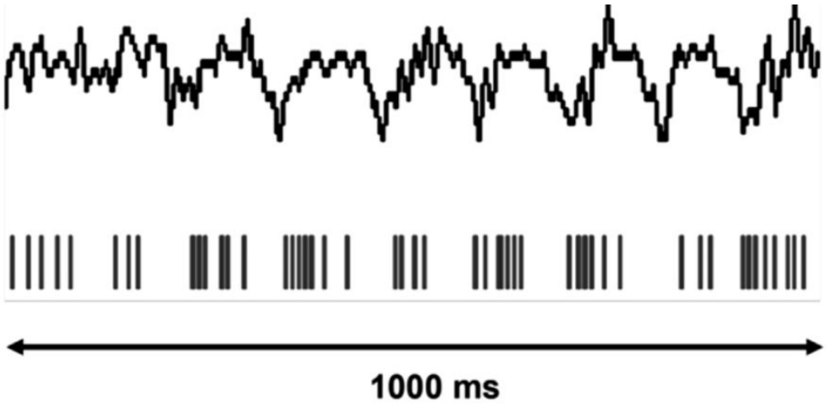
Lecture 7

Review of last lecture

Two Types of Data: Spikes and Local Field Potentials

Two different types of data:

- Spikes (list of times, point process)
- Local Field Potentials (continuous)



Discrete time Fourier transform

Sound familiar? $$X_j = \sum_{n=1}^{N} x_n \exp(-2\pi i f_j t_n)$$ Fourier transform of the data *x*.

replace with Euler's formula

$$\cos(-2\pi f_j t_n) + i \sin(-2\pi f_j t_n).$$

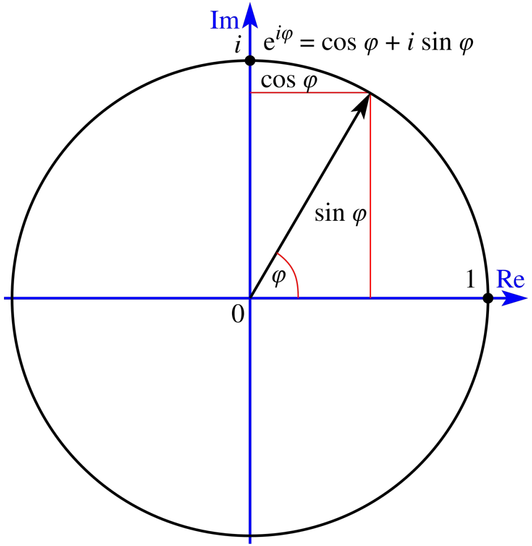
$$X_j = \boxed{\left(\sum_{n=1}^{N} x_n \cos(-2\pi f_j t_n)\right)} + i \boxed{\left(\sum_{n=1}^{N} x_n \sin(-2\pi f_j t_n)\right)}$$

Looks like $\boxed{A_k} = \frac{2}{T}\int_0^T V[t]\cos(2\pi f_k t)dt$ $\boxed{B_k} = \frac{2}{T}\int_0^T V[t]\sin(2\pi f_k t)dt$

Same idea: compare data to sinusoids and see how well they match

Why do we need sine and cosine?

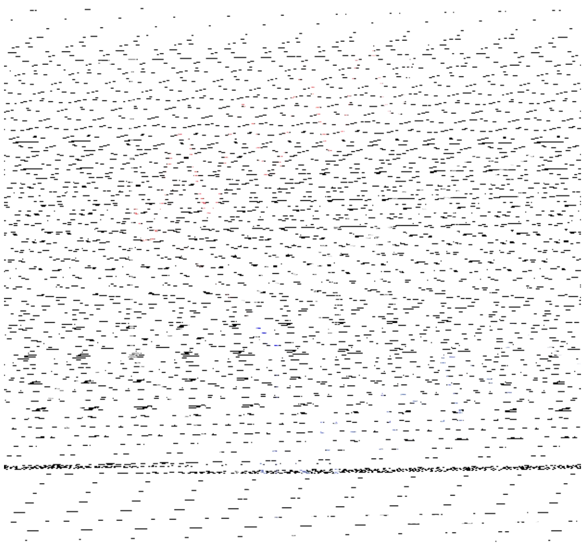
We can’t properly represent a full oscillation if we only use cosine or sine.



With only sine (or cosine), you lose phase—so you can’t represent an arbitrary oscillation

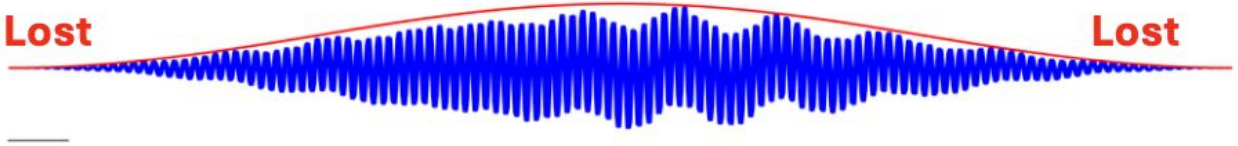
Phasors (pew pew)

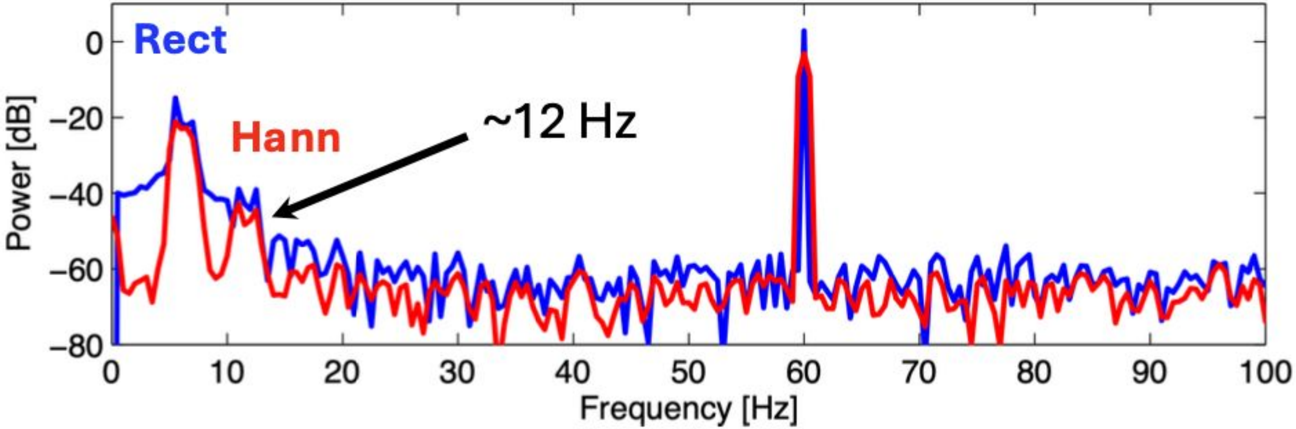
This is sometimes known as a complex phasor



Spectrum: tapers

Ex:





Good: Reduced sidelobes reveals a new peak.

Bad: Broader peaks & lose data at edges.

"More lives have been lost looking at the [rectangular tapered spectrum] than by any other action involving time series." [Tukey 1980]

Multitaper Spectral Estimation

- Select a time window width T (temporal resolution).
- Select a time-bandwidth product p=WT (i.e. set the frequency resolution).
- Compute the set of set of dpss tapers using T and p=WT
- Estimate the spectrum using each of the k= 2*p-1 tapers

$$\hat{S}_n(f) = \left| \sum_{t=1}^{N} w_n(t) y(t) e^{-i2\pi f t} \right|^2$$

- Average the estimates to get the spectrum!

$$S(f) = \frac{1}{k} \sum_{n=1}^{k} \hat{S}_n(f)$$

- You get multiple spectral estimates from the same piece of data.

Key things to remember

Your sampling rate must by twice the max frequency you want to represent (Nyquist)

You must subtract the mean to estimate the spectrum or spectrogram.

You should know your temporal and frequency resolution and tailor it to the frequency bands you want to resolve.

Don’t use rectangular tapers.

The number of time points in a window controls the number of frequency bins, not the frequency resolution.

