One-dimensional calculus of variations

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1 Introduction

The main goal of this note is to study the following minimization problem. Let $u \in u_0 + W_0^{1,2}((a,b))$ be a minimizer of

$$\inf \left\{ I(u) = \int_{(a,b)} f(x, u(x), u'(x))\, dx : u \in u_0 + W_0^{1,2}((a,b)) \right\} = m, \tag{1}$$

where $u_0 \in W^{1,2}((a,b))$ with $I(u_0) < \infty$.

In section 3, we establish the existence of solutions within the Sobolev space $W^{1,2}$. Subsequently, in Section 4, we refine our analysis and demonstrate that the minimizer can, in fact, be found within the space of smooth functions (C^∞).

To facilitate the proof of the main results, we will first introduce a set of preliminary definitions.

Definition 1 (Definitions of L^p Spaces)**.** *Let $\Omega \subset \mathbb{R}$ be an open set. A measurable function $u : \Omega \to \mathbb{R}$ belongs to the $L^p(\Omega)$ space if the L^p norm is finite.*

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p\, dx\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \inf\{\alpha > 0 : |u(x)| \le \alpha \text{ a.e. in } \Omega\}, & \text{if } p = \infty. \end{cases}$$

Example 1. *The function $u(x) = x^{-\frac{1}{3}}$ on the interval $(0,1)$ is in $L^2((0,1))$.*

Example 2. *The function $u(x) = x^{-\frac{1}{2}}$ on the interval $(0,1)$ is not in $L^2((0,1))$.*

Definition 2 (Definition of Sobolev Spaces $W^{1,p}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}$ be an open set and $1 \leq p \leq \infty$. We define the Sobolev space $W^{1,p}(\Omega)$ as the set of functions $u : \Omega \to \mathbb{R}$ such that $u \in L^p(\Omega)$ and the derivative $u' \in L^p(\Omega)$. This space is endowed with the following norm:*

$$\|u\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\|u\|^p_{L^p(\Omega)} + \|u'\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max\left\{\|u\|_{L^\infty(\Omega)}, \|u'\|_{L^\infty(\Omega)}\right\}, & \text{if } p = \infty. \end{cases}$$

In the case where $p = 2$, the space $W^{1,2}(\Omega)$ is sometimes denoted by $H^1(\Omega)$.

Example 3. *The function $u(x) = \sqrt{x}$ on the interval $[0, 1]$ is in $W^{1,2}([0, 1])$.*

Example 4. *The function $u(x) = |x|^{-\frac{1}{3}}$ on the interval $[-1, 1]$ is not in $W^{1,2}([-1, 1])$ because as u' approaches 0, the function diverges, and therefore u' is not in $L^2([-1, 1])$.*

Definition 3. *Let $\Omega \subset \mathbb{R}$ be an open set and $1 \leq p \leq \infty$.*

(i) A sequence u_ν is said to (strongly) converge to u if $u_\nu, u \in L^p$ and if

$$\lim_{\nu\to\infty} \|u_\nu - u\|_{L^p} = 0.$$

We will denote this convergence by: $u_\nu \to u$ in L^p.

(ii) If $1 \leq p < \infty$, we say that the sequence u_ν weakly converges to u if $u_\nu, u \in L^p$ and if

$$\lim_{\nu\to\infty} \int_{(a,b)} [u_\nu(x) - u(x)]\, \varphi(x)\, dx = 0,\ \forall \varphi \in L^{p'}((a, b)).$$

This convergence will be denoted by: $u_\nu \rightharpoonup u$ in L^p.

(iii) If $p = \infty$, the sequence u_ν is said to weak$*$ converge to u if $u_\nu, u \in L^\infty$ and if

$$\lim_{\nu\to\infty} \int_{(a,b)} [u_\nu(x) - u(x)]\, \varphi(x)\, dx = 0,\ \forall \varphi \in L^1((a, b)).$$

and will be denoted by: $u_\nu \overset{*}{\rightharpoonup} u$ in L^∞.

