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# Experimental Validation of PID and LQR Control Techniques for Stabilization of Cart Inverted Pendulum System

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***Abstract*— This paper describes the design and implementation of control strategies for the stabilization of a Cart Inverted Pendulum system. An experimental model of the system is considered for the design of controllers. The real-time implementation is done by employing a two-loop PID controllers for the system where the gain values are pre-determined by using Frequency Response and Linear-Quadratic Regulator(LQR) methods. A comparison is made between the proposed design strategies in terms of steady state and transient performance parameters. It is observed that Linear-Quadratic method exhibits superior performance as compared to frequency response method.**

***Keywords*— Cart Inverted Pendulum system; PID Controller; LQR Controller; Full State Feedback**

## I. Introduction

The Inverted Pendulum System is among the most complex systems for its inherent instability and is one of the benchmark control problems given by international federation of automatic control theory committee (IFAC)[1]. The control of this system has been a thoroughly researched problem in the field of systems engineering and is now considered as a benchmark system to develop new control strategies[2]. The inverted pendulum belongs to a class of underactuated mechanical system which cannot be commanded to follow arbitrary trajectories. It has fewer inputs than the total degree-of-freedom (DOF), and it has two DOF, one for cart motion in the horizontal plane, and another for the angular motion of the pendulum in the vertical plane. However, only the cart position is actuated, and the pendulum is indirectly affected/controlled. This experimental study aims to stabilize the cart inverted pendulum system and pendulum erected in an upward direction such that the position of the cart on the track is controlled in short interval time. The setup of the experimental system[3] is shown in Fig.1, which can be used to determine control laws or control strategies to achieve the desired system response and performance. The performance of the dynamical systems being controlled is desired to be an optimal response by many optimal control strategies such as PID, LQR [4], etc.

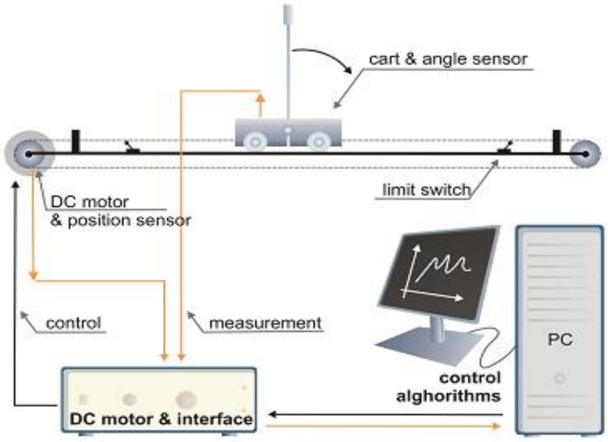


Fig. 1 Experimental setup of cart inverted pendulum system [3]

This paper uses the performance index criteria under the input voltage constraint, and implement the real-time experiment using the obtained optimal conditions. The real-time implementation is performed with two-loop PID controller, by using frequency response analysis and then using LQR optimal control. The two-loop PID controller[5] is designed by deriving the combined closed loop characteristic equation, and then the PID values are obtained by comparing it with the stable closed loop characteristic equation. This stable closed loop characteristic equation is obtained by adding a compensator and changing its parameters until to obtain the desired time and frequency response characteristics with a gain margin of greater than 2 and phase margin are higher than 60 degrees[6].

This paper organized in six sections commencing from Introduction followed by Section II which discusses the mathematical modeling of the system. Section III discusses the PID controller design using frequency response method. Section IV presents the Linear Quadratic Regulator design method using optimal control method. Section V includes the results of both methods and comparison is presented and ends with a conclusion as Section VI.

## II. Modeling of Cart Inverted Pendulum System

Mathematical equations are derived using Newtonian mechanics. The system free body diagram is represented in Fig.2, and the corresponding non-linear equations are derived. The inverted pendulum experimental set-up system parameters are given in Table.I.The set-up is computer controlled and any desired controller to stabilize the system can be implemented through SIMULINK in MATLAB. The feedback signals to the computer and control signals from the computer are all given through the PCI- 1711 A/D card manufactured by Advantech Technologies.

