ADMM for λ_{min} :

To find the λ_{min} with the optimization problem:

$$\hat{\lambda} = \min_{\Theta} \|\Sigma\Theta - \mathbb{1}_p\|_\infty \tag{1}$$

To make the optimization problem easier, I could split the problem (1) to each colunms of θ by the property of and Inf-norm.

$$\hat{\lambda_i} = \min_{x_i} \|\Sigma x_i \ - e_i\|_\infty \tag{2}$$

where x_i is the i^{th} colunm of θ and the e_i is the i^{th} column of $\mathbb{1}_p$. And $\lambda = \max\{\lambda_1, \lambda_2, ..., \lambda_p\}$

We will use ADMM method to solve this optimization problem for (2).

$$\hat{\lambda} = \min_{y} \|y\|_\infty \quad s.t \quad \Sigma x - v - y = 0 \tag{3}$$

To turn ADMM into ou problem, set v = e_i. In order to describe the ADMM iterations, then we introduce augmented Lagrangain function of problem (3):

$$L_\mu(x, y, u) = \|y\|_\infty + u^T(\Sigma x - v - y) + \frac{\mu}{2}\|\Sigma x - v - y\|_2^2 \tag{4}$$

for some $\mu > 0$.

Each iteration of the ADMM involves alternate minimization of L_μ with respect to x and y, followed by an update of u. Here is the outline for ADMM:

Start with $y^0, u^0 \in \mathbb{R}^p$ and $\mu > 0$.Then iterate with x and y followed by an update of u until converge. For k = 0,1,...

$$\begin{cases} y^{k+1} = \text{argmin}_y \, L_\mu(x^k, y, u^k) \\ x^{k+1} \in \text{argmin}_x \, L_\mu(x, y^{k+1}, u^k) \\ u^{k+1} = u^k + \mu(\Sigma x^{k+1} - v - y^{k+1}) \end{cases} \tag{5}$$

For the first subproblem in (5) can be rewritten as:

$$\begin{aligned} y^{k+1} &= \underset{y}{\text{argmin}} \, \|y - (\Sigma x^k - v + \frac{u^k}{\mu})\|_2^2 + \|y\|_\infty \\ y^{k+1} &= \underset{y}{\text{argmin}} \, \|y - b\| + \|y\|_\infty \quad \text{where } b = \Sigma x^k - v + \frac{u^k}{\mu} \end{aligned} \tag{6}$$

The solution y for the problem (6) always be a truncated version of the vector b. Hence, I need to search through all absolute value of b. And I will find out the T produced the lowest value of (6).

For each i = 1,...,length(b):

1. set T = $|b_i|$

2. Compute y with this level of T

3. Compute the value of the objective function (6)

Then I pick the T that produce the lowest value of function (6) and update y.

The second subproblem in (5) is harder to solve, it can be rewritten as,

$$x^{k+1} = \underset{x}{\operatorname{argmin}} \frac{\mu}{2} \|\Sigma x - v - y^{k+1} + \frac{u^k}{\mu}\|_2^2 \tag{7}$$

Then we introduce a precondition on (7) with α = largest absolute value eigenvalue of Σ * 1.01 and $A \in \mathbb{R}^p$ s.t $A^T A = \alpha^2 \mathbb{1}_p - \Sigma^T \Sigma$.

$$\begin{aligned}
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu}{2} \|\Sigma x - v - y^{k+1} + \frac{u^k}{\mu}\|_2^2 + \frac{\mu}{2} \|A(x - x^k)\|_2^2 \\
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu}{2} (x^T \Sigma^T \Sigma x + x^T A^T A x - 2x^T \Sigma v - 2x^T A^T A x^k) + C \\
&\quad \text{where } d = v + y^{k+1} - \frac{u^k}{\mu} \text{ and constant } C \\
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu \alpha^2}{2} [x^T x - \frac{2x^T}{\alpha^2} (\Sigma d + A^T A x^k)] + C \\
x^{k+1} &= \underset{x}{\operatorname{argmin}} \frac{\mu \alpha^2}{2} \|x - \frac{\Sigma d + A^T A x^k}{\alpha^2}\|_2^2 \qquad (8)
\end{aligned}$$

From the (8), it is clear that the solution of this quadratic optimization problem would be $x^{k+1} = \frac{\Sigma v + A^T A x^k}{\alpha^2}$