Signals and Systems - known past paper mistakes

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Contents

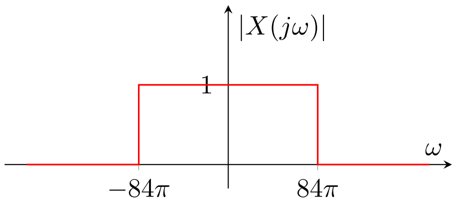
2017

Question 1.(j)(ii)

A signal is described by $x(t) = 84\,\text{sinc}(84\pi t)$. Sketch the Fourier transform of $x(t)$. Is a sampling period of $T = 0.01$ s small enough for aliasing to be avoided? Justify your answer.
Mistake: *Some versions of the correction have the wrong signal bandwidth.*

Note: The definition of the sinc function here is assumed to be **unnormalised**, i.e. $\text{sinc}(x) = \dfrac{\sin(x)}{x}$. The standard in digital signal processing (and in this module) is to use the **normalised** sinc defined as $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$.

Remember from the slides that $\text{sinc}(\frac{\omega_s}{2}t) \overset{FT}{\longleftrightarrow} \frac{2\pi}{\omega_s}\text{rect}(\frac{\omega}{\omega_s})$. Thus, the signal $x(t) = 84\,\text{sinc}_u(84\pi t)$ has Fourier transform $X(j\omega) = \text{rect}(\frac{\omega}{168\pi})$. The sketch of the Fourier transform is



and the minimum sampling rate required to satisfy the Nyquist criterion is $fs \geq 2 \cdot 42 = 84$ Hz. Since a sampling period of $T = 0.01$ s yields a sampling frequency of 100 Hz, there is no aliasing.

2018

Question 2.(c)(ii)

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt} - \frac{dy(t)}{dt} - 2y(t) = x(t)\,.$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system's impulse response. (i) Determine $H(s)$ as a ratio of polynomials. (ii) Determine $h(t)$ for each of the following cases

- The system is stable.
- The system is causal.
- The system is neither stable nor causal.

Mistake: *incorrect coefficients in the final solution.*

Part (i) gives

$$H(s) = \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}\,.$$

For a **stable** system, the ROC must include the $j\omega$ axis, thus

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)\,.$$

For a **causal** system, both ROCs must be right-handed, thus

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)\,.$$

For the system to be **neither** stable nor causal, choose either of

$$h(t) = \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(-t) \qquad \text{ROC doesn't exist!}$$
$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)\,.$$

Question 3.(b)(ii)

Consider the discrete signals $x_1[n] = 2^n$ and $x_2[n] = 3^n$ for $n \geq 0$. Find their convolution using their z-transforms and properties of convolution.

Mistake*: the final ROC was incorrect.*

Remember that the z-transform of $x[n] = a^n u[n]$ is given by

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \frac{1}{1-az^{-1}} \text{ if } |az^{-1}| < 1 \\ &= \frac{z}{z-a} \text{ if } |z| > |a|\,. \end{aligned}$$

Thus the convolution is obtained in the z-domain as

$$\begin{aligned} X_1(z)X_2(z) &= \frac{z}{z-2}\frac{z}{z-3} \text{ with } |z| > 3 \cap |z| > 2 \\ &= z\big(\frac{z}{z-3} - \frac{z}{z-2}\big) = z\big(\frac{z}{z-3} - x_1(0) + x_1(0) - \frac{z}{z-2}\big) \\ &= z\big(\frac{z}{z-3} - x_1(0) - \frac{z}{z-2} + x_2(0)\big) \text{ since } x_1(0) = x_2(0) \\ &= z\big(\frac{z}{z-3} - x_1(0)\big) - z\big(\frac{z}{z-2} - x_2(0)\big) \text{ with } |z| > 3\,. \end{aligned}$$

Using results from part (i), this gives $3^{n+1}u[n] - 2^{n+1}u[n]$ in the time domain.

Question 3.(c)

Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2]\,.$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its z-transform in the following three cases

- the system is causal
- the system is stable
- the system is neither causal nor causal.

Use the fact that the z-transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ in discrete time if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$.

