Chapter 1

Preliminaries

1.1 Delone set

Delone set is such a set that is both relatively dense and uniformly discrete. In order to characterize exactly what dense and discrete means, we define two parameters for any subset of $\mathbb{C}^n$.

Definition 1.1.1. Let $D \subset \mathbb{C}^n$. Then $R_P \in \mathbb{R}$

$$R_P = \frac{1}{2} \sup \{r_1 \in \mathbb{R} \,|\, \forall z_1, z_2 \in D, z_1 \neq z_2 : \; \|z_1 - z_2\| > r_1\}$$

is called **packing radius** of the set D.

Remark. Open balls of packing radius centered at the points of the set are disjoint.

Definition 1.1.2. Let $D \subset \mathbb{C}^n$. Then $R_C \in \mathbb{R}$

$$R_C = \inf \{r_2 > 0 \,|\, \forall z \in \mathbb{C}^n : \; B(z, r_2) \cap D \neq \emptyset\}$$

is called **covering radius** of the set D.

Remark. Union of closed balls of covering radius centered at the points of the set is the entire space $\mathbb{C}^n$.

Definition 1.1.3.

$D \subset \mathbb{C}^n$ which has positive packing radius R_P is **uniformly discrete**.
$D \subset \mathbb{C}^n$ which has finite covering radius R_C is **relatively dense**.
$D \subset \mathbb{C}^n$ which has both positive packing radius R_P and finite covering radius R_C is a **Delone set**.

1.2 Voronoi diagram

Definition 1.2.1. Let $P \subset \mathbb{R}^n$ be a discrete set and $x \in P$. Then

$$V_P(x) = \{y \in \mathbb{R}^n \,|\, \forall z \in P, z \neq x : \|y - x\| \leq \|y - z\|\}$$

is called **Voronoi polygon** or **Voronoi cell** or **Voronoi tile** of x on P.

Voronoi polygon $V_P(x)$ is said to belong to the point x and x is called the **center** of the Voronoi cell $V_P(x)$.

When there can be no confusion as to what set P is, it may be omitted: $V(x)$.

Definition 1.2.2. Let $P \subset \mathbb{R}^n$ be a discrete set. Then set of all Voronoi tiles

$$\{V(x) \,|\, x \in P\}$$

is called **Voronoi diagram** or **Voronoi tessellation**.

Definition 1.2.3. Let $P \subset \mathbb{R}^n$ be a discrete set and $x \in P$. Then

$$\sup_{y \in V(x)} \|y - x\|$$

is called **radius** of the Voronoi polygon.

Definition 1.2.4. Let $P \subset \mathbb{R}^n$ be a discrete set and $x \in P$. Then the set of points of P that directly shape the Voronoi polygon $V_P(x)$:

$$D_P(x) = \bigcap \left\{Q \subset P \,|\, V_Q(x) = V_P(x)\right\}$$

is called the **domain** of x or of $V_P(x)$.

1.3 Number theory

The study of quasicrystals relies heavily on number theory. Therefore this section list the definitions and their implications that are used further, we will however not show any proofs for our claims.

Definition 1.3.1. Let $P \subset \mathbb{C}$. Then $P[x]$ denotes the set of polynomials with coefficients in P.

Definition 1.3.2. Let $f \in \mathbb{C}[x]$ such that $f(x) = \sum_{k=0}^{m} \alpha_k x^k$. Then f is **monic polynomial** if $\alpha_m = 1$.

1.3.1 Algebraic numbers, minimal polynomial and degree

Definition 1.3.3.

Let $\alpha \in \mathbb{C}$. If there exists monic polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$, then α is an **algebraic number**. The set of algebraic numbers is denoted as $\mathbb{A}$.

Let $\beta \in \mathbb{C}$. If there exists monic polynomial $g \in \mathbb{Z}[x]$ such that $g(\beta) = 0$, then β is an **algebraic integer**. The set of algebraic integers is denoted as $\mathbb{B}$.

