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Section 6.3 Solutions

Exercise 6.22

Given the space 2^X and preimage $\pi_x^{-1}(\delta)$, where $\delta \in \{0,1\}$ and $\pi_x^{-1}(\delta) = \{f \in \{0,1\}^X | f(x) = \delta\}$, we have the subbasis $\mathcal{S} = \{\pi_x^{-1}(\delta) | x \in X, \delta \in \{0,1\}\}$. For any open cover $\mathcal{B} \subset \mathcal{S}$, let $B \in \mathcal{B}$ be given; then B is of the form $\pi_x^{-1}(\delta)$ for some $x \in X$. Let $\gamma \in \{0,1\}$ such that $\gamma \neq \delta$; for any $y \notin B$, $y \in \pi_x^{-1}(\gamma)$.

It remains to show that for some $\pi_x^{-1}(0)$ in $\mathcal{B}$, $\pi_x^{-1}(1)$ is also in $\mathcal{B}$. Suppose for the purpose of contradiction there is no $\pi_x^{-1}(0)$ in $\mathcal{B}$ where $\pi_x^{-1}(1)$ is also in $\mathcal{B}$; then $\mathcal{B} = \{\pi_x^{-1}(\delta) | x \in X \quad and \quad \delta \in \{0,1\} - \{\gamma\}\}$ and

$$\bigcap_{x \in X, \gamma \neq \delta} \pi_x^{-1}(\gamma)$$

is not in any subset of $\mathcal{B}$, but $\mathcal{B}$ covers 2^X, so we have a contradiction. So for any cover $\mathcal{B}$ of 2^X, there is an $x \in X$ such that $\pi_x^{-1}(0) \in \mathcal{B}$ and $\pi_x^{-1}(1) \in \mathcal{B}$, making a finite subcover of two subbasic sets.

Exercise 6.24

Let $U = [0, 2/3)$ and let $V = (1/3, 1]$. Then for countably infinite set ω, we have a cover $\mathcal{C} = \{U_i, V_i | i \in \omega\}$ of $[0,1]^\omega$ with the box topology. Each $C \in \mathcal{C}$ is of the form

$$\begin{aligned}
C &= \prod_{C_i \in \{U_i, V_i\}}^{\omega} C_i \\
&= \left(\prod_{C_i \in \{U_i, V_i\}, i \neq j}^{\omega} C_i \right) \times C_j.
\end{aligned}$$

For any $j \in \omega$ and any $D_j \in \{U_j, V_j\}$ where $D_j \neq C_j$ then, we have

$$\begin{aligned}
D &= \left(\prod_{C_i \in \{U_i, V_i\}, i \neq j}^{\omega} C_i \right) \times D_j \\
&\neq \left(\prod_{C_i \in \{U_i, V_i\}, i \neq j}^{\omega} C_i \right) \times C_j.
\end{aligned}$$

Suppose for the purpose of contradiction that D is not in $\mathcal{C}$. Then $\mathcal{C}$ is not a cover of $[0,1]^\omega$, a contradiction. So D must be in $\mathcal{C}$ for every $D_j \neq C_j$, and there are infinitely many necessary products for $\mathcal{C}$ to be a cover, so $\mathcal{C}$ has no finite subcover.