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NUMERICAL ANALYSIS OF CONTINUATION METHODS FOR NONLINEAR STRUCTURAL PROBLEMS†

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Abstract—Continuation methods are considered here in a broad sense as the collection of methods needed for the computational analysis of specified parts of the solution field of "under-determined" equations $Fx=c$ where $F: R^m \to R^n$, $m>n$, is given and any suitable $m-n$ of the variables x_i are designated as parameters. Such equations arise frequently in structural mechanics. In general, the solutions are $(m-n)$-dimensional manifolds in R^m. Some basic existence results for the case $m=n+1$ are presented, and a procedure for the computational trace of the corresponding one-dimensional solution manifolds in R^{n+1} is discussed in detail. Then a general approach is formulated which allows, under certain assumptions, the computation of the derivative of F, and which includes, the usual incremental formulations in structural mechanics. In finite element applications it is possible to combine the continuation procedure with adaptive mesh-refinements: for a model problem it is shown that such a combined process can be surprisingly effective. The article ends with some comments about the general case $m>n+1$ and the possibility of assessing numerically the structural stability of a structure.

1. INTRODUCTION

There appears to be little question that the socalled incremental methods represent by far the most popular procedures for the solution of problems in nonlinear structural mechanics. While these procedures were developed more or less independently in the engineering literature, it is now also recognized that they belong to the general class of continuation methods used for some time in mathematics in general and in numerical analysis in particular. The literature in this area is extensive; we refer only to [13] for a description of the incremental approach to structural problems as a continuation procedure, to [8] for a historical overview of uses of continuation techniques in mathematics, and to [24] for a more recent literature survey of some numerical aspects of continuation methods.

Basically, in structural mechanics, the discretized equations of equilibrium have the generic form

$$P(x)=p \tag{1.1}$$

with a given nonlinear mapping P: $D_P \subset R^n \to R^n$ and vector $p \in R^n$. In many cases, x represents here the displacement vector and p the load vector.

In order to focus the subsequent discussion we use as an illustration the very simple model of a plane structure shown in Fig. 1. Two equal, straight rods with longitudinal elastic modulus γ are pin-jointed at the two supports and at the tip where a dead-load p acts along the vertical symmetry axis. As indicated in Fig. 1, this symmetry axis is used as the x-axis with the origin at the unloaded position of the tip. Then the total potential energy of the structure under load p is given by

$$2[\tfrac{1}{2}\gamma(\sqrt{1+h^2}-\sqrt{1+(h-x)^2}))^2]-px$$

and hence the equilibrium equation (1.1) here has the form

$$P(x)\equiv\gamma\left[\sqrt{\frac{1+h^2}{1+(h-x)^2}}-1\right](h-x)=p, \tag{1.2}$$

where the nonlinear function P on the left is now a mapping of R^1 into itself.

In general, (1.1) has to be solved for a number of load vectors p in order to assess the behavior of the structure under different conditions. Often the linear set $\{tp;\ t \in R^1\}$ of loads is used and hence the family of equations

$$P(x)=tp \tag{1.3}$$

involving the real parameter $t \in R^1$ is considered in place of (1.1). The interest then centers on determining continuous paths in R^n

$$x: J \subset R^1 \to R^n, \tag{1.4}$$

such that $x(t)$ is a solution of (1.3) for each t in the interval of definition J. More specifically, a path (1.4) is to be found which passes for $t=t_0 \in J$ through a point $x^0=x(t_0)$ that is a known solution of (1.3) for t_0. Broadly speaking, a continuation method is now any procedure which, starting from x^0 at t_0, produces acceptable approximations x^i of $x(t_i)$ for a sequence of parameter values $t_1, \ldots, t_N$ in J.

In our simple example (1.2) the load p is already one-dimensional and thus p may be used in place of the parameter t. Clearly for $p=0$ we have the solution $x^0=0$ and hence we want here a real function $x=x(p)$ which solves (1.2) for all p in some interval J and satisfies $x(0)=0$, $0 \in J$. For $\gamma=1$, $h=0.5$ and $J=[0, 0.021457\ldots)$

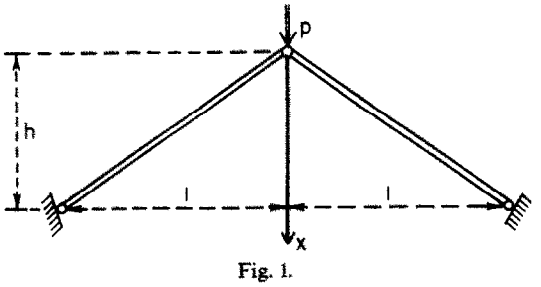

Fig. 1.

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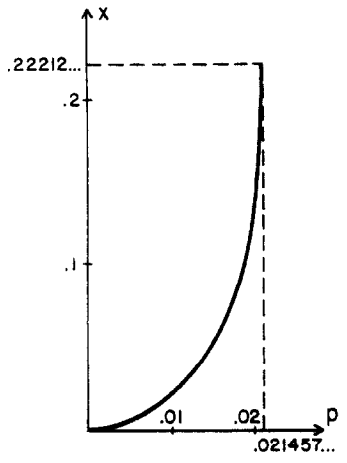

Fig. 2.

the solution is shown in Fig. 2. The right end point of J is a limit point where the derivative $x'(p)$ of the solution becomes infinite. This represents a buckling point of the structure.

The above description of continuation methods corresponds with the standard concept of these methods. However, it does not allow us to focus on two important aspects of our problem. Since a vector $x \in R^n$ can be said to solve (1.3) only if a corresponding value of $t \in R^1$ has been given, the solutions of (1.3) are in actuality points $(x_1, \ldots, x_n, t)$ of R^{n+1}. From differential geometry it is known, that, in general, the set of these solutions in R^{n+1} defines a one-dimensional manifold in the space. This means that we should look for parametric solutions $x_i = x_i(s)$, $i = 1, \ldots, n$, $t = t(s)$ involving a suitable parameter s. The choice of $s = t$ is only locally permissible. This is evident from Fig. 3 which shows for our model problem with $\gamma = 1$, $h = 0.5$, a portion of the solution manifold in R^2 through the origin. Clearly, the parametrization $s = t$ breaks down at the limit points. It would also fail at bifurcation points which our model does not possess. Both types of points signify a loss of stability and hence are of considerable importance in our assessment of the structure. This suggests that from the outset it may be well to consider instead of the solution paths (1.4) in R^n the manifold of solutions of (1.3) in R^{n+1}. This will be discussed further in the next Section.