Mistake*: the input $x[n-2]$ leads to a transfer function that does not contain $\frac{z}{z-a}$ terms. Instead, use $x[n]$ as input.*

The system can be written in the z-domain as

$$Y(z)[1 - 3.5z^{-1} + 2z^{-2}] = -7X(z)\,,$$

giving the transfer function

$$H(z) = \frac{-7}{(1-0.5z^{-1})(1-4z^{-1})} = \frac{1}{1-0.5z^{-1}} - \frac{8}{1-4z^{-1}} = \frac{z}{z-0.5} - 8\frac{z}{z-4}\,.$$

The system is causal

$$h[n] = (0.5)^n u[n] - 8(4)^n u[n],\ |z| > 4\,.$$

The system is stable

$$h[n] = (0.5)^n u[n] + 8(4)^n u[-n-1],\ 0.5 < |z| < 4\,.$$

The system is netiher stable nor causal

$$h[n] = -(0.5)^n u - [n-1] + 8(4)^n u[-n-1],\ |z| < 0.5\,.$$

There is no combination that allows the system to be both stable and causal.

2019

Question 1.(e)

The output of a continuous-time, LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t)\,.$$

Determine the frequency response of the system and sketch the asymptotic behaviour of its Bode plots.

***Mistake**: Incorrect numerator in the Fourier Transform.*

Applying the Fourier Transform on both sides gives

$$Y(\omega)j\omega + 2Y(\omega) = 2X(\omega)$$

thus yielding the frequency response

$$H(\omega) = \frac{2}{2 + j\omega}\,.$$

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1]\,.$$

Find the analytical expression and the ROC of the z-transform of the output if $x[n] = (\frac{1}{2})^n u[n]$.

***Mistake**: the final ROC was incorrect.*

From standard z-transforms we have

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{3}}\,.$$

Thus the z-transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{3})}\ \text{ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}\,.$$

Question 2.(c)(i)

Consider a continuous Linear Time-Invariant (LTI) system with input the signal $x(t) = \delta(t)$ (the Dirac function) and output the signal $y(t) = -\frac{2}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t)$. Determine the transfer function $H(s)$ of the system and its Region of Convergence (ROC).

Mistake: *The answer has a wrong coefficient ($\frac{1}{3}$ instead of $\frac{2}{3}$).*

Change the question so that the input is

$$y(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t),$$

and the answer is correct.

Question 2.(d)(ii)

The output $y(t)$ of an LTI system is related to the input $x(t)$ through the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + 2x(t)\,.$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of x(t) and y(t), respectively, and let $H(s)$ denote the Laplace transform of the system's impulse response $h(t)$.

- Determine $H(s)$ as a ratio of two polynomials.
- Determine $h(t)$ for each of the following cases
  - The system is stable.
  - The system is causal.

Mistake: *The answer for a stable system was incorrect.*

The system can be described in the Laplace domain as

$$\left(s^2 + s - 2\right)Y(s) = \left(s + 2\right)X(s)$$

giving $H(s)$ as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+s-2} = \frac{s+2}{(s+2)(s-1)} = \frac{1}{s+1}\,.$$

The last step occurs through pole-zero cancellation. This is beyond the scope of this course, but the analyticity of $H(s)$ at $s = 2$ is defined by Riemann's theorem on removable singularities.
For the system to be stable, the ROC must include the $j\omega$ axis. Thus $h(t) = -e^t u(-t)$ with ROC $\Re\{s\} < 1$. For the system to be causal, the ROC must be to the right of the pole, thus $h(t) = e^t u(t)$ with ROC $\Re\{s\} > 1$.

2020 v1

Question 1.(b)(ii)

Consider the signal

$$x(t) = \begin{cases} 1 - 2t, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $x(3t + 5)$ and describe briefly in words how it can be derived from the original signal $x(t)$.