Example 5. *Let $\Omega = (0, 2\pi)$ and $u_\nu(x) = \sin \nu x$, then we have*

$$\sin \nu x \nrightarrow 0 \text{ in } L^2$$

$$\sin \nu x \rightharpoonup 0 \text{ in } L^2.$$

Proof. First we need to check if $u_\nu(x) = sin(\nu x)$ tends to 0 in its norm. We compute

$$\|u_\nu(x)\|_{L^2} = \Big(\int_0^{2\pi} |sin(\nu x)|^2 dx\Big)^{\frac{1}{2}}.$$

Since the average value of $|sin(\nu x)|^2$ over one period is $\frac{1}{2}$, we have

$$\|u_\nu(x)\|_{L^2}^2 = \int_0^{2\pi} |sin(\nu x)|^2 dx = \frac{2\pi}{2} = \pi.$$

Thus, $\|u_\nu(x)\|_{L^2} = \sqrt{\pi}$, indicating that the norm does not tend to 0. Therefore, $sin(\nu x)$ does not strongly converge to 0 in L^2.

Next, we need to show that $sin(\nu x)$ weakly converges to 0 in L^2. The Riemann-Lebesgue Theorem (cf. theorem 8) states that if $g(x) \in L^1(\Omega)$, where Ω is a bounded domain, then the Fourier coefficients of $g(x)$ tends to 0 as their frequency increases. In this case, for any function $g(x) \in L^1(0, 2\pi)$ we have

$$\lim_{\nu \to \infty} \int_0^{2\pi} g(x) sin(\nu x) dx = 0. \tag{2}$$

To prove weak convergence of $sin(\nu x)$ to 0 in $L^2(0, 2\pi)$, consider any function $g(x) \in L^2(0, 2\pi)$. Since $L^2(0, 2\pi) \subset L^1(0, 2\pi)$, the function $g(x)$ is also in $L^1(0, 2\pi)$. By the Riemann-Lebesgue Theorem, we have (2), which directly implies that $sin(\nu x)$ weakly converges to 0 in L^2.

□

Example 6. *Let* $\Omega = (0,1)$, $\alpha, \beta \in \mathbb{R}$

$$u(x) = \begin{cases} \alpha & \text{if } x \in (0, 1/2) \\ \beta & \text{if } x \in (1/2, 1). \end{cases}$$

Extend u *by periodicity from* $(0,1)$ *to* $\mathbb{R}$ *and define*

$$u_\nu(x) = u(\nu x).$$

Note that u_ν *takes only the values* α *and* β *and the sets where it takes such values are, both, sets of measure* $1/2$. *It is clear that* $\{u_\nu\}$ *cannot be compact in any* L^p *spaces; however from Riemann-Lebesgue Theorem (cf. theorem 8), we will find*

$$u_\nu \rightharpoonup \frac{\alpha+\beta}{2} \text{ in } L^p, \quad \forall 1 \le p < \infty \quad \text{and} \quad u_\nu \overset{*}{\rightharpoonup} \frac{\alpha+\beta}{2} \text{ in } L^\infty.$$

Proof. We prove for the case $p = 2$. First we want to show that u_ν weakly converges to $\frac{\alpha+\beta}{2}$. That is, we want to show

$$\lim_{\nu \to \infty} \int_0^1 [u_\nu(x) - \frac{\alpha+\beta}{2}] \varphi(x) dx = 0, \forall \varphi \in L^2(0,1). \tag{3}$$

Define

$$\bar{u} = \int_0^1 u(x) dx = \int_0^{\frac{1}{2}} \alpha dx + \int_{\frac{1}{2}}^1 \beta dx = \frac{\alpha+\beta}{2}.$$

By Riemann-Lebesgue Theorem (cf. theorem 8), $u_\nu \rightharpoonup \bar{u}$ in L^2 (i.e. (3)). □

2 Basic theorems

Theorem 1. *Let $(a,b) \subset \mathbb{R}$ be open,and $u \in L^2((a,b))$. The following properties are then equivalent:*

(i) $u \in W^{1,2}((a,b))$;

