Chapter 1

Preliminaries

1.1 Delone set

Delone set is such a set that is both relatively dense and uniformly discrete. In order to characterize exactly what dense and discrete means, we define two parameters for any subset of $\mathbb{C}^n$.

Definition 1.1.1. Let $D \subset \mathbb{C}^n$. Then $R_P \in \mathbb{R}$

$$R_P = \frac{1}{2} \sup \{ r_1 \in \mathbb{R} \,|\, \forall z_1, z_2 \in D, z_1 \neq z_2 : \ \|z_1 - z_2\| > r_1 \}$$

is called **packing radius** of the set D.

Remark. Open balls of packing radius centered at the points of the set are disjoint.

Definition 1.1.2. Let $D \subset \mathbb{C}^n$. Then $R_C \in \mathbb{R}$

$$R_C = \inf \{ r_2 > 0 \,|\, \forall z \in \mathbb{C}^n : \ B(z, r_2) \cap D \neq \emptyset \}$$

is called **covering radius** of the set D.

Remark. Union of closed balls of covering radius centered at the points of the set is the entire space $\mathbb{C}^n$.

Definition 1.1.3.

$D \subset \mathbb{C}^n$ which has positive packing radius R_P is **uniformly discrete**.
$D \subset \mathbb{C}^n$ which has finite covering radius R_C is **relatively dense**.
$D \subset \mathbb{C}^n$ which has both positive packing radius R_P and finite covering radius R_C is a **Delone set**.

1.2 Voronoi diagram

Definition 1.2.1. Let $P \subset \mathbb{R}^n$ be a discrete set and $x \in P$. Then

$$V_P(x) = \{y \in \mathbb{R}^n \,|\, \forall z \in P, z \neq x : \|y - x\| \leq \|y - z\|\}$$

is called **Voronoi polygon** or **Voronoi cell** or **Voronoi tile** of x on P.

Voronoi polygon $V_P(x)$ is said to belong to the point x and x is called the **center** of the Voronoi cell $V_P(x)$.

When there can be no confusion as to what set P is, it may be omitted: $V(x)$.

Definition 1.2.2. Let $P \subset \mathbb{R}^n$ be a discrete set. Then the set of all Voronoi tiles

$$\{V(x) \,|\, x \in P\}$$

is called **Voronoi diagram** or **Voronoi tessellation**.

Definition 1.2.3. Let $P \subset \mathbb{R}^n$ be a discrete set. Then the set of centered Voronoi tiles

$$\{V(x) - x \,|\, x \in P\}$$

is called **list of Voronoi tiles**.

Remark. The Voronoi diagram can be viewed as an image whereas the list of Voronoi tiles can be viewed as a list of Voronoi polygon shapes

Definition 1.2.4. Let $P \subset \mathbb{R}^n$ be a discrete set and $x \in P$. Then

$$\sup_{y \in V(x)} \|y - x\|$$

is called **radius** of the Voronoi polygon.

Definition 1.2.5. Let $P \subset \mathbb{R}^n$ be a discrete set and $x \in P$. Then the set of points of P that directly shape the Voronoi polygon $V_P(x)$:

$$D_P(x) = \bigcap \left\{Q \subset P \,|\, V_Q(x) = V_P(x)\right\}$$

is called the **domain** of x or of $V_P(x)$.

1.3 Number theory

The study of quasicrystals relies heavily on number theory. Therefore this section list the definitions and their implications that are used further, we will however not show any proofs for our claims.

Definition 1.3.1. Let $P \subset \mathbb{C}$. Then $P[x]$ denotes the set of polynomials with coefficients in P.

Definition 1.3.2. Let $f \in \mathbb{C}[x]$ such that $f(x) = \sum_{k=0}^{m} \alpha_k x^k$. Then f is **monic polynomial** if $\alpha_m = 1$.

1.3.1 Algebraic numbers, minimal polynomial and degree

Definition 1.3.3.

Let $\alpha \in \mathbb{C}$. If there exists monic polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$, then α is an **algebraic number**. The set of algebraic numbers is denoted as $\mathbb{A}$.

Let $\beta \in \mathbb{C}$. If there exists monic polynomial $g \in \mathbb{Z}[x]$ such that $g(\beta) = 0$, then β is an **algebraic integer**. The set of algebraic integers is denoted as $\mathbb{B}$.

Such polynomial f or g is then called the **minimal polynomial** of α or β respectively and denoted as f_α or f_β respectively

The degree of the polynomial is also regarded as the **degree of the algebraic number**.

Remark. For each algebraic number there exists exactly one minimal polynomial.

