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MATH 4010
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Homework 1 Solutions

1

Claim: Let G be a finite Abelian group with identity e and $a_1, a_2, \cdots, a_n$ in G, and let $x = a_1 a_2 \cdots a_n$. Then $x^2 = e$.

Proof. Let some a_i in G be given. Since G is finite, there is some k_i in $\mathbb{Z}^+$ such that $a_i^{k_i} = e$. Consider the values of k_i.

Case 1: $k_i = 1$. Then a_i is the identity.

Case 2: $k_i = 2$. Since G is Abelian, we have

$$\begin{aligned} x^2 &= (a_1 a_2 \cdots a_i \cdots a_n)^2 \\ &= a_i^2 (a_1 a_2 \cdots a_n)^2 \\ &= e(a_1 a_2 \cdots a_n)^2, \end{aligned}$$

assuming without loss of generality that $2 < i < n$. The remaining factors of x^2 are considered in case 3.

Case 3: $k_i > 2$. Then there is some $a_i^{-1} \neq a_i$ in G such that $a_i^{-1} = a_i^{k-1}$. Since inverses are unique, a_i and a_i^{-1} are unique inverses of one another. G is Abelian, so every a_i and a_i^{-1} in x produce the identity.

$\square$

2

Claim: Let G be a group such that for every a, b, c in G where $ab = ca$ then $b = c$. Then G is Abelian.

Proof. Let a, b in G be given, and consider the product aba^{-1}. By the closure property, there is some c in G such that $aba^{-1} = c$. Multiplying both terms of this equation by a on the right side yields

$$\begin{aligned} aba^{-1}a &= ca \\ \rightarrow abe &= ca \\ \rightarrow ab &= ca, \end{aligned}$$

so $b = c$, and $ab = ba$. $\square$

3

Claim: Let G be a group where every nonidentity element a in G has order 2. Then G is Abelian.

Proof. Let elements a, b of G be given. Then for the product ab, we have

$$\begin{aligned} ab(ab)^{-1} &= ab(b^{-1}a^{-1}) \\ &= e. \end{aligned}$$

Since $a^2 = e$ and $b^2 = e$, we have $a^{-1} = a$ and $b^{-1} = b$ by uniqueness of inverses, so

$$\begin{aligned} ab(b^{-1}a^{-1}) &= e \\ \to ab(ba) &= e \\ \to ab(ba) &= (ab)^2. \end{aligned}$$

By the cancellation property, the equation $ab(ba) = ab(ab)$ implies $ba = ab$. $\square$