You need to account for the 1/f (pink noise) falloff in frequency and know that it can mask rhythms

Use multitaper spectral estimation

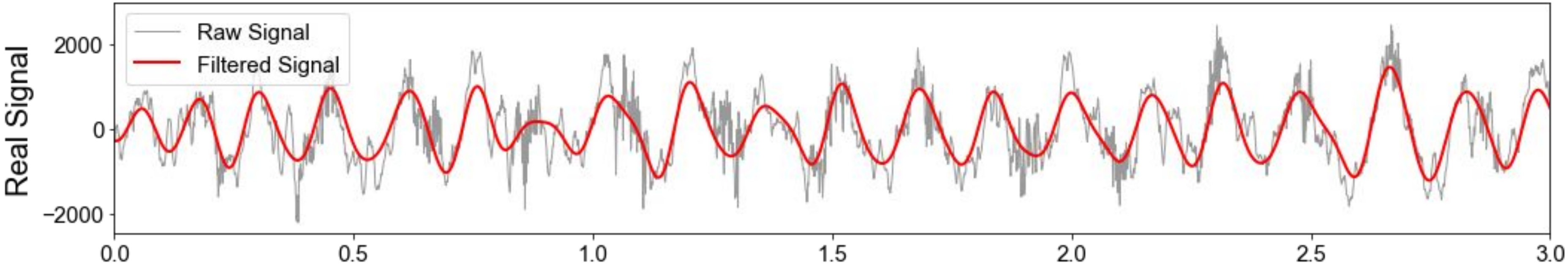
This time…

- Bandpass filters (and other types)
- Hilbert transform and instantaneous phase and power
- Spectral coherence (and related measures)
- Spike-field coherence (non-parametric)
- Spike-field coherence with Poisson regression (parametric)

Bandpass filters

Bandpass filters keep signal components in a frequency range (e.g., theta 4–12 Hz) and suppress components outside it.

There are also **highpass** (keep high frequencies), **lowpass** (keep low frequencies), **bandstop** (stop certain frequencies, aka notch).



A filter can be defined in terms of its difference equation in the time domain

$$y(n) = b_0 x(n) + \ldots + b_M x(n-M) - a_1 y(n-1) - \ldots - a_N y(n-N)$$

$$= \underbrace{\boxed{\sum_{k=0}^{M} b_k x(n-k)}}_{\text{current and past inputs}} - \underbrace{\boxed{\sum_{k=1}^{N} a_k y(n-k)}}_{\text{past outputs}}$$

current and past inputs

"weighted moving average”

past outputs

“autoregression”

x = input of filter

y = output of filter

Two kinds of filters: FIR and IIR

Finite Impulse Response (FIR)

$$\boxed{\sum_{k=0}^{M} b_k x(n-k)}$$

current and past inputs

Infinite Impulse Response (IIR)

$$\boxed{\sum_{k=0}^{M} b_k x(n-k)} - \boxed{\sum_{k=1}^{N} a_k y(n-k)}$$

current and past inputs

past outputs

FIR vs. IIR Filters

- A causal FIR filter can be linear-phase – i.e., the same time delay across all frequencies – whereas a causal IIR filter cannot. The phase and group delay characteristics are also usually better for FIR filters.
- IIR filters can generally have a steeper cutoff than an FIR filter of equivalent order.
- IIR filters are generally less numerically stable, in part due to accumulating error (due to recursive calculations).

Generally FIR filters are easier to deal with and IIR filters are more efficient.

https://mne.tools/stable/auto_tutorials/preprocessing/25_background_filtering.html

Causal vs. Non-causal (Zero-phase) Filtering

Causal filters

- Output at time *t* depends only on present/past input (works in real time).
- Usually introduce **phase delay** (group delay); IIR filters often have **frequency-dependent** delay.

Non-causal zero-phase filters (forward–backward)

- Use past *and future* samples (offline).
- **No phase distortion** (phase is preserved), but magnitude response is effectively squared and edges need care.
- **scipy.signal.filtfilt**

Rule of thumb

- Use **zero-phase** for offline LFP phase/amplitude analyses.
- Use **causal** filters for real-time/closed-loop or when causal timing matters.

Filters have time frequency tradeoffs

To be selective for a narrow range of frequencies, a filter must “listen” for many cycles.

That means the output at time n depends on a longer chunk of the signal—so timing gets smeared.

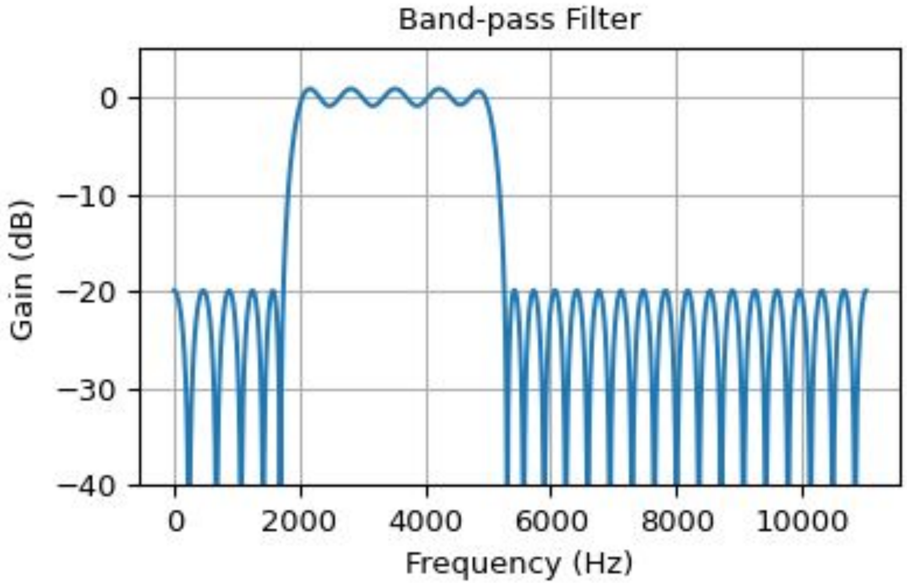
More “memory” gives better frequency selectivity, but it can’t respond instantly in time.

