Geometric set theory

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Preface

We wish to present to the reader a fresh and exciting new area of mathematics: geometric set theory. The purpose of this research direction is to compare transitive models of set theory with respect to their extensional agreement and definability. It turns out that many fracture lines in descriptive set theory, analysis, and model theory can be efficiently isolated and treated from this point of view. A particular success is the comparison of various Σ^2_1 consequences of the Axiom of Choice in unparalleled detail and depth.

The subject matter of the book was rather slow in coming. The initial work, restating Hjorth's turbulence in geometric terms and isolating the notion of a virtual quotient space of an analytic equivalence relation, existed in rudimentary versions since about 2013 in unpublished manuscripts of the second author. The joint effort [63] contained some independence results in choiceless set theory similar to those of the present book, but in decidedly suboptimal framework. It was not until the January 2018 discovery of balanced Suslin forcing that the flexibility and power of the geometric method fully asserted itself. The period after that discovery was filled with intense wonder–passing from one configuration of models of set theory to another and testing how they separate various well-known concepts in descriptive set theory, analysis, and model theory. At the time of writing, geometric set theory seems to be an area wide open for innumerable applications.

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Chapter 1

Introduction

1.1 Outline of the subject

Geometric set theory is the research direction which studies transitive models of set theory with respect to their extensional agreement. It turns out that numerous existing concepts in descriptive set theory and analysis have intuitive and informative geometric restatements, and the geometric point of view makes it possible to readily isolate many new concepts and generalize old ones. Surprising parallels appear between areas that to date did not have a discernible interface.

The oldest part of the subject deals with Borel equivalence relations on Polish spaces. Here, given an equivalence E on a space X and a configuration $\{M_i : i \in I\}$ of transitive models of set theory, one asks whether it is possible for an E-equivalence class to have representatives in some models of the configuration and fail to be represented in others. The answer to this question greatly varies with the nature of the equivalence relation and the configuration in question, and the resulting differences can be used with great effect to prove non-reducibility and ergodicity theorems for various Borel equivalence relations. One notable success in this area is a reformulation of Hjorth's turbulence in geometric terms, which is much more practicable than the original definition and allows endless generalizations and variations entirely divorced from the original context of Polish group actions.

Geometric set theory really started to blossom after it was applied in a seemingly entirely different direction: the independence results in choiceless Zermelo–Fraenkel (ZF) set theory. This is an area which saw a surge of results in early 1970's after which the interest in it waned. Both our purpose and methodology are quite different from the early practitioners though. We study almost exclusively ZF independence results between Σ^2_1 consequences of the Axiom of Choice which are intimately connected to various contemporary concerns of descriptive set theory and analysis. A very detailed structure appears, with some fracture lines running in parallel to existing combinatorial, algebraic, or analytic concepts and other fracture lines showing up in places quite unexpected. The main geometric motif is the following. Given a Σ^2_1 sentence $\phi = \exists A \subset X\ \psi(A)$, a consequence of the Axiom of Choice in which X is a Polish space and ψ is a formula quantifying only over elements of Polish spaces, and given a configuration $\{M_i : i \in I\}$ of transitive models of set theory with choice, is there a set $A \subset X$ such that in all (or many) models M in the configuration, $A \cap M \in M$ and $A \cap M$ is a witness to ϕ in the model M? The answer to this question is surprisingly varied and successful in separating consequences of the Axiom of Choice of this syntactical complexity.

The models of ZF we use for the independence results are nearly exclusively of the same form: they are extensions of the symmetric Solovay model by a simply definable σ-closed forcing. As a result, they are all models of DC, the Axiom of Dependent Choices. DC is instrumental for developing the basic concepts of mathematical analysis, thus highly desirable; at the same time, it is an axiom which is difficult to obtain in the symmetric models of 1970's. If suitable large cardinals are present, essentially identical effects can be obtained in generic extensions of the model $L(\mathbb{R})$ which have a very strong claim to canonicity. In addition, given Σ^2_1 sentences ϕ_0, ϕ_1, there is usually a quite canonical choice for a forcing which should generate a model of ZF+DC+$\neg\phi_0$+ϕ_1. The whole analysis of the forcing in question takes place in ZFC, using central concepts of the fields related to the Σ^2_1 sentences in question.

We conclude this brief outline of the area with the observation that the mass of natural questions that either seem to be treatable using the geometric method, or are generated by the inner workings of the geometric method, is entirely overwhelming. This book cannot be more than a relatively brief introduction to the subject. In many directions, our efforts faded due to sheer exhaustion and not because of insurmountable obstacles. We hope to motivate the reader to explore the enormous, inviting expanses of geometric set theory.

1.2 Equivalence relation results

The first group of results deals with the simplest geometric concern. If E is an analytic equivalence relation on a Polish space, and $V[H_0], V[H_1]$ are mutually generic extensions of V, is it possible to find an E-class which is represented in both $V[H_0]$ and $V[H_1]$ but not in V? Kanovei [48, Chapter 17], continuing the seminal work of Hjorth, defined an analytic equivalence E to be pinned just in case the answer is negative.

In Chapter 2, we develop the notion of the virtual quotient space for a given analytic equivalence relation on a Polish space. A virtual E-class is one which is represented in some mutually generic extensions of V; from a different angle, it is a class which may be represented only in some generic extension, but it does not depend on the choice of the generic filter. A formal definition uses the notion of an E-pin and an equivalence on the class of E-pins, which naturally extends the analytic equivalence relation E into the transfinite domain–Definitions 2.1.1 and 2.1.4. The virtual E-quotient space is then the class of all E-pins up to their equivalence. Every Borel structure on the E-quotient space has a natural virtual version on the virtual E-quotient space. Virtual structures appear in many places in this book. The main theorem governing their behavior:

Theorem 1.2.1. (Proposition 2.2.5) *Let E be a Borel equivalence relation on a Polish space X. Let $\mathcal{M}$ be a Borel structure on the E-quotient space X/E. The natural map from the E-quotient space to the virtual E-quotient space is a Π_1-elementary embedding from $\mathcal{M}$ to its virtual version.*

To size up the virtual E-quotient space, in Definition 2.5.1 we isolate cardinal invariants $\kappa(E)$ and $\lambda(E)$ of an equivalence relation E: $\lambda(E)$ is the cardinality of the virtual space, while $\kappa(E)$ is its forcing-theoretic complexity. The cardinals $\kappa(E), \lambda(E)$ respect the Borel reducibility order of equivalence relations, and as such serve as valuable tools for non-reducibility results. Their key feature: for Borel equivalence relations E, these cardinals are well-defined and obey an explicit upper bound. In particular, the virtual E-quotient space is a set, as opposed to a proper class as occurs for some analytic equivalence relations.

Theorem 1.2.2. (Theorem 2.5.6) *Let E be a Borel equivalence relation on a Polish space X. Then $\kappa(E), \lambda(E) < \beth_{\omega_1}$.*

As a result, the well-known Friedman–Stanley theorem [31] on nonreducibility of a jump E^+ of a Borel equivalence relation E to E is immediately translated to the Cantor theorem on the cardinality of the powerset: $\lambda(E^+) = 2^{\lambda(E)} > \lambda(E)$–Example 2.5.5. Other similar conceptual and brief nonreducibility results appear as well. The cardinal invariants $\kappa(E)$ and $\lambda(E)$ can attain many exotic and informative values, and inequalities between them can be manipulated by forcing. Most of Section 2.5 is devoted to examples of such manipulation. However, the main underlying concept is absolute, which justifies the parlance used throughout the book:

Theorem 1.2.3. (Corollary 2.7.3) *Let E be a Borel equivalence relation on a Polish space X. The statement "E is pinned" is absolute among all generic extensions.*

We stress that the above result holds within the context of ZFC. If one drops the Axiom of Choice but holds on to Dependent Choices, the class of Borel unpinned equivalence relations will shrink; in the symmetric Solovay model, it shrinks to the minimal possible extent: only the canonical unpinned relation $\mathbb{F}_2$ and the equivalence relations above it in the reducibility order are unpinned, see below. If one drops even the Axiom of Dependent Choices, the class of Borel unpinned equivalence relations becomes more erratic. We conjecture that it may expand to its largest possible extent: to the class of equivalence relations which are not reducible to F_σ equivalence relations.

As is usual in the realm of equivalence relations, we aim to prove various dichotomies.

Theorem 1.2.4. (Theorem 2.8.9) *Suppose that there is a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. Exactly one of the following occurs:*

1. *$\kappa(E), \lambda(E) < \infty$;*

2. *$\mathbb{E}_{\omega_1}$ is almost Borel reducible to E.*

Theorem 1.2.5. (Corollary 2.8.13) *In the symmetric Solovay model: let E be a Borel equivalence relation on a Polish space X. Exactly one of the following occurs:*

1. *E is pinned;*

2. *$\mathbb{F}_2$ is Borel reducible to E.*

In Chapter 3, we deal with a more sophisticated geometric concern. Let E be an analytic equivalence relation on a Polish space X. Is it possible to find two generic extensions $V[H_0]$ and $V[H_1]$ of V such that $V[H_0] \cap V[H_1] = V$ in which there is an E-class represented in both extensions $V[H_0]$ and $V[H_1]$, but not in V? Note that we do not ask the extensions to be mutually generic. In an initial approach, we develop a fruitful method for constructing separately Cohen-generic extensions $V[H_0]$ and $V[H_1]$ such that $V[H_0] \cap V[H_1] = V$. We define (Definition 3.1.3) a practical and easily checked notion of independence of continuous open maps between Polish spaces using a notion of a walk reminiscent of Hjorth's method of turbulence [48, Section 13.1] and prove:

Theorem 1.2.6. (Theorem 3.1.4) *Let X be a Polish space and $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ be continuous open maps to Polish spaces. The following are equivalent:*

1. *the maps f_0, f_1 are independent;*

2. *the Cohen forcing P_X of nonempty open subsets of X forces $V[f_0(\dot{x}_{gen})] \cap V[f_1(\dot{x}_{gen})] = V$.*

As one of the corollaries, we obtain a characterization of Hjorth's turbulence in geometric terms. This characterization is much more practical than the original statement for anyone familiar with the forcing relation.

Theorem 1.2.7. (Theorem 3.2.2) *Let Γ be a Polish group acting on a Polish space X with all orbits meager and dense. The following are equivalent:*

1. *the action is generically turbulent;*

2. *$V[x] \cap V[\gamma \cdot x] = V$ whenever $\gamma \in \Gamma$ and $x \in X$ are mutually generic points.*

To support a broad extension of ergodicity results due to Hjorth and Kechris [52, Theorem 12.5], we develop the classes of placid and virtually placid equivalence relations (the latter including all equivalence relations classifiable by countable structures) and prove:

Theorem 1.2.8. (Theorem 3.3.5) *Let E be the orbit equivalence relation on a Polish space X resulting from a turbulent Polish group action. Let F be an analytic, virtually placid equivalence relation on a Polish space Y and let $h\colon X \to Y$ be a Borel homomorphism of E to F. Then there is an F-class whose h-preimage is comeager in X.*

It turns out that there is a version of turbulence for measure which leads to many of the same ergodicity results. It uses the notion of concentration of measure quite close to the abstract whirly actions on measure algebras isolated in [33].

Theorem 1.2.9. (Theorem 3.6.2) *Let Γ be a Polish group continuously acting on a Polish space X with an invariant Borel probability measure and an invariant ultrametric. Suppose that the action has concentration of measure. Then $V[x] \cap V[\gamma \cdot x]$ whenever $x \in X$ is a random point and $\gamma \in \Gamma$ is a generic point mutually generic with x.*

As usual with the forcing reconceptualizations of various notions in descriptive set theory, we have to show that the resulting notions are suitably absolute and evaluate their complexity–Theorem 3.5.6.

Chapter 4 addresses infinite configurations of transitive models of ZFC. The most important case is in which there is an infinite inclusion-descending sequence $\langle M_n \colon n \in \omega\rangle$ of models. Under suitable, commonly satisfied coherence assumptions (Definition 4.2.1 or 4.3.1), the intersection $\bigcap_n M_n$ is a model of ZF or even ZFC. We spend some time developing a flexible technology of constructing nontrivial coherent sequences as in Theorem 4.3.5. The breakthough achieved in this chapter is the theorem showing that similar configurations detect the distinction between equivalence relations induced as orbit equivalences of continuous actions of Polish groups, and those which are not reducible to orbit equivalence relations:

Theorem 1.2.10. (Theorem 4.3.6) *Let $\langle M_n \colon n \in \omega\rangle$ be a choice-coherent sequence of models of ZFC. Let E be an orbit equivalence relation of a continuous Polish group action with a code in the intersection model $\bigcap_n M_n$. If a virtual E-class has a representative in every model M_n, then it has a representative in the intersection model $\bigcap_n M_n$.*

Recall that the canonical equivalence relation not reducible to an orbit equivalence relation is $\mathbb{E}_1$. It is not difficult to construct a choice-coherent sequence of models $\langle M_n \colon n \in \omega\rangle$ and an $\mathbb{E}_1$-class which is represented in each model on the sequence but not in the intersection model $\bigcap_n M_n$–Example 4.3.10.

1.3 Independence: by topic

Our concerns in the part of the book that deals with independence results in ZF+DC can be grouped into several large areas.

Cardinalities of quotient spaces. Mirroring the traditional concerns of descriptive set theory, we work on cardinalities of quotient spaces of Borel equivalence relations. Given Polish spaces X, Y and Borel equivalence relations E, F on each, descriptive set theorists study the question when there can be a Borel function $h\colon X \to Y$ which reduces E to F. This line of work has been very successful in the last two decades.

In the ZF+DC context, the E- and F-quotient spaces can have distinct cardinalities, and the existence of a Borel reduction implies the inequality $|E| \leq |F|$, where we abuse the notation to write $|E|$ for the cardinality of the E-quotient space. On the other hand, the nonexistence of a Borel reduction is often connected with the possibility that ZF+DC cannot prove the cardinal inequality $|E| \leq |F|$. A number of our results deal with the question whether a given Σ^2_1 statement is consistent with ZF+DC plus a statement of the type $|E| \not\leq |F|$ for some benchmark Borel equivalence relations E, F identified in Section 1.7.

The most commonly considered quotient space cardinal inequality is $|\mathbb{E}_0| > |2^\omega|$ and we spend a great deal of energy proving that various Σ^2_1 statements are consistent with it.

Theorem 1.3.1. *The following statements are separately consistent with ZF+DC and $|\mathbb{E}_0| > |2^\omega|$:*

1. ([21], Corollary 9.2.5) *There is a nonprincipal ultrafilter on ω;*
2. (Corollary 9.2.21)*there is a discontinuous homomorphism of $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$;*
3. (Corollary 9.2.12) *given a Borel equivalence relation E on a Polish space, the E-quotient space can be linearly ordered;*
4. *given a pinned Borel equivalence relation E, $|E| \leq |\mathbb{E}_0|$;*
5. (Corollary 11.4.12) *given a countable Borel equivalence relation E, E is the orbit equivalence relation of a (discontinuous) action of $\mathbb{Z}$.*

It is an interesting question whether some of the statements above can be combined; in general, we have

Question 1.3.2. Are there Σ^2_1-statements ϕ_0, ϕ_1 such that each of them is consistent with ZF+DC and $|\mathbb{E}_0| > |2^\omega|$, while their conjunction implies in ZF+DC that $|\mathbb{E}_0| \leq |2^\omega|$ holds?

We give a similar treatment to certain other well-known quotient cardinal inequalities. For example,

Theorem 1.3.3. *The following statements are separately consistent with ZF+DC and $|\mathbb{E}_1| \not\leq |F|$ for any orbit equivalence relation F of a continuous Polish group action:*

1. (Corollary 9.4.8) *There is a nonprincipal ultrafilter on ω;*
2. (Corollary 9.4.11) *there is a discontinuous homomorphism of $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$;*

3. (Corollary 9.4.22) *given a Borel graph Γ on a Polish space, Γ contains a maximal acyclic subgraph;*

4. (Corollary 9.4.10) *given a Borel equivalence relation E on a Polish space, the E-quotient space can be linearly ordered;*

5. (Corollary 9.4.15) *given a pinned orbit equivalence relation E of a continuous Polish group action, E has a transversal.*

Theorem 1.3.4. *The following statements are separately consistent with ZF+DC and $|E| \not\leq |F|$ for any orbit equivalence relation E of a turbulent Polish group action and an analytic equivalence relation F classifiable by countable structures:*

1. (Corollary 9.3.6) *Given a Polish vector space X over a countable field, X has a basis;*

2. (Corollary 9.3.7) *given a Borel graph G on a Polish space, G contains a maximal acyclic subgraph;*

3. (Corollary 9.3.10) *given a pinned equivalence relation G classifiable by countable structures, G has a transversal.*

There are many interesting open questions left in the area of quotient space cardinalities. For one, we do not know whether the various independence results we obtain can be in general combined in the spirit of Question 1.3.2. There are also more specific questions. Can the cardinalities of countable Borel equivalence relations can be manipulated in the context of ZF+DC? For example, are there countable Borel equivalence relations E, F such that both inequalities $|E| < |F|$ and $|F| < |E|$ are consistent with ZF+DC? In ZF+DC, does the existence of a nonprincipal ultrafilter on ω introduce new provable inequalities between quotient space cardinalities?

Locally countable structures. One particularly popular subject of descriptive set theory in the last quarter-century has been the study of countable Borel equivalence relations and the various structures on their equivalence classes. Using the methods of this book, one can identify many new fracture lines in this subject; various selection principles can be meaningfully stratified by the ZF+DC provability of implications between them. The following is a sampling of the results we obtain.

Theorem 1.3.5. (Corollary 11.4.17) *Let G be an analytic graph on a Polish space, with uncountable Borel chromatic number. Given a locally finite bipartite Borel graph H satisfying the Marks–Unger condition, it is consistent that ZF+DC holds, H has a perfect matching and the chromatic number of G is uncountable.*

Theorem 1.3.6. *Let G be an analytic hypergraph with all hyperedges finite, on a Polish space X, with uncountable Borel chromatic number. The following are consistent with ZF+DC plus the statement that the chromatic number of the hypergraph G is uncountable:*

1. (Corollary 11.4.12) *given a countable Borel equivalence relation E on a Polish space with all classes infinite, E is the orbit equivalence relation of a (discontinuous) action of $\mathbb{Z}$;*

2. (Corollary 11.4.10) *given a locally finite acyclic Borel graph Γ on a Polish space, the chromatic number of Γ is ≤ 3;*

3. (Corollary 11.6.7) *given a pinned Borel equivalence relation E on a Polish space, $|E| \leq |\mathbb{E}_0|$.*

An important fracture line among Borel locally countable graphs appears at the level of the diagonal Hamming graph $\mathbb{H}_{<\omega}$ which is the product of cliques on $2, 3, 4 \dots$:

Theorem 1.3.7. *The following are consistent with ZF+DC plus the statement that the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable:*

1. (Corollary 11.7.8) *there is a nonprincipal ultrafilter on ω;*

2. (Corollary 11.7.10) *given a locally finite bipartite Borel graph Γ satisfying the Hall's marriage condition, Γ has a perfect matching;*

3. (Corollary 11.7.9) *given a Borel equivalence relation E on a Polish space X, the E-quotient space is linearly ordered;*

4. (Corollary 11.6.3) *given a pinned Borel equivalence relation E on a Polish space, $|E| \leq |2^\omega|$.*

Note that in the case of the last consistency result for the case $E = \mathbb{E}_0$, the countable uniformization has to fail, as the uncountable chromatic number of $\mathbb{H}_{<\omega}$ implies among other things that $\mathbb{E}_0$ has no transversal.
Another fracture line occurs at the level of the Hamming graph $\mathbb{H}_\omega$ on ω^ω which is the product of infinitely many cliques on ω:

Theorem 1.3.8. (Corollary 11.5.10) *It is consistent with ZF+DC that the chromatic number of the diagonal Hamming graph $\mathbb{H}_{<\omega}$ is countable while the chromatic number of the Hamming graph H_ω on the Baire space is uncountable.*

Model theory. Given a pre-geometry on a Polish structure (for example, the pre-geometry of linear span on a Polish vector space), the axiom of choice yields a maximal independent set (a basis for the vector space). This broad stroke erases many fine combinatorial distinctions between the various pre-geometries. In the choiceless context, interesting distinctions become visible.

In our context, it is more convenient to work with Borel matroids, a combinatorial concept dual to pre-geometries. Matroids from a broad class give rise to balanced forcings (Theorem 6.5.2), providing theorems of the following kind:

Theorem 1.3.9. (Corollary 12.2.13) *It is consistent with ZF+DC that there is a transcendence basis for the reals over $\mathbb{Q}$ and there is no diffuse probability measure on ω.*

One traditional fracture line between matroids leads between the modular ones (like the matroid of linearly independent sets) and the non-modular ones (like the matroid of algeraically independent sets). We develop an abstract notion of modularity in Definition 6.4.1 and, improving on earlier work of [79, 12], we prove:

Theorem 1.3.10. (Corollary 12.3.8) *Let X be a Borel vector space over a countable field Φ. Let Y be an uncountable Polish field with a countable subfield Ψ. It is consistent with ZF+DC that there is a Hamel basis for X over Φ but no transcendence basis for Y over Ψ.*

Matroid theorists also distinguish between linear matroids (like the matroid of linearly independent sets) and graphoids (the matroids of finite acyclic subsets of a given graph). This distinction gives rise to the following theorem:

Theorem 1.3.11. *Let G be a Borel graph on a Polish space X. It is consistent with ZF+DC that G contains a maximal acyclic subgraph and there is no Hamel basis for the reals.*

The proof takes an unexpected turn: we prove that in ZF+DC, existence of a Hamel basis implies that $\mathbb{E}_1$ has a complete countable section, in particular $|\mathbb{E}_1| \leq |\mathbb{F}_2|$–Corollary 9.4.30. At the same time, existence of a maximal acyclic subgraph of any Borel graph G is consistent with ZF+DC+ $|\mathbb{E}_1| \not\leq |E|$ for any orbit equivalence relation E–Corollary 9.4.22.

In another direction connecting model theory with choiceless ZF+DC arguments, we study theories which can have models on the various quotient spaces. In a typical development, given a Borel equivalence relation E on a Polish space X and a Fraissé class $\mathcal{F}$ of relational structures with strong amalgamation, we can add a $\mathcal{F}$-structure (a structure all of whose finite induced substructures are in $\mathcal{F}$) on the E-quotient space in a highly controlled way. This results in many theorems of the following kind:

Theorem 1.3.12. (Corollary 13.3.2) *Let E be a Borel equivalence relation on a Polish space X. It is consistent with ZF+DC that there is a tournament on X/E while there is no linear ordering on $2^\omega/\mathbb{E}_0$.*

Combinatorics. A challenging part of set theory deals with chromatic numbers of various natural Borel graphs and hypergraphs. Non-locally countable graphs require an entirely different approach. For a sample result, for each number $n > 0$ consider the graph G_n on $\mathbb{R}^n$ connecting points of rational Euclidean distance. The work of Schmerl [80] shows that in ZF, the existence of a transcendence basis for the reals over the rationals implies that these graphs all have countable chromatic number. We can prove the following:

Theorem 1.3.13. *It is consistent with ZF+DC that the chromatic number of G_1 is countable while the chromatic number of G_2 is uncountable.*

In fact, it turns out that the existence of a Hamel basis for $\mathbb{R}$ implies in ZF+DC that G_1 has countable chromatic number, while it is consistent with ZF+DC plus the statement that G_2 has uncountable chromatic number (Theorem 12.3.5). The analysis of the graphs G_n becomes more difficult as the number n increases, and the following attractive question is left open for all $n > 1$:

Question 1.3.14. Let $n > 1$ be a number. Is it consistent with ZF+DC that the chromatic number of G_n is countable while the chromatic number of G_{n+1} is uncountable?

Hypergraphs of arity higher than 2 present enormous challenges. To illustrate the possibilities, call a coloring $c\colon \mathbb{R}^2 \to \omega$ an equilateral triangle-free decomposition if there is no equilateral triangle whose vertices are painted the same color. Ceder [17] showed that in ZF, the existence of a Hamel basis implies the existence of an equilateral tringle-free decomposition. The opposite implication does not go through:

Theorem 1.3.15. *It is consistent with ZF+DC that the equilateral triangle-free decomposition exists and yet there is no discontinuous homomorphism between Polish groups.*

Ultrafilters. The methods of the book can separate various types of ultrafilters on ω and other combinatorial objects in the context of ZF+DC.

Theorem 1.3.16. (Corollary 9.2.5, Proposition 9.2.22 and Corollary **??**) *In ZF+DC, there are no provable implications between the existence of a Hamel basis and existence of a nonprincipal ultrafilter on natural numbers.*

Improving [37], we have

Theorem 1.3.17. (Corollary 14.6.7) *In ZF+DC, the existence of a nonprincipal ultrafilter on natural numbers does not imply the existence of one which is disjoint from the summable ideal.*

We isolate several ways of adding an ultrafilter to the symmetric Solovay model in a controlled way, providing ultrafilters with various interesting partition properties: Ramsey ultrafilters and stable ordered union ultrafilters are just two examples. However, the comparison of the various models obtained as well as a classification of ultrafilters appearing in each model seems to be beyond our skill. Many other combinations of implications remain unresolved. For example, we do not know how to construct a model of ZF+DC in which there is a nonprincipal ultrafilter on ω, $|\mathbb{E}_1| \leq |\mathbb{E}_0|$ and yet $|\mathbb{E}_0| \not\leq |2^\omega|$.

Uniformization. Uniformization problems belong to the guiding lights of descriptive set theory. One of the most notorious versions of it is countable to one uniformization, the statement that every subset of the plane with nonempty countable vertical sections contains the graph of a total function. In Chapter 10, greatly improving the methods and results of [56], we develop a satisfactory criterion which guarantees that various strong uniformization principles (including the countable-to-one uniformization as the humblest special case) hold in nearly all models under study. Separating the various uniformization principles seems to be a difficult task beyond the techniques of the present book.

Limitations of the method. The balanced forcing offers a very flexible and powerful method for obtaining independence results in ZF+DC set theory. However, certain desirable types of independence results are outside of its reach. Certain basic limitations are encapsulated in the following theorems.

Theorem 1.3.18. (Corollary 9.1.2) *In every balanced extension of the symmetric Solovay model, there is no ω_1 sequence of distinct Borel sets of bounded Borel rank.*

Theorem 1.3.19. (Theorem 14.3.1) *In every nontrivial balanced extension of the symmetric Solovay model, there is a set of reals without the Baire property.*

Theorem 1.3.20. (Theorem 14.1.1) *In every balanced extension of the symmetric Solovay model, there is no maximal almost disjoint family of subsets of ω.*

Theorem 1.3.21. (Theorem 14.2.1 simplified) *In every balanced extension of the symmetric Solovay model, there is no linearly ordered, unbounded set of elements of ω^ω in the modulo finite domination ordering.*

Thus, the balanced forcing cannot be used to prove for example the consistency with ZF+DC of the statement that $\aleph_1 \leq |\mathbb{E}_1|$ but $\aleph_1 \not\leq |\mathbb{E}_0|$. For consistency results of this type, one has to reach to the weakly balanced forcing of Section 5.3. Another apparently difficult question which cannot be resolved using balanced forcing is the consistency of ZF+DC plus the statement that there is a non-Lebesgue measurable set of reals, every uncountable set of reals contains a perfect subset, and every set of reals has the Baire property [81]. A similar interesting question unresolvable by the balanced forcing technique was asked by Marks and Unger: is it consistent with ZF+DC that every set of reals has the Baire property and every countable Borel equivalence relation is an orbit equivalence relation of a (discontinuous) action of $\mathbb{Z}$?

1.4 Independence: by model

One appealing way to present the work in this book is to consider several more or less canonical generic extensions of the Solovay model and list the statements that we know hold in them. There are also test problems that we do not know how to resolve. The reader needs to keep in mind that all the models are balanced extensions of the Solovay model W and as such obey the general limitations described at the end of the previous section. The list below includes only models that we find of immediate interest for an introduction; there are innumerable other options.

The Ramsey ultrafilter model. Consider the poset of infinite subsets of ω ordered by inclusion and the associated extension of the Solovay model W. The generic filter is identified with a Ramsey ultrafilter, and well-known pre-existing results [26, 56] show that in a suitable sense, every Ramsey ultrafilter is generic over W for the poset $\mathcal{P}(\omega)$ modulo finite. The resulting model was investigated in numerous papers such as [63, 21].

Theorem 1.4.1. *In the model $W[U]$, the following statements hold:*

1. ([21], Corollary 9.2.5) $|\mathbb{E}_0| \not\leq |2^\omega|$;
2. (Corollary 9.4.8) $|\mathbb{E}_1| \not\leq |F|$ *for any orbit equivalence relation* F;
3. *the* $\mathbb{E}_0$- *and* $\mathbb{E}_1$*-quotient spaces are linearly orderable* (Theorem **??**)*, while the* $\mathbb{E}_2$- *and* $\mathbb{F}_2$*-quotient spaces* (Corollary 14.6.7 and Corollary 14.5.4) *do not carry even any tournament;*
4. (Example 12.1.9) *if E is a Borel equivalence relation and A is a subset of the E-quotient space then either $|A| \leq \aleph_0$ or $|2^\omega| \leq |A|$;*
5. (Corollary 14.6.7) *every nonprincipal ultrafilter on ω has nonempty intersection with the summable ideal;*
6. (Example 10.5.9) *countable-to-one uniformization.*

One can consider many variations, adding an ultrafilter which is not Ramsey. We study the cases of a stable ordered union ultrafilter and certain other ultrafilters. It is challenging to discern between the resulting models by sentences which do not mention ultrafilters.

There are many open questions about the Ramsey ultrafilter model. We do not know how to classify the ultrafilters on ω in it. We do not know if there can be Borel equivalence relations E, F such that $|E| \not\leq |F|$ holds in W and $|E| \leq |F|$ holds in $W[U]$. In particular, we do not know if this can occur for countable Borel equivalence relations E, F or in the situation where E is the orbit equivalence relation of a turbulent action of a Polish group and an equivalence F classifiable by countable structures.

The Hamel basis model. Consider the poset P of countable sets of reals which are linearly independent over the rationals. The generic filter yields a Hamel basis $B \subset \mathbb{R}$. It does not seem to be easy to provide a simple criterion which would guarantee that a given Hamel basis is P-generic over the Solovay model. Still, we understand the theory of the model $W[B]$ fairly well:

Theorem 1.4.2. *In the model $W[B]$, the following statements hold:*

1. (Proposition 9.2.22) $\mathbb{E}_0$ *has a transversal;*
2. (Corollary 9.4.30) $\mathbb{E}_1$ *has a complete countable section;*

3. (Corollary 12.3.8) *whenever X is an uncountable Polish field and Φ is a countable subfield, there is no transcendence basis for X over Φ;*
4. (Corollary 12.2.10) *there is no diffuse probability measure on ω;*
5. (Example 10.5.2) *countable-to-one uniformization.*

One can consider a different Polish vector space in place of $\mathbb{R}$ and add a basis to it; a natural example is $\mathcal{P}(\omega)$ viewed with the symmetric difference operation as a vector space over the binary field. The conclusions of the above theorem remain in force. We do not know how to discern between the resulting models in general.

If suitable large cardinals exist, one can find a Hamel basis generic over the model $L(\mathbb{R})$, independently of the size or structure of the continuum–Example 12.4.4. The model $L(\mathbb{R})[B]$ inherits all features quoted in Theorem 1.4.2.

Models with a linear ordering. Let E be a Borel equivalence relation on a Polish space X. Consider the poset P of linear orderings of countable subsets of the E-quotient space, ordered by reverse extension. The generic filter yields a linear ordering $\leq$ on the E-quotient space. The resulting model is a very humble extension of the symmetric Solovay model.

Theorem 1.4.3. *In the model $W[\leq]$, the following statements hold:*

1. (Corollary 9.2.12) $|\mathbb{E}_0| > |2^\omega|$;
2. (Corollary 9.4.10) $|\mathbb{E}_1| \not\leq |F|$ *for any orbit equivalence relation F;*
3. (Corollary 13.3.15) *there are no discontinuous homomorphism between Polish groups;*
4. (Corollary 12.2.15) *OCA holds;*
5. (Example 10.5.8) *countable-to-one uniformization.*

Other features may depend on the nature of the equivalence relation E. For example, if E is pinned, then in the model $W[\leq]$ one cannot find a linear ordering or even a tournament on the $\mathbb{F}_2$-quotient space–Corollary 14.5.5. If E is classifiable by countable structures, then in the model $W[\leq]$ cannot find a linear ordering on quotient spaces of turbulent group action orbit equivalence relations–Corollary 12.3.9. Many questions remain open. In particular, it is not clear whether in the model $W[\leq]$ there is a cardinal inequality between quotient spaces which does not appear already in W. If suitable large cardinals exist and E is pinned, one can find a linear ordering of the E-quotient space generic over the model $L(\mathbb{R})$, independently of the size or structure of the continuum–Example 12.4.6.

One can add other structures to quotient spaces. In particular, one can add a tournament instead of a linear ordering. In the resulting model, the $\mathbb{E}_0$-quotient space cannot be linearly ordered–Corollary 13.3.2.

The transversal models. Let E be a pinned Borel equivalence relation on a Polish space X. Consider the poset P of countable subsets of X consisting of pairwise E-unrelated elements, ordered by reverse inclusion. The generic filter yields an E-transversal $T \subset X$. The study of the resulting model is fairly straightforward with our methods:

Theorem 1.4.4. *In the model $W[T]$, the following statements hold:*

1. (Theorem 12.2.5) *the Open Coloring Axiom;*
2. (Corollary **??**) *there is no nonatomic probability measure on ω;*
3. (Corollary **??**) *there is no transcendence basis for $\mathbb{R}$;*
4. (Corollary 13.3.10) *there is no discontinuous homomorphism between Polish groups;*
5. (Example 10.5.7) *countable-to-one uniformization.*

Other features depend on the equivalence relation E in a rather predictable way. Thus, if E is classifiable by countable structures, then in $W[T]$, $|F| \not\leq |G|$ holds whenever F is an orbit equivalence relation of a turbulent group action and G is an equivalence relation classifiable by countable structures. If E is reducible to an orbit equivalence relation, then in $W[T]$, $|\mathbb{E}_1| \not\leq |G|$ holds whenever G is an orbit equivalence relation of a turbulent group action. If suitable large cardinals exist, one can find a transversal generic over the model $L(\mathbb{R})$, independently of the size or structure of the continuum. The model $L(\mathbb{R})[T]$ inherits all features quoted in Theorem 1.4.4.

1.5 Independence: by preservation theorem

For a practitioner, it is probably most useful to categorize the results we obtain by the technical preservation theorem for balanced forcing that leads to them. The spectrum of preservation theorems for balanced forcing rivals the proper forcing technology. All of the classes considered below (with the exception of nested balanced and weakly balanced forcings) are closed under countable, full support product.

In the broad class of balanced forcings, a number of preservation theorems are possible. The balanced extensions of symmetric Solovay model do not add any well-ordered sequences of elements of the Solovay model–Theorem 9.1.1. They do not contain any inequalities of the form $|E^+| \leq |E|$ where E is a Borel equivalence relation and E^+ is its Friedman–Stanley jump–Corollary 9.1.5. Certain more sophisticated objects are missing from these extensions, such as maximal almost disjoint families–Theorem 14.1.1. To overcome these limitations, one has to reach for the weakly balanced forcings. They do not add any sets of ordinals. A prominent example of a weakly balanced forcing which is not balanced is one which adds a maximal almost disjoint family with a certain type of approximations.

Compactly balanced forcings are those for which the space of balanced classes is naturally organized into a compact Hausdorff space–Definition 9.2.1. The most prominent example is the poset $\mathcal{P}(\omega)$ modulo finite–Example9.2.4. The extensions of the symmetric Solovay model by compactly balanced forcings satisfy $|\mathbb{E}_0| > |2^\omega|$–Theorem 9.2.2. It is even true that the chromatic number of a certain Hamming-type graph remains uncountable–Theorem 11.7.2.

Placid forcings are those in which placid virtual conditions exist: these are virtual conditions $\bar{p}$ such that for any pair of separately generic extensions $V[H_0], V[H_1]$ such that $V[H_0] \cap V[H_1] = V$ and for any conditions $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ such that p_0, p_1 are both stronger than $\bar{p}$, the conditions p_0, p_1 are compatible–Definition 9.3.1. Thus, the mutual genericity required in the definition of balanced forcing is replaced by the weaker demand $V[H_0] \cap V[H_1] = V$. The most prominent example is the poset adding a Hamel basis to the reals with countable approximations–Example 9.3.5. The extensions of the symmetric Solovay model by placid forcings do not contain any inequalities $|E| \leq |F|$ where E is an orbit equivalence relation of a turbulent Polish group action, and F is an equivalence relation classifiable by countable structures–Theorem 9.3.3. Another attractive property of placid forcings is that they do not add transcendence bases for Polish fields–Theorem 12.3.1. There are many other effects (such as the nonexistence of nonprincipal ultrafilters on ω) which are in fact consequences of preservation theorems for the broader class of Bernstein balanced forcings.

Nested balanced forcings are those which contain a sequence of balanced conditions for a suitable infinite nested sequence of generic extensions, as in Definition 9.4.1. All compactly balanced forcings are nested balanced–Example 9.4.7. A prominent example of a nested balanced but not compactly balanced forcing is the poset adding an $\mathbb{E}_0$-transversal by countable approximations–Example 9.4.14. The extensions of the symmetric Solovay model by nested balanced forcings do not contain any inequalities $|\mathbb{E}_1| \leq |E|$ where E is an orbit equivalence relation–Theorem 9.4.4.

Tethered forcings are those for which, to establish the balance of a pair $\langle Q, \tau \rangle$, it is enough to show that the pair decides virtual conditions of certain complexity–Definition 10.1.1. Most balanced forcings are tethered; a prominent example of an untethered poset is the collapse poset adding an injection of the $\mathbb{E}_0$-quotient space to 2^ω. In tethered extensions of the symmetric Solovay model, the countable-to-one uniformization (and many other uniformization principles) holds–Theorem 10.3.2. In the extension by the untethered collapse poset, there is no $\mathbb{E}_0$-transversal which means that the countable-to-one uniformization must fail.

The perfectly balanced forcings are the balanced Suslin forcings for which, whenever $\{V[H_x] \colon x \in 2^\omega\}$ is a perfect collection of mutually generic extensions of V and in each model $V[H_x]$ there is a condition $p_x \in P$ stronger than a given balanced condition $\bar{p}$, there is a condition in P stronger than uncountably many conditions p_x–Definition 12.1.1. Prominent examples include posets adding ultrafilters, like the poset of infinite subsets of ω ordered by inclusion. The perfectly balanced extensions of the symmetric Solovay model satisfy such features as the strong Silver dichotomy (Theorem 12.1.6) and a strong version of OCA (Theorem 12.1.8).

The Bernstein balanced forcings are those posets for which, whenever $\{V[H_x]\colon x \in 2^\omega\}$ is a perfect collection of mutually generic extensions of V and in each model $V[H_x]$ there is a condition $p_x \in P$ stronger than a given balanced condition $\bar{p}$, no condition in P below $\bar{p}$ is incompatible with more than countably many conditions p_x for $x \in 2^\omega$–Definition 12.2.1. While this definition may look obscure at first reading, many posets satisfy it and it has many consequences which are difficult to obtain otherwise. Bernstein balanced forcings add no finitely additive diffuse probability measures on ω, in particular non nonprincipal ultrafilters–Theorem 12.2.3. Among other effects, their generic extensions satisfy the Open Coloring Axiom–Theorem 12.2.5. For many of them, a filter generic over the model $L(\mathbb{R})$ exists in the theory ZFC plus a suitable large cardinal, independently of the status of the Continuum Hypothesis or similar issues–Theorem 12.4.2. The Bernstein balanced forcings should be viewed as dual to the perfectly balanced forcings, as generic filter for a perfectly balanced forcing and a generic filter for a Bernstein balanced forcing tend to be automatically mutually generic–Theorem 12.4.12. Prominent examples of Bernstein balanced forcings include all placid forcings and also the forcings adding transcendence bases to Polish fields.

$3,2$-balanced forcings form a class useful for ruling out discontinuous homomorphisms between Polish groups. A poset P is $3,2$-balanced if there are $3,2$-balanced virtual conditions $\bar{p}$ in it; these are virtual conditions such that for any triple $V[H_0], V[H_1], V[H_2]$ of pairwise mutually generic extensions of V and conditions $p_0 \in V[H_0], p_1 \in V[H_1]$, and $p_2 \in V[H_2]$ below $\bar{p}$ there is a common lower bound of $\{p_0, p_1, p_2\}$–Definition 13.1.1. A prominent example is the forcing adding a linear ordering of a given quotient space by countable approximations. $3,2$-balanced extensions of the symmetric Solovay model do not contain any discontinuous homomorphisms between Polish groups; in particular, they contain no nonprincipal ultrafilters on ω–Theorem 13.2.1. Among other effects, they do not contain non-Borel automorphisms between many quotient groups–Theorem 13.2.3. A much stronger form of this property is $3,2$-centeredness, in which any collection of three pairwise compatible conditions has a lower bound. Such posets for example do not add linear orderings of quotient spaces except for trivial reasons–Theorem 13.2.7.

There are a number of preservation properties concerned with locally countable structures. They all consider posets of the form $P_{\mathcal{KL}}$ (Definition 11.4.2) and start with a hypothesis on the simplicial complex $\mathcal{K}$. Thus, Theorem 11.4.5 shows that if $\mathcal{K}$ is Suslin σ-centered then uncountable Borel chromatic numbers of finitary hypergraphs are preserved by the poset $P_{\mathcal{KL}}$ and Theorem 11.4.13 proves the same for Suslin σ-linked complexes. There are a number of options, and many classes of posets concerned are closed under the countable support product. It seems difficult to get preservation theorems for locally countable structures without the technology of iterated very Suslin forcings of Chapter 11.

1.6 Navigation

The structure of the book is quite complicated and some navigation advice will greatly enhance the reader's experience. Part I contains results about reducibility of Borel and analytic equivalence relations and how amalgamation properties of models of set theory can be used to disprove it. Part II provides all of the choiceless independence results. It is difficult to appreciate the methods of Part II without having basic acquiantance with the concepts introduced in Part I. Still, Part I should be viewed as only the gate to Cantor's paradise, while Part II is already within the gates.

Chapter 2 defines and explores the notion of the virtual quotient space for an analytic equivalence relation E on a Polish space X. On an intuitive level, a virtual equivalence class is one which may only exist in some forcing extension but still we already have a sensible calculus for speaking about it.

Section 2.1 provides the basic definitions. If P is a partial order and τ is a P-name for an element of X, the name is called E-pinned if its E-class does not depend on the generic filter on P. There is a natural equivalence $\bar{E}$ on pinned names, and the $\bar{E}$ equivalence classes are referred to as the virtual E-classes. Section 2.2 shows that if one has a definable structure on the E-quotient space, it is possible to consider the virtual version of that structure on the virtual quotient space. This is a structure Π_1-elementarily equivalent to the original one. Structures such as virtual versions of partially ordered sets of Borel simplicial complexes will come handy later in the book.

Sections 2.3 and 2.4 deal with the most immediate concern: the classification problem for the virtual quotient spaces. The virtual equivalence classes should correspond to some immediately recognizable combinatorial objects, and in many important cases this hope is fulfilled. In a broad class of equivalence relations (the pinned equivalence relations of [48, Section 17]) the virtual quotient space is simply identical to the quotient space. Many jump-type operations on equivalence relations have a natural translation to operations on virtual quotient spaces; for example, the Friedman–Stanley jump [31] is translated to a powerset operation–Theorem 2.3.4. Virtual classes of isomorphism relations on countable structures naturally correspond to uncountable structures of the same type (Theorem 2.4.5), even though in some cases there may be mysterious virtual classes which are not classified in this way [50]. Still, there are many wide open questions. For example, we do not know how to classify the virtual quotient space for the measure equivalence. The virtual space of homomorphism of second countable compact Hausdorff spaces sems to be naturally classified by compact Hausdorff spaces, but there is no theorem to that effect.

Section 2.5 deals with the next most natural concern: the attempt to size up the virtual quotient space in terms of cardinality. The effort is surprisingly successful, generating cardinal invariants of analytic equivalence relations which can take all kinds of exotic and informative values. Given an analytic equivalence relation E, one defines the *pinned cardinal* $\kappa(E)$ as the minimal cardinal such that all virtual classes of E are represented by names on posets of size $< \kappa$. One also defines $\lambda(E)$ as the cardinality of the virtual quotient space, and $\lambda(E, P)$ as the cardinality of the set of virtual classes which are represented by names on the poset P. These cardinals respect the Borel reducibility order; therefore, they serve as viable tools for the Borel nonreducibility results. In addition, they interact well with a number of operations on equivalence relations. In a jarring development, these cardinals can connect Borel nonreducibility results to hard questions about singular cardinal arithmetic or similar concerns of transfinite set theory.

Section 2.6 limits the type of partial orders that can carry a nontrivial E-pinned name. Theorem 2.6.2 shows that, in particular, proper forcings do not carry any nontrivial E-pinned names. Theorem 2.6.3 shows that as soon as there are some nontrivial E-pinned names (i.e. E is not pinned), then one can find them in any partial order collapsing $\aleph_1$. The most curious result is Theorem 2.6.4: if E is Borel reducible to an orbit equivalence relation, then every nontrivial virtual E-class really comes from a collapse of some uncountable cardinality to $\aleph_0$. Compare this with the efforts of Section 2.4 where some virtual classes of equivalence relations classifiable by countable structures were classified by uncountable structures to be collapsed down to the countable size. This feature does not persist to arbitrary analytic equialence relations: Example 2.6.8 produces a nontrivial E-pinned name on the Namba forcing for a suitable equivalence relation E.

Section 2.7 proves two absoluteness results. First, in Corollary 2.7.3 we show that for a Borel equivalence relation E, the statement "E is pinned" is absolute among all forcing extensions. This greatly simplifies the language used to formulate later results. A similar statement for analytic equivalence relations requires large cardinal assumptions. Second, in Theorem 2.7.4, we show that at least for equivalence relations E reducible to orbit equivalence relations, transitive inner models of ZFC calculate the pinned cardinal $\kappa(E)$ predictably: their calculation cannot return a value larger than the actual value of $\kappa(E)$ in V.

It is not clear if the class of unpinned Borel equivalence relations allows a neat basis in ZFC. The main result of Section 2.8 shows that the answer is affirmative at least in the symmetric Solovay model. There, the basis for unpinned Borel equivalence relations is just $\{\mathbb{F}_2\}$ (a former conjecture of Kechris, which nevertheless fails badly in ZFC); for unpinned analytic equivalence relations we isolate the basis $\{\mathbb{F}_2, \mathbb{E}_{\omega_1}\}$–Corollary 2.8.13 and Theorem 2.8.11. We also provide a basis for the class of analytic equivalence relations E such that $\kappa(E) = \infty$. The basis is $\{\mathbb{E}_{\omega_1}\}$–Theorem 2.8.9.

Chapter 3 restates and greatly generalizes Hjorth's notion of turbulence in forcing terms. This development shows that nonturbulent equivalence relations are in fact parallel to pinned equivalence relations in a very precise sense. The forcing relation encapsulates many distracting estimates needed in the traditional treatment of turbulence, resulting in a clean and efficient general calculus.

Section 3.1 provides the motivating insight–namely, that building pairs of models $V[y_0], V[y_1]$ where $y_0 \in Y_0$ and $y_1 \in Y_1$ are separately Cohen generic elements of their respective Polish spaces such that $V[y_0] \cap V[y_1] = V$ depends on a suitable notion of a walk–Theorem 3.1.4. The comparison of generic extensions vis-a-vis their intersection is an important concern in the later parts of this book. Section 3.2 provides two basic examples. In the first, Hjorth's notion of turbulence of group actions is restated in geometric terms–Theorem 3.2.2. The second example provides a complete characterization of those analytic ideals on ω such that it is possible to find points $x_0, x_1 \in 2^\omega$ separately generic over V such that they are equal modulo I and $V[x_0] \cap V[x_1] = V$–Theorem 3.2.4.

Section 3.3 defines two classes of equivalence relations that are simple from the turbulent point of view. An equivalence relation E on a Polish space X is placid if any E-class represented in generic extensions with trivial intersection is represented in the ground model. E is virtually placid if any virtual E-class represented in such generic extensions is represented as a virtual class in the ground model. These definitions are preserved under Borel reducibility and under many natural operations on equivalence relations, as shown in Section 3.4. The main application is the generalization of an ergodicity theorem by Hjorth and Kechris [52, Theorem 12.5] (Theorem 3.3.5): any Borel homomorphism from a turbulent equivalence relation to a virtually placid one stabilizes on a comeager set.

Section 3.5 provides a necessary complement to the forcing development in this chapter: a theorem asserting that the notions introduced are absolute among all generic extensions and evaluating their descriptive complexity–Theorem 3.5.6. Finally, Section 3.6 develops turbulence of group actions for measure. It turns out that the appropriate concept uses the familiar notion of concentration of measure of [73].

Chapter 4 presents one of the entirely novel concepts of geometric set theory: that of a coherent nested sequence of models of ZFC, with emphasis on the intersection model. Thus, let $\langle M_n : n \in \omega \rangle$ be an inclusion-decreasing sequence of transitive models of ZFC and consider the transitive class $M_\omega = \bigcap_n M_n$. A simple coherence condition on the sequence guarantees that M_ω is a model of ZF (Theorem 4.2.9); a coherence condition with a suitable well-ordering parameter guarantees that V_ω is a model of ZFC–Theorem 4.3.2. In Section 4.2 we develop the technology of building coherent sequences of generic extensions (Theorem 4.2.8) and produce a case in which choice fails in the intersection model. In Section 4.3 we develop a machinery of building choice-coherent sequences of generic extensions–Theorem 4.3.5. In principle, there is a whole field of coherent set theory hiding behind these concepts and the intersection models can serve as new vehicles for obtaining independence results. However, in this chapter our emphasis is on the connection with Borel equivalence relations. Our main result shows that the choice-coherent sequences of models detect the difference between $\mathbb{E}_1$ and orbit equivalence relations: if E is an orbit equivalence relation and C is a virtual E-class represented in each model V_n, then it is represented in V_ω–Theorem 4.3.6. This typically fails for $\mathbb{E}_1$. The topic is revisited in Section 9.4 below.

Chapter 5 develops the basic general theory of balanced Suslin forcing. The treatment of Suslin forcing is quite different from that in [7] or similar treatments on the structure of the real line. Our partial orders are typically (but not necessarily) σ-closed, and serve to add combinatorial objects by explicit countable approximations in the context of the symmetric Solovay model. It is important to understand that the study of these partial orders takes place exclusively within set theory with choice.

Section 5.1 defines the notion of a virtual condition in a Suslin forcing P. A virtual condition is an element of the completion of the poset P which only exists in some generic extension, but for which one can develop a sensible calculus of comparison already in the ground model, quite in parallel to virtual equivalence classes developed in Chapter 2. There are natural equivalence and ordering relations on virtual conditions in P.

Section 5.2 defines the key notion of a balanced pair in a Suslin forcing P. This is a pair $\langle Q, \tau\rangle$ where Q is a poset and τ is a Q-name for an element of the forcing P which satisfies a natural amalgamation property in mutually generic extensions. There is a natural notion of equivalence on balanced pairs in P. The motherload feature exploited throughout the rest of the book is the simple Theorem 5.2.5: every balanced class contains exactly one virtual condition. The task of understanding a given Suslin forcing P in the context of geometric set theory is then nearly identical to the classification of balanced virtual conditions in a given Suslin forcing P. Balanced virtual conditions in balanced Suslin forcing are quite parallel to master conditions in proper forcing.

Section 5.3 defines weakly balanced Suslin forcing. This is a weakening of the notion of balanced Suslin forcing designed to produce some effects (such as MAD families) which are absent from balanced extensions of the symmetric Solovay model. The whole development is quite parallel to balanced Suslin forcing with suitable quantification adjustments. Some examples of weakly balanced forcings can be found in Section 8.9. This book's primary focus is on balanced forcing, and the potential of the weakly balanced variations is left wide open.

Chapter 6 analyzes a class of balanced forcings arising from simplicial complexes of various algebraic forms. Namely, let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X. Let $P_{\mathcal{K}}$ be the poset of countable subsets $p \subset X$ such that $[p]^{<\aleph_0} \subset \mathcal{K}$, ordered by reverse inclusion. Many useful posets in this book arise in this fashion, and the balance properties of the posets depend on model theoretic properties of $\mathcal{K}$. By coincidence or not, most natural examples are matroids. Section **??** introduces the notion of a modular simplicial complex (Definition 6.4.1), parallel to a modular matroid, with a number of examples and related notions. Another result (Theorem 6.5.2) shows that the poset adding a basis for a matroid is balanced as soon as the matroid satisfies a natural and common complexity requirement. Section 6.6 investigates simplicial complexes on quotient spaces, and shows that balanced conditions are naturally found in the related virtual quotient spaces. Particularly interesting cases concern the collapse of one quotient cardinality to another, Definition 6.6.2, and the transversal type posets which to each pair $E \subset F$ of Borel equivalence relations on a Polish space X add a maximal set $A \subset X$ such that $E \restriction A = F \restriction A$–Definition 6.6.5.

Chapter 7 analyzes various ways of adding a nonprincipal ultrafilter on ω to a choiceless model. The basic method, the poset of all infinite subsets of ω ordered by inclusion, is investigated in Section 7.1. This poset adds a Ramsey ultrafilter, and its balanced conditions are classified simply by ultrafilters. The resulting model has been investigated from many directions previously [21, 90]. The next simplest type of ultrafilter is added by a poset with Ramsey sequences of finite structures, discussed in Section 7.3. It turns out that balanced conditions in this type of forcing are classified by ω-sequences of ultrafilters of a certain type. Section 7.2 shows how to add an ultrafilter which is not a P-point in the simplest fashion: using the quotient poset of $\mathcal{P}(\omega \times \omega)$ modulo I where I is the Fubini power of the Fréchet ideal. The interesting development in this case is the complexity of balanced conditions–they are classified by ultrafilters on the set of ultrafilters on ω (Theorem 7.2.2). It is interesting to compare this situation with Section 6.6: there, the balanced conditions were also classified by complex sets, but the complexity seemed to be worked into the definitions of the posets from the beginning. Here, the increased complexity appeared without invitation and as a surprise, at least to the authors. Section 7.4 shows how to add a stable ordered union ultrafilter; the most interesting insight–the balanced conditions are classified by idempotent ultrafilters (Theorem 7.4.7).

Chapter 8 contains the analysis of balanced forcings which are designed to perform a specific task, and from the point of view of the previous chapters may seem somewhat ad hoc. In Section 8.1, we consider the problem of adding a coloring of a Borel graph by a fixed number of colors. It turns out that the task is closely related to the coloring number of graphs introduced by Erdős and Hajnal [25]. If a Borel graph has countable coloring number, then there is a natural balanced poset (Definition 8.1.1) adding a coloring by countably many colors (Theorem 8.1.2). The classical examples of such graphs appear on Euclidean spaces as distance graphs (Example 8.1.7 and 8.1.8) and have been studied among others by the Hungarian combinatorial school [58]. Adding colorings to hypergraphs seems to be much trickier business. We show how to add a countable decomposition of a vector space over a countable field such that no composant contains three linearly dependent points (Theorem 8.1.12), or a countable decomposition in which each composant is fully linearly independent (Theorem 8.1.19).

Section 8.2 considers the problem of adding a discontinuous homomorphism between given Polish groups. This task cannot have a positive resolution in general since many Polish groups have the automatic continuity property, for example the unitary group [95]. However, we provide a balanced forcing in the case of abelian groups, one of which is torsion free, and the other divisible (Theorem 8.2.2). Section 8.3 shows that it is possible to force a nontrivial automorphism of the Boolean algebra $\mathcal{P}(\omega)$ modulo finite with a balanced forcing. The forcing consists simply of countable approximations ordered by reverse inclusion, and balanced conditions correspond exactly to automorphisms of the algebra–Theorem 8.3.3. Section 8.4 considers the problem of adding a cofinal Kurepa family on a Polish space with a balanced poset. Again, the natural countable approximation poset works, and there is a very simple classification theorem for the balanced conditions–Theorem 8.4.3. Section 8.5 presents balanced forcings for adding set mappings on Polish spaces without nontrivial free sets.

Section 8.6 shows how one can add saturated models of first order theories to E-quotient spaces for a Borel equivalence relation E. An especially interesting case is adding a saturated limit of a Fraissé class with strong amalgamation; i.e. a linear ordering, a tournament, or similar structures (Theorem 8.6.4). In this section, the amalgamation problems familiar from model theory and the amalgamation questions arising in geometric set theory exhibit a particularly strong affinity.

Section 8.7 shows that there are balanced orderings which produce extensions of the symmetric Solovay model in which the Axiom of Dependent Choices fails. The seminal example is the finite-countable poset of Definition 8.7.1 which introduces a partition of an uncountable Polish space into two parts, one of them without a perfect subset and the other without countably infinite set.

Section 8.8 introduces several balanced posets which in the usual ZFC context would be viewed as "side condition" forcings. The general Definition 8.8.1 provides a general construction for adding an uncountable set whose intersection with every set in I is countable, where I is a given ideal generated by closed sets. Theorem 8.8.6 shows how to add a special type of a Lusin set. The forcings mostly serve as delimiting examples in Chapter **??**.

Section 8.9 provides two examples of weakly balanced forcings, achieving what balanced forcing cannot do. In the first case (Theorem 8.9.2) we construct a weakly balanced poset which collapses $|E|$ to $|\mathbb{E}_0$, for any given Borel equivalence relation E. This breaks the Friedman–Stanley jump barrier which is one of the basic features of the balanced extensions of the Solovay model by Corollary 9.1.5. The second example (Theorem 8.9.7) is a weakly balanced poset adding a maximal almost disjoint family, an object one cannot find in balanced extensions of the Solovay model by Theorem 14.1.1. In general, the weakly balanced arguments are much more difficult than the balanced ones.

Chapter 9 compares cardinalities of quotient spaces in the balanced extensions of the symmetric Solovay model W. On the surface, this is an enterprise very similar to the traditional forcing concerns in the context of the axiom of choice. However, it is important to understand that the non-well-ordered quotient cardinals offer many more opportunities for meaningful independence work and also for surprising ZF results preventing various patterns of cardinal collapses.

Section 9.1 shows that no new inequalities between well-ordered cardinalities and quotient cardinalities are added by balanced forcing. In fact, no new well-ordered sequences of elements of W are added–Theorem 9.1.1. This result is central for the rest of the book. Section 9.2 provides a very flexible tool for verification of the inequality $|\mathbb{E}_0| > |2^\omega|$ in certain classes of balanced extensions. Namely, it turns out that if the balanced classes of the forcing are naturally organized into a compact Hausdorff space, then the inequality holds in the P-extension of the symmetric Solovay model–Theorem 9.2.2. The compact Hausdorff space in question may certainly be nonmetrizable; in the case of the poset P of infinite subsets of ω ordered by inclusion, it is simply the Čech–Stone remainder of ω. Among the applications, it is possible to add linear ordering to any Borel quotient space and preserve $|\mathbb{E}_0| > |2^\omega|$–Corollary 9.2.12, or to add a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$ and preserve $|\mathbb{E}_0| > |2^\omega|$ as in Corollary 9.2.21. The inequality $|\mathbb{E}_0| > |2^\omega|$ can be verified in other ways when the balanced classes do not form a compact Hausdorff space, for example using Theorem 11.4.13 to maintain the chromatic number of the $\mathbb{G}_0$-graph uncountable which automatically implies $|\mathbb{E}_0| > |2^\omega|$ by Proposition 11.1.6(2).

Section 9.3 shows that the nonreducibility results obtained by Hjorth's turbulence method turn into cardinal inequalities in certain classes of balanced extensions. The main contribution here is Definition 9.3.1, isolating the relevant class of placid Suslin forcings. In placid extensions of the symmetric Solovay model, no inequalities of the type $|E| \leq |F|$ appear where E is the orbit equivalence relation of a turbulent Polish group action, and F is an equivalence relation classifiable by countable structures. A weakening of placidity is used in Chapter **??** to rule out combinatorial objects from generic extensions which have nothing to do with any group actions, such as nonprincipal ultrafilters.

Section 9.4 provides practical criteria for showing that the classical nonreducibility of $\mathbb{E}_1$ to any orbit equivalence relation translates into cardinal inequalities in certain classes of balanced extensions of the Solovay model. This turns out to require close study of nested sequences $\langle M_n \colon n \in \omega\rangle$ of models of ZFC as in Chapter 4. In particular, we must resolve the question whether, for a given Σ^2_1 sentence ϕ which is a consequence of the axiom of choice, one can produce a set A such that for each $n \in \omega$, the set $A \cap M_n$ belongs to M_n and is a witness for the sentence ϕ there. The resulting tricky combinatorial work can be used to separate for example the existence of a Hamel basis from existence of maximal acyclic subgraphs in Borel graphs.

Section 9.6 shows that the nonreducibility results between analytic equivalence relations obtained by the comparisons of their cardinal invariants κ and λ mostly turn into cardinal inequalities in the Solovay model W, and these inequalities survive unharmed into the balanced extension.

Chapter 10 deals with the question of whether countable-to-one uniformization and similar uniformization principles hold in the balanced forcing extensions. Section 10.1 provides the a practical criterion on a Suslin forcing P which implies that many strong uniformization principles hold in the forcing extension of the Solovay model by P. It turns out that the criterion is satisfied in a great number of interesting cases, and its verification mostly follows from the careful classification of balanced conditions for P. A number of examples appear in Section 10.5. Sections 10.2, 10.3, and 10.4 study various uniformization principles in turn and show that they in fact hold in tethered extensions of the symmetric

Solovay model. Lastly, Section 10.5 provides a great number of examples and non-examples.

Chapter 11 considers the independence results which appear when one attempts to select one type of structure to each E-class, and not select another type; here E is a given countable Borel equivalence relation on a Polish space. The problems considered are motivated by the present practice of descriptive graph theory. The upshot is a discovery of new and very pertinent fracture lines in the area. The main point of the whole section is that the definable c.c.c. posets have great influence on independence results in ZF+DC, even though heretofore they were used only for ZFC independence results.

The fracture lines and their critical objects are defined in Section 11.1. We define two variations of countable Borel chromatic number of hypergraphs on Polish spaces: the Borel σ-finite chromatic number (Definition 11.1.9) and the Borel σ-finite fractional chromatic number (Definition 11.1.18. The relevant classes of hypergraphs contain minimal objects obtained by the same operation of a skew product (Definition 11.1.1). There are several dichotomies in style very similar to the original $\mathbb{G}_0$-dichotomy of [54]. A number of examples illustrate the new concepts; among them, there is one obtained from the density version of the van der Waerden theorem (Example 11.1.20).

Section 11.2 contains information on the basic technical tool needed to prove independence results on locally finite structures: very Suslin c.c.c. forcing notions (Definition 11.2.3) and their iterations. The section is written in such a way that specialists on ZFC independence results can use its results for their own purposes. The basic result shows that the finite support iterations of very Suslin forcing notions are very Suslin again (Theorem 11.2.7). A long-standing concern of Stevo Todorcevic also makes appearance: for a given Borel simplicial complex $\mathcal{K}$, viewed as a poset with the reverse inclusion ordering, determining whether the poset is c.c.c. and calibrating its chain condition carefully. The section provides a novel angle though: the regularity properties of the posets must be exemplified in a simply definable way. Thus, Section **??** discusses posets which can be covered by countably many analytic linked pieces and their iterations, and proves an iteration and product preservation theorem. Sections **??**, **??**, and **??** provide the same treatment for posets which are definably σ-centered, have a definable version of a finitely additive measure, and are σ-liminf-centered, respectively.

Section 11.4 finally produces some independence results in ZF+DC. The common theme: there is a Borel locally countable simplicial complex $\mathcal{K}$ on a Polish space, and we study the consequences of the existence of a maximal $\mathcal{K}$-set of a certain type. Adding colorings for various locally countable hypergraphs, for example, can be viewed from this angle. In all cases, the definable forcing properties of $\mathcal{K}$ as a poset ordered by reverse inclusion play central role. We prove several results resonating with the current descriptive graph theory. Let G be an analytic hypergraph with finite hyperedges, with uncountable Borel chromatic number. Given a countable Borel equivalence relation E, we prove the consistency of ZF+DC plus "E is an orbit equivalence of a $\mathbb{Z}$-action" plus "the chromatic number of G is uncountable"–Corollary 11.4.10. Given a locally finite acyclic Borel graph H, we prove the consistency of ZF+DC plus "the chromatic number of H is not greater than three" plus "the chromatic number of G is uncountable"–Corollary 11.4.12.

Section 11.5 deals with the effects of coloring a larger, perhaps not locally countable, Borel graphs G on locally countable structures. If the graph G does not contain injective homomorphic copies of the bipartite graphs $K^{\rightarrow}_{\omega,\omega}$ and K_{n,ω_1} for some $n \in \omega$, then the canonical forcing for adding a G-coloring with countably many colors does not add an $\mathbb{E}_0$-transversal (Theorem 11.5.6). If the graph G contains no injective homomorphic copy of $K_{n,n}$, then the canonical coloring poset does not add a coloring of the diagonal Hamming graph $\mathbb{H}_{<\omega}$ with countably many colors (Theorem 11.5.8). Great examples of such graphs come from the Euclidean distance graphs considered at length by Erdős, Hajnal, and Komjáth among others. Let D be a Borel set of positive reals, and let G be the graph on $\mathbb{R}^2$ connecting points whose Euclidean distance belongs to the set D. The nature of the set D greatly influences the behavior of the graph. If D is a sequence converging to zero, then the graph G falls into the first category above, and so in ZF+DC the chromatic number of G can be equal to $\aleph_0$ without the existence of an $\mathbb{E}_0$-transversal. If the set D is algebraically independent, then the graph G falls into the second category and so its chromatic number can be equal to $\aleph_0$ without providing even a coloring to $\mathbb{H}_{<\omega}$. On the other hand, if the set D contains a dense subgroup of $\mathbb{Q}$, then the existence of a G-coloring with countably many colors easily implies the existence of an $\mathbb{E}_0$-transversal.

Section 11.6 studies the effects of collapses of various Borel cardinals to $|2^\omega|$ or to $\mathbb{E}_0$ on locally finite structures. Perhaps somewhat unexpectedly, collapsing a Borel cardinal such as $\mathbb{E}_0$ to 2^ω does not yield an $\mathbb{E}_0$-transversal; similarly, for many locally countable hypergraphs G, it does not make G have countable chromatic number. The notion of the fractional chromatic number of a hypergraph (Definition 11.1.18) makes a key apperance in the main result–Theorem 11.6.1. A collapse of a Borel cardinal to $\mathbb{E}_0$ has no discernible effect on locally countable structures, as Theorem 11.6.5 shows.

Section 11.7 revisits the class of compactly balanced forcing. We show that for many locally countable hypergraphs G, compactly balanced forcings do not make G have countable chromatic number. such forcings do not add countable coloring to a number of locally countable hypergraphs. The notion of Borel σ-finite chromatic number (Definition 11.1.9) is critical here; the diagonal Hamming graph $\mathbb{H}_{<\omega}$ is the most prominent example. In particular, it is consistent with ZF+DC that there be a nonprincipal ultrafilter on ω, yet the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.

Chapter 12 presents important two classes of posets which are in a precise sense dual to each other. Both of them offer extensive control over uncountable subsets of Polish spaces, but they approach the task from exactly orthogonal directions.

Section 12.1 isolates the classes of perfectly balanced and perfect forcings. The main inhabitant of these classes is the ordering of infinite subsets of natural numbers ordered by inclusion. However, it turns out that most natural partial orders for adding ultrafilters are perfectly balanced. Extensions of the symmetric Solovay model by perfectly balanced forcings satisfy a strong form of Silver dichotomy: every uncountable subset of a Borel quotient space contains an injective copy of 2^ω (Theorem 12.1.6). Among other effects, they also satisfy a strong form of the Open Coloring Axiom–Theorem 12.1.8.

Section 12.2 presents the dual class of Bernstein balanced forcings. It is very rich, including the placid posets, but also posets for adding transcendence bases for Polish fields, or posets adding Fraissé structures on Borel quotient spaces. The preservation theorems show in particular that their extensions do not contain nonprincipal ultrafilters (Theorem 12.2.3 or satisfy the Open Coloring Axiom (Theorem 12.2.5). The extensions also exhibit properties which are in certain sense dual to known ZFC independence results. For example, every unbounded subset of ω^ω under the modulo finite domination order contains a bounded uncountable subset–Theorem 12.2.7.

Section 12.3 contains preservation results that set the class of placid forcings apart from the much larger class of Bernstein balanced forcings. In particular, placid forcings do not add algebraic transcendence bases to uncountable Polish fields (Theorem 12.3.1). This yields one of the most striking results of this book: in ZF+DC, existence of Hamel bases for Borel vector spaces over countable fields does not imply existence of transcendence bases for Polish fields over countable subfields (Corollary 12.3.8). Placid forcings also do not add colorings to a great number of graphs which have countable coloring number in ZFC. Theorem 12.3.5 deals with an interesting class of graphs of this type, obtained from algebraic curves in $\mathbb{R}^2$.

The duality between perfectly balanced and Bernstein balanced posets comes to full view in Section 12.4. It turns out that the generic filters for Bernstein balanced forcings and perfectly balanced forcings are often automatically mutually generic–Theorem 12.4.12. This is the most precise and elegant way of framing intuitions like "ultrafilters and transcendence bases for fields have nothing to do with each other". At the same time, filters for Bernstein balanced forcings generic over $L(\mathbb{R})$ exist as a matter of ZFC plus large cardinals, without regard to the structure of continuum issues–Theorem 12.4.2. Thus, we get say generic Hamel bases for vector spaces or generic transcendence bases for Polish fields in ZFC plus large cardinals. This contrasts with the perfectly balanced ultrafilter posets, where a long line of ZFC independence results may be interpreted as showing that generic filters for these posets consistently fail to exist.

Chapter ?? presents the classes of m, n-balanced and centered forcings. These classes serve to separate combinatorial objects of different arity of organization. The reader has to keep in mind that the notion of arity used here is quite abstract or even somewhat ineffable.

Section 13.2 contains the main preservation results. The raison d'etre of the classes of forcings under discussion is Theorem 13.2.1 which shows that $n+1, n$-balanced forcings do not add discontinuous homomorphisms between Polish groups. This means that in particular no nonprincipal ultrafilters on ω

are added. A similar effect is produced by Theorem 13.2.3: $n+1, n$-balanced forcings do not add non-Borel homomorphisms between various Borel quotient groups studied in particular by Ilijas Farah [28]. The evaluation of arity comes into view with Theorem 13.2.4: it turns out that m, n-balanced forcings add no colorings of Polish groups which avoid monochromatic solutions to suitable equations, and the numbers m, n reflect the complexity of the equation. We also study the related and much more restrictive class of m, n-centered forcings. A particular success is Theorem 13.2.7 showing that $3, 2$-centered forcings do not add linear orderings of Borel quotient spaces except for a trivial reason. Section 13.3 provides a long list of examples and corollaries with particular emphasis on illustrating the differences between the forcing classes introduced in this chapter.

Chapter 14 contains results that for some reason do not fit into the previous chapters; some of our favorite theorems are here. Certain structures are impossible to add by balanced forcing; MAD families (Theorem 14.1.1) and linearly ordered unbounded subsets of many partial orderings (Theorem 14.2.1) fall into this category.

In Section 14.3, we prove one of the main constraints on the (nontrivial) balanced extensions of the Solovay model: there must be a set of reals in them which does not have the Baire property (Theorem 14.3.1). On the other hand, one does obtain certain models in which every set of real is Lebesgue measurable as per Theorem 14.3.4. An old result of Shelah concerning the consistency of ZF+DC plus every set of reals is Lebesgue measurable plus there is a set of reals without the Baire property can be presented using this theorem–Example 14.3.5.

In Section 14.4 we treat the very soft class of posets which possess definable balanced conditions. While at first sight it may seem that this class contains no particularly interesting posets, this is not the case: for example, the posets adding a discontinuous automorphism of $\mathcal{P}(\omega)$ modulo finite or a countable complete section to a given pinned Borel equivalence relation belong to it. We show that these posets preserve uncountable chromatic numbers of analytic hypergraphs (Theorem 14.4.7) and do not introduce inequalities of the form $|\mathbb{E}_1| \leq |E|$ where E is a pinned orbit equivalence relation. Many other preservation properties of posets in this class remain unstated.

1.7 Notation and terminology

General. In this book, as in many books on set theory, it is a frequently played gambit in the proofs to take an elementary submodel of a "large structure". The structure itself is rarely relevant. As a metadefinition, by a "large structure" we always mean the structure $\langle H_{<\kappa}, \in \rangle$ where $H_{<\kappa}$ is the set of all sets whose transitive closure has cardinality $< \kappa$, and κ is the smallest regular cardinal such that $H_{<\kappa}$ contains all objects named in the proof to that point, and also their powersets. Similarly, the phrase "a large fragment of ZFC" occurs in several places in the book. It is never truly informative to analyze precisely how large a fragment is needed. As a matter of convention, we mean the finite fragment of ZFC including all axioms except the schemas of comprehension and replacement, and the schemas of comprehension of replacement for all formulas of set theory of 10^100 many symbols or fewer.

Analytic equivalence relations. A number of concepts and results in this book are stated in terms of Borel equivalence relations on Polish spaces. As a matter of basic terminology, if E is an equivalence relation on a set X, a *E-transversal* is a set $T \subset X$ such that T has a singleton intersection which each equivalence class. If $x \in X$ is a point, then $[x]_E$ denotes the equivalence class containing x. If $A \subset X$ is any set, then $[A]_E$ denotes the *E-saturation* of A, the set $\{x \in X : \exists y \in A\ x\ E\ y\}$. The following definition records several benchmark relations which are used throughout the book.

Definition 1.7.1.

1. $\mathbb{E}_0$ is the *Vitali equivalence* on 2^ω, connecting $x, y \in 2^\omega$ if they differ at only finite set of entries;
2. $\mathbb{E}_1$ is the equivalence relation on $(2^\omega)^\omega$ connecting x, y if they differ at only finite number of entries;
3. $\mathbb{E}_2$ is the relation on 2^ω connecting x, y if the sum $\Sigma\{\frac{1}{n+1} : x(n) \neq y(n)\}$ is finite;
4. $\mathbb{F}_2$ is the equivalence relation on $(2^\omega)^\omega$ connecting x, y if $\mathrm{rng}(x) = \mathrm{rng}(y)$;
5. $\mathbb{HC}$ is the equivalence relation on $\mathcal{P}(\omega \times \omega)$ connecting relations x, y if either both are illfounded or fail the axiom of extensionality or fail to have a maximal element, or they are isomorphic;
6. $\mathbb{E}_{\omega_1}$ is the equivalence relation on $\mathcal{P}(\omega \times \omega)$ connecting relations x, y if either both are illfounded or are not linear orders, or they are isomorphic;
7. if Γ is a coanalytic class of structures on ω invariant under isomorphism, $\mathbb{E}_\Gamma$ is the equivalence relation on countable structures on ω connecting two such structures if they are both fail to belong to Γ or they are isomorphic;
8. if I is an ideal on ω then $=_I$ on 2^ω is the equivalence relation connecting x, y if $\{n \in \omega : x(n) \neq y(n)\} \in I$. There is an identically defined equivalence relation on $(2^\omega)^\omega$.

Borel equivalence relations are naturally ordered by Borel reducibility. In the case of analytic equivalence relations, the Borel reducibility relation exhibits certain pathologies, and it is best replaced by some of its strengthenings. This is the content of the following definition.

Definition 1.7.2. Let E and F be analytic equivalence relations on respective Polish spaces X and Y.

1. E is *Borel reducible to F*, in symbols $E \leq F$, if there is a Borel function $h : X \to Y$ such that $\forall x_0, x_1 \in X\ x_0\ E\ x_1$ iff $h(x_0)\ F\ h(x_1)$.

2. E is *almost Borel reducible to* F, in symbols $E \leq_a F$, if there exists a Borel function $h\colon X \to Y$ and a set $Z \subset X$ consisting of countably many E-classes such that $\forall x_0, x_1 \in X \setminus Z$ $x_0 \mathrel{E} x_1$ iff $h(x_0) \mathrel{F} h(x_1)$.

The most permissive comparison of equivalence relations is the one which compares just the cardinalities of the quotient spaces. The following abuse of notation is used throughout:

Definition 1.7.3. If E is an equivalence relation on a Polish space X then the E-*quotient space* is the set of all E-equivalence classes. Moreover, $|E|$ denotes the cardinality of the E-quotient space.

The Silver dichotomy shows that for a Borel equivalence relation E, $|E|$ is either countable or $|2^\omega| \leq |E|$. In the context of the axiom of choice, the latter disjunct implies that $|E| = |2^\omega|$ since the quotient space X/E is a surjective image of 2^ω. In choiceless context though, the dichotomies satisfying the latter disjunct may represent many different cardinalities, and this is the subject of study of several sections in this book. It is clear that $E \leq F$ implies that $|E| \leq |F|$, and in the context of the axiom of dependent choices, $E \leq_a F$ and $|F|$ is uncountable implies that $|E| \leq |F|$.

Analytic hypergraphs. A hypergraph on a set X is an arbitrary subset $\Gamma \subset \mathcal{P}(X)$ consisting of nonempty sets. As a matter of convention, our hypergraphs do not contain any singleton sets. The elements of Γ will be called *hyperedges* while the elements of X will be called *vertices*. A hypergraph is *finitary* if all of its hyperedges are finite sets. It is a *graph* if all its hyperedges have cardinality two. A Γ-*anticlique* is a set $A \subset X$ such that $\Gamma \cap \mathcal{P}(A) = 0$. A Γ-*coloring* is any function with domain X which is not constant on any hyperedge; the elements of the range of a coloring are referred to as *colors*. The *chromatic number* $\chi(\Gamma)$ of the hypergraph Γ is the smallest cardinal number κ such that there is a Γ-coloring with at most κ-many colors. This definition makes sense only in the context of the Axiom of Choice in which cardinalities are well-ordered; therefore, $\chi(\Gamma)$ must exist. In a choiceless context, only a limited version is available: we discern between countable and uncountable chromatic number, and the various values of the countable chromatic numbers.

We will be interested in analytic finitary hypergraphs. Say that $\Gamma \subset [X]^{<\aleph_0}$ is an analytic hypergraph if X is Polish and the set $\{y \in X^\omega \colon \mathrm{rng}(y) \in \Gamma\}$ is analytic. The *Borel chromatic number* of Γ is countable if X can be decomposed into countably many Borel sets $X = \bigcup_n X_n$ none of which contains all vertices of a Γ-edge.

Forcing. A great part of this book is devoted to forcing. The words forcing, poset, or partially ordered set are treated as synonyms. A condition is any element of a partially ordered set. We use the Boolean notation: $q \leq p$ means that the condition q is stronger, more informative than p. Now, let P be a partially ordered set. For a set $A \subset P$ and a condition $p \in P$, we write $p \leq \Sigma A$ if for every condition $q \leq p$ there is a condition $r \leq q$ which is stronger than some element of A. The formula $P \Vdash \phi$ denotes the statement that every condition $p \in P$, $p \Vdash \phi$. Whenever P is a partial ordering, τ is a P-name, and $G \subset P$ is a generic filter, the symbol τ/G denotes the valuation of the name τ according to the filter G. Every Polish space X and every analytic set $A \subset X$ have a canonical interpretation in every generic extension, which commutes with all usual descriptive set theoretic operations on Polish spaces. For the detailed theory of interpretations, see [102]; we will use it without explicit mention as is customary in the current forcing practice. The interpretations obey two basic absoluteness rules:

Fact 1.7.4. (Mostowski absoluteness) *If $M \subset N$ are transitive models of ZF+DC and $\phi(\vec{p})$ is a Σ^1_1-formula with parameters in M, then $M \models \phi(\vec{p})$ if and only if $N \models \phi(i\vec{p})$ where i is the interpretation operation.*

Fact 1.7.5. (Shoenfield absoluteness) *If $M \subset N$ are transitive models of ZF+DC such that $\omega_1 \subset M$ holds, and $\phi(\vec{p})$ is a Π^1_2-formula with parameters in M, then $M \models \phi(\vec{p})$ if and only if $N \models \phi(i\vec{p})$ where i is the interpretation operation.*

In particular, intepretations of analytic equivalence relations are equivalence relations again, and interpretations of Polish groups and continuous Polish group actions on Polish spaces are Polish groups and continuous Polish group actions again. Mutual relationships between forcing extensions of ZFC are captured in the following facts:

Fact 1.7.6. *If V is a model of ZFC, $V[G]$ is a forcing extension of V, and M is a model of ZFC such that $V \subset M \subset V[G]$, then M is a forcing extension of V and $V[G]$ is a forcing extension of M.*

The book is loaded with product forcing notions. This section provides basic information on products.

Fact 1.7.7. [45, Lemma 15.9] *Let P, Q be posets and in some generic extension, let $G \subset P, H \subset Q$ be filters separately generic over the ground model. The following are equivalent:*

1. *$G \times H \subset P \times Q$ is a filter generic over the ground model;*
2. *$G \subset P$ is generic over the model $V[H]$.*

In the affirmative case, $V[G] \cap V[H] = V$.

If $G \times H \subset P \times Q$ is a filter generic over the ground model, we say that the filters G, H (or their generic extensions) are *mutually generic*.

The following humble observation greatly simplifies the methodology of product forcing. It says that mutual genericity of forcing extensions can be characterized without an appeal to the specific generic filters and posets that were used to obtain the extensions, and indeed without any appeal to forcing at all.

Proposition 1.7.8. *Let $n \in \omega$ be a number and $\{P_i : i \in n\}$ be posets. Let $\{G_i : i \in n\}$ be filters separately generic over the ground model V over the respective posets. The following are equivalent:*

1. *$\prod_i G_i \subset \prod_i P_i$ is a filter generic over V;*

2. *whenever $\{a_i : i \in n\}$ are subsets of the ground model in the respective models $V[G_i]$ such that $\bigcap_i a_i = 0$ then there are sets $\{b_i : i \in n\}$ in the ground model V such that for all $i \in n$ $a_i \subset b_i$ and $\bigcap_i b_i = 0$ holds.*

Proof. Suppose first that (1) holds. Move to the ground model V. Suppose that $\{\dot{a}_i : i \in n\}$ are P_i-names for subsets of the ground model and $\langle p_i : i \in n\rangle \in \prod_i P_i$ is a condition forcing that $\bigcap_i a_i = 0$. Let $b_i = \{x \in V : \exists p' \leq p_i \; p' \Vdash_{P_i} \check{x} \in \dot{a}_i\}$ for each $i \in n$. It is immediate that $\bigcap_i b_i = 0$ holds, and for all $i \in n$ and $p_i \Vdash_{P_i} \dot{a}_i \subset \check{b}_i$ holds. (2) then follows by the forcing theorem.

Suppose now that (2) holds. To confirm (1), suppose towards a contradiction that it fails. There must be an open dense set $D \subset \prod_i P_i$ in the ground model such that $\prod_i G_i \cap D = 0$. For every number $i \in n$, in the model $V[G_i]$ consider the set $a_i = \{\langle p_j : j \in n\rangle \in D : p_i \in G_i\} \subset \prod_j P_j$. The contradictory assumption gives that $\bigcap_i a_i = 0$; the assumption (2) yields sets $b_i \subset \prod_i P_i$ in the ground model such that $\forall i \in n \; a_i \subset b_i$ and $\bigcap_i b_i = 0$. In each model $V[G_j]$, a genericity argument shows that there must be a condition $r_j \in G_j$ such that every tuple $\langle p_i : i \in n\rangle \in D$ with $p_j \leq r_j$ must in fact belong to the set b_j. Use the density of the set D to find a tuple $\langle p_i : i \in n\rangle \in D$ such that for all $i \in n$, $p_i \leq r_i$ holds. Then $\langle p_i : i \in n\rangle \in \bigcup_i b_i$, contradicting the choice of the sets b_i for $i \in n$. □

The following corollary is easy to prove without the proposition, but its present proof is much more appealing:

Corollary 1.7.9. *Let $n \in \omega$ be a natural number. Suppose that $\{P_i : i \in n\}$ are posets, $\{Q_i : i \in n\}$ are posets, $\prod_i G_i \subset \prod_i P_i$ is a generic filter, and for each $i \in n$ $H_i \in V[G_i]$ is a filter generic over V on the poset Q_i. Then $\prod_i H_i \subset \prod_i Q_i$ is a filter generic over V.*

Proof. The criterion (2) of Proposition 1.7.8 is preserved when passing to smaller models, in particular from $V[G_i]$ to $V[H_i]$. □

The final remark on the product forcing provides a perfect set of generics over any countable model–a standard trick which comes handy in several places in the book.

Proposition 1.7.10. *Let M be a countable transitive model of set theory and $P \in M$ be a poset. Then there is a continuous map $h : 2^\omega \to \mathcal{P}(P)$ such that for every finite set $a \subset 2^\omega$, the product $\prod_{x \in a} h(x)$ is a filter on the poset $P^{|a|}$ generic over the model M.*

Proof. Let $\langle D_n : n \in \omega\rangle$ be an enumeration of all open dense subsets of various finite powers of P which appear in the model M. By induction on $m \in \omega$ build maps $g_m : 2^m \to P$ satisfying the following demands:

- $g_0 : \{0\} \to P$ is arbitrary;

- for every $m \in \omega$, every $s \in 2^m$ and every $t \in 2^{m+1}$ such that $s \subset t$, $g_{m+1}(t) \leq g_m(s)$ holds;
- for every $m \in \omega$ and every $n \leq m$, if $D_n \subset P^k$ is a set for some $k \leq 2^{m+1}$ and $\langle t_i \colon i \in k \rangle$ is a sequence of distinct elements of 2^{m+1}, then $\langle g_{m+1}(t_i) \colon i \in k \rangle \in D$ holds.

This is easy to arrange as the last item requires meeting only finitely many open dense sets. In the end, let $h \colon 2^\omega \to \mathcal{P}(\omega)$ be the continuous function defined by setting $h(x)$ to be the filter generated by the set $\{g_m(x \restriction m) \colon m \in \omega\}$, and check that the demands of the theorem are satisfied. $\square$

There are several standard forcing notions used throughout:

Definition 1.7.11.

1. If X is a set then $\mathrm{Coll}(\omega, X)$ is the poset of finite partial functions from ω to X ordered by reverse extension;
2. if κ is a cardinal then $\mathrm{Coll}(\omega, <\kappa)$ is the finite support product of the posets $\mathrm{Coll}(\omega, \alpha)$ for all $\alpha \in \kappa$;
3. if X is a topological space then P_X is the poset of all nonempty open subsets of X ordered by inclusion. If X is in addition Polish, then $\dot{x}_{gen}$ denotes the P_X-name for the *generic point* of P_X–the unique element of the interpretation of the Polish space X in the P_X-extension which belongs to all open sets in the generic filter.

The collapse poset obeys an important factoring rule:

Fact 1.7.12. [45, Corollary 26.11] *Let λ be a cardinal and let P be a poset of size $< \lambda$. Suppose that $G \subset \mathrm{Coll}(\omega, \lambda)$ is a generic filter and in $V[G]$, $H \subset P$ is a filter generic over V. Then there is a filter $K \subset \mathrm{Coll}(\omega, \lambda)$ generic over $V[H]$ such that $V[G] = V[H][K]$.*

Let κ be an inaccessible cardinal. The *symmetric Solovay model derived from κ* is obtained as follows. Let $G \subset \mathrm{Coll}(\omega, <\kappa)$ be a generic filter and in $V[G]$, form the model $\mathrm{HOD}^{V[G]}_{V \cup \mathcal{P}(\omega)}$ of all sets hereditarily definable from reals and elements of the ground model. This is the symmetric Solovay model; we will always denote it by W, neglecting the dependence on κ and the filter G in the notation. The theory of the Solovay model has been thoroughly investigated throughout the years. We note the following:

Fact 1.7.13. [87] *In W, ZF+DC holds, every set has the Baire property and is Lebesgue measurable, and there is no uncountable well-ordered sequence of distinct Borel sets of bounded Borel rank.*

During the investigation of the symmetric Solovay model, the following technical fact and its corollary about the symmetric Solovay model will be used without mention.

Fact 1.7.14. [45, Section 26] *Let κ be an inaccessible cardinal and let W be the symmetric Solovay extension of V associated with κ. Then*

1. *every set in W is definable from parameters in $V \cup 2^\omega$;*
2. *in W, whenever M is a generic extension of V by a partial order of size $< \kappa$ then W is a symmetric Solovay extension of M.*

Corollary 1.7.15. *Suppose that κ is an inaccessible cardinal, X is a Polish space, $\phi(x, \vec{y})$ is a formula of set theory with all free variables displayed, and $\vec{p}$ is a sequence of sets of the same length as $\vec{y}$. The following are equivalent:*

1. $\mathrm{Coll}(\omega, <\kappa) \Vdash \exists x \in X\ \phi(x, \vec{p})$*;*
2. *there exist a poset R of size $< \kappa$ and an R-name σ for an element of X such that $R \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \phi(\sigma, \vec{p})$.*

Many models of set theory we investigate are extensions of the symmetric Solovay model by a suitably definable, typically Suslin, forcing. Since the instrumental properties of the Suslin forcing in question may not be absolute between forcing extensions, we use the following key convention to shorten the statements of the results we obtain.

Convention 1.7.16. Let ψ be a property of partial orders, and let ϕ be a sentence in the language of set theory.

1. For a Suslin partial order P and an inaccessible cardinal κ, the statement "P is ψ below κ" means that $V_\kappa \models \psi(P)$ holds in every forcing extension". The statement "P is ψ cofinally below κ" means that $V_\kappa \models$every forcing extension has a further extension in which $\psi(P)$ holds".
2. The phrase "In ψ extensions of the Solovay model, ϕ holds" denotes the following long statement. Let P be a Suslin forcing and let κ be an inaccessible cardinal such that P is ϕ below κ. Let W be a symmetric Solovay model derived from κ. Let $G \subset P$ be a filter generic over W. Then $W[G] \models \phi$ holds.
3. The phrase "In cofinally ψ extensions of the Solovay model, ϕ holds" denotes the following statement. Let P be a Suslin forcing. Let κ be an inaccessible cardinal. Suppose that P is ψ cofinally below κ. Let W be a symmetric Solovay model derived from κ. Let $G \subset P$ be a filter generic over W. Then $W[G] \models \phi$ holds.

In several cases, we will need the basics of the stationary tower forcing.

Definition 1.7.17. [64] Let κ be an inaccessible cardinal. The symbol $\mathbb{Q}_\kappa$ denotes the (countably based) stationary tower up to κ. That is, $\mathbb{Q}_\kappa$ consists of sets S such that $S \subset [\mathrm{dom}(S)]^{\aleph_0}$ is stationary and $\mathrm{dom}(S) \in V_\kappa$. The ordering is defined by $T \leq S$ if $\mathrm{dom}(S) \subset \mathrm{dom}(T)$ and $\{x \cap \mathrm{dom}(S) \colon x \in T\} \subset S$. If $G \subset \mathbb{Q}_\kappa$ is a generic filter, $j \colon V \to M$ denotes the generic ultrapower derived from G.

Fact 1.7.18. [64] *Let κ be an inaccessible cardinal, $G \subset \mathbb{Q}_\kappa$ be a generic filter, and $j\colon V \to M$ be the generic embedding.*

1. *If κ is a Woodin cardinal in the ground model, then $\kappa = \omega_1^{V[G]}$ and $M^\omega \subset M$ in $V[G]$. In particular, M is well-founded.*
2. *If κ is a weakly compact Woodin cardinal in the ground model, then for every $z \in 2^\omega$ in $V[G]$ there is a Woodin cardinal $\lambda < \kappa$ such that $G \cap \mathbb{Q}_\lambda$ is generic over V and $z \in V[G \cap \mathbb{Q}_\lambda]$. In particular, the model $W = V(\mathbb{R}^{V[G]})$ is a symmetric Solovay extension of V derived from κ.*

As a matter of notation, if the generic ultrapower model M is well-founded, it is always identified with its transitive collapse.

Part I

Equivalence relations

Chapter 2

The virtual realm

There are many quotient structures in mathematics. It turns out that a typical quotient structure allows a useful canonical extension to its *virtual version*. The purpose of this chapter is to lay the foundations of the theory of virtual structures.

2.1 Virtual equivalence classes

The basis of any quotient structure of interest in the present book is a Polish space X. A quotient structure worth its salt will also use an equivalence relation on the underlying Polish space E. In this section we indicate how to extend the E-quotient space in a canonical way to a potentially much larger set or class.

Definition 2.1.1. [48, Definition 17.1.2] Let E be an analytic equivalence relation on a Polish space X. Let P be a poset and τ a P-name for an element of X.

1. The name τ is *E-pinned* if $P \times P \Vdash \tau_{\text{left}}\ E\ \tau_{\text{right}}$;
2. if τ is E-pinned, then the pair $\langle P, \tau \rangle$ will be called an *E-pin*.

The definition may puzzle a novice reader. Its meaning is best illustrated by the following proposition. An E-pinned name is one which in all forcing extensions points at the same E-class, even though that E-class may not have any representative in the ground model.

Proposition 2.1.2. *Let E be an analytic equivalence relation on a Polish space X. Let P be a poset and τ a P-name for an element of X. The following are equivalent:*

1. *$\langle P, \tau \rangle$ is an E-pin;*
2. *in every forcing extension, if $G_0, G_1 \subset P$ are filters separately generic over the ground model, then $\tau/G_0\ E\ \tau/G_1$ holds.*

Proof. (2) immediately implies (1) by considering the $P \times P$ extension. To see how (1) implies (2), suppose that $V[H]$ is a forcing extension and in $V[H]$ there are filters $G_0, G_1 \subset P$ separately generic over V. Let $G_2 \subset P$ be a filter generic over $V[H]$. By the product forcing theorem, $G_0, G_2 \subset P$ are mutually generic filters, and so are $G_1, G_2 \subset P$. Applying the assumption (1), we see that $\tau/G_0\ E\ \tau/G_2\ E\ \tau/G_1$, so $\tau/G_0\ E\ \tau/G_1$ by the transitivity of the equivalence relation E. □

The following is the archetypal example of a non-trivial pinned name.

Example 2.1.3. Consider the poset $P = \mathrm{Coll}(\omega, 2^\omega)$ and its name τ for the generic surjection from ω to $(2^\omega)^V$. The name τ is $\mathbb{F}_2$-pinned since no matter which generic filter $G \subset P$ one selects, the range of τ/G is the same: it is the set $(2^\omega)^V$. Clearly, it is a nontrivial name since the set 2^ω is uncountable, so there is no ground model element of $(2^\omega)^\omega$ that can enumerate it and be equivalent to τ.

Given an analytic equivalence relation E on a Polish space X, the E-pinned names form a seemingly inexhaustible and complicated class. However, this class admits a natural equivalence relation which usually greatly clarifies matters:

Definition 2.1.4. Let E be an analytic equivalence relation on a Polish space X. Suppose that $\langle P, \tau\rangle$ and $\langle Q, \sigma\rangle$ are E-pins. Define $\langle P, \tau\rangle\ \bar{E}\ \langle R, \sigma\rangle$ if $P \times Q \Vdash \tau\ E\ \sigma$.

Proposition 2.1.5. *The relation $\bar{E}$ is an equivalence. If P, Q are posets with E-pinned names τ, σ on them, the following are equivalent:*

1. *$\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$;*
2. *in every forcing extension, if $G \subset P$ and $H \subset Q$ are filters separately generic over the ground model, then $\tau/G\ E\ \sigma/H$ holds.*

Proof. $\bar{E}$ is clearly symmetric and reflexive by its definition. To see the transitivity, suppose that $\langle P_0, \tau_0\rangle\ \bar{E}\ \langle P_1, \tau_1\rangle\ \bar{E}\ \langle P_2, \tau_2\rangle$. This means that $P_0 \times P_1 \times P_2 \Vdash \tau_0\ E\ \tau_1\ E\ \tau_2$, and by the transitivity of the equivalence relation E, $P_0 \times P_1 \times P_2 \Vdash \tau_0\ E\ \tau_2$. By the Mostowski absoluteness between the $P_0 \times P_1 \times P_2$ extension and $P_0 \times P_2$ extension, it is the case that $P_0 \times P_2 \Vdash \tau_0\ E\ \tau_2$ and consequently $\langle P_0, \tau_0\ \bar{E}\ \langle P_2, \tau_2\rangle$.

Now, (2) immediately implies (1) by considering the $P \times Q$ extension. To see how (1) implies (2), suppose that $V[K]$ is a forcing extension and in $V[K]$ there are filters $G \subset P, H \subset Q$ separately generic over V. Let $H' \subset Q$ be a filter generic over $V[K]$. By the product forcing theorem, G, H' are mutually generic filters, and so are H, H'. Applying the assumption (1), we see that $\tau/G\ E\ \sigma/H'\ E\ \sigma/H$, so $\tau/G\ E\ \sigma/H$ by the transitivity of the equivalence relation E. □

Definition 2.1.6. Let E be an analytic equivalence relation on a Polish space X.

1. The $\bar{E}$-classes are referred to as the *virtual E-classes*;
2. if z is a virtual E-class and in some generic extension $V[G]$ y is an E-class, we say that y is a *realization of z* if for some (equivalently, all) representatives $\langle P, \tau\rangle \in z$ and $x \in y$, $V[G] \models P \Vdash \tau\ E\ \check{x}$ holds.

The following proposition is used throughout this book. It says that E-classes represented in mutually generic extensions must be realizations of a virtual E-class in the ground model.

Proposition 2.1.7. *Let E be an analytic equivalence relation on a Polish space X. Let P_0, P_1 be partial orders and $G_0 \subset P_0$ and $G_1 \subset P_1$ be mutually generic filters. If $x_0 \in V[G_0]$ and $x_1 \in V[G_1]$ are E-equivalent points then $[x_0]_E$ is the realization of some virtual E-class from the ground model.*

Proof. Suppose that $p_0 \in P_0$, $p_1 \in P_1$, τ_0 is a P_0-name and τ_1 is a P_1-name such that $\langle p_0, p_1\rangle \Vdash \tau_0\ E\ \tau_1$. It immediately follows that τ_0 must be an E-pinned name on the poset $P \restriction p_0$ so p_0 forces $[\tau_0]_E$ to be the realization of the virtual E-class represented by the pair $\langle P \restriction p_0, \tau_0\rangle$. $\square$

The main question surrounding the virtual E-classes is whether they can be classified in some informative way. Is there a proper class of virtual E-classes or just a set? If there is just a set, what is its cardinality? Do virtual E-classes correspond to some more tangible combinatorial objects? This chapter contains many good answers to similar questions, even though many problems remain unsolved.

2.2 Virtual structures

It is now possible to define virtual versions of quotient structures on Polish spaces.

Definition 2.2.1. An *analytic quotient structure* is a tuple $\mathcal{M} = \langle X, E, R_i \colon i \in \omega, f_j \colon j \in \omega\rangle$ where

1. X is a Polish space;
2. E is an analytic equivalence relation on X;
3. for every $i \in \omega$, $R_i \subset X^{n_i}$ is an analytic relation which is invariant under E;
4. for every $j \in \omega$, $f_j \subset X^{m_j+1}$ is an analytic relation which is invariant under E, and in the E-quotient space it is a graph of a function.

The quotient structure $\mathcal{M}$ is *Borel* if all the relations above including E are Borel.

There are many popular examples of analytic quotient structures. If $\langle G, \cdot\rangle$ is a Polish group and $H \subset G$ is an analytic normal subgroup, one can form the quotient group G/H. If $\langle X, \leq\rangle$ is an analytic partial ordering, one can form the separative quotient under the assumption that the quotient equivalence relation is analytic. Embeddability of countable structures forms an ordering on the quotients space of all countable structures modulo the equivalence relation of biembeddability etc.

If $\mathcal{M} = \langle X, E, R_i \colon i \in \omega, f_j \colon j \in \omega\rangle$ is an analytic quotient structure, then we write $\mathcal{M}^* = \langle X^*, R_i^* \colon i \in \omega, f_j^* \colon j \in \omega\rangle$ for its associated structure on the actual quotient space X/E of all E-classes.

Definition 2.2.2. Let $\mathcal{M} = \langle X, E, R_i \colon i \in \omega, f_j \colon j \in \omega\rangle$ be an analytic quotient structure. The *virtual version* of $\mathcal{M}$ is the tuple $\mathcal{M}^{**} = \langle X^{**}, R_i^{**} \colon i \in \omega, f_j^{**} \colon j \in \omega\rangle$ where

1. X^{**} is the set or class of all virtual E-classes;
2. for each $i \in \omega$, R_i^{**} is the relation on X^{**} of arity n_i given by $\langle\langle Q_k, \tau_k\rangle \colon k \in n_i\rangle \in R_i^{**}$ if $\prod_k Q_k \Vdash \langle \tau_k \colon k \in n_i\rangle \in \dot{R}_i$;
3. for each $j \in \omega$, f_j^{**} is the relation on X^{**} of arity $m_j + 1$ given by $\langle\langle Q_k, \tau_k\rangle \colon k \in m_j + 1\rangle \in f_j^{**}$ if $\prod_k Q_k \Vdash \langle \tau_k \colon k \in m_j + 1\rangle \in \dot{f}_j$.

The first proposition says that Definition 2.2.2 is sound and that we receive a structure with the same signature as the original one:

Proposition 2.2.3. *The definition of R_i^{**} and f_j^{**} does not depend on the choice of representatives of the virtual classes. Moreover, f_j is a graph of a (total) function.*

Proof. The statements "R_i, f_j are E-invariant relations" and "f_j is a graph of a total function in the quotient" are $\mathbf{\Pi}^1_2$; therefore, they are absolute between V and all forcing extensions. The first sentence of the proposition immediately follows. For the second sentence, suppose that the function f_j has arity m_j and $\langle\langle Q_k, \tau_k\rangle \colon k \in m_j\rangle$ is a tuple of E-pins. Let $Q = \prod_k Q_k$ and let τ be a Q-name for an element of X such that $f_j(\tau_k \colon k \in m_j, \tau)$ is forced to hold. Since in the $Q \times Q$-extension, f_j is a graph of a function on the quotient and the names τ_k for $k \in m_j$ are E-pinned, it follows that the name τ is E-pinned as well. Clearly, the tuple $\langle\langle Q_k, \tau_k\rangle \colon k \in m_j, \langle Q, \tau\rangle\rangle$ belongs to f_j^{**} and the second sentence of the proposition follows. $\square$

In the case of an analytic quotient structure, it is possible that its virtual version is a proper class. In particular, it is possible to express the whole ordinal axis as an isomorph of a virtual version of an analytic quotient structure– Example 2.4.6. However, if the equivalence relation E is Borel then the virtual version is a set of size $< \beth_{\omega_1}$ by Theorem 2.5.6. In any case, it is possible to stratify $\mathcal{M}^{**}$ into set sized pieces:

Definition 2.2.4. Let $\mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle$ be an analytic quotient structure. $\mathcal{M}^{**}_\kappa$ is the substructure of $\mathcal{M}^{**}$ consisting of the virtual E-classes represented by names on $\mathrm{Coll}(\omega, \kappa)$.

A proof identical to that of Proposition 2.2.3 shows that $\mathcal{M}^{**}_\kappa$ is closed under all functions of $\mathcal{M}^{**}$ so it is truly a substructure of $\mathcal{M}^{**}$. An elementary name-counting argument shows that for each infinite κ, the structure $\mathcal{M}^{**}_\kappa$ has size at most 2^κ. If $\kappa \leq \lambda$ are cardinals then $\mathrm{Coll}(\omega, \kappa)$ is regularly embedded in $\mathrm{Coll}(\omega, \lambda)$ so $\mathcal{M}^{**}_\kappa \subseteq \mathcal{M}^{**}_\lambda$. Every poset is regularly embedded in $\mathrm{Coll}(\omega, \kappa)$ for some κ so $\mathcal{M}^{**}$ decomposes into a monotone union $\bigcup_\kappa \mathcal{M}^{**}_\kappa$.

For each analytic quotient structure $\mathcal{M}$, there is a canonical embedding $\pi \colon \mathcal{M}^* \to \mathcal{M}^{**}$ which maps each class $[x]_E$ to the virtual E-class of $\langle Q, \check{x} \rangle$ for a trivial poset Q. The most important fact about the virtual structures is that there is some degree of elementarity:

Proposition 2.2.5. *The canonical embedding $\pi \colon \mathcal{M}^* \to \mathcal{M}^{**}$ is Π_1-elementary.*

Proof. Let κ be a cardinal, and let $G \subset \mathrm{Coll}(\omega, \kappa)$ be a generic filter. In the model $V[G]$, let $\chi \colon (\mathcal{M}^{**}_\kappa)^V \to (\mathcal{M}^*)^{V[G]}$ be the map sending a virtual E-class in $(\mathcal{M}^{**}_\kappa)^V$ to its realization. Thus, we have maps $(\mathcal{M}^*)^V \xrightarrow{\pi} (\mathcal{M}^{**}_\kappa)^V \xrightarrow{\chi} (\mathcal{M}^*)^{V[G]}$. The composition $\chi \circ \pi$ sends each equivalence class in V to its interpretation in $V[G]$.

Let ϕ be a Π_1 formula in $\mathcal{L}_{\omega_1\omega}$ logic with possible parameters such that $\mathcal{M}^* \models \phi$. The statement $\mathcal{M}^* \models \phi$ is a $\boldsymbol{\Pi}^1_2$ sentence about the structure $\mathcal{M}$, and by Shoenfield absoluteness it transfers from the ground model V to the generic extension $V[G]$. Thus, the map $\chi \circ \pi$ is a Π_1 elementary embedding from $(\mathcal{M}^*)^V$ to $(\mathcal{M}^*)^{V[G]}$. It follows that the map $\pi \colon \mathcal{M}^* \to \mathcal{M}^{**}_\kappa$ in V must be a Π_1-elementary embedding. Since M^{**} is an increasing union $\bigcup_\kappa \mathcal{M}^{**}_\kappa$, the proposition follows. $\square$

In particular, if the original quotient structure was a group, a partial order, or an acyclic graph, its virtual version maintains these properties. However, it is important to understand that the embedding does not have to be Σ_2-elementary, so virtual versions of connected graphs may become disconnected, virtual versions of divisible groups may not be divisible anymore, and virtual versions of nonatomic partial orders may have atoms.

Example 2.2.6. Let R be the relation on $X = (2^\omega)^\omega$ defined by $x_0 \; R \; x_1$ if $\mathrm{rng}(x_0) \subseteq \mathrm{rng}(x_1)$. Clearly, the tuple $\langle X, \mathbb{F}_2, R \rangle$ is an analytic quotient structure. The relation R^* is a partial order on X^* without largest element. At the same time, the relation R^{**} on X^{**} does have the largest element, namely the $\mathbb{F}_2$-pin presented in Example **??**: the pin corresponding to the name for a generic enumeration of 2^ω in ordertype ω. This follows immediately from the classification of $\mathbb{F}_2$-pins in Example 2.3.5 below.

2.3 Classification: general theorems

In this section, we provide a number of general classification theorems for virtual equivalence classes. The theorems are all of the same type: if F is an analytic equivalence relation which is obtained from another equivalence relation E using a certain operation, then all virtual F-classes are obtained from virtual E-classes using a similar operation. This will provide a suitable background to the investigation of specific cases in Section 2.4.

To start with, a great many analytic equivalence relations yield only utterly uninteresting virtual classes: only those already realized in the ground model. This phenomenon is isolated in the following definition.

Definition 2.3.1. [48, Definition 17.1.2] Let E be an analytic equivalence relation on a Polish space X. A virtual E-class represented by $\langle P, \tau\rangle$ is *trivial* if there is $x \in X$ such that $P \Vdash \tau \mathrel{E} \check{x}$. The equivalence relation E is *pinned* if it has only trivial virtual classes.

The class of pinned equivalence relations has been investigated for a number of years. The basic pre-existing knowledge about this class is subsumed in the following fact.

Fact 2.3.2. [48, Theorem 17.1.3] *The analytic equivalence relations in the following classes are pinned:*

1. *orbit equivalence relations generated by actions of Polish cli groups;*
2. *Borel equivalence relations with all classes $\mathbf{\Sigma}^0_3$;*
3. *equivalence relations Borel reducible to pinned ones.*

The operation on equivalence relations which has the most informative translation into virtual classes is that of the Friedman–Stanley jump.

Definition 2.3.3. Let E be an analytic equivalence relation on a Polish space X. The *Friedman–Stanley jump* of E is the equivalence relation E^+ on the space $Y = X^\omega$ defined by $y_0 \mathrel{E^+} y_1$ if $[\mathrm{rng}(y_0)]_E = [\mathrm{rng}(y_1)]_E$.

Here, the classification of pinned names is right at hand: a pinned name for the jump is essentially just a set of pinned names for the original equivalence relation. For a nonempty set $S = \{\langle P_i, \tau_i\rangle : i \in I\}$ of representatives of virtual E-classes, let τ_S be the name on the poset $Q_S = \prod_i P_i \times \mathrm{Coll}(\omega, I)$ for an element of X^ω enumerating the set $\{\tau_i : i \in I\}$.

Theorem 2.3.4. *Let E be an analytic equivalence relation on a Polish space X.*

1. *If S is a set of E-pinned names then $\langle Q_S, \tau_S\rangle$ is an E^+-pinned name;*
2. *$\langle Q_S, \tau_S\rangle \mathrel{\bar{E}^+} \langle Q_T, \tau_T\rangle$ iff the sets S, T represent the same set of virtual E-classes;*

3. *whenever $\langle P, \tau\rangle$ is an E^+-pinned name, there is a set S of E-pinned names such that $\langle P, \tau\rangle\ \bar{E}^+\ \langle Q_S, \tau_S\rangle$.*

Proof. Items (1) and (2) are immediate. To prove (3), suppose that τ is an E^+-pinned name on a poset P. For every virtual E-class y, the statement $\phi(y) =$"rng(τ) contains a realization of the class y" must be decided in the same way by every condition in P. Let S be any set of E-pinned names which collects representatives from all virtual E-classes y such that $\forall p \in P\ p \Vdash \phi(\check{y})$. We claim that the set S works.

Indeed, suppose that $G_0, G_1 \subset P$ are mutually generic filters. The sets $[\mathrm{rng}(\tau/G_0)]_E$ and $[\mathrm{rng}(\tau/G_1)]_E$ are equal, and by Proposition 2.1.7 they contain only realizations of ground model virtual E-classes. By the choice of the set S, these sets contain exactly realizations of virtual E-classes represented by names in S. Since the generic ultrafilter G_0 was arbitrary, $\langle P, \tau\rangle\ \bar{E}^+\ \langle Q_S, \tau_S\rangle$ as desired. $\square$

Example 2.3.5. The relation $\mathbb{F}_2$ is the Friedman–Stanley jump of the identity on $X = 2^\omega$. The identity is pinned by Fact 2.3.2 so its virtual classes can be identified with elements of X. The $\mathbb{F}_2$-classes then correspond to subsets of 2^ω.

Countable products of equivalence relations translate to the virtual realm without change.

Definition 2.3.6. For each $n \in \omega$, let E_n be an analytic equivalence relation on a Polish space X_n. The product $\Pi_n E_n$ is the equivalence relation E on $Y = \prod_n X_n$ defined by $y_0\ E\ y_1$ if for every $n \in \omega$, $y_0(n)\ E_n\ y_1(n)$.

For every function g such that $\mathrm{dom}(g) = \omega$ and for all $n \in \omega$ $g(n)$ is some E_n-pinned name $\langle Q_n, \tau_n\rangle$, let τ_g be the name on the poset $Q_g = \prod_n Q_n$ (the support applied in the product is irrelevant for the equivalence class of the resulting $\Pi_n E_n$-pin) for the sequence $\langle \tau_n \colon n \in \omega\rangle$, which is forced to be an element of Y. The following is nearly immediate.

Theorem 2.3.7. *For each $n \in \omega$, let E_n be analytic equivalence relations on respective Polish spaces X_n and let $E = \prod_n E_n$.*

1. *If g is a sequence of E_n-pinned names, then $\langle Q_g, \tau_g\rangle$ is an $\prod_n E_n$-pin;*
2. *$\langle Q_g, \tau_g\rangle\ \bar{E}\ \langle Q_h, \tau_h\rangle$ iff for each $n \in \omega$, $g(n)\ \bar{E}_n\ h(n)$;*
3. *whenever $\langle P, \tau\rangle$ is an E-pinned name, there is a function g such that $\langle P, \tau\rangle\ \bar{E}\ \langle Q_g, \tau_g\rangle$.*

Countable increasing unions of equivalence relations translate to the virtual realm without change as well.

Theorem 2.3.8. *Let $\{E_n \colon n \in \omega\}$ be an increasing sequence of analytic equivalence relations on a Polish space X, and let $E = \bigcup_n E_n$.*

1. *Whenever $\langle Q, \sigma\rangle$ is an E_n-pinned name for some $n \in \omega$ then it is also E-pinned;*

2. *whenever $\langle P, \tau\rangle$ is an E-pinned name, there exists a number $n \in \omega$ and an E_n-pinned name $\langle Q, \sigma\rangle$ such that $\langle P, \tau\rangle \bar{E} \langle Q, \sigma\rangle$.*

Proof. (1) is immediate as $E_n \subset E$. For (2), by the forcing theorem there must be conditions $p_0, p_1 \in P$ and a number n such that $\langle p_0, p_1\rangle \Vdash_{P\times P} \tau_{\text{left}} \; E_n \; \tau_{\text{right}}$. The transitivity of the relation E_n then shows that τ on the poset $P \restriction p_0$ is an E_n-pinned name. The initial assumptions show that $\langle P, \tau\rangle \; \bar{E} \; \langle P \restriction p_0, \tau\rangle$. as desired. □

Example 2.3.9. The Louveau jump survives into the virtual realm without change. The *Louveau jump* of an analytic equivalence relation E on a Polish space X is the equivalence relation E^{+L} on $Y = X^\omega$ connecting $y_0, y_1 \in Y$ if for all but finitely many $n \in \omega$, $y_0(n) \; E \; y_1(n)$. The Louveau jump can be written as a countable increasing union of countable products of E, which by Theorems 2.3.7 and 2.3.8 yields a complete analysis of its virtual classes in terms of virtual E-classes. In particular, if E is pinned then its Louveau jump is pinned.

The virtual realm also correctly reflects the situation in which the equivalence classes of one relation consist of countably many equivalence classes of another one.

Definition 2.3.10. Let E, F be analytic equivalence relations on a Polish space X. We say that F *is countable over* E if $E \subset F$ and every F-class consists of countably many E-classes.

Theorem 2.3.11. *Let E, F be analytic equivalence relations on a Polish space X, with F countable over E.*

1. *If $\langle Q, \sigma\rangle$ is an E-pinned name then it is F-pinned as well;*

2. *if $\langle P, \tau\rangle$ is an F-pinned name then there is an E-pinned name $\langle Q, \sigma\rangle$ such that $\langle P, \tau\rangle \; \bar{F} \; \langle Q, \sigma\rangle$.*

Proof. (1) is immediate. For (2), it will be enough to show that there is a condition $p \in P$ such that τ is an E-pinned name on $P \restriction p$, for then $\langle P, \tau\rangle \; \bar{F} \; \langle P \restriction p, \tau\rangle$ as desired.

Suppose towards a contradiction that there is no condition $p \in P$ such that τ is E-pinned on the poset $P \restriction p$. It follows by the forcing theorem that $P \times P \Vdash \neg\tau_{\text{left}} \; E \; \tau_{\text{right}}$ holds. Let M be a countable elementary submodel of a large structure containing P, τ and the codes for E, F. Use Proposition 1.7.8 to find an uncountable collection $\{g_i : i \in I\}$ of filters on $M \cap P$ pairwise mutually generic over the model M. By the Mostowski absoluteness between the models $M[g_i, g_j]$ and V for $i \neq j \in I$, the elements $\tau/g_i \in X$ for $i \in I$ are pairwise F-related, but pairwise E-unrelated, contradicting the initial assumptions on the relations E, F. □

Example 2.3.12. The Clemens jump survives into the virtual realm without change. Here the *Clemens jump* of an analytic equivalence relation E on a Polish space X is the equivalence relation E^{+C} on $Y = X^{\mathbb{Z}}$ connecting $y_0, y_1 \in Y$ if there is $n \in \mathbb{Z}$ such that for every $m \in \mathbb{Z}$ $y_0(m)\ E\ y_1(m+n)$. The Clemens jump is countable over the product of $\mathbb{Z}$-many copies of E, which yields a complete analysis of its virtual classes by Theorems 2.3.7 and 2.3.8. In particular, if E is pinned, then so is its Clemens jump.

2.4 Classification: specific examples

There are many analytic equivalence relations for which the virtual space can be classified by more tangible combinatorial objects, but which do not fit into the context of the theorems of Section 2.3. The purpose of this section is to investigate these more difficult, but also more informative, possibilities.

The most interesting issues arise in equivalence relations classifiable by countable structures. Among these, the Borel equivalence relations are Borel reducible to an iterated jump of the identity [48, Theorem 12.5.2], so can be handled by Theorems 2.3.4 and 2.3.8. For example, for all equivalence relations E Borel reducible to $\mathbb{F}_2$, the virtual E-classes are classifiable by subsets of 2^ω by Example 2.3.5.

More interesting issues arise with analytic equivalence relations. Let E be the equivalence relation of isomorphism of structures on ω. In this case, the virtual classes correspond to potential Scott sentences in the sense of [96]. In some cases, it is possible to classify virtual classes of E by uncountable structures as in the following definition.

Definition 2.4.1. Let M be a (possibly uncountable) structure of a countable signature. τ_M is a $\mathrm{Coll}(\omega, M)$-name for some structure on ω isomorphic to M.

It is immediate that the pair $\langle \mathrm{Coll}(\omega, M), \tau_M \rangle$ is an E-pin, and its $\bar{E}$-equivalence relation does not depend on the choice of the name τ_M. It turns out that the $\bar{E}$-equivalence relation on the E-pins obtained in this way coincides with a familiar concept of model theory:

Definition 2.4.2. [67, Section 2.4] Let M, N be structures with the same signature. Say that M, N are *Ehrenfeucht–Fraissé*-equivalent if Player II has a winning strategy in the *Ehrenfeucht–Fraissé game*. In the EF-game, the two players take turns, at round $i \in \omega$ Player I starting with an element $n_i \in N$ or $m_i \in M$ and Player II responding with an element $m_i \in M$ or $n_1 \in N$ respectively. After all rounds indexed by $i \in \omega$ have been completed, Player II wins if the map $n_i \mapsto m_i$ for $i \in \omega$ preserves all relations and functions of N, M in the signature.

Theorem 2.4.3. *Let M, N be models with the same countable signature. The following are equivalent:*

1. *M, N are Ehrenfeucht–Fraissé equivalent;*

2. $\langle \mathrm{Coll}(\omega, M), \tau_M\rangle \ \bar{E} \ \langle \mathrm{Coll}(\omega, N), \tau_N\rangle$.

Proof. For the (1)→(2) direction, if M is Ehrenfeucht–Fraissé equivalent to N as witnessed by a winning strategy σ for Player I in the EF-game, then $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N) \Vdash \tau_M \ E \ \tau_N$, since a generic run of the EF-game in which Player II follows the strategy σ will generate an isomorphism between M and N in the extension. For the (2)→(1) direction, suppose that $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N) \Vdash \tau_M \ E \ \tau_N$ and let $\pi \colon \check{M} \to \check{N}$ be a product name for the isomorphism. The winning strategy for Player II can be described as follows: as the game develops, Player II also maintains on the side conditions $q_i \in \mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N)$ such that $q_0 \geq q_1 \geq \dots$ and $q_i \Vdash \pi(\check{m}_i) = \check{n}_i$. It is immediate that this is possible and Player II must win in the end. $\square$

In the language of [96], a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ is *grounded* if every virtual equivalence class is represented by a collapse name for a possibly uncountable model of ϕ. In general, there are virtual E-classes which are not represented by a straightforward collapse name as in Definition 2.4.1, as [50, Section 4] shows. However, a collapse name can be found for certain classes of structures.

Definition 2.4.4. Let Γ be a coanalytic set of structures on ω, closed under isomorphism. A (possibly uncountable) structure M is a Γ^{**}-*structure* if $\mathrm{Coll}(\omega, M) \Vdash \tau_M \in \Gamma$.

We proceed to show that for some interesting coanalytic classes Γ, every $\bar{\mathbb{E}}_\Gamma$-class is represented by a collapse name of a Γ^{**}-structure as in Definition 2.4.1. In the classification results, we always ignore the trivial class of structures which do not belong to Γ. Results similar to the following theorem appear in [57] and [65].

Theorem 2.4.5. *Let Γ be a coanalytic class of countable structures on ω, invariant under isomorphism, consisting of rigid structures only.*

1. *For Γ^{**}-structures, Ehrenfeucht–Fraissé equivalence and isomorphism coincide;*
2. *for every $\mathbb{E}_\Gamma$-pin $\langle P, \sigma\rangle$ there is a Γ^{**}-structure M such that $\langle P, \sigma\rangle \ \bar{E} \ \langle \mathrm{Coll}(\omega, M), \tau_M\rangle$.*

Proof. Before we begin the argument, note that the statement that every structure in the set A is rigid is $\mathbf{\Pi}^1_2$; therefore, it holds also in all generic extensions by the Shoenfield absoluteness.

For (1), it is clear that isomorphic structures are Ehrenfeucht–Fraissé equivalent. For the opposite implication, suppose that M, N are Γ^{**} structures which are EF-equivalent. By Theorem 2.4.3, $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N) \Vdash \tau_M \ E \ \tau_N$ must hold. As Γ consists of rigid structures even in the collapse extension, $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N)$ forces that there is a unique isomorphism $\pi \colon M \to N$. Since $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N)$ is a homogeneous notion of forcing, for each $m \in M$ the value of $\pi(\check{m})$ is decided by the largest condition to be some

$h(m) \in N$. The function $h\colon M \to N$ is an isomorphism of M to N present already in V.

For (2), let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over V. In the model $V[G_0, G_1]$, let $N_0 = \tau/G_0$ and $N_1 = \tau/G_1$. To define the model $M \in V$, let $x_0 = \{s\colon s$ is the Scott sentence of the model $\langle N_0, a\rangle$ for some $a \in N_0\} \in V[G_0]$ and $x_1 = \{s\colon s$ is the Scott sentence of the model $\langle N_1, a\rangle$ for some $a \in N_1\} \in V[G_1]$. Since N_0 is isomorphic to N_1, it follows from Karp's theorem [32, Lemma 12.1.6] that $x_0 = x_1$, so $x_0 = x_1 \in V[G_0] \cap V[G_1] \in V$. The set x_0 will be the universe of the model M. Note that since the model N_0 is rigid, the elements of N_0 are in one-to-one correspondence with x_0 by Karp's theorem again and the unique isomorphism between N_0 and N_1 factors through the identity on the set $x_0 = x_1$. To construct the realizations of relational and functional symbols of the model M, for every relational symbol R (say binary) of the language of the models and $s, t \in x_0$, let $s\ R^M\ t$ if for the unique $a, b \in N_0$ such that s is the Scott sentence of a and t is a Scott sentence of b, $N_0 \models s\ R\ t$. The same definition using the model N_1 yields the same relation, so $R^M \in V[G_0] \cap V[G_1] = V$. Define the realizations of all functional and relational symbols of the model M in this way. As a result, M is a model in V and the map sending each $a \in M$ to the Scott sentence of $\langle M, a\rangle$ is an isomorphism of N_0 and M in the model $V[G_0]$. Thus, (2) follows. $\square$

Theorem 2.4.5 makes it possible to describe some class-sized virtual spaces explicitly:

Example 2.4.6. The virtual $\mathbb{E}_{\omega_1}$-classes are precisely classified by ordinals, since $\mathbb{E}_{\omega_1} = \mathbb{E}_\Gamma$ where Γ is the class of all well-orderings on ω. Well-orderings are rigid, and up to isomorphism are classified by ordinals.

Example 2.4.7. The virtual $\mathbb{HC}$-classes are classified by transitive sets with the $\in$-relation. Note that all extensional, well-founded relations are rigid and uniquely isomorphic to a unique transitive set with the $\in$-relation by the Mostowski collapse theorem [45, Theorem 6.15].

In the case of non-rigid structures, the classification may become more complicated. We will treat the case of acyclic graphs, which has the virtue of being Borel-complete among the equivalence relations classifiable by countable structures. Note that the Ehrenfeucht–Fraissé equivalence on uncountable acyclic graphs is distinct from isomorphism, as the case of an empty graph on $\aleph_1$ or $\aleph_2$ vertices shows.

Theorem 2.4.8. *Let Γ be the class of all acyclic graphs on ω. For every $\mathbb{E}_\Gamma$-pin $\langle P, \sigma\rangle$ there is an acyclic graph H such that $\langle P, \sigma\rangle\ \bar{E}\ \langle \mathrm{Coll}(\omega, H), \tau_H\rangle$.*

Proof. The point of the proof is that an acyclic graph can be explicitly built from automorphism orbits of its elements. This procedure is captured in the following observation. Suppose x is a set, $f\colon x^2 \to \omega + 1$ is a function such that $f(s,t) > 0 \leftrightarrow f(t,s) > 0$, and $g\colon x^2 \to \omega + 1$ is a function such that $f(s,t) > 0$ implies $g(s) = g(t)$. Then there is, up to an isomorphism unique, acyclic graph

$H(x, f, g)$ together with an onto map $h\colon y \to x$, where y is the set of vertices of $H(x, f, g)$, such that

- for every $s, t \in x$ and every vertex $v \in y$, if $h(v) = s$ then the set of all neighbors of v mapped to t has size $f(s, t)$;
- for every $s \in x$ there are $g(s)$ many connected components of the graph H containing a vertex v with $h(v) = s$.

The construction of the graph $H(x, f, g)$ is straightforward. Note that whenever $u, v \in y$ are two vertices such that $h(u) = h(v)$ then there is an automorphism of the graph $H(x, f, g)$ sending u to v.

Now, suppose that σ is an E-pinned name on a poset P and let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over V. In the model $V[G_0, G_1]$, let $H_0 = \sigma/G_0$ and $H_1 = \sigma/G_1$. To define the graph $H \in V$, let $x_0 = \{s\colon s$ is the Scott sentence of the model $\langle H_0, v\rangle$ for some vertex v of $H_0\} \in V[G_0]$ and $x_1 = \{s\colon s$ is the Scott sentence of the model $\langle H_1, v\rangle$ for some vertex v in $H_1\} \in V[G_1]$. Since H_0 is isomorphic to H_1, it follows from Karp's theorem [32, Lemma 12.1.6] that $x_0 = x_1$, so $x = x_0 = x_1 \in V[G_0] \cap V[G_1] \in V$. Let $f\colon x^2 \to \omega + 1$ be the function defined by $f(s, t) = i$ if every vertex of H_0 of type s has i-many neighbors of type t when $i \in \omega$, and $f(s, t) = \omega$ if every vertex of H_0 of type s has infinitely many neighbors of type t. Let $g\colon x \to \omega + 1$ be the function defined by $f(s) = i$ if there are i-many connected components of H_0 containing a node of type s when $i \in \omega$, and $g(s) = \omega$ if there are infinitely many connected components of H_0 containing a node of type s. Note that these functions are well-defined and the graph H_0 is isomorphic to $H(x, f, g)$ in the model $V[G_0]$. Similar definitions using the grap H_1 yield the same functions, so $f, g \in V[G_0] \cap V[G_1] = V$. Working in V, consider the graph $H = H(x, f, g)$. This graph is isomorphic to H_0 in $V[G_0]$, so $\tau_H \bar{E} \sigma$. $\square$

In the recent years, model theorists proved several other theorems regarding the classification of virtual isomorphism classes by possibly uncountable structures. [96] provides further examples of grounded sentences beyond those obtained by Theorems 2.4.5 and 2.4.8. The first item of the following theorem is related to classical arguments going at least as far back as the unpublished work of Leo Harrington in the 1980's. Similar results also appear in [6] and [65].

Fact 2.4.9. [50, Section 4] *Let E be the equivalence relation of isomorphism of structures on ω.*

1. *If P is a poset forcing $|\aleph_2^V| > \aleph_0$ then every E-pin $\langle P, \sigma\rangle$ is $\bar{E}$-equivalent to $\langle \mathrm{Coll}(\omega, M), \tau_M\rangle$ for some structure M of size $\leq \aleph_1$;*
2. *if P forces $|\aleph_2^V| = \aleph_0$, then there is an E-pin $\langle P, \sigma\rangle$ which is not $\bar{E}$-equivalent to $\langle \mathrm{Coll}(\omega, M), \tau_M\rangle$ for any structure M.*

Virtual spaces for equivalence relations which are not classifiable by countable structures are often quite difficult to understand. To conclude this section, we state another classification theorem and a couple of open questions.

Theorem 2.4.10. *Let E be the equivalence relation on $X = (\mathcal{P}(\omega))^\omega$ connecting x_0, x_1 if $\operatorname{rng}(x_0)$ and $\operatorname{rng}(x_1)$ generate the same filter on ω. The virtual E-classes are classified by filters on ω.*

Proof. On one hand, if F is a filter on ω, one can consider the $\operatorname{Coll}(\omega, F)$-name τ_F for a generic enumeration of the filter F. It is immediate that the name τ_F is E-pinned, and distinct filters yield inequivalent names.

For the more difficult part, suppose that P is a partial order and τ is an E-pinned name on P; we must find a filter F on ω such that τ is equivalent to τ_F. To do this, let $F = \{a \subset \omega \colon \exists p\ p \Vdash \check{a}$ belongs to the filter generated by $\operatorname{rng}(\tau)\}$. By the pinned property of the name τ, the existential quantifier in the definition of F can be replaced by universal without changing the resulting set F. It follows immediately that the set F is a filter; we must show that τ is equivalent to τ_F.

Suppose towards a contradiction that this fails; then it must be the case that for some condition $p \in P$ and some number $n \in \omega$, p forces $\tau(\check{n})$ to have no subset in the filter F. Since the name τ is E-pinned, there must exist conditions $p_0, p_1 \leq p$ and a finite set $a \subset \omega$ such that $\langle p_0, p_1\rangle \Vdash \bigcap_{m \in \check{a}} \tau_{\text{right}}(m) \subset \tau_{\text{left}}(\check{n})$. Let $b = \{k \in \omega \colon \exists r \leq p_1\ r \Vdash \check{k} \in \bigcap_{m \in a} \tau(m)\}$ and let $c = \{k \in \omega \colon p_0 \Vdash \check{k} \in \tau(\check{n})\}$. Observe that $b \in F$ (since $p_1 \Vdash \bigcap_{m\in a} \tau(m) \subset \check{b}$) and $c \notin F$ (since $p_0 \Vdash \check{c} \subset \tau$). Thus, there has to be a number $k \in b \setminus c$, and for this number k there are conditions $r_0 \leq p_0$ and $r_1 \leq p_1$ such that $r_1 \Vdash \check{k} \in \bigcap_{m \in a} \tau(m)$ and $r_0 \Vdash \check{k} \notin \tau(\check{n})$. This, however, contradicts the choice of the conditions p_0, p_1. $\square$

Consider the equivalence relation E of homeomorphism of compact metrizable spaces. This is known to be the largest equivalence relation reducible to an orbit equivalence in the sense of Borel reducibility [105]. In an attempt to describe its virtual space, consider any compact Hausdorff space X with a topology basis of size κ, and consider the $\operatorname{Coll}(\omega, \kappa)$-name τ_X for the interpretation of the space X in the extension in the sense of [102]. The interpretation of X will have a countable basis, so the interpretation will be a compact metrizable space. The basic theory of interpretations shows that $\langle \operatorname{Coll}(\omega, \kappa), \tau_X\rangle$ is an E-pin. The most natural question is open:

Question 2.4.11. Is every virtual E-class represented by a compact Hausdorff space?

The measure equivalence E is one of the hardest equivalence relations to understand. It connects two Borel probability measures μ, ν on the Cantor space if they share the same ideal of null sets. The relation $\mathbb{F}_2$ Borel reduces to E, so E is not pinned.

Question 2.4.12. Classify the virtual space for the measure equivalence.

2.5 Cardinal invariants

There are several cardinal invariants of equivalence relations which are associated with the concept of virtual equivalence classes. They respect the Borel reducibility order; therefore, they are useful as tools for nonreducibility results. They also are conceptually useful in several places in this book.

2.5a Basic definitions

The following definition records the most natural cardinal invariants associated with the virtual spaces.

Definition 2.5.1. Let E be an analytic equivalence relation on a Polish space X.

1. $\kappa(E)$, the *pinned cardinal* of E, is the smallest κ (if it exists) such that every virtual E-class has a representative on a poset of size $<\kappa$. If E is pinned, we let $\kappa(E)=\aleph_1$. If $\kappa(E)$ does not exist, we write $\kappa(E)=\infty$;
2. $\lambda(E)$ is the cardinality of the set of all virtual E-classes. If the virtual E-classes form a proper class, we let $\lambda(E)=\infty$;
3. $\lambda(E,P)$ is the cardinality of the set of all virtual E-classes represented on the poset P.

Note that $\kappa(E)=\infty$ just in case $\lambda(E)=\infty$; at the same time, $\lambda(E,P)<\infty$ holds for all E,P. The rather mysterious demand that $\kappa(E)=\aleph_1$ for all pinned equivalence relations E is explained by a reference to Theorem 2.6.2: there are no nontrivial pinned names on countable posets for any analytic equivalence relation. It turns out that the cardinals $\kappa(E)$ and $\lambda(E,P)$ can attain all kinds of exotic and informative values. We will start with the two archetypal and somewhat boring computations.

Example 2.5.2. $\kappa(\mathbb{F}_2)=\mathfrak{c}^+$ and $\lambda(\mathbb{F}_2)=2^{\mathfrak{c}}$.

Proof. This follows immediately from the classification of pinned names for the Friedman–Stanley jump. Every virtual $\mathbb{F}_2$ class is represented by a subset of 2^ω, and distinct subsets of 2^ω give rise to distinct virtual $\mathbb{F}_2$ classes. □

Example 2.5.3. $\kappa(\mathbb{E}_{\omega_1})=\infty$. To see this, note that by Example 2.4.6, the virtual $\mathbb{E}_{\omega_1}$-classes are classified by ordinals, so there is a proper class of them. Similarly, $\kappa(\mathbb{HC})=\infty$, as by Example 2.4.7 the virtual $\mathbb{HC}$-classes are classified by transitive sets, so there is a proper class of them.

Theorem 2.8.9 shows that in fact $\mathbb{E}_{\omega_1}$ is a minimal example of an equivalence relation E with $\kappa(E)=\infty$.

The most appealing fact about the cardinal invariants $\kappa(E)$ and $\lambda(E,P)$ is that they respect the Borel reducibility order; therefore, they can be used to prove Borel nonreducibility results. In a good number of instances, the comparison of the cardinal invariants is the fastest and most intuitive way of proving nonreducibility. One of the main features of this style of argumentation is that it automatically survives the transfer to nonreducibility by functions more complicated than Borel.

Theorem 2.5.4. *Let E, F be analytic equivalence relations on Polish spaces X, Y respectively. If $E \leq_a F$ then $\kappa(E) \leq \kappa(F)$ and $\lambda(E, P) \leq \lambda(F, P)$ holds for every partial order P.*

Proof. Suppose that $h\colon X \to Y$ is a Borel function witnessing the reduction of E to F everywhere except for a set $Z \subset X$ consisting of countably many E-classes. By a Shoenfield absoluteness argument, these properties of the function h transfer to all generic extensions. If P is a partial ordering and τ is an E-pinned name on P which is not a name for one of the classes in the set Z, then $\dot{h}(\tau)$ is an F-pinned name on P, and the map $\tau \mapsto \dot{h}(\tau)$ respects virtual E- and F-classes, and it is an injection from the former to the latter. It follows that $\lambda(E, P) \leq \lambda(F, P)$.

Now, suppose that Q is another poset and that $h(\tau)$ has a $\bar{F}$-equivalent name σ on Q. By the Shoenfield absoluteness, $Q \Vdash \exists x \in X \setminus Z\ \dot{h}(x) = \tau$; any Q-name for such an x is E-pinned and the name for $h(x)$ is $\bar{F}$-equivalent to $h(\tau)$. A brief bout of diagram chasing now shows that $\kappa(E) \leq \kappa(F)$ and the theorem follows. $\square$

The following neat application has been observed by [96] for the case of Borel equivalence relations classified by countable structures.

Example 2.5.5. Friedman and Stanley proved that the Friedman–Stanley jump of a Borel equivalence relation E is not Borel reducible to E [31]. The most appealing proof of this fact uses the cardinal invariant $\lambda(E)$. Theorem 2.5.6 below shows that for Borel equivalence relations, the value of $\lambda(E)$ is an actual cardinal as opposed to ∞. Theorem 2.3.4 shows that virtual E^+-classes are classified by sets of virtual E-classes, in other words $\lambda(E^+) = 2^{\lambda(E)}$. Theorem 2.5.4 then completes the argument.

2.5b Estimates

The key fact about the pinned cardinal is the following theorem. It shows that there are a priori bounds on the size of the pinned cardinal; in particular, if the equivalence relation E is Borel, then $\kappa(E) \leq \beth_{\omega_1}$ and the virtual space of E is a set. Recall that the $\beth$ function is defined by recursion: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\alpha = \sup_{\beta\in\alpha} \beth_\beta$ if α is a limit ordinal.

Theorem 2.5.6. *Let E be a Borel equivalence relation on a Polish space X of rank $\mathbf{\Pi}^0_\alpha$. Then $\kappa(E) \leq (\beth_\alpha)^+$.*

Proof. Let τ be an E-pinned name on a poset P; we must produce a $\mathrm{Coll}(\omega, \beth_\alpha)$-name σ which is $\bar{E}$-related to τ. Note that $[\tau]_E$ is a P-name for a Borel set of rank $\leq \alpha$. As is the case for every name for a Borel set, [87, Corollary 2.9] shows that in the $\mathrm{Coll}(\omega, \beth_\alpha)$ extension $V[G]$ there is a Borel code for a Borel set $B \subset X$ such that in every further forcing extension $V[G][H]$ and every $x \in X \cap V[G][H]$ in that extension, $x \in B$ if and only if $V[x] \models P \Vdash \check{x} \in [\tau]_E$. Note that if $H \subset P$ is generic over $V[G]$, then the set B is nonempty in $V[G][H]$, containing the point τ/H; this follows from the fact that τ is E-pinned. Thus, the set B is nonempty already in $V[G]$ by the Mostowski absoluteness between $V[G]$ and $V[G][H]$. Back in V, let σ be any $\mathrm{Coll}(\omega, \beth_\alpha)$-name for an element of the set B. This clearly works. $\square$

Example 2.5.7. For any given countable ordinal α, let Γ_α be the class of binary relations on ω which are extensional and wellfounded of rank $< \alpha$; let E_α be the isomorphism relation. It is not difficult to check (Theorem 2.4.5) that for each countable ordinal α, the relation E_α is Borel, and its pinned names are collapse names for isomorphs of the membership relation on sets in V_α. Thus, the cardinals $\kappa(E_\alpha)$ converge to $\beth_{\omega_1}$.

Theorem 2.5.8. *Let E be an analytic equivalence relation almost reducible to an orbit equivalence relation of a continuous Polish group action. If $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first ω_1-Erdős cardinal.*

Proof. Let κ be the first ω_1-Erdős cardinal, and suppose that $\kappa(E) > \kappa$; we must show that $\kappa(E) = \infty$. Since the cardinal κ is the Hanf number for the class of wellfounded models of first order sentences, for every cardinal λ there is a wellfounded model M such that $M \models \kappa(E) > \lambda$. Now, since E is almost reducible to an orbit equivalence relation, Corollary 2.7.4 shows that the wellfounded model M is correct about $\kappa(E)$ to the extent that $|\kappa(E)^M| \leq \kappa(E)$. It follows that $\kappa(E) > \lambda$, and since λ was arbitrary, $\kappa(E) = \infty$. $\square$

Example 2.5.9. For every countable ordinal α there is a coanalytic class Γ of structures on ω, invariant under isomorphism, such that $\kappa(\mathbb{E}_\Gamma)$ is equal to the first α-Erdős cardinal.

Proof. Let Γ be the class of all binary relations on ω which are extensional, wellfounded, and do not admit a sequence of indiscernibles of ordertype α; we claim that this class works.

Clearly, Γ is a coanalytic set of rigid structures invariant under isomorphism. Write E for $\mathbb{E}_\Gamma$ and κ for the first α-Erdős cardinal. By Theorem 2.4.5, every virtual E-class is represented by a transitive set A without indiscernibles of ordertype α. It must be the case that $|A| < \kappa$ so $\kappa(E) \leq \kappa$. On the other hand, whenever $\lambda < \kappa$ is an ordinal, then the structure $\langle V_\lambda, \in \rangle$ has no indiscernibles of ordertype α, and it remains such in every forcing extension by a wellfoundedness argument. Thus, the $\mathrm{Coll}(\omega, V_\lambda)$-name for the generic isomorph of this structure is E-pinned, and it is not equivalent to any E-pinned name on a poset of size $< |V_\lambda|$ since it entails the collapse of $|V_\lambda|$ to $\aleph_0$. Thus, $\kappa(E) = \kappa$ as desired. $\square$

Theorem 2.5.10. *Let E be an analytic equivalence relation on a Polish space X. Let κ be a measurable cardinal. If $\kappa(E) < \infty$ then $\kappa(E) < \kappa$.*

Proof. Suppose that there is a poset P and an E-pinned name τ on P which is not $\bar{E}$-related to any name on a poset of size $< \kappa$. We will produce a proper class of pairwise non-$\bar{E}$-related E-pinned names.

First note that the poset P and the name τ can be selected so that $|P| = \kappa$. Simply take an elementary submodel M of size κ of large structure with $V_\kappa \subset M$

and consider $Q = P \cap M$ and $\sigma = \tau \cap M$; so $|Q| = \kappa$. As M is correct about pinned names and the equivalence $\bar{E}$ by a Shoenfield absoluteness argument, σ is an E-pinned name on Q and it is not $\bar{E}$-equivalent to any pinned names on posets of size $< \kappa$.

Thus, assume that the poset P has size κ. Let $j : V \to N$ be any elementary embedding into a transitive model with critical point equal to κ. Note that $H(\kappa) \subset N$, so both P, τ are (isomorphic to) elements of N. Let $\langle N_\alpha, j_{\beta\alpha} : \beta \in \alpha \rangle$ be the usual system of iteration of the elementary embedding j along the ordinal axis. Let $P_\alpha = j_{0\alpha}(P)$ and $\tau_\alpha = j_{0,\alpha}(\tau)$. It will be enough to show that the pairs $\langle P_\alpha, \tau_\alpha \rangle$ for $\alpha \in \mathrm{Ord}$ are pairwise $\bar{E}$-unrelated. To see this, pick ordinals $\alpha \in \beta$. As the original poset had size κ, it is the case that $P_\alpha, \tau_\alpha, P_\beta, \tau_\beta$ are in the model N_β. By the elementarity of the embedding $j_{0\beta}$, $N_\beta \models \neg \langle P_\alpha, \tau_\alpha \rangle \ \bar{E} \ \langle P_\beta, \tau_\beta \rangle$. The wellfounded model N_β is correct about $\bar{E}$ by a Shoenfield absoluteness argument, so $\langle P_\alpha, \tau_\alpha \rangle \ \bar{E} \ \langle P_\beta, \tau_\beta \rangle$ fails also in V as required. □

Unlike the previous theorems in this section, we do not have a complementary example showing that the measurable cardinal bound is, at least to some extent, optimal.

The last theorem in this section provides an estimate of the λ cardinal for unpinned equivalence relations. The well-known Silver dichotomy says that every Borel equivalence relation with uncountably many classes has in fact $2^{\aleph_0}$ many classes. We would like to show that every unpinned Borel equivalence relation has at least $2^{\aleph_1}$ many virtual classes. However, in ZFC this is still open. The best we can do is the following.

Theorem 2.5.11. *Let E be an unpinned Borel equivalence relation. Let κ be an inaccessible cardinal. Then* $\mathrm{Coll}(\omega, <\kappa) \Vdash \lambda(E, \mathrm{Coll}(\omega, \omega_1)) = 2^{\aleph_1}$.

Proof. Let $A \subset \mathcal{P}(\kappa)$ be a set of size 2^κ such that all elements of A have cardinality κ and any two distinct elements of A have intersection of cardinality $< \kappa$. Recall that the poset $P = \mathrm{Coll}(\omega, <\kappa)$ is a finite support product of posets $\mathrm{Coll}(\omega, \alpha)$ for $\alpha \in \kappa$. For every set $a \subset \kappa$ write $P_a = \Pi_{\alpha \in a} \mathrm{Coll}(\omega, \alpha) \subset P$. Let $G \subset \mathrm{Coll}(\omega, <\kappa)$ be a generic filter. Use Theorem 2.7.1 to argue that for every $a \in A$, $V[G \cap P_a] \models E$ is unpinned, and let $\langle Q_a, \tau_a \rangle \in V[G \cap P_a]$ be a nontrivial E-pinned name with its attendant poset. It will be enough to show that the pairs $\langle Q_a, \tau_a \rangle$ for $a \in A$ represent pairwise distinct virtual classes.

