Linear Algebra Review

Matrices

1. Matrices are rectangular arrays of numbers.
2. Example of a matrix of dimension 2×3 :

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

3. $\mathbf{A}_{ij}$ denotes the entry in i^{th} row and j^{th} column.
4. Column vectors of dimension n are matrices of dimension $n \times 1$.
5. Row vectors of dimension n are matrices of dimension $1 \times n$.
6. Totality of row vectors of a fixed dimension n is denoted as $\mathbb{R}^n$.
7. One can only add or subtract matrices of the **same size**.
8. The zero matrix $\mathbf{0}$ is a matrix each of whose entries is a zero.
9. Identity matrix $\mathbf{I}_n$ of size n is a matrix with 1's on the diagonal and zeroes elsewhere.
10. In this class, matrices and vectors will be written with boldface letters, like $\mathbf{A}, \mathbf{x}, \mathbf{b}$ etc. Unlike numeric variables, which will be written like x, y, z etc.

Matrix Multiplication

Multiplying $a \times b$ and $b \times c$ matrices produces a matrix of size $a \times c$. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -14 \\ 10 & -32 \end{bmatrix}$$

Here, $(2 \times 3) \cdot (3 \times 2) = (2 \times 2)$.

One can also multiply a matrix by a scalar as follows:

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Properties of Matrix Multiplication

Matrix multiplication is quite different from multiplication of numbers, and in general,

1. $A \cdot B \neq B \cdot A$.
2. $A \cdot C = B \cdot C$ does not always imply $A = B$.
3. $A \cdot B = 0$ without either being equal to 0.

Transpose of a Matrix

1. Flips rows and columns of a matrix.
2. From a $m \times n$ matrix, produces a $n \times m$ matrix.

3. Example:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ \\ 4 & 5 & 6 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 0 & 8 \end{bmatrix}$$

Properties:

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$. Note the change in the order.

Question: If $u, v \in \mathbf{R}^n$, what sort of an object is $u^T v$?

Linear Systems

$$\begin{array}{ccccccc} 3x & & +2z & +2w & = & -8 \\ 2x & +3y & +5z & -w & = & 4 \\ x & & & +w & = & 6 \end{array}$$

This system can be written in a matrix form as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 2 & 3 & 5 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -8 \\ 4 \\ 6 \end{bmatrix}.$$

The ***augmented matrix*** for this system is:

$$\left[\begin{array}{cccc|c} 3 & 0 & 2 & 2 & -8 \\ 2 & 3 & 5 & -1 & 4 \\ 1 & 0 & 1 & 1 & 6 \end{array}\right].$$

RREF

Stands for Row Reduced Echelon Form.

Example:

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 4 \\ 0 & \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- A boxed number indicates a ***leading entry*** of a row.
- Columns containing a leading entry are called ***pivot columns***.
- Column 1, 2, 4 are pivot columns.
- ***Rank*** = No. of pivot columns = No. of leading entries

What is RREF?

- All "0 rows" are at the bottom of the matrix.
- Leading entries of the successive rows keep "moving right".
- All pivot columns have exactly one 1, and rest of their entries are 0s.

It is easy to solve linear systems once their augmeted matrix is put in an RREF. Example on the board.

Therefore, the first (and the biggest) step in solving a linear system is transforming its augmented matrix in the RREF!

Elementary row operations

- Interchange rows i and j. Denoted as $\mathbf{R}_i \leftrightarrow \mathbf{R}_j$.
- Multiply row i by a non-zero scalar c : Denoted as $\mathbf{R}_i \leftarrow c\mathbf{R}_i$.
- Add a non-zero multiple of a row i to row j and store the result in row j : Denoted as $\mathbf{R}_j \leftarrow \mathbf{R}_j + c\mathbf{R}_i$.

More about RREF

- Any matrix can be brought into a (unique) RREF using a sequence of elementary row operations. –> ***Gauss-Jordan Elimination***
- Each elementary row operation corresponds to multiplying the given matrix by an ***Elementary Matrix*** on the left.
- Elementary matrices are square matrices, whose inverse is the elementary matrix corresponding to the "opposite" row operation.
- Gauss-Jordan Elimination is perhaps *the most important* topic from math 250.
- Example of Gauss-Jordan Elimination on the board (if there is time).

Inverse of a matrix

1. We talk of inverses only for square matrices.
2. ***Inverse*** of an $n \times n$ matrix $\mathbf{A}$ (if it exists) is another $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{I}_n = \mathbf{B} \cdot \mathbf{A}.$$

3. Inverse, if it exists is unique.
4. Matrices without inverses are called **singular**, else are called **invertible** or **non-singular**.
5. If $\mathbf{A}$ is invertible, then $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a *unique* solution for any vector $\mathbf{b}$ of appropriate size, and that solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
6. A matrix is invertible if and only if its determinant is non-zero.

Questions: What is the RREF of an invertible matrix? What is the rank of an invertible matrix of size $n \times n$?

Finding inverse

1. Make an augmented matrix $[\mathbf{A}|\mathbf{I}_n]$.
2. Now perform row operations on this augmented matrix so that the "left block" gets transformed into an identity matrix.
3. Say you arrive at $[\mathbf{I}_n|\mathbf{B}]$, then $\mathbf{B}$ is the inverse of $\mathbf{A}$.
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Examples:

(1) Invertible example:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} \xrightarrow{\text{Augment}} \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row Ops}} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$$

So $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}.$$

(2) A singular example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -4 \\ 2 & 3 & 7 \end{bmatrix} \xrightarrow{\text{Augmentation, Row ops}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right].$$

Subspaces

Fix a value of n, we will talk of subspaces of $\mathbb{R}^n$.