(ii) There exists a constant $c = c(u,(a,b))$ so that

$$\left| \int_{(a,b)} u(x)\varphi'(x)\,dx \right| \leq c\|\varphi\|_{L^2} \quad \forall \varphi \in C_0^\infty((a,b)), \forall i = 1, 2, \ldots, n.$$

Theorem 2. *Let $(a,b) \subset \mathbb{R}$ be a bounded open interval. Then $W^{1,2}(a,b) \subset C^{0,\alpha}(a,b)$ for every $\alpha \in [0, \frac{1}{2}]$. In particular, there exists a constant $c = c(a,b)$ such that*

$$\|u\|_{L^\infty} \leq c\|u\|_{W^{1,2}}.$$

Theorem 3. *Let $(a,b) \subset \mathbb{R}$ be a bounded open interval. Let $f \in C^1((a,b) \times \mathbb{R} \times \mathbb{R})$, $f = f(x,u,\xi)$, satisfy*

$$\text{(H3)} \quad \exists\, \beta \geq 0 \text{ so that for every } (x,u,\xi) \in (a,b) \times \mathbb{R} \times \mathbb{R},$$
$$|f_u(x,u,\xi)|, |f_\xi(x,u,\xi)| \leq \beta\,(1 + |u| + |\xi|)\,,$$

where $f_\xi = (f_{\xi_1}, \ldots, f_{\xi_n})$, $f_{\xi_i} = \frac{\partial f}{\partial \xi_i}$ and $f_u = \frac{\partial f}{\partial u}$. Then u satisfies the weak form of the Euler-Lagrange equation

$$(Ew) \qquad \int_{(a,b)} \left[f_u(x,u,\nabla u)\varphi + \langle f_\xi(x,u,\nabla u), \nabla\varphi \rangle \right] dx = 0, \quad \forall \varphi \in W_0^{1,2}((a,b)).$$

Conversely, if $(u,\xi) \to f(x,u,\xi)$ is convex for every $x \in (a,b)$ and if u is a solution of either (Ew), then it is a minimizer of Equation (1).

Theorem 4 (Fundamental Theorem of Calculus)**.** *If f is a real-valued continuous function on $[a,b]$ and F is an antiderivative of f in $[a,b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Theorem 5 (Poincaré inequality)**.** *There exists a constant C, depending only on $[a,b]$, so that, for every function u of the Sobolev space $\mathrm{W}_0^{1,2}([a,b])$ of zero-trace (i.e. zero on the boundary) functions, we have*

$$\|u\|_{L^2([a,b])} \leq C\|u'\|_{L^2([a,b])}.$$

Proof. We will prove this for the case f is a $W_0^{1,2}$ function, thus satisfying $f(a) = f(b) = 0$. Here the statement becomes

$$\int_a^b f^2 \leq kb^2 \int_a^b (f')^2.$$

By the Fundamental Theorem of Calculus (cf. theorem 4)

$$f(s) = \int_a^s f'(x)dx, \ \forall s \in (a,b).$$

Therefore

$$|f(s)| \leq \int_a^s |f'(x)|dx, \ \forall s \in (a,b).$$

Recall the Cauchy-Schwarz inequality $\left(\int hg \leq \left(\int h^2\right)^{1/2}\left(\int g^2\right)^{1/2}\right)$. Apply this with $h=1$, $g=|f'|$ to get

$$|f(s)| \leq \left(\int_a^s (f')^2\right)^{1/2} (b+s)^{1/2} \leq \left(\int_a^b (f')^2\right)^{1/2} (2b)^{1/2}.$$

Squaring both sides gives

$$|f(s)|^2 \leq 2b\int_a^b (f'(s))^2,$$

and finally we integrate over $[a,b]$ to give

$$\int_a^b |f(s)|^2 \leq 4b^2 \int_a^b |f'(s)|^2,$$

as required. □

Theorem 6 (Fundamental lemma of the calculus of variations)**.** *Let $\Omega \subset \mathbb{R}$ be an open set and $u \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\,\psi(x)\,dx = 0, \forall \psi \in C_0^\infty(\Omega),$$

then $u=0$ almost everywhere in Ω.

