Curves & Splines

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1 Bézier Curves

A)

$$\begin{aligned} B(u) &= (1-u)^2\mathbf{p}_0 + 2(1-u)\mathbf{p}_1 + u^2\mathbf{p}_2 \\ &= (1-0.5)^2\mathbf{p}_0 + 2(0.5)(1-0.5)\mathbf{p}_1 + (0.5)^2\mathbf{p}_2 \\ &= \frac{1}{4}\mathbf{p}_0 + \frac{1}{2}\mathbf{p}_1 + \frac{1}{4}\mathbf{p}_2 \end{aligned}$$

$$\begin{aligned} \mathbf{p}_2 = (1,0), \mathbf{p}_1 &= (1,1), \mathbf{p}_2 = (0,1) \\ x &= \frac{1}{4}(1) + \frac{1}{2}(1) + \frac{1}{4}(0) = 0.75 \\ y &= \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(1) = 0.75 \end{aligned}$$

B) The shape is a squircle, which approximates a circle. It is not quite a circle since the radius is not constant.

C) The 4 control points are $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. Since we are given the end points we know that $\mathbf{p}_0 = (0,1)$ and $\mathbf{p}_1 = (1,0)$. We are also given the tangent of those points, so we know $\mathbf{p}_1 = (x_1, 1)$ and $\mathbf{p}_2 = (1, y_2)$. Since this is a cubic Bézier and we are given the midpoint $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, we know that $\frac{\sqrt{2}}{2}$ is the midpoint of of a midpoint of a midpoint. We can solve for x_1 albegraically:

$$\frac{\sqrt{2}}{2} = \frac{\left(\left(\frac{\frac{0+x_1}{2} + \frac{x_1+1}{2}}{2}\right) + \left(\frac{\frac{1+1}{2} + \frac{1+x_1}{2}}{2}\right)\right)}{2}$$

Solving for x_1, we get $x_1 = \frac{4\sqrt{2}}{3} - \frac{4}{3} = 0.55228$. Since this curve is symmetric about $y = x$, the reflected y value will be the same, therefore $y_2 = 0.55228$. Answer: the control points are (0, 1), (0.55228, 1), (01, 0.55228), (1, 0).

D) No, the curve does not perfectly form part of a circular arc, only approximate a circular arc. The equation is not exactly equivalent to that of a circle as the radius is not constant.

$$
\begin{aligned}
u &= \frac{2}{3} \\
p_x(\frac{2}{3}) &= (1-\frac{2}{3})^3 p_{0,x} + 3(\frac{2}{3})(1-\frac{2}{3})^2 p_{1,x} + 3(\frac{2}{3})^2(1-\frac{2}{3}) p_{2,x} + (\frac{2}{3})^3 p_{3,x} \\
&\approx 0.027(0) + 0.1889(0.55223) + 0.441(1) + 0.343(1) \\
&= 0.863469\ldots \\
p_y(\frac{2}{3}) &= (1-\frac{2}{3})^3 p_{0,y} + 3(\frac{2}{3})(1-\frac{2}{3})^2 p_{1,y} + 3(\frac{2}{3})^2(1-\frac{2}{3}) p_{2,y} + (\frac{2}{3})^3 p_{3,y} \\
&\approx 0.027(1) + 0.1889(1) + 0.441(0.55223) + 0.343(0) \\
&= 0.504717\ldots
\end{aligned}
$$

To calculate the radius:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{0.863469^2 + 0.504717^2} \\
&= 1.00015953 \\
&\neq 1
\end{aligned}
$$

This is close to, but not exactly 1, meaning that the arc is not that of a perfect circle.

E) The four control points of the of the cubic Bézier that form a curve identical to that in 1A can be found as it is close to that of 1C. However, the midpoint is now $(\frac{3}{4}, \frac{3}{4})$. We can safely conclude that the midpoints maintain the same form:

$$
\begin{aligned}
p_0 &= (0, 1) \\
p_1 &= (c, 1) \\
p_2 &= (1, c) \\
p_3 &= (1, 0)
\end{aligned}
$$

To find c:

$$
\begin{aligned}
\frac{3}{4} = p_x(0.5) &= (1-0.5)^3 p_{0,x} + 3(0.5)(1-0.5)^2 p_{1,x} + 3(0.5)^2(1-0.5) * p_{2,x} + (0.5)^3 p_{3,x} \\
\frac{3}{4} &= \frac{1}{8}(0) + \frac{3}{8}c + \frac{3}{8}(1) + \frac{1}{8}(1) \\
\frac{3}{4} &= \frac{4}{8} + \frac{3}{8}c \\
6 &= \frac{4}{8} + \frac{3}{8}c \\
c &= \frac{2}{3}
\end{aligned}
$$

So therefore, our points are:

- $\mathbf{p}_0 = (0, 1)$
- $\mathbf{p}_1 = (\frac{2}{3}, 1)$
- $\mathbf{p}_2 = (1, \frac{2}{3})$
- $\mathbf{p}_3 = (1, 0)$

2 Rendering Fonts

A) All the knots are C^0 continuous since the shape as a whole is continuous. In other words, there are no gaps between adjacent knots.

B) 4, 11, 2, 3, 5, 6, 9, 10, 12, 13 are G_1 continuous because the tangents are collinear. (The corner knots 1, 7, 8, and 14 are not G_1 continuous since the tangents are not collinear).

C) 5, 10 are C_1 continuous since they have collinear tangents, as needed to be G_1 continuous, and the adjacent off-curve control points are of equivalent magnitude.

3 Scrolling Fonts

A) If a 1000 px wide window can fit 10 letters, then a letter is 100 px wide. So to move 3 pixels every frame, we would need to 300 px every frame. So we need 300 px/s at 60 fps. $\frac{300px}{60\text{ frame}} = 5px/\text{frame}$. Therefore we would have to move 5 pixels per frame.

B) $\frac{300px}{24\text{frame}} = 12.5px/\text{frame}$. We would have to move 12.5 pixels per second. Since we can only move whole integer values of pixels, one possible solution to this problem would be to alternate between moving 12 and 13 pixels per frame to keep the average of 12.5 pixels per second.