Signature-based Gröbner Basis Algorithms

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Buchberger's algorithm

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G of I

1. $G = \emptyset$
2. $G := G \cup \{f_i\}$ for all $i \in \{1, \ldots, m\}$
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   (a) If $p \xrightarrow{G} 0$ ▶▶ **no new information**
   Go on with the next element in P.
   (b) If $p \xrightarrow{G} h \neq 0$ ▶▶ **new information**
   Build new S-pair with h and add them to P.
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How to predict zero reductions?

Signatures

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4. **A signature** of f is given by $\mathfrak{s}(f) = \mathrm{lt}_{\prec}(\alpha)$ where $f = \overline{\alpha}$.

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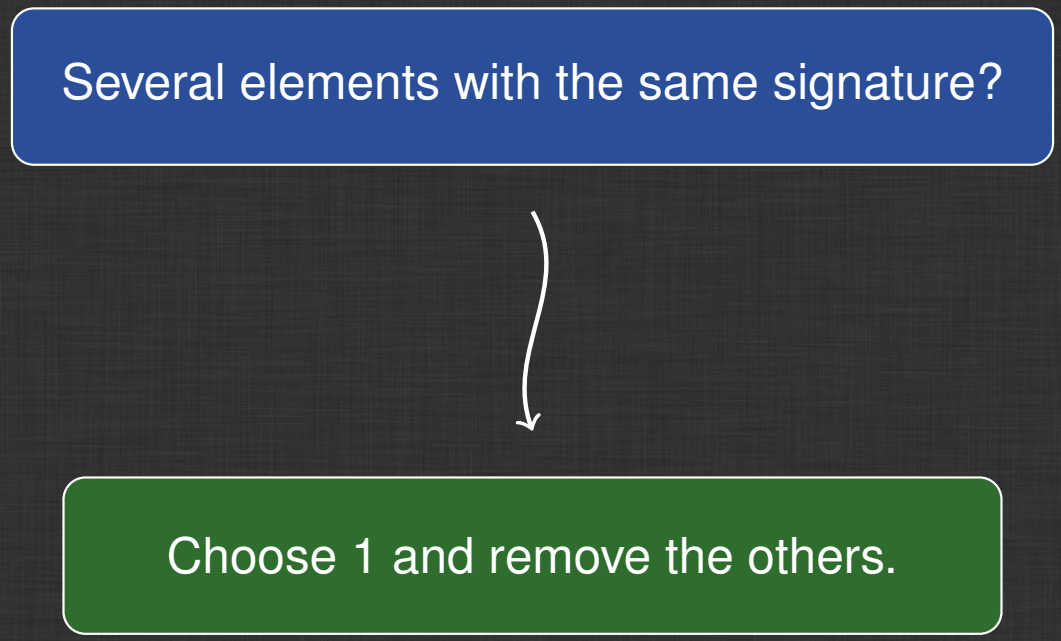

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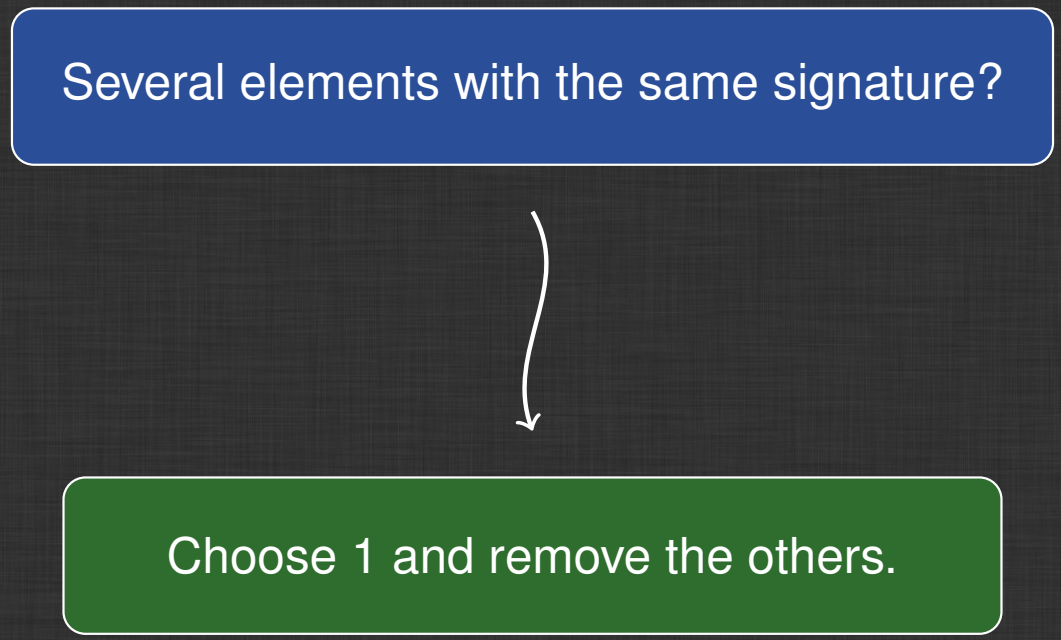

