

# Signals and Systems - known past paper mistakes

Emilie d'Olne

April 25, 2023

2017

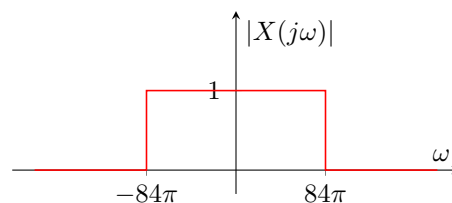
## Question 1.(j)(ii)

A signal is described by  $x(t) = 84 \operatorname{sinc}(84\pi t)$ . Sketch the Fourier transform of  $x(t)$ . Is a sampling period of  $T = 0.01$  s small enough for aliasing to be avoided? Justify your answer.

*Mistake: Some versions of the correction have the wrong signal bandwidth.*

Note: The definition of the sinc function here is assumed to be **unnormalised**, i.e.  $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$ . The standard in digital signal processing (and in this module) is to use the **normalised** sinc defined as  $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ .

Remember from the slides that  $\operatorname{sinc}(\frac{\omega_s}{2}t) \xleftrightarrow{FT} \frac{2\pi}{\omega_s} \operatorname{rect}(\frac{\omega}{\omega_s})$ . Thus, the signal  $x(t) = 84 \operatorname{sinc}_u(84\pi t)$  has Fourier transform  $X(j\omega) = \operatorname{rect}(\frac{\omega}{168\pi})$ . The sketch of the Fourier transform is



and the minimum sampling rate required to satisfy the Nyquist criterion is  $f_s \geq 2 \cdot 42 = 84$  Hz. Since a sampling period of  $T = 0.01$  s yields a sampling frequency of 100 Hz, there is no aliasing.

2018

## Question 2.(c)(ii)

The output  $y(t)$  of a causal LTI system is related to the input  $x(t)$  by the differential equation

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

Let  $X(s)$  and  $Y(s)$  denote Laplace transforms of  $x(t)$  and  $y(t)$ , respectively, and let  $H(s)$  denote the Laplace transform of  $h(t)$ , the system's impulse response. (i) Determine  $H(s)$  as a ratio of polynomials. (ii) Determine  $h(t)$  for each of the following cases

- The system is stable.
- The system is causal.
- The system is neither stable nor causal.

*Mistake: incorrect coefficients in the final solution.*

Part (i) gives

$$H(s) = \frac{1}{3} \frac{1}{s-2} - \frac{1}{3} \frac{1}{s+1}.$$

For a **stable** system, the ROC must include the  $j\omega$  axis, thus

$$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t).$$

For a **causal** system, both ROCs must be right-handed, thus

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t).$$

For the system to be **neither** stable nor causal, choose either of

$$\begin{aligned} h(t) &= \frac{1}{3}e^{2t}u(t) + \frac{1}{3}e^{-t}u(-t) && \text{ROC doesn't exist!} \\ h(t) &= -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t). \end{aligned}$$

### Question 3.(b)(ii)

Consider the discrete signals  $x_1[n] = 2^n$  and  $x_2[n] = 3^n$  for  $n \geq 0$ . Find their convolution using their  $z$ -transforms and properties of convolution.

*Mistake: the final ROC was incorrect.*

Remember that the  $z$ -transform of  $x[n] = a^n u[n]$  is given by

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \frac{1}{1 - az^{-1}} \text{ if } |az^{-1}| < 1 \\ &= \frac{z}{z - a} \text{ if } |z| > |a|. \end{aligned}$$

Thus the convolution is obtained in the  $z$ -domain as

$$\begin{aligned} X_1(z)X_2(z) &= \frac{z}{z-2} \frac{z}{z-3} \text{ with } |z| > 3 \cap |z| > 2 \\ &= z \left( \frac{z}{z-3} - \frac{z}{z-2} \right) = z \left( \frac{z}{z-3} - x_1(0) + x_1(0) - \frac{z}{z-2} \right) \\ &= z \left( \frac{z}{z-3} - x_1(0) - \frac{z}{z-2} + x_2(0) \right) \text{ since } x_1(0) = x_2(0) \\ &= z \left( \frac{z}{z-3} - x_1(0) \right) - z \left( \frac{z}{z-2} - x_2(0) \right) \text{ with } |z| > 3. \end{aligned}$$

Using results from part (i), this gives  $3^{n+1}u[n] - 2^{n+1}u[n]$  in the time domain.

