1 Preliminaries

This section provides initial definitions and brief introduction to one-dimensional and two-dimensional quasicrystals.

1.1 Initial definitions

Definition 1.1. Roots of the following quadratic equation are denoted as β and β'.

$$x^2 = 4x - 1 \qquad \beta = 2 + \sqrt{3} \doteq 3.732 \qquad \beta' = 2 - \sqrt{3} \doteq 0.268$$

Remark 1. The number β as defined in Definition **??** will represent the same value in the entire text.

Being roots of the same quadratic equation, β and β' have some interesting properties that are often used in work with quasicrystals.

Theorem 1.2. *Properties of the roots β and β'.*

$$\beta\beta' = 1 \qquad \beta^{k+2} = 4 \cdot \beta^{k+1} - \beta^k \qquad \frac{1}{\beta} = \beta' = 4 - \beta$$
$$\beta + \beta' = 4 \qquad \beta'^{k+2} = 4 \cdot \beta'^{k+1} - \beta'^k \qquad \frac{1}{\beta'} = \beta = 4 - \beta'$$

Definition 1.3. Symbol $\mathbb{Z}[\beta]$ denotes the smallest ring containing integers $\mathbb{Z}$ and the irrationality β. Since β is quadratic the ring has the following simple form.

$$\mathbb{Z}[\beta] = \{a + b\beta | a, b \in \mathbb{Z}\}$$

Remark 2. Similarly, ring $\mathbb{Z}[\beta']$ can be defined. According to the Theorem **??** the two rings are equivalent: $\mathbb{Z}[\beta] = \mathbb{Z}[\beta']$.

1.2 One-dimensional quasicrystals

To define a quasicrystal one more definition is needed. Function connecting a space of the quasicrystal with a space of the acceptance set called window.

Definition 1.4. Function $' : \mathbb{Z}[\beta] \to \mathbb{Z}[\beta']$ is defined as $(a + b\beta)' = a + b\beta'$.

Remark 3. Notation is consistent with Definition **??**: $(\beta)' = \beta'$.

Definition 1.5. Let $\Omega \subset \mathbb{R}$ be a bounded set with non-empty interior. Then **one-dimensional quasicrystal** with the window Ω is denoted by $\Sigma(\Omega)$ and defined as:

$$\Sigma(\Omega) = \{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\}$$

Remark 4. $\Sigma(\Omega)$ where $\Omega \subset \mathbb{R}$ always denotes one-dimensional quasicrystal.

Some properties of one-dimensional quasicrystals are crucial for the algorithms used later.

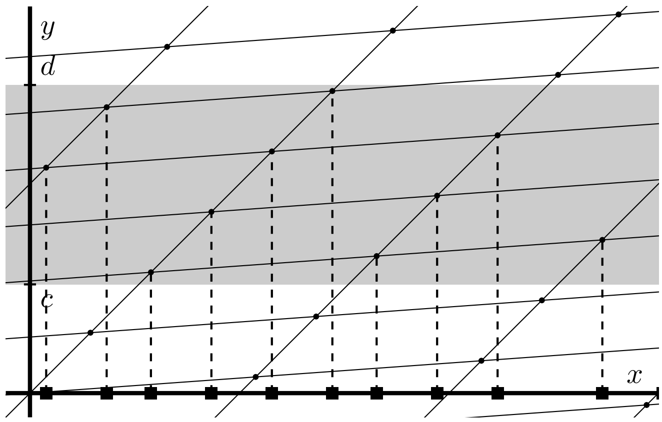


Figure 1: Illustration of one-dimensional quasicrystal. Grid intersections are defined as a set $\{(\lambda, \lambda')|\lambda \in \mathbb{Z}[\beta]\}$. There is a window $\Omega = [c, d)$ on the y axis and finally the squares on the x axis are points of the quasicrystal $\Sigma(\Omega)$.

