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Homework 8, Sylow Theory Solutions

1

Let $G = S_9$. Elements in the centralizer send conjugates of x to x; in other words, for elements g in the centralizer of x, $gxg^{-1} = x$. Permutation cycles preserve their structure under conjugation, so the centralizer of $h = (1 \quad 2 \quad 3)(4 \quad 5 \quad 6)(7 \quad 8 \quad 9)$ will preserve its structure. Then the centralizer of $(1 \quad 2 \quad 3)(4 \quad 5 \quad 6)(7 \quad 8 \quad 9)$ is the set of permutations in G that permute the elements within each disjoint cycle, i.e.
$g(1 \quad 2 \quad 3)(4 \quad 5 \quad 6)(7 \quad 8 \quad 9)g^{-1} = g(1 \quad 2 \quad 3)g^{-1}g(4 \quad 5 \quad 6)g^{-1}g(7 \quad 8 \quad 9)g^{-1}$. The cycles in the conjugation must therefore be of the same disjoint form with elements $1, 2, 3$ in a cycle, $4, 5, 6$ in a cycle, and $7, 8, 9$ in a cycle. For example, $(1 \quad 2 \quad 3)(1 \quad 2 \quad 3)(4 \quad 5 \quad 6)(7 \quad 8 \quad 9)(1 \quad 3 \quad 2) = (1 \quad 2 \quad 3)(4 \quad 5 \quad 6)(7 \quad 8 \quad 9)$, so $(1 \quad 2 \quad 3)$ is in the centralizer. We find that the three disjoint cycles generate the centralizer, or $C_{S_9}(h) =< (1 \quad 2 \quad 3), (4 \quad 5 \quad 6), (7 \quad 8 \quad 9) >$. Each of these dijoint cycles has order 3, and there are $3(3) = 9$ combinations of these elements, so the centralizer contains a Sylow 3-subgroup of G.

2

The three-cycles generating the centralizer in part 1 are even permutations, so are in A_9. Again, they have order three and there are 9 combinations of the generators, so the centralizer is a Sylow 3-subgroup of G.

3

We will use the approach used in a lemma on interactions of p-Sylows. Let Q be a Sylow-p subgroup of a group G, and let P be a p-subgroup of G. For the normalizer N of Q, let $H = P \cap N$. Q is a normal subgroup of N, so HQ is a subgroup on N. Then HQ/Q is isomorphic to $H/(H \cap Q)$ by the Second Isomorphism Theorem. Q is a Sylow-p subgroup of G, so its order is the highest power of p that divides the order of G; as a result, p does not divide the index $[HQ : Q]$. H is a subgroup of P, so p is the only prime dividing the index $[H : H \cap Q]$. Since these two quotient groups are isomorphic, and p divides one and not the other, their order must be 1. It follows that $HQ = Q$, so any elements of P that are in the normalizer of Q are also in Q.

4

For Sylow-p subgroup Q of G and p-subgroup P of G, let $H = \{gQg^{-1} | g \in G\}$, or the set of conjugates of Q in G. For $P \cap H = \{pQp^{-1} | p \in P\}$, the orbits have orders of p^n for integer n. As with the proof of the Second Sylow Theorem given in our notes, there is a least one orbit of length 1, so there is some $g \in G$ where for every $x \in P$, $x(gQg^{-1})x^{-1} = gQg^{-1}$. The conjugate of P is in the normalizer of Q, and so it is also in Q.