of structures under epistemic uncertainty, we redefine structural uncertainty reliability based on the uncertain measure[15] and belief reliability[30] as follows.

Definition 6 (*Structural uncertainty reliability*). Given an uncertainty space $(\Gamma, \mathcal{L}, M)$, without loss of generality, we consider the basic problem in structural reliability, containing only one generalized stress S resisted by one general strength R. Both S and R are independent uncertain variables. Let the limit state function $G = R - S$, where $G = R - S > 0$ indicates the state of being safe, and $G = R - S \leqslant 0$ indicates the state of being failure. Then the uncertainty of occurrence of a failure event is defined as

$$M_{\text{failure}} = M\{R \leqslant S\} = M\{G = R - S \leqslant 0\} \tag{15}$$

From the duality axiom of the uncertainty theory, the uncertainty reliability of a structural system where $G = R - S > 0$ can be formulated as

$$M_{\text{reliability}} = 1 - M_{\text{failure}} \tag{16}$$

Following the probability of reliability, we denote $M_{\text{reliability}}$ as the "structural uncertainty reliability" to quantify the uncertainty of a safe or failure event in a structural system with the numerical value of [0, 1], and M_{failure} describes the confidence how we believe the occurrence of a failure event. It is clear that due to the similarity with the probability of failure and reliability, the numerical value of M has been used to represent the confidence with which it is believed that the event will occur instead of "frequency". When $M_{\text{failure}} = 1 (M_{\text{reliability}} = 0)$, it means that we completely believe the failure event will happen and the event of structural reliability is completely impossible. When the failure event and its complementary (reliability) event are equally likely, then $M_{\text{reliability}} = M_{\text{failure}} = 0.5$. The higher $M_{\text{reliability}}$ is, the more strongly we believe the reliability event will happen.

If the uncertainty distribution Φ_G is given, the failure uncertainty with Eq. (15) can be obtained by

$$M_{\text{failure}} = M\{G(R, S) \leqslant 0\} = \Phi_G(0) \tag{17}$$

Meanwhile, the structural uncertainty reliability Eq. (18) turns into

$$\begin{aligned} M_{\text{reliability}} &= M\{G(R, S) > 0\} = 1 - M\{G(R, S) \leqslant 0\} \\ &= 1 - \Phi_G(0) \end{aligned} \tag{18}$$

Remark 1. In case when Φ_S and Φ_R are provided and Φ_G is unknown, we establish a model for solving the uncertain reliability of a structure, as depicted in the following theorems.

Theorem 2. *If both R and S have continuous uncertainty distributions denoted by Φ_R and Φ_S, respectively, then the uncertainty of failure and uncertainty reliability of a structure system can be calculated as*

$$M_{\text{failure}} = \sup_{x_1 - x_2 = 0} (\min \Phi_R(x_1) \wedge \min(1 - \Phi_S(x_2))) \tag{19}$$

$$M_{\text{reliability}} = 1 - \sup_{x_1 - x_2 = 0} (\min \Phi_R(x_1) \wedge \min(1 - \Phi_S(x_2))) \tag{20}$$

Proof. It is not difficult to see the function $G(R, S) = R - S$ is strictly increasing with respect to R and strictly decreasing with respect to S. According to Theorem 1 and Eq. (10), ξ_G has the following continuous uncertainty distribution $\Phi_G(x)$:

$$\Phi_G(x) = \sup_{x_1 - x_2 = x} (\min \Phi_R(x_1) \wedge \min(1 - \Phi_S(x_2))) \tag{21}$$

Then, by the definition of the uncertainty of failure and uncertainty reliability, the following equations hold:

$$\begin{aligned} M_{\text{failure}} &= M\{G(S, R) \leqslant 0\} \\ &= \Phi_G(0) \\ &= \sup_{x_1 - x_2 = 0} (\min \Phi_R(x_1) \wedge \min(1 - \Phi_S(x_2))) \end{aligned} \tag{22}$$

$$\begin{aligned} M_{\text{reliability}} &= 1 - M_{\text{failure}} \\ &= 1 - \sup_{x_1 - x_2 = 0} (\min \Phi_R(x_1) \wedge \min(1 - \Phi_S(x_2))) \end{aligned} \tag{23}$$

