MINIMAL SURFACES AND SOAP FILMS

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CONTENTS

1. INTRODUCTION

Soap film is an ultra-thin liquid membrane formed by soap solution, whose shape is determined by surface tension and boundary conditions. In mathematics and physics, soap films serve as a classic real-world example of minimal surfaces.

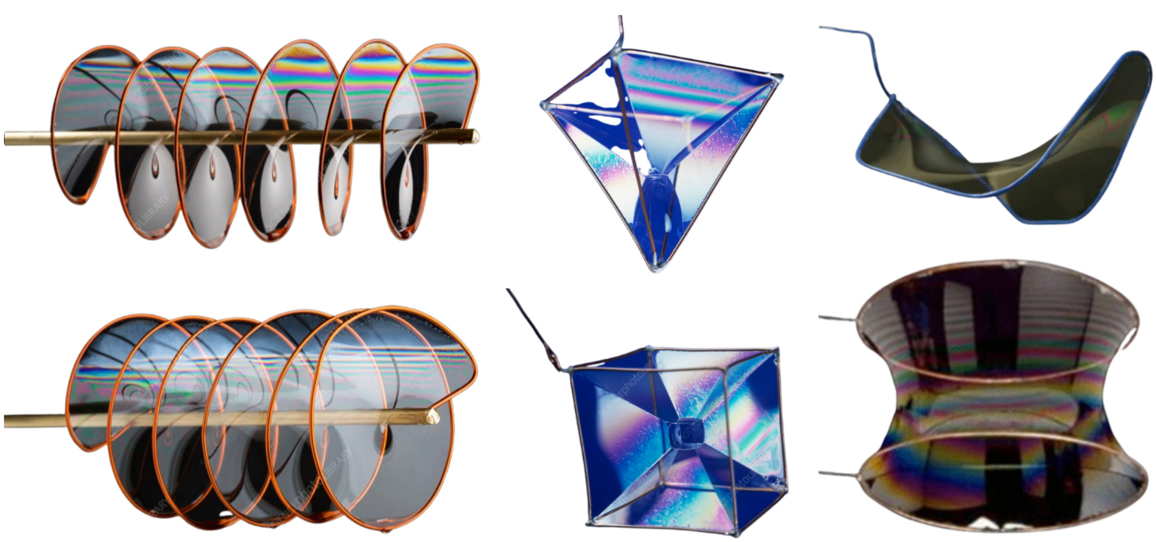


FIGURE 1. Soap film configurations

When a wire frame (boundary curve) is dipped into soapy water, surface tension causes the liquid film to automatically contract to the smallest possible area, forming a stable shape. Mathematically, soap films are surfaces with zero mean curvature.

Definition 1. A *minimal surface* is a surface whose mean curvature is zero everywhere.

Equivalently, if a surface S has least area among all surfaces with the same boundary curve, then S is a minimal surface.

The main goal of this note is to study the Plateau's Problem: finding a surface of minimal area with a fixed curve as its boundary. To numerically compute minimal surfaces, we need to discretize the continuous problem. Geodesic patches provide a natural discrete framework: by decomposing the surface into local patches with geodesic boundaries, we can transform the infinite-dimensional surface optimization problem into a finite-dimensional mesh optimization problem.

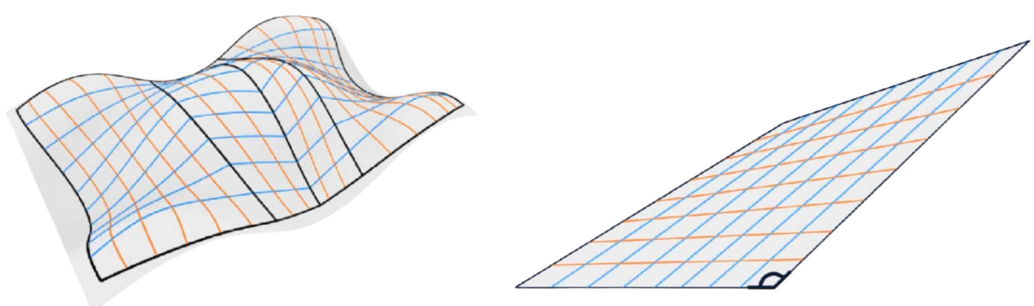


FIGURE 2. Geodesic grid structures

Definition 2. A curve γ on a surface S is called a *geodesic* if $\gamma''(t)$ is zero or perpendicular to the tangent plane of the surface at the point $\gamma(t)$, i.e., parallel to its unit normal, for all values of the parameter t.