The second of the indicated two aspects lacking in our earlier formulation concerns the degree of controllability of the structure. The variable t in (1.3) represents an albeit somewhat artificial control parameter in the description of the system. In general, there are other parameters entering into the specification of P which are in some sense under our control and may have a strong influence on the behavior of the structure. Even if no such parameters are readily available, more general load condition than those of the set $\{tp;\ t \in R^1\}$ may need to be considered. In other words, the description of the system usually involves not only the "behavior" variables $x_1, \ldots, x_n$ but also a certain number $q > 0$ of assignable "control" parameters $u_1, u_2, \ldots, u_q$. Hence instead of (1.3) our problem has the generic form

$$F(x, u) = c \tag{1.5}$$

where $F: D_F \subset R^n \times R^q \to R^n$ is again a given mapping and $c \in R^n$ a fixed vector. In our example, we have the control parameters $u_1 = p$, $u_2 = h$ and hence the mapping

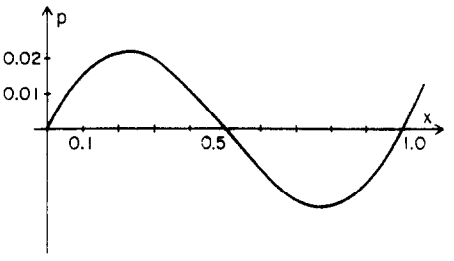

Fig. 3.

$$F(x, u_1, u_2) = \gamma\left[\sqrt{\frac{1+u_2^2}{1+(u_2-x)^2}} - 1\right](u_2 - x) - u_1, \tag{1.6}$$

from R^3 into R^1. The right side c of (1.5) is here zero.

In analogy to the case of the equation (1.3) the set of all solutions (x, y) of (1.5) is a q-dimensional manifold in R^{n+q}. For our model problem (with $\gamma = 1$), Fig. 4 shows the contour lines $p = \text{constant}$ of this 2-dimensional "surface" in (x, p, h)-space.

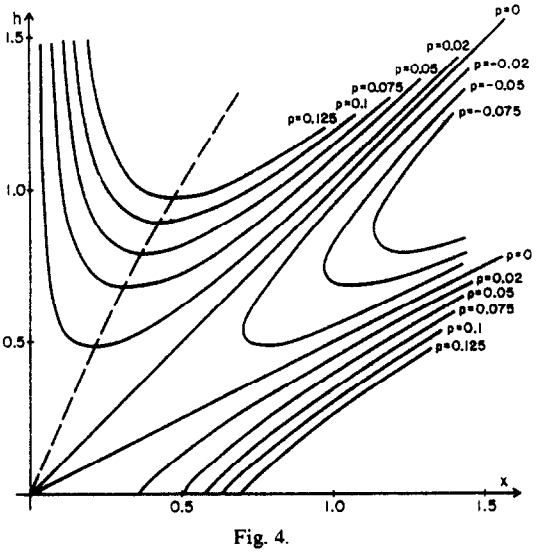

Fig. 4.

M. J. Sewell [19] called the solution manifold of (1.5) in R^{n+q} the equilibrium surface of the structure. In [21] several simple, but instructive examples are given for model structures with $n = 1$, $q = 2$, including, in essence, our model problem (1.2). The equilibrium surface provides global information about the behavior of the system under changes of the parameters. For instance, the limit points (and bifurcation points) are those points on the structure for which the matrix $F_x(x, u)$ of partial derivatives of F with respect to x is singular. Figure 4 shows one line of limit points for our model problems; it gives immediate information about the buckling loads and deformations for different initial heights h.

Since the advent of catastrophe theory the interest in a deeper study of such equilibrium surfaces has certainly intensified (see, e.g. [20]). But, the computational analysis of these surfaces is still very much in its infancy. More general continuation methods are required which permit a trace of any specific path on the surface. For example, we may wish to compute directly the line of limit points in Fig. 4. Some comments about this will be given later.

The aim of this paper is to present an overview of certain recent results as well as of various open questions about continuation methods in the setting described above. More specifically, Section 2 gives a summary of some of the basic properties of the set of solutions in R^{n+1} of the equation (1.5) in the case $q=1$, that is, when there is only one parameter. Then in Section 3 we discuss in reasonable detail a complete algorithm for the numerical trace of these solutions in R^{n+1}. Finally, Section 4 covers a number of aspects arising in connection with structural problems and leading to modifications or extensions of the method most of which involve as yet unresolved questions.

2. THE SOLUTION MANIFOLD FOR ONE-PARAMETER PROBLEMS

This section presents a brief summary of some of the basic properties of the set of solutions (x, u_1) in R^{n+1} of the equations (1.5) when $q=1$. This includes the case of the original equations (1.3). But in order to simplify the notation, it will be useful to set $x_{n+1}=u_1$ in (1.5) (or $x_{n+1}=t$ in (1.3)) and hence to write the equation in the "under-determined" form

$$Fx=c \tag{2.1}$$

involving a function $F: R^{n+1}\to R^n$ from R^{n+1} into R^n and a given vector $c\in R^n$. For ease of discussion we assume that F is defined and twice continuously differentiable on all of R^{n+1}. As before some specific solution $x^0\in R^{n+1}$ of (2.1) is supposed to be known.

A (non-trivial, parametric) solution of (2.1) is defined as any continuously differentiable function

$$x: J\subset R^1\to R^{n+1}, \quad x'(s)\neq 0, \qquad s\in J, \tag{2.2}$$

on some open interval J of R^1 such that $Fx(s)=c$ for all s in J. Evidently, such a solution remains one under any continuously differentiable parameter transformation with non-zero derivative. Hence there is no restriction to assume that in (2.2) the parameter s is the arc-length. This choice has certain advantages and has been used by various authors. Without attempting any historical survey, we mention only the articles [10, 11, 16, 18].

The set

$$\mathbb{R}(F)=\{x\in R^{n+1}; \text{rank } F'(x)=n\} \tag{2.3}$$

will be termed the regularity set of the function F. Here $F'(x)$ is the derivative of F represented by the $n\times(n+1)$ Jacobian matrix of all partial deterivatives. This set excludes essentially only the bifurcation points. For instance consider the case of our simple model problem (1.6) when x_1 is the displacement, $x_2=u_1=p$ and the height $u_2=h$ is a fixed constant. Then (with $\gamma=1$) we have

$$F'(x)=\left(1-\frac{(1+h^2)^{1/2}}{[1+(h-x_1)^2]^{3/2}}, -1\right), \qquad x=\begin{pmatrix}x_1\\ x_2\end{pmatrix},$$

and hence $\mathbb{R}(F)=R^2$. On the other hand, if we use the displacement x_1, $x_2=u_2=h$, and keep the load $u_1=p$ constant, then

$$F'(x)=\left(1-\frac{(1+x_2^2)^{1/2}}{[(1+(x_2-x_1)^2]^{3/2}}, \frac{(1+x_2^2)^{1/2}}{[1+(x_2-x_1)^2]^{3/2}} + \frac{x_2(x_2-x_1)}{[(1+x_2^2)(1+(x_2-x_1)^2]}-1\right) \tag{2.5}$$

and for $x_1=x_2=0$ the rank of $F'(x)$ reduces to zero. In other words, here we have $\mathbb{R}(F)=\{x\in R^2; x\neq 0\}$ and, in fact, Fig. 4 shows that in this case the origin is a bifurcation point.

A proof of the following result is given in [16]:

Theorem 2.1. For any $x^0\in\mathbb{R}(F)$ with $Fx^0=b$ there exists a unique solution (2.2) of (2.1) in $\mathbb{R}(F)$ for which the interval of definition J is maximal under set inclusion, the parameter s is the arclength, and $0\in J$, $x(0)=x^0$. If $\alpha\in\hat{c}J$ is finite then $x(s)\to\hat{c}\mathbb{R}(F)$ or $\|x(s)\|\to\infty$ as $s\to\alpha$, $s\in J$.

