Intergenerational Welfare Assessments*

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Abstract

This paper studies welfare assessments in economies with rich demographics. We introduce the notion of demographically disconnected economies, those with no date at which all individuals are concurrently alive. We identify the unique class of units that enables meaningful welfare comparisons in such economies: those based on perpetual consumption, that is, consumption at all dates. Using this metric, we uncover a novel possibility: feasible perturbations of Pareto efficient allocations can yield Kaldor-Hicks efficiency gains. We also introduce a decomposition that attributes intertemporal-sharing efficiency gains to financial frictions or demographic differences. These results allow us to derive new insights in three workhorse intergenerational models: (i) Samuelson (1958) two-date-life model, offering a novel rationale for social security; (ii) Diamond (1965) growth model, providing a new theory for capital taxation and capital over-/under-accumulation; and (iii) Samuelson (1958) three-date-life model, decomposing the efficiency gains from intergenerational transfers into frictional and demographic sources.

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1 Introduction

Making interpersonal welfare comparisons across individuals who are not concurrently alive is conceptually challenging but essential for evaluating policies in areas such as social security, public debt, education, and climate change. Yet, existing work on these questions typically avoids welfare comparisons across individuals from different generations. The alternative is to rely on the Pareto criterion, which — though widely accepted — is often inconclusive.

In this paper, we systematically explore how welfarist planners — those who evaluate outcomes using a social welfare function — conduct welfare assessments in economies in which individuals are born and die at different times. We introduce our results in an environment with a flexible demographic structure that allows for arbitrary birth and death patterns. A key notion in our analysis is that of *demographically disconnected* economies, which we define as economies where there is no date at which all individuals are concurrently alive.

Valid Unit for Interpersonal Welfare Comparisons. Meaningful interpersonal welfare comparisons require choosing a unit, that is, a welfare numeraire, since individual utilities are ordinal and not directly comparable across individuals. Our first main result identifies the *unique* class of units that always enables such comparisons in demographically disconnected economies: those based on *perpetual consumption*, that is, consumption at all dates. Intuitively, only bundles that include consumption at all dates are intrinsically valued by all individuals, alive and yet-to-be-born. For instance, individuals born at date 1 derive no value from date-0 consumption, while individuals dead with certainty by date 7 derive no value from date-9 consumption. But each individual has positive value for a bundle that includes consumption at all dates. This result contrasts with demographically connected economies, like those with infinitely-lived agents. In those economies, other units — such as date-0 consumption, date-t consumption, or bundles of consumption spanning multiple dates — can also be used to make interpersonal comparisons.

Kaldor-Hicks Improvements from Pareto Efficient Allocations. Our second main result shows that there exist *feasible* perturbations of Pareto efficient allocations in demographically disconnected economies that feature Kaldor-Hicks efficiency gains.[1] That is, it is possible to move away from a Pareto efficient allocation in a demographically disconnected economy in a way that the sum of individual welfare changes is strictly positive. This result

[1]A perturbation features Kaldor-Hicks efficiency gains if the sum of individual welfare changes, expressed in a common valid unit, is strictly positive. This implies that the winners could hypothetically compensate the losers if transfers in the chosen unit were feasible.

critically hinges on expressing Kaldor-Hicks efficiency gains in terms of perpetual consumption — a bundle whose elements, namely consumption at specific dates, are not valued by every individual. But there is no way around this: as already established, demographically disconnected economies require welfare numeraires based on perpetual consumption.

This seemingly paradoxical result contrasts with economies where all individuals consume at every date. In such settings, no feasible perturbation of a Pareto efficient allocation can yield Kaldor-Hicks efficiency gains, regardless of the chosen welfare numeraire. In general, when a perturbation generates Kaldor-Hicks efficiency gains, the compensation principle calls for reallocating resources from winners to losers to engineer a Pareto improvement. Yet such transfers are *not feasible* in demographically disconnected economies because they would require transferring consumption away from the dead, violating consumption non-negativity constraints. That is, while welfare changes can be expressed and aggregated in terms of perpetual consumption, consumption non-negativity constraints prevent perpetual consumption from actually being transferred.

This result opens the door to reconsidering the role of government intervention in settings where demographics are relevant. Even when the first welfare theorem holds, finding feasible Kaldor-Hicks improvements from Pareto efficient allocations suggests that there exist untapped sources of value in demographically disconnected economies that only government policies can potentially capture. Our applications present two scenarios in which this insight can be used to rationalize specific policies, such as social security or capital taxation.

Intertemporal-Sharing Decomposition: Frictions vs. Demographics. Intertemporal-sharing efficiency gains arise from reallocating consumption toward individuals who value it more at specific dates. In economies where all individuals consume at all times, such differences in valuations can only reflect financial frictions. But in demographically disconnected economies, demographic differences also lead to valuation gaps. Even when all financial trades are feasible — so that individuals value consumption identically whenever they are concurrently alive — differences in lifespans lead them to value perpetual consumption bundles differently. As a result, valuations for consumption at a given date relative to a bundle based on perpetual consumption typically differ, creating scope for intertemporal-sharing gains.

Our third main result introduces a decomposition that systematically separates the sources of intertemporal-sharing efficiency gains into two components: i) *frictional intertemporal-sharing* (FIS), which captures the gains from reallocating consumption towards individuals who value consumption differently across dates due to financial frictions and ii) *demographic* *intertemporal-sharing* (DIS), which captures the gains from reallocating consumption towards individuals whose valuations differ due to demographic differences. To our knowledge, this is the first decomposition that systematically distinguishes whether efficiency gains arise because consumption is reallocated towards individuals who i) face financial frictions, addressing inefficiencies that individuals could in principle correct via contracts or markets, or ii) differ demographically, where gains require policy intervention.

Applications. We present three applications that demonstrate the practical relevance of our results. To highlight that our findings are not driven by double-infinity arguments (Shell, 1971; Geanakoplos, 1989), all three applications feature a finite terminal date.

Our first application conducts a welfare assessment of intergenerational transfers in the simplest overlapping generations (OLG) economy: the *two-date-life* version of Samuelson (1958)'s endowment economy. A central takeaway from this application is that a policy of young-to-old transfers — modeled to mimic a pay-as-you-go social security system — generates Kaldor-Hicks efficiency gains over the unique competitive equilibrium, which is Pareto efficient. Since this is a pure endowment economy with complete markets, all efficiency gains stem from demographic intertemporal-sharing (DIS), by reallocating consumption towards individuals whose valuations differ due to demographic differences.

While the social security policy that we study does not constitute a Pareto improvement, our analysis provides a justification for it on the grounds that the sum of willingness-to-pay for it among all individuals, expressed in units of perpetual aggregate consumption, is strictly positive. In our calibration, the efficiency-maximizing transfer is roughly equivalent to a permanent 12.5% increase in aggregate consumption evenly distributed across all individuals. This application illustrates Proposition 1, as we use perpetual aggregate consumption as welfare numeraire, Proposition 2, by showing that social security is a policy that generates Kaldor-Hicks efficiency gains from a Pareto efficient allocation, and Proposition 3, by establishing that such gains are solely due to demographic differences (DIS), as markets are complete and there are no financial frictions.

Our second application establishes that in the finite-horizon version of Diamond (1965)'s OLG growth model, the efficiency-maximizing capital tax is positive in low-capital-share economies, which over-accumulate capital, and negative (a subsidy) in high-capital-share economies, which under-accumulate capital. These results hold even though the competitive equilibrium is Pareto efficient, providing a new illustration of Proposition 2 by showing that a production economy at a Pareto-efficient allocation can still be perturbed to yield

Kaldor–Hicks efficiency gains. We show that intertemporal-sharing considerations — again solely of demographic nature (DIS) — work against aggregate-efficiency effects, dampening the optimal tax relative to what would be implied by investment efficiency alone, while redistribution gains are positive in both economies for an undiscounted utilitarian planner.

This application revisits the question of the efficiency of capital accumulation. Prior work, including Diamond (1965) and Abel et al. (1989), has focused on identifying conditions under which Pareto improvements are possible, with Abel et al. (1989) calling for the use of social welfare functions to evaluate alternative dynamic paths.[2] Our analysis takes up this challenge by moving beyond the Pareto criterion and using welfare and efficiency gains to assess alternative capital-accumulation paths in a production economy. In doing so, it connects to the Golden Rule literature initiated by Phelps (1961), which asks whether an economy accumulates "too much" or "too little" capital. Our decomposition addresses this classic question in two ways: i) aggregate-efficiency accounts for the entire transition path and moves beyond steady-state comparisons, and ii) intertemporal-sharing captures efficiency gains arising from heterogeneity in valuations across generations.

Our third and last application revisits the welfare assessment of intergenerational transfers in the simplest OLG economy in which financial markets can be used to smooth consumption: the *three-date-life* version of Samuelson (1958)'s endowment economy. Starting from a hump-shaped consumption profile and absent financial markets, we consider a policy that transfers resources from middle-aged to old, mimicking again a pay-as-you-go social security system. A central takeaway is that the justification for this policy differs fundamentally from that of the young-to-old transfer in Application 1. Both interventions generate intertemporal-sharing gains, but for entirely different reasons: in Application 3, frictional intertemporal-sharing (FIS) gains outweigh demographic intertemporal-sharing (DIS) losses, whereas in Application 1 all gains are demographic, with no contribution from FIS. This application underscores the importance of decomposing intertemporal-sharing gains into frictional and demographic components to understand the justification for policies in economies with incomplete markets and rich demographics, as we do in Proposition 3.

[2] Abel et al. (1989) conclude their study of efficiency in OLG economies with this observation:

> *"The most important direction for future research is the evaluation of alternative dynamic paths using stronger criteria than the dynamic efficiency criterion. Our criterion is the dynamic analogue of the standard Pareto criterion. (...) The use of social welfare functions would make possible the evaluation of alternative social decision rules for determining the level of investment."*

Related Literature. Our analysis contributes to the literature that studies the efficiency of OLG economies, following Samuelson (1958) and Diamond (1965), the two models which underpin our applications. These ideas have made their way to macroeconomics textbooks, including Blanchard and Fischer (1989), De La Croix and Michel (2002), Romer (2006), and Acemoglu (2009), among others. Spear and Young (2023) comprehensively review the development of these ideas.

A key normative insight of the OLG literature is the potential suboptimality of competitive equilibria. Geanakoplos (1989) concludes that this inefficiency result necessitates i) a double infinity of individuals and goods, and ii) overlapping lifespans. Our results are orthogonal to the question of suboptimality, and apply to finite-horizon economies, in which the first welfare theorem holds. Therefore, our results should be read as characterizing special normative properties of Arrow-Debreu economies in which there is no commodity liked by all individuals.

Prior work has also recognized the importance of the demographic structure in OLG economies Weil (1989), in particular, points out that the arrival of new individuals (dynasties) not linked to older cohorts is necessary to generate asset bubbles, dynamic inefficiency, and violations of Ricardian equivalence. While the continuous arrival of new individuals makes an economy demographically disconnected, the notion of demographically disconnected economies that we introduce in this paper is, to our knowledge, not present in prior work.[3]

A separate body of work studies intergenerational welfare criteria, including for instance, the work by Calvo and Obstfeld (1988) and Eden (2023), among others. Our approach takes the social objective as given, but our emphasis on making interpersonal comparisons in a common unit leads us to establish that valid welfare numeraires must be based on perpetual consumption. A different approach is pursued by Aguiar, Amador and Arellano (2023), who characterize Robust Pareto Improvements (RPI) in OLG economies, ensuring that the budget set of any agent is guaranteed to be weakly expanded at any state and time. An advantage of the RPI criterion is that it ensures a Pareto improvement regardless of how agents trade off consumption intertemporally or across states. Our results are complementary, as they are most useful for evaluating perturbations that do not constitute Pareto improvements.

The importance of population growth for welfare gains has been emphasized by Jones and Klenow (2016) and Adhami et al. (2024), among others. We hope that the results of this paper motivate future research on the welfare implications of population dynamics.

[3] McKenzie (1959)'s notion of a connected economy refers to the interdependence of goods in preferences or production, ensuring that no group of goods is isolated from the rest of the economy. Demographically disconnected economies are typically connected in the sense of McKenzie (1959).

2 Environment

Our notation closely follows Chapter 8 of Ljungqvist and Sargent (2018). We consider a deterministic discrete-time setting with dates indexed by $t \in \{0, \ldots, T\}$, where $T \leq \infty$.

Demographics. The economy is populated by a countable number of individuals, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$, where $1 \leq I \leq \infty$. Each individual i is associated with a date of birth $\tau_b^i \in \{-\infty, \ldots, T\}$, where a negative date of birth means that the individual is already alive at date 0, and a date of death $\tau_d^i \in \{1, \ldots, T\}$, where $\tau_b^i \leq \tau_d^i$

Preferences. An individual i derives utility from consumption, with a lifetime utility

$$\text{(Preferences)} \qquad V^i = \sum_{t=\tau_b^i}^{T} \left(\beta^i\right)^{t-\tau_b^i} u_t^i\left(c_t^i\right), \tag{1}$$

where $\beta^i \in [0, 1)$ denotes individual i's discount factor, and $u_t^i(\cdot)$ and c_t^i respectively correspond to individual i's instantaneous utility and consumption at date t. Whenever individual i is not alive, $u_t^i\left(c_t^i\right) = 0$. Whenever individual i is alive, $u_t^i\left(c_t^i\right)$ is well behaved, so that $\frac{\partial u_t^i(\cdot)}{\partial c_t^i} > 0$ and an Inada condition applies.[4] We refer to the unit of V^i as individual i's utils.

Social Welfare Function. We study welfare assessments for welfarist planners, those who evaluate outcomes using a social welfare function given by

$$\text{(Social Welfare Function)} \qquad W = \mathcal{W}\left(V^1, \ldots, V^i, \ldots, V^I\right), \tag{2}$$

where individual lifetime utilities V^i are defined in (1). We assume that $\frac{\partial \mathcal{W}}{\partial V^i} > 0$, $\forall i$, so that the planner values all individuals alive and yet-to-be-born at date 0. We refer to the units of W as social utils.

Defining social welfare functions when there are multiple generations is analogous to doing so in static environments or environments with infinitely-lived agents, as explained, for instance, in Blanchard and Fischer (1989). However, it may be necessary to restrict the form of $\mathcal{W}(\cdot)$ in some infinite-horizon economies to ensure that W is finite.[5] The welfarist approach is widely used because it is Paretian, that is, it concludes that every Pareto-improving

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[4]We can equivalently express individual preferences as $V^i = \sum_{t=\tau_b^i}^{t=\tau_d^i} \left(\beta^i\right)^{t-\tau_b^i} u_t^i\left(c_t^i\right)$.

[5]The most common intergenerational social welfare functions are i) discounted utilitarian, which exponentially discounts future generations' utility, and ii) undiscounted utilitarian, which gives all individuals equal utility weight. We use both in our applications.

perturbation is desirable, and because nonwelfarist approaches violate the Pareto principle (Kaplow and Shavell, 2001).

Remarks on the Environment. To simplify the exposition, in the body of the paper we study a deterministic environment with a single consumption good and a countable number of individuals. In the Online Appendix, we show how to accommodate demographic uncertainty, which is useful to model random deaths, as in Yaari (1965) and Blanchard (1985), non-demographic uncertainty, multiple consumption goods, and factor supplies. It is straightforward to accommodate a continuum of individuals, dates, and histories. While much of the existing work on economies with rich demographics emphasizes the case where T is infinite, our results equally apply when T is finite or infinite.

Demographic Disconnect. Throughout the paper, we rely on the notion of demographically disconnected economies, which we introduce here.

Definition. *(Demographic Disconnect) An economy is demographically disconnected if there is no date at which all individuals are concurrently alive.*

The key property of demographically disconnected economies is the absence of a date at which all individuals — both currently alive and those yet-to-be-born — assign positive value to consumption. Demographic disconnect is a realistic assumption and a common feature of economies with rich demographics. We illustrate this notion through an example that compares two economies with different demographic structures: see Table 1.

Example 1. *(Illustrating Demographic Disconnect)* Economy A has two individuals: individual $i = 1$, who is exclusively alive at date 0, and individual $i = 2$, who is alive at both dates 0 and 1. Economy A is demographically connected because all individuals are alive at date 0. Economy B is identical to economy A, except for the presence of a third individual $i = 3$, who is exclusively alive at date 1. The presence of this third individual makes Economy B demographically disconnected, as there is no date at which all individuals are concurrently alive: individual $i = 2$ is not alive at date 1, and individual $i = 3$ is not alive at date 0. The demographics of Economy B are the same as in the two-date-lives OLG model, which we study in Application 1.

It is worth making three remarks on the notion of demographic disconnect.

Table 1: Illustrating Demographic Disconnect

	Economy A		Economy B	
	$t=0$	$t=1$	$t=0$	$t=1$
$i=1$	$\checkmark$	$\times$	$\checkmark$	$\times$
$i=2$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$i=3$			$\times$	$\checkmark$

Note: Economy A has two individuals: $i \in \{1,2\}$. Economy B has three individuals: $i = \{1,2,3\}$. Checks ($\checkmark$) indicate dates when an individual is alive, while crosses ($\times$) indicate dates when an individual is not alive. Economy A is demographically connected since all individuals are concurrently alive at date 0. Economy B is demographically disconnected since there is no date at which all individuals are concurrently alive: individual $i=2$ is not alive at date 1 and individual $i=3$ is not alive at date 0.

Remark 1. *(Preference Disconnect)* Static economies with multiple consumption goods can exhibit patterns formally analogous to demographically disconnected economies when individuals have preferences that assign zero value to consuming some goods. In Section C.3 of the Online Appendix, we develop this analogy and refer to it as *"preference disconnect"*. Once suitably adapted, our results extend to such settings. However, despite the formal analogy, real-world economies are demographically disconnected at least when individuals are not altruistic — see next remark. In contrast, real-world economies seem preference connected, as every individual has positive value for, say, water.

Remark 2. *(Intergenerational Altruism)* Our definition of demographic disconnect assumes that individuals are self-interested and do not value the consumption of future generations. In some circumstances, intergenerational altruism can potentially transform demographically disconnected economies into demographically connected economies because individuals who care about their descendants assign positive value to consumption that occurs after their own death. The notion of demographic disconnect can be extended to economies with intergenerational altruism by requiring that there is no date at which all individuals have positive value for consumption, either privately or altruistically.

Remark 3. *(Demographic Disconnect Requires Births)* Demographic disconnect always requires births after date 0: if all individuals are alive at date 0 — even if they die over time — the economy is demographically connected at date 0. When the time horizon is finite ($T < \infty$), deaths are also necessary for disconnect since otherwise all born individuals would be concurrently alive at the terminal date T. But if $T = \infty$, this logic does not apply: demographic disconnect can arise even if all individuals live forever, provided that new individuals continue to be born over time.

