Tracking The Tangency Portfolio With Alpha*

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Abstract

We propose an investment strategy aimed at resolving well-known problems with practical implementation of tracking the risky-securities-only mean-variance optimal portfolio (the "tangency portfolio" TP). The strategy uses the alphas of available investments relative to one's past portfolio in order to gradually move towards, and track, the TP. Back-testing on monthly U.S. data demonstrates that this alpha-based TP tracking strategy actually improves upon TP when the latter is estimated from the same rolling window of past return data. The strategy also produces higher Sharpe ratios than a standard market index, but not necessarily higher average returns.

The mean-variance optimising investor faces a number of challenges. Here, we focus on one of them: tracking the so-called Tangency Portfolio (TP), which is the mean-variance optimal portfolio consisting only of risky portfolios. At least four issues arise when attempting to construct the TP.

The first issue is computational. To find the composition of the TP, one has to solve an ill-defined problem, namely, to solve a system of linear equations the solution of which is known to be sensitive to small changes in the coefficient matrix. In the present case, the coefficient matrix is the covariance matrix of returns on the candidate investments. Estimates of the covariance matrix tend to have a low conditioning number, which is the cause of the sensitivity. In the past, this has prompted the use of numerical regularization (a numerical technique to solve ill-defined problems; Brodie, Daubechies, De Mol, Giannone, and Loris 2009) and statistical "shrinkage" (a technique to improve multivariate estimation; Ledoit and Wolf 2004). There are many approaches to implement shrinkage and regularization, and their merits are still hotly debated. Neither address the additional issues with constructing the TP; see, e.g., Bollerslev, Patton, and Quaedvlieg 2016.

The second issue is theoretical. Almost fifty years of empirical research has demonstrated that the historical mean-variance frontier can be best approximated by combining a number of factor funds, such as the Fama-French five-factor funds (Fama and French 2016). As a result, the industry has started to offer funds that track these and other factors. It is left to the investor to determine the combination of available funds into an optimal portfolio, the TP portfolio. Unfortunately, weights that worked in the past hardly work in the future (Cooper, Gutierrez, and Marcum 2005). In this sense, the investor is provided with the right funds but is left with the most difficult part of the exercise: to determine how to combine the funds.

The third issue is cognitive. Many investors hold substantially under-diversified portfolios (Goetzmann and Kumar 2008). The TP provides far more diversification. By definition, it provides the best diversification for the expected return it generates. Financial advisors who recommend investors to move from their under-diversified portfolio to a more optimal portfolio run the risk of disappointing the advisee in the short run. This is because risky investments are volatile (indeed), and the distribution of outcomes of the advisee's original portfolio has far heavier tails – including to the right – than the TP. As such, there is a high chance that the TP under-performs for a while, causing the advisee to doubt the expertise of the advisor, and returning to the original, sub-optimal portfolio. In general, investors prefer to stay with the status quo (Samuelson and Zeckhauser 1988).

The fourth issue concerns implementation. Weights of the TP are estimated, and these estimates are known to vary substantially over time, even after properly adjusting for estimation error. The corresponding portfolio adjustments lead to substantial transaction costs. The variability of weights is related to the first issue discussed above, though it also reflects movements in the mean-variance frontier.

Behind these four issues is a common cause: the insistence on immediate, full implementation of the TP. We suggest that a more gradual approach may do better. Such an approach would directly address the third issue above, allowing the advisor to stay closer to the status quo—which the investors prefer—and recommend only marginal changes. Those marginal changes also provide better control of transaction costs. Whether a gradual approach addresses the computational and theoretical issues above depends on the nature of the marginal adjustments, and is ultimately an empirical question, which we investigate here.

We propose a TP tracking strategy based on locally optimal weight adjustments. In a nutshell, the strategy works as follows. The TP is defined to be the portfolio of risky securities that maximizes the return-to-risk ratio, i.e., the *Sharpe ratio*. It provides the highest expected excess return per unit of volatility (standard deviation). As Blume 1984 and Dybvig and Ross 1985 demonstrated before, to improve the Sharpe ratio of any given portfolio, one computes the alphas of all available investments with respect to the existing portfolio, and increases weights on those investments that have a positive alpha, while decreasing weights on investments with a negative alpha. Our strategy entails computation of alphas with respect to the existing portfolio and executing weight changes that are proportional to alpha.

It is important to point out that in the approach we proposed, the alphas are computed with respect to the existing portfolio, that is, the status quo, and not some other benchmark portfolio. This is in stark contrast with the so-called "Jensen's alpha", where the alpha is computed with regards to some benchmark portfolio (or a collection of benchmark portfolios). This approach is typically used to evaluate the performance of portfolio managers. A positive alpha indicates out-performance, while a negative alpha indicates under-performance. The aim of the strategy we proposed is to improve an existing portfolio by identifying appropriate assets. Therefore, we are interested in alphas with respect to the existing portfolio and not some other benchmark. Thus, "Jensen's alpha" would not be a suitable measure for our analysis.

Practical implementation of a locally optimal procedure to track TP requires one to address a number of issues. Foremost, one is immediately confronted with the question: When is an adjustment local? Equivalently: What is the maximum weight change before the adjustment can no longer be considered local? While the answer to this question should in part be determined by the investor – how much she is willing to adjust her portfolio in view of the status-quo bias – changes that are too small may cause the investments to never come close to performing like the TP, while large changes may invalidate the mathematics, which in principle only guides infinitesimal changes. Here, we propose a reasonable strategy, namely, to change at most five (5) percent of one's portfolio at a time; we leave it to future work to determined the value of different parameters, and which values best accommodate investors' status-quo bias. For ease of reference, we call our investment strategy the *alpha-based TP tracker*.