Definition 3. Let γ be a curve on a surface S and let v be a tangent vector field along γ. The *covariant derivative* of v along γ is the orthogonal projection $\nabla_\gamma v$ of dv/dt onto the tangent plane $T_{\gamma(t)}S$ at a point $\gamma(t)$. v is said to be *parallel along* γ if $\nabla_\gamma v = 0$ at every point of γ.

Equivalently, γ is a geodesic if and only if its tangent vector γ' is parallel along γ.

2. EXAMPLES OF MINIMAL SURFACES

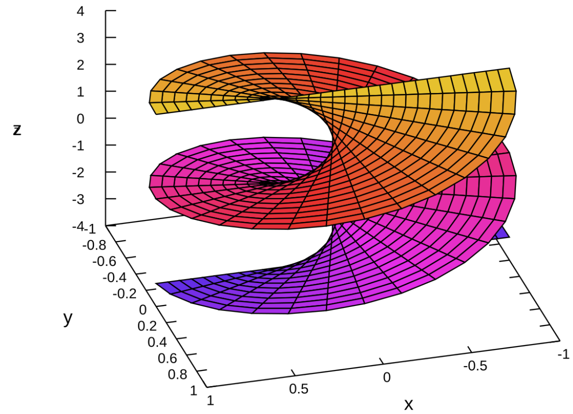
Example 4. (Helicoid) The helicoid can be parametrized as

$$\sigma(u, v) = (v \cos u, v \sin u, \lambda u),$$

where λ is a constant. We will show that every helicoid is a minimal surface.

Proof. First Derivatives:

$$\begin{aligned}\sigma_u &= (-v \sin u,\ v \cos u,\ \lambda),\\ \sigma_v &= (\cos u,\ \sin u,\ 0).\end{aligned}$$



Second Derivatives:

$$\begin{aligned}\sigma_{uu} &= (-v\cos u,\ -v\sin u,\ 0),\\ \sigma_{uv} &= (-\sin u,\ \cos u,\ 0),\\ \sigma_{vv} &= (0,\ 0,\ 0).\end{aligned}$$

The first fundamental form has coefficients:

$$\begin{aligned}E &= \sigma_u\cdot\sigma_u = (-v\sin u)^2 + (v\cos u)^2 + \lambda^2 = v^2+\lambda^2,\\ F &= \sigma_u\cdot\sigma_v = (-v\sin u)(\cos u) + (v\cos u)(\sin u) + \lambda\cdot 0 = 0,\\ G &= \sigma_v\cdot\sigma_v = (\cos u)^2 + (\sin u)^2 + 0^2 = 1,\end{aligned}$$

$$\mathbf{u} = \frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|} = \frac{1}{\sqrt{\lambda^2+v^2}}(-\lambda\sin u,\ \lambda\cos u,\ -v).$$

The second fundamental form has coefficients:

$$\begin{aligned}L &= \mathbf{u}\cdot\sigma_{uu} = \frac{1}{\sqrt{\lambda^2+v^2}}(-\lambda\sin u(-v\cos u) + \lambda\cos u(-v\sin u) + (-v)\cdot 0) = 0,\\ M &= \mathbf{u}\cdot\sigma_{uv} = \frac{1}{\sqrt{\lambda^2+v^2}}(-\lambda\sin u(-\sin u) + \lambda\cos u(\cos u) + (-v)\cdot 0) = \frac{\lambda}{\sqrt{\lambda^2+v^2}},\\ N &= \mathbf{u}\cdot\sigma_{vv} = 0.\end{aligned}$$