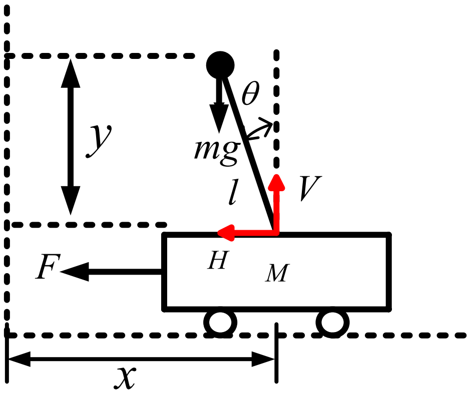

Fig. 2 Free body diagram of Cart Inverted Pendulum System

TABLE I. Cart Inverted Pendulum System Parameters

| Parameter | Value |
|---|---|
| $F$ - Force applied to cart | ±24N |
| $x$ - Cart position from the reference | ±0.5 m |
| $\theta$ - Pendulum angle w.r.t verticle axis | 0.1 rad |
| $g$ - Gravity | 9.81m/s$^2$ |
| $l$ - Length of the pole | 0.38m |
| $M$ - Cart mass | 2.4 Kg |
| $m$ - Pole mass | 0.23 Kg |
| $I$ - Moment of inertia of pole | 0.099Kg/m$^2$ |
| $b$ - Cart friction coefficient | 0.05Ns/m |
| $d$ - Pendulum damping coefficient | 0.05 s /rad |

Newtonian mechanics is used to developing the nonlinear equations for the system. Forces along translational motion and rotational motion are taken into consideration, and thus two coordinate systems are taken into account from Fig.2.

$H$ -Net horizontal force on the cart

$V$ - Net vertical force on the pendulum

$\theta$ - Pendulum angle w.r.t vertical axis

$mg$ - Weight of the bob

$\tau$ - Torque about the center of gravity of the pendulum

$$m\frac{d^2}{dt^2}\left(x - l\sin\theta\right) = -H \qquad (1)$$

$$m\frac{d^2}{dt^2}\left(l\cos\theta\right) = V - mg \qquad (2)$$

Taking moments about the center of gravity of the pendulum,

$$\tau_{net} = I\frac{d^2\theta}{dt^2} \qquad (3)$$

$$-H\cos\theta + V\sin\theta - d\dot{\theta} = I\ddot{\theta} \qquad (4)$$

From the equations (1) and (2), the pole equation is derived as (5)

$$\left(-ml\cos\theta\right)\ddot{x} + \left(I + ml^2\right)\ddot{\theta} + d\dot{\theta} - mgl\sin\theta = 0 \qquad (5)$$

The cart dynamic equations are expressed as

$$F_{net} = M\ddot{x} \qquad (6)$$

$$F + H - b\dot{x} = M\ddot{x} \qquad (7)$$

$$(M+m)\ddot{x} + b\dot{x} - (ml\cos\theta)\ddot{\theta} + (ml\sin\theta)(\dot{\theta})^2 = F \qquad (8)$$

Where equations (5) and (8) represent the nonlinear equations of the system and can be written as equation (9),

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix}-ml\cos\theta & -(I+ml^2)\\ -(M+m) & -ml\cos\theta\end{bmatrix}\begin{bmatrix}mgl\sin\theta - d\dot{\theta}\\ F - \left(ml\sin\theta\right)\left(\dot{\theta}\right)^2 - b\dot{x}\end{bmatrix} \qquad (9)$$

Where $\Delta = m^2l^2\cos\theta - \left(M+m\right)\left(I+ml^2\right)$

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix}ml\cos\theta\left(d\dot{\theta} - mgl\sin\theta\right) + \left(I+ml^2\right)\left(ml\sin\theta(\dot{\theta}^2) + b\dot{x}\right)\\ \left(M+m\right)\left(d\dot{\theta} - mgl\sin\theta\right) + \left(ml\cos\theta\right)\left(ml\sin\theta(\dot{\theta}^2) + b\dot{x}\right)\end{bmatrix} + \frac{1}{\Delta}F\begin{bmatrix}-\left(I+ml^2\right)\\ -\left(ml\cos\theta\right)\end{bmatrix} \qquad (10)$$

