DATA 609 - Homework 7: Nonconvex Optimization and Deep Learning

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Instructions

For this problem I recommend submitting a python notebook or a quarto file. It might be easier with a python notebook. You can use google colab if you do not have enough computational power in your personal computer.

Problem 1: Comparing Optimization Algorithms

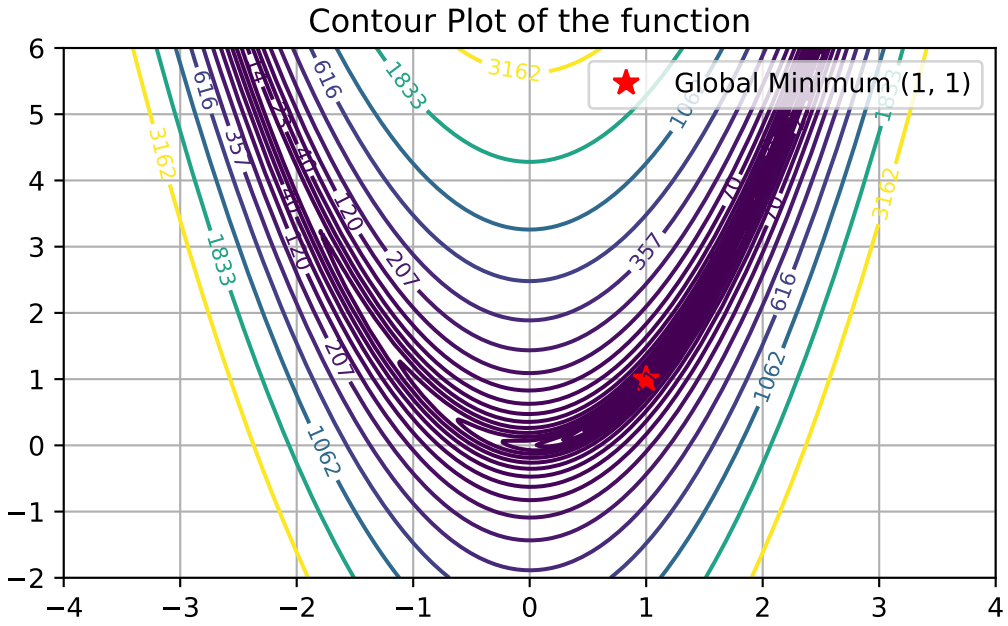
Consider the function, $f(x, y) = (1 - x^2) + 100(y - x^2)^2)$ which has a global minimum at, $x = 1$. For this problem, you are going to explore using different optimization algorithms to find the global minimum, to gain an intuitive understanding of their different strengths and weaknesses.

(a) Make a contour plot of this function. You should observe a that the contour lines are "banana-shaped" around the global minimum point, which lies in a deep valley. In technical terms, we would say that the gradient of this function is strongly anisotropic, a fact that can cause slow or no convergence for optimization algorithms.

Problem 1(a) Solution

```
# load dependencies
import numpy as np
import matplotlib.pyplot as plt

# define function
def prob_func(x, y):
    return (1 - x)**2 + 100 * (y - x**2)**2
```

```
# Create a grid of (x, y) values
x = np.linspace(-4, 4, 400)
y = np.linspace(-2, 6, 400)
X, Y = np.meshgrid(x, y)
Z = prob_func(X, Y)

# Plot the contour
contours = plt.contour(X, Y, Z, levels=np.logspace(-1, 3.5, 20), cmap='viridis')
plt.clabel(contours, inline=True, fontsize=8)
plt.plot(1, 1, 'r*', markersize=10, label='Global Minimum (1, 1)')
plt.title('Contour Plot of the function')
plt.legend()
plt.grid(True)

plt.tight_layout()
plt.show()
```



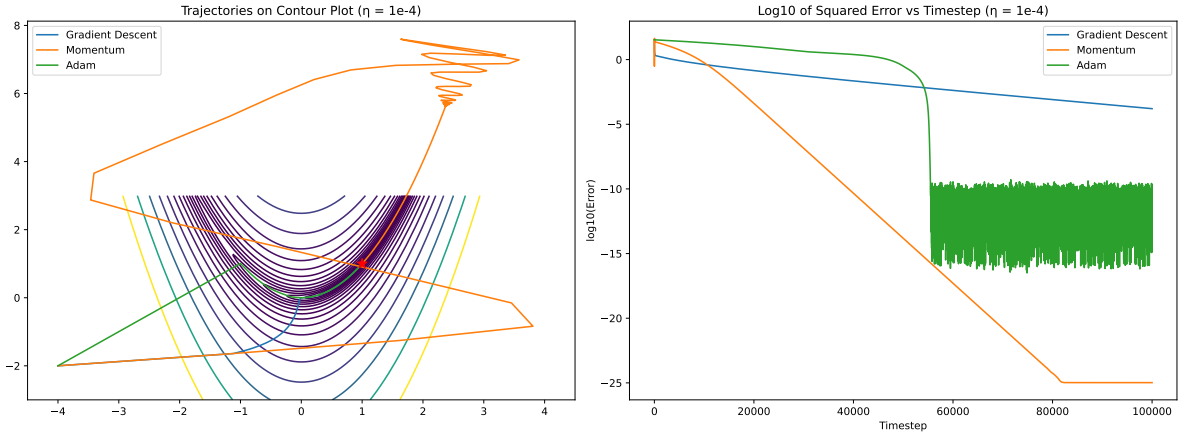
(b) In the code chunk below I have python code for three different optimization algorithms,(1) stochastic gradient descent; (2) stochastic gradient descent with momentum, and (3) ADAM (ADAptive Moment Estimation). Starting at the initial point $x = -4, y = -2$, use each algorithm to find the minimum of the function f. Start with a learning rate of $\kappa = 10^{-4}$ for all three algorithms, and run the algorithm for 10^5 timesteps. Plot the trajectories of each algorithm and the log base 10 of the error rate as