Suppose towards a contradiction that for some $a \neq b$ in the set A the E-pinned names τ_a, τ_b are equivalent. Let $c = a \cap b$ and work in the model $V[G \cap P_c]$. The assumptions imply that the poset $(P_{a \setminus c} * \dot{Q}_a) \times (P_{b \setminus c} * \dot{Q}_b)$ forces (at least below some condition) that $\tau_a \ E \ \tau_b$. It follows that the name τ_a on the poset $P_{a \setminus c} * \dot{Q}_a$ is E-pinned. Since the equivalence relation E is Borel, it follows that that the virtual class of $\langle P_{a \setminus c} * \dot{Q}_a, \tau_a \rangle$ is also represented by some pair $\langle R, \sigma \rangle$ by some poset of size $< \beth_{\omega_1} < \kappa$ by Theorem 2.5.6. However, in the model $V[G \cap P_a]$, there is a filter $H \subset R$ generic over the model $V[G \cap P_c]$. Thus, it would have to be the case that in the model $V[G \cap P_a]$, $Q_a \Vdash \tau_a \ E \ \sigma/H$, contradicting the nontriviality of the name τ_a. □

Example 2.5.12. As with the Silver dichotomy, the conclusion of Theorem 2.5.11 fails for analytic equivalence relations. Consider the case of the equivalence relation $\mathbb{E}_{\omega_1}$ and an inaccessible cardinal κ such that $2^\kappa > \kappa^+$. In the $\mathrm{Coll}(\omega, <\kappa)$ extension, $\aleph_2 < 2^{\aleph_1}$ will hold; at the same time, $\mathbb{E}_{\omega_1}$-names realized on $\mathrm{Coll}(\omega, \omega_1)$ are classified by ordinals of cardinality $\aleph_1$, so there are only $\aleph_2$-many of them.

2.5c Cardinal arithmetic examples

The cardinal invariants $\kappa(E)$ and $\lambda(E)$ provide a basis on which equivalence relations can be compared with uncountable cardinals of all sorts. In this section, we will introduce several jump operations which have direct translations into cardinal arithmetic operations. Using this approach, one can formally encode statements such as the failure of the singular cardinal hypothesis into reducibility results between Borel or analytic equivalence relations. For brevity, given a coanalytic class Γ of structures on ω invariant under isomorphism, we write $\kappa(\Gamma)$ for $\kappa(\mathbb{E}_\Gamma)$ in this section.

The constructions in this section depend on certain types of jump operators on structures and equivalence relation. They are all provisionally denoted by the + sign, not to be confused with the Friedman–Stanley jump. The first jump operation on equivalence relations we will consider translates into the powerset operation using the pinned cardinal:

Definition 2.5.13. Let Γ be a coanalytic class of structures on ω, invariant under isomorphism. Γ^+ is the class of structures on ω of the following description: there is a partition $\omega = a \cup b$ into two infinite sets, there is a Γ-structure on a, and there is an extra relation R on $b \times a$ such that the vertical sections R_m for $m \in b$ are pairwise distinct subsets of a.

Proposition 2.5.14. *Let Γ be a coanalytic class of structures on ω, invariant under isomorphism.*

1. *Γ^+ is coanalytic, and if Γ is Borel then so is Γ^+;*
2. *if Γ consists of rigid structures, then so does Γ^+;*
3. *if Γ consists of rigid structures and $\kappa(\Gamma) = \lambda^+$ then $\kappa(\Gamma^+) = (2^\lambda)^+$.*

Proof. The first two items are obvious. For (3), suppose that $\kappa(\Gamma) = \lambda^+$. Then there must be a Γ^{**} structure M of size λ and no Γ^{**} structures of larger size. By Theorem 2.4.5 and (2), every virtual $\mathbb{E}_\Gamma$-class is associated with a Γ^{**} structure. Every $(\Gamma^+)^{**}$-structure consists of a Γ^{**}-structure and some family of its pairwise distict subsets; the largest such structure then is of size 2^λ exactly. This completes the proof. □

Corollary 2.5.15. *For every countable ordinal α there is a Borel class Γ of rigid structures such that $\kappa(\mathbb{E}_\Gamma) = \beth_\alpha^+$.*

Proof. By transfinite recursion on α define Borel classes Γ_α consisting of rigid structures as follows. Let Γ_0 be the class of structures isomorphic to $\langle\omega,\in\rangle$. Let $\Gamma_{\alpha+1}=(\Gamma_\alpha)^+$. For a limit cardinal α let Γ_α be the class of structures which consist of exactly one copy of a structure in class Γ_β for each $\beta\in\alpha$. It is not difficult to prove by induction on α using the Proposition 2.5.14 at the successor stage and Theorem 2.4.5 at the limit stage that $\kappa(\mathbb{E}_\Gamma)=\beth_\alpha^+$ as desired. □

Definition 2.5.16. Let Γ be a coanalytic class consisting of structures on ω invariant under isomorphism. Γ^+ is the class of structures on ω of the following description. A structure $M\in\Gamma^+$ has a linear order $\leq_M$. Moreover, writing L_n for the set $\{m\in M\colon m\leq_M n\}$, the structure M induces a Γ-structure M_n on L_n for coboundedly many $n\in M$ such that for distinct elements $n_0,n_1\in M$, the structures $\langle M_{n_0},\leq_M\rangle$ and $\langle M_{n_1},\leq_M\rangle$ are nonisomorphic.

Proposition 2.5.17. *Let Γ be a coanalytic class of structures on ω, invariant under isomorphism.*

1. *Γ^+ is coanalytic;*
2. *if Γ is Borel and consists of rigid structures, then Γ^+ is Borel and consists of rigid structures;*
3. *if Γ consists of rigid structures, then $\kappa(\Gamma^+)=\kappa(\Gamma)^+$.*

Proof. (1) is clear. For (2), the rigidity conclusion is clear. To see the Borelness of the class Γ^+ note that the statement "$\langle M,\leq_M\rangle$ is isomorphic to $\langle N,\leq_N\rangle$" for structures $M,N\in\Gamma$ is Borel by the rigidity of the structures in Γ and the Lusin–Suslin theorem [55, Theorem 15.1]. For (3), write $\kappa=\kappa(\Gamma)$. To show that $\kappa(\Gamma^+)\leq\kappa^+$, use Theorem 2.4.5 to argue that every virtual E_{Γ^+} class is classified by $(\Gamma^+)^{**}$-structure. Such a structure contains a linear ordering, and on cofinally many initial segments of the ordering there is a Γ^{**}-structure. It follows that every initial segment of the ordering is of cardinality less than κ, so the underlying set has size at most κ.

To show that $\kappa(\Gamma^+)\geq\kappa^+$, treat first the case that κ is a limit cardinal. Let M be the structure on κ which includes the usual $\in$-ordering on κ and on each cardinal $\lambda<\kappa$ on which there is a Γ^{**}-structure, M contains one. This is a $(\Gamma^+)^{**}$-structure of size κ, so $\kappa(\Gamma^+)\geq\kappa^+$.

Suppose now that κ is a successor cardinal, $\kappa=\lambda^+$. This means that there is a Γ^{**}-structure N of size λ. Let M be the structure on κ which includes the usual $\in$-ordering of κ and on each ordinal $\mu<\kappa$ of cardinality λ, it contains a copy of N. This is a $(\Gamma^+)^{**}$-structure of size κ, so $\kappa(\Gamma^+)\geq\kappa^+$ in this case as well. The proof is complete. □

Corollary 2.5.18. *For every countable ordinal $\alpha>0$ there is a Borel equivalence relation E such that (provably) $\kappa(E)=\aleph_\alpha$.*

Proof. Now, by recursion on a countable ordinal α construct a Borel class Γ_α of rigid structures as follows. Γ_0 is the set of structures isomorphic to $\langle\omega,\in\rangle$, containing just one isomorphism class. Then let $\Gamma_{\alpha+1} = \Gamma_\alpha^+$ and $\Gamma_\alpha = \bigcup_{\beta\in\alpha}\Gamma_\beta$ if α is limit. Proposition 2.5.17 can be used to show by transfinite induction that $\kappa(\Gamma_\alpha) = \aleph_{1+\alpha}$ as desired. $\square$

Note that an equivalence relation E as above for $\alpha \geq 2$ cannot be reducible to $\mathbb{F}_2$ and $\mathbb{F}_2$ cannot be reducible to it. This answers a question of Kechris [48, Question 17.6.1] in the negative as well as some related questions of Simon Thomas. To see that $\mathbb{F}_2$ cannot be Borel reducible to any E, suppose for contradiction that $h\colon \mathrm{dom}(E) \to \mathrm{dom}(\mathbb{F}_2)$ is a Borel reduction. Pass to a generic extension in which $\mathfrak{c} > \aleph_{\omega_1}$. There, h is still a reduction of E to F, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 2.5.4. To see that E cannot be reducible to $\mathbb{F}_2$ for any $\alpha > 2$, pass to a generic extension in which the Continuum Hypothesis holds instead.

The following two examples deal with jump operations designed to mimic cardinal exponentiation. They lead to nonreducibility results which, at least on the face of it, use the failure of singular cardinal hypothesis in various situations. This means that the proofs presented use large cardinal assumptions, as they are needed to get the failure of the singular cardinal hypothesis. We make no claim as to whether the large cardinal assumptions are necessary for the conclusion.

Definition 2.5.19. Let Γ, Δ be coanalytic classes of structures on ω, invariant under isomorphism. The symbol Γ^Δ stands for the coanalytic class of structures M on ω of the following form: ω is partitioned into infinite sets $\omega = a \cup b \cup c$, on $M \restriction a$ is a structure in class Γ, $M \restriction b$ is a structure in class Δ, and there is an extra relation $R \subset c \times b \times a$ such that for every $m \in c$, the vertical section R_m is a function from b to a, and for $m_0 \neq m_1 \in c$ the vertical sections R_{m_0} and R_{m_1} are distinct.

Proposition 2.5.20. *Suppose that the classes Γ, Δ consist of rigid structures only. Then*

1. *Γ^Δ consists of rigid structures only;*
2. *$\kappa(\Gamma^\Delta) = \sup\{\kappa^\lambda\colon \kappa < \kappa(\Gamma), \lambda < \kappa(\Delta)\}$.*

Proof. The first item is nearly trivial. For the second item, use Theorem 2.4.5 to note that a $(\Gamma^\Delta)^{**}$ structure is represented by a Γ^{**}-structure, a Δ^{**}-structure, and an infinite set of functions from the latter to the former. $\square$

Example 2.5.21. Let Δ be the class of structures isomorphic to $\langle\omega, \in\rangle$ (consisting of just one equivalence class). For every countable ordinal α, let Γ_α be the class derived in Corollary 2.5.18. Let E be the equivalence relation of isomorphism on the class $(\Gamma_\omega)^\Delta$ and F be the isomorphism equivalence relation on the class $\mathbb{E}_{\Gamma_\alpha}$. Then E is not Borel reducible to $\mathbb{F}_2 \times F$.

Proof. Move to a model of ZFC where $\mathfrak{c} = \aleph_1$ and $\aleph_\omega^{\aleph_0} > \aleph_\alpha$. Proposition 2.5.20 shows that $\kappa(E) = \aleph_\omega^{\aleph_0} > \mathfrak{c} \cdot \aleph_\alpha = \kappa(\mathbb{F}_2 \times F)$. The conclusion of the example follows from Theorem 2.5.4. $\square$

Definition 2.5.22. Let E be an analytic equivalence relation on a Polish space X and F be a Borel equivalence relation on a Polish space Y. E^F is the equivalence relation $(F \times E)^+$ on the space $(Y \times X)^\omega$ restricted to the Borel set $\{z \in (Y \times X)^\omega \colon \mathrm{rng}(z)$ is a partial function from Y to X whose domain consists of pairwise F-unrelated elements$\}$.

It seems to be impossible to formulate this concept in an analytic form without the additional demand that F be Borel.

Proposition 2.5.23. *Let E be an analytic equivalence relation on a Polish space X and F be a Borel equivalence relation on a Polish space Y. Then*

1. $\lambda(E^F) = \lambda(E)^{\lambda(F)}$*;*
2. *if P is a partial order such that $P \Vdash |\lambda(F)| = |\kappa(F)| = \aleph_0$ then $\lambda(E^F, P) = \lambda(E, P)^{\lambda(F)}$.*

Note that as F is assumed to be Borel, the value of $\lambda(F)$ is not ∞.

Proof. Both statements follow from the classification of virtual classes for product and Friedman–Stanley jump in Theorem 2.3.4. An E^F-virtual class is represented by a function from F-virtual classes to E-virtual classes. □

Example 2.5.24. Let F_0 be any Borel equivalence relation with countably many classes and let F_1 be the identity on 2^ω. For every Borel equivalence relation E, $\mathbb{E}_{\omega_1}^{F_1}$ is not Borel reducible to $E \times \mathbb{E}_{\omega_1}^{F_0}$.

Proof. Move to a model of set theory where $\mathfrak{c} = \aleph_1$ and there is a cardinal $\kappa > \beth_{\omega_1}$ such that $(\kappa^+)^{\aleph_1} > (\kappa^+)^{\aleph_0}$. The first such a cardinal must be in violation of the singular cardinal hypothesis at cofinality ω_1. Let $P = \mathrm{Coll}(\omega, \kappa)$. Clearly, $\lambda(\mathbb{E}_{\omega_1}, P) = \kappa^+$, since the pinned names on P correspond to ordinals below κ^+. Since E is Borel, $\lambda(E), \kappa(E) < \beth_{\omega_1} < \kappa$ by Theorem 2.5.6, so Proposition 2.5.23 shows that $\lambda(E \times \mathbb{E}_{\omega_1}^{F_0}) = \lambda(E) \cdot (\kappa^+)^{\aleph_0} = (\kappa^+)^{\aleph_0} < (\kappa^+)^{\aleph_1} = \lambda(\mathbb{E}_{\omega_1}^{F_1})$. The argument is concluded by a reference to Theorem 2.5.4. □

A similar argument with a failure of the singular cardinal hypothesis at larger cofinalities yields

Example 2.5.25. Let F be any Borel equivalence relation. For every Borel equivalence relation E, $\mathbb{E}_{\omega_1}^{F^+}$ is not Borel reducible to $E \times \mathbb{E}_{\omega_1}^{F}$.

2.5d Hypergraph examples

The previous examples were to some extent artificial in the sense that the values of the pinned cardinal were directly built into them. The following examples, all Borel reducible to $\mathbb{F}_2$, are connected with combinatorics of small uncountable cardinals via the pinned cardinal, even though the connection is not at first sight obvious.

Definition 2.5.26. A *hypergraph* on a set X is a subset of $X^{\leq\omega}$. If $\mathcal{G}$ is a hypergraph on X, a *$\mathcal{G}$-anticlique* is a set $A \subset X$ such that $A^{\leq\omega} \cap \mathcal{G} = 0$. If $\mathcal{G}$ is an analytic hypergraph on a Polish space X, write $E_\mathcal{G}$ for the equivalence relation on X^ω connecting y, z if the sets $\operatorname{rng}(y), \operatorname{rng}(z)$ either both fail to be $\mathcal{G}$-anticliques or they are equal.

It is clear that the equivalence relations of the form $E_\mathcal{G}$ are all analytic and almost Borel reducible to $\mathbb{F}_2$. In the common case when $\mathcal{G}$ is Borel and contains only finite edges, the equivalence relation is in fact Borel. The following proposition provides a simple combinatorial characterization of the pinned cardinal of equivalence relations of this form.

Proposition 2.5.27. *Let $\mathcal{G}$ be an analytic hypergraph on a Polish space X. Then $\kappa(E_\mathcal{G})$ is equal to the larger of $\aleph_1$ and the minimal cardinal κ such that there is no $\mathcal{G}$-anticlique of size κ.*

Proof. Write $E = E_\mathcal{G}$ and write κ for the minimum cardinality in which there is no $\mathcal{G}$-anticlique. To exclude trivial cases, assume that κ is uncountable. For the $\leq$-inequality in the proposition, let Q be a poset and σ be an E-pinned Q-name. By Example 2.3.5, there must be a set $A \subset X$ such that $Q \Vdash \operatorname{rng}(\sigma) = \check{A}$. If A is not an anticlique then the E-pin $\langle Q, \sigma\rangle$ is trivial as it is forced to belong to the single E-class consisting of enumerations of sets which are not anticliques. If A is an anticlique, then the E-pin $\langle Q, \sigma\rangle$ is equivalent to a pin on the $\operatorname{Coll}(\omega, |A|)$-poset which has size less than κ.

For the $\geq$-inequality, suppose that $A \subset X$ is an anticlique. A simple wellfoundedness argument shows that A remains an anticlique in any generic extension. Thus, the $\operatorname{Coll}(\omega, A)$-name τ_A for a generic enumeration of A is E-pinned an the E-pin $\langle \operatorname{Coll}(\omega, A), \tau_A\rangle$ can be equivalent only to E-pins on posets which make the cardinality of A countable. $\square$

The values of cardinals defined in this way are subject to forcing manipulations and transfinite combinatorics. The interesting examples are always connected with a universality feature of the hypergraph in question. Our first two examples use Borel hypergraphs of finite arity.

Definition 2.5.28. Let X be a Polish space. A Borel relation $R \subset [X]^{<\aleph_0} \times X$ is *combinatorially universal* if it has countable vertical sections and for every cardinal κ and every relation $T \subset \kappa^{<\aleph_0} \times \kappa$ with countable vertical sections, there is a c.c.c. poset P adding an injective function $\pi\colon \kappa \to X$ which is a homomorphism of T to R.

It is not clear whether combinatorial universality of this sort is actually a property absolute among transitive models of ZFC, but all universal examples found in this section are absolutely universal.

Theorem 2.5.29. *Let X be a Polish space. There is a combinatorially universal Borel relation on $[X]^{<\aleph_0} \times X$ with countable vertical sections.*

Proof. Let $X = \mathcal{P}(\omega)$. We will show that the relation $R \subset [X]^{<\aleph_0} \times X$ defined by $\langle a, x\rangle \in R$ if x is computable from a is universal. Let κ be a cardinal and let $T \subset [\kappa]^n \times \kappa$ be a relation with countable vertical sections. Let $\langle k_m : m \in \omega\rangle$ be a recursive sequence of increasing functions in ω^ω with disjoint ranges. For a finite set $b \subset \mathcal{P}(\omega)$ let e_b be the increasing enumeration of the set $\bigcap b$, for every $m \in \omega$ let $h_m(b) \subset \mathcal{P}(\omega)$ be the set of all $l \in \omega$ such that $e_b \circ k_m(l)$ is an odd number. We will produce a forcing which adds an injection $\pi : \kappa \to X$ such that for every finite set $a \subset \kappa$ and every $\alpha \in T_a$, there is a number $m \in \omega$ such that $\pi(\alpha)$ is modulo finite equal to $h_m(\pi'' a)$.

Let P be the poset of all tuples $p = \langle n_p, \pi_p, \nu_p\rangle$ so that

- $n_p \in \omega$, π_p is a partial function from κ to $\mathcal{P}(n_p)$ with finite domain $\mathrm{dom}(p)$;
- ν_p is a finite partial function from $\mathcal{P}(\mathrm{dom}(p)) \times \omega$ to $\mathrm{dom}(p)$ such that $(a, \nu_p(a, m)) \in T$ whenever $\langle a, m\rangle \in \mathrm{dom}(\nu_p)$.

The ordering on P is defined by $q \leq p$ if $n_p \leq n_q$, $\mathrm{dom}(p) \subset \mathrm{dom}(q)$, $\forall \alpha \in \mathrm{dom}(p)\ \pi_p(\alpha) = \pi_q(\alpha) \cap n_p$, $\nu_p \subset \nu_q$, and for every $\langle a, m\rangle \in \mathrm{dom}(\nu_p)$, whenever l is a number in the domain of $(e_{\pi_q'' a} \setminus e_{\pi_q'' a}) \circ k_m$ then $e_{\pi_q'' a} \circ k_m(l)$ is odd if and only if $l \in \pi_q(\nu_p(a, m))$. It is not difficult to see that P is indeed an ordering.

Claim 2.5.30. *The poset P is c.c.c.*

Proof. Let $\langle p_\alpha : \alpha \in \omega_1\rangle$ be conditions in P. The usual Δ-system and counting arguments can be used to thin down the collection if necessary so that the sets $\mathrm{dom}(p_\alpha)$ for $\alpha \in \omega_1$ form a Δ-system with root b and for all $a \in [b]^n$ and all $\alpha \in \omega_1$, $T_a \cap \mathrm{dom}(p_\alpha) \subset b$. Moreover, we can require that the increasing bijection between $\mathrm{dom}(p_\alpha)$ and $\mathrm{dom}(p_\beta)$ extends to an isomorphism of p_α and p_β for every $\alpha, \beta \in \omega_1$.

We claim that any two conditions in such a collection are compatible. Indeed, whenever $\alpha, \beta \in \omega_1$, then the condition q defined by $n_q = n_{q_\alpha}$, $\pi_q = \pi_{p_\alpha} \cup \pi_{p_\beta}$ and $\nu_q = \nu_{p_\alpha} \cup \nu_{p_\beta}$ is easily checked to be a common lower bound of the conditions p_α, p_β. ☐

Claim 2.5.31. *Whenever $a \in [\kappa]^n$ and $\beta \in T_a$, the set $D_{a,\beta} = \{p \in P : a \cup \{\beta\} \subset \mathrm{dom}(p), \exists m\ \nu_p(a, m) = \beta\}$ is dense in P.*

Proof. Let $p \in P$; we must find a condition $q \leq p$ in the set $D_{a,\beta}$. For definiteness assume that $\beta \notin \mathrm{dom}(p)$. Choose $m \in \omega$ such that $\langle a, m\rangle \notin \mathrm{dom}(\nu_p)$. Consider the condition $q \leq p$ defined by $n_q = n_p$, $\pi_q = \pi_p \cup \{\langle \alpha, 0\rangle : \alpha \in a \setminus \mathrm{dom}(p), \langle \beta, 0\rangle\}$, $\nu_q = \nu_p \cup \{\langle a, m, \beta\rangle\}$. The condition $q \leq p$ is in the set $D_{a,\beta}$ as required. ☐

Claim 2.5.32. *For every finite set $a \subset \kappa$ and every $k \in \omega$, the set $D_{a,k} = \{p \in P : a \subset \mathrm{dom}(p)$ and the set $\bigcap \pi_p'' a$ has at least k elements$\}$ is dense in P.*

Proof. Fix a, k and let $p \in P$ be an arbitrary condition. We must find a condition $q \leq p$ in the set $D_{a,k}$. First of all, the previous claim shows that one can strengthen p to include all ordinals in a. Increasing n_p if necessary, we may also assume that $k < n_p$.

Consider the set $b = \pi_p'' a$ and the function e_b; write $k' = \text{dom}(e_b)$. If $k \leq k'$ then $q = p$ will work. Otherwise, it is easy to find an increasing sequence $d = \langle m_i \colon k' \leq i < k \rangle$ of numbers larger than n_p such that, writing $e = e_b \cup d$, for every natural number m such that $\langle a, m \rangle \in \text{dom}(\nu_p)$ and every l such that $k' \leq k_m(l) < k$, $m_{k_m(l)}$ is odd if and only if $l \in p(\nu_r(a, m))$. The condition $q \leq p$ defined by $n_q = m_{k-1} + 1$, $\text{dom}(\pi_q) = \text{dom}(\pi_p)$, $\forall \beta \in a\ \pi_q(\beta) = \pi_p(a) \cup \{m_i \colon k' \leq i < k\}$, $\forall \beta \in \text{dom}(\pi_p) \setminus a\ \pi_q(\beta) = \pi_p(\beta)$, and $\nu_q = \nu_p$, is in the set $D_{a,k}$ as desired. $\square$

The last two claims show that the function $\pi \colon \kappa \to X$ defined as $\pi(\alpha) = \bigcup\{\pi_p \colon p \text{ is in the generic filter}\}$ is forced by P to be the desired homomorphism. $\square$

Recall that if $R \subset [X]^{\aleph_0} \times X$ is a relation then $a \subset X$ is *R-free* if for every $x \in a$ $\langle a \setminus \{x\}, x \rangle \notin R$.

Example 2.5.33. Let X be a Polish space and let $R \subset [X]^{<\aleph_0} \times X$ be a combinatorially universal Borel relation with countable vertical sections. Let $n \geq 1$ be a number. Let $\mathcal{G}_n$ be the hypergraph of all sets $a \in [X]^n$ which are R-free. Then

1. $\kappa(E_{\mathcal{G}_n}) \leq \aleph_{n+1}$;
2. if Martin's Axiom for $\aleph_n$ holds then equality is attained.

Proof. Fix the number $n \in \omega$. The argument depends on an old theorem of Sierpiński [35]: $\aleph_{n+1}$ is the smallest cardinal κ such that every relation $T \subset [\kappa]^n \times \kappa$ with countable vertical sections has a T-free $n+1$-tuple. To prove (1), apply the Sierpiński theorem to show that there is no $\mathcal{G}_n$-anticlique of size $\geq \aleph_{n+1}$. To prove (2), let $\kappa = \aleph_n$ and use the Sierpiński theorem again to find a relation $T \subset \kappa^{<\aleph_0} \times \kappa$ with countable vertical sections and no T-free n-tuple. Then use the universality assumption to find an injective homomorphism $\pi \colon \kappa \to X$ which is a homomorphism of T to R. Observe that $\text{rng}(\pi)$ is a $\mathcal{G}_n$-anticlique of size κ. Proposition 2.5.27 then implies that $\kappa(E_{\mathcal{G}_n}) \geq \kappa^+ = \aleph_{n+1}$. $\square$

Definition 2.5.34. Let X be a Polish space. A Borel equivalence relation R on $[X]^{<\aleph_0}$ with countably many classes is *combinatorially universal* if for every cardinal κ and every equivalence relation T on $[\kappa]^{<\aleph_0}$ with countably many classes, there is a c.c.c. poset P adding an injection $\pi \colon \kappa \to X$ which is a homomorphism of $\neg T$ to $\neg R$.

Theorem 2.5.35. *Let X be a Polish space. There is a combinatorially universal Borel equivalence relation on $[X]^{<\aleph_0}$ with countably many classes.*

Proof. Without loss of generality assume $X = [\omega]^{\aleph_0}$. Consider the following relation R on $[X]^{<\aleph_0}$. Define a Borel function $g\colon [X]^{<\aleph_0} \to [\omega]^{<\aleph_0}$ by $g(a) = \{\min(x \setminus m+1)\colon x \in a\}$ if a is a set of size at least two and consists of pairwise almost disjoint sets and m is the largest number which appears in at least two of them; $g(a) = \min(x)$ if $a = \{x\}$ is a singleton; and otherwise $g(a) = 0$. Let R be the equivalence relation induced by the function g. We will show that R is universal.

Let κ be a cardinal, T an equivalence relation on $[\kappa]^{<\aleph_0}$ with countably many classes, and let $f\colon [\kappa]^{<\aleph_0} \to \omega$ be a map inducing the equivalence relation T. Let $\nu\colon [\omega]^{<\aleph_0} \to \omega$ be a sufficiently generic map such that $\nu(g(0)) = f(0)$. Let P be the poset of all maps p such that

- $\mathrm{dom}(p) \subset \kappa$ is a finite set;
- for every $\alpha \in \mathrm{dom}(p)$ the value $p(\alpha)$ is a nonempty subset of ω;
- for every $\alpha \in \mathrm{dom}(p)$, $\nu(\min(p(\alpha))) = f(\alpha)$;
- for every set $a \subset \mathrm{dom}(p)$ of size at least 2 there is a number which belongs to at least two sets $p(\alpha), p(\beta)$ for $\alpha \neq \beta \in a$, and writing m for the largest such number, $p(\alpha) \setminus m+1 \neq 0$ holds for every $\alpha \in a$, and $\nu(\{\min(p(\alpha) \setminus m+1)\colon \alpha \in a\}) = f(a)$.

The ordering is defined by $q \leq p$ if for every $\alpha \in \mathrm{dom}(p)$, $q(\alpha)$ end-extends $p(\alpha)$, and the sets $q(\alpha) \setminus p(\alpha)$ are pairwise disjoint for $\alpha \in \mathrm{dom}(p)$.

Claim 2.5.36. *P is c.c.c.*

Proof. In fact, P is semi-Cohen in the sense of [4], but we will not need that fact here. By the usual Δ-system arguments, it is enough to show that any two conditions $p, q \in P$ such that $p \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q) = q \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q)$, are compatible. To find the lower bound, enumerate $\mathrm{dom}(p) \cup \mathrm{dom}(q)$ as β_i for $i \in k$, enumerate $(\mathrm{dom}(p) \setminus \mathrm{dom}(q)) \times (\mathrm{dom}(q) \setminus \mathrm{dom}(p))$ as u_j for $j \in l$. Use the genericity of the function ν to build numbers $m_0 < m_1 < \cdots < m_{l-1}$ and pairwise distinct numbers n_i^j for $i \in k$ and $j \in l$ so that

- $m_0 > \max(\bigcup \mathrm{rng}(p) \cup \bigcup \mathrm{rng}(q))$;
- $m_j < n_i^j < m_{j-1}$ for every $j \in l$;
- for every set $a \subset \mathrm{dom}(p) \cup \mathrm{dom}(q)$, $\nu(\{n_i^j\colon \beta_i \in a\}) = f(a)$.

The lower bound is then a function r defined by $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$, for $\alpha \in \mathrm{dom}(p)$, $\alpha = \beta_i$ set $r(\alpha) = p(\alpha) \cup \{n_i^j\colon j \in l\} \cup \{m_j\colon \alpha$ appears in the pair $u_j\}$. Similarly, for $\alpha \in \mathrm{dom}(q)$, $\alpha = \beta_i$ set $r(\alpha) = q(\alpha) \cup \{n_i^j\colon j \in l\} \cup \{m_j\colon \alpha$ appears in the pair $u_j\}$. It is not difficult to check that $r \leq p, q$ as required. □

Claim 2.5.37. *The set $D_\alpha = \{p \in P\colon \alpha \in \mathrm{dom}(p)\}$ is dense in P for every $\alpha \in \kappa$.*

Proof. Let $\alpha \in \kappa$ and $p \in P$; we must produce $q \leq p$ such that $\alpha \in \mathrm{dom}(q)$. Enumerate $\mathrm{dom}(p)$ as β_i for $i \in k$ and write $\alpha = \beta_k$. Use the genericity of the function ν to find numbers $m < m_0 < m_1 < \dots m_{k-1}$ and pairwise distinct numbers n_i^j for $i \in k, j \in k+1$ so that

- $\nu(m) = f(\alpha)$ and $m_0 > \max \bigcup \mathrm{rng}(p)$;
- $m_i < n_i^j$ for every $j \in k+1$;
- for every nonempty set $a \subset \mathrm{dom}(p) \cup \{\alpha\}$ and every $i \in k$, $\nu(\{n_i^j \colon \beta_j \in a\}) = f(a)$.

Once this is done, just consider the function q defined by $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\alpha\}$, $s(\alpha) = \{m, m_i \colon i \in k, n_i^k \colon i \in k\}$, and for every $i \in k$, $q(\beta_i) = p(\beta_i) \cup \{m_i, n_j^i \colon j \in k\}$. It is not difficult to observe that $q \in P$ and $q \leq p$ as desired. □

Now, if $G \subset P$ is a generic filter, then in $V[G]$ let $\pi \colon \kappa \to \mathcal{P}(\omega)$ be defined by $\pi(\alpha) = \bigcup_{p \in G} p(\alpha)$. The claims show that $f = \nu \circ g \circ \pi$, in particular π is a homomorphism of $\neg T$ to $\neg R$. □

Example 2.5.38. Let X be a Polish space and R a combinatorially universal equivalence relation on $[X]^{<\aleph_0}$ with countably many classes. Let $n > 0$ be a number. Let $\mathcal{G}_n$ be the Borel hypergraph on X consisting of all finite sets $a \subset X$ which can be written in more than $2^n - 1$ ways as a union of two distinct R-equivalent sets. Then

1. $\kappa(E_{\mathcal{G}_n}) \leq \aleph_{n+1}$;
2. if Martin's Axiom for $\aleph_n$ holds, then the equality is attained.

Proof. Fix the number n. The computations depend on and are motivated by the following results of Komjáth and Shelah [60]. Let $\kappa = \aleph_n$. For every equivalence relation T on $[\kappa]^{<\aleph_0}$ with countably many classes, there is a finite set $a \subset \kappa$ which can be written in at least $2^n - 1$ ways as a union of two distinct T-related sets. In addition, if Martin's Axiom for κ holds then there is an equivalence relation T on $[\kappa]^{<\aleph_0}$ with countably many classes such that every finite set $a \subset \kappa$ can be written in at most $2^n - 1$ ways as a union of two distinct T-related sets.

For (1), the first part of this result shows that there is no $\mathcal{G}_n$-anticlique of size κ^+. For (2), use Martin's Axiom and the second part of the Komjáth–Shelah result to find the equivalence relation T on $[\kappa]^{<\aleph_0}$ with countably many classes as above, and use the universality of the relation R to find an injective homomorphism $\pi \colon \kappa \to X$ of $\neg T$ to $\neg R$. It is immediate that $\mathrm{rng}(\pi) \subset X$ is a $\mathcal{G}_n$-anticlique of size κ. Proposition 2.5.27 then implies that $\kappa(E_{\mathcal{G}_n}) \geq \kappa^+ = \aleph_{n+1}$. □

Much more complicated effects can be realized if the hypergraph $\mathcal{G}$ is allowed to have infinite edges. We conclude this section with an example of this type.

Definition 2.5.39. Let X be a Polish space. A Borel equivalence relation R on X^2 with countably many classes is *combinatorially universal* if for every ordinal κ and every equivalence relation T on κ^2 with countably many classes, there is a c.c.c. poset P adding an injective function $\pi\colon \kappa \to X$ which is a homomorphism of $\neg T$ to $\neg R$.

Theorem 2.5.40. *There is a combinatorially universal Borel equivalence relation with countably many classes on X^2 for every Polish space X.*

Proof. Without loss of generality, assume $X = \mathcal{P}(\omega)$. Let $\omega = \bigcup_{n,m\in\omega} a_{n,m}$ be a partition of ω into infinite sets. For almost disjoint sets $b, c \subset \omega$ such that b is lexicographically less than c define $f(b, c) = n$ and $f(c, b) = m$ if $\max(b \cap c) \in a_{n,m}$, in other cases define $f(b, c) = 0$. Let R be the equivalence relation induced by the function f. We will show that R is combinatorially universal.

Fix a cardinal κ and a function $g\colon \kappa^2 \to \omega$ which induces an equivalence relation T with countably many classes. Define the poset P as the collection of all functions p such that

- $\mathrm{dom}(p) \subset \kappa$ is a finite set;
- $\mathrm{rng}(p)$ consists of finite subsets of ω such that neither of them is an initial segment of another;
- for every $\alpha \neq \beta$ such that $p(\alpha)$ is lexicographically smaller than $p(\beta)$, the set $p(\alpha) \cap p(\beta)$ is nonempty, and its maximum belongs to the set $a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$.

The ordering on P is defined by $q \leq p$ if $\mathrm{dom}(p) \subset \mathrm{dom}(q)$, for every $\alpha \in \mathrm{dom}(p)$ the set $p(\alpha)$ is an initial segment of $q(\alpha)$, and the sets $\{q(\alpha) \setminus p(\alpha)\colon \alpha \in \mathrm{dom}(p)\}$ are pairwise disjoint. The following routine claims complete the proof of the theorem.

Claim 2.5.41. *The poset P is c.c.c.*

Proof. By the usual Δ-system arguments it is only necessary to show that any two conditions $p, q \in P$ such that $p \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q) = q \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q)$ are compatible in the poset P. Strengthening the conditions p, q on $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$ and $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$ respectively if necessary, we may assume that no set in $\mathrm{rng}(p) \cup \mathrm{rng}(q)$ is an initial segment of another. Enumerate $(\mathrm{dom}(p) \setminus \mathrm{dom}(q)) \times (\mathrm{dom}(q) \setminus \mathrm{dom}(p))$ as u_i for $i \in j$ and find pairwise distinct numbers m_i for $i \in j$ such that

- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically smaller than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$;
- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically greater than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = n$ and $g(\beta, \alpha) = m$;
- all numbers m_i are greater than $\max(\bigcup \mathrm{rng}(p) \cup \bigcup \mathrm{rng}(q))$.

In the end, let r be the function defined by $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$, for all $\alpha \in \mathrm{dom}(p)$ let $r(\alpha) = p(\alpha) \cup \{m_i \colon \alpha \text{ appears in } u_i\}$, and for all $\beta \in \mathrm{dom}(q)$ let $r(\beta) = q(\beta) \cup \{m_i \colon \beta \text{ appears in } u_i\}$. It is not difficult to check that r is a common lower bound of the conditions p, q as desired. $\square$

Claim 2.5.42. *For every $\alpha \in \kappa$ the set $D_\alpha = \{p \in P \colon \alpha \in \mathrm{dom}(r)\}$ is dense in P.*

Proof. Let $\alpha \in \kappa$ be an ordinal and $p \in P$ be a condition; we must produce a condition $q \leq p$ such that $\alpha \in \mathrm{dom}(q)$. It will be the case that $q(\alpha) \cap \max(\bigcup \mathrm{rng}(p)) + 1 = 0$; this way, $q(\alpha)$ will be lexicographically smaller than all $q(\beta)$ for $\beta \in \mathrm{dom}(p)$. List $\mathrm{dom}(p)$ as β_i for $i \in j$, and find pairwise distinct numbers m_i for $i \in j$ so that

- $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\alpha, \beta) = n$;
- each m_i is greater than $\max(\bigcup \mathrm{rng}(p))$.

Then, let q be the function defined by $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\alpha\}$ and $q(\beta_i) = p(\beta_i) \cup \{m_i\}$ and $q(\alpha) = \{m_i \colon i \in j\}$. It is immediate that the condition q works. $\square$

Now it is easy to see that if $G \subset P$ is a generic filter, the function $\pi \colon \kappa \to \mathcal{P}(\omega)$ defined by $\pi(\alpha) = \bigcup \{p(\alpha) \colon p \in G\}$ induces a homomorphism of $\neg T$ to $\neg R$ as desired. $\square$

Example 2.5.43. Let X be a Polish space and let R be a combinatorially universal Borel equivalence relation on X^2 with countably many classes. Let $\mathcal{G}$ be the hypergraph on X consisting of all unions $b_0 \cup b_1$ where b_0, b_1 are infinite sets such that $b_0 \times b_1$ is a subset of a single R-class.

1. if Chang's conjecture holds, then $\kappa(E_{\mathcal{G}}) \leq \aleph_2$;
2. if Martin's Axiom for $\aleph_2$ holds then Chang's conjecture is equivalent to $\kappa(E_{\mathcal{G}}) \leq \aleph_2$.

Proof. The argument is based on and motivated by two results of Todorcevic [92]. Namely, if Chang's conjecture holds, then for every partition of ω_2^2 into countably many pieces, one piece of the partition contains a product of infinite sets. In addition, if Martin's Axiom for $\aleph_2$ holds and Chang's conjecture fails, then there is a partition of ω_2^2 into countably many pieces such that no piece of the partition contains a product of infinite sets.

Now, for (1) use the first result of Todorcevic to argue that under Chang's conjecture there is no $\mathcal{G}$-anticlique of size $\aleph_2$, so by Proposition 2.5.27, $\kappa(E_{\mathcal{G}}) \leq \aleph_2$. For (2), suppose that Martin's Axiom holds and Chang's conjecture fails. By the second result of Todorcevic, find an equivalence relation T on ω_2^2 with countably many classes without any monochromatic infinite rectangles. Use the universality of the equivalence relation R to find an injection $\pi \colon \omega_2 \to X$ which is a homomorphism of $\neg T$ to $\neg R$. Observe that $\mathrm{rng}(\pi)$ is a $\mathcal{G}$-anticlique of size $\aleph_2$. By Proposition 2.5.27, $\kappa(E_{\mathcal{G}}) > \aleph_2$. $\square$

2.6 Restrictions on partial orders

Given an analytic equivalence relation E on a Polish space X, it may be informative to investigate which posets can carry nontrivial E-pinned names. After all, essentially all pinned names discussed in this chapter naturally live on collapse posets, so one may easily (and wrongly) assume that no forcing sophistication is needed when it comes to the investigation of the virtual realm. This section contains several theorems on this topic.

First of all, there are some partial orders which can never carry a nontrivial pinned name.

Definition 2.6.1. [29] A poset P is *reasonable* if for every ordinal λ and for every function $f\colon \lambda^{<\omega} \to \lambda$ in the P-extension there is a set $a \subset \lambda$ which is closed under f, belongs to the ground model, and is countable in the ground model.

In particular, all c.c.c. and all proper forcings are reasonable. Good examples of unreasonable forcings are posets which collapse $\aleph_1$, Namba forcing and Prikry forcing.

Theorem 2.6.2. *Let E be an analytic equivalence relation on a Polish space X. If P is a reasonable forcing and τ is an E-pinned name on P, then τ is trivial.*

Proof. Suppose that P is a reasonable poset and τ is an E-pinned name on P. We will produce a condition $p \in P$ and a point $x \in X$ such that $p \Vdash \tau\ E\ \check{x}$. Towards this end, choose a large structure and use the reasonability of P to find a countable elementary submodel M of it containing P, E and τ and a condition $p \in P$ such that $p \Vdash \dot{G} \cap \check{M}$ is generic over $\check{M}$, where $\dot{G}$ is the canonical P-name for its generic ultrafilter. As M is countable, there is a filter $H \subset P \cap M$ generic over M in the ground model V. Let $x = \tau/H \in X$. Proposition 2.1.5 applied to the model M and the filters H and $\dot{G} \cap M$ now says that $p \Vdash \check{x}\ E\ \tau$, completing the proof. □

The key feature of partial orders from the point of view of existence of pinned names is collapsing $\aleph_1$, as the following theorem shows:

Theorem 2.6.3. *Let E be an analytic equivalence relation on a Polish space X. Exactly one of the following occurs:*

1. *E is pinned;*
2. *for every poset P collapsing $\aleph_1$, P carries a nontrivial E-pinned name.*

Proof. Clearly (1) implies the negation of (2). For the difficult direction, suppose that (1) fails and work to confirm (2). Let τ be a nontrivial E-pinned name on some poset P. Let $\langle M_\alpha \colon \alpha \in \omega_1 \rangle$ be a continuous $\in$-tower of countable elementary submodels of a large structure containing X, E, τ, and P. Let $M = \bigcup_\alpha M_\alpha$, let $Q = P \cap M$ and let $\sigma = \tau \cap M$.

First, observe that the pair $\langle Q, \sigma\rangle$ is an E-pin: by elementarity, $M \models Q\times Q \Vdash \sigma_{\text{left}}\ E\ \sigma_{\text{right}}$, and by the Mostowski absoluteness between the generic extensions of M and V, $V \models Q \times Q \Vdash \sigma_{\text{left}}\ E\ \sigma_{\text{right}}$ as well. We will now prove that the pair $\langle Q, \tau\rangle$ is a non-trivial E-pin.

Assume towards a contradiction that there exists a point $x \in X$ such that $Q \Vdash \sigma\ E\ \check{x}$. We will show that there then must be $y \in M\cap X$ which is E-related to x. Then $Q \Vdash \sigma\ E\ \check{y}$, by the Mostowski absoluteness between the Q-extensions of M and V $M \models P \Vdash \tau\ E\ \check{y}$, and this will contradict the elementarity of the model M and the nontriviality of the name τ.

To find the point $y \in M \cap X$, let N be a countable elementary submodel of a large structure containing $\langle M_\alpha \colon \alpha \in \omega_1\rangle, Q, x$. Since the tower of models $\langle M_\alpha \colon \alpha \in \omega_1\rangle$ is continuous, there is a limit ordinal $\alpha \in \omega_1$ such that $M_\alpha = N\cap M$. Let $Q_\alpha = Q\cap M_\alpha = P\cap M_\alpha$ and $\sigma_\alpha = \sigma\cap M_\alpha = \tau\cap M_\alpha$. By elementarity of the model N and analytic absoluteness between the Q_α-extension of N and V, $Q_\alpha \Vdash \sigma_\alpha\ E\ \check{x}$. Since $Q_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \tau \cap M_\alpha$, both Q_α, τ_α belong to the model $M_{\alpha+1}$. By the elementarity of the model $M_{\alpha+1}$, there must be a point $y \in X \cap M_{\alpha+1}$ such that $Q_\alpha \Vdash \sigma_\alpha\ E\ \check{y}$ (since the point x is such). By the transitivity of E, it follows that $x\ E\ y$. The point $y \in M_{\alpha+1} \subset M$ works.

Now, suppose that R is a poset collapsing $\aleph_1$. Since $|M| = \aleph_1$, in the R-extension there is a filter Q-generic over M. Let $\dot{H}$ be an R-name for such a filter and let ν be the R-name for $\sigma/\dot{H}$. By the Mostowski absoluteness between the generic extensions of M and V again, it follows that $Q \times R \Vdash \sigma\ E\ \nu$. We have just proved that ν is a nontrivial E-pinned name on R. □

Theorem 2.6.3 does not rule out the possibility that some $\aleph_1$ preserving posets carry nontrivial E-pinned names. This does not occur for orbit equivalence relations. In them, every pinned name is inescapably connected with a collapse of a certain cardinal to $\aleph_0$. To state this in the most informative way, we introduce a piece of notation. Let E be an analytic equivalence relation on a Polish space X, and let c be a virtual E-class. We write $\kappa(c)$ for the smallest cardinality of a poset P such that there is an E-pinned P-name σ such that the E-pin $\langle P, \sigma\rangle$ belongs to c.

Theorem 2.6.4. *Let E be an equivalence relation on a Polish space X, Borel reducible to an orbit equivalence relation of a Polish group action. Let c be a virtual E-class and write $\kappa = \kappa(c)$. The following are equivalent for every poset P:*

1. *there is an E-pinned P-name σ such that the E-pin $\langle P, \sigma\rangle$ belongs to c;*
2. *$P \Vdash |\kappa| = \aleph_0$.*

Proof. (2) implies (1) is the easier direction, and it does not use the assumption on the equivalence relation E. Suppose that $P \Vdash |\kappa| = \aleph_0|$. Let $\langle R, \tau\rangle$ be an E-pin in c such that $|R| = \kappa$. Let M be an elementary submodel of a large structure such that $|M| = \kappa$, $R \subset \kappa$, and $R, \tau \in M$. By the collapsing assumption, there is a P-name η for a filter on R which is generic over the model

M. Let σ be a P-name for τ/η. We claim that σ is E-pinned and moreover $\langle P, \sigma\rangle \in c$.

To this end, let $G \subset P$ and $H \subset R$ be mutually generic filters; we must show that $\tau/G\ E\ \sigma/H$. To see this, observe that the filters $\eta/G \subset R$ and $H \subset R$ are mutually generic over the model M, so $M[\eta/G, H] \models \sigma/\eta/G\ E\ \sigma/H$ as the name σ is E-pinned and M is an elementary submodel. $\sigma/\eta/G = \tau/G\ E\ \sigma/H$ then follows by the Mostowski absoluteness between the models $M[\eta/G, H]$ and $V[G, H]$.

For the harder direction (1) implies (2), first fix a Polish group Γ, an action of Γ on a Polish space Y, and a Borel function $h\colon X \to Y$ which is a reduction of E to the orbit equivalence relation F induced by the action of Γ. Suppose that σ is an E-pinned name on P such that the E-pin $\langle P, \sigma\rangle$ belongs to c. Consider the Cohen poset P_Γ of nonempty open subsets of Γ, with its associated name $\dot\gamma_{gen}$ for a generic element of Γ. Consider the product $P_\Gamma \times P$ and the poset Q, the subset of the complete Boolean algebra of $P_\Gamma \times P$ which is generated by the name τ for $\dot\gamma_{gen} \cdot \dot h(\sigma)$, an element of the space Y. Since the poset P_Γ is countable, it preserves all cardinals. Therefore, to show that P collapses κ, it will be enough to show that Q collapses κ.

Claim 2.6.5. *Let $H \subset Q$ be a generic filter. In $V[H]$, P_Γ forces the point $\dot\gamma_{gen} \cdot \tau/H \in Y$ to be Q-generic over $V[H]$.*

The difficulty resides in the assertion that the point is forced to be Q-generic over $V[H]$ and not just over V.

Proof. Let $\gamma_0, \gamma_1, \gamma_2 \in \Gamma$ and $K_0, K_1 \subset P$ be P_Γ-generic points and generic filters on P respectively, which are in addition mutually generic. Since σ is an E-pinned name, $\sigma/K_0\ E\ \sigma/K_1$ holds. Since h is a reduction of E to F, $h(\sigma/K_0)\ F\ h(\sigma/K_1)$ holds. Writing $y_0 = h(\sigma/K_0)$ and $y_1 = h(\sigma/K_1)$, there must be a group element $\delta \in \Gamma$ such that $\delta\gamma_0 \cdot y_0 = \gamma_1 \cdot y_1$ in the model $V[\gamma_0, K_0][\gamma_1, K_1]$. Observe that both points $\gamma_2\delta$ and $\gamma_2\gamma_1$ are P_Γ-generic over the model $V[\gamma_0, K_0][\gamma_1, K_1]$ since γ_2 is, and multiplication by δ (or γ_1) from the right induces an automorphism of the poset P_Γ.

It follows that $\gamma_2\delta$ is P_Γ-generic, and the pair $\gamma_2\gamma_1, K_1$ is $P_\Gamma \times P$-generic, both over the model $V[\gamma_0, K_0]$. Observe that $\gamma_2\delta\gamma_0 \cdot y_0 = \gamma_2\gamma_1 \cdot y_1$ by the initial choice of δ. The claim now follows by considering the filter $H \subset Q$ (generic over V) obtained from the point $\gamma_0 \cdot y_0 \in Y$, and the point $\gamma_2\delta \in \Gamma$ (P_Γ-generic over $V[H]$), and the forcing theorem. $\square$

Now, let $H \subset Q$ be a generic filter, and work in $V[H]$. Since the poset P_Γ has a countable dense subset and it introduces a generic for Q, it must be the case that Q has a countable dense subset as well. Back in V, the forcing theorem says that Q forces $\check Q$ to have a countable dense subset. Fix a Q-name η for a function from ω to Q whose range is forced to be dense, and let $D = \{q \in Q\colon \exists r\ \exists m \in \omega\ r \Vdash \eta(m) = \check q\}$. It will be enough to show that $|D| \geq \kappa$ and $Q \Vdash |\check D| = \aleph_0$.

The latter statement is easier: Q forces D to be the range of the partial function f on $\omega \times \omega$ defined by $f(n,m) = q$ if $\eta(n) \Vdash \eta(m) = \check{q}$. For the former statement, first note that $D \subset Q$ is dense since D is forced to contain the range of η. It follows that D and Q viewed as posets give the same extensions and τ is really a D-name. By the Mostowski absoluteness between the $P_\Gamma \times P$-extension and the D-extension, there must be a D-name χ for an element of X such that $D \Vdash \dot{h}(\chi)\ F\ \tau$. Since h is a reduction, it follows that $\langle Q, \chi\rangle$ is an E-pin equivalent to $\langle P, \sigma\rangle$. By the initial choice of the cardinal κ, $|Q| \geq \kappa$ follows. $\square$

Corollary 2.6.6. *Let E be an analytic equivalence relation on a Polish space X, Borel reducible to an orbit equivalence relation. If E is not pinned then the following are equivalent for any partial order P:*

1. *P carries a nontrivial E-pinned name;*
2. *P collapses $\aleph_1$ to $\aleph_0$.*

Proof. (2) implies (1) by Theorem 2.6.3. For the opposite implication, let τ be a nontrivial E-pinned name on P. Let κ be the cardinal of Theorem 2.6.4 associated with the virtual E-class containing $\langle P, \tau\rangle$. Note that κ cannot be $\aleph_0$ since then the Cohen poset would carry an E-pinned name equivalent to τ; however, all E-pinned names on the Cohen poset are trivial by Theorem 2.6.2. Note also that P collapses κ to $\aleph_0$ since it realizes the virtual E-class associated with $\langle P, \tau\rangle$. In conclusion, P collapses some uncountable cardinal to $\aleph_0$, in particular it must collapse $\aleph_1$. $\square$

Example 2.6.7. Consider the equivalence relation $\mathbb{F}_2$ on $(2^\omega)^\omega$. This is clearly an orbit equivalence relation. Every E-pinned name is represented by a set $A \subset 2^\omega$ by Example 2.3.5. The cardinal κ of Theorem 2.6.4 is equal to $|A|$.

Example 2.6.8. Let E be the equivalence relation on $X = (\mathcal{P}(\omega))^\omega$ defined by $x_0\ E\ x_1$ if $\mathrm{rng}(x_0)$ and $\mathrm{rng}(x_1)$ generate the same filter on ω. Suppose that in V, there is a modulo finite strictly decreasing sequence $a = \langle a_\alpha \colon \alpha \in \omega_2\rangle$ of subsets of ω. Let P be the Namba forcing. It is well-known that P preserves $\aleph_1$ and adds a cofinal sequence $\sigma \colon \omega \to \omega_2^V$ to ω_2. Let τ be the P-name for $a \circ \sigma$. It is immediate that τ is a nontrivial E-pinned name on P. Thus, the assumption that E be reducible to an orbit equivalence in Corollary 2.6.6 cannot be omitted.

2.7 Absoluteness

This section compiles the available information regarding the absoluteness of various notions regarding the virtual equivalence classes. The most substantial and useful statement is Corollary 2.7.3: if E is a Borel equivalence relation then its pinned status is absolute among all generic extensions.

Theorem 2.7.1. *Suppose that E is a Borel equivalence relation on a Polish space X. The following are equivalent:*

1. *E is pinned;*

2. *For every ω-model M of ZFC containing the code for E, $M \models E$ is pinned.*

Proof. For simplicity assume $X = \omega^\omega$. The implication (2)→(1) is trivial. If (1) fails, then E is not pinned, and the failure of (2) is witnessed by $M = V$.

The implication (1)→(2) is more difficult. We start with an abstract claim which can be used in several other absoluteness results, and which appears in several less efficient versions in existing literature. Say that an ω-model N of ZFC is *correct about stationary sets* if for all sets $a, b \in N$, if $N \models a$ is uncountable then in fact a is uncountable, and if $N \models b \subset [a]^{\aleph_0}$ is a stationary system of countable sets, then in fact b is a stationary system of countable sets. Note that as N is an ω-model, for any set c, if $N \models c$ is countable then in fact c is countable.

Claim 2.7.2. [5, Theorem 1.12], [51, Section 4] *Let M be a countable ω-model of a large fragment of ZFC. Then there is an elementary embedding $j\colon M \to N$ such that N is an ω-model correct about stationary sets.*

Proof. We start with setting up a bookkeeping tool. By a classical result of Solovay, ω_1 can be decomposed into $\aleph_1$ many stationary sets. Thus, there is a surjective function $f\colon \omega_1 \to \omega_1 \times \omega$ such that preimages of singletons are stationary.

Now, by recursion on $\alpha \in \omega_1$ build countable ω-models M_α of ZFC and maps $j_{\beta\alpha}\colon M_\beta \to M_\alpha$ for all $\beta \in \alpha$. With each model M_α we also pick a surjection $\pi_\alpha\colon \omega \to M_\alpha$ arbitrarily. The following are the recursive demands on the construction:

- $M = M_0$ and $\langle j_{\gamma\beta}\colon M_\gamma \to M_\beta\colon \gamma \in \beta \in \alpha\rangle$ is a commuting system of elementary embeddings;

- if $\alpha \in \omega_1$ is limit then M_α is the direct limit of the previous models and embeddings;

- whenever $\alpha \in \omega_1$ is an ordinal such that $f(\alpha) = \langle \beta, n\rangle$ and $\beta \in \alpha$ and $M_\beta \models b = \pi_\beta(n)$ is a stationary system of countable sets on $a = \bigcup b$, then writing $c = j''_{\alpha\alpha+1} j_{\beta\alpha}(a)$ we get that $M_{\alpha+1}$ contains c and $M_{\alpha+1} \models c \in j_{\beta\alpha+1}(b)$.

It is only necessary to show how the last item is arranged. Thus, suppose that its assumptions are satisfied at $\alpha \in \omega_1$. Work in the model M_α and consider the poset P of all stationary subsets of b, ordered by inclusion. Since the model M_α is countable, there is a filter $G \subset P$ generic over the model M_α. Let $j_{\alpha\alpha+1}\colon M_\alpha \to M_{\alpha+1}$ be its associated generic ultrapower as in [45, Lemmas 22.13 and 14]. Then $M_{\alpha+1}$ will be again an ω-model. Note that the set c belongs to the model $M_{\alpha+1}$ as it is represented by the identity function; it also has the requested properties by the Łoś theorem.

In the end, let $j\colon M \to N$ be the direct limit of the system of elementary embeddings produced during the recursion; we claim that this embedding works as required. Suppose then that $M \models b$ is a stationary system of countable sets on $a = \bigcup b$. To show that $\bar{b}$ is indeed stationary, let d be a closed unbounded set of countable subsets of a, and work to show that $d \cap \bar{b} \neq 0$ holds. Fix an ordinal $\beta \in \omega_1$ and a number $n \in \omega$ such that $b = j_{\beta\omega_1}(\pi_\beta(n))$. Let K be a countable elementary submodel of a large substructure, containing in particular the set d, the ordinal β, and the system of elementary embeddings constructed above, and such that $f(\alpha) = \langle \beta, n \rangle$ where $\alpha = K \cap \omega_1$. Such a model exists by the initial choice of the function f. Clearly $K \cap a \in d$; we will be done once we show that $K \cap a \in b$.

To see this, let $c = j''_{\alpha\alpha+1} j_{\beta\alpha}(\bigcup \pi_\beta(n))$. By the elementarity of the embedding $j_{\alpha+1\omega_1}$ and the last item above, it is the case that $j_{\alpha+1\omega_1}(c) \in b$. Also, $j_{\alpha+1\omega_1}(c) = j''_{\alpha+1\omega_1} c = j''_{\alpha\omega_1} j_{\beta\alpha}(\bigcup \pi_\beta(n))$, By the elementarity of the model K and the second item above, the latter expression is exactly equal to $K \cap j_{\beta\omega_1}(\bigcup \pi_\beta(n)) = K \cap a$. This completes the proof. $\square$

Now, back to the theorem. Suppose that (2) fails. Fix a ω-model M of ZFC containing the code for E such that $M \models E$ is unpinned; taking an elementary submodel if necessary we may assume that M is countable. Let $j \colon M \to N$ be an elementary embedding from the claim. By elementarity, $N \models E$ is unpinned, so there is a poset P and a P-name τ such that $N \models \tau$ is a nice nontrivial E-pinned name. We will show that in fact τ really is a nontrivial E-pinned name.

First of all, τ indeed is E-pinned. If $G_0, G_1 \subset P$ are mutually generic filters then $N[G_0, G_1] \models \tau/G_0 \; E \; \tau/G_1$ holds, and by the Borel absoluteness between $N[G_0, G_1]$ and $V[G_0, G_1]$, $\tau/G_0 \; E \; \tau/G_1$ holds in $V[G_0, G_1]$ as required. To show that τ is a nontrivial E-pinned name, assume towards a contradiction that there is a point $x \in X$ such that $P \Vdash \tau \; E \; \check{x}$; the difficulty resides in the possibility that $x \notin N$.

Let κ be some large cardinal in the model N. Let K be a countable elementary submodel of a large structure containing in particular N, P, τ, and κ, such that $c = K \cap N \cap V_\kappa$ belongs to the model N and is countable there. Such a model K exists since the model N is correct about the stationarity of the set it views as $[V_\kappa]^{\aleph_0}$. Work in the model N. Since c is an elementary submodel of V_κ, it follows that $\sigma = \tau \cap c$ is an E-pinned name on the poset $Q = P \cap c$. Since the poset Q is countable, by Theorem 2.6.2 below applied in the model N, there must be a point $y \in N$ such that $Q \Vdash \sigma \; E \; \check{y}$. Now, work in the model K and observe that $Q \Vdash \sigma \; E \; \check{x}$. It follows that $x \; E \; y$. Now, back in the model N it must be the case that $P \Vdash \tau \; E \; \check{y}$, contradicting the assumption that τ is a nontrivial E-pinned name. $\square$

Corollary 2.7.3. *Let E be a Borel equivalence relation on a Polish space X. The statement "E is pinned" is absolute between all generic extensions.*

Proof. The validity of item (2) of Theorem 2.7.1 does not change if one only considers countable models. This follows from an immediate downward Löwenheim-Skolem argument. The countable model version of (2) is a coanalytic statement, and as such it is absolute among all forcing extensions by the Mostowski absoluteness. ☐

The absoluteness of the pinned status of analytic equivalence relations does not allow a similar sweeping statement. Consider a Π^1_1 set $A \subset 2^\omega$ which contains no perfect subset, but is uncountable in the constructible universe L [45, Corollary 25.37]. Let E be the analytic equivalence relation on $X = (2^\omega)^\omega$ defined by $x_0 \mathrel{E} x_1$ if either both $\mathrm{rng}(x_0), \mathrm{rng}(x_1)$ have nonempty intersection with the complement of A, or $\mathrm{rng}(x_0) = \mathrm{rng}(x_1)$. In L, the equivalence relation E is not pinned, as witnessed by the $\mathrm{Coll}(\omega, A)$-name for the generic enumeration of A. At the same time, in the $\mathrm{Coll}(\omega, 2^\omega)$-extension of L, the (reinterpretation of the) set A is countable, containing only constructible elements, and in consequence the equivalence relation E is smooth, so pinned. On the other hand, in the presence of sufficiently large cardinals the pinned status of every analytic equivalence relation is absolute by a proof similar to the proof of Theorem 2.7.1 where we adjust Claim 2.7.2 to input a well-founded model M of ZFC plus the statement that there is a propoer class of Woodin cardinals, and output a well-founded model N correct about stationary sets. The adjustment needs the classical well-foundedness results regarding the well-foundedness of the stationary tower ultrapower in [99, 64]. We refrain from providing further detail.

Another natural question concerns the computation of the various cardinal invariants of equivalence relations in inner models. In particular, we would like to see that if $M \subset N$ are transitive models of ZFC containing the code for an analytic equivalence relation E, then $(\kappa(E))^M \leq (\kappa(E))^N$. Interestingly enough, this is not clear, and we only have a positive answer in the case of orbit equivalence relations and their relatives.

Theorem 2.7.4. *Let E be an analytic equivalence relation on a Polish space X almost reducible to an orbit equivalence relation. Let M be a transitive model of large fragment of set theory containing the codes for E, the group action, and the almost reduction. Then $\kappa(E)^M \leq \kappa(E)$ holds.*

Proof. To begin, let Γ be a Polish group continuously acting on a Polish space Y, inducing an orbit equivalence relation F. Let $h\colon X \to Y$ be a Borel reduction of E to F. Choose Γ, Y, h so that they belong to the model M.

Suppose that the conclusion of the theorem fails. Then, writing $\kappa = (\kappa(E))^V$, in the model M there exists an E-pin $\langle Q, \tau\rangle$ such that $M \models \langle Q, \tau\rangle$ is not equivalent to any E-pin on a poset of cardinality less than κ. By Theorem 2.6.3 applied in the model M, this means that if N is a generic extension of M, then $N \models (\exists x \in X\ Q \Vdash \tau \mathrel{E} \check{x}) \to |\kappa| = \aleph_0)$.

Let $\langle R, \sigma\rangle$ be an E-pin in V such that $|R| < \kappa$ and $Q \times R \Vdash \tau \mathrel{E} \sigma$. Let $G \subset Q$, $H \subset R$ and $K \subset P_\Gamma$ be filters mutually generic over V. Write $y_0 = h(\tau/G)$, $y_1 = h(\sigma/H)$, write $\gamma \in \Gamma$ for the generic point associated with the filter K, and let $y_2 = \gamma \cdot y_1$. By the assumptions, y_0, y_1, and $y_2 \in Y$ are mutually orbit-equivalent points.

Claim 2.7.5. *The point $y_2 \in Y$ is generic over the model M, and in the model $M[y_2]$ there exists a point $x \in X$ such that $M[y_2] \models Q \Vdash \tau\ E\ \check{x}$.*

Proof. There is a point $\delta \in \Gamma$ in the model $V[G][H]$ such that $y_1 = \delta \cdot y_0$. The point $\gamma \in \Gamma$ is P_Γ-generic over $V[G][H]$ by the product forcing theorem. Now, multiplication by δ from the right is an automorphism of the poset P_Γ, so the point $\gamma\delta \in \Gamma$ is P_Γ-generic over $V[G][H]$. Therefore, the point $\gamma\delta$ is also P_Γ-generic over the smaller model $M[G]$, and by the product forcing theorem, G and $\gamma\delta$ are mutually generic objects over M. The point $y_2 = \gamma\delta \cdot y_0 \in M[G, \gamma\delta]$ belongs to a generic extension of M, so is generic over M itself.

Now, the model $M[y_2]$ is transitive, so by Mostowski absoluteness it must contain a point $x \in X$ such that $h(x)\ F\ y_2$, since one such point σ/H exists in the model $V[H][K]$. We claim that $M[y_2] \models Q \Vdash \tau\ E\ \check{x}$. To see this, observe that the filter $G \subset Q$ is generic over the model $V[H][K]$, so also generic over the smaller model $M[y_2]$. Since $M[G][y_2] \models y_0 = h(\tau/G)\ F\ y_2$, the forcing theorem implies that $M[y_2] \models Q \Vdash h(\tau)\ F\ \check{y}_2$, so $\tau\ E\ \check{x}$. □

By the second paragraph of the proof, it must be the case that $M[y_2] \models |\kappa| = \aleph_0$; therefore, $V[H][K] \models |\kappa| = \aleph_0$. This, however, contradicts the assumption that in V, $|R| < \kappa$, which is equivalent to $|R \times P_\Gamma| < \kappa$. □

2.8 Dichotomies

This section presents three theorems which characterize various complex features of the virtual realm in terms of dichotomies. It turns out that each of the dichotomies uses a preparatory proposition or two on the descriptive theory of forcing and model theory. These preparatory propositions are perhaps more difficult to state properly than to prove. Nevertheless, they are unavoidably needed and they do not seem to be present in published literature. We gather them in the first subsection, while the second subsection contains the dichotomies themselves.

2.8a Preliminaries

The first two auxiliary propositions concern pure model theory; they show that the satisfaction relation and the construction of models from indiscernibles are in a suitable sense Borel affairs. The proofs are straightforward and left to the reader. For both of them, fix a countable language and let X be the Polish space of models of that language whose universe is ω.

Proposition 2.8.1. *Suppose that $f\colon 2^\omega \to X$ is a Borel function, ϕ is a formula of the language with n free variables, and $g_i\colon 2^\omega \to \omega$ are Borel functions for every $i \in n$. The set $\{x \in X\colon f(x) \models \phi(g_0(x), g_1(x) \dots g_{n-1}(x))\}$ is Borel.*

Let T be a Skolemized complete consistent theory containing a linear ordering. Let S be the theory of an infinite ordered set of indiscernibles in the linear order of T. Let Y be the space of all linear orderings on ω.

Proposition 2.8.2. *There are Borel functions $f\colon Y \to X$ and $g\colon Y \times \omega \to \omega$ such that for each $y \in Y$,*

1. *$f(y)$ is a model of T;*
2. *the function $g_y\colon \omega \to \omega$ is an order-preserving map from the ordering y to the ordering of $f(y)$ and its range consists of indiscernibles with theory S;*
3. *the model $f(y)$ is a Skolem hull of $\mathrm{rng}(g_y)$.*

The rest of the subsection deals with models of set theory, showing that the construction of their generic extensions is a Borel affair. Thus, let X be the Polish space of all models for the language with one binary relation whose universe is ω. Let Y be any uncountable Polish space, serving as an index space. Let $M\colon Y \to X$ and $P\colon Y \to \omega$ be Borel functions such that for each $y \in Y$, $M(y)$ is a model of (a large fragment of) ZF and $M(y) \models P(y)$ is a partially ordered set.

Proposition 2.8.3. *There is a Borel function $G\colon Y \to \mathcal{P}(\omega)$ such that for every $y \in Y$, $G(y)$ is a filter on $P(y)$ which is generic over $M(y)$.*

Proof. By induction on $n \in \omega$ define Borel functions $f_n\colon Y \to \omega$ so that

- for every $y \in Y$, $M(y) \models f_0(y)$ is the largest element of the poset $P(y)$;
- for every $y \in Y$ and $n \in \omega$, if $M(y) \models n$ is an open dense subset of the poset $P(y)$, then $f_{n+1}(y)$ is the smallest number m such that $M(y) \models m$ is an element of n and it is smaller that $f_n(y)$ in the poset $P(y)$; otherwise, $f_{n+1}(y) = f_n(y)$.

The functions defined in this way are Borel by Proposition 2.8.1. Define the function $G\colon Y \to \mathcal{P}(\omega)$ as follows: for every $y \in \omega$, $G(y)$ is the set of all m such that $M(y) \models m$ is an element of the poset $P(y)$ and for some $n \in \omega$, $M(y) \models f_n(y)$ is smaller than m in the poset $P(y)$. It is clear that the function G works. $\square$

Now, suppose that a Borel function $G\colon Y \to \mathcal{P}(\omega)$ as in Proposition 2.8.3 is given; we want to produce the generic extensions. To wit, we have to produce the generic extensions as Borel functions on Y and also the valuation functions for names in the ground models as Borel functions.

Proposition 2.8.4. *There are Borel functions $M[G]\colon Y \to X$ and $v\colon Y \times \omega \to \omega$ such that for every $y \in Y$, $M[G](y)$ is a generic extension of $M(y)$ by $G(y)$ and for every $n \in \omega$, $v(y)(n) = n/G(y)$ whenever $M(y) \models n$ is a $P(y)$-name.*

Proof. Let E be the Borel equivalence relation on $Y \times \omega$ connecting pairs $\langle y_0, n_0\rangle$ and $\langle y_1, n_1\rangle$ if $y_0 = y_1$ and (denoting their common value by y) either $M(y) \models$ neither n_0, n_1 is a $P(y)$-name or $M[y] \models$ both n_0, n_1 are $P(y)$-names and there is some condition $p \in G(y)$ such that $M(y) \models p \Vdash n_0 = n_1$. Let $v\colon Y \times \omega \to \omega$

be the Borel function defined in the following way: $f(y)(n) = 0$ if $M(y) \models n$ is not a $P(y)$-name, and $f(y)(n) = m$ if $M(y) \models n$ is a $P(y)$-name and, writing n' for the smallest name E-related to n, there are exactly m-many E-classes of $P(y)$-names represented by numbers smaller than n'. Let $M[G]$ be the Borel function defined in the following way: $M[G](y)$ connects the pair $\langle m_0, m_1\rangle$ just in case there are numbers $n_0, n_1 \in \omega$ such that $M(y) \models n_0, n_1$ are $P(y)$-names, $f(y)(n_0) = m_0 + 1$, $f(y)(n_1) = m_1 + 1$, and there is a condition $p \in G(y)$ such that $M(y) \models p \Vdash n_0 \in n_1$. The Borelness of the functions $M[G]$ and v follows from Proposition 2.8.1. $\square$

The construction of the generic filters may be subject to an additional constraint. Suppose that E is an equivalence relation on a Polish space Z, coded in each model $M(y)$. Suppose that $\tau\colon Y \to \omega$ is a Borel function such that for each $y \in Y$, $M(y) \models \tau(y)$ is a $P(y)$-name of an element of Z. Suppose that $z\colon Y \to Z$ is a Borel function. Can we find a Borel function $G\colon Y \to \mathcal{P}(\omega)$ such that for each $y \in Y$, $G(y)$ is a filter on $P(y)$ generic over $M(y)$, such that the valuation of $\tau(y)$ over $G(y)$ is E-related to $z(y)$? To come to a useful affirmative conclusion, we assume that the models in the range of the function M are all well-founded, and the equivalence relation E is induced as an orbit equivalence relation of a continuous action of a Polish group Γ on the space Z. In such circumstances, we have the following:

Proposition 2.8.5. *Given E, M, P, τ as above:*

1. *the set $B = \{y \in Y\colon$ there is a filter $G \subset P(y)$ generic over the model $M(y)$ such that $\tau(y)/G$ is E-related to $z(y)\}$ is Borel;*
2. *there is a Borel function $G\colon B \to \mathcal{P}(\omega)$ such that for every $y \in B$, $G(y) \subset P(y)$ is a filter generic over $M(y)$ such that $\tau(y)/G(y)$ is E-related to $z(y)$.*

Proof. Let P_Γ be the poset of all nonempty open subsets of Γ ordered by inclusion, with its name $\dot\gamma$ for a generic element of Γ. Use Proposition 2.8.1 to find Borel functions $Q, \sigma\colon Y \to \omega$ so that for every $y \in Y$, $M(y) \models Q(y)$ is the complete subalgebra of the complete Boolean algebra of $P(y) \times P_\Gamma$ generated by the name $\sigma(y) = \dot\gamma \cdot \tau(y)$.

Let $D = \{\langle y, \delta, G\rangle \in Y \times \Gamma \times \mathcal{P}(\omega)\colon G \subset Q(y)$ is a filter generic over the model $M(y)$ such that $\nu(y)/G = \delta \cdot z(y)\}$. This is a Borel set by Proposition 2.8.1. The projection of D into the Y coordinate is the set B.

Claim 2.8.6. *For each pair $\langle y, \delta\rangle \in Y \times \Gamma$, the vertical section $D_{y\delta}$ is either empty or a singleton.*

Proof. This uses the wellfoundedness of the model $M(y)$. Suppose that the section $D_{y\delta}$ is nonempty, containing some filter G. As $M(y) \models$"$Q(y)$ is completely generated by the name $\sigma(y)$", the filter G can be recovered by transfinite induction using infinitary Boolean expressions in M applied to $\delta \cdot z(y)$; therefore, it is unique. $\square$

Write $C \subset Y \times \Gamma$ for the projection of D into the first two coordinates. Since one-to-one projections of Borel sets are Borel [55, Theorem 15.1], the set C is Borel.

Claim 2.8.7. *For each point $y \in Y$, the section C_y either empty or else comeager in Γ.*

Proof. To simplify the notation, fix $y \in Y$ and omit the argument y from the expressions like $M(y), \sigma(y) \dots$. Suppose that the Γ-section C_y is nonempty. Thus, there is a filter $G \subset P$ be a filter generic over M such that τ/G is E-related to z. Let $A \subset \Gamma$ be the set of all elements of the group Γ which are P_Γ-generic over the model $M[G]$; this is a co-meager set. Let $\delta \in \Gamma$ be an element of the group such that $\tau/G = \delta \cdot z$. Then the vertical section C_y contains the set $A \cdot \delta$ which is co-meager as the right multiplication by δ is a self-homeomorphism of the group Γ. □

As the category quantifier yields Borel sets [55, Theorem 16.1], the set B as the projection of C into the first coordinate is Borel. Borel sets with nonmeager vertical sections allow Borel uniformizations [55, Theorem 18.6], so there is a Borel uniformization $f \colon B \to \Gamma$ of C. As the set D has singleton vertical sections, it is itself its uniformization $g \colon C \to \mathcal{P}(\omega)$. Let $H \colon B \to \mathcal{P}(\omega)$ be the function defined by $H(y) = g(y, f(y))$. Thus, for every $y \in B$, $H(y) \subset Q(y)$ is a filter generic over $M(y)$ such that $\sigma(y)/H(y)$ is E-related to $z(y)$.

Now, let $R \colon Y \to \omega$ be a Borel function such that for every $y \in 2^\omega$, $M(y) \models R(y)$ is a name for the quotient poset $P(y) \times P_\Gamma / Q(y)$. Use Propositions 2.8.3 and 2.8.4 to find a Borel function $K \colon B \to \mathcal{P}(\omega)$ such that $K(y) \subset R(y)$ is a filter generic over the model $M(y)[H(y)]$. Let $G \colon B \to \mathcal{P}(\omega)$ be a Borel function indicating a filter on $P(y)$ which is the $P(y)$-coordinate the composition of $H(y) * K(y)$. The function G has the required properties. □

Corollary 2.8.8. *Suppose that Γ is a Polish group continuously acting on a Polish space X, inducing an orbit equivalence relation E. Let M be a countable transitive model of set theory containing a code for the action, let $P \in M$ be a poset and $\tau \in M$ be a P-name for an element of the space X. Then*

1. *the set $B = \{x \in X \colon \exists G \subset P$ generic over M such that $\tau/G = x\}$ is Borel;*
2. *the equivalence relation $E \restriction B$ is Borel.*

Proof. (1) is immediate from the proposition. To see (2), consider the set $C = \{\langle x, y\rangle \in X^2 \colon \exists G \times H \subset P \times P_\Gamma$ generic over M such that $x = \tau/G$ and $y = (\dot\gamma_{\mathrm{gen}}/H) \cdot x\}$. The set C is Borel by the proposition again. Now, for $x, z \in B$, $x \mathrel{E} z$ just in case the set $A_{xz} = \{\gamma \in \Gamma \colon \langle x, \gamma \cdot z\rangle \in C\}$ is comeager in Γ. To see this, note that if $x \mathrel{E} z$ fails then the set $A_{xz} \subset \Gamma$ is actually empty, and if $x \mathrel{E} z$ holds with $\delta \cdot x = z$, then the $A_{xz} \supset \{\gamma \in \Gamma \colon \gamma \cdot \delta^{-1}$ is P_Γ-generic over $M[x]\}$ and the latter set is comeager since it is a shift of the comeager set of all points in Γ which are P_Γ-generic over the model $M[x]$.

Finally, observe that the equivalence relation $E \restriction B$ is obtained by an application of the category quantifier to the Borel set $\{\langle x, z, \gamma\rangle \in B \times B \times \Gamma \colon \gamma \in A_{xz}\rangle\}$, so is Borel by [55, Theorem 16.1]. $\square$

2.8b Results

The first dichotomy characterizes the existence of the pinned cardinal in analytic equivalence relations.

Theorem 2.8.9. *Assume that there is a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. $\kappa(E) = \infty$;
2. $\mathbb{E}_{\omega_1} \leq_a E$.

Proof. (2) implies (1) by Example 2.4.6 and Theorem 2.5.4. The large cardinal assumption is not needed for this direction. For the (1)$\to$(2) implication, suppose that $\kappa(E) = \infty$. Let κ be a measurable cardinal.

Claim 2.8.10. *There exist a poset P of size κ and an E-pinned P-name τ such that τ is not $\bar{E}$-related to any name on a poset of size less than κ.*

Proof. Since $\kappa(E) = \infty$, there exist a poset Q and an E-pinned Q-name σ which is not $\bar{E}$-equivalent to any name on a poset of size less than κ. Choose an elementary submodel M of a large structure such that $|M| = \kappa$, $V_\kappa \subset M$ and $Q, \sigma \in M$. Let $P = Q \cap M$ and let $\tau = \sigma \cap M$. We claim that P, τ works as required.

Indeed, if $G \times H \subset P \times P$ is a filter generic over V, then it is also generic over M, by the elementarity of M and the forcing theorem in M $M[G, H] \models \tau/G\ E\ \tau/H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, $V[G, H] \models \tau/G\ E\ \tau/H$. This proves that the name τ is E-pinned. If R is a poset in V of size $< \kappa$ and ν is an E-pinned R-name and $G \times H \subset P \times R$ is a generic filter over V, then $R, \nu \in M$, the filter $G \times H$ also generic over M, by the elementarity of M and the forcing theorem in M $M[G, H] \models \neg\tau/G\ E\ \nu/H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, $V[G, H] \models \neg\tau/G\ E\ \sigma/H$. This proves that $\nu\ \bar{E}\ \tau$ fails and completes the proof of the claim. $\square$

Choose a poset P on κ and an E-pinned name τ as in the claim. Choose a large enough cardinal θ and skolemize the structure $\langle H_\theta, \in, P, \tau\rangle$ of sets whose transitive closures have cardinality less than θ. Let T be the theory of the skolemized structure. Let $j \colon V = M_0 \to M_1$ be an ultrapower embedding associated with some normal measure on the measurable cardinal κ, and let $j_n \colon V \to M_n$ be its iterations of length n for $n \leq \omega$. The ordinals $\{j_n(\kappa) \colon n \in \omega\}$ form a sequence of indiscernibles in the model $j_\omega(H_\theta)$. Let S be the theory of the indiscernibles in this structure. Let Y be the space of all linear orderings on ω, and let Z be the space of all binary relations on ω. By Propositions 2.8.2

and 2.8.3, there are Borel functions $M\colon Y \to Z$, $G\colon Y \times M \to \mathcal{P}(\omega)$, $P\colon Y \to \omega$ and $\tau\colon Y \to \omega$ such that whenever $y \in Y$ is a linear ordering then $M(y)$ is a model of T which is a Skolem hull of indiscernibles of ordertype y satisfying the theory S, $P(y)$ is its version of the poset P, and $\tau(y)$ is its version of the name τ, and $G(y) \subset P(y)$ is a filter generic over the model $M(y)$. Note that all models $M(y)$ are ω-models, so compute the Polish space X correctly. Let $k\colon Y \to X$ be the function defined by $k(y) = \tau(y)/G(y)$, which is Borel by Proposition 2.8.4. It will be enough to show that k is a Borel reduction of $\mathbb{E}_{\omega_1}$ to E on the set of $y \in Y$ which are well-orders.

Suppose first that $y_0, y_1 \in Y$ are well-orders of the same length. Then the models $M(y_0), M(y_1)$ are wellfounded and isomorphic. Write N for their common transitive isomorph and Q, σ for its version of the poset P and the name τ. Thus, $N \models \sigma$ is an E-pinned Q-name. The filters $G(y_0), G(y_1) \subset Q$ are separately generic over N. Let $G \subset Q$ be a filter generic over both countable models $N[G(y_0)]$ and $N[G(y_1)]$. By the product forcing theorem, the filters $G(y_0) \times G$ and $G(y_1) \times G$ are both $Q \times Q$-generic over N. As $N \models \sigma$ is an E-pinned name, it follows that $N[G(y_0), G] \models \sigma/G(y_0)\ E\ \sigma/G$ and $N[G(y_1), G] \models \sigma/G(y_1)\ E\ \sigma/G$. By the Mostowski absoluteness between these two models and V, and the transitivity of the relation E, $\sigma/G(y_0) = k(y_0)\ E\ k((y_1) = \sigma/G(y_1)$ follows.

Suppose now that $y_0, y_1 \in Y$ are well-orders of different lengths; say that y_0 is shorter than y_1. Let N_0 be the transitive isomorph of $M(y_0)$, let κ_0, P_0, σ_0 be the versions of κ, P, and τ in the model N_0, and let $G_0 \subset P_0$ be the filter generic over N_0 indicated by $G(y_0)$. Similar notation prevails for subscript 1. Let $i\colon N_0 \to N_1$ be the elementary embedding obtained by sending the indiscernibles of N_0 to an initial segment of the indiscernibles of N_1. Note that the critical point of i is exactly κ_0, so $P_0 \in N_1$ as $P_0 = P_1 \restriction \kappa_0$. Also, $N_1 \models |P_0| < \kappa_1$. Find a filter $G \subset P_0$ generic over both the countable models $N_0[G_0]$ and $N_1[G_1]$. Since $N_0 \models \sigma_0$ is E-pinned, the Mostowski absoluteness between V and $N_0[G, G_0]$ implies that $\sigma_0/G_0\ E\ \sigma_0/G$ holds. Since $N_1 \models \sigma_1$ is not $\bar{E}$-equivalent to any name on a poset of size $< \kappa_1$, in particular to σ_0, it follows from the Mostowski absoluteness between V and $N_1[G, G_1]$ that $\sigma_0/G\ E\ \sigma_1/G_1$ fails. In conclusion, $k(y_0) = \sigma_0/G_0\ E\ \sigma_1/G_1 = k(y_1)$ fails as required. □

The status of a Borel equivalence relation as pinned/unpinned may be absolute between models of ZFC by Corollary 2.7.3. However, if one dares to look at choiceless models, a much more colorful picture comes to sight. The simplest description of the pinned status occurs in the Solovay model, where it actually obeys a former conjecture of Kechris [48, Question 17.6.1].

Theorem 2.8.11. *The following holds in the Solovay model derived from a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is unpinned;*
2. *$\mathbb{F}_2 \leq E$ or $\mathbb{E}_{\omega_1} \leq_a E$.*

Proof. Let κ be a measurable cardinal and let W be the derived Solovay model. In W, (2) certainly implies (1) as the proofs that $\mathbb{F}_2, E_{\omega_1}$ are unpinned work in ZF, and the proof that pinned equivalence relations persist downwards in the reducibility orderings works in ZF+DC.

For the implication (1)$\to$ (2), assume that $W \models E$ is unpinned. There must be a poset P and a nontrivial E-pinned name τ on the poset P, both in W. Both P and τ must be definable in W from a ground model parameter $z \in 2^\omega$. For simplicity of the notation assume that $z \in V$. Return to V. Let Q be the two-step iteration $\mathrm{Coll}(\omega, <\kappa) * \dot{P}$, and write σ for the Q-name obtained from the $\dot{P}$-name τ. There are two cases.

Case 1. There is a condition $q \in Q$ such that the $Q \restriction q$-name σ is E-pinned. In this case, we will conclude that $\mathbb{E}_{\omega_1} \leq_a E$ and use the Shoenfield absoluteness to transfer the almost reducibility to the Solovay model. To simplify the notation assume that q is the largest element of the poset Q.

Observe that the name σ cannot be $\bar{E}$-equivalent to any name on a poset of size $< \kappa$. Suppose for contradiction that R is a poset of size $< \kappa$ and χ an E-pinned R-name such that $\langle R, \chi\rangle \ \bar{E}\ \langle Q, \sigma\rangle$. In W, let $H \subset R$ be a filter generic over V. By Proposition 2.1.5 applied over V, $W \models P \Vdash \tau\ E\ \chi/H$. This contradicts the assumption that $V[G] \models P \Vdash \forall x \in X \cap V[G]\ \neg\tau\ E\ x$.

Now, it follows that $\kappa(E) \geq \kappa$. By Theorem 2.5.6(2), since κ is a measurable cardinal, $\kappa(E) = \infty$. By Theorem 2.8.9, $\mathbb{E}_{\omega_1} \leq_a E$ as desired.

Case 2. For every condition $q \in Q$, the $Q \restriction q$-name σ is not E-pinned. In this case, we will conclude that $\mathbb{F}_2$ is reducible to E. Then, a Shoenfield absoluteness argument shows that the Borel reduction of $\mathbb{F}_2$ to E remains a Borel reduction also in the Solovay model.

Fix a countable elementary submodel M of a large structure containing the code for E, the posets P, Q and the name τ. Write $R = \mathrm{Coll}(\omega, <\kappa) \cap M$. Let $Z = 2^\omega$, and use Proposition 1.7.8 to find a Borel function G from Z to $\mathcal{P}(M)$ such that for every finite set $a \subset Z$ the filters $G(z) \subset R$ for $z \in a$ are mutually generic over M. For every set $a \subset Z$ write $2^\omega_a = \bigcup\{2^\omega \cap M[\prod_{z\in b} G(z)] \colon b \subset a$ finite$\}$, $M_a = M(2^\omega_a)$, P_a and τ_a for the poset and name in M_a defined in the model $M(2^\omega_a)$ by the formulas ϕ_P and ϕ_τ. Similar usage will prevail for functions $y \colon \omega \to I$, writing $P_y = P_{\mathrm{rng}(y)}$ etc.

Lemma 2.8.12.

1. *whenever $a \subset Z$ is a nonempty set then there exists a filter $h \subset R$ generic over M such that $2^\omega \cap M[h] = 2^\omega_a$;*
2. *whenever $a, b, c \subset I$ are pairwise disjoint countable nonempty sets then there is a filter $h_a \times h_b \times h_c \subset R^3$ generic over M such that $2^\omega_a = 2^\omega \cap V[h_a]$ and similarly for 2^ω_b and 2^ω_c;*
3. *whenever a, b are distinct countable subsets of I then $P_a \times P_b \Vdash \neg\tau_a\ E\ \tau_b$.*

Proof. For (1), let $S = \{k \colon \exists\alpha \in \kappa \cap M\ \exists b \subset a$ b is finite and $k \subset \mathrm{Coll}(\omega, <\alpha) \cap M$ is a filter generic over M and $k \in M[G(z) \colon z \in b]\}$ and order S by inclusion. Let $K \subset S$ be a sufficiently generic filter; we claim that $h = \bigcup K$

works as desired. Indeed, a simple density argument shows that $h \subset R$ is an ultrafilter all of whose proper initial segments are generic over M. By the κ-c.c. of $\mathrm{Coll}(\omega, < \kappa)$, the filter h is in fact generic over M itself. A straightforward genericity argument using Fact 1.7.12 then shows that $2^\omega_a = 2^\omega \cap M[h]$ as desired.

(2) follows easily from (1). Let $h_a, h_b, h_c \subset R$ be any filters obtained from (1); we will show that these filters are in fact mutually generic over the model V. Since $\mathrm{Coll}(\omega, < \kappa)^3$ has κ-c.c.c., it is enough to show that for every ordinal $\alpha \in \kappa$, the filters $h^\alpha_a = h_a \cap \mathrm{Coll}(\omega, < \alpha)$, $h^\alpha_b = h_b \cap \mathrm{Coll}(\omega, < \alpha)$, and $h^\alpha_c = h_c \cap \mathrm{Coll}(\omega, < \alpha)$ are mutually generic over V. Since the filters h^α_a, h^α_b and h^α_c are coded by reals in the models $M[h_a]$, $M[h_b]$, and $M[h_c]$,there are finite subsets a', b', c' of a, b, c respectively such that $h^\alpha_a \in M[\prod_{i \in a'} g_i]$ etc. The mutual genericity now follows from the general Corollary 1.7.9 about product forcing.

(3) is proved in several parallel cases depending on the mutual position of the sets a, b vis-a-vis inclusion. We will treat the case in which all three sets $a \cap b$, $a \setminus b$, $b \setminus a$ are nonempty. Suppose for contradiction that $P_a \times P_b \Vdash \tau_a \mathrel{E} \tau_b$. From (2), it follows that in V, the triple product $\mathrm{Coll}(\omega, < \kappa)^3$ forces $\dot P_{\{0,1\}} \times \dot P_{\{1,2\}} \Vdash \tau_{\{0,1\}} \mathrel{E} \tau_{\{1,2\}}$. Then, the quadruple product $\mathrm{Coll}(\omega, < \kappa)^4$ forces in V that $\dot P_{\{0,1\}} \times \dot P_{\{1,2\}} \times \dot P_{\{2,3\}} \Vdash \tau_{\{0,1\}} \mathrel{E} \tau_{\{1,2\}} \mathrel{E} \tau_{\{2,3\}}$, in particular $\dot P_{\{0,1\}} \times \dot P_{\{2,3\}} \Vdash \tau_{\{0,1\}} \times \tau_{\{2,3\}}$. In view of (2) again, this means that the product $\mathrm{Coll}(\omega, < \kappa) \times \mathrm{Coll}(\omega, < \kappa)$ forces $\dot P_{\mathrm{left}} \times \dot P_{\mathrm{right}} \Vdash \tau_{\mathrm{left}} \mathrel{E} \tau_{\mathrm{right}}$. In other words, $(\mathrm{Coll}(\omega, < \kappa) * \dot P) \times (\mathrm{Coll}(\omega, < \kappa) * \dot P)$ forces $\sigma_{\mathrm{left}} \mathrel{E} \sigma_{\mathrm{right}}$, contradicting the case assumption. $\square$

Write $Y = (2^\omega)^\omega = \mathrm{dom}(\mathbb{F}_2)$. Use the Lusin–Novikov theorem to find a Borel map $g\colon Y \to (2^\omega)^\omega$ such that for every $y \in Y$, $g(y)$ enumerates the set 2^ω_y. Use Lemmas 2.8.12(1) and 2.8.5 to find a Borel map $h\colon Y \to \mathcal{P}(Q \cap M)$ such that for every $y \in Y$, $h(y) \subset Q$ is a filter generic over M and $\mathrm{rng}(g(y)) = 2^\omega \cap V[h_0(y)]$, where $h_0(y) \subset R$ is the filter generic over M obtained from $h(y)$. Let $k\colon Y \to X$ be given by $k(y) = \tau/h(y)$; this is a Borel map by Lemma 2.8.4. We will show that k is a reduction of $\mathbb{F}_2$ to E.

First, assume that $y_0, y_1 \in Y$ are $\mathbb{F}_2$-related. Then $\mathrm{rng}(y_0) = \mathrm{rng}(y_1)$, $\mathrm{rng}(g(y_0)) = \mathrm{rng}(g(y_1))$, and so $M_{y_0} = M_{y_1}$, $P_{y_0} = P_{y_1}$ and $\tau_{y_0} = \tau_{y_1}$. Let $H \subset P_{y_0}$ be a filter generic over both countable models $M_{y_0}[k(y_0)]$ and $M_{y_1}[k(y_1)]$ and let $x = \tau_{y_0}/H$. By the forcing theorem applied in the model $M_{y_0} = M_{y_1}$ and the fact that τ_{y_0} is an E-pinned name, conclude that $x \mathrel{E} k(y_0)$ and $x \mathrel{E} k(y_1)$, so $k(x_0) \mathrel{E} k(x_1)$ as desired.

Second, assume that $y_0, y_1 \in Y$ are not $\mathbb{F}_2$-related. Choose a sufficiently generic filter $H_0 \times H_1 \subset P_{y_0} \times P_{y_1}$ so that H_0 is generic over $M_{y_0}[k(y_0)]$ and H_1 is generic over $M_{y_1}[k(y_1)]$. As the names τ_{y_0} and τ_{y_1} are E-pinned, the forcing theorem in the models M_{y_0} and M_{y_1} implies that $k(y_0) \mathrel{E} \tau_{y_0}/H_0$ and $k(y_1) \mathrel{E} \tau_{y_1}/H_1$. Now, $\tau_{y_0}/H_0 \mathrel{E} \tau_{y_1}/H_1$ fails by Lemma 2.8.12(3), so $k(y_0) \mathrel{E} k(y_1)$ must fail as well. This completes the proof. $\square$

Corollary 2.8.13. *The following holds in the symmetric Solovay model derived from a measurable cardinal. Let E be a Borel equivalence relation on a Polish space X. E is unpinned if and only if $\mathbb{F}_2 \leq E$.*

Proof. This follows from Theorem 2.8.11 once we show that the option $\mathbb{E}_{\omega_1} \leq_a E$ is not available for any Borel equivalence relation E. This in turn follows easily from results on the pinned cardinal $\kappa(E)$ obtained in Section 2.5: $\kappa(E) < \infty$ by Theorem 2.5.6(1), $\kappa(E_{\omega_1}) = \infty$ by Example 2.4.6, and the pinned cardinal is monotone with respect to the reducibility ordering $\leq_a$–Theorem 2.5.4. $\square$

Chapter 3

Turbulence

In this chapter, we investigate pairs of generic extensions $V[H_0]$ and $V[H_1]$ such that $V[H_0] \cap V[H_1] = V$. In a particularly successful application, this allows us to restate Hjorth's concept of turbulence in geometric terms and produce many generalizations of the associated ergodicity theorem [52, Theorem 12.5].

3.1 Independent functions

The purpose of this section is to find a practical way of producing many extensions which satisfy the condition $V[H_0] \cap V[H_1] = V$ without being mutually generic. We need a folkloric observation. Recall that for a Polish space X, the poset P_X consists of nonempty open subsets of X ordered by inclusion; $\dot{x}_{gen}$ is its name for the unique element of X belonging to all open sets in the generic filter.

Proposition 3.1.1. *Let X, Y be Polish spaces and $f\colon X \to Y$ be a continuous open function. Then P_X forces $f(\dot{x}_{gen})$ to be a P_Y-generic element of Y.*

Proof. It is just necessary to show that for every open dense set $D \subset Y$, $P_X \Vdash f(\dot{x}_{gen}) \in D$ holds. To this end, let $O \in P_X$ be a condition. The set $f''O \subset Y$ is open; therefore, it has nonempty intersection with D. Consider the nonempty open set $O' = (f^{-1}(f''O \cap D)) \cap O \subset O$. For every point $x \in O'$, $f(x) \in D$ holds, so $O' \Vdash f(\dot{x}_{gen}) \in D$ as required. □

Now suppose that X, Y_0, Y_1 are Polish spaces and $f_0\colon X \to Y_0$ and $f_1\colon X \to Y_1$ are continuous open maps. We want to find a criterion implying that $P_X \Vdash V[f_0(\dot{x}_{gen})] \cap V[f_1(\dot{x}_{gen})] = V$. The following turbulence–like definition is central.

Definition 3.1.2. Let X, Y_0, Y_1 be Polish spaces and $f_0\colon X \to Y_0$ and $f_1\colon X \to Y_1$ be continuous open maps. Let $k \in \omega$ be a number. A *walk* (of points) of length k is a sequence $\langle x_i : i \leq k\rangle$ of points in X such that for each $i \in k$, either $f_0(x_i) = f_0(x_{i+1})$ or $f_1(x_i) = f_1(x_{i+1})$ holds.

Definition 3.1.3. Let X, Y_0, Y_1 be Polish spaces and $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ be continuous open maps. The functions f_0, f_1 are *independent* if for every nonempty open set $O \subset X$ there is a nonempty open set $A \subset Y_0$ such that for all nonempty open subsets $B_0, B_1 \subset A$ there is a walk consisting of points in O starting in $f_0^{-1}B_0$ and ending in $f_0^{-1}B_1$.

It may appear that the definition is not symmetric with respect to the maps f_0, f_1. In fact, the definition is symmetric, and this follows as a small corollary from the following central theorem.

Theorem 3.1.4. *Suppose that X, Y_0, Y_1 are Polish spaces and $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ are continuous open maps. The following are equivalent:*

1. *f_0, f_1 are independent;*
2. *$P_X \Vdash V[f_0(\dot{x}_{gen})] \cap V[f_1(\dot{x}_{gen})] = V$ where $\dot{x}_{gen}$ is the P_X-name for the generic element of X.*

Proof. For the (1)→(2) implication, we need a preliminary definition and a claim. Let $k \in \omega$ be a number. A *walk of open sets* of length k is a sequence $\langle O_i \colon i \leq k\rangle$ of nonempty open subsets of X such that for each $i \in k$, either $f_0''O_i = f_0''O_{i+1}$ or $f_1''O_i = f_1''O_{i+1}$.

Claim 3.1.5. *Let $k \in \omega$ be a number. Suppose that $\langle x_i \colon i \leq k\rangle$ is a walk of points, $O_i \subset X$ are open sets such that $x_i \in O_i$, and D is a finite collection of open subsets of X. Then there is a walk $\langle P_i \colon i \leq k\rangle$ of open sets such that $P_i \subset O_i$ holds for each $i \leq k$ and for each $U \in D$ and each $i \leq k$, P_i is either disjoint from U or a subset of U.*

Proof. By induction on k. The case $k = 0$ is trivial. Suppose that the claim has been proved for k and $\langle x_i \colon i \leq k+1\rangle$ is a walk of points, O_i are open sets, and D is a finite collection of open subsets of X. Suppose for definiteness that $f_0(x_k) = f_0(x_{k+1})$. Shrinking O_k and O_{k+1} if necessary, we may assume that $f_0''O_k = f_0''O_{k+1}$. For each open set $U \in D$ let $U' = \bigcup\{V \subset X \colon V$ is open and $f_0^{-1}f_0''V \subset U\}$ and let $E = \{U' \colon U \in D\}$. By the induction hypothesis, there is a walk $\langle P_i \colon i \leq k\rangle$ of open sets such that $P_i \subset O_i$ and P_i is either disjoint from or a subset of each element of $D \cup E$. Let $P_{k+1} = f_0^{-1}f_0''P_k \cap O_{k+1}$ and observe that the walk $\langle P_i \colon i \leq k+1\rangle$ of open sets works as required. $\square$

Assume now that the functions f_0, f_1 are independent. Since the models in (2) satisfy ZFC, it is enough to show that every set of ordinals in the intersection is actually in V. To this end, suppose that τ_0, τ_1 are P_{Y_0} and P_{Y_1}-names for sets of ordinals. We will abuse notation somewhat and write τ_0 also for the P_X-name for $\tau_0/\dot{f}_0(\dot{x}_{gen})$ and τ_1 for the P_X-name for $\tau_1/f_1(\dot{x}_{gen})$. Suppose that $O \subset X$ is a nonempty open set forcing $\tau_0 = \tau_1$. We must find a nonempty open set $O' \subset O$ deciding the statement $\check{\alpha} \in \tau_0$ for every ordinal α.

Let $A \subset Y_0$ be a nonempty open set standing witness to the independence of the functions f_0, f_1. We claim that for every ordinal α, in the poset P_{Y_0},

$A \Vdash \check{\alpha} \in \tau_0$ or $A \Vdash \check{\alpha} \notin \tau_0$. Once this is proved, let $O' = O \cap f_0^{-1}A$ and note that O' decides the membership of every ordinal in τ_0 as desired.

Suppose towards a contradiction that there exist an ordinal α and nonempty open sets $B_0, B_1 \subset A$ such that $B_0 \Vdash \check{\alpha} \in \tau_0$ and $B_1 \Vdash \check{\alpha} \notin \tau_0$. Let $\langle x_i : i \leq k \rangle$ be a walk of points in O such that $f_0(x_0) \in B_0$ and $f_0(x_k) \in B_1$. Let $U_0 = \bigcup\{U : U \subset X \text{ and } U \Vdash \check{\alpha} \in \tau_0\}$ and $U_1 = \bigcup\{U : U \subset X \text{ and } U \Vdash \check{\alpha} \in \tau_1\}$. By the claim, there must be a walk of open sets $\langle O_i : i \leq k \rangle$ such that each set O_i is a subset of O, $O_0 \subset f_0^{-1}B_0$, $O_k \subset f_0^{-1}B_1$, and each O_i is either disjoint from or a subset of U_0, and either disjoint from or a subset of U_1. This means in particular that each O_i decides both statements $\check{\alpha} \in \tau_0$ and $\check{\alpha} \in \tau_1$.

Now, since $O_0 \Vdash \check{\alpha} \in \tau_0$ and $O_k \Vdash \check{\alpha} \notin \tau_0$, there must be a number $i \in k$ such that $O_i \Vdash \check{\alpha} \in \tau_0$ and $O_{i+1} \Vdash \check{\alpha} \notin \tau_0$. It then must be the case that $f_0''O_i, f_0''O_{i+1} \subset Y_0$ are disjoint; therefore, the sets $f_1''O_i, f_1''O_{i+1} \subset Y_1$ are equal. This means that the sets O_i, O_{i+1} must either both force $\check{\alpha} \in \tau_1$ or both force $\check{\alpha} \notin \tau_1$. In conclusion, one of the sets O_i, O_{i+1} forces $\tau_0 \neq \tau_1$, a contradiction.

To see why the negation of (1) implies the negation of (2), let $O \subset X$ be a nonempty open set witnessing the failure of (1). For each $y \in Y_0$, define the *orbit* of y to be the set of all elements $z \in Y_0$ such that there is a walk $\langle x_i : i \leq 2k \rangle$ in O such that $f_0(x_0) = y$ and $f_0(x_{2k}) = z$. Let $x \in O$ be a P_X-generic and work in the model $V[x]$.

Claim 3.1.6. *The orbit of $f_0(x)$ is nowhere dense in Y_0.*

Proof. If the orbit were dense in some nonempty open set $B \subset Y_0$, then in the ground model, the set B would show that O is not a witness to the failure of (1). To see this, suppose that $B_0, B_1 \subset B$ are nonempty open subsets of B. Working in $V[x]$, find walks w_0, w_1 in O which lead from $f_0(x)$ to B_0 and B_1, invert w_0 and concatenate with w_1 and get a walk from B_0 to B_1. By the Mostowski absoluteness between $V[x]$ and V, there must be such a walk in V as well. □

Now look at the set $A = \{B \subset Y_0 : B \text{ is a basic open subset of } Y_0 \text{ and } B \text{ contains no element of the orbit of } f_0(x)\}$.

Claim 3.1.7. $A \in V[f_0(x)] \cap V[f_1(x)]$.

Proof. In $V[x]$, the set A is defined as the set of all basic open sets $B \subset Y_0$ such that there is no walk $\langle x_i : i \leq 2k \rangle$ in O such that $f_0(x_0) = f_0(x)$ and $f_0(x_{2k}) \in B$. This is a Π^1_1 definition with parameter $f_0(x)$ which therefore yields the same set in the model $V[f_0(x)]$.

In $V[x]$, the same set A also has an alternative definition: it is the set of all basic open sets $B \subset Y_0$ such that there is no walk $\langle x_i : i \leq 2k \rangle$ in O such that $f_1(x_0) = f_1(x)$ and $f_0(x_{2k}) \in B$. To see this, note that a walk with $f_0(x_0) = f_0(x)$ can be transformed into a walk with $f_1(x_0) = f_1(x)$ (and also vice versa) by simply adding the point x twice to the beginning of the walk. This alternative definition of the set A is again Π^1_1 with parameter $f_1(x)$, so Mostowski absoluteness yields the same set in the model $V[f_1(x)]$. □

To show that (2) fails, it is enough to argue that $A \notin V$. However, if $A \in V$ then $\bigcup A \in V$ is an open dense subset of Y_0 by Claim 3.1.6, and $f_0(x) \notin \bigcup A$. This contradicts Proposition 3.1.1: the point $f_0(x) \in Y$ is forced to be P_Y-generic over the ground model, so to belong to every open dense set in the ground model. □

3.2 Examples

Several groups of results in this book depend on an identification of a suitable pair of independent maps. The most important example comes from Hjorth's notion of turbulence of group actions. Recall the standard definition:

Definition 3.2.1. [48, Section 13.1] Let Γ be a Polish group continuously acting on a Polish space Y.

1. If $U \subset \Gamma, O \subset Y$ are sets then a U, O-*walk* is a sequence $\langle y_i \colon i \leq k, \gamma_i \colon i < k\rangle$ such that for each $i \leq k$ $y_i \in O$ holds, and for each $i < k$ $\gamma_i \in U$ and $y_{i+1} = \gamma_i \cdot y_i$ both hold;
2. if $y \in O$ then U, O-*orbit* of y is the set of all $z \in O$ such that there is a U, O-walk starting at y and ending at z;
3. the action is *turbulent at* $y \in Y$ if for all open sets $U \subset \Gamma$ and $O \subset X$ with $1 \in U$ and $y \in O$ the U, O-orbit of y is somewhere dense;
4. the action is *generically turbulent* if its orbits are meager and dense and the set of points of turbulence is comeager.

Now, suppose that Γ is a Polish group acting continuously on a Polish space Y. Let $X = \{\langle \gamma, y_0, y_1\rangle \in \Gamma \times Y \times Y \colon \gamma \cdot y_0 = y_1\}$; this is a closed subset of $\Gamma \times Y \times Y$ and Polish in the inherited topology. Let $f_0 \colon X \to Y$ be the projection into the second coordinate, $f_1 \colon X \to Y$ be the projection into the third coordinate, and let $f_2 \colon X \to \Gamma \times Y$ be the projection into the first two coordinates. Since the set X can be viewed as a graph of a continuous function of any pair of coordinates into the remaining one, these mappings are continuous and open. With this notation in hand, we prove:

Theorem 3.2.2. *Let Γ be a Polish group continuously acting on a Polish space Y such that all orbits are meager and dense. The following are equivalent:*

1. *the action is generically turbulent;*
2. *$f_0, f_1 \colon X \to Y$ is an independent pair of functions;*
3. *$P_\Gamma \times P_Y$ forces $V[\dot{x}_{gen}] \cap V[\dot{\gamma}_{gen} \cdot \dot{x}_{gen}] = V$.*

Proof. For the implication (1)→(2), suppose that the action is generically turbulent. Let $O \subset X$ be a nonempty open set. Find a point $\delta \in \Gamma$ and open sets

$U \subset \Gamma$ and $O_0, O_1 \subset Y$ such that $1 \in U$, $U = U^{-1}$, and $(\delta \cdot U \times O_0 \times O_1) \cap X \subset O$. Find a point $y \in O_0$ such that the U, O_0-orbit of y is somewhere dense; let $A \subset Y$ be a nonempty open set such that the U, O_0-orbit of y is dense in A. We claim that $A \subset Y$ is an open set witnessing the independence of functions f_0 and f_1.

To see this, let $B_0, B_1 \subset A$ be nonempty open sets. Concatenating U, O_0-walks from y to B_0 and B_1, we can find a U, O_0-walk $\langle y_i : i \leq k, \gamma_i : i < k\rangle$ which starts in B_0 and ends in B_1. Consider the sequence $\langle x_i : i \leq 2k\rangle$ of points in the open set $O \subset X$ given by $x_{2i} = \langle \delta, y_i, \delta \cdot y_i\rangle$ and $x_{2i+1} = \langle \delta\gamma_i, y_i, \delta y_{i+1}\rangle$ if $i < k$. It is immediate that this is an walk of points in the set O in the sense of Definition 3.1.2, confirming the independence of functions f_0, f_1.

For the implication (2)→(1), suppose that the functions f_0, f_1 are independent. Let $U \subset \Gamma$ be an open neighborhood of 1 and $O_0 \subset Y$ be a nonempty open set. Choose an open neighborhood $U_0 \subset \Gamma$ of the unit such that $U_0^{-1} \cdot U_0 \subset U$ and let $O = \{\langle \gamma, y_0, y_1\rangle \in X : \gamma \in U_0 \text{ and } y_0, y_1 \in O_0\}$. Suppose that $\langle x_i : i \leq k\rangle$ is a walk of points in the set O in the sense of Definition 3.1.2 and $x_i = \langle \delta_i, z_i, z'_i\rangle$; then the sequence $\langle z_i : i \leq k, \gamma_i : i < k\rangle$ is an U, O_0-walk in the sense of Definition 3.2.1, where $\gamma_i = 1$ if $z_i = z_{i+1}$ and $\gamma_i = \delta_{i+1}^{-1}\delta_i$ if $z_i \neq z_{i+1}$ (in which case $z'_i = z'_{i+1}$).

Now, if $A \subset O_0$ is an open set witnessing the independence of f_0, f_1 for O and $B \subset A$ is a nonempty open set, then the set $\{y \in A :$ there is a walk of points in O from y to $B\}$ is open dense in A, and by the previous paragraph it is a subset of the set $\{y \in A :$ there is an U, O_0-walk from y to $B\}$. Thus, the set $C_{U,O_0} = \{y \in Y :$ either $y \notin A$ or the U, O_0-orbit of y is dense in $A\}$ is comeager. The set of turbulent points then contains the intersection $\bigcap\{C_{U,O_0} : U \subset \Gamma$ is a basic open neighborhood of 1 and $O_0 \subset Y$ is a basic open set$\}$ and is comeager.

The equivalence (2)↔(3) follows from Theorem 3.1.4 and the fact that the function f_2 is continuous and open, so the first two coordinates of the P_X-generic point are generic for $P_\Gamma \times P_Y$ by Proposition 3.1.1. □

A somewhat similar characterization theorem can be proved for analytic ideals on ω.

Definition 3.2.3. [13] A set $I \subset \mathcal{P}(\omega)$ is *ω-hitting* if for every countable collection of infinite subsets of ω there is a set $b \in I$ which has nonempty intersection with each set in the collection.

A typical analytic ω-hitting ideal is I_p, where p is a partition of ω into finite sets and I_p is the collection of all sets $a \subset \omega$ such that for some number $n \in \omega$, for every set $b \in p$, $|a \cap b| \leq n$. In fact, [43] shows that every Borel ω-hitting ideal contains one of the ideals I_p as a subset in a suitable sense. ω-hitting ideals give rise to pairs of independent functions in the proof of the following theorem:

Theorem 3.2.4. *Let I be an analytic ideal on ω. The following are equivalent:*

1. *I is ω-hitting;*
2. *in some forcing extension, there are points $y_0, y_1 \in 2^\omega$ separately Cohen-generic over V such that $y_0 =_I y_1$ and $V[y_0] \cap V[y_1] = V$.*

Proof. For the direction (1)→(2), assume that the ideal I is ω-hitting. Let $h\colon \omega^\omega \to \mathcal{P}(\omega)$ be a continuous map such that $I = \operatorname{rng}(h)$. Let $T = \{t \in \omega^{<\omega}\colon h''[t]$ is ω-hitting$\}$. The set $T \subset \omega^{<\omega}$ is clearly closed under initial segment.

Claim 3.2.5. *For all $t \in T$ there is $n \in \omega$ such that for all $m > n$ there is $z \in [T \restriction t]$ such that $m \in h(z)$.*

Proof. Suppose towards a contradiction that the conclusion fails for some node $t \in T$. Let $a = \omega \setminus \bigcup h''[T \restriction t]$; the set $a \subset \omega$ is infinite. For each $s \in \omega^{<\omega} \setminus T$ let c_s be a countable collection of infinite subsets of ω such that for all $z \in [s]$, $h(z)$ has finite intersection with one element of c_s. Let $c = \bigcup_{s \notin T} c_s \cup \{a\}$; it is not difficult to see that for every $z \in [t]$, $h(z)$ has finite intersection with an element of c, contradicting the assumption that $t \in T$. □

Now, let X be the closed set of all triples $\langle y_0, y_1, z\rangle \subset 2^\omega \times 2^\omega \times [T]$ such that $\{n \in \omega\colon y_0(n) \neq y_1(n)\} \subset h(z)$. Let $f_0, f_1\colon X \to 2^\omega$ be the projections into the first and second coordinate. It is not difficult to see that both of these maps are continuous and open. In view of Theorem 3.1.4, the following claim concludes the proof of (2).

Claim 3.2.6. *The maps f_0, f_1 are independent.*

Proof. Let $O \subset X$ be an open set. Find $u_0, u_1 \in 2^{<\omega}$ and $t \in T$ such that $([u_0] \times [u_1] \times [t]) \cap X$ is a nonempty subset of O. Thinning out further, we may assume that the binary strings u_0, u_1 have the same length and for all $z \in [T \restriction t]$ and every l such that $u_0(l) \neq u_1(l)$, $n \in h(z)$ holds. Let $n \in \omega$ be such that for all $m > n$ there is $z \in [T \restriction t]$ such that $m \in h(z)$. Let $v \in 2^{<\omega}$ be any binary string extending u_0 of length $> n$ and $A = [v]$. We claim that the open set A witnesses the definition of independence for f_0, f_1 and O.

Indeed, suppose that w_0, w_1 are two binary strings of the same length extending v. To produce a walk from $[w_0]$ to $[w_1]$, list all entries on which the strings w_0, w_1 differ by $\{m_j\colon j \in i\}$. Let $y \in [w_0]$ be any point, and let $y_j \in 2^\omega$ be the point obtained from y by flipping the outputs at all m_k for $k \in j$. Thus, y_j differs from y_{j+1} only at entry m_j and $y_i \in [w_1]$. Let $y'_j \in 2^\omega$ be the point obtained by replacing the initial segment $u_0 \subset y_j$ with u_1. Also, let $z_j \in [T \restriction t]$ be any point such that $m_j \in h(z_j)$. Now, define points $x_{2j+1}, x_{2j+2} \in X$ for $j \in i$ as follows:

- $x_0 = \langle y_0, y'_0, z\rangle$ for any $z \in [T \restriction t]$;
- $x_{2j+1} = \langle y_j, y'_{j+1}, z_j\rangle$;
- $x_{2j+2} = \langle y_{j+1}, y'_{j+1}, z\rangle$ for any $z \in [T \restriction t]$.

Clearly, the sequence $\langle x_k\colon k \leq 2i\rangle$ is a walk from $[w_0]$ to $[w_1]$ as required. □

To prove that (2) implies (1), assume that (1) fails, as witnessed by a countable collection $\{a_n\colon n \in \omega\}$ of infinite subsets of ω. Suppose that in some generic extension, $y_0, y_1 \in 2^\omega$ are $=_I$-related points which are separately Cohen-generic over V. The set $b = \{n \in \omega \colon y_0(n) \neq y_1(n)\}$ belongs to the ideal I; therefore, there is a number $n \in \omega$ such that $b \cap a_n$ is finite. It follows that the functions $y_0 \restriction a_n$ and $y_1 \restriction a_n$ are modulo finite equal; therefore, they belong to $V[y_0] \cap V[y_1]$. By a genericity argument, $y_0 \restriction a_n \notin V$ holds, so (2) fails. $\square$

Another example of an independent pair of functions deals with Polish fields.

Theorem 3.2.7. *Let Φ be an uncountable Polish field. Let $X = \{\langle u_0, u_1\rangle \in (\Phi^2)^2 \colon u_0 \cdot u_1 = 1\}$ with the topology inherited from Φ^4. The projection functions from X to the first and the second coordinate are independent.*

Proof. Write $f_0 \colon X \to \Phi^2$ and $f_1 \colon X \to \Phi^2$ for the two projection functions. It is not difficult to see that the maps f_0, f_1 are continuous and open. Towards the independence, let $O \subset (\Phi^2)^2$ be an open set with nonempty intersection with X. Let $\langle u_0, u_1\rangle$ be a point in $X \cap O$. Let $v_0 \in \Phi^2$ be any nonzero vector such that $u_0 \cdot v_0 = 0$. By the continuity of multiplication, there must be a nonzero field element $y \in \Phi$ such that the pair $\langle u_0, u_1 + yv_0\rangle$ belongs to O; it also belongs to X. The determinant of the 2×2 matrix consisting of the vectors u_1 and $v_1 = u_1 + yv_0$ is nonzero. It follows that there are open neighborhoods $O_0, O_1 \subset \Phi^2$ of u_1, v_1 respectively such that for each choice of $w_0 \in O_0$ and $w_1 \in O_1$, the system of equations $x \cdot w_0 = 1$ and $x \cdot w_1 = 1$ has a unique solution, and the solution $x \in \Phi^2$ is moreover such that $\langle x, w_0\rangle$, $\langle x, w_1\rangle$ are both elements of O.

We claim that the set $O_0 \subset \Phi^2$ stands witness to the independence of the functions f_0, f_1. Indeed, if $w_0, w_0' \in O_0$ are arbitrary points, then choose any point $w_1 \in O_1$, find vectors x_0 and x_1 such that $x_0 \cdot w_0 = 1$, $x_0 \cdot w_1 = 1$, $x_1 \cdot w_0' = 1$ and $x_1 \cdot w_1 = 0$, and observe that $\langle x_0, w_0\rangle$, $\langle x_0, w_1\rangle$, $\langle x_1, w_1\rangle$,, $\langle x_1, w_0'\rangle$ is a walk of points in O starting at w_0 and ending at w_0'. $\square$

The following theorem is closely related to the previous one in view of the fact that its statement is intended to be applied with algebraic curves.

Theorem 3.2.8. *Let $C \subset \mathbb{R}$ be a smooth curve containing no straight segments, which is closed as a subset of $\mathbb{R}^2$. Let $X = \langle u_0, u_1\rangle \in \mathbb{R}^2 \colon u_0 - u_1 \in C\}$. The projection functions from X to the first and second coordinate are independent.*

Proof. Since the set C is closed, $X \subset (\mathbb{R}^2)^2$ is closed and equipped with the topology inherited from $(\mathbb{R}^2)^2$. Let $f_0 \colon X \to \mathbb{R}^2$ and $f_1 \colon X \to \mathbb{R}^2$ be the projection functions. Clearly, f_0 and f_1 are continuous. They are also open: if $O \subset (\mathbb{R}^2)^2$ is an open set, $\langle u_0, u_1\rangle \in X \cap O$ is a point, and $\varepsilon > 0$ is a real number such that for each vector $v \in \mathbb{R}^2$ of norm smaller than ε the point $\langle u_0 + v, u_1 + v\rangle$ belongs to O, then the open neighborhood $\{u_0 + v \colon |v| < \varepsilon\}$ of u_0 is a subset of the f_0-image of $X \cap O$, as the points $\langle u_0 + v, u_1 + v\rangle$ belong not only to O but also to X as $u_0 - u_1 = (u_0 + v) - (u_1 + v)$.

To see the independence, fix an open set $O \subset (\mathbb{R}^2)^2$ and a point $\langle u_0, u_1\rangle \in X \cap O$. Translating if necessary, we may assume that $u_1 = 0$. Find an open ball $B \subset \mathbb{R}^2$ close to u_0 such that $B \times (B - B) \subset O$ and $C \cap B$ is a simple curve connecting two distinct points at the boundary of B. As the curve C contains no straight segments, there must be points $p_0, p_1 \in B \cap C$ such that the tangent vectors at p_0 and p_1 are nonzero and distinct. An inspection reveals that there must be an open set $P \subset \mathbb{R}^2$ close to 0 such that $p_0 - P \subset B$ and for every vector $v \in P$, the points $p_0 + v$ and $p_1 + v$ end up on the opposite sides of the curve C and in the ball B, and the v-shift of the part of the curve C between the points p_0, p_1 is a subset of B. By the Jordan curve theorem, the curves C and $C + v$ must intersect in the ball B.

We claim that the set $p_0 - P \subset \mathbb{R}^2$ witnesses the independence of the maps f_0, f_1. Let $v_0, v_1 \in P$ be two vectors and $q_0, q_1 \in B$ be respective points in the intersections $C \cap (C + v_0)$ and $C \cap (C + v_1)$. Then $\langle p_0 - v_0, p_0 - q_0\rangle$, $\langle p_0, p_0 - q_0\rangle$, $\langle p_0, p_0 - q_1\rangle$, and $\langle p_0 - v_1, p_0 - q_1\rangle$ constitutes a walk from $p_0 - v_0$ to $p_0 - v_1$ using points in O. $\square$

The assumption that C contains no straight segments is necessary in Theorem 3.2.8. Suppose that C is a line passing through the origin and argue that the projection functions are not independent. To see this, consider the P_X-generic pair $\langle u_0, u_1\rangle$. The line in the direction of C through u_0 passes through u_1 and vice versa. Let B be the collection of all basic open subsets of $\mathbb{R}^2$ which have empty intersection with this line; thus, $B \in V[u_0] \cap V[u_1]$. The set $\bigcup B$ is open dense in $\mathbb{R}^2$ and contains neither u_0 nor u_1. Since u_0, u_1 are Cohen-generic points, they meet all open dense subsets of $\mathbb{R}^2$ coded in the ground model; it follows that $B \notin V$. By Theorem 3.1.4, the projection functions are not independent.

3.3 Placid equivalence relations

The characterization of turbulence in Theorem 3.2.2 leads to many Borel nonreducibility results and cardinal preservation results in the generic extensions of the Solovay model. To state these results succinctly, the following definitions are helpful.

Definition 3.3.1. Let E be an analytic equivalence relation on a Polish space X. We say that E is *placid* if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of V (inside some ambient generic extension) such that $V[H_0] \cap V[H_1] = V$ and $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ are E-related points in the space X, then they are E-related to some point in V.

Definition 3.3.2. Let E be an analytic equivalence relation on a Polish space X. We say that E is *virtually placid* if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of V (inside some ambient generic extension) such that $V[H_0] \cap V[H_1] = V$ and $\langle Q_0, \tau_0\rangle \in V[H_0]$ and $\langle Q_1, \tau_1\rangle \in V[H_1]$ are $\bar{E}$-related E-pins, then they are $\bar{E}$-related to some E-pin in V.

In other words, an equivalence relation E is placid if disjoint generic extensions do not share any E-classes which are not represented in V. An equivalence relation E is virtually placid if disjoint generic extensions do not share any virtual E-classes which are not represented in V. The following propositions encapsulate the basic properties of the two classes of equivalence relations.

Proposition 3.3.3. *Let E be an analytic equivalence relation on a Polish space X. E is virtually placid if and only if for any separately generic extensions $V[H_0], V[H_1]$ such that $V[H_0] \cap V[H_1] = V$ and E-related points $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$, x_0 and x_1 are realizations of some virtual E-class in V.*

Proof. The left-to-right implication is immediate from the definitions. For the right-to-left implication, suppose that $V[H_0]$ and $V[H_1]$ are separately generic extensions such that $V[H_0] \cap V[H_1] = V$ and $\langle Q, \tau_0 \rangle \in V[H_0]$ and $\langle Q_1, \tau_1 \rangle \in V[H_1]$ are $\bar{E}$-related pins. Let $K_0 \subset Q_0$ and $K_1 \subset Q_1$ be filters mutually generic over the model $V[H_0, H_1]$ and let $x_0 = \tau_0/K_0$ and $x_1 = \tau_1/K_1$. By the mutual genericity, $V[H_0][K_0] \cap V[H_1][K_1] = V$, and the assumption on E gives that x_0, x_1 are realizations of some virtual E-class from the ground model. It follows that both E-pins are $\bar{E}$-related to some E-pin from V. □

Proposition 3.3.4. *An analytic equivalence relation E on a Polish space X is placid if and only if it is pinned and virtually placid.*

Proof. The right-to-left implication is immediate. If E is a pinned and virtually placid equivalence relation and $V[H_0], V[H_1]$ are separately generic extensions such that $V[H_0] \cap V[H_1] = V$, then any E-class represented in both must be represented as a virtual class in V by the virtual placid assumption, and this virtual class must be trivial by the pinned assumption. This means that E is placid. For the left-to-right implication, if E is placid then it must be pinned because if $V[H_0], V[H_1]$ are mutually generic extensions then $V[H_0] \cap V[H_1] = V$ by the product forcing theorem. To show that E must be virtually placid, suppose that $V[H_0]$, $V[H_1]$ are separately generic extensions. By Proposition 3.3.3, it is enough to verify that every E-class represented in both extensions is a realization of some virtual E-class in V. However, the placid assumption even implies that it is a realization of a trivial virtual E-class in V. □

A good example of an equivalence relation which is virtually placid but not placid is $\mathbb{F}_2$. It is not placid since it is not pinned.

The following ergodicity theorem is a great generalization of the ergodicity theorems of Hjorth and Kechris [52, Theorem 12.5], and it is the main motivation behind the definition of placid and virtually placid classes of equivalence relations.

Theorem 3.3.5. *Suppose that a Polish group Γ acts continuously and in a generically turbulent way on a Polish space X such that the resulting orbit equivalence relation E is analytic, with all orbits dense. Suppose that F is a virtually placid equivalence relation on a Polish space Y. Suppose that $h\colon X \to Y$ is a homomorphism of E to F. Then there is a comeager set $B \subset X$ which is mapped into a single F-class.*

Proof. Let $\gamma \in \Gamma, x \in X$ be a mutually generic pair of points for the posets P_Γ and P_X. Theorem 3.2.2 implies the instrumental equality: $V[x] \cap V[\gamma \cdot x] = V$.

Since $x \mathrel{E} \gamma \cdot x$ holds, it must be the case that $h(x) \mathrel{F} h(\gamma \cdot x)$ must hold. By the virtual placidity of the equivalence relation F, it must be the case that $h(x)$ is a realization of a ground model virtual F-class. However, the poset P_X is countable, so all virtual classes realized in its extension are already represented in the ground model by Theorem 2.6.2. Thus, there must be a point $y \in Y \cap V$ such that $h(x) \mathrel{F} y$. Since the generic point $x \in X$ avoids all ground model coded meager sets, it must be the case that the set $B = h^{-1}([y]_E) \subset X$ is nonmeager. Since this is an analytic set which is invariant under the continuous group action all of whose orbits are dense, it follows that the set B is comeager. $\square$

3.4 Examples and operations

The extent of the classes of placid and virtually placid equivalence relations is best explored using the following closure theorems.

Theorem 3.4.1. *The class of placid equivalence relations is closed under the following operations:*

1. *Borel almost reduction;*
2. *countable product;*
3. *countable increasing union;*
4. *countable factor;*

The class of virtually placid equivalence relations is closed under the same operations and the Friedman–Stanley jump.

Proof. In view of Proposition 3.3.4, it is enough to show the closure of virtual placidity under these operations, since the class of pinned equivalence relations is closed under (1–4) by the work of Chapter 2. For virtual placidity, we will prove (1) and the closure under the Friedman–Stanley jump.

For (1), suppose that E, F are analytic equivalence relations on Polish spaces X, Y and $h \colon X \to Y$ is a Borel function which is a reduction of E to Y everywhere except for a set $Z \subset X$ consisting of countably many E-classes. Suppose that F is virtually placid and work towards the conclusion that E is virtually placid. Let $V[H_0]$, $V[H_1]$ be generic extensions such that $V[H_0] \cap V[H_1] = V$ and let $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ be E-related points in X; we need to show that they are realizations of some virtual E-class in V. If $x_0 \in Z$ then this certainly occurs as V contains a countable set of points whose E-classes cover Z, and this feature of the countable set persists to $V[H_0]$ by a Shoenfield absoluteness argument. If $x_0 \notin Z$ then $x_1 \notin Z$; look at the points $h(x_0) \in V[H_0]$ and $h(x_1) \in V[H_1]$. These are F-related points; since F is placid they realize some virtual F-class represented by some F-pin $\langle Q, \sigma \rangle$. By a Shoenfield absoluteness argument, $Q \Vdash \exists x \in X \setminus Z \; h(x) \mathrel{E} \sigma$; let τ be a Q-name for such a point $x \in X$. It is not difficult to see that $\langle Q, \tau \rangle$ is an E-pin and x_0, x_1 realize the virtual class of $\langle Q, \sigma \rangle$.

For the Friedman–Stanley jump, suppose that E is a virtually placid equivalence relation on a Polish space X, $V[H_0]$ and $V[H_1]$ are separately generic extensions of V such that $V[H_0] \cap V[H_1] = V$ and $y_0, y_1 \in X^\omega$ are elements in the respective models such that $[\mathrm{rng}(y_0)]_E = [\mathrm{rng}(y_1)]_E$. Since E is virtually placid, every element of $\mathrm{rng}(y_0)$ is a realization of a virtual E-class from V. Let B be the set of all virtual E-classes in V which have realizations in $\mathrm{rng}(y_0)$. Since B is also the set of all virtual E-classes in V which have realizations in $\mathrm{rng}(y_1)$, it is clear that $B \in V[H_0] \cap V[H_1] = V$. Thus, the points $y_0, y_1 \in X^\omega$ are realizations of the virtual E^+-class represented by the set B. By Proposition 3.3.3, we conclude that E^+ is virtually placid. $\square$

Theorem 3.4.2. *Suppose that Γ is a Polish group continuously acting on a Polish space X and E is the resulting orbit equivalence relation. The following are equivalent:*

1. *E is virtually placid;*
2. *for every Borel set $B \subset X$ such that $E \restriction B$ is Borel, $E \restriction B$ is virtually placid.*

Proof. (1) immediately implies (2) since $E \restriction B$ is reducible to E by the identity map on B. Now suppose that (1) fails, and let P be a poset and τ be a name for an element of X such that $P \times P \Vdash \tau_{\mathrm{left}}\ E\ \tau_{\mathrm{right}}$ fails, and let Q be a poset with names $\dot{H}_0$, $\dot{H}_1$ for filters generic over the ground model such that $Q \Vdash V[\dot{H}_0] \cap V[\dot{H}_1] = V$ and $\tau/\dot{H}_0 = \tau/\dot{H}_1$.

Let M be a countable elementary submodel of a large structure containing all objects mentioned so far. Let $\bar{M}$ be the transitive collapse of M, and $\bar{P}, \bar{Q}$ etc. be the images of P, Q etc. under the transitive collapse map. By Proposition 3.5.5, $\bar{Q} \Vdash V[\bar{H}_0] \cap V[\bar{H}_1] = V$ holds even in V (as opposed to $\bar{M}$). Let $B = \{x \in X \colon \exists H \subset \bar{P}$ generic over $\bar{M}$ such that $x = \bar{\tau}/H\}$. We have just proved that $E \restriction B$ is not virtually placid. The proof is completed by a reference to Corollary 2.8.8–the set $B \subset X$ is Borel and the equivalence relation $E \restriction B$ is Borel as well. $\square$

The theorem allows the transfer of the virtual placid property from Borel fragments of a given orbit equivalence relation to the whole equivalence relation. Consider the following attractive corollary:

Corollary 3.4.3. *Every equivalence relation classifiable by countable structures is virtually placid.*

Proof. In view of Theorem 3.4.1(1), it is only necessary to show that every orbit equivalence relation E obtained from an action of S_∞ is virtually placid. By Theorem 3.4.2, it is only necessary to show that all Borel fragments of E are virtually placid. To this end, consider the transfinite sequence $\langle \mathbb{F}_\alpha \colon \alpha \in \omega_1\rangle$ obtained from $\mathbb{F}_1$ =identity on 2^ω by repeated application of Friedman–Stanley jump, at limit stages taking disjoint unions. It is well-known [48, Theorem 12.5.2] that every Borel equivalence relation classifiable by countable structures is Borel reducible to $\mathbb{F}_\alpha$ for some countable ordinal α. Theorem 3.4.1 iterated transfinitely shows that each $\mathbb{F}_\alpha$ is virtually placid. Thus, every Borel fragment of E is virtually placid, and so is E. □

There are many Borel placid equivalence relations which cannot be obtained from the identity by repeated application of the operations indicated in Theorem 3.4.1. The following examples deal with equivalence relations associated with ideals on countable sets. If a is a countable set and J is an ideal on a, write $=_J$ for the equivalence relation on the space $X = (2^\omega)^a$ connecting points x_0, x_1 if the set $\{n \in a\colon x_0(n) \neq x_1(n)\}$ belongs to J.

Example 3.4.4. Let $a = 2^{<\omega}$ and let J be the *branch ideal*: the ideal generated by the subsets of a linearly ordered by inclusion. The equivalence relation $=_J$ is placid.

Proof. Write $X = (2^\omega)^{2^{<\omega}}$. For every node $t \in \omega^{<\omega}$ write $[t]$ for the set of all nodes in $2^{<\omega}$ extending t. Let $V[G_0], V[G_1]$ be two generic extensions containing respective $=_J$-related points $x_0, x_1 \in X$. Assume that $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $x \in X$ which is $=_J$-related to both x_0, x_1.

Let $T = \{t \in 2^{<\omega}\colon x_0 \restriction [t]$ is not $=_J$-equivalent to any point in the ground model$\}$. This is a subtree of $\omega^{<\omega}$. If $0 \notin T$ then the proof is complete; thus, it is only necessary to derive a contradiction from the assumption $0 \in T$. First, observe that the tree T cannot have any terminal nodes: if t were a terminal node of T then one could combine the ground model witnesses for $t^\frown 0 \notin T$ and $t^\frown 1 \notin T$ to find a witness for $t \notin T$. Second, observe that the definition of the tree T depends only on the $=_J$-class of x_0, so $T \in V[G_0] \cap V[G_1] = V$. Since T is a nonempty ground model tree without endnodes, it must have an infinite branch $y \in 2^\omega$ in the ground model. Since $x_0 =_J x_1$, there is a number $n \in \omega$ such that for every $t \in 2^{<\omega}$ such that $x_0(t) \neq x_1(t)$, either t is incompatible with $y \restriction n$ or else t is an initial segment of y. Let $e = [y \restriction n] \setminus \{y \restriction m\colon m \geq n\}$ and observe that $e \in V$ and x_0 and x_1 coincide on e, therefore $x_0 \restriction e \in V$. If $z \in V$ is any function in $2^{[y \restriction n]}$ extending $x_0 \restriction e$, then $z =_J x_0 \restriction [t]$, contradicting the assumption that $y \restriction n \in T$. □

A whole class of ideals with placid equivalence relations is isolated in the following definition.

Definition 3.4.5. An ideal I on ω is *countably separated* if there is a countable collection A of subsets of ω such that for every $b \in I$ and $c \notin I$ there is $a \in A$ such that $b \cap a = 0$ and $c \cap a \notin I$.

Countably separated ideals include for example the ideal of nowhere dense subsets of the rationals; a characterization theorem is proved in [62]. Note that if I is analytic then the instrumental property of the set A is coanalytic, and by Mostowski absoluteness it will hold in all forcing extensions.

Theorem 3.4.6. *The equivalence relation $=_I$ is placid for every countably separated Borel ideal I on ω.*

Proof. Let $\{a_n : n \in \omega\}$ be a countable collection of subsets of ω witnessing the countable separation of the ideal I. Write $X = (2^\omega)^\omega$. Let $V[G_0], V[G_1]$ be generic extensions containing respective $=_I$-related points $x_0, x_1 \in X$. Assume $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $x \in X$ which is $=_I$-related to both x_0, x_1.

Consider the set $b = \{n \in \omega : x_0 \restriction a_n$ is I-equivalent to some element of the ground model$\}$. The definition of the set b depends only on the $=_I$-equivalence class of x_0, therefore the set b belongs to both $V[G_0]$ and $V[G_1]$, and by the initial assumptions, to V. Let f be the map with domain b which to each $n \in b$ identifies the $=_I$-class in V which contains $x_0 \restriction a_n$. Again, the definition of f depends only on the $=_I$-class of x_0, so $f \in V[G_0]$ and $f \in V[G_1]$, therefore $f \in V$. By the Mostowski absoluteness between V and $V[G_0]$, there is a point $x \in X$ such that for all $n \in b$, $x \restriction a_n \in f(n)$. We will show that $x =_I x_0$ holds.

Suppose towards a contradiction that this fails. Consider the set $c = \{i \in \omega : x(i) \neq x_0(i)\} \notin I$ and the set $d = \{i \in \omega : x_0(i) \neq x_1(i)\} \in I$. By the countable separation of the ideal I, there must be a number $n \in \omega$ such that $c \cap a_n \notin I$ and $d \cap a_n = 0$. However, the latter equality shows that $x_0 \restriction a_n = x_1 \restriction a_n$, so $x_0 \restriction a_n \in V$ and $n \in b$. But then $\{i \in a_n : x(i) \neq x_0(i)\} \in I$ by the choice of the point x. This contradicts the fact that $c \cap a_n \notin I$. The proof of the theorem is complete. $\square$

There is a natural operation on analytic ideals which seeks a countably separated closure. For the discussion below, let $\{a_n : n \in \omega\}$ be a fixed countable collection of subsets of ω.

Definition 3.4.7. Let I be an ideal on ω. I^+ is the ideal on ω consisting of all sets $b \subset \omega$ such that there is $c \in I$ such that $\forall n \in \omega\ b \cap a_n \notin I \to c \cap a_n \neq 0$.

We will start with some simple observations. It is clear that I^+ is an ideal containing I as a subset. If I is analytic, then so is I^+. Moreover, I is separated by the sequence $\{a_n : n \in \omega\}$ just in case $I = I^+$. We will show that under suitable additional assumption, the jump preserves placidity of the equivalence relation $=_I$.

Definition 3.4.8. Let I be an ideal on ω. I is *skew* if for every set $c \subset \omega$ there is a subset $c' \subset c$ such that for every finite set $d \subset \omega$, if $c \cap \bigcap_{n\in d} a_n \notin I$ then $c' \cap \bigcap_{n\in d} a_n \notin I$ and if $c \cap \bigcap_{n\in d} a_n \in I$ then $c' \cap \bigcap_{n\in d} a_n$ is finite.

It is not dificult to see that every F_σ ideal is skew and every analytic P-ideal is skew.

Theorem 3.4.9. *If I is a skew analytic ideal on ω then I^+ is a skew analytic ideal as well. If in addition the equivalence relation $=_I$ is placid then the equivalence relation $=_{I^+}$ is placid as well.*

Proof. For the first sentence, if $c \subset \omega$ is a set and $c' \subset c$ witnesses the skew property for I and c, then c' also witnesses the skew property for I^+ and c. The proof of the second sentence is more involved. Write $X = (2^\omega)^\omega$. Let $V[G_0], V[G_1]$

be generic extensions containing respective $=_{I^+}$-related points $x_0, x_1 \in X$. Assume $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $x \in X$ which is $=_{I^+}$-related to both x_0, x_1.

Consider the set $b = \{n \in \omega\colon x_0 \restriction a_n$ is I^+-equivalent to some element of the ground model$\}$. The definition of the set b depends only on the $=_{I^+}$-equivalence class of x_0, therefore the set b belongs to both $V[G_0]$ and $V[G_1]$, and by the initial assumptions, to V. Let f be the map with domain b which to each $n \in b$ identifies the $=_{I^+}$-class in V which contains $x_0 \restriction a_n$. Again, the definition of f depends only on the $=_{I^+}$-class of x_0, so $f \in V[G_0]$ and $f \in V[G_1]$, therefore $f \in V$. By the Mostowski absoluteness between V and $V[G_0]$, there is a point $x \in X$ such that for all $n \in b$, $x \restriction a_n \in f(n)$. We will show that $x =_{I^+} x_0$ holds.

Suppose towards a contradiction that this fails. Consider the set $c = \{i \in \omega\colon x(i) \neq x_0(i)\} \notin I^+$ and the set $d = \{i \in \omega\colon x_0(i) \neq x_1(i)\} \in I^+$. It is now time to use the skew assumption on the ideal I. Find a set $c' \subset c$ witnessing the skew property of I^+. Use the definition of the jump to find a set $e \in I$ such that for all $n \in \omega$, $d \cap a_n \notin I$ implies $e \cap a_n \neq 0$. Since the set c' is positive in the jump, there must be a number $n \in \omega$ such that $c' \cap a_n \notin I$ and $e \cap a_n = 0$. The choice of the set c' shows that $c' \cap a_n$ is in fact I^+-positive. The choice of the set e shows that $d \cap a_n \in I$. This means that $x_0 \restriction a_n =_I x_1 \restriction a_n$, and by the placid assumption on the equivalence relation $=_I$, $n \in b$ must hold. This stands in contradiction with the fact that $c' \cap a_n$, so $c \cap a_n$ is I^+-positive. The proof of the theorem is complete. $\square$

Example 3.4.10. Let I be the branch ideal on 2^ω, and let $\{a_n\colon n \in \omega\}$ be an enumeration of the basic open subsets of $2^{<\omega}$. The ideal I is skew. One can start iterating the jump for countable ordinals α, at limit stages taking unions. The resulting ideals consist of subsets of $\omega^{<\omega}$ whose closure in 2^ω is countable with Cantor–Bendixson rank $\leq \alpha$. All the resulting equivalence relations are placid.

3.5 Absoluteness

The placid and virtually placid classes of equivalence relations are defined in such a way that it is not clear whether the membership in them is absolute, and what the actual complexity is. This section provides a satisfactory resolution to these questions.

To reach the absoluteness result, one has to perform a computation of intersections of forcing extensions of independent interest. The computation starts with several definitions:

Definition 3.5.1. Let B be a Boolean algebra. A subalgebra $A \subset B$ is *projective* if the *projection function* $\pi\colon B \to A$, assigning to each $b \in B$ the smallest element of A which is $\geq b$, is defined for every b.

A good example of a projection is any complete subalgebra of a complete Boolean algebra. The point of the current definition is that the property of being projective is first order in A, B, so (unlike completeness) transfers from model to model without damage. Note that if $A \subset B$ is a projective subalgebra, then every maximal antichain of A is also a maximal antichain of B; therefore, the intersection of a generic filter on B with A will be a generic filter on A. To see this, let $D \subset A$ be a maximal antichain of A and $b \in B$ be a condition. Let $d \in D$ be an element of A compatible with $\pi(b)$. Then $\pi(b) - d \in A$ is strictly smaller element of A than $\pi(b)$, so $\pi(b) - d \not\geq b$ holds. This means that d and b must be compatible.

Definition 3.5.2. Let B be a Boolean algebra and $A_0, A_1 \subset B$ be projective subalgebras with projection functions π_0, π_1.

1. The *projection sequence* starting at $b \in B$ is the sequence $\langle b_n : n \in \omega \rangle$ defined by $b_0 = b$, $b_{2n+1} = \pi_0(b_{2n})$ and $b_{2n+2} = \pi_1(b_{2n+1})$;
2. the pair $\{A_0, A_1\}$ is *projective* if for each $b \in B$, the supremum of the projection sequence starting at b exists in B and belongs to $A_0 \cap A_1$.

Again, a good example of a projective pair is a pair of complete subalgebras of a complete Boolean algebra. The notion of a projective pair is no longer a first order statement about A_0, A_1, B, but it still transfers between ω-models of ZFC without damage.

Proposition 3.5.3. *Let B be a Boolean algebra and $\{A_0, A_1\}$ be a projective pair of subalgebras. Then*

1. *$A_0 \cap A_1 \subset B$ is a projective subalgebra of B;*
2. *if τ_0, τ_1 are A_0- and A_1-names for sets of ordinals and $B \Vdash \tau_0 = \tau_1$, then there is a $A_0 \cap A_1$-name τ_2 such that $B \Vdash \tau_0 = \tau_1 = \tau_2$.*

Proof. Let π_0, π_1 be the projections of B to A_0, A_1. Let $\pi \colon B \to A_0 \cap A_1$ be the function which assigns to each $b \in B$ the supremum of the projection sequence starting from b. It is clear from the definitions that $\pi(b)$ is the smallest element of $A_0 \cap A_1$ above b and (1) follows.

For (2), first note that for every $b \in B$ and every ordinal α, $b \Vdash \check{\alpha} \in \tau_0$ just in case $\pi(b) \Vdash \check{\alpha} \in \tau$. To see this, by induction on $n \in \omega$ argue that $b_n \Vdash \check{\alpha} \in \tau_0$. If this holds for $n = 2k$, then use the fact that τ_0 is an A_0-name, so $b_{n+1} = \pi_0(b_n)$ must force $\check{\alpha} \in \tau$; if this holds for $n = 2k + 1$, then use the fact that τ_1 is an A_1-name forced to be equal to τ_0, so $b_{n+1} = \pi_1(b_n)$ must force $\check{\alpha} \in \tau$. It follows that $b \Vdash \check{\alpha} \in \tau$ just in case $\pi(b) \Vdash \check{\alpha} \in \tau$, so one can let $\langle \pi(b), \check{\alpha} \rangle \in \tau_2$ if $b \Vdash \check{\alpha} \in \tau_0$. □

The above notions have the slightly unpleasant feature that they do not necessarily survive localization well, so an additional local definition is needed.

Definition 3.5.4. Let B be a Boolean algebra and $\{A_0, A_1\}$ be a pair of subalgebras. The pair is *locally projective* if for every nonzero element $b \in B$, in the algebra $B \restriction b$ the subalgebras $A_0 \restriction b = \{b \wedge a : a \in A_0\}$ and $A_1 \restriction b = \{b \wedge a : a \in A_1\}$ form a projective pair.

