Signature-based Gröbner Basis Algorithms in SINGULAR

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Buchberger's algorithm

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G of I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{f_i\}$ for all $i \in \{1, \ldots, m\}$
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   Go on with the next element in P.
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How to predict zero reductions?

How to detect zero reductions in advance?

Let $I = \langle g_1, g_2 \rangle \in \mathbb{Q}[x, y, z]$ and let $<$ denote DRL. Let

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$$\begin{aligned} \text{spol}(g_2, g_1) &= xg_2 - yg_1 = \mathbf{xy^2} - xz^2 - \mathbf{xy^2} + yz^2 \\ &= -xz^2 + yz^2. \end{aligned}$$

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We can reduce further using $z^2 g_2$:

$$\boxed{-y^2z^2 + z^4 + y^2z^2 - z^4 = 0.}$$

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- **A signature** of f is given by $\mathfrak{s}(f) = \mathrm{lt}_{\prec}(\alpha)$ where $f = \overline{\alpha}$.
- An element $\alpha \in \mathscr{R}^m$ with $\overline{\alpha} = 0$ is called **a syzygy**.

Our example again – with signatures and $\prec_{\text{pot}}$

$$g_1 = xy - z^2,\ \mathfrak{s}(g_1) = e_1,$$
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Note that $\mathfrak{s}(\operatorname{spol}(g_3, g_1)) = xy\,e_2$ and $\operatorname{lm}(g_1) = xy$.

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Our task: Keep signatures correct.

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$\alpha \in R^m \implies$ polynomial $\overline{\alpha}$ with $\operatorname{lt}(\overline{\alpha})$, signature $\mathfrak{s}(\alpha) = \operatorname{lt}(\alpha)$

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S-pairs/S-polynomials:

$$\mathrm{spol}\left(\overline{\alpha}, \overline{\beta}\right) = a\overline{\alpha} - b\overline{\beta} \implies \mathrm{spair}(\alpha, \beta) = a\alpha - b\beta$$

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Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from P by increasing signature.

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Sketch of proof

1. $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta)$.
2. All S-pairs are handled by increasing signature.
   $\Rightarrow$ All relatons $\prec \mathfrak{s}(\alpha)$ are known:

$$\alpha = \beta + \text{ elements of smaller signature}$$

$\square$

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S-pairs in signature T

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What are all possible configurations to reach signature T?

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$$\mathfrak{R}_T = \left\{ a\alpha \mid \alpha \text{ handled by the algorithm and } \mathfrak{s}(a\alpha) = T \right\}$$

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Define an order on $\mathfrak{R}_T$ and choose the maximal element.

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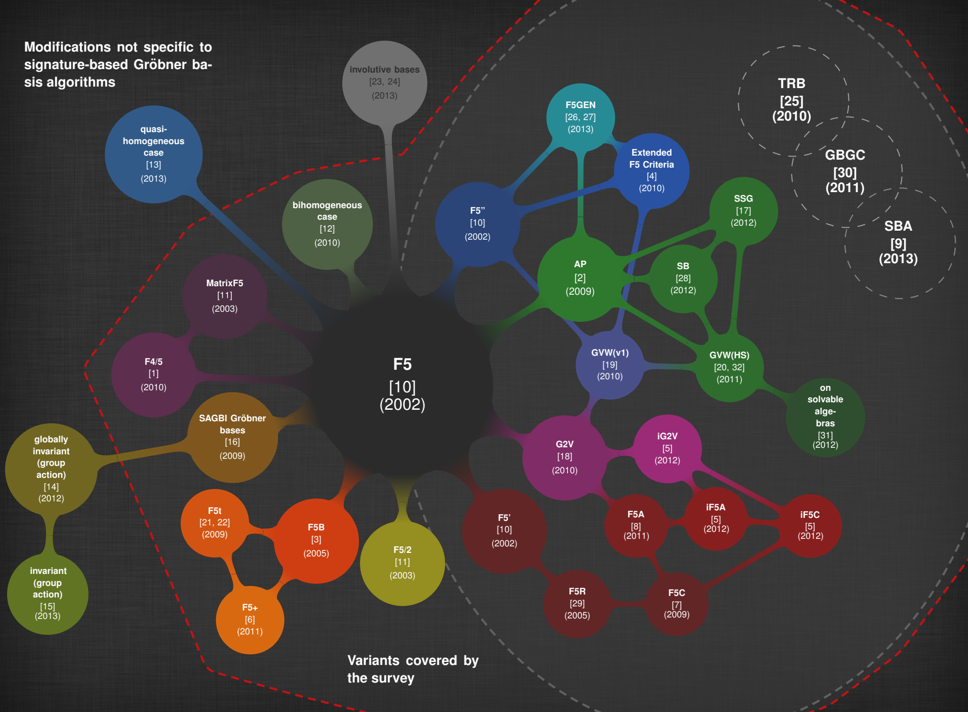
Revisiting our example with $\prec_{\text{pot}}$

