Probability Pricing*

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Abstract

This paper develops probability pricing, extending cash flow pricing to quantify the willingness-to-pay for changes in probabilities. We show that the value of any marginal change in probabilities can be expressed as a standard asset-pricing formula with hypothetical cash flows derived from changes in the survival function. This equivalence between probability and cash flow valuation allows us to construct hedging strategies and systematically decompose individual and aggregate willingness-to-pay. Four applications examine the valuation of changes in the distribution of aggregate consumption, the efficiency effects of varying performance precision in principal-agent problems, and the welfare implications of public and private information.

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1 Introduction

Many economic problems involve quantifying the value of changes in probabilities. How much is an individual willing to pay to mitigate the risk of a potential disaster? How much should a firm value having access to a more precise technology? What are the private and social values of better information about the state of an economy?[1] Despite the importance of these questions, the general tools for valuing changes in probabilities are less developed than those for valuing cash flows.

In this paper, we introduce *probability pricing*, a new unified framework for calculating the value of changes in the probabilities of different states of the world. We introduce probability pricing in a two-date environment in which the terminal consumption of an expected utility agent depends on a random state. We then consider an arbitrary marginal perturbation to the probability distribution of the state and define the *probability price* associated with it as the agent's willingness-to-pay for that perturbation.

A central contribution of this paper is to show that an agent's willingness-to-pay for a marginal change in probabilities is equivalent to the agent's willingness-to-pay for a hypothetical asset. The cash flows of this asset represent the state-by-state consumption equivalent of the probability perturbation. In other words, we establish a formal equivalence between changes in probabilities (probability pricing) and changes in consumption (cash flow pricing). In each state, the appropriate hypothetical cash flow is the product of i) the change in the normalized survival function, and ii) the sensitivity of consumption to the state.[2] As we explain in detail, the change in the survival function is the relevant object to compute hypothetical cash flows because perturbations that increase (decrease) probability mass to the right of a particular state are effectively adding up (subtracting) the marginal utility at that state, by virtue of the fundamental theorem of calculus.

It is useful to contrast our approach with existing studies of changes in probabilities or information. Prominent examples include Lucas (1987) on the value of dampening aggregate fluctuations, Barro (2009) and Martin and Pindyck (2015) on the probability of rare disasters, as well as Hirshleifer (1971), Morris and Shin (2002) and Pavan, Sundaresan, and Vives (2022) on the value of information. The typical approach involves solving a class of models with parametric utilities and probability distributions, and then directly

[1] Changes in information — whether private or public — can be represented as changes in the probabilities that agents assign to different states.

[2] The survival function of a probability distribution, $1 - F(s)$, is the complement of the cumulative distribution function, $F(s)$.

evaluating utility/welfare as a function of the parameter of interest (e.g., variance, probability of disaster, or signal precision). By contrast, our approach is non-parametric and works for general preferences, distributions, and perturbations to probabilities.

Perhaps more importantly, the equivalence between cash flow and probability pricing has several practical advantages that have not been previously exploited. In particular, the hypothetical cash flows that we characterize are useful for hedging/immunization purposes. That is, an investor who wants to be immunized against (initially unanticipated) changes in the probabilities of different scenarios can use our result to identify the cash flows that a hedging strategy must be designed to replicate. Moreover, once changes in probabilities are expressed in terms of equivalent cash flows, a standard stochastic decomposition can be used to attribute probability prices to changes in expected (equivalent) payoffs and to risk compensation, providing additional economic insights into the deep sources of willingness-to-pay. Relatedly, probability pricing can also be useful in equilibrium settings with heterogeneous agents and incomplete markets to compute cross-sectional welfare decompositions, separating gains due to changes in aggregate consumption (aggregate/production efficiency) from those due to reallocating consumption across agents (risk-sharing).

Further insights arise when we examine parametric perturbations to specific distributions. We initially highlight properties of probability prices by considering perturbations to a distribution characterized in terms of a mean parameter and a standard deviation parameter. In particular, we show that the willingness-to-pay for a perturbation that marginally increases consumption unconditionally at all future states (risk-free asset) must be the same as the willingness-to-pay for a perturbation that shifts all probabilities uniformly to the right. We also show that the probability price of a shift in the standard deviation parameter is entirely driven by risk compensation.

The probability pricing formula relates to the classic literature on preferences over monetary lotteries that follows Pratt (1964), Rothschild and Stiglitz (1970), and Arrow (1971). Formally, we confirm that first-order stochastically dominant perturbations have a positive probability price, while second-order stochastically dominant perturbations have a negative probability price. Probability pricing can thus also be understood as a way to generalize classic results about gambles, because it determines whether an individual is willing to pay a positive (or negative) price for any perturbation, not only those that satisfy particular dominance conditions.

While our initial focus is on changes in physical probabilities, probability pricing is uniquely suited to studying the private and social values of information, because changes in information are effectively changes in (subjective) probabilities. We illustrate this use in two of our four applications.

Applications. The probability pricing approach has broad implications and multiple uses, which we illustrate in four applications. Our first application leverages the probability pricing result to study the willingness-to-pay of a representative agent for changes in the distribution of aggregate consumption in a canonical consumption-based asset pricing model. This exercise is the closest application of our single-agent results in a general equilibrium setting. It shows how our framework can be used to calculate and decompose the welfare costs of changes in risk, such as those induced by climate change, disasters, or macroeconomic volatility. Our results allow us to separate the welfare cost of a given perturbation into an expected payoff effect and a risk-compensation effect, providing a transparent way to identify and assess the channels through which changes in risk affect both valuation and welfare.

Our second application studies the welfare impact of changes in the precision of output performance (given effort) in a canonical principal-agent problem. We show that the efficiency gains from increasing the precision of output performance arise entirely from the probability pricing channel: endogenous adjustments to equilibrium consumption profiles for given signal realizations cancel out because the optimal contract is constrained efficient, so production efficiency gains from higher effort are exactly offset by risk-sharing losses. Indeed, this point extends beyond the particular application — in any constrained efficient environment, probability prices are sufficient for the efficiency gains for changes in primitive probability distributions. In contrast, the direct effect of changing probabilities — reducing aggregate consumption risk for given allocations — is strictly positive, combining aggregate-efficiency gains from less volatile aggregate consumption similar to those in Application 1 with risk-sharing effects whose sign depends on the performance sensitivity of the contract. As a result, overall efficiency gains from increasing the precision of output performance are always positive, with both aggregate-efficiency and risk-sharing gains when the precision of output is sufficiently high.

Our final two applications illustrate the role of probability pricing to study the value of information in equilibrium settings with risk-averse heterogeneous agents and incomplete markets. Our third application revisits Hirshleifer (1971)'s result that less precise public information can improve welfare in an endowment economy with incomplete markets by enhancing risk-sharing. Using probability pricing, we show that this "Hirshleifer effect" is the sum of two distinct, previously unexplored phenomena with opposite welfare implications. First, changes in public information alter the probabilities of different signal realizations for given consumption allocation mappings. In particular, less precise public information increases the likelihood of more extreme signals — scenarios in which consumption is more unequal ex-post — worsening risk-sharing and reducing welfare. Second, changes in public information trigger endogenous adjustments to equilibrium consumption profiles for given signal realizations, as agents trade and asset prices adjust. In particular, less precise public information makes consumption profiles more similar conditional on a signal realization. This phenomenon improves risk-sharing and increases welfare, because the equilibrium is constrained inefficient, and — when it dominates — drives the overall welfare loss pointed out by Hirshleifer (1971). Hence, probability pricing reveals that better public information carries an inherently positive social value, even in a pure endowment economy. The counterintuitive adverse effects of better public information are solely due to the constrained inefficient nature of the environment. The adverse force happens to dominate in the case highlighted by Hirshleifer (1971), but need not do so in all economies, so that our decomposition provides a useful new perspective on the sources of welfare effects. We also extend the analysis to a production economy, separating the production-efficiency and risk-sharing effects of changes in public information.

Our final application uses probability pricing to assess the welfare effects of changes in the precision of private information in a canonical competitive model of financial trading with dispersed information and noise traders. In the noisy rational expectations equilibrium of this model, as in the extensive literature on information aggregation following Grossman and Stiglitz (1980), the price acts as a public signal that partially aggregates the private signals received by investors. We study the welfare effects of changes in the precision of private information. Using probability pricing, we derive a detailed decomposition that exposes novel economic effects. Two challenges for efficiency assessments arise in these models: i) how to account for the welfare of behavioral "noise traders" and ii) how to identify the consequences of changes in the endogenous price signal. Using probability pricing, we present a decomposition of welfare effects that deals with both issues. Indeed, noise traders only affect one term — distributive pecuniary effects — in our decomposition, while the remaining effects can be precisely signed. Better private information increases the welfare of investors due to the private benefits of better information and pecuniary effects on learning, but decreases it due to signal compression (in contrast to the "Hirshleifer effect" discussed above) and increased price informativeness. Overall, efficiency is minimized at intermediate levels of information precision.

Related Literature. At its core, the idea of probability pricing is most related to the classic work characterizing notions of risk and risk aversion that follows Pratt (1964), Rothschild and Stiglitz (1970), and Arrow (1971), among many other contributions. These results have by now made their way to graduate textbooks, see e.g., Ingersoll (1987), Mas-Colell, Whinston, and Green (1995), Gollier (2001), or Campbell (2017). To the best of our knowledge, both the probability pricing formula and its associated consumption equivalent characterization of probability changes are novel contributions to this literature. Two reasons may explain this. First, our goal is to compute the willingness-to-pay for general perturbations given a preference specification, rather than trying to derive relations or stochastic orders for specific probability perturbations for arbitrary preference specifications. Second, following the successful logic behind cash flow pricing, we focus on marginal perturbations, which allows us to connect our results to cash flow pricing.[3] That said, probability pricing can be useful to derive well-known properties of distributions and preferences of risk-averse agents.[4]

The question of how to value changes in probabilities has arisen in specific settings. For instance, Barro (2009) computes welfare changes from changing disaster probabilities and consumption volatility in a consumption-based asset pricing model. In fact, our first application builds directly on his analysis; see also Martin and Pindyck (2015). Our general results focus on i) valuing changes in probabilities generally, making minimal assumptions on preferences, distributions, or perturbations, and ii) establishing a general analogy between cash flow and probability pricing. It is also useful to note that there is a clear difference between probability pricing — expressing the effect of probability changes as an asset price — and the literature that studies the effect of probability changes on asset prices themselves (e.g., Martin, 2013). We elaborate on this distinction in Appendix D.4.

The closest antecedent to our principal-agent application is Rantakari (2008), who studies how changes in uncertainty affect the optimal intensity of incentives — see also Holmström (1979) and Grossman and Hart (1983). In contrast to this work, we use probability pricing to determine the private and social values of changes in uncertainty, not the change in the

[3] See, e.g., Alvarez and Jermann (2004) and Hansen and Souganidis (2024) for related applications of marginal valuation methods.

[4] Integration by parts, which is key to establishing our main result, is widely used in screening models and mechanism design (Mirrlees, 1971; Baron and Myerson, 1982; Pavan, Segal, and Toikka, 2014) — see Bolton and Dewatripont (2005) for a textbook treatment. While there is an evident high-level relation between those papers and ours, our focus and theirs are completely different.

form of the optimal contract.

Finally, the question of whether information improves efficiency pervades the literature on the efficiency of stock markets (e.g., Grossman and Stiglitz, 1980; Hellwig, 1980; Diamond and Verrecchia, 1981; Vives, 2017), strategic trade (e.g., Kyle, 1989; Vives, 2011; Rostek and Weretka, 2012), and public disclosures (e.g., Hirshleifer, 1971; Diamond, 1985; Diamond and Verrecchia, 1991; Morris and Shin, 2002; Angeletos and Pavan, 2007; Goldstein and Yang, 2019). The conventional tools of valuation and welfare analysis are, at first glance, of limited use in these problems: information affects not only prices and allocations, but also agents' statistical inferences and the probabilities that they attach to different states of the economy. Perhaps for this reason, and despite several important contributions to this area — including those of Morris and Shin (2002); Angeletos and Pavan (2007, 2009); Veldkamp (2009); Gottardi and Rahi (2014); Vives (2017); Kadan and Manela (2019); Bond and Garcia (2022); Pavan, Sundaresan, and Vives (2022) — the welfare analysis of models with imperfect information, in particular when agents are risk averse, remains understudied. Our results illustrate how probability pricing is helpful to understand the value of information, as it allows us to distinguish endogenous consumption adjustments that take place in equilibrium from direct changes in probabilities. We hope that the results in this paper can spur further efforts in this area, both theoretically and empirically, connecting to the results in Ai and Bansal (2018), Kadan and Manela (2019), and Veldkamp (2023), among others. Since our approach is based on consumption equivalents, it is also uniquely suited to serve as the foundation of quantitative work.

2 Probability Pricing

This section introduces probability pricing in a single-agent setting. Sections 3 and 4 apply this framework to equilibrium models with multiple/heterogeneous agents.

2.1 Environment

We consider a single-agent environment with two dates, $t \in \{0, 1\}$. At date 1, there is a continuum of possible states indexed by s with (potentially unbounded) support on $[\underline{s}, \bar{s}]$. We denote the cumulative distribution function (cdf) of the state by $F(s) \in [0, 1]$, and its probability density function (pdf) by $f(s) > 0$.

The agent has standard expected utility preferences, given by

$$V = u(c_0) + \beta \int_{\underline{s}}^{\bar{s}} u(c_1(s)) f(s)\, ds, \tag{1}$$

where c_0 denotes consumption at date 0, $c_1(s)$ denotes consumption at date 1 in state s, and $\beta \in [0,1]$ is the time discount factor. We assume that the flow utility function $u(\cdot)$ is twice differentiable, strictly increasing, and concave. To maintain focus, we restrict attention to expected utility preferences, although the results can be extended to more general preference specifications.

2.2 Cash Flow Pricing

To fix ideas, it is useful to first consider the standard problem of asset/cash flow pricing. Suppose the agent can purchase q units of an asset that delivers state-contingent cash flows $x(s)$, at a price p_x. The budget constraints are

$$\begin{aligned} c_0 &= \ \ldots \ - p_x q \\ c_1(s) &= \ \ldots \ + x(s) q, \end{aligned}$$

where the ellipses $(\ldots)$ represent other elements in the agent's budget constraints. The agent's willingness-to-pay for a marginal unit of the asset satisfies the familiar asset pricing formula:

$$p_x = \int_{\underline{s}}^{\bar{s}} \omega(s) x(s)\, ds, \tag{2}$$

where $\omega(s) = \frac{\beta u'(c_1(s))}{u'(c_0)} f(s)$ defines a state-price. One can also express p_x in terms of a stochastic discount factor $m(s) = \frac{\beta u'(c_1(s))}{u'(c_0)}$, which satisfies $\omega(s) = m(s) f(s)$. In this case, asset/cash flow pricing uncovers the willingness-to-pay at date 0 for changes in consumption at date 1 in different states induced by the asset's cash flows. Asset prices are higher for assets with higher payoffs $x(s)$, in particular in states with high state-prices $\omega(s)$.

2.3 Probability Pricing

We now show that the logic behind cash flow pricing extends to characterize the agent's willingness-to-pay at date 0 for changes in probabilities. We refer to this alternative thought experiment as *probability pricing*.

To formalize changes in probabilities, we introduce a perturbation parameter θ that shifts the cdf (and pdf) over states, assuming that $F(s;\theta)$ and $f(s;\theta)$ are differentiable functions of θ. Our goal is to characterize the probability price, p_θ: the agent's willingness-to-pay for a marginal change $d\theta$, reflected in the budget constraint as

$$c_0 = \ldots - p_\theta \theta.$$

To isolate the novel effects that arise from perturbing probabilities, we initially assume that the agent's consumption profile is predetermined and does not depend on θ. We relax this assumption below.

The agent's willingness-to-pay for a marginal change in probabilities satisfies the formula

$$p_\theta = \int_{\underline{s}}^{\bar{s}} \frac{\beta u(c_1(s))}{u'(c_0)} \frac{df(s;\theta)}{d\theta} ds, \tag{3}$$

where it must be that $\int_{\underline{s}}^{\bar{s}} \frac{df(s;\theta)}{d\theta} ds = 0$.[5] This expression has an intuitive interpretation: the willingness-to-pay for a change in probabilities is high, all else equal, if increases in density $\frac{df(s)}{d\theta}$ coincide with high-consumption states, i.e., those with large $u(c(s))$. However, directly comparing p_x and p_θ is not possible because Equation (3) is expressed in terms of utility levels $u(\cdot)$, while (2) is expressed in terms of marginal utilities $u'(\cdot)$.[6]

Our main result, Proposition 1, transforms changes in probabilities into consumption equivalents, yielding a probability pricing formula that parallels the traditional cash flow pricing formula. While the proof of Proposition 1 relies on integration by parts and thus makes use of the continuity of the underlying distribution of s, the same logic applies with discrete states, as shown in Appendix D.1.

Proposition 1 (Probability Pricing)**.** *The willingness-to-pay, or probability price, p_θ, for a*

[5]Formally, this perturbation can be interpreted as taking a variational/Gateaux derivative (Luenberger, 1969). To ensure that the perturbed cdf remains valid without making parametric assumptions, $F(s;\theta)$ can always be formulated as

$$F(s;\theta) = \theta \underline{F}(s) + (1-\theta)\overline{F}(s),$$

where $\underline{F}(s)$ denotes the cdf of the "initial" distribution, and $\overline{F}(s)$ denotes the cdf of the "final" distribution, or equivalently the "direction" of the perturbation. In general, the parameter θ can be mapped to a parameter of a particular distribution; see e.g., Section 2.4. See Dávila and Walther (2023) for an application of variational derivatives to leverage regulation with distorted beliefs.

[6]One may be tempted to view the ratio $\frac{\beta u(c_1(s))}{u'(c_0)}$ as analogous to the state-price for cash flow pricing. However, this ratio is not invariant to preference-preserving transformations — in particular, additive transformations of $u(\cdot)$ such as $u(\cdot) \to u(\cdot) + a$ — which makes it unsuitable as a foundation for a theory of valuation of changes in probabilities.

marginal perturbation in probabilities indexed by θ is given by

$$p_{\theta}=\int_{\underline{s}}^{\bar{s}}\omega\left(s\right)x_{\theta}\left(s\right)ds, \tag{4}$$

where $\omega\left(s\right)=\frac{\beta u'(c_1(s))}{u'(c_0)}f\left(s;\theta\right)$ defines a state-price, and where

$$x_{\theta}\left(s\right)=\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f\left(s;\theta\right)}\frac{dc_1\left(s\right)}{ds} \tag{5}$$

defines a consumption-equivalent cash flow for state s.

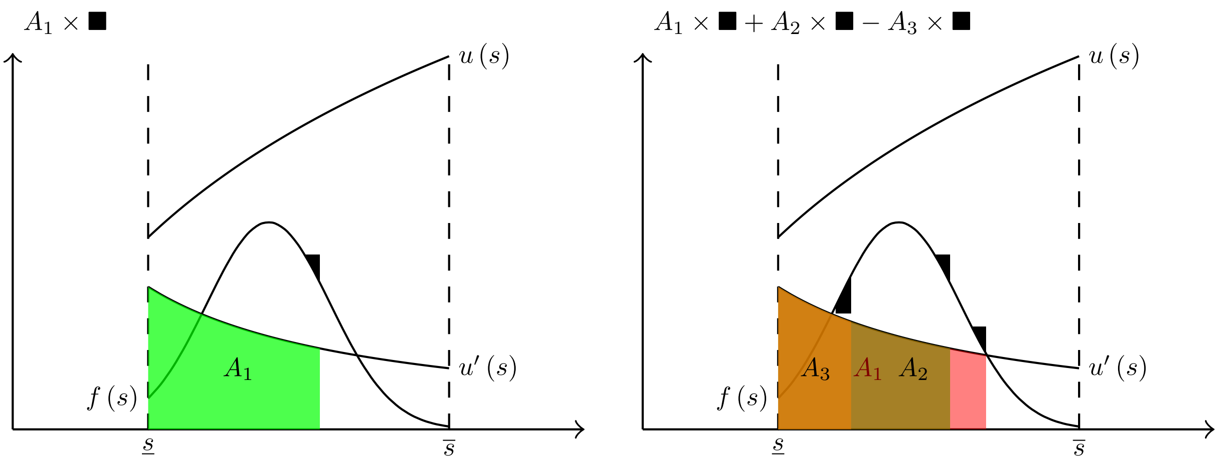
Equation (4) shows that computing the agent's willingness-to-pay for a marginal change in probabilities is equivalent to pricing a hypothetical asset that delivers cash flows $x_{\theta}\left(s\right)$ in state s. These are the appropriate hypothetical cash flows that translate changes in probabilities into consumption equivalent changes.

A basic economic intuition behind this transformation is as follows. Note that $\frac{d(1-F(s;\theta))}{d\theta}$ represents the change in the survival function, i.e., the total probability mass that the perturbation shifts from states to the left of s to states to the *right* of s. Dividing by the density $f\left(s;\theta\right)$ expresses this change relative to the likelihood of state s, yielding the *normalized survival change*, $\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)}$. Probability pricing says the cash flow equivalent of a perturbation in probabilities is large for an agent if probabilities are shifted towards higher realizations of s. Moreover, the equivalent cash flow is particularly large if those shifts occur in states in which the sensitivity $\frac{dc_1(s)}{ds}$ of consumption to the state is large.[7]

For a more detailed understanding of Equation (5), note that gaining the utility flow $u\left(\cdot\right)$ at a given state is equivalent to gaining the marginal utility $\frac{du(c_1(s))}{ds}=u'\left(c_1\left(s\right)\right)\frac{dc_1(s)}{ds}$ in all states to the *left* of that state, by virtue of the fundamental theorem of calculus. Next, note that the normalized survival change $\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)}$ aggregates all the (net) changes in densities to the *right* of state s, where each of these changes induces a welfare gain valued at $u'\left(c_1\left(s\right)\right)\frac{dc_1(s)}{ds}$. Aggregating these gains across all states yields the probability pricing formula in Equation (4). Figure 1 illustrates this logic.

In principle, both $\frac{d(1-F(s;\theta))}{d\theta}$ and $\frac{dc_1(s)}{ds}$ can take negative values, so the consumption-equivalent defined in Equation (5) can be negative, so can the probability price p_{θ}. For instance, if the normalized survival change is positive, then the distribution of s becomes

[7]On a more technical note, the hypothetical cash flows in our analysis are related to *weak derivatives*, and the associated probability price is often well-defined even when standard regularity conditions fail (see, for example, Valli, 2023).