Theorem 7. *Let $\Omega \subset \mathbb{R}$ be a bounded open set. If $u_\nu \rightharpoonup u$ in L^2, then there exists a constant $\gamma > 0$ so that $\|u_\nu\|_{L^2} \leq \gamma$, moreover $\|u\|_{L^2} \leq \liminf_{\nu\to\infty} \|u_\nu\|_{L^2}$.*

Theorem 8 (Riemann-Lebesgue Theorem)**.** *Let $\Omega = (a,b)$ and $u \in L^2(a,b)$. Let u be extended by periodicity from Ω to $\mathbb{R}$ and define*

$$u_\nu(x) = u(\nu x) \text{ and } \bar{u} = \frac{1}{b-a}\int_a^b u(x)dx,$$

then $u_\nu \rightharpoonup \bar{u}$ in L^2.

3 Existence theorem

Theorem 9. *Let $(a,b) \subset \mathbb{R}$ be a bounded open interval. Let $f \in \mathcal{C}^1([a,b] \times \mathbb{R} \times \mathbb{R})$, where $f = f(x,u,u')$ satisfies the following conditions:*

1. *$(u,u') \to f(x,u,u')$ is convex for every $x \in [a,b]$;*
2. *there exist $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ such that*
$$f(x,u,u') \geq \alpha_1 |u'|^2 + \alpha_2, \forall (x,u,u') \in [a,b] \times \mathbb{R} \times \mathbb{R}.$$
3. *there exists a constant $\beta \geq 0$ so that for every $(x,u,u') \in [a,b] \times \mathbb{R} \times \mathbb{R}$*
$$|\frac{\partial f}{\partial u}|, |\frac{\partial f}{\partial u'}| \leq \beta \left(1 + |u| + |u'|\right).$$
4. *there exists $\bar{u} \in u_0 + \mathrm{W}_0^{1,2}([a,b])$ a minimizer of (1).*

Furthermore if $(u,u') \to f(x,u,u')$ is strictly convex for every $x \in [a,b]$, then the minimizer is unique.

Proof. **Part 1 (Existence).** The proof is divided into three steps.

Step 1 (Compactness). Recall that by assumption on u_0 and by *item* 2 we have
$$-\infty < m \leq I(u_0) < \infty.$$
Let $u_\nu \in u_0 + W_0^{1,2}([a,b])$ be a minimizing sequence of (1), i.e.
$$I(u_\nu) \to \inf\{I(u)\} = m, \text{ as } \nu \to \infty.$$
We therefore have from *item* 2 that for ν large enough
$$m + 1 \geq I(u_\nu) \geq \alpha_1 \int_a^b (u'_\nu)^2 dx - |\alpha_2|(b-a)$$
and hence there exists $\alpha_3 > 0$ so that
$$\Big(\int_a^b (u'_\nu)^2 dx \Big)^{\frac{1}{2}} \leq \alpha_3.$$
Appealing to Poincaré inequality (cf. theorem 5) we can find constants $\alpha_4, \alpha_5 > 0$ so that
$$\alpha_3 \geq \Big(\int_a^b (u'_\nu)^2 dx \Big)^{\frac{1}{2}} \geq \alpha_4 \|u_\nu\|_{W^{1,2}} - \alpha_5 \|u_0\|_{W^{1,2}}$$
and hence we can find $\alpha_6 > 0$ so that
$$\|u_\nu\|_{W^{1,2}} \leq \alpha_6.$$
By Theorem 7 we deduce that there exists $\overline{u} \in u_0 + W_0^{1,2}([a,b])$ and a subsequence (still denoted u_ν) such that
$$u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,2}, \text{ as } \nu \to \infty.$$

Step 2 (Lower semicontinuity). We now show that I is (sequentially) weakly lower semicontinuous; this means that

$$u_\nu \to \overline{u} \text{ in } W^{1,2} \implies \liminf_{\nu\to\infty} I(u_\nu) \geq I(\overline{u}). \tag{4}$$

