Signals and Systems - known past paper mistakes

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May 3, 2022

2018

Question 2.(c)(ii)

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt} - \frac{dy(t)}{dt} - 2y(t) = x(t)\,.$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system's impulse response. (i) Determine $H(s)$ as a ratio of polynomials. (ii) Determine $h(t)$ for each of the following cases

- The system is stable.
- The system is causal.
- The system is neither stable nor causal.

Mistake*: incorrect coefficients in the final solution.*

Part (i) gives

$$H(s) = \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}\,. \tag{1}$$

For a **stable** system, the ROC must include the $j\omega$ axis, thus

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)\,.$$

For a **causal** system, both ROCs must be right-handed, thus

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)\,.$$

For the system to be **neither** stable nor causal, choose either of

$$h(t) = \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(-t) \qquad \text{ROC doesn't exist!}$$
$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t)\,.$$

Question 3.(b)(ii)

Consider the discrete signals $x_1[n] = 2^n$ and $x_2[n] = 3^n$ for $n \geq 0$. Find their convolution using their z-transforms and properties of convolution.

Mistake*: the final ROC was incorrect.*

Remember that the z-transform of $x[n] = a^n u[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \frac{1}{1 - az^{-1}} \text{ if } |az^{-1}| < 1$$
$$= \frac{z}{z-a} \text{ if } |z| > |a| \,.$$

Thus the convolution is obtained in the z-domain as

$$\begin{aligned}
X_1(z)X_2(z) &= \frac{z}{z-2}\frac{z}{z-3} \text{ with } |z| > 3 \cap |z| > 2 \\
&= z\big(\frac{z}{z-3} - \frac{z}{z-2}\big) = z\big(\frac{z}{z-3} - x_1(0) + x_1(0) - \frac{z}{z-2}\big) \\
&= z\big(\frac{z}{z-3} - x_1(0) - \frac{z}{z-2} + x_2(0)\big) \text{ since } x_1(0) = x_2(0) \\
&= z\big(\frac{z}{z-3} - x_1(0)\big) - z\big(\frac{z}{z-2} - x_2(0)\big) \text{ with } |z| > 3 \,.
\end{aligned}$$

Using results from part (i), this gives $3^{n+1}u[n] - 2^{n+1}u[n]$ in the time domain.

Question 3.(c)

Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2] \,.$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its z-transform in the following three cases

- the system is causal
- the system is stable
- the system is neither causal nor causal.

Use the fact that the z-transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ in discrete time if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$.

Mistake: *the input $x[n-2]$ leads to a transfer function that does not contain $\frac{z}{z-a}$ terms. Instead, use $x[n]$ as input.*

The system can be written in the z-domain as

$$Y(z)[1 - 3.5z^{-1} + 2z^{-2}] = -7X(z) \,,$$

giving the transfer function

$$H(z) = \frac{-7}{(1-0.5z^{-1})(1-4z^{-1})} = \frac{1}{1-0.5z^{-1}} - \frac{8}{1-4z^{-1}} = \frac{z}{z-0.5} - 8\frac{z}{z-4} \,.$$

The system is causal

$$h[n] = (0.5)^n u[n] - 8(4)^n u[n], \; |z| > 4 \,.$$

The system is stable

$$h[n] = (0.5)^n u[n] + 8(4)^n u[-n-1], \; 0.5 < |z| < 4 \,.$$

The system is netiher stable nor causal

$$h[n] = -(0.5)^n u - [n-1] + 8(4)^n u[-n-1], \; |z| < 0.5 \,.$$

There is no combination that allows the system to be both stable and causal.

2019

Question 1.(e)

The output of a continuous-time, LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t)\,.$$

Determine the frequency response of the system and sketch the asymptotic behaviour of its Bode plots.

