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MATH 4200
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Section 4.2 Solutions

Theorem 4.16

Let topological spaces X and Y be Hausdorff. Then $X \times Y$ is Hausdorff.

Proof. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be points in $X \times Y$ where a_x, $b_x \in X$ and a_y, $b_y \in Y$. If $a \neq b$, then $a_x \neq b_x$ or $a_y \neq b_y$.

Suppose $a_x \neq b_x$; since X is Hausdorff, there exist open sets $U_a \ni a_x$ and $U_b \ni b_x$ such that $U_a \cap U_b = \emptyset$, so for any open sets V_a, $V_b \subset Y$ where $V_a \ni a_y$ and $V_b \ni b_y$, we have

$$\begin{aligned}(U_a \times V_a) \cap (U_b \times V_b) &= \left[\pi^{-1}(U_a) \cap \pi^{-1}(V_a)\right] \cap \left[\pi^{-1}(U_b) \cap \pi^{-1}(V_b)\right] \\ &= \emptyset.\end{aligned}$$

If $a_y \neq b_y$, then by the same reasoning as above we have open sets $V_a \ni a_y$ and $V_b \ni b_y$ where $V_a \cap V_b = \emptyset$, and sets $(U_a \times V_a)$ and $(U_b \times V_b)$ are disjoint, so $X \times Y$ is Hausdorff. $\square$

Theorem 4.17

Let topological spaces X and Y be regular. Then $X \times Y$ is regular.

Proof. Let $p = (p_x, p_y)$ be a point in $X \times Y$, and let $U_x \times U_y$ be an open set in $X \times Y$ where U_x is a basic element in X and U_y is a basic element in Y. Then $p_x \in U_x$ and $p_y \in U_y$. X and Y are each regular, so by Theorem 4.8, there exist open sets $V_x \ni p_x$, $V_y \ni p_y$ where $V_x \subset U_x$ and $V_y \subset U_y$ such that $\overline{V_x} \subset U_x$ and $\overline{V_y} \subset U_y$. The point p is in $V_x \times V_y$, and $\overline{V_x} \times \overline{V_y} \subset U_x \times U_y$.

It remains to show that $\overline{V_x} \times \overline{V_y}$ is closed. Let $q = (q_x, q_y)$ be a limit point of $V_x \times V_y$, and let $Q_x \times Q_y \ni q$ be open where $Q_x \subset X$ and $Q_y \subset Y$ are basic elements. Then $q_x \in Q_x$ and $q_y \in Q_y$, and $(Q_x - q_x) \cap V_x$ is nonempty, and $(Q_y - q_y) \cap V_y$ is nonempty, so $q_x \in \overline{V_x}$ and $q_y \in \overline{V_y}$. It follows that $q \in \overline{V_x} \times \overline{V_y}$ and $\overline{V_x} \times \overline{V_y}$ is closed. By Theorem 4.8, $X \times Y$ is regular. $\square$

Exercise 4.18

Let the set $A \in \mathbb{R}_{LL} \times \mathbb{R}_{LL}$ be the function $f(x) = y = 1 - x$ from $x \in [0,1]$, and $g(x) = y = -1 + x$ from $x \in [0,1]$. Let $U \subset \mathbb{R}_{LL} \times \mathbb{R}_{LL}$ be open where $U = [0,2) \times [-1,2)$; then U contains A. It is not true that