3 Intergenerational Welfare Assessments

3.1 Unique Class of Valid Welfare Numeraires

Our goal is to understand why a particular welfarist planner finds a given perturbation desirable. Formally, a perturbation corresponds to a smooth change in c_t^i as a function of a perturbation parameter $\theta \in [0,1]$, so derivatives such as $\frac{dc_t^i}{d\theta}$ are well-defined.[6] A perturbation may capture changes in policies, technologies, endowments, or any other primitive of a fully specified model. It can also capture changes in feasible allocations directly selected by a planner, as in the solution to a planning problem.

A welfarist planner finds a perturbation $d\theta$ desirable (undesirable) if

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} > (<)\, 0. \tag{3}$$

However, (3) cannot be immediately used to understand how a welfarist planner makes tradeoffs in meaningful units across individuals, because individual utilities are ordinal and not directly comparable. Meaningful comparisons across individuals require choosing a common unit to express individual welfare gains — a welfare numeraire. The only requirement for a welfare numeraire to be valid is that all individuals — alive and yet-to-be-born — must have a positive value for it.

In demographically disconnected economies, there is a unique class of units in which it is always possible to make interpersonal welfare comparisons: those based on perpetual consumption, that is, consumption at all dates. Identifying this class is the first main result of the paper, formalized in Proposition 1.

Proposition 1. *(Unique Class of Valid Welfare Numeraires) Only welfare numeraires based on perpetual consumption are always valid in demographically disconnected economies.*

Intuitively, only bundles that include consumption at all dates are intrinsically valued by all individuals, alive and yet-to-be-born. For instance, individuals born at date 1 derive no value from date-0 consumption, while individuals dead with certainty by date 7 derive no value from date 9 consumption. But each individual has positive value for a bundle that includes consumption at all dates, even when they do not value every element of the bundle. In contrast, in demographically connected economies — like those with infinitely-lived agents — there is an

[6] A perturbation is strictly (weakly) Pareto-improving when $\frac{dV^i}{d\theta} > (\geq)\, 0$, $\forall i$, with at least one strict inequality. Equation (3) and $\frac{\partial \mathcal{W}}{\partial V^i} > 0$ imply that $\frac{dW}{d\theta} > 0$ for Pareto-improving perturbations.

abundance of possible welfare numeraires. For example, consumption at any connected date or over a subset of those dates are valid welfare numeraires in those economies.[7]

While Propositions 2 and 3 are valid for any numeraire in the class identified in Proposition 1, to actually compute welfare gains it is necessary to specify a particular bundle of perpetual consumption. In the spirit of Lucas (1987) and Alvarez and Jermann (2004), it seems natural to choose *perpetual aggregate consumption* — the bundle that consists of aggregate consumption at each date — as welfare numeraire.[8] Hence, we can express $\frac{dW}{d\theta}$ using perpetual aggregate consumption as welfare numeraire as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \lambda^i = \sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t, \tag{4}$$

and where $c_t = \sum_i c_t^i$ denotes date t aggregate consumption. The normalizing factor λ^i corresponds the individual i's marginal valuation for perpetual aggregate consumption consumption. Hence, in (4), $\frac{\frac{dV^i}{d\theta}}{\lambda^i}$ corresponds to individual i's willingness-to-pay for the perturbation in units of the chosen welfare numeraire. That is, if $\frac{\frac{dV^i}{d\theta}}{\lambda^i} = 0.05$, then individual i's willingness-to-pay for the perturbation is 5% of perpetual aggregate consumption. The upshot of using this numeraire is that welfare and efficiency gains are easily interpretable, as we further explain below.

Normalized Welfare Gains and Weights. Just as we have expressed lifetime welfare gains in terms of aggregate consumption, we can also express welfare gains at a particular date in terms of aggregate consumption at that date. By expressing welfare changes in common units, the following Lemma — the counterpart to Lemma 1 in Dávila and Schaab (2024) using aggregate consumption as numeraire — facilitates the understanding of the sources of welfare gains. Notationally, variables with a λ superscript are expressed in the appropriate numeraire.

Lemma 1. *(Normalized Welfare Gains and Weights) A normalized welfare assessment for a*

[7] In competitive economies, one may consider expressing welfare gains in terms of monetary values or prices, that is, using "money-metric" welfare numeraires. While these welfare numeraires may be useful in particular scenarios, such as when markets are complete, they are not always well-defined in the absence of markets or when markets are frictional, in contrast to consumption-based welfare numeraires.

[8] In the Online Appendix, we derive (4) starting from a global representation of the aggregate welfare gain. We also show how to use unit perpetual consumption — the bundle that pays one unit of consumption at each date — as welfare numeraire. In economies without growth, unit and perpetual aggregate consumption yield identical results. In practice, we find minimal quantitative differences between the two.

welfarist planner can be represented as

$$\frac{dW^{\lambda}}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}, \tag{5}$$

where $\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i}$ *and* $\frac{dV_t^{i|\lambda}}{d\theta} = \frac{\frac{dV_t^i}{d\theta}}{\lambda_t^i}$ *respectively denote (normalized) lifetime and date welfare gains, given by*

$$\frac{dV^{i|\lambda}}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} \qquad \text{(Lifetime Welfare Gains)} \tag{6}$$

$$\frac{dV_t^{i|\lambda}}{d\theta} = \frac{1}{c_t}\frac{dc_t^i}{d\theta}, \qquad \text{(Date Welfare Gains)} \tag{7}$$

where $\lambda_t^i = \frac{\partial u_t^i}{\partial c_t^i} c_t$, *and where* ω^i *and* ω_t^i *respectively denote (normalized) individual and dynamic weights, given by*

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t} \qquad \text{(Individual Weight)} \tag{8}$$

$$\omega_t^i = \frac{\left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t}. \qquad \text{(Dynamic Weight)} \tag{9}$$

This lemma is useful because it shows that welfare assessments can be interpreted as weighted sums across individuals and dates of date-individual-specific payoffs $\frac{dV_t^{i|\lambda}}{d\theta}$.[9] The individual weight ω^i defines how a planner trades off welfare gains across individuals. For example, if $\omega^i = 1.3$, the planner perceives providing individual i with a 1% increase in perpetual aggregate consumption as equivalent to providing everyone a 1.3% increase.[10] The dynamic weight ω_t^i measures how individual i trades off date t aggregate consumption against perpetual aggregate consumption. For example, if $\omega_t^i = 0.1$, the planner perceives a 1% increase in aggregate consumption assigned to individual i at date t as equivalent to a 0.1% increase in perpetual aggregate consumption assigned to that individual.

[9] In this case, defining individual i's consumption share by $\chi_t^i = \frac{c_t^i}{c_t}$, we have that

$$\frac{dV_t^{i|\lambda}}{d\theta} = \frac{1}{c_t}\frac{dc_t^i}{d\theta} = \frac{d\chi_t^i}{d\theta} + \chi_t^i \frac{d\ln c_t}{d\theta} = \chi_t^i \frac{d\ln c_t^i}{d\theta}.$$

[10] Note that individual weights average to one across individuals, so $\frac{1}{I}\sum_i \omega^i = 1$. The dynamic weights sum to one over time for each individual, defining a normalized discount factor: $\sum_t \omega_t^i = 1$, $\forall i$.

Efficiency vs. Redistribution. After expressing welfare assessments in comparable units, Lemma 2 derives Dávila and Schaab (2024) efficiency/redistribution decomposition in our environment. This is the unique decomposition in which a normalized welfare assessment can be expressed as Kaldor-Hicks efficiency and its complement.

Lemma 2. *(Efficiency/Redistribution Decomposition) A normalized welfare assessment for a welfarist planner can be — given a welfare numeraire — uniquely decomposed into (Kaldor-Hicks) efficiency and redistribution components, Ξ^E and Ξ^{RD}, as follows:*

$$\frac{dW^\lambda}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \ (Efficiency)} + \underbrace{\mathbb{C}ov_i^\Sigma \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \ (Redistribution)}, \tag{10}$$

where $\mathbb{C}ov_i^\Sigma [\cdot,\cdot] = I \cdot \mathbb{C}ov_i [\cdot,\cdot]$ denotes a cross-sectional covariance-sum among individuals.

The efficiency component Ξ^E corresponds to the sum of individual willingness-to-pay for a given perturbation, expressed in units of the welfare numeraire. Hence, perturbations with $\Xi^E > 0$ can be turned into Pareto improvements provided that transfers of the welfare numeraire are feasible and costless. This property typically makes Kaldor-Hicks efficiency — which is is invariant to i) the choice of social welfare function and ii) preference-preserving utility transformations — a useful notion to characterize Pareto frontiers.

The redistribution component Ξ^{RD} captures the equity concerns embedded in a particular social welfare function: Ξ^{RD} is positive when the individuals relatively favored in a perturbation are those relatively preferred by the planner, that is, have a higher individual weight ω^i. Because the precise efficiency/redistribution split depends on the choice of welfare numeraire, using perpetual aggregate consumption is natural because it yields a clear interpretation: for example, if $\frac{dW^\lambda}{d\theta} = 0.12$, then this is equivalent to 12% perpetual increase in aggregate consumption distributed equally across all individuals.

3.2 Kaldor-Hicks Improvements from Pareto Efficient Allocations

In economies where all individuals consume at every date, no feasible perturbation from a Pareto efficient allocation can feature Kaldor-Hicks efficiency gains, regardless of the welfare numeraire.[11] In these economies, every Pareto efficient allocation is a solution to a planning

[11]Pareto efficient allocations are those that are feasible — satisfying resources constraints and possibly production and accumulation technologies — and solve the Pareto problem; see e.g. Ljungqvist and Sargent (2018). It is possible to find perturbations of *constrained* Pareto efficient allocations that feature $\Xi^E > 0$.

problem supported by Pareto weights such that individual weights ω^i's are equalized across individuals for every welfare numeraire. This occurs because the planner can freely reallocate consumption at the margin across all individuals at all dates. Hence, since the solution to the planning problem requires that $\frac{dW^\lambda}{d\theta} \leq 0$ for any feasible perturbation and ω^i's are equalized across individuals ensuring that $\Xi^{RD} = 0$, then $\Xi^E \leq 0$ for every feasible perturbation.

Proposition 2, the second main result of the paper, shows that the logic outlined above does not extend to demographically disconnected economies.

Proposition 2. *(Kaldor-Hicks Improvements from Pareto Efficient Allocations) There exist feasible perturbations from Pareto efficient allocations in demographically disconnected economies that feature Kaldor-Hicks efficiency gains ($\Xi^E > 0$).*

To explain Proposition 2, it is useful to characterize the set of Pareto efficient allocations by solving the Pareto problem. Formally, this involves maximizing a weighted sum of individual utilities

$$\max_{\chi_t^i} \sum_i \alpha^i V^i,$$

where $\alpha^i > 0$ are Pareto weights, V^i is defined in (1), and $\chi_t^i = \frac{c_t^i}{c_t}$ denotes individual i's share of aggregate consumption $c_t = \sum_i c_t^i$ at date t. This problem is subject to resource constraints

$$\sum_i \chi_t^i = 1, \quad \forall t, \tag{11}$$

and consumption non-negativity constraints

$$\chi_t^i \geq 0, \quad \forall i, t. \tag{12}$$

Denoting by $\eta_t > 0$ the (normalized) Lagrange multiplier on the aggregate resource constraint for date-t consumption, the solution to the planning problem must satisfy

$$\omega^i \omega_t^i = \eta_t > 0 \quad \text{if} \quad c_t^i > 0, \quad \text{and} \quad \omega^i \omega_t^i < \eta_t \quad \text{if} \quad c_t^i = 0, \tag{13}$$

where ω^i and ω_t^i are defined in (8) and (9).[12] Intuitively, the planner reallocates consumption to equalize the social marginal value of consumption — given by $\omega^i \omega_t^i$ — across individuals

[12]While the full planning problem may include additional resource or constraints or production and accumulation technologies, it is sufficient to consider the constraints in (11) and (12) to establish our result.

alive at each date. Aggregating the optimality conditions for individual i yields

$$\omega^i \underbrace{\sum_t \omega_t^i}_{=1} = \omega^i = \sum_t \eta_t \mathbb{I}\left[\omega_t^i > 0\right], \tag{14}$$

where $\mathbb{I}[\cdot]$ denotes the indicator function. Equation (14) implies that individual weights ω^i differ across individuals with different lifespans, although they are equalized across individuals with identical lifespans. Consequently, (13) implies that dynamic weights ω_t^i also differ across individuals with different lifespans, which immediately implies that feasible perturbations — reallocating consumption at a specific date — with $\Xi^E > 0$ exist, yielding Proposition 2.[13] We summarize the implications of these findings in four remarks.

Remark 4. *(ω^i not Equalized: Demographic Pecking-order at Planning Solutions)* Because individual weights ω^i are not equalized, the planner has a pecking order over individuals with different lifespans after solving a Pareto problem. That is, the planner would prefer to assign a unit of the welfare numeraire to some individuals relative to others, depending on their lifespans. Equation (14) implies that the length of an individual's lifespan is a key determinant of the planner's pecking order, with longer-lived individuals carrying a higher individual weight all else equal, as illustrated in our applications. No pecking order exists after solving the planning problem in economies with infinitely-lived agents or in connected economies with a connected numeraire.

Remark 5. *(ω_t^i not Equalized: Efficiency Gains from Pareto Efficient Allocations)* Because individual weights ω_t^i are not equalized, reallocating consumption from individuals with low to high dynamic weights at a given date is a feasible perturbation that generates Kaldor-Hicks efficiency gains. Indeed, it seems paradoxical that it is possible to increase the sum of individual willingness-to-pay away from a Pareto efficient allocation. This and the other paradoxical results presented in this section are ultimately due to using as welfare numeraire a bundle whose elements, namely consumption at specific dates, are not valued by every individual. This is unavoidable in demographically disconnected economies — see Proposition (1) — but can be avoided in connected economies by choosing a welfare numeraire based on consumption at dates when all individuals are concurrently alive. In fact, no feasible perturbation can generate efficiency gains away from Pareto efficient allocations in economies with infinitely-lived agents or in connected economies with a connected numeraire.

[13]By the same logic, dynamic weights ω_t^i are equalized across individuals with identical lifespans. This implies that the efficiency gains generated by reallocating resources away from Pareto efficient allocations are inherently due to demographics.

Remark 6. *(Infeasible Compensating Transfers)* When a perturbation generates Kaldor-Hicks efficiency gains, the compensation principle calls for reallocating resources from winners to losers to engineer a Pareto improvement. Yet the necessary compensating transfers that would turn the perturbations with $\Xi^E > 0$ identified in Proposition 2 into a Pareto improvement with transfers are *not feasible*. After all, the planner can freely transfer resources across all individuals at all times when solving the planning problem: why are the compensating transfers unfeasible? Such transfers are *not feasible* in demographically disconnected economies because they would require transferring perpetual consumption away from the dead, violating consumption non-negativity constraints at dates when individuals are dead and already have zero consumption. That is, while welfare changes can be expressed and aggregated in terms of perpetual consumption, consumption non-negativity constraints prevent perpetual consumption from actually being transferred away from dead individuals.

Remark 7. *(Is Proposition 2 a Negative Result for Kaldor-Hicks Efficiency?)* One may interpret Proposition 2 as a negative result about Kaldor-Hicks efficiency, in the sense that it cannot be used as a statistic that identifies potential Pareto improvements with transfers in some economies. This is a perfectly valid conclusion, as it highlights the unique normative properties of demographically disconnected economies that we identify in this paper. We prefer to follow a more constructive approach, by treating this feasible Kaldor-Hicks improvements from Pareto efficient allocations as a symptom that there are untapped sources of value in demographically disconnected economies that only government policies can potentially capture, as we further discuss in our applications.

3.3 Intertemporal-Sharing: Incompleteness vs. Demographics

Aggregate-Efficiency vs. Intertemporal-Sharing. In dynamic economies with heterogeneous agents, the efficiency gains defined in (10) can be due to i) *aggregate efficiency*, which captures the gains from changes in discounted aggregate consumption, and ii) *intertemporal-sharing*, which captures the gains from reallocating consumption toward individuals who value it more at specific dates. Lemma 3 derives Dávila and Schaab (2024) aggregate-efficiency/intertemporal-sharing decomposition in our environment.[14]

[14]This is the unique decomposition in which the efficiency component can be expressed as the discounted sum — using an aggregate discount factor — of aggregate date welfare gains, Ξ^{AE}, and its complement, Ξ^{IS}.

Lemma 3. *(Aggregate-Efficiency/Intertemporal-Sharing Decomposition) The efficiency component of a normalized welfare assessment Ξ^E can be decomposed into aggregate-efficiency and intertemporal-sharing components, Ξ^{AE} and Ξ^{IS}, as follows:*

$$\Xi^E = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \underbrace{\sum_t \omega_t \sum_i \frac{dV_t^{i|\lambda}}{d\theta}}_{\Xi^{AE} \text{ (Aggregate-Efficiency)}} + \underbrace{\sum_t \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS} \text{ (Intertemporal-Sharing)}}, \tag{15}$$

where the averages of dynamic weights $\omega_t = \frac{1}{I_t}\sum_i \omega_t^i$ define aggregate time discount factors with $I_t = \sum_i \mathbb{I}\{i \mid \omega_t^i > 0\}$, and where $\mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}[\cdot,\cdot] = I \cdot \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}[\cdot,\cdot]$ denotes a cross-sectional covariance-sum among individuals alive at date t.

Intertemporal-sharing efficiency gains arise from reallocating consumption toward individuals who value it more — have higher dynamic weights ω_t^i — at specific dates. In economies where all individuals consume at all times, cross-sectional differences in dynamic weights ω_t^i can only stem from financial frictions. In these economies, if all individuals can frictionlessly borrow and save, $\omega_t^i = \omega_t$, $\forall i$, and $\Xi^{IS} = 0$.

In contrast, cross-sectional differences in dynamic weights ω_t^i can arise in demographically disconnected economies even without financial frictions — see Remark 5.[15] Even when all financial trades are feasible — so that individuals value consumption identically whenever they are concurrently alive — differences in lifespan lead them to value perpetual consumption bundles differently. Consequently, their valuations for consumption at a given date relative to a bundle based on perpetual consumption will typically differ, resulting in intertemporal-sharing gains. Example 2 illustrates this possibility.