(a) Single State Perturbation

(b) General Perturbation

Figure 1: Illustrating Probability Pricing

Note: In this figure, $c_1(s) = s$, so $u(c_1(s)) \equiv u(s)$. The left panel illustrates the change in date-1 utility, $\int_{\underline{s}}^{\bar{s}} u(s) f(s)\, ds$, induced by perturbing the pdf in a particular state. In this case, the fundamental theorem of calculus allows us to express the utility change $u(s) \frac{df(s)}{d\theta}$ as $A_1 \times \blacksquare$, assuming without loss that $u(\underline{s}) = 0$. The right panel illustrates the change in date-1 utility induced by perturbing the pdf in multiple states. The probability price of the general perturbation, p_θ, corresponds to the sum over the perturbed states of the products of the areas A_1 to A_3 with the density changes:

$$p_\theta = A_1 \times \blacksquare + A_2 \times \blacksquare + A_3 \times \blacksquare,$$

where we exploit the fact that $\int_{\underline{s}}^{\bar{s}} \frac{df(s)}{d\theta} ds = 0$. But this sum can be equivalently calculated by adding up (integrating) at every state the product of marginal utility $u'(\cdot)$ with the change in densities to the right $\frac{d(1-F(\cdot))}{d\theta}$: this is exactly the probability pricing formula in Equation (4).

locally more "optimistic", since probability mass shifts from low to high realizations of s. A negative value of p_θ simply indicates that the perturbation makes the agent worse off, so the willingness-to-pay for it is negative.[8]

On the Usefulness of Probability Pricing. The following remarks highlight the benefits of expressing probability prices in terms of marginal utilities and consumption equivalents, as in Equation (4), rather than in terms of utility levels, as in (3).

[8] Computing an agent's willingness-to-pay for a marginal change in probabilities (probability pricing, p_θ) is different from computing how an agent's willingness-to-pay for an asset changes in response to a marginal change in probabilities (comparative statics of cash flow pricing, $\frac{dp_x}{d\theta}$). The answer to the latter question can be expressed as

$$\frac{dp_x}{d\theta} = \int_{\underline{s}}^{\bar{s}} \omega(s) \frac{\frac{df(s;\theta)}{d\theta}}{f(s;\theta)} x(s)\, ds.$$

It is thus evident that $\frac{dp_x}{d\theta}$ and p_θ differ. We further elaborate on this difference in Appendix D.4.

Remark 1. *(Hedging/Immunization against Changes in Probabilities)* The consumption-equivalent cash flows defined in (5) offer a direct way to construct a trading strategy that would immunize an investor against (initially unanticipated) changes in probabilities. In particular, an investor who seeks insurance against changes in the probabilities of different scenarios can use Equation (5) to identify the cash flows that a hedging strategy must be designed to replicate. This strategy ensures that the investor's welfare is immunized against changes in probabilities.[9] Notice that the replicating portfolio does not depend on the agent's preferences, and depends only on changes in probabilities and consumption sensitivities $\frac{dc_1(s)}{ds}$.

Remark 2. *(Stochastic and Cross-Sectional Decompositions)* Equation (4) admits a useful stochastic decomposition into two components, an expected (equivalent) payoff component and a risk compensation component, as follows:

$$p_\theta = \underbrace{\frac{1}{1+r^f}\frac{d\mathbb{E}\left[c_1\left(s\right)\right]}{d\theta}}_{\text{Expected Payoff}} + \underbrace{\mathbb{C}ov\left[m\left(s\right), \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f\left(s;\theta\right)}\frac{dc_1\left(s\right)}{ds}\right]}_{\text{Risk Compensation}}, \tag{6}$$

where $1+r^f = 1/\mathbb{E}\left[m\left(s\right)\right]$ denotes the risk-free rate and where, as shown in the Appendix, $\frac{d\mathbb{E}[c_1(s)]}{d\theta} = \mathbb{E}\left[\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)}\frac{dc_1(s)}{ds}\right]$. This decomposition clarifies whether a perturbation to probabilities is valuable because it changes expected consumption or because it reshuffles consumption across states with different valuations. We illustrate this use in Application 1.

Probability pricing can also be useful in economies with heterogeneous agents and incomplete markets to compute cross-sectional welfare decompositions of the form introduced in Dávila and Schaab (2025). These decompositions require expressing the impact of perturbations in consumption-equivalent units, which would not be possible without the probability pricing formula introduced in Equation (4). We illustrate this use in Applications 2 and 3.

Remark 3. *(Consumption Adjustments and Probability Pricing)* In general, we can allow the agent's consumption profile at date 1 to vary with the perturbation parameter θ by writing $c_1\left(s;\theta\right)$; we could do the same with date-0 consumption. In particular, the mapping $c_1\left(s;\theta\right)$ can capture endogenous consumption adjustments that take place in equilibrium in response to changes in probabilities. In this case, the agent's welfare gain/willingness-to-pay, expressed in date-0 consumption units, can be written as a combination of a cash flow pricing

[9] This application of the probability pricing formula for hedging/immunization is analogous to how Greeks — in particular, Vega — are employed in option pricing. However, note that Equation (5) explicitly provides the replicating payoffs needed to immunize against any change in probabilities.

term for consumption adjustments and a probability pricing term, as follows:

$$\frac{\frac{dV}{d\theta}}{u'(c_0)} = \int_{\underline{s}}^{\bar{s}} \omega(s) \left(\underbrace{\frac{\partial c_1(s;\theta)}{\partial \theta}}_{\text{Consumption}} + \underbrace{\frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{\partial c_1(s;\theta)}{\partial s}}_{\text{Probability}} \right) f(s;\theta)\, ds, \tag{7}$$

where $\omega(s)$ is the state-price, as in (4). Equation (7) is agnostic about the exact mechanism through which the consumption mapping $c_1(s;\theta)$ depends on changes in probabilities. In our applications, consumption varies in response to a change in probabilities conditional on a state because agents' decisions endogenously adjust. This consumption adjustment may be due to changes in optimal contracts (Application 2) or in competitive equilibrium allocations (Applications 3 and 4). Similar forces would also apply to economies with strategic trade (e.g., Kyle, 1989; Vives, 2011; Rostek and Weretka, 2012). In Appendix D.6, we further extend this formula to show how it can be applied in dynamic economies.

Moreover, Equation (7) also implies that one can construct changes in consumption holding probabilities fixed and, symmetrically, changes in probabilities holding the consumption mapping fixed that yield equivalent valuation/welfare implications. Appendix D.5 elaborates on this equivalence, contrasting the approaches to evaluating the cost of business cycles in Lucas (1987) and Alvarez and Jermann (2004).

2.4 Probability Pricing for Specific Perturbations

We now illustrate the implications of the probability pricing result by examining specific perturbations to probability distributions. This is useful to better understand the form that normalized survival changes — and consequently consumption-equivalent cash flows $x_\theta(s)$ and probability prices p_θ — take in practical scenarios.

2.4.1 Mean/Variance Perturbations

First, we highlight properties of probability prices by considering perturbations to a distribution that is characterized in terms of a mean parameter μ and a standard deviation parameter σ. Suppose that the state is defined in affine form, as in

$$s = \mu + \sigma n, \tag{8}$$

where μ and $\sigma \geq 0$ are parameters, and where n is a random variable distributed according to a cdf $H(n)$.[10] In this case, the normalized survival change $\frac{\frac{d(1-F(s))}{d\theta}}{f(s)}$ can be expressed in closed form:

i) A marginal increase in the mean of s, $d\mu$, induces a normalized survival change given by

$$\frac{\frac{d(1-F(s))}{d\mu}}{f(s)} = 1. \tag{9}$$

ii) A marginal increase in the standard deviation of s, $d\sigma$, induces a normalized survival change given by

$$\frac{\frac{d(1-F(s))}{d\sigma}}{f(s)} = \frac{s-\mu}{\sigma}. \tag{10}$$

Equation (9) shows that the hypothetical cash flow determining probability prices induced by a marginal increase in μ is simply the consumption sensitivity $\frac{dc(s)}{ds}$, so

$$p_\mu = \int_{\underline{s}}^{\bar{s}} \omega(s) \frac{dc_1(s)}{ds} ds \quad \underset{c_1(s)=s}{\Longrightarrow} \quad p_\mu = \int_{\underline{s}}^{\bar{s}} \omega(s)\, ds. \tag{11}$$

If $c_1(s) = s$, the distribution $F(s)$ is directly defined over consumption (s is a lottery), and Equation (11) implies that the probability price of a marginal increase in μ is the same as the price of the risk-free asset. Intuitively, the willingness-to-pay for an asset that increases consumption unconditionally in all future states (risk-free asset) must be the same as the willingness-to-pay for a perturbation that shifts all probabilities uniformly to the right when $c_1(s) = s$ (marginal increase in μ).

Similarly, Equation (10) implies that the hypothetical cash flow determining probability prices induced by a marginal increase in σ is given by $\left(\frac{s-\mu}{\sigma}\right)\frac{dc_1(s)}{ds}$, so

$$p_\sigma = \int_{\underline{s}}^{\bar{s}} \omega(s) \left(\frac{s-\mu}{\sigma}\right) \frac{dc_1(s)}{ds} ds \quad \underset{c_1(s)=s}{\Longrightarrow} \quad p_\sigma = \int_{\underline{s}}^{\bar{s}} \omega(s) \left(\frac{s-\mu}{\sigma}\right) ds. \tag{12}$$

Intuitively, a marginal increase in σ shifts mass to the tails, reducing survival probabilities in states below the mean ($s < \mu$), and increasing them in states above the mean ($s > \mu$). It thus follows from Equation (6) that the probability price of a shift in σ is exclusively driven by a risk compensation. Interestingly, if $c_1(s) = s$, Equation (12) implies that the willingness-to-pay for a forward contract with forward price μ must be the same as the willingness-to-pay

[10] If we further assume that $\mathbb{E}[n] = 0$ and $\mathbb{V}ar[n] = 1$, then $\mathbb{E}[s] = \mu$ and $\mathbb{V}\mathrm{ar}[s] = \sigma^2$, but this is not necessary for Equations (9) and (10) to hold.

for a perturbation that increases the volatility of consumption σ, after normalizing by $1/\sigma$.

2.4.2 Mixture Distributions

A second useful scenario is one in which the state is defined as a mixture of two distributions. Formally, suppose that with probability $1-h$ the state follows a distribution with cdf $\overline{F}(s)$, and with probability h the state follows a distribution with cdf $\underline{F}(s)$. Such mixtures are often used to model discrete jumps, large shocks, or disasters, as in Application 1 below.

In this case, a marginal increase in the mixing probability h leads to a normalized survival change given by

$$\frac{\frac{d(1-F(s))}{dh}}{f(s)} = \frac{\overline{F}(s) - \underline{F}(s)}{(1-h)\,\overline{f}(s) + h\underline{f}(s)}. \tag{13}$$

The sign of the normalized survival change — and hence the hypothetical consumption-equivalent cash flow in state s — is entirely determined by the difference between cdf's $\overline{F}(s) - \underline{F}(s)$. Moreover, note that under this mixture structure p_h is invariant to the level of h. This is a property that we exploit in Application 1.

2.4.3 Stochastic Dominance

Finally, we show how the probability pricing formula relates to the classic literature on preferences over lotteries.[11] Suppose that consumption is $c_1(s) = s$, so that the state s and the distribution $F(s)$ define a lottery over consumption. In this case, probability prices obey the following properties:

i) *First-order stochastic dominance:* The probability price for perturbations such that $\frac{dF(s)}{d\theta} \leq 0$ — equivalently, $\frac{d(1-F(s))}{d\theta} \geq 0$ — for all s satisfies $p_\theta \geq 0$.

ii) *Second-order stochastic dominance/mean-preserving spreads:* The probability price for perturbations such that $\frac{d\mathbb{E}[s]}{d\theta} = 0$ and $\int_{\underline{s}}^{s} \frac{dF(t)}{d\theta} dt \geq 0$ for all s satisfies $p_\theta \leq 0$.

The two properties just highlighted confirm that probability prices can be used to derive well-known properties of the preferences of risk-averse agents. The first property shows that an agent is always willing to pay a positive price for a perturbation that implies "good news" in the sense of first-order stochastic dominance. The second property, as in Rothschild and Stiglitz (1970), shows that a risk-averse agent is never willing to pay a positive price for a

[11] See, for example, Pratt (1964), Rothschild and Stiglitz (1970) and Arrow (1971). More recent work includes, among others, Aumann and Serrano (2008), Foster and Hart (2009), and, outside of the expected utility framework, Ai, Bansal, Guo, and Yaron (2024).

perturbation that implies “more risk” in the sense of a mean-preserving spread. Note that these properties rely on assuming $c_1(s) = s$. Outside of the special case in which $\frac{dc_1(s)}{ds} = 1$, the sensitivity of consumption to the state $\frac{dc_1(s)}{ds}$ can have non-trivial implications that go beyond the classic literature on lotteries.

More broadly, Proposition 1 can be viewed as a generalization of this classical literature. Instead of insisting on deriving stochastic orders over the distributions of particular gambles, probability pricing provides the exact willingness-to-pay — positive or negative — of an agent for arbitrary marginal changes in the distribution of outcomes, not only those changes that satisfy particular dominance properties.

3 Applications: Physical Probabilities

Our objective in the remainder of this paper is to show that probability pricing has broad applications and multiple use cases. In this section, we initially present two scenarios in which we directly vary physical probabilities. In Section 4, we use probability pricing to study changes in the distribution of signals, that is, the consequences of changes in information. We highlight across all four applications how being able to transform changes in probabilities into consumption-equivalents is critical to generate new insights.

3.1 Application 1: Consumption-Based Asset Pricing

Our first application leverages the probability pricing result to study the willingness-to-pay for changes in the distribution of aggregate consumption in a canonical consumption-based asset pricing model. This exercise is the closest application of our single-agent results in a general equilibrium setting. It shows how our framework can be used to quantify and decompose the welfare costs of changes in risk, such as those induced by disasters (e.g., climate change) or macroeconomic volatility. While the model is standard, our results allow us to separate the welfare cost of a given perturbation into an expected payoff effect and a risk-compensation effect — as in (6) — offering a clear lens through which to assess the channels through which risk impacts valuation and welfare.

3.1.1 Environment

Consider a two-date fruit-tree economy with exogenous output.[12] We assume that the representative agent has time-additive utility with isoelastic preferences parametrized by γ, so

$$V = u(c_0) + \beta \int_{\underline{s}}^{\bar{s}} u(c_1(s)) f(s) \, ds, \qquad \text{where} \qquad u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Following Barro (2009), we assume that date-1 output is distributed as $y_1(s) = e^s y_0$, where the physical state s is a random variable whose distribution is a mixture of normal distributions. With probability $1-h$, s is normally distributed with mean $\overline{\mu}$ and standard deviation $\overline{\sigma}$, and with probability h, s is normally distributed with mean $\underline{\mu}$ and standard deviation $\underline{\sigma}$. Since the economy is closed and all output is consumed, consumption equals output at all times. This setting can be used to capture disasters or large shocks — see Section 2.4.2 The parameters $\overline{\mu}$ and $\overline{\sigma}$ can be interpreted as defining the distribution of consumption in normal times, while h has the interpretation of the discrete likelihood of a disaster materializing, in which case the distribution of consumption is defined by $\underline{\mu}$ and $\underline{\sigma}$.

In this model, the normalized survival change is given by (13), while the assumed specification of uncertainty implies that

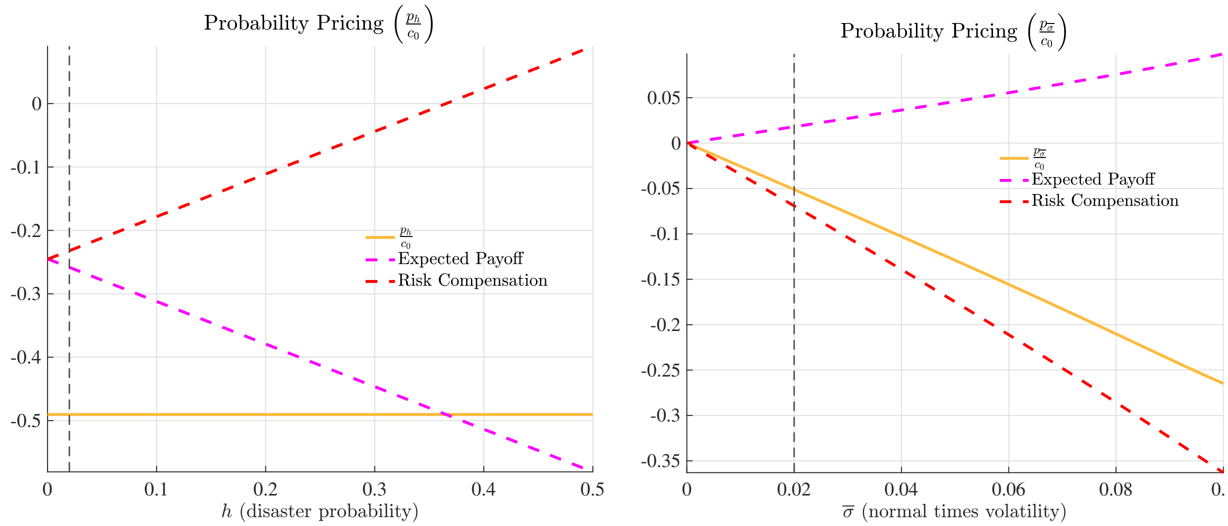
$$\frac{dc_1(s)}{ds} = c_1(s).$$

3.1.2 Value of Changes in Probabilities

We now compute the willingness-to-pay for different changes in the underlying distribution of consumption. Interpreting a date in the model as a year, we use a rate of time preference $\beta = 0.95$ and a risk aversion coefficient of $\gamma = 4$, again consistent with Barro (2009). We set $h = 0.02$ to capture a 2% yearly disaster probability, with a distribution of consumption growth in normal times given by $\overline{\mu} = 0.025$ and $\overline{\sigma} = 0.02$. If a disaster takes place, consumption falls on average by roughly 30%, with $\underline{\mu} = -0.3$ and $\underline{\sigma} = 0.02$. By normalizing $y_0 = 1$, we can interpret all values as relative to the level of initial consumption.

The left panel in Figure 2 shows the willingness-to-pay for a marginal change in the yearly probability of disaster h, p_h, for different values of h. In terms of magnitudes, increasing the probability of disaster by one percentage point (from 2% to 3%, so $\Delta h = 0.01$) is associated with a consumption loss of roughly half that amount, since $p_h/c_0 \approx -0.5$. Although p_h is

[12]To more easily illustrate the results, we consider a two-date economy. The results straightforwardly extend to multi-period economies.



(a) Varying Disaster Probability h

(b) Varying Volatility $\overline{\sigma}$

Figure 2: Probability Pricing (Application 1)

Note: This figure shows the willingness-to-pay (relative to initial consumption) for changes in the disaster probability h for different values of h (left panel), p_h/c_0, and for changes in the volatility of consumption in normal times $\overline{\sigma}$ for different values of $\overline{\sigma}$ (right panel), $p_{\overline{\sigma}}/c_0$. Preference parameters are $\beta = 0.95$ and $\gamma = 4$. The baseline distribution is defined by $\overline{\mu} = 0.025$, $\underline{\sigma} = 0.02$, $\underline{\mu} = -0.3$, $\overline{\sigma} = 0.02$ and $h = 0.02$, indicated by dashed vertical black lines in both figures.

constant across value of h — a property of mixture distributions discussed in Section 2.4.2 — our decomposition reveals that this constancy masks two offsetting effects. First, since $\frac{d\mathbb{E}[c_1(s)]}{dh}$ is constant in h, the expected payoff component in (6) must become more negative as h increases. This reflects stronger precautionary motives, which lower the interest rate. Second, to maintain a constant total p_h, the risk-compensation component in (6) must rise in magnitude with h, offsetting the growing impact of expected consumption. Thus, the probability pricing formula uncovers that, as h increases, the balance between precautionary savings and risk compensation shifts.

The right panel in Figure 2 shows the willingness-to-pay for a marginal change in the volatility of consumption during normal times $\overline{\sigma}$, $p_{\overline{\sigma}}$, for different values of $\overline{\sigma}$. In terms of magnitudes, increasing the volatility $\overline{\sigma}$ also by one percentage point yields a much smaller loss since $p_{\overline{\sigma}}/c_0 \approx -0.05$ at $\overline{\sigma} = 0.02$. Because $y_1(s)$ is log-normally distributed, increasing $\overline{\sigma}$ raises expected aggregate consumption via Jensen's inequality effect, so $\frac{d\mathbb{E}[c_1(s)]}{d\overline{\sigma}} > 0$. Yet despite this rise in expected consumption, the willingness-to-pay is negative. This reflects a welfare loss due to the fact that changes in consumption take place in states with different valuations, with reductions in consumption taking place in bad states and vice versa. Thus, the probability pricing formula can be used to show that the net loss results from a trade-off between a higher average payoff and greater exposure to low-value states.

3.2 Application 2: Principal-Agent Problem

This second application studies the welfare/willingness-to-pay impact of changes in output uncertainty in a canonical principal-agent problem — as in, e.g., chapter 4 of Bolton and Dewatripont (2005), whose notation we follow whenever possible. Probability pricing allows us to formalize new insights about this widely studied environment. In particular, this application highlights how the probability pricing effects that we study in this paper are the key driver of overall efficiency gains in a constrained efficient environment, in which endogenous consumption adjustments have no overall marginal impact on efficiency.

3.2.1 Environment

We consider an environment in which a principal, indexed by $i = B$ (boss), contracts with an agent, indexed by $i = A$. The principal is risk-neutral, with preferences given by

$$V^B = \int c^B(s) f(s)\, ds,$$

while the agent is risk-averse, with preferences given by

$$V^A = \int u\left(c^A(s)\right) f(s)\, ds,$$

where $c^i(s)$ denotes consumption of agent $i \in \{A, B\}$ in state s. We assume that the agent has constant absolute risk aversion preferences, with $u(c) = -e^{-\eta c}$, where η is the coefficient of absolute risk aversion. The agent makes a costly effort decision e, which yields a normally distributed random output/performance

$$y(s) = e + s \quad \text{with} \quad s \sim \mathcal{N}(0, \sigma),$$

where $\tau = 1/\sigma^2$ denotes the precision of output uncertainty. The agent receives a compensation $w(s)$, which is linear in output, so the consumption of the agent in state s is given by

$$c^A(s) = w(s) - \psi(e), \quad \text{where} \quad w(s) = t + \alpha y(s),$$

where t denotes an uncontingent transfer and where we refer to α as the performance sensitivity to output. We assume that the cost function for effort is $\psi(e) = \frac{\kappa}{2}e^2$. The consumption of the principal is thus given by

$$c^B(s) = y(s) - w(s).$$

In this economy, aggregate consumption in state s is simply given by $e + s - \psi(e)$, so the first-best level of effort is $e = \frac{1}{\kappa}$.

Optimal Contract. The optimal linear contract, fully characterized in the Appendix, maximizes the utility of the principal subject to participation and incentive constraints for the agent. The optimal contract features a sensitivity to output $\alpha^\star$, which in turn induces an effort decision $e^\star$, given by

$$\alpha^\star = \frac{1}{1 + \frac{\eta\kappa}{\tau}} = \frac{\tau}{\tau + \eta\kappa}, \quad \text{and} \quad e^\star = \frac{\alpha}{\kappa}.$$

As the output uncertainty vanishes ($\tau \to \infty$), the performance sensitivity becomes maximal $\alpha^\star \to 1$, as if there were no incentive constraint, while effort approaches the first-best level $e^\star \to \frac{1}{\kappa}$. Because we consider linear contracts, the solution to the principal-agent problem when $\tau \to \infty$ features production efficiency but is not the first-best solution, which would require the risk-neutral principal to fully insure the risk-averse agent. However, an important property of the optimal contract is that it is constrained efficient, in the sense that no alternative (linear) contract can generate a Pareto improvement for principal and agent.