Theorem 1.6. *Let* $\Omega, \tilde{\Omega} \subset \mathbb{R}$ *and* $\lambda \in \mathbb{Z}[\beta]$.

$$\Omega \subset \tilde{\Omega} \Rightarrow \Sigma(\Omega) \subset \Sigma\left(\tilde{\Omega}\right) \qquad \Sigma(\Omega) \cap \Sigma\left(\tilde{\Omega}\right) = \Sigma\left(\Omega \cap \tilde{\Omega}\right)$$
$$\Sigma\left(\Omega + \lambda'\right) = \Sigma(\Omega) + \lambda \qquad \Sigma(\Omega) \cup \Sigma\left(\tilde{\Omega}\right) = \Sigma\left(\Omega \cup \tilde{\Omega}\right)$$
$$\Sigma(\beta\Omega) = \frac{1}{\beta}\Sigma(\Omega)$$

Remark 5. Further only left-closed right-open intervals will be analyzed as windows. That is justified by theorem **??** and following analysis.

$$\Sigma((c, d)) = \begin{cases} \Sigma([c, d)) & c \notin \mathbb{Z}[\beta] \\ \Sigma([c, d)) \setminus \{c'\} & c \in \mathbb{Z}[\beta] \end{cases} \qquad \Sigma([c, d]) = \begin{cases} \Sigma([c, d)) & d \notin \mathbb{Z}[\beta] \\ \Sigma([c, d)) \cup \{d'\} & d \in \mathbb{Z}[\beta] \end{cases}$$
$$\Sigma((c, d]) = \Sigma((c, d)) \cap \Sigma([c, d])$$

Theorem 1.7. *Let* $\Omega \subset \mathbb{R}$ *then* $\forall k \in \mathbb{Z} : \Sigma\left(\frac{1}{\beta^k}\Omega\right) = \beta^k \Sigma(\Omega)$.

Corollary 1.8. *From remark* **??** *and theorem* **??** *follows that only windows* $\Omega = [c, d)$ *where* $d - c \in \left(\frac{1}{\beta}, 1\right]$ *need to be analyzed. Such windows will be called* ***base windows****. Quasicrystals for all other windows can be acquired through the analysis of quasicrystals with base windows by scaling and operations from remark* **??**.

1.2.1 One-dimensional quasicrystal structure

Figure **??** suggests that the one-dimensional quasicrystal is a sequence of points. This subsection presents an analysis of spacing and distribution of these points.

Definition 1.9. Strictly increasing sequence $(y_n^\Omega)_{n\in\mathbb{Z}}$ defined as $\left\{y_n^\Omega \,|\, n \in \mathbb{Z}\right\} = \Sigma(\Omega)$ where $\Omega \subset \mathbb{R}$ is called **sequence of quasicrystal** $\Sigma(\Omega)$.

Theorem 1.10. *Let $\Omega = [c, d)$ where $d - c \in \left[\frac{1}{\beta}, 1\right]$ then all possible distances between two immediately following points of sequence of quasicrystal $\Sigma([c, d))$ $\left(y_{n+1}^\Omega - y_n^\Omega\right)$ are listed in table* **??**.

$\frac{1}{\beta}$	$\left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right)$	$\frac{\beta-2}{\beta}$	$\left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right)$	$\frac{\beta-1}{\beta}$	$\left(\frac{\beta-1}{\beta}, 1\right)$	1
$4\beta - 1$	$4\beta - 1$					
$3\beta - 1$	$3\beta - 1$	$3\beta - 1$	$3\beta - 1$			
			$2\beta - 1$	$2\beta - 1$	$2\beta - 1$	
	β	β	β	β	β	β
					$\beta - 1$	$\beta - 1$

Table 1: All possible distances between two immediately following points of sequence of quasicrystal with window of given size.

Definition 1.11. Distances $y_{n+1}^\Omega - y_n^\Omega$ will be denoted: $A = 4\beta - 1$, $B = 3\beta - 1$, $C = 2\beta - 1$, $D = \beta$ and $E = \beta - 1$.

Definition 1.12. Function $f^\Omega : \Omega \to \Omega$ for $\Omega = [c, d)$ defined as

$$d - c \in \left(\frac{1}{\beta}, \frac{\beta - 2}{\beta}\right] : \quad f^\Omega(x) = \begin{cases} x + (\beta)' & x \in [c, d - \frac{1}{\beta}) \\ x + (4\beta - 1)' & x \in [d - \frac{1}{\beta}, c + \frac{\beta-3}{\beta}) \\ x + (3\beta - 1)' & x \in [c + \frac{\beta-3}{\beta}, d) \end{cases}$$

