

# Optimal Monetary Policy with Heterogeneous Agents: A Timeless Ramsey Approach

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## Abstract

This paper characterizes optimal monetary policy in a canonical heterogeneous-agent New Keynesian (“HANK”) model with wage rigidity. We develop a timeless Ramsey approach to study optimal long-run policy, time inconsistency, and optimal stabilization policy, as well as optimal policy under discretion. We show that *i*) zero inflation is the optimal long-run policy in our baseline model, *ii*) the standard inflation target is augmented by distributional considerations, *iii*) monetary policy in HANK faces a second source of time inconsistency that requires a new distributional target, *iv*) Divine Coincidence fails in heterogeneous-agent economies due to pecuniary externalities even in the absence of cost-push shocks, and *v*) there are gains from commitment relative to discretion even in the absence of cost-push shocks. We compute optimal stabilization policy in response to productivity, demand, and cost-push shocks both non-linearly and using sequence-space perturbation methods, which we extend to Ramsey problems and welfare analysis.

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# 1 Introduction

There is large heterogeneity in households' exposure to business cycle fluctuations. At the same time, there is now a growing consensus that monetary policy has distributional consequences — a view supported by mounting empirical evidence (Doepke and Schneider, 2006; Coibion et al., 2017; Ampudia et al., 2018) and the burgeoning heterogeneous-agent New Keynesian ("HANK") literature (McKay et al., 2016; Kaplan et al., 2018; Auclert, 2019; Auclert et al., 2020). Household heterogeneity may therefore be an important determinant of the welfare impact of monetary policy and should inform the study of optimal policy design.<sup>1</sup> Accounting for rich heterogeneity and incomplete markets in dynamic optimal policy problems has remained challenging, however, because the planner must internalize the effects of policy on an evolving cross-sectional distribution.

In this paper, we adopt a *timeless Ramsey approach* to study optimal monetary policy in heterogeneous-agent environments. This approach allows us to isolate three important dimensions of monetary policy design: long-run policy, time consistency and targeting rules, and stabilization policy. Household heterogeneity may interact with optimal policy considerations along each of these dimensions. We leverage the timeless Ramsey approach to systematically revisit the canonical New Keynesian consensus on optimal monetary policy (Clarida et al., 1999; Woodford, 2003; Galí, 2015) and study — both analytically and quantitatively — the implications of household heterogeneity. We introduce our approach and develop our analytical results in a one-asset HANK economy with wage rigidity, which represents a minimal departure from the representative-agent New Keynesian ("RANK") model. In our quantitative analysis, we compute optimal monetary policy both non-linearly and using sequence-space perturbation methods (Boppert et al., 2018; Auclert et al., 2021), which we extend to Ramsey problems and welfare analysis.<sup>2</sup>

The timeless Ramsey approach comprises three steps. Each step allows us to isolate one important dimension of monetary policy design: long-run policy, time consistency and targeting rules, and stabilization policy. Our goal in this paper is to study the implications of household heterogeneity for optimal policy along each of these dimensions.

In the first step, after identifying the implementability conditions that characterize equilibrium and serve as constraints for a Ramsey planner, we define a standard primal Ramsey problem and characterize its solution, which we refer to as a Ramsey plan. At this stage, Ramsey plans conflate three distinct economic motives that govern optimal policy dynamics — the incentive to respond to long-run distortions, time consistency problems, and the stabilization motive in response to shocks. We first solve for the stationary equilibrium that the Ramsey planner perceives

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<sup>1</sup> In fact, the Federal Reserve has started taking into account "distributional considerations." After concluding a long-run strategic review in August, 2020, Chairman Jerome Powell remarked: "*Our revised statement emphasizes that maximum employment is a broad-based and inclusive goal. This change reflects our appreciation for the benefits of a strong labor market, particularly for many in low- and moderate-income communities.*" The full speech can be found at: [https://www.federalreserve.gov/news\\_events/speech/powell20200827a.htm](https://www.federalreserve.gov/news_events/speech/powell20200827a.htm).

<sup>2</sup> In ongoing work, we extend our quantitative analysis to a state-of-the-art two-asset HANK model, augmented to capture economic forces that are important drivers of risk-sharing and redistribution considerations.

as optimal in the long run — a stationary Ramsey plan — and characterize the policy that supports it — a stationary Ramsey or optimal long-run policy. Crucially, the planner’s incentives to respond to transient shocks or a time consistency problem dissipate in the long run. Characterizing the stationary Ramsey plan consequently allows us to isolate those policy considerations governed by the distortions and inefficiencies that the planner perceives in the long run.

In the second step, after showing that Ramsey plans feature multiple dimensions of time inconsistency, we extend the approach of [Marcel and Marimon \(2019\)](#) to our setting (i.e., continuous-time heterogeneous-agent economies). We define a timeless Ramsey problem that augments the standard Ramsey problem with *timeless penalties* for each forward-looking implementability condition, and show that it yields time-consistent planning solutions. That is, the timeless Ramsey planner no longer has an incentive to deviate from the stationary Ramsey plan in the absence of shocks. We develop both a primal and a dual representation of the timeless Ramsey problem. Characterizing the timeless Ramsey plan and the timeless penalties that support it allows us to isolate the planner’s incentives to deviate from the stationary Ramsey policy in the short run due to time consistency problems in the presence of heterogeneity.

In the third and final step, we study optimal stabilization policy under the timeless Ramsey plan around the stationary Ramsey plan. Optimal policy dynamics in response to shocks under the appropriate timeless penalties are now driven solely by stabilization motives and are no longer confounded by considerations of long-run distortions and time inconsistency.

The timeless Ramsey approach is designed to find a stationary point and a set of penalties so that the Ramsey planner perceives no gain from time-inconsistent deviations in the absence of shocks. Importantly, it is not enough to characterize and compute such a stationary equilibrium and find the optimal long-run policy that supports it; it is also necessary to define an appropriate Lagrangian that is time-consistent around the stationary Ramsey plan, which requires computing the associated timeless penalties. Our approach then allows us to compute optimal stabilization policy, which is driven solely by stabilization motives in response to shocks.

While our approach allows us to characterize and compute timeless Ramsey plans non-linearly, an important contribution of this paper is to bring perturbation methods to bear on the question of optimal stabilization policy in heterogeneous-agent economies. Building on the increasingly popular literature on sequence-space methods, we develop a sequence-space representation of timeless Ramsey plans. Under this representation, a Ramsey plan is a system of equations that takes as inputs the time paths of aggregate allocations and prices, aggregate multipliers, policies, and shocks. While we can also leverage this representation to compute Ramsey plans non-linearly, we develop sequence-space perturbation methods — in both a primal and dual form — to approximate optimal stabilization policy to first order.<sup>3</sup> In the primal representation, we compute an extended set of sequence-space Jacobians and solve for the time paths of the multipliers that comprise a

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<sup>3</sup> Our paper presents three alternatives to compute optimal policy dynamics in response to shocks: a non-linear solution, as well as first-order approximations in a primal and dual form. In Section 3.4, we contrast the advantages and disadvantages of the different approaches. As we discuss there, no single method dominates.

Ramsey plan. In the dual representation, we avoid having to compute the time paths of multipliers. However, approximating optimal policy in the dual is no longer possible in terms of sequence-space Jacobians and instead requires a second-order analysis. To that end, we introduce sequence-space Hessians as the natural, second-order generalization of sequence-space Jacobians. Our paper therefore builds on [Boppart et al. \(2018\)](#) and [Auclert et al. \(2021\)](#) and extends the sequence-space apparatus to Ramsey problems and welfare analysis in heterogeneous-agent environments.

In the second half of our paper, we leverage the timeless Ramsey approach to systematically revisit — both analytically and quantitatively — five important results of the canonical New Keynesian consensus on optimal monetary policy. We characterize our economy’s representative-agent limit as the main comparison benchmark for our analysis. While we adopt a utilitarian objective as our baseline, we also consider the implications for optimal policy of alternative welfare criteria and distinct normative considerations in economies with individual heterogeneity, following [Dávila and Schaab \(2021\)](#).

First, we show that the optimal long-run policy in both the HANK and RANK versions of our model features zero optimal long-run inflation. This result critically hinges on the fact that the nominal interest rate and inflation enter the optimal policy problem symmetrically, which can be seen as a relevant benchmark. That is, in environments in which inflation and the nominal interest rate have a differential impact on different individuals in the economy, we should expect that optimal policy features non-zero inflation in the long run, even without other considerations that are known to affect the optimal long-run rate of inflation (e.g., a demand for fiat money).

Second, we also show that two different sources of time inconsistency — inflation and distributional — emerge in our model. The first source of time inconsistency arises because future inflation enters the Ramsey problem via the forward-looking New Keynesian wage Phillips curve. This source of time inconsistency has been widely studied in RANK economies and has been the subject of a vast literature following [Barro and Gordon \(1983\)](#). Our second result is that the standard inflation target, while taking the same linear penalty form, is now shaped by distributional considerations. In particular, the benchmark result of the representative-agent literature is that no inflation target is necessary in the absence of cost-push shocks when the planner can set an appropriate employment subsidy to offset markup distortions in steady state. No time consistency problem emerges in this case. This important benchmark result breaks down in heterogeneous-agent environments. We show that a utilitarian planner in our setting has an incentive to use inflation to depress the real interest rate and alleviate the financial burden of indebted households. Whenever a planner (central bank) values distributional considerations, the inflation target necessary to conduct time-consistent policy is shaped by and must account for this distributional motive.

A third insight of our paper is that a planner (central bank) that adopts a welfare criterion (mandate) that is not the aggregate efficiency one must also adopt a new *distributional target* in addition to the standard inflation target in order to implement time-consistent policy. In the presence of household heterogeneity, a second time consistency problem emerges that is entirely

absent from RANK economies (Acharya et al., 2020). The implementability conditions comprise a cross section of forward-looking equations that characterize individual consumption and savings decisions. The Ramsey planner in a HANK economy finds that households in general save too much or too little, in principle to different degrees. While the planner can use the nominal interest rate to control consumption on average, the monetary policy instrument is insufficient to adjust the consumption and savings decisions of different households to the degree the planner would like. Under commitment, the Ramsey planner therefore still finds it optimal to make promises about future interest rates, which in turn manifest via promises over household continuation values. It is through this mechanism that a planner generates distributional promises over time, opening the door to time inconsistency. Formally, we show that the distribution of promises associated with this form of time inconsistency evolves according to a particular Kolmogorov forward equation. We leverage this equation to characterize the distributional target that a planner must adopt to conduct time-consistent policy.

Our fourth insight summarizes the departures of optimal monetary stabilization policy from the representative-agent benchmark. In a RANK economy, the Divine Coincidence result establishes that no tradeoff emerges between inflation and output in the absence of cost-push shocks; the planner finds it optimal to close both the inflation and output gap at the same time (Blanchard and Galí, 2007). In HANK economies, on the other hand, Divine Coincidence generically fails even in the absence of cost push shocks. Due to a pecuniary externality, the planner always perceives a tradeoff between aggregate stabilization, i.e., inflation and output, on the one hand, and distributional considerations on the other hand. Accounting for such distributional considerations comes at the cost of aggregate efficiency. Indeed, from the perspective of a planner (central bank) that only values aggregate efficiency, the Divine Coincidence benchmark is restored in our setting.

Fifth and finally, we study optimal policy under discretion and characterize the gains from commitment. An important and well-understood insight from representative-agent analysis is that there are no gains from commitment in the absence of cost-push shocks when the planner sets an appropriate employment subsidy in steady state. No time consistency problem emerges in this case, and there are no gains from rules relative to discretion. This important benchmark result again fails in an environment with heterogeneous agents. In particular, we analytically characterize optimal policy under discretion in terms of a simple distributional wedge that we can sign unambiguously. Under discretion, a utilitarian planner in our baseline HANK economy always has an incentive to raise output above natural output, i.e., engineer a positive output gap, even in the absence of cost-push shocks. Intuitively, by raising inflation the planner can temporarily depress the real interest rate, which benefits indebted households. The welfare gains from commitment and rules consequently become larger in heterogeneous-agent environments.

We provide a quantitative analysis of our model in Section 5, where we leverage our timeless Ramsey approach to compute optimal monetary policy in response to three types of shocks — demand, productivity, and cost-push shocks. In the presence of heterogeneous households, the

departure of optimal stabilization policy from the representative-agent benchmark depends on the nature of the underlying shock. In response to a productivity shock, we find that the optimal stabilization policy broadly follows the same principles as in the RANK benchmark, with minor quantitative departures. The planner still stabilizes the output and (wage) inflation gaps, but not fully. In response to a demand shock, as in RANK, the planner optimally leans against the shock but partially accommodates it, allowing the economy to overheat in the short run. In response to a cost-push shock, optimal monetary policy eases substantially less than in RANK, allowing a sizable negative output gap to open.

In ongoing work, we show how to solve Ramsey problems for welfare criteria defined with dynamic-stochastic generalized welfare weights (Dávila and Schaab, 2021). The implications of household heterogeneity for optimal monetary policy depend crucially on the relative weights that a given welfare criterion attributes to different normative considerations — namely aggregate efficiency, risk-sharing, intertemporal-sharing, and redistribution. In the context of our model, we study how these different normative considerations interact with the three dimensions of optimal monetary policy design.

**Related literature.** Our paper contributes to multiple branches of the literature on optimal monetary policy. First and foremost, our paper contributes to the growing literature on optimal policy in HANK economies. This literature includes the work of Bhandari et al. (2021), who introduce a small-noise expansion method to compute optimal monetary and fiscal policy in HANK models; Acharya et al. (2020), who study optimal monetary policy in closed form in a HANK economy with constant absolute risk aversion (CARA) preferences and normally distributed shocks; Le Grand et al. (2021), who study optimal monetary and fiscal policy keeping heterogeneity finite-dimensional by truncating idiosyncratic histories; González et al. (2021), who study optimal monetary policy in an environment with firm heterogeneity; and McKay and Wolf (2022), who study optimal monetary policy with heterogeneous households in linear-quadratic environments.

Within this literature, our continuous-time formulation of the optimal monetary policy problem, treating the cross-sectional distribution of households as a control, is most closely related to the work of Nuño and Thomas (2020), on which we build.<sup>4</sup> Nuño and Thomas (2020) study optimal monetary policy in a small open economy, in which short-term real interest rates and output are unaffected by monetary policy.<sup>5</sup> A contribution of our paper is to study optimal monetary policy in a closed economy that features the classic output-inflation trade-off, which is central to the New Keynesian literature and allows us to characterize a new source of time inconsistency due to the presence of individual heterogeneity. As in their paper, we characterize fully dynamic,

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<sup>4</sup> See also Nuño and Moll (2018), who solve constrained-efficiency problems treating the cross-sectional distribution as a control.

<sup>5</sup> Formally, the open economy setup in Nuño and Thomas (2020) immediately implies that both the Lagrange multipliers of the households' HJB equation and their optimality condition — which correspond to  $\phi_t(a, z)$  and  $\chi_t(a, z)$  in our paper, see equation (20) — are zero by construction. Hence, in their model, the planner would make the same savings decisions as the households. Characterizing and computing these multipliers is one contribution of our paper.

non-linear Ramsey plans. We also develop sequence-space perturbation methods to compute optimal stabilization policy around the stationary Ramsey plan.

Both in the body of the paper (see Section 4) and in the Appendix, we purposefully relate our results to those of the vast literature on monetary policy in RANK models (Clarida et al., 1999; Woodford, 2003; Gali, 2015). By doing so we are able to provide clear analytical insights into the form of the optimal policy in a HANK environment and how it relates to the well-understood results in RANK economies. At an abstract level, our approach is closest to the work of Khan et al. (2003), who initially characterize standard and timeless Ramsey optimal policy using the exact structural equations and utility function, and then use perturbation methods to characterize the optimal responses to shocks. Schmitt-Grohé and Uribe (2010) and Woodford (2010) systematically study and review optimal long-run policy and optimal stabilization policy in RANK economies.

Our characterization of timeless penalties builds on the recursive multiplier approach of Marcer and Marimon (2019). We provide a novel application of their approach in a continuous-time environment where the planner must keep track of a full cross-sectional distribution and faces a continuum of individual forward-looking constraints. A central contribution of our paper is to show that the evolution of the distribution of promises associated with such forward-looking constraints under the optimal Ramsey plan satisfies a Kolmogorov forward equation that accounts for the “births” and “deaths” of promises.

Relative to the existing literature on optimal monetary policy in HANK, based on the results of Dávila and Schaab (2021), we provide a novel characterization of optimal policy for specific planners outside of the utilitarian case, such as the aggregate efficiency planner and the no-redistribution planner. As that paper explains, a planner who follows these welfare criteria only values specific motives in a four-way decomposition of welfare assessments (aggregate efficiency, risk-sharing, intertemporal-sharing, and redistribution).

Finally, we contribute to recent work on computational methods in heterogeneous-agent environments by extending the sequence-space apparatus to Ramsey problems and welfare analysis (Boppert et al., 2018; Auclert et al., 2021). We develop a sequence-space representation of timeless Ramsey plans, which we can compute non-linearly or using sequence-space perturbation methods. In particular, we extend the fake-news algorithm of Auclert et al. (2021) to compute Ramsey problems in both primal and dual form.

**Outline.** Section 2 presents the baseline model, defines and characterizes an equilibrium, and discusses the sources of suboptimality in the model. Section 3 introduces our timeless Ramsey approach to optimal monetary policy with heterogeneous agents. Section 4 presents a variety of analytical results and discusses qualitatively the departures of optimal policy in our setting from the representative-agent benchmark. Section 5 presents our quantitative results and Section 6 concludes. All proofs and derivations are in the Appendix. The Appendix also includes several extensions and additional results.

## 2 Baseline Model

Our baseline model is a one-asset heterogeneous-agent New Keynesian (“HANK”) model with wage rigidity. This baseline model is deliberately stylized to make our characterization of optimal monetary policy in Sections 3 and 4 accessible. Our baseline model moreover represents a minimal departure from a representative-agent New Keynesian (“RANK”) model (Clarida et al., 1999; Woodford, 2003; Galí, 2015), which we use as a benchmark for our analytical and quantitative results. In ongoing work, we also study optimal monetary policy in a state-of-the-art two-asset HANK model augmented to capture economic forces that are important drivers of risk-sharing and redistribution considerations.

We cast our baseline model in continuous time. The time horizon is infinite with  $t \in [0, \infty)$ . There is no aggregate uncertainty and we focus on one-time, unanticipated shocks. In particular, following much of the standard New Keynesian literature, we allow for three types of shocks: demand, productivity, and cost-push shocks.

### 2.1 Households

The economy is populated by a unit mass of households who consume and work. Household preferences are given by

$$\mathbb{E}_0 \int_0^\infty e^{-\int_0^t \rho_s ds} \left[ u(c_t) - \Phi \left( \left\{ n_{k,t}, \pi_{k,t}^w \right\}_{k \in [0,1]} \right) \right] dt, \quad (1)$$

where  $c_t$  is the rate of consumption and  $u(\cdot)$  captures the instantaneous utility flow from consumption.  $\rho_t$  is a common but potentially time-varying discount rate and represents a source of demand shocks. The function  $\Phi(\cdot)$  captures the household’s disutility from work and its details depend on the wage bargaining and labor market structure, which we discuss below. In particular,  $n_{k,t}$  denotes the hours that the household supplies to union  $k$  and  $\pi_{k,t}^w$  is union  $k$ ’s wage inflation, which enters  $\Phi(\cdot)$  as an additional cost as we describe below.

Each household supplies labor to all of  $k \in [0, 1]$  unions. We denote a household’s total hours of work by

$$n_t = \int_0^1 n_{k,t} dk.$$

Each union pays the household a nominal wage  $W_{k,t}$ . The household budget constraint therefore corresponds to

$$\dot{a}_t = r_t a_t + z_t \frac{1}{P_t} \int_0^1 W_{k,t} n_{k,t} dk + \tau_t(z_t) - c_t, \quad (2)$$

where  $a_t$  denotes bond holdings,  $r_t$  the real interest rate, and  $P_t$  is the price of the consumption good. Households’ non-financial income comprises labor income, which is proportional to idiosyncratic labor productivity  $z_t$ , and lump-sum rebates  $\tau_t(z_t)$ , which may also depend on  $z_t$ . Finally,

households face a borrowing constraint given by

$$a_t \geq \underline{a},$$

where  $\underline{a} \leq 0$ .

Even though there is no aggregate uncertainty, households face idiosyncratic earnings risk. We assume that labor productivity  $z_t$  follows an exogenous Markov process that we further specialize below. Since households are only heterogeneous *ex post*, we can index individual households by their idiosyncratic state variables  $(a, z)$ . We denote the mass of households with idiosyncratic states  $(a, z)$  by  $g_t(a, z)$ , which we also refer to as the cross-sectional distribution. And since our economy is populated by a measure 1 of infinitely-lived households, we have  $\iint g_t(a, z) da dz = 1$ .

## 2.2 Labor Market

As is standard in the New Keynesian sticky-wage literatures without heterogeneity (Erceg et al., 2000; Schmitt-Grohé and Uribe, 2005) and with heterogeneity (Auclert et al., 2020), labor unions determine work hours. Each union  $k \in [0, 1]$  transforms hours supplied by households into a differentiated labor service according to the linear aggregation technology

$$N_{k,t} = \iint z n_{k,t} g_t(a, z) da dz,$$

where  $N_{k,t}$  is expressed in units of effective labor. Each union also rations labor, so that all households work the same hours. In particular, this implies  $N_{k,t} = n_{k,t} \iint z g_t(a, z) da dz = n_{k,t}$ , after normalizing cross-sectional average labor productivity to 1.

**Labor packer.** Unions sell their differentiated labor services to an aggregate labor packer. The packer operates the CES aggregation technology

$$N_t = \left( \int_0^1 N_{k,t}^{\frac{\epsilon_t-1}{\epsilon_t}} dk \right)^{\frac{\epsilon_t}{\epsilon_t-1}},$$

where the elasticity of substitution  $\epsilon_t$  is potentially time-varying. We interpret time variation in the desired wage mark-up of unions as a source of cost-push shocks, following standard practice in the literature (see, e.g., Galí, 2015). The packer sells the aggregate labor bundle to firms at nominal wage rate  $W_t$ . The labor packer's cost-minimization problem is standard and yields the demand

function and wage index

$$N_{k,t} = \left( \frac{W_{k,t}}{W_t} \right)^{-\epsilon_t} N_t \quad (3)$$

$$W_t = \left( \int_0^1 W_{k,t}^{1-\epsilon_t} dk \right)^{\frac{1}{1-\epsilon_t}}, \quad (4)$$

where  $W_{k,t}$  is the nominal wage rate charged by union  $k$ .

**Wage rigidity.** Nominal wages are sticky in our model. Each union  $k$  faces an adjustment cost to change its wage. Formally, the union takes  $W_{k,t}$  as a state variable and controls how the wage evolves by setting wage inflation  $\pi_{k,t}^w$ , with

$$\pi_{k,t}^w = \frac{\dot{W}_{k,t}}{W_{k,t}}. \quad (5)$$

In our baseline model, the union's adjustment cost is directly passed to union members as a quadratic utility cost, so  $\Phi(\cdot)$  in equation (1) is explicitly given by

$$\Phi\left(\left\{n_{k,t}, \pi_{k,t}^w\right\}_{k \in [0,1]}\right) = v\left(\int_0^1 n_{k,t} dk\right) + \frac{\delta}{2} \int_0^1 (\pi_{k,t}^w)^2 dk,$$

where  $v(\cdot)$  captures pure disutility from working and  $\delta$  modulates the strength of the wage rigidity.

**Employment subsidy.** As is standard in the New Keynesian literature, we allow for an employment subsidy. Given union wage receipts  $z_t W_{k,t} n_{k,t}$  to a household with labor productivity  $z_t$ , the government pays the household a proportional income subsidy  $\tau^L z_t W_{k,t} n_{k,t}$ , which the union internalizes when setting wages.

**Wage Phillips curve.** We now formalize the union's wage setting problem to derive a New Keynesian wage Phillips curve. We assume that the union chooses wages in order to maximize stakeholder value — the sum of stakeholders', i.e., union members', utilities.<sup>6</sup> That is, union  $k$  solves

$$\max_{\pi_{k,t}^w} \int_0^\infty e^{-\int_0^t \rho_s ds} \left( \iint \left[ u(c_t(a, z; W_{k,t})) - v\left(\int_0^1 n_{k,t} dk\right) - \frac{\delta}{2} \int_0^1 (\pi_{k,t}^w)^2 dk \right] g_t(a, z) da dz \right) dt \quad (6)$$

subject to equations (3) and (5). The union further internalizes the effect of its wage policy on its members' consumption — hence the explicit dependence of  $c_t$  on  $W_{k,t}$  in equation (6). However, since union  $k$  is small, it takes as given all macroeconomic aggregates, including the cross-sectional

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<sup>6</sup> The choice of objective by a union under incomplete markets is subject to the same caveats as the choice of objective by a firm. It is straightforward to consider alternative objectives.

household distribution. We solve the union's problem in Section C of the Appendix, where we also derive wage Phillips curves under alternative assumptions on wage adjustment costs.<sup>7</sup>

There, we show that the wage policies that result from the union's problem give rise to a symmetric equilibrium, where wages and labor allocations are equalized across unions, i.e.,  $W_{k,t} = W_t$  and  $N_{k,t} = N_t$ . In such a symmetric equilibrium, the non-linear New Keynesian wage Phillips curve is given by

$$\dot{\pi}_t^w = \rho_t \pi_t^w + \frac{\epsilon_t}{\delta} \left[ \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) w_t \Lambda_t - v'(N_t) \right] N_t, \quad (7)$$

where we define  $\Lambda_t$  as

$$\Lambda_t = \iint z u'(c_t(a, z)) g_t(a, z) da dz. \quad (8)$$

Equation (7) highlights that demand and cost-push shocks directly affect inflation dynamics.

### 2.3 Final Good Producer

A representative firm produces the final consumption good, operating the linear production technology

$$Y_t = A_t N_t, \quad (9)$$

where the aggregate labor bundle is the only input to production. We refer to  $A_t$  as total factor productivity (TFP), which is potentially time-varying and represents a source of productivity shocks. Under perfect competition and flexible prices, the real marginal cost of labor is equal to its marginal product, with

$$w_t = A_t \quad (10)$$

where  $w_t = \frac{W_t}{P_t}$  denotes the real wage. Moreover, profits are zero, which is consistent with the absence of profits in equation (2).

### 2.4 Government

We keep the role of the fiscal authority deliberately minimal in our baseline model. Our focus is on the monetary authority, which optimally sets the nominal interest rate.

**Fiscal policy.** There is no government spending and no debt, with bonds in zero net supply. The fiscal authority pays an employment subsidy to households. Running a balanced budget, it pays for these outlays with a lump-sum tax based on aggregate employment. We assume that both the

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<sup>7</sup> There are three natural ways to model wage adjustment costs: as an explicit resource cost that is passed on to households, as labor productivity distortions, or as a direct utility cost. In the main text, we adopt the utility cost specification largely because it is more tractable. For robustness, in the Appendix, we derive alternative Phillips curves under different assumptions, and discuss how they impact our conclusions.

subsidy and the tax are proportional to a household's labor productivity. That is, the net fiscal rebate that a household with idiosyncratic labor productivity  $z$  receives is zero, with

$$P_t \tau_t(z) = \int_0^1 \tau^L z W_{k,t} n_{k,t} dk - \tau^L z W_t N_t = 0.$$

Given this form of fiscal policy and the structure of the labor market, we can simplify and rewrite the household budget constraint as

$$\dot{a}_t = r_t a_t + z_t w_t N_t - c_t.$$

**Monetary policy.** The central bank sets the nominal interest rate  $i_t$  optimally as we describe in the next section. A Fisher relation holds in our economy, with

$$r_t = i_t - \pi_t, \quad (11)$$

where  $\pi_t$  is consumer price index (CPI) inflation. Finally, we can relate price inflation to wage inflation by differentiating equation (10), which yields

$$\pi_t = \pi_t^w - \frac{\dot{A}_t}{A_t}. \quad (12)$$

## 2.5 Equilibrium

**Definition 1. (Competitive Equilibrium)** Given an initial distribution over household bond holdings and idiosyncratic labor productivities,  $g_0(a, z)$ , a symmetric initial nominal wage distribution,  $W_{k,0} = W_0$ , and given predetermined sequences of monetary policy  $\{i_t\}$  and shocks  $\{A_t, \rho_t, \epsilon_t\}$ , an equilibrium is defined as paths for prices  $\{\pi_t^w, \pi_t, w_t, r_t\}$ , aggregates  $\{Y_t, N_t, C_t, B_t\}$ , individual allocation rules  $\{c_t(a, z)\}$ , and for the joint distribution over household bond holdings and idiosyncratic labor productivities  $\{g_t(a, z)\}$ , such that households optimize, unions optimize, final good producers optimize, and markets clear, that is,

$$Y_t = C_t = \iint c_t(a, z) g_t(a, z) da dz \quad (13)$$

$$0 = B_t = \iint a g_t(a, z) da dz. \quad (14)$$

In Lemma 13 in the Appendix, we provide a complete description of each of the equations that characterize an equilibrium and formally describe the implementability conditions that act as constraints for a Ramsey planner. It is helpful to separate the constraints into two blocks: an individual or “micro” block of three equations, which characterizes household consumption and utility, as well as the law of motion of the distribution of households, and a “macro block”, of two equations, which characterizes the economy’s aggregate resource constraint and the evolution

of inflation. While the macro block is almost identical to the RANK version of our model, as we explain below, the individual block is richer.

**Micro block.** The individual block consists of three equations. First, there is the households' Hamilton-Jacobi-Bellman equation, given by

$$\rho_t V_t(a, z) = \partial_t V_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2}(\pi_t^w)^2 + \mathcal{A}_t V_t(a, z), \quad (15)$$

where  $V_t(a, z)$  denotes the lifetime utility of a household with bond holdings and idiosyncratic labor productivity  $(a, z)$ , and where  $\mathcal{A}_t$  denotes the infinitesimal generator of the process  $(a_t, z_t)$ , formally defined in equation (53) in Appendix A.<sup>8</sup> Second, there is the savings optimality condition that relates the household's consumption policy function to the value function, and is given by

$$u'(c_t(a, z)) = \partial_a V_t(a, z). \quad (16)$$

Finally, there is the Kolmogorov forward equation (KFE), which describes the time evolution of the cross-sectional distribution of households over the possible individual states,  $(a, z)$ , given by

$$\partial_t g_t(a, z) = \mathcal{A}_t^* g_t(a, z), \quad (17)$$

where  $\mathcal{A}_t^*$  denotes the adjoint of  $\mathcal{A}_t$ .<sup>9</sup> Intuitively, the operator  $\mathcal{A}_t$  accounts for the fact that households idiosyncratic states vary over time. In particular, it encapsulates how a household's value function accounts for the path of future idiosyncratic productivity shocks and bond holdings. In turn, the adjoint  $\mathcal{A}_t^*$  keeps track of how the distribution of households over the set of idiosyncratic states varies over time. Hence, it is natural for  $\mathcal{A}_t^*$  to be the adjoint of  $\mathcal{A}_t$ .

**Macro block.** The macro block comprises the aggregate resource constraint of the economy, which we obtain by combining the goods market clearing condition (13) with the aggregate production function (9),

$$\iint c_t(a, z) g_t(a, z) da dz = A_t N_t, \quad (18)$$

and the New Keynesian wage Phillips curve (7).

## 2.6 Sources of Suboptimality

Before characterizing optimal policy, it is worth describing the sources of suboptimality in our baseline HANK model with wage rigidity. This economy features four sources of inefficiency,

<sup>8</sup> We adopt the more compact notation  $\partial_x f$  to represent  $\frac{\partial f}{\partial x}$ .

<sup>9</sup> The adjoint of an operator can be seen as a generalization of the transpose of a matrix.

whose implications shape different dimensions of optimal policy, as we substantiate below.<sup>10</sup>

First, the model features monopolistic competition. Labor unions are monopolistically competitive and charge a wage markup relative to the perceived (utility) marginal cost of work hours by members. This wage markup drives a wedge between the marginal rate of transformation (MRT),  $A_t$ , and the economy's average marginal rate of substitution (MRS),  $\Lambda_t$ . When we set the employment subsidy so that  $\frac{\epsilon-1}{\epsilon}(1 + \tau^L) = 1$ , the stationary equilibrium with constant elasticity of substitution  $\epsilon$  features no wage markups.

Second, the model has a nominal rigidity. Nominal wages are sticky and can only be adjusted gradually in our model. This imposes two separate costs from a welfare perspective. First, assuming a steady state employment subsidy, the economy's average MRS converges only gradually to the MRT in response to shocks. Second, wage adjustments are directly associated with a deadweight (utility) cost.

Third, our model features labor rationing. We assume that unions ration work hours across their members, imposing that all households supply the same number of hours. While an appropriate notion of *average* MRS is equal to the MRT in our economy whenever the employment subsidy is in place, individual MRS are not equalized across households.

Finally, there are incomplete markets for risk. Noncontingent bonds are the only financial asset in this economy. Households furthermore face a borrowing constraint and are consequently not able to fully insure against idiosyncratic earnings risk. As a result, households' marginal rates of substitution are not equalized across periods and states.

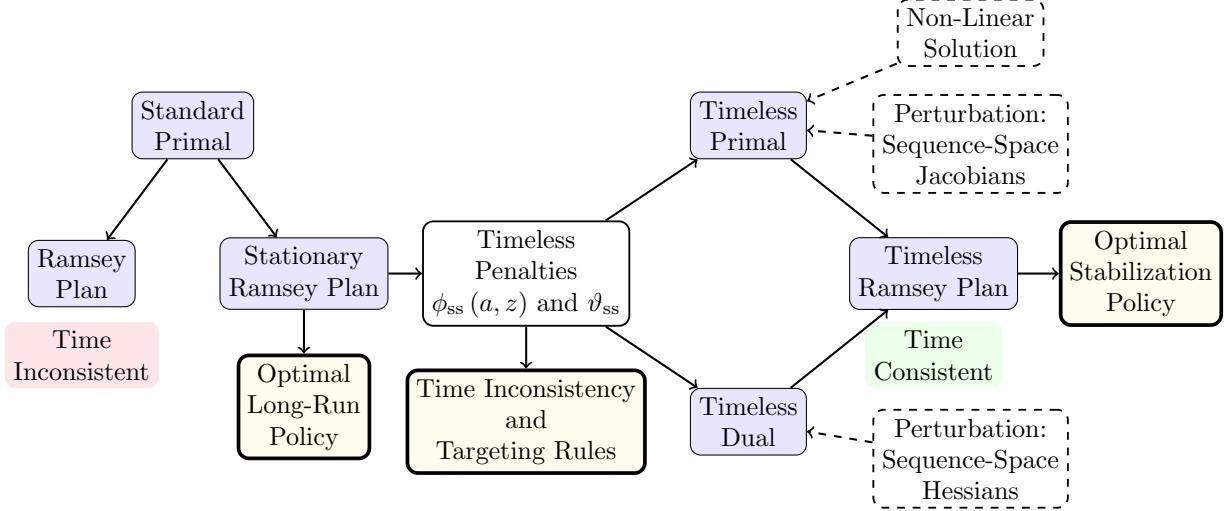
The first two sources of inefficiency are also present in the representative-agent version of our economy, as we show below in Section 4. In fact, these two distortions exactly mirror the sources of inefficiency in the standard RANK model (Clarida et al., 1999; Galí, 2015). The latter two sources of inefficiency are unique to the heterogeneous-agent environment. Due to these two inefficiencies, the flexible-price allocation with an employment subsidy is no longer first-best in a HANK economy, as we show below. Consequently, the celebrated Divine Coincidence result (Blanchard and Galí, 2007) fails in our environment, as we show in Section 4.

### 3 Optimal Monetary Policy: A Timeless Ramsey Approach

In this section, we develop a *timeless Ramsey approach* to characterizing optimal monetary policy in heterogeneous-agent economies. We proceed in three steps. First, in Section 3.1, we define a standard primal Ramsey problem, which in turn allows us to define and characterize Ramsey plans and stationary Ramsey plans. Second, in Sections 3.2 and 3.3, we show that Ramsey plans feature multiple dimensions of time inconsistency. In order to find time-consistent planning solutions, we extend the approach of Marimon (2019) to our setting (i.e., continuous-

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<sup>10</sup> This subsection is meant to parallel Section 4 of Khan et al. (2003) and Chapter 4.2 of Galí (2015), which discuss sources of suboptimality in RANK economies.



**Figure 1.** Illustration of the Timeless Ramsey Approach

**Note:** Figure 1 illustrates the approach to optimal policy developed in this paper. First, we define a standard primal Ramsey problem, which allows us to characterize optimal Ramsey plans, which are time inconsistent, and stationary Ramsey plans, which define the long-run optimal policy. By constructing appropriate timeless penalties —  $\phi_{ss}(a, z)$  for the distribution of individual savings and  $\vartheta_{ss}$  for inflation — which guarantee that a planner does not want to deviate from the stationary Ramsey plan in the absence of shocks, we study the question of time inconsistency and the determination of optimal targeting rules. Taking the timeless penalties as an input, we define timeless primal and dual Ramsey problems, which in turn allow us to characterize a timeless Ramsey plan, which is time-consistent. Finally, we study optimal stabilization policy in response to specific shocks (demand, productivity, and cost-push shocks). We compute Ramsey plans using three different methods, whose advantages and disadvantages we discuss in Section 3.4.4. First, fully nonlinearly in the primal. Second, using sequence-space Jacobians in the primal. Third, using sequence-space Hessians in the dual.

time heterogeneous-agent economies) by introducing timeless penalties for each forward-looking implementability condition. We then define a timeless Ramsey problem, which augments the standard Ramsey problem with the timeless penalty, and prove that it is time-consistent. That is, the timeless Ramsey planner no longer has an incentive to deviate from the stationary Ramsey plan in the absence of shocks. We develop both a primal and a dual representation of the timeless Ramsey problem. Third, in Section 3.4, we study optimal stabilization policy under the timeless Ramsey problem around the stationary Ramsey plan. We work in the sequence-space representation of our model and show how to extend the sequence-space apparatus (Boppart et al., 2018; Auclert et al., 2021) to Ramsey problems and welfare analysis.

Each of the three steps of our approach allows us to isolate one important dimension of optimal monetary policy design. Characterizing the stationary Ramsey plan allows us to solve for optimal long-run policy, with which the planner addresses distortions and inefficiencies she perceives in a stationary equilibrium. Characterizing the timeless Ramsey plan and the timeless penalties that support it allows us to isolate the planner’s incentives to deviate from the stationary Ramsey plan in the short run due to time consistency problems. And finally, by characterizing optimal stabilization policy with the appropriate timeless penalty around the stationary Ramsey

plan, we isolate the planner's stabilization motive and no longer confound it with considerations of long-run distortions and time inconsistency.