3.2.2 Value of Changes in Performance Precision

We are interested in computing the welfare/willingness-to-pay impact of changes in the precision of output uncertainty, parametrized by τ. Formally, agent i's welfare gains induced by a marginal change in the precision of output uncertainty τ, expressed in units of date-1 uncontingent consumption,[13] are given by the following augmented probability pricing

[13] This choice of units is natural because, unlike in our general theory, we cannot use date-0 consumption in the classical principal-agent models, since all consumption happens at date 1.

formula:

$$\frac{dV^{i|\lambda}}{d\tau} = \frac{\frac{dV^i}{d\tau}}{\lambda^i} = \int \omega^i(s) \left(\underbrace{\frac{\partial c^i(s)}{\partial \tau}}_{\text{Consumption}} + \underbrace{\frac{\frac{d(1-F(s))}{d\tau}}{f(s)} \frac{\partial c^i(s)}{\partial s}}_{\text{Probability}} \right) ds, \tag{14}$$

where $\lambda^i = \int \frac{\partial u^i(c^i(s))}{\partial c^i(s)} f(s)\, ds$ and where agent i's (shadow) state prices are given by

$$\omega^i(s) = \frac{1}{\lambda^i} \frac{\partial u^i(c^i(s))}{\partial c^i(s)} f(s). \tag{15}$$

The state price $\omega^i(s)$ defines agent i's marginal rate of substitution between consumption in state s and uncontingent consumption.[14] If both agents could perfectly insure each other, $\omega^i(s)$ would be equal across agents, but in this economy full insurance is not possible. Analogously to (7), Equation (14) shows that individual welfare gains consist of i) a consumption component, which captures impact of the endogenous change in the optimal contract between principal and agent — and the associated changes in consumption — induced by changes in the precision of output uncertainty τ for particular realizations of the state s, and ii) a probability component, which captures the direct impact of changing the probabilities of different states, for given consumption allocations in each state s. In the Appendix, we analytically characterize each of the components of Equation (14).

Cross-Sectional Efficiency Decomposition. Because probability pricing allows us to translate changes in probabilities into consumption equivalents, we are able to decompose the sources of efficiency gains in this economy. Formally, we abstract from redistributional considerations and focus on characterizing (Kaldor-Hicks) efficiency gains — that is, the sum of individual gains/willingness-to-pay — given by

$$\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\tau} = \underbrace{\sum_i \int \omega^i(s) \frac{\partial c^i(s)}{\partial \tau} ds}_{\Xi^E_c \text{ (Consumption)}} + \underbrace{\sum_i \int \omega^i(s) \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} \frac{\partial c^i(s)}{\partial s} ds}_{\Xi^E_s \text{ (Probability)}}, \tag{16}$$

[14] Other numeraire choices to express welfare gains — such as date-0 consumption if it existed, as in Section 2 — would require to define a different normalizing factor λ^i, but would yield similar insights. Note that

$$\omega^i(s) = \frac{\frac{\partial u^i(c^i(s))}{\partial c^i(s)} f(s)}{\int \frac{\partial u^i(c^i(s))}{\partial c^i(s)} f(s)\, ds},$$

so the normalized states prices for each agent add up to 1. Formally, $\int \omega^i(s)\, ds = 1$, $\forall i$.

where Ξ_c^E and Ξ_s^E denote the sum of individual welfare gains due to consumption adjustments and probabilities, respectively. The consumption and probability terms aggregate the individual effects described in (14). As explained above, the consumption term has the interpretation of changes in consumption given a state is realized, while the probability term has the interpretation of changes in the probabilities of different states being realized, for given consumption allocations at each state.

To gain further insight, we decompose the efficiency gains from a perturbation, using a similar method to Dávila and Schaab (2025). First, we re-write the consumption term as

$$\begin{aligned}\Xi_c^E &= \int \omega(s) \frac{\partial C(s)}{\partial \tau} ds + \int \omega(s) \mathbb{C}ov_i^{\Sigma} \left[\frac{\omega^i(s)}{\omega(s)}, \frac{\partial c^i(s)}{\partial \tau} \right] ds \\ &\equiv \Xi_c^{AE} + \Xi_c^{RS},\end{aligned} \tag{17}$$

where $\omega(s) = \frac{1}{I} \sum_i \omega^i(s)$ is an average state price and $C(s) = \sum_i c^i(s)$ denotes aggregate consumption.[15] This formula highlights that Ξ_c^E can be decomposed into i) the aggregate-efficiency component Ξ_c^{AE}, capturing the gains from changes in the value of aggregate consumption, and ii) the risk-sharing component Ξ_c^{RS}, capturing the gains from reallocating consumption towards agents with higher marginal valuations, $\omega^i(s)$.

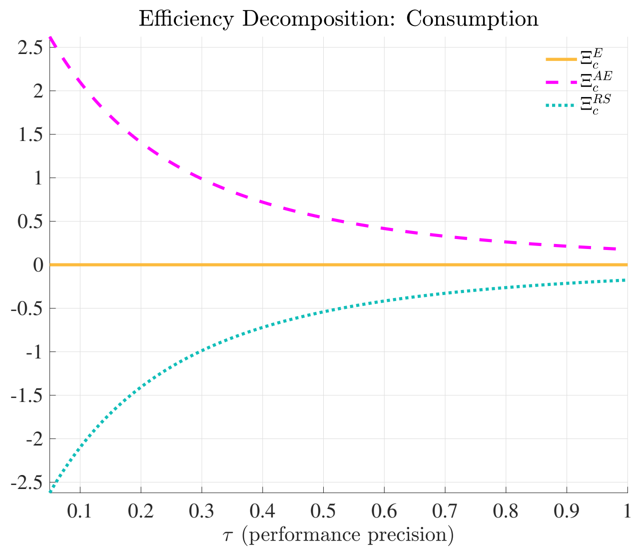
Second, we decompose the probability term as follows:

$$\begin{aligned}\Xi_s^E &= \int \omega(s) \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} \frac{\partial C(s)}{\partial s} ds + \int \omega(s) \mathbb{C}ov_i^{\Sigma} \left[\frac{\omega^i(s)}{\omega(s)}, \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} \frac{\partial c^i(s)}{\partial s} \right] ds \\ &\equiv \Xi_s^{AE} + \Xi_s^{RS}.\end{aligned} \tag{18}$$

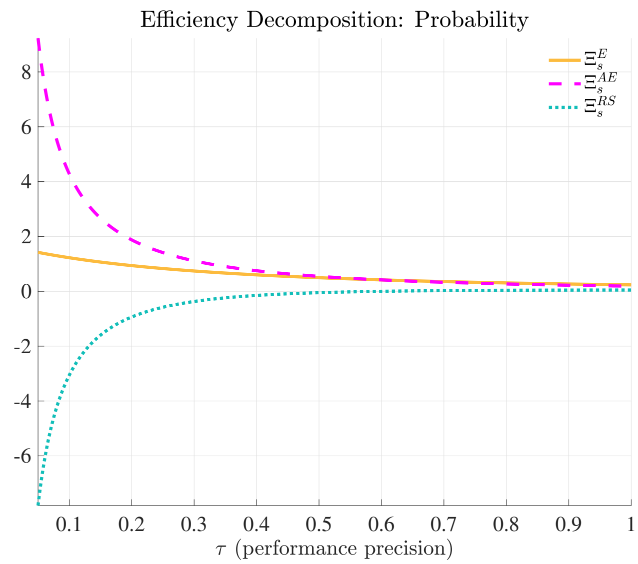
The aggregate-efficiency component Ξ_s^{AE} in this expression is a probability price in the sense of Proposition 1: it measures the (hypothetical) willingness-to-pay of an average agent, who consumes $C(s)$, for the perturbation in probabilities. The risk-sharing component Ξ_s^{RS} again measures the gains that arise due to reallocating (hypothetical) consumption gains towards agents with high marginal valuations $\omega^i(s)$. We provide explicit definitions and analytical characterizations of each of the components in the Appendix.[16]

[15] The operator $\mathbb{C}ov_i^{\Sigma}[\cdot,\cdot] = I \cdot \mathbb{C}ov_i[\cdot,\cdot]$ denotes a cross-sectional covariance-sum among agents, where $I = 2$ in this two-agent model.

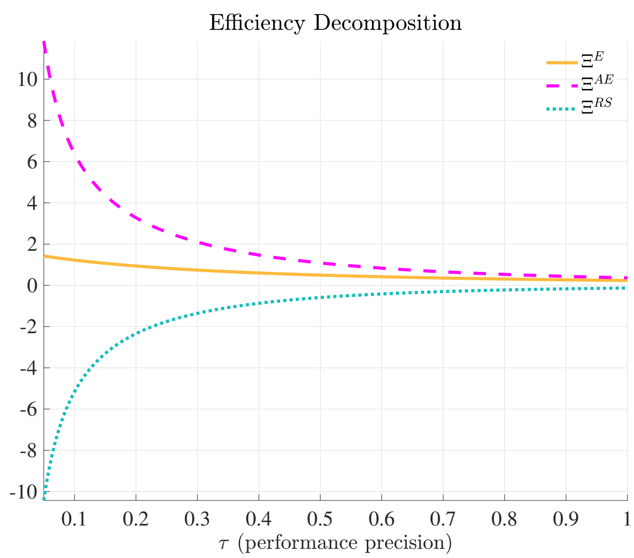
[16] While the split in (16) does not require the probability pricing result, it is necessary for (17) and (18), and more generally for any stochastic and cross-sectional decomposition.



(a) Consumption Decomposition:
$\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS}$



(b) Probability Decomposition:
$\Xi_s^E = \Xi_s^{AE} + \Xi_s^{RS}$



(c) Efficiency Decomposition:
$\Xi^E = \Xi^{AE} + \Xi^{RS}$

Figure 3: Varying Precision of Output Uncertainty (Application 2)

Note: This figure illustrates the implications for efficiency gains induced by changing the precision of output uncertainty. The top left panel illustrates that the efficiency gains due to consumption adjustments induced by a change in the precision of output uncertainty are zero, with positive aggregate-efficiency and negative risk-sharing gains exactly compensating each other. The top right panel illustrates that the efficiency gains due to directly changing probabilities induced by an increase in the precision of output uncertainty are strictly positive, even though initially there are risk-sharing losses. The bottom panel illustrates that overall efficiency gains induced by an increase in the precision of output uncertainty are strictly positive, initially trading off aggregate-efficiency gains with risk-sharing losses. The parameters in this figure are $\eta = 1.2$, $\kappa = 0.5$, and $\bar{V} = -0.25$.

Results. Our analysis yields three main takeaways, which we collect in Proposition 2 and illustrate in Figure 3.

Proposition 2. (Principal-Agent Problem) *In response to an increase in the precision of output uncertainty* τ*:*

i) Efficiency gains due to the equilibrium consumption response are zero: $\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS} = 0$*, where* $\Xi_c^{AE} > 0$ *and* $\Xi_c^{RS} < 0$*;*

ii) Efficiency gains due to the direct probability change are strictly positive: $\Xi_s^E = \Xi_s^{AE} + \Xi_s^{RS} > 0$*, where* $\Xi_s^{AE} > 0$ *and* $\Xi_s^{RS} \gtreqless 0$*;*

iii) Overall efficiency gains are strictly positive: $\Xi^E = \Xi^{AE} + \Xi^{RS} > 0$*, where* $\Xi^{AE} > 0$ *and* $\Xi^{RS} < (>)\, 0$.

First, we show that the efficiency gains due to the endogenous consumption response to the change in the precision of output uncertainty are zero, that is,

$$\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS} = 0. \tag{19}$$

This result arises because the contract is optimally chosen to trade off effort/production decisions and risk-sharing considerations, and is a reflection of the fact that this economy is constrained efficient.[17] Intuitively, as the precision of output uncertainty τ increases, the performance sensitivity of the contract α increases, which induces the agent to exert more effort e. This increase in effort increases aggregate net consumption in each state since the agent's effort is too little — below first-best — to begin with, generating an aggregate-efficiency gain. But the increase in the performance sensitivity of the contract makes the agent's consumption more volatile, generating a risk-sharing loss. Formally,

$$\Xi_c^{AE} = (1 - \psi'(e)) \frac{de}{d\tau} > 0 \quad \text{and} \quad \Xi_c^{RS} < 0, \tag{20}$$

where the optimal contract is such both channels exactly cancel out and Equation (19) holds. The top left panel in Figure 3 illustrates this result, with the dashed line showing the

[17] As Hart and Holmström (1987) put it:

> *"The agency problem is not an inference problem in a strict statistical sense; conceptually, the principal is not inferring anything about the agent's action from the output because he already knows what action is being implemented. Yet, the optimal sharing rule reflects precisely the pricing of inference."*

aggregate-efficiency gains and the dotted line the risk-sharing losses. A consequence of this result is that the efficiency gains of perturbing τ are solely driven by the direct impact of changing probabilities, that is, $\Xi^E = \Xi^E_s$.

Second, we show that the efficiency gains attributed to directly changing probabilities induced by an increase in the precision of output uncertainty τ are strictly positive, that is,

$$\Xi^E_s = \Xi^{AE}_s + \Xi^{RS}_s > 0. \tag{21}$$

This occurs for two reasons. First, an increase in the precision of output uncertainty τ reduces aggregate consumption risk purely for technological reasons, for given effort and consumption allocations. This change in the probability distribution of aggregate consumption generates an aggregate-efficiency gain, since the average agent is risk-averse. Second, the reduction in aggregate consumption disproportionally benefits the principal or the agent depending on the value of the (endogenously determined) performance sensitivity α. Intuitively, when $\alpha > \frac{1}{2}$ the principal is relatively less exposed to aggregate consumption, so a reduction in aggregate consumption risk disproportionally benefits the agent. But the agent — who is risk-averse — values less volatile consumption relatively less than the risk-neutral principal, generating risk-sharing gains. The opposite occurs when $\alpha < \frac{1}{2}$. Formally,

$$\Xi^{AE}_s > 0 \quad \text{and} \quad \Xi^{RS}_s = \begin{cases} < 0 & \text{if } \ \alpha < \frac{1}{2} \\ = 0 & \text{if } \ \alpha = \frac{1}{2} \\ > 0 & \text{if } \ \alpha \geq \frac{1}{2} \end{cases}, \tag{22}$$

where the aggregate-efficient gains always dominate so that Equation (21) holds.

At last, we show that the overall efficiency gains induced by an increase in the precision of output uncertainty are strictly positive. This is because aggregate-efficiency gains are strictly positive, and are always higher than risk-sharing losses, even for low values of τ. As τ increases, there are both aggregate-efficiency and risk-sharing gains, yielding an unambiguous overall efficiency gain. Formally,

$$\Xi^E = \Xi^{AE} + \Xi^{RS} > 0,$$

where $\Xi^{AE} > 0$ and $\Xi^{RS} < (>)\, 0$ for low (high) values of τ.

4 Applications: Value of Information

So far, we have used probability pricing to study changes in physical probabilities. However, probability pricing is particularly well suited to characterizing the (private and social) value of changes in information, as we explain next.

Remark 4. *(Changes in Information Are Changes in Probabilities).* Changes in information can be represented as changes in the conditional probabilities that agents assign to different states. Thus, modifying the informational environment amounts to perturbing these conditional probabilities, typically via changes in a signal's likelihood, that is, the conditional distribution of the signal given other variables. This equivalence allows changes in information to be analyzed directly through the lens of probability pricing. The overall marginal value (willingness-to-pay) for a change in the informational environment can be decomposed, as in Equation (7), into two parts: one reflecting *consumption* adjustments conditional on a state/signal being realized, and another reflecting changes in the *probability* of different constellations of signals/states taking place, for given consumption allocation mappings.

We now illustrate these ideas in two equilibrium models: Application 3 examines changes in the precision of public information, while Application 4 examines changes in the precision of private information.

4.1 Application 3: Public Information

This application studies the welfare impact of changing public information in a version of Hirshleifer (1971)'s classic model. Hirshleifer (1971) shows that worse (i.e., less precise) public information can be welfare-improving, by improving risk-sharing in an endowment economy. Using probability pricing, we show that the "Hirshleifer effect" is the sum of two distinct, previously unexplored phenomena with opposite welfare implications.

First, changes in public information alter the probabilities of different signal realizations for given consumption allocation mappings. In particular, less precise public information increases the likelihood of more extreme signals — scenarios in which consumption is more unequal ex-post — a phenomenon we refer to as *signal dispersion* (and, conversely, *signal compression* when public information becomes more precise). We show that signal dispersion worsens risk-sharing and reduces welfare. Second, changes in public information trigger adjustments to equilibrium consumption profiles for given signal realizations. In particular, less precise public information makes consumption profiles more similar conditional on a signal realization. This phenomenon improves risk-sharing and increases welfare, because the equilibrium is constrained inefficient, and — when it dominates — drives the overall welfare loss pointed out by Hirshleifer (1971).[18] Hence, a key takeaway from this application is that more precise public information would be beneficial if agents were not allowed to adjust their consumption in a constrained inefficient environment.

Below, we also extend our analysis to a production economy. This extension demonstrates how probability pricing can be used to separately study and quantify the production efficiency and risk-sharing implications of changes in public information.

4.1.1 Environment: Endowment Economy

We initially consider an endowment economy with three dates $t = \{0, 1, 2\}$, a single consumption good, and two types of individuals indexed by $i = \{A, B\}$. Individuals are ex-ante identical at date 0 and consume at date 2. At date 2, there is a continuum of states, indexed by s. The aggregate endowment of consumption at date 2 is normalized to $n_2 = 1$, with type i receiving a share $\chi^i(s)$, with $\chi^A(s) + \chi^B(s) \equiv 1$. Following Hirshleifer (1971), there is no aggregate endowment risk and the state s affects only the distribution of endowments.

At date 1, there is a public signal, denoted by ξ, which provides information about the date-2 state. We write $f(\xi)$ for the marginal distribution of the signal, $f(s|\xi)$ for the conditional distribution (or posterior) of the date-2 state given the signal, and $f(\xi|s)$ for the conditional distribution (or likelihood) of the date-1 signal given the state.

Type-i individuals have expected utility preferences that can be expressed recursively as follows:

$$V^i = \int V_1^i(\xi) f(\xi) \, d\xi \quad \text{where} \quad V_1^i(\xi) = \int u\left(c_2^i(s, \xi)\right) f(s|\xi) \, ds,$$

In this formulation, V^i stands for type-i's ex ante expected utility at date 0. This value is given by the expected continuation value $V_1^i(\xi)$ at date 1 when the public signal is ξ. In turn, $V_1^i(\xi)$ is computed as the individual's expected utility from consumption $c_2^i(s, \xi)$ at date 2 when the state is s and the date-1 signal is ξ.[19]

[18] The constrained inefficiency — in the sense of Geanakoplos and Polemarchakis (1986) — arises because agents cannot trade securities contingent on signal realizations, resulting in market incompleteness.

[19] This formulation also implies the ex-ante expected utility $V^i = \iint u\left(c_2^i(s, \xi)\right) f(s, \xi) \, ds d\xi$, where $f(s, \xi)$ is the joint distribution of s and ξ. In models with information, the assumption of expected utility rules out preferences for early or late resolution of uncertainty, which in turn generate a "psychic" value on the timing of information. There is scope to further explore these additional sources of value in future work.

As in Hirshleifer (1971), we assume that individuals have no access to financial markets or contracting opportunities at date 0. At date 1, after the public signal is released, individuals have access to complete markets against the realization of the state s, so a type-i individual faces a date-1 budget constraint given by

$$\int q_1 (s|\xi)\, x_1^i (s|\xi)\, ds = 0, \ \forall \xi,$$

where $q_1 (s|\xi)$ denotes the price of an Arrow-Debreu security that pays at date 2 in state s, and $x_1^i (s|\xi)$ denotes type-i's holdings of that security. At date 2, consumption is given by

$$c_2^i (s, \xi) = \chi^i (s)\, n_2 + x_1^i (s|\xi)\,, \ \forall (s, \xi)\,.$$

The definition of competitive equilibrium is standard. Individuals make date-1 decisions over their portfolio of Arrow-Debreu securities $x_1^i (s|\xi)$. Then financial markets clear competitively. In particular, since markets are complete from date 1 onwards, portfolio decisions must satisfy

$$\frac{u' (c_2^i (s, \xi))\, f (s|\xi)}{u' (c_2^i (s', \xi))\, f (s'|\xi)} = \frac{q_1 (s|\xi)}{q_1 (s'|\xi)},$$

so marginal rates of substitution equal relative state prices for all individuals. However, notice that risk-sharing is incomplete ex ante, because individuals cannot insure each other against different realizations of the signal ξ.

While our method can be applied to this class of models regardless of functional forms, we make some additional assumptions for concreteness. First, we assume that $\chi^A (s) = \Phi (s)$, where $\Phi (s)$ denotes the normal cumulative distribution function, so that high realizations of s are "good news" for type-A individuals. Second, we assume that individuals have constant relative risk aversion with $u (c) = \frac{c^{1-\gamma}}{1-\gamma}$. Finally, the public signal takes the form:

$$\xi = s + \varepsilon, \text{ where } s \sim \mathcal{N} \left(0, \sigma_0^2\right) \quad \text{and} \quad \varepsilon \sim \mathcal{N} \left(0, \sigma^2\right).$$

Hence, changes in the information content of the public signal are parametrized by the volatility/noise parameter σ, with higher values of σ capturing less precise (worse) public information. Varying σ in this context contrasts with our first two applications since this change has no impact on technologies and endowments — it is purely about information.[20]

[20] After observing the signal ξ, the posterior distribution $f(s|\xi)$ is a normal distribution $\mathcal{N} \left(\mu_{s|\xi}, \sigma_{s|\xi}\right)$, with moments given by $\mu_{s|\xi} = \alpha\xi$ and $\sigma_{s|\xi}^2 = \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\sigma_0^2}}$, where $\alpha = \frac{\sigma_0^2}{\sigma^2 + \sigma_0^2}$.