□

Theorem 3. *If the uncertainty distribution Φ_R of R and Φ_S of S are regular, then the uncertainty of failure is λ and the uncertainty reliability $M_{\text{reliability}} = 1 - \lambda$, where λ satisfies the following condition:*

$$f(\Phi_R^{-1}(\lambda), \Phi_S^{-1}(1 - \lambda)) = \Phi_R^{-1}(\lambda) - \Phi_S^{-1}(1 - \lambda) = 0 \tag{24}$$

Proof. The results can be directly derived from Theorem 1 and Eq. (12). □

Here, λ can be estimated by solving Eq. (24) via the interpolation method.[15]

3. Reliability index based on uncertain measure

Given a limit state function $G(R, S) = R - S$ in the uncertainty space, a new reliability index can be defined by the uncertain expected value and variance operator,[15] which is similar to the "Cornell reliability index" with the probability measure.[2,5] It can be seen as an uncertain indicator that a failure (reliability) event will happen in the structural system.

Definition 7 (*Cornell uncertainty reliability index*). Let $(\Gamma, \mathcal{L}, M)$ be an uncertainty space, and $G(R, S) = R - S$ be the limit state function with two uncertain variables R and S. The uncertainty reliability index γ is defined as

$$\gamma = \frac{E(G)}{\sqrt{V(G)}} = \frac{E(R - S)}{\sqrt{V(R - S)}} \tag{25}$$

Remark 2. If R and S are independent with regular uncertainty distributions Φ_R and Φ_S, γ can be equivalently formulated as

$$\gamma = \frac{E(G)}{\sqrt{V(G)}} = \frac{\int_0^1 \left(\Phi_R^{-1}(\alpha) - \Phi_S^{-1}(1-\alpha)\right)\mathrm{d}\alpha}{\sqrt{\int_0^1 \left(\Phi_R^{-1}(\alpha) - \Phi_S^{-1}(1-\alpha) - e\right)^2 \mathrm{d}\alpha}} \tag{26}$$

where $e = E(G) = \int_0^1 \left(\Phi_R^{-1}(\alpha) - \Phi_S^{-1}(1-\alpha)\right)\mathrm{d}\alpha$. According to the linearity of the uncertain expected value operator,[28] we can deduce that $e = E(R) - E(S)$.

From the Cornell formulation of the structural reliability index,[3] γ can be denoted as "Cornell uncertainty reliability index". Based on the uncertain measure, in order to estimate structural reliability with hybrid subjective randomness and fuzziness, it has been achieved by the subadditivity axiom of the uncertainty theory[15] dealing with the fuzzy parameters, and the duality axiom dealing with the random parameters. Following this way, our formulation can overcome these drawbacks in the probabilistic and possibilistic measures. It can be seen that the Cornell uncertainty reliability index γ is the same as the definition of "Cornell reliability index", of which a larger γ indicates a higher belief degree on the occurrence of a reliability event.

4. Numerical experiments

4.1. Reliability problem in strengths of materials

Reliability analysis for the strengths of materials shown in Fig. 1 is a fundamental reliability problem in structural systems.[2,3] In the viewpoint of the uncertainty theory, variables in benchmark problems usually consist of the fuzzy stress S and the random strength R (MPa), which are treated as uncertain variables. This numerical example is used to evaluate the significance of the Cornell uncertainty reliability index and uncertainty reliability based on the expected value and variance as R changes.