### 3.1 Ramsey Plan and Stationary Ramsey Plan

In general, a Ramsey planner chooses policy in order to maximize a particular objective subject to a set of conditions that define equilibria. Here, we assume that a planner seeks to maximize a utilitarian social welfare function. In ongoing work, we extend our analysis to compute welfare under general dynamic-stochastic weights (Dávila and Schaab, 2021), which allows us to consider welfare criteria that exclusively value a subset of normative considerations among aggregate efficiency, risk-sharing, intertemporal-sharing, and redistribution.<sup>11</sup> Lemma 13 in the Appendix formally describes the set of implementability conditions that constrain the standard primal Ramsey problem, which we define next. In this definition, the superscript SP stands for standard primal.<sup>12</sup>

**Definition 2. (Standard Primal Ramsey Problem / Ramsey Plan)**

- a) *A standard primal Ramsey problem solves*

$$\min_{\{\phi_t(a,z), \chi_t(a,z), \lambda_t(a,z), \mu_t, \vartheta_t\}} \max_{\{c_t(a,z), V_t(a,z), g_t(a,z), N_t, \pi_t^w, i_t\}} L^{\text{SP}}(g_0(\cdot)), \quad (19)$$

where  $L^{\text{SP}}(g_0(\cdot))$  denotes the planner's Lagrangian, given an initial distribution of bond holdings and

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<sup>11</sup> As shown in Dávila and Schaab (2021), a planner with a conventional utilitarian objective of the form assumed here values aggregate efficiency, risk-sharing, intertemporal-sharing, and redistribution motives. In ongoing work, we compute optimal policy for the welfare criteria introduced in that paper that exclusively value a subset of these motives, e.g., aggregate efficiency planner or no-redistribution planner.

<sup>12</sup> Throughout the paper, we say that a planning problem is in primal form when allocations or prices are explicit control variables for a planner, perhaps in addition to policy instruments. Alternatively, we say that a planning problem is in dual form when the only explicit control variables for a planner are policy instruments. This terminology is consistent with standard use in related environments, e.g., Chari and Kehoe (1999) and Ljungqvist and Sargent (2018).

idiosyncratic labor productivity  $g_0(a, z)$ :

$$\begin{aligned}
L^{\text{SP}}(g_0(\cdot)) = & \int_0^\infty e^{-\int_0^t \rho_s ds} \left\{ \iint \left\{ \left[ u(c_t(a, z)) - v(N_t) - \frac{\delta}{2}(\pi_t^w)^2 \right] g_t(a, z) \right. \right. \\
& + \phi_t(a, z) \begin{bmatrix} -\rho_t V_t(a, z) + \partial_t V_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2}(\pi_t^w)^2 \\ +(r_t a + z A_t N_t - c_t(a, z)) \partial_a V_t(a, z) + \mathcal{A}^z V_t(a, z) \end{bmatrix} \\
& + \chi_t(a, z) \left[ u'(c_t(a, z)) - \partial_a V_t(a, z) \right] \\
& + \lambda_t(a, z) \left[ -\partial_t g_t(a, z) - \partial_a \left( (r_t a + z A_t N_t - c_t(a, z)) g_t(a, z) \right) + \mathcal{A}^{z,*} g_t(a, z) \right] \left. \right\} da dz \\
& - \mu_t \left[ \iint c_t(a, z) g_t(a, z) da dz - A_t N_t \right] \\
& \left. + \vartheta_t \left[ -\partial_t \pi_t^w + \rho_t \pi_t^w + \frac{\epsilon_t}{\delta} \left( \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) A_t \Lambda_t - v'(N_t) \right) N_t \right] \right\} dt, \quad (20)
\end{aligned}$$

where the real interest rate satisfies  $r_t = i_t - \pi_t^w + \frac{\dot{\Lambda}_t}{\Lambda_t}$ , where  $\Lambda_t = \iint z u'(c_t(a, z)) g_t(a, z) da dz$ , and where we decompose the operators  $\mathcal{A}_t$  and  $\mathcal{A}_t^*$  as follows:  $\mathcal{A}_t = \mathcal{A}_t^a + \mathcal{A}^z$  and  $\mathcal{A}_t^* = \mathcal{A}_t^{a,*} + \mathcal{A}^{z,*}$ , where each component is defined in Appendix A.

- b) A Ramsey plan, which corresponds to the solution to this problem, is given by i) paths for prices,  $\pi_t^w$ , aggregates,  $N_t$ , individual consumption allocations and value functions,  $c_t(a, z)$  and  $V_t(a, z)$ , and for the distribution of bond holdings and idiosyncratic labor productivities,  $g_t(a, z)$ , that satisfy the competitive equilibrium conditions given a path of interest rate policy  $i_t$  and shocks ( $A_t, \rho_t, \epsilon_t$ ), as well as an initial distribution  $g_0(a, z)$ ; ii) a path of interest rate policy  $i_t$ , and iii) a sequence of multiplier functions,  $\phi_t(a, z)$ ,  $\chi_t(a, z)$ ,  $\lambda_t(a, z)$ ,  $\mu_t$ , and  $\vartheta_t$  that solve (19) for a given initial distribution  $g_0(a, z)$ .

The implementability constraints incorporate all of the conditions that characterize the competitive equilibrium of the model. In particular, the first three sets of constraints map to equations (15) through (17), described above. Importantly, these constraints apply to every pair of individual states  $(a, z)$ . The last two constraints are of aggregate nature: one is the aggregate resource constraint and the other one is the New Keynesian wage Phillips curve, introduced in equations (18) and (7), respectively. It is worth highlighting that this is a minimal characterization of the Ramsey problem, in the sense that there are no redundant constraints that can be easily deduced from other constraints.<sup>13</sup>

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<sup>13</sup> We could have defined an alternative Ramsey problem that includes households' consumption Euler equations instead of their HJB equations and consumption optimality conditions. As it will become clear shortly, our formulation provides valuable analytical insights, by virtue of having two sets of multipliers associated with households' lifetime value and consumption-savings decisions,  $\phi_t(a, z)$  and  $\chi_t(a, z)$ . Our formulation is also computationally more tractable.

In Proposition 1 we characterize the optimality conditions of the Standard Primal Ramsey Problem. Our derivation relies on a variational approach, formally developed in Appendix A.

**Proposition 1. (Standard Primal Ramsey Problem: Optimality Conditions)** *The optimality conditions for the standard primal Ramsey problem are given by*

$$\partial_t \phi_t(a, z) = -\mathcal{A}_t^* \phi_t(a, z) + \partial_a \chi_t(a, z) \quad (21)$$

$$\chi_t(a, z) = -\frac{1}{u''(c_t(a, z))} \left( \begin{array}{l} u'(c_t(a, z)) - \partial_a \lambda_t(a, z) - \mu_t \\ + \vartheta_t \frac{\epsilon_t}{\delta} \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) A_t N_t z u''(c_t(a, z)) \end{array} \right) g_t(a, z) \quad (22)$$

$$\begin{aligned} \rho_t \lambda_t(a, z) &= \partial_t \lambda_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2} (\pi_t^w)^2 - \mu_t c_t(a, z) \\ &\quad + \vartheta_t \frac{\epsilon_t}{\delta} \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) A_t N_t z u'(c_t(a, z)) + \mathcal{A}_t \lambda_t(a, z) \end{aligned} \quad (23)$$

$$\begin{aligned} 0 &= \mu_t - \frac{v'(N_t)}{A_t} \left( 1 + \iint \phi_t(a, z) da dz \right) \\ &\quad + \iint \left( z \phi_t(a, z) \partial_a V_t(a, z) + z \partial_a \lambda_t(a, z) g_t(a, z) \right) da dz \\ &\quad + \vartheta_t \frac{\epsilon_t}{\delta} \left( \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) A_t \Lambda_t - v'(N_t) - v''(N_t) N_t \right) \end{aligned} \quad (24)$$

$$\dot{\vartheta}_t = \delta \pi_t^w + \iint \left( a \phi_t(a, z) \partial_a V_t(a, z) + a \partial_a \lambda_t(a, z) g_t(a, z) \right) da dz \quad (25)$$

$$0 = \iint \left( a \phi_t(a, z) \partial_a V_t(a, z) + a \partial_a \lambda_t(a, z) g_t(a, z) \right) da dz \quad (26)$$

as well as a set of initial conditions for the multipliers on forward-looking implementability conditions

$$0 = \vartheta_0 \quad (27)$$

$$0 = \phi_0(a, z). \quad (28)$$

These optimality conditions hold everywhere in the interior of the idiosyncratic state space. For a formal treatment of boundary conditions, see Appendices A.3 through A.6.

Equations (21), (22), (23), (24), and (25) respectively correspond to the optimality conditions for the value function, consumption, the cross-sectional distribution of bond holdings and idiosyncratic labor productivities, aggregate labor, and wage inflation. Finally, equation (26) corresponds to the optimality condition for the nominal interest rate, i.e., monetary policy. We present the proof of Proposition 1 first in continuous time in Appendix A.2 for the interior of the idiosyncratic state space. We then provide a formal treatment of boundary conditions in Appendices A.3 through A.6.

Here, we briefly describe several insights that directly emerge from the six optimality conditions of the Ramsey problem, introduced in Proposition 1. Additional insights will emerge in Section 4, where we compare in detail optimal Ramsey policies in HANK economies with those in RANK economies.

First, note that the multipliers on the forward-looking constraints,  $\phi_t(a, z)$  for households' lifetime utility and  $\vartheta_t$  for the Phillips curve, have the dual interpretation of i) promises that a planner must satisfy or ii) penalties that a planner faces when choosing households' utility or inflation, respectively. Throughout the paper, we refer to  $\vartheta_t$  as an inflation penalty or target, and to  $\phi_t(a, z)$  as a distributional penalty or target.

Second, the sign of  $\chi_t(a, z)$ , which is the multiplier on the households' optimal consumption-savings decisions, determines whether a particular household is over-borrowing or under-borrowing from a planner's perspective. When  $\chi_t(a, z) < 0$ , households with bond holdings  $a$  and labor productivity  $z$  are over-borrowing (equivalently, under-saving). Alternatively, when  $\chi_t(a, z) > 0$ , such households are under-borrowing (over-saving). In Section 4 — see in particular Figure 2 — we further describe the properties and determinants of  $\chi_t(a, z)$ . We will find that all households under-save in our model, although to different degrees.

Third, note that equation (21), which is central to this paper, has the form of a Kolmogorov forward equation augmented to account for births and deaths. This equation tightly connects the multipliers on households' lifetime utility  $\phi_t(a, z)$ , which encode how all *lifetime* policies determine households' lifetime utility with the multipliers on households' optimal consumption-savings decisions,  $\chi_t(a, z)$ , which capture whether the planner perceives that a household is *currently* over- or under-saving. Intuitively, equation (21) is the key promise-keeping relation that a Ramsey planner — which has commitment — must satisfy. In particular, the evolution of the distribution of distributional penalties must be consistent with the evolution of households across idiosyncratic states, via  $\mathcal{A}^*$ , but also account for the birth of promises, captured by the term  $\partial_a \chi_t(a, z)$ .

Fourth, the multiplier  $\lambda_t(a, z)$  that solves equation (23) can be interpreted as the social lifetime value of an individual with idiosyncratic state  $(a, z)$ . When compared to the individual HJB — in equation (15) — the social HJB includes two additional flow terms, associated with the terms multiplying  $\mu_t$  and  $\vartheta_t$ . These terms appear in the social lifetime value but not the private one because they capture the general equilibrium impact of private decisions, via aggregate consumption and inflation, which the planner internalizes.

Fifth, equation (24) corresponds to the optimality condition for aggregate labor, or equivalently, aggregate output. This condition equals the social marginal benefit of aggregate output, given by  $\mu_t$ , with the marginal cost of doing so. Such cost includes the marginal cost of labor,  $\frac{\sigma'(N_t)}{A_t}$ , but also include the distributional impact of higher output on households and on the evolution of inflation. This equation is the foundation of our conclusion regarding optimal stabilization policy in Section 3.4.

Sixth, note that the optimality condition for inflation (25) simplifies to

$$\dot{\vartheta}_t = \delta \pi_t^w \quad (29)$$

whenever the average of the distributional penalties adds up to zero, that is,

$$\iint \phi_t(a, z) da dz = 0, \forall t. \quad (30)$$

While we have formally proven that this condition is satisfied at a stationary Ramsey plan (defined below), we strongly conjecture that it holds everywhere, as we have verified numerically. Moreover, whenever the distributional penalties cancel on average, equation (29) will be identical to the analogous equation in RANK, as we show in Section 4. In turn, equation (30) has the interpretation that a planner does not over- or under-promise on aggregate in terms of lifetime utilities.

Finally, the optimality condition for the interest rate (26) captures the marginal social value of increasing interest rates. Its second term,

$$\iint a \partial_a \lambda_t(a, z) g_t(a, z) da dz, \quad (31)$$

is central to how a planner conducts policy in this model. This term has the interpretation of the distributive pecuniary effect of an interest rate change in this model. If the planner valued a dollar across households identically, that is,  $\partial_a \lambda_t(a, z) g_t(a, z)$  is constant, this term is exactly zero.<sup>14</sup> In general, we expect the planner to find it desirable to redistribute resources from households who save, with  $a > 0$  and a low social marginal value of wealth, to households who borrow, with  $a < 0$  and a high social marginal value of wealth. This force will create a force towards lowering interest rates, to make borrowing cheaper. The first term of equation (26) captures how changing interest rates affects the dynamics of households across the wealth distribution.

We are ready to formally define a stationary Ramsey plan, towards which a Ramsey plan may converge when all shocks,  $\{A_t, \rho_t, \epsilon_t\}$ , converge as  $t \rightarrow \infty$ . Finding a stationary Ramsey plan is critical to be able to use perturbation methods to compute optimal stabilization policy.

**Definition 3. (Stationary Ramsey Plan)** A stationary Ramsey plan, with  $(A_t, \rho_t, \epsilon_t) = (A_{ss}, \rho_{ss}, \epsilon_{ss})$  constant, is given by (i) an inflation rate,  $\pi_{ss}^w$ , aggregate hours,  $N_{ss}$ , stationary individual consumption allocations and value functions,  $c_{ss}(a, z)$  and  $V_{ss}(a, z)$ , and a stationary cross-sectional distribution of bond holdings and idiosyncratic labor productivities,  $g_{ss}(a, z)$ ; (ii) a stationary Ramsey policy,  $i_{ss}$ ; and (iii) a set of stationary multipliers,  $\phi_{ss}(a, z)$ ,  $\lambda_{ss}(a, z)$ ,  $\chi_{ss}(a, z)$ ,  $\mu_{ss}$ , and  $\vartheta_{ss}$ , such that the optimality conditions and the implementability conditions for a Ramsey plan are satisfied.

In Lemma 21 in the Appendix, we describe the conditions that explicitly characterize a stationary

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<sup>14</sup> As in Dávila and Korinek (2018), we use the terminology distributive pecuniary effects because they are zero-sum across households when measured in dollars basis.

Ramsey plan, which are in turn derived from our characterization of the Ramsey plan in Proposition (1). From the stationary counterpart of equation (21), it immediately follows that there is a key necessary condition for a stationary Ramsey plan to exist.

**Lemma 2. (Necessary Condition for Existence of Stationary Ramsey Plan)** *A necessary condition for the existence of a stationary Ramsey plan is that*

$$\iint \partial_a \chi_{ss}(a, z) da dz = 0. \quad (32)$$

We highlight this condition because it has an important economic interpretation in the context of equation (21), which we further explain below. Equation (32) implies that the “births” and “deaths” of distributional promises must average out to zero at a stationary Ramsey plan. In the Appendix, we show that this condition is satisfied in our baseline HANK model. This result has the interpretation that a planner does not want to over- or under-promise in the aggregate in terms of lifetime utilities.

### 3.2 Time Consistency and Timeless Penalties

Our definition of the standard primal Ramsey problem assumes commitment by the planner from time 0 onwards. Importantly, as we describe next, standard Ramsey plans will generically be time-inconsistent. Motivated by this observation, we introduce and define *timeless penalties* in this subsection and use them to formalize a timeless Ramsey plan, which we show is time-consistent.

Our formulation of the standard primal Ramsey problem includes two sets of forward-looking conditions: individual Hamilton-Jacobi-Bellman equations and a New Keynesian Phillips curve. Each of these implementability conditions is a source of time inconsistency.

First, consider equation (21), which characterizes the evolution of  $\phi_t(a, z)$ . In our continuous-time setup, time inconsistency materializes as follows: We know from the standard Ramsey problem that optimality requires an initial condition

$$\phi_0(a, z) = 0$$

for all  $(a, z)$ . But any stationary Ramsey plan will generically feature  $\phi_{ss}(a, z) \neq 0$ , which follows directly from the stationary version of equation (21), as long as  $\partial_a \chi_{ss}(a, z) \neq 0$ . Hence, even if we initialize the economy at the allocation that obtains at the stationary Ramsey plan, i.e.,  $g_0(a, z) = g_{ss}(a, z)$ , the planner will generically not set policy to  $i_t = i_{ss}$  in the absence of shocks, which would keep the economy at the stationary Ramsey plan. This would violate the initial condition of the standard Ramsey problem that  $\phi_0(a, z) = 0$  and precisely formalizes the first form of time inconsistency in our setting.<sup>15</sup>

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<sup>15</sup> Note that Acharya et al. (2020) have already identified a version of this form of time inconsistency in a HANK

There is a second source of time inconsistency that emerges from the Phillips curve. While the first source of time inconsistency, encoded in  $\phi_0(a, z) = 0$ , emerges from the planner's incentive to deviate from the relative consumption and redistribution promises associated with the stationary Ramsey plan, the second time consistency problem manifests as the planner has an incentive to deviate from the promised value of inflation. Optimality in the standard Ramsey problem requires an initial condition

$$\vartheta_0 = 0.$$

As in the case of  $\phi_{ss}(a, z)$ , the stationary Ramsey plan generically features  $\vartheta_{ss} \neq 0$ . This second source of time inconsistency is present in RANK models and has been widely studied, as we explain detail in Section 4.

Before we can formally introduce the timeless Ramsey problem in the next subsection, we now define *timeless penalties*, which are at the heart of our approach. Building on [Marcat and Marimon \(2019\)](#), we introduce a penalty or initial promise for each forward-looking implementability condition that the Ramsey planner faces.

**Definition 4. (Timeless Penalties)** We define the timeless penalty as

$$\mathcal{T}(\phi, \vartheta) = \underbrace{\iint \phi(a, z) V_0(a, z) da dz}_{\text{Distributional Penalty}} - \underbrace{\vartheta \pi_0^w}_{\text{Inflation Penalty}}$$

where we refer to  $\phi(a, z)$  as the initial distributional promise and  $\vartheta$  the initial inflation promise.

In principle, the timeless penalty is defined for any pair of initial promises  $\phi(\cdot)$  and  $\vartheta$ . As we show next, when we confront the Ramsey planner with the timeless penalty for a specific set of initial promises, the penalty exactly offsets the two time consistency problems.

### 3.3 The Timeless Ramsey Problem

In order to characterize optimal time-consistent monetary policy, we now define the timeless Ramsey problem ([Woodford, 1999, 2003, 2010](#)) in both a primal and dual form.<sup>16</sup> Formally, we build on and extend [Marcat and Marimon \(2019\)](#) to our setting — continuous-time heterogeneous-agent economies — by augmenting the standard Ramsey problem with the timeless penalties defined above. The resulting Ramsey problem is indexed by and defined for any set of initial promises  $\phi(\cdot)$  and  $\vartheta$ , which act as state variables. The main result of this section is that, under an

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economy. More generally, in incomplete market environments in which pecuniary externalities are present, it is understood that optimal policies are typically time-inconsistent — see, e.g., [Bianchi and Mendoza \(2018\)](#) and [Jeanne and Korinek \(2020\)](#).

<sup>16</sup> A timeless policy, as defined by [Woodford \(2010\)](#), represents a policy that, “even if not what the policy authority would choose if optimizing afresh at a given date  $t$ , ... it should have been willing to commit itself to follow from that date  $t$  onward if the choice had been made at some indeterminate point in the past, when its choice would have internalized the consequences of the policy for expectations prior to date  $t$ .”

appropriate penalty, the timeless Ramsey problem is time-consistent: a timeless Ramsey planner has no incentive to deviate from the stationary Ramsey plan in the absence shocks.

**Definition 5. (Timeless Primal Ramsey Problem)** A timeless primal Ramsey problem solves

$$\min_{\{\phi_t(a,z), \chi_t(a,z), \lambda_t(a,z), \mu_t, \vartheta_t\}} \max_{\{c_t(a,z), V_t(a,z), g_t(a,z), N_t, \pi_t^w, j_t\}} L^{\text{TP}}(g_0(\cdot), \phi(\cdot), \vartheta),$$

where  $L^{\text{TP}}(g_0(\cdot), \phi(\cdot), \vartheta)$  denotes the timeless primal Lagrangian, given an initial distribution  $g_0(\cdot)$  as well as two new state variables: a distributional promise  $\phi(\cdot)$  and an inflation promise  $\vartheta$ . The Lagrangian is defined as

$$L^{\text{TP}}(g_0(\cdot), \phi(\cdot), \vartheta) = L^{\text{SP}}(g_0(\cdot)) + \mathcal{T}(\phi(\cdot), \vartheta) \quad (33)$$

where  $L^{\text{SP}}(g_0(\cdot))$  is the standard primal Lagrangian defined in equation (20).

While we define the Lagrangian  $L^{\text{TP}}(g_0(\cdot), \phi(\cdot), \vartheta)$  for an arbitrary set of initial promises  $\phi(\cdot)$  and  $\vartheta$ , it will become clear in the following that the timeless primal Ramsey problem is time-consistent only if we set  $\phi(a,z) = \phi_{ss}(a,z)$  and  $\vartheta = \vartheta_{ss}$  for all  $(a,z)$ . Also note that the timeless Ramsey problem nests the standard one, which is time-inconsistent, when we set  $\phi(a,z) = 0$  and  $\vartheta = 0$ , implying  $L^{\text{TP}}(g_0(\cdot), 0, 0) = L^{\text{SP}}(g_0(\cdot))$ .

A key insight of our approach is that we can transform the standard primal Ramsey problem into a time-consistent, timeless problem by augmenting it with the timeless penalty. Intuitively, by confronting the Ramsey planner with a set of penalties for each forward-looking implementability condition, we introduce an artificial cost that, on the margin, exactly offsets the marginal benefit of time-inconsistent deviations from the stationary Ramsey plan.

Technically, the timeless penalty enforces a set of initial conditions on the promise-keeping multipliers of the primal Ramsey problem. In continuous time, the choice of initial lifetime values  $V_0(\cdot)$  and inflation  $\pi_0^w$  is free under the standard Ramsey problem, which gives rise to the initial conditions (28) and (27) of the standard Ramsey plan. Indeed,  $\phi_0(\cdot) = 0$  and  $\vartheta_0 = 0$  is precisely an expression of the fact that the standard Ramsey planner is not bound by past promises on lifetime values and inflation at time 0, even though she would like to bind her future self by making such promises. By augmenting the Ramsey problem with the timeless penalty, we technically enforce new initial conditions,

$$\phi_0(a,z) = \phi(a,z) \quad (34)$$

$$\vartheta_0 = \vartheta, \quad (35)$$

which in turn constrain the planner's choice of initial lifetime values and inflation. Formally, the first-order optimality conditions of the timeless primal Ramsey problem, which define the timeless Ramsey plan, comprise exactly the same equations as in Proposition 1, i.e., equations (21) through

(26), except that the initial conditions for the multipliers are now given by (34) and (35). In economic terms, it is for this reason that we refer to  $\phi(\cdot)$  and  $\vartheta$  as *initial promises*. And when we initialize the timeless penalty at  $\phi(a, z) = \phi_{ss}(a, z)$  and  $\vartheta = \vartheta_{ss}$ , it is as if the timeless Ramsey planner is confronted with the same promises at time 0 that she herself would like to make in the long run.

The timeless Ramsey problem also admits a dual representation, which we introduce next. The distinction between the primal and dual problems lies in the treatment of the constraints that a planner faces. In the primal approach, the planner optimizes over allocations, prices, and instruments given a set of constraints or implementability conditions. In the dual approach, the planner explicitly optimizes over the policy instrument, in this case, interest rates, using the implementability conditions to characterize the comparative statics of endogenous variables to policy.<sup>17</sup> The primal and dual representations of the timeless Ramsey problem have their distinct advantages and we leverage both in our analysis.

**Definition 6. (Timeless Dual Ramsey Problem)** A timeless dual Ramsey problem solves

$$\max_{\{i_t\}} L^{\text{TD}}(g_0(\cdot), \phi(\cdot), \vartheta),$$

where  $L^{\text{TD}}(g_0(\cdot), \phi(\cdot), \vartheta)$  denotes the timeless dual Lagrangian, given an initial distribution  $g_0(\cdot)$  as well as initial promises  $\phi(\cdot)$  and  $\vartheta$ . The Lagrangian is defined as

$$L^{\text{TD}}(g_0(\cdot), \phi(\cdot), \vartheta) = \int_0^\infty e^{-\rho t} \iint \left[ u(c_t(a, z)) - v(N_t) - \frac{\delta}{2} (\pi_t^w)^2 \right] g_t(a, z) da dz dt + \mathcal{T}(\phi(\cdot), \vartheta), \quad (36)$$

where all endogenous variables are understood as functions of the policy path  $\{i_t\}$ .

Having introduced the timeless Ramsey problem in both its primal and dual form, we now show that it is time-consistent in Proposition 3, which is the main result of this subsection.

At an abstract level, we can interpret all endogenous variables as functions of (i) the policy path, which we denote by  $\theta = \{i_t\}$ , the (ii) exogenous shocks, which we denote by  $Z = \{A_t, \rho_t, \epsilon_t\}$ , and (iii) initial conditions for the distribution  $g_0(\cdot)$  and promises  $(\phi(\cdot), \vartheta)$ . This already anticipates the sequence-space representation of our model, which we introduce in Section 3.4. Hence, given an exogenous sequence of shocks, an initial cross-sectional distribution, as well as initial promises, the planner chooses among different equilibria by choosing a sequence of policy. We can therefore evaluate the timeless Ramsey planner's objective — in both the primal and the dual form — for a given policy path  $\theta$ . The two timeless Lagrangians then directly imply a local efficiency criterion, according to which the Lagrangian's derivative with respect to the policy path  $\theta$  must be 0 for any locally efficient policy.

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<sup>17</sup> In simple terms, a useful analogy may be to interpret the dual approach as substituting constraints into the objective of an optimization problem, and the primal approach as accounting for constraints as additional terms in a Lagrangian.

**Proposition 3. (Time-Consistency of the Timeless Ramsey Problem)** *Optimal policy under the timeless primal and dual Lagrangians is time-consistent around the stationary Ramsey plan under the timeless penalty  $\mathcal{T}(\phi_{ss}(\cdot), \vartheta_{ss})$ . That is,*

$$\frac{d}{d\theta} L^{\text{TP}}(g_{ss}(\cdot), \phi_{ss}(\cdot), \vartheta_{ss}, \theta_{ss}, Z_{ss}) = \frac{d}{d\theta} L^{\text{TD}}(g_{ss}(\cdot), \phi_{ss}(\cdot), \vartheta_{ss}, \theta_{ss}, Z_{ss}) = 0. \quad (37)$$

Proposition 3 is an important pillar of our timeless Ramsey approach, and we state its proof in Appendix A.7. It formally establishes that a planner who sets policy according to the timeless Lagrangian in either the primal or the dual representation has no incentive to deviate from the stationary Ramsey plan in the absence of shocks. Equation (37) says that, when we initialize the economy at the stationary Ramsey plan, i.e.,  $g_0(\cdot) = g_{ss}(\cdot)$ , and set the timeless penalties of the timeless Lagrangians equal to the stationary multipliers, i.e.,  $\phi(\cdot) = \phi_{ss}(\cdot)$  and  $\vartheta = \vartheta_{ss}$ , then in the absence of shocks, i.e.,  $Z = Z_{ss}$ , the stationary Ramsey policy is optimal, i.e.,  $\frac{d}{d\theta} L^{\text{TP}}(\cdot) = \frac{d}{d\theta} L^{\text{TD}}(\cdot) = 0$  when we set  $\theta = \theta_{ss}$ .

We have now presented two of the three stages of our timeless Ramsey approach to optimal monetary policy. By characterizing the stationary Ramsey plan, we can solve for optimal long-run policy, with which the planner addresses the distortions and inefficiencies she perceives in a stationary equilibrium. By defining the timeless Ramsey problem, we can isolate the planner's incentives to deviate from the stationary Ramsey plan in the short run due to time consistency problems. And when we now characterize optimal stabilization policy around the stationary Ramsey plan under the timeless Ramsey problem, we no longer confound the planner's stabilization motive with the other two motives.

### 3.4 Optimal Stabilization Policy

To characterize optimal stabilization policy, we work with an abstract sequence-space representation of our model. The results we develop in this section build on recent work by Boppart et al. (2018) and Auclert et al. (2021), and extend the sequence-space approach to Ramsey problems and welfare analysis in heterogeneous-agent economies.

Equilibria in our baseline HANK economy can be summarized by an *equilibrium map* that take as inputs the time paths of aggregates,

$$\mathcal{H}(X, \theta, Z) = 0, \quad (38)$$

where  $\theta = \{i_t\}_{t \geq 0}$  denotes the path of policy,  $Z = \{A_t, \rho_t, \epsilon_t\}_{t \geq 0}$  the path of exogenous shocks, and  $X$  the path of macroeconomic aggregates. Given an initial cross-sectional distribution  $g_0(\cdot)$ , which is implicitly encoded in  $\mathcal{H}(\cdot)$ , the equilibrium map (38) characterizes macroeconomic aggregates in terms of policy  $\theta$  and shocks  $Z$ ,

$$X = X(\theta, Z).$$

The sequence-space representation (38) of general equilibrium in our model is as in [Auclert et al. \(2021\)](#), except that  $\mathcal{H}(\cdot)$  here also takes the path of policy  $\theta$  as an input, which is set optimally by the planner. Optimal policy, in turn, is determined as part of a Ramsey plan, whose sequence-space representation we characterize next.

**Proposition 4. (Sequence-Space Representation of Ramsey Plans)** *Given an initial distribution  $g_0(\cdot)$ , initial promises  $\phi(\cdot)$  and  $\vartheta$ , as well as a path for exogenous shocks  $Z$ , a Ramsey plan jointly characterizes macroeconomic allocations and prices  $X$ , optimal policy  $\theta$ , and multipliers  $M$ . Its sequence-space representation is given by*

$$\mathcal{R}(X, M, \theta, Z) = 0, \quad (39)$$

where we leave implicit the dependence of the Ramsey map  $\mathcal{R}(\cdot)$  on  $g_0(\cdot)$ ,  $\phi(\cdot)$ , and  $\vartheta$ .

We prove the sequence-space representations of equilibrium (38) and Ramsey plans (39) in Appendices [B.1](#) and [B.2](#).

Our sequence-space representation of Ramsey plans in heterogeneous-agent economies is valid for any initial distribution  $g_0(\cdot)$  and initial promises  $\phi(\cdot)$  and  $\vartheta$ . Equation (39) therefore recovers the standard Ramsey plan of Proposition [1](#) when we set  $\phi(\cdot) = 0$  and  $\vartheta = 0$ . Similarly, it follows directly from Proposition [3](#) that the Ramsey plan characterized by (39) is time-consistent when instead evaluated at  $(g_{ss}(\cdot), \phi_{ss}(\cdot), \vartheta_{ss})$ . In that case, we refer to it as a timeless Ramsey plan. In the following, we always initialize the timeless penalty at  $\phi(\cdot) = \phi_{ss}(\cdot)$  and  $\vartheta = \vartheta_{ss}$ , and focus on characterizing the response of optimal policy,  $d\theta$ , to exogenous shocks,  $dZ$ , under the timeless Ramsey plan.

The Ramsey plan representation (39) consists of two sets of equations. The first block is the system of equations (38), which characterizes aggregate allocations and prices  $X$  given policy  $\theta$  and shocks  $Z$ . The second block comprises the first-order optimality conditions of the Ramsey problem that solve for aggregate multipliers  $M$  and policy  $\theta$ . Crucially, the Ramsey equations that characterize optimal policy are coupled with those that describe the evolution of multipliers. Unlike the equilibrium map  $\mathcal{H}(\cdot)$ , which suffices to solve for transition dynamics given policy, the Ramsey map  $\mathcal{R}(\cdot)$  takes as inputs the aggregate multipliers  $M$  and features the equations that characterize them.<sup>[18](#)</sup>

In this sequence-space representation, we refer to a (timeless) Ramsey plan as the time paths of aggregates,  $R = (X, M, \theta)$ .<sup>[19](#)</sup> The system of equations (39) characterizes a Ramsey plan as a

<sup>[18](#)</sup> In this sense, Proposition [4](#) and the sequence-space representation of Ramsey plans (39) are closely associated with the timeless primal Ramsey problem, rather than its dual form. This discussion already anticipates that solving the Ramsey plan in its primal sequence-space representation requires solving for the transition dynamics of multipliers, as we further discuss below.

<sup>[19](#)</sup> In Section [3.1](#), we defined a Ramsey plan as the time paths of both aggregates and individual objects — namely, individual allocations, the cross-sectional distribution, and individual multipliers. In the sequence-space representation of our economy, we can express these individual objects as functions of the time paths of aggregates, as we formally show in Appendices [B.1](#) and [B.2](#). We thus loosely refer to a Ramsey plan in sequence-space form as the time paths of

function of the exogenous shocks, i.e.,

$$\mathbf{R} = \mathbf{R}(\mathbf{Z}),$$

implicitly taking as given an initial distribution  $g_0(\cdot)$  as well as initial promises  $\phi(\cdot)$  and  $\vartheta$ . The sequence-space representation of Ramsey plans  $\mathbf{R}$  in Proposition 4 is not unique. One minimal representation of our baseline economy, which we use for numerical implementation in Section 5, is  $\mathbf{X} = \{\Lambda_t, N_t\}_{t \geq 0}$ ,  $\mathbf{M} = \{\mu_t, \vartheta_t\}_{t \geq 0}$ , and  $\boldsymbol{\theta} = \{i_t\}_{t \geq 0}$ . In that case, the Ramsey plan representation (39) becomes a system of five equations: the definition of  $\Lambda_t$  in (8), the resource constraint (18), as well as the three aggregate optimality conditions (24), (25), and (26). Together, they solve for the Ramsey plan as a function of shocks, i.e.,  $\mathbf{X}(\mathbf{Z})$ ,  $\mathbf{M}(\mathbf{Z})$ , and  $\boldsymbol{\theta}(\mathbf{Z})$ , taking as given an initial cross-sectional distribution  $g_0(\cdot)$ , as well as initial promises  $\phi(\cdot)$  and  $\vartheta$ .

Finally, notice that while Proposition 4 develops a sequence-space representation of Ramsey plans in heterogeneous-agent economies, it is a non-linear, i.e., exact, representation. We will not make use of sequence-space perturbation methods, i.e., approximate representations, until Section 3.4.2.

### 3.4.1 Non-Linear Optimal Policy

The sequence-space representation of a timeless Ramsey plan in our economy is a system of non-linear equations. For a given initial distribution  $g_0(\cdot)$ , we can directly solve (39) non-linearly to compute optimal stabilization policy around the stationary Ramsey plan for any sequence of shocks  $\mathbf{Z}$  that reverts back to  $\mathbf{Z}_{ss}$ .

Computing timeless Ramsey plans non-linearly is tractable and fast in our baseline HANK economy. Using an efficient quasi-Newton algorithm, we can solve the system (39) non-linearly in less than 10 seconds.<sup>20</sup> However, computing non-linear transition paths in more complex HANK economies with richer cross-sectional heterogeneity can become cumbersome. Local perturbation methods, on the other hand, are fast and oftentimes very accurate in the context of canonical HANK environments.<sup>21</sup>

In the remainder of this section, we develop sequence-space perturbation methods to approxi-

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aggregates  $(\mathbf{X}, \mathbf{M}, \boldsymbol{\theta})$  with the understanding that the remaining individual objects can be expressed and easily obtained as functions of these.

<sup>20</sup> We use the quasi-Newton algorithm developed by Schaab and Zhang (2021) and Schaab (2020) to compute non-linear transition paths in heterogenous-agent economies. The code is available at <https://github.com/schaab-lab/SparseEcon>. Using this solver, computing non-linear Ramsey plans for our baseline HANK economy takes less than 10 seconds on a personal computer for discretized time grids with up to 100 nodes. For this experiment, we use a 2020 13-inch MacBook Pro with an M1 chip and 16 GB memory.

<sup>21</sup> When computing Ramsey plans non-linearly, we use quasi-Newton rather than standard Newton methods. This means that we compute the Jacobians involved in the algorithm once and subsequently use a recursive approximation. In practice, the algorithm never has to recompute the Jacobians and converges quickly, precisely because first-order perturbation solutions are typically very accurate approximations in canonical HANK economies. Therefore, the objects we need are precisely those we also compute below in Section 3.4.2, i.e.,  $\mathcal{R}_R$  and  $\mathcal{R}_Z$  evaluated around the stationary Ramsey plan, using a fake-news algorithm. As long as the quasi-Newton algorithm does not require that we recompute the Jacobian matrix, computing the non-linear solution is just as fast as the fake-news algorithm for the perturbation approach, requiring the computation of only a single column of the Jacobians  $\mathcal{R}_R$  and  $\mathcal{R}_Z$ .

mate optimal policy in a neighborhood around the stationary Ramsey plan. In principle, we can take either the primal or the dual representation of our Ramsey problem as a starting point to approximate optimal policy. In Sections 3.4.2 and 3.4.3, we present both approaches and argue that they have distinct advantages and disadvantages in different contexts. We conclude with a discussion in Section 3.4.4.

### 3.4.2 Optimal Policy Perturbations in the Primal

To approximate optimal policy in the primal representation of the Ramsey problem, we take as our starting point the system of equations (39).

**Proposition 5. (Optimal Policy Perturbations in the Primal)** Consider the primal Ramsey problem and the associated Ramsey plan, which is characterized by (39) and solves  $\mathcal{R}(\cdot) = 0$ . Suppose we initialize the Ramsey plan at the cross-sectional distribution  $g_0(\cdot) = g_{ss}(\cdot)$  and with initial timeless penalties  $\phi(\cdot) = \phi_{ss}(\cdot)$  and  $\vartheta = \vartheta_{ss}$ . To first order, optimal stabilization policy is then characterized as part of the Ramsey plan by

$$d\mathbf{R} = -\mathcal{R}_R^{-1}\mathcal{R}_Z d\mathbf{Z} \quad (40)$$

where  $d\mathbf{Z} = \mathbf{Z} - \mathbf{Z}_{ss}$  is the exogenous shock,  $d\mathbf{R} = (dX, dM, d\theta)$  denotes the response of the Ramsey plan, and  $\mathcal{R}_R$  and  $\mathcal{R}_Z$  are Jacobians of the Ramsey plan map.

We prove Proposition 5 in Appendix B.3.

Note that the characterization of stabilization policy in Proposition 5 relies on initializing the Ramsey problem with the proper timeless penalties, so that  $\mathcal{R}(\cdot)$  characterizes a timeless Ramsey plan. With the appropriate timeless penalties,  $d\mathbf{R}$  only captures the planner's stabilization motive in response to shocks  $d\mathbf{Z}$ . Without them,  $d\mathbf{R}$  conflates the stabilization motive with time consistency problems and is consequently no valid solution of optimal stabilization policy to first order.