The state vector taken as $X = \left[x_1, x_2, x_3, x_4\right]^T = \left[x, \dot{x}, \theta, \dot{\theta}\right]^T$ and linearizing the non-linear differential equation for the cart inverted pendulum system at the unstable equilibrium point $\theta = 0$, and by taking $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and $\dot{\theta}^2 \approx 0$. Thus, the linearized state-space model can be written as,

$$\dot{X} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \frac{\left(I+ml^2\right)b}{\Delta} & \frac{-m^2gl^2}{\Delta} & \frac{mld}{\Delta}\\ 0 & 0 & 0 & 1\\ 0 & \frac{mlb}{\Delta} & \frac{-mgl\left(M+m\right)}{\Delta} & \frac{d\left(M+m\right)}{\Delta}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + F\begin{bmatrix}0\\ -\frac{(I+ml^2)}{\Delta}\\ 0\\ -\frac{ml}{\Delta}\end{bmatrix} \qquad (11)$$

Where $\Delta$ is $\left(ml\right)^2 - \left(I+ml^2\right)\left(M+m\right)$. The state-variables of the continuous-time linear dynamical system are chosen as $x, \dot{x}, \theta, \dot{\theta}$ where $x$ is the position of the cart and $\theta$ is the angle of the pendulum, which are described by equations (12) and (13).

$$\dot{x} = Ax + Bu \tag{12}$$

$$y = Cx \tag{13}$$

By substituting the cart inverted pendulum system parameters, the obtained $A, B, C$ and $D$ values are given in (14), (15),(16) and (17).

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.02 & 0.22 & -0.0013 \\ 0 & 0 & 0 & 1 \\ 0 & -0.013 & 6.632 & -0.034 \end{bmatrix} \tag{14}$$

$$B = \begin{bmatrix} 0 \\ 0.388 \\ 0 \\ 0.257 \end{bmatrix} \tag{15}$$

$$C = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16}$$

$$D = \begin{bmatrix} 0 \end{bmatrix} \tag{17}$$

The transfer functions of the cart and pendulum are shown in equation (18) and (19)

$$T_x = \frac{X}{F} = \frac{0.132\left(s^2 + 0.0378s - 12.98\right)}{0.339\left(s^3 + 0.058s^2 - 13.27s - 0.253\right)} \tag{18}$$

$$T_\theta = \frac{\theta}{F} = \frac{0.0874s^2}{0.339\left(s^3 + 0.058s^2 - 13.27s - 0.253\right)} \tag{19}$$

## III. Design of PID Controllers for Cart Inverted Pendulum System

The designed two loop control strategy structure diagram for cart inverted pendulum system is shown in Fig.3.

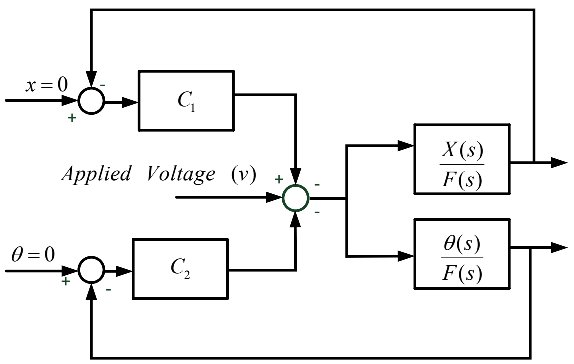


Fig. 3 Closed-loop PID structure diagram

$$C_1 = K_{p1} + \frac{K_{i1}}{s} + K_{d1}s \tag{20}$$

$$C_2 = K_{p2} + \frac{K_{i2}}{s} + K_{d2}s \tag{21}$$

Moreover, the closed loop transfer functions are obtained from Fig.3.

$$\begin{bmatrix} \frac{X}{F} \\ \frac{\theta}{F} \end{bmatrix} = \frac{\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}}{1 + T_1C_1 + T_2C_2} \tag{22}$$

Where F is the applied force, $T_1$ and $T_2$ are the respective transfer functions of $x$ and $\theta$ respectively, $1 + T_1C_1 + T_2C_2$ is the characteristic equation of the closed loop system. After substituting the equations (20) and (21) in (22), the net characteristic equation for the closed-loop is rewritten in equation (23).