a function of the time step. What do you notice about the performance of the difference algorithms, both in terms of convergence speed and ultimate accuracy?

```
# available defined function
def gd(grad, init, n_epochs=1000, eta=10**-4):

    params=np.array(init)
    param_traj=np.zeros([n_epochs+1,2])
    param_traj[0,]=init
    v=0;
    for j in range(n_epochs):
        v=eta*(np.array(grad(params)))
        params=params-v
        param_traj[j+1,]=params
    return param_traj

def gd_with_mom(grad, init, n_epochs=5000, eta=10**-4, beta=0.9,gamma=0.9):
    params=np.array(init) # Start with initial condition
    param_traj=np.zeros([n_epochs+1,2]) # Save the entire trajecotry
    param_traj[0,]=init # Also save the initial condition to the trajectory

    v=0 # Starting with 0 momentum

    # Epochs is borrowing term from machine learning
    # Here it means timestep

    for j in range(n_epochs):
        v=gamma*v+(np.array(grad(params))) # Compute v
        params=params-eta*v  # Update the location
        param_traj[j+1,]=params # Save the trajectory
    return param_traj

def adams(grad, init, n_epochs=5000, eta=10**-4, gamma=0.9, beta=0.99,epsilon=10**-8):
    params=np.array(init)
    param_traj=np.zeros([n_epochs+1,2])
    param_traj[0,]=init
    v=0;
    grad_sq=0;
    for j in range(n_epochs):
        g=np.array(grad(params))
        v=gamma*v+(1-gamma)*g
        grad_sq=beta*grad_sq+(1-beta)*g*g
        v_hat=v/(1-gamma**(j+1))
```

```
        grad_sq_hat=grad_sq/(1-beta**(j+1))
        params=params-eta*np.divide(v_hat,np.sqrt(grad_sq_hat+epsilon))
        param_traj[j+1,]=params
    return param_traj

# load dependencies
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# define function
def prob_func(x, y):
    return (1 - x)**2 + 100 * (y - x**2)**2

def prob_func_grad(xy):
    x, y = xy
    dfdx = -2 * (1 - x) - 400 * x * (y - x**2)
    dfdy = 200 * (y - x**2)
    return np.array([dfdx, dfdy])

# intitalize the optimizer
init_point = [-4.0, -2.0]
eta = 1e-4
n_epochs = 100000

# run the optimizer
gd_traj = gd(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta)
mom_traj = gd_with_mom(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta)
adam_traj = adams(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta)

# compute log10 error
true_min = np.array([1, 1])
gd_error = np.log10(np.sum((gd_traj - true_min)**2, axis=1))
mom_error = np.log10(np.sum((mom_traj - true_min)**2, axis=1))
adam_error = np.log10(np.sum((adam_traj - true_min)**2, axis=1))

# create a contour plot of the function
x = np.linspace(-4.5, 4.5, 400)
y = np.linspace(-3, 3, 400)
X, Y = np.meshgrid(x, y)
Z = prob_func(X, Y)
```

```
# plot results
fig, axs = plt.subplots(1, 2, figsize=(16, 6))

# plot the contour with trajectories
axs[0].contour(X, Y, Z, levels=np.logspace(-1, 3.5, 20), cmap='viridis')
axs[0].plot(gd_traj[:, 0], gd_traj[:, 1], label='Gradient Descent')
axs[0].plot(mom_traj[:, 0], mom_traj[:, 1], label='Momentum')
axs[0].plot(adam_traj[:, 0], adam_traj[:, 1], label='Adam')
axs[0].plot(1, 1, 'r*', markersize=10)
axs[0].set_title("Trajectories on Contour Plot (  = 1e-4)")
axs[0].legend()

# plot the log error
axs[1].plot(gd_error, label='Gradient Descent')
axs[1].plot(mom_error, label='Momentum')
axs[1].plot(adam_error, label='Adam')
axs[1].set_title("Log10 of Squared Error vs Timestep (  = 1e-4)")
axs[1].set_xlabel("Timestep")
axs[1].set_ylabel("log10(Error)")
axs[1].legend()

plt.tight_layout()
plt.show()
```



(c) Perform the same experiment for the learning rate $\kappa = 10^{-3}$, only comparing ADAM and gradient descent with momentum. You will likely observe that one of the methods does not converge, keep the same range of values for your trajectory/contour plot as you did in (b). Which method worked better with $\kappa = 10^{3}$?

(d) Now perform a comparison between ADAM with $\kappa = 10^{-2}$ against gradient descent with momentum using . What are the trade-offs between the two methods for these values of the learning rate?

Problem 1(c) and 1(d) Solution

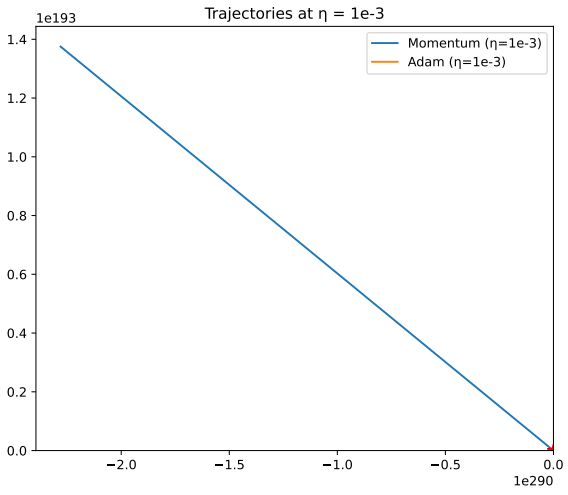
```
# load dependencies
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# define function
def prob_func(x, y):
    return (1 - x)**2 + 100 * (y - x**2)**2

def prob_func_grad(xy):
    x, y = xy
    dfdx = -2 * (1 - x) - 400 * x * (y - x**2)
    dfdy = 200 * (y - x**2)
    return np.array([dfdx, dfdy])

# set the applied the function
def adams(grad, init, n_epochs=5000, eta=10**-4, gamma=0.9, beta=0.99,epsilon=10**-8):
    params=np.array(init)
    param_traj=np.zeros([n_epochs+1,2])
    param_traj[0,]=init
    v=0;
    grad_sq=0;
    for j in range(n_epochs):
        g=np.array(grad(params))
        v=gamma*v+(1-gamma)*g
        grad_sq=beta*grad_sq+(1-beta)*g*g
        v_hat=v/(1-gamma**(j+1))
        grad_sq_hat=grad_sq/(1-beta**(j+1))
        params=params-eta*np.divide(v_hat,np.sqrt(grad_sq_hat+epsilon))
        param_traj[j+1,]=params
    return param_traj

def gd_with_mom(grad, init, n_epochs=5000, eta=10**-4, beta=0.9,gamma=0.9):
    params=np.array(init) # Start with initial condition
    param_traj=np.zeros([n_epochs+1,2]) # Save the entire trajecotry
```