Thus,

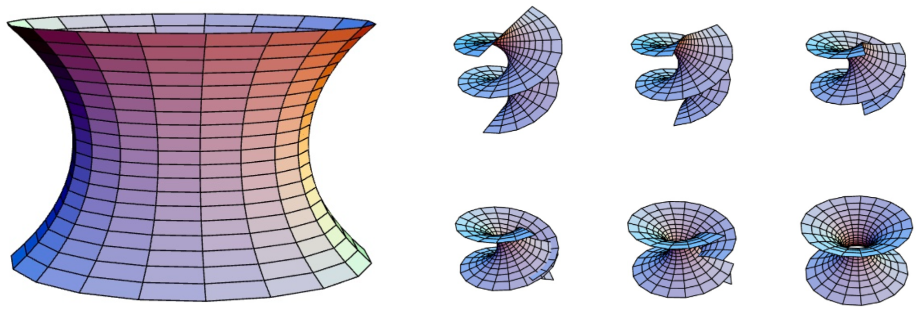
$$K = \frac{LN-M^2}{EG-F^2} = \frac{0\cdot 0 - \left(\frac{\lambda}{\sqrt{\lambda^2+v^2}}\right)^2}{(v^2+\lambda^2)\cdot 1 - 0^2} = -\frac{\lambda^2}{(\lambda^2+v^2)^2},$$

$$H = \frac{GL+EN-2FM}{2(EG-F^2)} = \frac{1\cdot 0 + (v^2+\lambda^2)\cdot 0 - 2\cdot 0\cdot\frac{\lambda}{\sqrt{\lambda^2+v^2}}}{2(v^2+\lambda^2)} = 0,$$

$$k_1 = H+\sqrt{H^2-K} = \frac{\lambda}{\lambda^2+v^2},\quad k_2 = H-\sqrt{H^2-K} = -\frac{\lambda}{\lambda^2+v^2}.$$

So the Helicoid is a minimal surface. ■

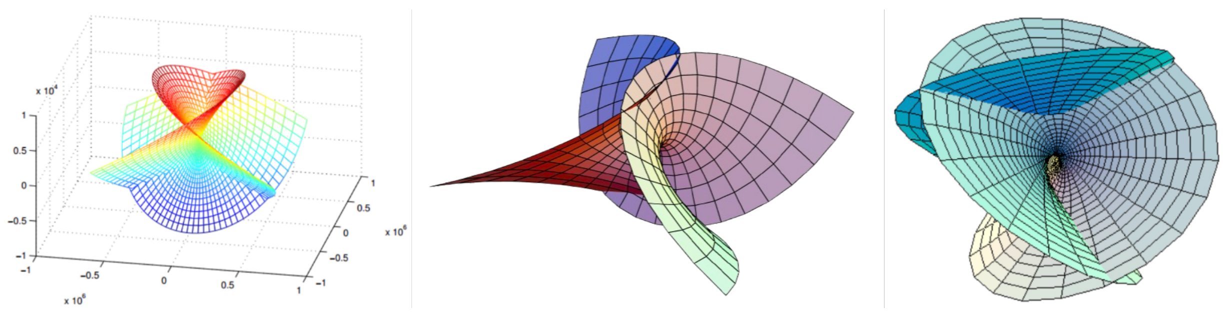
Note that: the Helicoid can be continuously deformed into a catenoid with $c=1$ (parametrized by $\sigma(u,v) = (c\cosh(\frac{v}{c})\cos u, c\cosh(\frac{v}{c})\sin u), v))$, which is also a minimal surface:



Example 5. (Enneper's surface) *Enneper's surface* is

$$\sigma(u, v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2 v,\ u^2 - v^2\right).$$

We will show that this is a minimal surface.



Proof. First Derivatives:

$$\begin{aligned} \sigma_u &= \left(1 - u^2 + v^2,\ 2uv,\ 2u\right), \\ \sigma_v &= \left(2uv,\ 1 - v^2 + u^2,\ -2v\right). \end{aligned}$$

Second Derivatives:

$$\begin{aligned} \sigma_{uu} &= (-2u,\ 2v,\ 2), \\ \sigma_{uv} &= (2v,\ 2u,\ 0), \\ \sigma_{vv} &= (2u,\ -2v,\ -2). \end{aligned}$$

The first fundamental form has coefficients:

$$\begin{aligned} E &= \sigma_u \cdot \sigma_u = (1 - u^2 + v^2)^2 + (2uv)^2 + (2u)^2 = (1 + u^2 + v^2)^2, \\ F &= \sigma_u \cdot \sigma_v = (1 - u^2 + v^2)(2uv) + (2uv)(1 - v^2 + u^2) + (2u)(-2v) = 0, \\ G &= \sigma_v \cdot \sigma_v = (2uv)^2 + (1 - v^2 + u^2)^2 + (-2v)^2 = (1 + u^2 + v^2)^2, \end{aligned}$$

$$\mathbf{u} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{(1 + u^2 + v^2)^{3/2}}\left(-2u(1 + u^2 + v^2),\ 2v(1 + u^2 + v^2),\ (1 + u^2 + v^2)^2\right).$$

