עדן קשת – 305156531 שי מחנאי – 308009356

מבוא לאנליזה נומרית – עבודה 4

1. $||x-y|| \ge |\,||x|| - ||y||\,|$

מהגדרת נורמה וכן מאי שיוויון המשולש מתקיים לכל x, y:

$$||x+y|| \le ||x|| + ||y||$$

ולכן:

$$||x|| = ||x+y-y|| = ||(x-y)+y|| \le ||x-y|| + ||y||$$

נקבל מהעברת אגף:

$$||x|| - ||y|| \le ||x-y||$$

כאמור, מתקיים לכל x, y, וכמובן שגם:

$$||y|| - ||x|| \le ||y-x||$$

נסתכל על חוקי נורמה: (הומוגניות) . במקרה הכללי:

$$||c\,x|| = |c| * ||x||$$

אז נקבל:

$$||y-x|| = ||(-1)(x-y)|| = |-1| * ||(x-y)|| = ||x-y||$$

נציב במה שקיבלנו לפני כן:

$$||y-x|| = ||x-y|| \ge ||x|| - ||y||,$$
$$\ge ||y|| - ||x||$$

אז קיבלנו בסך הכל:

$$||x-y|| \ge ||x|| - ||y|| \quad \text{וגם} \quad ||x-y|| \ge ||y|| - ||x||$$

ולכן:

$$||x-y|| \ge |\ ||x|| - ||y||\ |$$

2. $\|AB\| \leq \|A\|\|B\|; A, B \in R^n x R^n$

$$\|AB\| = \sup_{x\neq 0} \frac{\|ABx\|}{||x||} = \sup_{x\neq 0, Bx\neq 0} \frac{\|ABx\|}{||Bx||} \cdot \frac{\|Bx\|}{\big||x|\big|} \leq$$

$$\sup_{Bx\neq 0} \frac{\|ABx\|}{||Bx||} \cdot \sup_{x\neq 0} \frac{\|Bx\|}{\big||x|\big|} \leq \frac{\|ABx\|}{||Bx||} \cdot \frac{\|Bx\|}{\big||x|\big|} \leq (from\ defintion)\ \|A\| \cdot \|B\|.$$

Then: $\|AB\| \leq \|A\|\|B\|; A, B \in R^n x R^n$

For the matrix A given by $\begin{pmatrix} 9.7 & 6.6 \\ 4.1 & 2.8 \end{pmatrix}$ estimate the $cond(A)$. Use the 2-norm.

3. מתקיים: $cond(A) = ||A||\ ||A^{-1}||$

נורמה 2 של מטריצה A הוא השורש של הערך העצמי הגדול ביותר של A^TA.

$$A = \begin{pmatrix} 9.7 & 6.6 \\ 4.1 & 2.8 \end{pmatrix},\ A^T = \begin{pmatrix} 9.7 & 4.1 \\ 6.6 & 2.8 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 9.7 & 4.1 \\ 6.6 & 2.8 \end{pmatrix} * \begin{pmatrix} 9.7 & 6.6 \\ 4.1 & 2.8 \end{pmatrix} = \begin{pmatrix} 110.9 & 75.5 \\ 75.5 & 51.4 \end{pmatrix}$$

כעת נחפש פולינום אופייני: $|A^TA - \gamma I| = 0$

$$\left|\begin{pmatrix} 110.9 & 75.5 \\ 75.5 & 51.4 \end{pmatrix} - \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}\right| = \begin{vmatrix} 110.9-\gamma & 75.5 \\ 75.5 & 51.4-\gamma \end{vmatrix} = \gamma^2 - 102.3\gamma + 0.01$$

$$\gamma^2 - 162.3\gamma + 0.01 = 0 \rightarrow \qquad \gamma_1 = 162.2999, \gamma_2 = 0.0001$$

אז קיבלנו: $||A|| = \sqrt{162.2999}$

כעת,

$$cond(A) = ||A||\ ||A^{-1}||$$

נחשב את A^{-1}:

$$\left(\begin{array}{cc|cc} 9.7 & 6.6 & 1 & 0 \\ 4.1 & 2.8 & 0 & 1 \end{array}\right) \xrightarrow[R2=R2-\frac{4.1}{9.7}R1]{} \left(\begin{array}{cc|cc} 9.7 & 6.6 & 1 & 0 \\ 0 & 0.010309 & -0.42268 & 1 \end{array}\right) \xrightarrow[R1=\frac{1}{9.7}R1,\ \ R2=\frac{1}{0.010309}R2]{}$$

$$\left(\begin{array}{cc|cc} 1 & 0.68041 & 0.010309 & 0 \\ 0 & 1 & -40.99709 & 96.992310 \end{array}\right) \xrightarrow[R1=R1-0.68041\,R2]{} \left(\begin{array}{cc|cc} 1 & 0 & 27.999 & -65.9945 \\ 0 & 1 & -40.99709 & 96.992310 \end{array}\right)$$

קיבלנו: $A^{-1} = \begin{pmatrix} 28 & -66 \\ -41 & 97 \end{pmatrix}$

$$(A^{-1})^T = \begin{pmatrix} 28 & -41 \\ -66 & 97 \end{pmatrix}$$

$$(A^{-1})^TA^{-1} = \begin{pmatrix} 28 & -41 \\ -66 & 97 \end{pmatrix}\begin{pmatrix} 28 & -66 \\ -41 & 97 \end{pmatrix} = \begin{pmatrix} 2465 & -5825 \\ -5825 & 13765 \end{pmatrix}$$