Our procedure was first proposed in Bossaerts and Yang 2016 and successfully back-tested on U.S. mutual fund data, without consideration, however, of transaction costs or short-selling constraints. In the back-testing reported here, we did account for transaction costs, barred short-selling, and made sure prices used in return calculations to estimate alphas were not re-used in re-balancing calculations. The latter means that we left one period (one month) between estimation of alphas, and implementation of alpha-based improvements.

Levy and Roll 2015 also back-tested an alpha-based strategy. Their goal was to improve on buying-and-holding an index by adjusting weights of component securities. They found little evidence of an improvement, which Bossaerts and Yang 2016 attributes to high error in estimating alphas when using individual securities. Here, we use diversified funds. In addition, we do not limit investments to securities already in the benchmark portfolio, but consider adding new – sometimes vastly different – investments. Indeed, the purpose of our analysis is not to improve weights of an existing portfolio as in Levy and Roll 2015, but track the TP when the TP could have changed because of expansion of one's investment opportunity set with the inclusion of new risky assets.

Our TP tracking policy avoids the computational problem with direct determination of the TP because weight changes do not depend on the solution to an ill-defined problem. Estimation error is also minimized because, with N investments, only N parameters have to be estimated (the N alphas), as opposed to $(N(N-1))/2 + N$ [namely, the $(N(N-1))/2$ return variances and covariances, plus the N average returns].

Importantly, the alpha-based TP tracker should not be compared to dynamic asset allocation, also referred to as tactical asset allocation. We are not forecasting which funds would do better temporarily, switching in and out of, say, equity funds and cash. Instead, the alpha-based TP tracking strategy is a way to implement strategic asset allocation, and as such aims at obtaining the right mix of investments rather than to time investments. Our benchmark will be a particular portfolio, namely, the TP portfolio. To the extent that the investment opportunity set changes over time, TP will change weights. Because of this, our strategy does acquire dynamic features, since its target is a moving object in mean-variance space.

The remainder of this paper is organized as follows. The next section describes the methodology. Section 2 summaries the back-testing results. Section 3 concludes.

1 Methods

1.1 Using alpha to track the Tangency Portfolio

The goal of our strategy is to track the tangency portfolio TP. We start from a current portfolio p of risky securities only and consider adding investments. The current portfolio can be viewed as a tangency portfolio itself when restricting the investment space to only p and its constituent assets, and a risk free investment (the "old efficient frontier" in Figure 1). The additional investments expand the frontier to a "new frontier" (see Figure 1). The goal is to move from the old TP to the new TP in a stepwise manner, whereby each step is locally optimal. Locally optimal moves need not be on the shortest path from the old to the new TP. See green arrows in Figure 1.

In Appendix 1 we prove that optimal moves into any additional investment should be proportional to the α (intercept) in a projection of the excess returns on the new investment onto those of p. Specifically, if α is positive, the optimal strategy is to increase the weight of the new investment proportional to α. If α is negative, and if there are short-sales constraints, the only way to implement the locally optimal move is when the new investment is part of the existing p, in which case one reduces its weight in p.

The link between alpha and locally optimal portfolio improvements is not new; to our knowledge, the first formal proofs appeared in Blume 1984 and Dybvig and Ross 1985. The first rudimentary tests of the practical relevance of the result were reported in Bossaerts and Yang 2016 and Levy

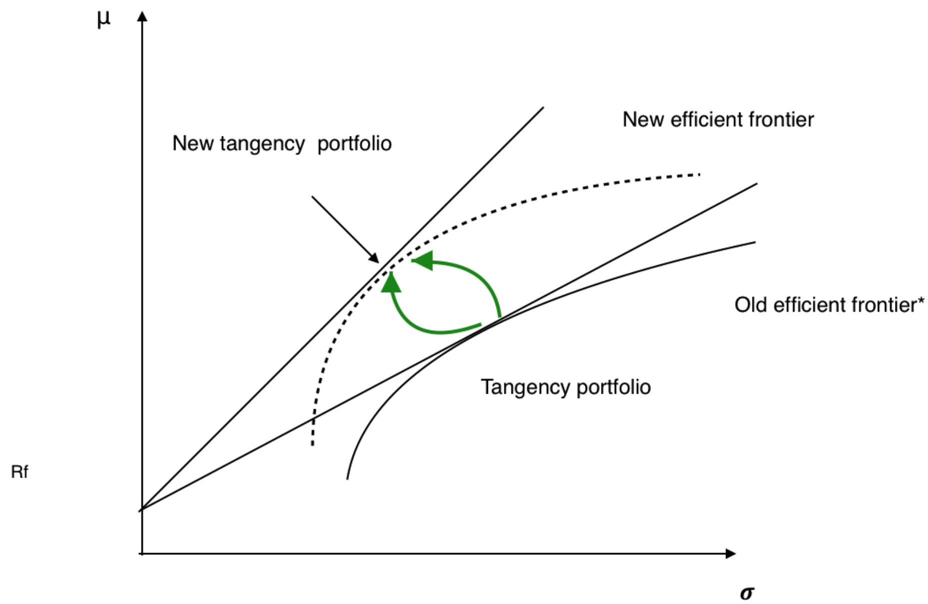

Figure 1.
Graphical illustration of the alpha-based TP tracker.
Addition of investment opportunities moves the tangency portfolio (TP) up in mean-variance space. The alpha-based TP tracker moves gradually from the old TP to the new TP (green arrows). Each step is locally optimal. The path may not be the shortest path in mean-variance space; it could initially reduce volatility more than increasing expected return (left green arrow), or first increase expected return while keeping volatility constant (right green arrow).

and Roll 2015.

1.2 Implementation

1.2.1 Asset Space. We back-tested the alpha-based TP tracker on U.S. data, starting with a portfolio that is 100% invested in a broad index (Vanguard S&P500 index fund, code VFINX) and gradually adjusting this portfolio by contemplating movements in and out of mutual funds. S&P500 is an American stock market index based on the market capitalisation of 500 large companies listed on the NYSE or NASDAQ.

