Intergenerational Welfare Assessments*

Sergi Barcons† Eduardo Dávila‡ Andreas Schaab§

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Abstract

This paper studies welfare assessments in economies with rich demographic structures. First, we show that perpetual consumption is the only unit that enables interpersonal comparisons in demographically disconnected economies, in which there is no date in which all individuals are concurrently alive. Second, we show that there exist feasible perturbations of Pareto efficient allocations in demographically disconnected economies that feature positive Kaldor-Hicks efficiency gains. Third, we show how welfare gains from reallocating consumption can be separately attributed to an incomplete markets and a demographic component. We use our results to derive new insights in three workhorse intergenerational models: i) Samuelson (1958) two-date-life model, exploring the desirability of young-to-old transfers; ii) Diamond (1965) growth model with capital, exploring capital taxation and the question of over-/under-accumulation of capital, and iii) Samuelson (1958) three-date-life model, decomposing the efficiency gains from intergenerational transfers into markets and demographic components.

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†Toulouse School of Economics. `sergi.barcons@tse-fr.eu`

‡Yale University and NBER. `eduardo.davila@yale.edu`

§University of California, Berkeley. `schaab@berkeley.edu`

1 Introduction

Understanding how to make interpersonal welfare comparisons across individuals who are not concurrently alive is far from trivial and particularly important for addressing pressing global challenges, such as climate change, public debt, and social security sustainability. However, existing work on these questions typically sidesteps making interpersonal welfare comparisons across individuals from different generations. The alternative is to rely on the Pareto criterion, which while widely accepted is often inconclusive.

In this paper, we study how welfarist planners — those who use a social welfare function — make welfare assessments in demographically-rich economies.[1] We introduce our results in a single-good perfect foresight economy with a flexible demographic structure in which individuals have arbitrary birth and death dates. Several of our results rely on the notion of *demographic disconnect*, which describes economies with no date in which all individuals are concurrently alive.

Interpersonal Welfare Comparisons. We first must determine the units in which it is possible to make interpersonal welfare comparisons across individuals, because individual utilities are ordinal and interpersonal comparisons based on utils are meaningless. Our first main contribution is to identify the *unique* class of units in which it is possible to make interpersonal comparisons in demographically disconnected economies: those based on perpetual consumption. Comparisons of welfare gains across individuals must be made in terms of a claim that every individual — alive and yet-to-be-born — values. And claims based on perpetual consumption, that is, consumption at all dates, are the only ones that all individuals from all generations value. This result contrasts with conventional, demographically connected economies, in which there are many more units in which interpersonal comparisons can be made.

Feasible Kaldor-Hicks Improvements. Our second main contribution is to show that there exist *feasible* perturbations of Pareto efficient allocations in demographically disconnected economies in which the sum of individual willingness-to-pay for the perturbation (Kaldor-Hicks efficiency gains) is strictly positive. This arguably paradoxical result contrasts once again with conventional, demographically connected economies, in which no feasible perturbation of Pareto efficient allocations can generate Kaldor-Hicks efficiency gains. If there exists a perturbation that feature Kaldor-Hicks efficiency gains, the conventional logic — based on the compensation principle — calls for reshuffling resources from winners to losers so that a Pareto improvement is achieved. However, we show that in demographically disconnected economies the transfers that would be necessary to actually implement the compensation principle are *not feasible* since they would hit non-negativity constraints.

[1] We use the expression demographically-rich economies to refer to economies in which individuals are born and die.

Reallocating consumption towards individuals with higher valuations for consumption at particular dates is a source of efficiency gains in dynamic economies with heterogeneous individuals: we refer to these gains as intertemporal-sharing gains. In conventional, non-demographically-rich economies, the only reason why individuals' valuations for consumption are not equalized is because they face financing frictions, i.e., because markets are incomplete. However, in demographically rich economies, a second reason may lead to differences in valuations even when all possible financial markets are open: demographic differences. Intuitively, any two individuals may value consumption at different dates relative to perpetual consumption — already established to be only valid unit for intertemporal comparisons — differently simply because their different lifespans impact their valuation for permanent consumption, even when they share the value for consumption at all dates in which both are concurrently alive.

Incomplete Markets vs. Demographics. Our third main contribution is to show that intertemporal-sharing efficiency gains can be further decomposed into i) *intertemporal-market-sharing*, a term that captures how perturbations reallocate consumption towards individuals with different valuations for consumption across dates due to market incompleteness and ii) *intertemporal-demographic-sharing*, a term that captures how perturbations reallocate consumption towards individuals with different valuations for consumption across dates due to demographic differences.

Applications. In the remainder of the paper, we use our results to derive new insights in three workhorse intergenerational models. To highlight that none of our conclusions hinge on having an infinity of periods — an issue widely studied in overlapping generation economies — all three applications feature a finite horizon.

Our first application conducts a welfare assessment of intergenerational transfers in the simplest OLG economy: the two-date-life version of Samuelson (1958)'s endowment economy. In particular, we focus on a policy that implements young-to-old transfers across all generations starting from the unique competitive equilibrium, which is Pareto efficient. This is a policy that has been widely studied and that provides the foundation of the study of social security and money. As a first step, we show that utilitarian planners put higher weight on the gains of individuals who i) live longer, ii) have lower share of consumption (are poorer), iii) are born later if the planner is a discounted utilitarian planner, and iv) are born later (earlier) if the economy features growth and individual preferences have an elasticity of intertemporal substitution less than one. We also show that individuals and the planner initially assign a higher value to the consumption of the old relative to the young at any given date since the former are poorer. This difference in valuation — solely given by the demographic structure and the pattern of endowments since in this economy financial markets do not operate between generations — motivate the desire for the planner to implement transfers from young-to-old individuals.

The study of the welfare consequences of implementing young-to-old transfers across all generations yields several results. First, we show that a young-to-old transfer generates welfare gains for all generations, with the exception of the final young generation at the terminal date. This occurs because the initial old generation simply consumes more, while all other generations have smoother consumption paths. These transfers are initially associated with an efficiency gain, solely due to intertemporal-sharing, in turn only due to intertemporal-demographic-sharing. That is, consistent with our main results, we establish that young-to-old transfers generate Kaldor-Hicks efficiency gains starting from a competitive equilibrium allocation, which is Pareto efficient. Also consistent with our theoretical results, these Kaldor-Hicks gains cannot be transformed into Pareto improvement since there is no feasible way of transferring the gain from the initial old generation to terminal young generation.

Our second application studies the desirability of capital taxation in the classic Diamond (1965) model, which is the simplest environment used to understand over- and under-accumulation of capital in OLG economies. In low-capital-share economies, we show that the efficiency maximizing capital tax is strictly positive. In this case, an increase in the capital tax increases aggregate consumption at all dates, with the exception of the last one, contributing positively to efficiency gains via the its aggregate efficiency component, which capture the net present value of the change in aggregate consumption. This gain overcomes the intertemporal-sharing losses induced by a capital tax which increases the consumption of the young, who start with a higher consumption. In high-capital-share economies, the efficiency maximizing capital tax is negative (a subsidy). In this case, an increase in the capital tax reduces aggregate consumption at all dates, with the exception of the first one, contributing negatively to efficiency gains via the its aggregate efficiency component. This gain overcomes intertemporal-sharing gains from increasing the consumption of the young, who initially consume less than the old.

A central takeaway from our second application is that there may be Kaldor-Hicks efficiency gains from reducing the aggregate capital stock in low-capital share economies. Our results illustrate once again Proposition 2, but in a production economy, showing how it is possible to find efficiency gains associated with the allocation of capital at a Pareto efficient allocation of a demographically disconnected. More broadly, our results can be used to formalize notions of over- and under-accumulation of capital in Pareto efficient economies, addressing in a precise sense the challenge laid out by Abel et al. (1989) in their conclusion.[2]

[2] Abel et al. (1989) conclude their study of Pareto efficiency in OLG economies with the following observation:

> *"The most important direction for future research is the evaluation of alternative dynamic paths using stronger criteria than the dynamic efficiency criterion. Our criterion is the dynamic analogue of the standard Pareto criterion. (...) The use of social welfare functions would make possible the evaluation of alternative social decision rules for determining the level of investment."*

Our third application conducts a welfare assessment of transfers between middle-aged and old individuals in the simplest OLG economy in which individuals use financial markets to smooth consumption: the three-date-life version of Samuelson (1958)'s endowment economy. Starting from a status quo with no financial markets, we consider a perturbation that transfers resources from middle-aged to old, partially playing the role of a missing financial market for savings.

This application allows us to illustrate the intertemporal-sharing decomposition introduced in Proposition 3, distinguishing between the intertemporal-sharing efficiency gains that arise due to the fact that markets are incomplete and the fact that individuals are demographically different. While this transfer initially generates intertemporal-sharing welfare gains, these are due to the markets component and a markets-demographic component, whereas the contribution of the demographic intertemporal-sharing component is negative.

Related Literature. Our analysis contributes to the literature that follows classic studies of efficiency in overlapping generation (OLG) economies, such as Samuelson (1958) and Diamond (1965) the two models in which we base our applications. These ideas have made their way to most macroeconomics textbooks, including Blanchard and Fischer (1989), De La Croix and Michel (2002), Romer (2006), and Acemoglu (2009), among others. Spear and Young (2023) presents a historical account of the development of these ideas.

A central normative result of the OLG literature has been the suboptimality of competitive equilibria, which has spurred large amounts of research. Geanakoplos (1989) highlights that this inefficiency result necessitates i) a double infinity of individuals and goods, and ii) the overlapping nature of individual lives. Our focus on i) understanding the sources of welfare gains for welfarist planners and ii) characterizing properties of the planning problem, are largely orthogonal to the question of suboptimality of competitive equilibria. To highlight the differences, we focus on economies with a terminal date, which eliminates the pathological properties of competitive equilibria associated with the double infinity.

We are not the first to identify particular demographic features that are important in OLG economies. For instance, Weil (1989) highlights that the arrival into the economy of new individuals (dynasties) not linked to older cohorts is necessary to generate asset bubbles, dynamic inefficiency, and violations of Ricardian equivalence. While the continuous arrival of new individuals makes an economy demographically disconnected, we are not aware of any existing work presenting the notion of demographically disconnected economies that we introduce in this paper.

A separate body of work explores the properties of intergenerational social welfare functions or other welfare criteria, including for instance, the work by Calvo and Obstfeld (1988) and Eden (2023), among others. Our approach takes the social objective as given, but our emphasis on making interpersonal comparisons in a common a unit (the lifetime welfare numeraire) leads us to establish that the unique feasible (class of) lifetime welfare numeraire(s) must be based on perpetual consumption. A different approach is followed by Aguiar, Amador and Arellano (2023), who characterize Robust Pareto Improvements (RPI) in OLG economies, ensuring that the budget set of any agent is guaranteed to be weakly expanded at any state and time. An advantage of the RPI criteria is that it ensures a Pareto improvement regardless of how agents trade off consumption intertemporally or across uncertain states. Our results are complementary to these, in the sense that our result are most useful to consider perturbations that are not Pareto improvements.

While, simply to keep the paper focused, we have abstracted from explicitly dealing with population growth, the importance of population growth for welfare gains has been emphasized by Jones and Klenow (2016) and Adhami et al. (2024), among others. We hope that the results of this paper spur future work on these important issues.

2 Environment

Demographics. We consider a perfect foresight economy populated by a countable number of individuals, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$, where $1 \leq I \leq \infty$. There are $t \in \{0, \ldots, T\}$ dates, where $T \leq \infty$, in which an individual i (potentially) consumes a single good. Each individual i is associated with a date of birth $\tau_b^i \in \{-\infty, \ldots, T\}$ — where a negative date of birth means that the individual is already alive at date 0 — and a date of death $\tau_d^i \in \{1, \ldots, T\}$. While much of the existing work on demographically-rich economies emphasizes the $T = \infty$ case, our results apply when T is finite or infinite. To more easily contrast our results to existing work, we focus on the finite T case.

Preferences. An individual i derives utility from consumption, with a lifetime utility representation given by

$$\text{(Preferences)} \qquad V^i = \sum_{t=\tau_b^i}^{T} \left(\beta^i\right)^{t-\tau_b^i} u_t^i\left(c_t^i\right), \tag{1}$$

where $\beta^i \in [0, 1)$ denotes individual i's discount factor, and $u_t^i(\cdot)$ and c_t^i respectively correspond to individual i's instantaneous utility and consumption at date t. Whenever individual i is not alive, $u_t^i\left(c_t^i\right) = 0$. Whenever individual i is alive, $u_t^i\left(c_t^i\right)$ is well behaved, so that $\frac{\partial u_t^i(\cdot)}{\partial c_t^i} > 0$ and an Inada condition applies. We refer to the unit of V^i as individual i utils.

Social Welfare Function. Whenever we study welfare assessments, we do so for welfarist planners, that is, planners with a social welfare function given by

$$\text{(Social Welfare Function)} \qquad W = \mathcal{W}\left(V^1, \ldots, V^i, \ldots, V^I\right), \tag{2}$$

where individual lifetime utilities V^i are defined in (1). We assume that $\frac{\partial \mathcal{W}}{\partial V^i} > 0$, $\forall i$, which means that the planner values both alive and yet-to-be-born individuals at date 0. We refer to the units of W as social utils.

Defining social welfare functions when there are multiple generations is analogous to doing so in static environments or environments with infinitely lived agents — this is noted, for instance, in Blanchard and Fischer (1989). However, at times, it may be necessary to restrict the form of $\mathcal{W}(\cdot)$ to ensure that W is finite.[3] The welfarist approach is widely used because it is Paretian, that is, it concludes that every Pareto-improving perturbation is desirable, and because nonwelfarist approaches violate the Pareto principle (Kaplow and Shavell, 2001).

Remarks on the Environment. To simplify the exposition, we study a perfect foresight environment with a single consumption good and a continuum of individuals in the body of the paper. In the Online Appendix, we show how to accommodate demographic and non-demographic uncertainty and multiple consumption goods and factors. In particular, considering demographic uncertainty is useful to model random death environments, as in Yaari (1965) and Blanchard (1985). It is straightforward to generalize the results to economies with a continuum of individuals, dates, and histories.

Demographic Disconnect. Throughout the paper, we rely on the notion of demographically disconnected economies, which we introduce here.

Definition. *(Demographic Disconnect) An economy is demographically disconnected if there is no date in which all individuals are concurrently alive.*

The key property of demographically disconnected economies is that there is no single date at which all individuals have positive value for consumption. This property has a significant implications for welfare assessments and planning solutions. Demographic disconnect is a realistic assumption and a widespread feature in demographically-rich economies. Demographically disconnected economies must feature individuals born after date 0; if all individuals were alive at date 0 — even if they died gradually — the economy would be demographically connected at date 0. However, births are necessary but not sufficient to generate demographic disconnect.

It is useful to illustrate the notion of demographic disconnect with an example that compares two two-date ($T = 1$) economies with different demographic structures. Table 1 illustrates this example.

[3] The most common intergenerational social welfare functions are i) the discounted utilitarian social welfare function, in which the lifetime utility of future generations is exponentially discounted at a time-invariant rate, and ii) the undiscounted utilitarian social welfare function, in which all individuals are given the same utility weight. We use both in our applications.

Table 1: Illustrating Demographically Disconnected Economies

	Economy A		Economy B	
	$t=0$	$t=1$	$t=0$	$t=1$
$i=1$	✓	✓	✓	✓
$i=2$	✓	$\times$	✓	$\times$
$i=3$			$\times$	✓

Note: Economy A has two individuals: $i=1$ and $i=2$. Economy B has three individuals, $i=1$, $i=2$, and $i=3$. Checks (✓) represent dates in which an individual is alive, while crosses ($\times$) represent dates in which an individual is not alive. Economy A is demographically connected since all individuals are concurrently alive at date 0. Economy B is demographically disconnected since there is no date in which all individuals are concurrently alive: individual $i=2$ is not alive at date 1 and individual $i=3$ is not alive at date 0.

Example 1. *(Illustrating Demographic Disconnect)* In Economy A, there are two individuals: a first individual who is alive at dates 0 and 1 and a second individual, who is exclusively alive at date 0. Given our definition, Economy A is not demographically disconnected, because both individuals are alive at date 0. Economy B is identical to economy A, but for the fact that there is a third individual who is exclusively alive at date 1. The presence of this third individual makes Economy B demographically disconnected. At dates 0 and 1 there are two concurrently alive individuals, but there is no date in which all three individuals are concurrently alive.

It is worth making two remarks on the notion of demographic disconnect.