The second fundamental form has coefficients:

$$
\begin{aligned}
L &= \mathbf{u}\cdot\sigma_{uu} = \frac{1}{(1+u^2+v^2)^{3/2}}\big(-2u(-2u)+2v(2v)+2(1+u^2+v^2)\big) = 2,\\
M &= \mathbf{u}\cdot\sigma_{uv} = \frac{1}{(1+u^2+v^2)^{3/2}}(-2u(2v)+2v(2u)+0) = 0,\\
N &= \mathbf{u}\cdot\sigma_{vv} = \frac{1}{(1+u^2+v^2)^{3/2}}\big(-2u(2u)+2v(-2v)+(1+u^2+v^2)(-2)\big) = -2.
\end{aligned}
$$

Thus,

$$K = \frac{LN-M^2}{EG-F^2} = \frac{(2)(-2)-0^2}{(1+u^2+v^2)^4} = -\frac{4}{(1+u^2+v^2)^4},$$

$$H = \frac{GL+EN-2FM}{2(EG-F^2)} = \frac{(1+u^2+v^2)^2\cdot 2 + (1+u^2+v^2)^2\cdot(-2)}{2(1+u^2+v^2)^4} = 0,$$

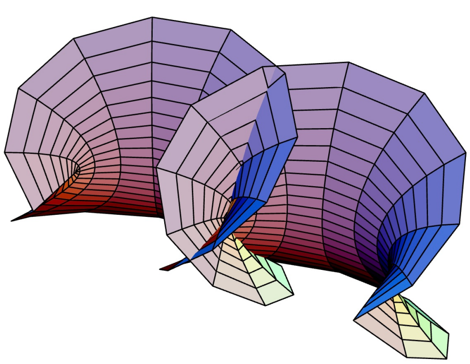
$$k_1 = H+\sqrt{H^2-K} = \frac{2}{(1+u^2+v^2)^2},\quad k_2 = H-\sqrt{H^2-K} = -\frac{2}{(1+u^2+v^2)^2}.$$

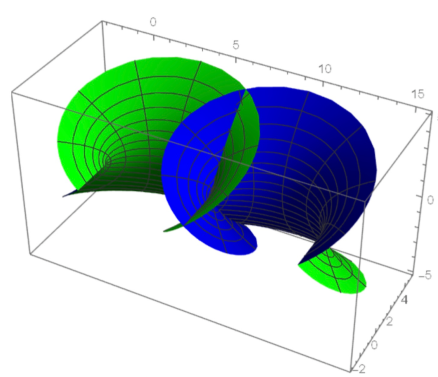
So Enneper's surface is a minimal surface. ■

Example 6. (Catalan's surface) *Catalan's surface* is

$$\sigma(u,v) = \Big(u-\sin u\cosh v,\ 1-\cos u\cosh v,\ -4\sin\frac{u}{2}\sinh\frac{v}{2}\Big).$$

We will show that this is a minimal surface.





Proof. First derivatives:

$$\sigma_u = \Big(1-\cos u\cosh v,\ \sin u\cosh v,\ -2\cos\frac{u}{2}\sinh\frac{v}{2}\Big),$$

$$\sigma_v = \Big(-\sin u\sinh v,\ -\cos u\sinh v,\ -2\sin\frac{u}{2}\cosh\frac{v}{2}\Big).$$

Second Derivatives:

$$\sigma_{uu} = \Big(\sin u \cosh v,\ \cos u \cosh v,\ \sin\frac{u}{2}\sinh\frac{v}{2}\Big),$$

$$\sigma_{uv} = \Big(-\cos u \sinh v,\ -\sin u \sinh v,\ -\cos\frac{u}{2}\cosh\frac{v}{2}\Big),$$

$$\sigma_{vv} = \Big(-\sin u \cosh v,\ -\cos u \cosh v,\ -\sin\frac{u}{2}\sinh\frac{v}{2}\Big).$$