We selected mutual funds rather than individual securities (stocks, bonds) because the volatility of mutual funds is lower than that of individual stocks (Bansal and Somani 2002), which means that the alphas of the mutual funds with respect to one's current portfolio could be estimated with more precision.

1.2.2 Trading Restrictions. To further increase relevance for retail investors, we restrict short sales, which are costly and cumbersome for private clients to execute, and in fact impossible for our asset class, mutual funds. At no time is the position in any mutual fund allowed to go negative. Leveraged positions are also excluded. As such, the portfolio is a non-negatively weighted investment in the available risky assets.

Transaction costs are hugely relevant for retail investors, all the more so because our strategy may lead to continuous transacting, because it only gradually adjusts to the TP, and the TP may itself move. It is assumed that the transaction costs for mutual funds are symmetric in the sense that they do not depend on whether the investor is buying or selling. The size of the transaction costs are assumed to be proportional to the sum traded. Transaction costs of mutual funds (the asset universe considered here) vary depending on circumstances. They may be zero, but in such circumstances one is not allowed excessive trading. Whether our alpha-based TP tracker strategy generates excessive trading is an empirical matter, which we investigate here. Our monthly rebalancing frequency should avoid concerns of over-trading. Nevertheless, we shall not consider transaction costs to be zero, and instead use an estimate from trading a related asset class, namely, Exchange-Traded Funds (ETFs). For ETFs, Poterba and Shoven (2002) suggested that the average bid–ask spread as a percentage of the price is 0.096 percent. We therefore follow Pesaran and Timmermann (1994), and take transaction costs to be 0.1 percent of volume.

The costs are accounted for at the time of transaction, as follows. Let N_t denote the number of units bought or sold, and P_t denote the price of the mutual fund at time t. Then transaction costs are computed as $0.1\% \times N_t \times P_t$.

The alpha-based PT tracker uses the alpha of available assets (here: mutual funds) relative to the current asset portfolio. Alphas are estimated as the intercept in a time-series projection (OLS): $R_i = \alpha_i + \beta_i R_p + \epsilon$, where $R_i = r_i - r_f$ and $R_p = r_p - r_f$; r_i denotes the return on mutual fund i, r_f the risk free rate, and r_p the return on the current portfolio.

The investor is assumed to start with $100,000 entirely invested in a benchmark index (S&P 500). The alphas of the alternative investments are estimated each month from the excess returns over a fixed prior window (sixty months). Projections are onto the excess returns of a portfolio with weights equal to those of the current-period holding portfolio. A mutual fund is then acquired, its weight in the current portfolio increased or reduced, based on the estimated alpha. If the alpha is positive and significant at the 10-percent level, then the mutual fund weight is increased. If the alpha is significantly negative, we sell the mutual fund, provided it is in the current portfolio; that is, we reduce its weight.

Importantly, purchases and sales are implemented at prices valid for the *end* of the month. That is, alphas (and hence, potential purchases and sales) are based on returns computed from prices up to and including the beginning-of-month observations, but re-balancing is not implemented until the end of the month. If we implemented purchases and sales at beginning-of-month prices, practical application of our strategy could quite rightly be questioned.

The rebalancing amount is set to be maximally 5% of the current wealth of the portfolio. If there are multiple assets to be bought or sold, assets are bought equally weighted (regardless of the alphas), and sold based on their respective weights in the current portfolio (i.e., mutual funds with higher weights are sold more than those with lower weights, regardless of alpha).

Our decision to not change weights proportional to alphas, as in the theory, is meant to reduce estimation error. Bond mutual funds tend to have lower estimated alphas, merely because they are less volatile; equity mutual funds can have bigger alphas (in absolute value), but a large fraction of the estimated alphas reflects estimation error. Our equal weighting of adjustments (for purchases) or value weighting (for selling) effectively implements a statistical "shrinkage" strategy, often used in a multivariate context to reduce estimation error Ledoit and Wolf (2004). We reduced the impact of estimation error even more, through imposition of a minimum significance level before we adjusted weights, namely, 10% ($p < 0.10$).

1.2.3 Historical Data. U.S. return data cover the period from January 1996 to June 2016. The asset space consists of fifteen large U.S. mutual funds that include equity, fixed income, natural resources and health care sectors (see Appendix 2 for details).

We only selected a small number of mutual funds. The choice was deliberately arbitrary, except that they spanned asset classes beyond U.S. large-firm common stock. There are thousands of mutual funds to choose from, while retail investors tend to consider only a dozen or so, or are offered no more. To add realism (for small investors), we stayed within one mutual fund family, namely Vanguard, and selected 15 funds (including the benchmark S&P 500 fund). We did require the funds to be in existence at the beginning of our sample (January 1996).[1]

The return observation interval is monthly, and the fixed window on which we estimate alphas is sixty months (five years) long. The 30-day Treasury bill rate approximates the risk-free rate.

The data, obtained from CRSP,[2] include the Global Financial Crisis (GFC). To better gauge the impact of an unusual and extreme event such as the GFC, in a robustness analysis we re-started the investment just prior to the GFC, in February 2007. This allowed us to determine whether our strategy avoided the dramatic reduction in value of the index, while at the same time re-covering together with the index after the GFC.

1.2.4 Performance evaluation. Our performance measure is the Sharpe ratio (Sharpe 1994). It measures the reward-to-risk ratio of the portfolio under consideration, and is defined as $[E(R_p) - r_f]/\sigma_p$, where $E(R_p)$ stands for the mean return of portfolio p, r_f denotes the risk-free interest rate, and σ_p is the standard deviation of the return on portfolio p. Portfolio p is then said to dominate the benchmark portfolio in the mean-variance sense if its Sharpe index is higher. As benchmarks, we used the S&P 500, and the estimated TP. The weights of the TP were determined from the estimated mean excess returns on the available investments, and their estimated covariances, computed from the same fixed window as the alphas (sixty months). We impose short-sale constraints: all weights of the TP are restricted to be non-negative.

