Elliott Daimler
MATH 4010
Dr. Chastkofsky
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Homework 12, Fundamental Theorem Solutions

Claim: Let K/F be Galois, let $G = \mathrm{Gal}(K/F)$, let E be an intermediate field, and let $H = \mathrm{Gal}(K/E)$ so that $E = \mathrm{Inv}(H)$. Then E/F is Galois if and only if $\mathrm{Inv}(E)$ is a normal subgroup.

Proof. We'll prove the claim in three sections.

1

Let H be any subgroup of G, and define $g(E) = \{ga | g \in G, a \in E\}$ and $gHg^{-1} = \{ghg^{-1} | g \in G, h \in H\}$. Then for $b = ga$, $ghg^{-1}b = ghg^{-1}(ga) = gha$. Since a is fixed by h, $gha = ga = b$, so $ghg^{-1}b = b$ for any $b \in g(E)$.

Conversely, let some $b \in \mathrm{Inv}(gHg^{-1})$ be given. There is some $a \in K$ such that $g^{-1}b = a$, so $b = ga$ and $ghg^{-1}b = ghg^{-1}ga = gha = b$. But $ga = b$, so $ga = gha$ and a must therefore be fixed by h. So a is in E and $b \in g(E)$. It follows that $\mathrm{Inv}(gHg^{-1}) = g(E)$.

2

Assume that $g(E) = E$ for all $g \in G$. Let $f_g(j) = gj$ for $j \in \mathrm{Gal}(E/F)$, and define a mapping $\varphi : \mathrm{Gal}(K/F) \to \mathrm{Gal}(E/F)$ by $\varphi(g) = f_g(j)$ For $g \in \mathrm{Gal}(K/E)$, g fixes elements of E so $f_g(j) = j = f_{id}(j)$ where id is the identity permutation. This implies that $\mathrm{Gal}(K/E)$ is the kernel of φ and $j \in \mathrm{Gal}(E/F)$ are mapped to non-identity permutations of E.

Now let $a, b \in G$ be given. Then $\varphi(ab) = f_{ab}(j) = abj$. Similarly, $\varphi(a)\varphi(b) = f_a \circ f_b(j) = f_a(bj) = abj$, so φ is a homomorphism of the Galois group.

3

Suppose that E is Galois over F; then E is the splitting field of some $f(x) \in F[x]$ and all roots of $f(x)$ are contained in E. For any $g \in G$, g permutes roots of $f(x)$ by Theorem 12.2 of Hungerford. Since all roots of $f(x)$ are in E, it follows that for any $e \in E$, $ge \in E$ and thus $g(E) = E$.

We have shown that $\mathrm{Inv}(E)$ being normal in G implies that E is Galois over F, and conversely, if E is Galois over F, then $g(E) = E = gHg^{-1}$ and $\mathrm{Inv}(E)$ is normal in G.

$\square$