Let the limit state function be $G = R - S$, the uncertainty distribution of R be $\Phi_R(x_1) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x_1} \mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}} \mathrm{d}t$ ($x_1 \in (-\infty, +\infty)$), and the uncertainty distribution of S be

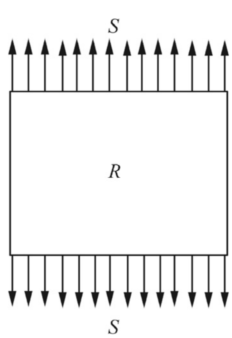

Fig. 1 Stress versus strength.

$$\Phi_S(x_2) = \begin{cases} 0 & x_2 < 100 \\ \dfrac{x_2 - 100}{40} & 100 \leqslant x_2 < 120 \\ \dfrac{x_2 - 90}{60} & 120 \leqslant x_2 < 150 \\ 1 & x_2 \geqslant 150 \end{cases} \tag{27}$$

By using the uncertainty expected value operator Eq. (5) and the variance operator Eq. (8), we can derive that the expected value of R is μ and its variance is σ^2. Similarly, the expected value of S is 122.5 and its variance is 210.4167.

Supposing that the variance σ^2 of R is 25, the Cornell uncertainty reliability index, uncertainty of failure, and uncertainty reliability can be obtained by using different values of μ. The results are summarized in Table 1 and Fig. 2.

From Fig. 2 and Table 1, it can be observed that when the mean value of R increases with a constant variance, the trend of uncertainty reliability is consistent with that of the Cornell uncertainty reliability index. The larger the uncertainty reliability is, the higher the Cornell uncertainty reliability index will be, which takes on the same trend as the probability reliability. Intuitively, when the mean value of the strength increases with a constant variance, the probability that the stress is larger than the strength of a material (i.e., a failure event) will decrease. Conversely, the reliability and the reliability index increase in the viewpoint of the probability theory, due to the uncertainty of the stress invariant. Similarly, as the mean value of strength increases, the belief degree about the material reliability becomes higher with an increasing value of the Cornell uncertainty reliability index based on the uncertainty theory. Therefore, the definition of the Cornell uncertainty reliability index proposed in this paper is reasonable and can partially reflect the law of structural reliability.

Following the same way, when the variance of R increases and the mean value is constant, the probability that the stress is greater than the strength of the material (i.e., a failure event) will increase, and the reliability index will decrease in the viewpoint of the probability theory due to the dispersity of strength increasing. It should be consistent with the trend of the uncertainty reliability based on the uncertainty theory. To validate this intuitive observation, we calculate the trend of the uncertainty reliability by using different variances (25, 36, 100, and 200) with a constant mean value of 160. The results are shown in Fig. 3 and Table 2.

From Fig. 3 and Table 2, it can been seen that the trends of the uncertainty reliability and its index are consistent with our intuitive observations. We can therefore conclude that the uncertainty reliability and the Cornell uncertainty reliability index are able to provide constructive results in hybrid uncertain structural reliability analysis, if the uncertain variables are reasonably acceptable. They can be utilized to help designers make the most appropriate choice. Additionally, the presented reliability index can be solved for nonlinear functions. Let the

Table 1 Calculated results with different μ ($\sigma^2 = 25$).

μ	150	155	160	165	170
Uncertainty of failure M_{failure}	0.1046	0.0521	0.0147	1.2829×10^{-3}	6.3245×10^{-5}
Uncertainty reliability $M_{\text{reliability}}$	0.8954	0.9479	0.9853	0.9987	9999.3675×10^{-4}
Cornell uncertainty reliability index γ	1.4172	1.6749	1.9326	2.1902	2.4479

