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MATH 4010
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MWF 10:20am
April 19, 2022

Homework 10, Chapter 11 Solutions

1

The minimum polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is x^2-2; since this polynomial is degree 2, $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2$. Similarly, the minimum polynomial of $\sqrt{5}$ over $\mathbb{Q}$ is x^2-5. The roots here are not in $\mathbb{Q}(\sqrt{2})$, so this polynomial is irreducible over $\mathbb{Q}(\sqrt{2})$ and $[\mathbb{Q}(\sqrt{5}):\mathbb{Q}]=2$. The minimum polynomial of $\sqrt{10}$ over $\mathbb{Q}$ is x^2-10; this polynomial has roots in $\mathbb{Q}(\sqrt{2})(\sqrt{5})$, namely $(\pm\sqrt{2}\sqrt{5})$. The result is $[\mathbb{Q}(\sqrt{2},\sqrt{5},\sqrt{10}):\mathbb{Q}]=4$.

2

The polynomial x^4-4x^2-5 can be factored into $(x^2-5)(x^2+1)$; each of these polynomials can be further factored into $x^2-5=(x+\sqrt{5})(x-\sqrt{5})$ and $x^2+1=(x+i)(x-i)$. Since the polynomial is degree 4 and we've factored into four distinct polynomials of degree one, we have all of the roots: $\pm i$, $\pm\sqrt{5}$. Since negative coefficients are in the field extensions $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(i)$, the splitting field is $\mathbb{Q}(\sqrt{5},i)$.

3

a

Claim: If $f(x)=cx^n\in F[x]$ and $g(x)=b_0+b_1x^1+\cdots+b_kx^k\in F[x]$, then $(fg)'(x)=f(x)g'(x)+f'(x)g(x)$.

Proof. We have $(fg)(x)=\sum_{i=0}^k cb_ix^{n+i}$, so $(fg)'(x)=(n+i)\sum_{i=1}^k cb_ix^{n+i-1}$.

Taking the individual functions' derivatives, we have $f'(x)=cnx^{n-1}$ and $g'(x)=\sum_{i=1}^k b_ix^{i-1}$. So $fg'(x)=\sum cb_ix^{n+i-1}(i)$ and $f'g(x)=\sum_{i=1}^k cnb_ix^{n+i-1}$, and $fg'(x)+f'g(x)=(n+i)\sum_{i=1}^k cb_ix^{n+i-1}=(fg)'(x)$. ☐

b

Claim: If $f(x)$, $g(x)$ are any polynomials in $F[x]$, then $(fg)'(x)=f(x)g'(x)+f'(x)g(x)$.

Proof. Let $f(x)$ be of the form $f(x)=a_0+a_1x^1+\cdots+a_nx^n$. Then this is the same form of $g(x)$ from part *a*. So $(fg)(x)$ can be written as $(fg)(x)=\sum_{i=0}^n a_ix^ig(x)$. We then have for each term $(a_ix^ig(x))'=ia_ix^{i-1}g(x)+a_ix^ig'(x)$, so by the summation rule, $(fg)'(x)=(\sum_{i=0}^n a_ix^ig(x))'=\sum_{i=1}^n ia_ix^{i-1}g(x)+\sum_{i=0}^n a_ix^ig'(x)=fg'(x)+f'g(x)$. ☐