The following is the central motivation of the notion of a projective pair of subalgebras:

Proposition 3.5.5. *Let B be a Boolean algebra and A_0, A_1 be a locally projective pair of subalgebras. The following are equivalent:*

1. *$B \Vdash V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] = V$;*
2. *for every $b \in B$, the intersection algebra $A_0 \restriction b \cap A_1 \restriction b$ has an atom.*

Proof. Suppose that (2) fails for some $b \in B$. Write $C = A_0 \restriction b \cap A_1 \restriction b$ and use Proposition 3.5.3 to argue that C is a projective subalgebra of $B \restriction b$, so $b \Vdash \dot{G} \cap \check{C}$ is a C-generic filter, and since C is nonatomic, $\dot{G} \cap \check{C} \notin V$. At the same time, $\dot{G} \cap \check{C}$ belongs to $V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1]$, so (1) fails.

Suppose now that (2) holds. Suppose that τ_0, τ_1 are A_0- and A_1-names for sets of ordinals and some condition $b \in B$ forces $\tau_0 = \tau_1$. Write $C = A_0 \restriction b \cap A_1 \restriction b$ and let c be an atom of C. To verify (1), it is enough to argue that c decides the membership of any given ordinal in τ_0. However, this follows immediately from Proposition 3.5.3 (2) applied below b. $\square$

Finally, we are in a position to give a succinct and principled proof of the main absoluteness result of this section.

Theorem 3.5.6. *Let E be a Borel equivalence relation on a Polish space X. The statement "E is virtually placid" is absolute among transitive models of set theory containing the code for E.*

Proof. We will show that the statement "E is not virtually placid" is in ZFC equivalent to the statement "there is a countable ω-model M of a large fragment of ZFC containing the code for E such that $M \models E$ is not virtually placid". This is an analytic statement; therefore, it is absolute among transitive models of set theory.

Now, the left-to-right implication is immediate: one just needs to take a countable elementary submodel of a large enough structure to get the requisite M. The right-to-left direction is more interesting. Suppose that M is a countable ω-model containing the code for E which satisfies that E is not virtually placid. Working in the model M, there must be complete algebra B, complete subalgebras $A_0, A_1 \subset B$, and respective A_0, A_1-names τ_0, τ_1 for elements of the underlying space X such that first, $B \times B \Vdash \neg(\tau_0)_{\text{left}}\ E\ (\tau_0)_{\text{right}}$, and second, $B \Vdash V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] = V$ and $\tau_0\ E\ \tau_1$. The subalgebras $A_0, A_1 \subset B$ form a locally projective pair in M since they are complete. By Proposition 3.5.5 applied in M in the (1)$\to$(2) direction, for every $b \in B$, the algebra $A_0 \restriction b \cap A_1 \restriction b$ has an atom. Stepping out of the model M, we see that the pair $A_0, A_1 \subset B$ is a projective pair and for every $b \in B$, the algebra $A_0 \restriction b \cap A_1 \restriction b$ has an atom. By Proposition 3.5.5 applied in the (2)$\to$(1) direction, $B \Vdash V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] = V$ holds. Moreover, $B \Vdash \tau_0\ E\ \tau_1$, and $B \times B \Vdash \neg(\tau_0)_{\text{left}}\ E\ (\tau_0)_{\text{right}}$, since E is absolute between generic extensions of V and generic extensions of M by Borel absoluteness. In conclusion, E is not virtually placid in V. $\square$

Corollary 3.5.7. *Let E be a Borel equivalence relation on a Polish space X. The statement "E is placid" is absolute among transitive models of set theory containing the code for E.*

Proof. The placidity of E is a conjunction of virtual placidity and the pinned property of E by Proposition 3.3.4. The conjuncts are absolute by Theorem 3.5.6 and 2.7.1, and so is the conjunction. □

3.6 A variation for measure

The purpose of this section is to introduce a parallel for turbulence for measure, with attendant ergodicity results for measure. Curiously enough, the measure theoretic parallel for turbulence is intimately connected to the concentration of measure phenomenon. The following definition is close to the whirly actions of [33]:

Definition 3.6.1. Let X be a Polish space with a Borel probability measure μ and a metric d. Let Γ be a Polish group acting on X in a measure preserving and distance preserving fashion. We say that the action has *concentration of measure* if for every open neighborhood $U \subset \Gamma$ containing the unit and every $\varepsilon > 0$ there is $\delta > 0$ such that for every d-ball $B \subset X$ of radius $< \delta$ and every Borel set $C \subset B$ of relative μ-mass $> \varepsilon$, the set $(U \cdot C) \cap B$ has relative μ-mass $> 1/2$.

To formulate the main results of this section, let P_Γ be the Cohen forcing on Γ and P_μ be the random forcing with μ, i.e. the forcing with μ-positive Borel subset of X, ordered by inclusion. The poset P_Γ adds a generic point $\dot\gamma_{gen} \in \Gamma$ while the poset P_μ adds a random point $\dot x_{gen} \in X$.

Theorem 3.6.2. *Suppose that Γ is a Polish group acting on a Polish space X with a Borel probability measure μ and an ultrametric d in a measure preserving and distance preserving fashion. Suppose that the action has concentration of measure. Then $P_\Gamma \times P_\mu \Vdash V[\dot x_{gen}] \cap V[\dot\gamma_{gen} \cdot \dot x_{gen}] = V$.*

The proof of Theorem 3.6.2 hinges on a new walk concept and a proposition.

Definition 3.6.3. Let $U \subset \Gamma$ be an open neighborhood of the unit and $\delta > 0$. A U, δ-*walk* is a sequence $\langle x_i : i \leq j\rangle$ of points in X such that for every $i \in j$, either $d(x_i, x_{i+1}) < \delta$ or there is $\gamma \in U$ such that $\gamma \cdot x_i = x_{i+1}$.

Definition 3.6.4. Let $U \subset \Gamma$ be an open neighborhood of the unit and $D \subset X$ be a closed set. We say that the set D is U-*connected* if for any two points $x_0, x_1 \in D$ and any $\delta > 0$ there is a U, δ-walk from x_0 to x_1 using only points from D.

The main import of the concentration of measure assumption is that for any open neighborhood U of the unit, closed U-connected μ-positive sets can be found under every stone.

Proposition 3.6.5. *Suppose that the action of Γ has concentration of measure and d is an ultrametric. Then for every open neighborhood $U \subset \Gamma$ of the unit and every μ-positive Borel set $C \subset X$ there is a μ-positive U-connected compact set $D \subset C$.*

Proof. We will first fix useful terminology. For a symmetric neighborhood $V \subset \Gamma$ containing the unit and d-balls $B_0, B_1 \subset X$ of the same radius we say that B_0 is V-*related* to B_1 if there is an element $\gamma \in V$ such that $\gamma \cdot B_0 = B_1$. Since d is an ultrametric, this is equivalent to the statement that there is $\gamma \in V$ such that $\gamma \cdot B_0 \cap B_1 \neq 0$. A set $\mathcal{B}$ consisting of d-balls of the same radius will be called V-*connected* if for any two balls $B_0, B_1 \in \mathcal{B}$ one can find a sequence of balls in $\mathcal{B}$ which starts with B_0, ends in B_1, and successive balls in it are V-related. Lastly, for Borel sets $B, D \subset X$ write $\mu_B(D)$ for the relative measure of D in B, i.e. the ratio $\frac{\mu(B\cap D)}{\mu(B)}$.

Let $U \subset \Gamma$ be an open neighborhood of the unit, and let $C \subset X$ be a Borel μ-positive set. Find a symmetric open neighborhood $V \subset \Gamma$ of the unit such that $V^4 \subset U$, and thin down C if necessary to a compact μ-positive set. Let $\varepsilon_0 = 1/8$ and for each $n \in \omega$ let $\varepsilon_{n+1} = 2^{-n-2}\varepsilon_n$. Let $\delta_n > 0$ be the numbers witnessing the concentration of measure of the action for V and ε_n. Thinning down C if necessary, use the Lebesgue density theorem to find a d-ball B_{ini} of radius $< \delta_0$ such that $C \subset B_{\text{ini}}$ and $\mu_{B_{\text{ini}}}(C) > 1/2$.

By recursion on $n \in \omega$ find finite families $\mathcal{B}_n$ of pairwise disjoint d-balls of radius $< \delta_n$ such that for every number $n \in \omega$ $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$, and writing $Y(B) = \{A \in \mathcal{B}_{n+1} \colon A \subset B$ for every ball $B \in \mathcal{B}_n$, the following hold:

- for every $A \in Y(B)$, $\mu_A(C) > 2^{n+2}\varepsilon_{n+1}$;
- $\mu_B(C \cap \bigcup Y(B)) > (1 - 2^{-n})\mu_B(C)$.
- the set $Y(B)$ consists of balls of the same radius $< \delta_{n+1}$ and it is V^2-connected.

To begin, set $\mathcal{B}_0 = \{B_{\text{ini}}\}$. Now, suppose that $\mathcal{B}_n$ has been constructed and $B \in \mathcal{B}_n$ is a ball; we shall show how to construct the set $Y(B)$ and therefore $\mathcal{B}_{n+1}$. First, use the compactness of the set C to find a finite set Y_0 of d-balls (subsets of B) of the same radius $< \delta_{n+1}$ such that $C \cap B \subset \bigcup Y_0$. Let $Y_1 = \{A \in Y_0 \colon \mu_A(C) > 2^{n+2}\varepsilon_{n+1}\}$. Thus, $\mu_B(C \cap \bigcup(Y_0 \setminus Y_1)) \leq 2^{n+2}\varepsilon_{n+1} = \varepsilon_n$.

Claim 3.6.6. *There is a V^2-connected component $Y_2 \subset Y_1$ such that $\mu_B(C \cap \bigcup(Y_1 \setminus Y_2)) < \varepsilon_n$.*

Proof. We first show that there is a V^2-connected component $Y_2 \subset Y_1$ such that $\mu_B(C \cap \bigcup Y_2) > \varepsilon_n$. If this were not the case, it would be possible to divide Y_1 into sets Y' and Y'' which are both invariant under V^2-connections, and $\mu_B(C \cap \bigcup Y')$ and $\mu_B(C \cap \bigcup Y'')$ are both greater than ε_n. By the concentration of measure assumption, the sets $V \cdot (C \cap \bigcup Y')$ and $V \cdot (C \cap \bigcup Y'')$ are both of μ_B-mass greater than 1/2 and therefore intersect. It follows that $V^2 \cdot (C \cap$

$\bigcup Y') \cap (C \cap \bigcup Y'') \neq 0$, contradicting the invariance of the sets Y' and Y'' under V^2-connections.

Now, the V^2-connected component Y_2 found in the first paragraph must in fact be such that $\mu_B(C \cap \bigcup(Y_1 \setminus Y_2)) < \varepsilon_n$, by an argument identical to the one in the previous paragraph with $Y' = Y_2$ and $Y'' = Y_1 \setminus Y_2$. This completes the proof of the claim. □

Letting $Y(B) = Y_2$ completes the construction. The V^2-connectedness is clear from the choice of Y_2, the first item is clear from the choice of $Y_1(B)$, and the second item follows from some arithmetic: $\mu_B(C \cap \bigcup(Y \setminus Y_2)) < 2\varepsilon_n = 2^{-n}2^{n+1}\varepsilon_n \leq 2^{-n}\mu_B(C)$.

Now, let $D = \bigcap_n \bigcup \mathcal{B}_n$. The set D is closed as the d-balls in the sets $\mathcal{B}_n$ are clopen. It is also a subset of C, because each of the balls in $\mathcal{B}_n$ has nonempty intersection with C, and C is closed. The set D also has positive μ-mass; the mass distribution of D is governed by the following claim:

Claim 3.6.7. *For every $n \geq 2$ and every ball $B \in \mathcal{B}_n$, $\mu_B(D) > \varepsilon_n$.*

Proof. To get the set $D \cap B$, we subtracted from $C \cap B$ a set of size at most $\Sigma_{m>n} 2^{-m}\mu_B(C)$ by the second item above, so $\mu_B(D) > 1/2\mu_B(C) > \varepsilon_n$ by the first item above. □

Towards the connectedness of the set D, consider the following claim:

Claim 3.6.8. *Let $m \leq n$ be natural numbers, $B \in \mathcal{B}_m$ a ball, and $x_0, x_1 \in B \cap D$ be any points. Then there is a U, δ_n-walk from x_0 to x_1 using only points in $B \cap D$.*

Proof. This is proved by induction on $n - m$, which is the reason for the convoluted statement of the claim. The case $n - m = 0$ is trivial since then $d(x_0, x_1) < \delta_n$. Now suppose that the statement is known for $m + 1 \leq n$ and proceed to show that it is also true for $m \leq n$. Let $B_0, B_1 \in \mathcal{B}_{m+1}$ be balls such that $x_0 \in B_0$ and $x_1 \in B_1$. If $B_0 = B_1$ then the induction hypothesis immediately applies. Otherwise, by the third item above, the set $Y(B)$ is V^2-connected, so there must be a sequence of balls in $Y(B)$ starting at B_0 and ending at B_1 such that successive balls are V^2-connected. Suppose for simplicity that this sequence has length 2–i.e. B_0 and B_1 are V^2-connected. Fix an element $\gamma \in V^2$ such that $\gamma \cdot B_0 = B_1$.

Since $\mu_{B_0}(D)$ and $\mu_{B_1}(D)$ are both greater than ε_{m+1}, the concentration assumption yields that the numbers $\mu_{B_0}(V \cdot (D \cap B_0)) = \mu_{B_1}(V \cdot (D \cap B_1))$ are both greater than $1/2$. It follows that there is a point $x \in (V \cdot (D \cap B_0)) \cap B_0$ such that $\gamma \cdot x \in (V \cdot (D \cap B_1))$. Find points $x_0' \in D \cap B_0$ and $x_1' \in D \cap B_1$ and group elements $\beta_0, \beta_1 \in V$ such that $\beta_0 \cdot x_0' = x$ and $\beta_1 \cdot (\gamma \cdot x) = x_1'$. Use the induction hypothesis to find a U, δ_n-walk from x_0 to x_0'. Follow it by acting by $\beta_0\gamma\beta_1 \in U$ to get to the point x_1', and then use the induction hypothesis again to extend the walk from x_1' to x_1. The claim follows. □

Since the numbers δ_n tend to 0 as n tends to infinity and $D \subset B_{\mathrm{ini}}$, the claim shows that the set D is U-connected and completes the proof of the proposition. $\square$

Proof of Theorem 3.6.2. Suppose towards a contradiction that $\langle U, C\rangle$ is a condition in the product $P_\Gamma \times P_\mu$ which forces $V[\dot{x}_{gen}] \cap V[\dot{\gamma}_{gen} \cdot \dot{x}_{gen}] \neq V$. The following claim follows from abstract forcing considerations.

Claim 3.6.9. $\langle U, C\rangle \Vdash 2^\omega \cap V[\dot{x}_{gen}] \cap V[\dot{\gamma}_{gen} \cdot \dot{x}_{gen}] \neq 2^\omega \cap V$.

Proof. It will be enough to show that P_μ forces that for every ordinal α and every function $f : \alpha \to 2$ in the extension, if $f \notin V$ then there is a ground model countable set $b \subset \alpha$ such that $f \restriction b \notin V$. It will follow immediately that $P_\Gamma \times P_\mu$ forces that if $f \in V[\dot{x}_{gen}] \cap V[\dot{\gamma}_{gen} \cdot \dot{x}_{gen}] \setminus V$ then $f \restriction b \in V[\dot{x}_{gen}] \cap V[\dot{\gamma}_{gen} \cdot \dot{x}_{gen}] \setminus V$ for some ground model countable set b. This will prove the claim.

The forcing property of P_μ in question is well-known; we include a complete proof. Suppose towards a contradiction that $B \in P_\mu$ and $B \Vdash \tau : \check{\alpha} \to 2$ is a function which is not in the ground model, and for every countable set $b \in V$, $\tau \restriction b \in V$. Let $\langle M_\beta : \beta \in \omega_1\rangle$ be an $\in$-tower of countable elementary submodels of a large structure containing B, τ as elements. For each $\beta \in \omega_1$ use the contradictory assumption to find a function $g_\beta : M_\beta \cap \alpha \to 2$ in the model $M_{\beta+1}$ such that some condition below B forces $\tau \restriction M_\beta = \check{g}_\beta$. Let $B_\beta \subset B$ be the Borel set representing the nonzero Boolean value of the latter statement; $B_\beta \in M_{\beta+1}$ holds by elementarity of the model $M_{\beta+1}$, but also $B_\beta \notin M_\beta$ since τ is forced not to belong to the ground model. Use a counting argument and the Lebesgue density theorem to find a basic open set $O \subset X$ such that the set $C = \{\beta \in \omega_1 : \mu(B_\beta \cap O) > \frac{1}{2}\mu(O)\}$ is uncountable. Since the conditions $\{B_\beta : \beta \in C\}$ are pairwise compatible, the functions $\{g_\beta : \beta \in C\}$ must form an increasing chain, so in fact the conditions $\{B_\beta : \beta \in C\}$ form a strictly decreasing chain in P_μ. This contradicts the countable chain condition of P_μ. $\square$

Strengthening the condition $\langle U, C\rangle$ if necessary, we may find a continuous function $f : C \to 2^\omega$ and a name τ in the complete Boolean algebra generated by the name for $\dot{\gamma}_{gen} \cdot \dot{x}_{gen}$ such that the fibers of f are μ-null and $\langle U, C\rangle \Vdash \dot{f}(\dot{x}_{gen}) = \tau/\dot{\gamma}_{gen} \cdot \dot{x}_{gen}$. Let M be a countable elementary submodel of a large structure containing f, τ in particular, and let $C' \subset C$ be a set of points P_μ-generic over the model M. The set C' is Borel and μ-positive. Find open subsets V, U' of Γ such that $1 \in V$, $U' \subset U$, and $U' \cdot V \subset U$. Use Proposition 3.6.5 to find a V-connected compact μ-positive subset $D \subset C'$.

Since the fibers of f are μ-null, there must be points $x_0, x_1 \in D$ such that $f(x_0) \neq f(x_1)$. Find a number $n \in \omega$ such that $f(x_0)(n) \neq f(x_1)(n)$. Let $O_0 = \{x \in D : f(x)(n) = f(x_0)(n)\}$ and $O_1 = \{x \in D : f(x)(n) = f(x_1)(n)\}$. These are complementary relatively open subsets of the compact set D, so they are separated by some distance $\delta > 0$. Use the connectedness of the set D to produce a V, δ-walk from x_0 to x_1. There must be successive points x'_0, x'_1 on the walk such that $f(x'_0)(n) \neq f(x'_1)(n)$. The two points are at a distance $> \delta$ by the choice of δ, so there must be a group element $\beta \in V$ such that $\beta \cdot x'_0 = x'_1$.

By the Baire category theorem, there must be an element $\gamma \in U'$ which belongs to no meager subset of Γ coded in the model $V[x'_1]$ and also to no right β^{-1}-shift of any meager subset of Γ coded in the model $V[x'_0]$. As a result, the point γ is P_Γ-generic over $V[x'_1]$ and the point $\gamma \cdot \beta$ is P_Γ-generic over the model $V[x'_0]$. Both of these points belong to the set U. By the product forcing theorem, the pairs $\langle \gamma\beta, x'_0 \rangle$ and $\langle \gamma, x'_1 \rangle$ are both $P_\Gamma \times P_\mu$-generic over the model M, meeting the condition $\langle U, C \rangle$. However, $f(x'_0) \neq f(x'_1)$ while $\tau/\gamma\beta x'_0 = \tau/\gamma x'_1$, violating the forcing theorem in view of the initial contradictory assumption. Theorem 3.6.2 follows! □

The main corollaries are encapsulated in the following ergodicity result.

Corollary 3.6.10. *Suppose that Γ is a Polish group acting on a Polish space X with a Borel probability measure μ and an ultrametric d in a measure preserving and distance preserving fashion. Suppose that the action has concentration of measure. Suppose that E is the orbit equivalence relation, Y is a Polish space, F on Y is an analytic virtually placid equivalence relation, and $h\colon X \to Y$ is a Borel homomorphism from X to Y. Then there is an F-equivalence class with μ-positive h-preimage.*

Proof. Let $\gamma \in \Gamma$ and $x \in X$ be mutually P_Γ-generic and P_μ-generic points, and look at the models $V[x]$ and $V[\gamma \cdot x]$. Since h is a homomorphism of E to F and $x \mathrel{E} \gamma \cdot x$, $h(x) \mathrel{F} h(\gamma \cdot x)$ must hold. Since F is virtually placid and $V[x] \cap V[\gamma \cdot x] = V$ holds per Theorem 3.6.2, there must be a virtual F-class in the ground model such that $h(x)$ and $h(\gamma \cdot x)$ realize it. Since the poset $P_\Gamma \times P_\mu$ is c.c.c., Theorem 2.6.2 shows that all virtual classes realized in its extension are in fact repreented in the ground model. Thus, there is a ground model element $y \in Y$ such that $h(x) \mathrel{F} y$ holds. Since x is a P_μ-generic point, it belongs to no analytic ground model coded μ-small sets. Thus, $\mu(h^{-1}[y]_F) > 0$ as desired. □

Examples of actions with concentration of measure are not easy to identify. The following examples use F_σ P-ideals I on ω (which are Polish groups with the symmetric difference operation and a suitable topology by a result of Solecki [85]) and their standard action on 2^ω ($a \cdot x = y$ just in case $\{n \in \omega\colon x(n) \neq y(n)\} = a\}$), inducing the equivalence relation $=_I$. The action preserves the usual Borel probability measure μ on 2^ω and also the usual minimum difference metric d on 2^ω.

Example 3.6.11. Let $\{a_n\colon n \in \omega\}$ be positive real numbers such that $\Sigma_n a_n$ is infinite while $\Sigma_n a_n^2$ is finite. Let I be the ideal of all sets $b \subset \omega$ such that $\Sigma_{n\in b} a_n$ is finite. One can view I as a Polish group Γ with the complete metric $e(\gamma, \delta) = \Sigma\{a_n\colon \gamma(n) \neq \delta(n)\}$, continuously acting on the space $X = 2^\omega$ by coordinatewise Boolean addition. The action exhibits the concentration of measure.

To see this, let U be a neighborhood of the unit in Γ, and $\varepsilon > 0$ be a real number. Find a real number $\eta > 0$ such that the η-ball in the metric e around the unit is a subset of U, and find a number $m \in \omega$ such that $2\exp(-\eta^2/8\Sigma_{n=m}^\infty a_n^2) < \varepsilon$. The concentration of measure formula in [73, Theorem 4.3.19] then shows that $\delta = 2^{-m}$ works as required in Definition 3.6.1.

Corollary 3.6.12. *Let I be the usual summable ideal on ω. Let $h\colon 2^\omega \to X$ be a Borel homomorphism of $=_I$ to a virtually placid analytic equivalence relation F on a Polish space X. Then there is an F-class whose h-preimage has full μ-mass.*

Proof. By Corollary 3.6.10, there is an F-class whose h-preimage has positive μ-mass. However, the h-preimage is an $=_I$-invariant set, the equivalence relation $=_I$ includes $\mathbb{E}_0$ as a subset, and by the usual $\mathbb{E}_0$-ergodicity considerations, the h-preimage must in fact have full μ-mass. ☐

The concentration of measure for actions fails in many cases. Typically, there is a homomorphism of the orbit equivalence relation which violates the conclusion of Corollary 3.6.10 and therefore witnesses the failure of the concentration of measure in a strong sense.

Example 3.6.13. There is a summable-type ideal I on ω and a Borel homomorphism $h\colon 2^\omega \to 2^\omega$ of $=_I$ to $\mathbb{E}_0$ such that preimages of $\mathbb{E}_0$-equivalence classes are μ-null.

Proof. The key tool is the following:

Claim 3.6.14. *For every $i \in \omega$ and every $\varepsilon > 0$ there is a number $n \in \omega$ and sets $a, b \subset 2^n$ of the same relative size $> \frac{1-\varepsilon}{2}$ each, such that for every $x \in a$ and $y \in b$ the set $\{m \in n\colon x(m) \neq y(m)\}$ contains at least i many elements.*

Proof. Fix i and ε. Stirling's approximation formula shows that there is $n \in \omega$ such that the size of the set $\{a \subset n : ||a| - \frac{n}{2}| < i+1\}$ is less than $\varepsilon 2^n$. Let $a = \{x \in 2^n :$ the set $\{m \in n\colon x(m) = 1\}$ contains at most $\frac{n}{2} - i$ many elements$\}$ and $b = \{x \in 2^n :$ the set $\{m \in n\colon x(m) = 1\}$ contains at least $\frac{n}{2} + 1$ many elements$\}$. This works. ☐

Towards the proof of the example, find a partition $\omega = \bigcup_n I_n$ into finite sets such that for every $n \in \omega$, the set 2^{I_n} contains subsets a_n, b_n of the same size such that their relative size in 2^{I_n} is greater than $1/2 - 2^{-n}$, and if $x \in a_n$ and $y \in b_n$ are arbitrary elements, then the set $\{i \in I_n\colon x(n) \neq y(n)\}$ has size at least n. Now, define $w(m) = \frac{1}{n+1}$ if $m \in I_n$ and $I = \{a \subset \omega\colon \Sigma_{n\in a} w(n) < \infty\}$. Define $B = \{x \in 2^\omega\colon \forall^\infty n\ x \restriction n \in a_n \cup b_n\}$; this is a Borel set of full mass. For $x \in B$, define $h_0(x) \in 2^\omega$ by $h_0(x)(n) = 0 \leftrightarrow x \restriction I_n \in a_n$. It is not difficult to check that $h_0\colon B \to 2^\omega$ is a continuous homomorphism from B to $\mathbb{E}_0$ such that preimages of $\mathbb{E}_0$-classes are of zero mass. The rest of the proof only adjusts h_0 to a Borel homomorphism h defined on the whole space.

To this end, let C_n for $n \in \omega$ be inclusion increasing compact subsets of B whose mass converges to 1. The set $\bigcup_n C_n$ is K_σ and the equivalence relation $=_I$ is K_σ, so the saturation $D = [\bigcup_n C_n]_{=_I}$ is K_σ as well. Let $F \subset D \times 2^\omega$ be the Borel set of all pairs $\langle x, y\rangle$ such that for some (equivalently, for all) $x' \in B$ such that $x' =_I x$, $h_0(x')$ is $\mathbb{E}_0$-related to y. F is a Borel set with nonempty countable sections, and by the Lusin–Novikov theorem, it has a Borel uniformization h. Extend h to all of 2^ω by defining $h(x)$ for $x \notin D$ to be an arbitrary fixed element of 2^ω. It is not difficult to check that h has the required properties. ☐

Example 3.6.15. There is a Tsirelson-type [27] ideal I on ω and a Borel homomorphism $h\colon B \to 2^\omega$ of $=_I$ to $\mathbb{E}_0$ such that preimages of $\mathbb{E}_0$-equivalence classes are μ-null.

Proof. We will deal with a certain special kind of Tsirelson submeasures. Let $\alpha > 0$ be a real number and $f\colon \omega \to \mathbb{R}^+$ be a nonincreasing function converging to 0. By induction on $n \in \omega$ define submeasures ν_n on ω by setting $\nu_0(a) = \sup_{i\in a} f(i)$, and $\nu_{n+1}(a) = \sup\{\nu_n(a), \alpha \sum_{b\in\vec{b}} \nu_n(b)\}$ where the variable $\vec{b}$ ranges over all finite sequences $\langle b_0, b_1, \dots b_j\rangle$ of finite subsets of a such that $j < \min(b_0) \leq \max(b_0) < \min(b_1) \leq \max(b_1) < \dots$. In the end, let the submeasure ν be the supremum of ν_n for $n \in \omega$. Some computations are necessary to verify that ν is really a lower semicontinuous submeasure on ω [27]. The Tsirelson ideal $I = \{a \subset \omega\colon \lim_m \nu(a \setminus m) = 0\}$ turns out to be an F_σ P-ideal [27].

By induction on $i \in \omega$ choose intervals $I_i \subset \omega$ such that $\max(I_i) < \min(I_{i+1})$ and such that $\min(I_i) > i/\alpha$ and there are sets $a_i, b_i \subset 2^{I_i}$ of the same relative size $\geq \frac{1-2^{-i}}{2}$ such that for any elements $x \in a_i, y \in b_i$ the set $\{m \in I_i\colon x(m) \neq y(m)\}$ has size at least i/α. This is easily possible by Claim 3.6.14. Now, consider the function f defined by $f(m) = 1/i$ for $m \in (\max(I_{i-1}), \max(I_i)]$ and let ν be the derived submeasure and I the derived Tsirelson ideal. Observe that with this choice of the function f, for any $i \in \omega$ and elements $x \in a_i, y \in b_i$ the set $\{m \in I_i\colon x(m) \neq y(m)\}$ has ν-mass at least 1, since it has ν_1-mass at least 1. The rest of the proof follows the lines of Example 3.6.13. $\square$

Example 3.6.16. Let J be the Rado graph ideal on ω. There is a Borel homomorphism $h\colon 2^\omega \to 2^\omega$ from $=_J$ to $\mathbb{E}_0$ such that preimages of $\mathbb{E}_0$-classes are μ-null.

Proof. Let G be the Rado graph, interpreted so that ω is the set of its vertices; then J is the ideal on ω generated by J-cliques and J-anticliques. To construct h, by induction on $n \in \omega$ find pairwise disjoint finite sets $I_n \subset \omega$ and sets $a_n, b_n \subset 2^{I_n}$ such that

- each I_n is a G-anticlique;
- if $n \neq m$ then $I_n \times I_m \subset G$;
- for every $x \in a_n$ and every $y \in b_n$ the set $\{t \in I_n\colon x(t) \neq y(t)\}$ has size at least n;
- the sets $a_n, b_n \subset 2^{I_n}$ have the same relative size larger than $1/2 - 2^{-n}$.

This is easy to do using the universality properties of the Rado graph and Claim 3.6.14 repeatedly. Let $B = \{x \in 2^\omega\colon \forall^\infty n\ x \restriction I_n \in a_n \cup b_n\}$ and let $h_0\colon B \to 2^\omega$ be the Borel map defined by $h_0(x) = 0 \leftrightarrow x \restriction I_n \in a_n$. It is immediate that the set B has full μ-mass and the function h is a homomorphism from $=_J$ to $\mathbb{E}_0$. The rest of the proof follows the lines of Example 3.6.13. $\square$

There are numerous questions left open by the development of this section, of which we quote two.

Question 3.6.17. Can the assumption that d be an ultrametric be eliminated from the assumptions of Theorem 3.6.2?

Question 3.6.18. Is there a Tsirelson ideal whose natural action on 2^ω exhibits concentration of measure?

Chapter 4

Nested sequences of models

4.1 Prologue

The purpose of this chapter is to set up a calculus for infinite nested sequences of models of ZFC, which turn out to be critical for the treatment of the $\mathbb{E}_1$ cardinal. As a motivation, we include a simple proof of the fact that $\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation. It is quite different from the standard one [48, Theorem 11.8.1], and it has the advantage that it can be easily adapted to the context of inequalities between cardinalities of quotient spaces.

Theorem 4.1.1. *$\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation.*

Proof. Let Y be the Polish space $(2^\omega)^\omega$ and Γ be a Polish group continuously acting on a Polish space Z, inducing the orbit equivalence relation F. Suppose towards a contradiction that there is a Borel reduction $h\colon Y \to Z$ of $\mathbb{E}_1$ to F. Let $\langle x_n : n \in \omega\rangle$ be a sequence of mutually Cohen generic points in 2^ω. For each $n \in \omega$ let y_n denote the element of Y for which $y(m)$ is the zero sequence in 2^ω if $m < n$, and x_m if $m \geq n$; write $M_n = V[y_n]$.

Work in the model M_0. The reinterpretation of the Borel map h in M_0 is still a reduction of $\mathbb{E}_1$ to F. For each number $n > 0$ fix a group element $\gamma_n \in \Gamma$ such that $\gamma_n \cdot h(y_n) = h(y_0)$. Let $\gamma \in \Gamma$ be a point Cohen-generic over $V[y_0]$ and look into the model $V[\gamma \cdot h(y_0)]$. By a Mostowski absoluteness argument, there must be a point $y \in V[\gamma \cdot h(y_0)]$ such that $h(y)\ F\ \gamma \cdot h(y_0)$. Since the function h is a reduction of $\mathbb{E}_1$ to F even in the model $M_0[\gamma]$, this point $y \in Y$ must be $\mathbb{E}_1$-related to y_0, so for all but finitely many numbers n it must be the case that $y(n) = x_n$. Choose a number $n \in \omega$ such that $y(n) = x_n$ and look at the model M_{n+1}. The point $\gamma\gamma_{n+1} \in \Gamma$ is Cohen generic over M_0 by the invariance of the meager ideal under right shift, and by the product forcing theorem it follows that the models M_0 and $M_{n+1}[\gamma\gamma_{n+1}]$ are mutually generic over the model M_{n+1}. Now, the points y_{n+1} and $\gamma\gamma_{n+1} \cdot h(y_{n+1}) = \gamma \cdot h(y_0)$ belong to the model $M_{n+1}[\gamma\gamma_{n+1}]$ and so does y. Thus, even the point $y(n) = x_n \in 2^\omega$ belongs to this model; however, it is a point of M_0 Cohen generic over M_{n+1}. This contradicts the product forcing theorem. $\square$

4.2 Coherent sequences of models

It is clear from the proof of Theorem 4.1.1 that its generalizations will require codification of decreasing ω-sequences of generic extensions. In addition to the approach from Theorem 4.1.1, we pay close attention to the intersection model. This is the content of the following definitions and theorems.

Definition 4.2.1. Let $\langle M_n : n \in \omega \rangle$ be an inclusion decreasing sequence of transitive models of ZFC. We say that the sequence is *coherent* if for every ordinal λ and every natural number n, the sequence $\langle V_\lambda \cap M_m : m \geq n \rangle$ belongs to M_n. Given a coherent sequence of models $\langle M_n : n \in \omega \rangle$, a sequence $\langle v_n : n \in \omega \rangle$ is *coherent* if for every number $m \in \omega$, $\langle v_n : n \geq m \rangle \in M_m$ holds.

Example 4.2.2. Let R_m for $m \in \omega$ be any partial orders and let $\langle G_m : m \in \omega \rangle$ be a sequence of generic filters on the respective posets R_m added by the countable support product $\prod_m R_m$. Let $M_n = V[G_m : m \geq n]$. Then $\langle M_n : n \in \omega \rangle$ is a coherent sequence of models.

Example 4.2.3. Let x be a set. By recursion on $n \in \omega$, define models M_n by letting M_0 be V and each M_{n+1} be the collection of all sets hereditarily definable from ordinal parameters and the parameter $x \cap M_n$ in the model M_n. The sequence $\langle M_n : n \in \omega \rangle$ is coherent.

Example 4.2.4. Let κ be a measurable cardinal, U a measure on it, and $j : V \to M$ the U-ultrapower, with iterands denoted by j_α for every ordinal α. For each $n \in \omega$ let $M_n = j_n(V)$. The sequence $\langle M_n : n \in \omega \rangle$ is coherent.

Most of our choice-coherent sequences are sequences of generic extensions in the following sense:

Definition 4.2.5. A sequence $\langle M_n : n \in \omega \rangle$ is *generic over* V if V is a model of ZFC contained in all M_n for $n \in \omega$ and M_0 is a generic extension of M.

The usual abstract forcing arguments (Fact 1.7.6) show that if the sequence of models is generic over V then all models M_n are generic extensions of V and if $n \in m$ are numbers then M_n is a generic extension of M_m. Coherent sequences of models are most often formed as generic extensions of the constant sequence $\langle M_n = V : n \in \omega \rangle$ using the following definitions and theorem.

Definition 4.2.6. Let P, Q be posets. A *projection* of Q to P is a pair of order-preserving functions $\pi : Q \to P$ and $\xi : P \to Q$ such that

1. $\pi \circ \xi$ is the identity on P;
2. whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
3. whenever $p \leq \pi(q)$ then there is $q' \leq q$ such that $\pi(q') \leq p$.

As the simplest initial example, let P, R be any posets, let R have a largest element 1_R and let $Q = P \times R$. Then one can consider the projection of Q to P by setting $\pi(p, r) = p$ and $\xi(p) = \langle p, 1_R \rangle$. An important effect of the demand (3) is that the π-image of the generic filter on Q is a generic filter on P.

Definition 4.2.7. Let $\langle M_n : n \in \omega \rangle$ be a coherent sequence of models of ZFC. A *coherent sequence of posets* is a sequence $\langle P_n, \pi_{nm}, \xi_{mn} : n \leq m \in \omega \rangle$ such that

1. For all numbers $n \leq m$ the maps $\pi_{nm} : P_n \to P_m$ and $\xi_{mn} : P_m \to P_n$ form a projection of P_n to P_m;
2. the functions π_{nm} form a commutative system, the same for ξ_{mn}, and for all $n \in \omega$ the functions π_{nn}, ξ_{nn} are the identity on P_n;
3. for every number $k \in \omega$, the sequence $\langle P_n, \pi_{nm}, \xi_{mn} : k \leq n \leq m \in \omega \rangle$ belongs to the model M_k.

In particular, every commutative sequence of projections is coherent over the constant coherent sequence $\langle M_n = V : n \in \omega \rangle$. As the simplest initial example, let $\langle R_n : n \in \omega \rangle$ be a sequence of posets with respective largest elements 1_n, let P_n be the finite (or countable) support product of R_m for $m \geq n$, and let $\pi_{nm}(p) = p \restriction [m, \omega)$ and $\xi_{nm}(p) = \langle 1_k : k \in [n, m) \rangle^\frown p$. The sequence $\langle P_n, \pi_{nm}, \xi_{mn} : n \leq m \in \omega \rangle$ is coherent over $\langle M_n = V : n \in \omega \rangle$.

Theorem 4.2.8. *Let $\langle M_n : n \in \omega \rangle$ be a coherent sequence of models of ZFC and $\langle P_n, \pi_{nm} : P_n \to P_m, \xi_{mn} : P_m \to P_n : n \leq m \in \omega \rangle$ be a coherent sequence of posets. Let $G \subset P_0$ be a filter generic over M_0, and for each $n \in \omega$ let $G_n = \xi_{n0}^{-1} G$. The sequence $\langle M_n[G_n] : n \in \omega \rangle$ is a coherent sequence of models of ZFC again.*

Proof. Observe that for every number $k \in \omega$, in the model $M_k[G_k]$, one can form the sequence $\langle G_n : n \geq k \rangle$ as $\langle \xi_{nk}^{-1} G_k : n \geq k \rangle$ by the commutativity of the ξ-maps. Thus, if λ is a limit ordinal larger than the rank of all the posets on the coherent sequence, one can form the relation $\{\langle n, x \rangle : n \geq k, x \in M_n[G_n]$ has rank $< \lambda\}$ in the model $M_k[G_k]$ as the set $\{\langle n, \tau/G_n \rangle : n \geq k$ and $\tau \in M_n$ is a P_n-name of rank $< \lambda\}$ by the coherence of the original sequence $\langle M_n : n \in \omega \rangle$. $\square$

The critical object for understanding a coherent sequence of models is the intersection model $M_\omega = \bigcap_n M_n$. In Example 4.2.2, the intersection model is a model of ZFC, and it has been discussed in [49, Theorem 9.3.4]; we will come back to it below–Theorem 4.3.5. In the context of general coherent sequences, the intersection model is a transitive model of ZF, and the Axiom of Choice may fail in it. This is the content of the following theorem and example.

Theorem 4.2.9. *If $\langle M_n : n \in \omega \rangle$ is a coherent sequence of generic extensions of V, then $M_\omega = \bigcap_n M_n$ is a class in all models M_n, and it is a model of ZF.*

Proof. We will only show that M_ω is a class in M_0; it then follows by the same argument that M_ω is a class in each M_n for $n \in \omega$. To show that M_ω is a model of ZF, by [45, Theorem 13. 9], it is enough to show that M_ω is closed under the Gödel operations and it is universal: for every set $a \subset M_\omega$ there is a set $b \in M_\omega$ such that $a \subset b$. The closure under the Gödel operations follows from the fact that these operations are evaluated in the same way in each model M_n. For the universality, suppose that $a \subset M_\omega$ is a set, choose an ordinal α such that all sets in a have rank smaller than α, and form the set $b = V_\alpha \cap M_\omega$. Since M_ω is a class in each model M_n, the set b is in all models M_n and therefore in M_ω. Clearly, $a \subset b$, concluding the proof of universality.

To show that M_ω is a class in M_0, let λ be a large limit cardinal in V so that M_0 is a generic extension of V by a poset in V_λ, and such that V_λ satisfies a large fragment of ZFC. Note that then, all the models M_n are also obtained from V as generic extensions by posets in V_λ. Move to the model M_0. Let f be the function from ω to $M_0 \cap V_\lambda$ such that $f(n) = \{\langle P, G\rangle \colon P \in V \cap V_\lambda,\ G \in M_n$ is a filter on P generic over V, and whenever $\langle Q, H\rangle$ is a pair consisting of a poset in $V \cap V_\lambda$ and a filter on Q in M_n generic over V, there is a P-name σ in V such that $\sigma/G = H\}$. The coherence of the sequence $\langle M_n \colon n \in \omega\rangle$ shows that the function f can be formed in M_0. Then, M_ω is exactly the collection of all sets x such that for every $n \in \omega$ and every pair $\langle P, G\rangle \in f(n)$, $x \in V[G]$. Since V is a class in M_0, this shows that M_ω is a class in M_0 as well. $\square$

Example 4.2.10. Let κ be a measurable cardinal, with a normal measure U on κ and the associated ultrapower $j \colon V \to M$. Let $j_{nm} \colon M_n \to M_m$ be the iterands of j for $n \leq m \leq \omega$. Then $\bigcap_{n\in\omega} M_n = M_\omega[c]$ holds where $c = \langle j_{0n}(\kappa) \colon n \in \omega\rangle$ [20, 15]. It is well-known and follows from the geometric description of Prikry genericity by Mathias [69] that the set c is generic over the model M_ω for the Prikry forcing associated with the measure $j_{0\omega}U$.

Example 4.2.11. Let $c_0 \colon \omega_1 \times \omega \to 2$ be a Cohen-generic map, and let $c_n = c_0 \restriction \omega_1 \times (\omega \setminus n)$. Let $M_n = V[c_n]$. In the model $M_\omega = \bigcap_n M_n$, the chromatic number of $\mathbb{G}_0$ is greater than 2; thus, the Axiom of Choice must fail in M_ω.

Proof. Work in V. For each number $n \in \omega$, let P_n be the poset of all finite functions from $\omega_1 \times (\omega \setminus n)$ to 2 ordered by extension. Note that for each $n \in \omega$ the map c_n is P_n-generic over V. For each ordinal $\alpha \in \omega_1$ and a number $n \in \omega$, let $\dot{d}_{\alpha n}$ be a name for the function defined by letting $d_{\alpha n}(m)$ be 0 if $m \in n$ and the unique value of $p(\alpha, m)$ for all conditions p in the generic filter with the pair (α, m) in their domain if $m \geq n$. Note that $d_{\alpha n}$ is really a P_n-name and it is forced to belong to the intersection model M_ω.

Now, let $p \in P$ be a condition and σ be a name such that $p \Vdash \sigma \colon 2^\omega \to 2$ is a function in M_ω; we will find an ordinal $\alpha \in \omega_1$, a number $n \in \omega$ and a condition strengthening p which forces $\dot{d}_{\alpha 0}$ and $\dot{d}_{\alpha n}$ to differ in an even number of entries if and only if $\sigma(\dot{d}_{\alpha 0}) \neq \sigma(\dot{d}_{\alpha n})$. This cannot occur if σ is a coloring of $\mathbb{G}_0$.

By a standard Δ-system argument, strengthening p if necessary, we may find an infinite set $S \subset \omega_1$, conditions $p_\alpha \in P$ for $\alpha \in S$ and a number $n \in \omega$ such that the conditions p_α for $\alpha \in S$ form a Δ-system with root p, $\mathrm{dom}(p_\alpha) \subset \omega_1 \times n - 1$, and each p_α decides the value of $\sigma(\dot{d}_{\alpha 0})$ to be some bit $b_\alpha \in 2$. Find a condition $q \leq p$ and a P_n-name τ such that $q \Vdash \sigma = \tau$; this is possible as σ is forced to belong to M_ω. Since the set S is infinite, it is possible to find an ordinal $\alpha \in S$ such that p_α is compatible with q. Find a condition $r \in P_n$ such that $r \leq q \restriction \omega_1 \times (\omega \setminus n)$ and r decides the value of $\tau(\dot{d}_{\alpha n})$ to be some specific bit $b \in 2$. Note that p_α and r are compatible in P, and the pair

$\langle \alpha, n-1\rangle$ does not belong to $\mathrm{dom}(p_\alpha \cup r)$. Thus, it is possible to strengthen the condition $p_\alpha \cup r$ to some $s \in P$ such that $\{\alpha\} \times n \subset \mathrm{dom}(s)$, and cardinality of the set $\{m \in n \colon s(\alpha, m) = 1\}$ is even if and only if $b_\alpha \neq b$. This completes the proof. $\square$

4.3 Choice-coherent sequences of models

In most of our examples, we will want to look at sequences of models which have a greater degree of coherence. Certain constructions arising from the axiom of choice will have to be performed in a coherent way. The following definition records the demands:

Definition 4.3.1. Let $\langle M_n \colon n \in \omega\rangle$ be an inclusion decreasing sequence of transitive models of ZFC. We say that the sequence is *choice-coherent* if it is coherent and for every ordinal λ there is a well-ordering $\leq_\lambda$ of $V_\lambda \cap M_0$ such that its intersection with each model M_n belongs to M_n.

In the common case of generic coherent sequences, the choice-coherence can be detected from the theory of the intersection model as follows:

Theorem 4.3.2. *Suppsoe that $\langle M_n \colon n \in \omega\rangle$ is a generic coherent sequence of generic extensions of V. The following are equivalent:*

1. *$\langle M_n \colon n \in \omega\rangle$ is choice-coherent;*
2. *$M_\omega = \bigcap_n M_n$ is a model of ZFC.*

Proof. For the (1)$\to$(2) direction, assume the choice coherence. Let λ be any ordinal. In view of Theorem 4.2.9, we only need to produce a well-ordering $\leq^*$ of $V_\lambda \cap M_\omega$ such that $\leq \in M_\omega$. Fix a wellordering $\leq$ witnessing the fact that $\langle M_n \colon n \in \omega\rangle$ is a choice-coherent decreasing sequence, and note that $\leq^* = \leq \cap M_\omega$ works as desired. The genericity assumption is not needed for this direction.

For the (2)$\to$(1) direction, that M_ω is a model of ZFC. Let λ be any ordinal. Let $\kappa > \lambda$ be a cardinal such that each model M_n is a generic extension of V by a poset of cardinality smaller than λ. Let $\prec$ be a well-ordering of V_κ in V. By recursion on $n \in \omega$ build a sequence $\langle P_n, G_n, \tau_n \colon n \in \omega\rangle$ such that P_n is $\prec$-least poset in $V \cap V_\kappa$ such that M_n is P_n-generic extension of V, $G_n \subset P_n$ is a filter generic over V such that $M_n = V[G_n]$, τ_n is the $\prec$-least P_n-name in V such that $\tau_n/G_n \subset P_{n+1}$ is a filter generic over V such that $M_{n+1} = V[\tau_n/G_n]$, and $G_{n+1} = \tau_n/G_n$. It is not difficult to see that the tail $\langle P_n, G_n, \tau_n \colon n \geq m\rangle$ of the sequence can be recovered from G_m, and therefore the sequence is coherent.

Now, let $\leq_\lambda$ be the following well-ordering of $V_\lambda \cap M_0$: first come the elements of $V_\lambda \cap M_\omega$, then the elements of $V_\lambda \cap M_0 \setminus M_1$, and then the elements of $V_\lambda \cap M_n \setminus M_{n+1}$ in turn. The elements of $V_\lambda \cap M_\omega$ are ordered by some well-ordering in M_ω which is available as M_ω is a model of ZFC. The elements of $M_n \setminus M_{n+1}$ are ordered by σ_n/G_n where σ_n is the $\prec$-least name in $V \cap V_\kappa$ such that σ_n/G_n is a well-ordering of $V_\lambda \cap M_n$. It is not difficult to see that $\leq_\lambda$ is a well-ordering of $V_\lambda \cap M_0$ and $\leq_\lambda \cap M_n \in M_n$ holds for all $n \in \omega$. $\square$

Most examples of choice-coherent sequences are generic and obtained from the trivial one $\langle M_n = V : n \in \omega\rangle$ by a coherent forcing which satisfies a certain degree of completeness.

Definition 4.3.3. Let $\langle P_n, \pi_{nm}, \xi_{mn} : n \leq m \in \omega\rangle$ be a commutative system of projections from posets P_n to P_m for $n \leq m$.

1. The *diagonal game* is the following infinite game between Players I and II, in round n Player I plays $p_n \in P_n$ and Player II responds by $q_n \leq p_n$. Additionally, $p_{n+1} \leq \pi_{nn+1}(q_n)$. In the end, Player II wins if there is a condition $r \in P_0$ such that $\pi_{0n}(r) \leq q_n$ holds for all $n \in \omega$.

2. The sequence is *diagonally distributive* if Player I has no winning strategy in the diagonal game.

Example 4.3.4. Suppose that $\langle Q_m : m \in \omega\rangle$ are arbitrary posets, and let $P_n = \prod_{m \geq n} Q_m$ be the countable support product with the natural projection maps from P_n to P_m for $n \leq m$. Player II has a simple winning strategy in the diagonal game in this setup: set $q_n = p_n$.

Theorem 4.3.5. *Let $\langle M_n : n \in \omega\rangle$ be a choice-coherent sequence of models of ZFC. Let $\langle P_n, \pi_{nm}, \xi_{mn} : n \leq m \in \omega\rangle$ be a coherent sequence of posets which is diagonally distributive in M_0. Let $G \subset P_0$ be a filter generic over M_0, and let $G_n = \xi_{n0}^{-1} G$. Then*

1. *the sequence $\langle M_n[G_n] : n \in \omega\rangle$ is choice-coherent;*

2. *the models $\bigcap_n M_n$ and $\bigcap_n M_n[G_n]$ contain the same ω-sequences of ordinals.*

Proof. Write $M_\omega = \bigcap_n M_n$. We start with (1). The main task is to find a poset $P_\omega \in M_\omega$ and a filter $G_\omega \subset P_\omega$ generic over M_ω such that $\bigcap_n M_n[G_n] = M_\omega[G_\omega]$.

Using the choice coherence of the original sequence $\langle M_n : n \in \omega\rangle$, we may assume that there is a sequence $\langle \alpha_n : n \in \omega\rangle$ such that the underlying set of each poset P_n is exactly α_n. For each condition $p \in P_0$, the ordinal ω-sequence $\langle \pi_{0n}(p) : n \in \omega\rangle$ belongs to M_ω, since for each number $k \in \omega$, the tail $\langle \pi_{0n}(p) : n \geq k\rangle$ is reconstructed as $\langle \pi_{kn}(\pi_{0k}(p)) : n \geq k\rangle$ in the model M_k. Similarly, the set $P_\omega = \{q \in \prod_n P_n : \exists k \in \omega\ \forall n \geq k\ q(n) = \pi_{kn}(q(k))\}$ belongs to the model M_ω. For elements $q_0, q_1 \in P_\omega$ let $q_1 \leq q_0$ if for all but finitely many numbers $n \in \omega$, $q_1(n) \leq q_0(n)$ in the poset P_n, and conclude that the poset $\langle P_\omega, \leq\rangle$ belongs to the model M_ω.

Define a function $\pi_{0\omega} : P_0 \to P_\omega$ by $\pi_{0\omega}(p) = q$ where $q(n) = \pi_{0n}(p)$, and a function $\xi_{\omega 0} : P_\omega \to P_0$ by $\xi_{\omega 0}(q) = \xi_{k0}(q(k))$ where $k \in \omega$ is such that for all $n \geq k$, $\pi_{kn}(q(k)) = q(n)$. One can also similarly define maps $\pi_{n\omega} : P_n \to P_\omega$

and $\xi_{\omega n} \colon P_\omega \to P_n$. It is a matter of trivial diagram chasing to show that the maps form a commuting system of projections from P_n to P_ω and moreover $\pi_{n\omega}, \xi_{\omega n} \in M_n$. Thus, letting $G_\omega = \xi_{\omega 0}^{-1} G_0$, one can conclude that $G_\omega \subset P_\omega$ is a filter generic over M_0 and therefore over M_ω. Also $G_\omega \in \bigcap_n M_n[G_n]$ since G_ω can be reconstructed in $M_n[G_n]$ as $G_\omega = \xi_{\omega n}^{-1} G_n$. In conclusion, $G_\omega \in \bigcap_n M_n[G_n]$.

Finally, we have to prove that every element of the intersection $\bigcap_n M_n[G_n]$ belongs to $M_\omega[G_\omega]$. This is where the diagonal distributivity of the original poset sequence is used. Suppose that $\tau \in M_0$ is a P_0-name for a set of ordinals and $p \in P_0$ is a condition forcing $\tau \in \bigcap_n M_n[G_n]$; we must produce a condition $p' \leq p$ and a P_ω-name $\tau_\omega \in M_\omega$ such that $p' \Vdash \tau = \tau_\omega/G_\omega$. Consider a strategy by Player I in the diagonalization game in which he plays p_n so that $p_0 \leq p$, $\tau_0 = \tau$, and there is a P_{n+1}-name $\tau_{n+1} \in M_{n+1}$ such that $p_n \Vdash_{P_n} \tau_n = \tau_{n+1}/G_{n+1}$. This is possible by the assumption on the name τ. By the diagonalization assumption, Player II has a counterplay with conditions $q_n \leq p_n$ such that there is a condition $p' \leq p$ for which $\pi_{0n}(p) \leq q_n$ for all $n \in \omega$. Let τ_ω be the $P_\omega \restriction \pi_{0\omega}(p')$-name defined by $q \Vdash \check{\alpha} \in \tau_\omega$ just in case $\xi_{\omega 0}(q) \Vdash_{P_0} \check{\alpha} \in \tau$. The name τ_ω can be reconstructed in every model M_n by the definition $q \Vdash \check{\alpha} \in \tau_\omega$ just in case $\xi_{\omega n}(q) \Vdash_{P_n} \check{\alpha} \in \tau_n$ by the choice of the strategy for Player I in the diagonalization game. As a result, $\tau_\omega \in M_\omega$. It is immediate from the definition of τ_ω that $p' \Vdash \tau = \tau_\omega/G_\omega$ as desired.

Now we are ready to construct the requisite well-orderings verifying the choice-coherence of the models $\langle M_n[G_n] \colon n \in \omega \rangle$. Let λ be an ordinal larger than the ranks of all the posets P_n for $n \in \omega$. Let $\leq$ be a coherent well-ordering of $V_\lambda \cap M_0$. We will now describe a coherent well-ordering $\leq'$ of sets of rank $< \lambda$ in the model $M_0[G_0]$. In this well-ordering, the sets in $M_\omega[G_\omega]$ come first, ordered by some well-ordering in the model $M_\omega[G_\omega]$. The sets in $M_0[G_0] \setminus M_1[G_1]$ come next, well-ordered by their $\leq$-first P_0-name in the model M_0 representing them. The sets in $M_1[G_1] \setminus M_2[G_2]$ come next with a similar well-order, and so on. The coherence of the resulting well-ordering $\leq'$ is due to the fact that for each $k \in \omega$, the sequence $\langle G_n \colon n \geq k \rangle$ belongs to the model $M_k[G_k]$.

(2) is much easier. Suppose that $\tau \in M_0$ is a P_0-name for an ω-sequence of ordinals in the model $M_\omega[G_\omega]$ and $p \in P_0$ is a condition; we must find a condition $r \leq p$ and an ω-sequence $z \in M_\omega$ such that $r \Vdash \tau = \check{z}$. Consider a strategy for Player I in the diagonal game in which he plays conditions $p_n \in P_n$ and on the side produces P_n-names $\tau_n \in M_n$ so that $p_0 \leq p$, $\tau_0 = \tau$ and $p_n \Vdash_{P_n} \tau_n = \tau_{n+1}$ evaluated by the π_{nn+1}-image of the generic filter on P_n, and also p_n decides the value $\tau_n(n)$ to be some ordinal $z(n)$. The assumptions on the name τ shows that this is a valid strategy. The initial assumptions on the coherent sequence of posets show that this is not a winning strategy, so there must be a play against it such that in the end there is a condition $r \leq p$ with $\pi_{0n}(r) \leq p_n$ for all $n \in \omega$. Let $\tau_n \in M_n$ be the names produced during that counterplay, and let z be the ω-sequence of ordinals obtained. The definitions show that for all $n \in \omega$, $\pi_{0n}(r) \Vdash_{P_n} \tau_n = \check{z}$. It follows that $z \in M_n$ for all $n \in \omega$, and therefore r, z are as required in (2). $\square$

The main feature of choice-coherent sequences of models we use later is the following theorem connecting them with orbit equivalence relations:

Theorem 4.3.6. *Let $\langle M_n : n \in \omega \rangle$ be a generic choice-coherent sequence of models. Let E be an orbit equivalence relation on a Polish space X with code in $M_\omega = \bigcap_n M_n$. If a virtual E-class is represented in M_n for every $n \in \omega$, then it is represented in M_ω.*

Note that a virtual E-class is an equivalence class of E-pins. Thus, the theorem says that if there are pairwise equivalent E-pins $\langle P_n, \tau_n \rangle \in M_n$ for all $n \in \omega$, then there is an E-pin equivalent to them in the intersection model.

Proof. Let Γ be a Polish group continuously acting on the space X, inducing the equivalence relation E. Let d be a compatible right-invariant metric on Γ. Let $\langle P_0, \tau_0 \rangle \in M_0$ be an E-pin which has an equivalent in the model M_n for every $n \in \omega$. Let λ be a cardinal so large that for each $n \in \omega$, M_0 is a generic extension of M_n by a poset of size $< \lambda$, and M_n contains an E-pin on a poset of size $< \lambda$ equivalent to the pin $\langle P_0, \tau_0 \rangle$.

Let P_Γ be Cohen forcing on the Polish group Γ, with its name $\dot{\gamma}_{\text{gen}}$ for the generic point. Let $\gamma \in \Gamma$ be a Cohen-generic point, $H \subset P_0$ be a generic filter and $K \subset \text{Coll}(\omega, \lambda)$ be a generic filter, mutually generic over M_0; let $x_0 = \tau_0/H$. In the model $M_0[\gamma, H, K]$, form the model N as the class of all sets hereditarily definable from $\gamma \cdot x_0$ and parameters in M_ω. The model N is an intermediate model of ZFC between M_ω and $M_0[\gamma, H]$, so by Fact 1.7.6, the model N is a forcing extension of M_ω. We will argue that N and $M_0[H]$ are mutually generic extensions of M_ω.

First note that this will prove the theorem. Let $Q, \tau \in M_\omega$ be a poset and a name and $L \subset Q$ be a filter generic over the model $M_0[H]$ such that $M_\omega[L] = N$ and $\tau/L = \gamma \cdot x_0$. By the forcing theorem in the model M_0, there have to be conditions $p \in H$ and $q \in L$ such that $\langle p, q \rangle \Vdash \tau_0 \mathrel{E} \tau$. It is immediate that τ as a name on $Q \restriction q$ is E-pinned, and the E-pin $\langle Q \restriction q, \tau \rangle$ is equivalent to $\langle P_0, \tau_0 \rangle$. This confirms the conclusion of the theorem.

To argue that N and $M_0[H]$ are mutually generic extensions of M_ω, we use the criterion of Proposition 1.7.8. In other words, if $a \in M_0[H]$ and $b \in N$ are disjoint subsets of some ordinal κ, we must find a set $c \in M_\omega$ of ordinals such that $a \subset c$ and $b \cap c = 0$. Towards this end, move back to the model M_0. Suppose that $O \subset \Gamma$ is a nonempty open set, $p \in P_0$ is a condition, $\dot{a}$ is a P_0-name for a set of ordinals, and ϕ is a formula with parameters in M_ω such that in the poset $P_\Gamma \times P_0$, $\langle O, p \rangle \Vdash \text{Coll}(\omega, \lambda) \Vdash \forall \beta \in \dot{a}\ \phi(\beta, \dot{\gamma}_{\text{gen}} \cdot \tau_0)$ holds. Due to the definition of the model N, it will be enough to find a set $c \in M_\omega$ and a condition $\langle O', p' \rangle \leq \langle O, p \rangle$ which forces $\dot{a} \subset \check{c}$ and $\text{Coll}(\omega, \lambda) \Vdash \forall \beta \in \check{c}\ \phi(\beta, \dot{\gamma}_{\text{gen}} \cdot \tau_0)$ holds.

Finally, we are in a position to use some coherence arguments. Let $\prec$ be a coherent well-ordering of $M_0 \cap V_\lambda$; i.e. such that the restriction of $\prec$ to each M_n belongs to M_n. We will use the ordering to perform some coherent constructions. A typical construction of a coherent sequence (in the sense of Definition 4.2.1) proceeds by induction on $n \in \omega$. If $\langle v_n : n \in \omega \rangle$ is coherent, $w_0 \in M_0$, and ϕ is some formula with parameters in M_ω, one can select the $\prec$-least $w_{n+1} \in M_{n+1}$

such that $M_n \models \phi(v_n, w_n, w_{n+1})$ if it exists; then, the sequence $\langle w_n : n \in \omega \rangle$ is coherent. The routine details of these constructions will be suppressed below.

Find a coherent sequence $\langle P_n, \tau_n : n \in \omega \rangle$ of pairwise equivalent E-pins on posets in V_λ starting with $\langle P_0, \tau_0 \rangle$; i.e. for every number $n \in \omega$ it is the case that $P_n \times P_{n+1} \Vdash \sigma_n \; E \; \sigma_{n+1}$. Find a coherent sequence $\langle \dot\gamma_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\dot\gamma_n$ is a $P_n \times P_{n+1}$-name for an element of the group Γ such that $\tau_n = \dot\gamma_n \cdot \tau_{n+1}$. Let $D \subset \Gamma$ be a fixed countable dense set in the model M_ω, and let $\delta_0 \in D$ and $\varepsilon > 0$ be such that the open d-ball $B(\delta_0, \varepsilon) \subset \Gamma$ is a subset of the open set O. Find a coherent sequence $\langle p_n, \delta_n : n \in \omega \rangle$ such that $p_0 \leq p$, $p_n \in P_n$, $\delta_n \in D$, and in the poset $P_n \times P_{n+1}$, $\langle p_n, p_{n+1} \rangle \Vdash d(\delta_n \cdot \dot\gamma_n, \delta_{n+1}) < \varepsilon \cdot 2^{-n-3}$. Let $O_n = B(\delta_n, \varepsilon/2)$. The point of these definitions is the following claim:

Claim 4.3.7. *Let $n > 0$. The condition $\langle p_i : i \leq n \rangle$ forces in the product $\prod_{i \leq n} P_i$ the following:*

1. $O_n \subset B(\delta_0, \varepsilon) \cdot \dot\gamma_0 \dot\gamma_1 \dots \dot\gamma_{n-1}$;
2. $B(\delta_0, \varepsilon/4) \cdot \dot\gamma_0 \dot\gamma_1 \dots \dot\gamma_{n-1} \subset O_n$.

Proof. Use the right invariance of the metric d to argue by induction on $i \in n$ that $d(\delta_{i+1}, \delta_0 \dot\gamma_0 \dot\gamma_1 \dots \dot\gamma_i)$ is forced to be smaller than $\varepsilon \cdot \Sigma_{j \leq i} 2^{-j-3}$. In conclusion, $d(\delta_n, \delta_0 \dot\gamma_0 \dot\gamma_1 \dots \dot\gamma_{n-1})$ is forced to be smaller than $\varepsilon/4$. The two items then follow immediately by the right invariance of the metric d again. □

Now, for every number $n \in \omega$, in the model M_n form the set $c_n = \{\beta \in \kappa :$ in the poset $P_\Gamma \times P_n$, $\langle O_n, p_n \rangle \Vdash \mathrm{Coll}(\omega, \lambda) \Vdash \phi(\check\beta, \dot\gamma_{\mathrm{gen}} \cdot \tau_n)\}$. Finally, let $c = \limsup_n c_n = \{\beta \in \kappa : \exists^\infty n \; \beta \in c_n\}$. It is immediate that the sequence $\langle c_n : n \in \omega \rangle$ is coherent and therefore the set c belongs to the model M_ω. Let $O' = B(\delta_0, \varepsilon/4)$ and $p' = p_0$. The following two claims stated in the model M_0 complete the proof of the theorem.

Claim 4.3.8. *In the poset P_0, $p' \Vdash \dot a \subset \check c$.*

Proof. Let $p'' \leq p'$ be a condition and $\beta \in \kappa$ an ordinal such that $p'' \Vdash \check\beta \in \dot a$. It will be enough to show that for all $n > 0$, $\beta \in c_n$. To this end, fix a number $n > 0$ and let $\langle H_i : i \leq n \rangle$ be a tuple of filters on the respective posets P_i mutually generic over the model M_0 such that $p_i \in H_i$ and moreover $p'' \in H_0$. Write $x_i = \tau_i / H_i$ and $\gamma_i = \dot\gamma_i / H_i, H_{i+1}$; so $x_0 = \gamma_0 \gamma_1 \dots \gamma_{n-1} x_n$.

Let $\gamma \in O_n$ be a point P_Γ-generic over the model $M_0[H_i : i \leq n]$. Let $\gamma' = \gamma \gamma_{n-1}^{-1} \gamma_{n-2}^{-1} \dots \gamma_0^{-1}$. By the invariance of the meager ideal on Γ under right translations, $\gamma' \in \Gamma$ is a point Cohen generic over the model $M_0[H_i : i \leq n]$. By Claim 4.3.7(1), $\gamma' \in B(\delta_0, \varepsilon) \subset O$; moreover, $\gamma \cdot x_n = \gamma' \cdot x_0$.

Let $K \subset \mathrm{Coll}(\omega, \lambda)$ be a filter generic over the model $M_0[H_i : i \leq n][\gamma]$. The model $M_0[H_i : i \leq n][\gamma][K]$ is a $\mathrm{Coll}(\omega, \lambda)$-extension of both $M_0[\gamma', H_0]$ and $M_n[\gamma, H_n]$ by the choice of λ and Fact 1.7.12. By the forcing theorem in the model M_0 and the initial assumptions on the name $\dot a$ and the formula ϕ, $M_0[H_i : i \leq n][\gamma][K] \models \phi(\beta, \gamma' \cdot x_0)$. By the forcing theorem in the model M_n, the filter on $P_\Gamma \times P_n$ given by γ, H_n must contain a condition forcing

$\mathrm{Coll}(\omega, \lambda) \Vdash \phi(\check{\beta}, \gamma \cdot \tau_n)$. However, γ, H_n were arbitrary generics meeting the condition $\langle O_n, p_n \rangle$, so it must be the case that this condition forces $\mathrm{Coll}(\omega, \lambda) \Vdash \phi(\check{\beta}, \dot{\gamma}_{\mathrm{gen}} \cdot \tau_n)$. This means that $\beta \in c_n$ as required. □

Claim 4.3.9. *In the poset $P_\Gamma \times P_0$, for every ordinal $\beta \in c$, $\langle O', p' \rangle \Vdash \mathrm{Coll}(\omega, \lambda) \Vdash \phi(\check{\beta}, \dot{\gamma} \cdot \tau_0)$.*

Proof. Find a number $n > 0$ such that $\beta \in c_n$. Let $H_i \subset P_i$ for $i \leq n$ be filters mutually generic over M_0 containing the conditions p_i respectively, with $p' \in H_0$. Write $x_i = \tau_i / H_i$ and $\gamma_i = \dot{\gamma}_i / H_i, H_{i+1}$; so $x_0 = \gamma_0 \gamma_1 \dots \gamma_{n-1} x_n$.

Let $\gamma' \in O'$ be a point P_Γ-generic over the model $M_0[H_i : i \leq n]$. Let $\gamma = \gamma' \gamma_0 \gamma_1 \cdot \gamma_{n-1}$; by the invariance of the meager ideal on Γ under right translations, $\gamma \in \Gamma$ is a point Cohen generic over the model $M_0[H_i : i \leq n]$. By Claim 4.3.7(2), $\gamma \in O_n$; moreover, $\gamma \cdot x_n = \gamma' \cdot x_0$.

Let $K \subset \mathrm{Coll}(\omega, \lambda)$ be a filter generic over the model $M_0[H_i : i \leq n][\gamma]$. The model $M_0[H_i : i \leq n][\gamma'][K]$ is a $\mathrm{Coll}(\omega, \lambda)$-extension of both $M_0[\gamma', H_0]$ and $M_n[\gamma, H_n]$ by Fact 1.7.12. By the forcing theorem in the model M_n and the definition of the set c_n, $M_0[H_i : i \leq n][\gamma'][K] \models \phi(\beta, \gamma \cdot x_n)$. By the forcing theorem in the model M_0, the filter on $P_\Gamma \times P_0$ given by γ', H_n must contain a condition forcing $\mathrm{Coll}(\omega, \lambda) \Vdash \phi(\check{\beta}, \dot{\gamma}_{\mathrm{gen}} \cdot \tau_0)$. However, γ', H_0 were arbitrary meeting the condition $\langle O', p' \rangle$, so it must be the case that this condition forces $\mathrm{Coll}(\omega, \lambda) \Vdash \phi(\check{\beta}, \dot{\gamma}_{\mathrm{gen}} \cdot \tau_n)$ as required. □

□

Example 4.3.10. Consider the equivalence relation $\mathbb{E}_1$ on $X = (2^\omega)^\omega$; it is well-known not to be reducible to any orbit equivalence relation [48, Theorem 11.8.1]. The conclusion of Theorem 4.3.6 fails for $\mathbb{E}_1$. To see this, choose any partial order Q which adds a new point $\dot{y} \in 2^\omega$. Let P be the full support product of ω-many copies of Q, and let P_n be the product of the copies of the copies of Q indexed by natural numbers $\geq n$. The posets P_n for $n \in \omega$ naturally form a coherent sequence. Let $G \subset P$ be a generic filter, and for each $n \in \omega$ let $G_n \subset P_n$ be the restriction of G to conditions in P_n. Theorem 4.3.5 shows that $\langle V[G_n] : n \in \omega \rangle$ is a choice-coherent sequence of models, and that $\bigcap_n V[G_n]$ contains no new reals compared to V. In $V[G_n]$, let $x_n \in X$ be the sequence defined by letting $x_n(m)$ be the zero sequence if $m \in n$ and the evaluation of $\dot{y}$ by the n-th coordinate of the generic filter G otherwise. It is clear that the points x_n all represent the same $\mathbb{E}_1$-class, which is not represented in V and therefore in $\bigcap_n V[G_n]$.

As a final remark, in general it is necessary to consider virtual E-classes as opposed to just E-classes in the statement of Theorem 4.3.6. To see this, start with the trivial coherent sequence $\langle M_n = V : n \in \omega \rangle$ and let P_n be the countable support product of copies of the poset $\mathrm{Coll}(\omega, 2^\omega)$ indexed by natural numbers $\geq n$, with the natural projections from P_n to P_m added. This is a diagonally complete sequence of posets as in Example **??**, and by Theorem 4.2.8 it induces a choice-coherent sequence of models $\langle M_n[G_n] : n \in \omega \rangle$ such that the model

$\bigcap_n M_n[G_n]$ contains only ground model ω-sequences of ordinals. Now, every model $M_n[G_n]$ contains an enumeration of the set $(2^\omega \cap V)$ by natural numbers, and all of these enumerations are $\mathbb{F}_2$-related. Clearly, there is no $\mathbb{F}_2$-equivalent of them in the intersection model. Yet, there is a *virtual* $\mathbb{F}_2$-class related to these enumerations in the intersection model, and even in the ground model V.

Part II

Balanced extensions of the Solovay model

Chapter 5

Balanced Suslin forcing

5.1 Virtual conditions

We look at the class of Suslin posets from an angle quite distinct from the standard treatment in [7]; in particular, the center of attention is on σ-closed Suslin forcings as opposed to c.c.c. or proper forcings adding reals. Recall:

Definition 5.1.1. A preorder $\langle P, \leq \rangle$ is *Suslin* if there is a Polish space X such that

1. P is an analytic subset of X;
2. the ordering $\leq$ is an analytic subset of X^2;
3. the incompatibility relation is an analytic subset of X^2.

It is important to understand that Suslin forcings may fail to be separative. Recall that conditions p_0, p_1 are called *inseparable* if for every condition $q \leq p_0$, q, p_1 have a lower bound, and vice versa, for every condition $q \leq p_1$, q, p_0 have a lower bound. The inseparability relation E_P is an equivalence; the partial order P is separative if this equivalence relation is the identity. A simple yet informative example of a nonseparative Suslin forcing is in order. Consider the poset of countable partial functions from 2^ω to 2 ordered by reverse extension. As stated, it is not a Suslin forcing, since it is not an analytic subset of a Polish space. One has to make an innocuous adjustment: P is in fact the set of all functions from ω to $2^\omega \times 2$ whose range is a function, and order P by setting $q \leq p$ if $\mathrm{rng}(p) \subseteq \mathrm{rng}(q)$. This adjustment sacrifices the separability for Suslinness. Note that the equivalence relation E_P is unpinned in this case.

Throughout the rest of the book, we will make use of *virtual conditions* in Suslin posets. Similar to virtual equivalence classes, these are conditions which may not exist in the present model of set theory and appear only in some generic extension, and yet we have a sensible calculus for dealing with them in the ground model. We want the space of virtual conditions not to depend on a particular presentation of a given Suslin poset, and to be rich enough to harvest certain critical features. This leads to the following definition:

Definition 5.1.2. Let $\langle P, \leq \rangle$ be a Suslin forcing.

1. Let Q be a poset and τ a Q-name for an analytic subset of P. We say that the name τ is *P-pinned* if $Q \times Q \Vdash \Sigma\tau_{\text{left}} = \Sigma\tau_{\text{right}}$ in the completion of the separative quotient of the poset P. If τ is P-pinned, then the pair $\langle P, \tau \rangle$ is called a *P-pin*.

2. Let $\langle Q_0, \tau_0 \rangle$ and $\langle Q_1, \tau_1 \rangle$ be P-pins. Define $\langle Q_0, \tau_0 \rangle \leq \langle Q_1, \tau_1 \rangle$ if $Q_0 \times Q_1 \Vdash \Sigma\tau_0 \leq \Sigma\tau_1$ in the separative quotient of the poset P, and $\langle Q_0, \tau_0 \rangle \equiv \langle Q_1, \tau_1 \rangle$ if $\langle Q_0, \tau_0 \rangle \leq \langle Q_1, \tau_1 \rangle$ and $\langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle$.

3. The *virtual conditions* of P are the equivalence classes of $\equiv$.

For the last item of the above definition, we must verify that $\equiv$ is in fact an equivalence relation. This is the content of the following propositions. As the first technical remark, note that for analytic sets $A_0, A_1 \subset P$, the statement $\Sigma A_1 \leq \Sigma A_0$ is equivalent to $\forall p \in P\ \forall p_1 \in A_1\ p \leq p_1 \to \exists q \leq p\ \exists p_0 \in A_0\ q \leq p_0$. Since the ordering $\leq$ on P is analytic, this is a $\mathbf{\Pi}^1_2$ statement and therefore absolute among all generic extensions by the Shoenfield absoluteness. As another remark, since any two generic extensions are mutually generic with a third, $\langle Q, \tau \rangle$ is a P-pin just in case in every forcing extension $V[H]$ and every pair of filters $G_0, G_1 \subset Q$ in $V[H]$ separately generic over V, $\Sigma\tau/G_0 = \Sigma\tau/H_1$ holds in the completion of the separative quotient of P. Similarly, it is possible to restate the definitions of $\leq$ and $\equiv$ without mutual genericity.

Proposition 5.1.3. *Let $\langle P, \leq \rangle$ be a Suslin forcing. Let $\langle Q_0, \tau_0 \rangle$, $\langle Q_1, \tau_1 \rangle$ be posets and names for analytic subsets of P. The following statements are absolute among all forcing extensions:*

1. *$\langle Q_0, \tau_0 \rangle$ is a P-pin;*

2. *$\langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle$;*

3. *$\langle Q_0, \tau_0 \rangle \equiv \langle Q_1, \tau_1 \rangle$.*

Proof. We will prove (2), the other items are parallel. Suppose first that $\langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle$ holds, and $V[H]$ is a generic extension of V; we must verify that the $\leq$-relation transfers to $V[H]$. To see this, suppose that $G_0 \subset Q_0, G_1 \subset Q_1$ are filters mutually generic over $V[H]$, and let $A_0 = \tau_0/G_0$ and $A_1 = \tau_1/G_1$. Thus, $A_0, A_1 \subset P$ are analytic sets with codes in $V[G_0]$ and $V[G_1]$, respectively. By the assumption, $V[G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0$ in the completion of the separative quotient of the poset P. This $\mathbf{\Pi}^1_2$ statement transfers to $V[H][G_0, G_1]$ by the Shoenfield absoluteness. In conclusion, $V[H][G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0$ and therefore $V[H] \models \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle$.

For the other direction, if $\langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle$ fails in V and $V[H]$ is a generic extension of V, we must show that the failure of the $\leq$ relation transfers to $V[H]$.

Fix conditions $q_0 \in Q_0, q_1 \in Q_1$ such that in V, $\langle q_0, q_1\rangle \Vdash_{Q_0\times Q_1} \Sigma\tau_1 \not\leq \Sigma\tau_0$. This latter property of q_0, q_1 readily transfers to the model $V[H]$ by the same Shoenfield absoluteness argument as the one used in the previous paragraph. $\square$

Proposition 5.1.4. *$\equiv$ is an equivalence relation on P-pins and $\leq$ is an ordering on $\equiv$-equivalence classes.*

Proof. We will show that $\leq$ is transitive, the other statements are similar. Suppose that $\langle Q_2, \tau_2\rangle \leq \langle Q_1, \tau_1\rangle \leq \langle Q_0, \tau_0\rangle$; we need to show that $\langle Q_2, \tau_2\rangle \leq \langle Q_0, \tau_0\rangle$ holds. Let $G_0 \subset Q_0, G_2 \subset Q_2$ are mutually generic filters and $A_2 = \tau_2/G_2$ and τ_0/G_0; we must show that $V[G_0, G_2] \models \Sigma A_2 \leq \Sigma A_0$. Let $G_1 \subset Q_1$ be a filter generic over $V[G_0, G_2]$ and let $A_1 = \tau_1/G_1$. By the assumption, $V[G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0$ and $V[G_1, G_2] \models \Sigma A_2 \leq \Sigma A_1$. By the Shoenfield absoluteness, $V[G_0, G_1, G_2] \models \Sigma A_2 \leq \Sigma A_1 \leq \Sigma A_0$, in particular $\Sigma A_2 \leq \Sigma A_0$. By another application of the Shoenfield absoluteness, the inequality $\Sigma A_2 \leq \Sigma A_0$ transfers from $V[G_0, G_1, G_2]$ to $V[G_0, G_2]$. $\square$

It is now time to exhibit some virtual conditions for familiar partial orders.

Example 5.1.5. Let P be the poset of infinite subsets of ω, ordered by inclusion. Let F be a nonprincipal filter on ω. Let τ be a $\mathrm{Coll}(\omega, F)$-name for the set of all conditions $p \in P$ which diagonalize the filter F, i.e. $\forall a \in F\ p \setminus a$ is finite. The name τ is P-pinned, since its valuation does not depend on the choice of the generic filter on $\mathrm{Coll}(\omega, F)$. It is not difficult to see that distinct nonprincipal filters generate distinct virtual conditions.

Example 5.1.6. Let P be the poset of all countable functions from 2^ω to 2, ordered by reverse inclusion. Let f be any function from 2^ω to 2, perhaps uncountable. Let τ be a $\mathrm{Coll}(\omega, f)$-name for the set of all conditions $p \in P$ such that $f \subset p$. Again, the name τ is P-pinned. It is clear that distinct functions f generate distinct virtual conditions in P.

For many Suslin orders there exist virtual conditions which can be classified by natural combinatorial objects. In particular, this holds quite often for balanced virtual conditions, which are introduced in the following section and are the central topic of this book.

Proposition 5.1.7. *Suppose that $\langle P, \leq\rangle$ is a σ-closed Suslin poset such that below any element $p \in P$ there are two incompatible ones. Then the equivalence $\equiv$ has proper class many equivalence classes.*

Proof. Let $h\colon 2^{<\omega} \to P$ be a function such that for every $t \in \omega^{<\omega}$, the conditions $h(t^\frown 0)$ and $h(t^\frown 1)$ are incompatible and stronger than $h(t)$. Let $g\colon \omega \to \omega^2$ be a bijection. For every ordinal α, consider the $\mathrm{Coll}(\omega, \alpha)$ name τ_α for the set of all conditions $p \in P$ such that for every $n \in \omega$ there is (exactly one) string $t(p, n) \in 2^n$ such that $p \leq h(t)$, and the binary relation $g''\{n \in \omega\colon t(p, n+1)(n) = 1\}$ is isomorphic to α. It is not difficult to see that the pair $\langle \mathrm{Coll}(\omega, \alpha), \tau_\alpha\rangle$ is a P-pin; in fact the evaluation of τ_α yields the same analytic set no matter what the generic filter on $\mathrm{Coll}(\omega, \alpha)$ is. The P-pins obtained in this way are $\equiv$-inequivalent; in fact, for distinct ordinals $\alpha \neq \beta$, $\mathrm{Coll}(\omega, \alpha) \times \mathrm{Coll}(\omega, \beta) \Vdash \Sigma\tau_\alpha$ and $\Sigma\tau_\beta$ are incompatible elements of the completion of the separative quotient of P. $\square$

However, certain type of virtual conditions is entirely central to the technology developed in this book, and normally allows a neat classification by natural combinatorial objects. These are the balanced virtual conditions of the next section.

5.2 Balanced conditions

This section isolates the notion of a balanced condition in a given Suslin poset and a natural equivalence of balanced conditions. It turns out that every balanced class is represented by a virtual condition.

Definition 5.2.1. Let $\langle P, \leq \rangle$ be a Suslin forcing. Let Q be a poset and τ a Q-name for an analytic subset of P. We say that $\langle Q, \tau \rangle$ is a *balanced pair* in P if for all posets R_0, R_1 and all $R_0 \times Q$- and $R_1 \times Q$- names σ_0, σ_1 for elements of P such that $R_0 \times Q \Vdash \sigma_0 \leq \Sigma\tau$ and $R_1 \times Q \Vdash \sigma_1 \leq \Sigma\tau$, it is the case that $(R_0 \times Q) \times (R_1 \times Q) \Vdash \sigma_0, \sigma_1$ are compatible conditions in the poset P.

Note that the definition of a balanced pair depends on the Suslin poset P. The Suslin poset is not mentioned as it will be always understood from the context and no opportunity for confusion arises. The expressions of the type $\sigma \leq \Sigma\tau$ (meaning that every condition extending σ is compatible with a condition in τ) will be shortened to $\sigma \leq \tau$ below as there can be no confusion. Balanced pairs and virtual conditions are quite different things; the main topic of interest in this book are pairs $\langle Q, \tau \rangle$ which are simultaneously balanced and virtual conditions. The following proposition restates the notion of a balanced pair in terms of the generic extension as opposed to the forcing relation.

Proposition 5.2.2. *Let $\langle P, \leq \rangle$ be a Suslin forcing. Let Q be a poset and τ a Q-name for an analytic subset of P. The following are equivalent:*

1. *$\langle Q, \tau \rangle$ is a balanced pair;*
2. *whenever $V[H_0]$ and $V[H_1]$ are mutually generic extensions, $G_0, G_1 \subset Q$ are filters generic over the ground model in the respective extensions and $p_0 \leq \tau/G_0$, $p_1 \leq \tau/G_1$ are conditions in P in the respective extensions $V[H_0], V[H_1]$, then p_0, p_1 are compatible in P.*

The class of balanced conditions is greatly simplified by introducing the following equivalence relation on it:

Definition 5.2.3. Let $\langle P, \leq \rangle$ be a Suslin forcing. If $\langle Q_0, \tau_0 \rangle$ and $\langle Q_1, \tau_1 \rangle$ are balanced pairs, we say that $\langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle$ if for all posets R_0, R_1 and all $R_0 \times Q_0$- and $R_1 \times Q_1$- names σ_0, σ_1 for elements of P such that $R_0 \times Q_0 \Vdash \sigma_0 \leq \tau$

and $R_1 \times Q_1 \Vdash \sigma_1 \leq \tau$, it is the case that $(R_0 \times Q_0) \times (R_1 \times Q_1) \Vdash \sigma_0, \sigma_1$ are compatible conditions in the poset P.

Proposition 5.2.4. *The relation $\equiv_b$ is an equivalence on balanced pairs. Moreover, if $\langle Q, \tau \rangle$ is a balanced pair, then*

1. *if $Q \Vdash \sigma \leq \tau$ then $\langle Q, \sigma \rangle$ is a balanced pair $\equiv_b$-related to $\langle Q, \tau \rangle$;*
2. *if Q is a regular subposet of R then $\langle R, \tau \rangle$ is a balanced pair $\equiv_b$-related to $\langle Q, \tau \rangle$.*

Proof. We argue for the first sentence; items (1) and (2) are immediate consequences of the definition. The relation $\equiv_b$ is clearly symmetric and contains the identity; we will show that it is transitive. Let $\langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \equiv_b \langle Q_2, \tau_2 \rangle$; it must be shown that $\langle Q_0, \tau_0 \rangle \equiv_b \langle Q_2, \tau_2 \rangle$ follows. To this end, let R_0, R_2 be posets and σ_0, σ_2 be $R_0 \times Q_0$- and $R_2 \times Q_2$-names for conditions in P stronger than τ_0, τ_1 respectively. Since $\langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle$ holds, there is a $(R_0 \times Q_0) \times (R_1 \times Q_1)$-name σ_1 for an element of P which is a lower bound of σ_0 and τ_1. Since $\langle Q_1, \tau_1 \rangle \equiv_b \langle Q_2, \tau_2 \rangle$ holds, there is a $(R_0 \times Q_0) \times (R_1 \times Q_1) \times (R_2 \times Q_2)$-name η which is a lower bound of σ_1 and σ_2. Thus, in the long product extension $(R_0 \times Q_0) \times (R_1 \times Q_1) \times (R_2 \times Q_2)$, the conditions σ_0, σ_2 are compatible, and by Mostowski absoluteness this statement transfers to the $(R_0 \times Q_0) \times (R_2 \times Q_2)$-extension. This completes the proof of the transitivity of $\equiv_b$. $\square$

It is now time for a simple and informative example. Let P be the poset of countable functions from 2^ω to 2, ordered by reverse inclusion. On one hand, if $f\colon 2^\omega \to 2$ is a total function, then $\langle \mathrm{Coll}(\omega, 2^\omega), \check{f} \rangle$ is a balanced pair. On the other hand, if $\langle Q, \tau \rangle$ is a balanced pair, then for every point $x \in 2^\omega$ in the ground model, it must be the case that Q forces $\check{f}(x) \in \mathrm{dom}(\tau)$ and in fact Q has to decide the value of $\tau(\check{x})$ as well–otherwise it would be easy to violate the balance of the pair. Let $f\colon 2^\omega \to 2$ be the total function given by $\forall x \in 2^\omega\ Q \Vdash \tau(\check{x}) = \check{f}(\check{x})$ and note that $\langle Q, \tau \rangle \equiv_b \langle \mathrm{Coll}(\omega, 2^\omega), \check{f} \rangle$. Thus, the balanced classes for P are exactly classified by total functions from 2^ω to 2.

One of the main concerns of this book is the classification of balanced classes for various Suslin posets $\langle P, \leq \rangle$. The following theorem is the basic contribution in this direction.

Theorem 5.2.5. *Let $\langle P, \leq \rangle$ be a Suslin forcing. Every $\equiv_b$-class contains a virtual condition. The condition is unique up to virtual condition equivalence.*

Proof. Let $\langle Q, \tau \rangle$ be a balanced pair; by Proposition 5.2.4 we may assume that τ is a name for a single element of P. Consider the poset $Q' = \mathrm{Coll}(\omega, \mathcal{P}(Q))$ and the name τ' for the analytic set $\{p \in P\colon$ for some filter $G \subset Q$ meeting all open dense sets enumerated by the Q'-generic, $p = \tau/G\}$. We will first show that the pair $\langle Q', \tau' \rangle$ is balanced via Proposition 5.2.2.

To this end, suppose that $V[H_0], V[H_1]$ are mutually generic extensions, $G'_0, G'_1 \subset Q'$ are filters in the respective extensions generic over V, and $r_0 \leq \tau'/G'_0$ and $r_1 \leq \tau'/G'_1$ are conditions in the poset P; we must show that $r_0, r_1 \in$

P are compatible conditions. By the definition of the name τ', in $V[H_0]$ it is possible to find a filter $G_0 \subset Q$ generic over V such that $p_0 = \tau/G_0$ is compatible with r_0, and similarly in the model $V[G_1]$. Let $s_0, s_1 \in P$ be lower bounds of r_0 and p_0 in $V[G_0]$ and r_1 and p_1 in $V[G_1]$ respectively. Since $\langle Q, \tau\rangle$ was a balanced condition in P, the conditions s_0, s_1 must be compatible in P, and their lower bound is a lower bound of r_0 and r_1 as well.

To show that $\langle Q, \tau\rangle \equiv_b \langle Q', \tau'\rangle$, use both parts (1) and (2) of Proposition 5.2.4 with the intermediate pair $\langle Q', \tau'\rangle$. To show that the pair $\langle Q', \tau'\rangle$ is a virtual condition, suppose that $G'_0, G'_1 \subset Q'$ are mutually generic filters. Looking at the model $V[G_0, G_1]$, it in fact turns out that τ'/G'_0 and τ'/G'_1 are in fact equal as analytic subsets of P.

To show the uniqueness of the virtual condition $\langle Q', \tau'\rangle$ up to the virtual condition equivalence, suppose that $\langle Q'', \tau''\rangle$ is another virtual condition which is an element of the $\equiv_b$-class of the balanced class of $\langle Q, \tau\rangle$; we must show that $Q' \times Q'' \Vdash \Sigma\tau' = \Sigma\tau''$. For the $\leq$ direction, suppose that $G' \subset Q'$ and $G'' \subset Q''$ are mutually generic filters and $r \in P$ is a condition in $V[G', G'']$ which is below $\Sigma\tau'/G'$; we must show that it is compatible with some element of τ''/G''. Let $H \subset Q''$ be a filter generic over the model $V[G', G'']$. Since the pairs $\langle Q', \tau'\rangle$ and $\langle Q'', \tau''\rangle$ come from the same balanced class, the condition r must be compatible with every element of τ''/H. Let $s \in V[G', G'', H]$ be such that $s \leq r$ and $s \leq \Sigma\tau''/H$. Since the pair $\langle Q'', \tau''\rangle$ is a virtual condition, the condition $s \leq \Sigma\tau''/H$ must be compatible with some condition of τ''/G''. By the Mostowski absoluteness between the models $V[G', G'']$ and $V[G', G'', H]$, r is compatible with some element of τ''/G'' in the model $V[G', G'']$ as required. □

Question 5.2.6. Let $\langle P, \leq\rangle$ be a Suslin forcing. Is it necessarily the case that the equivalence relation $\equiv_b$ has only set many classes? Is it necessarily the case that every balance class has a representative on a poset of size $< \beth_{\omega_1}$?

It is now time to state the central definition of this book.

Definition 5.2.7. Let P be a Suslin poset. P is *balanced* if for every condition $p \in P$ there is a balanced virtual condition below p.

A definition of this sort immediately raises a question: which Suslin posets are balanced? We should immediately douse the flames of entirely misguided hopes:

Proposition 5.2.8. *The following Suslin posets do not have any balanced virtual conditions and therefore are not balanced:*

1. *nonatomic c.c.c. posets;*
2. *nonatomic tree posets;*
3. *posets of the form $\mathcal{P}(\omega)/I$ where I is a countably separated Borel ideal on ω.*

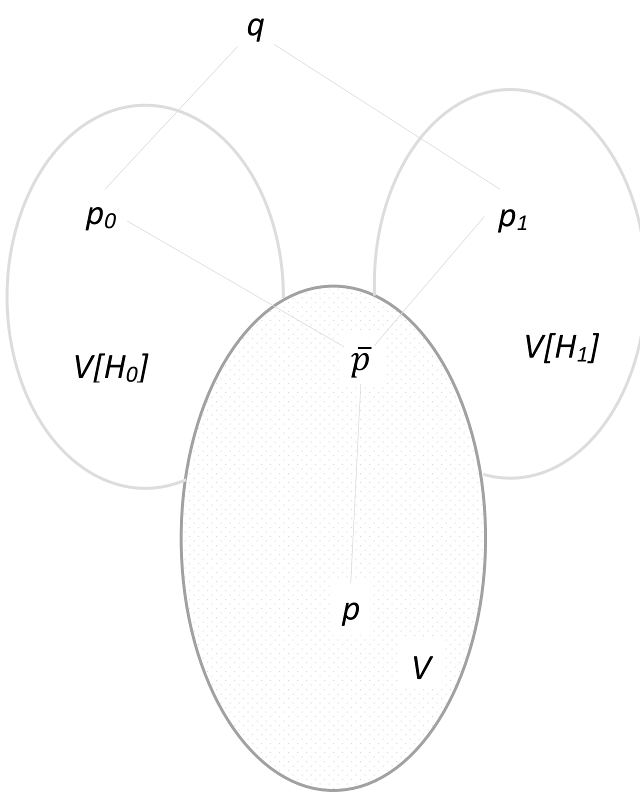

Figure 5.1: A balanced virtual condition $\bar{p} \leq p$.

Here, a *tree poset* on a Polish space X is an analytic family of closed subsets of X closed under nonempty intersections with closures of basic open sets, ordered by inclusion.

Proof. For (1), suppose towards a contradiction that $\langle P, \leq\rangle$ is a Suslin c.c.c. poset and $\langle Q, \tau\rangle$ is a balanced pair. For every maximal antichain $A \subset P$, its maximality is a coanalytic statement and therefore absolute to the Q-extension. Thus, $Q \Vdash \tau$ is compatible with some element of A. The balance of τ immediately shows that there can be only one element of A with which τ is compatible, and the largest condition in Q identifies this element. It follows that the set $\{p \in P \colon Q \Vdash \tau \leq \check{p}$ in the separative quotient of $P\}$ is a filter on P which meets all maximal antichains, an impossibility in nonatomic posets.

For (2), suppose towards a contradiction that $\langle P, \leq\rangle$ is a tree poset on a Polish space X and $\langle Q, \tau\rangle$ is a balanced pair. For every basic open set $O \subset X$, the statement $\tau \cap \bar{O} \neq 0$ must be decided by the largest condition in Q by the balance of τ. This decision cannot be positive for two disjoint basic open subsets of X by the balance of τ again. Thus, τ would have to be forced by Q to be a singleton (even a specific singleton in the ground model), an impossibility in nonatomic tree posets.

For (3), suppose towards a contradiction that I is a countably separated ideal on ω as witnessed by a countable separating set $A \subset \mathcal{P}(\omega)$, meaning that for every $b \in I$ and $c \notin I$ there is $a \in A$ such that $b \cap a = 0$ and $c \cap a \notin I$. Suppose that $\langle Q, \tau\rangle$ is a balanced pair in the poset $\mathcal{P}(\omega)/I$; so τ is a name for an I-positive subset of ω. By the balance of τ, for every set $a \in A$ it must be decided by the largest condition in Q whether $\tau \cap \check{a} \in I$ or not. Let $B = \{a \in A \colon Q \Vdash \tau \cap \check{a} \notin I\}$. Use the balance of the name τ to argue that any intersection of finitely many elements of B is an infinite set. Let $b \subset \omega$ be an infinite set such that for every $a \in B$, $b \setminus a$ is finite. Use the density of the ideal I to argue that thinning out the set b if necessary, we may assume that $b \in I$. It is now immediate that no set $a \in A$ can separate the I-small set b from the I-positive set τ in the Q-extension: if $a \in A \setminus B$ then $\tau \cap a \in I$ is forced, and if $a \in B$ then $b \cap a \neq 0$. This is a contradiction. $\square$

Most balanced Suslin posets used in this book are σ-closed. There are σ-closed posets which are not balanced, such as the one which adds a maximal almost disjoint family in $\mathcal{P}(\omega)$ by countable approximations, cf. Theorem 14.1.1. There are some posets which are balanced and in ZFC even collapse $\aleph_1$, cf. Theorem 8.7.2. Even such posets are valuable; remember that they prove their worth in the choiceless symmetric Solovay extension. The balanced status of certain posets is nonabsolute, see for example Theorem 8.1.19 or 8.5.5. Thus, even though typically the balanced conditions correspond to traditional objects of combinatorial set theory, the balanced status is a complicated matter. There is only one general preservation theorem, which is nevertheless extremely useful for obtaining consistency results:

Theorem 5.2.9. *Let $P = \prod_n P_n$ be a countable support product of Suslin forcing notions. Balanced conditions in P are exactly classified by sequences $\langle p_n : n \in \omega\rangle$ where for every $n \in \omega$, p_n is a balanced condition in P_n.*

Proof. On one hand, if $\langle Q_n, \tau_n\rangle$ are balanced pairs for each $n \in \omega$, the name for the sequence $\langle \tau_n : n \in \omega\rangle$ in the product $Q = \prod_n Q_n$ is balanced for the poset P essentially by the definitions. The choice of the support in the product Q is immaterial. On the other hand, if Q is a poset and $\tau = \langle \tau_n : n \in \omega\rangle$ is a balanced Q-name for the poset P, it must be the case that each of the names τ_n for $n \in \omega$ is balanced for the poset P_n. It equally easy to see that equivalent balanced names for P give equivalent balanced names on each coordinate, and sequences of balanced names on the posets P_n which are coordinatewise equivalent yield equivalent balanced names for the product forcing. $\square$

Corollary 5.2.10. *The countable support product of balanced Suslin forcings is balanced.*

One issue that is constantly present in this book is the lack of absoluteness of the notions surrounding balance. As long as Question 5.2.6 remains open, it will also be necessary to relativize the definition of a balanced poset to V_κ where κ is an inaccessible cardinal. One may think that with suitable large cardinal hypothesis on κ, one could use reflection to show that relativization is unnecessary. The best result we have in this direction is

Proposition 5.2.11. *Let $\langle P, \leq\rangle$ be a Suslin forcing. Let κ be a strong cardinal. The following are equivalent:*

1. *P is balanced;*
2. *$V_\kappa \models P$ is balanced.*

Proof. The argument uses two simple absoluteness claims of independent interest:

Claim 5.2.12. *If M is a transitive model of ZFC, $\langle Q, \tau\rangle$ is a pair in M and the pair is balanced in V, then it is balanced in M.*

Proof. Immediate by Mostowski absoluteness between generic extensions of M and V. $\square$

Claim 5.2.13. *If M is a transitive model of ZFC, $\langle Q, \tau\rangle$ is a pair in M, $M \models \langle Q, \tau\rangle$ is balanced, and $\mathcal{P}(Q) \subset M$, then the pair $\langle Q, \tau\rangle$ is balanced in V.*

Proof. Work in V. Suppose that the conclusion fails, as witnessed by posets R_0, R_1 and names σ_0, σ_1 on $R_0 \times Q$ and $R_1 \times Q$ respectively. Take an elementary submodel N of a large enough structure such that $Q \subset N$ and $|N| = |Q|$. The posets $R_0 \cap N, R_1 \cap N$ and names $\sigma_0 \cap N$ and $\sigma_1 \cap N$ still witness the failure of of the balance of the pair $\langle Q, \tau\rangle$: if $G_0 \subset (R_0 \cap N) \times Q$ and $G_1 \subset (R_1 \cap N) \times Q$ are mutually generic filters, $p_0 = \sigma_0/G_0$ and $p_1 = \sigma_1/G_1 \in P$, then $N[G_0, G_1] \models$

$p_0, p_1 \in P$ are incompatible conditions by the forcing theorem applied in the model N, and by the Mostowski absoluteness between $N[G_0, G_1]$ and $V[G_0, G_1]$, this is still true in $V[G_0, G_1]$. Now, the assumption $\mathcal{P}(Q) \subset M$ shows that the posets $R_0 \cap N, R_1 \cap N$ and names $\sigma_0 \cap N$ and $\sigma_1 \cap N$ have isomorphic copies in the model M, obtaining the failure of balance of the pair $\langle Q, \tau\rangle$ in M. $\square$

Now, Claim 5.2.13 applied to $M = V_\kappa$ immediately yields the implication (2)$\to$(1). For the converse, suppose that P is balanced, $p \in P$, and find a balanced pair $\langle Q, \tau\rangle$ below p. We have to deal with the unseemly possibility that $|Q| > \kappa$ holds. Use the large cardinal hypothesis to find an elementary embedding $j\colon V \to M$ with critical point κ such that $\langle Q, \tau\rangle \in M \cap V_{j(\kappa)}$. Claim 5.2.12 shows that $M \models \langle Q, \tau\rangle$ is a balanced pair below p. By elementarity of the embedding j, there must be a balanced pair $\langle Q', \tau'\rangle \in V_\kappa$ below p. Applying Claim 5.2.12 again with $M = V_\kappa$, we see that $V_\kappa \models \langle Q', \tau'\rangle$ is a balanced pair below p. Since the condition $p \in P$ was arbitrary, (2) follows. $\square$

5.3 Weakly balanced Suslin forcing

There is an interesting generalization of balanced Suslin forcing which can realize additional effects in extensions of the symmetric Solovay model. The basic definitions can be stated as a minor variation of the work done in the previous sections.

Definition 5.3.1. Let P be a Suslin forcing. A pair $\langle Q, \tau\rangle$ is *weakly balanced* if Q forces τ to be an analytic subset of P, and whenever R_0, R_1 are posets, σ_0, σ_1 are $R_0 \times Q$ and $R_1 \times Q$-names for elements of P below $\Sigma\tau$ and $\langle r_0, q_0\rangle \in R_0 \times Q$ and $\langle r_1, q_1\rangle \in R_1 \times Q$ are conditions then in some forcing extension there are filters $H_0 \subset R_0 \times Q$ and $H_1 \subset R_1 \times Q$ which are separately generic over V, $\langle r_0, q_0\rangle \in H_0$, $\langle r_1, q_1\rangle \in H_1$, and $\sigma_0/H_0, \sigma_1/H_1$ are compatible elements of P.

Definition 5.3.2. Let P be a Suslin forcing and $\langle Q_0, \tau_0\rangle$ and $\langle Q_1, \tau_1\rangle$ are weakly balanced pairs. Say that the pairs are *equivalent* and write $\langle Q_0, \tau_0\rangle \equiv_{wb} \langle Q_1, \tau_1\rangle$ if whenever R_0, R_1 are posets, σ_0, σ_1 are $R_0 \times Q_0$ and $R_1 \times Q_1$-names for elements of P below τ_0 and τ_1 respectively and $\langle r_0, q_0\rangle \in R_0 \times Q_0$ and $\langle r_1, q_1\rangle \in R_1 \times Q_1$ are conditions then in some forcing extension there are filters $H_0 \subset R_0 \times Q_0$ and $H_1 \subset R_1 \times Q_1$ which are separately generic over V, $\langle r_0, q_0\rangle \in H_0$, $\langle r_1, q_1\rangle \in H_1$, and $\sigma_0/H_0, \sigma_1/H_1$ are compatible elements of P.

Proposition 5.3.3. *The relation $\equiv_{wb}$ is an equivalence relation on weakly balanced pairs.*

Proof. The relation is clearly symmetric and contains the identity as a subset. For the transitivity, let $\langle Q_0, \tau_0\rangle$, $\langle Q_1, \tau_1\rangle$ and $\langle Q_2, \tau_2\rangle$ be weakly balanced pairs such that $\langle Q_0, \tau_0\rangle \equiv_{wb} \langle Q_1, \tau_1\rangle$ and $\langle Q_1, \tau_1\rangle \equiv_{wb} \langle Q_2, \tau_2\rangle$. To show that $\langle Q_0, \tau_0\rangle \equiv_{wb} \langle Q_2, \tau_2\rangle$ holds, let R_0, R_2 be posets and σ_0, σ_2 be $R_0 \times Q_0$- and $R_2 \times Q_2$-names for elements of P which are forced to be smaller than some element of τ_0 and τ_2 respectively, and let $\langle r_0, q_0\rangle$ and $\langle r_2, q_2\rangle$ be conditions in the two product posets. Let κ be a cardinal bigger than $|\mathcal{P}(R_0 \times Q_0)|$. Since $\langle Q_0, \tau_0\rangle \equiv_{wb} \langle Q_1, \tau_1\rangle$ holds, an absoluteness argument shows that in the $\mathrm{Coll}(\omega, \kappa) \times Q_0$-extension, there is a filter $\dot{H}_0 \subset R_0 \times Q_0$ generic over the ground model containing $\langle r_0, q_0\rangle$ such that σ_0/H_0 and τ_1 as conditions in P have a lower bound; call the $\mathrm{Coll}(\omega, \kappa) \times Q_0$-name for the bound σ_1. Since $\langle Q_1, \tau_1\rangle \equiv_{wb} \langle Q_2, \tau_2\rangle$ holds, in some forcing extension there are filters $H_1 \subset \mathrm{Coll}(\omega, \kappa) \times Q_1$ and $H_2 \subset R_2 \times Q_2$ separately generic over the ground model, such that $\langle r_2, q_2\rangle \in H_2$ and $\sigma_1/H_1, \sigma_2/H_2$ are compatible elements of P. Let $H_0 = \dot{H}_0/H_1 \subset R_0 \times Q_0$; this is a filter generic over the ground model, containing the condition $\langle r_0, q_0\rangle$ such that $\sigma_0/H_0 \leq \sigma_1/H_1$.

In total, the conditions $\sigma_0/H_0, \sigma_2/H_2 \in P$ are compatible, and the relation $\langle Q_0, \tau_0\rangle \equiv_{wb} \langle Q_2, \tau_2\rangle$ has been verified. □

The notion of a weakly balanced pair is a faithful extension of the notion of a balanced pair. This is the content of the following proposition.

Proposition 5.3.4. *Let P be a Suslin forcing.*

1. *Every balanced pair is weakly balanced;*
2. *the class of balanced pairs is invariant under the $\equiv_{wb}$ equivalence;*
3. *the relations $\equiv_b$ and $\equiv_{wb}$ coincide on balanced pairs.*

Proof. (1) is immediate from the definitions. For (2), suppose towards a contradiction that $\langle Q_0, \tau_0\rangle$ and $\langle Q_1, \tau_1\rangle$ are $\equiv_{wb}$-equivalent pairs and $\langle Q_0, \tau_0\rangle$ is balanced, while $\langle Q_1, \tau_1\rangle$ is not. The latter statement is witnessed by some posets R_0, R_1, names σ_0, σ_1, and conditions $\langle r_0, q_0\rangle \in R_0 \times Q_1$ and $\langle r_1, q_1\rangle \in R_1 \times Q_1$ which force in the product $(R_0 \times Q_0) \times (R_1 \times Q_1)$ that σ_0, σ_1 are incompatible elements of P.

Use the $\equiv_{wb}$-equivalence assumption to find posets S_0, S_1 and S_0, S_1-names $\dot{G}_0, \dot{H}_0, \dot{K}_0$ and $\dot{G}_1, \dot{H}_1, \dot{K}_1$ respectively so that $S_0 \Vdash \dot{G}_0 \subset Q_0$ and $\dot{H}_0 \times \dot{K}_0 \subset R_0 \times Q_1$ are filters generic over V such that $\check{r}_0 \in \dot{H}_0$, $\check{q}_0 \in \dot{K}_0$, and $\tau_0/\dot{G}_0$, $\sigma_0/\dot{H}_0 \times \dot{K}_0$ are conditions compatible in P, with a lower bound $\chi_0 \in P$. Similar objects exist on the S_1-side.

Now, let $L_0 \subset S_0, L_1 \subset S_1$ be filters mutually generic over V. The balance of the pair $\langle Q_0, \tau_0\rangle$ shows that $\chi_0/L_0, \chi_1/L_1$ are compatible conditions in the poset P. Consider the filters $H_0 = \dot{H}_0/L_0 \subset R_0$, $K_0 = \dot{K}_0/L_0 \subset Q_1$, and $H_1 = \dot{H}_1/L_1 \subset R_0$, $K_1 = \dot{K}_1/L_1 \subset Q_1$. The filters $H_0 \times K_0 \subset R_0 \times Q_1$ in $V[L_0]$ and $H_1 \times K_1 \subset R_1 \times Q_1$ in $V[L_1]$ are generic over V. Since the filters L_0, L_1 are mutually generic, so are $H_0 \times K_0$ and $H_1 \times K_1$ by Corollary 1.7.9. At the same time, the conditions $\sigma_0/H_0 \times K_0$ and $\sigma_1/H_1 \times K_1$ in P must be compatible, because they are weaker than the compatible conditions χ_0/L_0 and χ_1/L_1 respectively. This is a contradiction with the initial choice of σ_0, σ_1.

(3) is proved in a similar way; the argument is left to the patient reader. □

Proposition 5.3.5. *Let P be a Suslin forcing. Every $\equiv_{wb}$-class contains a virtual condition, which is unique up to $\equiv$-equivalence.*

Proof. Let $\langle Q, \tau\rangle$ be a weakly balanced condition; strengthening τ if necessary, we may assume that τ is in fact a name for an element of P. Let κ be an ordinal such that $\mathcal{P}(Q) \subset V_\kappa$ and let σ be a $\mathrm{Coll}(\omega, V_\kappa)$-name for the set $\{p \in P \colon \exists G \subset Q\ G$ is a filter meeting all the dense subsets of Q in V_κ^V such that $p = \tau/G\}$. It is clear that the pair $\langle \mathrm{Coll}(\omega, V_\kappa), \sigma\rangle$ is a P-pin. We will show that the pair $\langle \mathrm{Coll}(\omega, V_\kappa), \sigma\rangle$ is $\equiv_{wb}$-related to $\langle Q, \tau\rangle$.

Suppose that R_0, R_1 are posets and σ_0, σ_1 are $R_0 \times \mathrm{Coll}(\omega, V_\kappa)$- and $R_1 \times Q$-names for elements of P stronger than σ and τ respectively, and $\langle r_0, q_0\rangle, \langle r_1, q_1\rangle$ are conditions in the respective products. Without loss of generality we may assume that there is a cardinal $\lambda > |V_\kappa|$ such that $R_0 = \mathrm{Coll}(\omega, \lambda)$. Let $\dot K$ be the $R_0 \times \mathrm{Coll}(\omega, V_\kappa)$-name for a filter on Q which is generic over V and such that $\sigma_0 \leq \tau/\dot K$ is forced. By abstract forcing theory, strengthening the condition $\langle r_0, q_0\rangle$ if necessary, we may cast the product $R_0 \times \mathrm{Coll}(\omega, V_\kappa)$ below $\langle r_0, q_0\rangle$ as the poset $\mathrm{Coll}(\omega, \lambda) \times Q$ below some condition $\langle r_0', q_0'\rangle$ so that the name $\dot K$ is a name for the generic filter on the second coordinate of the product. By the weak balance of the name τ, in some generic extension there are filters$H_0' \subset \mathrm{Coll}(\omega, \lambda) \times Q$ and $H_1 \subset R_1 \times Q$ containing the conditions $\langle r_0', q_0'\rangle$ and $\langle r_1, q_1\rangle$ such that the condition $\sigma/H_0', \sigma_1/H_1$ are compatible in P. The filter H_0' can be cast as a filter $H_0 \subset R_0 \times \mathrm{Coll}(\omega, V_\kappa)$ meeting the condition $\langle r_0, q_0\rangle$. This completes the proof of the $\equiv_{wb}$-equivalence.

For the uniqueness part, suppose that $\langle Q_0, \tau_0\rangle$ and $\langle Q_1, \tau_1\rangle$ are P-pins which are both $\equiv_{wb}$-related to $\langle Q, \tau\rangle$; we must show that they are $\equiv$-related. Suppose towards a contradiction that this fails. Then, in the $Q_1 \times Q_0$-extension, there must be an element $p \in P$ which is below some element of τ_0 and incompatible with any element of τ_1 (or vice versa). By a Mostowski absoluteness argument, this element p will maintain its incompatibility property in every further forcing extension. Let σ_0 be a $Q_1 \times Q_0$-name for this element, let σ_1 be a Q_1-name for any element of τ_1, and observe that $\langle Q_1 \times Q_0, \sigma_0\rangle$ and $\langle Q_1, \sigma_1\rangle$ witness that the the pairs $\langle Q_0, \tau_0\rangle$ and $\langle Q_1, \tau_1\rangle$ are $\equiv_{wb}$-unrelated, contradicting the initial assumptions. □

Unlike the balanced conditions, the weakly balanced virtual conditions can be actually recognized in the ordering of virtual conditions by a natural first order property.

Proposition 5.3.6. *Let P be a Suslin forcing. Let $\langle Q, \tau\rangle$ be a P-pin. The following are equivalent:*

1. *$\langle Q, \tau\rangle$ is weakly balanced;*
2. *$\langle Q, \tau\rangle$ is an atom in the ordering of virtual conditions.*

Recall that an element $\bar p$ of a partial order is an atom if every element compatible with $\bar p$ is in fact above $\bar p$.

Proof. For (1)→(2) direction, let $\langle R, \sigma\rangle$ be a P-pin which is not above $\langle Q, \tau\rangle$; we must show that it is incompatible with $\langle Q, \tau\rangle$. Suppose towards a contradiction that it is compatible, with a lower bound $\langle S, \chi\rangle$. In the $R \times Q$ extension, the inequality $\Sigma\tau \leq \Sigma\sigma$ must fail, so there is a condition $\dot{p}_0$ for an element of P which is below $\Sigma\tau$ but incompatible with $\Sigma\sigma$. In the $S \times R \times Q$-extension, there is a condition $\dot{p}_1$ of P which is below $\Sigma\chi$, and therefore also below $\Sigma\sigma$ and $\Sigma\tau$. Use the weak balance of the pair $\langle Q, \tau\rangle$ to find, in some generic extension, filters $H_0 \times G_0 \subset R \times Q$ and $K_1 \times H_1 \times G_1 \subset S \times R \times Q$ separately generic over V such that the conditions $\dot{p}_0/H_0 \times G_0$ and $\dot{p}_1/K_1 \times H_1 \times G_1$ are compatible in P, with a lower bound p. Note that the sums $\Sigma\sigma/H_0$ and $\Sigma\sigma/H_1$ in the completion of the poset P must coincide, as $\langle R, \sigma\rangle$ is a P-pin. However, the condition p should be incompatible with the former and below the latter by the forcing theorem. This is a contradiction.

For the (2)$\to$(1) direction, suppose that $\langle Q, \tau\rangle$ is a P-pin which is an atom in the ordering of virtual conditions. To prove the weak balance, suppose that R_0, R_1 are posets, σ_0, σ_1 are $R_0 \times Q$ and $R_1 \times Q$-names for elements of P stronger than τ, and let $\langle r_0, q_0\rangle$ and $\langle r_1, q_1\rangle$ be conditions in the products. To find the instrumental generic filters, let κ be an ordinal such that $\mathcal{P}(R_0 \times Q)$ and $\mathcal{P}(R_1 \times Q)$ are both subsets of V_κ, and consider the $\mathrm{Coll}(\omega, V_\kappa)$-names χ_0, χ_1 for analytic subsets of P defined by $\chi_0 = \{p \in P \colon \exists G \subset R_0 \times Q$ such that G is a filter meeting all open dense subsets of $R_0 \times Q$ in V_κ^V such that $\langle r_0, q_0\rangle \in G$ and $p = \sigma/G\}$. The name χ_1 is defined in the same way.

It is not difficult to see that $\langle \mathrm{Coll}(\omega, V_\kappa), \chi_0\rangle$ and $\langle \mathrm{Coll}(\omega, V_\kappa), \chi_1\rangle$ are both P-pins. They are also both $\leq \langle Q, \tau\rangle$ by their definition. By the assumption on the P-pin $\langle Q, \tau\rangle$, they must both be $\equiv$-equivalent to $\langle Q, \tau\rangle$. This means that in the $\mathrm{Coll}(\omega, V_\kappa)$-extension, there must be conditions $p_0 \in \chi_0$ and $p_1 \in \chi_1$ which are compatible in P. Reviewing the definition of the names χ_0 and χ_1, we get the filters $G_0 \subset H_0 \times K_0$ and $G_1 \subset H_1 \times K_1$ separately generic over V such that $\langle r_0, q_0\rangle \in G_0$, $\langle r_1, q_1\rangle \in G_1$, and σ_0/G_0 and σ_1/G_1 are compatible conditions in P as desired. $\square$

Finally, we record the central definition of this section.

Definition 5.3.7. A Suslin poset P is *weakly balanced* if below every condition $p \in P$ there is a weakly balanced virtual condition stronger than p.

Chapter 6

Simplicial complex forcings

6.1 Basic concepts

Many examples of σ-closed Suslin partially ordered sets in this book are presented in the same way:

Definition 6.1.1. Let X be a set.

1. A set $\mathcal{K}$ of finite subsets of X is *a simplicial complex* if it is closed under subset;
2. a set $A \subset X$ is a $\mathcal{K}$-*set* if $[A]^{<\aleph_0} \subset \mathcal{K}$. It is *maximal* if it is not a proper subset of another $\mathcal{K}$-set;
3. the poset $P_{\mathcal{K}} \subset X^\omega$ consists of countable $\mathcal{K}$-sets ordered by reverse inclusion;
4. $\dot{A}_{gen}$ is the $P_{\mathcal{K}}$-name for the union of all sets in the generic filter.

Unless specifically stated otherwise, we will tacitly assume that every singleton belongs to $\mathcal{K}$. When the simplicial context $\mathcal{K}$ is understood from the context, we put $P = P_{\mathcal{K}}$ in this section. The poset P is obviously σ-closed. By an elementary density argument, the set $\dot{A}_{gen}$ is forced to be a maximal $\mathcal{K}$-set. The poset P can be naturally presented as a Suslin forcing by replacing the countable $\mathcal{K}$-sets with their enumerations by natural numbers, and replacing the reverse inclusion ordering by reverse inclusion of ranges of the enumerations. We will neglect this innocuous step in this section as it merely complicates the notation.

Nearly every poset considered in this book can be presented as a poset of the form $P_{\mathcal{K}}$ for a Borel simplicial complex $\mathcal{K}$ on a Polish space X. Namely, for a poset Q let $\mathcal{K}$ be the simplicial complex of the finite subsets of Q which have a common lower bound. Under suitable assumptions on definability and existence of lower bounds (which are invariably satisfied), the posets Q and $P_{\mathcal{K}}$ are naturally forcing equivalent. However, this point of view rarely brings any new insight. In this chapter, we deal with simplicial complexes that are in some way algebraically natural, and their algebraic structure leads to the classification of the balanced conditions. We discovered a number of possibilities. Many of them are categorized by the properties of the following central concepts associated with any simplicial complex whatsoever.

Definition 6.1.2. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X. $\Gamma_{\mathcal{K}}$ is the Borel graph on $\mathcal{K}$ consisting of all pairs $\langle a, b\rangle$ where $a, b \in \mathcal{K}$, $a \cup b \notin \mathcal{K}$, and a, b are inclusion minimal such: if $a' \subseteq a$ and $b' \subseteq b$ then $a' \cup b' \notin \mathcal{K}$ iff $a' = a$ and $b' = b$.

In particular, if $\langle a, b\rangle \in \Gamma_{\mathcal{K}}$ then any other disjoint partition of the set $a \cup b$ into nonempty pieces $c \cup d$ must have $c, d \in \mathcal{K}$ and $\langle c, d\rangle \in \Gamma_{\mathcal{K}}$.

Definition 6.1.3. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X.

1. A $\mathcal{K}$-*walk* is a finite sequence $\langle a_i : i \in j\rangle$ of elements of $\mathcal{K}$ such that for every $k \in j \setminus \{0\}$ there is a set $b \subset \bigcup_{i \in k} a_i$ such that $\langle a_k, b\rangle \in \Gamma_{\mathcal{K}}$;
2. A $\mathcal{K}$-walk $\langle a_i : i \in j\rangle$ is said to *start* at a point $x_0 \in X$ if $a_0 = \{x_0\}$ and *reach* a point $x_1 \in X$ if for some $i \in j$ $x_1 \in a_i$;
3. $E_{\mathcal{K}}$ is the set of all pairs $\langle x_0, x_1\rangle$ such that there is a $\mathcal{K}$-walk starting from x_0 and reaching x_1 and a $\mathcal{K}$-walk starting from x_1 and reaching x_0.

Proposition 6.1.4. *Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X. $E_{\mathcal{K}}$ is an analytic equivalence relation. It is the smallest equivalence relation E on X such that for every finite set $a \subset X$, $a \in \mathcal{K}$ if and only if for every E-class c, $a \cap c \in \mathcal{K}$.*

Proof. Write $\Gamma = \Gamma_{\mathcal{K}}$ and $E = E_{\mathcal{K}}$. It is immediate that the relation E is an equivalence. The complexity assertion is also immediate from the definitions. For the optimality assertion, for an equivalence relation F on X write $\phi(F)$ for the statement that for every finite set $a \subset X$, $a \in \mathcal{K}$ if and only if for every F-class c, $a \cap c \in \mathcal{K}$. We first need to verify that $\phi(E)$ holds. Suppose towards a contradiction that it does not, and let $a \subset X$ be an inclusion-minimal counterexample. Note that for every E-class c such that $a \cap c \neq 0$ it must be the case that $\langle a \cap c, a \setminus c\rangle \in \Gamma$. Now let $x \in a$ be any point and let c be the E-class of x. Produce a $\mathcal{K}$-walk from x which reaches all the finitely many points in $c \cap a$. The walk can be extended by the set $a \setminus c$, reaching an arbitrary element of a. This means that all points of a belong to the same E-class, a contradiction.

Now suppose that F be an equivalence relation on X for which $\phi(F)$ holds; we must argue that $E \subseteq F$. Suppose towards a contradiction that this fails and let $x_0, x_1 \in X$ be E-related points which are not F-related. Let $\langle a_i : i \in j\rangle$ be a $\mathcal{K}$-walk starting from x_0 and reaching x_1. Let $k \in j$ be the smallest number such that $a_k \not\subset [x_0]_F$. There must be a set $b \subset \bigcup_{i \in k} a_i$ such that $\langle b, a_k\rangle \in \Gamma$. Applying $\phi(F)$ to the union $a_k \cup b$, we see that it must be the case that $(a_k \cup b) \cap [x_0]_F \notin \mathcal{K}$. Thus, considering the proper subset $a'_k = a_k \cap [x_0]_F$, we get $b \cup a'_k \notin \mathcal{K}$, contradicting the assumption that $\langle b, a_k\rangle \in \Gamma$. ◻

To conclude this section, we mention two operations on simplicial complexes. Both of them may seem trivial from the simplicial complex view, but they generate quite unpredictable operations on the associated posets $P_\mathcal{K}$. Several classes of complexes introduced below are nearly trivially closed under these operations.

Definition 6.1.5. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X, and let $B \subset X$ be a Borel set. The *restriction* $\mathcal{K} \restriction B$ is just the set $\mathcal{K} \cap [B]^{<\aleph_0}$.

Definition 6.1.6. Let X be a Polish space and for each $n \in \omega$, let $\mathcal{K}_n$ be a Borel complex on a Borel set $B_n \subset X$. The product $\mathcal{K} = \bigwedge_n \mathcal{K}_n$ is the simplicial complex of all finite sets $a \subset X$ such that $\forall n\ a \cap B_n \in \mathcal{K}_n$.

One way to use the latter operation appears is when the Borel sets B_n for $n \in \omega$ are pairwise disjoint. In this case, the poset $P_\mathcal{K}$ corresponds to the countable support product of the posets $P_{\mathcal{K}_n}$ for $n \in \omega$. However, if the Borel sets B_n overlap, the forcing effects of this operation become much more difficult to analyze.

6.2 Locally countable complexes

In this section, we discuss the simples class of simplicial complexes to handle.

Definition 6.2.1. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X. We say that $\mathcal{K}$ is *locally countable* if the graph $\Gamma_\mathcal{K}$ is locally countable.

It follows from a routine application of the Lusin–Novikov theorem that if a Borel simplicial complex $\mathcal{K}$ on a Polish space X is locally countable, then $E_\mathcal{K}$ is a countable Borel equivalence relation. In particular, a Borel simplicial complex $\mathcal{K}$ is locally countable if and only if there is a countable Borel equivalence relation E on X such that for every finite set $a \subset X$, $a \in \mathcal{K}$ if and only if $a \cap c \in \mathcal{K}$ for every E-class c. The classification of balanced pairs in the case of locally countable simplicial complexes is particularly simple: they correspond to maximal $\mathcal{K}$-sets. This is the content of the following theorem:

Theorem 6.2.2. *Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space X.*

1. *For every maximal $\mathcal{K}$-set $A \subset X$, the pair $\langle \mathrm{Coll}(\omega, X), \check{A}\rangle$ is balanced in the poset $P = P_\mathcal{K}$;*

2. *for every balanced pair $\langle Q, \tau\rangle$, there is a maximal $\mathcal{K}$-set $A \subset X$ such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{A}\rangle$ are equivalent;*

3. *distinct maximal $\mathcal{K}$-sets yield inequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. For (1), suppose that $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are conditions extending A; we must show that $p_0 \cup p_1$ is a $\mathcal{K}$-set. Suppose towards a contradiction that $a_0 \subset p_0$ and $a_1 \subset p_1$ are finite sets such that $a_0 \cup a_1 \notin \mathcal{K}$. Choosing a_0, a_1 inclusion-minimal, we conclude that $\{a_0, a_1\} \in \Gamma_{\mathcal{K}}$. Since the graph $\Gamma_{\mathcal{K}}$ is locally countable, by the Mostowski absoluteness between $V[H_0]$ and $V[H_0, H_1]$ it must be the case that as $a_0 \in V[H_0]$, $a_1 \in V[H_0]$ must hold as well. Thus, $a_1 \in V[H_0] \cap V[H_1] = V$. Since A was a maximal $\mathcal{K}$-set and $A \cup a_1 \subset p_1$ is a $\mathcal{K}$-set, it follows that $a_1 \subset A$ and $a_0 \cup a_1 \subset p_0$, contradicting the assumption that $a_0 \cup a_1 \notin \mathcal{K}$.

For (2), let $\langle Q, \tau \rangle$ be a balanced pair; strengthening τ if necessary we may assume that it is a Q-name for a single $\mathcal{K}$-set. Replacing Q with $Q \times \mathrm{Coll}(\omega, X)$ and strengthening the condition τ if necessary, we may assume that $Q \Vdash \forall x \in X^V\ x \in \tau$ or there is $a \subset \tau$ such that $\{x\} \cup a \notin \mathcal{K}$. For each point $x \in X$, either $Q \Vdash \check{x} \in \tau$ or $Q \Vdash \exists a \subset \tau \in \Gamma_{\mathcal{K}}$ and $a \subset \tau$ by a balance argument. Let $A \subset X$ be the set of all points $x \in X$ for which the first option prevails. Then $Q \Vdash \check{A} = \tau \cap V$, so A is a $\mathcal{K}$-set; in view of (1) and Proposition 5.2.4, it is enough to show that A is a maximal $\mathcal{K}$-set. Let $x \in X \setminus A$. Then $Q \Vdash \exists a \subset \tau\ \{x\} \cup a \notin \mathcal{K}$; choosing a inclusion minimal, we conclude that $Q \Vdash \{\{x\}, a\} \in \Gamma_{\mathcal{K}}$. By the local countability of the graph $\Gamma_{\mathcal{K}}$ and a Mostowski absoluteness between the ground model and its Q-extension, it must be that $Q \Vdash a \subset V$ and then $a \subset \check{A}$. It follows that $A \cup \{x\}$ is not a $\mathcal{K}$-set as desired.

Finally, (3) is obvious. For the last sentence, any condition $p \in P$ can be extended to a maximal $\mathcal{K}$-set, which then represents a balanced condition stronger than p by (1). $\square$

Example 6.2.3. Let E be a countable Borel equivalence relation on a Polish space X. Let $\mathcal{K}$ be the set of all finite sets consisting of pairwise E-unrelated elements. Then $\Gamma_{\mathcal{K}}$ consists of all edges $\{\{x_0\}, \{x_1\}\}$ where $x_0, x_1 \in X$ are distinct E-related points, so $\mathcal{K}$ is locally countable. The balanced pairs are classified by maximal $\mathcal{K}$-sets, which are precisely the E-transversals.

For the following example, recall the notion of a *perfect matching* of a graph; this is just a set of edges such that each vertex gets exactly one edge in the set adjacent to it. For a locally finite bipartite graph Γ, the existence of a perfect matching is equivalent to *Hall's marriage condition* [38]: for every finite set a of vertices on one side of the bipartition, the set of neighbors of a has cardinality at least that of a.

Example 6.2.4. Let Γ be a locally finite bipartite Borel graph on a Polish space X satisfying the Hall's marriage condition; view Γ as a subset of $[X]^2$. Let $\mathcal{K}$ be the simplicial complex of all finite subsets of Γ which can be completed to a perfect matching. Then $\mathcal{K}$ is a Borel locally countable simplicial complex: given a finite set $a \subset \Gamma$, the membership of a in $\mathcal{K}$ can be detected by checking all fragments of a in the components of Γ which are countable. The balanced pairs are classified by maximal $\mathcal{K}$-sets which are precisely the perfect matchings of Γ.

For the following example, recall the notion of an *end* of an infinite connected graph [36]. If the graph in question is in addition acyclic, an end can be identified with an orientation of the graph in which every vertex gets exactly one edge flowing out of it.

Example 6.2.5. Let Γ be a locally finite acyclic Borel graph on a Polish space X with all components infinite; view Γ as a symmetric subset of X^2. Let $\mathcal{K}$ be the simplicial complex of all finite subsets $a \subset \Gamma$ consisting of finite sets a such that for each Γ-path-connectedness class $c \subset X$ the set $a \cap c^2$ can be extended to an end. It is immediate that $\mathcal{K}$ is a Borel locally countable simplicial complex. The balanced conditions are classified by maximal $\mathcal{K}$-sets, which are orientations of the graph Γ in which every vertex has exactly one point in its outflow.

Example 6.2.6. Let Γ be a locally countable Borel graph on a Polish space X of chromatic number $\leq n$ for some finite $n \in \omega$. Let $\mathcal{K}$ be the simplicial complex of all finite partial Γ-colorings by at most n colors which can be completed to a total coloring by at most n colors. Then $\mathcal{K}$ is a locally countable simplicial complex: given a finite coloring a, the membership of a in $\mathcal{K}$ can be detected by checking all fragments of a in the components of Γ which are countable. The balanced pairs are classified by maximal $\mathcal{K}$-sets which are precisely the total Γ-colorings.

6.3 Complexes of Borel coloring number $\aleph_1$

The next class of examples is obtained as a generalization of the locally countable simplicial complex using a definable version of a concept due to Erdős and Hajnal [25].

Definition 6.3.1.
1. Let Γ be a Borel graph on a Polish space X. We say that the graph has *Borel coloring number* $\aleph_1$ if there is a Borel directed graph $\vec{\Gamma}$ (an *orientation* of Γ) such that for each edge $\{x_0, x_1\} \in \Gamma$, either $\langle x_0, x_1 \rangle \in \vec{\Gamma}$ or $\langle x_1, x_0 \rangle \in \vec{\Gamma}$, and the $\vec{\Gamma}$-outflow of each point is countable .
2. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X. We say that $\mathcal{K}$ has *Borel coloring number* $\aleph_1$ if the graph $\Gamma_{\mathcal{K}}$ has Borel coloring number $\aleph_1$.

In particular, every locally countable Borel simplicial complex has Borel coloring number $\aleph_1$. A Mostowski absoluteness argument shows that if a Borel graph has Borel coloring number $\aleph_1$ then this feature will persist to all generic extensions.

To prove the balance for the posets associated with Borel simplicial complexes of Borel coloring number $\aleph_1$ and to classify the balanced conditions, we need a technical weakening of maximality.

Definition 6.3.2. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X. A $\mathcal{K}$-set $A \subset X$ is *weakly maximal* if for every point $x \in X$ either $x \in A$ or for every countable set $b \subset X$ there is $c \in \mathcal{K}$ disjoint from b such that $(A \cap b) \cup c$ is a $\mathcal{K}$-set, but $(A \cap b) \cup c \cup \{x\}$ is not.

In particular, every maximal $\mathcal{K}$-set is weakly maximal. For a weakly maximal set $A \subset X$, let τ_A be the $\mathrm{Coll}(\omega, X)$-name for the analytic set of all $p \in P$ such that for all $x \in X \cap V$, if $x \in A$ then $x \in p$ and if $x \notin A$ then $p \cup \{x\}$ is not a $\mathcal{K}$-set. In particular, if A is a maximal $\mathcal{K}$-set, then $\mathrm{Coll}(\omega, X) \Vdash \Sigma\tau_A = \check{A}$.

Theorem 6.3.3. *Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X of Borel coloring number $\aleph_1$.*

1. *For every weakly maximal $\mathcal{K}$-set $A \subset X$, the pair $\langle \mathrm{Coll}(\omega, X), \tau_A \rangle$ is balanced in the poset $P = P_{\mathcal{K}}$;*
2. *for every balanced pair $\langle Q, \tau \rangle$, there is a weakly maximal $\mathcal{K}$-set $A \subset X$ such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \mathrm{Coll}(\omega, X), \tau_A \rangle$ are equivalent;*
3. *distinct weakly maximal $\mathcal{K}$-sets yield inequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. For (1), we must first show that $\mathrm{Coll}(\omega, X) \Vdash \tau_A \neq 0$. By a Mostowski absoluteness argument, it is just enough to produce some generic extension in which there is a $\mathcal{K}$-set $p \subset X$ such that for all $x \in X \cap V$, if $x \in A$ then $x \in p$ and if $x \notin A$ then $p \cup \{x\}$ is not a $\mathcal{K}$-set. To do this, let Q be the poset of nonstationary subsets of $[X]^{\aleph_0}$ and let $j \colon V \to M$ be the Q-name for the associated generic ultrapower. In particular, M is an ω-model of ZFC containing $j''X$ as an element, represented by the identity function; this set is forced to be countable in M. Now, consider a product of copies Q_x of Q indexed by elements $x \in X \setminus A$, with mutually generic filters $G_x \subset Q_x$ and in the model $V[G_x \colon x \in X \setminus A]$ consider the generic ultrapower embeddings $j_x \colon V \to M_x$. By the weak maximality of A and elementarity, there is a set $c_x \in M_x$ disjoint from V such that $A \cup c_x$ is a $\mathcal{K}$-set, and $A \cup c_x \cup \{x\}$ is not. We claim that $p = A \cup \bigcup_x c_x$ is a $\mathcal{K}$-set; this will complete the proof. Suppose towards a contradiction that this is not the case, and find finite sets $a \subset A$ and $a_x \subset c_x$ for x in some finite index set I such that $a \cup \bigcup_x a_x \notin \mathcal{K}$ and a_x are inclusion-minimal possible. Pick x_0 be any element of I such that $a_{x_0} \neq 0$, let $b = a_{x_0}$, and let $c = a \cup \bigcup_{x \neq x_0} a_x$. It is clear that $\{c, d\} \in \Gamma$. Let $\vec{\Gamma}$ be a Borel orientation of Γ with countable outflows, and assume for definiteness that $\langle c, b \rangle \in \vec{\Gamma}$. By a Mostowski absoluteness argument between $V[G_x \colon x \neq x_0]$ and $V[G_x \colon x \in X \setminus A]$, it must be the case that b, as much as all other points of the countable $\vec{\Gamma}$-outflow of c, belongs to $V[G_x \colon x \neq x_0]$, so $b \in V[G_x \colon x \neq x_0] \cap V[G_{x_0}] = V$. This contradicts the initial choice of the sets $a_x \subset c_x$.

Now, for the balance part, suppose that $V[H_0]$ and $V[H_1]$ are mutually generic extensions of V, and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are conditions such that for all $x \in X \cap V$, if $x \in A$ then $x \in p_0$ and if $x \notin A$ then $p_0 \cup \{x\}$ is not a $\mathcal{K}$-set, and similarly for p_1. We must show that $p_0 \cup p_1$ is a $\mathcal{K}$-set. Suppose towards a contradiction that this fails, and let $a_0 \subset p_0$ and $a_1 \subset p_1$ be finite sets such that $a_0 \cup a_1 \notin \mathcal{K}$. Passing to an inclusion-minimal example of this form if necessary, we may assume that $\{a_0, a_1\} \in \Gamma$. Let $\vec{\Gamma}$ be a Borel orientation of Γ with countable outflows, and assume for definiteness that $\langle a_0, a_1\rangle \in \vec{\Gamma}$. By a Mostowski absoluteness argument between $V[H_0]$ and $V[H_0, H_1]$, it must be the case that a_1, as much as all other points of the countable $\vec{\Gamma}$-outflow of a_0, belongs to $V[H_0]$, so $a_1 \in V[H_1] \cap V[H_0] = V$. It follows that $a_1 \subset A \subset p_0$, and this contradicts the assumption that $a_0 \cup a_1 \notin \mathcal{K}$.

For (2), let $\langle Q, \tau\rangle$ be a balanced pair; strengthening τ if necessary we may assume that it is a Q-name for a single $\mathcal{K}$-set. Replacing Q with $Q \times \mathrm{Coll}(\omega, X)$ and strengthening the condition τ if necessary, we may assume that $Q \Vdash \forall x \in X^V\ x \in \tau$ or there is $a \subset \tau$ such that $\{x\} \cup a \notin \mathcal{K}$. For each point $x \in X$, either $Q \Vdash \check{x} \in \tau$ or $Q \Vdash \exists a \subset \tau \in \Gamma_{\mathcal{K}}$ and $a \subset \tau$ by a balance argument. Let $A \subset X$ be the set of all points $x \in X$ for which the first option prevails. Then $Q \Vdash \check{A} = \tau \cap V$ and therefore A is a $\mathcal{K}$-set; in view of (1) and Proposition 5.2.4, it is enough to show that A is a weakly maximal $\mathcal{K}$-set. Let $x \in X \setminus A$ and $b \subset X$ be countable. Observe that $Q \Vdash \exists c \subset \tau\ \{x\} \cup c \notin \mathcal{K}$; note that for such set c, $(A \cap b) \cup c$ is still a $\mathcal{K}$-set. Use the Mostowski absoluteness argument to find a set c with these properties in the ground model; this confirms that A is a weakly maximal $\mathcal{K}$-set.

Finally, (3) is obvious. For the last sentence, any condition $p \in P$ can be extended to a maximal $\mathcal{K}$-set, which then represents a balanced condition stronger than p by (1). $\square$

Example 6.3.4. Let $g\colon 2^\omega \to (2^\omega)^{\aleph_0}$ be any Borel function. Let $\mathcal{K}$ be the simplicial complex of g-free sets on $X = 2^\omega$. It is immediate that $\mathcal{K}$ has Borel coloring number $\aleph_1$: the graph $\Gamma_{\mathcal{K}}$ consists of all pairs $\{x_0, x_1\}$ such that either $x_0 \in f(x_1)$ or $x_1 \in f(x_0)$. The Borel orientation of $\Gamma_{\mathcal{K}}$ with countable outflows can be defined by considering which of the two cases occurs. To consider a case in which the classification of balanced conditions is particularly simple, suppose that every point in $(2^\omega)^{\aleph_0}$ has uncountable preimage. Then, any g-independent set $A \subset X$ is weakly maximal: for every countable set $b \subset X$ and every point $x \in X \setminus A$ there is an uncountable set of points $y \in X$ such that $x \in g(y)$ and $g(y) \cap (A \cap b) = 0$, and uncountably many such points y can be found also outside of the countable set $\bigcup\{g(z)\colon z \in A \cap b\}$, making the set $(A \cap b) \cup \{y\}$ free, while $(A \cap b) \cup \{y\} \cup \{x\}$ is not.

6.4 Modular complexes

Another class of simplicial complexes in this section is not as closely tied to the graph $\Gamma_{\mathcal{K}}$. The definition is motivated by concerns of geometric model theory.

Definition 6.4.1. A Borel simplicial complex $\mathcal{K}$ on a Polish space X is *modular* if there is a Borel *modular* function $f\colon \mathcal{K} \to [Y]^{\aleph_0}$ for some Polish space Y such that

1. *(monotonicity)* $f(0) = 0$ and $a \subset b$ implies $f(a) \subset f(b)$;

2. *(modularity)* for all $a, b \in \mathcal{K}$, $a \cup b \in \mathcal{K}$ if and only if $f(a) \cap f(b) = f(a \cap b)$.

For a $\mathcal{K}$ set $A \subset X$ we set $f(A) = \bigcup\{f(a) \colon a \in [A]^{<\aleph_0}\}$.

Note that the properties of the function f are coanalytic, and therefore transfer to any transitive model of set theory by the Mostowski absoluteness, in particular to all generic extensions. It is not easy to see which Borel simplicial complexes are modular and which are not. As the simplest initial example, let $X = \mathbb{R}$ viewed as a vector space over the rationals, and let $\mathcal{K}$ be the simplicial complex on X consisting of finite linearly independent sets. The simplicial complex is modular as witnessed by the modular function f which to each finite set of reals assigns its linear span.

The balanced virtual conditions in the poset $P = P_{\mathcal{K}}$ for modular complexes $\mathcal{K}$ are again classified by certain $\mathcal{K}$-sets, as recorded in the following definition:

Definition 6.4.2. Let $\mathcal{K}$ be a simplicial complex on a Polish space X. A $\mathcal{K}$-set $A \subset X$ is *strongly maximal* if for every Borel collection $B \subset \mathcal{K}$ at least one of the following occurs:

1. there are $a_0 \neq a_1$ in B such that $a_0 \cup a_1 \in \mathcal{K}$;
2. there is $b \in B$ such that $b \subset A$;
3. there is a countable set $p \subset A$ such that for no $b \in B$, $p \cup b$ is a $\mathcal{K}$-set.

Fittingly, a strongly maximal set $A \subset X$ must be a maximal $\mathcal{K}$-set. For this, given $x \in X$ consider the set $B = \{\{x\}\}$ and consult the three options of Definition 6.4.3. The first option is impossible, the second option yields $x \in A$, and the third option shows gives that $A \cup \{x\}$ is not a $\mathcal{K}$-set. The maximality of the set A follows.

Theorem 6.4.3. *Suppose that $\mathcal{K}$ is a modular Borel simplicial complex on a Polish space X and write $P = P_{\mathcal{K}}$.*

1. *Whenever $A \subset X$ is a strongly maximal set then the pair $\langle \mathrm{Coll}(\omega, X), A\rangle$ is balanced in P;*
2. *every balanced pair is equivalent to $\langle \mathrm{Coll}(\omega, X), A\rangle$ for some strongly maximal set $A \subset X$;*
3. *distinct strongly maximal sets give rise to inequivalent balanced pairs.*

In particular, under the Continuum Hypothesis the poset P is balanced.

We do not know if the poset P is balanced in ZFC, even though this does occur in all specific cases under consideration in this book.

Proof. Let $f \colon \mathcal{K} \to [Y]^{\aleph_0}$ be a Borel modular function. For (1), suppose that $V[H_0]$ and $V[H_1]$ are mutually generic extensions of the ground model, and $a_0 \in V[H_0]$ and $a_1 \in V[H_1]$ are elements of $\mathcal{K}$ such that $A \cup a_0, A \cup a_1$ are

$\mathcal{K}$-sets; we must show that $a_0 \cup a_1 \in \mathcal{K}$. Suppose towards a contradiction that this fails. Using the maximality of the set A, enlarge the sets a_0, a_1 so that $a_0 \cap a_1 = a_0 \cap V = a_1 \cap V$. The modularity of the function f implies that the set $f(a_0) \cap f(a_1) \setminus f(a_0 \cap a_1)$ is nonempty, containing some element $y \in Y$. Since $y \in V[H_0] \cap V[H_1]$, the product forcing theorem implies that $y \in V$.

Back in V, consider the set $B \subset \mathcal{K}$, $B = \{b \in \mathcal{K} \colon y \in f(b)$ and for no proper subset $c \subset b$, $y \in f(c)\}$. Apply Definition 6.4.2 to A, B. The first option is impossible by the modularity of the function f. The second option gives a set $b \in \mathcal{K}$ such that $b \subset A$ and $y \in f(b)$. Increasing the set b if necessary, we may arrange that $a_0 \cap V \subset b$. But then, $y \in f(a_0) \cap f(b)$ while $y \notin f(a_0 \cap b) = f(a_0 \cap V) = f(a_0 \cap a_1)$. The modularity of the function f shows that $b \cup a_0 \notin \mathcal{K}$ and so $A \cup a_0$ is not a $\mathcal{K}$-set, contradicting the initial asumptions. In the third option of the trichotomy, there is (in V) a countable set $p \subset A$ such that no element of B can be added to it. Let $b \subset a_0$ be an inclusion-minimal subset of a_0 such that $y \in f(b)$. Then $b \in B$ and $p \cup b$ is a $\mathcal{K}$-set, contradicting the choice of p. (1) has been proved.