$$\boxed{\sum_{k=0}^{M} b_k x(n-k)} - \boxed{\sum_{k=1}^{N} a_k y(n-k)}$$

current and past inputs

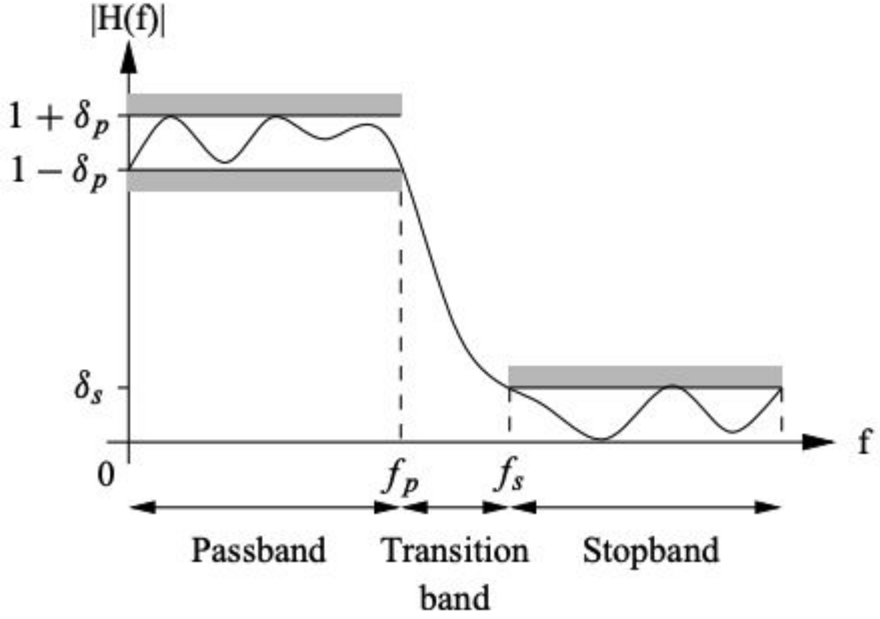
past outputs

Example of a filter



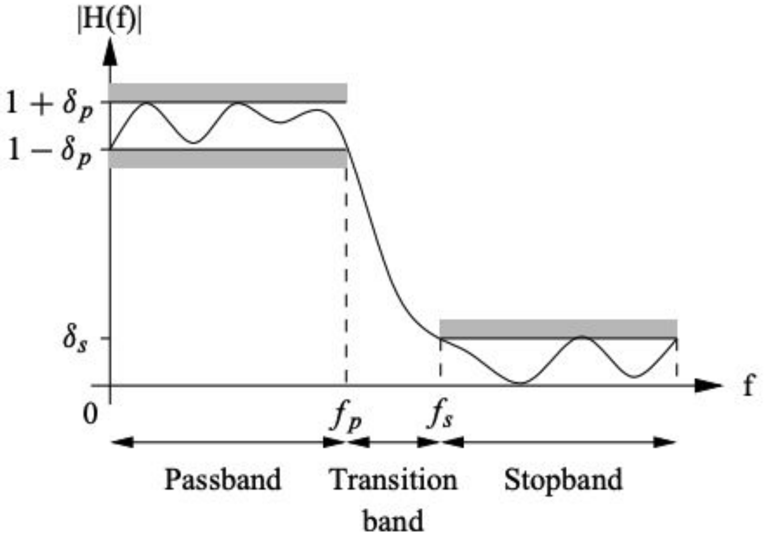
frequency magnitude plot

Characteristics of the filter



Filter design tradeoffs

1. Ripple in the pass-band
2. Attenuation of the stop-band
3. Steepness of roll-off
4. Filter order (i.e., length for FIR filters)
5. Time-domain ringing



Filter design isn’t easy and should be done carefully

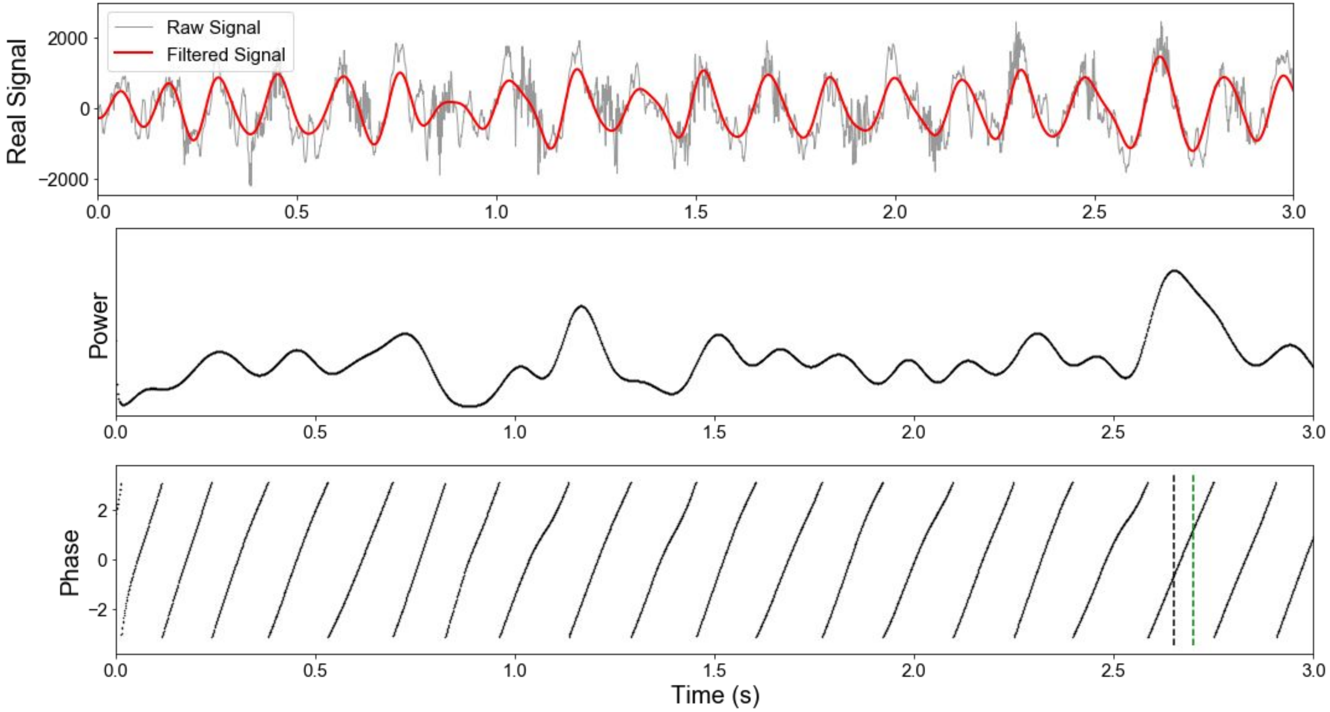
For FIR filters, there are tools to help you design them:

1. The Remez algorithm (**scipy.signal.remez()**)
2. Windowed FIR design (**scipy.signal.firwin2()**, **scipy.signal.firwin()**)
3. Least squares design (**scipy.signal.firls()**)
4. Frequency-domain design (construct filter in Fourier domain and use an **IFFT** to invert it)

Filter summary

- A filter is a frequency-selective operation with a time footprint (its impulse response / “memory”).
- **Time–frequency tradeoff**: sharper frequency selectivity (narrow transitions / narrow bands) requires longer time support → more delay/smearing and more ringing.
- So filter design is choosing: passband, transition width, stopband attenuation, and how much time-domain distortion you can tolerate.
- **FIR vs IIR**:
  - **FIR** can be designed with linear phase (simple, predictable delay; safe for timing/phase work).
  - **IIR** is computationally efficient but often has frequency-dependent phase delay (needs more care for phase interpretation).
- In practice: use well-tested routines (FIR design, stable IIR designs, filtfilt for zero-phase offline), inspect frequency response, and always sanity-check results on raw snippets (edge artifacts + transient-induced ringing).

Sometimes we want the instantaneous phase and power of a signal in a particular frequency band



Contrast to Fourier Transform

In the Fourier Transform, you only get the phase at the beginning of the window (or whatever time you've referenced to in the window) and the average power in the window (time-frequency bin)

Instantaneous phase and power tell you for each time point, what the phase and power are **for a given narrowband signal**.

Hilbert transform

$$y = H(x)$$

$$H(x) = \begin{cases} -\pi/2 \text{ phase shift} & \text{if } f > 0, \\ 0 \text{ phase shift} & \text{if } f = 0, \\ \pi/2 \text{ phase shift} & \text{if } f < 0. \end{cases}$$

The Hilbert transform $H(x)$ of the signal x produces a phase shift of ± 90 degrees for $\mp$ frequencies of x.

Hilbert transform

Define: Analytic signal z

$$z = x + i\,y \;= x + i\,H(x)$$

Impact: remove negative frequencies from z

Q. How?

Hilbert transform

Q. What does it do?

<u>Ex.</u>

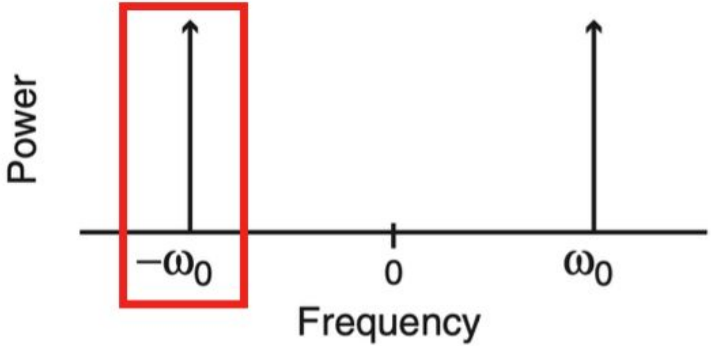
$$x_0 = 2\cos(2\pi f_o t) = 2\cos(\omega_0 t) \quad \text{where} \quad \omega_0 = 2\pi f_o$$

Euler's formula

$$x_0 = e^{i\omega_0 t} + e^{-i\omega_0 t}$$

positive frequency   negative frequency

we usually ignore this one



<u>Note</u>: The spectrum has <u>two peaks</u>

Hilbert transform

Apply the Hilbert transform to x_0.

