

Investigating Gaussian Integers

Problem 153



As we all know the equation $x^2=-1$ has no solutions for real x .
If we however introduce the imaginary number i this equation has two solutions: $x=i$ and $x=-i$.
If we go a step further the equation $(x-3)^2=-4$ has two complex solutions: $x=3+2i$ and $x=3-2i$.
 $x=3+2i$ and $x=3-2i$ are called each others' complex conjugate.
Numbers of the form $a+bi$ are called complex numbers.
In general $a+bi$ and $a-bi$ are each other's complex conjugate.

A Gaussian Integer is a complex number $a+bi$ such that both a and b are integers.
The regular integers are also Gaussian integers (with $b=0$).
To distinguish them from Gaussian integers with $b \neq 0$ we call such integers "rational integers."
A Gaussian integer is called a divisor of a rational integer n if the result is also a Gaussian integer.

If for example we divide 5 by $1+2i$ we can simplify $\frac{5}{1+2i}$ in the following manner:
Multiply numerator and denominator by the complex conjugate of $1+2i$: $1-2i$.
The result is $\frac{5}{1+2i} = \frac{5}{1+2i} \frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$.

So $1+2i$ is a divisor of 5.
Note that $1+i$ is not a divisor of 5 because $\frac{5}{1+i} = \frac{5}{2} - \frac{5}{2}i$.
Note also that if the Gaussian Integer $(a+bi)$ is a divisor of a rational integer n , then its complex conjugate $(a-bi)$ is also a divisor of n .

In fact, 5 has six divisors such that the real part is positive: $\{1, 1+2i, 1-2i, 2+i, 2-i, 5\}$.
The following is a table of all of the divisors for the first five positive rational integers:

n	Gaussian integer divisors with positive real part	Sum $s(n)$ of these divisors
1	1	1
2	1, $1+i$, $1-i$, 2	5
3	1, 3	4
4	1, $1+i$, $1-i$, 2, $2+2i$, $2-2i$, 4	13
5	1, $1+2i$, $1-2i$, $2+i$, $2-i$, 5	12

For divisors with positive real parts, then, we have: $\sum_{n=1}^5 s(n) = 35$.

For $\sum_{n=1}^{10^5} s(n) = 17924657155$.

What is $\sum_{n=1}^{10^8} s(n)$?