$$\begin{aligned} &s^5 + \left(0.388K_{d1} + 0.257K_{d2} + 0.0581\right)s^4 + \left(0.0147K_{d1} + 0.388K_{p1} + 0.257K_{p2} - 6.63\right)s^3 \\ &+ \left(0.388K_{i1} + 0.257K_{i2} - 2.52K_{d1} + 0.0147K_{p1} - 0.126\right)s^2 + \left(0.0147K_{i1} - 2.52K_{p1}\right)s \\ &-2.52K_{i1} = 0 \end{aligned} \tag{23}$$

The Nyquist plot of the pendulum is shown in Fig. 4, which is used to tune the PID controller, where a compensator is assumed to be in parallel, the poles and zeros are added to the compensator until we get the desired frequency and time response. Nyquist stability criterion, that is Z=P+N, where Z gives the closed loop poles in the right half plane, P is the number of zeros of open loop transfer function, and N is the number of encirclements of -1 of the Nyquist plot in the clockwise direction. Z should be zero for the closed loop system to be stable.

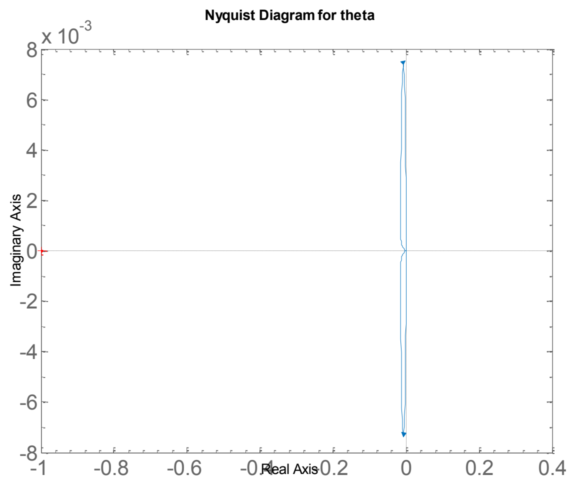


Fig.4 Nyquist plot of pendulum angle

The desired compensator transfer function is obtained as

$$T_{ctr} = \frac{200s^2 + 920s + 133.5}{s} \tag{24}$$

The net transfer function of the closed loop is calculated from the above compensator which gives a stable system. The obtained characteristic equation can be compared with equation (23) to obtain the PID values for both the controllers. The net closed loop characteristic equation of the closed loop system is assumed as

$$s^5 + p_1s^4 + p_2s^3 + p_3s^2 + p_4s + p_5 = 0 \tag{25}$$

To find the least square norm solution, let coefficient matrix be M, controller parameter matrix be P, and the stable characteristic equation matrix be $S = MP$ .

$$P = \left(M^T M\right)^{-1}\left(M^T S\right) \tag{26}$$

Rearranging the equations (23),(25) and (26), we obtain equation (27) as

$$\begin{bmatrix} 0 & 0 & 0.388 & 0 & 0 & 0.257 \\ 0.388 & 0 & 0.0147 & 0.257 & 0 & 0 \\ 0.0147 & 0.388 & -2.52 & 0 & 0.257 & 0 \\ -2.52 & 0.0147 & 0 & 0 & 0 & 0 \\ 0 & -2.52 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} K_{p1} \\ K_{i1} \\ K_{d1} \\ K_{p2} \\ K_{i2} \\ K_{d2} \end{bmatrix} = \begin{bmatrix} p_1 - 0.0581 \\ p_2 + 6.63 \\ p_3 + 0.126 \\ p_4 \\ p_5 \end{bmatrix} \tag{27}$$

The net characteristic equation of the stable closed loop system after solving equation (27), we obtain

$$s^5 + 50.1s^4 + 229s^3 + 362s^2 + 1399s + 201 = 0 \tag{28}$$

By comparing the equations (28) and (23), the PID gain values are obtained as:

$$K_{p1} = -520; K_{i1} = -79.7; K_{d1} = -78; K_{p2} = 1711; K_{i2} = 794; \\ K_{d2} = 312 .$$