4.1.2 Value of Changes in Public Signal Precision: Endowment Economy

We focus on characterizing the welfare impact of changes in the information content of the public signal, here parametrized by the volatility parameter σ. As shown in the Appendix, type-i's welfare gains induced by a marginal change in σ are given by the augmented probability pricing formula:

$$\frac{dV^{i|\lambda}}{d\sigma} \equiv \frac{\frac{dV^{i}}{d\theta}}{\lambda^{i}} = \iint \omega_2^i\left(s,\xi\right)\left(\underbrace{\frac{\partial c_2^i\left(s,\xi\right)}{\partial\sigma}}_{\text{Consumption}} + \underbrace{\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f\left(\xi\mid s\right)}\frac{\partial c_2^i\left(s,\xi\right)}{\partial\xi}}_{\text{Probability}}\right)d\xi ds, \tag{23}$$

where $\lambda^i = \iint u'\left(c_2^i\left(s,\xi\right)\right)f\left(s,\xi\right)dsd\xi$ denotes type-i's value of uncontingent date-2 consumption consumption,[21] and individual i's (shadow) state prices are given by

$$\omega_2^i\left(s,\xi\right) = \frac{u'\left(c_2^i\left(s,\xi\right)\right)f\left(s,\xi\right)}{\lambda^i}.$$

As in the previous application, we characterize (Kaldor-Hicks) efficiency gains — that is, the sum of individual gains/willingness-to-pay — as follows:

$$\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\sigma} = \underbrace{\sum_i \iint \omega_2^i\left(s,\xi\right)\frac{\partial c_2^i\left(s,\xi\right)}{\partial\sigma}d\xi ds}_{\Xi_c^E \text{ (Consumption)}} + \underbrace{\sum_i \iint \omega_2^i\left(s,\xi\right)\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f\left(\xi\mid s\right)}\frac{\partial c_2^i\left(s,\xi\right)}{\partial\xi}d\xi ds}_{\Xi_\xi^E \text{ (Probability)}}. \tag{24}$$

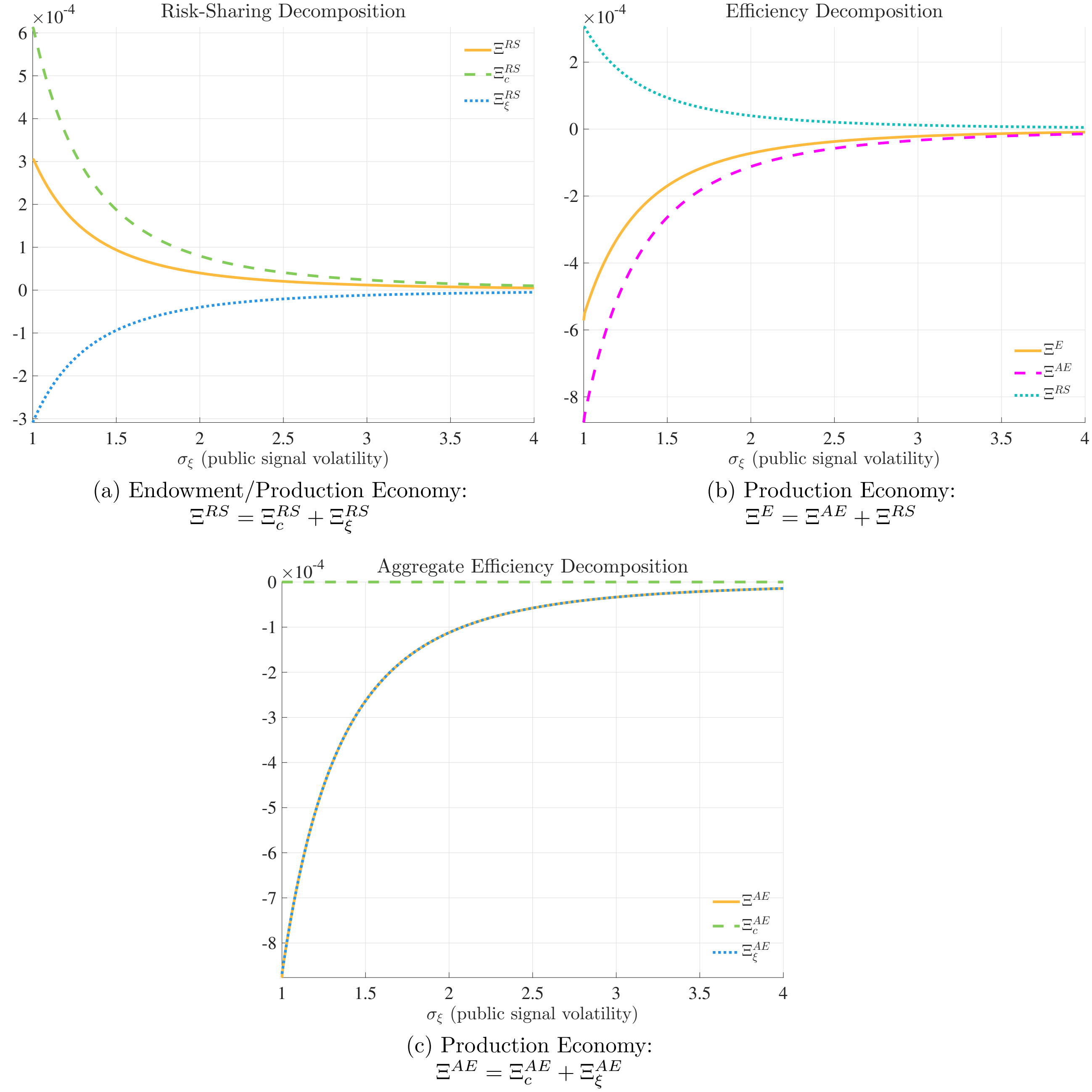
Here Ξ_c^E and Ξ_ξ^E denote the sum of individual welfare gains due to consumption adjustments and to the direct effect of changing probabilities. Since individuals are ex ante identical, welfare and efficiency coincide in this case. Notice that all efficiency gains in this endowment economy are necessarily driven by risk-sharing . Below, we further consider a production economy in which the quality of public information can affect aggregate-efficiency.

Results. Our analysis yields several takeaways, which we collect in Proposition 3 and illustrate in Figure 4.

Proposition 3. (Value of Public Information) *In response to an increase in the volatility of public information σ:*

i) Efficiency gains due to the direct probability change are strictly negative: $\Xi_\xi^E < 0$;

[21] As in the previous application, we use uncontingent date-2 consumption as the numeraire because there is no date-0 consumption in this model.



(a) Endowment/Production Economy:
$\Xi^{RS} = \Xi^{RS}_c + \Xi^{RS}_\xi$

(b) Production Economy:
$\Xi^{E} = \Xi^{AE} + \Xi^{RS}$

(c) Production Economy:
$\Xi^{AE} = \Xi^{AE}_c + \Xi^{AE}_\xi$

Figure 4: Varying Volatility of Public Signal (Application 3)

Note: This figure shows the welfare/efficiency implications induced by changing the volatility of the public signal. The top left panel illustrates that increasing the volatility of public information (worsening information) is associated with efficiency gains, $\Xi^{RS}_c > 0$, due to consumption adjustments — because the equilibrium is constrained inefficient — that are larger in magnitude than the efficiency losses due to probability pricing, $\Xi^{RS}_\xi < 0$. The top right panel shows that increasing volatility (worsening information) of public information can be welfare reducing due to an aggregate/production-efficiency loss even when this generates risk-sharing gains, although this result hinges on our specific parameterization. The bottom panel illustrates that increasing the volatility of public information (worsening information) generates efficiency losses because states with extreme signals in which production choices are less accurate become more likely, $\Xi^{AE}_\xi < 0$, but has no impact on the net consumption adjustment conditional on signal realizations, $\Xi^{AE}_c > 0$, as production decisions are constrained efficient. The parameters in this figure are $\mu_0 = 0$, $\sigma_0 = 0.15$, $\gamma = 2$, $\kappa = 1$, and $n_2 = 1$.

ii) Efficiency gains due to the equilibrium consumption response are strictly positive: $\Xi_c^E > 0$.

Our first insight comes from the direct probability effect, Ξ_ξ^E, which captures the welfare impact of changing the likelihood of different signal realizations while holding the consumption allocation mapping fixed. Increasing the volatility parameter σ makes extreme public signal realizations more likely. In the endowment economy, extreme signal realizations correspond to more unequal ex-post consumption, so this channel — what we term signal compression— reduces welfare: $\Xi_\xi^E < 0$, where the negative sign means that increases in the volatility parameter σ reduce welfare or, conversely, that worse public information *reduces* welfare at the margin.

In contrast, the consumption adjustment effect, Ξ_c^E, is strictly positive: $\Xi_c^E > 0$. Because markets are incomplete with respect to signal realizations, the competitive equilibrium is constrained inefficient (Geanakoplos and Polemarchakis, 1986; Lorenzoni, 2008; Dávila and Korinek, 2018). Increasing σ changes the prices of Arrow–Debreu securities in a way that reallocates consumption toward those who are initially worse off given the signal, thereby improving risk-sharing. This is the force emphasized in Hirshleifer (1971)'s original result.

Under our functional form assumptions, combining these two effects — as illustrated in the top left panel of Figure 4 — we find that the overall effect of increasing σ is positive: $\Xi^E > 0$, so the positive consumption adjustment channel dominates the losses from signal dispersion. While this overall result reproduces Hirshleifer's qualitative finding, our decomposition shows that better public information also has a countervailing welfare effect — via changes in the probability distribution — that could, in other settings, overturn the net result.

In summary, probability pricing allows us to uncover some novel economics behind the "Hirshleifer effect". On one hand, the welfare losses associated with better public information are driven exclusively by consumption adjustments, which are in turn generated by the underlying incompleteness of markets. On the other hand, not all the effects of better public information in Hirshleifer's economy are negative. Indeed, the pure effect of changing probabilities suggests that *better information increases welfare* due to signal compression. Hence, probability pricing shows that improving public information has — as one would expect — an immediate positive social value even in an endowment economy. While the consumption adjustment force dominates in the case we have displayed, one could envision scenarios in which this is not the case.

4.1.3 Environment: Production Economy

It is worth considering an extension of the model in which public information also influences production decisions. In this case, observing a more precise public signal before investing can improve efficiency by tailoring investment decisions. The setup is the same as in the endowment economy, except for the following features. First, at date 2, type-i's consumption is given by

$$c_2^i(s,\xi) = n_2^i(s) + x_1^i(s|\xi) + \Pi^i(s|\xi),\ \forall (s,\xi),$$

where $\Pi_1^i(s|\xi)$ denotes the proceeds from operating a "backyard" technology:

$$\Pi^i(s|\xi) = e^s k_1^i(\xi) - \frac{\kappa}{2}\left(k_1^i(\xi)\right)^2,$$

where each individual chooses $k_1^i(\xi)$ at date 1 after observing the public signal ξ. The equilibrium definition is standard, with individuals choosing Arrow-Debreu portfolios $x_1^i(s|\xi)$ and making production decisions $k_1^i(\xi)$.

Market completeness from date 1 onwards implies that all individuals make identical production decisions, given by

$$k_1^i(\xi) = k_1(\xi) = \frac{1}{\kappa}\int \omega_2(s|\xi)\, e^s ds,\ \forall i,$$

where $\omega_2(s|\xi) = \frac{u'\left(c_2^i(s,\xi)\right)f(s|\xi)}{\int u'\left(c_2^i(s,\xi)\right)f(s|\xi)ds}$.

4.1.4 Value of Changes in Public Signal Precision: Production Economy

As in Application 2 — and as shown formally in the Appendix — the efficiency gains Ξ^E from a change in information can be decomposed as

$$\Xi^E = \Xi^{AE} + \Xi^{RS},$$

where Ξ^{AE} captures gains from changes in the value of aggregate consumption (aggregate-efficiency) and Ξ^{RS} captures gains due to reallocating consumption across individuals with different valuations (risk-sharing). In the endowment economy, aggregate consumption is fixed, so $\Xi^{AE} = 0$, and welfare effects arise solely from risk-sharing. In the production economy, by contrast, changes in information can also generate aggregate-efficiency effects whenever they alter the efficiency of investment decisions.

As shown in the top right panel of Figure 4, increasing the volatility of public information (worsening information) generates welfare losses due to aggregate/production efficiency ($\Xi^{AE} < 0$), but welfare gains due to risk-sharing ($\Xi^{RS} > 0$), as in the endowment economy. The aggregate-efficiency losses arise because production decisions are now carried out with less precise information.

Following our approach in Application 2 — and as shown formally in the Appendix — we further decompose the aggregate-efficiency loses into a component driven by consumption adjustments and a probability price, that is

$$\Xi^{AE} = \Xi_c^{AE} + \Xi_\xi^{AE},$$

where Ξ_c^{AE} captures the adjustment in aggregate consumption given signal realizations and Ξ_ξ^{AE} captures the direct effect of changes in conditional probabilities/likelihoods $f(\xi|s)$. As illustrated in the bottom panel of Figure 4, aggregate-efficiency losses from increased volatility do not operate through consumption adjustments. Formally,

$$\Xi_c^{AE} = 0. \tag{25}$$

This result is due to the fact that production decisions are constrained efficient, because at date 1, when individuals choose their investments, they have access to complete markets against date-2 risks. Hence, endogenous adjustments in production induced by changes in signal volatility do not directly change the value of aggregate net consumption, via an envelope theorem/optimality argument. In that sense, production efficiency in Equation (25) captures analogous forces to constrained efficiency in Application 2, in which $\Xi_c^E = 0$, and it contrasts with constrained inefficiency due to imperfect risk-sharing in the presence of incomplete markets, where we have $\Xi_c^{RS} \neq 0$. Therefore, the aggregate-efficiency losses due to the increased signal volatility instead arise because more extreme signals — where production choices are less accurate — become more likely, $\Xi_\xi^{AE} < 0$, a different manifestation of the signal dispersion phenomenon explained above.

In the parametrization displayed in Figure 4, aggregate-efficiency losses outweigh the overall risk-sharing from increased signal volatility, yielding $\Xi^E = \Xi^{RS} + \Xi^{AE} < 0$. This result is, however, parameter-dependent. More generally, this application has illustrated how probability pricing can be employed to clearly identify and potentially quantify the various sources of the welfare effects of public information in incomplete markets.

4.2 Application 4: Private Information (REE)

Our final application shows how probability pricing is useful to understand the private and social value of changes in the precision of private information in a canonical competitive model of financial trading with dispersed information. This application is the simplest setting that allows us to illustrate how probability pricing helps analyze noisy rational expectations equilibrium (REE), in which the price acts as a public signal that partially aggregates the private signals received by investors.[22]

4.2.1 Environment

We consider an economy with three dates $t = \{0, 1, 2\}$ and two types of agents: a continuum of utility-maximizing investors in unit measure, and a single noise trader who trades inelastically. Agents receive signals and trade at date 1 and consume at date 2. All agents, including the noise trader, have identical constant absolute risk aversion (CARA) expected utility preferences over their date-2 consumption, with flow utility given by

$$u(c) = -e^{-\gamma c},$$

where $\gamma > 0$ denotes the coefficient of absolute risk aversion.

There are two assets: a risky asset and a riskless asset. The risky asset pays a normally distributed payoff δ at date 2 given by

$$\delta \sim N\left(\mu_\delta, \tau_\delta^{-1}\right).$$

The risky asset is competitively traded at date 1 at a price q and is in fixed supply $\bar{a} = 0$. The riskless asset pays a gross interest rate normalized to one. Assuming that the aggregate endowment of consumption at date 1 is zero ensures that the riskless market also clears. For simplicity, we assume that all agents have the same initial endowment of the risky asset $a_0 = \bar{a}$. At date 1, each investor receives a private signal ξ about the asset payoff δ, where

$$\xi = \delta + u_\xi \quad \text{with} \quad u_\xi \sim N\left(0, \tau_\xi^{-1}\right),$$

and where the realizations of u_ξ are independent across investors.

[22] In ongoing work (Dávila, Parlatore, and Walther, 2025), we explore the substantive role that different assumptions about noise trading have on welfare.

The noise trader trades a random amount of the risky asset

$$n \sim N\left(\mu_n, \tau_n^{-1}\right),$$

and their consumption is given by $c_2^n = \delta\left(n + a_0\right) - qn$.

Equilibrium The definition of a rational expectations equilibrium in this model is standard. Investors choose their asset holdings for the risky and riskless assets to maximize their expected utility, subject to their information and taking prices as given, and goods and asset markets clear. We focus on the unique equilibrium in linear strategies in which the optimal risky asset demand is linear in the private signal and the price, and the price are linear functions of the date-1 aggregate state, which is given by the average signal of investors, $\overline{\delta} = \int \xi f\left(\xi|\, \delta\right)$, and the amount traded by the noise trader, n.

The welfare of investors at date-0 is given by

$$V_0 = \iiint V_1\left(\xi, \overline{\delta}, n; \tau_\xi\right) f\left(\xi, \overline{\delta}, n; \tau_\xi\right) d\xi d\overline{\delta} dn, \tag{26}$$

where V_1 is the expected utility at date 1 of an investor that receives a signal ξ when the aggregate state at date 1 is $\left(\overline{\delta}, n\right)$, given by

$$V_1\left(\xi, \overline{\delta}, n; \tau_\xi\right) = \int u\left(c_2\left(\delta, \xi, \overline{\delta}, n\right)\right) f\left(\delta|\, \xi, q\left(\overline{\delta}, n\right); \tau_\xi, \tau_{\hat{q}}\left(\tau_\xi\right)\right) d\delta, \tag{27}$$

where $\tau_{\hat{q}}$ denotes price informativeness and is given by the precision of the unbiased signal about the payoff δ contained in the price (Dávila and Parlatore, 2021, 2025). In Appendix E, we show formally that the implied definition of V_0 is equivalent to ex-ante expected utility.[23]

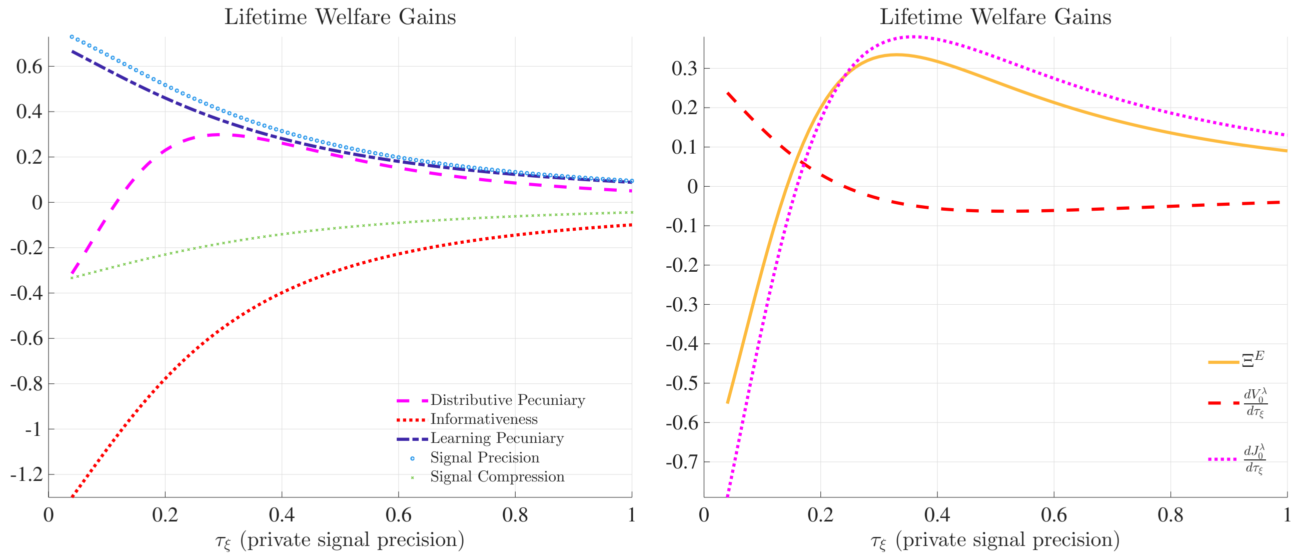
For the noise trader, the welfare at date-0 is given by

$$J_0 = \iint u\left(c_2\left(\overline{\delta}, q^\star\left(\overline{\delta}, n\right), n\right)\right) f\left(\overline{\delta}\right) f\left(n\right) d\overline{\delta} dn,$$

because the noise n and the price $q^\star$ jointly perfectly reveal $\overline{\delta} = \delta$, which implies

$$\int u\left(c_2^n\left(\delta, n, q^\star\right)\right) f\left(\delta | q^\star, n; \tau_{\hat{q}}\left(\tau_\xi\right)\right) d\delta = u\left(c_2\left(\overline{\delta}, q^\star, n\right)\right).$$

[23] This is not immediately obvious because the density under the integral in Equation (27) conditions on the price $q\left(\overline{\delta}, n\right)$ rather than the full set of states $\left(\xi, \overline{\delta}, n\right)$ over which we integrate in Equation (26). However, the fact that the price is a deterministic function of $\left(\overline{\delta}, n\right)$ implies that V_0 as defined in Equation (26) equals ex ante expected utility, as shown formally in the Appendix.



(a) Total Welfare Decomposition

(b) Probability Pricing

Figure 5: Varying Private Signal Precision (Application 4)

Note: This figure shows the different welfare decompositions in Application 4 as a function of the precision of the private signals. Benchmark parameters are $\gamma = 1.2$, $a_0 = 0$, $\mu_\delta = 10$, $\mu_n = 0$, $\tau_\delta = 1$, and $\tau_n = 9$.

4.2.2 Value of Changes in Private Signal Precision

We are interested in the effect of a change in the precision τ_ξ of the private signal received by the investors on welfare. As in the previous section, we measure an agent's welfare gains induced by a marginal increase in the precision of the private signals τ_ξ expressed in units of date-2 uncontingent consumption. Formally, this value for investors is given by $\frac{dV_0^\lambda}{d\tau_\xi} \equiv \frac{dV_0}{d\tau_\xi}\frac{1}{\lambda}$, where

$$\frac{dV_0^\lambda}{d\tau_\xi} = \iiint \left(\underbrace{\frac{1}{\lambda}\frac{dV_1\left(\xi, \bar{\delta}, n; \tau_\xi\right)}{d\tau_\xi}}_{\text{Date-1 Continuation}} + \underbrace{\frac{1}{\lambda}\frac{dV_1\left(\xi, \bar{\delta}, n; \tau_\xi\right)}{d\xi}\frac{\frac{d\left(1-F\left(\xi|\bar{\delta};\tau_\xi\right)\right)}{d\tau_\xi}}{f\left(\xi|\bar{\delta};\tau_\xi\right)}}_{\text{Date-1 Probability/Signal Compression}} \right) f\left(\xi|\bar{\delta};\tau_\xi\right) d\xi f\left(\bar{\delta}\right) d\bar{\delta} f\left(n\right) dn,$$

and $\lambda = \iiint \left[\int u'\left(c_2\left(\delta, \xi, \bar{\delta}, n\right)\right) f\left(\delta \middle| \xi, q\left(\bar{\delta}, n\right); \tau_\xi, \tau_{\hat{q}}\left(\tau_\xi\right)\right) d\delta\right] f\left(\xi, \bar{\delta}, n; \tau_\xi\right) d\xi d\bar{\delta} dn$. The first term in the expression above represents the effects on date-1 welfare, hence the label "date-1 continuation", while the second term captures the probability pricing effect coming from the change in the conditional distribution of the private signal being compressed as τ_ξ increases.

To understand the effect on the date-1 continuation, it is helpful to isolate the informational content captured by $\hat{q}$. To do so, we can define the date-1 expected utility of investors as follows

$$V_1\left(\xi,\overline{\delta},n;\tau_\varepsilon\right)=\tilde{V}_1\left(\xi,q^\star\left(\overline{\delta},n;\tau_\xi,\tau_{\hat{q}}\left(\tau_\xi\right)\right),\hat{q}^\star\left(\overline{\delta},n;\tau_\xi\right);\tau_\xi,\tau_{\hat{q}}\left(\tau_\xi\right)\right),$$

where

$$\tilde{V}_1\left(\xi,q,\hat{q};\tau_\xi,\tau_{\hat{q}}\right)=\int u\left(\tilde{c}_2\left(\delta,a_1^\star\left(\xi,q;\tau_\xi,\tau_{\hat{q}}\right),q\right)\right)f\left(\delta|\,\xi,\hat{q};\tau_\xi,\tau_{\hat{q}}\right)d\delta$$

and $\hat{q}^\star\left(\overline{\delta},n;\tau_\xi\right)$ is the unbiased signal about the payoff δ contained in the price in equilibrium. For notational convenience, we denote the arguments of the equilibrium mappings by *. Then, the effect on the date-one continuation can be further decomposed into the pecuniary effects and signal distribution effects, as follows:

$$\frac{dV_1}{d\tau_\xi}=\underbrace{\overbrace{\frac{\partial\tilde{V}_1}{\partial q}\frac{dq^\star}{d\tau_\xi}}^{\text{distributive pecuniary}}+\overbrace{\frac{\partial\tilde{V}_1}{\partial\hat{q}}\frac{d\hat{q}^\star}{d\tau_\xi}}^{\text{learning pecuniary}}}_{\text{pecuniary effects}}+\underbrace{\overbrace{\frac{\partial\tilde{V}_1}{\partial\tau_\xi}}^{\text{private signal}}+\overbrace{\frac{\partial\tilde{V}_1}{\partial\tau_{\hat{q}}}\frac{d\tau_{\hat{q}}}{d\tau_\xi}}^{\text{informativeness}}}_{\text{signal distributions effects}},$$

where the partial derivatives keep all other arguments in the corresponding function fixed. Since the price aggregates all private signals received by investors, the pecuniary effects contain the usual distributive pecuniary effects and the learning pecuniary effects that come from the mapping between the aggregate states and the information contained in the price changing with τ_ξ. The effect on the distribution of signals can also be decomposed into the effect of the change in the precision of an investor's private signal, $\frac{\partial\tilde{V}_1}{\partial\tau_\xi}$, and the effect of the change in the rest of the investors' precisions which is captured by the change in price informativeness, $\frac{\partial\tilde{V}_1}{\partial\tau_{\hat{q}}}\frac{d\tau_{\hat{q}}}{d\tau_\xi}$.

