Department of Mathematics
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5-th Exercise Sheet in „Computer Algebra“

Deadline: Thursday, 24 November 2011, 10.00 h

Exercise 1. Let $R = K[x]$, and let $M = (m_{ij})$ be an $n \times n$–matrix with entries in R. Consider the matrix (M, E) obtained by concatenating M with the $n \times n$–unit matrix E, and let $v_1, \ldots, v_n \in R^{2n}$ be the rows of (M, E). On the free R–module $R^{2n} = \oplus_{i=1}^{2n} Re_i, e_1 = (1, 0, \ldots, 0), \ldots, e_{2n} = (0, \ldots, 1)$, consider the ordering defined by $x^\alpha e_i < x^\beta e_j$ if $i > j$ or $i = j$ and $x^\alpha < x^\beta$.

Let $\{w_1, \ldots, w_m\} \subset R^{2n}$ be the reduced standard basis of $\langle v_1, \ldots, v_n \rangle$, with $\mathrm{LM}(w_1) > \ldots > \mathrm{LM}(w_m)$. Prove that M is invertible if and only if $m = n$ and $\mathrm{LM}(w_i) = e_i$ for $i = 1, \ldots, m$, and then $w_1, \ldots, w_m$ are rows of (E, M^{-1}).

Exercise 2. Let $I \subset K[x]^r$ be a submodule, $x = (x_1, \ldots, x_n)$, and let $>$ be a global module ordering on $K[x]^r$. Prove that

$$K[x]^r \cong I \oplus \left(\bigoplus_{m \notin L(I)} K \cdot m \right).$$

Exercise 3. Compute the normal form of $\begin{pmatrix} x+y \\ y-1 \end{pmatrix}$ w.r.t. the module $M \subset K[x, y]^2$ generated by the vectors $\begin{pmatrix} x^2 \\ xy \end{pmatrix}$, $\begin{pmatrix} x \\ y^2 \end{pmatrix}$, and the ordering `(c,dp)`.

Exercise 4. Change your SINGULAR procedure `standard` computing a standard basis algorithm in such a way that

(1) the output S is sorted in ascending order with respect to $\mathrm{LM}(f)$, $f \in S$, and

(2) it takes an optional parameter such that if this optional parameter is the string „reduced“, the procedure returns a reduced standard basis and if this optional parameter is missing, the procedure just returns some standard basis as before.