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MATH 4200
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Sections 7.1 Through 7.4 Solutions

Exercise 7.12

Since $h = f$ and $h = g$ where $A \cap B \neq \emptyset$, we can define h as

$$h(x) = \left\{ \begin{array}{ll} f(x), & \text{if } x \in X - B \\ g(x), & \text{if } x \in B \end{array} \right\}.$$

So for an open set U, we have

$$h(U) = \{f(x_1) | x_1 \in (X - B) \cap U\} \cup \{g(x_1) | x_2 \in B \cap U\}$$

and for an open set $V \in Y$, we have

$$h^{-1}(V) = \{z_1 | z_1 \in (X - B) \cap f^{-1}(V)\} \cup \{z_2 | z_2 \in B \cap g^{-1}(V)\}.$$

Taking $(X - B)$ as a subspace of A, any open $W \subset (X - B)$ is equal to some $U \cap (X - B)$ where U is open in A. Since $U - W$ is in $g^{-1}(V)$ and W is in $f^{-1}(V)$, U is in $h^{-1}(V)$ and the preimage $h^{-1}(V)$ is the union of these open preimages $f^{-1}(V)$ and $g^{-1}(V)$. So $h^{-1}(V)$ is open and h is continuous.

Exercise 7.20

1

Let $\mathbb{R}_{disc}$ be the set $\mathbb{R}$ with the discrete topology, and let $\mathbb{R}_{std}$ be the set $\mathbb{R}$ with the standard topology. Then Then a mapping $f : \mathbb{R}_{std} \to \mathbb{R}_{disc}$ defined by $f(x) = x$ is open because any open set in $\mathbb{R}_{std}$ is open in $\mathbb{R}_{disc}$, but f is not continuous because any $x \in \mathbb{R}_{disc}$ is open, but $f^{-1}(x)$ is not open in $\mathbb{R}_{std}$.

2

Same as above; the mapping $f : \mathbb{R}_{std} \to \mathbb{R}_{disc}$ defined by $f(x) = x$ is closed because for every closed $A \subset \mathbb{R}_{std}$, $f(A)$ is closed, but f is not continuous.

3

Let $g : \mathbb{R}_{disc} \to \mathbb{R}_{std}$ be defined by $g(x) = f^{-1}(x)$ where f is the mapping defined in parts 1 and 2. Then g is continuous, but is not open nor closed.

4

Let $A = (0, 1)$ be a subspace of $\mathbb{R}_{std}$. Then a mapping $h : A \to \mathbb{R}_{std}$ defined by $h(x) = x$ (inclusion map) is continuous, because for any open V in $\mathbb{R}_{std}$, $h^{-1}(V) = U \cap A$ for some open U in $\mathbb{R}_{std}$, and $U \cap A$ is the intersection of two open intervals. any open set in A is open in $\mathbb{R}_{std}$, but A is closed in the subspace, while A is not closed in $\mathbb{R}_{std}$.

5

Let $\mathbb{Z}_{comp}$ be the set of integers with the countable complement topology. Then for a mapping $j : \mathbb{Z}_{disc} \to \mathbb{R}_{std}$ defined by $j(x) = x$, j is continuous because every set in j is open. A closed set in $\mathbb{Z}_{comp}$ is finite, so its image onto $\mathbb{R}_{std}$ is a set of finite points, and is closed.

Exercise 7.37

Given inclusion maps $i_{X,y} : X \to X \times Y$ defined by $i_{X,y}(x) = (x, y)$, and a mapping $f : X \times Y \to Z$ where $f \circ i_{X,y}$ is continuous, then for every open $W \subset Z$, $i_{X,y}^{-1} \circ f^{-1}(W) = U_y$ where U_y is open in X. There is some $V \subset X \times Y$ where $f^{-1}(W) = V$ and $i_{X,y}^{-1}(V) = U_y$.

Following in a similar manner for inclusion maps $i_{Y,x} : Y \to X \times Y$ defined by $i_{Y,x}(y) = (x, y)$, we have $f^{-1}(W) = V$ and $i_{Y,x}^{-1}(V) = U_x \subset Y$. for every $U_y \subset X$ and $U_x \subset Y$, we can construct a basic open set in $X \times Y$; then V is equal to the union of all such subbasic open sets contained in V, so V is open.

Theorem 7.42

The Cantor set is homeomorphic to the product $\prod_{n\in\mathbb{N}}\{0,1\}$ where $\{0,1\}$ has the discrete topology.

Proof. For $C_0 = [0,1]$, $C_1 = [0,1/3]\cup[2/3,1]$, $\cdots$, C_n where C_n is C_{n-1} with the middle-thirds of each of its intervals removed, we have the Cantor set

$$C = \bigcap_{i=1}^{\infty} C_i.$$

The n^{th} iteration of the set has 2^n intervals, and as $n \to \infty$, these intervals become points at the limit.

The product $\prod_{n\in\mathbb{N}}\{0,1\}$ is bijective to the power set $2^{\mathbb{N}}$, so it is sufficient to show that C is homeomorphic to $2^{\mathbb{N}}$ where $2^{\mathbb{N}}$ has the product topology induced by $\prod_{n\in\mathbb{N}}\{0,1\}$. Let some element a of $\mathbb{N}$ be given; its power set is $\{\emptyset, a\}$. For this one element, we can associate the two intervals in $C_1 = [0,1/3]\cup[2/3,1]$ with the two elements of the power set of a. Then for another element b of $2^{\mathbb{N}}$, the power set of a, b is $\{\{\emptyset\},\{a\},\{b\},\{a,b\}\}$ which has a cardinality of 4. So for the set $\{a,b\}$ of cardinality 2, we can associate the 4 intervals in C_2 with the 4 elements in the power set of $\{a,b\}$. Continuing in this manner, for any interval I in C_n and any element $n\in\mathbb{N}$, we can define a mapping $f_n : C_n \to 2^n$ as $f_n(I) = y \in 2^n$. For $\mathbb{N}$, the Cantor set C contains isolated points, so we then have $f_{\mathbb{N}} : C \to 2^{\mathbb{N}}$ and

$$\begin{aligned} f_{\mathbb{N}}(I) &= f(x) \\ &= y \in 2^{\mathbb{N}}, \end{aligned}$$

where each $x \in C$ corresponds to exactly one $y \in 2^{\mathbb{N}}$. So $f_{\mathbb{N}}$ is injective and surjective.

Every $x \in C$ is a limit point, and every open set U in the subspace C contains infinitely many points, so $f_{\mathbb{N}}(U) = V$ where V is open in $2^{\mathbb{N}}$. $\square$