1.3.2 Galois isomorphism

Definition 1.3.4. The $(m-1)$ other roots of f_α for $\alpha \in \mathbb{A}$ of degree m are called **conjugate roots** of α and denoted as α', α'', ..., $\alpha^{(m-1)}$.

Remark. Consistently with the notation of its conjugate roots, α may be denoted as $\alpha^{(0)}$ or $\alpha^{(m)}$.

Definition 1.3.5. The ring $\mathbb{Z}(\alpha) \subset \mathbb{C}$:

$$\mathbb{Z}(\alpha) = \left\{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{m-1}\alpha^{m-1} \mid a_i \in \mathbb{Z}\right\}$$

is called the **extension ring** of the number $\alpha \in \mathbb{A}$ of degree m.

Definition 1.3.6. The number field $\mathbb{Q}(\alpha) \subset \mathbb{C}$:

$$\mathbb{Q}(\alpha) = \left\{b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{m-1}\alpha^{m-1} \mid b_i \in \mathbb{Q}\right\}$$

is called the **extension field** of the number $\alpha \in \mathbb{A}$ of degree m.

Definition 1.3.7. Let $\alpha \in \mathbb{A}$ of degree m and α', α'', ..., $\alpha^{(m-1)}$ its conjugate roots. Then $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\alpha')$, ..., $\mathbb{Q}(\alpha^{(m-1)})$ are isomorphic and the isomorphisms:

$$\sigma_i(\mathbb{Q}(\alpha) = \mathbb{Q}\big(\alpha^{(i)}\big) \qquad i \in \widehat{m-1}$$

are called **Galois isomorphisms**.

Galois isomorphisms are significant part of the definition of the quasicrystals, so they surely deserve an example.

The Galois isomorphism σ_0 is always identity.

In general the Galois isomorphism σ_i exchanges α of degree m with its ith conjugate root.

$$\sigma_i\big(b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{m-1}\alpha^{m-1}\big) = b_0 + b_1\alpha^{(i)} + b_2\big(\alpha^{(i)}\big)^2 + \cdots + b_{m-1}\big(\alpha^{(i)}\big)^{m-1}$$

Since further we will mostly work only with quadratic algebraic numbers (of degree 2), there will only be two roots and two Galois isomorphisms, identity and $\sigma_1(\alpha) = \alpha'$. Thus it is often denoted only as $'$, as in $(\alpha)' = \sigma_1(\alpha) = \alpha'$.

1.3.3 Root of unity, cyclotomic polynomial

Definition 1.3.8. Every $\zeta \in \mathbb{C}$ such that $\zeta^n - 1 = 0$ for $n \in \mathbb{N}$ is called **nth root of unity** or just **root of unity** if n is not given. Minimal $d \in \mathbb{N}$ for which $\zeta^d - 1 = 0$ is the **order** of ζ.

Nontrivial root of unity is a root of unity $\zeta \neq 1$.

Remark. Nontrivial root of unity is a root of polynomial $\frac{x^n-1}{x-1}$.

Remark. nth root of unity may be written as $\zeta = e^{2k\pi i/n}$ for $k \in \{0, 1, \ldots, n-1\}$.

Theorem 1.3.9. *Degree of nth root of unity ζ is $\varphi(n)$. Where φ is the Euler's function.*

1.3.4 Pisot numbers

Definition 1.3.10. Let $\beta \in \mathbb{B}$ be an algebraic integer of degree m, $\beta > 1$ and for all conjugate roots β', β'', …, $\beta^{(m-1)}$ it holds

$$|\beta^{(i)}| < 1 \qquad \forall\, i \in \widehat{m-1}$$

Then β is called **Pisot**.

As we will see in section 2.2, Pisot numbers another crucial part of our quasicrystal model.

1.3.5 Vieta's formulas

Since we will work a lot with roots of quadratic equations we wnat to, just for completeness, show the Vieta's formulas in the form that we will use.

The roots $x_1, x_2 \in \mathbb{C}$ of quadratic polynomial $ax^2 + bx + c$ satisfy:

$$x_1 + x_2 = -\frac{b}{a} \qquad \text{and} \qquad x_1x_2 = \frac{c}{a}$$

Even better for a monic quadratic polynomial $(a = 1)$ we have:

$$x_1 + x_2 = -b \qquad \text{and} \qquad x_1x_2 = c$$

Lastly since we are interested in expressing one root in terms of the other:

$$x_1 = -b - x_2 \qquad \text{and} \qquad x_1 = \frac{c}{x_2}$$

1.4 Cut-and-project scheme

We are using cut-and-project scheme to model the quasicrystals. Here is a brief introduction into its workings.

Roughly speaking cut-and-project is a way of selecting a subset from a larger set, in our case this larger set is a $\mathbb{Z}[\beta]$-module.

