RHYTHMIC AND MELODIC MOTZKIN NUMBERS

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ABSTRACT. We show an interpretation of the Euclidean rhythm $E(k,n)$ on a subset of elements enumerated by the Motzkin numbers $\mathcal{M}_n$.

1. Introduction

Given n equidistant points on a circle, the Motzkin numbers enumerate $\mathcal{M}_n-$ways to draw chords between those $n-$points, without intersecting.

Inspired by the geometric representation of Euclidean rhythms [DGMM+09] we find that this type of circular combinatorial object is apt to be interpreted musically, associating rhythms and musical notes with it.

2. Idea

Definition 1. We call the Motzkin-Euclidean number $\mathcal{M}_{E(k,n)}$ the subset of elements enumerated by the Motzkin numbers $\mathcal{M}_n$ that represent a Euclidean rhythm.

Example 1. We list the following Euclidean elements in subset rhythmic notation [DGMM+09, p.430] and bits for a single chord: $\mathcal{M}_{E(2,3)} = \{0,2\}_3 = (1,0,1)$, $\mathcal{M}_{E(2,4)} = \{0,2\}_4 = (1,0,1,0)$, $\mathcal{M}_{E(2,5)} = \{0,2\}_5 = (1,0,1,0,0)$, $\mathcal{M}_{E(2,7)} = \{0,3\}_7 = (1,0,0,1,0,0,0)$.

In $\mathcal{M}_{E(2,3)}$ we could consider the rotations $\{1,2\}_3$ y $\{0,1\}_3$; in $\mathcal{M}_{E(2,4)}$, $\{1,3\}_4 = (0,1,0,1)$; in $\mathcal{M}_{E(2,5)}$, $\{1,4\}_5 = (0,1,0,0,1)$, $\{0,3\}_5 = (1,0,0,1,0)$, $\{2,4\}_5 = (0,0,1,0,1)$, $\{1,3\}_5 = (0,1,0,1,0)$.

Example 2. We can associate to each Motzkin-Euclidean element a melodic interpretation, that is, an $n-$scale. If we take $n = 5$, we can associate a major pentatonic scale over C, and have $(C,0,E,0,0)$ and its possible rotations. For $n = 7$, we would associate a diatonic or modal scale. For example, a Lydian scale on D, $\mathcal{M}_{E(2,7)} = \{0,3\}_7 = (D,0,0,G\sharp,0,0,0)$.

For the case of two chords $k = 4$ we have the rhythm-melody or, equivalently, an Amaj7:

$$\mathcal{M}_{E(4,7)} = \{0,3\}_7 = (1,0,1,0,1,0,1) = (A,0,C\sharp,0,E,0,G\sharp)$$

In total, the following Motzkin-Euclidean elements are configured, where r is the rotation on Sonic Pi:

$$\begin{aligned}
\mathcal{M}_{E(4,7)} &= \{0,2,4,6\}_7, r = 0,\\
\mathcal{M}_{E(4,7)} &= \{0,1,3,5\}_7, r = 6,\\
\mathcal{M}_{E(4,7)} &= \{1,2,4,6\}_7, r = 5,\\
\mathcal{M}_{E(4,7)} &= \{0,2,3,5\}_7, r = 4,\\
\mathcal{M}_{E(4,7)} &= \{1,3,4,6\}_7, r = 3,\\
\mathcal{M}_{E(4,7)} &= \{0,2,4,5\}_7, r = 2,\\
\mathcal{M}_{E(4,7)} &= \{1,3,5,6\}_7, r = 1.
\end{aligned}$$

Following the constructive pattern of chords over a greater number of points, we have the following

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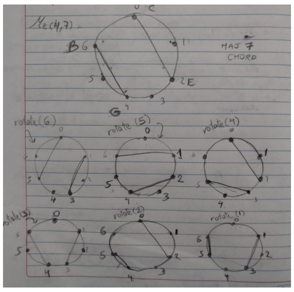
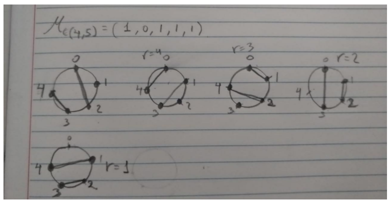
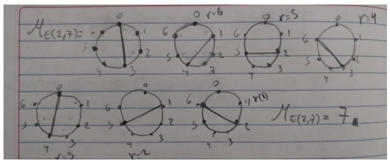
Proposition 1. Let $k, n \in \mathbb{N}$, with k even. Given a Motzkin-Euclidean number $\mathcal{M}_{E(k,n)}$, there is an euclidean rhythm for every $k < n$, except rotations.

Proof. See previous lists. Geometric proof for the cases $k = 2, 4$ and $n = 5, 7$ are given below:







□

3. Conclusions and future work

The limitations of the object are that although we could have m-chords for a rhythm of $n-$pulses, the number k is restricted to even numbers. On the other hand, according to the constructive premise of $\mathcal{M}_n$ we can see that some operations such as complementation [GMTT09, p.19] need the intersection of chords, so from this point of view they remain disabled.

In future work it would be worth investigating which operations are feasible to apply from this geometric approach and its computational implementation.

References

[DGMM+09] Erik D Demaine, Francisco Gomez-Martin, Henk Meijer, David Rappaport, Perouz Taslakian, Godfried T Toussaint, Terry Winograd, and David R Wood. The distance geometry of music. *Computational geometry*, 42(5):429–454, 2009.

[GMTT09] Francisco Gómez-Martín, Perouz Taslakian, and Godfried Toussaint. Interlocking and euclidean rhythms. *Journal of Mathematics and Music*, 3(1):15–30, 2009.