Therefore, the willingness-to-pay for a marginal change in τ_ξ for investors is

$$\frac{dV_0^\lambda}{d\tau_\xi}=\iiint\overbrace{\frac{\partial\tilde{V}_1^\lambda}{\partial q}\frac{dq^\star}{d\tau_\xi}}^{\text{distributive pecuniary}}f\left(\xi|\overline{\delta};\tau_\xi\right)d\xi f\left(\overline{\delta}\right)d\overline{\delta}f\left(n\right)dn+$$
$$\iiint\left(\overbrace{\frac{\partial\tilde{V}_1^\lambda}{\partial\hat{q}}\frac{d\hat{q}^\star}{d\tau_\xi}}^{\substack{\text{learning}\\\text{pecuniary}}}+\overbrace{\frac{\partial\tilde{V}_1^\lambda}{\partial\tau_\xi}}^{\substack{\text{private}\\\text{signal}}}+\overbrace{\frac{\partial\tilde{V}_1^\lambda}{\partial\tau_{\hat{q}}}\frac{d\tau_{\hat{q}}}{d\tau_\xi}}^{\text{informativeness}}+\overbrace{\frac{dV_1^\lambda}{d\xi}\frac{\frac{d\left(1-F\left(\xi|\overline{\delta};\tau_\xi\right)\right)}{d\tau_\xi}}{f\left(\xi|\overline{\delta};\tau_\xi\right)}}^{\text{signal compression}}\right)f\left(\xi|\overline{\delta};\tau_\xi\right)d\xi f\left(\overline{\delta}\right)d\overline{\delta}f\left(n\right)dn,$$

where we define $\frac{\partial \tilde{V}_1^\lambda}{\partial x} \equiv \frac{1}{\lambda}\frac{\partial \tilde{V}_1}{\partial x}$ for any x. The distributive pecuniary effects are the only channel through which changes in the private signal precision affect consumption. The rest of the effects work by affecting the date-1 probability of the aggregate state at date-2 payoff δ (learning, pecuniary, private signal, and informativeness), and the distribution of the idiosyncratic state at date 1 (signal compression).

Since the noise and the price jointly perfectly reveal $\overline{\delta} = \delta$, there are only pecuniary effects for the noise trader. The noise trader always learns the dividend perfectly from the price (due to the continuum of investors and the LLN), regardless of the precision of the private signals. Therefore, the date-0 welfare effect for the noise trader expressed in units of their date-2 uncontingent consumption is

$$\frac{dJ_0^{\lambda^n}}{d\tau_\xi} \equiv \frac{1}{\lambda^n}\frac{dJ_0}{d\tau_\xi} = \frac{1}{\lambda^n}\iint \underbrace{u'\left(c_2^n\left(\overline{\delta}, n, q^\star\right)\right)\frac{\partial c_2}{\partial q}\frac{dq^\star}{d\tau_\xi}}_{\text{distributive pecuniary}} f\left(\overline{\delta}\right) f\left(n\right) d\overline{\delta} dn,$$

where $\lambda^n \equiv \iint u'\left(c_2^n\left(\overline{\delta}, n, q^\star\right)\right) f\left(\overline{\delta}\right) f\left(n\right) d\overline{\delta} dn$.

While our decomposition of $\frac{dV_0^\lambda}{d\tau_\xi}$ appears complex at first glance, its key advantage is that each term has a natural economic interpretation, and that all but one term can be precisely signed. We demonstrate this formally in Proposition 4:

Proposition 4. (Value of Private Information) *In response to an increase in the precision of private information τ_ξ:*

i) Efficiency gains due to learning pecuniary effects are positive;

ii) Efficiency gains due to signal compression effects are negative;

iii) Efficiency gains due to private signal precision effects are positive;

iv) Efficiency gains due to informativeness effects are negative;

v) Efficiency gains due to distributive pecuniary effects are ambiguous;

vi) Overall efficiency gains are ambiguous.

Proposition 4 shows that, while the overall effect of an increase in precision of private information is ambiguous, the probability pricing effects are not. On the one hand, the learning pecuniary and private signal precision effects are positive. To see this, note that buyers benefit from decreases in the price and sellers benefit from increases in the price.

Moreover, investors are buyers when their private signal is above the (public) signal contained in the price and sellers otherwise. The weight the public signal $\hat{q}$ puts on noise is lower when private information is more precise. Hence, ceteris paribus, when $n < 0$, $\hat{q}$ increases and there are more sellers as τ_ξ increases, while when $n > 0$, $\hat{q}$ decreases and there are more buyers as τ_ξ goes up. This implies that the learning pecuniary channel is positive. The private signal precision effect is intuitive: as the precision of private information goes up, investors' posterior uncertainty is reduced and they can trade more accurately, increasing investors' welfare.

On the other hand, the signal compression and the informativeness effects are negative. Intuitively, the farther away from the price the investor's beliefs are, the more extreme their valuations for the asset and the more the investor trades (either buying or selling), which leads to a higher expected utility. The signal compression effect shifts probability away from these extreme states and, hence, has a negative effect on welfare. While an increase in the precision of private information increases the precision of the signal contained in the price by reducing the overall uncertainty faced by investors, it also makes the posterior belief distribution more concentrated around the public signal contained in the price, reducing the gains from trade. This last effect dominates, leading to the informativeness channel being negative. Even though the results are intuitive, we can only show this because of our composition.

As is always the case, distributive pecuniary effects can be positive or negative depending on whether sellers or buyers have a higher valuation for the asset. Finally, aggregating the welfare effects for investors and the noise trader, we do a final decomposition, thinking about the effects of changes in one's private signal and the effects of changes in everyone else's private signals. More specifically, the "private" effects are given by the private signal and signal compression effects, while the "social" effects—those happening through pecuniary and information externalities—are captured by the distributive pecuniary, learning pecuniary, and informativeness effects. As the corollary below shows, the private effects of an increase in the precision of private information are always positive, while the social effects are ambiguous.

Corollary. (Private Value of Private Information) *In response to an increase in the precision of private information τ_ξ, efficiency gains from investors' private effects are positive, while the social effects are ambiguous.*

Figure 5, shows the different welfare components discussed above as a function of the precision of the private signals. Interestingly, as can be seen from the left panel of Figure

5, the change in overall welfare is non-monotonic and has a minimum where the solid line crosses the horizontal axis, which implies welfare is higher either at the no information or full information limits. Consistent with our results above, we see positive signal precision and learning pecuniary effects, negative signal compression and informativeness effects, and positive and negative distributive pecuniary effects. The right panel in Figure 5 shows that changes in welfare for investors and the noise trader can often go in different directions, with investors benefiting from increases in precision when precision is low and the noise trader benefiting from them when precision is high.

5 Conclusion

This paper extends traditional cash flow pricing to analyze the willingness-to-pay for changes in probabilities, that is, probability pricing. We show that an agent's willingness-to-pay for a marginal change in probabilities is equivalent to pricing an asset with hypothetical cash flows derived from changes in the survival function. This result establishes a formal equivalence between changes in probabilities (probability pricing) and changes in consumption (cash flow pricing), which is useful to construct hedging strategies, decompose probability prices into expected-payoff and risk-compensation components, and understand the sources of welfare gains in equilibrium models.

Our applications, which illustrate the multiple uses of probability pricing, study the valuation of changes in the distribution of aggregate consumption, the efficiency effects of changes in performance noise in principal-agent problems, and the private and social values of information in markets with public or dispersed information. Beyond theoretical insights, our results show how to calculate and interpret the impact of changes in probabilities, paving the way for further quantitative applications in asset pricing, information economics, and welfare analysis.

Appendix

Proof of Proposition 1. (Probability Pricing)

Proof. Let $U(s) = \beta u(c_1(s))$, so

$$\frac{dU(s)}{ds} = \beta u'(c_1(s)) \frac{dc_1(s)}{ds}.$$

Using integration by parts, Equation (3) can be expressed as

$$\begin{aligned}
p_\theta &= \frac{1}{u'(c_0)} \int_{\underline{s}}^{\bar{s}} U(s) \frac{df(s;\theta)}{d\theta} ds = \frac{1}{u'(c_0)} \left[U(s) \frac{dF(s;\theta)}{d\theta}\Big|_{\underline{s}}^{\bar{s}} - \int_{\underline{s}}^{\bar{s}} \frac{dU(s)}{ds} \frac{dF(s;\theta)}{d\theta} ds \right] \\
&= \frac{1}{u'(c_0)} \left[U(\bar{s}) \underbrace{\frac{dF(\bar{s};\theta)}{d\theta}}_{=0} - U(\underline{s}) \underbrace{\frac{dF(\underline{s};\theta)}{d\theta}}_{=0} - \int_{\underline{s}}^{\bar{s}} \frac{dU(s)}{ds} \frac{dF(s;\theta)}{d\theta} ds \right] \\
&= \frac{1}{u'(c_0)} \left[\int_{\underline{s}}^{\bar{s}} \frac{dU(s)}{ds} \frac{d(1-F(s;\theta))}{d\theta} ds \right] = \int_{\underline{s}}^{\bar{s}} \frac{\beta u'(c_1(s))}{u'(c_0)} \frac{dc_1(s)}{ds} \frac{d(1-F(s;\theta))}{d\theta} ds,
\end{aligned}$$

which corresponds to Equation (4) in the text. We use the fact that for all θ, we have $F(\bar{s};\theta) = 1$ and $F(\underline{s};\theta) = 0$. □

Additional Results. Note that defining expected consumption as

$$\mathbb{E}[c_1(s)] = \int_{\underline{s}}^{\bar{s}} c_1(s) f(s;\theta) ds,$$

it is the case that

$$\frac{d\mathbb{E}[c_1(s)]}{d\theta} = \int_{\underline{s}}^{\bar{s}} c_1(s) \frac{df(s;\theta)}{d\theta} ds = \int_{\underline{s}}^{\bar{s}} \frac{dc_1(s)}{ds} \frac{d(1-F(s;\theta))}{d\theta} ds = \int_{\underline{s}}^{\bar{s}} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds} f(s;\theta) ds.$$

Note also that

$$\begin{aligned}
p_\theta &= \int_{\underline{s}}^{\bar{s}} m(s) \frac{dc_1(s)}{ds} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} f(s;\theta) ds \\
&= \int_{\underline{s}}^{\bar{s}} m(s) f(s;\theta) ds \int_{\underline{s}}^{\bar{s}} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \frac{dc_1(s)}{ds} f(s;\theta) ds + \mathbb{C}ov \left[m(s), \frac{dc_1(s)}{ds} \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f(s;\theta)} \right],
\end{aligned}$$

which corresponds to Equation (6) in the text.

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Online Appendix

A Proofs and Derivations: Section 2

Mean/Variance Perturbations. We assume that $s = \mu + \sigma n$. Therefore, the cdf $F(s)$ is given by

$$F(s) = H\left(\frac{s-\mu}{\sigma}\right),$$

where $H(\cdot)$ denotes the cdf over n. Therefore, the pdf $f(s)$ is given by

$$f(s) = \frac{d}{ds} H\left(\frac{s-\mu}{\sigma}\right) = \frac{1}{\sigma} h\left(\frac{s-\mu}{\sigma}\right).$$

A marginal increase in μ implies that

$$\frac{dF(s)}{d\mu} = \frac{d}{d\mu} H\left(\frac{s-\mu}{\sigma}\right) = -\frac{1}{\sigma} h\left(\frac{s-\mu}{\sigma}\right), \quad \text{so} \quad \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} = 1.$$

A marginal increase in σ implies that

$$\frac{dF(s)}{d\sigma} = \frac{d}{d\sigma} H\left(\frac{s-\mu}{\sigma}\right) = -\frac{s-\mu}{\sigma}\frac{1}{\sigma} h\left(\frac{s-\mu}{\sigma}\right), \quad \text{so} \quad \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} = \frac{s-\mu}{\sigma}.$$

Mixture Distributions. In this case, note that $f(s;h) = (1-h)\overline{f}(s) + h\underline{f}(s)$ and $F(s;h) = (1-h)\overline{F}(s) + h\underline{F}(s)$, so we can express the survival change as

$$\frac{d(1-F(s;h))}{dh} = \overline{F}(s) - \underline{F}(s),$$

implying Equation (13) in the text. Note that

$$p_h = \int_{\underline{s}}^{\bar{s}} \frac{\beta u'(c_1(s))}{u'(c_0)} \frac{dc_1(s)}{ds} \left(\overline{F}(s) - \underline{F}(s)\right) ds,$$

which is invariant to the level of h, as stated in the text.

Stochastic Dominance. First, we consider a perturbation that satisfies first-order stochastic dominance. In the case of a lottery, $c_1(s) = s$, and $\frac{dc_1(s)}{ds} = 1$, so Equation

(4) simplifies as follows

$$p_\theta = \int_{\underline{s}}^{\bar{s}} \frac{\beta u'(c_1(s))}{u'(c_0)} \frac{d(1-F(s;\theta))}{d\theta} ds.$$

It is immediate that any first-order stochastic dominance shift, with $\frac{d(1-F(s;\theta))}{d\theta} \geq 0$, leads to $p_\theta \geq 0$, since utility is increasing in consumption with $u'(c_1(s)) > 0$.

Second, we consider a perturbation that satisfies second-order stochastic dominance. Define the cumulative perturbation $H(s;\theta) = \int_{\underline{s}}^{s} \frac{dF(t;\theta)}{d\theta} dt$, where we have assumed that $H(s;\theta) \geq 0$ for all s. Notice that, integrating by parts, the change in the expected value of s as a result of the perturbation can be written as

$$\begin{aligned}\frac{d\mathbb{E}[s]}{d\theta} &= \int_{\underline{s}}^{\bar{s}} s \frac{df(s;\theta)}{d\theta} ds = \bar{s}\frac{dF(\bar{s};\theta)}{d\theta} - \underline{s}\frac{dF(\underline{s};\theta)}{d\theta} - \int_{\underline{s}}^{\bar{s}} \frac{dF(s;\theta)}{d\theta} ds \\ &= -\int_{\underline{s}}^{\bar{s}} \frac{dF(s;\theta)}{d\theta} ds = -H(\bar{s};\theta),\end{aligned}$$

so that, by assumption, we have $H(\bar{s};\theta) = 0$. Now integrating by parts again, we have

$$\begin{aligned}p_\theta &= -\int_{\underline{s}}^{\bar{s}} \frac{\beta u'(c_1(s))}{u'(c_0)} \frac{d(1-F(s;\theta))}{d\theta} ds \\ &= -\frac{1}{u'(c_0)} \left\{ u'(\bar{s}) \underbrace{H(\bar{s};\theta)}_{=0} - u'(\underline{s};\theta) \underbrace{H(\underline{s};\theta)}_{=0} - \int u''(s) H(s;\theta) ds \right\} \\ &= \frac{1}{u'(c_0)} \int u''(s) H(s;\theta) ds.\end{aligned}$$

Then, since utility is strictly concave with $u''(s) < 0$, we obtain $p_\theta \leq 0$.

B Proofs and Derivations: Section 3

B.1 Application 1: Consumption-Based Asset Pricing

In this case, the probability price for a change in the disaster probability h can be written as

$$p_h = \int_{\underline{s}}^{\overline{s}} \beta \frac{u'(c_1(s))}{u'(c_0)} \frac{dc_1(s)}{ds} \frac{d(1-F(s))}{d\theta} ds = c_0 \int_{\underline{s}}^{\overline{s}} \beta e^{(1-\gamma)s} f(s) \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} ds,$$

where we used the fact that $\frac{u'(c_1(s))}{u'(c_0)} = \left(\frac{c_1(s)}{c_0}\right)^{-\gamma} = e^{-\gamma(c_1(s)/c_0)} = e^{-\gamma s}$ and $\frac{dc_1(s)}{ds} = c_1(s)$. Since

$$\frac{d(1-F(s))}{d\theta} = \overline{F}(s) - \underline{F}(s),$$

it is straightforward to conclude that $\frac{p_h}{c_0}$ is invariant to the level of h.

A similar logic yields an equivalent formulation for

$$\frac{p_{\overline{\sigma}}}{c_0} = \int_{\underline{s}}^{\overline{s}} \beta e^{(1-\gamma)s} f(s) \frac{\frac{d(1-F(s))}{d\theta}}{f(s)} ds,$$

where in this case

$$\frac{\frac{d(1-F(s))}{d\theta}}{f(s)} = \frac{s-\mu}{\sigma}.$$

A direct application of Equation (6) generates the decomposition shown in Figure 2.

B.2 Application 2: Principal-Agent Problem

The principal, with full bargaining power, solves the following problem

$$\max_{\{e,t,\alpha\}} \int c^B(s) f(s) ds,$$

subject to participation and incentive constraints for the agent, given by

$$\int u\left(c^A(s)\right) f(s) ds = \overline{V}$$
$$e \in \arg\max_{\hat{e}} \int u\left(c^A(s)\right) f(s) ds.$$

It is well-understood that the restriction to linear contracts is not innocuous, as explained in Bolton and Dewatripont (2005), although these considerations are orthogonal to our results in this paper.

Note that the objective function can be expressed as

$$V^B = \int c^B(s) f(s)\, ds = (1-\alpha) e - t,$$

and the utility of the agent as

$$V^A = \int u\left(c^A(s)\right) f(s)\, ds = -\exp\left[-\eta\left(t + \alpha e - \frac{1}{2}\kappa e^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau}\right)\right].$$

Therefore, the principal-agent problem can be reformulated as

$$\max_{\{e,t,\alpha\}} e - \frac{1}{2}\kappa e^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau}$$

subject to

$$e \in \arg\max_{\hat{e}} \left\{-\exp\left[-\eta\left(t + \alpha \hat{e} - \frac{1}{2}\kappa \hat{e}^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau}\right)\right]\right\}.$$

Optimal Contract. If effort is observable, in the absence of the incentive constraint, the principal chooses effort e to solve

$$\max_{\hat{e}} \hat{e} - \frac{1}{2}\kappa \hat{e}^2,$$

implying a level of effort $e = \frac{1}{\kappa}$. This solution implies a performance sensitivity $\alpha = 1$ and a non-contingent transfer is given by $t = -\frac{1}{\eta}\ln\left(-\overline{V}\right) + \frac{1}{2\kappa}$.

If effort is not observable, the agent's optimality condition determines the optimal effort

$$e = \frac{\alpha}{\kappa}.$$

So the principal's problem becomes

$$\max_{\{t,\alpha\}} \frac{\alpha}{\kappa} - \frac{1}{2}\kappa\left(\frac{\alpha}{\kappa}\right)^2 - \frac{\eta}{2}\alpha^2 \frac{1}{\tau},$$

which yields the following solution for the optimal performance sensitivity α:

$$\alpha = \frac{1}{1 + \frac{\eta\kappa}{\tau}} = \frac{\tau}{\tau + \eta\kappa}.$$

The non-contingent transfer t is given by

$$t = -\frac{1}{\eta}\ln\left(-\overline{V}\right) - \frac{\alpha^2}{2\kappa}\left(1 - \frac{\eta\kappa}{\tau}\right).$$

Note that we can write

$$c^B(s) = (1-\alpha)(e+s) - t$$
$$c^A(s) = \alpha(e+s) - \psi(e) + t.$$

Hence, equilibrium changes in consumption for both individuals are given by

$$\frac{\partial c^B(s)}{\partial \tau} = (1-\alpha)\frac{de}{d\tau} - (e+s)\frac{d\alpha}{d\tau} - \frac{dt}{d\tau} \qquad \text{and} \quad \frac{\partial c^B(s)}{\partial s} = 1-\alpha$$
$$\frac{\partial c^A(s)}{\partial \tau} = (e+s)\frac{d\alpha}{d\tau} + (\alpha - \psi'(e))\frac{de}{d\tau} + \frac{dt}{d\tau} \quad \text{and} \quad \frac{\partial c^A(s)}{\partial s} = \alpha,$$

where the contract changes according to

$$\frac{de}{d\tau} = \frac{1}{\kappa}\frac{d\alpha}{d\tau}, \qquad \frac{d\alpha}{d\tau} = \frac{\eta\kappa}{(\tau+\eta\kappa)^2}, \quad \text{and} \quad \frac{dt}{d\tau} = -\frac{\alpha}{\kappa}\left(1 - \frac{\eta\kappa}{\tau}\right)\frac{d\alpha}{d\tau} - \frac{\eta}{2}\left(\frac{\alpha}{\tau}\right)^2.$$

In this case, $f(s) = \sqrt{\tau}\phi(\sqrt{\tau}s)$ and $F(s) = \Phi(\sqrt{\tau}s)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of the standard normal. Consistent with our result in Section 2.4.1, $\frac{\frac{d(1-F(s))}{d\tau}}{f(s)} = -\frac{1}{2\tau}s$.