The first fundamental form has coefficients:

$$E = \sigma_u \cdot \sigma_u = 2\cosh^2\left(\frac{1}{2}v\right)(\cosh v - \cos u),$$

$$F = \sigma_u \cdot \sigma_v = 0,$$

$$G = \sigma_v \cdot \sigma_v = 2\cosh^2\left(\frac{1}{2}v\right)(\cosh v - \cos u),$$

$$\mathbf{u} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{\cosh v - \cos u}}(\sin\frac{u}{2}, \cos\frac{u}{2}, \sinh\frac{v}{2}).$$

The second fundamental form has coefficients:

$$L = \mathbf{u} \cdot \sigma_{uu} = -\cosh\left(\frac{1}{2}v\right)\sin\left(\frac{1}{2}u\right),$$

$$M = \mathbf{u} \cdot \sigma_{uv} = \cos\left(\frac{1}{2}u\right)\sinh\left(\frac{1}{2}v\right),$$

$$N = \mathbf{u} \cdot \sigma_{vv} = \cosh\left(\frac{1}{2}v\right)\sin\left(\frac{1}{2}u\right).$$

Thus,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\operatorname{sech}^4\left(\frac{1}{2}v\right)}{8(\cos u - \cosh v)},$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = 0,$$

$$k_1 = H + \sqrt{H^2 - K} = \frac{\operatorname{sech}^2\left(\frac{1}{2}v\right)}{\sqrt{8(\cosh v - \cos u)}},$$

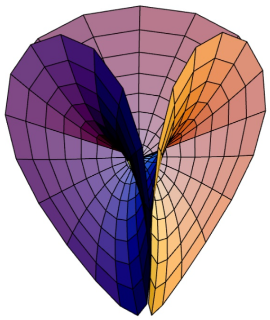
$$k_2 = H - \sqrt{H^2 - K} = -\frac{\operatorname{sech}^2\left(\frac{1}{2}v\right)}{\sqrt{8(\cosh v - \cos u)}}.$$

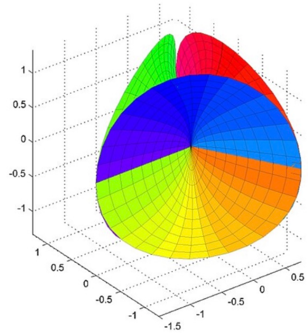
So Catalan's surface is a minimal surface. ■

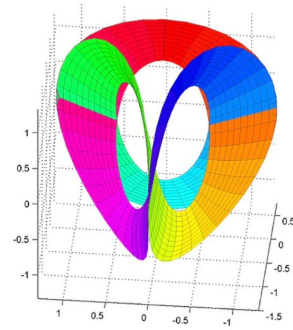
Example 7. (Bour's Surface) *Bour's Surface* is

$$\sigma(r,\theta) = \left(r\cos\theta - \frac{1}{2}r^2\cos(2\theta),\ -r\sin\theta - \frac{1}{2}r^2\sin(2\theta),\ \frac{4}{3}r^{3/2}\cos\left(\frac{3}{2}\theta\right)\right).$$

We will show that this is a minimal surface.







Proof. First Derivatives:

$$\sigma_r = \left(\cos\theta - r\cos(2\theta),\ -\sin\theta - r\sin(2\theta),\ 2r^{1/2}\cos\left(\frac{3}{2}\theta\right)\right),$$

$$\sigma_\theta = \left(-r\sin\theta + r^2\sin(2\theta),\ -r\cos\theta - r^2\cos(2\theta),\ -2r^{3/2}\sin\left(\frac{3}{2}\theta\right)\right).$$

Second Derivatives:

$$\sigma_{rr} = \left(-\cos(2\theta),\ -\sin(2\theta),\ r^{-1/2}\cos\left(\frac{3}{2}\theta\right)\right),$$

$$\sigma_{r\theta} = \left(-\sin\theta + 2r\sin(2\theta),\ -\cos\theta - 2r\cos(2\theta),\ -3r^{1/2}\sin\left(\frac{3}{2}\theta\right)\right),$$

$$\sigma_{\theta\theta} = \left(-r\cos\theta + 2r^2\cos(2\theta),\ r\sin\theta + 2r^2\sin(2\theta),\ -3r^{3/2}\cos\left(\frac{3}{2}\theta\right)\right).$$