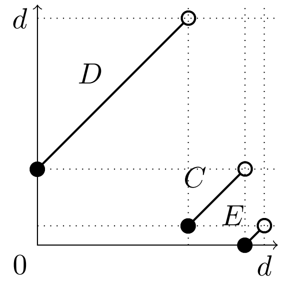
$$d - c \in \left(\frac{\beta - 2}{\beta}, \frac{\beta - 1}{\beta}\right] : \quad f^\Omega(x) = \begin{cases} x + (\beta)' & x \in [c, d - \frac{1}{\beta}) \\ x + (3\beta - 1)' & x \in [d - \frac{1}{\beta}, c + \frac{\beta-2}{\beta}) \\ x + (2\beta - 1)' & x \in [c + \frac{\beta-2}{\beta}, d) \end{cases}$$

$$d - c \in \left(\frac{\beta - 1}{\beta}, 1\right] : \quad f^\Omega(x) = \begin{cases} x + (\beta)' & x \in [c, d - \frac{1}{\beta}) \\ x + (2\beta - 1)' & x \in [d - \frac{1}{\beta}, c + \frac{\beta-1}{\beta}) \\ x + (\beta - 1)' & x \in [c + \frac{\beta-1}{\beta}, d) \end{cases}$$

is called **stepping function** of quasicrystal $\Sigma(\Omega)$.

Remark 6. Stepping function takes $(\cdot)'$ image of a point of quasicrystal and returns $(\cdot)'$ image of immediately following point.

Figure 2: Graph of stepping function for quasicrystal $\Sigma(\Omega)$ where $\Omega = [c, d)$, $c = 0$, $d = 12 - 3\beta$. $C = 2\beta - 1$, $D = \beta$ and $E = \beta - 1$ (as in Definition ??).



Stepping function is valuable tool in theoretical quasicrystal analysis and has direct practical use in quasicrystal generation. Therefore the following theorem lists several key properties of this function.

Theorem 1.13. *Let $\Omega \subset \mathbb{R}$:*

- $f^{\Omega}\left((y_n^{\Omega})' \right) = (y_{n+1}^{\Omega})' \quad \forall n \in \mathbb{N}$
- $\left(f^{\Omega}\right)^{-1}\left((y_{n+1}^{\Omega})' \right) = (y_n^{\Omega})' \quad \forall n \in \mathbb{N}$
- *f^{Ω} is piece-wise translation*
- *Discontinuities of f^{Ω} divide the window Ω to intervals. After preimages of points of one interval there is the same distance to the next point of the quasicrystal.*

Definition 1.14. Discontinuities of stepping function of quasicrystal $\Sigma(\Omega)$ are denoted as a^{Ω} and b^{Ω}. Let $\Omega = [c, d)$

$$
\begin{aligned}
d - c \in \left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right] &: \quad \begin{aligned} a^{\Omega} &= d - \tfrac{1}{\beta} \\ b^{\Omega} &= c + \tfrac{\beta-3}{\beta} \end{aligned} \\
d - c \in \left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right] &: \quad \begin{aligned} a^{\Omega} &= d - \tfrac{1}{\beta} \\ b^{\Omega} &= c + \tfrac{\beta-2}{\beta} \end{aligned} \\
d - c \in \left(\frac{\beta-1}{\beta}, 1\right] &: \quad \begin{aligned} a^{\Omega} &= d - \tfrac{1}{\beta} \\ b^{\Omega} &= c + \tfrac{\beta-1}{\beta} \end{aligned}
\end{aligned}
$$

Remark 7. Notation from previous definition will be often used to divide window $\Omega = [c, d)$ where $d - c \in \left(\frac{1}{\beta}, 1\right]$ to three disjunct intervals.

$$\Omega = \left[c, a^{\Omega}\right) \cup \left[a^{\Omega}, b^{\Omega}\right) \cup \left[b^{\Omega}, d\right)$$

For singular cases where $d - c \in \left\{\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}, 1\right\}$, $a^{\Omega} = b^{\Omega}$ and the window is then divided only in to two intervals $\left[c, a^{\Omega}\right) \cup \left[a^{\Omega}, d\right)$.

