1 Preliminaries

Definition 1.1. Roots of the following quadratic equation are denoted as β and β'.

$$x^2 = 4x - 1 \qquad \beta = 2 + \sqrt{3} \doteq 3.732 \qquad \beta' = 2 - \sqrt{3} \doteq 0.268$$

Remark 1. The number β as defined in Definition 1.1 will represent the same value in the entire text.

Being roots of the same quadratic equation, β and β' have some interesting properties that are often used while working with quasicrystals.

Theorem 1.2. *Properties of the roots β and β'.*

$$\beta\beta' = 1 \qquad \beta^{k+2} = 4 \cdot \beta^{k+1} - \beta^k \qquad \frac{1}{\beta} = \beta' = 4 - \beta$$

$$\beta + \beta' = 4 \qquad \beta'^{k+2} = 4 \cdot \beta'^{k+1} - \beta'^k \qquad \frac{1}{\beta'} = \beta = 4 - \beta'$$

Definition 1.3. Symbol $\mathbb{Z}[\beta]$ denotes the smallest ring containing integers $\mathbb{Z}$ and the irrationality β. Since β is quadratic the ring has the following simple form.

$$\mathbb{Z}[\beta] = \{a + b\beta | a, b \in \mathbb{Z}\}$$

Remark 2. Similarly, ring $\mathbb{Z}[\beta']$ can be defined. According to the Theorem 1.2 the two rings are equivalent: $\mathbb{Z}[\beta] = \mathbb{Z}[\beta']$.

2 One-dimensional quasicrystals

To define a quasicrystal one more definition is needed. Function connecting a space of the quasicrystal with a space of the acceptance set called acceptance window.

Definition 2.1. Function $' : \mathbb{Z}[\beta] \to \mathbb{Z}[\beta']$ is defined as $(a + b\beta)' = a + b\beta'$.

Remark 3. Notation is consistent with the Definition 1.1: $(\beta)' = \beta'$.

Definition 2.2. Let $\Omega \subset \mathbb{R}$ be a bounded set with non-empty interior. Then **one-dimensional quasicrystal** with the window Ω is denoted by $\Sigma(\Omega)$ and defined as:

$$\Sigma(\Omega) = \{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\}$$

Remark 4. $\Sigma(\Omega)$ where $\Omega \subset \mathbb{R}$ always denotes one-dimensional quasicrystal.

Some properties of one-dimensional quasicrystals are crucial for the algorithms used for the analysis.

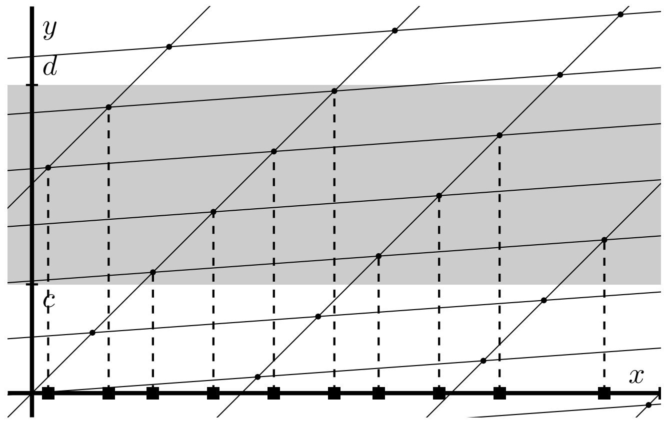

Figure 1: Illustration of one-dimensional quasicrystal. Grid intersections are defined as a set $\{(\lambda, \lambda')|\lambda \in \mathbb{Z}[\beta]\}$. There is a window $\Omega = [c, d)$ on the y axis and finally the squares on the x axis are points of the quasicrystal $\Sigma(\Omega)$.

Theorem 2.3. *Let* $\Omega, \tilde{\Omega} \subset \mathbb{R}$ *and* $\lambda \in \mathbb{Z}[\beta]$.

$$\Omega \subset \tilde{\Omega} \Rightarrow \Sigma(\Omega) \subset \Sigma(\tilde{\Omega}) \qquad \Sigma(\Omega) \cap \Sigma(\tilde{\Omega}) = \Sigma(\Omega \cap \tilde{\Omega})$$
$$\Sigma(\Omega + \lambda') = \Sigma(\Omega) + \lambda \qquad \Sigma(\Omega) \cup \Sigma(\tilde{\Omega}) = \Sigma(\Omega \cup \tilde{\Omega})$$
$$\Sigma(\beta\Omega) = \frac{1}{\beta}\Sigma(\Omega)$$

Remark 5. Further only left-closed right-open intervals will be analyzed as windows. That is justified by theorem 2.3 and following analysis.

$$\Sigma((c,d)) = \begin{cases} \Sigma([c,d)) & c \notin \mathbb{Z}[\beta] \\ \Sigma([c,d)) \setminus \{c'\} & c \in \mathbb{Z}[\beta] \end{cases} \qquad \Sigma([c,d]) = \begin{cases} \Sigma([c,d)) & d \notin \mathbb{Z}[\beta] \\ \Sigma([c,d)) \cup \{d'\} & d \in \mathbb{Z}[\beta] \end{cases}$$
$$\Sigma((c,d]) = \begin{cases} \Sigma((c,d)) & d \notin \mathbb{Z}[\beta] \\ \Sigma((c,d)) \cup \{d'\} & d \in \mathbb{Z}[\beta] \end{cases}$$

Theorem 2.4. *Let* $\Omega \subset \mathbb{R}$ *then* $\forall k \in \mathbb{Z} : \Sigma(\frac{1}{\beta^k}\Omega) = \beta^k\Sigma(\Omega)$.

