1 Quasicrystal

In two dimensions a quasicrystal can be viewed as a subset of complex numbers $\Lambda \subset \mathbb{C}$ following these five properties:

1. rotational symmetry:
$$\exists\, \zeta = e^{\frac{2\pi i}{n}} \,:\, \zeta\Lambda = \Lambda$$
2. dilation:
$$\exists\, \beta \in \mathbb{R} \setminus \{-1, 1\} :\, \beta\Lambda \subset \Lambda$$
3. uniform discreteness:
$$\exists \epsilon_1 > 0,\ \forall z_1, z_2 \in \Lambda, z_1 \neq z_2 :\ |z_1 - z_2| > \epsilon_1$$
4. relative density:
$$\exists \epsilon_2 > 0,\ \forall z \in \mathbb{C} :\ B(z, \epsilon_2) \cap \Lambda \neq \emptyset$$
5. finite local complexity:
$$\forall\, \rho > 0 :\ \big|\{\Lambda \cap B(x, \rho) \mid \forall x \in \Lambda\}\big| < \infty$$

Remark. Properties 3. and 4. together make quasicrystal to be a Delone set.

It stems form these properties alone, that among other constants a quasicrystal is linked to a root of unity ζ and to a number $\beta \in \mathbb{R} \setminus \{-1, 1\}$. Of course not every pair (ζ, β) is associated with a quasicrystal.

In the next section we will go through which numbers are associated with a quasicrystal and where do they come from.

2 Pisot-cyclotomic numbers

Pisot-cyclotomic numbers are Pisot and are algebraically related to roots of unity. We will use these numbers in place of β from previous section.

Definition 2.1. Let $\rho = 2\cos(2\pi/n)$ for a given $n > 4$ and its associate extension ring $\mathbb{Z}[\rho]$. A Pisot-cyclotomic number of degree m, of order n associated to ρ is a Pisot number $\beta \in \mathbb{Z}[\rho]$ such that
$$\mathbb{Z}[\beta] = \mathbb{Z}[\rho]$$