Dynamic Analysis of Structures with Interval Uncertainty

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Abstract: A new method for dynamic response spectrum analysis of a structural system with interval uncertainty is developed. This interval finite-element–based method is capable of obtaining the bounds on dynamic response of a structure with interval uncertainty. The present method is performed using a set-theoretic (interval) formulation to quantify the uncertainty present in the structure's parameters, such as material properties and cross-sectional geometry. Independent and/or dependent variations for each element of the structure are considered. At each stage of analysis, the existence of variation is considered as presence of the perturbation in a pseudodeterministic system. Having this consideration, first, a linear interval eigenvalue problem is performed using the concept of monotonic behavior of eigenvalues for symmetric matrices subjected to nonnegative definite perturbations. Then, using the procedures for perturbation of invariant subspaces of matrices, the bounds on directional deviation (inclination) of each mode shape are obtained. Following this, the interval response spectrum analysis is performed considering the effects of input variation in terms of the structure's total response, which includes maximum modal coordinates, modal participation factors, and mode shapes. Using this method, for the problems considered, it is shown that calculating the bounds on the dynamic response is more computationally efficient than the combinatorial or Monte Carlo–solution procedures. Several problems that illustrate the behavior of the method and comparison with combinatorial and Monte Carlo–simulation results are presented. **DOI:** 10.1061/(ASCE)EM.1943-7889.0000660.

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Introduction

The dynamic analysis of a structure is an essential procedure to design a reliable structure subjected to dynamic loads (such as earthquake excitations). The objective of dynamic analysis is to determine the structure's response and interpret those theoretical results to design the structure. Dynamic response spectrum analysis is one method of dynamic analysis that predicts the structure's response using the combination of modal maxima.

Throughout conventional dynamic response spectrum analysis, the possible existence of any uncertainty present in the structure's geometric and/or material characteristics is not considered. However, in the design process, the presence of uncertainty is accounted for by considering a combination of load-amplification and strength-reduction factors that are obtained by modeling of historic data. In the presence of uncertainty in the geometric and/or material properties of the system, an uncertainty analysis should be performed to obtain bounds on the structure's response.

Uncertainty analysis on the dynamics of a structure requires two major considerations: first, modifications on the representation of the characteristics attributable to the existence of uncertainty; and second, development of schemes that are capable of considering the presence of uncertainty throughout the solution process. Those developed schemes must be consistent with the system's physical behavior and must also be computationally feasible.

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The set-theoretic (unknown but bounded) or interval representation of vagueness is one possible method to quantify the uncertainty present in a physical system. The interval representation of uncertainty in the parametric space has been motivated by the lack of detailed probabilistic information on possible distributions of parameters and/or computational issues in obtaining solutions.

In this work, a new method for dynamic response spectrum analysis of a structural system with interval uncertainty, entitled interval response spectrum analysis (IRSA), is developed. The IRSA enhances the deterministic dynamic response spectrum analysis by including the presence of uncertainty at each step of the analysis procedure. In this finite-element–based method, uncertainty in the elements is viewed by a closed-set representation of element parameters that can vary within intervals defined by extreme values. This representation transforms the point values in the deterministic system to inclusive sets of values in the system with interval uncertainty.

The concepts of matrix perturbation theories are used to find the bounds on the intervals of the terms involved in the modal contributions to the total structure's response, including circular natural frequencies, mode shapes, and modal coordinates. Having the bounds on those terms, the bounds on the total response are obtained using interval calculations. Functional dependency and/or independency of intervals of uncertainty are considered to attain sharper results. The IRSA can calculate the bounds on the dynamic response without combinatorial or Monte Carlo–simulation procedures. This computational efficiency makes IRSA an attractive method to introduce uncertainty into dynamic analysis.

This work represents the synthesis of two historically independent fields: structural dynamics and interval analysis. To represent the background for this work, a review of development of both fields is presented. First, the analytical procedure for deterministic dynamic analysis is presented. Next, the fundamentals of uncertainty analyses with emphasis on the interval method are discussed. Subsequently, the method of IRSA is introduced. Following that, the bounds on variations of natural frequencies and mode shapes are obtained using matrix perturbation theories for eigenvalues and eigenvectors. Then, the bounds on the total dynamic response of the structure are determined. Finally, exemplars and numerical results are presented that are followed by observations and conclusions.

Historical Background

Structural Dynamics

Theories of structural dynamics were introduced mostly in the mid–twentieth century. Biot (1932) introduced the concept of earthquake response spectra, and Housner (1959) was instrumental in the widespread acceptance of this concept as a practical means of characterizing ground motions and their effects on structures. Newmark (1962) introduced computational methods for structural dynamics and earthquake engineering. In 1959, he developed a family of time-stepping methods based on variation of acceleration over a time step. Anderson et al. (1952) developed methods for considering the effects of lateral forces on structures induced by earthquake and wind, and Looney (1954) studied the behavior of structures subjected to forced vibrations. Also, Hudson (1956) developed techniques for response spectrum analysis in engineering seismology. Veletsos and Newmark (1957) determined natural frequencies of continuous flexural members. Moreover, they investigated the deformation of nonlinear systems caused by dynamic loads. Rosenblueth and Bustamente (1962) introduced methods for combining modal responses and characterizing earthquake analysis. Biggs (1964) developed dynamic analyses for structures subjected to blast loads. Moreover, Clough and Penzien (1993) further developed numerical methods for dynamics of structures and modal analysis.

Review of Response Spectrum Analysis

The method of response spectrum analysis for computing the dynamic response of a multiple-degree-of-freedom (DOF) structure to a system of dynamic loads can be sequenced as follows:

1. Define the structural properties:
   - Determine the stiffness matrix $[K]$ and mass matrix $[M]$; and
   - Assume the modal damping ratio ζ_n.
2. Perform a generalized eigenvalue problem between the stiffness and mass matrices:
   - Determine natural circular frequencies (ω_n); and
   - Determine mode shapes $\{\varphi_n\}$.
3. Compute the maximum modal response:
   - Determine the maximum modal coordinate $D_{n,\max}$ using the excitation response spectrum for the corresponding natural circular frequency and modal damping ratio;
   - Determine the modal participation factor Γ_n; and
   - Compute the maximum modal response as a product of maximum modal coordinate, modal participation factor, and mode shape.
4. Combine the contributions of all maximum modal responses to determine the maximum total response using square root of sum of squares (SRSS) or other combination methods (Clough and Penzien 1993).

