Numerically Computed Derivatives:

- Exact formulas can be very complicated ~ easier to approximate functions

Starting with first order derivatives

Approximations:

- $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
- Assuming $h > 0$
- This is the exact value for linear functions ($y = 2x$)
- Other functions introduce error ($y = x^2 + 9 * x^3$)

Taylor Series EQ

- $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + ...$
- $f(x) \approx \sum_{n=0}^{n} \frac{f^n(x_0)}{n!}(x - x_0)^n$

Computing Error of Approximation

- $f(x + h) \approx f(x) + f'(x)(x + h - x) + \frac{f''(x)}{2!}(x + h - x)^2 + ...$
- (AKA) $f(x + h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + ...$
- $f(x) \approx \sum_{n=0}^{n} \frac{f^n(x)}{n!}(h)^n$
- So actually....
  - We Approximated $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
  - Actualy (Order 2 Sol) $f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{h^2}{2}f''(x)$, where $-\frac{h^2}{2}f''(x)$ is the error in the previous assumption and removiing this is called a "truncation error"

Forward/Backward Differencing

- *Forward*: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
- *Backward*: $f'(x) \approx \frac{f(x)-f(x-h)}{h}$

Explaining the Equations:

The derivative is effectively a measure of some rate of change. Here we have x which can be a scalar or a array. To approximate the derivative using continous values, the rate of change must be computed at each point on the function with respect to x. f(x) or y is what is being derived with respect. As such, we measure f(x) and difference it to a point behind (f(x-h)) or in front (f(x+h)) with respect the change in x = (x+h-x = h). The best solution is for h → 0. We see this is the second order error term $-\frac{h^2}{2}f''(x)$ where a lower h significantly reduces error (rate of convergence).

Truncation error is measured in terms of error. Nth order derivatives have error of the order of $O(h^n)$

To reduce error, we combine back and forward differencing into central differencing.

- *Center*: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$

Showing the lesser Error (2nd order case)

- $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x)$
- $f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x)$

Therefore...

- $f'(x) = \frac{f(x+h)-f(x-h)}{h^2} - \frac{h^2}{12}f''(x)$, with a truncation error = $-\frac{h^2}{12}f''(x)$

Moving to 2nd Order Derivative...

Looking at the taylor expansions of $f(x \pm h)$ we can derive ....

- $f(x+h) + f(x-h) = 2f(x) + h^2 f''(x)$
- $f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$

And 3rd Order Derivative...

Looking at the taylor expansions of $f(x \pm h)$ we can derive ....

- $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x)$
- $f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x)$

Remember Taylor EQ:

- $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + ...$

Applying to +2h

- $f(x+2h) \approx f(x) + f'(x)(x - x + 2h) + \frac{f''(x)}{2!}(x - x + 2h)^2 + \frac{f'''(x)}{3!}(x - x + 2h)^3 + ...$
- $f(x+2h) \approx f(x) + 2hf'(x) + \frac{f''(x)}{2!}(2h)^2 + \frac{f'''(x)}{3!}(2h)^3 + ...$
- $f(x+2h) \approx f(x) + 2hf'(x) + \frac{f''(x)}{2!}(2h)^2 + \frac{f'''(x)}{3!}(2h)^3 + ...$
- $f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4h^3}{3}f'''(x)$
- $f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4h^3}{3}f'''(x)$

- $f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}$
- $f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3}$
- $f(x+2h) - f(x-2h) - 2(f(x+h) - f(x-h)) = 4hf'(x) + \frac{8h^3}{3}f'''(x) - 4hf'(x) - 2\frac{h}{3}$
- $f(x+2h) - f(x-2h) - 2(f(x+h) - f(x-h)) = 2h^3 f'''(x)$
- $f'''(x) = \frac{f(x+2h)-f(x-2h)-2(f(x+h)-f(x-h))}{2h^3}$

And 4th Order Derivative...

- $f^4(x) = \frac{f(x+2h)+f(x-2h)+6f(x)-4(f(x+h)+f(x-h))}{h^4}$

Looking at all Approxs

- $f^1(x) \approx \frac{f(x+h)-f(x-h)}{2h}$
- $f^2(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$
- $f^3(x) \approx \frac{f(x+2h)-f(x-2h)-2(f(x+h)-f(x-h))}{2h^3}$
- $f^4(x) \approx \frac{f(x+2h)+f(x-2h)+6f(x)-4(f(x+h)+f(x-h))}{h^4}$

http://www2.math.umd.edu/~dlevy/classes/amsc466/lecture-notes/differentiation-chap.pdf
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Differentiation Via Interpolation

Generate differentiation formulas by differentiating interpolant

How Can this be reduced in complexity?

As mentioneed before, a change in the function is defined as $\Delta f = f^1(x)$ Therefore logically we can say

- $f^1(x) \approx \Delta f$
- $f^2(x) \approx \Delta\Delta f = \Delta f^1$
- $f^3(x) \approx \Delta\Delta\Delta f = \Delta f^2$
- $f^4(x) \approx \Delta\Delta\Delta\Delta f = \Delta f^3$

Δ = change or more specifically a shift operator Shift Operator --> $T_h = e^{h\frac{d}{dx}}$
$\Delta = \frac{T_h - T_{-h}}{2h} = \frac{e^{h\frac{d}{dx}} - e^{-h\frac{d}{dx}}}{2h}$

Review of Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k$$

Where $\binom{n}{k}$ is the number of k combinations from a total of n

This formula makes it easy to explan binomials ie two summed or subtracted variables

Back to Complexity Reduction

We have a binomial expansion problem

$\Delta^{2k} = (\frac{T_h - T_{-h}}{2h})^{2k}$

= $\frac{1}{h^{2k}} \sum_{m=-k}^{k} \frac{2k!}{(k+m)!(2k-k-m)!}(-1)^{m+k} f(x+mh)$$

with vars k=order/2,m=,h=1e-2