

# Signals and Systems - known past paper mistakes

Emilie d'Olne

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## Question 3.(b)(ii)

Consider the discrete signals  $x_1[n] = 2^n$  and  $x_2[n] = 3^n$  for  $n \geq 0$ . Find their convolution using their  $z$ -transforms and properties of convolution.

*Mistake: the final ROC was incorrect.*

Remember that the  $z$ -transform of  $x[n] = a^n u[n]$  is given by

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \frac{1}{1 - az^{-1}} \text{ if } |az^{-1}| < 1 \\ &= \frac{z}{z - a} \text{ if } |z| > |a|. \end{aligned}$$

Thus the convolution is obtained in the  $z$ -domain as

$$\begin{aligned} X_1(z)X_2(z) &= \frac{z}{z-2} \frac{z}{z-3} \text{ with } |z| > 3 \cap |z| > 2 \\ &= z \left( \frac{z}{z-3} - \frac{z}{z-2} \right) = z \left( \frac{z}{z-3} - x_1(0) + x_1(0) - \frac{z}{z-2} \right) \\ &= z \left( \frac{z}{z-3} - x_1(0) - \frac{z}{z-2} + x_2(0) \right) \text{ since } x_1(0) = x_2(0) \\ &= z \left( \frac{z}{z-3} - x_1(0) \right) - z \left( \frac{z}{z-2} - x_2(0) \right) \text{ with } |z| > 3. \end{aligned}$$

Using results from part (i), this gives  $3^{n+1}u[n] - 2^{n+1}u[n]$  in the time domain.

## Question 3.(c)

Consider a LTI system with input  $x[n]$  and output  $y[n]$  related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2].$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its  $z$ -transform in the following three cases

- the system is causal
- the system is stable
- the system is neither causal nor stable.

Use the fact that the  $z$ -transform  $\frac{z}{z-a}$  corresponds to the function  $a^n u[n]$  in discrete time if  $|z| > |a|$  and the function  $-a^n u[-n-1]$  if  $|z| < |a|$ .

*Mistake: the input  $x[n-2]$  leads to a transfer function that does not contain  $\frac{z}{z-a}$  terms. Instead, use  $x[n]$  as input.*

The system can be written in the  $z$ -domain as

$$Y(z)[1 - 3.5z^{-1} + 2z^{-2}] = -7X(z),$$

giving the transfer function

$$H(z) = \frac{-7}{(1 - 0.5z^{-1})(1 - 4z^{-1})} = \frac{1}{1 - 0.5z^{-1}} - \frac{8}{1 - 4z^{-1}} = \frac{z}{z - 0.5} - 8\frac{z}{z - 4}.$$

**The system is causal**

$$h[n] = (0.5)^n u[n] - 8(4)^n u[n], |z| > 4.$$

**The system is stable**

$$h[n] = (0.5)^n u[n] + 8(4)^n u[-n - 1], 0.5 < |z| < 4.$$

**The system is neither stable nor causal**

$$h[n] = -(0.5)^n u[-n - 1] + 8(4)^n u[-n - 1], |z| < 0.5.$$

There is no combination that allows the system to be both stable and causal.

## 2019

### Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input  $x[n]$  and output  $y[n]$  related with the difference equation

$$y[n] - \frac{5}{6}y[n - 1] + \frac{1}{6}y[n - 2] = x[n] - \frac{1}{2}x[n - 1].$$

Find the analytical expression and the ROC of the  $z$ -transform of the output if  $x[n] = (\frac{1}{2})^n u[n]$ .

*Mistake: the final ROC was incorrect.*

From standard  $z$ -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{3}}.$$

Thus the  $z$ -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{3})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}.$$

### Question 2.(d)(ii)

The output  $y(t)$  of an LTI system is related to the input  $x(t)$  through the differential equation

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Let  $X(s)$  and  $Y(s)$  denote the Laplace transforms of  $x(t)$  and  $y(t)$ , respectively, and let  $H(s)$  denote the Laplace transform of the system's impulse response  $h(t)$ .

- Determine  $H(s)$  as a ratio of two polynomials.
- Determine  $h(t)$  for each of the following cases
  - The system is stable.
  - The system is causal.

*Mistake: The answer for a stable system was incorrect.*

The system can be described in the Laplace domain as

$$(s^2 + s - 2)Y(s) = (s + 2)X(s)$$

giving  $H(s)$  as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s + 2}{s^2 + s - 2} = \frac{s + 2}{(s + 2)(s - 1)} = \frac{1}{s - 1}.$$

The last step occurs through pole-zero cancellation. This is beyond the scope of this course, but the analyticity of  $H(s)$  at  $s = 2$  is defined by Riemann's theorem on removable singularities.

For the system to be stable, the ROC must include the  $j\omega$  axis. Thus  $h(t) = -e^t u(-t)$  with ROC  $\Re\{s\} < 1$ . For the system to be causal, the ROC must be to the right of the pole, thus  $h(t) = e^t u(t)$  with ROC  $\Re\{s\} > 1$ .

## 2020 v2

### Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input  $x[n]$  and output  $y[n]$  related with the difference equation

$$y[n] - \frac{5}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - \frac{1}{8}x[n-1].$$

Find the analytical expression and the ROC of the  $z$ -transform of the output if  $x[n] = (\frac{1}{4})^n u[n]$ .

*Mistake: the final ROC was incorrect.*

From standard  $z$ -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{4}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{2}}.$$

Thus the  $z$ -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{4} = |z| > \frac{1}{2}.$$

## 2020 v3

**Question 1.(a)(i)**

Consider  $x(t) = |\sin(2\pi t + \frac{\pi}{3})|$ . Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

*Mistake: the answer said this signal is even, when it is neither even or odd.*

Non causal, periodic with period  $T = 0.5$ , neither even or odd.

**Question 3.(d)(iii)**

Verify the results in part (d)(i) for the sequence  $u[n+2] - u[-n+2]$ .

*Mistake: wrong power of  $z$  in the first step.*

Taking the  $z$ -transform of  $x[n] = u[n+2] - u[-n+2]$

$$X(z) = z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}$$

The expression for  $X(z^{-1})$  is

$$\begin{aligned} X(z^{-1}) &= z^{-2} \frac{z^{-1}}{z^{-1}-1} - z^2 \frac{z}{z-1} \\ &= -\left( z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1} \right) \\ &= -X(z) \end{aligned}$$

Thus proving the results in (d)(i).