# Lecture 2

# Reference Frame Mathematics & Rotor-Frame Modeling of PM Machines

## Objectives:

- Discuss the Clarke Transformation, alpha-beta reference frame, and space vector notation of rotating fields, currents, and voltages.
- Introduce the Park Transformation, and describe the idea behind heterodyning or frequency warping space-vector signals.
- Transform the stationary-frame stator-voltage equations into the rotating-frame, and explain all of the resulting terms.

## Keywords:

| | |
|---|---|
| Clarke Transformation | Stationary frame |
| Park Transformation | Rotating frame |
| Heterodyne process | Direct-axis |
| Space vectors | Quadrature-axis |

# Coordinate Transformations & Space Vectors

The three phase currents are not linearly independent because they are connected in a Y-configuration, i.e.

$$i_1 + i_2 + i_3 = 0$$

We can transform the balanced three-phase system (in the abc or 123 frame) into an equivalent two-phase system, called the alpha-beta frame, using the Clarke Transformation.

$$\begin{bmatrix} i_\alpha \\ i_\beta \\ i_\gamma \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

This basically converts the three-phase system to the Cartesian (x,y) coordinate system. If unbalanced, we get a non-zero γ-term.

Ignoring the γ-term, since α and β are oriented at 0° and 90°, respectively, we can use a single complex number to describe both components. This now becomes a space vector!

$$\mathbf{i}_\mathbf{s} = \left(i_\alpha + j\,i_\beta\right) = \frac{2}{3}\left(i_1\,e^{j0} + i_2\,e^{j2\pi/3} + i_3\,e^{j4\pi/3}\right)$$

It is common to confuse space vectors with phasors

- Phasors describe ONE voltage or current, operating at sinusoidal steady state – magnitude and phase are usually constant.
- Space vectors describe SEVERAL voltages or currents, varying in space and time – magnitude and phase are spatial and temporal.

## Reference Frames

Discuss the “moving train” analogy and apply to the rotor

- If standing on the road, and you see a train go by, you could say the speed of the train is 50MPH. This is stator frame-of-reference.
- If riding in your car at 50MPH, and you look at the train, it will appear to be motionless! Its speed is 0MPH. Rotor frame-of-reference.

Viewing the machine in the rotor reference frame has some advantages:

- If we always maintain our 3ϕ currents to have the same rotational speed as the rotor, the space vector components of the current in the rotor reference frame will appear as CONSTANT DC CURRENTS! This allows us to use simple PI controllers in the control loop.
- When analyzing the electromagnetics in the rotor frame of reference, the equation for torque simplifies greatly:

$$T_{em} = \frac{3}{2}\frac{p}{2} i_q \Lambda_{pm}$$

To place ourselves into the rotor flux frame-of-reference, we need to know the position of the rotor flux!

- Magnets are mechanically fixed to the rotor; we can determine the flux position by knowing the rotor position. This is NOT true in induction machines, where the rotor field spins at a different speed than the rotor.
- Thus, in PM machines, a rotor angular position sensor can be used to determine the flux position.

## The Park Transformation

- We can convert our stator currents (in space-vector form) from the stator (stationary) frame-of-reference into the rotor (rotating) frame-of-reference using the Park Transformation.

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}$$

where subscripts "d" and "q" stand for the "*direct*" and "*quadrature*" axes.

- Let's assume that the rotor flux rotates at $\omega t$, starting from angle 0°, and the space vector of the current is aligned with the rotor flux (zero torque):

$$i_\alpha = I_m \cos(\omega t)\mathrm{A} \quad \text{and} \quad i_\beta = I_m \sin(\omega t)\mathrm{A}$$

Then we can obtain the values for the rotor frame currents: $i_d = I_m$ & $i_q = 0$.

$$\begin{aligned} i_d &= I_m \cos(\omega t)\cos(\theta) + I_m \sin(\omega t)\sin(\theta) \\ &= \tfrac{1}{2} I_m [1 + \cos(2\omega t) + 1 - \cos(2\omega t)] = I_m \end{aligned}$$

$$\begin{aligned} i_q &= -I_m \cos(\omega t)\sin(\theta) + I_m \sin(\omega t)\cos(\theta) \\ &= \tfrac{1}{2} I_m [-\sin(2\omega t) - 0 + \sin(2\omega t) - 0] = 0 \end{aligned}$$

- What if the currents are advanced by 90°? Recall that this is the maximum torque condition. Here, we find that $i_q$ is maximum and $i_d$ is zero.