Such polynomial f or g is then called the **minimal polynomial** of α or β respectively and denoted as f_α or f_β respectively

The degree of the polynomial is also regarded as the **degree of the algebraic number**.

Remark. For each algebraic number there exists exactly one minimal polynomial.

1.3.2 Galois isomorphism

Definition 1.3.4. The $(m-1)$ other roots of f_α for $\alpha \in \mathbb{A}$ of degree m are called **conjugate roots** of α and denoted as α', α'', ..., $\alpha^{(m-1)}$.

Remark. Consistently with the notation of its conjugate roots, α may be denoted as $\alpha^{(0)}$ or $\alpha^{(m)}$.

Definition 1.3.5. The ring $\mathbb{Z}(\alpha) \subset \mathbb{C}$:

$$\mathbb{Z}(\alpha) = \left\{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{m-1}\alpha^{m-1} \mid a_i \in \mathbb{Z}\right\}$$

is called the **extension ring** of the number $\alpha \in \mathbb{A}$ of degree m.

Definition 1.3.6. The number field $\mathbb{Q}(\alpha) \subset \mathbb{C}$:

$$\mathbb{Q}(\alpha) = \left\{b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{m-1}\alpha^{m-1} \mid b_i \in \mathbb{Q}\right\}$$

is called the **extension field** of the number $\alpha \in \mathbb{A}$ of degree m.

Definition 1.3.7. Let $\alpha \in \mathbb{A}$ of degree m and α', α'', ..., $\alpha^{(m-1)}$ its conjugate roots. Then $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\alpha')$, ..., $\mathbb{Q}(\alpha^{(m-1)})$ are isomorphic and the isomorphisms:

$$\sigma_i(\mathbb{Q}(\alpha) = \mathbb{Q}\big(\alpha^{(i)}\big) \qquad i \in \widehat{m-1}$$

are called **Galois isomorphisms**.

Galois isomorphisms are significant part of the definition of the quasicrystals, so they surely deserve an example.

The Galois isomorphism σ_0 is always identity.

In general the Galois isomorphism σ_i exchanges α of degree m with its ith conjugate root.

$$\sigma_i\big(b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{m-1}\alpha^{m-1}\big) = b_0 + b_1\alpha^{(i)} + b_2\big(\alpha^{(i)}\big)^2 + \cdots + b_{m-1}\big(\alpha^{(i)}\big)^{m-1}$$

Since further we will mostly work only with quadratic algebraic numbers (of degree 2), there will only be two roots and two Galois isomorphisms, identity and $\sigma_1(\alpha) = \alpha'$. Thus it is often denoted only as $'$, as in $(\alpha)' = \sigma_1(\alpha) = \alpha'$.

1.3.3 Root of unity, cyclotomic polynomial

Definition 1.3.8. Every $\zeta \in \mathbb{C}$ such that $\zeta^n - 1 = 0$ for $n \in \mathbb{N}$ is called **nth root of unity** or just **root of unity** if n is not given. Minimal $d \in \mathbb{N}$ for which $\zeta^d - 1 = 0$ is the **order** of ζ.

Nontrivial root of unity is a root of unity $\zeta \neq 1$.

Remark. Nontrivial root of unity is a root of polynomial $\frac{x^n-1}{x-1}$.

Remark. nth root of unity may be written as $\zeta = e^{2k\pi i/n}$ for $k \in \{0, 1, \ldots, n-1\}$.

Theorem 1.3.9. *Degree of nth root of unity ζ is $\varphi(n)$. Where φ is the Euler's function.*

1.3.4 Pisot numbers

Definition 1.3.10. Let $\beta \in \mathbb{B}$ be an algebraic integer of degree m, $\beta > 1$ and for all conjugate roots β', β'', ..., $\beta^{(m-1)}$ it holds

$$|\beta^{(i)}| < 1 \qquad \forall\, i \in \widehat{m-1}$$

Then β is called **Pisot**.

