Chapter 4

Predicate Logic or "Predicate Calculus"

Two commonly-used definitions of what a predicate is, exist:

1. A predicate is a truth-valued statement which may contain variables
2. A predicate is a statement whose truth-value depends on a value or values.

Note that the variable(s) in definition 1 takes on the value(s) referred to in definition 2, so the two definitions intend the same thing.

When a predicate is applied to a definite value it becomes a proposition.

Examples:

$x < 6$
Fraction(k)
Four-legged(A)
t1 defeated t2
B OnLoanTo N
A loves B

Fraction(0.7) is a true proposition.
Fraction(π) and Four-legged(whale) are false propositions.

The Existential Quantifier $\exists$

The quantifier $\exists$, read as "there exists" is used in the construction
$\exists$ declaration | constraint (or "predicate") • predicate

The declaration introduces a variable which is then constrained (optional). The predicate applies to this variable. The constraint acts like a filter through which all candidates for the variable are "screened" on their way to taking part in the final predicate.
The "constraint" is a truth valued statement.

For example, the statement "There exists an RSI Number for each named adult in this state" can be expressed as

$\exists$ x: $\mathbb{N}$ | $x > 0$ • x = RSI Number for Sean Mac Oibre

Other examples

$\exists$ x: : $\mathbb{Z}$ | x less than 10 and positive • $30 < x^2 < 40$
$\exists$ y : Person | "Up there" • y loves me

If the set in the declaration, suitably constrained, is finite, then $\exists$ can be replaced by **Disjunction** $\vee$:

$\exists\ x: : \ \mathbb{Z} \mid \ x$ less than 10 and positive $\bullet \ 30 < x^2 < 40$

can be replaced by

$30 < 1^2 < 40 \quad \vee \quad 30 < 2^2 < 40 \quad \vee \quad 30 < 3^2 < 40 \ \vee \ .... \ \vee \quad 30 < 9^2 < 40$

Unique Existential Quantifier $\exists_1$

The Unique Quantifier symbolises "There exists one and only one".

$\exists_1$ h: Types of Hare | NativeIrishSpecies(h)

This is read as "There exists one and only one hare type which is a Native Irish Species"

Careful : It is only in very specific circumstances that this is used.

Universal Quantifier $\forall$

$\forall$ is read as "For all" or "for every" and its definition has a similar structure to $\exists$:

$\forall$ declaration | constraint (or "predicate") $\bullet$ predicate

Example

$\forall\ x: : \ \mathbb{N} \mid \ x > 10 \ \bullet \ x^2 > 100$

means "For every x that is a Natural number greater than 10, its square is greater than 100".

With a finite set, $\forall$ can be represented by **Conjunction** $\wedge$:

$\forall$ d: : Dwarfs | d lives in the little house in the wood $\bullet$ d LovesSnowWhite

is the same as

Sneezy LovesSnowWhite $\wedge$ Dopey LovesSnowWhite $\wedge$ ... $\wedge$ Doc LovesSnowWhite

Both $\forall$ and $\exists$ are used closely together in predicate construction

"Every cloud has a silver lining" can be written as

$\forall$ c: Clouds $\bullet$ ($\exists$ l : Linings | l is silver $\bullet$ l IsHadBy c)

Set Comprehension

Sets have been defined so far by listing their elements. This is called set enumeration. This enumeration or listing the elements may always not be possible or convenient.

An alternative definition-mode is Set Comprehension. It is a mode known in Mathematics by the names "Definition by Rule", or "Set-building Notation". In this, the typical set-member is declared and if necessary constrained before the "rule" or defining expression is given.

The structure is below:

Set-Exp:: = {declaration | constraint (or "predicate") ● expression or "term" }

Example

$Sq == \{ x: \mathbb{Z} \mid 1 \leqslant x \leqslant 20 \bullet x^2 \}$

is the set of squares of every integer from 1 to 20.

The members of the set { Decl. | Pred. • Exp. } are the values taken by the expression Exp. when the variables introduced by Decl. take all possible values which make the Pred. true.

The above set Sq might be read as: "select all those integers which lie between 1 and 20 inclusive, and form the set of their squares".

Omissions

If the predicate P is omitted it is taken to be True i.e. it does not constrain the variables in the expression.

$T == \{ x: \mathbb{Z} \bullet x^2 \}$

is the set of squares of all integers.

If the expression E is omitted, it is taken to be the variable(s) in the Declaration.

The set $\{ i: \mathbb{N} \mid i \leqslant 30 \}$

$= \{ i: \mathbb{N} \mid i \leqslant 30 \bullet i \}$

You should always use the latter format.

Tuples and Product Types

We will meet tuples later in a discussion on relations.

A tuple is an ordered collection of two or more objects. Examples are common in records : (Name, Address) is an ordered couple; and in such mathematical constructs as vectors and co-ordinate points . If to the couple we add a third component PhoneNo we get an ordered triple (Name, Address, PhoneNo), and so on.

Characteristic Tuple

The characteristic tuple is the tuple formed from the variables in the declaration part of a set comprehension, in the order of declaration.

In $K == \{ a: \mathbb{N},\ b: \mathbb{Z} \mid a \leqslant 10 \wedge |b| \leqslant 10 \}$ the characteristic tuple is (a,b).

The default Expression in a set comprehension is the characteristic tuple of the set:

$K = \{ a: \mathbb{N},\ b: \mathbb{Z} \mid a \leqslant 10 \wedge |b| \leqslant 10 \bullet (a,b) \}$