$$i_\alpha = -I_m \sin(\omega t)\mathrm{A} \quad \text{and} \quad i_\beta = I_m \cos(\omega t)\mathrm{A}$$

$$\begin{aligned} i_d &= -I_m \sin(\omega t)\cos(\theta) + I_m \cos(\omega t)\sin(\theta) \\ &= \tfrac{1}{2} I_m [-\sin(2\omega t) - 0 + \sin(2\omega t) - 0] = 0 \end{aligned}$$

$$\begin{aligned} i_q &= +I_m \sin(\omega t)\sin(\theta) + I_m \cos(\omega t)\cos(\theta) \\ &= \tfrac{1}{2} I_m [1 - \cos(2\omega t) + 1 + \cos(2\omega t) - 0] = I_m \end{aligned}$$

## Space Vector Park Transform

- The Park transformation can also be written in space-vector form, using a complex exponential function.

$$\mathbf{i}_{\mathbf{s}}^{\mathbf{r}} = \mathbf{i}_{\mathbf{s}}^{\mathbf{s}} e^{-j\theta}$$

Recall Euler's identity, which relates the complex exponential to sin & cos.

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\begin{aligned} e^{-j\theta} &= \cos(-\theta) + j\sin(-\theta) \\ &= \cos(\theta) - j\sin(\theta) \end{aligned}$$

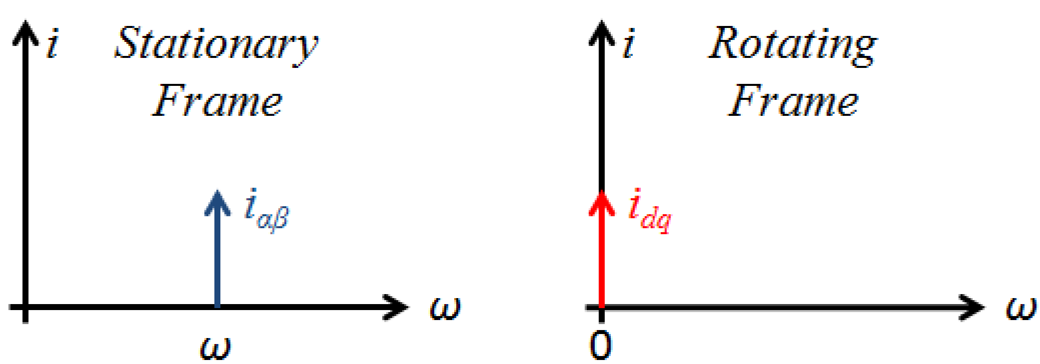
If we expand the space vectors on the right side of the equation, we get:

$$\mathbf{i}_{\mathbf{s}}^{\mathbf{r}} = (i_d + j\,i_q) = (i_\alpha + j\,i_\beta)(\cos(\theta) - j\sin(\theta))$$

FOILing the two binomials on the right will yield the same results as before.

$$\mathbf{i}_{\mathbf{s}}^{\mathbf{r}} = [i_\alpha \cos(\theta) + i_\beta \sin(\theta)] + j[i_\beta \cos(\theta) - i_\alpha \sin(\theta)]$$

- The multiplication of sinusoids at different frequencies is sometimes called *heterodyning*, and can be thought of as "frequency warping" or shifting.




The αβ currents at frequency $\omega$ get shifted down to DC – this is why the angle of the heterodyne function is negative.

## Stator Voltages in the Rotor Frame

- It is much easier to analyze a machine in the rotor frame, since all of the voltages and currents will be DC quantities. To properly obtain the voltage equation, we can use space-vector notation. Recall from before, we had:

$$\mathbf{v}_\mathbf{s}^\mathbf{s} = R_s \mathbf{i}_\mathbf{s}^\mathbf{s} + L_s \frac{d\,\mathbf{i}_\mathbf{s}^\mathbf{s}}{dt} + \frac{d\,\boldsymbol{\lambda}_\mathbf{pm}^\mathbf{s}}{dt}$$

  Multiplying by our Park Transform operator, the stator voltage and current terms transform straight through. However, for the derivative terms, we must use the inverse of the Product Rule – each will yield two terms.