Interval Analysis

The concept of representation of an imprecise real number by its bounds is quite old. In fact, Archimedes (Heath 1897) defined the irrational number (π) by an interval, $[3+(10/71)<\pi<3+(1/7)]$, which he found by approximating the circle with the inscribed and circumscribed 96-side regular polygons. Early work in modern interval analysis was performed by Young (1931), who developed a formal algebra of multivalued numbers. Also, the special case of multivalued functions with closed intervals was discussed by Dwyer (1951). The introduction of digital computers in the 1950s provided impetus for further interval analysis as discrete representations of real numbers with associated truncation error. Interval mathematics was further developed by Sunaga (1958), who introduced the theory of interval algebra and its applications in numerical analysis. Also, Moore (1966) introduced interval analysis, interval vectors, and interval matrices as a set of techniques that provides error analyses for computational results.

Interval analysis provides a powerful set of tools with direct applicability to important problems in scientific computing. Alefeld and Herzberger (1983) presented an extensive treatment of interval linear and nonlinear algebraic equations and interval methods for systems of equations. Moreover, Neumaier (1990) investigated the methods for solution of interval systems of equations.

The concept of interval systems has been further developed in analysis of structures with interval uncertainty. Qiu and Elishakoff (1998) developed a method for antioptimization of structures with nonrandom uncertain parameters using interval analysis. Muhanna and Mullen (1999) developed fuzzy FEMs for solid mechanics problems. For the solution of interval FEM (IFEM) problems, Muhanna and Mullen (2001) introduced an element-by-element interval finite-element formulation, in which a guaranteed enclosure for the solution of interval linear systems of equations was achieved. The research in interval eigenvalue problem began to emerge as its wide applicability in science and engineering was realized. Dief (1991) presented a method for computing interval eigenvalues of an interval matrix based on an assumption of invariance properties of eigenvectors. Using Dief's method, the lower eigenvalues have a wider range of uncertainty than the exact results. The concept of the interval eigenvalue problem has been applied to dynamics of structures with uncertainty. Modares et al. (2006) have introduced a method for the solution of the parametric interval eigenvalue problem resulting from semidiscretization of structural dynamics that determines the exact bounds of the natural frequencies of a structure. Moens and Vandepitte (2007) applied interval sensitivity theory to frequency response envelope analysis of uncertain structures.

The interval approach to the uncertain problems is to model the structural parameters as interval quantities. In this method, uncertainty in the elements is viewed by a closed-set representation of element parameters that can vary within intervals between extreme values. Then, structural analysis is performed using interval operations.

Interval (Convex) Number

A real interval is a closed set defined by extreme values as (Fig. 1)

$$\tilde{Z}=\left[z^{l},z^{u}\right]=\left\{z\in\Re\middle|z^{l}\leq z\leq z^{u}\right\} \tag{1}$$

One interpretation of an interval number is a random variable for which the probability density function is unknown but nonzero only in the range of interval. Another interpretation of an interval number includes intervals of confidence for α-cuts of fuzzy sets. The interval

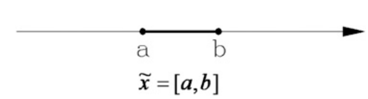

Fig. 1. Interval quantity

representation transforms the point values in the deterministic system to inclusive set values in the system with bounded uncertainty. Therefore, interval arithmetic is a computational tool that can be used to represent uncertainty (1) as a set of probability density functions; (2) in Dempster-Shafer models for epistemic probability; and (3) as α-cuts in fuzzy sets, for example. In this work, the symbol ($\sim$) represents an interval quantity. Although interval methods ensure all solutions are contained within the computed interval, their usefulness in engineering requires that an interval value not be wider than needed to enclose the possible results. There are several factors that can produce such overestimation of interval widths. For example, the effect of functional dependency of interval operations is considered. If $\tilde{X} = [-2,2]$ and $\tilde{Y} = [-2,2]$ are two independent interval numbers, the functional dependent interval multiplication results in

$$\tilde{X} \times \tilde{X} = [0,4]$$

In contrast, the functional independent interval multiplication results in

$$\tilde{X} \times \tilde{Y} = [-4,4]$$

The calculation of exact sharp bounds to the interval system of equations resulting from linear static analysis using the FEM has been proved to be computationally a nondeterministic polynomial-time hard (NP-hard) problem. However, even the 2^n combinations of upper and lower bounds do not always yield the bounds. In problems with narrow intervals associated with truncation errors, the naïve implementation of interval arithmetic will yield acceptable bounds. However, for wider intervals representing uncertainty in parameters, the naïve method will overestimate the bounds by several orders of magnitude. Successful applications of the interval method in the linear static problem have required the development of new algorithms that are computationally feasible yet still provide nearly sharp bounds (Muhanna and Mullen 2001).

Formulation of IRSA

The method for interval dynamic analysis named as IRSA is composed of the following steps:

1. Define the uncertain physical or geometrical characteristics with closed intervals:
   - Determine the interval stiffness matrix $[\tilde{K}]$ and interval mass matrix $[\tilde{M}]$; and
   - Assume the modal damping ratio ζ_n or $\tilde{\zeta}_n$.
2. Perform an interval eigenvalue problem between the interval stiffness and interval mass matrices:
   - Determine the bounds on natural circular frequencies $\tilde{\omega}_n$ (interval natural frequencies); and
   - Determine the bounds on mode shapes $\{\tilde{\varphi}_n\}$ (interval mode shapes).
3. Compute the maximum modal response:
   - Determine the interval modal coordinate $\tilde{D}_n$ and the maximum modal coordinate $D_{n,\max}$ using the excitation response spectrum for the bounds of corresponding natural circular frequency and assumed modal damping ratio or interval modal damping ratio;
   - Determine the interval modal participation factor $\tilde{\Gamma}_n$; and
   - Compute the maximum modal response as the product of the maximum modal coordinate, the interval modal participation factor, and the interval mode shape.
4. Combine the contributions of all maximum modal responses to determine the maximum total response using SRSS or other combination methods.

Interval Stiffness and Mass Matrices

The structure's noninterval global stiffness matrix can be viewed as a linear summation of the element contributions to the global stiffness matrix

$$[K] = \sum_{i=1}^{n} [L_i][K_i][L_i]^T \tag{2}$$

where $[L_i]$ = element Boolean connectivity matrix; and $[K_i]$ = element stiffness matrix in the global coordinate system (a geometric second-order tensor transformation may be required from the element local coordinates to the structure's global coordinates). Considering the presence of uncertainty in the stiffness characteristics, the interval element stiffness matrix is expressed as

$$\left[\tilde{K}_i\right] = ([l_i, u_i])[K_i] \tag{3}$$

where $[l_i, u_i]$ = interval number that premultiplies the noninterval element stiffness matrix.

