

Signals and Systems - known past paper mistakes

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2017

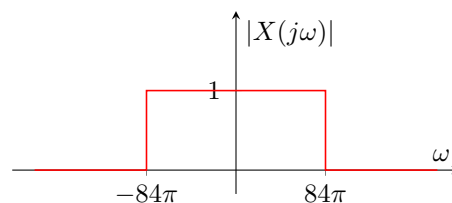
Question 1.(j)(ii)

A signal is described by $x(t) = 84 \operatorname{sinc}(84\pi t)$. Sketch the Fourier transform of $x(t)$. Is a sampling period of $T = 0.01$ s small enough for aliasing to be avoided? Justify your answer.

Mistake: Some versions of the correction have the wrong signal bandwidth.

Note: The definition of the sinc function here is assumed to be **unnormalised**, i.e. $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$. The standard in digital signal processing (and in this module) is to use the **normalised** sinc defined as $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

Remember from the slides that $\operatorname{sinc}(\frac{\omega_s}{2}t) \xleftrightarrow{FT} \frac{2\pi}{\omega_s} \operatorname{rect}(\frac{\omega}{\omega_s})$. Thus, the signal $x(t) = 84 \operatorname{sinc}_u(84\pi t)$ has Fourier transform $X(j\omega) = \operatorname{rect}(\frac{\omega}{168\pi})$. The sketch of the Fourier transform is



and the minimum sampling rate required to satisfy the Nyquist criterion is $f_s \geq 2 \cdot 42 = 84$ Hz. Since a sampling period of $T = 0.01$ s yields a sampling frequency of 100 Hz, there is no aliasing.

2018

Question 2.(c)(ii)

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system's impulse response. (i) Determine $H(s)$ as a ratio of polynomials. (ii) Determine $h(t)$ for each of the following cases

- The system is stable.
- The system is causal.
- The system is neither stable nor causal.

Mistake: incorrect coefficients in the final solution.

Part (i) gives

$$H(s) = \frac{1}{3} \frac{1}{s-2} - \frac{1}{3} \frac{1}{s+1}.$$

For a **stable** system, the ROC must include the $j\omega$ axis, thus

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t).$$

For a **causal** system, both ROCs must be right-handed, thus

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t).$$

For the system to be **neither** stable nor causal, choose either of

$$\begin{aligned} h(t) &= \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(-t) && \text{ROC doesn't exist!} \\ h(t) &= -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t). \end{aligned}$$

Question 3.(b)(ii)

Consider the discrete signals $x_1[n] = 2^n$ and $x_2[n] = 3^n$ for $n \geq 0$. Find their convolution using their z -transforms and properties of convolution.

Mistake: the final ROC was incorrect.

Remember that the z -transform of $x[n] = a^n u[n]$ is given by

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \frac{1}{1 - az^{-1}} \text{ if } |az^{-1}| < 1 \\ &= \frac{z}{z - a} \text{ if } |z| > |a|. \end{aligned}$$

Thus the convolution is obtained in the z -domain as

$$\begin{aligned} X_1(z)X_2(z) &= \frac{z}{z-2} \frac{z}{z-3} \text{ with } |z| > 3 \cap |z| > 2 \\ &= z \left(\frac{z}{z-3} - \frac{z}{z-2} \right) = z \left(\frac{z}{z-3} - x_1(0) + x_1(0) - \frac{z}{z-2} \right) \\ &= z \left(\frac{z}{z-3} - x_1(0) - \frac{z}{z-2} + x_2(0) \right) \text{ since } x_1(0) = x_2(0) \\ &= z \left(\frac{z}{z-3} - x_1(0) \right) - z \left(\frac{z}{z-2} - x_2(0) \right) \text{ with } |z| > 3. \end{aligned}$$

Using results from part (i), this gives $3^{n+1}u[n] - 2^{n+1}u[n]$ in the time domain.

Question 3.(c)

Consider a LTI system with input $x[n]$ and output $y[n]$ related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2].$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its z -transform in the following three cases

- the system is causal
- the system is stable
- the system is neither causal nor stable.

Use the fact that the z -transform $\frac{z}{z-a}$ corresponds to the function $a^n u[n]$ in discrete time if $|z| > |a|$ and the function $-a^n u[-n-1]$ if $|z| < |a|$.

Mistake: the input $x[n-2]$ leads to a transfer function that does not contain $\frac{z}{z-a}$ terms. Instead, use $x[n]$ as input.

