Probability Pricing*

Eduardo Dávila[†] Cecilia Parlatore[‡] Ansgar Walther[§]

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Abstract

This paper extends traditional cash-flow pricing to analyze the willingness-to-pay for changes in probabilities, that is, probability pricing. We show that an agent's willingness-to-pay for an arbitrary probability perturbation can be expressed as a cash-flow pricing formula for hypothetical cash flows linked to changes in the survival function. Our cash-flow equivalent formulation provides a way to compute hedging strategies and decompose probability prices into expected payoffs and risk compensation. We extend the analysis to environments with heterogeneous agents and public and private information, characterizing the value of changes in private and public signals. We include four applications that study the valuation of changes in the distribution of aggregate consumption, the efficiency effects of performance noise in principal-agent problems, and the welfare implications of changes in public and private information.

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†Yale University and NBER. Email: eduardo.davila@yale.edu

‡New York University, Stern School of Business, NBER, and CEPR. Email: cparlato@stern.nyu.edu

§Imperial College London and CEPR. Email: a.walther@imperial.ac.uk

1 Introduction

The central question in asset pricing is finding the value of claims to uncertain cash flows. However, it is not evident how traditional cash-flow pricing can be used to value changes in uncertainty itself. This paper fills this gap by studying probability pricing, that is, we study the value of changes in the probabilities of different states.

We introduce probability pricing in a two-date environment in which the terminal consumption of an expected utility agent depends on a random state. We then consider a marginal perturbation to the probability distribution of the state and define the *probability price* associated with it as the agent's willingness-to-pay for the perturbation. While the direct computation of a variational derivative yields an expression for the probability price that values changes in probabilities by utility flows, we explain why this characterization is not useful. Instead, our main result establishes an equivalence between probability prices and the prices derived from a standard cash-flow pricing formula.

The main contribution of this paper lies in showing that an agent's willingness-to-pay for a marginal change in probabilities is equivalent to pricing an asset with hypothetical cash flows that represent the state-by-state value of the probability perturbation. The hypothetical cash flow at a given state is in turn given by the product of i) the change in the normalized survival function at that state, and ii) the sensitivity of consumption to the state.[1] As we explain in detail, the change in the survival function is the relevant object to compute hypothetical cash flows because perturbations that increase (decrease) probability mass to the right of a particular state are effectively adding up (subtracting) the marginal utility at that state, by virtue of the fundamental theorem of calculus.

Establishing a precise equivalence between cash-flow and probability pricing and expressing the probability pricing formula in terms of cash-flow equivalents is useful for several reasons. First, the hypothetical cash flows that we characterize are useful for hedging purposes. That is, an investor who wants to be hedged against changes in the probabilities of different scenarios can use our result to identify the cash flows that a hedging strategy must be designed to replicate. Second, once changes in probabilities are expressed in terms of equivalent cash flows, it is possible to use a standard stochastic decomposition to attribute part of the probability price to changes in expected (equivalent) payoffs and to a risk compensation. Relatedly, probability pricing can also be useful in economies with heterogeneous agents and incomplete markets to compute cross-sectional

[1] The survival function of a probability distribution, $1 - F(s)$, is the complement to the cumulative distribution function, $F(s)$.

welfare decompositions.

By considering particular probability perturbations we can derive additional results. First, we highlight properties of probability prices by considering perturbations to a distribution characterized in terms of a mean parameter and a standard deviation parameter. In particular, we show that the willingness-to-pay for a perturbation that marginally increases consumption unconditionally at all future states (risk-free asset) must be the same as the willingness-to-pay for a perturbation that shifts all probabilities uniformly to the right. We also show that the probability price of a shift in a standard deviation parameter is exclusively driven by a risk compensation. Second, we show how the probability pricing formula relates to the classical literature on preferences over monetary lotteries. Formally, we show that first-order stochastically dominant perturbations feature a positive probability price, while second-order stochastically dominant perturbations feature a negative probability price. Probability pricing can be understood as a way to generalize these classic results about gambles, because it determines whether an individual is willing to pay a positive (or negative) price for any gamble, not only those that satisfy particular dominance properties.

While we initially use probability pricing to study changes in physical probabilities, probability pricing is particularly well-suited to study the private and social values of information since changes in information are effectively changes in probabilities. The second part of our analysis introduces heterogeneous agents who face uncertainty and, in addition, receive private and/or public signals about the state of the economy. We represent the informational environment by the likelihood, i.e., the conditional probability distribution of a signal given physical states, and extend probability pricing to characterize agents' willingness-to-pay for perturbations to the likelihood that represent marginal changes in information.

Our general approach does not require these perturbations to represent either an increase or a decrease in the informativeness of signals, e.g., in the sense of the Blackwell (1953) order. However, as we illustrate in our applications, our formulae are particularly tractable for standard truth-noise signal structures in which a perturbation corresponds to an increase/decrease in noise. We show that the relevant probability price, which can now differ across agents, is again the discounted value of a hypothetical cash flow. In this case, the cash flow depends on i) the local sensitivity of agents' consumption to *signals*, ii) a local measure of the perturbation in probabilities, and iii) the direct effects of the perturbation on consumption which arise because equilibrium prices and allocations change in response to the perturbation. Together, these three components are sufficient to characterize the valuation/welfare impact of changes in an informational environment.

Applications. The probability pricing approach has broad implications and multiple use cases, which we illustrate in four applications. Our first application leverages probability pricing to compute the willingness-to-pay for changes in the distribution of aggregate consumption in a canonical consumption-based asset pricing model. This exercise illustrates how our results can be useful to compute and decompose, for instance, the cost of changes in climate risks, disaster probabilities, or other shocks that alter the distribution of aggregate consumption.

Our second application studies the welfare/willingness-to-pay impact of changes in the precision of the performance noise in a canonical principal-agent problem. Probability pricing allow us to formalize new insights about this well studied constrained Pareto efficient contracting environment. In particular, we show that the efficiency gains of perturbing the precision of the performance noise are solely driven by the probability pricing terms, not by changes in consumption. We also show that increases in the performance precision are always associated with aggregate-efficiency gains, both because the contract adjusts to make production more efficient, but also because aggregate consumption risk is reduced. At last, we show that increases in the performance precision have ambiguous risk-sharing implications. While the contract endogenously adjust to an increase in the performance precision to give more high-powdered incentives to the agent, hence worsening risk-sharing, there may be a countervailing force when the performance sensitivity of contract is sufficiently large since the agent relatively benefits from the smoother consumption, generating risk-sharing gains.

Our final two applications illustrate the role of probability pricing to study the value of information in settings with risk-averse heterogeneous agents and incomplete markets. Our third application explores the welfare implication of changing public information in a version of Hirshleifer (1971)'s model that allows for production. Hirshleifer (1971) shows that more precise public information can make all agents *worse off* in an endowment economy (with incomplete markets) by worsening risk-sharing, a result a priori seen as counterintuitive. Here we use probability pricing to show show that there is a channel through which public information is welfare-improving, even in an endowment economy. We also show how probability pricing can be used to separately study the production and risk-sharing implications of changes in public information.

Our final application shows how probability pricing is useful to understand the welfare impact of changes in the precision of private information in a canonical competitive model of financial trading with dispersed information and noise traders. In the noisy rational expectations equilibrium (REE) of this model, the price acts as a public signal that partially aggregates the private signals received by investors. The repeated use of probability pricing allows us to show that changes in the precision of private information exclusively impact consumption through distributive pecuniary effects. The remaining welfare effects are due to four distinct probability pricing channels, which we carefully identify. In particular, we show that i) efficiency is minimized at intermediate levels of information, and ii) the private value of more precise signals is strictly positive.

Related Literature. At its core, the idea of probability pricing is most related to the classic work characterizing notions of risk and risk aversion that follows Pratt (1964), Rothschild and Stiglitz (1970) and Arrow (1971), among many other contributions. These results have by now made their way to PhD textbooks, see e.g., Ingersoll (1987), Mas-Colell, Whinston, and Green (1995), Gollier (2001), or Campbell (2017). To the best of our knowledge, the probability pricing formula and its associated consumption equivalent characterization of probability changes are novel contributions to this literature. Two reasons may explain this. First, our goal is to compute the willingness-to-pay for general perturbations, rather than trying to derive orders or relations for particular perturbations or utility specifications. Second, following the widely successful cash-flow pricing literature, we focus on marginal perturbations, which allows us to connect our results to cash-flow pricing. That said, as explained in the text, probability pricing can be useful to derive well-known properties of distributions and preferences of risk-averse agents.[2]

The question of how to value changes in probabilities has been asked in specific contexts. For instance, Barro (2009) computes welfare changes from changing disaster probabilities and consumption volatility in a consumption-based asset pricing model. In fact, we model our first application after his results. See also Martin and Pindyck (2015). Our general results focus on i) valuing changes in probabilities generally, making minimal assumptions on preferences, distributions, or perturbations, and ii) establishing a general analogy between cash-flow and probability pricing.

Arguably the closest contribution to our principal-agent application is Rantakari (2008),

[2]Integration by parts, which is key to establish our main result, is widely used in screening models and mechanism design (Mirrlees, 1971; Baron and Myerson, 1982; Segal and Whinston, 2002) — see Bolton and Dewatripont (2005) for a textbook treatment. While there is an evident high-level relation between those papers and ours, our focus and theirs is completely different.

who explores the impact of changes in uncertainty on the optimal strength of incentives. See also Holmström (1979) and Grossman and Hart (1983). In contrast to this work, we leverage probability pricing to determine the individual and social values of changes in uncertainty, not the change in the form of the optimal contract.

The question of whether information improves efficiency pervades the literature on the efficiency of stock markets (e.g., Grossman and Stiglitz, 1980; Hellwig, 1980; Diamond and Verrecchia, 1981; Vives, 2016), strategic trade (e.g., Kyle, 1989; Vives, 2011; Rostek and Weretka, 2012), and public disclosures (e.g., Hirshleifer, 1971; Diamond, 1985; Diamond and Verrecchia, 1991; Morris and Shin, 2002; Angeletos and Pavan, 2007; Goldstein and Yang, 2019). The classical tools of valuation and welfare analysis are, at first glance, of limited use in these problems: information affects not only prices and allocations, but also agents' statistical inferences and the probabilities that they attach to different states of the economy. Perhaps for this reason, and despite several important contributions to this area — including those of Morris and Shin (2002); Angeletos and Pavan (2007); Veldkamp (2009); Gottardi and Rahi (2014); Vives (2016); Kadan and Manela (2019); Pavan, Sundaresan, and Vives (2022) — the welfare analysis of models with information, in particular when agents are risk averse, remains understudied. Our results illustrate how probability pricing is helpful to understand the value of information. We hope that the results in this paper can spur further efforts in this area, both theoretically and empirically, connecting to the results in Ai and Bansal (2018), Kadan and Manela (2019), and Veldkamp (2023), among others. Since our approach is based on consumption equivalents, it is uniquely suited to serve as the foundation of measurement efforts.

2 Probability Pricing

2.1 Environment

We initially consider a single-agent environment with two dates, $t \in \{0, 1\}$. At date 1, there is a continuum of possible states indexed by s with (potentially unbounded) support on $[\underline{s}, \bar{s}]$. We denote the cumulative distribution function (cdf) of the state by $F(s) \in [0, 1]$, and its probability density function (pdf) by $f(s) > 0$.

The agent has standard expected utility preferences, given by

$$V = u(c_0) + \beta \int_{\underline{s}}^{\bar{s}} u(c_1(s)) f(s) \, ds, \tag{1}$$

where c_0 denotes consumption at date 0, $c_1(s)$ denotes consumption at date 1 in state s, and $\beta \in [0,1]$ denotes the agent's time discount factor. We assume throughout that the flow utility function $u(\cdot)$ is twice differentiable, increasing, and concave.

2.2 Cash-Flow Pricing

To fix ideas, it is useful to first consider the standard problem of asset/cash-flow pricing. Suppose the agent is able to purchase q units of an asset that delivers state-contingent cash flows $x(s)$ at a price p_x, with budget constraints given by

$$\begin{aligned} c_0 &= \ \ldots \ - p_x q \\ c_1(s) &= \ \ldots \ + x(s) q, \end{aligned}$$

where the ellipses $(\ldots)$ capture any other elements in the agent's budget constraints. The agent's willingness-to-pay for a marginal unit of the asset satisfies the following well-known asset pricing formula:

$$p_x = \int_{\underline{s}}^{\bar{s}} \omega(s) x(s) ds, \quad \text{where} \quad \omega(s) = \frac{\beta u'(c_1(s))}{u'(c_0)} f(s) \tag{2}$$

defines a state-price and $m(s) = \frac{\beta u'(c_1(s))}{u'(c_0)}$ defines a stochastic discount factor. In this case, asset/cash-flow pricing uncovers the willingness-to-pay at date 0 for changes in consumption at date 1 in different states induced by the asset's cash flows. Asset prices are higher for assets with higher payoffs $x(s)$, in particular in states with high state-prices $\omega(s)$.

2.3 Probability Pricing

We now show that the logic behind cash-flow pricing can be extended to characterize the willingness-to-pay at date 0 for changes in probabilities. We refer to this alternative thought experiment as *probability pricing*.

In order to consider changes in probabilities, we introduce a perturbation parameter θ that determines the cdf (and pdf) over states, assuming that $F(s;\theta)$ and $f(s;\theta)$ are differentiable functions of θ. Our objective is to characterize the agent's willingness-to-pay, or probability price, p_θ, for a marginal change $d\theta$ in this parameter, so that

$$c_0 = \ldots - p_\theta \theta.$$

To isolate the novel effects that arise from perturbations to probabilities, we initially assume that the agent's consumption profile does not depend on θ. We relax this assumption below.

The agent's willingness-to-pay for a marginal change in probabilities satisfies the formula

$$p_\theta = \int_{\underline{s}}^{\bar{s}} \frac{\beta u\left(c_1\left(s\right)\right)}{u'\left(c_0\right)} \frac{df\left(s;\theta\right)}{d\theta} ds, \tag{3}$$

where it must be that $\int_{\underline{s}}^{\bar{s}} \frac{df(s;\theta)}{d\theta} ds = 0$.[3] This expression is intuitive. The agent's willingness-to-pay for a change in probabilities is high, all else equal, if increases in density $\frac{df(s)}{d\theta}$ coincide with high-consumption states, i.e., those with large $u\left(c\left(s\right)\right)$.

However, this characterization has two undesirable properties. First, it is difficult to directly compare p_x and p_θ, because the standard asset pricing formula in Equation (2) expresses p_x in terms of marginal utilities $u'\left(\cdot\right)$, while Equation (3) is written in terms of utility levels $u\left(\cdot\right)$. Second, one may think at first that the ratio $\frac{\beta u(c_1(s))}{u'(c_0)}$ plays an analogous role to the stochastic discount factor or the state-price for cash-flow pricing. However, this ratio is unfortunately not invariant to preference-preserving transformations — in particular, additive transformations of $u\left(\cdot\right)$ such as $u\left(\cdot\right) \to u\left(\cdot\right) + a$ — which makes it unsuitable as a foundation for a theory of valuation of changes in probabilities.

Our main result, Proposition 1, transforms changes in probabilities into consumption equivalents, yielding a probability pricing formula that parallels the traditional cash-flow formula. While the proof of Proposition 1 relies on integration by parts and, hence, makes use of the continuity of the underlying distribution of s, the same logic applies with discrete states, as we show in Appendix D.1.

Proposition 1 (Probability Pricing)**.** *The willingness-to-pay, or probability price, p_θ, for a marginal perturbation in probabilities indexed by θ is given by*

$$p_\theta = \int_{\underline{s}}^{\bar{s}} \omega\left(s\right) x_\theta\left(s\right) ds, \quad \textit{where} \quad \omega\left(s\right) = \frac{\beta u'\left(c_1\left(s\right)\right)}{u'\left(c_0\right)} f\left(s;\theta\right), \tag{4}$$

[3] Formally, this perturbation can be interpreted as taking a variational/Gateaux derivative (Luenberger, 1969). To ensure that the perturbed cdf distribution remains a valid cdf without making parametric assumptions, $F\left(s;\theta\right)$ can always be formulated as

$$F\left(s;\theta\right) = \theta \underline{F}\left(s\right) + \left(1-\theta\right) \overline{F}\left(s\right),$$

where $\underline{F}\left(s\right)$ denotes the cdf of the "initial" distribution, and $\overline{F}\left(s\right)$ denotes the cdf of the "final" distribution, or equivalently the "direction" of the perturbation. In general, the parameter θ can be mapped to a parameter of a particular distribution; see e.g., Section 2.4. See Dávila and Walther (2023) for an application of variational derivatives to leverage regulation with distorted beliefs.

defines a state-price, and where

$$x_\theta(s) = \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds} \tag{5}$$

defines a consumption-equivalent cash flow for state s.

Equation (4) shows that computing an agent's willingness-to-pay for a marginal change in probabilities is equivalent to pricing an asset with hypothetical cash flows in state s given by $x_\theta(s)$. But why are these the appropriate hypothetical cash flows that translate changes in probabilities into consumption equivalent changes?