Considering the variation as a multiplier outside of the stiffness matrix preserves the element physical characteristics, such as real natural frequencies and rigid body modes, as well as stiffness matrix properties, such as symmetry and positive semidefiniteness. In terms of the physics of the system, this means that the stiffness within each element is unknown but bounded and has a unique value that can vary from the stiffness of other elements.

This parametric form must be used to preserve sharp interval bounds. The uncertainty in each element's stiffness is assumed to be an independent value contained within the prescribed interval. For coupled elements, matrix decompositions can be used. For instance, in a beam-column, if functional independent values of axial and bending properties are uncertain, the axial and bending components can be additively decomposed as

$$\left[\tilde{K}_i\right] = \left([l_i, u_i]_{\text{axial}}\right)[K_i]_{\text{axial}} + \left([l_i, u_i]_{\text{bending}}\right)[K_i]_{\text{bending}} \tag{4}$$

Likewise, for continuum problems with functional independent uncertain properties at integration points, the contribution of each integration point can be assembled independently.

The structure's global stiffness matrix in the presence of interval uncertainty is the linear summation of the contributions of nondeterministic interval element stiffness matrices

$$\left[\tilde{K}\right] = \sum_{i=1}^{n} [L_i]\left([l_i, u_i]\right)[K_i][L_i]^T \tag{5}$$

or

$$\left[\tilde{K}\right] = \sum_{i=1}^{n} ([l_i, u_i])[L_i][K_i][L_i]^T = \sum_{i=1}^{n} ([l_i, u_i])\left[\overline{K}_i\right] \tag{6}$$

where $\left[\overline{K}_i\right]$ = deterministic element stiffness contribution to the global stiffness matrix. Similarly, the structure's deterministic global mass matrix is viewed as a linear summation of the element contributions to the global mass matrix as

$$[M] = \sum_{i=1}^{n} [L_i][M_i][L_i]^T \tag{7}$$

where $[M_i]$ = element stiffness matrix in the global coordinate system.

Considering the presence of uncertainty in the mass properties, the nondeterministic element mass matrix is

$$\left[\tilde{M}\right]_i = ([l_i, u_i])[M_i] \tag{8}$$

where $[l_i,\ u_i]$ = interval number that premultiplies the deterministic element mass matrix. Considering the variation as a multiplier outside of the mass matrix preserves the element physical properties. Analogous to the interval stiffness matrix, this procedure preserves the physical and mathematical characteristics of the mass matrix.

The structure's global mass matrix in the presence of any uncertainty is the linear summation of the contributions of nondeterministic interval element mass matrices

$$\left[\tilde{M}\right] = \sum_{i=1}^{n} [L_i]([l_i, u_i])[M_i][L_i]^T \tag{9}$$

or

$$\left[\tilde{M}\right] = \sum_{i=1}^{n} ([l_i, u_i])[L_i][M_i][L_i]^T = \sum_{i=1}^{n} ([l_i, u_i])\left[\overline{M}_i\right] \tag{10}$$

where $[\overline{M}_i]$ = deterministic element mass contribution to the global mass matrix.

Interval Eigenvalue Problem for Structural Dynamics

For dynamics problems, the interval generalized eigenvalue problem between the interval stiffness and mass matrices can be set up by using the interval global stiffness and mass matrices [Eqs. (7) and (10)]. Therefore, the nondeterministic interval eigenvalue problem is obtained as

$$\left(\sum_{i=1}^{n} ([l_i, u_i])\left[\overline{K}_i\right]\right)\left\{\tilde{\varphi}\right\} = \left(\tilde{\omega}^2\right)\left(\sum_{i=1}^{n} ([l_i, u_i])\left[\overline{M}_i\right]\right)\left\{\tilde{\varphi}\right\} \tag{11}$$

Hence, determination of bounds on natural frequencies in the presence of uncertainty can be mathematically interpreted as performing an interval eigenvalue problem on the interval-set-represented nondeterministic stiffness and mass matrices. Two solutions of interest are as follows:

1. $(\tilde{\omega})$: Interval natural frequencies or bounds on variation of circular natural frequencies; and
2. $\{\tilde{\varphi}\}$: Interval mode shapes or bounds on directional deviation of mode shapes.

Although the element mass matrix contribution can also have interval uncertainty, in this work only problems with interval stiffness properties are addressed.

The interval eigenvalue problem for a structure with stiffness properties expressed as interval values is

$$\left(\sum_{i=1}^{n} ([l_i, u_i])\left[\overline{K}_i\right]\right)\left\{\tilde{\varphi}\right\} = \left(\tilde{\omega}^2\right)[M]\left\{\tilde{\varphi}\right\} \tag{12}$$

This interval eigenvalue problem can be transformed to a pseudodeterministic eigenvalue problem subjected to a matrix perturbation. Introducing the central and radial (perturbation) stiffness matrices as

$$[K_C] = \sum_{i=1}^{n}\left(\frac{l_i + u_i}{2}\right)\left[\overline{K}_i\right] \tag{13}$$

$$\left[\tilde{K}_R\right] = \sum_{i=1}^{n}\left(\tilde{\varepsilon}_i\right)\left(\frac{u_i - l_i}{2}\right)\left[\overline{K}_i\right], \quad \tilde{\varepsilon}_i = [-1,1] \tag{14}$$

Using Eqs. (13) and (14), the nondeterministic interval eigenpair problem, Eq. (12), becomes

$$\left([K_C] + \left[\tilde{K}_R\right]\right)\left\{\tilde{\varphi}\right\} = \left(\tilde{\omega}^2\right)[M]\left\{\tilde{\varphi}\right\} \tag{15}$$

Therefore, the determination of bounds on natural frequencies and bounds on mode shapes of a system in the presence of uncertainty in the stiffness properties is mathematically interpreted as an eigenvalue problem on a central stiffness matrix $([K_C])$ that is subjected to a radial perturbation stiffness matrix $([\tilde{K}_R])$. This perturbation is, in fact, a linear summation of nonnegative definite deterministic element stiffness contribution matrices that are scaled with bounded real numbers $(\tilde{\varepsilon}_i)$.

