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MATH 4010
Dr. Chastkofsky
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Bonus Assignment on Section 9.5 Solutions

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Claim: Let G be a simple group, and let K be a subgroup of G with index n. Then The order of G divides $n!$.

Proof. Let φ be a mapping $\varphi : G \to A(T)$ defined as $\varphi(a) = f_{a^{-1}}(Kb) = Kba^{-1}$ for coset Kb in $A(T)$. We must first show that φ is a homomorphism. Let c, d in G be given; then $\varphi(cd) = f_{(cd)^{-1}}(Kb) = Kb(cd)^{-1} = Kbd^{-1}c^{-1}$. Likewise, $\varphi(c) \circ \varphi(d) = f_{c^{-1}} \circ f_{d^{-1}}(Kb) = f_{c^{-1}}(Kbd^{-1}) = Kbd^{-1}c^{-1} = \varphi(cd)$, so φ is a homomorphism. For any b in Kb and c, d in G, suppose $Kbc = Kbd$. Then $f_c(Kb) = f_d(Kb)$ and $\varphi(c^{-1}) = \varphi(d^{-1})$. Since φ is a homomorphism, $\varphi(c^{-1}) = \varphi(c)^{-1}$ and $\varphi(d^{-1}) = \varphi(d)^{-1}$, so $\varphi(c)^{-1} = \varphi(d)^{-1}$ and $\varphi(c) = \varphi(d)$. It follows that $c = d$, and φ is injective, so the kernel of φ is the identity e, so $A(T)$ is isomorphic to S_n. The cosets of T are closed under multiplication and inverses, so φ maps G to a subgroup of $A(T)$ and thus to a subgroup of S_n. By Lagrange's Theorem, G divides $|S_n| = n!$.

$\square$