Corollary 2.5. *From remark 5 and theorem 2.4 follows that only windows* $\Omega = [c, d)$ *where* $d - c \in \left(\frac{1}{\beta}, 1\right]$ *need to be analyzed. Such windows are called* ***base windows*** *or* ***windows in the base form****. Quasicrystals for all other windows can be acquired from the quasicrystals with the base windows by scaling and operations from remark 5.*

2.1 One-dimensional quasicrystal structure

Figure 1 suggests that the one-dimensional quasicrystal is a sequence of points. This subsection presents an analysis of spacing and distribution of these points.

Definition 2.6. Strictly increasing sequence $(y_n^\Omega)_{n\in\mathbb{Z}}$ defined as $\left\{y_n^\Omega \mid n \in \mathbb{Z}\right\} = \Sigma(\Omega)$ where $\Omega \subset \mathbb{R}$ is called the **sequence of quasicrystal** $\Sigma(\Omega)$.

Theorem 2.7. *Let $\Omega = [c, d)$ is a base window, then all possible distances between two immediately following points of the sequence of the quasicrystal $\Sigma(\Omega)$, $\left(y_{n+1}^\Omega - y_n^\Omega\right)$ are listed in the table 1.*

$\frac{1}{\beta}$	$\left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right)$	$\frac{\beta-2}{\beta}$	$\left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right)$	$\frac{\beta-1}{\beta}$	$\left(\frac{\beta-1}{\beta}, 1\right)$	1
$4\beta - 1$	$4\beta - 1$					
$3\beta - 1$	$3\beta - 1$	$3\beta - 1$	$3\beta - 1$			
			$2\beta - 1$	$2\beta - 1$	$2\beta - 1$	
	β	β	β	β	β	β
					$\beta - 1$	$\beta - 1$

Table 1: All possible distances between two immediately following points of the sequence of the quasicrystal with a window of the given size.

Remark 6. Please notice that the cases for window sizes $\frac{1}{\beta}$, $\frac{\beta-2}{\beta}$, $\frac{\beta-1}{\beta}$ and 1 each have only two different distances, therefore windows of these sizes are regarded as **singular**. Also distances for the size $\frac{1}{\beta}$ are β multiples of the distances for the size 1.

Definition 2.8. The distances $y_{n+1}^\Omega - y_n^\Omega$ are denoted: $A = 4\beta - 1$, $B = 3\beta - 1$, $C = 2\beta - 1$, $D = \beta$ and $E = \beta - 1$.

Definition 2.9. Function $f^\Omega : \Omega \to \Omega$ for $\Omega = [c, d)$ defined as

$$d - c \in \left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right] : \quad f^\Omega(x) = \begin{cases} x + (\beta)' & x \in [c, d - \frac{1}{\beta}) \\ x + (4\beta - 1)' & x \in [d - \frac{1}{\beta}, c + \frac{\beta-3}{\beta}) \\ x + (3\beta - 1)' & x \in [c + \frac{\beta-3}{\beta}, d) \end{cases}$$

$$d - c \in \left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right] : \quad f^\Omega(x) = \begin{cases} x + (\beta)' & x \in [c, d - \frac{1}{\beta}) \\ x + (3\beta - 1)' & x \in [d - \frac{1}{\beta}, c + \frac{\beta-2}{\beta}) \\ x + (2\beta - 1)' & x \in [c + \frac{\beta-2}{\beta}, d) \end{cases}$$

$$d - c \in \left(\frac{\beta-1}{\beta}, 1\right] : \quad f^\Omega(x) = \begin{cases} x + (\beta)' & x \in [c, d - \frac{1}{\beta}) \\ x + (2\beta - 1)' & x \in [d - \frac{1}{\beta}, c + \frac{\beta-1}{\beta}) \\ x + (\beta - 1)' & x \in [c + \frac{\beta-1}{\beta}, d) \end{cases}$$

is called the **stepping function** of the quasicrystal $\Sigma(\Omega)$.

Remark 7. Stepping function takes $(\cdot)'$ image of a point of the quasicrystal and returns $(\cdot)'$ image of immediately following point.

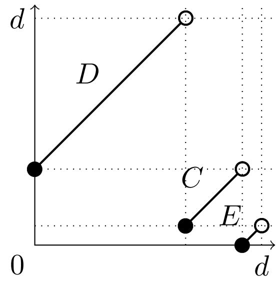

Figure 2: Graph of stepping function for quasicrystal $\Sigma(\Omega)$ where $\Omega = [c, d)$, $c = 0$, $d = 12 - 3\beta$. $C = 2\beta - 1$, $D = \beta$ and $E = \beta - 1$ (as in Definition 2.8).

