1 Delone set

Delone set is such a set that is both relatively dense and uniformly discrete. In order to characterize exactly what dense and discrete means, we define two parameters for any subset of $\mathbb{C}^n$.

Definition 1.1. Let $D \subset \mathbb{C}^n$. Then $R_P \in \mathbb{R}$

$$R_P = \frac{1}{2} \sup \{ r_1 \in \mathbb{R} \,|\, \forall z_1, z_2 \in D, z_1 \neq z_2 : \; \|z_1 - z_2\| > r_1 \}$$

is called **packing radius** of the set D.

Remark. Open balls of packing radius centered at the points of the set are disjoint.

Definition 1.2. Let $D \subset \mathbb{C}^n$. Then $R_C \in \mathbb{R}$

$$R_C = \inf \{ r_2 > 0 \,|\, \forall z \in \mathbb{C}^n : \; B(z, r_2) \cap D \neq \emptyset \}$$

is called **covering radius** of the set D.

Remark. Union of closed balls of covering radius centered at the points of the set is the entire space $\mathbb{C}^n$.

Definition 1.3. $D \subset \mathbb{C}^n$ which has positive packing radius R_P is **uniformly discrete**.
$D \subset \mathbb{C}^n$ which has finite covering radius R_C is **relatively dense**.
$D \subset \mathbb{C}^n$ which has both positive packing radius R_P and finite covering radius R_C is a **Delone set**.

2 number theory

3 Cut-and-project scheme

We are using cut-and-project scheme to model quasicrystals. Here is a brief introduction into its workings.

Roughly speaking cut-and-project is a way of selecting a subset from a larger set, in our case this larger set is a $\mathbb{Z}[\beta]$-module.

Definition 3.1. Let $\beta \in \mathbb{R}$ be Pisot, $\mathbb{Z}[\beta]$ its extension ring and $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ be a basis of $\mathbb{R}^d$ for $d \in \mathbb{N}$.

$$L = \bigoplus_{j=1}^{d} \mathbb{Z}[\beta] \, \mathbf{e}_j$$

is d dimensional $\mathbb{Z}[\beta]$**-module**.

The cut-and-project scheme utilizes $2n$ dimensional $\mathbb{Z}[\beta]$-module $L \subset \mathbb{R}^{2n}$ and two more n dimensional subspaces $V_1, V_2 \subset \mathbb{R}^{2n}$.

Further we define two projections $\pi_1 : \mathbb{R}^{2n} \to V_1$ and $\pi_2 : \mathbb{R}^{2n} \to V_2$ such that $\pi_1|_L$ is injection and $\pi_2(L)$ is dense in V_2.

That is where the 'project' part of cut-and-project comes from. The 'cut' part comes from a bounded subset $\Omega \subset V_2$ with nonempty interior usually referred to as **window**.

All put together the cut-and-project scheme produces a subset $Q \subset V_1$:

$$Q = \{\pi_1(x) \mid \pi_2(x) \in \Omega,\ x \in L\}$$

Put in words the set Q are π_1 projections of those points of L whose π_2 projections fit in the window Ω.

The notation can be somewhat simplified by composing a bijection between V_1 and V_2: $\pi_2 \circ \pi_1^{-1}$, usually denoted as $*$ and referred to as a **star map**. Q then becomes:

$$Q = \{x \in V_1 \mid x^* \in \Omega\}$$

This is the form in which we will use the cut-and-project scheme.

4 Quasicrystal

In two dimensions a quasicrystal can be viewed as a subset of complex numbers $\Lambda \subset \mathbb{C}$ following these five properties:

1. rotational symmetry:
$$\exists\, \zeta = e^{2\pi i/n} :\ \zeta\Lambda = \Lambda$$
2. dilation:
$$\exists\, \beta \in \mathbb{R} \setminus \{-1, 1\} :\ \beta\Lambda \subset \Lambda$$
3. uniform discreteness:
$$\exists\, r_1 > 0,\ \forall z_1, z_2 \in \Lambda, z_1 \neq z_2 :\ |z_1 - z_2| > r_1$$
4. relative density:
$$\exists\, r_2 > 0,\ \forall z \in \mathbb{C} :\ B(z, r_2) \cap \Lambda \neq \emptyset$$
5. finite local complexity:
$$\forall\, \rho > 0 :\ \big|\{\Lambda \cap B(x, \rho) \mid \forall x \in \Lambda\}\big| < \infty$$

Remark. Properties 3. and 4. together make quasicrystal to be a Delone set.