$$\mathfrak{s}\left(\text{spol}(g_3, g_1)\right) = xye_2$$

$$\left.\begin{aligned} g_1 &= xy - z^2 \\ g_2 &= y^2 - z^2 \end{aligned}\right\} \Rightarrow \text{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \ldots$$



Modifications not specific to signature-based Gröbner basis algorithms
involutive bases [23, 24] (2013)
quasi-homogeneous case [13] (2013)
bihomogeneous case [12] (2010)
MatrixF5 [11] (2003)
F4/5 [1] (2010)
SAGBI Gröbner bases [16] (2009)
globally invariant (group action) [14] (2012)
invariant (group action) [15] (2013)
F5t [21, 22] (2009)
F5B [3] (2005)
F5+ [6] (2011)
F5/2 [11] (2003)
F5 [10] (2002)
F5GEN [26, 27] (2013)
Extended F5 Criteria [4] (2010)
F5” [10] (2002)
AP [2] (2009)
SSG [17] (2012)
SB [28] (2012)
GVW(v1) [19] (2010)
GVW(HS) [20, 32] (2011)
on solvable algebras [31] (2012)
G2V [18] (2010)
iG2V [5] (2012)
F5’ [10] (2002)
F5A [8] (2011)
iF5A [5] (2012)
iF5C [5] (2012)
F5R [29] (2005)
F5C [7] (2009)
TRB [25] (2010)
GBGC [30] (2011)
SBA [9] (2013)
Variants covered by the survey

zero reductions (Singular-4-0-0, $\mathbb{F}_{32003}$)

Benchmark	STD	SBA $<_{\text{pot}}$	SBA $<_{\text{d-pot}}$	SBA $<_{\text{lt}}$
cyclic-8	4,284	243	243	671
cyclic-8-h	5,843	243	243	671
eco-11	3,476	0	749	749
eco-11-h	5,429	502	502	749
katsura-11	3,933	0	0	353
katsura-11-h	3,933	0	0	353
noon-9	25,508	0	0	682
noon-9-h	25,508	0	0	682
Random$(11,2,2)$	6,292	0	0	590
HRandom$(11,2,2)$	6,292	0	0	590
Random$(12,2,2)$	13,576	0	0	1,083
HRandom$(12,2,2)$	13,576	0	0	1,083

Time in seconds (Singular-4-0-0, $\mathbb{F}_{32003}$)

Benchmark	STD	SBA $<_{\text{pot}}$	SBA $<_{\text{d-pot}}$	SBA $<_{\text{lt}}$
cyclic-8	32.480	44.310	100.780	38.120
cyclic-8-h	38.300	35.770	98.440	32.640
eco-11	28.450	3.450	27.360	13.270
eco-11-h	20.630	11.600	14.840	7.960
katsura-11	54.780	35.720	31.010	11.790
katsura-11-h	51.260	34.080	32.590	17.230
noon-9	29.730	12.940	14.620	15.220
noon-9-h	34.410	17.850	20.090	20.510
Random$(11,2,2)$	267.810	77.430	130.400	28.640
HRandom$(11,2,2)$	22.970	14.060	39.320	3.540
Random$(12,2,2)$	2,069.890	537.340	1,062.390	176.920
HRandom$(12,2,2)$	172.910	112.420	331.680	22.060

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  - **SBA** for modules
  - **SBA** over $\mathbb{Z}$, $\mathbb{Z}_n$
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- New C library for GBs using fast special linear algebra:
  - Structured Gaussian Elimination
  - Parallelization on CPU, GPU, network
  - Use group actions on ideals
  - Specific implementations for dense resp. sparse polynomial systems
  - Aiming towards crypto applications (aiming on $\mathbf{F}_2$ and $\mathbf{F}_{2^e}$ for small e)

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