Determination of Eigenvalue Bounds (Interval Natural Frequencies)

In the presence of any interval uncertainty in the characteristics of structural elements, the solutions to only two noninterval eigenvalue problems, Eqs. (16) and (17), are sufficient to bound the natural frequencies of the structure (Modares et al. 2006). The noninterval eigenvalue problems corresponding to the maximum and minimum natural frequencies are obtained as

$$\left(\sum_{i=1}^{n} (u_i)\left[\overline{K}_i\right]\right)\left\{\tilde{\varphi}\right\} = \left(\omega_{\max}^2\right)[M]\left\{\tilde{\varphi}\right\} \tag{16}$$

$$\left(\sum_{i=1}^{n} (l_i)\left[\overline{K}_i\right]\right)\left\{\tilde{\varphi}\right\} = \left(\omega_{\min}^2\right)[M]\left\{\tilde{\varphi}\right\} \tag{17}$$

This means that in the presence of any interval uncertainty in the stiffness of structural elements, the exact upper bounds of natural frequencies are obtained by using the upper values of stiffness for all elements in a deterministic generalized eigenvalue problem. Similarly, the exact lower bounds of natural frequencies are obtained by using the lower values of stiffness for all elements in another deterministic generalized eigenvalue problem. The methodology is developed using the concept of monotonic behavior of eigenvalues for symmetric matrices subjected to nonnegative definite perturbation, which leads to a computationally efficient procedure to determine the bounds on a structure's natural frequencies. Based on the given mathematical proof, the obtained bounds on natural frequencies are exact.

Determination of Eigenvector Bounds (Interval Mode Shapes)

If $[\tilde{A}]$ is an interval real symmetric matrix $(\tilde{A} \in \mathfrak{R}^{n\times n})$, and $[A]$ is a member of the interval matrix $(A \in \tilde{A})$ or in terms of components $(a_{ij} \in \tilde{a}_{ij})$, the interval eigenvalue problem is shown as

$$\left([A]-\lambda[I]\right)\{x\}=0,\left(A\in\tilde{A}\right) \tag{18}$$

The solution of interest to the real interval eigenvalue problem for bounds on each eigenvector is defined as an inclusive set of real values of vector $\{\tilde{x}\}$ such that for any member of the interval matrix, the eigenvector solution to the problem is a member of the solution set. Thus, the solution to the interval eigenvalue problem for each eigenvector is

$$\left\{\{x\}\in\{\tilde{x}\}\middle|\forall A\in\tilde{A},\lambda:\left([A]-\lambda[I]\right)\{x\}=0\right\} \tag{19}$$

To bound the eigenvectors, the concept of perturbation of invariant subspaces is employed. A perturbation of an eigenvector problem will result in perturbation being contained in an invariant subspace of the matrix.

Invariant Subspace

The subspace χ is defined to be an invariant subspace of matrix $[A]$ if (Stewart and Sun 1990)

$$A\chi\subset\chi \tag{20}$$

This means that if χ is an invariant subspace of $[A]_{n\times n}$ and, also, columns of $[X_1]_{n\times m}$ form a basis for χ, then there is a unique matrix $[L_1]_{m\times m}$ such that

$$[A][X_1]=[X_1][L_1] \tag{21}$$

The matrix $[L_1]$ is the representation of $[A]$ on χ with respect to the basis $[X_1]$, and the eigenvalues of $[L_1]$ are a subset of eigenvalues of $[A]$. Therefore, for the invariant subspace, $(\{v\},\lambda)$ is an eigenpair of $[L_1]$ if and only if $(\{[X_1]\{v\}\},\lambda)$ is an eigenpair of $[A]$.

Theorem of Invariant Subspaces

For a real symmetric matrix $[A]$, considering the subspace χ with the linearly independent columns of $[X_1]$ forming a basis for χ and the linearly independent columns of $[X_2]$ spanning the complementary subspace $\chi^{\perp}$, then χ is an invariant subspace of $[A]$ if and only if

$$[X_2]^T[A][X_1]=[0] \tag{22}$$

Therefore, invoking the necessary and sufficient condition and postulating the definition of invariant subspaces, the symmetric matrix $[A]$ can be reduced to a diagonalized form using a unitary similarity transformation as

$$[X_1X_2]^T[A][X_1X_2]=\begin{bmatrix}[X_1]^T[A][X_1] & [X_1]^T[A][X_2]\\ [X_2]^T[A][X_1] & [X_2]^T[A][X_2]\end{bmatrix}=\begin{bmatrix}[L_1] & [0]\\ [0] & [L_2]\end{bmatrix} \tag{23}$$

where $[L_i]=[X_i]^T[A][X_i]$; and $i=1,2$.

An invariant subspace is simple if the eigenvalues of its representation $[L_1]$ are distinct from other eigenvalues of $[A]$. Thus, using the reduced form of $[A]$ with respect to the unitary matrix $[[X_1][X_2]]$, χ is a simple invariant subspace if the eigenvalues of $[L_1]$ and $[L_2]$ are distinct:

$$\lambda\left([L_1]\right)\cap\lambda\left([L_2]\right)=\varnothing \tag{24}$$

The symmetric matrix $[A]$ can be decomposed as the summation of contributions of simple invariant subspaces χ and $\chi^{\perp}$ as

$$[A]=[X_1][L_1][X_1]^T+[X_2][L_2][X_2]^T \tag{25}$$

which is the spectral resolution of the matrix $[A]$ into two complementary invariant subspaces.

Considering the projection matrices $[P_i]=[X_i][X_i]^T$, $i=1,2$, with properties as

$$[P_i]^2=[P_i]\quad(i=1,2) \tag{26a}$$
$$[P_1][P_2]=[P_2][P_1]=[0] \tag{26b}$$
$$[A]=[P_1][A][P_1]+[P_2][A][P_2] \tag{26c}$$

Hence, any vector $\{z\}$ can be decomposed into the sum of two vectors, $\{z\}=\{x_1\}+\{x_2\}$ and $(\{x_1\}\in\chi,\{x_2\}\in\chi^{\perp})$, in which the decomposed component vectors are obtained using projection matrices as

$$\{x_1\}=[P_1]\{z\} \tag{27}$$

$$\{x_2\}=\left([I]-[P_1]\right)\{z\}=[P_2]\{z\} \tag{28}$$

which are known as spectral projections of simple invariant subspaces.

Perturbation of Simple Invariant Subspaces

Considering the column spaces of $[X_1]$ and $[X_2]$ to span two complementary simple invariant subspaces, the perturbed orthogonal subspaces are defined as (Stewart and Sun 1990)

$$\left[\hat{X}_1\right]=[X_1]+[X_2][P] \tag{29}$$

$$\left[\hat{X}_2\right]=[X_2]+[X_1][P]^T \tag{30}$$

where $[P]$ = a matrix to be determined. Thus, each perturbed subspace is defined as a summation of the exact subspace and the contribution of the complementary subspace.

Performing inner products on each perturbed subspace, Eqs. (29) and (30), as

$$\left[\hat{X}_1\right]^T\left[\hat{X}_1\right]=\left([I]+[P]^T[P]\right) \tag{31}$$

$$\left[\hat{X}_2\right]^T\left[\hat{X}_2\right]=\left([I]+[P][P]^T\right) \tag{32}$$

the perturbed complementary subspaces can be orthonormalized as

$$\left[\hat{X}_1\right]=([X_1]+[X_2][P])\left([I]+[P]^T[P]\right)^{-1/2} \tag{33}$$

$$\left[\hat{X}_2\right]=\left([X_2]+[X_1][P]^T\right)\left([I]+[P][P]^T\right)^{-1/2} \tag{34}$$

where the redefined perturbed subspaces have orthonormal columns.