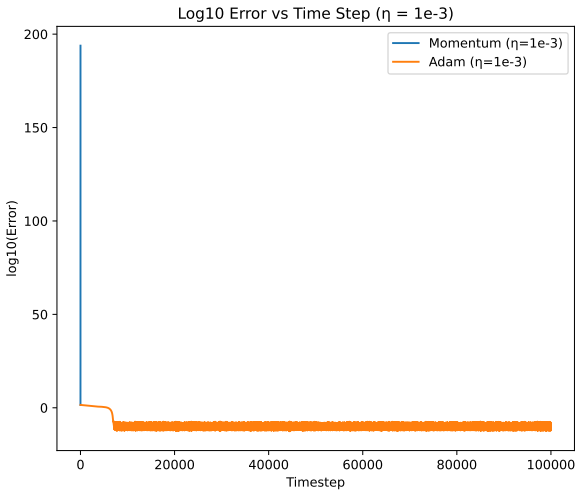
```
    param_traj[0,]=init # Also save the initial condition to the trajectory

    v=0 # Starting with 0 momentum

    # Epochs is borrowing term from machine learning
    # Here it means timestep

    for j in range(n_epochs):
        v=gamma*v+(np.array(grad(params))) # Compute v
        params=params-eta*v  # Update the location
        param_traj[j+1,]=params # Save the trajectory
    return param_traj

# intitalize the optimizer
init_point = [-4.0, -2.0]
n_epochs = 100000

# define the eta for part c
eta_c = 1e-3

# run the optimizer for the momentum and ADAM with higher learning rate
mom_traj_c = gd_with_mom(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta_c)
adam_traj_c = adams(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta_c)

# compute the log 10 error
true_min = np.array([1, 1])
mom_error_c = np.log10(np.sum((mom_traj_c - true_min)**2, axis=1))
adam_error_c = np.log10(np.sum((adam_traj_c - true_min)**2, axis=1))

# plot
fig, axs = plt.subplots(1, 2, figsize=(16, 6))

# plot the contour with trajectories
axs[0].contour(X, Y, Z, levels=np.logspace(-1, 3.5, 20), cmap='viridis')
axs[0].plot(mom_traj_c[:, 0], mom_traj_c[:, 1], label='Momentum ( =1e-3)')
axs[0].plot(adam_traj_c[:, 0], adam_traj_c[:, 1], label='Adam ( =1e-3)')
axs[0].plot(1, 1, 'r*', markersize=10)
axs[0].set_title("Trajectories at   = 1e-3")
axs[0].legend()

# plot the log error
axs[1].plot(mom_error_c, label='Momentum ( =1e-3)')
```

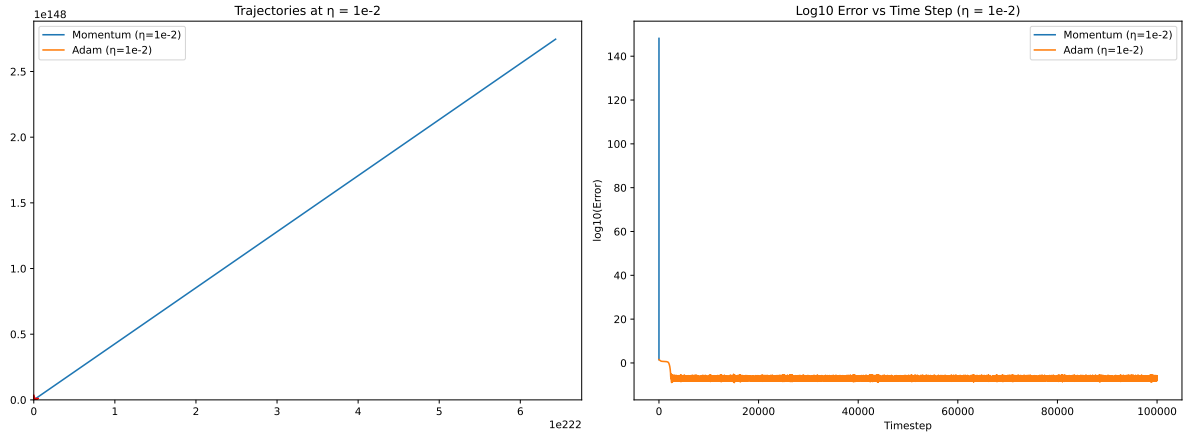
```
axs[1].plot(adam_error_c, label='Adam ( =1e-3)')
axs[1].set_title("Log10 Error vs Time Step (  = 1e-3)")
axs[1].set_xlabel("Timestep")
axs[1].set_ylabel("log10(Error)")
axs[1].legend()

# define the eta for part d
eta_d = 1e-2

# Run with even higher learning rate
mom_traj_d = gd_with_mom(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta_d)
adam_traj_d = adams(prob_func_grad, init_point, n_epochs=n_epochs, eta=eta_d)

# calculate errors
mom_error_d = np.log10(np.sum((mom_traj_d - true_min)**2, axis=1))
adam_error_d = np.log10(np.sum((adam_traj_d - true_min)**2, axis=1))

# plot
fig, axs = plt.subplots(1, 2, figsize=(16, 6))

# plot the contour with trajectories
axs[0].contour(X, Y, Z, levels=np.logspace(-1, 3.5, 20), cmap='viridis')
axs[0].plot(mom_traj_d[:, 0], mom_traj_d[:, 1], label='Momentum ( =1e-2)')
axs[0].plot(adam_traj_d[:, 0], adam_traj_d[:, 1], label='Adam ( =1e-2)')
axs[0].plot(1, 1, 'r*', markersize=10)
axs[0].set_title("Trajectories at   = 1e-2")
axs[0].legend()


# plot the log error
axs[1].plot(mom_error_d, label='Momentum ( =1e-2)')
axs[1].plot(adam_error_d, label='Adam ( =1e-2)')
axs[1].set_title("Log10 Error vs Time Step (  = 1e-2)")
axs[1].set_xlabel("Timestep")
axs[1].set_ylabel("log10(Error)")
axs[1].legend()

plt.tight_layout()
plt.show()
```

```
/var/folders/h4/zjq554hs0b57vqfcrc5738wh0000gn/T/ipykernel_40271/3856364375.py:12: RuntimeWa
```

```
overflow encountered in scalar power

/var/folders/h4/zjq554hs0b57vqfcrc5738wh0000gn/T/ipykernel_40271/3856364375.py:13: RuntimeWa

overflow encountered in scalar power

/var/folders/h4/zjq554hs0b57vqfcrc5738wh0000gn/T/ipykernel_40271/3856364375.py:12: RuntimeWa

invalid value encountered in scalar subtract

/var/folders/h4/zjq554hs0b57vqfcrc5738wh0000gn/T/ipykernel_40271/3856364375.py:13: RuntimeWa

invalid value encountered in scalar subtract

/var/folders/h4/zjq554hs0b57vqfcrc5738wh0000gn/T/ipykernel_40271/3856364375.py:62: RuntimeWa

overflow encountered in square

/var/folders/h4/zjq554hs0b57vqfcrc5738wh0000gn/T/ipykernel_40271/3856364375.py:92: RuntimeWa

overflow encountered in square
```







Problem 2: Shallow Nets and MNIST

For this exercise, we will work on one of the standard model problems in Machine Learning, classifying handwritten digits. We will use an adaptation of the neural network code from your reading assignment to pytorch, which is one of the leading frameworks for training neural networks. pytorch is fairly flexible, you can use it with the CPU on your personal computer, with GPUs, and even on computing clusters.