For (2), assume that $\langle Q, \tau \rangle$ is a balanced pair. Replacing Q with $Q \times \mathrm{Coll}(\omega, X)$ and strengthening τ if necessary, we may assume that τ is in fact a name for an element of $P_{\mathcal{K}}$, and $Q \Vdash \forall x \in X \cap V\ x \in \tau$ or $\{x\} \cup \tau$ is not a $\mathcal{K}$-set. By the balance, for each $x \in X$ it has to be the case that $Q \Vdash \check{x} \in \tau$ or $Q \Vdash \{\check{x}\} \cup \tau$ is not a $\mathcal{K}$-set; let $A \subset X$ be the set of all points $x \in X$ for which the former alternative occurs. We will show that $A \subset X$ is a strongly maximal $\mathcal{K}$-set. Then, $\langle Q, \tau \rangle$ is equivalent to $\langle \mathrm{Coll}(\omega, X), A \rangle$. Since $Q \Vdash \tau \leq A$ holds, in view of (1) and Proposition 5.2.4 $\langle Q, \tau \rangle$ is equivalent to $\langle \mathrm{Coll}(\omega, X), A \rangle$ as desired.

Now, it is clear that A is a $\mathcal{K}$-set, since it is forced to be a subset of the $\mathcal{K}$-set τ. Suppose towards a contradiction that $A \subset X$ is not strongly maximal, as witnessed by some Borel set $B \subset \mathcal{K}$.

Claim 6.4.4. *There is a poset R and an R-name $\dot{b}$ such that $R \Vdash \dot{b} \in \mathcal{K}$ and $\check{A} \cup b$ is a $\mathcal{K}$-set.*

Proof. Let R be the poset of all stationary subsets of $[X]^{\aleph_0}$ ordered by inclusion. Now, if $G \subset R$ is a generic filter and $j \colon V \to M$ is the associated generic ultrapower, the set $X \cap V$ belongs to M and it is countable in M, as it is represented by the identity function. Thus, $A = j(A) \cap (X \cap M)$ is a countable subset of $j(A)$ in the model M, and by the failure of the third option in Definition 6.4.2 in V and elementarity of j, $M \models \exists b\ b \in B$ and $A \cup b$ is a $\mathcal{K}$-set. Since M is an ω-model of set theory, these two statements about b transfer without change to the generic extension $V[G]$. $\square$

The treatment now divides into two cases.

Case 1. For some condition $q \leq Q$ and a Q-name $\dot{b}$ for an element of $\dot{B}$, $q \Vdash \tau \cup \dot{b}$ is a $\mathcal{K}$ set. In this case, note that since the second option in Definition 6.4.2 fails for A, B, it must be the case that $\dot{b}$ is forced not to be a subset of the ground model. Let $H_0, H_1 \subset Q$ be mutually generic filters meeting the condition q. The sets $\dot{b}/H_0, \dot{b}/H_1 \in B$ must be distinct by the product forcing theorem. Since the first option of Definition 6.4.2 fails, it must be that $\dot b/H_0 \cup \dot b/H_1 \notin \mathcal{K}$. Thus, $\tau/H_0 \cup \dot b/H_0$ and $\tau/H_1 \cup \dot b/H_1$ are incompatible conditions in the poset P in the respective models $V[H_0]$ and $V[H_1]$ contradicting the balance of the pair $\langle Q, \tau\rangle$.

Case 2. $Q \Vdash \forall b \in B\ \tau \cup b$ is not a $\mathcal{K}$-set. Let R and $\dot b$ be a poset and a name as in the claim. Note that $Q \times R \Vdash \tau \cup \dot b$ is not a $\mathcal{K}$-set by the Mostowski absolutenes between the Q- and $Q \times R$-extension. By passing to a condition in Q and R if necessary, find a Q-name $\dot c$ and a finite set $d \subset A$ such that $Q \Vdash \dot c \subset \tau, \check d = \dot c \cap V$, and $Q \times R \Vdash \dot c \cup \dot b \notin \mathcal{K}$. Note that $R \Vdash \check d \cup \dot b \in \mathcal{K}$. By the modularity of the function f, $Q \times R \Vdash f(d \cup \dot b) \cap f(\dot c \setminus V) \neq 0$ holds, and by the product forcing theorem, all elements in the intersection must be in the ground model. In particular, there must be a point $y \in Y$ and a condition $q \in Q$ forcing $\check y \in \dot f(\dot c \setminus V)$. Let $H_0, H_1 \subset Q$ be filters mutually generic over the ground model, containing the condition $q \in Q$. Then $y \in f(\dot c/H_0 \setminus V) \cap f(\dot c/H_1 \setminus V)$ while $y \notin f(\dot c/H_0 \cap \dot H_1 \setminus V) = f(0) = 0$. The modularity of the function f shows that $\dot c/H_0 \cup \dot c/H_0 \setminus V \notin \mathcal{K}$, in particular $\tau/H_0, \tau/H_1$ are incompatible conditions in the poset P. This contradicts the balance of the pair $\langle Q, \tau\rangle$.

(2) has just been proved. (3) is obvious. For the last sentence, assume that the Continuum Hypothesis holds and let $p \in P$ be an arbitrary condition. Let $\langle B_\alpha \colon \alpha \in \omega_1$ enumerate all Borel subsets of $\mathcal{K}$. By recursion on $\alpha \in \omega_1$ construct a descending chain of conditions $p_\alpha \in P$ so that $p_0 = p$, $p_\alpha = \bigcup_{\beta\in\alpha} p_\beta$ for limit ordinals α, and $p_{\alpha+1}$ contains some element of B_α as a subset if such an extension of p_α exists at all. Then $\bigcup_{\alpha\in\omega_1} p_\alpha$ is a strongly maximal $\mathcal{K}$-set which by (1) yields a balanced virtual condition stronger than p. □

The following examples provide several interesting classes of modular complexes.

Example 6.4.5. Every locally countable Borel simplicial complex is modular.

Proof. Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space X. Consider the function $f \colon \mathcal{K} \to (\Gamma_{\mathcal{K}})^{\aleph_0}$ defined by $\{c, d\} \in f(a)$ just in case either $c \subseteq a$ or $d \subseteq a$. The values of the function f are countable sets by the locally countable assumption on the simplicial complex; it is not difficult to see that the function f is Borel in a suitable sense. The monotonicity of the function f is immediate; this leaves us with verifying the modularity of the function f.

Suppose that $a, b \in \mathcal{K}$ are finite sets. Suppose first that $a \cup b \in \mathcal{K}$ and $\{c, d\} \in f(a) \cap f(b)$; we need to show that $\{c, d\} \in f(a) \cap f(b)$. By the definition of the function f, either $c \subseteq a$ or $d \subseteq a$, and $c \subseteq b$ or $d \subseteq b$ must hold. Now, $c \subseteq a$ and $d \subseteq b$ is impossible since $c \cup d \notin \mathcal{K}$ while $a \cup b \in \mathcal{K}$. For the same reason, the conjunction $d \subseteq a$ and $c \subseteq b$ is impossible as well. This means that one of c, d must be a subset of both a and b, so of $a \cap b$. The definition of the function f then shows that $\{c, d\} \in f(a \cap b)$ as required.

Now, suppose that $a \cup b \notin \mathcal{K}$; we must find an edge $\{c, d\} \in f(a) \cap f(b)$ which does not belong to $f(a \cap b)$. To this end, find an inclusion-minimal set $e \subset a \cup b$ such that $e \notin \mathcal{K}$, and let $c = e \cap a$ and $d = e \setminus a$. The minimal choice of e shows that $\{c, d\} \in \Gamma_{\mathcal{K}}$. Since $c \subset a$ and $d \subset b$, it is clear that $\{c, d\} \in f(a) \cap f(b)$.

Since $e \not\subset a$ and $e \not\subset b$, it follows that neither c nor d is a subset of $a \cap b$, so $\{c, d\} \notin f(a \cap b)$ as required. □

Many simplicial complexes that we consider, and in particular modular simplicial complexes, are in fact matroids. For completeness, we include the definition of matroids and several basic facts.

Definition 6.4.6. [2, Chapters VI and VII] A simplicial complex $\mathcal{K}$ on a set X is a *matroid* if for any sets $a, b \in \mathcal{K}$, if $|b| > |a|$ then there is a point $x \in b \setminus a$ such that $a \cup \{x\} \in \mathcal{K}$.

It is immediate that for a Borel simplicial complex $\mathcal{K}$ on a Polish space X, the statement that $\mathcal{K}$ is a matroid is coanalytic and therefore absolute among transitive models of set theory. Matroid theory introduces several useful basic notions that apply to every matroid. For a set $a \subset X$ let $\mathtt{rk}(a)$ be the largest cardinality of a set $b \subset a$ such that $b \in \mathcal{K}$, if such exists; otherwise $\mathtt{rk}(a) = \infty$. For a set $a \subset X$ let $\bar{a} = \{x \in X : \mathtt{rk}(a) = \mathtt{rk}(a \cup \{x\})\}$. This is a closure operation satisfying the properties from the following fact, which may serve as an alternative definition of a matroid [2, 6.9].

Fact 6.4.7. *Let $\mathcal{K}$ be a matroid on a set X and let $a, b \subset X$ be finite sets. Then*

1. $a \subset \bar{a}$;
2. (idempotence) $b \subseteq \bar{a}$ *implies* $\bar{b} \subseteq \bar{a}$;
3. (exchange principle) *if* $x \in \overline{a \cup \{y\}} \setminus \bar{a}$ *then* $y \in \overline{a \cup \{x\}}$.

The set X with the closure operation satisfying the above properties is a *pre-geometry*. For a finite set $a \subset X$, $a \in \mathcal{K}$ just in case for any set $b \subset a$ and any point $x \in a \setminus b$, $x \notin \bar{b}$ holds–$\mathcal{K}$ is the complex of *free sets* of the pre-geometry. The closed sets (the sets $a \subset X$ such that $\bar{a} = a$) form a lattice under inclusion; the pre-geometry is *modular* if the lattice of its closed sets is modular. This can be equivalently characterized by $x \in \overline{a \cup b}$ iff there are points $x_a \in \bar{a}$ and $x_b \in \bar{b}$ such that $x \in \overline{\{x_a, x_b\}}$. The pre-geometry is *locally countable* if the closure of any finite set is countable. The following example justifies our choice of terminology.

Example 6.4.8. Let $\mathcal{K}$ be a Borel matroid on a Polish space X whose pre-geometry is locally countable and modular. Then $\mathcal{K}$ is a modular simplicial complex and the closure operation is a modular function on $\mathcal{K}$. Every maximal $\mathcal{K}$-set is strongly maximal.

Proof. Let $f : \mathcal{K} \to [X]^{\aleph_0}$ be the closure operation; it is immediate from its definition that it is a Borel function. To verify the modularity of f, suppose that $a, b \in \mathcal{K}$ are finite sets. Suppose first that $f(a) \cap f(b) \neq f(a \cap b)$ and work to show that $a \cup b$ is free. Let $x \in X$ be a point in $f(a) \cap f(b)$ which is not in $f(a \cap b)$. Let $a' \subset a$ be inclusion minimal superset of $a \cap b$ such that $x \in f(a')$. Note that $a' \not\subset a \cap b$, choose a point $y \in a' \setminus (a \cap b)$, and let $a'' = a' \setminus \{y\}$. By the exchange property of the pre-geometry, it follows that $y \in f(a'' \cup \{x\})$. By the monotonicity and idempotence of the pre-geometry, $y \in f(a'' \cup b)$ and since $y \in a \cup b$ and $y \notin a'' \cup b$, this means that $a \cup b \notin \mathcal{K}$. This direction does not use the modularity assumption on the pre-geometry.

Now suppose that $a \cup b \notin \mathcal{K}$ and work to show that $f(a) \cap f(b) \neq f(a \cap b)$. Let c be an inclusion minimal subset of $a \cup b$ such that $a \cap b \subset c$, c is free, and for some $x \in a \cup b$, $c \cup \{x\}$ is not free; pick the witness x. First argue that $x \in f(c)$. To this end, let $d \subset c \cup \{x\}$ be minimal such that there is $y \in (c \cup \{x\}) \setminus d$ which is in $f(d)$. If $x \notin d$ then y must be equal to x by the freeness of the set c; and if $x \in d$, then by the exchange property $x \in f(d \cup \{y\} \setminus \{x\})$, so $x \in f(c)$ again.

Now, assume for definiteness that $x \in b$. Let $a' = c \setminus b$ and $b' = c \cap b$. Since $x \in f(a' \cup b')$, the modularity assumption on the pre-geometry yields points $x_0 \in f(a')$ and $x_1 \in f(b')$ such that $x \in f(x_0, x_1)$. Note that $\{x_0, x_1\} \in \mathcal{K}$: if for example $x_1 \in f(\{x_0\})$, then by idempotence $x_1 \in f(a')$, by the exchange property there would have to be some element $z \in a'$ such that $z \in f(\{x_1\} \cup (a' \setminus \{z\}))$, and by idempotence again $z \in f(c \setminus \{z\})$, contradicting the freeness of the set c. By the freeness of the set b, $x \notin f(\{x_1\})$. By the exchange property applied again, $x_0 \in f(\{x_1, x\}) \subset f(b)$. It follows that $x_0 \in f(a) \cap f(b)$. At the same time, $x_0 \notin f(a \cap b)$, since then (as $a \cap b \subset c$ holds) $x_0 \in f(b')$ and therefore $x \in f(b')$, violating the freeness assumption on b.

To show that every maximal $\mathcal{K}$-set is strongly maximal, let $A \subset X$ be a maximal $\mathcal{K}$-set and let $B \subset \mathcal{K}$ be a set. Let M be a countable elementary submodel of a large structure containing B and A. If the third option of Definition 6.4.2 fails for B, there must be a set $b \in B$ such that $(M \cap A) \cup b$ is a $\mathcal{K}$-set. If $b \in M$ then $b \subset A$ by the maximality of A and the elementarity of the model M and the second option of Definition 6.4.2 holds. Suppose then that $b \notin M$. By the elementarity of the model M, there must be a set $a \in B \cap M$ such that $b \cap M \subset a$. We claim that $a \cup b \in \mathcal{K}$, confirming the first option of Definition 6.4.2. Suppose towards a contradiction that this fails; by the exchange property, there have to be sets $a' \subset a$, $M \cap b \subset b' \subset b$ and a point $x \in b \setminus b'$ such that $x \in f(a' \cup b')$. By the modularity, there have to be points y, z such that $y \in f(a')$, $z \in f(b')$ and $x \in f(y, z)$. By the exchange property, $y \in f(z, x) \subset f(b)$. At the same time, $y \in M$ and so by the maximality of the set A and the elementarity of the model M, there has to be $d \subset A$ in the model M such that $y \in f(d)$. In total, $y \in f(b) \cap f(d)$, by the initial assumption on the set b $b \cup d \in \mathcal{K}$ holds, and by the modularity of the function f, $y \in f(b \cap d) \subset f(b')$. Then $x \in f(y, z) \subset f(b')$, contradicting the freeness of the set b. □

Example 6.4.9. (A linear matroid) Let X be a Borel vector space over a countable field, and let $\mathcal{K}$ be the matroid of finite linearly independent subsets of X. The pre-geometry of $\mathcal{K}$ is well-known to be modular, so the simplicial complex $\mathcal{K}$ is modular. Let P be the poset of countable linearly independent sets. The balanced conditions in P are classified by bases of X over the field.

One important point of the present section is that there are interesting matroids modular in the sense of Definition 6.4.1 even though their pre-geometries are not modular.

Example 6.4.10. (A graphic matroid) Let G be a Borel graph on a Polish space X. Let $\mathcal{K}$ be the simplicial complex on G consisting of finite acyclic subsets of G. Then $\mathcal{K}$ is modular. Every maximal $\mathcal{K}$-set is strongly maximal.

Proof. Consider the (non-Polishable) group Y of finite subsets of X with the symmetric difference operation, viewed as a vector space over the binary field. The simplicial complex $\mathcal{L}$ of linearly independent finite subsets of Y is modular by Example 6.4.8. It is easy to see that $\mathcal{K}$ is just the restriction of $\mathcal{L}$ to G, where every edge of G is viewed as a two-element subset of X. The modular function for $\mathcal{L}$ restricted to $\mathcal{L}$ witnesses the modularity of $\mathcal{K}$.

To show that every maximal $\mathcal{K}$-set is strongly maximal, let $A \subset X$ be a maximal $\mathcal{K}$-set and let $B \subset \mathcal{K}$ be a set. Let M be a countable elementary submodel of a large structure containing B and A. If the third option of Definition 6.4.2 fails for B, there must be a set $b \in B$ such that $(M \cap A) \cup b$ is a $\mathcal{K}$-set. If $b \in M$ then $b \subset A$ by the maximality of A and the elementarity of the model M and the second option of Definition 6.4.2 holds. Suppose then that $b \notin M$. By the elementarity of the model M, there must be a set $a \in B \cap M$ such that $b \cap M \subset a$. We claim that $a \cup b \in \mathcal{K}$, confirming the first option of Definition 6.4.2. Suppose towards a contradiction that this fails, and let $c \subset a \cup b$ be a cycle. The cycle has to contain some edges in $b \setminus M$; let $c' \subset c$ be a maximal contiguous part of c containing only edges in $b \setminus M$. The beginning and ending vertex of c' (denoted by v_0, v_1 respectively) must belong to M. Thus, v_0, v_1 are connected by a G-path, they are also connected by A-path by the maximality of A, and such a path $d \subset A$ must be found in the model M by the elementarity of M. Then $d \cup c'$ is a cycle in $(M \cap A) \cup b$, contradicting the choice of the set b. □

Example 6.4.11. (A transversal matroid) Let X be a Polish space partitioned into Borel sets B_0, B_1, and let G be a Borel bipartite graph between B_0 and B_1. Let $\mathcal{K}$ be the matroid on G consisting of finite sets of edges in which no two distinct edges share a vertex. Then $\mathcal{K}$ is modular. If every vertex in X has uncountable degree, then a $\mathcal{K}$-set $A \subset G$ is maximal if and only if it is a perfect matching.

Proof. To exhibit the modular function, let $f(a) = \{x \in X\colon \text{there is an edge } e \in a \text{ such that } x \text{ is one of the two vertices of } e\}$. Now suppose that every vertex in X has uncountable degree and proceed towards the classification of strongly maximal sets. Let $A \subset X$ be any $\mathcal{K}$-set.

First, assume that A is strongly maximal. To see that A has to be a perfect matching, for every vertex $x \in X$ consider the set $B_x = \{\{e\}\colon x \text{ is one of the vertices in } e\}$. Consider the alternatives of Definition 6.4.2 for B_x. (1) fails and (3) is impossible as the vertex x has uncountable degree in G; therefore, (2) has to hold, meaning that the set A contains an edge with the vertex x on it and so A is a perfect matching.

Second, suppose that A is a perfect matching and $B \subset \mathcal{K}$ is a Borel set. To verify Definition 6.4.2 for A, B, let M be a countable elementary submodel of a large structure containing G, A, B. If the third option of Definition 6.4.2 fails, there must be $b \in B$ such that $(M \cap A) \cup b$ is a matching. Let $b' = b \cap M$ and use the elementarity of the model M to find $a \in B \cap M$ such that $b' \subset a$. There are several possibilities now. If $a = b = b'$ then $a \cup A$ is a matching by the elementarity of M, and by the maximality of A, $a \subset A$ and the option (2) has been verified. If $a \neq b$ and $a \cup b$ is a matching, then option (1) follows. Otherwise, there has to be an edge $e_0 \in b$ connecting some vertex x mentioned in a with a vertex outside of M. Since A is a perfect matching and M is elementary, there is an edge $e_1 \in A \cap M$ containing x; since the other vertex of e_1 is in M, it must be the case that $e_0 \neq e_1$ and so $(M \cap A) \cup b$ is not a matching, contradicting the choice of the set b. □

Not every modular simplicial complex needs to be a matroid.

Example 6.4.12. Let X be a Polish space and let Γ be a Borel collection of finite subsets of X. Let $\mathcal{K}$ be the simplicial complex on Γ consisting of finite subsets of Γ which consist of pairwise disjoint sets. It is immediate that the simplicial complex $\mathcal{K}$ is modular, with the modular function $f(a) = \bigcup a$. The simplicial complex $\mathcal{K}$ is typically not a matroid: for example, if there are points x_0, x_1 which are together contained in a single element of Γ and are also contained in respective disjoint elements of Γ, then the definitory property of a matroid immediately fails.

Finally, we present several examples of simplicial complexes which are not modular. To rule out the existence of a function with the modular property, we use the following simple criterion:

(*) there are pairwise disjoint sets $\{a, b_\alpha \colon \alpha \in \omega_1\}$ in $\mathcal{K}$ such that for any $\alpha, \beta \in \omega_1$ $b_\alpha \cup b_\beta \in \mathcal{K}$ holds and for every $\alpha \in \omega_1$, $a \cup b_\alpha \notin \mathcal{K}$ holds.

It is clear that if (*) holds, then no function f can have the modularity property: the values of $f(b_\alpha)$ for $\alpha \in \omega_1$ would have to be pairwise disjoint in view of the first demand in (*), and then there would have to be an ordinal $\alpha \in \omega_1$ such that $f(a) \cap f(b_\alpha) = 0$, which violates the modularity property in view of the second demand in (*).

Example 6.4.13. (An algebraic matroid) Let X be an uncountable Polish field, and let $\mathcal{K}$ be the simplicial complex on X consisting of sets algebraically free over some countable subfield $F \subset X$. The simplicial complex $\mathcal{K}$ satisfies (*) and therefore is not modular. To see this, let $B \subset X$ be an uncountable algebraically free set, let $a \subset B$ be an arbitrary set of size 2, $a = \{s, t\}$, and for each $x \in B \setminus a$ let $b_x = \{x + s, xt\}$. We will show that $\{a, b_x \colon x \in X \setminus a\}$ exemplify (*). Clearly, for each $x \in B \setminus a$, $a \cup b_x$ is not free. Also, for distinct elements $x, y \in B \setminus a$, the set $b_x \cup b_y$ is free. To see that, it is enough to show that the field $Q(b_x, b_y)$ contains s, t, x, y and therefore has transcendence degree 4. To recover s, t, x, y from $b_x \cup b_y$, write $w = x - y$ and $z = xy^{-1}$; both clearly belong to $Q(b_x, b_y)$. Then the recovery follows from the string of equalities $y = w(z-1)^{-1}$, $x = zy$, $t = (xt)x^{-1}$ and $s = (x + s) - x$. (This elegant argument was pointed out to us by Peter Sin.)

Example 6.4.14. (A gammoid) Let X be a Polish space. Let $\mathcal{K}$ be the matroid on finite subsets of X containing those finite sets $a \subset [X]^{<\aleph_0}$ which allow the existence of an injective function $g\colon a \to X$ such that $\forall u \in a\ f(u) \in u$. To verify that (*) is satisfied, select pairwise distinct points y_0, y_1 and x_α for $\alpha \in \omega_1$ of the underlying space X, let $a = \{\{y_0\}, \{y_1\}\}$ and $b_\alpha = \{\{y_0, x_\alpha\}, \{y_1, x_\alpha\}\}$.

Example 6.4.15. Let X be an uncountable Polish space and let $\mathcal{K}$ be the simplicial complex on $[X]^2$ consisting of those finite sets $a \subset [X]^2$ such that there is no set $b \in [X]^3$ such that $[b]^2 \subset a$. The associated poset $P_\mathcal{K}$ adds a maximal triangle-free graph on X. The simplicial complex $\mathcal{K}$ satisfies (*); therefore, it is not modular. To see this, let $x_0, x_1 \in X$ be any distinct points and let $a = \{\{x_0, x_1\}\}$. For each $z \in X$ distinct from x_0, x_1 let $b_z = \{\{x_0, z\}, \{x_1, z\}\}$. It is easy to check that $\{a, b_z \colon z \in X\}$ exemplify (*).

Example 6.4.16. Let $X = \mathcal{P}(\omega)$ and let $\mathcal{K}$ be the simplicial complex of all finite sets $a \subset X$ such that $\bigcap a$ is infinite. The associated poset $P_\mathcal{K}$ adds a Ramsey ultrafilter. The simplicial complex $\mathcal{K}$ satisfies (*) and so is not modular. To see this, let F be any filter on ω which is not generated by countably many sets. Let $a \subset \omega$ be an infinite set such that the complement of a is in F, and let b_α for $\alpha \in \omega_1$ be pairwise distinct elements of F disjoint from a. It is not difficult to verify that the sets $\{a\}, \{b_\alpha\} \colon \alpha \in \omega_1$ exemplify (*).

The examples above are selected so that the associated poset $P_\mathcal{K}$ is always balanced. Since the criterion (*) rules out an arbitrary modular function as opposed to a Borel modular function, and in general the modular functions seem to be very difficult to construct by transfinite recursion procedures, the following question suggests itself:

Question 6.4.17. Is there a Borel simplicial complex which possesses a modular function, but no Borel modular function?

We conclude this section with the best general result on the existence of strongly maximal sets we can prove in ZFC.

Theorem 6.4.18. *Let $\mathcal{K}$ be a Borel modular simplicial complex on a Polish space X such that its modular function takes only finite values. Then every countable $\mathcal{K}$-set can be extended to a strongly maximal $\mathcal{K}$-set.*

Proof. Let $C \subset X$ be a countable $\mathcal{K}$-set. Let $f\colon \mathcal{K} \to [Y]^{<\aleph_0}$ be a Borel modular function. Let κ be the cardinality of the continuum, and let $\{B_\alpha \colon \alpha \in \kappa\}$ be an enumeration of all Borel subsets of $\mathcal{K}$. By recursion on $\alpha \in \kappa$ construct $\mathcal{K}$-sets C_α so that

- $C = C_0 \subset C_1 \subset \dots$, $|C_\alpha| = |\alpha| + \aleph_0$;
- for every ordinal α, either there is a set $b \in B_\alpha$ such that $b \subset C_{\alpha+1}$ or there is a countable set $c \subset C_\alpha$ such that for no $b \in B_\alpha$, $c \cup b$ is a $\mathcal{K}$-set.

To perform the recursion, for a limit ordinal α let $C_\alpha = \bigcup_{\beta\in\alpha} C_\beta$. Now suppose that the set C_α has been constructed and work to find $C_{\alpha+1}$. Let M be a countable elementary submodel of a large structure containing C_α. If there is no $b \in B_\alpha$ such that $(C_\alpha \cap M) \cup b$ is a $\mathcal{K}$-set, then let $C_{\alpha+1} = C_\alpha$ and proceed to the next ordinal. If, on the other hand, there is a set $b \in B_\alpha$ such that $(C_\alpha \cap M) \cup b$ is a $\mathcal{K}$-set, we will work to produce a set $a \in B_\alpha$ such that $C_\alpha \cup a$ is a $\mathcal{K}$-set. Setting $C_{\alpha+1} = C_\alpha \cup a$ will complete the recursion step in this case as well.

Use Proposition 1.7.10 to find a perfect collection $\{H_z \colon z \in 2^\omega\}$ of filters on $\mathrm{Coll}(\omega, <\kappa)$ pairwise mutually generic over the model M. The product forcing theorem guarantees that for distinct points $z_0, z_1 \in 2^\omega$, $M[H_{z_0}] \cap M[H_{z_1}] = M$. Since the set C_α has size smaller than continuum, a counting argument shows that there is a point $z \in 2^\omega$ such that $C_\alpha \cap M[H_z]$ and $f(C_\alpha) \cap M[H_z]$ are both subsets of M. Note that $b \cap M$, $f(b) \cap M$, and $X \cap M$ are all sets in $M[H_z]$ and use a Mostowski absoluteness argument for the model $M[H_z]$ to find a set $a \in B_\alpha$ in the model $M[H_z]$ such that $a \cap M = b \cap M$ and $f(a) \cap M = f(b) \cap M$. We claim that $C_\alpha \cup a$ is a $\mathcal{K}$-set as desired.

To see this, suppose that $d \subset C_\alpha$ is a finite set. Adding some points of $C_\alpha \cap M$ to d if necessary, we may assume that $f(d) \cap f(a) \subset f(d \cap M) \cap f(a)$. The choice of the point z then implies that $f(d) \cap f(a) = f(d) \cap f(a) \cap M = f(d \cap M) \cap f(b)$. The latter set is equal to $f(d \cap M \cap b)$ since $(d \cap M) \cup b \in \mathcal{K}$ by the initial choice of b and the fact that f is a modular function. Finally, $d \cap M \cap b = d \cap a$ by the choice of the point z again, and so $f(d) \cap f(a) = f(d \cap a)$ and $a \cup d \in \mathcal{K}$ by the modularity of the function f.

Once the recursion has been performed, let $A = \bigcup_\alpha C_\alpha$ and use the second item of the recursion demands to conclude that $A \subset X$ is a strongly maximal $\mathcal{K}$-set extending C. ☐

6.5 G_δ matroids

The final class of simplicial complexes investigated in this section deals with complexes associated with pre-geometries of a form particularly simple from the descriptive set theoretic view.

Definition 6.5.1. A simplicial complex $\mathcal{K}$ on a Polish space X is G_δ if it is a G_δ subset of $[X]^{<\aleph_0}$, where the latter set is equipped with the topology inherited from the hyperspace $K(X)$ of compact subsets of X with the Vietoris topology.

Theorem 6.5.2. *Let $\mathcal{K}$ be a G_δ matroid on a Polish space X. Write $P = P_{\mathcal{K}}$.*

1. *If $A \subset X$ is a maximal $\mathcal{K}$-set, then the pair $\langle \mathrm{Coll}(\omega, A), \check{A}\rangle$ is balanced in the poset P;*
2. *whenever $\langle Q, \tau\rangle$ is a balanced pair in P then there is a maximal $\mathcal{K}$-set such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, A), \check{A}\rangle$ are equivalent;*
3. *distinct maximal $\mathcal{K}$-sets yield inequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. For (1), suppose that $R_0 \Vdash \sigma_0 \subset X$ is a finite set such that $A \cup \sigma_0$ is a $\mathcal{K}$-set, and similarly for R_1, σ_1; we must show that $R_0 \times R_1 \Vdash \sigma_0 \cup \sigma_1 \in \mathcal{K}$. Suppose towards a contradiction that this fails; rearranging the names σ_0, σ_1 and passing to a condition in R_0, R_1 if necessary, we may assume that there is a finite set $a \subset A$ and $R_0 \Vdash \sigma_0 \cap V = 0$ and $R_1 \Vdash \sigma_1 \cap V = 0$ and $R_0 \times R_1 \Vdash \check{a} \cup \sigma_0 \cup \sigma_1 \notin \mathcal{K}$.

Let M be a countable elementary submodel of a large enough structure, containing all objects named so far. Let $R_0' = M \cap R_0$, $\sigma_0' = M \cap \sigma_0$, $R_1' = M \cap R_1$ and $\sigma_1' = M \cap \sigma_1$. Perhaps more precisely, σ_0' is the name on R_0' such that for every basic open set $O \subset K(X)$ and every condition $r \in R_0'$, if $r \Vdash_{R_0} \sigma_0 \in O$ then $r \Vdash_{R_0'} \sigma_0' \in O$; similarly for σ_1'. It is not difficult to see that σ_0' is an R_0'-name for a finite subset of X disjoint from the ground model. The following claim is key.

Claim 6.5.3. $R_0' \Vdash \check{A} \cup \sigma_0'$ *is a $\mathcal{K}$-set.*

Proof. Suppose towards a contradiction that this fails. Then there must be a finite set $b \subset A$ and a condition $r \in R_0'$ such that $r \Vdash_{R_0'} \check{b} \cup \sigma_0' \notin \mathcal{K}$. By the complexity assumption on $\mathcal{K}$, strengthening the condition r if necessary we can find a closed set $C \subset K(X)$ in the model M such that $C \cap \mathcal{K} = 0$ and $r \Vdash \check{b} \cup \sigma_0' \in C$. Now, step out of the model M. The poset R_0 forces $\check{b} \cup \sigma_0 \notin C$; thus, there is a basic open set $O \subset K(X)$ in the Vietoris topology disjoint from C and a condition $s \leq r$ such that $s \Vdash_{R_0} \check{b} \cup \sigma_0 \in O$. The set O is of the form $O = \{K \in K(X) \colon K \subset U \text{ and } K \cap V_i \neq 0 \text{ for all } i \in n\}$ for some choice of basic open sets $U, V_i \subset X$ and some number $n \in \omega$. Let $a = \{i \in n \colon b \cap V_i = 0\}$; then $s \Vdash \sigma_0 \subset U$ and $\sigma_0 \cap V_i \neq 0$ for all $i \in a$. By the elementarity of the model M, there is a condition $t \leq r$ in M such that $t \Vdash_{R_0} \sigma_0 \subset U$ and $\sigma_0 \cap V_i \neq 0$ for all $i \in a$; then $t \Vdash_{R_0'} \sigma_0' \subset U$ and $\sigma_0 \cap V_i \neq 0$ for all $i \in a$, in other words $t \Vdash_{R_0'} \check{b} \cup \sigma_0' \in O$. This contradicts the assumption that $r \Vdash_{R_0'} \check{b} \cup \sigma_0' \in C$. □

Now, let $H_1 \subset R_1'$ be a filter generic over the model M, in V. Let $H_0 \subset R_0'$ be a filter generic over V. By the product forcing theorem, the filters $H_0 \subset R_0'$ and $H_1 \subset R_1'$ are mutually generic over the model M. Write $b_0 = \sigma_0'/H_0$ and $b_1 = \sigma_1'/H_1$; so $b_1 \in V$. By the elementarity of the model M and the forcing theorem applied in it to the poset $R_0 \times R_1$, $M[H_0, H_1] \models a \cup b_1 \in \mathcal{K}$ and $a \cup b_1 \cup b_0 \notin \mathcal{K}$. By the Mostowski absoluteness $V[H_0] \models a \cup b_1 \in \mathcal{K}$ and $a \cup b_1 \cup b_0 \notin \mathcal{K}$.

By the maximality of the initial set $A \subset X$, find a finite set $c \subset A$ such that $a \cup b_1$ is a subset of the algebraic closure of c. By the claim, $c \cup b_0 \in \mathcal{K}$. Find an inclusion maximal set $d \subset a \cup b_1 \cup b_0$ containing $a \cup b_1$ as a subset and such that $d \in \mathcal{K}$. Let $x \in (a \cup b_1 \cup b_0) \setminus d$ be an arbitrary point; clearly $x \in b_0$. Consider the pre-geometry associated with the matroid $\mathcal{K}$. Denote its closure operation by *algebraic closure* to eliminate possible confusion with topological closure in the space X. By the exchange property of the pre-geometry, x belongs to the algebraic closure of d. By the idempotence of the pre-geometry, x belongs to the algebraic closure of $c \cup (b_0 \cap d)$. However, this contradicts the fact that $c \cup b_0 \in \mathcal{K}$ holds. (1) has just been proved.

For (2), let $\langle Q, \tau\rangle$ be a balanced pair. Replacing Q with $Q \times \mathrm{Coll}(\omega, X)$ and strengthening τ if necessary, we may assume that $Q \Vdash \forall x \in X \cap V\ x \in \tau$ or $\{x\} \cup \tau$ is not a $\mathcal{K}$-set. By a balance argument, for each $x \in X$ it must be the case that either $Q \Vdash \check{x} \in \tau$ or $Q \Vdash \{\check{x}\} \cup \tau$ is not a $\mathcal{K}$-set. Let $A \subset X$ be the set of those points $x \in X$ for which the former case prevails. Since $Q \Vdash \tau \leq \check{A}$, in view of (1) and Proposition 5.2.4 it will be enough to show that $A \subset X$ is a maximal $\mathcal{K}$-set.

Certainly, $A \subset X$ is a $\mathcal{K}$-set since it is forced to be a subset of a $\mathcal{K}$-set τ. Now suppose that $A \subset X$ is not maximal and pick a point $x \notin A$ such that $A \cup \{x\}$ is still a $\mathcal{K}$-set. Passing to a condition of Q if necessary, we may find a finite set $a \subset A$ and a Q-name σ for a finite subset of τ such that Q forces $\check{x}$ to belong to the algebraic closure of $\check{a} \cup \sigma$ and not to belong to the closure of any of its proper subsets. Let η be a Q-name for an inclusion-maximal subset of $a \cup \{x\} \cup \sigma$ which is in $\mathcal{K}$ and contains $a \cup \{x\}$. It is not equal to $a \cup \{x\} \cup \sigma$, and it contains every element of $\sigma \setminus \eta$ in its algebraic closure by the exchange property of the pre-geometry. Let $H_0, H_1 \subset Q$ be filters mutually generic over V and write $p_0 = \tau/H_0$, $b_0 = \sigma/H_0$, and $c_0 = \eta/H_0$, and similarly for p_1, b_1, c_1. It will be enough to show that $a \cup b_0 \cup b_1 \notin \mathcal{K}$, since this will contradict the initial assumption of balance of the name τ and the resulting compatibility of conditions $p_0, p_1 \in P$.

Note that x belongs to the algebraic closure of $a \cup b_0$. By the idempotence of the pre-geometry, it follows that every element of the nonempty set $b_1 \setminus c_1$ belongs to the algebraic closure of $a \cup b_0 \cup c_1$. Since $a \cup b_0 \cup c_1 \subset a \cup b_0 \cup b_1$, this shows that the latter set is not free and completes the proof of (2).

Finally, (3) is obvious. The last sentence of the theorem follows from a straighforward application of Kuratowski–Zorn lemma–every $\mathcal{K}$-set can be extended to a maximal $\mathcal{K}$-set. □

Example 6.5.4. [42] (An algebraic matroid) Let X be a Polish field over a countable subfield F. Consider the Borel simplicial complex $\mathcal{K}$ of finite subsets of X which are algebraically free over F. $\mathcal{K}$ is well-known to be generated by the Borel pre-geometry of algebraic closure. Also, a finite set $a \subset X$ is not in $\mathcal{K}$ just in case it is a solution to a nontrivial polynomial with coefficients in F. There are only countably many such polynomials and for each of them, the set of solutions is F_σ. Thus, the simplicial complex $\mathcal{K}$ is G_δ and the poset $P_{\mathcal{K}}$ is balanced by Theorem 6.5.2. The poset adds a transcendence basis to the field X as the union of the generic filter.

6.6 Quotient variations

Many simplicial complex forcings are naturally connected with a Borel equivalence relation.

Definition 6.6.1. Let E be a Borel equivalence relation on a Polish space X. A Borel simplicial complex $\mathcal{K}$ on X is an *E-quotient complex* if the membership of any finite set $a \subset X$ in $\mathcal{K}$ depends only on $[a]_E$. The letter E is left out if E is understood from the context or will be specified later.

In this section we identify two general schemas for defining useful quotient simplicial complexes and their associated posets: collapse posets for quotient cardinals and uniformization posets for quotient spaces.

Definition 6.6.2. Let E, F be Borel equivalence relations on Polish spaces X, Y with uncountably many classes each. Let $\mathcal{K}_{E,F} = \mathcal{K}$ be the simplicial complex on the set $(X \times Y) \cup Y$ consisting of finite sets a such that for all $\langle x_0, y_0\rangle, \langle x_1, y_1\rangle \in a$, $x_1 \mathrel{E} x_0$ if and only if $y_1 \mathrel{F} y_0$ holds, and for all $\langle x_0, y_0\rangle, y_2 \in a$, $y_0 \mathrel{F} y_2$ fails. Clearly, $\mathcal{K}$ is an $(E \times F) \cup F$-quotient complex. The associated poset $P_{\mathcal{K}}$ is referred to as the *E, F-collapse poset*.

If $A \subset (X \times Y) \cup Y$ is a generic set, the part $A \cap (X \times Y)$ is an injection from X/E to Y/F, while the part $A \cap Y$ is the complement of the range of $A \cap (X \times Y)$. The balanced conditions are neatly classified by injections from the virtual E-quotient space X^{**} to the virtual F-quotient space Y^{**}. If $g\colon X^{**} \to Y^{**}$ is an injection, let τ_g be the $\mathrm{Coll}(\omega, \beth_{\omega_1})$-name for the set of those elements $p \in P_{\mathcal{K}}$ such that for each pair $\langle c, d\rangle \in g$, there is a pair $\langle x, y\rangle \in p$ such that $x \in X$ is a realization of the virtual E-class c and $y \in Y$ is a realization of the virtual F-class d. Moreover, if d is a virtual F-class not in the range of g, then p is required to contain an element $y \in Y$ which is a realization of the virtual F-class d.

Theorem 6.6.3. *Let E, F be Borel equivalence relations on respective Polish spaces X, Y. Let P be the E, F-collapse poset.*

1. *For every total injection $g\colon X^{**} \to Y^{**}$, the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_g\rangle$ is a balanced P-pin;*
2. *for every balanced pair $\langle Q, \tau\rangle$ there is a total injection $g\colon X^{**} \to Y^{**}$ such the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_g\rangle$ is equivalent to $\langle Q, \tau\rangle$;*
3. *distinct injections as in (1) yield inequivalent balanced conditions.*

Proof. Write $R = \mathrm{Coll}(\omega, \beth_{\omega_1})$. For (1), the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_g\rangle$ is clearly a P-pin. For the balance, let $V[H_0], V[H_1]$ be mutually generic extensions of the ground model and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be conditions in the analytic set $\tau_g/H_0 = \tau_g/H_1$; we must show that $p_0, p_1 \in P$ are compatible.

To see this, suppose for example that $\langle x_0, y_0\rangle \in p_0$, $\langle x_1, y_1\rangle \in p_1$, and $x_0 \mathrel{E} x_1$; we must show that $y_0 \mathrel{E} y_1$. By a mutual genericity argument, the points x_0, x_1 must be realizations of the same virtual E-class c in the ground model. Since both p_0, p_1 are $\mathcal{K}$-sets, the definition of the name τ_g shows that both y_0, y_1 must be realizations of the virtual F-class $g(c)$ and therefore $y_0 \mathrel{F} y_1$ as desired.

Similarly, if $\langle x_0, y_0\rangle \in p_0$ and $y_2 \in p_1$, we need to show that $y_0 \mathrel{F} y_2$ fails. Suppose towards a contradiction that $y_0 \mathrel{F} y_2$ holds. By a mutual genericity argument, the points y_0, y_2 are representations of the same virtual F-class d in the ground model. Now, if $d \in \text{rng}(g)$ then p_1 cannot be a $\mathcal{K}$-set, and if $d \notin \text{rng}(g)$ then p_0 cannot be a $\mathcal{K}$-set. There are no other options, and this contradiction completes the proof of (1).

For (2), let $g\colon X^{**} \to Y^{**}$ be the collection of all pairs $\langle c, d\rangle \in X^{**} \times Y^{**}$ so that $R \times Q \Vdash \langle x, y\rangle \in \tau$ for some realizations x, y of the virtual classes c, d. It will be enough to show that g is a total injection from X^{**} to Y^{**} and the pair $\langle Q, \tau\rangle$ is equivalent to $\langle R, \tau_g\rangle$.

To see that $\text{dom}(g) = X^{**}$ and g is an injection, let $c \in X^{**}$ be an arbitrary virtual E-class. Use the balance of the pair $\langle Q, \tau\rangle$ to show that there must be a unique virtual F-class $d \in Y^{**}$ such that $R \times Q \Vdash \langle x, y\rangle \in \dot{A}_{gen}$ for some realizations x, y of the virtual classes c, d. Use the balance of the pair $\langle Q, \tau\rangle$ again to show that for every virtual F-class $d \in Y^{**}$, there either must be a virtual E-class $c \in X^{**}$ such that $R \times Q \Vdash \langle x, y\rangle \in \dot{A}_{gen}$ for some realizations x, y of the virtual classes c, d, or it must be the case that $R \times Q \Vdash y \in \tau$ for some realization y of the virtual class d.

To see that $\langle R, \tau_g\rangle$ is equivalent to $\langle Q, \tau\rangle$, strengthen τ if necessary so it is a name for an actual element of P and notice that in the $R \times Q$ extension, $\tau \in \tau_g$ holds. The equivalence then follows from Proposition 5.2.4.

Finally, (3) is obvious. $\square$

Corollary 6.6.4. *Let E, F be Borel equivalence relations on respective Polish spaces X, Y with uncountably many classes. The E, F-collapse poset is balanced if and only if $\lambda(E) \leq \lambda(F)$.*

The collapse posets exemplify an important phenomenon: a σ-closed Suslin forcing whose balanced status cannot be decided in ZFC. There are Borel equivalence relations E, F for which the status of the inequality $\lambda(E) \leq \lambda(F)$ cannot be decided in ZFC (as can be seen for example from the combination of Corollaries 2.5.18 and 2.5.15) and for them the balanced status of the corresponding collapse forcing is undecidable as well.

Definition 6.6.5. Let $E \subset F$ be Borel equivalence relations on a Polish space X. Let $\mathcal{K} = \mathcal{K}_{E,F}$ be the simplicial complex of all finite sets $a \subset X$ such that for $x_0, x_1 \in X$, $x_0\ E\ x_1$ is equivalent to $x_0\ F\ x_1$. We will refer to the associated poset $P_{\mathcal{K}}$ as the *E, F-transversal poset.*

It is immediate that the simplicial complex $\mathcal{K}_{E,F}$ is a Borel E-quotient complex. The terminology comes from the examples below; the posets of this type are used to uniformize various relations in products of quotient spaces. The first order of business is to classify the balanced conditions. Call a total function f from the F-quotient space to the virtual E-quotient space a *virtual selection function* if for every F-class c the functional value $f(c)$ is a virtual E-class which is (forced to be) a subset of c. For a virtual selection function f let τ_f be the $\text{Coll}(\omega, \beth_{\omega_1})$-name for the set of all conditions p in the uniformization poset such that for each class c represented in the ground model, p contains some point $x \in X$ which is a realization of the virtual E-class $f(c)$. It is not difficult to see that the pair $\langle \text{Coll}(\omega, \beth_{\omega_1}), \tau_f\rangle$ is a P-pin.

Theorem 6.6.6. *Let $E \subset F$ be Borel equivalence relations on a Polish space X and suppose that F is pinned. Let P be the E, F-transversal poset. Then*

1. *for every virtual selection function f, the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_f \rangle$ is balanced;*
2. *for every balanced pair $\langle Q, \tau \rangle$ there is a virtual selection function f such that the pairs $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_f \rangle$ and $\langle Q, \tau \rangle$ are equivalent;*
3. *distinct virtual selection functions yield nonequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. To see (1), let $V[H_0]$ and $V[H_1]$ be mutually generic extensions of the ground model and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be conditions in the set $\tau_f/H_0 = \tau_f/H_1$; we have to show that $p_0, p_1 \in P$ are compatible. To this end, suppose that $x_0 \in p_0$ and $x_1 \in p_1$ are points; we must show that if $x_0 \mathrel{F} x_1$ then $x_0 \mathrel{E} x_1$ holds. A mutual genericity argument shows that x_0, x_1 are realizations of the same virtual F-class c. By the definition of the name τ_f, it must be the case that x_0, x_1 are both realizations of the virtual E-class $f(c)$ and therefore E-related.

For (2), suppose that $\langle Q, \tau \rangle$ is a balanced pair. To find the total function f, let c be an F-class. There must be a virtual E-class $f(c)$ such that $Q \Vdash$ for some point $x \in \tau \cap \dot{c}$ x is a realization of $f(c)$; otherwise, one could find conditions $q_0, q_1 \in Q$ and names for strengthenings τ_0, τ_1 of τ and names $\dot{x}_0, \dot{x}_1$ for elements of $\dot{c}$ such that $\langle q_0, q_1 \rangle \Vdash_{Q \times Q} \neg (\dot{x}_0)_{\mathrm{left}} \mathrel{F} (\dot{x}_1)_{\mathrm{right}}$. This would violate the balance of the pair $\langle Q, \tau \rangle$ as $\langle q_0, q_1 \rangle \Vdash \tau_0, \tau_1 \in P$ are incompatible conditions. Let $f \colon c \mapsto d_c$ be the resulting function.

To see that $\langle R, \tau_f \rangle$ is a balanced pair equivalent to $\langle Q, \tau \rangle$, strengthen τ if necessary to ensure that τ is a name for an actual element of P, and observe that $Q \times \mathrm{Coll}(\omega, \beth_{\omega_1}) \Vdash \tau \in \tau_f$. The equivalence of the two balanced pairs then follows from Proposition 5.2.4.

Finally, (3) is obvious. The balance of the poset P is now a trivial application of the Axiom of Choice: every condition $p \in P$ can be extended to a maximal set $A \subset X$ such that $E \restriction A = F \restriction A$. Let f be the function which to each F-class c assigns the E-class $c \cap A$, and observe that the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_f \rangle$ is balanced by (1) and is below p. □

Corollary 6.6.7. *Let $E \subset F$ be Borel equivalence relations on a Polish space X such that F is pinned. Let P be the E, F-uniformization poset. Then P is balanced.*

Example 6.6.8. A poset introducing a transversal to a pinned Borel equivalence relation F on a Polish space X. Just let E be the identity on the space X and consider the E, F-transversal poset. The generic set is an F-transversal. By Theorem 6.6.6, the balanced conditions are classified by F-transversals.

Example 6.6.9. A poset introducing a complete countable section to a pinned Borel equivalence relation G on a Polish space Y. Let $X = Y^\omega$ and let $B \subset X$

be the Borel set of all elements $x \in X$ whose range consists of pairwise G-related points. Let F be the Borel equivalence relation on X connecting points x_0, x_1 if either they are equal or else they are both in B and their ranges are subsets of the same G-class. Let $E = \mathbb{F}_2 \cap F$. It is not difficult to check that $E \subset F$ are Borel equivalence relations on X and F is pinned. Consider the associated E, F-transversal poset. The generic set is a countable complete section of the relation E. In view of the fact that virtual $\mathbb{F}_2$-classes are classified by nonempty subsets of Y, Theorem 6.6.6 says that the balanced conditions are classified by arbitrary sets $A \subset Y$ which have nonempty intersection with every G-class.

Numerous examples of uniformization posets arise in the context of countable Borel equivalence relations. If E is a countable Borel equivalence relation on a Polish space Y and P is a poset adding a choice of a structure of some type on each E-class with countable approximations, then P can be represented as a transversal poset in the sense of Definition 6.6.5. The following example spells out the details.

Example 6.6.10. Let G be a countable Borel equivalence relation on a Polish space Y with all equivalence classes infinite. Let P be the poset of all countable functions on the G-quotient space assigning to each equivalence class a $\mathbb{Z}$-ordering on it; the poset P serves to introduce a (discontinuous) $\mathbb{Z}$-action on Y such that G is its orbit equivalence relation. It is not difficult to cast P as a transversal poset in the sense of Definition 6.6.5. Let $X = (Y^2)^\omega$ and let $B \subset X$ be the Borel set of all elements $x \in X$ which enumerate a linear ordering of a whole single G-class of ordertype $\mathbb{Z}$. Let F be the equivalence relations on X connecting points $x_0, x_1 \in X$ if either they are equal or else they both belong to B and enumerate a linear ordering of the same G-class. Let $E = \mathbb{F}_2 \cap F$. It is not difficult to check that both E, F are pinned Borel equivalence relations. Consider the associated E, F-transversal poset. The generic set selects one $\mathbb{Z}$-type ordering for each G-class. Theorem 6.6.6 says that the balanced conditions are classified by all functions f assigning each G-class a linear ordering of its elements isomorphic to $\mathbb{Z}$.

Example 6.6.11. Let $X = ((2^\omega)^\omega)^2$ and consider the Borel set $B \subset X$ consisting of all pairs $\langle y_0, y_1 \rangle \in X$ such that $\operatorname{rng}(x_0) \cap \operatorname{rng}(x_1) = 0$. Let F be the equivalence relation connecting points $\langle y_0, y_1 \rangle$ and $\langle z_0, z_1 \rangle$ if they are either equal or both in B and $y_0 \mathrel{\mathbb{F}_2} z_0$. Let E be the equivalence relation $\mathbb{F}_2 \times \mathbb{F}_2$ intersected with F. The resulting E, F-transversal poset is designed to add a function assigning to each nonempty countable set a single nonempty countable set disjoint from it. However, the equivalence relation F is not pinned and so Theorem 6.6.6 does not apply to show that the uniformization poset is balanced. Indeed, in ZF one can prove that existence of a maximal $\mathcal{K}_{E,F}$-set implies the existence of an ω_1-sequence of pairwise distinct E-classes. To see this, let $A \subset X$ be such a maximal set, suppose $c_0 \subset 2^\omega$ is an arbitrary nonempty countable set, and by transfinite recursion on $\alpha \in \omega_1$ define c_α as the unique nonempty countable set such that for some enumerations y_0 of $\bigcup_{\beta \in \alpha} c_\beta$ and y_1 of c_α it is the case that $\langle y_0, y_1 \in B$. It is clear that the sets c_α for $\alpha \in \omega_1$ are nonempty and pairwise disjoint and therefore distinct. Finally, note (Theorem 9.1.1) that in balanced extensions of the symmetric Solovay model there are no uncountable sequences of distinct equivalence classes of Borel equivalence relations such as $\mathbb{F}_2$.

If the larger equivalence relation F in Definition 6.6.5 is not pinned, the circumstances must be quite peculiar for the E, F-transversal poset to be balanced. One class of examples of this type is captured by the following theorem. For Borel equivalence relations $E \subset F$ on a Polish space X write X_E^{**}, X_F^{**} to be their respective virtual quotient spaces. A function $f \colon X_F^{**} \to X_E^{**}$ is *a virtual E, F-selection function* if for every virtual F-class c, the virtual E-class $f(c)$ is forced to be a subset of c. For each E, F-selection function f let τ_f be the $\mathrm{Coll}(\omega, \beth_{\omega_1})$-name for the set of all countable subsets $a \subset X$ such that for each virtual F-class c, a contains a realization of the virtual E-class $f(c)$.

Theorem 6.6.12. *Let $E \subset F$ be Borel equivalence relations on a Polish space X such that each F-class consists of countably many E-classes. Let P be the E, F-transversal poset. Then*

1. *for every virtual E, F-selection function $f \colon X_F^{**} \to X_E^{**}$, the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_f \rangle$ is balanced;*
2. *for every balanced pair $\langle Q, \tau \rangle$ there is a virtual selection function f such that the balanced pairs $\langle Q, \tau \rangle$, $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_f \rangle$ are equivalent;*
3. *distinct virtual selection functions yield nonequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. The argument hinges on a simple claim of independent merit:

Claim 6.6.13. *Let $V[H]$ be a generic extension of V, and in $V[H]$ let c be an F-class which is a realization of a ground model virtual F-class. Then c is a union of realizations of ground model virtual E-classes.*

Proof. Suppose towards a contradiction that this fails. Then in V there has to be a poset P and a P-name τ for an element of the space X such that τ is F-pinned, but below no condition of P, τ is E-pinned. Let M be a countable elementary submodel of a large structure containing E, F, P, τ and use Proposition 1.7.10 to find a perfect collection $\{g_y \colon y \in 2^\omega\}$ of filters on $P \cap M$ mutually generic over M. Let $x_y = \tau/g_y \in X$. For distinct binary sequences $y, z \in 2^\omega$, $M[g_y, g_z] \models x_y \mathrel{F} x_z$ and $\neg x_y \mathrel{E} x_z$ by the forcing theorem applied in M. By the Mostowski absoluteness for the model $M[g_y, g_z]$, it follows that $x_y \mathrel{F} x_z$ and $\neg x_y \mathrel{E} x_z$ holds in V. However, this means that the unique F-class of the points x_y for $y \in 2^\omega$ contains perfectly many pairwise E-unrelated points, contradicting the initial assumptions on the equivalence relations E, F. $\square$

Now, for (1) suppose that $V[H_0], V[H_1]$ are mutually generic extensions containing respective conditions p_0, p_1 below τ_f; we must show that p_0, p_1 are compatible. This is to say, $p_0 \cup p_1$ is a $\mathcal{K}_{E,F}$-set; in other words, if $x_0 \in p_0$ and

$x_1 \in p_1$ are F-related points then they are in fact E-related. To see this, by a mutual genericity argument x_0, x_1 are realizations of some virtual F-class c; by the definition of the virtual condition τ_f, they must be realizations of the virtual E-class $f(c)$ and therefore must be F-related.

For (2), first use a balance argument and Claim 6.6.13 to argue that for each virtual F-class c there must be a unique virtual E-class d which is forced to be a subclass of c and such that $Q \Vdash \tau$ contains a realization of d. Let $f\colon X_F^{**} \to X_E^{**}$ be the function recording the correspondence $c \mapsto d$ and observe that $Q \times \mathrm{Coll}(\omega, \beth_{\omega_1}) \Vdash \tau \leq \tau_f$ holds. (2) then follows by a reference to Proposition 5.2.4.

Finally, (3) is obvious. The balance of the poset P follows from the axiom of choice: every condition $p \in P$ can be extended to a virtual selection function by virtua of Claim 6.6.13. $\square$

Example 6.6.14. Let Γ be a Borel graph on a Polish space X which contains no odd length cycles such that the G-path connectedness equivalence relation F on X is Borel. We wish to add a 2-coloring of the graph Γ. To this end, let E be the Borel equivalence relation on X relating x to y if $x \mathrel{E} y$ holds and some (equivalently, every) Γ-path from x to y is of even length. It is clear that every F-class consists of exactly two E-classes. The E, F-transversal poset selects exactly one E-class from each F-class and therefore adds a bipartization of the graph Γ. The balanced conditions are classified by functions which select exactly one virtual E-class from each virtual F-class.

Example 6.6.15. Let F be the $\mathbb{F}_2$-equivalence relation on the space $X = (2^\omega)^\omega$. Let E be the equivalence relation on X connecting points x_0 and x_1 if $\mathrm{rng}(x_0) = \mathrm{rng}(x_1)$ and $x_0(0) = x_1(0)$. Clearly, every F-class consists of only countably many E-classes. An E, F-transversal can be viewed as a function which selects a single element from each nonempty countable subset of 2^ω. It is instructive to inspect the balanced virtual conditions provided by Theorem 6.6.12. An inspection reveals that a balanced virtual condition is a function which to each nonempty subset of X assigns one of its elements, i.e. a selection function on $\mathcal{P}(X)$. In consequence, the existence of balanced conditions for P is in ZF equivalent to the statement that the space 2^ω can be well-ordered.

Chapter 7

Ultrafilter forcings

Many applications of the axiom of choice rely on nonprincipal ultrafilters on ω and their combinatorial properties. In this section, we will show that in several cases, natural attempts to add an ultrafilter result in balanced forcings and moreover the resulting generic ultrafilter can be characterized in a simple way. The following definition and proposition will play a central role in several arguments.

Definition 7.0.1. Let U be a nonprincipal ultrafilter on a set $\mathrm{dom}(U)$. An infinite set $a \subset \mathrm{dom}(U)$ in some forcing extension *diagonalizes* U if for every $b \in U$, $a \setminus b$ is finite.

Proposition 7.0.2. *Let U be a nonprincipal ultrafilter on* $\mathrm{dom}(U)$. *Suppose that Q_0, Q_1 are posets and τ_0, τ_1 are respective names for infinite subsets of* $\mathrm{dom}(U)$ *which diagonalize U. Then $Q_0 \times Q_1 \Vdash \tau_0 \cap \tau_1$ is infinite.*

Proof. The most appealing argument uses the general Proposition 1.7.8 on product forcing extensions. Let $H_0, H_1 \subset Q_0, Q_1$ be mutually generic filters and let $a_0 = \tau_0/H_0$ and $a_1 = \tau_1/H_1$. Suppose towards a contradiction that the intersection $a_0 \cap a_1$ is finite. By Proposition 1.7.8, there are sets $b_0, b_1 \subset \mathrm{dom}(U)$ in the ground model such that $a_0 \subset b_0$, $a_1 \subset b_1$, and $b_0 \cap b_1$ is finite. At most one of the sets b_0, b_1 belongs to the ultrafilter U. If, say, $b_0 \notin U$, then $\omega \setminus b_0 \in U$, violating the assumption that a_0 diagonalizes the ultrafilter U. $\square$

7.1 A Ramsey ultrafilter

The most elementary attempt at adding a nonprincipal ultrafilter is the poset P of infinite subsets of ω ordered by inclusion. Here, the classification of balanced conditions is particularly appealing and simple.

Definition 7.1.1. Let U be a nonprincipal ultrafilter on ω. The symbol τ_U denotes the $\mathrm{Coll}(\omega, U)$-name for the analytic collection of all infinite sets $a \subset \omega$ which diagonalize the ultrafilter U.

Theorem 7.1.2. *Let P be the partial order of infinite subsets of ω, ordered by inclusion.*

1. *The pair $\langle \mathrm{Coll}(\omega, U), \tau_U\rangle$ is a balanced virtual condition for P for every nonprincipal ultrafilter U on ω;*
2. *If $\langle Q, \tau\rangle$ is a balanced pair for P, then there is a nonprincipal ultrafilter U such that $\langle Q, \tau\rangle$ is equivalent to $\langle \mathrm{Coll}(\omega, U), \tau_U\rangle$;*
3. *distinct nonprincipal ultrafilters yield inequivalent balanced virtual conditions.*

In particular, the poset P is balanced.

Proof. For (1), note that the pair in fact is a virtual condition: in $\mathrm{Coll}(\omega, U)$, the set U is countable and so τ_U is an analytic subset of P, and the evaluation of τ_U does not depend on the particular generic filter on the collapse poset. The balance of the pair follows immediately from Proposition 7.0.2 applied to U.

For (2), first use the balance of the name τ to note that for every set $a \subset \omega$, it must be the case that either $Q \Vdash \tau \subset \check{a}$ up to a finite set, or $Q \Vdash \tau \cap \check{a}$ is finite. Let U be the set of all $a \subset \omega$ for which the first option occurs. It is immediate that U is a nonprincipal ultrafilter on ω. The equivalence of $\langle Q, \tau\rangle$ with $\langle \mathrm{Coll}(\omega, U), \tau_U\rangle$ is immediately clear from Proposition 7.0.2.

For (3), not that if U_0, U_1 are distinct ultrafilters on ω then there is a set $a \subset \omega$ such that $a \in U_0$ and $\omega \setminus a \in U_1$. Thus the balanced names τ_{U_0} and τ_{U_1} represent incompatible virtual conditions as the former is below a and the other is below $\omega \setminus a$.

To prove the last sentence of the theorem, for any infinite set $a \subset \omega$ there is a nonprincipal ultrafilter U containing a as an element. A reference to item (1) then completes the argument. $\square$

7.2 Fubini powers of the Fréchet ideal

The most natural attempt to force an ultrafilter which is not a P-point with a σ-closed subset is encapsulated in the following definition.

Definition 7.2.1. For the duration of this section, write I to be the ideal on $\omega \times \omega$ which is the Fubini product of the Fréchet ideal on ω with itself. That is, a set $a \subset \omega \times \omega$ belongs to the ideal I just in case a has only finitely many infinite vertical sections. The *Fin×Fin poset* P isthe partial order of I-positive sets, ordered by $q \leq p$ if $q \setminus p \in I$.

It is not difficult to see that P is a σ-closed forcing; the sets in the generic filter on P form an ultrafilter disjoint from I. This ultrafilter, by virtue of the partition of the domain set $\omega \times \omega$ into the I-small vertical sections such that every transversal is again in I, is not a P-point; its rich combinatorial properties were investigated for example in [11, 89].

It turns out that the poset P is balanced, and the balanced virtual conditions allow a simple classification. Let ω^* be the set of all nonprincipal ultrafilters on ω. Let W be an ultrafilter on the set $\omega \times \omega^*$ which contains no set $n \times \omega^*$ for any number $n \in \omega$. Write τ_W for the name on $\mathrm{Coll}(\omega, \mathcal{P}(\omega \times \omega^*))$ for the set of all conditions $p \in P$ such that there is a set $\{\langle n_i, U_i \rangle : i \in \omega\} \subset (\omega \times \omega^*)^V$ diagonalizing the ultrafilter W, the numbers n_i for $i \in \omega$ are pairwise distinct, and the set $p \subset \omega \times \omega$ has the following property: only the vertical sections p_{n_i} for some $i \in \omega$ are nonempty, and each vertical section p_{n_i} is infinite and diagonalizes the ultrafilter U_i.

Theorem 7.2.2. *Let P be the Fin×Fin poset.*

1. *For every ultrafilter W on the set $\omega \times \omega^*$ containing no set $n \times \omega^*$ for $n \in \omega$, the pair $\langle \mathrm{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)), \tau_W \rangle$ is balanced in P;*
2. *for every balanced pair $\langle Q, \sigma \rangle$ there is an ultrafilter W on the set $\omega \times \omega^*$, containing no set $n \times \omega^*$ for $n \in \omega$, such that the balanced pairs $\langle Q, \sigma \rangle$ and $\langle \mathrm{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)), \tau_W \rangle$ are equivalent;*
3. *distinct ultrafilters give rise to inequivalent balanced conditions.*

In particular, the poset P is balanced.

Proof. Towards (1), suppose that $V[H_0], V[H_1]$ are mutually generic extensions containing respective sets $p_0, p_1 \subset \omega \times \omega$ such that there are sets $a_0 = \{\langle n_i^0, U_i^0 \rangle : i \in \omega\}$ and $a_1 = \{\langle n_i^1, U_i^1 \rangle : i \in \omega\}$ in the respective models which diagonalize the ultrafilter W, and the vertical sections $(p_0)_{n_i^0}$ diagonalize the ultrafilter U_i^0 for all $i \in \omega$, and the vertical sections $(p_1)_{n_i^1}$ diagonalize the ultrafilter U_i^1 for all $i \in \omega$ as well. It will be enough to show that p_0, p_1 are compatible in the poset P; i.e. the set $p_0 \cap p_1$ has infinitely many infinite vertical sections.

To this end, use Proposition 7.0.2 for the sets a_0, a_1 and the ultrafilter W to see that $a_0 \cap a_1$ is infinite. This means, that the set $b = \{n \in \omega : \exists U\ \langle n, U \rangle \in a_0 \cap a_1\}$ is infinite. For each number $n \in b$, the vertical sections $(p_0)_n$ and $(p_1)_n$ diagonalize the same ultrafilter U on ω for which $\langle n, U \rangle \in a_0 \cap a_1$ holds. By another application of Proposition 7.0.2, the set $(p_0)_n \cap (p_1)_n$ is infinite. It follows that $p_0 \cap p_1 \notin I$ as desired.

Towards (2), strengthen σ if necessary so that σ is a name for an actual single condition in the poset P as opposed to an analytic set of conditions. Replacing Q with a poset collapsing the cardinality of $\mathcal{P}(\omega)^V$ to $\aleph_0$ and strengthening the name σ if necessary we may assume that Q forces that for every $n \in \omega$, if the vertical section σ_n is infinite, then for every set $a \subset \omega$ in the ground model, either $\sigma_n \subset a$ or $\sigma_n \cap a = 0$ modulo finite. Let $\dot{U}_n$ be the name for the set of all sets $a \subset \omega$ in the ground model for which the former alternative prevails; define $\dot{U}_n = 0$ if the vertical section σ_n is finite. Passing to a condition in Q and strengthening σ again, we may assume that either Q forces that for all $n \in \omega$ such that σ_n is infinite $\dot{U}_n \notin V$ holds, or Q forces that for all $n \in \omega$ such that σ_n is infinite $\dot{U}_n \in V$ holds.

Now, the former alternative is impossible as it contradicts the balance of the pair $\langle Q, \tau \rangle$. To see this, let $H_0, H_1 \subset Q$ be mutually generic filters and let $p_0 = \sigma/H_0$ and $p_1 = \sigma/H_1$. To reach the contradiction, we will show that all vertical sections of $p_0 \cap p_1$ are finite. Suppose not, and let $n \in \omega$ be such that $(p_0)_n \cap (p_1)_n$ is infinite. Then $\dot{U}_n/H_0 = \dot{U}_n/H_1$, which by the product forcing theorem implies that $\dot{U}_n/H_0 \in V$ and violates the former alternative assumption.

Thus, the latter alternative prevails, and the set $\eta = \{\langle n, \dot{U}_n \rangle : \sigma_n \text{ is infinite}\}$ is forced to be a subset of $(\omega \times \omega^*)^V$. Strengthening σ again, we may assume that Q forces that for every set $b \subset \omega \times \omega^*$ in the ground model, either $\eta \subset b$ or $\eta \cap b = 0$ modulo finite.

Claim 7.2.3. *For every set $b \subset \omega \times \omega^*$, either $Q \Vdash \eta \subset \check{b}$ modulo finite, or $Q \Vdash \eta \cap \check{b}$ is finite.*

Proof. Suppose towards a contradiction that the conclusion fails for some set $b \subset \omega \times \omega^*$. Then, there must be conditions $q_0, q_1 \in Q$ forcing the former and latter alternative respectively. Let $H_0, H_1 \subset Q$ be mutually generic filters containing the conditions q_0, q_1 respectively. Write $p_0 = \sigma/H_0$ and $p_1 = \sigma_1/H_1$; we will reach a contradiction by showing that all but finitely many vertical sections of the set $p_0 \cap p_1$ are finite, violating the balance assumption on σ.

Write $c_0 = \eta/H_0$ and $c_1 = \eta/H_1$. By the contradictory assumption, the intersection $c_0 \cap c_1$ is finite. Let $n \in \omega$ be a natural number which is not in the finite projection of the set $c_0 \cap c_1$ to ω; we will show that $(p_0 \cap p_1)_n$ is finite. Either $(p_0)_n$ or $(p_1)_n$ are finite sets, in which case we are done, or both $(p_0)_n$ and $(p_1)_n$ are infinite. In the latter case, there are ultrafilters U_0, U_1 on ω in the ground model such that $\langle n, U_0 \rangle \in c_0$ and $\langle n, U_1 \rangle \in c_1$. Since the number n is not in the projection of $c_0 \cap c_1$ to ω, the ultrafilters U_0, U_1 must be distinct. Since $(p_0)_n$ diagonalizes U_0 and $(p_1)_n$ diagonalizes U_1, the intersection $(p_0)_n \cap (p_1)_n$ must be finite in this case as well. $\square$

Let W be the set of all subsets of $b \subset \omega \times \omega^*$ for which $Q \Vdash \eta \subset \check{b}$ modulo finite; by the claim, this is an ultrafilter on $\omega \times \omega^*$. The definitions show that $\text{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)) \times Q \Vdash \sigma \in \tau_F$. Then, Proposition 5.2.4 shows that the balanced pairs $\langle Q, \sigma \rangle$ and $\langle \text{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)) \rangle$ are equivalent as desired.

Item (3) is immediate. For the last sentence, suppose that $p \in P$ is a condition. Let $a = \{n \in \omega : p_n \text{ is infinite}\}$; the set a is infinite. For each $n \in \omega$ let U_n be a nonprincipal ultrafilter on ω containing p_n as an element. Let W be a nonprincipal ultrafilter on the set $\omega \times \omega^*$ containing the set $\{\langle n, U_n \rangle : n \in a\}$. It is not difficult to see that the balanced pair $\langle \text{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)), \tau_W \rangle$ is below the condition p, proving the balance of the poset P. $\square$

7.3 Ramsey sequences of structures

There is a family of forcings present in several papers concerning the Rudin–Keisler order on ultrafilters with strong Ramsey-type properties [23]. The family is parametrized by sequences of structures tied by a partition property as in the following definition.

Definition 7.3.1. A *Ramsey sequence of finite structures* is a sequence $A = \langle A_n \colon n \in \omega\rangle$ of finite structures in the same language, with pairwise disjoint domains, such that

1. for every natural number $n \in \omega$, A_{n+1} contains an isomorphic copy of A_n;
2. for every $k < n \in \omega$ there is $m > n$ such that the structural Ramsey property $A_m \to (A_n)^{A_k}_2$ holds.

For each Ramsey sequence $A = \langle A_n \colon n \in \omega\rangle$ of finite structures, we will use the following notation. $\mathrm{dom}(A)$ stands for $\bigcup_n \mathrm{dom}(A_n)$. Given a number $n \in \omega$, the symbol D_n stands for the set of all finite sets d such that for some $m \in \omega$, $d \subset \mathrm{dom}(A_m)$ and $A_m \restriction d$ is isomorphic to A_n. For every set $p \subset \mathrm{dom}(A)$, the symbol p^{A_n} stands for $\mathcal{P}(p) \cap D_n$. Each Ramsey sequence of finite structures has a natural σ-closed poset associated to it.

Definition 7.3.2. Let $A = \langle A_n \colon n \in \omega\rangle$ be a Ramsey sequence of finite structures. Let P_A denote the poset of all sets $p \subset \mathrm{dom}(A)$ such that p^{A_n} is nonempty for each $n \in \omega$. The ordering is defined by $q \leq p$ if $q \subseteq p$ modulo finite.

It is not difficult to use the Ramsey property of the sequence A to see that a generic filter on P_A is in fact an ultrafilter on $\mathrm{dom}(A)$; the combinatorial properties of this generic ultrafilter are the reason for the study of these posets in [8]. The posets of the form P_A are balanced. The balanced virtual conditions are parametrized by sequences of ultrafilters as opposed to single ultrafilters in the case of $\mathcal{P}(\omega)$ modulo finite.

Definition 7.3.3. Let $A = \langle A_n \colon n \in \omega\rangle$ be a Ramsey sequence of finite structures. An *A-sequence of filters* is a sequence $\langle F_n \colon n \in \omega\rangle$ such that

1. for every number $n \in \omega$, F_n is a filter on the set D_n;
2. for all numbers $n, m \in \omega$ and every selection $\langle a_i \colon i \in n\rangle$ of sets in the respective filters F_i for $i \in n$, there is a set $d \in D_m$ such that $d^{A_i} \subset a_i$ holds for all $i \in n$.

Whenever $F = \langle F_n \colon n \in \omega\rangle$ is an A-sequence of ultrafilters, define the $\mathrm{Coll}(\omega, 2^\omega)$-name τ_F for the set of all conditions $p \in P$ such that for every $n \in \omega$ and every set $a \in F_n$, $p^{A_n} \subset a$ modulo finite.

Theorem 7.3.4. *Let $A = \langle A_n \colon n \in \omega\rangle$ be a Ramsey sequence of finite structures.*

1. *For every A-sequence F of ultrafilters, the pair $\langle \mathrm{Coll}(\omega, 2^\omega), \tau_F\rangle$ is balanced in the poset P_A;*

2. *for every balanced pair $\langle Q, \sigma\rangle$ there is a A-sequence F of ultrafilters such that the balanced pairs $\langle Q, \sigma\rangle$ and $\langle \mathrm{Coll}(\omega, 2^\omega), \tau_F\rangle$ are equivalent;*

3. *distinct A-sequences of ultrafilters yield inequivalent balanced pairs.*

In particular, the poset P_A is balanced.

Proof. Write $P = P_A$. For the first item, it is necessary to argue that in fact $\mathrm{Coll}(\omega, \tau_F) \Vdash \tau_F \neq 0$. To see this, work in the $\mathrm{Coll}(\omega, 2^\omega)$ extension, for each number $n \in \omega$ let $D_n = \mathrm{dom}(F_n)$ and fix an enumeration $\langle a^i_n : i \in \omega\rangle$ of all elements of F_n. By induction on $n \in \omega$, repeatedly using item (2) of Definition 7.3.3 build sets $d_n \in D_n$ such that for every $i \in n$ the set $d_n^{A_i}$ is a subset of $\bigcap_{j\in n} a^j_i$. It is immediate that the condition $p = \bigcup_n d_n \in P$ is an element of τ_F.

For the balance, suppose that R_0, R_1 are posets and σ_0, σ_1 are their respective names for conditions in P such that R_0 forces that for every $n \in \omega$ and every set $a \in F_n$, $\sigma_0^{A_n} \subset a$ modulo finite holds, and similarly for R_1, σ_1. We have to show that $R_0 \times R_1$ forces the conditions σ_0, σ_1 to be compatible in P, which is the same as to say that $R_0 \times R_1$ forces the intersection $\sigma_0 \cap \sigma_1$ to contain a copy of A_n for every number $n \in \omega$. To this end, note that the sets $\sigma_0^{A_n}$ and $\sigma_1^{A_n}$ are forced to diagonalize the ultrafilter F_n. By Proposition 7.0.2, their intersection is forced to be infinite, in particular nonempty.

For the second item, we use a simple claim.

Claim 7.3.5. *Let $p \in P$ be a condition, $n \in \omega$ be a number, and let $a \subset D_n$ be a set. Then there is $q \leq p$ such that either $q^{A_n} \subset a$ or $q^{A_n} \cap a = 0$.*

Proof. For each $k \in \omega$ find a number $m_k \in \omega$ such that $A_{m_k} \to (A_k)^{A_n}_2$. Find pairwise disjoint finite sets $d_k \subset p$ in D_{m_k} for each $k \in \omega$. Use the partition property to find sets $d'_k \subset d_k$ such that $d'_k \in D_k$ and $(d'_k)^{A_n}$ is either a subset of a or disjoint from a. One of the options prevails for infinitely many numbers k. For definiteness, assume that the set $b = \{k \in \omega : (d'_k)^{A_n} \subset a\}$ is infinite. Let $q = \bigcup_{k\in b} d'_k$ and observe that $q \leq p$ works. □

Now let $\langle Q, \sigma\rangle$ be a balanced pair. Replacing Q with $Q \times \mathrm{Coll}(\omega, 2^\omega)$ and repeatedly strengthening p using the claim, we may assume that for every $n \in \omega$ and every set $a \subset D_n$, σ^{A_n} is forced to be either a subset of a or disjoint from a modulo finite. By a balance argument, it must be the case that either $Q \Vdash \sigma^{A_n} \subset \check{a}$ modulo finite, or $Q \Vdash \sigma^{A_n} \cap \check{a} = 0$ modulo finite. Let F_n be the collection of all sets $a \subset D_n$ for which the former option prevails. It is clear that F_n is an ultrafilter and that $F = \langle F_n : n \in \omega\rangle$ is an A-sequence. Since $Q \times \mathrm{Coll}(\omega, 2^\omega) \Vdash \sigma \in \tau_F$, by the first item and Proposition 5.2.4 it must be the case that the balanced pairs $\langle Q, \sigma\rangle$ and $\langle \mathrm{Coll}(\omega, 2^\omega), \tau_F\rangle$ are equivalent balanced pairs.

The third item is immediate. The last sentence of the theorem is not an entirely formal consequence of the previous work. It needs the following application of the axiom of choice.

Claim 7.3.6. *Every A-sequence of filters can be extended to a A-sequence of ultrafilters.*

Proof. Order the A-sequences of filters by coordinatewise inclusion: $F \leq G$ if for every $n \in \omega$, $F_n \subseteq G_n$ holds. It is immediate from the definitions that every $\leq$-chain has an upper bound–the coordinatewise union of the A-sequences of filters in the chain. By an application of Kuratowski–Zorn lemma, every A-sequence of filters can be extended into a $\leq$-maximal one. It only remains to prove that any $\leq$-maximal A-sequence of filters is in fact a sequence of ultrafilters.

Let F be a $\leq$-maximal A-sequence of filters, let $n \in \omega$ be a number, and let $b \subset D_n$ be a set; we claim that either b or its complement belongs to F_n. Suppose towards a contradiction that neither is the case. By the maximality of F it must be the case that the sequences G^0, G^1 of filters obtained from F by adding b (or the complement of b, respectively) to the filter F_n are not A-sequences. This must be witnessed by some sets $\langle a_i^0 \colon i \in k\rangle$ and $\langle a_i^1 \colon i \in k\rangle$ for some number $k \in \omega$ respectively, and some number $m \in \omega$: these are sets in F_i such that there is no $d \in D_m$ such that for all $i \in k, i \neq n$ $d^{A_i} \subset a_i^0$ and $d^{A_n} \subset a_n^0 \cap b$, and there is no $d \in D_m$ such that for all $i \in k, i \neq n$ $d^{A_i} \subset a_i^1$ and $d^{A_n} \subset a_n^1 \setminus b$. Since F is a sequence of filters, for all $i \in k$ $a_i^0 \cap a_i^1 \in F_i$ holds. Let $m' \in \omega$ be a number such that $A_{m'} \to (A_m)^{A_n}$ holds. Since F is a A-sequence, there is $d' \in D_{m'}$ such that for all $i \in k$, $(d')^{A_i} \subset a_i^0 \cap a_i^1$ holds. Use the partition assumption to find a a set $d \subset d'$ in D_m such that either $d^{A_n} \subset b$ or $d^{A_n} \cap b = 0$. The condition q violates the choice of either the sets $\langle a_0^i \colon i \in k\rangle$ or the sets $\langle a_1^i \colon i \in k\rangle$. □

Now, suppose that $p \in P$ is a condition. Consider the sequence F of filters F_n generated by sets cofinite in p^{A_n}. Clearly, F is an A-sequence. By the claim, F can be extended to a A-sequence G of ultrafilters. Then $\langle \mathrm{Coll}(\omega, 2^\omega), \tau_G\rangle$ is a balanced pair which is easily seen to be below p as required. □

For every Ramsey sequence $A = \langle A_n \colon n \in \omega\rangle$, the poset P_A introduces an ultrafilter on the set D_0 as the set of all subsets $a \subset D_0$ such that for some condition $p \in P$ in the generic ultrafilter, $p^{A_0} \subset a$. Claim 7.3.5 together with an obvious density argument show that this formula indeed defines an ultrafilter U on D_0, which will be referred to as the P_A-*generic ultrafilter*. It is clear that the generic filter on P_A can be reconstructed from U, and therefore the potential for confusion is minimal. The combinatorial and Rudin–Keisler properties of the generic ultrafilter have been the main reason for the study of the partial orders of the type P_A [8]. It is not difficult to see that it is a P-point.

Example 7.3.7. Let A be the sequence $\langle A_n \colon n \in \omega\rangle$ where A_n is the linear order on n many elements. The Ramsey theorem implies that this is in fact a Ramsey sequence of structures. The P_A-generic ultrafilter is a P-point which is weakly Ramsey and not a Q-point [8, Theorem 4.9].

Example 7.3.8. Let $k \in \omega$ and let A be a sequence enumerating all ordered finite graphs which contain no clique of size k. The fact that this is a Ramsey sequence of structures follows from [72]. The P_A-generic ultrafilter is a P-point U such that $U \to (U, k)^2$ but not $U \to (U, k+1)^2$ [8, Theorem 4.11]

7.4 Semigroup ultrafilters

In this section, we show that in a natural forcing connected with a countable semigroup, the balanced conditions exist and are classified by idempotent ultrafilters. This connects the theory of balanced forcing extensions with Ramsey theory and dynamics.

Definition 7.4.1. Let Γ be a countable semigroup. The poset P_Γ is defined as follows. The conditions of P_Γ are elements of Γ^ω. The ordering is defined by $q \leq p$ if there are nonempty finite sets $a_n \subset \omega$ for all $n \in \omega$ such that $\max(a_n) < \min(a_{n+1})$ and $\Pi_{m \in a_n} p(m) = q(n)$, where the products are always taken in the increasing order.

It is not difficult to see that given a condition in P_Γ, shifting its entries to the left and/or changing finitely many entries does not change the separative quotient equivalence class of the condition. It follows that the separative quotient of P_Γ is a σ-closed poset. The purpose of the poset P_Γ is clear from the following definition and proposition.

Definition 7.4.2. Let A be a collection of subsets of Γ. A sequence $p \in P_\Gamma$ *diagonalizes* A if for every set $b \in A$ there is $n \in \omega$ such that for every nonempty finite set $a \subset \omega \setminus n$, $\Pi_{m \in a} p(m) \in b$ holds. The sequence $p \in P_\Gamma$ *sorts out* A if for every set $b \in A$ there is $n \in \omega$ such that for every nonempty finite set $a \subset \omega \setminus n$, $\Pi_{m \in a} p(m) \in b$ holds, or for every nonempty finite set $a \subset \omega \setminus n$, $\Pi_{m \in a} p(m) \notin b$ holds. If $b \in A$ and the former alternative occurs, we say that p *accepts* b, if the latter alternative occurs then p *declines* b.

Proposition 7.4.3. *For every condition $p \in P_\Gamma$ and every countable set $A \subset \mathcal{P}(\Gamma)$, there is a condition $q \leq p$ which sorts out A.*

Proof. Let $A = \{b_n : n \in \omega\}$. By induction on $n \in \omega$ build a descending sequence p_n of conditions in P_Γ such that $p = p_0$ and for all $n \in \omega$, for all nonempty finite sets $a \subset \omega$ it is the case that $\prod_{m \in a} p_{n+1}(m) \in b_n$ holds, or for all nonempty finite sets $a \subset \omega$ it is the case that $\prod_{m \in a} p_{n+1}(m) \in b_n$ holds. To perform the induction step, use Hindman's theorem on the partition π_n of nonempty finite subsets of ω defined by $\pi_n(a) = 1$ if $\prod_{m \in a} p_n(m) \in b_n$ holds.

In the end, let $q \in P_\Gamma$ be defined by $q(n) = p_n(n)$ and observe that the conclusion of the proposition is satisfied. □

Suppose that $G \subset P_\Gamma$ is a generic filter. By Proposition 7.4.3 and a density argument, the set $U \subset \mathcal{P}(\Gamma)$ of all sets $b \subset \Gamma$ such that some condition $p \in G$ accepts b, is an ultrafilter. Also, G can be recovered from U as the set of all conditions $p \in P_\Gamma$ such that for all $n \in \omega$, the set $\{\prod_{m \in a} p(m) : a \subset \omega$ is a nonempty finite set and $\min(a) > n\}$ belongs to the ultrafilter U.

It turns out that the balanced conditions in the poset P_Γ correspond to a well-known concept from topological dynamics and Ramsey theory.

Definition 7.4.4. [40, Section 4.1] Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup.

1. For ultrafilters U_0, U_1 on Γ, let $U_0 \cdot U_1 = \{A \subset \Gamma \colon \{\gamma \in \Gamma \colon \{\delta \in \Gamma \colon \gamma \cdot \delta \in A\} \in U_1\} \in U_0\}$;
2. an ultrafilter U on Γ is an *idempotent* if $U \cdot U = U$.

It turns out that $\cdot$ is a semi-continuous operation on the space $\beta\Gamma$ of all ultrafilters on Γ. The fundamental Ellis–Numakura theorem [40, Section 2.4] says that every closed subsemigroup of $\beta\Gamma$ contains an idempotent ultrafilter. The idempotent ultrafilters have an alternative diagonalization description.

Proposition 7.4.5. *Let U be an ultrafilter on Γ. The following are equivalent:*

1. *in* $\mathrm{Coll}(\omega, \mathcal{P}(\Gamma))$ *extension there is a condition in P_Γ diagonalizing U;*
2. *for every countable subset of U, there is a condition in P_Γ diagonalizing it;*
3. *U is an idempotent.*

Proof. For the implication (1)$\to$(2), if $U' \subset U$ is a countable set, then in some forcing extension there is a condition $p \in P$ which diagonalizes U and therefore U'. By a Mostowski absoluteness argument between the ground model and the forcing extension, there must be a condition $p \in P$ diagonalizing U'.

For the implication (2)$\to$(1), consider the partial order $Q = [\mathcal{P}(U)]^{\aleph_0}$ modulo the nonstationary ideal. Let $G \subset Q$ be a filter generic over V and let $j \colon V \to M$ be the generic ultrapower. Then M is an ω-model which is possibly illfounded. Moreover, M contains the set $j''U$ as the equivalence class of the identity function on $[\mathcal{P}(U)]^{\aleph_0}$, and $M \models j''U \subset U$ is a countable set by the Łoś theorem. Thus, if the ground model satisfies (2), then by elementarity M (and so $V[G]$) contains a condition $p \in P$ which diagonalizes $j''U$ and therefore U. The existence of such a condition then transfers to any $\mathrm{Coll}(\omega, \mathcal{P}(\Gamma))$ extension of the ground model. This confirms (1).

(3) implies (2): this is the content of Galvin–Glazer theorem or its folkloric variation for countable sets, [40, Chapter 5]. Finally, the negation of (3) implies the negation of (2). To see this, if U is not an idempotent, there must be a set $A \in U$ such that $A \notin U \cdot U$. This means that the set $B = \{\gamma \in \Gamma \colon \{\delta \in \Gamma \colon \gamma \cdot \delta \notin A\} \in U\}$ belongs to U. Consider the countable set including A, B, as well as all the sets $C_\gamma = \{\delta \in \Gamma \colon \gamma \cdot \delta \notin A\}$ for $\gamma \in B$; we will show that it cannot be diagonalized. Suppose towards a contradiction that some condition $p \in P_\Gamma$ does diagonalize it. This means that there is $m_0 \in \gamma$ such that for all $k > m \geq m_0$, $p(m) \in B$ and $p(m) \cdot p(k) \in A$. There must be also some $k_0 \in \omega$ such that for all $k \geq k_0$, $p(k) \in C_{p(m_0)}$. Then $p(m_0) \cdot p(k_0)$ should belong simultaneously to A and to the complement of A, which is a contradiction. $\square$

Finally, the statement of the classification theorem for the balanced virtual conditions in the poset P_Γ is at hand.

Definition 7.4.6. Whenever $\langle \Gamma, \cdot \rangle$ is a countable semigroup and U is an ultrafilter on Γ, let τ_U denote the $\mathrm{Coll}(\omega, \mathcal{P}(\Gamma))$-name for the (nonempty analytic) set of all conditions diagonalizing U.

Theorem 7.4.7. *Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup.*

1. *Whenever U is an idempotent ultrafilter then $\langle \mathrm{Coll}(\omega, \mathcal{P}(\Gamma)), \tau_U \rangle$ is a balanced virtual condition;*
2. *every balanced pair is equivalent to $\langle \mathrm{Coll}(\omega, \mathcal{P}(\Gamma)), \tau_U \rangle$ for some idempotent ultrafilter U;*
3. *distinct idempotent ultrafilters give rise to inequivalent balanced virtual conditions.*

In particular, the poset P_Γ is balanced.

Proof. For (1), note that by Proposition 7.4.5, τ_U is a name for a nonempty set jsut in case U is an idempotent ultrafilter. (1) then follows immediately from the following claim:

Claim 7.4.8. *Let U be an ultrafilter on Γ and $\langle Q_0, \tau_0 \rangle$, $\langle Q_1, \tau_1 \rangle$ are posets and their respective names for elements of P_Γ diagonalizing U. Then $Q_0 \times Q_1 \Vdash \tau_0, \tau_1$ are compatible in P_Γ.*

Proof. Let $q_0 \in Q_0$ and $q_1 \in Q_1$ be conditions and $n \in \omega$ be a natural number. We must find an element $\gamma \in \Gamma$ and conditions $q_0' \leq q_0$ and $q_1' \leq q_1$ and finite nonempty sets $a_0, a_1 \subset \omega$ with minimum larger than n such that $q_0' \Vdash \check{\gamma} = \Pi_{m \in a_0} \tau_0(m)$ and $q_1' \Vdash \check{\gamma} = \Pi_{m \in a_1} \tau_1(m)$. The compatibility of τ_0, τ_1 is then granted by a genericity argument.

Let $A_0 \subset \Gamma$ be the set of all $\gamma \in \Gamma$ such that there exists a finite nonempty set $a \subset \omega$ with minimum greater than n and a condition $q' \leq q_0$ in Q_0 forcing $\check{\gamma} = \Pi_{m \in a} \tau_0(m)$. Since τ_0 is forced to diagonalize U, it must be the case that $A_0 \in U$. Similarly, let $A_1 \subset \Gamma$ be the set of all $\gamma \in \Gamma$ such that there exists a finite nonempty set $a \subset \omega$ with minimum greater than n and a condition $q' \leq q_1$ in Q_1 forcing $\check{\gamma} = \Pi_{m \in a} \tau_1(m)$. Since τ_1 is forced to diagonalize U, it must be the case that $A_1 \in U$. Choose $\gamma \in A_0 \cap A_1$ and choose $q_0' \leq q_0$, $q_1' \leq q_1$ and sets a_0, a_1 witnessing the membership of γ in A_0, A_1. This completes the proof. $\square$

For (2), suppose that $\langle Q, \tau \rangle$ is a balanced pair. Without loss of generality we may assume that Q collapses the size of $\mathcal{P}(\Gamma) \cap V$ to $\aleph_0$. Strengthening τ in the Q-extension repeatedly by an application of Proposition 7.4.3, we may assume that τ sorts out $\mathcal{P}(\Gamma) \cap V$. Let σ be the Q-name for the collection of all sets in $\mathcal{P}(\Gamma) \cap V$ which τ accepts.

Claim 7.4.9. *The membership of every set $A \subset \Gamma$ in σ is decided by the largest condition in Q.*

Proof. If not, then there is a set $A \subset \Gamma$ and conditions $q_0, q_1 \in Q$ such that $q_0 \Vdash \check{A} \in \sigma$ and $q_1 \Vdash \Gamma \setminus \check{A} \in \sigma$. Plainly, the condition $\langle q_0, q_1 \rangle$ forces in the product $Q \times Q$ that $\tau_{\text{left}}, \tau_{\text{right}}$ are conditions incompatible in P_Γ, contradicting the balance assumption on the name τ. $\square$

Let $U = \{A \subset \Gamma \colon Q \Vdash \check{A} \in \sigma\}$. This is an ultrafilter on Γ. By Proposition 7.4.5, it is an idempotent ultrafilter. By Claim 7.4.8, the pair $\langle Q, \tau \rangle$ is equivalent to $\langle \mathrm{Coll}(\omega, \mathcal{P}(\Gamma)), \tau_U \rangle$. (2) follows.

(3) is immediate. To prove the last sentence, note that for every condition $p \in P_\Gamma$, the Ellis–Numakura theorem yields an idempotent ultrafilter U such that for all $n \in \omega$, the set $\{\prod_{m \in a} p(m) \colon a \subset \omega$ is a nonempty finite set and $\min(a) > n\}$ belongs to U. Then $\mathrm{Coll}(\omega, \mathcal{P}(\Gamma)) \Vdash \Sigma\tau_U \leq p$ by the definition of the ordering P_Γ and so $\langle \mathrm{Coll}(\omega, \mathcal{P}(\Gamma)), \tau_U \rangle$ is a balanced virtual condition below p. $\square$

Example 7.4.10. Let Γ be the group of finite subsets of ω with the symmetric difference operation. Consider the poset P_Γ below the natural initial condition $p \in \Gamma^\omega$ which assigns the singleton set $\{n\}$ to each number $n \in \omega$. The generic ultrafilter added by the poset is known as the stable ordered union ultrafilter as introduced in [10]. The stable union ultrafilters are studied in many places in the literature, including the study of the P_Γ-extension of the symmetric Solovay model [22].

Chapter 8

Other forcings

In this chapter we gather partial orders which do not fit in the previous chapters.

8.1 Chromatic numbers

Given a Polish space X, a Borel hypergraph $\Gamma \subset [X]^{<\aleph_0}$, one can consider the task of forcing a Γ-coloring $c\colon X \to \omega$. In principle, this is a very difficult task and we do not have a general way of resolving it. The first poset one can consider is the poset $P_{\mathcal{K}}$ where $\mathcal{K}$ is the Borel simplicial complex on $X \times \omega$ consisting of finite partial Γ-colorings. The basic disadvantage of this poset is that not all maximal $\mathcal{K}$-sets are total Γ-colorings, and there are elements of $P_{\mathcal{K}}$ which cannot be extended to total Γ-colorings. Thus, we have to restrict the posets $P_{\mathcal{K}}$ in some way to get the desired effect. This is not always straightforward, and the resulting posets are usually not simplicial complex posets in the sense of Definition 6.1.1. The most natural attempt is the following.

Definition 8.1.1. Let X be a Polish space and let $\Gamma \subset [X]^{<\aleph_0}$ be a Borel graph on a Polish space X. The Γ-*coloring forcing* P_Γ is the partially ordered set of all partial countable functions $p\colon X \to \omega$ which can be extended to a Γ-coloring of the whole space X. The ordering is that of reverse inclusion.

It is immediate that the poset P_Γ, if nonempty, forces the union of the conditions in the generic filter to be a total Γ-coloring with countably many colors. However, it is not at all clear whether the poset P_Γ is Suslin, σ-closed, or balanced. In an interesting large class of Borel graphs Γ, these questions have a smooth resolution. Recall that a graph Γ on vertex set X has *coloring number* κ if there is a well-ordering $\prec$ of X such that for each vertex $x \in X$ the set $\{y \in X\colon y \prec x \text{ and } y\ \Gamma\ x\}$ has cardinality smaller than κ [25]. This may seem like a very complex notion especially in the case of infinite cardinals κ, but in fact there is a simple characterization of analytic graphs of uncountable coloring number [1].

Theorem 8.1.2. *Let* Γ *be a Borel graph on a Polish space* X *with countable coloring number. Then* P_Γ *contains a dense subset which is a Suslin* $\aleph_0$*-distributive poset. Moreover,*

1. *for every total* Γ*-coloring* $c\colon X \to \omega$ *the pair* $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ *is balanced;*
2. *for every balanced pair* $\langle Q, \tau\rangle$ *there is a total coloring* c *such that* $\langle Q, \tau\rangle$ *is equivalent to* $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$*;*
3. *distinct colorings yield inequivalent balanced conditions.*

In particular, the poset P_Γ *is balanced.*

Proof. Fix a well-ordering $\prec$ on the space X such that for every element $x \in X$, the set $\{y \in X : y \prec x \text{ and } \{x, y\} \in \Gamma\}$ is finite. We start with a humble observation which will be used in several places below.

Claim 8.1.3. *Let* M *be a countable elementary submodel of a large countable structure containing* $\prec$*. Then every* $y \in X \setminus M$ *is* Γ*-connected with at most finitely many elements of* $X \cap M$*.*

Proof. Suppose that $y \in X \setminus M$ is a point. Then for every $x \in M$ $\prec$-greater than y, $\{x, y\} \notin \Gamma$ holds: if $\{x, y\} \in \Gamma$ then y belongs to the finite set of all points $\prec$-smaller than x and connected with x and by elementarity it would have to belong to the model M, which is not the case. Thus, the only points of M connected to y in Γ are $\prec$-smaller than y, and there are only finitely many points like that by the definitory property of the ordering $\prec$. □

Let P be the poset of all partial countable Γ-colorings $p\colon X \to \omega$ such that for all points $y \in X \setminus \mathrm{dom}(p)$, the set $\{x \in \mathrm{dom}(p) : \{x, y\} \in \Gamma\}$ is finite. The ordering on P is that of reverse extension. The following two claims show that P has the properties required in (1).

Claim 8.1.4. *P is a Suslin* $\aleph_0$*-distributive poset.*

Proof. We first need to show that the set of all (enumerations of) conditions in P is Borel. Use Claim 8.1.3 to conclude that for every countable set $a \subset X$, the set $b = \{y \in X : \exists^\infty x \in a\ \{x, y\} \in \Gamma\}$ is countable. By the Lusin–Novikov theorem, there are Borel functions $\{f_i : i \in \omega\}$ from X^ω to X such that for every $z \in X^\omega$, the list $\{f_i(z) : i \in \omega\}$ includes all points in X which are Γ-connected with infinitely many points of $\mathrm{rng}(z)$. Now we can evaluate the complexity of the set of enumerations of conditions in the poset P. A function $r\colon \omega \to X \times \omega$ is an enumeration of a condition in P just in case when $\mathrm{rng}(r)$ is a function and a Γ-coloring, and, writing $s\colon \omega \to X$ for the first component function of r, for every $i \in \omega$ if $f_i(s)$ is Γ-connected to infinitely many points in $\mathrm{rng}(s)$, then $f_i(s) \in \mathrm{rng}(s)$. This is a Borel condition.

To show that the poset P is Suslin, note that two conditions $p_0, p_1 \in P$ are compatible just in case they are compatible as functions: then $p_0 \cup p_1$ will be their lower bound. To argue for the $\aleph_0$-distributivity, let $p \in P$ be a condition and let $\{D_i : i \in \omega\}$ be a countable collection of open dense subsets of P. To find a condition $q \leq p$ in the intersection $\bigcap_i D_i$, let M be a countable elementary submodel of a large structure containing $p, \Gamma, \prec$ and $\{D_i : i \in \omega\}$ and use the elementarity of M to build a sequence p_i for $i \in \omega$ by recursion so that

- $p = p_0 \geq p_1 \geq p_1 \geq \dots$ are all conditions in M;
- $p_{i+1} \in D_i$ and $\mathrm{dom}(p_{i+1})$ contains the i-th element of $M \cap X$ in some fixed enumeration.

In the end, let $q = \bigcup_i p_i$ and use Claim 8.1.3 to argue that $q \in P$ as desired. □

Claim 8.1.5. *P is a dense subset of P_Γ.*

Proof. It is first necessary to show that P is a subset of P_Γ. To this end, let $p \in P$. Let $\{b_k : k \in \omega\}$ be a collection of pairwise disjoint infinite subsets of ω and $c : X \to \omega$ be a total Γ-coloring. Let $d : X \to \omega$ be the function defined as follows: $d(x) = p(x)$ if $x \in \mathrm{dom}(p)$, and otherwise let $d(x)$ be the smallest number in $b_{c(x)}$ which is different from the finitely many numbers $\{p(y) : y \in \mathrm{dom}(p)$ and $\{x, y\} \in \Gamma\}$. It is not difficult to verify that d is a total Γ-coloring extending p, and so $p \in P_\omega(\Gamma)$.

For the density, suppose that $p \in P_\Gamma$ is a condition. Let $c : X \to \omega$ be a total Γ-coloring extending p. Let M be a countable elementary submodel of a large structure containing p, c, and $\prec$. By Claim 8.1.3, the set $\{y \in X : \exists^\infty x \in M\ \{x, y\} \in \Gamma\}$ is a subset of M. Thus, the function $q = c \restriction M$ is a condition in P extending p as desired. □

For the classification of the balanced virtual conditions, we need a claim:

Claim 8.1.6. *Let R_0, R_1 be posets and η_0, η_1 be respective R_0- and R_1-names for elements of $X \setminus V$. Then*

1. *R_0 forces η_0 to be Γ-connected with at most finitely many elements of the ground model;*
2. *$R_0 \times R_1 \Vdash \{\eta_0, \eta_1\} \notin \Gamma$.*

Proof. For the first item, suppose towards a contradiction that this fails as forced by some condition $r \in R$. Let M be a countable elementary submodel of a large structure containing all relevant information, let $g \subset R_0 \cap M$ be a filter generic over the model M, and let $y = \eta_0/g \in X$. By the forcing theorem, $M[g]$ satisfies that y is Γ-connected with infinitely many elements of $X \cap M$. By the Mostowski absoluteness for the model $M[g]$, y is indeed connected with infinitely many elements of M, and this contradicts the conclusion of Claim 8.1.3.

For the second item, suppose towards a contradiction that this fails as forced by some condition $\langle r_0, r_1 \rangle$ in the product. Let $M \in N$ be countable elementary submodels of a large structure containing all relevant information. In the model N, there are filters $\{g_i : i \in \omega\}$ on the poset $R_0 \cap M$ which are mutually generic over the model M. By a mutual genericity argument applied in M, the points

$x_i = \eta_0/g_i \in X$ are pairwise distinct elements of N. Now, let $h \subset R_1 \cap M$ be a filter generic over the model N containing the condition r_1 and let $y = \eta_1/h \in X \setminus N$. By the forcing theorem in M, it is the case that $M[G_i, h] \models \{x_i, y\} \in \Gamma$ for each $i \in \omega$. By the Mostowski absoluteness for the models $M[g_i, h]$, it is in fact the case that y is Γ-connected with each x_i for $i \in \omega$. This, however, contradicts Claim 8.1.3 applied to N. $\square$

Now, for item (1) of the theorem, let $c\colon X \to \Gamma$ be a total Γ-coloring. Note that $\mathrm{Coll}(\omega, X) \Vdash \check{c} \in P$ by Claim 8.1.6(1). To see that the pair $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ is balanced, suppose that R_0, R_1 are posets and σ_0, σ_1 are R_0- and R_1-names for elements of P extending c; we must show that the product $R_0 \times R_1$ forces σ_0, σ_1 to be compatible in P; in other words, $\sigma_0 \cup \sigma_1$ is a function and a Γ-coloring. To verify that $\sigma_0 \cup \sigma_1$ is a function, use the product forcing theorem to conclude that $\mathrm{dom}(\sigma_0) \cap \mathrm{dom}(\sigma_1)$ is forced to be a subset of the ground model, and both $\sigma_0 \restriction V$ and $\sigma_1 \restriction V$ are forced to be equal to $\check{c}$. To see that $\sigma_0 \cup \sigma_1$ is a Γ-coloring, use Claim 8.1.3(2). This completes the proof of (1).

To argue for item (2), suppose that $\langle Q, \tau\rangle$ is a balanced pair. Strengthening τ if necessary we may assume that $Q \Vdash X \cap V \in \mathrm{dom}(p)$. By a balance argument, for each $x \in X$ there must be a specific number $c(x) \in \omega$ such that $Q \Vdash \tau(\check{x}) = c(x)$. It is not difficult to check that $c\colon X \to \omega$ is a total Γ-coloring and $Q \times \mathrm{Coll}(\omega, X) \Vdash \tau \leq \check{c}$. (2) then follows from Proposition 5.2.4.

(3) is immediate. The last sentence now easily follows: if $p \in P_\Gamma$ is a condition then by the definition of P_Γ it can be extended to a total Γ-coloring $c\colon X \to \omega$ which then yields a balanced virtual condition below p by (1). $\square$

Example 8.1.7. Let $n \leq 3$ be a number, let $D \subset \mathbb{R}$ be a countable set of positive reals converging to 0, and let Γ_{Dn} be the graph connecting two points of $\mathbb{R}^n$ if their distance belongs to D. The graph Γ_{Dn} has countable coloring number by [58, Theorem 7], and therefore the poset $P_{\Gamma_{Dn}}$ is balanced.

Example 8.1.8. Let $D \subset \mathbb{R}$ be a countable set of positive reals and consider the graph Γ_D connecting two points of the plane if their distance belongs to D. The graph Γ_D does not contain 2 times uncountable as a subgraph; by [25] it must have countable coloring number and so the poset P_{Γ_D} is balanced. If D is the set of all positive rationals then the graph contains infinite cliques and so its chromatic number is infinite. If D is an algebraically independent set, then it is not known whether the chromatic number is finite or infinite, and Bukh [14] conjectured the former.

Chromatic numbers of hypergraphs offer additional challenges. We provide elegant coloring posets for two natural classes of hypergraphs. They are less canonical than the rest of the posets in this book as they need a choice of an auxiliary tool: a Borel ideal J on ω containing all finite sets and such that J is not generated by a countable subset of J. There are many such ideals, one possible choice is the summable ideal.

Definition 8.1.9. Let X be a Borel abelian group. Let Φ be a countable field whose multiplicative group acts on X in a Borel way, making X into a vector space. The 3-*Hamel hypergraph* is the set $\Gamma \subset [X]^3$ consisting of triples of linearly dependent nonzero elements of X. A Γ-coloring $c\colon X \to \omega$ is called an 3-*Hamel decomposition.*

Proposition 8.1.10. *Let X be a Borel vector space over a countable field Φ. Then there is a 3-Hamel decomposition of X.*

Proof. Let $A \subset X$ be a Hamel basis. Let $\mathcal{B}$ a countable basis of the topology of the ambient Polish space of which X is a Borel subset. We define a Γ-coloring whose range is the countable set $C = (\Phi \times \mathcal{B})^{<\omega}$. For each nonzero element $x \in X$, let $x = \Sigma_{i \in n} \phi_i y_i$ be the unique expression of x as a linear combination of basis vectors with nonzero coefficients, in which the basis vectors are listed in an increasing order with respect to some fixed Borel linear ordering of X. Let $\langle O_i \colon i \in n\rangle$ be the first sequence (in some fixed enumeration) of pairwise disjoint basic open sets in the basis $\mathcal{B}$ such that $y_i \in O_i$ holds for every $i \in n$. Define the coloring $c\colon X \to C$ by setting $c(0) = 0$ and $c(x) = \langle \phi_i, O_i \colon i \in n\rangle$ where ϕ_i, O_i are as above.

We must show that there is no monochromatic Γ-edge. Suppose towards a contradiction that $x^0, x^1, x^2 \in X$ are distinct nonzero elements of X, all with the same color $\langle \phi_i, O_i \colon i \in n\rangle$, such that some nontrivial linear combination of them yields zero. Let $y_i^0 \in O_i$ be the basis vectors yielding x^0, similarly for y_i^1 and y_i^2. Since the three vectors are distinct, there must be $i \in n$ and $j \in 3$ such that y_i^j is distinct from y_i^k for all $k \in 3$ different from j. But then, the basis vector y_i^j can be expressed as a linear combination of all the remaining basis vectors used to obtain the points x^0, x^1, x^2. Since the remaining vectors are all distinct from y_i^j, this is a contradiction. □

We now provide a balanced forcing which adds a 3-Hamel coloring to a given Borel vector space over a countable field.

Definition 8.1.11. Let X be a Borel vector space over a countable field Φ. The 3-*Hamel decomposition forcing* P is the set of all functions p such that $\mathrm{dom}(p)$ is a countable subspace of X, $\mathrm{rng}(p) \subset \omega$, and p is a Γ-coloring on $\mathrm{dom}(p)$ where Γ is the 3-Hamel hypergraph. The ordering is defined by $q \leq p$ if $p \subset q$, and for every $\mathrm{dom}(p)$-coset $e \subset \mathrm{dom}(q)$ distinct from $\mathrm{dom}(p)$, the image $q'' \bigcup_{\phi \in \Phi} \phi \cdot e$ belongs to the ideal J.

It is not difficult to see that P is a σ-closed Suslin poset. We now classify its balanced conditions.

Theorem 8.1.12. *Let X be a Borel vector space over a countable field Φ. Let Γ be the 3-Hamel hypergraph and let P be the 3-Hamel decomposition forcing.*

1. *If $c\colon X \to \omega$ is a Γ-coloring, the pair $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ is balanced;*
2. *if $\langle Q, \tau\rangle$ is a balanced pair, then there is a Γ-coloring $c\colon X \to \omega$ such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ are equivalent;*
3. *distinct Γ-colorings yield inequivalent balanced pairs.*

In particular, P is balanced.

Proof. For (1), let $c\colon X \to \omega$ be a Γ-coloring. Let $V[G_0], V[G_1]$ be mutually generic extensions in which the set $X \cap V$ is countable and let $p_0 \in V[G_0]$, $p_1 \in V[G_1]$ be conditions below c; we must produce a lower bound $q \leq p_0, p_1$ in the poset P in the model $V[G_0, G_1]$. Write $d_0 = \mathrm{dom}(p_0$ and $d_1 = \mathrm{dom}(p_1)$. For every point $x \in X \setminus (d_0 \cup d_1)$, write $e_0(x) = \{y_0 \in d_0 \colon \exists y_1 \in d_1\ x$ is a linear combination of $y_0, y_1\}$ and $e_1(x) = \{y_1 \in d_1 \colon \exists y_0 \in d_0\ x$ is a linear combination of $y_0, y_1\}$. The following claim is key.

Claim 8.1.13. *Let $x \in X \setminus (d_0 \cup d_1)$ be any point. If the set $e_0(x)$ is nonempty, then there is a V-coset f distinct from V such that $e_0 = \bigcup_{\phi\in\Phi} \phi\cdot f$, and similarly for $e_1(x)$.*

Proof. Fix the point x and write $e_0(x) = e_0$ and $e_1(x) = e_1$. We argue for e_0, the case of e_1 is symmetric. First note that if $y_0 \in e_0$ then y_0 cannot belong to V, as in such a case x would be a linear combination of elements of d_1 and so an element of d_1, contradicting the initial choice of x.

It is clear that the set e_0 is closed under scalar multiplication and addition of vectors in V. To show that e_0 is obtained from a single V-coset, suppose that $y_0, z_0 \in e_0$ are distinct points; we must show that there is a linear combination of y_0, z_0 which yields a point in V. Then there must be points $y_1, z_1 \in e_1$ and field elements $\phi_0, phi_1, \psi_0, \psi_1$ such that $x = \phi_0 y_0 + \phi_1 y_1$ and $x = \psi_0 z_0 + \psi_1 z_1$. It follows that $\phi_0 y_0 - \psi_0 z_0 = \psi_1 z_1 - \phi_1 y_1$. The left-hand side of the equation belongs to $V[G_0]$, the right hand side belongs to $V[G_1]$, and by the product forcing theorem, their common value v must belong to V. But then $v = \phi_0 y_0 - \psi_0 z_0$ is the desired linear combination. $\square$

Now, let d be any countable subspace of X containing both d_0, d_1 as subsets. Write $a = d \setminus (d_0 \cup d_1)$. Let I be the set $\{p_0'' e_0(x) \colon x \in a\}$ united with $\{p_1'' e_1(x) \colon x \in a\}$. By the claim and the definition of the ordering on P, I is a countable set of sets in the ideal J. By the initial assumptions on the ideal J, there is a set $b \in J$ which cannot be covered by finitely many elements of I and a finite set. The set a can be easily decomposed into infinitely many sets with the same property, and so there are pairwise disjoint sets $b(x) \subset \omega$ for $x \in a$ such that none of them can be covered by finitely many elements of I and a finite set, and such that $\bigcup_x b(x) \in J$. Let $q\colon d \to \omega$ be any function such that $p_0, p_1 \subset q$, and for every point $x \in a$ we have $q(x) \in b(x) \setminus (p_0'' e_0(x) \cup p_1'' e_1(x))$. We claim that $q \leq p_0, p_1$ is the desired lower bound.

To see that $q \in P$, we must show that q is a Γ-coloring. Suppose towards a contradiction that $\{x_0, x_1, x_2\} \subset d$ is a monochromatic triple of nonzero linearly dependent pairwise distinct points. The triple cannot be a subset of $d(p_0)$ or $d(p_1)$ since p_0, p_1 are Γ-colorings. The triple cannot use more than one point in a since $q \restriction a$ is an injection. We are left with only one case, where (after re-indexing if necessary) $x_2 \in a$ and $x_0 \in d_0$, $x_1 \in d_1$. In such a case, $x_0 \in e_0(x)$ and $x_1 \in d_1(x)$ holds and by the initial choice of the function q, $q(x_2)$ is distinct from both $q(x_0)$ and $q(x_1)$.

To see that $q \leq p_0$, let e be a d_0-coset distinct from d_0; we must show that the set $q'' \bigcup_{\phi\in\Phi} \phi \cdot e$ belongs to J. By the product forcing theorem, $e \cap d_1$ is in fact a V-coset distinct from V, and so $q'' \bigcup_{\phi\in\Phi} \phi \cdot e \cap d_1 = p_1'' \bigcup_{\phi\in\Phi} \phi \cdot e \cap d_1 \in J$ since $p_1 \leq c$. Thus, the set $q'' \bigcup_{\phi\in\Phi} \phi \cdot e$ is covered by the union of the J-small sets $q'' \bigcup_{\phi\in\Phi} \phi \cdot e \cap d_1$ and b and therefore belongs to J. The case $q \leq p_1$ is symmetric.

For (2), suppose that $\langle Q, \tau\rangle$ is a balanced pair. Strengthening τ if necessary, we may assume that $Q \Vdash (X \cap V) \subset \mathrm{dom}(\tau)$. By a balance argument, for each point $x \in X \cap V$ there is a number $c(x) \in \omega$ such that $Q \Vdash \tau(\check{x}) = c(x)$. It is immediately clear that the map c is a Γ-coloring. We will show that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ are equivalent. To this end, it is enough to argue that $Q \Vdash \tau \leq \check{c}$. Suppose towards a contradiction that some condition $q \in Q$ forces the opposite. Let $G_0, G_1 \subset Q$ be mutually generic filters meeting the condition q, and write $p_0 = \tau/G_0$ and $p_1 = \tau/G_1$. To reach the contradiction with the balance of the pair $\langle Q, \tau\rangle$, argue that p_0, p_1 are conditions incompatible in P. To see this, use the forcing theorem and the assumption that $p_0 \not\leq c$ to find a V-coset $e \subset \mathrm{dom}(p_0)$ distinct from V such that $p_0'' \bigcup_{\phi\in\Phi} \phi \cdot e \notin J$. Now, the set e is a subset of a single $\mathrm{dom}(p_1)$-coset distinct from $\mathrm{dom}(p_1)$ by the product forcing theorem. Therefore, it prevents finding a common lower bound of p_0, p_1.

(3) is clear. For the last sentence, we prove a claim which will come handy later.

Claim 8.1.14. *Let $p \in P$ be a condition and $a \subset \omega$ be an infinite set. Then there is a Γ-coloring $c \colon X \to \omega$ such that $p \subset c$ and all points not in* $\mathrm{dom}(p)$ *get a c-color from the set a.*

Proof. Use Proposition 8.1.10 to find a coloring $d \colon [X]^2 \to \omega$. Fix a collection $\{a_n \colon n \in \omega\}$ of pairwise disjoint infinite sets such that $\bigcup_n a_n \in a$. Consider the graph Δ on $X \setminus \mathrm{dom}(p)$ connecting points x, y if there is a point $z \in \mathrm{dom}(p)$ such that the set $\{x, y, z\}$ is linearly dependent. Note that Δ is a locally countable graph, and so its path-connectedness equivalence relation E has all classes countable. Therefore, one can find a function $c \colon X \to \omega$ such that $p \subset c$, for all $x \in X \setminus \mathrm{dom}(p)$ $c(x) \in a_{d(x)}$ holds, and c restricted to any E-class is an injection. To check that c is a Γ-coloring, assume that $\{x_0, x_1, x_2\} \in \Gamma$ is a triple of pairwise distinct points. If all three of them belong to $\mathrm{dom}(p)$ then the the triple is not monochromatic as p is a Γ-coloring. If two of them belong to $\mathrm{dom}(p)$ then so does the third one by the closure properties of $\mathrm{dom}(p)$ and we are in the previous case. If exactly one of the points belongs to $\mathrm{dom}(p)$ then the other two are E-related and assigned a different color by c. Finally, if none of the points are in $\mathrm{dom}(p)$ then the triple is not monochromatic since d is a Γ-coloring. □

Now, let $p \in P$ be a condition. Choose an infinite set $a \in J$ and find a Γ-coloring $c \colon X \to \omega$ such that $p \subset c$ and all points not in $\mathrm{dom}(p)$ get a c-color from the set a. Clearly, $\mathrm{Coll}(\omega, X) \Vdash \check{c} \leq \check{p}$ and so c is the desired balanced condition stronger than p. □

Example 8.1.15. Ceder [17] provided a countable subfield Φ of the complex plane $\mathbb{C}$ such that any 3-Hamel decomposition of the complex plane over Φ consists of pieces which contain no equilateral triangles.

A very similar poset can produce a decomposition of a vector space into countably many internally linearly independent sets.

Definition 8.1.16. Let X be a Borel vector space over a countable field Φ. The *Hamel hypergraph* is the set $\Gamma \subset [X]^{<\aleph_0}$ consisting of linearly dependent finite sets of nonzero elements of X. A Γ-coloring $c\colon X \to \omega$ is called an *Hamel decomposition.*

Proposition 8.1.17. *Let X be an uncountable Borel vector space over a countable field Φ. Then there is a Hamel decomposition of X if and only if the Continuum Hypothesis holds.*

Proof. Assume first that the Continuum Hypothesis holds. Exhaust X by an inclusion increasing sequence of countable subspaces, $X = \bigcup_{\alpha\in\omega_1} X_\alpha$. Let $c\colon X \to \omega$ be a map which is an injection on each set $X_\alpha \setminus \bigcup_{\beta\in\alpha} X_\beta$. We claim that c is a Hamel decomposition. Indeed, suppose that $a \subset X$ is a finite linearly dependent set. Fix a linear combination of elements of a adding up to 0. Let $\alpha \in \omega_1$ be the largest ordinal such that the set $X_\alpha \setminus \bigcup_{\beta\in\alpha} X_\beta$ contains some element of a with a nonzero coefficient in the linear combination. This element cannot be unique, since then it would be expressible as a linear combination of the subspace $\bigcup_{\beta\in\alpha} X_\beta$, an impossibility. So a contains at least two elements in the set $X_\alpha \setminus \bigcup_{\beta\in\alpha} X_\beta$; thus, a cannot be monochromatic, as c is injective on this set.

On the other hand, suppose that the Continuum Hypothesis does not hold, and let $c\colon X \to \omega$ is a map; we must produce a nontrivial monochromatic linear combination adding up to zero. Let M be an elementary submodel of cardinality $\aleph_1$ of a large structure, containing the map c. Use the failure of the Continuum Hypothesis to produce a point $x_0 \in X \setminus M$. Use a counting argument to find distinct points $y_0, y_1 \in X \cap M$ and a number $n \in \omega$ such that $c(x_0 - y_0) = c(x_0 - y_1) = n$. Use the elementarity of the model M to find a point $x_1 \in X \cap M$ such that $c(x_1 - y_0) = c(x_1 - y_1) = n$. Note that the points $z_0 = x_0 - y_0, z_1 = x_0 - y_1$ and $z_2 = x_1 - y_0, z_3 = x_1 - y_1$ are pairwise distinct, they attain the same color n, and $z_0 - z_1 - z_2 + z_3 = 0$. The proof is complete. □

Definition 8.1.18. Let X be a Borel vector space over a countable field Φ. The *Hamel decomposition forcing* P is the set of all functions p such that $\mathrm{dom}(p)$ is a countable subspace of X, $\mathrm{rng}(p) \subset \omega$, and p is a Γ-coloring on $\mathrm{dom}(p)$ where Γ is the Hamel hypergraph. The ordering is defined by $q \leq p$ if $p \subset q$, and there is no nontrivial linear combination of a q-monochromatic set elements of $\mathrm{dom}(q)$ with result in $\mathrm{dom}(p)$, and for every $\mathrm{dom}(p)$-coset $e \subset \mathrm{dom}(q)$ distinct from $\mathrm{dom}(p)$, the set $\{n \in \omega\colon$ there exists a linear combination of a q-monochromatic set with result in $\bigcup_{\phi\in\Phi} \phi \cdot e\}$ belongs to the ideal J.

It is not difficult to see that P is a σ-closed Suslin poset. We now classify its balanced conditions.

Theorem 8.1.19. *Let X be a Borel vector space over a countable field Φ. Let Γ be the Hamel hypergraph and let P be the Hamel decomposition forcing.*

1. *If $c\colon X \to \omega$ is a Γ-coloring, the pair $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ is balanced;*
2. *if $\langle Q, \tau\rangle$ is a balanced pair, then there is a Γ-coloring $c\colon X \to \omega$ such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ are equivalent;*
3. *distinct Γ-colorings yield inequivalent balanced pairs.*

In particular, the Continuum Hypothesis is equivalent to the statement that P is balanced.

As the proof of Theorem 8.1.19 is nearly literally the same as the one for Theorem 8.1.12, we dare to leave it to the patient reader.

Example 8.1.20. Let X be the (non-Polishable) group of finite subsets of an uncountable Polish space X with the symmetric difference operation. View X as a vector space over the binary field. Let $c\colon X \to \omega$ be a Hamel decomposition of X. The restriction $c \restriction [Y]^2 \to \omega$ is a decomposition of the clique graph on Y into countably many sets with no monochromatic cycle.

8.2 Discontinuous homomorphisms

Another task which can be handled by balanced posets derived from simplicial complexes is the adding of discontinuous homomorphisms between topological structures. In this section, we treat the case of Polish groups. The problem of the existence of discontinuous homomorphisms from one Polish group to another has been studied for decades. In ZFC, some groups such as the unitary group [95] or the group of homeomorphisms of the unit circle [77] cannot be homomorphically mapped to another Polish group in a discontinuous way. In a ZF+DC context, existence of discontinuous homomorphisms has interesting consequences: in general, it implies that the chromatic number of the Hamming graph $\mathbb{H}_2$ is 2 [76], for some specific groups it implies the existence of a nonprincipal ultrafilter [90, Theorem 4.1]. For an exposition, see [75]. In this section, we provide a balanced forcing adding a discontinuous homomorphism of Polish groups in a certain common special case.

Definition 8.2.1. Let Γ and Δ be Polish groups. The simplicial complex $\mathcal{K}(\Gamma, \Delta)$ on $X \times Y$ consists of all finite sets $a \subset X \times Y$ which are subsets of a homomorphism from X to Y. We will write $P(\Gamma, \Delta)$ for the associated Γ, Δ-*homomorphism* poset of countable $\mathcal{K}(\Gamma, \Delta)$-sets ordered by inclusion.

As it was the case in Section 8.1, it seems to be difficult to evaluate the complexity of the complex $\mathcal{K}(\Gamma, \Delta)$ without having additional information on the nature of the two groups. In this section, we will deal with a very special case which nevertheless resolves some interesting problems.

Theorem 8.2.2. *Let Γ, Δ be abelian Polish groups on respective Polish spaces X, Y. Suppose that Δ is divisible. Then the poset $P(\Gamma, \Delta)$ is Suslin and*

1. *for every homomorphism $h\colon \Gamma \to \Delta$, the pair $\langle \mathrm{Coll}(\omega, X \times Y), \check{h}\rangle$ is balanced;*
2. *for every balanced pair $\langle Q, \tau\rangle$ there is a homomorphism $h\colon \Gamma \to \Delta$ such that the pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X \times Y), \check{h}\rangle$ are equivalent;*
3. *distinct homomorphisms yield inequivalent balanced virtual conditions.*

In particular, the poset $P(\Gamma, \Delta)$ is balanced.

Proof. Write $\mathcal{K} = \mathcal{K}(\Gamma, \Delta)$ and $P = P(\Gamma, \Delta)$. For the Suslinness of P, it is enough to argue that the simplicial complex $\mathcal{K}$ is Borel. For that, note that a finite partial function $a \subset X \times Y$ belongs to $\mathcal{K}$ if and only if for every $\mathbb{Z}$-combinations $\Sigma_i n_i \gamma_i = \Sigma_j n_j \gamma_j$ of elements of P, it is the case that $\Sigma_i n_i a(\gamma_i) = \Sigma_j n_j a(\gamma_j)$. The left-to-right implication is clear, since the right hand side holds for every homomorphism from Γ to Δ. For the right-to-left implication, note that the assumption on the right hand side implies that a can be extended to a homomorphism from the subgroup generated by $\mathrm{dom}(a)$ to Δ, which then can be extended to a homomorphism of all of Γ to Δ by the divisibility assumption on Δ and Baer's criterion [3].

For (1), suppose that $V[H_0], V[H_1]$ are mutually generic extensions of the ground model V and $a_0, a_1 \in \mathcal{K}$ are finite functions in the respective models such that $a_0 \cup h$ and $a_1 \cup h$ are $\mathcal{K}$-sets. We need to show that $a_0 \cup a_1 \in \mathcal{K}$. Towards a contradiction, suppose that this fails, and write $a = a_0 \cup a_1$. By the work of the previous paragraph, there must be a $\mathbb{Z}$-combinations $\Sigma_i n_i \gamma_i = \Sigma_j n_j \gamma_j$ such that $\Sigma_i n_i a(\gamma_i) \neq \Sigma_j n_j a(\gamma_j)$. Note that elements of both a_0, a_1 must be used in these combinations. Rearranging the combinations, we may assume that the left combination uses only elements from a_0 and the right one uses only elements of a_1. By the product forcing theorem, $\gamma = \Sigma_i n_i \gamma_i = \Sigma_j n_j \gamma_j$ must belong to V. Since both a_0 and a_1 form a $\mathcal{K}$-set with the homomorphism h, the outputs $\Sigma_i n_i a(\gamma_i), \Sigma_j n_j a(\gamma_j)$ must both be equal to $h(\gamma)$, and therefore cannot be distinct, a contradiction.

This also means that P forces that the union of the generic filter is a total homomorphism from Γ to Δ; the name for the union we will denote by $\dot{h}_{\mathrm{gen}}$. For (2), let $\langle Q, \tau\rangle$ be a balanced pair. Strengthening τ if necessary, we may assume that for each point $\gamma \in \Gamma \cap V$ $Q \Vdash \tau(\check{\gamma}) \in \mathrm{dom}(\tau)$. Now, fix a point $\gamma \in \Gamma$. By a balance argument, for every basic open set $O \subset \Delta$ it must be the case that either $Q \Vdash \tau(\gamma) \in \dot{O}$ or $Q \Vdash \tau(\gamma) \notin \dot{O}$. This means that there is a single point $\delta(\gamma) \in \Delta$ which is in the intersection of all open sets $O \subset \Delta$ for which the former option prevails; it must be the case that $Q \Vdash \tau(\gamma) = \delta(\gamma)$. The function $h\colon \gamma \mapsto \delta(\gamma)$ must be a homomorphism from Γ to Δ. Clearly $Q \Vdash \tau \leq \check{h}$; by Proposition 5.2.4, the pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X \times Y), \check{h}\rangle$ are equivalent.

Finally, (3) is obvious. For the last sentence, note that every condition $p \in P$ can be extended to a homomorphism from Γ to Δ by the work of the first paragraph. $\square$

Example 8.2.3. Suppose that Γ, Δ are abelian Polish groups, with Γ torsion free and uncountable and Δ divisible–the case $\Gamma = \mathbb{R}$ and $\Delta = \mathbb{R}/\mathbb{Z}$ is of particular interest. Then the poset $P = P(\Gamma, \Delta)$ forces the generic homomorphism to be discontinuous. To see this, let $p \in P$ be a condition. Let $\langle \gamma_i \colon i \leq \omega \rangle$ be a sequence of elements of Γ which are linearly independent over $\mathrm{dom}(p)$ and such that $\lim \gamma_i = \gamma_\omega$. Pick distinct elements δ, δ_ω and let $q = p \cup \{\langle \gamma_i, \delta \rangle \colon i \in \omega, \langle \gamma_\omega, \delta_\omega \rangle\}$. It is immediate that q is a $\mathcal{K}(\Gamma, \Delta)$-set, and it forces γ_ω to be a point of discontinuity of the generic homomorphism.

8.3 Automorphisms of $\mathcal{P}(\omega)$ modulo finite

Automorphisms of the Boolean algebra $\mathcal{P}(\omega)$ modulo finite have been investigated for decades. The simply definable ones are trivial [98] and the Proper Forcing Axiom implies that downright all automorphisms of the algebra are trivial [83]. It turns out that there is a natural balanced poset for adding a nontrivial automorphism of the Boolean algebra $B = \mathcal{P}(\omega)$ modulo finite.

Definition 8.3.1. The *automorphism poset* P consists of pairs $p = \langle B_p, \pi_p \rangle$ where $B_p \subset B$ is a countable subalgebra and π_p is its automorphism. The ordering is that of coordinatewise reverse inclusion.

The automorphism poset is clearly σ-closed and Suslin. By a simple density argument, it adds a nontrivial automorphism of the algebra B. This is an immediate consequence of the following observation.

Fact 8.3.2. [9, Theorem 2.3] *Let $C \subset B$ be a countable algebra and $\pi \colon C \to C$ be an automorphism. Let $\chi \colon B \to B$ be an automorphism. Then there is an automorphism $\eta \colon B \to B$ extending π and not equal to χ.*

The balanced virtual conditions in the poset P are naturally classified by automorphisms of the algebra B. Let $\pi \colon B \to B$ be such an automorphism, and consider the $\mathrm{Coll}(\omega, B)$-name τ_π for the (set of all conditions stronger than the) condition $\langle B^V, \check{\pi} \rangle \in P$. The following theorem provides the key classification information.

Theorem 8.3.3. *Let P be the automorphism poset.*

1. *For every automorphism π of B, the pair $\langle \mathrm{Coll}(\omega, B), \tau_\pi \rangle$ is balanced in P;*
2. *for every balanced pair $\langle Q, \tau \rangle$ there is an automorphism π of B such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \mathrm{Coll}(\omega, B), \tau_\pi \rangle$ are equivalent;*
3. *distinct automorphisms yield inequivalent balanced conditions.*

In particular, the poset P is balanced.

Proof. Item (1) is actually the most demanding part. Suppose that $V[H_0]$, $V[H_1]$ are mutually generic extensions of V, and $p_0 = \langle B_0, \pi_0\rangle \in V[H_0]$ and $p_1 = \langle B_1, \pi_1\rangle \in V[H_1]$ are Boolean algebras extending B^V and automorphisms extending π respectively. Let C be the subalgebra of $B^{V[H_0,H_1]}$ generated by $B_0 \cup B_1$. We will find an automorphism of C extending $\pi_0 \cup \pi_1$, providing a common lower bound of the conditions p_0, p_1.

Claim 8.3.4.

1. *If $a_0 \in B_0$, $a_1 \in B_1$ and $c \in B^V$ are such that $a_0 \wedge a_1 \leq c$, then $\pi_0(a_0) \wedge \pi_1(a_1) \leq \pi(c)$ holds;*

2. *If $a_0, b_0 \in B_0$ and $a_1, b_1 \in B_1$ and $a_0 \wedge a_1 \leq b_0 \wedge b_1$, then $\pi_0(a_0) \wedge \pi_1(a_1) \leq p_0(b_0) \wedge \pi_1(b_1)$ holds;*

3. *If a_0, b_0^i for $i \in n$ belong to B_0, a_1, b_0^i for $i \in n$ belong to B_1, and $a_0 \wedge a_1 \leq \bigvee_i (b_0^i \wedge b_1^i)$ holds, then $\pi_0(a_0) \wedge \pi_1(a_1) \leq \bigvee(\pi_0(b_0^i) \wedge \pi_1(b_1^i))$ holds.*

Proof. For (1), note that $a_0 - c$ and $a_1 - c$ are disjoint sets in the mutually generic models $V[H_0]$ and $V[H_1]$, and so by Proposition 1.7.8 there are ground model sets d_0, d_1 such that $a_0 - c \leq d_0$, $a_1 - c \leq d_1$, and $d_0 \wedge d_1 = 0$. Then $\pi_0(a_0) - \pi(c) \leq \pi(d_0)$ since π_0 is an automorphism of B_0 extending π. For the same reason $\pi_1(a_1) - \pi(c) \leq \pi(d_1)$ holds; moreover, $\pi(d_0) \wedge \pi(d_1) = 0$ follows from the fact that π is an automorphism. For (2), the inequality $a_0 \wedge a_1 \leq b_0 \wedge b_1$ yields $(a_0 \wedge b_0) \wedge (a_1 - b_1) = 0$. Since $a_0 \wedge b_0 \in V[H_0]$ and $a_1 - b_1 \in V[H_1]$ and the two extensions are mutually generic, by Proposition 1.7.8 there has to be a ground model set $c \subset \omega$ such that $a_0 \wedge b_0 \leq c$ and $c \wedge (a_1 - b_1) = 0$. For the same reason, there is a ground model set $d \subset \omega$ such that $a_1 \wedge b_1 \leq d$ and $d \wedge (a_0 \setminus b_0) = 0$. The choice of c shows that $a_0 \wedge a_1 \leq c$ and moreover $a_1 \wedge c \leq b_1$; for the same reason, $a_0 \wedge a_1 \leq d$ and $a_0 \wedge d \leq b_0$ holds. In total, writing $e = c \wedge d$, we have $a_0 \wedge a_1 \leq e$, $a_0 \wedge e \leq b_0$, and $a_1 \wedge e \leq b_1$. By the first item, $\pi_0(a_0) \wedge \pi_1(a_1) \leq \pi(e)$ follows. Moreover, since π_0 and π_1 are automorphisms, the inequalities $\pi_0(a_0) \wedge \pi(e) \leq \pi_0(b_0)$ and $\pi_1(a_1) \wedge \pi(e) \leq \pi_1(b_1)$ hold. The conclusion of (2) is then at hand.

For (3), for each nonempty set $t \subset n$ let $c^t = \bigwedge_{i\in t} b_0^i - \bigvee_{i\notin t} b_0^i$; these are pairwise disjoint sets in $V[H_0]$. Write also $d^t = \bigvee_{i\in t} b_1^i$; these are sets in $V[H_1]$. Now, $a_0 \wedge a_1 \subset \bigvee_t c^t$, and by (2) $\pi_0(a_0) \wedge \pi_1(a_1) \leq \bigvee_t \pi_0(c^t)$ holds. For each $t \subset n$, $a_0 \wedge a_1 \wedge c^t \subset d^t$ holds, and by (2) $\pi_0(a_0) \wedge \pi_1(a_1) \wedge \pi_0(c_t) \subset \pi_1(d_t)$ holds. The conclusion of (3) follows. □

Now, express each element of the algebra C as a disjunction of conjunctions of elements of B_0 and B_1, and define $\chi(\bigvee_i (a_0^i \wedge a_1^i)) = \bigvee_i (\pi_0(a_0^i) \wedge \pi_1(a_1^i))$ where $a_0^i \in B_0$ and $a_1^i \in B_1$. Claim 8.3.4(3) shows that the definition of $\chi(c)$ does not depend on the choice of the representation of the element $c \in C$ as a disjunction of conjunctions. The map $\chi\colon C \to C$ preserves ordering by (3) of Claim 8.3.4 and therefore is an automorphism of the algebra C as desired.

For item (2) of the theorem, extend τ if necessary so that $Q \Vdash \tau = \langle C, \chi\rangle$ for some algebra C containing B^V and some automorphism χ of C. By a balance argument, for each element $b \in B$ there must be an element $\pi(b) \in B$ such that $Q \Vdash \chi(\check{b}) = \pi(b)$. It is immediate that π is an automorphism of B and $Q \Vdash \langle C, \chi\rangle \leq \langle B^V, \check{\pi}\rangle$. Item (2) then follows from (1) and Proposition 5.2.4.

Finally, item (3) is obvious. For the last sentence, apply (1) with Fact 8.3.2. □

8.4 Kurepa families

The notion of a Kurepa family on a set is an old one [61]; it appears intermittently in modern set theory [94], [24, Section 7]. In this section, we show how to force a cofinal Kurepa family on a Polish space with a balanced poset.

Definition 8.4.1. Let X be a set.

1. A *Kurepa family* is a set $A \subset [X]^{\aleph_0}$ such that for every countable set $b \subset X$, the set $\{a \cap b \colon a \in A\}$ is countable;
2. the family is *cofinal* if for every countable set $b \subset X$ there is a set $a \in A$ such that $b \subset a$.

The main task of this section is to show how a cofinal Kurepa family for a given Polish space can be added by balanced forcing.

Definition 8.4.2. Let X be an uncountable Polish space. The *Kurepa poset* P is a poset of all countable sets $p \subset [X]^{\aleph_0}$ closed under finite intersections. The ordering is defined by $q \leq p$ if $p \subset q$ and for every $a \in q$ and $b \in p$, $a \cap b \in p$.

Clearly the poset P does not depend on the choice of the uncountable Polish space X up to isomorphism. Clearly, P is a σ-closed Suslin forcing. If $G \subset P$ is a generic filter, then a simple genericity argument shows that $\bigcup G$ is a cofinal Kurepa family on X which is closed under finite intersections. For every set $A \subset \mathcal{P}(X)$ closed under finite intersections note that $\check{A}$ is a $\mathrm{Coll}(\omega, \mathcal{P}(X))$-name for an element of the Kurepa poset. It turns out that if $X \in A$ then the pair $\langle \mathrm{Coll}(\omega, \mathcal{P}(\omega)), \check{A})\rangle$ is balanced, and all balanced pairs are up to equivalence of this form. This is the content of the following theorem.

Theorem 8.4.3. *Let X be an uncountable Polish space and let P be the Kurepa poset on X.*

1. *if $A \subset \mathcal{P}(X)$ is a set containing X and closed under finite intersections, then the pair $\langle \mathrm{Coll}(\omega, \mathcal{P}(X)), \check{A}\rangle$ is balanced;*
2. *if $\langle Q, \tau\rangle$ is a balanced pair for P then there is a set $A \subset \mathcal{P}(X)$ closed under finite intersections and containing X such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, \mathcal{P}(X)), \check{A}\rangle$ are equivalent;*
3. *distinct sets as in (1) yield inequivalent balanced conditions.*

In particular, the poset P is balanced.

Proof. For (1), fix a set $A \subset \mathcal{P}(X)$ closed under finite intersections and containing the set X. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of the ground model V in which the set $\mathcal{P}(X) \cap V$ is countable. Suppose that $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are conditions stronger than A; we must show that $p_0, p_1 \in P$ are compatible, in other words that $p_0 \cup p_1 \in P$. To this end, let $a_0 \in p_0$ and $a_1 \in p_1$ be sets; we must show that $a_0 \cap a_1 \in p_0 \cap p_1$. To see this, first the product forcing theorem shows that $a_0 \cap a_1 \subset V \cap X$ holds. Second, both sets $a_0 \cap V \cap X$ and $a_1 \cap V \cap X$ must belong to A, since $V \cap X \in A$ and $A \leq p_0, p_1$. Since A is closed under finite intersections, we see that $a_0 \cap a_1 = (a_0 \cap V) \cap (a_1 \cap V) \in A \subset p_0 \cap p_1$ holds. Therefore, the conditions p_0, p_1 are compatible as desired.

(2) is more challenging. Let $\langle Q, \tau \rangle$ be a balanced pair in the poset P. Without loss of generality we may assume that $Q \Vdash |\mathcal{P}(X) \cap V| = \aleph_0$. Strengthening the condition τ repeatedly in the Q-extension if necessary, we may assume that for each set $a \subset X$ in the ground model, either $a \in \tau$ or there is $b \in \tau$ such that $a \cap b \notin \tau$. A balance argument shows that for each set $a \subset X$ in the ground model, the largest condition in Q decides which alternative prevails. Let $A \subset \mathcal{P}(X)$ be a set of all sets $a \subset X$ for which the former alternative prevails. We will show that A is a set closed under finite intersections containing X, and $Q \Vdash \check{A} \geq \tau$. The desired equivalence of $\langle Q, \tau \rangle$ and $\langle \mathrm{Coll}(\omega, \mathcal{P}(X)), \check{A} \rangle$ then follows from Proposition 5.2.4.

The closure of the set A under finite intersections follows immediately from the fact that Q forces τ to be closed under finite intersections. To see that $X \in A$ holds, suppose towards a contradiction that it fails; so there must be a Q-name σ for a subset of X such that $Q \Vdash \sigma \in \tau$ and $\sigma \cap V \notin \tau$. Strengthen the name τ if necessary to contain a set which contains $X \cap V$ as a subset, and let η be a Q-name for such a set in τ. Let $H_0, H_1 \subset Q$ be mutually generic filters, let $p_0 = \tau/H_0$ and $p_1 = \tau/H_1$, and let $a_0 = \eta/H_0$ and $a_1 = \sigma/H_1$. By a mutual genericity argument, $a_0 \cap a_1 = (a_0 \cap V) \cap (a_1 \cap V) = a_1 \cap V$. It follows that $a_0 \cap a_1 \notin p_1$ and so p_0, p_1 are incompatible conditions in P, contradicting the balance assumption.

Finally, to show that $Q \Vdash \tau \leq \check{A}$, suppose towards a contradiction that this fails. Passing to a condition in Q if necessary we may find a set $b \in A$ and a Q-name σ such that $Q \Vdash \sigma \in \tau$ and $\sigma \cap \check{b} \notin \check{A}$. Now proceed similarly to the previous paragraph. Strengthen the name τ if necessary to contain a set which contains $X \cap V$ as a subset, and let η be a Q-name for such a set in τ. Let $H_0, H_1 \subset Q$ be mutually generic filters, let $p_0 = \tau/H_0$ and $p_1 = \tau/H_1$, and let $a_0 = \eta/H_0$ and $a_1 = \sigma/H_1$. By a mutual genericity argument, $a_0 \cap b \cap a_1 = (a_0 \cap b) \cap (a_1 \cap b) = a_1 \cap b$. Now, if $a_1 \cap b \in V$, then $a_1 \cap b \in A$, contradicting the choice of the name σ. Thus, it must be the case that $a_1 \cap b \notin V$, and by a mutual genericity argument $a_1 \cap b \notin V[H_0]$. It follows that $a_0 \cap b \cap a_1 \notin p_0$ and so p_0, p_1 are incompatible conditions in P, contradicting the balance assumption.

Finally, (3) is obvious. The balance of the poset P follows from (1): if $p \in P$ is a condition then $\langle \mathrm{Coll}(\omega, X), p \cup \{X \cap V\} \rangle$ is a balanced pair below p. $\square$

8.5 Set mappings

In this section, we develop posets for adding interesting set mappings. Let $n \in \omega$. A *set mapping* (of arity n) on a set X is just a function $f\colon [X]^n \to \mathcal{P}(X)$ such that $f(a) \cap a = 0$ holds for every set $a \in [X]^n$. A *free set* for f is a set $b \subset X$ such that $f(a) \cap b = 0$ holds for all $a \in [b]^n$. Set mappings without large free sets have been investigated by Komjáth ??? and others. It is possible to add such mappings on Polish spaces by balanced forcing. We start with an interesting limiting result.

Proposition 8.5.1. *(ZF) Let X be an uncountable Polish space. If there is a set mapping $f\colon [X]^2 \to X^{<\aleph_0}$ without a free triple, then there is a countable-to-one function from X to ω_1.*

Proof. For each $x \in X$ let M_x be the model of sets hereditarily ordinally definable from x and f. Let $\leq$ be the pre-ordering on the space X defined by $y \leq x$ if $y \in M_x$.

First, we claim that $\leq$ is linear. To prove it, suppose towards a contradiction that $x_0, x_1 \in X$ are $\leq$-incomparable elements. Let $x_2 \in X$ be an ordinally definable point which does not belong to the finite set $f(x_0, x_1)$. Then $x_1 \notin f(x_0, x_2)$ since x_1 is not definable from x_0, and $x_0 \notin f(x_1, x_2)$ since x_0 is not definable from x_1. Thus, the set $\{x_0, x_1, x_2\}$ is a free triple for f, contradicting the initial choice of f.

Second, we claim that if $y < x$ are points in X then $M_x \models M_y \cap X$ is a countable set. Suppose towards a contradiction that this fails. Working in the model M_x, let N be a countable elementary submodel of a large structure containing x, y and $f \restriction M_x$. Since M_y contains uncountably many reals from the point of view of M_x, there is a point $x_0 \in M_y \setminus N$. Let $x_1 \in M_y \cap N$ be any point which does not belong to $f(x, x_0)$. As in the first paragraph, $\{x, x_0, x_1\}$ is a free triple for f, contradicting the initial choice of f.