Remark 1. *(Preference Disconnect)* It is possible to construct static economies with multiple consumption goods that look formally similar to demographically disconnected economies by making specific assumptions on individual preferences for different consumption goods. In Section C.3 of the Online Appendix, we describe this analogy and refer to this phenomenon as *"preference disconnect"*. Despite the formal analogy, real-world economies are demographically disconnected as long as individuals are not altruistic — see next remark. At the same time, real-world economies are preference connected, e.g., every individual has positive value for some good, say, water. This justifies our focus on demographic disconnect.

Remark 2. *(Role of Intergenerational Altruism)* Our definition of demographic disconnect assumes that all individuals are self-interested and do not incorporate the preferences of future generations in their own. Intergenerational altruism can transform demographically disconnected economies into demographically connected economies. Individuals who are altruistic towards future generations have positive value for consumption after their death. Hence, demographic disconnect in economies with intergenerational altruism requires that there is no date in which all individuals have positive value for consumption, either privately or altruistically.

3 Intergenerational Welfare Assessments

3.1 Lifetime Welfare Numeraire

Our goal is to understand why a planner finds a perturbation desirable in demographically-rich economies. Formally, a perturbation is a smooth change in c_t^i as a function of a perturbation parameter $\theta \in [0,1]$, so derivatives such as $\frac{dc_t^i}{d\theta}$ are well-defined. A perturbation may capture changes in technologies, endowments, policies, or any other primitive of a fully specified model. It can also capture changes in feasible allocations directly chosen by a planner.

A welfarist planner finds a perturbation $d\theta$ desirable (undesirable) if

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} > (<)\, 0. \tag{3}$$

It follows directly from (3) that $\frac{dW}{d\theta} > 0$ whenever a perturbation is a Pareto improvement.[4] However, understanding how a welfarist planner makes tradeoffs in meaningful units across individuals is not straightforward, because direct comparisons based on utils are meaningless and because individuals have different lifespans.

Meaningful comparisons across individuals require a common unit in which to express individual welfare gains: we refer to this unit as the lifetime welfare numeraire. The only requirement for a lifetime welfare numeraire to be valid is that all individuals — alive and yet-to-be-born — must have a positive value for it. In demographically connected economies there is an abundance of possible lifetime welfare numeraires: for instance, consumption at any single date or consumption at a subset of different dates are valid lifetime welfare numeraires in those economies. However, in demographically disconnected economies there is a single class of units in which it is possible to make interpersonal welfare comparisons: those based on perpetual consumption, that is, consumption at all dates. Identifying this class is the first main result of the paper, which we formalize in Proposition 1.

Proposition 1. *(Unique Class of Valid Lifetime Welfare Numeraires) Only lifetime welfare numeraires based on perpetual consumption are always valid in demographically disconnected economies.*

In economies in which there is no date in which all individuals are concurrently alive, it is only possible to make interpersonal comparisons across all individuals in terms of consumption at all dates, that is, perpetual consumption. By construction, all individuals — alive and yet-to-be-born — have positive value for bundles of consumption at all dates, which makes any of these bundles a valid lifetime welfare numeraire to make interpersonal welfare comparisons.

[4]Formally, a perturbation is strictly (weakly) Pareto-improving when $\frac{dV^i}{d\theta} > (\geq)\, 0$, $\forall i$.

Within this class of numeraires based on perpetual consumption, there are two natural choices: *unit perpetual consumption*, the bundle that pays one unit of the consumption good at each date, and *aggregate perpetual consumption*, the bundle that pays aggregate consumption at each date.[5] Because some of our applications feature positive growth, we adopt aggregate perpetual consumption as the lifetime welfare numeraire in the body of the paper, and discuss and justify this choice in more detail in the Appendix.

Hence, with aggregate perpetual consumption as lifetime welfare numeraire, we can express $\frac{dW}{d\theta}$ as follows:

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \lambda^i = \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t, \tag{4}$$

and where aggregate consumption is given by $c_t = \sum_i c_t^i$. The normalizing factor λ^i denotes the marginal increase in utility associated with a proportional increase in aggregate perpetual consumption. In equation (4), $\frac{\frac{dV^i}{d\theta}}{\lambda^i}$ denotes the lifetime welfare gains of the perturbation for individual i, expressed in units of the chose lifetime welfare numeraire. That is, $\frac{\frac{dV^i}{d\theta}}{\lambda^i}$ can be interpreted as individual i's willingness to pay for the perturbation expressed in units of a proportional increase of aggregate perpetual consumption. Formally, Proposition 1 ensures that $\lambda^i > 0$, $\forall i$, for the lifetime welfare numeraire chosen.

Lemma 1 allows us to express welfare assessments in terms of normalized lifetime and date welfare gains and normalized individual and dynamic weights. This result allows us to write express welfare changes in common units, which in turn is useful to understand the sources of welfare gains. In terms of notation, variables with a λ superscript are expressed in the appropriate numeraire.

Lemma 1. *(Normalized Welfare Gains and Normalized Weights) A normalized welfare assessment for a welfarist planner can be represented as*

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}, \tag{5}$$

where $\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i}$ *and* $\frac{dV_t^{i|\lambda}}{d\theta}$ *respectively denote lifetime, and date welfare gains, given by*

$$\frac{dV^{i|\lambda}}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} \qquad \text{(Normalized Lifetime Welfare Gains)} \tag{6}$$

$$\frac{dV_t^{i|\lambda}}{d\theta} = \frac{1}{c_t}\frac{dc_t^i}{d\theta} = \chi_{t,c}^i \frac{d\ln c_t^i}{d\theta}, \qquad \text{(Normalized Date Welfare Gains)} \tag{7}$$

where $\chi_{t,c}^i = \frac{c_t^i}{c_t}$ *denotes individual* i*'s consumption share, and where* ω^i *and* ω_t^i *respectively denote*

[5] In economies without growth, there is no distinction between both choices.

normalized individual and dynamic stochastic weights, given by

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\frac{1}{I} \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} \qquad \text{(Normalized Individual Weight)} \qquad (8)$$

$$\omega_t^i = \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}. \qquad \text{(Normalized Dynamic Weight)} \qquad (9)$$

This result is the counterpart to Lemma 1 in Dávila and Schaab (2024) with two differences. First, aggregate perpetual consumption is now the lifetime welfare numeraire. In this case, as shown in Proposition 1, only lifetime welfare numeraires based on perpetual consumption are valid. Second, we use aggregate consumption as date welfare numeraire.

The normalized individual weight ω^i defines how a welfarist planner trades off lifetime welfare gains across individuals in terms of proportional changes to aggregate consumption. The normalized dynamic weight ω_t^i defines a marginal rate of substitution for individual i between a consumption change proportional to aggregate consumption at date t and a perpetual consumption change proportional to aggregate consumption.[6] Note that normalized individual weights average to one in the cross-section, so $\frac{1}{I} \sum_i \omega^i = 1$, and that normalized dynamic weights add up to one when aggregated over time, defining a *normalized discount factor*, so $\sum_i \omega_t^i = 1$, $\forall i$.

Lemma 1 is useful, because it shows that every welfare assessment can be interpreted as a weighted sum across all individuals, with weights given by ω^i, of the discounted values — using ω_t^i as normalized discount factors — of $\frac{dV_t^{i|\lambda}}{d\theta}$, which play the role of date-individual-specific payoffs. Because we have used numeraires based on aggregate consumption, $\frac{dV_t^{i|\lambda}}{d\theta}$ correspond to the change in individual consumption relative to aggregate consumption. This in turn can be written as the product of the proportional change in individual i's consumption at a given date $\frac{d \ln c_t^i}{d\theta}$ with the individual's consumption share $\chi_{t,c}^i$.

3.2 Efficiency vs. Redistribution

After expressing welfare assessments in comparable units, Lemma 2 derives Dávila and Schaab (2024)'s efficiency/redistribution decomposition in our environment. This is the unique decomposition in which a normalized welfare assessment can be expressed as Kaldor-Hicks efficiency and its complement.

[6] For instance, if $\omega^i = 1.3$, a welfarist planner finds the welfare gain associated with a proportional increase in aggregate perpetual consumption distributed to individual i equivalent to the welfare gain associated with a 1.3 times proportional increase in aggregate perpetual consumption distributed equally across all individuals. For instance, if $\omega_t^i = 0.1$, a welfarist planner finds the welfare gain associated with a proportional increase in aggregate consumption distributed to individual i at date t equivalent to a 0.1 times proportional increase in aggregate perpetual consumption at all times distributed to that individual.

Lemma 2. *(Efficiency/Redistribution Decomposition) A normalized welfare assessment for a welfarist planner can be decomposed into efficiency and redistribution components, Ξ^E and Ξ^{RD}, as follows:*

$$\frac{dW^\lambda}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{\mathbb{C}ov_i^\Sigma \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{ (Redistribution)}}, \tag{10}$$

where $\mathbb{C}ov_i^\Sigma [\cdot, \cdot] = I \cdot \mathbb{C}ov_i [\cdot, \cdot]$ denotes a cross-sectional covariance-sum among individuals.

The efficiency component Ξ^E corresponds to Kaldor-Hicks efficiency, that is, it is the unweighted sum of individual willingness-to-pay for the perturbation in units of the lifetime welfare numeraire. Hence, perturbations in which $\Xi^E > 0$ can be turned into Pareto improvements if transfers of the lifetime welfare numeraire are feasible and costless. The redistribution component Ξ^{RD} captures the equity concerns embedded in a particular social welfare function: Ξ^{RD} is positive when the individuals relatively favored in a perturbation are those relatively preferred by the planner, that is, have a higher ω^i.

A perturbation in which $\Xi^E < 0$ cannot yield a Pareto improvement since there must be at least one individual who is worse off. If $\Xi^E > 0$ instead, a Pareto improvement with transfers would be possible if transfers of the lifetime welfare numeraire were feasible and costless. However, such transfers may not be feasible, as we explain next and we illustrate in our applications. Hence, $\Xi^E > 0$ is only a necessary condition for a perturbation to potentially become a Pareto improvement.

Regardless of whether economies are demographically connected or disconnected, the efficiency-redistribution decomposition in Lemma 2 satisfies several desirable properties. In particular, the efficiency component defined in (10) is invariant to i) the choice of social welfare function and ii) preference-preserving utility transformations.

3.3 Planning Solution in Demographically Disconnected Economies

No feasible perturbation from a Pareto efficient allocation can feature Kaldor-Hicks efficiency gains ($\Xi^E > 0$) in single-good economies with infinitely-lived agents.[7] In those economies, every Pareto efficient allocation can be interpreted as the solution to a planning problem supported by Pareto weights that ensure that ω^i is equalized across all individuals for every lifetime welfare numeraire. This occurs because the planner can freely transfer consumption at the margin across individuals at all dates. Hence, since the solution to the planning problem requires that $\frac{dW^\lambda}{d\theta} \leq 0$ for any feasible perturbation, and Pareto weights can be chosen so that ω^i is equalized across individuals ensuring that $\Xi^{RD} = 0$, then $\Xi^E \leq 0$ for every feasible perturbation.

[7]This result is valid for any lifetime welfare numeraire and solely assumes that individual preferences are well-behaved and feature Inada conditions. Whenever we refer to Pareto efficient allocations, we take as the set of feasible allocations those defined by resources constraints and (possibly) production and accumulation technologies. In other words, we refer to allocations that solve the Pareto problem, as defined in, for instance, Ljungqvist and Sargent (2018). It is possible to find perturbations of *constrained* Pareto efficient allocations that feature $\Xi^E > 0$.

Proposition 2, the second main result of the paper, shows that the logic just explained need not apply in demographically-rich economies.

Proposition 2. *(Feasible Perturbations from Pareto Efficient Allocations with Kaldor-Hicks Efficiency Gains)*

a) (Demographically Disconnected Economies) There exist feasible perturbations from Pareto efficient allocations in demographically disconnected economies that feature Kaldor-Hicks efficiency gains ($\Xi^E > 0$).

b) (Demographically Connected Economies) There exist feasible perturbations from Pareto efficient allocations in demographically connected economies that feature Kaldor-Hicks efficiency gains ($\Xi^E > 0$) only when the lifetime welfare numeraire is based on consumption at dates in which not all individuals are concurrently alive.

In order to explain Proposition 2, it is useful to characterize the set of Pareto efficient allocations by solving the Pareto problem. Formally, the Pareto problem characterizes the set of Pareto efficient allocations by varying the set of Pareto weights $\alpha^i > 0$ and computing the solution to

$$\max_{\{\chi_t^i\}} \sum_i \alpha^i V^i,$$

where $\chi_t^i = \frac{c_t^i}{c_t}$ denotes individual i's consumption share at date t with $c_t = \sum_i c_t^i > 0$ denoting aggregate consumption and V^i is defined in (1); subject to aggregate resource constraints for consumption, given by

$$\sum_i \chi_t^i = 1, \ \forall t, \tag{11}$$

and individual non-negativity constraints for consumption at each date, given by

$$\chi_t^i \geq 0, \ \forall i, t. \tag{12}$$

While in general the planning problem may include other constraints, such as production technologies, accumulation technologies, or other resource constraints, it is sufficient to consider the constraints in (11) and (12) to establish our result.

Denoting by $\eta_t > 0$ the (normalized) Lagrange multiplier in the aggregate resource constraint for consumption at date t, the solution to the planning problem must satisfy

$$\omega^i \omega_t^i = \eta_t > 0 \quad \text{if} \quad c_t^i > 0, \quad \text{and} \quad \omega^i \omega_t^i < \eta_t \quad \text{if} \quad c_t^i = 0, \tag{13}$$

where ω^i and ω_t^i are defined in (8) and (9). Intuitively, a planner optimally reallocates consumption to equalize the social marginal value of consumption across all individuals alive at a given date.

Aggregating the optimality conditions pertaining individual i in those periods in which i is alive allows us to obtain an explicit characterization of ω^i:

$$\omega^i \underbrace{\sum_t \omega_t^i}_{=1} = \omega^i = \sum_t \eta_t \mathbb{I}\left[\omega_t^i > 0\right], \tag{14}$$

where $\mathbb{I}[\cdot]$ denotes the indicator function. From (14), it follows that normalized individual weights ω^i are equalized across individuals at Pareto efficient allocations of economies with infinitely-lived agents — since the indicators equal 1 at all dates — but are not equalized in demographically disconnected economies. As shown in the Appendix, a similar argument applies to demographically connected economies in which the lifetime welfare numeraire is based on consumption at dates in which not all individuals are concurrently alive.[8] Once ω^i's are not equalized across individuals, then it follows from (13) that dynamic weights ω_t^i are also not equalized across individuals at date t.[9] This means that reallocating consumption from alive individuals with low to high dynamic weights at a particular date is a feasible perturbation that features Kaldor-Hicks efficiency gains, $\Xi^E > 0$, concluding the argument.

The fact that the solution of the planning problem features normalized individual weights ω^i that are not equalized across individuals implies that a planner has a *pecking-order* over which individual should receive a unit of the lifetime welfare numeraire, as we state in Remark 3. In economies with infinitely lived agents no pecking-order exists, and a planner is indifferent about which individual should receive a marginal unit of the lifetime welfare numeraire.

Remark 3. *(Individual pecking-order at Pareto efficient allocations)* At the allocation that solves the planning problem for particular Pareto weights, the planner has a pecking-order over which individual should receive a marginal unit of the lifetime welfare numeraire to maximize welfare in demographically disconnected economies or demographically connected economies in which the lifetime welfare numeraire is based on consumption at dates in which not all individuals are concurrently alive.

It may seem surprising that there are *feasible* perturbations to Pareto efficient allocations that increase the sum of individual willingness-to-pay. After all, when solving the planning problem, the planner can apparently transfer resources across all individuals at all times, without restrictions. Why doesn't the planner implement such perturbations? When non-negativity constraints are binding for an individual at some date t — as they are at all dates whenever individuals are not alive since it is optimal for them to have zero consumption — the planner cannot further reduce the

[8] Proposition 2b) implies that choosing lifetime numeraires based on consumption at dates in which all individuals are concurrently alive (demographically connected dates) is a good idea, insofar as it makes the properties of the planning problem similar to those in the standard infinitely-lived model.

[9] Note that $\omega^i \omega_t^i = \eta_t$ implies that individuals with high normalized individual weights ω^i must have relatively low dynamic weights ω_t^i at date t, and vice versa.

consumption of that individual. Hence, even though there may exist feasible perturbations that feature strictly positive Kaldor-Hicks efficiency gains, it is not possible to transform such gains into a feasible Pareto improvement, because the needed compensating transfers would violate the non-negativity constraints on consumption for some individual. We highlight this fact in Remark 4.

Remark 4. *(Infeasible compensating transfers)* The transfers that would be necessary to implement the compensating transfers that would turn the perturbations with $\Xi^E > 0$ identified in Proposition 2 into a Pareto improvement with transfers are not feasible.