$$\begin{aligned}
\mathbf{v}_\mathbf{s}^\mathbf{r} = \mathbf{v}_\mathbf{s}^\mathbf{s}\, e^{-j\theta} &= \left\{ R_s \mathbf{i}_\mathbf{s}^\mathbf{s} + L_s \frac{d\,\mathbf{i}_\mathbf{s}^\mathbf{s}}{dt} + \frac{d\,\boldsymbol{\lambda}_\mathbf{pm}^\mathbf{s}}{dt} \right\} e^{-j\theta} \\
&= R_s \mathbf{i}_\mathbf{s}^\mathbf{r} + L_s \left[ \frac{d}{dt}\left\{ \mathbf{i}_\mathbf{s}^\mathbf{s}\, e^{-j\theta} \right\} + j\omega\, \mathbf{i}_\mathbf{s}^\mathbf{s}\, e^{-j\theta} \right] + \left[ \frac{d}{dt}\left\{ \boldsymbol{\lambda}_\mathbf{pm}^\mathbf{s}\, e^{-j\theta} \right\} + j\omega\, \boldsymbol{\lambda}_\mathbf{pm}^\mathbf{s}\, e^{-j\theta} \right] \\
&= R_s \mathbf{i}_\mathbf{s}^\mathbf{r} + L_s \frac{d\,\mathbf{i}_\mathbf{s}^\mathbf{r}}{dt} + j\omega L_s\, \mathbf{i}_\mathbf{s}^\mathbf{r} + 0 + j\omega\, \boldsymbol{\lambda}_\mathbf{pm}^\mathbf{r}
\end{aligned}$$

  The 4$^{th}$ term drops to zero because the derivative of the magnet flux is zero (we're using permanent magnets) in the rotor frame. The field strength of the magnets is constant and should not change with time.

- Recall that space-vectors are complex numbers. The equation above is really two voltage equations: the d-axis (real parts) and q-axis (imaginary parts) of the stator voltages, in the rotor frame-of-reference.

$$\mathbf{v}_\mathbf{s}^\mathbf{r} = \left( v_d + j v_q \right)$$

$$v_d = R_s i_d + L_s \frac{d\,i_d}{dt} - \omega L_s i_q + 0$$

$$v_q = R_s i_q + L_s \frac{d\,i_q}{dt} + \omega L_s i_d + \omega \Lambda_{pm}$$

# Analyzing the Rotor-Frame Stator Voltage Equations

- From the previous page, we have:

$$v_d = R_s i_d + L_s \frac{d\,i_d}{dt} - \omega L_s i_q + 0$$
$$v_q = R_s i_q + L_s \frac{d\,i_q}{dt} + \omega L_s i_d + \omega \Lambda_{pm}$$

  - The first *R·i* terms represent the stator copper losses of the windings, just as they did in the stator-frame stator-voltage equations.
  - The second *L·di/dt* terms represent the voltage needed to grow or shrink the magnitude of the AC component of the current. This can be modeled as an inductance in the rotor-frame.
  - The third *ω·L·i* terms represent the actual voltage on the leakage inductances due to a constant-magnitude sinusoidal current flowing through them (this is the phasor-voltage term appearing across an inductance in the stationary-frame, like what we're used to from our traditional Circuits classes).
  - The last term is the BEMF term – the BEMF is only in the q-axis. The whole definition of "rotor-frame" was to align our coordinate system with the rotor flux. All the flux is in the d-axis.

$$\boldsymbol{\lambda}^{\mathbf{s}}_{\mathbf{pm}} = \Lambda_{pm}\left(\cos(\theta) + j\sin(\theta)\right)$$
$$\boldsymbol{\lambda}^{\mathbf{r}}_{\mathbf{pm}} = \boldsymbol{\lambda}^{\mathbf{s}}_{\mathbf{pm}}\, e^{-j\theta} = \left(\Lambda_{pm} + j0\right)$$

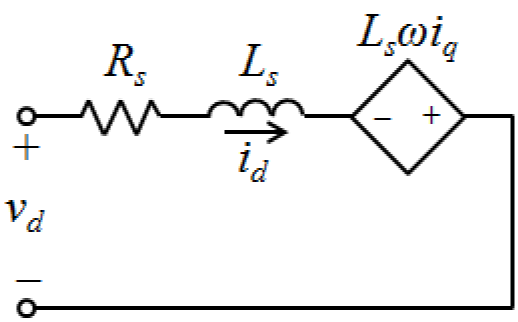
  So it makes sense, due to Faraday's Law, that all of the voltage due to the permanent magnets should show up in the q-axis, which is 90° leading the d-axis (where the magnet flux is).

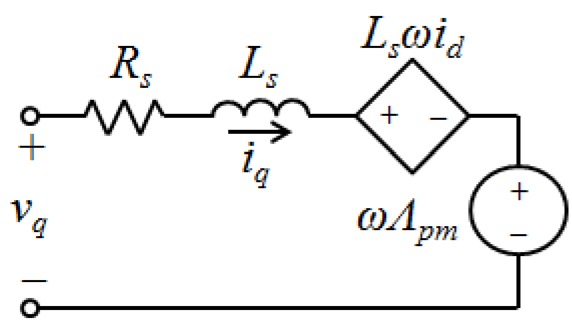
$$v(t) = \frac{d\lambda_{pm}(t)}{dt} \quad \text{or we can say} \quad \mathbf{V} = j\omega\boldsymbol{\Lambda}_{\mathbf{pm}}$$

