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Exercise 4.5

Claim: The set of rationals $\mathbb{R}$ with the lower limit topology is normal.

Proof. The approach used below employs the fact that for $x, y \in \mathbb{R}_{LL}$ where $x < y$, there exist disjoint open sets $[x, y)$ and $[y, z)$ for some $z > y$.

Let A, B be disjoint closed sets in $\mathbb{R}_{LL}$, and let a be any point in A. Then for any points $b_i \in B$ where $a < b_i$, there is some minimum point $\text{Min}(b_i) = b_{min} \in B$ where $a < b_{min}$. It follows that there is some open set $U_a = [a, b_{min})$ containing a that contains no points in B, and the union of open sets

$$\bigcup_{a \in A} U_a \supset A$$

is open and contains no points in B. Repeating this approach for open sets $V_b = [b, a_{min})$ containing points b in B, we have the union of opens sets

$$\bigcup_{b \in B} V_b \supset B$$

which is open, and the unions are disjoint:

$$\left(\bigcup_{a \in A} U_a\right) \bigcap \left(\bigcup_{b \in B} V_b\right) = \emptyset.$$

□