First, note that gaining the utility flow $u(\cdot)$ at a given state is equivalent to gaining the marginal utility $\frac{du(c_1(s))}{ds} = u'(c_1(s))\frac{dc_1(s)}{ds}$ at all states to the *left* of that state, by virtue of the fundamental theorem of calculus. Notice also that $\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s)}$ is the normalized amount of probability mass that the perturbation shifts from states to the left of s to states to the *right* of s: we refer to this term as the *normalized survival change* since the complement function to the cdf, $1 - F(s)$, is commonly referred to as the "survival function". Therefore, the survival change $\frac{d(1-F(s;\theta))}{d\theta}$ aggregates all the (net) density changes to the *right* of state s, and each of these changes induces a welfare gain valued at $u'(c_1(s))\frac{dc_1(s)}{ds}$. Aggregating these gains over all states yields Equation (4). Figure 1 illustrates this logic.

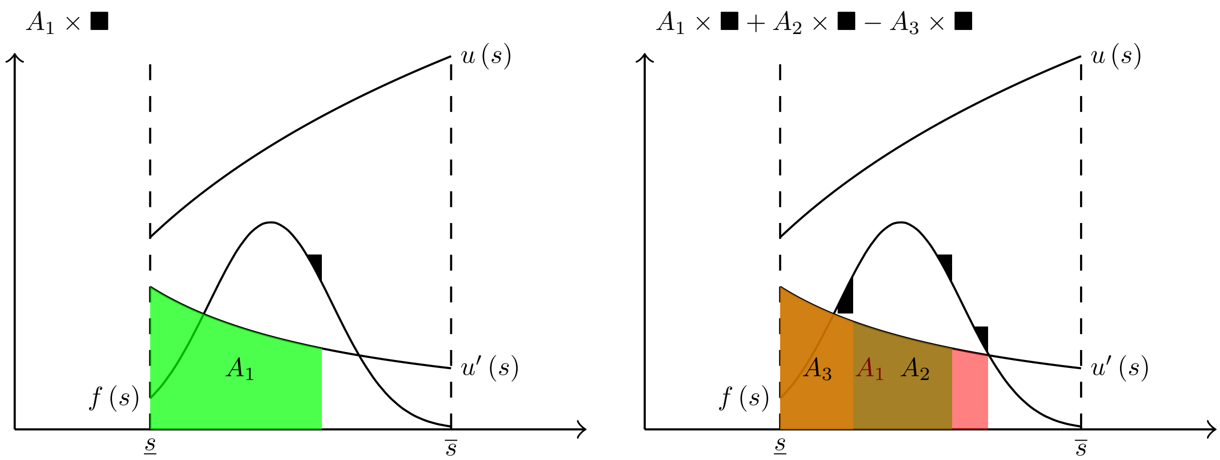
In principle, $\frac{d(1-F(s;\theta))}{d\theta}$ and $\frac{dc_1(s)}{ds}$ can take negative values, so the consumption-equivalent defined in Equation (5) can be negative, as well as p_θ. A negative value of p_θ simply indicates that the perturbation makes the agent worse off, so the willingness-to-pay for it is negative.[4] The following remarks elaborate on why formulating probability pricing in terms of marginal utilities and consumption equivalents, as in Equation (4), rather than in terms of utility flows, as in (3), is desirable.

Remark 1. *(Hedging)* Equation (5) is useful for the purposes of constructing a hedging strategy against changes in probabilities. In particular, an investor who wants to be insured against changes in the probabilities of different scenarios can use Equation (5) to identify the cash flows that a hedging strategy must be designed to replicate. This strategy ensures

[4] Notice that computing an agent's willingness-to-pay for a marginal change in probabilities (probability pricing) is different from computing the change in an agent's willingness-to-pay for an asset given a marginal change in probabilities (comparative statics of cash-flow pricing). The answer to the latter question can be expressed as

$$\frac{dp_x}{d\theta} = \int_{\underline{s}}^{\bar{s}} \omega(s) \frac{\frac{df(s;\theta)}{d\theta}}{f(s;\theta)} x(s)\, ds,$$

which obviously yields a different answer than (4).



(a) Perturbation at a Single State

(b) General Perturbation

Figure 1: Illustrating Probability Pricing

Note: In this figure, $u(c(s)) \equiv u(s)$. The left figure illustrates the change in date-1 utility induced by perturbing the pdf at a particular state. In this case, the utility change is given by $u(\cdot)$, which in turn can be written as $u(0) + A_1$. The right figure illustrates the change in date-1 utility induced by perturbing the pdf at multiple states. The probability price of the general perturbation, p_θ, corresponds to the sum (difference when the pdf decreases) of the areas A_1 to A_3 multiplied by the changes in the density: $p_\theta = A_1 \times \blacksquare + A_2 \times \blacksquare + A_3 \times \blacksquare$, where we exploit the fact that $\int_{\underline{s}}^{\bar{s}} \frac{df(s;\theta)}{d\theta} ds = 0$. But this sum can be equivalent calculated by adding up (integrating) marginal utilities $u'(\cdot)$ multiplied by the changes in densities to the right $\frac{d(1-F(s;\theta))}{d\theta}$ of every state: this is exactly the probability pricing formula in Equation (4).

that the investor's welfare is hedged against changes in probabilities. The central result in Proposition 1 is arguably showing how to construct state-by-state equivalent cash-flows, to in turn be able to leverage readily available used cash-flow pricing insights.

Remark 2. *(Stochastic Decomposition)* A standard stochastic decomposition of Equation (4) allows us to attribute part of the probability price to changes in expected (equivalent) payoffs and to a risk compensation, as follows:

$$p_\theta = \underbrace{\frac{1}{1+r^f} \frac{d\mathbb{E}\left[c_1(s)\right]}{d\theta}}_{\text{Expected Payoff}} + \underbrace{\mathbb{C}ov\left[m(s), \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds}\right]}_{\text{Risk Compensation}}, \tag{6}$$

where $1 + r^f = 1/\mathbb{E}[m(s)]$ denotes the risk-free rate and where, as shown in the Appendix, $\frac{d\mathbb{E}[c_1(s)]}{d\theta} = \mathbb{E}\left[\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds}\right]$. This decomposition is useful to understand whether a perturbation to probabilities is valuable because it changes expected consumption or because the changes in consumption take place in states with different valuations.

Remark 3. *(Cash-Flow and Probability Pricing)* In general, we can allow the agent's consumption profile at date 1, $c_1(s;\theta)$, to be affected by the perturbation parameter θ (same with date-0 consumption). In particular, the mapping $c_1(s;\theta)$ can capture equilibrium or off-equilibrium effects of any sort, either in competitive, strategic, contracting environments, etc. In this case, the agent's welfare gain expressed in date-0 consumption units can be written as

$$\frac{\frac{dV}{d\theta}}{u'(c_0)} = \int_{\underline{s}}^{\bar{s}} \omega(s) \left(\underbrace{\frac{dc_1(s;\theta)}{d\theta}}_{\text{Consumption}} + \underbrace{\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s;\theta)}{ds}}_{\text{Probability}} \right) f(s;\theta)\, ds, \tag{7}$$

where $\omega(s)$ is defined as in (4). Equation (7) is agnostic about the exact mechanism through which consumption mapping $c_1(s;\theta)$ depends on changes in probabilities. In our applications, consumption will vary in response to a change in probabilities conditional on a state because agents' decisions endogenously adjust. This adjustment may be due to changes in optimal contracts (Application 2) or in competitive equilibrium allocations (Applications 3 and 4). Similar forces would also apply to economies with strategic trade (e.g., Kyle, 1989; Vives, 2011; Rostek and Weretka, 2012).

Remark 4. *(Cross-Sectional Welfare Decomposition)* Probability pricing can also be useful in economies with heterogeneous agents and incomplete markets to compute cross-sectional welfare decompositions of the form introduced in Dávila and Schaab (2024). These decompositions require to express the impact of perturbations in consumption equivalents, relying again on (4). We illustrate this use in Applications 2 and 3.

2.4 Probability Pricing for Particular Perturbations

Here we develop several implications of the probability pricing result for particular perturbations.

2.4.1 Mean/Variance Perturbations

First, we highlight properties of probability prices by considering perturbations to a distribution that is characterized in terms of a mean parameter μ and a standard deviation parameter σ. Suppose that the state is defined in affine form, as in

$$s = \mu + \sigma m, \tag{8}$$

where μ and $\sigma \geq 0$ are parameters, and where m is random variable distributed according to a cdf $H(m)$.[5] In this case, the normalized survival change $\frac{\frac{d(1-F(s))}{d\theta}}{f(s)}$ can be expressed in tractable forms:

i) A marginal increase in the mean of s, $d\mu$, induces a normalized survival change given by

$$\frac{\frac{d(1-F(s))}{d\mu}}{f(s)} = 1. \tag{9}$$

ii) A marginal increase in the standard deviation of s, $d\sigma$, induces a normalized survival change given by

$$\frac{\frac{d(1-F(s))}{d\sigma}}{f(s)} = \frac{s-\mu}{\sigma}. \tag{10}$$

Equation (9) shows that the hypothetical cash-flow determining probability prices induced by a marginal increase in μ is simply the consumption sensitivity $\frac{dc(s)}{ds}$, so

$$p_\mu = \int_{\underline{s}}^{\bar{s}} \omega(s) \frac{dc_1(s)}{ds} ds \quad \underset{c_1(s)=s}{\Longrightarrow} \quad p_\mu = \int_{\underline{s}}^{\bar{s}} \omega(s) \, ds.$$

If $c_1(s) = s$, the distribution $F(s)$ is directly defined over consumption (s is a lottery), and then the probability price of a marginal increase in μ is the same as the price of the risk-free asset. Intuitively, the willingness-to-pay for a perturbation that marginally increases consumption unconditionally at all future states (risk-free asset) must be the same as the willingness-to-pay for a perturbation that shifts all probabilities uniformly to the right when $c_1(s) = s$ (marginal increase in μ). Another related implication of probability pricing for a general $c_1(s)$ is that it is always possible to construct a perturbation of probabilities whose price is the risk-free rate by ensuring that $\frac{d(1-F(s))}{f(s)} \frac{dc_1(s)}{ds} = 1$.

Equation (10) shows that the hypothetical cash-flow determining probability prices induced by a marginal increase in σ is given by $\left(\frac{s-\mu}{\sigma}\right) \frac{dc_1(s)}{ds}$, so

$$p_\sigma = \int_{\underline{s}}^{\bar{s}} \omega(s) \left(\frac{s-\mu}{\sigma}\right) \frac{dc_1(s)}{ds} ds.$$

Intuitively, a marginal increase in σ shifts mass to the tails, reducing survival probabilities in states below the mean ($s < \mu$), and increasing them in states above the mean ($s > \mu$). It thus follows from Equation (6) that the probability price of a shift in σ is exclusively driven

[5] If we further assume that $\mathbb{E}[m] = 0$ and $\mathbb{V}ar[m] = 1$, then $\mathbb{E}[s] = \mu$ and $\mathbb{V}\mathrm{ar}[s] = \sigma^2$, but this is not necessary for Equations (9) and (10) to hold.

by a risk compensation.

2.4.2 Mixture Distributions

A second scenario is one in which the state is defined as a mixture of two distributions. Formally, suppose that with probability $1-h$ the state follows a distribution $\overline{F}(s)$, and with probability h the state follows a distribution with cdf $\underline{F}(s)$. Mixture distributions are often used to capture discrete jumps, large shocks, or disasters, as in Application 1.

In this case, a marginal increase in the probability h induces a normalized survival change given by

$$\frac{\frac{d(1-F(s))}{dh}}{f(s)} = \frac{\overline{F}(s) - \underline{F}(s)}{(1-h)\overline{f}(s) + h\underline{f}(s)}. \tag{11}$$

In this case, the difference between cdf's $\overline{F}(s) - \underline{F}(s)$ directly determines the sign of the normalized survival change and of the hypothetical consumption equivalent at s. So if the distribution $\underline{F}(s)$ first-order stochastically dominates $\overline{F}(s)$, that is $\overline{F}(s) > \underline{F}(s)$, then $p_h > 0$ and vice versa. We further elaborate on stochastic dominance next. Moreover, note that p_h in this case is invariant to the level of h.

2.4.3 Stochastic Dominance

Finally, we show how the probability pricing formula relates to the classical literature on preferences over monetary lotteries.[6] Suppose that consumption is $c_1(s) = s$, so that the state s and the distribution $F(s)$ define a lottery over consumption. In this case, we have the following properties:

i) *First-order stochastic dominance:* The probability price for perturbations such that $\frac{dF(s)}{d\theta} \leq 0$ (or equivalently, $\frac{d(1-F(s))}{d\theta} \geq 0$) for all s satisfies $p_\theta \geq 0$.

ii) *Second-order stochastic dominance/mean-preserving spreads:* The probability price for perturbations such that $\frac{d\mathbb{E}[s]}{d\theta} = 0$ and $\int_{\underline{s}}^{s} \frac{dF(t)}{d\theta} dt \geq 0$ for all s satisfies $p_\theta \leq 0$.

The two properties highlighted here confirm that probability prices can be used to derive well-known properties of the preferences of risk-averse agents. The first property shows that an agent is always willing to pay a positive price for a perturbation that implies "good news" in the sense of first-order stochastic dominance. The second property, as in Rothschild and

[6] See, for example, Pratt (1964), Rothschild and Stiglitz (1970) and Arrow (1971). For an overview of the literature, see Mas-Colell, Whinston, and Green (1995, Chapter 6) or Gollier (2001).

Stiglitz (1970), shows that a risk-averse agent is never willing to pay a positive price for a perturbation that implies "more risk" in the sense of a mean-preserving spread. We must emphasize, however, that these properties do not extend to the general probability prices derived in Equation (4). This is because, outside of the special case in which $\frac{dc_1(s)}{ds} = 1$, the sensitivity of consumption to the state $\frac{dc_1(s)}{ds}$ can have non-trivial implications that go beyond the classical literature on lotteries.

In fact, Proposition 1 can be understood as a way to generalize these classic results about gambles. Probability pricing can be used to determine whether an individual is willing to pay a positive (or negative) price for any gamble, not only those that satisfy particular dominance properties.

3 Physical Probabilities

Our objective in the remainder of this paper is to show that probability pricing has broad implications and multiple use cases. In this section, we initially present two scenarios in which we directly vary physical probabilities. This contrasts with our results in Section 4, in which we use probability pricing to study changes in the distribution of signals, that is, to study the value of information.

3.1 Application 1: Consumption-Based Asset Pricing

Our first application leverages probability pricing to compute the willingness-to-pay for changes in the distribution of aggregate consumption in a canonical consumption-based asset pricing model. This exercise illustrates how our results can be useful to compute and decompose, for instance, the cost of changes in climate risks or the risk of other disasters, as well as changes in the distribution of individual consumption.

3.1.1 Environment

Consider a two-date representative-agent, fruit-tree economy with exogenous output.[7] We assume that the representative individual has time-additive utility with isoelastic preferences

[7] To more easily illustrate the results, we consider a two-date economy. The results straightforwardly extend to multi-period economies.

parametrized by γ, so

$$V = u(c_0) + \beta \int_{\underline{s}}^{\bar{s}} u(c_1(s)) f(s) \, ds, \qquad \text{where} \qquad u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Similar to Barro (2009), we assume that date 1 output is distributed as $y_1(s) = e^s y_0$, where s is given by a mixture of normals, which can be used to capture disasters or large shocks — see Section 2.4.2. With probability $1-h$, s is normally distributed with mean $\overline{\mu}$ and standard deviation $\overline{\sigma}$, and with probability h, s is normally distributed with mean $\underline{\mu}$ and standard deviation $\underline{\sigma}$. Since the economy is closed and all output is consumed, consumption equals output at all times. The parameters $\overline{\mu}$ and $\overline{\sigma}$ can be interpreted as defining the distribution of consumption in normal times, while h has the interpretation of the discrete likelihood of a disaster materializing, in which case the distribution of consumption is defined by $\underline{\mu}$ and $\underline{\sigma}$.