Example 2. *(Illustrating Demographic-Driven Differences in ω_t^i)* Consider an economy with three-dates, $T = 2$, and two individuals, $I = 2$, in which individual $i = 1$ is alive at dates 0, 1, and 2, and individual $i = 2$ is alive at dates 1 and 2. Both individuals have identical discount factor β and utility $u(\cdot)$ when alive. In this case, dynamic weights at date 1, ω_1^i, are given by

$$\omega_1^1 = \frac{\beta u'(c_1^1) c_1}{u'(c_0^1) c_0 + \beta u'(c_1^1) c_1 + (\beta)^2 u'(c_2^1) c_2} = \overbrace{\frac{\beta u'(c_1^1) c_1}{u'(c_1^1) c_1 + (\beta)^2 u'(c_2^1) c_2}}^{\text{Frictional}} \times \overbrace{\frac{u'(c_1^1) c_1 + (\beta)^2 u'(c_2^1) c_2}{u'(c_0^1) c_0 + \beta u'(c_1^1) c_1 + (\beta)^2 u'(c_2^1) c_2}}^{\text{Demographic}} \tag{16}$$

$$\omega_1^2 = \frac{\beta u'(c_1^2) c_1}{\beta u'(c_1^2) c_1 + (\beta)^2 u'(c_2^2) c_2} = \underbrace{\frac{\beta u'(c_1^2) c_1}{\beta u'(c_1^2) c_1 + (\beta)^2 u'(c_2^2) c_2}}_{\text{Frictional}} \times \underbrace{1}_{\text{Demographic}}, \tag{17}$$

[15] Although we focus on demographically disconnected economies, the same results extend to connected economies when the welfare numeraire is based on dates when not all individuals are alive.

where the labels are consistent with (19) below. Hence, while frictionless borrowing and saving implies that ω_1^2 equals the term labeled frictional in (16), in general $\omega_1^1 \neq \omega_1^2$, so consumption reallocations at date 1 generate efficiency gains.

Therefore, it is evident that both financial frictions and demographics are key determinants of dynamic weights and, in turn, of intertemporal-sharing gains.

Intertemporal-Sharing Decomposition. In the remainder of this section, to distinguish the sources of intertemporal-sharing efficiency gains, we introduce a decomposition into two components: i) *frictional-intertemporal-sharing* (FIS), which captures the gains from reallocating consumption towards individuals who value consumption differently across dates due to financial frictions and ii) *demographic-intertemporal-sharing* (DIS), which captures the gains from reallocating consumption towards individuals whose valuations differ due to demographic differences.

To our knowledge, this is the first decomposition that distinguishes whether efficiency gains arise because consumption is reallocated towards individuals who i) face financial frictions, addressing inefficiencies that individuals could in principle correct via contracts or markets, or ii) differ demographically, where gains are attainable through policy intervention.

Our approach is based on three steps.

Step #1: First, we express the cross-sectional covariance-sum over all individuals at date t as a sum of pairwise covariance-sums across any two pairs of individuals.[16] To do so, we define all the ordered pairs of alive individuals at date t by $\mathcal{A}_t = \{(i,j) \mid \{i,j\} \text{ are alive at } t\}$. Formally, we can write the date-t component of intertemporal-sharing as

$$\underbrace{\mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\text{Cross-Sectional Covariance at } t} = \underbrace{\frac{1}{I_t}\sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{(i,j)}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]}_{\text{Sum of Pairwise Covariances at } t}, \tag{18}$$

where $\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}[\cdot,\cdot]$ denotes a pairwise-covariance-sum (with just two elements). Working with pairwise covariances is useful since it allows us to define common lifespans among any two individuals.

Step #2: Second, we multiplicatively decompose the dynamic weight of individual i in the

[16] Expressing covariances in terms of pairwise covariances is common, for instance, in the study of portfolio choice.

pair (i,j) at date t as into a "frictional" and a "demographic" component:

$$\underbrace{\omega_t^{i,(i,j)}}_{\substack{\text{Pairwise Dynamic}\\\text{Weight}}} = \underbrace{\omega_t^{i,(i,j)|f}}_{\text{Frictional}} \underbrace{\omega^{i,(i,j)|d}}_{\text{Demographic}}, \tag{19}$$

where we define friction-dynamic weights and demographic-dynamic weights as follows:[17]

$$\omega_t^{i,(i,j)|f} = \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_{t|\omega_t^i,\omega_t^j>0} \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} \qquad \text{(Frictional Dynamic Weight)}$$

$$\omega^{i,(i,j)|d} = \frac{\sum_{t|\omega_t^i,\omega_t^j>0} \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}. \qquad \text{(Demographic Dynamic Weight)}$$

Intuitively, individual i's frictional dynamic weight normalizes the welfare gain at date t to units of a pairwise welfare numeraire, given by the value of aggregate consumption at all dates when *both* individuals i and j are jointly alive. Individual i's demographic dynamic weight corresponds to individual i's value of aggregate consumption in all periods in which i and j are jointly alive, relative the value of perpetual aggregate consumption for individual i. Note that the demographic dynamic weight is time-independent and only relies on the marginal value of consumption during the common lifespan of the two individuals considered.

When i and j have access to frictionless financing opportunities over their common lifespans, it is evident that $\omega_t^{i,(i,j)|d} = \omega_t^{j,(i,j)|d}$ for all pairs of individuals since their valuations of consumption across all dates are equalized, justifying the "frictional" label. Even if i and j have access to frictionless financing opportunities over their common lifespans, it will typically be the case that $\omega^{i,(i,j)|d} \neq \omega^{j,(i,j)|d}$, justifying the "demographics" label. In particular, when any two individuals overlap for a single period $\omega_t^i = \omega_t^{i,(i,j)|f} = 1$, and the dynamic weight exclusively features the demographic component, as there is no role for financial trading in that economy. This will be the case in Applications 1 and 2.

Step #3: Third, we exploit Bohrnstedt and Goldberger (1969)'s formula to compute the covariance of products of random variables to decompose what part of the pairwise covariance in (18) is due to cross-sectional variation that comes from the frictional component vs. the

[17] Equation (19) can be informally described as

$$\omega_t^{i,(i,j)} = \underbrace{\frac{i's \text{ date-}t \text{ value}}{i'\text{s value over overlapping dates } (i,j)}}_{\omega_t^{i,(i,j)|f} \text{ (Frictional)}} \times \underbrace{\frac{i'\text{s value over overlapping dates } (i,j)}{i'\text{s lifetime value}}}_{\omega^{i,(i,j)|d} \text{ (Demographic)}},$$

where each element corresponds to marginal valuations over the appropriate dates.

demographic component. We present this decomposition, which is the third main result of the paper, in Proposition 3.

Proposition 3. *(Intertemporal-Sharing Decomposition: Frictional Intertemporal-Sharing vs. Demographic Intertemporal-Sharing) Intertemporal-sharing gains can be decomposed into frictional and demographic components, as follows:*

$$\Xi^{IS} = \underbrace{\sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{i,(i,j)|f}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]}_{\Xi^{FIS}\ (\textit{Frictional Intertemporal-Sharing})} + \underbrace{\sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega^{i,(i,j)|d}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]}_{\Xi^{DIS}\ (\textit{Demographic Intertemporal-Sharing})}, \tag{20}$$

where $\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}[\cdot,\cdot]$ *denotes a pairwise-covariance-sum.*

This decomposition separates the economic mechanisms behind intertemporal-sharing efficiency gains. The first term, Ξ^{FIS}, isolates the gains attributable to cross-sectional differences in valuation across dates in which individuals are concurrently alive. These are gains that could have been potentially achieved by improved financial arrangements between any two individuals. The second term, Ξ^{DIS}, captures the gains that arise from demographic differences. These are gains that *cannot* be realized by rearranging consumption between any two individuals. By explicitly separating these two forces, Proposition (3) allows us to assess whether gains from specific reallocations stem from correctable market failures or from intrinsic features of the demographic structure that require collective action.

We illustrate the mechanics behind Proposition 3 in Application 3. Proposition 4 presents useful properties of the intertemporal-sharing decomposition.

Proposition 4. *(Properties of Intertemporal-Sharing Decomposition)*

a) (Financial Frictions) When any pair of individuals alive at any date t have different valuations over consumption at the dates when they are concurrently alive, which occurs when there are financial frictions, then $\Xi^{FIS} \neq 0$.

b) (Demographic Differences) When all individuals have different birth and death dates, the demographic component is generically non-zero, even when all individuals can frictionlessly borrow and save, so $\Xi^{DIS} \neq 0$.

c) (Frictionless Borrowing/Saving) When any pair of individuals alive at any date t have identical valuations over consumption at the dates when they are concurrently alive — a *condition satisfied when individuals frictionlessly borrow and save — then* $\Xi^{FIS} = 0$ *and* $\Xi^{IS} = \Xi^{DIS}$.

d) (Common Demographics) When all individuals have identical birth and death dates, then $\Xi^{DIS} = 0$ *and* $\Xi^{IS} = \Xi^{FIS}$.

e) (Frictionless Borrowing/Saving and Common Demographics) When any pair of individuals alive at any date t *have identical valuations over consumption at the dates when they are concurrently alive — a condition satisfied when all individuals frictionlessly borrow and save — and all individuals have identical birth and death dates, then* $\Xi^{IS} = \Xi^{FIS} = \Xi^{DIS} = 0$.

Proposition 4 establishes key properties of the intertemporal-sharing decomposition. Part a) establishes that financial frictions lead to a non-zero frictional component Ξ^{FIS}, while Part c) concludes that frictional gains vanish under frictionless borrowing and saving. Part b) establishes that demographic differences frictions lead to a non-zero demographic component Ξ^{DIS}, while Part d) concludes that demographic gains vanish under common demographics. Lastly, when financial arrangements are frictionless and all individuals share identical lifespans, there are no intertemporal-sharing gains of any kind, so the total, frictional, and demographic components all vanish, and $\Xi^{IS} = \Xi^{FIS} = \Xi^{DIS} = 0$.

4 Application 1: Social Security

We now present three applications that demonstrate the practical relevance of our results. The first application assesses the welfare implications of intergenerational transfers in the simplest OLG economy: the two-date-life version of Samuelson (1958)'s endowment economy. The main takeaway from this application is that a policy of young-to-old transfers — modeled to mimic a pay-as-you-go social security system — is a Kaldor-Hicks improvement over the unique competitive equilibrium, which is Pareto efficient. Moreover, we show that all efficiency gains are solely due to demographic intertemporal-sharing, that is, $\Xi^{E} = \Xi^{IS} = \Xi^{DIS} > 0$.

While the social security policy that we study does not constitute a Pareto improvement, our analysis provides a novel justification for it: the sum of the willingness-to-pay in units of perpetual aggregate consumption among all individuals from all generations for the social security policy is strictly positive. This application illustrates Proposition 1, as we use perpetual aggregate consumption as welfare numeraire, Proposition 2, by showing that social security is a policy that generates Kaldor-Hicks efficiency gains from a Pareto efficient allocation, and Proposition 3, by establishing that such gains are solely due to demographic differences, as markets are complete and there are no financial frictions.

4.1 Environment, Equilibrium, and Policy

We consider a deterministic, single-good endowment economy with dates $t \in \{0, \ldots, T\}$, where $T < \infty$. A single individual i is born at each date, where $i \in \{-1, \ldots, T\}$. Since individuals are uniquely associated with a date of birth, $i \leftrightarrow t$, we also use t — as a superscript — to index individuals. Individuals born at dates $t \in \{0, \ldots, T-1\}$ are alive at two dates, first as "young" and then as "old". Individuals born at dates $t \in \{-1, T\}$ are alive at one date: these are the "initial old" and the "terminal young", respectively.[18]

Preferences and Endowments. The lifetime utility of an individual born at date $t \in \{0, \ldots, T-1\}$ is

$$V^t = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right), \tag{21}$$

where c_t^t and c_{t+1}^t denote individual t's consumption at dates t and $t+1$, respectively. The lifetime utilities of individuals born at dates $t \in \{-1, T\}$ are given by

$$V^{-1} = \beta u\left(c_0^{-1}\right) \qquad \text{and} \qquad V^T = u\left(c_T^T\right), \tag{22}$$

respectively. Individuals have endowments of the consumption good, denoted $\left\{e_t^t, e_{t+1}^t\right\}$ for individuals $t \in \{0, \ldots, T-1\}$, as well as e_0^{-1} and e_T^T for individuals $t \in \{-1, T\}$, respectively.

Competitive Equilibrium. To highlight that there are no financial frictions and markets are complete, we adopt and an Arrow-Debreu style formulation with once-and-for-all trading. As it is well-understood, there exists an equivalent formulation in which individuals sequentially trade a risk-free asset at all dates. The budget constraints of individuals $t \in \{0, \ldots, T-1\}$ are

$$p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t, \tag{23}$$

[18] All three applications omit population growth for simplicity; adding it is straightforward.

where p_t denotes the price of date-t consumption. Individuals $t \in \{-1, T\}$ face the budget constraints

$$p_0 c_0^{-1} = p_0 e_0^{-1} \qquad \text{and} \qquad p_T c_T^T = p_T e_T^T, \tag{24}$$

respectively. The resource constraint for date-t consumption can be written as

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t. \tag{25}$$

Definition. *(Competitive Equilibrium) Given endowments, a competitive equilibrium comprises an consumption allocations and prices such that i) individuals maximize lifetime utility* (21) – (22) *subject to* (23) – (24)*, and ii) consumption markets at each date clear, so* (25) *holds at each date.*

Autarky, $c_t^i = e_t^i$, is the unique competitive equilibrium of this economy. The ratios of prices (interest rates) that support the autarky allocation must satisfy

$$\frac{p_{t+1}}{p_t} = \beta \frac{u'\left(e_{t+1}^t\right)}{u'\left(e_t^t\right)}, \tag{26}$$

where when needed we normalize $p_0 = 1$. Intuitively, there is no intertemporal trade between young and old individuals at any date since the old are no longer alive at future dates and there is a terminal date. This is an Arrow-Debreu economy since $T < \infty$, so the first welfare theorem holds and the competitive equilibrium is Pareto efficient.

Shares, Growth, and Young-to-Old Transfer. It is useful to formulate the model in terms of consumption and endowment shares. At date t, individual i's consumption and endowment shares are $\chi_{t,c}^i = \frac{c_t^i}{c_t}$ and $\chi_{t,e}^i = \frac{e_t^i}{e_t}$, where aggregate consumption and endowment are $c_t = c_t^{t-1} + c_t^t$ and $e_t = e_t^{t-1} + e_t^t$. By construction, shares add up to 1.

The aggregate endowment — and, in equilibrium, aggregate consumption — grows at a constant rate g, that is,

$$e_t = (1+g)^t e_0.$$

Finally, to model a pay-as-you-go social security system, we consider a young-to-old (YTO) transfer. Formally, we parametrize the model by initial endowment shares given by $\bar{\chi}_{t,e}^t$ and $\bar{\chi}_{t,e}^{t-1}$, and then consider a policy — indexed by a perturbation parameter $\theta \geq 0$ — that transfers endowments from young to old individuals at each date according to

$$\text{(Young-to-Old Transfer)} \qquad \chi_{t,e}^t = \bar{\chi}_{t,e}^t - \theta \qquad \text{and} \qquad \chi_{t,e}^{t-1} = \bar{\chi}_{t,e}^{t-1} + \theta. \tag{27}$$

Calibration. Individual preferences are $u(c) = \log(c)$. We interpret one date in the model as 25 years, so $\beta = (0.98)^{25} = 0.60$. We normalize the date 0 aggregate endowment to $e_0 = 1$, and set the initial endowment shares to

$$\text{young:} \quad \bar{\chi}^{t}_{t,e} = 0.75 \qquad \text{and} \qquad \text{old:} \quad \bar{\chi}^{t-1}_{t,e} = 0.25,$$

so young individuals consume more than old at each date in the absence of policy. The economy grows at a rate $g = 0.64 = (1.02)^{25} - 1$, which implies an annual growth rate of 2%. For ease of visualization, we assume that $T = 5$, so the economy runs for 125 years.

4.2 Welfare Assessments and Weights

Welfare Assessments. We study welfare assessments for utilitarian planners, for whom the welfare assessment of the policy is

$$\frac{dW}{d\theta} = \sum_{t=-1}^{T} \alpha^{t} \frac{dV^{t}(\theta)}{d\theta}, \tag{28}$$

where α^t denotes the Pareto weight associated with the individual born at date t. We focus on two cases: i) an undiscounted utilitarian planner (our benchmark), who puts the same weight on the lifetime utility of all individuals, so $\alpha^t = 1$, and ii) a discounted utilitarian planner, who exponentially discounts the utilities of future generations, with $\alpha^t = (\bar{\alpha})^t$ for a constant $\bar{\alpha} \in (0, 1)$, which we take to be $\bar{\alpha} = \beta$, the individuals' discount factor.

We use perpetual aggregate consumption — the bundle that pays aggregate consumption at each date — as the welfare numeraire. The normalizing factor λ^t, defined in (4), for individuals $t \in \{0, \ldots, T-1\}$ is therefore

$$\lambda^{t} = u'\left(c^{t}_{t}\right) c_{t} + \beta u'\left(c^{t}_{t+1}\right) c_{t+1}, \tag{29}$$

as well as $\lambda^{-1} = \beta u'\left(c^{-1}_{0}\right) c_0$ and $\lambda^{T} = u'\left(c^{T}_{T}\right) c_T$ for individuals $t \in \{-1, T\}$, where c_t denotes aggregate consumption at date t.

Individual Weights. Individual weights uncover the value assigned by a planner to the lifetime welfare gains of different individuals in units of aggregate perpetual consumption. Therefore, a policy that reallocates aggregate perpetual consumption from individuals with low to high individual weights generates redistribution gains.

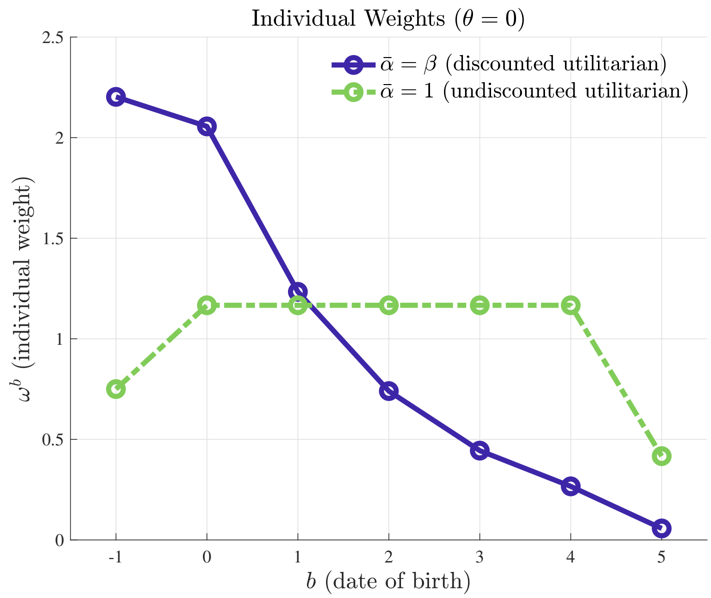


Figure 1: Individual Weights (Application 1)

Note. This figure shows individual weights as a function of an individual's date of birth for the undiscounted (dashed green line) and discounted (solid blue line) utilitarian planners in the absence of policy ($\theta = 0$).