Summing up, the results of this section imply that while finding perturbations with Kaldor-Hicks efficiency gains $\Xi^E > 0$ in economies with infinitely lived agents ensures that an allocation is not Pareto efficient, this need not be the case in demographically-rich economies. Our applications will illustrate this phenomena in both endowment and production economies.

3.4 Intertemporal-Sharing Decomposition: Markets vs. Demographics

The efficiency component of a normalized welfare assessment presented in Lemma 1 can be decomposed into an an *aggregate efficiency* component, which corresponds to the discounted sum — using a common discount factor ω_t — of aggregate normalized date welfare gains $\sum_i \frac{dV_t^{i|\lambda}}{d\theta}$ and an *intertemporal-sharing* component, which captures the differential impact of a perturbation towards individuals with different valuations ω_t^i over consumption at different dates.[10] The decomposition in Lemma 3 is the unique decomposition in which the efficiency component can be expressed as the discounted sum — using an aggregate time discount factor — of aggregate date welfare gains, Ξ^{AE}, and its complement, Ξ^{IS}.

Lemma 3. *(Aggregate Efficiency/Intertemporal-Sharing Decomposition) The efficiency component of a normalized welfare assessment can be decomposed into aggregate efficiency and intertemporal-sharing components, Ξ^{AE} and Ξ^{IS}, as follows:*

$$\Xi^E = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \underbrace{\sum_t \omega_t \sum_i \frac{dV_t^{i|\lambda}}{d\theta}}_{\Xi^{AE}\ \textit{(Aggregate Efficiency)}} + \underbrace{\sum_t \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}\ \textit{(Intertemporal-Sharing)}}, \tag{15}$$

where the averages of normalized dynamic weights $\omega_t = \frac{1}{I_t}\sum_i \omega_t^i$ define aggregate time discount factors, where $I_t = \sum_i \mathbb{I}\left\{i \mid \omega_t^i > 0\right\}$, and where $\mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}[\cdot,\cdot] = I \cdot \mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}[\cdot,\cdot]$ denotes a cross-sectional covariance-sum among individuals alive at date t.

In economies with infinitely-lived individuals, Dávila and Schaab (2024) show that $\Xi^{IS} = 0$ if all individuals can frictionlessly borrow and save. This occurs because ω_t^i is then identical across all

[10] When $\sum_i \frac{dV_t^{i|\lambda}}{d\theta} > 0$, the winners of the perturbation at date t could hypothetically compensate the losers in terms of the date welfare numeraire at that date (a unit of aggregate date-t consumption).

individuals since the marginal value over consumption at all dates is equalized under frictionless financial conditions. However, in demographically-rich economies, dynamic weights ω_t^i may differ across individuals even when all individuals have access to frictionless financial markets. This occurs because even if any two individuals can equalize the relative valuation of consumption over the dates in which both are concurrently alive by frictionlessly borrowing and saving, their individual valuation over consumption at a particular date relative to the lifetime welfare numeraire — which must be based on perpetual consumption as per Proposition 1 — will typically differ. From this observation, it is natural to conclude that welfare gains from intertemporal-sharing have two sources: one related to the presence of financial market frictions when borrowing and saving and the other due to demographic differences among individuals.

Example 2. *(Illustrating Differences in ω_t^i in Demographically-Rich Economies with Frictionless Financial Conditions)* Consider a three-date ($T = 2$) two-individual ($I = 2$) economy in which individual $i = 1$ is alive at dates 0, 1, and 2, and individual $i = 2$ is alive at dates 1 and 2. Let's assume that both individuals have a common discount factor β, and the same instantaneous utility $u(\cdot)$ when alive. In this case

$$\omega_1^1 = \frac{\beta u'\left(c_1^1\right) c_1}{u'\left(c_0^1\right) c_0 + \beta u'\left(c_1^1\right) c_1 + (\beta)^2 u'\left(c_2^1\right) c_2} = \frac{u'\left(c_1^1\right) c_1 + (\beta)^2 u'\left(c_2^1\right) c_2}{u'\left(c_0^1\right) c_0 + \beta u'\left(c_1^1\right) c_1 + (\beta)^2 u'\left(c_2^1\right) c_2} \frac{\beta u'\left(c_1^1\right) c_1}{u'\left(c_1^1\right) c_1 + (\beta)^2 u'\left(c_2^1\right) c_2} \tag{16}$$

$$\omega_1^2 = \frac{\beta u'\left(c_1^2\right) c_1}{\beta u'\left(c_1^2\right) c_1 + (\beta)^2 u'\left(c_2^2\right) c_2}. \tag{17}$$

Hence, while frictionless borrowing and saving implies that ω_1^2 equals the right-most fraction in (16), in general $\omega_1^1 \neq \omega_1^2$ in demographically-rich economies.

In the remainder of this section, we introduce a decomposition that allows us to separate how much of the intertemporal-sharing efficiency gains— that is, due to the fact that a perturbation differentially impacts individual with different valuations over consumption at different dates — are due to i) imperfect financial conditions, or ii) different demographics. Our approach is based on first multiplicatively decomposing the pairwise cross-sectional variation of the dynamic weights at a given date ω_t^i into a "markets" term and a "demographic" term, and then exploiting a formula to the compute covariance of the product of random variables, as we explain next.

First, we define all the ordered pairs (including self-pairs) of alive individuals at date t by $\mathcal{A}_t = \{(i,j) \,|\, \{i,j\} \in I_t\}$, and use the fact that the cross-sectional covariance-sum over all individual at date t can be expressed as a sum of pairwise covariance-sums across any two pairs of individuals. Formally, we can write the date-t component of intertemporal-sharing as

$$\underbrace{\mathbb{C}ov^{\Sigma}_{i|\omega_t^i>0}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\text{Cross-Sectional Covariance at } t} = \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \underbrace{\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{(i,j)}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]}_{\text{Pairwise Covariances at } t},$$

where $\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}[\cdot,\cdot]$ denotes a pairwise-covariance-sum (with just two elements).

Second, we multiplicatively decompose the dynamic weight of individual i in the pair (i,j) at date t as

$$\underbrace{\omega_t^{i,(i,j)}}_{\substack{\text{Pairwise Dynamic}\\ \text{Weight}}} = \underbrace{\omega_t^{i,(i,j)|m}}_{\text{Markets}} \underbrace{\omega^{i,(i,j)|d}}_{\text{Demographics}}, \tag{18}$$

where we define normalized pairwise-market-dynamic and pairwise-demographic-dynamic weights as follows:

$$\omega_t^{i,(i,j)|m} = \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_{t|\omega_t^i,\omega_t^j>0}\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} \quad \text{(Normalized Pairwise-Market-Dynamic Weight)}$$

$$\omega^{i,(i,j)|d} = \frac{\sum_{t|\omega_t^i,\omega_t^j>0}\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} \quad \text{(Normalized Pairwise-Demographic-Dynamic Weight).}$$

Intuitively, individual i's market-dynamic weight normalizes the welfare gain at date t to units of a common welfare numeraire, given by the value of aggregate consumption at all dates in which *both* individuals i and j are jointly alive. Individual i's demographic-dynamic weight corresponds to valuation of marginal unit of aggregate consumption of individual in all periods in which i and j are jointly alive, relative the value of a perpetual aggregate consumption for individual i. Note that the demographic-dynamic weight is time-independent and only relies on the marginal value of consumption during the common lifespan of the two individuals considered.

When i and j have access to unrestricted financial markets over their common lifespans, it is evident that $\omega_t^{i,(i,j)|d} = \omega_t^{j,(i,j)|d}$ for all pairs of individuals since their valuations of consumption across all dates are equalized, justifying the "market" label. Even if i and j have access to unrestricted financial markets over their common lifespans, it will typically be the case that $\omega^{i,(i,j)|d} \neq \omega^{j,(i,j)|d}$, justifying the "demographics" label. In particular, when any two individuals overlap for a single period $\omega_t^i = \omega_t^{i,(i,j)|m} = 1$, and the dynamic weight is exclusively due to demographics since there is no role for financial markets in that economy.

Third, we can use the formula for the product of random variables from Bohrnstedt and Goldberger (1969) to decompose what part of the pairwise covariances is due to cross-sectional variation at date t in the markets component or the demographic component. We present this decomposition as our third main result in Proposition 3.

Proposition 3. *(Intertemporal-Sharing Decomposition: Intertemporal-Market-Sharing vs. Intertemporal-Demographic-Sharing) The intertemporal-sharing component can be decomposed into intertemporal-*

market-sharing and intertemporal-demographic-sharing, as follows:

$$\Xi^{IS}=\underbrace{\sum_t \frac{1}{I_t}\sum_{(i,j)\in\mathcal{A}_t}\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega_t^{(i,j)|m},\frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]}_{\Xi^{IMS}\ \textit{(Intertemporal-Market-Sharing)}}+\underbrace{\sum_t \frac{1}{I_t}\sum_{(i,j)\in\mathcal{A}_t}\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}\left[\omega^{(i,j)|d},\frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]}_{\Xi^{IDS}\ \textit{(Intertemporal-Demographic-Sharing)}},\quad (19)$$

where $\mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t}[\cdot,\cdot]$ *denotes a pairwise-covariance-sum.*

Proposition 4 presents three properties of the intertemporal-sharing decomposition.

Proposition 4. *(Properties of Intertemporal-Sharing Decomposition)*

a) (Frictionless Borrowing/Saving) When any two pairs of individuals alive at any date t have identical marginal rates of substitution over consumption at the dates in which they are concurrently alive — a condition satisfied when all individuals frictionlessly borrow and save — then $\Xi^{IMS}=0$.

b) (Demographically-rich economies) In demographically-rich economies, the demographic component is generically non-zero, even when all individuals can frictionlessly borrow and save, so $\Xi^{IDS}\neq 0$.

c) (Frictionless Borrowing/Saving and Common Demographics) When any two pairs of individuals alive at any date t have identical marginal rates of substitution over consumption at the dates in which they are concurrently alive — a condition satisfied when all individuals frictionlessly borrow and save — and all individuals have identical birth and death dates, then $\Xi^{IS}=\Xi^{IMS}=\Xi^{IDS}=0$.

Finally, it is important to highlight that both markets and demographics components contribute to intertemporal-sharing in any demographically-rich economy, regardless of whether the economy is demographically connected or not, or whether a lifetime welfare numeraire based on consumption at dates in which not all individuals are concurrently alive is chosen. We illustrate the mechanics behind Proposition 3 in Application 3.

4 Application 1: Young-to-Old Transfer

This application analyzes the welfare impact of transfers across individuals from different generations in the simplest OLG economy: the two-date-life version of Samuelson (1958)'s endowment economy. This application allows us to illustrate how a planner makes interpersonal and intertemporal welfare comparisons in a demographically disconnected economy. In this economy, we show that a young-to-old transfer — a policy that has been widely analyzed and that provides the foundation for the study of social security and money — is a perturbation of a Pareto efficient allocation that generates efficiency gains, illustrating Proposition 2.

4.1 Environment

We consider a single-good deterministic endowment economy with $t \in \{0, \ldots, T\}$ dates, where $T < \infty$. A single representative individual is born at each date and typically lives for two dates, so there are two individuals alive at any date — one "young" and the other "old".[11] Since individuals are associated with a date of birth, we use t as superscript to index individuals $\{-1, \ldots, T\}$.

Preferences and Endowments. The lifetime utility of an individual born at date $t \in \{0, \ldots, T-1\}$ is

$$V^t = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right), \tag{20}$$

where c_t^t and c_{t+1}^t respectively denote individual t's consumption at dates t and $t+1$. Individuals born at dates $t = -1$ and $t = T$ are alive for only one date. Their lifetime utility is given by

$$V^{-1} = \beta u\left(c_0^{-1}\right) \qquad \text{and} \qquad V^T = u\left(c_T^T\right). \tag{21}$$

Individuals have endowments of the consumption good, denoted $\{e_t^t, e_{t+1}^t\}$ for individuals $t \in \{0, \ldots, T-1\}$, as well as e_0^{-1} and e_T^T for individuals $t \in \{-1, T\}$, respectively.

Competitive Equilibrium. Using an Arrow-Debreu style formulation, we express the budget constraints of individuals $t \in \{0, \ldots, T-1\}$ as

$$p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t, \tag{22}$$

where p_t denotes the price of date t consumption. There exists an equivalent formulation in which individuals sequentially trade a risk-free asset at all dates. Individuals $t \in \{-1, T\}$ face the budget constraints

$$p_0 c_0^{-1} \leq p_0 e_0^{-1} \qquad \text{and} \qquad p_T c_T^T \leq p_T e_T^T. \tag{23}$$

The resource constraint for date t consumption can be written as

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t. \tag{24}$$

Definition. *(Competitive Equilibrium) Given endowments $(e_{-1}^0, \{e_t^t, e_{t+1}^t\}_{t=0}^{T-1}, e_T^T)$, a competitive equilibrium comprises an allocation $(c_{-1}^0, \{c_t^t, c_{t+1}^t\}_{t=0}^{T-1}, c_T^T)$ and prices $\{p_t\}_{t=0}^T$ such that i) individuals maximize lifetime utility (20) – (21) subject to (22) – (23), and ii) consumption markets at each date clear (24).*

The unique competitive equilibrium in this economy features autarky ($c_t^i = e_t^i$). The ratios of prices

[11] We abstract from population growth in all three applications purely for simplicity. It is possible to introduce population growth in each application.

(interest rates) that support the autarky allocation must satisfy

$$\frac{p_{t+1}}{p_t} = \beta \frac{u'\left(e_{t+1}^t\right)}{u'\left(e_t^t\right)}, \tag{25}$$

where when needed we normalize $p_0 = 1$. Intuitively, there is no intertemporal trade between young and old individuals at any date since the old are no longer alive at future dates and there is a terminal date. When $T < \infty$, standard arguments imply that the first welfare theorem holds, so the competitive equilibrium is Pareto efficient.

Shares, Growth, and Young-to-Old Transfer. It is useful to formulate the model in terms of consumption and endowment shares. First, we define aggregate consumption and aggregate endowment at date t as $c_t = c_t^{t-1} + c_t^t$ and $e_t = e_t^{t-1} + e_t^t$, respectively. Hence, individual t's and $t-1$'s shares of date t aggregate consumption are $\chi_{t,c}^t = \frac{c_t^t}{c_t}$ and $\chi_{t,c}^{t-1} = \frac{c_t^{t-1}}{c_t}$, and of aggregate endowment $\chi_{t,e}^t = \frac{e_t^t}{e_t}$ and $\chi_{t,e}^{t-1} = \frac{e_t^{t-1}}{e_t}$. Both consumption and endowment shares add up to 1 by construction.

We assume that the aggregate endowment grows at a constant rate g, that is,

$$e_t = (1+g)^t e_0.$$

In equilibrium, aggregate consumption also grows a constant rate g.

Finally, we formulate a young-to-old (YTO) transfer as a smooth perturbation of endowment shares at all dates. Formally, we parametrize the model by initial endowment shares given by $\bar{\chi}_{t,e}^t$ and $\bar{\chi}_{t,e}^{t-1}$, and then consider a policy — indexed by a perturbation parameter $\theta \geq 0$ — that transfers endowments from young to old individuals at each date according to

$$\text{(Young-to-Old Transfer)} \qquad \chi_{t,e}^t = \bar{\chi}_{t,e}^t - \theta \qquad \text{and} \qquad \chi_{t,e}^{t-1} = \bar{\chi}_{t,e}^{t-1} + \theta, \tag{26}$$

In our calibration, the endowment share of the young is larger than the old, featuring $\bar{\chi}_{t,e}^t > \bar{\chi}_{t,e}^{t-1}$.

Calibration. Individual preferences are $u(c) = \log(c)$. We interpret one date in the model as 25 years, so $\beta = (0.98)^{25} = 0.60$. We normalize the date 0 aggregate endowment to $e_0 = 1$, and set the initial endowment shares of the young and old respectively to

$$\bar{\chi}_{t,e}^t = 0.75 \quad \text{and} \quad \bar{\chi}_{t,e}^{t-1} = 0.25.$$

Therefore, in the competitive equilibrium, individuals consume more when young than when old. The economy grows at a rate $g = 0.64 = (1.02)^{25} - 1$, which implies an annual growth rate of 2%. For ease of visualization, we assume that $T = 5$, so the economy runs for 125 years.