If you have trouble getting pytorch to work on your own computer, I recommend trying in on google colab, or alternatively you are welcome to develop your own implementation. This and the next assignment have helper code in the provided ipython notebook Lab 7 Helper Noteook

First, you should acquire the MNIST dataset. This can be downloaded automatically using pytorch via the following code chunk:

```
# # Here is some code that automatically downlaods the MNIST data. Technically it will also i
# # data in if you have already downloaded and the path points to the folder where you have t
# # There will be 4 binary files which together contain the testing and training examples and
# # for the testing and training examples.

# from torchvision import datasets, transforms

# # Load MNIST


# # transform defines a function which takes an image file, converts the analog bits into flo

# transform = transforms.Compose([transforms.ToTensor(), transforms.Lambda(lambda x: x.view(-
```

```
# # The first line downloads the entire MNIST dataset to the data directory (or whereever you
# # If the data is already there, this won't download it. THis downloads both the training an
# # the transform keyword applies the transform defined above, the train dataset has 60,000 e
# # the test dataset has 10,000 examples. The train and test data is loaded in the variables

# train_dataset = datasets.MNIST('data/', train=True, download=True, transform=transform)
# test_dataset = datasets.MNIST('data/', train=False, transform=transform)
```

We are going to train a simple neural network to classify the MNIST images. The neural network has an input layer of 784 neurons (one for each pixel), a 30 neuron hidden layer, and a 10 neuron output layer, which provided a weight that corresponds to the predictions of the neural network for each class. Initially we will use sigmoidal neurons to process the inputs.

Throughout the rest of the assignment, you will use and improve the code in this file to study the performance of different combinations of network structure, optimization algorithm choice, hyperparameters, and activation functions.

The initial configuration of the neural network uses stochastic gradient descent without momentum. The following code sets several key hyperparameters and trains the neural network:

```
# # batch_size determines the minibatch size
# import torch
# import torch.nn as nn
# import torch.nn.functional as F
# import torch.optim as optim
# from torch.utils.data import DataLoader

# train_loader = DataLoader(train_dataset, batch_size=10, shuffle=True)
# test_loader = DataLoader(test_dataset, batch_size=10, shuffle=False)

# # Initialize network
# # net = Network([784, 30, 10])

# # Train
# # sol = train(net, train_loader, epochs=30, eta=0.001, test_data=test_loader)
```

Additional parameters can be changed within the code of the function itself (and you can organize your code in any way you see fit), in the train function:

```
    # optimizer = optim.SGD(network.parameters(),momentum=0.8,nesterov=True, lr=eta,weight_de

    # This uncomment this (and comment the above line) if you want to use ADAM. The betas are
```

```
    # parameters, you can experiment with these hyperparaeters if you like:
    #optimizer = optim.Adam(network.parameters(),betas = (0.9,0.999), lr=eta,weight_decay=1e-

    # Here is code for using learning rate scheduling. You might find this helpful
    #step_size = 2
    #gamma = 0.7
    #scheduler = optim.lr_scheduler.StepLR(optimizer, step_size=step_size, gamma=gamma)
```

and also in the class function:

```
# class Network(nn.Module):
#     def __init__(self, sizes):
#         super(Network, self).__init__()
#         self.sizes = sizes
#         self.num_layers = len(sizes)

#         self.layers = nn.ModuleList()
#         for i in range(self.num_layers - 1):
#             layer = nn.Linear(sizes[i], sizes[i+1])
#             nn.init.xavier_normal_(layer.weight)   # Good initialization for shallow/sigmoi
#             #nn.init.kaiming_normal_(layer.weight, mode='fan_out', nonlinearity='relu') ini
#             #nn.init.kaiming_uniform_(layer.weight, mode='fan_out', nonlinearity='relu') in

#             nn.init.zeros_(layer.bias)              # initialize the bias to 0
#             self.layers.append(layer)

#     # Forward is the method that calculates the value of the neural network. Basically we i
#     # layer

#     def forward(self, x):
#         for layer in self.layers[:-1]:
#             x = F.sigmoid(layer(x))   # sigmoid layers
#             # x = F.relu(layer(x)) # You will try the relu layer in the last problem
#         x = self.layers[-1](x)
#         return x
```

The default set of hyperparameters lead to a neural network that successfully trains, and has an ultimate out of sample accuracy of just below 0.95 after 30 epochs of training on my computer.

(a) Validate this result by executing the train command with the parameters described above. Next, the learning rate is one of the most important hyperparameters in machine

learning. Train the same structure of neural network with a range of different learning rates both higher and lower. Make sure you find a learning rate high enough so that the neural network performance is poor. What was the learning rate that led to the best accuracy at the end of 30 epochs? Plot the test accuracy as a function of epochs (it is one of the outputs of the train function) for all of the learning rates that you tested.

(b) How does the ADAM optimizer handle this problem? Modify the neural network code so that the optimizer is ADAM optimizer (you will see commented code in the train function). Then train a neural network using ADAM again with a range of different learning rates (including the same starting point as in part(a)). Compare the behavior of the learning curves, the final accuracy, and the values of the learning rates that were most successful with part (a).

(c) How good can you make your 3-layer network? You don't need to do an exhaustive search of all possible options (which would take forever) but experiment with the optimization algorithm, the learning rate and the other hyper-parameters. For example, you could include more or fewer neurons in the hidden layer, change the values of the betas in ADAM or the momentum in SGD, alter the batch size in the data loader, change the learning rate, increase or decrease the weight decay (which is L2 regularization), or change the number of training epochs.