The weights on the alpha-based TP tracker and on the estimated TP change over time, and as such the corresponding investment strategy incurs transaction costs. We account for transaction costs in the computation of the Sharpe ratio, as follows. In each period, we compute the return from wealth changes accounting for transaction costs. Then we calculate average excess returns and standard deviations over a fixed window in the past (sixty months), including the current period. From average excess returns and sample standard deviations, we compute the Sharpe ratio.

We also construct expected Sharpe ratios, meant to be reward-to-risk ratios that gauge the promise of the current portfolio (alpha-based TP tracker; estimated TP) going forward. Let w denote the weight vector of the portfolio at hand (alpha-based TP tracker; estimated TP). We estimated the average excess return vector R and the covariance matrix Ω over the same fixed window from which the weights were determined, and computed the Sharpe ratio as $wR/(w\Omega w')^{0.5}$.

Note that this forward-looking Sharpe ratio does not account for transaction costs.

1.2.5 Programs. Python (2.7) was used to program the algorithm of the strategy and Matlab was used for data analysis. Descriptions of the code can be found in Appendix 3.

2 Results

In terms of expected Sharpe ratios, the alpha-based TP tracking portfolio beat buying-and-holding the S&P 500 every month but two during our sample period (Figure 2, left). The TP does even better, but the alpha-based TP tracker faithfully follows the evolution of the Sharpe ratio of the TP. Initially, and after a major shock (the GFC), the alpha-based TP tracker takes about a year to catch up with the TP. This suggests that (i) initially, the TP has a very different composition than the portfolio the investor started out with, which is 100% in the S&P 500; (ii) after a major shock, such as the GFC, the investment opportunity set changes substantially, and hence, the composition of the TP changes as well. It takes time for the TP tracker to catch up.

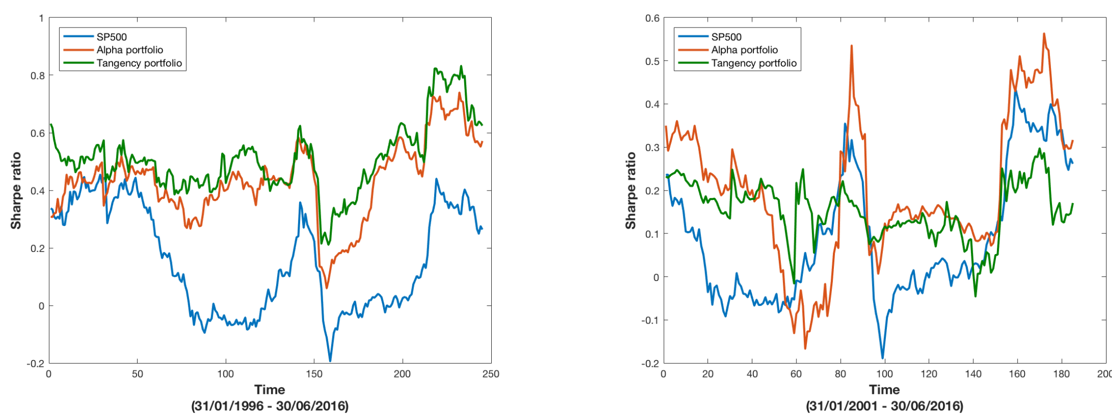

Figure 2.
Evolution of the Sharpe Ratios of the S&P 500 index, the alpha-based TP tracker and the TP.
Left: Ex-ante month-by-month Sharpe ratios, ignoring transaction costs; Right: ex-post Sharpe ratios, computed from realised, after-transaction-costs returns over prior 60 months including current month.

The after-transaction-costs, realized Sharpe ratio of the alpha-based TP tracker is substantially below the expected Sharpe ratios, suggesting that transaction costs, estimation error, and movements in the investment opportunity set have major effects. Still, the alpha-based TP tracker beats buying-and-holding of the S&P 500 in all but two months. (The time series start 60 months later than that of the expected Sharpe ratios because 60 months of performance data need to be recorded before the first ex-post Sharpe ratio can be computed.)

Importantly, the alpha-based TP tracker beats the TP over long episodes. We attribute this to the enormous variability of the estimated weights (see Figure 2, right). This is not unexpected: the root cause is the ill-defined nature of the weight determination problem, as explained in the introductory remarks. Indeed, the erratic behavior of the weights is *prima facie* proof that the problem is ill-defined: small changes in parameters (estimated over a sixty-month rolling window) cause major shifts in weights.

In contrast, the weights of the alpha-based TP tracker change far less abruptly (see Figure 3, left). Significantly, the tracker portfolio tends to be invested in only a few mutual funds, and the mix of invested mutual funds changes over time, but slowly. Figure 4 shows a time plot where dots indicate months when the alpha-based TP tracker strategy bought (top) and when it sold (bottom). Purchases are infrequent, and sales even less frequent.

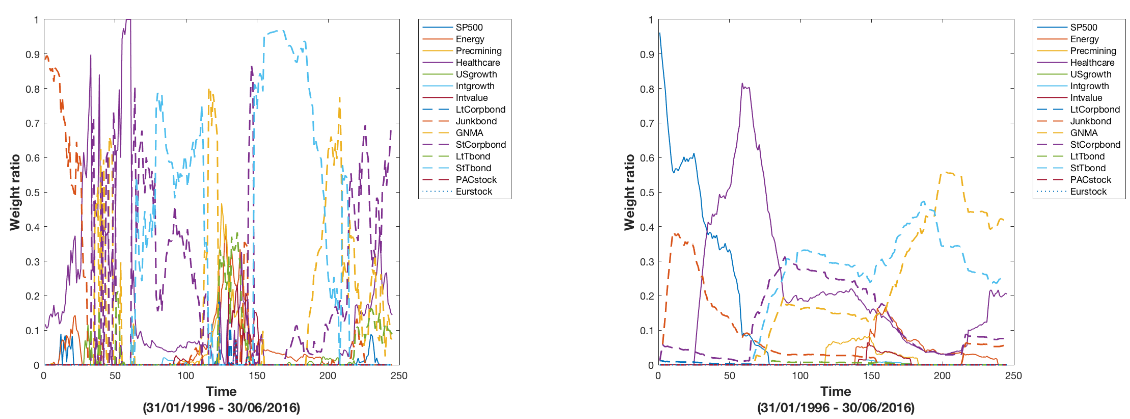

Figure 3.
Evolution of the weights on mutual funds in the investment opportunity set.
Left: TP weights. Right: alpha-based TP tracker, starting from 100% S&P 500 index.