To approximate Ramsey plans to first order in the primal, we have to compute two first-order derivative matrices,  $\mathcal{R}_R$  and  $\mathcal{R}_Z$ . These matrices can in turn be constructed from sequence-space Jacobians, which allows us to leverage the power of sequence-space perturbation methods. In Appendix B, we extend the fake-news algorithm developed by Auclert et al. (2021) for sequence-space Jacobians to compute optimal policy and Ramsey plans.

Auclert et al. (2021) show how to use the equilibrium map (38) to efficiently compute transition dynamics for a given path of policy to first order. They develop a general model representation of the standard micro block of competitive equilibria in heterogeneous-agent economies, i.e., the set of equations that characterize the allocations and behavior of individual agents. We show in Appendix B that computing optimal policy using the sequence-space Ramsey plan representation (39) requires a second "micro block," namely the set of individual multiplier equations. We develop a general sequence-space representation for this multiplier block and show that the same

principles underlying [Auclert et al. \(2021\)](#)'s fake-news algorithm can be used to efficiently compute sequence-space Jacobians for multipliers.

### 3.4.3 Optimal Policy Perturbations in the Dual

In this section, we develop a sequence-space perturbation approach to solve optimal stabilization policy in the dual. We take as our starting point not equation (39) but a sequence-space representation of the timeless dual Lagrangian, which we now introduce.

The timeless dual Lagrangian defined in Section 3.3 takes as its inputs (i) the time paths of allocations and prices, (ii) an initial distribution, and (iii) initial timeless penalties. Unlike in the primal form, it does not explicitly feature the time paths of multipliers. For a given path of policy  $\theta$  and shocks  $Z$ , we can directly use the equilibrium map (38) to solve out for allocations and prices, i.e.,  $X = X(\theta, Z)$ . A sequence-space representation of the timeless dual Lagrangian is then given by

$$L^{\text{TD}}(X(\theta, Z), \theta, Z), \quad (41)$$

where we again leave implicit the dependence of  $L^{\text{TD}}(\cdot)$  on  $g_0(\cdot)$  as well as  $\phi(\cdot)$  and  $\vartheta$ .

The timeless dual Lagrangian (41) implies the local efficiency criterion for optimal policy

$$\mathcal{F}(\theta, Z) = \frac{d}{d\theta} L^{\text{TD}}(X(\theta, Z), \theta, Z) = 0, \quad (42)$$

where  $\mathcal{F}(\cdot)$  implicitly takes as given an initial distribution as well as initial promises. Equation (42) represents the planner's necessary first-order optimality condition in sequence-space form. We can use it to directly characterize optimal policy in the dual in terms of exogenous shocks, i.e.,

$$\theta = \theta(Z).$$

Importantly, given policy  $\theta$ , the timeless penalty  $(\phi(\cdot), \vartheta)$  does not affect competitive equilibrium, summarized by (38). It only influences the planner's assessment of optimal policy.

**Proposition 6. (Optimal Policy Perturbations in the Dual)** Consider the dual Ramsey problem, under which a locally efficient policy is characterized by  $\mathcal{F}(\cdot) = 0$ . Suppose we initialize the Ramsey problem at the stationary Ramsey plan, with  $g_0(\cdot) = g_{ss}(\cdot)$ , and with initial timeless penalties  $\phi(\cdot) = \phi_{ss}(\cdot)$  and  $\vartheta = \vartheta_{ss}$ . To first order, optimal stabilization policy is then characterized by

$$d\theta = -\mathcal{F}_\theta^{-1} \mathcal{F}_Z dZ, \quad (43)$$

where  $dZ = Z - Z_{ss}$  is the exogenous shock, and where  $\mathcal{F}_\theta$  and  $\mathcal{F}_Z$  denote Hessians of the timeless dual Lagrangian.

We prove Proposition 6 in Appendix B.4.

In the dual, approximating optimal policy using sequence-space perturbation methods requires computing second-order total derivatives of the timeless dual Lagrangian. These are in turn given by the Jacobians of the planner’s first-order condition, i.e.,  $\mathcal{F}_\theta$  and  $\mathcal{F}_Z$ . The recent literature on perturbation methods in heterogeneous-agent economies has shown that sequence-space Jacobians, i.e., first-order derivatives of model objects in sequence-space representation, are sufficient to characterize transition dynamics to first order. Similarly, we have shown in Section 3.4.2 that computing optimal policy and Ramsey plans in the primal also only requires sequence-space Jacobians. Computing optimal policy and welfare in the dual representation of our Ramsey problem requires a second-order analysis, however.

To that end, we introduce *sequence-space Hessians* as the natural, second-order generalization of sequence-space Jacobians. In Appendix B.5, we formally define sequence-space Hessians, and we show both how to efficiently compute and leverage them to characterize optimal policy in the dual. We extend the methodology developed by Auclert et al. (2021) to problems that require second-order derivatives, i.e., sequence-space Hessians. While we focus on their use in the context of computing Ramsey plans in the dual, we argue that sequence-space Hessians are useful more broadly whenever a second-order analysis is required.

### 3.4.4 Advantages and Disadvantages of Primal and Dual Formulations

Our approach allows for a representation of Ramsey problems in either the primal or the dual form, and we show how to characterize and compute optimal policy in both cases. We conclude this section with a discussion of the advantages and disadvantages of both approaches.

The primal form and, in particular, the associated Ramsey plan representation (39) are more conducive to computing optimal policy non-linearly. Computing the Ramsey plan in the primal requires solving a system of non-linear equations. Fully optimized quasi-Newton methods and other non-linear equation solvers can be leveraged for this task.

In the primal approach, however, we have to compute multipliers and their transition dynamics. In the Ramsey plan representation (39), multipliers  $M$  enter as an explicit argument. It is not generically possible to characterize Ramsey plans in the spirit of (39) without multipliers. The dual approach, on the other hand, takes as its starting point the timeless dual Lagrangian, which does not explicitly depend on multipliers. Consequently, it is not necessary to compute the time paths of multipliers to characterize optimal policy in the dual.

The relative disadvantage of the dual approach, however, is that a first-order approximation of optimal policy requires second-order total derivatives of the timeless dual Lagrangian. Unlike in the primal approach, which only requires computing sequence-space Jacobians, in the dual we have to compute sequence-space Hessians. Computationally, this is a more complex task both in terms of compute time and memory demands. In summary, therefore, the main tradeoff between the primal and dual approaches is that the former requires computing the time paths of multipliers, while the latter requires sequence-space Hessians instead of only Jacobians.

Another advantage of the dual representation is that it provides an easily implementable local efficiency criterion. In particular, assessing whether a policy  $\theta$  is locally efficient in the dual only requires computing first-order derivatives of the timeless dual Lagrangian, which is possible in terms of sequence-space Jacobians. Unlike in the primal, however, efficiency assessments in the dual do not require computing the derivatives of multipliers. In practice, even if optimal policy is computed in the primal, verifying local efficiency in the dual is a cheap but helpful exercise.

Lastly, sequence-space Hessians and, more broadly, second-order sequence-space perturbation methods likely have many useful applications beyond computing Ramsey plans in the dual. We therefore view our treatment of sequence-space Hessians as a standalone contribution of this paper.

## 4 Analytical Insights

In this section, we provide multiple results that allow us to describe the drivers of optimal monetary policy with heterogeneous agents. First, in Section 4.1, we present the RANK limit of our model and characterize Ramsey optimal policy in that limit. The results in the RANK benchmark are helpful for Sections 4.2 through 4.5, where we analytically characterize optimal long-run optimal policy, time consistency, and optimal stabilization policy in HANK economies. Finally, in Section 4.6, we revisit the question of rules versus discretion in our setting. We contrast the results that emerge from the standard and timeless Ramsey plans, which embed commitment, with optimal monetary policy under full discretion.

In the main text, we assume that households' preferences are isoelastic, with  $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$  and  $v(n) = \frac{n^{1+\eta}}{1+\eta}$ . With isoelastic preferences, our analytical characterization of targets and targeting rules becomes particularly tractable and can be expressed in reference to output gaps. In Appendix E, we derive the results of this section for general preferences, where we reinterpret targets and targeting rules in reference to labor wedges.

### 4.1 RANK Benchmark

In order to relate to the vast literature on optimal monetary policy in the standard New Keynesian model and to better pinpoint how individual heterogeneity affects the conduct of optimal policy, it is helpful to characterize the representative-agent limit of our model as a benchmark for comparison. Starting from the baseline model, the RANK limit obtains when *i*) households' idiosyncratic labor productivity converges to a constant value, that is,  $z_t \rightarrow \bar{z}, \forall t$ , and *ii*) the economy is initialized with a cross-sectional distribution of bond holdings and productivities that is degenerate at  $a = 0$  and  $z = \bar{z}$ , that is,  $g_0^{\text{RA}}(a, z) = \delta(a = 0, z = \bar{z})$ , where here  $\delta(\cdot)$  denotes a two-dimensional Dirac delta function. In Appendix D, we include a self-contained characterization of the competitive equilibrium and optimal policy in this RANK benchmark. Here we describe the main takeaways.

The micro block of the implementability conditions of the Ramsey problem in RANK economies collapses to a single equation, the dynamic IS condition, while the aggregate macro block remains

largely unchanged. Unlike in HANK, we can use the resource constraint and goods market clearing condition in RANK,  $C_t^{\text{RA}} = A_t N_t^{\text{RA}}$  and  $Y_t^{\text{RA}} = C_t^{\text{RA}}$ , to simplify the implementability conditions to the standard system of two equations:

$$\dot{Y}_t^{\text{RA}} = \frac{1}{\gamma} \left( i_t^{\text{RA}} - \pi_t^{w,\text{RA}} + \frac{\dot{A}_t}{A_t} - \rho_t \right) Y_t^{\text{RA}} \quad (44)$$

$$\dot{\pi}_t^{w,\text{RA}} = \rho_t \pi_t^{w,\text{RA}} + \frac{\epsilon_t}{\delta} \left[ \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) Y_t^{\text{RA}} u'(Y_t^{\text{RA}}) - v' \left( \frac{Y_t^{\text{RA}}}{A_t} \right) \frac{Y_t^{\text{RA}}}{A_t} \right]. \quad (45)$$

In our baseline HANK economy, the micro block comprises the three equations (15) through (17), which characterize the household's first-order condition for consumption as well as the evolution of the private value function and the cross-sectional distribution. In the representative-agent model, on the other hand, the micro block collapses to a single equation, the aggregate Euler or dynamic IS equation (44). While the macro block takes the same form in both models, unions' perceived marginal value of higher wages is now proportional to  $u'(Y_t^{\text{RA}})$ , the representative household's marginal utility of aggregate consumption, whereas it was governed by  $\Lambda_t = \iint z u'(c_t(a, z)) g_t(a, z) da dz$  in HANK, an earnings-weighted average of marginal utilities of consumption.

We formalize the Ramsey problem in RANK in Appendix D.2 and summarize the resulting optimality conditions here.<sup>22</sup> Since the planner chooses output, inflation, and the nominal interest rate, the Ramsey plan in RANK comprises three optimality conditions.<sup>23</sup> The condition for optimal monetary policy, i.e.,  $i_t$ , is given by

$$0 = \frac{1}{\gamma} \phi_t^{\text{RA}} Y_t^{\text{RA}}, \quad (46)$$

where  $\phi_t^{\text{RA}}$  is the representative-agent analog of the promise-keeping multiplier  $\phi_t(a, z)$  in HANK. Since  $Y_t^{\text{RA}} > 0$ , equation (46) directly implies  $\phi_t^{\text{RA}} = 0$ . This result implies that the planner agrees with the representative household on consumption, which in turn implies that no time consistency problem emerges from the household's consumption and savings decisions in a RANK economy.

Using  $\phi_t^{\text{RA}} = 0$ , the optimality condition for aggregate economic activity is

$$\begin{aligned} u'(Y_t^{\text{RA}}) - v' \left( \frac{Y_t^{\text{RA}}}{A_t} \right) \frac{1}{A_t} = \\ - \vartheta_t^{\text{RA}} \frac{\epsilon_t}{\delta} \left[ \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) (Y_t^{\text{RA}} u''(Y_t^{\text{RA}}) + u'(Y_t^{\text{RA}})) - v' \left( \frac{Y_t^{\text{RA}}}{A_t} \right) \frac{1}{A_t} - v'' \left( \frac{Y_t^{\text{RA}}}{A_t} \right) \frac{1}{A_t^2} \right] \end{aligned} \quad (47)$$

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<sup>22</sup> We denote the Lagrange multiplier of the Euler equation by  $\phi_t^{\text{RA}}$ , which is the same notation as the multiplier we used for the HJB in the HANK case. When comparing optimality conditions, it is evident that this is the right choice. There is no need for multipliers  $\lambda_t$  and  $\chi_t$  in the RANK model.

<sup>23</sup> While it is common to drop the dynamic IS equation when studying optimal policy, here it is instructive to preserve it to understand why  $\phi_t^{\text{RA}} = 0$  in HANK economies.

and that for inflation is

$$\dot{\vartheta}_t^{\text{RA}} = \delta \pi_t^{w,\text{RA}}, \quad (48)$$

which takes exactly the same form as in HANK.

We structure the rest of our discussion in this section by comparing the HANK and RANK benchmarks according to the three dimensions of optimal monetary policy: optimal long-run inflation, time consistency, and optimal stabilization policy.

## 4.2 Optimal Long-Run Policy

We first investigate the implications of household heterogeneity for optimal long-run inflation in our baseline environment. Comparing equations (25) and (48), the Ramsey plan optimality condition for inflation takes the same form in our baseline HANK model as it does in the RANK limit. In particular, any stationary Ramsey plan must feature  $\dot{\vartheta}_t = 0$ , which implies that optimal long-run wage inflation is zero,  $\pi_{ss}^w = \pi_{ss}^{w,\text{RA}} = 0$ .

In the HANK model, inflation and the nominal interest rate affect households' financial income, which itself is proportional to the real interest rate  $r_t = i_t - \pi_t^w + \frac{\dot{A}_t}{A_t}$ , symmetrically. Therefore, the Ramsey planner can in principle use either to affect consumption promises. Maintaining non-zero inflation, however, is costly due to wage adjustment costs. The planner thus finds it optimal to adjust the nominal interest rate while keeping inflation at zero. An alternative way to explain this result is that, in any stationary equilibrium, the real interest rate is pinned down by real forces, so the only choice that the planner effectively has is the split between nominal interest rate and nominal price inflation for a given real interest rate. Therefore, since inflation is directly costly while adjusting the nominal rate is not, it is optimal to set  $\pi_{ss}^w = 0$ .

In the RANK model, we have the exact parallel set of equations. However, the optimality condition for the interest rate collapses to  $\phi_t^{\text{RA}} = 0$ , which automatically removes any desire to adjust policy to change the distribution of consumption promises. Importantly, the logic behind the optimal long-run policy in RANK and HANK is identical. That is, the Ramsey planner also has in principle a distributional "target," but it turns out that this target becomes degenerate in the representative-agent case.

**Proposition 7. (Optimal Long-Run Policy)** *Optimal long-run price inflation in both HANK and RANK is zero. That is,*

$$\pi_{ss} = \pi_{ss}^{\text{RA}} = 0.$$

It is useful to make two remarks regarding Proposition 7. First, given our assumptions, it should not be surprising that zero-inflation is the optimal long-run policy in RANK. A rich literature, including Chari and Kehoe (1999), Khan et al. (2003), and Schmitt-Grohé and Uribe (2010), has studied environments in which the optimal long-run policy features non-zero inflation.

Second, a corollary of our analysis is that household heterogeneity can only be a source of non-zero optimal long-run inflation if the nominal interest rate and inflation affect individual households asymmetrically. That is, in environments in which inflation and the nominal interest rate have a differential impact on the distribution of individuals in the economy, we should expect an optimal long-run policy that features non-zero inflation.

### 4.3 Distributional Wedges

We now introduce a set of *distributional wedges*, which are important inputs to our results on time inconsistency, target rules, and stabilization policy below.

**Definition 7. (Distributional Wedges)** We define the two distributional wedges

$$\Omega_t^1 = \iint \left( \underbrace{\frac{zu'(c_t(a, z))}{u'(Y_t)}}_{\text{Marginal Value of Consumption}} + \underbrace{\frac{zu'(c_t(a, z)) \phi_t(a, z)}{u'(Y_t) g_t(a, z)}}_{\text{Promise-Keeping}} + \underbrace{\frac{zu''(c_t(a, z)) \chi_t(a, z)}{u'(Y_t) g_t(a, z)}}_{\text{Over-Borrowing}} \right) g_t(a, z) da dz$$

$$\Omega_t^2 = \iint \frac{1}{1-\gamma} \underbrace{\left( \frac{zu'(c_t(a, z))}{u'(Y_t)} - \gamma \frac{z^2 u''(c_t)}{u''(Y_t)} \right)}_{\text{Labor Rationing}} g_t(a, z) da dz.$$

In the RANK limit of our economy, these distributional wedges vanish, with  $\Omega_t^1, \Omega_t^2 \rightarrow 1$ . In the HANK economy, however, we generically have  $\Omega_t^1, \Omega_t^2 \neq 1$ . As we show below, these distributional wedges summarize the implications of household heterogeneity for optimal monetary policy.

The first distributional wedge consists of three terms. The first term captures differences in the planner's valuation of a marginal dollar of aggregate resources. In RANK, a dollar of resources is valued according to the representative household's marginal utility of (aggregate) consumption,  $u'(C_t)$ . In a HANK economy, however, marginal utilities of consumption differ across households, and the planner's valuation of a marginal unit of resource is consequently given by a weighted average of marginal utilities. The second term represents the Ramsey planner's valuation of consumption promises made in the past.

Finally, the third term captures the planner's static incentive to change households' consumption behavior in order to influence the real interest rate. This is a distributive pecuniary externality force. In particular, the planner perceives uniform over-borrowing in our baseline HANK environment: Under a utilitarian criterion, the planner is particularly concerned with supporting indebted households. By depressing the real interest rate, the planner can lower the effective financial burden of indebted households.

The second distributional wedge captures labor rationing. While the labor union targets a weighted average of labor wedges, individual households cannot equalize their marginal rates

of substitution between consumption and labor with their marginal rates of transformation due to labor rationing. When the planner considers perturbations that change households' hours of work and consumption in the cross section, the union's valuation of wage adjustments changes as well. In particular,  $(1 - \gamma)\Omega_t^2$  represents the net effect on the union's perceived marginal benefit of higher wages under a perturbation that increases households' hours by one unit and distributes out the associated resource gain in proportion to idiosyncratic labor productivity.

## 4.4 Time Inconsistency

Next, we compare the sources of time inconsistency in the two models. We start by studying time inconsistency in the planner's inflation choice, which emerges due to the forward-looking Phillips curve in the set of implementability conditions, and connect to the vast RANK literature on this topic. A key insight is that distributional considerations meaningfully interact with and become an important driver of the classical time consistency problem on inflation. Next, we turn to the second and novel source of time inconsistency in HANK economies, which emerges from the individual forward-looking implementability conditions in the standard Ramsey problem. Not only is optimal monetary policy in HANK economies subject to a second source of time inconsistency, but the two sources of time inconsistency — aggregate and individual — meaningfully interact now.

### 4.4.1 Inflation Target

The presence of time inconsistency in the representative-agent New Keynesian model has been the source of a vast literature starting with [Barro and Gordon \(1983\)](#) — see [Clarida et al. \(1999\)](#) and, in particular, [Woodford \(2010\)](#), who, as we do, works with exact structural equations, for modern treatments.<sup>24</sup> The nature of time inconsistency in RANK models is due to the fact that the planner faces a forward-looking constraint, the Phillips curve, which makes promises over future inflation useful to achieve the planner's objective. In the following, we establish that our timeless penalty on inflation in the RANK limit of our model, i.e.,  $\vartheta_{ss}^{RA}\pi_0^{w,RA}$ , takes the exact same form as the inflation target derived in the literature that follows [Barro and Gordon \(1983\)](#). The same time consistency problem appears in HANK, where the Phillips curve also incentivizes the planner to make promises about future inflation.

[Lemma 31](#) in the Appendix provides an exact characterization of the marginal benefit a Ramsey planner perceives from time-inconsistent deviations from the stationary Ramsey plan at time 0. We also prove that the timeless inflation penalty,  $\vartheta_{ss}^{RA}\pi_0^{w,RA}$ , exactly offsets this benefit and can therefore be interpreted as the marginal cost of such time-inconsistent deviations. In other

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<sup>24</sup> It is well understood that RANK economies suffer from a time consistency problem on inflation whenever i) there are cost-push shocks or ii) the appropriate employment subsidy is not in place. Here we focus on ii), leaving the study of cost-push shocks to the analysis of optimal stabilization.

words, a Ramsey planner that sets interest rate policy in RANK according to

$$L^{\text{TD}, \text{RA}}(\vartheta) = \int_0^\infty e^{-\int_0^t \rho_s ds} \left\{ u(Y_t^{\text{RA}}) - v\left(\frac{Y_t^{\text{RA}}}{A_t}\right) - \frac{\delta}{2} (\pi_t^{w, \text{RA}})^2 \right\} dt - \vartheta \pi_0^{w, \text{RA}}$$

has no incentive to deviate from the stationary Ramsey plan in the absence of shocks when we confront the planner with the appropriate inflation target, i.e., when we set  $\vartheta = \vartheta_{ss}^{\text{RA}}$ .<sup>25</sup> This result is helpful because it allows us to study how the inflation time consistency problem in HANK differs from the well-understood benchmark case in RANK.

**Proposition 8. (Inflation Targets)** *The timeless penalty on inflation in both RANK and HANK economies is given by*

$$\vartheta_{ss} = -\frac{\delta}{\epsilon} \frac{Y_{ss}^{-\gamma} \Omega_{ss}^1 - Y_{ss}^\eta}{\frac{\epsilon-1}{\epsilon}(1+\tau^L)(1-\gamma)Y_{ss}^{-\gamma} \Omega_{ss}^2 - (1+\eta)Y_{ss}^\eta}. \quad (49)$$

Hence, distributional considerations shape the inflation target in HANK economies, where  $\Omega_{ss}^1, \Omega_{ss}^2 \neq 1$ , but not in the RANK limit, where  $\Omega_{ss}^1, \Omega_{ss}^2 \rightarrow 1$ .

The inflation target of Proposition 8 generalizes the standard “Barro-Gordon” inflation target to environments with heterogeneous households. In particular, equation (49) nests the inflation target in both HANK and RANK. The only difference is that the distributional wedges disappear in the RANK limit, i.e.,  $\Omega_{ss}^1, \Omega_{ss}^2 \rightarrow 1$ .

In RANK, therefore, the inflation target simplifies to  $\vartheta_{ss} = -\frac{\delta}{\epsilon} \frac{Y_{ss}^{-\gamma} - Y_{ss}^\eta}{\frac{\epsilon-1}{\epsilon}(1+\tau^L)(1-\gamma)Y_{ss}^{-\gamma} - (1+\eta)Y_{ss}^\eta}$ , which is the standard linear penalty form of the inflation target that emerges from the vast literature following Barro and Gordon (1983). It is well understood that time inconsistency only emerges under a distorted steady state so that the planner perceives an incentive from raising inflation and engineering a positive output gap in the short run. In our setting, this is the case whenever the employment subsidy is not large enough to offset the steady state markup distortion. In fact, it is easy to see that no inflation target is required under the appropriate employment subsidy, with  $\frac{\epsilon-1}{\epsilon}(1+\tau^L) = 1$ , because no time consistency problem emerges in that case.<sup>26</sup>

This is no longer true in HANK. In other words, even with the correct employment subsidy to address the markup distortion arising from monopolistic competition in steady state, the time consistency problem on inflation does not disappear. We show that the marginal benefit the Ramsey planner in HANK perceives from time-inconsistent deviations in the choice of inflation, relative to the stationary Ramsey plan featuring  $\pi_{ss}^w = 0$ , now features a set of distributional motives that are

<sup>25</sup> Formally, this means

$$\frac{dL^{\text{TD}}(\vartheta_{ss}^{\text{RA}})}{d\vartheta} = 0.$$

<sup>26</sup> To see this, plug in for  $\frac{\epsilon-1}{\epsilon}(1+\tau^L) = 1$  and set the distributional wedges to 1. Then the steady state with  $Y_{ss}^{\text{RA}} = \tilde{Y}_{ss}^{\text{RA}} = A$  implies  $\vartheta_{ss}^{\text{RA}} = 0$ .

absent in RANK. Specifically, comparing  $\vartheta_{ss}$  in the RANK and HANK models shows that the two sources of time inconsistency in HANK — inflation and individual value — interact. In particular, the planner now has an additional incentive to “cheat” on inflation because, by overheating the economy with lower nominal interest rates, higher inflation, and a positive output gap, the planner can depress real interest rates in the short run, and consequently make borrowing cheaper for indebted households.

An important corollary of this result is that the choice of aggregate inflation targets takes on a distributional dimension whenever the planner has a welfare criterion (or the central bank has a mandate) that goes beyond aggregate efficiency considerations. The standard time consistency problem associated with inflation consequently interacts with the novel time consistency problem due to heterogeneity, which we discuss next. Crucially, even with the correct employment subsidy to address the markup distortion arising from monopolistic competition in steady state, there remains a time consistency problem on inflation in HANK economies.

#### 4.4.2 Distributional Target

We now characterize the second source of time inconsistency in HANK economies, which is completely absent from RANK economies. It arises because the Ramsey planner disagrees with households on their consumption-savings decisions given the current nominal rate. In particular, a Ramsey planner in a HANK economy finds that households in general save too much or too little, in principle to different degrees. The planner can use the nominal interest rate to adjust consumption-savings decisions on average, and this is sufficient in a RANK economy to fully correct the representative household’s consumption-savings decisions. In a HANK economy, however, the planner would still like to modify households’ *relative* consumption-savings decisions. Under commitment, therefore, a Ramsey planner finds it optimal to make promises about future interest rates, which in turn manifests via promises on households continuation values. It is through this mechanism that a planner generates promises  $\phi_t(a, z)$  over time, opening the door to time inconsistency.

In a RANK economy, it follows from equation (46) that the Ramsey planner sets  $\phi_t^{\text{RA}} = 0$  at all times. This implies that it is sufficient for the planner to adjust the current nominal interest rate to implement any desired level of aggregate consumption-savings. That is, given that the planner can adjust the nominal interest rate, there is no need to further distort the consumption-savings decision of the representative household. And certainly not by making future promises, which eliminates time consistency problems in RANK economies.

In a HANK economy, on the other hand, two equations determine the multiplier  $\phi_t(a, z)$ . First, the optimality condition for the nominal interest rate, equation (26), implies that a planner seeks to distort individual decisions by setting a weighted average of  $\phi_t(a, z)$  equal to

$$\iint ag_t(a, z) \partial_a \lambda_t(a, z) da dz, \quad (50)$$

which captures the *distributive pecuniary effects* of a change in the nominal interest rate.<sup>27</sup> Intuitively, an increase in interest rates benefits savers (with  $a > 0$ ) and hurts borrowers (with  $a < 0$ ). When a planner values dollars in the hand of borrowers and savers differently at the margin — it does so according to the social marginal value of wealth,  $\partial_a \lambda_t(a, z)$  — this provides a rationale to distort individual decisions.

Second, the optimality condition for the households' HJB shows that the evolution of the distribution of individual penalties satisfies the Kolmogorov forward equation (21), which we restate here for convenience:

$$\partial_t \phi_t(a, z) = -\mathcal{A}_t^* \phi_t(a, z) + \partial_a \chi_t(a, z)$$

Intuitively, this KF equation implies that the distribution of penalties must be consistent with the law of motion of households across the different individual states, summarized by the operator  $\mathcal{A}_t^*$ .<sup>28</sup> The term  $\partial_a \chi_t(a, z)$  is a forcing term and thus a source of time variation in the distribution of promises  $\phi_t(a, z)$ . If individuals' optimal consumption and savings decisions change as they transition between different wealth states, then  $\partial_a \chi_t(a, z)$  can be interpreted as "births" and "deaths" of relative promises in the cross section. Solving for  $\phi_t$  and  $\chi_t$  jointly and characterizing how they are linked via the Kolmogorov forward equation (21) is one the major contributions of this paper.<sup>29</sup>

### Proposition 9. (Distributional Target)

- a) A utilitarian planner faces a distributional time consistency problem, with distributional penalties that evolve via the Kolmogorov forward equation defined in equation (21).
- b) An Aggregate Efficiency planner — see Dávila and Schaab (2021) — does not face a distributional time consistency problem.

An important conclusion from this section is that if a planner (central bank) adopts a welfare criterion (mandate) that is not the aggregate efficiency one, and if the planner (central bank) would like to implement a time-consistent policy, then it must also adopt what we call a *distributional target* in addition to the standard inflation target. In other words, outside of the aggregate efficiency criterion, the distributional target becomes necessary for optimal policy.

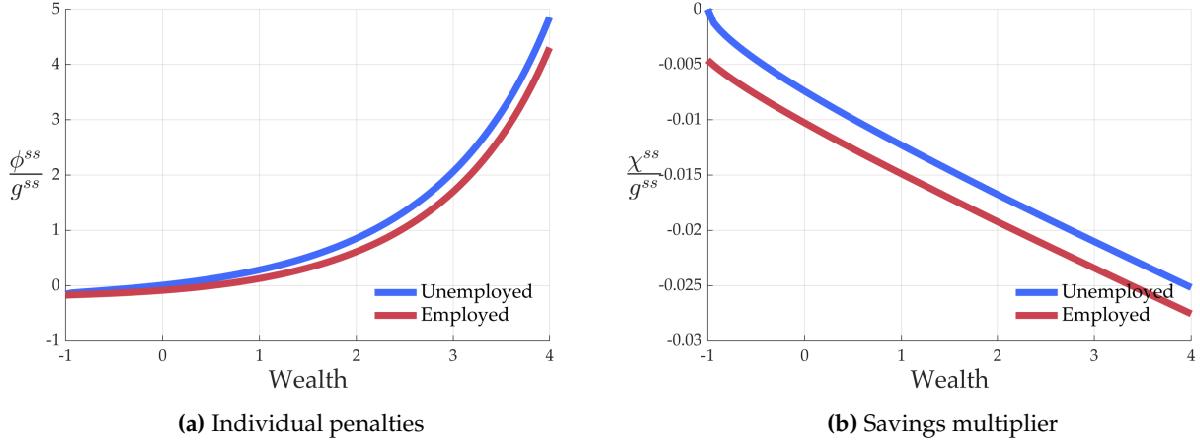
Finally, it may be helpful to illustrate the form of the distributional penalties, as we do in Figure 2. The left panel of Figure 2 plots the distributional penalties associated with the stationary

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<sup>27</sup> We use the terminology distributive pecuniary externalities as in Dávila and Korinek (2018). That paper shows that distributive pecuniary externalities are characterized by i) change in net asset positions ( $a$ ) and ii) differences in valuation, here  $\partial_a \lambda_t(a, z)$ . As shown in that paper, if all  $\partial_a \lambda_t(a, z)$  are constant across individuals, then distributive pecuniary externalities, so  $\iint a g_t(a, z) da dz = 0$ , based on market clearing, and equation (50) is zero.

<sup>28</sup> Note that the operator  $\mathcal{A}_t^*$  is mass-preserving, that is,  $\iint \mathcal{A}_t^* \phi_t(a, z) da dz = 0$ , which allows us to interpret  $\phi_t(a, z)$  as a distribution.

<sup>29</sup> It should be evident that if households made multiple individual decisions in addition to the consumption and savings decisions, these would enter equation (21) as additional forcing terms.



**Figure 2.** Distributional Target

**Note.** The left panel of Figure 2 shows the steady state values of the individual target,  $\phi_{ss}(a, z)$ , normalized by the mass of households,  $g_{ss}(a, z)$ . The right panel of Figure 2 shows the steady state values of the savings multiplier,  $\chi_{ss}(a, z)$ , also normalized by the mass of households,  $g_{ss}(a, z)$ . From the right panel, it immediately follows that the planner perceives that all agents are under-saving. Note that we formally show that  $\iint \phi_{ss}(a, z) g_{ss}(a, z) da dz = 0$ . Note that the planner perceives that a household is under-saving when  $\chi_{ss}(a, z) < 0$ .

Ramsey plan,  $\phi_{ss}(a, z)$ , normalized by the mass of households,  $g_{ss}(a, z)$ . The right panel of Figure 2 displays the stationary consumption-savings multiplier,  $\chi_{ss}(a, z)$ , also normalized by the mass of households,  $g_{ss}(a, z)$ . From the right panel, it immediately follows that the planner perceives that all households are under-saving and over-consuming, since  $\chi_{ss}(a, z) < 0$ , with poorer and unemployed households less so.

## 4.5 Optimal Stabilization Policy

Finally, we compare optimal monetary stabilization policy in HANK and RANK. In the classical representative-agent analysis, prescriptions for monetary stabilization policy have commonly been summarized in the form of *targeting rules* (Clarida et al., 1999; Galí, 2015). We follow this tradition in our analysis. In fact, our main result in this subsection is a non-linear, i.e., exact, targeting rule for optimal monetary stabilization policy in response to demand, productivity, and cost-push shocks that nests both RANK and HANK.

Before stating our targeting rule, we introduce the notion of *natural output*, i.e., the level of output that obtains in the flexible-price allocation. In RANK, natural output is given by

$$\tilde{Y}_t^{\text{RA}} = \left( \frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) A_t^{1+\eta} \right)^{\frac{1}{\gamma+\eta}},$$

which varies with productivity and cost-push shocks, but not with demand shocks. Natural output

in HANK is affected by labor rationing and given by

$$\tilde{Y}_t = \tilde{Y}_t^{\text{RA}} \iint \frac{zu'(c_t(a, z))}{u'(Y_t)} g_t(a, z) da dz.$$

**Proposition 10. (Targeting Rule for Optimal Stabilization)** *We summarize optimal monetary stabilization policy with the targeting rule*

$$Y_t = \tilde{Y}_t^{\text{RA}} \left\{ \frac{\frac{\epsilon_t}{\epsilon_t-1}(1+\tau^L)\Omega_t^1 + \vartheta_t(1-\gamma)\frac{\epsilon_t}{\delta}\Omega_t^2}{1+\vartheta_t(1+\eta)\frac{\epsilon_t}{\delta}} \right\}^{\frac{1}{\gamma+\eta}}. \quad (51)$$

We start by revisiting the classical results on monetary stabilization policy in RANK through the lens of targeting rule (51). In RANK, we have  $\Omega_t^1 = \Omega_t^2 = 1$ . Suppose that we also allow for the appropriate steady state employment subsidy, so that  $\frac{\epsilon-1}{\epsilon}(1+\tau^L) = 1$  and there is no steady state distortion due to monopolistic competition.

Consider first the case of demand and TFP shocks. Suppose the planner closes the wage inflation gap so that  $\pi_t^{w,\text{RA}} = 0$ . This implies  $\vartheta_t^{\text{RA}} = 0$  from equation (48) because, under zero inflation, the planner never builds up inflation promises. Consequently, the term in brackets in equation (51) simply becomes 1, and we have  $Y_t^{\text{RA}} = \tilde{Y}_t^{\text{RA}}$ . In response to demand and TFP shocks, the Ramsey planner in RANK wants to close both the inflation and output gap. This is the seminal Divine Coincidence benchmark (Blanchard and Galí, 2007). In RANK, there is no tradeoff between inflation and output in the absence of cost-push shocks.

In the case of cost-push shocks, Divine Coincidence breaks down, even in RANK. Suppose again that the planner closes the wage inflation gap, implying  $\pi_t^{w,\text{RA}} = \vartheta_t^{\text{RA}} = 0$ . With  $\epsilon_t \neq \epsilon$ , we have  $Y_t^{\text{RA}} = \tilde{Y}_t^{\text{RA}} \left\{ \frac{\epsilon_t}{\epsilon_t-1}(1+\tau^L) \right\}^{\frac{1}{\gamma+\eta}}$ . The planner consequently does not find it optimal to close the inflation and output gaps at the same time.

In a HANK economy, we generically have  $\Omega_t^1, \Omega_t^2 \neq 1$ . In the presence of these distributional wedges, Divine Coincidence fails even with the appropriate employment subsidy: the Ramsey planner never finds it optimal to close both output and inflation gaps at the same time. In other words, the Ramsey planner always perceives a tradeoff between aggregate stabilization, i.e., the inflation and output gaps, on the one hand, and distributional considerations on the other hand.

At the same time, Proposition 10 tells us accounting for distributional considerations in stabilization policy generically comes at the cost of aggregate efficiency. For an Aggregate Efficiency planner — see Dávila and Schaab (2021) — Divine Coincidence is restored. The targeting rule for optimal monetary stabilization policy for such a planner is the same as the RANK targeting rule.

## 4.6 Rules versus Discretion According to HANK

In this section, we revisit the classical analysis of “rules versus discretion” in our baseline HANK model. For a classical treatment in the context of RANK, see for example Clarida et al. (1999) and Galí (2015).

In ongoing work, we formalize the planning problem under full discretion in continuous time in the spirit of Harris and Laibson (2013). From time  $t$  onwards, the planner has commitment over the finite horizon  $[t, t + \tau]$ , where  $\tau$  is a stopping time that arrives with a Poisson intensity  $\alpha$ . The planning problem under full discretion then corresponds to the limit as  $\alpha \rightarrow \infty$ . We leverage our results in Section 3 as well as the insights of Marcer and Marimon (2019), which we extend to continuous-time heterogeneous-agent settings, to characterize the resulting planning problem recursively. Over the horizon with commitment, the problem resembles that of Section 3. A recursive representation therefore requires that we add the timeless penalties  $\vartheta$  and  $\phi$  as state variables. We then characterize the value function  $V(g, \vartheta, \phi)$  with an extended Hamilton-Jacobi-Bellman (HJB) equation. The key insight is the following: Before the stopping time  $\tau$ , the planner accumulates promises, so that  $\vartheta$  and  $\phi$  evolve much like they do in Section 3. The extended HJB of the problem under discretion then features the additional term  $-\alpha[V(g, \vartheta, \phi) - V(g, 0, 0)]$ , which captures the hazard of a “new” planner taking control, resetting promises to 0 because the new planner has no commitment to honor the promises of the current planner. The limit of  $\alpha \rightarrow \infty$  then formally represents the planning problem under full discretion in continuous time.

For now, we proceed instead by directly discretizing the continuous-time problem as we did in Appendix A and then solving the planning problem under discretion much like in discrete time. At a given time step  $t_n$ , the current planner optimizes under discretion, taking as given all decisions of her future self at time step  $t_{n+1}$ , over which she has no commitment. The resulting problem resembles the analogous one in discrete time, except that the time intervals over which the planner effectively has commitment become vanishingly small. Moreover, while the problem under discretion in the standard New Keynesian model features no endogenous state variables, in HANK the entire cross-sectional distribution becomes an endogenous state variable. We formalize this discretized planning problem in Appendix F, which also contains detailed derivations and the proofs for this section. The next Proposition characterizes optimal policy under discretion.

