Homework 5 TTIC 31250

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Exercise 1

[Zero-sum games]

(1a)

What is the value to Alice of the strategy "with probability 1/2 hide a nickel and with probability 1/2 hide a quarter? (The value of a strategy is its value assuming that the opponent knows it and plays a best response)
If Bob knows Alice's strategy is to choose probabilities 1/2 nickel and 1/2 quarter, then his best response is to always choose probabilities 0 nickel and 1 quarter. Thus Alice's expected value (in cents) is

$$(.5 * 15) + (.5 * -25) = -5$$

(1b)

What is Alice's minimax optimal strategy, and what is its value?
Let p be the probability that Alice chooses nickel and $1-p$ the probability that Alice chooses quarter. Let q be the probability that Bob chooses nickel and $1-q$ the probability that Bob chooses quarter. The expected value of Alice's payoff can be calculated as such:

$$\begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -5 & 15 \\ 15 & -25 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

$$= \begin{bmatrix} -20p+15 & 40p-25 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

$$= -60pq + 40q + 40p - 25$$

$$= 5(-12pq + 8q + 8p - 5)$$

$$= 5(-12(p - \frac{8}{12})(q - \frac{8}{12}) + \frac{1}{3})$$

$$= -60(p - \frac{2}{3})(q - \frac{2}{3}) + \frac{5}{3}$$

For Alice to maximize this EV, she should choose $p = \frac{2}{3}$ (meaning choose nickel 2/3 of the time and quarter 1/3 of the time) in order to grab a payoff of 1/3. Any other value of p will allow Bob to adversely choose a q that will lower Alice's EV away from 5/3.

(1c)

What is Bob's minimax optimal strategy, and what its its value to Bob?
As shown in part (b), we know that Bob's optimal strategy is to choose $q = \frac{2}{3}$, meaning choose nickel 2/3 of the time and quarter 1/3 of the time in order to receive an EV of 1/3 (Bob wants negative EV since he is the opponent) Any other value of q means Alice can choose a p such that Bob's EV is increased.

(1d)

Is it better to be Alice or Bob in this game?
It's better to be Alice since the minimax optimal strategies for both Alice and Bob evaluate to 5/3 (positive means favoring Alice).

Problem 2

[On approximate Nash equilibria] Assume we have a game which all payoffs are in range [0,1]. Define a pair of distributions p, q to be an ϵ-Nash equilibrium if each player has at most ϵ incentive to deviate. That is, the expected payoff to the row player for each row i with $p_i > 0$ is within ϵ of the maximum payoff out of all the rows, and vice-versa for the column player. Using the fact that Nash equilibria must exist, show that there must exist an ϵ-Nash equilibrium in which each player has positive probability on at most $O(\frac{1}{\epsilon^2}\log n)$ actions (rows or columns).

To approach this problem, I read up on the following reference[1].

As given by definition, a Nash equilibrium is when the following holds: assuming the column player plays at random from distribution q, the expected payoff to the row player for each row i with $p_i > 0$ is equal to the max payoff out of all the rows ($e_i^T Rq = \max_{i'} e_{i'}^T Rq$) and assuming the row player plays at random from distribution p, the expected payoff to the column player for each column j with $q_j > 0$ is equal to the max payoff out of all columns ($p^T Ce_j = \max_{j'} p^T Ce_{j'}$) where e_i denotes the column vector with a 1 in position i and 0 everywhere else.

[**Defining Approximation Lemma**] To approach the ϵ-Nash equilibrium problem, I will first use the following lemma, then also prove the lemma.

Let A denote the payoff matrix (which can be defined as R for the row player or C for the column player), where real numbered $a_{ij} \in [0, 1]$ denotes the i^{th} action by the row player and j^{th} action from the column player. Let $p = (p_1, ..., p_n)$ be a probability vector. We can create another probability vector $z = z_1, ..., z_m$ with at most $k = \log 2n/2\epsilon^2 = O(\frac{1}{\epsilon^2} \log n)$ positive terms z_i such that:

$$|\sum_{i=1}^{n} p_i a_{ij} - \sum_{i=1}^{n} z_i a_{ij}| \leq \epsilon \text{ for all } j = 1, ..., n$$

The vector z represents an ϵ-approximation to p, meaning the player at hand has at most ϵ incentive to deviate from p. Since z_i is a probability vector itself, we can also represent $z_i = k_i/k$, where k is a natural number.

[**Applying Approximation Lemma**] With this vector z, we can create an ϵ-approximation to the optimal min-max strategy of our game. Since we are assuming that a Nash equilibrium holds in our game, define v as the expected value that both players arrive at when playing their optimal strategy with the assumption that the other player also plays optimally (the expected value that follows from min-max playing, namely $\sum_{i=1}^{n} p_i a_{ij} \geq v$ and $\sum_{j=1}^{n} q_i a_{ij} \leq v$). Here we can create our ϵ-approximation to min-max strategy by substituting z in, resulting in:

$$\sum_{i=1}^{n} z_i a_{ij} \geq v - \epsilon$$

[1] On Sparse Approximations to Randomized Strategies and Convex Combinations, Ingo Althofer

Using the properties stated in the lemma about z, we know that we need at most $O(\frac{1}{\epsilon^2}\log n)$ positive terms q_i if all entries of A are within [0,1]. Thus, if we can prove that strategy q exists as an ϵ-approximation to the min-max strategy, we have our desired ϵ-Nash equilibrium. The following is the proof for the approximation lemma:

[Proving Approximation Lemma]

We can use Hoeffding bounds here in order to grab our upper bounds on the deviations in sums of independent and bounded random variables.