Notice that the tracker portfolio gradually divests of the S&P 500 index and never returns to this benchmark portfolio (see Figure 3, right). This is quite surprising, especially after the GFC, when the S&P 500 produced high Sharpe ratios. Nevertheless, the alpha-based TP tracker beats the S&P 500 in all but two months after the GFC.

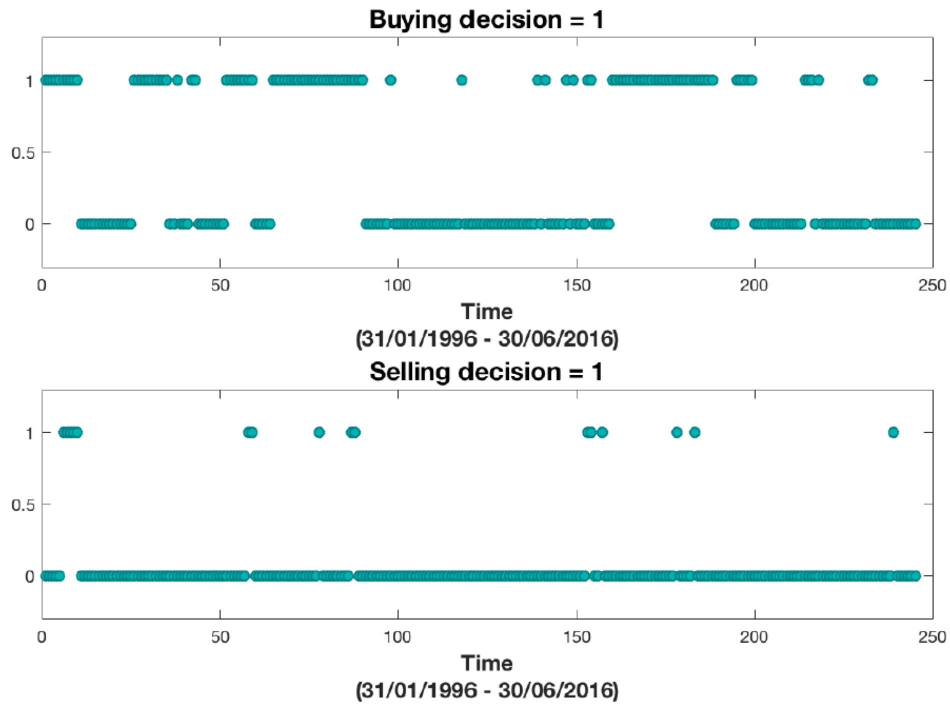

Figure 4.
Evolution of purchases and sales decisions, alpha-based TP tracker.
Alpha-based TP tracker: Buy (Top) or sell (Bottom) decisions were made in months with a green dot at 1; otherwise no purchasing/sales were made (green dot at zero).

The evolution of the wealth ratio (current wealth, after accounting for transaction costs, divided by initial wealth) provides a clue as to what happened after the GFC. Figure 5. Wealth in the alpha-based TP tracker and in the TP both increase more slowly than that in the S&P 500. One can also discern far higher volatility in the S&P 500. Together, this means that the TP and its tracker became high-return, low-volatility investments. The S&P 500 generated far higher average returns, but at too high a cost in terms of volatility: it generated a Sharpe ratio that is *below* that of the TP tracker (but not below that of the TP; see Figure 2). This is reflected in the composition of the TP tracker after the GFC: besides natural resources, it is virtually exclusively invested in fixed-income securities (Figure 3). In 2013, the tracker strategy started unloading Natural Resources and Short-term T-Bonds and instead bought health care stocks.

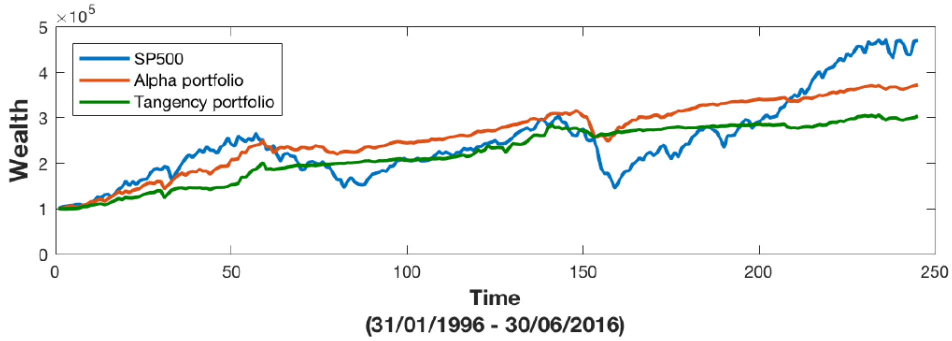

Figure 5.
Evolution of wealth, alpha-based TP tracker.
Evolution of wealth, after transaction costs, from investing 10,000 dollars in the S&P 500 (blue), alpha-based TP tracker (red) and TP (green).

To gauge to what extent the alpha-based TP tracker strategy is robust to initial conditions (initial portfolio), we submitted it to an extreme test. We re-started TP tracking just before the GFC (February 2007) and studied its performance compared to that of the S&P 500 and the TP itself. The conjecture was that the strategy is insensitive to this rather extreme switch in initial conditions. We did expect a reduction in performance, if only because of transaction costs: the alpha-based TP tracker would initially divest of the S&P 500 and move into the assets that best replicate TP.