As we will see in section 2.2, Pisot numbers another crucial part of our quasicrystal model.

1.4 Cut-and-project scheme

We are using cut-and-project scheme to model the quasicrystals. Here is a brief introduction into its workings.

Roughly speaking cut-and-project is a way of selecting a subset from a larger set, in our case this larger set is a $\mathbb{Z}[\beta]$-module.

Definition 1.4.1. Let $\beta \in \mathbb{R}$ be Pisot, $\mathbb{Z}[\beta]$ its extension ring and $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ be a basis of $\mathbb{R}^d$ for $d \in \mathbb{N}$.

$$L = \bigoplus_{j=1}^{d} \mathbb{Z}\,\mathbf{e}_j$$

is **crystallographic lattice** in $\mathbb{R}^d$.

The cut-and-project scheme utilizes $2n$ dimensional crystallographic lattice $L \subset \mathbb{R}^{2n}$ and two more n dimensional subspaces $V_1, V_2 \subset \mathbb{R}^{2n}$.

Further we define two projections $\pi_1 : \mathbb{R}^{2n} \to V_1$ and $\pi_2 : \mathbb{R}^{2n} \to V_2$ such that $\pi_1|_L$ is injection and $\pi_2(L)$ is dense in V_2.

These projections are where the 'project' part of the cut-and-project scheme comes from. The 'cut' part comes from a bounded subset $\Omega \subset V_2$ with nonempty interior usually referred to as **window**.

All put together the cut-and-project scheme produces a subset $Q \subset V_1$:

$$Q = \{\pi_1(x) \mid \pi_2(x) \in \Omega, \, x \in L\}$$

Put in words the set Q are π_1 projections of those points of L whose π_2 projections fit in the window Ω.

The notation can be somewhat simplified by composing a bijection between V_1 and V_2: $\pi_2 \circ \pi_1^{-1}$, usually denoted as $*$ and referred to as a **star map**. Q then becomes:

$$Q = \{x \in V_1 \mid x^* \in \Omega\}$$

This is the form in which we will use the cut-and-project scheme.

Chapter 2

Quasicrystal

2.1 Quasicrystal

In two dimensions a quasicrystal can be viewed as a subset of complex numbers $\Lambda \subset \mathbb{C}$ following these five properties:

1. rotational symmetry:
$$\exists\, \zeta = e^{2\pi i/n} : \ \zeta\Lambda = \Lambda$$

2. dilation:
$$\exists\, \beta \in \mathbb{R} \setminus \{-1, 1\} : \ \beta\Lambda \subset \Lambda$$

3. uniform discreteness:
$$\exists\, r_1 > 0, \ \forall z_1, z_2 \in \Lambda, z_1 \neq z_2 : \ |z_1 - z_2| > r_1$$

4. relative density:
$$\exists\, r_2 > 0, \ \forall z \in \mathbb{C} : \ B(z, r_2) \cap \Lambda \neq \emptyset$$

5. finite local complexity:
$$\forall\, \rho > 0 : \ \big|\{\Lambda \cap B(x, \rho) \mid \forall x \in \Lambda\}\big| < \infty$$

Remark. Properties 3. and 4. together make quasicrystal to be a Delone set.

It stems form these properties alone, that among other constants a quasicrystal is linked to a root of unity ζ and to a number $\beta \in \mathbb{R} \setminus \{-1, 1\}$. Of course not every pair (ζ, β) is associated with a quasicrystal.

In the next section we will go through which numbers are associated with a quasicrystal and where do they come from.

2.2 Pisot-cyclotomic numbers

Pisot-cyclotomic numbers are Pisot and are algebraically related to roots of unity. We will use these numbers in place of β from previous section.