**Proposition 11. (Optimal Monetary Policy under Discretion in HANK)** *Under full discretion, the Lagrange multipliers on the two forward-looking constraints are 0, capturing the planner’s inability to commit, i.e.,*

$$0 = \phi_t(a, z)$$

$$0 = \vartheta_t.$$

The remaining conditions that characterize optimal policy under full discretion are

$$\begin{aligned}\chi_t(a, z) &= -\frac{1}{u''(c_t(a, z))} \left[ u'(c_t(a, z)) + \mu_t - \partial_a \lambda_t(a, z) \right] g_t(a, z) \\ \rho \lambda_t(a, z) &= \partial_t \lambda_t(a, z) + \mathcal{A}_t \lambda_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2} (\pi_t^w)^2 + \mu_t c_t(a, z) \\ 0 &= -v'(N_t) - \mu_t A_t + \iint z g_t(a, z) \partial_a \lambda_t(a, z) da dz \\ 0 &= \iint a g_t(a, z) \partial_a \lambda_t(a, z) da dz\end{aligned}$$

Relative to the Ramsey plan of Proposition 1, the two multipliers that encode promise-keeping under commitment are now 0 at all times because the planner cannot commit to future actions. We again characterize optimal stabilization policy under discretion in the form of a targeting rule, which we then compare to Proposition 10.

**Proposition 12. (Targeting Rule for Optimal Stabilization under Discretion)** *Under full discretion, optimal monetary stabilization policy is governed by the targeting rule*

$$Y_t = \tilde{Y}_t \left\{ \frac{\epsilon_t}{\epsilon_t - 1} \frac{1}{1 + \tau^L} \underbrace{\left( 1 + \frac{\iint z u''(c_t(a, z)) \chi_t(a, z) da dz}{\iint z u'(c_t(a, z)) g_t(a, z) da dz} \right)}_{\substack{\text{Distributional Wedge} \\ >0}}^{\frac{1}{\gamma+\eta}} \right\}^{\frac{1}{\gamma+\eta}}, \quad (52)$$

which implies that a HANK planner always finds it optimal to raise output above natural output.

In RANK, the targeting rule under discretion simplifies to  $Y_t^{\text{RA}} = \tilde{Y}_t^{\text{RA}} \left\{ \frac{\epsilon_t}{\epsilon_t - 1} \frac{1}{1 + \tau^L} \right\}^{\frac{1}{\gamma+\eta}}$ . In the absence of cost-push shocks and with the appropriate employment subsidy, policy under discretion in RANK is simply characterized by  $Y_t^{\text{RA}} = \tilde{Y}_t^{\text{RA}}$ , as it would be under commitment. This well-understood insight echoes our earlier observation that, as long as  $\frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) = 1$ , there is no time consistency problem in RANK and, consequently, no gain from commitment. In other words, there is no welfare benefit from rules versus discretion.

This important benchmark result breaks down in environments with heterogeneous households. The targeting rule under discretion in HANK differs from that in RANK by a simple wedge. Crucially, we can sign this wedge unambiguously. Under discretion, the planner in HANK always wants to raise output above natural output, i.e., engineer a positive output gap, even in the absence of cost-push shocks and with the appropriate employment subsidy. Intuitively, by raising inflation and output, the planner can depress the real interest rate, which benefits indebted households. In our HANK environment, a utilitarian planner always has a *static* incentive to do so, and this policy motive consequently survives in the case of full discretion. In heterogeneous-agent envi-

ronments, there are welfare gains from commitment even in the important benchmark case where  $\frac{\epsilon_t - 1}{\epsilon_t} (1 + \tau^L) = 1$ .

## 5 Quantitative Analysis

In this section, we compute and characterize optimal monetary stabilization policy in response to different types of shocks, comparing results for our HANK baseline economy against the classical RANK benchmark. The primary goal of our quantitative analysis is to further shed light on the key departures from the classical consensus on optimal monetary policy as, for example, summarized in [Clarida et al. \(1999\)](#), [Woodford \(2003\)](#), and [Galí \(2015\)](#). We have purposively designed our one-asset HANK model to represent a minimal departure from the classical RANK benchmark, and we consequently try to stay close to the latter in our calibration exercise.

In ongoing work, we leverage our method to compute and characterize optimal monetary policy in a state-of-the-art two-asset HANK model in the spirit of [Kaplan et al. \(2018\)](#) and [Auclert et al. \(2020\)](#).

**Calibration.** We adopt isoelastic preferences and use standard parameters for household preferences in both model benchmarks, setting the discount rate to a quarterly  $\rho = 0.02$ , the elasticity of intertemporal substitution to  $\gamma = 2$ , and the inverse Frisch elasticity to  $\eta = 2$ . We set the elasticity of substitution between labor varieties to  $\epsilon = 10$  and the nominal wage adjustment cost to  $\delta = 100$ , following standard practice in the wage rigidity literature (see, e.g., [Auclert et al., 2020](#)).

The main difference between our representative- and heterogeneous-agent benchmarks is the earnings process that households face. In our HANK model, we assume that  $z_t$  follows a two-state Markov chain, with  $z_t \in \{z^L, z^H\}$ , where  $z^L = 0.8$  and  $z^H = 1.2$ . We set the quarterly Poisson transition rate out of both states to 0.33. Our RANK benchmark can be seen as the limit as  $z^L, z^H \rightarrow \bar{z} = 1$ , and using as initial condition for the cross-sectional distribution a Dirac mass at  $(a, z) = (0, \bar{z})$ .

Finally, in the following exercises we assume that the planner sets the employment subsidy  $\tau^L$  to address the wage-markup distortion in stationary equilibrium. That is,  $(1 + \tau^L)^{\frac{\epsilon-1}{\epsilon}} = 1$ . Since much of the analysis of monetary policy in RANK has made use of such an employment subsidy, we also report results in the main text under this assumption to facilitate comparison to the classical results.

**Shocks.** In this section, we study optimal stabilization policy in response to three types of shocks — demand, productivity, and cost-push shocks — following the classical analysis of optimal monetary policy in RANK as, for instance, in [Clarida et al. \(1999\)](#), [Woodford \(2003\)](#), and [Galí \(2015\)](#). We assume that the three shocks follow mean-reverting AR(1) processes. In continuous time, this

implies that

$$\dot{A}_t = \xi_A(A - A_t), \quad \dot{\epsilon}_t = \xi_\epsilon(\epsilon - \epsilon_t), \quad \text{and} \quad \dot{\rho}_t = \xi_\rho(\rho - \rho_t)$$

where  $A$ ,  $\epsilon$ , and  $\rho$  denote the steady-state constant levels. We study one-time, unanticipated (“MIT”) shocks at time  $t = 0$ , so that  $A_0$ ,  $\epsilon_0$ , and  $\rho_0$  jump and subsequently revert back to their steady-state levels following the above laws of motion. We set the initial shock levels to  $A_0 = 1.005A$ ,  $\epsilon_0 = 1.25\epsilon$ , and  $\rho_0 = 1.5\rho$ , and calibrate the shock’s persistence in each case to a half-life of one quarter.

**Transition dynamics with interest rate rules.** For reference, we also compute the transition dynamics of both HANK and RANK models for the case where monetary policy is not set optimally but instead follows a standard interest rate rule. We report these results in Appendix G.3.

**Sensitivity analysis.** We perform additional sensitivity and robustness analysis in Appendix G.4. In particular, we replicate the following numerical experiments for economies featuring differing degrees of earnings and consumption inequality. These exercises underscore that uninsurable earnings risk and incomplete markets are the key economic force that motivates departures of optimal policy from the classical targeting rules in RANK.

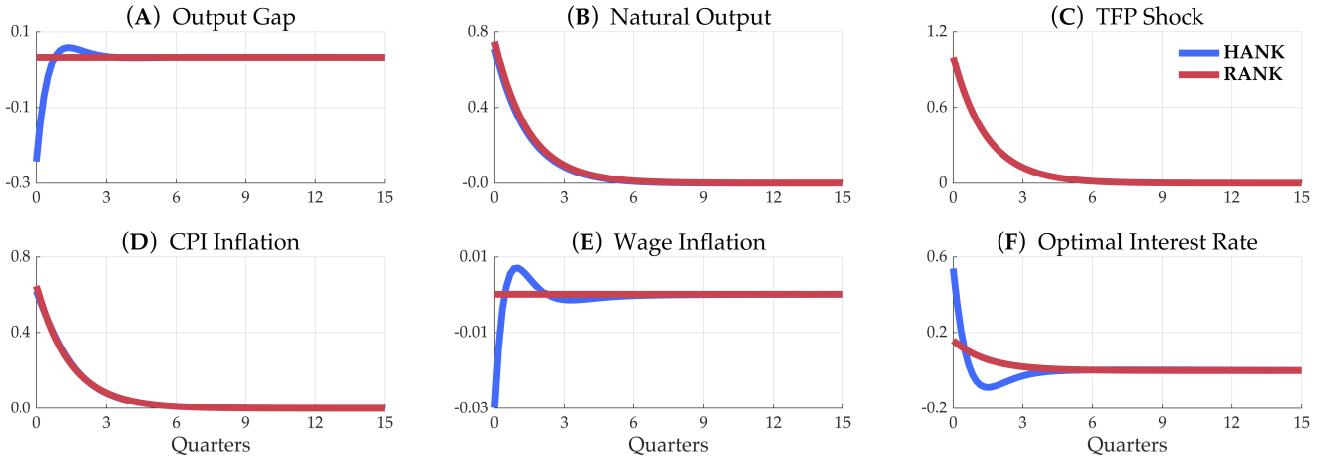
## 5.1 Productivity Shocks

We start by discussing optimal stabilization policy in response to a TFP shock. Figure 3 reports the transition dynamics of the economy under optimal policy, while Figure 8 in Appendix G.3 reports those under a Taylor rule for comparison.

In both model benchmarks, natural output, defined as the level of output in the flexprice allocation, increases in response to a positive productivity shock. Natural output increases less than one-for-one, primarily due to diminishing marginal utility from consumption and convex disutility from labor. Equation (69) in Appendix D, provides an exact and non-linear relationship between natural output and productivity in RANK. In HANK, natural output increases slightly less than in RANK as a result of union wage bargaining, which now features a distributional consideration.

The classical result on optimal monetary stabilization policy in RANK is that Divine Coincidence obtains in the face of productivity shocks: the planner perfectly stabilizes both output and (wage) inflation gaps. This sharp benchmark result requires the appropriate employment subsidy, of course, which we assume here. While costly wage inflation is perfectly stabilized, CPI inflation is allowed to track the productivity shock. To support this desired allocation with no output or (wage) inflation gap, we raise the interest rate by about 10 basis points in response to a 1% TFP shock.

Optimal stabilization policy in HANK follows the same principles, with minor quantitative departures. The planner largely stabilizes both output and (wage) inflation gaps, but not fully. The planner allows both to become briefly negative on impact, before becoming positive and



**Figure 3.** Optimal Policy Transition Dynamics — TFP Shock

**Note.** Transition dynamics after a positive TFP shock in both RANK (red) and HANK (blue) models under optimal monetary stabilization policy. Initial shock is 1% of steady state TFP and mean-reverts with a half-life of 2 quarters. Panels (A) through (C) report the dynamics of the output gap,  $\frac{Y_t - \bar{Y}_t}{\bar{Y}_t}$ , natural output, and the shock, all in percent deviations from the stationary Ramsey plan. Panels (D) through (F) report CPI inflation, wage inflation, and the optimal interest rate, all in percentage point deviations from the stationary Ramsey plan.

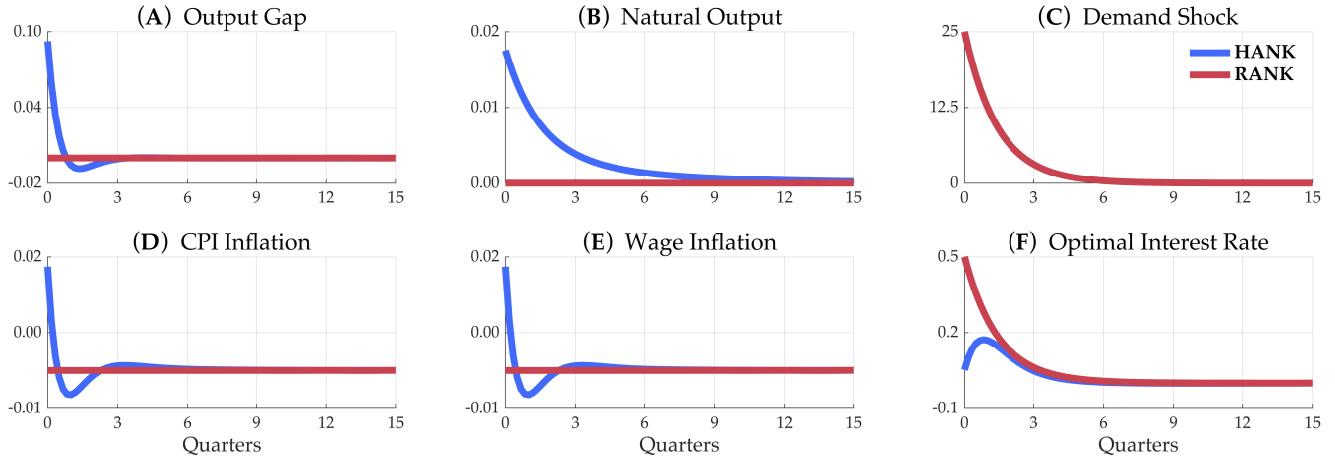
overshooting, yielding a hump-shaped response. The wage inflation gap on impact is small, reaching only  $-0.02\%$ , and consequently not meaningfully different from 0. Compared to the response of wage inflation under a Taylor rule, where the wage inflation gap opens up to  $0.4\%$  under the same shock, this deviation from the Divine Coincidence benchmark of RANK should be viewed as minimal. Similarly, while optimal policy stabilizes the output gap substantially relative to policy under the Taylor rule, the planner allows a small negative output gap to open up. The on-impact negative output gap under optimal policy is less than 20% of the size of the output gap under the Taylor rule.

## 5.2 Demand Shocks

We next turn to optimal stabilization policy in response to a demand shock. We display the associated transition dynamics in Figure 4. See Figure 9 in Appendix G.3 for the analogous impulse responses under a Taylor rule.

Optimal policy in response to a demand shock in RANK again features Divine Coincidence: the planner leans strongly against the shock and fully closes the output and wage inflation gaps. This allocation is supported by a 50bps increase in the nominal interest rate.

In HANK, we see natural output increase slightly after a positive demand shock, again because union bargaining now has a distributional term that leads to time variation in natural output even in the absence of cost-push shocks. The planner again leans against the demand shock, stabilizing



**Figure 4.** Optimal Policy Transition Dynamics — Demand Shock

**Note.** Transition dynamics after a positive discount rate shock in both RANK (red) and HANK (blue) models under optimal monetary stabilization policy. Discount rate shock is initialized at  $\rho_0 = 0.025$  and mean-reverts to its steady state value  $\rho = 0.02$ , with a half-life of 2 quarters. Panels (A) through (C) report the dynamics of the output gap,  $\frac{Y_t - \bar{Y}_t}{\bar{Y}_t}$ , natural output, and the shock, all in percent deviations from the stationary Ramsey plan. Panels (D) through (F) report CPI inflation, wage inflation, and the optimal interest rate, all in percentage point deviations from the stationary Ramsey plan.

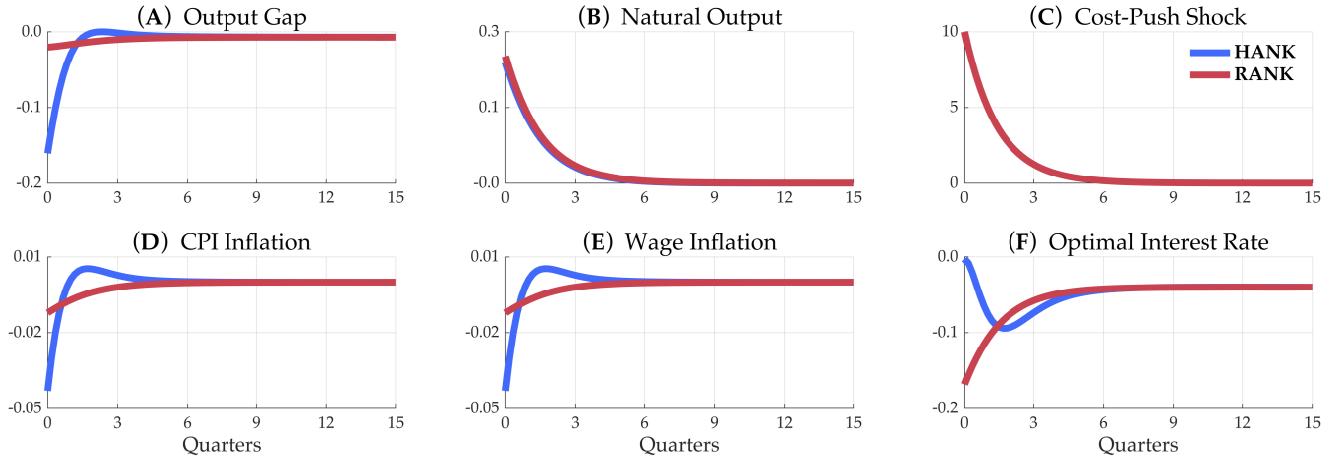
output and inflation gaps, but not as strongly as in RANK. Especially the output gap is allowed to open up meaningfully. The on-impact output gap response under optimal policy is only dampened by 50% relative to the Taylor rule case. The inflation gap, on the other hand, is stabilized almost entirely. Unlike in RANK, the path of interest rates that supports this allocation features a hump, where the planner only gradually increases the nominal rate.

### 5.3 Cost-Push Shocks

Finally, we consider a cost-push shock under which the desired wage mark-up of labor unions changes and natural output increases by 0.25%. We report the transition dynamics under optimal policy in Figure 5, and also report the analogous transition dynamics under a Taylor rule in Figure 10 in Appendix G for comparison.

In RANK, Divine Coincidence fails in the presence of cost-push shocks and the planner now faces a tradeoff between inflation and output. Optimal stabilization policy is accommodative, lowering the nominal interest rate, but a small negative output gap still opens up.

In HANK, natural output again increase but slightly less due to distributional concerns in union bargaining. Monetary policy eases substantially less than in RANK, allowing a sizable negative output gap to open up. However, there is still substantial stabilization relative to the Taylor rule case. Especially inflation is again stabilized substantially.



**Figure 5.** Optimal Policy Transition Dynamics — Cost-Push Shock

**Note.** Transition dynamics after a positive cost-push shock in both RANK (red) and HANK (blue) models under optimal monetary stabilization policy. The cost-push shock is modeled as an increase in labor union's desired wage mark-up. The shock is initialized at  $\epsilon_0 = 11$  and mean-reverts to its steady state value  $\epsilon = 10$ , with a half-life of 2 quarters. Panels (A) through (C) report the dynamics of the output gap,  $\frac{Y_t - \hat{Y}_t}{\hat{Y}_t}$ , natural output, and the shock, all in percent deviations from the stationary Ramsey plan. Panels (D) through (F) report CPI inflation, wage inflation, and the optimal interest rate, all in percentage point deviations from the stationary Ramsey plan.

## 6 Conclusion

In this paper, we have characterized optimal monetary policy in a HANK economy in which households face uninsurable idiosyncratic risk and wages are rigid. Following a timeless Ramsey approach, we have studied optimal long-run policy and optimal stabilization policy under commitment, analyzed the question of time consistency, and analyzed optimal policy under discretion. We show i) that zero inflation is the optimal long-run policy (in our baseline model), ii) that a planner faces two sources of time inconsistency (inflation and distributional) that non-trivially interact with each other, and iii) that the Divine Coincidence does not hold in HANK economies in response to non-cost-push shocks, even when it would in RANK economies.

Lastly, our paper establishes that time-consistent, optimal monetary stabilization policy generically requires a distributional target — alongside the classical inflation target — whenever a welfare criterion is used that has distributional considerations beyond aggregate efficiency. If societies consider expanding central banks' mandates in the future to include distributional considerations, they will also confront a new set of time consistency problems that may warrant a revision of the classical targeting framework.

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## A Proofs and Derivations for Section 3

In this Appendix, we present the proofs and main derivations for the results of Section 3. We start in Appendix A.1 by stating formally the implementability conditions for the Ramsey problem of Section 3 in continuous time. We also provide additional details on the generator and adjoining,  $\mathcal{A}_t$  and  $\mathcal{A}_t^*$ , which we use in the main text.

We then formally state and solve the standard primal Ramsey problem in Appendix A.2. The resulting optimality conditions, which we report in the main text in Proposition 1 hold everywhere in the interior of the state space over  $(a, z)$ .

A key challenge in solving Ramsey problems with heterogeneous agents is to formally account for boundary conditions, in particular the borrowing constraint at  $a$ . We find it convenient to derive all proofs that explicitly account for the boundary of the state space in a discretized version of our model. We follow Achdou et al. (2021) and work with a consistent finite-difference discretization of our continuous-time heterogeneous-agent equations, which of course converge in the limit to our baseline HANK economy. We follow this approach in the remainder of this appendix.

In recent work, González et al. (2021) follow a similar approach, first casting the optimal policy problem in continuous time, and then discretizing the resulting Ramsey plan conditions. The main difference between our paper and theirs is that they directly take their discretized system of equations to Dynare to obtain a numerical characterization of the Ramsey plan. We leverage the discretized equations to prove the main results on time consistency for our timeless Ramsey approach in Section 3. Our primary interest in discretizing the Ramsey plan conditions is to properly take into account the borrowing constraint faced by households, as well as the distribution mass point that emerges at the borrowing constraint.

### A.1 Competitive Equilibrium and Implementability

A competitive equilibrium of our baseline HANK model can be characterized by three blocks of equations. First, there is an individual block, explained in the text, which corresponds to the households' HJB, their optimality condition for consumption, and the Kolmogorov forward equation:

$$\begin{aligned} \rho V_t(a, z) &= \partial_t V_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2} (\pi_t^w)^2 + \mathcal{A}_t V_t(a, z), \\ u'(c_t(a, z)) &= \partial_a V_t(a, z) \\ \partial_t g_t(a, z) &= \mathcal{A}_t^* g_t(a, z). \end{aligned}$$

$\mathcal{A}_t$  is the infinitesimal generator of the process  $(a_t, z_t)$ . Intuitively, it captures an agent's perceived law of motion of the process  $d(a_t, z_t)$ . It is analogous to a transition matrix in discrete time, and it is

defined by

$$\mathcal{A}_t f_t(a, z) = \left( r_t a + z w_t N_t - c_t(a, z) \right) \partial_a f_t(a, z) + \mathcal{A}_z f_t(a, z), \quad (53)$$

for any function  $f_t(a, z) : \mathbb{R}^3 \rightarrow \mathbb{R}$ , where  $\mathcal{A}_z$  is an additively separable component that captures perceived transition dynamics of earnings risk. We leave the structure of  $\mathcal{A}_z$  fully general in our derivations, except that we assume it to be independent from policy. Our baseline results currently do not apply to the case of counter-cyclical earnings risk that responds to monetary policy, for example, but extending our approach to this more general case is straightforward.

We denote the adjoint of the infinitesimal generator by  $\mathcal{A}_t^*$ . The adjoint is defined by

$$\mathcal{A}_t^* f_t(a, z) = -\partial_a \left[ \left( r_t a + z w_t N_t - c_t(a, z) \right) f_t(a, z) \right] + \mathcal{A}_z^* f_t(a, z), \quad (54)$$

where  $\mathcal{A}_z^*$  is the adjoint of  $\mathcal{A}_z$ .

Second, there is an aggregate block, which includes the New Keynesian wage Phillips curve, the production technology, the wage equation, the Fisher equation, and an equation that relates price and wage inflation:

$$\dot{\pi}_t^w = \rho \pi_t^w + \frac{\epsilon - 1}{\delta} \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) w_t \Lambda_t - v'(N_t) \right] N_t$$

$$Y_t = A_t N_t$$

$$w_t = A_t$$

$$r_t = i_t - \pi_t$$

$$\pi_t = \pi_t^w - \frac{\dot{A}_t}{A_t}.$$

Finally, we have the market clearing conditions in the goods and bond markets, given by

$$Y_t = C_t = \iint c_t(a, z) g_t(a, z) da dz$$

$$0 = B_t = \iint a g_t(a, z) da dz.$$

The following Lemma defines the set of implementability conditions that act as constraints for a Ramsey planner.

**Lemma 13. (Implementability conditions)** *The set of equations that define an equilibrium can be*

expressed as implementability conditions for a standard primal Ramsey problem as follows:

$$\begin{aligned}
\rho V_t(a, z) &= \partial_t V_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2}(\pi_t^w)^2 + \mathcal{A}_t V_t(a, z) \\
u'(c_t(a, z)) &= \partial_a V_t(a, z) \\
\partial_t g_t(a, z) &= \mathcal{A}_t^* g_t(a, z) \\
0 &= A_t N_t - \iint c_t(a, z) g_t(a, z) da dz \\
r_t &= i_t - \pi_t^w + \frac{\dot{A}_t}{A_t} \\
\dot{\pi}_t^w &= \rho \pi_t^w + \frac{\epsilon - 1}{\delta} \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_t \iint z u'(c_t(a, z)) g_t(a, z) da dz - v'(N_t) \right] N_t
\end{aligned}$$

## A.2 Derivation of Ramsey Plan in Continuous Time

In this section, we prove Proposition 1 and derive the continuous-time optimality conditions that characterize the optimal Ramsey plan in the interior of the state space. We defer a formal treatment of the boundary and associated boundary conditions to the following sections.

It is useful to adopt more compact notation for this derivation. In particular, we drop time subscripts and make implicit the dependence of individual variables on states, so that  $c_t(a, z)$  simply becomes  $c$ . Furthermore, we now reserve subscripts to denote partial derivatives, so that  $\partial_t c_t(a, z)$  will simply become  $c_t$ .

The functional Lagrangian associated with the standard primal Ramsey problem is given by

$$\begin{aligned}
L^{\text{SP}}(g_0) = \int_0^\infty e^{-\rho t} \Bigg\{ &\iint \left\{ \left[ u(c) - v(N) - \frac{\delta}{2}(\pi^w)^2 \right] g \right. \\
&+ \phi \left[ -\rho V + V_t + u(c) - v(N) - \frac{\delta}{2}(\pi^w)^2 + \mathcal{A}V \right] \\
&+ \chi \left[ u'(c) - V_a \right] \\
&+ \lambda \left[ -g_t + \mathcal{A}^* g \right] \Big\} da dz \\
&- \mu \left[ \iint c g da dz - AN \right] \\
&+ \vartheta \left[ -\dot{\pi}^w + \rho \pi^w + \frac{\epsilon - 1}{\delta} \left( \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A \Lambda - v'(N) \right) N \right] \Bigg\} dt
\end{aligned}$$

We now use the following auxilliary results, integrating various partial derivatives in the

above Lagrangian by parts. We have

$$\begin{aligned} \int_0^\infty \iint \left[ e^{-\rho t} \phi V_t \right] d(a, z) dt &= \int \left[ -\nu(0, a, z) V(0, a, z) + \rho \int_0^\infty e^{-\rho t} \phi V dt - \int_0^\infty e^{-\rho t} \phi_t V dt \right] d(a, z) \\ \int_0^\infty \iint \left[ e^{-\rho t} \lambda g_t \right] d(a, z) dt &= \int \left[ -\lambda(0, a, z) g(0, a, z) + \rho \int_0^\infty e^{-\rho t} \lambda g dt - \int_0^\infty e^{-\rho t} \lambda_t g dt \right] d(a, z) \\ \int_0^\infty \left[ e^{-\rho t} \vartheta \pi_t \right] dt &= -\vartheta(0) \pi(0) + \rho \int_0^\infty e^{-\rho t} \vartheta \pi dt - \int_0^\infty e^{-\rho t} \vartheta_t \pi dt. \end{aligned}$$

Next, for the adjoint, we have

$$-\int_0^\infty e^{-\rho t} \iint \lambda \mathcal{A}^* g d(a, z) dt = -\int_0^\infty e^{-\rho t} \iint (\mathcal{A} \lambda) g d(a, z) dt,$$

where we drop boundary terms, which we consider formally in the following subsections. And for the generator, we have

$$\int_0^\infty e^{-\rho t} \int \phi \mathcal{A} V d a d z d t = \int_0^\infty e^{-\rho t} \iint V \mathcal{A}^* \phi d a d z d t$$

Finally, for the consumption FOC, we simply have

$$-\int_0^\infty e^{-\rho t} \iint \chi V_a d a d z d t = \int_0^\infty e^{-\rho t} \iint \chi_a V d a d z d t,$$

where we also drop boundary terms.

The functional Lagrangian can thus be rewritten as

$$\begin{aligned} L^{\text{SP}}(g_0) = \int_0^\infty e^{-\rho t} \left\{ \iint \left\{ \begin{aligned} &\left[ u(c) - \mu c - v(N) - \frac{\delta}{2} (\pi^w)^2 \right] g \\ &- V \phi_t + V \mathcal{A}^* \phi + \phi \left[ u(c) - v(N) - \frac{\delta}{2} (\pi^w)^2 \right] \\ &+ \chi u'(c) + \chi_a V \\ &+ g \lambda_t - \rho \lambda g + g \mathcal{A} \lambda \end{aligned} \right\} d a d z \right. \\ &+ \mu A N \\ &\left. + \vartheta_t \pi^w + \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A \Lambda - v'(N) \right) N \right\} d t \end{aligned}$$

We now consider a general functional perturbation around a candidate optimal Ramsey plan, and parametrize this perturbation by  $\alpha \in \mathbb{R}$ . Since  $\alpha$  is a scalar, the maximum principle then implies

that our candidate plan can only be optimal if  $L_\alpha^{\text{SP}}(g_0, \alpha) |_{\alpha=0} = 0$ .

We have

$$\begin{aligned}
L^{\text{SP}}(g_0, \alpha) = & \int_0^\infty e^{-\rho t} \left\{ \iint \left\{ \left[ u(c + \alpha h_c) - \mu(c + \alpha h_c) - v(N + \alpha h_N) - \frac{\delta}{2}(\pi^w + \alpha h_\pi)^2 \right] (g + \alpha h_g) \right. \right. \\
& - (V + \alpha h_V)\phi_t + (V + \alpha h_V)\mathcal{A}^*(\alpha)\phi \\
& + \phi \left[ u(c + \alpha h_c) - v(N + \alpha h_N) - \frac{\delta}{2}(\pi^w + \alpha h_\pi)^2 \right] \\
& + \chi u'(c + \alpha h_c) + \chi_a(V + \alpha h_V) \\
& \left. \left. + (g + \alpha h_g)\lambda_t - \rho\lambda(g + \alpha h_g) + (g + \alpha h_g)\mathcal{A}(\alpha)\lambda \right\} dadz \right. \\
& + \mu A(N + \alpha h_N) + \vartheta_t(\pi^w + \alpha h_\pi) \\
& \left. + \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A \iint u'(c + \alpha h_c)(g + \alpha h_g) dadz - v'(N + \alpha h_N) \right) (N + \alpha h_N) \right\} dt
\end{aligned}$$

We now differentiate and take the limit  $\alpha \rightarrow 0$ . Setting the resulting expression to 0, we have the following first-order necessary condition for optimality:

$$\begin{aligned}
0 = & \int_0^\infty e^{-\rho t} \left\{ \iint \left\{ \left[ u'(c)h_c - \mu h_c - v'(N)h_N - \delta\pi^w h_\pi \right] g + h_g \left[ u(c) - \mu c - v(N) - \frac{\delta}{2}(\pi^w)^2 \right] \right. \right. \\
& - h_V\phi_t + h_V\mathcal{A}^*(0)\phi + V \frac{d}{d\alpha} \mathcal{A}^*(0)\phi + \phi \left[ u'(c)h_c - v'(N)h_N - \delta\pi^w h_\pi \right] \\
& + \chi u''(c)h_c + \chi_a h_V \\
& \left. \left. + h_g\lambda_t - \rho\lambda h_g + h_g\mathcal{A}(0)\lambda + g \frac{d}{d\alpha} \mathcal{A}(0)\lambda \right\} dadz \right. \\
& + \mu Ah_N + \vartheta_t h_\pi \\
& \left. + \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A \iint [u'(c)h_g + u''(c)gh_c] dadz - v'(N)h_N \right) N \right. \\
& \left. + h_N \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A \Lambda - v'(N) \right) \right\} dt
\end{aligned}$$

where we have

$$\frac{d}{d\alpha} \mathcal{A}(0) = (ah_r + zh_w N + zw h_N - h_c) \partial_a$$

and, again dropping boundary terms,

$$\begin{aligned} V \frac{d}{d\alpha} \mathcal{A}^*(0) \phi &= \phi \frac{d}{d\alpha} \mathcal{A}(0) V \\ &= \phi(ah_r + zh_w N + zwh_N - h_c) V_a. \end{aligned}$$

Finally, we group terms by  $h_c$ ,  $h_g$ , etc., and invoke the fundamental lemma of the calculus of variations. We directly obtain the optimality conditions that characterize the optimal Ramsey plan of Proposition 1 in the interior of the state space.

In the following, we provide a formal treatment of boundary conditions. To that end, we start by discretizing the conditions for competitive equilibrium in the spirit of Achdou et al. (2021).

### A.3 Discretized Competitive Equilibrium Conditions

We now prove a discretized representation of competitive equilibria in our baseline HANK model. This will elucidate how boundary conditions are treated formally by the Ramsey planner. For any function  $c_t(a, z)$ , we discretize both in the state space and in time, so that we write  $c_n$  for  $n = 0, \dots, N$ . In particular,  $c_n$  is a  $J \times 1$  vector, so that  $c_{i,n} = c_{t_n}(a_i, z_i)$  associated with grid point  $i$ . We also use notation  $c_{n,[2:J]}$ , for example, to denote the  $(J-1) \times 1$  vector consisting of elements 2 through  $J$  in  $c_n$ .

We summarize the discretized competitive equilibrium conditions of our model in the following Lemma, using a finite-difference discretization given a policy path  $\theta = \{\theta_n\}_{n \geq 0}$ . The proof follows along the lines of Achdou et al. (2021) and Schaab and Zhang (2021), and we refer the interested reader to those papers. This characterization will justify setting up the Ramsey problem using the following discretized equations as implementability conditions.

**Lemma 14.** *A consistent finite-difference discretization of the implementability conditions of our baseline HANK model is as follows. For the Hamilton-Jacobi-Bellman equation, we have*

$$\begin{aligned} \rho V_n &= \frac{V_{n+1} - V_n}{dt} + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{c_{n,[2:J]}} \right) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 \\ &\quad + \left( \frac{0}{i_n a_{[2:J]} - \pi_n^w a_{[2:J]} + \frac{A_{n+1} - A_n}{dt A_n} a_{[2:J]} + z_{[2:J]} A_n N_n - c_{n,[2:J]}} \right) \cdot \frac{D_a}{da} V_n + A^z V_n \end{aligned}$$

For the consumption first-order condition of the household, we simply have

$$u'(c_{n,[2:J]}) = \left( \frac{D_a}{da} V_{n+1} \right)_{[2:J]}$$

For the Kolmogorov forward equation, we have

$$\frac{\mathbf{g}_{n+1} - \mathbf{g}_n}{dt} = (\mathbf{A}^z)' \mathbf{g}_n + \frac{\mathbf{D}'_a}{da} \left[ \begin{pmatrix} 0 \\ i_n \mathbf{a}_{[2:J]} - \pi_n^w \mathbf{a}_{[2:J]} + \frac{A_{n+1} - A_n}{dtA_n} \mathbf{a}_{[2:J]} + \mathbf{z}_{[2:J]} A_n N_n - \mathbf{c}_{n,[2:J]} \end{pmatrix} \cdot \mathbf{g}_n \right]$$

Finally, for the resource constraint we simply have

$$A_n N_n = \mathbf{c}'_n \mathbf{g}_n d\mathbf{x}$$

and for the Phillips curve

$$\frac{\pi_{n+1}^w - \pi_n^w}{dt} = \rho \pi_n^w + \frac{\epsilon}{\delta} \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot u'(\mathbf{c}_n))' \mathbf{g}_n d\mathbf{x} - v'(N_n) \right] N_n$$

and we have already used  $c_{n,1} = i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dtA_n} a_1 + z_1 A_n N_n$ .

Crucially, the discretized system of equations in the above Lemma properly accounts for the household borrowing constraint, leveraging results from Achdou et al. (2021). In particular, they prove that in the simple Huggett economy with two earnings states the only point in the state space where the borrowing constraint binds is  $(a, z^L)$ . We use this result here to plug in the borrowing constraint directly at that discretized point. While we have not formally proven that their representation extends to our HANK economy, we verify its validity numerically ex post. And since the stationary equilibrium of our model is almost identical to theirs, there is little reason to expect any sharp discrepancies in this context.

#### A.4 Discretized Standard Ramsey Problem

The standard primal Ramsey problem in our baseline HANK model is associated with the discretized Lagrangian

$$\begin{aligned}
L^{\text{SP}}(\mathbf{g}_0) = & \min_{\{\phi_n, \chi_n, \lambda_n, \mu_n, \vartheta_n\}} \max_{\{\mathbf{V}_n, \mathbf{c}_{n,[2:J]}, \mathbf{g}_n, \pi_n^w, N_n, i_n\}} \sum_{n=0}^{N-1} e^{-\rho t_n} \left\{ \right. \\
& + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{\mathbf{c}_{n,[2:J]}} \right)' \mathbf{g}_t - v(N_n) \mathbf{1}' \mathbf{g}_t - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1}' \mathbf{g}_t \\
& + \boldsymbol{\nu}'_n \left[ -\rho \mathbf{V}_n + \frac{\mathbf{V}_{n+1} - \mathbf{V}_n}{dt} + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{\mathbf{c}_{n,[2:J]}} \right) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 \right] \\
& + \boldsymbol{\nu}'_n \mathbf{A}^z \mathbf{V}_n + \sum_{i \geq 2} \nu_{i,n} \left( i_n a_i - \pi_n^w a_i + \frac{A_{n+1} - A_n}{dt A_n} a_i + z_i A_n N_n - c_{n,i} \right) \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_n}{da} \\
& + \chi'_{n,[2:J]} \left[ u'(\mathbf{c}_{n,[2:J]}) - \left( \frac{\mathbf{D}_a \mathbf{V}_{n+1}}{da} \right)_{[2:J]} \right] \\
& - \omega'_n \frac{\mathbf{g}_{n+1} - \mathbf{g}_n}{dt} + \omega'_n (\mathbf{A}^z)' \mathbf{g}_n \\
& + \sum_{i \geq 2} \omega_{n,i} \frac{\mathbf{D}'_{a,[i,:]} \mathbf{V}_n}{da} \left[ \left( \frac{0}{i_n \mathbf{a}_{[2:J]} - \pi_n^w \mathbf{a}_{[2:J]} + \frac{A_{n+1} - A_n}{dt A_n} \mathbf{a}_{[2:J]} + \mathbf{z}_{[2:J]} A_n N_n - \mathbf{c}_{n,[2:J]}} \right) \cdot \mathbf{g}_n \right] \left. \right\} d\mathbf{x} \\
& + \mu_n \left[ \mathbf{c}'_n \mathbf{g}_n d\mathbf{x} - A_n N_n \right] \\
& + \vartheta_n \left[ -\frac{\pi_{n+1}^w - \pi_n^w}{dt} + \rho \pi_n^w + \frac{\epsilon - 1}{\delta} \left( \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot u'(\mathbf{c}_n))' \mathbf{g}_n d\mathbf{x} - v'(N_n) \right) N_n \right] \left. \right\} dt
\end{aligned}$$

where the planner takes as given an initial condition for the cross-sectional distribution,  $\mathbf{g}_0$ .