Definition 1.15. Let $\Omega = [c, d)$. The word $\left(t_n^\Omega\right)_{n\in\mathbb{Z}}$ over the alphabet $\{A, B, C, D, E\}$ is called **word** of quasicrystal $\Sigma(\Omega)$.

$$d - c \in \left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right] : \qquad t_n^\Omega = \begin{cases} D & y_{n+1}^\Omega - y_n^\Omega = \beta \\ A & y_{n+1}^\Omega - y_n^\Omega = 4\beta - 1 \\ B & y_{n+1}^\Omega - y_n^\Omega = 3\beta - 1 \end{cases}$$

$$d - c \in \left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right] : \qquad t_n^\Omega = \begin{cases} D & y_{n+1}^\Omega - y_n^\Omega = \beta \\ B & y_{n+1}^\Omega - y_n^\Omega = 3\beta - 1 \\ C & y_{n+1}^\Omega - y_n^\Omega = 2\beta - 1 \end{cases}$$

$$d - c \in \left(\frac{\beta-1}{\beta}, 1\right] : \qquad t_n^\Omega = \begin{cases} D & y_{n+1}^\Omega - y_n^\Omega = \beta \\ C & y_{n+1}^\Omega - y_n^\Omega = 2\beta - 1 \\ E & y_{n+1}^\Omega - y_n^\Omega = \beta - 1 \end{cases}$$

Remark 8. Word of a quasicrystal describes the distribution of points of the quasicrystal.

Definition 1.16. Function $\mathcal{C}_\ell : \mathbb{N} \to \mathbb{N}$, that assigns to $n \in \mathbb{N}$ number of different sub-words of length n in the word of quasicrystal $\left(t_n^\Omega\right)_{n\in\mathbb{Z}}$ where $|\Omega| = \ell$ is called **complexity** of quasicrystal.

Definition 1.17. Set $\mathcal{L}_\ell(n)$ containing all different sub-words of length n in the word of quasicrystal $\left(t_n^\Omega\right)_{n\in\mathbb{Z}}$ where $|\Omega| = \ell$ is called **language** of quasicrystal.

Remark 9. Please note that complexity and language of quasicrystal are defined dependent only on the length of the window. Correctness of such Definition is verified in the following theorem.

Theorem 1.18. *$\left(f^\Omega\right)^n$ denotes the n-th iteration of stepping function of quasicrystal $\Sigma(\Omega)$. Set $D_n = \{z_1 < z_2 < \cdots < z_{m-1}\}$ contains all discontinuities of $\left(f^\Omega\right)^n$, $z_0 = c$ and $z_m = d$.*
Then $(\forall i \in \widehat{m} \cup \{0\})(\forall \left(y_l^\Omega\right)', \left(y_k^\Omega\right)' \in (z_i, z_{i+1}))$ are words $t_l^\Omega t_{l+1}^\Omega \dots t_{l+n-1}^\Omega$ and $t_k^\Omega t_{k+1}^\Omega \dots t_{k+n-1}^\Omega$ the same.

Remark 10. In other words theorem **??** states that discontinuities of n-th iteration of stepping function divide window into intervals of images of points of quasicrystal after which the same sequence of distances of length n follow.

Theorem 1.19. *Let $\Omega = [c, c + \ell)$.*
If $\ell \notin \mathbb{Z}[\beta]$ then $\mathcal{C}_\ell(n) = 2n + 1$, $\forall n \in \mathbb{N}$.
If $\ell \in \mathbb{Z}[\beta]$ then $\exists^1 k \in \mathbb{N}$ such that $\left(\left(f^\Omega\right)^k(a^\Omega) = b^\Omega\right)$ or $\left(\left(f^\Omega\right)^{k+1}(b^\Omega) = a^\Omega\right)$ and

$$\mathcal{C}_\ell(n) = \begin{cases} 2n+1 & \forall n \le k \\ n+k+1 & \forall n > k \end{cases}$$

Theorem 1.20. *Let* $\Omega = [c, c+\ell)$.

$$\mathcal{D}_n = \left\{ \ell \,\middle|\, \ell \in \left(\frac{1}{\beta}, 1\right] \wedge \mathcal{C}_\ell(n) < 2n+1 \right\}$$

Then elements of $\mathcal{D}_n$ *divide interval* $I := \left(\frac{1}{\beta}, 1\right]$ *into finite amount of disjoint sub-intervals* $(I_m)_{m\in\hat{N}}$ *such that* $\mathcal{L}_{\ell_1}(n) = \mathcal{L}_{\ell_2}(n)$ $\forall \ell_1, \ell_2 \in I_m$, $\forall m \in \hat{N}$.

Remark 11. Please note that $\mathcal{D}_n$ from theorem **??** divides base windows into sets of same language whereas D_n from theorem **??** divides specific window into intervals by the sequences of points that follow.