$$x_0 = e^{i\omega_0 t} + e^{-i\omega_0 t}$$

$$H(x) = \begin{cases} -\pi/2 \text{ phase shift} \quad \text{if } f > 0, \\ \end{cases}$$

$\boxed{\rightarrow \text{multiply positive frequency part of } x \text{ by } -i}$

Q. Really?

Consider $e^{i\omega_0 t}$

positive frequency part of x

Shift $e^{i\omega_0 t}$ by $-\frac{\pi}{2}$

$\rightarrow e^{i(\omega_0 t - \frac{\pi}{2})} \rightarrow e^{i\omega_0 t} e^{-i\pi/2} \rightarrow e^{i\omega_0 t}(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}) \rightarrow \boxed{e^{i\omega_0 t}(-i)}$

Hilbert transform

Apply the Hilbert transform to x_0.

$$x_0 = e^{i\omega_0 t} + e^{-i\omega_0 t}$$

$$H(x) = \begin{cases} -\pi/2 \text{ phase shift} & \text{if } f > 0, \\ 0 \text{ phase shift} & \text{if } f = 0, \\ \pi/2 \text{ phase shift} & \text{if } f < 0. \end{cases} \quad \rightarrow \quad H(x) = \begin{cases} -ix & \text{if } f > 0, \\ x & \text{if } f = 0, \\ ix & \text{if } f < 0. \end{cases}$$

So, $$x_0 = \underset{\text{multiply by } -i}{e^{i\omega_0 t}} + \underset{\text{multiply by } i}{e^{-i\omega_0 t}}$$

$$y_0 = H(x_0) = -ie^{i\omega_0 t} + ie^{-i\omega_0 t}$$

Euler's formula

$$\boxed{= 2\sin(\omega_0 t)}$$

Hilbert Transform of x_0 (a cosine function) is a sine function.

Hilbert transform

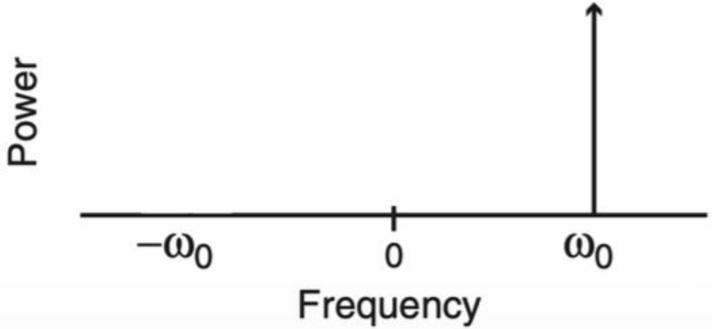
Analytic signal z

$$z = x + iy = x + iH(x)$$

$$2\cos(\omega_0 t) \qquad 2\sin(\omega_0 t)$$

$$= 2\cos(\omega_0 t) + i\,2\sin(\omega_0 t)$$

$$\boxed{= 2e^{i\omega_0 t}}$$

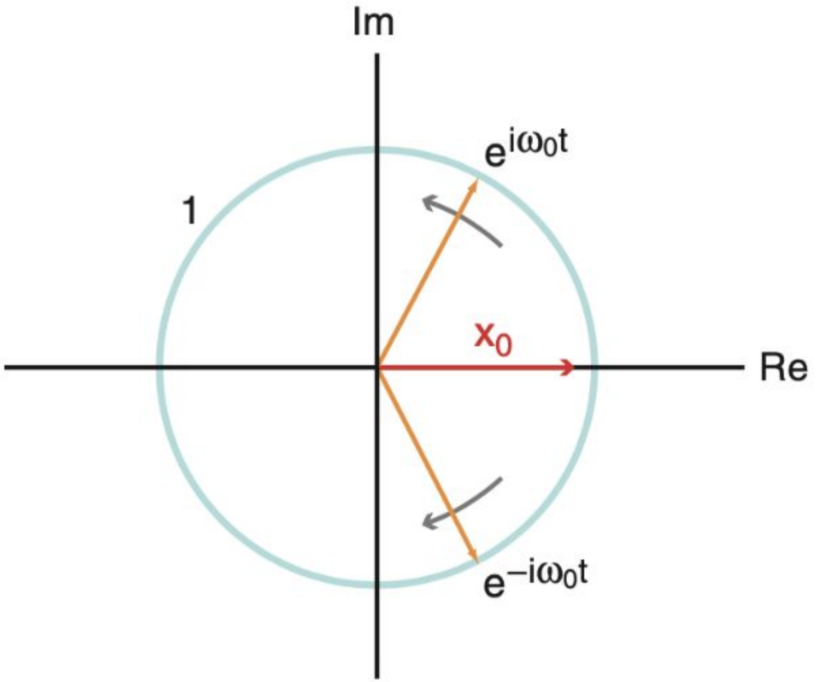


The analytic signal contains
no negative frequencies

Hilbert transform

<u>Original signal</u> $x_0 = 2\cos(2\pi f_o t) = e^{i\omega_0 t} + e^{-i\omega_0 t}$

Complicated (2 complex exponentials)



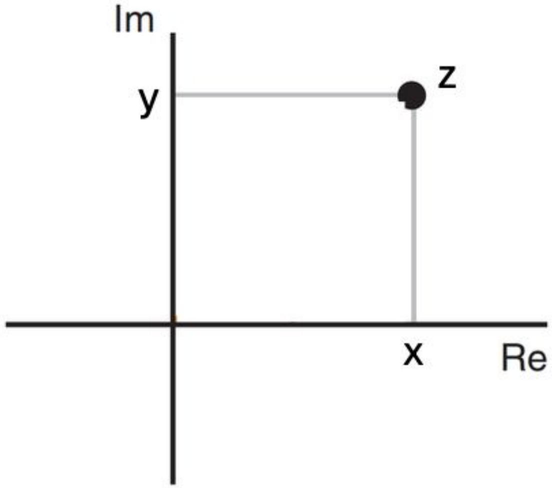
<u>Analytic signal</u> $z_0 = 2e^{i\omega_0 t}$

Simple (1 complex exp)

A point in the complex plane

Hilbert transform

Analytic signal $z = x + iy$ A point in the complex plane



$$z(t) = A(t)\, e^{i\,\phi(t)}$$

amplitude phase

Get the **amplitude** and **phase** from the analytic signal

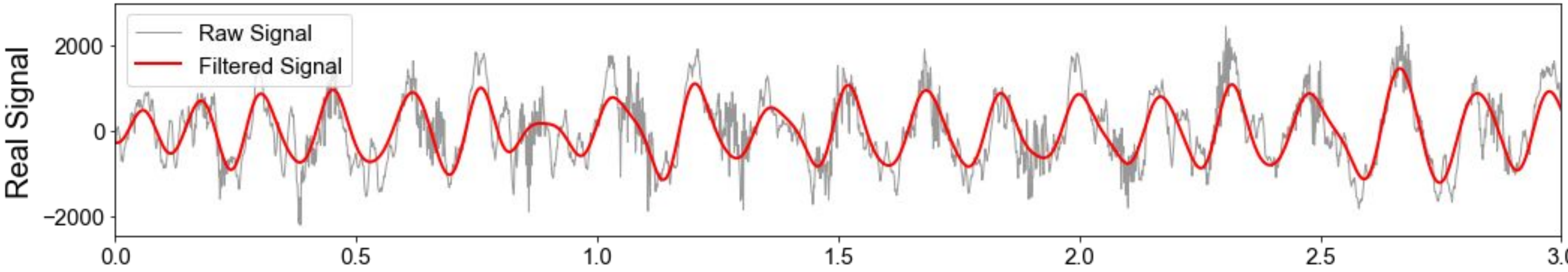
Ex.