Perturbation Problem

Considering a symmetric perturbation $[E]$, the perturbed matrix is defined as

$$\left[\hat{A}\right]=[A]+[E] \tag{35}$$

Performing the similarity transformation on the symmetric perturbed matrix $[\tilde{A}]$ using the unitary matrix $[[\hat{X}_1][\hat{X}_2]]$ obtained from the orthonormalized perturbed subspaces as

$$\left[\hat{X}_1\hat{X}_2\right]^T\left[\hat{A}\right]\left[\hat{X}_1\hat{X}_2\right] = \begin{bmatrix} \left[\hat{L}_1\right] & \left[\hat{G}\right]^T \\ \left[\hat{G}\right] & \left[\hat{L}_2\right] \end{bmatrix} \tag{36}$$

Then, using the theorem of invariant subspaces, $[\hat{X}_1]$ is an invariant subspace if and only if

$$\left[\hat{G}\right] = \left[\hat{X}_2\right]^T\left[\hat{A}\right]\left[\hat{X}_1\right] = [0] \tag{37}$$

Substituting the perturbed matrix and perturbed subspaces, Eqs. (33)–(35), and linearizing the result attributable to a small perturbation compared with the unperturbed matrix (which may result in conservative overestimations), Eq. (37) is rewritten as

$$[P][L_1] - [L_2][P] = [X_2]^T[E][X_1] \tag{38}$$

This perturbation problem is an equation for unknown $[P]$ in the form of a Sylvester's equation.

Sylvester's Equation

A Sylvester's equation (Stewart and Sun 1990) is of the form

$$[A][X] - [X][B] = [C] \tag{39}$$

where $[A]_{n\times n}$, $[B]_{m\times m}$, and $[C]_{n\times m}$ = known matrices; and $[X]_{n\times m}$ = unknown matrix to be determined. Equivalently, a linear operator $[T]$ can be defined as

$$[T] = [X]_{n\times m} \to \left([A][X] - [X][B]\right)_{n\times m} \tag{40}$$

The uniqueness of the solution to the Sylvester's equation is guaranteed when the operator $[T]$ is nonsingular. The operator $[T]$ is nonsingular if and only if the eigenvalues of $[A]$ and $[B]$ are distinct

$$\lambda\left([A]\right) \cap \lambda\left([B]\right) = \varnothing \tag{41}$$

Thus, for the perturbation problem, Eq. (38), the uniqueness of the solution matrix $[P]$ is guaranteed if the eigenvalues of $[L_1]$ and $[L_2]$ are distinct, and hence, for the uniqueness, the column spaces of $[X_1]$ and $[X_2]$ must span two simple invariant subspaces [Eq. (24)].

Perturbation of Eigenvectors

The perturbation of the first eigenvector, using Eq. (29), is defined as

$$\left\{\hat{x}_1\right\} = \{x_1\} + [X_2]\{p\} \tag{42}$$

Thus, the perturbation problem, Eq. (38), is considerably simplified as

$$\{p\}\lambda_1 - [L_2]\{p\} = [X_2]^T[E]\{x_1\} \tag{43}$$

because the operator $[T]$ is specialized as $\left(\lambda_1[I] - [L_2]\right)$. If (λ_1) is a simple eigenvalue, the solution for $[p]$ exists and is unique as

$$\{p\} = \left(\lambda_1[I] - [L_2]\right)^{-1}[X_2]^T[E]\{x_1\} \tag{44}$$

Therefore, the perturbed eigenvector is

$$\left\{\hat{x}_1\right\} = \{x_1\} + [X_2]\left(\lambda_1[I] - [L_2]\right)^{-1}[X_2]^T[E]\{x_1\} \tag{45}$$

Determination of Eigenvector Bounds (Interval Mode Shapes)

The perturbed generalized eigenvalue problem for structural dynamics, Eq. (15), can be transformed to a perturbed classic eigenpair problem as

$$\begin{aligned}&\left([M]^{-1/2}[K_C][M]^{-1/2} + [M]^{-1/2}\left[\tilde{K}_R\right][M]^{-1/2}\right)\left\{\tilde{\varphi}\right\}\\&\quad = \left(\tilde{\omega}^2\right)\left\{\tilde{\varphi}\right\}\end{aligned} \tag{46}$$

Hence, the symmetric perturbation matrix is

$$[E] = [M]^{-1/2}\left[\tilde{K}_R\right][M]^{-1/2} \tag{47}$$

Substituting for radial stiffness $[\tilde{K}_R]$, Eq. (14), in Eq. (47), the error matrix becomes

$$[E] = [M]^{-1/2}\left(\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i\right)\left(\frac{u_i - l_i}{2}\right)\left[\overline{K}_i\right]\right)[M]^{-1/2} \tag{48}$$

Using the obtained error matrix in eigenvector perturbation equation for the first eigenvector, Eq. (45), yields the dynamic perturbed mode shape as

$$\begin{aligned}\left\{\tilde{\varphi}_1\right\} &= \left\{\tilde{\varphi}_1\right\} + \left([\Phi_2]\left(\omega_1^2[I] - [\Omega_2]^2\right)[\Phi_2]^T\right)^{-1}\\&\quad\times\left([M]^{-1/2}\left(\sum_{i=1}^{n}\left(\tilde{\varepsilon}_i\right)\left(\frac{u_i - l_i}{2}\right)\left[\overline{K}_i\right]\right)[M]^{-1/2}\right)\left\{\tilde{\varphi}_1\right\}\end{aligned} \tag{49}$$

where $\{\tilde{\varphi}_1\}$ = first mode shape; (ω_1) = first natural circular frequency; $[\Phi_2]$ = matrix of remaining mode shapes; and $[\Omega_2]$ = diagonal matrix of remaining natural circular frequencies obtained from the unperturbed eigenvalue problem. For simplicity, Eq. (49) can be written as

$$\left\{\tilde{\varphi}_1\right\} = \left\{\tilde{\varphi}_1\right\} + [C_1]\left(\sum_{i=1}^{n}(\varepsilon_i)[E_i]\right)\left\{\tilde{\varphi}_1\right\} \tag{50}$$

where $[C_1] = [\Phi_2](\omega_1^2[I] - [\Omega_2]^2)^{-1}[\Phi_2]^T$; and $[E_i] = \left(\frac{u_i - l_i}{2}\right)[M]^{-1/2}[\overline{K}_i][M]^{-1/2}$, $i = 1, \ldots, n$. The procedure is then applied to obtain subsequent eigenvectors.

