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MATH 4010
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Homework 9, Section 9.4 Solutions

1

Let N be a normal subgroup of G, let $a \in G$, and let C be the conjugacy class of a in G.

a)

Claim: $a \in N$ if and only if $C \subseteq N$.

Proof. Suppose $a \in N$. Since N is normal, for any $h = g^{-1}ag$ for $g \in G$, $h \in N$, so all conjugates of a are in N.
Conversely, suppose $C \subseteq N$. Then there is some $h \in N$ where $h = g^{-1}ag$. Again, since N is normal, for any $x \in G$, $xNx^{-1} = N$, so $xhx^{-1} = xx^{-1}axx^{-1} = a \in N$. ☐

b)

Claim: If C_i is any conjugacy class in G, prove that $C_i \subseteq N$ or $C_i \cap N = \emptyset$.

Proof. Assume there is some non-identity $h \in N$ and $h \in C_i$. Now assume for the purpose of contradiction that $C_i \nsubseteq N$. Then there is some $x \in G$ such that for any $y \in C_i$, $g^{-1}hg = y$ for $g \in G$. Since $g^{-1}Ng = N$, $g^{-1}hg \in N$ which implies that $y \in N$. So all elements in C_i are in N, and we have a contradiction. ☐

c)

The class equation shows that for a group G, $|G| = |C_1| + |C_2| + \cdots + |C_i|$. We've shown that each class equation belongs to one and only one normal subgroup of G, so it follows that $|N| = |C_1| + |C_2| + \cdots + |C_i|$ where $C_1, C_2, \cdots, C_i$ are the conjugacy classes contained in N.

2

Claim: If $N \neq < e >$ is a normal subgroup of G and $|G| = p^n$, $N \cap Z(G) \neq < e >$.

Proof. By the First Sylow Theorem, G is a Sylow p-group, so its subgroups are p-subgroups and thus have non-trivial centers. Consider a normal subgroup M of one of these p-subgroups P; the conjugation of M by $g \in G$ then maps $a \in Z(P)$ to itself, and a is in M so the intersection is non-trivial. ☐

3

Claim: If K is a Sylow p-subgroup of G and H is a subgroup that contains the normalizer $N(K)$, then $[G : H] \equiv 1 \mod p$.

Proof. By Theorem 9.25, the number of distinct H-conjugates of K is $[H : H \cap N(K)]$. Since H contains $N(K)$, $H \cap N(K) = N(K)$. The index $[G : N(K)]$ is equal to $1 \mod p$, and $[H : N(K)] = 1 \mod p$, so $[G : N(K)] = [G : H][H : N(K)] = [G : H] \cdot 1 \mod p = 1 \mod p$.

□

4

Claim: If K is a Sylow p-subgroup of a group G, then its $N(N(K)) = N(K)$ where $N()$ is a normalizer.

Proof. Since K is a Sylow p-subgroup of both $N(K)$ and $N(N(K))$, then for some $g \in N(N(K))$, $g^{-1}Kg$ is a subgroup of $g^{-1}N(K)g$. $N(K)$ is normal, so $g^{-1}N(K)g = N(K)$. By the conjugation property of Sylow p-groups, there is some $h \in N(K)$ where $g^{-1}Kg = h^{-1}Kh$. Since h is in the normalizer of K, its conjugation of K remains in K, or $g^{-1}Kg = h^{-1}Kh = K$. It follows that g is also in $N(K)$. □