Figure 6 displays the evolution of the expected Sharpe ratio post-GFC. Within 10 months, the alpha-based TP tracker catches up with (though remaining below) the performance of the TP, substantially beating the S&P 500. Ex-post, and after transaction costs, it takes 40 months before the alpha-based TP tracker beats the S&P 500 (see Figure 6, right). The TP never manages to recover.

The evolution of the composition of the alpha-based TP tracker proves its robustness. By the end of the sample (2016), the tracker portfolio is invested essentially in the same mutual funds, regardless of the starting point, 01/1996 or 02/2007 (compare Figure 3, Right Panel, to 7). The

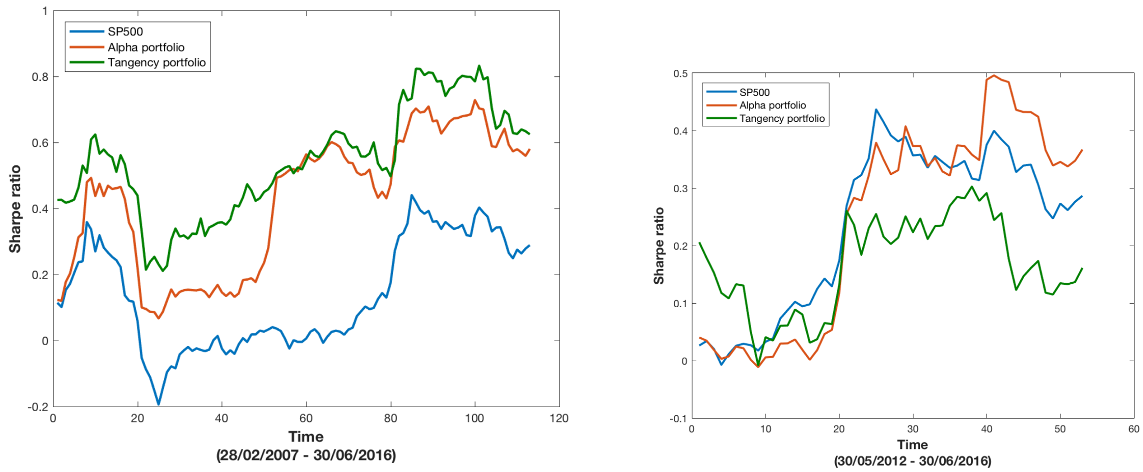

Figure 6.
Post-GFC evolution of the Sharpe Ratios of the S&P 500, the alpha-based TP tracker and the TP.
Alpha-based TP tracker starts 100% invested in S&P 500 on 28 February 2007. Left: Ex-ante month-by-month Sharpe ratios, ignoring transaction costs; Right: ex-post Sharpe ratios, computed from realised, after-transaction-costs returns over prior 60 months including current month.

impact on volatility is noticeable: return variance dropped substantially as the tracker portfolio moved into bonds.

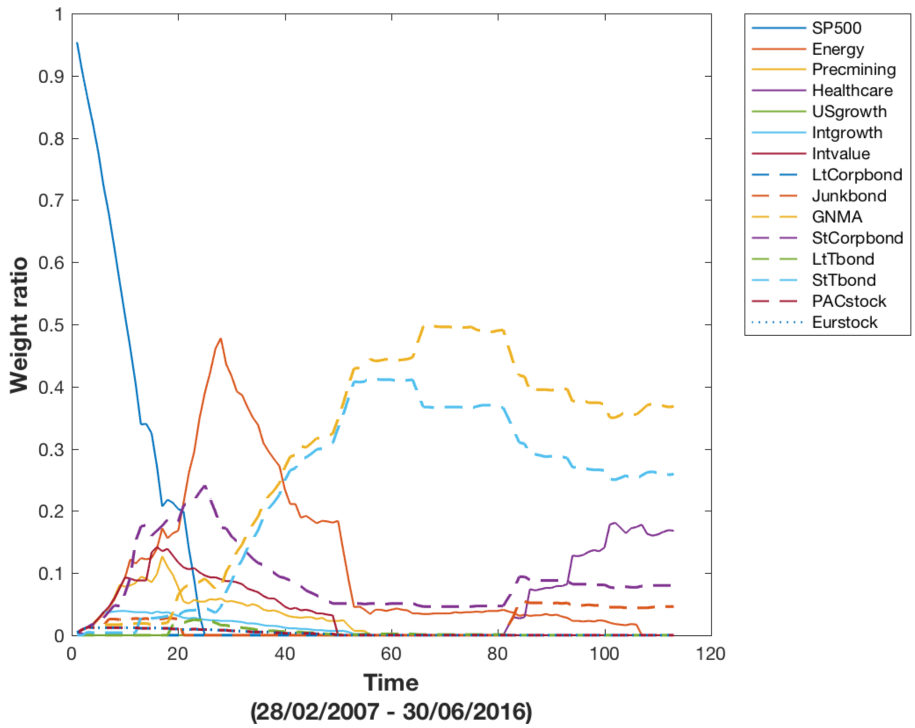

Figure 7.
Post-GFC evolution of weights on mutual funds in the investment opportunity set, alpha-based TP tracker.

3 Discussion

We propose here an optimal, gradual approach to tracking the TP (Tangency Portfolio) and back-tested the approach on monthly U.S. data. The approach continuously tracks the alphas of candidate investments with respect to the current portfolio, increasing weights on investments with a positive alpha while decreasing those with a negative alpha. In the back-testing, we set a adjustment limit of 5% of the value of the current portfolio. The alpha-based TP tracking strategy closely follows the ex-ante Sharpe ratio of the TP, and beats the ex-post, after transaction-costs Sharpe ratio of both the TP and the local benchmark (S&P 500).