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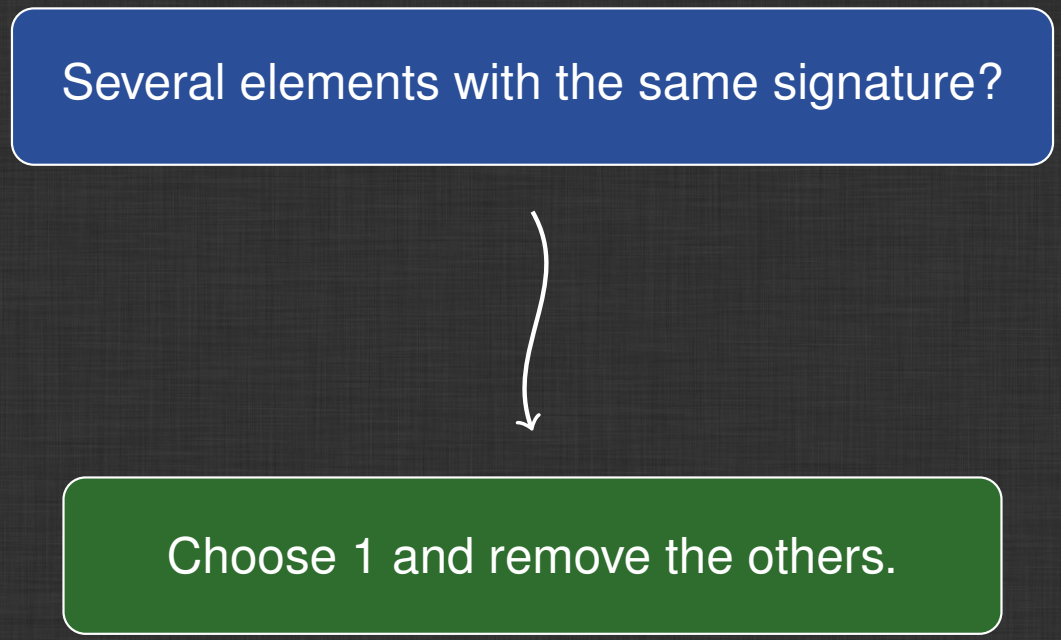

Our goal: Make good choices.

Our task: Keep signatures correct.

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$\alpha \in R^m \implies$ polynomial $\overline{\alpha}$ with $\mathrm{lt}(\overline{\alpha})$, signature $\mathfrak{s}(\alpha) = \mathrm{lt}(\alpha)$

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$$\mathrm{spol}\left(\overline{\alpha}, \overline{\beta}\right) = a\overline{\alpha} - b\overline{\beta} \implies \mathrm{spair}(\alpha, \beta) = a\alpha - b\beta$$

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Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from P by increasing signature.

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Sketch of proof

1. $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta)$.
2. All S-pairs are handled by increasing signature.
   $\Rightarrow$ All relatons $\prec \mathfrak{s}(\alpha)$ are known:

$$\alpha = \beta + \text{ elements of smaller signature}$$

□

Signature-based criteria

S-pairs in signature T

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S-pairs in signature T

$$\mathfrak{R}_T = \left\{ a\alpha \mid \alpha \text{ handled by the algorithm and } \mathfrak{s}(a\alpha) = T \right\}$$

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What are all possible configurations to reach signature T?

Define an order on $\mathfrak{R}_T$ and choose the maximal element.