This step is independent of the fact that $\{u_\nu\}$ is a minimizing sequence. Using the convexity of f and the fact that it is C^1 we get

$$f(x, u_\nu, u_\nu') \geq f(x, \overline{u}, \overline{u}') + |\frac{\partial f}{\partial \overline{u}}|(u_\nu - \overline{u}) + \langle |\frac{\partial f}{\partial \overline{u}'}|, u_\nu' - \overline{u}'\rangle. \tag{5}$$

Before proceeding further we need to show that the combination of *item* 3 and $\overline{u} \in W^{1,2}([a,b])$ leads to

$$|\frac{\partial f}{\partial \overline{u}}| \in L^2([a,b]) \text{ and } |\frac{\partial f}{\partial \overline{u}'}| \in L^2([a,b];\mathbb{R}) \tag{6}$$

Indeed let us prove the first statement, the other one being shown similarly. We have (β_1 being a constant)

$$\int_a^b |\frac{\partial f}{\partial \overline{u}}|^2 \, dx \leq \beta^2 \int_a^b \left(1 + |\overline{u}| + |\overline{u}'|\right)^2 dx \leq \beta_1 \left(1 + \|\overline{u}\|^2_{W^{1,2}}\right) < \infty.$$

Using Hölder inequality and (6) we find that for $u_\nu \in W^{1,2}([a,b])$

$$|\frac{\partial f}{\partial \overline{u}}| \, (u_\nu - \overline{u}) \, , \, \left\langle |\frac{\partial f}{\partial \overline{u}'}|; u_\nu' - \overline{u}' \right\rangle \in L^1([a,b]).$$

We next integrate (5) to get

$$I\left(u_\nu\right) \geq I\left(\overline{u}\right) + \int_a^b |\frac{\partial f}{\partial \overline{u}}| \left(u_\nu - \overline{u}\right) dx + \int_a^b \left\langle |\frac{\partial f}{\partial \overline{u}'}|; u_\nu' - \overline{u}' \right\rangle dx. \tag{7}$$

Since $u_\nu - \overline{u} \rightharpoonup 0$ in $W^{1,2}$ (i.e. $u_\nu - \overline{u} \to 0$ in L^2 and $u_\nu' - \overline{u}' \to 0$ in L^2) and (6) holds, we deduce, from the definition of weak convergence in L^2, that

$$\lim_{\nu\to\infty} \int_a^b |\frac{\partial f}{\partial \overline{u}}| \left(u_\nu - \overline{u}\right) dx = \lim_{\nu\to\infty} \int_a^b \left\langle |\frac{\partial f}{\partial \overline{u}'}|; u_\nu' - \overline{u}' \right\rangle dx = 0.$$

Therefore returning to (7) we have indeed obtained that

$$\liminf_{\nu\to\infty} I(u_\nu) \geq I(\overline{u}).$$

Step 3. We now combine the two steps. Since $\{u_\nu\}$ was a minimizing sequence (i.e. $I(u_\nu) \to \inf\{I(u)\} = m$) and for such a sequence we have lower semicontinuity (i.e. $\liminf I(u_\nu) \geq I(\overline{u})$) we deduce that $I(\overline{u}) = m$, i.e. $\overline{u}$ is a minimizer of (1).

Part 2 (Uniqueness). Assume that there exist $\overline{u}, \overline{v} \in u_0 + W_0^{1,2}([a,b])$ so that

$$I(\overline{u}) = I(\overline{v}) = m$$

and we prove that this implies $\overline{u} = \overline{v}$. Denote by $\overline{w} = (\overline{u} + \overline{v})/2$ and observe that $\overline{w} \in u_0 + W_0^{1,2}([a,b])$. The function $(u,u') \to f(x,u,u')$ being convex, we can infer that $\overline{w}$ is also a minimizer since

$$m \leq I(\overline{w}) \leq \frac{1}{2}I(\overline{u}) + \frac{1}{2}I(\overline{v}) = m,$$

which readily implies that

$$\int_a^b \left[\frac{1}{2}f\left(x,\overline{u},\overline{u}'\right) + \frac{1}{2}f\left(x,\overline{v},\overline{v}'\right) - f\left(x, \frac{\overline{u}+\overline{v}}{2}, \frac{\overline{u}'+\overline{v}'}{2}\right)\right] dx = 0.$$