Definition 1.4.1. Let $\beta \in \mathbb{R}$ be Pisot, $\mathbb{Z}[\beta]$ its extension ring and $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be a basis of $\mathbb{R}^d$ for $d \in \mathbb{N}$.

$$L = \bigoplus_{j=1}^{d} \mathbb{Z}\,\mathbf{e}_j$$

is **crystallographic lattice** in $\mathbb{R}^d$.

The cut-and-project scheme utilizes $2n$ dimensional crystallographic lattice $L \subset \mathbb{R}^{2n}$ and two more n dimensional subspaces $V_1, V_2 \subset \mathbb{R}^{2n}$.

Further we define two projections $\pi_1 : \mathbb{R}^{2n} \to V_1$ and $\pi_2 : \mathbb{R}^{2n} \to V_2$ such that $\pi_1|_L$ is injection and $\pi_2(L)$ is dense in V_2.

These projections are where the 'project' part of the cut-and-project scheme comes from. The 'cut' part comes from a bounded subset $\Omega \subset V_2$ with nonempty interior usually referred to as **window**.

All put together the cut-and-project scheme produces a subset $Q \subset V_1$:

$$Q = \{\pi_1(x) \mid \pi_2(x) \in \Omega,\ x \in L\}$$

Put in words the set Q are π_1 projections of those points of L whose π_2 projections fit in the window Ω.

The notation can be somewhat simplified by composing a bijection between V_1 and V_2: $\pi_2 \circ \pi_1^{-1}$, usually denoted as $*$ and referred to as a **star map**. Q then becomes:

$$Q = \{x \in V_1 \mid x^* \in \Omega\}$$

This is the form in which we will use the cut-and-project scheme.

Chapter 2

Quasicrystal

2.1 Quasicrystal

Unfortunately there is so far no established mathematical definition of quasicrystal, in the most basic terms it is just a set that is ordered but not periodic. Therefore we are going to introduce our attributes that a set $\Lambda \subset \mathbb{C}$ needs to have to be a quasicrystal.

Although quasicrystals exist in any dimension, we are only interested in two dimensional ones. Thus we specify just for this case which will ease our work.

First we require Λ to be not too dense but also not too discrete. In other words to have uniform discreetness as well as finite density.

1. uniform discreteness:

$$\exists\, r_1 > 0,\ \forall z_1, z_2 \in \Lambda, z_1 \neq z_2 :\ |z_1 - z_2| > r_1$$

2. relative density:

$$\exists\, r_2 > 0,\ \forall z \in \mathbb{C} :\ B(z, r_2) \cap \Lambda \neq \emptyset$$

Next we want finite local complexity. Sometimes this attribute is written as finiteness of a set of intersections of Λ with balls centered at any point in $\mathbb{C}$ of fixed but arbitrary radius:

$$\forall\, \rho > 0 :\ \big|\{\Lambda \cap B(x, \rho) \mid \forall x \in \Lambda\}\big| < \infty$$

However since we are going to study the Voronoi diagram of Λ we directly require the list of Voronoi tiles of Λ to be finite.

3. finite local complexity:

$$\big|\{V(x) - x \,|\, x \in \Lambda\}\big| < \infty$$

So far we have achieved the orderedness part but even periodic crystallographic lattices have these properties. To break the periodicity we are going to require rotational symmetry and nontrivial dilation.

4. rotational symmetry:
$$\exists\, \zeta = e^{2\pi i/n} :\ \zeta\Lambda = \Lambda$$

5. dilation:
$$\exists\, \beta \in \mathbb{R} \setminus \{-1, 1\} :\ \beta\Lambda \subset \Lambda$$

It stems form these properties alone, that among other constants a quasicrystal is linked to a root of unity ζ and to a number $\beta \in \mathbb{R} \setminus \{-1, 1\}$. Of course not every pair (ζ, β) is associated with a quasicrystal.

In the next section we will go through which numbers are associated with a quasicrystal and where do they come from.

2.2 Pisot-cyclotomic numbers

Pisot-cyclotomic numbers are Pisot and are algebraically related to roots of unity. We will use these numbers in place of β from previous section.