The first fundamental form has coefficients:

$$E = \sigma_r \cdot \sigma_r = (1+r^2),$$
$$F = \sigma_r \cdot \sigma_\theta = 0,$$
$$G = \sigma_\theta \cdot \sigma_\theta = r^2(1+r^2),$$

$$\mathbf{u} = \frac{\sigma_r \times \sigma_\theta}{\|\sigma_r \times \sigma_\theta\|} = \frac{1}{r^{1/2}(1+r^2)^{3/2}}\left(-2r^{3/2}(1+r^2)\sin\left(\frac{3}{2}\theta\right),\ -2r^{3/2}(1+r^2)\cos\left(\frac{3}{2}\theta\right),\ r(1+r^2)\right).$$

The second fundamental form has coefficients:

$$L = \mathbf{u}\cdot\sigma_{rr} = -r^{-1/2}\cos\left(\frac{3}{2}\theta\right),$$

$$M = \mathbf{u}\cdot\sigma_{r\theta} = \sqrt{r}\sin\left(\frac{3}{2}\theta\right),$$

$$N = \mathbf{u}\cdot\sigma_{\theta\theta} = r^{3/2}\cos\left(\frac{3}{2}\theta\right).$$

Thus,

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{r(1+r^2)^4},$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = 0,$$

$$k_1 = H + \sqrt{H^2 - K} = \frac{1}{r^{1/2}(1+r^2)^{3/2}}, \quad k_2 = H - \sqrt{H^2 - K} = -\frac{1}{r^{1/2}(1+r^2)^{3/2}}.$$

So Bour's surface is a minimal surface. ■

3. Geodesics and Variational Principles

3.1. **Geodesics as shortest paths.** Let S be a surface. Let σ be a surface patch of S.

Definition 8. The *arc-length* of a curve γ starting at the point $\gamma(t_0)$ is the function $s(t)$ given by

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| \, du.$$

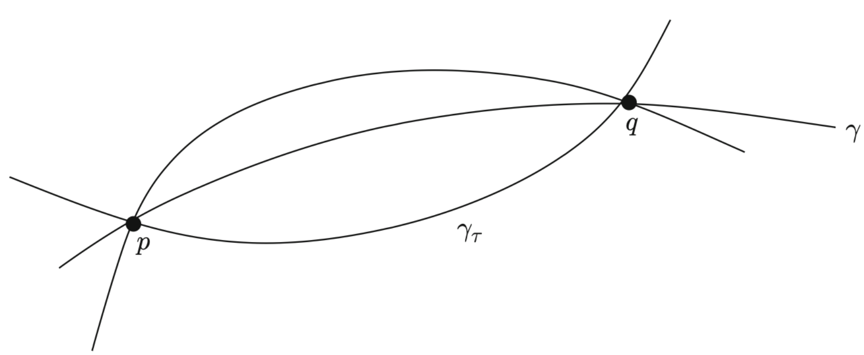
A shortest path between two points on a surface is a curve minimizing the arc-length among all admissible curves joining those points.

If γ is a shortest path on S from p to q, then the part of γ contained in any surface patch σ of S must be the shortest path between any two of its points. For if p' and q' are any two points of γ in (the image of) σ, and if there were a shorter path in σ from p' to q' than γ, we could replace the part of γ between p' and q' by this shorter path, thus giving a shorter path from p to q in S. We may therefore consider a path γ entirely contained in σ.

Let γ be a unit-speed curve on S, γ is entirely contained in σ and passing through two fixed points $p, q \in S$. We will see that if γ is a shortest path on σ from p to q, then γ is a geodesic. However, the converse may not hold. If γ is a geodesic, it need not be a shortest path from p to q. Moreover, a shortest path joining two points on a surface may not exist.

To test whether γ has smaller length than any other path in σ passing through two fixed points p, q on σ; we embed γ in a smooth family of curves on σ passing through p and q. By such a family, we mean a curve γ^τ on σ, for each τ in an open interval $(-\delta, \delta)$, such that

(1) There is an $\epsilon > 0$ such that $\gamma^\tau(t)$ is defined for all $t \in (-\epsilon, \epsilon)$ and all $\tau \in (-\delta, \delta)$;
(2) For some a, b with $-\epsilon < a < b < \epsilon$, we have
$$\gamma^\tau(a) = p \quad \text{and} \quad \gamma^\tau(b) = q \quad \text{for all} \quad \tau \in (-\delta, \delta);$$
(3) The map from the rectangle $(-\delta, \delta) \times (-\epsilon, \epsilon)$ into $\mathbb{R}^3$ given by
$$(\tau, t) \mapsto \gamma^\tau(t)$$
is smooth;
(4) $\gamma^0 = \gamma$.