S is called a ***subspace*** of $\mathbb{R}^n$ if:

1. S is a collection of vectors from $\mathbb{R}^n$.
2. Zero vector is a member of S.
3. If $\mathbf{u} \in S$ and c is any scalar, then $c\mathbf{u} \in S$.
4. If $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

Basic Examples

1. $S = \{\mathbf{0}\}$.
2. $S = \mathbb{R}^n$.

Geometric examples:

1. Any line passing through the origin
2. Any plane passing through the origin
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(Examples on the board)

Examples Continued

"Spock, take me through that subspace fracture" — Kirk.

"Ever took linear algebra, Sir?" — Spock.

Examples continued

Given a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k$, the **span** of these vectors is the collection of all possible vectors $\mathbf{v}$ which can be written as $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.

Spans are always subspaces!

Collection of vectors of the form $\begin{bmatrix} 2a+b \\ b-c \\ a \end{bmatrix}$ is the span of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, since

$$\begin{bmatrix} 2a+b \\ b-c \\ a \end{bmatrix} = a\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Question: What about collection of vectors of the form $\begin{bmatrix} a \\ b+a \\ a-3 \end{bmatrix}$?

Linear Independence

Caution: An extremely important concept.

- A set $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is called ***Linearly Dependent*** if there exist scalars $c_1, c_2, \ldots, c_k$, **not all zero** such that

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

  Otherwise, the set is called ***Linearly Independent***
- In other words, if the only scalars which satisfy $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k = \mathbf{0}$ are $c_1 = 0, c_2 = 0, \cdots, c_k = 0$, then V is linearly independent.
- Any set V which contains the $\mathbf{0}$ vector is linearly dependent.
- Btw, a vector of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is called a ***Linear Combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Example

$$V = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

Of course, $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. So $c_1 = c_2 = c_3 = 1$ in the definition of linear dependence => V is L.D.

How do we see it in a non-ad-hoc way? RREFs!

Essentially, we are trying to solve the system

$$\begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \mathbf{0}.$$

$$\mathbf{A} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- If $\mathbf{A}$ has any non-pivot columns, (the 3^{rd} column above), then the columns of $\mathbf{A}$ form a set of linearly dependent vectors.
- In fact, the pivot columns of $\mathbf{A}$, (the 1^{st} and 2^{nd} columns) form a set of linearly independent vectors. So, $L = \{\mathbf{v}_1, \mathbf{v_2}\}$ is a linearly independent subset of $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- In other words, if Rank of $\mathbf{A}$ (= the number of pivot columns) equals the number of columns of $\mathbf{A}$, then the columns of $\mathbf{A}$ form a linearly independent set.

Bases

Let us talk of bases of $\mathbb{R}^n$. In general, one can talk of bases for any subspace for $\mathbb{R}^n$.

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A set of vectors V in $\mathbb{R}^n$ is called a ***Basis*** for $\mathbb{R}^n$ if:

1. V spans $\mathbb{R}^n$, i.e., any n-dimensional vector can be expressed as a linear combination of vectors in V.
2. V forms a linearly independent set.

Notes:

1. A basis for $\mathbb{R}^n$ will have exactly n vectors.
2. If V has size less than n, then V won't span $\mathbb{R}^n$.
3. If V has size greater than n, then V will necessarily be linearly dependent, as the matrix $\mathbf{A} = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$ will have a non-pivot column.

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First form the matrix $\mathbf{A} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$.

- If RREF of $\mathbf{A}$ has no non-zero rows, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$.
- If each column of $\mathbf{A}$ is a pivot column, then the columns of $\mathbf{A}$ are linearly independent.

Therefore, - If the RREF of $\mathbf{A}$ satisfies both the conditions, then the columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$. - If the RREF of $\mathbf{A}$ satisfies both the conditions, then the RREF equals the identity matrix $\mathbf{I}_n$!

Examples

In each of the following cases, answer:

1. If the columns of $\mathbf{A}$ span $\mathbb{R}^n$, for some appropriate value of n.
2. What is this appropriate value of n?
3. Are columns of $\mathbf{A}$ linearly independent?
4. Which columns of $\mathbf{A}$ form a linearly independent set?
5. Do columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$ for some appropriate value of n?

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Analysis

First form the matrix $\mathbf{A} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$.

- If RREF of $\mathbf{A}$ has no non-zero rows, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$.
- If each column of $\mathbf{A}$ is a pivot column, then the columns of $\mathbf{A}$ are linearly independent.

Therefore, - If the RREF of $\mathbf{A}$ satisfies both the conditions, then the columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$. - If the RREF of $\mathbf{A}$ satisfies both the conditions, then the RREF equals the identity matrix $\mathbf{I}_n$!

Examples

In each of the following cases, answer:

1. If the columns of $\mathbf{A}$ span $\mathbb{R}^n$, for some appropriate value of n.
2. What is this appropriate value of n?
3. Are columns of $\mathbf{A}$ linearly independent?
4. Which columns of $\mathbf{A}$ form a linearly independent set?
5. Do columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$ for some appropriate value of n?

- $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$
- $\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & -1 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
- $\mathbf{C} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & -1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$