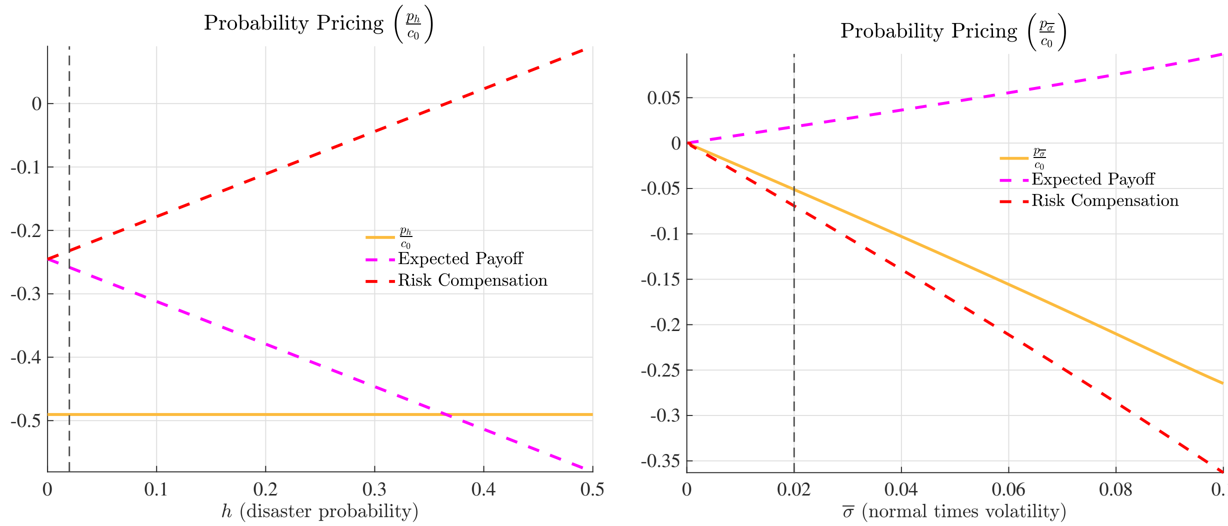
In this model, the normalized survival change is given by (11), while the assumed specification of uncertainty implies that

$$\frac{dc_1(s)}{ds} = c_1(s).$$

3.1.2 Value of Changes in Probabilities

We now compute the willingness-to-pay for different changes in the underlying distribution of consumption. Interpreting a date in the model as a year, we use a rate of time preference $\beta = 0.95$ and a risk aversion coefficient of $\gamma = 4$, again consistent with Barro (2009). We set $h = 0.02$ to capture a 2% yearly disaster probability, with a distribution of consumption growth in normal times given by $\overline{\mu} = 0.025$ and $\overline{\sigma} = 0.02$. If a disaster takes place, consumption falls on average by roughly 30%, with $\overline{\mu} = -0.3$ and $\overline{\sigma} = 0.02$. By normalizing $y_0 = 1$, we can interpret all values as relative to the level of initial consumption.

The left panel in Figure 2 shows the willingness-to-pay for a marginal change in the yearly probability of disaster h, p_h, for different values of h. As noted above in Section 2.4.2, the value of p_h is invariant to h for mixture distributions. Interestingly, this constant value masks two different forces. First, since $\frac{d\mathbb{E}[c_1(s)]}{dh}$ is also invariant to h, it follows from Equation (6) that the expected payoff component must increase in magnitude as h grows, since the interest rate decreases through a standard precautionary savings effect. Hence, it has to be the case that the risk-compensation component of the perturbation — defined in (6) — increases with h. This decomposition provides a practical illustration of how the



(a) Varying Disaster Probability h

(b) Varying Volatility $\overline{\sigma}$

Figure 2: Probability Pricing (Application 1)

Note: This figure shows the willingness-to-pay (relative to initial consumption) for changes in the disaster probability h for different values of h (left panel), p_h/c_0, and for changes in the volatility of consumption in normal times $\overline{\sigma}$ for different values of $\overline{\sigma}$ (right panel) $p_{\overline{\sigma}}/c_0$. Preference parameters are $\beta = 0.95$ and $\gamma = 4$. The baseline distribution is defined by $\underline{\mu} = 0.03$, $\underline{\sigma} = 0.02$, $\overline{\mu} = -0.3$, $\overline{\sigma} = 0.02$ and $h = 0.02$, indicated by dashed vertical black lines in both figures.

probability pricing formula can be useful to decompose a particular willingness-to-pay for a perturbation.

The right panel in Figure 2 shows the willingness-to-pay for a marginal change in the volatility of consumption in normal times $\overline{\sigma}$, $p_{\overline{\sigma}}$, for different values of $\overline{\sigma}$. Since $y_1(s)$ is log-normally distributed, an increase in $\overline{\sigma}$ increases expected aggregate consumption through a Jensen's inequality effect, so $\frac{d\mathbb{E}[c_1(s)]}{d\overline{\sigma}} > 0$. Therefore, the negative willingness-to-pay for a marginal increase in $\overline{\sigma}$ combines a welfare gain due to a higher expected payoff and a welfare loss due to the fact that the change in consumption takes place in states with different valuations.

In terms of magnitudes, an increase in the probability of disaster by one percentage point ($\Delta h = 0.01$) is associated with consumption loss of roughly half that amount, since $\frac{p_h}{c_0} \approx -0.5$. An increase in the volatility $\overline{\sigma}$ of similar magnitude around the baseline calibration is associated with a consumption equivalent loss an order of magnitude smaller, since $\frac{p_{\overline{\sigma}}}{c_0} \approx -0.05$ at $\overline{\sigma} = 0.02$.

3.2 Application 2: Principal-Agent Problem

This second application studies the welfare/willingness-to-pay impact of changes in output uncertainty in a canonical principal-agent problem — as in, e.g., Bolton and Dewatripont (2005, Chapter 4), whose notation we follow whenever possible. Probability pricing will allow us to formalize new insights about this constrained Pareto efficient contracting environment.

3.2.1 Environment

We consider an environment in which a principal, indexed by $i = B$ (boss), contracts with an agent, indexed by $i = A$. The principal is risk-neutral, with preferences given by

$$V^B = \int c^B(s) f(s) \, ds,$$

while the agent is risk-averse, with preferences given by

$$V^A = \int u\left(c^A(s)\right) f(s) \, ds,$$

where $c^i(s)$ denotes consumption of individual $i \in \{A, B\}$ in state s. We assume that the agent has constant absolute risk aversion preferences, with $u(c) = -e^{-\eta c}$, where η is the coefficient of absolute risk aversion. The agent makes a costly effort decision e, which generates a random output/performance $y(s) = e + s$, where s is normally distributed with $s \sim \mathcal{N}\left(0, \frac{1}{\tau}\right)$ and where we denote the precision of output uncertainty by $\tau = \frac{1}{\sigma^2}$. The agent receives a compensation $w(s)$, which is linear in output, so the consumption of the agent in state s is given by

$$c^A(s) = w(s) - \psi(e), \quad \text{where} \quad w(s) = t + \alpha q(s),$$

where t denotes an uncontingent transfer and where we refer to α as the contract sensitivity. We assume that the cost function for effort is $\psi(e) = \frac{\kappa}{2} e^2$. The consumption of the principal is thus given by

$$c^B(s) = y(s) - w(s).$$

In this economy, aggregate consumption at state s is simply given by $e + s - \psi(e)$, so the first-best level of effort is $e = \frac{1}{\kappa}$.

Optimal Contract. We provide a step-by-step derivation of all results in the Appendix. The optimal contract features a sensitivity to output α, which in turn induces an effort decision e, given by

$$\alpha = \frac{1}{1+\frac{\eta\kappa}{\tau}} = \frac{\tau}{\tau+\eta\kappa}, \quad \text{and} \quad e = \frac{\alpha}{\kappa}.$$

As the performance noise vanishes ($\tau \to \infty$), the contract sensitivity becomes maximal $\alpha \to 1$, and effort approaches its first-best level $e \to \frac{1}{\kappa}$. Because we consider linear contracts, the solution to the principal-agent problem when $\tau \to \infty$ feature production efficiency but is not the first-best solution, which would require the principal fully insuring the agent.

3.2.2 Value of Changes in Performance Precision

Our goal is to compute the welfare/willingness-to-pay impact of changes in the precision of output uncertainty, parametrized by τ. Formally, individual i's welfare gains induced by a marginal change in the precision of output uncertainty τ, expressed in units of date-1 uncontingent consumption, are given by the following augmented probability pricing formula:

$$\frac{dV^{i|\lambda}}{d\tau} = \frac{\frac{dV^i}{d\tau}}{\lambda^i} = \int \omega^i(s) \left(\frac{dc^i(s)}{d\tau} + \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} \frac{dc^i(s)}{ds} \right) ds, \tag{12}$$

where $\lambda^i = \int \frac{\partial u^i\left(c^i(s)\right)}{\partial c^i(s)} f(s)\, ds$ and where individual i's (shadow) state prices are given by

$$\omega^i(s) = \frac{\frac{\partial u^i\left(c^i(s)\right)}{\partial c^i(s)} f(s)}{\int \frac{\partial u^i(c^i(s))}{\partial c^i(s)} f(s)\, ds}.$$

The state price $\omega^i(s)$ defines a marginal rate of substitution for individual i between consumption at state s and uncontingent consumption.[8] If both individuals could perfectly insure each other, $\omega^i(s)$ would be equal across individuals, but in this economy full insurance is not possible.

Similar to (7), Equation (12) shows that individual welfare gains consist of a consumption and probability components. In the Appendix, we analytically characterize all of the components of Equation (12).

[8] Using the language of Dávila and Schaab (2024), we have chosen date-1 uncontingent consumption as the lifetime welfare numeraire. Other numeraire choices to express welfare gains — such as date-0 consumption, as in Section 2 — would require to define a different λ^i, but would yield similar insights.

Cross-Sectional Efficiency Decomposition. Because probability pricing allows us to translate changes in probabilities into consumption equivalents, we are able to decompose the sources of efficiency gains in this economy. Formally, we abstract from redistributional considerations and focus on characterizing (Kaldor-Hicks) efficiency gains, given by

$$\Xi^{E}=\sum_{i}\frac{dV^{i|\lambda}}{d\tau}=\underbrace{\sum_{i}\int\omega^{i}\left(s\right)\frac{dc^{i}\left(s\right)}{d\tau}ds}_{\Xi_{c}^{E}\text{ (consumption)}}+\underbrace{\sum_{i}\int\omega^{i}\left(s\right)\frac{\frac{d\left(1-F\left(s\right)\right)}{d\tau}}{f\left(s\right)}\frac{dc^{i}\left(s\right)}{ds}ds}_{\Xi_{s}^{E}\text{ (probability)}},\tag{13}$$

where Ξ_c^E and Ξ_s^E denote the sum of individual welfare gains due to changes in consumption and probabilities, respectively. As explained above, the consumption term has the interpretation of changes in consumption given a state is realized, while the probability term has the interpretation of changes in the probabilities of different states being realized, for given consumption allocations at each state. Moreover, we can implement Dávila and Schaab (2024)'s welfare decomposition to further decompose Ξ_c^E and Ξ_s^E into i) aggregate-efficiency and ii) risk-sharing components. Aggregate-efficiency gains arise because of changes in the social value of *aggregate* consumption (or consumption-equivalents), while risk-sharing gains arise because consumption (or consumption-equivalents) is *reshuffled* towards individuals with higher marginal valuations, $\omega^i(s)$. Formally, we can express Ξ_c^E and Ξ_s^E as

$$\Xi_{c}^{E}=\Xi_{c}^{AE}+\Xi_{c}^{RS}\quad\text{and}\quad\Xi_{s}^{E}=\Xi_{s}^{AE}+\Xi_{s}^{RS},\tag{14}$$

where we provide explicit definitions and analytical characterizations of each of the components in the Appendix.[9] Our analysis yields three main takeaways.

First, we show that the efficiency gains of perturbing τ are solely driven by the probability terms. Formally, we show that

$$\Xi_{c}^{E}=\Xi_{c}^{AE}+\Xi_{c}^{RS}=0.$$

This result may seem intuitive, since the optimal contract is precisely structured to tradeoff effort/production decisions and risk-sharing considerations that operate via changes in consumption. This result is due to the fact that we are perturbing probabilities, but other perturbations could feature Ξ_c^E since this economy is constrained efficient but not Pareto efficient. Figure (3) illustrates this result. An increase in τ optimally increases the

[9]While the split in (13) does not require the probability pricing result, it is necessary for both stochastic and cross-sectional decompositions.

performance sensitivity of the contract, which induces the agent to supply more effort while making his consumption more volatile. The additional effort induces an aggregate-efficiency gain (dashed line), in turn fully compensated by a risk-sharing loss (dotted line).[10]

Second, we show that increases in the performance precision τ are always associated with aggregate-efficiency gains. This occurs for two reasons. First, since the agent's effort increases with τ because of the increase in the contract's performance sensitivity, aggregate consumption increases, since the agent was working too little to begin with. Formally,

$$\Xi_c^{AE} = (1 - \psi'(e)) \frac{de}{d\tau} > 0. \tag{15}$$

And even in the absence of an endogenous consumption response, increasing τ reduces aggregate consumption risk, which generates an aggregate efficiency gain similar to the one in Application 1, where we reduced the volatility of aggregate consumption. Formally, this smoothing of aggregate consumption implies that

$$\Xi_s^{AE} > 0.$$

At last, we show that increases in the performance precision τ have ambiguous risk-sharing implications. We first show that the risk-sharing component due to consumption is always strictly negative. Formally,

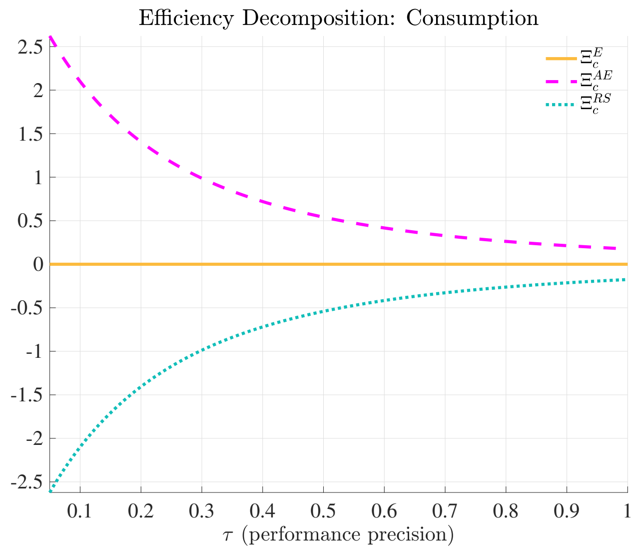
$$\Xi_c^{RS} < 0.$$

Consistent with our discussion above, this is a reflection of the fact that the contract adjusts optimally, trading off this risk-sharing loss with the aggregate-efficiency gain in (15). We then show, perhaps surprisingly, that the risk-sharing component due to information can take positive and negative values. Formally, we show that

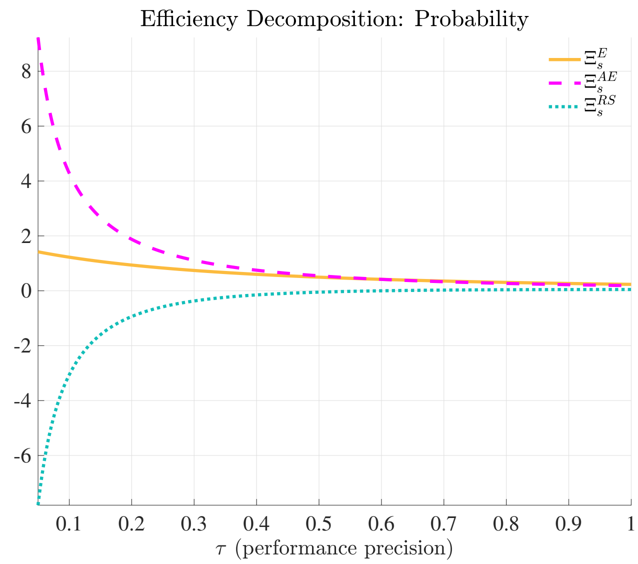
$$\Xi_s^{RS} = \begin{cases} < 0 & \text{if } \alpha < 0.5 \\ \geq 0 & \text{if } \alpha \geq 0.5. \end{cases}$$

For low values of the performance sensitivity α, smoother aggregate consumption due to an increase in τ disproportionally benefits the principal since she receives the bulk $(1 - \alpha)$ of

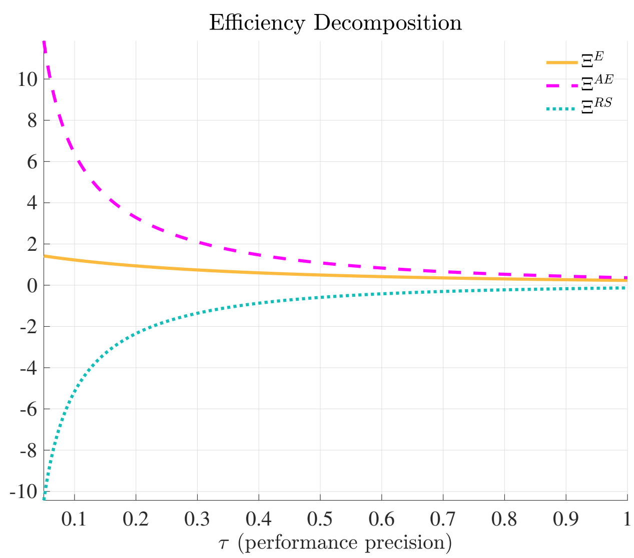
[10]As Hart and Holmström (1987) put it: *"The agency problem is not an inference problem in a strict statistical sense; conceptually, the principal is not inferring anything about the agent's action from the output because he already knows what action is being implemented. Yet, the optimal sharing rule reflects precisely the pricing of inference."*



(a) Consumption Decomposition: $\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS}$



(b) Probability Decomposition: $\Xi_s^E = \Xi_s^{AE} + \Xi_s^{RS}$



(c) Efficiency Decomposition: $\Xi^E = \Xi^{AE} + \Xi^{RS}$

Figure 3: Varying Performance Precision (Application 2)

Note: This figure shows the efficiency/willingness-to-pay implications induced by changing the precision of output uncertainty. The top left figure illustrates how efficiency gains are solely driven by the probability terms. It also shows that the optimal contract is such that aggregate-efficiency and risk-sharing gains/losses exactly compensate each other. The top right figure illustrates that increases in performance precision generate aggregate efficiency gains, but risk-sharing gains or losses. The bottom figure shows that increasing output precision increases efficiency, always through aggregate-efficiency, while risk-sharing can take any sign. In this figure, the parameters are $\eta = 1.2$, $\kappa = 0.5$, and $\bar{V} = -0.25$.

the change. But the agent values smoother consumption less in relative terms. Instead, for high values of the performance sensitivity α, the agent relatively benefits from the smoother consumption, generating risk-sharing gains.

