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MATH 4010
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Homework 5, Section 8.2 Solutions

1

Claim: Let G be an abelian group. Then for $K = \{a \in G | |a| <= 2\}$, $H = \{x^2 | x \in G\}$, the quotient group G/K is isomorphic to H.

Proof. We will complete this proof in three parts.

a

Let $a, b \in K$ be given. Since G is abelian, we have $(ab)^2 = abab = a^2b^2 = e(e)$. So $|ab| \leq 2$ and $ab \in K$, and K is closed under the group operation. For any $a \in K$, its inverse $a^{-1} = a$. So $a^{-1} \in K$ and K is closed under inverses, and K is a subgroup of G.

b

Let $c, d \in G$ be given. The squares c^2 and d^2 are in H, and $c^2d^2 = cdcd = (cd)^2$ where $cd = g$ for some $g \in G$. So H is closed under the group operation. For some $x \in G$, $x^2 \in H$ and for every $x^{-1} \in G$, $(x^{-1})^2 = x^{-2} \in H$. Taking the product of x^2 and x^{-2}, we have $x^2x^{-2} = xx^{-1}xx^{-1} = e^2$, so H is closed under inverses.

c

Define a map $f : G \to H$ given by $f(a) = a^2$. Since H is the set of all a^2 for $a \in G$, the map f is surjective, and for any $a, b \in G$, $f(ab) = (ab)^2 = abab = a^2b^2 = f(a)f(b)$, so f is a homomorphism. Then K is the kernel of f and by the First Isomorphism Theorem, G/K is isomorphic to H. □

2

Claim: Let M be a normal subgroup of a group G and let N be a normal subgroup of a group H. Then $M \times N$ is a normal subgroup of $G \times H$ and $(G \times H)/(M \times N)$ is isomorphic to $G/M \times H/N$.

Proof. Define a map $f : G \times H \to G/M \times H/N$ given by $f(g, h) = (Mg, Nh)$ for $g \in G, h \in H$. The coset Mg is a subgroup of G, and Nh is a subgroup of H, so f is surjective. For any $a, b \in G$, $c, d \in H$, $f(ab, cd) =$

$(Mab, Ncd) = (MaMb, NcNd) = f(a,c)f(b,d)$ by the normality of M and N, so f is a homomorphism. Then the kernel of f is $M \times N$ so $M \times N$ is a normal subgroup of $G \times H$. By the First Isomorphism Theorem, $(G \times H)/(M \times N)$ is isomorphic to $G \times H/M \times N$. □

3

Claim: Let N be a normal subgroup of a group G and let $f : G \to H$ be a homomorphism of groups such that the restriction of f to N is an isomorphism $N \cong H$. Then $G \cong N \times K$, where K is the kernel of f.

Proof. Since $f : N \to H$ is an isomorphism, it is surjective on H. N is a subgroup of G, so $f : G \to H$ is also a surjective homomorphism. Then by the First Isomorphism Theorem, $G/K \cong H \cong N$. The product $(G/K \times K)$ is clearly isomorphic to G since it is (Kg, k) for all $Kg \in G/K$ and $k \in K$. Then $G \cong (G/K \times K) \cong N \times K$.

□