The task of the desired continuation process is now to compute an approximation of this solution for a given $x^0\in\mathbb{R}(F)$. A possible process of this type is discussed in the next Section. Here we mention only a few properties of the solutions characterized by the Theorem.

Since $F'(x)$ is an $n\times(n+1)$ matrix which for $x\in\mathbb{R}(F)$ has n linearly independent columns, there exists a non-zero vector $v\in R^{n+1}$ for which $F'(x)v=0$. Obviously, v has the direction of the tangent of the solution curve of (2.1) through x. In order to obtain a unique vector v we need to fix the sign of its direction and its length. For theoretical purposes this may be accomplished by requiring that

$$F'(x)v=0, \quad v^Tv=1, \quad \det\begin{pmatrix}F'(x)\\ v^T\end{pmatrix}>0. \tag{2.6}$$

Then

$$T: \mathbb{R}(F)\to R^{n+1}, \quad v=Tx \tag{2.7}$$

is a well defined mapping from $\mathbb{R}(F)$ into R^{n+1}. For instance in the particular case of our model problem when $F'(x)$ is given by (2.5), we have

$$Tx=\frac{1}{[1+(1-A)^2]^{1/2}}\begin{pmatrix}1\\ 1-A\end{pmatrix}, \quad A=\frac{(1+h^2)^{1/2}}{(1+(h-x_1)^2)^{3/2}}, \qquad x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\in\mathbb{R}(F). \tag{2.8}$$

It is easily seen that in $\mathbb{R}(F)$ any solution of the (autonomous) system of differential equations $x'=Tx$ is a parametric solution of (2.1) with the arc-length as parameter.

A point $x\in\mathbb{R}(F)$ is a limit-point (or turning point) of (2.1) with respect to the ith variable x_i, $1\leq i\leq n+1$, if the ith-component of Tx is zero. In the case of (2.8) this may happen only for $i=2$ when $A=1$, that is, when

$$x_1^*=h\pm\sqrt{(1+h^2)^{1/3}-5} \tag{2.9}$$

This is the heavy-dashed curve in Fig. 4. It represents the location of the buckling points of the structure of Fig. 1 as a function of the height. The corresponding buckling loads are found by a simple evaluation of the function P of (1.3) and hence are given by $p=[(1+h^2)^{1/3}-1]^{3/2}$.

In the case when $F'(x)$ is given by (2.5) there are limit points with respect to both variables x_1, x_2. In Fig. 4 these are the points where the tangent of the curve is parallel to the x_2-axis or x_1-axis, respectively. Then, for example, the x_2-coordinate of an x_1-limit point respresents the critical height for the given load p such that for all lower heights the structure is instable at that load.

3. NUMERICAL DETAILS OF A CONTINUATION PROCESS

Let $x^0\in\mathbb{R}(F)$ with $Fx^0=c$ be a given point and

$$x: J \subset R^1 \to \mathbb{R}(F), \quad \|x'(s)\|_2 = 1, \quad s \in J, \quad x(0) = x^0. \tag{3.1}$$

the solution of (2.1) through x^0 which is guaranteed to exist in $\mathbb{R}(F)$ by Theorem 2.1. By requiring that the Euclidean norm of the tangent equals one we automatically ensure that the parameter s is the arc-length.

Our task is now to compute a suitable sequence of points

$$x^j \doteq x(s_j), \quad j = 0, 1, \ldots, N, \quad s_0 < s_1 < \ldots < s_N, \tag{3.2}$$

approximating (3.1). Essentially all numerical continuation processes for doing this now are of the predictor-corrector type. Suppose that the points $x^0, x^1, \ldots, x^k$ have already been computed for some $k \geq 0$. Then a suitable predictor

$$y = \pi(t), \quad \pi: J_\pi \subset R^1 \to R^{n+1}, \quad \pi(0) = x^k, \tag{3.3}$$

is calculated which approximates (3.1) on some small interval $[s_k, s_k + \delta]$, $\delta > 0$ beyond s_k. For example, π may be the polynomial of degree at most m defined by the interpolatory requirements $\pi(s_j) = x^j$, $j = k, k-1, \ldots, k-m$. Alternatively, if the tangents Tx^j at these points are also available, Hermite-interpolation may be applied as well. Other than in the case of the multistep ODE-solvers for the computer solution of initial value problems for ordinary differential equations, we appear to gain very little from the use of high order interpolation polynomials for the predictor π of our continuation process. Some reasons for this will be given later. Generally, the simple Euler predictor

$$\pi(t) = x^k + tTx^k, \quad t \geq 0 \tag{3.4}$$

has been found to be computationally optimal.

Before we can use (3.4), the tangent vector Tx^k needs to be computed, that is, we have to determine the vector v specified by (2.6) at the point $x = x^k$. For this the $n \times (n+1)$ system of equations $F'(x^k)v = 0$ has to be augmented by some $(n+1)$st scalar equation. There are various possible choices for this (see, e.g. [1]), but from a practical viewpoint an affine equation $w^T v = \xi$ has obvious advantages. The vector $w \in R^{n+1}$ has to be such that the resulting $(n+1) \times (n+1)$ system of equations is nonsingular. In order to reduce the computational work it is natural to consider the choice $w = e^i$ where e^i is the ith natural basis vector of R^{n+1} with a suitable index i, $1 \leq i \leq n+1$. It is easily seen that

$$\det \begin{pmatrix} F'(x^k) \\ (e^i)^T \end{pmatrix} = ((Tx^k)^T e^i) \det \begin{pmatrix} F'(x^k) \\ (Tx^k)^T \end{pmatrix} \tag{3.5}$$

and that the determinant on the right will be non-singular for any $x^k \in \mathbb{R}(F)$. Hence, the index i should be chosen such that $|(Tx^k)^T e^i|$ is as large as possible. Generally, let

$$|(Tx^j)^T e^{i_j}| = \max \{|(Tx^j)^T e^l|, l = 1, \ldots, n+1\}, \quad j = 0, 1, \ldots. \tag{3.6}$$

Then it is reasonable to use $w = e^i$ with $i = i_{k-1}$. Of course, this applies only for $k \geq 1$; for $k = 0$ a suitable index i is assumed to have been given with x^0. Recall that x^{k-1} is a limit point with respect to the ith variable if $(Tx)^{k-1} e^i = 0$. Hence for $k \geq 1$ we choose the index $i = i_{k-1}$ for which Tx^{k-1} is in some sense furthest from a limit point with respect to x_i.

Our basic task is now to solve the system of equations

$$\begin{pmatrix} F'(x^k) \\ (e^i)^T \end{pmatrix} v = e^{n+1}, \tag{3.7}$$

which will be non-singular if $(Tx^k)^T e^i \neq 0$ and hence if x^k is not too far from x^{k-1}. Basically, any standard method for the solution of a non-singular system of linear equations is applicable here. However, $F'(x)$ often has a special structure and the addition of the last row may lead to complications. For example, when F is derived from (1.3), that is, when

$$Fx = P\bar{x} - x_{n+1} p, \quad x = \begin{pmatrix} \bar{x} \\ x_{n+1} \end{pmatrix} \in R^{n+1}, \quad \bar{x} \in R^n \tag{3.8}$$

we frequently find in applications that $P'(\bar{x})$ is a banded, symmetric matrix. Obviously, the matrix of (3.7) has lost these advantageous properties and hence some special consideration is required to prevent an increase in the complexity of the solution process.