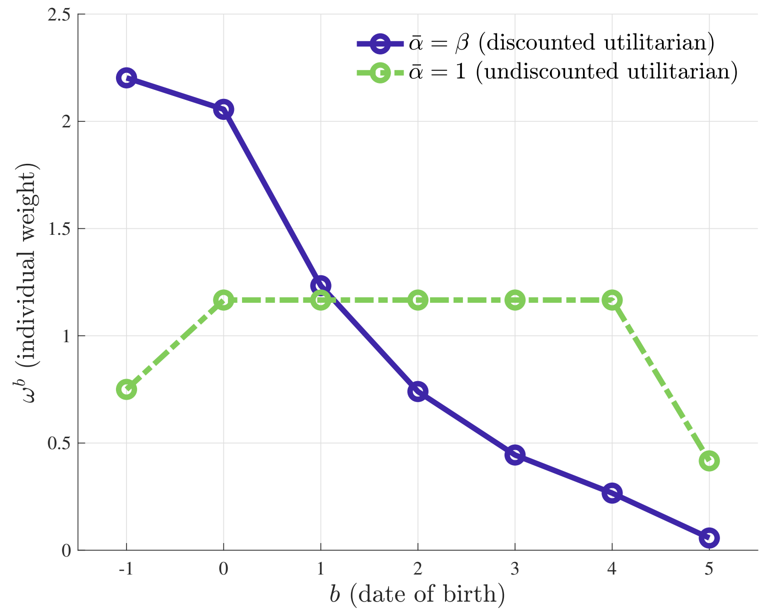


Figure 1: Normalized Individual Weights (Application 1)

Note. This figure shows normalized individual weights for all individuals as a function of their date of birth for the undiscounted and discounted utilitarian planners when $\theta = 0$.

4.2 Normalized Welfare Weights

Welfare Assessments. We study welfare assessments for utilitarian planners, for whom the welfare assessment of a perturbation $d\theta$ is given by

$$\frac{dW}{d\theta} = \sum_{t=-1}^{T} \alpha^t \frac{dV^t(\theta)}{d\theta}, \tag{27}$$

where α^t denotes the Pareto weight associated with the individual born at date t. To simplify the exposition, we focus on two cases: i) an undiscounted utilitarian planner (our benchmark), who puts the same weight on the lifetime utility of all individuals, so $\alpha^t = 1$, and ii) a discounted utilitarian planner, who exponentially discounts the utilities of future generations, with $\alpha^t = (\bar{\alpha})^t$ for a constant $\bar{\alpha} \in (0,1)$, which we take to be $\bar{\alpha} = \beta$, the individuals' discount factor.

As in Section 3, we use aggregate perpetual consumption — the bundle that pays aggregate consumption at each date — as the lifetime welfare numeraire. The normalizing factor λ^t for individuals $t \in \{0, \ldots, T-1\}$ is therefore

$$\lambda^t = u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1}, \tag{28}$$

as well as $\lambda^{-1} = \beta u'\left(c_0^{-1}\right) c_0$ and $\lambda^T = u'\left(c_T^T\right) c_T$ for individuals $t \in \{-1, T\}$, where c_t denotes aggregate consumption at date t.[12]

[12] Choosing unit perpetual consumption — the other natural choice in this environment — does not change any of

Normalized Individual Weights. By computing normalized individual weights, we uncover the value assigned by different planners to the lifetime welfare gains of different individuals in units of the lifetime welfare numeraire. Defined in equation (8), normalized individual weights are given by

$$\omega^t = \frac{\alpha^t \lambda^t}{\frac{1}{I}\sum_{t=-1}^{T} \alpha^t \lambda^t}, \tag{29}$$

where λ^t is defined in (28) and where the number of individuals in this economy is $I = T + 2$. Normalized individual weights are thus shaped by i) marginal utilities of aggregate perpetual consumption and ii) Pareto weights. In general, the former depends on individual lifespans, discount factors, and aggregate and individual consumption levels. With isoelastic utility, individual t's marginal utility of aggregate consumption at date t can be written as

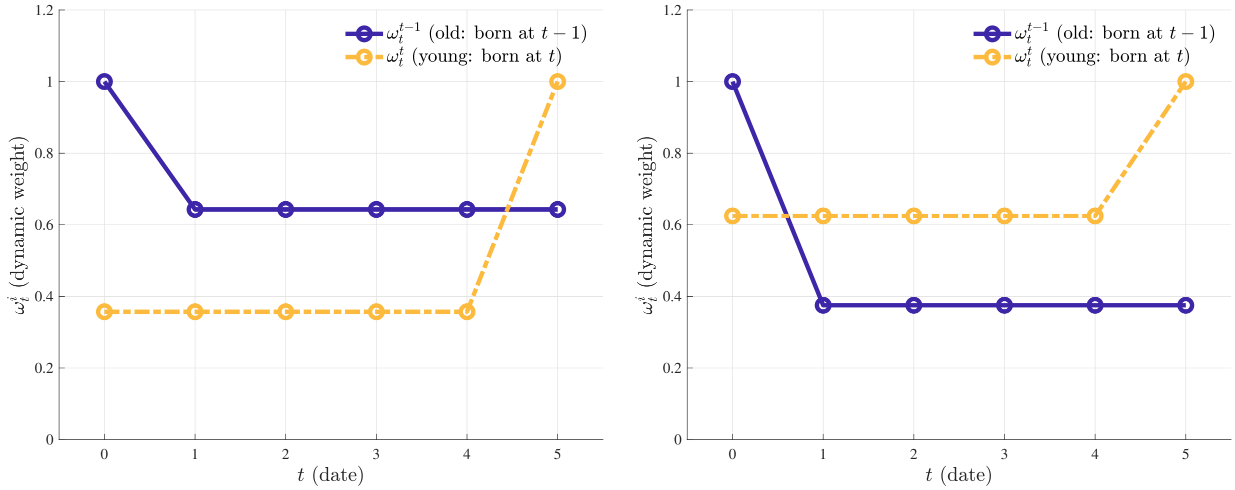
$$u'\left(c_t^t\right) c_t = \frac{1}{\left(\chi_t^t\right)^{\frac{1}{\psi}}} \left(c_t\right)^{1-\frac{1}{\psi}}, \tag{30}$$

which is useful to separate the impact of consumption shares relative to aggregate consumption. Since we focus on log preferences, equation (30) implies that λ^t is exclusively a function of individual consumption shares.[13]

Figure 1 illustrates the forces that determine the normalized individual weights ω^t. The left panel illustrates the form that individual weights take for undiscounted and discounted utilitarian planners. The undiscounted utilitarian planner assigns lower weights to shorter lived individuals $t \in \{-1, T\}$. When individuals live longer, all else equal, they value a marginal unit of perpetual aggregate consumption more because they are alive to receive the consumption flow in more periods, which increases their individual weight. The date-0-old individual has a higher weight than the date-T-young because in our calibration young individuals consume more. If the planner discounts individuals born in the future, so $\bar{\alpha} = \beta < 1$, the forces just discussed remain, but normalized individual weights become lower for individuals born later in time. Figure 1 implicitly defines a *pecking-order* over different individuals by each planner. While the undiscounted utilitarian planner ranks highest the longer lived individuals, then the date-0-old, and at last the date-0-young, the discounted utilitarian planner simple ranks individuals by their date of birth.

the qualitative insights. In fact, in economies without growth, the results are quantitatively identical regardless of whether we adopt unit or aggregate perpetual consumption as lifetime welfare numeraires.

[13] In general, the marginal utility of aggregate consumption $u'\left(c_t^t\right) c_t$ depends on aggregate consumption in addition to consumption shares. As aggregate consumption c_t increases, for a given consumption share χ_t^t, two effects materialize: on the one hand, there is more consumption (substitution effect), but consumption is valued less (income/valuation effect), with the net of both effects captured by $1 - \frac{1}{\psi}$ exponent. If $\psi > 1$, individuals have a high willingness to substitute intertemporally, so the first effect dominates. If $\psi < 1$, individuals are unwilling to substitute intertemporally, so the second effect dominates. In economies with growth, these effects can impact both individual and dynamic weights, even when consumption shares remain fixed.



(a) No transfer ($\theta = 0$)

(b) Positive transfer ($\theta = 0.25$)

Figure 2: Normalized Dynamic Weights (Application 1)

Note. This figure illustrates the normalized dynamic weights before the young-to-old transfer is implemented (left panel, $\theta = 0$, so $\chi_{t,c}^t = 0.75$ and $\chi_{t,c}^{t-1} = 0.25$), and once the welfare-maximizing transfer for the undiscounted utilitarian planner is implemented (right panel, $\theta = 0.25$, so $\chi_{t,c}^t = \chi_{t,c}^{t-1} = \frac{1}{2}$).

Normalized Dynamic Weights. For individuals $t \in \{0, \dots, T-1\}$, the normalized dynamic weights, defined in equation (9), are given by

$$\omega_t^t = \frac{u'\left(c_t^t\right) c_t}{u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1}} \quad \text{and} \quad \omega_{t+1}^t = \frac{\beta u'\left(c_{t+1}^t\right) c_{t+1}}{u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1}}. \tag{31}$$

For individuals $t \in \{-1, T\}$, who are only alive at one date, $\omega_0^{-1} = \frac{\beta u'\left(c_0^{-1}\right) c_0}{\beta u'\left(c_0^{-1}\right) c_0} = 1$ and $\omega_T^T = \frac{u'\left(c_T^T\right) c_T}{u'\left(c_T^T\right) c_T} = 1$. Intuitively, normalized dynamic weights define marginal rates of substitution for each individual between date-t and perpetual aggregate consumption, and capture whose consumption the planner values more at each date. A policy that shifts consumption from individuals with low to high normalized dynamic weights at a given date will generate efficiency gains due to intertemporal-sharing. Individuals who only consume at one date are willing to exchange one-for-one a unit of aggregate consumption at the date in which they are alive for a unit of perpetual aggregate consumption.

Figure 2 shows the dynamic weights (vertical axis) of the individuals alive at a given date t (horizontal axis). In this calibration, an individual's endowment is relatively large when young and relatively small when old, so old individuals have a higher normalized dynamic weight than young individuals at all dates, with the exception of the last date. In the left panel of Figure 2, when $\theta = 0$, the solid blue line is above the dashed yellow line for all individuals alive for two dates. Concretely, a transfer of consumption from the young to the old at, for instance, date $t = 2$ is desirable because $\omega_2^1 = 0.64 > 0.36 = \omega_2^2$. Hence, at each date in which the weight of the old is above the weight of the young, a young-to-old transfer generates efficiency gains. The only date in which such a transfer would lead to efficiency losses is the final date $T = 5$, where the dynamic weight of the young is larger than that of the old.

The right panel of Figure 2 shows the normalized dynamic weight once the welfare-maximizing transfer for the discounted utilitarian planner is implemented ($\theta = 0.25$). In this case, consumption shares are equalized across agents alive at period t. However, due to private discounting ($\beta < 1$), individuals alive for two dates value consumption more when young. Therefore, a marginal young-to-old transfer at $\theta = 0.25$ generates efficiency losses at each date but the initial date $t = 0$, which justifies the negative sign of the efficiency component of the welfare decomposition when $\theta = 0.25$, shown in Figure 4.

4.3 Policy Experiment: Young-to-Old Transfer

Individual Welfare Gains. To understand the aggregate welfare implications of a young-to-old (YTO) transfer in the economy considered here, it is useful to first explore the impact of the policy on individual welfare. Figure 3 plots normalized welfare gains of the YTO transfer for each individual. The initial old (individual $t = -1$) always gains from the perturbation, whereas the final young (individual $t = T$) always loses from the perturbation, simply because their consumption is higher and lower, respectively. All other individuals initially benefit from the YTO transfer since their consumption becomes smoother. Starting from $\theta = 0.125$, further increases in the YTO transfer hurt consumption smoothing. Note that these smoothing gains would disappear if individual utilities were linear in consumption.

Figure 3 shows that while the YTO transfer initially benefits all but one individual, it is not a Pareto improvement: the final young is always worse off. Turning this policy into a Pareto improvement requires compensating the final young for her welfare loss, but such a compensation is not feasible. A technology that could turn a YTO transfer into a Pareto improvement would require transferring resources to the initial old: this is an investment technology not available in this economy. Note that, while the YTO transfer can never be a Pareto improvement for finite T, the welfare gains from such a perturbation increase with T because more generations of individuals benefit from smoother consumption, while only the final young loses.

Welfare Decomposition. Figure 4 plots the efficiency/redistribution decomposition (10) for this application. Since this is an endowment economy, in which aggregate consumption remains fixed, $\Xi^E = \Xi^{IS}$, so all efficiency gains due to a YTO transfer are due to intertemporal-sharing. Also, since individuals overlap for a single period, all intertemporal-sharing gains are exclusively due to intertemporal-demographic-sharing, defined in Proposition 3, so $\Xi^{IS} = \Xi^{IDS}$.

Initially, the YTO transfer generates both efficiency and redistribution gains. Intuitively,

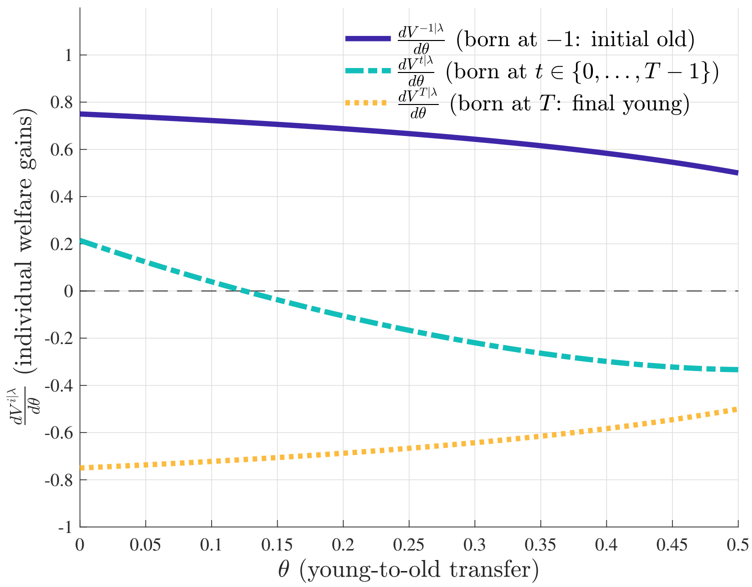
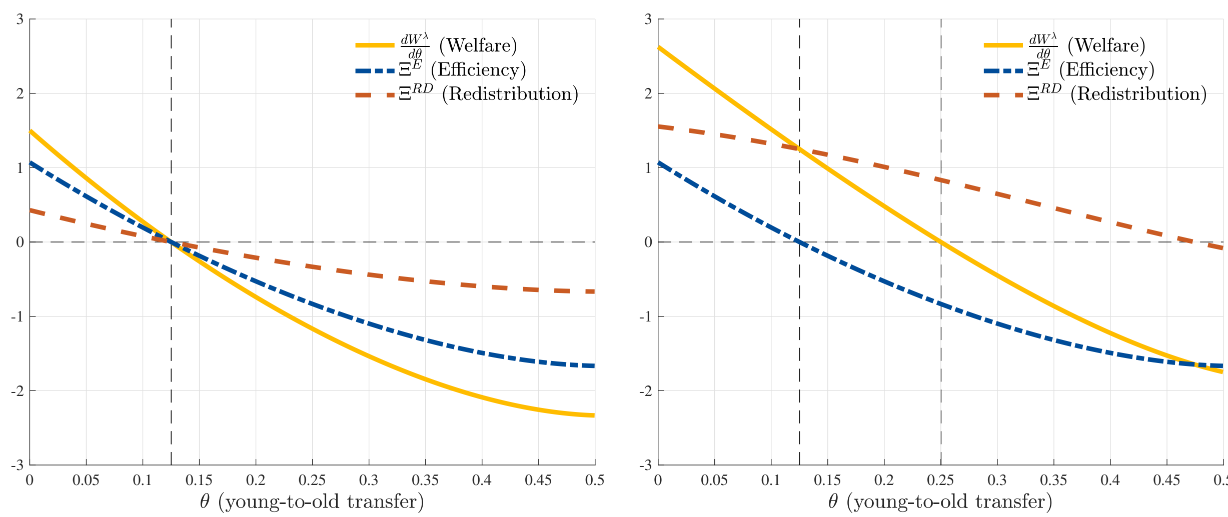


Figure 3: Normalized Individual Welfare Gains (Application 1)

Note. This figure shows the normalized individual welfare gains, defined by $\frac{dV^{i|\lambda}}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta}$. The light green dashed line corresponds to the normalized individual welfare gains of all individuals alive for 2 periods. The dark blue solid line represents the gains of the initial old born at date -1. The yellow dotted line corresponds to the gains of the final young born at date T.



