Solution to Mar 18th 2022 Riddler Classic

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Problem Statement:
From Rolfe Petschek comes a puzzle that's good fun in snow, rain, heat or gloom of night. (Okay, maybe not in gloom of night.)

A postal worker and his customer joke about the various ways the customer could mathematically encode her post office box number.

The customer realizes that every integer greater than 1 can be encoded via at least one Fibonacci-like sequence using an ordered triple (m, n, q). The encoded number is the qth member of the sequence after the first two positive integers m and n, where each term is the sum of the previous two terms. For example, 7 has the encodings (3, 4, 1) and (1, 3, 2).

In an attempt to stump the postal worker, the customer prefers encodings with a maximal value of q. What encoding should she use for the number 81?

Extra credit: What encoding should she use for the number 179?

Solution:
This problem is easier to solve by working backwards from the target number. Suppose that the target number is N and the preceding number in the sequence is αN for some α. Then the sequence of numbers, starting from the target and going backwards, is $v_0 = N$, $v_{-1} = \alpha N$, $v_{-2} = (1-\alpha)N$, $v_{-3} = (2\alpha - 1)N$, $v_{-4} = (2-3\alpha)N$, $v_{-5} = (5\alpha - 3)N$, $v_{-6} = (5-8\alpha)N$, ….

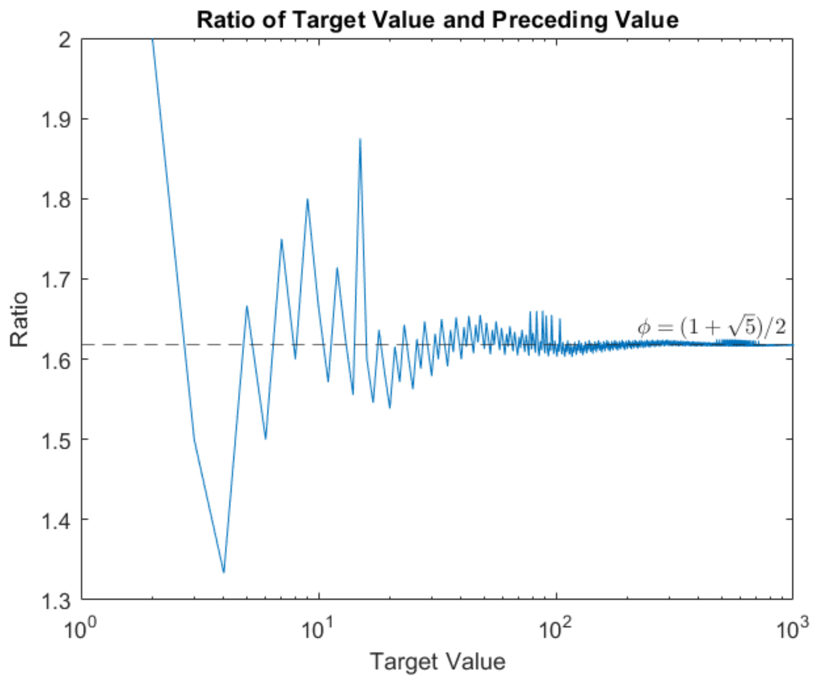
As can be seen, the pattern involves the Fibonacci numbers. Let $F_1 = F_2 = 1$ be the first two Fibonacci numbers. Then, the general pattern is:

$$v_{-k} = \begin{cases} (F_{k-1} - F_k\alpha)N, & \text{if } k \text{ is even} \\ (F_k\alpha - F_{k-1})N, & \text{if } k \text{ is odd} \end{cases} \tag{1}$$

Clearly, if αN is an integer, then all members of the sequence will be integers. For a particular value of $v_{-1} = \alpha N$, the corresponding value of q will be two less than the number of terms in the sequence $v_0, v_{-1}, v_{-2}, \ldots$ which are positive.

It follows that α should be chosen such that $2\alpha - 1 > 0$, $2 - 3\alpha > 0$, $5\alpha - 3 > 0$, …. This sequence of inequalities imposes increasingly tight bounds on α, bounds which are all satisfied when $\alpha = 1/\phi$, where $\phi = (1+\sqrt{5})/2$, known as the golden ratio.

To solve this problem for a given target N, it therefore suffices to consider v_{-1} as the floor and ceiling of N/ϕ. One can see this in the following plot, which shows the value of $1/\alpha$ as a function of the target number. As shown in the plot, $1/\alpha$ converges to the golden ratio ϕ.



The optimal encoding for $N = 81$ is $(3, 2, 7)$ and the optimal encoding for $N = 179$ is $(11, 7, 6)$.