Stepping function is a valuable tool in theoretical quasicrystal analysis and has direct practical use in quasicrystal generation. Therefore the following theorem lists several key properties of this function.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}$:*

- $f^{\Omega}(\,(y_n^{\Omega})'\,) = (y_{n+1}^{\Omega})' \quad \forall n \in \mathbb{N}$
- $\left(f^{\Omega}\right)^{-1}(\,(y_{n+1}^{\Omega})'\,) = (y_n^{\Omega})' \quad \forall n \in \mathbb{N}$
- f^{Ω} *is piece-wise translation*
- *Discontinuities of f^{Ω} divide the window Ω into intervals. After the preimages of the points of one interval there is the same distance to the next point of the quasicrystal.*

Definition 2.11. Discontinuities of the stepping function of the quasicrystal $\Sigma(\Omega)$, where $\Omega = [c, d)$ in the base form, are denoted as a^{Ω} and b^{Ω}.

$$d - c \in \left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right] : \qquad \begin{aligned} a^{\Omega} &= d - \tfrac{1}{\beta} \\ b^{\Omega} &= c + \tfrac{\beta-3}{\beta} \end{aligned}$$

$$d - c \in \left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right] : \qquad \begin{aligned} a^{\Omega} &= d - \tfrac{1}{\beta} \\ b^{\Omega} &= c + \tfrac{\beta-2}{\beta} \end{aligned}$$

$$d - c \in \left(\frac{\beta-1}{\beta}, 1\right] : \qquad \begin{aligned} a^{\Omega} &= d - \tfrac{1}{\beta} \\ b^{\Omega} &= c + \tfrac{\beta-1}{\beta} \end{aligned}$$

Remark 8. Notation from previous definition will be often used to divide a base window $\Omega = [c, d)$ into three disjunct intervals.

$$\Omega = \left[c, a^{\Omega}\right) \cup \left[a^{\Omega}, b^{\Omega}\right) \cup \left[b^{\Omega}, d\right)$$

For singular cases where $d - c \in \left\{\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}, 1\right\}$, $a^{\Omega} = b^{\Omega}$ and the window is then divided only in to two intervals $\left[c, a^{\Omega}\right) \cup \left[a^{\Omega}, d\right)$.

Definition 2.12. Let $\Omega = [c, d)$. The word $\left(t_n^\Omega\right)_{n\in\mathbb{Z}}$ over the alphabet $\{A, B, C, D, E\}$ is called the **word** of the quasicrystal $\Sigma(\Omega)$.

$$d - c \in \left(\frac{1}{\beta}, \frac{\beta-2}{\beta}\right] : \qquad t_n^\Omega = \begin{cases} D & y_{n+1}^\Omega - y_n^\Omega = \beta \\ A & y_{n+1}^\Omega - y_n^\Omega = 4\beta - 1 \\ B & y_{n+1}^\Omega - y_n^\Omega = 3\beta - 1 \end{cases}$$

$$d - c \in \left(\frac{\beta-2}{\beta}, \frac{\beta-1}{\beta}\right] : \qquad t_n^\Omega = \begin{cases} D & y_{n+1}^\Omega - y_n^\Omega = \beta \\ B & y_{n+1}^\Omega - y_n^\Omega = 3\beta - 1 \\ C & y_{n+1}^\Omega - y_n^\Omega = 2\beta - 1 \end{cases}$$

$$d - c \in \left(\frac{\beta-1}{\beta}, 1\right] : \qquad t_n^\Omega = \begin{cases} D & y_{n+1}^\Omega - y_n^\Omega = \beta \\ C & y_{n+1}^\Omega - y_n^\Omega = 2\beta - 1 \\ E & y_{n+1}^\Omega - y_n^\Omega = \beta - 1 \end{cases}$$

Remark 9. Word of the quasicrystal describes the distribution of the points of the quasicrystal.

Definition 2.13. Function $\mathcal{C}_\ell : \mathbb{N} \to \mathbb{N}$, that assigns to $n \in \mathbb{N}$ number of different sub-words of the length n in the word of the quasicrystal $\left(t_n^\Omega\right)_{n\in\mathbb{Z}}$ where $|\Omega| = \ell$ is called the **complexity** of the quasicrystal.

Definition 2.14. Set $\mathcal{L}_\ell(n)$ containing all different sub-words of the length n in the word of the quasicrystal $\left(t_n^\Omega\right)_{n\in\mathbb{Z}}$ where $|\Omega| = \ell$ is called the **language** of the quasicrystal.

Remark 10. Please note that the complexity and the language of the quasicrystal are defined dependent only on the length of the window.

That concludes the analysis of the one-dimensional quasicrystal for now. Additional findings will be presented later.

Delone set and voronoi tessellation

Section provides the definitions of a delone set, a covering radius and a voronoi tessellation.