Individual weights, defined in (8), are given by

$$\omega^t = \frac{\alpha^t \lambda^t}{\frac{1}{I}\sum_{t=-1}^{T} \alpha^t \lambda^t}, \tag{30}$$

where λ^t is defined in (29) and where the number of individuals in this economy is $I = T + 2$. Individual weights are thus shaped by i) marginal utilities of perpetual aggregate consumption and ii) Pareto weights. In general, the former depend on individual lifespans, discount factors, and aggregate and individual consumption levels. With log utility, we have that

$$u'\left(c_t^t\right) c_t = \left(\chi_{t,c}^t\right)^{-\frac{1}{\psi}},$$

so λ^t depends inversely on individual consumption shares.[19]

Figure 1 shows individual weights as a function of an individual's date of birth in the

[19] With isoelastic utility, individual t's marginal utility of aggregate consumption at date t is

$$u'\left(c_t^t\right) c_t = \left(\chi_{t,c}^t\right)^{-\frac{1}{\psi}} \left(c_t\right)^{1-\frac{1}{\psi}},$$

which depends on aggregate consumption in addition to consumption shares. An increase in aggregate consumption c_t holding the consumption share χ_t^t fixed, has two effects: a direct increase in consumption (substitution effect), and reduction in the value of consumption (income/valuation effect), with the net effect captured $1 - \frac{1}{\psi}$. When $\psi > (<)\, 1$, individuals have a high (low) willingness to substitute intertemporally, so the first (second) effect dominates. With growth, these effects shape both individual and dynamic weights, even when consumption shares are constant.

absence of policy ($\theta = 0$). First, note that the undiscounted utilitarian planner assigns lower individual weights to individuals $t \in \{-1, T\}$, because they have shorter lives. Longer lived individuals have higher individual weights, as they derive utility from consumption over more dates. Second, note that the initial-old has a higher individual weight than the terminal young, because young individuals have a higher consumption share in our calibration. For the discounted utilitarian planner, who discounts individuals born in the future by $\bar{\alpha} = \beta < 1$, the forces just discussed remain, but individual weights decline over time simply because individuals born later are discounted more heavily. Therefore, the undiscounted utilitarian planner places the highest value on the welfare gains by longer-lived individuals, followed by the initial old and, finally, the terminal young. The discounted utilitarian planner simply ranks individuals strictly by birth date, favoring earlier generations.

Dynamic Weights. Dynamic weights define marginal rates of substitution between date-t and perpetual aggregate consumption for each individual. They capture whose consumption the planner values more at each date. Consequently, a policy that reallocates consumption from individuals with lower to higher dynamic weights at a given date generates intertemporal-sharing efficiency gains.

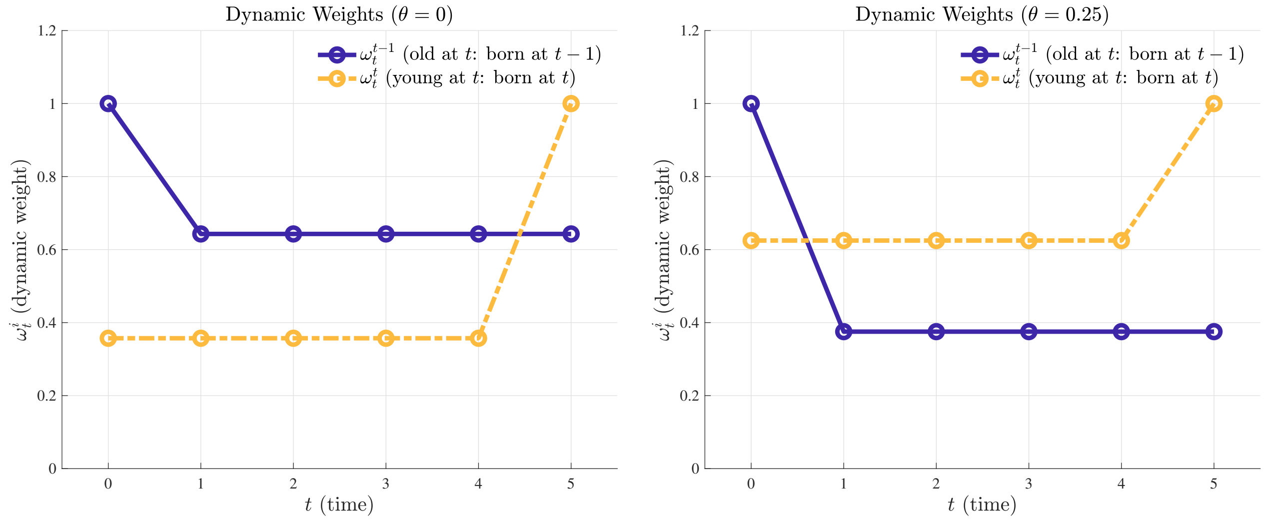
For individuals $t \in \{0, \ldots, T-1\}$, dynamic weights, defined in (9), are given by

$$\omega_t^t = \frac{u'\left(c_t^t\right) c_t}{u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1}} \quad \text{and} \quad \omega_{t+1}^t = \frac{\beta u'\left(c_{t+1}^t\right) c_{t+1}}{u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1}}. \tag{31}$$

For individuals only alive at one date, $t \in \{-1, T\}$, dynamic weights are $\omega_0^{-1} = \omega_T^T = 1$, as they are willing to exchange one unit of perpetual aggregate consumption for one unit of consumption when alive.

The left panel of Figure 2 shows the dynamic weights in the absence of policy ($\theta = 0$). In this case, old individuals have a higher dynamic weight than young individuals whenever both have two-date-lives. This reflects that young individuals have a higher consumption share in our calibration. Therefore, reallocating consumption from young to old at, say, date $t = 2$ generates intertemporal-sharing efficiency gains because $\omega_2^1 = 0.64 > 0.36 = \omega_2^2$. This pattern holds at all dates, except the terminal one.

The right panel of Figure 2 shows the dynamic weights after implementing the welfare-maximizing transfer for the undiscounted utilitarian planner ($\theta = 0.25$). In this case, young individuals have a higher dynamic weight than old individuals whenever both have two-date-lives. While consumption shares are now equal, as every individual consumes one-half of the



(a) No transfer ($\theta = 0$)

(b) Positive transfer ($\theta = 0.25$)

Figure 2: Dynamic Weights (Application 1)

Note. The left panel of this figure shows the dynamic weights in the absence of policy ($\theta = 0$, so $\chi_{t,c}^{t} = 0.75$ and $\chi_{t,c}^{t-1} = 0.25$). The right panel of this figure shows the dynamic weights after implementing the welfare-maximizing transfer for the undiscounted utilitarian planner ($\theta = 0.25$, so $\chi_{t,c}^{t} = \chi_{t,c}^{t-1} = \frac{1}{2}$). The dashed yellow lines denote the dynamic weights of young individuals and the solid blue lines denote the dynamic weights of old individuals at each date t (horizontal axis).

aggregate endowment, consumption when old is valued less due to time discounting ($\beta < 1$). As a result, a young-to-old transfer initially generate efficiency gains, but beyond a point, further transfers reduce efficiency. The shape of the dynamic weights described here explains the pattern of intertemporal-sharing efficiency gains shown in Figure 4.

4.3 Policy Experiment: Young-to-Old Transfer

Individual Welfare Gains. To understand the aggregate welfare implications of the young-to-old (YTO) transfer, it is useful to first examine its welfare impact on each individual. Figure 3 shows lifetime welfare gains, expressed in units of perpetual aggregate consumption, for each individual as a function of the size of the YTO transfer. As is well-known in this context, the initial old benefits from the policy while the terminal young is worse off, simply because their consumption increases and decreases, respectively. All other individuals initially benefit from the policy, as it improves consumption smoothing, but only up to $\theta = 0.125$.

Four takeaways emerge from Figure 3. First, it is evident that a YTO transfer is not a Pareto improvement: the terminal young is always worse off. Turning this policy into a Pareto

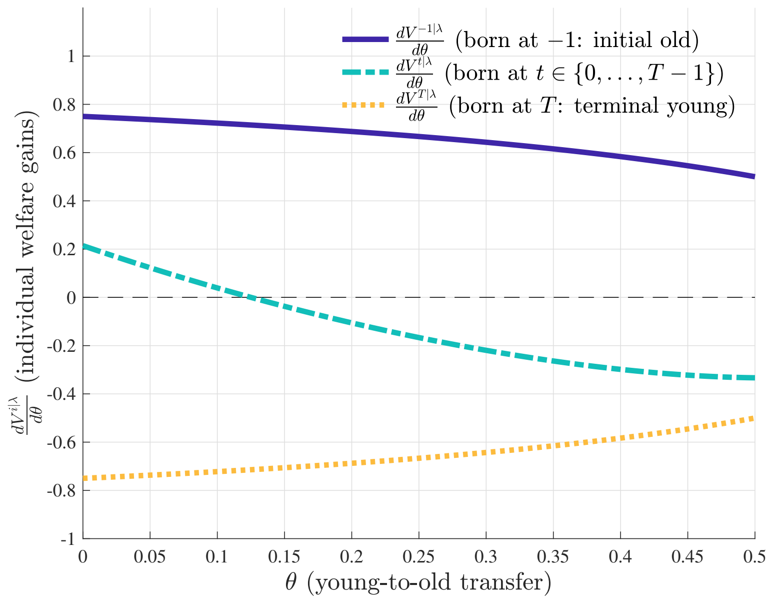


Figure 3: Individual Welfare Gains (Application 1)

Note. This figure shows normalized lifetime welfare gains, defined in (6), for different individuals as a function of the size of the young-to-old transfer. The dashed green line corresponds to the normalized individual welfare gains of all individuals with two-date-lives. The solid blue line represents the gains of the initial old born at date -1. The dotted yellow line corresponds to the gains of the terminal young born at date T.

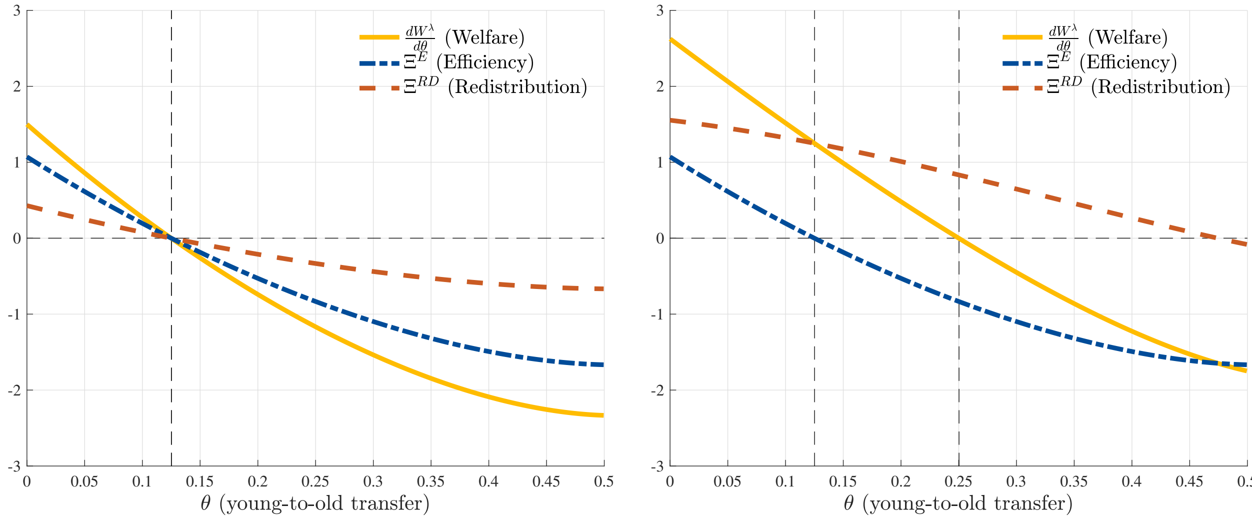
improvement would require transferring perpetual consumption to the terminal young, but individuals born at dates $\{-1, \ldots, T-2\}$ already consume zero at the terminal date, and transfers from the terminal old to the terminal young would eliminate consumption smoothing gains for all. Ideally, one would transfer resources from the initial old to the terminal young, but this economy lacks a technology to do so.[20]

Second, while the YTO transfer can never constitute a Pareto improvement for any finite T, its aggregate gains grow with T, as more generations benefit from smoother consumption while the losses to the terminal young remain fixed. It is well-understood that if $T = \infty$, the terminal young "disappear", enabling a Pareto improvement.

Third, note that the consumption smoothing gains identified here, which underpin the efficiency gains in Figure 4, would disappear if individual utilities were linear in consumption, as in Samuelson (1958). This fact further differentiates our results from double-infinite arguments, which are also valid with linear utilities.

Finally, we would like to highlight that the contribution in Figure 4 relative to existing work is the ability to express individual welfare gains in units of aggregate perpetual consumption,

[20]Introducing a linear storage technology fundamentally changes both the characterization of the competitive equilibrium and the assessment of policies. Detailed results for this case are available upon request.



(a) Undiscounted Utilitarian

(b) Discounted utilitarian

Figure 4: Welfare Assessment and Sources of Welfare Gains (Application 1)

Note. This figure shows aggregate welfare gains as a function of the size of the YTO transfer, decomposed into efficiency and redistribution components, defined in (10). The left panel shows the welfare assessment of an undiscounted utilitarian planner, with $\bar{\alpha} = 1$, and the right panel shows the welfare assessment of a discounted utilitarian planner, with $\bar{\alpha} = \beta$. Since this is an endowment economy with fixed aggregate consumption, all efficiency gains arise from intertemporal-sharing, so $\Xi^E = \Xi^{IS}$. Because individuals overlap for a single date, these intertemporal-sharing gains are entirely demographic, so $\Xi^{IS} = \Xi^{DIS}$.

leveraging Proposition 1. This allows us to add up individual gains to make aggregate assessments, as we do next.

Welfare Assessment and Sources of Welfare Gains. Figure 4 shows aggregate welfare gains as a function of the size of the YTO transfer, decomposed into efficiency and redistribution components — following Lemma 2. Since this is an endowment economy with fixed aggregate consumption, all efficiency gains arise from intertemporal-sharing. Moreover, because individuals overlap for a single date, these intertemporal-sharing gains are entirely demographic, as defined in Proposition 3. Therefore, $\Xi^E = \Xi^{IS} = \Xi^{DIS}$.

Initially, the YTO transfer generates both efficiency and redistribution gains. Adding up the individual welfare gains shown in Figure 3 corresponds to the efficiency gains Ξ^E displayed in Figure 4. The welfare gain for the initial old exactly offsets the loss for the terminal young, so the overall efficiency gains are driven by the consumption-smoothing improvements experienced by individuals $t \in \{0, \dots, T-1\}$.[21] Quantitatively, note that $\Xi^E\big|_{\theta=0} \approx -0.8 + 0.2 \times 5 + 0.8 = 1$,

[21] The sources of welfare gains shift discontinuously at $T = \infty$. In a finite-horizon economy, aggregate

so a back-of-the-envelope calculation implies that the efficiency gains from a YTO transfer of size $\theta = 0.125$ are equivalent to a $1 \times 0.125 \approx 12.5\%$ increase in aggregate perpetual consumption equally distributed across individuals. Because efficiency gains are invariant to the choice of social welfare function, $\theta = 0.125$ is the size of the efficiency-maximizing transfer for *any* welfarist planner.

The YTO transfer also generates redistribution gains initially for both discounted and undiscounted utilitarian planners. The undiscounted utilitarian planner assigns the lowest individual weight to the terminal young, so the YTO transfer, which relatively favors all other individuals, yields positive redistribution gains. For this planner, the redistribution and efficiency components are jointly maximizing at a size $\theta = 0.125$ of the YTO transfer. A discounted utilitarian planner assigns even lower individual weights to individuals born at later dates, thus finding larger redistribution gains. This planner finds a larger YTO transfer of size $\theta = 0.25$ optimal.

This application shows that a social security policy — modeled as young-to-old transfers — generates Kaldor-Hicks efficiency gains over the unique competitive equilibrium, which is Pareto efficient. It thus offers a concrete illustration of Proposition 2 applied to one of the foundational OLG models. This phenomenon arises because the economy is demographically disconnected, making it infeasible for the winners of the YTO transfer to compensate the losers in units of aggregate perpetual consumption despite the aggregate gain in value. While the social security policy that we study does not constitute a Pareto improvement, our analysis offers a normative justification for a limited level of social security. This is grounded on the fact that the willingness-to-pay — expressed in units of perpetual aggregate consumption — across all individuals from all generations is strictly positive.

5 Application 2: Capital Taxation

This application analyzes the welfare consequences of taxing capital in the finite-horizon version of Diamond (1965)'s overlapping-generations growth model — the canonical OLG economy with capital. Even though the competitive equilibrium in this economy is Pareto efficient, we show that appropriately chosen capital taxes or subsidies can generate Kaldor-Hicks efficiency

consumption is fixed, so efficiency gains arise only only from intertemporal-sharing. When $T = \infty$, the YTO transfer also increases aggregate consumption at infinity, yielding aggregate efficiency gains. Thus gains in double-infinite economies can be due to i) intertemporal-sharing, if individuals value consumption-smoothing, and ii) aggregate-efficiency, even with linear utilities.

gains.

The main takeaway from this applications is that the efficiency-maximizing capital tax is positive in low-capital-share economies but negative (i.e. a subsidy) in high-capital-share economies. This result illustrates that it is possible to perturb Pareto efficient allocations in a production economy to generate efficiency gains, providing a new illustration of Proposition 2. More broadly, this application illustrates how the results of the paper can be used to define notions of over- and under-accumulation of capital in economies that are Pareto optimal.

5.1 Environment, Equilibrium, and Policy

We consider a deterministic economy with dates $t \in \{0, \ldots, T\}$, where $T < \infty$. A single individual i is born at each date, where $i \in \{-1, \ldots, T\}$. As in Application 1, since individuals are uniquely associated with a date of birth, we also use t — as a superscript — to index individuals. Individuals born at dates $t \in \{0, \ldots, T-1\}$ are alive at two dates, first as "young" and then as "old". Individuals born at dates $t \in \{-1, T\}$ are alive at one date: these are the "initial old" and the "terminal young", respectively.

Individuals. The lifetime utilities of all individuals are the same as in Application 1: see (21) and (22). When young, individuals supply one unit of labor inelastically and decide how much to consume (c_t^t) and save. The only savings vehicle available is capital. Young individuals can either purchase k_t units of capital on the secondary market at a price q_t or invest ι_t in new capital. The budget constraint of a young individual at date t is

$$c_t^t + \iota_t + q_t k_t = w_t, \tag{32}$$

where w_t denotes the wage. Capital accumulates neoclassically according to

$$k_{t+1} = (1 - \delta) k_t + \iota_t, \tag{33}$$

where δ is the depreciation rate.

When old, individuals no longer work. They rent out the capital they have accumulated to firms and subsequently sell it on the secondary market.[22] The budget constraint of an old

[22]The timing assumption is therefore as follows: At the beginning of period t, old individuals rent their capital to firms and receive rental income. At the end of period t, old individuals sell their capital to young individuals at price q_t. Since capital was already used in production that period, young individuals have to wait until period $t+1$ before they can make use of their newly acquired capital by renting it to firms. And in the meantime, a fraction δ of this capital will have depreciated.

individual at date $t+1$ is therefore given by

$$c_{t+1}^{t} = (1-\tau)\left(q_{t+1} + d_{t+1}\right) k_{t+1} + T_{t+1}, \tag{34}$$

where d_{t+1} is the rental rate of capital, τ is the tax on capital, and T_{t+1} is a lump-sum rebate.

