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MATH 4010
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Homework 1 Solutions

1

(a)

To evaluate $(x+1)^3$ in $\mathbb{Z}_3[x]$, we'll first expand the polynomial:

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

Now we evaluate in $\mathbb{Z}_3$. In $\mathbb{Z}_3$, each coefficient in the polynomial is congruent to some integer a for $0 \leq a < 3$, so we have $x^3 + 3x^2 + 3x + 1 = x^3 + 1 \bmod 3$.

(b)

Similar to above, to evaluate $(x-1)^5$ in $\mathbb{Z}_5[x]$ we'll expand the polynomial:

$$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$$

In $\mathbb{Z}_5$, each coefficient in the polynomial is congruent to some integer a for $0 \leq a < 5$, so we have $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 = x^5 - 1 \bmod 5$.

2

(a)

Given $f(x) = x^4 - 7x + 1$ and $g(x) = 2x^2 + 1$ in $\mathbb{Q}[x]$, we can choose $q(x) = (\frac{1}{2}x^2 - \frac{1}{4})$ and $r(x) = 7x + \frac{5}{4}$. The degree of $r(x)$ is nonzero and less than the degree of $f(x)$. Furthermore,

$$\left(2x^2+1\right)\left(\frac{1}{2}x^2 - \frac{1}{4}\right) + \left(7x + \frac{5}{4}\right) = x^4 - 7x + 1.$$

The left side of the above equation is $g(x)q(x)+r(x)$ and the right side is $f(x)$, so we have $g(x)q(x)+r(x) = f(x)$.

(b)

Given $f(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$ and and $g(x) = 3x^2 + 2$ in $\mathbb{Z}_7[x]$, we can choose $q(x) = (6x^2 + x)$ and $r(x) = -x^3 + x^2 + 2x + 5$. The degree of $r(x)$ is nonzero and less than the degree of $f(x)$. Furthermore,

$$\begin{aligned}(3x^2+2)(6x^2+x) + -x^3 + x^2 + 2x + 5 &= 18x^4 + 3x^3 + 12x^2 + 2x + (-x^3 + x^2 + 2x + 5) \bmod 7 \\ &= 4x^4 + 3x^3 5x^2 + 2x + (-x^3 + x^2 + 2x + 5) \bmod 7. \\ &= 4x^4 + 2x^3 + 6x^2 + 4x + 5\end{aligned}$$

The left side of the above equation is $g(x)q(x)+r(x)$ and the right side is $f(x)$, so we have $g(x)q(x)+r(x) = f(x)$.

3

For the polynomial $3x+1$, we can multiply by $6x+1$ to get the multiplicative identity in $\mathbb{Z}_9[x]$:

$$\begin{aligned}(3x+1)(6x+1) &= 18x + 9x + 1 \\ &= 1 \bmod 9.\end{aligned}$$

The resulting polynomial above is not a constant polynomial, but this does not contradict Corrolary 4.5 because $\mathbb{Z}_9[x]$ is not an integral domain.

4

The map $D : \mathbb{R}[x] \to \mathbb{R}[x]$ is not a ring homomorphism, because it does not preserve multiplication or the multiplicative identity. Consider the values $(x+1)$ and $(x+2)$. Then

$$\begin{aligned}D[(x+1)(x+2)] &= D(x^2 + 3x + 2) \\ &= 2x + 3,\end{aligned}$$

and alternatively,

$$\begin{aligned}D(x+1)D(x+2) &= 1 \cdot 1 \\ &= 1.\end{aligned}$$

Now consider an identity element of the particular ring of polynomials, say a_0. Then

$$\begin{aligned} D(a_0 x) &= a_0 \cdot 1 \\ &= a_0 \end{aligned}$$

and alternatively,

$$\begin{aligned} D(a_0)D(x) &= 0 \cdot 1 \\ &= 0. \end{aligned}$$

The counterexamples above show that the map is not a ring homomorphism.

5

Suppose there is some polynomial $f[x]$ in $F[x]$ that divides $x+a$ and $x+b$. Then $f(x)$ must have a degree less than $x+a$ and $x+b$; these two expressions are both degree one, so $f(x)$ is degree zero, meaning it is the multiplicative identity. So $x+a$ and $x+b$ are relatively prime.

6

(a)

For the polynomial $x^4 - x^3 - x^2 + 1$, an easy root is 1, so we can factor out $(x-1)$. This gives us $(x^4 - x^3 - x^2 + 1)/(x-1) = (x^3 - x - 1)$. The result has no more factors in $\mathbb{Q}[x]$, so we are done. For the polynomial $x^3 - 1$, an easy root is again 1, so we can factor out $(x-1)$. This gives us $(x^3-1)/(x-1) = (x^2+x+1)$. The result has no more factors in $\mathbb{Q}[x]$, so we are done. The gcd for both polynomials is $(x-1)$.

(b)

For the polynomial $x^4 + 3x^3 + 2x + 4$, one root is -1, so we can factor out $(x+1)$. Then we have $(x^4 + 3x^3 + 2x + 4)/(x+1) = (x^3 + 2x^2 - 2x + 4)$. The result has no more factors in $\mathbb{Z}_5[x]$, so we are done. For the polynomial $x^2 - 1$, one root is 1, so we can factor out $(x-1)$. Then we have $(x^2-1)/(x+1) = (x-1)$. The result has no more factors in $\mathbb{Z}_5[x]$, so we are done.

7

For the factor $(x-1)$ in part (a) above, we compare the coefficients in the equation

$$x - 1 = a(x^4 - x^3 - x^2 + 1) + b(x^3 - 1)$$

which gives $2a - 2b = -2$ for $x = -1$, $a - b = -1$ for $x = 0$, and $5a + 7b = 1$ for $x = 2$. The solution is $a = -1/2$, $b = 1/2$ and

$$-\frac{1}{2}(x^4 - x^3 - x^2 + 1) + \frac{1}{2}(x^3 - 1) = (x - 1).$$

For the factor $(x + 1)$ in part (b) above, we compare the coefficients in the equation

$$x + 1 = a(x^4 + 3x^3 + 2x + 4) + b(x^2 - 1)$$

which gives $4a - b = 1$ for $x = 0$, and $10a = 2$ for $x = 1$. The solution is $a = 1/5$, $b = -1/5$ and

$$\frac{1}{5}(x^4 + 3x^3 + 2x + 4) - \frac{1}{5}(x^2 - 1) = x + 1.$$