The alpha-based TP tracking strategy solves a number of problems encountered when implementing the TP in the traditional way. First, determination of optimal weights does not involve solving an ill-defined problem. Here, we provided – once more – evidence of how badly defined the problem of computing TP weights is: the estimated weights vary enormously from one period to another, and the resulting transactions generate costs that ultimately reduce the performance of TP from *ex ante* "better than," to *ex-post* "worse than," our alpha-based TP tracking strategy.

Second, the alpha-based TP tracking strategy provides the investor with an automatic, yet fruitful way to determine the most difficult task in investing, which the industry has left her to do on her own, namely, to determine the correct weights on the different funds/mutual funds that the industry has come to offer.

Third, the fact that the proposed strategy adjusts only gradually, though successfully, towards the TP, makes it attractive to retail investors, who tend to prefer the status quo, or would be willing to make only marginal changes in the face of substantial uncertainty about investment returns and mistrust of advisors.

Fourth, the gradual approach allows the investor to better control transaction costs. In our back-testing, transaction costs were accounted for only in a rudimentary way. Control could be dramatically improved by explicit modulation of a fund's weight changes depending on that fund's transaction costs.

Further fine-tuning of the alpha-based TP tracking strategy could be envisaged, by means of more sophisticated estimation of alphas (accounting for, e.g., GARCH effects).

The automatic and gradual approach of our strategy makes it highly suitable for robot advice that can be tailored to both the client (e.g., periodic adjustments can be reduced or increased depending on the anxiety of the client) and the fund provider (with each provider offering its own set of funds). Before practical implementations are offered, the alpha-based TP tracker needs to be more thoroughly tested for robustness. In particular, sensitivity of performance to initial holdings and to periodic re-balancing maximum needs to be documented.

We left an entire period (one month) between estimation of alphas, determination of potential improvements in the alpha-based tracker portfolio, and implementation of those improvements. We did so in order to make our strategy implementable in practice. The long period between calculation of new weights and implementation of those weights ensures that our analysis is relevant for the retail customer, who will take more time to apply the alpha-based re-balancing. Efforts to reduce this lead time should of course improve the performance of the proposed strategy.

Back-testing of the alpha-based TP tracking strategy once more proved how hard it is to beat buying and holding a broad index such as the S&P 500. There is a large gap between ex-ante promise (Sharpe ratio) of strategic asset allocation proposals such as ours and their ex-post performance. Still, ours did better than the index overall, and it tended to generate returns at a lower level of volatility. The bias of the alpha-based TP tracking strategy towards lower volatility investments is one more reason why it is suitable especially for retail customers.

Appendix 1: Proof of Main Result

We are interested in finding a locally optimal change towards the new tangency portfolio (TP) given the availability of a "new security." With "locally" we mean "small," or to use a technical term, "infinitesimal." The main result is:

> *The locally optimal change towards TP equals α. This means that, if α is positive, one has to increase exposure to the new security; if negative, one decreases exposure/short-sells.*

Remark: If one cannot find any "new" security with a non-zero alpha, the above result implies that one's existing portfolio is optimal, which means that one is holding the new TP. This is an implication of a celebrated result in portfolio theory, namely: a portfolio is (mean-variance) optimal if and only if all alphas computed relative to it are zero (Roll 1977).

Rather than directly proving this when the criterion is to optimize the Sharpe ratio subject to a constraint to be all in risky securities (no investment in the risk free security), we here take an alternative route.

Our starting point will be the current TP p. It is the tangency portfolio (TP) for the universe of assets consisting of p and the risk-free security only. To simplify derivation of locally optimal moves once we add a risky security to the universe of assets, we first determine the mean-variance trade-off parameter (the penalty for risk b) so that it is optimal to hold TP, and hence, it is optimal to be 100% invested in p. We can always find such a b, except in the pathological case that the expected return on p is equal to or below the risk-free rate.

We then introduce a new risky security and ask whether we should add it. It could already be a component of p, in which case the question boils down to: is it optimal to increase the weight on it? If it is indeed a component of p, we can even entertain the possibility to reduce its weight, which would be equivalent to keeping p and short-selling the new investment. Once the adjustment is done, we can repeat the procedure. We find the penalty b that makes it optimal to invest in the new combination of the original p and the new investment. We then determine whether more of the new investment needs to be added/sold short. Or maybe another investment. Etc.

Let p denote the existing portfolio with expected *excess* return μ_p and return variance σ_p^2. The risk-free rate is denoted by R_F.

Let z denote the proportion of wealth allocated to risk free securities and $1-z$ (the complement) allocated to p. Find the penalty b by evaluating the maximum of the following mean-variance performance criterion at $z=0$:

$$zR_F + (1-z)(R_F + \mu_p) - b(1-z)^2\sigma_p^2.$$

Here, the first two terms add up to the expected return on a portfolio of weight z in the risk-free security and weight $(1-z)$ in p; the third term is the penalty for risk when risk is measured as return variance (penalty coefficient: b). The first-order condition reads equals:

$$R_F - (R_F + \mu_p) + 2b(1-z)\sigma_p^2 = 0.$$

The penalty b that satisfies this condition for $z=0$ can readily be obtained from the following equality:

$$\mu_p - 2b\sigma_p^2 = 0. \tag{1}$$

Assuming that $\mu_p \neq 0$ a solution always exists. (For the second-order condition to be satisfied, $\mu_p > 0$; it's obvious to see why: if $\mu_p < 0$ then $b < 0$ which means that the risk penalty coefficient is negative.) We'll use this condition on b later.

Now contemplate re-allocating a tiny part of the current portfolio p to a new security. The new security has expected excess return equal to μ, return variance equal to σ^2, and covariance with the return on your existing portfolio equal to $\beta\sigma_p^2$, where β is the slope in a linear regression (projection) of excess returns on this new security onto excess returns of your current portfolio (p).

