Elliott Daimler
MATH 4200
Dr. Gay
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Section 2.2 Solutions

Exercise 2.4

Claim: The "standard topology" $\mathcal{T}_{std}$ on $\mathbb{R}^n$ satisfies the four conditions of a topology.

Proof. It is vacuously true that $\emptyset$ is open. Next, let $\epsilon = 1$; then for every point $p \in \mathbb{R}$, $B(p, \epsilon) \subset \mathbb{R}$, so $\mathbb{R}$ is open.

For any U, $V \in \mathcal{T}_{std}$, if $U \cap V = \emptyset$, then $U \cap V$ is open. If $U \cap V \neq \emptyset$, then there is a point $p \in U \cap V$. Let $B_U(p, \epsilon_U)$ be the open ball in U around center p, and let $B_V(p, \epsilon_V)$ be the open ball in V around center p. Since B_U and B_V are concentric, at least one of the following is true:

Case 1: $B_U \subset B_V$. Then $B_U \subset U$ and $B_U \subset V$.

Case 2: $B_V \subset B_U$. Then $B_V \subset V$ and $B_V \subset U$.

So for every U, $V \in \mathcal{T}_{std}$, $U \cap V \in \mathcal{T}_{std}$.

Last, let λ be an infinite subset of $\mathcal{T}_{std}$; then for every point $p \in \cup_{\alpha\in\lambda} U_\alpha$ there is a $U_\beta \in \lambda$ where $p \in U_\beta$. It follows that there is an open ball $B(p, \epsilon) \subset U_\beta$ for $\epsilon > 0$, so $B(p, \epsilon) \subset \cup_{\alpha\in\lambda} U_\alpha$ and $\cup_{\alpha\in\lambda} U_\alpha \in \mathcal{T}_{std}$. $\square$

Exercise 2.11

1

For the standard topology on $\mathbb{R}$ with $a, b \in \mathbb{R}$, a closed interval $A = [a, b]$ has limit points a and b, because any open interval containing a or b also contains other elements of A. The elements a, b are also contained in A.

For the finite complement topology on $\mathbb{R}$, the open set $A = (-\infty, 0) \cup (0, \infty)$ has a limit point $p = 1$; this is because for any open set $B \subset \mathbb{R}$ where $p \in B$, B has infinite points and therefore must contain other points in A.

2

For the standard topology on $\mathbb{R}$ with $a, b \in \mathbb{R}$, an open interval $A = (a, b)$ has limit points a and b because any open set containing these two points also contains other points in A, but a, b are not contained in A.

For the finite complement topology on $\mathbb{R}$, the open set $A = (-\infty, 0) \cup (0, \infty)$ has a limit point $p = 0$. It is a limit point because for any open set $B \subset \mathbb{R}$ where $p \in B$, B has infinite points so it must contain other points in A.

3

Any set A in the topology on $\mathbb{R}$ has no limit points, because for any point p in $\mathbb{R}$, the set $\{p\}$ is open, so $\{p\} - \{p\} \cap A = 0$. It follows that any point in A is an isolated point.

A closed set A in the finite complement topology on $\mathbb{R}$ is finite, so for any point $p \in A$, there is an open set B containing p for which $B - \{p\} \cap A = \emptyset$. It follows that A has no limit points and p is an isolated point of A.

4

As explained in part 3), a closed set A in the finite complement topology on $\mathbb{R}$ has no limit points, so for any point $p \notin A$, p is not a limit point of A.

In the set of integers $\mathbb{Z}$ with the standard topology on $\mathbb{R}$, any non-integer rational $p \in (\mathbb{Q} - \mathbb{Z})$ is not contained in $\mathbb{Z}$; neither is it a limit point of $\mathbb{Z}$, because for any non-integer rational p, there is an integer n such that $n < p < n+1$, so there exists an open interval $(n, n+1)$ containing p where $(n, n+1) - \{p\} \cap \mathbb{Z} = \emptyset$.

Exercise 2.12

1

Every set with the discrete topology is closed, because every set $B \subset X$ is open, therefore $X - B$ is closed.

2

Both sets $X, \emptyset$ in the indiscrete topology $\{X, \emptyset\}$ are closed, because both sets are open, so the complements $X - \emptyset = X$ and $X - X = \emptyset$ are closed.

3

A set $A \subset X$ is open in the finite complement topology when its complement $X - A$ is finite, so every finite set in this topology is closed, including the empty set $\emptyset$. The entire set X is also closed, because the empty set is open, and its complement $X - \emptyset = X$ is closed.

4

A set $A \subset X$ is open in the countable complement topology on X when its complement is countable, so every countable set in X is closed, including the empty set $\emptyset$. The entire set X is also closed, because the empty set is open, and its complement $X - \emptyset = X$ is closed.

Exercise 2.18

1

In the standard topology $\mathcal{T}_{std}$ on $\mathbb{R}$, the finite union $\cup_{i=1}^{n}[a_i, b_i]$ of any finite set of closed intervals $[a_i, b_i] \in \mathbb{R}$ is closed, but not open.

Any finite, nonempty subset $A \subset \mathbb{R}$ with the finite complement topology on $\mathbb{R}$ is closed, but not open.

2

For any finite, nonempty set $A \subset \mathbb{R}$ with the finite complement topology on $\mathbb{R}$, its complement $\mathbb{R} - A$ is open, but not closed.

Let λ be a collection of open intervals $(a, b) \in \mathbb{R}$ with the standard topology on $\mathbb{R}$; then if the union $A = \cup_{(a,b)\in\lambda}(a, b)$ is open if $A \subsetneq \mathbb{R}$.