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1. If $b\beta$ is a syzygy $\Longrightarrow$ Go on to next signature.

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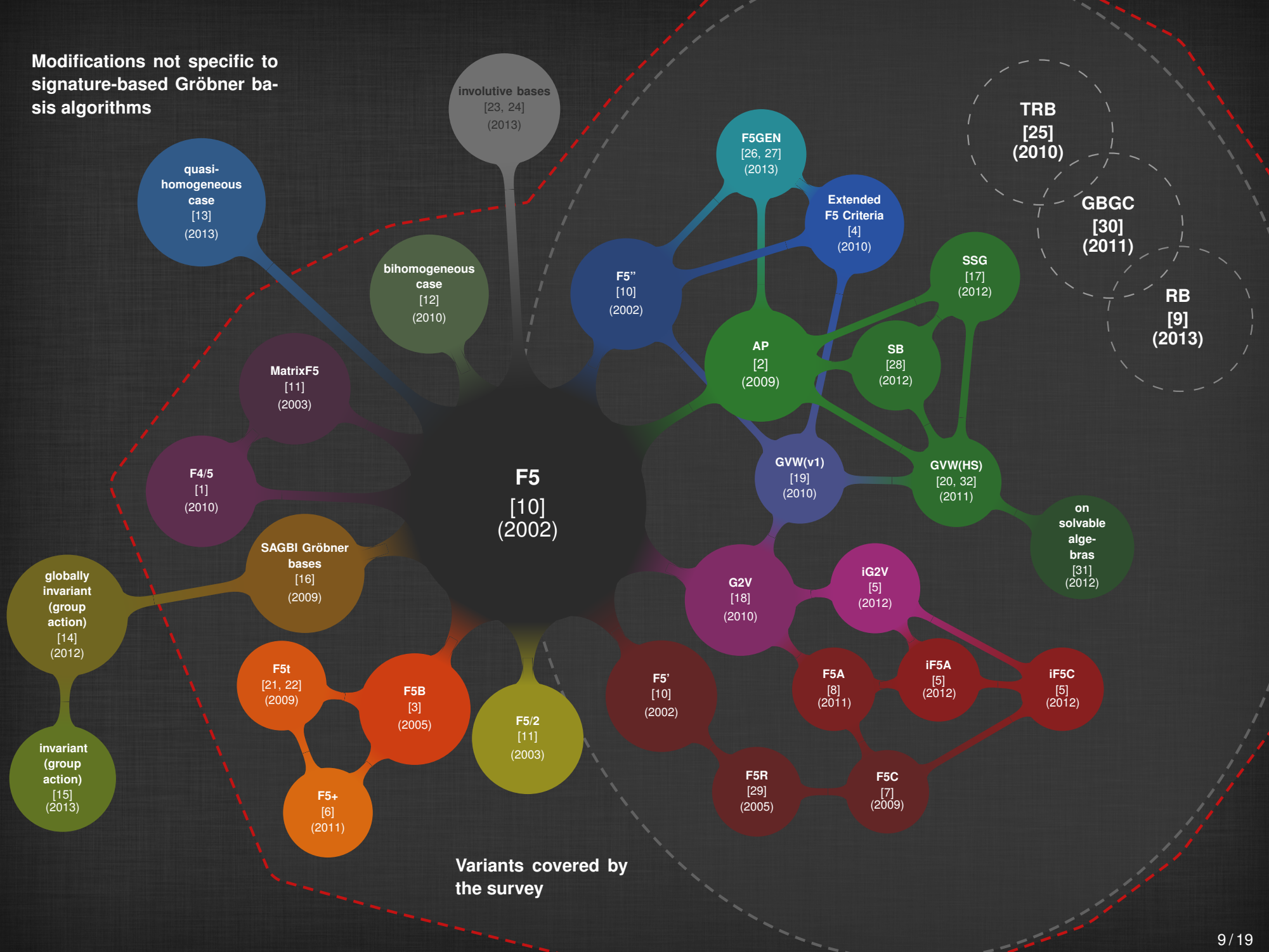