By definition 8, the length of the part of γ^τ between p and q is

$$\mathcal{L}(\tau) = \int_a^b \|\dot{\gamma}^\tau\| \, dt.$$

Theorem 9. *The unit-speed curve γ is a geodesic if and only if*

$$\frac{d}{d\tau}\mathcal{L}(\tau)\bigg|_{\tau=0} = 0$$

for all families of curves γ^τ with $\gamma^0 = \gamma$.

The proof appears in [2] (page 237 - 239). We omit it here for brevity.

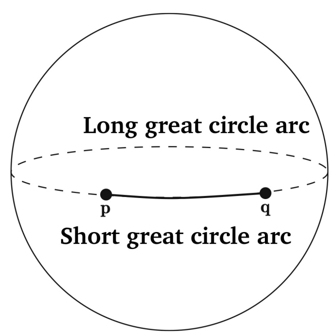
Intuitively, $\frac{d}{d\tau}$ calculates the rate of change in the curve's length as τ varies infinitesimally from 0. If γ^0 is the shortest/longest path in the family, then the length $\mathcal{L}(\tau)$ attains an extremum at $\tau = 0$ and $\frac{d}{d\tau}\mathcal{L}(\tau)\big|_{\tau=0} = 0$ (analogous to the derivative of a single-variable function being zero).

Thus, if γ is a shortest path on σ from p to q, then $\mathcal{L}(\tau)$ must have an absolute minimum when $\tau = 0$. This implies that

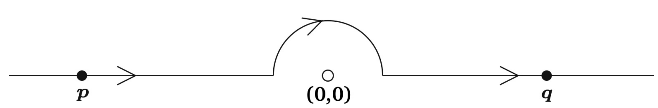
$$\frac{d}{d\tau}\mathcal{L}(\tau)\bigg|_{\tau=0} = 0,$$

and hence by Theorem 9 that γ is a geodesic.

However, if γ is a geodesic on σ passing through p and q, then $\mathcal{L}(\tau)$ attains an extremum at $\tau = 0$, but this need not be an absolute minimum, or even a local minimum, so γ need not be a shortest path from p to q. For example, if p and q are two nearby points on a sphere, the short great circle arc joining p and q is the shortest path from p to q, but the long great circle arc joining p and q is also a geodesic.



Now, if we consider the surface S consisting of the xy-plane with the origin removed, we see that there is no shortest path on the surface from the point $p = (-a, 0)$ to the point $q = (a, 0)$ for any $a \in \mathbb{R}^+$.



3.2. **Surface variations.** Similar to the study of the family of curves, we now want to study a family of surface patches $\sigma^\tau : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$ independent of τ, and τ lies in some open interval $(-\delta, \delta)$, for some $\delta > 0$. Let $\sigma = \sigma^0$. The family is required to be *smooth*, in the sense that the map $(u, v, \tau) \mapsto \sigma^\tau(u, v)$ from the open subset $\{(u, v, \tau) \mid (u, v) \in U,\ \tau \in (-\delta, \delta)\}$ of $\mathbb{R}^3$ to $\mathbb{R}^3$ is smooth. The *surface variation* of the family is the function $\varphi : U \to \mathbb{R}^3$ given by

$$\varphi = \dot{\sigma}^\tau|_{\tau=0},$$

where here and elsewhere in this section, a dot denotes $d/d\tau$.

Let π be a simple closed curve that is contained, along with its interior int(π), in U (see Section 3.1). Then π corresponds to a closed curve $\gamma^\tau = \sigma^\tau \circ \pi$ in the surface patch σ^τ, and we define the area function $A(\tau)$ to be the area of the part of σ^τ inside γ^τ:

$$A(\tau) = \int_{\text{int}(\pi)} d\mathcal{A}_{\sigma^\tau}.$$

Note that, if we are considering a family of surfaces with a fixed boundary curve γ, then $\gamma^\tau = \gamma$ for all τ, and hence $\varphi^\tau(u, v) = 0$ when (u, v) is a point on the curve π.