Definition 2.2.1. Let $\rho = 2\cos(2\pi/n)$ for a given $n > 4$, its extension ring $\mathbb{Z}[\rho]$ and m order of ρ. A **Pisot-cyclotomic** number of degree m, of order n associated to ρ is a Pisot number $\beta \in \mathbb{Z}[\rho]$ such that

$$\mathbb{Z}[\beta] = \mathbb{Z}[\rho]$$

Nontrivial nth root of unity $\zeta = e^{2\pi i/n}$ is by definition a solution to equation

$$\frac{\zeta^n - 1}{\zeta - 1} = \zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta + 1 = 0$$

further for $\rho = 2\cos(2\pi/n)$ it holds

$$\rho = \zeta + \bar{\zeta} \quad \Rightarrow \quad \zeta^2 = \rho\zeta - 1$$

Therefore for extension rings $\mathbb{Z}[\zeta]$ and $\mathbb{Z}[\rho]$ we have

$$\mathbb{Z}[\zeta] = \mathbb{Z}[\rho] + \mathbb{Z}[\rho]\zeta$$

and finally for Pisot-cyclotomic β associated to ρ we acquire

$$\mathbb{Z}[\zeta] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$$

Such countable ring is of course n-fold rotationally invariant

$$\zeta^k \mathbb{Z}[\zeta] \subset \mathbb{Z}[\zeta] \qquad k \in \widehat{n-1}$$

To summarize β is real and a Pisot and it can be used to decompose n-fold rotationally invariant complex ring $\mathbb{Z}[\zeta]$ as $\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$.

For further work we will need actual values of Pisot-cyclotomic numbers and so in the next section we will show a method for finding quadratic Pisot-cyclotomic numbers, whose quasicrystals we will later analyze. The method could of course be generalized to different degrees.

2.2.1 Quadratic Pisot-cyclotomic numbers

As stated in preliminaries, the degree of root of unity of order n is $\varphi(n)$ (where φ is the Euler's function). From the decomposition in the previous section we can easily infer that the degree of ζ is double of degree of β (or ρ). Moreover we are looking for β of degree 2. Together we acquire following equation.

$$\varphi(n) = 2 \cdot 2 = 4$$

With help from the Euler's product formula we can show that such equation holds only for $n \in \{5, 8, 10, 12\}$.

For each $n \in \{5, 8, 10, 12\}$ there is $\rho = 2\cos(2\pi/n)$ and for each such ρ there are $\beta \in \mathbb{Z}[\rho]$ following the definition 2.2.1.

To summarize, quadratic Pisot-cyclotomic numbers β can only be associated to quasicrystals with 5-fold, 8-fold or 12-fold rotational symmetries. Even though we are mainly focusing on two dimensional quasicrystals, that is not dictated by the quadratic-ness of β. Quadratic Pisot-cyclotomic numbers are also associated to arbitrarily dimensional quasicrystals. For example we will also study one dimensional quasicrystals.

We close this section with a list of quadratic Pisot-cyclotomic numbers (table 2.1).

n	ρ	β	ζ
5	$2\cos\left(\frac{2\pi}{5}\right)$	$\frac{1+\sqrt{5}}{2}$	$e^{2i\pi/5}$
8	$2\cos\left(\frac{2\pi}{8}\right)$	$1+\sqrt{2}$	$e^{2i\pi/8}$
12	$2\cos\left(\frac{2\pi}{12}\right)$	$2+\sqrt{3}$	$e^{2i\pi/12}$

Table 2.1: Pisot-cyclotomic numbers of degree 2, of order n, associated to ρ.

2.3 Quasicrystal model

There certainly are many ways to acquire a set that follows the properties listed in section 2.1. We utilize the cut-and-project scheme described in section 1.4.

The cut-and-project scheme requires among others crystallographic lattice L, the sets V_1 and V_2, and suitable projections π_1 and π_2 or $*$. Finding these is outside of scope of this work. For now we will continue using general notation with the promise of specifying these sets and projections once necessary.