Let α denote the corresponding intercept of this regression. It follows: $\mu = \alpha + \beta\mu_p$.

Proof

A celebrated result in calculus teaches us that locally optimal changes should be in the direction of the gradient of the performance criterion. So, let us compute the gradient (vector of derivatives of the criterion function w.r.t. portfolio weights). We only accept changes that satisfy the budget constraint. That is, we are interested in re-allocating the existing portfolio to a tiny proportion

x in the "new security," a tiny proportion z in the risk-free security, and leave the remainder $1-x-z$ in the existing portfolio. The gradient is two-dimensional, consisting of the derivative of the performance criterion w.r.t. x and the derivative w.r.t. z.

The performance criterion equals:

$$\begin{aligned} P &= zR_F + x(R_F+\mu) + (1-z-x)(R_F+\mu_p) \\ & \quad -b\{x^2\sigma^2 + (1-z-x)^2\sigma_p^2 + 2x(1-z-x)\beta\sigma_p^2\}. \end{aligned}$$

We substitute $\alpha + \beta\mu_p$ for μ and simplify:

$$\begin{aligned} P &= R_F + x(\alpha+\beta\mu_p) + (1-z-x)\mu_p \\ & \quad -b\{x^2\sigma^2 + (1-z-x)^2\sigma_p^2 + 2x(1-z-x)\beta\sigma_p^2\}. \end{aligned}$$

The derivatives we are interested in are:

$$\frac{\partial P}{\partial x} = (\alpha+\beta\mu_p) - \mu_p - b\{2x\sigma^2 - 2(1-z-x)\sigma_p^2 + 2(1-z-x)\beta\sigma_p^2 - 2x\beta\sigma_p^2\},$$

$$\frac{\partial P}{\partial z} = -\mu_p - b\{-2(1-z-x)\sigma_p^2 - 2x\beta\sigma_p^2\}.$$

Evaluate these derivatives at the status-quo, i.e., $x = z = 0$:

$$\frac{\partial P}{\partial x} = (\alpha+\beta\mu_p) - \mu_p - b\{-2\sigma_p^2 + 2\beta\sigma_p^2\},$$

$$\frac{\partial P}{\partial z} = -(\mu_p - b\{2\sigma_p^2\}).$$

Because b satisfies (1), the second derivative is zero, so one remains invested in risky securities only:

$$\frac{\partial P}{\partial z} = -(\mu_p - b\{2\sigma_p^2\}) = 0.$$

Inspection of the first derivative reveals that this last result simplifies things dramatically:

$$\frac{\partial P}{\partial x} = \alpha - (1-\beta)(\mu_p - b\{2\sigma_p^2\}) = \alpha.$$

In words: the optimal infinitesimal adjustment to your current portfolio is to add the new investment in proportion to alpha.

Appendix 2: Mutual Fund Data

U.S.	31/01/1996 - 30/06/2016
Name	Code
S&P 500	VFINX
US Growth	VWUSX
International Growth	VWIGX
International Value	VTRIX
Pacific Stock Index (PacStock)	VPACX
European Stock Index (Eurstock)	VEURX
Long-Term (Lt) Corporate Bonds	VWESX
Junk Bonds	VWEHX
GNMA Bonds	VFIIX
Short-Term (StT) Corporate Bonds	VFSTX
Long-Term (LtT) Treasury Bonds	VUSTX
Short-Term (StT) Treasury Bonds	VSGBX
Energy	VGENX
Precious Metals and Mining (PrecMining)	VGPMX
Healthcare	VGHCX

Table 1. Identification of mutual funds used

Appendix 3: Pseudo Code

1. Input the data and date to start;
2. Select 60 months of prior data for regression and find return series of them; construct past returns of the current portfolio as if the weights had been the same throughout the prior 60 months.
3. Take regressions of each of the mutual fund excess returns with respect to the current portfolio excess return (current portfolio excess return = independent return; individual mutual fund excess returns = dependent variable).
4. Select the intercept (the alpha value) and its p-value.
5. Make rebalancing decisions dependent on the alpha value and p-value. If the alpha is positive and p-value is lower than 10%, add the particular mutual fund to the buy list; If alpha is

negative, the p-value is lower than 10%, and the mutual fund is in the current portfolio, add the particular mutual fund into the sell list.

6. Buy the mutual funds on the buy list and sell the mutual funds in the sell list if these mutual funds are current holdings in one's portfolio.

   Cases:

   6.1. If there is a need to buy mutual funds and there are no negative alpha mutual funds in the portfolio, sell the current mutual funds with insignificant alphas in the current portfolio in proportion to their value.

   6.2. If there is a need to buy mutual funds and there are negative alpha mutual funds in the current portfolio, sell these negative alpha mutual funds to meet the rebalancing requirement.

   Cases:

   6.2.1. If, in selling all the negative alpha mutual funds, the proceeds satisfy the rebalancing requirement, sell the negative alpha mutual funds value-weighted.

   6.2.2. If the proceeds do not satisfy the rebalancing requirement, sell the negative alpha mutual funds and sell mutual funds in the current portfolio with insignificant alphas equally weighted.

   6.3. If there are negative alpha mutual funds in the current portfolio and no positive alpha mutual funds to buy, sell the rest of the mutual funds (with insignificant alphas) in the portfolio equally-weighted.

7. The buying and selling decision are taken at beginning-of-period prices of the next period (month), to avoid data-snooping issues with using the same data for both portfolio construction and implementation.

8. Each time the rebalancing amount is 5% on the current portfolio value.

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Notes

[1]It is worth emphasizing that the problem of choosing an optimal subset of mutual funds from the universe of U.S. funds is computationally extremely challenging. Indeed, it is an NP hard problem that even in the best of situations (absence of ambiguity about performance) is beyond human capacity. See Murawski and Bossaerts 2016.

[2]We based all calculations on gross return data. Prices were computed from those returns. This effectively means that we assumed continuous, costless reinvestment of cash dividends.