Bounding Dynamic Response

The interval modal coordinate $\tilde{D}_n$ is determined using the excitation response spectrum evaluated for the corresponding interval of natural circular frequency $\tilde{\omega}_n$ and assumed modal damping ratio (Fig. 2). Having the interval modal coordinate, the maximum (upper bound) modal coordinate $D_{n,\max}$ is determined as

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$$D_{n,\max} = \max\left(\tilde{D}_n\right) \tag{51}$$

In addition, in the presence of interval uncertainty in modal damping ratio, interval modal damping ratio, $\tilde{\zeta}_n$, is obtained using a cluster of response spectra corresponding to the bounds of interval modal damping ratio (Fig. 3).

If excitation is proportional, the interval modal participation factor is obtained as

$$\tilde{\Gamma}_n = \frac{\left\{\tilde{\varphi}_n\right\}^T\{P\}}{M_n} = \frac{\left\{\tilde{\varphi}_n\right\}^T\{P\}}{\left\{\tilde{\varphi}_n\right\}^T[M]\left\{\tilde{\varphi}_n\right\}} \tag{52}$$

The maximum modal response is determined as the maximum of the product of the maximum modal coordinate, the interval modal participation factor, and the interval mode shape as

$$\left\{U_{n,\max}\right\} = \left\{\max\left(D_{n,\max}\tilde{\Gamma}_n\left\{\tilde{\varphi}_n\right\}\right)\right\} \tag{53}$$

To achieve sharper results, functional dependency of intervals in the multiplicative terms must be considered. Maximum modal response, Eq. (53), is expanded using the definitions of the interval mode shapes and the interval modal participation factor, Eqs. (49) and (52), as

$$\begin{aligned}&\left\{U_{n,\max}\right\}\\ &= \left\{\max\left[\left(D_{n,\max}\right)\frac{\{P\}^T\{\varphi_n\}[I] + \sum_{i=1}^{N}(\varepsilon_i)\left(\{P\}^T\{\varphi_n\}\right)[C_n][E_i] + \sum_{i=1}^{N}(\varepsilon_i)\left(\{P\}^T[C_n][E_i]\{\varphi_n\}\right)[I] + \sum_{i=1}^{N}\sum_{j=1}^{N}(\varepsilon_i)(\varepsilon_j)\left(\{P\}^T[C_n][E_i]\{\varphi_n\}\right)[C_n][E_j]}{\{\varphi_n\}^T[M]\{\varphi_n\} + \{\varphi_n\}^T\left(\sum_{i=1}^{N}(\varepsilon_i)[M][C_n][E_i]\right)\{\varphi_n\} + \{\varphi_n\}^T\left(\sum_{i=1}^{N}(\varepsilon_i)[C_n][E_i][M]\right)\{\varphi_n\} + \{\varphi_n\}^T\left(\sum_{i=1}^{N}\sum_{j=1}^{N}(\varepsilon_i)(\varepsilon_j)[C_n][E_i][M][C_n][E_j]\right)\{\varphi_n\}}\{\varphi_n\}\right]\right\}\end{aligned} \tag{54}$$

Thus, considering the dependency of the intervals of uncertainty for each element, $(\tilde{\varepsilon}_i)$, the sharper results for maximum modal response are obtained.

Finally, the contributions of all maximum modal responses are combined to determine the maximum total response using SRSS [or other combination methods such as complete quadratic combination (CQC)] is

$$\{U_{\max}\} = \sqrt{\sum_{n=1}^{N}\left\{U_{n,\max}^2\right\}} \tag{55}$$

where upper bound of response attributable to presence of uncertainty is obtained.

Numerical Examples

Problem 1: Uncertainty in Stiffness

This example obtains the bounds on dynamic responses for a spring-mass system with fixed supports at both ends with interval uncertainty in the elements' stiffness (Fig. 4). The individual element interval stiffness matrices are

$$\tilde{K}_1 = \tilde{K}_2 = \tilde{K}_3 = \tilde{K}_4 = ([0.99, 1.01])\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}k$$

The system's stiffness mass matrix is

$$M = \mathrm{diag}(1, 1, 1)m$$

The excitation is in the form of a suddenly applied proportional constant load as

$$\{P(t)\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}p$$

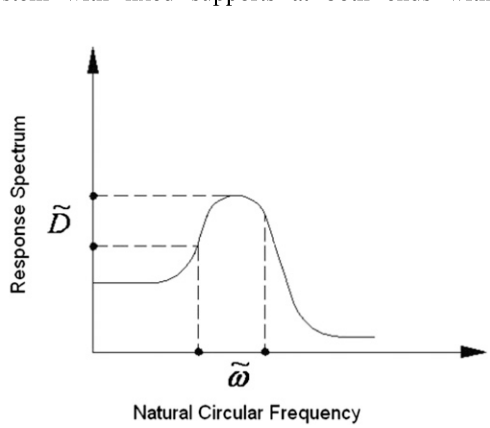

Fig. 2. Determination of $\tilde{D}_n$ corresponding to $\tilde{\omega}_n$ for generic response spectrum

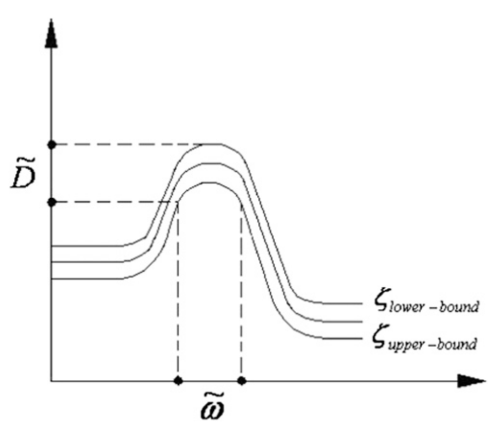

Fig. 3. Determination of $\tilde{D}_n$ corresponding to $\tilde{\omega}_n$ for cluster of response spectra attributable to uncertainty in $\tilde{\zeta}_n$

The response spectrum for this proportional loading is shown in Fig. 5.

The problem is solved using the method of IRSA presented in this work. For comparison, this problem is also solved with two additional methods.

1. Combinatorial analysis (all combination of endpoints): solution to $2^n = 2^4 = 64$ deterministic problems representing the interval bounds; and
2. Monte Carlo simulation: performing 10^7 simulations using independent uniformly distributed random variables.

Also, the convergence behavior of Monte Carlo simulation to the combinatorial for the bounds on displacement of the first node is depicted in Fig. (6). Clearly, the Monte Carlo method starts converging around 10^3-10^4 simulations.