Definition 2.3.1. Let β be a Pisot-cyclotomic number of order n, associated to $\rho = 2\cos(2\pi/n)$ (and $\zeta = e^{2\pi i/n}$).

Further let $M, N \subset \mathbb{R}^d$ and projection $* : M \to N$ follow the conditions of the cut-and-project scheme.

Lastly let $\Omega \subset N$ be bounded with nonempty interior.

Then d **dimensional quasicrystal linked to irrationality** β **and window** Ω is the set:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\}$$

So far the entire theses was purely abstract. To give you an example of what quasicrystal might look like, there is the image 2.1 which shows finite sections of a two dimensional quasicrystal with 8-fold rotational symmetry.

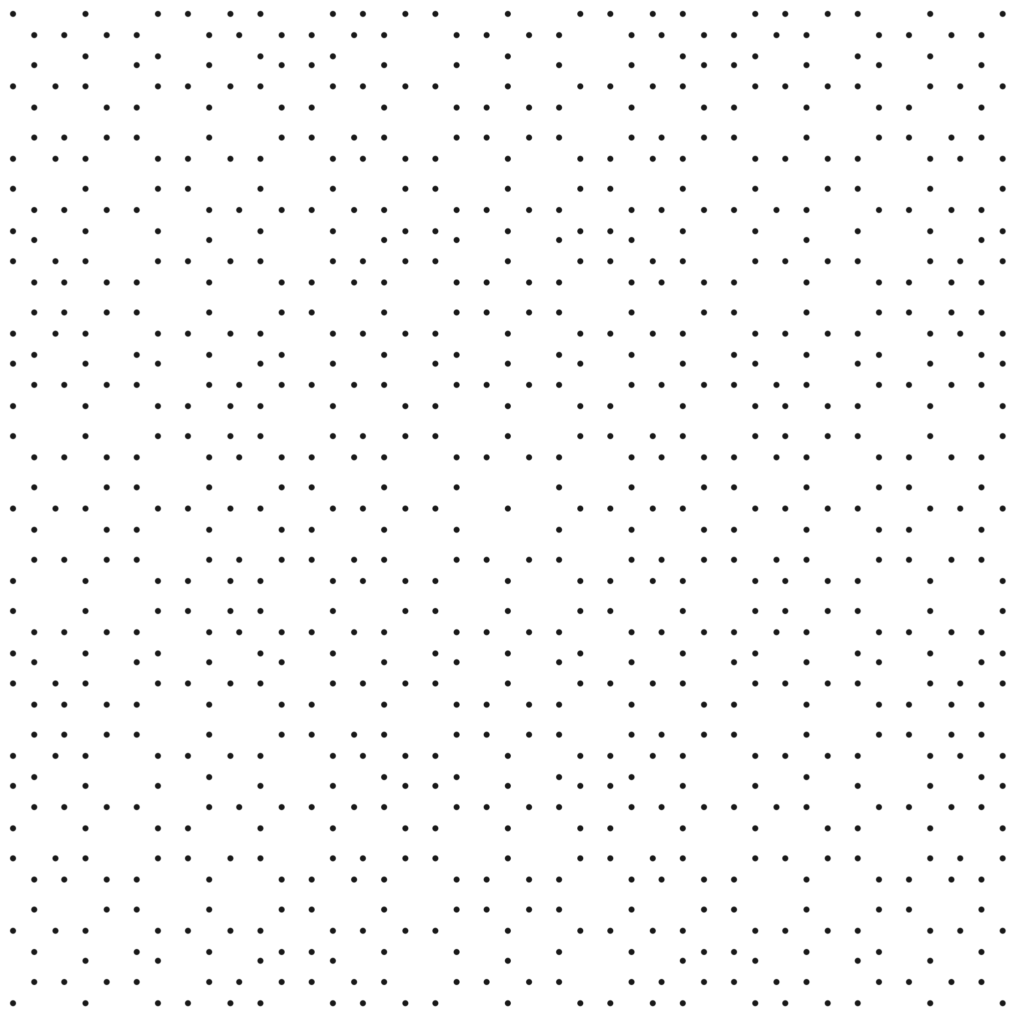

Figure 2.1: Example of a two dimensional quasicrystal.