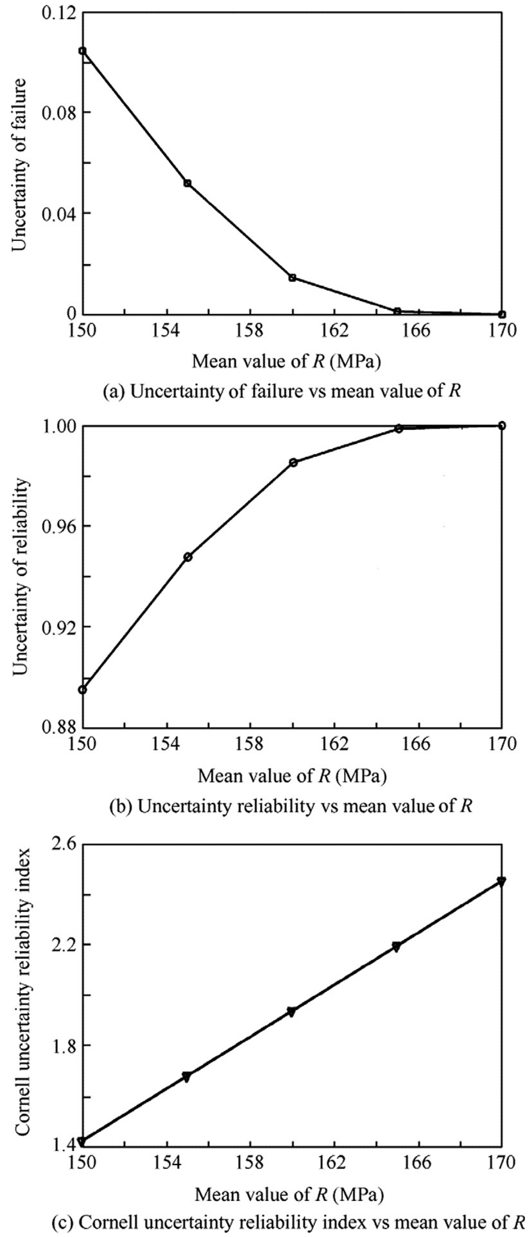

Fig. 2 Calculated results vs mean value of strength R.

nonlinear limit state function be $G(x_1, x_2) = \mathrm{e}^{x_1} - x_2^3 - x_2$, the uncertainty distribution of x_1 be $\Phi(x_1) = \left(1 + \exp\left(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}\right)\right)^{-1}$ $(x_1 \in (-\infty, +\infty))$, and the uncertainty distribution of x_2 be

$$\Phi(x_2) = \begin{cases} 0 & x_2 < 2 \\ \dfrac{x_2 - 2}{4} & 2 \leqslant x_2 < 6 \\ 1 & x_2 \geqslant 6 \end{cases}$$

Table 2 Calculated results (μ = 160).

σ^2	25	36	100	144	400
M_{failure}	0.0147	0.0266	0.0742	0.0952	0.1620
$M_{\text{reliability}}$	0.9853	0.9734	0.9258	0.9048	0.8380
γ	1.9326	1.8392	1.5405	1.4245	1.0941