## IV. Design of LQR controllers for cart inverted pendulum system

To design a full-state feedback regulator [7] for the plant described in equation (12) such that the control input vector is chosen as equation (29)

$$u(t) = -Kx(t) \tag{29}$$

For an infinite final time, the quadratic objective function is given in equation (30).

$$J_\infty = \int_{t_o}^{\infty} [x^T Q x + u^T R u] dt \tag{30}$$

Here $u^T Ru$ is control energy in quadratic form, $R$ is a cost matrix which puts a penalty on the control input, $x^T Qx$ is transient energy and $Q$ is a state weighting matrix, which, puts a penalty on the state matrix. Both matrices $Q$ and $R$ are diagonal matrices whose entries put a penalty on respective states and input energy. The feedback gain matrix ( $K$ ) is solved for the scalar objective function ( $J$ ) by substituting equation (29) into (12) to obtain equation (31)

$$\dot{x} = \left[A - BK\right]x(t) = \left[A_{Closedloop}\right]x(t) \tag{31}$$

The optimal feedback gain matrix (K) in equation (32) is calculated by using MATLAB command, $K = lqr(A, B, Q, R)$ .

$$K = \begin{bmatrix} -5 & -9.22 & 107 & 41.6 \end{bmatrix} \tag{32}$$

The diagonal entries of matrix Q are chosen by considering the range of each state and taking the inverse square. The position is bounded within 5cm, while the angle is bounded by 0.1 rad, thus giving the respective entries as 25 and 25 respectively. Derivatives of position and angle function in the $Q$ matrix are penalized by unity. As the system only has a single input, the $R$ matrix will only contain a single entry which is taken as 0.1 by considering the inverse square of the range of dc motor voltage. Closed-loop responses are presented with $R = 0.1$ , and $Q = diag(1, 25, 1, 25)$ . Controllability and observability matrices with full rank are given in equations (33) and (34), which implies that all states are simultaneously controllable and an observer design is possible.

$$C = \begin{bmatrix} 0 & 0.377 & -0.0071 & 0.2003 \\ 0.377 & -0.0071 & 0.2003 & -0.0075 \\ 0 & 0.367 & -0.0068 & 3.6885 \\ 0.3670 & -0.0068 & 3.6885 & -0.0718 \end{bmatrix} \tag{33}$$

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.019 & 0.2026 & 0 \\ 0 & -0.019 & 6.43 & 0 \\ 0 & 0.0004 & -0.002 & 0 \\ 0 & 0.0002 & -0.02 & 6.4 \end{bmatrix} \tag{34}$$

Using equation (32), eigenvalues of a matrix $(A - BK)$ are evaluated which gives the four poles of the desired closed loop system as in equation (35)

$$M = \begin{bmatrix} -2.585 + i0.206 \\ -2.585 - i0.206 \\ -0.992 + i0.942 \\ -0.992 - i0.942 \end{bmatrix} \tag{35}$$

For appropriate controller gain values, the fifth pole is now placed at -10. The characteristic equation derived from newly obtained poles is

$$s^5 + 17.2s^4 + 92.2s^3 + 232s^2 + 279s + 157 = 0 \tag{36}$$

By comparing the coefficients of equation (35) with (23), the PID values are obtained as $K_{p1} = -111$ , $K_{i1} = -62$ , $K_{d1} = -44$ , $K_{p2} = 555$ , $K_{i2} = 563$ and $K_{d2} = 135$ .

It is observed that the values obtained of controller gain values are considerably low as compared to what we got using frequency response tuning, implying that the LQR method gives the minimum energy gain values for the PID controllers.

## V. Comparative analysis and results

### *A. PID tuning using frequency response*

The experimental response of cart and pendulum is shown in Fig.5 and Fig.6 respectively. The cart inverted pendulum system stabilized by tuning of PID controllers with frequency response, it is observed that the cart and pendulums are stabilized after 63 seconds but with more noise.

