Homework 1

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Task 1

To prove:

$$\rho(A) \leq ||A|| \tag{1}$$

From the book we already know that the spectral radius ρ(A) of A (square matrix) is the largest absolute value $|\lambda|$ of all eigenvalues λ of A. If we take the equation 3.6 from the book, that the operator norm of a matrix is:

$$||A||_{(m,n)} = \sup_{x \neq 0} \frac{||Ax||_{(m)}}{||x||_{(n)}} \tag{2}$$

where m and n are any symbols for vector norms. We can give a lower estimate to the expression by replacing the vector x by an arbitrary vector, for example an eigenvector u_i (associated with the eigenvalue λ_i):

$$\sup_{x \neq 0} \frac{||Ax||_{(m)}}{||x||_{(n)}} \geq \frac{||Au_i||}{||u_i||} = \frac{||\lambda_i \cdot u_i||}{||u_i||} = |\lambda_i| \frac{||u_i||}{||u_i||} = |\lambda_i|, \ i = 1, \ldots, n. \tag{3}$$

This is true for all eigenvalues of A, in particular the largest in absolute value, $|\lambda| = \rho(A)$. This means that the spectral radius of A is never larger than the operator norm of A.

Task 2

To prove:

$$||A|| < 1 \Rightarrow \lim_{n \to \infty} A^n = 0 \tag{4}$$

wherein A is a square matrix and 0 is a zero matrix.

We don't know if A can be diagonalized, that's why we use the Jordan normal form to transform the matrix. With the Jordan normal we take a matrix Q so that $A = QJQ^{-1}$. Then A^n can be expressed as:

$$\lim_{n \to \infty} A^n = \lim_{n \to \infty} (QJQ^{-1})^n = QJ \underbrace{Q^{-1}Q}_{1} J \underbrace{Q^{-1}Q}_{1} JQ^{-1} ... = QJ^nQ^{-1} \tag{5}$$

The Jordan normal form has all eigenvectors on the diagonal axis and it's an upper triangular matrix. Because of Task 1 we already know, that all the absolute values of the eigenvalues are smaller than 1 with that fact J^n will be 0 when the limes of n is infinity. If $\lim_{n\to\infty} J^n$ is 0 then also $\lim_{n\to\infty} A^n$ is 0.

Task 3

1. $$||x||_\infty \leq ||x||_2 \tag{6}$$

   the infinity norm is defined as:

   $$||x||_\infty = \max_i |x_i| \tag{7}$$

   In our example we name $\max_i |x_i| = x_1$ (it is not the first entry of the vector only the largest of the absolute values). If we insert this in the formula then is looks like the following:

   $$||x||_\infty = x_1 = |\sqrt{x_1^2}| \tag{8}$$

   we can extend the equation above with the absolute values of the other entries (except x_1).

   $$|\sqrt{x_1^2}| \leq |\sqrt{x_1^2 + x_2^2 + .. + x_n^2}| = ||x||_2 \tag{9}$$

   This shows that the infinity norm of a vector is smaller or equal than the 2-norm of the same vector.

2. $$||x||_2 \leq \sqrt{n}||x||_\infty \tag{10}$$

   The 2-norm is defined as:

   $$||x||_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \tag{11}$$

   it can be also written as:

   $$\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \sqrt{x_1^2 + \ldots + x_n^2}. \tag{12}$$

   If we take instead of the different vector values only the max of all absolute vector values then:

   $$\sqrt{x_1^2 + \ldots + x_n^2} \leq \sqrt{n \cdot max(x_i)^2} = \sqrt{n} \cdot max|x_i| = \sqrt{n}||x||_\infty \tag{13}$$

   Thus the 2-norm of a n vector is smaller than the infinity norm of that vector multiplied by square root of n.

3.
$$||A||_\infty \leq \sqrt{n}||A||_2 \tag{14}$$

For this proof we use the knowledge from above, that:

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty \tag{15}$$

The $\infty-$ matrix norm is by definition:

$$||A||_\infty = \sup_{x\neq 0} \frac{||Ax||_\infty}{||x||_\infty} = \sup_{x\neq 0} \frac{\sqrt{n}||Ax||_\infty}{\sqrt{n}||x||_\infty} \leq \sup_{x\neq 0} \frac{\sqrt{n}||Ax||_\infty}{||x||_2} \leq \sup_{x\neq 0} \frac{\sqrt{n}||Ax||_2}{||x||_2} = \sqrt{n}||A||_2 \tag{16}$$

4.
$$||A||_2 \leq \sqrt{n}||A||_\infty \tag{17}$$

For the last proof we use the same principle as in 3:

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} \leq \sup_{x\neq 0} \sqrt{n}\frac{||Ax||_\infty}{||x||_2} \leq \sup_{x\neq 0} \sqrt{n}\frac{||Ax||_\infty}{||x||_\infty} = \sqrt{n}||A||_\infty \tag{18}$$

Task 4

To prove:
$$||A||_2 = ||a||_2, \tag{19}$$

if A is a row matrix and has exactly the same entries as the vector a.