$z_0(t) = 2e^{i\omega_0 t}$ $A(t) = 2$ $\phi(t) = \omega_0\, t$

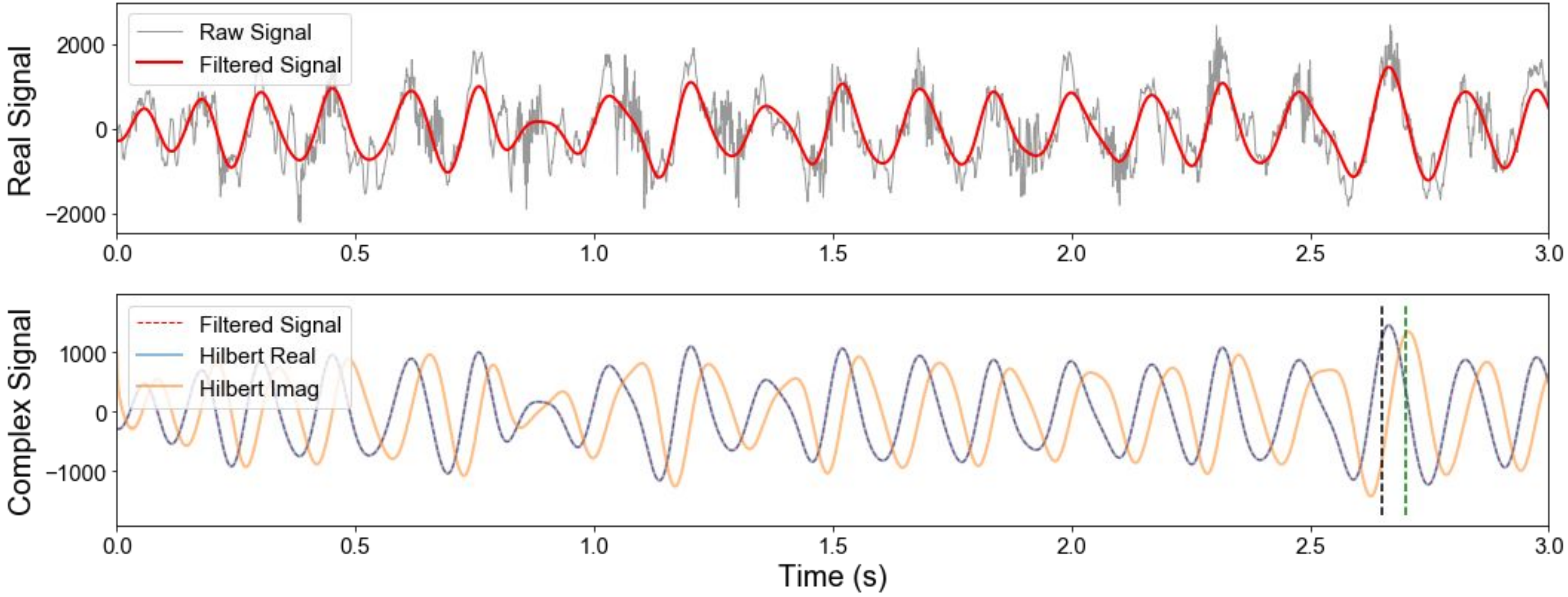
Step 1: Isolate one rhythm with a bandpass

$$x_{\mathrm{bp}}(t) = \mathrm{Bandpass}\{x(t) \text{ around } [f_1, f_2]\}$$

Why: phase only makes sense if the signal is (approximately) narrowband



Analytic signal



http://www.rdgao.com/roemerhasit_Hilbert_Transform/

Step 3: Compute instantaneous phase and power

Instantaneous amplitude (envelope)

$$A(t) = |z(t)| = \sqrt{x_{\mathrm{bp}}(t)^2 + \mathcal{H}\{x_{\mathrm{bp}}(t)\}^2}$$

Instantaneous power

Common choices (pick one and state it):

- **Power as squared amplitude**

$$P(t) = |z(t)|^2 = A(t)^2$$

- **Power normalized to match sinusoid mean-square convention**
  (if $x(t) = A\cos(\cdot)$, mean power over a cycle is $A^2/2$)

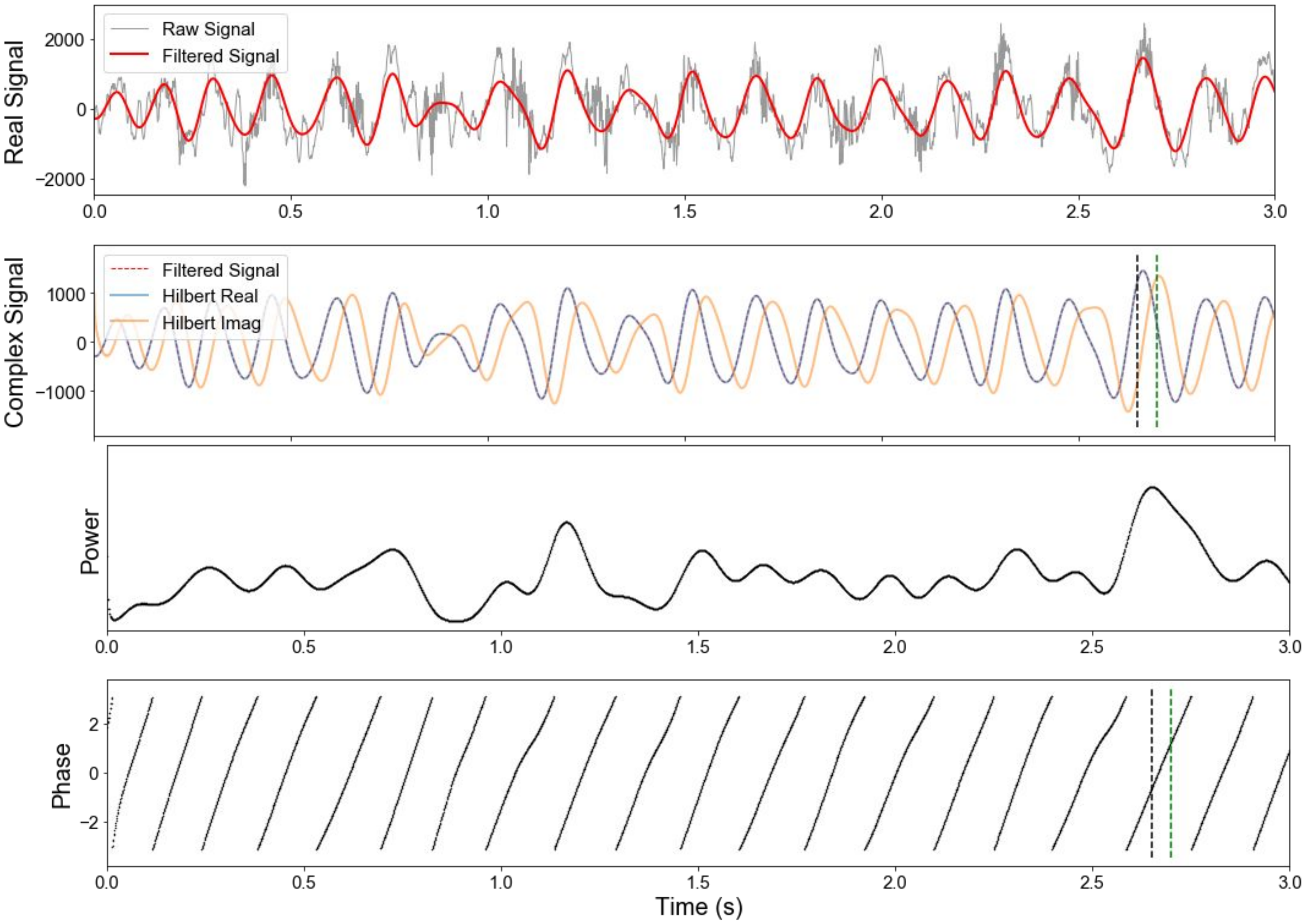
$$P_{\mathrm{ms}}(t) = \frac{|z(t)|^2}{2} = \frac{A(t)^2}{2}$$

Instantaneous phase

$$\phi(t) = \arg(z(t)) = \mathrm{atan2}\Big(\mathcal{H}\{x_{\mathrm{bp}}(t)\},\ x_{\mathrm{bp}}(t)\Big)$$

(If you want instantaneous frequency too:)

$$f_{\mathrm{inst}}(t) = \frac{1}{2\pi}\frac{d}{dt}\,\mathrm{unwrap}(\phi(t))$$



Raw Signal
Filtered Signal
Real Signal
2000
0
−2000
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Filtered Signal
Hilbert Real
Hilbert Imag
Complex Signal
1000
0
−1000
Power
Phase
2
0
−2
Time (s)