Definition 2.15. Let $P \subset \mathbb{R}^n$ and $\exists R > 0, \exists r > 0$:

$$\forall x, y \in P,\, x \neq y : r \leq \|x - y\|$$

$$\forall z \in \mathbb{R}^n \exists x \in P : \|z - x\| \leq R$$

Then P is called **delone** set.
For each delone set P **covering radius** is defined as:

$$R_c = \inf\{R > 0 \,|\, z \in \mathbb{R}^n \exists x \in P : \|z - x\| \leq R\}$$

Definition 2.16. Let $P \subset \mathbb{R}^n$, P is a discrete set and $x \in P$. Then

$$V(x) = \{y \in \mathbb{R}^n \,|\, \forall z \in P, z \neq x :\, \|y - x\| < \|y - z\|\}$$

is called **voronoi tile** (polygon) of x on P.

Remark 11. Example of a delone set with the voronoi tessellation can be seen in the Figure 5.

Theorem 2.17. *Let $P \subset \mathbb{R}^n$ is a delone set and R_c it's covering radius. For any $x \in P$:*

$$N_x = \{z \in P \,|\, z \neq x \wedge \|z - x\| \leq 2R_c\}$$

Then voronoi tile of x on P is

$$V(x) = \bigcap_{z \in \boldsymbol{N}} \{y \in \mathbb{R}^n \,|\, \|y - x\| < \|y - z\|\}$$

3 Two-dimensional quasicrystals

In the following section a two-dimensional quasicrystal is defined and analyzed. Thanks to the Theorem 3.4, analysis of one-dimensional quasicrystals can be in some way applied to the two-dimensional quasicrystals as well.

Definition 3.1. Vectors α_1, α_2, α_3 and the set M denote the following.

$$\alpha_1 = (1,0) \quad \alpha_2 = \left(\frac{2-\beta}{2}, \frac{1}{2}\right) \quad \alpha_3 = \left(\frac{\beta-2}{2}, \frac{1}{2}\right)$$

$$M = \mathbb{Z}[\beta]\,\alpha_1 + \mathbb{Z}[\beta]\,\alpha_2$$

Remark 12. The vectors and the set from previous definition are key to two-dimensional quasicrystal definition. The set M is used as a two-dimensional equivalent to $\mathbb{Z}[\beta]$ from the one-dimensional quasicrystal. Function from following definition is used as a two-dimensional equivalent to $'$.

Definition 3.2. Function $* : M \to M$ is called **star** function:

$$v^* = (a\alpha_1 + b\alpha_2)^* = a'\alpha_1 + b'\alpha_3 \;\forall a, b \in \mathbb{Z}[\beta]$$

Remark 13. Simple consequence of the Theorem 3.2 is that $\alpha_1^* = \alpha_1$ and $\alpha_2^* = \alpha_3$.

Definition 3.3. Let $\Omega \subset \mathbb{R}^2$ be bounded set with nonempty interior. Then **two-dimensional quasicrystal** with the window Ω is defined as:

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\}$$

Remark 14. $\Sigma(\Omega)$ where $\Omega \subset \mathbb{R}^2$ always denotes two-dimensional quasicrystal.

Remark 15. The same properties from the Theorem 2.3 for the one-dimensional quasicrystals apply to the two-dimensional quasicrystals as well.

To analyze the two-dimensional quasicrystals again only windows of a certain shape will be considered. That is sufficient because of the Remark 15. The chosen window shape is a rhombus.

Theorem 3.4. *Let $I = [c, d)$, then for the rhombus $\Omega = I\alpha_1^* + I\alpha_2^*$ and the quasicrystal $\Sigma(\Omega)$:*

$$\Sigma(\Omega) = \Sigma(I)\alpha_1 + \Sigma(I)\alpha_2$$

Remark 16. Note in the previous theorem that while $\Omega \subset \mathbb{R}^2$ and so $\Sigma(\Omega)$ is a two-dimensional quasicrystal, $I \subset \mathbb{R}$ and so $\Sigma(I)$ is a one-dimensional quasicrystal. Illustration of the construction is in the Figure 3. The same Theorem also applies to parallelogram shaped windows.

From the analysis of one-dimensional quasicrystals and the Theorem 3.4 follows that the two-dimensional quasicrystals are delone sets.

To analyze distribution of the points of a two-dimensional quasicrystal, voronoi tessellation is used. The goal is to catalog shapes of all voronoi tiles that appear in a quasicrystal with a rhombus window.

First an algorithm for generation of a finite section of a quasicrystal is presented.

