Elliott Daimler
MATH 4010
Dr. Chastkofsky
MWF 10:20am
March 3, 2022

Homework 5, Section 8.2 Solutions

1

Claim: Suppose that N is a subgroup of G and that N has the property that if a and b are any elements of G, then ab belongs to N if and only if ba belongs to N. Then N is a normal subgroup.

Proof. Let $c \in N$, $b \in G$ be given. By the closure of G, there is some $a \in G$ such that $cb^{-1} = a$. Then $c = cb^{-1}b = ab$, so $ab \in N$, which implies $ba \in N$. Since $a = cb^{-1}$, it follows that $ba = bcb^{-1} \in N$. □

2

Claim: The inner automorphisms of a group G are a normal subgroup of $\mathrm{Aut}(G)$.

Proof. Let φ_x denote the inner automorphism $\varphi_x(g) = xgx^{-1}$, and let f denote any automorphism of G. Then $\varphi_{f(x)}$ is also in the automorphism group. Now let some $g \in G$ be given; applying $\varphi_{f(x)}$ to g, we have

$$\varphi_{f(x)}(g) = f(x)gf^{-1}(x),$$

and likewise,

$$f \circ \varphi_x \circ f^{-1}(g) = f(xf^{-1}(g)x^{-1}).$$

Since f is an isomorphism,

$$\begin{aligned} f(xf^{-1}(g)x^{-1}) &= f(x)f \circ f^{-1}(g)f(x^{-1}) \\ &= f(x)gf(x^{-1}) \\ &= f(x)gf^{-1}(x) \end{aligned}$$

which is equal to $\varphi_{f(x)}(g)$, so the inner automorphisms of G are a normal subgroup of $\mathrm{Aut}(G)$. □

3

Claim: If K is a characteristic subgroup of N and N is a normal subgroup of G, then K is a normal subgroup of G.

Proof. Let $g \in G$ and $n \in N$ be given, and define the inner automorphism $f_g(n) := gng^{-1}$. N is a normal subgroup of G, so $f_g(n) = gng^{-1} \in N$. Moreover, K is a subgroup of N, so for any $k \in K$, $f_g(k) = gkg^{-1} \in N$. Since K is a characteristic subgroup of N and f_g is in the automorphism group $\mathrm{Aut}(N)$, it follows that $f_g(k) = gkg^{-1} \in K$, so K is a normal subgroup of G. $\square$

4

Claim: Suppose that G is a group all of whose subroups are normal. If a, b, are elements of G, then $ab = ba^k$ for some integer k.

Proof. Let $a_1 \in G$ be given. Then a_1 is a generator for the cyclic group $\langle a_1 \rangle \leq G$. Since all subgroups of G are normal, $\langle a_1 \rangle$ is normal, so for any $b \in G$, $b\langle a_1 \rangle = \langle a_1 \rangle b$ where $ba_1 = a_2 b$ for some $a_1, a_2 \in \langle a_1 \rangle$. The group $\langle a_1 \rangle$ is also cyclic, so for each $a_1, a_2 \in \langle a_1 \rangle$, $a_2 = a_1^k$ for some integer k, where k may be positive or negative. It follows that $a_2 b = ba_1 = a_1^k b$. $\square$