Case 1. There is a point x such that M_x contains uncountably many elements of X. By the previous two paragraphs, M_x contains all points of X. As ZFC proves that existence of a set mapping with finite values and no free triple is equivalent to the continuum hypothesis and M_x is a model of ZFC containing f as an element, we conclude that M_x contains an injection from X to $\omega_1^{M_x} = \omega_1$. This proves the proposition in this case.

Case 2. Case 1 fails. In this case, let h be the map on X defined by $h(x) = \omega_1^{M_x}$. By the failure of Case 1, the range of h is a subset of ω_1. It is also a countable-to-one map. Suppose towards a contradiction that for some ordinal $\alpha \in \omega_1$, the set $A = \{y \in X\colon h(y) = \alpha\}$ is uncountable. Let $y \in A$ be any point, and use the case assumption to find $y \in A$ such that $x \notin M_y$. By the linearity of $\prec$, it follows that $y < x$. By the previous work, the reals of M_y are a countable set in M_x, violating the assumption that the two models have the same ω_1. The proposition follows in this case as well. $\square$

Proposition 8.5.1 shows that we cannot hope to produce a set mapping with finite values with no free triple, as such a mapping yields an uncountable sequence of $\mathbb{F}_2$-classes. However, other options for set mappings are achievable in balanced extensions.

Definition 8.5.2. Let X be an uncountable Polish space. The *fat set mapping forcing* is the poset P of all functions p such that for some countable set $d(p) \subset X$, $p\colon [d(p)]^2 \to \mathcal{P}(d(p))$ is a set mapping without free triples. The ordering is defined by $q \leq p$ if $d(p) \subset d(q)$ and $p \subset q$.

It is clear that P is a σ-closed Suslin forcing. The union of the P-generic set if a set mapping with countable values and no free triples.

Theorem 8.5.3. *Let P be the fat set mapping forcing.*

1. *If $f\colon [X]^2 \to \mathcal{P}(X)$ is any set mapping without free triples, then $\langle \mathrm{Coll}(\omega, X), \check{f}\rangle$ is a balanced pair;*
2. *if $\langle Q, \tau\rangle$ is any balanced pair, there is a set mapping $f\colon [X]^2 \to \mathcal{P}(X)$ without free triples such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{f}\rangle$ are equivalent;*
3. *distinct set mappings on X without free triples yield inequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. For (1), it is clear that $\mathrm{Coll}(\omega, X) \Vdash \check{f} \in P$ is a condition. Let $V[G_0], V[G_1]$ be mutually generic extensions of V, and let $p_0 \in V[G_0], p_1 \in V[G_1]$ be conditions containing f as a subset. To find a lower bound of p_0, p_1 in $V[G_0, G_1]$, let p be the function defined by $d(p) = d(p_0) \cup d(p_1)$, $p_0 \cup p_1 \subset p$, and if $\{x_0, x_1\} \subset d(p)$ is a pair such that $x_0 \in d(p_0) \setminus V$ and $x_1 \in d(p_1) \setminus V$ then $p(x_0, x_1) = d(p) \setminus \{x_0, x_1\}$. We must show that p is a common lower bound of p_0, p_1.

First of all, $p_0 \cup p_1$ is indeed a function, since any pair a in $\mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$ is already in the ground model V by the product forcing theorem, and then $p_0(a) = p_1(a) = f(a)$. Second, we must verify that p has no free triple. Let $b \subset d(p)$ be a triple. If $b \subset d(p_0)$ or $b \subset d(p_1)$ holds, then b is not free because p_0, p_1 contain no free triples. Otherwise, b can be listed as $\{x_0, x_1, x_2\}$ such that $x_0 \in d(p_0) \setminus V$ and $x_1 \in d(p_1) \setminus V$; then $x_2 \in p(x_0, x_1)$ by the definition of p and b is not free either.

For (2), let $\langle Q, \tau\rangle$ be a balanced pair. Strengthening Q and τ if necessary, we may assume that $Q \Vdash V \cap X \subset d(\tau)$ holds. By a balance argument, for any points $x_0, x_1, y \in X$ in the ground model, $Q \Vdash \check{y} \in \tau(\check{x}_0, \check{x}_1)$ or $Q \Vdash \check{y} \notin \tau(\check{x}_0, \check{x}_1)$ holds. Let $f\colon [X]^2 \to \mathcal{P}(X)$ be the function defined by $y \in f(x_0, x_1)$ if the former alternative in the previous sentence prevails. It is clear that f contains no free triple since τ is forced to contain none. Thus, it will be enough to show that $Q \Vdash \check{f} \subset \tau$.

Suppose towards a contradiction that this fails. Then there must be a condition $q \in Q$ and points $x_0, x_1 \in X$ such that $q \Vdash \tau(\check{x}_0, \check{x}_1) \neq \check{f}(\check{x}_0, \check{x}_1)$.

Let $G_0, G_1 \subset Q$ be mutually generic filters containing the condition q and let $p_0 = \tau/G_0, p_1 = \tau/G_1$. By the initial choice of the function f, there must be points $y_0, y_1 \in X \setminus V$ such that $y_0 \in p_0(x_0, x_1)$ and $y_1 \in p_1(x_0, x_1)$. By the product forcing theorem, $y_0 \notin V[G_1]$ and $y_1 \notin V[G_0]$ holds. Thus, $p_0(x_0, x_1) \neq p_1(x_0, x_1)$ and p_0, p_1 are incompatible, contradicting the balance of the pair $\langle Q, \tau \rangle$.

(3) is obvious. For the last sentence, suppose that $p \in P$ is a condition. Consider the set mapping $f \colon [X]^2 \to \mathcal{P}(X)$ defined by $f(x_0, x_1) = p(x_0, x_1)$ if both $x_0, x_1 \in X$ belong to $d(p)$, and $f(x_0, x_1) = X \setminus \{x_0, x_1\}$ otherwise. It is not difficult to check that f is a set mapping without free triples. By item (1), f represents a balanced virtual condition below p. □

Definition 8.5.4. Let X be an uncountable Polish space. A *thin set mapping forcing* is a poset P of all functions p such that for some countable set $d(p) \subset X$, $p \colon [d(p)]^3 \to [d(p)]^{<\aleph_0}$ is a set mapping without a free quadruple. The ordering is defined by $q \leq p$ if $d(p) \subset d(q)$ and $p \subset q$.

Theorem 8.5.5. *Let P be the thin set mapping forcing.*

1. *If $f \colon [X]^3 \to [X]^{<\aleph_0}$ is any set mapping without free quadruples, then $\langle \mathrm{Coll}(\omega, X), \check{f} \rangle$ is a balanced pair;*
2. *if $\langle Q, \tau \rangle$ is any balanced pair, there is a set mapping $f \colon [X]^3 \to [X]^{<\aleph_0}$ without free quadruples such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{f} \rangle$ are equivalent;*
3. *distinct set mappings on X without free quadruples yield inequivalent balanced pairs.*

In particular, the poset P is balanced if and only if $2^{\aleph_0} \leq \aleph_2$.

Proof. For (1), it is clear that $\mathrm{Coll}(\omega, X) \Vdash \check{f} \in p$ holds. For the balance, suppose that $V[G_0], V[G_1]$ are mutually generic extensions and $p_0 \in V[G_0]$, $p_1 \in V[G_1]$ are conditions extending f. To construct the lower bound p of p_0, p_1, in the model $V[G_0, G_1]$ choose an enumeration $\{z_n \colon n \in \omega\}$ of $d(p_0) \cup d(p_1)$. Let p be the function on $[d(p_0) \cup d(p_1)]^3$ defined by $p_0, p_1 \subset p$ and if $x_0, x_1, x_2 \in d(p_0) \cup d(p_1)$ are pairwise distinct points such that $x_0 \in d(p_0) \setminus V$ and $x_1 \in d(p_1) \setminus V$, then $p(x_0, x_1, x_2)$ is the set of those points of $d(p_0) \cup d(p_1)$ which are enumerated before one of the points x_0, x_1, x_2. We must show that p is a lower bound of p_0, p_1.

First of all, $p_0 \cup p_1$ is indeed a function, since any pair a in $\mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$ is already in the ground model V by the product forcing theorem, and then $p_0(a) = p_1(a) = f(a)$. Second, we must verify that p has no free quadruple. Let $b \subset d(p)$ be a quadruple. If $b \subset d(p_0)$ or $b \subset d(p_1)$ holds, then b is not free because p_0, p_1 contain no free qudruples. Otherwise, b can be listed as $\{x_0, x_1, x_2, x_3\}$ such that $x_0 \in d(p_0) \setminus V$, $x_1 \in d(p_1) \setminus V$ and x_2 is enumerated after x_3; then $x_3 \in p(x_0, x_1, x_2)$ by the definition of p and b is not free either.

The proof of (2) is literally copied from the proof of Theorem 8.5.3(2). (3) is obvious. For the last sentence, use an old result of Kuratowski and Sierpiński [34]: for every number $n \geq 2$, $2^{\aleph_0} \leq \aleph_{n-1}$ holds if and only if there is a set mapping $f\colon [2^\omega]^n \to [2^\omega]^{<\aleph_0}$ without a free set of size $n+1$. Thus, if $2^{\aleph_0} > \aleph_2$, by (2) there are no balanced conditions and P is not balanced. On the other hand, suppose that $2^{\aleph_0} \leq \aleph_2$ holds, and let $p \in P$ be a condition. To produce a balanced virtual condition stronger than p, let $g\colon [X]^3 \to [X]^{<\aleph_0}$ be a set mapping without a free set of size 4. Enumerate the set $d(p)$ by $\langle x_i \colon i \in \omega\rangle$ and let $f\colon [X]^3 \to [X]^{<\aleph_0}$ be a function defined by $f(a) = p(a)$ if $a \subset d(p)$, $f(a) = g(a)$ if $a \cap d(p) = 0$, and $f(a) = g(a) \cup \{x_i \colon i \in n\}$ where n is the smallest number such that $x_n \in \mathrm{dom}(a)$, for sets a such that $a \not\subset d(p)$ and $a \cap d(p) \neq 0$. It is not difficult to see that f is a set mapping without a free set of size 4 such that $p \subset f$; therefore, by (1) it represents a balanced condition stronger than p. $\square$

8.6 Saturated models on quotient spaces

One can use quotient simplicial complex forcings to add structures to the various quotient spaces as long as the structures satisfy a well-known amalgamation property from model theory. We will first review the definitions. For first order structures M, N in the same language, a map $\pi\colon M \to N$ is an *morphism* if it is an injection from $\mathrm{dom}(M)$ to $\mathrm{dom}(N)$ which transports all relations and functions of M to the corresponding relations and functions in N.

Definition 8.6.1. [30] A *Fraissé class* is a class $\mathcal{F}$ of finite structures in a fixed finite first order language such that

1. $\mathcal{F}$ is closed under isomorphism and under induced substructures;
2. (*joint embedding property*) whenever $N_0, N_1 \in \mathcal{F}$ are structures then there is $M \in \mathcal{F}$ containing both N_0, N_1 as induced substructures;
3. (*amalgamation*) whenever $M, N_0, N_1 \in \mathcal{F}$ are structures and $\pi_0\colon M \to N_0$ and $\pi_1\colon M \to N_1$ are morphisms, then there is a structure $K \in \mathcal{F}$ and morphisms $\chi_0\colon N_0 \to K$ and $\chi_1\colon N_1 \to K$ such that $\chi_0 \circ \pi_0 = \chi_1 \circ \pi_1$.

We will tacitly assume that our Fraissé classes contain arbitrarily large finite structures. Fraissé classes are prominent in model theory [41, Theorems 5.3.3 and 5.3.5], Ramsey theory [71, 23], and topological dynamics [53, 106]. The classes we can handle satisfy a well-known stronger version of amalgamation:

Definition 8.6.2. A Fraissé class $\mathcal{F}$ satisfies *strong amalgamation* if for whenever $M, N_0, N_1 \in \mathcal{F}$ are structures and $\pi_0\colon M \to N_0$ and $\pi_1\colon M \to N_1$ are morphisms, then there is a structure $K \in \mathcal{F}$ and morphisms $\chi_0\colon N_0 \to K$ and $\chi_1\colon N_1 \to K$ such that $\chi_0 \circ \pi_0 = \chi_1 \circ \pi_1$, and $\mathrm{rng}(\chi_0) \cap \mathrm{rng}(\chi_1) = (\chi_0 \circ \pi)''M$.

Definition 8.6.3. Let $\mathcal{F}$ be a Fraissé class in a finite relational language.

1. An $\mathcal{F}$-structure is a structure M in the same language as $\mathcal{F}$ such that for every finite set $a \subset \mathrm{dom}(M)$, $M \restriction a \in \mathcal{F}$;

2. Let E be a Borel equivalence relation on a Polish space X. The *$E, \mathcal{F}$-Fraissé poset* P consists of conditions p where p is an $\mathcal{F}$-structure whose domain is a countable subset of the quotient space X/E. The ordering is defined by $q \leq p$ if $\mathrm{dom}(p) \subset \mathrm{dom}(q)$ and $p = q \restriction \mathrm{dom}(p)$.

It is obvious that the poset P introduces a $\mathcal{F}$-structure on the space X/E, denoted by $\dot{M}_{\mathrm{gen}}$. It is not difficult, but at the same time also not particularly natural, to describe the poset $P(\mathcal{F}, E)$ as a quotient simplicial complex poset: the vertices of the simplicial complex $\mathcal{K}$ will be finite $\mathcal{F}$-structures on the E-quotient space, and a finite set a of vertices belongs to $\mathcal{K}$ if all structures in a are induced substructures of a single finite $\mathcal{F}$-structure on the E-quotient space. It is obvious that the posets $P(\mathcal{F}, E)$ and $P_{\mathcal{K}}$ are naturally co-dense; however, we will never use the simplicial complex presentation of $P(\mathcal{F}, E)$. To describe the balanced conditions in the poset $P(\mathcal{F}, E)$, suppose that M is an $\mathcal{F}$-structure on the virtual E-quotient space X^{**}. Let τ_M be a $\mathrm{Coll}(\omega, \beth_{\omega_1}))$ name for the condition $\pi'' M$ where π is the map from the virtual space $(X^{**})^V$ to the E-quotient space of the extension which maps each virtual class to its realization. Theorem 2.5.6 provides the necessary assurance that the map π is well-defined and its range is a countable set.

Theorem 8.6.4. *Let $\mathcal{F}$ be a Fraissé class in a finite relational language which satisfies strong amalgamation. Let E be a Borel equivalence relation on a Polish space X. Then in the $E, \mathcal{F}$-Fraissé poset P,*

1. *for every $\mathcal{F}$-structure M on X^{**}, the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_M \rangle$ is balanced;*

2. *for every balanced pair $\langle Q, \tau \rangle$ there is a $\mathcal{F}$-structure M on X^{**} such that the pairs $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_M \rangle$ and $\langle Q, \tau \rangle$ are equivalent;*

3. *distinct $\mathcal{F}$-structures on X^{**} yield inequivalent balanced conditions.*

In particular, P is balanced.

Proof. For (1), suppose that M is a $\mathcal{F}$-structure on X^{**}. Let R_0, R_1 be arbitrary posets and σ_0, σ_1 are respective $R_0 \times \mathrm{Coll}(\omega, \beth_{\omega_1})$- and $R_1 \times \mathrm{Coll}(\omega, \beth_{\omega_1})$-names for elements of P such that in the respective posets $\sigma_0 \leq \tau_M$ and $\sigma_1 \leq \tau$ is forced. Let H_0, H_1 be mutually generic filters on the respective posets and let $p = \tau_M / H_0$, $p_0 = \sigma_0 / H_0$ and $p_1 = \sigma_1 / H_1$; we need to show that p_0, p_1 are compatible in the poset P.

For this, first note that by the mutual genericity, $\mathrm{dom}(p_0) \cap \mathrm{dom}(p_1) = \mathrm{dom}(p)$. By the assumption on the names σ_0, σ_1 it is the case that $p_0 \restriction \mathrm{dom}(p) = p_1 \restriction \mathrm{dom}(p) = p$. Write $c = \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$. By the strong amalgamation property of the Fraissé class $\mathcal{F}$, for each finite set $a \subset c$ there is a structure $N_a \in \mathcal{F}$ such that $\mathrm{dom}(N_a) = a$ and writing $a_0 = a \cap \mathrm{dom}(p_0)$ and $a_1 \cap \mathrm{dom}(p_1)$, $N_a \restriction a_0 = p_0 \restriction a_0$ and $N_1 \restriction a_1 = p_1 \restriction a_1$ both hold. Let U be an ultrafilter on

$[c]^{<\aleph_0}$ such that for each finite set $a \subset c$ the set $\{b : a \subset b\}$ is in the ultrafilter U. Let N be the structure on c which is the U-integral of the structures N_a. It is immediate that $N \restriction \mathrm{dom}(p_0) = p_0$ and $N \restriction p_1 = p_1$; the closure of the Fraissé class under substructures shows that N is a $\mathcal{F}$-structure. It is the desired lower bound of the conditions p_0, p_1.

For (2), note that for each relation R in the language of the Fraissé class $\mathcal{F}$ and every tuple a of virtual E-classes, it must be the case that either $Q \Vdash \tau \Vdash a$ belongs to the relation R in the generic structure $\dot{M}_{\mathrm{gen}}$, or $Q \Vdash \tau \Vdash a$ does not to the relation R in the generic structure $\dot{M}_{\mathrm{gen}}$ by the balance of the pair $\langle Q, \tau\rangle$. Let M be the structure on the virtual E-quotient space consisting of those tuples for which the former alternative occurs. It is immediate that $Q \times \mathrm{Coll}(\omega, \beth_{\omega_1}) \Vdash \tau \leq \tau_M$ in the separative quotient of the poset P. The equivalence of the two pairs follows from Proposition 5.2.4.

Now, (3) is obvious. For the last sentence, suppose that $p \in P$ is a condition. For each finite set $a \subset X^{**}$, choose a structure $N_a \in \mathcal{F}$ such that $\mathrm{dom}(N_a) = a$ and $N_a \restriction (\mathrm{dom}(p) \cap a) = p \restriction (\mathrm{dom}(p) \cap a)$. Let U be an ultrafilter on $[X^{**}]^{<\aleph_0}$ such that for each finite set $a \subset X^{**}$ the set $\{b : a \subset b\}$ is in the ultrafilter U. Let N be the structure on X^{**} which is the U-integral of the structures N_a. It is immediate that $N \restriction \mathrm{dom}(p) = p$; the closure of the Fraissé class under substructures shows that N is a $\mathcal{F}$-structure. The pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_N\rangle$ is a balanced virtual condition below p. $\square$

Example 8.6.5. Let E be a Borel equivalence relation on a Polish space X. Let P be the poset of all linear orderings on countable subsets of the E-quotient space. The poset is designed to add a linear ordering on the quotient space. Since the Fraissé class of finite linear orderings has the strong amalgamation property, the poset P is balanced, and its balanced conditions are classified by linear orders on the virtual E-quotient space.

Example 8.6.6. Let E be a Borel equivalence relation on a Polish space X. Let P be the poset of all tournaments on countable subsets of the E-quotient space. The poset is designed to add a tournament on the quotient space. Since the Fraissé class of finite tournaments has the strong amalgamation property, the poset P is balanced, and its balanced conditions are classified by tournaments on the virtual E-quotient space.

The Fraissé posets can be viewed as a particularly well-behaved subclass of a significantly larger class obtained from standard amalgamation constructions in model theory.

Definition 8.6.7. Let T be a complete first-order theory in countable language, with infinite models. Let E be a Borel equivalence relation on a Polish space X. The poset P_{TE} consists of structures p satisfying the theory T whose domain is a countable subset of the E-quotient space. The ordering is that of elementary substructure: $q \leq p$ if the domain of p is a subset of the domain of q, and p is an elementary substructure of q.

It is immediate that the union of the structures in the generic filter is a countably saturated model of the theory T on the E-quotient space. To classify the balanced virtual conditions, we use the following definition.

Definition 8.6.8. A *balanced theory* is a complete consistent theory S in the language of T plus a constant for each virtual E-class which contains T and asserts than no elements other than the virtual E-class constants are algebraic over those constants.

Suppose that S is a balanced theory. Let τ_S be the $\mathrm{Coll}(\omega, \beth_{\omega_1})$-name for the set of all countable models $p \in P_{TE}$ such that $\mathrm{dom}(p)$ contains the set p_0 of realizations of all ground model virtual E-classes and $p \models S$. It is not difficult to see that τ_S is a name for an analytic, nonempty, and open subset of the poset P_{TE}.

Theorem 8.6.9. *Let T be a complete first-order theory in countable language, with infinite models. Let E be a Borel equivalence relation on a Polish space X.*

1. *For every balanced theory S, the pair $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_S\rangle$ is balanced in P_{TE};*
2. *for every balanced pair $\langle Q, \tau\rangle$ there is a balanced theory S such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_S\rangle$ are equivalent;*
3. *distinct balanced theories yield inequivalent balanced pairs.*

In particular, the poset P_{TE} is balanced.

Proof. For (1), suppose that S is a balanced theory, $V[H_0], V[H_1]$ are mutually generic extensions of the ground model, and $p_0, p_1 \in P_{TE}$ are conditions in the respective models in the set given by the name τ_S. We must show that there is a model $q \in P_{TE}$ in which both models p_0, p_1 of T are elementary. Note that $\mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$ is exactly the set of all realizations of ground model virtual E-classes by the mutual genericity assumption. Use the algebraicity clause of Definition 8.6.8 and a textbook amalgamation theorem [41, Theorem 5.3.5] to conclude that in $V[H_0, H_1]$ there is a countable model q of T and elementary embeddings $j_0 \colon p_0 \to q$ and $j_1 \colon p_1 \to q$ such that j_0, j_1 coincide on the set r of realizations of virtual E-classes from the ground model, and the sets $j_0''(p_0 \setminus r)$ and $j_1''(p_1 \setminus r)$ are disjoint. It is easy to see that such a model q can be realized in such a way that its domain is a set of E-classes and the embeddings j_0, j_1 are both identity maps. Then q is the sought lower bound of the conditions p_0, p_1.

For (2), let $\langle Q, \sigma\rangle$ be a balanced pair. Strengthening the name σ and the poset Q if necessary, we may assume that σ is in fact a name for a single element of P_{TE} as opposed to an analytic subset of P_{TE} and that σ is forced to contain the realization of every virtual E-class in the ground model. By a balance argument, for every formula $\phi(\vec{x})$ and every tuple $\vec{c}$ of virtual E-classes of the same length as $\vec{x}$, it must be the case that either $Q \Vdash \sigma \models \phi(\vec{c})$ or $Q \Vdash \sigma \models \neg\phi(\vec{c})$. Let S be the set of all formulas and tuples of virtual E-classes $\phi(\vec{c})$ for which the former option prevails. We need to show that S is a balanced theory and the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_S\rangle$ are equivalent.

Claim 8.6.10. *Q forces that no element of σ which is not a realization of a virtual E-class is algebraic over a tuple of virtual E-classes from the ground model.*

Proof. Suppose towards a contradiction that this fails, and let $q \in Q$ be a condition, let η be a Q-name for an element of σ which is not a realization of a virtual E-class, and let ϕ be a formula, $\vec{c}$ a tuple of virtual E-classes and $n \in \omega$ be a natural number such that $q \Vdash \sigma \models$ there are precisely n many x such that $\phi(x, \vec{c})$ holds, and η is one of them. Let $H_0, H_1 \subset Q$ be mutually generic filters over the ground model, both containing q, and let $p_0 = \sigma/H_0$ and $p_1 = \sigma/H_1$. We will show that p_0, p_1 have no lower bound in the poset P_{TE}, reaching a contradiction with the balance assumption on the pair $\langle Q, \sigma \rangle$.

Suppose then that q is such a lower bound. By elementaricity, q must see exactly n many solutions to $\phi(x, \vec{c})$, and all solutions in p_0 and in p_1 are solutions in q. Now, p_0 already sees n many solutions, and η/H_1 is another solution in p_1 which does not belong to p_0 or even to $V[H_0]$. Thus, q in fact sees at least $n + 1$ many solutions,a contradiction. □

It follows that S is a balanced theory, and also that $\mathrm{Coll}(\omega, \beth_{\omega_1}) \times Q \Vdash \sigma \in \tau_S$. The equivalence of the balanced pairs $\langle Q, \tau \rangle$ and $\langle \mathrm{Coll}(\omega, \beth_{\omega_1}), \tau_S \rangle$ then follows from Proposition 5.2.4.

(3) is obvious. For the last sentence, suppose that $p \in P_{TE}$ is a condition–a countable model of the theory T. By the upwards Löwenheim–Skolem theorem, there is a model M of the theory T on the virtual E-quotient space in which p is an elementary submodel. Let S be the diagram of M and observe that S is a balanced theory and its corresponding balanced condition is below p. □

Example 8.6.11. Let T be the theory of dense linear orders without endpoints. The notion of algebraicity is trivial in the theory T, and T has elimination of quantifiers. It follows that the balanced theories are exactly the linear orders on the set of virtual E-classes. This should be compared to Example 8.6.5.

8.7 Non-DC variations

All of the partial orders exhibited so far are either σ-closed or $\aleph_0$-distributive, and therefore their corresponding extensions of the symmetric Solovay model satisfy DC, the Axiom of Dependent Choices. This is normaly viewed as highly desirable, as DC is a key tool for developing mathematical analysis and descriptive set theory as we know them today. However, balanced forcing can be used to generate extensions of the symmetric Solovay model in which DC fails. We include one striking example.

Definition 8.7.1. Let X be an uncountable Polish space. The *finite-countable* poset P associated with X consists of pairs $p = \langle a_p, b_p \rangle$ where $a_p \subset X$ is a finite set, $b_p \subset X$ is a countable set, and $a_p \cap b_p = 0$. The ordering is that of coordinatewise reverse inclusion.

The finite-countable poset is rather worthless in the ZFC context. The lack of control over the finite part means that it collapses $\aleph_1$. Namely, for every sequence $\langle x_\alpha : \alpha \in \omega_1\rangle$ of distinct points of X, the set $\{\alpha \in \omega_1 : \langle\{x_\alpha\}, 0\rangle$ belongs to the generic filter on $P\}$ is forced to be cofinal in ω_1 of ordertype ω. However, the finite-countable poset provides a particularly easy answer to an old question of Woodin: is there a Suslin forcing which is not c.c.c. and yet has no perfect antichain? The finite-countable poset is not c.c.c. as it collapses $\aleph_1$. On the other hand, it does satisfy $\aleph_2$-c.c. in ZFC, and so by a straightforward absoluteness argument cannot contain a perfect antichain. This should be contrasted with the convoluted answer to Woodin's question given in [46].

In the ZF+DC context, the finite-countable poset is much better behaved, as is clear from the following theorem. The generic filter will add a partition of the space X into two sets, one of which does not contain an infinite countable subset and the other contains no perfect subset. Such a partition clearly violates DC.

In order to classify the balanced conditions in the finite-countable poset, we need a piece of notation. Whenever $a \subset X$ is a finite set, let τ_a be the $\mathrm{Coll}(\omega, X)$-name for a condition in the poset P which is the pair $\langle a, (X \cap V) \setminus a\rangle$.

Theorem 8.7.2. *Let X be a Polish space and P the associated finite-countable poset.*

1. *For every finite set $a \subset X$, the pair $\langle \mathrm{Coll}(\omega, X), \tau_a\rangle$ is balanced in P;*
2. *for every balanced pair $\langle Q, \tau\rangle$ there is a finite set $a \subset X$ such that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \tau_a\rangle$ are equivalent;*
3. *distinct finite sets yield inequivalent balanced pairs.*

Proof. For (1), suppose that $V[H_0], V[H_1]$ are mutually generic extensions of the ground model V and $p_0 = \langle a_0, b_0\rangle \in V[H_0]$ and $p_1 = \langle a_1, b_1\rangle \in V[H_1]$ are conditions stronger than $\bar{p} = \langle a, (X \cap V) \setminus a$. We have to show that p_0, p_1 are compatible, which is to say that $a_0 \cap b_1 = a_1 \cap b_0 = 0$ holds. Suppose towards a contradiction that for example $x \in a_0 \cap b_1$ is a point. By the product forcing theorem, $x \in V[H_0] \cap V[H_1] = V$. If $x \in a$ then $x \in b_1$ is impossible as $p_1 \leq \bar{p}$, and if $x \notin a$ then $x \in a_0$ is impossible as $p_0 \leq \bar{p}$. A contradiction!

For (2), let $\tau = \langle \dot{a}, \dot{b}\rangle$ where $\dot{a}, \dot{b}$ are Q-names for disjoint subsets of X. A balance argument show that for each point $x \in X$, either $Q \Vdash \check{x} \in \dot{a}$ or $Q \Vdash \check{x} \in \dot{b}$. Let $a \subset X$ be the set of all points $x \in X$ for which the first alternative prevails. It is immediate that a is a finite set and $Q \times \mathrm{Coll}(\omega, X) \Vdash \tau \leq \tau_a$. Proposition 5.2.4 shows then that the balanced pairs $\langle Q, \tau\rangle$ and $\langle \mathrm{Coll}(\omega, X), \tau_a\rangle$ are equivalent, proving (2). (3) is immediate. □

8.8 Side condition forcings

Attempts to force uncountable subsets of Polish spaces with a lack of internal structure often converge to a kind of a side condition poset. Posets of this kind are not covered by the previous sections. In this section, we outline several examples.

Definition 8.8.1. Let X be a Polish space. Let $F(X)$ denote the Effros Borel space of all closed subsets of X [55, Section 12.C]. Let $I \subset F(X)$ be a Borel set such that $\bigcup I = X$ and I is not a countable union of elements of I. The poset P_I consists of pairs $p = \langle a_p, b_p \rangle$ where $a_p \subset X$ and $b_p \subset I$ are countable sets. The ordering on P_I is defined by $q \leq p$ if $a_p \subset a_q$, $b_p \subset b_q$, and $(\bigcup b_p) \cap a_q \setminus a_p = 0$.

It is immediate that P_I is a σ-closed Suslin forcing. It is designed to add an uncountable set which has countable intersection with every element of I. For the classification of balanced conditions, whenever $a \subset X$ is a set, write τ_a for the $\mathrm{Coll}(\omega, X)$-name for the pair $\langle \check{a}, \dot{b} \rangle$ where $\dot{b}$ is the name for the set of all ground model coded sets in I.

Theorem 8.8.2. *Let X be a Polish space and let $I \subset F(X)$ be a Borel set such that $\bigcup I = X$. Then*

1. *for every set $a \subset X$, the pair $\langle \mathrm{Coll}(\omega, X), \tau_a \rangle$ is balanced;*
2. *for every balanced pair $\langle Q, \sigma \rangle$ there is a set $a \subset X$ such that the pairs $\langle Q, \sigma \rangle$ and $\langle \mathrm{Coll}(\omega, X), \tau_a \rangle$ are equivalent;*
3. *distinct subsets of X yield nonequivalent balanced pairs.*

In particular, the poset P_I is balanced.

Proof. Towards (1), unraveling the definitions, it is sufficient show the following. Suppose that R_0, R_1 are posets and σ_0, σ_1 are respective names for countable subsets of X such that $R_0 \Vdash \sigma_0 \cap \bigcup (I \cap V) = 0$ and $R_1 \Vdash \sigma_1 \cap \bigcup (I \cap V) = 0$; then $R_0 \times R_1 \Vdash \sigma_0 \cap \bigcup (I \cap V[\dot{G}_1]) = 0$ and $\sigma_1 \cap \bigcup (I \cap V[\dot{G}_0]) = 0$ where $\dot{G}_0, \dot{G}_1$ are the respective product names for the generic filters on the 0th and 1st coordinate.

This, however, is immediate. We will show that $R_0 \times R_1 \Vdash \sigma_0 \cap \bigcup (I \cap V[\dot{G}_1]) = 0$; the proof of the other assertion is symmetric. Suppose towards a contradiction that η is an R_0-name for an element of σ_0, χ is an R_1-name for an element of I, and $\langle r_0, r_1 \rangle \in R_0 \times R_1$ is a condition forcing that $\eta \in \chi$ holds. Let M be a countable elementary submodel of a large structure containing all objects named so far, let $g_1 \subset R_1 \cap M$ be a filter generic over M containing the condition r_1, and let $F = \chi / g_1$. Since F is a closed set in $I \cap V$, there must be a basic open set $O \subset X$ and a condition $r_0' \leq r_0$ such that $O \cap F = 0$ and $r_0' \Vdash_{R_0} \eta \in \dot{O}$. By the genericity of the filter g_1 and the forcing theorem applied in the model M, there must be a condition $r_1' \leq r_1$ such that $r_1' \Vdash_{R_1} O \cap \chi = 0$. Clearly, $\langle r_0', r_1' \rangle \Vdash \eta \notin \chi$, contradicting the initial choice of $\langle r_0, r_1 \rangle$.

Towards (2), let $\langle Q, \sigma \rangle$ be a balanced pair; enlarging the poset Q if necessary we may assume that Q collapses the size of the ground model continuum to $\aleph_0$; strengthening the condition σ we may assume that b_σ is forced to contain all ground model coded elements of I. By a balance argument, for every point $x \in X$ it is the case that either $Q \Vdash \check{x} \in a_\sigma$ or $Q \Vdash \check{x} \notin a_\sigma$. Let $a \subset X$ be the set of all those elements of X for which the former alternative prevails. We will show that $Q \times \mathrm{Coll}(\omega, X) \Vdash \sigma \leq \tau_a$, which implies the equivalence of σ and τ_a by Proposition 5.2.4.

To prove the inequality $\sigma \leq \tau_a$, it is enough to show that $Q \Vdash (a_\sigma \setminus V) \cap (\bigcup(I \cap V)) = 0$. Suppose towards a contradiction that this fails, and let $q \in Q$ be a condition, η be a Q-name for an element of $a_\sigma \setminus V$ and $F \in I \cap V$ be a closed set such that $q \Vdash \eta \in \dot{F}$. Let $G_0, G_1 \subset Q$ be mutually generic filters containing the condition q. By a mutual genericity argument, the point η/G_0 does not belong to a_{σ/G_1}, while it does belong to F and $F \in b_{\sigma/G_1}$. This means the conditions $\sigma/G_0, \sigma/G_1$ are incompatible in P, violating the balance assumption on $\langle Q, \sigma\rangle$.

Finally, (3) is immediate from the assumption that $X = \bigcup I$. $\square$

Example 8.8.3. Let $X = 2^\omega$, let Γ be the graph connecting points $x, y \in X$ if the least number such that $x(n) \neq y(n)$ is even. Let I be the collection of closed Γ-cliques and closed Γ-anticliques. The poset P_I adds a generic uncountable set such that every Γ-clique or anticlique is countable. This exhibits a balanced extension in which OCA fails, cf. Example 12.2.16.

Example 8.8.4. Let $X = \omega^\omega$ and let I be the collection $\{x \leq y \colon x \in X\}$ for all functions $y \in \omega^\omega$, where $\leq$ is the everywhere domination ordering on ω^ω. The poset P_I adds a dominating subset of ω^ω which has only countably many elements below any given function $y \in \omega^\omega$. A set like this must fail to have the Baire property. In a rather different language, Shelah [81] showed that in the P_I-extension of the Solovay model every set is Lebesgue measurable, showing that ZF+DC plus the statement "every set of reals is Lebesgue measurable" does not imply that every set has the Baire property.

In the case when I is the collection of closed nowhere dense subsets of X, there is a more sophisticated variation of the poset P_I. For a natural number $n \in \omega$, a set $s \subset (2^\omega)^n$, and a partial function $h \colon n \to 2^\omega$ write $s_h = \{g \in (2^\omega)^{n \setminus \mathrm{dom}(h)} \colon h \cup g \in s\}$.

Definition 8.8.5. The *Lusin poset* P is the partial order of all pairs $p = \langle a_p, b_p\rangle$ where $a_p \subset 2^\omega$ is a countable $\mathbb{E}_0$-invariant set and b_p is a countable set of pairs $\langle s, a\rangle$ where $s \subset (2^\omega)^n$ is a closed nowhere dense subset of $(2^\omega)^n$ for some $n \in \omega$, $a \subset a_p$ is an $\mathbb{E}_0$-invariant set, and for each partial function $h \colon n \to a_p \setminus a$ whose range consists of pairwise non-$\mathbb{E}_0$-related elements, if $\mathrm{dom}(h) = n$ then $h \notin s$, and if $\mathrm{dom}(h) \neq n$ then the set s_h is nowhere dense in the space $(2^\omega)^{n \setminus \mathrm{dom}(h)}$. The order is that of coordinatewise reverse inclusion.

It is not difficult to check that P is a σ-closed Suslin partial ordering. The poset adds an $\mathbb{E}_0$-invariant Lusin subset of 2^ω–an uncountable set which intersects every nowhere dense set in a countable set. The generic set has a similar property in all finite powers though, and in Example 12.2.17 it serves an even more refined purpose. The balance of P is an immediate consequence of the following classification theorem. For any $\mathbb{E}_0$-invariant set $a \subset 2^\omega$, let τ_a be the $\mathrm{Coll}(\omega, 2^\omega)$-name for the condition $\langle \check{a}, b\rangle \in P$ where b is the set of all pairs $\langle s, a\rangle$

where s is a ground model coded closed nowhere dense subset of $(2^\omega)^n$ in the ground model for some $n \in \omega$.

Theorem 8.8.6. *Let P be the Lusin poset.*

1. *For every $\mathbb{E}_0$-invariant set $a \subset X$, the pair $\langle \mathrm{Coll}(\omega, X), \tau_a \rangle$ is balanced;*
2. *for every balanced pair $\langle Q, \sigma \rangle$ there is a set $a \subset X$ such that the pairs $\langle Q, \sigma \rangle$ and $\langle \mathrm{Coll}(\omega, X), \tau_a \rangle$ are equivalent;*
3. *distinct subsets of X yield nonequivalent balanced pairs.*

In particular, the poset P is balanced.

Proof. Write $S = \mathrm{Coll}(\omega, 2^\omega)$. For (1), let R_0, R_1 be posets and σ_0, σ_1 be $R_0 \times S$ and $R_1 \times S$-names respectively such that $R_0 \times S \Vdash \sigma_0 \leq \tau_a$ and $R_1 \times S \Vdash \sigma_1 \leq \tau_a$. Let $G_0 \subset R_0 \times S$ and $G_1 \subset R_1 \times S$ be mutually generic filters; we must show that the conditions $p_0 = \sigma_0/G_0, p_1 = \sigma_1/G_1 \in P$ are compatible.

To this end, write $p_0 = \langle a_0, b_0 \rangle$ and $p_1 = \langle a_1, b_1 \rangle$. Let $n \in \omega$, let $s \in (2^\omega)^n$ be a nowhere dense set, let $\langle a, s \rangle \in b_0 \cup b_1$ be a pair, and let $h\colon n \to a_0 \cup a_1$ be a partial function whose range consists of pairwise non-$\mathbb{E}_0$-related elements of $(a_0 \cup a_1) \setminus a$. We must show that either (if $\mathrm{dom}(h) = n$) $h \notin s$, or (if $\mathrm{dom}(h) \neq n$) the set s_h is nowhere dense.

For definiteness assume that $\langle a, s \rangle \in b_0$. Let $h_0 = h \cap V[G_0]$ and $h_1 = h \setminus h_0$. Since $p_0 \in P$ holds, the set s_{h_0} is nowhere dense. By the Kuratowski–Ulam theorem, the set $t = \{k \in (2^\omega)^{\mathrm{dom}(h_1)}\colon s_{h_0 \cup k}$ is somewhere dense$\}$ is meager and in the model $V[G_0]$. Now, the function h_1 is product-Cohen generic over V by the definition of the name τ_a. By a mutual genericity argument, h_1 is also product-Cohen generic over $V[G_0]$ and therefore does not belong to the set t. Thus the set $s_{h_0 \cup h_1}$ is nowhere dense as required.

For (2), suppose that $\langle Q, \sigma \rangle$ is a balanced pair. Strengthening the poset Q or the condition σ repeatedly, we may assume that $Q \Vdash 2^\omega \cap V$ is countable, and

- for each $x \in 2^\omega$ in V either $x \in a_\sigma$ or for no extension $p \leq \sigma$ it is the case that $x \in a_p$;
- for each $n \in \omega$ and each closed nowhere dense set $s \subset (2^\omega)^n$ in the ground model, either $\langle (a_\sigma \cap V), s \rangle \in b_\sigma$ or for no extension $p \leq \sigma$ it is the case that $\langle (a_\sigma \cap V), s \rangle \in b_p$;
- for each $n \in \omega$ and each closed nowhere dense set $s \subset (2^\omega)^n$ in the ground model there is a such that $\langle a, s \rangle \in b_\sigma$.

By a balance argument, for each $x \in 2^\omega$ in the ground model, either Q forces the first clause of the first item to prevail for $\check{x}$, or Q forces the second clause of the first item to prevail for $\check{x}$. Let $a \subset 2^\omega$ be the set of all points for which the first option prevails; we will show that $Q \times S \Vdash \sigma \leq \tau_a$, showing that the balanced pairs $\langle Q, \sigma \rangle$ and $\langle \mathrm{Coll}(\omega, 2^\omega), \tau_a \rangle$ are equivalent by Proposition 5.2.4. To see the inequality, it is only necessary to show that the second clause of the second item above is impossible for any closed nowhere dense set $s \subset (2^\omega)^n$ in the ground model. Suppose towards a contradiction that the second clause occurs. This can only happen if there is a Q-name $\dot{h}$ such that Q forces $\dot{h}\colon n \to 2^\omega$ to be a partial map such that $\mathrm{rng}(h)$ consists of pairwise non-$\mathbb{E}_0$-equivalent, non-ground model elements of a_σ such that either (if $\mathrm{dom}(h) = n$) $h \in s$ or (if $\mathrm{dom}(h) \neq n$) the set s_h contains a nonempty open set. In either case, let $G_0, G_1 \subset Q$ be mutually generic filters, and use the third item to find a set $a \in V[G_0]$ such that $\langle a, s\rangle \in b_{\sigma/G_0}$. Then by mutual genericity the set $\mathrm{rng}(\dot{h}/G_1)$ consists of points which are not in a. This means that the conditions $\sigma/G_0, \sigma/G_1$ are incompatible in P, contradicting the balance assumption on the pair $\langle Q, \sigma\rangle$.

(3) is immediate. For the last sentence, if $p \in P$ is a condition, it is not difficult to see that S forces $\check{p}$ to be compatible with τ_{a_p}. The name for a lower bound is a balanced virtual condition below p. $\square$

8.9 Weakly balanced variations

There are a number of restrictions on the balanced extensions of the Solovay model–Corollaries 9.1.2, 9.1.5 or Theorems 14.1.1 or 14.2.1 are good examples. Transcending these restrictions requires reaching for a weakly balanced forcing. The weakly balanced arguments are invariably more complicated than the balanced ones, and we never obtain a classification of weakly balanced classes. We include two examples reminiscent of the side condition forcings of Section 8.8. The first poset adds an injection from the E-quotient space to the $\mathbb{E}_0$-quotient space, where E is any give Borel equivalence relation on a Polish space. For a pinned equivalence relation E this is possible to do with a balanced forcing, see Theorem 6.6.3; in the general case, this is impossible to do with balanced forcing by Theorem 9.1.1.

Definition 8.9.1. Let E be a Borel equivalence relation on a Polish space X. The *Lusin collapse forcing* P_E is the poset of all triples $p = \langle a_p, b_p, f_p\rangle$ where $\langle a_p, b_p\rangle$ is a condition in the Lusin forcing of Definition 8.8.5, and f_p is an injection from the E-space to the set of $\mathbb{E}_0$-classes represented in a_p. The order is that of coordinatewise reverse inclusion.

It is not difficult to see that P is a σ-closed Suslin partial order and if $G \subset P$ is a generic filter then $f = \bigcup_{p\in G} f_p$ is a total injection from the E-quotient space to the $\mathbb{E}_0$-quotient space. It is instructive to view the poset P_E as a natural two-step iteration. The first step is the (balanced) Lusin poset of Definition 8.8.5 adding a subset A of the $\mathbb{E}_0$-quotient space. In the second step, an injection of the E-quotient space into A is added by straightforward countable approximations. The projection from P_E into the Lusin poset is given by the projection of conditions in p into the first two coordinates.

Theorem 8.9.2. *Let E be a Borel equivalence relation on a Polish space X. The Lusin collapse forcing of $|E|$ to $|\mathbb{E}_0|$ is weakly balanced.*

Proof. Write $P = P_E$. Let $p \in P$ be a condition; we must find a virtual weakly balanced condition $\bar{p} \leq p$. Let $\bar{p} = \langle \mathrm{Coll}(\omega, 2^\omega), \langle \check{a}_p, \check{b}, \check{f}_p \rangle \rangle$ where b is the set of all pairs $\langle a_p, s \rangle$ where $s \subset (2^\omega)^n$ is a closed nowhere dense set coded in the ground model for some $n \in \omega$. It is immediate that $\bar{p}$ is a virtual condition; it will be enough to show that $\bar{p}$ is weakly balanced.

Before we proceed, we must fix some notation and terminology. Let $V[G]$ be a generic extension of V, and in $V[G]$ let f be a function from the virtual E-quotient space to the quotient $\mathbb{E}_0$-space. If M is an intermediate model of ZF between V and $V[G]$, write $f|M$ for the following set. Let γ be the smallest ordinal such that in M, every E-pin is equivalent to an E-pin of set-theoretic rank smaller than γ; such an ordinal exists since by Theorem 2.5.6 there are only set many virtual E-classes in $V[G]$. Let $f|M$ be the set of all pairs $\langle \langle Q, \tau \rangle, y \rangle$ such that $\langle Q, \tau \rangle \in M$ is an E-pin of rank smaller than γ, $y \in 2^\omega$, and for some virtual E-class $c \in \mathrm{dom}(f)$, $y \in f(c)$ and $\langle Q, \tau \rangle \in c$. The *transfinite analysis* of f is the sequence of models M_α of ZF given by the recursive formula $M_\alpha = V(\langle f|M_\beta \colon \beta \in \alpha \rangle)$; note that always $M_0 = V$ and the models M_α form an inclusion-increasing sequence. The function f *has a heart* if there is an ordinal α such that

- for every ordinal $\beta \in \alpha$ and every E-pin $\langle Q, \tau \rangle \in M_\beta$ there is a class $c \in \mathrm{dom}(f)$ with $\langle Q, \tau \rangle \in c$;
- for every virtual E-class $c \in \mathrm{dom}(f)$, if $c \cap M_\alpha \neq 0$ then $c \cap \bigcup_{\beta \in \alpha} M_\beta \neq 0$.

Note that once such an ordinal α is reached then $M_\delta = M_\alpha$ for all ordinals $\delta \geq \alpha$. If the function f has a heart, then the least ordinal α as above is its *depth*. The *heart of* f is the pair $\langle M, h \rangle$ where $M = M_\alpha$ and $h = f|M$.

We apply the above transfinite analysis also to conditions in P. Any E-class c will be identified with the virtual E-class of all E-pins $\langle Q, \tau \rangle$ such that for some (all) $x \in c$, $Q \Vdash \tau \; E \; \check{x}$; thus, a partial function on the E-quotient space is viewed as a partial function on the virtual E-quotient space. The heart of a condition in P is the heart of its last coordinate. The following two key claims control the behavior of hearts of conditions below $\bar{p}$.

Claim 8.9.3. *Every condition $q \in P$ can be strengthened in some generic extension to a condition $r \leq q$ which has a heart.*

Proof. Let $V[G]$ be a generic extension of V, and let $q \in V[G]$ be a condition in P. Work in $V[G]$. By transfinite recursion on $\beta \leq \omega_1$ define a finite support iteration $\langle R_\beta \colon \beta \leq \omega_1, \dot{Q}_\beta \colon \beta < \omega_1 \rangle$ of c.c.c. forcings and R_β-names τ_β so that R_β forces the following:

- $\tau_0 = 0$, $\gamma \in \beta$ implies $\tau_\gamma \subset \tau_\beta$, and if β is limit then $\tau_\beta = \bigcup_{\gamma \in \beta} \tau_\gamma$;
- τ_β is a partial function from the virtual E-quotient space to the $\mathbb{E}_0$-quotient space which has a heart $\langle M_\beta, \tau_\beta | M_\beta \rangle$ of depth β and $\mathrm{dom}(\tau_\beta)$ contains exactly those virtual E-classes represented in $\mathrm{dom}(\tau_\beta | M_\beta)$;

- $\dot{Q}_\beta$ is the finite support product of copies of Cohen forcing on 2^ω indexed by all the virtual E-classes in M_β which are not represented in $\bigcup_{\gamma\in\beta}\dot{M}_\gamma$ and do not belong to $\mathrm{dom}(f_q)$, and $\tau_{\beta+1}$ is the function τ_β together with the function sending each virtual E-class in M_β as above to the $\mathbb{E}_0$-class of the corresponding Cohen real, and sending every virtual E-class in M_β which is $\mathrm{dom}(f_q)$ to the $\mathbb{E}_0$-class indicated by f_q.

Write $R = R_{\omega_1}$. Let $H \subset R$ be a filter generic over $V[G]$; thus, the models $V[G]$ and $V[G][H]$ have the same ω_1. In $V[G][H]$, write $f_\beta = \tau_\beta/H$ for all $\beta \leq \omega_1$ and $f = f_{\omega_1}$. Consider the transfinite analysis $\langle M_\beta \colon \beta \in \omega_1\rangle$ of f, starting with $M_0 = V$. Since the function f_q is countable, a counting argument shows that there has to be an ordinal $\beta \in \omega_1$ such that every virtual E-class represented in $M_{\beta+1}$ and in $\mathrm{dom}(f_q)$ is already represented in M_β. Thus, the transfinite analysis of the function $f_q \cup f_\beta$ equals to $\langle M_\gamma \colon \gamma \leq \alpha\rangle$ and $\langle M_\beta, f_\beta | M_\beta\rangle$ is its heart. Consider the triple $\langle a_q \cup \mathrm{rng}(f_\beta), b_q, f_q \cup f_\beta\rangle$. It is not difficult to see that it is a virtual condition in P stronger than q. In a suitable generic extension, it turns into a condition $r \leq q$ with heart $\langle M_\beta, f_\beta | M_\beta\rangle$. □

Claim 8.9.4. *Let $r \leq \bar{p}$ be a condition in P in some generic extension with heart $\langle M, h\rangle$. Then*

1. *the theory of the model M with parameters in V and the parameter h depends only on the depth of r;*
2. *every real in M belongs to some $V[b]$ where b is a finite subset of $\bigcup \mathrm{rng}(h)$.*

Proof. Fix an ordinal α. It turns out that the hearts of conditions $\leq \bar{p}$ of depth α are all generated in the same way. Work in V. By transfinite recursion on $\beta \leq \alpha$ define a finite support iteration $\langle R_\beta \colon \beta \leq \alpha, \dot{Q}_\beta \colon \beta < \alpha\rangle$ of c.c.c. forcings and R_β-names τ_β so that R_β forces the following:

- $\tau_0 = 0$, $\gamma \in \beta$ implies $\tau_\gamma \subset \tau_\beta$, and if β is limit then $\tau_\beta = \bigcup_{\gamma\in\beta}\tau_\gamma$;
- τ_β is a partial function from the virtual E-quotient space to the $\mathbb{E}_0$-quotient space which has a heart $\langle M_\beta, \tau_\beta | M_\beta\rangle$ of depth β and $\mathrm{dom}(\tau_\beta)$ contains exactly those virtual E-classes represented in $\mathrm{dom}(\tau_\beta | M_\beta)$;
- if $\beta = 0$ then Q_0 is the finite support product of copies of Cohen forcing on 2^ω indexed by all the virtual E-classes in $M_0 = V$ which do not belong to $\mathrm{dom}(f_p)$, and τ_1 is the function f_p together with the function sending each virtual E-class in V as above to the $\mathbb{E}_0$-class of the corresponding Cohen real;
- if $\beta > 0$ then $\dot{Q}_\beta$ is the finite support product of copies of Cohen forcing on 2^ω indexed by all the virtual E-classes in M_β which are not represented in $\bigcup_{\gamma\in\beta}\dot{M}_\gamma$, and $\tau_{\beta+1}$ is the function τ_β together with the function sending each virtual E-class in M_β as above to the $\mathbb{E}_0$-class of the corresponding Cohen real.

We will show that if $r \leq \bar{p}$ is a condition with heart $\langle M, h\rangle$ of depth α, then in some forcing extension there is a filter $H \subset R_\alpha$ generic over V such that writing $\langle N, k\rangle$ for the heart of τ_α/H, then $\langle M, h\rangle = \langle N, k\rangle$. This will prove the first item since the theory of $\langle M, h\rangle$ (with parameter h and other parameters in the ground model) is then exactly the collection of those statements which R_α forces to be true in the heart of τ_α. The second item of the claim ???

$\square$

The following claim just restates the information from the previous claim in a form that will be useful later.

Claim 8.9.5. *Let $r \leq \bar{p}$ be a condition in P in some generic extension with heart $\langle M, h\rangle$. Then*

1. *every E-class in* $\mathrm{dom}(f_r)$ *is either a realization of a virtual E-class in* $\mathrm{dom}(h)$ *or it is not a realization of any virtual E-class in M;*
2. *every finite tuple c of pairwise $\mathbb{E}_0$-unrelated elements of $a_r \setminus \bigcup \mathrm{rng}(h))$ is product Cohen-generic over M.*

Proof. The first item is an immediate consequence of the definition of the heart. For the second item, we use the fact that $r \leq \bar{p}$. Let $n = |c|$. Every closed nowhere dense subset $s \subset (2^\omega)^n$ in the model M belongs to some model $V[d]$ where d is a finite tuple of pairwise $\mathbb{E}_0$-unrelated elements of $\bigcup \mathrm{rng}(h) \setminus \bigcup \mathrm{rng}(f_p))$ by Claim 8.9.4(2). By the choice of the virtual condition $\bar{p}$, the tuple $c \cup d$ is product Cohen-generic over V, and by the product forcing theorem $c \notin s$. We have just shown that c does not belong to any closed nowhere dense subset of $(2^\omega)^n$ in the model M; that is, c is product Cohen-generic over M. $\square$

Finally, we are ready to conclude the proof of Theorem 8.9.2. Let Q_0, Q_1 be posets in V, with names σ_0, σ_1 for conditions in P stronger than $\bar{p}$. We must find, in some generic extension, filters $G_0 \subset Q_0$ and $G_1 \subset Q_1$ separately generic over the ground model such that the conditions σ_0, σ_1 are compatible in P. By Claim 8.9.3, we may assume that $Q_0 \Vdash \sigma_0$ has heart of depth $\check{\alpha}_0$ and $Q_1 \Vdash \sigma_1$ has heart of depth $\check{\alpha}_1$. For definiteness, assume that $\alpha_0 \leq \alpha_1$. Move to some forcing extension (such as the Q_1-extension) where there is a condition $r \leq \bar{p}$ with a heart of depth α_1. Let M_β be the models arising in the transfinite analysis of r, and let $N_0 = M_{\alpha_0}, h_0 = f_r | N_0$ and $N_1 = M_{\alpha_1}, h_1 = f_r | N_1$. Thus, $\langle N_1, h_1\rangle$ is a heart of the condition r; note that $\langle N_0, h_0\rangle$ is also a heart of a condition $\leq \bar{p}$, namely the condition obtained from r by restricting the function f_r to the set of E-classes which are realizations of the virtual classes in $\mathrm{dom}(h_0)$.

Note that in the model N_0, there exists a poset S_0 and a name η_0 for a filter on Q_0 generic over V such that S_0 forces the heart of σ_0/η_0 to be $\langle N_0, h_0\rangle$. This occurs because this statement is true in the heart of σ_0 and the theory of the hearts depends only on the height by Claim 8.9.4(1). Similarly, in the model N_1 there exists a poset S_1 and a name η_1 for a filter on Q_1 generic over V such that S_1 forces the heart of σ_1/η_1 to be $\langle N_1, h_1\rangle$. Let $H_0 \subset S_0$ and $H_1 \subset S_1$ be filters mutually generic over the model N_1. Let $G_0 = \eta_0/H_0$ and $G_1 = \eta_1/H_1$; we claim that the conditions σ_0/G_0 and σ_1/G_1 are compatible in P as desired.

To see this, write $\sigma_0/G_0 = \langle a_0, b_0, f_0\rangle$ and $\sigma_1/G_1 = \langle a_1, b_1, f_1\rangle$. To show that these two conditions are compatible, we must argue that $\langle a_0, b_0\rangle$ and $\langle a_1, b_1\rangle$ are compatible as conditions in the Lusin forcing, and then show that $f_0 \cup f_1$ is an injection from $\mathbb{E}$-quotient space to the $\mathbb{E}_0$-space. The compatibility in the Lusin forcing is nearly identical to the proof of Theorem 8.8.6(1) and we omit it. To show that $f_0 \cup f_1$ is an injection from the E-quotient space to the $\mathbb{E}_0$-quotient space, suppose that $c_0 \in \operatorname{dom}(f_0)$ and $c_1 \in \operatorname{dom}(f_1)$ are E-classes and work to prove the equivalence $f_0(c_0) = f_1(c_1) \leftrightarrow c_0 = c_1$. By Claim 8.9.5(1) there are two cases:

- c_0 is a realization of a virtual E-class $d \in \operatorname{dom}(h_0)$. Then $f_0(c_0) = h_0(d) = h_1(d)$ and either $c_1 = c_0$ in which case $f_1(c_1) = h_1(d) = f_0(c_0)$, or $c_1 \neq c_0$ and then $f_1(c_1) \neq h_1(d)$ since the function f_1 is an injection.
- c_0 is not a realization of a virtual E-class in N_0. By a mutual genericity argument then, c_0 is not an E-class represented in the model $N_1[H_1]$; in particular $c_0 \neq c_1$. By Claim 8.9.5(2), the $\mathbb{E}_0$-class $f_0(c_0)$ consists of points Cohen generic over N_0. By a mutual genericity argument it must be distinct from the $\mathbb{E}_0$-classes in $N_1[H_1]$; in particular, $f_0(c_0) \neq f_1(c_1)$.

This completes the proof. $\square$

The following poset introduced in [63] adds an infinite maximal disjoint family, a task impossible to perform with balanced forcing by Theorem 14.1.1. It yields a model of ZF+DC where there is an infinite MAD family and every set of reals is Lebesgue measurable by Example 14.3.6.

Definition 8.9.6. The *MAD forcing* P consists of all pairs $p = \langle a_p, b_p\rangle$ such that $a_p \subset [\omega]^{\aleph_0}$ is an infinite countable almost disjoint family, and b_p is a countable set of pairs $\langle s, a\rangle$ such that s is a partition of ω into finite sets and $a \subset a_p$ is a countable set. Moreover, for every pair $\langle s, a\rangle \in b_p$ and every finite set $d \subset a_p \setminus a$, there are infinitely many sets $e \in s$ such that $\bigcup d \cap e = 0$. The set P is ordered by coordinatewise reverse inclusion.

It is clear that the MAD forcing is Suslin and σ-closed. The union of the first coordinates of conditions in the generic filter is forced to be a maximal almost disjoint family by an elementary density argument.

Theorem 8.9.7. *The MAD forcing is weakly balanced.*

Proof. Suppose that $p \in P$ is a condition, and write $b = b_p \cup \{\langle s, a_p\rangle : s$ is a ground model partition of ω into finite sets$\}$. It is clear that for any poset Q collapsing the size of the continuum to $\aleph_0$, $Q \Vdash \langle \check{a}_p, \check{b}\rangle$ is a condition in P stronger than p. It will be enough to show that the pair $\langle Q, \langle \check{a}_p, \check{b}\rangle\rangle$ is weakly balanced in P.

To this end, suppose that R_0, R_1 are partial orders and σ_0, σ_1 are respective R_0- and R_1-names for conditions in P extending $\langle a_p, b\rangle$; We must produce, in some generic extension, filters $H_0 \subset R_0$ and $H_1 \subset R_1$ separately generic over the ground model such that the conditions σ_0/H_0 and σ_1/H_1 are compatible in P. The following claim is key.

Claim 8.9.8. *For every $i \in 2$, the following holds. Whenever $r \in R_i$ is a condition and η_j for $j \in \omega$ are names for elements of $a_{\sigma_i} \setminus a_p$, there is a number $n \in \omega$ such that for every $m \in \omega$ there is a strengthening of r forcing $\bigcup_{j\in m} \eta_j \cap [n, m) = 0$.*

Proof. For definiteness let $i = 0$. Suppose towards a contradiction that the claim fails for a condition $r \in R_0$ and names η_j for $j \in m$. Then, there is a partition s of ω into finite intervals such that $r \Vdash \bigcup_{j\in b} \eta_j \cap b \neq 0$ for every $b \in s$. This shows that $r \Vdash \sigma_0 \not\leq \langle a_p, b\rangle$, contradicting the initial assumptions. $\square$

Let $V[K]$ be a generic extension collapsing a sufficiently large cardinal and work in $V[K]$. An inductive application of the claim makes it possible to find filters $H_0 \subset R_0$ and $H_1 \subset R_1$ separately generic over the ground model, and a partition $\omega = \bigcup_j c_j$ of ω into finite sets such that, writing $\sigma_0/H_0 = \langle a_0, b_0\rangle$ and $\sigma_1/H_1 = \langle a_1, b_1\rangle$, we have:

- for every $x \in a_0 \setminus a_p$, $x \subseteq \bigcup\{c_j : j \text{ even}\}$ up to finitely many exceptions and similarly for every $x \in a_1 \setminus a_p$, $x \subseteq \bigcup\{c_j : j \text{ odd}\}$ up to finitely many exceptions;
- for every infinite collection $s \in V[H_0]$ of pairwise disjoint subsets of ω, there is $e \in s$ which is a subset of some c_j for j even, and similarly for every infinite collection $s \in V[H_1]$ of pairwise disjoint subsets of ω, there is $e \in s$ which is a subset of some c_j for j odd.

We claim that the filters work as required. This is clearly the same as showing that the pair $\langle a_0 \cup a_1, b_0 \cup b_1\rangle$ belongs to P, since then it is a lower bound of both $\langle a_0, b_0\rangle$ and $\langle a_1, b_1\rangle$. First of all, the first item above immediately shows that $a_0 \cup a_1$ is an almost disjoint family. To confirm the "moreover" demand in the definition of the poset P for the pair $\langle a_0 \cup a_1, b_0 \cup b_1\rangle$, suppose that $\langle s, a\rangle \in b_0 \cup b_1$; for definiteness assume that $\langle s, a\rangle \in b_0$. Let $d \subset (a_0 \cup a_1) \setminus a$ be a finite set; we must produce infinitely many $e \in s$ such that $\bigcup d \cap e = 0$.

Write $d_0 = d \cap a_0$ and $d_1 = d \setminus d_0$. Since the pair $\langle a_0, b_0\rangle$ is a condition in the poset P, there are infinitely many sets $e \in s$ such that $\bigcup d_0 \cap e = 0$. By the second item above, there must be infinitely many $e \in s$ such that $\bigcup d_0 \cap e = 0$ and $e \subset c_j$ for some even number $j \in \omega$. By the first item above, there must be infinitely many sets e among them such that $\bigcup d_1 \cap e = 0$. For all such $e \in s$, it is the case that $\bigcup d \cap e = 0$. This completes the proof. $\square$

The definition of the MAD forcing may seem odd, since it skips the most obvious choice: the poset Q of all infinite countable almost disjoint families ordered by reverse inclusion. However, the following remains open:

Question 8.9.9. Is the poset Q weakly balanced?

Chapter 9

Preserving cardinalities

9.1 The well-ordered divide

The main feature of the balanced Suslin forcing is that it does not add any well-ordered sequences of elements of the symmetric Solovay model. In particular, the balanced extensions of the Solovay model are barren in the sense of [39, 22]. This has a number of cardinality corollaries for the resulting extensions. The following theorem is stated in terms of Convention 1.7.16.

Theorem 9.1.1. *In cofinally balanced extensions of the symmetric Solovay model W, every well-ordered sequence of elements of W belongs to W.*

Proof. Let P be a Suslin forcing. Let κ be an inaccessible cardinal such that P is balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ and work in W. For a formula ϕ of the language of set theory, element $y \in 2^\omega$, and a set $v \in V$, write $u(\phi, y, v)$ for the unique set x such that $W \models \phi(x, y, v)$ if such a unique x exists; otherwise, write $u(\phi, y, v) = 0$. Suppose towards a contradiction that there is a condition $p \in P$, an ordinal α and a P-name τ such that p forces τ to be an α-sequence of elements of W which does not belong to W. The name τ is definable from some real parameter $z \in 2^\omega$ and some elements of the ground model. Find an intermediate model $V[K]$ which is an extension of the ground model by a poset of size $< \kappa$ such that $p, z \in V[K]$ and P is balanced in $V[K]$.

Work in $V[K]$. Let $\bar{p}$ be a balanced virtual condition in P below p. Since in W, τ is forced not to belong to W, there must be in $V[K]$ an ordinal $\beta \in \alpha$ and a posets R of size $< \kappa$, R-names σ_0, σ_1 for conditions in P stronger than $\bar{p}$, R-names η_0, η_1 for elements of 2^ω, ground models elements v_0, v_1, and formulas ϕ_0, ϕ_1 such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash$ the following:

- $\sigma_0 \Vdash_P \tau(\check{\beta}) = u(\phi_0, \eta_0, v_0)$;
- $\sigma_1 \Vdash_P \tau(\check{\beta}) = u(\phi_1, \eta_1, v_1)$;
- $u(\phi_0, \eta_0, v_0) \neq u(\phi_1, \eta_1, v_1)$.

Working in the model W, let $H_0, H_1 \subset R$ be mutually generic filters, and for bits $b, c \in 2$ let $p_{bc} = \sigma_b/H_c \in P$ and $y_{bc} = \eta_b/H_c \in 2^\omega$. By the third item above, in the model $V[K][H_0, H_1]$ there must be a bit $c \in 2$ such that $\text{Coll}(\omega, <\kappa) \Vdash u(\phi_0, y_{00}, v_0) \neq u(\phi_1, y_{1c}, v_1)$. The conditions $p_{00}, p_{1c} \in P$ are compatible in W by the balance of the condition $\bar{p}$, with a lower bound $q \in P$. Since W is the symmetric Solovay extension of the model $V[K][H_0]$, the forcing theorem applied with that model yields that $q \Vdash \tau(\check{\beta}) = u(\phi_0, \eta_0, v_0)$. Since W is the symmetric Solovay extension of $V[K][H_1]$, the forcing theorem applied with $V[K][H_1]$ yields that $q \Vdash \tau(\check{\beta}) = u(\phi_1, y_{1c}, v_1)$. Finally, since W is the symmetric Solovay extension of $V[K][H_0][H_1]$, $u(\phi_0, y_{00}, v_0) \neq u(\phi_1, y_{1c}, v_1)$ holds in W. Thus, the condition q forces two distinct values to the name $\tau(\check{\beta})$, an impossibility. $\square$

Corollary 9.1.2. *In a cofinally balanced extension of a symmetric Solovay model, there is no transfinite uncountable sequences of pairwise distinct Borel sets of bounded rank.*

Proof. Since cofinally balanced extensions add no countable sequences of elements of the Solovay model W by Theorem 9.1.1, all Borel sets in $W[G]$ belong to W and have the same Borel rank there as in $W[G]$. Thus, an uncountable sequence of distinct Borel sets of bounded Borel rank in $W[G]$ would have to belong to W by Theorem 9.1.1 again. However, there are no such sequences in the Solovay model W by a result of Stern [87]. $\square$

Corollary 9.1.3. *Let E be a Borel equivalence relation on a Polish space X. In cofinally balanced extensions of the symmetric Solovay model, $\aleph_1 \not\leq |E|$ holds.*

Proof. An ω_1-sequence of distinct E-classes would constitute an ω_1-sequence of distinct Borel sets of Borel rank bounded by the rank of E. Such sequences are ruled out by the previous corollary. $\square$

Corollary 9.1.3 guarantees that in balanced extensions of the symmetric Solovay model, the Friedman–Stanley jump divide is preserved. This follows from a humble ZF result of independent interest.

Proposition 9.1.4. (ZF) *Let X be a set and E an equivalence relation on X with all classes countable. If $|X^{\aleph_0}| \leq |E|$ then $|\mathbb{HC}| \leq |E|$.*

Proof. Let g be an injection from the set of all countable subsets of X to E-classes. Define a function h from the collection of hereditarily countable sets to E-classes by $\in$-recursion: $h(a) = g(\bigcup h''a)$. By induction on the minimum of the rank a and b argue that $a \neq b$ implies that $h(a) \neq h(b)$. Thus the function h is an injection and the statement of the proposition follows. $\square$

Corollary 9.1.5. *Let E be a Borel equivalence relation on a Polish space X. In cofinally balanced extensions of a symmetric Solovay model, $|E^+| \not\leq |E|$ holds.*

Proof. Suppose towards a contradiction that $|E^+| \leq |E|$ holds in the extension. The Proposition yields $|\mathbb{HC}| \leq |E|$. Clearly, $\aleph_1 \leq |\mathbb{HC}|$ holds in ZF, and the concatenation of the cardinal inequalities contradicts the conclusion of Corollary 9.1.3. $\square$

If one wishes to add well-ordered sequences of some objects in the Solovay model and maintain control, it is possible to use weakly balanced forcing. The following theorem, stated in terms of Convention 1.7.16, shows that there will be no uncountable sequences of reals in the resulting extension.

Theorem 9.1.6. *In weakly balanced extensions of the symmetric Solovay model W, every set of ordinals belongs to W.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset such that P is weakly balanced below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Let $p \in P$ be a partial order and let τ be a P-name for a set of ordinals; we have to find a strengthening of the condition p which decides the membership of every ordinal in the set τ. To this end, note that both p, τ are definable from a parameter $z \in 2^\omega$ and some parameters in the ground model.

Let $V[K]$ be an intermediate generic extension obtained by a poset of size $< \kappa$ such that $z \in V[K]$. Working in the model $V[K]$, let $\bar{p}$ be a weakly balanced virtual condition in the poset P, below the condition P. We claim that for every ordinal α, it is the case that $\mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p}$ decides in P the membership of the ordinal $\check{\alpha}$ in the set τ. Suppose towards a contradiction that this is not the case. Then there must be an ordinal α, posets R_0, R_1 of size $< \kappa$, and R_0- and R_1-names σ_0 and σ_1 for elements of p stronger than $\bar{p}$ such that $R_0 \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_0 \Vdash \check{\alpha} \notin \tau$ and $R_1 \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_1 \Vdash \check{\alpha} \in \tau$.

Work in W again. Use the weak balance of $\bar{p}$ to find filters $H_0 \subset R_0$ and $H_1 \subset R_1$ separately generic over $V[K]$ such that the conditions $p_0 = \sigma_0/H_0$ and $p_1 = \sigma_1/H_1$ are compatible in P. Since W is a symmetric Solovay extension of both models $V[K][H_0]$ and $V[K][H_1]$, $p_0 \Vdash \check{\alpha} \notin \tau$ and $p_1\check{\alpha} \in \tau$. Thus, the common lower bound of p_0, p_1 must force two contradictory statements, which is impossible. $\square$

It is impossible to strengthen the conclusion of Theorem 9.1.6 to sequences of elements of the Solovay model. For any Borel equivalence relation E on a Polish space X, Theorem 8.9.2 produces a weakly balanced extension of the symmetric Solovay model in which $|E| \leq \mathbb{E}_0$. In the natural case $E = \mathbb{F}_2$, Proposition 9.1.4 applied in the resulting model shows that $|\mathbb{HC}| \leq |\mathbb{E}_0|$ there, in particular there is an ω_1-sequence of $\mathbb{E}_0$-classes. Note that the conclusion of Corollary 9.1.5 fails as well in the model.

As the last remark in this section, many preservation theorems in this book can be combined since the various properties of the posets concerned are preserved under product. In view of the many mutual consistency results concerning ZFC such as those contained in [100], one can ask for example whether there can Σ^2_1 sentences ϕ_0 and ϕ_1 such that both ZF+DC+ϕ_0 and ZF+DC+ϕ_1 are consistent with the statement ψ asserting the nonexistence of an uncountable sequence of pairwise distinct reals, while ZF+DC+ϕ_0+ϕ_1 implies $\neg\psi$. The following two examples provide an affirmative answer, in the second case ϕ_0+ϕ_1 even implies that the reals are well-ordered in ordertype ω_1.

Example 9.1.7. Let ϕ_0 be the statement "there is an uncountable sequence of $\mathbb{E}_0$-classes" and let ϕ_1 be the statement "there is an $\mathbb{E}_0$-transversal". The conjunction $\psi \wedge \phi_0$ holds in the abovementioned weakly balanced extension of the Solovay model. The conjunction $\psi \wedge \phi_1$ holds in the balanced extension of the Solovay model adding an $\mathbb{E}_0$-transversal with countable approximations by Corollary 9.1.2. Finally, the conjunction $\phi_0 \wedge \phi_1$ immediately implies $\neg\psi$ in ZF. This example shows that a product of a balanced and a weakly balanced forcing is not necessarily weakly balanced.

Example 9.1.8. Let ϕ_0 be the statement "there is an acyclic decomposition of 2^ω" as in Definition **??**. Let $X = (2^\omega)^\omega$ and let $A \subset X$ be the set of all $x \in X$ such that Turing reducibility linearly orders $\mathrm{rng}(x)$. Let $E = \mathbb{F}_2 \restriction A$ and let ϕ_1 be the statement "$|E| \leq |2^\omega|$". The theories ZF+DC+ϕ_0 and ZF+DC+ϕ_1 are separately consistent with ψ, but the theory ZF+DC+ϕ_0+ϕ_1 implies that the reals are well-ordered of ordertype ω_1.

Proof. Let P_0 be the acyclic decomposition forcing of Definition **??**. It is balanced if CH holds by Theorem **??**, and so by Theorem 9.1.1 the P_0-extension of the symmetric Solovay model contains an acyclic decomposition of 2^ω while it does not contain any uncountable sequence of pairwise distinct reals. Let P_1 be the collapse poset of E to 2^ω of Definition 6.6.2. Since the virtual E-classes are classified by subsets of 2^ω linearly ordered by Turing reducibility and each such set can have size at most $\aleph_1$, it follows that $\lambda(E) = 2^{\aleph_1}$ and by Corollary 6.6.4 the poset P_1 is balanced if $2^{\aleph_0} = 2^{\aleph_1}$ holds. By Theorem 9.1.1, the P_1 extension of the symmetric Solovay model satisfies $|E| \leq |2^\omega|$ while it does not contain any uncountable sequence of pairwise distinct reals. Note that the posets P_0, P_1 are balanced under incompatible assumptions additional to ZFC, and as a result ZFC proves that the product $P_0 \times P_1$ is not balanced.

Now, argue in the theory ZF+DC+ϕ_0+ϕ_1. Let $c\colon [2^\omega]^2 \to \omega$ be an acyclic decomposition, and let f be an injection from the E-quotient space to 2^ω. Let M be the model of sets hereditarily ordinally definable from the parameters c, f. As is well-known, M is a model of ZFC. Since $c \restriction M \in M$ is an acyclic decomposition in M, by Proposition **??** M satisfies CH. The argument will be complete if we show that $2^\omega \subset M$.

We first show that M contains uncountably many reals. Suppose towards contradiction that this fails. Working in M, write $\kappa = \omega_1^M$ and find an uncountable sequence $\langle x_\alpha \colon \alpha \in \kappa\rangle$ of elements of 2^ω which is increasing in the Turing reducibility order. Consider the injective map $g\colon \mathcal{P}(\kappa) \cap M \to 2^\omega$ defined by $g(b) = f(\{x_\alpha \colon \alpha \in b\})$; note that the definition is enabled by the initial contradictory assumption. The map g is definable, therefore belongs to M and in M, it witnesses $2^{\aleph_1} \leq 2^{\aleph_0}$, in contradiction with the Continuum Hypothesis.

Now, we are ready to show that $2^\omega \subset M$. Suppose that $x \in 2^\omega$ is a point. Since M contains uncountably many elements of 2^ω, there must be distinct points $y_0, y_1 \in 2^\omega \cap M$ such that $c(x, y_0) = c(x, y_1) = n$ for some number $n \in \omega$. There cannot be any other point $x' \in 2^\omega$ such that $c(x', y_0) = c(x', y_1) = n$, because the sequence x, y_0, x', y_1 would form a monochromatic cycle. Thus, x is defined by the demand $c(x, y_0) = c(x, y_1) = n$ and so it is an element of M as desired. $\square$

9.2 The smooth divide

A basic result in the theory of analytic equivalence relations says that there is no Borel reduction from $\mathbb{E}_0$ to the identity on 2^ω. A basic concern of our theory then is when $|\mathbb{E}_0| > |2^\omega|$ holds in the balanced extensions of the symmetric Solovay model. The take home lesson of this section is that if P is a balanced Suslin poset whose balanced conditions are naturally organized into a compact Hausdorff space, then in the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$ indeed holds. There are other ways to preserve the inequality $|\mathbb{E}_0| > |2^\omega|$ discussed in Chapter 11, but the compactness arguments are the most elegant and cover a lot of ground.

Definition 9.2.1. A Suslin poset P is *compactly balanced* if there is a definable compact Hausdorff topology T on the set B of all equivalence classes of balanced virtual conditions such that

1. for every $p \in P$ the set $\{\bar{p} \in B \colon \bar{p} \leq p\} \subset B$ is nonempty and T-closed.

Moreover, if $V[H_0] \subset V[H_1]$ are generic extensions of V then

2. for every balanced virtual condition $\bar{p}_0 \in V[H_0]$ there is a balanced virtual condition $\bar{p}_1 \in V[H_1]$ such that $\bar{p}_1 \leq \bar{p}_0$ holds;
3. the relation $\{\langle \bar{p}_0, \bar{p}_1\rangle \in B^{V[H_0]} \times B^{V[H_1]} \colon \bar{p}_1 \leq \bar{p}_0\}$ is closed in $T^{V[H_0]} \times T^{V[H_1]}$.

The definition seems to be obscure at the first reading, but it is fully justified by the natural compact Hausdorff topologies present in the examples. One rather (intentionally) unclear feature is the definability of the topology T. It is central in that otherwise the definition would have no content and the key item (3) could not be stated without it. The definition is allowed to have real parameters. However, the nature of the definition of T is left undetermined. In the examples, the topology T is typically nonseparable, and the compact Hausdorff space $\langle B, T\rangle$ can be in principle very complicated. The following theorem is the main result of this section. Its statement uses Convention 1.7.16.

Theorem 9.2.2. *In compactly balanced extensions of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$ holds.*

The proof uses a technical tool which has been studied in its own right: the Vitali forcing, also called $\mathbb{E}_0$-forcing in [48, Section 10.9] or in [101, Section

4.7.1], where it was isolated for the first time. To define it, consider the σ-ideal I on 2^ω generated by Borel partial $\mathbb{E}_0$-transversals, and let Q be the poset of Borel I-positive sets ordered by inclusion. As every poset of this form, Q adds a generic point in 2^ω–the unique point which belongs to all Borel sets in the generic filter [101, Proposition 2.1.2].

Fact 9.2.3. *Let Q be the Vitali forcing and let $G \subset Q$ be a generic filter. Then*

1. [101, Theorem 4.7.3] *every real in $V[G]$ is the image of the generic point $x_{\rm gen}$ under a ground model Borel function (Q is proper);*
2. [84, Proposition 4.5] *for every set $a \subset \omega$ in $V[G]$, either a or its complement contain a ground model infinite set (Q adds no independent reals);*
3. [103, Theorem 2.11] *every real definable in $V[G]$ from $[x_{\rm gen}]_{\mathbb{E}_0}$ and ground model parameters belongs to the ground model.*

Proof of Theorem 9.2.2. Let κ be an inaccessible cardinal. Suppose that P is a Suslin forcing such that P is compactly balanced below κ. Let W be a symmetric Solovay model derived from κ and work in W. Towards a contradiction, assume that $p \in P$ is a condition and τ is a P-name for an injection from the $\mathbb{E}_0$-classes to 2^ω. Both p, τ are definable from some real parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate generic extension of V obtained by a forcing of size $< \kappa$ such that $z \in V[K]$ holds, and work in the model $V[K]$.

Let Q_0 be the poset of infinite subsets of ω, ordered by inclusion. Let Q_1 be the Vitali forcing of Borel subsets of 2^ω on which $\mathbb{E}_0$ is not smooth, ordered by inclusion. Let $\langle U, y\rangle$ be an object $Q_0 \times Q_1$-generic over the model $V[K]$; that is, U is an ultrafilter on ω and $y \in 2^\omega$ is a point. By the σ-closure of Q_0, the models $V[K]$ and $V[K][U]$ have the same reals, in particular they evaluate the definition of the Vitali forcing in the same way and so y is a Vitali-generic point over $V[K][U]$ by the product forcing theorem. By Fact 9.2.3(2), and a genericity argument, the ultrafilter U still generates an ultrafilter on ω in the model $V[K][U][y]$.

Now, work in the model $V[K][U]$. Let $\bar{p}_0 \leq p$ be a balanced virtual condition below p. Let also χ be any Q_1-name for a balanced virtual condition in P in the model $V[K][U][y]$ below $\bar{p}_0$. This is possible by (2) of Definition 9.2.1. Note that the model $V[K][U][y]$ has more reals than $V[K][U]$; thus, the balance of the condition $\bar{p}_0$ does not necessarily transfer from $V[K][U]$ to $V[K][U][y]$ and $\bar{p}_0$ may have to be improved in a nontrivial way to get a balanced condition in $V[K][U][y]$. For each $n \in \omega$, let $y_n \in 2^\omega$ be the binary sequence obtained from y by replacing its first n entries by 0. Note that for each $n \in \omega$, y_n is still Q_1-generic over $V[K][U]$, and $V[K][U][y] = V[K][U][y_n]$. In the model $V[K][U][y]$, let $\bar{p}_1$ be the U-limit of the sequence χ/y_n for $n \in \omega$. By (3) of Definition 9.2.1, it is the case that $\bar{p}_1 \leq \bar{p}_0 \leq p$. Note also that the definition of $\bar{p}_1$ does not depend on y per se, but only on the $\mathbb{E}_0$-class of y. The treatment divides into two cases according to the fate of the value $\tau([y]_{\mathbb{E}_0})$ as forced by $\bar{p}_1$, and a contradiction is reached in each case.

Case 1. In the model $V[K][U][y]$, there is no point $u \in 2^\omega$ such that $\mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p}_1 \Vdash_P \tau([y]_{\mathbb{E}_0}) = \check{u}$. In such a case, there have to be disjoint basic open subsets O_0, O_1 of 2^ω, posets R_0, R_1 of size $< \kappa$, and respective R_0, R_1-names σ_0, σ_1 for conditions in P stronger than $\bar{p}$ such that $R_0 \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau([y]_{\mathbb{E}_0}) \in O_0$, and similarly for subscript 1. In the model W, let $H_0 \subset R_0$ and $H_1 \subset R_1$ be filters mutually generic over the model $V[K][U][y]$ and consider the conditions $\sigma_0/H_0, \sigma_1/H_1 \in P$. By the balance of the condition $\bar{p}_1$, the conditions $\sigma_0/H_0, \sigma_1/H_1$ are compatible. The forcing theorem applied in both models $V[K][U][y][H_0]$ and $V[K][U][y][H_1]$ now says that in W, their lower bound forces $\tau([y]_{\mathbb{E}_0})$ to belong simultaneously to O_0 and to O_1, an impossibility.
Case 2. There is a point $u \in 2^\omega$ in $V[K][U][y]$ such that $\mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p}_1 \Vdash_P \tau([y]_{\mathbb{E}_0}) = \check{u}$. This point u is clearly unique, and since $\bar{p}_1$ is definable from $[y]_{\mathbb{E}_0}$, so is u. By Fact 9.2.3(3), $u \in V[K][U]$ holds. Working in $V[K][U]$, let $q \in Q_1$ be a condition forcing $\mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p}_1 \Vdash_P \tau([x_{\mathrm{gen}}]_{\mathbb{E}_0}) = \check{u}$, where x_{gen} is the Q_1-name for its generic point. Let $H_0, H_1 \subset Q$ be filters mutually generic over $V[K][U]$ containing the condition q. Let $y^0, y^1 \in 2^\omega$ be the respective generic points; observe that y^0, y^1 are not $\mathbb{E}_0$-related by mutual genericity. Write also $p^0 = \bar{p}_1/H_0$ and $p^1 = \bar{p}_1/H_1$. These are two virtual conditions in P strengthening $\bar{p}_0$. By the balance of $\bar{p}_0$, they are compatible in the poset P. Then, in the model W, their lower bound forces that $\tau([\check{y}^0]_{\mathbb{E}_0}) = \tau([\check{y}^1]_{\mathbb{E}_0}) = \check{u}$. This contradicts the assumption that τ was forced to be an injection. $\square$

Example 9.2.4. Let P be the poset of $\mathcal{P}(\omega)$ modulo finite. By Theorem 7.1.2, the balanced virtual conditions are classified by nonprincipal ultrafilters on ω. Equip the set of balanced classes by the topology of the remainder $\beta\omega$. It is immediate that the topology witnesses the fact that P is compactly balanced.

Corollary 9.2.5.

1. *Let P be the poset $\mathcal{P}(\omega)$ modulo finite. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Ramsey ultrafilter on ω, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Example 9.2.6. Let A be a Ramsey sequence of finite structures and P_A the associated σ-closed poset as in Definition 7.3.1. Theorem 7.3.4 provides a classification of the balanced virtual conditions: they correspond to Ramsey sequences of ultrafilters on the respective domain sets D_n for $n \in \omega$. Sequences of ultrafilters of this type form a closed subset of the product $\prod_n D_n^*$ of the respective compact sequences of ultrafilters, and so form a compact Hausdorff space. Claim 7.3.6 confirms the extension property Definition 9.2.1(2). It follows that the poset P_A is compactly balanced.

The conjunction of Examples 9.2.6 and 7.3.8 now yields the following:

Corollary 9.2.7. *Let $k \in \omega$ be a number.*

1. *Let P be the poset P_A where A is a sequence enumerating all finite ordered graphs with no clique of size k. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is an ultrafilter U on ω such that $U \to (U, k)^2$ and $U \not\to (U, k+1)^2$, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Example 9.2.8. Let P be the Fin×Fin poset of Definition 7.2.1. The poset P is compactly balanced.

Proof. The balanced virtual conditions of P were classified in Theorem 7.2.2. They correspond to ultrafilters on $\omega\times\omega^*$ which do not contain the set $n\times\omega^*$ for any number $n \in \omega$. Thus, the space of balanced virtual conditions is naturally organized into a definable compact Hausdorff space. It is necessary to verify the conditions (2, 3) of Definition 9.2.1. Let $V[H_0] \subset V[H_1]$ be generic extensions.

We first evaluate the complexity of the order between the balanced virtual conditions in $V[H_0]$ and $V[H_1]$. Consider the partial map $f\colon (\omega \times \omega^*)^{V[H_1]} \to (\omega \times \omega^*)^{V[H_0]}$ defined by $f(n, U) = \langle n, U \cap V[H_0]\rangle$; if $U \cap V[H_0] \notin V[H_0]$, the functional value is left undefined. Unraveling the definitions, a balanced virtual condition in $V[H_0]$ correspoding to an ultrafilter W_0 on $\omega \times \omega^*$ is weaker than the balanced virtual condition in $V[H_1]$ corresponding to some ultrafilter W_1 on $\omega \times \omega^*$ just in case for every set $A \in W_0$, the set $f^{-1}A$ belongs to W_1. In view of the topology on the ultrafilter spaces, this is a closed relation.

To see that every balanced virtual condition in $V[H_0]$ can be extended to a balanced virtual condition in $V[H_1]$, suppose that W_0 is an ultrafilter on $\omega\times\omega^*$ in the model $V[H_0]$. Considering the function f from the previous paragraph, it is obvious that the collection $\{f^{-1}A\colon A \in W_0\}$ of subsets of $\omega\times\omega^*$ in the model $V[H_1]$ has the finite intersection property, and therefore can be extended to an ultrafilter W_1. The ultrafilter W_1 corresponds to a balanced virtual condition in the model $V[H_1]$ stronger than the one corresponding to W_0. □

Corollary 9.2.9.

1. *Let P be the Fin×Fin poset of Definition 7.2.1. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is an ultrafilter U on ω such that ???, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Question 9.2.10. Is there a Borel filter F on ω such that ZF+DC proves that existence of a nonprincipal ultrafilter extending F implies $|\mathbb{E}_0| \leq |2^\omega|$?

Example 9.2.11. Let $\mathcal{F}$ be a Fraissé class in a finite relational language with strong amalgamation. Let E be a Borel equivalence relation on a Polish space X. Let $P(\mathcal{F}, E)$ be the poset of Definition 8.6.3 for adding a $\mathcal{F}$-structure to the E-quotient space. Then $P(\mathcal{F}, E)$ is compactly balanced: the balanced conditions are classified by $\mathcal{F}$-structures on the virtual quotient space X^{**}, which form a compact subspace of $\mathcal{P}([X^{**}]^n)$ where n corresponds to the arity of the relations in the language for $\mathcal{F}$. It is easy to verify that the demands of Definition 9.2.1 are satisfied with the inherited topology.

Corollary 9.2.12. *Let E be a Borel equivalence relation on a Polish space X.*

1. *Let P be the linearization poset for E of Example 8.6.5. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > 2^\omega$;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, the E-quotient space can be linearly ordered, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Another class of examples stems from posets which select a single structure on each E-class from a compact class of structures, where E is a countable Borel equivalence relation.

Example 9.2.13. Let X be a Polish space and let Γ be a locally finite bipartite Borel graph on X satisfying the Hall's marriage condition. Consider the poset P adding a perfect Γ-matching as in Example 6.2.4. The balanced virtual conditions are classified by perfect matchings. The set of perfect matchings is naturally viewed as a compact subset of $\mathcal{P}(\Gamma)$. It is not difficult to check that the inherited compact topology satisfies the demands of Definition 9.2.1.

Corollary 9.2.14. *Let X be a Polish space and let Γ be a locally finite bipartite Borel graph on X satisfying Hall's marriage condition.*

1. *Let P be the complete matching poset for Γ. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, Γ has a perfect matching, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Example 9.2.15. Let X be a Polish space and let Γ be a Borel, acyclic, locally finite graph on X in which every vertex has degree at least 2; view Γ as a symmetric subset of X^2. Consider the poset P of Example 6.2.5, adding an orientation of Γ in which every vertex has exactly one point in its outflow, or in other words which selects an end to each connected component of Γ. The balanced conditions are classified by all such orientations, which form a compact subset of $\mathcal{P}(\Gamma)$. It is not difficult to check that the inherited compact topology satisfies the demands of Definition 9.2.1.

This example is somewhat singular in that it is the only compactly balanced poset for which we are able to confirm that it introduces some new cardinal inequalities between quotient cardinals. Namely, writing E for the Γ-path-connectedness equivalence relation, in the P-extension of the symmetric Solovay model $|E| \leq |\mathbb{E}_0|$ must hold. To see this, consider the equivalence relation F on the set $D \subset X^\omega$ of all sequences consisting of pairwise E-equivalent points, making two such sequences $\vec{x}_0, \vec{x}_1$ F-equivalent if some tail of $\vec{x}_0$ is equal to some tail of $\vec{x}_1$. It is not difficult to see that F is a hypersmooth equivalence relation [48, Theorem 8.3.1] with all classes countable and so by a standard result [48, Theorem 8.1.1], F is Borel reducible to $\mathbb{E}_0$. Consider the P-generic orientation $\vec{\Gamma}$ of the graph Γ; each vertex has exactly one point in its outflow. For each point $x \in X$ let $h(x) = \langle x_i : i \in \omega \rangle$ where $x_0 = x$ and x_{i+1} is the unique point in the $\vec{\Gamma}$-outflow of x_i. Then h is a (non-Borel) reduction of E to F and so in the P-extension $|E| \leq |F| \leq |\mathbb{E}_0|$ holds.

Corollary 9.2.16. *Let X be a Polish space and let Γ be a Borel, acyclic, locally finite bipartite Borel graph on X in which every vertex has degree at least 2.*

1. *Let P be the poset selecting an end to each connected component of Γ. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a function assigning exactly one end to each connected component of Γ, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Example 9.2.17. Let $n \in \omega$ be a number, let X be a Polish space and let Γ be a locally countable Borel graph on X such that every finite subgraph of Γ has chromatic number $\leq n$. Consider the poset P adding a Γ-coloring by n colors as described in Example 6.2.6. The balanced virtual conditions are classified by total Γ-colorings by $\leq n$-colors; these form a compact subset of n^X. It is not difficult to see that the inherited compact product topology satisfies the demands of Definition 9.2.1.

Corollary 9.2.18. *Let X be a Polish space and let Γ be a locally countable Borel graph on X such that each finite subset of Γ has chromatic number $\leq n$ for some fixed number $n \in \omega$.*

1. *Let P be the coloring poset for Γ. In the P extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, Γ has chromatic number $\leq n$, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Question 9.2.19. Is there a Borel graph G on a Polish space X such that each finite subgraph of G has chromatic number $\leq n$, and ZF+DC proves that if G has finite chromatic number then $|\mathbb{E}_0| \leq |2^\omega|$?

Example 9.2.20. Let Γ, Δ be abelian Polish groups, with Δ divisible and compact. Let $P(\Gamma, \Delta)$ be the poset adding a homomorphism from Γ to Δ as isolated in Definition 8.2.1. As proved in Theorem 8.2.2, the balanced conditions are classified by homomorphisms from Γ to Δ. The space of homomorphisms is a closed subset of Δ^Γ equipped with the product topology. It is not difficult to see that the demands of Definition 9.2.1 are met. Let us elaborate on the extension property (2). If $h\colon \Gamma \to \Delta$ is a homomorphism in some generic extension $V[H_0]$ then it can be extended to a homomorphism in any larger forcing extension $V[H_1]$ by the divisibility of Δ and Baer's criterion [3]. Note that the group Δ remains abelian and divisible in all generic extensions by Mostowski absoluteness.

Corollary 9.2.21. *Let Γ, Δ be Polish abelian groups, with Γ uncountable and torsion free, and Δ divisible and compact.*

1. *In the $P(\Gamma, \Delta)$ extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a discontinuous homomorphism from Γ to Δ, and yet $|\mathbb{E}_0| > |2^\omega|$.*

Proof. Recal from Example 8.2.3 that the generic homomorphism added by the poset $P(\Gamma, \Delta)$ is discontinuous. Other than that, the corollary follows from Theorem 9.2.2. □

The assumption that Δ be compact cannot be dropped entirely from the statement of the corollary by the following ZF observation:

Proposition 9.2.22. (ZF+DC) *Let Y be a separable Banach space. If there is a discontinuous homomorphism $h\colon Y \to Y$ then there is an $\mathbb{E}_0$-transversal, in particular $|\mathbb{E}_0| \leq |2^\omega|$.*

Proof. The discontinuity of the homomorphism and the DC assumption yield a sequence $\langle y_n \colon n \in \omega\rangle$ of elements of Y such that $|y_n| > \Sigma_{m>n}|y_m|$ and $|h(y_{n+1}) > 2|h(y_n)|$. Now, for every $x \in 2^\omega$ let $g(x) = \Sigma\{y_n \colon x(n) = 1\} \in Y$. Let $d \subset 2^\omega$ be any $\mathbb{E}_0$-class. The homomorphism assumptions on h show that the function $g \restriction d$ is injective and also, the norms of the points in $h \circ g(d)$ diverge to infinity. Thus, d contains a finite subset of points whose $h \circ g$-images have the smallest possible norm, and one can let $x_d \in d$ be the lexicographically smallest point in d the norm of whose $h \circ g$-image is as small as possible. The set $\{x_d \colon d$ is an $\mathbb{E}_0$-class$\}$ is an $\mathbb{E}_0$-transversal. □

Example 9.2.23. The Kurepa poset P on a Polish space X of Definition 8.4.2 is compactly balanced. To see this, equip 2^X, identified with $\mathcal{P}(X)$, with the usual product topology, refer to Theorem 8.4.3 to argue that the balanced conditions are classified by subsets of $\mathcal{P}(X)$ closed under intersections and containing X, and observe, that sets of this type form a closed and therefore compact subset of $\mathcal{P}(X)$. The properties (1-3) of Definition 9.2.1 are easily verified for this topology.

Corollary 9.2.24. *Let X be a Polish space.*

1. *In the Kurepa poset extension of the symmetric Solovay model, $|\mathbb{E}_0| > |2^\omega|$;*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Kurepa family on X, and yet $|\mathbb{E}_0| > |2^\omega|$.*

As a final remark in this section, the consistency results obtained here can be combined using the countable support product. The balanced virtual conditions in a countable product of posets are simply sequences of balanced virtual conditions in each coordinate (Theorem 5.2.9). A product of Hausdorff compact spaces is Hasudorff compact again; thus, a product of countably many compactly balanced forcings with full support is compactly balanced again. However, the machinery of Chapter 11 (which can also be used to show that the smooth divide is preserved in certain extensions) seems to be incompatible with the compactly balanced approach.

9.3 The turbulent divide

We wish to transfer the ergodicity theorem 3.3.5 to cardinal inequalities in generic extensions of the Solovay model. The following variation of balance will be central in this effort.

Definition 9.3.1. Let P be a Suslin forcing.

1. A pair $\langle Q, \tau\rangle$ is *placid* if $Q \Vdash \tau \subset P$ is an analytic set and whenever R_0, R_1 are posets and in some ambient forcing extension, $H_0 \subset R_0 \times Q$ and $H_1 \subset R_1 \times Q$ are filters separately mutually generic over V such that $V[H_0] \cap V[H_1] = V$, then any two conditions $p_0 \leq \Sigma\tau/H_0$ and $p_1 \leq \Sigma\tau/H_1$ in the respective models $V[H_0], V[H_1]$ are compatible in P.
2. The poset P is *placid* if below every condition $p \in P$ there is a virtual condition $\bar{p} \leq p$ which is placid.

As an initial example, consider the poset P of countable functions from 2^ω to 2 ordered by reverse inclusion. The balanced virtual conditions are classified by total functions from 2^ω to 2. It turns out that every such a virtual condition $\bar{p}$ is placid. If $V[H_0]$ and $V[H_1]$ are generic extensions such that $V[H_0] \cap V[H_1] = V$ and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are conditions stronger than $\bar{p}$, then $p_0 \cup p_1$ is a function since $\mathrm{dom}(p_0) \setminus V$ and $\mathrm{dom}(p_1) \setminus V$ are disjoint sets and the functions p_0, p_1 agree on the entries from V. Therefore, the conditions p_0, p_1 are compatible as desired.
Placid posets share many preservation properties. The following theorems are stated using the standard Convention 1.7.16. The first theorem in addition uses a standard parlance.

Definition 9.3.2. (ZF) The phrase "the turbulent divide is preserved" denotes the following statement: Let E be an analytic equivalence relation on a Polish space induced as an orbit equivalence relation of a turbulent Polish group action. Let F be a virtually placid analytic equivalence relation on a Polish space. Then $|E| \not\leq |F|$.

Theorem 9.3.3. *In cofinally placid extensions of the symmetric Solovay model, the turbulent divide is preserved.*

Proof. Let Γ be a Polish group, turbulently acting on a Polish space X, resulting in the equivalence relation E. Let F be a virtually placid orbit equivalence relation on a Polish space Y. Let P be a Suslin forcing and let κ be an inaccessible cardinal such that P is cofinally placid below κ. Let W be a symmetric

Solovay model derived from κ and work in the model W. Suppose towards a contradiction that there is a condition $p \in P$ and a P-name for a function which is an injection from the E-quotient space to the F-quotient space. The condition p as well as the name τ must be definable from some ground model parameters together with a parameter $z \in 2^\omega$. Use the assumptions to find an intermediate model $V[K]$, which is obtained from V by a poset of cardinality smaller than κ, contains z and in which the poset P is placid.

Work in the model $V[K]$. Let $\bar{p} \leq p$ be a virtual condition in P which is weakly placid. Let P_X be the poset for adding a Cohen point of the space X. That is, P_X is the poset of all nonempty open subsets of X ordered by inclusion, adding a point $\dot{x} \in X$. There must be a poset R of size $< \kappa$ and $P_X \times R$-names σ for an element of the poset P stronger than $\bar{p}$ and η for an element of Y such that $P_X \times R \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau([\dot{x}]_E) = [\eta]_F$. There are two cases.

Case 1. There is a nonempty open set $O \subset X$ and a condition $r \in R$ such that the name η is F-pinned below $\langle O, r\rangle$. In this case, let $x_0, x_1 \in X$ and $H_0, H_1 \subset R$ be points in O and filters on R containing r mutually generic over the model $V[K]$. Let $p_0 = \sigma/x_0, H_0 \in P \cap V[K][x_0][H_0]$ and $p_1 = \sigma/x_1, H_1 \in P \cap V[K][x_1][H_1]$ and let $y_0 = \eta/x_0, H_0$ and $y_1 = \eta/x_1, H_1$.

The points $x_0, x_1 \in X$ are mutually Cohen generic, and since their respective E-classes are meager, it must be the case that they are unrelated. By the case assumption, the points $y_0, y_1 \in F$ are F-related. By the balance of the condition $\bar{p}$, the conditions p_0, p_1 are compatible in the poset P. Now, the forcing theorem applied in the model $V[K]$ shows that in the model W, the lower bound of p_0, p_1 forces in P that $\tau([x_0]_E)$ and $\tau([x_1]_E)$ are both equal to $[y_0]_F$, contradicting the injectivity assumption on τ.

Case 2. Case 1 fails. This is to say that η is forced not to be a realization of any F-pinned class of $V[K]$. Let P_Γ be the Cohen poset on the Polish group Γ and let $x_0 \in X$ and $\gamma \in \Gamma$ be points mutually generic for P_X, P_Γ over the model $V[K]$, and write $x_1 = \gamma \cdot x_0$.

Since the action of the group is a continuous open map from $\Gamma \times X$ to X, the point x_y is P_X-generic over the model $V[K]$ by Proposition 3.1.1 applied in $V[K]$. By Theorem 3.2.2 (this is the only place where we use the turbulence assumption), $V[K][x_0] \cap V[K][x_1] = V[K]$ holds. Let $H_0, H_1 \subset R$ be filters mutually generic over the model $V[K][\gamma, x_0]$. By the mutual genericity, $V[K][x_0][H_0] \cap V[K][x_1][H_1] = V[K]$ holds. Write $p_0 = \sigma/x_0, H_0$ and $p_1 = x_1, H_1$, and also $y_0 = \eta/x_0, H_0$ and $y_1 = \eta/x_1, H_1$.

By their initial choice, the points $x_0, x_1 \in X$ are E-related. By the virtual placidity assumption on the equivalence relation F, the points $y_0, y_1 \in Y$ are not F-related: if they were, they would be realizations of some virtual F-class in $V[K]$, contradicting the case assumption. By the placidity assumption on the virtual condition $\bar{p}$, the conditions p_0, p_1 have a common lower bound. Now, the forcing theorem applied in $V[K]$ shows that in the model W, the lower bound of p_0, p_1 forces in P that $\tau([x_0]_E)$ must be equal simultaneously to $[y_0]_F$ and $[y_1]_F$. This is a contradiction, as τ is forced to be a function. $\square$

Other preservation properties are proved in Section 12.3, respectively in Section 12.2 based on the observation (Example 12.2.8) that placid forcings are included in the much wider class of Bernstein balanced forcings.

It is now time to provide a long but not exhaustive list of examples with their associated corollaries.

Example 9.3.4. Suppose that $\mathcal{K}$ is a Borel simplicial complex on a Polish space X of Borel coloring number $\aleph_1$. Then the poset $P_{\mathcal{K}}$ is placid, and every balanced virtual condition is placid. To see this, revisit Theorem 6.3.3 and observe that its proof uses the product forcing theorem only to ascertain that if $V[H_0], V[H_1]$ are mutually generic extensions of the ground model, then $V[H_0] \cap V[H_1] = V$.

Example 9.3.5. Suppose that $\mathcal{K}$ is a modular Borel complex on a Polish space X. Every balanced virtual condition in the poset $P_{\mathcal{K}}$ is placid. To see this, revisit the proof of Theorem 6.4.3 and note that $V[H_0] \cap V[H_1] = V$ was the only feature used of the mutually generic extensions $V[H_0], V[H_1]$.

Corollary 9.3.6. *Let X be a Borel vector space over a countable field Φ.*

1. *Let P be the poset of countable subsets of X linearly independent over Φ, ordered by reverse inclusion–Example 6.4.9. In the P-extension of the symmetric Solovay model, the turbulent divide is preserved;*
2. *it is consistent relatively to an inaccessible cardinal that ZF+DC holds, X has a basis, and yet the turbulent divide is preserved.*

Corollary 9.3.7. *Let Γ be a Borel graph on a Polish space X.*

1. *Let P be the poset of acyclic countable subsets of Γ, ordered by reverse inclusion–Example 6.4.10. In the P-extension of the symmetric Solovay model, the turbulent divide is preserved;*
2. *it is consistent relatively to an inaccessible cardinal that ZF+DC holds, Γ contains a maximal acyclic subgraph, and yet the turbulent divide is preserved.*

Example 9.3.8. Suppose that E, F are Borel virtually placid equivalence relations on the respective Polish spaces X, Y. Suppose that $\lambda(E) \leq \lambda(F)$. Then the E, F-collapse poset of Definition 6.6.2 is placid, and every balanced condition is placid.

Example 9.3.9. Suppose that $E \subset F$ are Borel equivalence relation on a Polish spaces X, with F placid. The E, F-transversal poset of Definition 6.6.5 is placid, and every balanced condition is placid.

Proof. The proof of Theorem 6.6.6 uses only one consequence of mutual genericity: the two mutually generic extensions $V[H_0], V[H_1]$ do not share any F-class which is not represented in the ground model. In the case of a placid equivalence relation E, this is implied by the assumption that $V[H_0] \cap V[H_1] = V$ by the definition of placidity. $\square$

Corollary 9.3.10. *Suppose that E is a Borel placid equivalence relation on a Polish space X..*

1. *In the extension of the symmetric Solovay model by the transversal poset for E of Example 6.6.8, the turbulent divide is preserved;*
2. *it is consistent relatively to an inaccessible cardinal that ZF+DC holds, E has a transversal, and yet the turbulent divide is preserved.*

The posets which select a structure on each E-class for a countable Borel equivalence relation E are placid by Example 9.3.9, resulting of many corollaries of the following kind.

Corollary 9.3.11. *Suppose that E is a countable Borel equivalence relation on a Polish space X with infinite classes.*

1. *In the extension of the symmetric Solovay model by the $\mathbb{Z}$-action poset of Example 6.6.10, the turbulent divide is preserved;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, E is an orbit equivalence of a (discontinuous) action of $\mathbb{Z}$, and yet the turbulent divide is preserved.*

Example 9.3.12. Suppose that $\mathcal{F}$ is a Fraissé class of structures in finite relational language, with strong amalgamation. Let E be a virtually placid equivalence relation on a Polish space X. Let P be the poset for adding an $\mathcal{F}$-structure on the quotient space X with countable approximations of Definition 8.6.3. Then P is placid, and every balanced condition is placid.

Corollary 9.3.13. *Let E be a Borel virtually placid equivalence relation on a Polish space X.*

1. *In the extension of the symmetric Solovay model by the linearization poset for E of Example 8.6.5, the turbulent divide is preserved;*
2. *it is consistent relatively to an inaccessible cardinal that ZF+DC holds, the E-quotient space is linearly ordered and yet the turbulent divide is preserved.*

Example 9.3.14. Suppose that X is a Polish space and P is the Kurepa poset on it as isolated in Definition 8.4.2. Then P is placid, and every balanced condition is placid. To see this observe that the proof of Theorem 8.4.3 used only the consequence $V[H_0] \cap V[H_1] = V$ for mutually generic filters H_0, H_1.

Corollary 9.3.15. *Let X be a Polish space.*

1. *In the extension of the symmetric Solovay model by the Kurepa poset, the turbulent divide is preserved;*

2. *it is consistent relatively to an inaccessible cardinal that ZF+DC holds, there is a Kurepa family on X, and yet the turbulent divide is preserved.*

Example 9.3.16. Let X be an uncountable Polish space and P be the associated acyclic decomposition forcing of Definition **??**. Then, under CH, P is placid and every balanced condition is placid. To see this observe that the proof of Theorem **??** used only the consequence $V[H_0] \cap V[H_1] = V$ for mutually generic filters H_0, H_1.

Corollary 9.3.17. *Let X be an uncountable Polish space.*

1. *In the extension of the symmetric Solovay model by the acyclic decomposition poset, the turbulent divide is preserved;*
2. *it is consistent relatively to an inaccessible cardinal that ZF+DC holds, there is an acyclic decomposition of $[X]^2$, and yet the turbulent divide is preserved.*

We conclude this section with several non-examples.

Example 9.3.18. Let E be the equivalence relation on 2^ω connecting sets x, y if $\Sigma\{\frac{1}{n+1} \colon n \in x \Delta y\}$ is finite. E is a Borel orbit equivalence relation of a turbulent Polish group action; it is well-known to be pinned. Let P be the poset of countable partial E-transversals ordered by reverse inclusion. The poset P is balanced by Corollary 6.6.4, but not placid: in fact, it is designed to collapse the turbulent divide as it forces $|E| \leq |2^\omega|$.

Example 9.3.19. Let X be an uncountable Polish field and $F \subset X$ be a countable subfield. The poset P of countable subsets of X which are algebraically independent over F, with the reverse inclusion ordering is balanced. It is not placid by Theorem 12.3.1 below. We do not know if in the P-extension of the symmetric Solovay model, the turbulent divide is preserved.

Example 9.3.20. The poset P of infinite subsets of ω ordered by inclusion is balanced. It is not placid by Theorem 12.2.3 below. We do not know if in the P-extension of the symmetric Solovay model, the turbulent divide is preserved.

9.4 The orbit divide

It is well-known that $\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation. It is then tempting to think that in many models which we study, $|\mathbb{E}_1|$ cannot be smaller than $|E|$ where E is an orbit equivalence relation. This question is with some success addressed in the present section. The following definition connects coherent sequences of generic extensions with Suslin forcings.

Definition 9.4.1. Let P be a Suslin forcing. A *nest* below a condition $p \in P$ is a choice-coherent sequence $\langle M_n \colon n \in \omega \rangle$ of generic extensions of V and a sequence $\langle \bar{p}_n \colon n \leq \omega \rangle$ so that

1. $2^\omega \cap M_n \neq 2^\omega \cap M_{n+1}$ for all $n \in \omega$;
2. $\bar{p}_0 \leq \bar{p}_1 \leq \cdots \leq \bar{p}_\omega \leq p$ where $\bar{p}_n$ for each $n \in \omega$ is a balanced virtual condition in M_n and $\bar{p}_\omega$ is a balanced virtual condition in the intersection model $M_\omega = \bigcap_n M_n$.

Definition 9.4.2. Let P be a Suslin forcing. The poset P is *nested balanced* below κ if it has a nest below every condition.

As a simple initial example, let P be the poset of all countable functions from 2^ω to 2, ordered by reverse inclusion. The balanced conditions are classified by total functions from 2^ω to 2. Now, let $\langle M_n : n \in \omega \rangle$ be a coherent sequence of models and $p \in M_\omega = \bigcap_n M_n$ be a condition in P. Then the functions $\bar{p}_n \in M_n$ for $n \leq \omega$ obtained in the model M_n from p by extending it by zero values at every possible point $x \in 2^\omega$ correspond to balanced virtual conditions in every model M_n and $p \geq \bar{p}_\omega \geq \cdots \geq \bar{p}_1 \geq \bar{p}_0$.
To state the results of this section succintly, we establish a standard parlance.

Definition 9.4.3. (ZF) The phrase "orbit divide is preserved" denotes the following statement. Let E be an orbit equivalence relation on a Polish space induced as an orbit equivalence relation of a Polish group action. Then $|\mathbb{E}_1| \not\leq |E|$.

The central theorem can now be stated easily using Convention 1.7.16.

Theorem 9.4.4. *In nested balanced extensions of the symmetric Solovay model, the orbit divide is preserved.*

Proof. Suppose that Y is a Polish space, Γ is a Polish group and Γ continuously acts on Y, inducing an orbit equivalence relation E. Suppose that κ is an inaccessible cardinal and P is a Suslin poset which is nested balanced below κ. Suppose also that W is a symmetric Solovay model derived from κ and in W, τ is a P-name and $p \in P$ is a condition forcing τ to be a function from the $\mathbb{E}_1$-quotient space to the E-quotient space. We must find a stronger condition and two distinct $\mathbb{E}_1$ classes which are forced by the stronger condition to be mapped to the same E-class.

The objects Γ, Y, p and τ are all definable in the model W from parameters in the ground model plus a parameter $z \in 2^\omega$. Use the assumptions to find an intermediate forcing extension $V[K]$ by a poset of size $< \kappa$ such that $z \in V[K]$ and $V[K]$ has a nest below p. Still working in W, find the nest. This is a sequence $\langle M_n : n \in \omega \rangle$ of generic extensions of $V[K]$ together with a sequence $\langle \bar{p}_n : n \leq \omega \rangle$ of balanced virtual conditions below p satisfying the demands of Definition 9.4.1.

Use the coherence of the sequence $\langle M_n : n \in \omega \rangle$ to find a well-ordering $\prec$ of 2^ω in M_0 such that for each $n \in \omega$, $\prec \restriction M_n \in M_n$ holds. Let $z_n \in 2^\omega$ be the $\prec$-least element of $M_n \setminus M_{n+1}$ for every $n \in \omega$ and let $x_n \in X = (2^\omega)^\omega$ be the point such that $x_n(m)$ is the zero binary sequence if $m < n$ and $x_n(m) = z_m$ if $m \geq n$; thus $x_n \in M_n$. The points $x_n \in X$ for $n \in \omega$ are pairwise $\mathbb{E}_1$-equivalent;

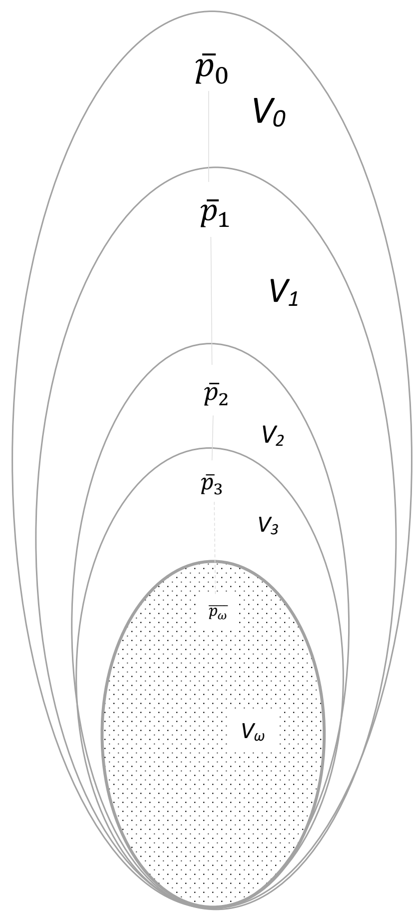

Figure 9.1: A nest of models and balanced virtual conditions.

at the same time, they have no $\mathbb{E}_1$-equivalent in the model $M_\omega = \bigcap_n M_n$, since none of the points z_n belong to M_ω.

By the forcing theorem, for each number $n \in \omega$, in the model M_n there must be a poset R_n of size $< \kappa$, an R_n-name σ_n for a condition in P stronger than $\bar{p}_n$ and an R_n-name η_n for an element of Y such that $R_n \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_n \Vdash_P \tau([\check{x}_n]_{\mathbb{E}_1}) = [\eta_n]_E$.

Claim 9.4.5. *For $n \in \omega$, the pair $\langle R_n, \eta_n\rangle$ is an E-pin.*

Proof. If this failed for some number $n \in \omega$, then in the model W there would be filters $H_0, H_1 \subset R_n$ mutually generic over the model M_n such that the points $y_0 = \sigma_n/H_0, y_1 = \sigma_n/H_1 \in Y$ are E-unrelated. Let $p_0 = \sigma_n/H_0$ and $p_1 = \sigma_n/H_1$ be conditions in the poset P; they are compatible by the balance of $\bar{p}_n$. By the forcing theorem applied in the models $M_n[H_0]$ and $M_n[H_1]$, their lower bound forces in P that $\tau([\check{x}_n]_{\mathbb{E}_1})$ is equal simultaneously to $[\check{y}_0]_E$ and $[\check{y}_1]_E$. This is impossible. □

Claim 9.4.6. *For $n \in m \in \omega$, the E-pins $\langle R_n, \eta_n\rangle$ and $\langle R_m, \eta_m\rangle$ are equivalent.*

Proof. If this failed for some numbers $n \in m \in \omega$, then in the model W there would be filters $H_0 \subset R_n, H_1 \subset R_m$ mutually generic over the model M_n such that the points $y_0 = \sigma_n/H_0, y_1 = \sigma_m/H_1 \in Y$ are E-unrelated. Let $p_0 = \sigma_n/H_0$ and $p_1 = \sigma_m/H_1$ be conditions in the poset P; they are compatible by the balance of $\bar{p}_m$ since $p_0 \leq \bar{p}_n \leq \bar{p}_m$ by demand (2) of Definition 9.4.1. By the forcing theorem applied in the models $M_n[H_0]$ and $M_m[H_1]$, their lower bound forces in P that $\tau([\check{x}_n]_{\mathbb{E}_1}) = [\check{y}_0]_E$ and $\tau([\check{x}_m]_{\mathbb{E}_1}) = [\check{y}_1]_E$. This is impossible since $x_n \; \mathbb{E}_1 \; x_m$ holds. □

By Theorem 4.3.6, there is an E-pin $\langle R, \eta\rangle \in M_\omega$ which is equivalent to all the pins $\langle R_n, \eta_n\rangle$ for $n \in \omega$. Still in M_ω, find a poset Q generating the model M_0, a Q-name $\dot{x}$ for the sequence x_0, and a Q-name $\dot{R}_0$ for the poset R_0 . By the forcing theorem, there must be a condition $q \in Q$ which forces the following: $\dot{x}$ has no $\mathbb{E}_1$-equivalent in M_ω, $\dot{R}_0 \Vdash \sigma_0 \leq p_\omega$ and $\check{R} \times \dot{R}_0 \Vdash \eta \; E \; \eta_0$. Note that the last statement means that the name η_0 is E-pinned on the iteration $Q \restriction q * \dot{R}_0$. In the model W, let $H_0, H_1 \subset Q \restriction q * \dot{R}_0$ be filters mutually generic over the model M_ω, and write $p_0 = \sigma_0/H_0$, $p_1 = \sigma_0/H_1$, $\bar{x}_0 = \dot{x}/H_0$, $\bar{x}_1 = \dot{x}/H_1$, $y_0 = \eta_0/H_0$ and $y_1 = \eta_0/H_1$. The balance of the condition p_ω implies that $p_0, p_1 \in P$ are compatible conditions. Since $\mathbb{E}_1$ is a pinned equivalence relation, the points $\bar{x}_0, \bar{x}_1 \in X$ are $\mathbb{E}_1$-unrelated. Since the name η_0 was pinned on the iteration $Q \restriction q * \dot{R}_0$, the points y_0, y_1 are E-related. In total, in the model W, the lower bound of the conditions p_0, p_1 forces that $\tau([\bar{x}_0]_{\mathbb{E}_1}) = \tau([\bar{x}_1]_{\mathbb{E}_1}) = [y_0]_E$, completing the proof. □

It turns out that one large class of nested balanced posets has already been isolated in Definition 9.2.1:

Example 9.4.7. Every compactly balanced Suslin poset is nested balanced.

Proof. Let P be a compactly balanced Suslin poset and $p \in P$ be a condition. Let $U \subset \mathcal{P}(\omega)$ modulo finite and $\langle x_n : n \in \omega \rangle$ be mutually generic ultrafilter and Sacks reals added with full support product. Let $\langle M_n : n \in \omega \rangle$ be the sequence of models determined by $M_n = V[U, y_m : m \geq n]$; this is a choice-coherent sequence by Theorem 4.3.5 or Example 4.3.4 applied in the model $V[U]$. We will produce a descending sequence of balanced virtual conditions as required by Definition 9.4.1.

It is well known [84] that the Sacks real product adds no independent reals, and therefore, by a genericity argument, U generates an ultrafilter on ω in the model M_0. The generating set of U is present already in the model $M_\omega = \bigcap_n M_n$. For each number $n \in \omega$, use the extension property (2) of Definition 9.2.1 repeatedly to find a sequence $\langle \bar{p}(\omega, n), \bar{p}(m, n) : m \leq n \rangle$ such that $\bar{p}(\omega, n)$ is a balanced virtual condition in M_ω, each $\bar{p}(m, n)$ is a balanced virtual condition in the model M_m, and $p \geq \bar{p}(\omega, n) \geq p(n, n) \geq \bar{p}(n-1, n) \geq \bar{p}(n-2, n) \geq \dots$; also, demand that each virtual condition $\bar{p}(m, n)$ is selected $\prec$-least where $\prec$ in M_0 is some coherent well-ordering of the set of virtual conditions on P. It is not difficult to see that the resulting system is coherent in the sense that for each $n \leq \omega$, the system $\langle p(m, k) : m = \omega \text{ and } k \in \omega \text{ or } k \geq m \geq n \rangle$ belongs to the model M_n.

Finally, use the compactness of the space of balanced virtual conditions to define $\bar{p}_n$ for $n \leq \omega$ to be the U-limit of the sequence $\langle \bar{p}(n, k) : k \geq n \rangle$. Each $\bar{p}_n$ is a balanced virtual condition in the model M_n, $m \leq n \leq \omega$ implies that $\bar{p}_m \leq \bar{p}_n \leq p$ by the closedness demands (1) and (3) of Definition 9.2.1. It follows that the sequence $\langle \bar{p}_n : n \leq \omega \rangle$ is as required in Definition 9.4.1. ☐

Corollary 9.4.8.

1. *Let P be the poset $\mathcal{P}(\omega)$ modulo finite. In the P-extension of the symmetric Solovay model, the orbit divide is preserved.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Ramsey ultrafilter on ω, and the orbit divide is preserved.*

Proof. This is a conjunction of Example 9.2.4 and Theorem 9.4.4. ☐

Question 9.4.9. Is there a Borel (or even F_σ) ideal I on ω such that the existence of an ultrafilter disjoint from I implies in ZF+DC that $|\mathbb{E}_1| \leq |E|$ for some orbit equivalence relation E?

Corollary 9.4.10. *Let E be a Borel equivalence relation on a Polish space X.*

1. *In the extension of the Solovay model by the E-linearization poset of Example 8.6.5, the orbit divide is preserved.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the E-quotient space is linearly ordered, and the orbit divide is preserved.*

Proof. This is a conjunction of Example 8.6.5, Example 9.2.11 and Theorem 9.4.4. ☐

Corollary 9.4.11. *Let Γ, Δ be abelian Polish groups, with Γ uncountable and torsion-free and Δ compact, nontrivial, and divisible.*

1. *Let $P(\Gamma, \Delta)$ be the poset of Definition 8.2.1. In the $P(\Gamma, \Delta)$-extension of the symmetric Solovay model, the orbit divide is preserved.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a discontinuous homomorphism from Γ to Δ, the orbit divide is preserved.*

Proof. This is a conjunction of Example 9.2.20 and Theorem 9.4.4. □

Question 9.4.12. Does the existence of a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$ imply in ZF+DC that $|\mathbb{E}_1| \leq |E|$ holds for some orbit equivalence relation E, or for $E = \mathbb{F}_2$ in particular? That $\mathbb{E}_1$ has a countable complete section?

Corollary 9.4.13. *Let X be a Polish space.*

1. *In the extension of the Solovay model by the Kurepa poset of Definition 8.4.2, the orbit divide is preserved.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Kurepa family on X, and the orbit divide is preserved.*

Proof. This is a conjunction of Example 9.2.23 and Theorem 9.4.4. □

Another group of nested balanced posets is obtained from orbit equivalence relations:

Example 9.4.14. Let $E \subset F$ be Borel equivalence relations on a Polish space X such that F is pinned and Borel reducible to an orbit equivalence relation. Let P be the E, F-transversal poset of Definition 6.6.5. Then P is nested balanced.

Proof. By Theorem 6.6.6, the balanced virtual conditions are classified by functions selecting, for each F-class, a single virtual E-class which is forced to be a subset of the F-class. Let $p \in P$ be a condition. To find a nest below p, consider any choice-coherent sequence $\langle M_n : n \in \omega\rangle$ consisting of generic extensions of the ground model such that for every $n \in \omega$, $2^\omega \cap M_n \setminus M_{n+1} \neq 0$–the choice-coherent sequence of models obtained from the infinite product of Sacks reals will do. Let $\prec$ be a coherent well-ordering of the space X. To find a coherent sequence of balanced virtual conditions, in the model M_0 consider the function f on the F-quotient space defined as follows: if c is an F-class mentioned in p, let $f(c)$ be the E-class $d \subset c$ which p selects. If c is a class represented in M_ω but not mentioned in p, then let $f(c)$ be the E-class of the $\prec$-first representative of c in the model M_ω. If $n \in \omega$ is the largest number such that c is represented in M_n, then let $f(c)$ be the E-class of the $\prec$-first representative of c in the model M_n. Now, by Theorem 4.3.6, this defines the function f on all F-equivalence classes represented in M_0. It is also clear that for each $n \leq \omega$, $f \restriction M_n \in M_n$. For each $n \leq \omega$, let $\bar{p}_n$ be the virtual balanced condition in the model M_n associated with the function $f \restriction M_n$ by Theorem 6.6.6. The system $\langle M_n : n \in \omega, \bar{p}_n : n \leq \omega\rangle$ is the sought nest below p. □

Corollary 9.4.15. *Let E be a Borel pinned orbit equivalence relation on a Polish space X.*

1. *Let P be the transversal poset of Example 6.6.8. In the P-extension of the symmetric Solovay model, the orbit divide is preserved.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, E has a transversal, and the orbit divide is preserved.*

Proof. This is a conjunction of Example 9.4.14 and Theorem 9.4.4. □

The posets selecting a structure on each E-class for a countable Borel equivalence relation E are always nested balanced by virtue of Example 9.4.14. The following is a sample corollary.

Corollary 9.4.16. *Let E be a countable Borel equivalence relation on a Polish space X, with all orbits infinite.*

1. *Let P be the $\mathbb{Z}$-action poset of Example 6.6.10. In the P-extension of the symmetric Solovay model, the orbit divide is preserved.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, E is induced as an orbit equivalence relation of a discontinuous $\mathbb{Z}$-action, and the orbit divide is preserved.*

The determination of the nested balanced status of the posets adding a maximal $\mathcal{K}$-set for a Borel simplicial complex $\mathcal{K}$ is much more challenging. The ultimate limitation is the surprising Corollary 9.4.30 below, showing in ZF+DC that if $\mathbb{R}$ has a Hamel basis, then $\mathbb{E}_1$ has a complete countable section, and therefore $|\mathbb{E}_1| \leq |\mathbb{F}_2|$ holds. Thus, we have to resort to partial results.

Example 9.4.17. Let Γ be a Borel graph on a Polish space X with Borel coloring number $\aleph_1$. Let $P = P_\Gamma$ be the poset of countable Γ-anticliques ordered by reverse inclusion. Then P is nested balanced.

Proof. Let $p \in P$ be a condition; we must produce a nest below $p \in P$. Let $\langle M_n \colon n \in \omega\rangle$ be a choice-coherent sequence of generic extensions such that M_0 is an extension of the ground model by proper forcing, and for all $n \in \omega$ the set $2^\omega \cap M_n \setminus M_{n+1}$ is nonempty. We will show that there is a sequence $\langle p_n \colon n \in \omega\rangle$ of balanced conditions satisfying the demands of Definition 9.4.1.

Theorem 6.3.3 classified the balanced conditions in the poset P in terms of weakly maximal Γ-anticliques. Thus, it will be enough to produce a set $A \subset X$ in the model M_0 which contains p as a subset and such that for each $n \leq \omega$, the set $A \cap M_n$ belongs to M_n and is a weakly maximal Γ-anticlique there. To cut the notational clutter, we ignore the condition $p \in P$. Let $\vec{\Gamma}$ be a Borel orientation of Γ such that the $\vec{\Gamma}$-outflow $o(y)$ of every vertex $y \in X$ is countable. Consider the set $B \subset X$ of all points $x \in X$ for which there is a countable set $a \subset X$ such that

- either for every countable set $b \subset X$ there is $y \notin b$ such that $x \in o(y) \subset a$;
- or for every countable set $b \subset X$ there is $y \in X$ such that $\langle y, x\rangle \in \vec{\Gamma}$ and $o(y) \setminus a$ is a nonempty set disjoint from b.

A careful complexity computation shows that the set B is $\mathbf{\Sigma}^1_2$. The following claim uses the properness assumption:

Claim 9.4.18. *Whenever $n \in \omega$ is a number and $y \in X \cap M_n \setminus M_{n+1}$ is a point and $\langle y, x\rangle \in \vec{\Gamma}$, then $x \in B$ or $x \in M_n \setminus M_{n+1}$.*

Proof. Suppose that $x \in M_{n+1}$; we must show that $x \in B$. The set $o(y) \cap M_{n+1}$ is countable in M_n; by the properness assumption, it is covered by a set $a \in M_{n+1}$ which is countable in M_{n+1}. We claim that a is a witness to $x \in B$. Indeed, if $o(y) \subset M_{n+1}$ then $x \in B$ by the first item in the definition of B–the point y is not in M_{n+1} and so it does not belong to any countable set in M_{n+1}. On the other hand, if $o(y) \subset M_{n+1}$ then $x \in B$ by the second item in the definition of B–the nonempty set $o(y) \setminus a$ is disjoint from any countable set in M_{n+1}. □

Now, work in M_0. Let $\prec$ be a coherent well-ordering of $\mathcal{P}(X)$, and for every number $n \in \omega$ let $C_n \in M_n$ be the $\prec$-least maximal Γ-anticlique in the set $(X \setminus B) \cap M_n \setminus M_{n+1}$. Let also $C_\omega \in M_\omega$ be the $\prec$-maximal Γ-anticlique in $(X \cap M_\omega) \setminus B$ extending p. Let $A = \bigcup_{n \leq \omega} C_n$; we claim that for each $n \leq \omega$, the set $A \cap M_n$ belongs to M_n and is weakly maximal there. Note that $A \cap M_n \in M_n$ follows immediately from the coherence of the choices above. The claim implies that $A \cap M_n$ is a Γ-anticlique and so a maximal Γ-anticlique in the set $(X \setminus B) \cap M_n$. It follows directly from the definition of a weakly maximal set (Definition 6.3.2) that $A \cap M_n$ is a weakly maximal set in M_n as desired. □

Corollary 9.4.19. *Let Γ be a Borel graph on a Polish space X of Borel coloring number $\aleph_1$.*

1. *In the P_Γ extension of the symmetric Solovay model and the orbit divide is preserved;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a maximal Γ-anticlique and the orbit divide is preserved.*

Example 9.4.20. Let X be a Polish space and let G be a Borel graph on X. Let $\mathcal{K}$ be the Borel simplicial complex on G of finite acyclic subsets of G, and let $P = P_{\mathcal{K}}$. Then P is nested balanced.

Proof. Let Q_m for each $m \in \omega$ be the countable support product of $\aleph_1$ many Sacks posets, and for each $n \in \omega$ let R_n be the countable support product $\Pi_{m \geq n} Q_m$, with the natural projection maps. Let $H_0 \subset R_0$ be a generic filter, and for each $n \in \omega$ let $H_n = H \cap R_n$. By Example 4.3.4 and Theorem 4.3.5, the sequence $\langle V[H_n] \colon n \in \omega\rangle$ is a choice-coherent sequence of models of ZFC; write $M_n = V[H_n]$ and $M_\omega = \bigcap_n M_n$. In view of Theorem **??**, to complete the proof of the example it will be enough for an arbitrary condition $p \in P \cap V$ to find sets $A_m \subset G$ for $m \leq \omega$ forming an inclusion-descending sequence such that $p \subset A_\omega$ and $M_m \models A_m \subset G$ is a maximal acyclic subset of G for every $m \leq \omega$.

Write $X_\omega = X \cap M_\omega$, and for each $m < n$ write $X_{mn} = X_\omega \cup (X \cap M_m \setminus M_n)$. The following elusive claim is the main reason for the choice of the posets Q_m.

Claim 9.4.21. *Let $l < m < n$ be natural numbers and $x_0, x_1 \in X_{mn}$ be vertices. If x_0, x_1 are connected by a path in the graph $G \restriction X_{ln}$, then they are connected by a path in the graph $G \restriction X_{mn}$.*

Proof. It will be enough to show that if x_0, x_1 are connected by a path in the graph $G \restriction X_{ln}$ whose vertices except for x_0, x_1 all belong to X_{lm}, then they are connected by a path in the graph $G \restriction X_{mn}$. To this end, work in M_m and let $A = \{a \in [X]^{<\aleph_0} :$ there is a G-path connecting x_0, x_1 using only the vertices in $a\}$.

The set $A \subset [X]^{<\aleph_0}$ is Borel. It cannot be punctured by a countable set by a Mostowski absoluteness argument: in the model M_l there is a G-path between x_0, x_1 using no vertices in M_m and therefore no vertices in any given countable set in M_m. By the work on puncture sets [16, Theorem 21] applied in the model M_m, there is a perfect set $B \subset A$ in the model M_m consisting of pairwise disjoint sets. By a standard fusion argument with the product of Sacks forcing, there is a countable set c of Sacks reals added by the filter H_m such that B has a code in the model $V[c] \subset M_m$. Since the poset Q_m is a product of uncountably many copies of Sacks forcing, in M_m there has to be an element $a \in B$ which does not belong to the model $M_n[c] \subset M_m$. Since the sets in B are pairwise disjoint, every element of a would reconstruct a over the model $M_n[c]$. Therefore, no element of a belongs to the model $M_n[c]$; in particular, $a \cap M_n = 0$ and a yields the desired G-path between x_0 and x_1 using only vertices in X_{mn}. $\square$

Let $\prec$ be a coherent well-ordering of all subsets of G in M_0. Let $A_\omega \subset G$ be a maximal acyclic subset of G in the model M_ω such that $p \subset A_\omega$. For each $m \in \omega$, let $A_{mm+1} \in M_m$ be the $\prec$-first set in the model M_m which is a maximal acyclic subset of $G \restriction X_{mm+1}$ and extends A_ω.

Now, by induction on $n - m$, for $m < n$ define $A_{mn} \in M_m$ to be the $\prec$-first set in the model M_m which is a maximal acyclic subset of $G \restriction X_{mn}$ and extends both $A_{m+1,n}$ and $A_{m,n-1}$. To see that this is possible, simultaneously argue by induction on $n - m$ that if $m \leq m' < n' \leq n$ then $A_{m'n'} \subset A_{mn}$ and moreover, the set $A_{m+1,n} \cup A_{m,n-1}$ does not contain a cycle. The former statement is clear. The latter statement requires the claim. A putative cycle $c \subset A_{m+1,n} \cup A_{m,n-1}$ has to contain some edges from both sets by the acyclicity induction hypothesis. Choose an inclusion-maximal contiguous part $d \subset c$ of the cycle consisting of edges in $A_{m,n-1} \setminus A_{m+1,n}$. The end-nodes of d, denote them by x_0, x_1, must be distinct because the cycle must use some edges from the set $A_{m+1,n}$ as well. It must also be the case that $x_0, x_1 \in X_{m,n-1} \cap X_{m+1,n} = X_{m+1,n-1}$. The path d connects the nodes x_0, x_1 in the graph $G \restriction X_{m,n-1}$; by the claim then, they have to be connected by a path in the graph $G \restriction X_{m+1,n-1}$ as well, and therefore by a path e in the set $A_{m+1,n-1}$. Then $e \cup d$ forms a cycle in the set $A_{m,n-1}$, violating the induction hypothesis.

In the end, let $A_m = \bigcup_{n>m} A_{mn}$. It is clear from the coherence of the well-ordering $\prec$ that $A_m \in M_m$. The construction also guarantees that $M_m \models A_m \subset G$ is a maximal acyclic subset, and $m < n$ implies $A_n \subset A_m$. The proof is complete. □

Corollary 9.4.22. *Let X be a Polish space and let G be a Borel graph on X.*

1. *Let P be the poset of Example 6.4.10 for adding a maximal acyclic subset of G. In the P-extension of the symmetric Solovay model, the orbit divide is preserved;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, G has a maximal acyclic subset, and the orbit divide is preserved.*

Example 9.4.23. Let Γ be a Borel graph on a Polish space X of countable coloring number. The coloring poset $P = P_\Gamma$ of Definition 8.1.1 is nested balanced.

Proof. We identify P with its dense Suslin subset identified in Theorem 8.1.2. Let $p \in P$ be a condition, and let $\langle M_n \colon n \in \omega \rangle$ be a nontrivial choice coherent sequence of generic extensions of V, with $M_\omega = \bigcap_n M_n$. We will produce a decreasing sequence $\langle \bar{p} \colon n \leq \omega \rangle$ of balanced virtual conditions below p for the respective models M_n. Let $\langle a_n \colon n \leq \omega \rangle$ be a recursive sequence of pairwise disjoint infinite subsets of ω.

Claim 9.4.24. *Fix $n \leq \omega$. In the model M_n, there is a total Γ coloring $d \colon X \to \omega$ which extends p and at all points $x \in X \setminus \mathrm{dom}(p)$, $d(x) \in a_n$.*

Proof. Work in M_n. Fix a coloring $e \colon X \to \omega$. Let $a_n = \bigcup_i b_i$ be a partition of a_n into infinitely many sets of size m. Let $d \colon X \to \omega$ be a function such that $p \subset d$ and for all $x \notin X \setminus \mathrm{dom}(p)$, $d(x)$ is a number in $e(x)$ which is not one of the (fewer than m many) colors $\{p(y) \colon \langle x, y \rangle \in \Gamma\}$. This is the desired coloring. □

Let $\prec$ be a coherent wellordering of $\mathcal{P}(X \times \omega)$, and for each $n \leq \omega$ let d_n be the $\prec$-least coloring as in the claim in the model M_n. Finally, in the model M_0, define a map $c \colon X \to \omega$ as follows: $c(x) = d_n(x)$ if $x \notin \mathrm{dom}(p)$ and $n \leq \omega$ is the largest number such that $x \in M_n$, and $c(x) = p(x)$ if $x \in \mathrm{dom}(p)$. It is not difficult to check that c is a Γ-coloring extending p. Moreover, the coherence of the well-ordering $\prec$ guarantees that for each $n \leq \omega$, $c \restriction M_n \in M_n$ holds. In each model M_n for $n \leq \omega$, the coloring $c \restriction M_n$ is associated with a balanced virtual condition $\bar{p}_n$ for the poset P_Γ by Theorem 8.1.2. The sequence $\langle \bar{p}_n \colon n \leq \omega \rangle$ is as desired. □

Corollary 9.4.25. *Let Γ be the graph on $\mathbb{R}^2$ connecting points x_0, x_1 if they have a nonzero rational distance.*

1. *In the P_Γ-extension of the symmetric Solovay model, the orbit divide is preserved;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable, and the orbit divide is preserved.*

Example 9.4.26. Let X be a Borel vector space over a countable field Φ. Let P be the 3-Hamel decomposition forcing of Definition 8.1.11. The poset P is nested balanced.

Proof. Let J be the ideal on ω from which the poset P is constructed. Let $p \in P$ be any condition. Let $\langle M_n : n \leq \omega\rangle$ be a generic choice-coherent sequence such that for every $n \in \omega$, the set $2^\omega \cap M_n \setminus M_{n+1}$ is nonempty. Let $\prec$ be a coherent well-ordering of all functions from X to ω. Let $\{a_n : n \leq \omega\} \in V$ be a collection of pairwise disjoint infinite sets such that $\bigcup_n a_n \in J$. Use Claim 8.1.14 in each model M_n to show that in M_n, there is a 3-Hamel decomposition extending p such that all points not in $\mathrm{dom}(p)$ get a color in the set a_n; let d_n be the $\prec$-least such a coloring. Now, working in M_n, let $c_n : X \to \omega$ be the map defined by $c_n(x) = d_\omega(x)$ if $x \in M_\omega$, and $c_n(x) = d_m(x)$ if $x \in M_m \setminus M_{m+1}$.

We claim that c_n is a 3-Hamel decomposition. To see this, suppose that x_0, x_1, x_2 are distinct nonzero points points in a nontrivial linear combination resulting in 0. Let $m \leq \omega$ be the largest index such that one of the points, say x_0, is in M_m. Either all three points are in M_m but not in M_{m+1} and then the triple is not monochromatic as d_m is a Γ-coloring. Or, the other two points are in $M_k \setminus M_{k+1}$ for some $k < m$. In such a case the triple is again not monochromatic: either $x_0 \in \mathrm{dom}(p)$ and then the failure of monochromaticity follows from the assumption that d_k is a 3-Hamel decomposition, or $x_0 \notin \mathrm{dom}(p)$ and then $c_n(x_0) \in a_m$ and $c_n(x_1) \in a_k$ and the sets a_m, a_k are disjoint.

Now, it is not difficult to see that $\langle c_n : n \in \omega\rangle$ is a coherent sequence of 3-Hamel decompositions. A 3-Hamel decomposition is a balanced condition for P by Theorem **??**. Finally, $p \geq c_\omega \geq \cdots \geq c_2 \geq c_1 \geq c_0$ holds in the poset P as the sets a_n for $n \leq \omega$ all belong to the ideal J. Thus, $\langle M_n, c_n : n \leq \omega\rangle$ is the required nest below the condition p. ☐

Corollary 9.4.27. *Let X be a Borel vector space over a countable field Φ. Let P be the 3-Hamel decomposition forcing of Definition 8.1.11.*

1. *Let P be the 3-Hamel decomposition forcing. In the P-extension of the symmetric Solovay model, the orbit divide is preserved;*

2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, there is 3-Hamel decomposition of X, and the orbit divide is preserved.*

In view of Example 8.1.15 it is consistent relative to an inaccessible cardinal that ZF+DC holds, the orbit divide is preserved, and there is a decomposition of $\mathbb{R}^2$ into countably many pieces neither of which contains the vertices of an equilateral triangle.

Now it is time to show that the existence of certain combinatorial objects implies that the orbit divide breaks. In the following theorem and corollaries, we neglect the real parameter necessary to define the Polish spaces in question. For the rest of the section, put $X = (2^\omega)^\omega$.

Theorem 9.4.28. (ZF) *Suppose that there is a Polish space Y and a set $A \subset Y$ such that the preordering $\leq$ on X, defined by $x_1 \leq x_0$ if $x_1 \in L[x_0, A]$, is well-founded. Then there is a countable complete section for $\mathbb{E}_1$.*

Proof. For each point $x \in X$ and each number $m \in \omega$, write $x \setminus m$ for the element of X defined by $(x \setminus m)(n) = x(n)$ if $n \geq m$, and $(x \setminus m)(n) =$the infinite sequence of zeroes if $m < n$. For each number $m \in \omega$ write $M_m(x)$ for the model $L[x \setminus m, A]$. By the well-foundedness assumption, there must be $n \in \omega$ such that for all $m \geq n$ the models $M_m(x)$ are the same. Note that if $x_0, x_1 \in X$ are $\mathbb{E}_1$-related points, the stable value of the models $M_m(x_0)$ and $M_m(x_1)$ is the same. We would like to define the complete section $D \subset X$ by setting $x \in D$ if x belongs to $\bigcap_m M_m(x)$. The stabilization feature shows that D has nonempty intersection with each $\mathbb{E}_1$-class. However, a more precise definition of the complete section is necessary to make the conclusion that the intersection with each $\mathbb{E}_1$-class is countable.

Observe first that by a standard condensation argument the model $M_m(x)$ is a model of ZFC+CH, and its constructibility ordering orders its elements of X in ordertype $\omega_1^{M_m(x)}$. For natural numbers $n \leq m \in \omega$ define $\rho_{nm}(x)$ to be the index of $x \setminus n$ in the constructibility well-ordering of $M_m(x)$ if $x \setminus n \in M_m(x)$, and let $\rho_{nm}(x) = 0$ otherwise. Let $\alpha(x) = \limsup_n \sup_{m \geq n} \rho_{nm}(x)$. The definition of the ordinal $\alpha(x)$ does not depend on the choice of x within its equivalence class by the stabilization assumption. Also, $\alpha(x)$ is a countable ordinal. To see this, work in the model $M_0(x)$ and evaluate the ordinal $\alpha(x)$ there. Since $M_0(x)$ is a model of choice, its ω_1 is regular and so $\alpha(x) \in \omega_1^{M_0(x)}$. Since $\omega_1^{M_0(x)} \leq \omega_1$, the countability of $\alpha(x)$ follows.

Finally, let $C = \{x \in X \colon \exists n\ x = x \setminus n$ and for all $m \geq n$, $x \setminus n \in M_m(x)$ and $\rho_{nm}(x) \leq \alpha(x)\}$. To see that the set C meets every $\mathbb{E}_1$ class in a nonempty countable set, let $z \in X$ be arbitrary. The definition of the ordinal $\alpha(z)$ shows that there is a number $n \in \omega$ such that for all $m \geq n$, $z \setminus n \in M_m(z)$ and is enumerated before $\alpha(z)$ there; clearly, letting $x = z \setminus n$ we get $x \in C \cap [z]_{\mathbb{E}_1}$. Also, note that $C \cap [z]_{\mathbb{E}_1} \subset \{x \in X \colon \exists n\ x \in M_n(z)$ and the index of x in the constructibility well-ordering of $M_n(z)$ is $\leq \alpha(z)\}$ and observe that the latter set is countable as the ordinal $\alpha(z)$ is countable. □

Corollary 9.4.29. (ZF) *Let Y be an uncountable Polish space. If there is an acyclic decomposition of $[Y]^2$ then $\mathbb{E}_1$ has a countable complete section.*

In particular, in ZF the existence of an acyclic decomposition implies $|\mathbb{E}_1| \leq |\mathbb{F}_2|$.

