Gröbner Basis Computations

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Conventions

- $\mathscr{R} = \mathscr{K}[x_1, \ldots, x_n]$, $\mathscr{K}$ field, $<$ well-ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$
- $f \in \mathscr{R}$ can be represented in a unique way by $<$.
  $\Rightarrow$ Definitions as $\mathrm{lc}(f)$, $\mathrm{lm}(f)$, and $\mathrm{lt}(f)$ make sense.
- An ideal I in $\mathscr{R}$ is an additive subgroup of $\mathscr{R}$ such that for $f \in I$, $g \in \mathscr{R}$ it holds that $fg \in I$.
- $G = \{g_1, \ldots, g_s\} \subset \mathscr{R}$ is a Gröbner basis for $I = \langle f_1, \ldots, f_m \rangle$ w.r.t. $<$

$$:\Longleftrightarrow$$

$$G \subset I \text{ and } L_<(G) = L_<(I)$$

Polynomial reduction

Definition

Let $f \in \mathscr{R}$ and let t be a term of f.

- We can **reduce** t by $g \in \mathscr{R}$ if $\exists\, b$ such that $\mathrm{lt}(bg) = t$.
  Outcome: $f - bg$.
- Reduce $f \in \mathscr{R}$ to $h \in \mathscr{R}$ by a sequence of reduction steps.
- Reductions always w.r.t. a finite subset $G \subset \mathscr{R}$

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$\Longrightarrow$ Another characterisation of Gröbner bases:

Gröbner basis (1)

$G \subset \mathscr{R}$ finite is

1. a **Gröbner basis up to degree** d **for** I if
$$G \subset I \text{ and for all } f \in I \text{ with } \deg(f) \leq d: \; f \xrightarrow{G} 0;$$
2. a **Gröbner basis for** I if G is a Gröbner basis in all degrees.

Even more characterizations!

Standard representation

Let $f \in \mathscr{R}$ and $G \subset \mathscr{R}$ finite. A representation

$$f = \sum_{i=1}^{k} m_i g_i$$

with $m_i \neq 0$, $g_i \in G$ pairwise different is called a **standard representation** if

$$\max_{\leq} \{\mathrm{lt}(m_i g_i) \mid 1 \leq i \leq k\} \leq \mathrm{lt}(f).$$

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Gröbner basis (2)

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S-polynomials

Let $f \neq 0, g \neq 0 \in \mathscr{R}$ and let $\lambda = \operatorname{lcm}(\operatorname{lt}(f), \operatorname{lt}(g))$ be the least common multiple of $\operatorname{lt}(f)$ and $\operatorname{lt}(g)$. The **S-polynomial** between f and g is given by

$$\operatorname{spol}(f, g) := \frac{\lambda}{\operatorname{lt}(f)} f - \frac{\lambda}{\operatorname{lt}(g)} g.$$

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Buchberger's criterion [7]

Let $I = \langle f_1, \ldots, f_m \rangle$ be an ideal in $\mathscr{R}$. A finite subset $G \subset \mathscr{R}$ is a **Gröbner basis up to degree d for I** if $G \subset I$ and for all $f, g \in G$ with $\deg(\operatorname{spol}(f,g)) \leq d$: $\operatorname{spol}(f,g) \xrightarrow{G} 0$.

Buchberger's algorithm

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G for I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{f_i\}$ for all $i \in \{1, \ldots, m\}$
3. Set $P \leftarrow \{\text{spol}(f_i, f_j) \mid f_i, f_j \in G, i > j\}$

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4. Choose $p \in P$, $P \leftarrow P \setminus \{p\}$
   (a) If $p \xrightarrow{G} 0$ ▶▶ **no new information**
   Go on with the next element in P.
   (b) If $p \xrightarrow{G} h \neq 0$ ▶▶ **new information**
   Build new S-pair with h and add them to P.
   Add h to G.
   Go on with the next element in P.
5. When $P = \emptyset$ we are done and G is a Gröbner basis of I.

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In theory:

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In practice:

- Quite OK

How to improve computations?

- Modular computations (modStd et al.)
- Predict zero reductions (Buchberger, Gebauer-Möller, Möller-Mora-Traverso, Faugère.)
- Sort pair set (Buchberger, Giovini et al., Möller et al.)
- Homogenize: d-Gröbner bases
- Change of ordering (FGLM, Gröbner Walk)
- Linear Algebra: Gauss Elimination (Lazard, Faugère)
- ...

- **Predicting zero reductions**
- **Pair set sorting and homogenization**
- **Finding better reducers**
- **Fast linear algebra for computing Gröbner bases**
- **Modular computations**
- **Change of ordering**

How to detect zero reductions in advance?

Let $I = \langle g_1, g_2 \rangle \in \mathbb{Q}[x, y, z]$ and let $<$ denote the reverse lexicographical ordering. Let

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$$\begin{aligned} \text{spol}(g_2, g_1) &= xg_2 - yg_1 = \mathbf{xy^2} - xz^2 - \mathbf{xy^2} + yz^2 \\ &= -xz^2 + yz^2. \end{aligned}$$

$$\Longrightarrow \mathbf{g_3 = xz^2 - yz^2}.$$

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We can reduce further using $z^2 g_2$:

$$-y^2z^2 + z^4 + y^2z^2 - z^4 = 0.$$

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For all $u \in \operatorname{support}(\operatorname{lot}(g_3))$ we can reduce with ug_2:

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So we can reduce this to zero by vg_3 by all $v \in \operatorname{support}(\operatorname{lot}(g_2))$.

Product criterion [8, 9]

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$$\operatorname{spol}(g_3, g_2) = y \underbrace{\operatorname{spol}(g_3, g_1)}_{\xrightarrow{G} 0} - z^2 \underbrace{\operatorname{spol}(g_2, g_1)}_{\xrightarrow{G} -g_3}$$

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Standard representations of $\operatorname{spol}(g_2, g_1)$ and $\operatorname{spol}(g_3, g_1)$
$\Longrightarrow$ Standard representation of $\operatorname{spol}(g_3, g_2)$.

Buchberger's criteria

Chain criterion [10]

Let $f, g, h \in \mathscr{R}$, $G \subset \mathscr{R}$ finite. If

1. $\mathrm{lt}(h) \mid \mathrm{lcm}(\mathrm{lt}(f), \mathrm{lt}(g))$, and
2. $\mathrm{spol}(f, h)$ and $\mathrm{spol}(h, g)$ have a standard representation w.r.t. G respectively,

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Note

Do not remove too much information! If $\lambda = 1$ and

$$\mathrm{spol}(f, g) = \lambda \mathrm{spol}(f, h) + \sigma \mathrm{spol}(h, g),$$

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How to combine Product and Chain criterion?

Gebauer-Möller installation [33]

We add a new element h to G and generate new pairs $P' := \{(f, h) \mid f \in G\}$.