In the particular case of the mapping (3.8) our system (3.7) has the particular form

$$Av = c, \quad A = \begin{pmatrix} B & p \\ (e^i)^T & \end{pmatrix}, \quad B = P'(\bar{x}^k). \tag{3.9}$$

For $i = n+1$ we need to solve only the subsystem with the matrix B and hence no complications arise. Suppose therefore that $1 \leq i \leq n$. Then A is a bordered matrix and our basic approach will be to introduce a decomposition

$$A = A_0 + ab^T \tag{3.10}$$

of A into the sum of a suitable non-singular matrix A_0 and a rank-one matrix ab^T. Once the two systems

$$A_0 y = r, \quad A_0 z = a \tag{3.11}$$

with the same matrix A_0 have been solved, the solution of (3.9) itself is given by

$$v = y - \frac{b^T y}{1 + b^T z} z. \tag{3.12}$$

Here, it follows from the well-known Sherman–Morrison formula that $b^T z \neq -1$ as long as A and A_0 are non-singular.

Among the various possible decompositions (3.9) the following one has been found very effective in applications:

$$A = A_0 + a(e^{n+1})^T, \quad A_0 = \begin{pmatrix} B & e_n^i \\ (e_n^i)^T & 0 \end{pmatrix}, \quad a = \begin{pmatrix} p \\ 0 \end{pmatrix} - e^i \tag{3.13}$$

Here e_n^i denote the natural basis vectors of R^n while as before, e^i are those of R^{n+1}. It turns-out that A_0 is generally non-singular although there are situations when this is not true. Without entering into a theoretical discussion about conditions ensuring the non-singularity of A_0, we consider here only the solution of the systems (3.11) for this particular A_0. These systems have the common form

$$\left(\begin{array}{c|c|c|c} B_1 & b_1 & C & 0 \\ \hline b^T & \beta & b_2^T & 1 \\ \hline C^T & b_2 & B_2 & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} w_1 \\ \eta_1 \\ w_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ \delta_1 \\ d_2 \\ \delta_2 \end{pmatrix} \tag{3.14}$$

where B_1, B_2 are square symmetric matrices. Hence we

need to solve merely the banded, symmetric $(n-1)\times(n-1)$ system

$$\left(\begin{array}{c|c} B_1 & C \\ \hline C^T & B_2 \end{array}\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}=\begin{pmatrix} d \\ d_2 \end{pmatrix}$$

which has a non-singular matrix exactly if A_0 has the property and for which the band-width does not exceed that of B itself. Since always $\eta_1=\delta_2$ the remaining unknown of the system (3.14) is given by $\eta_2=\delta_1-\beta\delta_2-b_1^Tw_1-b_2^Tw_2$.

Evidently, if $B=P'(\bar{x}^k)$ has been stored in some banded format, then it is unnecessary to store the matrix of (3.15) separately. In fact, after saving the vectors b_1, b_2 and the scalar β of (3.14) we need only zero-out the locations occupied by b_1, b_2 and store 1 in the location of β. Then we may work with this $n\times n$ banded, symmetric matrix in place of the matrix of (3.15).

Once the—necessarily non-zero—solution v of the system (3.7) has been obtained, the tangent vector

$$Tx^k=\sigma\frac{v}{\|v\|_2} \tag{3.16}$$

can be computed. Here $\sigma=\pm1$ is the direction specified in (2.6). By (3.5) we have

$$\sigma_{k,i}=\sigma_k\ \text{sign}\,(Tx^k)^Te^i,\quad \sigma_{k,i}=\text{sign det}\begin{pmatrix} F'(x^k) \\ (e^i)^T \end{pmatrix},$$

$$\sigma_k=\text{sign det}\begin{pmatrix} F'(x^k) \\ (Tx^k)^T \end{pmatrix} \tag{3.17}$$

Here $\sigma_{k,i}$ is the sign of the determinant of our system (3.7) which is easily computed during the solution of that system. By (2.6) we want $\sigma_k=+1$ and hence we should set $\sigma=\sigma_{k,i}$ sign (v^Te^i). As long as the solution curve remains in $\mathbb{R}(F)$ this approach is satisfactory. However, it does not permit us to detect when we may have passed a bifurcation point. More specifically, suppose that $x^*\in\mathbb{R}(F)$ is a bifurcation point where several solutions x^j: $J_j\subset R^1\to\mathbb{R}(F)$, $j=1, 2,\ldots,$ of (2.1) (for different c) terminate, that is, for which $x^j(s)$ tends to x^* when s tends to one of the endpoints of J_k. In structural applications it frequently happens that there are pairs of these solutions, say, $x=x^1(s)$, $x=x^2(s)$ for which $\lim Tx^1(s)=-\lim Tx^2(s)$, (see Fig. 5). In other words, if we disregard the direction of the solutions, they appear to form one smooth curve through x^*. In such a case, when the process moves along $x^1(s)$ toward x^* it usually "jumps" over x^* onto $x^2(s)$. Then, unless we reverse the sign of σ_k we will not move along $x^2(s)$ away from x^* but again toward x^* instead. In order to avoid this problem the factor σ in (3.16) is defined by

$$\sigma=\begin{cases} +1 & \text{if sign } v^Te^i_{n+1}=\text{sign }(Tx^{k-1})^Te^i_{n+1} \\ -1 & \text{otherwise} \end{cases} \tag{3.18}$$

and then σ_k is computed from (3.17). Now for $\sigma_k\neq\sigma_{k-1}$ we have a situation as in Fig. 5 and hence we change the sign of Tx^k. Otherwise Tx^k is used as computed. Note, however, that in this case we may well have jumped over a bifurcation point of even multiplicity.

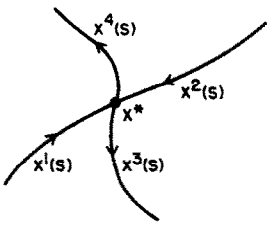

Fig. 5.

With Tx^k we now know the predictor (3.4). In order to define the corrector, some $(n+1)$st equation has to be added again to the system (2.1). As in the case of the tangent calculation it is natural to use for this purpose the following augmented system

$$\hat{F}x=\begin{pmatrix} Fx-c \\ (e^{i_k})^T(x-\pi(t_{k+1}))-\gamma \end{pmatrix}=0. \tag{3.19}$$

Here i_k is defined by (3.6), t_{k+1} represents the step to be taken along the predictor line (3.4) and the scalar γ will be fixed shortly. Any locally convergent iterative process for solving (3.19) may now be used as the corrector. For instance, if Newton's method is chosen then the corrector has the form

$$\begin{pmatrix} F'(y^j) \\ (e^{i_k}_{n+1})^T \end{pmatrix}(y^j-y^{j+1})=\hat{F}y^j,\ y^0=\pi(t_{k+1}),\quad j=0, 1\ldots \tag{3.20}$$

and the systems of equations to be solved have exactly the same form as those in the tangent computations.