Mistake: *multiplication by 2 is missing in the answer.*

$$x(3t+5) = \begin{cases} 1-2(3t+5) = -9-6t, & 0 \le 3t+5 \le 1 \Leftrightarrow -5 \le 3t \le -4 \Leftrightarrow -\frac{5}{3} \le t \le -\frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}$$

2020 v2

Question 1.(a)(i)

Consider $x(t) = |\cos(2\pi t + \pi/6)|$. Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

Mistake: *The answers states this signal is even, when it is neither even nor odd.*

Non causal, periodic with period $T = 0.5$, neither even nor odd.

Question 1.(b)(ii)

Consider the signal

$$x(t) = \begin{cases} 1-3t, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $x(3t+4)$ and describe briefly in words how it can be derived from the original signal $x(t)$.

Mistake: *multiplication by 3 is missing in the answer.*

$$x(3t+4) = \begin{cases} 1-3(3t+4) = -11-9t, & 0 \le 3t+4 \le 1 \Leftrightarrow -4 \le 3t \le -3 \Leftrightarrow -\frac{4}{3} \le t \le -1 \\ 0, & \text{otherwise.} \end{cases}$$

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - \frac{1}{8}x[n-1]\,.$$

Find the analytical expression and the ROC of the z-transform of the output if $x[n] = (\frac{1}{4})^n u[n]$.

Mistake: *the final ROC was incorrect.*

From standard z-transforms we have

$$X(z) = \frac{z}{z-\frac{1}{4}}$$

and from part (i) we have

$$H(z) = \frac{z}{z-\frac{1}{2}}\,.$$

Thus the z-transform of the output is

$$Y(z) = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{4} = |z| > \frac{1}{2}\,.$$

2020 v3

Question 1.(a)(i)

Consider $x(t) = |\sin(2\pi t + \frac{\pi}{3})|$. Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

Mistake: *the answer said this signal is even, when it is neither even nor odd.*

Non causal, periodic with period $T = 0.5$, neither even nor odd.

Question 3.(d)(iii)

Verify the results in part (d)(i) for the sequence $u[n+2] - u[-n+2]$.

Mistake: *wrong power of z in the first step.*

Taking the z-transform of $x[n] = u[n+2] - u[-n+2]$

$$X(z) = z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}$$

The expression for $X(z^{-1})$ is

$$\begin{aligned} X(z^{-1}) &= z^{-2} \frac{z^{-1}}{z^{-1}-1} - z^2 \frac{z}{z-1} \\ &= -\big(z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}\big) \\ &= -X(z) \end{aligned}$$

Thus proving the results in (d)(i).

2022

Question 1.(a)(ii)

For the following continuous-time signal, state whether it is causal or non-causal, periodic or nonperiodic, and even, odd, or non-symmetric. Give justifications for your statements. If periodic, state the period

$$x(t) = \cos^3(3t + \pi/2)\,.$$

Mistake: *The signal is odd symmetric rather than non-symmetric.*

The signal is non-causal, has a period $T = 2\pi/3$ and is **odd** symmetric.

Question 2.(c)(ii)

Consider a continuous time system with impulse response

$$h(t) = \delta(t+2) + 2\delta(t)\,.$$

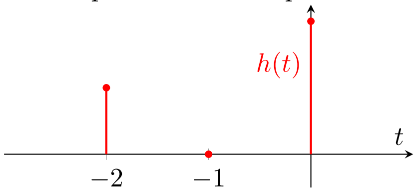
The input signal to this system is

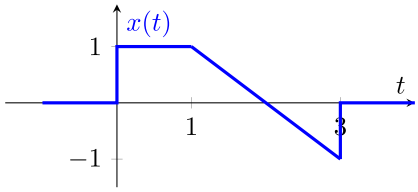
$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 3 \\ 0, & t < 0 \text{ and } t > 3\,. \end{cases}$$

Find an expression for the output, $y(t)$, of the system when the input is $x(t)$.

Mistake: *the answer gives the wrong expression for $y(t)$.*

The impulse response and the input to the system are given by

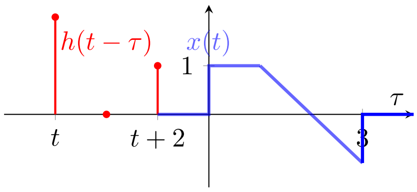


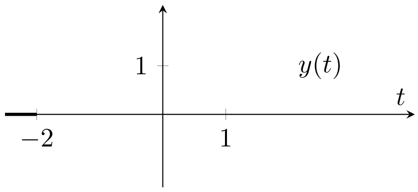


The output of the system is given by the plots below.