Intuition

- Think of the Hilbert transform as a coordinate transformation that is convenient for making the analytic signal (time domain to phasor circular coordinates)
- We are not changing any information (because real signals are symmetric in the frequency domain and we can take the real part of the analytic signal to recover)
- We are just transforming a 1D time series to 2D complex time series
- We can also think of it as zeroing out the negative frequency Fourier coefficients and doubling the positive frequency Fourier coefficients
- NB: **scipy.signal.hilbert** does not compute the Hilbert transform, it computes the analytic signal

Coherence: words

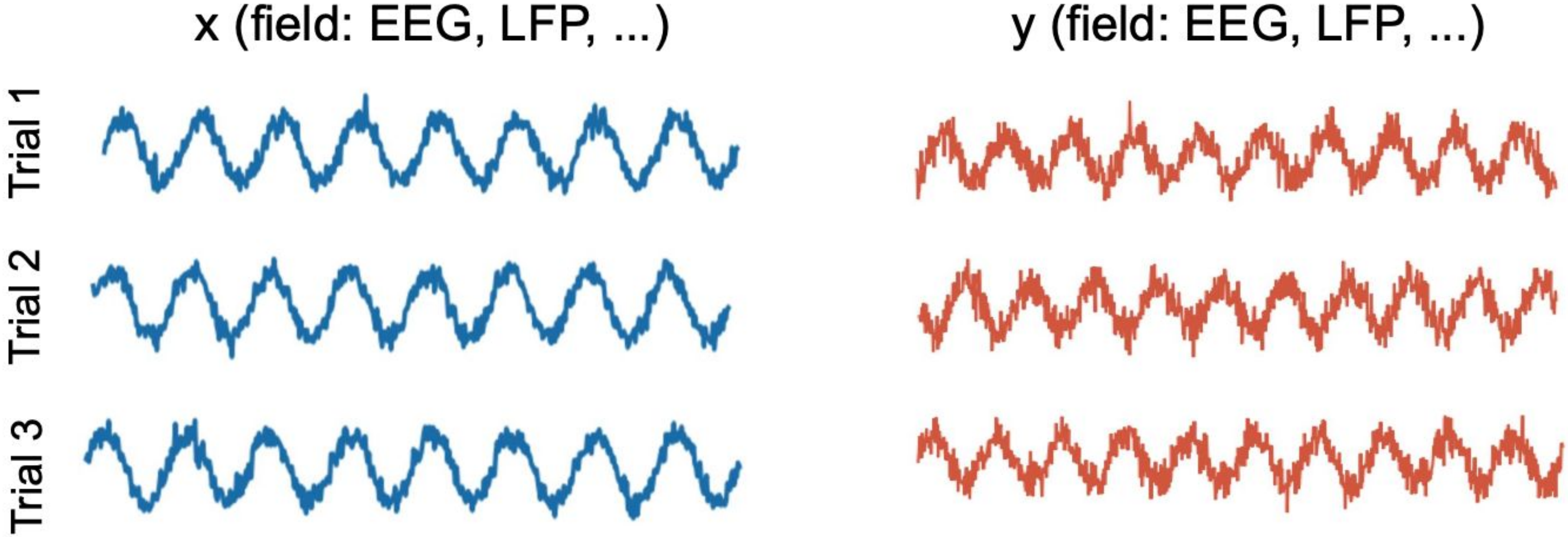
A constant phase relationship between two signals, at the same frequency, across trials.

Note

- "*same frequency*"
- "*across trials*"

Coherence: idea

Ex: Record data simultaneously from two sensors, across multiple trials

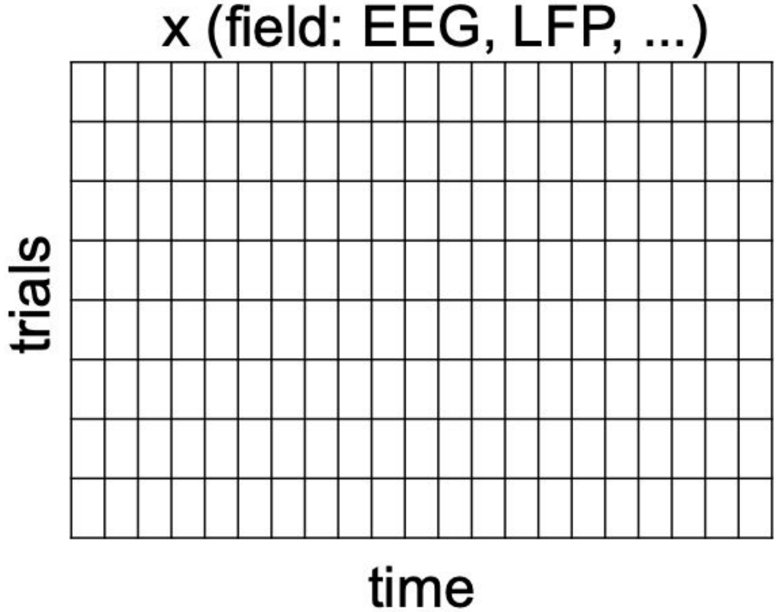


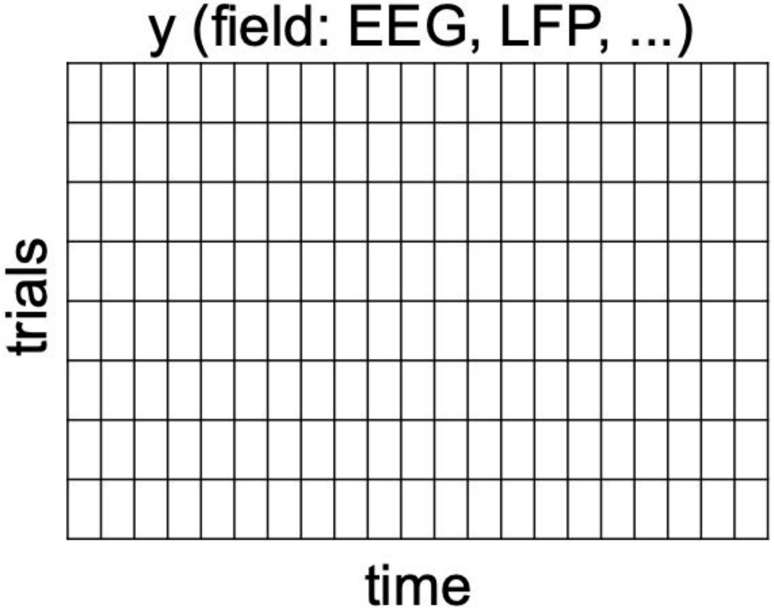
Is there a *constant phase relationship between x & y, at the same f, across trials?*

Coherence: idea

<u>Ex</u>: Record data simultaneously from two sensors, across multiple trials

Organize the data ...





Each row is a trial, each column is a time point, organize data in matrices.

Coherence: equations

This is what we'll compute:

$$\kappa_{xy,\,j} = \frac{|< S_{xy,\,j} >|}{\sqrt{< \boxed{S_{xx,\,j}} >}\sqrt{< S_{yy,\,j} >}}$$

$S_{xy,j}$ = Cross-spectrum at frequency index j

$S_{xx,j}, S_{yy,j}$ = Auto-spectra at frequency index j ✓

$< S >$ = Average of S over trials

Define each piece ...

Spectrum: intuition

$$S_{xx,j} = \frac{2\Delta^2}{T} X_j X_j^*$$

More spectrum intuition ...

Fourier transform of the data *x*.