Definition 2.2.1. Let $\rho = 2\cos(2\pi/n)$ for a given $n > 4$, its extension ring $\mathbb{Z}[\rho]$ and m order of ρ. A **Pisot-cyclotomic** number of degree m, of order n associated to ρ is a Pisot number $\beta \in \mathbb{Z}[\rho]$ such that
$$\mathbb{Z}[\beta] = \mathbb{Z}[\rho]$$

Nontrivial nth root of unity $\zeta = e^{2\pi i/n}$ is by definition a solution to equation
$$\frac{\zeta^n - 1}{\zeta - 1} = \zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta + 1 = 0$$
further for $\rho = 2\cos(2\pi/n)$ it holds
$$\rho = \zeta + \bar{\zeta} \quad \Rightarrow \quad \zeta^2 = \rho\zeta - 1$$
Therefore for extension rings $\mathbb{Z}[\zeta]$ and $\mathbb{Z}[\rho]$ we have
$$\mathbb{Z}[\zeta] = \mathbb{Z}[\rho] + \mathbb{Z}[\rho]\zeta$$
and finally for Pisot-cyclotomic β associated to ρ we acquire
$$\mathbb{Z}[\zeta] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\,\zeta$$
Such countable ring is of course n-fold rotationally invariant
$$\zeta^k\mathbb{Z}[\zeta] \subset \mathbb{Z}[\zeta] \qquad k \in \widehat{n-1}$$

To summarize β is real and a Pisot and it can be used to decompose n-fold rotationally invariant complex ring $\mathbb{Z}[\zeta]$ as $\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$.

For further work we will need actual values of Pisot-cyclotomic numbers and so in the next section we will show a method for finding quadratic Pisot-cyclotomic numbers, whose quasicrystals we will later analyze. The method could of course be generalized to different degrees.

2.2.1 Quadratic Pisot-cyclotomic numbers

As stated in preliminaries, the degree of root of unity of order n is $\varphi(n)$ (where φ is the Euler's function). From the decomposition in the previous section we can easily infer that the degree of ζ is double of degree of β (or ρ). Moreover we are looking for β of degree 2. Together we acquire following equation.

$$\varphi(n) = 2 \cdot 2 = 4$$

With help from the Euler's product formula we can show that such equation holds only for $n \in \{5, 8, 10, 12\}$.

For each $n \in \{5, 8, 10, 12\}$ there is $\rho = 2\cos(2\pi/n)$ and for each such ρ there are $\beta \in \mathbb{Z}[\rho]$ following the definition 2.2.1. Each of these numbers β are associated with the same quasicrystals and so it is sufficient to only pick one representative.

Moreover since $2\cos(2\pi/5) = \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2} - 1 = 2\cos(2\pi/10) - 1$ the extension rings $\mathbb{Z}[2\cos(2\pi/5)]$ and $\mathbb{Z}[2\cos(2\pi/10)]$ are identical and by extension the quasicrystals associated are also the same. Therefore we can skip the 5-fold rotational symmetry.

To summarize, quadratic Pisot-cyclotomic numbers β can only be associated to quasicrystals with 8-fold, 10-fold or 12-fold rotational symmetries.

Even though we are mainly focusing on two dimensional quasicrystals, that is not dictated by the quadratic-ness of β. Quadratic Pisot-cyclotomic numbers are also associated to arbitrarily dimensional quasicrystals.

We close this section with a list of quadratic Pisot-cyclotomic numbers (table 2.1).

n	ρ	β	ζ
8	$2\cos\left(\frac{2\pi}{8}\right)$	$1+\sqrt{2}$	$e^{2i\pi/8}$
10	$2\cos\left(\frac{2\pi}{5}\right)$	$\frac{1+\sqrt{5}}{2}$	$e^{2i\pi/5}$
12	$2\cos\left(\frac{2\pi}{12}\right)$	$2+\sqrt{3}$	$e^{2i\pi/12}$

Table 2.1: Pisot-cyclotomic numbers of degree 2, of order n, associated to ρ.

2.3 Quasicrystal model

There certainly are many ways to acquire a set that follows the attributes listed in section 2.1. We utilize the cut-and-project scheme described in section 1.4.

Even though we specified the two dimensional quasicrystal as subset of $\mathbb{C}$ it is sometimes preferable to treat it as subset of $\mathbb{R}^2$; the two coordinates are of course real and imaginary parts.

The cut-and-project scheme requires among others crystallographic lattice L, the sets V_1 and V_2, and suitable projections π_1 and π_2 or $*$. Finding these is outside of scope of this work. For now we will continue using general notation with the promise of specifying these sets and projections once necessary.

Definition 2.3.1. Let β be a Pisot-cyclotomic number of order n, associated to $\rho = 2\cos(2\pi/n)$ (and $\zeta = e^{2\pi i/n}$).

Further let $M, N \subset \mathbb{R}^2$ and projection $* : M \to N$ follow the conditions of the cut-and-project scheme.

Lastly let $\Omega \subset N$ be bounded with nonempty interior.

Then **model of two dimensional quasicrystal linked to irrationality** β **and window** Ω is the set:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\}$$

So far the entire theses was purely abstract. To give you an example of what quasicrystal might look like, there is the image 2.1 which shows finite section of a two dimensional quasicrystal with 8-fold rotational symmetry.