(a) Undiscounted Utilitarian

(b) Discounted utilitarian

Figure 4: Welfare Decomposition (Application 1)

Note. This figure illustrates the efficiency/redistribution decomposition, defined in equation (10). The left panel shows the welfare assessment by an undiscounted utilitarian planner ($\bar{\alpha} = \beta$), and the right panel shows the welfare assessment by a discounted utilitarian planner ($\bar{\alpha} = 1$). Because this is an endowment economy, efficiency is exclusively due to intertemporal-sharing, so $\Xi^E = \Xi^{IS}$. Because individuals overlap for a single date, intertemporal-sharing is exclusively due to demographics, so $\Xi^{IS} = \Xi^{IDS}$.

the efficiency term is solely driven by the sum of consumption-smoothing gains from individuals $t \in \{0, \ldots, T-1\}$, with the gains of the initial old exactly compensating the losses of the final young in units of the lifetime welfare numeraire. The efficiency gains are positive until $\theta = 0.125$, at which point consumption-smoothing gains are exhausted. Since efficiency gains are invariant to the choice of social welfare function, $\theta = 0.125$ is the size of the efficiency-maximizing transfer for *any* welfarist planner.[14]

The YTO transfer also generates redistribution gains initially for the two planners considered. An undiscounted utilitarian planner assigns the lowest normalized individual weight to the final young, so the YTO transfer, which relatively favors all other individuals, generates positive gains from redistribution. It turns out that $\theta = 0.125$ is the size of YTO transfer that maximizes both redistribution and efficiency for this planner. A discounted utilitarian planner values those individuals born in later dates even less, thus finding larger gains from redistribution. This planner finds a larger transfer of $\theta = 0.25$ optimal.

Broader Takeaway. This application shows that a YTO transfer is a perturbation of a Pareto efficient allocation that generates (Kaldor-Hicks) efficiency gains. This application therefore shows that Proposition 2 applies to one of the foundational OLG models. As explained when introducing Proposition 2, this phenomenon can occur because the economy is demographically disconnected, which makes it infeasible for the winners of the YTO transfer to compensate the losers despite the aggregate gain in value.

5 Application 2: Capital Taxation

This application analyzes the welfare consequences of taxing capital in the simplest OLG model with capital: the finite horizon version of Diamond (1965)'s growth model. Even though the competitive equilibrium of this economy is Pareto efficient, we show that capital taxes or subsidies can generate efficiency gains. In particular, we show that the efficiency-maximizing capital tax is positive in low-capital-share economies but negative (i.e. a subsidy) in high-capital-share economies. This application shows that it is possible to perturb Pareto efficient allocations in a production economy to generate efficiency gains, providing a new illustration of Proposition 2. More broadly, this application illustrates how the results of the paper can be used to define notions of over- and under-accumulation of capital in economies that are Pareto optimal.

[14] It is worth highlighting that the welfare decomposition jumps in the limit when $T = \infty$. In every finite T economy, aggregate consumption is fixed and efficiency gains are due to intertemporal-sharing. But when $T = \infty$, the YTO transfer also generates aggregate efficiency gains due to the increase in aggregate consumption at infinity. This is the type of $T = \infty$ phenomenon we have purposefully stayed away from in this paper.

5.1 Environment

We study the finite horizon textbook version by Blanchard and Fischer (1989) of the overlapping generations model with capital of Diamond (1965). This is a perfect foresight economy with $t \in \{0, \ldots, T\}$ dates, where $T < \infty$.

Individuals. A single representative individual is born at each date and typically lives for two dates, so there are two individuals alive at any date, young and old. Since individuals are associated with a date of birth, we use t as superscript to index individuals $\{-1, \ldots, T\}$.

The lifetime utility of an individual born at date $t \in \{0, \ldots, T-1\}$ is

$$V^t = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right), \tag{32}$$

where c_t^t and c_{t+1}^t respectively denote individual t's consumption at dates t and $t+1$.

Individuals born at dates $t = -1$ and $t = T$ are alive for only one date. Their lifetime utility is given by

$$V^{-1} = \beta u\left(c_0^{-1}\right) \qquad \text{and} \qquad V^T = u\left(c_T^T\right). \tag{33}$$

When young, individuals supply one unit of labor inelastically and decide how much of their income to consume (c_t^t) and save. The only savings vehicle available is capital. Young individuals can either purchase a quantity k_t of capital on a secondary market at a price q_t or invest ι_t in new capital. The budget constraint of a young individual at date t is therefore

$$c_t^t + \iota_t + q_t k_t = w_t, \tag{34}$$

where w_t is the wage. Capital accumulates neoclassically between periods t and $t+1$, according to

$$k_{t+1} = (1-\delta)\, k_t + \iota_t, \tag{35}$$

where δ is the depreciation rate.

When old, individuals no longer work, but they rent the capital they have accumulated to firms and subsequently sell it on the secondary market.[15] The budget constraint of an old individual at date $t+1$ is therefore given by

$$c_{t+1}^t = (1-\tau)\left(q_{t+1} + d_{t+1}\right) k_{t+1} + T_{t+1}, \tag{36}$$

[15] The timing assumption is therefore as follows: At the beginning of period t, old individuals rent their capital to firms and receive rental income. At the end of period t, old individuals sell their capital to young individuals at price q_t. Since capital was already used in production that period, young individuals have to wait until period $t+1$ before they can make use of their newly acquired capital by renting it to firms. And in the meantime, a fraction δ of this capital will have depreciated.

where d_{t+1} is the rental rate. We allow for a tax on capital τ and a lump-sum rebate T_{t+1}.

Firms. At each date t, a representative firm produces y_t units of the final consumption good with a constant-returns production function $y_t = f(k_t, n_t)$ that uses capital and labor. The firm maximizes profit $y_t - w_t n_t - d_t k_t$.

Government. We assume that the government budget must balance in each period. Revenue from the capital tax is therefore fully rebated to the old individuals, which requires that

$$T_t = \tau (q_t + d_t) k_t.$$

Competitive equilibrium.

Definition. *(Competitive Equilibrium)* Given an initial capital stock k_0 and a capital tax τ, a competitive equilibrium comprises allocations $\{y_t, c_t, \iota_t, n_t, k_t, T_t\}$ and prices $\{w_t, d_t, q_t\}$ such that (i) individuals maximize lifetime utility (32) – (33) subject to (34), (35), and (36), (ii) firms maximize profits, (iii) the government budget balances, and (iv) markets clear, that is,

$$y_t = c_t + \iota_t \tag{37}$$

$$1 = n_t, \tag{38}$$

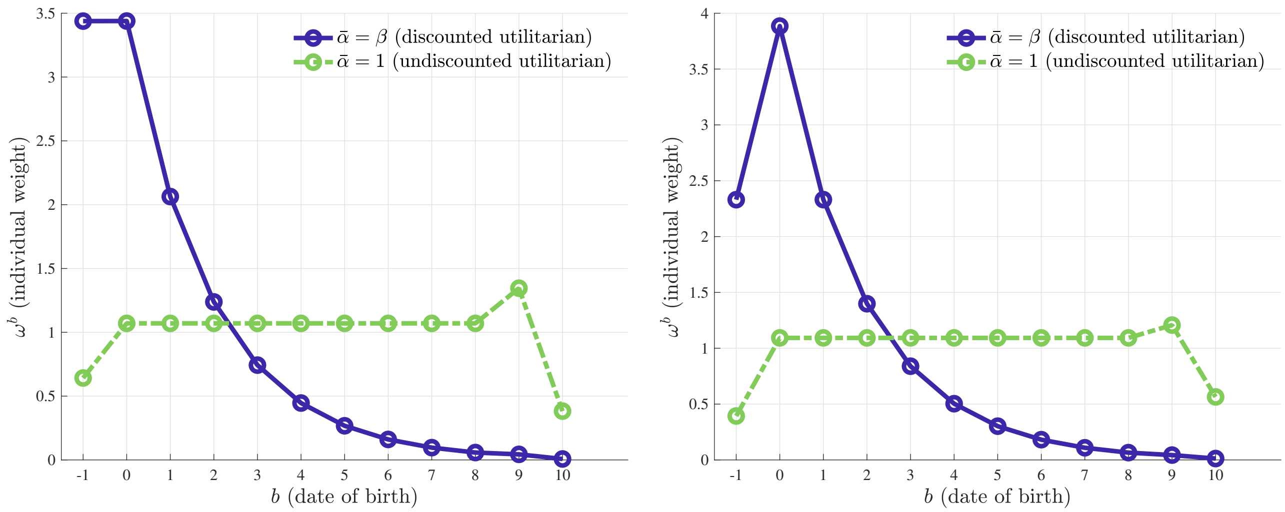
where $c_t = c_t^{t-1} + c_t^t$ denotes aggregate consumption at date t.

In equilibrium, young individuals must be indifferent between acquiring capital on the secondary market or via new investment. The secondary market price must therefore satisfy $q_t = 1 - \delta$ for dates $t \leq T - 1$. At the final date, $q_T = 0$ since capital has no future use. Firm optimization implies $\frac{\partial f}{\partial k_t} = d_t$ and $\frac{\partial f}{\partial n_t} = w_t$.

Calibration. Individual preferences are $u(c) = \log(c)$. We interpret one date in the model as 25 years, so $\beta = (0.98)^{25} = 0.60$. Consistent with this choice, we set $\delta = 1$, so capital depreciates fully between dates. The production technology is Cobb-Douglas, $f(k, n) = k^a n^{1-a}$, where a maps to the capital share. We contrast calibrations with two different values of the capital share a:

$$a = \begin{cases} 0.2 & \Rightarrow \text{low-capital-share/high-labor-share} \\ 0.4 & \Rightarrow \text{high-capital-share/low-labor-share} \end{cases}$$

By assuming that $T = 10$ the economy runs for 250 years. We assume that $k_{-1} = 0.01$, so this economy initially experiences growth before reaching a steady state in both calibrations.



(a) Low-capital-share ($a = 0.2$)

(b) High-capital-share ($a = 0.4$)

Figure 5: Normalized Individual Weights (Application 2)

Note. This figure shows the normalized individual weights as a function of the date of birth for the undiscounted and discounted utilitarian planners when $\tau = 0$ for the low-capital-share economy ($a = 0.2$, left panel) and the high-capital-share economy ($a = 0.4$, right panel).

5.2 Normalized Welfare Weights

Welfare Assessments. We study welfare assessments for utilitarian planners, for whom the welfare assessment of a perturbation $d\theta$ is given by

$$\frac{dW}{d\theta} = \sum_{t=-1}^{T} \alpha^t \frac{dV^t(\theta)}{d\theta}, \tag{39}$$

where α^t denotes the Pareto weight associated with the individual born at date t. Once again, we focus on undiscounted and discounted utilitarian planners and we use aggregate perpetual consumption as the lifetime welfare numeraire, so equation (28) and its counterpart for individuals $t \in \{-1, T\}$ also apply here.

Normalized Individual Weights. Normalized individual weights capture the value assigned by a planner to the lifetime welfare gains of different individuals in units of the lifetime welfare numeraire. They are defined as in equation (29).

Two forces shape the normalized individual weights — shown in Figure 5 — for the undiscounted utilitarian planner. First, as in Application 1, the initial old and the final young live less, so the planner assigns lower individual weights to them. Second, the planner assigns a higher individual weight to individuals born at $T-1$ since they have a lower consumption share when old at date T because the date-T young do not invest. Moreover, the discounted utilitarian planner assign lower normalized individual weights to individuals born later in time purely because she directly discounts the utility of individuals born in the future. Qualitatively, normalized individual weights are similar in high- and low-capital-share economies.

Normalized Dynamic Weights. The normalized dynamic weights define marginal rates of substitution for each individual between date-t and perpetual aggregate consumption, and capture whose consumption the planner values more at each date. They are defined as in equation (28) and its counterpart for individuals $t \in \{-1, T\}$ also applies here.

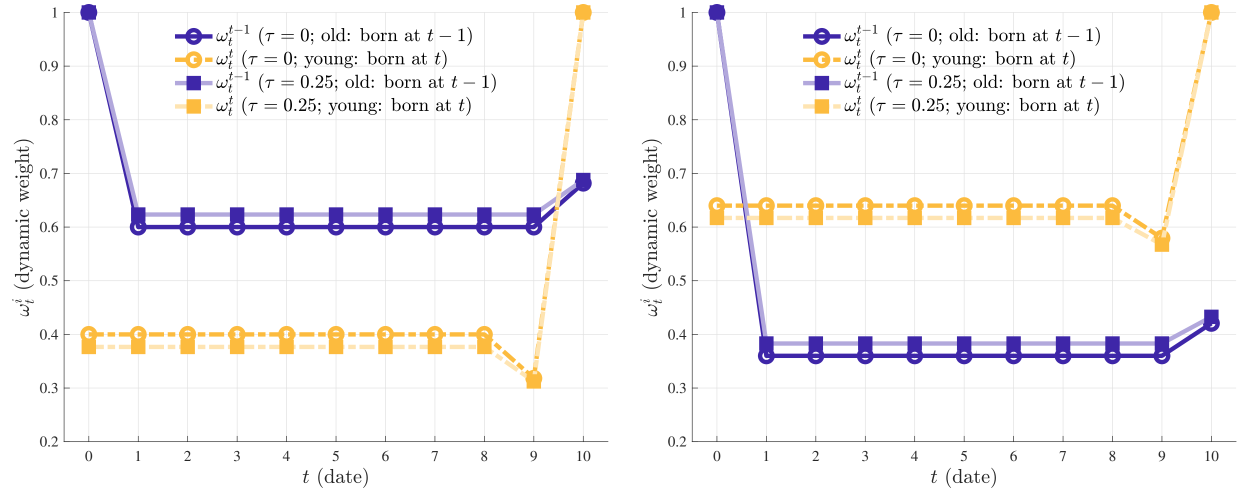
Figure 6 shows the dynamic weights (vertical axis) of the individuals alive at a given date t (horizontal axis). In the baseline low-capital-share economy (left panel), young individuals consume more than old individuals in the absence of the tax, so old individuals have higher normalized dynamic weights than young individuals at all dates, with the exception of the last date. As the capital tax increases, the young invest less and consume more, further widening the consumption gap relative to the old and further widening the gap between normalized dynamic weights. Therefore, increasing the capital tax features negative intertemporal-sharing, consistent with Figure 7.

In the high-capital-share economy (right panel), old individuals consume more than young individuals in the absence of the tax, so young individuals have higher normalized dynamic weights than young individuals at all dates, with the exception of the first date. As the capital tax increases, the young invest less and consume more, narrowing the consumption gap relative to the old as well as the gap between normalized dynamic weights. Therefore, increasing (reducing) the capital tax starting from $\tau = 0$ features a negative intertemporal-sharing component in the low-capital-share economy but a positive one in the high-capital-share economy, consistent with Figure 7. Note that all intertemporal-sharing gains and losses in this economy are entirely due to demographics since there is no date in which any two individuals would enter in financial arrangements.

5.3 Policy Experiment: Capital Tax

Welfare Decomposition. Figure 7 plots the welfare/redistribution decomposition (10) and the decomposition of efficiency in aggregate efficiency and intertemporal-sharing (15) for this application. In contrast to the previous two applications, efficiency gains now comprise both aggregate efficiency and intertemporal-sharing gains since we are considering a production economy rather than an endowment economy.