As in Appendix A.3, we fix from the beginning that unemployed households at the borrowing constraint always consume their income, that is

$$c_{n,1} = i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n$$

for all  $n$ . The planner takes this as given and does not get to consider perturbations in  $c_{n,1}$  for any  $n$ .

We want to emphasize at this point how important it is exactly which finite-difference stencils are used for the discretization. For discretization in the time dimension, for example, the above Lagrangian assumes a *semi-implicit backwards* discretization of  $\partial_t V_t$  in the HJB. And it assumes an *explicit forwards* discretization of  $\partial_t g_t$  in the KF equation. For the aggregates, it assumes an *explicit forwards* discretization for  $\dot{A}_t$  and also an *explicit forwards* discretization for  $\dot{\pi}_t^w$ . These assumptions

also correspond to the appropriate stencils we use numerically to implement our results.

We also want to echo Achdou et al. (2021) at this point, recalling that the correct discretization stencil for the KF equation in the wealth dimension is given by

$$(A^a)'g = \frac{1}{da}(s \cdot D_a)'g = \frac{1}{da}D_a'(s \cdot g).$$

That is, the correct stencil uses the transpose  $D_a'$  rather than, as one might have expected,  $-D_a(s \cdot g)$ .

## A.5 Auxilliary Results

Before tackling the main proof of this appendix, we state several auxilliary results that will be helpful below. Most of these results follow trivially by applying well-known properties of matrix algebra. We consequently provide only some of the proofs explicitly.

**Lemma 15.** *The following matrix algebra tricks will be useful. Let  $x, y$  and  $z$  be  $J \times 1$  vectors and  $A$  a  $J \times J$  matrix. Transposition satisfies*

$$(Ax)' = x'A'.$$

We also have

$$x'Ay = \sum_i x_i A_{[i,:]} y = \sum_i x_i \sum_j A_{[i,j]} y_j = \sum_j y_j \sum_i A'_{[j,i]} x_i = y'A'x.$$

We also have

$$x'(y \cdot A)z = x'(y \cdot (Az)) = (x \cdot y)'Az = (Az)'(x \cdot y) = z'A'(x \cdot y) = z'(y \cdot A)'x.$$

Taking derivatives, we have

$$\begin{aligned} \frac{d}{dx}x'Ay &= Ay \\ \frac{d}{dx}y'Ax &= (y'A)' = A'y \end{aligned}$$

**Lemma 16.** In the Lagrangian, the HJB term can be rearranged as follows:

$$\begin{aligned}
\frac{1}{da} \sum_{i \geq 2} v_i s_i D_{a,[i,:]} V &= \frac{1}{da} \sum_{i \geq 1} v_i s_i D_{a,[i,:]} V \\
&= \frac{1}{da} \boldsymbol{\nu}' (\boldsymbol{s} \cdot \boldsymbol{D}_a) V \\
&= \frac{1}{da} V' (\boldsymbol{s} \cdot \boldsymbol{D}_a)' \boldsymbol{\nu} \\
&= \frac{1}{da} V' \boldsymbol{D}'_a (\boldsymbol{s} \cdot \boldsymbol{\nu})
\end{aligned}$$

where  $\boldsymbol{D}_a$  is the upwind finite-difference matrix in the  $a$  dimension. We sometimes use  $\boldsymbol{s} \cdot \boldsymbol{D}_a = \boldsymbol{A}^a$ .

*Proof.* We have

$$\begin{aligned}
\sum_{i \geq 2} v_i s_i D_{a,[i,:]} V &= \sum_{i \geq 2} v_i s_i \sum_{j \geq 1} D_{a,[i,j]} V_j \\
&= \sum_{i \geq 2} v_i s_i \sum_{j \geq 1} \boldsymbol{D}'_{a,[j,i]} V_j \\
&= \sum_{j \geq 1} V_j \sum_{i \geq 2} \boldsymbol{D}'_{a,[j,i]} v_i s_i \\
&= \sum_{j \geq 1} V_j \sum_{i \geq 1} \boldsymbol{D}'_{a,[j,i]} v_i s_i \\
&= \sum_{j \geq 1} V_j \boldsymbol{D}'_{a,[j,:]} (\boldsymbol{s} \cdot \boldsymbol{\nu}) \\
&= V' \boldsymbol{D}'_a (\boldsymbol{s} \cdot \boldsymbol{\nu})
\end{aligned}$$

where  $D'_{a,[j,:]}$  denotes the  $j$ th row of the matrix  $\boldsymbol{D}'_a$ . ■

**Lemma 17.** The correct adjoint operation, i.e., the one we use to define  $\mathcal{A}^* \approx \boldsymbol{A}'$ , is given by

$$\boldsymbol{D}'_a (\boldsymbol{s} \cdot \boldsymbol{\nu}) = (\boldsymbol{A}^a)' \boldsymbol{\nu}.$$

In particular, we have

$$\begin{aligned}
\omega'(A^a)'g &= \omega'D_a'(s \cdot g) \\
&= (s \cdot g)'D_a\omega \\
&= g'(s \cdot D_a)\omega \\
&= (D_a\omega)'(s \cdot g) \\
&= (D_a\omega)' \left[ \begin{pmatrix} 0 \\ r\alpha_{[2:J]} + z_{[2:J]} - c_{[2:J]} - G \end{pmatrix} \cdot g \right].
\end{aligned}$$

**Lemma 18.** We can “integrate by parts” the FOC term in the Lagrangian to arrive at

$$\frac{1}{da}\chi'_{t(n),[2:J]}\left(D_aV_{t(n+1)}\right)_{[2:J]} = \frac{1}{da}V'_{t(n+1)}D'_a \begin{pmatrix} 0 \\ \chi_{[2:J],t(n)} \end{pmatrix}.$$

*Proof.* We have

$$\begin{aligned}
\frac{1}{da} \sum_{i \geq 2} \chi_{i,t(n)} D_{a,[i,:]} V_{t(n+1)} &= \frac{1}{da} \sum_{i \geq 2} \chi_{i,t(n)} \sum_{j \geq 1} D_{a,[i,j]} V_{j,t(n+1)} \\
&= \frac{1}{da} \sum_{j \geq 1} V_{j,t(n+1)} \sum_{i \geq 2} D'_{a,[j,i]} \chi_{i,t(n)} \\
&= \frac{1}{da} \sum_{j \geq 1} V_{j,t(n+1)} D'_{a,[j,:]} \begin{pmatrix} 0 \\ \chi_{[2:J],t(n)} \end{pmatrix} \\
&= \frac{1}{da} V'_{t(n+1)} D'_a \begin{pmatrix} 0 \\ \chi_{[2:J],t(n)} \end{pmatrix}.
\end{aligned}$$

It is important to note that we *cannot* roll the sum  $\sum_{i \geq 2}$  forward to simply read  $\sum_{i \geq 1}$ . This is only possible for the terms that include savings, using the fact that  $s_1 = 0$ . ■

**Lemma 19.** We can “integrate by parts” in the time dimension as follows. For any  $x_n$ , we have

$$\sum_{n=0}^{N-1} e^{-\rho t_n} x_{n+1} = e^{\rho dt} \sum_{n=0}^{N-1} e^{-\rho t_n} x_n - e^{\rho dt} x_0 + e^{\rho dt} e^{-\rho t_N} x_N.$$

We prove the following results below. In particular, this implies

$$\sum_{n=0}^{N-1} e^{-\rho t_n} \boldsymbol{\nu}'_n \mathbf{V}_{n+1} = \sum_{n=0}^{N-1} e^{-\rho t_n} e^{\rho dt} \boldsymbol{\nu}'_{n-1} \mathbf{V}_n - e^{\rho dt} \boldsymbol{\nu}'_{-1} \mathbf{V}_0 + e^{\rho dt} e^{-\rho t_N} \boldsymbol{\nu}_{N-1} \mathbf{V}_N$$

as well as

$$\begin{aligned} - \sum_{n=0}^{N-1} e^{-\rho t_n} \boldsymbol{\omega}'_n \frac{\mathbf{g}_{n+1} - \mathbf{g}_n}{dt} &= \sum_{n=0}^{N-1} e^{-\rho t_n} \frac{\boldsymbol{\omega}'_n - e^{\rho dt} \boldsymbol{\omega}'_{n-1}}{dt} \mathbf{g}_n \\ &\quad + \frac{1}{dt} e^{\rho dt} \boldsymbol{\omega}'_{-1} \mathbf{g}_0 - \frac{1}{dt} e^{\rho dt} e^{-\rho t_N} \boldsymbol{\omega}'_{N-1} \mathbf{g}_N \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{N-1} e^{-\rho t_n} \boldsymbol{\chi}'_{n,[2:J]} \left( \frac{\mathbf{D}_a}{da} \mathbf{V}_{n+1} \right)_{[2:J]} &= \sum_{n=0}^{N-1} e^{-\rho t_n} e^{\rho dt} \boldsymbol{\chi}'_{n-1,[2:J]} \left( \frac{\mathbf{D}_a}{da} \mathbf{V}_n \right)_{[2:J]} \\ &\quad - e^{\rho dt} \boldsymbol{\chi}'_{-1,[2:J]} \left( \frac{\mathbf{D}_a}{da} \mathbf{V}_0 \right)_{[2:J]} + e^{\rho dt} e^{-\rho t_N} \boldsymbol{\chi}'_{N-1,[2:J]} \left( \frac{\mathbf{D}_a}{da} \mathbf{V}_N \right)_{[2:J]} \end{aligned}$$

Finally, we have

$$- \sum_{n=0}^{N-1} e^{-\rho t_n} \vartheta_n \frac{\pi_{n+1}^w - \pi_n^w}{dt} = \sum_{n=0}^{N-1} e^{-\rho t_n} \frac{\vartheta_n - e^{\rho dt} \vartheta_{n-1}}{dt} \pi_n^w + \frac{1}{dt} e^{\rho dt} \vartheta_{-1} \pi_0^w - \frac{1}{dt} e^{\rho dt} e^{-\rho t_N} \vartheta_{N-1} \pi_N^w$$

*Proof.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-\rho t(n)} \boldsymbol{\nu}'_{t(n)} \frac{1}{dt} \mathbf{V}_{t(n+1)} &= \sum_{n=0}^{\infty} e^{-\rho t(n)} e^{\rho t(n+1)} e^{-\rho t(n+1)} \boldsymbol{\nu}'_{t(n)} \frac{1}{dt} \mathbf{V}_{t(n+1)} \\ &= \sum_{n=0}^{\infty} e^{-\rho t(n+1)} e^{\rho t(n+1) - \rho t(n)} \boldsymbol{\nu}'_{t(n)} \frac{1}{dt} \mathbf{V}_{t(n+1)} \\ &= \sum_{n=1}^{\infty} e^{-\rho t(n)} e^{\rho dt} \boldsymbol{\nu}'_{t(n-1)} \frac{1}{dt} \mathbf{V}_{t(n)} \\ &= \sum_{n=1}^{\infty} e^{-\rho t(n)} e^{\rho dt} \boldsymbol{\nu}'_{t(n-1)} \frac{1}{dt} \mathbf{V}_{t(n)} + e^{-\rho t(0)} e^{\rho dt} \boldsymbol{\nu}'_{-1} \frac{1}{dt} \mathbf{V}_{t(0)} - e^{-\rho t(0)} e^{\rho dt} \boldsymbol{\nu}'_{-1} \frac{1}{dt} \mathbf{V}_{t(0)} \\ &= \sum_{n=0}^{\infty} e^{-\rho t(n)} e^{\rho dt} \boldsymbol{\nu}'_{t(n-1)} \frac{1}{dt} \mathbf{V}_{t(n)} - e^{-\rho t(0)} e^{\rho dt} \boldsymbol{\nu}'_{-1} \frac{1}{dt} \mathbf{V}_{t(0)}. \end{aligned}$$

Similarly, we can rearrange

$$\begin{aligned}
\sum_{n=0}^{\infty} e^{-\rho t(n)} \chi'_{t(n),[2:J]} \left( \frac{D_a}{da} V_{t(n+1)} \right)_{[2:J]} &= \sum_{n=0}^{\infty} e^{-\rho t(n)} e^{\rho t(n+1)} e^{-\rho t(n+1)} \chi'_{t(n),[2:J]} \left( \frac{D_a}{da} V_{t(n+1)} \right)_{[2:J]} \\
&= \sum_{n=0}^{\infty} e^{-\rho t(n+1)} e^{\rho t(n+1)-\rho t(n)} \chi'_{t(n),[2:J]} \left( \frac{D_a}{da} V_{t(n+1)} \right)_{[2:J]} \\
&= \sum_{n=1}^{\infty} e^{-\rho t(n)} e^{\rho dt} \chi'_{t(n-1),[2:J]} \left( \frac{D_a}{da} V_{t(n)} \right)_{[2:J]} \\
&= \sum_{n=0}^{\infty} e^{-\rho t(n)} e^{\rho dt} \chi'_{t(n-1),[2:J]} \left( \frac{D_a}{da} V_{t(n)} \right)_{[2:J]} - e^{-\rho t(0)} e^{\rho dt} \chi'_{-1,[2:J]} \left( \frac{D_a}{da} V_{t(0)} \right)_{[2:J]}
\end{aligned}$$

Finally, notice that

$$\begin{aligned}
e^{\rho dt} \nu'_{t(n-1)} \frac{1}{dt} V_{t(n)} &= (1 + \rho dt) \nu'_{t(n-1)} \frac{1}{dt} V_{t(n)} \\
&= \nu'_{t(n-1)} \frac{1}{dt} V_{t(n)} + \rho \nu'_{t(n-1)} V_{t(n)}
\end{aligned}$$

Lastly,

$$\begin{aligned}
-\sum_{n=0}^{\infty} e^{-\rho t(n)} \omega'_{t(n)} \frac{g_{t(n+1)} - g_{t(n)}}{dt(n)} &= -\sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n+1)} + \sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n)} \\
&= -\frac{1}{dt} \sum_{n=0}^{\infty} e^{-\rho t(n)} e^{\rho t(n+1)} e^{-\rho t(n+1)} \omega'_{t(n)} g_{t(n+1)} + \sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n)} \\
&= -\frac{1}{dt} e^{\rho dt} \sum_{n=0}^{\infty} e^{-\rho t(n+1)} \omega'_{t(n)} g_{t(n+1)} + \sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n)} \\
&= -\frac{1}{dt} e^{\rho dt} \sum_{n=1}^{\infty} e^{-\rho t(n)} \omega'_{t(n-1)} g_{t(n)} + \sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n)}
\end{aligned}$$

And so we get

$$\begin{aligned}
&= -\frac{1}{dt} e^{\rho dt} \sum_{n=1}^{\infty} e^{-\rho t(n)} \omega'_{t(n-1)} g_{t(n)} + \frac{1}{dt} e^{\rho dt} e^{-\rho t(0)} \omega'_{t(-1)} g_{t(0)} - \frac{1}{dt} e^{\rho dt} e^{-\rho t(0)} \omega'_{t(-1)} g_{t(0)} + \sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n)} \\
&= -\frac{1}{dt} e^{\rho dt} \sum_{n=0}^{\infty} e^{-\rho t(n)} \omega'_{t(n-1)} g_{t(n)} + \frac{1}{dt} e^{\rho dt} e^{-\rho t(0)} \omega'_{t(-1)} g_{t(0)} + \sum_{n=0}^{\infty} e^{-\rho t(n)} \frac{1}{dt} \omega'_{t(n)} g_{t(n)} \\
&= \sum_{n=0}^{\infty} e^{-\rho t(n)} \left( \frac{1}{dt} \omega'_{t(n)} - \frac{1}{dt} e^{\rho dt} \omega'_{t(n-1)} \right) g_{t(n)} + \frac{1}{dt} e^{\rho dt} e^{-\rho t(0)} \omega'_{t(-1)} g_{t(0)} \\
&= \sum_{n=0}^{\infty} e^{-\rho t(n)} \left( \frac{\omega'_{t(n)} - \omega'_{t(n-1)}}{dt} - \rho \omega'_{t(n-1)} \right) g_{t(n)} + \frac{1}{dt} e^{\rho dt} e^{-\rho t(0)} \omega'_{t(-1)} g_{t(0)}
\end{aligned}$$

Finally, we drop the second term on the RHS because  $g_{t(0)}$  is fixed as an initial condition and so it does not respond to  $\frac{d}{d\theta}$ , which is precisely why the KFE is not a forward-looking constraint. ■

**Lemma 20.** *In the continuous time limit as  $dt \rightarrow 0$ , we have*

$$e^{\rho dt} \approx 1 + \rho dt.$$

## A.6 Proof of Proposition 1 with Boundary Condition

We are now ready to present our main proof. We use the auxilliary results above to rewrite the discretized Lagrangian that corresponds to the standard primal Ramsey problem of Section 3 as

$$\begin{aligned}
L^{\text{SP}}(\mathbf{g}_0) = & \min_{\{\phi_n, \chi_n, \lambda_n, \mu_n, \vartheta_n\}} \max_{\{V_n, c_{n,[2:J]}, g_n, \pi_n^w, N_n, i_n\}} \sum_{n=0}^{N-1} e^{-\rho t_n} \left\{ \begin{array}{c} \left( i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1}-A_n}{dt A_n} a_1 + z_1 A_n N_n \right)' \mathbf{g}_n \\ \mathbf{c}_{n,[2:J]} \end{array} \right\} \\
& + \mu_n \left( i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1}-A_n}{dt A_n} a_1 + z_1 A_n N_n \right)' \mathbf{g}_n - v(N_n) \mathbf{1}' \mathbf{g}_n - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1}' \mathbf{g}_n \\
& - \frac{\nu'_n - e^{\rho dt} \nu'_{n-1}}{dt} V_n + \nu'_n \left[ -\rho V_n + u \left( i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1}-A_n}{dt A_n} a_1 + z_1 A_n N_n \right) \mathbf{c}_{n,[2:J]} - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 \right] \\
& + \nu'_n A^z V_n + \frac{1}{da} V'_n (\nu_n \cdot D_a)' \begin{pmatrix} 0 \\ i_n \mathbf{a}_{[2:J]} - \pi_n^w \mathbf{a}_{[2:J]} + \frac{A_{n+1}-A_n}{dt A_n} \mathbf{a}_{[2:J]} + z_{[2:J]} A_n N_n - \mathbf{c}_{n,[2:J]} \end{pmatrix} \\
& + \chi'_{n,[2:J]} u'(\mathbf{c}_{n,[2:J]}) - e^{\rho dt} \frac{1}{da} V'_n D'_a \begin{pmatrix} 0 \\ \chi_{n-1,[2:J]} \end{pmatrix} \\
& + \frac{\omega'_n - e^{\rho dt} \omega'_{n-1}}{dt} g_n + \omega'_n (A^z)' \mathbf{g}_n \\
& + \frac{1}{da} (D_a \omega_t)' \left[ \begin{pmatrix} 0 \\ i_n \mathbf{a}_{[2:J]} - \pi_n^w \mathbf{a}_{[2:J]} + \frac{A_{n+1}-A_n}{dt A_n} \mathbf{a}_{[2:J]} + z_{[2:J]} A_n N_n - \mathbf{c}_{n,[2:J]} \end{pmatrix} \cdot \mathbf{g}_n \right] \} d\mathbf{x} \\
& - \mu_n A_n N_n \\
& + \frac{\vartheta_n - e^{\rho dt} \vartheta_{n-1}}{dt} \pi_n^w + \vartheta_n \rho \pi_n^w - \vartheta_n \frac{\epsilon}{\delta} v'(N_n) N_n \\
& + \vartheta_n \frac{\epsilon}{\delta} N_n \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot \mathbf{g}_n)' u' \left( i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1}-A_n}{dt A_n} a_1 + z_1 A_n N_n \right) \mathbf{c}_{n,[2:J]} \} dt \\
& - e^{\rho dt} \nu'_{-1} V_0 d\mathbf{x} + e^{\rho dt} e^{-\rho t_N} \nu_{N-1} V_N d\mathbf{x} \\
& + e^{\rho dt} \frac{1}{da} V'_0 D'_a \begin{pmatrix} 0 \\ \chi_{-1,[2:J]} \end{pmatrix} dx dt - e^{\rho dt} e^{-\rho t_N} V'_N D'_a \begin{pmatrix} 0 \\ \chi_{N-1,[2:J]} \end{pmatrix} dx dt \\
& + e^{\rho dt} \omega'_{-1} g_0 d\mathbf{x} - e^{\rho dt} e^{-\rho t_N} \omega'_{N-1} g_N d\mathbf{x} \\
& + e^{\rho dt} \vartheta_{-1} \pi_0^w - e^{\rho dt} e^{-\rho t_N} \vartheta_{N-1} \pi_N^w
\end{aligned}$$

In the spirit of [Marcel and Marimon \(2019\)](#), we have reordered the forward-looking constraints — this corresponds to integration by parts in the time dimension in the fully continuous case. The resulting “boundary” terms in the last few lines of the above Lagrangian are the key objects at the

heart of the time consistency problems we discuss in Sections 3 and 4.

We are now ready to take derivatives and characterize necessary first-order conditions for the standard Ramsey plan.

**Derivative  $V_n$ .** We have

$$0 = -\frac{\boldsymbol{v}'_n - e^{\rho dt} \boldsymbol{v}'_{n-1}}{dt} - \rho \boldsymbol{v}_n + (\boldsymbol{A}^z)' \boldsymbol{v}_n + \frac{1}{da} (\boldsymbol{v}_n \cdot \boldsymbol{D}_a)' \boldsymbol{s}_n - e^{\rho dt} \frac{1}{da} \boldsymbol{D}'_a \begin{pmatrix} 0 \\ \chi_{n-1,[2:J]} \end{pmatrix}$$

Using our auxilliary results, we have  $(\boldsymbol{v}_n \cdot \boldsymbol{D}_a)' \boldsymbol{s}_n = (\boldsymbol{s}_n \cdot \boldsymbol{D}_a)' \boldsymbol{v}_n = (\boldsymbol{A}^a)' \boldsymbol{v}_n$ , and so

$$0 = -\frac{\boldsymbol{v}'_n - e^{\rho dt} \boldsymbol{v}'_{n-1}}{dt} - \rho \boldsymbol{v}_n + \boldsymbol{A}' \boldsymbol{v}_n - e^{\rho dt} \frac{1}{da} \boldsymbol{D}'_a \begin{pmatrix} 0 \\ \chi_{n-1,[2:J]} \end{pmatrix}.$$

**Derivative  $\boldsymbol{g}_n$ .** We have

$$\begin{aligned} 0 = & u(\boldsymbol{c}_n) + \mu_n \boldsymbol{c}_n - v(N_n) \mathbf{1} - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1} + \frac{\boldsymbol{\omega}'_n - e^{\rho dt} \boldsymbol{\omega}'_{n-1}}{dt} + (\boldsymbol{\omega}'_n (\boldsymbol{A}^z)')' \\ & + \frac{d}{d\boldsymbol{g}_n} \left[ \frac{1}{da} (\boldsymbol{D}_a \boldsymbol{\omega}_n)' [\boldsymbol{s}_n \cdot \boldsymbol{g}_n] \right] + \vartheta_n N_t \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_n \boldsymbol{z} \cdot u'(\boldsymbol{c}_n) \end{aligned}$$

Now we work out the remaining derivative,

$$\begin{aligned} \frac{d}{d\boldsymbol{g}_n} \left[ \frac{1}{da} (\boldsymbol{D}_a \boldsymbol{\omega}_n)' [\boldsymbol{s}_n \cdot \boldsymbol{g}_n] \right] &= \frac{1}{da} \frac{d}{d\boldsymbol{g}_n} \left[ (\boldsymbol{s}'_n \cdot (\boldsymbol{D}_a \boldsymbol{\omega}_n))' \boldsymbol{g}_n \right] \\ &= \frac{1}{da} \frac{d}{d\boldsymbol{g}_n} \left[ \boldsymbol{g}'_n (\boldsymbol{s}_n \cdot (\boldsymbol{D}_a \boldsymbol{\omega}_n)) \right] \\ &= \frac{1}{da} \frac{d}{d\boldsymbol{g}_n} \left[ \boldsymbol{g}'_n ((\boldsymbol{s}_n \cdot \boldsymbol{D}_a) \boldsymbol{\omega}_n) \right] \\ &= \frac{1}{da} (\boldsymbol{s}_n \cdot \boldsymbol{D}_a) \boldsymbol{\omega}_n \\ &= \boldsymbol{A}^a \boldsymbol{\omega}_n. \end{aligned}$$

Thus, we have

$$0 = u(\boldsymbol{c}_n) + \mu_n \boldsymbol{c}_n - v(N_n) \mathbf{1} - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1} + \frac{\boldsymbol{\omega}'_n - e^{\rho dt} \boldsymbol{\omega}'_{n-1}}{dt} + \boldsymbol{A} \boldsymbol{\omega}_n + \vartheta_n N_t \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_n \boldsymbol{z} \cdot u'(\boldsymbol{c}_n)$$

**Derivative  $c_{n,[2:J]}$ .** We now take the derivative with respect to  $c_{n,i}$  for  $i \geq 2$ . We have

$$0 = u'(c_{n,i})g_{n,i} + \mu_n g_{n,i} + u'(c_{n,i})v_{n,i} + u''(c_{n,i})\chi_{n,i} + \vartheta_n N_n \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_n z_i u''(c_{n,i}) g_{n,i} \\ + \frac{d}{dc_{n,i}} \left[ \frac{1}{da} \mathbf{s}'_n (\boldsymbol{\nu}_n \cdot \mathbf{D}_a) \mathbf{V}_n \right] + \frac{d}{dc_{n,i}} \left[ \frac{1}{da} \mathbf{s}'_n \cdot (\mathbf{D}_a \boldsymbol{\omega}_n)' \mathbf{g}_n \right]$$

Working out the remaining derivatives, we have

$$\begin{aligned} \frac{d}{dc_{n,i}} \left[ \frac{1}{da} \mathbf{s}'_n (\boldsymbol{\nu}_n \cdot \mathbf{D}_a) \mathbf{V}_n \right] &= \frac{1}{da} \left( (\boldsymbol{\nu}_n \cdot \mathbf{D}_a) \mathbf{V}_n \right)_{[i]} \frac{ds_{n,i}}{dc_{n,i}} \\ &= -\frac{1}{da} \left( (\boldsymbol{\nu}_n \cdot \mathbf{D}_a) \mathbf{V}_n \right)_{[i]} \\ &= -\frac{1}{da} v_{n,i} \left( \mathbf{D}_a \mathbf{V}_n \right)_{[i]} \\ &= -\frac{1}{da} v_{n,i} \mathbf{D}_{a,[i,:]} \mathbf{V}_n. \end{aligned}$$

And similarly,

$$\begin{aligned} \frac{d}{dc_{n,i}} \left[ \frac{1}{da} \mathbf{s}'_n \cdot (\mathbf{D}_a \boldsymbol{\omega}_n)' \mathbf{g}_n \right] &= \frac{d}{dc_{n,i}} \left[ \frac{1}{da} \mathbf{g}'_n \left( \mathbf{s}_n \cdot (\mathbf{D}_a \boldsymbol{\omega}_n) \right) \right] \\ &= \frac{ds_{n,it}}{dc_{n,i}} \frac{1}{da} g_{n,i} (\mathbf{D}_a \boldsymbol{\omega}_n)_{[i]} \\ &= -\frac{1}{da} g_{n,i} \mathbf{D}_{a,[i,:]} \boldsymbol{\omega}_n. \end{aligned}$$

Thus, we have

$$0 = u'(c_{n,i})g_{n,i} + \mu_n g_{n,i} + u'(c_{n,i})v_{n,i} + u''(c_{n,i})\chi_{n,i} + \vartheta_n N_n \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_n z_i u''(c_{n,i}) g_{n,i} \\ - \frac{1}{da} v_{n,i} \mathbf{D}_{a,[i,:]} \mathbf{V}_n - \frac{1}{da} g_{n,i} \mathbf{D}_{a,[i,:]} \boldsymbol{\omega}_n.$$

**Derivative  $\pi_n^w$ .** We have

$$\begin{aligned}
0 = & \left[ -u'(c_{n,1})g_{n,1}a_1 - \mu_n g_{n,1}a_1 - \delta\pi_n^w \mathbf{1}' \mathbf{g}_n - \nu_{n,1} u'(c_{n,1})a_1 - \delta\pi_n^w \boldsymbol{\nu}'_n \mathbf{1} \right] d\mathbf{x} \\
& - \vartheta_n \frac{\epsilon \epsilon - 1}{\epsilon} (1 + \tau^L) A_n N_n z_1 u''(c_{n,1}) g_{n,1} a_1 d\mathbf{x} \\
& + \left[ - \sum_{i \geq 2} \nu_{i,n} a_i \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_n}{da} + \sum_{i \geq 2} \omega_{n,i} \frac{\mathbf{D}'_{a,[i,:]} \mathbf{V}_n}{da} \left[ \begin{pmatrix} 0 \\ -\boldsymbol{a}_{[2:J]} \end{pmatrix} \cdot \mathbf{g}_n \right] \right] d\mathbf{x} \\
& + \frac{\vartheta_n - e^{\rho dt} \vartheta_{n-1}}{dt} + \rho \vartheta_n
\end{aligned}$$

Alternatively, we have

$$\begin{aligned}
\frac{d}{d\pi_n^w} \frac{1}{da} (\mathbf{D}_a \boldsymbol{\omega}_n)' [\mathbf{s}_n \cdot \mathbf{g}_n] &= \frac{d}{d\pi_n^w} \frac{1}{da} \left( \mathbf{s}_n \cdot \mathbf{D}_a \boldsymbol{\omega}_n \right)' \mathbf{g}_n \\
&= \frac{d}{d\pi_n^w} \frac{1}{da} \mathbf{g}'_n \left( \mathbf{s}_n \cdot \mathbf{D}_a \boldsymbol{\omega}_n \right) \\
&= \frac{1}{da} \mathbf{g}'_n \left( \frac{d\mathbf{s}_n}{d\pi_n^w} \cdot \mathbf{D}_a \boldsymbol{\omega}_n \right) \\
&= \frac{1}{da} \mathbf{g}'_n \left( \begin{pmatrix} 0 \\ -\boldsymbol{a}_{[2:J]} \end{pmatrix} \cdot \mathbf{D}_a \boldsymbol{\omega}_n \right) \\
&= \sum_{i \geq 1} g_{n,i} \left( \begin{pmatrix} 0 \\ -\boldsymbol{a}_{[2:J]} \end{pmatrix} \cdot \frac{\mathbf{D}_a}{da} \boldsymbol{\omega}_n \right)_{[i]} \\
&= - \sum_{i \geq 2} g_{n,i} a_i \frac{\mathbf{D}_{a,[i,:]} \boldsymbol{\omega}_n}{da}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
0 = & \left[ -u'(c_{n,1})g_{n,1}a_1 - \mu_n g_{n,1}a_1 - \delta\pi_n^w \mathbf{1}' \mathbf{g}_n - \nu_{n,1} u'(c_{n,1})a_1 - \delta\pi_n^w \boldsymbol{\nu}'_n \mathbf{1} \right] d\mathbf{x} \\
& - \vartheta_n \frac{\epsilon \epsilon - 1}{\epsilon} (1 + \tau^L) A_n N_n z_1 u''(c_{n,1}) g_{n,1} a_1 d\mathbf{x} \\
& - \sum_{i \geq 2} \nu_{i,n} a_i \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_n}{da} d\mathbf{x} - \sum_{i \geq 2} g_{n,i} a_i \frac{\mathbf{D}_{a,[i,:]} \boldsymbol{\omega}_n}{da} d\mathbf{x} + \frac{\vartheta_n - e^{\rho dt} \vartheta_{n-1}}{dt} + \rho \vartheta_n
\end{aligned}$$

**Derivative  $i_n$ .** The nominal interest rate derivative is very easy because it's parallel to wage inflation, except in the Phillips curve. That is, we have

$$0 = u'(c_{n,1})g_{n,1}a_1 + \mu_n g_{n,1}a_1 + \nu_{n,1}u'(c_{n,1})a_1 + \sum_{i \geq 2} \nu_{i,n}a_i \frac{D_{a,[i,:]}}{da} V_n + \sum_{i \geq 2} g_{n,i}a_i \frac{D_{a,[i,:]}}{da} \omega_n \\ + \vartheta_n \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_n N_n z_1 u''(c_{n,1}) g_{n,1}a_1$$

**Derivative  $N_n$ .** Finally, we take the derivative for aggregate labor. This yields

$$0 = \left[ u'(c_{n,1})g_{n,1}z_1 A_n + \mu_n g_{n,1}z_1 A_n + \nu_{n,1}u'(c_{n,1})z_1 A_n + \sum_{i \geq 2} \nu_{i,n}z_i A_n \frac{D_{a,[i,:]}}{da} V_n + \sum_{i \geq 2} g_{n,i}z_i A_n \frac{D_{a,[i,:]}}{da} \omega_n \right] d\mathbf{x} \\ + \vartheta_n \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_n N_n z_1 u''(c_{n,1}) g_{n,1}z_1 A_n d\mathbf{x} \\ - v'(N_n) \mathbf{1}' \mathbf{g}_n d\mathbf{x} - v'(N_n) \mathbf{v}'_n \mathbf{1} d\mathbf{x} \\ - \mu_n A_n + \vartheta_n \frac{\epsilon}{\delta} \left( \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot u'(\mathbf{c}_n))' \mathbf{g}_n d\mathbf{x} - v'(N_n) \right) - \frac{\epsilon}{\delta} \vartheta_n v''(N_n) N_n$$

These derivations conclude our proof. In particular, the first-order conditions we have now derived are the exact, discretized analogs of the conditions we present in Proposition 1. For convenience, we formally state the discretized characterization of the stationary Ramsey plan here, so that interested readers can follow the mapping more easily. In the following representation of the stationary Ramsey plan, we use the fact that, in any stationary equilibrium, we simply have

$$u'(\mathbf{c}_i) = \frac{1}{da} D_{a,[i,:]} V$$

for  $i \geq 2$ .

**Lemma 21.** (*Discretized Stationary Ramsey Plan*) A consistent discretization of the stationary Ramsey plan, with  $A_{ss} = 1$ , is given by the following equations. For the value function, we have

$$0 = -\frac{1 - e^{\rho dt}}{dt} \nu - \rho \nu + \mathbf{A}' \nu - e^{\rho dt} \frac{1}{da} D'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix}$$

and for the distribution

$$0 = \frac{1 - e^{\rho dt}}{dt} \omega + \mathbf{A} \omega + u(\mathbf{c}) + \mu \mathbf{c} - v(N) - \frac{\delta}{2} (\pi^w)^2 + \vartheta N \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) \mathbf{z} \cdot u'(\mathbf{c})$$

For consumption, for  $i \geq 2$ , we have

$$-u''(c_i)\chi_i = \left[ u'(c_i) + \mu + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) z_i u''(c_i) - \frac{1}{da} D_{a,[i,:]} \omega \right] g_i$$

The optimality condition for monetary policy, i.e., the nominal interest rate, is given by

$$0 = \left( u'(c_1) + \mu - \frac{1}{da} D_{a,[1,:]} \omega + \vartheta \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) N z_1 u''(c_1) \right) g_1 a_1 + \sum_{i \geq 1} v_i a_i u'(c_i) + \sum_{i \geq 1} g_i a_i \frac{D_{a,[i,:]} \omega}{da}$$

We see here nicely how we need a boundary correction at the borrowing constraint. For inflation, we have

$$0 = -\delta \pi^w - \delta \pi^w \mathbf{v}' \mathbf{1} dx + \frac{1 - e^{\rho dt}}{dt} \vartheta + \rho \vartheta$$

where we used the optimality condition for monetary policy to drop terms. Finally, the optimality condition for aggregate labor, i.e., aggregate economic activity, is given by

$$\begin{aligned} 0 = & \left[ \left( u'(c_1) + \mu - \frac{1}{da} D_{a,[1,:]} \omega + \vartheta \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) N z_1 u''(c_1) \right) g_1 z_1 + \sum_{i \geq 1} v_i z_i u'(c_i) + \sum_{i \geq 1} g_i z_i \frac{D_{a,[i,:]} \omega}{da} \right] dx \\ & - v'(N) - v'(N) \mathbf{v}' \mathbf{1} dx - \mu + \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) (\mathbf{z} \cdot \mathbf{u}'(\mathbf{c}))' \mathbf{g} dx - v'(N) \right) - \frac{\epsilon}{\delta} \vartheta v''(N) N \end{aligned}$$

**Remark 22.** Crucially, this discretized representation of the Ramsey plan provides a formal treatment of boundary conditions. We see exactly how the planner takes into account the borrowing constraint that households face. And we see exactly where the corresponding boundary terms enter the optimality conditions and targeting rules for optimal monetary policy.

## A.7 Proof of Proposition 3

Our goal is to show that

$$\frac{dL^{\text{TD}}(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}_{ss}, \mathbf{Z}_{ss})}{d\boldsymbol{\theta}} = F(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}_{ss}, \mathbf{Z}_{ss}) = 0. \quad (55)$$

In the following, we will prove that this perturbation is 0 for a given  $\frac{d}{d\theta_k}$ , and we can then simply “stack” up to arrive at any perturbation  $\frac{d}{d\theta}$ . For our baseline HANK model,  $\frac{dL^{\text{TD}}}{d\theta_k}$  takes the form

$$0 = \frac{d}{d\theta_k} \left\{ \sum_{n=0}^{\infty} e^{-\rho t} \left\{ u \begin{pmatrix} i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n \\ \mathbf{c}_{n,[2:J]} \end{pmatrix}' \mathbf{g}_n - v(N_n) \mathbf{1}' \mathbf{g}_n - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1}' \mathbf{g}_n \right\} d\mathbf{x} dt \right. \\ \left. + \underbrace{\frac{1}{dt} e^{\rho dt} \boldsymbol{\phi}' V_0 dx - \frac{1}{dt} e^{\rho dt} \vartheta \pi_0^w}_{\text{Timeless Penalties}} \right\} \Big|_{g_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \theta_{ss}, Z_{ss}}$$

for all  $k \geq 0$ . We start by evaluating the derivative for any arbitrary set of inputs to  $F(\cdot)$ . This yields

$$0 = \sum_{n=0}^{\infty} e^{-\rho t} \left\{ \left[ u(\mathbf{c}_n) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 \right]' \frac{d\mathbf{g}_t}{d\theta_k} + (\mathbf{g}_n \cdot u'(\mathbf{c}_n))' \frac{d\mathbf{c}_n}{d\theta_k} - (v'(N_n) \mathbf{1} + \delta \pi_n^w \mathbf{1})' \mathbf{g}_n \frac{dN_n}{d\theta_k} \right\} dt \\ + \frac{1}{dt} e^{\rho dt} \boldsymbol{\phi}' \frac{dV_0}{d\theta_k} - \frac{1}{dt} e^{\rho dt} \vartheta \frac{d\pi_0^w}{d\theta_k} \frac{1}{dx}$$

where we note that we always have  $\frac{dc_{1,n}}{d\theta_k} = 0$  because the planner is constrained by the same boundary condition that the household faces when considering policy perturbations.