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2. Fix $(f, h) \in P'$. If $(g, h) \in P' \setminus \{(f, h)\}$ s.t.
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4. If $(f, h) \in P'$ s.t. $\mathrm{lcm}(\mathrm{lt}(f), \mathrm{lt}(h)) = \mathrm{lt}(f)\,\mathrm{lt}(h)$
   $\Longrightarrow$ Remove (f, h) from P'. [Product criterion done]

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Use more structure of $I \Longrightarrow$ **Signatures**

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4. A **signature** of f is given by $\mathfrak{s}(f) = \mathrm{lt}_{\prec}(\alpha)$ where $f = \overline{\alpha}$.

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3. Each $f \in I$ can be represented via some $\alpha \in \mathscr{R}^m$: $f = \overline{\alpha}$

4. A **signature** of f is given by $\mathfrak{s}(f) = \mathrm{lt}_{\prec}(\alpha)$ where $f = \overline{\alpha}$.

5. An element $\alpha \in \mathscr{R}^m$ such that $\overline{\alpha} = 0$ is called a **syzygy**.

Our example again – with signatures and $\prec_{\mathbf{pot}}$

$$g_1 = xy - z^2,\ \mathfrak{s}(g_1) = e_1,$$
$$g_2 = y^2 - z^2,\ \mathfrak{s}(g_2) = e_2.$$

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Note that $\mathfrak{s}(\mathrm{spol}(g_3, g_1)) = xye_2$ and $\mathrm{lm}(g_1) = xy$.

How to use signatures?

General idea: Only 1 element per signature.

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Several elements with the same signature?

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Choose 1 and remove the others.

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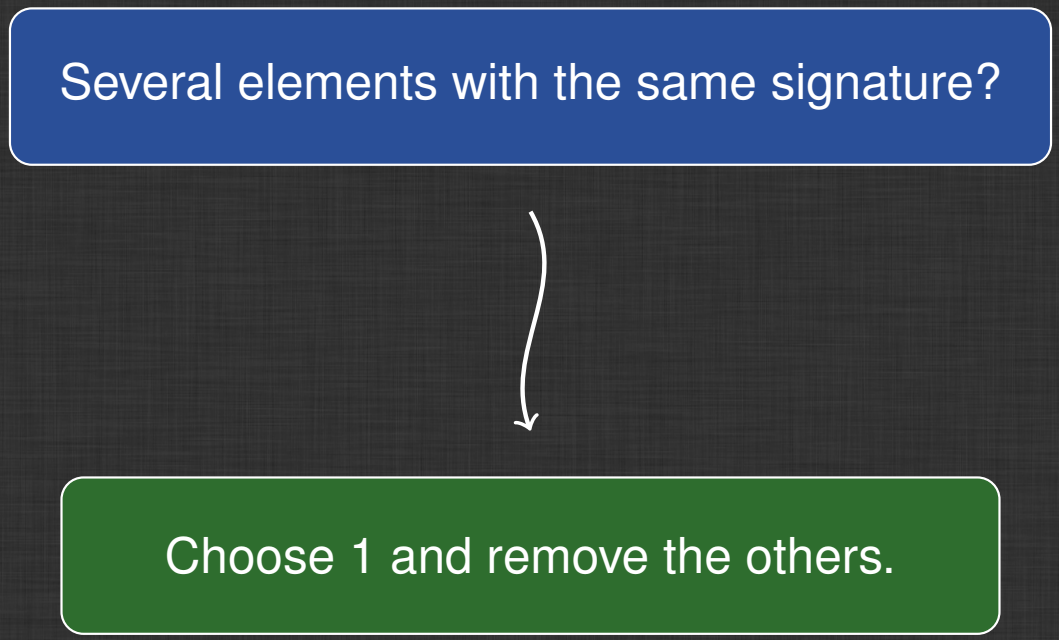

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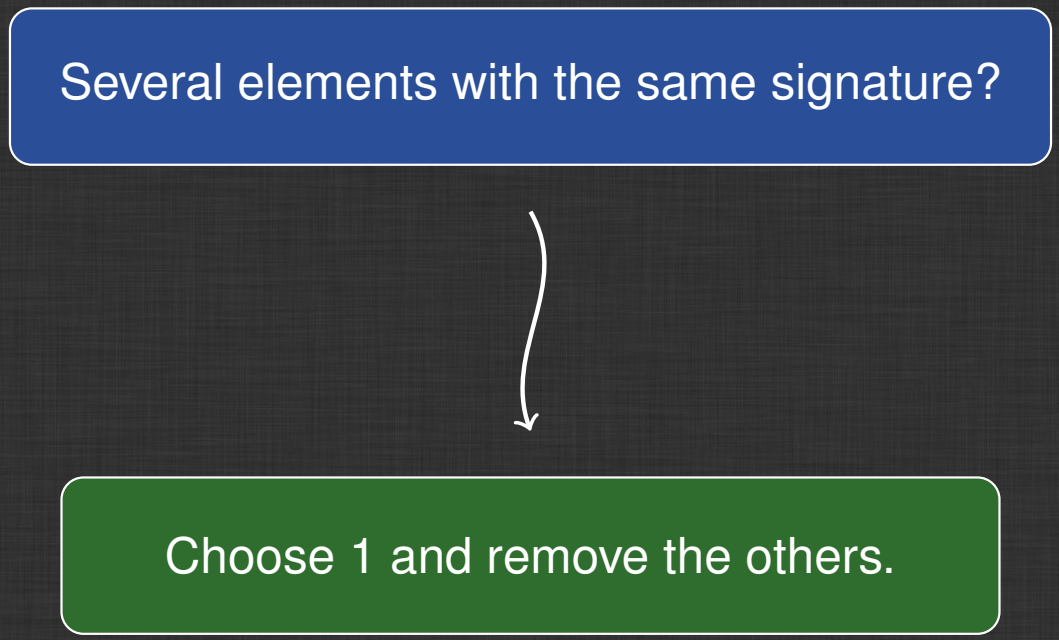

Our goal: Make good choices.

Our task: Keep signatures correct.

Think in the module

$\alpha \in \mathscr{R}^m \implies$ polynomial $\overline{\alpha}$ with $\mathrm{lt}(\overline{\alpha})$, signature $\mathfrak{s}(\alpha) = \mathrm{lt}(\alpha)$

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$\mathfrak{s}$-reductions:

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Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from P by increasing signature.

Signature-based criteria

$\mathfrak{s}(\alpha) = \mathfrak{s}(\beta) \implies$ Compute 1, remove 1.

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Sketch of proof

1. $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta)$.
2. All S-pairs are handled by increasing signature.
   $\Rightarrow$ All relatons $\prec \mathfrak{s}(\alpha)$ are known:

$$\alpha = \beta + \text{ elements of smaller signature}$$

□

Signature-based criteria

S-pairs in signature T

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What are all possible configurations to reach signature T?

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$$\mathfrak{R}_T = \left\{ a\alpha \mid \alpha \text{ handled by the algorithm and } \mathfrak{s}(a\alpha) = T \right\}$$

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What are all possible configurations to reach signature T?

Define an order on $\mathfrak{R}_T$ and choose the maximal element.