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\,.
\end{aligned}
$$

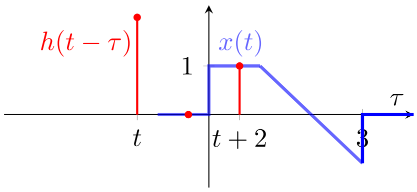
For $t < -2$: There is no overlap between the two functions, thus $y(t)$ is zero.

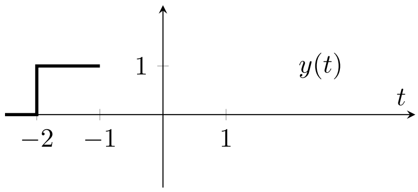




For $-2 < t < -1$: Only one δ function overlaps with $x(\tau)$, thus $y(t)$ is given by

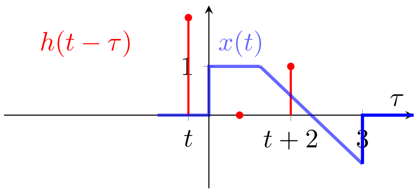
$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(t+2-\tau)x(\tau)d\tau \\
&= x(t+2) \qquad \text{for } -2 < t < -1 \\
&= x(\tau) \qquad \text{for } 0 < \tau < 1 \\
&= 1\,.
\end{aligned}
$$

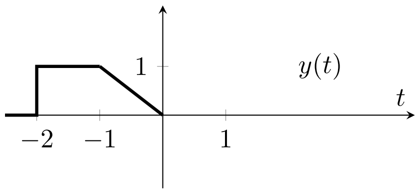




For $-1 < t < -0$: Only one δ function overlaps with $x(\tau)$, thus $y(t)$ is given by

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(t+2-\tau)x(\tau)d\tau \\
&= x(t+2) \qquad \text{for } -1 < t < 0 \\
&= x(\tau) \qquad \text{for } 1 < \tau < 2 \\
&= 2-\tau = -t\,.
\end{aligned}
$$

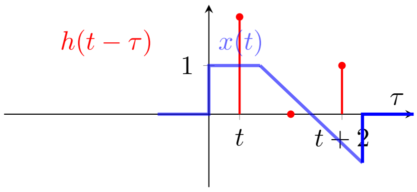


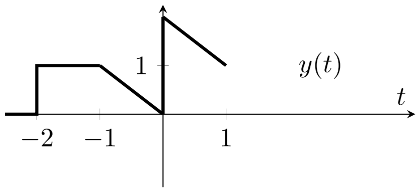


For $0 < t < 1$: Both δ functions overlaps with $x(\tau)$, thus $y(t)$ is given by

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} 2\delta(t-\tau)x(\tau) + \delta(t+2-\tau)x(\tau)d\tau
\end{aligned}
$$

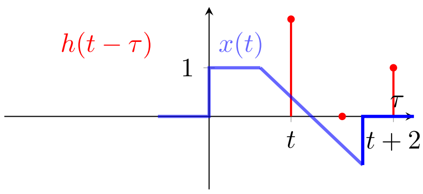
$$
\begin{aligned}
&= 2x(t) + x(t+2) \qquad \text{for } 0 < t < 1 \\
&= 2 - t\,.
\end{aligned}
$$

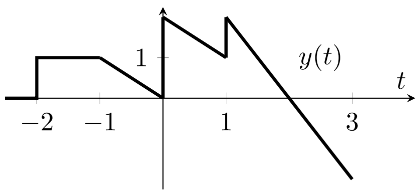




For $1 < t < 3$: Only one δ function overlaps with $x(\tau)$, thus $y(t)$ is given by

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} 2\delta(t-\tau)x(\tau) \\
&= 2x(t) \qquad \text{for } 1 < t < 3 \\
&= 4 - 2t\,.
\end{aligned}
$$

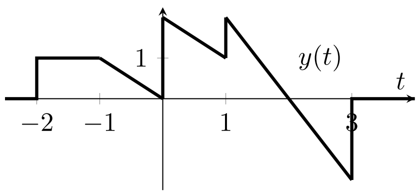




For $t > 3$: There is no overlap between the two functions, thus $y(t)$ is zero.