Some tips: - For this problem I don't think it is likely to have an accuracy above 0.99 without expanding the data - Larger batch sizes have less "noise" so might need more regularization. They also sometimes require larger learning rates - The learning rate scheduler will decrease the learning rate by a factor of $gamma$ every steps number of epochs

Problem 2 Solution

```
import torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
from torchvision import datasets, transforms
from torch.utils.data import DataLoader
import matplotlib.pyplot as plt

# run the already defined functions
transform = transforms.Compose([
    transforms.ToTensor(),
    transforms.Lambda(lambda x: x.view(-1))
])
```

```
train_dataset = datasets.MNIST('data/', train=True, download=True, transform=transform)
test_dataset = datasets.MNIST('data/', train=False, transform=transform)

train_loader = DataLoader(train_dataset, batch_size=10, shuffle=True)
test_loader = DataLoader(test_dataset, batch_size=10, shuffle=False)

class Network(nn.Module):
    def __init__(self, sizes, activation='sigmoid'):
        super(Network, self).__init__()
        self.activation = activation
        self.layers = nn.ModuleList()
        for i in range(len(sizes) - 1):
            layer = nn.Linear(sizes[i], sizes[i+1])
            nn.init.xavier_normal_(layer.weight) if activation == 'sigmoid' else nn.init.kai
            nn.init.zeros_(layer.bias)
            self.layers.append(layer)

    def forward(self, x):
        for layer in self.layers[:-1]:
            x = F.sigmoid(layer(x)) if self.activation == 'sigmoid' else F.relu(layer(x))
        return self.layers[-1](x)

# define the train function
def train(network, train_loader, test_loader, epochs=30, eta=0.001, optimizer_type='SGD'):
    criterion = nn.CrossEntropyLoss()
    if optimizer_type == 'SGD':
        optimizer = optim.SGD(network.parameters(), lr=eta, momentum=0.8, nesterov=True, wei
    elif optimizer_type == 'Adam':
        optimizer = optim.Adam(network.parameters(), betas=(0.9, 0.999), lr=eta, weight_deca
    else:
        raise ValueError("Unsupported optimizer type.")

    test_acc_list = []

    for epoch in range(epochs):
        network.train()
        for images, labels in train_loader:
            outputs = network(images)
            loss = criterion(outputs, labels)

            optimizer.zero_grad()
            loss.backward()
```

```
            optimizer.step()

        # Evaluate on test set
        network.eval()
        correct = 0
        total = 0
        with torch.no_grad():
            for images, labels in test_loader:
                outputs = network(images)
                _, predicted = torch.max(outputs.data, 1)
                total += labels.size(0)
                correct += (predicted == labels).sum().item()
        acc = correct / total
        test_acc_list.append(acc)
        print(f"Epoch {epoch+1}/{epochs}, Accuracy: {acc:.4f}")

    return test_acc_list

# initiate the optimzers and run it
learning_rates = [0.0001, 0.001, 0.01, 0.05]
optimizers = ['SGD', 'Adam']
results = {}

for opt in optimizers:
    for lr in learning_rates:
        print(f"\nTraining with {opt}, Learning Rate = {lr}")
        model = Network([784, 30, 10], activation='sigmoid')
        acc = train(model, train_loader, test_loader, epochs=30, eta=lr, optimizer_type=opt)
        results[(opt, lr)] = acc

# plot
plt.figure(figsize=(12, 7))
for key, acc in results.items():
    label = f"{key[0]} (lr={key[1]})"
    plt.plot(acc, label=label)

plt.xlabel("Epoch")
plt.ylabel("Test Accuracy")
plt.title("Test Accuracy vs Epochs for Different Optimizers and Learning Rates")
plt.legend()
plt.grid(True)
plt.show()
```

```
Training with SGD, Learning Rate = 0.0001
Epoch 1/30, Accuracy: 0.5373
Epoch 2/30, Accuracy: 0.6801
Epoch 3/30, Accuracy: 0.7212
Epoch 4/30, Accuracy: 0.7525
Epoch 5/30, Accuracy: 0.7780
Epoch 6/30, Accuracy: 0.7979
Epoch 7/30, Accuracy: 0.8108
Epoch 8/30, Accuracy: 0.8216
Epoch 9/30, Accuracy: 0.8298
Epoch 10/30, Accuracy: 0.8357
Epoch 11/30, Accuracy: 0.8443
Epoch 12/30, Accuracy: 0.8517
Epoch 13/30, Accuracy: 0.8575
Epoch 14/30, Accuracy: 0.8625
Epoch 15/30, Accuracy: 0.8664
Epoch 16/30, Accuracy: 0.8693
Epoch 17/30, Accuracy: 0.8721
Epoch 18/30, Accuracy: 0.8742
Epoch 19/30, Accuracy: 0.8763
Epoch 20/30, Accuracy: 0.8786
Epoch 21/30, Accuracy: 0.8800
Epoch 22/30, Accuracy: 0.8824
Epoch 23/30, Accuracy: 0.8847
Epoch 24/30, Accuracy: 0.8863
Epoch 25/30, Accuracy: 0.8877
Epoch 26/30, Accuracy: 0.8899
Epoch 27/30, Accuracy: 0.8917
Epoch 28/30, Accuracy: 0.8926
Epoch 29/30, Accuracy: 0.8932
Epoch 30/30, Accuracy: 0.8947

Training with SGD, Learning Rate = 0.001
Epoch 1/30, Accuracy: 0.8323
Epoch 2/30, Accuracy: 0.8808
Epoch 3/30, Accuracy: 0.8936
Epoch 4/30, Accuracy: 0.9016
Epoch 5/30, Accuracy: 0.9070
Epoch 6/30, Accuracy: 0.9125
Epoch 7/30, Accuracy: 0.9163
Epoch 8/30, Accuracy: 0.9181
Epoch 9/30, Accuracy: 0.9216
```