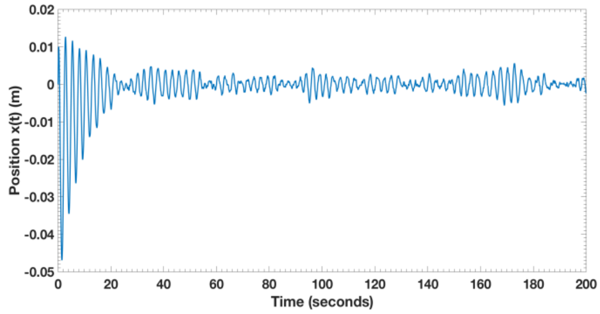


Fig.5 Experimental response of cart position

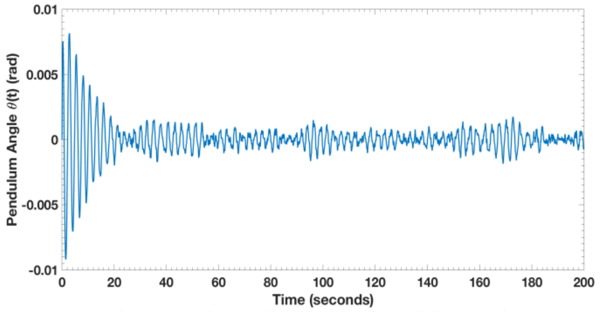


Fig.6 Experimental response of pendulum angle

### B. *PID tuning using LQR method*

The experimental response of cart and pendulum is shown in Fig.7 and Fig.8 respectively.

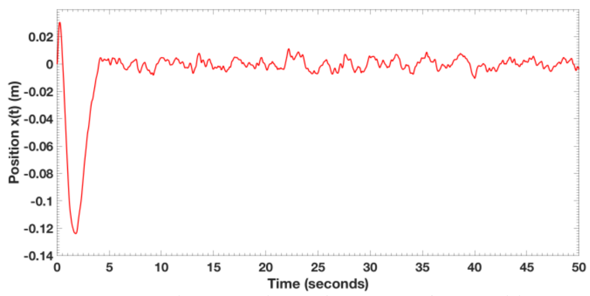


Fig.7 Experimental response of cart position

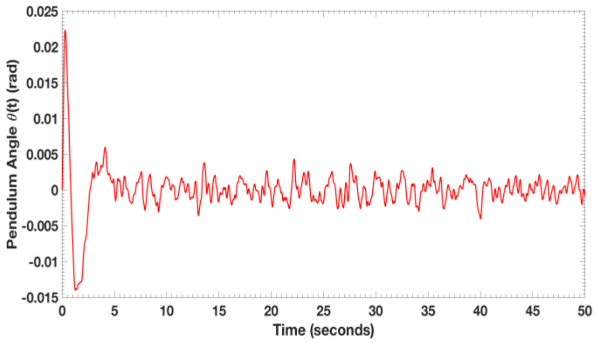


Fig. 8 Experimental response of Pendulum angle

The cart inverted pendulum system stabilized by tuning of PID controllers with LQR method, it is observed that the cart and pendulums are stabilized after 6 seconds. The comparative results of PID tuning with frequency response and LQR methods are given in Table.II.

TABLE II. COMPARATIVE RESULTS

| Performance Parameters | Frequency Response | | LQR | |
|---|---|---|---|---|
| | x(t) | Θ(t) | x(t) | Θ(t) |
| **Settling Time (sec)** | 56.5 | 62.9 | 5.9 | 5.5 |
| **Overshoot (%)** | 13.21 | 8.45 | 23.48 | 18.28 |
| **Gain Margin (dB)** | 28.5 | 9.49 | 36.6 | 61.3 |
| **Phase Margin (dB)** | 82 | 68.7 | 50.2 | 62 |
| **Gain Crossover frequency (rad/sec)** | 3.94 | 1.41 | 0.789 | 0.791 |
| **Phase Crossover frequency (rad/sec)** | 4.876 | 2.43 | 1.21 | 1.35 |

## VI. CONCLUSION

The stabilization of the Cart inverted pendulum system by two loops PID controllers tuned using frequency response and Linear Quadratic Regulator. While all design methodologies met the goal, the Linear Quadratic methods certainly lead to a better designed closed loop system. All methods lead to satisfactory transient response, but based on the stability margins and settling time, a system designed by frequency response gives an inferior closed loop system. Between these methods, the LQR gives the minimum energy gain values for the PID controllers, making it practical for real-time implementation.


## ACKNOWLEDGMENT

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