ADMM Part :

To estimate the $\hat{\theta}$, the precision matrix, with Σ, the correlatoin matrix by CLIME method, here is the optimization problem:

$$\hat{\theta} = \operatorname*{argmin}_{\theta} \|\theta\|_1 \quad s.t \quad \|\Sigma\Theta - \mathbb{1}_p\|_\infty \le \lambda \tag{1}$$

To make the optimization problem easier, I could split the problem (1) to each colunms of θ by the property of 1-norm and Inf-norm.

$$\hat{x_i} = \operatorname*{argmin}_{x} \|x\|_1 \quad s.t \quad \|\Sigma x - e_i\|_\infty \le \lambda \tag{2}$$

where x_i is the i^{th} colunm of θ and the e_i is the i^{th} column of $\mathbb{1}_p$.
At first place, we attempt to solve this problem (2) in MATLAB using a convex optimization package like CVX, which employ accurate but slow methos. Fortunately, however, there is a much more efficient way to do this, here is the Alternating Direction Method of Multipliers (ADMM):

$$\hat{\beta} = \operatorname*{argmin}_{\beta} \|\beta\|_1 \quad s.t \quad X^TX\beta - X^Ty - z = 0, \quad \|D^{-1}z\|_\infty \le \delta \tag{3}$$

To turn ADMM into ou problem, set D = $\mathbb{1}_p$, and wherever they have X^TX we write Σ, and wherever they have X^Ty we write e_i. In order to describe the ADMM iterations, we introduce the augmented Lagrangian function for (3):

$$L_\mu(z, \beta, \lambda) = \|\beta\|_1 + \lambda^T(X^TX\beta - X^Ty - z) + \frac{\mu}{2}\|X^TX\beta - X^Ty - z\|_2^2 \tag{4}$$

for some $\mu > 0$.

Each iteration of the ADMM involves alternate minimization of L_μ with respect to z and β, followed by an update of λ. Here is the outline for ADMM:

Start with $\beta^0, \lambda^0 \in \mathbb{R}^p$ and $\mu > 0$.Then iterate with z and β followed by an update of λ until converge(Actually, I do have the question of the convergence condition). For k = 0,1,...

$$\begin{cases} z^k + 1 = \operatorname{argmin}_z L_\mu(z, \beta^k, \lambda^k) \quad s.t \quad \|D^{-1}z\|_\infty \le \delta \\ \beta^{k+1} \in \operatorname{argmin}_\beta L_\mu(z^{k+1}, \beta, \lambda^k) \\ \lambda^{k+1} = \lambda^k + \mu(X^TX\beta^{k+1} - X^Ty - z^{k+1}) \end{cases} \tag{5}$$

For the augmented Lagrangian function (4), it is easy to observe that the first subproblem in (5) has a close-form solution.

$$z^{k+1} = \operatorname*{argmin}_{z} \|z - (X^TX\beta^k - X^Ty - \frac{\lambda^k}{\mu})\|_2^2 \quad s.t \quad \|D^{-1}z\|_\infty \le \delta$$

$$z_i^{k+1} = min\{max\{(X^TX\beta^k - X^Ty - \frac{\lambda^k}{\mu})_i, -\delta d_i\}, \delta d_i)\} \tag{6}$$

where d_i is the i^{th} diagonal entry of D.

The second subproblem in (5) is harder to solve, it can be rewritten as,

$$\beta^{k+1} = \underset{\beta}{\operatorname{argmin}} \frac{\mu}{2} \|X^T X \beta^k - X^T y - z^{k+1} + \frac{\lambda^k}{\mu}\|_2^2 + \|\beta\|_1 \tag{7}$$

However, the problem (7) is obvious Lasso question, which cannot be solved in a single iteration. For our problem, their z, β, λ, $X^T X$ and $X^T y$ are equivalent to our y, x, u Σ and e_i.

$$x^{k+1} = \underset{x}{\operatorname{argmin}} \frac{\mu}{2} \|\Sigma x^k - e_i - y^{k+1} + \frac{u^k}{\mu}\|_2^2 + \|\beta\|_1 \tag{8}$$

where e_i is the i^{th} column of $\mathbb{1}_p$.

Then we introduce a precondition on (8) with α = largest absolute value eigenvalue of Σ + 0.01 and $A \in \mathbb{R}^p$ s.t $A^T A = \alpha^2 \mathbb{1}_p - \Sigma^T \Sigma$.

$$\begin{aligned}
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu}{2} \|\Sigma x - e_i - y^{k+1} + \frac{u^k}{\mu}\|_2^2 + \|x\|_1 + \frac{\mu}{2}\|A(x - x^k)\|_2^2 \\
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu}{2}(x^T \Sigma^T \Sigma x + x^T A^T A x - 2x^T \Sigma v - 2x^T A^T A x^k) + \|x\|_1 + C \\
&\quad \text{where } v = e_i + y^{k+1} - \frac{u^k}{\mu} \text{ and constant } C \\
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu\alpha^2}{2}[x^T x - \frac{2x^T}{\alpha^2}(\Sigma v + A^T A x^k)] + \|x\|_1 + C \\
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu\alpha^2}{2} \|x - \frac{\Sigma v + A^T A x^k}{\alpha^2}\|_2^2 + \|x\|_1 \qquad (9)
\end{aligned}$$

For the problem (9), it is still a Lasso problem. But it could be solved by a simple soft thresholding method.

$$x = \underset{x}{\operatorname{argmin}} \|x - b\|_2^2 + \lambda\|x\|_1$$

with solution:

$$S_\lambda(b) = \begin{cases} b - \frac{\lambda}{2} & \text{if } b > \frac{\lambda}{2} \\ b + \frac{\lambda}{2} & \text{if } b < -\frac{\lambda}{2} \\ 0 & otherwise \end{cases} \tag{10}$$

In our case, we substitude those value below into (10):

$$b = \frac{\Sigma v + A^T A x^k}{\alpha^2} \text{ and } \lambda = \frac{2}{\mu\alpha^2}$$