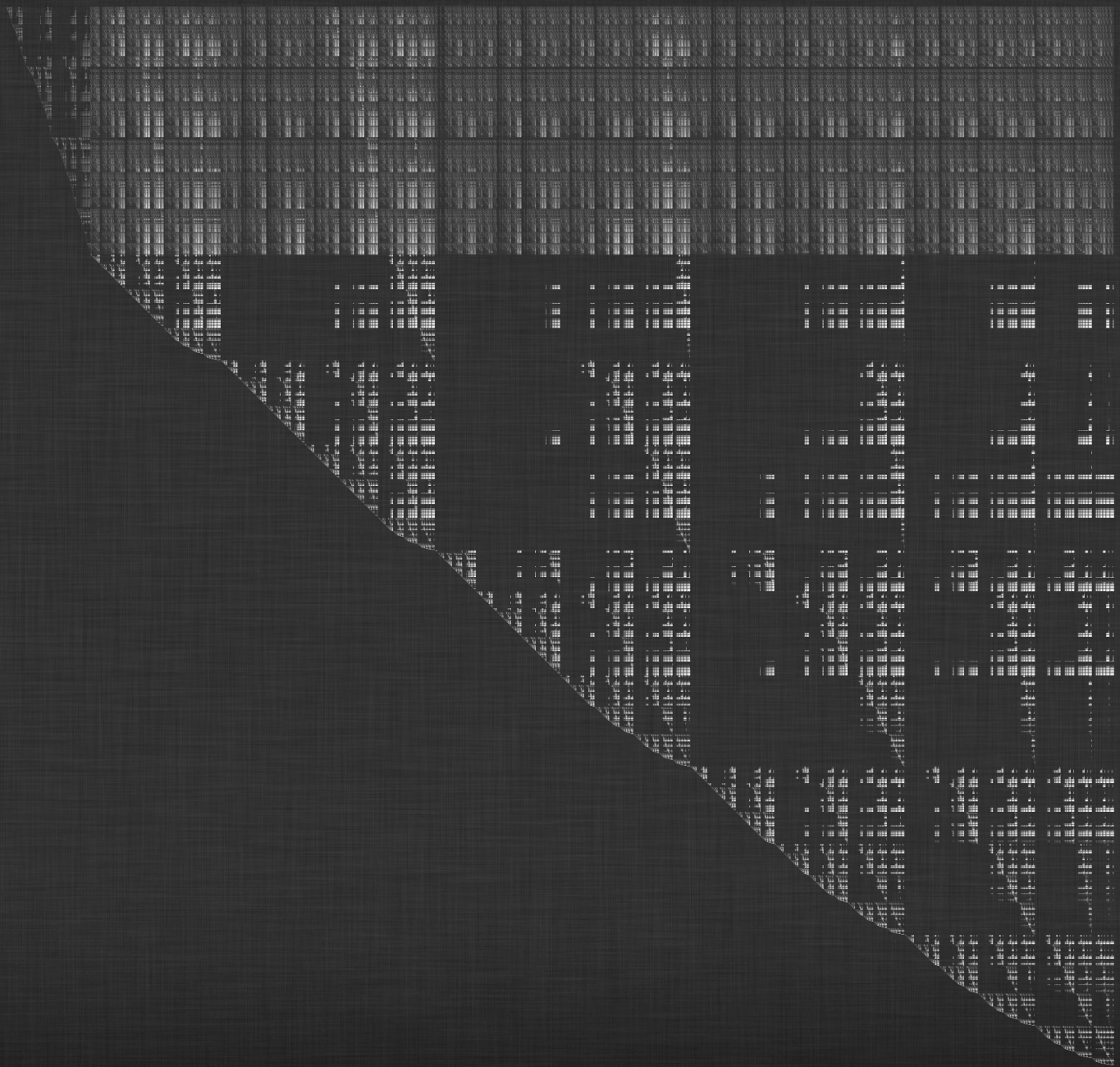
1. If $b\beta$ is a syzygy $\Longrightarrow$ Go on to next signature.
2. If $b\beta$ is not part of an S-pair $\Longrightarrow$ Go on to next signature.