To validate our quasicrystal model we of course have to show that it follows the properties in section 2.1. However first we list few properties of our model which will help us show that.

Since such general approach won't be necessary for our work, from now on we limit our scope to only quadratic Pisot-cyclotomic numbers (that greatly simplifies the scaling property).

- Inclusion property:
$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad \Sigma(\Omega_1) \subset \Sigma(\Omega_2)$$
- Union property:
$$\Sigma(\Omega_1 \cup \Omega_2) \; = \; \Sigma(\Omega_1) \cup \Sigma(\Omega_2)$$

- Translation property:

$$\Sigma(\Omega + x^*) \quad = \quad \Sigma(\Omega) + x \qquad \text{for } x \in M$$

- Scaling property:

$$\Sigma(\beta\Omega) = \beta'\Sigma(\Omega)$$

And now to validate our model.

1. rotational symmetry:

   $$\exists\, \zeta = e^{2\pi i/n} :\ \zeta\Lambda = \Lambda$$

   Rotational symmetry appears in quasicrystal if the window Ω has the same rotational symmetry.

2. dilation:

   $$\exists\, b \in \mathbb{R} \setminus \{-1, 1\} :\ b\Lambda \subset \Lambda$$

   Using the scaling property and the inclusion property:

   $$\beta\Sigma(\Omega) = \Sigma(\beta'\Omega) \subset \Sigma(\Omega)$$

3. uniform discreteness:

   $$\exists\, r_1 > 0,\ \forall z_1, z_2 \in \Lambda, z_1 \neq z_2 :\ |z_1 - z_2| > r_1$$

4. relative density:

   $$\exists\, r_2 > 0,\ \forall z \in \mathbb{C} :\ B(z, r_2) \cap \Lambda \neq \emptyset$$

   The result of cut-and-project scheme is a Delone set if the window Ω is bounded with nonempty interior.

5. finite local complexity:

   $$\forall\, \rho > 0 :\ \big|\{\Lambda \cap B(x, \rho) \mid \forall x \in \Lambda\}\big| < \infty$$

   It has been shown that our model has even this property, however since this work explores this finite complexity it essentially proves this property as well.

Having quasicrystal defined and our model validated, we will present our plan for the analysis.

2.4 General quasicrystal analysis

In this section we will outline general method of analysis of any quasicrystal. Later we will use this method to analyze two dimensional and one dimensional quasicrystals.

Analysis should reveal the structure of the points of the quasicrystal, for us that means listing all Voronoi tiles that appear in the qusicrystal's Voronoi diagram.

To acquire such list we follow these steps:

1. **Acquire arbitrary finite section of the quasicrystal**

   In other words this means creating algorithm that for finite section $P \subset \mathbb{R}^d$ returns $P \cap \Sigma(\Omega)$.

2. **Estimate covering radius of the quasicrystal** R_C

   We are specifically interested in the upper bound $\hat{R}_C$ of the covering radius.

3. **Generate superset of all finite sections spanning** $B(2\hat{R}_C)$

   Each of these finite sections represents one Voronoi tile that appears in the quasicrystal's voronoi diagram.

4. **Filter the superset to the final list of Voronoi tiles**

These steps are general enough to analyze any quasicrystal associated with any Pisot-cyclotomic number and in any dimension. Unfortunately they are also too general and need to be specified for a specific quasicrystal.

We close this chapter and our general overview of quasicrystals with discussion of window shapes. It is obviously impossible to analyze quasicrystal for arbitrary bounded $\Omega \subset N$ with nonempty interior. There however is a way to limit our scope to few window shapes that in some way represent all possible windows.