The choice of the step t_{k+1} to be taken along the predictor depends on the selection of γ. This is equivalent with the selection of the point $x(s_k+\Delta s)$ on the solution (3.1) which is to be approximated by the corrector. In fact, as long as t_{k+1} is not too large the equation

$$(e^{i_k})^Tx(s_k+\Delta s)=\gamma+(e^{i_k})^T\pi(t_{k+1}) \tag{3.21}$$

defines a one-to-one correspondence between Δs and γ for all sufficiently small Δs. Evidently, $\Delta s=t_{k+1}$ is a natural but by no means required choice for Δs.

In order to compute γ for $\Delta s=t_{k+1}$ and to estimate the distance between the predicted point $\pi(t_{k+1})$ and the desired solution $x(s_k+t_{k+1})$ of (3.18) we proceed as in the case of the ODE solvers. The quadratic polynomial

$$q(s)=x^k+(s-s_k)Tx^k+(s-s_k)^2w^k \tag{3.22}$$

$$w^k=\frac{1}{h_k}\left[Tx^k-\frac{1}{h_k}(x^k-x^{k-1})\right],$$

$$h_k=\|x^k-x^{k-1}\|_2\doteq s_k-s_{k-1}$$

represents a "better" approximation of $x(s)$ than the predictor (3.4). In fact, for three-times continuously differentiable F the error $\|q(s)-x(s)\|$ is asymptotically of order h^3 where $h=\max(h_k, |s-s_k|)$ while for the predictor it is only of order h^2. This suggests that for sufficiently small h we may replace $x(s)$ by $q(s)$. If this is done in (3.21) then we obtain for $\Delta s=t_{k+1}$ the value of γ as

$$\gamma=\frac{t_{k+1}^2}{h_k}\left[(e^{i_k})^TTx^k-\frac{1}{h_k}(e^{i_k})^T(x^k-x^{k-1})\right]. \tag{3.23}$$

At the same time, the requirement, that the distance between $\pi(t_{k+1})$ and $x(s_k+t_{k+1})$ is below some tolerance $\rho_{k+1}>0$, may be approximated by

$$\|\pi(t_{k+1})-q(s_k+t_{k+1})\|\leq\rho_{k+1}. \tag{3.24}$$

This leads immediately to the estimate

$$t_{k+1}\leq\sqrt{\frac{\rho_{k+1}}{\|w^k\|_2}}. \tag{3.25}$$

Before we discuss the choice of ρ_{k+1} it may be useful to comment on the computation of the quantities in

(3.23) and (3.24). The norm of w^k represents an estimate of the curvature of $x(s)$ at x^k and hence it is small when the curve is fairly straight. Since w^k is proportional to the difference of two vectors of length one, the explicit computation of this vector may lead to severe loss of accuracy due to subtractive cancellation. In (3.23) only one component of w^k is needed, and it should be computed with care and preferably in double precision. For the computation of $\|w^k\|_2$ in (3.25) we use

$$\|w^k\|_2=\frac{1}{h_k}\sqrt{2(1-\cos\alpha_k)}=\frac{2}{h_k}\left|\sin\frac{\alpha_k}{2}\right|.$$

$$\alpha_k=\arccos\left((Tx^k)^T\frac{x^k-x^{k-1}}{h_k}\right), \qquad (3.26)$$

which is a numerically reliable formula as long as the last step h_k was not overly small.

We turn now to the choice of the tolerance $\rho_{k+1}>0$. Ideally, ρ_{k+1} should be chosen such that the corrector iteration, when started from $\pi(t_{k+1})$, is known to converge to a point on the solution curve. In [16] it was shown that for any closed segment $\{x(s),\ s\leq s\leq\bar{s}\}$ of the solution curve which is completely contained in $\mathbb{R}(F)$ there exists a fixed positive tolerance $\rho>0$ such that the Newton method (3.20) converges to a point on the curve as long as the predicted point has at most a distance ρ from it. This will be the case for all sufficiently small steps t_{k+1} along the predictor. But then the achievable error $\|x^{k+1}-x(s_{k+1})\|$ is solely determined by the termination criterion for the corrector iteration. In contrast to this the standard multistep ODE-solvers involve a corrector equation which is obtained by interpolation and hence for which the solutions are not, in general, on the exact curve. As a consequence the achievable error for the ODE-solvers depends on the history of the process up to that point, and this in turn has a strong influence on the selection of the steps. On the other hand, in the case of the continuation process any step t_{k+1} is acceptable in principle if only the corrector converges from $\pi(t_{k+1})$. This is the basic reason why it has little sense to use higher order predictors; in fact, they rarely provide consistently better starting points to save sufficient work in the corrector iteration to balance the added effort and storage requirements needed for their computation.

This still leaves us with the question how to choose the tolerance $\rho_{k+1}>0$ in (3.25). By definition, for any locally convergent corrector iteration at $x(s_{k+1})$ there exists a convergence radius $r_{k+1}>0$ such that for all starting points within the distance r_{k+1} from $x(s_{k+1})$ the process is guaranteed to converge. A natural idea is then to estimate $r_1,\ldots\ r_k$ from the performance of the corrector iterates at the computed points. In [6] it is also $x^1,\ldots,x^k$ and to select ρ_{k+1} by extrapolation from these r^j. Various schemes of this type have been discussed in the literature (see e.g. [1, 16, 18]), but the results appear to be very sensitive to the properties of the problem at hand. Now a recent result in [6] has provided new insight into the reason for this.

It is natural to attempt to estimate any convergence radius r_j on the basis of the corrector iterates at x^j and our knowledge of F and F' at these points. In [6] it is proved that this information is insufficient to obtain any lower or upper bound of r_j. In other words, we would need more global information about the problem to obtain such bounds for r_j and this would require more computational efforts than is reasonably justifiable. The quantities used in the cited articles do not represent bounds of the convergence radii but of the quality of the convergence of the particular sequence of corrector iterates at the compute points. In [6] it is also shown that for Newton's method such assessments of the convergence quality can be used effectively for the determination of suitable tolerances ρ_j even though such ρ_j are not estimates of the radii r_j.

We proceed here in a more straightforward way which is not restricted to a particular corrector process. As in the ODE-solvers we need to control both the relative growth and the absolute size of the predictor step t_{k+1}. Thus we require that

$$\text{(i)}\ \frac{1}{p}h_k\leq t_{k+1}\leq ph_k,\quad \text{(ii)}\ t_{\min}\leq t_{k+1}\leq t_{\max}, \qquad (3.27)$$

where p is some factor, say $p=3$, and $t_{\min}$, $t_{\max}$ depend on the machine as well as on the requirements of the problem. A natural tolerance at the last computed point x^k is the distance $\delta_k=\|(x^{k-1}+t_kTx^{k-1})-x^k\|$ between the points where the corrector started and ended. However, it turns out that the corresponding tentative step

$$t_{k+1}=\sqrt{\frac{\delta_k}{\|w^k\|}}=h_k\sqrt{\frac{\delta_k/h_k}{2|\sin\alpha_k/2|}},$$

tends to lead to numerical instabilities even if it is adjusted to satisfy (3.27). For instance, for small δ_k the step will be small and the next distance δ_{k+1} is likely to be even smaller. Thus, especially when the curvature does not decrease, the algorithm ends up taking minimal steps only. Clearly we need to introduce thresholds for the quantities in (3.28).

