Lecture 14

Digital Filter and PI Controller Design

Objectives:

- Discuss software design of digital control systems and introduce the problem of numerical integration for digitized differential equations.
- Work through the three basic approximations for numerical integration.
- Use the bilinear transform to relate digital frequency to analog frequency.
- Compute the digital filter coefficients for a low-pass Butterworth filter.
- Build a prototype function of a PI controller using C-code with Euler's rule.

Keywords:

Numerical Integration
Euler's method
Forward Euler rule
Backward Euler rule
Trapezoidal rule
Bilinear Transform
Tustin's method
Butterworth Filter
Frequency pre-warping

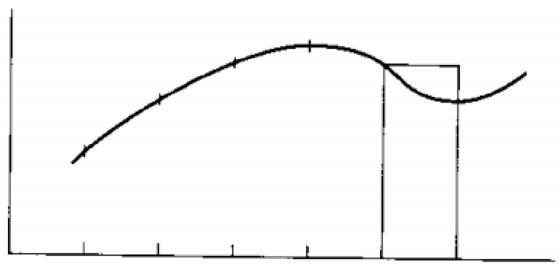
Digitizing a Control System

Control is performed sample-by-sample in discrete domain.

If we have differential equations in our control (e.g. the “I” in a PI controller), we run into a problem.

Common Numerical Integration Methods

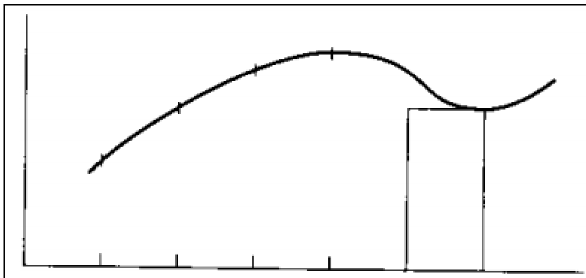
$$y(t+\Delta t) = \int_0^{t+\Delta t} x(\tau)\,d\tau = \underbrace{\int_0^{t} x(\tau)\,d\tau}_{y(t)} + \underbrace{\int_t^{\Delta t} x(\tau)\,d\tau}_{\text{approximate}}$$



$$y[kT] = y[(k-1)T] + Tx[(k-1)T]$$

$$Y(z)\left(1-z^{-1}\right) = TX(z)z^{-1}$$

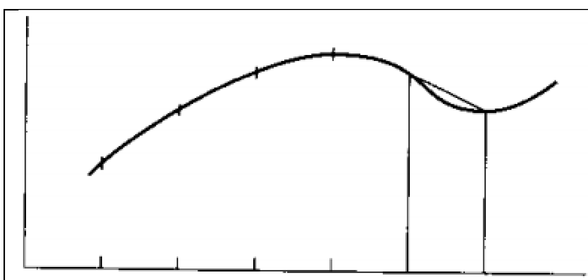
$$H(z) = \frac{Tz^{-1}}{1-z^{-1}} = \frac{T}{z-1} \qquad s \leftrightarrow \frac{z-1}{T}$$



$$y[kT] = y[(k-1)T] + Tx[kT]$$

$$Y(z)\left(1-z^{-1}\right) = TX(z)$$

$$H(z) = \frac{T}{1-z^{-1}} = \frac{Tz}{z-1} \qquad s \leftrightarrow \frac{z-1}{Tz}$$



$$y[kT] = y[(k-1)T] + \tfrac{T}{2}\left(x[kT] + x[(k-1)T]\right)$$

$$Y(z)\left(1-z^{-1}\right) = \tfrac{T}{2}X(z)\left(1+z^{-1}\right)$$

$$H(z) = \frac{T}{2}\frac{z+1}{z-1} \qquad s \leftrightarrow \frac{2}{T}\frac{z-1}{z+1}$$

An Example with a Simple DE (1 of 3)

$$4\frac{dx(t)}{dt}+5x(t)=3 \quad \rightarrow \quad \frac{dx(t)}{dt}+\frac{5}{4}x(t)=\frac{3}{4} \quad \text{and} \quad IF=e^{\int\frac{5}{4}dt}=e^{\frac{5}{4}t}$$

$$\frac{dx(t)}{dt}e^{\frac{5}{4}t}+\frac{5}{4}x(t)e^{\frac{5}{4}t}=\frac{3}{4}e^{\frac{5}{4}t} \quad \rightarrow \quad \frac{d}{dt}\left[x(t)e^{\frac{5}{4}t}\right]=\frac{3}{4}e^{\frac{5}{4}t}$$

$$x(t)e^{\frac{5}{4}t}=\frac{3}{4}\left(\frac{4}{5}e^{\frac{5}{4}t}+c_o\right) \quad \rightarrow \quad x(t)=\frac{3}{5}+\frac{3}{4}e^{-\frac{5}{4}t}c_o$$

$$x(0)=0=\frac{3}{5}+\frac{3}{4}e^{0}c_o \quad \rightarrow \quad c_o=-\frac{4}{5}$$

$$x(t)=\frac{3}{5}-\frac{3}{5}e^{-\frac{5}{4}t}$$

An Example with a Simple DE (2 of 3)