4 Information

So far, we have explored how probability pricing is useful to study changes in physical probabilities. However, probability pricing is a particularly useful approach to characterize the private and social values of information. We now consider a general economy with heterogeneous individuals who experience perturbations to the information environment. This environment is sufficiently general to nest our two new applications: Application 3 considers changes in public information, while Application 4 considers changes in private information.

4.1 Changes in Information are Changes in Probabilities

General Environment. We consider an economy populated by a finite number of types of individuals $i \in \{1, \ldots, I\} \equiv \mathcal{I}$. There is a unit measure of agents of each type. The aggregate state of the economy is denoted by $z \in Z \subset \mathbb{R}^n$, where n is the number of state variables, and drawn from an absolutely continuous distribution with density $\pi(z)$. Each individual experiences an idiosyncratic state $\varepsilon \in E \subset \mathbb{R}^m$, where m is the number of idiosyncratic state variables, which are drawn from an absolutely continuous distribution with density $\pi^i(\varepsilon|z)$, and are independent across agents within each type.[11] We write $s = (z, \varepsilon) \in \mathbb{R}^{n+m}$ for the overall vector of states faced by an agent, whose density is defined as

$$\pi^i(s) \equiv \pi(z)\, \pi^i(\varepsilon|z)\,.$$

All agents derive utility from consumption, with preferences given by

$$V^i = \int u_i\left(c^i(s)\right) \pi^i(s)\, ds,$$

where $c^i(s) = c^i(z, \varepsilon)$ denotes the consumption of an agent of type i in state $s = (z, \varepsilon)$. All types' utility functions $u_i(\cdot)$ are twice differentiable, strictly increasing, and strictly concave.

[11] We adopt a law of large numbers convention throughout: Conditional on aggregate state z, $\pi^i(\varepsilon|z)$ also describes the *measure* of agents of type i who experience a given shock ε.

When convenient, we partition the vector of state variables using the notation $s = (\xi, s_{-\xi})$, where $\xi \in \left[\underline{\xi}, \bar{\xi}\right]$ is a single/scalar state variable, and $s_{-\xi} \in \mathbb{R}^{n+m-1}$ denotes the vector of all states except ξ. We accordingly write the distribution of state variables as $\pi^i(s) = \pi^i(s_{-\xi}) \times f^i(\xi|s_{-\xi})$, where $f^i(\xi|s_{-\xi})$ denotes the conditional distribution (or likelihood) of ξ given all other states. We denote the associated cumulative distribution by $F^i(\xi|s_{-\xi})$. One interpretation of this partition is that ξ is an informative signal of physical state variables that is observed by agents. We then interpret the likelihood $f^i(\xi|s_{-\xi})$ as capturing agents' information environment.

Our notation captures both public and private information. First, the signal ξ can represent *public information.* In this case, ξ is an element of the *aggregate* state vector z, whose realization is common to all agents in the economy. Another implication of public information is that the likelihood $f(\xi|s_{-\xi})$ does not depend on the agent's type i, nor on any idiosyncratic element of $s_{-\xi}$. Second, the signal ξ can represent *private information.* In this case, ξ is an idiosyncratic state variable, whose realization is specific to each individual agent. Notice that this case allows different types of agents to have different conditional distributions of informative signals. For instance, one type i of agent may be uninformed, while another type $j \neq i$ is informed. [12]

Note that our analysis can be applied to settings in which there is more than one signal, for instance, economies with both private and public information. In terms of our notation above, we have defined ξ as a scalar, so that it represents a single informative signal. We adopt this convention because the welfare effects of changes in the information environment are clearest when one perturbs the conditional distribution of one signal at a time. However, since the state vector $s = (z, \varepsilon)$ is vector-valued, it is always possible to interpret elements of z or ε as additional public or private informative signals, whose conditional distribution is being held constant in the perturbations we analyze.

Probability Pricing and The Value of Information. We now analyze an agent's willingness-to-pay for a marginal change in the likelihood $f^i(\xi|s_{-\xi})$, which affects the informational environment that agents face in our model. We follow a parallel approach to our analysis of probability pricing in Section 2. We introduce a perturbation parameter

[12] Some additional notation allows us to make the distinction more explicit when needed. We can define the residual state as $s_{-\xi} = (z_{-\xi}, \varepsilon_{-\xi})$, where $z_{-\xi}$ and $\varepsilon_{-\xi}$, respectively, denote all aggregate and idiosyncratic states that are not equal to ξ, and write the distribution of state variables as $\pi^i(s) = \pi(z_{-\xi})\, \pi^i(\varepsilon_{-\xi}|z_{-\xi})\, f^i(\xi|z_{-\xi}, \varepsilon_{-\xi})$. For example, in the case where ξ is public information, and is therefore only contained in the aggregate state vector, we have $z = (z_{-\xi}, \xi)$, and $\varepsilon = \varepsilon_{-\xi}$, while in the case of private information, where ξ is an element of the idiosyncratic state vector, we have $z = z_{-\xi}$ and $\varepsilon = (\xi, \varepsilon_{-\xi})$.

θ, assume that $f^i(\xi|z)$ is a differentiable function of θ, and study the following normalized welfare gain of type i:

$$\frac{1}{\lambda^i}\frac{dV^i}{d\theta}, \tag{16}$$

where λ^i denotes the marginal value of an arbitrary numeraire.[13] This normalized gain is the probability price (augmented with the change in consumption), defined in analogy to Section 2, associated with a marginal change in information.

We also allow the agent's consumption profile $c^i(s)$ to be affected by the perturbation parameter θ. Thus, we can capture the equilibrium effects of a change in the information environment. However, we continue to assume that the other primitives of the model – in particular, agents' preferences and the marginal distribution $\pi^i(s_{-\xi})$ of state variables – are held constant and do not depend on θ. Differentiating an individual's expected utility in Equation (1), we obtain a two-part decomposition of normalized welfare gains:

$$\frac{1}{\lambda^i}\frac{dV^i}{d\theta} = \underbrace{\int \frac{u_i'(c^i(s))}{\lambda^i}\frac{dc^i(s)}{d\theta}\pi^i(s)\,ds}_{\text{consumption}} + \underbrace{\int \frac{u_i(c^i(s))}{\lambda^i}\frac{d\pi^i(s)}{d\theta}ds}_{\text{probabilities/likelihood}}. \tag{17}$$

The first term in Equation (17) measures the change in expected utility that is caused by changes in the agent's *consumption* conditional on a state taking place. The second term expresses the change in expected utility that is due to changes in the *probability* of different constellations of signals and states. This term is the exact equivalent to Equation (3) in the context of probability pricing. Once again, the agent values changes in probabilities that shift mass towards states in which she enjoys higher consumption and utility. In the context of changes to information, the changes in probabilities are driven by changes in the likelihood $\frac{df^i(\xi|z)}{d\theta}$ of the signal. Indeed, since we have assumed that the marginal distribution of other states $s_{-\xi}$ is constant, the relevant change in probabilities is

$$\frac{d\pi^i(s)}{d\theta} = \pi^i(s_{-\xi})\frac{df^i(\xi|s_{-\xi})}{d\theta}.$$

We can now derive the counterpart of Equation (5) in which probability pricing can be used to quantify the value of information.

[13] As noted in Section 2, the choice of this numeraire does not materially influence our results as long as it expresses different agents' utility in common units and satisfies $\lambda^i > 0$ for all i. For instance, if the numeraire is an asset that pays one dollar unconditionally, then we have $\lambda^i = \int u_i'\left(c^i(s)\right)\pi^i(s)\,ds$. In Section 2, we used $\lambda^i = u'(c_0)$.

Proposition 2 (Probability Pricing: Information)**.** *The normalized effect of a marginal change in information on the utility of agent type i is given by:*

$$\frac{1}{\lambda^i}\frac{dV^i}{d\theta} = \int \omega^i(s)\left(\underbrace{\frac{dc^i(s)}{d\theta}}_{\text{consumption}} + \underbrace{\frac{\frac{d\left(1-F^i\left(\xi|s_{-\xi}\right)\right)}{d\theta}}{f^i\left(\xi|s_{-\xi}\right)}\frac{\partial c^i(s)}{\partial \xi}}_{\text{likelihood}}\right)ds, \tag{18}$$

where $\omega^i(s) = \frac{u_i'\left(c^i(s)\right)}{\lambda^i}\pi^i(s)$ *is a type-specific state-price, and where* $\frac{d\left(1-F^i\left(\xi|s_{-\xi}\right)\right)}{d\theta}$ *denotes the perturbation to the cumulative likelihood of signal* ξ*, conditional on the vector of other state variables* $s_{-\xi}$*.*

The willingness-to-pay of a marginal change in information for an individual consists of the weighted sum — with weights given by individual state-prices $\omega^i(s)$ — of two elements. On one hand, the agent benefits in a given state when her *consumption* in this state is adjusted upwards after the perturbation, with $\frac{dc^i(s)}{d\theta} > 0$. On the other hand, the agent benefits if the perturbation shifts *likelihood* towards signal realizations ξ for which her consumption is high. The latter effect is measured by $\frac{\frac{d\left(1-F^i\left(\xi|s_{-\xi}\right)\right)}{d\theta}}{f^i\left(\xi|s_{-\xi}\right)}\frac{dc^i(s)}{d\xi}$, which has an analogous interpretation to Equation (5) in the context of probability pricing. Equation (18) highlights that the sensitivity of the consumption profile $c^i(\cdot)$ to changes in θ and the signal ξ are critical to understand the individual and social welfare implications of changes in information, as we illustrate next in our applications.

4.2 Application 3: Public Information

This application explores the welfare implication of changing public information in a version of Hirshleifer (1971)'s model that allows for production. Hirshleifer (1971) shows that better (more precise) public information can make all agents *worse off* in an endowment economy (with incomplete markets) by generating risk-sharing losses, a result a priori seen as counterintuitive. Here we use probability pricing to show that there is a channel through which more precise public information generates efficiency gains, even in an endowment economy. We also show how probability pricing can be used to separately study and quantify the production and risk-sharing implications of changes in public information.

4.2.1 Environment

We consider an economy with three dates $t = \{0, 1, 2\}$ and two types of individuals, indexed by $i = \{A, B\}$. Individuals are ex-ante identical at date 0 and exclusively consume at date 2. At date 2, there is a continuum of states, indexed by $s \in [\underline{s}, \overline{s}]$. At date 1, there is a public signal over the date 2 state, denoted by $\xi \in \left[\underline{\xi}, \overline{\xi}\right]$. Hence, the pair (s, ξ) defines a history at date 2.

Individual i expected utility preferences can be formulated recursively as follows:

$$V_0^i = \int V_1^i(\xi) f(\xi) d\xi \quad \text{where} \quad V_1^i(\xi) = \int u\left(c_2^i(s, \xi)\right) f(s|\xi) ds,$$

where $c_2^i(s, \xi)$ denotes the consumption of individual i at date 2 when the date-2 state is s and the signal realized at date-1 is ξ.[14] We assume that individuals have no access to financial markets or contracting opportunities at date 0. At date 1, individuals have access to complete markets against the realization of the state s, so individual faces date-1 budget constraints given by

$$\int q_1(s|\xi) x_1^i(s|\xi) ds = 0, \ \forall \xi,$$

where $q_1(s|\xi)$ denotes the prices of an Arrow-Debreu security that pays at date 2 in state s, and $x_1^i(s|\xi)$ denotes individual i's position in that security. At date 2, individual i's consumption is given by

$$c_2^i(s, \xi) = n_2^i(s) + x_1^i(s|\xi) + \Pi^i(s|\xi), \ \forall (s, \xi),$$

where $n_2^i(s)$ denotes individual i's endowment of the consumption good, and $\Pi_1^i(s|\xi)$ denotes the individual i's proceeds from operating a technology. We assume that investor i manages a backyard quadratic technology and chooses $k_1^i(\xi)$ at date 1 after observing the public signal ξ, where

$$\Pi^i(s|\xi) = e^s k_1^i(\xi) - \frac{\kappa}{2}\left(k_1^i(\xi)\right)^2.$$

Note that by setting $k_1^i(\xi) = 0$, this economy nests the pure endowment economy case in Hirshleifer (1971).

We further assume that $n_2^A(s) = \chi^A(s) \bar{n}_2$ and $n_2^B(s) = \left(1 - \chi^A(s)\right) \bar{n}_2$, where $\bar{n}_2$ denotes a (predetermined) aggregate endowment, and where the share of investor A's endowment

[14] In models with information, the assumption of expected utility rules out preferences for early or late resolution of uncertainty, which in turn generate a "psychic" value on the timing of information. There is scope to further explore these additional sources of value in future work.

has sigmoid/logistic form:

$$\chi^A(s) = \frac{e^s}{1+e^s}.$$

This functional form ensures that $\lim_{s\to\infty}\chi^A(s) = 1$ and $\lim_{s\to-\infty}\chi^A(s) = 0$. Hence, high s states are states in which investors A owns most of the endowment: these are "good" states from the perspective of A.

Finally, we assume that the public signal takes the following form:

$$\xi = s + \varepsilon,$$

where $s \sim \mathcal{N}(\mu_0, \sigma_0^2)$ and $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. After observing the signal ξ, the distribution of posterior beliefs $s|\xi$ follows a normal distribution $\mathcal{N}\left(\mu_{s|\xi}, \sigma_{s|\xi}\right)$, with moments given by

$$\mu_{s|\xi} = \alpha\xi + (1-\alpha)\mu_0 \quad \text{and} \quad \sigma^2_{s|\xi} = \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\sigma_0^2}},$$

with $\alpha = \frac{\sigma_0^2}{\sigma^2+\sigma_0^2}$. By considering changes in the public signal volatility, σ, we can illustrate the pure impact of a change in information. This exercise contrasts with our first two applications since this change has no impact on technologies and endowments.

Equilibrium. The definition of competitive equilibrium is standard and provided in the Appendix. Individuals make date-1 decisions over i) their portfolio of Arrow-Debreu securities, $x_1^i(s|\xi)$, and ii) their production, $k_1^i(\xi)$. Then financial markets clear.