Our proof strategy will be to fix five sets of auxiliary terms to this equation, each of which evaluates to 0, and then use these additional terms to rearrange. In particular, the expressions we add correspond to the discretized competitive equilibrium conditions. And our goal will be to then evaluate the corresponding expression at the stationary equilibrium, group terms, and show that everything evaluates to 0.

**Equation 1.** We have

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \boldsymbol{\phi}' \left[ -\rho V_n + \frac{V_{n+1} - V_n}{dt} + u(\mathbf{c}_n) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 + \mathbf{A}^z V_n + \frac{1}{da} (\mathbf{s}_n \cdot \mathbf{D}_a V_n) \right]$$

where we use  $\boldsymbol{\phi} = \boldsymbol{\phi}_{ss}$ . We now use auxiliary results and derivations from before to rewrite this equation as

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \boldsymbol{\phi}' \left[ u(\mathbf{c}_n) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 + \mathbf{A}^z V_n + \frac{1}{da} (\mathbf{s}_n \cdot \mathbf{D}_a V_n) \right] - e^{\rho dt} \boldsymbol{\phi}' \frac{1}{dt} V_0$$

Differentiating, we obtain

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \left[ (\boldsymbol{\phi} \cdot u'(\mathbf{c}_n)) \frac{d\mathbf{c}_n}{d\theta_k} - \boldsymbol{\phi}' \mathbf{1} \left( v'(N_t) \frac{dN_n}{d\theta_k} + \delta \pi_n^w \frac{d\pi_n^w}{d\theta_k} \right) \right. \\ \left. + \boldsymbol{\phi}' A^z \frac{dV_n}{d\theta_k} + \frac{1}{da} (\boldsymbol{\phi} \cdot \mathbf{D}_a V_n)' \frac{ds_n}{d\theta_k} + \frac{1}{da} \boldsymbol{\phi}' s_n \cdot \mathbf{D}_a \frac{dV_n}{d\theta_k} \right] - e^{\rho dt} \boldsymbol{\phi}' \frac{1}{dt} \frac{dV_0}{d\theta_k}.$$

This is the first auxilliary equation that we will add to our desired expression.

**Equation 2.** We obtain the second auxilliary condition by simply differentiating the consumption first-order condition. We rewrite the equation as

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \left[ \chi'_{[2:J]} u''(\mathbf{c}_{n,[2:J]}) - e^{\rho dt} \frac{1}{da} V'_n \mathbf{D}'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix} \right] + e^{\rho dt} \frac{1}{da} V'_0 \mathbf{D}'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix}$$

where we use  $\chi = \chi_{ss}$ , and then differentiate to obtain

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \left[ \left( \chi_{[2:J]} \cdot u''(\mathbf{c}_{n,[2:J]}) \right)' \frac{d\mathbf{c}_{n,[2:J]}}{d\theta_k} - e^{\rho dt} \frac{1}{da} \left[ \mathbf{D}'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix} \right]' \frac{dV_n}{d\theta_k} \right] + e^{\rho dt} \frac{1}{da} \left[ \mathbf{D}'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix} \right]' \frac{dV_0}{d\theta_k}$$

**Equation 3.** For our third auxilliary equation, we differentiate the discretized Kolmogorov forward equation. From before, we have

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \left[ -\rho \lambda' g_n + \lambda' (A^z)' g_n - \frac{1}{da} (s_n \cdot g_n)' \mathbf{D}'_a \lambda \right] + \frac{1}{dt} e^{\rho dt} \lambda' g_0$$

Differentiating with respect to  $\theta_k$ , we obtain

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \left[ -\rho \lambda' \frac{dg_n}{d\theta_k} + \lambda' (A^z)' \frac{dg_n}{d\theta_k} + (g_n \cdot \mathbf{D}_a \lambda)' \frac{ds_n}{d\theta_k} + (s_n \cdot \mathbf{D}_a \lambda)' \frac{dg_n}{d\theta_k} \right] + \frac{1}{dt} e^{\rho dt} \lambda' \frac{dg_0}{d\theta_k}$$

where we again use  $\lambda = \lambda_{ss}$ .

**Equation 4.** We have the aggregate resource constraint with

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \mu \left[ \frac{1}{dx} A_n N_n - c'_n g_n \right],$$

where we use  $\mu = \mu_{ss}$ . Differentiating, we have

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \mu \left[ \frac{1}{dx} A_n \frac{dN_n}{d\theta_k} - \mathbf{c}'_n \frac{d\mathbf{g}_n}{d\theta_k} - \mathbf{g}'_n \frac{d\mathbf{c}_n}{d\theta_k} \right].$$

**Equation 5.** And finally, we use the Phillips curve, which we rewrite using previous results as

$$0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot u'(\mathbf{c}_n))' \mathbf{g}_n dx - v'(N_n) \right) N_n + \frac{1}{dt} e^{\rho dt} \vartheta \pi_0^w$$

Differentiating, we obtain

$$\begin{aligned} 0 = \sum_{n=0}^{\infty} e^{-\rho t_n} \vartheta \frac{\epsilon}{\delta} & \left[ \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_n \left( (\mathbf{z} \cdot u'(\mathbf{c}_n))' \frac{d\mathbf{g}_n}{d\theta_k} + (\mathbf{z} \cdot u''(\mathbf{c}_n) \cdot \mathbf{g}_n)' \frac{d\mathbf{c}_n}{d\theta_k} \right) dx - v''(N_n) \frac{dN_n}{d\theta_n} \right) N_n \right. \\ & \left. + \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot u'(\mathbf{c}_n))' \mathbf{g}_n dx - v'(N_n) \right) \frac{dN_n}{d\theta_k} + \frac{1}{dt} e^{\rho dt} \vartheta \frac{d\pi_0^w}{d\theta_k} \right] \end{aligned}$$

**Evaluate at stationary Ramsey plan.** Crucially, each of our five auxilliary equations must necessarily also hold when evaluated at a stationary Ramsey plan. The key step now, is to evaluate each of the first-order derivatives we taken at the stationary Ramsey plan.

**Putting everything together.** Having evaluated all derivatives around the stationary Ramsey plan, we add the five auxilliary equations we have derived to the expression for  $\frac{dL^{\text{TD}}}{d\theta_k}$  which we started

out with, where we now also evaluate the latter at the stationary Ramsey plan. This yields

$$\begin{aligned}
0 = \sum_{n=0}^{\infty} e^{-\rho t} & \left\{ \left[ u(\mathbf{c}) - v(N) - \frac{\delta}{2} (\pi^w)^2 \right]' \frac{d\mathbf{g}_t}{d\theta_k} + (\mathbf{g} \cdot u'(\mathbf{c}))' \frac{d\mathbf{c}_n}{d\theta_k} - (v'(N)\mathbf{1} + \delta\pi^w\mathbf{1})' \mathbf{g} \frac{dN_n}{d\theta_k} \right. \\
& + (\boldsymbol{\phi} \cdot u'(\mathbf{c})) \frac{d\mathbf{c}_n}{d\theta_k} - \boldsymbol{\phi}' \mathbf{1} \left( v'(N) \frac{dN_n}{d\theta_k} + \delta\pi_n^w \frac{d\pi_n^w}{d\theta_k} \right) \\
& + \boldsymbol{\phi}' A^z \frac{dV_n}{d\theta_k} + \frac{1}{da} (\boldsymbol{\phi} \cdot D_a V)' \frac{ds_n}{d\theta_k} + \frac{1}{da} \boldsymbol{\phi}' s \cdot D_a \frac{dV_n}{d\theta_k} \\
& + \left( \chi_{[2:J]} \cdot u''(\mathbf{c}_{[2:J]}) \right)' \frac{d\mathbf{c}_{n,[2:J]}}{d\theta_k} - e^{\rho dt} \frac{1}{da} \left[ D'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix} \right]' \frac{dV_n}{d\theta_k} \\
& - \rho\lambda' \frac{d\mathbf{g}_n}{d\theta_k} + \lambda' (A^z)' \frac{d\mathbf{g}_n}{d\theta_k} + (\mathbf{g} \cdot D_a \lambda)' \frac{ds_n}{d\theta_k} + (s \cdot D_a \lambda)' \frac{d\mathbf{g}_n}{d\theta_k} \\
& + \mu \frac{1}{dx} A \frac{dN_n}{d\theta_k} - \mu c' \frac{d\mathbf{g}_n}{d\theta_k} - \mu g' \frac{d\mathbf{c}_n}{d\theta_k} \\
& + \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1+\tau^L) A \left( (z \cdot u'(\mathbf{c}))' \frac{d\mathbf{g}_n}{d\theta_k} + (z \cdot u''(\mathbf{c}) \cdot \mathbf{g})' \frac{d\mathbf{c}_n}{d\theta_k} \right) dx - v''(N) \frac{dN_n}{d\theta_n} \right) N_n \\
& + \vartheta \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1+\tau^L) A (z \cdot u'(\mathbf{c}))' \mathbf{g} dx - v'(N) \right) \frac{dN_n}{d\theta_k} \Big\} dt \\
& + e^{\rho dt} \boldsymbol{\phi}' \frac{dV_0}{d\theta_k} - e^{\rho dt} \vartheta \frac{d\pi_0^w}{d\theta_k} \frac{1}{dx} \\
& - e^{\rho dt} \boldsymbol{\phi}' \frac{dV_0}{d\theta_k} + e^{\rho dt} \lambda' \frac{d\mathbf{g}_0}{d\theta_k} + e^{\rho dt} \vartheta \frac{d\pi_0^w}{d\theta_k} \frac{1}{dx} \\
& \left. + dt e^{\rho dt} \frac{1}{da} \left[ D'_a \begin{pmatrix} 0 \\ \chi_{[2:J]} \end{pmatrix} \right]' \frac{dV_0}{d\theta_k} \right)
\end{aligned}$$

where every term that does not have a time step subscript  $n$  is understood to have been evaluated at the stationary Ramsey plan.

Our proof is now almost complete. First, note how the timeless penalties *exactly offset* the “boundary terms” that resulted from rearranging the forward looking implementability conditions. In particular, notice that  $\frac{d\mathbf{g}_0}{d\theta_k} = 0$  and the term in the very last line goes to 0 as  $dt \rightarrow 0$ . The remaining boundary (or initial condition) terms exactly cancel out.

Second, we plug in for

$$\frac{ds_n}{d\theta_k} = \frac{dr_n}{d\theta_k} a + zw \frac{dN_n}{d\theta_k} + zN \frac{dw_n}{d\theta_k} - \frac{d\mathbf{c}_n}{d\theta_k}$$

when evaluated at the stationary Ramsey plan.

Third and finally, we group all terms by *derivatives*. After this last step, we see that the grouped

expressions correspond *exactly* to the optimality conditions that define the stationary Ramsey plan. Consequently, they must be 0. This concludes the proof: We started with an expression for  $\frac{dL^{\text{TD}}}{d\theta_k}$ , and added five auxilliary expressions, each of which itself evaluated to 0. Then we evaluated the resulting expression around the stationary Ramsey plan and showed that it was 0. Consequently, we have shown that

$$\frac{dL^{\text{TD}}}{d\theta_k} = 0$$

when evaluated at the stationary Ramsey plan. And since  $k$  was arbitrary, we have our desired result for any policy perturbation around the stationary Ramsey plan. We have thus shown that Ramsey policy according to the timeless dual Lagrangian  $L^{\text{TD}}$  is indeed time consistent.

## B Optimal Policy and Ramsey Plans in Sequence Space

In this Appendix, we discuss how to operationalize our method and compute optimal policy numerically. Following much of the recent literature on computational methods in heterogeneous-agent economies, we work with a sequence-space representation of our model. In the interest of accessibility, we follow the notation and conventions of [Auclert et al. \(2021\)](#) as closely as possible, extending their work on sequence-space Jacobians to Ramsey problems and welfare analysis. While they work in discrete time, we show below that continuous-time heterogeneous-agent models are nested by the same general model representation they propose. To establish this relationship, we first discretize our model following the same steps that would also be required for numerical implementation.

**Discretization.** We first discretize the equations that characterize competitive equilibrium and optimal policy in both time and space. We use a finite-difference discretization scheme building on [Achdou et al. \(2021\)](#).<sup>30</sup> In particular, we discretize the time dimension over a finite horizon,  $t \in [0, T]$  where  $T$  can be arbitrarily large, using  $N$  discrete time steps, which we denote by  $n = 1, \dots, N$ . With a step size  $dt = \frac{T}{N-1}$ , we have  $t_n = dt(n - 1)$ . We similarly discretize the idiosyncratic state space over  $(a, z)$  using  $J$  grid points. Using bold-faced notation, we denote the discretized consumption policy function of the household at time  $t_n$  as the  $J \times 1$  vector  $\mathbf{c}_n$ , where the  $i$ th element corresponds to  $c_{t_n}(a_i, z_i)$ .

### B.1 Sequence-Space Representation of Equilibrium

After discretizing our model, the resulting equations satisfy the general model representation of heterogeneous-agent economies presented in [Auclert et al. \(2021\)](#). To facilitate comparison, we follow their notation in this Appendix. We consider a general representation of a heterogeneous-agent problem as a mapping from time paths of aggregate inputs  $(X, \theta, Z)$  to time paths of aggregate outputs  $Y$ . We use bold-faced notation here to indicate time paths, with  $\theta = \{\theta_n\}_{n=1}^N$ . It will be useful to explicitly distinguish between the time paths for policy  $\theta$  and the exogenous shock  $Z$  on the one hand, and the time paths for other aggregate inputs  $X$  on the other hand. To simplify the exposition, we assume that there is only one aggregate input variable other than policy and the shock, so that  $X_n \in \mathbb{R}$ .

Denoting the discretized cross-sectional distribution by the  $J \times 1$  vector  $\mathbf{g}_n$ , our main focus will be on outcome variables that take the form  $Y_n = \mathbf{y}'_n \mathbf{g}_n$ , where  $\mathbf{y}_n$  is a  $J \times 1$  vector that represents an individual outcome.<sup>31</sup> For example, aggregate consumption takes the form  $C_n = \mathbf{c}'_n \mathbf{g}$ . Given an

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<sup>30</sup> For a detailed description of the discretization procedure, see [Achdou et al. \(2021\)](#) or [Schaab and Zhang \(2021\)](#). We also leverage the adaptive sparse grid method developed by [Schaab and Zhang \(2021\)](#) and [Schaab \(2020\)](#) to solve dynamic programming problems in continuous time.

<sup>31</sup> We normalize the discretized distribution representation so that  $\mathbf{g}_n$  sums to 1, i.e.,  $\mathbf{1}' \mathbf{g}_n = 1$ , where  $\mathbf{1}$  is a  $J \times 1$  vector of 1s.

initial distribution  $g_0$ , aggregate outcomes  $\mathbf{Y}$  then solve the system of equations

$$\mathbf{V}_n = \nu(\mathbf{V}_{n+1}, \mathbf{X}_n, \boldsymbol{\theta}_n, \mathbf{Z}_n) \quad (56)$$

$$\mathbf{g}_{n+1} = \Lambda(\mathbf{V}_{n+1}, \mathbf{X}_n, \boldsymbol{\theta}_n, \mathbf{Z}_n) \mathbf{g}_n \quad (57)$$

$$\mathbf{Y}_n = \gamma(\mathbf{V}_{n+1}, \mathbf{X}_n, \boldsymbol{\theta}_n, \mathbf{Z}_n)' \mathbf{g}_n. \quad (58)$$

The implementability conditions of our baseline HANK economy can be expressed in terms of the time paths of macroeconomic aggregates as well as those of aggregate outcomes  $\mathbf{Y}$ . Using the above representation, aggregate outcomes  $\mathbf{Y}$  can in turn be expressed in terms of the time paths of aggregate allocations and prices  $\mathbf{X}$ , policy  $\boldsymbol{\theta}$ , and shocks  $\mathbf{Z}$ . In summary, equilibria given policy and shocks can be expressed in terms of the *equilibrium map*  $\mathcal{H}(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Z}) = 0$ .

## B.2 Sequence-Space Representation of Ramsey Plans

We now show how to express the Ramsey plan optimality conditions, which characterize the multipliers and optimal policy, in a general model representation akin to equations (56) through (58). In general, Ramsey plans in heterogeneous-agent economies feature three types of multipliers: aggregate multipliers, individual forward-looking multipliers, and individual backward-looking multipliers. In our baseline environment, the aggregate multipliers are  $\vartheta_t$  and  $\mu_t$ , the individual forward-looking multiplier is  $\lambda_t(a, z)$ , and the (system of) individual backward-looking multipliers is  $\phi_t(a, z)$  and  $\chi_t(a, z)$ .

The Ramsey plan representation (39) summarizes the optimality conditions of the timeless Ramsey plan in sequence-space form. In particular, the Ramsey map  $\mathcal{R}(\cdot)$  takes the time paths of aggregate multipliers  $\mathbf{M}$  as explicit inputs. Our goal now is to show that the optimality conditions of the timeless Ramsey plan can be written in terms of  $\mathbf{R} = (\mathbf{X}, \mathbf{M}, \boldsymbol{\theta})$  and  $\mathbf{Z}$ .

Forward-looking individual multipliers take the form

$$\lambda_n = f(\lambda_{n+1}, \mathbf{V}_n, \mathbf{X}_n, \mathbf{M}_n, \boldsymbol{\theta}_n, \mathbf{Z}_n), \quad (59)$$

which is analogous to equation (56), which characterizes individual forward-looking behavior. For example, it is straightforward to verify that equation (23) satisfies this form: it expresses today's multiplier  $\lambda_n(a, z)$  in terms of today's aggregate multipliers, individual allocations, aggregate allocations and prices, as well as the future multiplier  $\lambda_{n+1}(a, z)$ .

Analogously to equation (56), the recursive structure of forward-looking individual multipliers allows us to efficiently compute their first-order derivatives. We summarize this observation in the following Lemma.

**Lemma 23.** *For any  $k \geq 1$ , we have*

$$\frac{\partial \lambda_n}{\partial \theta_k} = \begin{cases} 0 & \text{if } n > k \\ \frac{\partial \lambda_{n-s}}{\partial \theta_{k-s}} & \text{else for } s < n \end{cases}$$

and likewise for first-order derivatives in  $X_k$ ,  $M_n$ , and  $Z_n$ .

Backward-looking individual multipliers typically correspond to promises that the Ramsey planner makes to individuals. They are characterized by a particular kind of Kolmogorov forward equation. In particular, promise-keeping Kolmogorov forward equations feature a forcing term that captures the “births” and “deaths” of promises, captured by  $\partial_a \chi_t(a, z)$  in equation (21). Consequently, backward-looking multipliers can be represented as

$$\boldsymbol{\phi}_{n+1} = \Lambda(V_{n+1}, X_n, \theta_n, Z_n) \boldsymbol{\phi}_n + b(\boldsymbol{\phi}_n, \lambda_n, V_n, g_n, X_n, M_n, \theta_n, Z_n). \quad (60)$$

The multiplier representations (59) and (60), together with equations (56) through (58) and the equilibrium map (38), let us conclude that timeless Ramsey plans admit the sequence-space representation (39).

### B.3 Proof of Proposition 5

Denote the Ramsey plan by  $\mathbf{R} = (X, M, \theta)$ . Using a first-order Taylor expansion around the stationary Ramsey plan, we have

$$\mathcal{R}(\mathbf{R}, \mathbf{Z}) \approx \mathcal{R}(\mathbf{R}_{ss}, \mathbf{Z}_{ss}) + \mathcal{R}_R(\mathbf{R}_{ss}, \mathbf{Z}_{ss})d\mathbf{R} + \mathcal{R}_Z(\mathbf{R}_{ss}, \mathbf{Z}_{ss})d\mathbf{Z}.$$

Notice that we have  $\mathcal{R}(\mathbf{R}, \mathbf{Z}) = 0$  by the definition of  $\mathbf{R}$  as a Ramsey plan, i.e., as solving  $\mathcal{R}(\cdot) = 0$  for a given  $\mathbf{Z}$  as in (39).

We now show that  $\mathcal{R}(\mathbf{R}_{ss}, \mathbf{Z}_{ss}) = 0$  as well, assuming, as we do in Proposition 5, that the Ramsey problem is initialized at  $(g_0(\cdot), \phi(\cdot), \vartheta) = (g_{ss}(\cdot), \phi_{ss}(\cdot), \vartheta_{ss})$ . The Ramsey plan map  $\mathcal{R}(\cdot)$  is a system of equations that comprises two sets of conditions, those for competitive equilibrium as well as the first-order conditions associated with the timeless Ramsey problem in the primal representation. By definition, the stationary Ramsey plan comprises a feasible competitive equilibrium. Consequently, when evaluated at  $(\mathbf{R}, \mathbf{Z}) = (\mathbf{R}_{ss}, \mathbf{Z}_{ss})$ , those conditions in  $\mathcal{R}(\cdot)$  associated with competitive equilibrium are 0.

That leaves the first-order conditions associated with the timeless primal Ramsey problem. It follows from Proposition 3 that the timeless primal Ramsey problem is time-consistent, so that the planner does not want to deviate from the stationary Ramsey plan when  $\mathbf{Z} = \mathbf{Z}_{ss}$ . It also follows from our duality proof for the primal and dual representations that  $\frac{d}{d\theta} L^{TP}(\theta_{ss}, \mathbf{Z}_{ss}) = 0$  also implies that each of the associated first-order conditions of the timeless primal problem are 0 when

evaluated at  $(\mathbf{R}_{ss}, \mathbf{Z}_{ss})$ . Putting these observations together implies  $\mathcal{R}(\mathbf{R}_{ss}, \mathbf{Z}_{ss}) = 0$ .

We are then simply left with

$$0 \approx \mathcal{R}_R(\mathbf{R}_{ss}, \mathbf{Z}_{ss})d\mathbf{R} + \mathcal{R}_Z(\mathbf{R}_{ss}, \mathbf{Z}_{ss})d\mathbf{Z}.$$

Rearranging and inverting  $\mathcal{R}_R$  yields the desired result.

#### B.4 Proof of Proposition 6

We proceed as follows. A first-order Taylor approximation of  $F(\cdot)$  (in  $\theta$  and  $\mathbf{Z}$ ) around the stationary Ramsey plan yields

$$\begin{aligned} F(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}, \mathbf{Z}) &= F(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}_{ss}, \mathbf{Z}_{ss}) \\ &\quad + F_\theta(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}_{ss}, \mathbf{Z}_{ss})(\boldsymbol{\theta} - \boldsymbol{\theta}_{ss}) + F_Z(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}_{ss}, \mathbf{Z}_{ss})(\mathbf{Z} - \mathbf{Z}_{ss}). \end{aligned}$$

First, we must have

$$F(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}, \mathbf{Z}) = 0$$

by construction because that's our definition for optimal policy  $\boldsymbol{\theta}(\mathbf{Z})$ . Second, we also have

$$F(\mathbf{g}_{ss}, \boldsymbol{\phi}_{ss}, \vartheta_{ss}, \boldsymbol{\theta}_{ss}, \mathbf{Z}_{ss}) = 0,$$

which is the main result of Section 3.4 and whose proof was just presented in the previous Appendix subsection. Denoting  $d\boldsymbol{\theta} = \boldsymbol{\theta} - \boldsymbol{\theta}_{ss}$  and  $d\mathbf{Z} = \mathbf{Z} - \mathbf{Z}_{ss}$ , we thus have

$$0 = F_\theta d\boldsymbol{\theta} + F_Z d\mathbf{Z},$$

where the Jacobians of  $F(\cdot)$  are evaluated at the stationary Ramsey plan.

#### B.5 Sequence-Space Hessians

To compute optimal policy to first order using Proposition 6, we effectively need to differentiate  $L^{\text{TD}}(\cdot)$  twice. In particular,  $F(\cdot) = \frac{d}{d\boldsymbol{\theta}} L^{\text{TD}}$  features first-order derivative terms, which can be cast as sequence-space Jacobians (Auclet et al., 2021). Therefore, computing the total derivatives  $\frac{d}{d\boldsymbol{\theta}} F(\cdot)$  and  $\frac{d}{d\mathbf{Z}} F(\cdot)$ , which are used in equation (43) to characterize optimal policy  $d\boldsymbol{\theta}$ , we require second-order derivatives. Consequently, computing optimal stabilization policy using our approach requires that we compute both first- and second-order total derivatives of all objects that feature in the timeless dual Lagrangian.<sup>32</sup>

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<sup>32</sup> An alternative approach is to work in the primal representation of our timeless Ramsey problem and solve for the time-consistent Ramsey plan — including all multipliers — to first order using the system of equations that characterizes the Ramsey plan. We develop this alternative approach in ongoing work. The dual approach avoids having to compute the multipliers but requires sequence-space Hessians. For the primal approach, we have to compute an extended set

In a sequence-space representation of our model, these objects are all functions of the time paths of aggregate inputs, i.e.,  $(X, \theta, Z)$ , where  $X = X(\theta, Z)$ . Moreover, the timeless dual Lagrangian itself can be represented in terms of aggregate outcomes  $Y$ , using the general model representation above. Consequently, computing the matrices  $\frac{d}{d\theta} F$  and  $\frac{d}{dZ} F$  requires taking total derivatives of specific aggregate outcomes  $Y$ .

We define *sequence-space Hessians* as the matrices of mixed partial derivatives of model objects that can be represented as functions of aggregate sequences around the stationary Ramsey plan. We discuss these mixed partial derivative matrices in detail in Section B.5.1. Subsequently, in Section B.5.2, we show how to build up the second-order total derivatives of the timeless dual Lagrangian, i.e.,  $\frac{d^2}{d\theta^2} F$  and  $\frac{d^2}{dZ^2} F$ , from sequence-space Hessians.

### B.5.1 A Fake-News Algorithm to Compute Sequence-Space Hessians

We now extend the fake-news algorithm of Auclert et al. (2021) to compute sequence-space Hessians, i.e., the matrices of mixed partial derivatives  $\frac{\partial^2}{\partial\theta_k\partial\theta_l} Y_n$  in a sequence-space representation of the model. The results we present below hold for any mixed partial derivative of  $Y_n(X, \theta, Z)$ , but to ease notation we focus specifically on the mixed derivative  $\frac{\partial^2}{\partial\theta_k\partial\theta_l} Y_n$  for some given  $k, l \in \{1, \dots, N\}$ .

Using equation (58), we can rewrite the mixed derivative of aggregate outcome  $Y_n$  at time  $t_n$  as

$$\frac{\partial^2 Y_n}{\partial\theta_k\partial\theta_l} = \frac{\partial}{\partial\theta_l} \left( \mathbf{y}'_n \frac{\partial g_n}{\partial\theta_k} + \mathbf{g}'_n \frac{\partial y_n}{\partial\theta_k} \right) = \mathbf{y}'_n \frac{\partial^2 g_n}{\partial\theta_k\partial\theta_l} + \frac{\partial g_n'}{\partial\theta_k} \frac{\partial y_n}{\partial\theta_l} + \mathbf{g}'_n \frac{\partial^2 y_n}{\partial\theta_k\partial\theta_l} + \frac{\partial y_n'}{\partial\theta_k} \frac{\partial g_n}{\partial\theta_l}$$

This derivation underscores that we generally need both the first-order and second-order mixed partial derivatives of individual outcomes  $y_n$  and the distribution  $g_n$  to compute aggregate sequence-space Hessians  $\frac{\partial^2}{\partial\theta_k\partial\theta_l} Y_n$ . Our method leverages several useful properties of these first- and second-order derivatives. In the following, we prove key properties of the second-order mixed derivatives  $\frac{\partial^2}{\partial\theta_k\partial\theta_l} \mathbf{y}_n$  and  $\frac{\partial^2}{\partial\theta_k\partial\theta_l} \mathbf{g}_n$ , and we refer the reader to Auclert et al. (2021) for the properties of the first-order partial derivatives.

First, notice that mixed partial derivatives are symmetric, or interchangeable, by the standard continuity argument. That is

$$\frac{\partial^2 \mathbf{y}_n}{\partial\theta_k\partial\theta_l} = \frac{\partial^2 \mathbf{y}_n}{\partial\theta_l\partial\theta_k} \quad \text{and} \quad \frac{\partial^2 \mathbf{g}_n}{\partial\theta_k\partial\theta_l} = \frac{\partial^2 \mathbf{g}_n}{\partial\theta_l\partial\theta_k}.$$

Second, the recursive structure of the system (56) - (58) gives rise to the following key property of mixed partial derivatives in sequence space.

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of sequence-space Jacobians, solving to first order for the transition paths of i) allocations, ii) optimal policy, and iii) multipliers.

**Lemma 24.** *We have*

$$\frac{\partial^2 \mathbf{y}_n}{\partial \theta_k \partial \theta_l} = \begin{cases} 0 & \text{if } n > \min\{k, l\} \\ \frac{\partial^2 \mathbf{y}_{n-s}}{\partial \theta_{k-s} \partial \theta_{l-s}} & \text{else for } s < n \end{cases}$$

Leveraging these first two properties of mixed derivatives of individual outcomes in sequence space, we can construct sequence-space Hessian matrices using the following shortcut: Instead of computing all  $N^2$  numerical derivatives, we simply compute

$$\frac{\partial^2 \mathbf{y}_n}{\partial \theta_k \partial \theta_N}$$

for  $1 \leq k \leq N$ , which requires only  $N$  numerical derivative evaluations.<sup>33</sup>

Third, we exploit the fact that the transition matrix  $\Lambda$ , which describes the law of motion of the cross-sectional distribution in equation (58), has a particular structure in continuous time. In particular, we have

$$\Lambda(\mathbf{V}_{n+1}, X_n, \theta_n, Z_n) = 1 + dt \mathbf{A}(\mathbf{V}_{n+1}, X_n, \theta_n, Z_n)',$$

where  $\mathbf{A}$  is the  $J \times J$  matrix that discretizes the HJB operator  $\mathcal{A}$ , and  $\mathbf{A}'$ , its transpose, is the analog for the adjoint  $\mathcal{A}^*$ . In our baseline HANK model, the discretized transition matrix takes the form

$$\mathbf{A}_n = \mathbf{s}_n \cdot \mathbf{D}_a + \mathbf{A}^z \quad (61)$$

where  $\mathbf{A}^z$  is given exogenously, and its derivatives with respect to  $\theta_k$ ,  $X_k$  and  $Z_k$  are therefore 0. The matrix  $\mathbf{D}_a$  discretizes the partial derivative  $\partial_a$  as we show in Appendix A, and it is also invariant to perturbations in aggregate inputs as long as the step size used for the numerical derivative is fine enough. In particular, the key insight here is that taking derivatives of the general transition matrix  $\Lambda$  in equation (58) simply amounts to differentiating  $\mathbf{s}_n$  in equation (61).<sup>34</sup> We record this observation in the following Lemma.

**Lemma 25.** *The first- and second-order mixed partial derivatives of the transition matrix  $\Lambda_n$  in our setting are given by*

$$\frac{\partial \Lambda_n}{\partial \theta_k} = dt \frac{\partial \mathbf{s}_n}{\partial \theta_k} \cdot \mathbf{D}_a \quad \text{and} \quad \frac{\partial^2 \Lambda_n}{\partial \theta_k \partial \theta_l} = dt \frac{\partial^2 \mathbf{s}_n}{\partial \theta_k \partial \theta_l} \cdot \mathbf{D}_a.$$

Fourth, we characterize the properties of the mixed derivatives of the cross-sectional distribution. We assume for simplicity that the economy is initialized at the cross-sectional distribution that

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<sup>33</sup> For other mixed derivatives, such as  $\frac{\partial^2 \mathbf{y}_n}{\partial \theta_k \partial Z_l}$ , we require  $2N$  evaluations, i.e., both  $\frac{\partial^2 \mathbf{y}_n}{\partial \theta_k \partial Z_N}$  and  $\frac{\partial^2 \mathbf{y}_n}{\partial \theta_N \partial Z_l}$ .

<sup>34</sup> Notice that equation (61) is specific to our baseline model and consequently breaks with the spirit of generality otherwise adopted in this section. However, an equation like (61) will generally hold in any continuous-time heterogeneous-agent model. We think there is some value to highlighting how to leverage this equation when constructing sequence-space Jacobians and Hessians, and we therefore use equation (61) in the following while otherwise maintaining our general notation.

corresponds to the stationary Ramsey plan, that is,  $g_1 = g_{ss}$ , where we recall that  $n$  starts at 1 and  $t_1 = 0$ . The initial distribution is given exogenously and does not adjust on impact. That is,  $\frac{\partial^2 g_1}{\partial \theta_k \partial \theta_l} = 0$ . Using equation (58) and Lemma 25, the response of the cross-sectional distribution at time step  $n = 2$  is thus

$$\frac{\partial^2 g_2}{\partial \theta_k \partial \theta_l} = \frac{\partial^2 \Lambda_1}{\partial \theta_k \partial \theta_l} g_{ss} = dt \left( \frac{\partial^2 s_1}{\partial \theta_k \partial \theta_l} \cdot D_a \right) g_{ss}$$

We now exploit the recursive structure of equation (58) to derive two alternative expressions for the mixed derivatives  $\frac{\partial^2 g_n}{\partial \theta_k \partial \theta_l}$ , for  $n \geq 3$ . We summarize in the next Lemma.

**Lemma 26.** *The mixed partial derivatives of the cross-sectional distribution  $g_n$  at time steps  $n \geq 3$  can be computed recursively using*

$$\begin{aligned} \frac{\partial^2 g_n}{\partial \theta_k \partial \theta_l} &= \Lambda_{ss} \frac{\partial^2 g_{n-1}}{\partial \theta_k \partial \theta_l} + \frac{\partial^2 g_2}{\partial \theta_{k-(n-2)} \partial \theta_{l-(n-2)}} \mathbb{1}_{\min\{k-(n-2), l-(n-2)\} \geq 1} \\ &\quad + dt \left( \frac{\partial s_1}{\partial \theta_{l-(n-2)}} \cdot D_a \right) \frac{\partial g_{n-1}}{\partial \theta_k} \mathbb{1}_{l-(n-2) \geq 1} + dt \left( \frac{\partial s_1}{\partial \theta_{k-(n-2)}} \cdot D_a \right) \frac{\partial g_{n-1}}{\partial \theta_l} \mathbb{1}_{k-(n-2) \geq 1} \end{aligned}$$

or non-recursively using

$$\begin{aligned} \frac{\partial^2 g_n}{\partial \theta_k \partial \theta_l} &= \sum_{r=1}^{R_1} (\Lambda_{ss})^{n-r-1} \frac{\partial^2 g_2}{\partial \theta_k \partial \theta_l} \\ &\quad + dt \sum_{r=1}^{R_2} (\Lambda_{ss})^{n-r-2} \left( \frac{\partial s_1}{\partial \theta_{k-r}} \cdot D_a \right) \frac{\partial g_{1+r}}{\partial \theta_l} + dt \sum_{r=1}^{R_3} (\Lambda_{ss})^{n-r-2} \left( \frac{\partial s_1}{\partial \theta_{l-r}} \cdot D_a \right) \frac{\partial g_{1+r}}{\partial \theta_k} \end{aligned}$$

where  $R_1 = \min\{k, l, n - 1\}$ ,  $R_2 = \min\{k - 1, n - 2\}$ , and  $R_3 = \min\{l - 1, n - 2\}$ .

Fifth and finally, we discuss how to efficiently compute a given mixed partial derivative numerically. The most popular finite-difference stencil to compute second-order mixed derivatives is given by

$$\frac{\partial^2 y_n}{\partial \theta_k \partial \theta_l} = \frac{y_n^{++} - y_n^{+-} - y_n^{-+} + y_n^{--}}{4h^2} \tag{62}$$

where  $y_n^{++} = y_n(\dots, \theta_k + h, \dots, \theta_l + h, \dots)$ ,  $y_n^{+-} = y_n(\dots, \theta_k + h, \dots, \theta_l - h, \dots)$ ,  $y_n^{-+} = y_n(\dots, \theta_k - h, \dots, \theta_l + h, \dots)$ , and  $y_n^{--} = y_n(\dots, \theta_k - h, \dots, \theta_l - h, \dots)$ . This stencil requires 4 function evaluations for every mixed derivative and is therefore very costly.

An alternative and, in our case, substantially more efficient stencil is

$$\frac{\partial^2 y_n}{\partial \theta_k \partial \theta_l} = \frac{y_n^{++} - y_n^{+\cdot} - y_n^{+\cdot} + 2y_n - y_n^{-\cdot} - y_n^{-\cdot} + y_n^{--}}{2h^2} \tag{63}$$

where  $y_n^{+\cdot} = y_n(\dots, \theta_k + h, \dots, \theta_l, \dots)$ ,  $y_n^{+\cdot} = y_n(\dots, \theta_k, \dots, \theta_l + h, \dots)$ ,  $y_n^{-\cdot} = y_n(\dots, \theta_k - h, \dots, \theta_l, \dots)$ , and  $y_n^{-\cdot} = y_n(\dots, \theta_k, \dots, \theta_l - h, \dots)$ . Stencil (63) requires only 2 new function evaluations compared

to stencil (62)'s 4. The additional terms  $y_n$ ,  $y_n^+$ , and  $y_n^-$  are already available from constructing the first-order sequence-space Jacobians. And the terms  $y_n^-$  and  $y_n^+$  can be computed very cheaply using the standard fake-news algorithm for first-order derivatives.

**Comparison to fake-news algorithm of Auclert et al. (2021).** In their seminal contribution, Auclert et al. (2021) develop a highly efficient algorithm to compute sequence-space Jacobians, showing that computing a single column of the Jacobian suffices to derive all other columns from it. For sequence-space Hessians, on the other hand, we need to evaluate one “block” of the Hessian, which requires  $N$  numerical derivatives, and is consequently substantially more expensive than computing a sequence-space Jacobian.

Why does the Hessian matrix have a higher information requirement? For Jacobians, Auclert et al. (2021) show that we only require a single piece of information to evaluate the impact of shocks on household behavior: How far in the future is the shock, i.e., what is the distance from the present to the shock. For Hessians, on the other hand, we need two pieces of information: How far in the future is the (later of the two) shock(s), and, in addition, what is the relative distance between the two shocks. We therefore cannot obtain all required information with a single numerical derivative as in the case of the Jacobian.

Nonetheless, our fake-news algorithm for sequence-space Hessians represents a substantial improvement over computing the Hessian matrices directly, which would require the evaluation of  $N^2$  numerical derivatives.

### B.5.2 Total Derivatives and General Equilibrium

Our perturbation approach to optimal stabilization policy in the dual requires the two total derivatives  $\frac{d}{d\theta}F$  and  $\frac{d}{dZ}F$ . In particular, the  $[k, l]$ th entry of the  $N \times N$  matrix  $F_\theta$  is given by

$$(F_\theta)_{[k,l]} = \sum_{n=1}^N e^{-\rho t_n} \frac{d^2 U_n}{d\theta_k d\theta_l} dt + (\phi_{ss})' \frac{d^2 V_1}{d\theta_k d\theta_l} + \vartheta_{ss} \frac{d^2 \pi_1^w}{d\theta_k d\theta_l}, \quad (64)$$

where  $U_n = (u(c_n) - v(N_n) - \frac{\delta}{2}(\pi_n^w)^2)' g_n$ . The first term in equation (64) thus captures the present discounted sum of future aggregate social welfare flows, and the second and third terms capture the timeless penalties.