We can approximate this solution by rewriting the derivative as a difference and solving for the future sample value of x(t).

$$\frac{dx(t)}{dt} = \frac{3}{4} - \frac{5}{4}x(t)$$

Forward Euler	Backward Euler
$\frac{x(t+\Delta t)-x(t)}{\Delta t} \approx \frac{3}{4} - \frac{5}{4}x(t)$ $x(t+\Delta t) \approx x(t) + \frac{\Delta t}{4}\left[3-5x(t)\right]$ $x(t+\Delta t) \approx \frac{\Delta t}{4}\left[3 + x(t)\left(\frac{4}{\Delta t}-5\right)\right]$	$\frac{x(t)-x(t-\Delta t)}{\Delta t} \approx \frac{3}{4} - \frac{5}{4}x(t)$ $\frac{x(t+\Delta t)-x(t)}{\Delta t} \approx \frac{3}{4} - \frac{5}{4}x(t+\Delta t)$ $x(t+\Delta t)\left[\frac{1}{\Delta t}+\frac{5}{4}\right] \approx \frac{3}{4} + \frac{1}{\Delta t}x(t)$ $x(t+\Delta t) \approx \frac{4\Delta t}{4+5\Delta t}\left[\frac{3}{4}+\frac{x(t)}{\Delta t}\right]$

Trapezoidal

$$X\left(\frac{2}{T}\frac{z-1}{z+1}\right) \approx \frac{3}{4}U - \frac{5}{4}X \quad \rightarrow \quad X\frac{2}{T}(z-1) \approx U\frac{3(z+1)}{4} - X\frac{5(z+1)}{4}$$

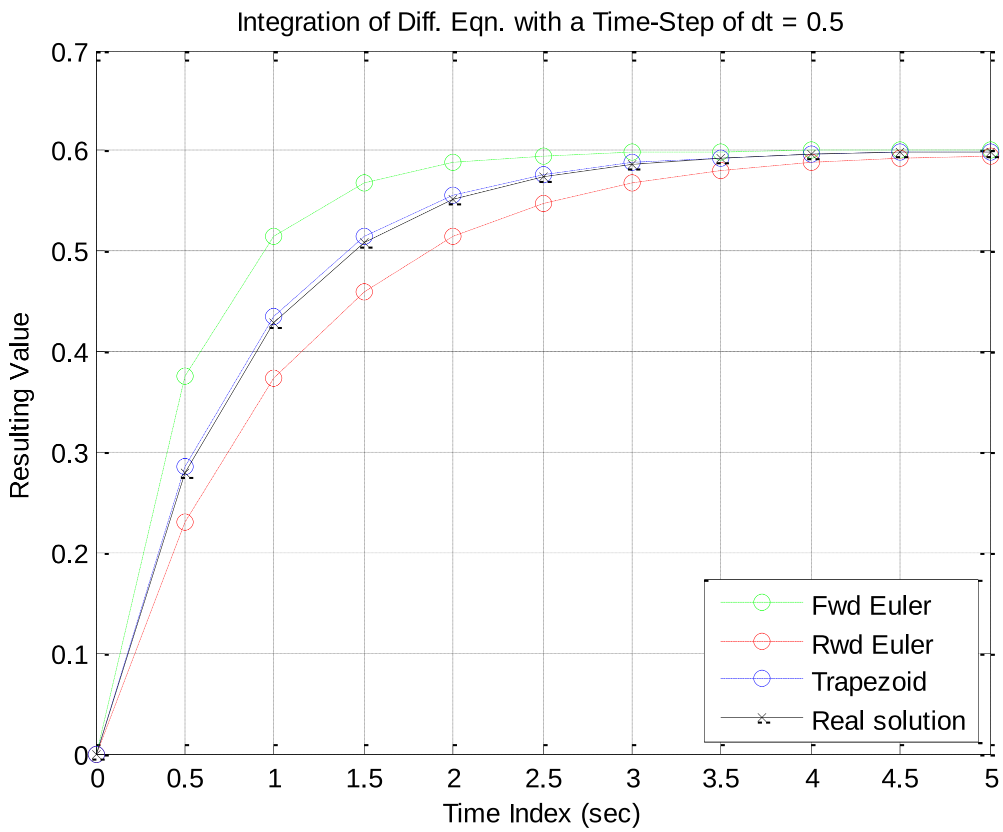
$$\frac{x(t+\Delta t)-x(t)}{\Delta t/2} \approx \frac{3}{4} + \frac{3}{4} - \frac{5}{4}\left[x(t+\Delta t)+x(t)\right]$$