Welfare Gains. Normalized individual welfare gains are given by

$$\frac{dV^{i|\lambda}}{d\tau} = \frac{\frac{dV^i}{d\tau}}{\lambda^i} = \int \omega^i(s)\left(\frac{\partial c^i(s)}{\partial \tau} + \frac{\partial c^i(s)}{\partial s}\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}\right)ds,$$

where $\omega^i(s)$ is defined in Equation (15) in the text. We can respectively express them for the principal and the agent as follows:

$$\frac{dV^{B|\lambda}}{d\tau} = \int \omega^B(s)\left(-s\frac{d\alpha}{d\tau} + (1-\alpha)\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}\right)ds + (1-\alpha)\frac{de}{d\tau} - e\frac{d\alpha}{d\tau} - \frac{dt}{d\tau}$$
$$\frac{dV^{A|\lambda}}{d\tau} = \int \omega^A(s)\left(s\frac{d\alpha}{d\tau} + \alpha\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}\right)ds + (\alpha - \psi'(e))\frac{de}{d\tau} + e\frac{d\alpha}{d\tau} + \frac{dt}{d\tau}.$$

Efficiency, defined by $\Xi^{E}=\sum_i \frac{dV^{i|\lambda}}{d\tau}=\frac{dV^{B|\lambda}}{d\tau}+\frac{dV^{A|\lambda}}{d\tau}$, can be thus decomposed into aggregate-efficiency and risk-sharing as follows: $\Xi^{E}=\Xi^{AE}+\Xi^{RS}$, where

$$\begin{aligned}\Xi^{AE} &= \int \omega(s)\sum_i \frac{\partial c^i(s)}{\partial \tau}ds+\int \omega(s)\sum_i \frac{\partial c^i(s)}{\partial s}\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}ds\\ \Xi^{RS} &= \int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)},\frac{\partial c^i(s)}{\partial \tau}\right]ds+\int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)},\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}\right]\frac{\partial c^i(s)}{\partial s}ds,\end{aligned}$$

where $\omega(s)=\frac{1}{I}\sum_i \omega^i(s)$, which implies that $\int \omega(s)\,ds=1$, and where $\mathbb{C}ov_i^{\Sigma}[\cdot,\cdot]=I\cdot\mathbb{C}ov_i[\cdot,\cdot]$. Note that

$$\sum_i \frac{\partial c^i(s)}{\partial s}=1 \quad\text{and}\quad \sum_i \frac{\partial c^i(s)}{\partial \tau}=(1-\psi'(e))\frac{de}{d\tau},\ \forall s.$$

We can alternatively decompose efficiency and its components as follows: $\Xi^{E}=\Xi^{E}_c+\Xi^{E}_s$, where

$$\begin{aligned}\Xi^{E}_c &= \underbrace{\int \omega(s)\sum_i \frac{\partial c^i(s)}{\partial \tau}ds}_{=\Xi^{AE}_c}+\underbrace{\int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)},\frac{\partial c^i(s)}{\partial \tau}\right]ds}_{=\Xi^{RS}_c}\\ \Xi^{E}_s &= \underbrace{\int \omega(s)\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}ds}_{=\Xi^{AE}_s}+\underbrace{\int \omega(s)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)},\frac{\partial c^i(s)}{\partial s}\right]\frac{\frac{d(1-F(s))}{d\tau}}{f(s)}ds}_{=\Xi^{RS}_c}.\end{aligned}$$

Proof of Proposition 2. (Principal-Agent Problem)

Proof. i) First, we show that the efficiency gains due to consumption adjustments are zero, that is,

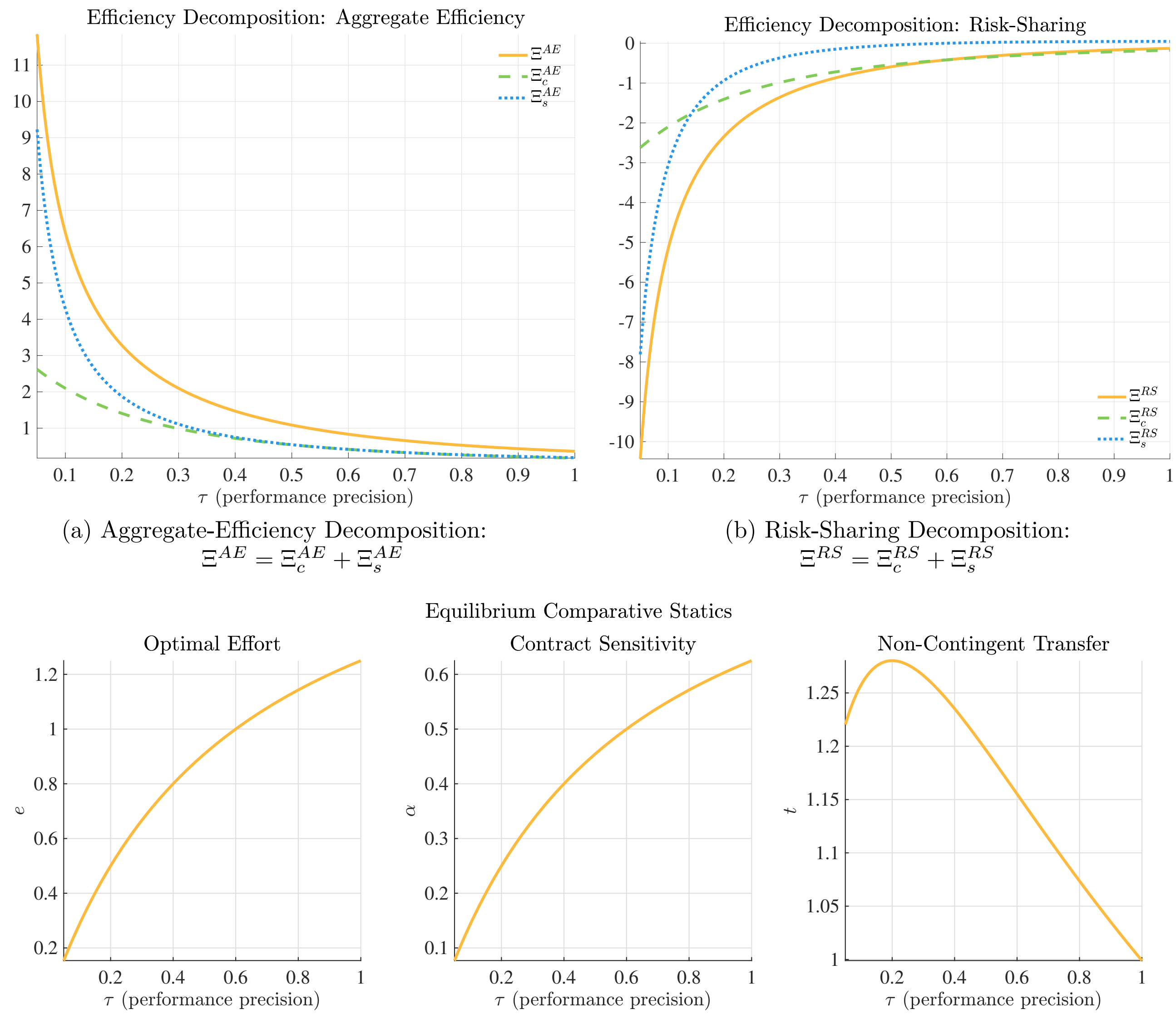
$$\Xi^{E}_c=\Xi^{AE}_c+\Xi^{RS}_c=0.$$

This is a result that follows from the optimality of the contract. Note that

$$\Xi^{AE}_c=(1-\psi'(e))\frac{de}{d\tau}=(1-\alpha)\frac{1}{\kappa}\frac{d\alpha}{d\tau}>0,$$

so it must be that

$$\Xi^{RS}_c=-\Xi^{AE}_c=-(1-\psi'(e))\frac{de}{d\tau}<0.$$



(a) Aggregate-Efficiency Decomposition: $\Xi^{AE} = \Xi_c^{AE} + \Xi_s^{AE}$

(b) Risk-Sharing Decomposition: $\Xi^{RS} = \Xi_c^{RS} + \Xi_s^{RS}$

(c) Equilibrium Comparative Statics

Figure OA.1: Varying Performance Precision: Ξ^{AE} vs. Ξ^{RS} (Application 2)

Note: This figure shows the efficiency/willingness-to-pay and equilibrium comparative statics induced by changing the performance precision τ.

Note also that

$$\Xi_c^{RS}=\Xi_c^{E}-\Xi_c^{AE}=\sum_i\int\omega^i\left(s\right)\frac{\partial c^i\left(s\right)}{\partial\tau}ds-\int\omega\left(s\right)\sum_i\frac{\partial c^i\left(s\right)}{\partial\tau}ds.$$

Hence, it is sufficient to show that $\sum_i\int\omega^i\left(s\right)\frac{\partial c^i(s)}{\partial\tau}ds=0$. We can write the principal's problem as

$$\mathcal{L}^B=\max_{\{\alpha,t\}}\left\{\int c^B\left(s,\alpha,e^\star\left(\tau\right),t\right)f\left(s\right)ds+\phi\left(\int u\left(c^A\left(s,\alpha,e^\star\left(\tau\right),t\right)\right)f\left(s\right)ds-\overline{V}\right)\right\},$$

once the agent's optimal effort choice is embedded in $e^\star\left(\tau\right)$, highlighting its dependence on τ. Note that Ξ_c^E corresponds to the change in $\mathcal{L}^B$ that is induced by changes in α, t, and $e^\star$ in response to a change in τ, abstracting from the direct impact of τ on $f(s)$.

First, note that $\frac{dc^B(s)}{dt}=-\frac{dc^A(s)}{dt}=-1$, which implies that

$$\frac{\partial\mathcal{L}^B}{\partial t}=-1+\phi\int\frac{\partial u^A\left(c^A\left(\cdot\right)\right)}{\partial c^A}f\left(s\right)ds=0\Rightarrow\phi=\frac{1}{\int\frac{\partial u^A(c^A(\cdot))}{\partial c^A}f\left(s\right)ds}.$$

Therefore

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\alpha}&=\int\frac{dc^B\left(s\right)}{d\alpha}f\left(s\right)ds+\phi\int\frac{\partial u^A\left(c^A\left(s\right)\right)}{\partial c^A\left(s\right)}\frac{dc^A\left(s\right)}{d\alpha}f\left(s\right)ds\\&=\int\frac{dc^B\left(s\right)}{d\alpha}f\left(s\right)ds+\int\frac{\frac{\partial u^A\left(c^A(s)\right)}{\partial c^A(s)}f\left(s\right)}{\int\frac{\partial u^A(c^A(\cdot))}{\partial c^A}f\left(s\right)ds}\frac{dc^A\left(s\right)}{d\alpha}ds\\&=\sum_i\int\omega^i\left(s\right)\frac{\partial c^i\left(s\right)}{\partial\tau}ds=0,\end{aligned}$$

where $\omega^A\left(s\right)=\frac{\frac{\partial u^A\left(c^A(s)\right)}{\partial c^A(s)}f(s)}{\int\frac{\partial u^A(c^A(\cdot))}{\partial c^A}f(s)ds}$ and where $\omega^B\left(s\right)=f\left(s\right)$, which is sufficient to establish the result.

ii) Second, we show that the efficiency gains attributed to directly changing probabilities induced by an increase in the precision of output uncertainty τ are strictly positive, that is,

$$\Xi_s^E=\Xi_s^{AE}+\Xi_s^{RS}>0.$$

Note that

$$\Xi_s^{AE} = \int \omega(s) \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} ds = -\frac{1}{2\tau} \int \omega(s)\, s ds = -\frac{1}{2\tau} \mathbb{C}ov\left[s, \omega(s)\right] > 0,$$

where we use the fact that $\frac{\frac{d(1-F(s))}{d\tau}}{f(s)} = -\frac{1}{2\tau} s$ and also that $\mathbb{C}ov\left[s, \omega(s)\right]$, that is, low s realizations are associated with higher aggregate valuation $\omega(s) < 0$. It also follows from the optimality of the contract that

$$\Xi_s^{AE} = \frac{1}{4\tau} \left(1 - \psi'(e)\right) \frac{de}{d\alpha} > 0,$$

as each term is strictly positive. Second, note that the risk-sharing term can take different signs:

$$\begin{aligned}
\Xi_s^{RS} &= \int \omega(s)\, \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega^i(s)}{\omega(s)}, \frac{dc^i(s)}{ds} \frac{\frac{d(1-F(s))}{d\tau}}{f(s)}\right] ds \\
&= (1-\alpha) \int \omega^1(s) \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} ds + \alpha \int \omega^2(s) \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} ds - \int \omega(s) \frac{\frac{d(1-F(s))}{d\tau}}{f(s)} ds \\
&= \frac{1}{2\tau} \left(\alpha - \frac{1}{2}\right) \left(1 - \psi'(e)\right) \frac{de}{d\alpha}.
\end{aligned}$$

This shows that

$$\Xi_s^{RS} = \begin{cases} < 0 & \text{if } \alpha < \frac{1}{2} \\ = 0 & \text{if } \alpha = \frac{1}{2} \\ > 0 & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Therefore

$$\begin{aligned}
\Xi_s^{E} &= \Xi_s^{AE} + \Xi_s^{RS} \\
&= \left(\frac{1}{2} \frac{1}{2\tau} + \frac{1}{2\tau} \left(\alpha - \frac{1}{2}\right)\right) \left(1 - \psi'(e)\right) \frac{de}{d\alpha} \\
&= \frac{\alpha}{2\tau} \left(1 - \psi'(e)\right) \frac{de}{d\alpha} > 0,
\end{aligned}$$

which establishes the result.

iii) At last, we show that the overall efficiency gains induced by an increase in the precision of output uncertainty are strictly positive. As we have shown that

$$\Xi_c^E = \Xi_c^{AE} + \Xi_c^{RS} = 0 \quad \text{and} \quad \Xi_s^E = \Xi_s^{AE} + \Xi_s^{RS} > 0,$$

which directly implies that

$$\Xi^E = \Xi_c^E + \Xi_s^E.$$

It is also immediate that $\Xi^{AE} = \Xi_c^{AE} + \Xi_s^{AE} > 0$, as both constituents are strictly positive. While it is obvious that Ξ^{RS} is negative when $\alpha < \frac{1}{2}$, risk-sharing can become positive for high values of α. We can define a threshold for sensitivity α above which risk-sharing is positive. Formally, risk-sharing is given by

$$\Xi^{RS} = \Xi_c^{RS} + \Xi_s^{RS} = \left(\frac{1}{2\tau}\left(\alpha - \frac{1}{2}\right) - 1\right)(1-\alpha)\frac{1}{\kappa}.$$

Let $\overline{\alpha}$ denote the sensitivity threshold where $\Xi^{RS} = 0$, which is given by

$$\overline{\alpha} = \frac{1}{2} + 2\tau.$$

Then, risk-sharing is negative whenever $\alpha < \overline{\alpha}$, and is positive whenever $\alpha > \overline{\alpha}$. $\square$

C Proofs and Derivations: Section 4

C.1 Application 3: Public Information

Equilibrium Characterization Because markets are complete, it must be that each type consumes a share of consumption at date 2 that exclusively depends on the date-1 signal. Hence,

$$c_2^i(s,\xi) = c_2^i(\xi) = \psi^i(\xi)\underbrace{n_2}_{=1} = \psi^i(\xi).$$

Therefore, $\psi^i(\xi) = \int \chi^i(s) f(s|\xi)\, ds$, so

$$c_2^A(\xi) = \psi^A(\xi) = \mathbb{E}\left[\Phi(s) \mid \xi\right] = \Phi\left(\frac{\mu_{s|\xi}}{\sqrt{1+\sigma_{s|\xi}^2}}\right) = \Phi\left(\frac{\alpha\xi}{\sqrt{1+\sigma_{s|\xi}^2}}\right),$$

where $\mu_{s|\xi} = \alpha\xi$, $\sigma_{s|\xi}^2 = \frac{1}{\frac{1}{\sigma^2}+\frac{1}{\sigma_0^2}} = \alpha\sigma^2 = (1-\alpha)\sigma_0^2$, with $\alpha = \frac{\sigma_0^2}{\sigma^2+\sigma_0^2}$ and $1-\alpha = \frac{\sigma^2}{\sigma^2+\sigma_0^2}$.

It is useful to compute:

$$\frac{\partial c_2^A(s,\xi)}{\partial \xi} = \frac{\partial \psi^A(\xi)}{\partial \xi} = \phi\left(\frac{\alpha\xi}{\sqrt{1+\sigma_{s|\xi}^2}}\right)\overbrace{\frac{\alpha}{\sqrt{1+\sigma_{s|\xi}^2}}}^{>0} > 0 \tag{OA.1}$$

$$\frac{\partial c_2^A(s,\xi)}{\partial \sigma} = \frac{\partial \psi^A(\xi)}{\partial \sigma} = \phi\left(\frac{\alpha\xi}{\sqrt{1+\sigma_{s|\xi}^2}}\right)\underbrace{\frac{\sqrt{1+\sigma_{s|\xi}^2}+\alpha\sigma_0^2}{1+\sigma_{s|\xi}^2}}_{>0}\underbrace{\frac{d\alpha}{d\sigma}}_{<0}\xi, \tag{OA.2}$$

where we use the fact that

$$\frac{d\alpha}{d\sigma} = -\frac{\sigma_0^2}{\left(\sigma^2+\sigma_0^2\right)^2}2\sigma = -\alpha\frac{1}{\sigma^2+\sigma_0^2}2\sigma = -2\alpha(1-\alpha)\frac{1}{\sigma} < 0.$$

Note that $\frac{\partial c_2^A(s,\xi)}{\partial \xi}$ is always strictly positive, while $\frac{\partial c_2^A(s,\xi)}{\partial \sigma}$ has the opposite sign of the signal realization ξ.

Welfare Note that

$$\frac{dV^i}{d\sigma} = \iint u'\left(c_2^i(s,\xi)\right)\frac{\partial c_2^i(s,\xi)}{\partial \sigma} f(s,\xi)\, d\xi ds + \iint u'\left(c_2^i(s,\xi)\right)\frac{\partial c_2^i(s,\xi)}{\partial \xi}\frac{d(1-F(\xi|s))}{d\sigma} f(s)\, d\xi ds,$$

where $f(s,\xi) = f(\xi|s)\, f(s)$. The normalized individual welfare gain can be written as

$$\frac{dV^{i|\lambda}}{d\sigma} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \iint \omega_2^i(s,\xi)\,\frac{\partial c_2^i(s,\xi)}{\partial \sigma}d\xi ds + \iint \omega_2^i(s,\xi)\,\frac{\partial c_2^i(s,\xi)}{\partial \xi}\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f(\xi|s)}d\xi ds \quad ,$$

where we multiply and divide the second term by $f(\xi|s)$, and where we define

$$\omega_2^i(s,\xi) = \frac{u'\left(c_2^i(s,\xi)\right) f(s,\xi)}{\iint u'\left(c_2^i(s,\xi)\right) f(s,\xi)\, ds d\xi}.$$

Kaldor-Hicks efficiency $\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\sigma}$, is given in (24), and can be decomposed as

$$\Xi^E = \underbrace{\iint \omega_2(s,\xi)\overbrace{\left(\sum_i \frac{\partial c_2^i(s,\xi)}{\partial \sigma}\right)}^{=\frac{\partial C(s,\xi)}{\partial \sigma}}d\xi ds}_{\Xi_c^{AE}} + \underbrace{\iint \omega_2(s,\xi)\left(\overbrace{\sum_i \frac{\partial c_2^i(s,\xi)}{\partial \xi}}^{\frac{\partial C(s,\xi)}{\partial \xi}}\left(\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f(\xi|s)}\right)\right)d\xi ds}_{\Xi_\xi^{AE}}$$

$$+ \underbrace{\iint \omega_2(s,\xi)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_2^i(s,\xi)}{\omega_2(s,\xi)}, \frac{\partial c_2^i(s,\xi)}{\partial \sigma}\right]d\xi ds}_{\Xi_c^{RS}} + \underbrace{\iint \omega_2(s,\xi)\,\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_2^i(s,\xi)}{\omega_2(s,\xi)}, \frac{\partial c_2^i(s,\xi)}{\partial \xi}\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f(\xi|s)}\right]d\xi ds}_{\Xi_\xi^{RS}}.$$

Note that $\frac{\partial C(s,\xi)}{\partial \sigma} = \frac{\partial C(s,\xi)}{\partial \xi} = 0$ in the endowment economy without aggregate risk considered here.

Proof of Proposition 3. (Value of Public Information)

Proof. i) We first show that $\Xi_c^{RS} > 0$. First, note that both the left and the right constituents of $\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_2^i(s,\xi)}{\omega_2(s,\xi)}, \frac{\partial c_2^i(s,\xi)}{\partial \sigma}\right]$ are independent of s. Second, note that $\frac{\omega_2^A(s,\xi)}{\omega_2(s,\xi)}$ is monotonically decreasing in ξ: type-A's marginal utility of consumption is low for high realizations of the signal ξ. From (OA.1) above, note that $\frac{\partial c_2^i(s,\xi)}{\partial \sigma}$ can be written as $a \cdot \xi$, where $a < 0$. Hence,

$$\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_2^i(s,\xi)}{\omega_2(s,\xi)}, \frac{\partial c_2^i(s,\xi)}{\partial \sigma}\right] = \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_2^i(\xi)}{\omega_2(\xi)}, \frac{\partial c_2^i(\xi)}{\partial \sigma}\right] > 0,$$

for every signal ξ, as both components of the covariance are decreasing functions of ξ. This immediately implies that $\Xi_c^{RS} > 0$.

ii) We next show that $\Xi_{\xi}^{RS}<0$. First, as explained in Section 2, note that

$$\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f\left(\xi\mid s\right)}=\frac{\xi-s}{\sigma}.$$

Hence, we can focus on

$$\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{2}^{i}\left(s,\xi\right)}{\omega_{2}\left(s,\xi\right)},\frac{\partial c_{2}^{i}\left(s,\xi\right)}{\partial\xi}\frac{\frac{d(1-F(\xi|s))}{d\sigma}}{f\left(\xi|s\right)}\right]=\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{2}^{i}\left(\xi\right)}{\omega_{2}\left(\xi\right)},\frac{\partial c_{2}^{i}\left(\xi\right)}{\partial\xi}\right]\left(\frac{\xi-s}{\sigma}\right).$$

Note that it is sufficient to compute the term $\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{2}^{i}(\xi)}{\omega_{2}(\xi)},\frac{\partial c_{2}^{i}(\xi)}{\partial\xi}\right]\xi$, as the term multiplied by s integrates to zero. But also note that $\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{2}^{i}(\xi)}{\omega_{2}(\xi)},\frac{\partial c_{2}^{i}(\xi)}{\partial\xi}\right]<0$ when $\xi>0$ and $\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{2}^{i}(\xi)}{\omega_{2}(\xi)},\frac{\partial c_{2}^{i}(\xi)}{\partial\xi}\right]>0$ when $\xi<0$. This occurs because always $\frac{\partial c_{2}^{A}(\xi)}{\partial\xi}>0>\frac{\partial c_{2}^{B}(\xi)}{\partial\xi}$, but $\omega^{A}\left(\xi\right)>\omega^{B}\left(\xi\right)$ when $\xi<0$ while $\omega^{A}\left(\xi\right)<\omega^{B}\left(\xi\right)$ when $\xi>0$. But this immediately implies that $\Xi_{\xi}^{RS}<0$, as it is a sum over negative elements. □

C.2 Application 4: Private Information (REE)

Linear Equilibrium in CARA-Normal. The FOC for the investors is

$$a_{1}^{\star}\left(\xi,q;\tau_{\xi},\tau_{\hat{q}}\right)=\frac{\mathbb{E}\left[\delta|\,\xi,q;\tau_{\xi},\tau_{\hat{q}}\right]-q}{\gamma\mathbb{V}ar\left[\delta|\,\xi,q;\tau_{\xi},\tau_{\hat{q}}\right]}. \tag{OA.3}$$

In a symmetric equilibrium in linear strategies we have

$$a_{1}^{\star}\left(\xi,q;\tau_{\xi},\tau_{\hat{q}}\right)=\alpha_{\xi}\xi-\alpha_{q}q+\psi, \tag{OA.4}$$

where $\tau_{\hat{q}}=\left(\frac{\tau_{\xi}}{\gamma}\right)^{2}\tau_{n}$ and

$$\alpha_{\xi}=\frac{\tau_{\xi}}{\gamma},\quad\alpha_{q}=\frac{\tau_{\delta}+\tau_{\xi}+\tau_{\hat{q}}}{\gamma}\frac{\tau_{\xi}}{\tau_{\xi}+\tau_{\hat{q}}},\quad\text{and}\quad\psi=\frac{\tau_{\xi}}{\tau_{\xi}+\tau_{\hat{q}}}\frac{\tau_{\delta}\mu_{\delta}}{\gamma}. \tag{OA.5}$$

This implies that the equilibrium price and the unbiased signal contained in it are respectively given by

$$q^{\star}\left(\bar{\delta}, n; \tau_{\xi}, \tau_{\hat{q}}\right) = \frac{\alpha_{\xi}}{\alpha_q}\bar{\delta} + \frac{1}{\alpha_q}n + \frac{\psi}{\alpha_q} \tag{OA.6}$$

$$\hat{q}^{\star}\left(\bar{\delta}, n; \tau_{\xi}\right) = \frac{\alpha_q}{\alpha_{\xi}}\left(q^{\star} - \frac{\psi}{\alpha_q}\right) = \bar{\delta} + \frac{\gamma}{\tau_{\xi}}n. \tag{OA.7}$$

For notational convenience, we'll omit the arguments of $a_1^{\star}$, $q^{\star}$, and $\hat{q}^{\star}$ and the dependence of $\tau_{\hat{q}}$ on τ_{ξ} in the remainder of this section.

