Elliott Daimler
MATH 4010
Dr. Chastkofsky
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Bonus Assignment, Galois Correspondence Solutions

1

Claim: Let $f(x) = x^3 - 2$ in $\mathbb{Q}[x]$, and let $\omega = e^{2\pi i/3}$. Then for σ, $\tau \in \mathrm{Gal}(\mathbb{Q}[2^{1/3}, \omega 2^{1/3}, \omega^2 2^{1/3}]/\mathbb{Q})$ where $\sigma : 2^{1/3} \to \omega 2^{1/3}$, $\omega \to \omega$ and $\tau : 2^{1/3} \to 2^{1/3}$, $\omega \to \omega^2$, the fixed field of $< \sigma\tau >$ is $\mathbb{Q}[\omega^2 2^{1/3}]$ and that of $< \sigma^2\tau >$ is $\mathbb{Q}[\omega 2^{1/3}]$.

Proof. For the subgroup $< \sigma\tau >$, we have

$$\begin{aligned}
\sigma\tau(\omega^2 2^{1/3}) &= \sigma(\omega^4 2^{1/3}) \\
&= \sigma(\omega\omega^3 2^{1/3}) \\
&= \sigma(\omega 2^{1/3}) \\
&= \omega^2 2^{1/3},
\end{aligned}$$

The base field is $\mathbb{Q}$, so $\sigma\tau$ acts as an identity permutation on $\mathbb{Q}[\omega^2 2^{1/3}]$. The subgroup is generated by $\sigma\tau$ so each $a \in < \sigma\tau >$ is of the form $a = (\sigma\tau)^k$ for some $k \in \mathbb{Z}$, and a acts as a product of identity permutations on $\mathbb{Q}[\omega^2 2^{1/3}]$ and thus fixes the field.

For the subgroup $< \sigma^2\tau >$, we have

$$\begin{aligned}
\sigma^2\tau(\omega 2^{1/3}) &= \sigma^2(\omega^2 2^{1/3}) \\
&= \sigma(\omega^2\omega 2^{1/3}) \\
&= \omega^2\omega\omega 2^{1/3} \\
&= \omega^3\omega 2^{1/3} \\
&= \omega 2^{1/3}.
\end{aligned}$$

Again, the base field is $\mathbb{Q}$, and $\sigma^2\tau$ acts as an identity permutation on $\mathbb{Q}[\omega 2^{1/3}]$. Using an argument similar to that used for $< \sigma\tau >$, we have that the subgroup $< \sigma^2\tau >$ fixes $\mathbb{Q}[\omega 2^{1/3}]$. $\square$

2

Let $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ and let $\omega = e^{2\pi i/7}$ be a 7th root of unity. The polynomial $x^7 - 1$ has ω as a root, so must have $x - 1$ as a factor. We then have

$$\frac{x^7 - 1}{x - 1} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

Let $a = \omega^6 + \omega$. Then $a^2 = \omega^5 + \omega^2 + 2$ and $a^3 = \omega^4 + \omega^3 + 3\omega^6 + 3\omega$. We can rewrite this into the form $a^3 + a^2 - 2a - 1 = 0$, which has no rational roots in $\mathbb{Q}$, so is not reducible in $\mathbb{Q}$. So the minimal polynomial has degree 6 and must be the above polynomial.

The degree of $f(x)$ is 6, so it must have 6 roots. Define $\sigma \in \mathrm{Gal}\mathbb{Q}[\omega]$ as $\sigma(\omega) = \omega^3$. Then $\sigma(\omega) = \omega^3$; $\sigma(\omega^3) = \omega^9 = \omega^2$; $\sigma(\omega^2) = \omega^6$; $\sigma(\omega^6) = \omega^{18} = \omega^4$; $\sigma(\omega^4) = \omega^{12} = \omega^5$; $\sigma(\omega^5) = \omega^{15} = \omega$. So σ generates all roots of the polynomial.

We also have $\sigma^3(\omega) = \omega^{27} = \omega^6$ and $\sigma^3(\omega^6) = \sigma^2(\omega^4) = \sigma(\omega^5) = \omega$, so $< \sigma^3 >$ is a proper subgroup of order 2 of $< \sigma >$, and $\sigma^2(\omega) = \omega^9 = \omega^2$, $\sigma^2(\omega^2) = \omega^4$, and $\sigma^2(\omega^4) = \omega$ so $< \sigma^2 >$ is a proper subgroup of order 3 of $< \sigma >$.

We permute $\omega^6 + \omega$ by σ^3 to get $\sigma^3(\omega^6 + \omega) = \sigma^3(\omega^6) + \sigma^3(\omega) = \omega + \omega^6$. $\mathbb{Q}$ is the base field and is fixed by σ, so $\mathbb{Q}[\omega^6 + \omega]$ is fixed by σ^3.

Similarly, we permute $\omega^4 + \omega^2 + \omega$ by σ^2 to get $\sigma^2(\omega^4 + \omega^2 + \omega) = \sigma^2(\omega^4) + \sigma^2(\omega^2) + \sigma^2(\omega) = \omega + \omega^4 + \omega^2$. Again, σ fixes the base field $\mathbb{Q}$, so $\mathbb{Q}[\omega^4 + \omega^2 + \omega]$ is fixed by σ^2.