# Solution to Feb 25th 2022 Riddler Classic

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**Problem Statement**:
From Sean Sweeney, Chris Nho and Eli Luberoff comes a Classic that is for anyone who ever thought even more deeply about functions in high school algebra. Suppose you have two distinct points on the x-axis of the coordinate plane. If I tell you a parabola passes through those two points, where on the plane could that parabola's vertex be? Spoiler alert: The vertex can be anywhere on the perpendicular bisector of those two points. (Neat!)
Now, suppose the two distinct points are anywhere on the coordinate plane. If I tell you that a parabola with a vertical line of symmetry passes through those two points, where on the plane could that parabola's vertex be?

**Solution**:
Let's suppose the two points are $(x_1, y_1)$ and $(x_2, y_2)$. If $x_1 = x_2$, then there is no solution. If $y_1 = y_2$, then the solution is as given in the problem statement. So suppose that $x_1 \neq x_2$ and $y_1 \neq y_2$.

The equation of a line that passes through the two points is:

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}. \tag{1}$$

To simplify notation in what follows, let:

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
k &= \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} \\
\bar{x} &= (x_1 + x_2)/2 \\
\bar{y} &= (y_1 + y_2)/2
\end{aligned}$$

The equation of a parabola that passes through the two points is therefore:

$$\begin{aligned}
y &= mx + k + a(x - x_1)(x - x_2) \\
&= ax^2 + [m - a(x_1 + x_2)]x + (k + a x_1 x_2) \qquad (2)
\end{aligned}$$

For a quadratic of the form $y = ax^2 + bx + c$, the extreme point of the parabola is at $x = -b/(2a)$, $y = -b^2/(4a) + c$.

Therefore, the locus of points for the extreme point of the parabola is given by:

$$\begin{aligned}
x(a) &= -\frac{m}{2a} + \bar{x} && (3)\\
y(a) &= -\frac{m^2}{4a} + m\bar{x} - \bar{x}^2 a + k + a x_1 x_2 && (4)\\
&= -\frac{m^2}{4a} + \bar{y} - \frac{a}{4}(x_2 - x_1)^2 && (5)
\end{aligned}$$

(The simplification between Eqs (4) and (5) has been skipped.) This set of equations has two asymptotes. Taking the limit as $a$ goes to plus or minus infinity gives $x = \bar{x}$ whereas $y$ goes to plus or minus infinity. Thus the first asymptote is $x = \bar{x}$. As $a$ converges to 0, we have that $y - \bar{y}$ converges to $m(x - \bar{x})/2$, so that the second asymptote is $y = m(x - \bar{x})/2 + \bar{y}$.

Moreover, taking $a = m/(x_2 - x_1)$ gives $(x, y) = (x_1, y_1)$, whereas taking $a = -m/(x_2 - x_1)$ gives $(x, y) = (x_2, y_2)$.

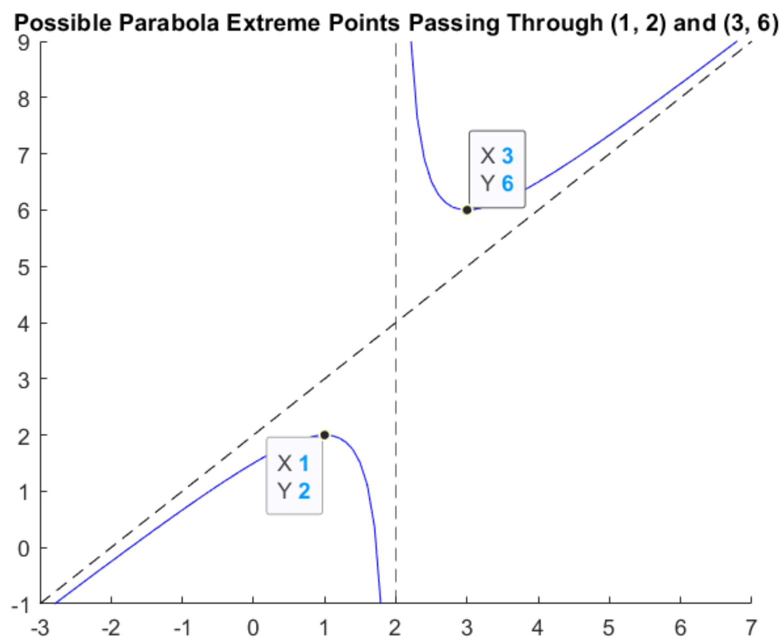
Although not obvious from the above equations, these are the equations of a hyperbola. This can be seen by solving for $a$ in terms of $x$, plugging this into the equation for $y$, and re-arranging:

$$\begin{aligned}
a &= -\frac{m}{2(x - \bar{x})} && (6)\\
y &= \frac{1}{2}m(x - \bar{x}) + \bar{y} + \frac{1}{8}m\frac{(x_2 - x_1)^2}{x - \bar{x}} && (7)\\
0 &= 4m(x - \bar{x})^2 - 8(x - \bar{x})(y - \bar{y}) + m(x_2 - x_1)^2 && (8)
\end{aligned}$$

Thus, the locus of points for the extreme point of the parabola is the hyperbola that is uniquely defined by the following properties:

1. It passes through both given points
2. It has an asymptote at $x = \bar{x}$
3. It has an asymptote given by $y = m(x - \bar{x})/2 + \bar{y}$.

For example, if the given points are $(1, 2)$ and $(3, 6)$, the locus of points would be the following hyperbola:


Possible Parabola Extreme Points Passing Through (1, 2) and (3, 6)
X 3
Y 6
X 1
Y 2
9
8
7
6
5
4
3
2
1
0
-1
-3
-2
-1
0
1
2
3
4
5
6
7