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MATH 4010
Dr. Chastkofsky
MWF 10:20am
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Bonus Assignment on 7.5 Solutions

1

Claim: If $G = S_3$ then $\text{Inn}(G) = \text{Aut}(G)$.

Proof. We know that $\text{Inn}(G) \subseteq \text{Aut}(G)$, so we must show that $\text{Aut}(G) \subseteq \text{Inn}(G)$. Since S_3 has only 6 elements, we will list the elements explicitly:

$(1\ 2\ 3) = (1\ 2)(1\ 3\ 2)(1\ 2)$
$(1\ 3\ 2) = (1\ 2)(1\ 2\ 3)(1\ 2)$
$(1\ 2) = (1\ 2)(1\ 2)(1\ 2)$
$(1\ 3) = (1\ 2)(2\ 3)(1\ 2)$
$(2\ 3) = (1\ 2)(1\ 3)(1\ 2)$
$e = (1\ 2)e(1\ 2)$.

We have shown that every automorphism of G can be written as an inner automorphism, and is thus an element of the inner automorphism group. So $\text{Inn}(G) \subseteq \text{Aut}(G)$ and $\text{Aut}(G) \subseteq \text{Inn}(G)$, and as a result, $\text{Inn}(G) = \text{Aut}(G)$. $\square$

2

Claim: The symmetric group S_n is isomorphic to a subgroup of the alternating group A_{n+2}.

Proof. We define a map $f : S_n \to A_{n+2}$ as follows: for every permutation τ_j in S_n for even j, $f(\tau_j) = \tau_j$, and for every permutation τ_k in S_n for odd k, $f(\tau_k) = \tau_k\sigma$ where σ is the transposition $(n\ \ n+1)$. Note that σ and any τ are disjoint, and note that σ is order 2. We must show that this map is an isomorphism.

First, let two permutations τ_a, τ_b in S_n be given. Their product $\tau_a\tau_b$ is either even or odd; if even, $f(\tau_a\tau_b) = \tau_a\tau_b$. If the product is odd, $f(\tau_a\tau_b) = \tau_a\tau_b\sigma$, which produces an even permutation. Now suppose τ_a is even and τ_b is even. Then $f(\tau_a)f(\tau_b) = \tau_a\tau_b$. If one of τ_a, τ_b is even and the other odd, then the product of the mappings $f(\tau_a)f(\tau_b)$ is either $\tau_a\sigma\tau_b$ or $\tau_a\tau_b\sigma$, and since σ is disjoint, these two terms are equal. If both τ_a and τ_b are odd, then $f(\tau_a)f(\tau_b) = \tau_a\sigma\tau_b\sigma = \tau_a\sigma\sigma\tau_b = \tau_a\tau_b$. So $f(\tau_a\tau_b) = f(\tau_a)f(\tau_b)$, and the mapping f is a homomorphism.

Next, let two permutations τ_a, τ_b be given, and suppose $f(\tau_a) = f(\tau_b)$. We have two cases for $f(\tau_a)$. Either $f(\tau_a) = \tau_a$, or $f(\tau_a) = \tau_a\sigma$. In the former case, this implies $\tau_a = \tau_b$. In the latter case, this implies $\tau_a\sigma = \tau_b\sigma$. Then with σ being order 2, we have

$$
\begin{aligned}
\tau_a \sigma \sigma^{-1} &= \tau_b \sigma \sigma^{-1} \\
\rightarrow \tau_a \sigma \sigma &= \tau_b \sigma \sigma \\
\rightarrow \tau_a &= \tau_b,
\end{aligned}
$$

so the mapping f is injective.

To show that f is surjective, let ϕ in $B \subseteq A_{n+2}$ be given. Then ϕ is either of the form τ_a for even a, or $\tau_a\sigma$ for odd a. Since τ_a is a permutation of at most length n, it is in the group S_n and we can choose τ_a from S_n to map $f(\tau_a) = \phi$, so the mapping f is surjective.

By the properties of homomorphisms, f maps the identity in S_n to the identity in $B \subseteq A_{n+2}$, and likewise it maps all inverses to B. Since σ is of order 2, the product of any two elements ϕ and α in B are of the form τ_a for even a, or $\tau_a\sigma$ for odd a, so B is closed under the group operation and is a subgroup of A_{n+2}. $\square$