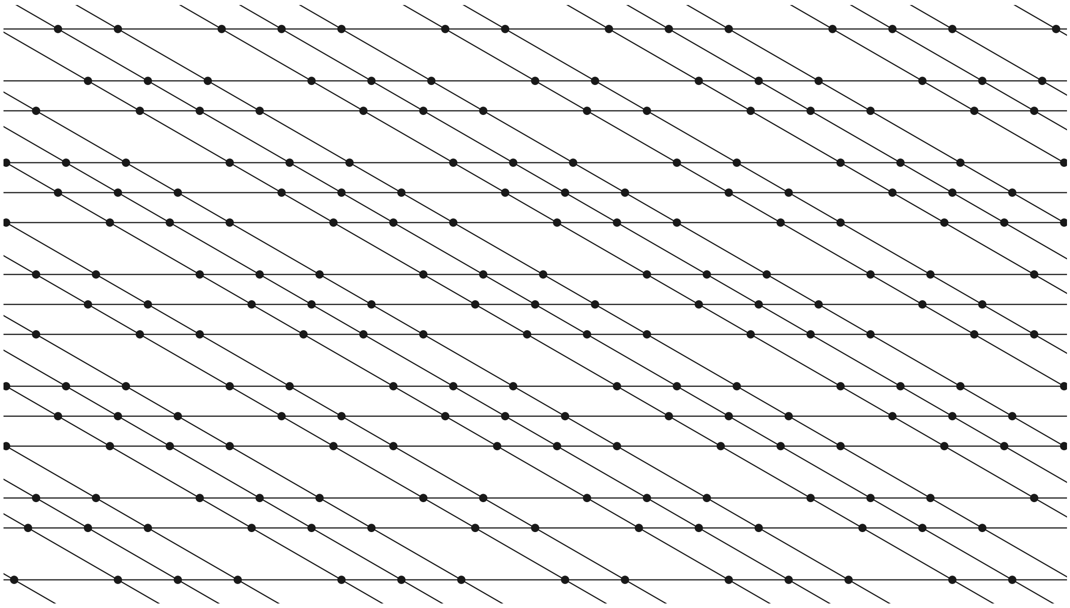

Figure 3: Illustration of the construction of the two-dimensional quasicrystal with a rhombus window from two one-dimensional quasicrystals. Horizontal lines mark copies of $\Sigma(I)\alpha_1$ and the skewed lines mark copies of $\Sigma(I)\alpha_2$.

4 Generation of a finite section of a quasicrystal with a rhombus window

The algorithm is rather simple. It uses the stepping function and the Theorem 3.4.

Algorithm definition The algorithm receives as an input a rhombus window $\Omega = I\alpha_1^* + I\alpha_2^*$ and bounds $x_1, x_2, y_1, y_2 \in \mathbb{Z}[\beta]$. The algorithm returns a subset of the quasicrystal $\Sigma(\Omega)$ bounded by the given bounds.

$$\Sigma(\Omega) \cap ([x_1, x_2] \times [y_1, y_2])$$

First the one-dimensional interval $I = [c, d)$ needs to be scaled and moved in such a way, that it becomes a base window and contains 0.

$$(\exists k \in \mathbb{Z})(\exists \lambda \in \mathbb{Z}[\beta]) : \left(I = \beta^k \tilde{I} + \lambda\right) \wedge \left(|\tilde{I}| \in \left(\frac{1}{\beta}, 1\right]\right) \wedge \left(0 \in \tilde{I}\right)$$

Now the stepping function can be used to iterate from 0 and generate enough points of the quasicrystal $\Sigma(\tilde{I})$ to cover the bounds. However since the bounds are for the quasicrystal $\Sigma(\Omega)$ they need to be transformed to be applicable to the quasicrystal $\Sigma(\tilde{I})$.

$$\begin{aligned} \tilde{x_1} &= x_1 & \tilde{y_1} &= 2y_1 \\ \tilde{x_2} &= x_2 + (\beta - 2)(y_2 - y_1) & \tilde{y_2} &= 2y_2 \end{aligned}$$

The stepping function is then used to acquire two sections of the quasicrystal $\Sigma(\tilde{I})$: $\Sigma(\tilde{I}) \cap [\tilde{x_1}, \tilde{x_2}]$ and $\Sigma(\tilde{I}) \cap [\tilde{y_1}, \tilde{y_2}]$.

Each section needs to be transformed back:

$$\Sigma(I) = \beta^{-k}\Sigma(\tilde{I}) + \lambda'$$

and finaly the finite section of the quasicrystal $\Sigma(\Omega)$ is constructed:

$$\Sigma(\Omega) = \Sigma(I)\alpha_1 + \Sigma(I)\alpha_2$$

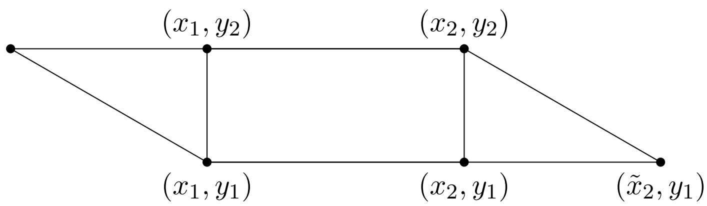

Figure 4: Illustration of how big parallelogram section of the quasicrystal (not a window) is needed to acquire a rectangular one.

Remark 17. Due to the way the two-dimensional quasicrystal is constructed, the result will contain more points then requested (Figure 4). However the excess points can be easily discarded.

The next goal is now to catalog all different voronoi polygons that appear in a quasicrystal for a single window.