2.4.1 Analyzed window shapes

We will use the properties of quasicrystals to gradually limit the set of analyzed windows. We start with all of the windows:

$$\big\{\Omega \,|\, \Omega \subset N, \text{ bounded with nonempty interior}\big\}$$

Thanks to the union property we can limit our scope to a continuous convex bounded windows with nonempty interior since most more complicated windows are unions of continuous convex bounded window with nonempty interior:

$$\big\{\Omega \,|\, \Omega \subset N, \text{ continuous convex bounded with nonempty interior}\big\}$$

Further thanks to the translation property we can limit our scope to a continuous convex bounded windows with nonempty interior centered around the origin:

$$\left\{\Omega - \overline{\Omega} \,|\, \Omega \subset N, \text{ continuous convex bounded with nonempty interior}\right\}$$

where $\overline{\Omega}$ is the arithmetic mean (or "center of mass") of the window Ω.

Lastly thanks to the scaling property we can limit our scope to a continuous convex bounded windows with nonempty interior centered around the origin of diameter in $(1/\beta, 1]$:

$$\left\{\Omega - \overline{\Omega} \,|\, \Omega \subset N,\ d(\Omega) \in (1/\beta, 1], \text{ continuous convex bounded with nonempty interior}\right\}$$

where $d(\Omega)$ is the set diameter $d(\Omega) = \sup\{d(x, y) \,|\, x, y \in \Omega\}$

Remark. We could of course also pick any other β multiple of $(1/\beta, 1]$.

Of course even after all this limiting, the set of windows is still infinite. Therefore we will further limit our scope to three basic window shapes: rhombus, regular n-gon and a circle. The rhombus is not really a valid window since it is not sufficiently rotationally symmetrical, it is however fundamental to our method, more on this later. The regular n-gon represents the simplest window with sufficient rotational symmetry and the circle interesting for its circular symmetry.

To summarize we will analyze quasicrystals for windows in shape of rhombus, regular n-gon and a circle centered around the origin of diameter in $(1/\beta, 1]$. Technically there is an infinite amount of these windows as well however as we will see later it is already manageable.

Finally we need to discuss the boundaries of the windows. Generally we will assume open windows i. e. we will exclude the boundary. The exception is the rhombus (and its one dimensional equivalent – interval) for reasons we will also explain later.

Now we have quasicrystal defined, model explored and windows limited. In the next chapter we will finally start the analysis.

Chapter 3

Analysis

The first step of analysis of two dimensional quasicrystal is to create algorithm for acquiring arbitrary finite section of the quasicrystal. That is however not so simple. Luckily there is a workaround. Using specific shape of a window it is possible to decompose two dimensional quasicrystal into two one dimensional quasicrystals. We will explain exactly what that means later. However that is the motivation for analysis of the one dimensional quasicrystal.

3.1 One dimensional quasicrystal

First we need to specify our general quasicrystal model for one dimension.

Let β be a Pisot-cyclotomic number of order n, associated to $\rho = 2\cos(2\pi/n)$ (and $\zeta = e^{2\pi i/n}$).

Further let $M = \mathbb{Z}[\beta]$ extension ring of β and $M^* = \mathbb{Z}[\beta']$ extension ring of its conjugate root.

The projection $* : M \to M^*$ is the Galois isomorphism σ_1 (often denoted as $'$).

Lastly let $\Omega \subset M^*$ be bounded with nonempty interior.

Then one dimensional quasicrystal linked to irrationality β and window Ω is the set:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\} = \big\{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\big\}$$

To state the obvious, one dimensional quasicrystal is a Delone set of points on a line.

The one dimensional window shape for which we will analyze is left-closed right-open interval $\Omega = \big[-\frac{\ell}{2}, \frac{\ell}{2}\big)$ where $\ell \in (1/\beta, 1]$.

According to our plan, first step of analysis is generating arbitrary finite section of one dimensional quasicrystal.

3.1.1 Arbitrary finite section