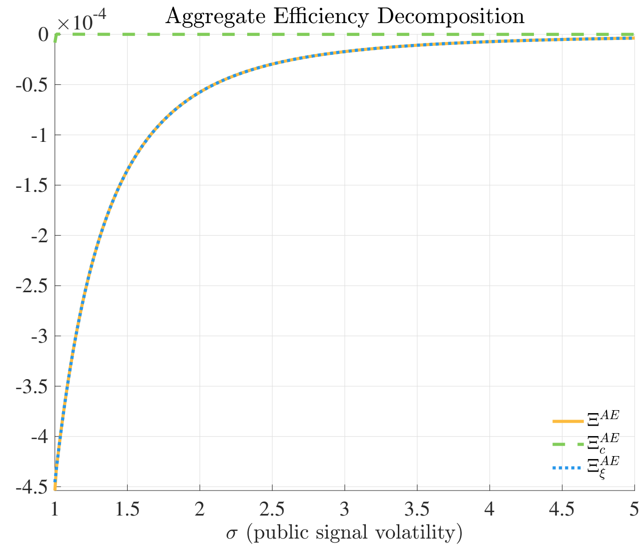
Since markets are complete, individuals' portfolio decisions must satisfy

$$\frac{u'\left(c_2^i(s,\xi)\right) f(s|\xi)}{u'\left(c_2^i(s',\xi)\right) f(s'|\xi)} = \frac{q_1(s|\xi)}{q_1(s'|\xi)},$$

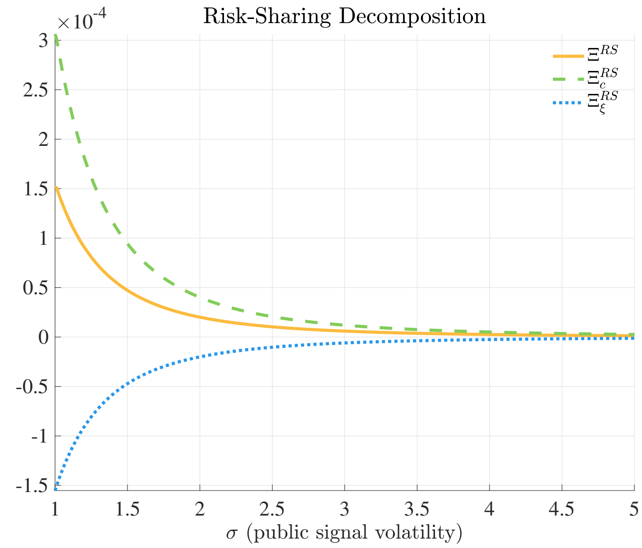
for any two state s and s' given the signal realization ξ. Market completeness further implies that both individuals will make identical production decisions, given by

$$k_1^i(\xi) = k_1(\xi) = \frac{1}{\kappa}\int \omega_2(s|\xi)\, e^s ds, \ \forall i,$$

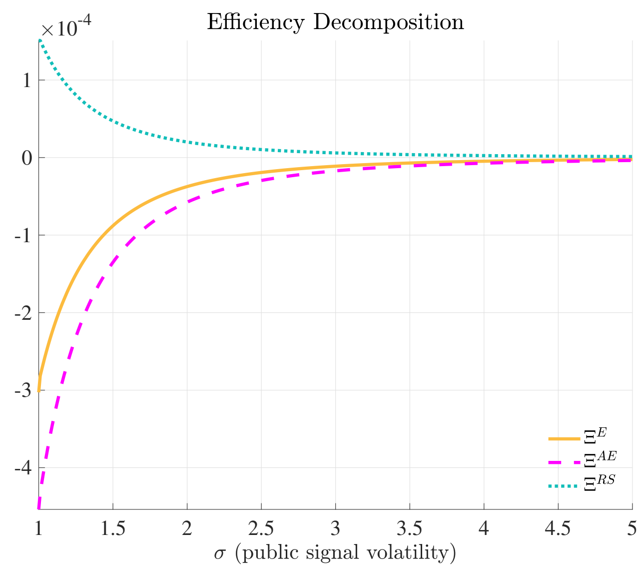
where $\omega_2(s|\xi) = \frac{u'\left(c_2^i(s,\xi)\right)f(s|\xi)}{\int u'\left(c_2^i(s,\xi)\right)f(s|\xi)ds} = \frac{q_1(s|\xi)}{\int q_1(s|\xi)ds}$.



(a) Aggregate Efficiency Decomposition



(b) Risk-Sharing Decomposition



(c) Efficiency Decomposition

Figure 4: Varying Signal Precision (Application 3)

Note: This figure shows the efficiency/willingness-to-pay implications induced by changing the public signal volatility. The top left figure shows that aggregate efficiency is exclusively driven by the probability pricing term, not consumption. The top right figure shows that increasing volatility (reducing precision) of public information generates risk-sharing gains, due to the consumption component, with the probability pricing component in the opposite direction. The bottom figure shows that increasing volatility (reducing precision) of public information can be welfare reducing due to an aggregate efficiency loss even when this generates risk-sharing gains. The baseline parameters are $\sigma = 0.15$, $\gamma = 2$, and $\kappa = 2$.

4.2.2 Value of Changes in Public Signal Precision.

We focus on characterizing the welfare/willingness-to-pay impact of changes in the precision of the public signal, here parametrized by the volatility parameter σ. Formally, individual i's welfare gains induced by a marginal change in the volatility of the signal σ, expressed in units of date-1 uncontingent consumption, are given by the following formula that combines consumption and probability pricing:

$$\frac{dV^{i\lambda}}{d\sigma} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \iint \omega_2^i(s,\xi)\left(\frac{dc_2^i(s,\xi)}{d\theta} + \frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f(\xi \mid s)}\frac{dc_2^i(s,\xi)}{d\xi}\right)d\xi ds, \tag{19}$$

where $\lambda^i = \iint u_i'\left(c_2^i(s,\xi)\right)f(s,\xi)\,dsd\xi$ and where individual i's (shadow) state prices are given by

$$\omega_2^i(s,\xi) = \frac{u_i'\left(c_2^i(s,\xi)\right)f(s,\xi)}{\iint u_i'\left(c_2^i(s,\xi)\right)f(s,\xi)\,dsd\xi}.$$

Similar to (7) and (12), Equation (19) implies that individual welfare gains consist of a consumption and a probability component.

And similar to Application 2, we can express (Kaldor-Hicks) efficiency gains as follows:

$$\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\sigma} = \underbrace{\sum_i \iint \omega_2^i(s,\xi)\left(\frac{dc_2^i(s,\xi)}{d\theta} + \frac{\frac{d(1-F(\xi|s))}{d\theta}}{f(\xi \mid s)}\frac{dc_2^i(s,\xi)}{d\xi}\right)d\xi ds}_{\Xi_c^E \text{ (consumption)}} \tag{20}$$

$$+\underbrace{\sum_i \iint \omega_2^i(s,\xi)\frac{\frac{d(1-F(\xi|s))}{d\theta}}{f(\xi \mid s)}\frac{dc_2^i(s,\xi)}{d\xi}d\xi ds,}_{\Xi_\xi^E \text{ (probability)}} \tag{21}$$

where Ξ_c^E and Ξ_ξ^E denote the sum of individual welfare gains due to changes in consumption and probabilities, respectively. As explained above, these terms can be further decomposed in the social value of *aggregate* consumption (or consumption-equivalents), while risk-sharing gains arise because consumption (or consumption-equivalents) is *reshuffled* towards individuals with higher marginal valuations, $\omega^i(s)$. Formally, we can express Ξ_c^E and Ξ_ξ^E as

$$\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS} \quad \text{and} \quad \Xi_\xi^E = \Xi_\xi^{AE} + \Xi_\xi^{RS}, \tag{22}$$

where we provide explicit definitions and analytical characterizations of each of the components in the Appendix. Our analysis yields three main takeaways.

First, we discuss our results on risk-sharing, which would be qualitatively identical if we had considered an endowment economy. The top right panel of Figure 4 illustrates these results. We show that while increasing the precision of public information (lower σ) generates risk-sharing losses, this is exclusively due to the consumption component. In fact, we show that the probability pricing component goes in the opposite direction. Intuitively, for a given realization of the signal ξ, an increase in the precision of public information damages the consumption of those individuals who are already doing worse off. This effect, captured by

$$\Xi_c^{RS} > 0,$$

is the driver of the Hirshleifer (1971) effect. In this competitive environment, this loss is caused by the distributive pecuniary effects of the perturbation, combined with the fact that markets are incomplete against the signal.

However, probability pricing is countervailing force, since

$$\Xi_\xi^{RS} < 0.$$

By reducing the likelihood of extreme signals, an increase in the precision of public information (lower σ) disproportionally benefits those individuals who are worse off, even for a given mapping between signals and consumption. In other words, even when the counterintuitive Hirshleifer effect materializes, reducing the likelihood of extreme signal realizations is a benefit from increasing the precision of public information, even in endowment economies.

Second, we show that increases in the precision of the public signal (lower σ) are associated with aggregate-efficiency gains, $\Xi^{AE} > 0$, but only due to the probability pricing term, not consumption. The top left panel of Figure 4 illustrates these results. Intuitively, a more volatile public signal makes production decisions less efficient via

$$\Xi_\xi^{AE} > 0.$$

However, it is important to notice that this increase in volatility has no impact on the aggregate-efficiency consumption term since

$$\Xi_c^{AE} = 0.$$

This result is due to the fact that production in this economy is optimally chosen, so changes in production induced by changes in the signals do not directly change the value of aggregate output (an envelope theorem/optimality argument). This occurs because production is carried out under complete markets. Overall, our results are helpful to understand the welfare impact of public information in both endowment and production economies.

4.3 Application 4: Private Information (REE)

Our last applications shows how probability pricing is useful to understand the private and social value of changes in the precision of private information in a canonical competitive model of financial trading with dispersion information. In the noisy rational expectations equilibrium (REE) of this model, the price acts as a public signal that partially aggregates the private signals received by investors.

4.3.1 Environment

We consider an economy with three dates $t = \{0, 1, 2\}$ and two types of agents: investors and noise traders. Agents receive signals and trade at date 1 and consume at date 2. All agents have identical constant absolute risk aversion (CARA) expected utility preferences over their date-2 consumption, with flow utility given by

$$u(c) = -e^{-\eta c},$$

where $\eta > 0$ denotes the coefficient of absolute risk aversion. There is a continuum of utility maximizing investors in unit measure. There is also a single noise trader who trades inelastically.

There are two assets: a risky asset and a riskless asset. The risky asset pays a normally distributed payoff δ at date 2 given by

$$\delta \sim N\left(\mu_\delta, \frac{1}{\tau_\delta}\right).$$

The risky asset is competitively traded at date 1 at a price q and is in fixed supply $\bar{a} \geq 0$. The riskless asset pays a gross interest rate normalized to one. Assuming that the aggregate endowment of consumption at date 1 is zero ensures that riskless market also clears. For simplicity, we assume that all agents initial endowment of the risky asset is $a_0 = \bar{a}$. At date

1, each investor receives a private signal ξ about the asset payoff δ, where

$$\xi = \delta + u_\xi \quad \text{with} \quad u_\xi \sim N\left(0, \frac{1}{\tau_\xi}\right),$$

and where the realizations of u_ξ are independent across investors.

The noise trader (with measure one) trades a random amount of the risky asset

$$n \sim N\left(\mu_n, \tau_n^{-1}\right),$$

and their consumption is given by $c_2^n = \delta\left(n + a_0\right) - qn$.

Equilibrium The definition of a rational expectations equilibrium in this model is standard. Investors choose their asset holdings for the risky and riskless assets to maximize their expected utility subject to their information and taking prices as given, and goods and asset markets clear. We focus on the unique equilibrium in linear strategies in which the optimal risky asset demand is linear in the private signal and the price, and the price are linear functions of the date-1 aggregate state, which is given by the average signal of investors, $\overline{\delta} = \int \xi f\left(\xi | \delta\right)$, and the amount traded by the noise trader, n.

The welfare of investors at date 0 is given by

$$V_0 = \iiint V_1\left(\xi, \overline{\delta}, n; \tau_\xi\right) f\left(\xi, \overline{\delta}, n; \tau_\xi\right) d\xi d\overline{\delta} dn,$$

where V_1 is the expected utility at date 1 of an investor that receives a signal ξ when the aggregate state at date 1 is $\left(\overline{\delta}, n\right)$, given by

$$V_1\left(\xi, \overline{\delta}, n; \tau_\xi\right) = \int u\left(c_2\left(\delta, \xi, \overline{\delta}, n\right)\right) f\left(\delta | \xi, q\left(\overline{\delta}, n\right); \tau_\xi, \tau_{\hat{q}}\left(\tau_\xi\right)\right) d\delta,$$

where $\tau_{\hat{q}}$ measures price informativeness and is given by the precision of the unbiased signal about δ contained in the price.

For the noise trader, the welfare at date 0 is given

$$J_0 = \iint u\left(c_2\left(\overline{\delta}, q^\star\left(\overline{\delta}, n\right), n\right)\right) f\left(\overline{\delta}\right) f\left(n\right) d\overline{\delta} dn,$$

because the noise trader's noise n and the price $q^\star$ jointly perfectly reveal $\overline{\delta} = \delta$, which implies

$$\int u\left(c_2^n\left(\delta,n,q^\star\right)\right)f\left(\delta|q^\star,n;\tau_{\hat{q}}\left(\tau_\xi\right)\right)d\delta = u\left(c_2\left(\overline{\delta},q^\star,n\right)\right).$$

4.3.2 Value of Change in Private Signal Precision

We are interested in the willingness-to-pay for a change in the precision τ_ξ of the private signal received by the investors has the following effects on investor welfare. As in the previous section, we measure an agent's welfare gains induced by a marginal increase in the precision of the private signals τ_ξ expressed in units of date-2 uncontingent consumption. Formally, this value for investors is given by $\frac{dV_0^\lambda}{d\tau_\xi} \equiv \frac{dV_0}{d\tau_\xi}\frac{1}{\lambda}$, where

$$\frac{dV_0^\lambda}{d\tau_\xi} = \iiint \left(\underbrace{\frac{1}{\lambda}\frac{dV_1\left(\xi,\overline{\delta},n;\tau_\xi\right)}{d\tau_\xi}}_{\text{Date-1 Continuation}} + \underbrace{\frac{1}{\lambda}\frac{dV_1\left(\xi,\overline{\delta},n;\tau_\xi\right)}{d\xi}\frac{\frac{d\left(1-F\left(\xi|\overline{\delta};\tau_\xi\right)\right)}{d\tau_\xi}}{f\left(\xi|\overline{\delta};\tau_\xi\right)}}_{\text{Date-1 Probability/Signal Compression}} \right) f\left(\xi|\overline{\delta};\tau_\xi\right)d\xi f\left(\overline{\delta}\right)d\overline{\delta}f\left(n\right)dn,$$

and $\lambda = \iiint\left[\int u'\left(c_2\left(\delta,\xi,\overline{\delta},n\right)\right)f\left(\delta|\,\xi,q\left(\overline{\delta},n\right);\tau_\xi,\tau_{\hat{q}}\left(\tau_\xi\right)\right)d\delta\right]f\left(\xi,\overline{\delta},n;\tau_\xi\right)d\xi d\overline{\delta}dn$. The first term in the expression above represents the effects on date-1 welfare, hence the label "date-1 continuation", while the second term captures the probability pricing effect coming from the change in the conditional distribution of the private signal being compressed as τ_ξ increases.

To understand the effect on the date-1 continuation, it is helpful to isolate the informational content captured by $\hat{q}$. To do so, we can define the date-1 expected utility of investors as follows

$$V_1\left(\xi,\overline{\delta},n;\tau_\varepsilon\right) = \tilde{V}_1\left(\xi,q^\star\left(\overline{\delta},n;\tau_\xi,\tau_{\hat{q}}\left(\tau_\xi\right)\right),\hat{q}^\star\left(\overline{\delta},n;\tau_\xi\right);\tau_\xi,\tau_{\hat{q}}\left(\tau_\xi\right)\right),$$

where

$$\tilde{V}_1\left(\xi,q,\hat{q};\tau_\xi,\tau_{\hat{q}}\right) = \int u\left(\tilde{c}_2\left(\delta,a_1^\star\left(\xi,q;\tau_\xi,\tau_{\hat{q}}\right),q\right)\right)f\left(\delta|\,\xi,\hat{q};\tau_\xi,\tau_{\hat{q}}\right)d\delta$$

and $\hat{q}^\star\left(\overline{\delta},n;\tau_\xi\right)$ is the unbiased signal about δ contained in the price in equilibrium. For notational convenience, we denote the arguments of the equilibrium mappings by *. Then, the effect on the date-one continuation can be further decomposed into the pecuniary effects and signal distribution effects, as follows.

$$\frac{dV_1}{d\tau_\xi} = \underbrace{\overbrace{\frac{\partial \tilde{V}_1}{\partial q}\frac{dq^\star}{d\tau_\xi}}^{\text{distributive pecuniary}} + \overbrace{\frac{\partial \tilde{V}_1}{\partial \hat{q}}\frac{d\hat{q}^\star}{d\tau_\xi}}^{\text{learning pecuniary}}}_{\text{pecuniary effects}} + \underbrace{\overbrace{\frac{\partial \tilde{V}_1}{\partial \tau_\xi}}^{\text{private signal}} + \overbrace{\frac{\partial \tilde{V}_1}{\partial \tau_{\hat{q}}}\frac{d\tau_{\hat{q}}}{d\tau_\xi}}^{\text{informativeness}}}_{\text{signal distributions}}.$$

where the partial derivatives keep all other arguments in the corresponding function fixed. Since the price aggregates all private signals received by investors, the pecuniary effects contain the usual distributive pecuniary effects and the learning pecuniary effects that come from the mapping between the aggregate states and the information contained in the price changing with τ_ξ. The effect on the distribution of signals can also be decomposed into the effect of the change in an investors private signal, $\frac{\partial \tilde{V}_1}{\partial \tau_\xi}$, and the effect of the change in the rest of the investors' precisions which is captured by the change in price informativeness, $\frac{\partial \tilde{V}_1}{\partial \tau_{\hat{q}}}\frac{d\tau_{\hat{q}}}{d\tau_\xi}$.