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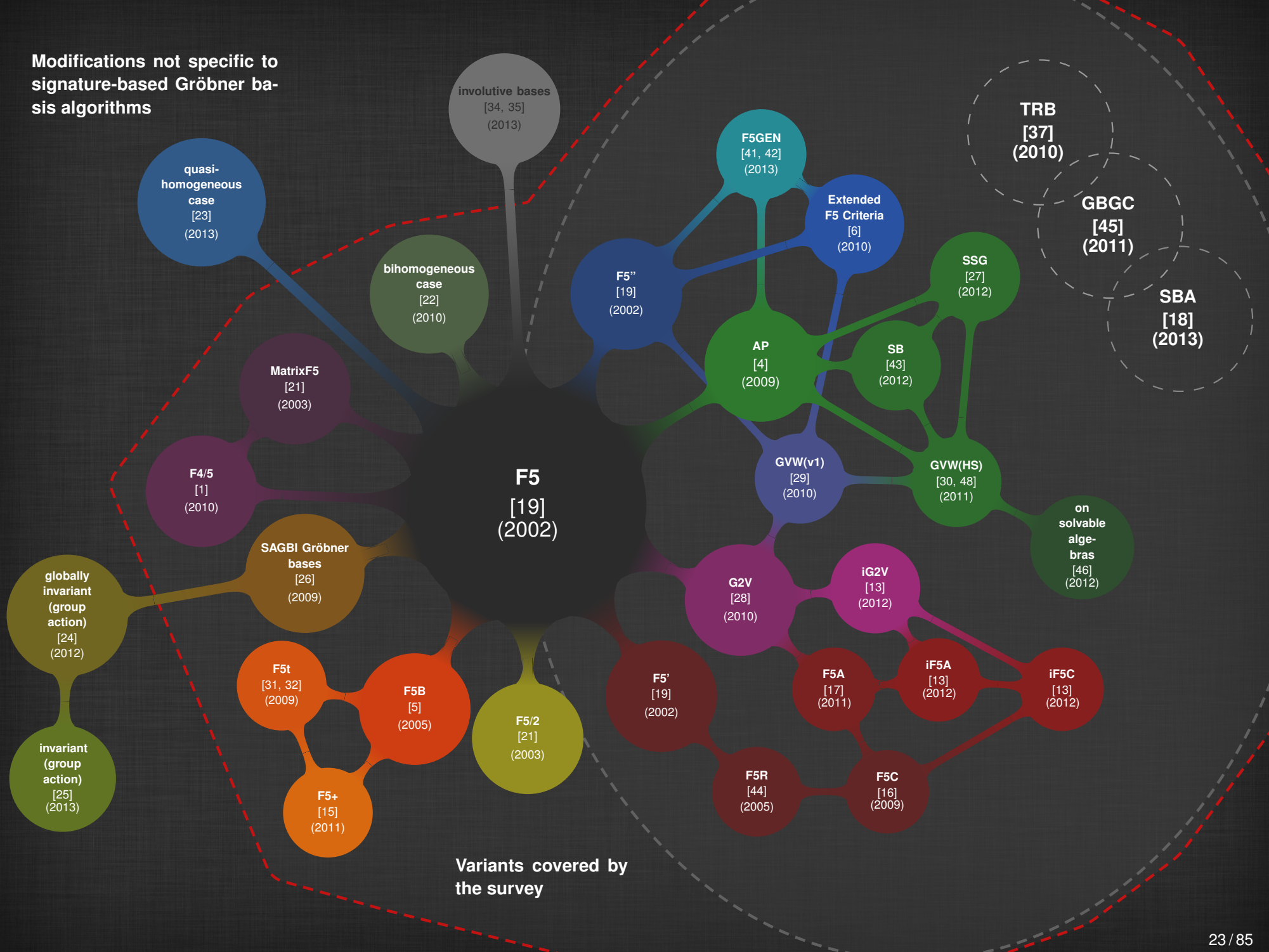
Revisiting our example with $\prec_{\text{pot}}$

$$\mathfrak{s}\left(\text{spol}(g_3, g_1)\right) = xye_2$$

$$\left.\begin{aligned} g_1 &= xy - z^2 \\ g_2 &= y^2 - z^2 \end{aligned}\right\} \Rightarrow \text{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \ldots$$


Modifications not specific to signature-based Gröbner basis algorithms
involutive bases [34, 35] (2013)
quasi-homogeneous case [23] (2013)
bihomogeneous case [22] (2010)
F5GEN [41, 42] (2013)
Extended F5 Criteria [6] (2010)
F5" [19] (2002)
SSG [27] (2012)
AP [4] (2009)
SB [43] (2012)
TRB [37] (2010)
GBGC [45] (2011)
SBA [18] (2013)
MatrixF5 [21] (2003)
F4/5 [1] (2010)
F5 [19] (2002)
GVW(v1) [29] (2010)
GVW(HS) [30, 48] (2011)
on solvable algebras [46] (2012)
SAGBI Gröbner bases [26] (2009)
globally invariant (group action) [24] (2012)
G2V [28] (2010)
iG2V [13] (2012)
F5t [31, 32] (2009)
F5B [5] (2005)
F5/2 [21] (2003)
F5' [19] (2002)
F5A [17] (2011)
iF5A [13] (2012)
iF5C [13] (2012)
invariant (group action) [25] (2013)
F5+ [15] (2011)
F5R [44] (2005)
F5C [16] (2009)
Variants covered by the survey

zero reductions (Singular-4-0-0, $\mathbb{F}_{32003}$)

Benchmark	STD	SBA $<_{\text{pot}}$	SBA $<_{\text{d-pot}}$	SBA $<_{\text{lt}}$
cyclic-8	4,284	243	243	671
cyclic-8-h	5,843	243	243	671
eco-11	3,476	0	749	749
eco-11-h	5,429	502	502	749
katsura-11	3,933	0	0	353
katsura-11-h	3,933	0	0	353
noon-9	25,508	0	0	682
noon-9-h	25,508	0	0	682
Random$(11,2,2)$	6,292	0	0	590
HRandom$(11,2,2)$	6,292	0	0	590
Random$(12,2,2)$	13,576	0	0	1,083
HRandom$(12,2,2)$	13,576	0	0	1,083

Time in seconds (Singular-4-0-0, $\mathbb{F}_{32003}$)

Benchmark	STD	SBA $<_{\text{pot}}$	SBA $<_{\text{d-pot}}$	SBA $<_{\text{lt}}$
cyclic-8	32.480	44.310	100.780	38.120
cyclic-8-h	38.300	35.770	98.440	32.640
eco-11	28.450	3.450	27.360	13.270
eco-11-h	20.630	11.600	14.840	7.960
katsura-11	54.780	35.720	31.010	11.790
katsura-11-h	51.260	34.080	32.590	17.230
noon-9	29.730	12.940	14.620	15.220
noon-9-h	34.410	17.850	20.090	20.510
Random(11,2,2)	267.810	77.430	130.400	28.640
HRandom(11,2,2)	22.970	14.060	39.320	3.540
Random(12,2,2)	2,069.890	537.340	1,062.390	176.920
HRandom(12,2,2)	172.910	112.420	331.680	22.060

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Yes, rather easily:

- Chain criterion included in Rewritten criterion
- Product criterion not completely included

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Note

There is a conjecture that for $<_{\text{pot}}$ the Product criterion is also included in the Rewritten criterion [14].