## Machine Model in the Rotor Frame

From the stator voltage equations, we can build a DC circuit model of the machine parameters. Notice that there is a cross-dependence between the two currents and voltages.

$$v_d = R_s i_d + L_s \frac{d\,i_d}{dt} - \omega L_s i_q \qquad\qquad v_q = R_s i_q + L_s \frac{d\,i_q}{dt} + \omega L_s i_d + \omega \Lambda_{pm}$$




We can start to analyze the dynamics of the motor's electrical parameters from this model. Using super-position with the independent sources, we can derive the transfer function between voltage and current in each of the two axes. Let's assume we first set $v_q$ and $\omega\Lambda_{pm}$ to zero. The current $i_q$ and $i_d$ can be expressed as

Let $v_q = \omega\Lambda_{pm} = 0\text{V}$ …

$$i_q(s) = -\frac{\omega L_s i_d(s)}{sL_s + R_s} \qquad\qquad i_d(s) = \frac{v_d(s) + \omega L_s\left(-\dfrac{\omega L_s i_d(s)}{sL_s + R_s}\right)}{sL_s + R_s}$$

If we simplify the expression for $i_d$, we can obtain the transfer function between the voltage and current in the d-axis.

$$i_d(s)\left[1 + \frac{\omega^2 L_s^{\,2}}{(sL_s + R_s)^2}\right] = \frac{v_d(s)}{sL_s + R_s} \qquad\qquad \frac{i_d(s)}{v_d(s)} = \frac{sL_s + R_s}{(sL_s + R_s)^2 + \omega^2 L_s^{\,2}}$$

$$\frac{i_d(s)}{v_d(s)} = \frac{sL_s + R_s}{s^2 L_s^{\,2} + 2sL_sR_s + R_s^{\,2} + \omega^2 L_s^{\,2}}$$

## Machine Electrical Dynamics

Using the same super-position technique on the q-axis current, we actually get the same transfer function in the q-axis as we did in the d-axis.

$$\frac{i_d(s)}{v_d(s)} = \frac{i_q(s)}{v_q(s)} = \frac{\frac{1}{R_s}\left(s\frac{L_s}{R_s}+1\right)}{s^2+2s\frac{R_s}{L_s}+\left(\frac{R_s}{L_s}\right)^2+\omega^2}$$

Examining the roots of the denominator, we see that we have two poles which vary with the machine speed. Thus, the electrical dynamics are time-varying.

$$s^2+2s\frac{R_s}{L_s}+\left(\frac{R_s}{L_s}\right)^2+\omega^2=0$$
$$\left(s+\frac{R_s}{L_s}\right)^2-\left(\frac{R_s}{L_s}\right)^2+\left(\frac{R_s}{L_s}\right)^2=-\omega^2$$

$$s=-\frac{R_s}{L_s}\pm j\omega$$

The natural frequency of the machine's electrical parameters can give us a sense of how fast our control algorithm needs to be to effectively regulate the current.

- Typical inverse time-constants or natural frequencies (the $R_s$ / $L_s$ term) for a machine might range from around 200rad/s to ~600rad/s.
- The rotor speeds (the $\omega$ term) can vary from 0rad/sec up to ~2000rad/s for typical machines, but this can be much larger for high-speed machines.
- The overall natural frequency of the machine would be

$$\omega_{elec}=\sqrt{\left(\frac{R_s}{L_s}\right)^2+\omega^2}$$

A controller should be able to sample at least around 10x this frequency to maintain a good regulation of the current.

## Getting Back to Stator Frame

The Park and Clarke transformations have inverse operations as well. To obtain the ABC components from our dq or αβ components, we can apply the appropriate inverse transform.

- The inverse of the Park Transform is the same as multiplying by a positive angle in the complex exponential. This frequency-warps back to $\omega$.

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix}$$

- We can find the inverse of the Clarke Transformation just by inverting the original matrix.

$$\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 2/3 & 0 \\ -1/3 & 1/\sqrt{3} \\ -1/3 & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix}$$

We can summarize all of the transforms as just their matrices:

$$\mathrm{T_C} = \frac{2}{3} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\mathrm{T_C}^{-1} = \frac{3}{2} \begin{bmatrix} 2/3 & 0 \\ -1/3 & 1/\sqrt{3} \\ -1/3 & -1/\sqrt{3} \end{bmatrix}$$

$$\mathrm{T_P} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathrm{T_P}^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

If we multiply the two transform matrices together, we can directly jump from ABC to dq, and from dq back to ABC. This is common in software implementation.

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \mathrm{T_P}\,\mathrm{T_C} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \mathrm{T_C}^{-1}\,\mathrm{T_P}^{-1} \begin{bmatrix} v_d \\ v_q \end{bmatrix}$$