To validate our quasicrystal model we of course have to show that it follows the attributes from section 2.1. However first we list few properties of our model which will help us show that.

Since such general approach won't be necessary for our work, from now on we limit our scope to only quadratic Pisot-cyclotomic numbers (that greatly simplifies the scaling property).

- Inclusion property:
$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \Sigma(\Omega_1) \subset \Sigma(\Omega_2)$$
- Union property:
$$\Sigma(\Omega_1 \cup \Omega_2) \;=\; \Sigma(\Omega_1) \cup \Sigma(\Omega_2)$$
- Translation property:
$$\Sigma(\Omega + x^*) \quad = \quad \Sigma(\Omega) + x \qquad \text{for } x \in M$$
- Scaling property:
$$\Sigma(\beta\Omega) = \beta'\Sigma(\Omega)$$

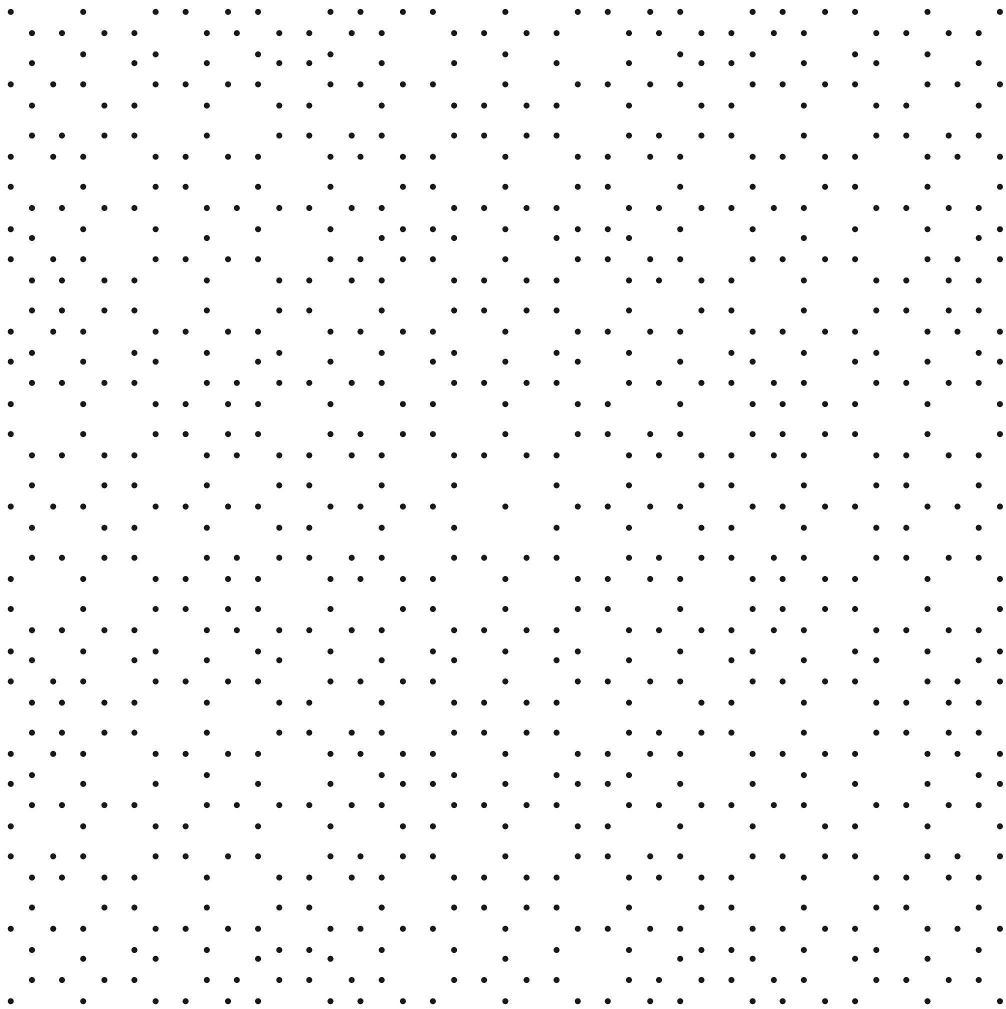

Figure 2.1: Example of a two dimensional quasicrystal.

And now to validate our model.

1. uniform discreteness:

$$\exists\, r_1 > 0,\ \forall z_1, z_2 \in \Lambda, z_1 \neq z_2 :\ |z_1 - z_2| > r_1$$

2. relative density:

$$\exists\, r_2 > 0,\ \forall z \in \mathbb{C} :\ B(z, r_2) \cap \Lambda \neq \emptyset$$

   The result of the cut-and-project scheme is a Delone set if the window Ω is bounded with nonempty interior, which is how we defined it.