- Predicting zero reductions
- Pair set sorting and homogenization
- Finding better reducers
- Fast linear algebra for computing Gröbner bases
- Modular computations
- Change of ordering

Why is homogeneous input nice?

If $\deg(\operatorname{spol}(f,g)) = d$ and $\operatorname{spol}(f,g) \xrightarrow{G} h \neq 0$

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All newly generated S-polynomials have degree $> d$.
All possible lower-degree reducers are already in G.

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Still, choosing by the normal selection strategy is quite good.

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   - ▷ Better behaviour
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Sugar-degree [36]
Let $I = \langle f_1, \ldots, f_m \rangle$.

$$\begin{aligned} \text{s-deg}(f_i) &:= \deg(f_i) \\ \text{s-deg}(tg) &:= \deg(t) + \deg(g) \text{ for } t \in M \text{ and } g \in G \\ \text{s-deg}(g+h) &:= \max\{\text{s-deg}(g), \text{s-deg}(h)\} \text{ for } g, h \in G. \end{aligned}$$

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If f has **better properties** then exchange g with f in G.

And if $\lambda > 1$?

If f has better properties than $\lambda g \Rightarrow f \in R$ (special reducer set)

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Example

$$\mathrm{lt}(f) = \mathrm{lt}(\lambda g)$$

Let h be another intermediate S-polynomial such that

- $\exists\, \lambda' \in \mathrm{Mon}$ with $\lambda' \mathrm{lt}(g) = \mathrm{lt}(h)$
- $\lambda \mid \lambda'$

Then we replace the reduction

$$h - \lambda' g = h - \frac{\lambda'}{\lambda} \lambda g$$

by

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by

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In the same way:
Exchanging multiples of λg when generating S-polynomials.

How to implement this?

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3. How many special reducers do we store?

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There are three algorithms using this:

- **slimGB** (SINGULAR); to some extend also **std**
- Faugère's **F4** algorithm (everywhere but not in SINGULAR)
- Signature-based algorithms (**sba** in SINGULAR, **GVW**, **F5** in FGB)
  Checking reducers with the Rewritten criterion

- Predicting zero reductions
- Pair set sorting and homogenization
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- **Fast linear algebra for computing Gröbner bases**
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Buchberger's algorithm - revisited

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G for I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{f_i\}$ for all $i \in \{1, \ldots, m\}$
3. Set $P \leftarrow \{\text{spol}(f_i, f_j) \mid f_i, f_j \in G, i > j\}$
4. Choose $p \in P$, $P \leftarrow P \setminus \{p\}$
   (a) If $p \xrightarrow{G} 0$ ▶▶ **no new information**
   Go on with the next element in P.
   (b) If $p \xrightarrow{G} h \neq 0$ ▶▶ **new information**
   Build new S-pair with h and add them to P.
   Add h to G.
   Go on with the next element in P.
5. When $P = \emptyset$ we are done and G is a Gröbner basis of I.

Faugère's F4 algorithm

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Output: Gröbner basis G for I

1. $G \leftarrow \emptyset$
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4. $d \leftarrow 0$
5. while $P \neq \emptyset$:

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4. $d \leftarrow 0$
5. while $P \neq \emptyset$:
   (a) $d \leftarrow d + 1$
   (b) $P_d \leftarrow \textbf{Select}(P)$, $P \leftarrow P \setminus P_d$
   (c) $L_d \leftarrow \{af, bg \mid (af, bg) \in P_d\}$
   (d) $L_d \leftarrow \textbf{Symbolic Preprocessing}(L_d, G)$
   (e) $F_d \leftarrow \textbf{Reduction}(L_d, G)$
   (f) for $h \in F_d$:
      - If $\mathrm{lt}(h) \notin L(G)$ (all other h are useless):
        - $P \leftarrow P \cup \{\text{new pairs with } h\}$
        - $G \leftarrow G \cup \{h\}$
6. Return G

Differences to Buchberger

1. Select a subset P_d of P, not only one element.
2. Do a symbolic preprocessing:
   Search and store reducers, but do not reduce.
3. Do a full reduction of P_d at once:
   Reduce a subset of $\mathscr{R}$ by a subset of $\mathscr{R}$

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If **Select**(P) selects only 1 pair F4 is just Buchberger's algorithm.
Usually one chooses the normal selection strategy,
i.e. all pairs of lowest degree.

Symbolic preprocessing

Input: L, G finite subsets of $\mathscr{R}$
Output: a finite subset of $\mathscr{R}$

1. $F \leftarrow L$
2. $D \leftarrow L(F)$ (S-pairs already reduce lead terms)
3. while $T(F) \neq D$:
   (a) Choose $m \in T(F) \setminus D$, $D \leftarrow D \cup \{m\}$.
   (b) If $m \in L(G) \Rightarrow \exists\, g \in G$ and $\lambda \in$ Mon such that $\lambda\, \mathrm{lt}(g) = m$
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We optimize this soon!

Reduction

Input: L, G finite subsets of $\mathscr{R}$
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1. $M \leftarrow$ Macaulay matrix of L
2. $M \leftarrow$ Gaussian Elimination of M (Linear algebra)
3. $F \leftarrow$ polynomials from rows of M
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Macaulay matrix

columns $\hat{=}$ monomials (sorted by monomial order $<$)

rows $\hat{=}$ coeffs of polynomials in L

Example: Cyclic-4

$\mathscr{R} = \mathbb{Q}[a, b, c, d]$, $<$ denotes DRL and we use the normal selection strategy for **Select**(P). $I = \langle f_1, \ldots, f_4 \rangle$, where

$$
\begin{aligned}
f_1 &= abcd - 1, \\
f_2 &= abc + abd + acd + bcd, \\
f_3 &= ab + bc + ad + cd, \\
f_4 &= a + b + c + d.
\end{aligned}
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$$\begin{aligned} f_1 &= abcd - 1, \\ f_2 &= abc + abd + acd + bcd, \\ f_3 &= ab + bc + ad + cd, \\ f_4 &= a + b + c + d. \end{aligned}$$

We start with $G = \{f_4\}$ and $P_1 = \{(f_3, bf_4)\}$, thus $L_1 = \{f_3, bf_4\}$.