Modifications not specific to signature-based Gröbner basis algorithms
involutive bases [23, 24] (2013)
quasi-homogeneous case [13] (2013)
bihomogeneous case [12] (2010)
F5GEN [26, 27] (2013)
Extended F5 Criteria [4] (2010)
F5" [10] (2002)
SSG [17] (2012)
AP [2] (2009)
SB [28] (2012)
TRB [25] (2010)
GBGC [30] (2011)
RB [9] (2013)
MatrixF5 [11] (2003)
F4/5 [1] (2010)
F5 [10] (2002)
GVW(v1) [19] (2010)
GVW(HS) [20, 32] (2011)
on solvable algebras [31] (2012)
SAGBI Gröbner bases [16] (2009)
globally invariant (group action) [14] (2012)
G2V [18] (2010)
iG2V [5] (2012)
F5t [21, 22] (2009)
F5B [3] (2005)
F5/2 [11] (2003)
F5' [10] (2002)
F5A [8] (2011)
iF5A [5] (2012)
iF5C [5] (2012)
invariant (group action) [15] (2013)
F5+ [6] (2011)
F5R [29] (2005)
F5C [7] (2009)
Variants covered by the survey

How our matrices look like



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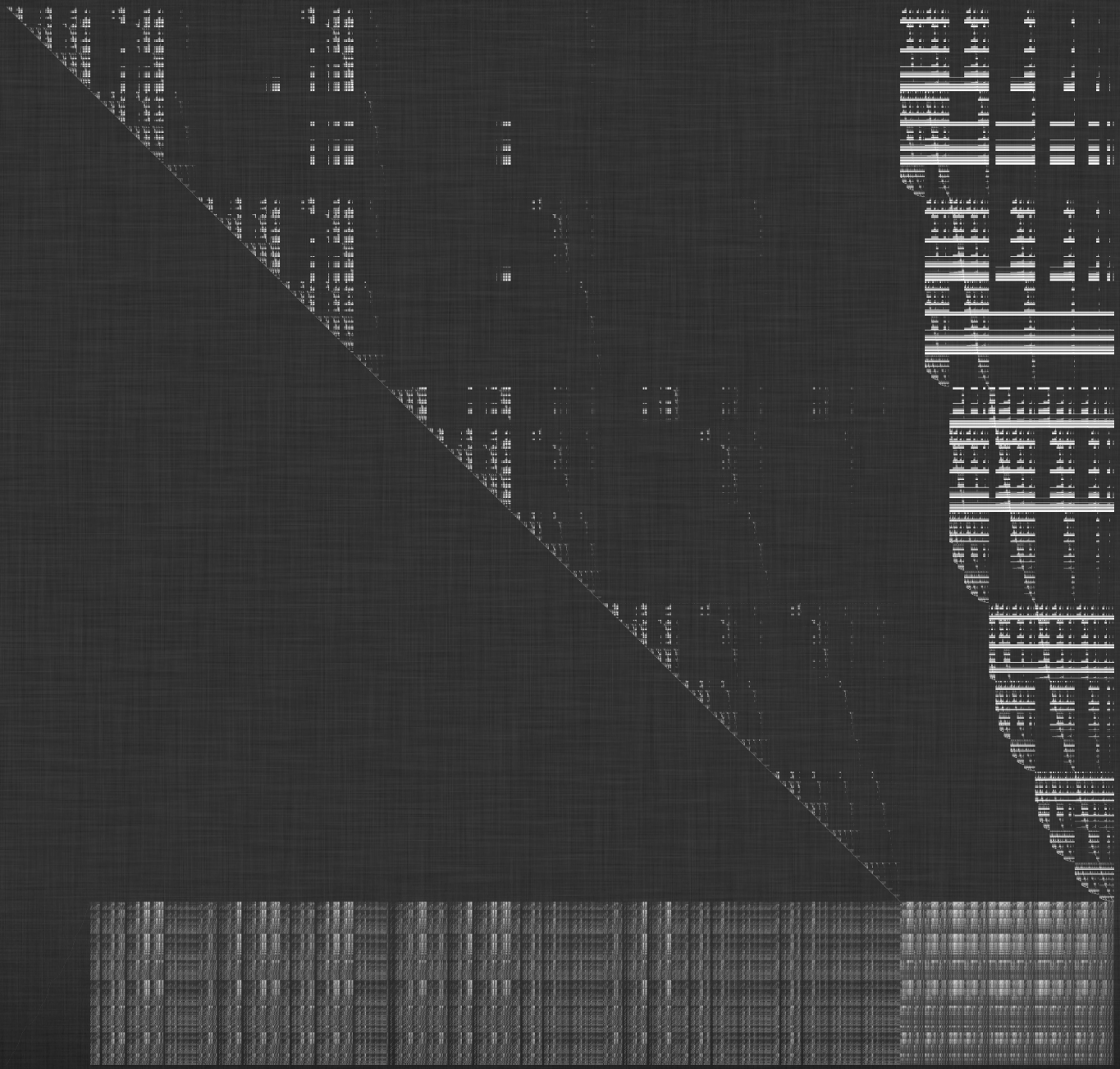
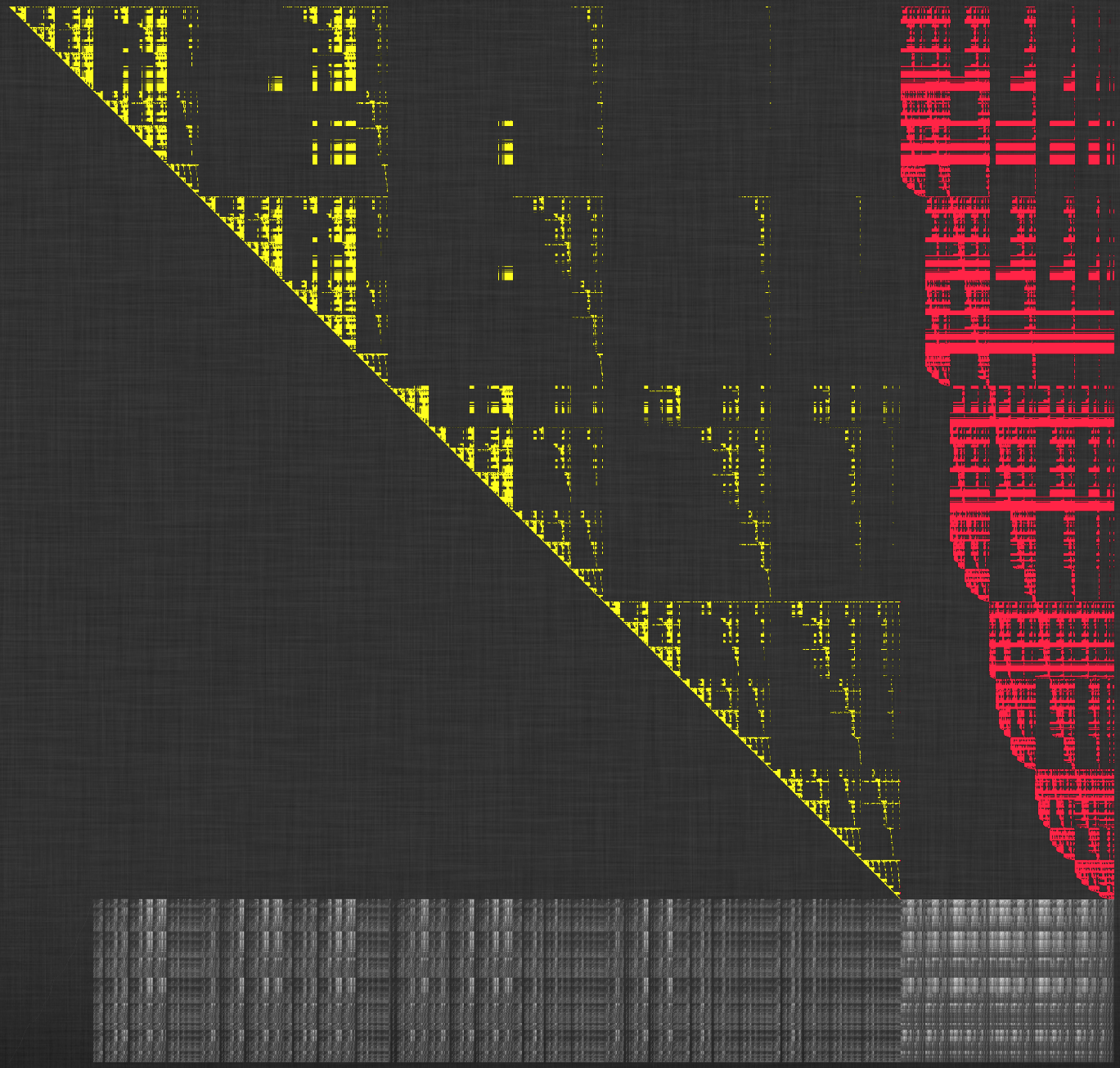
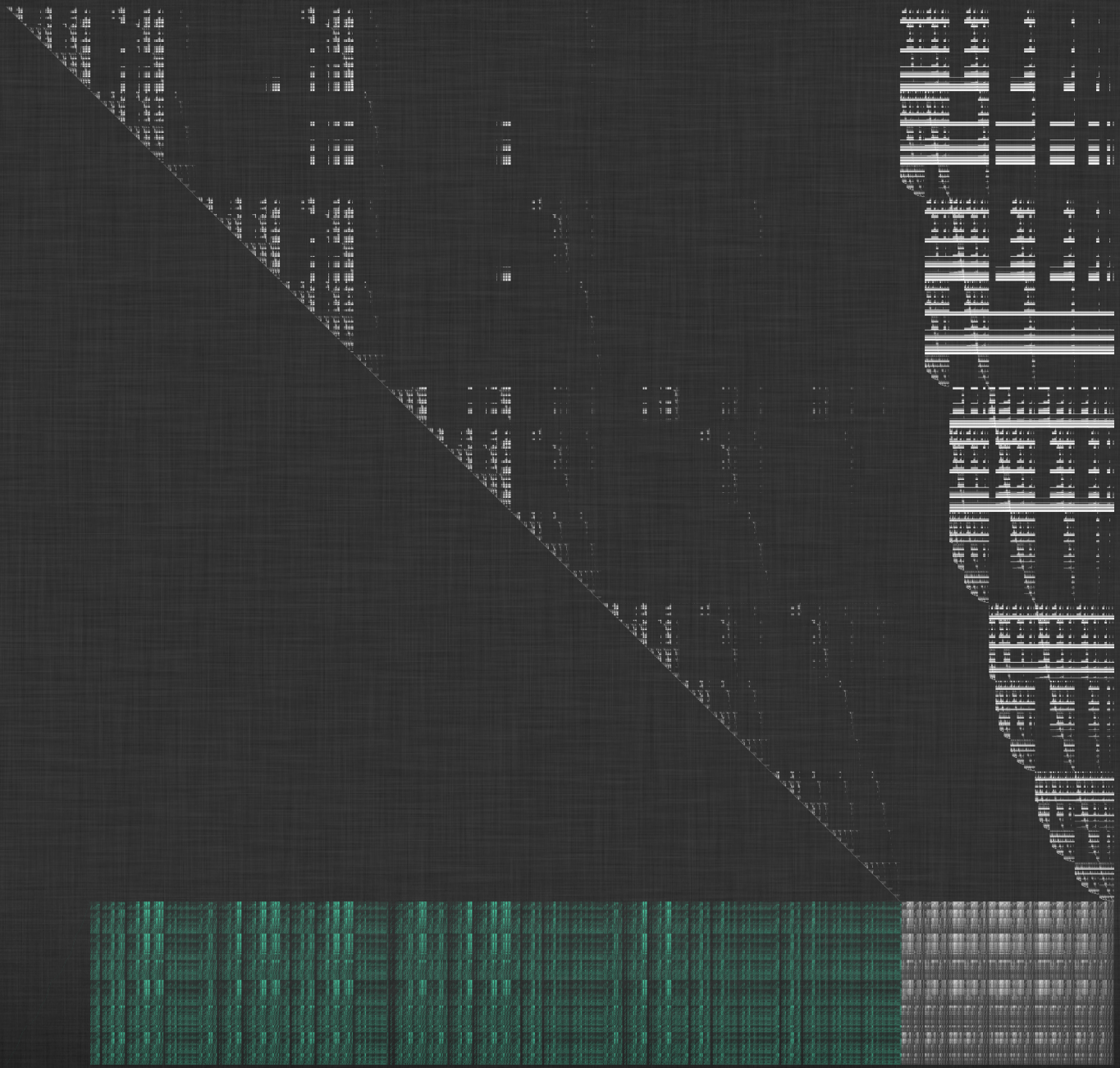