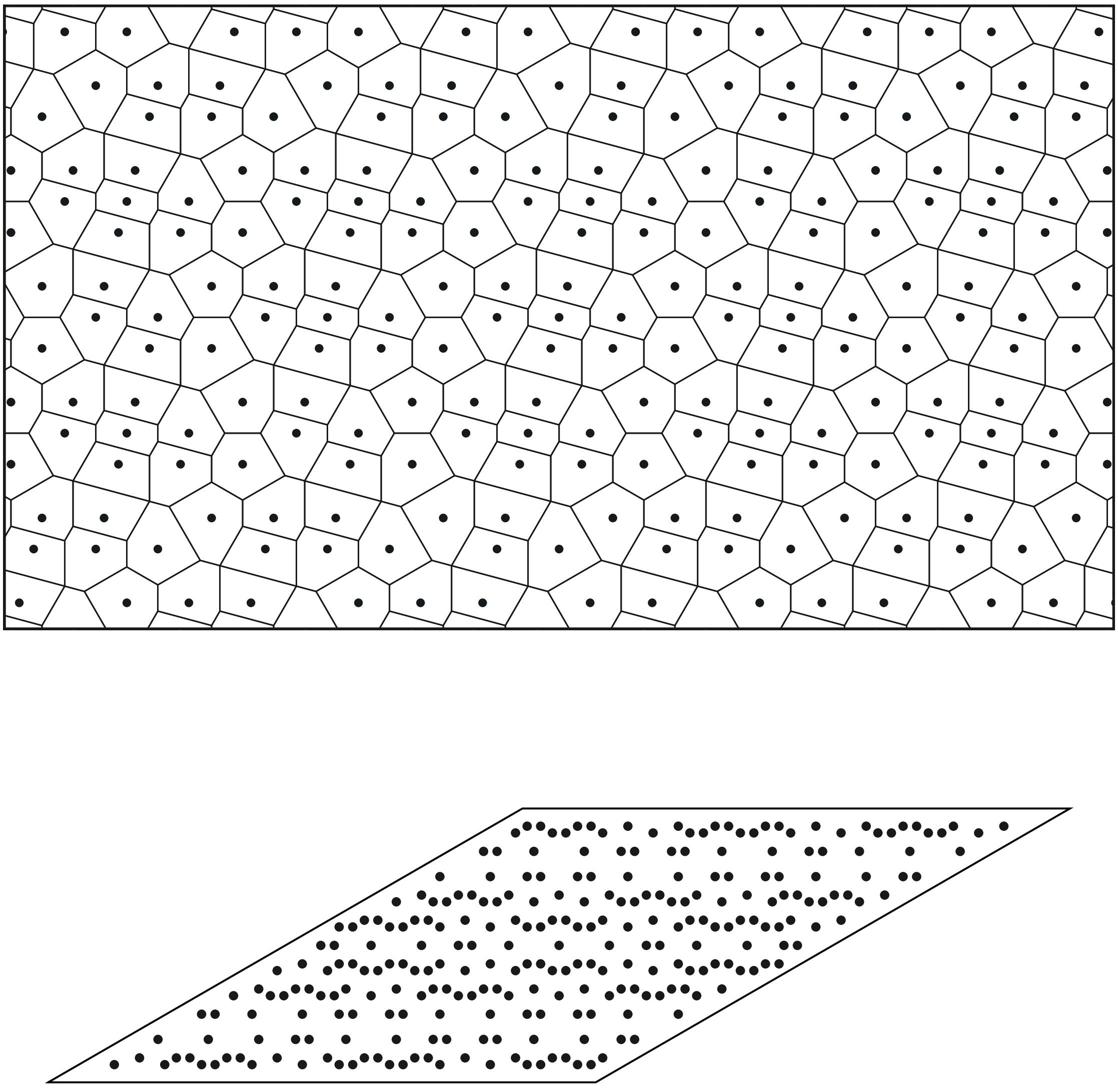

Figure 5: Finite section of the quasicrystal $\Sigma(I\alpha_1^* + I\alpha_2^*)$ where $|I| = \frac{6\beta-22}{7}$ with the voronoi tessellation and the rhombus window $I\alpha_1^*+I\alpha_2^*$ with $*$ images of the points from the finite section.

5 Estimate of the covering radius

To catalog all different tiles that appear in a quasicrystal for a single window, all possible local configurations of the points of the quasicrystal need to be generated.

That is achieved by generating the language of the quasicrystal $\mathcal{L}_n$ of a sufficient length that the finite sections corresponding to the words from the language $\mathcal{L}_n$ cover the disk of the radius $2R_c$ (in more detail in the next section), where R_c is the covering radius of the quasicrystal (Definition 2.15 and Theorem 2.17).

Since the precise value of R_c is difficult to evaluate, an upper bound estimate is used instead. As a reminder, here is the definition of the covering radius R_c, as is in Definition 2.15.

$$R_c = \inf\{R > 0 \,|\, z \in \mathbb{R}^n \exists x \in P : \|z - x\| \leq R\}$$

The estimate is derived from an artificial quasicrystal with only the largest distances between points (largest for the given window). Such quasicrystal has, for given window, certainly larger covering radius than any other. Since all such artificial quasicrystals are different only in scale and translation, the estimate is derived form a "normalized" one (a point in the origin and a unitized distance between points).

