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MATH 4200
Dr. Gay
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Section 5.1 and Order Topology Solutions

Theorem 5.5

If X and Y are separable spaces, then $X \times Y$ is separable.

Proof. Both X and Y have countable dense subsets; call these subsets $A \subset X$ and $B \subset Y$. The product $A \times B$ is countable. There is no closed proper subset of X that contains A.

Let $p = (x, y) \in X \times Y$ be given, where $x \in A$ and $y \in B$. Then for every open set $U \subset X$ where $U \ni x$, $(U - x) \cap A$ is nonempty. Similarly, for every open set $V \subset Y$ where $V \ni y$, $(V - y) \cap B$ is nonempty. It follows that $U \times V$ contains p, and $(U \times V - p) \cap A \times B$ is nonempty, so $A \times B$ is separable.

□

Order Topology Theorem

The order topology is normal.

Proof. Let $(X, \mathcal{T}_{order})$ be the topological space made by the order topology with set X, and let A, B be disjoint closed sets in X. Let C be the complement of $A \cup B$ in X, or $C = X - (A \cup B)$. Let $a \in A$ be given.

For some $b_1 \in B$, there is a set $(a, b_1) = \{x | a < x < b_1 \quad for \quad x \in X\}$ where $(a, b_1) \cap A = \emptyset$ and $(a, b_1) \cap B = \emptyset$. Note that this set may be empty. The set (a, b_1) is open. Remove some point c from this set, and we have two open intervals, either of which may be empty: (a, c) and (c, b_1). For some other $b_2 \in B$, there is a set $(b_2, a) = \{x | b_2 < x < a \quad for \quad x \in X\}$ where $(b_2, a) \cap A = \emptyset$ and $(b_2, a) \cap B = \emptyset$. Again, this set may be empty. Removing a point e from this set and proceeding in a similar manner as above, we have two open intervals (b_2, e) and (e, a), either of which may be empty. Then taking the union $(e, a) \cup \{a\} \cup (a, c)$, we have an open set $(e, c) = \{x | e < x < c \quad for \quad x \in X\}$ that contains a and contains no points in B. Assuming without loss of generality that (e, a) is empty, the union above becomes $(b_2, a) \cup \{a\} \cup (a, c)$, and still contains no points in B and is open.

For b_1, we can construct an open set (possibly empty) (b_1, f) using the method above and take a union $(c, b_1) \cup \{b\}(b_1, f)$ which contains no points in A. Proceeding in this manner for d and all other elements of A and B, we can construct an open set containing every point in A and every point in B, where each of these sets are disjoint. Taking the union of all of these sets that contain points in A, we have an open set containing all of A, and taking the union of all of these sets that contain points in B, we have an open set containing all of B. These open sets containing A and B are disjoint. □