The convexity of $(u,\xi) \to f(x,u,\xi)$ implies that the integrand is non-negative, while the integral is 0. This is possible only if

$$\frac{1}{2}f\left(x,\overline{u},\overline{u}'\right) + \frac{1}{2}f\left(x,\overline{v},\overline{v}'\right) - f\left(x, \frac{\overline{u}+\overline{v}}{2}, \frac{\overline{u}'+\overline{v}'}{2}\right) = 0 \quad \text{a.e. in } [a,b].$$

We now use the strict convexity of $(u,u') \to f(x,u,u')$ to obtain that $\overline{u} = \overline{v}$ and $\overline{u}' = \overline{v}'$ a.e. in $[a,b]$, which implies the desired uniqueness, namely $\overline{u} = \overline{v}$ a.e. in $[a,b]$. □

4 Regularity theorem

Theorem 10. *Let $(a,b) \subset \mathbb{R}$ be a bounded open interval. Let $f \in \mathcal{C}^1([a,b] \times \mathbb{R} \times \mathbb{R})$, where $f = f(x,u,u')$ satisfies the following conditions:*

1. *$(u,u') \to f(x,u,u')$ is convex for every $x \in [a,b]$;*
2. *there exist $\alpha_1 > 0$, $\alpha_3 \in \mathbb{R}$ such that*

$$f(x,u,u') \geq \alpha_1|u'|^2 + \alpha_3, \forall (x,u,u') \in [a,b] \times \mathbb{R} \times \mathbb{R}.$$

3. *there exists a constant $\beta \geq 0$ so that for every $(x,u,u') \in [a,b] \times \mathbb{R} \times \mathbb{R}$*

$$|\frac{\partial f}{\partial u}|, |\frac{\partial f}{\partial u'}| \leq \beta\left(1 + |u| + |u'|\right).$$

Let $g \in C^\infty([a,b] \times \mathbb{R})$ satisfy

$$f(x,u,\xi) = \frac{1}{2}\xi^2 + g(x,u).$$

Then there exists $u \in C^\infty([a,b])$, a minimizer of (P). If, in addition, $u \to g(x,u)$ is convex for every $x \in [a,b]$, then the minimizer is unique.

Proof. We have from existence theorem (Theorem 9) that a minimizer $\overline{u} \in W^{1,2}(a,b)$ exists and if g is convex, $\overline{u}$ is unique. Using Theorem 3, $\overline{u}$ satisfies the weak form of the Euler-Lagrange equation:

$$\int_\Omega \left[f_u(x,u,\nabla u)\varphi + \langle f_\xi(x,u,\nabla u), \nabla\varphi\rangle\right] dx = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Since

$$f(x,u,\xi)=\frac{1}{2}\xi^2+g(x,u),$$

The partial derivatives of $f(x,\overline{u},\xi)$ are given by:

$$f_u(x,\overline{u},\xi)=g_u(x,\overline{u}) \text{ and } f_\xi(x,\overline{u},\xi)=\xi.$$

Substituting these into the weak form of the Euler-Lagrange equation, we obtain:

$$\int_a^b \left[g_u(x,\overline{u})v+\overline{u}'v'\right]dx=0, \quad \forall v\in C_0^\infty(a,b),$$

then

$$\int_a^b \overline{u}'v'\,dx=-\int_a^b g_u(x,\overline{u})v\,dx, \quad \forall v\in C_0^\infty(a,b).$$

We now want to show that $\overline{u}\in W^{2,2}(a,b)$. We know $\overline{u}\in W^{1,2}$. First we show that $\overline{u}\in L^\infty$ and then $g_u(x,\overline{u})\in L^2$.