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$\mathscr{R} = \mathbb{Q}[a,b,c,d]$, $<$ denotes DRL and we use the normal selection strategy for **Select**(P). $I = \langle f_1, \ldots, f_4 \rangle$, where

$$\begin{aligned} f_1 &= abcd - 1, \\ f_2 &= abc + abd + acd + bcd, \\ f_3 &= ab + bc + ad + cd, \\ f_4 &= a + b + c + d. \end{aligned}$$

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Now reduction:
Convert polynomial data L_1 to Macaulay Matrix M_1

$$\begin{array}{c} \\ df_4 \\ f_3 \\ bf_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} ab & b^2 & bc & ad & bd & cd & d^2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

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Gaussian Elimination of M_1:

$$\begin{array}{c} \\ df_4 \\ f_3 \\ bf_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} ab & b^2 & bc & ad & bd & cd & d^2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 \end{pmatrix} \end{array}$$

Example: Cyclic-4

Convert matrix data back to polynomial structure F_1:

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$$F_1 = \left\{ \underbrace{ad + bd + cd + d^2}_{f_5}, \underbrace{ab + bc - bd - d^2}_{f_6}, \underbrace{b^2 + 2bd + d^2}_{f_7} \right\}$$

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$\mathrm{lt}(f_5), \mathrm{lt}(f_6) \in L(G)$, so

$\mathbf{G} \leftarrow \mathbf{G} \cup \{\mathbf{f_7}\}$.

Example: Cyclic-4

Next round:

$$G = \{f_4, f_7\},\ P_2 = \{(f_2, bcf_4)\},\ L_2 = \{f_2, bcf_4\}.$$

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f_6 possibly better reduced than f_4.(f_6 is not in G!)

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Let us investigate this in more detail.

Interlude – Simplify

Idea

Try to replace $u \cdot f$ by a product $(wv) \cdot g$ where vg corresponds to an already computed row in the Gauss. Elim. of a previous matrix M_i.

$\Rightarrow$ Reuse rows that are reduced but not “in” G.

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1. $d \leftarrow$ current index in the F4 algorithm
2. $D(u) \leftarrow \{\text{list of divisors of } u\}$
3. for $w \in D(u)$
   (a) if $\exists j \in \{1, \ldots, d-1\}$ such that $w \cdot f$ corresponds to row in M_j
      ▷ $\exists_1 g \in F_j$ such that $\mathrm{lt}(g) = \mathrm{lt}(w \cdot f)$
      ▷ if $w \neq u$: Return **Simplify** $\left(\frac{u}{w}, g, \mathscr{F}\right)$ (recursive call)
      ▷ else: Return $1 \cdot g$
4. else: Return $u \cdot f$

Interlude – Simplify

Note

- Tries to reuse all rows from old matrices.
  $\Rightarrow$ We need to keep them in memory.
- We also simplify generators of S-pairs, as we have done in our example: $(f_2, bcf_4) \Longrightarrow (f_2, cf_6)$.
- One can also choose “better” reducers by other properties, not only “last reduced one”.
- Without **Simplify** the F4 algorithm is rather slow.

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- Without **Simplify** the F4 algorithm is rather slow.

In our example:
Choose bf_5 as reducer, not bdf_4.

Example: Cyclic-4

Symbolic preprocessing - now with simplify:

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Now try to exploit the special structure of the Macaulay matrices.

Improve Gaussian Elimination

Use **Linear Algebra** for reduction steps in GB computations.

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$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

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Knowledge of underlying GB structure

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Knowledge of underlying GB structure

Idea

Do a static **reordering before** the Gaussian Elimination to achieve a better initial shape. **Reorder afterwards**.

Faugère-Lachartre Idea

1st step: Sort pivot and non-pivot columns

$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

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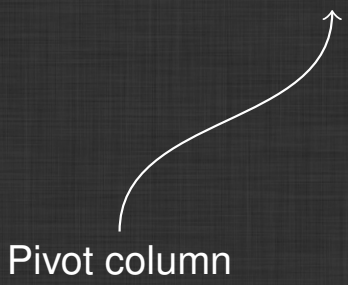

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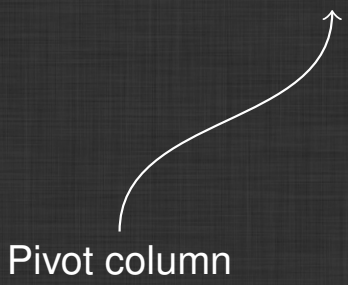

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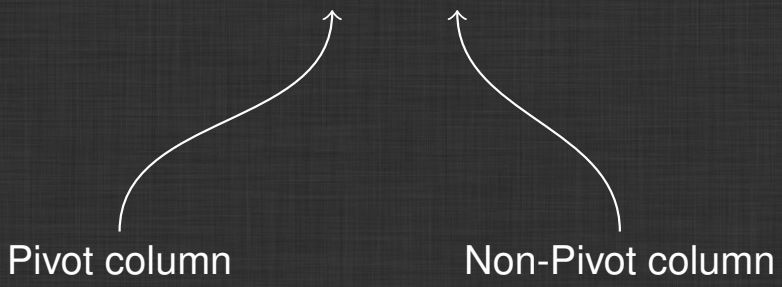

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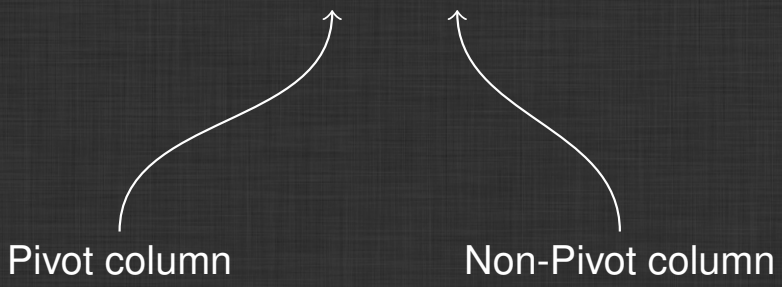