Let $\alpha_0>0$ be a small lower threshold for $|\alpha_k|$ and set

$$\omega_k=2\min\left(\tfrac{1}{2}\sqrt{2},\ \max\left(\left|\sin\frac{\alpha_0}{2}\right|,\left|\sin\frac{\alpha_k}{2}\right|\right)\right). \qquad (3.29)\text{(a)}$$

Moreover, let

$$\eta_{\max}=2p^2\left|\sin\frac{\alpha_0}{2}\right|,\quad \eta_{\min}=\frac{2}{p^2}\sin\frac{\pi}{4}=\frac{\sqrt{2}}{p^2} \qquad (3.29)\text{(b)}$$

and introduce the relative tolerance

$$\eta_{k+1}=\begin{cases}\eta_{\max}, & \text{if } \left|\sin\frac{\alpha_k}{2}\right|\leq\sin\frac{\alpha_0}{2} \text{ or } \delta_k\leq h_k\eta_{\min},\\ \eta_{\min} & \text{if } \left|\sin\frac{\alpha_k}{2}\right|\geq\sqrt{2} \text{ or } \delta_k\leq h_k\eta_{\min}\\ \delta_k/h_k, & \text{otherwise.}\end{cases} \qquad (3.29)\text{(c)}$$

Clearly we need $\alpha_0>\underline{\alpha}_0=2\arcsin[(\sqrt{2}p^2)^{-1}]$ to ensure that $\eta_{\min}<\eta_{\max}$. Then for $\underline{\alpha}_0<\alpha_0<\pi/2$ the tentative step

$$\tilde{t}_{k+1}=h_k\sqrt{\frac{\eta_{k+1}}{\omega_k}}$$

satisfies (3.27)(i), and hence for the step

$$t_{k+1}=\min\{\max(\tilde{t}_{k+1},t_{\min}),t_{\max}\}. \qquad (3.30)$$

both conditions (3.27) hold. With $p=3$ and $\alpha_0^*=0.05$ and various values of $t_{\min}$ and $t_{\max}$ this step-length has been used extensively with excellent results.

Once the predictor step t_{k+1} has been chosen, the corrector iteration is started from $\pi(t_{k+1})$. This process has to incorporate provisions for monitoring the convergence and for aborting the iteration as soon as divergence is detected. These provisions depend of course on the type of corrector used. In the case of Newton's

method (3.20) it has been found satisfactory to declare non-convergence if either one of the following the three conditions is true

$$\begin{aligned}&\text{(i)}\ \ \|\hat{F}y^j\| \geqslant \theta \|\hat{F}y^{j-1}\| \text{ for some } j \geqslant 1\\ &\text{(ii)}\ \ \|y^j - y^{j-1}\| \geqslant \theta \|y^{j-1} - y^{j-2}\| \text{ for some } j \geqslant 2\\ &\text{(iii)}\ \ j \leqslant j_{\max}\end{aligned} \tag{3.31}$$

with a suitable constant θ, say $\theta = 1.05$, and an integer $j_{\max}$, say $j_{\max} = 6$. On the other hand, the iterate y^j is accepted as the next point x^{k+1} if either one of the two conditions holds

$$\begin{aligned}&\text{(i)}\ \ \|\hat{F}y^0\| \leqslant \varepsilon_1\\ &\text{(ii)}\ \ (\|\hat{F}y^j\| \leqslant \varepsilon_2) \text{ and } (\|y^j - y^{j-1}\| \leqslant \varepsilon_3 + \varepsilon_4 \|y^j\|)\\ &\qquad\qquad \text{for some } j \geqslant 1\end{aligned} \tag{3.32}$$

with given tolerances $\varepsilon_1, \ldots, \varepsilon_4$ which depend on the machine and on the problem.

In the case of non-convergence the predictor step is halved and the corrector is restarted from $\pi(\frac{1}{2}t_{k+1})$ with the corresponding value (3.23) of γ. However, it is required here that $\frac{1}{2}t_{k+1} \geqslant t_{\min}$ otherwise some user dependent action needs to be taken to modify $t_{\min}$ or to stop the overall process. On the other hand, if the convergence is declared with $x^{k+1} = y^j$ as the last iterate then we approximate the exact arc-length by

$$s_{k+1} = s_k + \|x^{k+1} - x^k\|_2 \tag{3.33}$$

and, if required, proceed to take a new predictor-corrector step.

The continuation procedure described in this section has been used extensively with excellent success on many problems from structural mechanics and other fields. For space reasons we forego including here some graphs of the solution of a particular, large problem obtained by the procedure. Only a few of the many data characterizing any practically meaningful problem can ever be sketched and this does provide little or no insight into the way the continuation procedure actually works. Therefore, in the interest of giving a more detailed picture of the operation of the process, we present here computational results for a small, simple problem for which all relevant data can be exhibited.

More specifically, for a two-bar structure of the form of Figure 1 with $h = 2$ we use the following model derived in [13], p. 232:

$$Fx = \begin{pmatrix} (x_1 - 2)^3 + (x_1 - 2)(x_2^2 - 4) - 2x_3 \cos\alpha \\ x_2(x_1 - 2)^2 + x_2(x_2^2 - 2) - 2x_3 \sin\alpha \end{pmatrix} = 0. \tag{3.34}$$

The load of size $2x_3$ is here tilted by an angle α from the vertical direction. Table 1 gives results computed on a PDP-10 for the case $\alpha = 0$, that is, for a vertically downward load. All data are rounded to three digits. In this case the horizontal component x_2 of the displacement vector x and of the tangent vector Tx is always zero and hence is not given. The change of the sign of the determinant σ_k (see (3.17)) between steps 1 and 2 signifies that a bifurcation point has been passed and that the direction of the trace had to be reversed. Between steps 2 and 3 a limit point with respect to the load variable x_3 is encountered. Note that the number of corrector steps is essentially constant except—as expected—near the bifurcation point where of course the Newton method used here becomes singular. The value $t_{\max} = 1$ was used; the decrease of the predicted steps near the limit points is caused solely by the increased curvature: otherwise limit points have no effect whatsoever on the process.

Table 2 gives results for the case $\alpha = \arcsin 0.75 \doteq 0.8481$. Here the limit point for the load variable occurs between steps 3 and 4 and a limit point for the horizontal deformation component x_2 is encountered between steps 4 and 5. If the run is continued a limit point with respect to the first variable x_1 will be found between steps 6 and 7. The solution is here more strongly curved and hence the steps are smaller and the number of corrector steps is larger.