3. finite local complexity:

$$\big|\{V(x) - x \,|\, x \in \Lambda\}\big| < \infty$$

It has been shown that our model has even this property, however since our work explores the list of Voronoi tiles we essentially prove this in the process.

4. rotational symmetry:
$$\exists\, \zeta = e^{2\pi i/n} :\ \zeta\Lambda = \Lambda$$
Rotational symmetry appears in quasicrystal if the window Ω has the same rotational symmetry.

5. dilation:
$$\exists\, b \in \mathbb{R} \setminus \{-1, 1\} :\ b\Lambda \subset \Lambda$$
Using the scaling property and the inclusion property:
$$\beta\Sigma(\Omega) = \Sigma(\beta'\Omega) \subset \Sigma(\Omega)$$

Having quasicrystal defined and our model validated, we shall present our plan for the analysis.

2.4 General quasicrystal analysis

In this section we will outline general method of analysis of any quasicrystal. Later we will use this method to analyze two dimensional quasicrystals.

Analysis should reveal the structure of the points of the quasicrystal, for us that means the list of Voronoi tiles for each quasicrystal.

To acquire such list we follow these steps:

1. **Acquire arbitrary finite section of the quasicrystal**

   In other words this means creating algorithm that for finite section $P \subset \mathbb{R}^d$ returns $P \cap \Sigma(\Omega)$.

2. **Estimate covering radius of the quasicrystal** R_C

   We are specifically interested in the upper bound $\hat{R}_C$ of the covering radius. This is necessary since on a Delone set Voronoi tile's domain's points can be no further from the center then double of the covering radius.

3. **Generate superset of all finite sections spanning** $B(2\hat{R}_C)$

   Each of these finite sections represents one Voronoi tile that appears in the quasicrystal's voronoi diagram.

4. **Filter the superset to the final list of Voronoi tiles**

   Due to technical constrains the previous step may have created more tiles than actually are in the list of Voronoi tiles and so it needs to be filtered.

These steps are general enough to analyze any quasicrystal associated with any Pisot-cyclotomic number and in any dimension. Unfortunately they are also too general and we will need to specify them for each quasicrystal.

We close this chapter and our general overview of quasicrystals with discussion of window shapes. It is obviously impossible to analyze a quasicrystal for arbitrary bounded $\Omega \subset N$ with nonempty interior. That is however not necessary. Not every two windows generate different quasicrystals, especially since we do not consider translated and/or β inflated quasicrystals to be different.

2.4.1 Analyzed window shapes

We will use the properties of quasicrystals to gradually limit the set of analyzed windows. We start with all of the windows:

$$\big\{\Omega \,|\, \Omega \subset N, \text{ bounded with nonempty interior}\big\}$$

To maintain our sanity we first limit our scope to convex bounded windows with nonempty interior.

$$\big\{\Omega \,|\, \Omega \subset N, \text{ convex bounded with nonempty interior}\big\}$$

Further thanks to the translation property we can limit our scope to convex bounded windows with nonempty interior centered around the origin:

$$\big\{\Omega - C_\Omega \,|\, \Omega \subset N, \text{ convex bounded with nonempty interior}\big\}$$

where C_Ω is the centroid or geometric center of the window Ω.

Lastly thanks to the scaling property we can limit our scope to convex bounded windows with nonempty interior centered around the origin of diameter in $(1/\beta, 1]$:

$$\big\{\Omega - C_\Omega, \,|\, \Omega \subset N, \ d(\Omega) \in (1/\beta, 1], \text{ convex bounded with nonempty interior}\big\}$$

where $d(\Omega)$ is the set diameter $d(\Omega) = \sup\{d(x,y) \,|\, x, y \in \Omega\}$, where $d(x,y)$ is the distance between x and y.

Remark. We could of course also pick any other β multiple of $(1/\beta, 1]$.

Of course even after all this limiting, the set of windows is still infinite. Therefore we will further limit our scope to three basic window shapes: rhombus, regular n-gon and a circle. The rhombus is not really a valid window since it is not sufficiently rotationally symmetrical, it is however fundamental to our method, more on this later. The regular n-gon represents the simplest window with sufficient rotational symmetry and the circle is interesting for its circular symmetry.

To summarize we will analyze quasicrystals for windows in shape of rhombus, regular n-gon and a circle centered around the origin of diameter in $(1/\beta, 1]$. Technically there is an infinite amount of these windows as well however as we will see later it is already manageable.