Firms. At each date t, a representative firm produces y_t units of the final consumption good with a constant-returns production function $y_t = f(k_t, n_t)$ that uses capital and labor. The firm choose inputs to maximize profits: $y_t - w_t n_t - d_t k_t$.

Government. The government budget must balance in each period. Revenues from the tax on capital are rebated lump-sum to the old individuals, which requires that $T_t = \tau\left(q_t + d_t\right) k_t$.

Competitive Equilibrium.

Definition. *(Competitive Equilibrium) Given an initial capital stock k_0 and a capital tax τ, a competitive equilibrium comprises allocations $\{y_t, c_t, \iota_t, n_t, k_t, T_t\}$ and prices $\{w_t, d_t, q_t\}$ such that (i) individuals maximize lifetime utility (21) – (22) subject to (32), (33), and (34), (ii) firms maximize profits, (iii) the government budget balances, and (iv) markets clear, that is, $y_t = c_t + \iota_t$ and $1 = n_t$ hold, where $c_t = c_t^{t-1} + c_t^t$ denotes aggregate consumption at date t.*

In equilibrium, young individuals must be indifferent between acquiring capital via new investment on purchasing it on the secondary market. The secondary market price must therefore satisfy $q_t = 1 - \delta$ for dates $t \leq T-1$. At the final date, capital has no productive use, so $q_T = 0$. Firm optimization requires $\frac{\partial f}{\partial k_t} = d_t$ and $\frac{\partial f}{\partial n_t} = w_t$.

Calibration. Individual preferences are $u(c) = \log(c)$. We interpret one date in the model as 25 years, so $\beta = (0.98)^{25} = 0.60$. Consistent with this choice, we set $\delta = 1$, so capital depreciates fully between dates. The production technology is Cobb-Douglas, $f(k, n) = k^a n^{1-a}$, where a maps to the capital share. We contrast calibrations with two different values of the capital share a:

$$a = \begin{cases} 0.2 & \Rightarrow \text{low-capital-share/high-labor-share} \\ 0.4 & \Rightarrow \text{high-capital-share/low-labor-share.} \end{cases}$$

By assuming that $T = 10$ the economy runs for 250 years. The initial capital stock is set to $k_{-1} = 0.01$, so this economy initially undergoes a transition phase with growth before converging to a steady state under both calibrations.

Equilibrium Characterization. In the Appendix, we display the equilibrium paths of the economy under both calibrations. Note that a *higher* capital share parameter a leads to *lower* capital accumulation, both in the steady state and along the transition. This occurs because, as a increases, a larger portion of output gets allocated to capital income (received by the old), reducing labor income (received by the young). Since only the young save in this economy, this reduces aggregate savings and, consequently, investment and capital accumulation:

$$a \uparrow \quad \Rightarrow \quad \text{labor share} \downarrow \quad \Rightarrow \quad \text{savings} \downarrow \quad \Rightarrow \quad \text{investment} \downarrow \quad \Rightarrow \quad k \downarrow \ .$$

5.2 Welfare Assessments and Weights

Welfare Assessments. We study welfare assessments for utilitarian planners, for whom the welfare assessment of a change in the capital tax rate $d\tau$ is given by

$$\frac{dW}{d\tau} = \sum_{t=-1}^{T} \alpha^t \frac{dV^t(\tau)}{d\tau}, \tag{35}$$

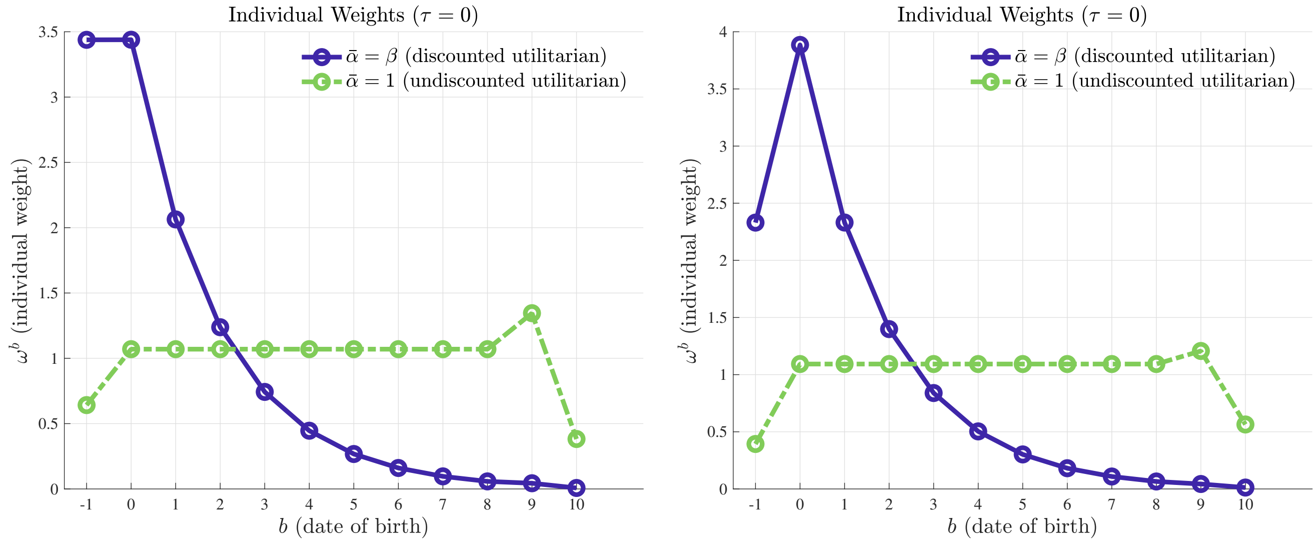
where α^t denotes the Pareto weight associated with the individual born at date t. As in Application 1, we focus on undiscounted and discounted utilitarian planners. We also use perpetual aggregate consumption as the welfare numeraire, so (29) and its counterpart for individuals $t \in \{-1, T\}$ apply here unchanged.

Individual Weights. Individual weights uncover the value assigned by a planner to the lifetime welfare gains of different individuals in units of the aggregate perpetual consumption. They are defined exactly as in (30).

Figure 5 shows individual weights as a function of an individual's date of birth in the absence of the capital tax ($\tau = 0$). Qualitatively, individual weights are similar in high- and low-capital-share economies. As in Application 1, shorter-lived individuals have lower individual weights, all else equal. A discounted utilitarian planner assigns lower weights to individuals born later because of discounting. In contrast to Application 1, individuals born at $T-1$ have higher individual weights than other long-lived individuals since they have a lower consumption share when old at date T, because the date-T young do not invest.

Dynamic Weights. Dynamic weights — defined exactly as in (29) — define marginal rates of substitution between date-t and perpetual aggregate consumption for each individual.

The left panel of Figure 6 shows the dynamic weights for the low-capital-share economy



(a) Low-capital-share (b) High-capital-share

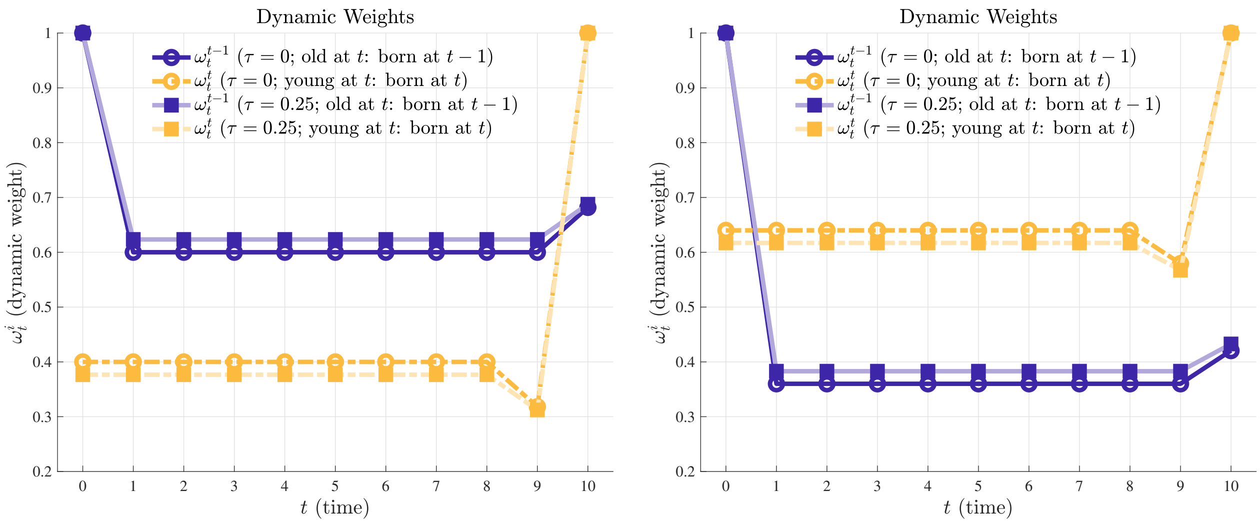
Figure 5: Individual Weights (Application 2)

Note. This figure shows individual weights as a function of an individual's date of birth for the undiscounted (dashed green line) and discounted (solid blue line) utilitarian planners when $\tau = 0$ for the low-capital-share economy ($a = 0.2$, left panel) and the high-capital-share economy ($a = 0.4$, right panel).

($a = 0.2$). In the absence of a capital tax ($\tau = 0$), old individuals have a higher dynamic weight than young individuals whenever both have two-date-lives. This reflects that the labor share is high in this calibration, so young individuals, who earn labor income, consume more than old individuals. As the capital tax increases, young individuals save and invest less, consuming more when young. This widens the consumption gap between the young and the old, further increasing the gap in dynamic weights.

The right panel of Figure 6 shows the dynamic weights for the high-capital-share economy ($a = 0.4$). In the absence of a capital tax ($\tau = 0$), old individuals have a lower dynamic weight than young individuals whenever both have two-date-lives. This reflects that the labor share is low in this calibration, so young individuals, who earn labor income, consume less than old individuals. As the capital tax increases, young individuals save less and increase their current consumption, narrowing the consumption gap between the young and the old and thereby reducing the gap in dynamic weights.

Therefore, increasing (reducing) the capital tax starting from $\tau = 0$ generates intertemporal-sharing efficiency losses in the low-capital-share economy and gains in the high-capital-share economy, as further explained below and consistent with Figure 7. As in Application 1, intertemporal-sharing gains and losses are entirely demographic, as defined in Proposition



(a) Low-capital-share

(b) High-capital-share

Figure 6: Dynamic Weights (Application 2)

Note. The left panel of this figure shows the dynamic weights for the low-capital-share economy ($a = 0.2$) in the absence of a capital tax ($\tau = 0$) and after the efficiency-maximizing capital tax for the low-capital-share economy is implemented ($\tau = 0.25$). The right panel of this figure shows the dynamic weights for the high-capital-share economy ($a = 0.4$) in the absence of a capital tax ($\tau = 0$) and after the efficiency-maximizing capital tax for the low-capital-share economy is implemented ($\tau = 0.25$).

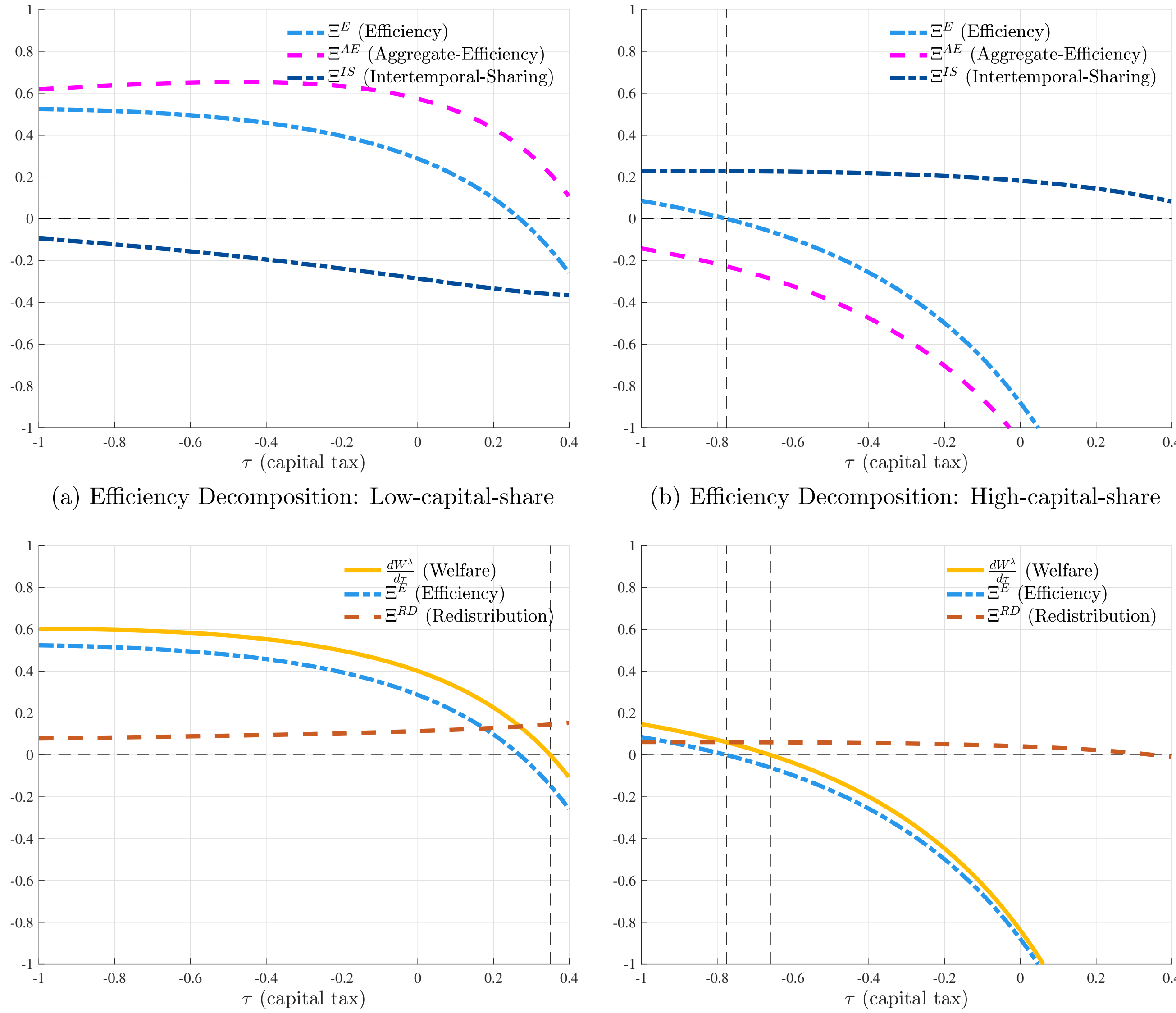
3, because individuals overlap for a single date. Hence, $\Xi^{IS} = \Xi^{DIS}$.

5.3 Policy Experiment: Capital Tax

Figure 7 shows aggregate welfare gains from capital taxation, decomposed into efficiency and redistribution components (bottom panels) and, within efficiency, into aggregate-efficiency and intertemporal-sharing gains (top panels). Unlike in Application 1, both sources of efficiency are present here because capital taxes affect the path of aggregate consumption.

Three main conclusions emerge. First, the efficiency-maximizing tax is positive in the low-capital-share economy ($\tau^\star = 0.25$) and negative in the high-capital-share economy ($\tau^\star = -0.7$), with the sign in each case driven by aggregate (investment) efficiency considerations. In the low-capital-share economy, higher capital taxes raise aggregate consumption at all but the final date, indicating over-investment and over-accumulation of capital. In the high-capital-share economy, the opposite holds: higher capital taxes lower aggregate consumption at all but the initial date, reflecting under-investment and under-accumulation of capital.

Second, intertemporal-sharing and aggregate efficiency gains move in opposite directions in both economies, so intertemporal-sharing considerations dampen the optimal capital tax



(a) Efficiency Decomposition: Low-capital-share

(b) Efficiency Decomposition: High-capital-share

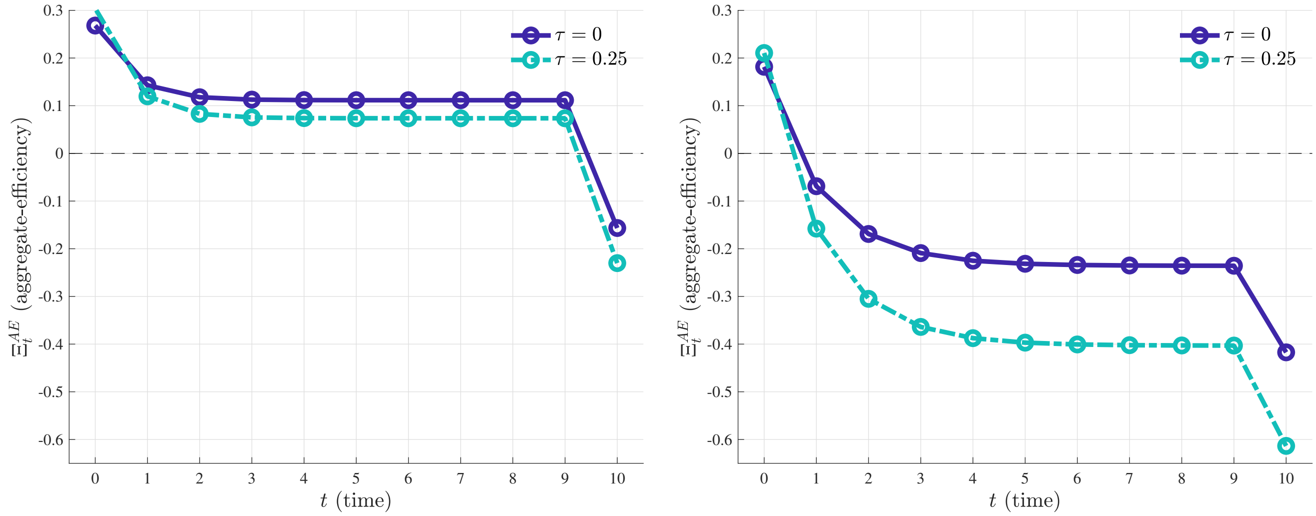
(c) Welfare Decomposition: Low-capital-share

(d) Welfare Decomposition: High-capital-share

Figure 7: Welfare Assessment and Sources of Welfare Gains (Application 2)

Note. The top panels of this figure show efficiency gains as a function of the size of the the tax, decomposed into aggregate-efficiency and intertemporal-sharing components, defined in (15). The bottom panels of this figure show aggregate welfare gains as a function of the size of the the tax, decomposed into efficiency and redistribution components, defined in (10). The bottom panels show the welfare assessments of an undiscounted utilitarian planner, with $\bar{\alpha} = 1$. The left panels correspond to the low-capital-share economy ($a = 0.2$) and the right panels to the high-capital-share economy ($a = 0.4$). Because individuals overlap for a single date, intertemporal-sharing gains are entirely demographic, so $\Xi^{IS} = \Xi^{DIS}$.

relative to what would be implied by aggregate-efficiency effects alone. In the low-capital-share economy, raising the capital tax increases the consumption of the young—who already consume more than the old—thereby widening the consumption gap and generating intertemporal-sharing losses. In the high-capital-share economy, the opposite occurs: raising the capital tax



(a) Aggregate Efficiency: Low-capital-share

(b) Aggregate Efficiency: High-capital-share

Figure 8: Term Structure of Aggregate Efficiency Gains (Application 2)

Note. This figure shows the term structure of aggregate-efficiency gains under two scenarios: no capital tax, $\tau = 0$, and a positive tax, $\tau = 0.25$. Formally, we can write $\Xi^{AE} = \sum_t \omega_t \Xi_t^{AE}$, where $\omega_t = \frac{1}{I} \sum_i \omega_t^i$.

increases the consumption of the old relative to the young, narrowing the gap and generating intertemporal-sharing gains. As in Application 1, these intertemporal-sharing effects are purely demographic, since individuals overlap for a single date, so $\Xi^{IS} = \Xi^{DIS}$.