### Question 3.(c)

Consider a LTI system with input  $x[n]$  and output  $y[n]$  related by the difference equation

$$2y[n] - 9y[n-1] + 4y[n-2] = -14x[n-2].$$

Investigate whether the above system can be both stable and causal. Justify your answer. Determine the impulse response and its  $z$ -transform in the following three cases

- the system is causal
- the system is stable
- the system is neither causal nor stable.

Use the fact that the  $z$ -transform  $\frac{z}{z-a}$  corresponds to the function  $a^n u[n]$  in discrete time if  $|z| > |a|$  and the function  $-a^n u[-n-1]$  if  $|z| < |a|$ .

*Mistake: the input  $x[n-2]$  leads to a transfer function that does not contain  $\frac{z}{z-a}$  terms. Instead, use  $x[n]$  as input.*

The system can be written in the  $z$ -domain as

$$Y(z)[1 - 3.5z^{-1} + 2z^{-2}] = -7X(z),$$

giving the transfer function

$$H(z) = \frac{-7}{(1 - 0.5z^{-1})(1 - 4z^{-1})} = \frac{1}{1 - 0.5z^{-1}} - \frac{8}{1 - 4z^{-1}} = \frac{z}{z - 0.5} - 8\frac{z}{z - 4}.$$

**The system is causal**

$$h[n] = (0.5)^n u[n] - 8(4)^n u[n], |z| > 4.$$

**The system is stable**

$$h[n] = (0.5)^n u[n] + 8(4)^n u[-n-1], 0.5 < |z| < 4.$$

**The system is neither stable nor causal**

$$h[n] = -(0.5)^n u[-n-1] + 8(4)^n u[-n-1], |z| < 0.5.$$

There is no combination that allows the system to be both stable and causal.

2019

### Question 1.(e)

The output of a continuous-time, LTI system is related to the input  $x(t)$  by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t).$$

Determine the frequency response of the system and sketch the asymptotic behaviour of its Bode plots.

*Mistake: Incorrect numerator in the Fourier Transform.*

Applying the Fourier Transform on both sides gives

$$Y(\omega)j\omega + 2Y(\omega) = 2X(\omega)$$

thus yielding the frequency response

$$H(\omega) = \frac{2}{2 + j\omega}.$$

### Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input  $x[n]$  and output  $y[n]$  related with the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

Find the analytical expression and the ROC of the  $z$ -transform of the output if  $x[n] = (\frac{1}{2})^n u[n]$ .

*Mistake: the final ROC was incorrect.*

From standard  $z$ -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{3}}.$$

Thus the  $z$ -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{3})} \text{ ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}.$$

### Question 2.(d)(ii)

The output  $y(t)$  of an LTI system is related to the input  $x(t)$  through the differential equation

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Let  $X(s)$  and  $Y(s)$  denote the Laplace transforms of  $x(t)$  and  $y(t)$ , respectively, and let  $H(s)$  denote the Laplace transform of the system's impulse response  $h(t)$ .

- Determine  $H(s)$  as a ratio of two polynomials.
- Determine  $h(t)$  for each of the following cases
  - The system is stable.
  - The system is causal.

*Mistake: The answer for a stable system was incorrect.*

The system can be described in the Laplace domain as

$$(s^2 + s - 2)Y(s) = (s + 2)X(s)$$

giving  $H(s)$  as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s + 2}{s^2 + s - 2} = \frac{s + 2}{(s + 2)(s - 1)} = \frac{1}{s - 1}.$$

The last step occurs through pole-zero cancellation. This is beyond the scope of this course, but the analyticity of  $H(s)$  at  $s = 2$  is defined by Riemann's theorem on removable singularities.