Using Theorem 2, we know that if $\overline{u}\in W^{1,2}(a,b)$, then $\overline{u}$ is also in $C^{0,\alpha}(a,b)$, which means $\overline{u}$ is Hölder continuous.

Since $\overline{u}$ is continuous over the compact set $[a,b]$, the image $\overline{u}([a,b])$ is also compact. This implies that there exists some bound $[c,d]$ such that $\overline{u}([a,b])\subseteq[c,d]$, meaning that $\overline{u}$ is finite and therefore belongs to $L^\infty(a,b)$. .

Now we want to show $g_u(x,\overline{u})\in L^2$ We need to check that the following integral is finite:

$$\int_a^b |g_u(x,\bar{u})|^2\,dx<\infty.$$

From $u\in L^\infty(a,b)$, we can define $\sup\bar{u}=c$. Then, we have:

$$|\bar{u}(x)|\leq c \quad \text{for all } x\in[a,b].$$

Next, define the set K as:

$$K=[a,b]\times[-c,c].$$

This set K represents the domain $[a,b]$ for x and the range $[-c,c]$ for $\bar{u}(x)$.

Since $g(x,u)$ is C^∞ and K is a compact set, it follows that $g_u(x,u)$ is bounded on K. Therefore, there exists a constant T such that:

$$|g_u(x,u)|\leq T \quad \text{for all } (x,u)\in K.$$

We can now bound the square of the integral of $g_u(x,\bar{u})$ over (a,b) as follows:

$$\int_a^b |g_u(x,\bar{u})|^2\,dx\leq\int_a^b (T)^2\,dx=(T)^2\int_a^b 1\,dx=(T)^2(b-a).$$

Since $(T)^2(b-a)$ is finite, it follows that:

$$\int_a^b |g_u(x,\bar{u})|^2\,dx < \infty,$$

which implies that $g_u(x,\bar{u}) \in L^2(a,b)$.

We have already established that $\bar{u} \in L^\infty(a,b)$ and $g_u(x,\bar{u}) \in L^2(a,b)$. Now, we aim to establish the following inequality:

$$\left|\int_a^b u'v'\,dx\right| \leq \|g_u(x,\bar{u})\|_{L^2}\|v\|_{L^2}, \quad \forall v \in C_0^\infty(a,b).$$

Recall from the weak form of the Euler-Lagrange equation that:

$$\int_a^b g_u(x,\bar{u})v\,dx = -\int_a^b u'v'\,dx.$$

To establish this inequality, we apply the Cauchy-Schwarz inequality in L^2 space, which states that for any functions $f(x)$ and $g(x)$ in $L^2(a,b)$:

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \|f\|_{L^2}\|g\|_{L^2}.$$

Let $f(x) = g_u(x,\bar{u})$ and $g(x) = v(x)$. Applying the Cauchy-Schwarz inequality, we get:

$$\left|\int_a^b g_u(x,\bar{u})v(x)\,dx\right| \leq \|g_u(x,\bar{u})\|_{L^2}\|v\|_{L^2}.$$

Substituting the earlier integral relation $\int_a^b g_u(x,\bar{u})v(x)\,dx = -\int_a^b \overline{u}'(x)v'(x)\,dx$, we have:

$$\left|\int_a^b \overline{u}'v'\,dx\right| = \left|-\int_a^b g_u(x,\bar{u})v\,dx\right| \leq \|g_u(x,\bar{u})\|_{L^2}\|v\|_{L^2}.$$

Therefore,

$$\left|\int_a^b \overline{u}'v'\,dx\right| \leq \|g_u(x,\overline{u})\|_{L^2}\|v\|_{L^2}, \quad \forall v \in C_0^\infty(a,b).$$

Now, to show that $u \in W^{2,2}(a,b)$, we will use Theorem 1.