We know, that $||A|| = \sqrt{max\lambda(A^T A)}$ is true. If we multiply the transposed matrix A with matrix A then we'll get a scalar. This scalar is the same as the 2 norm for the vector a.

$$\sqrt{max\lambda(A^T A)} = \sqrt{a_1^2 + \cdots + a_n^2} = ||a||_2 \tag{20}$$

The equation above works because the eigenvalue of a 1x1 matrix is always the entry itself. For the infinity and the one norm you can find a counterexample. The vector a and the matrix have the entries $a_1 = 1$ and $a_2 = 2$. If we take the infinity norm of the matrix A we will get 3 if we take the infinity norm of the vector a we will get 2. If we take the one norm of A we will get 2 and if we take the one norm of a we will get 3, which means if we change the definition so that the infinity norm of the matrix is equal to the one norm of the vector and the other way around it will work.

Task 5

We want to show that the function $f: x \mapsto \|x\|_\beta$ is continuous (at a point x^*) with respect to $\|\cdot\|_1$, i.e.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, \|x - x^*\|_1 < \delta \implies |f(x) - f(x^*)| < \epsilon.$$

We fix $\epsilon > 0$. By the triangular inequality we get

$$|f(x) - f(x^*)| = |\|x\|_\beta - \|x^*\|_\beta| \leq \|x - x^*\|_\beta.$$

Since $\mathbb{R}^n$ is a vector space of finite dimension n, we can write x, x^* as a linear combinasion of the basis vectors, as $x = \sum_{i=1}^n a_i e_i,\ x^* = \sum_{i=1}^n a_i^* e_i$ so that:

$$\|x - x^*\|_\beta = \left\| \sum_{i=1}^n a_i e_i - \sum_{i=1}^n a_i^* e_i \right\|_\beta = \left\| \sum_{i=1}^n (a_i - a_i^*) e_i \right\|_\beta \leq \sum_{i=1}^n |a_i - a_i^*| \cdot \|e_i\|_\beta$$

By denoting $\max\limits_i \|e_i\|_\beta = M$ we can estimate the expression

$$\sum_{i=1}^n |a_i - a_i^*| \cdot \|e_i\|_\beta \leq \sum_{i=1}^n |a_i - a_i^*| \cdot M = M \sum_{i=1}^n |a_i - a_i^*| = M \|x - x^*\|_1.$$

Thus, if we choose $\delta = \frac{\epsilon}{M}$, we have that

$$\|x - x^*\|_1 < \delta \implies |f(x) - f(x^*)| < \epsilon.$$

Task 6

If we consider the vectors u, v as $n \times 1$-matrices, the product of u, v^T is a $(n \times 1) \times (1 \times n) = n \times n$-matrix. The rank is necessarily smaller than or equal to the ranks of u and v^T, that is 1.

Task 7

We know form Task 6 that uv^T is a nxn-matrix of rank 1. We search the three $n \times n$–matrices U, Σ, V^T, where U, V^T are orthogonal and Σ diagonal (with the singular values as entries). Because of its rank, we know there is only one non-zero singular value, thus the matrix will look like

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We construct the orthogonal matrices U and V^T with the normed vectors u, v^T, so it looks like:

$$U = \begin{bmatrix} \frac{u_1}{||u||} & a_{1,1} & \dots & a_{(n-1),1} \\ \frac{u_2}{||u||} & a_{1,2} & \dots & a_{(n-1),2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{u_n}{||u||} & a_{1,n} & \dots & a_{(n-1),n} \end{bmatrix}$$

```python
# -*- coding: utf-8 -*-
"""
Created on Wed Sep  6 10:24:24 2017

@author: edadagasan
"""
from scipy import *
from pylab import *
import sys
import scipy.linalg as sl
import numpy as np

u=[3,0,0]
v=[0,3,0]

UVt=np.outer(u,v)


U,s,Vh = sl.svd(UVt)

print(s,U,Vh)

```

Figure 1: Construction of a numerical example of two vectors u, v, determining the outer product of the vectors and returning the singular value decomposition.

where the vectors $a_1, \ldots, a_{n-1}$ are chosen to be orthogonal to each other and to u. The matrix V is constructed in the same way, but with the vector v. Then it is transposed. We find that $\sigma_1 = ||u||||v||$.

We finally look at an example:

```
In [11]: runfile('/Users/edadagasan/Desktop/NLA_H1.py',
wdir='/Users/edadagasan/Desktop')
[ 9.  0.  0.] [[ 1.  0.  0.]
 [ 0.  0.  1.]
 [ 0.  1.  0.]] [[ 0.  1.  0.]
 [ 0.  0.  1.]
 [-1.  0.  0.]]
```

Figure 2: The output of the code in Figure 1. The first vector contains the singular values, and the two matrices are U and V^T.