The system can be written in the z -domain as

$$Y(z)[1 - 3.5z^{-1} + 2z^{-2}] = -7X(z),$$

giving the transfer function

$$H(z) = \frac{-7}{(1 - 0.5z^{-1})(1 - 4z^{-1})} = \frac{1}{1 - 0.5z^{-1}} - \frac{8}{1 - 4z^{-1}} = \frac{z}{z - 0.5} - 8\frac{z}{z - 4}.$$

The system is causal

$$h[n] = (0.5)^n u[n] - 8(4)^n u[n], |z| > 4.$$

The system is stable

$$h[n] = (0.5)^n u[n] + 8(4)^n u[-n-1], 0.5 < |z| < 4.$$

The system is neither stable nor causal

$$h[n] = -(0.5)^n u[-n-1] + 8(4)^n u[-n-1], |z| < 0.5.$$

There is no combination that allows the system to be both stable and causal.

2019

Question 1.(e)

The output of a continuous-time, LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t).$$

Determine the frequency response of the system and sketch the asymptotic behaviour of its Bode plots.

Mistake: Incorrect numerator in the Fourier Transform.

Applying the Fourier Transform on both sides gives

$$Y(\omega)j\omega + 2Y(\omega) = 2X(\omega)$$

thus yielding the frequency response

$$H(\omega) = \frac{2}{2 + j\omega}.$$

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

Find the analytical expression and the ROC of the z -transform of the output if $x[n] = (\frac{1}{2})^n u[n]$.

Mistake: the final ROC was incorrect.

From standard z -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{3}}.$$

Thus the z -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{3})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}.$$

Question 2.(d)(ii)

The output $y(t)$ of an LTI system is related to the input $x(t)$ through the differential equation

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of the system's impulse response $h(t)$.

- Determine $H(s)$ as a ratio of two polynomials.
- Determine $h(t)$ for each of the following cases
 - The system is stable.
 - The system is causal.

Mistake: The answer for a stable system was incorrect.

The system can be described in the Laplace domain as

$$(s^2 + s - 2)Y(s) = (s + 2)X(s)$$

giving $H(s)$ as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s + 2}{s^2 + s - 2} = \frac{s + 2}{(s + 2)(s - 1)} = \frac{1}{s - 1}.$$

The last step occurs through pole-zero cancellation. This is beyond the scope of this course, but the analyticity of $H(s)$ at $s = 2$ is defined by Riemann's theorem on removable singularities.

For the system to be stable, the ROC must include the $j\omega$ axis. Thus $h(t) = -e^t u(-t)$ with ROC $\Re\{s\} < 1$. For the system to be causal, the ROC must be to the right of the pole, thus $h(t) = e^t u(t)$ with ROC $\Re\{s\} > 1$.

2020 v1

Question 1.(b)(ii)

Consider the signal

$$x(t) = \begin{cases} 1 - 2t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $x(3t + 5)$ and describe briefly in words how it can be derived from the original signal $x(t)$.

Mistake: multiplication by 2 is missing in the answer.

$$x(3t + 5) = \begin{cases} 1 - 2(3t + 5) = -9 - 6t, & 0 \leq 3t + 5 \leq 1 \Leftrightarrow -5 \leq 3t \leq -4 \Leftrightarrow -\frac{5}{3} \leq t \leq -\frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}$$

2020 v2

Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - \frac{1}{8}x[n-1].$$

Find the analytical expression and the ROC of the z -transform of the output if $x[n] = (\frac{1}{4})^n u[n]$.

Mistake: the final ROC was incorrect.

From standard z -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{4}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{2}}.$$

Thus the z -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \quad \text{ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{4} = |z| > \frac{1}{2}.$$

2020 v3

Question 1.(a)(i)

Consider $x(t) = |\sin(2\pi t + \frac{\pi}{3})|$. Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

Mistake: the answer said this signal is even, when it is neither even or odd.

Non causal, periodic with period $T = 0.5$, neither even or odd.

Question 3.(d)(iii)

Verify the results in part (d)(i) for the sequence $u[n+2] - u[-n+2]$.

Mistake: wrong power of z in the first step.

Taking the z -transform of $x[n] = u[n+2] - u[-n+2]$

$$X(z) = z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}$$

The expression for $X(z^{-1})$ is

$$\begin{aligned} X(z^{-1}) &= z^{-2} \frac{z^{-1}}{z^{-1}-1} - z^2 \frac{z}{z-1} \\ &= -\left(z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1} \right) \\ &= -X(z) \end{aligned}$$

Thus proving the results in (d)(i).