Table 1.

k	s_k	x_1^k $(Tx^k)_1$	x_3^k $(Tx^k)_3$	i_k	h_k t_{k+1}	Corr. Steps	α
0	0	0 .243	0 .970	3	- .500	3	16.5
1	.503	.135 .297	.485 .955	3	.503 1.00	4	9.98
2	1.55	.595 .721	1.42 .693	1	1.04 .703	3	-.0697
3	2.20	1.23 .666	1.31 -.746	3	.650 .540	3	-4.24
4	2.99	1.66 .479	.649 -.878	3	.787 1.00	2	-7.89
5	4.03	2.14 .453	-.282 -.892	3	1.05		-8.75

Table 2.

	s_k	x_1^k $(Tx^k)_1$	x_2^k $(Tx^k)_2$	x_3^k $(Tx^k)_3$	τ_k	η_k t_{k+1}	# of Corr. Steps	ε_e
0	0	0 .131	0 .595	0 .793	3	.500	5	20.2
1	.500	.181 .309	.404 .745	.397 .591	2	.595 .829	4	14.9
2	1.55	.568 .709	1.10 .682	.921 .140	1	.951 .544	7	13.5
3	2.21	1.05 .894	1.50 .198	1.15 -.206	1	.665 .376	7	13.1
4	2.66	1.42 .856	1.59 .140	.910 -.498	1	.452 .496	6	12.4
5	3.45	1.86 .631	1.50 -.183	.260 -.754	3	.790 1.00		10.9

4. EXTENSIONS AND MODIFICATIONS

In this section we consider various special aspects of our problem, in particular, as they arise in structural applications. Up to now, the function F of (1.5) was simply assumed to be given. In practice, however, this mapping is derived in a more or less complicated manner from the original problem formulation and it depends strongly on the form of this derivation how much information is actually available about F during the solution process.

In structural applications the mapping F is typically obtained by finite element techniques. As a consequence, each evaluation of the vector Fx for a given point $x \in R^{n+1}$ involves computations of elemental stiffness data, their assembly into global form, etc. Hence there is no readily apparent way of computing the derivative matrix $F'(x)$, and without it the process of the previous section requires substantial modifications.

The derivative F' is an integral part of the computation of the predictor and it may or may not be needed in the corrector iteration. While the various forms of Newton's method certainly are well suited as corrector processes, there are many other choices available which do not depend on an explicit knowledge of F'. For instance, quasi-Newton-(update)-methods have been proposed for this purpose (see e.g. [23]). Another possibility arises when F has the quasi-linear form

$$Fx = A(\bar{x})\bar{x} - x_{n+1}p, \quad x = \begin{pmatrix} \bar{x} \\ x_{n+1} \end{pmatrix} \in R^{n+1}, \quad \bar{x} \in R^n. \tag{4.1}$$

This is frequently the case for geometrically nonlinear problems. Then the "corrector equation" (3.19) has the same generic form.

$$\hat{A}(x)x = d, \quad \hat{A}(x) = \begin{pmatrix} A(\bar{x}) & -p \\ \hline (e^i)^T & \end{pmatrix}, \tag{4.2}$$

and this suggests the well-known iterative process

$$\hat{A}(y^{j+1})y^j = d, \quad y^0 = x^k, \quad j = 0, 1, \ldots \tag{4.3}$$

as a corrector method.

Evidently, the derivative F' may also be avoided in the definition of the predictor. For instance, as indicated in the previous section we may use the standard Lagrangian interpolation polynomial π based on the data $\pi(s_j) = x^j$, $j = k, k-1, \ldots, k-m$. Unfortunately, this requires that the last $m+1$ points are kept in storage. Moreover, it is well-known that the quality of the approximation of the solution by π deteriorates rapidly outside the interpolation interval $[s_{k-m}, s_k]$.

Generally, continuation processes not involving the derivative perform noticeably more poorly than those based on it. As indicated, the predictions tend to be less reliable, and, in most cases, derivative-free corrector processes, such as (4.3), converge more slowly or may fail to converge altogether in parts of $\mathbb{R}(F)$. Fortunately, there is a technique which often allows us to obtain F' without unreasonable effort. It involves what in structural mechanics is usually called the incremental formulation of the basic equations.

For simplicity we consider the case of the equations (1.3). Moreover, before the discretization the original equilibrium problem is assumed to have the form

$$Hu = tw, \quad t \in R^1, \tag{4.4}$$

involving a mapping $H: X \to X$ on some infinite dimensional space X. Usually, X is a Banach- or Hilbert-space of suitable functions on a given finite-dimensional domain. The finite-element approximation then introduces a discretization mapping

$$\phi: X \to R^n \tag{4.5}$$

from X onto R^n and with it the discretized version $P: R^n \to R^n$ of H defined by $P(x) = \phi(Hu)$ for all $x = \phi(u)$, $u \in X$. Note, however, that P is only well-defined if ϕ is compatible with ϕ in the sense that $\phi(u) = \phi(v)$ for any $u, v \in X$ implies that $\phi(Hu) = \phi(Hv)$. Now, if, say, $\phi(w) = p$, then our discretized problem assumes the form (1.3).

In most applications, H possesses a (Frechet) derivative H' which maps X into the space $L(X)$ of

bounded linear operators on X. Moreover, it is usually rather straightforward to obtain H' from H. The incremental forms, discussed, for instance, in [13], are simply H' in the differential form $H'(u)v$. Now our discretization (4.5) induces also the matrix valued mapping $M: R^n \to L(R^n)$ with $M(x)y = \phi(H'(u)v)$, $x = \phi(u)$, $y = \phi(v)$, provided only that H' satisfies the appropriate compatibility condition. Evidently, the same mechanism used in the computation of Px also allows us to compute $M(x)y$. The question then arises under what conditions this discretization M of the derivative H' turns out to be the derivative P' of the discretization P. In other words we have the (Fig. 6) diagram

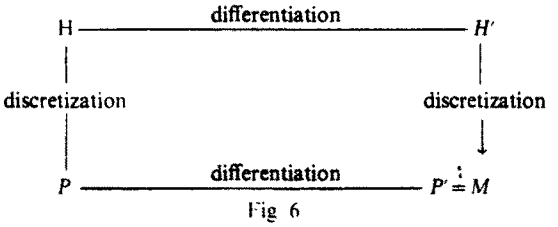

Fig 6

and the question is whether the operations of differentiation and discretization are commutative. Since M can be computed this would provide us with the desired method for computing the derivative P' of P.

In [14] it was shown that, under suitable assumptions, the answer is indeed affirmative. Unfortunately, in practical situations these assumptions—and, in particular, the mentioned compatibility conditions—are difficult to verify. There are also simple examples where the commutativity does not hold. However, it is conjectured that in most of the finite element applications to structural problems the result does hold and hence that in these cases P' can be computed as the discretization of H'. A formal proof is expected to be given elsewhere. In addition, the general "incremental approach" sketched here is beginning to find applications in fluid-flow problems, etc., where formerly derivative-based continuation methods were thought to be out of the question. There is certainly a need for a broader study of this incremental approach to the computation of the derivative of discretizations of practically important operators H.

Generally, in the application of the finite element method a very critical decision involves the choice of the mesh and the elements. Various approaches have been proposed for the design of meshes which are optimal in some sense. In particular, in [2–5] a theory of *a posteriori* error estimates for finite elements solutions has been developed which permits the construction of an adaptive mesh refinement algorithm that has been found to be highly effective for a range of problems.

Suppose again that in its basic form our problem is given by the equation (4.4). Then the desired solution may be expected to have the form

$$u: J \to X, \quad t: J \to R^1 \tag{4.6}$$

involving suitable functions u and t on some open interval J. Evidently, now, our principal aim should be to approximate this solution (4.6) of the infinite-dimensional problem. In order to accomplish this we construct first the discretization (1.3) of (4.4) and in turn compute its solution curve in R^{n+1}. This curve depends on the discretization (4.5) we have used; moreover the approximation-error between it and (4.6) is likely to vary considerably when we move along the solution (4.6). Since our main aim is to approximate (4.6), it is natural to consider different discretizations (4.5) (and hence dimensions n) in different parts of the parameter interval J.