Interestingly, in both low- and high-capital-share economies, intertemporal-sharing and aggregate efficiency take opposite signs at $\tau = 0$. In the low-capital-share economy (left panel), an increase in the capital tax increases aggregate consumption at all dates, with the exception of the last one, thus featuring a positive aggregate efficiency component. This gain overcomes the intertemporal-sharing losses described when introducing the dynamic weights. Overall, the efficiency-maximizing tax in the low-capital-share economy is $\tau^\star = 0.25$. In the high-capital-share economy (right panel), an increase in the capital tax reduces aggregate consumption at all dates,



(a) Low-capital-share ($a = 0.2$)　(b) High-capital-share ($a = 0.4$)

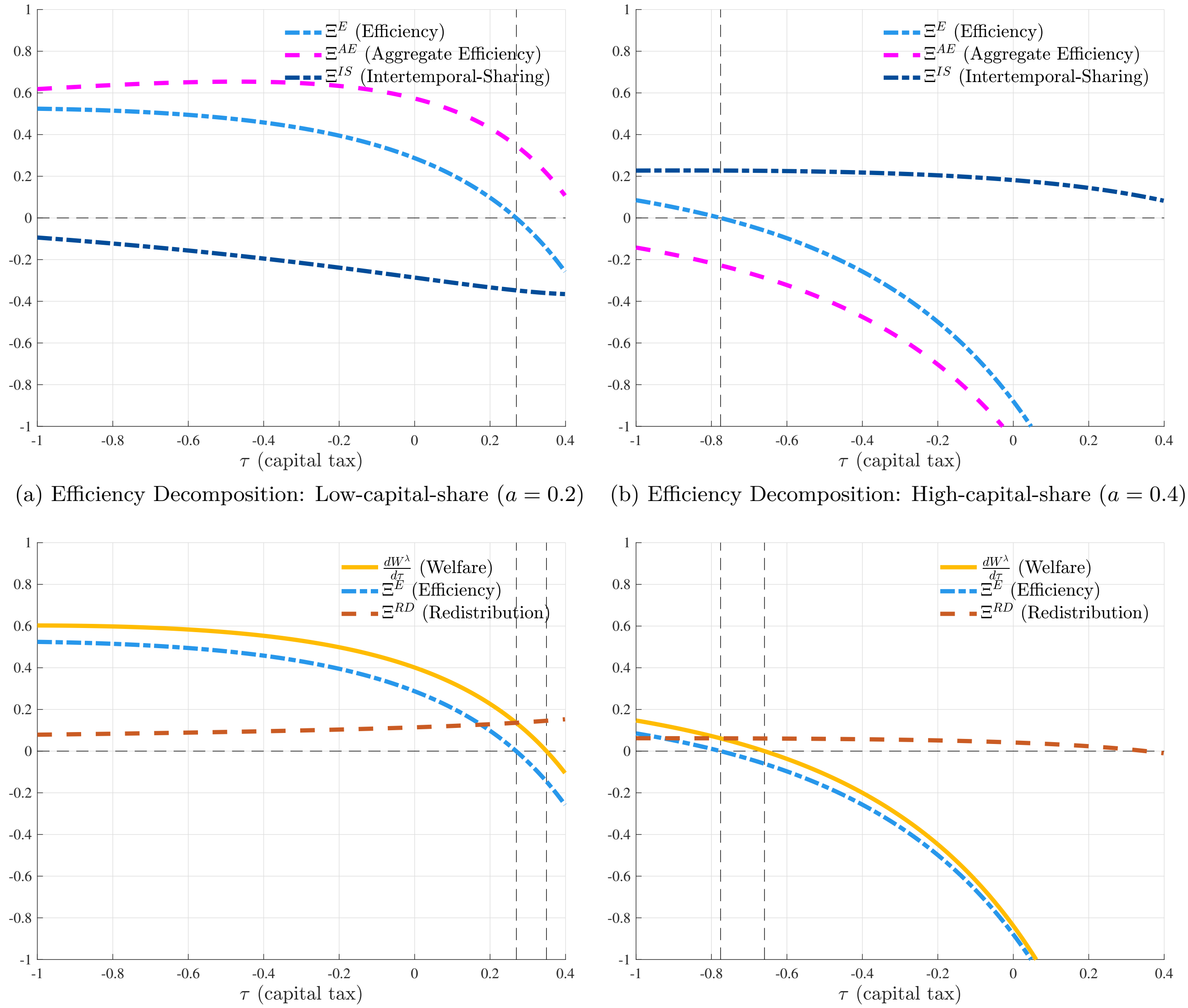
Figure 6: Normalized Dynamic Weights (Application 2)

Note. This figure illustrates the normalized dynamic weights for the low-capital-share economy (left panel) and the high-low-capital share economy (right panel) in the absence of the capital tax (left panel, $\tau = 0$), and when a capital tax of $\tau = 0.25$ is implemented: this is the welfare-maximizing tax for the undiscounted utilitarian planner in the low-capital-share economy.

with the exception of the first one, thus featuring a negative aggregate efficiency component — see also Figure 8 below. This loss overcomes the intertemporal-sharing gains described when introducing the dynamic weights. Overall, the efficiency-maximizing tax in the low-capital-share economy is negative (a subsidy): $\tau^\star = -0.7$.

The bottom panels in Figure 7 illustrate the role played by the choice of social welfare function, undiscounted utilitarian in this case. In both the low- and high-capital-share economies, a tax increase starting from $\tau = 0$ always impacts most negatively the last young individual, but this is the individual with the lowest normalized individual weight, as shown in Figure 5. Hence, regardless of the calibration, higher capital taxes are associated with a positive redistribution component.

Term Structure. At last, Figure 8 brings additional insight into aggregate efficiency by showing the term structure of the aggregate efficiency gains. This is useful to determine the horizon in which the aggregate consumption gains that shape aggregate efficiency materialize. In both low- and high-capital-share economies, imposing a capital tax discourages investment and increases aggregate consumption at impact at date 0, which is reflected as a positive date-0 aggregate efficiency component, Ξ_0^{AE}. In the low-capital-share economy, an increase in τ starting from $\tau = 0$ increases consumption at all dates but the last date. One could say that this is an economy which overaccumulates capital, in the sense that discouraging investment increases aggregate consumption at (almost) all times. The opposite occurs in a high-capital-share economy. One could say that this is an economy underaccumulates capital, in the sense that discouraging investment reduces



(a) Efficiency Decomposition: Low-capital-share ($a = 0.2$)

(b) Efficiency Decomposition: High-capital-share ($a = 0.4$)

(c) Welfare Decomposition: Low-capital-share ($a = 0.2$):

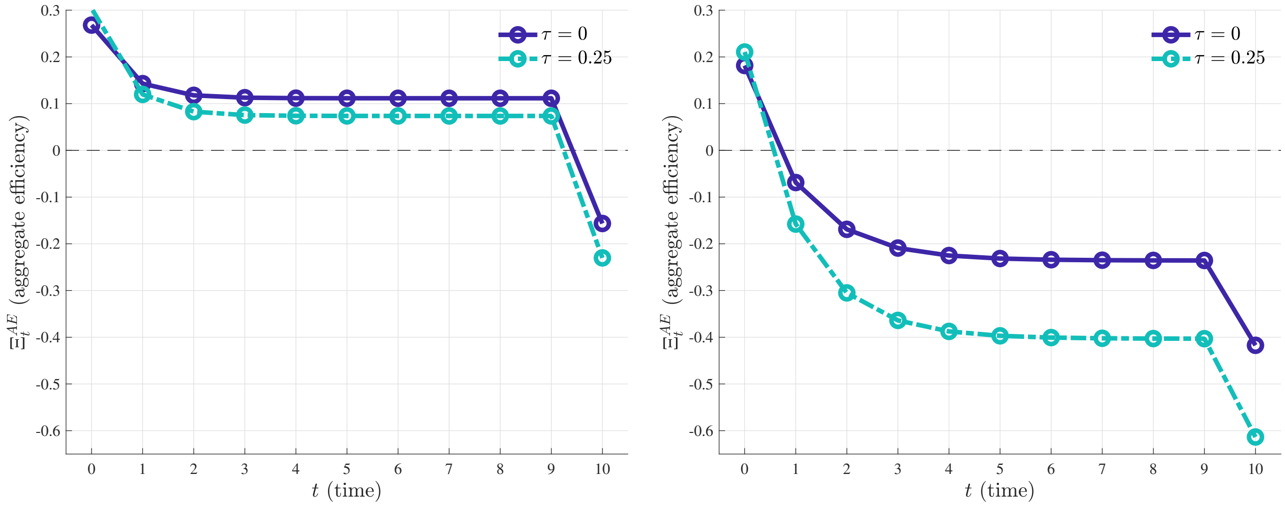
(d) Welfare Decomposition: High-capital-share ($a = 0.4$)

Figure 7: Efficiency and Welfare Decompositions (Application 2)

Note. This figure shows the welfare assessment and its components.. The left panel shows the welfare assessment by a discounted utilitarian planner ($\bar{\alpha} = \beta$), and the right panel shows the welfare assessment by an undiscounted utilitarian planner ($\bar{\alpha} = 1$).

aggregate consumption at almost all times. In both cases, increasing τ reduces capital overall accumulation, which reduces aggregate consumption at the terminal date T, in which the available capital stock is consumed.

Broader Takeaway. What is therefore the broader takeaway from this application? The prior literature (Diamond, 1965; Abel et al., 1989) has focused on characterizing the conditions under which Pareto improvements are possible. These are rare and rely on an infinite time horizon. The economy we characterize here is Pareto efficient, since a capital tax always makes the individual



(a) Aggregate Efficiency: Low-capital-share ($a = 0.2$) (b) Aggregate Efficiency: High-capital-share ($a = 0.4$)

Figure 8: Term Structure of Aggregate Efficiency Gains (Application 2)

Note. This figure shows the term structure of welfare assessments, illustrating the policy's impact under two scenarios: no capital tax, $\tau = 0$, and a positive tax, $\tau = 0.25$. Panel (a) to (c) show the three components (Ξ_t^{AE}, Ξ_t^{IS} and Ξ_t^{RD}), while Panel (d) shows their sum, i.e., $\frac{dW_t^\lambda}{d\tau} = \Xi_t^{AE} + \Xi_t^{IS} + \Xi_t^{RD}$.

born in the last period strictly worse off. And yet, there may be Kaldor-Hicks efficiency gains from reducing the aggregate capital stock in low-capital share economies. Our results illustrate once again Proposition 2, but in a production economy — showing how it is possible to find efficiency gains at a Pareto efficient allocation of a demographically disconnected.

6 Application 3: Markets vs. Demographics

This application analyzes the welfare impact of transfers between middle-aged and old individuals in the simplest OLG economy in which individuals use financial markets to smooth consumption: the three-date-life version of Samuelson (1958)'s endowment economy. This application allows us to illustrate the intertemporal-sharing decomposition introduced in Proposition 3, distinguishing between the intertemporal-sharing efficiency gains that arise due to the fact that markets are incomplete and the fact that individuals are demographically different.

6.1 Environment

We consider a single-good deterministic endowment economy with $t \in \{0, \ldots, T\}$ dates, where $T < \infty$. A single representative individual is born at each date and typically lives for three dates, so there are three individuals alive at any date — one "young", one "middle-aged", and one "old". Since individuals are associated with a date of birth, we use t as superscript to index individuals $\{-2, \ldots, T\}$.

Preferences and Endowments. The lifetime utility of an individual born at date $t \in \{0, \dots, T-2\}$ is

$$V^t = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right) + \beta^2 u\left(c_{t+2}^t\right), \tag{40}$$

where c_t^t, c_{t+1}^t, and c_{t+2}^t respectively denote individual t's consumption at dates t, $t+1$, and $t+2$. Individuals born at dates $t = -1$ and $t = T-1$ are alive for two dates. Their lifetime utility is given by

$$V^{-1} = \beta u\left(c_0^{-1}\right) + \beta^2 u\left(c_1^{-1}\right) \quad \text{and} \quad V^{T-1} = u\left(c_{T-1}^{T-1}\right) + \beta u\left(c_T^{T-1}\right). \tag{41}$$

Individuals born at dates $t = -2$ and $t = T$ are alive for only one date. Their lifetime utility is given by

$$V^{-2} = \beta^2 u\left(c_0^{-2}\right) \qquad \text{and} \qquad V^T = u\left(c_T^T\right). \tag{42}$$

Individuals have endowments of the consumption good, denoted $\{e_t^t, e_{t+1}^t, e_{t+2}^t\}$ for individuals $t \in \{0, \dots, T-2\}$, $\left\{e_0^{-1}, e_1^{-1}\right\}$ and $\left\{e_{T-1}^{T-1}, e_T^{T-1}\right\}$ for individuals $t \in \{-1, T-1\}$, as well as e_0^{-2} and e_T^T for individuals $t \in \{-2, T\}$.

Competitive Equilibrium. Individuals $t \in \{0, \dots, T-2\}$ face the sequence of budget constraints

$$c_t^t = e_t^t + b_t^t, \quad c_{t+1}^t + (1 + r_{t+1})\, b_t^t = b_{t+1}^t + e_{t+1}^t, \quad \text{and} \quad c_{t+2}^t + (1 + r_{t+2})\, b_{t+1}^t = e_{t+2}^t, \tag{43}$$

where b_t^t and b_{t+1}^t denote individual t's borrowing at dates t and $t+1$ respectively, and $1 + r_{t+1}$ denotes the interest rate between dates t and $t+1$. Analogous budget constraints hold for individuals that live for only one or two dates.

We assume that individuals face a borrowing constraint at all dates of the form

$$b_t^i \leq \bar{b}. \tag{44}$$

The resource constraint for the consumption good at date t is given by

$$c_t^{t-2} + c_t^{t-1} + c_t^t = e_t^{t-2} + e_t^{t-1} + e_t^t, \tag{45}$$

while market clearing in the borrowing market at date t requires that

$$b_t^{t-1} + b_t^t = 0. \tag{46}$$

Definition. *(Competitive Equilibrium)* Given endowments, a competitive equilibrium comprises consumption allocations and interest rates such that i) individuals maximize lifetime utility (40) – (42) subject to (43) and (44), and ii) markets clear, that is, (45) and (46) hold at each date.

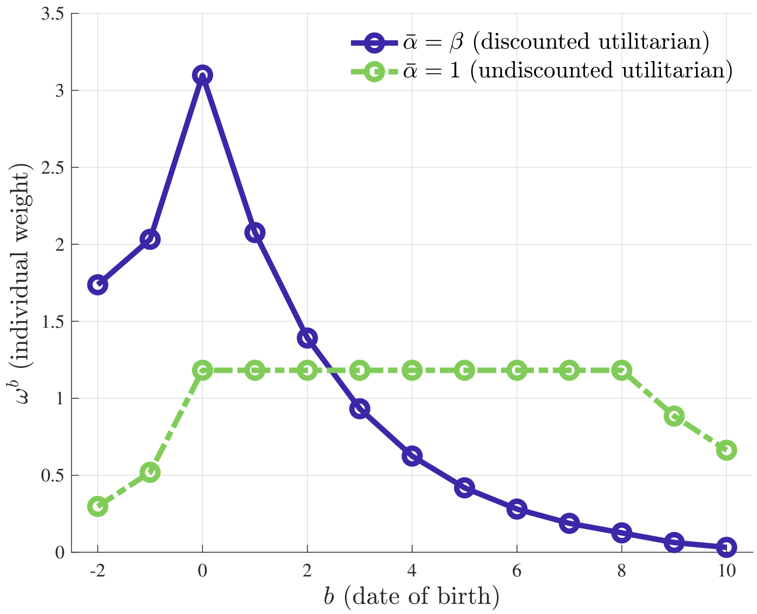


Figure 9: Normalized Individual Weights (Application 3)

Note. This figure shows the normalized individual weights as a function of the date of birth for the undiscounted and discounted utilitarian planners when $\theta = 0$.

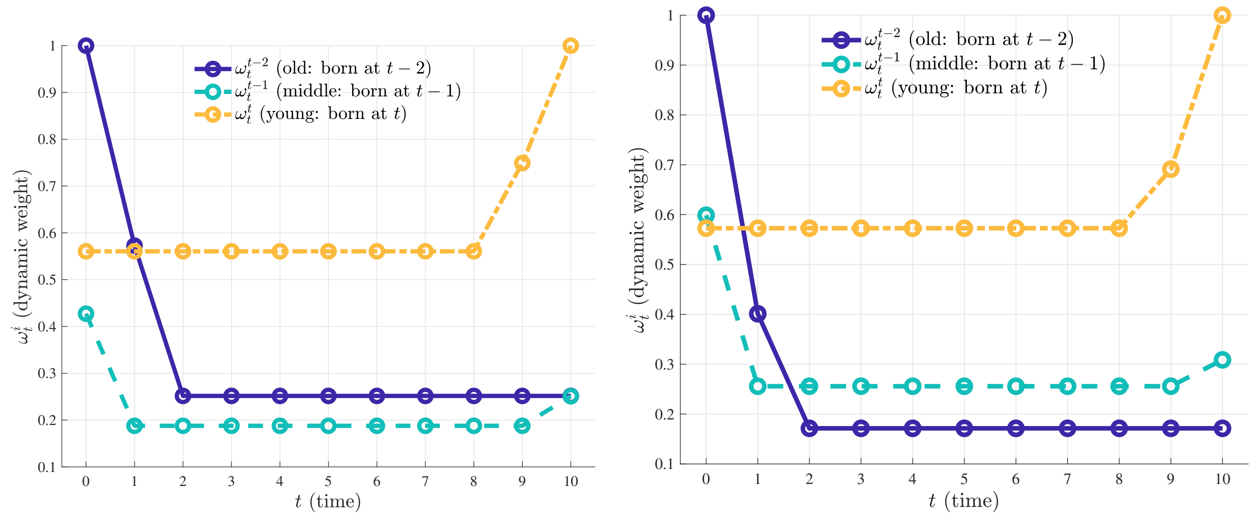
Shares and Middle-Aged-to-Old Transfer. It is useful to formulate the model in terms of consumption and endowment shares. First, we define aggregate consumption and aggregate endowment at date t as $c_t = c_t^{t-2} + c_t^{t-1} + c_t^t$ and $e_t = e_t^{t-2} + e_t^{t-1} + e_t^t$, respectively. Hence, individual t's, $t-1$'s, and $t-2$'s shares of date t aggregate consumption are $\chi_{t,c}^t = \frac{c_t^t}{c_t}$, $\chi_{t,c}^{t-1} = \frac{c_t^{t-1}}{c_t}$, and $\chi_{t,c}^{t-2} = \frac{c_t^{t-2}}{c_t}$ and of aggregate endowment $\chi_{t,e}^t = \frac{e_t^t}{e_t}$, $\chi_{t,e}^{t-1} = \frac{e_t^{t-1}}{e_t}$, and $\chi_{t,e}^{t-2} = \frac{e_t^{t-2}}{e_t}$. Both consumption and endowment shares add up to 1 by construction. We assume no growth of the aggregate endowment.