Finally we need to discuss the boundaries of the windows. Generally we will assume open windows i. e. we will exclude the boundary. The exception is the rhombus for reasons we will also explain later.

Now we have quasicrystal defined, model explored and windows limited. In the next chapter we will finally start the analysis.

Chapter 3

Analysis

The first step of analysis of the two dimensional quasicrystal is to create algorithm for acquiring arbitrary finite section of the quasicrystal. That is however not so simple. Luckily there is a workaround. Using specific window shape it is possible to decompose two dimensional quasicrystal into two quasicrystals with one dimensional windows. We will explain exactly what that means later, for now let's explore two dimensional quasicrystals with one dimensional windows.

By definition the quasicrystal with one dimensional window is a set of points whose Galois isomorphism images fit on a bounded section of a line. Since the Galois isomorphism inverse image of a line is again a line then two dimensional quasicrystal with a line segment for a window is in fact one dimensional (i.e. set of points on a line).

We present this connection between two dimensional and one dimensional quasicrystal mainly to avoid the need to explicitly define the attributes and show the properties of one dimensional quasicrystal. This way aside from the rotational symmetry all the attributes and all the properties apply to the one dimensional quasicrystal as well.

To summarize the motivation behind the analysis of one dimensional quasicrystal is the eventual analysis of two dimensional quasicrystal. To take full advantage of previous work we view one dimensional quasicrystal as a special case of two dimensional quasicrystal, that is two dimensional quasicrystal with a line segment for a window.

3.1 One dimensional quasicrystal

First we need to show exactly what our model of one dimensional quasicrystal is. As stated before that means specifying the sets and projection for cut-and-project scheme.

Let β be a quadratic Pisot-cyclotomic number of order n, associated to $\rho = 2\cos(2\pi/n)$.

Further let $M = \mathbb{Z}[\beta]$ extension ring of β and $N = \mathbb{Z}[\beta']$ extension ring of β's conjugate root.

The projection $*: M \to N$ is the Galois isomorphism σ_1 (often denoted as $'$).

Lastly let $\Omega \subset N$ be bounded with nonempty interior.

Then model of one dimensional quasicrystal linked to irrationality β and window Ω is the set:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\} = \big\{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\big\}$$

Convex bounded one dimensional window is a line segment, which is in one dimension represented by an interval, specifically we will use left-closed right-open interval $\Omega = \big[-\frac{\ell}{2}, \frac{\ell}{2}\big)$ where $\ell \in (1/\beta, 1]$.

As we see in the following breakdown, quasicrystals with different openness or closeness differ only by at most a single point.

$$\Sigma((c,d)) = \begin{cases} \Sigma([c,d)) & c \notin \mathbb{Z}[\beta] \\ \Sigma([c,d)) \setminus \{c'\} & c \in \mathbb{Z}[\beta] \end{cases}$$

$$\Sigma([c,d]) = \begin{cases} \Sigma([c,d)) & d \notin \mathbb{Z}[\beta] \\ \Sigma([c,d)) \cup \{d'\} & d \in \mathbb{Z}[\beta] \end{cases}$$

$$\Sigma((c,d]) = \begin{cases} \Sigma((c,d)) & d \notin \mathbb{Z}[\beta] \\ \Sigma((c,d)) \cup \{d'\} & d \in \mathbb{Z}[\beta] \end{cases}$$

Unfortunately the addition or the removal of the single point causes occurrence of local configurations that appear only once in the entire quasicrystal. That would needlessly complicate our work and therefore we chose to only analyze left-closed right-open interval, which does not suffer from these zero density occurrences.

According to our plan, first step of the analysis is generating arbitrary finite section of one dimensional quasicrystal.

3.1.1 Arbitrary finite section

The picture 3.1 illustrates well, what we want to acquire – the sequence of points on the x axis. For this purpose we define the sequence of the quasicrystal.

Definition 3.1.1. Strictly increasing sequence $(y_n^\Omega)_{n\in\mathbb{Z}}$ defined as $\big\{y_n^\Omega \,|\, n \in \mathbb{Z}\big\} = \Sigma(\Omega)$ where $\Sigma(\Omega)$ is one dimensional quasicrystal is called the **sequence of the quasicrystal** $\Sigma(\Omega)$.