Data as a function of time index n

$$X_j = \sum_{n=1}^{N} \boxed{x_n} \boxed{\exp(-2\pi i f_j t_n)}\,.$$

Replace with Euler's formula

Sinusoids at frequency f_j

$$X_j = \left(\sum_{n=1}^{N} x_n \cos(-2\pi f_j t_n)\right) + i\left(\sum_{n=1}^{N} x_n \sin(-2\pi f_j t_n)\right)$$

Real

Imaginary

Spectrum: intuition

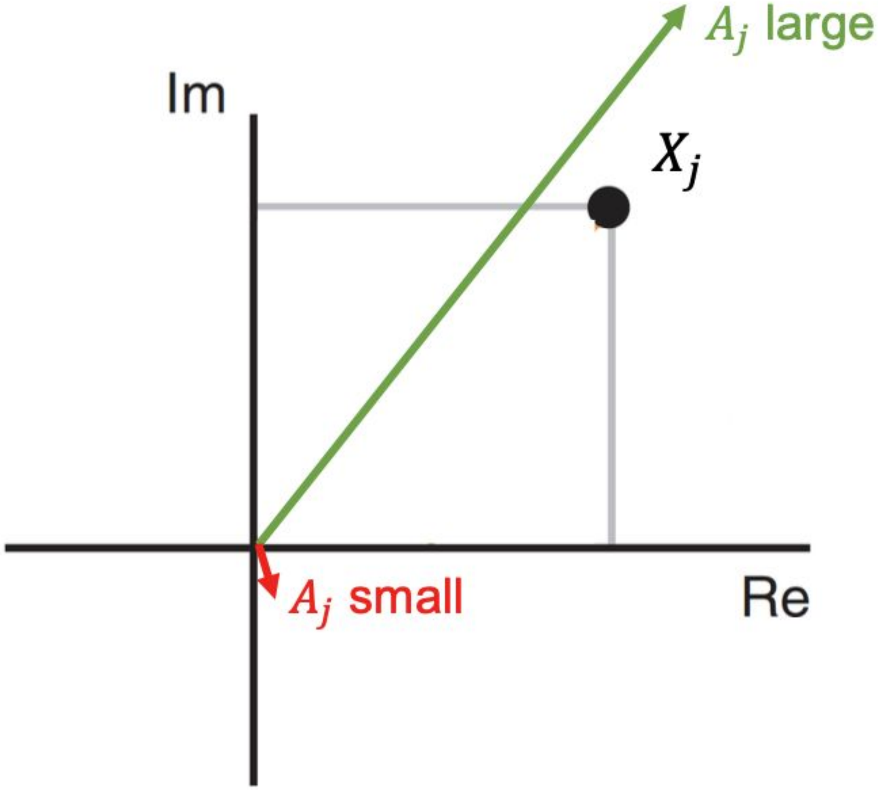
X_j lives in the complex-plane:

Express X_j in polar coordinates:

$$X_j = A_j \exp(i\phi_j)$$

A_j = Amplitude at frequency index j

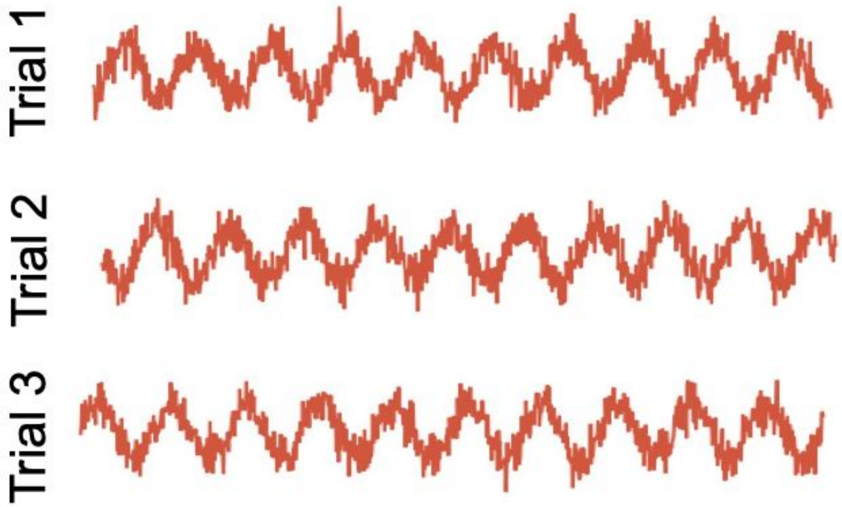
ϕ_j = Phase at frequency index j



Match:	A_j at frequency f_j is large
Mismatch:	A_j at frequency f_j is small

Spectrum from multiple trials

Ex: Record data across multiple trials



K total trials

Q. Spectrum?

Spectrum from multiple trials

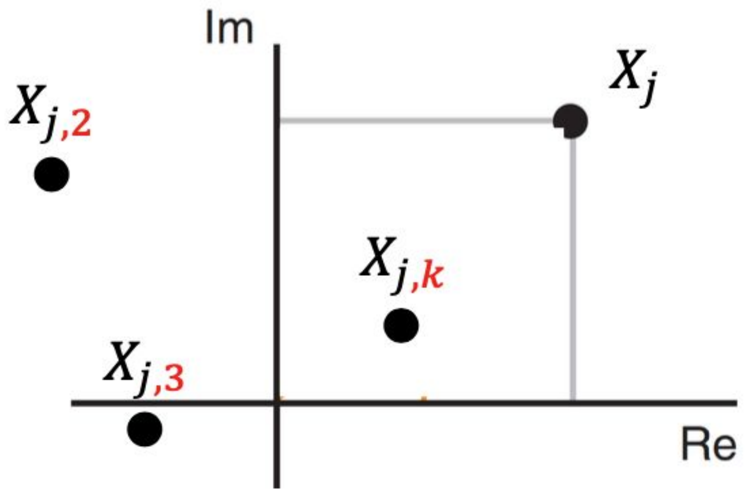
X_j lives in the complex-plane:

Fourier transform for each trial lives in the complex-plane:

In polar coordinates:

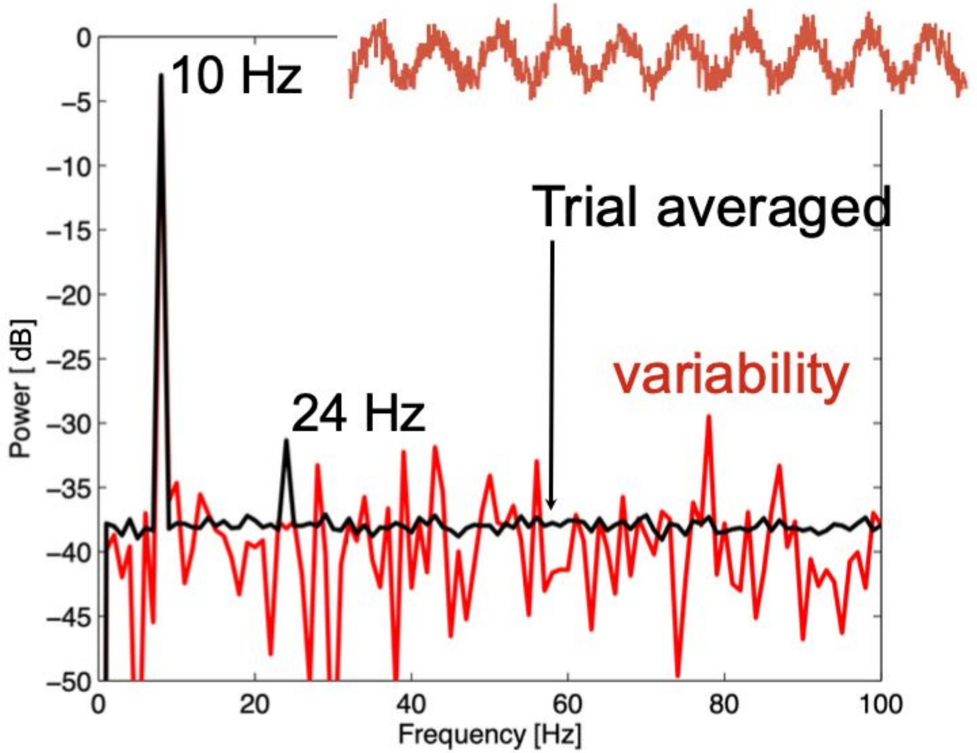
$A_{j,k}$ = Amplitude at frequency index j and trial index k

$\phi_{j,k}$ = Phase at frequency index j and trial index k



Spectrum from multiple trials

Single trial:



Trial averaged spectrum:

reduced variability.
reveals another peak . . .