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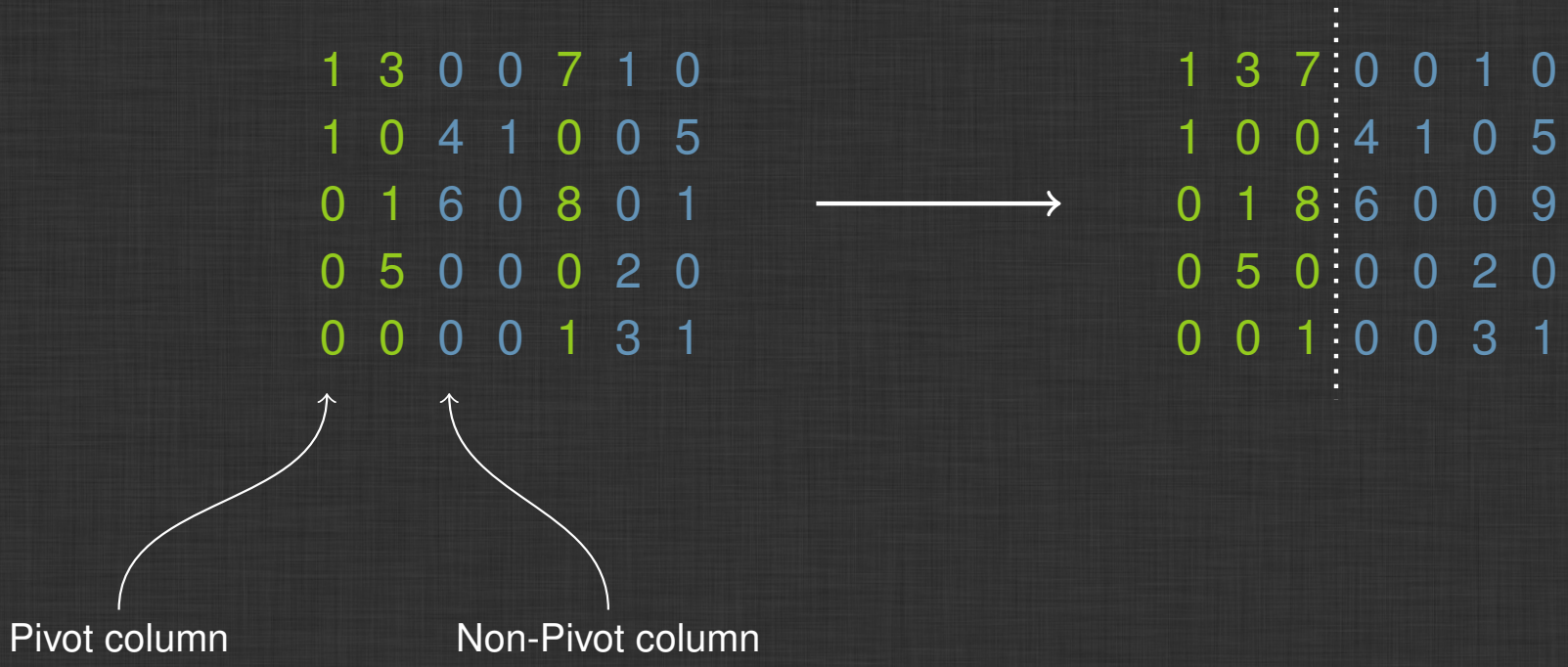

Faugère-Lachartre Idea

2nd step: Sort pivot and non-pivot rows

$$\left(\begin{array}{ccc:cccc} 1 & 3 & 7 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 1 & 8 & 6 & 0 & 0 & 9 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \end{array}\right)$$

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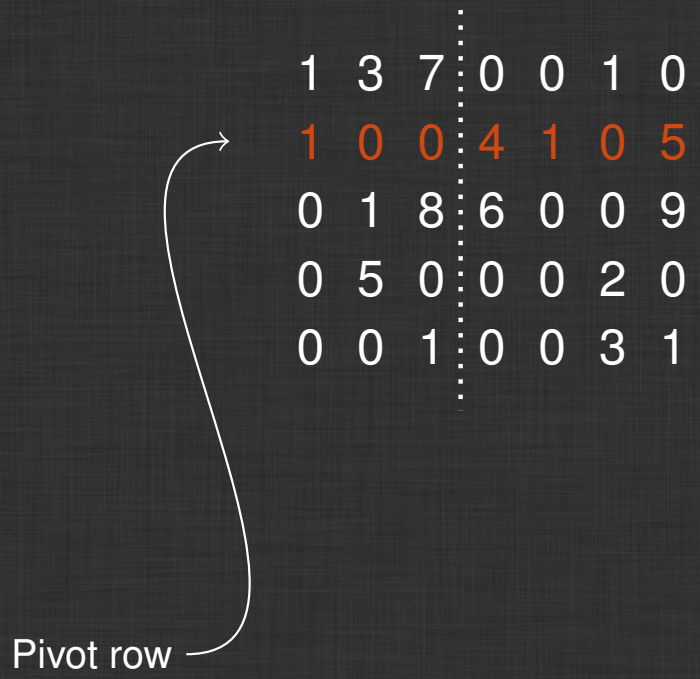

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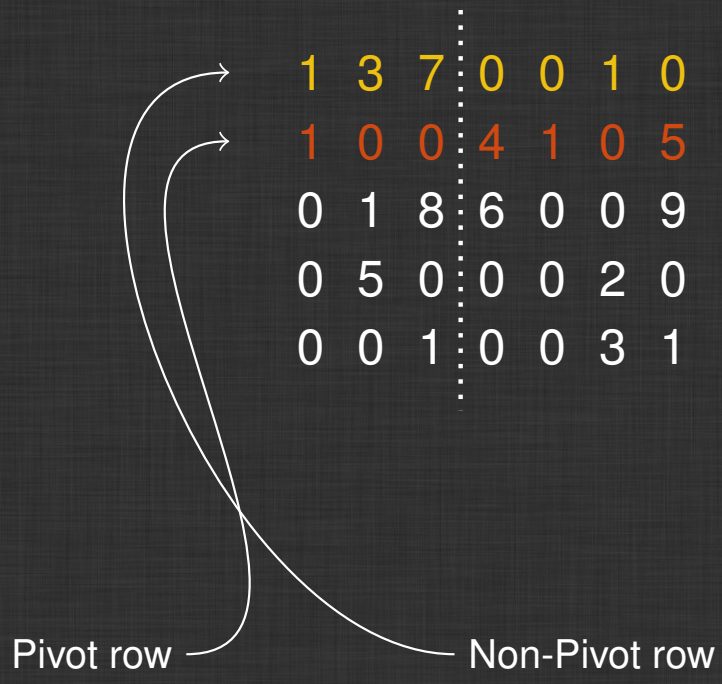

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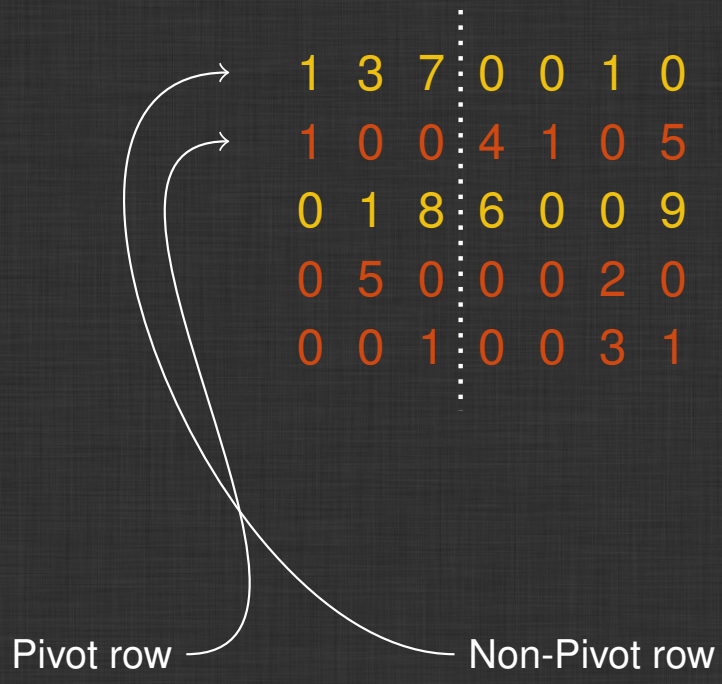

