

# Multidimensional Scaling

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# Overview

- Classical Problem
  - Why is it a problem?
  - Principle Coordinate Analysis
  - Relationship to Principle Component Analysis
- General Problem
  - Metric vs Non-metric

# Classical Multidimensional Scaling

# A Tale of Three Cities

Imagine we have three cities:  $A$ ,  $B$ , and  $C$

We wish to make a map of our cities, *but* we only know *distances* and not *locations*.

How do we place them on the map?

→ How do we represent distances?

## Distance Matrix

We can introduce the distance matrix  $D$ , where, given a measure of distance  $d$ ,

$$D = \{d_{ij}\} = \{d(\vec{r}_i, \vec{r}_j)\}. \quad (1)$$

For our city example, where  $\vec{r}_i = (x_i, y_i)^T$ , we'll use

$$d(\vec{r}_B, \vec{r}_A) = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2} \quad (2)$$

$$= \|\vec{r}_B - \vec{r}_A\| \quad (3)$$

## Example - City Distances

For our cities  $d_{ab} = d(\vec{r}_A, \vec{r}_B)$ , so

$$D = \begin{bmatrix} d(\vec{r}_A, \vec{r}_A) & d(\vec{r}_A, \vec{r}_B) & d(\vec{r}_A, \vec{r}_C) \\ d(\vec{r}_B, \vec{r}_A) & d(\vec{r}_B, \vec{r}_B) & d(\vec{r}_B, \vec{r}_C) \\ d(\vec{r}_C, \vec{r}_A) & d(\vec{r}_C, \vec{r}_B) & d(\vec{r}_C, \vec{r}_C) \end{bmatrix}. \quad (4)$$

If  $\vec{r}_A = (1, 1)^T$ ,  $\vec{r}_B = (1, 2)^T$ ,  $\vec{r}_C = (2, 1)^T$ , then

$$D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 0 \end{bmatrix}. \quad (5)$$

## Rephrase the Problem

Instead of looking for the original vectors  $\{\vec{r}_i\}_{i=1}^n$ , we instead wish to find the set of vectors  $\{\vec{z}_i\}_{i=1}^n$  s.t. they minimize

$$Stress_D = \sum_{i,j} (d_{ij} - d(\vec{z}_i, \vec{z}_j))^2. \quad (6)$$

In other words, find a set of vectors which have the same distances as the original vectors.

## Difficulties Arise

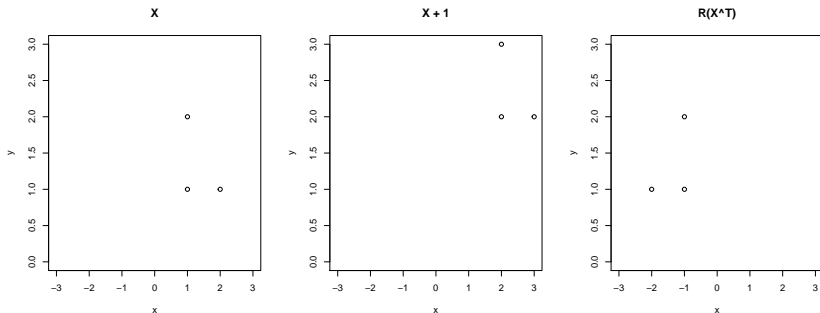
Depending on the definition of  $d(\vec{z}_i, \vec{z}_j)$ , Eq.9 can be difficult to solve analytically.

Additionally. . .



# Unique Selections

... translations and rotations do not affect the distances!



Multiple solutions for a single distance matrix.

## Altered Problem

Consider instead the distance matrix  $B$  where

$$B = \{b_{ij}\} = \{\langle \vec{x}_i - \bar{x}, \vec{x}_j - \bar{x} \rangle\}. \quad (7)$$

In otherwise  $B$  is a matrix of mean centered inner products of  $\vec{x}_i$ .

Alternatively, for  $X \in \mathbb{R}^{n \times p}$  and  $X_{i\cdot} = \vec{x}_i - \bar{x}$

$$B = XX^T \quad (8)$$

## Altered Problem Cont.

Let review our goal – minimize

$$\sum_{i,j} (d_{ij} - d(\vec{z}_i, \vec{z}_j))^2. \quad (9)$$

For  $B$

$$\text{Stress} = \sum_{i,j} (b_{ij} - d(\vec{z}_i, \vec{z}_j))^2 \quad (10)$$

$$= \sum_{i,j} (\langle \vec{x}_i - \bar{x}, \vec{x}_j - \bar{x} \rangle - d(\vec{z}_i, \vec{z}_j))^2 \quad (11)$$

## Simple Minimum

It becomes evident that given

$$d(\vec{z}_i, \vec{z}_j) = \langle \vec{z}_i, \vec{z}_j \rangle \quad (12)$$

the minimum of

$$Strain_B = \sum_{i,j} (\langle \vec{x}_i - \bar{x}, \vec{x}_j - \bar{x} \rangle - d(\vec{z}_i, \vec{z}_j))^2 \quad (13)$$

occures when  $\{\vec{z}_i\}_{i=1}^n = \{\vec{x}_i - \bar{x}\}$ .

## Decompose Our Solution

So for  $B = XX^T$  we know finding  $X$  directly is a solution.

Since  $B$  is symmetric and semi-definite

$$B = E\Lambda E^{-1} \tag{14}$$

$$XX^T = E\Lambda^{1/2}\Lambda^{1/2}E^{-1}, \tag{15}$$

$E_m$  is the matrix of eigenvectors of  $B$  and  $\Lambda$  is the diagonal matrix of eigenvalues.

Using Eq. 14, we find  $X = E\Lambda^{1/2}$ .

## Our Current Solution

We now know how to find a solution for  $B$  given  $d(\vec{z}_i, \vec{z}_j) = \langle \vec{z}_i, \vec{z}_j \rangle$ .

However, original problem was  $D = \{\|\vec{r}_i - \vec{r}_j\|\}$

We now ask if there exists a relationship between  $B$  and  $D$ .

## Law of Cosines

Recall

$$\|\vec{r}_i - \vec{r}_j\|^2 = \|\vec{r}_i - \bar{r}\|^2 + \|\vec{r}_j - \bar{r}\|^2 - 2\langle \vec{r}_i - \bar{r}, \vec{r}_j - \bar{r} \rangle \quad (16)$$

Rearranging we find

$$\langle \vec{r}_i - \bar{r}, \vec{r}_j - \bar{r} \rangle = -\frac{\|\vec{r}_i - \vec{r}_j\|^2 - \|\vec{r}_i - \bar{r}\|^2 - \|\vec{r}_j - \bar{r}\|^2}{2}. \quad (17)$$

Perform this transform on  $D$ ?

## Centering Matrix

$C_n$  is the centering matrix defined by

$$C_n = I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \quad (18)$$

where  $\mathbf{1}_n$  is the column vectors of 1s.

For a vector  $v$

$$C_n v = I_n v - \frac{1}{n} \mathbf{1}_n v \mathbf{1}_n^T \quad (19)$$

$$= v - \bar{v} \quad (20)$$



## Solution to Classical MDS

Now we can relate  $B$  to  $D$  by

$$B = -\frac{1}{2} C_n D^2 C_n. \quad (21)$$

So the  $\vec{z}_i$  that minimize

$$Stress_B = \sum_{ij} (b_{ij} - \langle \vec{z}_i, \vec{z}_j \rangle)^2 \quad (22)$$

are given by  $X = E\Lambda^{1/2}$ .