Now we would like to explore the set of all possible distances between two consecutive points of the sequence of the quasicrystal:

$$\big\{y_{n+1}^\Omega - y_n^\Omega \,|\, n \in \mathbb{N}\big\}$$

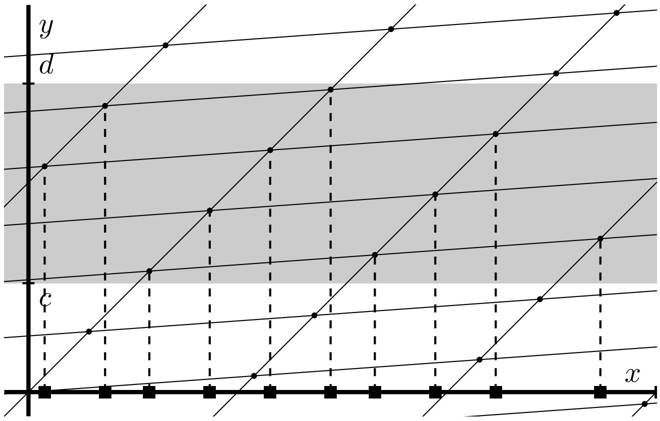

Figure 3.1: Illustration of one-dimensional quasicrystal. The grid is $M \times N$. On the y axis there is a window $\Omega = [c, d)$. The squares on the x axis are points of the quasicrystal $\Sigma(\Omega)$.

For that we need an expression for y_n^Ω. Let's start with the simplest window: $[-\frac{1}{2}, \frac{1}{2})$. The key here is the length of the window – i.e. 1. First we do a little algebraic exercise with the expression for quasicrystal:

$$\begin{aligned}
\Sigma\left(\left[-\frac{1}{2}, \frac{1}{2}\right)\right) &= \left\{ x \in \mathbb{Z}[\beta] \,\middle|\, x' \in \left[-\frac{1}{2}, \frac{1}{2}\right) \right\} \\
&= \left\{ a + b\beta \,\middle|\, a + b\beta' \in \left[-\frac{1}{2}, \frac{1}{2}\right),\ a, b \in \mathbb{Z} \right\} \\
&= \left\{ a + b\beta \,\middle|\, -\frac{1}{2} \leq a + b\beta' < \frac{1}{2},\ a, b \in \mathbb{Z} \right\} \\
&= \left\{ a + b\beta \,\middle|\, -\frac{1}{2} - b\beta' \leq a < \frac{1}{2} - b\beta',\ a, b \in \mathbb{Z} \right\} \\
&= \left\{ \left\lceil -\frac{1}{2} - b\beta' \right\rceil + b\beta \,\middle|\, b \in \mathbb{Z} \right\}
\end{aligned}$$

Thus we can express the sequence of points of the quasicrystal as:

$$y_n^{[-\frac{1}{2}, \frac{1}{2})} = \left\lceil -\frac{1}{2} - n\beta' \right\rceil + n\beta$$

And for the set of distances between two consecutive points we have:

$$\left\{ \left\lceil -\frac{1}{2} - (n+1)\beta' \right\rceil + (n+1)\beta - \left\lceil -\frac{1}{2} - n\beta' \right\rceil - n\beta \,\middle|\, n \in \mathbb{Z} \right\}$$

$$\left\{ \left\lceil -\frac{1}{2} - (n+1)\beta' \right\rceil - \left\lceil -\frac{1}{2} - n\beta' \right\rceil + \beta \,\middle|\, n \in \mathbb{Z} \right\}$$

$$\left\{ \left\lceil -\frac{1}{2} - n\beta' - \beta' \right\rceil - \left\lceil -\frac{1}{2} - n\beta' \right\rceil + \beta \;\middle|\; n \in \mathbb{Z} \right\}$$

Because β is Pisot we have $|\beta'| < 1$. Therefore the difference between the ceils is either 0 or 1 and the set of distances between two consecutive points thus collapses to simple $\{\beta, \beta + 1\}$.

With a little thought and with use of the scaling property of a qusicrystal we can expand this to any window of size β^k where $k \in \mathbb{Z}$.

$$\Sigma\left(\beta\left[-\frac{1}{2}, \frac{1}{2}\right)\right) = \beta'\Sigma\left(\left[-\frac{1}{2}, \frac{1}{2}\right)\right)$$

Thus for window $\left[-\frac{\beta^k}{2}, \frac{\beta^k}{2}\right)$ we have the set of distances between two consecutive points $\left\{\left|(\beta')^k\beta\right|, \left|(\beta')^k(\beta+1)\right|\right\}$.

Applying Vieta's formulas we have:

$$\left\{ \left|\frac{c^k}{\beta^{k-1}}\right|, \left|\frac{c^k}{\beta^{k-1}} + \frac{c^k}{\beta^k}\right| \right\}$$

where c is the constant coefficient of β's minimal polynomial.

In the scope of our interest we now know the set of distances between two consecutive points for window of size 1 and just outside of our scope for window of size $\frac{1}{\beta}$. Now we need to expand this knowledge to the entire interval $(\frac{1}{\beta}, 1]$.

For that we utilize the inclussion property.