The estimate is then evaluated as the radius of a circumscribed circle or the circumradius R of a triangle with vertices $(0,0)$, $(-1,0)$ and $\left(\frac{2-\beta}{2}, \frac{1}{2}\right)$, as in Figure 6.

$$R_c \leq R = \frac{a}{2sin(\alpha)} = \frac{1}{2\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)} = \frac{\sqrt{2}(\sqrt{3}-1)}{2}$$

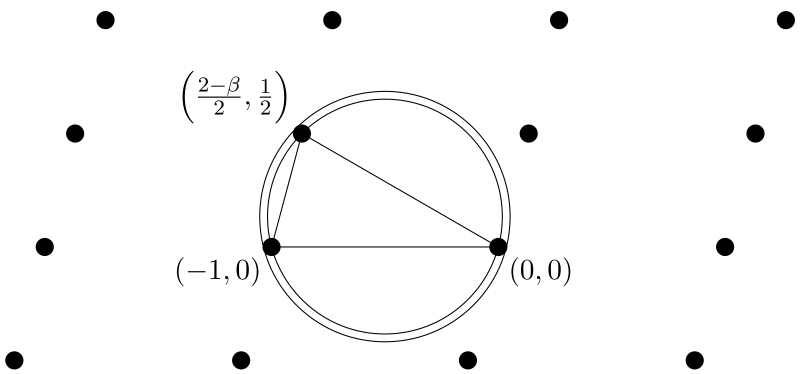

Figure 6: Section of the artificial quasicrystal with the circumcircle and a circle of the estimated radius $\hat{R}_c$.

Because the estimate is used in comparison with coordinates of the points of the quasicrystal, it is advantageous if it is also from $\mathbb{Z}[\beta]$. For that the estimate $1.414 \doteq \sqrt{2} < 32\beta - 118 \doteq 1.426$ is used.

$$\frac{\sqrt{2}(\sqrt{3}-1)}{2} < \frac{(32\beta - 118)(\beta - 3)}{2} = 161 - 43\beta = \hat{R}_c \doteq 0.522$$

Since a unitized quasicrystal was used for the estimate, the value used in computation is the larges distance for a given window times $\hat{R}_c$.

Remark 18. There is an easier way that removes the need for such deriving. Simply estimate the covering radius with the largest distance itself. That is at first sufficient, but computational complexity of quasicrystals with a general window forced us to use all optimizations available.

6 Division of window

Previous section has established that for each point of the quasicrystal, the shape of the associated voronoi polygon is only influenced by the points of the quasicrystal that are closer then $2L \cdot \hat{R}_c$, where L is the largest distance for a given window.

In this section we describe the algorithm to divide one-dimensional window to parts by the same corresponding words. That is vital for two-dimensional quasicrystal analysis.

Theorem 6.1. *Function $(f^{\Omega})^n$ denotes the n-th iteration of the stepping function of the quasicrystal $\Sigma(\Omega)$. Set $D_n = \{z_1 < z_2 < \cdots < z_{m-1}\}$ contains all discontinuities of $(f^{\Omega})^n$, $z_0 = c$ and $z_m = d$.*
Then $(\forall i \in \widehat{m} \cup \{0\})(\forall (y_l^{\Omega})', (y_k^{\Omega})' \in (z_i, z_{i+1}))$ are words $t_l^{\Omega} t_{l+1}^{\Omega} \dots t_{l+n-1}^{\Omega}$ and $t_k^{\Omega} t_{k+1}^{\Omega} \dots t_{k+n-1}^{\Omega}$ the same.

Remark 19. In other words the Theorem 6.1 states that the discontinuities of the n-th iteration of the stepping function divide the window into intervals of images of the points of the quasicrystal after which the same sequence of distances of the length n follow.

Theorem 6.2. *Let $\Omega = [c, c + \ell)$.*
If $\ell \notin \mathbb{Z}[\beta]$ then $\mathcal{C}_\ell(n) = 2n + 1$, $\forall n \in \mathbb{N}$.
If $\ell \in \mathbb{Z}[\beta]$ then $\exists^1 k \in \mathbb{N}$ such that $\left((f^{\Omega})^k (a^{\Omega}) = b^{\Omega}\right)$ or $\left((f^{\Omega})^{k+1} (b^{\Omega}) = a^{\Omega}\right)$ and

$$\mathcal{C}_\ell(n) = \begin{cases} 2n + 1 & \forall n \leq k \\ n + k + 1 & \forall n > k \end{cases}$$

Theorem 6.3. *Let $\Omega = [c, c + \ell)$.*

$$\mathcal{D}_n = \left\{ \ell \,\middle|\, \ell \in \left(\frac{1}{\beta}, 1\right] \wedge \mathcal{C}_\ell(n) < 2n + 1 \right\}$$

Then elements of $\mathcal{D}_n$ divide interval $I := \left(\frac{1}{\beta}, 1\right]$ into finite amount of disjoint subintervals $(I_m)_{m \in \hat{N}}$ such that $\mathcal{L}_{\ell_1}(n) = \mathcal{L}_{\ell_2}(n)$ $\forall \ell_1, \ell_2 \in I_m$, $\forall m \in \hat{N}, \forall n \in \mathbb{N}$.

Remark 20. Please note that $\mathcal{D}_n$ from theorem 6.3 divides base windows into sets of same language whereas D_n from theorem 6.1 divides specific window into intervals by the sequences of points that follow.

