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MATH 4200
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Section 3.5 Solutions

Exercise 3.39

For rationals q in $\mathbb{Q}$, the set Z consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate and 0 or 1 on any irrational coordinate can be written as

$$Z = \bigcap_{q \in \mathbb{Q}} \pi_q^{-1}(0).$$

Any open set in the product space $2^{\mathbb{R}}$ is a finite intersection of preimages $\pi_x^{-1}(\delta)$ for x in $\mathbb{R}$, and open sets containing some point p in Z must contain an element that is 0 for every rational number and 0 or 1 for every irrational. Any set in $2^{\mathbb{R}}$ that does not contain multiple points in Z must have the form

$$\bigcap_{q \in \mathbb{Q}} \pi_q^{-1}(1),$$

which is an infinite intersection of subbasis elements, and is not an open set. So every set in $2^{\mathbb{R}}$ contains multiple points in Z, and the closure of Z is the entire space $2^{\mathbb{R}}$.

Exercise 3.42

The set $2^{\mathbb{N}}$ with the box topology has basis $\prod_{\alpha \in \lambda} U_\alpha$ for each open set U_α in X_α. Each U_α takes the form $U_\alpha(0)$ or $U_\alpha(1)$, where $U_\alpha(0) = \{f \in \{0,1\}^{\mathbb{N}} | f(\alpha) = 0\}$, and $U_\alpha(1) = \{f \in \{0,1\}^{\mathbb{N}} | f(\alpha) = 1\}$ for functions f mapping each α to 0 or 1. Taking the product of open sets U_α for every α in X produces a single set $\{\delta_\alpha\}_{\alpha \in \mathbb{N}}$ where each $\delta = 0$ or $\delta = 1$. This set is a point in in $2^{\mathbb{N}}$, and the point is a basis element in the box topology. Since every point in the topological space can be generated this way, the space is discrete.

The set $2^{\mathbb{N}}$ with the product topology is generated by subbasis elements $\pi_\alpha^{-1}\{\delta\}$ where $\delta = 0$ or $\delta = 1$. Each $\pi_\alpha^{-1}\{\delta\}$ contains every set $\{\delta_i | i \in \mathbb{N}\} \cup \{\delta_\alpha\}$, or all sets in $2^{\mathbb{N}}$ containing δ_α. Each basis element in this topology is generated by the finite intersections of these preimages, $\cap_{i=1}^n \pi_i^{-1}\{\delta\}$, so for each of these intersections, there are infinitely many preimages $\pi_\alpha^{-1}\{\delta\}$ that are not included in the intersection. For some basis element $\cap_{i=1}^n \pi_i^{-1}\{\delta\}$, let C be the collection of preimages $\pi_\alpha^{-1}\{\delta\}$ that are not included in the intersection, or $\alpha \neq i$ for any α, i. Then the product $\pi_\alpha^{-1}\{\delta\} \times \cap_{i=1}^n \pi_i^{-1}\{\delta\}$ contains all sets containing δ_α and each value δ_i. Since there are infinitely many $\pi_\alpha^{-1}\{\delta\} \in C$, there are infinitely many such sets in each open set in the product topology, so there are no isolated points in $2^{\mathbb{N}}$ with the product topology.