$$x(t+\Delta t)\left[\frac{2}{\Delta t}+\frac{5}{4}\right] \approx \frac{3}{2} + x(t)\left[\frac{2}{\Delta t}-\frac{5}{4}\right]$$

$$x(t+\Delta t) \approx \frac{4\Delta t}{8+5\Delta t}\left[\frac{3}{2} + x(t)\left(\frac{8-5\Delta t}{4\Delta t}\right)\right]$$

An Example with a Simple DE (3 of 3)

The following image was generated using MATLAB code, to evaluate the differential equation using the three integration methods discussed.



The real solution is also shown in the figure. The bilinear transform does the best job at approximating the solution.

```
%Prof. Taylor
%Summer 2016
clear; clc;

dt = 0.5;  %define time step
N = 5/dt;  %Let's define "N" as the number of points to get to 5sec

f = 0; r = 0; b = 0;%define initial conditions
for k = 1:N %Evaluate N points of this equation with time step "dt"
    f(k+1) = (3    + f(k)*(4/dt-5))    *dt/4; %Forward methtod
    r(k+1) = (3/4  + r(k)*(1/dt))    *4*dt/(4+5*dt); %Rearward method
    b(k+1) = (1.5  + b(k)*(2/dt-5/4))*4*dt/(8+5*dt); %Bilinear method
end

%We know the time step was "dt"... we could label the horizontal axis
t = 0:dt:N*dt; %Time starts at 0, increments by dt, and stops at N*dt
z = 3/5*(1-exp(-t/(4/5))); %solution to diff. eqn.

figure(1);
plot(t,f,'go-.',t,r,'ro-.',t,b,'bo-.',t,z,'kx-');
grid on;
xlabel('Time Index (sec)');
ylabel('Resulting Value');
title(['Integration of Diff. Eqn. with a Time-Step of dt = ' num2str(dt)]);
legend('Fwd Euler','Rwd Euler','Trapezoid','Real solution','Location','SE')
```

Applying Trapezoidal Rule to Filters

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

$$H(z) = \frac{\omega_c}{\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)\right] + \omega_c} = \frac{T\omega_c(z+1)}{2(z-1) + T\omega_c(z+1)} = \frac{z(T\omega_c) + T\omega_c}{z(T\omega_c + 2) + (T\omega_c - 2)}$$

$$H(z) = \frac{z\,\frac{T\omega_c}{T\omega_c+2} + \frac{T\omega_c}{T\omega_c+2}}{z + \frac{T\omega_c-2}{T\omega_c+2}} \quad \rightarrow \quad H(z) = \frac{zb_1 + b_0}{za_1 + a_0}$$

Relating Digital Frequency to Analog Frequency

$$H(z) = \frac{\omega_c}{\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)\right] + \omega_c}$$

$$H\left(e^{j\Omega T}\right) = \frac{\omega_c}{\left[\frac{2}{T}\left(\frac{e^{j\Omega T}-1}{e^{j\Omega T}+1}\right)\right] + \omega_c} = \frac{\omega_c}{\left[\frac{2}{T}\left(\frac{e^{j\Omega T/2}-e^{-j\Omega T/2}}{e^{j\Omega T/2}+e^{-j\Omega T/2}}\right)\right] + \omega_c}$$

$$H\left(e^{j\Omega T}\right) = \frac{\omega_c}{j\left[\frac{2}{T}\tan\left(\frac{\Omega T}{2}\right)\right] + \omega_c}$$

$\omega = \frac{2}{T}\tan\left(\frac{\Omega T}{2}\right)$, when using Bilinear Transform

$$\Omega = \frac{2}{T}\tan^{-1}\left(\frac{\omega T}{2}\right)$$

Digital Filter Design Steps

A Digital Low-Pass Filter Example

A Butterworth filter has a maximally flat passband.

Introduce the “butter” command in MATLAB.