Coherence: equations

$$\kappa_{xy,\,j} = \frac{\left|\boxed{< S_{xy,\,j} >}\right|}{\sqrt{< S_{xx,\,j} >}\sqrt{< S_{yy,\,j} >}}$$

$\checkmark$ $\checkmark$

$$< S_{xx,j} > = \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} A_{j,k}^2 \qquad < S_{yy,j} > = \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} B_{j,k}^2$$

Consider the trial averaged cross-spectrum ...

Coherence: equations

The trial averaged cross-spectrum at frequency index j:

Like the auto-spectrum, but use X and Y.

$$< S_{xy,j} >= \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} X_{j,k}Y_{j,k}^*$$

In polar coordinates:

$$< S_{xy,j} >= \frac{2\Delta^2}{T}\frac{1}{K}\sum_{k=1}^{K} A_{j,k}B_{j,k}\exp\left(i\Phi_{j,k}\right)$$

where $\underbrace{\Phi_{j,k} = \phi_{j,k}}_{\text{Phase of x}} - \underbrace{\theta_{j,k}}_{\text{Phase of y}}$ is the phase difference between the two signals, at frequency index j and trial k.

Coherence: equations

Put it all together ...

$$\kappa_{xy,\,j} = \frac{|< S_{xy,\,j} >|}{\sqrt{< S_{xx,\,j} >}\sqrt{< S_{yy,\,j} >}}$$

In polar coordinates

cross-spectrum of x & y, depends on trial averaged amplitudes, phase differences.

$$= \frac{\left|\sum_{k=1}^{K} A_{j,k} B_{j,k} \exp\left(i\Phi_{j,k}\right)\right|}{\sqrt{\sum_{k=1}^{K} A_{j,k}^2}\sqrt{\sum_{m=1}^{K} B_{j,m}^2}}$$

x trial averaged spectrum, at frequency index j

y trial averaged spectrum, at frequency index j

Coherence: intuition

To build intuition, assume: the amplitude is <u>identical</u> for both signals and all trials.

$$A_{j,k} = B_{j,k} = C_j$$

Note: no trial dependence

then

$$\kappa_{xy,j} = \frac{\left| \sum_{k=1}^{K} A_{j,k} B_{j,k} \exp\left(i\Phi_{j,k}\right) \right|}{\sqrt{\sum_{k=1}^{K} A_{j,k}^2} \sqrt{\sum_{m=1}^{K} B_{j,m}^2}}$$

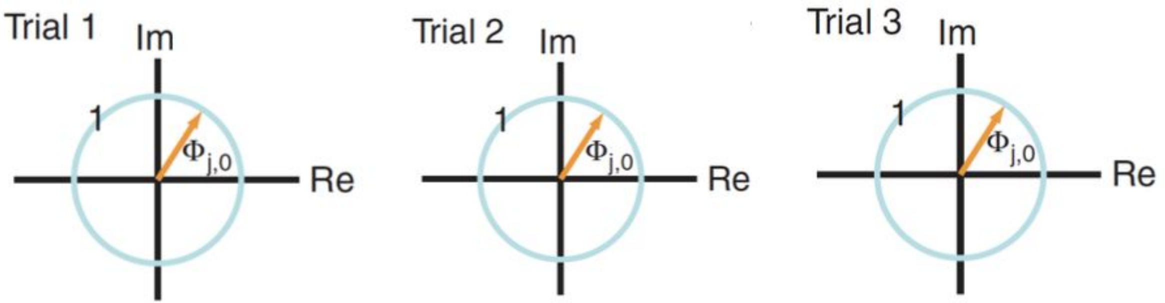
only involves the <u>phase difference</u> between the two signals averaged across trials.

Coherence: intuition

$$\kappa_{xy,j} = \frac{1}{K}\left|\sum_{k=1}^{K}\exp\left(i\Phi_{j,k}\right)\right|$$

Case 1: Phases align across trials. $\boxed{\Phi_{j,k} = \Phi_{j,0}}$

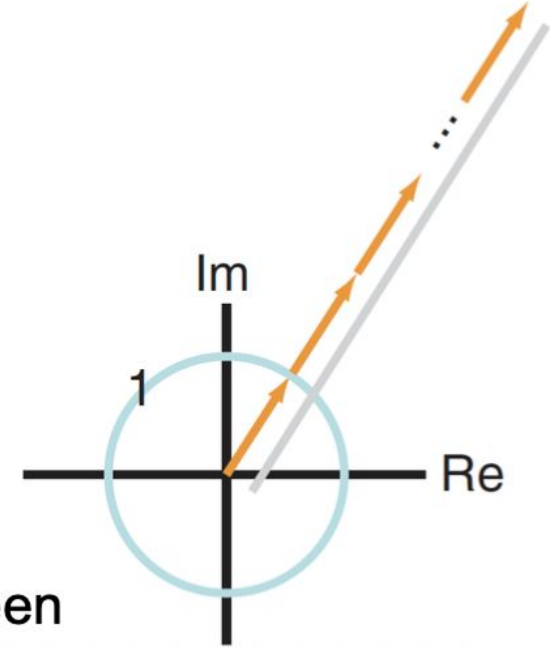
Plot $\exp\left(i\Phi_{j,k}\right)$ in the complex plane.



sum these vectors end to end across trials

divide by K

$$\kappa_{xy,j} \approx 1$$



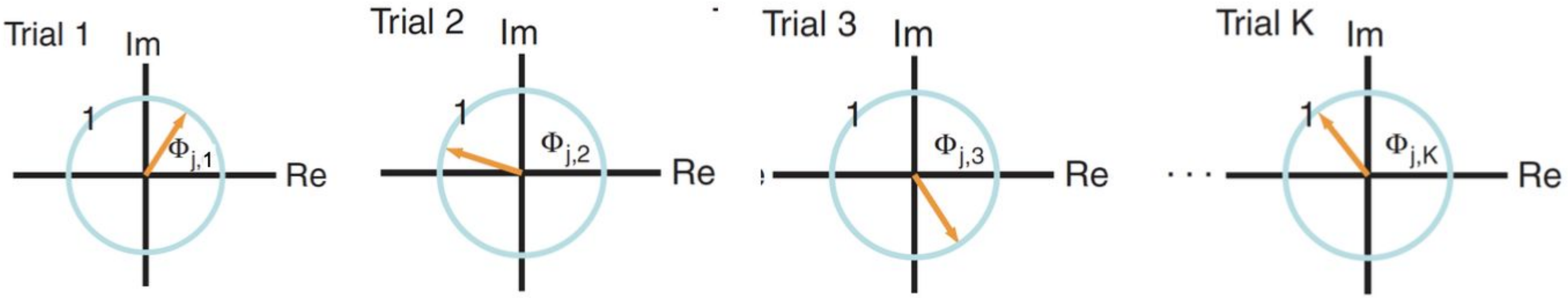
strong coherence - constant phase relation between the two signals across trials at frequency index j.

Coherence: intuition

$$\kappa_{xy,j} = \frac{1}{K}\left|\sum_{k=1}^{K} \exp\left(i\Phi_{j,k}\right)\right|$$

Case 2: Random phase differences across trials.

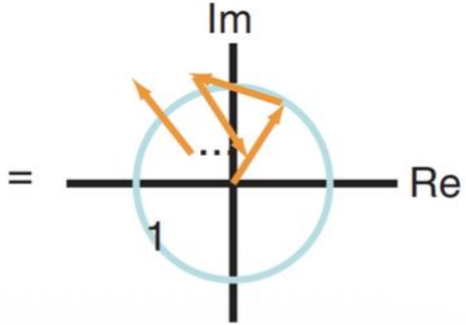
Plot $\exp\left(i\Phi_{j,k}\right)$ in the complex plane.



sum these vectors end to end across trials

divide by K

$$\kappa_{xy,j} \approx 0$$



weak coherence - random phase relation between the two signals across trials at frequency index j.

Coherence: summary

$$0 \leq \kappa_{xy,j} \leq 1$$

0: no coherence between signals x and y at frequency index j

1: strong coherence between signals x and y at frequency index j .

The coherence is a measure of the phase consistency between two signals at frequency index j across trials.