Intermediate results and calculations. The mean and variance of the posterior distribution of the asset payoff given a private signal ξ and the equilibrium price $q^{\star}$ are respectively given by

$$\mu_{\delta|\xi,q} \equiv \mathbb{E}\left[\delta|\, \xi, q^{\star}; \tau_{\xi}, \tau_{\hat{q}}\right] = \frac{\tau_{\delta}\mu_{\delta} + \tau_{\xi}\xi + \tau_{\hat{q}}\hat{q}^{\star}}{\tau_{\delta} + \tau_{\xi} + \tau_{\hat{q}}} \tag{OA.8}$$

$$\sigma^2_{\delta|\xi,q} \equiv \mathbb{V}ar\left[\delta|\, \xi, q^{\star}; \tau_{\xi}, \tau_{\hat{q}}\right] = \left(\tau_{\delta} + \tau_{\xi} + \tau_{\hat{q}}\right)^{-1}. \tag{OA.9}$$

Let $\omega\left(\delta, \xi, \bar{\delta}, n; \tau_{\xi}\right) \equiv u'\left(\tilde{c}_2\left(\delta, a_1^{\star}, q^{\star}\right)\right) f\left(\delta|\xi, q^{\star}; \tau_{\xi}, \tau_{\hat{q}}\right)$, where

$$\tilde{c}_2\left(\delta, a_1, q\right) = \overline{y} + \left(\delta - q\right) a_1. \tag{OA.10}$$

Then, using the CARA-normal structure of the model, we have

$$\overline{\omega}_{\delta}\left(\xi, \bar{\delta}, n; \tau_{\xi}\right) \equiv \int \omega\left(\delta, \xi, \bar{\delta}, n; \tau_{\xi}\right) d\delta = \gamma \exp\left\{-\gamma\overline{y}\right\} \exp\left\{-\frac{\gamma^2}{2}\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right\}. \tag{OA.11}$$

Similarly, using the CARA-normal structure we have

$$\overline{\omega}_{\delta}\left(\xi, \bar{\delta}, n; \tau_{\xi}\right) f\left(\xi|\, \bar{\delta}\right) = -\gamma \exp\left\{-\gamma\overline{y}\right\} \sqrt{\frac{\tau_{\xi}}{\left(\tau_{\xi} + \tau_{\xi}^2\sigma^2_{\delta|\xi,q}\right)}} \exp\left\{-\frac{1}{2}\frac{\tau_{\xi}\sigma^2_{\delta|\xi,q}}{\left(\tau_{\xi} + \tau_{\xi}^2\sigma^2_{\delta|\xi,q}\right)}\gamma^2 n^2\right\} \tilde{f}\left(\xi|\bar{\delta}, n\right) \tag{OA.12}$$

$$\int \overline{\omega}_{\delta}\left(\xi, \bar{\delta}, n; \tau_{\xi}\right) a_1^{\star} f\left(\xi|\, \bar{\delta}\right) d\xi = \gamma \exp\left\{-\gamma\overline{y}\right\} \left(\frac{\tau_{\xi}}{\tau_{\xi} + \tau_{\xi}^2\sigma^2_{\delta|\xi,q}}\right)^{\frac{3}{2}} \exp\left\{-\frac{1}{2}\frac{\sigma^2_{\delta|\xi,q}\tau_{\xi}}{\sigma^2_{\delta|\xi,q}\tau_{\xi}^2 + \tau_{\xi}}\gamma^2 n^2\right\} n, \tag{OA.13}$$

where $\tilde{f}\left(\xi|\bar{\delta},n\right)$ is the pdf of a normal random variable with mean $\tilde{\mu}_{\xi|\bar{\delta},n} = \bar{\delta} + \frac{\tau_\xi^2\sigma^2_{\delta|\xi,q}}{\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}}\frac{\gamma}{\tau_\xi}n$ and variance $\tilde{\sigma}^2_{\xi|\bar{\delta},n} = \left(\tau_\xi + \tau_\xi^2\sigma^2_{\delta|\xi,q}\right)^{-1}$, and

$$\int \omega\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\left(\delta - \mu_{\delta|\xi,q}\right)d\delta = -\exp\left\{-\gamma\overline{y}\right\}\gamma\sigma^2_{\delta|\xi,q}a_1^\star\exp\left\{-\frac{1}{2}\gamma^2\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right\}. \quad \text{(OA.14)}$$

Proof of Proposition 4. (Value of Private Information)

Proof. i) Learning pecuniary effects. Using that $F\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)$ is the cdf of a normal random variable with mean $\mu_{\delta|\xi,q}$ and variance $\sigma^2_{\delta|\xi,q}$, as given in Equation (OA.8) and (OA.9) respectively, with associated pdf $f\left(\delta|\,\xi,\hat{q};\tau_\xi,\tau_{\hat{q}}\right)$ and that consumption is given by Equation (OA.10), the learning pecuniary channel can be written as

$$\begin{aligned}\text{LP} &\equiv \frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int \omega\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\frac{\partial\tilde{c}_2}{\partial\delta}\frac{\frac{d\left[1-F\left(\delta|\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)\right]}{d\hat{q}^\star}}{f\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)}\frac{d\hat{q}^\star}{d\tau_\xi}d\delta\right] \\ &= \mathbb{E}_{\xi,\bar{\delta},n}\left[\overline{\omega}\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\frac{\left(\mu_{\delta|\xi,q}-q^\star\right)}{\gamma}\tau_{\hat{q}}\frac{d\hat{q}^\star}{d\tau_\xi}\right].\end{aligned}$$

Using Equation (OA.11), the expression for $\hat{q}^\star$ in Equation (OA.7) and Equation (OA.13) we have

$$\text{LP} = \frac{1}{\lambda}\exp\left\{-\gamma\overline{y}\right\}\gamma^2\sigma^2_{\delta|\xi,q}\frac{\tau_{\hat{q}}}{\tau_\xi^2}\left(\frac{\tau_\xi}{\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}}\right)^{\frac{3}{2}}\mathbb{E}_n\left[\exp\left\{-\frac{1}{2}\frac{\sigma^2_{\delta|\xi,q}\tau_\xi}{\sigma^2_{\delta|\xi,q}\tau_\xi^2+\tau_\xi}\gamma^2n^2\right\}n^2\right] > 0.$$

ii) Signal compression. Using that $F\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)$ is the cdf of a normal random variable with mean $\mu_{\delta|\xi,q}$ and variance $\sigma^2_{\delta|\xi,q}$, as given in Equation (OA.8) and (OA.9) respectively, , with associated pdf $f\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)$, that $F\left(\xi|\bar{\delta};\tau_\xi\right)$ is the cdf of a normal random variable with mean $\bar{\delta}$ and variance τ_ξ^{-1} with associated pdf $f\left(\xi|\bar{\delta};\tau_\xi\right)$, and that consumption is given by Equation (OA.10) we have

$$\begin{aligned}\text{SC} &\equiv \frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int \omega\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\frac{\partial\tilde{c}_2}{\partial\delta}\frac{\frac{\partial\left[1-F\left(\delta|\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)\right]}{\partial\xi}}{f\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)}\frac{\frac{d\left(1-F\left(\xi|\bar{\delta};\tau_\xi\right)\right)}{d\tau_\xi}}{f\left(\xi|\bar{\delta};\tau_\xi\right)}d\delta\right] \\ &= -\frac{1}{2\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\overline{\omega}\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)a_1^\star\sigma^2_{\delta|\xi,q}\left(\xi-\bar{\delta}\right)\right]. \quad \text{(OA.15)}\end{aligned}$$

Moreover, using Equation (OA.4) and Equation (OA.6) we can write $a_1^\star =$

$\frac{1}{\gamma}\left(\tau_\xi\left(\xi-\bar{\delta}\right)-\gamma n\right)$, which gives

$$\mathrm{SC}=-\frac{\gamma}{2\lambda}\exp\left\{-\gamma\bar{y}\right\}\sigma^2_{\delta|\xi,q}\tau_\xi\mathbb{E}_{\bar{\delta},n}\left[\int\exp\left\{-\frac{\gamma^2}{2}\sigma^2_{\delta|\xi,q}a_1^2\right\}\frac{1}{\gamma}\left(-\left(\xi-\bar{\delta}\right)\frac{\gamma}{\tau_\xi}n+\left(\xi-\bar{\delta}\right)^2\right)f\left(\xi|\,\bar{\delta}\right)d\xi\right].$$

Using (OA.12) and that

$$\exp\left\{-\frac{1}{2}\frac{\tau_\xi^2\sigma^2_{\delta|\xi,q}}{\left(\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}\right)}\frac{\gamma^2}{\tau_\xi}n^2\right\}f\left(n\right)=\sqrt{\frac{\tau_n}{\left(\tau_n+\frac{\tau_\xi^2\sigma^2_{\delta|\xi,q}}{\left(\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}\right)}\frac{\gamma^2}{\tau_\xi}\right)}}\hat{f}\left(n\right),$$

where $\hat{f}\left(n\right)$ is the pdf of a normal random variable with mean zero and variance $\hat{\sigma}_n^2=\left(\tau_n+\frac{\tau_\xi^2\sigma^2_{\delta|\xi,q}}{\left(\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}\right)}\frac{\gamma^2}{\tau_\xi}\right)^{-1}$ we get

$$\mathrm{SC}=-\frac{1}{2\lambda}\exp\left\{-\gamma\bar{y}\right\}\sigma^2_{\delta|\xi,q}\left(\frac{\tau_n}{\left(\tau_n+\frac{\tau_\xi^2\sigma^2_{\delta|\xi,q}}{\left(\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}\right)}\frac{\gamma^2}{\tau_\xi}\right)}\frac{\tau_\xi}{\left(\tau_\xi+\tau_\xi^2\sigma^2_{\delta|\xi,q}\right)}\right)^{\frac{3}{2}}<0.$$

iii) Private Signal. Using that $F\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)$ is the cdf of a normal random variable with mean $\mu_{\delta|\xi,q}$ and variance $\sigma^2_{\delta|\xi,q}$, as given in Equation (OA.8) and (OA.9) respectively, with associated pdf $f\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)$, that consumption is given by Equation (OA.10) and the FOC in (OA.3), we can write

$$\begin{aligned}\mathrm{PSP}&\equiv\frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\int\left[\omega\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\frac{\partial\tilde{c}_2}{\partial\delta}\frac{\frac{\partial\left[1-F\left(\delta|\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)\right]}{\partial\tau_\xi}}{f\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)}d\delta\right]\\&=\frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int\omega\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)a_1^\star\sigma^2_{\delta|\xi,q}\left(\left(\xi-\mu_{\delta|\xi,q}\right)-\frac{1}{2}\left(\delta-\mu_{\delta|\xi,q}\right)\right)d\delta\right].\end{aligned}\tag{OA.16}$$

Using Equation (OA.3) and (OA.7), and the definition of $\mu_{\delta|\xi,q}$ in Equation (OA.8) we can write

$$\begin{aligned}\sigma^2_{\delta|\xi,q}a_1^\star=\frac{\mu_{\delta|\xi,q}-\hat{q}^\star}{\gamma}&=\sigma^2_{\delta|\xi,q}\left(\frac{\tau_\xi}{\gamma}\left(\xi-\bar{\delta}\right)-n\right)\\\xi-\mu_{\delta|\xi,q}&=\sigma^2_{\delta|\xi,q}\left(\tau_\delta\left(\xi-\mu_\delta\right)+\frac{\tau_\xi}{\gamma}\tau_n\left(\frac{\tau_\xi}{\gamma}\left(\xi-\bar{\delta}\right)-n\right)\right).\end{aligned}$$

Then, using these expressions and Equation (OA.11) we can write

$$\lambda \text{PSP} = \sigma^4_{\delta|\xi,q}\tau_\xi \mathbb{E}_{\xi,\bar{\delta},n}\left[\overline{\omega}\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\left(\tau_\delta\left(\xi-\mu_\delta\right)\left(\left(\xi-\bar{\delta}\right)-\frac{\gamma}{\tau_\xi}n\right)+\left(\frac{\tau_\xi}{\gamma}\right)^2\tau_n\left(\left(\xi-\bar{\delta}\right)-\frac{\gamma}{\tau_\xi}n\right)^2\right)\right] \tag{OA.17}$$

$$-\frac{\sigma^2_{\delta|\xi,q}}{2}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int\omega\left(\delta,\xi,\bar{\delta},n;\tau_\xi\right)\left(\delta-\mu_{\delta|\xi,q}\right)d\delta\left(\frac{\tau_\xi}{\gamma}\left(\xi-\bar{\delta}\right)-n\right)\right]. \tag{OA.18}$$

The term in (OA.17) can be written as $\sigma^4_{\delta|\xi,q}\tau_\xi\exp\left\{-\gamma\overline{y}\right\}\sqrt{\frac{\tau_\xi}{\left(\sigma^2_{\delta|\xi,q}\tau^2_\xi+\tau_\xi\right)}}\times$

$$\begin{aligned}&\left(\mathbb{E}_{\bar{\delta},n}\left[\exp\left\{-\frac{1}{2}\left(\frac{\sigma^2_{\delta|\xi,q}\tau^2_\xi}{\sigma^2_{\delta|\xi,q}\tau^2_\xi+\tau_\xi}\right)\frac{\gamma^2}{\tau_\xi}n^2\right\}\int\tau_\delta\left(\xi-\mu_\delta\right)\left(\left(\xi-\bar{\delta}\right)-\frac{\gamma}{\tau_\xi}n\right)\tilde{f}\left(\xi|\bar{\delta},n\right)d\xi\right]\right.\\&\left.+\mathbb{E}_{\bar{\delta},n}\left[\exp\left\{-\frac{1}{2}\left(\frac{\sigma^2_{\delta|\xi,q}\tau^2_\xi}{\sigma^2_{\delta|\xi,q}\tau^2_\xi+\tau_\xi}\right)\frac{\gamma^2}{\tau_\xi}n^2\right\}\int\left(\frac{\tau_\xi}{\gamma}\right)^2\tau_n\left(\left(\xi-\bar{\delta}\right)-\frac{\gamma}{\tau_\xi}n\right)^2\tilde{f}\left(\xi|\bar{\delta},n\right)d\xi\right]\right),\end{aligned}$$

where $\tilde{f}\left(\xi|\bar{\delta},n\right)$ is the pdf of a normal random variable with mean $\tilde{\mu}_{\xi|\bar{\delta},n}=\bar{\delta}+\frac{\tau^2_\xi\sigma^2_{\delta|\xi,q}}{\tau_\xi+\tau^2_\xi\sigma^2_{\delta|\xi,q}}\frac{\gamma}{\tau_\xi}n$ and variance $\tilde{\sigma}^2_{\xi|\bar{\delta},n}=\left(\tau_\xi+\tau^2_\xi\sigma^2_{\delta|\xi,q}\right)$. The second line in the expression above is positive. Solving the expectations over ξ and $\bar{\delta}$, and using that $\bar{\delta}\perp n$ the first line can be written as

$$-\sigma^4_{\delta|\xi,q}\exp\left\{-\gamma\overline{y}\right\}\tau_\delta\left(\frac{\tau_\xi}{\tau_\xi+\tau^2_\xi\sigma^2_{\delta|\xi,q}}\right)^{\frac{3}{2}}\int\left(\left(\frac{\tau^2_\xi\sigma^2_{\delta|\xi,q}}{\tau_\xi+\tau^2_\xi\sigma^2_{\delta|\xi,q}}\right)\frac{\gamma^2}{\tau_\xi}n^2-1\right)\hat{f}\left(n\right)dn,$$

where $\hat{f}\left(n\right)$ is the pdf of a normal random variable with mean zero and variance $\hat{\sigma}^2_n\equiv\left(\tau_n+\left(\frac{\sigma^2_{\delta|\xi,q}\tau^2_\xi}{\sigma^2_{\delta|\xi,q}\tau^2_\xi+\tau_\xi}\right)\frac{\gamma^2}{\tau_\xi}\right)^{-1}$. Then, the first term is

$$\sigma^4_{\delta|\xi,q}\exp\left\{-\gamma\overline{y}\right\}\tau_\delta\left(\frac{\tau_\xi}{\tau_\xi+\tau^2_\xi\sigma^2_{\delta|\xi,q}}\right)^{\frac{3}{2}}\frac{\tau_n}{\tau_n+\left(\frac{\sigma^2_{\delta|\xi,q}\tau^2_\xi}{\sigma^2_{\delta|\xi,q}\tau^2_\xi+\tau_\xi}\right)\frac{\gamma^2}{\tau_\xi}}>0.$$

Finally, using (OA.14), the term in (OA.18) can be written as

$$\exp\left\{-\gamma\overline{y}\right\}\frac{\sigma^2_{\delta|\xi,q}}{2}\gamma\mathbb{E}_{\xi,\bar{\delta},n}\left[\exp\left\{-\frac{1}{2}\gamma^2\sigma^2_{\delta|\xi,q}a^{\star 2}_1\right\}\gamma\sigma^2_{\delta|\xi,q}a^{\star 2}_1\right]>0,$$

which shows the private signal precision effect is positive.
iv) Informativeness. Using that $F\left(\delta|\,\xi,\hat{q}^\star;\tau_\xi,\tau_{\hat{q}}\right)$ is the cdf of a normal random variable with mean $\mu_{\delta|\xi,q}$ and variance $\sigma^2_{\delta|\xi,q}$, as given in Equation (OA.8) and Equations (OA.9)

respectively, with associated pdf $f\left(\delta|\,\xi,\hat{q}^{\star};\tau_{\xi},\tau_{\hat{q}}\right)$, that $\frac{\partial\tilde{c}_2}{\partial\delta}=a_1^{\star}$, and Equations (OA.14) and (OA.12), we can write

$$\begin{aligned}\mathrm{I}&=\frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int\omega\left(\delta,\xi,\bar{\delta},n;\tau_{\xi}\right)\frac{\partial\tilde{c}_2}{\partial\delta}\frac{\frac{\partial\left[1-F\left(\delta|\xi,\hat{q}^{\star};\tau_{\xi},\tau_{\hat{q}}\right)\right]}{\partial\tau_{\hat{q}}}}{f\left(\delta|\,\xi,\hat{q}^{\star};\tau_{\xi},\tau_{\hat{q}}\right)}\frac{d\tau_{\hat{q}}}{d\tau_{\xi}}d\delta\right]\\&=\frac{1}{\lambda}\exp\left\{-\gamma\overline{y}\right\}\sigma^2_{\delta|\xi,\hat{q}}\mathbb{E}_{\xi,\bar{\delta},n}\left[\frac{d\tau_{\hat{q}}}{d\tau_{\xi}}\exp\left\{-\frac{\gamma^2}{2}\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right\}\left(\gamma a_1^{\star}\left(\hat{q}^{\star}-\mu_{\delta|\xi,\hat{q}}\right)+\frac{1}{2}\gamma^2\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right)\right].\end{aligned}$$

Since $\overline{\delta}\perp n\perp\left(\xi-\overline{\delta}\right)$, and using that $\hat{q}^{\star}-\mu_{\delta|\xi,\hat{q}}=\left(\hat{q}^{\star}-q^{\star}\right)-\gamma\sigma^2_{\delta|\xi,q}a_1^{\star}$, $a_1^{\star}=\frac{\tau_{\xi}}{\gamma}\left(\xi-\overline{\delta}\right)-n$, and $\left(\hat{q}^{\star}-q^{\star}\right)=\overline{\delta}\left(1-\frac{\alpha_{\xi}}{\alpha_q}\right)+\left(\frac{\gamma}{\tau_{\xi}}-\frac{1}{\alpha_q}\right)n-\frac{\psi}{\alpha_q}$, plus the equilibrium coefficients in (OA.5), the term above becomes

$$\begin{aligned}\mathrm{I}&=\frac{1}{\lambda}\exp\left\{-\gamma\overline{y}\right\}\sigma^2_{\delta|\xi,\hat{q}}\frac{d\tau_{\hat{q}}}{d\tau_{\xi}}\mathbb{E}_{\xi,\bar{\delta},n}\left[\exp\left\{-\frac{\gamma^2}{2}\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right\}\left(\gamma a_1^{\star}\left(\hat{q}^{\star}-q^{\star}\right)-\frac{1}{2}\gamma^2\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right)\right]\\&=\frac{1}{\lambda}\exp\left\{-\gamma\overline{y}\right\}\sigma^2_{\delta|\xi,\hat{q}}\frac{d\tau_{\hat{q}}}{d\tau_{\xi}}\mathbb{E}_{\xi-\bar{\delta},n}\left[\exp\left\{-\frac{\gamma^2}{2}\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right\}\left(\sigma^2_{\delta|\xi,q}a_1^{\star}\tau_{\delta}\frac{\gamma^2}{\tau_{\xi}}n-\frac{1}{2}\gamma^2\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right)\right].\end{aligned}$$

Moreover, since

$$\exp\left\{-\frac{\gamma^2}{2}\sigma^2_{\delta|\xi,q}a_1^{\star 2}\right\}f\left(\xi-\overline{\delta}|n\right)=\exp\left\{-\frac{1}{2}\frac{\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}{\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}\frac{\gamma^2}{\tau_{\xi}}n^2\right\}\tilde{f}\left(\xi-\overline{\delta}|n\right),$$

where $\tilde{f}\left(\xi-\overline{\delta}|n\right)$ is the pdf of a normal random variable with mean $\tilde{\mu}_{\xi-\bar{\delta}}\equiv\frac{\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}{\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}\frac{\gamma}{\tau_{\xi}}n$ and variance $\tilde{\sigma}^2_{\xi-\bar{\delta}}\equiv\left(\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}\right)^{-1}$ we have

$$\begin{aligned}\mathrm{I}=&-\frac{1}{\lambda}\exp\left\{-\gamma\overline{y}\right\}\frac{d\tau_{\hat{q}}}{d\tau_{\xi}}\left(\frac{\tau_{\xi}}{\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}\right)^{\frac{3}{2}}\mathbb{E}_n\left[\exp\left\{-\frac{1}{2}\frac{\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}{\left(\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}\right)}\frac{\gamma^2}{\tau_{\xi}}n^2\right\}\left(\tau_{\delta}\tau_{\xi}+\tau_{\xi}^2\right)\sigma^2_{\delta|\xi,q}\left(\frac{\gamma}{\tau_{\xi}}\right)^2n^2\right]\\&-\frac{1}{\lambda}\exp\left\{-\gamma\overline{y}\right\}\frac{d\tau_{\hat{q}}}{d\tau_{\xi}}\sqrt{\frac{\tau_{\xi}}{\left(\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}\right)}}\mathbb{E}_n\left[\exp\left\{-\frac{1}{2}\frac{\tau_{\xi}^2\sigma^2_{\delta|\xi,q}}{\left(\tau_{\xi}+\tau_{\xi}^2\sigma^2_{\delta|\xi,q}\right)}\frac{\gamma^2}{\tau_{\xi}}n^2\right\}\left(\frac{1}{2}\sigma^2_{\delta|\xi,q}\tau_{\xi}^2\tilde{\sigma}^2_{\xi-\bar{\delta}}\right)\right]<0,\end{aligned}$$

showing that the informativeness channel is negative.