Let $Y_1, ..., Y_k$ be independent random variables, with $0 \leq Y_t \leq 1$ and $\bar{Y} = (1/k)\sum_{t=1}^{k} Y_t, and \mathbb{E}[\bar{Y}]$ its expectation. The Hoeffding bounds states:

$$Pr[\bar{Y} - \mathbb{E}[\bar{Y}] \geq \epsilon] \leq e^{-2\epsilon^2 k}$$

$$Pr[\bar{Y} - \mathbb{E}[\bar{Y}] \leq -\epsilon] \leq e^{-2\epsilon^2 k}$$

From here, lets grab our original input matrix A and fix its column j, which sets $\mathbb{E}[\bar{Y_t}] = \sum_{i=1}^{m} p_i a_{ij}$. Using Hoeffding, we can grab the upper bound for the prob that the deviation for column j is greater than ϵ:

$$Pr\left(|\sum_{i=1}^{m} p_i a_{ij} - \frac{1}{k}\sum_{t=1}^{k} Y_t| \geq \epsilon\right) < 2e^{-2\epsilon^2 k}$$

Applying this to all n columns, we get:

$$\begin{aligned} Pr\left(|\sum_{i=1}^{m} p_i a_{ij} - \frac{1}{k}\sum_{t=1}^{k} Y_t| > \epsilon\right) &\text{ for at least one column } j \\ \leq \sum_{j=1}^{n} Pr\left(|\sum_{i=1}^{m} p_i a_{ij} - \frac{1}{k}\sum_{t=1}^{k} Y_t| > \epsilon\right) & \\ < 2ne^{-2\epsilon^2 k} & \end{aligned}$$

By choosing $k \geq \log 2n/2\epsilon^2$, we can force the last term to be ≤ 1. Thus, we've proved that if we choose k rows with at most $O(\frac{1}{\epsilon^2}\log n)$ positive terms from distribution p, we can create a resulting distribution z that is an ϵ-approximation to p.

Problem 3

[Compression bounds]

(3a)

Use $x_1, ..., x_n$ to denote the examples in S. Suppose you are given a sequence of indices $i_1, ..., i_k$. Define $A_{i_1,...i_k}$ to be the event that $h_{(x_{i1},...,x_{ik})}$ has zero error on all examples $x_j \in S$ such that $j \notin \{i_1, ..., i_k\}$ and yet true error of $h_{(x_{i1},...,x_{ik})}$ is more than ϵ. Prove that if $S \sim D^n$, the probability of event $A_{i_1,...,i_k}$ is at most $(1-\epsilon)^{n-k}$.

Since we desire our incoming test data x to be labeled with true error less than ϵ, we know that the probability that we get a new data observation correct is at least $(1-\epsilon)$. Thus to get an event $A_{i_1,...,i_k}$ where we get k indices with error greater than ϵ but $|j| = n-k$ indices with zero error on a total of n examples, the amount we need correct is at least $n-k$ examples (correct meaning error less than ϵ), thus this probability is simply (1-ϵ)$^{n-k}$. Thus $Pr(A_{i_1,...,i_k}) \leq (1-\epsilon)^{n-k}$.

(3b)

Use (a) to show that $\underset{S\sim D^n}{Pr}(\exists S' \subseteq S, |S'| = k$, such that $h_{S'}$ has 0 error on $S - S'$ but true error $> \epsilon) \leq \delta$, so as long as $n \geq \frac{1}{\epsilon}(k\ln n + \epsilon k + \ln\frac{1}{\delta})$

Using the same structure given in (3a), define $x_1, ..., x_n$ as examples in S. We know that $Pr(A_{i_1,...,i_k}) \leq (1-\epsilon)^{n-k}$. We know the number of ways to choose a subset size k from a set of size n is nCk (sampling without replacement), which is bounded above by n^k (sampling with replacement). Taking the union bound over the probability given in (3a) and knowing that $\underset{S\sim D^n}{Pr}(\exists S' \subseteq S) \leq \delta$ that we can infer:

$$\underset{S\sim D^n}{Pr}(\exists S' \subseteq S) \leq n^k(1-\epsilon)^{n-k} = \delta$$

Using the known fact from lecture 1 that $(1-\epsilon) \leq e^{-\epsilon}$, we have:

$$n^k((e^{-\epsilon})^{n-k}) \leq \delta$$

$$n^k(e^{-\epsilon n + \epsilon k}) \leq \delta$$

$$klnn - \epsilon n + \epsilon k \leq ln\delta$$

$$klnn + \epsilon k - ln\delta \leq \epsilon n$$

$$n \geq \frac{1}{\epsilon}klnn + \epsilon k + ln\frac{1}{\delta}$$