We formulate a middle-aged-to-old (MTO) transfer as a smooth perturbation of endowment shares at all dates. Formally, we parametrize the model by initial endowment shares given by $\bar{\chi}_{t,e}^t$, $\bar{\chi}_{t,e}^{t-1}$, and $\bar{\chi}_{t,e}^{t-2}$ and then consider a policy — indexed by a perturbation parameter $\theta \geq 0$ — that transfers endowments from middle-aged to old individuals at each date according to

$$\text{(Middle-Aged-to-Old Transfer)} \qquad \chi_{t,e}^{t-1} = \bar{\chi}_{t,e}^{t-1} - \theta \qquad \text{and} \qquad \chi_{t,e}^{t-2} = \bar{\chi}_{t,e}^{t-2} + \theta, \tag{47}$$

In our calibration, the endowment share of the middle-aged is larger than the young and old, featuring $\bar{\chi}_{t,e}^{t-1} > \bar{\chi}_{t,e}^t = \bar{\chi}_{t,e}^{t-2}$.

Calibration. Individual preferences are $u(c) = \log(c)$. We interpret a date in the model as 15 years, so $\beta = (0.98)^{20} = 0.67$. We normalize the aggregate endowment to $e_t = 1$ at all times and calibrate individuals' endowments consistently with the life-cycle profile of earnings, setting the



(a) No transfer ($\theta = 0$)

(b) Positive transfer ($\theta = 0.125$)

Figure 10: Normalized Dynamic Weights (Application 3)

Note. This figure illustrates the normalized dynamic weights before the young-to-old transfer is implemented (left panel, $\theta = 0$, so $\chi_{t,c}^{t} = 0.25$, $\chi_{t,c}^{t-1} = 0.5$, and $\chi_{t,c}^{t-1} = 0.25$), and once the welfare-maximizing transfer for the undiscounted utilitarian planner is implemented (right panel, $\theta = 0.125$, so $\chi_{t,c}^{t} = 0.25$, $\chi_{t,c}^{t-1} = 0.375$, and $\chi_{t,c}^{t-1} = 0.375$).

initial endowment shares of the young, middle-aged, and old respectively to

$$\bar{\chi}_{t,e}^{t} = 0.25, \quad \bar{\chi}_{t,e}^{t-1} = 0.5, \quad \text{and} \quad \bar{\chi}_{t,e}^{t-2} = 0.25.$$

To more clearly illustrate our results, we set $\bar{b} = 0$, but similar results obtain as long as the borrowing constraint binds. We assume that individuals $i = -2$ and -1 have no savings at date 0. For ease of visualization, we assume that $T = 10$, so the economy runs for 200 years.

6.2 Normalized Welfare Weights

Welfare Assessments. We study welfare assessments for utilitarian planners, for whom the welfare assessment of a perturbation $d\theta$ is given by

$$\frac{dW}{d\theta} = \sum_{t=-2}^{T} \alpha^t \frac{dV^t(\theta)}{d\theta}, \tag{48}$$

where α^t denotes the Pareto weight associated with the individual born at date t. Once again, we focus on undiscounted and discounted utilitarian planners and we use aggregate perpetual consumption as the lifetime welfare numeraire, so the counterpart of equation (28) is now

$$\lambda^t = u'\left(c_t^t\right) c_t + \beta u'\left(c_{t+1}^t\right) c_{t+1} + \beta^2 u'\left(c_{t+2}^t\right) c_{t+2},$$

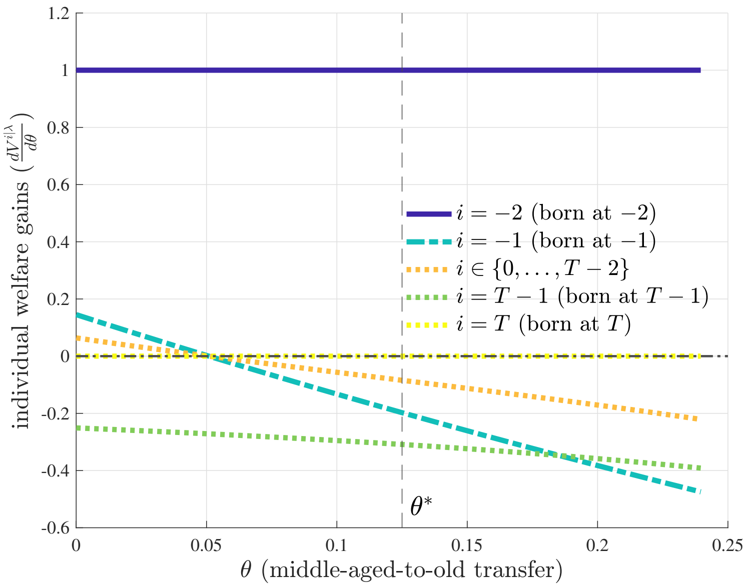


Figure 11: Normalized Individual Welfare Gains (Application 3)

Note. This figure shows the normalized individual welfare gains, given by $\frac{dV^{i|\lambda}}{d\theta}$. The left panel focuses on the first three individuals alive for three consecutive periods — that is, individuals born at dates 0, 1 and 2. The right panel shows the normalized individual welfare gains at the steady-state.

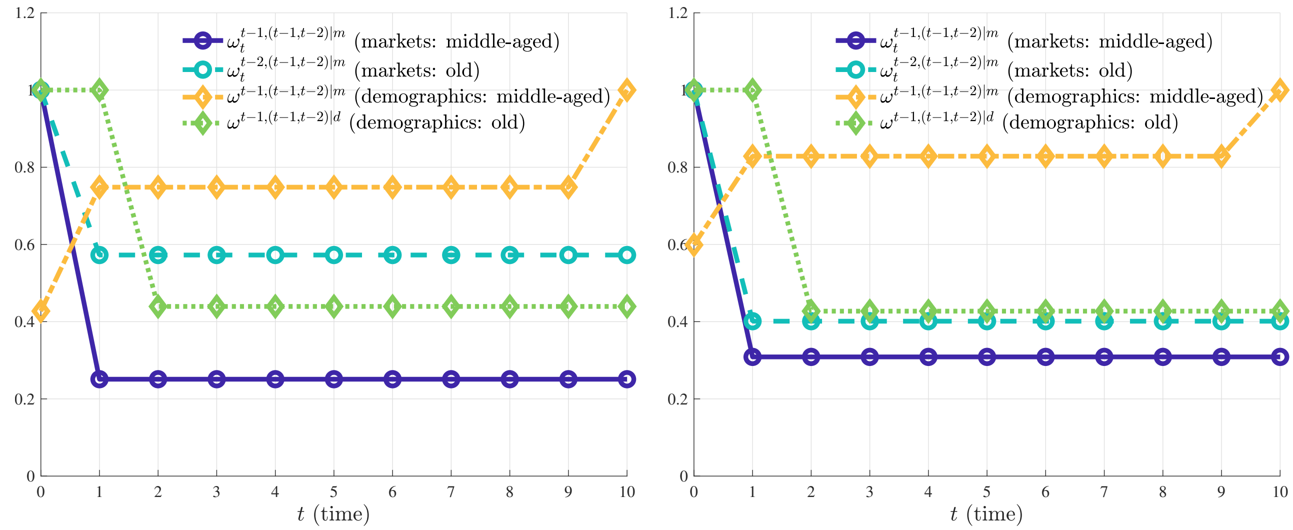
with equivalent expressions for shorter-lived individuals. Since aggregate consumption is constant over time, this is equivalent to using unit perpetual consumption as lifetime welfare numeraire.

Normalized Individual Weights. Normalized individual weights capture the value assigned by a planner to the lifetime welfare gains of different individuals in units of the lifetime welfare numeraire. They are defined as in equation (29).

As in Application 1, normalized individual weights are shaped by perpetual marginal utility differences, individual lifespans, and generational discounting. Longer-lived individuals in general have higher individual weights. The initial old and middle-aged have a higher individual weight than the final middle-aged and young because of our calibration and the definition of individuals utilities in (41) and (42) discounts the flow utilities of the latter. The discounted utilitarian planner assign lower normalized individual weights to individuals born later in time purely because she directly discounts the utility of individuals born in the future. This planner has clear pecking order that favors the date 0 young. Individual weights barely change with θ.

Normalized Dynamic Weights. The normalized dynamic weights define marginal rates of substitution for each individual between date-t and perpetual aggregate consumption, and capture whose consumption the planner values more at each date. They are defined as in (9), and we provide explicit expressions in the Appendix.

Figure 10 shows the normalized dynamic weights (vertical axis) of the individuals alive at a



(a) No transfer ($\theta = 0$)

(b) Positive transfer ($\theta = 0.125$)

Figure 12: Multiplicative Decomposition of Normalized Dynamic Weights (Application 3)

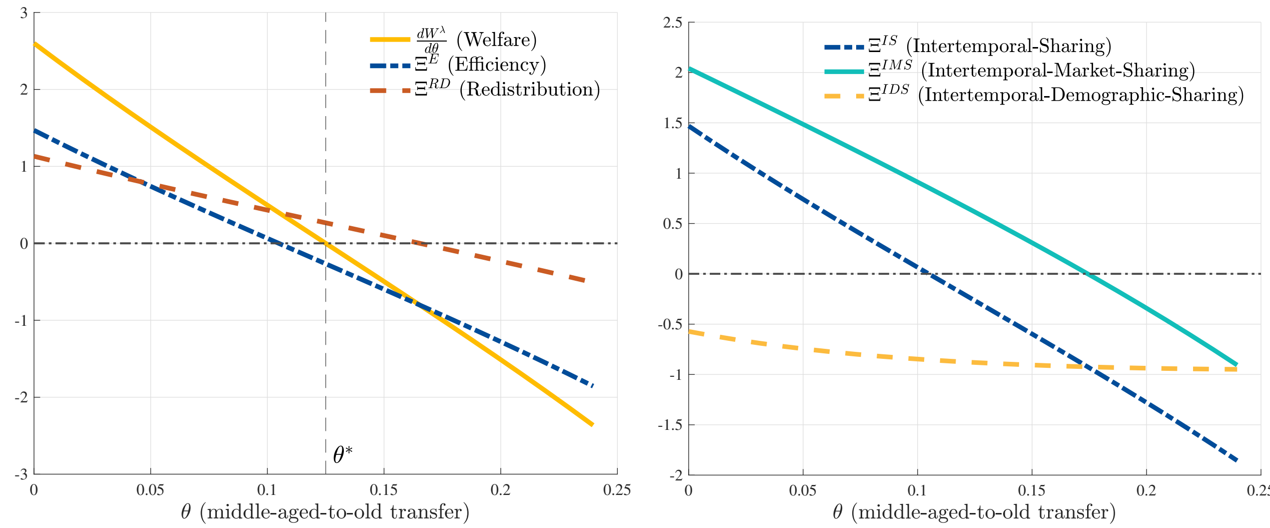
Note. This figure illustrates the multiplicative decomposition of the normalized dynamic weights, as defined in equation (18).

given date t (horizontal axis). In general, shorter-lived individuals — old and middle-aged born at date 0 and middle-aged and old at date T — have larger dynamic weights. At dates in which all individuals have identical lifespans, in the absence of the transfer young individuals have higher dynamic weights than old, because of discounting. Middle-aged in turn have the lowest dynamic weights because they have the highest consumption.

As the transfer is implemented, the consumption of middle-aged and old is equalized, making the dynamic weight of the middle-aged higher than the old due to discounting. Therefore, while a marginal middle-aged-to-old transfer initially generates intertemporal-sharing efficiency gains, at $\theta = 0.125$ this transfer generates efficiency losses, consistent with Figure 13.

Multiplicative Decomposition of Normalized Dynamic Weights. Figure 12 illustrates the multiplicative decomposition of the normalized dynamic weights, introduced in (18). This figure shows that the normalized pairwise-market-dynamic weight is higher at $\theta = 0$ for old individuals relative to middle-aged at any given date, with the exception of initial and final dates. Note that middle-aged and old individuals overlap when the former are young and middle-aged and the latter are middle-aged and old, so the value of consumption at a date relative to the value when they overlap is mostly driven by their consumption at that date. Since old individuals consume less than middle-aged, they a have a relatively higher valuation for consumption, justifying why their market component is higher.

This figure also shows why — again with the exception of initial and final dates — the



(a) Efficiency/Redistribution Decomposition

(b) Intertemporal-Sharing Decomposition

Figure 13: Welfare Decomposition (Application 3)

Note. The left panel of this figure illustrates the efficiency/redistribution decomposition, defined in equation (10) for an undiscounted utilitarian planner ($\bar{\alpha} = \beta$). Because this is an endowment economy, efficiency is exclusively due to intertemporal-sharing, so $\Xi^E = \Xi^{IS}$. The right panel of this figure illustrates intertemporal-sharing (Ξ^{IS}) and its components (intertemporal-market-sharing, Ξ^{IMS}, and intertemporal-demographic-sharing, Ξ^{IDS}), as defined in Proposition 3.

normalized pairwise-demographic-dynamic weight is higher at $\theta = 0$ for middle-aged individuals. The demographic component captures the relative valuation for an individual to a unit of aggregate consumption over the overlapping period between middle-aged and old, and permanent consumption. In this case, because of discounting, a unit of aggregate consumption when young and middle-aged is more valuable than when middle-and and old, justifying why the demographic component is higher.

As θ grows, old individuals consume more and the the marginal value of consumption decreases, explaining why the differences between these weights narrow. These patterns explain the intertemporal-sharing decomposition in Figure 13.

6.3 Policy Experiment: Middle-Aged-to-Old Transfer

Individual Welfare Gains. Figure 11 shows normalized welfare gains of the MTO transfer for each individual. Similar to Application 1, the initial old (individual $t = -2$) always gains from the perturbation, whereas the final middle-aged (individual $t = T - 1$) always loses, with the final young remaining indifferent. All other individuals initially benefit from the MTO transfer since their consumption becomes smoother. Figure 11 also shows that the MTO transfer is not a Pareto improvement since the final middle-aged are worse off.

Welfare and Intertemporal-Sharing Decompositions. The left panel of Figure 13 plots the efficiency/redistribution decomposition defined in (10) for an undiscounted utilitarian planner. Similar to Application 1, since this is an endowment economy, in which aggregate consumption remains fixed, $\Xi^E = \Xi^{IS}$, so all efficiency gains due to a MTO transfer are due to intertemporal-sharing. In contrast to Application 1, in which intertemporal-sharing gains were purely demographic in nature, both the market and demographic components of the intertemporal-sharing decomposition introduced in Proposition 3 are non-zero in this application.

The right panel of Figure 13 plots the intertemporal-sharing decomposition introduced in Proposition 3. As explained above, the market-dynamic weight is higher for old than for middle-aged individuals, explaining the positive sign of Ξ^{IMS}, and why it decreases with θ. The demographic-dynamic weight is instead relatively higher for middle-aged, justifying the negative-sign of Ξ^{IDS}. As the transfer increases, the Ξ^{IMS} becomes smaller and eventually turns negative. Since the demographic-dynamic weight are not that sensitive to the value of the transfer, Ξ^{IDS} continuous to be negative. This application provides an illustration of a scenario in which the demographic structure reduces the gains from implementing a policy that seeks to "complete markets".

7 Conclusion

This paper studies welfare assessments in economies with rich demographics, providing a systematic way to evaluate the efficiency and redistribution implications of policies. Three main results are worth highlighting. First, we introduce the concept of demographically disconnected economies, where no single date exists when all individuals are alive at the same time. In such economies, we show that interpersonal welfare comparisons can only be made in terms of perpetual consumption. Second, we show that there exist feasible perturbations of Pareto efficient allocations in demographically disconnected economies that generate positive Kaldor-Hicks efficiency gains. This is a phenomenon that cannot occur in conventional infinitely-lived agent economies. Third, we introduce a decomposition of intertemporal-sharing welfare gains into market and demographic components, providing insights into how financial frictions and demographic heterogeneity separately drive efficiency gains.

We use our framework to revisit three foundational OLG models in economics, yielding novel insights into the desirability of intergenerational transfers and the implications of capital taxation. We hope that our results spur the study of policies addressing intergenerational challenges such as social security, public debt, and climate change.

