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MATH 4010
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MWF 10:20am
April 27, 2022

Homework 11, Section 12.1 Solutions

1

Claim: If K is an extension field of $\mathbb{Q}$ and σ is an automorphism of K, then σ is a $\mathbb{Q}$-automorphism.

Proof. K is characteristic 0, so $1 \cdot n = \underbrace{1 + 1 + \cdots + 1}_{\text{n}}$ and $\sigma(n) = \underbrace{\sigma(1) + \sigma(1) + \cdots + \sigma(1)}_{\text{n}} = n$. Elements of the field $\mathbb{Q}$ have the form $n/m = n(m^{-1})$ where $n, m \in \mathbb{Z}$ and m^{-1} is the multiplicative inverse of m.

Now let $a \in \mathbb{Q}$ be given. Then there are some $n, m \in \mathbb{Z}$ where $a = n(m^{-1})$ and

$$\begin{aligned} \sigma(a) &= \sigma[n(m^{-1})] \\ &= \sigma(n)\sigma(m^{-1}) \\ &= \sigma(n)\sigma(m)^{-1}. \end{aligned}$$

Making further use of the properties of automorphism, we have

$$\begin{aligned} \sigma(n)\sigma(m)^{-1} &= \underbrace{\sigma(1) + \sigma(1) + \cdots + \sigma(1)}_{\text{n}} \underbrace{(\sigma(1) + \sigma(1) + \cdots + \sigma(1))^{-1}}_{\text{m}} \\ &= n(m^{-1}). \end{aligned}$$

So σ maps the rational a to itself, and is a $\mathbb{Q}$-automorphism.

$\square$

2

a

For $\omega = (-1+\sqrt{3}i)/2$, we have

$$\begin{aligned}
\omega^2 &= (-1+\sqrt{3}i)(-1+\sqrt{3}i)/4 \\
&= (1-2\sqrt{3}i-3)/4 \\
&= (-2-2\sqrt{3}i)/4 \\
&= (-1-\sqrt{3}i)/2,
\end{aligned}$$

so ω^2 is the complex conjugate of ω. Using a similar computation to the above, we have $(\omega^2)^2 = \omega$. The minimum polynomial of ω over $\mathbb{Q}$ is x^2+x+1:

$$\begin{aligned}
\omega^2+\omega+1 &= (-1-\sqrt{3}i)/2+(-1+\sqrt{3}i)/2+1 \\
&= \frac{-1}{2}-\frac{\sqrt{3}i}{2}+\frac{-1}{2}+\frac{\sqrt{3}i}{2}+1 \\
&= -1+1 \\
&= 0.
\end{aligned}$$

Since $(\omega^2)^2 = \omega$, if we use ω^2 in the above polynomial, the two complex conjugates simply switch positions and we get the same result:

$$\begin{aligned}
(\omega^2)^2+(\omega^2)+1 &= (-1+\sqrt{3}i)/2+(-1-\sqrt{3}i)/2+1 \\
&= \frac{-1}{2}+\frac{\sqrt{3}i}{2}+\frac{-1}{2}-\frac{\sqrt{3}i}{2}+1 \\
&= 0.
\end{aligned}$$

So ω and ω^2 are the roots of this polynomial.

b

The minimal polynomial in part **a** has degree 2, so by the Fundamental Theorem of Algebra, there are two roots. We have already identified these roots as $\omega = (-1+\sqrt{3}i)/2$ and $\omega^2 = (-1-\sqrt{3}i)/2$, so the Galois group $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\omega)$ has one automorphism, say σ, that permutes these two roots, and one identity automorphism, say τ:

$$\begin{aligned}
\sigma(\omega) &= \omega^2 \\
\sigma(\omega^2) &= \omega \\
\tau(\omega) &= \omega \\
\tau(\omega^2) &= \omega^2.
\end{aligned}$$

The above mappings obey the properties of automorphism; for example, with $\sigma(\omega^2)$ we have $\sigma(\omega^2) = \sigma(\omega)^2 = \sigma(\omega)\sigma(\omega) = \omega^2\omega^2$, and as shown in part **a**, $(\omega^2)^2 = \omega$. So $\mathrm{Gal}_{\mathbb{Q}}\mathbb{Q}(\omega)$ has the two elements described above.