2022

Question 2.(c)(ii)

Consider a continuous time system with impulse response

$$h(t) = \delta(t+2) + 2\delta(t).$$

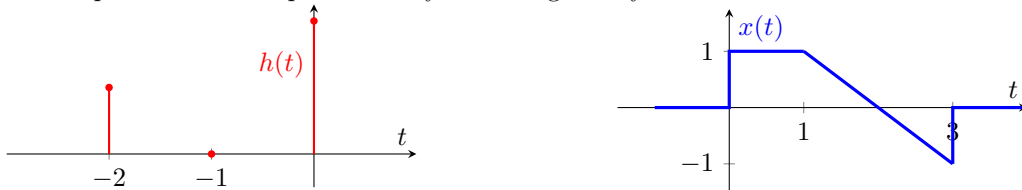
The input signal to this system is

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 2-t, & 1 < t \leq 3 \\ 0, & t < 0 \text{ and } t > 3. \end{cases}$$

Find an expression for the output, $y(t)$, of the system when the input is $x(t)$.

Mistake: the answer gives the wrong expression for $y(t)$.

The impulse response and the input to the system are given by



The output of the system is given by the plots below.

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau. \end{aligned}$$

For $t < -2$:

There is no overlap between the two functions, thus $y(t)$ is zero.



For $-2 < t < -1$:

Only one δ function overlaps with $x(\tau)$, thus $y(t)$ is given by

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} \delta(t+2-\tau)x(\tau)d\tau \\
 &= x(t+2) \quad \text{for } -2 < t < -1 \\
 &= x(\tau) \quad \text{for } 0 < \tau < 1 \\
 &= 1.
 \end{aligned}$$



For $-1 < t < 0$:

Only one δ function overlaps with $x(\tau)$, thus $y(t)$ is given by

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} \delta(t+2-\tau)x(\tau)d\tau \\
 &= x(t+2) \quad \text{for } -1 < t < 0 \\
 &= x(\tau) \quad \text{for } 1 < \tau < 2 \\
 &= 2 - \tau = -t.
 \end{aligned}$$



For $0 < t < 1$:

Both δ functions overlaps with $x(\tau)$, thus $y(t)$ is given by

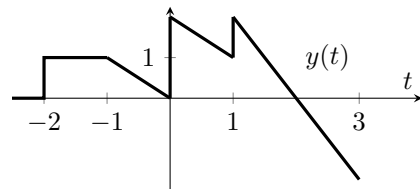
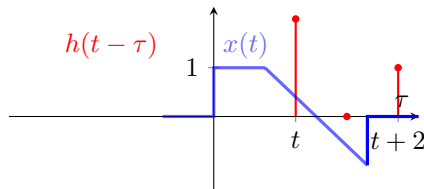
$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} 2\delta(t-\tau)x(\tau) + \delta(t+2-\tau)x(\tau)d\tau \\
 &= 2x(t) + x(t+2) \quad \text{for } 0 < t < 1 \\
 &= 2 - t.
 \end{aligned}$$



For $1 < t < 3$:

Only one δ function overlaps with $x(\tau)$, thus $y(t)$ is given by

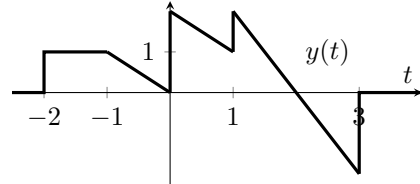
$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} 2\delta(t-\tau)x(\tau) \\
 &= 2x(t) \quad \text{for } 1 < t < 3 \\
 &= 4 - 2t.
 \end{aligned}$$



For $t > 3$: There is no overlap between the two functions, thus $y(t)$ is zero.

The output, $y(t)$, should be

$$y(t) = \begin{cases} 0, & t < -2 \\ 1, & -2 \leq t < -1 \\ -t, & -1 \leq t < 0 \\ 2-t, & 0 \leq t < 1 \\ 4-2t, & 1 \leq t \leq 3 \\ 0, & t > 3. \end{cases}$$



Question 3.(c)(iv)

Consider a discrete-time LTI system with impulse response $h[n]$ and system function $H(z)$ with

$$h[n] = [1, 0.1, -0.12, 0.04] .$$

Find the magnitude and phase response of $H(z)$ at normalized angular frequency $\pi/2$.

Mistake: there is a j term missing in the answer.

The system function is given by

$$H(z) = 1 + 0.1z^{-1} - 0.12z^{-2} + 0.04z^{-3} .$$

To evaluate the frequency response at normalized angular frequency $\Omega = \pi/2$, let $z = e^{j\Omega} = e^{j\pi/2} = j$. Therefore,

$$\begin{aligned} H(j) &= 1 + 0.1j^{-1} - 0.12j^{-2} + 0.04j^{-3} \\ &= 1 - 0.1j + 0.12 + 0.04j = 1.12 - 0.06j . \end{aligned}$$

Giving a magnitude $|H(j)| = 1.122$ and a phase $\angle H(j) = -0.053$.