So far, we have discussed how to construct the first- and second-order partial derivatives of the economic variables that comprise  $F$ . To compute total derivatives, we start with a discussion of general equilibrium.

**General equilibrium.** General equilibrium considerations in our model can be summarized in terms of the equilibrium map (38), which is a system of  $N$  equations, assuming for now that  $X_n \in \mathbb{R}$ . Given paths for policy  $\theta$  and the exogenous shock  $Z$ , we can solve equation (38) for  $X = X(\theta, Z)$ .

To compute the total derivative  $F_\theta$ , i.e., the response in the planner's first-order condition to a perturbation in the policy path, we must take into account both the direct effect of the policy via its partial derivative and the indirect general equilibrium effects. We use the first-order derivatives  $\mathbf{X}_{\theta_k} = -\mathbf{H}_X^{-1}\mathbf{H}_{\theta_k}$ . Likewise, the mixed partial derivatives are given by

$$\mathbf{X}_{\theta_k \theta_l} = -\mathbf{H}_X^{-1}\mathbf{H}_{\theta_k \theta_l} + \mathbf{H}_X^{-1}\mathbf{H}_{X \theta_l}\mathbf{H}_X^{-1}\mathbf{H}_{\theta_k}. \quad (65)$$

**Total derivatives.** We now summarize how total derivatives of  $Y_n$  relate to the partial derivatives we have discussed so far. For notational convenience, we drop the  $n$  subscript and instead use subscripts to denote partial derivatives. Recall that  $Y$  depends on the time paths of all aggregate inputs,  $Y(\mathbf{X}, \boldsymbol{\theta}, \mathbf{Z})$ .

**Lemma 27.** *The total derivatives of  $Y$  are given by*

$$\frac{d^2 Y}{d\theta_k d\theta_l} = \begin{pmatrix} Y_{X_1 X} \mathbf{X}_{\theta_l} & \dots & Y_{X_N X} \mathbf{X}_{\theta_l} \end{pmatrix} \mathbf{X}_{\theta_k} + Y_{X \theta_l} \mathbf{X}_{\theta_k} + Y_X \mathbf{X}_{\theta_k \theta_l} + Y_{\theta_k X} \mathbf{X}_{\theta_l} + Y_{\theta_k \theta_l} \quad (66)$$

where subscripts denote partial derivatives, and likewise for the total derivatives  $\frac{d^2 Y}{d\theta_k dZ_l}$ .

The total derivatives for  $V_1$  and  $\pi_1^w$  take the same form and can be computed via their second-order partial derivatives together with the general equilibrium maps, i.e., the partial derivatives of  $\mathbf{X}$ . We now have all the objects we need to implement our perturbation approach in the dual and compute optimal stabilization policy numerically.

### B.5.3 Algorithm to Compute Optimal Policy in the Dual

We summarize in Algorithm 1 our fake-news algorithm to compute sequence-space Hessians and, with them, optimal stabilization policy to first order in the dual representation of the timeless Ramsey problem.

### B.5.4 Accuracy and Performance

We test the accuracy of our method in a number of ways. In Appendix G.2, we show that the numerical solution of optimal policy in RANK using our perturbation method based on sequence-space Hessians is highly accurate. In RANK, we can compute optimal policy analytically. We compare this exact analytical solution to the first-order approximation of optimal policy given by  $d\boldsymbol{\theta} = -F_\theta^{-1}F_Z d\mathbf{Z}$ . For demand shocks, we show that the difference in optimal CPI inflation, for example, is on the order of  $10^{-6}$ . In the case of TFP shocks, the remaining discrepancy is slightly larger, with the two optimal interest rate paths differing by about 1 basis point.

In ongoing work, we use our perturbation method to compute optimal monetary policy numerically in the analytical HANK model of Acharya et al. (2020). Since their model admits

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**Algorithm 1** Optimal Stabilization Policy using Sequence-Space Hessians

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- 1: Compute stationary Ramsey plan
  - 2: Compute sequence-space Jacobians around the stationary Ramsey plan using fake-news algorithm of [Auclert et al. \(2021\)](#)
  - 3: Compute  $N$  numerical mixed partial derivatives, and
    - a: construct policy Hessians ▷ use stencil (63)
    - b: construct distribution Hessians ▷ use Lemma 24▷ use Lemmas 25 and 26
  - 4: Use Hessians to compute mixed derivatives of  $H$  and  $X$  ▷ use equations (38) and (65)
  - 5: Compute total derivatives for  $F_\theta$  and  $F_Z$  ▷ use equations (64) and (66)
  - 6: Compute optimal stabilization policy as  $d\theta = -F_\theta^{-1} F_Z dZ$
- 

an analytical solution for optimal policy in a heterogeneous-agent context, it represents a useful environment to benchmark the accuracy of our method.

## C Labor Market Structure and Wage Phillips Curves

### C.1 Wage Phillips Curve with Utility Adjustment Cost

The union's problem is associated with the Lagrangian

$$L = \int_0^\infty e^{-\rho t} \int \left[ u\left(c_t(a, z; W_{k,t})\right) - v\left(\int_0^1 \left(\frac{W_{k,t}}{W_t}\right)^{-\epsilon} N_t dk\right) - \frac{\delta}{2} \int_0^1 \left(\pi_{k,t}^w\right)^2 dk \right] g_t(a, z) d(a, z) dt \\ + \int_0^\infty e^{-\rho t} \left[ \mu_t \pi_{k,t}^w W_{k,t} - \rho \mu_t W_{k,t} + W_{k,t} \dot{\mu}_t \right] dt + \mu_0 W_{k,0},$$

where in the second line we already integrated by parts. Thus, the two first-order conditions are given by

$$0 = \int u'(c_t) \frac{\partial c_t(a, z; W_{k,t})}{\partial W_{k,t}} g_t(a, z) d(a, z) + \epsilon v'(N_t) \frac{N_t}{W_t} + \mu_t \pi_{k,t}^w - \rho \mu_t + \dot{\mu}_t \\ 0 = -\delta \pi_{k,t}^w + \mu_t W_{k,t},$$

as well as the initial condition  $\mu_0 = 0$ . By the envelope theorem, we have

$$\frac{\partial c_t(a, z; W_{k,t})}{\partial W_{k,t}} = \frac{1}{P_t} (1 + \tau^L) (1 - \epsilon) z_t N_t.$$

Defining

$$\Lambda_t = \int z u'(c_t(a, z)) g_t(a, z) d(a, z),$$

the first FOC becomes

$$0 = (1 + \tau^L) (1 - \epsilon) w_t N_t \Lambda_t + \epsilon v'(N_t) N_t + \mu_t \dot{W}_t - \rho W_t \mu_t + W_t \dot{\mu}_t.$$

Differentiating the second FOC yields

$$\mu_t \dot{W}_t + W_t \dot{\mu}_t = \delta \dot{\pi}_t^w.$$

Plugging back into the first FOC, we arrive at

$$0 = (1 + \tau^L) (1 - \epsilon) w_t N_t \Lambda_t + \epsilon v'(N_t) N_t - \rho \delta \pi_t^w + \delta \dot{\pi}_t^w$$

which yields the result after rearranging.

## D RANK with Wage Rigidity

In this Appendix, we present a self-contained treatment of optimal monetary policy in RANK. We leverage our timeless Ramsey approach to give an exact, non-linear characterization, which we leverage in Section 4 of the main text to compare optimal policy in HANK to the RANK benchmark.

### D.1 Model

The representative household has preferences over consumption and labor. We assume that the household's labor decision is intermediated by a continuum of  $k \in [0, 1]$  unions, which we further describe below. Preferences are thus given by

$$\int_0^\infty e^{-\rho t} \left[ u(C_t) - V(N_{k,t}, \pi_{k,t}^w) \right] dt,$$

where  $C_t$  denotes consumption of the final consumption good and  $V(n_{k,t}, \pi_{k,t}^w)$  denotes disutility from work. The representative household's budget constraint is given by

$$\dot{A}_t = r_t A_t + E_t + \tau_t - C_t,$$

where  $A_t$  is the aggregate bond position,  $r_t$  the real interest rate, and  $E_t$  denotes real labor income. Finally,  $\tau_t$  denotes a real lump-sum rebate to households.

**Labor market structure.** The labor market structure and union problem in the RANK benchmark are unchanged relative to HANK except for one important difference: Union  $k$  maximizes stakeholder value, which in HANK is an appropriately weighted average of its members' utility flows. In RANK, on the other hand, there is a single representative household. Consequently, the marginal cost term in the New Keynesian wage Phillips curve that obtains in RANK simply features the marginal utility of consumption of the representative household, i.e.,

$$\dot{\pi}_t^w = \rho \pi_t^w + \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} w_t u'(C_t) - v'(N_t) \right] N_t.$$

instead of the weighted average of marginal utilities,  $\Lambda_t$ , which features in the Phillips curve in HANK.

**Equilibrium.** The characterization of firms in RANK is identical to that in HANK. As are the details of fiscal and monetary policy. To formally define equilibrium and RANK and state the implementability conditions for the Ramsey problem, we first characterize the representative household's optimal consumption behavior in terms of an aggregate consumption Euler equation. Under isoelastic preferences, which we assume in this Appendix, we simply arrive at the standard

differential equation

$$\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho}{\gamma}.$$

The equilibrium conditions of the RANK economy are then given by

$$\dot{C}_t = \frac{r_t - \rho}{\gamma} C_t$$

$$Y_t = A_t N_t$$

$$Y_t = C_t$$

$$B_t = 0$$

$$r_t = i_t - \pi_t$$

$$w_t = A_t$$

$$\pi_t = \pi_t^w - \frac{\dot{A}_t}{A_t}$$

$$\dot{\pi}_t^w = \rho \pi_t^w + \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} w_t u'(C_t) - v'(N_t) \right] N_t$$

We drop the bond market clearing condition. We substitute in for the real wage as well as CPI inflation. And we drop output. We summarize the resulting implementability conditions in the following Lemma.

**Lemma 28.** *The implementability conditions that a Ramsey planner faces in RANK can be summarized as*

$$\dot{C}_t = \frac{r_t - \rho}{\gamma} C_t$$

$$C_t = A_t N_t$$

$$r_t = i_t - \pi_t^w + \frac{\dot{A}_t}{A_t}$$

$$\dot{\pi}_t^w = \rho \pi_t^w + \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A_t u'(C_t) - v'(N_t) \right] N_t$$

It is important to note that the implementability conditions of Lemma 28 could be simplified further. We refrain from doing so in order to maintain as parallel a structure to the Ramsey problem in HANK as possible.

## D.2 Optimal Policy

We associate the standard Ramsey problem in the primal with the following Lagrangian, where we drop time subscripts for convenience,

$$\begin{aligned}
L = & \int_0^\infty e^{-\rho t} \left\{ \frac{1}{1-\gamma} C^{1-\gamma} - v(N) - \frac{\delta}{2} (\pi^w)^2 \right. \\
& + \phi \left[ \frac{1}{\gamma} \left( i - \pi^w + \frac{\dot{A}}{A} - \rho \right) C - \dot{C} \right] \\
& + \mu \left[ AN - C \right] \\
& \left. + \vartheta \left[ \rho \pi^w + \frac{\epsilon}{\delta} \left( (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u'(C) - v'(N) \right) N - \dot{\pi}^w \right] \right\} dt
\end{aligned}$$

Crucially, both  $C_0$  and  $\pi_0^w$  are *free* from the planner's perspective.

**Lemma 29.** *The first-order conditions for optimal monetary policy in RANK are given by*

$$\begin{aligned}
0 &= C^{-\gamma} - \mu + \phi \frac{1}{\gamma} \left( i - \pi^w + \frac{\dot{A}}{A} - \rho \right) - \rho \phi + \dot{\phi} + \vartheta \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u''(C) N \\
0 &= -\delta \pi^w - \phi \frac{1}{\gamma} C + \vartheta \rho - \rho \vartheta + \dot{\vartheta} \\
0 &= -v'(N) + \mu A + \vartheta \frac{\epsilon}{\delta} \left( (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u'(C) - v'(N) \right) - \vartheta \frac{\epsilon}{\delta} v''(N) N \\
0 &= \phi \frac{1}{\gamma} C,
\end{aligned}$$

with initial conditions

$$0 = \phi_0$$

$$0 = \vartheta_0.$$

We see that we must have  $\phi = 0$  for all  $t$ . This allows us to simplify the first-order conditions and arrive at

$$\begin{aligned}
0 &= C^{-\gamma} - \mu + \vartheta \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u''(C) N \\
\dot{\vartheta} &= -\delta \pi^w \\
0 &= -v'(N) + \mu A + \vartheta \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u'(C) - \vartheta \frac{\epsilon}{\delta} v'(N) - \vartheta \frac{\epsilon}{\delta} v''(N) N
\end{aligned}$$

with initial condition  $\vartheta_0 = 0$ .

*Proof.* We now integrate by parts and consider a general functional perturbation, yielding

$$\begin{aligned}
L = \int_0^\infty e^{-\rho t} & \left\{ \frac{1}{1-\gamma} (C + \alpha h_C)^{1-\gamma} - v(N + \alpha h_N) - \frac{\delta}{2} (\pi^w + \alpha h_\pi)^2 \right. \\
& + \phi \left[ \frac{1}{\gamma} \left( i + \alpha h_i - \pi^w - \alpha h_\pi + \frac{\dot{A}}{A} - \rho \right) (C + \alpha h_C) \right] \\
& + \mu \left[ A(N + \alpha h_N) - C - \alpha h_C \right] \\
& + \vartheta \left[ \rho(\pi^w + \alpha h_\pi) + \frac{\epsilon}{\delta} \left( (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u'(C + \alpha h_C) - v'(N + \alpha h_N) \right) (N + \alpha h_N) \right] \\
& \left. - \rho \phi(C + \alpha h_C) + (C + \alpha h_C) \dot{\phi} - \rho \vartheta(\pi^w + \alpha h_\pi) + (\pi^w + \alpha h_\pi) \dot{\vartheta} \right\} dt + \phi_0(C_0 + \alpha h_{C,0}) + \vartheta_0(\pi_0^w + \alpha h_{\pi,0})
\end{aligned}$$

Working out the Gateaux derivatives and employing the fundamental lemma of the calculus of variations, we arrive at the following

$$\begin{aligned}
0 = \int_0^\infty e^{-\rho t} & \left\{ C^{-\gamma} h_C - v'(N) h_N - \delta \pi^w h_\pi \right. \\
& + \phi \left[ \frac{1}{\gamma} \left( h_i - h_\pi \right) C + \frac{1}{\gamma} \left( i - \pi^w + \frac{\dot{A}}{A} - \rho \right) h_C \right] \\
& + \mu \left[ A h_N - h_C \right] \\
& + \vartheta \left[ \rho h_\pi + \frac{\epsilon}{\delta} \left( (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u''(C) h_C - v''(N) h_N \right) N \right. \\
& \left. + \frac{\epsilon}{\delta} \left( (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u'(C) - v'(N) \right) h_N \right] \\
& \left. - \rho \phi h_C + h_C \dot{\phi} - \rho \vartheta h_\pi + h_\pi \dot{\vartheta} \right\} dt + \phi_0 h_{C,0} + \vartheta_0 h_{\pi,0}
\end{aligned}$$

Grouping terms,

$$\begin{aligned}
0 = \int_0^\infty e^{-\rho t} & \left\{ C^{-\gamma} h_C - \mu h_C + \phi \frac{1}{\gamma} \left( i - \pi^w + \frac{\dot{A}}{A} - \rho \right) h_C - \rho \phi h_C + h_C \dot{\phi} + \vartheta \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u''(C) N h_C \right. \\
& - \delta \pi^w h_\pi - \phi \frac{1}{\gamma} C h_\pi + \vartheta \rho h_\pi - \rho \vartheta h_\pi + h_\pi \dot{\vartheta} \\
& - v'(N) h_N + \mu A h_N + \vartheta \frac{\epsilon}{\delta} \left( (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A u'(C) - v'(N) \right) h_N - \vartheta \frac{\epsilon}{\delta} v''(N) h_N N \\
& \left. + \phi \frac{1}{\gamma} C h_i \right\} dt + \phi_0 h_{C,0} + \vartheta_0 h_{\pi,0}
\end{aligned}$$

The fundamental lemma of the calculus of variations yields the desired result. ■

**Lemma 30.** *The stationary Ramsey plan in RANK satisfies*

$$\begin{aligned}
\pi_{ss}^w &= 0 \\
i_{ss} &= \rho \\
N_{ss} &= \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right]^{\frac{1}{\gamma + \eta}} \\
C_{ss} &= N_{ss} \\
\vartheta_{ss} &= \frac{1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon}}{(\gamma + \eta) \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon}} \\
\mu_{ss} &= \frac{\gamma}{\gamma + \eta} (N_{ss})^\eta + \frac{\eta}{\gamma + \eta} (C_{ss})^{-\gamma}.
\end{aligned}$$

Importantly, we see that  $\vartheta_{ss} = 0$  if and only if an appropriate employment subsidy is in place, so that  $(1 + \tau^L) \frac{\epsilon - 1}{\epsilon} = 1$ .

### D.3 Timeless Dual Lagrangian

In the following, we leverage our timeless Ramsey approach to give a novel, non-linear characterization of optimal monetary policy in RANK. We associate the timeless Ramsey problem in the dual with the Lagrangian

$$L^{\text{TD}}(\vartheta) = \int_0^\infty e^{-\rho t} \left\{ \frac{1}{1 - \gamma} C_t^{1-\gamma} - v(N_t) - \underbrace{\frac{\delta}{2} (\pi_t^w)^2}_{\text{Inflation Target}} \right\} dt$$

**Lemma 31.** *The timeless dual Ramsey problem in RANK is time consistent. In the absence of shocks, the Ramsey planner has no incentive to deviate from the stationary Ramsey plan. That is,*

$$\frac{d}{d\theta} L^{TD}(\vartheta) \Big|_{ss} = 0.$$

*Proof.* Suppose we differentiate

$$\frac{d}{d\theta} L^{TD}(\vartheta) = \int_0^\infty e^{-\rho t} \left\{ C_t^{-\gamma} \frac{d}{d\theta} - N_t^\eta \frac{dN_t}{d\theta} - \delta \pi_t^w \frac{d\pi_t^w}{d\theta} \right\} dt - \vartheta \frac{d\pi_0^w}{d\theta}$$

Next, we evaluate at the stationary Ramsey plan. This yields

$$\begin{aligned} \frac{d}{d\theta} L^{TD}(\vartheta) &= \int_0^\infty e^{-\rho t} \left\{ C^{-\gamma} \frac{dC_t}{d\theta} - N^\eta \frac{dN_t}{d\theta} \right\} dt - \vartheta \frac{d\pi_0^w}{d\theta} \\ &= \int_0^\infty e^{-\rho t} \left\{ C^{-\gamma} \left[ \frac{dC_t}{d\theta} - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \frac{dN_t}{d\theta} \right] \right\} dt - \vartheta \frac{d\pi_0^w}{d\theta} \end{aligned}$$

Next, from  $C_t = A_t N_t$ , we have when evaluated at the stationary Ramsey plan that

$$\frac{dC_t}{d\theta} = A \frac{dN_t}{d\theta}.$$

Thus,

$$\begin{aligned} \frac{d}{d\theta} L^{TD}(\vartheta) &= \int_0^\infty e^{-\rho t} \left\{ C^{-\gamma} \frac{dC_t}{d\theta} - N^\eta \frac{dN_t}{d\theta} \right\} dt - \vartheta \frac{d\pi_0^w}{d\theta} \\ &= C^{-\gamma} \left[ 1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right] \int_0^\infty e^{-\rho t} \frac{dC_t}{d\theta} dt - \vartheta \frac{d\pi_0^w}{d\theta}. \end{aligned}$$

Next, we use the Phillips curve. With  $\lim_{T \rightarrow \infty} \pi_T^w = 0$ , we have in integral form

$$\begin{aligned} \dot{\pi}_t^w &= \rho \pi_t^w + \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A_t u'(C_t) - v'(N_t) \right] N_t \\ \pi_t^w &= - \int_t^\infty e^{-\rho(s-t)} \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} A_s u'(C_s) - v'(N_s) \right] N_s \end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{d\pi_0^w}{d\theta} &= - \int_0^\infty e^{-\rho(s-0)} \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} (1 - \gamma) C^{-\gamma} \frac{dC_s}{d\theta} - (1 + \eta) N^\eta \frac{dN_s}{d\theta} \right] ds \\
&= - \frac{\epsilon}{\delta} \left[ (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} (1 - \gamma) C^{-\gamma} - (1 + \eta) N^\eta \right] \int_0^\infty e^{-\rho t} \frac{dC_t}{d\theta} dt \\
&= \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} C^{-\gamma} (\gamma + \eta) \int_0^\infty e^{-\rho t} \frac{dC_t}{d\theta} dt
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{d}{d\theta} L^{\text{TD}}(\vartheta) &= \overbrace{C^{-\gamma} \left[ 1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right] \int_0^\infty e^{-\rho t} \frac{dC_t}{d\theta} dt}^{\text{Marginal benefit from time-inconsistent deviations}} \\
&\quad - \underbrace{\vartheta \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} C^{-\gamma} (\gamma + \eta) \int_0^\infty e^{-\rho t} \frac{dC_t}{d\theta} dt}_{\text{Marginal cost of time-inconsistent deviations under timeless penalty / inflation target}}
\end{aligned}$$

Finally, we now have

$$\begin{aligned}
0 &= C^{-\gamma} \left[ 1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right] - \vartheta \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} C^{-\gamma} (\gamma + \eta) \\
&= \left[ 1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right] - \frac{1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon}}{(\gamma + \eta) \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon}} \frac{\epsilon}{\delta} (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} (\gamma + \eta) \\
&= \left[ 1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right] - \left[ 1 - (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} \right],
\end{aligned}$$

which concludes the proof. ■

**Remark 32.** Our constructive proof of Lemma 31 characterizes clearly the *marginal benefit* from time-inconsistent deviations from the stationary Ramsey plan. And it also shows clearly how the timeless penalty, the *marginal cost* of deviations, exactly offsets the marginal benefit. Importantly, we see here in closed-form what the economic determinants are of the marginal benefit and the timeless penalty.

#### D.4 Retracing Classical RANK Results

We are now ready to use our apparatus to retrace the classical analysis of optimal monetary stabilization policy in RANK. In this subsection, we restate several of the classical results in an exact, non-linear form.

To that end, we now consider a version of the baseline RANK model with demand, productivity, and cost-push shocks. Formally, we assume the following exogenous processes

$$\dot{A}_t = \xi_A (A - A_t)$$

$$\dot{\epsilon}_t = \xi_\epsilon (\epsilon - \epsilon_t)$$

$$\dot{\rho}_t = \xi_\rho (\rho - \rho_t)$$

where  $A$ ,  $\epsilon$ , and  $\rho$  are the steady-state constant levels. We will consider one-time unanticipate (“MIT”) shocks at time  $t = 0$ , so that  $A_0$ ,  $\epsilon_0$ , and  $\rho_0$  jump and subsequently revert back to their steady-state levels following the above laws of motion. With this enriched structure, the implementability conditions for the optimal policy Ramsey problem become

$$\begin{aligned}\dot{C}_t &= \frac{1}{\gamma} \left( i_t - \pi_t^w + \frac{\dot{A}_t}{A_t} - \rho_t \right) C_t \\ C_t &= A_t N_t - G_t \\ \dot{\pi}_t^w &= \rho_t \pi_t^w + \frac{\epsilon_t}{\delta} \left[ (1 + \tau^L) \frac{\epsilon_t - 1}{\epsilon_t} A_t u'(C_t) - v'(N_t) \right] N_t\end{aligned}$$

In much of the standard RANK literature, e.g., Clarida et al. (1999), optimal policy analysis drops the IS equation as an implementability condition and then proceeds to derive *targeting rules* for inflation and output (gaps). In the following, our goal is to retrace this classical analysis in our setting. We will leverage the results we derive below in Section 4 of the main text to compare optimal policy and targeting rules across RANK and HANK.

We drop the IS equation and use the resource constraint to solve out for  $N_t$ . Following Galí (2015), we define the *natural level of output*, denoted  $Y_t^n$ , as the equilibrium level of output under flexible prices. From the Phillips curve, which is in our setting given by

$$Y_t^n = \left[ (1 + \tau^L) \frac{\epsilon_t - 1}{\epsilon_t} A_t^{1+\eta} \right]^{\frac{1}{\gamma+\eta}}. \quad (67)$$

Going back to the Phillips curve and using the resource constraint with  $Y_t = A_t N_t = C_t$ , we have

$$\dot{\pi}_t^w = \rho_t \pi_t^w + \frac{\epsilon_t}{\delta} \left[ \left( Y_t^n \right)^{\gamma+\eta} - Y_t^{\gamma+\eta} \right] Y_t^{1-\gamma} A_t^{-1-\eta} \quad (68)$$

which is our sole remaining implementability condition and features all three shocks:  $A_t$ ,  $\epsilon_t$ , and  $\rho_t$ .

The planner's Ramsey problem can now be associated with the Lagrangian

$$L = \int_0^\infty e^{-\int_0^t \rho_s ds} \left\{ u(Y_t) - v\left(\frac{Y_t}{A_t}\right) - \frac{\delta}{2} (\pi_t^w)^2 + \vartheta_t \left[ \rho_t \pi^w + \frac{\epsilon_t}{\delta} \left( (1 + \tau^L) \frac{\epsilon_t - 1}{\epsilon_t} A_t u'(Y_t) - v'\left(\frac{Y_t}{A_t}\right) \right) \frac{Y_t}{A_t} - \dot{\pi}_t^w \right] \right\} dt$$

We now state the main result of this appendix: a non-linear targeting rule for optimal monetary policy in RANK under demand, TFP, and cost-push shocks.

**Proposition 33. (Optimal Policy Targeting Rules / Divine Coincidence in RANK)**

a) (*Targeting Rule*) Optimal monetary policy in RANK is fully characterized by the non-linear targeting rule

$$Y_t = Y_t^n \left( \frac{\frac{1}{1+\tau^L} \frac{\epsilon_t}{\epsilon_t-1} + \frac{\epsilon_t}{\delta} \vartheta_t (1-\gamma)}{1 + \frac{\epsilon_t}{\delta} \vartheta_t (1+\eta)} \right)^{\frac{1}{\gamma+\eta}} \quad (69)$$

b) (*Divine Coincidence*) Suppose there are no cost-push shocks, i.e.,  $\epsilon_t = \epsilon$ , and we implement an employment subsidy so that  $(1 + \tau^L) \frac{\epsilon-1}{\epsilon} = 1$ . We have

$$Y_t = Y_t^n \left( \frac{1 + \frac{\epsilon}{\delta} \vartheta_t (1-\gamma)}{1 + \frac{\epsilon}{\delta} \vartheta_t (1+\eta)} \right)^{\frac{1}{\gamma+\eta}}. \quad (70)$$

A solution to the non-linear Ramsey plan is then given by  $Y_t = Y_t^n$ ,  $\vartheta_t = 0$ , and  $\pi_t^w = 0$ .

*Proof.* Crucially, both  $Y_0$  and  $\pi_0^w$  are free from the planner's perspective. We start by integrating by parts, yielding

$$\begin{aligned} L = \int_0^\infty e^{-\int_0^t \rho_s ds} & \left\{ u(Y_t) - v\left(\frac{Y_t}{A_t}\right) - \frac{\delta}{2} (\pi_t^w)^2 \right. \\ & + \vartheta_t \left[ \rho_t \pi^w + \frac{\epsilon_t}{\delta} \left( (1 + \tau^L) \frac{\epsilon_t - 1}{\epsilon_t} A_t u'(Y_t) - v'\left(\frac{Y_t}{A_t}\right) \right) \frac{Y_t}{A_t} \right. \\ & \left. \left. - \rho_t \vartheta_t \pi_t^w + \pi_t^w \dot{\vartheta}_t \right] \right\} dt + \vartheta_0 \pi_0^w \end{aligned}$$

The two first-order conditions are then given by

$$0 = u'(Y_t) - v'\left(\frac{Y_t}{A_t}\right)\frac{1}{A_t} + \frac{\epsilon_t}{\delta}\vartheta_t \left[ (1 + \tau^L)\frac{\epsilon_t - 1}{\epsilon_t} A_t u''(Y_t) - v''\left(\frac{Y_t}{A_t}\right)\frac{1}{A_t} \right] \frac{Y_t}{A_t}$$

$$+ \frac{\epsilon_t}{\delta}\vartheta_t \left( (1 + \tau^L)\frac{\epsilon_t - 1}{\epsilon_t} A_t u'(Y_t) - v'\left(\frac{Y_t}{A_t}\right) \right) \frac{1}{A_t}$$

for output and  $\dot{\vartheta}_t = \delta\pi_t^w$  for the multiplier.

We now simplify the first condition, which will take the form of a targeting rule, as discussed in much of the classical optimal policy analysis in RANK. With isoelastic preferences, we have

$$0 = Y_t^{-\gamma} - Y_t^\eta A_t^{-\eta-1} + \frac{\epsilon_t}{\delta}\vartheta_t \left( (1 + \tau^L)\frac{\epsilon_t - 1}{\epsilon_t}(1 - \gamma)Y_t^{-\gamma} - (1 + \eta)Y_t^\eta A_t^{-\eta-1} \right)$$

Further rearranging yields

$$0 = A_t^{1+\eta} - Y_t^{\gamma+\eta} + \frac{\epsilon_t}{\delta}\vartheta_t(1 + \tau^L)\frac{\epsilon_t - 1}{\epsilon_t}(1 - \gamma)A_t^{1+\eta} - \frac{\epsilon_t}{\delta}\vartheta_t(1 + \eta)Y_t^{\gamma+\eta}$$

or simply

$$\left[1 + \frac{\epsilon_t}{\delta}\vartheta_t(1 + \eta)\right]^{\frac{1}{\gamma+\eta}} Y_t = \left[\left(\frac{1}{1 + \tau^L}\frac{\epsilon_t}{\epsilon_t - 1} + \frac{\epsilon_t}{\delta}\vartheta_t(1 - \gamma)\right)(1 + \tau^L)\frac{\epsilon_t - 1}{\epsilon_t} A_t^{1+\eta}\right]^{\frac{1}{\gamma+\eta}}$$

Using the definition of natural output, we therefore have

$$\left[1 + \frac{\epsilon_t}{\delta}\vartheta_t(1 + \eta)\right]^{\frac{1}{\gamma+\eta}} Y_t = \left(\frac{1}{1 + \tau^L}\frac{\epsilon_t}{\epsilon_t - 1} + \frac{\epsilon_t}{\delta}\vartheta_t(1 - \gamma)\right)^{\frac{1}{\gamma+\eta}} Y_t^n$$

or simply

■

Importantly, the targeting rule of Proposition 33 echoes the seminal result of the standard New Keynesian framework, that Divine Coincidence obtains unless there are cost-push shocks. In the presence of only productivity and demand shocks, the planner perceives no tradeoff between inflation and output.

## E Analytical Results for Separable Preferences

Here, we derive the counterpart of equation (51) for general preferences. We present a targeting rule in terms of labor wedges.

Note that by integrating equation (22), using an arbitrary set of weights  $\omega_t^U(a, z)$ , we can find the following expression for  $\mu_t$ :

$$\mu_t = \frac{\iint \omega_t(a, z) [u'(c_t(a, z)) - \partial_a \lambda_t(a, z) + \vartheta_t \frac{\epsilon}{\delta} \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_t N_t z u''(c_t(a, z)) + u''(c_t(a, z)) \tilde{\chi}_t(a, z)] g_t(a, z) da dz}{\iint \omega_t(a, z) g_t(a, z) da dz}$$

Combining this expression with equation (24), and setting  $\omega_t(a, z) = 0 = z$ , we find that

$$\begin{aligned} 0 &= \Lambda_t - \frac{v'(N_t)}{A_t} \\ &+ \iint z u''(c_t(a, z)) \tilde{\chi}_t(a, z) g_t(a, z) da dz \\ &+ \iint z \tilde{\phi}_t(a, z) \partial_a V_t(a, z) g_t(a, z) da dz \\ &+ \vartheta_t \frac{\epsilon}{\delta} \left[ \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_t \Lambda_t - v'(N_t) + \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_t N_t \iint z^2 u''(c_t(a, z)) - v''(N_t) N_t \right], \end{aligned}$$

where we define  $\tilde{\phi}_t(a, z) = \frac{\phi_t(a, z)}{g_t(a, z)}$  and  $\tilde{\chi}_t(a, z) = \frac{\chi_t(a, z)}{g_t(a, z)}$ . This expression is the exact counterpart of equation (51) in the text.

## F Optimal Policy under Discretion

The optimal planning problem under discretion at time  $s$  is formally associated with the Lagrangian

$$\begin{aligned}
L^D(\mathbf{g}_s) = & \min_{\phi_s, \chi_s, \lambda_s, \mu_s, \vartheta_s} \max_{V_s, \mathbf{c}_{s,[2:J]}, \mathbf{g}_{s+1}, \pi_s^w, N_s, i_s} \sum_{n=s}^{N-1} e^{-\rho t_n} \left\{ \right. \\
& + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{\mathbf{c}_{n,[2:J]}} \right)' \mathbf{g}_t - v(N_n) \mathbf{1}' \mathbf{g}_t - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1}' \mathbf{g}_t \\
& + \boldsymbol{\phi}'_n \left[ -\rho V_n + \frac{V_{n+1} - V_n}{dt} + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{\mathbf{c}_{n,[2:J]}} \right) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 \right] \\
& + \boldsymbol{\phi}'_n A^z V_n + \sum_{i \geq 2} \phi_{i,n} \left( i_n a_i - \pi_n^w a_i + \frac{A_{n+1} - A_n}{dt A_n} a_i + z_i A_n N_n - c_{n,i} \right) \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_n}{da} \\
& + \boldsymbol{\chi}'_{n,[2:J]} \left[ u'(\mathbf{c}_{n,[2:J]}) - \left( \frac{\mathbf{D}_a}{da} \mathbf{V}_{n+1} \right)_{[2:J]} \right] \\
& - \lambda'_n \frac{\mathbf{g}_{n+1} - \mathbf{g}_n}{dt} + \lambda'_n (A^z)' \mathbf{g}_n \\
& + \sum_{i \geq 2} \lambda_{n,i} \frac{\mathbf{D}'_{a,[i,:]} \left[ \left( i_n \mathbf{a}_{[2:J]} - \pi_n^w \mathbf{a}_{[2:J]} + \frac{A_{n+1} - A_n}{dt A_n} \mathbf{a}_{[2:J]} + z_{[2:J]} A_n N_n - \mathbf{c}_{n,[2:J]} \right) \cdot \mathbf{g}_n \right]}{da} \\
& + \mu_n \left[ \mathbf{c}'_n \mathbf{g}_n d\mathbf{x} - A_n N_n \right] \\
& \left. + \vartheta_n \left[ -\frac{\pi_{n+1}^w - \pi_n^w}{dt} + \rho \pi_n^w + \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_n (\mathbf{z} \cdot u'(\mathbf{c}_n))' \mathbf{g}_n d\mathbf{x} - v'(N_n) \right) N_n \right] \right\} dt
\end{aligned}$$

where the superscript  $D$  denotes the planning problem under discretion. The planner takes as given an initial condition for the cross-sectional distribution,  $\mathbf{g}_s$ .

Unlike in the Ramsey problem with commitment, we only integrate by parts the *state variables* of the problem, and not those terms associated with forward-looking constraints. That is, we use

$$\begin{aligned}
-\sum_{n=s}^{N-1} e^{-\rho t_n} \lambda'_n \frac{\mathbf{g}_{n+1} - \mathbf{g}_n}{dt} &= \sum_{n=s}^{N-1} e^{-\rho t_n} \frac{\lambda'_n - e^{\rho dt} \lambda'_{n-1}}{dt} \mathbf{g}_n \\
&+ \frac{1}{dt} e^{\rho dt} \lambda'_{s-1} \mathbf{g}_s - \frac{1}{dt} e^{\rho dt} e^{-\rho t_N} \lambda'_{N-1} \mathbf{g}_N
\end{aligned}$$

and rewrite the Lagrangian as

$$\begin{aligned}
L^D(\mathbf{g}_s) = & \min_{\phi_s, \chi_s, \lambda_s, \mu_s, \vartheta_s} \max_{V_s, c_{s,[2:J]}, g_{s+1}, \pi_s^w, N_s, i_s} \sum_{n=s}^{N-1} e^{-\rho t_n} \left\{ \left\{ \right. \right. \\
& + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{c_{n,[2:J]}} \right)' \mathbf{g}_t - v(N_n) \mathbf{1}' \mathbf{g}_t - \frac{\delta}{2} (\pi_n^w)^2 \mathbf{1}' \mathbf{g}_t \\
& + \boldsymbol{\phi}'_n \left[ -\rho V_n + \frac{V_{n+1} - V_n}{dt} + u \left( \frac{i_n a_1 - \pi_n^w a_1 + \frac{A_{n+1} - A_n}{dt A_n} a_1 + z_1 A_n N_n}{c_{n,[2:J]}} \right) - v(N_n) - \frac{\delta}{2} (\pi_n^w)^2 \right] \\
& + \boldsymbol{\phi}'_n \mathbf{A}^z V_n + \sum_{i \geq 2} \phi_{i,n} \left( i_n a_i - \pi_n^w a_i + \frac{A_{n+1} - A_n}{dt A_n} a_i + z_i A_n N_n - c_{n,i} \right) \frac{\mathbf{D}_{a,[i,:]} V_n}{da} \\
& + \chi'_{n,[2:J]} \left[ u'(\mathbf{c}_{n,[2:J]}) - \left( \frac{\mathbf{D}_a}{da} V_{n+1} \right)_{[2:J]} \right] \\
& + e^{-\rho t_n} \frac{\lambda'_n - e^{\rho dt} \lambda'_{n-1}}{dt} \mathbf{g}_n + \lambda'_n (\mathbf{A}^z)' \mathbf{g}_n \\
& + \sum_{i \geq 2} \lambda_{n,i} \frac{\mathbf{D}'_{a,[i,:]} \left[ \left( \frac{0}{i_n \mathbf{a}_{[2:J]} - \pi_n^w \mathbf{a}_{[2:J]} + \frac{A_{n+1} - A_n}{dt A_n} \mathbf{a}_{[2:J]} + z_{[2:J]} A_n N_n - \mathbf{c}_{n,[2:J]}} \right) \cdot \mathbf{g}_n \right]}{da} \\
& + \mu_n \left[ \mathbf{c}'_n \mathbf{g}_n d\mathbf{x} - A_n N_n \right] \\
& + \vartheta_n \left[ -\frac{\pi_{n+1}^w - \pi_n^w}{dt} + \rho \pi_n^w + \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon} (1 + \tau^L) A_n (z \cdot u'(\mathbf{c}_n))' \mathbf{g}_n d\mathbf{x} - v'(N_n) \right) N_n \right] \\
& + e^{\rho dt} \lambda'_{s-1} \mathbf{g}_s d\mathbf{x} - e^{\rho dt} e^{-\rho t_N} \lambda'_{N-1} \mathbf{g}_N d\mathbf{x}
\end{aligned}$$

We now characterize the first-order optimality conditions associated with the planning problem under discretion.