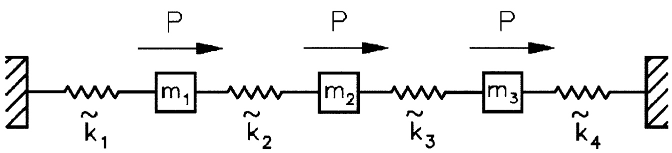

Fig. 4. Structure of multiple-DOF spring-mass system

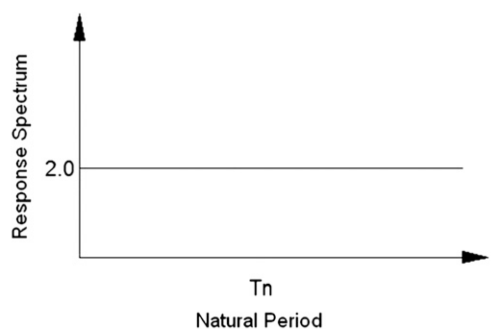

Fig. 5. Response spectrum for external excitation

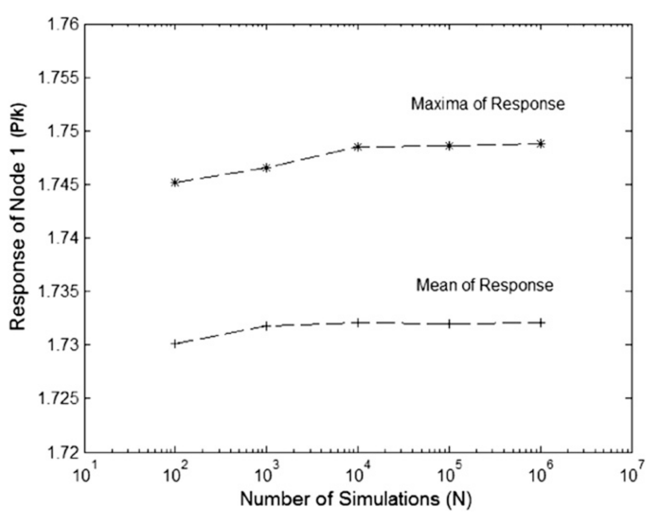

Fig. 6. Convergence of Monte Carlo simulation

The results for nodal displacements are summarized in Table 1. Problem 1 is redefined in different ways and solved using IRSA to investigate the behavior of the algorithm as subsequently described.

Computation Time. Three problems similar to Problem 1 with 3, 4, and 5 DOF, using IRSA and the elapsed time for each problem (using a computer with 1.7-GHz processor and 512-MB memory), are recorded and shown in Table 2 and plotted in logarithmic scale in Fig. (7).

The slope of the diagram in Fig. (7) is approximately 1.2. This means that computation time for this problem using IRSA method increases between linear to quadratic with increasing the number of DOF.

Output Width as Function of Initial Width. Problem 1 is solved with different input variations in elements' stiffness, and the results are compared with the combinatorial solution. The overestimation in IRSA, measured as the difference of IRSA value with combinatorial value divided by combinatorial value, is depicted in Fig. (8). This shows a linear increase in overestimation of output results for IRSA compared with the combinatorial solution.

Problem 2: Uncertainty in Material

This example problem solves for the dynamic response a two-dimensional (2D) cross-braced truss system subjected to an earthquake excitation with uncertainty in the modulus of elasticity (Fig. 9). The cross-sectional area $A = 6.4516 \times 10^{-3}\ \mathrm{m}^2$ (10 sq in.); the floor load $= 5.7456\ \mathrm{N/m}^2$ (0.120 kip/sq in.); the length for horizontal and vertical members $L = 3.6575$ m (12 ft); the Young's moduli E for all elements are $\tilde{E} = [0.99, 1.01]$ 199,948.04 MPa ([0.99, 1.01] 29,000 ksi); and modal damping is $\zeta = 0.02$.

Table 1. Solution to Problem 1

Displacement	IRSA	Combination	Simulation
$U_{1,\max}(P/k)$	1.7993	1.7493	1.7491
$U_{2,\max}(P/k)$	2.4997	2.4577	2.4575
$U_{3,\max}(P/k)$	1.7993	1.7493	1.7493

Table 2. Computation Time of IRSA Method for Problem 1

Degree of freedom	Elapsed time (s)
3	0.797
5	1.452
6	1.797

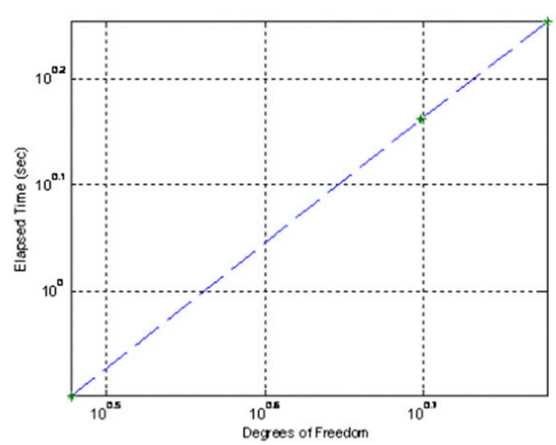

Fig. 7. Computation time for IRSA

The Newmark-Blume-Kapur (Newmark et al. 1973) design spectra are used to obtain interval modal coordinates. The problem is solved using IRSA, and, for comparison, is solved with two alternate methods.

1. Combinatorial analysis (all combination of endpoints): solution to $2^n = 2^{10} = 1{,}024$ deterministic problems; and
2. Monte Carlo simulation: performing 10^4 simulations using independent uniformly distributed random variables.

The results for roof lateral displacements are summarized in Table 3.

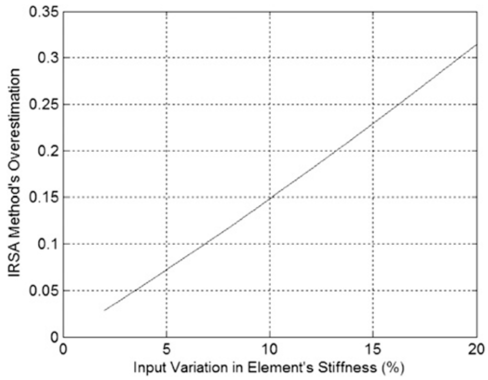

Fig. 8. Comparison of output variation for IRSA with combinatorial solution versus input variation

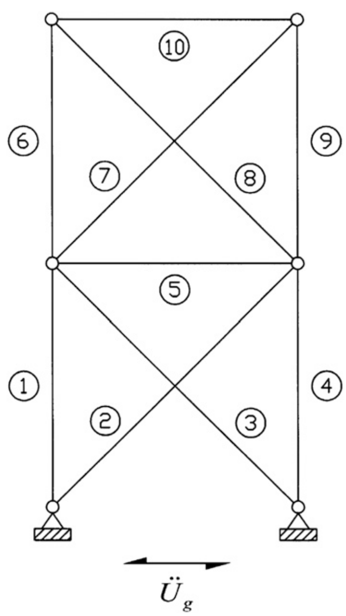

Fig. 9. Structure of 2D cross-braced truss

Observation

Comparing the results for output width as a function of problem size, obtained by two example problems (Problem 1 with overestimation of 2.86%, and Problem 2 with overestimation of 2.36%), it is observed that the overestimation of IRSA method in output results does not increase with increasing the number of elements and DOF.