v) Distributive pecuniary effects are ambiguous. The total pecuniary effects are given by

$$\mathrm{DP}=-\frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\overline{\omega}_{\delta}\left(\xi,\bar{\delta},n;\tau_{\xi}\right)a_1^{\star}\frac{dq}{d\tau_{\xi}}\right]+\frac{1}{\lambda^n}\frac{dJ_0}{d\tau_{\xi}},$$

which using the FOC for investors and market clearing can be written as

$$\mathrm{DP} = -\mathbb{E}_{\bar{\delta},n}\left[\mathbb{E}_{\xi}\left[\left(\frac{1}{\lambda}\overline{\omega}\left(\xi,\bar{\delta},n;\tau_{\xi}\right) - \frac{1}{\lambda^{n}}u'\left(c_{2}\left(\delta,q,n\right)\right)\right)a_{1}^{\star}\left(\xi,q\right)\right]\frac{dq}{d\tau_{\xi}}\right],$$

which can be positive or negative depending on parameters. □

Proof of Corollary. (Private Value of Private Information)

Proof. Using Eq. (OA.15) and (OA.16), the total private effects can be written as

$$\begin{aligned}
\mathrm{P} &= \frac{1}{\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int\omega\left(\delta,\xi,\bar{\delta},n;\tau_{\xi}\right)a_{1}^{\star}\sigma_{\delta|\xi,q}^{2}\left(\left(\xi-\mu_{\delta|\xi,q}\right)-\frac{1}{2}\left(\delta-\mu_{\delta|\xi,q}\right)-\frac{1}{2}\left(\xi-\bar{\delta}\right)\right)d\delta\right]\\
&= \frac{1}{2\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\int\omega\left(\delta,\xi,\bar{\delta},n;\tau_{\xi}\right)a_{1}^{\star}\sigma_{\delta|\xi,q}^{2}\left(\left(\xi-\mu_{\delta|\xi,q}\right)-\left(\delta-\mu_{\delta|\xi,q}\right)+\left(\bar{\delta}-\mu_{\delta|\xi,q}\right)\right)d\delta\right].
\end{aligned}$$

Note that this expression can be written as half the term in (OA.17) plus the term in (OA.17), which are positive form the proof of Part iii) of Proposition 4, plus

$$\frac{1}{2\lambda}\mathbb{E}_{\xi,\bar{\delta},n}\left[\overline{\omega}\left(\delta,\xi,\bar{\delta},n;\tau_{\xi}\right)a_{1}^{\star}\sigma_{\delta|\xi,q}^{2}\left(\bar{\delta}-\mu_{\delta|\xi,q}\right)\right]=0.$$

The last equality can be shown using (OA.12), and that $\bar{\delta}-\mu_{\delta|\xi,q}=\sigma_{\delta|\xi,q}^{2}\left(\tau_{\xi}\left(\bar{\delta}-\xi\right)+\tau_{\delta}\left(\bar{\delta}-\mu_{\delta}\right)-\tau_{\hat{q}}\frac{\gamma}{\tau_{\xi}}n\right)$, $a_{1}^{\star}=\frac{\tau_{\xi}}{\gamma}\left(\xi-\bar{\delta}\right)-n$, and

$$\exp\left\{-\frac{1}{2}\frac{\tau_{\xi}\sigma_{\delta|\xi,q}^{2}}{\left(\tau_{\xi}+\tau_{\xi}^{2}\sigma_{\delta|\xi,q}^{2}\right)}\gamma^{2}n^{2}\right\}f\left(n\right)=\exp\left\{-\frac{1}{2}\left(\frac{\tau_{\xi}\sigma_{\delta|\xi,q}^{2}}{\left(\tau_{\xi}+\tau_{\xi}^{2}\sigma_{\delta|\xi,q}^{2}\right)}\gamma^{2}+\tau_{n}\right)n^{2}\right\}\equiv\hat{f}\left(n\right),$$

where $\hat{f}\left(n\right)$ is the pdf of a normal distribution with mean zero and variance $\left(\frac{\tau_{\xi}\sigma_{\delta|\xi,q}^{2}}{\left(\tau_{\xi}+\tau_{\xi}^{2}\sigma_{\delta|\xi,q}^{2}\right)}\gamma^{2}+\tau_{n}\right)^{-1}$. □

D Extensions and Comparisons

D.1 Discrete States

The counterpart to Equation (1) in the discrete case is given by

$$V = u\left(c_0\right) + \beta \sum_s \pi\left(s;\theta\right) u\left(c_1\left(s\right)\right),$$

where $\pi\left(s\right)$ denotes the probability of one of the countably many states $s \in \{1, \ldots, S\}$. The agent's willingness-to-pay for a marginal change in probabilities satisfies the formula

$$p_\theta = \sum_s \frac{d\pi\left(s;\theta\right)}{d\theta} \frac{\beta u\left(c_1\left(s\right)\right)}{u'\left(c_0\right)},$$

where it must be that $\sum_s \frac{d\pi(s)}{d\theta} = 0$. Summation by parts (Courant and John, 1989) implies that

$$\sum_s u\left(c_1\left(s\right)\right) \frac{d\pi\left(s\right)}{d\theta} = \sum_{s=0}^{S-1} \left(u\left(c_1\left(s\right)\right) - u\left(c_1\left(s+1\right)\right)\right) \frac{d\Pi\left(s;\theta\right)}{d\theta},$$

where $\frac{d\Pi(s;\theta)}{d\theta} = \sum_{u=1}^{s} \frac{d\pi(u;\theta)}{d\theta}$ denotes the change in cdf, and where we use the fact that $\frac{d\Pi(S)}{d\theta} = 0$. Therefore, we can write express p_θ as

$$p_\theta = \beta \sum_{s=0}^{S-1} \pi\left(s;\theta\right) \left(\frac{u\left(c_1\left(s+1\right)\right) - u\left(c_1\left(s\right)\right)}{u'\left(c_0\right)}\right) \frac{\frac{d(1-\Pi(s;\theta))}{d\theta}}{\pi\left(s;\theta\right)},$$

which is the exact counterpart of Equation (4). Formally as the difference between $s+1$ and s becomes small,

$$u\left(c_1\left(s+1\right)\right) - u\left(c_1\left(s\right)\right) \to u'\left(c_1\left(s\right)\right) \frac{dc_1\left(s\right)}{ds}.$$

Binomial Case. In the binomial case, with states denoted by $s \in \{L, H\}$, the probability price can be written as

$$p_\theta = \beta\pi\left(L;\theta\right) \left(\frac{u\left(c_1\left(H\right)\right) - u\left(c_1\left(L\right)\right)}{u'\left(c_0\right)}\right) \frac{\frac{d(1-\Pi(L))}{d\theta}}{\pi\left(L;\theta\right)} = -\beta \left(\frac{u\left(c_1\left(H\right)\right) - u\left(c_1\left(L\right)\right)}{u'\left(c_0\right)}\right) \frac{d\pi\left(L\right)}{d\theta},$$

where the second equality uses the fact that $\Pi\left(L\right) = \pi\left(L\right)$ in the binomial case. Even though the binomial case is the easiest to compute, it does not clearly identify that the shift in the cdf is the relevant object in general, since a change in the probability of one state necessarily implies an opposite change in the other. For this reason, we do not use it as a benchmark in the paper.

D.2 Leisure

When preferences include both consumption and leisure, then $U(s) = \beta u(c_1(s), n_1(s))$, so

$$\frac{dU(s)}{ds} = \beta \frac{\partial u(c_1(s))}{\partial c_1(s)} \left(\frac{dc_1(s)}{ds} + \frac{\frac{\partial u(n_1(s))}{\partial n_1(s)}}{\frac{\partial u(c_1(s))}{\partial c_1(s)}} \frac{dn_1(s)}{ds} \right). \tag{OA.19}$$

The term in parentheses in Equation (OA.19) corresponds to a consumption-equivalent, expressing changes in $n_1(s)$ in consumption units. Hence, all results in the paper apply using this leisure-augmented consumption-equivalent. The same logic applies to environments with multiple consumption goods.

D.3 Redistributional Concerns

In Applications 2 and 3, we have focused on characterizing (Kaldor-Hicks) efficiency. It is however straightforward to compute welfare gains for any welfarist social welfare function, given by $W = \mathcal{W}\left(V^1, \ldots, V^I\right)$, using the efficiency/redistribution decomposition in Dávila and Schaab (2025). Formally, the normalized marginal social welfare effects of any perturbation for any welfarist social welfare function can be expressed as

$$\frac{dW^\lambda}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{I \cdot \mathbb{C}ov_i \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{(Redistribution)}}, \tag{OA.20}$$

where $\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i}$ and $\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}$. This is the unique decomposition in which a normalized welfare assessment can be expressed as Kaldor-Hicks efficiency, Ξ^E, and its complement, Ξ^{RD}. The choice of λ^i simply accounts for the chosen unit to make interpersonal comparisons (welfare numeraire). Hence, perturbations in which $\Xi^E > 0$ can be turned into Pareto improvements if transfers are feasible and costless. Given (OA.20), it is thus straightforward to augment our analysis to also draw insights over Ξ^{RD}.

D.4 Probability Pricing vs. Comparative Statics of Cash Flow Pricing

It should be clear that the willingness-to-pay for a marginal change in probabilities (probability pricing) differs from the change in the willingness-to-pay for an asset induced by a change in probabilities (comparative statics of cash-flow pricing). This distinction arises even in the canonical representative-agent, consumption-based asset-pricing model a la Lucas (1978), when the asset's payoff coincides with aggregate consumption.

Formally, interpreting the environment in Section 2 as a representative-agent economy a la Lucas (1978), the price of a claim to aggregate consumption satisfies

$$\frac{p_x}{c_0} = \int_{\underline{s}}^{\overline{s}} \frac{\beta u'\left(c_1\left(s\right)\right)}{u'\left(c_0\right)} \frac{c_1\left(s\right)}{c_0} f\left(s\right) ds,$$

since $x\left(s\right) = c_1\left(s\right)$ in equilibrium. Parametrizing the pdf by θ, the sensitivity of the price of this claim to a marginal change in probabilities is

$$\frac{d\left(\frac{p_x}{c_0}\right)}{d\theta} = \int_{\underline{s}}^{\overline{s}} \frac{\beta u'\left(c_1\left(s\right)\right)}{u'\left(c_0\right)} \frac{c_1\left(s\right)}{c_0} \frac{\partial \ln f\left(s;\theta\right)}{\partial\theta} f\left(s;\theta\right) ds. \tag{OA.21}$$

This expression contrast with the probability pricing exercise — computing the willingness-to-pay for a marginal change in probabilities — which yields

$$\frac{p_\theta}{c_0} = \int_{\underline{s}}^{\overline{s}} \frac{\beta u'\left(c_1\left(s;\theta\right)\right)}{u'\left(c_0\right)} \frac{c_1\left(s\right)}{c_0} \frac{\frac{\partial\left(1-F\left(s;\theta\right)\right)}{d\theta}}{f\left(s\right)} f\left(s\right) ds, \tag{OA.22}$$

after assuming that $c_1\left(s\right) = e^s c_0$, so $\frac{\partial c_1(s)}{\partial s} = c_1\left(s\right)$ — this is similar to Application 1.

The difference between OA.21 and OA.22 becomes especially transparent under log utility. In that case, $\frac{p_x}{c_0} = \beta$ and $\frac{d\left(\frac{p_x}{c_0}\right)}{d\theta} = 0$ for any change in the distribution $f\left(s\right)$. For example, increasing the mean of aggregate consumption growth leaves the willingness-to-pay for the consumption claim, p_x/c_0, unchanged, as cash-flow and discount-rate effects exactly offset. By contrast, such a shift in the mean implies a positive probability price, $p_\theta/c_0 > 0$, since $\frac{d(1-F(s;\theta))}{d\theta} > 0$ by Equation (9) in the text. This reasoning, which highlights the differences between computing an individual's willingness-to-pay for a change in probabilities (probability pricing) and the comparative statics of cash-flow pricing with respect to changes in probabilities, also applies beyond the log-utility case

D.5 Varying Consumption vs. Probabilities

A central implication of our results is that changes in consumption allocations (holding probabilities fixed) are equivalent to changes in probabilities (holding the consumption–state mapping fixed). Formally, as in Equation (7) in the text,

$$\frac{dV}{d\theta} = \beta \int_{\underline{s}}^{\overline{s}} u'\left(c_1\left(s;\theta\right)\right) \left(\frac{\partial c_1\left(s;\theta\right)}{\partial \theta} + \frac{\frac{d(1-F(s;\theta))}{d\theta}}{f\left(s;\theta\right)} \frac{\partial c_1\left(s;\theta\right)}{\partial s} \right) f\left(s;\theta\right) ds.$$

This expression makes clear that willingness-to-pay for changes in either the consumption mapping $c_1(s;\theta)$ or the probability distribution $F(s;\theta)$ can be identical. This equivalence is useful to contrast, for example, two different ways of formulating the willingness-to-pay for changes in the business cycle. First, consider the approach of Alvarez and Jermann (2004), which holds probabilities fixed — so that $\frac{\partial(1-F(s;\theta))}{\partial\theta} = 0$ — and instead parameterizes changes in the consumption mapping:

$$c_1\left(s;\theta\right) = \left(1-\theta\right) \underline{c}_1\left(s\right) + \theta \overline{c}_1\left(s\right),$$

where $\underline{c}_1(s)$ denotes the status-quo consumption allocation and $\overline{c}_1(s)$ a counterfactual allocation without fluctuations. In this case,

$$\frac{\partial c_1\left(s;\theta\right)}{\partial \theta} = \overline{c}_1\left(s\right) - \underline{c}_1\left(s\right) \neq 0.$$

Alternatively, one can consider perturbations that shift the probabilities of states while keeping the consumption mapping fixed, that is, $\frac{\partial c_1(s;\theta)}{\partial\theta} = 0$ but $\frac{\partial(1-F(s;\theta))}{\partial\theta} \neq 0$. The language in Lucas (1987) appears to describe such a change in the *distribution* of consumption, although the distinction between a change in the consumption mapping and a change in probabilities is not explicit — since Lucas's results are derived for a fully parameterized model in which expectations are explicitly computed. More generally, for marginal willingness-to-pay computations, our results provide an exact framework to ensure that changes in consumption allocations and in probabilities are accounted for, yielding conclusions that do not depend on the approach taken.

D.6 Probability Pricing in Dynamic Economies

Consider a dynamic stochastic economy with agent types $i = \{1, \ldots, I\}$, each in unit measure, and time periods $t = \{1, \ldots, T\}$. Each agent's ex ante expected utility is

$$V^i = \sum_t \beta_i^t \int u_i \left(c_t^i \left(s^t; \theta\right)\right) \pi_t^i \left(s^t; \theta\right) ds^t,$$

where $s^t = (s_1, \ldots, s_t)$ is the sequence of states up to date t, occurring with probability $\pi_t^i (s^t; \theta)$, and s_t is the (vector-valued) state of the economy at date t. We allow the probabilities to be type-specific. For instance, if the state s_t includes idiosyncratic shocks as well as aggregate states, then different agent types can face different distributions of idiosyncratic shocks. The marginal response of agent i's utility to a change in probabilities, in terms of a numeraire asset with marginal value λ^i, is given by

$$\begin{aligned}
\frac{1}{\lambda^i}\frac{dV^i}{d\theta} &= \sum_t \beta_i^t \int \frac{1}{\lambda^i} u_i \left(c_t^i \left(s^t; \theta\right)\right) \frac{d\pi_t^i (s^t; \theta)}{d\theta} ds^t \\
&+ \sum_t \beta_i^t \int \frac{1}{\lambda^i} u_i' \left(c_t^i \left(s^t; \theta\right)\right) \frac{dc_t^i (s^t; \theta)}{d\theta} \pi_t^i \left(s^t; \theta\right) ds^t \\
&\equiv \mathcal{P}_\theta^i + \mathcal{C}_\theta^i,
\end{aligned}$$

where $\mathcal{P}_\theta^i$ is a probability price and $\mathcal{C}_\theta^i$ captures endogenous consumption adjustments.

To derive a formula for probability prices that mirrors the static case in the paper, write one element of s_τ as ξ for some time τ, assuming that the probabilities of states at times $t < \tau$ are unchanged. For $t \geq \tau$, write states as

$$s^t = \left(\hat{s}^t, \xi\right)$$

and decompose probabilities as

$$\pi_t^i \left(s^t; \theta\right) = \pi_t^i \left(\hat{s}^t\right) f_t^i \left(\xi | \hat{s}^t; \theta\right),$$

where we allow only the likelihood of ξ given $\hat{s}^t$ to depend on θ. In other words, our notation here focuses on shocks that leave the marginal distribution of other states unchanged – shocks that affect the distribution of several states can be analyzed by repeatedly applying this technique. Following the same steps as in Proposition 1 yields the equivalent probability pricing formula:

$$\mathcal{P}_{\theta}^{i}=\sum_{t}\left(\beta^{i}\right)^{t}\int\frac{1}{\lambda^{i}}u_{i}^{\prime}\left(c_{t}^{i}\left(s^{t};\theta\right)\right)\frac{\partial c_{t}^{i}\left(s^{t};\theta\right)}{\partial\xi}\frac{\frac{d\left(1-F_{t}^{i}\left(\xi|\hat{s}^{t};\theta\right)\right)}{d\theta}}{f_{t}^{i}\left(\xi|\hat{s}^{t};\theta\right)}\pi_{t}^{i}\left(s^{t};\theta\right)ds^{t},$$

where $\frac{\partial c_{t}^{i}\left(s^{t};\theta\right)}{\partial\xi}$ denotes the sensitivity of consumption to the state ξ whose likelihood is being shocked. Notice that this probability price is, once again, the same as the valuation of a hypothetical asset, whose cash flows are given by $\frac{\partial c_{t}^{i}\left(s^{t};\theta\right)}{\partial\xi}\frac{\frac{d\left(1-F_{t}^{i}\left(\xi|\hat{s}^{t};\theta\right)\right)}{d\theta}}{f_{t}^{i}\left(\xi|\hat{s}^{t};\theta\right)}$.

E Additional Derivations: Section 4

In our final application to noisy REE, we use the recursive representation of expected utility presented in Equations (26) and (27). This appendix shows that our formulation for V_0 is equivalent to standard ex ante expected utility. Recall that an agent's expected utility in this application is given by

$$V_{0}=\iiiint u\left(c_{2}\left(\delta,\xi,\bar{\delta},n\right)\right)f\left(\delta,\xi,\bar{\delta},n\right)d\delta d\xi d\bar{\delta}dn, \tag{OA.23}$$

where $f\left(.\right)$ denotes the joint density of all (aggregate and idiosyncratic) state variables. We start with the following canonical result:

Lemma OA-1. *Consider an agent who enjoys utility $u\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)$, where $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ are realizations of random vectors with joint density $f\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)$. Then the following definitions of her ex ante expected utility, denoted V_0, are equivalent:*

1. $V_{0}=\iint u\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)f\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)d\boldsymbol{x}_{1}d\boldsymbol{x}_{2}$, *and*
2. $V_{0}=\iint V_{1}\left(\boldsymbol{x}_{1}\right)f\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)d\boldsymbol{x}_{1}d\boldsymbol{x}_{2}$,

where $V_{1}\left(\boldsymbol{x}_{1}\right)=\int u\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)f\left(\boldsymbol{x}_{2}|\boldsymbol{x}_{1}\right)d\boldsymbol{x}_{2}$ is her expected continuation value conditional on observing $\boldsymbol{x}_{1}$.

Proof. Let $f\left(\boldsymbol{x}_{1}\right)$ be the marginal density of $\boldsymbol{x}_{1}$, which satisfies $f\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)=f\left(\boldsymbol{x}_{1}\right)f\left(\boldsymbol{x}_{2}|\boldsymbol{x}_{1}\right)$ as well as $f\left(\boldsymbol{x}_{1}\right)=\int f\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)d\boldsymbol{x}_{2}$. The first definition of V_0 now implies that

$$\begin{aligned}V_{0}&=\int f\left(\boldsymbol{x}_{1}\right)V_{1}\left(\boldsymbol{x}_{1}\right)d\boldsymbol{x}_{1}\\&=\iint f\left(\boldsymbol{x}_{1},\boldsymbol{x}_{2}\right)V_{1}\left(\boldsymbol{x}_{1}\right)d\boldsymbol{x}_{1},\end{aligned}$$

as required. $\square$

Next, we apply this result to our model to show that Equations (26) and (OA.23) are equivalent.

Lemma OA-2. *The two following definitions of an investor's ex ante expected utility in Application 4 are equivalent:*

1. $V_0 = \iiiint u\left(c_2\left(\delta, \xi, \bar{\delta}, n\right)\right) f\left(\delta, \xi, \bar{\delta}, n\right) d\delta d\xi d\bar{\delta} dn$
2. $V_0 = \iiint V_1\left(\xi, \bar{\delta}, n\right) f\left(\xi, \bar{\delta}, n\right) d\xi d\bar{\delta} dn,$

where $V_1\left(\xi, \bar{\delta}, n\right)$ is her expected continuation value after observing her private signal ξ and a price equal to $q = q\left(\bar{\delta}, n\right)$.

Proof. In our setting, let $\boldsymbol{x}_1 = (q, \xi)$, where q is the random variable representing the equilibrium price, and $\boldsymbol{x}_2 = \left(\delta, \bar{\delta}, n\right)$. We can write the first definition of V_0 as

$$V_0 = \iint u\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right) f\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right) d\boldsymbol{x}_1 d\boldsymbol{x}_2$$

and apply our preliminary lemma to get

$$\begin{aligned} V_0 &= \iint V_1\left(\boldsymbol{x}_1\right) f\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right) d\boldsymbol{x}_1 d\boldsymbol{x}_2 \\ &= \int \cdots \int V_1\left(q, \xi\right) f\left(\delta, \xi, \bar{\delta}, n\right) f\left(q | \bar{\delta}, n\right) dq d\delta d\xi d\bar{\delta} dn, \end{aligned}$$

where $V_1\left(q, \xi\right)$ is the agent's continuation value after price q and private signal ξ. In equilibrium, $f\left(q | \bar{\delta}, n\right)$ is degenerate at $q = q\left(\bar{\delta}, n\right)$ and $f\left(\delta, \xi, \bar{\delta}, n\right)$ is degenerate at $\delta = \bar{\delta}$. Hence the quintuple integral above simplifies to the triple integral

$$V_0 = \iiint V_1\left(q\left(\bar{\delta}, n\right), \xi\right) f\left(\xi, \bar{\delta}, n\right) d\xi d\bar{\delta} dn$$

and the result now follows by changing the notation for the continuation value to $V_1\left(\xi, \bar{\delta}, n\right) \equiv V_1\left(q\left(\bar{\delta}, n\right), \xi\right)$. $\square$