Theorem 10. *With the above notation, assume that the surface variation φ^τ vanishes along the boundary curve π. Then,*

$$\dot{A}(0) = -2 \int_{int(\pi)} H(EG - F^2)^{1/2} \alpha \, dudv,$$

where H is the mean curvature of σ, E, F and G are the coefficients of its first fundamental form, and $\alpha = \varphi \cdot \mathbf{u}$ where $\mathbf{u}$ is the standard unit normal of σ.

Proof. The proof appears in [2] (page 309 - 311). We omit it here for brevity. ∎

One last thing to mention is, a minimal surface need not be surface area minimizing. We'll see this through the example of catenoid.

The surface obtained by revolving the curve $x = \cosh z$ in the xz-plane around the z-axis is called a *catenoid*. It can be parametrized by:

$$\sigma(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$

The first derivatives are:

$$\sigma_u = (\sinh u \cos v, \sinh u \sin v, 1),$$
$$\sigma_v = (-\cosh u \sin v, \cosh u \cos v, 0).$$

The unit normal vector and second derivatives are:

$$\sigma_u \times \sigma_v = (-\cosh u \cos v, -\cosh u \sin v, \sinh u \cosh u),$$
$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (-\text{sech}\, u \cos v, -\text{sech}\, u \sin v, \tanh u),$$
$$\sigma_{uu} = (\cosh u \cos v, \cosh u \sin v, 0),$$
$$\sigma_{uv} = (-\sinh u \sin v, \sinh u \cos v, 0),$$
$$\sigma_{vv} = (-\cosh u \cos v, -\cosh u \sin v, 0).$$

This gives the coefficients of the fundamental forms:

$$E = G = \cosh^2 u, \quad F = 0, \quad L = -1, \quad M = 0, \quad N = 1.$$

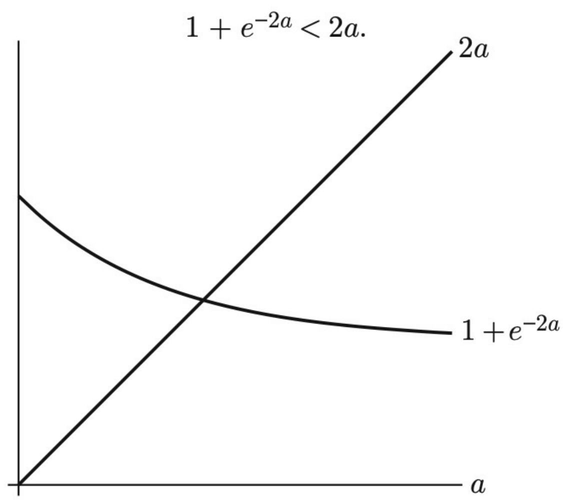
Thus,

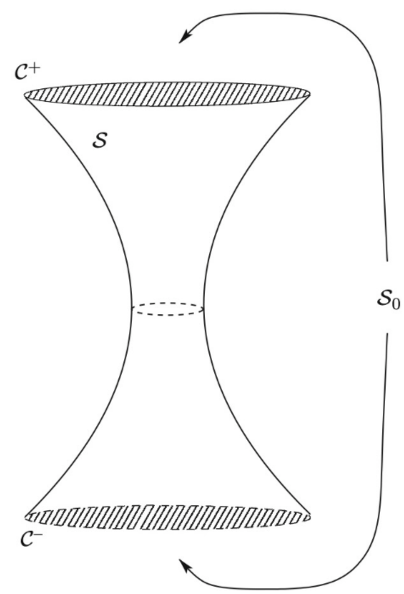
$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{-\cosh^2 u + \cosh^2 u}{2\cosh^4 u} = 0,$$

which shows that the catenoid is a minimal surface.

Fix $a > 0$, and let $b = \cosh a$. Consider:

- The catenoid part S with $|z| < a$ bounded by two circles $C^{\pm}$
- The comparison surface S_0 consisting of two discs





The areas are:

$$\text{Area}(S) = 2\pi(a + \sinh a \cosh a),$$
$$\text{Area}(S_0) = 2\pi \cosh^2 a.$$

There exists a critical value a_0 such that:

- For $a < a_0$: Catenoid has least area
- For $a > a_0$: Catenoid is *not* area-minimizing

This shows that while all area-minimizing surfaces are minimal, the converse is not true: minimal surfaces may only represent local rather than global area minima.

Similarly, a minimal surface spanning a given boundary may not exist. The existence is not guaranteed and depends on the boundary's geometry and topology.

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