

Signals and Systems - known past paper mistakes

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Question 2.(c)(ii)

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system's impulse response. (i) Determine $H(s)$ as a ratio of polynomials. (ii) Determine $h(t)$ for each of the following cases

- The system is stable.
- The system is causal.
- The system is neither stable nor causal.

Mistake: incorrect coefficients in the final solution.

Part (i) gives

$$H(s) = \frac{1}{3} \frac{1}{s-2} - \frac{1}{3} \frac{1}{s+1}.$$

For a **stable** system, the ROC must include the $j\omega$ axis, thus

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t).$$

For a **causal** system, both ROCs must be right-handed, thus

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t).$$

For the system to be **neither** stable nor causal, choose either of

$$\begin{aligned} h(t) &= \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(-t) && \text{ROC doesn't exist!} \\ h(t) &= -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t). \end{aligned}$$

Question 3.(b)(ii)

Consider the discrete signals $x_1[n] = 2^n$ and $x_2[n] = 3^n$ for $n \geq 0$. Find their convolution using their z -transforms and properties of convolution.

Mistake: the final ROC was incorrect.

Remember that the z -transform of $x[n] = a^n u[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$\begin{aligned}
&= \frac{1}{1 - az^{-1}} \text{ if } |az^{-1}| < 1 \\
&= \frac{z}{z - a} \text{ if } |z| > |a|.
\end{aligned}$$

Thus the convolution is obtained in the z -domain as

$$\begin{aligned}
X_1(z)X_2(z) &= \frac{z}{z-2} \frac{z}{z-3} \text{ with } |z| > 3 \cap |z| > 2 \\
&= z \left(\frac{z}{z-3} - \frac{z}{z-2} \right) = z \left(\frac{z}{z-3} - x_1(0) + x_1(0) - \frac{z}{z-2} \right) \\
&= z \left(\frac{z}{z-3} - x_1(0) - \frac{z}{z-2} + x_2(0) \right) \text{ since } x_1(0) = x_2(0) \\
&= z \left(\frac{z}{z-3} - x_1(0) \right) - z \left(\frac{z}{z-2} - x_2(0) \right) \text{ with } |z| > 3.
\end{aligned}$$

Using results from part (i), this gives $3^{n+1}u[n] - 2^{n+1}u[n]$ in the time domain.

Question 3.(c)

Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2].$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its z -transform in the following three cases

- the system is causal
- the system is stable
- the system is neither causal nor causal.

Use the fact that the z -transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ in discrete time if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$.

***Mistake:** the input $x[n-2]$ leads to a transfer function that does not contain $\frac{z}{z-a}$ terms. Instead, use $x[n]$ as input.*

The system can be written in the z -domain as

$$Y(z)[1 - 3.5z^{-1} + 2z^{-2}] = -7X(z),$$

giving the transfer function

$$H(z) = \frac{-7}{(1 - 0.5z^{-1})(1 - 4z^{-1})} = \frac{1}{1 - 0.5z^{-1}} - \frac{8}{1 - 4z^{-1}} = \frac{z}{z - 0.5} - 8 \frac{z}{z - 4}.$$

The system is causal

$$h[n] = (0.5)^n u[n] - 8(4)^n u[n], |z| > 4.$$

The system is stable

$$h[n] = (0.5)^n u[n] + 8(4)^n u[-n-1], 0.5 < |z| < 4.$$

The system is neither stable nor causal

$$h[n] = -(0.5)^n u[-n-1] + 8(4)^n u[-n-1], |z| < 0.5.$$

There is no combination that allows the system to be both stable and causal.

Question 1.(e)

The output of a continuous-time, LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t).$$

Determine the frequency response of the system and sketch the asymptotic behaviour of its Bode plots.

Mistake: Incorrect numerator in the Fourier Transform.

Applying the Fourier Transform on both sides gives

$$Y(\omega)j\omega + 2Y(\omega) = 2X(\omega)$$

thus yielding the frequency response

$$H(\omega) = \frac{2}{2 + j\omega}.$$

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

Find the analytical expression and the ROC of the z -transform of the output if $x[n] = (\frac{1}{2})^n u[n]$.

Mistake: the final ROC was incorrect.

From standard z -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{3}}.$$

Thus the z -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{3})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}.$$

Question 2.(d)(ii)

The output $y(t)$ of an LTI system is related to the input $x(t)$ through the differential equation

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of the system's impulse response $h(t)$.

- Determine $H(s)$ as a ratio of two polynomials.
- Determine $h(t)$ for each of the following cases
 - The system is stable.
 - The system is causal.

Mistake: The answer for a stable system was incorrect.

The system can be described in the Laplace domain as

$$(s^2 + s - 2)Y(s) = (s + 2)X(s)$$

giving $H(s)$ as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s + 2}{s^2 + s - 2} = \frac{s + 2}{(s + 2)(s - 1)} = \frac{1}{s - 1}.$$

The last step occurs through pole-zero cancellation. This is beyond the scope of this course, but the analyticity of $H(s)$ at $s = 2$ is defined by Riemann's theorem on removable singularities.

For the system to be stable, the ROC must include the $j\omega$ axis. Thus $h(t) = -e^t u(-t)$ with ROC $\Re\{s\} < 1$. For the system to be causal, the ROC must be to the right of the pole, thus $h(t) = e^t u(t)$ with ROC $\Re\{s\} > 1$.

2020 v1

Question 1.(b)(ii)

Consider the signal

$$x(t) = \begin{cases} 1 - 2t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $x(3t + 5)$ and describe briefly in words how it can be derived from the original signal $x(t)$.

Mistake: multiplication by 2 is missing in the answer.

$$x(3t + 5) = \begin{cases} 1 - 2(3t + 5) = -9 - 6t, & 0 \leq 3t + 5 \leq 1 \Leftrightarrow -5 \leq 3t \leq -4 \Leftrightarrow -\frac{5}{3} \leq t \leq -\frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}$$

2020 v2

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - \frac{1}{8}x[n-1].$$

Find the analytical expression and the ROC of the z -transform of the output if $x[n] = (\frac{1}{4})^n u[n]$.

Mistake: the final ROC was incorrect.

From standard z -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{4}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{2}}.$$

Thus the z -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{4} = |z| > \frac{1}{2}.$$

2020 v3**Question 1.(a)(i)**

Consider $x(t) = |\sin(2\pi t + \frac{\pi}{3})|$. Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

Mistake: the answer said this signal is even, when it is neither even or odd.

Non causal, periodic with period $T = 0.5$, neither even or odd.

Question 3.(d)(iii)

Verify the results in part (d)(i) for the sequence $u[n+2] - u[-n+2]$.

Mistake: wrong power of z in the first step.

Taking the z -transform of $x[n] = u[n+2] - u[-n+2]$

$$X(z) = z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}$$

The expression for $X(z^{-1})$ is

$$\begin{aligned} X(z^{-1}) &= z^{-2} \frac{z^{-1}}{z^{-1}-1} - z^2 \frac{z}{z-1} \\ &= -\left(z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}\right) \\ &= -X(z) \end{aligned}$$

Thus proving the results in (d)(i).