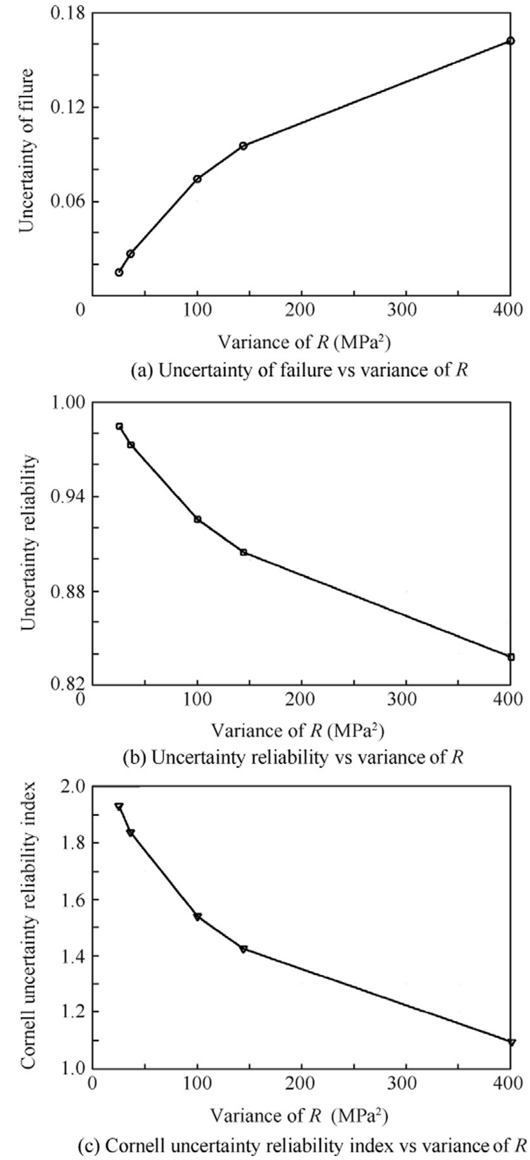

Fig. 3 Calculated results vs variance of strength R.

Supposing that the variance σ^2 of x_1 is 1, the Cornell uncertainty reliability index, uncertainty of failure, and uncertainty reliability can be obtained by using different values of μ. The results are summarized in Table 3.

To validate this intuitive observation, we calculate the trend of the uncertainty reliability by using different variances (1^2, 1.5^2, and 2^2) with a constant mean value of 5. The results are shown in Table 4.

From the results in Tables 3 and 4, it can be seen that the trends of the uncertainty reliability and its index are also consistent with the intuitive observations. It is indicated that the presented reliability index can be solved for nonlinear functions.

4.2. Reliability analysis in a beam structure

In this numerical example, the reliability measure of a beam associated with its bending moment is calculated in Fig. 4.

Table 3 Calculated results with different μ ($\sigma^2 = 1$).

μ	5	6	8
M_{failure}	0.3115	0.1577	0.0086
$M_{\text{reliability}}$	0.6885	0.8423	0.9914
γ	0.3112	0.4200	0.4765

Table 4 Calculated results ($\mu = 5$).

σ	1	1.5	2
M_{failure}	0.3115	0.3920	0.4331
$M_{\text{reliability}}$	0.6885	0.6080	0.5669
γ	0.3112	0.1152	0.0481

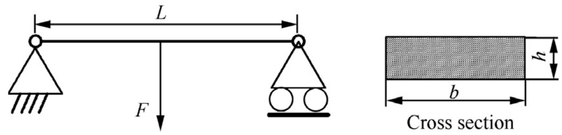

Fig. 4 Illustration of beam structure.

The uncertain numerical values are summarized in Tables 5 and 6.

Here, we assume that the length of the beam $L = 1300$ mm, beam height $h = 8$ mm, and the force density $\rho_{st} = 78.5 \times 10^{-6}$ kN/mm^3. The load F is a fuzzy variable applied at mid span as shown in Fig. 4 and Table 6, and the beam breadth b is a random variable as shown in Table 5.

The bending moment at mid-span associated with random loads is subject to the following equation:

$$m = \frac{FL}{4} + \frac{\rho bhL^2}{8} \tag{28}$$

In order to analyze the reliability state of the beam, it is necessary to define the ultimate bending moment. In our work, as shown in Fig. 5, the ultimate bending moment can be categorized into "low", "normal", and "high" acceptable, according to the subjective judgment with incomplete test data.[3]

Here,

$$\begin{cases} m_u^1 = (1.9 \times 10^5/2.0 \times 10^5/2.1 \times 10^5) \\ m_u^2 = (2.0 \times 10^5/2.05 \times 10^5/2.1 \times 10^5) \\ m_u^3 = (2.05 \times 10^5/2.1 \times 10^5/2.15 \times 10^5) \end{cases}$$

Table 5 Random variable.

Parameter	Mean value	Standard deviation	Type of distribution
b (mm)	40	4	Normal

Table 6 Fuzzy variable.

