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MATH 4010
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Bonus Assignment, Galois's Criterion Solutions

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Let $f(x) = x^5 - 5x^3 + 5x + 2$ and $\alpha = \omega^9 + \omega$ where $\omega = e^{2\pi i/10}$. The polynomial can be factored into the form $f(x) = (x+2)(x^2 - x - 1)^2$, has a root -2, and has two repeating (irrational) real roots for $(x^2 - x - 1)$, so all 5 roots are real. The trigonometric form of α is

$$\begin{aligned}\alpha &= e^{18\pi i/10} + e^{2\pi i/10} \\ &= \cos\left(\frac{18\pi}{10}\right) + i\sin\left(\frac{18\pi}{10}\right) + \cos\left(\frac{2\pi}{10}\right) + i\sin\left(\frac{2\pi}{10}\right).\end{aligned}$$

Using the sum-to-product identities of sine and cosine, this becomes

$$\begin{aligned}\cos\left(\frac{18\pi}{10}\right) + i\sin\left(\frac{18\pi}{10}\right) + \cos\left(\frac{2\pi}{10}\right) + i\sin\left(\frac{2\pi}{10}\right) &= 2\cos\left(\frac{20\pi}{20}\right)\cos\left(\frac{16\pi}{20}\right) + 2i\sin\left(\frac{20\pi}{20}\right)\sin\left(\frac{16\pi}{20}\right) \\ &= -2\cos\left(\frac{4\pi}{5}\right) + 0 \\ &= -2\cos\left(\frac{4\pi}{5}\right).\end{aligned}$$

The values of α^3 and α^5 are then

$$\begin{aligned}\alpha^3 &= \left[-2\cos\left(\frac{4\pi}{5}\right)\right]^3 \\ &= -4\cos\left(\frac{4\pi}{5}\right) - 4\cos\left(\frac{8\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) \\ &= -4\cos\left(\frac{4\pi}{5}\right) - 2\cos\left(\frac{4\pi}{5}\right) - 2\cos\left(\frac{12\pi}{5}\right),\end{aligned}$$

$$\begin{aligned}\alpha^5 &= \left[-2\cos\left(\frac{4\pi}{5}\right)\right]^5 \\ &= -12\cos\left(\frac{4\pi}{5}\right) - 8\cos\left(\frac{4\pi}{5}\right) - 8\cos\left(\frac{12\pi}{5}\right) - 2\cos\left(\frac{12\pi}{5}\right) - 2 \\ &= -20\cos\left(\frac{4\pi}{5}\right) - 10\cos\left(\frac{12\pi}{5}\right) - 2,\end{aligned}$$

and $f(\alpha)$ becomes

$$\begin{aligned} f(\alpha) &= -20\cos\left(\frac{4\pi}{5}\right) - 10\cos\left(\frac{12\pi}{5}\right) - 2 - 5\left[-4\cos\left(\frac{4\pi}{5}\right) - 2\cos\left(\frac{4\pi}{5}\right) - 2\cos\left(\frac{12\pi}{5}\right)\right] + 5\left[-2\cos\left(\frac{4\pi}{5}\right)\right] \\ &= -20\cos\left(\frac{4\pi}{5}\right) - 10\cos\left(\frac{12\pi}{5}\right) - 2 + 20\cos\left(\frac{4\pi}{5}\right) + 10\cos\left(\frac{4\pi}{5}\right) + 10\cos\left(\frac{12\pi}{5}\right) - 10\cos\left(\frac{4\pi}{5}\right) + 2 \\ &= 0. \end{aligned}$$

So α is a root of $f(x)$. Since α is real and all 5 roots are real, $\mathbb{Q}[\alpha]$ splits $f(x)$.