Elliott Daimler
MATH 4010
Dr. Chastkofsky
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Homework 8, Sylow Theory Solutions

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Claim:

Proof. □

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Claim: Let G be a finite abelian group, and let p be a prime number. If p divides the order of G, then G has an element of order p.

Proof. By the Fundamental Theorem of Finite Abelian Groups, G is isomorphic to a direct sum of cyclic groups of prime power order, $G \cong P_1 \oplus P_2 \oplus \cdots \oplus P_k$. Since each group P_i in the direct sum is cyclic, it has an element a of order p_i, where p_i is prime and $p_i = |P_i|$. The order of the direct sum $P_1 \oplus P_2 \oplus \cdots \oplus P_k$ is $|P_1| \cdot |P_2| \cdot \cdots \cdot |P_k| = p_1 p_2 \cdots p_k = |G|$, so any prime p that divides G is equal to the order of some P_i in $P_1 \oplus P_2 \oplus \cdots \oplus P_k$, and consequently is equal to the order of the generator a of the cyclic group P_i.

Assume without loss of generality that there is an element $b \in P_1 \oplus P_2 \oplus \cdots \oplus P_k$ of the form $b = (0, 0, \cdots, a, \cdots, 0)$ where a is in the i^{th} tuple position and all other positions have a value of zero, the identity for each cyclic group. Then under the group operation of the direct sum, the order of b is also p. Since $P_1 \oplus P_2 \oplus \cdots \oplus P_k$ is isomorphic to G, there must be an element of order p in G. So if there is a prime p that divides G, that prime is the order of an element in $P_1 \oplus P_2 \oplus \cdots \oplus P_k$ and thus is the order of an element in G. □