Proof. Let $c \colon [Y]^2 \to \omega$ be the acyclic decomposition. Fix a countable basis for the space Y and let $A = \{\langle z_0, z_1, O, m\rangle \colon z_0, z_1 \in Z$ are distinct points, $O \subset Y$ is a basic open set, $m \in \omega$, and there is $y \in O$ such that $c(y, z_0) = c(y, z_1) = m\}$.

Let $\leq$ be the preordering on X defined by $x_1 \leq x_0$ if $x_1 \in L[x_0, A]$. In view of Theorem 9.4.28 it will be enough to show that $\leq$ is well-founded.

To do this, consider the map $\pi\colon X \to \omega_1$ defined by $\pi(x) = \omega_1^{L[x,A]}$. It will be enough to show that $x_1 < x_0$ implies $\pi(x_1) \in \pi(x_0)$. Suppose towards contradiction that this fails for some $x_1 < x_0$, and write $M_1 = L[x_1, A]$ and $M_0 = L[x_0, A]$. Since M_1 is a class of M_0, it must be the case that $\omega_1^{M_1} \leq \omega_1^{M_0}$; by the contradictory assumption, the equality in fact prevails. Thus, in the model M_0 the set $Y \cap M_1$ is an uncountable proper subset of $Y \cap M_0$. Let $y \in Y \cap (M_1 \setminus M_0)$ be an arbitrary point.

By a counting argument in the model M_0 there must be a number $m \in \omega$ and distinct points $z_0, z_1 \in Y \cap M_1$ such that $c(y, z_0) = c(y, z_1) = m$. If $y \in Y$ was the unique point satisfying these equalities then it can be constructed from A and the points z_0, z_1; as such, it would belong to the model M_1, which is not the case. Thus, there is a point $y' \in Y$ distinct from y such that $c(y', z_0) = c(y', z_1) = m$. However, then the 4-cycle consisting of edges connecting the vertices z_0, y, z_1, y', z_0 in this order is monochromatic, a contradiction. □

Corollary 9.4.30. (ZF) *Let Y be an uncountable Polish vector space over a countable field Φ. If there is a basis for Y over Φ then there is a countable complete section for $\mathbb{E}_1$.*

In particular, in ZF the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ implies $|\mathbb{E}_1| \leq |\mathbb{F}_2|$.

Proof. Let d be an invariant complete metric on Y. Let $B \subset Y$ be a basis for Y over F. Let $A = \{\langle y, O_i \colon i \in k\rangle \colon y \in Y, O_i \subset Y$ are basic open sets and there are points $y_i \in O_i \cap b$ such that y is a linear combination of $y_i \colon i \in k$ with nonzero coefficients in $\Phi\}$. Define a preordering on the space X by setting $x_1 \leq x_0$ if $x_1 \in L[x_0, A, B]$. It will be enough to show that the preordering $\leq$ is well-founded. A reference to Theorem 9.4.28 then concludes the proof.

Suppose towards contradiction that the ordering $\leq$ is ill-founded. The first order of business is to produce a sequence $\langle x_n \colon n \in \omega\rangle$ such that for each $m \in \omega$, $\langle x_n \colon n \geq m\rangle \in L[x_m, A, B]$ and $x_m \notin L[x_{m+1}, A, B]$. To do this, let $x_0 \in X$ be any element such that there is a strictly decreasing infinite sequence in $\leq$ below x_0, and by recursion define $x_{n+1} \in X$ to be the first (in the constructibility ordering of $L[x_n, A, B]$) point in $L[x_n, A, B]$ such that $x_n \notin L[x_{n+1}, A, B]$ and there is a strictly decreasing sequence in $\leq$ below x_{n+1}. The point here is that the ordering is defined in an absolute way, and its well-foundedness is absolute into the models concerned.

Write M_n for the model $L[x_n, A, B]$ and M_ω for $\bigcap_n M_n$. As an initial observation, note that the set $B \cap M_n$ is a basis in M_n: for every $y \in Y \cap M_n$, the unique linear combination of elements of B with nonzero coefficients yielding y can be constructed from y and C, so belongs to M_n. Now, by recursion on $i \in \omega$ build points $y_i \in Y$ and numbers $n_i \in \omega$ so that $n_0 = 0$ and for every $i \in \omega$, $y_i \in M_{n_i}$ is the first point in the constructibility well-ordering of M_{n_i} which is within d-distance 2^{-i} of the zero element of Y and does not belong to M_{n_i+1}. Such a point must exist since the 2^{-i}-neighborhood of the zero element is an uncountable open subset of the Polish space Y, and M_{n_i+1} does not contain all reals of M_{n_i}. The point y_i is expressed as a unique linear combination ϕ_i of elements of $B \cap M_{n_i}$. Finally, let $n_{i+1} \in \omega$ be the first number n greater than n_i such the linear combination ϕ_i uses no elements of the set $M_n \setminus M_\omega$.

Note that for each $i \in \omega$ the sequence $\langle y_j, n_j : j \geq i\rangle$ belongs to the model M_{n_i}. Let $z_i = \Sigma_{j \geq i} y_j = \lim_k \Sigma_{k \geq j \geq i} y_k$. The limit exists as the metric d is invariant and complete and the points y_i converge to zero fast. It is also clear that $z_i \in M_{n_i}$, and $z_i - z_{i+1} = y_i$. Since $z_{i+1} \in M_{n_{i+1}}$ and $y_i \notin M_{n_{i+1}}$ both hold, we conclude that $z_i \notin M_{n_{i+1}}$ holds.

Now, the point $z_0 \in Y$ can be expressed as a linear combination ψ of elements of $B \cap M_0$. Let $i \in \omega$ be a number so large that the combination ψ uses no elements of $M_{n_i} \setminus M_\omega$. Observe that $z_i = z_0 - \Sigma_{j<i} y_j$ and the linear combinations ϕ_j yielding the points y_j for $j \in i$ use no elements of $M_{n_i} \setminus M_\omega$. In other words, the point $z_k \in M_k$ can be expressed as a linear combination of elements of $B \setminus M_0$ which uses no elements of $M_{n_i} \setminus M_\omega$. This combination (after cancellations) is unique as B is a basis, and it uses only elements of M_{n_i} since $B \cap M_{n_i}$ is a basis in M_{n_i}. Thus, it uses only elements of M_ω, so $z_i \in M_\omega$. This ontradicts the last sentence of the previous paragraph. $\square$

Note that the Polish assumption on the vector space Y is necessary in Corollary 9.4.30. One can consider the (non-Polishable) group Y of finite subsets of 2^ω equipped with the symmetric difference operation as a vector space over the binary field. This vector space has a basis in ZF, namely the set of all singletons. Thus, the existence of the basis for this particular vector space does not imply in ZF the existence of a complete countable section for $\mathbb{E}_1$.

9.5 The $\mathbb{E}_{K_\sigma}$ divide

Recall that $\mathbb{E}_{K_\sigma}$ is the equivalence relation on the space of all functions in ω^ω pointwise dominated by the identity function, connecting functions x_0, x_1 if the function $x_0 - x_1$ is bounded. The central position of this equivalence relation is documented by the fact that it is K_σ, and that every K_σ-equivalence relation is Borel reducible to it by a result of Rosendal [74]. It is well-known that the equivalence relation $\mathbb{E}_0^\omega$ is not Borel reducible to $\mathbb{E}_{K_\sigma}$ [48, Lemma 6.1.1], and among the equivalence relations classifiable by countable structures, it even has a central place in this regard. In this section, we show that in certain extensions of the symmetric Solovay model, this non-reducibility result is translated to a cardinality preservation result. The reader should note that in ZF, the inequality $|\mathbb{E}_0| \leq |2^\omega|$ implies $|\mathbb{E}_0^\omega| \leq |(2^\omega)^\omega| = |2^\omega|$, and therefore preserving the $\mathbb{E}_{K_\sigma}$-divide is strictly more difficult than preserving the smooth divide.

The preservation of the relation $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$ uses the following generalization of the mutual genericity motivated by Proposition 1.7.8.

Definition 9.5.1. Let $V[H_0], V[H_1]$ be generic extensions in some ambient extension of V. We say that $V[H_0], V[H_1]$ are *mutually generic over a pod* if

1. for every ordinal α, $V_\alpha \cap V[H_0] \cap V[H_1] \in V[H_0] \cap V[H_1]$;
2. for any disjoint sets $a_0 \in V[H_0]$ and $a_1 \in V[H_1]$ which are subsets of $V[H_0] \cap V[H_1]$, there are disjoint sets $b_0, b_1 \in V[H_0] \cap V[H_1]$ such that $a_0 \subset b_0$ and $a_1 \subset b_1$.

The *pod* is the intersection $V[H_0] \cap V[H_1]$.

It is not hard to see that the first item implies that the intersection $V[H_0] \cap V[H_1]$ is a model of ZF, since it is weakly universal and closed under the Gödel functions [45, Theorem 13. 9]. If the intersection is in fact a model of ZFC, then the extensions $V[H_0], V[H_1]$ are mutually generic over it by Proposition 1.7.8. However, we will be interested exactly in the situations where the intersection fails to satisfy even DC. The following is a simple observation which will be critical later.

Proposition 9.5.2. *Let $V[H_0], V[H_1]$ be extensions mutually generic over a pod. Every $\mathbb{E}_{K_\sigma}$-class represented in both $V[H_0]$, $V[H_1]$ is represented in $V[H_0] \cap V[H_1]$.*

Proof. Let $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ be points in ω^ω below the identity function which are $\mathbb{E}_{K_\sigma}$-related. Find a natural number $m \in \omega$ such that for all $k \in \omega$, $|x_0(k) - x_1(k)| \leq m$. Let $a_0 \subset \omega \times \omega$ be the set x_0, and let $a_1 \subset \omega \times \omega$ be the set $\{\langle k, l\rangle : |l - x_1(k)| > m\}$. Thus, $a_0 \in V[H_0]$ and $a_1 \in V[H_1]$ are disjoint sets, and therefore can be separated by some disjoint sets $b_0, b_1 \subset \omega \times \omega$ in $V[H_0] \cap V[H_1]$ by the mutual genericity assumption. The vertical sections of the set b_0 are all nonempty and the function $y \in 2^\omega$, which for each $n \in \omega$ indicates the smallest element of the vertical section $(b_0)_n$, belongs to $V[H_0] \cap V[H_1]$. Since $y \cap a_1 = 0$, it must be the case that y is $\mathbb{E}_{K_\sigma}$-related to both x_0, x_1 as desired. $\square$

The conclusion of the proposition fails for such simple pinned equivalence relations as $\mathbb{E}_0^\omega$, and this is exactly the point exploited in this section. We will need a notion of pod balance.

Definition 9.5.3. A Suslin poset P is *pod balanced* if for every pair $V[H_0]$, $V[H_1]$ of extensions mutually generic over a pod and every pair of conditions $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$, if $p_0, p_1 \in P$ are incompatible, then there are analytic sets $A_0, A_1 \subset P$ coded in $V[H_0] \cap V[H_1]$ such that ΣA_0, ΣA_1 are incompatible in P and $p_0, \Sigma A_0$ are compatible and $p_1, \Sigma A_1$ are compatible.

Theorem 9.5.4. *In compactly balanced, pod balanced extensions of the symmetric Solovay model, $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$ holds.*

Proof. The main technical tool used in the proof is the countable support product Q of countably many copies of the Vitali forcing. We first study the product Q in its own right.

Fact 9.5.5. *The poset Q is proper, bounding, and adds no independent reals.*

Proof. The first two assertions are well-known, and follow for example from [**?**, Theorem 5.6] or [101, Theorem 5.2.6]. The third assertion is more difficult. It can be derived from the work of [84, Section 4], and it appears explicitly in the forthcoming [104]. $\square$

The poset Q is designed to add an interesting model of ZF. In the Q-extension, let $\langle x_n : n \in \omega \rangle$ be the sequence of points Vitali-generic over the ground model added by the product Q. Let $A \subset \omega \times 2^\omega$ be the relation consisting of all pairs $\langle n, x \rangle$ such that x is E_0-related to x_n. For each number $n \in \omega$, let V_n be the model of all sets hereditarily definable from x_m for $m \in n$, A, and some parameters in the ground model. Also, let M be the model of all sets hereditarily definable from the set A and its elements, and parameters in the ground model. We have the following:

Claim 9.5.6. *1. $V_0 \subset V_1 \subset V_2 \subset \cdots \subset M$;*

2. the sequence $\langle V_n : n \in \omega \rangle$ belongs to M;

3. every set of ordinals in M belongs to $\bigcup_n V_n$;

4. the model M does not contain any uniformization of the set A.

Note also that the models V_n satisfy the axiom of choice, while the model M fails even DC, as item (3) shows.

Proof. The first item follows directly from the definitions of the models V_n and M: if $m \in n \in \omega$ are numbers then there are fewer definitions of elements of the model V_m than of elements of the model V_n, and certainly fewer than the definitions of elements the model M. For (2), the sequence $\langle V_n : n \in \omega \rangle$ is definable from the set A, since in the definition of the model V_n it certainly does not matter which $\mathbb{E}_0$-equivalents of the points x_m for $m \in n$ one uses as the parameters. For (3), any definition of a set of ordinals in the model M uses only finitely many parameters, and then must belong to the model V_n for a number n large enough so that the parameters of the definition are definable from x_m for $m \in n$, the set A, and some elements of the ground model.

Lastly, for (4), return to the ground model and assume towards a contradiction that some condition in Q forces (4) to fail. Then, in the poset Q, there must be a condition q, numbers $n, k \in \omega$ and a formula ϕ such that q forces that the formula ϕ uses only parameters $\dot{x}_m$ for $m \in n$, A, and some parameters in the ground model, it defines a uniformization y of the set A, and $y(n)$ agrees with $\dot{x}_n$ from k on. Let $\langle x_m : m \in \omega \rangle$ be a Q-generic sequence of points meeting the condition q. By a basic analysis of the Vitali forcing in [101, Section 4.7.1], there is also a point $x'_n \in 2^\omega$ $\mathbb{E}_0$-related to x_n which differs from x_n at some point past k and such that $\langle x_m : m \neq n, x'_n \rangle$ is a Q-generic sequence meeting the condition q. Applying the forcing theorem to the two sequences $\langle x_m : m \in \omega \rangle$ and $\langle x_m : m \neq n, x'_n \rangle$, we see that the formula ϕ should define a uniformization y of the set A such that $y(n)$ is equal to both x_n and x'_n beyond k. This is impossible though as the two sequences x_n, x'_n do differ at some point past k. $\square$

Now, towards the proof of the theorem. Write $E = \mathbb{E}_0^\omega$ and $F = \mathbb{E}_{K\sigma}$. Let κ be an inaccessible cardinal. Let P be a Suslin forcing which is compactly balanced and pod balanced below κ. Let W be a symmetric Solovay model derived from κ and work in the model W. Let $p \in P$ be a condition and τ be a P-name such that p forces τ to be a function from the E-quotient space to the F-quotient space. We must find a condition $q \leq p$ and distinct E-classes which are forced by q to have the same τ-image.

The condition p and the name τ must be definable from a parameter $z \in 2^\omega$ and additional parameters from the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of size $< \kappa$ such that $z \in V[K]$. Working in the model $V[K]$, let Q_0 be the poset $\mathcal{P}(\omega)$ modulo finite, and let Q_1 be the product of countably many copies of the Vitali forcing as discussed in the preamble of the present proof. Let $\langle U, x_n \colon n \in \omega\rangle$ be objects generic over $V[K]$ for the poset $Q_0 \times Q_1$; in particular, U is an ultrafilter on ω, and $\langle x_n \colon n \in \omega\rangle$ is a sequence of points in 2^ω. Since the poset Q_0 is σ-closed, the models $V[K]$ and $V[K][U]$ compute the poset Q_1 in the same way, in particular $V[K][U][x_n \colon n \in \omega]$ is a Q_1-extension of $V[K][U]$ and $2^\omega \cap V[K][U][x_n \colon n \in \omega] = 2^\omega \cap V[K][x_n \colon n \in \omega]$. Since (Fact 9.5.5) the poset Q_1 adds no independent reals, a density argument shows that U generates an ultrafilter in the model $V[K][U][x_n \colon n \in \omega]$.

Let $A \subset \omega \times 2^\omega$ be the relation consisting of all pairs $\langle n, x\rangle$ such that x is $\mathbb{E}_0$-related to x_n. In the model $V[K][U][x_n \colon n \in \omega]$ form the models V_n of all sets hereditarily definable from parameters x_m for $m \in n$, A, and parameters in the model $V[K][U]$, and let M be the model of all sets hereditarily definable from the set A and its elements and some other parameters in $V[K][U]$. Note that by Vopěnka's theorem [45, page 249], the sequence $\langle x_n \colon n \in \omega\rangle$ is generic over the model M and the forcing adding it restores the axiom of choice in its generic extension of M.

The following claim makes a critical use of the compact balance assumption.

Claim 9.5.7. *In the model M:*

1. *there is a sequence $\langle \bar{p}_n \colon n \in \omega\rangle$ such that $\bar{p}_n$ is a balanced virtual condition for V_n, and $p \geq \bar{p}_0 \geq \bar{p}_1 \geq \bar{p}_2 \geq \dots$;*
2. *any sequence as in (1) must have a lower bound.*

Proof. Start with item (1). Working in $V[K][U]$, fix R_n-names χ_n so that R_n is the product of the first n-many copies of the Vitali forcing in the product Q_1 and χ_n is a name for a virtual balanced condition stronger than p and than all χ_m for $m \in n$. This is possible by Definition 9.2.1(2). Now, in the model $V[K][U][x_n \colon n \in \omega]$ for each $n \in \omega$ let $\bar{p}_n$ be the U-limit of the sequence $\langle \chi_n / \langle x^i_m \colon m \in n\rangle \colon i \in \omega\rangle$ in the compact space of virtual balanced conditions in the model V_n, where $x^i_m \in 2^\omega$ is the binary sequence obtained from x_m by replacing its first i many entries with 0. The limit exists since U generates an ultrafilter in all models V_n. The balanced virtual conditions $\bar{p}_n$ for $n \in \omega$ form a decreasing sequence by Definition 9.2.1(3). Also, the definition of the limit

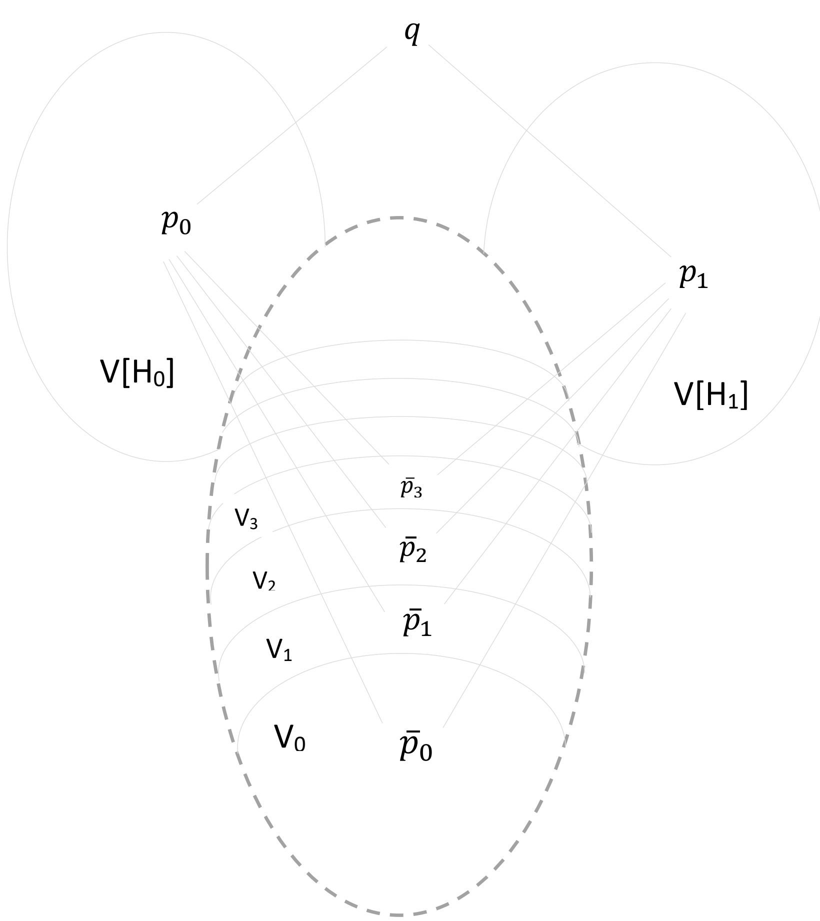

Figure 9.2: A pod at work.

does not really depend on the sequence $\langle x_n : n \in \omega\rangle$ but only on the set A, and therefore $\langle \bar{p}_n : n \in \omega\rangle \in M$ as desired.

Item (2) is somewhat trickier. Let N be some generic extension of M which restores the axiom of choice. In the model N, let B be the compact Hausdorff space of balanced conditions for P. For each number $n \in \omega$, consider the set $C_n = \{\bar{p} \in B : \bar{p} \leq \bar{p}_n\}$. This is a nonempty and closed subset of B by Definition 9.2.1(2) and (3). Since the closed sets C_n for $n \in \omega$ form a decreasing sequence, a compactness argument shows that there must be some $\bar{p} \in \bigcap_n C_n$. This is the required lower bound of the sequence $\langle \bar{p}_n : n \in \omega\rangle$. $\square$

Work in the model M. By the forcing theorem, there must be a poset R of rank $< \kappa$ restoring the axiom of choice, an R-name σ for a condition in the poset P stronger than all the balanced virtual conditions $\bar{p}_n$ for $n \in \omega$ and an R-name η for an element of ω^ω such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau([y]_E) = [\eta]_F$ for all functions y uniformizing the relation A. The following claim uses the pod balance assumption.

Claim 9.5.8. *There is a condition $r \in R$ and a point $u \in \omega^\omega \cap M$ such that $r \Vdash \eta \ F \ \check{u}$.*

Proof. Suppose towards a contradiction that this fails. Let $H_0, H_1 \subset R$ be filters mutually generic over the model M. Observe that the models $M[H_0], M[H_1]$ are generic extensions of V mutually generic over the pod M: if $a_0 \in M[H_0]$ and $a_1 \in M[H_1]$ are disjoint sets of ordinals, they must be separated by disjoint sets b_0, b_1 of ordinals in the model M by the mutual genericity over M and Proposition 1.7.8. Now, let $p_0 = \sigma/H_0$, $u_0 = \eta/H_0$, $p_1 = \sigma/H_1$ and $u_1 = \eta/H_1$. Note that for every condition $\hat{p} \in P \cap M$, either both p_0, p_1 are below $\hat{p}$ or both p_0, p_1 are incompatible with $\hat{p}$. This occurs because by Claim 9.5.6, there is a number $n \in \omega$ such that $\hat{p} \in V_n$, the virtual condition $\bar{p}_n$ is either below $\hat{p}$ or incompatible with it by its balance, and $p_0, p_1 \leq \bar{p}_n$. By the pod balance of the poset P, the conditions p_0, p_1 have a lower bound in the poset P. By the forcing theorem in M and the initial contradictory assumption, u_0, u_1 are not F-related to any element of M. By Proposition 9.5.2, u_0 cannot be F-related to u_1. Thus, the lower bound of p_0, p_1 would have to force the unique E-class of all uniformizations of the set A to be mapped by τ simultaneously to $[u_0]_F$ and to $[u_1]_F$, which is impossible. $\square$

Now, the claim together with the forcing theorem means that there is a natural number $n \in \omega$, a point $u \in V_n$ in ω^ω and in the model V_n a poset S of size $< \kappa$ and an S-name χ for a condition in the poset P stronger than $\bar{p}$ and an S-name ξ for an element of $(2^\omega)^\omega$ which is not E-related to any point in V_n such that $S \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \chi \Vdash_P \tau([\xi]_E) = [\check{u}]_F$. Namely, the poset S is a remainder of the iteration of Q and R after the first n-many Q-generic reals are added. Let $H_0, H_1 \subset S$ be filters mutually generic over the model $V_n = V[K][U][x_m : m \in n]$ and write $p_0 = \chi/H_0$, $y_0 = \xi/H_0$, $p_1 = \chi/H_1$, and $y_1 = \xi/H_1$. By the balance of the virtual condition $\bar{p}_n$, the conditions p_0, p_1 have a lower bound. Since the equivalence relation E is pinned, the points y_0, y_1 are not E-related. By the forcing theorem, the lower bound of the conditions p_0, p_1 forces $\tau([y_0]_E) = \tau([y_1]_E) = [u]_F$. This shows that τ cannot be an injection and completes the proof of the theorem. □

Example 9.5.9. The poset $P = \mathcal{P}(\omega)$ modulo finite is compactly balanced and pod balanced. Compact balance was proved in Example 9.2.4. Towards the pod balance, suppose that $V[H_0], V[H_1]$ are generic extensions mutually generic over a pod and $p_0, p_1 \in P$ are incompatible conditions in the respective models, i.e. almost disjoint subsets of ω. By the mutual genericity there must be almost disjoint sets $p_0', p_1' \in V[H_0] \cap V[H_1]$ of natural numbers such that $p_0 \subset p_0'$ and $p_1 \subset p_1'$. These sets as conditions in P exemplify the pod balance.

Corollary 9.5.10.

1. *Let $P = \mathcal{P}(\omega)$ modulo finite. Then in the P-extension of the symmetric Solovay model, $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$ holds.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a nonprincipal ultrafilter on ω and yet $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$.*

Example 9.5.11. Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space X. Then the poset $P = P_\mathcal{K}$ is pod balanced. To see this, let $V[H_0], V[H_1]$ be models mutually generic over a pod and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are incompatible conditions. Write $E = E_\mathcal{K}$ as in Definition 6.1.3; this is a countable Borel equivalence relation on X. The only way how p_0, p_1 can fail to be compatible is that there is an E-class c represented in both $V[H_0]$ and $V[H_1]$ and finite sets $a_0, a_1 \subset b$ such that $a_0 \subset p_0$ and $a_1 \subset p_1$ and $a_0 \cup a_1 \notin \mathcal{K}$. Use the Mostowski absoluteness between $V[H_0], V[H_1]$ and $V[H_0, H_1]$ to see that $c \subset V[H_0] \cap V[H_1]$ holds. Thus, the conditions $p_0' = a_0$ and $p_1' = a_1$ witness the pod balance of the poset P.

Many posets associated with locally countable simplicial complexes are compactly balanced; we get for example the following.

Corollary 9.5.12. *Let Γ be a Borel locally finite graph on a Polish space X satisfying the Hall's marriage condition.*

1. *Let P be the poset adding a perfect matching to Γ with countable approximations. Then in the P-extension of the symmetric Solovay model, $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$ holds.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, Γ has a perfect matching, and yet $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$.*

Example 9.5.13. Let P be the poset adding a linear ordering on the $\mathbb{E}_{K_\sigma}$-quotient space as in Example 8.6.5. Then the poset P is compactly balanced and pod balanced. Compact balance was proved in Example 9.2.11. For the pod balance, suppose that $V[H_0], V[H_1]$ are extensions mutually generic over a pod, and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. Passing to stronger conditions if necessary, we may assume that p_0, p_1 are linear orders on some subsets of the $\mathbb{E}_{K_\sigma}$-quotient space. The only way how they can fail to be compatible is that there are $\mathbb{E}_{K_\sigma}$-classes c, d represented in both $V[H_0], V[H_1]$ such that $\langle c, d\rangle \in p_0$ and $\langle d, c\rangle \in p_1$. In view of Proposition 9.5.2, c, d are represented in $V[H_0] \cap V[H_1]$ and so the conditions $p_0' = \{\langle c, d\rangle\}$ and $p_1' = \{\langle d, c\rangle\}$ belong to $V[H_0] \cap V[H_1]$ and exemplify the pod balance.

Corollary 9.5.14.

1. *Let P be the $\mathbb{E}_{K_\sigma}$-linearization poset. Then in the P-extension of the symmetric Solovay model, $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$ holds.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the $\mathbb{E}_{K_\sigma}$-quotient space is linearly ordered and yet $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$.*

Example 9.5.15. Let Γ, Δ be abelian Polish groups, with Δ divisible. Let P be the poset for adding a discontinuous homomorphism from Δ to Γ as in Definition 8.2.1. Then the poset P is compactly balanced and pod balanced. Compact balance was proved in Example 9.2.20. For the pod balance, suppose that $V[H_0], V[H_1]$ are extensions mutually generic over a pod, and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. Passing to stronger conditions if necessary, we may assume that $\Gamma \cap V[H_0] \cap V[H_1] \subset \mathrm{dom}(p_0), \mathrm{dom}(p_1)$ holds. Since $\Gamma \cap V[H_0] \cap V[H_1]$ is a subgroup of Γ, the only way how p_0, p_1 can fail to be compatible is that there is a point $\gamma \in V[H_0] \cap V[H_1]$ with $p_0(\gamma) \neq p_1(\gamma)$. Let $O_0, O_1 \subset \Delta$ be basic open sets separating the values $p_0(\gamma)$ and $p_1(\gamma)$ and let $A_0 = \{q \in P \colon \gamma \in \mathrm{dom}(q) \text{ and } q(x) \in O_0\}$ and $A_1 = \{q \in P \colon \gamma \in \mathrm{dom}(q) \text{ and } q(x) \in O_1\}$. Clearly, $A_0, A_1 \subset P$ are analytic sets coded in $V[H_0] \cap V[H_1]$, the sums $\Sigma A_0, \Sigma A_1$ are incompatible in P, and $p_0 \leq \Sigma A_0$ and $p_1 \leq \Sigma A_1$ holds.

Corollary 9.5.16. *Let Γ, Δ be abelian Polish groups, with Δ divisible.*

1. *Let P be the homomorphism poset. Then in the P-extension of the symmetric Solovay model, $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$ holds.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a discontinuous homomorphism from Γ to Δ and yet $|\mathbb{E}_0^\omega| \not\leq |\mathbb{E}_{K_\sigma}|$.*

This section leaves perhaps more questions open than it resolves. We content ourselves with quoting one glaring case:

Question 9.5.17. Does the conclusion of Theorem 9.5.4 stay in force if the assumption of pod balance on P is dropped?

9.6 The pinned divide

No unpinned equivalence relation can be reduced by a Borel function to a pinned one, see Fact 2.3.2. This feature persists to the cardinality computations in balanced extensions of the choiceless Solovay model, with a small proviso.

Theorem 9.6.1. *Let E, F be Borel equivalence relations on respective Polish spaces X, Y, E pinned and F unpinned. In a balanced extension of a symmetric Solovay model derived from an inaccessible limit of inaccessibles, $|F| \not\leq |E|$ holds.*

We do not know if the increase in the large cardinal strength of the large cardinal hypothesis is necessary. We do know though that one has to consider Suslin posets which are balanced everywhere below κ as opposed to just cofinally balanced. This is clear from Example 9.6.2 below.

Proof. Let κ be an inaccessible cardinal which is a limit of inaccessibles. Let P be a Suslin poset which is balanced below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Towards a contradiction, suppose that $p \in P$ is a condition and τ a P-name for an injection from F-classes to E-classes. Both p, τ are definable from some parameter $z \in 2^\omega$ and parameters in the ground model. The assumptions imply that there is an inaccessible cardinal $\lambda < \kappa$ of V and a filter $K \subset \text{Coll}(\omega, <\lambda)$ generic over V such that $z \in V[K]$.

Work in the model $V[K]$. The balanced assumption on the poset P implies that there is a balanced virtual condition $\bar{p} \leq p$ in P. Theorem 2.7.1 shows that E is pinned in $V[K]$ and therefore has at most $\mathfrak{c} = \aleph_1$ many virtual classes, while Theorem 2.5.11 shows that F has at least $2^{\aleph_1}$ many virtual classes. The argument splits into two cases:

Case 1. For every poset R of size $< \kappa$, every R-name for a condition $\sigma \leq \bar{p}$ in the poset P, every F-pinned R-name η for an element of X and every R-name χ for an element of X such that $R \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau([\eta]_F) = [\chi]_E$, the name χ is E-pinned.

In this case, by a counting argument with the virtual E- and F-classes, it must be the case that there are posets R_0, R_1 and respective names σ_0, η_0, χ_0 and σ_1, η_1, χ_1 on them such that η_0, η_1 are F-pinned, χ_0, χ_1 are E-pinned, $R_0 \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma_0 \Vdash_P \tau_0([\eta]_F) = [\chi_0]_E$ and $R_1 \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma_1 \Vdash_P \tau([\eta_1]_F) = [\chi_1]_E$ and $\langle R_0, \chi_0\rangle \bar{E} \langle R_1, \chi_1\rangle$ holds while $\langle R_0, \eta_0\rangle \bar{F} \langle R_1, \eta_1\rangle$ fails. Let $H_0 \subset R_0, H_1 \subset R_1$ be filters generic over $V[K]$, and write $r_0 = \sigma_0/H_0 \in P$, $r_1 = \sigma_1/H_1 \in P$, $x_0 = \chi_0/H_0 \in X$, $x_1 = \chi_1/H_1 \in X$, and $y_0 = \eta/H_0 \in Y$ and $y_1 = \eta_1/H_1 \in Y$. The balance of the virtual condition $\bar{p}$ in P implies that r_0, r_1 are compatible in P with some lower bound $r \in P$, and the pinned assumptions imply that $x_0 \mathrel{E} x_1$ holds and $y_0 \mathrel{F} y_1$ fails. Since W is the symmetric Solovay extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem in these two models implies that in W, $r \Vdash_P \tau([\check{y}_0)_F) = [\check{x}_0]_E$ and $\tau([\check{y}_1]_F) = [\check{x}_1]_E$; this contradicts the assumption that τ is forced to be an injection.

Case 2. Case 1 fails. Then, there has to be poset R of size $< \kappa$, every R-name for a condition $\sigma \leq \bar{p}$ in the poset P, every F-pinned R-name η for an element of X and every R-name χ for an element of X such that $R \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau([\eta]_F) = [\chi]_E$, such that the name χ is not E-pinned. In such a case, there must be filters $H_0, H_1 \subset R$ mutually generic over $V[K]$ such that the points $x_0 = \chi/H_0, x_1 = \chi/H_1 \in X$ are E-unrelated. Let $r_0 = \sigma/H_0, r_1 = \sigma/H_1$, and let $y_0 = \eta/H_0, y_1 = \eta/H_1 \in Y$. The balance of the virtual condition $\bar{p}$ implies that the conditions $r_0, r_1 \in P$ are compatible with a lower bound r, and the pinned assumptions imply that $y_0 \mathrel{F} y_1$ holds. Since W is the symmetric Solovay extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem in these two models implies that in W, $r \Vdash_P \tau([\check{y}_0]_F) = [\check{x}_0]_E$ and $\tau([\check{y}_1]_F) = [\check{x}_1]_E$ this contradicts the assumption that τ is forced to be a function. $\square$

Example 9.6.2. Let $A \subset X = (2^\omega)^\omega$ be the set of all elements $x \in X$ such that $\operatorname{rng}(x)$ is linearly ordered by Turing reducibility, and let $E = \mathbb{F} \restriction A$. Note that the E-quotient space is classified by subsets of 2^ω which are linearly ordered by Turing reducibility. There are uncountable sets of this form, and therefore E is unpinned. At the same time, such sets have size at most $\aleph_1$ and so the virtual E-quotientspace has size $2^{\aleph_1}$.

Now, let P be the collapse of $E = \mathbb{F}^2 \restriction A$ to $|2^\omega|$. The poset P is balanced if and only if $2^{\aleph_0} = 2^{\aleph_1}$ by Theorem 6.6.3; in particular, it is cofinally balanced below any inaccessible cardinal. Thus, the P-extension of the Solovay model is a cofinally balanced extension in which the (in ZFC) unpinned equivalence relation E has the same cardinality as 2^ω.

Chapter 10

Uniformization

The question whether various forms of uniformization hold in the models within purview of this book is one of the more slippery issues we set out to resolve.

10.1 Tethered Suslin forcing

In order to prove all forms of uniformization in a clean sweep, the following definition will be central.

Definition 10.1.1. Let P be a Suslin forcing and λ be an infinite cardinal. The poset P is λ-*tethered* if whenever $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions, then there are incompatible virtual conditions $p'_0, p'_1 \in V$ represented on posets of size $< \lambda$ such that p_0, p'_0 are compatible and p_1, p'_1 are compatible. The poset P is *tethered* if it is λ-tethered for some cardinal λ.

One immediate corollary of tether is that it places an upper bound on the number of balanced classes, which we do not know how to obtain in general.

Proposition 10.1.2. *Let P be a Suslin forcing and λ be an infinite cardinal. If P is λ-tethered then there are at most 2^{2^λ} many balanced classes.*

Proof. Let A be the set of all virtual conditions of P represented on posets of size $< \lambda$. It is a matter of elementary cardinal arithmetic to conclude that $|A| \leq 2^\lambda$. Whenever $\bar{p}$ is a balanced virtual condition, write $B_{\bar{p}} = \{q \in A \colon \bar{p} \leq q\}$. Note that for each $q \in A$, if $q \in B_{\bar{p}}$ then $\bar{p} \leq q$ and if $q \notin B_{\bar{p}}$ then $\bar{p}, q$ are incompatible. To prove the proposition, it will be enough to show that the set $B_{\bar{p}}$ characterizes the balanced condition $\bar{p}$.

Now suppose that $\bar{p}_0, \bar{p}_1$ are balanced virtual conditions such that $B_{\bar{p}_0} = B_{\bar{p}_1}$; denote the common value by B and work to show that $\bar{p}_0 = \bar{p}_1$. To this end, suppose that R_0, R_1 are arbitrary posets, $H_0 \subset R_0$ and $H_1 \subset R_1$ are mutually generic filters and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are conditions below $\bar{p}_0, \bar{p}_1$ respectively. In view of the definition of the equivalence of balanced pairs, it will be enough to show that p_0, p_1 are compatible. However, if they were incompatible, by the tether assumption there would have to be incompatible virtual conditions $p'_0, p'_1 \in A$ such that p'_0, p_0 are compatible and p'_1, p_1 are compatible. By the note from the previous paragraph, it must be the case that $p'_0, p'_1 \in B$ holds. But then, p'_0, p'_1 are not incompatible because $\bar{p}_0, \bar{p}_1$ are both their common lower bounds. $\square$

10.2 Pinned uniformization

In this section, we show that sets whose vertical sections are E-classes for a suitably regular equivalence relation E can be uniformized.

Definition 10.2.1. Let E be an equivalence relation on a Polish space X. E-*uniformization* is the statement: if $B \subset 2^\omega \times X$ is a set whose vertical sections are E-classes, then there is a function $f \subset B$ such that for every $y \in 2^\omega$, if $B_y \neq 0$ then $f(y)$ is defined (and is an element of B_y).

The central theorem of this section can now be stated using Convention 1.7.16.

Theorem 10.2.2. *Let E be a pinned Borel equivalence relation on a Polish space X. In tethered, cofinally balanced extensions of the symmetric Solovay model, E-uniformization holds.*

To parse the statement of the theorem correctly, note that P needs to be tethered in all forcing extensions and balanced only in cofinally many forcing extensions below the inaccessible cardinal which gives rise to the symmetric Solovay model.

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is balanced cofinally below κ and tethered below κ. Let W be a symmetric Solovay model derived from κ. Let $G \subset P$ be a filter generic over W and work in $W[G]$. The following claim contains the key use of the tether assumption.

Claim 10.2.3. *Let $y \in W[G]$ be any set and let M be the class of all sets hereditarily definable from G, y, and parameters in the ground model. Then G contains a realization of a balanced virtual condition in M.*

A tacit part of the statement of the claim is that M is a model of ZFC, and therefore it belongs to W by Theorem 9.1.1 by the cofinal balance assumption on the poset P; in fact, it is a generic extension of the ground model by a poset of size $< \kappa$.

Proof. Use the tether assumption to find a cardinal $\lambda \in \kappa$ such that $M \models P$ is λ-tethered. Working in M, let D be the set of all virtual conditions in P represented on posets of size $< \lambda$. In $W[G]$, let $C \subset D$ be the set of all virtual conditions in the set D whose realization belongs to the generic filter G. The set C has just been defined from the ordinal λ and the ultrafilter G, so $C \in M$. By the genericity of the filter G, there must be a condition $p \in G$ which is below all conditions in C and incompatible with all the conditions in $D \setminus C$. By the forcing theorem, in M there have to be a poset Q of cardinality less than κ and a Q-name τ for a condition in P such that Q forces τ to be below all conditions in C and incompatible with all conditions in $D \setminus C$, and such that there is a filter $H \subset Q$ generic over the model M with $p = \tau/H$. We claim that in the model M, the pair $\langle Q, \tau\rangle$ is balanced.

To see this, suppose towards contradiction that R_0, R_1 are posets and $K_0 \times H_0 \subset R_0 \times Q$ and $K_1 \times H_1 \subset R_1 \times Q$ are filters mutually generic over the model M and p_0, p_1 are conditions in the respective models below τ/H_0 and τ/H_1 respectively. We must show that p_0, p_1 are compatible. This, however, is immediately clear from the λ-tether assumption and the fact that for every virtual condition $\bar{p}$ on P represented on a poset of cardinality less than λ and for every $i \in 2$, $\bar{p} \geq p_i \leftrightarrow \bar{p} \in C$ and $\bar{p}$ is incompatible with p_i if and only if $\bar{p} \notin C$.

Still working in the model M, let $\bar{p}$ be a virtual balanced condition in the balance class of $\langle Q, \sigma\rangle$, obtained from Theorem 5.2.5. Clearly, the filter G contains a realization of it. $\square$

Return to the model W for a moment. Let $p \in G$ be a condition and let τ be a P-name for a subset of $2^\omega \times X$ such that each vertical section of τ is a single E-class. Let $z \in 2^\omega$ be a point such that both p, τ are definable from z and a ground model parameter. Back in the model $W[G]$, write $B = \tau/G$. Let $y \in 2^\omega$ be an arbitrary point and let M_y be the model of all sets hereditarily definable from G, y, and parameters in the ground model. By Claim 10.2.3, there is a balanced virtual condition $\bar{p}$ in M_y whose realization belongs to the filter G. Necessarily, $\bar{p} \leq p$.

In the model M_y, we will show that $\mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p} \Vdash_P \tau_y \cap M_y \neq 0$. Once this is done, the set B can be uniformized by the set of all pairs $\langle y, x\rangle$ such that $x \in M_y$ is the first point in the canonical well-ordering of the model M_y which belongs to B_y.

To prove the forcing statement in the previous paragraph, suppose towards a contradiction that it fails. Move to the model M_y. There have to be a poset R of size $< \kappa$ and R-names σ for a condition in P stronger than $\bar{p}$ and η for an element of X which has no E-equivalent in the model M_y such that $R \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \langle \check{y}, \eta\rangle \in \tau$. In W, let $H_0, H_1 \subset R$ be filters mutually generic over the model M_y. Write $p_0 = \sigma/H_0$, $p_1 = \sigma/H_1$, $x_0 = \eta/H_0$ and $x_1 = \eta/H_1$. The balance of the condition $\bar{p}$ implies that $p_0, p_1 \in P$ are compatible conditions with a lower bound q. The pinned assumption on the equivalence relation implies that $x_0, x_1 \in X$ are not E-related.

Since W is also a symmetric Solovay extension of both $M_y[H_0]$ and $M_y[H_1]$, the forcing theorem applied in these two models implies that $q \Vdash \check{x}_0, \check{x}_1 \in \tau_y$. This is impossible as τ is forced to be an E-class and x_0, x_1 are not E-related. $\square$

Theorem 10.2.2 is the strongest possible result of its kind, as the following observation shows.

Theorem 10.2.4. *Let E be an unpinned Borel equivalence relation on a Polish space X. Then E-uniformization fails in balanced extensions of the Solovay* *model.*

Proof. Let $W[G]$ be a balanced extension of the Solovay model. For each parameter $z \in 2^\omega$ the model $M_z = \mathrm{HOD}_{V,z,G}$ is well-ordered and therefore by Theorem 9.1.1, it is a well-ordered subclass of W and so an extension of V by a poset $< \kappa$. The Borel equivalence relation E is unpinned in M_z by Theorem 2.7.1. By Theorem 2.6.3, $M_z \models$there is a nontrivial E-pinned name on the poset $\mathrm{Coll}(\omega, \omega_1)$. Note that $(\mathcal{P}(\omega_1))^{M_z}$ is a countable set in W. Thus, one can successfully define the set $B \subset 2^\omega \times X$ by setting $\langle z, x\rangle \in B$ if there is a filter $g \subset \mathrm{Coll}(\omega, \omega_1^{M_z})$ generic over M_z such that $\tau_z/g \mathrel{E} x$, where τ_z is the first E-unpinned name in the canonical well-ordering of the model M_z. Every vertical section of the set B then consists of precisely one E-equivalence class.

To see that the set B cannot be uniformized, note that every P-name in the model W is definable from a real parameter and some additional parameters in the ground model, and therefore every element of $W[G]$ is definable from a real parameter, the generic filter G, and some additional parameters in the ground model. Thus, if $f\colon 2^\omega \to X$ is a putative uniformization of the set B, one can find a real parameter $z \in 2^\omega$ such that f is definable from z, G and some elements of the ground model. Then $f(z)$ should be an element of the model M_z by the definition of the model M_z. At the same time, the vertical sections B_z contains no elements of M_z by the definition of the set B. $\square$

10.3 Well-orderable uniformization

In this section, we will prove that a strong version of the countable-to-one uniformization statement holds in the extensions of the Solovay model by tethered forcings.

Definition 10.3.1. Let E be an equivalence relation on a Polish space X. *E-well-orderable uniformization* is the statement: if E is a Borel equivalence relation on a Polish space X and $B \subset 2^\omega \times X$ is a set, then the following are equivalent:

1. for every $y \in 2^\omega$, B_y is a union of a well-orderable collection of E-classes;
2. $B = \bigcup_\alpha B_\alpha$ where for each ordinal α, $B_\alpha \subset 2^\omega \times X$ is a set whose vertical sections are either empty or E-classes and the variable α ranges over ordinals.

Clearly, only the implication (1)$\to$(2) has nontrivial content. As a very special case, one can consider a set $B \subset 2^\omega \times X$ with all vertical sections countable and E the identity on X. Then (2) yields in particular a function uniformizing the set B: for each y with B_y nonempty find the least $\alpha = \alpha_y$ for which the vertical section $(B_\alpha)_y$ is nonempty and then let $f(y) = x$ for the unique $x \in X$ such that $\langle y, x\rangle \in B_{\alpha_y}$. This is the familiar countable-to-one uniformization.

The central theorem of this section can now be stated using Convention 1.7.16.

Theorem 10.3.2. *Let E be a Borel equivalence relation on a Polish space X. In tethered, cofinally balanced extensions of the symmetric Solovay model, E-well-orderable uniformization holds.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is balanced cofinally below κ and tethered below κ. Let W the symmetric Solovay model derived from κ. In the model W, let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau \subset 2^\omega \times X$ is a set whose vertical sections are well-orderable unions of E-classes. Let $z \in 2^\omega$ be a parameter such that p, τ are both definable from z and some additional ground model parameters. Let $G \subset P$ be a generic filter meeting the condition p and let $B = \tau/G$. Work in the model $W[G]$.

Fix a point $y \in 2^\omega$ and consider the model M_y of all sets hereditarily definable from parameters y, z, G and parameters in the ground model. Note that M_y is a model of ZFC and therefore finds itself already in the model W by Theorem 9.1.1; in fact, it is a generic extension of the ground model by a poset of size $< \kappa$. We claim that the vertical section B_y consists of realizations of virtual E-classes of the model M_y. Once this is proved, one can decompose the set B into the union $B = \bigcup_\alpha B_\alpha$ by setting $\langle y, x\rangle \in B_\alpha$ if $\langle y, x\rangle \in B$ and x belongs to the realization of α-th virtual E-class in M_y in the canonical well-ordering of the model M_y.

To this end, use Claim 10.2.3 to argue that there is a balanced virtual condition $\bar{p}$ in the model M_y such that its realization belongs to the generic filter G. Necessarily $\bar{p} \leq p$ must hold. Now, move to the model M_y and argue that in this model, $\mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \tau_{\check{y}}$ consists of realizations of virtual E-classes of the model M_y. The proof of the theorem is then concluded by an appeal to the forcing theorem applied in M_y.

Suppose towards a contradiction that the forcing statement in the previous paragraph fails and work in M_y. In view of Corollary 9.1.3, there exist a poset R_0 of cardinality less than κ, an R_0-name η for an element of X^ω and an R_0-name σ_0 for an element of P stronger than $\bar{p}$ such that $R_0 \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash_P \forall x \in X\ \langle \check{y}, x\rangle \in \eta \leftrightarrow \exists i\ x\ E\ \eta(i)$. Also there must be a poset R_1 of size $< \kappa$, an R_1-name χ for an element of X which is forced not to realize any virtual E-class in M_y, and an R_1-name σ_1 for a condition in P stronger than $\bar{p}$ such that $R_1 \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash \langle \check{y}, \chi\rangle \in \tau$.

Working in W, let $H_0 \subset R_0$ and $H_1 \subset R_1$ be filters mutually generic over V, let $u = \eta/H_0 \in X^\omega$, $x = \chi/H_1 \in X$, $p_0 = \sigma_0/H_0 \in P$ and $p_1 = \sigma_1/H_1 \in P$. The balance of the condition $\bar{p}$ implies that p_0, p_1 are compatible conditions in the poset P with a lower bound q. The choice of the name χ implies that x is not E-related to any element on the sequence u. Since W is a symmetric Solovay extension of the model $M_y[H_0]$, the forcing theorem applied in that model says that $q \Vdash \tau = [\mathrm{rng}(u)]_E$. Since W is a symmetric Solovay extension of the model $M_y[H_1]$, the forcing theorem applied in that model says that $q \Vdash \check{x} \in \tau$. Since $x \notin [\mathrm{rng}(u)]_E$, this is a contradiction. $\square$

10.4 Saint Raymond uniformization

In this section, we will prove that a strong version of Saint Raymond uniformization holds in generic extensions of the Solovay model by tethered balanced forcing.

Definition 10.4.1. Let I be a downward closed collection of closed subsets of a Polish space X. *I-Saint Raymond uniformization* is the following statement: for every set $B \subset 2^\omega \times X$, the following are equivalent:

1. for every $y \in 2^\omega$, B_y is a union of a well-ordered collection of sets in I;
2. $B = \bigcup_\alpha B_\alpha$ where for each ordinal α and each $y \in 2^\omega$, the vertical section $(B_\alpha)_y \subset X$ is closed and belongs to I.

As in the previous section, only the (1)$\to$(2) implication has content. In the case of I =the collection of singletons (plus the empty set), we can again derive the familiar countable-to-one uniformization as a very special case. The following theorem is stated using Convention 1.7.16.

Theorem 10.4.2. *Let I be an analytic collection of closed subsets of a Polsh space X. In tethered, cofinally balanced extensions of the symmetric Solovay model, the I-Saint-Raymond uniformization holds.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin forcing which is balanced cofinally below κ and tethered below κ. Let W be the symmetric Solovay model derived from κ and work in W. Let $p \in P$ be a condition and τ a P-name for a subset of $2^\omega \times X$ whose vertical sections are well-ordered unions of sets in I. Let $z \in 2^\omega$ be a parameter such that p, τ are both definable from z and some additional ground model parameters. Let $G \subset P$ be a generic filter meeting the condition p and let $B = \tau/G$.

Work in $W[G]$. Fix a point $y \in 2^\omega$ and consider the model M_y of sets hereditarily definable from parameters G, z, y and parameters in the ground model. Note that M_y is a model of choice and therefore belongs to W already; in fact, it is an extension of the ground model by a poset of size $< \kappa$. We claim that every nonempty vertical section B_y is a union of sets in I which are coded in the model M_y. Once this is proved, one can decompose the set B into the union $B = \bigcup_\alpha B_\alpha$ by setting $\langle y, x\rangle \in B_\alpha$ if $\langle y, x\rangle \in B$ and x belongs to the α-th set in I in the model M_y in the canonical well-ordering of the model M_y, and this α-th set is a subset of B.

To prove this, first use Claim 10.2.3 to find a balanced virtual condition $\bar{p}$ in M_y whose realization belongs to the generic filter G. Work in the model M_y and argue that $\mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \tau_y$ is a union of a collection of ground model coded elements of I. By the forcing theorem applied in the model M_y, this will complete the proof.

To prove the forcing statement in the previous paragraph, work in M_y. Suppose towards a contradiction that it fails. By the assumptions on the name τ, there must be a poset R_0 of size $< \kappa$ and R_0-name η for a countable sequence of elements of I and an R_0-name σ_0 for a condition in P stronger than $\bar{p}$ such that $R_0 \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_0 \Vdash_p \langle \check{y}, x\rangle \in \eta \leftrightarrow \exists i\ x \in \eta(i)$. By the contradictory assumption, there also must be a poset R_1 of size $<\kappa$ and R_1-name χ for an element of X and an R_1-name σ_1 for a condition in P stronger than $\bar{p}$ such that $R_1 \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_1 \Vdash \langle \check{y}, \chi\rangle \in \tau$ and χ does not belong to any element of I which belongs to M_y and is a subset of $\tau_{\check{y}}$.

Move to the model W. Let $H_0, H_1 \subset R_0, R_1$ be filters mutually generic over M_y, and let $p_0 = \sigma_0/H_0$ and $p_1 = \sigma_1/H_1$. By the balance of the condition $\bar{p}$, $p_0, p_1 \in P$ are compatible conditions with a lower bound $q \in P$. Let $x = \chi/H_1 \in X$. The contradiction is now reached by a split into cases.

Case 1. There exist a condition $r \in H_1$ and a closed set $C \in \mathrm{rng}(\eta/H_0)$ such that the ground model closed set $D = X \setminus \bigcup\{O \colon O \subset X$ is a basic open set such that $r \Vdash \chi \notin O\}$ is a subset of C. By the closure of the collection I under subsets, $D \in I$ holds; by the definition of the set D, $x \in D$ holds. Now, W is the symmetric Solovay extension of both models $M_y[H_0]$ and $M_y[H_1]$. The forcing theorem applied in $M_y[H_0]$ shows that $W \models q \Vdash D \subset C \subset \tau$. The forcing theorem applied in $M_y[H_1]$ shows that $W \models q \Vdash \check{x} \in D$ and $\check{x}$ belongs to no ground model closed set in I which is a subset of τ_y. Thus, the same condition q forces two contradictory statements.

Case 2. Case 1 fails. Then, by the mutual genericity of the filters H_0, H_1 it must be the case that $x \notin \bigcup \mathrm{rng}(\eta/H_0)$. Now, W is the symmetric Solovay extension of both models $M_y[H_0]$ and $M_y[H_1]$. The forcing theorem applied in $M_y[H_0]$ shows that $W \models q \Vdash \tau_y = \bigcup \mathrm{rng}(\eta/H_0)$. The forcing theorem applied in $M_y[H_1]$ shows that $W \models q \Vdash \check{x} \in \tau_y \setminus \bigcup \mathrm{rng}(\eta/H_0)$. So again, the same condition q forces two contradictory statements. The proof is complete. □

10.5 Examples

In this section we present several examples of tethered and untethered partial orders. In all affirmative examples below, the conclusion is that in view of Theorems 10.2.2, 10.3.2, and 10.4.2, the extensions of the symmetric Solovay model by the posets in question satisfy the pinned uniformization, the well-orderable uniformization, and the Saint Raymond uniformization.

We start with simplicial complex posets which naturally live on Polish spaces as opposed to quotient spaces. These posets are $\aleph_0$-tethered in all cases that we can check.

Example 10.5.1. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space X of Borel coloring number $\aleph_1$. Then the poset $P = P_{\mathcal{K}}$ is $\aleph_0$-tethered.

Proof. Let $\vec{\Gamma}$ be a Borel orientation of the graph $\Gamma = \Gamma_{\mathcal{K}}$ of Definition 6.1.2 in which every node gets countable outflow. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. There must be finite sets $a_0 \subset p_0$ and $a_1 \in p_1$ such that $a_0 \cup a_1 \notin \mathcal{K}$; choose a_0, a_1 to be inclusion-minimal. Then they are connected in the graph Γ; assume for definiteness that a_1 is in the $\vec{\Gamma}$-outflow of a_0.

Since the $\vec{\Gamma}$-outflow of a_0 is countable, Mostowski absoluteness shows that it is a subset of $V[H_0]$; in particular, $a_1 \in V[H_0]$. By the product forcing theorem, $a_1 \in V$ must hold. Let $A_0 = \{q \in P\colon$ there is a set $c \subset q$ such that a_1 belongs to the $\vec{\Gamma}$-outflow of $c\}$, and $A_1 = \{q \in P\colon a_1 \subset C\}$. The sets A_0 and A_1 are analytic subsets of P coded in V, ΣA_0 and ΣA_1 are incompatible in P, and $p_0 \leq \Sigma A_0$, $p_1 \leq \Sigma A_1$ holds. ☐

Example 10.5.2. Let $\mathcal{K}$ be a modular Borel simplicial complex on a Polish space X. Then the poset $P = P_{\mathcal{K}}$ is $\aleph_0$-tethered.

Proof. Let Y be a Polish space and $f\colon \mathcal{K} \to [Y]^{\aleph_0}$ be a Borel function witnessing the modularity of $\mathcal{K}$. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. There must be finite sets $a_0 \subset p_0$ and $a_1 \in p_1$ such that $a_0 \cup a_1 \notin \mathcal{K}$. Write $b = a_0 \cap a_1$ and observe that $f(a_0) \cap f(a_1) \neq f(b)$ by the modularity of the function f.

Let $O_0, O_1 \subset X$ be basic open sets separating the finite disjoint sets $a_0 \setminus b$, $a_1 \setminus b$. Use the mutual genericity to argue that $b \in V$ and $f(a_0) \cap f(a_1) \subset V$. Find a point $x \in f(a_0) \cap f(a_1)$ which is not in $f(b)$, and let $A_0 = \{q \in P\colon$ there is a set $c \subset q$ such that $b \subset c$, $c \setminus b \subset O_0$, and $x \in f(c)\}$, and similarly $A_1 = \{q \in P\colon$ there is a set $c \subset q$ such that $b \subset c$, $c \setminus b \subset O_1$, and $x \in f(c)\}$. The sets A_0, A_1 are analytic subsets of P coded in V, ΣA_0 and ΣA_1 are incompatible in P by the modularity of the function f, and $p_0 \leq \Sigma A_0$, $p_1 \leq \Sigma A_1$ holds. ☐

Example 10.5.3. Let $\mathcal{K}$ be a G_δ-matroid on a Polish space X as in Definition 6.5.1. Then the poset $P = P_{\mathcal{K}}$ is $\aleph_0$-tethered.

Proof. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. Expanding the models $V[H_0], V[H_1]$ and strengthening the conditions p_0, p_1 if necessary, we may assume that for every point $x \in X \cap V$, either $x \in p_0$ or $p_0 \cup \{x\}$ is not a $\mathcal{K}$-set, and similarly for p_1.

Let $q_0 = p_0 \cap V$ and $q_1 = p_1 \cap V$. There are several cases. If $q_0 \neq q_1$, with say a point $x \in \bar{q}_0 \setminus q_1$, then let $A_0 = \{q \in P\colon x \in q\}$ and $A_1 = \{q \in P\colon \exists a \in [q]^{<\aleph_0}\ a \cup \{x\} \notin \mathcal{K}\}$. It is clear that both sets $A_0, A_1 \subset P$ are analytic and their suprema ΣA_0 and ΣA_1 are incompatible in P, and $p_0 \leq \Sigma p_0'$ and $p_1 \leq \Sigma p_1'$. Suppose then that $q_0 = q_1$, denote the common value $\bar{p}$ and use the product forcing theorem to show that $\bar{p} \in V$ holds. Suppose first that $\bar{p}$ is not a maximal $\mathcal{K}$-set. Then there must be $x \in X \setminus \bar{p}$ such that $\bar{p} \cup \{x\}$ is a $\mathcal{K}$-set and inclusion-minimal finite sets $a_0 \subset p_0$ and $a_1 \subset p_1$ such that $a_0 \cup \{x\} \notin \mathcal{K}$ and $a_1 \cup \{x\} \notin \mathcal{K}$. By the assumption on x, $a_0 \not\subset V$ and $a_1 \not\subset V$ must hold. Let $b = a_0 \cap a_1$ and let $O_0, O_1 \subset X$ be basic open sets separating the nonempty sets $a_0 \setminus b$ and $a_1 \setminus b$; note that $b \in V$ holds by the product forcing theorem. Let $A_0 = \{q \in P\colon b \subset q$ and there is a finite set $c \subset q \cap O_0$ such that $b \cup c \cup \{x\} \notin \mathcal{K}\}$ and $A_1 = \{q \in P\colon b \subset q$ and there is a finite set $c \subset q \cap O_0$ such that $b \cup c \cup \{x\} \notin \mathcal{K}\}$. Clearly, $A_0, A_1 \subset P$ are analytic sets coded in the ground model, ΣA_0 and ΣA_1 are incompatible in P by the exchange property of the matroid $\mathcal{K}$, and $p_0 \leq \Sigma A_0$ and $p_1 \leq \Sigma A_1$.

We come to the last case, where $\bar{p}$ is a maximal $\mathcal{K}$-set in the ground model. This is impossible though: $\bar{p}$ is then a balanced virtual condition in P by Theorem 6.5.2, $p_0 \leq \bar{p}$ and $p_1 \leq \bar{p}$, and so p_0, p_1 must be compatible by the balance of $\bar{p}$. This contradicts the initial choice of p_0, p_1. $\square$

Example 10.5.4. Let Γ be a Borel graph on a Polish space X of countable coloring number. The Γ-coloring poset P of Definition 8.1.1 is $\aleph_0$-tethered.

Proof. We identify P with its Suslin dense subset isolated in Theorem 8.1.2. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. Expanding the models $V[H_0], V[H_1]$ and strengthening the conditions p_0, p_1 if necessary, we may assume that for every point $x \in X \cap V$, $x \in \mathrm{dom}(p_0)$ and $x \in \mathrm{dom}(p_1)$ both hold.

Let $q_0 = p_0 \restriction V$ and $q_1 = p_1 \restriction V$. Suppose first that $q_0 \neq q_1$. Then there must be a point $x \in V$ and distinct numbers $n_0, n_1 \in \omega$ such that $q_0(x) = n_0$ and $q_1(x) = n_1$. Let $A_0 = \{q \in P \colon x \in \mathrm{dom}(q) \wedge q(x) = n_0\}$ and $A_1 = \{q \in P \colon x \in \mathrm{dom}(q) \wedge q(x) = n_1\}$. Clearly, $A_0, A_1 \subset P$ are analytic sets coded in the ground model, ΣA_0 and ΣA_1 are incompatible in P and $p_0 \leq \Sigma A_0$ and $p_1 \leq \Sigma A_1$.

Now suppose that $q_0 = q_1$ and work towards a contradiction. Write $\bar{p}$ for the common value and note that by the product forcing theorem, $\bar{p} \in V$ holds. Now $\bar{p}$ is a balanced virtual condition in P by Theorem 8.1.2, $p_0 \leq \bar{p}$ and $p_1 \leq \bar{p}$, and so p_0, p_1 must be compatible by the balance of $\bar{p}$. This contradicts the initial choice of p_0, p_1. $\square$

Example 10.5.5. Let Γ, Δ be abelian Polish groups, with Δ divisible. The homomorphism poset of Definition 8.2.1 is $\aleph_0$-tethered.

Proof. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of V and $p_0 \in V[H_0]$, $p_1 \in V[H_1]$ are incompatible conditions. Expanding the models $V[H_0], V[H_1]$ and strengthening the conditions p_0, p_1 if necessary, we may assume that for every point $\gamma \in \Gamma \cap V$, $\gamma \in \mathrm{dom}(p_0)$ and $\Gamma \in \mathrm{dom}(p_1)$ both hold.

Let $q_0 = p_0 \restriction V$ and $q_1 = p_1 \restriction V$. Suppose first that $q_0 \neq q_1$. Then there must be a point $\gamma \in V$ and disjoint basic open sets $O_0, O_1 \subset \Delta$ such that $p_0(\gamma) \in O_0$ and $p_1(\gamma) \in O_1$. Let $A_0 = \{q \in P \colon \gamma \in \mathrm{dom}(q) \wedge q(\gamma) \in O_0\}$ and $A_1 = \{q \in P \colon \gamma \in \mathrm{dom}(q) \wedge q(\gamma) = O_1\}$. Clearly, $A_0, A_1 \subset P$ are analytic sets coded in the ground model, ΣA_0 and ΣA_1 are incompatible in P and $p_0 \leq \Sigma A_0$ and $p_1 \leq \Sigma A_1$.

Now suppose that $q_0 = q_1$ and work towards a contradiction. Write $\bar{p}$ for the common value and note that by the product forcing theorem, $\bar{p} \in V$ holds. Now $\bar{p}$ is a balanced virtual condition in P by Theorem 8.2.2, $p_0 \leq \bar{p}$ and $p_1 \leq \bar{p}$, and so p_0, p_1 must be compatible by the balance of $\bar{p}$. This contradicts the initial choice of p_0, p_1. $\square$

The following two examples deal with quotient simplicial complex posets.

Example 10.5.6. Let $E \subset F$ be Borel equivalence relations on a Polish space X such that every F-class consists of countably many E-classes. The E, F-transversal poset P of Definition 6.6.5 is tethered.

Proof. We will show that P is λ-tethered where $\lambda = \beth_{\omega_1}$. Let $V[H_0], V[H_1]$ be mutually generic extensions of the ground model V, and let $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be incompatible conditions in P. There must be elements $x_0 \in p_0$ and $x_1 \in p_1$ which are F-related but not E-related. In view of Proposition 2.1.7, the points x_0, x_1 must both be realizations of a virtual F-class in V and by Claim 6.6.13 there are virtual E-classes $d_0 \neq d_1$ in V such that x_0 is a realization of d_0 and x_1 is a realization of d_1.

In V, let $p_0' = \{q \in P \colon q$ contains some realization of the virtual E-class $d_0\}$ and $p_1' = \{q \in P \colon q$ contains some realization of the virtual E-class $d_1\}$. These are virtual conditions on the posets which carry the virtual E-classes d_0, d_1, and these are of size $< \lambda$ by Theorem 2.5.6. In addition, p_0', p_1' are incompatible, and $p_0 \leq p_0'$ and $p_1 \leq p_1'$ as required. □

The following example covers all posets which associate a structure to each G-class where G is a countable Borel equivalence relation on a Polish space, as in Example 6.6.10.

Example 10.5.7. Let X be a Polish space and let $E \subset F$ be Borel equivalence relations on X such that F is pinned and for each F-equivalence class $C \subset X$, $E \restriction C$ is smooth. Then the E, F-transversal poset P of Definition 6.6.5 is $\aleph_0$-tethered.

Proof. Let $V[H_0], V[H_1]$ be mutually generic extensions of the ground model V, and let $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be incompatible conditions in P. There must be elements $x_0 \in p_0$ and $x_1 \in p_1$ which are F-related but not E-related. Since the equivalence relation F is pinned, there must be a point $x \in X \cap V$ which is F-related to both x_0, x_1. By the initial assumptions on E, F, in the ground model there also must be a Borel map $h \colon [x]_F \to 2^\omega$ reducing E to the identity. Since $h(x_0) \neq h(x_1)$, there must be distinct binary strings $s_0, s_1 \in 2^n$ for some $n \in \omega$ such that $s_0 \subset h(x_0)$ and $s_1 \subset h(x_1)$. Let $A_0 = \{q \in P \colon p \cap h^{-1}[s_0] \neq 0\}$ and $A_1 = \{q \in P \colon p \cap h^{-1}[s_1] \neq 0\}$. It is clear that A_0, A_1 are analytic subsets of P coded in the ground model, ΣA_0 is incompatible with ΣA_1 in P, and $p_0 \leq \Sigma A_0$ and $p_1 \leq \Sigma A_1$ as required. □

Example 10.5.8. Let $\mathcal{F}$ be a Fraissé class in a finite relational language with strong amalgamation. Let E be a Borel equivalence relation on a Polish space X. Then the $E, \mathcal{F}$-Fraissé poset P of Definition 8.6.3 is tethered. In addition, if E is pinned, then P is $\aleph_0$-tethered.

Proof. Start with the general case. Let $\lambda = \beth_{\omega_1}$. We will show that the poset P is λ-tethered. For brevity, assume that $\mathcal{F}$ has just one relational symbol of its language of some finite arity $n \in \omega$. Thus, a condition $p \in P$ is just a single n-ary relation on a countable set $\operatorname{dom}(p)$ which respects the equivalence E, and such that its E-quotient is a $\mathcal{F}$-structure. Let $V[H_0], V[H_1]$ be mutually generic extensions of the ground model V and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be incompatible conditions. In view of the strong amalgamation assumption, there must be n-tuples $\vec{x}_0$ and $\vec{x}_1$ in $\mathrm{dom}(p_0)$ and $\mathrm{dom}(p_1)$ respectively such that for each $i \in n$, $\vec{x}_0(i)\ E\ \vec{x}_1(i)$ holds, and $\vec{x}_0 \in p_0$ and $\vec{x}_1 \notin p_1$ (or vice versa). By Proposition 2.1.7, there is an n-tuple $\vec{d}$ of virtual E-classes in V such that for each $i \in n$, $\vec{x}_0(i), \vec{x}_1(i)$ are realizations of $\vec{d}(i)$. Let $p_0' = \{q \in P\colon \mathrm{dom}(q)$ contains an n-tuple $\vec{x}$ which is a realization of $\vec{d}$ and $\vec{x} \in q\}$ and $p_1' = \{q \in P\colon \mathrm{dom}(q)$ contains an n-tuple $\vec{x}$ which is a realization of $\vec{d}$ and $\vec{x} \notin p\}$. These are virtual conditions on the posets which carry the virtual E-classes on the tuple $\vec{d}$, and these are of size $< \lambda$ by Theorem 2.5.6. In addition, p_0', p_1' are incompatible, and $p_0 \leq p_0'$ and $p_1 \leq p_1'$ as required.

The case of a pinned equivalence relation E proceeds in the same way, noting that in this case the$E$-quotient space and the virtual E-quotient space coincide. □

The ultrafilter posets typically do satisfy the tether demands.

Example 10.5.9. The poset P of all infinite subsets of ω ordered by inclusion is $\aleph_0$-tethered.

Proof. Let $V[H_0], V[H_1]$ be mutually generic extensions of V, containing the respective conditions $p_0, p_1 \in P$ which are incompatible in P. Removing finitely many numbers from each, we may assume that in fact $p_0 \cap p_1 = 0$. By Proposition 1.7.8, there are disjoint sets $p_0', p_1' \subset \omega$ in the ground model such that $p_0 \subset p_0'$ and $p_1 \subset p_1'$. These sets as conditions in P exemplify the tether of the poset P. □

Example 10.5.10. Let A be a Ramsey sequence of finite structures. The poset $P = P_A$ of Definition 7.3.2 is $\aleph_0$-tethered.

Proof. We use the terminology of Section 7.3. In particular, the sequence A is written as $\langle A_n \colon n \in \omega \rangle$ of structures on pairwise disjoint sets. Let $n \in \omega$ be a natural number. Write D_n for the set of all copies of A_n which are a subset of some A_m for $m \geq n$.

Let $V[H_0], V[H_1]$ be mutually generic extensions of the ground model V, and let $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be incompatible conditions in P. Expanding the models and strengthening the conditions if necessary, by Claim 7.3.5 we may assume that for each $n \in \omega$ and each set $b \subset D_n$ in the ground model, either $p_0^{A_n} \subset b$ or $p_0^{A_n} \cap b = 0$ modulo finite, and similarly for subscript 1. Let $(F_n)_0$ be the set of all ground model elements of $\mathcal{P}(D_n)$ for which the first option prevails, and similarly for subscript 1.

Suppose first that there is a number $n \in \omega$ for which $(F_n)_0 \neq (F_n)_1$. For definiteness, suppose that there is some set $b \in (F_n)_0 \setminus (F_n)_1$. Then let $p_0' = \{p \in P\colon p^{A_n} \subset b$ modulo finite$\}$ and $p_1' = \{p \in P\colon p^{A_n} \cap b = 0$ modulo finite$\}$. The sets $p_0', p_1' \subset P$ are analytic and coded in the ground model. Also, clearly $\Sigma p_0', \Sigma p_1'$ are incompatible in P, and $p_0 \leq \Sigma p_0'$ and $p_1 \leq \Sigma p_1'$ as desired.

Now suppose that the sets $(F_n)_0, (F_n)_1$ are equal for all $n \in \omega$. Write F_n for their common value. The product forcing theorem shows that $F_n \in V$ and even $\langle F_n : n \in \omega\rangle \in V$. It is not difficult to check that it is an A-sequence of ultrafilters. Theorem 7.3.4 then shows that it yields a virtual balanced condition $\bar{p}$, and $p_0, p_1 \leq \bar{p}$. Since $p_0, p_1 \leq \bar{p}$, the balance of $\bar{p}$ shows that p_0, p_1 are compatible conditions, contradicting the initial choice of p_0, p_1. □

Example 10.5.11. Let $\langle\Gamma, \cdot\rangle$ be a countable semigroup. The poset $P = P(\Gamma)$ of Section 7.4 is $\aleph_0$-tethered.

Proof. We adopt the terminology of Section 7.4. Let $V[H_0], V[H_1]$ be mutually generic extensions of V, containing the respective conditions $p_0, p_1 \in P$ which are incompatible in P. By Proposition 7.4.3, extending the models and strengthening the conditions if necessary, we may assume that both p_0, p_1 sort out the set $\mathcal{P}(\omega)\cap V$. Let $q_0 = \{a \subset \Gamma : a \in V$ and p_0 accepts $a\}$ and $q_1 = \{a \subset \Gamma : a \in V$ and p_1 accepts $a\}$.

Suppose first that $q_0 \neq q_1$, and for definiteness assume that there is a set $a \in q_0 \setminus a_1$. Let $p_0' = \{q \in P : q$ accepts $a\}$ and $p_1' = \{q \in P : q$ declines $a\}$. It is clear that $p_0', p_1' \subset P$ are analytic sets coded in V, $\Sigma p_0', \Sigma p_1'$ are incompatible, and $p_0 \leq \Sigma p_0'$, $p_1 \leq \Sigma p_1'$.

Suppose now that $q_0 = q_1$ and work towards a contradiction. Denote the common value by $\bar{p}$ and use the product forcing theorem to see that $\bar{p} \in V$. Since the conditions p_0, p_1 sort out $\mathcal{P}(\Gamma)\cap V$, $\bar{p}$ is an ultrafilter. In V, Proposition 7.4.5 then shows that $\bar{p}$ is an idempotent ultrafilter, and by Theorem 7.4.7, it yields a balanced condition in the poset P. Since $p_0, p_1 \leq \bar{p}$, the balance of $\bar{p}$ shows that p_0, p_1 are compatible conditions, contradicting the initial choice of p_0, p_1. □

Finally, we include two examples of balanced posets which are not tethered.

Example 10.5.12. Let E be a non-smooth Borel pinned equivalence relation on a Polish space X. The collapse poset P of $|E|$ to 2^ω of Definition 6.6.2 is not tethered. Countable-to-one uniformization fails in the resulting extension of the symmetric Solovay model.

Proof. We prove the last sentence. The poset P is balanced by the pinned assumption on E and Corollary 6.6.4. In the P-extension of the symmetric Solovay model, $|E| \leq |2^\omega|$ holds by the definition of the poset P. In addition, $|\mathbb{E}_0| \leq |E|$ holds by the Glimm–Effros dichotomy as E is assumed to be non-smooth. In sum, $|\mathbb{E}_0| \leq |2^\omega|$ holds, as witnessed by some function $g : 2^\omega \to 2^\omega$. In addition, Corollary 11.6.3 shows that in the P-extension of the symmetric Solovay model, $\mathbb{E}_0$ has no transversal, and therefore the function g has no left inverse. This feature of g stands witness to the failure of the countable-to-one uniformization. □

Example 10.5.13. Let X be an uncountable Polish space. The Kurepa poset P of Definition 8.4.2 is not tethered. In the P-extension of the symmetric Solovay model, countable-to-one uniformization fails.

Proof. For simplicity of notation, assume that $X = 2^\omega$. Let $W[F]$ be the symmetric Solovay model derived from an inaccessible cardinal κ, with the generic

Kurepa family F attached. In $W[F]$, let E be the equivalence relation on X connecting x, y if there is no set in F containing x but not y or vice versa. All classes of the equivalence relation E are countable as the family F consists of countable sets and is cofinal. We will show that in $W[F]$ there is no total function $f\colon X \to X$ such that for every $x \in X$, if $[x]_E \neq \{x\}$ then $f(x) \in [x]_E \setminus \{x\}$. This is clearly in violation of countable-to-one uniformization.

Working in W, let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau\colon X \to X$ is a function. We must find $x \in X$ and a stronger condition $\bar{p} \leq p$ which forces that $[x]_E \neq \{x\}$ and $\tau(x) \notin [x]_E \setminus \{x\}$. The condition p and the name τ are definable from some parameter $z \in 2^\omega$ and some ground model parameters. Let $V[K]$ be an intermediate model obtained by a forcing of size $< \kappa$ which contains the parameter z. Work in the model $V[K]$.

Let Q be the Cohen forcing on X, adding the generic point $\dot{x}_{gen}$. Let $H_0, H_1 \subset Q$ be mutually generic filters, let $x_0 = \dot{x}_{gen}/H_0$ and $x_1 = \dot{x}_{gen}/H_1$, and in the model $V[K][H_0][H_1]$ consider the virtual condition $\bar{p} \leq p$ which in addition to p contains the sets $\bigcup p \cup \{x_0\} \cup [x_1]_{\mathbb{E}_0}$ and $X \cap V[K][H_0][H_1]$. By Theorem 8.4.3, $\bar{p}$ is a balanced virtual condition. By a balance argument, it decides the value of $\tau(x_0)$ to be some specific point $y \in X$. An inspection of the condition $\bar{p}$ reveals that it forces $[x_0]_E = \{x_0\} \cup [x_1]_{\mathbb{E}_0}$. We will be finished if we show that y is not $\mathbb{E}_0$-related to x_1.

Towards contradiction, assume that it is. Then y is a Cohen generic point over $V[K][H_0]$, yielding the same extension as x_1. By the forcing theorem in $V[K][H_0]$ and the definability of τ, there would have to be a condition $q \in Q$ such that $V[K][H_0] \models q \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash \tau(\check{x}_0) = \dot{x}_{gen}$. By the forcing theorem again, in the model $V[K][H_0][H_1]$, the virtual condition $\bar{p}$ would force the value $\tau(\check{x}_0)$ to be each of the infinitely many elements of $[x_1]_{\mathbb{E}_0}$ in the open set q simultaneously, an impossibility. □

There are also partial orders where we are unable to decide the status of tether, such as the Fin×Fin poset of Definition 7.2.1.

Chapter 11

Locally countable structures

This chapter introduces a methodology for proving that certain models of ZF+DC we produce do not contain locally countable structures of certain type. In particular, we learn how to produce models of ZF+DC which contain locally countable structures of one type but not of another type. The whole enterprise should be viewed as a parallel to the extensive field of descriptive set theory of locally countable structures. There are many striking similarities present, and many others are sure to be found in the future.

11.1 Central objects and notions

In this section, we introduce certain basic locally countable graphs and hypergraphs and the associated concerns of the descriptive set theory of locally countable structures. Some of the critical issues deal with the comparison of chromatic and Borel chromatic numbers of various analytic hypergraphs on Polish spaces. An analytic hypergraph on a Polish space X is a set $G \subset [X]^{<\aleph_0}$ which is an analytic subset of the hyperspace $K(X)$ with the Vietoris topology. The elements of a hypergraph are referred to as *hyperedges*. Thus, our hypergraphs contain finite hyperedges only–they are *finitary*; it may occur though that there is no fixed $n \in \omega$ (*arity*) such that $G \subset [X]^n$. As a matter of convention, our hypergraphs contain no singleton sets. A *graph* is a hypergraph of arity 2. If G, H are hypergraphs on respectve spaces X, Y, a function $h\colon X \to Y$ is a *homomorphism* if the h-image of any G-hyperedge is an H-hyperedge. An *anticlique* is a set $a \subset X$ such that $[a]^{<\aleph_0} \cap G = 0$. The *chromatic number* of G is the smallest cardinality of a set of anticliques covering the whole space X; in the choiceless context, where cardinalities are not well-ordered, we distinguish only between countable and uncountable chromatic number and different values of countable chromatic numbers.

Most of the central analytic hypergraphs in this section are in fact products of hypergraphs on finite or countable domain. The following definition explains the typical construction.

Definition 11.1.1. Let $\langle a_n, H_n, t_n \colon n \in \omega\rangle$ be a sequence such that a_n is a nonempty countable (often finite) set, H_n is a hypergraph on a_n and $t_n \in \prod_{m\in n} a_m$ are finite strings such that $\{t_n \colon n \in \omega\}$ is a dense subset of $\bigcup_n \prod_{m\in n} a_m$. Let $X = \prod_n a_n$ and define:

1. (the *skew product*) $\prod_n H_n, t_n$ is the hypergraph on X containing all sets $b \subset X$ such that for some $n \in \omega$, $\forall m \neq n\ \forall x_0, x_1 \in b\ x_0(m) = x_1(m)$, $\forall x \in b\ t_n \subset b$, and $\{x(n) \colon x \in b\} \in H_n$;
2. (the *product*) $\prod_n H_n$ is the hypergraph on X containing all sets $b \subset X$ such that for some $n \in \omega$, $\forall m \neq n\ \forall x_0, x_1 \in b\ x_0(m) = x_1(m)$, and $\{x(n) \colon x \in b\} \in H_n$.

In all cases, the sets a_n are implicit in the notation. The definition of the product (as opposed to the skew product) does not depend on the strings t_n. Among the product graphs, the Hamming graphs are prominent.

Definition 11.1.2. Let $n \geq 2$ be a natural number. $\mathbb{H}_n$, the *Hamming graph on* n^ω, is the product of infinitely many cliques of size n. $\mathbb{H}_{<\omega}$, the *diagonal Hamming graph* on the space $\prod_n (n+1)$, is the product of cliques of all possible nonzero finite cardinalities. Finally, $\mathbb{H}_\omega$, the *Hamming graph on* ω^ω, is the product of infinitely many cliques on ω.

Skew products are useful as minimal examples of various phenomena. One standard example is the uncountable Borel chromatic number of graphs:

Definition 11.1.3. Let G be an analytic hypergraph on a Polish space X. G has *countable Borel chromatic number* if there are Borel G-anticliques $B_n \subset X$ for $n \in \omega$ such that $X = \bigcup_n B_n$.

Definition 11.1.4. For each $n \in \omega$, let $a_n = 2$, let H_n be the graph on a_n containing only the whole set a_n as an edge and let $t_n \in 2^n$ be a string such that $\{t_n \colon n \in \omega\} \subset 2^{<\omega}$ is dense. The graph $\mathbb{G}_0$ is defined as $\prod_n H_n, t_n$.

Fact 11.1.5. [54](ZF+DC) *For every analytic graph G on a Polish space X, exactly one of the following occurs:*

1. *the Borel chromatic number of G is countable;*
2. *there is a continuous map from 2^ω to X which is a homomorphism of $\mathbb{G}_0$ to G.*

Uncountable chromatic numbers of locally countable graphs can be used in ZF+DC to rule out seemingly unrelated phenomena. The following humble proposition will be useful at several points of this book:

Proposition 11.1.6. (ZF+DC) *If the chromatic number of $\mathbb{G}_0$ is greater than two then*

1. *there is no linear ordering of the $\mathbb{E}_0$-quotient space;*

2. *$|\mathbb{E}_0| \not\leq |2^\omega|$;*

3. *there is no discontinuous homomorphism between Polish groups.*

Proof. For the first item, consider the equivalence relation E on 2^ω connecting points $x, y \in 2^\omega$ if they differ in finite, even number of entries. Then $E \subset \mathbb{E}_0$; as a subset of a hyperfinite equivalence relation, it is itself hyperfinite and therefore Borel reducible to $\mathbb{E}_0$ [48, Theorem 8.1.1]. Now, suppose that the $\mathbb{E}_0$-quotient space is linearly orderable; then so is the E-quotient space, and one can define the function $c\colon 2^\omega \to 2$ by setting $c(x) = 0$ if among the two E-classes which constitute the $\mathbb{E}_0$-class of x, x belongs to the smaller one in the fixed linear ordering of the $\mathbb{E}$-quotient space. It is not difficult to show that c is a $\mathbb{G}_0$-coloring with two colors.

The second item follows from the first, since the inequality $|\mathbb{E}_0| \leq |2^\omega|$ yields a linear ordering on the $\mathbb{E}_0$-space by simply pulling back the usual lexicographical order on 2^ω to the $\mathbb{E}_0$-quotient space via the assumed injection from the $\mathbb{E}_0$-quotient space to 2^ω. The third item is much harder. [76] constructs, in ZF+DC, a coloring of the Hamming graph $\mathbb{H}_2$ (a superset of the $\mathbb{G}_0$-graph) with two colors from the assumption that there is a discontinuous homomorphism between Polish groups. □

It is not difficult to extend the $\mathbb{G}_0$-dichotomy to arbitrary finitary analytic hypergraphs. In this generality, one has to allow for uncountably many basis hypergraphs, which are nevertheless easy to describe:

Definition 11.1.7. Suppose that $\langle a_n, H_n, t_n \colon n \in \omega\rangle$ is a sequence such that each a_n is a a finite set of cardinality at least two, H_n is the hypergraph on it containing a_n as its only hyperedge, and $t_n \in \prod_{m\in n} a_m$ is a string such that the set $\{t_n \colon n \in \omega\}$ is dense in $\bigcup_n \prod_{m\in n} a_m$. Then the hypergraph $\prod_n H_n, t_n$ on $\prod_n a_n$ is called a *principal skew product.*

The following can be proved in the same way as the $\mathbb{G}_0$-dichotomy. We suppress the standard proof.

Fact 11.1.8. (ZF+DC) *Let G be a finitary analytic hypergraph on a Polish space X. Exactly one of the following occurs:*

1. *the Borel chromatic number of G is countable;*

2. *there exists a principal skew product and a continuous homomorphism of it to the hypergraph G.*

In this chapter, we will also use more involved variations of the uncountable Borel chromatic number.

Definition 11.1.9. Let G be an analytic finitary hypergraph on a Polish space X. The hypergraph G has *Borel σ-bounded chromatic number* if there are Borel sets $B_n \subset X$ for $n \in \omega$ such that $X = \bigcup_n B_n$ and for every $n \in \omega$, every finite subset of B_n has G-chromatic number less than $n + 2$.

It is immediate that countable Borel chromatic number implies Borel σ-bounded chromatic number. The opposite implication does not holds as the following examples show.

Example 11.1.10. The Hamming graph $\mathbb{H}_2$ has uncountable Borel chromatic number by Fact 11.1.5 but it does have Borel σ-bounded chromatic number. To see this, observe that it contains no odd length cycles and therefore $\mathbb{H}_2$ on each finite subset of 2^ω has chromatic number 2. The sequence B_n defined by $B_1 = 2^\omega$ and $B_n = 0$ for all $n \neq 1$ exemplifies the Borel σ-bounded chromatic number of $\mathbb{H}_2$.

Example 11.1.11. The diagonal Hamming graph $\mathbb{H}_{<\omega}$ does not have Borel σ-bounded chromatic number. To see this, by the Baire category theorem it is enough to show that any non-meager Borel subset of $X = \prod_n (n+1)$ contains arbitrarily large finite $\mathbb{H}_{<\omega}$-cliques. Thus, let $B \subset X$ be non-meager Borel and let $m \in \omega$ be a number. Let t be a finite string of natural numbers such that B is comeager in t and $|t| = n > m$. A simple construction yields a point $x \in X$ such that $t \subset x$ and for every $i \in n$ the point x_i, obtained from x by rewriting its n-th entry with i, belongs to the set B. Since the set $\{x_i : i \in n\} \subset B$ is an $\mathbb{H}_{<\omega}$-clique, the argument is complete.

The class of analytic finitary hypergraphs which do not have Borel σ-bounded chromatic number has a simple basis.

Definition 11.1.12. Let $\langle a_n, H_n, t_n : n \in \omega\rangle$ be a sequence such that a_n is a nonempty finite set, H_n is a hypergraph on a_n of chromatic number $> n$, $t_n \in \prod_{m \in n} a_m$, and the set $\{t_n : n \in \omega\}$ is dense in $\bigcup_n \prod_{m \in n} a_n$. The skew product $\prod_n H_n, t_n$ is then called a *large skew product*.

A straightforward Baire category argument as in Example 11.1.11 shows that a large skew product does not have Borel σ-bounded chromatic number. The following fact is proved in the same way as the $\mathbb{G}_0$-dichotomy, and we omit the standard argument.

Fact 11.1.13. *Let G be a finitary analytic hypergraph on a Polish space X. Exactly one of the following occurs:*

1. *G has Borel σ-bounded chromatic number;*
2. *there is a large skew product H on a Polish space Y and a continuous homomorphism $h\colon Y \to X$ of H to G.*

Many interesting natural examples of hypergraphs which do not have Borel σ-bounded chromatic number actually have a stronger property encapsulated in the following definitions.

Definition 11.1.14. Let a be a finite set and G be a hypergraph on a. The *fractional chromatic number* of G is the maximum of all numbers $\Sigma_{v \in a} f(v)$ where $f\colon a \to [0,1]$ ranges over all functions such that $\Sigma_{v \in b} f(v) \leq 1$ for all G-anticliques $b \subset a$.

A word about the terminology is in order. In fractional graph theory [78], the fractional chromatic number is typically defined for graphs only. In the graph context, the definition above corresponds to the fractional clique number of G, which is equal to the standard fractional chromatic number of G by a linear programming duality argument [78, Section 3.1]. In order to maintain coherent terminology, we neglect this important but for our ends irrelevant point. A trivial restatement of the fractional chromatic number will be used below: it is the largest real number r such that there is a probability measure on a in which every G-anticlique has mass $\leq 1/r$.

Note that the fractional chromatic number is no greater than the chromatic number. Just as in the case of the chromatic number, if G_0, G_1 are hypergraphs on a_0, a_1 respectively and $h\colon a_0 \to a_1$ is a homomorphism of G_0 to G_1 then the fractional chromatic number of G_1 is not smaller than the fractional chromatic number of G_0. The following examples further elucidate the relationship between the chromatic number and its fractional counterpart.

Example 11.1.15. For every $n \in \omega$ let G_n be the graph on $[n]^2$ connecting pairs a, b if the smaller element of a is equal to the larger element of b or vice versa. Then the chromatic numbers of G_n tend to infinity with n, but the fractional chromatic numbers remain bounded by 4.

For the former statement, note that if n is such that $n \to (3)^2_m$ then the chromatic number of G_n is greater than m: if $[n]^2 = \bigcup_{i\in m} B_i$, then by the Ramsey property of n there is $i \in m$ and a triple $c \subset n$ such that $[c]^2 \subset B_i$; however, $[c]^2$ clearly contains a G-edge.

For the latter statement, suppose that μ is a probability measure on $[n]^2$, and let λ be the normalized counting measure on $\mathcal{P}(n)$. Let $B = \{\langle a, x\rangle \in [n]^2 \times \mathcal{P}(n)\colon x$ contains the smaller number in a but not the larger number in $a\}$. The vertical sections of B have λ-mass $1/4$, so by the Fubini theorem there must be a horizontal section B^x whose μ-mass is $\geq 1/4$. It is not difficult to check that B^x is a G-anticlique.

Example 11.1.16. Let $\varepsilon > 0$ be a fixed real number. For each $n \in \omega$, let G_n be the graph on 2^n which connects x, y if the set $\{m \in n\colon x(m) = y(m)\}$ has size at most εn. The fractional chromatic numbers of G_n tend to infinity with n. To see this, consider the normalized counting measure μ_n on 2^n, the normalized Hamming metric d_n on 2^n, and the concentration of measure results for the Hamming cubes [73, Theorem 4.3.19]: for every $\delta > 0$ there is a number $k \in \omega$ such that for every $n \geq k$ and every set $b \subset 2^n$ of μ_n-mass greater than δ, the ε-neighborhood of b in 2^n in the sense of the metric d_n has mass greater than $1/2$. For each such number $n \in \omega$, every set $b \subset 2^n$ of μ_n-mass greater than δ contains a G_n-edge. To produce the edge, consider the automorphism $\pi_n\colon 2^n \to 2^n$ defined by $\pi_n(x)(m) = 1 - x(m)$. The set $\pi''b$ has μ-mass greater than δ again. The $\varepsilon/2$-neighborhoods of the sets b and $\pi''b$ in the sense of the metric d_n have both μ_n mass greater than $1/2$ and therefore intersect. It follows that there must be points $x \in b$ and $y \in \pi''b$ with $d_n(x, y) \leq \varepsilon_n$, and then $x, \pi^{-1}y$ are G_n-connected elements of the set b.

Example 11.1.17. For every $n \in \omega$, let H_n be the hypergraph on n of arity 3 consisting of all arithmetic progressions of length 3. Then the fractional chromatic numbers of H_n tend to infinity with n–just consider the normalized counting measure on n and the density van der Waerden's theorem [88].

Definition 11.1.18. Let G be an analytic finitary hypergraph on a Polish space X. We say that G has *Borel σ-bounded fractional chromatic number* if there are Borel sets $B_n \subset X$ such that $X = \bigcup_n B_n$ and for each $n \in \omega$, the hypergraph G has fractional chromatic number less than $n+2$ on every finite set $a \subset B_n$.

Clearly, Borel σ-bounded chromatic number implies Borel σ-bounded fractional chromatic number but the opposite implication fails. The most natural class of Borel graphs exemplifying the distinction is the following:

Example 11.1.19. Let E be a Borel non-smooth equivalence relation on a Polish space X. Let Q be the poset of all pairs $q = \langle a_q, b_q\rangle$ of finite subsets of X such that $(a_q \times b_q) \cap E = 0$; the ordering is that of coordinatewise reverse inclusion. Let G be the graph connecting two conditions in Q if they are incompatible. Then G does not have Borel σ-bounded chromatic number, yet it does have Borel σ-bounded fractional chromatic number.

Proof. To show that G does not have Borel σ-bounded chromatic number, first use the Glimm–Effros dichotomy to find a Borel reduction $h\colon 2^\omega \to X$ of $\mathbb{E}_0$ to E. For $\mathbb{E}_0$-unrelated points $y_0, y_1 \in 2^\omega$ let $r(y_0, y_1) = \langle \{h(y_0)\}, \{h(y_1)\}\rangle$ and note that $r(y_0, y_1) \in Q$. By the Baire category theorem, it will be enough, for any Borel nonmeager set $B \subset 2^\omega \times 2^\omega$ and every $n \in \omega$, to produce a finite subset $b \subset B$ such that the chromatic number of G on $q''b$ is greater than n. To this end, let $t_0, t_1 \in 2^{<\omega}$ be two binary strings of the same length such that $B \subset [t_0] \times [t_1]$ is comeager. Let $k \in \omega$ be a number such that $k \to (3)^2_n$. A simple construction yields points $y_i \in 2^\omega$ for $i \in k$ such that for any $i \in j \in k$, $\langle t_0^\frown y_i, t_1^\frown y_j\rangle \in B$ holds. It will be enough to show that the graph G on the set $c = \{q(t_0^\frown y_i, t_1^\frown y_j)\colon i \in j \in k\}$ has chromatic number greater than n. To see this, suppose that $c = \bigcup_{m\in n} d_m$. By the choice of the number k, there exist $m \in n$ and numbers $i_0 \in i_1 \in i_2$ such that any pair of them gives rise to a condition in the set d_m. Now note that the conditions $r(t_0^\frown y_{i_0}, t_1^\frown y_{i_1})$ and $r(t_0^\frown y_{i_1}, t_1^\frown y_{i_2})$ are incompatible in Q and so form a G-edge.

To show that G has Borel σ-bounded fractional chromatic number, let $B_n = \{q \in Q\colon |a_q \cup b_q| \leq n\}$, observe that the set B_n is Borel and $Q = \bigcup_n B_n$, and argue that for every finite set $c \subset B_n$, the fractional chromatic number on G on c is $\leq 2^{-n}$. Indeed, assume that μ is any probability measure on the set c; we will produce a set of μ-mass $\geq 2^{-n}$ consisting of pairwise compatible conditions, i.e. a G-anticlique. To do this, let e be the finite set of all E-classes with nonempty intersection with $\bigcup_{q\in c}(a_q \cup b_q)$ and let λ be the normalized counting measure on $\mathcal{P}(e)$. Let $A = \{\langle q, u\rangle \in c \times \mathcal{P}(e)\colon a_q \subset \bigcup u \wedge b_q \cap \bigcup u = 0\}$. Since $c \subset B_n$, all vertical sections of the set A have λ-mass $\geq 2^{-n}$. By the Fubini theorem, there is a horizontal section of A of μ-mass $\geq 2^{-n}$. This horizontal section is the desired G-anticlique of μ-mass $\geq 2^{-n}$. $\square$

Example 11.1.11 actually shows that the diagonal Hamming graph does not have Borel σ-bounded fractional chromatic number, since the density of a clique of size n is n. Many other examples arise from various density versions of Ramsey-type theorems.

Example 11.1.20. Suppose that $\mathbb{Z}$ acts in a Borel, free and measure-preserving way on a Polish probability measure space $\langle X, \mu\rangle$. Let G be the Borel hypergraph on X of arity 3 containing a triple $\{x, y, z\}$ if there is $n \in \omega$ such that $n \cdot x = y$ and $n \cdot y = z$. The hypergraph G does not have Borel σ-bounded fractional chromatic number.

Proof. Suppose towards a contradiction that $X = \bigcup_n B_n$ is a decomposition of X into Borel sets witnessing the Borel σ-bounded fractonal chromatic number of G. Pick $n \in \omega$ and $\varepsilon > 0$ such that $\mu(B_n) > \varepsilon$. Use the density version of van der Waerden theorem [88] to find a number $m_0 \in \omega$ such that in every subset of m_0 of size $> m_0/n$, one of the classes contains an arithmetic progression of length three. Use the density van der Waerden theorem again to find a number m_1 such that every subset of m_1 of size $> \varepsilon m_1$ contains an arithmetic progression of length m_0. Let $C \subset m_1 \times X$ be the set $C = \{\langle i, x\rangle : x \in i \cdot B_n\}$. As every vertical section of the set C has μ-mass $> \varepsilon$, by the Fubini theorem there must be a point $x \in X$ such that the horizontal section $C^x \subset m_1$ contains more than εm_1 many numbers and so an arithmetic progression $\{i + jk : k \in m_0\}$ for some choice of i, k. Let $x_j = (-i - jk) \cdot x$ for $j \in m_1$ and note that $\{x_j : j \in m_0\} \subset B_n$ holds. By the choice of m_0, the fractional chromatic number of G on the set $\{x_j : j \in m_0\}$ is larger than n, contradicting the choice of the set B_n. □

The class of analytic finitary hypergraphs which do not have Borel σ-bounded fractional chromatic number has a simple basis.

Definition 11.1.21. Suppose that $\langle a_n, H_n, t_n : n \in \omega\rangle$ is a sequence such that a_n is a nonempty finite set, H_n is a hypergraph on a_n such that every H_n-anticlique has fewer than $|a_n|/(n+2)$ many elements, and $t_n \in \prod_{m\in n} a_m$ and the set $\{t_n : n \in \omega\}$ is dense in $\bigcup_n \prod_{m\in n} a_n$. The skew product $\prod_n H_n, t_n$ is then called a *large measured skew product.*

A straightforward Baire category argument shows that a large measured skew product does not have Borel σ-bounded fractional chromatic number. The following fact is proved in the same way as the $\mathbb{G}_0$-dichotomy, and we omit the standard argument.

Fact 11.1.22. *Let G be a finitary analytic hypergraph on a Polish space X. Exactly one of the following occurs:*

1. *G has Borel σ-bounded fractional chromatic number;*
2. *there is a large measured skew product H on a Polish space Y and a continuous homomorphism $h : Y \to X$ of H to G.*

We now include two definitions and related dichotomies which deal with the clique number of Borel graphs.

Definition 11.1.23. Let G be an analytic graph on a Polish space X. We say that G has *Borel σ-bounded clique number* if there are Borel sets $B_n \subset X$ for $n \in \omega$ such that no B_n contains a G-clique of size $n+2$.

Fact 11.1.24. *Let G be an analytic graph on a Polish space X. Exactly one of the following occurs:*

1. *G has Borel σ-bounded clique number;*
2. *there is a continuous homomorphism from a skew product of infinitely many cliques of increasing finite size to G.*

Definition 11.1.25. Let G be an analytic graph on a Polish space X. We say that G has *Borel σ-finite clique number* if there are Borel sets $B_n \subset X$ for $n \in \omega$ such that no B_n contains an infinite G-clique.

Fact 11.1.26. *Let G be an analytic graph on a Polish space X. Exactly one of the following occurs:*

1. *G has Borel σ-finite clique number;*
2. *there is a continuous homomorphism from a skew product of infinitely many infinite cliques to G.*

11.2 Very Suslin forcings

The preservation arguments in this chapter are in spirit quite different than those in other chapters. All of them use definable c.c.c. *control forcings.* These are posets that serve to build interesting balanced conditions in the central σ-closed poset. We need to learn how to iterate the control forcings with finite support while preserving their definability properties. We also need to isolate some definable regularity properties of the control forcings and learn how to preserve those. This last task is quite reminiscent of Stevo Todorcevic's emphasis on regularity properties of c.c.c. posets as exhibited in numerous papers of his–[93, 92]. However, we must stress that in our case the regularity properties must be witnessed in a definable way, otherwise they are worthless. Similarly, our c.c.c. control forcings are nearly all isomorphic to the product of continuum many Cohen reals in ZFC; their worth for our purpose stems from their definability properties as opposed to their forcing properties.

Definition 11.2.1. [44] A pre-ordering $\langle P, \leq \rangle$ is a *Suslin forcing* if there is an ambient Polish space X such that $P \subset X$ is an analytic set, $\leq$ is an analytic subset of X^2, and the incompatibility relation $\perp$ on P is an analytic subset of X^2. To ease the notational clutter, in this section we only require that $\leq$ is a transitive relation on P containing the diagonal. In addition, the poset P is required to have a largest element.

A given Suslin poset $\langle P, \leq\rangle$ can be reinterpreted in every forcing extension. Note that the demands on the analytic definitions of $\leq$ and $\perp$ are $\mathbf{\Pi}^1_2$ and so the reinterpretation is again a Suslin poset. Importantly, the c.c.c. property of Suslin forcings persists to forcing extensions.

Fact 11.2.2. [44] *Let $\langle P, \leq\rangle$ be a Suslin c.c.c. poset. The reinterpretation of P in any forcing extension is c.c.c.*

One has to enter the much more restrictive class of *very Suslin forcings* to guarantee that the finite support iterations, separative quotients, and completions have desirable descriptive properties.

Definition 11.2.3. Suppose that P is a Suslin c.c.c. poset on an ambient Polish space X. The poset is *very Suslin* if the set $\{a \in P^\omega : \mathrm{rng}(a) \text{ is predense in } P\}$ is an analytic subset of X^ω.

Note that being a very Suslin partial order is a $\mathbf{\Pi}^1_2$ statement in the code for the analytic set of maximal antichains and therefore absolute among all forcing extensions. Note that if the poset P and the ordering $\leq$ are Borel, which is the case for all applications in this book and elsewhere as well, then both the incompatibility relation and the set of maximal antichains are coanalytic and therefore Borel by the Suslin theorem.

Example 11.2.4. The Cohen forcing, random forcing, and the eventually different real forcing are very Suslin.

Proof. The Suslinity of the definition of the posets is elementary and left for the reader to check; we only verify the very Suslin property. The Cohen forcing is the poset P of finite binary strings ordered by reverse inclusion. A set $a \subset P$ is predense if every condition in P is compatible with some element of a, which in view of the fact that P is countable is a Borel condition. If the random forcing P is realized as the collection of compact μ-positive mass subsets of X ordered by inclusion, where μ is some Borel probability measure on a Polish space X, then a countable set $a \subset P$ is pre-dense just in case $\mu(\bigcup a) = 1$, which is a Borel condition. The case of the eventually different real forcing is somewhat more involved, and addressed in [101, Proposition 3.8.12]. $\square$

Example 11.2.5. The Hechler forcing is Suslin c.c.c. but not very Suslin.

The Suslinness and c.c.c. of Hechler forcing are well-known and easily checked. The failure of the the very Suslin property follows from the basic distinction between Suslin and very Suslin c.c.c. posets: the latter cannot add dominating reals.

Proposition 11.2.6. *Let P be a very Suslin c.c.c. forcing. Then P does not add dominating reals.*

Proof. Let $p \in P$ and τ be a P-name for an element of ω^ω. Consider the set $A = \{T \subset \omega^{<\omega}:$ some $q \leq p$ forces $\check{T}$ to have no infinite branch modulo finite dominated by $\tau\}$. By the evaluation of the complexity of the forcing relation in Proposition 11.2.9 below, A is an analytic set of trees. At the same time, if p forces $\tau \in \omega^\omega$ to be a dominating real, then A is the set of all well-founded trees, which is coanalytic and not analytic. Since the conclusion is impossible, the assumption must be false as well. The proposition follows. $\square$

The main feature of the class of very Suslin c.c.c. posets is that it is in a suitable precise sense closed under finite support iterations of countable length.

Theorem 11.2.7. *Let $\alpha \in \omega_1$ be a countable ordinal and P a very Suslin c.c.c. forcing. The finite support iteration of P of length α is naturally isomorphic to a very Suslin c.c.c. forcing.*

We emphasise that no such iteration result is possible for general Suslin forcings. The complexity of the forcing relation on e.g. the Hechler forcing causes the complexity of the iterated posets to explode in ways that are unmanageable without large cardinal assumptions. In order to prove the iteration theorem for very Suslin c.c.c. posets succintly, we need several preliminary definitions and facts.

Definition 11.2.8. Suppose that P is a very Suslin c.c.c. poset and X is a Polish space.

1. A *layered P-name* for an element of X is a tuple $\tau = \langle A_n, f_n : n \in \omega\rangle$ such that $A_n \subset P$ is a maximal antichain, f_n is a map from A_n to basic open subsets of X of radius $< 2^{-n}$, and if $m < n$ then A_m refines A_n and if $p_n \in A_n$ and $p_m \in A_m$ are such that $p_m \leq p_n$, then the closure of $f_m(p_m)$ is a subset of $f_n(p_n)$.
2. X^P is the set of all layered P-names for elements of X.

Note that the set X^P is analytic in a suitable ambient space. The definition of X^P appears to depend on the choice of the metric for X, a dependence that we happily suppress. It is clear that for every name for an element of the space X there is a layered name which is forced to be equal to the original name. Now we are ready to show that the forcing relation of very Suslin c.c.c. forcings is $\mathbf{\Sigma}^1_1$ on $\mathbf{\Sigma}^1_1$ in a precise sense.

Proposition 11.2.9. *Suppose that P is a very Suslin c.c.c. poset, X is a Polish space, and $A \subset X$ is an analytic set. The set $\{\langle p, \tau\rangle \in P \times X^P : p \Vdash \tau \in \dot{A}\}$ is an analytic subset of $P \times X^P$.*

Proof. Let $\tau = \langle A_n, f_n : n \in \omega\rangle$. Let $h : \omega^\omega \to A$ be a continuous surjection. The statement $p \Vdash \tau \in \dot{A}$ is equivalent to the existence of a system $\langle B_n, g_n : n \in \omega\rangle$ where for each $n \in \omega$, B_n is a maximal antichain of P of conditions which are either below p or incompatible with p, $g_n : B_n \to \omega^{<\omega}$, B_{n+1} refines B_n and A_n, whenever $r \leq q$ are conditions in B_{n+1} and B_n respectively then $g_{n+1}(r)$ properly extends $g_n(q)$, and if $r \leq q$ is an element of B_{n+1} below p and an element of A_n respectively then $[g_{n+1}(r)] \subset h^{-1} f_n(q)$. This is an analytic statement. $\square$

In order to formulate the finite support iterations of very Suslin forcings, we must state the usual two-step iteration and direct limit definitions in the definable context. These definitions are verbose but contain no surprises.

Definition 11.2.10. Let P, Q be posets. A *projection* of Q to P is a pair of order-preserving functions $\pi\colon Q \to P$ and $\xi\colon P \to Q$ such that

1. $\pi \circ \xi$ is the identity on P;
2. whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
3. whenever $p \leq \pi(q)$ then there is $q' \leq q$ such that $\pi(q') \leq p$.

The following definition will be used at the successor stage of the iterations.

Definition 11.2.11. Let P be a very Suslin poset on a Polish space X and Q be a very Suslin poset on a Polish space Y. Define $P * \dot{Q}$ to be the set of all ordered pairs $\langle p, \dot{q}\rangle$ where $p \in P$ and q is a nice P-name for an element of Y such that $p \Vdash \dot{q} \in \dot{Q}$. The ordering is defined by $\langle p_1, \dot{q}_1\rangle \leq \langle p_0, \dot{q}_0\rangle$ if $p_1 \leq p_0$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_0$. The iteration $P * \dot{Q}$ also includes the iteration maps $\pi\colon P * \dot{Q} \to P$ defined by $\pi(p, \dot{q}) = p$ together with the function $\xi\colon P \to P * \dot{Q}$ defined by $\xi(p) = \langle p, 1_Q\rangle$.

The limit stages will be defined in the following way.

Definition 11.2.12. A *very Suslin system* is a tuple $\langle P_n\colon n \in \omega, \pi_{nm}, \xi_{mn}\colon m \leq n \in \omega\rangle$ where

1. each P_n is a very Suslin c.c.c. forcing;
2. for each $m \leq n \in \omega$, the functions $\pi_{nm}\colon P_n \to P_m$ and $\xi_{mn}\colon P_m \to P_n$ are analytic and form a projection of P_n to P_m, with π_{nn} and ξ_{nn} equal to the identity on P_n;
3. the functions π_{nm} commute, as do the functions ξ_{mn}.

The *limit* of the system is the poset P_ω of all pairs $\langle p, n\rangle$ where $p \in P_n$, ordered by $\langle q, n\rangle \leq \langle p, m\rangle$ if $m \leq n$ and $\pi_{nm}(q) \leq p$ in P_n. The limit also includes the limit maps $\pi_{\omega m}\colon P_\omega \to P_m$ and $\xi_{m\omega}\colon P_m \to P_\omega$ defined by $\xi_{m\omega}(p) = \langle p, m\rangle$ and $\pi_{\omega m}(\langle p, n\rangle) = \pi_{nm}(p)$ if $n > m$ and $\pi_{\omega m}(\langle p, n\rangle) = \xi_{nm}(p)$ if $n \leq m$.

Finally, we are ready to approach the proof of Theorem 11.2.7.

Proof of Theorem 11.2.7. The argument proceeds by a standard transfinite induction argument using the following claims:

Claim 11.2.13. *Let P, Q be very Suslin c.c.c. posets. Then*

1. *$P * \dot{Q}$ is a very Suslin c.c.c. poset;*
2. *the iteration maps are analytic and form a projection from $P * \dot{Q}$ to P.*

Proof. This is a routine complexity calculation. The underlying set $P * \dot{Q}$ is analytic by Proposition 11.2.9, since $p \Vdash \dot{q} \in \dot{Q}$ is an analytic statement. The ordering on $P * \dot{Q}$ is analytic for the same reason. To check the analyticity of the incompatibility relation, observe that conditions $\langle p_0, \dot{q}_0 \rangle$ and $\langle p_1, \dot{q}_1 \rangle$ are incompatible just in case there is a (countable) maximal antichain $A \subset P$ such that for every $p \in A$, either p is incompatible with p_0, or it is incompatible with p_1, or it is below both p_0, p_1, and in the latter case, $p \Vdash \dot{q}_0, \dot{q}_1$ are incompatible in the poset $\dot{Q}$. This is an analytic statement by Proposition 11.2.9 and the assumption that the incompatibility relation on Q is analytic.

The poset $P * \dot{Q}$ is c.c.c. because $\dot{Q}$ remains c.c.c. in the P-extension by Fact 11.2.2 and so $P * \dot{Q}$ is an iteration of two c.c.c. forcings. Finally, we have to check that for a countable set $B = \{\langle p_n, \dot{q}_n \rangle \colon n \in \omega\} \subset P * \dot{Q}$, the statement that B is predense is analytic. To see this, let Z be the Polish space resulting from adding an isolated point 0 to Y, and consider the following formula ϕ: there exists a name τ for an element of Z^ω and maximal antichains $A_n \subset P$ for $n \in \omega$ such that for each n, A_n consists of elements which are either incompatible with p_n or stronger than p_n, if they are stronger than p_n then they force $\tau(n) = \dot{q}_n$, if they are incompatible with p_n then they force $\tau(n) = 0$, and 1_P forces $\operatorname{rng}(\tau) \cap Y$ is predense in $\dot{Q}$. Parsing the formula ϕ, we see that it is analytic by Proposition 11.2.9, and that it says that $1_P \Vdash \{\dot{q}_n \colon p_n$ belongs to the generic filter$\}$ is predense in $\dot{Q}$, which is exactly equivalent to the set B being predense in $P * \dot{Q}$.

Checking the properties of the projection functions is routine and left to the reader. □

Claim 11.2.14. *Suppose that $\langle P_n \colon n \in \omega, \pi_{nm}, \xi_{mn} \colon m \leq n \in \omega \rangle$ is a very Suslin system. The limit P_ω is a very Suslin c.c.c. forcing. The limit maps $\pi_{\omega m}$ and $\xi_{m\omega}$ form analytic projections and commute with the projection maps of the very Suslin system.*

Proof. This is a straightforward complexity computation. Let X_n be the ambient Polish space of each poset P_n; the ambient space of the poset P_ω is then the disjoint union $\bigcup_n X_n$ in product with ω. The ordering of P_ω is analytic by the definition. The incompatibility relation is analytic as well: suppose that $\langle q, n \rangle$ and $\langle p, m \rangle$ are conditions in P_ω, say with $n \geq m$. These two conditions are incompatible just in case $\pi_{nm}(q)$ is incompatible with p in P_m by the definitory properties of projections.

The poset P_ω is c.c.c. since it is a direct limit of c.c.c. forcings. To check the very Suslin property of the poset P_ω, note that a countable set $A \subset P_\omega$ is predense if and only if for every number $n \in \omega$, there is a maximal antichain $B_n \subset P_\omega$ of conditions such that for each $p \in B_n$, either there is a condition $\langle q, m \rangle \in A$ such that $m \leq n$ and $\pi_{nm}(p) \leq q$ in P_m, or there is a condition $\langle q, m \rangle \in A$ such that $m \geq n$ and $p \leq \pi_{mn}(q)$ in P_n. This is an analytic statement as all the posets P_n are very Suslin.

Checking the projection properties of the functions $\pi_{\omega m}$ and $\xi_{m\omega}$ is routine and left to the reader. □

Theorem 11.2.7 follows. $\square$

The operation of the finite support product appears to be more difficult to handle. Here the main question remains open:

Question 11.2.15. The class of Suslin c.c.c. forcings is closed under countable, finite support product. Is this true for the class of very Suslin c.c.c. forcings as well?

11.3 Iteration theorems

The posets we use later in this chapter are fairly innocent from ZFC point of view. However, we need their regularity properties to be witnessed in a Suslin way, and to be preserved under finite support iterations of countable length. This section contains a number of rather routine, but still apparently novel, regularity properties and their associated preservation theorems.

Definition 11.3.1. Let P be a Suslin forcing.

1. A set $A \subset P$ is *linked* if any two conditions in A have a common lower bound;
2. P is *Suslin σ-linked* if P can be written as a countable union $\bigcup_n A_n$ of linked analytic sets.

Note that the definitory properties of the cover $P = \bigcup_n A_n$ are $\mathbf{\Pi}^1_2$ and therefore persist to all forcing extensions. The main general result of this section is the iteration preservation theorem for Suslin σ-linkedness:

Theorem 11.3.2. *Let P be a very Suslin c.c.c. forcing which is Suslin σ-linked. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support iteration of P of length α is a Suslin σ-linked forcing.*

Proof. The argument proceeds by a straightforward transfinite induction argument given the following two claims.

Claim 11.3.3. *Let P, Q be very Suslin c.c.c. forcings, both of which are Suslin σ-linked. The $P * \dot{Q}$ is Suslin σ-linked.*

Proof. Let $P = \bigcup_n A_n$ and $Q = \bigcup_m B_m$ be the covers of P, Q by analytic centered (or linked) sets. Let $C_{nm} \subset P * \dot{Q}$ be the set of all conditions $\langle p, \dot{q}\rangle$ such that there exists a condition $\langle p', \dot{q}'\rangle \leq \langle p, \dot{q}\rangle$ such that $p' \in A_n$ and $p' \Vdash \dot{q}' \in \dot{B}_m$. It is not difficult to check that $P * \dot{Q} = \bigcup_{nm} C_{nm}$, the sets C_{nm} are linked. Moreover, the sets C_{nm} are analytic by Proposition 11.2.9. $\square$

Claim 11.3.4. *Let $\langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega\rangle$ be a very Suslin system consisting of Suslin σ-linked forcings. Then the limit is Suslin σ-linked.*

Proof. Let P_ω be the limit of the system as described in Definition 11.2.12. Let $P_n = \bigcup_m A_{nm}\colon n \in \omega$ be a cover by analytic linked sets for each $n \in \omega$. For each $n, m \in \omega$ let $B_{nm} \subset P_\omega$ be the set $\{\langle p, n\rangle \in P_\omega \colon p \in A_{nm}\}$. It is immediate that these are linked analytic sets covering P_ω. □

The theorem follows. □

Theorem 11.3.5. *The finite support product of countably many very Suslin, Suslin σ-linked posets, if very Suslin, is Suslin σ-linked.*

Proof. Let P_n for $n \in \omega$ be the very Suslin posets, with their Suslin σ-linked property witnessed by sets A_{nm} for $m \in \omega$. Consider the finite support product $Q = \prod_n P_n$ and assume it is very Suslin. For each number $n_0 \in \omega$ and each function $h\colon n_0 \to \omega$ let $B_h = \{q \in Q \colon \mathrm{dom}(q) = n \wedge \forall m \in n_0\ q(m) \in A_{m,h(m)}\}$. It is not difficult to verify that the sets B_h witness the Suslin σ-linked property of the poset Q. □

Definition 11.3.6. Let P be a Suslin forcing.

1. A set $A \subset P$ is *centered* if any finite subset of A has a lower bound in P;
2. The poset P is *Suslin σ-centered* if P can be written as a countable union $\bigcup_n A_n$ and the sets A_n are analytic and centered.

The key feature of the Suslin σ-centered property is that it is preserved under the finite support iterations of very Suslin forcings. The proof of the following theorems are literally copied from Theorems 11.3.2 and 11.3.5, replacing the word "linked" with "centered".

Theorem 11.3.7. *Let P be a very Suslin c.c.c. forcing which is Suslin σ-centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support iteration of P of length α is a Suslin σ-centered forcing.*

Theorem 11.3.8. *The finite support product of countably many very Suslin, Suslin σ-centered posets, if very Suslin, is Suslin σ-centered.*

Next comes a regularity property of Suslin forcing notions more permissive than centeredness or linkedness, which should be compared to [86, Definition 3].

Definition 11.3.9. Let P be a Suslin forcing.

1. A set $A \subset P$ is *Ramsey-centered* if for every number $r \in \omega$ there is $k \in \omega$ such that every k-tuple (with possible repetitions) of elements of A contains an r-tuple with a common lower bound.
2. P is *Suslin σ-Ramsey-centered* if there is a cover $P = \bigcup_n A_n$ by analytic Ramsey-centered sets.

Note that the definitory properties of the covers are $\mathbf{\Pi}^1_2$ and therefore persist to all forcing extensions. The most important result of this section is that the Suslin Ramsey-centeredness is a property preserved under finite support iterations of very Suslin forcings.

Theorem 11.3.10. *Let P be a very Suslin c.c.c. forcing which is Suslin Ramsey-centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support iteration of P of length α is a Suslin Ramsey-centered forcing.*

Proof. The argument proceeds by a straightforward transfinite induction argument given the following two claims.

Claim 11.3.11. *Let P, Q be very Suslin c.c.c. forcings, both of which are Suslin Ramsey-centered. The $P * \dot{Q}$ is Suslin Ramsey-centered.*

Proof. Find a cover $Q = \bigcup_m B_m$ by analytic Ramsey-centered sets and a cover $P = \bigcup_n A_n$ by analytic Ramsey-centered sets. Let $C_{nm} \subset P * \dot{Q}$ be the set of all conditions $\langle p, \dot{q} \rangle$ such that there exists a condition $\langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle$ such that $p' \in A_{nm}$ and $p' \Vdash \dot{q}' \in \dot{B}_m$. It is not difficult to check that $P * \dot{Q} = \bigcup_{nm} C_{nm}$, the sets C_{nm} are Ramsey-centered: given $r \in \omega$ there is a $k \in \omega$ such that every k-tuple in the set B_m contains an r-tuple with a common lower bound, and there is an $l \in \omega$ such that every l-tuple in the set A_n contains a k-tuple with a common lower bound. Then, every l-tuple of conditions in C_{mn} contains an r-tuple with a common lower bound. Moreover, the sets C_{nm} are analytic by Proposition 11.2.9. $\square$

Claim 11.3.12. *Let $\langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega \rangle$ be a very Suslin system consisting of Suslin Ramsey-centered forcings. Then the limit is Suslin Ramsey-centered.*

Proof. Let P_ω be the limit of the system as described in Definition 11.2.12. Let $P_n = \bigcup_m A_{nm} : n \in \omega$ be a cover by analytic Ramsey-centered sets for each $n \in \omega$. For each $n, m \in \omega$ let $B_{nm} \subset P_\omega$ be the set $\{\langle p, n \rangle \in P_\omega : p \in A_{nm}\}$. It is immediate that these are Ramsey-centered analytic sets covering P_ω. $\square$

The theorem follows. $\square$

Theorem 11.3.13. *The finite support product of countably many very Suslin, Suslin σ-Ramsey-centered posets, if very Suslin, is Suslin σ-Ramsey-centered.*

Proof. Let P_n for $n \in \omega$ be the very Suslin posets, with their Suslin Ramsey-centered property witnessed by sets $A_{nm} \subset P_n$ for $m \in \omega$. Consider the finite support product $Q = \prod_n P_n$ and assume it is very Suslin. For each function $h \in \omega^{<\omega}$ let $B_h = \{q \in Q : \mathrm{dom}(q) = \mathrm{dom}(h) \wedge \forall m \in \mathrm{dom}(q)\ q(m) \in A_{m,h(m)}$. We proceed to verify that the sets B_h witness the Suslin Ramsey-centered property of the poset Q.

Fix the function h and let $r \in \omega$ be a number. By recursion on $n \in \mathrm{dom}(h)$ find numbers $k_n \in \omega$ so that every $k(0)$-tuple of elements of $A_{0,h(0)}$ of size $k(0)$ contains an r-tuple with a common lower bound in P_0, and every $k(n+1)$-tuple of elements of $A_{n+1,h(n+1)}$ contains a $k(n)$-tuple with a common lower bound in P_{n+1}. Let $n = \max(\mathrm{dom}(h))$ and $k = k(n)$. It is easy to see that every k-tuple of elements of B_h contains an r-tuple with a lower bound in the poset Q. $\square$

The existence of finitely additive measures on complete Boolean algebras is a useful tool in forcing theory, see [47]. The Suslin version of it needs to reflect the fact that finitely additive measures on uncountable sets are typically not Suslin in any sense. We use the following poor man's version of a finitely additive measure:

Definition 11.3.14. Let P be a very Suslin c.c.c. poset. We say that P is *Suslin measured* if there are analytic sets $A_n \subset P$ and positive rationals $\varepsilon_n > 0$ such that $P = \bigcup_n A_n$ and for every $n \in \omega$ and every finite sequence $\langle p_i : i \in j \rangle$ of elements of A_n with possible repetitions there is a set $b \subset j$ of size at least $\varepsilon_n j$ such that the set $\{p_i : i \in b\}$ has a lower bound in P.

Note that the definitory properties of the cover $P = \bigcup_n A_n$ are $\mathbf{\Pi}^1_2$ and therefore persist to all forcing extensions. Note also that every Suslin measured poset is Suslin Ramsey-centered as witnessed by the same analytic sets.

Theorem 11.3.15. *Let P be a very Suslin c.c.c. poset. If P is Suslin measured, then so are all of its finite support iterations of countable length.*

Proof. As in the previous subsections, we proceed with claims dealing with the successor and limit stage of the iteration.

Claim 11.3.16. *Suppose that P, Q are very Suslin, c.c.c. posets which are Suslin measured. Then $P * \dot{Q}$ is Suslin measured.*

Proof. Let A_n, α_n for $n \in \omega$ and B_n, β_n for $n \in \omega$ witness the Suslin measured property of P, Q respectively. Let $C_{nm} \subset P * \dot{Q}$ be the upward closure of the set of all pairs $\langle p, q \rangle$ such that $p \in A_n$ and $p \Vdash \dot{q} \in \dot{B}_m$. Let $\gamma_{nm} = \alpha_n \beta_m$. We claim that the sets C_{nm} and the numbers γ_{nm} witness the Suslin measured property of the iteration.

It is clear from Proposition 11.2.9 that the sets C_{nm} are analytic; they also clearly exhaust all of $P * \dot{Q}$. Now, fix numbers $n, m \in \omega$. Suppose that $\langle \langle p_i, \dot{q}_i \rangle : i \in j \rangle$ is a sequence of conditions in the set C_{nm}; strengthening the conditions we may assume that $p_i \in A_n$ and $p_i \Vdash \dot{q}_i \in \dot{B}_m$ holds for every $i \in j$. By the properties of the set A_n, there is a set $b \subset j$ such that $|b| > \alpha_n j$ and the conditions p_i for $i \in b$ have a lower bound p. By the properties of the set B_m applied in the generic extension, it is possible to strengthen p and identify a set $c \subset b$ such that $|c| > \beta_m |b|$ in such a way that p forces the set $\{\dot{q}_i : i \in c\}$ to have a lower bound in $\dot{Q}$. Then the conditions $\langle p_i, \dot{q}_i \rangle$ for $i \in c$ have a common lower bound in $P * \dot{Q}$. As the set $c \subset j$ has size $> \alpha_n \beta_m |j|$, this verifies the desired properties of the set C_{nm}. □

Claim 11.3.17. *Let $\langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega \rangle$ be a very Suslin system consisting of Suslin measured forcings. Then the limit is Suslin measured.*

Proof. Let P_ω be the limit of the system as described in Definition 11.2.12. For each $n \in \omega$, let A_{nm}, ε_{nm} for $m \in \omega$ be analytic subsets of P_n and positive rational numbers witnessing the measured property of P_n. For each $n, m \in \omega$ let $B_{nm} \subset P_\omega$ be the set $\{\langle p, n \rangle \in P_\omega : p \in A_{nm}\}$. It is immediate that the sets

B_{nm} with the numbers ε_{nm} witness the Suslin measured property of the limit P_ω. □

A routine transfinite induction argument now completes the proof. □

Theorem 11.3.18. *The finite support product of countably many very Suslin, Suslin measured posets, if very Suslin, is Suslin measured.*

Proof. Let P_n for $n \in \omega$ be the very Suslin posets, with their Suslin measured property witnessed by sets $A_{nm} \subset P_n$ and numbers ε_{nm} for $m \in \omega$. Consider the finite support product $Q = \prod_n P_n$ and assume it is very Suslin. For each function $h \in \omega^{<\omega}$ let $B_h = \{q \in Q\colon \mathrm{dom}(q) = \mathrm{dom}(h) \wedge \forall m \in \mathrm{dom}(q)\ q(m) \in A_{m,h(m)}$, and let $\varepsilon_h = \prod_{m \in \mathrm{dom}(h)} \varepsilon_{m,h(m)}\}$. It is not difficult to verify that the sets B_h and the numbers ε_h witness the Suslin measured property of the poset Q. □

As the last issue in this section, we elaborate on a notion which has been used in several contexts to guarantee that various posets do not add dominating reals [70, 1]; we will use it for a different purpose.

Definition 11.3.19. Let P be a Suslin poset.

1. A set $A \subset P$ is *liminf-centered* if for every collection $\{p_n\colon n \in \omega\} \subset A$ there is a condition $q \in P$ such that every condition stronger than q is compatible with p_n for infinitely many numbers $n \in \omega$;
2. P is *Suslin σ-liminf centered* if $P = \bigcup_n A_n$ where each set $A_n \subset P$ is analytic and liminf-centered.

It is not difficult to see that the property of the condition q demanded by the first item of the definition can be stated in the forcing language as $q \Vdash$ there are infinitely many $n \in \omega$ such that p_n belongs to the generic filter. It is important and instructive to note that if P is a very Suslin c.c.c. poset then its Suslin σ-liminf-centeredness will persist to all generic extensions. To see this, it is enough to observe that for an analytic set $A \subset P$, the statement "A is liminf-centered" is $\mathbf{\Pi}^1_2$; then, an application of Shoenfield absoluteness gives the required conclusion. Now, the statement "A is liminf-centered" is equivalent to the following: for all $\langle p_n\colon n \in \omega\rangle$, either for some $n \in \omega$, $p_n \notin A$, or there is $q \in P$ such that $q \Vdash$ there are infinitely many $n \in \omega$ such that p_n belongs to the generic filter. The forcing statement is analytic by Proposition 11.2.9 and the very Suslin assumption on P. Thus, the total statement is $\mathbf{\Pi}^1_2$ as required.

Theorem 11.3.20. *Let P be a very Suslin c.c.c. forcing which is Suslin σ-liminf centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support of P of length α is a very Suslin c.c.c. Suslin σ-liminf centered forcing.*

Proof. The argument is a routine variation of the proofs of previous preservation theorems.

Claim 11.3.21. *Let P, Q be very Suslin c.c.c. forcings, both of which are Suslin σ-liminf-centered. The $P * \dot{Q}$ is Suslin σ-liminf-centered.*

Proof. Find covers $Q = \bigcup_m B_m$ and $P = \bigcup_n A_n$ by analytic liminf-centered sets. Let $C_{nm} \subset P * \dot{Q}$ be the set of all conditions $\langle p, \dot{q} \rangle$ such that there exists a condition $\langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle$ such that $p' \in A_n$ and $p' \Vdash \dot{q}' \in \dot{B}_m$. It is not difficult to check that $P * \dot{Q} = \bigcup_{nm} C_{nm}$ and the sets C_{nm} are analytic by Proposition 11.2.9. To check the liminf-centered property, suppose that $\{\langle p_i, \dot{q}_i \rangle : i \in \omega\}$ are conditions in C_{nm}. Strengthening if necessary, we may assume that $p_i \in A_n$ and $p_i \Vdash \dot{q}_i \in \dot{B}_m$. Use the liminf-centeredness of A_n to find a condition p forcing that for infinitely many $i \in \omega$, p_i is in the generic filter. Using the liminf-centeredness of the set B_m, find a name $\dot{q}$ for a condition in Q such that $p \Vdash \dot{q} \Vdash$ for infinitely many $i \in \omega$ such that p_i belongs to the generic filter on P, $\dot{q}_i$ belongs to the generic filter on Q. The condition $\langle p, \dot{q} \rangle$ is as required. ☐

Claim 11.3.22. *Let $\langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega \rangle$ be a very Suslin system consisting of Suslin σ-liminf-centered forcings. Then the limit is Suslin σ-liminf-centered.*

Proof. Let P_ω be the limit of the system as described in Definition 11.2.12. Let $P_n = \bigcup_m A_{nm} : n \in \omega$ be a cover by analytic liminf-centered sets for each $n \in \omega$. For each $n, m \in \omega$ let $B_{nm} \subset P_\omega$ be the set $\{\langle p, n \rangle \in P_\omega : p \in A_{nm}\}$. It is immediate that these are analytic liminf-centered analytic sets exhausting P_ω. ☐

The theorem follows by a routine transfinite induction argument. ☐

The following question probably has a negative answer; still, products of the very Suslin σ-liminf-centered posets used later in the chapter are again Suslin σ-liminf-centered.

Question 11.3.23. Is the class of Suslin σ-liminf-centered posets closed under product?

11.4 Locally countable simplicial complexes

In this section, we concentrate on tasks of a very specific form common to many concerns of modern descriptive set theory. These tasks start with a countable Borel equivalence relation E on a Polish space X and attempt to assign a structure of some type to each E-class; at the same time, distinct E-classes do not communicate with each other in any way. Reviewing the tasks of this type, it becomes obvious that often there is a locally countable simplicial complex $\mathcal{K}$ on X such that the associated equivalence relation $E_{\mathcal{K}}$ (as identified in Definition 6.1.3) is E, and we seek a maximal $\mathcal{K}$-set. Sometimes, the domain of the simplicial complex $\mathcal{K}$ is instead the collection of finite subsets of X consisting of pairwise E-related points, and sometimes we actually seek a maximal $\mathcal{K}$-set with some additional largeness properties. The variations considered in this section all fall under the following definition:

Definition 11.4.1. A *locally countable pair* is a pair $\langle\mathcal{K}, \mathcal{L}\rangle$ such that

1. $\mathcal{K}$ is a Borel locally countable simplicial complex on a Polish space X;
2. $\mathcal{L} \subset \mathcal{K}$ is a cofinal Borel subset, i.e. $\forall a \in \mathcal{K}\ \exists b \in \mathcal{L}\ a \subset b$;
3. for every finite set $a \subset X$, $a \in \mathcal{L}$ just in case $a \cap c \in \mathcal{L}$ holds for every $E_{\mathcal{K}}$-class $c \subset X$.

A $\mathcal{K}$-set $a \subset X$ is $\mathcal{L}$-*regular* if every finite subset of a is a subset of a finite subset of a which is in $\mathcal{L}$.

The role of the set $\mathcal{L}$ deserves a comment. Note that item (3) holds with $\mathcal{K}$ replacing $\mathcal{L}$ by the definition of the equivalence relation $\mathcal{K}$; thus, distinct $E_{\mathcal{K}}$-classes do not communicate regarding the membership of finite sets in $\mathcal{K}$ and $\mathcal{L}$. In many cases, the equality $\mathcal{L} = \mathcal{K}$ will occur; some tasks seem to be impossible to achieve in this way though. For example, let G be a graph on ω and let $\mathcal{K}$ be the simplicial complex of all finite G-colorings with colors coming from ω. Not every maximal $\mathcal{K}$-set needs to be a total G-coloring. To alleviate this unwanted effect, let $\mathcal{L} \subset \mathcal{K}$ be the set of those finite colorings whose domain is a natural number; then every $\mathcal{L}$-regular maximal $\mathcal{K}$-set is in fact a total G-coloring.

Definition 11.4.2. Let $\langle\mathcal{K}, \mathcal{L}\rangle$ be a locally countable pair on a Polish space X. The poset $P_{\mathcal{K}\mathcal{L}}$ is the poset of all sets $p \subset \mathcal{K}$ for which there is a countable $E_{\mathcal{K}}$-invariant set $\mathrm{dom}(p) \subset X$ such that p is a $\mathcal{L}$-regular, maximal $\mathcal{K}$-subset of $\mathrm{dom}(p)$. The ordering is that of reverse inclusion. If $\mathcal{K} = \mathcal{L}$, we refer to the poset as $P_{\mathcal{K}}$ instead.

It is immediate that the poset $P_{\mathcal{K}\mathcal{L}}$ is Suslin and σ-closed; in fact, it is a special case of the uniformization posets of Definition 6.6.5 with the space $X \times X^\omega$, the equivalence relations E on X and $\mathbb{F}_2$ on X^ω, and the invariant Borel set $B \subset X \times X^\omega$ consisting of all pairs $\langle x, y\rangle$ where y enumerates an $\mathcal{L}$-regular maximal $\mathcal{K}$-subset of $[x]_E$. The classification of balanced conditions then follows from Theorem 6.6.6:

Theorem 11.4.3. *Let $\langle\mathcal{K}, \mathcal{L}\rangle$ be a locally countable pair on a Polish space X, with the associated poset $P_{\mathcal{K}\mathcal{L}}$.*

1. *For every $\mathcal{L}$-regular maximal $\mathcal{K}$-set $A \subset X$, the pair $\langle \mathrm{Coll}(\omega, X), \check{A}\rangle$ is balanced;*
2. *for every balanced pair $\langle Q, \sigma\rangle$ there is a set A as in (1) such that the balanced pairs $\langle Q, \sigma\rangle$ and $\langle \mathrm{Coll}(\omega, X), \check{A}\rangle$ are equivalent;*
3. *distinct sets as in (1) yield inequivalent balanced pairs.*

In order to identify properties of the poset $P_{\mathcal{KL}}$ in the Solovay model, we have to analyze its very Suslin c.c.c. *control poset* which in this case is simply the partial order $\mathcal{K}$ ordered by reverse inclusion. Note that the set $\mathcal{L}$ does not enter the definition of the control poset at all. The following proposition verifies the basic property of the control poset.

Proposition 11.4.4. *Let $\mathcal{K}$ be a Borel locally countable simplicial complex on a Polish space X. Then $\mathcal{K}$ is a very Suslin c.c.c. forcing.*

Proof. Write $E = E_{\mathcal{K}}$; this is a Borel countable equivalence relation on the space X. Observe that for an E-invariant set $C \subset X$, $\mathcal{K} \cap \mathcal{P}(C)$ is a regular subposet of $\mathcal{K}$. It is certainly closed under unions in $\mathcal{K}$. Now suppose that $a \in \mathcal{K}$ is any condition and let $b = a \cap C$. Whenever $c \supset b$ is some condition in $\mathcal{K} \cap \mathcal{P}(C)$, $a \cup c \in \mathcal{K}$ holds. To see this, recall that $a \cup c$ belongs to $\mathcal{K}$ just in case its intersection with every E-class is. Now, if d is an E-class, if $d \subset C$ then $(a \cup c) \cap d = c \in \mathcal{K}$, and if $d \cap C = 0$ then $(a \cup c) \cap d = a \in \mathcal{K}$ holds again. The regularity of $\mathcal{K} \cap \mathcal{P}(C)$ in $\mathcal{K}$ has just been proved.

To verify the c.c.c. part of the proposition, let $A \subset \mathcal{K}$ is a maximal antichain and let M be a countable elementary submodel of a large structure containing A, E, and $\mathcal{K}$. By the elementarity of M, the set $A \cap M$ is a maximal antichain in the poset $\mathcal{K} \cap M = \mathcal{K} \cap \mathcal{P}(X \cap M)$. Since E is an equivalence relation with all classes countable, the elementarity of M shows that $X \cap M$ is an E-invariant set. By the previous paragraph then, $A \cap M$ is a maximal antichain in $\mathcal{K}$. As a result, $A \cap M = A$ and so A is countable.

To verify the very Suslin part of the proposition, let $A \subset \mathcal{K}$ be a countable set. To check that A is a maximal antichain in $\mathcal{K}$, it is only necessary to check that it is a maximal antichain in the poset $\mathcal{K} \cap \mathcal{P}(C)$, where C is the E-saturation of $\bigcup A$ by the first paragraph of the proof. Such a verification is a Borel procedure as all E-classes are countable. □

The key point in all subsections below will be the identification of some Suslin forcing preservation properties of the control poset and their connection to the forcing properties of the poset $P_{\mathcal{KL}}$ in the Solovay model.

11.4a Suslin σ-centered complexes

The results of this section show that in certain extensions of the Solovay model, uncountable chromatic numbers of analytic hypergraphs of finite arity do not change at all. This is the content of the following theorem.

Theorem 11.4.5. *Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a locally countable pair. Suppose that the simplicial complex $\mathcal{K}$ is Suslin σ-centered. Suppose that G is a analytic finitary hypergraph of uncountable Borel chromatic number. Then in the $P_{\mathcal{KL}}$-extension of the Solovay model, the chromatic number of G is uncountable.*

Proof. As a preliminary consideration, it is clear from Fact 11.1.8 that it is enough to consider the case of G which is a principal skew product as in Definition 11.1.7. Let $\langle a_n, H_n, t_n : n \in \omega \rangle$ be a sequence such that $H_n = \{a_n\}$ and $G = \prod_n H_n, t_n$.

Write X for the Polish space which is the domain of the Borel simplicial complex $\mathcal{K}$, write $E = E_{\mathcal{K}}$ and $P = P_{\mathcal{K}\mathcal{L}}$. Let κ be an inaccessible cardinal, let W be the Solovay model derived from κ and work in the model W. Suppose that $p \in P$ is a condition and τ is a P-name for a function from $\prod_n a_n$ to ω. We must find a stronger condition $\bar{p} \leq p$, a number $m \in \omega$ and a G-hyperedge e such that $\bar{p} \Vdash \tau(y) = \check{m}$ for all vertices $y \in \check{e}$.

The condition $p \in P$ as well as the name τ are definable from some parameters in the ground model and a parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate model obtained by a poset of size $< \kappa$ containing the parameter z, and work in the model $V[K]$ until further notice. Consider the iteration $Q * \dot{R}$ where Q is the Cohen poset on $\prod_n a_n$ adding a generic point $\dot{y}_{gen} \in \prod_n a_n$, and in the Q-extension, $\dot{R}$ is the finite support iteration of length ω_1 of the poset $\mathcal{K}$. As usual, the poset $\mathcal{K}$ is reinterpreted at each stage of the iteration; we denote the model obtained after α-th stage by M_α, so M_0 is the Q-extension of $V[K]$. Thus, the $\dot{R}$ poset adds a sequence of $\mathcal{K}$-sets, $\langle \dot{A}_\alpha \colon \alpha \in \omega_1 \rangle$ derived from the iteration components of $\dot{R}$. By a density argument, for each E-class c represented in the model V_α, the set $\dot{A}_\alpha \cap c$ is a maximal $\mathcal{K}$-subset of c which is $\mathcal{L}$-regular. Consider the $Q * \dot{R}$-name $\bar{p}$ for the $\mathcal{K}$-set defined in the following way: if c is an E-class represented in $\mathrm{dom}(p)$, then $c \cap \bar{p} = c \cap p$, if c is an E-class represented in M_0 but not in $\mathrm{dom}(p)$ then $c \cap \bar{p} = \dot{A}_0 \cap c$, and if $\alpha \in \omega_1$ is an arbitrary nonzero ordinal and c is an E-class represented in M_α but not in $\bigcup_{\beta \in \alpha} M_\beta$ then $c \cap \bar{p} = c \cap \dot{A}_\alpha$. It is not difficult to see that $\bar{p}$ is a $Q * \dot{R}$-name for a maximal $\mathcal{K}$-set which is $\mathcal{L}$-large, and therefore a balanced condition for the poset P by Theorem 11.4.3. Note that the condition does not depend on the point $\dot{y}_{gen}$ but only on the model $V[K][\dot{y}_{gen}]$.

Now, by a standard balance argument, $Q * \dot{R} \Vdash \exists m\ \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \tau(\dot{y}_{gen}) = \check{m}$. Let $\langle q, \dot{r} \rangle$ be a condition in the iteration $Q * \dot{R}$ deciding the value of m; we abuse the notation and call the specific value m again. Strengthening the condition $q \in Q$ if necessary, we may assume that

- there is an ordinal $\alpha \in \omega_1$ such that $q \Vdash_Q \dot{r}$ is a name for a condition in the initial segment R_α of the iteration $\dot{R}$;
- by the iteration theorem 11.3.7, R_α is a very Suslin forcing which is Suslin σ-centered. Let $\{B_k \colon k \in \omega\}$ be analytic centered sets covering R_α. We may find a specific number $k \in \omega$ such that $q \Vdash \dot{r} \in \dot{B}_k$;
- $q = t_j$ for some number $j \in \omega$.

Let $y \in \prod_n a_n$ be a Cohen-generic element over the model $V[K]$ extending t_j and move into the model $V[K][y]$. For each $i \in a_j$ let $y_i \in \prod_n a_n$ be the element obtained from y by rewriting the j-th entry with i. Thus, each y_i is a Cohen generic element over $V[K]$ meeting the condition q, $V[K][y_i] = V[K][y]$, and $\{y_i \colon i \in n\}$ is a G-hyperedge. Let $r_i = \dot{r}/y_i$. Thus, each r_i is a condition in the poset R_α, and even in the analytic centered set B_k. Thus, the conditions r_i for $i \in a_n$ have a common lower bound. Now, let $H \subset R$ be a filter generic over the model $V[K][y]$ and containing all conditions r_i for $i \in a_n$. Consider the balanced condition $\bar{p} \leq p$ in the poset P in the model $V[K][y][H]$ described in the third paragraph of the present proof. The forcing theorem applied for each $i \in a_n$ shows that for each $i \in a_n$, $V[K][y][H] \models \mathrm{Coll}(\omega, < \kappa) \Vdash \dot{p} \Vdash_P \tau(\check{y}_i) = \check{m}$. Thus, $W \models \bar{p} \Vdash \tau(\check{y}_i) = \check{m}$ for each $i \in a_n$, completing the proof. □

The verification of Suslin σ-centeredness for a given locally countable Borel simplicial complex $\mathcal{K}$ may look like a formidable challenge. However, there is a standard trick, recorded in the following theorem, which makes short work of it in all cases we are aware of.

Theorem 11.4.6. *Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space X and write $E = E_{\mathcal{K}}$. Suppose that there is a Borel* support *function* supp: $\mathcal{K} \to [X]^{<\aleph_0}$ *such that*

1. $\mathrm{supp}(a) \subset [a]_E$;
2. *whenever $\{a_i : i \in j\}$ are sets in $\mathcal{K}$, subsets of the same E-class with pairwise disjoint supports, then $\bigcup_i a_i \in \mathcal{K}$.*

Then $\mathcal{K}$ is Borel σ-centered.