Third, because the last young generation—most harmed by a tax increase from $\tau = 0$—has the smallest normalized welfare weight, higher capital taxes generate positive redistribution gains in both economies under an undiscounted utilitarian criterion. Intuitively, raising the capital tax reduces capital accumulation, lowering the consumption of the final individual who consumes the remaining capital stock.

We would like to conclude this application with two remarks.

Remark 8. *(Term Structure of Aggregate Efficiency Gains)* Figure 8 sheds additional light on aggregate efficiency by displaying the term structure of aggregate-efficiency gains, clarifying when the aggregate consumption changes that drive efficiency actually occur. In both low- and high-capital-share economies, a capital tax discourages investment and raises aggregate consumption on impact (date 0), so $\Xi_0^{AE} > 0$. In the low-capital-share economy, increasing τ raises consumption at all but the final date, reflecting over-accumulation of capital: discouraging investment boosts aggregate consumption at almost all times. The high-capital-share economy shows the reverse pattern—an increase in τ lowers consumption at almost all dates—indicating under-accumulation of capital. In both cases, higher τ reduces total capital accumulation, lowering aggregate consumption at the terminal date T, when the remaining capital stock is consumed.

Remark 9. *(Social Welfare Functions, Pareto Criterion, and the Golden Rule)* Prior literature, including Diamond (1965) and Abel et al. (1989), has focused on the conditions under which Pareto improvements are possible in economies like the one studied here. The economy we analyze is Pareto efficient, since a capital tax hurts the terminal young and a subsidy the initial old. Nevertheless, our framework identifies Kaldor–Hicks efficiency gains from reducing (increasing) the aggregate capital stock in low- (high-) capital-share economies. This provides a concrete illustration of Proposition 2 in a production economy and shows how efficiency gains can arise even at a Pareto-efficient allocation in a demographically disconnected setting. Our results thus take up the challenge posed by Abel et al. (1989) — see Footnote 2 — by moving beyond the Pareto criterion to evaluate alternative capital accumulation paths using a social welfare function. In doing so, our results connect to the Golden Rule literature initiated by Phelps (1961) and others, which asks whether an economy accumulates "too much" or "too little" capital. The efficiency decomposition that we develop offers a more refined lens on the Golden Rule question in two dimensions. First, aggregate-efficiency extends beyond steady-state considerations to account for the entire transition path. Second, intertemporal-sharing captures efficiency gains arising from heterogeneity in valuations across different generations.

6 Application 3: Frictions vs. Demographics

This application revisits the welfare assessment of intergenerational transfers in the simplest OLG economy in which financial markets can be used to smooth consumption: the *three-date-life* version of Samuelson (1958)'s endowment economy. Starting from a hump-shaped consumption profile and absent financial markets, we consider a policy that transfers resources from middle-aged to old, mimicking again a pay-as-you-go social security system.

Our main result is that this policy differs fundamentally from that of the young-to-old transfer in Application 1. Although both policies are desirable, the underlying rationales supporting each policy are markedly distinct: the intertemporal-sharing gains in Application 3 arise solely from financial frictions (FIS), while the demographic intertemporal-sharing (DIS) component is negative, in sharp contrast to Application 1, in which all gains are demographic (DIS). This application underscores the importance of decomposing intertemporal-sharing gains into frictional and demographic components to understand the justification for policies in economies with incomplete markets and rich demographics, as we do in Proposition 3.

6.1 Environment, Equilibrium, and Policy

We consider a deterministic, single-good endowment economy with dates $t \in \{0, \ldots, T\}$ dates, where $T < \infty$. A single individual i is born at each date, where $i \in \{-1, \ldots, T\}$. Individual typically lives for three dates, so there are three individuals alive at any date — one "young", one "middle-aged", and one "old". As in the previous applications, we use t as a superscript to index individuals.

Preferences and Endowments. The lifetime utility of an individual born at date $t \in \{0, \ldots, T-2\}$ is

$$V^t = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right) + \beta^2 u\left(c_{t+2}^t\right), \tag{36}$$

where c_t^t, c_{t+1}^t, and c_{t+2}^t denote individual t's consumption at dates t, $t+1$, and $t+2$, respectively. These individuals have endowments of the consumption good, denoted $\left\{e_t^t, e_{t+1}^t, e_{t+2}^t\right\}$. The preferences and endowments of individuals born at dates $t \in \{-2, -1, T-1, T\}$ are defined analogously with zero endowments and preferences for consumption when not alive, as shown in the Appendix.

Competitive Equilibrium. The budget constraints of individuals $t \in \{0, \ldots, T-2\}$ are

$$c_t^t = e_t^t + b_t^t, \quad c_{t+1}^t + (1 + r_{t+1})\, b_t^t = b_{t+1}^t + e_{t+1}^t, \quad \text{and} \quad c_{t+2}^t + (1 + r_{t+2})\, b_{t+1}^t = e_{t+2}^t, \tag{37}$$

where b_t^t and b_{t+1}^t denote individual t's borrowing at dates t and $t+1$ respectively, and $1 + r_{t+1}$ denotes the interest rate between dates t and $t+1$. Analogous budget constraints hold for individuals that live for only one or two dates. Individuals face borrowing constraints of the form $b_t^i \leq \bar{b}$. To more clearly illustrate our results, we set $\bar{b} = 0$, but similar results obtain as long as the borrowing constraint binds.

The resource constraint for date-t consumption can be written as[23]

$$c_t^{t-2} + c_t^{t-1} + c_t^t = e_t^{t-2} + e_t^{t-1} + e_t^t. \tag{38}$$

Definition. *(Competitive Equilibrium) Given endowments, a competitive equilibrium comprises consumption allocations and interest rates such that i) individuals maximize lifetime*

[23] Walras' Law ensures that the borrowing market clears.

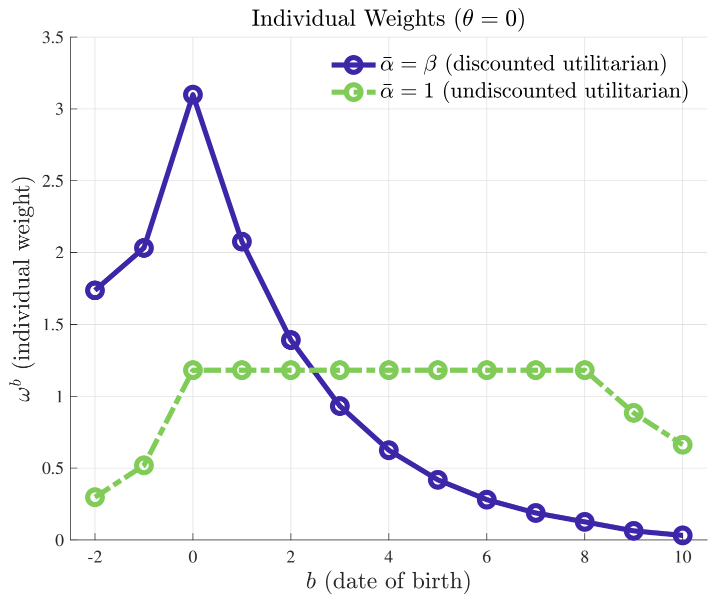


Figure 9: Individual Weights (Application 3)

Note. This figure shows individual weights as a function of an individual's date of birth for the undiscounted (dashed green line) and discounted (solid blue line) utilitarian planners in the absence of policy ($\theta = 0$).

utility subject to budget constraints, and ii) markets clear at each date.

Shares and Middle-Aged-to-Old Transfer. We model a pay-as-you-go social security system as a middle-aged-to-old (MTO) transfer at all dates. Formally, we parametrize the model by initial endowment shares given by $\bar{\chi}^{t}_{t,e}$, $\bar{\chi}^{t-1}_{t,e}$, and $\bar{\chi}^{t-2}_{t,e}$.[24] We then consider a policy — indexed by a perturbation parameter $\theta \geq 0$ — that transfers endowments from middle-aged to old individuals at each date according to

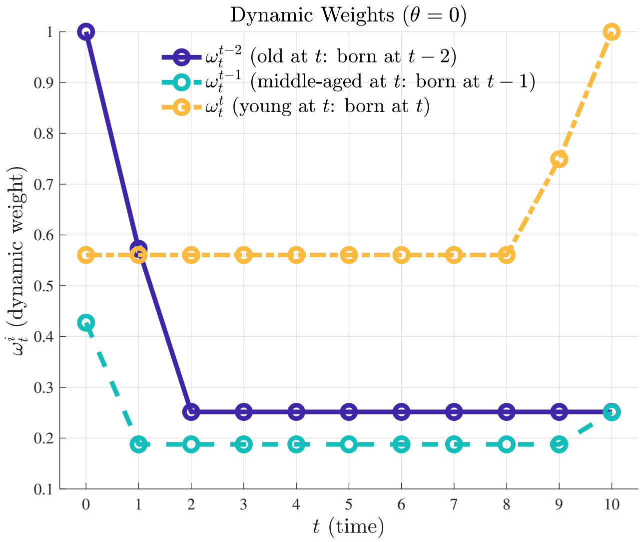
$$\text{(Middle-Aged-to-Old Transfer)} \qquad \chi^{t-1}_{t,e} = \bar{\chi}^{t-1}_{t,e} - \theta \qquad \text{and} \qquad \chi^{t-2}_{t,e} = \bar{\chi}^{t-2}_{t,e} + \theta. \tag{39}$$

Calibration. Individual preferences are $u(c) = \log(c)$. We interpret a date in the model as 20 years, so $\beta = (0.98)^{20} = 0.67$. We assume no growth of the aggregate endowment, normalizing $e_t = 1$, and set the initial endowment shares to

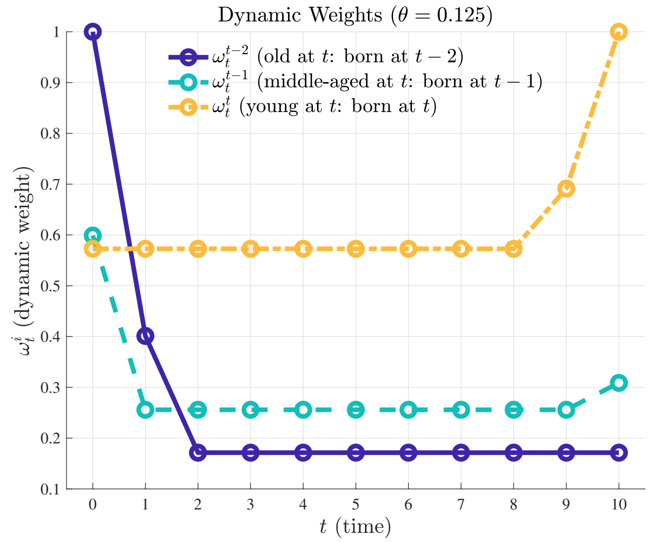
$$\text{young:} \quad \bar{\chi}^{t}_{t,e} = 0.25, \qquad \text{middle-aged:} \quad \bar{\chi}^{t-1}_{t,e} = 0.5 \qquad \text{and} \qquad \text{old:} \quad \bar{\chi}^{t-1}_{t,e} = 0.25,$$

so middle-aged individuals' consume more that young and old at each date in the absence of policy. We assume that individuals born at dates $t \in \{-2, -1\}$ have no savings at date 0. For

[24] As in Application 1, individual i's consumption and endowment shares at date t are $\chi^{i}_{t,c} = \frac{c^i_t}{c_t}$ and $\chi^{i}_{t,e} = \frac{e^i_t}{e_t}$, where aggregate consumption and endowment are $c_t = c^{t-2}_t + c^{t-1}_t + c^t_t$ and $e^{t-2}_t + e^{t-1}_t + e^t_t$, respectively.



(a) No transfer ($\theta = 0$)



(b) Positive transfer ($\theta = 0.125$)

Figure 10: Dynamic Weights (Application 3)

Note. The left panel of this figure shows the dynamic weights in the absence of policy ($\theta = 0$, so $\chi_{t,c}^{t} = 0.25$, $\chi_{t,c}^{t-1} = 0.5$, and $\chi_{t,c}^{t-1} = 0.25$). The right panel of this figure shows the dynamic weights after implementing the welfare-maximizing transfer for the undiscounted utilitarian planner ($\theta = 0.125$, so $\chi_{t,c}^{t} = 0.25$, $\chi_{t,c}^{t-1} = 0.375$, and $\chi_{t,c}^{t-1} = 0.375$). The dashed yellow lines denote the dynamic weights of young individuals and the solid blue lines denote the dynamic weights of old individuals at each date t (horizontal axis).

ease of visualization, we assume that $T = 10$, so the economy runs for 200 years.

6.2 Normalized Weights

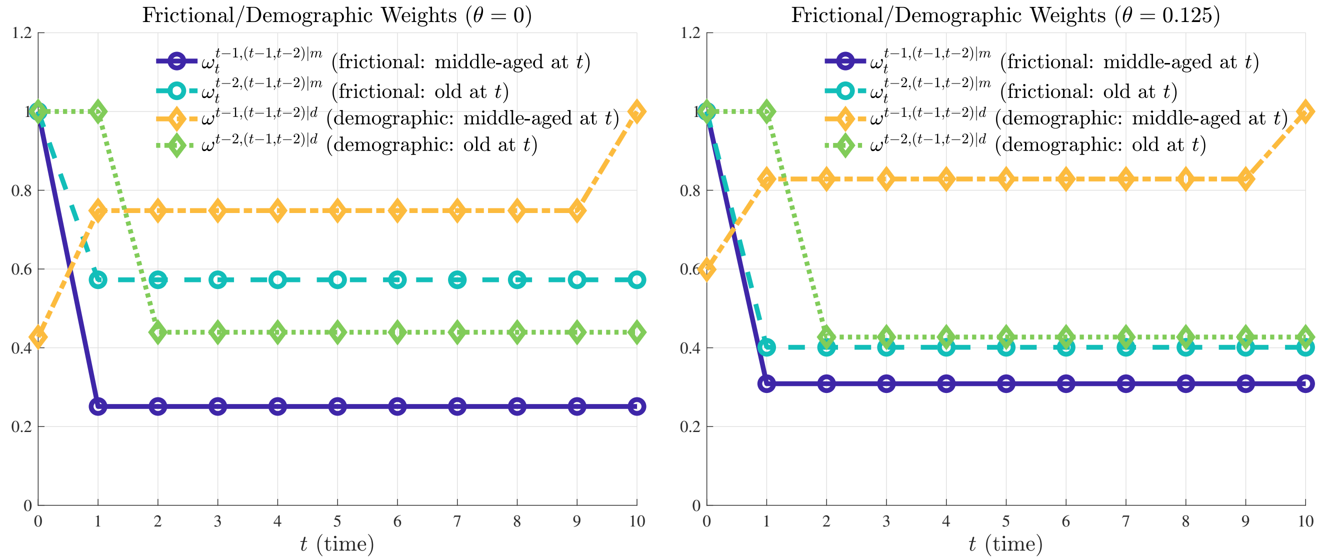
Welfare Assessments. We study welfare assessments for utilitarian planners, for whom the welfare assessment of a perturbation $d\theta$ is given by

$$\frac{dW}{d\theta} = \sum_{t=-2}^{T} \alpha^t \frac{dV^t(\theta)}{d\theta}, \tag{40}$$

where α^t denotes the Pareto weight associated with the individual born at date t. Once again, we focus on undiscounted and discounted utilitarian planners and use perpetual aggregate consumption as the welfare numeraire, so the counterpart of (29) is now

$$\lambda^t = u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1} + \beta^2 u'\left(c_{t+2}^t\right) c_{t+2},$$

with equivalent expressions for shorter-lived individuals.



(a) No transfer ($\theta = 0$) (b) Positive transfer ($\theta = 0.125$)

Figure 11: Frictional-Demographic Decomposition of Dynamic Weights (Application 3)

Note. This figure shows the multiplicative decomposition of pairwise dynamic weights between middle-aged and old, as defined in (19). The left panel of this figure is computed in the absence of policy ($\theta = 0$) while the right panel is computed after implementing the welfare-maximizing transfer for the undiscounted utilitarian planner ($\theta = 0.125$). The solid blue lines represent the frictional dynamic weights of the old, the dashed dark green lines represent the frictional dynamic weights of the middle-aged, the dash-dotted yellow lines represent the demographic dynamic weights of the young, and the dotted light green lines represent the demographic dynamic weights of the young at each date t (horizontal axis).

Individual Weights. Figure 9 displays individual weights — defined as in (30) — as a function of an individual's date of birth in the absence of policy. As in Application 1, longer-lived individuals have higher individual weights, all else equal. The initial old and middle-aged have a higher individual weight than the terminal middle-aged and young because the definition individuals utilities discounts the flow utilities of the latter. A discounted utilitarian planner assigns lower weights to individuals born later because of discounting.

Dynamic Weights. The left panel of Figure 10 displays dynamic weights in the absence of policy ($\theta = 0$). Among individuals with identical lifespans, the middle-aged have the lowest dynamic weights because they have the highest consumption, while the young have higher dynamic weights than the old, reflecting a preference for early-life consumption due to discounting. Hence, reallocating consumption from middle-aged to old (also to young) generates intertemporal-sharing efficiency gains, consistent with Figure 12.

The right panel of Figure 10 displays dynamic weights after implementing the welfare-maximizing transfer ($\theta = 0.125$). Once the transfer equalizes the consumption of middle-aged and old, discounting makes the dynamic weight of the middle-aged exceed that of the old. This reverses the earlier pattern, so at $\theta = 0.125$ a MTO transfer generates efficiency losses.