The algorithm uses the stepping function of a quasicrystal. As is apparent from Figure 7 and Theorem **??** DOPLNIT, stepping function is piece wise linear and after points of quasicrystal corresponding to one linear segment follows the same distance to the next point of the quasicrystal. Alternatively all the points of the sequence of the quasicrystal y_n whose images y_n' are in a single segment of linearity, have the same corresponding letter in the word of the quasicrystal. That is precisely what the algorithm uses.

First only non-singular windows are considered.

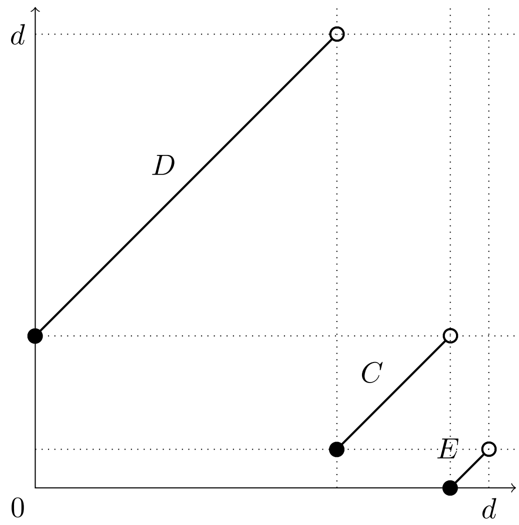

Figure 7: Graph of stepping function for quasicrystal $\Sigma(\Omega)$ where $\Omega = [c, d)$, $c = 0$, $d = 12 - 3\beta$. $C = 2\beta - 1$, $D = \beta$ and $E = \beta - 1$.

Algorithm definition Algorithm receives as an input an interval $\Omega = [c, d)$ representing the window of the quasicrystal and $n \in \mathbb{N}$ representing the desired length of the words.

As an output algorithm provides the division of Ω into disjunt intervals $[\omega_0, \omega_1)$, $[\omega_1, \omega_2), \ldots, [\omega_{m-1}, \omega_m)$ such that $\omega_0 = c$ and $\omega_m = d$.

$$\left(\forall y_j^\Omega, y_k^\Omega \in \left(y_n^\Omega\right)_{n\in\mathbb{Z}}\right)\left(\forall i \in \widehat{m-1}\right) : \left(\left(y_j^\Omega\right)', \left(y_k^\Omega\right)' \in [\omega_i, \omega_{i+1})\right) \Rightarrow \left(\left(t_n^\Omega\right)_j^{j+n} = \left(t_n^\Omega\right)_k^{k+n}\right)$$

The division is acquired by recursion.

For $n = 1$ is the division already known.

$$m = 3, \omega_1 = a^\Omega, \omega_2 = b^\Omega$$

For $n > 1$ is the division found from the division for $n - 1$. Intervals $[\omega_0^{n-1}, \omega_1^{n-1})$, $[\omega_1^{n-1}, \omega_2^{n-1}), \ldots, [\omega_{m-1}^{k-1}, \omega_k^{n-1})$ denote the division for $n - 1$.

For each interval $[\omega_i^{n-1}, \omega_{i+1}^{n-1})$ the stepping function image is evaluated.

$$f^\Omega\left([\omega_i^{n-1}, \omega_{i+1}^{n-1})\right) = [f^\Omega(\omega_i^{n-1}), f^\Omega(\omega_{i+1}^{n-1}))$$

Then the image is divided by the points a^Ω and b^Ω. If one or both of these points are inside the image, it gets divided into two or three disjunct intervals.

After all intervals for $i \in \widehat{k-1}$ are processed, all images or their divisions are sorted and denoted $[\omega_0, \omega_1), [\omega_1, \omega_2), \ldots, [\omega_{m-1}, \omega_m)$.

It may also be desirable to not only acquire the division of the window by the same words, but to also acquire the words themselves. That is done by a simple modification of the described algorithm. Each interval is marked with the corresponding letter A, B, C, D or E at the beginning of the recursion. While dividing the image of the interval by the points a^Ω and/or b^Ω, the mark is appended by an appropriate letter.

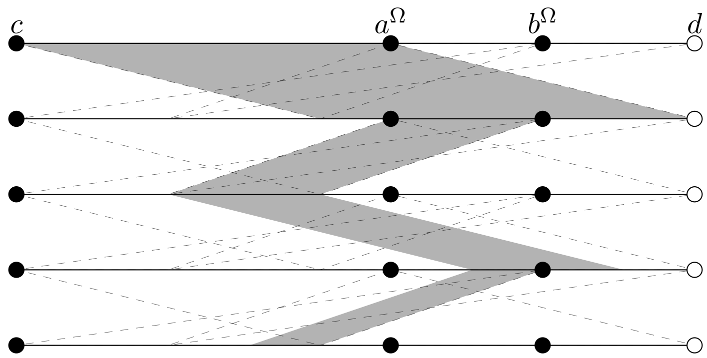

Figure 8: Iteration of the stepping function f^{Ω} where $|\Omega| = \frac{3\beta-10}{2}$. Dashed lines show the exchange of intervals of the stepping function and the gray area shows progression of division for one interval.