**Derivative for  $V_s$ .** We have

$$0 = -\rho \boldsymbol{\phi}_s - \frac{1}{dt} \boldsymbol{\phi}_s + (\mathbf{A}^z)' \boldsymbol{\phi}_s + \frac{1}{da} (\boldsymbol{\phi}_s \mathbf{D}_a)' \mathbf{s}_s - e^{\rho dt} \frac{\mathbf{D}'_a}{da} \begin{pmatrix} 0 \\ \chi_{s-1,[2:J]} \end{pmatrix}$$

or simply

$$0 = -\rho \boldsymbol{\phi}_s - \frac{1}{dt} \boldsymbol{\phi}_s + \mathbf{A}' \boldsymbol{\phi}_s - e^{\rho dt} \frac{\mathbf{D}'_a}{da} \begin{pmatrix} 0 \\ \chi_{s-1,[2:J]} \end{pmatrix}$$

Consider the last term in this equation. The household's consumption FOC says that consumption today is a function of "expected" future value, which therefore uses  $V_{n+1}$ . The planner under

discretion takes the future value  $V_{n+1}$  as given. And the planner is constrained by the competitive equilibrium condition that households make consumption decisions *purely* in terms of  $V_{n+1}$ . By the household's first-order condition, then,  $c_n$  is pinned down as a function of  $V_{n+1}$ .

We now see from this that, in the continuous-time limit with  $dt \rightarrow 0$ , we must have

$$\phi_s = 0.$$

This is the proper boundary condition for the formal continuous-time problem under discretion. Moreover, from the consumption FOC in the Lagrangian, we also have

$$0 = \frac{D'_a}{da} \begin{pmatrix} 0 \\ \chi_{s-1,[2:J]} \end{pmatrix}$$

for all  $s$ .

**Derivative for  $g_{s+1}$ .** We have

$$\begin{aligned} 0 = & u(c_{s+1}) + \mu_{s+1} c_{s+1} - v(N_{s+1}) \mathbf{1} - \frac{\delta}{2} (\pi_{s+1}^w)^2 \mathbf{1} + \frac{\lambda'_{s+1} - e^{\rho dt} \lambda'_s}{dt} + (\lambda'_{s+1}(A^z)')' \\ & + \frac{d}{dg_{s+1}} \left[ \frac{1}{da} (D_a \lambda_{s+1})' [s_{s+1} \cdot g_{s+1}] \right] + \vartheta_{s+1} N_{s+1} \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_{s+1} z \cdot u'(c_{s+1}) \end{aligned}$$

Now we work out the remaining derivative,

$$\frac{d}{dg_{s+1}} \left[ \frac{1}{da} (D_a \lambda_{s+1})' [s_{s+1} \cdot g_{s+1}] \right] = A^a \lambda_{s+1}.$$

Thus, we have

$$\begin{aligned} 0 = & u(c_{s+1}) + \mu_{s+1} c_{s+1} - v(N_{s+1}) \mathbf{1} - \frac{\delta}{2} (\pi_{s+1}^w)^2 \mathbf{1} + \frac{\lambda'_{s+1} - e^{\rho dt} \lambda'_s}{dt} \\ & + A \lambda_{s+1} + \vartheta_{s+1} N_{s+1} \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_{s+1} z \cdot u'(c_{s+1}) \end{aligned}$$

**Derivative  $c_{s,[2:J]}$ .** Notice that the planner under commitment also just solves a static problem for consumption at every time step. In other words, the choice of consumption today doesn't "bind" the planner tomorrow in any way under commitment. Therefore, we again have

$$\begin{aligned} 0 = & u'(c_{s,i}) g_{s,i} + \mu_s g_{s,i} + u'(c_{s,i}) \phi_{s,i} + u''(c_{s,i}) \chi_{s,i} + \vartheta_s N_s \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_s z_i u''(c_{s,i}) g_{s,i} \\ & - \frac{1}{da} \phi_{s,i} D_{a,[i,:]} V_s - \frac{1}{da} g_{s,i} D_{a,[i,:]} \lambda_s \end{aligned}$$

**Derivative  $\pi_n^w$ .** We have

$$\begin{aligned}
0 = & \left[ -u'(c_{s,1})g_{s,1}a_1 - \mu_s g_{s,1}a_1 - \delta \pi_s^w \mathbf{1}' \mathbf{g}_s - \phi_{s,1} u'(c_{s,1})a_1 - \delta \pi_s^w \boldsymbol{\phi}_s' \mathbf{1} \right] d\mathbf{x} \\
& - \vartheta_s \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_s N_s z_1 u''(c_{s,1}) g_{s,1} a_1 d\mathbf{x} \\
& + \left[ - \sum_{i \geq 2} \phi_{i,s} a_i \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_s}{da} + \sum_{i \geq 2} \lambda_{s,i} \frac{\mathbf{D}'_{a,[i,:]}}{da} \left[ \begin{pmatrix} 0 \\ -\mathbf{a}_{[2:J]} \end{pmatrix} \cdot \mathbf{g}_s \right] \right] d\mathbf{x} \\
& + \frac{1}{dt} \vartheta_s
\end{aligned}$$

Thus, we have

$$\begin{aligned}
0 = & \left[ -u'(c_{s,1})g_{s,1}a_1 - \mu_s g_{s,1}a_1 - \delta \pi_s^w \mathbf{1}' \mathbf{g}_s - \phi_{s,1} u'(c_{s,1})a_1 - \delta \pi_s^w \boldsymbol{\phi}_s' \mathbf{1} \right] d\mathbf{x} \\
& - \vartheta_s \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_s N_s z_1 u''(c_{s,1}) g_{s,1} a_1 d\mathbf{x} \\
& - \sum_{i \geq 2} \phi_{i,s} a_i \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_s}{da} - \sum_{i \geq 2} g_{s,i} a_i \frac{\mathbf{D}_{a,[i,:]}}{da} \lambda_s d\mathbf{x} + \frac{1}{dt} \vartheta_s
\end{aligned}$$

**Derivative  $i_n$ .** The nominal interest rate derivative is very easy because it's parallel to wage inflation, except in the Phillips curve. In particular, the choice of the nominal interest rate is again a fundamentally *static* problem, even in the case with commitment. We have

$$\begin{aligned}
0 = & u'(c_{s,1})g_{s,1}a_1 + \mu_s g_{s,1}a_1 + \phi_{s,1} u'(c_{s,1})a_1 + \sum_{i \geq 2} \phi_{i,s} a_i \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_s}{da} + \sum_{i \geq 2} g_{s,i} a_i \frac{\mathbf{D}_{a,[i,:]}}{da} \lambda_s \\
& + \vartheta_s \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_s N_s z_1 u''(c_{s,1}) g_{s,1} a_1
\end{aligned}$$

**Derivative  $N_n$ .** Finally, we take the derivative for aggregate labor. This is again a static problem. So, as before, we have

$$\begin{aligned}
0 = & \left[ u'(c_{s,1})g_{s,1}z_1 A_s + \mu_s g_{s,1}z_1 A_s + \phi_{s,1} u'(c_{s,1})z_1 A_s + \sum_{i \geq 2} \phi_{i,s} z_i A_s \frac{\mathbf{D}_{a,[i,:]} \mathbf{V}_s}{da} + \sum_{i \geq 2} g_{s,i} z_i A_s \frac{\mathbf{D}_{a,[i,:]}}{da} \lambda_s \right] d\mathbf{x} \\
& + \vartheta_s \frac{\epsilon \epsilon - 1}{\delta} (1 + \tau^L) A_s N_s z_1 u''(c_{s,1}) g_{s,1} z_1 A_s d\mathbf{x} \\
& - v'(N_s) \mathbf{1}' \mathbf{g}_s d\mathbf{x} - v'(N_s) \boldsymbol{\phi}_s' \mathbf{1} d\mathbf{x} \\
& - \mu_s A_s + \vartheta_s \frac{\epsilon - 1}{\delta} \left( \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A_s (\mathbf{z} \cdot u'(c_s))' \mathbf{g}_s d\mathbf{x} - v'(N_s) \right) - \frac{\epsilon}{\delta} \vartheta_s v''(N_s) N_s
\end{aligned}$$

We now summarize the resulting optimality conditions for the problem under discretion. We state these optimality conditions here for the fully discretized problem, which we have worked with thus far. For the main text, we bring these equations back to the continuous case.

We see immediately that  $\vartheta_s = 0$  and

$$\phi_s = 0$$

because the planner does not respect consumption promises from the past. These two conditions signify the lack of commitment. The optimality condition for the cross-sectional distribution still characterizes the evolution of the social lifetime value. We have

$$\begin{aligned} 0 = & u(c_{s+1}) + \mu_{s+1}c_{s+1} - v(N_{s+1})\mathbf{1} - \frac{\delta}{2}(\pi_{s+1}^w)^2\mathbf{1} + \frac{\lambda'_{s+1} - e^{\rho dt}\lambda'_s}{dt} + A\lambda_{s+1} \\ & + \vartheta_{s+1}N_{s+1}\frac{\epsilon\epsilon-1}{\delta}(1+\tau^L)A_{s+1}\mathbf{z} \cdot u'(c_{s+1}) \end{aligned}$$

The optimality condition for consumption becomes

$$-\chi u''(c) = \left[ u'(c) + \mu - \lambda_a + \vartheta N \frac{\epsilon\epsilon-1}{\delta}(1+\tau^L)A_z u''(c) \right] g.$$

The optimality condition for monetary policy now becomes

$$0 = \left[ u'(c_{s,1}) + \mu_s + \vartheta_s \frac{\epsilon\epsilon-1}{\delta}(1+\tau^L)A_s N_s z_1 u''(c_{s,1}) \right] g_{s,1} a_1 + \sum_{i \geq 2} g_{s,i} a_i \frac{D_{a,[i,:]}}{da} \lambda_s$$

And finally, the optimality condition for aggregate economic activity becomes

$$\begin{aligned} 0 = & \left[ u'(c_{s,1})g_{s,1}z_1A_s + \mu_s g_{s,1}z_1A_s + \sum_{i \geq 2} g_{s,i}z_iA_s \frac{D_{a,[i,:]}}{da} \lambda_s \right] dx \\ & + \vartheta_s \frac{\epsilon\epsilon-1}{\delta}(1+\tau^L)A_s N_s z_1 u''(c_{s,1})g_{s,1}z_1A_s dx \\ & - v'(N_s) - \mu_s A_s + \vartheta_s \frac{\epsilon}{\delta} \left( \frac{\epsilon-1}{\epsilon}(1+\tau^L)A_s (\mathbf{z} \cdot u'(\mathbf{c}_s))' \mathbf{g}_s dx - v'(N_s) \right) - \frac{\epsilon}{\delta} \vartheta_s v''(N_s) N_s \end{aligned}$$

Plugging in for  $\vartheta_s = 0$  and  $\phi_s = 0$ , we can leverage that several terms drop out and simplify to give an even sharper comparison to RANK. In RANK under discretion, the optimality condition for aggregate economic activity is given by

$$\begin{aligned} 0 = & -v'\left(\frac{Y_s}{A_s}\right) \frac{1}{A_s} + u'(Y_s) + \frac{\epsilon_s}{\delta} \vartheta_s (1+\tau^L) \frac{\epsilon_s-1}{\epsilon_s} u''(Y_s) Y_s \\ & + \frac{\epsilon_s}{\delta} \vartheta_s \left[ (1+\tau^L) \frac{\epsilon_s-1}{\epsilon_s} A_s u'(Y_s) - v'\left(\frac{Y_s}{A_s}\right) - v''\left(\frac{Y_s}{A_s}\right) \frac{Y_s}{A_s} \right] \frac{1}{A_s} \end{aligned}$$

which is unchanged from before because this is a fundamentally static optimization. However, as in HANK, we now have  $\vartheta_s = 0$  under discretion. So we can simplify the condition as

$$Y_s = A_s^{\frac{1+\eta}{\gamma+\eta}}$$

where, importantly, the RHS is *not* equal to the natural level of output — it is if and only if there are no cost-push shocks and we have the appropriate employment subsidy. Notice that we still have the same definition of natural output,  $Y_s^n = [(1 + \tau^L) \frac{\epsilon_s - 1}{\epsilon_s}]^{\frac{1}{\gamma+\eta}} A_s^{\frac{1}{\gamma+\eta}}$ . And so we get

$$Y_s = Y_s^n \left[ (1 + \tau^L) \frac{\epsilon_s - 1}{\epsilon_s} \right]^{-\frac{1}{\gamma+\eta}}$$

in RANK. This tell us that commitment is only useful when there are cost-push shocks or when there is no appropriate steady state employment subsidy in place — this insight is, of course, well known from the classical analysis, which has established that the time consistency problem associated with the Phillips curve only emerges under these two conditions. See, e.g., [Clarida et al. \(1999\)](#) for a detailed treatment.

In HANK, on the other hand, we now have

$$0 = -\frac{1}{A}v'(N) + \int z g \lambda_a d(a, z) - \mu + \vartheta \frac{\epsilon}{\delta} \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A \Lambda - v'(N) - v''(N) N \right] \frac{1}{A}$$

To proceed, we first state an auxilliary result.

**Lemma 34.** *The social marginal value of wealth can be expressed as*

$$\lambda_a = V_a + \left( \rho - r - \partial_t - \mathcal{A} + c_a \right)^{-1} \left\{ \mu c_a + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u''(c) c_a \right\}.$$

After discretizing in the space dimension, we can alternatively write this as

$$\lambda_a = V_a + \Gamma \left\{ \mu c_a + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z \cdot u''(c) c_a \right\}$$

*Proof.* The private and social lifetime value of a household in state  $(a, z)$  are, respectively, given by

$$\begin{aligned} \rho V &= (\partial_t + \mathcal{A})V + u(c) - v(N) - \frac{\delta}{2}(\pi^w)^2 \\ \rho \lambda &= (\partial_t + \mathcal{A})\lambda + u(c) - v(N) - \frac{\delta}{2}(\pi^w)^2 + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u'(c) + \mu c \end{aligned}$$

This should imply that we have

$$\lambda = V + \left( \rho - \partial_t - \mathcal{A} \right)^{-1} \left\{ \mu c + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u'(c) \right\}$$

or similarly

$$(\rho - \partial_t - \mathcal{A})(\lambda - V) = \left\{ \mu c + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u'(c) \right\}$$

Differentiating with respect to  $a$  yields

$$(\rho - \partial_t - \mathcal{A})(\lambda_a - V_a) - \left( \frac{d}{da} \mathcal{A} \right)(\lambda - V) = \mu c_a + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u''(c) c_a$$

or simply

$$(\rho - \partial_t - \mathcal{A})(\lambda_a - V_a) - (r - c_a)(\lambda_a - V_a) = \mu c_a + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u''(c) c_a.$$

Rearranging leads to the desired result. ■

Using our auxilliary Lemma 34 and plugging back into the optimality condition for aggregate economic activity in HANK, we obtain

$$0 = -\frac{1}{A} v'(N) + \int z g \left( V_a + \mu \kappa c_a + \kappa \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A z u''(c) c_a \right) d(a, z) - \mu + \vartheta \frac{\epsilon}{\delta} \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A \Lambda - v'(N) - v''(N) N \right] \frac{1}{A},$$

where

$$\kappa = \left( \rho - r - \partial_t - \mathcal{A} + c_a \right)^{-1}$$

while for RANK we had

$$0 = -\frac{1}{A} v'(N) + u'(C) + \frac{\epsilon}{\delta} \vartheta (1 + \tau^L) \frac{\epsilon - 1}{\epsilon} u''(C) C + \frac{\epsilon}{\delta} \vartheta \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A u'(C) - v'(N) - v''(N) N \right] \frac{1}{A}$$

We now rewrite HANK as

$$0 = -\frac{1}{A} v'(N) + \int z g V_a + \mu \int z \kappa c_a g + \vartheta N \frac{\epsilon}{\delta} \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A \int z u''(c) \kappa c_a d(a, z) - \mu + \vartheta \frac{\epsilon}{\delta} \left[ \frac{\epsilon - 1}{\epsilon} (1 + \tau^L) A \Lambda - v'(N) - v''(N) N \right] \frac{1}{A}$$

Now we set  $\vartheta = 0$  in both cases, which corresponds to the problem under full discretion. Rearranging, we finally obtain the two targeting rules

$$\begin{aligned} \text{RA : } \quad 0 &= -\frac{1}{A}v'(N) + u'(C) \\ \text{HA : } \quad 0 &= -\frac{1}{A}v'(N) + \int zu'(C)g + \mu(\Omega - 1), \end{aligned}$$

where

$$\Omega = \iint z\kappa\partial_a c_t(a, z)g_t(a, z)dadz$$

## G Quantitative Analysis: Sensitivity, Robustness, and Further Results

### G.1 Equilibrium Conditions with Shocks

We start by restating all relevant equilibrium conditions of our baseline HANK model when all three shocks are present.

The individual block is then given by

$$\rho_t V_t(a, z) = \partial_t V_t(a, z) + u(c_t(a, z)) - v(N_t) - \frac{\delta}{2}(\pi_t^w)^2 + \mathcal{A}_t V_t(a, z),$$

$$u'(c_t(a, z)) = \partial_a V_t(a, z)$$

$$\partial_t g_t(a, z) = \mathcal{A}_t^* g_t(a, z).$$

and the aggregate block by

$$\dot{\pi}_t^w = \rho_t \pi_t^w + \frac{\epsilon_t - 1}{\delta} \left[ (1 + \tau^L) w_t \Lambda_t - v'(N_t) \right] N_t$$

$$Y_t = A_t N_t$$

$$w_t = A_t$$

$$r_t = i_t - \pi_t$$

$$\pi_t = \pi_t^w - \frac{\dot{A}_t}{A_t}.$$

Importantly, household preferences must be modified to account for time variation in the discount rate. They are now given by

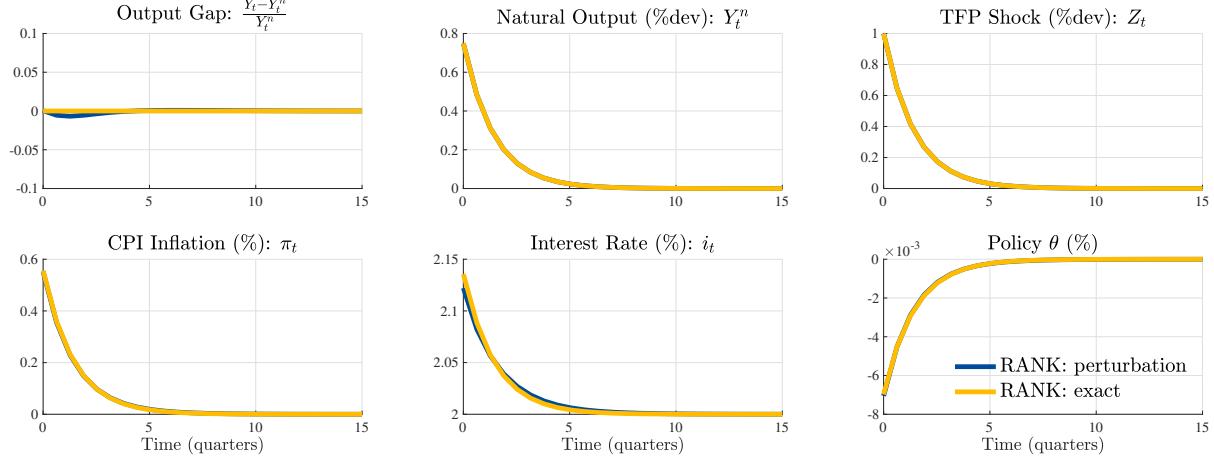
$$\int_0^\infty e^{-\int_0^t \rho_s ds} \left[ u(c_t) - v(N_t) - \frac{\delta}{2}(\pi_t^w)^2 \right] dt.$$

### G.2 Accuracy

In this section, we report a series of numerical tests to benchmark the accuracy of our perturbation method using sequence-space Hessians. In Figure 6, we compute the transition dynamics under optimal policy in RANK in response to a TFP shock using both our perturbation method and the exact analytical solution. The Figure underscores that our first-order perturbation method is highly accurate in the case of the baseline RANK model. The remaining error in the two solutions amounts to 0.01% in the output gap or, conversely, 1bps in the optimal interest rate response.

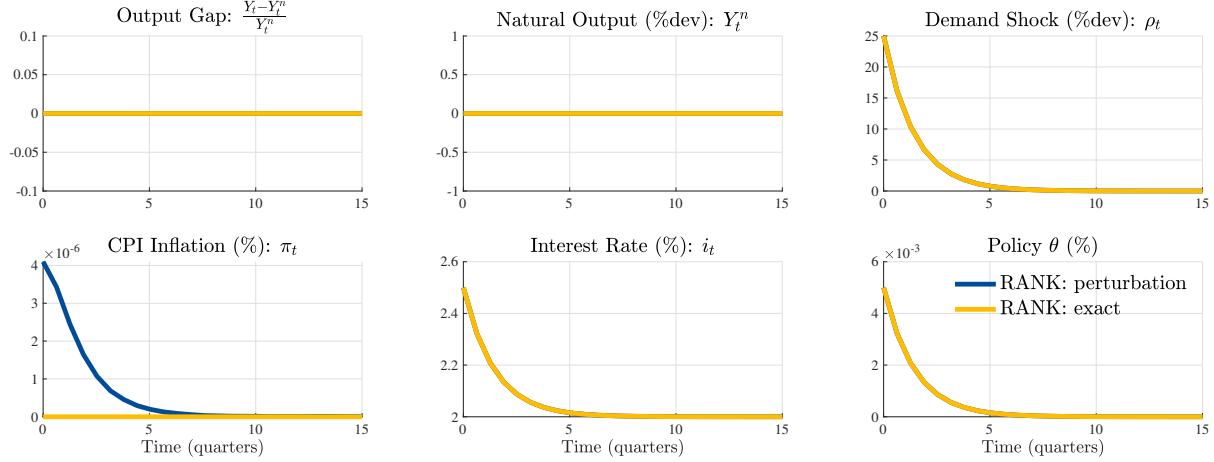
Likewise, Figure 7 reports the analogous comparison exercise for optimal policy in response to a demand shock in RANK. Here, the numerical error is even smaller. The discrepancy in optimal CPI inflation, for example, is on the order of  $10^{-6}$ .

**Figure 6.** Transition Dynamics with Optimal Policy — Perturbation vs. Exact Solution



**Note.** Impulse responses to positive TFP under optimal monetary policy in RANK. The Figure compares the exact analytical solution of optimal policy (yellow) against our numerical perturbation approach using sequence-space Hessians (blue).

**Figure 7.** Transition Dynamics with Optimal Policy — Perturbation vs. Exact Solution

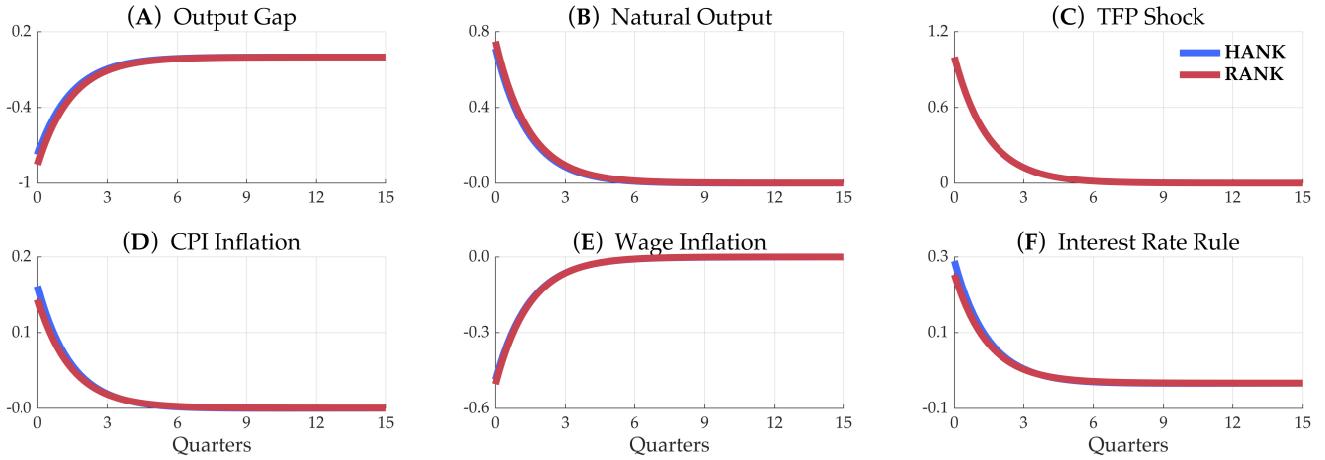


**Note.** Impulse responses to positive demand under optimal monetary policy in RANK. The Figure compares the exact analytical solution of optimal policy (yellow) against our numerical perturbation approach using sequence-space Hessians (blue).

### G.3 Transition Dynamics without Optimal Policy

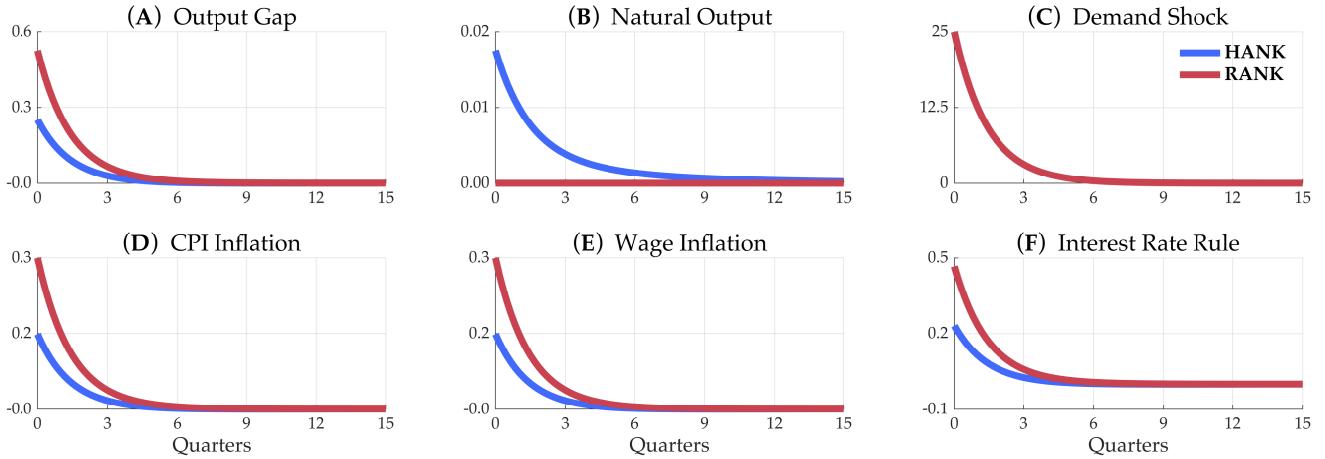
In this section, we present impulse response plots that display the transition dynamics of both RANK and HANK economies in response to TFP, demand, and cost-push shocks without optimal policy interventions. We model monetary policy instead as following a Taylor rule, with

$$i_t = r^{ss} + \lambda_\pi \pi_t, \quad (71)$$



**Figure 8.** Transition Dynamics under Taylor Rule — TFP Shock

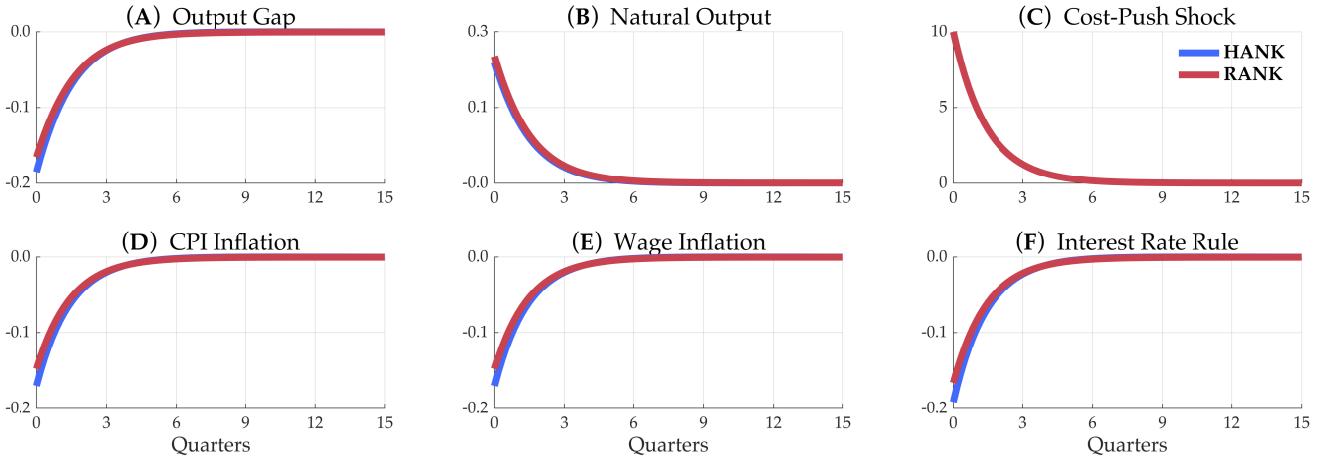
**Note.** Impulse responses to positive cost-push shock in both RANK (yellow) and HANK (blue) models. The nominal interest rate follows the Taylor rule (71) and is not set optimally. The cost-push shock is modeled as an increase in labor union's desired wage mark-up. The shock is initialized at  $\epsilon_0 = 11$  and mean-reverts to its steady state value  $\epsilon = 10$ , with a half-life of 2 quarters.



**Figure 9.** Transition Dynamics under Taylor Rule — Demand Shock

**Note.** Impulse responses to positive cost-push shock in both RANK (yellow) and HANK (blue) models. The nominal interest rate follows the Taylor rule (71) and is not set optimally. The cost-push shock is modeled as an increase in labor union's desired wage mark-up. The shock is initialized at  $\epsilon_0 = 11$  and mean-reverts to its steady state value  $\epsilon = 10$ , with a half-life of 2 quarters.

where we calibrate  $\lambda_\pi = 1.5$ .



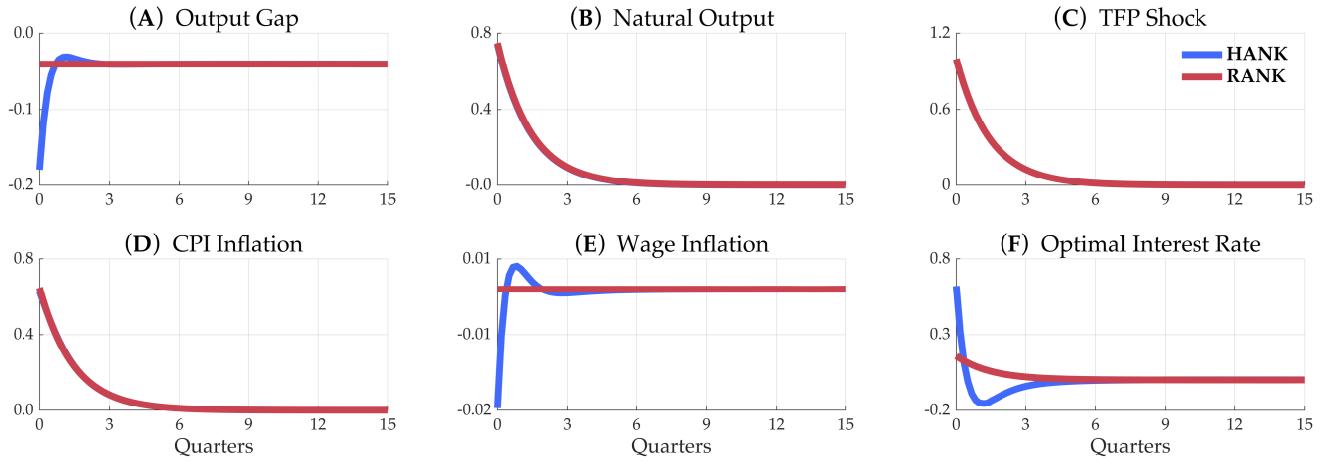
**Figure 10.** Transition Dynamics under Taylor Rule — Cost-Push Shock

**Note.** Impulse responses to positive cost-push shock in both RANK (yellow) and HANK (blue) models. The nominal interest rate follows the Taylor rule (71) and is not set optimally. The cost-push shock is modeled as an increase in labor union’s desired wage mark-up. The shock is initialized at  $\epsilon_0 = 11$  and mean-reverts to its steady state value  $\epsilon = 10$ , with a half-life of 2 quarters.

#### G.4 Sensitivity Analysis: Low Inequality

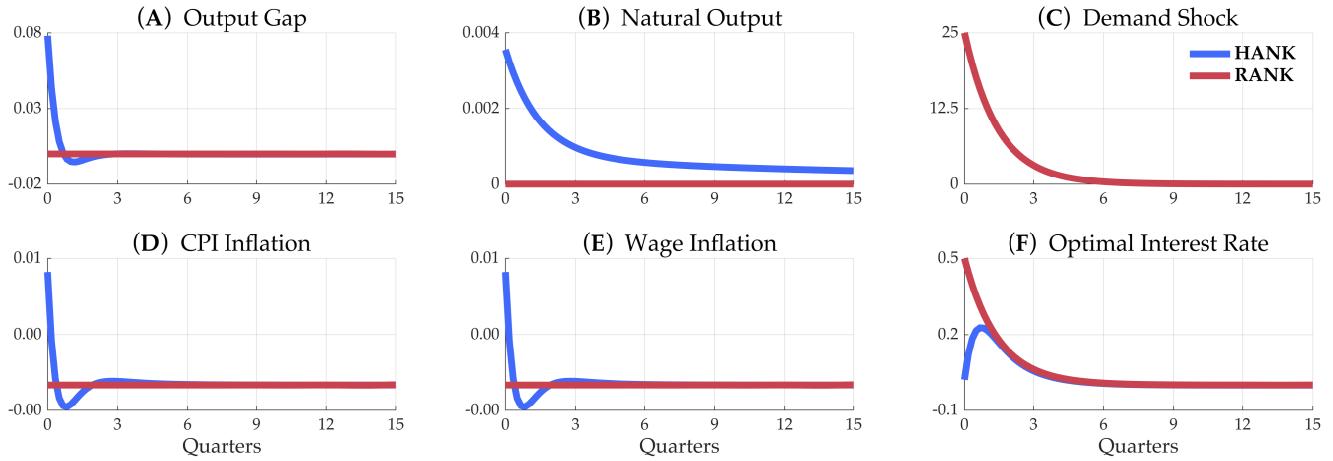
In this section, we explore the implications of inequality for optimal stabilization policy. In our benchmark calibration in Section 5, we set  $z^L = 0.8$  and  $z^H = 1.2$ . The RANK benchmark can be viewed instead as the limit where  $z^L, z^H \rightarrow \bar{z} = 1$ .

We again compute the impulse responses to TFP, demand, and cost-push shocks under optimal policy. However, we solve an alternative calibration with less uninsurable earnings risk, setting  $z^L = 0.95$  and  $z^H = 1.05$ . This calibration is between the RANK and HANK benchmarks of Section 5. We use this robustness exercise to confirm that the hump-shapes of the IRFs under optimal policy in Section 5 are driven by distributional considerations. We indeed find that less uninsurable risk and, consequently, lower cross-sectional consumption dispersion lead to less pronounced hump-shapes in the optimal policy IRFs.



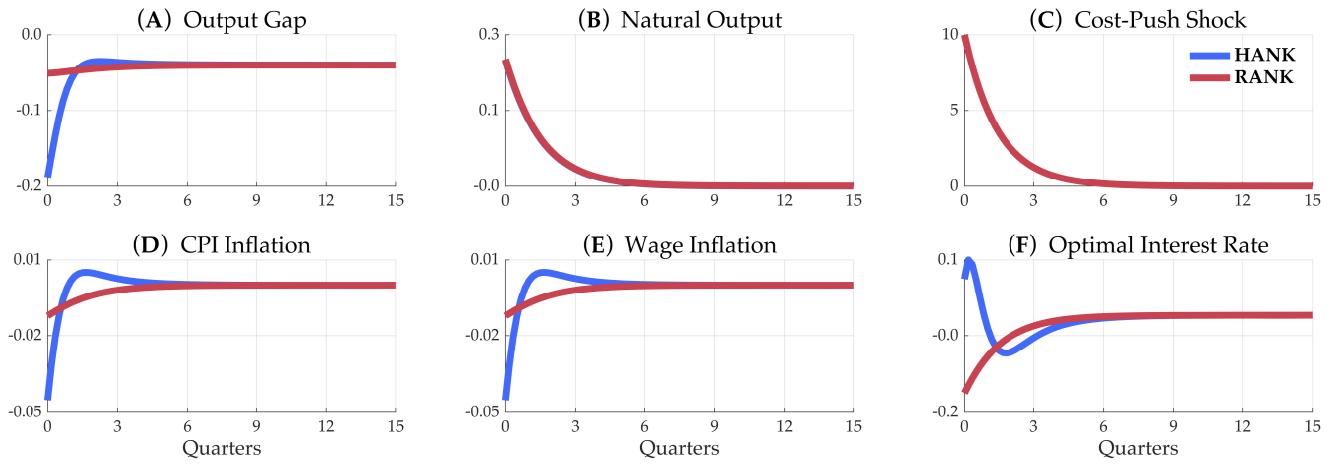
**Figure 11.** Optimal Policy Transition Dynamics with Low Earnings Risk — TFP Shock

**Note.** Impulse responses to positive TFP shock in both RANK (yellow) and HANK (blue) models under optimal monetary stabilization policy, using an alternative calibration with less uninsurable earnings risk with  $z^L = 0.95$  and  $z^H = 1.05$ . The initial shock is 1% of steady state TFP and mean-reverts with a half-life of 2 quarters.



**Figure 12.** Optimal Policy Transition Dynamics with Low Earnings Risk — Demand Shock

**Note.** Impulse responses to positive discount rate shock in both RANK (yellow) and HANK (blue) models under optimal monetary stabilization policy, using an alternative calibration with less uninsurable earnings risk with  $z^L = 0.95$  and  $z^H = 1.05$ . The discount rate shock is initialized at  $\rho_0 = 0.025$  and mean-reverts to its steady state value  $\rho = 0.02$ , with a half-life of 2 quarters.



**Figure 13.** Optimal Policy Transition Dynamics with Low Earnings Risk — Cost-Push Shock

**Note.** Impulse responses to positive cost-push shock in both RANK (yellow) and HANK (blue) models under optimal monetary stabilization policy, using an alternative calibration with less uninsurable earnings risk with  $z^L = 0.95$  and  $z^H = 1.05$ . The cost-push shock is modeled as an increase in labor union's desired wage mark-up. The shock is initialized at  $\epsilon_0 = 11$  and mean-reverts to its steady state value  $\epsilon = 10$ , with a half-life of 2 quarters.