Mistake: *Incorrect numerator in the Fourier Transform.*

Applying the Fourier Transform on both sides gives

$$Y(\omega)j\omega + 2Y(\omega) = 2X(\omega)$$

thus yielding the frequency response

$$H(\omega) = \frac{2}{2+j\omega}\,.$$

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1]\,.$$

Find the analytical expression and the ROC of the z-transform of the output if $x[n] = (\frac{1}{2})^n u[n]$.

Mistake: *the final ROC was incorrect.*

From standard z-transforms we have

$$X(z) = \frac{z}{z-\frac{1}{2}}$$

and from part (i) we have

$$H(z) = \frac{z}{z-\frac{1}{3}}\,.$$

Thus the z-transform of the output is

$$Y(z) = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}\,.$$

Question 2.(d)(ii)

The output $y(t)$ of an LTI system is related to the input $x(t)$ through the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + 2x(t)\,.$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of x(t) and y(t), respectively, and let $H(s)$ denote the Laplace transform of the system's impulse response $h(t)$.

- Determine $H(s)$ as a ratio of two polynomials.
- Determine $h(t)$ for each of the following cases
  - The system is stable.
  - The system is causal.

Mistake: *The answer for a stable system was incorrect.*

The system can be described in the Laplace domain as

$$\left(s^2 + s - 2\right)Y(s) = \left(s + 2\right)X(s)$$

giving $H(s)$ as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2+s-2} = \frac{s+2}{(s+2)(s-1)} = \frac{1}{s+1}\,.$$

The last step occurs through pole-zero cancellation. This is beyond the scope of this course, but the analyticity of $H(s)$ at $s = 2$ is defined by Riemann's theorem on removable singularities.
For the system to be stable, the ROC must include the $j\omega$ axis. Thus $h(t) = -e^t u(-t)$ with ROC $\Re\{s\} < 1$. For the system to be causal, the ROC must be to the right of the pole, thus $h(t) = e^t u(t)$ with ROC $\Re\{s\} > 1$.

2020 v1

Question 1.(b)(ii)

Consider the signal

$$x(t) = \begin{cases} 1 - 2t, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $x(3t+5)$ and describe briefly in words how it can be derived from the original signal $x(t)$.

Mistake: *multiplication by 2 is missing in the answer.*

$$x(3t+5) = \begin{cases} 1 - 2(3t+5) = -9 - 6t, & 0 \le 3t+5 \le 1 \Leftrightarrow -5 \le 3t \le -4 \Leftrightarrow -\frac{5}{3} \le t \le -\frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}$$

2020 v2

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - \frac{1}{8}x[n-1]\,.$$

Find the analytical expression and the ROC of the z-transform of the output if $x[n] = (\frac{1}{4})^n u[n]$.

Mistake: *the final ROC was incorrect.*

From standard z-transforms we have

$$X(z) = \frac{z}{z - \frac{1}{4}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{2}}\,.$$

Thus the z-transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \;\text{ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{4} = |z| > \frac{1}{2}\,.$$

2020 v3

Question 1.(a)(i)

Consider $x(t) = |\sin(2\pi t + \frac{\pi}{3})|$. Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

Mistake: *the answer said this signal is even, when it is neither even or odd.*

Non causal, periodic with period $T = 0.5$, neither even or odd.

Question 3.(d)(iii)

Verify the results in part (d)(i) for the sequence $u[n+2] - u[-n+2]$.

Mistake: *wrong power of z in the first step.*

Taking the z-transform of $x[n] = u[n+2] - u[-n+2]$

$$X(z) = z^2\frac{z}{z-1} - z^{-2}\frac{z^{-1}}{z^{-1}-1}$$

The expression for $X(z^{-1})$ is

$$\begin{aligned} X(z^{-1}) &= z^{-2}\frac{z^{-1}}{z^{-1}-1} - z^2\frac{z}{z-1} \\ &= -\big(z^2\frac{z}{z-1} - z^{-2}\frac{z^{-1}}{z^{-1}-1}\big) \\ &= -X(z) \end{aligned}$$

Thus proving the results in (d)(i).