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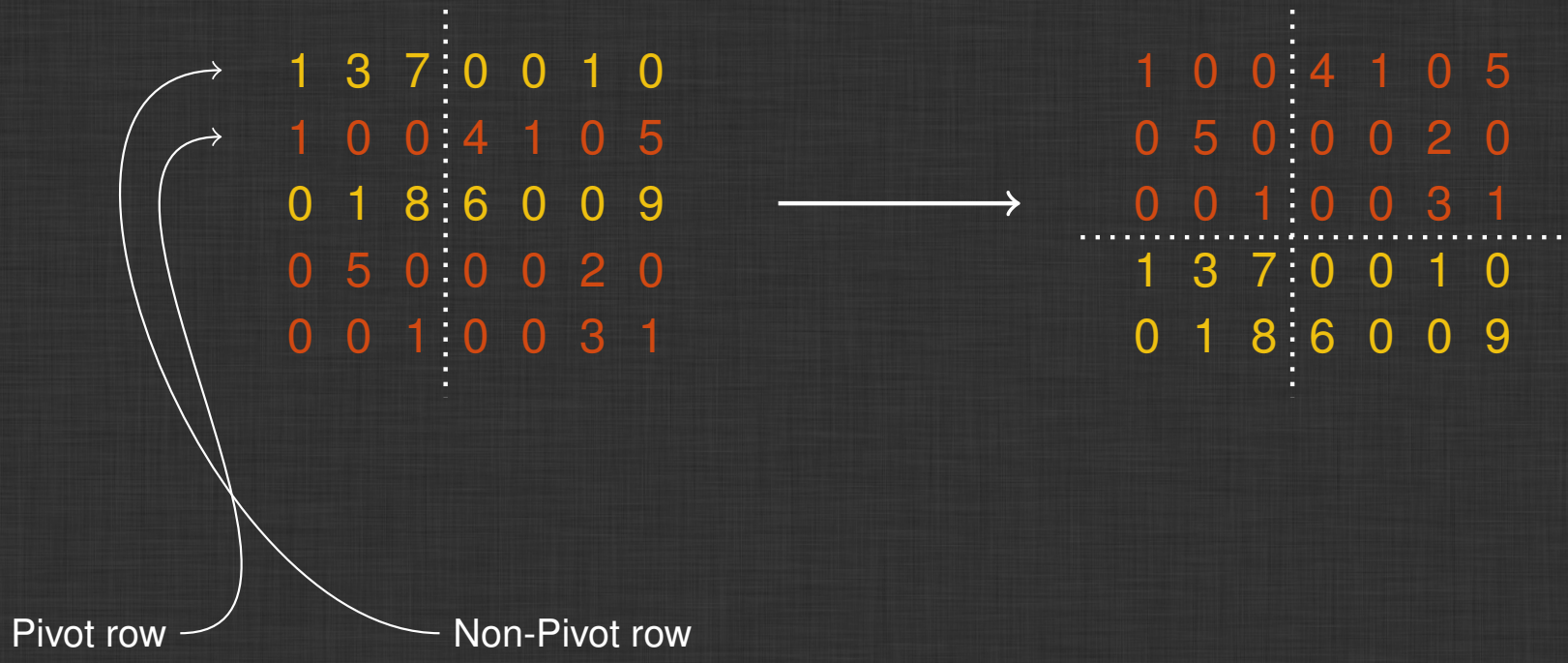

Faugère-Lachartre Idea

3rd step: Reduce lower left part to zero

$$\left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hdashline 1 & 3 & 7 & 0 & 0 & 1 & 0 \\ 0 & 1 & 8 & 6 & 0 & 0 & 9 \end{array}\right)$$

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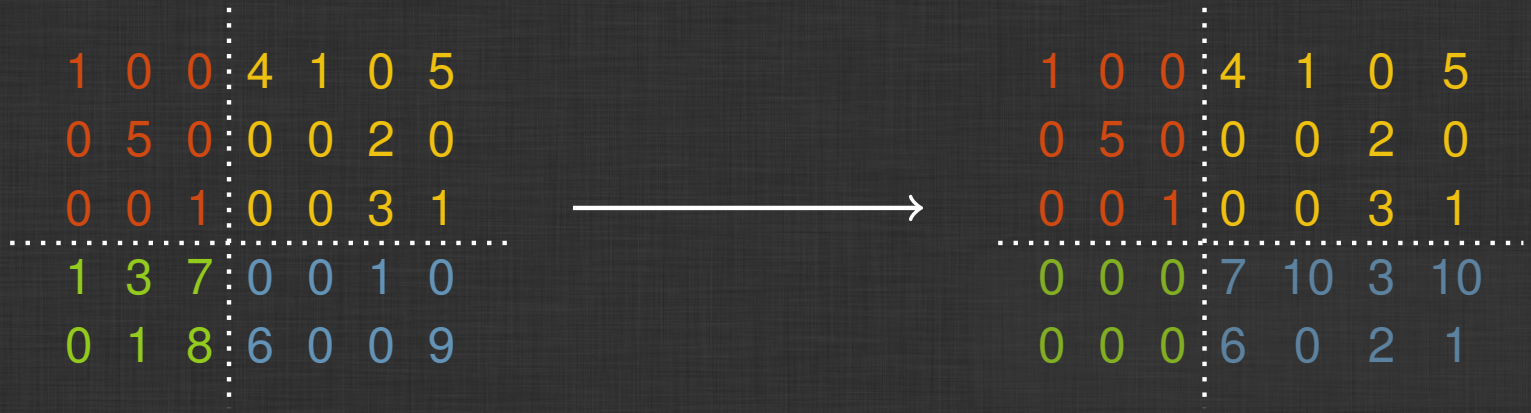


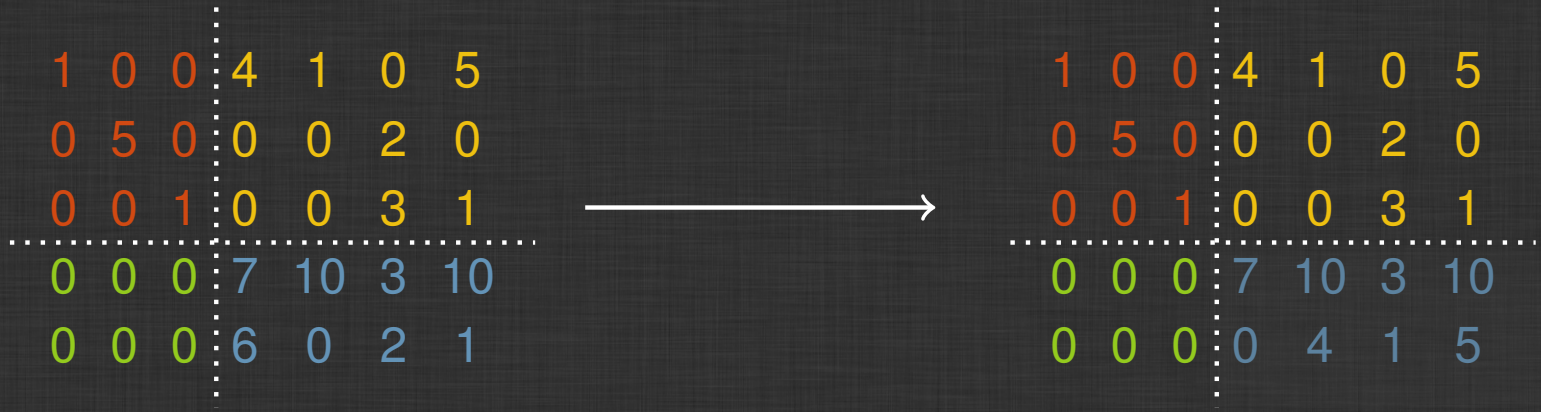
Faugère-Lachartre Idea

4th step: Reduce lower right part

$$\left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hdashline 0 & 0 & 0 & 7 & 10 & 3 & 10 \\ 0 & 0 & 0 & 6 & 0 & 2 & 1 \end{array}\right)$$