Delone set and voronoi tessellation

Subsection provides definitions of delone set, covering radius and voronoi tessellation.

Definition 1.21. Let $P \subset \mathbb{R}^n$ and $\exists R > 0, \exists r > 0$:

$$\forall x, y \in P,\, x \neq y : r \leq \|x - y\|$$

$$\forall z \in \mathbb{R}^n \exists x \in P : \|z - x\| \leq R$$

Then P is called **delone** set.
For each delone set P **covering radius** is defined as:

$$R_c = \inf\{R > 0 \,|\, z \in \mathbb{R}^n \exists x \in P : \|z - x\| \leq R\}$$

Definition 1.22. Let $P \subset \mathbb{R}^n$, P is a discrete set and $x \in P$. Then

$$V(x) = \{y \in \mathbb{R}^n \,|\, \forall z \in P, z \neq x :\, \|y - x\| < \|y - z\|\}$$

is called **voronoi tile** (polygon) of x on P.

Theorem 1.23. *Let* $P \subset \mathbb{R}^n$ *is a delone set and* R_c *it's covering radius. For any* $x \in P$:

$$N_x = \{z \in P \,|\, z \neq x \wedge \|z - x\| \leq 2R_c\}$$

Then voronoi tile of x *on* P *is*

$$V(x) = \bigcap_{z \in \boldsymbol{N}} \{y \in \mathbb{R}^n \,|\, \|y - x\| < \|y - z\|\}$$

1.3 Two-dimensional quasicrystals

In the following section two-dimensional quasicrystal is defined and analyzed. Thanks to the Theorem **??** analysis of one-dimensional quasicrystals can be in some way applied to two-dimensional quasicrystals as well.

Definition 1.24. Vectors α_1, α_2, α_3 and the set M denote the following.

$$\alpha_1 = (1,0) \quad \alpha_2 = \left(\frac{2-\beta}{2}, \frac{1}{2}\right) \quad \alpha_3 = \left(\frac{\beta-2}{2}, \frac{1}{2}\right)$$

$$M = \mathbb{Z}[\beta]\,\alpha_1 + \mathbb{Z}[\beta]\,\alpha_2$$

Remark 12. The vectors and the set from previous definition are key to two-dimensional quasicrystal definition. The set M is used as two-dimensional equivalent to $\mathbb{Z}[\beta]$ from on-dimensional quasicrystal. Function from the following definition is used as two-dimensional equivalent to $'$.

Definition 1.25. Function $* : M \to M$ is called **star** function:

$$v^* = (a\alpha_1 + b\alpha_2)^* = a'\alpha_1 + b'\alpha_3 \; \forall a, b \in \mathbb{Z}[\beta]$$

Remark 13. Simple consequence of the Theorem **??** is $\alpha_1^* = \alpha_1$ and $\alpha_2^* = \alpha_3$.

Definition 1.26. Let $\Omega \subset \mathbb{R}^2$ be bounded set with nonempty interior. Then **two-dimensional quasicrystal** with the window Ω is defined as:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\}$$

Remark 14. $\Sigma(\Omega)$ where $\Omega \subset \mathbb{R}^2$ always denotes two-dimensional quasicrystal.

To analyze two-dimensional quasicrystals again only windows of certain shape will be considered. That is sufficient again because of the Theorem **??**. Chosen window shape is parallelogram.

Theorem 1.27. *Let $I_1 = [c_1, d_1)$ and $I_2 = [c_2, d_2)$, then for parallelogram $\Omega = I_1\alpha_1^* + I_2\alpha_2^*$ and quasicrystal $\Sigma(\Omega)$ follows:*

$$\Sigma(\Omega) = \Sigma(I_1)\,\alpha_1 + \Sigma(I_2)\,\alpha_2$$

Remark 15. Note in previous theorem that while $\Omega \subset \mathbb{R}^2$ and so $\Sigma(\Omega)$ is two-dimensional quasicrystal, $I_1, I_2 \subset \mathbb{R}$ and so $\Sigma(I_1)$ and $\Sigma(I_2)$ are one-dimensional quasicrystals.

From the analysis of one-dimensional quasicrystals and Theorem **??** follows that two-dimensional quasicrystals are delone sets.

To analyze distribution of the points of two-dimensional quasicrystal voronoi tessellation is used. The goal is to catalog shapes of all voronoi tiles that appear in a quasicrystal with parallelogram window.

- first item in a list
- second item with yellow color