For the system to be stable, the ROC must include the  $j\omega$  axis. Thus  $h(t) = -e^t u(-t)$  with ROC  $\Re\{s\} < 1$ . For the system to be causal, the ROC must be to the right of the pole, thus  $h(t) = e^t u(t)$  with ROC  $\Re\{s\} > 1$ .

## 2020 v1

### Question 1.(b)(ii)

Consider the signal

$$x(t) = \begin{cases} 1 - 2t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Sketch  $x(3t + 5)$  and describe briefly in words how it can be derived from the original signal  $x(t)$ .

*Mistake: multiplication by 2 is missing in the answer.*

$$x(3t + 5) = \begin{cases} 1 - 2(3t + 5) = -9 - 6t, & 0 \leq 3t + 5 \leq 1 \Leftrightarrow -5 \leq 3t \leq -4 \Leftrightarrow -\frac{5}{3} \leq t \leq -\frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}$$

## 2020 v2

### Question 1.(g)(ii)

Consider the discrete-time, causal LTI system with input  $x[n]$  and output  $y[n]$  related with the difference equation

$$y[n] - \frac{5}{8}y[n-1] + \frac{1}{16}y[n-2] = x[n] - \frac{1}{8}x[n-1].$$

Find the analytical expression and the ROC of the  $z$ -transform of the output if  $x[n] = (\frac{1}{4})^n u[n]$ .

*Mistake: the final ROC was incorrect.*

From standard  $z$ -transforms we have

$$X(z) = \frac{z}{z - \frac{1}{4}}$$

and from part (i) we have

$$H(z) = \frac{z}{z - \frac{1}{2}}.$$

Thus the  $z$ -transform of the output is

$$Y(z) = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \quad \text{ROC: } |z| > \frac{1}{2} \cap |z| > \frac{1}{4} = |z| > \frac{1}{2}.$$

## 2020 v3

### Question 1.(a)(i)

Consider  $x(t) = |\sin(2\pi t + \frac{\pi}{3})|$ . Specify if the signal is causal/non causal, periodic/non-periodic, odd/even.

*Mistake: the answer said this signal is even, when it is neither even or odd.*

Non causal, periodic with period  $T = 0.5$ , neither even or odd.

### Question 3.(d)(iii)

Verify the results in part (d)(i) for the sequence  $u[n+2] - u[-n+2]$ .

*Mistake: wrong power of  $z$  in the first step.*

Taking the  $z$ -transform of  $x[n] = u[n+2] - u[-n+2]$

$$X(z) = z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1}$$

The expression for  $X(z^{-1})$  is

$$\begin{aligned} X(z^{-1}) &= z^{-2} \frac{z^{-1}}{z^{-1}-1} - z^2 \frac{z}{z-1} \\ &= -\left( z^2 \frac{z}{z-1} - z^{-2} \frac{z^{-1}}{z^{-1}-1} \right) \\ &= -X(z) \end{aligned}$$

Thus proving the results in (d)(i).

2022

### Question 2.(c)(ii)

Consider a continuous time system with impulse response

$$h(t) = \delta(t+2) + 2\delta(t).$$

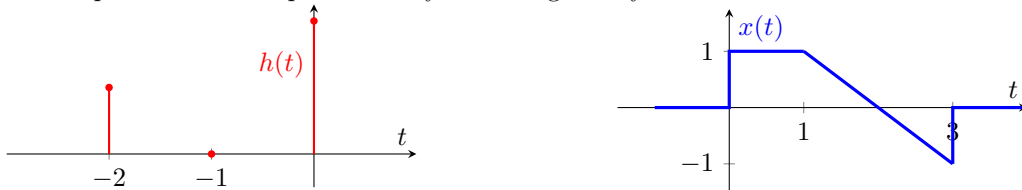
The input signal to this system is

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 2-t, & 1 < t \leq 3 \\ 0, & t < 0 \text{ and } t > 3. \end{cases}$$

Find an expression for the output,  $y(t)$ , of the system when the input is  $x(t)$ .

*Mistake: the answer gives the wrong expression for  $y(t)$ .*

The impulse response and the input to the system are given by



The output of the system is given by the plots below.