Therefore, the willingness-to-pay for a marginal change in τ_ξ for investors is

$$\begin{aligned}\frac{dV_0^\lambda}{d\tau_\xi} = &\iiint \underbrace{\overbrace{\frac{\partial \tilde{V}_1^\lambda}{\partial q}\frac{dq^\star}{d\tau_\xi}}^{\text{distributive pecuniary}}}_{\text{consumption pricing}} f\left(\xi|\overline{\delta};\tau_\xi\right)d\xi f\left(\overline{\delta}\right)d\overline{\delta} f(n)\,dn + \\ &\iiint \left(\underbrace{\overbrace{\frac{\partial \tilde{V}_1^\lambda}{\partial \hat{q}}\frac{d\hat{q}^\star}{d\tau_\xi}}^{\substack{\text{learning}\\\text{pecuniary}}} + \overbrace{\frac{\partial \tilde{V}_1^\lambda}{\partial \tau_\xi}}^{\substack{\text{private}\\\text{signal}}} + \overbrace{\frac{\partial \tilde{V}_1^\lambda}{\partial \tau_{\hat{q}}}\frac{d\tau_{\hat{q}}}{d\tau_\xi}}^{\text{informativeness}} + \overbrace{\frac{dV_1^\lambda}{d\xi}\frac{\frac{d\left(1-F\left(\xi|\overline{\delta};\tau_\xi\right)\right)}{d\tau_\xi}}{f\left(\xi|\overline{\delta};\tau_\xi\right)}}^{\text{signal compression}}}_{\text{probability pricing}}\right) f\left(\xi|\overline{\delta};\tau_\xi\right)d\xi f\left(\overline{\delta}\right)d\overline{\delta} f(n)\,dn\end{aligned}$$

where we define $\frac{\partial \tilde{V}_1^\lambda}{\partial x} \equiv \frac{1}{\lambda}\frac{\partial \tilde{V}_1}{\partial x}$ for any x. The distributive pecuniary effects are the only channel through which changes in the private signal precision affect consumption. The rest of the effects work by affecting the date-1 probability of the aggregate state at date-2 δ (learning, pecuniary, private signal, and informativeness), and the distribution of the idiosyncratic state at date 1 (signal compression).

Since the noise trader's noise and the price jointly perfectly reveal $\overline{\delta} = \delta$, there are only pecuniary effects for the noise traders. Noise traders always learn the dividend perfectly from the price (due to the continuum of investors and the LLN), regardless of the precision of the private signals. Therefore, the date-0 welfare effect for noise traders expressed in units of their date-2 uncontingent consumption is

$$\frac{dJ_0^{\lambda^n}}{d\tau_\xi} \equiv \frac{1}{\lambda^n}\frac{dJ_0}{d\tau_\xi} = \frac{1}{\lambda^n}\iint \underbrace{u'\left(c_2^n\left(\overline{\delta}, n, q^\star\right)\right)\frac{\partial c_2}{\partial q}\frac{dq^\star}{d\tau_\xi}}_{\text{distributive pecuniary}} f\left(\overline{\delta}\right) f\left(n\right) d\overline{\delta} dn,$$

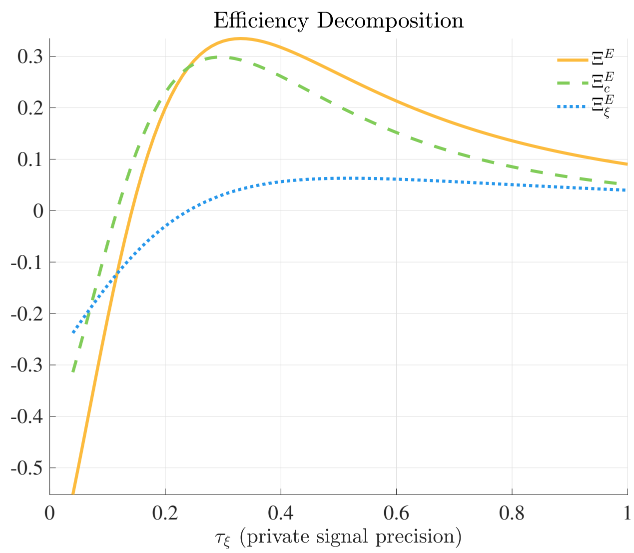
where $\lambda^n \equiv \iint u'\left(c_2^n\left(\overline{\delta}, n, q^\star\right)\right) f\left(\overline{\delta}\right) f\left(n\right) d\overline{\delta} dn$.

Aggregating the welfare effects for investors and the noise trader, we do a final decomposition thinking about the effects of changes in one's private signal and the effects of changes in everyone else's private signals. More specifically, the "private" effects are given by the private signal and signal compression effects, while the "social" effects—those happening through pecuniary and information externalities—are captured by the distributive pecuniary, learning pecuniary, and informativeness effects.

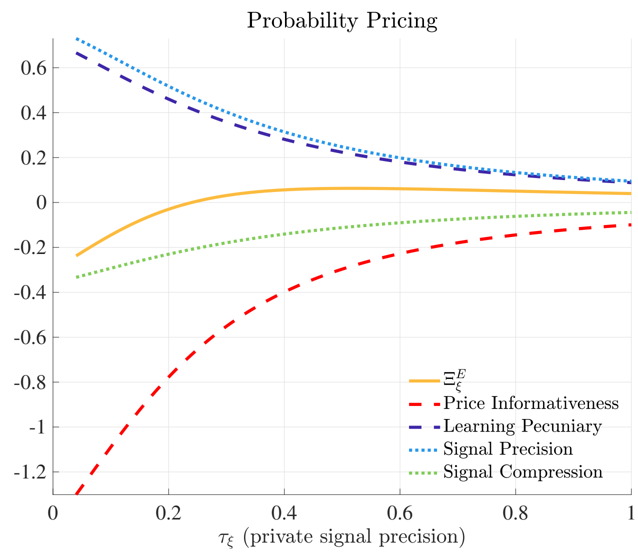
Figure 5, shows the different welfare components discussed above as a function of the precision of the private signals. Interestingly, as it can be seen from Figure 5a, the change in overall welfare is non-monotonic and has a minimum where the solid line crosses the horizontal axis, which implies welfare is higher either at the no information or full information limits. This non-monotonicity is present in both consumption and probability pricing effects. Moreover, within the probability pricing effects shown in Figure 5b, the signal precision and the learning pecuniary effects are positive, which is consistent with investors being better off when their information is more precise or the price reflects information better. Figure 5c shows that changes in welfare for investors and noise traders can often go in different directions with investors benefiting from increases in precision when precisions are low and the noise trader benefiting from them when precisions are high. Finally, Figure 5d shows that the private effect of information is positive while the social effect can be positive or negative.

5 Conclusion

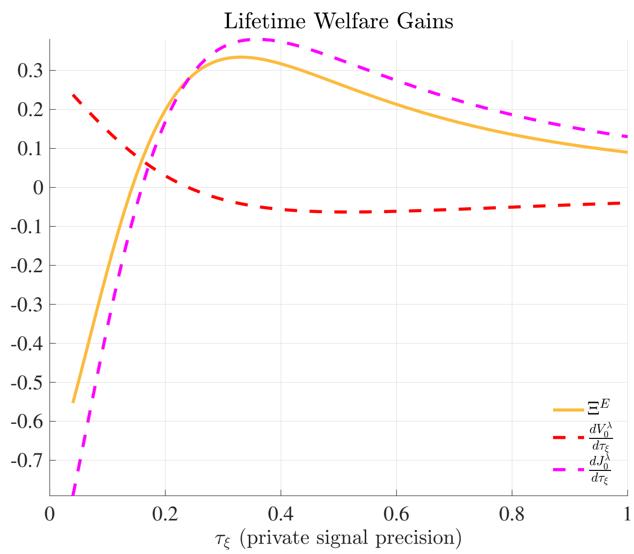
This paper extends traditional cash-flow pricing to analyze the willingness-to-pay for changes in probabilities, that is, probability pricing. We show that an agent's willingness-to-pay for a marginal change in probabilities is equivalent to pricing an asset with hypothetical cash flows that represent the state-by-state value of the probability perturbation. This result establishes a direct analogy with traditional cash-flow pricing and is useful to construct hedging strategies, among other things. Our applications illustrate the broad implications of



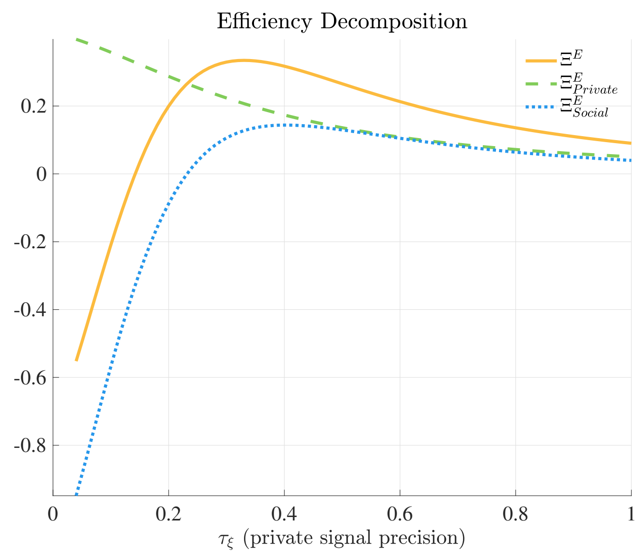
(a) Consumption and Probability Pricing



(b) Probability Pricing



(c) Investors vs. Noise Trader



(d) Private and Social

Figure 5: Varying Private Signal Precision (Application 4)

Note: This figure shows the different welfare decompositions in Application 4 as a function of the precision of the private signals. Benchmark parameters are $\gamma = 1.2$, $a_0 = 0$, $\mu_\delta = 10$, $\mu_n = 0$, $\tau_\delta = 1$, and $\tau_n = 9$.

probability pricing, from disaster risk valuation to principal-agent problems, to the study of the private and social values of information in financial markets with public or dispersed information. Beyond theoretical insights, our results show how quantify the impact of changes in probabilities, paving the way for further applications in asset pricing and welfare analysis.

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Appendix

A Proofs and Derivations: Section 2

Proof of Proposition 1. (Probability Pricing)

Proof. Let $U(s) = \beta u(c_1(s))$, so $\frac{dU(s)}{ds} = \beta u'(c_1(s)) \frac{dc_1(s)}{ds}$. Using integration by parts, Equation (3) can be expressed as

$$
\begin{aligned}
p_\theta &= \frac{1}{u'(c_0)} \int_{\underline{s}}^{\bar{s}} U(s) \frac{df(s;\theta)}{d\theta} ds = \frac{1}{u'(c_0)} \left[U(s) \frac{dF(s;\theta)}{d\theta} \Big|_{\underline{s}}^{\bar{s}} - \int_{\underline{s}}^{\bar{s}} \frac{dU(s)}{ds} \frac{dF(s;\theta)}{d\theta} ds \right] \\
&= \frac{1}{u'(c_0)} \left[U(\bar{s}) \underbrace{\frac{dF(\bar{s};\theta)}{d\theta}}_{=0} - U(\underline{s}) \underbrace{\frac{dF(\underline{s};\theta)}{d\theta}}_{=0} - \int_{\underline{s}}^{\bar{s}} \frac{dU(s)}{ds} \frac{dF(s;\theta)}{d\theta} ds \right] \\
&= \frac{1}{u'(c_0)} \left[\int_{\underline{s}}^{\bar{s}} \frac{dU(s)}{ds} \frac{d(1-F(s;\theta))}{d\theta} ds \right] = \int_{\underline{s}}^{\bar{s}} \frac{\beta u'(c_1(s))}{u'(c_0)} \frac{dc_1(s)}{ds} \frac{d(1-F(s;\theta))}{d\theta} ds,
\end{aligned}
$$

which corresponds to Equation (4) in the text. We use the fact that for all θ, we have $F(\bar{s};\theta) = 1$ and $F(\underline{s};\theta) = 0$. $\square$

Note that defining expected consumption as $\mathbb{E}[c_1(s)] = \int_{\underline{s}}^{\bar{s}} c_1(s) f(s;\theta) ds$, it is the case that

$$
\frac{d\mathbb{E}[c_1(s)]}{d\theta} = \int_{\underline{s}}^{\bar{s}} c_1(s) \frac{df(s;\theta)}{d\theta} ds = \int_{\underline{s}}^{\bar{s}} \frac{dc_1(s)}{ds} \frac{d(1-F(s;\theta))}{d\theta} ds = \int_{\underline{s}}^{\bar{s}} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds} f(s;\theta) ds.
$$

Note also that

$$
\begin{aligned}
p_\theta &= \int_{\underline{s}}^{\bar{s}} m(s) \frac{dc_1(s)}{ds} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} f(s;\theta) ds \\
&= \int_{\underline{s}}^{\bar{s}} m(s) f(s;\theta) ds \int_{\underline{s}}^{\bar{s}} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds} f(s;\theta) ds + \mathbb{C}ov\left[m(s), \frac{dc_1(s)}{ds} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \right],
\end{aligned}
$$

which corresponds to Equation (6) in the text.

Mean/Variance Perturbations. We assume that $s = \mu + \sigma m$. Therefore, the cdf $F(s)$ is given by

$$
F(s) = H\left(\frac{s-\mu}{\sigma}\right),
$$

where $H(\cdot)$ denotes the cdf over m. Therefore, the pdf $f(s)$ is given by

$$
f(s) = \frac{d}{ds} H\left(\frac{s-\mu}{\sigma}\right) = \frac{1}{\sigma} h\left(\frac{s-\mu}{\sigma}\right).
$$

A marginal increase in μ implies that:

$$
\frac{dF(s)}{d\mu} = \frac{d}{d\mu} H\left(\frac{s-\mu}{\sigma}\right) = -\frac{1}{\sigma} h\left(\frac{s-\mu}{\sigma}\right), \quad \text{so} \quad \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} = 1.
$$

A marginal increase in σ implies that:

$$\frac{dF(s)}{d\sigma}=\frac{d}{d\sigma}H\left(\frac{s-\mu}{\sigma}\right)=-\frac{s-\mu}{\sigma}\frac{1}{\sigma}h\left(\frac{s-\mu}{\sigma}\right),\quad \text{so}\quad \frac{\frac{d(1-F(s))}{d\theta}}{f(s)}=\frac{s-\mu}{\sigma}.$$

Mixture Distributions. In this case, note that $f(s;h)=(1-h)\overline{f}(s)+h\underline{f}(s)$ and $F(s;h)=(1-h)\overline{F}(s)+h\underline{F}(s)$, so we can express the survival change as

$$\frac{d(1-F(s;h))}{dh}=\overline{F}(s)-\underline{F}(s),$$

implying Equation (11) in the text. Note that

$$p_h=\int_{\underline{s}}^{\bar{s}}\frac{\beta u'(c_1(s))}{u'(c_0)}\frac{dc_1(s)}{ds}\left(\overline{F}(s)-\underline{F}(s)\right)ds,$$

which is invariant to the level of h, as stated in the text.

Stochastic Dominance. First, we consider a perturbation that satisfies first-order stochastic dominance. In the case of a lottery, $c_1(s)=s$, and $\frac{dc_1(s)}{ds}=1$, so Equation (4) simplifies as follows

$$p_\theta=\int_{\underline{s}}^{\bar{s}}\frac{\beta u'(c_1(s))}{u'(c_0)}\frac{d(1-F(s;\theta))}{d\theta}ds.$$

It is immediate that any first-order stochastic dominance shift, with $\frac{d(1-F(s;\theta))}{d\theta}\geq 0$, leads to $p_\theta\geq 0$, since utility is increasing in consumption with $u'(c_1(s))>0$.

Second, we consider a perturbation that satisfies second-order stochastic dominance. Define the cumulative perturbation $H(s;\theta)=\int_{\underline{s}}^{s}\frac{dF(t;\theta)}{d\theta}dt$, where we have assumed that $H(s;\theta)\geq 0$ for all s. Notice that, integrating by parts, the change in the expected value of s as a result of the perturbation can be written as

$$\begin{aligned}\frac{d\mathbb{E}[s]}{d\theta}&=\int_{\underline{s}}^{\bar{s}}s\frac{df(s;\theta)}{d\theta}ds=\bar{s}\frac{dF(\bar{s};\theta)}{d\theta}-\underline{s}\frac{dF(\underline{s};\theta)}{d\theta}-\int_{\underline{s}}^{\bar{s}}\frac{dF(s;\theta)}{d\theta}ds\\&=-\int_{\underline{s}}^{\bar{s}}\frac{dF(s;\theta)}{d\theta}ds=-H(\bar{s};\theta),\end{aligned}$$

so that, by assumption, we have $H(\bar{s};\theta)=0$. Now integrating by parts again, we have

$$\begin{aligned}p_\theta&=-\int_{\underline{s}}^{\bar{s}}\frac{\beta u'(c_1(s))}{u'(c_0)}\frac{d(1-F(s;\theta))}{d\theta}ds\\&=-\frac{1}{u'(c_0)}\left\{u'(\bar{s})\underbrace{H(\bar{s};\theta)}_{=0}-u'(\underline{s};\theta)\underbrace{H(\underline{s};\theta)}_{=0}-\int u''(s)H(s;\theta)ds\right\}\\&=\frac{1}{u'(c_0)}\int u''(s)H(s;\theta)ds.\end{aligned}$$

Then, since utility is strictly concave with $u''(s)<0$, we obtain $p_\theta\leq 0$.