3

Both elements $X, \emptyset$ in the indiscrete topology $\{X, \emptyset\}$ on X are open and closed.

Every set $A \subset \mathbb{R}$ with the discrete topology on $\mathbb{R}$ is both open and closed.

4

In the standard topology on $\mathbb{R}$ with $a, b \in \mathbb{R}$ with $a \neq b$, intervals $(a, b]$ that are left-open, right-closed, and intervals $[a, b)$ that are right-open, left-closed, are neither open nor closed.

Nonempty proper subsets $A \subsetneq X$ of a set X with the indiscrete topology $\{X, \emptyset\}$ are neither open nor closed.

Exercise 2.21

For $a, b, c \in \mathbb{R}$ with $a < b < c$, let $A = (a, b)$, $B = (a, b) \cup \{c\}$, and let $\mathbb{Q}$ be the rational numbers.

1

In the discrete topology on $\mathbb{R}$, every set is open and closed, so $\overline{A} = A$, $\overline{B} = B$, and $\overline{C} = C$.

2

For any nonempty subset U of X in the indiscrete topology $\{X, \emptyset\}$, every point in X is a limit point of U. So the closures of A, B, and $\mathbb{Q}$ are all X.

3

The sets A, B, and C are infinite sets with infinite complements in $\mathbb{R}$, so they are neither open nor closed in the finite complement topology. Any open set U in this topology will have a finite complement, so U must intersect A, B, and C at infinitely many points. It follows that any point in $\mathbb{R}$ is a limit point of A, B, and C, so the closure of each of these sets is $\mathbb{R}$.

4

In the standard topology on $\mathbb{R}$, the limit points of A include all points in (a, b) and the points a, b, so the closure of A is $[a, b]$.

Similarly, the limit points of B include all points in (a, b), and the points a, b. The point c is not a limit point of B, so the closure of B is $[a, b] \cup \{c\}$.

Any nonempty open subset in $\mathbb{R}$ has a nonempty intersection with $\mathbb{Q}$, so every point in $\mathbb{R}$ is a limit point of $\mathbb{Q}$, and the closure of $\mathbb{Q}$ is $\mathbb{R}$.

Exercise 2.29

As in Exercise 2.21, we will use sets $A = (a, b)$, $B = (a, b) \cup \{c\}$ for a, b, $c \in \mathbb{R}$ and $a < b < c$, and the rational numbers $\mathbb{Q}$.

1

In the discrete topology on $\mathbb{R}$, every subset of $\mathbb{R}$ is open, so the interiors of A, B, and $\mathbb{Q}$ are A, B, and $\mathbb{Q}$, respectively.

Every subset of $\mathbb{R}$ is also closed in this topology, so each subset along with its complement are closed. It follows that the boundaries of A, B, and $\mathbb{Q}$ are all empty.

2

The only open subset of A, B, and C in the indiscrete topology $\{\mathbb{R}, \emptyset\}$ is the empty set $\emptyset$, so the interior of each of these sets is empty. The only closed set containing each of these sets is the entire set $\mathbb{R}$, so $\overline{A} = \mathbb{R}$, $\overline{B} = \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. Likewise, the closure of each of their complements is also $\mathbb{R}$, so $\overline{A} \cap \overline{\mathbb{R} - A} = \mathbb{R}$, $\overline{B} \cap \overline{\mathbb{R} - B} = \mathbb{R}$, and $\overline{\mathbb{Q}} \cap \overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$. These are the boundaries of each set.

3

The only open set contained in A in the finite complement topology on $\mathbb{R}$ is the empty set $\emptyset$, and the same is true for sets B and $\mathbb{Q}$, so the interior of each of these sets is empty. Every point in this topology is a limit point, so $\overline{A} = \mathbb{R}$, $\overline{B} = \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$. The complement of each of these sets in $\mathbb{R}$ is also infinite, so $\overline{\mathbb{R} - A} = \mathbb{R}$, $\overline{\mathbb{R} - B} = \mathbb{R}$, and $\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$. It follows that $\overline{A} \cap \overline{\mathbb{R} - A} = \mathbb{R}$, $\overline{B} \cap \overline{\mathbb{R} - B} = \mathbb{R}$, and $\overline{\mathbb{Q}} \cap \overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$. These are the boundaries of each set.

4

In the standard topology on $\mathbb{R}$, A is open, so $\mathrm{Int}(A) = A$. The interval (a, b) is open, and the only open set contained in $\{c\}$ is the empty set $\emptyset$, so $\mathrm{Int}(B) = (a, b)$. The only open set contained in $\mathbb{Q}$ is the empty set $\emptyset$, so $\mathrm{Int}(\mathbb{Q}) = \emptyset$.

The closure of A in the standard topology on $\mathbb{R}$ is $[a, b]$, and the closure of $\mathbb{R} - A$ is $(-\infty, a] \cup [b, \infty)$, so $\overline{A} \cap \overline{\mathbb{R} - A} = \{a, b\}$ is the boundary. Similarly, the closure of B is $[a, b] \cup \{c\}$ and the closure of $\mathbb{R} - B$ is $(-\infty, a] \cup [b, \infty) \cup \{c\}$, so $\overline{B} \cap \overline{\mathbb{R} - B} = \{a, b, c\}$ is the boundary. The closure of $\mathbb{Q}$ is $\mathbb{R}$, and the closure of $\mathbb{R} - \mathbb{Q}$ is $\mathbb{R}$, so the boundary of $\mathbb{Q}$ is $\overline{\mathbb{Q}} \cap \overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$.