In the finite-element setting the discretization (4.5) represents a particular selection of the mesh and the elements. Thus we are interested in methods for adapting the mesh and/or the elements to ensure an approximation of (4.6) within a given tolerance throughout the entire parameter interval. This is at present still a largely unresolved problem. In order to measure the approximation error, a theory of *a posteriori* error estimates for nonlinear finite element problems is needed, and such a theory is as yet little developed. Moreover, each change of the discretization (4.5) represents a change of the finite-dimensional problem and its dimension. Hence each such change implies a restart of the continuation process for a modified problem in a different space, and at present the interaction between these problem changes and the stability of the continuation procedure is only barely understood.

Nevertheless, so far all experience with processes that combine continuation and mesh-adaptation have been almost startingly effective. As an example consider the elliptic two-point boundary-value problem

$$-\frac{d}{dx}\left(a\left(\frac{du}{dx}\right)\right) + b(u) = tf(x),\ x \in I = [0, 1],\ t \in R^1, \tag{4.7}$$

$$u(0) = u(1) = 0,$$

with suitable coefficient functions a, b, f. Let

$$\Delta: x_0 = 0 < x_1 < \ldots < x_n < x_{n+1} = 1, \quad n = n(\Delta), \tag{4.8}$$

be a given mesh on I and $S(\Delta)$ the set of continuous functions on I which are linear on each subinterval $I_j(\Delta) = [x_{j-1}, x_j]$, $j = 1, \ldots, n+1$, and zero at the endpoints of I. For any $t \in R^1$ the finite element solution $u(\Delta) \in \mathring{S}(\Delta)$ of (4.7) is defined by

$$\int_0^1 [a(u')v' + b(u)v]\,dx = t\int_0^1 fv\,dx, \quad v \in \mathring{S}(\Delta). \tag{4.9}$$

For fixed $j = 1, \ldots, n+1$ let

$$\omega_j(x) = u(\Delta)(x) + z_j q_j(x),\ q_j(x) = (x - x_{j-1})(x_j - x),$$

$$x \in I_j(\Delta), \tag{4.10}$$

where $z_j \in R^1$ is the solution of the scalar nonlinear equation

$$\int_{I_j(\Delta)} [a(\omega_j)q_j' + (b(\omega_j) - tf)q_j]\,dx = 0. \tag{4.11}$$

Then

$$\varepsilon(\Delta) = \left[\sum_{j=1}^{n} |z_j|^2\right]^{1/2} \tag{4.12}$$

represents an *a posteriori* estimate for the error between $u(\Delta)$ and the exact solution u^* of (4.7) under the seminorm $\|(u - u(\Delta))'\|_{L_2(I)}$ which—under proper conditions about a, b—is equivalent to the norm of the Sobolev space $\mathring{H}_2^1(I)$.

Evidently (4.9) defines the finite dimensional problem (1.3) for the particular mesh (4.8) to which we apply our continuation process. At each continuation step the error estimate (4.12) can be computed. Then, in line with the mesh-refinement algorithm discussed in [5] (see also [17]) those subintervals $I_j(\Delta)$ are halved for which $|z_j|$ exceeds a certain tolerance. On the new mesh

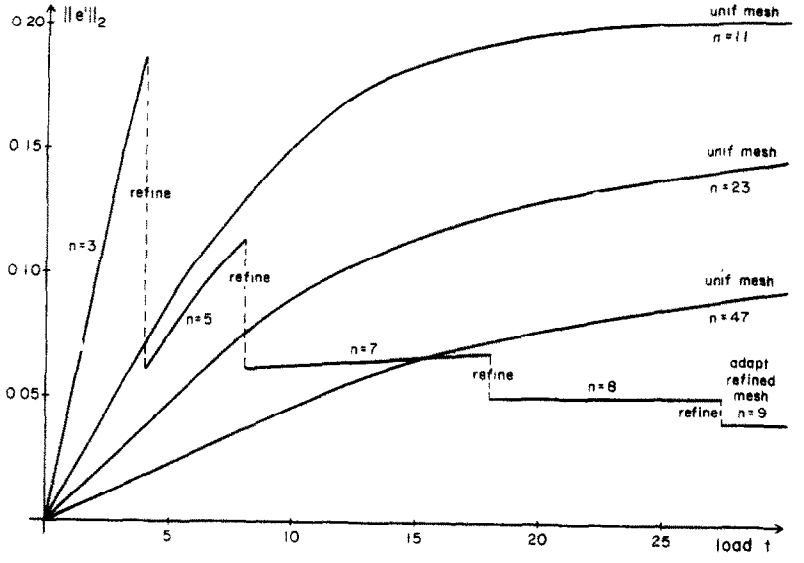

Fig. 6.

an approximate solution is obtained by interpolation and then corrected by means of a few Newton steps. Now the continuation process is applied again until a further refinement becomes necessary.

Some numerical results with this combined process have been given in [5] and are summarized in Fig. 6 for the problem (4.7) with

$$a(s)=\frac{s}{1+s}, \quad -1<s<\infty; \; b(s)\equiv 0, \quad s\in R^1$$

$$f(s)=\begin{cases}1, & \text{for } \frac{1}{4}<s<\frac{3}{4}\\ 0, & \text{otherwise}\end{cases}$$

This may be viewed as a model of a one-dimensional rod of length one which is clamped at the endpoints and subjected to a load in the direction of its axis specified by tf. The effectiveness of the procedure is certainly evident.

We end our discussion with some words about the general continuation problem (1.5) involving more than one parameter. As mentioned before, the solution then is a q-dimensional manifold in R^{n+q}, the equilibrium surface of the structure. Various authors have used perturbation techniques to analyse local features of this surface (see [19] and the many references given there, as well as [9, 12] for the same approach in a different context). In essence, these techniques allow for an analysis of the behavior of a given path on the equilibrium surface under certain perturbations of a parameter. In [19] these techniques are placed in the general context of the study of the equilibrium surface itself. This permitted a broader exploration of the close relationship between the shape of the surface and the structural stability of the mechanical system. In a more general setting these ideas are pursued further in [20, 21].

These studies certainly suggest the possibility of analyzing the structural stability of a system by exploring numerically the shape of the equilibrium surface, that is, by computing appropriate paths on the surface. A natural aim, for example, is to find and trace any paths on the surface consisting entirely of limit-points with respect to a given parameter. Since the continuation process discussed in Section 3 is not affected by such limit points, this task has certainly become feasible.

For small-dimensional problems with two parameters u_1, u_2 such a trace of the limit-point curves has been successfully accomplished. Basically, from a given starting point a primary curve is followed along which, say, u_2 is constant. As soon as a limit point with respect to u_1 is encountered the trace is turned to follow the curve of these limit points. The technical difficulty is here to specify an appropriate auxiliary equation which defines this curve on the equilibrium surface. It is easy to identify such equations, but in most cases their forms are numerically cumbersome. This was the reason for the restriction to small dimensional problems. Yet the results were in all cases highly interesting and encouraging. It is hoped that the indicated technical difficulties can be removed to provide a more general numerical approach to the study of the structural stability of complex mechanical systems.

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