B Proofs and Derivations: Section 3

B.1 Application 1: Consumption-Based Asset Pricing

In this case, the probability price for a change in the disaster probability h can be written as

$$p_h = \int_{\underline{s}}^{\overline{s}} \beta \frac{u'(c_1(s))}{u'(c_0)} \frac{dc_1(s)}{ds} \frac{d(1-F(s))}{d\theta} ds = c_0 \int_{\underline{s}}^{\overline{s}} \beta e^{(1-\gamma)s} f(s) \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} ds,$$

where we used the fact that $\frac{u'(c_1(s))}{u'(c_0)} = \left(\frac{c_1(s)}{c_0}\right)^{-\gamma} = e^{-\gamma(c_1(s)/c_0)} = e^{-\gamma s}$ and $\frac{dc_1(s)}{ds} = c_1(s)$. Since

$$\frac{d(1-F(s))}{d\theta} = \overline{F}(s) - \underline{F}(s),$$

it is straightforward to conclude that $\frac{p_\theta}{c_0}$ is invariant to the level of h.

A similar logic yields an equivalent formulation for

$$\frac{p_{\overline{\sigma}}}{c_0} = \int_{\underline{s}}^{\overline{s}} \beta e^{(1-\gamma)s} f(s) \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} ds,$$

where in this case

$$\frac{\frac{d(1-F(s))}{d\theta}}{f(s)} = \frac{s-\mu}{\sigma}.$$

A direct application of Equation (6) generates the decomposition shown in Figure 2.

B.2 Application 2: Principal-Agent Problem

The principal, with full bargaining power, solves the following problem

$$\max_{\{e,t,\alpha\}} \int c^1(s) f(s) ds,$$

subject to participation and incentive constraints for the agent, given by

$$\int u\left(c^2(s)\right) f(s) ds = \overline{V}$$

$$e \in \arg\max_e \int u\left(c^2(s)\right) f(s) ds.$$

Note that the objective function can be expressed as

$$V^1 = \int c^1(s) f(s) ds = (1-\alpha) e - t,$$

and the utility of the agent as

$$V^2 = \int u\left(c^2(s)\right) f(s) ds = -\exp\left[-\eta\left(t + \alpha e - \frac{1}{2}\kappa e^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau}\right)\right].$$

Therefore, the principal-agent problem can be re-written as

$$\max_{\{e,t,\alpha\}} e - \frac{1}{2}\kappa e^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau}$$

subject to

$$e \in \arg\max_{e} \left\{ -\exp\left[-\eta \left(t + \alpha e - \frac{1}{2}\kappa e^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau} \right) \right] \right\}.$$

If effort is observable, then the principal chooses an effort level e to solve

$$\max_{e} e - \frac{1}{2}\kappa e^2,$$

implying e level of effort $e = \frac{1}{\kappa}$. This solution implies a performance sensitivity of $\alpha = 1$ and a non-contingent transfer given by $t = \overline{w} + \frac{1}{2}\kappa e^2$.

Optimal Contract. If effort is not observable, the agent's optimality condition determines the optimal effort

$$e = \frac{\alpha}{\kappa}.$$

So the principal's problem becomes

$$\max_{\{t,\alpha\}} \frac{\alpha}{\kappa} - \frac{1}{2}\kappa \left(\frac{\alpha}{\kappa}\right)^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau},$$

which yields the following solution for the contract sensitivity α:

$$\alpha = \frac{1}{1 + \frac{\eta\kappa}{\tau}} = \frac{\tau}{\tau + \eta\kappa}.$$

The non-contingent transfer t is given by

$$t = -\frac{1}{\eta}\ln\left(-\overline{V}\right) - \frac{\alpha^2}{2\kappa}\left(\frac{\tau - \eta\kappa}{\tau}\right).$$

Note that equilibrium changes in consumption for both individuals are given by

$$\begin{aligned} \frac{dc^B(s)}{d\tau} &= (1-\alpha)\frac{da}{d\tau} - (e+s)\frac{d\alpha}{d\tau} - \frac{dt}{d\tau} \qquad \text{and} \qquad \frac{dc^B(s)}{ds} = 1-\alpha \\ \frac{dc^A(s)}{d\tau} &= \frac{dt}{d\tau} + (e+s)\frac{d\alpha}{d\tau} + (\alpha - \psi'(e))\frac{de}{d\tau} \quad \text{and} \quad \frac{dc^A(s)}{ds} = \alpha, \end{aligned}$$

where the contract changes according to

$$\frac{de}{d\tau} = \frac{1}{\kappa}\frac{d\alpha}{d\tau}, \qquad \frac{d\alpha}{d\tau} = \frac{\eta\kappa}{(\tau+\eta\kappa)^2}\frac{d\alpha}{d\tau} = \frac{\eta\kappa}{(\tau+\eta\kappa)^2}, \quad \text{and} \quad \frac{dt}{d\tau} = -\frac{\alpha}{\kappa}\left(\frac{\tau-\eta\kappa}{\tau}\right)\frac{d\alpha}{d\tau} - \frac{\eta}{2}\left(\frac{\alpha}{\tau}\right)^2.$$

In this case, $f(s) = \sqrt{\frac{\tau}{2\pi}}\exp\phi\left(\sqrt{\tau}s\right)$ and $F(s) = \Phi\left(\sqrt{\tau}s\right)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of the standard normal. Consistent with our result in Section 2.4.1, $\frac{\frac{d(1-F(s))}{d\tau}}{f(s)} = -\frac{1}{2\tau}s$.

Welfare Gains. Normalized individual welfare gains are given by

$$\frac{dV^{i|\lambda}}{d\tau}=\frac{\frac{dV^i}{d\tau}}{\lambda^i}=\int\omega^i\left(s\right)\left[\frac{dc^i\left(s\right)}{d\tau}+\frac{dc^i\left(s\right)}{ds}\frac{\frac{d(1-F(s))}{d\tau}}{f\left(s\right)}\right]ds.$$

We can respectively formulate them for the principal and the agent as follows:

$$\frac{dV^{1|\lambda}}{d\tau}=\int\omega^1\left(s\right)\left[\underbrace{-s\frac{d\alpha}{d\tau}+\left(1-\alpha\right)\frac{\frac{d(1-F(s))}{d\tau}}{f\left(s\right)}}_{\frac{d\tilde{c}^1(s)}{d\tau}}\right]ds+\left(1-\alpha\right)\frac{da}{d\tau}-e\frac{d\alpha}{d\tau}-\frac{dt}{d\tau}$$

$$\frac{dV^{2|\lambda}}{d\tau}=\int\omega^2\left(s\right)\left[\underbrace{s\frac{d\alpha}{d\tau}+\alpha\frac{\frac{d(1-F(s))}{d\tau}}{f\left(s\right)}}_{\frac{d\tilde{c}^2(s)}{d\tau}}\right]ds+\frac{dt}{d\tau}+e\frac{d\alpha}{d\tau}+\left(\alpha-\psi'\left(a\right)\right)\frac{da}{d\tau}.$$

Efficiency defined by $\Xi^E=\sum_i\frac{dV^{i|\lambda}}{d\tau}$, can be decomposed into aggregate-efficiency and risk-sharing as following

$$\Xi^{AE}=\left(1-\psi'\left(a\right)\right)\frac{da}{d\tau}+\int\omega\left(s\right)\left(-\frac{\frac{dF(s)}{d\tau}}{f\left(s\right)}\right)ds$$

$$\Xi^{RS}=\int\omega\left(s\right)\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i\left(s\right)}{\omega\left(s\right)},\frac{d\tilde{c}^i\left(s\right)}{d\tau}\right]ds,$$

where $\omega\left(s\right)=\frac{1}{I}\sum_i\omega^i\left(s\right)$.

To confirm our computations, let's proceed without further developing $\frac{dV^{i|\lambda}}{d\tau}$. Efficiency is given by the following

$$\Xi^E=\int\sum_i\omega^i\left(s\right)\left[\frac{dc^i\left(s\right)}{d\tau}+\frac{dc^i\left(s\right)}{ds}\left(-\frac{\frac{dF(s)}{d\tau}}{f\left(s\right)}\right)\right]ds,$$

which can be directly decomposed into aggregate-efficiency and risk-sharing as following

$$\Xi^{AE}=\left(1-\psi'\left(a\right)\right)\frac{da}{d\tau}+\int\omega\left(s\right)\left(-\frac{\frac{dF(s)}{d\tau}}{f\left(s\right)}\right)ds$$

$$\Xi^{RS}=\int\omega\left(s\right)\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i\left(s\right)}{\omega\left(s\right)},\frac{dc^i\left(s\right)}{d\tau}\right]ds+\int\omega\left(s\right)\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i\left(s\right)}{\omega\left(s\right)},\frac{dc^i\left(s\right)}{ds}\left(-\frac{\frac{dF(s)}{d\tau}}{f\left(s\right)}\right)\right]ds.$$

The aggregate-efficiency terms follow from the following steps

$$\Xi^{AE}=\int\omega\left(s\right)\left[\underbrace{\sum_i\frac{dc^i\left(s\right)}{d\tau}}_{=(1-\psi'(a))\frac{da}{d\tau}}\right]ds+\int\omega\left(s\right)\left[\underbrace{\sum_i\frac{dc^i\left(s\right)}{ds}}_{=1}\right]\left(-\frac{\frac{dF(s)}{d\tau}}{f\left(s\right)}\right)ds.$$

It's important to note that $\int \omega(s)\,ds = 1$. We can thus compute

$$\begin{aligned}
\Xi_c^{AE} &= \left(1-\psi'(a)\right)\frac{da}{d\tau} \\
\Xi_s^{AE} &= \int \omega(s)\,\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}ds \\
\Xi_c^{RS} &= \int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)},\frac{dc^i(s)}{d\tau}\right]ds \\
\Xi_s^{RS} &= \int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)},\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}\frac{dc^i(s)}{ds}\right]ds
\end{aligned}$$

Therefore, $\Xi^E = \Xi^c + \Xi^s$ where $\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS}$ and $\Xi_s^E = \Xi_s^{AE} + \Xi_s^{RS}$.

We can show that the efficiency gains are entirely driven by information, i.e. that $\Xi^c = 0$. This can be done by proving that $\Xi_c^{RS} = -\Xi_c^{AE} = -\left(1-\psi'(a)\right)\frac{da}{d\tau}$. Note that

$$\begin{aligned}
\Xi_c^{RS} &= \int \omega^1(s)\,\frac{dc^1(s)}{d\tau}ds + \int \omega^2(s)\,\frac{dc^2(s)}{d\tau}ds - \int \omega(s)\left(\sum_i \frac{dc^i(s)}{d\tau}\right)ds \\
&= \int \omega^1(s)\,\frac{dc^1(s)}{d\tau}ds + \int \omega^2(s)\,\frac{dc^2(s)}{d\tau}ds - \left(1-\psi'(a)\right)\frac{da}{d\tau}.
\end{aligned}$$

The proof boils down to proving that $\int \omega^1(s)\,\frac{dc^1(s)}{d\tau}ds + \int \omega^2(s)\,\frac{dc^2(s)}{d\tau}ds = 0$. Note the following

$$\int \omega^1(s)\,\frac{dc^1(s)}{d\tau}ds + \int \omega^2(s)\,\frac{dc^2(s)}{d\tau}ds = -\frac{d\alpha}{d\tau}\int \omega^1(s)\,s\,ds + \frac{d\alpha}{d\tau}\int \omega^2(s)\,s\,ds + \left(1-\psi'(a)\right)\frac{da}{d\tau}.$$

Now note that $\omega^i(s) = f(s)$ implying that $\int \omega^1(s)\,s\,ds = 0$. So we need to prove that $\frac{d\alpha}{d\tau}\int \omega^2(s)\,s\,ds = -\left(1-\psi'(a)\right)\frac{da}{d\tau}$. The principal's problem Lagrangian in the optimal contract case

$$\mathcal{L} = \max_{\{\alpha,t\}}\left\{\int u^1\left(c^1(s)\right)f(s)\,ds + \phi\left[\int u^2\left(c^2(s)\right)f(s)\,ds - \overline{V}\right]\right\}.$$

And let's focus on the normalized F.O.C with respect to α

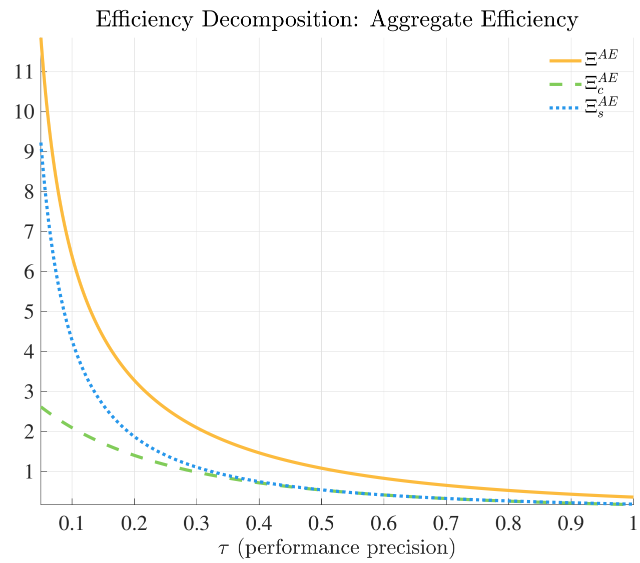
$$\frac{\partial\mathcal{L}}{\partial\alpha} = \int \frac{\partial u^1\left(c^1(s)\right)}{\partial c^1(s)}\frac{dc^1(s)}{d\alpha}f(s)\,ds + \phi\left[\int \frac{\partial u^2\left(c^2(s)\right)}{\partial c^2(s)}\frac{dc^2(s)}{d\alpha}f(s)\,ds\right] = 0.$$

Note the following

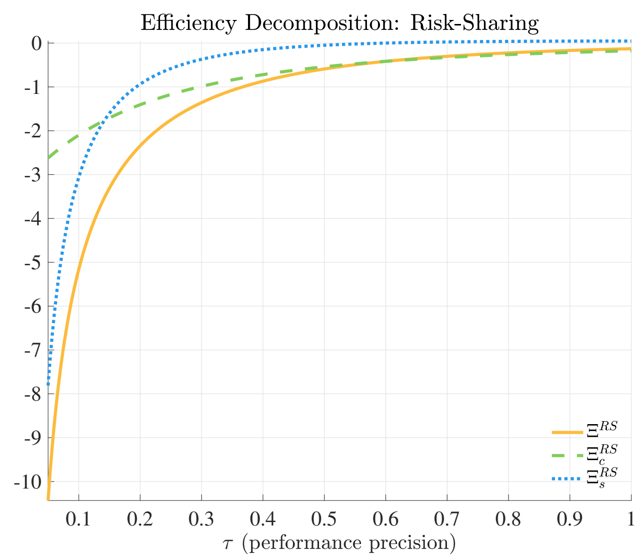
$$\begin{aligned}
\frac{dc^1(s)}{d\alpha} &= (1-\alpha)\frac{da}{d\alpha} - (a+s) - \frac{dt}{d\alpha} \\
\frac{dc^2(s)}{d\alpha} &= \frac{dt}{d\alpha} + \alpha\frac{da}{d\alpha} + (a+s) - \psi'(a)\frac{da}{d\alpha}.
\end{aligned}$$

Therefore,

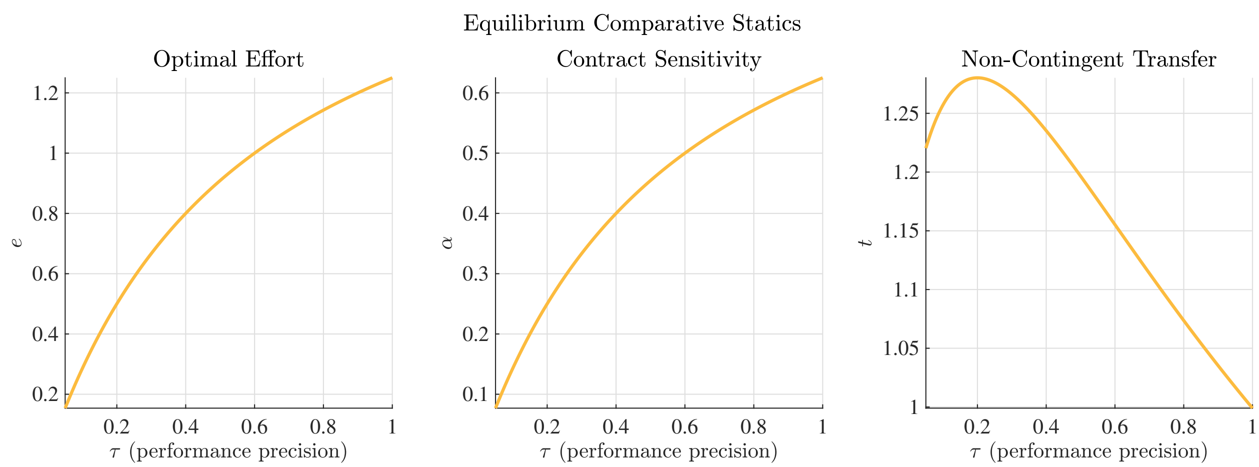
$$\begin{aligned}
&\int \frac{\partial u^1\left(c^1(s)\right)}{\partial c^1(s)}\frac{dc^1(s)}{d\alpha}f(s)\,ds = (1-\alpha)\frac{da}{d\alpha} - a - \frac{dt}{d\alpha} \\
&\int \frac{\partial u^2\left(c^2(s)\right)}{\partial c^2(s)}\frac{dc^2(s)}{d\alpha}f(s)\,ds = \lambda^2\left[\frac{dt}{d\alpha} + \alpha\frac{da}{d\alpha} + a + \int \omega^2(s)\,s\,ds - \psi'(a)\frac{da}{d\alpha}\right].
\end{aligned}$$



(a) Aggregate Efficiency Decomposition: $\Xi^{AE} = \Xi_c^{AE} + \Xi_s^{AE}$



(b) Risk-Sharing Decomposition: $\Xi^{RS} = \Xi_c^{RS} + \Xi_s^{RS}$



(c) Equilibrium Comparative Statics

Figure A.1: Varying Performance Precision: Ξ^{AE} vs. Ξ^{RS} (Application 2)

Note: This figure shows the efficiency/willingness-to-pay and equilibrium comparative statics induced by changing the output precision.