Decomposition of Dynamic Weights. The left panel of Figure 11 illustrates the multiplicative decomposition of dynamic weights into frictional and demographic components — introduced in (19) — for middle-aged and old individuals in the absence of policy ($\theta = 0$). Among individuals with identical lifespans, the frictional dynamic weight is higher for the old, who consume less than the middle-aged and hence value consumption more relative to consumption whenever both coexist.[25] Therefore, the MTO transfer generates frictional (FIS) gains. By contrast, the demographic dynamic weight is higher for the middle-aged: because of discounting, the value of consumption when young and middle-aged is valued more than when middle-aged and old, justifying why the middle-aged have a higher demographic dynamic weight. This explains why the MTO transfer generates demographic (DIS) losses.

When the transfer is implemented, old individuals consume more, lowering their value for consumption when old and narrowing the gap in frictional weights. At the same time, the MTO transfer leaves the value of consumption when middle-aged and old roughly unchanged while increasing the value of consumption when young and middle-aged, which widens the gap in demographic weights. These results are illustrated in the right panel of Figure 11 and explain the sources of intertemporal-sharing gains in Figure 12.

6.3 Policy Experiment: Middle-Aged-to-Old Transfer

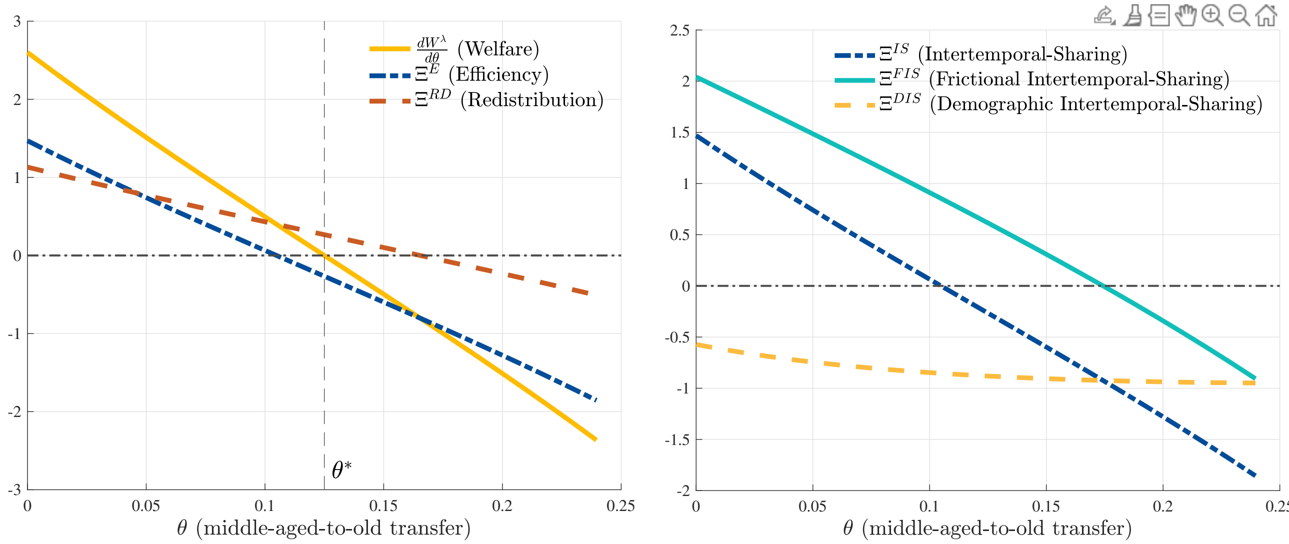
The left panel of Figure 12 displays the efficiency-redistribution decomposition in (10) for an undiscounted utilitarian planner. As in Application 1, all efficiency gains arise entirely from intertemporal-sharing since this is an endowment economy with fixed aggregate consumption,

[25] It is useful to display the multiplicative decomposition of dynamic weights for middle-aged and old at t:

$$\omega_t^{t-1} = \overbrace{\frac{u'\left(c_{t-1}^{t-1}\right) + \beta u'\left(c_t^{t-1}\right)}{u'\left(c_{t-1}^{t-1}\right) + \beta u'\left(c_t^{t-1}\right) + \beta^2 u'\left(c_{t+1}^{t-1}\right)}}^{\text{Demographic}} \times \overbrace{\frac{\beta u'\left(c_t^{t-1}\right)}{u'\left(c_{t-1}^{t-1}\right) + \beta u'\left(c_t^{t-1}\right)}}^{\text{Frictional}} \qquad \text{(Middle-Aged at } t\text{)}$$

$$\omega_t^{t-2} = \underbrace{\frac{\beta u'\left(c_{t-1}^{t-2}\right) + \beta^2 u'\left(c_t^{t-2}\right)}{u'\left(c_{t-2}^{t-2}\right) + \beta u'\left(c_{t-1}^{t-2}\right) + \beta^2 u'\left(c_t^{t-2}\right)}}_{\text{Demographic}} \times \underbrace{\frac{\beta u'\left(c_t^{t-2}\right)}{u'\left(c_{t-1}^{t-2}\right) + \beta u'\left(c_t^{t-2}\right)}}_{\text{Frictional}}, \qquad \text{(Old at } t\text{)}$$

where this expression makes use of the fact that $c_t = 1$, $\forall t$. Middle-aged and old coexist when the former are young and middle-aged and the latter are middle-aged and old, as reflected in the denominator of the frictional component and the numerator of the demographic component.



(a) Welfare Decomposition

(b) Intertemporal-Sharing Decomposition

Figure 12: Welfare Assessment and Sources of Welfare Gains (Application 3)

Note. The left panel of this figure shows aggregate welfare gains as a function of the size of the MTO transfer, decomposed into efficiency and redistribution components, defined in (10), for the undiscounted utilitarian planner. Since this is an endowment economy with fixed aggregate consumption, all efficiency gains arise from intertemporal-sharing, so $\Xi^E = \Xi^{IS}$. The right panel of this figure shows the decomposition of intertemporal-sharing gains Ξ^{IS} into frictional and demographic components, Ξ^{IMS} and Ξ^{IDS}, defined in (20).

so $\Xi^E = \Xi^{IS}$. Consistent with Figure (10), reallocating consumption from middle-aged individuals, who have lower dynamic weights, to old, who have higher, generates intertemporal-sharing gains. Redistribution gains arise because the MTO transfer harms only the terminal middle-aged, who carry lower individual weights, while benefiting others.

In contrast to Application 1, intertemporal-sharing gains now reflect both frictions and demographics, as illustrated in the right panel of Figure 12. The MTO transfer reallocates consumption from middle-aged individuals, who have lower frictional weights, to old, who have higher, thereby generating frictional gains, so $\Xi^{FIS} > 0$. This result shows that the MTO transfer effectively substitutes for the trade that middle-aged and old would undertake if financial markets or contracts were available. At the same time, the MTO transfer reallocates consumption from middle-aged individuals, who have higher demographic weights, to old, who have lower, thereby generating demographic losses, so $\Xi^{DIS} < 0$. Interestingly, the demographic structure is now such that a social-security like policy generates DIS losses, in contrast to Application 1, in which $\Xi^{DIS} > 0$. We elaborate on this point in our final remark.

Remark 10. *(Different Sources of Welfare Gains)* A central takeaway from this application is that the justification for a middle-aged-to-old transfer is fundamentally different from that of the young-to-old transfer in Application 1. Both policies generate intertemporal-sharing gains, but for entirely different reasons. In Application 1, markets are complete and the gains from the policy arise purely from demographic differences. In this application, the gains from the policy arise because markets are incomplete, with demographic differences actually contributing efficiency losses. This result underscores the importance of decomposing intertemporal-sharing gains into frictional and demographic components to understand the rationale for policies in economies with incomplete markets.

7 Conclusion

This paper shows that taking demographics seriously fundamentally reshapes welfare analysis. After establishing that intergenerational comparisons must be based on perpetual consumption, we show i) that feasible perturbations of Pareto efficient allocations in demographically disconnected economies can generate positive Kaldor-Hicks efficiency gains, a phenomenon absent when all individuals consume at every date, and ii) how to distinguish whether efficiency gains arise from financial frictions or demographic differences. Our applications quantify the sources of efficiency gains and identify new policy trade-offs faced by policymakers, illustrating the ability of the framework to yield new practical insights across multiple environments. We look forward to developing richer, quantitative applications to evaluate specific policy reforms in environments with complex demographics structures.

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Online Appendix

A Proofs and Derivations: Section 3

Proof of Proposition 1. (Unique Class of Welfare Numeraires)

Proof. The proof is constructive. Consumption at a given date or a bundle of consumption across different dates are valid welfare numeraires only when all individuals in the economy have a positive value for each of them. Formally, a welfare numeraire is valid if the individual normalizing factor λ^i, introduced in (4), is strictly positive, that is, $\lambda^i > 0$, $\forall i$. In demographically disconnected economies, consumption at a particular date cannot be a valid welfare numeraire, since there is always at least one individual who is not alive at that date. In principle, if $I < T$, one could choose a numeraire based on consumption over a subset of dates when all individuals are at least alive for one period. For instance, with three individuals and four dates, if $i = 1$ is alive at $\{0, 1\}$, $i = 2$ is alive $\{1, 2\}$, and $i = 3$ is alive at $\{2, 3\}$, numeraires based on bundles of consumption at $\{1, 2\}$, $\{0, 1, 2\}$, $\{1, 2, 3\}$, and $\{0, 1, 2, 3\}$ (perpetual consumption) are valid welfare numeraires. As I increases while the economy remain disconnected (in this example, say that $i = 4$ is alive only at $t = 0$ or $t = 4$), only numeraires based on perpetual consumption remain valid welfare numeraires. □

Proof of Lemma 1. (Normalized Welfare Gains and Weights)

Proof. We can express an (unnormalized) welfare assessment $\frac{dW}{d\theta}$ as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i},$$

where our choice of welfare numeraire is such that $\lambda^i = \sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t$. Hence, the normalized welfare assessment takes the form

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \sum_i \omega^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}.$$

We can then express individual i's normalized lifetime welfare gains as

$$\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_t \frac{\left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t} \frac{1}{c_t} \frac{dc_t^i}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta},$$

where

$$\omega_t^i = \frac{\left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i} c_t} \quad \text{and} \quad \frac{dV_t^{i|\lambda}}{d\theta} = \frac{1}{c_t} \frac{dc_t^i}{d\theta}.$$

Note that $c_t^i = \chi_t^i c_t$ implies that $\frac{dc_t^i}{d\theta} = \frac{d\chi_t^i}{d\theta} c_t + \chi_t^i \frac{dc_t}{d\theta}$, so

$$\frac{dV_t^{i|\lambda}}{d\theta} = \frac{1}{c_t} \frac{dc_t^i}{d\theta} = d\chi_t^i + \chi_t^i \frac{dc_t}{c_t}.$$

□

Proof of Lemma 2. (Efficiency/Redistribution Decomposition)

Proof. For any two random variables x_i and y_i, it follows that $\sum_i x_i y_i = \frac{1}{I} \sum_i x_i \sum_i y_i + \mathbb{C}ov_i^{\Sigma} [x_i, y_i]$, where $\mathbb{C}ov_i^{\Sigma} [x_i, y_i] = I \cdot \mathbb{C}ov_i [x_i, y_i]$. Equation (10) follows from

$$\frac{dW^{\lambda}}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E} + \underbrace{\mathbb{C}ov_i^{\Sigma} \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD}},$$

where we use the fact that $\frac{1}{I} \sum_i \omega^i = 1$. This is the unique decomposition of the weighted sum $\sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}$ into an unweighted sum and its complement. □

Proof of Proposition 2. (Kaldor-Hicks Improvements from Pareto Efficient Allocations)

Proof. The Pareto problem can be written as

$$\max_{\chi_t^i} \sum_i \alpha^i V^i,$$

subject to $\sum_i \chi_t^i = 1, \quad \forall t$ and $\chi_t^i \geq 0, \quad \forall i, t$. The Lagrangian associated with this problem can be written as

$$\mathcal{L} = \sum_i \alpha^i V^i - \tilde{\eta}_t \left(\sum_i \chi_t^i - 1 \right) + \phi_t^i \chi_t^i,$$

where the optimality condition for χ_t^i is given by

$$\omega^i \omega_t^i = \eta_t > 0 \quad \text{if} \quad c_t^i > 0, \quad \text{and} \quad \omega^i \omega_t^i < \eta_t \quad \text{if} \quad c_t^i = 0,$$

where $\eta_t = \frac{\tilde{\eta}_t}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}$. Aggregating the optimality conditions for individual i yields

$$\omega^i \underbrace{\sum_t \omega_t^i}_{=1} = \omega^i = \sum_t \eta_t \mathbb{I}\left[\omega_t^i > 0\right],$$

where $\mathbb{I}[\cdot]$ is an indicator function. This expression implies that ω^i is not equalized, and the optimality condition imply that ω_t^i are not equalized. But then note that

$$\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\theta} = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_t \sum_i \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta},$$

so as long as ω_t^i are not equalized, it is possible to reallocate consumption so that $\Xi^E > 0$. Note that a similar phenomenon occurs in demographically connected economies when using a "disconnected numeraire". In these economies, if the lifetime welfare is based on consumption at dates when all individuals are concurrently alive, then $\sum_{t\in\mathcal{T}} \omega_t^i = 1$, where $\mathcal{T}$is the set of dates when all individuals are concurrently alive. In this case, $\omega^i = \sum_{t\in\mathcal{T}} \eta_t$, which implies that ω_t^i is equalized across all individuals whenever they are alive. If instead the lifetime welfare is based on consumption at dates when not all individuals are concurrently alive, the same proof as in the demographically disconnected economies applies. □

Proof of Lemma 3. (Aggregate-Efficiency/Intertemporal-Sharing Decomposition)

Proof. It follows that

$$\Xi^E = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_t \sum_{i|\omega_t^i>0} \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \underbrace{\sum_t \omega_t \sum_{i|\omega_t^i>0} \frac{dV_t^{i|\lambda}}{d\theta}}_{\Xi^{AE}} + \underbrace{\sum_t \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}},$$

where $\omega_t = \frac{1}{I_t}\sum_i \omega_t^i$ with $I_t = \sum_i \mathbb{I}\{i \mid \omega_t^i > 0\}$, and where $\mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}[\cdot,\cdot] = I \cdot \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}[\cdot,\cdot]$. □

Proof of Proposition 3. (Intertemporal-Sharing Decomposition: Intertemporal-Market-Sharing vs. Intertemporal-Demographic-Sharing

Proof. It follows that

$$\Xi^{IS} = \sum_t \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right] = \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{(i,j)}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right],$$

where $\mathcal{A}_t$ denotes the set of all pairs of individuals alive at t and I_t is the number of alive individuals. Using the fact that $\omega_t^{i,(i,j)} = \omega_t^{i,(i,j)|f}\omega^{i,(i,j)|d}$, as defined in 19, we can apply the result in Bohrnstedt and Goldberger (1969) to write Ξ^{IS} as

$$\Xi^{IS} = \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{(i,j)|f}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right] + \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega^{(i,j)|d}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]$$
$$+ \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}osk^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{(i,j)|f}, \omega^{(i,j)|d}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right],$$

where, given three random variables X, Y, and Z, the coskewness operator is defined as $\mathbb{C}osk_i^{\Sigma}[X,Y,Z] = \mathbb{E}_i[(X-\mathbb{E}_i[X])(Y-\mathbb{E}_i[Y])(Z-\mathbb{E}_i[Z])]$ and $\mathbb{C}osk_i^{\Sigma}[\cdot,\cdot,\cdot] = I\cdot\mathbb{C}osk_i[\cdot,\cdot,\cdot]$. In this case, since all coskewness are computed pairwise (with two elements), they must be zero, which yields (20) in the text. □

Proof of Proposition 3. (Properties of Intertemporal-Sharing Decomposition)

Proof. a) In this case, the frictional dynamic weights $\omega_t^{i,(i,j)|f}$ are not equalized among individuals, so $\Xi^{FIS} \neq 0$ for generic perturbations.

b) In this case, the demographic dynamic weights $\omega_d^{i,(i,j)|d}$ are not equalized among individuals, so $\Xi^{DIS} \neq 0$ for generic perturbations.

c) In this case, the frictional dynamic weights $\omega_t^{i,(i,j)|f}$ is identical for all individuals, so $\Xi^{FIS} = 0$ for all perturbations.

d) In this case, the demographic dynamic weights $\omega_t^{i,(i,j)|d}$ is identical for all individuals, so $\Xi^{DIS} = 0$ for all perturbations. □

B Redistribution

The redistribution component captures the equity concerns embedded in a particular Social Welfare Function. Ξ^{RD} is positive when the individuals relatively favored in a perturbation are those relatively preferred by the planner, i.e., have higher normalized individual weights ω^i. At times, it is convenient to distinguish whether redistribution gains take place i) within individuals from the same generation or ii) across individuals from different generations. To this end, we show how further decompose the redistribution component into *intra*-generational redistribution and *inter*-generational redistribution.

Proposition 5. *(Redistribution Decomposition) The redistribution component, Ξ^{RD}, can be decomposed into i) intra-generational redistribution and inter-generational redistribution, as follows:*

$$\Xi_{RD} = \underbrace{\sum_t |\mathcal{G}_t| \cdot \mathbb{C}ov_{i\in\mathcal{G}_t}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD-intra} \text{ (Intra-Generational)}} + \underbrace{I \cdot \mathbb{C}ov_{\mathcal{G}}\left[\mathbb{E}_{i\in\mathcal{G}_t}\left[\omega^i\right], \mathbb{E}_{i\in\mathcal{G}_t}\left[\frac{dV^{i|\lambda}}{d\theta}\right]\right]}_{\Xi^{RD-inter} \text{ (Inter-Generational)}}, \tag{41}$$

where $\mathcal{G}_t$ is the set of individuals born at date t (generation-t) and $\mathcal{G}_0$ is the set of individual alive at date 0. $\mathbb{C}ov_{\mathcal{G}}[\cdot,\cdot]$ denotes a covariance across generations that weights each generation by the probability $\frac{|\mathcal{G}_t|}{I}$ of a given individual coming from a particular generation t. Finally, the cross-sectional conditional expectation is given by $\mathbb{E}_{i\in\mathcal{G}_t}[X_i] = \frac{1}{|\mathcal{G}_t|}\sum_{i\in\mathcal{G}_t} X_i$.

This decomposition illustrates that welfare gains from redistribution have two sources: a weighted sum of within generations redistribution; and a covariance between the per-generation average normalized individual weight, and the per-generation average normalized lifetime welfare gains. This last term, $\Xi^{RD-inter}$, is positive when the generations relatively favored in a perturbation are those relatively preferred by the planner, i.e., have higher per-generation average normalized individual weights. This decomposition of Ξ^{RD} separates the component that is zero if the planner can costlessly transfer across individuals alive, $\Xi^{RD-intra}$, an its complement, $\Xi^{RD-inter}$. Even if transfers are generally not possible across generations, especially when individuals do not overlap, certain policies can still redistribute across generations. For example, policies aimed at mitigating climate change require current generations to make sacrifices that benefit future individuals that are yet-to-be-born. We illustrate this trade-off and how our welfare decomposition rationalizes the implied welfare effects in the following example.