כעת נחפש פולינום אופייני: $|(A^{-1})^TA^{-1} - \gamma I| = 0$

$$\left|\begin{pmatrix} 2465 & -5825 \\ -5825 & 13765 \end{pmatrix} - \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}\right| = \begin{vmatrix} 2465-\gamma & 75-5825.5 \\ -5825.5 & 13765-\gamma \end{vmatrix} = \gamma^2 - 16230\gamma + 160$$

$$\gamma^2 - 16230\gamma + 160 = 0 \rightarrow \gamma_1 = 16229.99014, \gamma_2 = 0.00986$$

אז קיבלנו: $||A^{-1}|| = \sqrt{16229.99014}$

וכעת נחשב את $cond(A)$:

$$cond(A) = ||A|| * ||A^{-1}|| = \sqrt{162.2999} * \sqrt{16229.999} = 1622.99945$$

Solve the linear system: $\begin{cases} 4x_1 - x_2 + x_3 = 7 \\ 4x_1 - 8x_2 + x_3 = -21 \\ -2x_1 + x_2 + 5x_3 = 15 \end{cases}$

using:

- **Jacobi**
- **Gauss Seidel**

Starting in both from the initial guess: $\underline{x}^0 = (1,2,2)^T$

Calculate 10 iterations. It is recommended to solve this question using MATLAB. If you use MATLAB, submit your code and output.

הקוד במאט-לאב:

```
%Jacobi
x1 = 1;
x2 = 2;
x3 = 2;
itr = 10;
Jacobi = zeros(itr, 3);
for k=1:itr,
        xNew_1 = (7 + x2 -x3)/4;
        xNew_2 = (4*x1+x3+21)/8;
        xNew_3 = (2*x1-x2+15)/5;
        x1 = xNew_1;
        x2 = xNew_2;
        x3 = xNew_3;
 Jacobi(k, :) = [x1, x2, x3];
end

Jacobi

%Gauss-Seidel
x1 = 1;
x2 = 2;
x3 = 2;
itr = 10;
Gauss = zeros(itr, 3);
for k=1:itr,
        x1 = (7 + x2 -x3)/4;
        x2 = (4*x1+x3+21)/8;
        x3 = (2*x1-x2+15)/5;
 Gauss(k, :) = [x1, x2, x3];
end

Gauss
```

ה-output:

```
Jacobi =

   1.7500    3.3750    3.0000
   1.8438    3.8750    3.0250
   1.9625    3.9250    2.9625
   1.9906    3.9766    3.0000
   1.9941    3.9953    3.0009
   1.9986    3.9972    2.9986
   1.9996    3.9991    3.0000
   1.9998    3.9998    3.0000
   1.9999    3.9999    2.9999
   2.0000    4.0000    3.0000

Gauss =

   1.7500    3.7500    2.9500
   1.9500    3.9688    2.9863
   1.9956    3.9961    2.9990
   1.9993    3.9995    2.9998
   1.9999    3.9999    3.0000
   2.0000    4.0000    3.0000
   2.0000    4.0000    3.0000
   2.0000    4.0000    3.0000
   2.0000    4.0000    3.0000
   2.0000    4.0000    3.0000
```

The values $x_1 = x_2 = 1.000$ are the solutions to: $\begin{cases} 1.133x_1 + 5.281x_2 = 6.414 \\ 24.14x_1 - 1.210x_2 = 22.93 \end{cases}$

- Use four-digit arithmetic (with rounding) and ***Gaussian Elimination*** without pivoting to find a computed approximate solution to the system.
- Same as above, but use partial pivoting.

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א. נשתמש ב gaussian elimination ללא pivoting ועם עיגול של 4 ספרות:

$$1.133x_1 + 5.281x_2 = 6.414$$
$$24.14x_1 - 1.210x_2 = 22.93$$

$$\begin{pmatrix} 1.133 & 5.281 & : & 6.414 \\ 24.14 & -1.210 & & 22.93 \end{pmatrix} \xrightarrow[R2=R2-\frac{24.14}{1.133}R1]{} \begin{pmatrix} 1.133 & 5.281 & : & 6.414 \\ 0 & -113.7 & & -113.8 \end{pmatrix}$$

$$-113.7x_2 = -113.8 \ \rightarrow x_2 = 1.001$$
$$6.414 = 1.133x_1 + 5.281 * 1.001 \rightarrow x_1 = 0.996$$

ב. כעת נפתור באופן דומה, אבל עם partial pivoting:

$$\begin{pmatrix} 1.133 & 5.281 & : & 6.414 \\ 24.14 & -1.210 & & 22.93 \end{pmatrix} \xrightarrow[R2\leftrightarrow R1]{}$$
$$\begin{pmatrix} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{pmatrix} \xrightarrow[R2=R2-\frac{1.133}{24.14}R1]{}$$
$$\begin{pmatrix} 24.14 & -1.210 & : & 22.93 \\ 0 & 5.538 & & 5.536 \end{pmatrix}$$

$$5.338x_2 = 5.336 \rightarrow x_2 = 1.000$$
$$22.93 = 24.14x_1 + -1.210 * 1.000 \rightarrow x_1 = 1.000$$

Use the power method with 9 iterations to locate an eigenvalue and eigenvector for the matrix (written in Matlab notation): [[5,-1,7]; [-1,-1,1]; [7,1,5]].

If you decide to solve this manually, check with MATLAB and submit the code you wrote. Else, just submit the MATLAB code and output. .6

Math-lab code:

```
A;[[7,1,5] ;[1,1-,1-] ;[1,7-,5]] =

%power method
v;[1 ;1 ;0] =
for k=1:9,
        v = A * v;
        v = v/max(v);
end
v
lambda = (A*v)'*v/(v'*v)
```

output:

```
v =

   1
   0
   1

lambda =   12
```