The Lagrange multiplier here is the ratio of the individual numeraires, since

$$\frac{\partial \mathcal{L}}{\partial t} = \int \frac{\partial u^1\left(c^1(s)\right)}{\partial c^1(s)} \frac{dc^1(s)}{dt} f(s)\, ds + \phi \left[\int \frac{\partial u^2\left(c^2(s)\right)}{\partial c^2(s)} \frac{dc^2(s)}{dt} f(s)\, ds \right] = 0.$$

Note that $\frac{dc^1(s)}{dt} = -\frac{dc^2(s)}{dt} = -1$ which yields that

$$\phi = \frac{\int \frac{\partial u^1\left(c^1(s)\right)}{\partial c^1(s)} f(s)\, ds}{\int \frac{\partial u^2\left(c^2(s)\right)}{\partial c^2(s)} f(s)\, ds} = \frac{\lambda^1}{\lambda^2} = \frac{1}{\lambda^2}$$

Therefore,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \alpha} &= \int \frac{\partial u^1\left(c^1(s)\right)}{\partial c^1(s)} \frac{dc^1(s)}{d\alpha} f(s)\, ds + \phi \left[\int \frac{\partial u^2\left(c^2(s)\right)}{\partial c^2(s)} \frac{dc^2(s)}{d\alpha} f(s)\, ds \right] = 0 \\ &= \left(1-\psi'(a)\right) \frac{da}{d\alpha} + \int \omega^2(s)\, s ds = 0\end{aligned}$$

which yields that $\int \omega^2(s)\, s ds = -\left(1-\psi'(a)\right) \frac{da}{d\alpha}$.

Recall that we wanted to prove that $\frac{d\alpha}{d\tau} \int \omega^2(s)\, s ds = -\left(1-\psi'(a)\right) \frac{da}{d\tau}$ which is equivalent to

$$\frac{d\alpha}{d\tau} \int \omega^2(s)\, s ds = -\left(1-\psi'(a)\right) \frac{da}{d\alpha}\frac{d\alpha}{d\tau} \Rightarrow \frac{d\alpha}{d\tau} \left[\int \omega^2(s)\, s ds + \left(1-\psi'(a)\right) \frac{da}{d\alpha} \right] = 0$$

and we have already shown that $\int \omega^2(s)\, s ds + \left(1-\psi'(a)\right) \frac{da}{d\alpha} = 0$ due to the optimal contract formulation. This proves that the efficiency gains are mainly driven by information.

We can now establish that Ξ_c^{RS} takes the form

$$\begin{aligned}\Xi_c^{RS} &= \int \omega(s)\, \mathbb{C}ov_i^{\Sigma} \left[\frac{\omega^i(s)}{\omega(s)}, \frac{dc^i(s)}{d\tau} \right] ds \\ &= \int \omega^1(s) \frac{dc^1(s)}{d\tau} ds + \int \omega^2(s) \frac{dc^2(s)}{d\tau} ds - \left(1-\psi'(a)\right) \frac{da}{d\tau} \\ &= \frac{d\alpha}{d\tau} \left[\underbrace{\int \omega^1(s) \frac{dc^1(s)}{d\alpha} ds + \int \omega^2(s) \frac{dc^2(s)}{d\alpha} ds}_{=0} \right] - \left(1-\psi'(a)\right) \frac{da}{d\tau} \\ &= -\left(1-\psi'(a)\right) \frac{da}{d\tau}\end{aligned}$$

which proves that $\Xi^c = \Xi_c^{AE} + \Xi_c^{RS} = 0$.

We can now show that aggregate efficiency is always positive. First, note that

$$\Xi_c^{AE} = \left(1-\psi'(a)\right) \frac{da}{d\tau} = (1-\alpha) \frac{1}{\kappa} \frac{d\alpha}{d\tau} > 0.$$

Second, note that

$$\begin{aligned}
\Xi_s^{AE} &= \int \omega(s)\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right) ds \\
&= \frac{1}{2}\int \underbrace{\omega^1(s)}_{=f(s)}\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right) ds + \frac{1}{2}\int \omega^2(s)\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right) ds \\
&= \frac{1}{2}\int \left(-\frac{dF(s)}{d\tau}\right) ds + \frac{1}{4\tau}\int \omega^2(s)\left[\underline{s}\frac{f(\underline{s})}{f(s)} - s\right] ds \\
&= \frac{1}{4\tau}\underline{s}f(\underline{s})(\overline{s}-\underline{s}) + \frac{\underline{s}f(\underline{s})}{4\tau}\int \frac{\omega^2(s)}{f(s)} ds - \frac{1}{4\tau}\underbrace{\int \omega^2(s)\, s ds}_{=-(1-\psi'(a))\frac{da}{d\alpha}} \\
&= \frac{1}{4\tau}\underline{s}f(\underline{s})(\overline{s}-\underline{s}) + \frac{\underline{s}f(\underline{s})}{4\tau}\int \frac{\omega^2(s)}{f(s)} ds + \frac{1}{4\tau}(1-\psi'(a))\frac{da}{d\alpha} > 0,
\end{aligned}$$

where the t two terms are zero, since $f(\underline{s}) = 0$. Thus,

$$\Xi_s^{AE} = \frac{1}{4\tau}(1-\psi'(a))\frac{da}{d\alpha} > 0 \Rightarrow \Xi^{AE} > 0.$$

Finally, we show that risk-sharing can take different signs. First, note that

$$\Xi_c^{RS} = \int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)}, \frac{dc^i(s)}{d\tau}\right] ds = -(1-\psi'(a))\frac{da}{d\tau} = -(1-\alpha)\frac{1}{\kappa}\frac{d\alpha}{d\tau} < 0.$$

Second, note that

$$\begin{aligned}
\Xi_s^{RS} &= \int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)}, \frac{dc^i(s)}{ds}\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right)\right] ds \\
&= (1-\alpha)\int \omega^1(s)\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right) ds + \alpha\int \omega^2(s)\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right) ds - \int \omega(s)\left(-\frac{\frac{dF(s)}{d\tau}}{f(s)}\right) ds \\
&= \left(\frac{1}{2\tau}\right)\left[\alpha - \frac{1}{2}\right](1-\psi'(a))\frac{da}{d\alpha} \\
&= \left(\frac{1}{2\tau}\right)\left[\alpha - \frac{1}{2}\right](1-\alpha)\frac{da}{d\alpha} < 0 \Leftrightarrow \alpha < \frac{1}{2},
\end{aligned}$$

where we followed the same steps as in the proof of Ξ_s^{AE}. This shows that $\Xi_s^{RS} < 0$ as long as $\alpha < 0.5$. Therefore, whenever $\alpha < 0.5$, risk-sharing is negative. But risk-sharing can get slightly positive for high values of α. We can define a threshold for sensitivity α above which risk-sharing is positive. Formally, risk-sharing is given by

$$\Xi^{RS} = \left(\frac{1}{2\tau}\right)\left[\alpha - \frac{1}{2}\right](1-\alpha)\frac{1}{\kappa} - (1-\alpha)\frac{1}{\kappa}\frac{d\alpha}{d\tau}$$

Let $\overline{\alpha}$ denote the sensitivity threshold where $\Xi^{RS} = 0$, which is given by

$$\overline{\alpha} = \frac{1}{2} + 2\tau\frac{d\alpha}{d\tau}$$

Therefore, risk-sharing is negative whenever $\alpha < \overline{\alpha}$, and is positive whenever $\alpha > \overline{\alpha}$.

C Proofs and Derivations: Section 4

Proof of Proposition 2 We manipulate the first integral in Equation (17), which can be expanded as follows:

$$\int \frac{u_i\left(c^i(s)\right)}{\lambda^i}\frac{d\pi^i(s)}{d\theta}ds = \iint \frac{u_i\left(c^i(\xi, s_{-\xi})\right)}{\lambda^i}\frac{df^i(\xi|s_{-\xi})}{d\theta}\pi^i(s_{-\xi})\,d\xi ds_{-\xi}$$
$$= \int\left[\int_{\underline{\xi}}^{\bar{\xi}} \frac{u_i\left(c^i(\xi, s_{-\xi})\right)}{\lambda^i}\frac{df^i(\xi|s_{-\xi})}{d\theta}d\xi\right]\pi^i(s_{-\xi})\,ds_{-\xi} \tag{A.1}$$

For any fixed $s_{-\xi}$, let $U(\xi) = \frac{u_i\left(c^i(\xi, s_{-\xi})\right)}{\lambda^i}$ and $V(\xi) = \frac{dF^i(\xi|s_{-\xi})}{d\theta}$. Now the term in square brackets in (A.1) can be expressed as

$$\int_{\underline{\xi}}^{\bar{\xi}} U(\xi)\,V'(\xi)\,d\xi = U(\bar{\xi})\,V(\bar{\xi}) - U(\underline{\xi})\,V(\underline{\xi}) - \int_{\underline{\xi}}^{\bar{\xi}} U'(\xi)\,V(\xi)\,d\xi$$

using integration by parts. Notice that for all θ, we have $F^i\left(\bar{\xi}|s_{-\xi}\right) \equiv 1$ and $F^i\left(\underline{\xi}|s_{-\xi}\right) \equiv 0$. We then have $V\left(\bar{\xi}\right) = V\left(\underline{\xi}\right) = 0$, so we get

$$\int_{\underline{\xi}}^{\bar{\xi}} U(\xi)\,V'(\xi)\,d\xi = -\int_{\underline{\xi}}^{\bar{\xi}} U'(\xi)\,V(\xi)\,d\xi$$
$$= \int_{\underline{\xi}}^{\bar{\xi}} \frac{u_i'\left(c^i(\xi, s_{-\xi})\right)}{\lambda^i}\frac{dc^i(\xi, s_{-\xi})}{d\xi}\left(-\frac{dF^i(\xi|s_{-\xi})}{d\theta}\right)d\xi$$

Substituting into (A.1) we find that

$$\int \frac{u_i\left(c^i(s)\right)}{\lambda^i}\frac{d\pi^i(s)}{d\theta}ds = \int\left[\int_{\underline{\xi}}^{\bar{\xi}} \frac{u_i'\left(c^i(\xi, s_{-\xi})\right)}{\lambda^i}\frac{dc^i(\xi, s_{-\xi})}{d\xi}\left(-\frac{dF^i(\xi|s_{-\xi})}{d\theta}\right)d\xi\right]\pi^i(s_{-\xi})\,ds_{-\xi}$$
$$= \iint \frac{u_i'\left(c^i(\xi, s_{-\xi})\right)}{\lambda^i}\frac{dc^i(\xi, s_{-\xi})}{d\xi}\left(\frac{-\frac{dF^i(\xi|s_{-\xi})}{d\theta}}{f^i(\xi|s_{-\xi})}\right)f^i(\xi|s_{-\xi})\,\pi^i(s_{-\xi})\,d\xi ds_{-\xi}$$
$$= \int \frac{u_i'\left(c^i(s)\right)}{\lambda^i}\frac{dc^i(s)}{d\xi}\left(\frac{-\frac{dF^i(\xi|s_{-\xi})}{d\theta}}{f^i(\xi|s_{-\xi})}\right)\pi^i(s)\,ds$$

Finally, substituting into Equation (17), we obtain

$$\frac{1}{\lambda^i}\frac{dV^i}{d\theta} = \int \frac{u_i\left(c^i(s)\right)}{\lambda^i}\frac{d\pi^i(s)}{d\theta}ds + \int \frac{u_i'\left(c^i(s)\right)}{\lambda^i}\frac{dc^i(s)}{d\theta}\pi^i(s)\,ds$$
$$= \int \frac{u_i'\left(c^i(s)\right)}{\lambda^i}\left[\left(\frac{-\frac{dF^i(\xi|s_{-\xi})}{d\theta}}{f^i(\xi|s_{-\xi})}\right)\times\frac{dc^i(s)}{d\xi} + \frac{dc^i(s)}{d\theta}\right]\pi^i(s)\,ds$$

which establishes the result. The results for Applications 3 and 4 are to be included.

D Extensions

D.1 Discrete States

The counterpart to Equation (1) in the discrete case is given by

$$V = u(c_0) + \beta \sum_s \pi(s;\theta) u(c_1(s)),$$

where $\pi(s)$ denotes the probability of one of the countably many states $s \in \{1, \ldots, S\}$. The agent's willingness-to-pay for a marginal change in probabilities satisfies the formula

$$p_\theta = \sum_s \frac{d\pi(s;\theta)}{d\theta} \frac{\beta u(c_1(s))}{u'(c_0)}, \tag{A.2}$$

where it must be that $\sum_s \frac{d\pi(s)}{d\theta} = 0$. Summation by parts implies that

$$\sum_s u(c_1(s)) \frac{d\pi(s)}{d\theta} = \sum_{s=0}^{S-1} (u(c_1(s)) - u(c_1(s+1))) \frac{d\Pi(s;\theta)}{d\theta},$$

where $\frac{d\Pi(s;\theta)}{d\theta} = \sum_{u=1}^{s} \frac{d\pi(u;\theta)}{d\theta}$ denotes the change in cdf, and where we use the fact that $\frac{d\Pi(S)}{d\theta} = 0$. Therefore, we can write express p_θ as

$$p_\theta = \beta \sum_{s=0}^{S-1} \pi(s;\theta) \left(\frac{u(c_1(s+1)) - u(c_1(s))}{u'(c_0)} \right) \frac{\frac{d(1-\Pi(s;\theta))}{d\theta}}{\pi(s;\theta)},$$

which is the exact counterpart of Equation (4). Formally as the difference between $s+1$ and s becomes small,

$$u(c_1(s+1)) - u(c_1(s)) \to u'(c_1(s)) \frac{dc_1(s)}{ds}.$$

D.2 Leisure

When preferences include both consumption and leisure, then $U(s) = \beta u(c_1(s), n_1(s))$, so

$$\frac{dU(s)}{ds} = \beta \frac{\partial u(c_1(s))}{\partial c_1(s)} \left(\frac{dc_1(s)}{ds} + \frac{\frac{\partial u(n_1(s))}{\partial n_1(s)}}{\frac{\partial u(c_1(s))}{\partial c_1(s)}} \frac{dn_1(s)}{ds} \right). \tag{A.3}$$

The term in parentheses in Equation (A.3) corresponds to a consumption-equivalent, expressing changes in $n_1(s)$ in consumption units. Hence, all results in the paper apply using this leisure-augmented consumption-equivalent.

D.3 Redistributional Concerns

In the body of the paper, we have exclusively focused on characterizing (Kaldor-Hicks) efficiency. It is however straightforward to compute welfare gains for any welfarist social welfare function, given by $W = \mathcal{W}(V^1, \ldots, V^I)$, using the efficiency/redistribution decomposition in Dávila and Schaab (2024). Formally, the normalized marginal social welfare effects of any perturbation for any welfarist social welfare function can be expressed as

$$\frac{dW^{\lambda}}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{I \cdot \mathbb{C}ov_i \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{(Redistribution)}}, \tag{A.4}$$

where $\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i}$ and $\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}$. This is the unique decomposition in which a normalized welfare assessment can be expressed as Kaldor-Hicks efficiency, Ξ^E, and its complement, Ξ^{RD}. The choice of λ^i simply accounts for the chosen unit to make interpersonal comparisons (welfare numeraire). Hence, perturbations in which $\Xi^E > 0$ can be turned into Pareto improvements if transfers are feasible and costless. Given (A.4), it is thus straightforward to augment our analysis to also draw insights over Ξ^{RD}.