The output, $y(t)$, should be

$$
y(t) = \begin{cases}
0, & t < -2 \\
1, & -2 \leq t < -1 \\
-t, & -1 \leq t < 0 \\
2-t, & 0 \leq t < 1 \\
4-2t, & 1 \leq t \leq 3 \\
0, & t > 3\,.
\end{cases}
$$



Question 3.(c)(iv)

Consider a discrete-time LTI system with impulse response $h[n]$ and system function $H(z)$ with

$$h[n] = \big[1, 0.1, -0.12, 0.04\big]\ .$$

Find the magnitude and phase response of $H(z)$ at normalized angular frequency $\pi/2$.

Mistake*: there is a j term missing in the answer.*

The system function is given by

$$H(z) = 1 + 0.1z^{-1} - 0.12z^{-2} + 0.04z^{-3}\,.$$

To evaluate the frequency response at normalized angular frequency $\Omega = \pi/2$, let $z = e^{j\Omega} = e^{j\pi/2} = j$. Therefore,

$$
\begin{aligned}
H(j) &= 1 + 0.1j^{-1} - 0.12j^{-2} + 0.04j^{-3} \\
&= 1 - 0.1j + 0.12 + 0.04j = 1.12 - 0.06j\,.
\end{aligned}
$$

Giving a magnitude $|H(j)| = 1.122$ and a phase $\angle H(j) = -0.053$.

2023

Question 2.(c)(ii-iii)

The frequency response of a particular continuous-time system is

$$H(j\omega) = \frac{10}{j\omega + 10}$$

and the input signal is $x(t) = \cos(4\pi t)$. When $x(t)$ is used as the input signal, find the Fourier representation of the output signal, and write it in the form $A(R + Ij)$ where A is a scalar, and R and I are the real and imaginary terms respectively. Hence find the time domain output signal, $y(t)$.

Mistake: *The $A(R + Ij)$ form and consequent solution are incorrect.*

Using standard pairs, the Fourier representation of $x(t)$ is given by

$$X(j\omega) = \pi(\delta(\omega - 4\pi) + \delta(\omega + 4\pi)).$$

The Fourier representation of the output is then given by

$$\begin{aligned} Y(j\omega) &= \frac{10\pi}{j\omega + 10}(\delta(\omega - 4\pi) + \delta(\omega + 4\pi)) \\ &= \frac{10\pi}{\omega^2 + 100}\big(10(\delta(\omega - 4\pi) + \delta(\omega + 4\pi)) - j\omega(\delta(\omega - 4\pi) + \delta(\omega + 4\pi))\big), \end{aligned}$$

which satisfies the form required by question (ii). However, the inverse transform of such an expression is not directly obvious. Instead, follow the below procedure.
The input is re-written as the sum of two pure frequency components

$$x(t) = \cos(4\pi t) = \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}).$$

The response of the system at frequencies $\omega = \pm 4\pi$ is obtained as

$$\begin{aligned} H(j\omega)|_{\omega=4\pi} &= \frac{10}{4\pi j + 10} = |0.6227|\angle{-0.8986}\,\mathrm{rad}, \\ H(j\omega)|_{\omega=-4\pi} &= \frac{10}{-4\pi j + 10} = |0.6227|\angle 0.8986\,\mathrm{rad}, \end{aligned}$$

meaning the output is obtained as

$$\begin{aligned} y(t) &= \frac{1}{2}(e^{j4\pi t}0.6227e^{-j0.8986} + e^{-j4\pi t}0.6227e^{j0.8986}) \\ &= \frac{1}{2}0.6227(e^{j(4\pi t - 0.8986)} + e^{-j(4\pi t - 0.8986)}) \\ &= 0.6227\cos(4\pi t - 0.8986). \end{aligned}$$