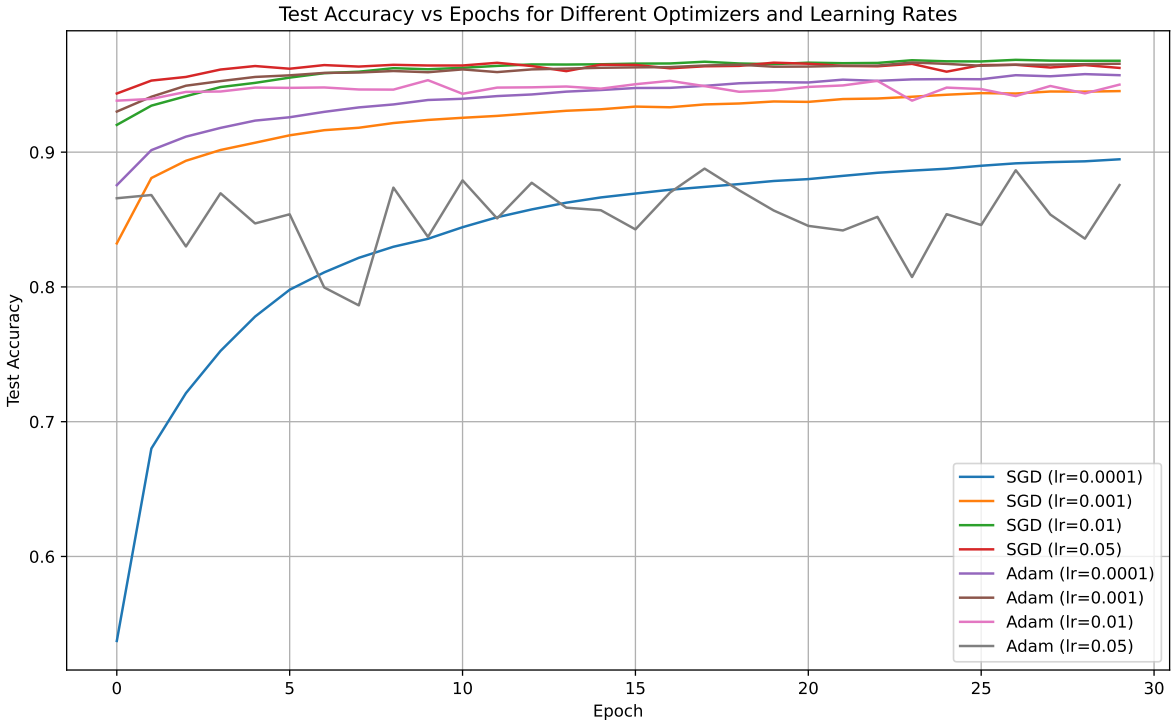
```
Epoch 10/30, Accuracy: 0.9239
Epoch 11/30, Accuracy: 0.9255
Epoch 12/30, Accuracy: 0.9269
Epoch 13/30, Accuracy: 0.9288
Epoch 14/30, Accuracy: 0.9307
Epoch 15/30, Accuracy: 0.9318
Epoch 16/30, Accuracy: 0.9338
Epoch 17/30, Accuracy: 0.9333
Epoch 18/30, Accuracy: 0.9354
Epoch 19/30, Accuracy: 0.9361
Epoch 20/30, Accuracy: 0.9376
Epoch 21/30, Accuracy: 0.9373
Epoch 22/30, Accuracy: 0.9394
Epoch 23/30, Accuracy: 0.9398
Epoch 24/30, Accuracy: 0.9411
Epoch 25/30, Accuracy: 0.9426
Epoch 26/30, Accuracy: 0.9438
Epoch 27/30, Accuracy: 0.9435
Epoch 28/30, Accuracy: 0.9450
Epoch 29/30, Accuracy: 0.9449
Epoch 30/30, Accuracy: 0.9453

Training with SGD, Learning Rate = 0.01
Epoch 1/30, Accuracy: 0.9202
Epoch 2/30, Accuracy: 0.9345
Epoch 3/30, Accuracy: 0.9415
Epoch 4/30, Accuracy: 0.9483
Epoch 5/30, Accuracy: 0.9514
Epoch 6/30, Accuracy: 0.9553
Epoch 7/30, Accuracy: 0.9586
Epoch 8/30, Accuracy: 0.9597
Epoch 9/30, Accuracy: 0.9623
Epoch 10/30, Accuracy: 0.9615
Epoch 11/30, Accuracy: 0.9626
Epoch 12/30, Accuracy: 0.9640
Epoch 13/30, Accuracy: 0.9651
Epoch 14/30, Accuracy: 0.9650
Epoch 15/30, Accuracy: 0.9652
Epoch 16/30, Accuracy: 0.9657
Epoch 17/30, Accuracy: 0.9658
Epoch 18/30, Accuracy: 0.9671
Epoch 19/30, Accuracy: 0.9658
Epoch 20/30, Accuracy: 0.9653
```

```
Epoch 21/30, Accuracy: 0.9664
Epoch 22/30, Accuracy: 0.9660
Epoch 23/30, Accuracy: 0.9662
Epoch 24/30, Accuracy: 0.9682
Epoch 25/30, Accuracy: 0.9674
Epoch 26/30, Accuracy: 0.9673
Epoch 27/30, Accuracy: 0.9685
Epoch 28/30, Accuracy: 0.9679
Epoch 29/30, Accuracy: 0.9678
Epoch 30/30, Accuracy: 0.9678

Training with SGD, Learning Rate = 0.05
Epoch 1/30, Accuracy: 0.9435
Epoch 2/30, Accuracy: 0.9531
Epoch 3/30, Accuracy: 0.9558
Epoch 4/30, Accuracy: 0.9613
Epoch 5/30, Accuracy: 0.9639
Epoch 6/30, Accuracy: 0.9619
Epoch 7/30, Accuracy: 0.9646
Epoch 8/30, Accuracy: 0.9635
Epoch 9/30, Accuracy: 0.9648
Epoch 10/30, Accuracy: 0.9643
Epoch 11/30, Accuracy: 0.9643
Epoch 12/30, Accuracy: 0.9663
Epoch 13/30, Accuracy: 0.9639
Epoch 14/30, Accuracy: 0.9601
Epoch 15/30, Accuracy: 0.9649
Epoch 16/30, Accuracy: 0.9647
Epoch 17/30, Accuracy: 0.9622
Epoch 18/30, Accuracy: 0.9637
Epoch 19/30, Accuracy: 0.9640
Epoch 20/30, Accuracy: 0.9664
Epoch 21/30, Accuracy: 0.9656
Epoch 22/30, Accuracy: 0.9640
Epoch 23/30, Accuracy: 0.9637
Epoch 24/30, Accuracy: 0.9655
Epoch 25/30, Accuracy: 0.9597
Epoch 26/30, Accuracy: 0.9646
Epoch 27/30, Accuracy: 0.9648
Epoch 28/30, Accuracy: 0.9628
Epoch 29/30, Accuracy: 0.9646
Epoch 30/30, Accuracy: 0.9625
```

```
Training with Adam, Learning Rate = 0.0001
Epoch 1/30, Accuracy: 0.8755
Epoch 2/30, Accuracy: 0.9015
Epoch 3/30, Accuracy: 0.9115
Epoch 4/30, Accuracy: 0.9180
Epoch 5/30, Accuracy: 0.9234
Epoch 6/30, Accuracy: 0.9259
Epoch 7/30, Accuracy: 0.9299
Epoch 8/30, Accuracy: 0.9332
Epoch 9/30, Accuracy: 0.9354
Epoch 10/30, Accuracy: 0.9387
Epoch 11/30, Accuracy: 0.9396
Epoch 12/30, Accuracy: 0.9416
Epoch 13/30, Accuracy: 0.9428
Epoch 14/30, Accuracy: 0.9449
Epoch 15/30, Accuracy: 0.9461
Epoch 16/30, Accuracy: 0.9476
Epoch 17/30, Accuracy: 0.9477
Epoch 18/30, Accuracy: 0.9492
Epoch 19/30, Accuracy: 0.9511
Epoch 20/30, Accuracy: 0.9519
Epoch 21/30, Accuracy: 0.9517
Epoch 22/30, Accuracy: 0.9538
Epoch 23/30, Accuracy: 0.9529
Epoch 24/30, Accuracy: 0.9540
Epoch 25/30, Accuracy: 0.9542
Epoch 26/30, Accuracy: 0.9541
Epoch 27/30, Accuracy: 0.9571
Epoch 28/30, Accuracy: 0.9563
Epoch 29/30, Accuracy: 0.9579
Epoch 30/30, Accuracy: 0.9571

Training with Adam, Learning Rate = 0.001
Epoch 1/30, Accuracy: 0.9301
Epoch 2/30, Accuracy: 0.9411
Epoch 3/30, Accuracy: 0.9493
Epoch 4/30, Accuracy: 0.9527
Epoch 5/30, Accuracy: 0.9558
Epoch 6/30, Accuracy: 0.9570
Epoch 7/30, Accuracy: 0.9588
Epoch 8/30, Accuracy: 0.9591
Epoch 9/30, Accuracy: 0.9602
Epoch 10/30, Accuracy: 0.9593
```

```
Epoch 11/30, Accuracy: 0.9614
Epoch 12/30, Accuracy: 0.9594
Epoch 13/30, Accuracy: 0.9614
Epoch 14/30, Accuracy: 0.9620
Epoch 15/30, Accuracy: 0.9626
Epoch 16/30, Accuracy: 0.9629
Epoch 17/30, Accuracy: 0.9631
Epoch 18/30, Accuracy: 0.9643
Epoch 19/30, Accuracy: 0.9649
Epoch 20/30, Accuracy: 0.9634
Epoch 21/30, Accuracy: 0.9635
Epoch 22/30, Accuracy: 0.9640
Epoch 23/30, Accuracy: 0.9642
Epoch 24/30, Accuracy: 0.9665
Epoch 25/30, Accuracy: 0.9655
Epoch 26/30, Accuracy: 0.9641
Epoch 27/30, Accuracy: 0.9649
Epoch 28/30, Accuracy: 0.9649
Epoch 29/30, Accuracy: 0.9653
Epoch 30/30, Accuracy: 0.9656

Training with Adam, Learning Rate = 0.01
Epoch 1/30, Accuracy: 0.9382
Epoch 2/30, Accuracy: 0.9395
Epoch 3/30, Accuracy: 0.9447
Epoch 4/30, Accuracy: 0.9450
Epoch 5/30, Accuracy: 0.9479
Epoch 6/30, Accuracy: 0.9477
Epoch 7/30, Accuracy: 0.9480
Epoch 8/30, Accuracy: 0.9465
Epoch 9/30, Accuracy: 0.9464
Epoch 10/30, Accuracy: 0.9534
Epoch 11/30, Accuracy: 0.9433
Epoch 12/30, Accuracy: 0.9479
Epoch 13/30, Accuracy: 0.9481
Epoch 14/30, Accuracy: 0.9487
Epoch 15/30, Accuracy: 0.9471
Epoch 16/30, Accuracy: 0.9504
Epoch 17/30, Accuracy: 0.9529
Epoch 18/30, Accuracy: 0.9490
Epoch 19/30, Accuracy: 0.9448
Epoch 20/30, Accuracy: 0.9458
Epoch 21/30, Accuracy: 0.9484
```

```
Epoch 22/30, Accuracy: 0.9495
Epoch 23/30, Accuracy: 0.9529
Epoch 24/30, Accuracy: 0.9382
Epoch 25/30, Accuracy: 0.9479
Epoch 26/30, Accuracy: 0.9468
Epoch 27/30, Accuracy: 0.9416
Epoch 28/30, Accuracy: 0.9491
Epoch 29/30, Accuracy: 0.9436
Epoch 30/30, Accuracy: 0.9501

Training with Adam, Learning Rate = 0.05
Epoch 1/30, Accuracy: 0.8658
Epoch 2/30, Accuracy: 0.8682
Epoch 3/30, Accuracy: 0.8300
Epoch 4/30, Accuracy: 0.8695
Epoch 5/30, Accuracy: 0.8471
Epoch 6/30, Accuracy: 0.8539
Epoch 7/30, Accuracy: 0.7996
Epoch 8/30, Accuracy: 0.7863
Epoch 9/30, Accuracy: 0.8737
Epoch 10/30, Accuracy: 0.8371
Epoch 11/30, Accuracy: 0.8791
Epoch 12/30, Accuracy: 0.8508
Epoch 13/30, Accuracy: 0.8773
Epoch 14/30, Accuracy: 0.8588
Epoch 15/30, Accuracy: 0.8569
Epoch 16/30, Accuracy: 0.8427
Epoch 17/30, Accuracy: 0.8700
Epoch 18/30, Accuracy: 0.8878
Epoch 19/30, Accuracy: 0.8717
Epoch 20/30, Accuracy: 0.8567
Epoch 21/30, Accuracy: 0.8453
Epoch 22/30, Accuracy: 0.8419
Epoch 23/30, Accuracy: 0.8520
Epoch 24/30, Accuracy: 0.8073
Epoch 25/30, Accuracy: 0.8540
Epoch 26/30, Accuracy: 0.8459
Epoch 27/30, Accuracy: 0.8866
Epoch 28/30, Accuracy: 0.8537
Epoch 29/30, Accuracy: 0.8358
Epoch 30/30, Accuracy: 0.8756
```