Proof. Let Γ be a countable group acting on X in a Borel way such that E is the resulting orbit equivalence relation. Let $a \in \mathcal{K}$ and write $\mathrm{supp}^*(a) = \bigcup\{\mathrm{supp}(a \cap c) : c$ is an E-class$\}$. We say that a tuple $\langle b, f \rangle$ is a *descriptor* of a if

- $b \subset \mathcal{B}$ is a finite set consisting of pairwise disjoint basic open subsets of X such that every element of $\mathrm{supp}^*(a)$ belongs to $\bigcup b$ and no element of b contains more than one element of $\mathrm{supp}^*(a)$;
- $f : b \to [\Gamma]^{<\aleph_0}$ is a function such that whenever $O \in b$ is an open set, c is an E-class, and $x \in O \cap \mathrm{supp}(a \cap c)$ is a point then $a \cap c = \{\gamma \cdot x : \gamma \in f(O)\}$.

Note that a indeed has a descriptor since the set $\mathrm{supp}^*(a)$ is finite (making the construction of b as in the first item possible), and for any E-class c the set $\mathrm{supp}(a \cap c)$ is a subset of c by (1) (making the construction of f as in the second item possible). We will show that if $\{a_i : i \in j\}$ are elements of $\mathcal{K}$ with the same descriptor $\langle b, f \rangle$ then $\bigcup_i a_i \in \mathcal{K}$. Since there are only countably many possible descriptors, and the set of elements of $\mathcal{K}$ with a given descriptor is Borel, this will conclude the proof of the theorem.

Let c be an E-class; we must show that $\bigcup_i (a_i \cap c) \in \mathcal{K}$. In view of property (2) of the supp function, it will be enough to show that whenever $i_0 \neq i_1$ are elements of j then either $a_{i_0} \cap c = a_{i_1} \cap c$ or $\mathrm{supp}(a_{i_0} \cap c)$ is disjoint from $\mathrm{supp}(a_{i_1} \cap c)$. Towards this end, suppose that $x \in \mathrm{supp}(a_{i_0} \cap c) \cap \mathrm{supp}(a_{i_1} \cap c)$ is a point; we have to show that $a_{i_0} \cap c = a_{i_1} \cap c$ holds. Let $O \in b$ be an open set such that $x \in O$, and use the second property of a descriptor to see that $a_{i_0} \cap c = a_{i_1} \cap c = \{\gamma \cdot x : \gamma \in f(O)\}$. The theorem follows. □

Our first example is motivated by a ZF+DC result of [19]: for every Borel locally finite graph on a Polish space X, if the chromatic number of G is $\leq n$, then there is a meager set $B \subset X$ and a decomposition of the set $X \setminus B$ into $2n-1$ many Borel G-anticliques. We have:

Example 11.4.7. Let G be a Borel locally finite graph on a Polish space X of chromatic number n. Let $\mathcal{K}$ be the simplicial complex of all finite partial G colorings with $\leq 2n-1$ colors which can be extended to a total coloring with $\leq 2n-1$ many colors in which at most n many colors are attained infinitely many times. Then $\mathcal{K}$ is a Borel locally countable simplicial complex which is Suslin σ-centered.

Proof. Note that $\mathcal{K}$ is in fact a simplicial complex on $X \times (2n-1)$. The various assertions of the example follow from two simple general graph theoretic claims. Let H be a locally finite graph on a set V of vertices and let $c\colon V \to m$ be a H-coloring with finitely many colors. Define $F(c) = \{i \in m\colon$ the set $\{v \in V\colon c(v) = i\}$ is finite$\}$.

Claim 11.4.8. *For every finite set $a \subset V$ and every set $e \subset m$ of size $|F(c)|$ there is a H-coloring $d\colon V \to m$ such that $c \restriction a = d \restriction a$ and $F(d) = e$.*

Proof. It will be enough, given $i \in F(c)$ and $j \in m \setminus F(c)$, to find a coloring $d\colon V \to m$ such that $c \restriction a = d \restriction a$ and $F(d) = (F(c) \cup \{j\}) \setminus \{i\}$. To this end, let $b = a \cup \{v \in V\colon c(v) = i\} \cup \{u \in V\colon \exists v\ \{u, v\} \in H$ and $c(v) = i\}$. Let d be the coloring equal to c except on the vertices $v \in V \setminus b$ such that $c(v) = j$, which will have $d(v) = i$. It is easy to verify that d works. $\square$

Claim 11.4.9. *For every finite set $a \subset V$ there is a finite partial H-coloring $d\colon V \to m$ such that $a \subset \mathrm{dom}(d)$, $c \restriction a = d \restriction a$, and for every edge $\{u, v\} \in H$ with $u \in \mathrm{dom}(d)$ and $v \notin \mathrm{dom}(d)$, $d(u) \notin |F(c)| + 1$ holds.*

Proof. By the previous claim, we can adjust c so that $F(c)$ is the set of the first $F(c)$-many natural numbers. Let a' be the union of the finite sets a, $\{v \in V\colon c(v) \in F(c)\}$, and $\{v \in V\colon \exists u\ \{u, v\} \in H$ and $c(u) \in F(c)\}$. Let b be the union of the finite sets a' and $\{v \in V\colon \exists u \in a'\ \{u, v\} \in H$ and $c(u) = |F(c)|\}$. Let $d = c \restriction b$ and observe that d works. $\square$

Now, we are ready to complete the proof of the example. Write E for the Borel equivalence relation of G-path connectedness on the space X. First of all, the simplicial complex $\mathcal{K}$ is Borel and locally countable. Applying the last claim to each connected component of the graph G, it is clear that a finite partial G-coloring a belongs to $\mathcal{K}$ just in case for every E-class $c \subset X$ there is a finite partial G-coloring d such that $\mathrm{dom}(a) \cap c \subset \mathrm{dom}(d) \subset c$ and for every G-edge $\{x, y\}$ with $x \in \mathrm{dom}(d)$ and $y \in c \setminus \mathrm{dom}(d)$ it is the case that $d(u) \notin n$ holds. (Such a coloring d can be completed by coloring the rest of the class c with the first n many colors, which is possible by the chromatic number assumption on G.) This is a Borel condition.

To prove the Suslin σ-centeredness of the simplicial complex $\mathcal{K}$, use the Lusin–Novikov theorem to find a Borel function g which, for each finite coloring $a \in \mathcal{K}$ whose domain consists of elements of a single E-class c, assigns a G-coloring $g(a) \supset a$ such that $\mathrm{dom}(a) \subset \mathrm{dom}(g(a)) \subset c$ and for every G-edge $\{x, y\}$ with $x \in \mathrm{dom}(g(a))$ and $y \in c \setminus \mathrm{dom}(g(a))$ it is the case that $g(a)(u) \notin n$ holds. Let $\mathrm{supp}(a)$ be the set of all pairs $\langle e, i\rangle$ such that $e \in \mathrm{dom}(g(a))$ and $i \in 2n-1$. In view of Theorem 11.4.6, it will be enough to show that if $\{a_i \colon i \in j\}$ are colorings in $\mathcal{K}$ whose domain is a subset of one and the same E-class c, with pairwise disjoint supports, then $\bigcup_i a_i \in \mathcal{K}$. This is nearly trivial, however. The colorings $g(a_i)$ for $i \in j$ have pairwise disjoint domains. Let $f \colon c \setminus \bigcup_i \mathrm{dom}(g(a_i)) \to n$ be any G-coloring–its existence is guaranteed by the chromatic number assumption on G. Then, $f \cup \bigcup_i g(a_i)$ is a G-coloring of the class c which extends $\bigcup_i a_i$ and uses no colors $\geq n$ infinitely many times, witnessing that $\bigcup_i a_i \in \mathcal{K}$. □

Corollary 11.4.10. *Let G be a locally finite Borel graph on a Polish space X of finite chromatic number n. Let $\mathcal{K}$ be the simplicial complex as in Example 11.4.7. Let H be a finitary analytic hypergraph of uncountable Borel chromatic number. Then*

1. *in the $P_{\mathcal{K}}$-extension of the Solovay model, the chromatic number of H is uncountable;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds and the chromatic number of G is $\leq 2n-1$ and the chromatic number of H is uncountable.*

The following example is motivated by a question of Marks and Unger (private communication): given a countable Borel equivalence relation E, is it consistent with ZF+DC that E is a union of an increasing countable sequence of equivalence relations with all equivalence classes finite and every set of reals has the Baire property? While that question remains open, we do have partial information:

Example 11.4.11. Let E be a countable Borel equivalence relation on a Polish space X with all classes infinite. Let $\mathcal{K}$ be the simplicial complex of all finite directed subgraphs of E which are acyclic and every vertex has in-degree ≤ 1 and out-degree ≤ 1. Note that $\mathcal{K}$ is in fact a simplicial complex on E rather than on X. The simplicial complex $\mathcal{K}$ is Suslin σ-centered by Theorem 11.4.6. To see this, let Γ be a countable group acting on X in a Borel way such that E is the resulting orbit equivalence relation. Let $\Gamma = \{\gamma_k \colon k \in \omega\}$ be an arbitrary enumeration. Whenever $a \in \mathcal{K}$ is a graph whose domain is a subset of a single E-class, let $n(a)$ be the smallest number such that for any two points $x, y \in \mathrm{dom}(a)$ there is $k \in n(a)$ such that $x = \gamma_k \cdot y$, and let $\mathrm{supp}(a) = \{\{\gamma_l \cdot x, \gamma_k \cdot x\} \colon x \in \mathrm{dom}(a), k, l \in n(a)\}$. If two graphs $a_0, a_1 \in \mathcal{K}$ have disjoint support, they must have disjoint domains. It follows that if $a_i \in \mathcal{K}$ are sets for $i \in j$ which are subsets of the same E-class and have pairwise disjoint supports, then $\bigcup_i a_i \in \mathcal{K}$ as required in Theorem 11.4.6.

We plan to use the simplicial complex $\mathcal{K}$ to introduce an ordering of ordertype $\mathbb{Z}$ to each E-class. However, not every maximal $\mathcal{K}$-set does that. Apparently, a cofinal subset $\mathcal{L} \subset \mathcal{K}$ is needed so that every $\mathcal{L}$-large maximal $\mathcal{K}$-set comes close the the purpose in mind. For one simple solution, let $\mathcal{L} \subset \mathcal{K}$ be the collection of all sets $a \in \mathcal{K}$ which in each E-class have only one vertex of in-degree 0 and only one vertex of out-degree 0. Note that $\mathcal{L} \subset \mathcal{K}$ is cofinal, and if $c \subset X$ is an E-class and $d \subset c^2$ is an $\mathcal{L}$-regular maximal $\mathcal{K}$-set, then d is a ray of ordertype $\mathbb{Z}$ or $\mathbb{N}$ or inverted $\mathbb{N}$.

Corollary 11.4.12. *Let E be a countable Borel equivalence relation with all classes infinite as in Example 11.4.11. Let H be a finitary analytic hypergraph of uncountable Borel chromatic number.*

1. *In the $P_{\mathcal{K}\mathcal{L}}$-extension of the Solovay model, the chromatic number of H is uncountable;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds and E is the orbit equivalence relation of a (discontinuous) $\mathbb{Z}$-action and the chromatic number of H is uncountable.*

Proof. The first item is the consequence of the fact that $\mathcal{K}$ is Suslin σ-centered as proved in Example 11.4.11. For the second item, let W be the Solovay model derived from some inaccessible cardinal and let $G \subset P_{\mathcal{K}\mathcal{L}}$ be a generic filter over the model W. A genericity argument shows that $\bigcup G$ is an $\mathcal{L}$-regular maximal $\mathcal{K}$-set. For every E-class c, $\bigcup G$ on the class c indicates either an ordering of type $\mathbb{Z}$ or of type $\mathbb{N}$ or of type inverted $\mathbb{N}$. It is a matter of minor surgery in ZF+DC to turn this into a relation which is an ordertype $\mathbb{Z}$ on every class.This immediately provides an action of $\mathbb{Z}$ on X which yields E as an orbit equivalence relation. The chromatic number part of the second item follows from the first item. $\square$

As a final observation, we show that the forcings of the type discussed in Theorem 11.4.5 can be placed in countable support product, and the conclusion of Theorem 11.4.5 remains in place. Suppose that $\langle X_i : i \in \omega \rangle$ are pairwise disjoint Polish spaces, and $\langle \mathcal{K}_i, \mathcal{L}_i \rangle$ are locally countable pairs on each. Suppose that each simplicial complex $\mathcal{K}_i$ is Suslin σ-centered. Let X be the Polish space which is the disjoint union of X_i for $i \in n$, let $\mathcal{K}$ be the simplicial complex of those finite sets $a \subset X$ such that $a \cap X_i \in \mathcal{K}_i$ holds for every $i \in \omega$, and let $\mathcal{L} = \{a \in \mathcal{K} \colon \forall i\ a \cap X_i \neq 0 \to a \cap X_i \in \mathcal{L}_i$. Then $P_{\mathcal{K}\mathcal{L}}$ is a countable support product of the posets $P_{\mathcal{K}_i \mathcal{L}_i}$, and the simplicial complex $\mathcal{K}$ is Suslin σ-centered by Theorem 11.3.8, since it is the finite support product of the posets $\mathcal{K}_i$ for $i \in \omega$. Therefore, by Theorem 11.4.5, in the $\prod_i P_{\mathcal{K}_i \mathcal{L}_i}$-extension of the Solovay model, the uncountable chromatic numbers of finitary analytic hypergraphs are preserved.

11.4b Suslin σ-linked complexes

In fairly common situations, one wishes to preserve uncountable chromatic numbers of analytic graphs while a task of higher arity is performed. This can be handled by Suslin σ-linked complexes as explained in the following theorem.

Theorem 11.4.13. *Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a locally countable pair. Suppose that the simplicial complex $\mathcal{K}$ is Suslin σ-linked. Let G be an analytic graph with uncountable Borel chromatic number. In the $P_{\mathcal{K}\mathcal{L}}$-extension of the Solovay model, the chromatic number of G is uncountable.*

The verification of the Suslin σ-linked property of Borel simplicial complexes is invariably achieved via the following result.

Theorem 11.4.14. *Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space X and write $E = E_{\mathcal{K}}$. Suppose that there is a Borel* support *function* $\operatorname{supp}\colon \mathcal{K} \to [X]^{<\aleph_0}$ *such that*

1. $\operatorname{supp}(a) \subset [a]_E$;
2. *whenever a_0, a_1 are sets in $\mathcal{K}$, subsets of the same E-class with disjoint supports, then $a_0 \cup a_1 \in \mathcal{K}$.*

Then $\mathcal{K}$ is Borel σ-linked.

The proofs of Theorems 11.4.13 and 11.4.14 are nearly identical to the proofs of Theorem 11.4.5 and 11.4.6, and we omit them.

Our first example is motivated by a result of Marks and Unger [68] which found perfect matchings with the Baire property in many Borel bipartite graphs. In order to do this, they isolated the following strengthening of the well-known Hall's condition of [38]:

Definition 11.4.15. Let Γ be a locally finite bipartite Borel graph on a Polish space X. We say that Γ satisfies the *Marks–Unger condition* if there is a real number $\varepsilon > 0$ such that for every finite set $a \subset X$ on one side of the bipartition, the number of Γ-neighbors of a is at least $(1+\varepsilon)|a|$.

Example 11.4.16. Let G be a locally finite bipartite Borel graph on a Polish space X satisfying the Marks–Unger condition. Let $\mathcal{K}$ be the simplicial complex on the edges of G consisting of all sets $b \subset G$ which can be completed to a perfect matching of G. Then $\mathcal{K}$ is Suslin σ-linked.

Proof. Let $\varepsilon > 0$ be a real number witnessing the Marks–Unger property of the graph G. Let $E = E_{\mathcal{K}}$ be the G-path equivalence relation on X. Let $a \in \mathcal{K}$ be a set which is a subset of a single E-class; we must calculate the support $\operatorname{supp}(a)$ such that the assumptions of Theorem 11.4.14 are satisfied.

Let $\operatorname{supp}(a)$ be the set of all G-edges which are within G-distance at most $8|a|/\varepsilon$ from some edge in a. We must verify the amalgamation condition, i.e. if $a, b \in \mathcal{K}$ are disjoint finite sets in the same E-class such that $\operatorname{supp}(a) \cap \operatorname{supp}(b) = 0$, then $a \cup b \in \mathcal{K}$. By Hall's marriage theorem [38], it is enough to verify that the

graph G' obtained from G by erasing all vertices mentioned in a or b satisfies the Hall's marriage condition: for each finite set c of vertices in the same equivalence class as a, b on one side of the bipartition, the set of G'-neighbors of c has size at least $|c|$. There are two cases:
Case 1. There is a G^2-path from some vertex in a to some vertex in b using only nodes in c as the intermediate steps. In this case, without loss of generality suppose that $|a| \geq |b|$, and use the definition of $\mathrm{supp}(a)$ to argue that the G^2-path from a to b must contain at least $4|a|/\varepsilon$-many steps. This means that $|c| > 4|a|/\varepsilon$. By the initial condition on the graph G, the set c has at least $(1+\varepsilon)|c|$ many G-neighbors and so at least $(1+\varepsilon)|c| - 2|a| - 2|b|$ many G'-neighbors. However, $2|a| + 2|b| < \varepsilon|c|$ and so the set c has at least $|c|$-many G'-neighbors as required.
Case 2. Case 1 fails. In this case, the set c can be written as a union of disjoint sets $c = c_a \cup c_b$ such that the set c_a has no common G-neighbors with c_b, the sets c_a and $\bigcup b$ are G-disconnected, and the sets c_b and $\bigcup a$ are G-disconnected. Then, since $a \in \mathcal{K}$, the set c_a has at least $|c_a|$-many G'-neighbors; since $b \in \mathcal{K}$, the set c_b has at least $|c_b|$-many G'-neighbors. The G'-neighborhoods of c_a, c_b do not overlap by the choice of c_a and c_b, and so the set c has at least $|c|$-many G'-neighbors as desired. $\square$

Note that as the graph G is locally finite, maximal $\mathcal{K}$-sets are in fact perfect G-matchings.

Corollary 11.4.17. *Let G be a locally finite Borel bipartite graph on a Polish space X satisfying the Marks–Unger condition. Let $\mathcal{K}$ be the simplicial complex of partial G-matchings that can be extended to a perfect matching. Let H be an analytic graph of uncountable Borel chromatic number.*

1. *In the $P_{\mathcal{K}}$-extension of the Solovay model, the chromatic number of H is uncountable;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a perfect G-matching, and the chromatic number of H is uncountable.*

Question 11.4.18. Is there a locally finite, bipartite Borel graph G satisfying the Marks–Unger condition such that the associated control poset is not Suslin σ-centered? Such that in ZF+DC, the existence of perfect matching for G is equivalent to some Borel hypergraph having countable chromatic number?

To many hypergraphs of arity greater than two it is possible to add a coloring while preserving uncountable chromatic numbers of analytic graphs purely as a matter of arity. Consider the following:

Example 11.4.19. Let $\mathbb{Z}$ act on a Polish space X in a Borel free way. Let G be the hypergraph of all triples $\{x_0, x_1, x_2\}$ such that there is $n \in \omega$ such that $n \cdot x_0 = x_1$ and $n \cdot x_1 = x_2$. Let $\mathcal{K}$ be the simplicial complex of partial finite colorings of the hypergraph G with colors coming from ω. The complex $\mathcal{K}$ is Suslin σ-linked.

Proof. Note that $\mathcal{K}$ is in fact a simplicial complex on $X \times \omega$. It is easy to observe that $E = E_{\mathcal{K}}$ is the equivalence relation on $X \times \omega$ connecting $\langle x, n\rangle$ and $\langle y, m\rangle$ just in case $x, y \in X$ are orbit-equivalent. Let $a \in \mathcal{K}$ be a nonempty finite partial G-coloring in a single E-class. We will compute a support $\operatorname{supp}(a)$ such that if $a, b \in \mathcal{K}$ are subsets of the same E-class and $\operatorname{supp}(a) \cap \operatorname{supp}(b) = 0$ then $a \cup b \in \mathcal{K}$. This will be enough by Theorem 11.4.14.

Let $n(a) \in \omega$ be the smallest number larger than all elements of $\operatorname{rng}(a)$ and all numbers n such that there are points $x_0, x_1 \in \operatorname{dom}(a)$ with $n \cdot x_0 = x_1$. Let $\operatorname{supp}(a) = \{\langle m \cdot x, k\rangle : x \in \operatorname{dom}(a), k \in n(a)$ and $m \in \mathbb{Z}$ is of absolute value smaller than $n(a)\}$. We claim that the function supp works as required in Theorem 11.4.14. Suppose that $a, b \in \mathcal{K}$ are colorings in the same E-class with disjoint supports; we must show that $a \cup b \in \mathcal{K}$. It is not hard to check that $a \cup b$ is in fact a function. To see that it is a G-coloring, let $\{x_0, x_1, x_2\} \in G$ be a hyperedge which is a subset of $\operatorname{dom}(a) \cup \operatorname{dom}(b)$; we must show that the coloring $a \cup b$ is not constant on it. If it is a subset of $\operatorname{dom}(a)$ or $\operatorname{dom}(b)$, then we are done as both a, b are G-colorings. Otherwise, two of the points (say x_0, x_1) would have to be in the domain of one of the colorings (say a) and the remaining point (z) has to be in the domain of the other coloring (b). If $b(z) \neq a(x_0)$ then we are done. However, $b(z) = a(x_0)$ is impossible: then $b(z) \in n(a)$ and so $\langle z, b(z)\rangle \in \operatorname{supp}(a) \cap \operatorname{supp}(b)$, violating the initial assumptions on a, b. □

In order to force a total G-coloring, we need to choose a cofinal subset $\mathcal{L} \subset \mathcal{K}$ such that $\mathcal{L}$-regular maximal $\mathcal{K}$-sets are in fact total colorings. One possible choice is $\mathcal{L} = \{a \in \mathcal{K}\colon$ for every infinite E-class $c \subset X$, $\operatorname{dom}(a) \cap c$ is a subinterval of the shift order on c, and the values of a at the endpoints of this interval are larger than the length of the interval$\}$. The verification of the requisite property of $\mathcal{L}$ is easy and left to the reader.

Corollary 11.4.20. *Let G be the hypergraph and $\langle \mathcal{K}, \mathcal{L}\rangle$ be the locally countable pair of Example 11.4.19. Let H be an analytic graph of uncountable chromatic number.*

1. *In the $P_{\mathcal{K}\mathcal{L}}$-extension of the Solovay model, the chromatic number of H is uncountable;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds and the chromatic number of G is countable while the chromatic number of H is uncountable.*

11.5 Larger graphs

In this section, we study the options for adding Γ-colorings with countably many colors to various, not necessarily locally countable Borel graphs Γ on Polish spaces while preserving the main features of locally countable structures. The main tool is the *control forcing* of finite approximations to such colorings, as recorded in the following definition.

Definition 11.5.1. Let Γ be a Borel graph on a Polish space X. $\mathcal{K}_\Gamma$ is the simplicial complex on $X \times \omega$ consisting of finite sets $a \subset X \times \omega$ which are partial Γ-colorings. $\mathcal{K}_\Gamma$ is viewed as a poset ordered by reverse inclusion.

In order to extract desirable properties of the poset $\mathcal{K}_\Gamma$, we deal with graphs with forbidden subgraphs. The following definition records the patterns we wish to prohibit later; all of them are bipartite graphs.

Definition 11.5.2. Let $n \in \omega$ be a natural number.

1. $K_{n,n}$ is the bipartite graph on $2 \times n$ connecting any pair of vertices with distinct first coordinates;
2. K_{n,ω_1} is the bipartite graph on the disjoint union of n and ω_1 connecting all elements of n to all elements of ω_1;
3. $K^{\rightarrow}_{\omega,\omega}$ is the graph on $2 \times \omega$ consisting of all pairs $\{\langle 0, m\rangle, \langle 1, k\rangle\}$ such that $m \in k$.

The first three theorems of this section deal with the definability and regularity properties of the control forcing.

Theorem 11.5.3. *Let Γ be a Borel graph on a Polish space X which for some $n \in \omega$ does not contain an injective homomorphic image of K_{n,ω_1}. Then $\mathcal{K}_\Gamma$ is a very Suslin c.c.c. partial ordering.*

Note that in this case the graph Γ has countable coloring number by a result of [25]. Observe also that the hypothesis on the graph is absolute throughout all forcing extensions. It simply says that for all injective tuple $\vec{x} \in X^n$, the set $\{y \in X \colon \forall i \in n\ \vec{x}(i)\ \Gamma\ y\}$ is countable. By the perfect set theorem for Borel sets, this is equivalent to the statement that for all injective tuples $\vec{x} \in X^n$ and all continuous maps $\pi \colon 2^\omega \to X$, either there are distinct points $z_0, z_1 \in 2^\omega$ such that $\pi(z_0) = \pi(z_1)$ or there is a point $z \in 2^\omega$ and a number $i \in n$ such that $\pi(z)\ \Gamma\ \vec{x}(i)$ fails. A closer reading of the formula reveals that it is $\not\succ^1_2$ and so the Shoenfield absoluteness applies to it.

Proof. Write $\mathcal{K} = \mathcal{K}_\Gamma$. It is clear that $\mathcal{K}$ is a Suslin poset. For the c.c.c. part of the theorem, let $A = \{a_\alpha \colon \alpha \in \omega_1\}$ be an uncountable set; we must produce two compatible elements in it. Let M be a countable elementary submodel of a large structure containing Γ and A. Let $\alpha \in \omega_1 \setminus M$ be any ordinal, and let $b = a_\alpha \restriction M$. By the elementarity of the model M, there have to be infinitely many ordinals β such that $b \subset a_\alpha$.

For each $y \in \mathrm{dom}(a_\alpha) \setminus b$, the set $u_y = \{x \in M \cap X \colon x\ \Gamma\ y\}$ must have size less than n: otherwise, one can find distinct elements $\{x_i \colon i \in n\} \subset y$ and place y in the set $\{z \in X \colon \forall i \in n\ x_i\ \Gamma\ z\}$. The latter set is countable by the initial assumption on Γ, it is an element of the model M and by elementarity of M it is a subset of M. This would contradict the choice of the point y.

Now, by elementarity of the model M, there is an ordinal $\beta \in M \cap \omega_1$ such that $b \subset \mathrm{dom}(a_\beta)$ and $\mathrm{dom}(a_\beta) \cap \bigcup_y u_y = \mathrm{dom}(b)$. It is not difficult to check that a_α, a_β are compatible as desired.

We turn to the proof of the very Suslin property. By the assumptions, the set $B = \{\langle \vec{x}, y\rangle \in X^n \times X : \vec{x}$ is injective and $\forall i \in n\ \vec{x}(i)\ \Gamma\ y\}$ is Borel and all of its vertical sections are countable by the initial assumption on the graph Γ. Use the Lusin–Novikov theorem to find Borel functions $\{f_i : i \in \omega\}$ from X^n to X such that $B \subset \bigcup_i f_i$.

Now, let $C \subset \mathcal{K}$ be a countable set. Write $\operatorname{supp}(C) \subset X$ for the closure of the set $\bigcup_{c\in C} \operatorname{dom}(c)$ under the functions f_i for $i \in \omega$. We claim that the following are equivalent:

- C is predense;
- for every $b \in \mathcal{K}$ with $\operatorname{dom}(b) \subset \operatorname{supp}(C)$ there is $c \in C$ such that $c \cup b \in \mathcal{K}$.

This will show that the collection of (enumerations of) countable predense subsets of $\mathcal{K}$ which are predense is Borel, since the second item clearly gives a Borel description of it.

Now, the first item certainly implies the second. The opposite implication is the heart of the matter. Suppose that the first item fails, and let $a \in \mathcal{K}$ be a condition such that for every $c \in C$ $c \cup a \notin \mathcal{K}$ holds. Note that for every $y \in \operatorname{dom}(a) \setminus \operatorname{supp}(C)$, the set $u_y = \{x \in \operatorname{supp}(C) : x\ \Gamma\ y\}$ must have size less than n, otherwise $y \in \operatorname{supp}(C)$ would hold by the initial choice of the functions f_i. Strengthening a if necessary then, we may assume that $\bigcup_y u_y \subset \operatorname{dom}(a)$. Let $b = a \restriction \operatorname{supp}(C)$ and argue that b witnesses the failure of the second item.

Let $c \in C$ be an arbitrary element. Since c is incompatible with a, there must be either some $x \in \operatorname{dom}(c) \cap \operatorname{dom}(a)$ such that $c(x) \neq a(x)$, or some Γ-connected $x \in \operatorname{dom}(c)$ and $y \in \operatorname{dom}(a)$ such that $c(x) = a(y)$. In the former case, $x \in \operatorname{supp}(C)$ and so $x \in \operatorname{dom}(b)$ witnesses the incompatibility of b, c. In the latter case, if $y \in \operatorname{supp}(C)$ then $y \in \operatorname{dom}(b)$ witnesses the incompatibility of b, c. If $y \notin \operatorname{supp}(C)$ then $x \in \operatorname{dom}(b)$ by the strengthening of a, and then $b(x) \neq a(y) = c(x)$ and so x witnesses the incompatibility of b, c again. □

Theorem 11.5.4. *Let Γ be a Borel graph on a Polish space X such that there is no injective homomorphism from $K^{\rightarrow}_{\omega,\omega}$ to Γ. Then $\mathcal{K}_\Gamma$ ordered by reverse inclusion is a Suslin σ-liminf-centered poset.*

Note that we are not asserting that the poset $\mathcal{K}_\Gamma$ must be very Suslin in this case; we do not know that.

Proof. For each number $n \in \omega$ let $A_n = \{a \in \mathcal{K} : |\operatorname{dom}(a)| < n$ and $\operatorname{rng}(a) \subset n\}$. It is clear that each set A_n is Borel and $\mathcal{K} = \bigcup_n A_n$. Assume that one of the sets A_n is not liminf-centered and work to produce an injective homomorphism from $K^{\rightarrow}_{\omega,\omega}$ to Γ. This will prove the theorem.

Suppose that $\{a_i : i \in \omega\}$ is a collection of conditions in A_n such that it is outright forced by $\mathcal{K}$ that a_i belongs to the generic filter for only finitely many values of $i \in \omega$. Let $b \in \mathcal{K}$ be inclusion-maximal such that $b \subset a_i$ holds for infinitely many $i \in \omega$; thinning out the original collection if necessary, we may assume that $b \subset a_i$ holds for all $i \in \omega$. Thinning out even further, we may assume that the sets $\operatorname{dom}(a_i \setminus b)$ for $i \in \omega$ are pairwise disjoint.

By recursion on $j \in \omega$ build conditions $c_j \in \mathcal{K}$ such that

- $b \subset c_j$;
- for all $x \in \bigcup_{k \in j} \operatorname{dom}(c_k \setminus b) \cup \bigcup_{k \in j} \operatorname{dom}(a_k \setminus b)$, $c_j(x) > n$;
- for all but finitely many numbers $i \in \omega$, $c_j \cup a_i \notin \mathcal{K}$.

This is possible by the initial assumptions on the collection $\{a_i : i \in \omega\}$. Note that the first and third item imply that for a given $j \in \omega$, for all but finitely many numbers $i \in \omega$ there are elements $x \in \operatorname{dom}(c_j)$ and $y \in \operatorname{dom}(a_i \setminus b)$ such that $x \; \Gamma \; y$ and $c_j(x) = a_i(y)$ both hold. In addition, the point $x \in X$ cannot belong to $\bigcup_{k \in j} \operatorname{dom}(c_k) \cup \bigcup_{k \in j} \operatorname{dom}(a_k)$ by the second item.

Now, let $\prec$ be any linear ordering of the space X. For numbers j, t write $x(j, t)$ for the t-th element of $\operatorname{dom}(c_j) \setminus \bigcup_{k \in j} \operatorname{dom}(c_k) \cup \bigcup_{k \in j} \operatorname{dom}(a_k)$ in the ordering $\prec$. Similarly, write $y(i, s)$ for the s-th element of $\operatorname{dom}(a_i)$ in the ordering $\prec$. Let U be a nonprincipal ultrafilter on ω. For each $j \in \omega$ there are numbers $t_j \in |\operatorname{dom}(c_j)|$ and $s_j \in n$ such that the set $a_j = \{i \in \omega : x(j, t_j) \; \Gamma \; y(i, s_j)\}$ belongs to the ultrafilter U. Also, there is a set $d \subset \omega$ in U and a number $s \in \omega$ such that for all $j \in d$, $s_j = s$ holds. Now by recursion on $k \in \omega$ define an increasing sequence of numbers j_k so that $j_k \in d \cap \bigcap_{l \in k} a_{j_l}$. Finally, define the map $\pi : 2 \times \omega \to X$ by $\pi(0, k) = x(j_{2k}, t_{j_{2k}})$ and $\pi(1, k) = y(j_{2k+1}, s)$. It is not difficult to see that π is an injective homomorphism of $K^{\to}_{\omega,\omega}$ to Γ. $\square$

Theorem 11.5.5. *Let Γ be a Borel graph on a Polish space X containing no injective homomorphic image of $K_{n,n}$ for some number $n \in \omega$. Then the poset $\mathcal{K}_\Gamma$ is Suslin σ-Ramsey-centered.*

Proof. Write $\mathcal{K} = \mathcal{K}_\Gamma$. For a number $m \in \omega$ let $A_m = \{c \in \mathcal{K} : |\operatorname{dom}(c)| \leq m$ and $\operatorname{rng}(m) \subset m\}$. Clearly, the sets A_m are all Borel and $\mathcal{K} = \bigcup_m A_m$. It will be enough to prove that each number $m \in \omega$, the set $A_m \subset \mathcal{K}$ is Ramsey-centered.

Let $k \in \omega$ be an arbitrary number greater than $2n$. We have to find $l \in \omega$ such that every set $\{c_i : i \in l\} \subset A_m$ contains a subset of size k with a common lower bound. We will show that any number l such that $l \to (k)^2_{2m^2+1}$ works. To see this, let $\{c_i : i \in l\} \subset A_m$ be a set. Let $\prec$ be any linear ordering of the space X, for each $i, u \in \omega$ write c_{iu} for the u-th element of $\operatorname{dom}(c_i)$ in the ordering $\prec$ if it exists, and define a partition $\pi : [l]^2 \to (m \times m \times 2) \cup \{\infty\}$ by requiring the following. If $i \in j \in l$ and $\pi(i, j) = \langle u, v, 0\rangle$ then $c_{iu} = c_{jv}$ and $c_i(c_{iu}) \neq c_j(c_{jv})$; if $\pi(i, j) = \langle u, v, 1$ then $c_{iu} \; \Gamma \; c_{jv}$ and $c_i(c_{iu}) = c_j(c_{jv})$; and if $\pi(i, j) = \infty$ then no u, v as in the previous items can be found.

Use the Ramsey property of the number l to find a set $a \subset l$ of size k which is homogeneous for the partition π. It is enough to argue that the set $\{c_i : i \in a\}$ has a common lower bound. To see this, inquire about the homogeneous partition value achieved. It cannot be of the form $\langle u, v, 0\rangle$ because then the coloring c_i has irreconcilable candidates for the value of $c_i(c_{iu})$, where i is the second number in the set a. The homogeneous partition value cannot be of the form $\langle u, v, 1\rangle$ since then the points c_{iu} where i ranges over the first n many elements of a, and c_{jv} where j ranges over the second n-many elements of a, form an injective homomorphic copy of $K_{n,n}$ in Γ, contradicting the initial assumptions on Γ. Thus, the homogeneous color is ∞, which precisely says that $\bigcup_{i\in a} c_i$ is a function which is a Γ-coloring and therefore a common lower bound of the conditions c_i for $i \in a$. $\square$

Finally, we are ready to prove some preservation results.

Theorem 11.5.6. *Let Γ be a Borel hypergraph on a Polish space X containing no injective homomorphic image of $K^{\to}_{\omega,\omega}$ and K_{n,ω_1} for some $n \in \omega$. Let Δ be a Borel graph which does not have Borel σ-finite clique number. In the P_Γ-extension of the symmetric Solovay model, the chromatic number of Δ is uncountable.*

Proof. By Fact 11.1.26 it is enough to prove this for the graph Δ on ω^ω which is the skew product of countably many cliques on ω. Write also $P = P_\Gamma$ and $\mathcal{K} = \mathcal{K}_\Gamma$. Let κ be an inaccessible cardinal, let W be the symmetric Solovay model derived from κ, and work in W. Suppose that $p \in P$ is a condition and τ is a P-name such that $p \Vdash \tau\colon \omega^\omega \to \omega$ is a function. We must find a condition $\bar{p} \leq p$, Δ-connected points $y_0, y_1 \in \omega^\omega$ and a number m such that $\bar{p} \Vdash \tau(y_0) = \tau(y_1) = \check{m}$.

Strengthening the condition p if necessary, we may assume that whenever x_i for $i \in n$ are distinct elements of $\operatorname{dom}(p)$ and $y \in X$ is a point Γ-connected to them all, then $y \in \operatorname{dom}(p)$ holds. The condition p and the name τ are all definable in the model W from ground model parameters and som additional parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate model obtained by forcing of size less than κ such that $z \in V[K]$ and work in $V[K]$. Consider the iteration $Q * \dot{R}$ where Q is the Cohen forcing Q on ω^ω, and $\dot{R}$ is (forced to be) the finite support iteration of length ω_1 whose iterands are the posets $\mathcal{K}$ evaluated in their respective models. Let M_α denote the model obtained after the initial segment of the iteration $\dot{R}$ of length α and let η_α be the union of the generic filter on $\mathcal{K}$ at the α-th stage of the iteration. Thus, R forces η_α to be a Γ-coloring on the domain $X \cap M_\alpha$.

To smooth out the various colorings η_α to a single total Γ-coloring extending p, we need a small claim:

Claim 11.5.7. *Let $y \in X \cap M_\alpha \setminus \bigcup_{\beta\in\alpha} M_\beta$ be a point. Then the set $\{x \in X \cap \bigcup_{\beta\in\alpha} M_\beta \colon x \mathrel{\Gamma} y\}$ has size less than n.*

Proof. If this failed, then (regardless of whether α is limit or successor) there would be a fixed ordinal $\beta \in \alpha$ and pairwise distinct points $x_i \in X \cap M_\beta$ for $i \in n$ such that $x_i \mathrel{\Gamma} y$ holds for all $i \in n$. But then, by the initial assumption on Γ the set of all points simultaneously connected to all x_i for $i \in n$ is countable, coded in M_β, and so a subset of M_β. This means that $y \in M_\beta$, contradicting the initial assumptions on the point y. $\square$

Let $\{b_k \colon k \in \omega\}$ be a recursive partition of ω into infinite sets. Now, define the $Q*\dot{R}$-name σ for a total Γ-coloring by the following recursive formula. $\sigma \restriction M_0$ is defined for a point $y \in M_0$ so that if $y \in \operatorname{dom}(p)$ then $\sigma(y) = p(y)$; if $y \notin \operatorname{dom}(p)$

then $\sigma(y)$ is the smallest element of $b_{\eta_0(y)}$ which is distinct from all colors $p(x)$ for points $x \in \mathrm{dom}(p)$ which are Γ-connected with y. If $\alpha > 0$ then let $\sigma(y)$ be the smallest element of $b_{\eta_\alpha(y)}$ which is distinct from all colors $p(x)$ for points $x \in X \cap \bigcup_{\beta\in\alpha} M_\beta$ which are Γ-connected with y. Note that the definition is correct as the number of colors in the set $b_{\eta_\alpha(y)}$ we must avoid is at most n by the claim and the initial assumption on p.

Now, observe that σ is really a name for a Γ-coloring. To see this, suppose that $y_0, y_1 \in X$ are Γ-connected points which appear in the models M_{α_0} and M_{α_1} respectively. For definiteness assume that $\alpha_0 \leq \alpha_1$ holds. If $\alpha_0 = \alpha_1$, then $\sigma(y_0) \neq \sigma(y_1)$ must hold as $\eta(y_0) \neq \eta(y_1)$ holds. If $\alpha_0 < \alpha_1$ then $\sigma(y_1)$ is distinct from all colors of points assigned to the points Γ-connected to y_1 and belonging to the models with smaller index, in particular $\sigma(y_1) \neq \sigma(y_0)$. Observe also that the name σ does not depend on the Q-generic point, but only on the model this point generates over $V[K]$.

Total Γ colorings are balanced virtual conditions in P by Theorem 8.1.2. Thus, writing χ for the Q-name for the generic element of ω^ω added by the poset Q, a balance argument shows that $Q * \dot{R} \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma$ decides in P the value of $\tau(\chi)$. Let $\langle q, \dot{r}\rangle$ be a condition in the iteration $Q * \dot{R}$ and let $m \in \omega$ be a number such that $\langle q, \dot{r}\rangle \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash \tau(\chi) = \check{m}$. By a chain condition argument, there is a countable ordinal α such that $q \Vdash \dot{r} \in \dot{R}_\alpha$, the intial segment of the iteration R of length α.

The poset $\mathcal{K}$ is very Suslin and Suslin σ-liminf-centered by Theorems 11.5.3 and 11.5.4. It follows by Theorem 11.3.20 that R_α is Suslin σ-liminf-centered; let $R_\alpha = \bigcup_k A_k$ be its cover by analytic liminf-centered pieces. Strengthening q if necessary, we may assume that there is $k \in \omega$ such that $q \Vdash \dot{r} \in A_k$; in addition, q is one of the finite sequences used in the definition in the skew product Δ. Let $y \in \omega^\omega$ be a point Cohen generic over $V[K]$ such that $q \subset y$, and for each number $l \in \omega$ let $y_l \in \omega^\omega$ be the point resulting from rewriting the first entry of y past q with l. Thus, the points y_l form a Δ-clique, they are all Cohen-generic over $V[K]$, and they all generate the model $V[K][y]$. By the liminf-centeredness of the set A_k, there is a condition $s \in R_\alpha$ which forces the set $\{l \in \omega\colon \dot{r}/y_l$ belongs to the generic filter$\}$ to be infinite. Let $H \subset R$ be a filter generic over $V[K][y]$ meeting the condition s. In $V[K][y][H]$, let $\bar{p} = \sigma/H$; this is a balanced condition for the poset P. Let $y_{l_0}, y_{l_1} \in \omega^\omega$ be two distict points such that the conditions $\dot{r}/y_{l_0}, \dot{r}/y_{l_1}$ both belong to the filter H. The forcing theorem applied in the model $V[K]$ shows that in the model W, $\bar{p} \Vdash_P \tau(\check{y}_{l_0}) = \tau(\check{y}_{l_1}) = \check{m}$ as desired. □

Theorem 11.5.8. *Let Γ be a Borel hypergraph on a Polish space X containing no injective homomorphic image of $K_{n,n}$ for some $n \in \omega$. Let Δ be a Borel graph which does not have Borel σ-bounded clique number. In the P_Γ-extension of the symmetric Solovay model, the chromatic number of Δ is uncountable.*

Proof. In view of Fact 11.1.24, it is enough to prove the theorem for a specific graph Δ. Let Y be the product $\prod_n (n+1)$ and let Δ be the graph on Y which is the skew product of cliques on the sets $n+1$ for $n \in \omega$. The proof now proceeds just as in Theorem 11.5.6; we only indicate the differences. The space ω^ω is naturally replaced with Y. The last paragraph is then amended to the following.

The poset $\mathcal{K}$ is very Suslin and Suslin σ-Ramsey-centered by Theorems 11.5.3 and 11.5.5. It follows by Theorem **??** that R_α is Suslin σ-Ramsey-centered; let $R_\alpha = \bigcup_k A_k$ be its cover by analytic Ramsey-centered pieces. Strengthening q if necessary, we may assume that there is $k \in \omega$ such that $q \Vdash \dot{r} \in A_k$. Let $m \in \omega$ be such that any m-tuple of elements of A_k contains two compatible conditions. Strengthening q further if necessary, we may assume that q is one of the finite sequences used in the definition in the skew product Δ, and the length of q is greater than m. Let $y \in Y$ be a point Cohen generic over $V[K]$ such that $q \subset y$, and for each number $l \in |q|$ let $y_l \in \omega^\omega$ be the point resulting from rewriting the first entry of y past q with l. Thus, the points y_l form a Δ-clique, they are all Cohen-generic over $V[K]$, and they all generate the model $V[K][y]$. By the choice of the number m, there are distinct numbers $l_0, l_1 \in |q|$ such that the conditions $\dot{r}/y_{l_0}$ and $\dot{r}/y_{l_1}$ are compatible, with a lower bound denoted by s. Let $H \subset R$ be a filter generic over $V[K][y]$ meeting the condition s. In $V[K][y][H]$, let $\bar{p} = \sigma/H$; this is a balanced condition for the poset P.The forcing theorem applied in the model $V[K]$ shows that in the model W, $\bar{p} \Vdash_P \tau(\check{y}_{l_0}) = \tau(\check{y}_{l_1}) = \check{m}$ as desired. $\square$

Example 11.5.9. Let Γ be the graph $\mathbb{H}_{<\omega}$. It does not contain an injective homomorphic copy of $K_{2,\omega}$, since for any point x, the points Γ-connected to it form a sequence converging to x; therefore, any two distinct points can have only finitely many common neighbors.

Theorem 11.5.6 now yields

Corollary 11.5.10. *Let Γ be the graph $\mathbb{H}_{<\omega}$.*

1. *In the P_Γ-extension of the Solovay model, the graph $\mathbb{H}_\omega$ has uncountable chromatic number. In particular, there is no $\mathbb{E}_0$-transversal;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds and the chromatic number of $\mathbb{H}_{<\omega}$ is countable and the chromatic number of $\mathbb{H}_\omega$ is uncountable.*

Example 11.5.11. Let $\vec{r}$ be a sequence of positive real numbers converging to zero. Let Γ be the graph on $X = \mathbb{R}^2$ connecting two points if their Euclidean distance is on the sequence $\vec{r}$. The graph Γ does not contain an injective homomorphic copy of $K_{3,\omega}$: if $x_0, x_1 \in X$ are distinct points and $\varepsilon > 0$ is some real number, then there are only finitely many points in the plane which are Γ-connected to both and of $> \varepsilon$-distance to both. Now, if $x_0, x_1, x_2 \in X$ are distinct points and $\varepsilon > 0$ is smaller than half of the minimum distance between two of the three, we see that every point of the plane is at a distance $> \varepsilon$ from two of the three, and so there are only finitely many points Γ-connected to all three.

Theorem 11.5.6 now yields

Corollary 11.5.12. *Let $\vec{r}$ be a sequence of positive real numbers converging to zero. Let Γ be the graph on $X = \mathbb{R}^2$ connecting two points if their Euclidean distance is on the sequence $\vec{r}$.*

1. *In the P_Γ-extension of the Solovay model, the graph $\mathbb{H}_\omega$ has uncountable chromatic number. In particular, there is no $\mathbb{E}_0$-transversal;*

2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable and the chromatic number of $\mathbb{H}_\omega$ is uncountable.*

Example 11.5.13. Let $D \subset \mathbb{R}$ be a Borel set algebraically independent over $\mathbb{Q}$, consisting of positive reals. Let Γ be the graph on $X = \mathbb{R}^2$ connecting two points if their Euclidean distance belongs to D. The graph Γ does not contain an injective homomorphic copy of $K_{n,n}$ for some large number n [59, Theorem 1]

Corollary 11.5.14. *Let $D \subset \mathbb{R}$ be a Borel set algebraically independent over $\mathbb{Q}$, consisting of positive reals. Let Γ be the graph on $X = \mathbb{R}^2$ connecting two points if their Euclidean distance belongs to D.*

1. *In the P_Γ-extension of the Solovay model, the diagonal Hamming graph $\mathbb{H}_{<\omega}$ has uncountable chromatic number;*

2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable and the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*

11.6 Collapses

In this section, we consider the problems of introducing an injection from $|F|$ to 2^ω or to $\mathbb{E}_0$, for a given pinned Borel equivalence relation F. It turns out that the natural forcings preserve uncountable chromatic numbers of many graphs and hypergraphs. In particular, they do not introduce a total $\mathbb{E}_0$-transversal. We start with the collapse to 2^ω.

Let E be a pinned equivalence relation on a Polish space X. Recall the definition of the poset collapsing $|E|$ to $|2^\omega|$ from Definition 6.6.2: P_E be the partial order of pairs $p = \langle a_p, b_p\rangle$ where $a_p \colon X \to 2^\omega$ is a countable partial function such that for $x_0, x_1 \in \mathrm{dom}(a_p)$, $x_0\ E\ x_1$ iff $a_p(x_0) = a_p(x_1)$, and $b_p \subset 2^\omega$ is a countable set disjoint from $\mathrm{rng}(a_p)$. The ordering is that of coordinatewise reverse inclusion. It is immediate in ZF+DC that the ordering P_E is Suslin and σ-closed, and the union of the P_E-generic filter is an injection from the set of E-classes to 2^ω. The balanced conditions are classified by all such injections as proved in Theorem 6.6.3.

Theorem 11.6.1. *Let E be a pinned Borel equivalence relation on a Polish space X. Suppose that G is an analytic finitary hypergraph on a Polish space Y which does not have Borel σ-bounded fractional chromatic number. In the P_E-extension of the Solovay model, the chromatic number of G is uncountable.*

Proof. In view of Fact 11.1.22 it is enough to consider the case of a large measured skew product G. Find a sequence $\langle a_n, H_n, t_n : n \in \omega\rangle$ which generates G as in Definition 11.1.21. Thus, $Y = \prod_n a_n$ and $G = \prod_n H_n, t_n$.

Now, consider the control poset Q: it conditions are finite functions $a : X \to 2^{<\omega}$ such that for some $n \in \omega$, $\text{rng}(a) \subset 2^n$, and for points $x_0, x_1 \in \text{dom}(a)$, $x_0\ E\ x_1$ iff $a(x_0) = a(x_1)$. The ordering is defined by $b \leq a$ if $\text{dom}(a) \subset \text{dom}(b)$ and for all $x \in \text{dom}(a)$, $a(x) \subset b(x)$. The following is the key observation.

Claim 11.6.2. *Q is a very Suslin c.c.c. poset. In addition, it is Suslin measured.*

Proof. It is clear that Q is very Suslin. To check the very Suslin property of Q, suppose that $a \subset Q$ is a countable set. It is not difficult to see that if there is a condition $r \in Q$ incompatible with all elements of a, then there is one whose support is a subset of $\bigcup_{q \in a} \text{supp}(q)$. The search for an incompatible condition among the countably many conditions of this type is a Borel procedure.

To verify the Suslin measured property of Q, consider the function $m : Q \to (0, 1]$ defined by $m(q) = 2^{-n}2^{-k}$ where $n \in \omega$ is such that $\text{rng}(q) \subset 2^n$, and $k \in \omega$ is such that $\text{dom}(q)$ has nonempty intersection with exactly k many E-classes. For each positive rational number $\varepsilon > 0$ let $A_\varepsilon = \{q \in Q : m(q) > \varepsilon\}$. Since the sets $A_\varepsilon \subset Q$ are all analytic and $Q = \bigcup_\varepsilon A_\varepsilon$, it is enough to show, given $\varepsilon > 0$ and a finite sequence $\langle q_i : i \in j\rangle$ of elements of the set A_ε, there is a set $b \subset j$ of size greater than εj such that the conditions p_i for $i \in b$ have a common lower bound.

To this end, let a be the set of all E-classes with nonempty intersection with $\bigcup_i \text{dom}(q_i)$. Let μ be the usual probability measure on 2^ω and let λ be the product measure on $Y = (2^\omega)^a$. Define the set $B \subset j \times Y$ as the set of all pairs $\langle i, y\rangle$ such that $\forall x \in \text{dom}(q_i)\ q_i(x) \subset y([x]_E)$. By the definition of the function m, the vertical sections B_i have λ-mass $> \varepsilon$ each. By the Fubini theorem applied to the counting measure on j and λ, there must be a point $y \in Y$ such that the horizontal section B^y has size greater than εj. It is easy to check that the set $\{q_i : i \in B^y\}$ has a common lower bound in Q. □

Note that a generic filter $G \subset Q$ induces a function defined by $f(x) = \bigcup\{p(x) : p \in G\}$ whenever $x \in X$ is a ground model point. It is clear that $\text{dom}(f)$ consists of the ground model elements of X, for all such points $x_0, x_1 \in X$ $x_0\ E\ x_1$ iff $f(x_0) = f(x_1)$ holds, and moreover the range of f consist of elements of 2^ω which are not in the ground model.

Now, suppose that $p \in P$ is a condition. Consider the poset R which is the finite support iteration of the posets Q of length ω_1. Let $\dot{f}_\alpha : X \to 2^\omega$ be the R-name for the partial function defined by the generic filter on the α-th iterand. Let M_α be the R-name for the model obtained from the first α many iterands. Consider the R-name σ for a function from E-classes to 2^ω such that if c is an

E-class in $\mathrm{dom}(p)$ then $\sigma(c) = p(c)$, and if c is an E-class not in σ and $\alpha \in \omega_1$ is the first ordinal such that c is represented in M_α, then $f(c) = y$ for the unique $y \in 2^\omega$ such that for all points $x \in c \cap V_\alpha$, $f_\alpha(x) = y$. It is not difficult to see that R forces σ to be an injection of the set of E-classes to 2^ω. By Theorem 6.6.3, σ represents a balanced condition in the collapse poset P, and it is not hard to see that this balanced condition is below p.

Finally, we are ready for the main body of the proof. Let κ be an inaccessible cardinal, let W be a Solovay model derived from κ, and work in W. Suppose that $p \in P_F$ is a condition and τ is a P-name such that $p \Vdash \tau\colon Y \to \omega$ is a function. We must find a condition $\bar{p} \leq p$, a number $n \in \omega$, and a hyperedge $e \in G$ such that for all $y \in e$, $\bar{p} \Vdash \tau(\check{y}) = \check{m}$. The condition $p \in P_E$ and the name τ are definable from parameters in the ground model and a parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate extension obtained by a poset of size $< \kappa$ which contains the parameter z and work in the model $V[K]$.

Consider the iteration $S * \dot{R}$ where S is the Cohen poset on the space Y and $\dot{R}$ is the finite support iteration of the control poset of length ω_1. Let $\dot{y}_{gen}$ be an S-name for the generic point in the space Y, and in the S-extension let σ be the R-name for the balanced condition in P_E below p as isolated in the preliminary part of this proof. Note that the name σ does not depend on $\dot{y}_{gen}$ per se, but only on the model generated by $\dot{y}_{gen}$. By a standard balance argument, $S * \dot{R} \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \exists m \in \omega\ \sigma \Vdash \dot{y}_{gen} \in B_m$. Thus, let $\langle s, \dot{r}\rangle \in S * \dot{R}$ be a condition and $m \in \omega$ be a number such that $\langle s, \dot{r}\rangle \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash \dot{y}_{gen} \in \dot{B}_m$. Now, strengthening s if necessary, we may assume that

- there is a specific countable ordinal $\alpha \in \omega_1$ such that $s \Vdash \dot{r}$ is a condition in the iteration R_α up to α;
- since the poset R_α is very Suslin and Suslin measured by Theorem 11.3.15, we may find analytic sets $A_n \subset R_\alpha$ and positive rationals $\varepsilon_n > 0$ for $n \in \omega$ witnessing the Suslin measured property of R_α. We may assume that there is a specific n such that $s \Vdash \dot{r} \in A_n$.
- there is a number $k \in \omega$ such that $1/k < \varepsilon_n$ and $s = t_k$.

Now, let $y \in Y$ be a point generic over $V[K]$ such that $s \subset y$ and work in $V[K][y]$. For each $i \in a_k$ let $y_i \in Y$ be the point obtained from y by rewriting the k-th entry of y with i. In the model $V[K][y]$, for each $i \in a_k$ let $r_i = \dot{r}/y_i \in R_\alpha$. Since $r_i \in A_n$ holds for all $i \in a_k$, there is a set $b \subset a_k$ of size at least $|a_k|/k + 2$ such that the conditions $\{r_i : i \in b\}$ have nonzero common lower bound. By the initial assumption on the hypergraph H_k, the set b is not an H_k-anticlique; let $c \subset b$ be an H_k-hyperedge. Thus, we have $\{y_i : i \in c\} \in G$. Let $H \subset R$ be a filter generic over $V[K][y]$ containing all the conditions r_i for $i \in c$ and $\bar{p} = \sigma/H$ its associated balanced condition. By the forcing theorem, in the model W $\bar{p} \Vdash \forall i \in c\ \tau(y_i) = \check{m}$. This completes the proof. □

Among the many preservation consequences of the theorem, we state the most striking one.

Corollary 11.6.3. *Let P be the collapse poset of $\mathbb{E}_0$ to 2^ω.*

1. *In the P-extension of the Solovay model, the chromatic number of the diagonal Hamming graph is uncountable;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds and $|\mathbb{E}_0| \leq |2^\omega|$ and yet the chromatic number of the diagonal Hamming graph is uncountable; in particular, there is no $\mathbb{E}_0$-transversal.*

Note that the collapse does change some chromatic numbers to their lowest possible value; for example, it forces the chromatic number of $\mathbb{H}_2$ to be equal to two, by Proposition 11.1.6. The following example shows a somewhat more dramatic case of this behavior, as the hypergraph in question does not have Borel σ-bounded chromatic number. The example also shows that the use of the fractional chromatic number in place of the usual chromatic number is to some extent necessary in the assumption of the theorem.

Example 11.6.4. Let E be a Borel equivalence relation on a Polish space X. Let Q be the poset of all pairs $q = \langle a_q, b_q\rangle$ of finite subsets of X such that $(a_q \times b_q) \cap E = 0$; the ordering is that of coordinatewise reverse inclusion. Let G be the graph connecting two conditions in Q if they are incompatible. Then in ZF, $|E| \leq |2^\omega|$ if and only if the chromatic number of G is countable.

Note that the graph G does not have Borel σ-bounded chromatic number as soon as the equivalence relation E is non-smooth as per Example 11.1.19. In particular, there is a Borel graph which does not have Borel σ-bounded chromatic number, yet the collapse of $|\mathbb{E}_0|$ to $|2^\omega|$ forces countable chromatic number to it.

Proof. For the left-to-right direction, let $h\colon X \to 2^\omega$ be a reduction of E to the identity on 2^ω. For every $n \in \omega$ and every set $c \subset 2^n$ let $B_c = \{q \in Q\colon \forall x \in a_q\ h(x) \restriction n \in c \wedge \forall x \in b_q\ h(x) \restriction n \notin c\}$. Since the function h is E-invariant, the sets $B_c \subset Q$ are G-anticliques. Now, if $q \in Q$ is a condition, there is $n \in \omega$ such that for all $x_0 \in a_q$ and $x_1 \in b_q$, $h(x_0) \restriction n \neq h(x_1) \restriction n$. Thus, writing $c = \{h(x) \restriction n\colon x \in a_q\}$, we conclude that $q \in B_c$. It follows that $Q = \bigcup_c B_c$ is a cover of Q by countably many G-anticliques.

For the right-to-left direction, let $Q = \bigcup_n B_n$ be a cover of Q by G-anticliques. For each $n \in \omega$, let $A_n = [\bigcup_{q \in B_n} a_q]_E$ and let $h\colon X \to 2^\omega$ be the function defined by $h(x)(n) = 0$ if $x \in A_n$. We claim that this is a reduction of E to the identity on 2^ω, thus inducing the desired inequality $|E| < |2^\omega|$. To see this, let $x_0, x_1 \in X$ be arbitrary points. If $x_0\ E\ x_1$ holds then $h(x_0) = h(x_1)$ since the sets A_n are E-invariant. On the other hand, if $x_0\ E\ x_1$ fails, then $q = \langle\{x_0\}, \{x_1\}$ is a condition in Q and there must be n such that $q \in B_n$. It is not difficult to see that then $h(x_0)(n) = 0 \neq h(x_1)(n)$ and so $h(x_0) \neq h(x_1)$. $\square$

Now we move to the case of a collapse of $|E|$ to $|\mathbb{E}_0|$ for a Borel pinned equivalence relation F. Again, let P_E be the collapse as defined in Definition 6.6.2. Recall that its balanced conditions are classified by injections from the E-quotient space to the $\mathbb{E}_0$-quotient space by Theorem 6.6.3.

Theorem 11.6.5. *Let E be a pinned Borel equivalence relation on a Polish space X. Let G be a finitary analytic hypergraph on a Polish space Y of uncountable Borel chromatic number. In the P_E-extension of the Solovay model, G has uncountable chromatic number.*

Proof. Consider the control poset Q: it consists of conditions q such that for some number $n_q \in \omega$, q is a finite partial function from X to 2^{n_q}. The ordering is defined by $r \leq q$ if $n_q \leq n_r$, $\mathrm{dom}(q) \subseteq \mathrm{dom}(r)$, for every $x \in \mathrm{dom}(q)$, $q(x) \subseteq r(x)$, and for every pair $x_0, x_1 \in \mathrm{dom}(q)$ of E-related points and for every $m \in n_r \setminus n_q$, $r(x_0)(m) = r(x_1)(m)$.

Claim 11.6.6. *Q is a very Suslin, Suslin σ-centered poset.*

Proof. It is immediate that Q is a Suslin poset. For the very Suslin part, suppose that $a \subset Q$ is a countable set. It is not difficult to check that if there is a condition $q \in Q$ which is incompatible with all conditions in a, then there must be such a condition q with $\mathrm{dom}(q) \subset \bigcup_{r\in a} \mathrm{dom}(r)$. There are only countably many conditions satisfying the latter formula, and the search in this countable set for one which is incompatible with all elements of a is a Borel procedure.

For the Suslin σ-centered property of the poset Q, a *descriptor* of a condition $q \in Q$ is a function h whose domain is a finite collection of pairwise disjoint basic open subsets of X such that for each $O \in \mathrm{dom}(h)$ there is exactly one point x_O of $\mathrm{dom}(q)$ in O, and for each point of $\mathrm{dom}(q)$ belongs to one set in $\mathrm{dom}(h)$; moreover, $h(O) = q(x_O)$. It is immediate that the set of all conditions with a given descriptor is Borel and σ-centered, every condition has many descriptors, and there are only countably many possible descriptors in all. Therefore, Q is Suslin σ-centered as required. □

Note that the poset Q adds an injection from the ground model E-classes to the collection of $\mathbb{E}_0$-classes which are not represented in the ground model. We denote the name for the injection by η; thus $\eta([\check{x}]_E)$ is forced to be the $\mathbb{E}_0$-class of the point $y = \bigcup\{r(x) \colon r \text{ is in the generic filter}\}$. Now, let R be the finite support iteration of the poset Q of length ω_1, adding functions η_α at each stage of the iteration; we denote by V_α the generic extension obtained after the α-th stage of the iteration. Let $p \in P_E$ be any condition. Let σ be the R-name for the injection from the E-quotient space to the $\mathbb{E}_0$-quotient space defined by $\sigma(c) = a_p(c)$ whenever c is an E-class in the domain of a_p, and $\sigma(c) = \eta_\alpha(c)$ whenever c is an E-class which is not in the domain of a_p and $\alpha \in \omega_1$ is the first ordinal such that c has a representative in the model V_α. It is clear that σ is a name for an injection from the E-quotient space to the $\mathbb{E}_0$-quotient space compatible with the condition p.

The rest of the proof follows closely the argument for Theorem 11.4.5 and we omit it. □

Corollary 11.6.7. *Let E be a pinned Borel equivalence relation. Let G be an analytic finitary hypergraph on a Polish space, with uncountable Borel chromatic number. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $|E| \leq |\mathbb{E}_0|$ and the chromatic number of G is uncountable.*

11.7 Compactly balanced posets

In this section we prove an additional preservation property of compactly balanced posets of Definition 9.2.1 which concerns the locally countable structures.

Definition 11.7.1. Let G be an analytic finitary hypergraph on a Polish space X. The hypergraph is *actionable* if there is a countable group Γ acting on a Polish space X such that all hyperedges of G consist of pairwise orbit-related elements and for every $\gamma \in \Gamma$, $\gamma \cdot G = G$.

The following theorem is stated using Convention 1.7.16.

Theorem 11.7.2. *Let G be an analytic, finitary, actionable hypergraph on a Polish space X which does not have Borel σ-bounded chromatic number. Then in compactly balanced extensions of the Solovay model, G has uncountable chromatic number.*

The actionable assumption is necessary, see the example below. The main difference between the general and actionable hypergraphs we exploit is the following routine strengthening of Fact 11.1.13; instead of skew products we can deal with the usual product.

Fact 11.7.3. *Let G be a finitary analytic actionable hypergraph on a Polish space X. Exactly one of the following occurs:*

1. *G has Borel σ-bounded chromatic number;*
2. *there is a large product H on a Polish space Y and a continuous homomorphism $h\colon Y \to X$ of H to G.*

Proof. In view of Fact 11.7.3, it is enough to prove the theorem for large product hypergraphs. Thus, let $\langle a_n, H_n\colon n \in \omega\rangle$ be a sequence such that for every $n \in \omega$, H_n is a hypergraph on the finite set a_n, and $|a_n| \geq 2$ and the chromatic number of H_n is at least n, and assume that G is in fact the product hypergraph $\prod_n H_n$ on the space $Y = \prod_n a_n$. With a large product of this form, we associate the poset R of all functions r with domain ω such that for every $n \in \omega$, $r(n) \subset a_n$ is a nonempty set and the chromatic numbers of H_n on $r(n)$ are unbounded as n tends to infinity. The ordering on R is that of coordinatewise inclusion. Clearly, the poset R adds a point $\dot{y}_{gen} \in Y$ defined by $\dot{y}_{gen}(n)$ is the only element of the set $\bigcap\{r(n)\colon r$ belongs to the generic filter$\}$. The following claim is key.

Claim 11.7.4. *The poset R is proper, bounding, and adds no independent reals.*

Proof. The first two assertions of the following claim are standard and proved by the usual fusion arguments. The last assertion is the heart of the present proof. Suppose that $r \in R$ is a condition and τ is an R-name for a subset of ω; we have to find a condition $s \leq r$ and an infinite set $c \subset \omega$ such that $s \Vdash \check{c} \subset \tau$ or $\check{c} \cap \tau = 0$. Strengthening the condition r if necessary, a standard fusion argument will yield a continuous function $f\colon \prod_n r(n) \to \mathcal{P}(\omega)$ such that $r \Vdash \tau = \dot{f}(\dot{y}_{gen})$.

Now, let $B \subset \prod_n r(n) \times \omega$ be the Borel set given by $\langle y, n\rangle \in B$ if $n \in f(y)$ holds. A partition result [84, Theorem 1.4] applied with an infinite subsequence of the sets $r(n)$ and the chromatic numbers of the hypergraphs H_n as submeasures on the sets $r(n)$ shows that there is a condition $s \leq r$ and an infinite set $c \subset \omega$ such that either $\prod_n s(n) \times c \subset B$ or $(\prod_n s(n) \times c) \cap B = 0$. In the former case, $s \Vdash \check{c} \subset \tau$; in the latter case, $s \Vdash \check{c} \cap \tau = 0$. The claim has just been proved. $\square$

Let κ be an inaccessible cardinal. Let P be a Suslin poset which is compactly balanced below κ. Let W be the Solovay model derived from κ and work in the model W. Let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau \colon X \to \omega$ is a function. We will find a number $m \in \omega$, finite sets $a_n \subset X$ for $n \in \omega$, and a condition $\bar{p} \leq p$ such that for all n, a_n cannot be covered by n-many G-anticliques, and $\bar{p} \Vdash \forall x \in \bigcup_n a_n\ \tau(x) = m$.

The condition $p \in P$ as well as the name τ are definable in the model W from parameters in the ground model and a parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate forcing extension obtained by a poset of size $< \kappa$, containing the point z. Work in $V[K]$ and consider the poset $Q \times R$ where Q is the poset of infinite subsets of ω ordered by inclusion. Let $U \subset Q$ and $y \in Y$ be mutually generic filter objects for the product; so U is in fact a nonprincipal ultrafilter on ω in the model $V[K][U]$. The poset Q is σ-closed, and therefore R computed in the model $V[K][U]$ is the same as R computed in the model $V[K]$ and y is in fact R-generic over the model $V[K][U]$. By the continuous reading of names for the poset R, the sets $\mathcal{P}(\omega) \cap V[K][U][y]$ and $\mathcal{P}(\omega) \cap V[K][y]$ are the same. By a mutual genericity argument and the fact that R adds no independent reals, U generates an ultrafilter on ω in the model $V[K][U][y]$. Work in the model $V[K][U]$.

Claim 11.7.5. *There is an R-name σ for a balanced condition in P such that $C \Vdash \forall y \in Y\ y = \dot{y}_{gen}$ up to finitely many entries$\to \sigma/\dot{y}_{gen} = \sigma/\gamma \cdot \dot{y}_{gen}$.*

To parse the claim correctly, note that any point of the space Y which is up to finitely many entries equal to $\dot{y}_{gen}$ is in fact R-generic again and yields the same forcing extension. The claim therefore says that the evaluation of the name σ does not depend on the specific generic point, but only on its modulo finite equivalence class.

Proof. Choose an arbitrary R-name χ for a balanced condition below the condition p. We now use the ultrafilter U and the compact balance of the poset P to integrate the name χ. Choose any point $y_\omega \in Y$. For each $n \in \omega$, let $\dot{y}_n$ be the R-name below C for the element of Y such that $\dot{y}_n \restriction n = y_\omega \restriction n$ and $\dot{y}_n \restriction [n, \omega) = \dot{y}_{gen} \restriction [n, \omega)$; note that $\dot{y}_n$ is forced to be a point R-generic over the model $V[K][U]$, generating the same model as $\dot{y}_{gen}$. Let χ_n be the name for the balanced condition $\chi/\dot{y}_n$. Let σ be the name for the U-limit of the sequence $\langle \dot{\chi}_n \colon n \in \omega\rangle$ in the definable compact Hausdorff topology on the space of all balance classes for P. It is immediate that the name σ works as desired. $\square$

Still working in the model $V[K][U]$, find a name σ as in the claim. By a standard balance argument, $R \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma$ decides in P the value of $\tau(\dot{y}_{gen})$. Passing to a condition r if necessary, we may assume that there is a specific number $m \in \omega$ such that $r \Vdash \text{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau(\dot{y}_{gen}) = \check{m}$. Now, let $y \in Y$ be a point R-generic over the model $V[K][U]$, meeting the condition r and let $\bar{p} = \sigma/y$. Find a number $n \in \omega$ such that the set $r(n)$ contains an H_n-hyperedge $b \in H_n$ and for each $i \in b$ let $y_i \in Y$ be the point obtained from y by replacing the n-th entry of y with i. Thus, the set $\{y_i : i \in b\}$ is a G-hyperedge (this is exactly the point which does not work if a skew product variation of G is considered) and for each $i \in n$ we have $\bar{p} = \sigma/y_i$ by the choice of the name σ. By the forcing theorem, in the model W $\bar{p} \Vdash_P \forall y \in a_n\ \tau(y) = \check{m}$, completing the proof. $\square$

Corollary 11.7.6. *Let G, H be analytic locally countable hypergraphs on respective Polish spaces X, Y such that G has Borel σ-bounded chromatic number and H is actionable and does not have Borel σ-bounded chromatic number. Then it is consistent relative to an inaccessible cardinal that ZF+DC holds, G has countable chromatic number and yet H has uncountable chromatic number.*

Proof. Let $X = \bigcup_n B_n$ be a partition into Borel sets witnessing the Borel σ-bounded chromatic number of G. Let E be a countable Borel equivalence relation on X such that all hyperedges of G consist of pairwise E-related elements. For each $n \in \omega$ let P_n be the poset of functions p whose domain is a countable relatively E-invariant subset of B_n, range is a subset of n, and p is a partial G-coloring. The ordering on P_n is that of reverse inclusion. Let $P = \prod_n P_n$. Then we see that in P_n, the balanced conditions are classified by total G-colorings of the set B_n whose range is a subset of n. Such colorings naturally form a compact subset of the compact Hausdorff space n^{B_n}, therefore the poset P_n is compactly balanced. As a result, even the full support product $P = \prod_n P_n$ is compactly balanced, and therefore in the P-extension of the symmetric Solovay model, the chromatic number of H is uncountable by Theorem 11.7.2. At the same time, the poset P_n adds a total G-coloring on the set B_n with n many colors and therefore the product P forces G to have countable chromatic number. $\square$

Corollary 11.7.7. *Let $\mathbb{Z}$ act freely and in a measure preserving Borel way on a Polish probability space $\langle X, \mu\rangle$. Let G be the hypergraph of arity three on X containing a triple $\{x_0, x_1, x_2\}$ if there is a number $n \in \omega$ such that $n \cdot x_0 = x_1$ and $n \cdot x_1 = x_2$. Then*

1. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of $\mathbb{G}_0$ is uncountable and the chromatic number of G is countable;*
2. *it is also consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of $\mathbb{G}_0$ is two and the chromatic number of G is uncountable.*

Proof. For the first item, use Corollary 11.4.20. For the second item, use Example 11.1.20 to show that the hypergraph G does not have Borel σ-bounded chromatic number; it is clearly actionable. Then, use Theorem 11.7.2 to the poset P adding a $\mathbb{G}_0$-coloring with two colors by countable approximations whose domain is $\mathbb{E}_0$-invariant. It is not difficult to see and follows from Theorem 6.2.2 that the balanced conditions for P are classified by total $\mathbb{G}_0$-colorings with two colors, which naturally form a closed subset of the compact Hausdorff space 2^{2^ω}. Thus the poset P is compactly balanced. □

Corollary 11.7.8. 1. *Let P be the poset of infinite subsets of ω ordered by inclusion. In the P-extension of the Solovay model, the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*

2. *It is consistent relative to an inaccessible cardinal that there is a nonprincipal ultrafilter on ω and the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*

This is immediate from the fact that the poset P is compactly balanced–Example 9.2.4. The corollary can be viewed as a commentary on a result of Rosendal [76]: in ZF+DC, if there is a discontinuous homomorphism between Polish groups then the chromatic numbers of the Hamming graphs $\mathbb{H}_n$ are finite. Since a nonprincipal ultrafilter on ω yields a discontinuous homomorphism from the Cantor group 2^ω to 2, this result cannot be extended to the diagonal Hamming graph.

The following two corollaries use compactly balanced posets from Examples 9.2.11 and 9.2.13.

Corollary 11.7.9. *Let E be a Borel equivalence relation on a Polish space X.*

1. *Let P be the linearization poset for the E-quotient space as in Example 8.6.5. In the P-extension of the Solovay model, the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*
2. *It is consistent relative to an inaccessible cardinal that there is a linear ordering on the E-quotient space and the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*

Corollary 11.7.10. *Let G be a locally finite Borel graph on a Polish space X satisfying the Hall condition.*

1. *Let P be the poset adding a perfect matching for G as in Example 6.2.4. In the P-extension of the Solovay model, the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*
2. *It is consistent relative to an inaccessible cardinal that G has a perfect matching and the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.*

The following example presents a (necessarily) non-invariant hypergraph which does not have Borel σ-bounded chromatic number, yet it has countable chromatic number in some compactly balanced extension of the Solovay model.

Example 11.7.11. The actionable assumption cannot be removed from the assumptions of Theorem 11.7.2.

Proof. Let Γ be the free group on two generators γ, δ. It acts on the space 2^Γ by shift; that is, $(\beta \cdot x)(\alpha) = x(\beta^{-1}\alpha)$ holds for all $\beta, \alpha \in \Gamma$ and $x \in 2^\Gamma$. Let $X \subset 2^\Gamma$ be the dense G_δ set on which the action is free. Let E_0 be the Γ-orbit equivalence relation on X. Let E_1 be the orbit equivalence relation induced by the subgroup of Γ generated by δ. It is not difficult to see that $E_1 \subset E_0$ are Borel equivalence relations. Let G be the hypergraph on X of arity three containing triples $\{x_0, x_1, x_2\}$ consisting of pairwise E_0-related points and containing two E_1-related and two E_1-unrelated points.

To see how a compactly balanced poset can make the chromatic number of G countable, consider the Cayley graph H on X: it connects points x_0, x_1 if $x_1 = \gamma \cdot x_0$ (the γ-edges of H) or $x_1 = \delta \cdot x_0$ (the δ-edges of H) or vice versa. This is an acyclic 4-regular graph whose connectedness components are exactly the E_0-classes. In ZFC, every acyclic graph without vertices of degree 0 or 1 has an orientation in which every vertex has out-degree one, constructed component by component. The following claim shows the impact of such an orientation on the hypergraph G:

Claim 11.7.12. *(ZF) Suppose that the graph H has an orientation in which every vertex has out-degree one. Then the chromatic number of G is countable.*

Proof. Let $\vec{H}$ be the orientation. For each $x \in X$ the color $h(x)$ is the pair $\langle b = b(x), k = k(x)\rangle$ such that k is the largest possible number such that $\vec{H}$ contains an oriented path from x to $\delta^{ck} \cdot x$ for some $c \in \{-1, 1\}$. If $k = 0$ then $b = 0$, otherwise $b = c$. If such k does not exist,then the color $h(x)$ is ∞. We claim that h is a G-coloring.

To see this, consider any two E_0-related points $x_0, x_1 \in X$. In the H-path between x_0, x_1 no vertex gets an out-degree two, and that leaves only three cases: either the whole path is oriented towards x_0, or towards x_1, or towards some point in the middle. Now, if $h(x_0) = h(x_1) = \infty$ then all three cases show that all the edges on the path must be δ-edges and so $h(x_0)\ E_1\ h(x_1)$. If x_0, x_1 are E_1-related, say $x_0 = \delta^m \cdot x_1$ for some $m > 0$, and $h(x_0) \neq \infty$, then in the first two cases $k(x_0) \neq k(x_1)$ and in the third case $b(x_0) = 1 \neq -1 = b(x_1)$. It follows that no G-hyperedge can be monochromatic: the homogeneous color cannot be ∞ on the account of the two E_1-unrelated points in the hyperedge, and the homogeneous color cannot be different from ∞ on the account of the two E_1-related points in the hyperedge. $\square$

Claim 11.7.13. *The hypergraph G does not have Borel σ-bounded fractional chromatic number.*

Proof. For a point $x \in X$ and a number $m \in \omega$, write $a_{xm} = \{\delta^{im}\gamma^{jm} \cdot x \colon i, j \in m\}$. Note that the set $a_{xm} \subset X$ is a subset of a single E_0-class and visits n many distinct E_1-classes, each in n many elements. A simple counting argument then shows that every subset of a_{xm} of size $m + 1$ contains a G-hyperedge, and so the fractional chromatic number of G on the set a_{xm} is not smaller than m as witnessed by the normalized counting measure on a_{xm}.

Now, suppose that $X = \bigcup_n B_n$ is a partition into Borel sets. By the Baire category theorem, there is a number $n \in \omega$ such that B_n is not meager. In view of the first paragraph, to prove the claim it will be enough that for all but finitely many $m \in \omega$ there is a point $x \in X$ such that $a_{xm} \subset B_n$. To this end, let $t \colon \Gamma \to 2$ be a finite partial function such that B_n is comeager in $[t]$. Let $m \in \omega$ be larger than the length of any word in t; we will find a point $x \in X$ such that $a_{xm} \subset B_n$. Just let $s = \bigcup\{t_{ij} \colon i, j \in m\}$ where $t_{ij} \colon \Gamma \to 2$ is a finite partial function given by the demand $t_{ij}(\beta) = t(\delta^{im}\gamma^{jm}\beta)$. By the choice of the number m, $s \colon \Gamma \to 2$ is a finite function. The set $C = \bigcap\{\gamma^{-jm}\delta^{-im}B_n \colon i, j \in m\}$ is comeager in $[s]$; let $x \in C$ be an arbitrary point. Reviewing the definitions, it is clear that $a_{xm} \subset B$ as required. □

Now, consider the poset P adding an orientation of H in which every vertex in X gets an out-degree one. A condition $p \in P$ is an orientation on countably many components of H in which every vertex in these components gets an out-degree one. The ordering is that of reverse inclusion. The poset P was analyzed in Example 9.2.15 It is not difficult to prove that the balanced conditions of P are classified by orientations of H in which every vertex gets an out-degree one, which naturally form a compact Hausdorff space. Thus, the poset P is compactly balanced, and it adds an H-coloring with countably many colors by Claim 11.7.12 while the hypergraph H does not have Borel σ-bounded chomatic or even fractional chromatic number by Claim 11.7.12. □

Chapter 12

The Silver divide

12.1 Perfectly balanced forcing

The perfect set property in the model $L(\mathbb{R})[U]$, where U is a Ramsey ultrafilter, was one of the first results in the literature about models of the type studied in this book [21]. In this section, we provide a general machinery for proving the perfect set property type of results, with much less effort than the original argument quoted above. We start with two key definitions.

Definition 12.1.1. Let P be a Suslin poset. A virtual condition $\bar{p}$ is *perfectly balanced* if in every generic extension $V[G]$, whenever

1. $Q \in V$ is a poset such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$;
2. $\sigma \in V$ is a Q-name for a condition in P stronger than $\bar{p}$;
3. $\mathcal{H} \subset \mathcal{P}Q$ is a perfect set such that every finite set $a \subset \mathcal{H}$ is a set of filters over Q mutually generic over V,

then there is a perfect set $C \subset \mathcal{H}$ such that the set of conditions $\{\sigma/H \colon H \in C\}$ has a lower bound in the separative quotient. A poset is *perfectly balanced* if below every condition $p \in P$ there is a virtual perfectly balanced condition.

This forcing property is often guaranteed by a simple feature which does not speak about any balance issues at all.

Definition 12.1.2. A Suslin forcing P is *perfect* if for every Borel function $f \colon 2^\omega \to P$, either there is a finite set $a \subset 2^\omega$ such that the set $f''a \subset P$ has no lower bound, or else there is a perfect set $C \subset 2^\omega$ such that $f''C$ has a lower bound in the separative quotient.

In general, the separative quotient of Suslin forcings is a $\mathbf{\Pi}^1_2$ ordering, making perfectness a rather complicated projective property of the poset P. In all particular posets considered in this book, the status of perfectness is absolute throughout all forcing extensions.

Proposition 12.1.3. *Every Suslin forcing which is perfect in all forcing extensions and balanced is perfectly balanced and every balanced virtual condition is perfectly balanced.*

Proof. Let $\bar{p}$ be a balanced virtual condition. Let Q be a partial order, σ a Q-name for a condition in P stronger than $\bar{p}$. Let $V[G]$ be a generic extension such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$. Let $\mathcal{H}$ be a perfect set of filters on Q which consists of filters in finite tuples mutually generic over V.

Claim 12.1.4. *For every finite set $a \subset \mathcal{H}$, the set $\{\sigma/H \colon H \in a\} \subset P$ has a common lower bound.*

Proof. Let $\langle H_j \colon j \in i\rangle$ enumerate the set a without repetitions. By induction on $j \in i$, construct a descending sequence $\langle r_j \colon j \in i\rangle$ of conditions in P such that for each $j \in i$ $r_j \in V[K][H_k \colon k \in j]$ and $r_{j+1} \leq \sigma/H_j$. This is easy to do using the balance of the condition $\bar{p}$ at every stage of the induction, noting that $\sigma/H_j, r_j$ are both strengthenings of $\bar{p}$ in mutually generic extensions of the model $V[K]$. In the end, the condition r_i is a lower bound of the set $\{\sigma/H \colon H \in a\}$. ☐

Now, let $h \colon 2^\omega \to \mathcal{H}$ be a continuous injection and let $f \colon 2^\omega \to P$ be the continuous function defined by $f(y) = \sigma/h(y)$. Apply the perfectness of the poset P to find a perfect set C such that for each $y \in C$, $r \leq f(y)$ holds in the separative quotient. The condition r witnesses the perfect balance of $\bar{p}$ in this instance. ☐

We prove two notable dichotomy type preservation properties of perfectly balanced posets.

Definition 12.1.5. The *full Silver dichotomy* is the following statement: If E is a Borel equivalence relation on a Polish space X and $A \subset X$ is an E-invariant set, then either A contains only countably many E-classes or A contains a perfect set consisting of pairwise E-unrelated elements.

As a consequence, the full Silver dichotomy implies that among all sets whose cardinality is smaller than a Borel quotient space, 2^ω is the set with the smallest uncountable cardinality. The terminology refers to the classical Silver dichotomy [48, Theorem 5.7.1], a theorem of ZF+DC which says in particular that the dichotomy holds for analytic sets $A \subset X$. The Solovay model satisfies the full Silver dichotomy. The following result is stated using Convention 1.7.16.

Theorem 12.1.6. *In the cofinally perfectly balanced extensions of the symmetric Solovay model, the full Silver dichotomy holds.*

Proof. Let κ be an inaccessible cardinal. Let P be a perfect Suslin forcing such that $V_\kappa \models P$ is balanced in every forcing extension. Let W be a symmetric Solovay model derived from κ and work in the model W. Let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau \subset X$ is an E-invariant set containing uncountably many E-classes. The condition p as well as the name τ have to be definable from parameters in V as well as some parameter $z \in 2^\omega$.

Use the assumptions to find an intermediate model $V[K]$ obtained as a generic extension of V by a poset of cardinality smaller than κ such that $z \in V[K]$ and P is perfectly balanced in $V[K]$.

Work in the model $V[K]$. Find a perfectly balanced virtual condition $\bar{p} \leq p$. Note that the equivalence relation E is Borel and therefore it has fewer than $\beth_{\omega_1}^{V[K]}$ many virtual classes. The cardinality $\beth_{\omega_1}^{V[K]}$ is countable in the model W, while the set τ is forced to contain uncountably many E-classes. It follows that in the model $V[K]$ there has to be a poset R of cardinality smaller than κ, an R-name σ for a condition in P stronger than $\bar{p}$ and an R-name η for an element of X which is forced not to be a realization of any virtual E-class in the model $V[K]$, and $R \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \eta \in \tau$.

Move to some generic extension $V[K][G]$ obtained by a poset of cardinality smaller than κ such that $\mathcal{P}(R) \cap V[K]$ is countable in $V[K][G]$. Work in $V[K][G]$. Use Proposition 1.7.10 to find a perfect set $\mathcal{H}$ of filters on R which are in finite tuples mutually generic over $V[K]$. Note that by the mutual genericity and the initial choice of the name η, the points η/H for $H \in \mathcal{H}$ are pairwise E-unrelated. By the perfect balance of the virtual condition $\bar{p}$, there is a perfect set $C \subset \mathcal{H}$ such that the set $\{\sigma/H \colon H \in C\}$ has a lower bound, say $q \in P$ in the separative quotient of P. By the forcing theorem applied in every model $V[K][H]$ for $H \in C$ it is the case that in the model W, the condition q forces the perfect set $\{\eta/H \colon H \in C\}$ consisting of pairwise E-unrelated elements to be a subset of τ. The proof is complete. □

The second preservation theorem of this section deals with a strong form of the well-known Open Coloring Axiom, OCA [91].

Definition 12.1.7. OCA+ is the following statement. Whenever X is a Polish space, $A \subset X$ is a set, and Γ is a graph on A which is open in the topology on $A \times A$ inherited from $X \times X$, then either A is a union of countably many Γ-anticliques, or A contains a perfect Γ-clique.

The following result is stated using Convention 1.7.16.

Theorem 12.1.8. *In every cofinally perfectly balanced extension of the symmetric Solovay model, OCA+ holds.*

Proof. Let X be a Polish space and let P be a perfect Suslin forcing. Let κ be an inaccessible cardinal such that in V_κ, in every forcing extension the poset P is balanced. Let W be a symmetric Solovay model derived from κ. In the model W, let $\Gamma \subset X^2$ be a symmetric open set, let $p \in P$ be a condition, and let τ be a P-name for a subset of X such that $p \Vdash \tau$ cannot be covered by countably many Γ-anticliques. We must find a perfect set $B \subset X$ such that any two points of B are Γ-related, and a condition $q \leq p$ in P which forces $\check{B} \subset \tau$.

To this end, choose a parameter $z \in 2^\omega$ such that p, τ, Γ are definable from z. Find an intermediate generic extension $V[K]$ of V by a poset of cardinality smaller than κ such that $z \in V[K]$ and P is perfectly balanced in $V[K]$. Work in the model $V[K]$. Let $\bar{p} \leq p$ be a perfectly balanced virtual condition in P.

Since $\mathrm{Coll}(\omega, <\kappa) \Vdash p \Vdash_P \tau$ is not covered by countably many Γ-anticliques, and a closure of a Γ-anticlique is still a Γ-anticlique, there must be a poset R of cardinality smaller than κ, an R-name η for an element of X which belongs to no closed Γ-anticlique coded in $V[K]$, and an R-name σ for a condition in P stronger than $\bar{p}$ such that $R \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \eta \in \tau$.

Let $V[K][G]$ be a generic extension obtained by a poset of cardinality smaller than κ such that $\mathcal{P}(R) \cap V[K]$ is countable in $V[K][G]$. Work in $V[K][G]$. Let $\{D_n : n \in \omega\}$ enumerate all open dense subsets of finite powers of the poset R in $V[K]$, with infinite repetitions. As in the proof of Theorem 12.2.5, by induction on $|t|$ build conditions $r_t \in R$ for $t \in 2^{<\omega}$ so that

- $t \subset s$ implies $r_s \leq r_t$;
- whenever $D_n \subset R^m$ is an open dense set for some $m < 2^n$, then every m-tuple of distinct elements from the set $\{r_t : t \in 2^n\}$ belongs to D_n;
- for all $t \in 2^{<\omega}$ there are open sets $O_{t0}, O_{t1} \subset X$ such that $O_{t0} \times O_{t1} \subset \Gamma$ and $r_{t^\frown 0} \Vdash \eta \in O_{t0}$ and $r_{t^\frown 1} \Vdash \eta \in O_{t1}$.

In the end, for every binary sequence $y \in 2^\omega$ let $H_y \subset R$ be the filter generated by the conditions $\{r_{y \restriction n} : n \in \omega\}$. Note that any finite tuple of distinct filters H_y for $y \in 2^\omega$ is mutually generic over the model $V[K]$ by the second item in the inductive construction above. The mutual genericity also shows that the function $y \mapsto \eta/H_y$ is a continuous injection from 2^ω to X, and its range is a Γ-clique by the third item above. By the perfect balance of the condition $\bar{p}$, there is a perfect set $C \subset 2^\omega$ such that the conditions $\{\sigma/H_y : y \in C\}$ have a lower bound q in the separative quotient of the poset P. By the forcing theorem applied in every model $V[K][H]$ for $H \in C$ it is the case that in the model W, the condition q forces the perfect Γ-clique $\{\eta/H : H \in C\}$ to be a subset of τ. The proof is complete. □

Now it is time for a list of perfect and perfectly balanced forcings.

Example 12.1.9. Let I be an F_σ-ideal on ω. The poset P of all I-positive subsets of ω ordered by inclusion is perfect.

As a special case, this includes the poset of infinite subsets of ω ordered by inclusion of Section 7.1 and the posets of Section 7.3, both adding ultrafilters on ω with various Ramsey properties.

Proof. Recall that $P(I)$ is the poset of all I-positive subsets of ω, ordered by inclusion. Write $I = \bigcup_n I_n$ as a countable union of closed sets, each of which is closed under taking subset. Let $f : 2^\omega \to \mathcal{P}(\omega)$ be a Borel function such that for any finite set $a \subset 2^\omega$, $\bigcap f''a \notin I$ holds; we must find a perfect set $B \subset 2^\omega$ such that the set $f''B \subset P$ has a lower bound. Thinning the domain of f if necessary we may assume that the function f is in fact continuous. By induction on $n \in \omega$ build nodes $u_t \in 2^{<\omega}$ for all $t \in 2^n$ and finite sets $b_n \subset \omega$ such that

- $b_n \notin I_n$;

- $s \subset t$ implies that $u_s \subset u_t$ and s is incompatible with t implies u_s is incompatible with u_t;
- for each $t \in 2^{n+1}$ and every $y \in [u_t]$ it is the case that $b_n \subset f(y)$.

Once the induction is performed, let $b = \bigcup_n b_n$, let $B \subset 2^\omega$ be the perfect set of all points $y \in 2^\omega$ such that $\forall n \exists t \in 2^n\ u_t \subset y$ and use the continuity of the function f to prove that b is the lower bound of the set $f''B$.

To start the induction, let $u_0 = 0$. Now suppose that the nodes $u_t \in 2^{<\omega}$ for $t \in 2^n$ as well as sets b_m for $m \in n$ have been constructed. For each $t \in 2^n$ choose distinct points $y_{t0}, y_{t1} \in [u_t]$ and use the initial assumption on the function f to observe that $c = \bigcap_{t \in 2^n} f(y_{t0}) \cap \bigcap_{t \in 2^n} f(y_{t1})$ is an I-positive set. Since the set $I_n \subset \mathcal{P}(\omega)$ is closed, there must be a finite subset $b_n \subset c$ which is not in I_n. Use the continuity of the function f to find initial segments $u_{t^\frown 0} \subset y_{t0}$ and $u_{t^\frown 1} \subset y_{t1}$ satisfying the second item of the induction hypothesis. This concludes the inductive step. □

Corollary 12.1.10. [21] *Let P be the poset of infinite subsets of ω ordered by inclusion.*

1. *In the P-extension of the symmetric Solovay model, the full Silver dichotomy holds, and OCA+ holds.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Ramsey ultrafilter on ω, the full Silver dichotomy holds, and OCA+ holds.*

Example 12.1.11. Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup. The poset $P = P(\Gamma)$ of Subsection 7.4 is perfect.

Proof. Recall that elements of P are just sequences in Γ^ω and the ordering is defined by $q \leq p$ if there are pairwise disjoint nonempty finite sets $a_n \subset \omega$ such that $q(n) = \prod_{m \in a_n} p(m)$. The proof of the perfect property of the poset P is another fusion argument. Let $f\colon 2^\omega \to P$ be a Borel function such that for every finite set $b \subset 2^\omega$, the conditions in the set $f''b$ have a common lower bound in P; we must find a perfect set $B \subset 2^\omega$ such that the set $f''B \subset P$ has a lower bound.

Thinning the domain of f if necessary we may assume that the function f is continuous. By induction on $n \in \omega$ build nodes $u_t \in 2^{<\omega}$ for all $t \in 2^n$, finite sets $a_t \subset \omega$ and elements γ_n so that

- $\min(a_{n+1}) > \max(a_n)$;
- $s \subset t$ implies that $u_s \subset u_t$ and s is incompatible with t implies u_s is incompatible with u_t;
- $a_0 = 0$ and if $s \subset t$ then $a_s \cap a_t = 0$;
- for each $t \in 2^{n+1}$ and each $y \in [u_t]$, $\gamma_n = \prod_{m \in a_t} f(y)(m)$.

Once the induction is performed, let $p = \langle \gamma_n : n \in \omega \rangle$, let $B \subset 2^\omega$ be the perfect set of all points $y \in 2^\omega$ such that $\forall n\ \exists t \in 2^n\ u_t \subset y$ and observe that p is the lower bound of the set $f''B$.

To start the induction, let $u_0 = 0$ and $a_0 = 0$. Now suppose that the nodes $u_t \in 2^{<\omega}$ for $t \in 2^n$ as well as sets a_s for $s \in 2^{\leq n}$ and elements γ_m for $m \in n$ have been constructed. For each $t \in 2^n$ choose distinct points $y_{t0}, y_{t1} \in [u_t]$ and use the initial assumption on the function f to observe that the set $c = \{f(y_{t0}), f(y_{t1}) : t \in 2^n\}$ has a lower bound in the poset P. This means that there are nonempty finite sets $a_{t^\frown 0}, a_{t^\frown 1} \subset \omega$ disjoint from $\bigcup_{s \in 2^{\leq n}} a_s$ and a semigroup element $\gamma_n \in \Gamma$ such that $\prod_{m \in a_{t^\frown 0}} f(y_{t0})(m) = \gamma_n$ and $\prod_{m \in a_{t^\frown 1}} f(y_{t1})(m) = \gamma_n$ holds for all $t \in 2^n$. Use the continuity of the function f to find initial segments $u_{t^\frown 0} \subset y_{t0}$ and $u_{t^\frown 1} \subset y_{t1}$ such that $f(y_{t0})(m) = f(y)(m)$ holds for all $m \in a_{t0}$ and all $y \in [u_{t^\frown 0}]$ and $f(y_{t1})(m) = f(y)(m)$ holds for all $m \in a_{t1}$ and all $y \in [u_{t^\frown 1}]$. This concludes the inductive step. $\square$

Corollary 12.1.12. *Let P be the poset of Subsection 7.4 designed to add a stable ordered union ultrafilter.*

1. *In the P-extension of the symmetric Solovay model, the full Silver dichotomy holds, and OCA+ holds.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a stable ordered union ultrafilter on ω, the full Silver dichotomy holds, and OCA+ holds.*

Example 12.1.13. The Fin$\times$Fin poset P of subsets of $\omega \times \omega$ with infinitely many infinite vertical sections, ordered by inclusion, is perfectly balanced but not perfect.

Proof. We first prove the failure of perfectness. For each $n \in \omega$ and each binary string $t \in 2^n$, it is easy to find a perfect set $A_t \subset \mathcal{P}(\omega)$ such that

- A_t consists of pairwise almost disjoint infinite sets;
- whenever $\{x_t : t \in 2^n\}$ is a collection such that $x_t \in A_t$, the set $\bigcap_t x_t \subset \omega$ is infinite.

Then, let $f_t : [t] \to \mathcal{P}(\omega)$ be a continuous injection into A_t for every $t \in 2^{<\omega}$. Now, for every $y \in 2^\omega$ let $g(y) \subset \omega \times \omega$ be the set of all pairs $\langle n, m \rangle$ such that $m \in f_{y \restriction n}(y)$. It is clear that the set $g(y) \subset \omega \times \omega$ has all vertical sections infinite and so is a condition in P. We will show that any finite subset of $g''2^\omega$ has a lower bound in P, yet no condition in P is below more than countably many elements of $g''2^\omega$.

For the first assertion, let $a \subset 2^\omega$ be a finite set. Let $m \in \omega$ be such that the strings $y \restriction m$ for $y \in a$ are pairwise distinct. Then, for all $n \geq m$ the second item implies that the set $\bigcap_{y \in a} f_{y \restriction n}(y)$ is infinite. In other words, the set $\bigcap_{y \in a} g(y)$ has all vertical sections beyond m infinite, and so is a lower bound for $g''a$. For the second assertion, suppose that $p \in P$ is a condition, and assume towards a contradiction that the set $\{y \in 2^\omega : p \leq g(y)\}$ is uncountable. By a counting argument, there must be a number $n \in \omega$ such that the set $\{y \in 2^\omega : p_n$ is infinite and modulo finite a subset of $g(y)_n\}$ is uncountable. Find two distinct elements y_0, y_1 of the latter set such that $y_0 \restriction n = y_1 \restriction n$; denote the common value by t. Then, the intersection $f_t(y_0) \cap f_t(y_1)$ should be finite by the choice of the function f_t, and at the same time it should modulo finite contain the infinite set p_n. This is a contradiction.

Now it is time to show that P is perfectly balanced. Let $p \in P$ be a condition. Let $a \subset \omega$ be an infinite set such that for each $n \in a$, the vertical section a_n is infinite. Let U be a nonprincipal ultrafilter on ω containing a, and for each $n \in \omega$ let U_n be a nonprincipal ultrafilter on ω such that if $n \in a$ then $a_n \in U_n$. Let $\bar{p}$ be the virtual condition on a collapse poset, standing for the analytic set A of all sets $b \subset a$ such that the set $c = \{n \in \omega : b_n \text{ is infinite}\}$ diagonalizes U, and for each $n \in c$, the set b_n diagonalizes U_n. The condition $\bar{p} \leq p$ is balanced; we will show that it is perfectly balanced.

Let Q be a poset and let σ be a Q-name for a condition in the set A. Let $V[G]$ be a generic extension such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$ and $\mathcal{H}$ is a perfect set of filters on Q in finite tuples mutually generic over V. We must find a condition $r \in P$ in the model $V[G]$ such that the set $\{H \in \mathcal{H} : r \leq \sigma/H\}$ is uncountable. ??? $\square$

12.2 Bernstein balanced forcing

There is a class of posets which is in a precise sense dual to the class of perfectly balanced forcings.

Definition 12.2.1. Let P be a Suslin poset. A virtual condition $\bar{p}$ is *Bernstein balanced* if in every generic extension $V[G]$, for every condition $p \leq \bar{p}$, every infinite poset $Q \in V$ such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$, and every perfect family $\mathcal{H} \subset \mathcal{P}(Q)$ such that each finite set $a \subset \mathcal{H}$ is a collection of filters mutually generic over V, there is a filter $H \in \mathcal{H}$ such that every condition in $P \cap V[H]$ below $\bar{p}$ is compatible with p. The poset P is *Bernstein balanced* if there is a Bernstein balanced virtual condition below every condition in P.

As the simplest initial example, consider the poset of countable partial functions from 2^ω to 2 ordered by reverse inclusion. The balanced conditions are classified by total functions from 2^ω to 2. Each such virtual condition $\bar{p}$ is in fact Bernstein balanced. To see this, note that any condition $p \leq \bar{p}$ in any extension $V[G]$ has countable domain. On the other hand, if $\mathcal{H} \in V[G]$ is a perfect set of filters mutually generic over V, then by the product forcing theorem, for each point $x \in \mathrm{dom}(p) \setminus V$ there is at most one filter $H \in \mathcal{H}$ such that $x \in V[H]$. By a counting argument then, there is a filter $H \in \mathcal{H}$ such that $\mathrm{dom}(p) \cap V[H] \setminus V = 0$ and then p is compatible with every condition in $V[H]$ which is stronger than $\bar{p}$.

Bernstein balanced extensions of the Solovay model share a number of regularity properties. The main technical tool used in all the theorems below is the following.

Proposition 12.2.2. *Let P be a Suslin poset and let $\bar{p}$ be a Bernstein balanced condition in P. Let Q be a partial order and σ a Q-name for a condition in P stronger than $\bar{p}$. Let W be a Solovay model derived from an inaccessible cardinal greater than κ. In the model W, if $\mathcal{H} \subset \mathcal{P}(Q)$ is a perfect family consisting of filters mutually generic over V, then $\bar{p}$ forces in P that the set $\{H \in \mathcal{H} \colon \sigma/H$ belongs to the P-generic filter$\}$ is uncountable.*

Proof. Work in the model W. Let $p \in P$ be a condition stronger than $\bar{p}$, and $a \subset \mathcal{H}$ be a countable set. We must find a filter $H \in \mathcal{H} \setminus a$ such that σ/H is compatible with p. Let $V[K]$ be an intermediate extension obtained by a poset of cardinality smaller than κ containing a code for $\mathcal{H}$, and enumeration of the set a, and the condition Q. Work in the model $V[K]$. The set $\mathcal{H} \setminus a$ is Borel and uncountable, and therefore contains a perfect subset. By the Bernstein balance of the virtual condition $\bar{p}$, there is a filter $H \in \mathcal{H} \setminus a$ such that p is compatible with all conditions in $P \cap V[H]$ stronger than $\bar{p}$, in particular with the condition σ/H. This concludes the proof. ☐

With the proposition in hand, we begin a line of preservation results. They are all stated using Convention 1.7.16.

Theorem 12.2.3. *In cofinally Bernstein balanced extensions of the symmetric Solovay model, there is no finitely additive diffuse probability measure on ω.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is Bernstein balanced cofinally below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Suppose towards a contradiction that $p \in P$ is a condition, τ is a P-name for a map from $\mathcal{P}(\omega)$ to $[0, 1]$, and p forces τ to be a finitely additive diffuse probability measure. The condition p and the name τ are definable from ground model parameters and an additional parameter $z \in 2^\omega$. Find an intermediate model $V[K]$ obtained using a poset of cardinality smaller than κ such that $z \in V[K]$ and P is Bernstein balanced in $V[K]$. Work in the model $V[K]$.

Let $\bar{p} \leq p$ be a Bernstein balanced virtual condition. Let Q be the poset of finite binary strings ordered by reverse end-extensions, with its name $\dot{x}$ for the set of those $n \in \omega$ for which there is a condition $q \in Q$ in the generic filter such that $q(n) = 1$. There must be a poset R of cardinality smaller than κ, a condition $\langle q, r\rangle \in P \times R$, and a $Q \times R$-name σ for a condition in P stronger than $\bar{p}$ such that either $\langle q, r\rangle \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau(\dot{x}) \geq 1/2$ or $\langle q, r\rangle \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \tau(\dot{x}) \leq 1/2$. For definiteness, assume that the former option prevails; with the latter option, replace $\dot{x}$ with the name for its complement and proceed in the same way.

Move to an extension $V[K][G]$ obtained with some poset of cardinality smaller than κ such that $\mathcal{P}(Q \times R) \cap V[K]$ is countable in $V[K][G]$, and work in the model $V[K][G]$. Let $\langle O_m \colon m \in \omega\rangle$ be an enumeration of all open dense subsets of all finite powers of $Q \times R$ that appear in the model $V[K]$, with infinite repetitions. By recursion on $m \in \omega$ build numbers $n_m \in \omega$ and conditions

$\langle q_{it}, r_{it}\rangle \leq \langle q, r\rangle$ for $i \in 3$ and $t \in 2^m$ so that $\mathrm{dom}(q_{it}) = n_m$ for all $i \in 3$ and $t \in 2^m$, and:

- if $i \in 3$ and $t \subset s$ are binary strings, then $\langle q_{is}, r_{is}\rangle \leq \langle q_{it}, r_{it}\rangle$;
- if O_m is an open dense set in the k-fold product of $Q \times R$ and k is smaller than m, then for every $i \in 3$ and every k-tuple $\langle t_l : l \in k$ of distinct elements of 2^{m+1}, the tuple $\langle\langle q_{it_l}, r_{it_l}\rangle : l \in k\rangle$ belongs to O_m;
- whenever i, j are distinct numbers in 3, $s, t \in 2^m$ and $k \in n_m \setminus \mathrm{dom}(q)$, then $q_{is}(k)$ and $q_{jt}(k)$ are not both simultaneously equal to 1.

To perform the recursion, start with $\langle q_{i0}, r_{i0}\rangle = \langle q, r\rangle$ for all $i \in 3$. Now, suppose that $n_m \in \omega$ and conditions $\langle q_{it}, r_{it}\rangle \leq \langle q, r\rangle$ for $i \in 3$ and $t \in 2^m$ have been found. First, work on subscript 0: for all $t \in 2^{m+1}$ find conditions $\langle q^0_{0t}, r_{0t}\rangle \in Q \times R$ so that the first two items are satisfied and let $n_{0m} = \max_t \mathrm{dom}(q^0_{0t})$. Then, work on subscript 1: for all $t \in 2^{m+1}$ find conditions $\langle q^0_{1t}, r_{1t}\rangle \in Q \times R$ so that the first two items are satisfied, for each t and each $j \in [n_m, n_{0m})$ $q^0_{1t}(j) = 0$ holds, and let $n_{1m} = \max_t \mathrm{dom}(q^0_{1t})$. Finally, work on subscript 2: for all $t \in 2^{m+1}$ find conditions $\langle q^0_{2t}, r_{2t}\rangle \in Q \times R$ so that the first two items are satisfied, for each t and each $j \in [n_m, n_{1m})$ $q^0_{2t}(j) = 0$ holds, and let $n_{2m} = \max_t \mathrm{dom}(q^0_{2t})$. To conclude the work, extend the binary strings q^0_{0t}, q^0_{1t}, and q^0_{2t} with zeroes only so that the resulting binary strings q_{0t}, q_{1t}, q_{2t} have the same domain n_{m+1}.

In the end, for each $i \in 3$ and each $y \in 2^\omega$ let $H_{iy} \subset Q \times R$ be the filter generated by the conditions $\langle q_{iy\restriction m}, r_{iy\restriction m}\rangle$ for $m \in \omega$. The first and second items above show that the family $\mathcal{H}_i = \{H_{iy} : y \in 2^\omega\}$ consists of filters on $Q \times R$ in finite tuples mutually generic over the model $V[K]$. The third item above shows that for distinct $i, j \in 3$ and points $y_0, y_1 \in 2^\omega$, the intersection $\dot{x}/H_{iy_0} \cap \dot{x}/H_{jy_1}$ is finite.

Now, a twice repeated use of the Bernstein balance of the virtual condition $\bar{p}$ in $V[K]$ with the families $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ yields points $y_0, y_1, y_2 \in 2^\omega$ such that the conditions $p_0 = \sigma/H_{0y_0}$, $p_1 = \sigma/H_{1y_1}$, and $p_2 = \sigma/H_{2y_2}$ have a common lower bound. Write $x_0 = \dot{x}/H_{0y_0}$, $x_1 = \dot{x}/H_{1y_1}$, and $x_2 = H_{2y_2}$. By the previous paragraph, the sets $x_0, x_1, x_2 \subset \omega$ are pairwise almost disjoint. At the same time, by the forcing theorem in $V[K]$, the common lower bound of the conditions p_0, p_1, p_2 forces the numbers $\tau(\check{x}_0)$, $\tau(\check{x}_1)$, and $\tau(\check{x}_2)$ to be at least 1/2 each. This contradicts the assumption that τ was forced to be a diffuse finitely additive measure. $\square$

The next preservation theorem deals with the well-known Open Coloring Axiom, OCA [91].

Definition 12.2.4. OCA is the following statement. Whenever X is a Polish space, $A \subset X$ is a set, and Γ is a graph on A which is open in the topology on $A \times A$ inherited from $X \times X$, then either A is a union of countably many Γ-anticliques, or A contains an uncountable Γ-clique.

Theorem 12.2.5. *In cofinally Bernstein balanced extensions of the symmetric Solovay model, OCA holds.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin forcing such that P is Bernstein balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ. In the model W, let X be a Polish space and let $\Gamma \subset X^2$ be a symmetric open set, let $p \in P$ be a condition, and let τ be a P-name for a subset of X such that $p \Vdash \tau$ cannot be covered by countably many Γ-anticliques. We will find a perfect set $C \subset X$ such that any two points of C are Γ-related, and a condition $q \leq p$ in P which forces $\check{C} \cap \tau$ is uncountable.

To this end, choose a parameter $z \in 2^\omega$ such that p, τ, Γ are definable from z. Find an intermediate generic extension $V[K]$ of V by a poset of size $< \kappa$ such that $z \in V[K]$ and P is Bernstein balanced in $V[K]$. Work in the model $V[K]$. Let $\bar{p} \leq p$ be a Bernstein balanced virtual condition. Since $\mathrm{Coll}(\omega, < \kappa) \Vdash p \Vdash_P \tau$ is not covered by countably many Γ-anticliques, and a closure of a Γ-anticlique is still a Γ-anticlique, there must be a poset R of size $< \kappa$, and R-name η for an element of X which belongs to no closed Γ-anticlique coded in $V[K]$, and an R-name σ for a condition in P stronger than $\bar{p}$ such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \eta \in \tau$.

Move into the model W. Let $\{D_n : n \in \omega\}$ enumerate all open dense subsets of finite powers of the poset R in $V[K]$, with infinite repetitions. By induction on $|t|$ build conditions $r_t \in R$ for $t \in 2^{<\omega}$ so that

- $t \subset s$ implies $r_s \leq r_t$;
- whenever $D_n \subset R^m$ is an open dense set for some $m < 2^n$, then every m-tuple of distinct elements from the set $\{r_t : t \in 2^n\}$ belongs to D_n;
- for all $t \in 2^{<\omega}$ there are open sets $O_{t0}, O_{t1} \subset X$ such that $O_{t0} \times O_{t1} \subset \Gamma$ and $r_{t^\frown 0} \Vdash \eta \in O_{t0}$ and $r_{t^\frown 1} \Vdash \eta \in O_{t1}$.

The construction is routine except for the last item; we just describe how the last item is obtained. Suppose that r_t has been found. Work in $V[K]$ and let $U \subset X$ be the union of all basic open sets $O \subset X$ such that $r_t \Vdash \eta \notin O$. Then $r_t \Vdash \eta \in X \setminus U$. Since the set $X \setminus U$ is closed, by the choice of the name η it cannot be a Γ-anticlique and therefore there are points $x_0, x_1 \in X \setminus U$ which are Γ-connected. Since the set $\Gamma \subset X^2$ is open, there are basic open sets $O_{t0}, O_{t1} \subset X$ such that $x_0 \in O_{t0}$, $x_1 \in O_{t1}$, and $O_{t0} \times O_{t1} \subset \Gamma$. Neither of the two open sets is a subset of U and therefore there must be conditions $r_{t^\frown 0}$ and $r_{t^\frown 1}$ below r_t such that the former forces $\eta \in O_{t0}$ and the latter forces $\eta \in O_{t1}$ as desired.

In the end, for every binary sequence $y \in 2^\omega$ let $H_y \subset R$ be the filter generated by the conditions $\{r_{y \restriction n} : n \in \omega\}$ and let $p_y = \sigma / H_y$. Note that any finite tuple of distinct filters H_y for $y \in 2^\omega$ is mutually generic over the model $V[K]$ by the second item in the inductive construction above. Proposition 12.2.2 applied in $V[K]$ shows that in W, $\bar{p}$ forces the set $a = \{y \in 2^\omega : p_y$ is in the generic filter$\}$ to be uncountable. The set $\{\eta / H_y : y \in a\}$ is then forced to be an uncountable Γ-clique. $\square$

Theorem 12.2.6. *Let E be a Borel equivalence relation on a Polish space X. In cofinally Bernstein balanced extensions of the symmetric Solovay model, every subset of X is either covered by countably many E-classes or contains an uncountable subset consisting of pairwise E-unrelated elements.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin forcing which is Bernstein balanced cofinally below κ. Let W be the symmetric Solovay model derived from κ, let $p \in P$ be a condition and let τ be a P-name such that $p \Vdash \tau \subset X$ is a set which is not covered by countably many E-classes. We will find a perfect set $A \subset X$ consisting of pairwise E-unrelated elements and a condition $q \leq p$, $q \Vdash \tau \cap \check{A}$ is uncountable. This will prove the theorem.

The condition p and the name τ are definable from a parameter $z \in 2^\omega$ and some parameters in the ground model. Let $V[K]$ be some intermediate extension of the ground model by a poset of size $< \kappa$ containing z and such that P is Bernstein balanced in $V[K]$; work in $V[K]$. Let $\bar{p} \leq p$ be a Bernstein balanced condition in the poset P. Since the equivalence relation E is Borel, there are fewer than $\beth_{\omega_1}$ many virtual E-classes in $V[K]$. Since the cardinal $\beth_{\omega_1}$ is countable in W, there must be in $V[K]$ a poset R of size $< \kappa$ and R-names σ for a condition in P stronger than $\bar{p}$ and η for an element of X which is not a realization of any virtual E-class in $V[K]$ such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \eta \in \tau$.

In the model W, use Proposition 1.7.10 to find a perfect set $\{H_y : y \in 2^\omega\}$ of filters on R which are mutually generic over the model $V[K]$. Write $p_y = \sigma/H_y$ and $x_y = \eta/H_y$. By the assumption on the name η and the mutual genericity, the set $A = \{\eta/H_y : y \in 2^\omega\}$ consists of pairwise E-unrelated elements. Proposition 12.2.2 applied in $V[K]$ shows that in W the condition $\bar{p}$ forces that for uncountably many $y \in 2^\omega$, p_y belongs to the generic filter. Then $\bar{p} \Vdash \tau \cap \check{A}$ is uncountable as required. □

Theorem 12.2.7. *Let I be an analytic P-ideal on ω. In cofinally Bernstein balanced extensions of the symmetric Solovay model, if $A \subset I$ is an uncountable set, then there is a set $b \in I$ such that the set $\{a \in A : a \subset b\}$ is uncountable.*

Proof. Suppose that κ is an inaccessible cardinal and P is a Suslin poset which is Bernstein balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ and in the model W, let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau \subset I$ is an uncountable set. We need to produce a condition $\bar{p} \leq p$ and a set $b \in I$ such that $q \Vdash \{a \in \tau : a \subset \check{b}\}$ is uncountable. The condition p and the name τ must be definable from parameters in the ground model and some real parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate forcing extension by a poset of size $< \kappa$ such that $z \in V[K]$ and P is Bernstein balanced in $V[K]$, and work in the model $V[K]$.

By the assumptions on the poset P, there must be a Bernstein balanced virtual condition $\bar{p} \leq p$ in the model $V[K]$. There must be a poset Q of size $< \kappa$, a Q-name η for an element of I which is not in $V[K]$, and a Q-name σ for a condition in P stronger than $\bar{p}$ such that $Q \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \eta \in \tau$. Let $V[K][G]$ be some generic extension by a poset of cardinality less than κ in which there is a perfect collection $\mathcal{H} \subset \mathcal{P}(Q)$ which consists of filters mutually generic over $V[K]$. Work in the model $V[K][G]$.

Use the result of Solecki [85] to fix a lower-semicontinuous submeasure μ on ω such that $I = \{a \subset \omega \colon \lim_n \mu(a \setminus n) = 0\}$. Consider the poset R of triples $\langle s, \varepsilon, a\rangle$ where $s \subset \omega$ is finite, $\varepsilon > 0$ is a real number, and $a \in I$ is a set such that $\mu(a \setminus s) < \varepsilon_q$. The ordering is defined by $\langle s_1, \varepsilon_1, a_1\rangle \leq \langle s_0, \varepsilon_0, a_0\rangle$ if $s_0 \subset s_1$, $\varepsilon_1 \leq \varepsilon_0$, and $a_0 \subset s_1 \cup a_0$. It is not difficult to see that R is a σ-linked poset, and the union of the first coordinates of conditions in the generic filter is forced to modulo finite contain every ground model element of I.

Let $a \subset \omega$ be a set generic over the model $V[K][G]$ for the poset R, and work in the model $V[K][G][a]$. For every filter $H \in \mathcal{H} \cap V[K][G]$, $\eta/H \subset a$ modulo finite holds, and the set $\mathcal{H} \cap V[K][G]$ is uncountable as the poset R is c.c.c. Thus, the set $\{H \in \mathcal{H} \colon \eta/H \subset a$ modulo finite$\}$ is analytic, uncountable, and therefore contains a nonempty perfect subset $\mathcal{H}'$. Proposition 12.2.2 applied in $V[K]$ shows that in the model W the condition $\bar{p}$ forces the set $A = \{H \in \mathcal{H}' \colon \sigma_y$ is in the generic filter$\}$ to be uncountable. The set $b = \{\eta/H \colon H \in A\}$ is then forced to be an uncountable subset of τ and each element of it is modulo finite included in the set a. □

Now we move to a rich list of examples of Bernstein balanced forcings and related corollaries.

Example 12.2.8. Every placid Suslin forcing P is Bernstein balanced and every placid virtual condition in P is Bernstein balanced.

Proof. Let $\bar{p}$ be a virtual placid condition in the poset P. Let $V[G]$ be a generic extension and in $V[G]$, let $p \leq \bar{p}$ be a condition in P, let $Q \in V$ be an infinite poset such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$, and suppose that $\mathcal{H} \subset \mathcal{P}(Q)$ is a perfect family of filters in finite tuples mutually generic over V. We must find a filter $H \in \mathcal{H}$ such that all conditions in $P \cap V[H]$ below $\bar{p}$ are compatible with P. By a Mostowski absoluteness argument, it is enough to find such a filter in some further generic extension of $V[G]$. Let R be any poset adding a new real, and let τ be an R-name for an element of $\mathcal{H}$ which is not in $V[G]$.

Claim 12.2.9. *$R \Vdash V[\tau] \cap V[G] = V$.*

Proof. Suppose towards a contradiction that this is not the case. Then in $V[G]$, there must be a condition $r \in R$, a set $a \notin V$ of ordinals, and a Q-name η such that $r \Vdash \check{a} = \eta/\tau$. Let $K_0, K_1 \subset R$ be filters mutually generic over $V[G]$, and let $H_0 = \tau/K_0$, $H_1 = \tau/K_1$. Since in the model $V[G]$, the perfect family $\mathcal{H}$ consisted of filters on Q mutually generic over V, by a Mostowski absoluteness argument this is also true in $V[G][K_0][K_1]$; in particular, $H_0, H_1 \subset Q$ are filters mutually generic over V. By the product forcing theorem applied in V then, $V[H_0] \cap V[H_1] = V$. However, the set $a \notin V$ belongs to both $V[H_0]$ and $V[H_1]$ as $a = \eta/H_0 = \eta/H_1$ by the initial assumptions on the set a. This is a contradiction. □

Now, let $K \subset R$ be a filter generic over the model $V[G]$ and let $H = \tau/K$; by the claim, $V[G] \cap V[H] = V$. The placidity of the virtual condition $\bar{p}$ applied with the models $V[G]$ and $V[H]$ now shows that p is compatible with all conditions inthe model $V[H]$ below $\bar{p}$ as desired. □

Corollary 12.2.10. *Let X be a Polish vector space over a countable field F.*

1. *Let P be the poset of countable subsets of X which are linearly independent over F, with the reverse inclusion ordering. In the P-extension of the symmetric Solovay model, there is no nonprincipal ultrafilter on ω and OCA holds.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, X has a linear basis over F, there is no nonprincipal ultrafilter on ω and OCA holds.*

Example 12.2.11. Let Γ be a Borel graph on a Polish space X such that for some $n \in \omega$, Γ does not contain an injective homomorphic copy of K_{n,ω_1}. Then the coloring poset P_Γ of Definition 8.1.1 is Bernstein balanced and every balanced virtual condition is Bernstein balanced.

Proof. The balanced virtual conditions are classified by Γ-colorings $c \colon X \to \omega$ by Theorem 8.1.2. We must prove that every such a coloring represents a Bernstein balanced condition.

To this end, let $V[G]$ be a generic extension and work in $V[G]$. Towards contradiction, suppose that there is a condition $p \in P$ such that $c \subset p$, an infinite poset $Q \in V$ such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$, and an uncountable family $\mathcal{H}$ consisting of filters on Q which are mutually generic over V such that for each $H \in \mathcal{H}$ there is a condition $p_H \in P \cap V[H]$ such that $c \subset p_H$ and p_H is incompatible with p. An examination of the incompatibility options and a counting argument reveal that there must be a point $x \in \mathrm{dom}(p) \setminus V$ such that the family $\mathcal{G} = \{H \in \mathcal{H} \colon \exists x_H \in X \cap V[H_y] \setminus V \ x \ \Gamma \ x_H\}$ is uncountable.

Let $a_0, a_1 \subset \mathcal{G}$ be disjoint sets of size n. By the initial mutual genericity demand, the generic extensions M_0 and M_1 obtained from V by attaching all filters in the set a_0 and a_1 respectively are mutually generic extensions of V. By the forcing theorem, the point $x \in X \setminus V$ belongs to at most one of them; say it does not belong to the model M_0. For each $H \in a_0$, let $x_H \in X \cap V[H] \setminus V$ be a point Γ-related to x; by the product forcing theorem, these points are pairwise distinct. By the initial assumptions on the graph Γ, the set $B = \{z \in X \colon \forall H \in a_0 \ z \ \Gamma \ x_H\}$ is countable. By the Shoenfield absoluteness between the models M_0 and $V[G]$, $B \subset M_0$ must hold. At the same time, x is an element of B which does not belong to the model M_0. A contradiction. □

Example 12.2.12. Let $\mathcal{K}$ be a G_δ matroid on a Polish space X and P be the poset of countable $\mathcal{K}$-sets as in Definition 6.5.1. The poset P is Bernstein balanced and every balanced virtual condition is Bernstein balanced.

Proof. The balanced virtual conditions are classified by maximal $\mathcal{K}$-sets by Theorem 6.5.2. We must prove that every such a maximal set $A \subset X$ represents a Bernstein balanced condition.

To this end, let $V[G]$ be a generic extension and work in $V[G]$. Towards contradiction, suppose that there is a condition $p \in P$ such that $A \subset p$, an infinite poset $Q \in V$ such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$, and an uncountable family $\mathcal{H}$ consisting of filters on Q which are mutually generic over V such that for each $H \in \mathcal{H}$ there is a condition $p_H \in P \cap V[H]$ such that $A \subset p_H$ and p_H is incompatible with p. An examination of the incompatibility options and a counting argument reveal that there must be a finite set $b \subset p \setminus V$ such that the family $\mathcal{G} = \{H \in \mathcal{H}\colon$ there is a finite set $a_H \subset X$ such that $a_H \cup A$ is a $\mathcal{K}$-set while $a_H \cup A \cup b$ is not$\}$ is uncountable.

Let $|b| = n$. Let $d \subset \mathcal{G}$ be a set of size greater than n. Let $c \subset A$ be a finite set such that for every filter $H \in d$, $b \cup c \cup a_H \notin \mathcal{K}$ holds. Note that $b \cup c \in \mathcal{K}$ and (by a repeated application of the balance of the set A) $\bigcup_{H \in d} a_H \cup c \in \mathcal{K}$. Let $e = b \cup c \cup \bigcup_{H \in d} a_H$ and let $|e| = m$. Since $\bigcup_{H \in d} a_H \cup c \subset e$ is a set in $\mathcal{K}$ of cardinality $m - n$, the exchange property of the matroid $\mathcal{K}$ guarantees that every subset of e maximal with respect to membership in $\mathcal{K}$ has cardinality at least $m - n$. Let $f \subset e$ be a maximal set in $\mathcal{K}$ extending the set $b \cup c$. Since $|f| \geq m - n$, there must be a filter $H \in d$ such that $a_H \subset f$. This contradicts the assumption that $b \cup c \cup a_H \notin \mathcal{K}$. ☐

Corollary 12.2.13. *Let X be a Polish field and $F \subset X$ a countable subfield.*

1. *Let P be the poset of countable subsets of X which are algebraically independent over F, with the reverse inclusion ordering. In the P-extension of the symmetric Solovay model, there is no nonprincipal ultrafilter on ω and OCA holds.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, X has a transcendence basis over F, there is no nonprincipal ultrafilter on ω and OCA holds.*

Many quotient simplicial complex forcings are Bernstein balanced.

Example 12.2.14. Let $\mathcal{F}$ be a Fraissé class with strong amalgamation. Let E be a Borel equivalence relation on a Polish space X. Then the $E, \mathcal{F}$-Fraissé forcing of Definition 8.6.3 is Berstein balanced and every balanced condition is Bernstein balanced.

Proof. The balanced conditions are classified by Theorem 8.6.4 as the $\mathcal{F}$-structures on the virtual E-quotient space. Now, let $\bar{p}$ be such a balanced virtual condition. Move to some generic extension and in it, suppose that there is a condition $p \in P$ such that $\bar{p} \leq p$, an infinite poset $Q \in V$ such that $\mathcal{P}(Q) \cap V$ is countable in $V[G]$, and an uncountable family $\mathcal{H}$ consisting of filters on Q which are mutually generic over V. By the mutual genericity, the only E-classes that occur in more than one model $V[H]$ for $H \in \mathcal{H}$ are the classes which are realizations of virtual E-classes in V. By a counting argument, there is a filter $H \in \mathcal{H}$ such that in the model $V[H]$ there are no E-classes represented in the domain of p other than the classes which are realizations of virtual E-classes in V. Then p is compatible with every condition in $V[H]$ which is stronger than $\bar{p}$. $\square$

Corollary 12.2.15. *Let E be a Borel equivalence relation on a Polish space X.*

1. *Let P be the poset of linear orders on countable subsets of the E-quotient space, with the reverse inclusion ordering. In the P-extension of the symmetric Solovay model, there is no nonprincipal ultrafilter on ω and OCA holds.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the E-quotient space is linearly ordered, there is no nonprincipal ultrafilter on ω and OCA holds.*

There are many posets which are neither Bernstein balanced nor perfectly balanced. The examples below are built to violate specific preservation properties of Bernstein balanced posets.

Example 12.2.16. Consider the clopen graph Γ on 2^ω connecting points $x_0 \neq x_1$ if the smallest $n \in \omega$ such that $x_0(n) \neq x_1(n)$ is even. Let P be the balanced poset of Example 8.8.3. In the P-extension of the Solovay model, there is an uncountable subset of 2^ω such that every Γ-clique and every Γ-anticlique in it is countable. Thus, in the P-extension of the Solovay model, OCA fails. In view of Theorem 12.2.5, P is not Bernstein balanced.

Example 12.2.17. Let P be the Lusin poset of Definition 8.8.5. In the P-extension of the symmetric Solovay model in which the conclusion of Theorem 12.2.6 fails. In particular, the Lusin poset is not Bernstein balanced.

Proof. Let κ be an inaccessible cardinal, let W be a symmetric Solovay model derived from κ, and move to the model W. Let $p = \langle a, b \rangle \in P$ be a condition and let τ be a P-name for an uncountable subset of $\dot{A}$. We must find a condition $q \leq p$ and two distinct $\mathbb{E}_0$-related points $x_0, x_1 \in 2^\omega$ such that $q \Vdash \check{x}_0, \check{x}_1 \in \tau$. To this end, find a parameter $z \in 2^\omega$ such that both p, τ are definable from some parameters in the ground model and the parameter z. Let $V[K]$ be an intermediate generic extension containing the parameter z, obtained as a forcing extension of the ground model by a poset of size less than κ. Work in the model $V[K]$.

Let $\bar{p} \leq p$ be a balanced virtual condition in P as in Theorem 8.8.6, so $a_{\bar{p}} = a$. Since τ is forced to be an uncountable set, there must be a poset R, an R-name σ for a condition in P stronger than $\bar{p}$, and an R-name η for a point in $a_\sigma \setminus a$ such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \eta \in \tau$. Note that η is forced to be a point Cohen-generic over the model $V[K]$, and it is also forced that any tuple $x \in (2^\omega)^n$ of points in the set $a_\sigma \setminus (a \cup [\eta]_{\mathbb{E}_0})$ is Cohen-generic over the model $V[K][\eta]$. By the general forcing theory, we may present the poset R as a two step iteration $S^0 * \dot{S}^1$, where S^0 is the Cohen poset $2^{<\omega}$ restricted to some finite binary string s, adding the Cohen-generic point η, and $\dot{S}^1$ is the S^0-name for the remainder forcing.

Move back to the model W. Choose a filter $H_0^0 \subset S^0$ generic over the model $V[K]$, and write $x_0 = \eta/H_0^0 \in 2^\omega$. Flip any value of x_0 past $\mathrm{dom}(s)$ to obtain another point $x_1 \in 2^\omega$. Then x_1 is $\mathbb{E}_0$-related to x_0 and also Cohen generic over the model $V[K]$, containing the string s as an initial segment. Write $H_1^0 \subset S^0$ for the filter generic over $V[K]$ obtained from x_1; thus, $V[K][H_0^0] = V[K][H_1^0]$. Find filters $H_0^1 \subset \dot{S}^1/H_0^0$ and $H_1^1 \subset \dot{S}/H_1^0$ mutually generic over the model $V[K][H_0^0]$, and write $p_0 = \sigma/H_0^0 * H_0^1$ and $p_1 = \sigma/H_1^0 * H_1^1$. The conditions p_0, p_1 are compatible in the poset P by a mutual genericity argument identical to Claim **??**. Let $q \in P$ be any lower bound for the conditions p_0, p_1. The initial choices of the names σ and η now imply that $q \Vdash \check{x}_0, \check{x}_1 \in \tau$ as desired. $\square$

Example 12.2.18. Let P be the balanced poset of Example 8.8.4. In the P-extension of the Solovay model there is a dominating subset of ω^ω such that the set of all functions in in dominated by any fixed $y \in \omega^\omega$ is countable. Thus, the conclusion of the theorem fails in the P-extension with the P-ideal I on $\omega \times \omega$ consisting of sets with all vertical sections finite. The poset P is not Bernstein balanced.

We conclude this section with an anti-preservation result and an example justifying the choice of the Berstein name for the class of posets in question. Recall that a Bernstein set is a subset of a Polish space such that neither it nor its complement contain a perfect set.

Proposition 12.2.19. *In nontrivial, cofinally Bernstein balanced extensions of the symmetric Solovay model, there is a Bernstein set.*

Proof. Suppose that κ is an inaccessible cardinal and P is a Suslin partial order which is Bernstein balanced cofinally below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Let $p \in P$ be a condition. Let $V[K]$ be an intermediate extension obtained with a poset of size $< \kappa$ such that $p \in V[K]$ and P is Bernstein balanced in $V[K]$. Working in $V[K]$, find a Bernstein balanced condition $\bar{p} \leq p$, and a poset Q of size $< \kappa$ and a Q-name σ for a condition such that $Q \Vdash \sigma \leq \bar{p}$. Since the poset P contains no atoms, it is also possible to find Q-names σ_0, σ_1 for incompatible elements of P stronger than σ.

Back in W, let $\{H_y : y \in 2^\omega\}$ be a perfect collection of filters on Q pairwise generic over the model $V[K]$, obtained by an application of Proposition 1.7.10. Let τ_0 be the P-name for the set $\{y \in 2^\omega : \sigma_0/H_y$ belongs to the generic filter on $P\}$ and let τ_1 be the P-name for the set $\{y \in 2^\omega : \sigma_1/H_y$ belongs to the generic filter on $P\}$. These are clearly forced to be disjoint subsets of 2^ω, and $\bar{p}$ forces that complement of neither contains a perfect set: any condition below $\bar{p}$ is incompatible with only countably many conditions in the set $\{\sigma_0/H_y, \sigma_1/H_y : y \in 2^\omega\}$ by the Bernstein balance of the virtual condition $\bar{p}$. Thus, $\bar{p}$ forces both τ_0, τ_1 to be Bernstein sets. $\square$

Unlike the perfectly balanced extensions, the Bernstein balanced extensions may exhibit chaotic structure of cardinalities below 2^ω. This concern was addressed in [97].

Example 12.2.20. Let P be the poset of all countable functions from 2^ω to 2, ordered by reverse extension. Let κ be an inaccessible cardinal, let W be the symmetric Solovay model derived from κ, and let $G \subset P$ be a filter generic over W. In $W[G]$, let $a_0 = \{y \in 2^\omega : \exists p \in G\ p(y) = 0\}$ and $a_1 = \{y \in 2^\omega : \exists p \in G\ p(y) = 1\}$. In $W[G]$, $|a_0| \neq |a_1|$.

Thus, 2^ω can be decomposed into two uncountable sets of distinct cardinalities, contradicting the full Silver dichotomy. In fact, one can prove all kinds of pathologies regarding the cardinalities of a_0 and a_1 respectively; for example $|a_0^2| \not\leq |a_0|$ holds in $W[G]$.

Proof. Work in W. Suppose towards a contradiction that $p \in P$ is a condition and τ is a P-name such that $p \Vdash \tau : \dot{a}_0 \to \dot{a}_1$ is a bijection. The condition p as well as the name τ have to be definable from some ground model parameters and some parameter $z \in 2^\omega$. Find an intermediate model $V[K]$ obtained as a generic extension of the ground model by a poset of size $< \kappa$ such that $z \in V[K]$.

Work in the model $V[K]$. Let $\bar{p}$ be the $\mathrm{Coll}(\omega, \mathfrak{c})$-name for the set $\{q \in P : p \subset q \wedge \forall x \in V[K] \cap 2^\omega \setminus \mathrm{dom}(p)\ q(x) = 0\}$. This is a balanced virtual condition in P for the model $V[K]$. For every point $x \in 2^\omega \setminus \mathrm{dom}(p)$ let $y_x \in 2^\omega$ be a point such that $\mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \tau(\check{x}) = \check{y}_x$, if it exists. As τ is forced to be an injection from $\dot{a}_0$ to $\dot{a}_1$, for each x there can be at most one y_x of this kind, and the function $g : x \mapsto y_x$ must be an injection from $2^\omega \setminus \mathrm{dom}(p)$ to $\mathrm{dom}(p)$. Since the former set is uncountable and the latter is countable, there must be a point $x \in 2^\omega \setminus \mathrm{dom}(p)$ which is not in the domain of g.

Still in the model $V[K]$, the statement that $x \notin \mathrm{dom}(g)$ means that $\mathrm{Coll}(\omega, < \kappa)$ forces that either there is a condition below $\bar{p}$ in P which forces $\tau(\check{x})$ out of $V[K]$, or there are two distinct conditions in P stronger than $\bar{p}$ which force the value $\tau(\check{x})$ to be two distinct points in $V[K]$. Suppose for definiteness that the former is the case. Then there must be a poset R of size $< \kappa$, an R-name σ for a condition in P stronger than $\bar{p}$ and an R-name η for an element of $2^\omega \setminus V[K]$ such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau(\check{x}) = \eta$.

In the model W, pick filters $H_0, H_1 \subset R$ mutually generic over the model $V[K]$. Let $p_0 = \sigma/H_0 \in P$ and $p_1 = \sigma/H_1 \in P$; by the balance of the condition $\bar{p}$, p_0 and p_1 are conditions compatible in P, with some lower bound $q \in P$. Let $y_0 = \eta/H_0 \in 2^\omega$ and $y_1 = \eta/H_1 \in 2^\omega$; by the product forcing theorem and the choice of the name η, $y_0 \neq y_1$ holds. But then, $q \Vdash \tau(\check{x}) = \check{y}_0$ and $\tau(\check{x}) = \check{y}_1$, an impossibility. $\square$

12.3 Placid forcing

Placid forcings have been introduced in Definition 9.3.1; as we have seen in Example 12.2.8, they are Bernstein balanced. In this section, we prove a couple of preservation theorems for placid forcing which show among other things that the matroid posets in Example 12.2.12 and the coloring posets in 12.2.11 are not placid. Thus, the placid forcings form a fairly small and tightly controlled subclass of Bernstein balanced forcings.

Theorem 12.3.1. *Let X be an uncountable Polish field and $F \subset X$ be a countable subfield. In cofinally placid extensions of the symmetric Solovay model, there is no transcendence basis of X over F.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is placid cofinally below κ. Let W be the symmetric Solovay model derived from κ, and work in W. Suppose towards contradiction that there is a condition $p \in P$ and a P-name τ such that p forces τ to be a transcendence basis of X over F. The name τ and the condition p are definable from ground model parameters and an additional parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate model obtained by a forcing of cardinality smaller than κ such that $z \in V[K]$ and P is placid in $V[K]$. Work in $V[K]$.

Let $\bar{p} \leq p$ be a placid virtual condition. Consider the closed set $Y = \{\langle x_0, x_1, x_2, x_3\rangle \in X^4 : x_0x_2 + x_1x_3 = 1\}$ equipped with the topology inherited from X^4, and consider the associated Cohen poset P_Y. Theorems 3.2.7 and 3.1.4 show that the poset P_Y adds a quadruple $\langle x_i : i \in 4\rangle$ of points in X such that x_0, x_1 are mutually Cohen generic elements of X, so are x_2, x_3, moreover $V[K][x_0, x_1] \cap V[K][x_2, x_3] = 0$ and $x_0x_2 + x_1x_3 = 1$. For each $i \in 4$, the model $V[K][x_i]$ contains a poset R_i of cardinality smaller than κ, an R_i-name σ_i for a condition in P stronger than $\bar{p}$, and a name η_i for a finite set such that the entries of the vector x_i is forced to be algebraic over η_i and $R_i \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_i \Vdash_P \eta_i \subset \tau$. Let $H_i \subset R_i$ for $i \in 4$ be filters mutually generic over the model $V[K][x_i : i \in 4]$. Write $p_i = \sigma_i/H_i$ and $a_i = \eta_i/H_i$.

Claim 12.3.2. *The conditions p_i for $i \in 4$ have a common lower bound.*

Proof. First, p_0, p_1 have a lower bound $p_{01} \in V[K][x_0, x_1][H_0, H_1]$ by the balance of $\bar{p}$. Similarly, p_2, p_3 have a lower bound in $V[K][x_2, x_3][H_2, H_3]$ by the balance of $\bar{p}$ again. Finally, $V[K][x_0, x_1][H_0, H_1] \cap V[K][x_2, x_3][H_2, H_3] = V[K]$ by the initial choice of the points x_i for $i \in 4$ and the mutual genericity of the filters H_i for $i \in 4$. The placidity of the virtual condition $\bar{p}$ then shows that p_{01} and p_{23} have a common lower bound. ☐

Claim 12.3.3. *The set $\bigcup_i a_i$ is not algebraically free.*

Proof. By the exchange property of the algebraic matroid, there is a point $y \in a_3 \setminus V$ which is in the algebraic closure of $(a_3 \cup \{x_3\}) \setminus \{y\}$. Since y is not an element of V, it is also not an element of $a_0 \cup a_1 \cup a_2$. Since x_3 is algebraic over $\{x_0, x_1, x_2\}$ and each of these points are in turn algebraic over a_0, a_1, and a_2, it follows that y is in the algebraic closure of $a_0 \cup a_1 \cup a_2 \cup (a_3 \setminus \{y\})$. This proves the claim. ☐

Thus, the lower bound of the conditions p_i for $i \in 4$ identifies a finite non-free set $\bigcup_i a_i \subset X$ forced to be a subset of τ. This is a contradiction. ☐

To state the following theorem in full generality, consider a definition.

Definition 12.3.4. A *curve graph* is a Borel graph Γ on $\mathbb{R}_2$ if there is a Borel set $A \subset \mathbb{R}^2$ consisting of graphs of some collection of smooth curves in $\mathbb{R}^2$ which are closed as subsets of $\mathbb{R}^2$ and contain no straight segments, such that A accumulates around 0 and $\Gamma = \{\{y_0, y_1\} \in [\mathbb{R}^2]^2 \colon y_0 - y_1 \in A \text{ or } y_1 - y_0 \in A\}$.

One good example of a curve graph is obtained as follows. Let $B \subset \mathbb{R}$ be a set of positive reals accumulating around 0. Let A be the collection of circles around the origin with radii belonging to B. Thus, the resulting curve graph Γ consists of pairs of points whose Euclidean distance belongs to B. If B is countable, then Γ does not contain an injective homomorphic copy of K_{2,ω_1} and therefore has countable coloring number by [25]. This is in effect Example 8.1.8.

Another example is obtained by choosing a closed smooth curve C containing no straight segments and passing though the origin such that for some number $n \in \omega$, n many distinct points in the plane belong to at most one translate of C–perhaps a circle through the origin with $n = 3$ comes to mind. Let $A = C$. The resulting curve graph Γ does not contain an injective homomorphic copy of $K_{m,m}$ whenever $m \in \omega$ is a number large enough so that $\binom{m}{m} \to \binom{n}{n}_2$ holds; therefore, it has countable coloring number by [25].

Theorem 12.3.5. *Let Γ be a curve graph on $\mathbb{R}^2$. In cofinally placid extensions of the Solovay model, the chromatic number of Γ is uncountable.*

Proof. Let $A \subset \mathbb{R}^2$ be the Borel set defining the curve graph Γ. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is placid cofinally below κ. Let W be the symmetric Solovay model derived from κ, and work in W. Suppose towards contradiction that there is a condition $p \in P$ and a P-name τ such that p forces τ to be a coloring from $\mathbb{R}^2$ to ω. We must find two Γ-connected points and a condition stronger than p which forces the two points to attain the same color in τ. The name τ and the condition p are definable from ground model parameters and an additional parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate model obtained by a forcing of cardinality smaller than κ such that $z \in V[K]$ and P is placid in $V[K]$. Work in $V[K]$.

Let $\bar{p} \leq p$ be a placid virtual condition. Consider the Cohen poset Q on $\mathbb{R}^2$ with its name $\dot{y}_{gen}$ for the generic point in $\mathbb{R}^2$. There must be a cardinal $\lambda \in \kappa$, a $Q \times \mathrm{Coll}(\omega, \lambda)$-name σ for a condition in the poset P stronger than $\bar{p}$, a nonempty open set $O \subset \mathbb{R}^2$ and a number $n \in \omega$ such that $O \Vdash_Q \mathrm{Coll}(\omega, \lambda) \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau(\dot{y}_{gen}) = \check{n}$. Let C be a smooth curve containing no straight segments, which is a closed subset of $\mathbb{R}^2$, such that $C \subset A$ and O contains some pair of points y_0, y_1 such that $y_0 - y_1 \in C$; this is possible as 0 is an accumulation point of A. Let $X = \{\langle y_0, y_1\rangle \in \mathbb{R}^2 \colon y_0, y_1 \in O \text{ and } y_0 - y_1 \in C\}$. Let $y_0, y_1 \in Y$ be a pair of points generic over the model $V[K]$ over the Cohen poset on the space X; so $\langle y_0, y_1\rangle \in \Gamma$. Theorems 3.2.8 and 3.1.4 show that the points $y_0, y_1 \in O$ are separately Q-generic over $V[K]$, and $V[K][y_0] \cap V[K][y_1] = V[K]$.

Let $H_0, H_1 \subset \mathrm{Coll}(\omega, < \lambda)$ be filters mutually generic over the model $V[K][y_0, y_1]$, and write $p_0 = \sigma/y_0, H_0$ and $p_1 = \sigma/y_1, H_1$. By the mutual genericity, the intersection $V[K][y_0][H_0] \cap V[K][y_1][H_1]$ equals to $V[K]$. By the placidity of the condition $\bar{p}$, the conditions p_0, p_1 are compatible. By the forcing theorem applied in the respective models $V[K][y_0][H_0]$ and $V[K][y_1][H_1]$, in the model W any common lower bound of the conditions p_0, p_1 forces in the poset P that $\tau(\check{y}_0) = \tau(\check{y}_1) = \check{n}$. The proof is complete. $\square$

Theorem 12.3.6. *Let E be an orbit equivalence on a Polish space resulting from a generically turbulent action of some Polish group. In the cofinally placid extensions of the Solovay model, there is no tournament on the E-quotient space.*

Proof. The argument depends on a simple abstract property of pairs of independent continuous open maps on Polish spaces of Definition 3.1.3. Let $m \in \omega$ be a number. For each natural number $n \in m$, let X_n be a Polish space, Y_n be Polish spaces and $f_n \colon X_n \to Y_n$ be continuous open maps. We define the *product map* $f = \prod_n f_n \colon \prod_n X_n \to \prod_n Y_n$ by $f(x)(n) = f_n(x(n))$. It is not difficult to check that the product map is continuous.