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau. \end{aligned}$$

For  $t < -2$ :

There is no overlap between the two functions, thus  $y(t)$  is zero.



**For  $-2 < t < -1$ :**

Only one  $\delta$  function overlaps with  $x(\tau)$ , thus  $y(t)$  is given by

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} \delta(t+2-\tau)x(\tau)d\tau \\
 &= x(t+2) \quad \text{for } -2 < t < -1 \\
 &= x(\tau) \quad \text{for } 0 < \tau < 1 \\
 &= 1.
 \end{aligned}$$



**For  $-1 < t < 0$ :**

Only one  $\delta$  function overlaps with  $x(\tau)$ , thus  $y(t)$  is given by

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} \delta(t+2-\tau)x(\tau)d\tau \\
 &= x(t+2) \quad \text{for } -1 < t < 0 \\
 &= x(\tau) \quad \text{for } 1 < \tau < 2 \\
 &= 2 - \tau = -t.
 \end{aligned}$$



**For  $0 < t < 1$ :**

Both  $\delta$  functions overlaps with  $x(\tau)$ , thus  $y(t)$  is given by

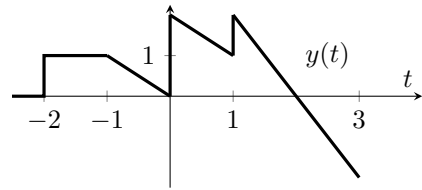
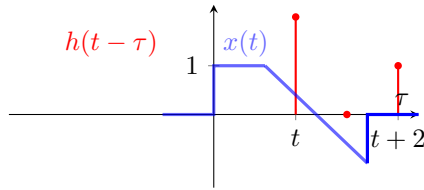
$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} 2\delta(-\tau)x(\tau) + \delta(t+2-\tau)x(\tau)d\tau \\
 &= x(t) + x(t+2) \quad \text{for } 0 < t < 1 \\
 &= 2 - t.
 \end{aligned}$$



**For  $1 < t < 3$ :**

Only one  $\delta$  function overlaps with  $x(\tau)$ , thus  $y(t)$  is given by

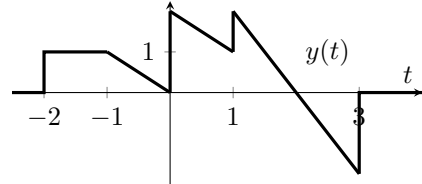
$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^{\infty} 2\delta(-\tau)x(\tau) \\
 &= 2x(t) \quad \text{for } 1 < t < 3 \\
 &= 4 - 2t.
 \end{aligned}$$



**For  $t > 3$ :** There is no overlap between the two functions, thus  $y(t)$  is zero.

The output,  $y(t)$ , should be

$$y(t) = \begin{cases} 0, & t < -2 \\ 1, & -2 \leq t < -1 \\ -t, & -1 \leq t < 0 \\ 2 - t, & 0 \leq t < 1 \\ 4 - 2t, & 1 \leq t \leq 3 \\ 0, & t > 3. \end{cases}$$



### Question 3.(c)(iv)

Consider a discrete-time LTI system with impulse response  $h[n]$  and system function  $H(z)$  with

$$h[n] = [1, 0.1, -0.12, 0.04] .$$

Find the magnitude and phase response of  $H(z)$  at normalized angular frequency  $\pi/2$ .

*Mistake: there is a  $j$  term missing in the answer.*

The system function is given by

$$H(z) = 1 + 0.1z^{-1} - 0.12z^{-2} + 0.04z^{-3} .$$

To evaluate the frequency response at normalized angular frequency  $\Omega = \pi/2$ , let  $z = e^{j\Omega} = e^{j\pi/2} = j$ . Therefore,

$$\begin{aligned} H(j) &= 1 + 0.1j^{-1} - 0.12j^{-2} + 0.04j^{-3} \\ &= 1 - 0.1j + 0.12 + 0.04j = 1.12 - 0.06j . \end{aligned}$$

Giving a magnitude  $|H(j)| = 1.122$  and a phase  $\angle H(j) = -0.053$ .