Parameter	Value	Type
F (kN)	(400/500/600)	Fuzzy triangular variable

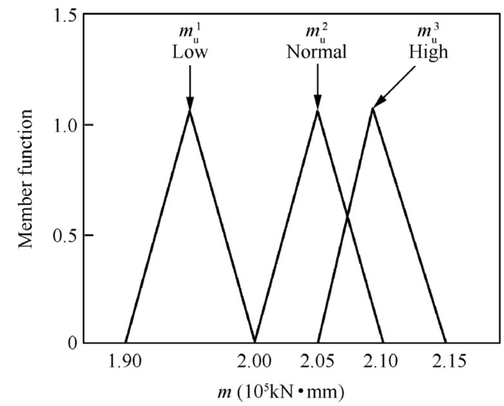

Fig. 5 Member function of ultimate bending moment.

Therefore, the limit state function with reference to the failure of the beam is formulated as

$$G(b, h, f, F, L) = m_u - \left(\frac{FL}{4} + \frac{\rho bhL^2}{8}\right) \tag{29}$$

In more general cases, by utilizing the design ultimate bending moment and the design bending moment at mid-span as shown in Table 7, the target Cornell uncertainty reliability index γ_T turns out to be 1.2469, by using Eq. (26). The results are listed in Table 8.

Firstly, if the target Cornell reliability index is estimated by the probability measure, γ_T will be 1.3800, greater than that based on the uncertain measure 1.2469. Thus, it will negatively affect the accuracy and belief degree of the results, when the probabilistic method is used to judge the reliability of the structure with uncertain variables.

Moreover, based on the uncertain measure-calculated results in Table 6, the reliability degree of the beam with different states for the ultimate bending moments can be obtained by the Cornell uncertainty reliability indices. For instance, when the state is "low", the reliability index is less than the target one with $\gamma = 0.9837\gamma_T$. It means that the state is not reliable. On the other hand, as the ultimate bending moment is changed from "normal" to "high", the reliability increases. Specifically, when the state is "normal", γ is $1.3455\gamma_T$. When the state is "high", γ is $1.3914\gamma_T$. Therefore, the ratio γ/γ_T can be assumed as "reliability indicator" for different states assumed by the uncertain variable ultimate bending moment.

Table 7 Random variables m_u and m.

Randomness	Mean value	Standard deviation	Type of distribution
m_u (kN·mm)	2.05×10^5	2.05×10^3	Normal
m (kN·mm)	1.8×10^5	1.8×10^4	Normal

Table 8 Cornell uncertainty reliability index in truss structure ($\gamma_T = 1.2469$).

Type of m_u	Low	Normal	High
γ	1.2266	1.6777	1.7349
Reliability state	No	Yes	Yes

In addition, comparing γ for "normal" with "high", they are not so much different; however, the uncertainty reliability for "normal" is 0.9812 and that for "high" is 9999.8754×10^{-4} which is close to 1. In other words, the uncertainty reliability can be used to compare the reliability of a structure with different states, when the uncertainty reliability indices in different states are closely valued.

5. Conclusions

In this paper, the Cornell uncertainty reliability index and the uncertainty reliability are used to deal with epistemic uncertainty, especially for subjective randomness and fuzziness mixed in structural reliability analysis. Uncertain variables together with the limit state function are used to represent subjective randomness and fuzziness. The uncertain measure is adopted to quantify the uncertainty of failure and reliability in structures. The following conclusions can be drawn:

(1) In the framework of the uncertainty theory, we have provided an alternative interpretation of the classic reliability index, and proposed new formulations to analyze reliability in a structural system from a viewpoint of the uncertain measure. With the proposed formulations, not only the duality of random events can be described, but also the subadditivity of fuzzy events can be explained.
(2) From two numerical experiments with linear and nonlinear functions, it has been proven that (a) the trend of uncertainty reliability is consistent with that of the Cornell uncertainty reliability index; (b) both of them are consistent with the ones in the viewpoint of the probability theory, respectively.

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