Example 3. *(Aggregate Efficiency v.s. Intergenerational Redistribution)* Consider a two-date endowment economy with two individuals ($I = 2$). Individual A is alive at date 0, with preferences $V^A = u\left(c_0^A\right)$; and individual B is alive at date 1, with preferences $V^B = u\left(c_1^B\right)$. We study a perturbation that reallocates consumption across periods as follows:

$$c_0^A = 1 - \theta \quad \text{and} \quad c_1^B = \frac{1}{2} + \varphi \cdot \theta; \qquad \theta \in (0, 1)\,,\ \varphi \in (0, \infty)\,.$$

We interpret the negative consequences of climate change as follows. In the absence of policy intervention ($\theta = 0$), the current generation, individual A, consumes more than the unborn generation, individual B. Individual A can make an effort, indexed by θ, that reduces their consumption levels and allows the future generation to consume more. This effort represents a costly climate change mitigation policy. Lastly, φ summarizes the effectiveness of the policy.

The efficiency gains, whose unique source is aggregate efficiency, are determined by the effectiveness parameter φ, and given by

$$\Xi^{AE} = \varphi - 1, \qquad \begin{cases} \Xi^{AE} \geq 0, & \text{if } \varphi \in [1, \infty) \\ \Xi^{AE} < 0, & \text{if } \varphi \in (0, 1)\,. \end{cases}$$

Only when $\varphi > 1$ does the perturbation result in an increase in the total aggregate consumption across periods, which is interpreted as an aggregate efficiency gain. The other source of welfare gains is intergenerational redistribution. For instance, if $\varphi < 1$, the planner may still consider the transfer worthwhile, especially if she places a high enough weight on the unborn, even though this transfer reduces efficiency. To highlight the importance of individual weights, consider the case with a constant social generational discount factor where $\alpha^A = 1$ and $\alpha^B = \bar{\alpha} \in (0, 1]$, set $\varphi = 1$, and use $u(c) = \log(c)$. Then,

$$\Xi^{RD} = \Xi^{RD-inter} > 0 \quad \text{if} \quad \begin{cases} \theta \in \left(0, \frac{1}{4}\right), & \text{if } \bar{\alpha} = 1 \\ \theta \in \left(0, \frac{\bar{\alpha} - 0.5}{\bar{\alpha} + 1}\right), & \text{if } \bar{\alpha} < 1. \end{cases}$$

In this special case, the only justification for implementing the policy is a redistribution motive. As the social generational discount rate rises — meaning the planner's discounting applied to unborn generations is smaller — the desirable level of effort to mitigate climate change also increases. In the extreme case where the planner attaches a very low social generational discount factor ($\bar{\alpha} \leq \frac{1}{2}$), mitigation of climate change is not desirable at all. Instead, the planner would prefer to redistribute away from unborn individuals by increasing the consumption of those currently alive, at the expense of future generations.

C Extensions

C.1 Stochastic Environment

Demographic and non-demographic uncertainty can be introduced using history-notation, as in Stokey, Lucas and Prescott (1989) or in Chapter 8 of Ljungqvist and Sargent (2018). In this case, let's consider an economy populated by a countable, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$, where $1 \leq I \leq \infty$. At each date $t \in \{0, \ldots, T\}$, where $0 \leq T \leq \infty$, there is a realization of a stochastic event $s_t \in S$. We denote the history of events up to date t by $s^t = (s_0, s_1, \ldots, s_t)$, and the probability of observing a particular sequence of events s^t by $\pi_t(s^t)$. The initial value of s_0 is predetermined, so $\pi_0(s_0) = 1$. At all dates and histories, individuals (potentially) consume a single good.

Births and deaths are now simply stochastic events captured by the realization of s^t. Formally, individual i *dies* at history s^t if $u_t^i(\cdot) = 0$ for all future histories. If births are random, individual i is *potentially born* at a date denoted $\tau_b^i \in \{-\infty, \ldots, T\}$, where this is the first date with is a history in which $u_t^i(\cdot) > 0$. This notion allows us to index individuals by the first time in which they are potentially born. In general, there are different reasonable stances one can take on how to formalize births and deaths, and this is sufficiently interesting to spur future work.

Formally, preferences in this case can be written as

$$V^i = \sum_{t=\tau_b^i}^{T} \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u_t^i\left(c_t^i\left(s^t\right); s^t\right),$$

where $\beta^i \in [0, 1)$ denotes individual i's discount factor, and $u_t^i(\cdot; s^t)$ and $c_t^i(s^t)$ respectively correspond to individual i's instantaneous utility and consumption at history s^t at date t. Whenever individual i is not alive, $u_t^i(\cdot; s^t) = 0$. Whenever individual i is alive, $u_t^i(\cdot; s^t)$ is well behaved, so that $\frac{\partial u_t^i(\cdot)}{\partial c_t^i} > 0$ and an Inada condition applies.

Lemma 1 applies unchanged in this case, after a few redefinitions. Normalized date welfare gains are now

$$\frac{dV_t^{i|\lambda}}{d\theta} = \sum_{s^t} \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta},$$

where the normalized stochastic weight $\omega_t^i(s^t)$ is given by

$$\omega_t^i\left(s^t\right)=\frac{\left(\beta^i\right)^t\pi_t\left(s^t\right)\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}c_t\left(s^t\right)}{\sum_{s^t}\left(\beta^i\right)^t\pi_t\left(s^t\right)\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}c_t\left(s^t\right)}.$$

Normalized history welfare gains are defined as

$$\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}=\frac{1}{c_t\left(s^t\right)}\frac{dc_t^i\left(s^t\right)}{d\theta}.$$

In this economy, efficiency can be decomposed into aggregate efficiency, risk-sharing, and intertemporal-sharing, as shown in Dávila and Schaab (2024).

C.2 Multiple Goods and Factors

It is straightforward to augment individual preferences to account for more goods and/or factors. Formally, we can consider preferences of the form

$$V^i=\sum_{t=\tau_b^i}^{T}\left(\beta^i\right)^t u_t^i\left(c_t^{ij},n_t^{if}\right),$$

where individual i now has preferences over J consumption, indexed by $j\in\{1,\ldots,J\}$, and F factors, goods and $f\in\{1,\ldots,F\}$ factors. In this case, a version of Proposition 1 still applies where the welfare numeraire needs to be defined over perpetual bundles of particular goods or factors.

C.3 Preference Disconnect

As explained in Remark 1, an analogous notion to demographic disconnect can be defined in static multi-good economies. Formally, consider a static economy populated by a finite number of individuals, indexed by $i\in\mathcal{I}=\{1,\ldots,I\}$ with preferences over J consumption goods, indexed by $j\in\{1,\ldots,J\}$. In this case, individual i preferences are given by

$$V^i=u^i\left(\left\{c^{ij}\right\}_j\right).$$

In this case, an economy is *preference disconnected* if there is no good j such that $\frac{\partial u^i}{\partial c^{ij}}>0$ for all individuals, assuming also an Inada condition for consumed goods. A version of Proposition 1 also applies to these economies: only welfare numeraires based on consumption of all goods are always valid in preference disconnected economies. A version of Proposition 2 also applies in this case.

C.4 Welfare Numeraire

C.4.1 Perpetual Aggregate Consumption: Global Formulation

Note that we can define an aggregate equivalent variation $\Lambda(\theta)$ as follows

$$\sum_i \alpha^i \sum_t \left(\beta^i\right)^t u_t^i \left(c_t^i(0) + \frac{\Lambda(\theta)}{I}\right) = \sum_i \alpha^i \sum_t \left(\beta^i\right)^t u_t^i \left(c_t^i(\theta)\right),$$

where $c_t^i(0)$ denotes consumption at a status-quo $\theta = 0$, and where $c_t^i(\theta)$ denotes the path of consumption for individual i as a function of perturbation parameter θ. Note that differentiation the definition of $\Lambda(\theta)$, we can express $\frac{d\Lambda(\theta)}{d\theta}$ in differential equation form, as follows:

$$\frac{1}{I} \sum_i \alpha^i \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i \left(c_t^i(0) + \frac{\Lambda(\theta)}{I}\right)}{\partial c_t^i} \frac{d\Lambda(\theta)}{d\theta} = \sum_i \alpha^i \underbrace{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i\left(c_t^i(\theta)\right)}{\partial c_t^i} \frac{dc_t^i(\theta)}{d\theta}}_{=\frac{dV^i(\theta)}{d\theta}},$$

which is equivalent to

$$\frac{d\Lambda(\theta)}{d\theta} = \frac{\sum_i \alpha^i \frac{dV^i(\theta)}{d\theta}}{\frac{1}{I} \sum_i \alpha^i \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i\left(c_t^i(0) + \frac{\Lambda(\theta)}{I}\right)}{\partial c_t^i}} = \frac{\frac{dW(\theta)}{d\theta}}{\frac{1}{I} \sum_i \alpha^i \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i\left(c_t^i(0) + \frac{\Lambda(\theta)}{I}\right)}{\partial c_t^i}}.$$

But note that as $\theta \to 0$, then

$$\frac{d\Lambda(0)}{d\theta} = \frac{\frac{dW(0)}{d\theta}}{\frac{1}{I} \sum_i \alpha^i \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i\left(c_t^i(0)\right)}{\partial c_t^i}} = \frac{dW^\lambda(0)}{d\theta},$$

which is the object that characterizes welfare gains from marginal perturbations.

C.4.2 Perpetual Unit Consumption

As discussed in Section 3, the two natural choices for welfare numeraire among those based on perpetual consumption are i) unit perpetual consumption, the bundle that pays one unit of the consumption good at each date, and ii) perpetual aggregate consumption, the bundle that pays aggregate consumption at each date. In the body of the text, we adopted perpetual aggregate consumption as welfare numeraire, but here we derive Lemma 1 for the unit perpetual consumption welfare numeraire.

In this case, we can express an (unnormalized) welfare assessment $\frac{dW}{d\theta}$ as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i},$$

where our choice of welfare numeraire is such that $\lambda^i = \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i}$. Hence, the normalized welfare assessment takes the form

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \sum_i \omega^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}.$$

We can then express individual i's normalized lifetime welfare gains as

$$\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_t \frac{\left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i}}{\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i}} \frac{dc_t^i}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta},$$

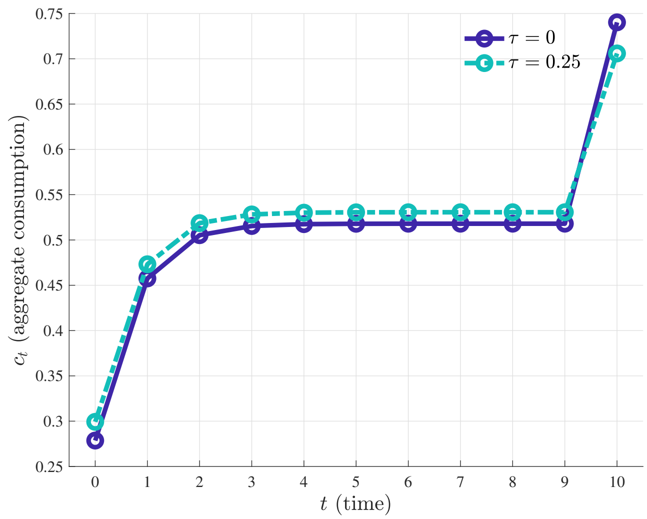
where

$$\omega_t^i = \frac{\left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i}}{\sum_t \left(\beta^i\right)^{t-\tau_b^i} \frac{\partial u_t^i}{\partial c_t^i}} \quad \text{and} \quad \frac{dV_t^{i|\lambda}}{d\theta} = \frac{dc_t^i}{d\theta}.$$

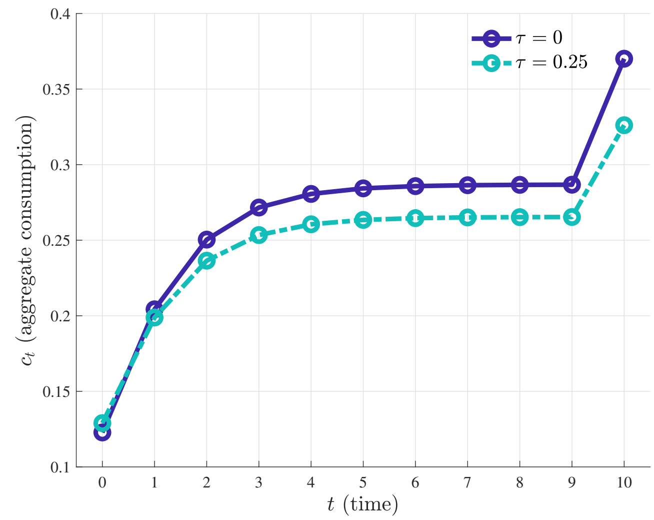
As explained in the text, in economies without aggregate consumption growth both numeraires yield identical results.

D Application 2: Additional Results

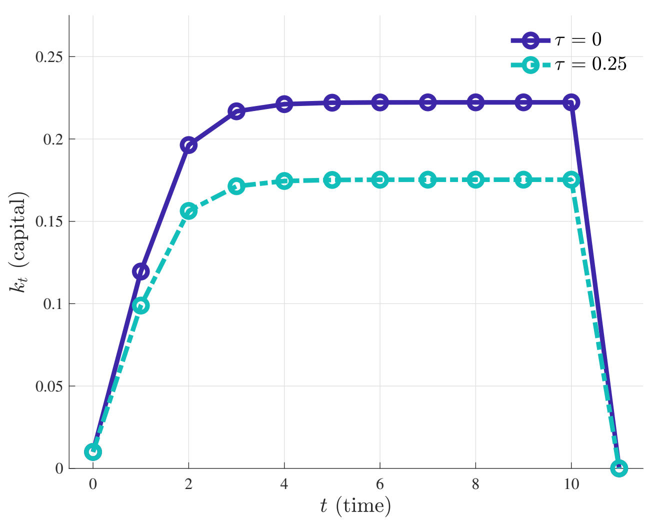
Figure 13 shows the equilibrium path for capital and aggregate consumption for both calibrations.



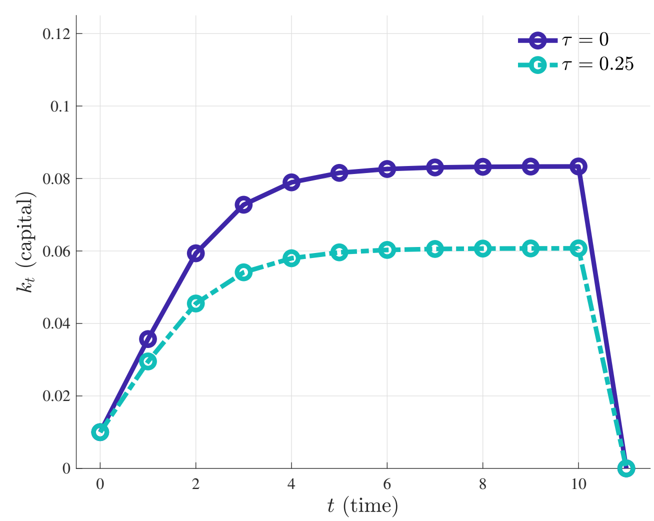
(a) Aggregate Consumption: Low-capital-share



(b) Aggregate Consumption: High-capital-share



(c) Capital: Low-capital-share



(d) Capital: High-capital-share

Figure 13: Equilibrium Dynamics (Application 2)

Note. This top panels of this figure show the equilibrium path of aggregate consumption, $c_t = \sum_i c_t^i$, for low-capital-share (left, $a = 0.2$) and high-capital-share (right, $a = 0.4$) economies. The bottom panels show the equilibrium path of capital for both calibrations.

E Application 3: Additional Results

Environment. Individuals born at dates $t = -1$ and $t = T-1$ are alive for two dates. Their lifetime utility is given by

$$V^{-1} = \beta u\left(c_0^{-1}\right) + \beta^2 u\left(c_1^{-1}\right) \quad \text{and} \quad V^{T-1} = u\left(c_{T-1}^{T-1}\right) + \beta u\left(c_T^{T-1}\right). \tag{42}$$

Individuals born at dates $t = -2$ and $t = T$ are alive for only one date. Their lifetime utility is given by

$$V^{-2} = \beta^2 u\left(c_0^{-2}\right) \qquad \text{and} \qquad V^{T} = u\left(c_T^{T}\right). \tag{43}$$

for individuals $t \in \{0, \ldots, T-2\}$, $\left\{e_0^{-1}, e_1^{-1}\right\}$ and $\left\{e_{T-1}^{T-1}, e_T^{T-1}\right\}$ for individuals $t \in \{-1, T-1\}$, as well as e_0^{-2} and e_T^T for individuals $t \in \{-2, T\}$.

Individual Welfare Gains. Figure 14 shows normalized welfare gains of the MTO transfer for each individual. Similar to Application 1, the initial old (individual $t = -2$) always gains from the perturbation, whereas the terminal middle-aged (individual $t = T-1$) always loses, with the terminal young remaining indifferent. All other individuals initially benefit from the MTO transfer since their consumption becomes smoother. Figure 14 also shows that the MTO transfer is not a Pareto improvement since the terminal middle-aged are worse off.

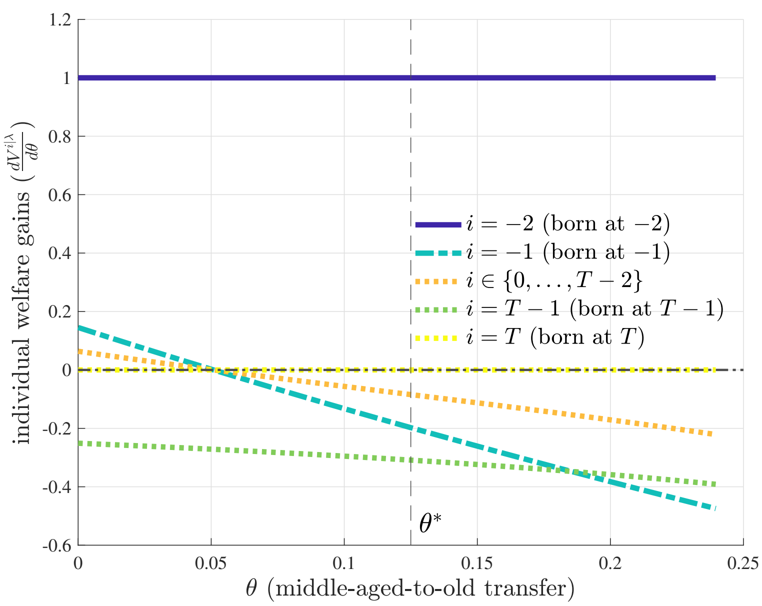


Figure 14: Individual Welfare Gains (Application 3)

Note. This figure shows the normalized individual welfare gains, given by $\frac{dV^{i|\lambda}}{d\theta}$.