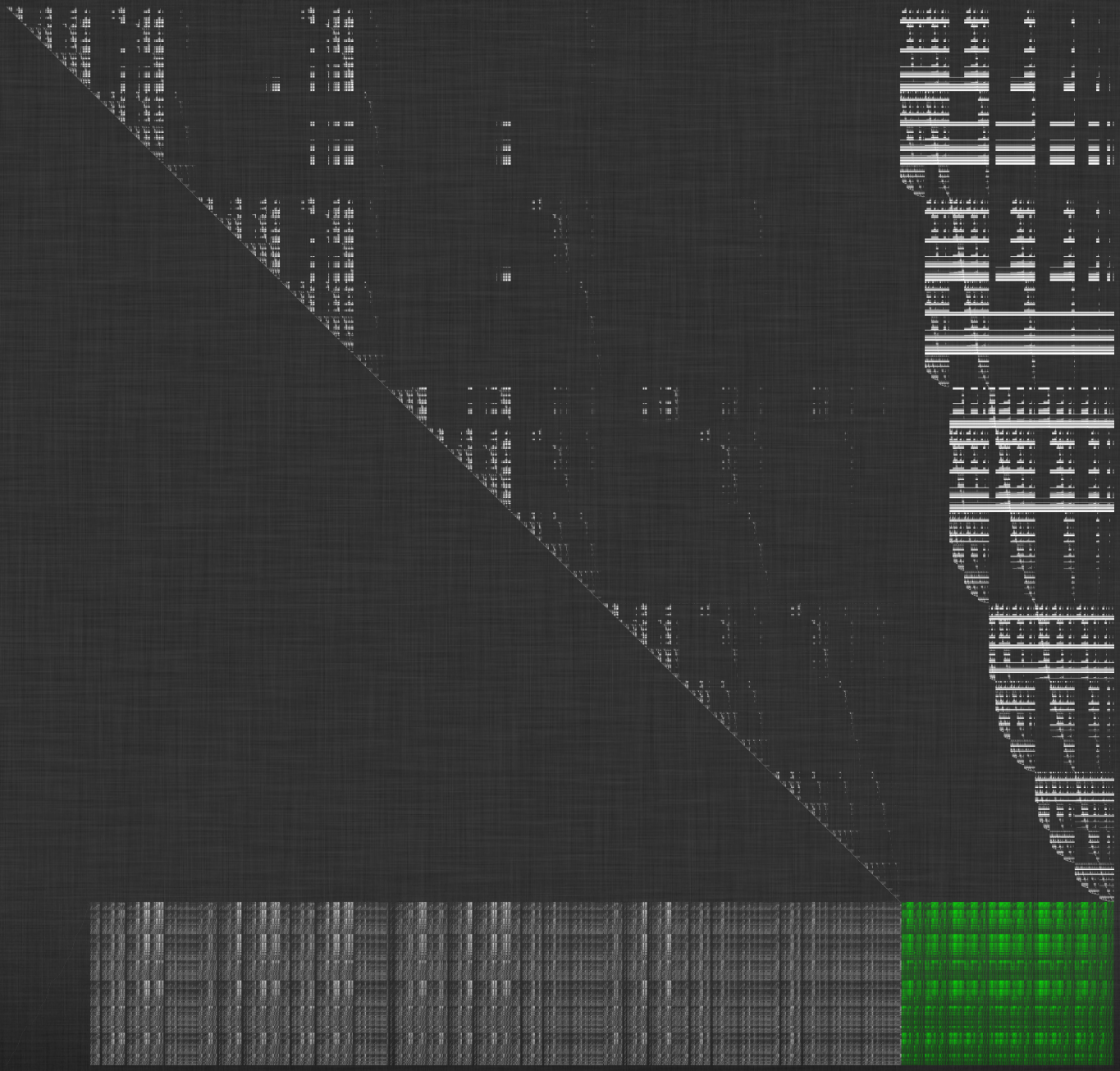
Hybrid Matrix Multiplication



Reduce to zero



New information



First attempts

2011 – University of Kaiserslautern
Bradford Hovinen – LELA
`https://github.com/Singular/LELA`

2012 – UPMC Paris 6, INRIA PolSys Team
Fayssal Martani – new implementation in LELA
`https://github.com/martani/LELA`

2012-2013 – University of Kaiserslautern
Bjarke Hammersholt Roune – MathicGB
`https://github.com/broune/mathicgb`

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