Problem 3: Deep Nets: Overcoming Gradients

(a) In deep networks, the gradients for the neural network weights can either vanish or explode due to the compositional nature of the network. This typically happens for weights near the input layer. In order to observe vanishing gradients we need to calculate the size of the gradients of the loss function with respect to each weight.

The neural network training routine has commented code that computes the norm of the gradients of the input layer and the norm of the gradients of the output layer and divides and averages them across all the batches in an epoch. Uncomment the code (and modify the print and return statement) so that the gradients are computed and the gradient ratio output and saved. Then try training a deep neural network.

I recommend beginning with the architecture net = Network([784,30,30,30,30,30,30,30,30,10]). Train this neural network. You will almost certainly find that the training does not succeed. What are the gradient ratios that you observe during the training?

If your computer cannot train this network, you can try this problem with a shallower network, or my recommendation is to use only a single epoch of training which should still demonstrate the vanishing gradient problem.

(b) There are several techniques to deal with vanishing (or exploding) gradients. These include using neurons with different activation functions or using different normalization schemes. The forward method in the Network class defines the activation functions. Change the activation function from sigmoid to relu and change the initialization from Xavier.normal_ to kaiming.uniform_ (which is more optimal for deep ReLU neurons) and train the deep neural network. How does the gradient ratio change? How does the test accuracy compare to the shallow net accuracy you achieved in problem 2?