Problem 3: Uncertainty in Damping

This example problem solves for the dynamic response a truss system subjected to an earthquake excitation with uncertainty in the modal damping ratios (Fig. 10). The cross-sectional area $A = 1.935 \times 10^{-3}\,\mathrm{m}^2$ (3 sq in.); the length for horizontal and vertical members $L = 3.6575\,\mathrm{m}$ (12 ft); the Young's moduli E for all elements are $E = 199{,}948.04\,\mathrm{MPa}$ (29,000 ksi); nodal mass is $m = 1{,}459.39\,\mathrm{kg}$ (100 slug); and interval modal damping ratio is $\tilde{\zeta} = [0.02, 0.07]$.

The Newmark-Blume-Kapur (Newmark et al. 1973) design spectra are used to obtain interval modal coordinates. The problem is solved using IRSA, and, for comparison, this problem is solved with alternate method of Monte Carlo simulation by performing 10^4 simulations using independent uniformly distributed random variables. The results for the structure's displacements response using IRSA and Monte Carlo simulation are summarized in Table 4.

Observation

Comparing the results obtained in Example 3 depicts that in the presence of uncertainty in damping obtained by IRSA and Monte Carlo simulations, the results from IRSA are sharp.

Table 3. Solution to Problem 2

Displacement	IRSA	Combination	Monte Carlo simulation
$U_{\max}$	0.0210 m (0.8294 in.)	0.0205 m (0.8103 in.)	0.0205 m (0.8103 in.)

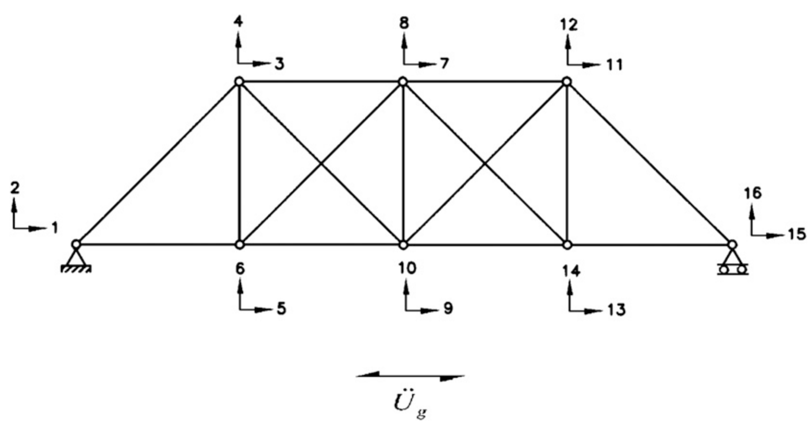

Fig. 10. Structure of truss

Table 4. Solution to Problem 3

Degree of freedom	IRSA Lower bound [10^{-5} m (10^{-3} in.)]	IRSA Upper bound [10^{-5} m (10^{-3} in.)]	Monte Carlo simulation Lower bound [10^{-5} m (10^{-3} in.)]	Monte Carlo simulation Upper bound [10^{-5} m (10^{-3} in.)]
U_3	7.8719 (3.0992)	12.2402 (4.8190)	7.8747 (3.1003)	12.2204 (4.8112)
U_4	9.7365 (3.8333)	15.2735 (6.0132)	9.7386 (3.8341)	15.2725 (6.0128)
U_5	4.4615 (1.7565)	6.7307 (2.6499)	4.46557 (1.7581)	6.7205 (2.6459)
U_6	10.3192 (4.0627)	16.1551 (6.3603)	10.3220 (4.0638)	16.1480 (6.3575)
U_7	7.0688 (2.7830)	10.7939 (4.2496)	7.0736 (2.78490)	10.7762 (4.2426)
U_8	13.0558 (5.1401)	20.4251 (8.0414)	13.0586 (5.1412)	20.4119 (8.0362)
U_9	7.6525 (3.0128)	11.6385 (4.5821)	7.6581 (3.015)	11.6202 (4.5749)
U_{10}	12.8541 (5.0607)	20.0789 (7.9051)	12.8574 (5.062)	20.0594 (7.8974)
U_{11}	7.1478 (2.8141)	10.6649 (4.1988)	7.1551 (2.817)	10.6525 (4.1939)
U_{12}	9.6875 (3.814)	15.1074 (5.9478)	9.6906 (3.8152)	15.0878 (5.9401)
U_{13}	9.5750 (3.7697)	14.7012 (5.7879)	9.5803 (3.7718)	14.6768 (5.7783)
U_{14}	10.5163 (4.1403)	16.3720 (6.4457)	10.5199 (4.1417)	16.3451 (6.4351)
U_{15}	10.7708 (4.2405)	16.6646 (6.5609)	10.7754 (4.2423)	16.6362 (6.5497)

Conclusions

A finite-element–based method for dynamic analysis of structures with interval uncertainty in structure's stiffness properties is presented. In the presence of any interval uncertainty in the characteristics of structural elements, the developed method of IRSA is capable to obtain the nearly sharp bounds on the structure's dynamic response. The IRSA is computationally feasible, and it shows that the bounds on the dynamic response can be obtained without combinatorial or Monte Carlo–simulation procedures.

The solutions to only two noninterval eigenvalue problems are sufficient to bound the natural frequencies of the structure. Based on the given mathematical proof, the obtained bounds on natural frequencies are exact. Computation time for the algorithm increases between linear to quadratic with increasing the number of DOF.

Some conservative overestimation in dynamic response occurs because of linearization in formation of bounds of mode shapes and, also, the dependency of intervals in the dynamic response formulation. These are the expected cause of loss of sharpness in the interval results. The overestimation of output results for IRSA method linearly increases with increasing the number of DOF in comparison with the combinatorial solution.

The solution of the solved problems for dynamic response indicates that the output overestimation does not increase as the problem size increases. In the presence of uncertainty in damping, IRSA yields sharp results. The computational efficiency of IRSA makes it an attractive method to introduce uncertainty into dynamic analysis.

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