Next, consider the derivative $\bar{u}'$. The inequality in Theorem 1 (ii) holds for $\bar{u}'$ because it satisfies the Cauchy-Schwarz inequality we derived earlier:

$$\left|\int_a^b \bar{u}'v'\,dx\right| \leq \|g_u(x,\bar{u})\|_{L^2}\|v\|_{L^2}, \quad \forall v \in C_0^\infty(a,b).$$

This inequality is of the same form as the one in Theorem 1 (ii), with $\bar{u}'$ playing the role of u in the theorem. Therefore, by Theorem 1, we conclude that $\bar{u}' \in W^{1,2}(a,b)$.

Since $\bar{u}' \in W^{1,2}(a,b)$, it follows that the second derivative $\bar{u}''$ exists almost everywhere, and $\bar{u}'' \in L^2(a,b)$. Consequently, $\bar{u} \in W^{2,2}(a,b)$.

We have already established that $\overline{u} \in W^{2,2}(a,b)$, meaning that $\overline{u}''$ exists and $\overline{u}'' \in L^2(a,b)$. Now, we want to show that $\overline{u}'' = g_u(x,\bar{u})$.

Starting from the weak form of the Euler-Lagrange equation, we have:

$$\int_a^b (g_u(x,\bar{u})v + \overline{u}'v')\,dx = 0, \quad \forall v \in C_0^\infty(a,b).$$

Next, we integrate by parts. The integration by parts formula is:

$$\int_a^b \overline{u}'v'\,dx = [\overline{u}'v]_a^b - \int_a^b \overline{u}''v\,dx.$$

and

$$[\overline{u}'v]_a^b \quad \text{evaluates to 0} \quad \text{since } v \in C_0^\infty(a,b) \quad \text{and } v(a) = v(b) = 0$$

then

$$\int_a^b \overline{u}'v'\,dx = -\int_a^b \overline{u}''v\,dx.$$

Substituting this into the weak form of the Euler-Lagrange equation, we get:

$$-\int_a^b \overline{u}''v\,dx + \int_a^b g_u(x,\bar{u})v\,dx = 0, \quad \forall v \in C_0^\infty(a,b).$$

This simplifies to:

$$\int_a^b (-u'' + g_u(x,\bar{u}))\,v\,dx = 0, \quad \forall v \in C_0^\infty(a,b).$$

Since this equality holds for all functions $v \in C_0^\infty(a,b)$, due to the Fundamental Theorem of Calculus of variations:

$$-\overline{u}''(x) + g_u(x,\bar{u}(x)) = 0 \quad \text{almost everywhere in } (a,b),$$

or equivalently,

$$\overline{u}''(x) = g_u(x,\bar{u}(x)) \quad \text{almost everywhere in } (a,b).$$

This completes the proof that $\overline{u}'' = g_u(x,\bar{u})$.

Now, since $\bar{u} \in W^{2,2}(a,b)$, we know that $\bar{u}' \in W^{1,2}(a,b)$. Applying Theorem 2 with $p = 2 > 1 = n$, we conclude that:

$$\bar{u}' \in C^{0,\alpha}(a,b) \text{ for some } \alpha \in [0,1].$$

This means that $\bar{u}$ is in $C^{1,\alpha}(a,b)$, and in particular $\bar{u}$ is $C^1(a,b)$.

Since $g(x,u)$ is C^∞, it follows that $g_u(x,\bar{u})$ is $C^1(a,b)$ as well. We previously established that $\bar{u}'' = g_u(x,\bar{u})$ a.e., which implies:

$$\bar{u}'' \in C^1(a,b).$$

Now, because $\bar{u}''$ is $C^1(a,b)$, it follows that $\bar{u}$ is in $C^3(a,b)$, meaning that the third derivative $\bar{u}'''$ exists.

Repeating this process, we observe:

1. Since $\bar{u} \in C^3(a,b)$, it follows that $\bar{u}'''$ exists and is continuous,

2. Since $g_u(x,u)$ is C^∞, the function $g_u(x,\bar{u})$ is C^3,

3. Since $\overline{u}''(x) = g_u(x,\bar{u}(x))$, $\bar{u}'' \in C^3$, so $\bar{u}$ is in C^5.

Continuing this iterative process, we eventually establishing that:

$$\bar{u} \in C^\infty(a,b).$$

□