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Online Appendix

A Proofs and Derivations: Section 3

Proof of Proposition 1. (Unique Class of Lifetime Welfare Numeraires)

Proof. The proof is constructive. Consumption at a given date or a bundle of consumption across different dates are valid lifetime welfare numeraires only when all individuals in the economy have a positive value for each of them. In demographically disconnected economies, consumption at a particular date cannot be a valid lifetime welfare numeraire, since there is always at least one individual who is not alive at that date. In principle, if $I < T$, one could choose a numeraire based on consumption over a subset of dates in which all individuals are at least alive for one period. For instance, with three individuals and four dates, if $i = 1$ is alive at $\{0, 1\}$, $i = 2$ is alive $\{1, 2\}$, and $i = 3$ is alive at $\{2, 3\}$, numeraires based on bundles of consumption at $\{0, 1, 2\}$, $\{1, 2, 3\}$, and $\{0, 1, 2, 3\}$ (perpetual consumption) are valid lifetime welfare numeraires. As I increases while the economy remain disconnected (in this example, say that $i = 4$ is alive only at $t = 0$ or $t = 4$), only numeraires based on perpetual consumption remain valid lifetime welfare numeraires. □

Proof of Lemma 1. (Normalized Welfare Gains and Normalized Weights)

Proof. We can express an (unnormalized) welfare assessment $\frac{dW}{d\theta}$ as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i},$$

where our choice of lifetime welfare numeraire is such that $\lambda^i = \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t$. Hence, the normalized welfare assessment takes the form

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I} \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i} = \sum_i \omega^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i} \lambda^i}{\frac{1}{I} \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i}.$$

We can then express individual i's normalized lifetime welfare gains as

$$\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_t \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} \frac{1}{c_t} \frac{dc_t^i}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta},$$

where

$$\omega_t^i = \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i} c_t} \quad \text{and} \quad \frac{dV_t^{i|\lambda}}{d\theta} = \frac{1}{c_t} \frac{dc_t^i}{d\theta}.$$

$\square$

Proof of Lemma 2. (Efficiency/Redistribution Decomposition)

Proof. For any two random variables x_i and y_i, it follows that $\sum_i x_i y_i = \frac{1}{I} \sum_i x_i \sum_i y_i + \mathbb{C}ov_i^{\Sigma}[x_i, y_i]$, where $\mathbb{C}ov_i^{\Sigma}[x_i, y_i] = I \cdot \mathbb{C}ov_i[x_i, y_i]$. Equation (10) follows from

$$\frac{dW^{\lambda}}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD}},$$

where we use the fact that $\frac{1}{I} \sum_i \omega^i = 1$. This is the unique decomposition of the weighted sum $\sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}$ into an unweighted sum and its complement. $\square$

Proof of Proposition 1. (Feasible Perturbations from Pareto Efficient Allocations with Kaldor-Hicks Efficiency Gains)

Proof. The proof is constructive. Part a) is proven in the text. Part b) pertains demographically connected economies. In these economies, if the lifetime welfare is based on consumption at dates in which all individuals are concurrently alive, then $\sum_{t \in \mathcal{T}} \omega_t^i = 1$, where $\mathcal{T}$is the set of dates in which all individuals are concurrently alive. In this case, $\omega^i = \sum_{t \in \mathcal{T}} \eta_t$, which implies that ω_t^i is equalized across all individuals whenever they are alive. If instead the lifetime welfare is based on consumption at dates in which not all individuals are concurrently alive, the same proof as in the demographically disconnected economies applies. $\square$

Proof of Lemma 3. (Aggregate Efficiency/Intertemporal-Sharing Decomposition)

Proof. It follows that

$$\Xi^E = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_t \sum_{i|\omega_t^i>0} \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \underbrace{\sum_t \omega_t \sum_{i|\omega_t^i>0} \frac{dV_t^{i|\lambda}}{d\theta}}_{\Xi^{AE}} + \underbrace{\sum_t \mathbb{C}ov_{i|\omega_t^i>0}^{\Sigma}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}},$$

where $\omega_t = \frac{1}{I_t} \sum_i \omega_t^i$ with $I_t = \sum_i \mathbb{I}\{i \mid \omega_t^i > 0\}$, and where $\mathbb{C}ov_{i|\omega_t^i>0}^{\Sigma}[\cdot,\cdot] = I \cdot \mathbb{C}ov_{i|\omega_t^i>0}^{\Sigma}[\cdot,\cdot]$. $\square$

Proof of Proposition 4. (Intertemporal-Sharing Decomposition: Intertemporal-Market-Sharing vs. Intertemporal-Demographic-Sharing

Proof. It follows that

$$\Xi^{IS} = \sum_t \mathbb{C}ov_{i|\omega_t^i>0}^{\Sigma}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right] = \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov_{(i,j)\in\mathcal{A}_t}^{\Sigma}\left[\omega_t^{(i,j)}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right].$$

Using the fact that $\omega_t^{i,(i,j)} = \omega_t^{i,(i,j)|m} \omega^{i,(i,j)|d}$, we can apply the result in Bohrnstedt and Goldberger (1969) to write Ξ^{IS} as

$$\Xi^{IS} = \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t} \left[\omega_t^{(i,j)|m}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right] + \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}ov^{\Sigma}_{(i,j)\in\mathcal{A}_t} \left[\omega^{(i,j)|d}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right]$$
$$+ \sum_t \frac{1}{I_t} \sum_{(i,j)\in\mathcal{A}_t} \mathbb{C}osk^{\Sigma}_{(i,j)\in\mathcal{A}_t} \left[\omega_t^{(i,j)|m}, \omega^{(i,j)|d}, \frac{dV_t^{(i,j)|\lambda}}{d\theta}\right],$$

where, given three random variables X, Y, and Z, the coskewness operator is defined as $\mathbb{C}osk_i^{\Sigma}[X,Y,Z] = \mathbb{E}_i\left[(X-\mathbb{E}_i[X])(Y-\mathbb{E}_i[Y])(Z-\mathbb{E}_i[Z])\right]$ and $\mathbb{C}osk_i^{\Sigma}[\cdot,\cdot,\cdot] = I \cdot \mathbb{C}osk_i[\cdot,\cdot,\cdot]$. In this case, since all coskewness are computed pairwise (with two elements), they must be zero, which yields equation (19) in the text. □

B Redistribution

The redistribution component captures the equity concerns embedded in a particular Social Welfare Function. Ξ^{RD} is positive when the individuals relatively favored in a perturbation are those relatively preferred by the planner, i.e., have higher normalized individual weights ω^i. At times, it is convenient to distinguish whether redistribution gains take place i) within individuals from the same generation or ii) across individuals from different generations. To this end, we show how further decompose the redistribution component into *intra*-generational redistribution and *inter*-generational redistribution.

Proposition 5. *(Redistribution Decomposition) The redistribution component, Ξ^{RD}, can be decomposed into i) intra-generational redistribution and inter-generational redistribution, as follows:*

$$\Xi_{RD} = \underbrace{\sum_t |\mathcal{G}_t| \cdot \mathbb{C}ov_{i\in\mathcal{G}_t}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD-intra} \text{ (Intra-Generational)}} + \underbrace{I \cdot \mathbb{C}ov_{\mathcal{G}}\left[\mathbb{E}_{i\in\mathcal{G}_t}\left[\omega^i\right], \mathbb{E}_{i\in\mathcal{G}_t}\left[\frac{dV^{i|\lambda}}{d\theta}\right]\right]}_{\Xi^{RD-inter} \text{ (Inter-Generational)}}, \tag{49}$$

where $\mathcal{G}_t$ is the set of individuals born at date t (generation-t) and $\mathcal{G}_0$ is the set of individual alive at date 0. $\mathbb{C}ov_{\mathcal{G}}[\cdot,\cdot]$ denotes a covariance across generations that weights each generation by the probability $\frac{|\mathcal{G}_t|}{I}$ of a given individual coming from a particular generation t. Finally, the cross-sectional conditional expectation is given by $\mathbb{E}_{i\in\mathcal{G}_t}[X_i] = \frac{1}{|\mathcal{G}_t|}\sum_{i\in\mathcal{G}_t} X_i$.

This decomposition illustrates that welfare gains from redistribution have two sources: a weighted sum of within generations redistribution; and a covariance between the per-generation average normalized individual weight, and the per-generation average normalized lifetime welfare gains. This last term, $\Xi^{RD-inter}$, is positive when the generations relatively favored in a perturbation are those relatively preferred by the planner, i.e., have higher per-generation average normalized individual weights. This decomposition of Ξ^{RD} separates the component that is zero if the planner can costlessly transfer across individuals alive, $\Xi^{RD-intra}$, an its complement, $\Xi^{RD-inter}$. Even if transfers are generally not possible across generations, especially when individuals do not overlap, certain policies can still redistribute across generations. For example, policies aimed at mitigating climate change require current generations to make sacrifices that benefit future individuals that are yet-to-be-born. We illustrate this trade-off and how our welfare decomposition rationalizes the implied welfare effects in the following example.

Example 3. *(Aggregate Efficiency v.s. Intergenerational Redistribution)* Consider a two-date endowment economy with two individuals ($I = 2$). Individual A is alive at date 0, with preferences $V^A = u\left(c_0^A\right)$; and individual B is alive at date 1, with preferences $V^B = u\left(c_1^B\right)$. We study a perturbation that reallocates consumption across periods as follows:

$$c_0^A = 1 - \theta \quad \text{and} \quad c_1^B = \frac{1}{2} + \varphi \cdot \theta; \qquad \theta \in (0, 1)\,,\ \varphi \in (0, \infty)\,.$$

We interpret the negative consequences of climate change as follows. In the absence of policy intervention ($\theta = 0$), the current generation, individual A, consumes more than the unborn generation, individual B. Individual A can make an effort, indexed by θ, that reduces their consumption levels and allows the future generation to consume more. This effort represents a costly climate change mitigation policy. Lastly, φ summarizes the effectiveness of the policy.

The efficiency gains, whose unique source is aggregate efficiency, are determined by the effectiveness parameter φ, and given by

$$\Xi^{AE} = \varphi - 1, \qquad \begin{cases} \Xi^{AE} \geq 0, & \text{if } \varphi \in [1, \infty) \\ \Xi^{AE} < 0, & \text{if } \varphi \in (0, 1)\,. \end{cases}$$

Only when $\varphi > 1$ does the perturbation result in an increase in the total aggregate consumption across periods, which is interpreted as an aggregate efficiency gain. The other source of welfare gains is intergenerational redistribution. For instance, if $\varphi < 1$, the planner may still consider the transfer worthwhile, especially if she places a high enough weight on the unborn, even though this transfer reduces efficiency. To highlight the importance of individual weights, consider the case with a constant social generational discount factor where $\alpha^A = 1$ and $\alpha^B = \bar{\alpha} \in (0, 1]$, set $\varphi = 1$, and use $u(c) = \log(c)$. Then,

$$\Xi^{RD} = \Xi^{RD-inter} > 0 \quad \text{if} \quad \begin{cases} \theta \in \left(0, \frac{1}{4}\right), & \text{if } \bar{\alpha} = 1 \\ \theta \in \left(0, \frac{\bar{\alpha} - 0.5}{\bar{\alpha} + 1}\right), & \text{if } \bar{\alpha} < 1. \end{cases}$$

In this special case, the only justification for implementing the policy is a redistribution motive. As the social generational discount rate rises — meaning the planner's discounting applied to unborn generations is smaller — the desirable level of effort to mitigate climate change also increases. In the extreme case where the planner attaches a very low social generational discount factor ($\bar{\alpha} \leq \frac{1}{2}$), mitigation of climate change is not desirable at all. Instead, the planner would prefer to redistribute away from unborn individuals by increasing the consumption of those currently alive, at the expense of future generations.

C Extensions

C.1 Stochastic Environment

Demographic and non-demographic uncertainty can be introduced using history-notation, as in Stokey, Lucas and Prescott (1989) or in Chapter 8 of Ljungqvist and Sargent (2018). In this case, let's consider an economy populated by a countable, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$, where $1 \leq I \leq \infty$. At each date $t \in \{0, \ldots, T\}$, where $0 \leq T \leq \infty$, there is a realization of a stochastic event $s_t \in S$. We denote the history of events up to date t by $s^t = (s_0, s_1, \ldots, s_t)$, and the probability of observing a particular sequence of events s^t by $\pi_t\left(s^t\right)$. The initial value of s_0 is predetermined, so $\pi_0\left(s_0\right) = 1$. At all dates and histories, individuals (potentially) consume a single good.

Births and deaths are now simply stochastic events captured by the realization of s^t. Formally, individual i *dies* at history s^t if $u_t^i\left(\cdot\right) = 0$ for all future histories. If births are random, individual i is *potentially born* at a date denoted $\tau_b^i \in \{-\infty, \ldots, T\}$, where this is the first date with is a history in which $u_t^i\left(\cdot\right) > 0$. This notion allows us to index individuals by the first time in which they are potentially born. In general, there are different reasonable stances one can take on how to formalize births and deaths, and this is sufficiently interesting to spur future work.

Formally, preferences in this case can be written as

$$V^i = \sum_{t=\tau_b^i}^{T} \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u_t^i\left(c_t^i\left(s^t\right); s^t\right),$$

where $\beta^i \in [0, 1)$ denotes individual i's discount factor, and $u_t^i\left(\cdot; s^t\right)$ and $c_t^i\left(s^t\right)$ respectively correspond to individual i's instantaneous utility and consumption at history s^t at date t. Whenever individual i is not alive, $u_t^i\left(\cdot; s^t\right) = 0$. Whenever individual i is alive, $u_t^i\left(\cdot; s^t\right)$ is well behaved, so that $\frac{\partial u_t^i(\cdot)}{\partial c_t^i} > 0$ and an Inada condition applies.

Lemma 1 applies unchanged in this case, after a few redefinitions. Normalized date welfare gains are now

$$\frac{dV_t^{i|\lambda}}{d\theta} = \sum_{s^t} \omega_t^i \left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta},$$

where the normalized stochastic weight $\omega_t^i\left(s^t\right)$ is given by

$$\omega_t^i \left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i} c_t\left(s^t\right)}{\sum_{s^t} \left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i} c_t\left(s^t\right)}.$$

Normalized history welfare gains are defined as

$$\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} = \frac{1}{c_t\left(s^t\right)} \frac{dc_t^i\left(s^t\right)}{d\theta}.$$

In this economy, efficiency can be decomposed into aggregate efficiency, risk-sharing, and intertemporal-sharing, as shown in Dávila and Schaab (2024).

C.2 Multiple Goods and Factors

It is straightforward to augment individual preferences to account for more goods and/or factors. Formally, we can consider preferences of the form

$$V^i = \sum_{t=\tau_b^i}^{T} \left(\beta^i\right)^t u_t^i \left(c_t^{ij}, n_t^{if}\right),$$

where individual i now has preferences over J consumption, indexed by $j \in \{1, \ldots, J\}$, and F factors, goods and $f \in \{1, \ldots, F\}$ factors. In this case, a version of Proposition 1 still applies where the lifetime welfare numeraire needs to be defined over perpetual bundles of particular goods or factors.

C.3 Preference Disconnect

As explained in Remark 1, an analogous notion to demographic disconnect can be defined in static multi-good economies. Formally, consider a static economy populated by a finite number of individuals, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$ with preferences over J consumption goods, indexed by $j \in \{1, \ldots, J\}$. In this case, individual i preferences are given by

$$V^i = u^i \left(\left\{c^{ij}\right\}_j\right).$$

In this case, an economy is *preference disconnected* if there is no good j such that $\frac{\partial u^i}{\partial c^{ij}} > 0$ for all individuals. A version of Proposition 1 also applies to these economies: only welfare numeraires based on consumption of all goods are always valid in demographically disconnected economies. A version of Proposition 2 also applies in this case.

C.4 Unit Perpetual Consumption as Lifetime Welfare Numeraire

As discussed in Section 3, the two natural choices for lifetime welfare numeraire among those based on perpetual consumption are i) unit perpetual consumption, the bundle that pays one unit of the consumption good at each date, and ii) aggregate perpetual consumption, the bundle that pays aggregate consumption at each date. In the body of the text, we adopted aggregate perpetual consumption as lifetime welfare numeraire, but here we derive Lemma 1 for the unit perpetual consumption welfare numeraire.

In this case, we can express an (unnormalized) welfare assessment $\frac{dW}{d\theta}$ as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i},$$

where our choice of lifetime welfare numeraire is such that $\lambda^i = \sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i}$. Hence, the normalized welfare assessment takes the form

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \sum_i \omega^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}.$$

We can then express individual i's normalized lifetime welfare gains as

$$\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_t \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i}}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i}} \frac{dc_t^i}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta},$$

where

$$\omega_t^i = \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i}}{\sum_t \left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^i}} \quad \text{and} \quad \frac{dV_t^{i|\lambda}}{d\theta} = \frac{dc_t^i}{d\theta}.$$

As explained in the text, in economies without aggregate consumption growth both numeraires yield identical results.