It stems form these properties alone, that among other constants a quasicrystal is linked to a root of unity ζ and to a number $\beta \in \mathbb{R} \setminus \{-1, 1\}$. Of course not every pair (ζ, β) is associated with a quasicrystal.

In the next section we will go through which numbers are associated with a quasicrystal and where do they come from.

5 Pisot-cyclotomic numbers

Pisot-cyclotomic numbers are Pisot and are algebraically related to roots of unity. We will use these numbers in place of β from previous section.

Definition 5.1. Let $\rho = 2\cos(2\pi/n)$ for a given $n > 4$, its associate extension ring $\mathbb{Z}[\rho]$ and m order of ρ. A **Pisot-cyclotomic** number of degree m, of order n associated to ρ is a Pisot number $\beta \in \mathbb{Z}[\rho]$ such that

$$\mathbb{Z}[\beta] = \mathbb{Z}[\rho]$$

Nontrivial nth root of unity $\zeta = e^{2\pi i/n}$ is by definition a solution to equation

$$\zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta + 1 = 0$$

further for $\rho = 2\cos(2\pi/n)$ it holds

$$\rho = \zeta + \bar{\zeta} \quad \Rightarrow \quad \zeta^2 = \rho\zeta - 1$$

Therefore for extension rings $\mathbb{Z}[\zeta]$ and $\mathbb{Z}[\rho]$ we have

$$\mathbb{Z}[\zeta] = \mathbb{Z}[\rho] + \mathbb{Z}[\rho]\zeta$$

and finally for Pisot-cyclotomic β associated to ρ we acquire

$$\mathbb{Z}[\zeta] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$$

Such countable ring is of course n-fold rotationally invariant

$$\zeta^k \mathbb{Z}[\zeta] \subset \mathbb{Z}[\zeta] \qquad k \in \widehat{n-1}$$

To summarize β is a real Pisot and it can be used to decompose n-fold rotationally invariant complex ring $\mathbb{Z}[\zeta]$ as $\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$.

Now we are going to show the link between Galois automorphism of ρ and a cyclic automorphism of $\{1, \zeta, \ldots, \zeta^{n-1}\}$.

The conjugate roots of ρ are in the form $\rho_k = 2\cos\left(\frac{2\pi}{n} n_k\right)$ for $k \in \widehat{m}$ where $n_k \leq \left[\frac{n-1}{2}\right]$ and $\rho = \rho_1$.

MORE HERE

We close this section with a list of quadratic ($m = 2$) Pisot-cyclotomic numbers.

6 Quasicrystal model

There certainly are many ways to acquire a set that follows the properties listed in section 4. We utilize the cut-and-project scheme described in section 3.

Definition 6.1. Let β be a Pisot-cyclotomic number of order n, associated to $\rho = 2\cos(2\pi/n)$ (and $\zeta = e^{2\pi i/n}$).

Further let $M = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$, $M^* = \mathbb{Z}[\beta'] + \mathbb{Z}[\beta']\sigma(\zeta)$ and projection $* : M \to M^*$ be defined as:

$$x = x_1 + x_2\zeta \to x^* = x_1' + x_2'\zeta^* \qquad x_1, x_2 \in \mathbb{Z}[\beta]$$

n	ρ	β	ζ
5	$2\cos\left(\frac{2\pi}{5}\right)$	$\frac{1+\sqrt{5}}{2}$	$e^{2i\pi/5}$
8	$2\cos\left(\frac{2\pi}{8}\right)$	$1+\sqrt{2}$	$e^{2i\pi/8}$
12	$2\cos\left(\frac{2\pi}{12}\right)$	$1+\sqrt{3}$	$e^{2i\pi/12}$
12	$2\cos\left(\frac{2\pi}{12}\right)$	$2+\sqrt{3}$	$e^{2i\pi/12}$

Table 1: Pisot-cyclotomic numbers of degree 2, of order n, associated to ρ.

Lastly let $\Omega \subset V_2$ be bounded with nonempty interior.

