Elliott Daimler
MATH 4200
Dr. Gay
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Homework 1 Solutions

For these solutions, I collaborated with Cole Wittbrodt and Dru Horne.

Exercise 1.3

Claim: For a function $f : X \to Y$ and sets $A, B \subset Y$, the following statements are true.

a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, and

b)$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Proof. For part a), for any $y \in A \cup B$, $y \in A$ or $y \in B$, so $f^{-1}(y) \subset f^{-1}(A) \cup f^{-1}(B)$, and $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.

Similarly, for any $x \in f^{-1}(A) \cup f^{-1}(B)$, $f(x) \in A$ or $f(x) \in B$ and $f(x) \in A \cup B$. It follows that $x \in f^{-1}(A \cup B)$, so $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$.

For part b), for any $y \in A \cap B$, $f^{-1}(y) \subset f^{-1}(A)$ and $f^{-1}(y) \subset f^{-1}(B)$, so $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.

For any $x \in f^{-1}(A) \cap f^{-1}(B)$, x is in $f^{-1}(A)$ and x is in $f^{-1}(B)$, so $f(x) \in A \cap B$ and $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$.

$\square$

Exercise 1.4

Claim: If $f : X \to Y$ is injective and $y \in Y$, then $f^{-1}(y)$ contains at most one point.

Proof. Let $a, b \in f^{-1}(y)$ be given. Then $f(a) = f(b) = y$, and by the definition of injectivity, $a = b$. $\square$

Exercise 1.5

Claim: If $f : X \to Y$ is surjective and $y \in Y$, then $f^{-1}(y)$ contains at least one point.

Proof. By the definition of surjectivity, there is some $x \in X$ such that $f(x) = y$, so $x \in f^{-1}(y)$. $\square$

Theorem 1.6

Claim: Let $2\mathbb{N}$ denote the even positive integers. Then $2\mathbb{N}$ has the same cardinality as $\mathbb{N}$, the natural numbers.

Proof. Define $f : \mathbb{N} \to 2\mathbb{N}$ as $f(x) = 2x$ for $x \in \mathbb{N}$. Then for every $y \in 2\mathbb{N}$, y is even, so $y/2 = x$ for some $x \in \mathbb{N}$ and $f(x) = y$, so f is surjective.

Let $a, b \in f^{-1}(y)$ be given. Then $f(a) = f(b) = y = 2a = 2b$, so $a = b = y/2$ and f is injective. $\square$

Theorem 1.7

Claim: The set $\mathbb{Z}$ has the same cardinality as $\mathbb{N}$.

Proof. Define $f : \mathbb{Z} \to \mathbb{N}$ as

$$f(x) = \left\{ \begin{array}{ll} 2x, & \text{for } x \geq 0 \\ -2x - 1, & \text{for } x < 0 \end{array} \right\}.$$

Then for every $x \geq 0$, $f(x)$ is even and for every $x < 0$, $f(x)$ is odd. Let $y \in \mathbb{N}$ and let $a, b \in f^{-1}(y)$ be given. If y is even, $f(a) = 2a = y$ and $f(b) = 2b = y$, so $a = b$. If y is odd, $f(a) = -2a - 1 = y$ and $f(b) = -2b - 1 = y$, so $-2a = -2b$ and $a = b$. So f is injective.

For every $y \in \mathbb{N}$, if y is even, there is some $x \in \mathbb{Z}$ where $x \geq 0$ and $x = y/2$, so $x \in f^{-1}(y)$. If y is odd, there is some $x \in \mathbb{Z}$ where $x < 0$ and $x = \frac{y+1}{-2}$, so f is surjective. $\square$

Theorem 1.8

Claim: Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.

Proof. For some $A \subset \mathbb{N}$, if $A = \{0\}$ then A is finite by definition. If $A \neq \{0\}$, let the least element $a \in A$ correspond with 1, the 2nd least element in A correspond with 2, repeating in this manner for the nth least element corresponding with $n \in \mathbb{N}$. This is a bijection from A to some $B \subset \mathbb{N}$.

For any $m \in \mathbb{N}$, if m is not in B, then B is finite so A is finite. If m is in B, then m corresponds with the mth least element of A, and the bijection holds. $\square$

Theorem 1.9

Claim: Every infinite set has a countably infinite subset

Proof. Let A be infinite and let $a \in A$ be given. Call this given element a_1, and repeat for unique elements $a_i \in A$. Then we have a bijection between the set $B = \{a_1, a_2, \cdots, a_n\}$ and A. Assume there is some $n \in \mathbb{N}$ for which $A - B = \{\emptyset\}$. Then A is finite, and we have a contradiction. So every $n \in \mathbb{N}$ has one corresponding element $a_n \in B$, and there is a bijection from $\mathbb{N}$ to B . □

Theorem 1.10

Claim: A set is infinite if and only if there is an injection from the set into a proper subset of itself.

Proof. Let B be a proper subset of a set A, and let $f : A \to B$ be injective. Assume B is finite; then for each element $y_i \in B$, we have $\{y_1 = f(x_1), y_2 = f(x_2), \cdots, y_n = f(x_n)\} = B$ for some $C = \{x_1, x_2, \cdots, x_n\} \subset A$. But there is some $x_m \in (A - B)$ where $f(x_m) \in B$ and $x_m \notin C$, a condradiction. So B is infinite, and therefore A is infinite.

Conversely, assume A is infinite. By Theorem 1.9, A has a countably infinite subset, which by definition means there is a subset B of A for which a bijection f exists between B and $\mathbb{N}$. By definition, f is injective. □

Theorem 1.11

Claim: The union of two countable sets is countable.

Proof. Let A, B be countable. Then for some subsets $C, D \subseteq \mathbb{N}$, there exist bijections $f : A \to C$ and $g : B \to D$, so $f(A) \cup g(B) = C \cup D \subseteq \mathbb{N}$. □

Theorem 1.12

Claim: The union of countably many countable sets is countable.

Proof. Let $A_1, A_2, \cdot, A_n$ be countably many countable sets. Then for each A_i there is a bijection $f_i : A_i \to B_i$ for some $B_i \subseteq \mathbb{N}$, so there are countably many subsets B_i, and their union is a subset of $\mathbb{N}$. Thus, $f^{-1}(A_1 \cup A_2 \cup \cdots \cup A_n)$ is countable. □