Claim 12.3.7. *The set of pairs of independent maps is closed under finite product.*

Proof. Let $m \in \omega$ be a number. For each number $n \in m$ let X_n be a Polish space, Y_{0n}, Y_{1n} be Polish spaces, and let $f_{0n} \colon X_n \to Y_{0n}$, $f_{1n} \colon X_n \to Y_{1n}$ be continuous open maps such that for all but finitely many $n \in \omega$ their ranges are the whole spaces Y_{0n}, Y_{1n} and such that each pair f_{0n}, f_{1n} is independent. Write $X = \prod_n X_n$, $Y_0 = \prod_n Y_{0n}$ and $Y_1 = \prod_n Y_{1n}$. We must prove that the product maps $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ are independent.

To this end, let $O \subset X$ be a nonempty open set; thinning out the set O if necessary, we may find open sets $O_n \subset X_n$ for $n \in m$ such that $O = \prod_n O_n$. Let $A_n \subset Y_{0n}$ be an open set witnessing the independence of the pair f_{0n}, f_{1n} for the set O_n. We claim that the open set $A = \prod_n A_n \subset Y_0$ witnesses the independence of the pair f_0, f_1 for O.

To see this, suppose that $B_0, B_1 \subset A$ are nonempty open sets; thinning them out if necessary, we may find open sets $B_{0n}, B_{1n} \subset A_n$ such that $B_0 = \prod_n B_{0n}$ and $B_1 = \prod_n B_{1n}$. For each $n \in m$ use the initial assumption on the set A_n to find a walk w_n of points in O_n such that for the initial point $x_{0n} \in w_n$ we have $f_{0n}(x_{0n}) \in B_{0n}$ and for the last point $x_{1n} \in w_n$ we have $f_{0n}(x_{1n}) \in B_{1n}$.

Now, each walk w_n is a finite string of points; shifting the indexation, we may find successive intervals I_n of natural numbers for $n \in m$ such that each walk w_n is indexed by numbers in I_n. Let $K = \bigcup_n I_n$ and consider the string of points in the open set O indexed by $k \in K$, defined in the following way: for each k write $n(k)$ for that $n \in m$ for which $k \in I_n$, and then define point $x_k \in O$ by $x_k(n) = x_{1n}$ if $n < n(k)$, $x_k(n) = w_k(n)$ if $n(k) = n$, and finally $x_k(n) = x_{0n}$ if $n(k) < n$. It is easy to see that the sequence $\langle x_k \colon k \in K \rangle$ is the desired walk in the set O from B_0 to B_1. $\square$

Now, towards the proof of Theorem 12.3.6, let Γ be a Polish group acting on a Polish space X in a generically turbulent way, so that E is the resulting orbit equivalence relation. Consider the space $\Gamma \times X \times X$, its closed subset

$C = \{\langle \gamma, x_0, x_1\rangle \colon \gamma \cdot x_0 = x_1\}$ and the projection maps $f_0 \colon C \to X$ into the x_0-coordinate and $f_1 \colon C \to X$ into the x_1 coordinate. The two maps are independent by Theorem 3.2.2. Let g_0, g_1 be the product maps $f_0 \times f_1$ and $f_1 \times f_0$. Thus, $g_0, g_1 \colon C \times C \to X \times X$ are continuous open maps, and they form an independent pair by the claim.

Let κ be an inaccessible cardinal and let P be a Suslin forcing which is placid cofinally below κ. Let W be the Solovay model derived from κ and work in the model W. Let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau$ is a directed graph on the E-quotient space and for each pair $\langle c, d\rangle$ of distinct E-equivalence classes, either $\langle c, d\rangle$ or $\langle d, c\rangle$ is in τ. We must find a condition $q \leq p$ and distinct E-classes c, d such that $q \Vdash \langle c, d\rangle \in \tau$ and $\langle d, c\rangle \in \tau$.

Both p, τ are definable in the model W from a parameter $z \in 2^\omega$ and parameters in the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of size $< \kappa$ containing z and such that P is placid in $V[K]$; work in the model $V[K]$. By the initial assumption on the poset P, there must be a placid virtual condition $\bar{p} \leq p$. Let Q be the Cohen forcing on $X \times X$, adding generic points $\dot{x}_0, \dot{x}_1$. There must be an open set $O \subset X \times X$ forcing the existence of a poset $\dot{S}$ adding a condition $\sigma \in P$ such that $\dot{S} \Vdash \sigma \leq \bar{p}$ and $\mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \langle [\dot{x}_0]_E, [\dot{x}_1]_E\rangle \in \tau$ (or the whole statement repeated with $\dot{x}_0, \dot{x}_1$ interchanged). Since Γ-orbits are dense, it is possible to find open sets $O_0, O_1 \subset X$ and $U_0, U_1 \subset \Gamma$ such that $O_0 \times O_1 \subset O$ and $(U_0 \cdot O_1) \times (U_1 \cdot O_0) \subset O$.

Now, consider the Cohen poset R on the space $C \times C$. Consider the condition $U \in Q$ consisting of all tuples $\langle \gamma, x_0, x_1, \delta, x_2, x_3\rangle \in C \times C$ such that $\gamma \in U_0$, $x_0 \in O_0$, $\delta \in U_1$ and $x_3 \in O_1$. Let $\langle \gamma, x_0, x_1, \delta, x_2, x_3\rangle$ be an R-generic tuple over the model $V[K]$. Note the following:

- $\langle x_0, x_3\rangle$ is a pair Q-generic over $V[K]$, meeting the condition O;
- similarly, $\langle x_2, x_1\rangle$ is a pair Q-generic over $V[K]$, meeting the condition O by the choice of the neighborhoods U_0, U_1;
- $x_0 \mathrel{E} x_1$ and $x_3 \mathrel{E} x_2$;
- $V[K][x_0, x_3] \cap V[K][x_2, x_1] = V[K]$ by the fact that g_0, g_1 are independent maps and Theorem 3.1.4.

Let $S_0 = \dot{S}/x_0, x_3$ and $S_1 = \dot{S}/x_2, x_1$. Let $H_0 \subset S_0$ and $H_1 \subset S_1$ be filters mutually generic over the model $V[K][x_i \colon i \in 4]$. By the third item above and a mutual genericity argument, $V[K][x_0, x_3][H_0] \cap V[K][x_2, x_1][H_1] = V[K]$. Let $p_0 = \sigma/H_0$ and $p_1 = \sigma/H_1$. By the placidity of the virtual condition $\bar{p}$, the conditions p_0, p_1 have a lower bound $q \in P$. The forcing theorem finally shows that in the model W, $q \Vdash$ both $\langle [x_0]_E, [x_2]_E\rangle$ and $\langle [x_2]_E, [x_0]_E\rangle$ belong to τ. $\square$

Corollary 12.3.8. *Let X be a Borel vector space over a countable field Φ. Let Y be an uncountable Polish field and let Ψ be a countable subfield.*

1. *Let P be the basis poset of Example 6.4.9. In the P-extension of the symmetric Solovay model, Y does not have a transcendence basis.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, X has a Hamel basis over Φ, and Y does not have a transcendence basis over Ψ.*

Corollary 12.3.9. *Let E be a Borel equivalence relation on a Polish space induced by a generically turbulent Polish group action. Let F be a Borel equivalence relation on a Polish space, classifiable by countable structures.*

1. *Let P be the linearization poset for F as in Example 8.6.5. In the P-extension of the Solovay model, there is no tournament on the E-quotient space;*

2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, the F-quotient space is linearly ordered, and yet there is no tournament on the E-quotient space.*

12.4 Existence of generic filters

Let P be a Suslin forcing. In the presence of suitable large cardinal axioms, one can replace the P-extension of the symmetric Solovay model with the P-extension of the model $L(\mathbb{R})$. A standard absoluteness argument shows that validity of $\mathbf{\Pi}^2_1$ sentences transfers from the former to the latter, and so the latter can serve as a vehicle for most independence results presented in this book. A natural question arises: does there exist a filter $G \subset P$ which is generic over $L(\mathbb{R})$?

The answer is an easy yes if the poset P in question is σ-closed and the Continuum Hypothesis holds, since then one can meet the dense subsets of the poset P appearing in the model $L(\mathbb{R})$ one by one in a straightforward transfinite recursion argument. An interesting feature of Bernstein balance is that it guarantees the existence of generic filters over inner models such as $L(\mathbb{R})$ independently of cardinal arithmetic and assumptions such as the Continuum Hypothesis, purely as a consequence of large cardinal axioms.

Definition 12.4.1. Let P be a Suslin forcing.

1. If $B \subset P$ is a set and $p \in P$ then write $p \leq B$ to denote that p is stronger than all conditions in B;

2. if $B \subset P$ then τ_B is the $\mathrm{Coll}(\omega, B)$-name for the analytic set $\{p \in P : p \leq B\}$.

3. We say that P is *soft* if for every centered set $B \subset P$ of cardinality less than $\mathfrak{c}$ there is a centered set $C \supset B$ such that the pair $\langle \mathrm{Coll}(\omega, C), \tau_C\rangle$ is a Bernstein balanced virtual condition.

Note that a soft poset P must have the following centeredness property: if $B \subset P$ is a countable centered set, then B has a lower bound. To see this, just extend it into an Bernstein balanced centered set $C \subset P$, note that in the $\mathrm{Coll}(\omega, C)$-extension, the set C has a lower bound by the definition, and by a Mostowski absoluteness argument the set B must have a lower bound already in the ground model.

Theorem 12.4.2. *Suppose that there is a weakly compact Woodin cardinal. Suppose that P is a soft Suslin forcing. Then there is a filter $G \subset P$ which is generic over the model $L(\mathbb{R})$.*

Proof. Let $B \subset P$ be a centered set of size $< \mathfrak{c}$, and let $D \subset P$ be an open dense set in $L(\mathbb{R})$. We will show that there is a condition $p \in D$ such that $\{p\} \cup B$ is a centered set. The theorem then follows by a straightforward transfinite recursion construction, given the fact that there are only $\mathfrak{c}$-many subsets of P in the model of $L(\mathbb{R})$ as soon as a measurable cardinal exists. So, let $C \supset B$ be an Bernstein balanced centered set witnessing the definitory properties of the soft poset P. The set D is definable from a real parameter in the model $L(\mathbb{R}^\#)$. Below, we identify the set D with its definition.

Let κ be a weakly compact Woodin cardinal, let $\mathbb{Q}_\kappa$ be the associated countably based stationary tower, let $K \subset \mathbb{Q}_\kappa$ be a generic filter and work in the model $V[K]$. Note that $\kappa = \aleph_1$ and every element of 2^ω is generic over V by a poset smaller than κ. Consider the poset $R = \mathrm{Coll}(\omega, C)$ and use Proposition 1.7.10 to find a perfect collection $\{H_y : y \in 2^\omega\}$ of filters on R pairwise mutually generic over the ground model V. Use Proposition **??** to see the following:

Claim 12.4.3. *For every finite set $a \subset P$ such that $a \cup C$ is a centered set, the set $Y_a = \{y \in 2^\omega : \exists p \in V[H_y]\ p \leq C$ and the set $\{p\} \cup a$ has no common lower bound$\}$ is countable.*

Let $j : V \to M$ be the generic ultrapower associated with K; the model M is closed under ω-sequences. Work in the model M. The closure properties of the model M guarantee that the following objects belong to the model M: the collection $\{H_y : y \in 2^\omega\}$, the collection $\{P \cap V[H_y] : y \in Y\}$, and the map $a \mapsto Y_a$ as a varies over finite subsets of P such that $a \cup C$ is a centered set. Also, the model M satisfies the statement that the range of this map consists of countable sets. Looking at the set $j(C)$ in the model M, we see that $|j(C)| < \mathfrak{c}$, and for each finite set $a \subset j(C)$ the set $a \cup C \subset j(C)$ is centered, and therefore the set $Y_a \subset 2^\omega$ is countable. By a cardinality argument, there must be a point $y \in 2^\omega \setminus \bigcup\{Y_a : a \in [j(C)]^{<\aleph_0}\}$.

Work in the model $V[H_y]$. Note that the theory of $L(\mathbb{R}^\#)$ with real parameters is invariant under forcing and so the set $D \subset P$ is still open dense in $V[H_y]$. Thus, there must be a condition $p \in D \cap V[H_y]$, $p \leq C$. By the choice of the point $y \in Y$, the set $\{p\} \cup j(C)$ is centered. Thus $M \models$ there is a condition $p \in D$ such that $\{p\} \cup j(C)$ is centered. By the elementarity of the embedding j, $M \models$ there is a condition $p \in D$ such that $\{p\} \cup C$ is centered; in particular, $\{p\} \cup B$ is centered as desired. $\square$

The following examples all use the Woodin cardinal assumption which is not spelled out.

Example 12.4.4. Let X be a Polish vector space over a countable field. There is a basis generic over $L(\mathbb{R})$ for the poset P of Example 6.4.9. This follows from Theorem 12.4.2 and Theorem **??**(1): every centered subset of P yields a linearly independent set, which can be extended to a maximal linearly independent set. A maximal linearly independent set yields a Bernstein balanced condition where E is simply the identity on 2^ω.

Example 12.4.5. Let Γ be a Borel graph on a Polish space X. There is a maximal acyclic subgraph $\Delta \subset \Gamma$ generic over $L(\mathbb{R})$ for the poset P of Example 6.4.10.This follows from Theorem 12.4.2 and Theorem **??**(2): every centered subset of P yields an acyclic subset of Γ, which can be extended to a maximal acyclic set. A maximal acyclic set yields a Bernstein balanced condition where E is simply the identity on 2^ω.

Example 12.4.6. Let E be a pinned Borel equivalence relation on a Polish space X. There is a linear ordering on the E-quotient space generic over $L(\mathbb{R})$ for the poset P of Example 8.6.5. This follows from Theorem 12.4.2 and Theorem 8.6.4: every centered subset of P yields a partial ordering on the E-quotient space, which can be extended to a linear order. A linear ordering on the E-quotient space yields a Bernstein balanced condition, by the pinned assumption on E and Theorem 8.6.4.

Example 12.4.7. Let E, F be pinned Borel equivalence relations on respective Polish spaces X, Y with uncountably many classes each. There is an injection from the E-quotient to the F-quotient space generic over $L(\mathbb{R})$ for the poset of Definition 6.6.2. This follows from Theorem 12.4.2 and Theorem 6.6.3: every centered subset of P of size $< \mathfrak{c}$ yields a partial injection from the E-quotient to the F-quotient space and also a set of F-classes which are the forbidden values, both of size $< \mathfrak{c}$. A counting argument provides a total injection from the E-quotient to the F-quotient space extending the given one and avoiding the given set of forbidden values. Such an injection yields a Bernstein balanced condition by Theorem 6.6.3, where $G = E \times F$.

Many other examples can be produced by the observation that the class of soft forcings is closed under countable product. The nonexamples are perhaps more interesting than the examples.

Example 12.4.8. Let P be the $\mathcal{P}(\omega)$ modulo finite poset. It is not Bernstein balanced by Theorem 12.2.3 above. The P-extension of $L(\mathbb{R})$ contains a Ramsey ultrafilter on ω, in particular a P-point. At the same time, it is consistent [82] and occurs in the product Silver model [18] that there are no P-points on ω. In such circumstances, there cannot be a filter on P generic over $L(\mathbb{R})$.

Example 12.4.9. Let P be the Lusin poset of Definition 8.8.5. It is not Bernstein balanced by Theorem 12.2.6 above. The P-extension of $L(\mathbb{R})$ contains a Lusin subset of 2^ω. At the same time, the negation of Martin's Axiom and the negation of the continuum hypothesis implies that every set of size $\aleph_1$ is meager ??? and so no Lusin set exists. In such circumstances, there cannot be a filter on P generic over $L(\mathbb{R})$.

Example 12.4.10. Consider the clopen graph Γ on 2^ω connecting points $x_0 \neq x_1$ if the smallest $n \in \omega$ such that $x_0(n) \neq x_1(n)$ is even. Let P be the balanced poset of Example 8.8.3. It is not Bernstein balanced by Example 12.2.16 above. In the P-extension of $L(\mathbb{R})$, there is an uncountable set $A \subset 2^\omega$ such that every Γ-clique and every Γ-anticlique in it is countable. Now, since the graph Γ is clopen, closures of both Γ-cliques and Γ-anticliques are cliques and anticliques respectively. Thus, the stated property of A holds also in V. At the same time, it is consistent that OCA holds in V [91]. In such circumstances, every uncountable subset of Γ must contain either a Γ-clique or an uncountable Γ-anticlique and therefore a P-generic filter over V cannot exist.

The classes of Bernstein balanced and soft Suslin forcings do not coincide, which leads to the following question about the existence of generic filters for a particularly simple Bernstein balanced Suslin poset which is not soft.

Question 12.4.11. (In the context of ZFC+suitable large cardinal assumption.) Is there a tournament on the $\mathbb{F}_2$-quotient space which is generic over $L(\mathbb{R})$ for the poset of Example 8.6.6?

We conclude this section with a mutual genericity result for Bernstein balanced and perfectly balanced forcings.

Theorem 12.4.12. *Let κ be a weakly compact Woodin cardinal. Let P_0, P_1 be σ-closed Suslin forcings such that*

1. *P_0, P_1 are $\aleph_0$-tethered;*
2. *P_0 is perfect;*
3. *every balanced virtual condition in P_1 is Bernstein balanced.*

If $G_0 \subset P_0$ and $G_1 \subset P_1$ are filters (in V) separately generic over $L(\mathbb{R})$, then they are mutually generic over $L(\mathbb{R})$.

Proof. We start with a preliminary observation. Let $\bar{p}_0$ be the virtual condition in P_0 consisting of a collapse name for the set of all lower bounds of the filter G_0; let $\bar{p}_1$ be defined similarly. By the σ-closure assumption, both $\bar{p}_0$, $\bar{p}_1$ are nonzero conditions. Since the filter G_0 is generic over $L(\mathbb{R})$, for every analytic set $A \subset P_0$ there is a condition in G_0 which is either incompatible with all elements of A or is stronger than some element of A. By the $\aleph_0$-tether of the poset P_0, it must then be the case that $\bar{p}_0$ is a balanced virtual condition in P_0. By the same reasoning, $\bar{p}_1$ is a balanced virtual condition in P_1, and it is even Bernstein balanced by the assumption (3).

Now, let X be any uncountable Polish space. In view of Proposition 1.7.8 applied in the model $L(\mathbb{R})$, it is enough to show that for any two disjoint sets $A_0, A_1 \subset X$ in the respective models $L(\mathbb{R})[G_0]$, $L(\mathbb{R})[G_1]$, there is a set $B \subset X$ in $L(\mathbb{R})$ containing A_1 and disjoint from A_0. To this end, let τ_0, τ_1 be respective P_0, P_1-names in $L(\mathbb{R})$ such that $A_0 = \tau_0/G_0$ and $A_1 = \tau_1$. The names τ_0, τ_1 can be viewed as sets of reals, and so each has a definition in $L(\mathbb{R})$ from real and ordinal parameters; below, we will evaluate them in other models using these definitions. We also evaluate P_0, P_1 using their definitions. Write $\Vdash_0$ for the forcing relation of P_0 and $\Vdash_1$ for the forcing relation of P_1.

Let $H \subset \mathbb{Q}_\kappa$ be a generic filter. Let $j: V \to M$ be the associated generic ultrapower and work in the model M. Note that $G_1 \subset j(G_1)$ is a countable set in the model M and by the σ-closure assumption on the poset P_1, the filter $j(G_1)$ must contain a lower bound of G_1. Consider the set $B = \{x \in X : \exists p \in P_1\ p$ is a lower bound of the filter G_1 and $p \Vdash_1 \check{x} \in \tau_1\}$. Clearly, the set $B \subset X$ belongs to $L(\mathbb{R})$ and it contains the set $j(A_1)$. By the elementarity of the embedding j, it will be enough to show that B is disjoint from the set $j(A_0)$.

To see this, we prove the following claim, which is also a statement of the model M. Call a Borel set $C \subset X$ *positive* if for every condition $p_1 \in P_1$ which is a lower bound of the filter G_1, $p_1 \Vdash_1 C \cap \tau_1 \neq 0$. By the forcing theorem applied to $j(G_1)$, $C \cap A_1 \neq 0$ holds for every positive Borel set C. Clearly, the collection of positive Borel sets is in $L(\mathbb{R})$.

Claim 12.4.13. *If $p_0 \in P_0$ is a condition which is a lower bound of the filter G_0 and $x \in B$ is a point such that $p_0 \Vdash_0 \check{x} \in \tau_0$, then there is a condition $p_0' \leq p_0$ and a positive Borel set $C \subset X$ such that $p_0' \Vdash_0 C \subset \tau_0$.*

The theorem immediately follows. By the claim and the genericity of the filter $j(G_0)$ over $L(\mathbb{R})$, there must be in $j(G_0)$ either a condition which forces $B \cap \tau_0 = 0$ or a condition which forces $C \subset \tau_0$ for some Borel positive set $C \subset X$. However, the latter alternative means that $j(A_0) \cap j(A_1) \neq 0$, contradicting our initial assumptions on A_0, A_1. The former alternative confirms that the set B separates $j(A_0)$ and $j(A_1)$ as required.

To prove the claim, fix $p_0 \in P_0$ and $x \in X$ as in the assumption. In view of the fact that the reals of the model M are the reals of some symmetric Solovay model derived from κ (Fact 1.7.18(3)), in the ground model there must be a poset Q of size $< \kappa$ and Q-names σ_0, σ_1, η for conditions in P_0, P_1 and a point in X respectively such that Q forces that σ_0 is a lower bound of G_0, σ_1 is a lower bound of G_1 and $\mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_0 \Vdash_0 \eta \in \tau_0$ and $\mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_1 \Vdash_1 \eta \in \tau_1$. Moreover, there has to be a filter $K \subset Q$ generic over V such that $p_0 = \sigma_0/K$ and $x = \eta/K$. Now, let $H \subset \mathrm{Coll}(\omega, \mathcal{P}(Q) \cap V)$ be a filter in M mutually generic over V with the filter K.

Work in the model $V[H]$. Use Proposition 1.7.10 to find a continuous function $y \mapsto H_y$ which to each element $y \in 2^\omega$ assigns a filter $H_y \subset Q$ so that for every finite set $a \subset 2^\omega$, the filters $\{H_y : y \in a\}$ are mutually generic over V. For every element $y \in 2^\omega$, write $p_{0y} = \sigma_0/K_y$, $p_{1y} = \sigma_1/K_y$, and $x_y = \eta/K_y$. Since the conditions $p_{0y} \in P_0$ are all stronger than the balanced virtual condition $\bar{p}_0$ and they are found in mutually generic models, each finite set of them has a lower bound as in Claim 12.1.4. By the perfectness assumption on the poset P_0, the model $V[H]$ contains a perfect set $D \subset 2^\omega$ and a condition $r \in P_0$ which is stronger than all p_{0y} for all $y \in D$. Since both conditions $p_0, r \in P_0$ are stronger than the balanced virtual condition $\bar{p}_0$ and are found in mutually generic models, they are compatible with a lower bound p_0'.

Now, consider the set $\{p_{1y} : y \in D\}$ and the perfect set $C = \{x_y : y \in D\}$. By the Bernstein balance of the virtual condition $\bar{p}_1$, every condition $p_1 \in P_1$ which is a lower bound of the filter G_1 is compatible with p_{1y} for all but countably many points $y \in D$. As a result, $C \subset X$ is a positive Borel set. The initial choice of the name η also implies that $p'_0 \Vdash C \subset \tau_0$. The claim has just been proved. $\square$

Chapter 13

The arity divide

13.1 m, n-centered and balanced forcings

In this chapter, we develop a tool which is particularly useful for ruling out combinatorial objects with high degree of organization. The following notion is central:

Definition 13.1.1. Let P be a Suslin poset and $n \in m$ be numbers.

1. A virtual condition $\bar{p}$ in P is *m, n-balanced* if in every generic extension, whenever $\{H_i : i \in m\}$ are filters generic over V such that for every set $a \subset m$ of size n the filters $\{H_i : i \in a\}$ are mutually generic over V, if $p_i \leq \bar{p}$ are conditions in P in $V[H_i]$, then the set $\{p_i : i \in m\}$ has a common lower bound in P.
2. The poset P is m, n-balanced if for every condition $p \in P$ there is a m, n-balanced virtual condition $\bar{p} \leq p$.

As an initial example, let P be the poset of linear orderings on countable sets of $\mathbb{E}_0$-classes, ordered by reverse extension. The balanced virtual conditions are classified by linear orderings of the $\mathbb{E}_0$-quotient space. Each such ordering $\bar{p}$ is in fact $m, 2$-balanced for every $m \in \omega$. To see this, in some ambient forcing extension let $V[G_i]$ for $i \in m$ be generic extensions which are pairwise mutually generic, and let $p_i \in V[G_i]$ be a condition stronger than $\bar{p}$ for each $i \in m$. The domains of the linear orderings p_i pairwise intersect in the set of ground model $\mathbb{E}_0$-classes; on this set, the orderings p_i agree and yield the ordering $\bar{p}$. Therefore, it is possible to find a common linearization of $\bigcup_i p_i$, which is their common lower bound.

Sometimes, a much simpler notion makes an appearance:

Definition 13.1.2. Let $n \in m$ be numbers. A poset P is *m, n-centered* if whenever $a \subset P$ is a set of size m such that any subset of a of size n has a common lower bound, then a has a common lower bound.

As an initial example, let P be the partial order of all tournaments on countable sets of $\mathbb{E}_0$-classes, ordered by reverse inclusion. In it, every countable set of pairwise compatible conditions has a common lower bound. It is immediately clear that every balanced poset which is m, n-centered is also m, n-balanced and each balanced virtual condition in it is m, n-balanced. An important part of this chapter is devoted to detecting distinctions between m, n-balanced and m, n-centered forcings. Here, we restrict ourselves to two initial observations. Note that m, n-centeredness (unlike the corresponding balance notion) is $\mathbf{\Pi}^1_2$ and therefore absolute between all generic extensions. Note also that m, n-centeredness is a combinatorial (as opposed to forcing) property of the the poset, in the sense that one can have two analytic dense subsets Q_0, Q_1 of a given Suslin poset P such that Q_0 is m, n-centered while Q_1 is not. On the other hand, m, n-balance is decidedly a forcing property of posets.

As a final remark, observe that both m, n-balance and m, n-centeredness are properties preserved by countable support product. Thus, the numerous independence results in this chapter can be combined.

13.2 Preservation theorems

The main reason for considering the m, n-balance is that it rules out discontinuous homomorphisms between Polish groups in balanced extensions of the symmetric Solovay model. One way of doing that is to prove that in the given model of ZF+DC, the chromatic number of the Hamming graph $\mathbb{H}_2$ is not two (say using Theorem 11.4.13 or 14.4.7), and then use a result of [76] to show that in ZF+DC this rules out discontinuous homomorphisms. The following theorem works in many cases which are not treatable using the chromatic number of the Hamming graph. Note that a nonprincipal ultrafilter on ω yields in ZF a discontinuous homomorphism from 2^ω to 2; thus, ruling out discontinuous homomorphisms automatically rules out nonprincipal ultrafilters on ω as well. The following theorems are stated using Convention 1.7.16.

Theorem 13.2.1. *Let $n \in \omega$ be a number. In cofinally $n+1, n$-balanced extensions of the symmetric Solovay model, all homomorphisms between Polish groups are continuous.*

Proof. Suppose that κ is an inaccessible cardinal. Suppose that P is a Suslin forcing which is is $n+1, n$-balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ and work in W. Suppose towards a contradiction that the conclusion of the theorem fails: thus, there are Polish groups Γ, Δ, a condition $p \in P$ and a P-name τ such that $p \Vdash \tau \colon \Gamma \to \Delta$ is a discontinuous homomorphism. Both p and τ must be definable in W from parameters in the ground model and some point $z \in 2^\omega$. Let $V[K]$ be some intermediate extension of V by a poset of cardinality smaller than κ such that $z \in V[K]$ and P is $n+1, n$-balanced in $V[K]$. Work in the model $V[K]$.

Let $\bar{p} \leq p$ be a $n+1, n$-balanced condition. Consider the poset P_Γ adding a generic element $\dot{\gamma}_{gen}$ of the group Γ. The following is a key claim:

Claim 13.2.2. *P_Γ forces that there are disjoint basic open sets $O_0, O_1 \subset \Delta$ such that* $\mathrm{Coll}(\omega, <\kappa) \Vdash \exists p_0 \leq \bar{p}\, p_0 \Vdash_P \tau(\dot{\gamma}_{gen}) \in O_0$ *and* $\exists p_1 \leq \bar{p}\, p_1 \Vdash \tau(\dot{\gamma}_{gen}) \in O_1$.

Proof. Suppose towards a contradiction that $q \in P_\Gamma$ forces the opposite. Then, q forces that in the $\mathrm{Coll}(\omega, <\kappa)$-extension there is a unique point $\delta \in \Delta$ such that there is a condition $p' \leq \bar{p}$ in the poset P forcing $\tau(\dot{\gamma}_{gen}) = \delta$. Since the point δ is in the symmetric Solovay model definable from $\tau, \bar{p}$, and $\dot{\gamma}_{gen}$, it must belong to the P_Γ-extension, and there is a P_Γ-name $\dot{\delta}$ for it. Note that then, $q \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p} \Vdash_P \dot{\delta} = \tau(\dot{\gamma}_{gen})$.

Now, pass to the symmetric Solovay model W and consider the set $B = \{\gamma \in \Gamma \colon \gamma$ is P_Γ-generic over $V[K]$ and $\gamma \in q\}$ and the function $f \colon B \to \Delta$ assigning to each point $\gamma \in B$ the point $\dot{\delta}/\gamma$. The function f is continuous on B, and the choice of the name $\dot{\delta}$ implies that $\bar{p} \Vdash \check{f} \subset \tau$ holds. The set $B \subset \Gamma$ is dense G_δ in q. A standard result in the theory of Polish groups [55, Theorems 9.9 and 9.10] now says that a homomorphism from Γ to Δ which is continuous on a set comeager in a nonempty open set is in fact continuous on the whole group Γ. This contradicts the initial choice of the name τ. ☐

Let $X = \{x \in \Gamma^{n+1} \colon \prod_{i \in n} x(i) = x(n)\}$; this is a closed subset of Γ^{n+1}, equipped with the topology inherited from Γ^{n+1}. If $a \subset n+1$ is a set of size n, it is clear that the remaining coordinate of a point $x \in X$ is a continuous function of the coordinates $x(i)$ for $i \in a$, and therefore the projection from X to Γ^a is a continuous and open surjection. Let $x \in X$ be a point P_X generic over $V[K]$. Use Proposition 3.1.1 to see that whenever $a \subset n+1$ are distinct numbers then $x \restriction a \in \Gamma$ is a sequence $(P_\Gamma)^n$-generic over $V[K]$.

Fix $i \in n$. Working in the model $V[K][x(i)]$ find a poset R_i of size $< \kappa$ and a name σ_i for a condition in the poset P stronger than $\bar{p}$ and a name $\dot{\delta}_i$ for an element of the group Δ such that $V[K][x(i)] \models R_i \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_i \Vdash_P \tau(x(i)) = \dot{\delta}_i$. In the model $V[K][x(n)]$, find the disjoint basic open sets $O_0, O_1 \subset \Delta$ as in Claim 13.2.2. Passing to a condition in the product $\prod_{i \in n} R_i$ and switching O_0, O_1 if necessary, we may assume that $V[K][x \restriction n] \models \prod_{i \in n} R_i \Vdash \prod_{i \in n} x(i) \notin O_0$. Use the choice of the set O_0 to find, in the model $V[K][x(n)]$, a poset R_n of size $< \kappa$, R_n-names σ_n for an element of P stronger than $\bar{p}$ and $\dot{\delta}_n$ for an element of $O_0 \subset \Delta$ such that $R_n \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_n \Vdash_P \tau(\check{x}(n)) = \dot{\delta}_n$.

Let $H_i \subset R_i$ for $i \in n+1$ be filters mutually generic over the model $V[K][x]$. For each $i \in n+1$ write $p_i = \sigma_i/H_i \in P$ and $\delta_i = \dot{\delta}_i/H_i$. Note that whenever $a \subset n$ is a set of size n, the models $V[K][x_i][H_i]$ for $i \in a$ are mutually generic extensions of the model $V[K]$. By the balance assumption on the virtual condition $\bar{p}$, the conditions $p_i \in P$ for $i \in n+1$ have a lower bound, call it $q \in P$. Let W be a symmetric Solovay extension of the model $V[K][x][H_i \colon i \in n+1]$ and work in W. The condition q forces that $\tau(x(i)) = \delta_i$ for all $i \in n+1$. Observe that $\prod_{i \in n} = x(n)$ and $\prod_{i \in n} \delta_i \neq \delta_n$ since $\prod_{i \in n} \delta_i \notin O_0$ while $\delta_n \in O_0$. This contradicts the assumption that τ is forced to be a homomorphism from Γ to Δ. ☐

Theorem 13.2.1 has a counterpart for certain quotient groups. Suppose that I

is a Borel ideal on ω. The quotient space $\mathcal{P}(\omega)/I$ will be viewed as a group with the (quotient of the) symmetric difference operation. A homomorphism $h\colon \mathcal{P}(\omega)/I \to \mathcal{P}(\omega)/J$ is *Borel* if the set $\{\langle x, y\rangle \in \mathcal{P}(\omega)\times\mathcal{P}(\omega)\colon h([x]_I) = [y]_J\}$ is Borel.

Theorem 13.2.3. *Let I, J be Borel ideals on ω containing all singletons and suppose that the modulo J equality is a pinned equivalence relation. Let $n\in\omega$ be a number. In cofinally $n+1, n$-balanced extensions of the symmetric Solovay model, all homomorphisms of $\mathcal{P}(\omega)/I$ to $\mathcal{P}(\omega)/J$ are Borel.*

The assumptions are satisfied in particular for the F_σ-ideals J (as then the modulo J equality is an F_σ-equivalence relation and as such pinned [48, Theorem 17.1.3 (iii)]) and for the analytic P-ideals J (as they are Borel and Polishable by [85]. The modulo J equality is an orbit equivalence relation of the action of the Polish abelian group J by the symmetric difference, and orbit equivalence relations of abelian Polish groups are pinned by [48, Theorem 17.1.3 (ii)]).

Proof. Write E for the modulo I equality and F for modulo J equality, both Borel equivalence relations on $X = \mathcal{P}(\omega)$. Suppose that κ is an inaccessible cardinal. Suppose that P is a Suslin forcing which is is $n+1, n$-balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ and work in W. Suppose towards a contradiction that the conclusion of the theorem fails. Thus, there is a condition $p\in P$ and a P-name τ such that p forces τ is a homomorphism. Both p and τ must be definable in W from parameters in the ground model and some point $z\in 2^\omega$. Let $V[K]$ be a generic extension of V by a poset of cardinality smaller than κ such that $z\in V[K]$ and P is $n+1, n$-balanced in $V[K]$.

Work in the model $V[K]$. Let $\bar p \le p$ be a $n+1, n$-balanced condition. Consider the Cohen poset Q on X adding a generic set $\dot x_{gen}\subset\omega$. There are two cases.

Case 1. $Q \Vdash \forall y\subset\omega\ \mathrm{Coll}(\omega, <\kappa) \Vdash \exists r\le \bar p\ r\Vdash_P \tau([\dot x_{gen}]_E) \neq [y]_F$ holds. This case in fact leads to a contradiction. Write Δ for the symmetric difference operation on X and utilize its associativity to apply it also to finite tuples of subsets of ω. Let $\bar X = \{\bar x\in X^{n+1}\colon \Delta_{i\in n} x(i) = x(n)\}$; this is a closed subset of X^{n+1}, equipped with the topology inherited from X^{n+1}. If $a\subset n+1$ is a set of size n, it is clear that the remaining coordinate of a point $x\in X$ is a continuous function of the coordinates $x(i)$ for $i\in a$, and therefore the projection from X to Γ^a is a continuous and open surjection. Let $\bar x\in\bar X$ be a point $P_{\bar X}$ generic over $V[K]$. Use Proposition 3.1.1 to see that whenever $a\subset n+1$ are distinct numbers then $\bar x\restriction a\in\Gamma$ is a sequence Q^n-generic over $V[K]$.

Fix $i\in n$. Working in the model $V[K][x(i)]$ find a poset R_i of cardinality smaller than κ and an R_i-name σ_i for a condition in the poset P stronger than $\bar p$ and an R_i-name η_i for a such that $V[K][x(i)] \models R_i \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_i \Vdash_P \tau([x(i)]_E) = [\eta_i]_F$. Let $H_i\subset R_i$ for $i\in n$ be filters mutually generic over the model $V[K][\bar x]$. Write $p_i = \sigma_i/H_i$ and $y_i = \eta_i/H_i$ for $i\in n$, and look at the set $y = \Delta_{i\in n} y_i \subset\omega$. There are two hopeless subcases.

Case 1a. The equivalence class $[y]_F$ is represented in the model $V[K][\bar{x}(n)]$. Work in the model $V[K][\bar{x}(n)]$. Use the initial case assumption to find a poset R_n, an R_n-name σ_n for a condition in P such that $R_n \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_n \Vdash \tau([\bar{x}(n)]_E) \neq [y]_F$. Let $H_n \subset R_n$ be a filter generic over the model $V[K][\bar{x}][H_i \colon i \in n]$ and let $p_n = \sigma_n/H_n$. Note that for every $j \in n+1$, the models $V[K][\bar{x}(i)][H_i]$ for $i \in n+1 \setminus \{j\}$ are mutually generic over $V[K]$. It follows from the $n+1, n$-balance assumption on the condition $\bar{p}$ that the conditions p_i for $i \in n+1$ have a common lower bound, say $r \in P$. The condition r forces $\tau([\bar{x}(i)]_E) = [y_i]_F$ for all $i \in n$, and $\tau(x_n)$ is not modulo J equivalent to $y = \Delta_{i \in n} y_i$. This contradicts the assumption that τ is forced to be a homomorphism, as $x_n = \Delta_{i \in n} x_i$.

Case 1b. Case 1a fails. Work in the model $V[K][\bar{x}(n)]$. Find a poset R_n, an R_n-name σ_n for a condition in P and an R_n-name η_n for a subset of ω such that $R_n \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_n \Vdash \tau([\bar{x}(n)]_E) \neq [\eta_n]_F$. Let $H_n \subset R_n$ be a filter generic over the model $V[K][\bar{x}][H_i \colon i \in n]$ and let $p_n = \sigma_n/H_n$ and $y_n = \eta_n/H_n$. Now, either the name η_n is F-pinned below some condition in the filter H_n, in which case the F-class $[y_n]_F$ is represented in $V[K][\bar{x}(n)]$ by the assumption that F is pinned, or the name η_n is not F-pinned below any condition in the filter H_n, in which case the F-class $[y_n]_F$ is not represented even in the model $V[K][\bar{x}(i) \colon i \in n][H_i \colon i \in n]$ by the mutual genericity. In either case, $y \; F \; y_n$ fails.

The rest of Case 1b is identical to Case 1a. For every index $j \in n+1$, the models $V[K][\bar{x}(i)][H_i]$ for $i \in n+1 \setminus \{j\}$ are mutually generic over $V[K]$. It follows from the $n+1, n$-balance assumption on the condition $\bar{p}$ that the conditions p_i for $i \in n+1$ have a common lower bound, say $r \in P$. The condition r forces $\tau([\bar{x}(i)]_E) = [y_i]_F$ for all $i \in n$, and $\tau(x_n) = y_n$ is not modulo J equivalent to $y = \Delta_{i \in n} y_i$. This contradicts the assumption that τ is forced to be a homomorphism, as $x_n = \Delta_{i \in n} x_i$.

Case 2. Case 1 fails. By the Borel reading of names for the poset Q, we can find a Borel set $B \subset X$ co-meager in some nonempty open set $O \subset X$ and a Borel function $f \colon B \to X$ such that $O \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \bar{p} \Vdash \tau([\dot{x}_{gen}]_E) = [\dot{f}(\dot{x}_{gen})]_F$. Now, in the model W let $C \subset B$ be the set of all points which are Q-generic over $V[K]$. Note that the set $C \subset B$ is comeager in O and Borel, and by the forcing theorem, $\bar{p} \Vdash_P \forall x \in C \; \tau([x]_E) = [f(x)]_E$. Since the set C is Borel and non-meager, for all infinite sets $y \in X$ there are sets $x_0, x_1 \in C$ such that $x_0 \Delta x_1 = y$ modulo finite (Pettis theorem, [55, Theorem 9.9]). It is then clear that $\bar{p}$ forces the homomorphism τ to be Borel: $\tau([y]_E) = [z]_F$ just in case there exist $x_0, x_1 \in C$ such that $x_0 \Delta x_1 = y$ modulo finite and $z \; F \; f(x_0) \Delta f(x_1)$, and $\tau([y]_E) = [z]_F$ just in case for all $x_0, x_1 \in C$ such that $x_0 \Delta x_1 = y$ modulo finite, $z \; F \; f(x_0) \Delta f(x_1)$. These are in turn analytic and coanalytic descriptions of τ. $\square$

The various degrees of $n+1, n$-balance are not difficult to separate. One particularly elegant way of doing so is to seek monochromatic solutions to equations in Polish groups. We state one prominent case, leaving the others to the patient reader.

Theorem 13.2.4. *Let $\langle \Gamma, \cdot \rangle$ be an uncountable Polish group such that the map $\gamma \mapsto \gamma \cdot \gamma$ is a self-homeomorphism of Γ. In cofinally $3, 2$-balanced extensions of the symmetric Solovay model, for every coloring $c \colon \Gamma \to \omega$ there is a solution to the equation $xy = zz$ consisting of monochromatic, pairwise distinct points in Γ.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is cofinally $3, 2$-balanced below κ. Let W be the symmetric Solovay model derived from κ and work in W. Let τ be a P-name and $p \in P$ be a condition forcing τ to be a map from Γ to ω. We must find a solution to the equation $xx = yz$ consisting of pairwise distinct points and a condition stronger than p which forces the point of the solution to have all the same color.

The name τ and the condition p must both be definable with parameters in the ground model and another parameters $z \in 2^\omega$. Let $V[K]$ be an intermediate model obtained by a poset of cardinality smaller than κ such that $z \in V[K]$ and P is $3, 2$-balanced in $V[K]$. Work in the model $V[K]$. Let $\bar{p} \leq p$ be a $3, 2$-balanced virtual condition. Let P_Γ be the Cohen poset on the group Γ with its name for the generic point $\dot{\gamma}_{\text{gen}}$ in Γ. There must be a cardinal $\lambda < \kappa$, a number $n \in \omega$, a $P_\Gamma \times \text{Coll}(\omega, \lambda)$-name σ and a condition $O \in P_\Gamma$ such that $O \Vdash \text{Coll}(\omega, \lambda) \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau(\dot{\gamma}_{\text{gen}}) = \check{n}$.

Let $\gamma \in O$ be an arbitrary point, and let $\delta \in \Gamma$ be a point so close (but not equal) to the unit that both $\gamma\delta$ and $\delta^{-1}\gamma$ belong to the open set $O \subset \Gamma$. Note that the points $\gamma\delta$, $\delta^{-1}\gamma$, and γ are pairwise distinct and solve the equation $xy = zz$. (If $\gamma\delta = \delta^{-1}\gamma$ then the square of this point would give the same result as $\gamma\gamma$, contradicting the assumption that the squaring function is a self-homeomorphism, in particular a bijective self-map, of Γ.) Let $O_0, O_1, O_2 \subset O$ be pairwise disjoint open sets separating these three points, and let $\bar{X} = \{\bar{x} \in O_0 \times O_1 \times O_2 \colon \bar{x}(0)\bar{x}(1) = \bar{x}(2)\bar{x}(2)\}$. This is a relatively closed subset of $O_0 \times O_1 \times O_2$. Moreover, each coordinate of a point in $\bar{X}$ is a continuous function of the other two (in the case of the third coordinate this again uses the assumption that the squaring function is a self-homeomorphism of Γ) and therefore the projection maps from $\bar{X}$ into any pair of coordinates are open maps from $\bar{X}$ to Γ^2.

Consider the Cohen poset $P_{\bar{X}}$. Let $\bar{x} \in \bar{X}$ be a triple of points which is $P_{\bar{X}}$-generic over the model $V[K]$. Use Proposition 3.1.1 to see that these points are pairwise mutually P_Γ-generic below the condition O. Let $H_i \subset \text{Coll}(\omega, \lambda)$ for $i \in 3$ be filters mutually generic over the model $V[X][\bar{x}]$, and let $p_i = \sigma_i/H_i$. Since the extensions $V[K][\bar{x}(i)][H_i]$ for $i \in 3$ are pairwise mutually generic, the balance assumption on the virtual condition $\bar{p}$ shows that the conditions p_i for $i \in 3$ have a common lower bound, say $r \in P$. By the forcing theorem applied in the respective models $V[K][\bar{x}(i)][H_i]$ for $i \in 3$, we see that r forces all three points in the triple $\bar{x}$ to have the same color n. At the same time, the triple solves the equation $xy = zz$ as required. $\square$

Finally, we move to the much better organized world of $3, 2$-centered forcings.

Theorem 13.2.5. *Let Γ be an acyclic Borel graph on a Polish space X such that the Γ-path connectedness relation is Borel and unpinned. In $3,2$-centered, cofinally balanced extensions of the symmetric Solovay model, Γ has no maximal acyclic subgraph.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin, $3,2$-centered poset which is balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ. Work in W. Suppose that $p \in P$ is a poset and τ is a P-name such that p forces τ to be a spanning subset of Γ. We will find a cycle in Γ and a condition stronger than p which forces the cycle to be a subset of τ.

The condition p and the name τ are definable from ground model parameters and an additional parameter in 2^ω. Let $V[K]$ be an intermediate extension containing z obtained by a poset of cardinality smaller than κ and such that P is balanced in $V[K]$. Work in the model $V[K]$. Let $\bar{p} \leq p$ be a balanced virtual condition in P. The equivalence relation E is not pinned in $V[K]$ by Corollary 2.7.3, and there is a nontrivial E-pinned name on the poset collapsing $\aleph_1$ to $\aleph_0$ by Theorem 2.6.3. By the cofinal balance assumption, there must be a poset Q of cardinality smaller than κ, a nontrivial E-pinned Q-name η for an element of X, and a Q-name σ for a balanced virtual condition in the Q-extension which is stronger than $\bar{p}$. Let $H_0, H_1, H_2 \subset Q$ be filters mutually generic over $V[K]$ and write $\bar{p}_i = \sigma/H_i$ and $x_i = \eta/H_i$ for $i \in 3$. For distinct indices $i, j \in 3$, we conclude by the balance of the condition $\bar{p}$ in $V[K]$ that the virtual conditions $\bar{p}_i$, $\bar{p}_j$ must be compatible. Thus, in the model $V[K][H_i][H_j]$, there must be a poset Q_{ij} and Q_{ij}-name σ_{ij} for a condition in P stronger than both $\bar{p}_i$ and $\bar{p}_j$, and a Q_{ij}-name χ_{ij} for a Γ-path between x_i and x_j such that $V[K][H_i][H_j] \models Q_{ij} \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_{ij} \Vdash \chi_{ij} \subset \tau$.

Let $G_{ij} \subset Q_{ij}$ be filters mutually generic over the model $V[K][H_i \colon i \in 3]$. Write $p_{ij} = \sigma_{ij}/G_{ij}$ and $u_{ij} = \chi_{ij}/G_{ij}$. Observe that whenever $i \in 3$ and $j_0, j_1 \in 3$ are the two indices distinct from i, then p_{ij_0} and p_{ij_1} are compatible conditions in P by the balance of the condition $\bar{p}_i$ in $V[K][H_i]$. By the $3,2$-centeredness assumption, the conditions p_{ij} for $i, j \in 3$ distinct have a common lower bound $r \in P$. Note also that after erasing some local loops if necessary $\bigcup_{ij} u_{ij}$ is a cycle. By the forcing theorem applied in each model $V[K][H_i, H_j][G_{ij}]$, it follows that $r \Vdash \bigcup_{ij} u_{ij} \subset \tau$ as desired. □

Theorem 13.2.6. *Let X be an uncountable Polish space. In $3,2$-centered, cofinally balanced extensions of the symmetric Solovay model, every set mapping $f \colon [X]^2 \to X^{\aleph_0}$ has a free triple.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin, $3,2$-centered poset which is balanced cofinally below κ. Let W be a symmetric Solovay model derived from κ. Work in W. Suppose that $p \in P$ is a poset and τ is a P-name such that p forces τ to be a function from $[X]^2$ to $[X]^{\aleph_0}$. We will find a triple of points in X and a condition stronger than p which forces the triple to be free for τ.

The condition p and the name τ are definable from ground model parameters and an additional parameter in 2^ω. Let $V[K]$ be an intermediate extension containing z obtained by a poset of cardinality smaller than κ and such that P is balanced in $V[K]$. Work in the model $V[K]$. Let $\bar{p} \leq p$ be a balanced virtual condition in P. By the cofinal balance assumption, there must be a poset Q of cardinality smaller than κ, a Q-name η for an element of X which does not belong to $V[K]$, and a Q-name σ for a balanced virtual condition in the Q-extension which is stronger than $\bar{p}$. Let $H_0, H_1, H_2 \subset Q$ be filters mutually generic over $V[K]$ and write $\bar{p}_i = \sigma/H_i$ and $x_i = \eta/H_i$ for $i \in 3$. For distinct indices $i, j \in 3$, we conclude by the balance of the condition $\bar{p}$ in $V[K]$ that the virtual conditions $\bar{p}_i$, $\bar{p}_j$ must be compatible. Thus, in the model $V[K][H_i][H_j]$, there must be a poset Q_{ij} and Q_{ij}-name σ_{ij} for a condition in P stronger than both $\bar{p}_i$ and $\bar{p}_j$, and a Q_{ij}-name χ_{ij} for an element of X^ω such that $V[K][H_i][H_j] \models Q_{ij} \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma_{ij} \Vdash \mathrm{rng}(\chi_{ij}) = \tau(\check{x}_i, \check{x}_j)$.

Let $G_{ij} \subset Q_{ij}$ be filters mutually generic over the model $V[K][H_i \colon i \in 3]$. Write $p_{ij} = \sigma_{ij}/G_{ij}$ and $u_{ij} = \chi_{ij}/G_{ij}$. Observe that whenever $i \in 3$ and $j_0, j_1 \in 3$ are the two indices distinct from i, then p_{ij_0} and p_{ij_1} are compatible conditions in P by the balance of the condition $\bar{p}_i$ in $V[K][H_i]$. By the $3,2$-centeredness assumption, the conditions p_{ij} for $i, j \in 3$ distinct have a common lower bound $r \in P$. Note also that for each $k \in 3$, the point x_k does not belong to the model $V[K][H_i, H_j][G_{ij}]$ where $i, j \in 3$ are the two indices distinct from k, and therefore does not belong to the range of u_{ij}. It follows that $r \Vdash \{x_i \colon i \in 3\}$ is a free set for τ as desired. □

Theorem 13.2.7. *Let E be a pinned Borel equivalence relation on a Polish space X. In $3,2$-centered, tethered, and cofinally balanced extensions of the symmetric Solovay model, the following are equivalent:*

1. *E has a transversal;*
2. *the E-quotient space is linearly ordered.*

Proof. The implication (1)→(2) is trivial. In ZF, the space X, as every Polish space, carries a Borel linear ordering $\leq$. If $A \subset X$ is an E-transversal, then one can order the E-quotient space by setting $c \prec d$ if the unique element of $A \cap c$ is $\leq$-smaller than the unique element of $A \cap d$. The implication (2)→(1) is the heart of the matter.

Let κ be an inaccessible cardinal. Let P be a Suslin forcing which is $3,2$-centered, balanced cofinally below κ and tethered below κ. Let W be a symmetric Solovay model derived from κ and work in W. Suppose that $p \in P$ is a condition forcing E not to have a transversal. Suppose that τ is a P-name for a tournament on the E-quotient space. We will find a 3-cycle on the E-quotient space and a condition stronger than p which forces the cycle to be a subset of τ. This will prove the theorem.

The condition p and the name τ are definable from some ground model parameters and an additional parameter $z \in 2^\omega$. As a provisional definition, call a tuple $\langle M, \bar{p}, Q, \eta, \sigma \rangle$ *symmetric* if M is an intermediate generic extension of V by a poset of cardinality smaller than κ such that $z \in M$, $\bar{p} \leq p$ is a balanced virtual condition in the model M, $Q \in M$ is a poset of cardinality smaller than κ, $\eta \in M$ is a Q-name for an element of X which is not E-related to any element of $M \cap X$, $\sigma \in M$ is a Q-name for a balanced virtual condition in the Q-extension of M which is stronger than $\bar{p}$, and

(*) for any conditions $q_0, q_1 \in Q$, in some generic extension there are filters $H_0, H_1 \subset Q$ separately generic over M such that $q_0 \in H_0$, $q_1 \in H_1$, η/H_0 is E-related to η/H_1, and σ/H_0 is compatible with σ/H_1 in the poset P.

The following claim is central.

Claim 13.2.8. *There is a symmetric tuple $\langle M, \bar{p}, Q, \eta, \sigma\rangle$.*

Proof. Let $G \subset P$ be a filter generic over the model W containing the condition p and work in $W[G]$. For each E-class c, consider the model M_c of sets hereditarily definable from parameters in the ground model and the additional parameters z, c, G. There must be an E-class c such that $c \cap M_c = 0$; otherwise, one could form an E-selector in $W[G]$ as the set $A = \{x \in X : x$ is the least element of $[x]_E$ in the canonical well-order of $M_{[x]_E}\}$, contradicting the initial assumptions on the condition p. Fix such an E-class c, an arbitrary element $x \in c$, and let M_x be the model of sets hereditarily definable from parameters in the ground model and the additional parameters z, x, G. Note that $M_c \subset M_x$ holds.

By the balance assumption, the models M_c and M_x are generic extensions of the ground model by posets of cardinality smaller than κ; in particular, M_x is a generic extension of M_c by a poset of cardinality smaller than κ. By the tether assumption, as in Proposition ???, there are virtual balanced conditions $\bar{p}_c$ and $\bar{p}_x$ in the respective models M_c and M_x such that their realizations belong to the generic filter G. Necessarily $\bar{p}_x \leq \bar{p}_c \leq p$ holds. Working in M_c, let Q be a poset of cardinality smaller than κ such that there is a filter $H \subset Q$ generic over M_c such that $M_x = M_c[H]$. Let $\eta \in M_c$ be a Q-name such that $x = \eta/H$ and let σ be a Q-name such that $\bar{p}_x = \sigma/H$. We claim that for some condition $q \in Q$, the tuple $\langle M, \bar{p}_c, Q \restriction q, \eta, \sigma\rangle$ is symmetric.

To see this, in $W[G]$ let $D \subset Q$ be the set of all conditions $r \in Q$ such that there is a filter $H' \subset Q$ which is generic over M_c and such that $\eta/H' \in c$ and σ/H' has a realization in the filter G. The set, having just been defined from c and G, belongs to the model M_c. It also contains the Q-generic filter $H \subset Q$ as a subset. By a density argument with the filter H, there must be a condition $q \in H$ such that D contains all conditions $r \in Q$ such that $r \leq q$. It is immediate from the definition of the set D that the condition $q \in Q$ works as required. $\square$

Now, fix the symmetric tuple $\langle M, \bar{p}, Q, \eta, \sigma\rangle$. Let $H_0 \subset Q_0$ and $H_1 \subset Q_1$ be filters mutually generic over M, and write $x_0 = \eta/H_0$, $x_1 = \eta/H_1$, $\bar{p}_0 = \sigma/H_0$ and $\bar{p}_1 = \sigma/H_1$.

Claim 13.2.9. *In some generic extension N of $M[H_0, H_1]$ by a poset of cardinality smaller than κ there is a condition $r \in P$ stronger than $\bar{p}_0, \bar{p}_1$ such that $N \models \mathrm{Coll}(\omega, < \kappa) \Vdash r \Vdash_P \langle [x_0]_E, [x_1]_E\rangle \in \tau$.*

Proof. To start with, note that $Q \times Q$ forces $\sigma_{\rm left}$ and $\sigma_{\rm right}$ to be compatible virtual conditions in P by the balance assumption on $\bar p$, and it also forces $\eta_{\rm left}$ and $\eta_{\rm right}$ to be E-unrelated elements of X by the pinned assumption on the equivalence relation E. Let $\langle q_0, q_1\rangle \in H_0, q_1 \in H_1$ be conditions and suppose towards a contradiction that in M, the condition $\langle q_0, q_1\rangle$ forces the negation of the statement of the claim.

Let $\lambda < \kappa$ be a cardinal larger than $|\mathcal{P}(Q) \cap M|$ and let $K_0, K_1 \subset \mathrm{Coll}(\omega, \lambda)$ be filters mutually generic over $M[H_0, H_1]$. By (*) and the forcing theorem, there are filters $H_0' \in M[H_0][K_0]$ and $H_1' \in M[H_1][K_1]$ which are generic over M, $q_0 \in H_1'$ and $q_1 \in H_0'$ holds, and moreover, writing $x_0' = \eta/H_0'$, $x_1' = \eta/H_1'$, $\bar p_0' = \sigma/H_0'$ and $\bar p_1' = \sigma/H_1'$, we have that $x_0 \mathrel{E} x_0'$ and $x_1 \mathrel{E} x_1'$ both hold, and $\bar p_0$ and $\bar p_0'$ are compatible virtual conditions in P, and $\bar p_1$ and $\bar p_1'$ are compatible virtual conditions in P. Note also that the filters $H_0', H_1' \subset Q$ are mutually generic over M since they are found in the respective mutually generic extensions $M[H_0][K_0]$ and $M[H_1][K_1]$–Corollary 1.7.9.

Now, observe that the conditions $\bar p_0, \bar p_0', \bar p_1, \bar p_1'$ have a common lower bound in the poset P. To see this, note that $\bar p_0, \bar p_0'$ are compatible with a lower bound $r_0 \in P$ in the model $M[H_0][K_0]$, and $\bar p_1, \bar p_1'$ are compatible with a lower bound $r_1 \in P_1$ in the model $M[H_1][K_1]$. Now use the balance of the condition $\bar p$ to see that r_0, r_1 are compatible conditions in P.

Finally, there must be a generic extension N of the model $M[H_0, H_1][K_0, K_1]$ by a poset of cardinality smaller than κ in which there is a condition $r \in P$ which is a common lower bound of r_0, r_1 and such that $N \models \mathrm{Coll}(\omega, <\kappa) \Vdash r \Vdash_P \langle [x_0]_E, [x_1]_E\rangle \in \tau$ or $N \models \mathrm{Coll}(\omega, <\kappa) \Vdash r \Vdash_P \langle [x_0]_E, [x_1]_E\rangle \in \tau$. The former option violates the initial contradictory assumption in view of the forcing theorem in the model M and the generic filters H_0, H_1, and the latter option violates the initial contradictory assumption in view of the generic filters H_0', H_1'. □

Finally, let $H_0, H_1, H_2 \subset Q$ be filters mutually generic over the model M. For each $i \in 3$ write $x_i = \eta/H_i$ and $\bar p_i$ for σ/H_i. Use the claim to find a poset $R_{01} \in M[H_0][H_1]$ of cardinality smaller than κ and an R-name χ_{01} for a common lower bound of $\bar p_0, \bar p_1$ in P such that $M[H_0][H_1] \models R_{01} \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \chi_{01} \Vdash \langle [\check x_0]_E, [\check x_1]_E\rangle \in \tau$. Similarly for R_{12}, χ_{12} and R_{20}, χ_{20}. Let $K_{01} \subset R_{01}$, $K_{12} \subset R_{12}$, and $K_{20} \subset R_{20}$ be filters mutually generic over the model $M[H_i \colon i \in 3]$. Write $p_{01} = \chi_{01}/K_{01}$ and similarly for p_{12}, p_{20}. Note that the conditions p_{01} and p_{12} are compatible in P by the balance of the virtual condition $\bar p_1 \in M[H_1]$: they are found in the respective extensions $M[H_1][H_0][K_{01}]$ and $M[H_1][H_2][K_{12}]$ mutually generic over $M[H_1]$, and they are both stronger than $\bar p_1$. Similarly, the conditions p_{12} and p_{20} are compatible, and the conditions p_{20} and p_{01} are compatible. Finally, the $3,2$-centeredness assumption shows that the conditions p_{01}, p_{12}, p_{20} have a common lower bound $q \leq p$. The forcing theorem applied in the respective models $M[H_0, H_1][K_{01}]$, $M[H_1][H_2][K_{12}]$ and $M[H_2][H_0][K_{20}]$ shows that in the model W, q forces the sequence $\langle [x_0]_E, [x_1]_E, [x_2]_E, [x_0]_E\rangle$ to form an oriented 3-cycle which is a subset of τ. The proof of the theorem is complete. □

13.3 Examples

The first batch of examples deals with the implications and limitations of Theorem 13.2.7. We use Theorem 13.2.1 to add the nonexistence of discontinuous homomorphisms of Polish groups to the conclusions for reference purposes.

Example 13.3.1. Let E be a Borel equivalence relation on a Polish space X. Let P be the poset for adding a tournament on the E-quotient space as in Example 8.6.6. Then P is $3, 2$-centered by its definition, balanced by Theorem 8.6.4, and tethered by Example 10.5.8. It is compactly balanced, so does not add a $\mathbb{E}_0$-transversal by Example 9.2.11. Thus, Theorem 13.2.7 applies to it, as do all other theorems of Section 13.2.

Corollary 13.3.2. *Let E be a Borel equivalence relation on a Polish space X.*

1. *Let P be the tournament poset associated with E as in Example 8.6.6. In the P-extension of a symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups, an there is no linear ordering on the $\mathbb{E}_0$-quotient space.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a tournament on the E-quotient space while there are no discontinuous homomorphisms of Polish groups and there is no linear ordering on the $\mathbb{E}_0$-quotient space.*

Example 13.3.3. Let E be a countable Borel equivalence relation on a Polish space X. Let P be the poset adding an action of $\mathbb{Z}$ on X inducing E as in Example 6.6.10. The poset P is $3, 2$-centered by its definition, balanced by Example 6.6.10, tethered by Example 10.5.7, and it does not add an $\mathbb{E}_0$-transversal by Corollary 11.4.12. Thus, Theorem 13.2.7 applies to it, as do all other theorems of Section 13.2.

Corollary 13.3.4. *Let E be a countable Borel equivalence relation on a Polish space X.*

1. *Let P be the poset adding an action of $\mathbb{Z}$ on X inducing E as in Example 6.6.10. In the P-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups, and there is no linear ordering on the $\mathbb{E}_0$-quotient space.*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, E is an orbit equivalence relation of a (discontinuous) $\mathbb{Z}$-action, there are no discontinuous homomorphisms of Polish groups, and there is no linear ordering on the $\mathbb{E}_0$-quotient space.*

Example 13.3.5. Let Γ be a Borel graph on a Polish space X Suppose that Γ does not contain an injective homomorphic copy of $K^{\rightarrow}_{\omega,\omega}$ and K_{n,ω_1} for some number $n \in \omega$. Let P be the coloring poset of Definition 8.1.1. The poset P is $3, 2$-centered by its definition, it is balanced by Theorem 8.1.2 and tethered by Example 10.5.4. It does not add a $\mathbb{E}_0$-trasnversal by Theorem 11.5.6. Thus, Theorem 13.2.7 applies to it, as do all other theorems of Section 13.2.

Corollary 13.3.6. *Let $A \subset \mathbb{R}$ be a countable set of positive relas converging to zero. Let Γ be the graph on $\mathbb{R}^2$ connecting two points if their Euclidean distance belongs to A.*

1. *Let P be the coloring poset of Definition 8.1.1. In the P-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups, and there is no linear ordering of the $\mathbb{E}_0$-quotient space.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable, and yet there is no discontinuous homomorphism between Polish groups and no linear ordering of the $\mathbb{E}_0$-quotient space.*

The next example shows that the tether assumption is necessary in Theorem 13.2.7.

Example 13.3.7. Let P be the collapse poset of $|\mathbb{E}_0|$ to $|2^\omega|$ as in Definition 6.6.2. The poset P is $3,2$-centered by its definition and balanced by Corollary 6.6.4. It does not add an $\mathbb{E}_0$-transversal by Corollary 11.6.3, but it does add a linear ordering of the $\mathbb{E}_0$-quotient space by mapping it injectively into the linearly ordered set 2^ω.

The following example shows that the assumption that E be pinned is necessary in Theorem 13.2.7.

Example 13.3.8. Let P be the poset adding a function which selects from each nonempty countable subset of $X = 2^\omega$ a single element as in Example 6.6.15. The poset is $3,2$-centered by its definition, balanced by Theorem 6.6.12, tethered by Example 10.5.6, and adds a linear ordering of the $\mathbb{F}_2$-quotient space without adding a $\mathbb{F}_2$-transversal.

Proof. The poset P does not add an $\mathbb{F}_2$-transversal since no balanced poset does by Corollary 9.1.5. At the same time, an existence of a function $f\colon [X]^{\aleph_0} \to X$ such that for each nonempty countable set $a \subset X$ $f(a) \in a$ holds implies $|\mathbb{F}_2| \leq |X^{<\omega_1}|$. To see the injection from the $\mathbb{F}_2$-space to $X^{<\omega_1}$, to each countable set $a \subset X$ assign a transfinite sequence $\langle x_\alpha \colon \alpha \in \beta$ by setting recursively $x_\alpha = f(a \setminus \{x_\gamma \colon \gamma \in \alpha\})$ unless the set $a \setminus \{x_\gamma \colon \gamma \in \alpha\}$ is empty, in which case we finish the recursion. The set $X^{<\omega_1}$ is linearly ordered by the lexicographic ordering, and so the poset P adds a linear ordering of the $\mathbb{F}_2$-space. $\square$

The following example shows that the pinned assumption in Theorem 13.2.5 is necessary.

Example 13.3.9. Let E be a pinned Borel equivalence relation on a Polish space X. Let P be the transversal poset of Example 6.6.8. The poset P is $3,2$-centered by its definition and it is balanced by Theorem 6.6.6. Letting Γ

be the graph on X connecting any two distinct E-related points, note that P adds a maximal acyclic subgraph to Γ–it is the graph consisting of all edges $\{x_0, x_1\} \in \Gamma$ such that one of the vertices x_0, x_1 belongs to the E-selector added by P.

Corollary 13.3.10. *Let E be a pinned Borel equivalence relation on a Polish space X.*

1. *Let P be the transversal poset of Example 6.6.8. In the P-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, E has a transversal, and yet there is no discontinuous homomorphism between Polish groups.*

The next batch of examples deals with partial orders which are m, n-balanced but not $3, 2$-centered, and violate the conclusions of various preservation theorems for $3, 2$-centered posets from Section 13.2.

Example 13.3.11. Let Γ be a Borel graph on a Polish space X. Let P be the poset of all countable acyclic subsets of Γ ordered by reverse inclusion. Whenever $n \in \omega$ is a natural number, the poset P is $n, 2$-balanced and every balanced virtual condition is $n, 2$-balanced.

The poset P adds a maximal acyclic subset of Γ, violating the conclusion of Theorem 13.2.5 for many graphs Γ. Therefore, in such cases the poset P is not $3, 2$-centered and even does not contain a dense analytic $3, 2$-centered subset.

Proof. It follows from Example 6.4.10 that balanced virtual conditions for P are classified by maximal acyclic subgraphs of Γ. Let $\bar{p}$ be any maximal acyclic subgraph of Γ. We will check that $\bar{p}$ is $n, 2$-balanced, completing the proof. To this end, let $\{V[H_i] \colon i \in n\}$ be pairwise mutually generic extensions of V, respectively containing some conditions $p_i \leq \bar{p}$ for each $i \in n$.

Claim 13.3.12. *For each $i \in n$ and vertices $x_0, x_1 \in X \cap V$, the following three formulas are equivalent:*

1. *x_0, x_1 are connected by a path in $\Gamma \cap V$;*
2. *x_0, x_1 are connected by a path in $\bar{p}$;*
3. *x_0, x_1 are connected by a path in p_i.*

Proof. (1)→(2) is implied by the maximality of $\bar{p}$. (2)→ (3) follows from the fact that $\bar{p} \subset p_i$, and the negation of (1) implies by the Mostowski absoluteness that x_0, x_1 are not connected by any path in the graph $\Gamma \cap V[H_i]$, which is larger than p_i and therefore (3) has to fail. ∎

The pairwise mutual genericity implies that if $i, j \in n$ are distinct numbers and $v_i, v_j \subset X$ are the sets of vertices mentioned in some edges in p_i, p_j respectively, then $v_i \cap v_j \subset V$. This, together with the claim, means that the union $\bigcup_i p_i$ cannot contain a cycle and so is the desired common lower bound of the conditions p_i for $i \in n$. □

Corollary 13.3.13. *Let Γ be a Borel graph on a Polish space X.*

1. *Write P for the poset adding a maximal acyclic subgraph of Γ as in Example 6.4.10. In the P-extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, Γ has a maximal acyclic subgraph and every homomorphism between Polish groups is continuous.*

Example 13.3.14. Let $\mathcal{F}$ be a Fraissé class of structures in finite relational language with strong amalgamation. Let E be a Borel equivalence relation on a Polish space X. Let $P = P(\mathcal{F}, E)$ be the poset of Definition 8.6.3. For each number $n \in \omega$, the poset P is $n, 2$-balanced and every balanced virtual condition is $n, 2$-centered.

Note that in case of the linearization poset P for the $\mathbb{E}_0$-quotient space, as isolated in Example 8.6.5, is tethered by Example 10.5.8 and does not add an $\mathbb{E}_0$-selector by Example 9.2.11. Therefore, P violates the conclusion of Theorem 13.2.7, so P contains no dense analytic $3, 2$-centered subset.

Proof. It follows from Theorem 8.6.4 that balanced virtual conditions in P are classified by $\mathcal{F}$-structures on the virtual quotient space X^{**}. Let $\bar{p}$ be such a structure. Let $V[H_i]$ for $i \in n$ be generic extensions of the ground model which are pairwise mutually generic. Let $p_i \in V[H_i]$ be a condition extending $\bar{p}$, for each $i \in n$. Note that the domains of the conditions pairwise intersect in the domain of $\bar{p}$. Let $b = \bigcup_i \operatorname{dom}(p_i)$. For each finite set $a \subset b$ use the strong amalgamation of the class $\mathcal{F}$ $n-1$-many times to find a structure $N_a \in \mathcal{F}$ on a so that for each $i \in n$, $N_a \restriction (\operatorname{dom}(p_i) \cap a) = p_i \restriction (\operatorname{dom}(p_i) \cap a)$. Let U be an ultrafilter on $[b]^{<\aleph_0}$ such that for every finite set $c \subset b$, the set $\{a \in [b]^{<\aleph_0} : c \subset a\}$ belongs to U. Let N be the structure on b which is the U-integral of the structures N_a. It is immediate that for all $N \restriction \operatorname{dom}(p_i) = p_i$ for each $i \in 3$, and the closure of the class $\mathcal{F}$ under substructures shows that N is a $\mathcal{F}$-structure. It is the desired lower bound of the conditions p_i for $i \in n$. □

Corollary 13.3.15. *Let E be a Borel equivalence relation on a Polish space X.*

1. *Let P be the E-linearization poset as in Example 8.6.5. In the P-generic extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;*

2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the E-quotient space carries a linear ordering and every homomorphism between Polish groups is continuous.*

Example 13.3.16. Let X be a Borel vector space over a countable field Φ. Let P be the 3-Hamel decomposition poset of Definition 8.1.11. The poset P is $4, 3$-balanced.

Theorem 13.2.4 shows that the poset is not $3, 2$-balanced as soon as the field Φ has characteristic distinct from 2.

Proof. By Theorem 8.1.12, balanced virtual conditions are classified by colorings $c\colon X \to \omega$ such that any monochromatic triple of pairwise distinct nonzero elements of X is linearly independent. For each such coloring c, the pair $\text{Coll}(\omega, X), \check{c}\rangle$ is balanced. We will show that the pair is in fact $4, 3$-balanced.

Suppose that in some ambient generic extension, $V[G_i]$ for $i \in 4$ are generic extensions and any three of them are mutually generic. Suppose that $p_i \in V[G_i]$ for $i \in 4$ are conditions below c; we must show that they have a common lower bound. Write $d_i = \text{dom}(p_i)$ for $i \in 4$. Let $d \subset X$ be a countable subspace containing $\bigcup_i d_i$ as a subset and let $a = d \setminus \bigcup_i d_i$. For each point $x \in a$ and distinct indexes $i, j \in 4$ let $e_{ij}(x) = \{y \in d_i \colon \exists z \in d_j \colon x, y, z$ are linearly dependent$\}$ and as in Claim 8.1.14 argue that each the set $e_{ij}(x)$, if nonempty, is a union of Φ-shifts of a single $X \cap V$-coset distinct from V.

Now, let J be the Borel ideal on ω used in the definition of P as in Definition 8.1.11. Let $b \in J$ be a set which cannot be covered by finitely many sets of the form $p_i''e_{ij}$ and a finite set. Let $b = \bigcup\{b_x \colon x \in a\}$ be a partition of the set b into sets with the same property. Let $q\colon d \to \omega$ be any map such that $\bigcup_i p_i \subset q$ and for each $x \in a$, $q(x) \in b_x \setminus \bigcup_{ij} p_i''e_{ij}(x)$. Note that the latter union consists of sets in the ideal J and so the set $b_x \setminus \bigcup_{ij} p_i''e_{ij}(x)$ is nonempty. We claim that q is a common lower bound of the conditions p_i for $i \in 4$.

We will only show that any triple $\{x_0, x_1, x_2\}$ of pairwise distinct nonzero linearly dependent points is not monochromatic. There are several cases to consider. If two points of the triple belong to one and the same set d_i then so does the third one, then we use the assumption that $p_i \in P$ is a coloring without such monochromatic triples. The case in which each point on the triple belongs to some d_i but never to the same one, say $x_0 \in d_0 \setminus V$, $x_1 \in d_1 \setminus V$, and $x_2 \in d_2 \setminus V$ is impossible in view of the assumption that the three extensions $V[G_0], V[G_1], V[G_2]$ are mutually generic. If one of the points, say x_0 belongs to a and the other two do not, say $x_1 \in d_i$ and $x_2 \in d_j$ for $i, j \in 4$ distinct, then $q(x_0)$ is distinct from all colors in the set $p_i''e_{ij}(x_0)$ and so is different from $p_i(x_1)$, and monochromaticity fails again. Finally, if more than one point of the triple is in the set a, then the triple cannot be monochromatic since the map $q \restriction a$ is an injection. $\square$

Corollary 13.3.17. *Let X be a Borel vector space over a countable field Φ.*

1. *Let P be the 3-Hamel decomposition forcing as in Definition 8.1.11. In the P-generic extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;*
2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, there exists a 3-Hamel decomposition of X, and every homomorphism between Polish groups is continuous.*

Chapter 14

Other combinatorics

The technology of balanced conditions can be applied to prove general theorems about lack of other combinatorial objects in the extensions under discourse. In this section, we include several theorems that are hopefully elegant axiomatizations and generalizations of earlier results in [63].

14.1 Maximal almost disjoint families

One rather surprising limitation of balanced extensions of the symmetric Solovay model is that they contain no maximal almost disjoint families of subsets of ω. The following theorem is stated using Convention 1.7.16.

Theorem 14.1.1. *In cofinally balanced extensions of the symmetric Solovay model there are no infinite maximal almost disjoint families of subsets of ω.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin forcing such that P is balanced below κ. Let W be a symmetric Solovay model derived from κ and work in W. Suppose towards a contradiction that there is a condition $p \in P$ and a name τ such that p forces τ to be an infinite maximal almost disjoint family. The condition p as well as the name τ must be definable from some parameter $z \in 2^\omega$ and some parameters from the ground model. Use Fact 1.7.14 and the assumptions to find an intermediate generic extension $V[K]$ such that $z \in V[K]$ and P is balanced in $V[K]$. Work in the model $V[K]$.

Let $\bar{p} \leq p$ be a balanced condition. Let I be the set $\{a \subset \omega\colon$ for some poset R_a of size $< \kappa$ and some R_a-name σ_a for a condition in P stronger than $\bar{p}$ such that $R_a \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \check{a} \in \tau\}$.

Claim 14.1.2. *ω cannot be covered by a finite set and finitely many elements of the set I.*

Proof. Suppose that $J \subset I$ is a finite set such that $\bigcup J \subset \omega$ is co-finite. Let $H_a \subset R_a$ for $a \in J$ be filters mutually generic over $V[K]$ and let $p_a = \sigma_a/H_a \in P$. By the balance of the condition $\bar{p}$, it follows that the set $\{p_a\colon a \in J\} \subset P$

has a lower bound, denote it by q. Since the model W is a symmetric Solovay extension of each of the models $V[K][H_a]$, the forcing theorem applied in the model $V[K][H_a]$ shows that in W, for each $a \in J$, the condition q forces $J \subset \tau$. However, since $\bigcup J \subset \omega$ is cofinite, it has to be the case that $q \Vdash \check{J} = \tau$, contradicting the initial assumptions on the name τ. $\square$

Let U be a nonprincipal ultrafilter on ω disjoint from I. There must be a poset R of size $< \kappa$, an R-name η for an infinite subset of ω which is modulo finite included in all sets in U, an R-name χ for a subset of ω and an R-name σ for an element of P stronger than $\bar{p}$ such that $R \Vdash \chi \cap \eta$ is infinite and $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \chi \in \tau$. This occurs since τ is forced to be a maximal almost disjoint family and therefore must contain an element with infinite intersection with η.

Now, let $H_0, H_1 \subset R$ be filters mutually generic over the model $V[K]$. Let $a_0 = \chi/H_0, a_1 = \chi/H_1 \in \mathcal{P}(\omega)$, and $p_0 = \sigma/H_0, p_1 = \sigma/H_1 \in P$. By the balance of the condition $\bar{p} \leq p$ it must be the case that $p_0, p_1 \in P$ have a lower bound, denote it by q. Since W is the symmetric Solovay extension of each of the models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem applied in these models shows that W satisfies that q forces both $\check{a}_0$ and $\check{a}_1$ into τ. The proof will be complete if we show that $a_0 \neq a_1$ and a_0, a_1 have infinite intersection.

For $a_0 \neq a_1$, observe that neither a_0, a_1 can belong to the model $V[K]$. If, say, $a_0 \in V[K]$ then R witnesses the fact that $a_0 \in I$, and consequently a_0 must have both finite and infinite intersection with η/H_0, a contradiction. Now, since $a_0 \in V[K][H_0]$ and $a_1 \in V[K][H_1]$, a mutual genericity argument shows that $a_0 \neq a_1$.

To establish that the set $a_0 \cap a_1$ is infinite, move back to the model $V[K]$, let $\langle s_0, s_1 \rangle$ be a condition in the product $R \times R$ and $n \in \omega$ be a number; we must find a number $m > n$ and conditions $t_0 \leq s_0$ and $t_1 \leq s_1$ such that $t_0 \Vdash \check{m} \in \chi$ and $t_1 \Vdash \check{m} \in \chi$. To this end, let $b_0 = \{m \in \omega \colon \exists t \leq s_0\ t \Vdash \check{m} \in \chi\}$ and $b_1 = \{m \in \omega \colon \exists t \leq s_1\ t \Vdash \check{m} \in \chi\}$. The sets $b_0, b_1 \subset \omega$ are both forced to have infinite intersection with η and therefore must belong to the ultrafilter U. This means that there is a natural number $m > n$ in the intersection $b_0 \cap b_1$, and then the desired conditions $t_0 \leq s_0$ and $t_1 \leq s_1$ are found by the definition of the sets b_0 and b_1. $\square$

Theorem 8.9.7 above produces a weakly balanced MAD forcing such that in its extension of the Solovay model, a maximal almost disjoint family exists. As a result of that example and Theorem 9.1.6, we have

Corollary 14.1.3. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is an infinite maximal almost disjoint family of subsets of ω, and there is no uncountable sequence of pairwise distinct reals.*

14.2 Unbounded linear suborders

Another general obstruction for reaching ZF independence results in balanced extensions is encapsulated in the following theorem, stated using Convention 1.7.16.

Theorem 14.2.1. *Let $\trianglelefteq$ be an F_σ-preorder on a Polish space X with no maximal element such that every countable linearly ordered set has an upper bound. In cofinally balanced σ-closed extensions of the symmetric Solovay model, $\trianglelefteq$ contains no unbounded linearly ordered sets.*

A typical preorder satisfying the assumptions is the Turing reducibility preorder or the modulo finite domination ordering on ω^ω.

Proof. We start with a simple claim which takes place in the context of ZFC. If R is a poset and η is an R-name for an element of X, say that η is *unbounded* if R forces that for every ground model element $x \in X$, $\eta \trianglelefteq x$ fails.

Claim 14.2.2. *Suppose that R_0, R_1 are posets and η_0, η_1 are respective unbounded names for elements of X. Then $R_0 \times R_1 \Vdash \tau_0, \tau_1$ are $\trianglelefteq$-incomparable.*

Proof. Let $\trianglelefteq = \bigcup_n F_n$ where for every $n \in \omega$, F_n is closed. If the conclusion failed, there would have to be a pair $\langle r_0, r_1\rangle \in R_0 \times R_1$ forcing say $\eta_0 \trianglelefteq \eta_1$ and then strengthening the pair if necessary there would have to be a number $n \in \omega$ such that $\langle \eta_0, \eta_1\rangle \in F_n$ is forced. Let M be a countable elementary submodel of a large structure containing R_1, r_1, η_1, let $g \subset R_1$ be a filter generic over the model M such that $r_1 \in g$, and let $x = \eta_1/g$. Since η_0 is an unbounded name, $R_0 \Vdash \eta_0 \trianglelefteq \check{x}$ fails, and so there must be a condition $r_0' \leq r_0$ and basic open sets $O_0, O_1 \subset X$ such that $(O_0 \times O_1) \cap F_n = 0$, $x \in O_1$, and $r_0' \Vdash \tau_0 \in O_0$. By the forcing theorem, there must be a condition $r_1' \in g$ below r_1 such that $r_1' \Vdash \tau_1 \in O_1$. Then the pair $\langle r_0', r_1'\rangle \leq \langle r_0, r_1\rangle$ forces $\langle \eta_0, \eta_1\rangle \in O_0 \times O_1$ and so $\langle \eta_0, \eta_1\rangle \notin F_n$, in contradiction with the initial assumptions. $\square$

Now, let κ be an inaccessible cardinal and P be a σ-closed Suslin forcing which is balanced cofinally below κ. Let W be the symmetric Solovay model derived from κ. Suppose towards a contradiction that the conclusion of the theorem fails in the P-extension of W. Then, in the model W there must be a condition $p \in P$ and a P-name τ such that $p \Vdash \tau \subset X$ is an $\trianglelefteq$-unbounded linearly ordered set. The condition p as well as the name τ must be definable from some parameter $z \in 2^\omega$ and some parameters in the ground model. Let $V[K]$ be a generic extension of the ground model by a poset of size $< \kappa$ such that $p, z \in V[K]$ and P is balanced in $V[K]$.

Work in the model $V[K]$. Let $\bar{p} \leq p$ be a balanced condition in the poset P. Since $\mathrm{Coll}(\omega, <\kappa) \Vdash p \Vdash \tau$ is an unbounded set in $\leq$, for every element $x \in X \cap V$ the set τ is forced to contain an element which is not $\trianglelefteq x$. Since $\mathrm{Coll}(\omega, <\kappa) \Vdash p \Vdash X \cap V[K]$ is a countable set and DC holds, τ is forced to contain a countable subset such that no element of $X \cap V[K]$ is an upper bound of it. By the initial assumptions on the preorder $\trianglelefteq$, this countable set is forced to have an upper bound in τ, which is then an element of τ which is not $\trianglelefteq x$

for any element $x \in X \cap V[K]$. In total, in $V[K]$ there must be a poset R of size $< \kappa$, an R-name σ for a condition in P such that $\sigma \leq \bar{p}$ and an unbounded R-name η for an element of X such that $R \Vdash \eta$ is not $\trianglelefteq$-below any element of $V[K]$ and $R \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \eta \in \tau$.

In the model W, let $H_0, H_1 \subset R$ be filters mutually generic over $V[K]$. Let $p_0 = \sigma/H_0 \in P$ and $p_1 = \sigma/H_1 \in P$; by the balance of the condition $\bar{p}$, the conditions p_0, p_1 are compatible with some lower bound $q \in P$. Let $x_0 = \eta/H_0 \in X$ and $x_1 = \eta/H_1 \in X$; by Claim 14.2.2, these are $\trianglelefteq$-incomparable elements of X. Since the model W is a symmetric extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem applied in these models shows that in W, $q \Vdash \check{x}_0, \check{x}_1 \in \tau$. This contradicts the assumption that τ is forced to be linearly ordered by $\trianglelefteq$. $\square$

14.3 Measure and category

In this section, we show that in balanced extensions of the Solovay model, there is a set of reals without the Baire property. We also provide a framework for showing that in certain circumstances, all sets of reals may be Lebesgue measurable. The following theorem is stated using Convention 1.7.16.

Theorem 14.3.1. *In nontrivial cofinally balanced extensions of the symmetric Solovay model, there is a set of reals without the Baire property.*

Proof. Let κ be an inaccessible cardinal. Let P be a nontrivial Suslin poset such that $V_\kappa \models P$ is balanced in every forcing extension. Let W be the symmetric Solovay model derived from κ and work in W. Let $p \in P$ be a condition. We must find a Polish space X, a condition $\bar{p} \leq p$ and a P-name τ for a subset of X such that $\bar{p} \Vdash \tau$ does not have the Baire property.

The condition $p \in P$ is definable from a real parameter $z \in 2^\omega$ and some parameters in the ground model. Let $V[K]$ be an intermediate extension using a poset of size $< \kappa$ such that $z \in V[K]$ and P is balanced in $V[K]$, and work in $V[K]$ for a moment. Let $\bar{p} \leq p$ be a balanced virtual condition stronger than P. Choose a poset Q of size $< \kappa$ and a name σ for a condition in P such that the pair $\langle Q, \sigma \rangle$ is balanced and in the balance equivalence class of $\bar{p}$. Move to the Q-extension. Since the poset P is balanced, it is not c.c.c. below σ by Proposition 5.2.8(1), in particular it is not Suslin σ-linked below σ. Thus, the analytic graph of incompatibility of conditions in P below σ has uncountable Borel chromatic number. By the $\mathbb{G}_0$-dichotomy, there is a continuous map $h \colon 2^\omega \to P$ which is a homomorphism of $\mathbb{G}_0$ to the incompatibility graph on P below σ. Let S be the Cohen forcing on 2^ω introducing a point $\dot{y}_{gen}$; let $\dot{p}_{\mathrm{gen}}$ be the $Q \times S$-name for $\dot{h}(\dot{y}_{gen})$.

Now, back in W, consider the space X of all filters on the two-step iteration $Q \times S$ generic over the model $V[K]$, viewed as a subset of $\mathcal{P}(Q \times S)$. Since the product is countable in W, its powerset gets the usual zero-dimensional compact topology. Since the model $V[K]$ contains only countably many open dense subsets of the product, the set $X \subset \mathcal{P}(Q \times S)$ is G_δ and therefore Polish. Let τ

be the P-name for the set of all filters $g \in X$ such that the condition $\dot{p}_{\mathrm{gen}}/g \in P$ belongs to the generic filter on P; more formally, $\tau = \{\langle g, \dot{p}_{\mathrm{gen}}/g\rangle : g \in X\}$. We claim that $\bar{p} \Vdash \tau$ does not have the Baire property.

To show this, first argue that $\bar{p} \Vdash \tau \subset X$ is not meager. Suppose towards a contradiction that this fails. Then in the model $V[K]$, there exists a poset R of size $< \kappa$ and R-names σ for a condition in P stronger than $\bar{p}$ and η for a dense G_δ subset of X such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau \cap \eta = 0$. Let $G \subset Q \times S$ and $H \subset R$ be mutually generic filters. By a mutual genericity argument, $G \in \eta/H$ holds. By a balance argument, the conditions σ/H and $\dot{p}_{\mathrm{gen}}/G$ are compatible in P. The common lower bound of these two conditions forces in P that $\check{G} \in \eta/H \cap \tau$. This is a contradiction.

We will now argue that $\bar{p} \Vdash \tau$ is not comeager in any nonempty open subset of X. Suppose towards a contradiction that this fails. Then in the model $V[K]$, there must be a condition $\langle q, s\rangle \in Q \times S$, a poset R of size $< \kappa$ and R-names σ for a condition in P stronger than $\bar{p}$ and η for a dense G_δ subset of X such that $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \forall g \in \eta\ \langle q, s\rangle \in g \to g \in \tau$. Let $L \subset Q$ and $H \subset R$ be mutually generic filters, with $q \in L$. A standard argument provides (in a further generic extension) two $\mathbb{G}_0$-related points $y_0, y_1 \in 2^\omega$ extending s which are separately generic over the model $V[K][H][L]$. Let $G_0, G_1 \subset Q \times S$ be the filters given by L, y_0 and L, y_1 respectively; both contain the condition $\langle q, s\rangle$. By the product forcing theorem, the filters G_0, H are mutually generic over $V[K]$ and so are the filters G_1, H. By a mutual genericity argument, both filters G_0, G_1 both belong to the set η/H. By a balance argument, the condition $\dot{p}_{\mathrm{gen}}/G_0$ is compatible with σ/H in the poset P. Their common lower bound forces in the poset P that $\check{G}_1 \in \eta/H \setminus \tau$. This is a contradiction. $\square$

Question 14.3.2. Does the conclusion of Theorem 14.3.1 remain in force in weakly balanced extensions of the symmetric Solovay model?

Other regularity properties of sets of reals may be preserved in the balanced extensions. Consider the following.

Definition 14.3.3. Let Q be a c.c.c. Suslin forcing. Let P be a Suslin forcing. We say that P is *Q-balanced* if for every condition $p \in P$ there is a balanced virtual condition $\bar{p} \leq p$ such that $Q \Vdash \bar{p}$ is still balanced. Similar terminology applies for Q-weakly balanced forcings.

For the following theorem, let X be a Polish space, let $\dot{x}_{gen}$ be a layered Q-name for an element of X, and let I be the σ-ideal of Borel subsets $B \subset X$ such that $Q \Vdash \dot{x}_{gen} \notin \dot{B}$. The theorem is stated using Convention 1.7.16.

Theorem 14.3.4. *In Q-weakly balanced extensions of the symmetric Solovay model, for every set $A \subset X$ there is a Borel set $B \subset X$ such that $A \Delta B \in I$.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset such that P is Q-balanced below κ. Let W be the symmetric Solovay model derived from κ and work in W. Let $p \in P$ be a condition and τ be a P-name such that

$p \Vdash \tau \subset X$. We must find a Borel set B and a stronger condition $\bar{p} \leq p$ in P such that $\bar{p} \Vdash \tau \Delta \dot{B} \in I$.

The condition $p \in P$ and the name τ are definable from a real parameter $z \in 2^\omega$ and some parameters in the ground model. Let $V[K]$ be an intermediate extension using a poset of size $< \kappa$ such that $z \in V[K]$ and work in the model $V[K]$. Since P is Q-balanced in $V[K]$, it must be the case that there is a weakly balanced condition $\bar{p} \leq p$ such that $Q \Vdash \bar{p}$ is weakly balanced. By a balance argument, $Q \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p}$ decides the statement $\dot{x}_{gen} \in \tau$. Let $A = A_0 \cup A_1$ be a maximal antichain of Q such that for every condition $q \in A_0$, $q \Vdash_Q \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \dot{x}_{gen} \notin \tau$, and for every condition $q \in A_1$, $q \Vdash_Q \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \dot{x}_{gen} \in \tau$. In W, let $B = \{x \in X \colon \exists g \subset Q^{V[K]} \colon g$ is generic over $V[K]$, contains a condition in A_1, and $x = \dot{x}_{gen}/g\}$. A complexity calculation in Proposition 2.8.5(1) shows that the set B is Borel since the model $V[K]$ contains only countably many maximal antichains of Q. It will be enough to show that $\bar{p} \Vdash \tau \Delta B \in I$.

On one hand, it is clear that $B \subset \tau$ must hold. Whenever $x \in B$ is a point and $g \subset Q$ is a filter generic over $V[K]$ containing a condition in A_1 and $\dot{x}_{gen}/g = x$, then by the forcing theorem in the model $V[K]$, $V[K][g] \models \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p}\check{x} \in \tau$, and then by the forcing theorem in $V[K][g]$, $W \models \bar{p} \Vdash \check{x} \in \tau$. By a similar reasoning, it must be the case that $\tau \setminus B \subset C$ where $C = \{x \in X \colon$ for no filter $g \subset Q^{V[K]}$ generic over $V[K]$, $\dot{x}_{gen}/g = x\}$. However, it is clear that $C \in I$ holds: if there were a condition $q \in Q$ forcing $\dot{x}_{gen} \in B$, taking any filter $G \subset Q$ generic over W containing the condition q, the point $\dot{x}_{gen}/G$ must belong to C. However, since the poset Q is c.c.c. and Suslin, the filter $G \cap V[K]$ is a filter on $Q^{V[K]}$ generic over $V[K]$ and so the point $x = \dot{x}_{gen}/G \cap V[K]$ does not belong to C, which is a contradiction. $\square$

Example 14.3.5. [81] It is consistent relative to an inaccessible cardinal that ZF+DC holds, every set of reals is Lebesgue measurable, and not every set of reals has the Baire property. For this, use the P-extension of the Solovay model poset where the Suslin poset P is as in Example 8.8.4 and the Suslin c.c.c. poset Q of closed subsets of $\mathbb{R}$ of positive Lebesgue measure ordered by inclusion, with its associated name for the generic real. It is clear that the associated ideal I is just the ideal of Lebesgue null sets. Now, the balanced conditions for P have been classified in Theorem 8.8.2. Let $p = \langle a_p, b_p \rangle$ be a condition. Then the (collapse name for the) pair $\bar{p} = \langle a_p, c \rangle$ is a balanced condition whenever c is a dominating set of functions in ω^ω in the ground model. Now, the poset Q is bounding, therefore the set b_p remains dominating in the Q-extension and $\bar{p}$ is a balanced condition in the Q-extension as well. By Theorem 14.3.4, in the P-extension of the Solovay model, every set of reals is Lebesgue measurable. In that extension, there must be a set of reals without the Baire property by Theorem 14.3.1; in fact, the generic set must fail to have the Baire property.

Example 14.3.6. It is consistent relative to an inaccessible cardinal that ZF+DC holds, every set of reals is Lebesgue measurable, and there is a maximal almost disjoint family. For this, use the P-extension of the symmetric Solovay model where P is the MAD forcing of Definition 8.9.6. It will be enough to show that P is Q-weakly balanced, where Q is the Suslin c.c.c. poset Q of closed subsets of $\mathbb{R}$ of positive Lebesgue measure ordered by inclusion. Now, certain weakly balanced conditions for P have been discovered in Theorem 8.9.7. Let $p = \langle a_p, b_p \rangle$ be a condition. Then any (collapse name for a) pair $\bar{p} = \langle a_p, c \rangle$ is a weakly balanced condition where c is the set of all pairs $\langle a_p, s \rangle$ where s is a partition of ω into finite intervals in the ground model. Now, let $G \subset Q$ be a generic filter, and in the extension $V[G]$, consider any (collapse name for a) pair $\tilde{p} = \langle a_p, d \rangle$ where d is the set of all pairs $\langle a_p, s \rangle$ where s is a partition of ω into finite intervals in $V[G]$. By Theorem 8.9.7 applied in $V[G]$, $\tilde{p}$ is a weakly balanced condition in $V[G]$. Since the poset Q is bounding, for every partition $s \in V[G]$ of ω into finite intervals there is a partition $t \in V$ of ω such that every interval in t contains a subinterval in s. It follows immediately from the definition of the poset P that $\bar{p}, \tilde{p}$ are inseparable in the poset P; in particular, $\bar{p}$ is still a weakly balanced condition in $V[G]$.

14.4 Definably balanced forcing

This section is devoted to the class of apparently the softest balanced Suslin extensions one can find: those in which the balanced virtual conditions can be found in an easily definable way.

Definition 14.4.1. Let P be a Suslin forcing. We say that P is *definably balanced* if for each condition $p \in P$ there is a balanced virtual condition $\bar{p} \leq p$ which is definable from a real parameter.

Example 14.4.2. Let E be a Borel equivalence relation on a Polish space X the poset P of Example 6.6.9 adding a countable complete section of E is definably balanced. Namely, let $Y = X^\omega$, let $B \subset Y$ be the set of those elements $y \in Y$ whose range consists of pairwise E-related points, and let P consist of all countable sets $p \subset B$ such that if $y_0, y_1 \in P$ satisfy $[\mathrm{rng}(y_0)]_E = [\mathrm{rng}(y_1)]_E$ then in fact $\mathrm{rng}(y_0) = \mathrm{rng}(y_1)$ holds. The ordering on P is that of inclusion. For every condition $p \in P$ one can look at the definable balanced virtual condition $\bar{p} \leq p$ represented by the pair $\langle \mathrm{Coll}(\omega, X), \tau \rangle$ where τ is the name for the set of all conditions $q \in Q$ such that $p \subset q$ and for every point $x \in V \setminus \bigcup_{y \in p} [\mathrm{rng}(y)]_E$, q contains an enumeration of the set $[x]_E \cap V$.

Example 14.4.3. Let P be the poset of Section 8.4 adding a cofinal Kurepa family on a fixed Polish space X. For every condition $p \in P$ (a countable set of countable subsets of X closed under intersection) one can find a balanced virtual condition $\bar{p} \leq p$ represented by the pair $\langle \mathrm{Coll}(\omega, X), \tau \rangle$ where τ is the name for the condition obtained from p by adding the set $X \cap V$. This is indeed a balanced pair as proved in Theorem 8.4.3, and it is definable from a real.

Example 14.4.4. Let P be the partial ordering introducing a nontrivial automorphism of the algebra $\mathcal{P}(\omega)$ modulo finite as in Section 8.3. As proved in

Theorem 8.3.3, the balanced virtual conditions are classified precisely by automorphisms of the algebra. At the same time, every condition (an automorphism of a countable subalgebra) can be extended into a total automorphism which is trivial, i.e. generated by a bijection between cofinite subsets of ω [9, Theorem 2.3]. Such a total automorphism is clearly definable from a real.

Example 14.4.5. Let Γ be the clopen graph on 2^ω connecting points $x_0 \neq x_1$ if the smallest number n such that $x_0(n) \neq x_1(n)$ is even. Let P be the partial order of Example 12.2.16 forcing a failure of OCA by adding an uncountable subset of 2^ω containing no uncountable Γ-clique or Γ-anticlique. For every condition $p \in P$ (which is a pair $\langle a_p, b_p\rangle$ where $a_p \subset 2^\omega$ is a countable set and b_p is a countable collection of closed Γ-cliques and closed Γ-anticliques) one can find a balanced virtual condition $\bar{p} \leq p$ represented by the pair $\langle \mathrm{Coll}(\omega, 2^\omega), \tau\rangle$ where τ is the name for the pair $\langle a_p, \bar{b}_p\rangle$ where $\bar{b}_p$ is the set of all closed Γ-cliques and all closed Γ-anticliques coded in the ground model. This is indeed a balanced pair below p as shown in Example 12.2.16 and it is definable from a real.

The preservation theorems we prove are best organized with a well-known factorization fact and its corollary.

Fact 14.4.6. *Let Q be the finite support product of ω_1-many copies of the Cohen forcing. Let $G \subset Q$ be a generic filter.*

1. *Whenever $x \in V[G]$ is an element of 2^ω, $V[G]$ is a Q-extension of $V[x]$.*
2. *Suppose that X, Y are Polish spaces in V, E is a pinned Borel equivalence relation on Y coded in V. In $V[G]$, if $x \in X$ is a point and $c \subset Y$ is an E-class definable from x and some elements of the ground model, then c has a representative in $V[x]$.*

The following results are stated using Convention 1.7.16.

Theorem 14.4.7. *In definably balanced extensions of the symmetric Solovay model, whenever Γ is an analytic hypergraph of countable, possibly infinite arity on a Polish space X of uncountable Borel chromatic number, then Γ has uncountable chromatic number.*

Proof. In view of the dichotomy theorem for uncountable Borel chromatic number of analytic hypergraphs of countable arity [66] it is enough to verify that the conclusion of the theorem holds for a specific analytic hypergraph Γ. Let X be the dense G_δ subspace of ω^ω consisting of all functions $x \in \omega^\omega$ such that for infinitely many $n \in \omega$, $x(n) > x(m)$ for all $m \in n$. Let $\{s_n : n \in \omega\}$ be a collection of finite strings of natural numbers such that $s_n \in \omega^n$ and $\{s_n : n \in \omega\} \subset \omega^{<\omega}$ is dense. Define the hypergraph Γ to consist of all tuples $\langle x_m : m \in \omega\rangle$ such that there is $n \in \omega$ such that s_n is an initial segment of all points x_m, $x_m(n) = m$ holds for all m, and the tails $x_m \restriction (n, \omega)$ are the same for all $m \in \omega$. [66] shows that the graph Γ has uncountable Borel chromatic numbers and moreover, it can be continuously homomorphically mapped to any analytic hypergraph of uncountable Borel chromatic number. Thus, it is enough to prove the conclusion of the theorem for Γ.

Suppose that κ is an inaccessible cardinal. Suppose that P is a Suslin forcing such that P is definably balanced below κ. Let W be the symmetric Solovay model derived from κ and work in W. Let $p \in P$ be a condition and let τ be a P-name for a function from X to ω; we must find a Γ-edge $\langle x_m : m \in \omega\rangle$, a number $k \in \omega$ and some strengthening of the condition p which forces $\tau(\check{x}_m) = \check{k}$ for all $m \in \omega$. The condition p as well as the name τ are defined from some parameters in the ground model and perhaps an additional parameter $z \in 2^\omega$. Find an intermediate extension $V[K]$ of the ground model using a poset of size $< \kappa$ such that $z \in V[K]$, and work in the model $V[K]$.

Let Q be the product of uncountably many copies of Cohen forcing. By the definable balance assumption, there must be a Q-name η for a real number and a condition $q \in Q$ which identifies a definition of a balanced virtual condition $\bar{p} \leq p$ which uses only η as a parameter. By Fact 14.4.6, passing to an intermediate extension if necessary we may assume that η is in fact a name for a specific element of the ground model. For the sake of brevity, we ignore η and q below. By a balance argument with $\bar{p}$, $Q \Vdash \forall x \in X\ \exists k \mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \tau(\check{x}) = \check{k}$. Let $\dot{h}$ be the Q-name for the map $x \mapsto k$; thus, $\dot{h}$ is forced to be a definable Γ-coloring.

Consider the Cohen poset P_X with its name $\dot{x}_{gen}$ for a generic point of the space X. Find a condition $O \in P_X$ and a number $k \in \omega$ such that $O \Vdash Q \Vdash \dot{h}(\dot{x}_{gen}) = \check{k}$. Now, let $x \in O$ be a point P_X-generic over $V[K]$. Use a genericity argument to find a number m large enough that s_m is an initial segment of x and $[s_m] \cap X \subset O$. For each number $n \in \omega$ let $x_n \in X$ be the point obtained from x by overwriting its m-th entry with n. Thus, each point x_n is Q_0-generic over $V[K]$ and belongs to the set O. Now, let $G \subset Q$ be a filter generic over the model $V[K]$ such that $x \in V[K][G]$, and consider the definable balanced condition $\bar{p} \leq p$ in $V[K][G]$. By the forcing theorem applied in the model $V[K]$, in the model W $\bar{p} \Vdash_P \tau(\check{x}_n) = \check{k}$ must hold for every number n. In view of the fact that the sequence $\langle x_n : n \in \omega\rangle$ is a Γ-hyperedge, the proof is complete. $\square$

Corollary 14.4.8. *1. Let P be the poset for adding a nontrivial automorphism of the algebra $\mathcal{P}(\omega)$ modulo finite of Section 8.3. In the P-extension of the Solovay model, there is no discontinuous homomorphism between Polish groups.*

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a nontrivial automorphism of $\mathcal{P}(\omega)$ modulo finite, and there is no discontinuous homomorphism between Polish groups.

Proof. The poset P is definably balanced by Example 14.4.4. It is clearly σ-closed, so in the P-extension of the Solovay model, ZF+DC holds. By Theorem 14.4.7, in the P-extension the chromatic number of the Hamming graph $\mathcal{H}_2$ is uncountable.. By a ZF+DC result of [76], this abstractly implies that there are no discontinuous homomorphisms between Polish groups. $\square$

Theorem 14.4.9. *In definably balanced extensions of the symmetric Solovay model, $|\mathbb{E}_1| \not\leq |F|$ holds for every pinned Borel orbit equivalence relation F.*

Proof. Let Γ be a Polish group continuously acting on a Polish space X, inducing an equivalence relation F. Suppose that F is pinned. Suppose that P is a Suslin forcing. Suppose further that κ is an inaccessible cardinal such that P is definably balanced below κ. Let W be the symmetric Solovay model derived from κ and work in W. Suppose that $p \in P$ is a condition and τ is a P-name such that $p \Vdash \tau$ is a function from the $\mathbb{E}_1$-quotient space to the F-quotient space. We must find a strengthening of the condition p and two distinct $\mathbb{E}_1$-classes such that their τ-images are forced to be the same.

The condition p as well as the name τ are definable from a parameter $z \in 2^\omega$ as well as some parameters from the ground model. Let $V[K]$ be an intermediate extension obtained from a poset of size $< \kappa$ such that $z \in V[K]$. Work in the model $V[K]$. Let Q be the finite support product of uncountably many copies of the Cohen forcing. Use the definable balance assumption to find a Q-name η for a real and a condition $q \in Q$ which identifies a definition of a virtual balanced condition $\bar{p} \leq p$ with a parameter η. By Fact 14.4.6, passing into a larger intermediate extension, we may assume that η is in fact a name for a specific element of $V[K]$. For the brevity of notation, we neglect η and q below.

Let $H \subset Q$ be a filter generic over $V[K]$ and work in the model $V[K][H]$ for the moment. Note that the equivalence relation F is pinned there by Theorem 2.7.1. By a balance argument with $\bar{p}$, for every $\mathbb{E}_1$-class c in the model $V[K][H]$ there is an F-class d such that $\mathrm{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \tau(c) = d$. Go back to $V[K]$. Fix a Q-name $\dot{h}$ for the map $c \mapsto d$. Note that $\dot{h}$ is forced to be a definable injection from the $\mathbb{E}_1$-quotient space to the F-quotient space.

Let $y \in (2^\omega)^\omega$ be a sequence of points in 2^ω generic over $V[K]$ for the finite support product of the usual Cohen forcing. Working in the model $V[K][y]$, by Fact 14.4.6, there is a point $x \in X$ such that $Q \Vdash \dot{h}([y]_{\mathbb{E}_1}) = [x]_F$. Let $\gamma \in \Gamma$ be a point generic over $V[K][y]$ for the poset P_Γ. As in the proof of Theorem 4.1.1, $V[K][\gamma \cdot x]$ contains no entries of the sequence y, and therefore no representatives of the $\mathbb{E}_1$-class of y. Since the class $[y]_{\mathbb{E}_1}$ is definable from $\gamma \cdot x$ as the class which is mapped by h to $[\gamma \cdot x]_F$, this contradicts Fact 14.4.6(2). $\square$

Using Example 14.4.2, we now get

Corollary 14.4.10.

1. *Let P be the partial ordering adding a countable complete section to $\mathbb{E}_1$. In the P-extension of the symmetric Solovay model, $|\mathbb{E}_1| \not\leq |F|$ holds for every pinned orbit equivalence relation F.*
2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, $|\mathbb{E}_1| \leq |\mathbb{F}_2|$ and yet $|\mathbb{E}_1| \not\leq |F|$ for any pinned Borel orbit equivalence relation F.*

14.5 $\mathbb{F}_2$ structurability

The following theorem is stated using Convention 1.7.16.

Theorem 14.5.1. *In σ-closed, balanced, $\aleph_0$-tethered extensions of a symmetric Solovay model there are no tournaments on the $\mathbb{F}_2$-quotient space.*

The σ-closure demand can be relaxed to include the formally not σ-closed posets of Section 8.1; we omit the details. The role of $\aleph_0$-tether is less clear. The linearization poset on the $\mathbb{F}_2$-quotient space of Example 8.6.5 is σ-closed, balanced, and adds a tournament, so clearly some additional assumption beyond the closure and balance is necessary. However, $\aleph_0$-tether is not preserved under taking a regular subposet, while the conclusion of the theorem is. Thus, there is a room for improvement in the statement of the theorem.

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin poset which is σ-closed, and balanced and $\aleph_0$-tethered below κ. Let W be a symmetric Solovay model derived from κ and work in W. Suppose towards a contradiction that $p \in P$ is a condition and τ is a P-name such that $p \Vdash \tau$ is a tournament on the $\mathbb{F}_2$-quotient space. Both p, τ must be definable from some real parameter $z \in 2^\omega$ and some parameters in the ground model. Let $V[K]$ is an intermediate extension by a poset of size $< \kappa$ such that $z \in V[K]$.

Work in the model $V[K]$ for a moment. Let $\bar{p} \leq p$ be a balanced virtual condition in the poset P in $V[K]$. Let $\lambda < \kappa$ be a cardinal such that $\bar{p}$ is represented on a poset of size $< \lambda$, let Q_0 be the finite support product of copies of $\mathrm{Coll}(\omega, \lambda)$ indexed by λ^+ with finite product of copies of the Cohen forcing indexed by $2 \times \lambda^+$. for any ordinal $\alpha \in \lambda^+$ write Q_0^α for the part of the product indexed by ordinals below α. Let η_0 and η_1 be the Q_0-names for the sets of Cohen generic reals indexed by $\{0\} \times \lambda^+$ and $\{1\} \times \lambda^+$ respectively. In the Q_0-extension of $V[K]$ consider the poset Q_1 which is the poset P of conditions which are stronger than some realization of $\bar{p}$. Let Q denote the iteration $Q_0 * \dot{Q}_1$, and let χ be a Q-name for the virtual condition in P consisting of all conditions in P stronger than all conditions in the generic filter on $\dot{Q}_1$.

Claim 14.5.2. *Q forces χ to be a balanced condition in P below $\bar{p}$.*

Proof. Let $G \subset Q_0$ and $H \subset Q_1$ be generic filters over $V[K]$ and work in $V[K][G][H]$. First, use the σ-closure assumption on P to conclude that in fact $\chi/G * H$ is forced to be a nonzero virtual condition. In some further collapse extension, the filter H contains a cofinal countable sequence, which then has a lower bound and that lower bound will be a condition below $\chi/G * H$.

Second, use the tether assumption to conclude that $\chi/G * H$ is a balanced condition. Observe that by a genericity argument with the filter H, for every analytic subset $A \subset P$ coded in the $V[K][G_0]$, H contains a condition which is either below some element of A or incompatible with all elements of A. Since the poset P (and so Q_1) is σ-closed in $V[K][G]$, it does not add any new analytic subsets of P. As a result, it is forced that $\chi/G * H$ is either stronger than or incompatible with any given virtual condition in P in the model $V[K][G][H]$ carried by a poset of size $< \aleph_0$ (i.e. an analytic subset of P). The balance of $\chi/G * H$ immediately follows from the definition of $\aleph_0$-tether. $\square$

Since χ is forced to be balanced, we also have $Q \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \chi$ decides the statement $\langle \eta_0, \eta_1 \rangle \in \tau$. Let $q \in Q$ be a condition such that $q \Vdash_Q \mathrm{Coll}(\omega, < \kappa) \Vdash \chi \Vdash_P \langle \eta_0, \eta_1 \rangle \in \tau$ (or vice versa). Let $q = \langle q_0, \dot{q}_1 \rangle$ be a breakdown of the condition q in the iteration $Q_0 * \dot{Q}_1$. Use the λ^+-c.c. of the poset Q_0 to find an ordinal $\alpha \in \lambda^+$ such that q_0 is a condition in Q_0^α and $\dot{q}_1$ is a Q_0^α-name.

Claim 14.5.3. *There are filters $G_0, G_1 \subset Q_0$ which are separately generic over $V[K]$ such that*

1. *$G_0 \cap Q_0^\alpha$ and $G_1 \cap Q_0^\alpha$ are mutually generic over $V[K]$, both containing q;*
2. *$V[K][G_0] = V[K][G_1]$*
3. *$\eta_0/G_0 = \eta_1/G_1$ and $\eta_1/G_0 = \eta_0/G_1$.*

Proof. Let $I = \lambda \cup (2 \times \lambda)$. Consider any bijection $\pi \colon I \to I$ such that $\pi''\lambda = \lambda$. Such a bijection induces an automorphism of the poset Q and the class of Q-names, which by an abuse of notation we denote by π again. If in addition we assume that $\pi''\{0\} \times \lambda = \{1\} \times \lambda$ and $\pi''\{1\} \times \lambda = \pi\{0\} \times \lambda$, then the automorphism switches the names η_0 and η_1. If in addition we assume that $\pi''\alpha$ is disjoint from α and $\pi''(2 \times \alpha)$ is disjoint from $2 \times \alpha$, then the conditions $q, \pi(q)$ are compatible. Let $G_0 \subset Q_0$ be a filter generic over $V[K]$ meeting the lower bound of q and $\pi(q)$, and let $G_1 = (\pi^{-1})''G_0$. The filters G_0, G_1 are as required. $\square$

Now, let G_0, G_1 be filters as in the claim. Let $A_0 = \eta_0/G_0 = \eta_1/G_1$ and $A_1 = \eta_1/G_0 = \eta_0/G_1$. The conditions $p_0 = \dot{q}/G_0$ and $p_1 = \dot{q}/G_1$ are compatible in the poset P by (1), and their lower bound must exist in the model $V[K][G_0] = V[K][G_1]$. Let $H \subset P$ be a filter generic over $V[K][G_0]$ meeting that lower bound and consider the balanced virtual condition $\bar{\bar{p}} = \chi/G_0 * H$, which is equal to $\chi/G_1 * H$. Now, since the filter $G_0 * H \subset Q$ contains the condition q, the forcing theorem shows that in the model W, $\bar{\bar{p}} \Vdash \langle \check{A}_0, \check{A}_1 \rangle \in \tau$. By the same argumentation, since the filter $G_1 * H \subset Q$ contains the condition q, the forcing theorem shows that in the model W, $\bar{\bar{p}} \Vdash \langle \check{A}_1, \check{A}_0 \rangle \in \tau$. This is a contradiction concluding the proof. $\square$

Corollary 14.5.4.
1. *Let $P = \mathcal{P}(\omega)$ modulo finite. In the P-extension of the symmetric Solovay model, there is no tournament on the quotient space of $\mathbb{F}_2$;*
2. *it is consistent relative to an inaccessible limit of inaccessibles that ZF+DC holds, there is a nonprincipal ultrafilter on ω, and yet there is no tournament on the quotient space of $\mathbb{F}_2$.*

Theorem 14.5.1 provides many other corollaries, among which we quote the following application of Example 10.5.8:

Corollary 14.5.5. *Let E be a pinned Borel equivalence relation on a Polish space X.*

1. *Let P be the linearization poset for E as in Example 8.6.5. In the P-extension of the Solovay model, there is no tournament on the $\mathbb{F}_2$-quotient space;*

2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, the E-quotient space is linearly ordered, and yet there is no tournament on the $\mathbb{F}_2$-quotient space.*

Proof. The linearization poset is certainly σ-closed. It is balanced by Example 8.6.5 and $\aleph_0$-tethered by Example 10.5.8. □

14.6 The Ramsey ultrafilter extension

Let P be the poset of infinite subsets of ω ordered by inclusion. Let κ be an inaccessible cardinal, and let W be the symmetric Solovay model derived from κ. Let $U \subset P$ be a filter generic over W. The model $W[U]$ is a balanced extension of W by Theorem 7.1.2. It is immediate that U is a nonprincipal ultrafilter on ω and in fact a Ramsey ultrafilter. The model $W[U]$ has been investigated for decades ??? We have provided new information about the model at several locations in this book. In this section, we outline several major areas of study of the model $W[U]$ and gather the available known results as well as critical open questions.

Question 14.6.1. Classify the Borel equivalence relations E, F on Polish spaces such that in $W[U]$, $|E| \leq |F|$ holds. In particular, are there Borel equivalence relations E, F such that $|E| \leq |F|$ fails in the symmetric Solovay model W and holds in $W[U]$?

We have proved that in $W[U]$, the smooth divide is preserved (Corollary 9.2.5, the orbit divide is preserved (Corollary 9.4.8), and the E_{K_σ}-divide is preserved (Corollary 9.5.10. However, our efforts stalled in other directions. We do not know if the turbulent divide is preserved. We do not know if there are any non-hyperfinite countable Borel equivalence relations E such that $|E| \leq \mathbb{E}_0|$ holds in $W[U]$; in principle, this inequality could hold for all countable Borel equivalence relations whatsoever.

Question 14.6.2. Classify the Borel equivalence relations E such that the E-quotient space is linearly ordered in $W[U]$.

As a basic affirmative result in this direction, we show that $\mathbb{E}_0$ and $\mathbb{E}_1$-quotient spaces are linearly ordered in $W[U]$. To see this, observe in ZF+DC plus the existence of a nonprincipal ultrafilter on ω that the class of linearly orderable equivalence relations is closed under countable increasing unions. For the proof, let X be a Polish space and $\langle E_n : n \in \omega \rangle$ be an increasing sequence of analytic equivalence relations on X, each with linearly orderable quotient space. Use the DC assumption to pick linear orders $\leq_n$ for each of them. Let U be a nonprincipal ultrafilter on ω. If $x, y \in X$ are E-unrelated elements, then let

$[x]_E \leq [y]_E$ if the set $a(x, y) = \{n \in \omega \colon [x]_{E_n} \leq_n [y]_{E_n}\}$ belongs to the ultrafilter U. If $x'\ E\ x$ and $y'\ E\ y$ are different representatives of the E-classes of x, y, then there is a number $m \in \omega$ such that $x\ E_m\ x'$ and $y\ E_m\ y'$; therefore, the symmetric difference $a(x, y)\Delta a(x', y')$ is a subset of m, and the definition of $\leq$ does not depend on the choice of the equivalence class representatives. It is easy to check that $\leq$ is a linear ordering on the E-classes.

As a basic negative result, the $\mathbb{F}_2$-quotient space is not linearly ordered in $W[U]$; it even carries no tournament by ???. Another negative result, Theorem 14.6.5 below, shows that the $\mathbb{E}_2$-quotient space cannot be linearly ordered. As a basic open instance, choose any non-hyperfinite countable Borel equivalence relation E.

Question 14.6.3. Let X be an uncountable Polish space. Classify the Borel sets $A \subset [X]^{\aleph_0}$ such that in $W[U]$, there is a function f assigning to each set $a \in A$ an ultrafilter on a.

As a basic affirmative result in this direction, if E is a hyperfinite countable Borel equivalence relation on X, then there is an assignment of ultrafilters to E-classes in $W[U]$: Let E_n for $n \in \omega$ be equivalence relations with all classes finite forming an inclusion-increasing sequence such that $E = \bigcup_n E_n$. For the first item, let $\leq$ be any linear ordering of X. For each point $x \in X$ let $x(n)$ be the $\leq$-least element of X which is E_n-related to X. Let $U_x = \{a \subset [x]_E \colon \{n \in \omega \colon x(n) \in a\} \in U\}$. Thus, U_x is an ultrafilter on $[x]_E$ and $x\ E\ y$ implies that $U_x = U_y$ since U is nonprincipal and for some $m \in \omega$ $x\ E_m\ y$ holds, and then for all $n > m$ $x(n) = y(n)$ holds.

As a basic negative result in this direction, there is no assignment of ultrafilters to all countable subsets of X; this follows from ???. As a basic open instance, let Γ be the free group on two generators acting on 2^Γ by shift and let $X \subset \Gamma$ be the G_δ set of points on which Γ acts freely. Let E be the orbit equivalence relation on X. Is there an assignment of ultrafilters to E-classes? This seemingly arbitrary question relates to the quotient cardinal question 14.6.1. It is well-known that E is not Borel reducible to $\mathbb{E}_0$???. However. the assignment of ultrafilters to Γ-orbits yields in ZF the cardinal inequality $|E| \leq |\mathbb{E}_0|$. To see this, let Δ be the acyclic locally finite Cayley graph on X and let $\vec{\Delta}$ be the orientation of the Cayley graph defined by $\langle x, y\rangle \in \vec{\Delta}$ if the set of $\{z \in X \colon$ the unique injective Cayley path from x to z includes $y\}$ belongs to the ultrafilter assigned to $[x]_E$. Clearly, every vertex gets outflow one in the orientation $\vec{\Delta}$. Now, apply the argument after Example 9.2.15 to conclude that $|E| \leq |\mathbb{E}_0|$ must hold.

Question 14.6.4. Classify ultrafilters on ω in $W[U]$. In particular, for which Borel ideals I on ω is there an ultrafilter F in $W[U]$ such that $I \cap F = 0$?

As the only nontrivial result in this section, we now prove that there is no ultrafilter in $W[U]$ which is disjoint from the summable ideal.

Theorem 14.6.5. *Let P be the poset of infinite subsets of ω ordered by inclusion. In the P-extension of the symmetric Solovay model, the complement graph associated with the summable ideal on ω cannot be oriented.*

Proof. Write I for the summable ideal on ω; i.e. I consists of sets $a \subset \omega$ such that the sum $\Sigma\{\frac{1}{n+1} : n \in a\}$ is finite. Towards a contradiction, let κ be an inaccessible cardinal, let W be a symmetric Solovay model derived from κ, let $p \in P$ be a condition and τ be a P-name such that $p \Vdash \tau$ is an orientation of the complement graph associated with I. Both p, τ must be definable from some real parameter $z \in 2^\omega$ and some ground model parameters. Let $V[K]$ be an intermediate extension of the ground model by a poset of size $< \kappa$ containing the parameter z, and work in the model $V[K]$.

We will produce a poset Q and a Q-name $\dot{x}_{gen}$ for an element of 2^ω such that

(I) below every condition $q \in Q$ there are conditions $q_0, q_1 \leq q$ and an automorphism $\pi\colon Q \restriction q_0 \to Q \restriction q_1$ such that $\pi(\dot{x}_{gen})$ is forced by q_1 to be equal to $1 - \dot{x}_{gen}$ modulo I;

(II) the ultrafilter added by P still generates an ultrafilter in the $P \times Q$-extension.

Once this is done, let $G \subset P$ and $H \subset Q$ be mutually generic filters over the model $V[K]$, with $p \in G$, and work in $V[K][G][H]$. Let $x = \dot{x}_{gen}/H$. Let U be the ultrafilter on ω generated by the filter G; this is possible by the item (II) above. By Theorem 7.1.2, U is a balanced virtual condition in the poset P in the model $V[K][G][H]$. The following is proved by a standard argument using the balance of U:

Claim 14.6.6. *In the model $V[K][G][H]$, either $\mathrm{Coll}(\omega, <\kappa) \Vdash U \Vdash_P \langle x, 1-x\rangle \in \tau$, or $\mathrm{Coll}(\omega, <\kappa) \Vdash U \Vdash_P \langle 1-x, x\rangle \in \tau$ holds.*

Suppose for definiteness that the former alternative prevails. Find a condition $q \in H$ which forces it, and find conditions $q_0, q_1 \leq q$ and an automorphism $\pi\colon Q \restriction q_0 \to Q \restriction q_1$ as in item (I). Find a filter $H_0 \subset Q$ generic over $V[K][G]$ meeting the condition q_0, and let $H_1 = \pi'' H_0$. Let $x_0, x_1 \in 2^\omega$ be the points associated with the filters H_0, H_1. Then the models $V[K][G][H_0]$ and $V[K][G][H_1]$ are equal, and by the forcing theorem applied in $V[K][G]$, it must be true in W that $U \Vdash_P \langle x_0, 1-x_0\rangle \in \tau$, $\langle x_1, 1-x_1\rangle \in \tau$. This is impossible though as $x_0 = 1 - x_1$ modulo I and τ is forced to be a tournament.

The remainder of the proof consists of the construction of the poset Q satisfying (I) and (II). This is a pure ZFC construction in which the concentration of measure phenomenon is a critical ingredient. Let $\langle J_n : n \in \omega\rangle$ be a very fast sequence of successive intervals on ω. The following is the concentration of measure type of demand on the intervals in question that we need. Let μ_n be the normalized counting measure on 2^{J_n}. Let d_n be the metric on 2^{J_n} defined by $d_n(x, y) = \Sigma\{\frac{1}{m+1} : x(m) \neq y(m)\}$. We demand that the following holds for every $n \in \omega$:

(III) for every set $a \subset 2^{J_n}$ of μ_n-mass $1/n$, the 2^{-n}-neigborhood of a in 2^{J_n} in the sense of the metric d_n has μ_n-mass greater than $1/2$.

The concentration of measure computations in [73, Theorem 4.3.19] show that such a fast sequence of intervals indeed exists. Now, let T_{ini} be the tree of all finite sequences t such that for each $n \in \mathrm{dom}(t)$, $t(n) \in 2^{J_n}$. Let Q be the partial order of all nonempty trees $q \subset T_{\mathrm{ini}}$ without endnodes such that for each $m \in \omega$ there is $n_m \in \omega$ such that for each $t \in q$ of length $n > n_m$, the set $\{x \in 2^{J_n} : t^\frown\langle x\rangle \in q\}$ has μ_n-mass $\geq m/n$. The ordering on Q is that of inclusion. Let $\dot{x}_{gen}$ be the Q-name for the concatenation of the trunks of the trees in the generic filter. We must show that items (I) and (II) above hold for Q and $\dot{x}_{gen}$.

Item (I) is where the concentration of measure shows up in force. We will in fact show that whenever $q_0, q_1 \in Q$ are arbitrary conditions then there are conditions $q_0' \leq q_0, q_1' \leq q_1$ and an isomorphism $\pi\colon Q \restriction q_0' \to Q \restriction q_1'$ such that $\pi(1 - \dot{x}_{gen})$ is forced by q_1' to be modulo I equivalent to $\dot{x}_{gen}$. To see this, let $t_0 \in q_0, t_1 \in q_1$ be nodes of the same length such that for every node $t \in q_0$ of length $n \geq |t_0|$, the set $a_0(t) = \{x \in 2^{J_n} : t^\frown\langle x\rangle \in q_0\}$ has μ_n-mass at least $2/n$, and for every node $t \in q_1$ of length $n \geq |t_0|$,the set $a_1(t) = \{x \in 2^{J_n} : t^\frown\langle x\rangle \in q_1\}$ has μ_n-mass at least $2/n$ as well. We will produce a tree $q_0' \leq q_0 \restriction t_0$ in Q and a level and order preserving injection $\pi\colon q_0' \to q_1 \restriction t_1$ so that

(IV) for each node $t \in q_0'$ and a number $n > |t_0|$ in $\mathrm{dom}(t)$, $d_n(t(n), 1 - \pi(t)(n)) < 2^{-n+1}$.

Write $q_1' = \pi''q_0'$. Since the measures μ_n are normalized counting measures, the map π naturally extends to an isomorphism $\pi\colon Q \restriction q_0' \to Q \restriction q_1'$. The demand (IV) then implies that $\pi(1 - \dot{x}_{gen})$ is forced by q_1' to be modulo I equivalent to $\dot{x}_{gen}$ as required.

The map π is obtained by a hungry algorithm. Among all level and order preserving injections from subsets of $q_0 \restriction t_0$ to $q_1 \restriction t_1$ which satisfy (IV), select an inclusion maximal one and call it π. It will be enough to show that $q_0' = \mathrm{dom}(\pi)$ belongs to Q. To see this, first of all q_0' is closed under initial segment by an obvious maximality argument. To verify the branching condition in the definition of Q for q_0', let $t \in q_0'$ of length some $n \geq |t_0|$. It will be enough to argue that the set $\{x \in 2^{J_n} : t^\frown\langle x\rangle \in q_0'\}$ has μ_n-mass at least $\min(\mu_n(a_0(t)), \mu_n(a_1(\pi(t))) - 1/n$.

To do this, note that if both sets $b_0 = \{x \in 2^{J_n} : t^\frown\langle x\rangle \in q_0' \setminus q_0\}$ and $b_1 = \{x \in 2^{J_n} : t^\frown\langle x\rangle \in \pi(q_0') \setminus q_1\}$ had μ_n-mass greater than $1/n$, then by (III) the 2^{-n}-neighborhood of b_0 and the set $\{1 - x : x \in b_1\}$ would both have μ_n-mass greater than $1/2$ and so they would intersect, making it possible to extend π while satisfying (IV) and contradicting the maximality of π.

Item (II) follows from a density argument on P and Q given the fact that the poset Q is proper and does not add independent reals [101, Theorem 4.4.8]. □

Corollary 14.6.7.

1. *Let $P = \mathcal{P}(\omega)$ modulo finite. In the P-extension of the symmetric Solovay model, there is no ultrafilter on ω disjoint from the summable ideal;*

2. *it is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a nonprincipal ultrafilter on ω, and yet there is no ultrafilter disjoint from the summable ideal.*

Bibliography

[1] Francis Adams and Jindřich Zapletal. Cardinal invariants of closed graphs. *Israel Journal of Mathematics*, 227:861–888, 2018.

[2] Martin Aigner. *Combinatorial theory.* Grundlehren der mathematischen Wissenschaften 234. Springer Verlag, New York, 1979.

[3] Reinhold Baer. Abelian groups that are direct summands of every containing abelian group. *Bulletin of the American Mathematical Society*, 46:800–807, 1940.

[4] Bohuslav Balcar, Thomas Jech, and Jindřich Zapletal. Semi-Cohen boolean algebras. *Annals of Pure and Applied Logic*, 87:187–208, 1997.

[5] John Baldwin and Paul B. Larson. Iterated elementary embeddings and the model theory of infinitary logic. *Annals of Pure and Applied Logic*, 167:309–334, 2016.

[6] John T. Baldwin, Sy D. Friedman, Martin Koerwien, and Michael C. Laskowski. Three red herrings around Vaught's conjecture. *Trans. Amer. Math. Soc.*, 368:3673–3694, 2016.

[7] Tomek Bartoszynski and Haim Judah. *Set Theory. On the structure of the real line.* A K Peters, Wellesley, MA, 1995.

[8] James Baumgartner and Adam Taylor. Partition theorems and ultrafilters. *Transactions of the American Mathematical Society*, 241:283–309, 1978.

[9] A. Bella, Alan Dow, Klaas-Pieter Hart, Michael Hrušák, Jan van Mill, and P. Ursino. Embeddings of $\mathcal{P}(\omega)/\mathrm{fin}$ and extension of automorphisms. *Fundamenta Mathematicae*, 174:271–284, 2002.

[10] Andreas Blass. Ultrafilters related to Hindmans finite-unions theorem and its extensions. In Stephen Simpson, editor, *Logic and Combinatorics*, Contemporary Mathematics 65, pages 89–124. American Mathematical Society, Providence, 1987.

[11] Andres Blass, Natasha Dobrinen, and Dilip Raghavan. The next best thing to a P-point. *Journal of Symbolic Logic*, 80:866–900, 2015.

[12] Jörg Brendle, Fabiana Castiblanco, Ralf Schindler, Liuzhen Wu, and Liang Yu. A model with everything except a well-ordering of the reals. 2018. arXiv:1809.10420.

[13] Jörg Brendle and Michael Hrušák. Countable Fréchet groups: An independence result. *Journal of Symbolic Logic*, 74:1061–1068, 2009.

[14] B. Bukh. Measurable sets with excluded distances. *Geometric and Functional Analysis*, 18:668–697, 2008.

[15] Lev Bukovský. Iterated ultrapower and Prikry's forcing. *Comment. Math. Univ. Carolinae*, 18:77–85, 1977.

[16] Andres Caicedo, John Clemens, Clinton Conley, and Benjamin Miller. Definability of small puncture sets. *Fundamenta Mathematicae*, 215:39–51, 2011.

[17] Jack Ceder. Finite subsets and countable decompositions of Euclidean spaces. *Rev. Roumaine Math. Pures Appl.*, 14:1247–1251, 1969.

[18] David Chodounský and Osvaldo Guzmán. There are no P-points in Silver extensions. Institute of Mathematics, Czech Academy of Sciences preprint 17-2017, 2017.

[19] Clinton Conley and Benjamin Miller. A bound on measurable chromatic numbers of locally finite Borel graphs. *Mathematical Research Letters*, 23:1633–1644, 2016.

[20] Patrick Dehornoy. Iterated ultrapowers and Prikry forcing. *Ann. Math. Logic*, 15:109–160, 1978.

[21] Carlos DiPrisco and Stevo Todorcevic. Perfect set properties in $L(R)[U]$. *Advances in Mathematics*, 139:240–259, 1998.

[22] Natasha Dobrinen and Daniel Hathaway. Forcing and the Halpern–Läuchli theorem. *Journal of Symbolic Logic*, 2019. to appear.

[23] Natasha Dobrinen, Jose G. Mijares, and Timothy Trujillo. Topological Ramsey spaces from Fraissé classes, Ramsey-classification theorems, and initial structures in the Tukey types of p-points. *Archive for Mathematical Logic*, 56:733–782, 2017.

[24] Alan Dow. Set theory in topology. In *Recent progress in general topology.* North Holland, New York, 1992.

[25] Paul Erdős and András Hajnal. On chromatic number of graphs and set systems. *Acta Math. Acad. Sci. Hung.*, 17:61–99, 1966.

[26] Ilijas Farah. Semiselective coideals. *Mathematika*, 45(1):79–103, 1998.

[27] Ilijas Farah. Ideals induced by Tsirelson submeasures. *Fundamenta Mathematicae*, 159:243–258, 1999.

[28] Ilijas Farah. *Analytic quotients.* Memoirs of AMS 702. American Mathematical Society, Providence, 2000.

[29] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Ann. Pure Appl. Logic*, 76:4797, 1995.

[30] R. Fraiss é. Sur lextension aux relations de quelques propriet és des ordres. *Ann. Sci. École Norm. Sup.*, 7:363–388, 1954.

[31] Harvey Friedman and Lee Stanley. A Borel reducibility theory for classes of countable structures. *J. Symbolic Logic*, 54:894–914, 1989.

[32] Su Gao. *Invariant Descriptive Set Theory.* CRC Press, Boca Raton, 2009.

[33] E. Glasner, B. Tsirelson, and B. Weiss. The automorphism group of the Gaussian measure cannot act pointwise. *Israel Journal of Mathematics*, 148:305–329, 2005.

[34] G. Grünwald. Egy halmazelméleti tételről. *Mathematikai és Fizikai Lapok*, 44:51–53, 1937.

[35] András Hajnal and A. Máté. Set mappings, partitions, and chromatic numbers. In *Logic Colloquium 1973*, pages 347–379. North Holland, Amsterdam, 1975.

[36] Rudolf Halin. Über unendliche Wege in Graphen. *Mathematische Annalen*, 157:125–137, 1964.

[37] E. Hall, K. Keremedis, and E. Tachtsis. The existence of free ultrafilters on ω does not imply the extension of filters on ω to ultrafilters. *Math. Logic Quarterly*, 59, 2013.

[38] Philip Hall. On representatives of subsets. *J. London Math. Soc.*, 10:26–30, 1935.

[39] James Henle, Adrian R. D. Mathias, and W. Hugh Woodin. A barren extension. In *Methods in mathematical logic*, Lecture Notes in Mathematics 1130, pages 195–207. Springer Verlag, New York, 1985.

[40] Neil Hindman and Donna Strauss. *Algebra in Stone–Čech compactification: theory and applications.* De Gruyter, New York, 1998. De Gruyter Expositions in Mathematics 27.

[41] Wilfrid Hodges. *A shorter model theory.* Cambridge University Press, Cambridge, 1997.

[42] Haim Horowitz and Saharon Shelah. Transcendence bases, well-orderings of the reals and the axiom of choice. ArXiv:1901.01508, 2019.

[43] M. Hrušák, D. Meza-Alcántara, E. Thümmel, and C. Uzcátegui. Ramsey type properties of ideals. *Annals of Pure and Applied Logic*, 168:2022–2049, 2017.

[44] Jaime Ihoda (Haim Judah) and Saharon Shelah. Souslin forcing. *The Journal of Symbolic Logic*, 53:1188–1207, 1988.

[45] Thomas Jech. *Set Theory.* Springer Verlag, New York, 2002.

[46] Haim Judah, Andrzej Roslanowski, and Saharon Shelah. Examples for Souslin Forcing. *Fundamenta Mathematicae*, 144:23–42, 1994. arxiv:math.LO/9310224.

[47] Anastasis Kamburelis. Iterations of boolean algebras with measure. *Arch. Math. Logic*, 29:21–28, 1989.

[48] Vladimir Kanovei. *Borel Equivalence Relations.* University Lecture Series 44. American Mathematical Society, Providence, RI, 2008.

[49] Vladimir Kanovei, Marcin Sabok, and Jindřich Zapletal. *Canonical Ramsey Theory on Polish Spaces.* Cambridge Tracts in Mathematics 202. Cambridge University Press, 2013.

[50] Itay Kaplan and Saharon Shelah. Forcing a countable structure to belong to the ground model. *Math. Log. Q.*, 62:530–546, 2016.

[51] M. Kaufmann. The quantifier "there exist uncountably many" and some of its relatives. In S. Barwise and J Feferman, editors, *Model theoretic logics*, pages 123–176. Springer-Verlag, New York, 1985.

[52] Alexander Kechris. *Actions of Polish groups and classification problems*, pages 115–187. London Mathematical Society Lecture Note Series 262. Cambridge University Press, Cambridge, 2003.

[53] Alexander Kechris, Vladimir Pestov, and Stevo Todorcevic. Frassé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geometric and Functional Analysis GAFA*, 15:106189, 2005.

[54] Alexander Kechris, Slawomir Solecki, and Stevo Todorcevic. Borel chromatic numbers. *Advances in Mathematics*, 141:1–44, 1999.

[55] Alexander S. Kechris. *Classical Descriptive Set Theory.* Springer Verlag, New York, 1994.

[56] Richard Ketchersid, Paul Larson, and Jindřich Zapletal. Ramsey ultrafilters and countable-to-one uniformization, 2016.

[57] Julia Knight, Antonio Montalbán, and Noah Schweber. Computable structures in generic extensions. *Journal of Symbolic Logic*, 81:814–832, 2016.

[58] Peter Komjáth. The list-chromatic number of infinite graphs defined on Euclidean spaces. *Discrete Comput. Geom.*, 45:497–502, 2011.

[59] Péter Komjáth and James Schmerl. Graphs on Euclidean spaces defined using trascendental distances. *Mathematika*, 58:1–9, 2019.

[60] Peter Komjath and Saharon Shelah. Coloring finite subsets of uncountable sets. *Proceedings of the American Mathematical Society*, 124:3501–3505, 1996. arxiv:math.LO/9505216.

[61] Georges Kurepa. Á propos d'une généralisation d'ue la notion d'ensembles bien ordonnés. *Acta Math.*, 75:139–150, 1943.

[62] Adam Kwela and Marcin Sabok. Topological representations. *J. Math. Anal. Appl.*, 422:14341446, 2015.

[63] Paul Larson and Jindřich Zapletal. Canonical models for fragments of the axiom of choice. *Journal of Symbolic Logic*, 82:489–509, 2017.

[64] Paul B. Larson. *The stationary tower*. University Lecture Series 32. American Mathematical Society, Providence, RI, 2004. Notes from Woodin's lectures.

[65] Paul B. Larson. Scott processes. In *Beyond first order model theory. Volume I.* CRC Press, New York, 2017.

[66] D. Lecomte and B. D. Miller. Basis theorems for non-potentially closed sets and graphs of uncountable Borel chromatic number. *J. Math. Log.*, 8:121–162, 2008.

[67] David Marker. *Model theory: An introduction*. Graduate Texts in Mathematics 217. Springer Verlag, 2002.

[68] Andrew Marks and Spencer Unger. Borel measurable paradoxical decompositions via matchings. *Advances in Mathematics*, 289:397–410, 2016.

[69] A. R. D. Mathias. On sequences generic in the sense of Prikry. *J. Austral. Math. Society*, 15:409–414, 1973.

[70] Diego Mejía. Matrix iterations with vertical support restrictions. In *Proceedings of the 15thAsian Logic Conference*. World Sci. Publ., Hackensack, NJ. to appear.

[71] Jaroslav Nešetřil. Ramsey classes and homogeneous structures. *Combinatorics, Probability and Computing*, 14:171–189, 2005.

[72] Jaroslav Nešetřil and Vojtěch Rödl. Partitions of finite relational sets and systems. *Journal of Combinatorial Theory Ser. A*, 22:289–312, 1977.

[73] Vladimir Pestov. *Dynamics of Infinite-Dimensional Groups*. University Lecture Series 40. Amer. Math. Society, Providence, 2006.

[74] Christian Rosendal. Cofinal families of Borel equivalence relations and quasiorders. *J. Symbolic Logic*, 70:1325–1340, 2005.

[75] Christian Rosendal. Automatic continuity of group homomorphisms. *Bull. Symbolic Logic*, 15:184–214, 2009.

[76] Christian Rosendal. Continuity of universally measurable homomorphisms. *Forum of Mathematics, Pi*, 2019. to appear.

[77] Christian Rosendal and Sławomir Solecki. Automatic continuity of homomorphisms and fixed points on metric compacta. *Israel Journal of Mathematics*, 162:349–371, 2007.

[78] Edward R. Scheinerman and Daniel H. Ullman. *Fractional graph theory*. Wiley Series in Discrete Mathematics and Optimization. John Wiley and sons, New York, 1997.

[79] Ralf Schindler, Liuzhen Wu, and Liang Yu. Hamel bases and the principle of dependent choice. 2018. preprint.

[80] James H. Schmerl. Countable partitions of Euclidean space. *Math. Proc. Camb. Phil. Soc.*, 120:7–12, 1996.

[81] Saharon Shelah. On measure and category. *Israel Journal of Mathematics*, 52:110–114, 1985.

[82] Saharon Shelah. *Proper and Improper Forcing*. Springer Verlag, New York, second edition, 1998.

[83] Saharon Shelah and Juris Steprans. PFA implies all automorphisms are trivial. *Proceedings of the American Mathematical Society*, 104:1220–1225, 1988.

[84] Saharon Shelah and Jindřich Zapletal. Ramsey theorems for product of finite sets with submeasures. *Combinatorica*, 31:225–244, 2011.

[85] Slawomir Solecki. Analytic ideals and their applications. *Annals of Pure and Applied Logic*, 99:51–72, 1999.

[86] Juris Steprans. Strong Q-sequences and variations o Martin's axiom. *Canadian Journal of Mathematics*, 37:730–746, 1985.

[87] Jacques Stern. On Lusin's restricted continuum problem. *Annals of Mathematics*, 120:7–37, 1984.

[88] Endre Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arithmetica*, 27:199245, 1975.

[89] A. Szymański and Zhou Hao Xua. The behaviour of ω^{2^*} under some consequences of Martin's axiom. In *General topology and its relations to modern analysis and algebra, V, Prague, 1981*, number 3 in Sigma Ser. Pure Math., pages 577–584. Heldermann, Berlin, 1983.

[90] Simon Thomas and Jindřich Zapletal. On the Steinhaus and Bergman properties for infinite products of finite groups. *Confluentes Math.*, 4, 2012. 1250002.

[91] Stevo Todorcevic. *Partition problems in topology.* Contemporary Mathematics 84. American Mathematical Society, Providence, RI, 1989.

[92] Stevo Todorcevic. Remarks on Martin's Axiom and the Continuum Hypothesis. *Canadian Journal of Mathematics*, 43:832–851, 1991.

[93] Stevo Todorcevic. Two examples of Borel partially ordered sets with the countable chain condition. *Proceedings of the American Mathematical Society*, 112:1125–1128, 1991.

[94] Stevo Todorcevic. *Walks on ordinals and their characteristics*. Birkhauser Verlag, Basel, 2007. Progress in Mathematics 263.

[95] Todor Tsankov. Automatic continuity for the unitary group. *Proceedings of American Mathematical Society*, 141:3673–3680, 2013.

[96] Douglas Ulrich, Richard Rast, and Michael C. Laskowski. Borel complexity and potential canonical Scott sentences. *Fund. Math.*, 239:101–147, 2017.

[97] Mark Brian VanLiere. *Splitting the reals into two small pieces.* PhD thesis, University of California, Berkeley, 1982.

[98] Boban Velickovic. Definable automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$. *Proceedings of the American Mathematical Society*, 96:130–135, 1986.

[99] Hugh Woodin. Supercompact cardinals, sets of reals and weakly homogeneous trees. *Proceedings of the National Academy of Sciences USA*, 85:6587–6591, 1988.

[100] Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms and the Nonstationary Ideal.* Walter de Gruyter, New York, 1999.

[101] Jindřich Zapletal. *Forcing Idealized.* Cambridge Tracts in Mathematics 174. Cambridge University Press, Cambridge, 2008.

[102] Jindřich Zapletal. Interpreter for topologists. *Journal of Logic and Analysis*, 7:1–61, 2015.

[103] Jindřich Zapletal. Hypergraphs and proper forcing. *Journal of Mathematical Logic*, 2019. to appear.

[104] Jindřich Zapletal. Subadditive families of hypergraphs. 2019. unpublished.

[105] Joseph Zielinski. The complexity of the homeomorphism relation between compact metric spaces. *Advances in Mathematics*, 291:635–645, 2016.

[106] Andy Zucker. Big Ramsey degrees and topological dynamics. *Groups, Geometry, and Dynamics*, 13:235–276, 2019.

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