Then m **dimensional quasicrystal linked to irrationality** β **and window** Ω is the set:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\}$$

So far the entire theses was purely abstract. To give you an example of what quasicrystal might look like, there is the image 1 which shows points of a two dimensional quasicrystal.

We close this section with a list of properties of quasicrystals which are crucial for our analysis but also very obvious from the definition.

- $\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \Sigma(\Omega_1) \subset \Sigma(\Omega_2)$
- $\Sigma(\Omega_1) \cap \Sigma(\Omega_2) \quad = \quad \Sigma(\Omega_1 \cap \Omega_2)$
- $\Sigma(\Omega_1) \cup \Sigma(\Omega_2) \quad = \quad \Sigma(\Omega_1 \cup \Omega_2)$
- $\Sigma(\Omega + x^*) \quad = \quad \Sigma(\Omega) + x \qquad$ for $x \in M$

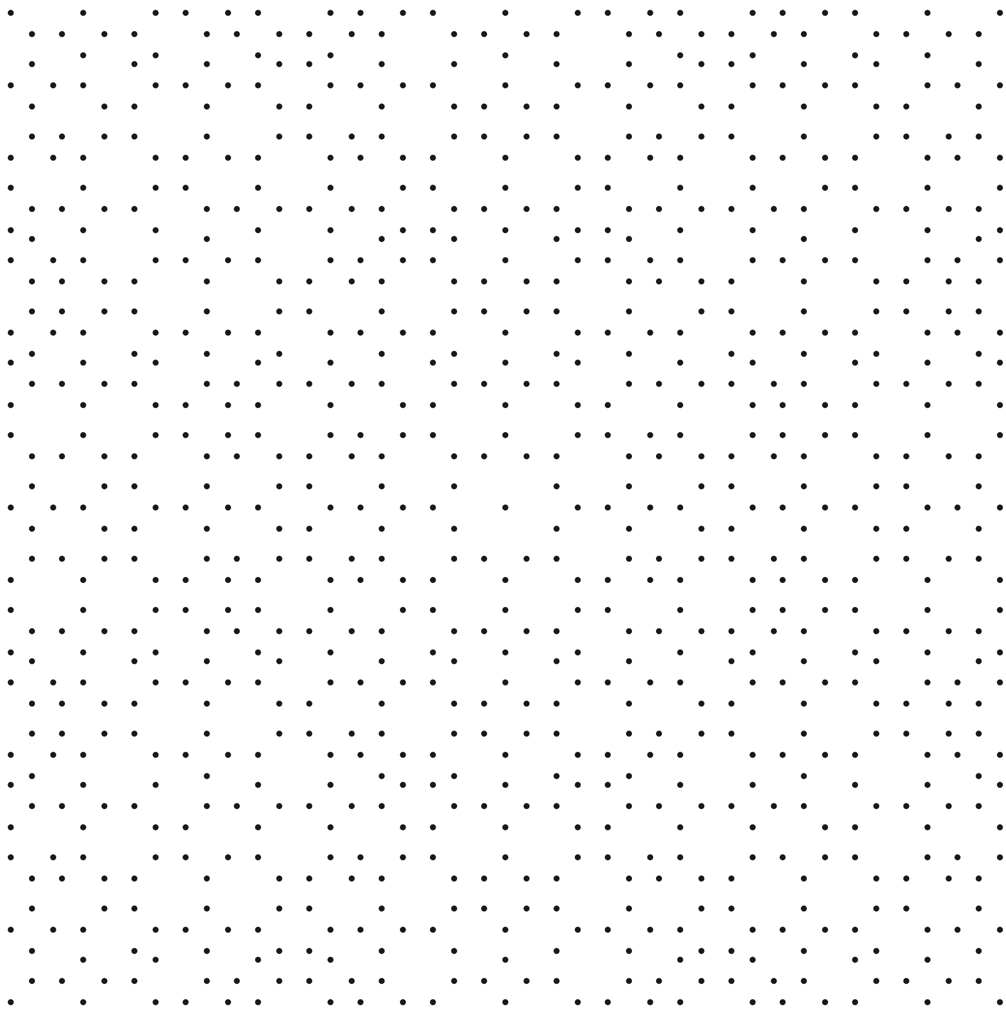

Figure 1: Example of a two dimensional quasicrystal.