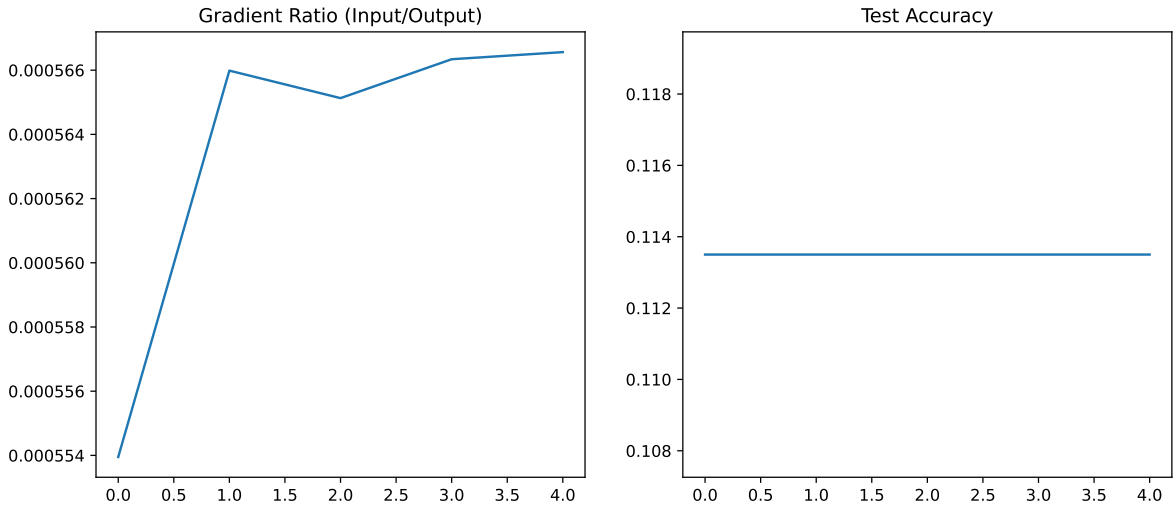
(c) Deep neural networks are a superior architecture of image classification problems than shallow networks, however typically the dense structure that we have implemented here is not used. Instead the neural networks usually have several convolutionary layers at the beginning. With some effort and experimentation, it should still be possible to achieve a very high accuracy with a dense neural network. Experiment with the network architecture and the hyperparameters and see how good you can make your deep/dense network. You can try a combination of increasing the number of neurons in the hidden layers or incrasing the depth of the network. A structure that often works is one which decreases the number of neurons per hidden layer steadily from the input layer to the final layer (i.e. starting at 784 and ending at 10). Can you improve upon the best shallow network that you constructed for problem 3?

```
# load dependencies
import torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
from torch.utils.data import DataLoader
from torchvision import datasets, transforms
import matplotlib.pyplot as plt

# load dataset
transform = transforms.Compose([transforms.ToTensor(), transforms.Lambda(lambda x: x.view(-1)
train_dataset = datasets.MNIST('data/', train=True, download=True, transform=transform)
test_dataset = datasets.MNIST('data/', train=False, transform=transform)
train_loader = DataLoader(train_dataset, batch_size=10, shuffle=True)
test_loader = DataLoader(test_dataset, batch_size=10, shuffle=False)

# use the applied function
class Network(nn.Module):
    def __init__(self, sizes, activation='sigmoid'):
        super(Network, self).__init__()
        self.activation = activation
        self.layers = nn.ModuleList()
        for i in range(len(sizes) - 1):
```

```
            layer = nn.Linear(sizes[i], sizes[i+1])
            nn.init.kaiming_uniform_(layer.weight, mode='fan_out', nonlinearity='relu') # ini
            nn.init.zeros_(layer.bias)
            self.layers.append(layer)

    def forward(self, x):
        for layer in self.layers[:-1]:
            x = F.sigmoid(layer(x)) if self.activation == 'sigmoid' else F.relu(layer(x))
        return self.layers[-1](x)

# Train function with gradient monitoring
def train(network, train_loader, test_loader, epochs=1, eta=0.001):
    optimizer = optim.SGD(network.parameters(), lr=eta)
    criterion = nn.CrossEntropyLoss()

    grad_ratios = []
    test_acc = []

    for epoch in range(epochs):
        network.train()
        input_grads, output_grads = 0.0, 0.0
        n_batches = 0

        for images, labels in train_loader:
            optimizer.zero_grad()
            outputs = network(images)
            loss = criterion(outputs, labels)
            loss.backward()

            # gradient norm
            input_grad_norm = network.layers[0].weight.grad.norm().item()
            output_grad_norm = network.layers[-1].weight.grad.norm().item()
            input_grads += input_grad_norm
            output_grads += output_grad_norm
            n_batches += 1

            optimizer.step()

        grad_ratio = (input_grads / n_batches) / (output_grads / n_batches + 1e-8)
        grad_ratios.append(grad_ratio)

        # review test data
```

```
        network.eval()
        correct, total = 0, 0
        with torch.no_grad():
            for images, labels in test_loader:
                outputs = network(images)
                predicted = torch.argmax(outputs, dim=1)
                total += labels.size(0)
                correct += (predicted == labels).sum().item()

        accuracy = correct / total
        test_acc.append(accuracy)

        print(f"Epoch {epoch+1}: Gradient Ratio = {grad_ratio:.6f}, Test Accuracy = {accuracy

    return grad_ratios, test_acc

# define the deep learning network
net = Network([784,30,30,30,30,30,30,30,30,10])
ratios_sigmoid, acc_sigmoid = train(net, train_loader, test_loader, epochs=5)

# plot
plt.figure(figsize=(12, 5))
plt.subplot(1, 2, 1)
plt.plot(ratios_sigmoid)
plt.title("Gradient Ratio (Input/Output)")
plt.subplot(1, 2, 2)
plt.plot(acc_sigmoid)
plt.title("Test Accuracy")
plt.show()
```

```
Epoch 1: Gradient Ratio = 0.000554, Test Accuracy = 0.1135
Epoch 2: Gradient Ratio = 0.000566, Test Accuracy = 0.1135
Epoch 3: Gradient Ratio = 0.000565, Test Accuracy = 0.1135
Epoch 4: Gradient Ratio = 0.000566, Test Accuracy = 0.1135
Epoch 5: Gradient Ratio = 0.000567, Test Accuracy = 0.1135
```


Gradient Ratio (Input/Output)
0.000566
0.000564
0.000562
0.000560
0.000558
0.000556
0.000554
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Test Accuracy
0.118
0.116
0.114
0.112
0.110
0.108
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0