Faugère-Lachartre Idea

4th step: Reduce lower right part



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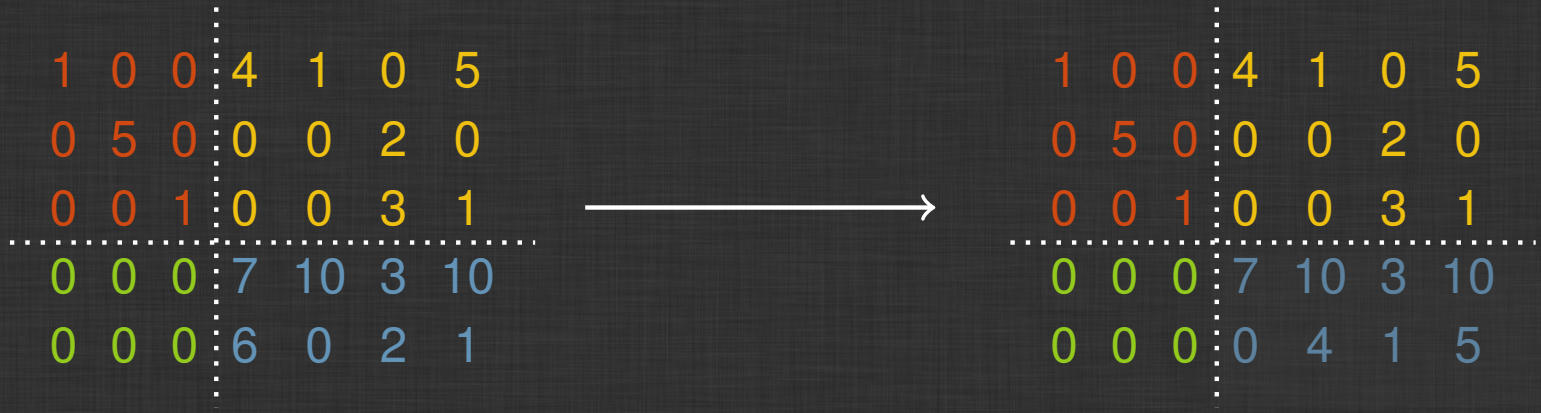

5th step: Remap columns of lower right part

How our matrices look like

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- F_4 computation of homogeneous KATSURA-12, degree 6 matrix

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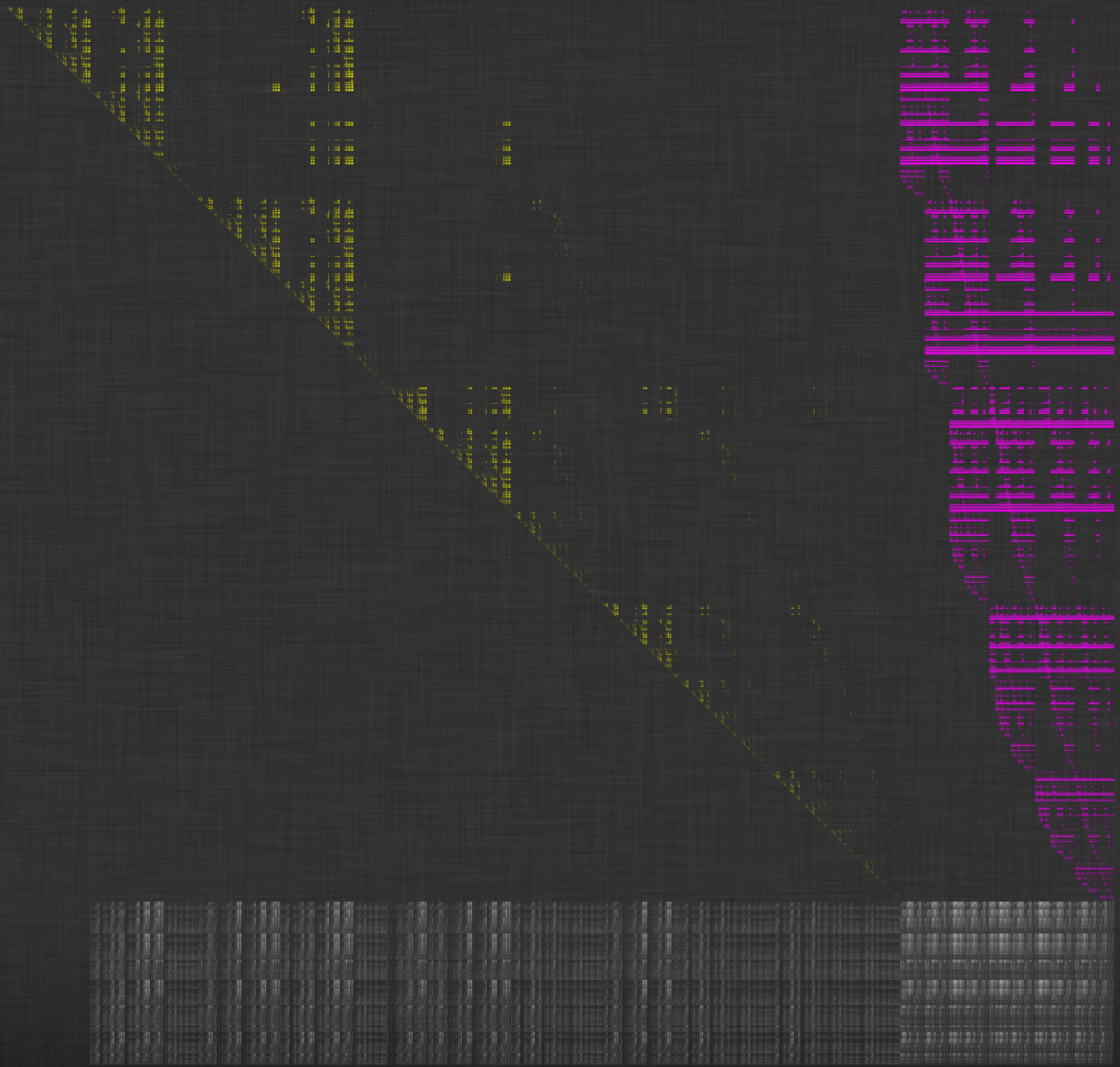
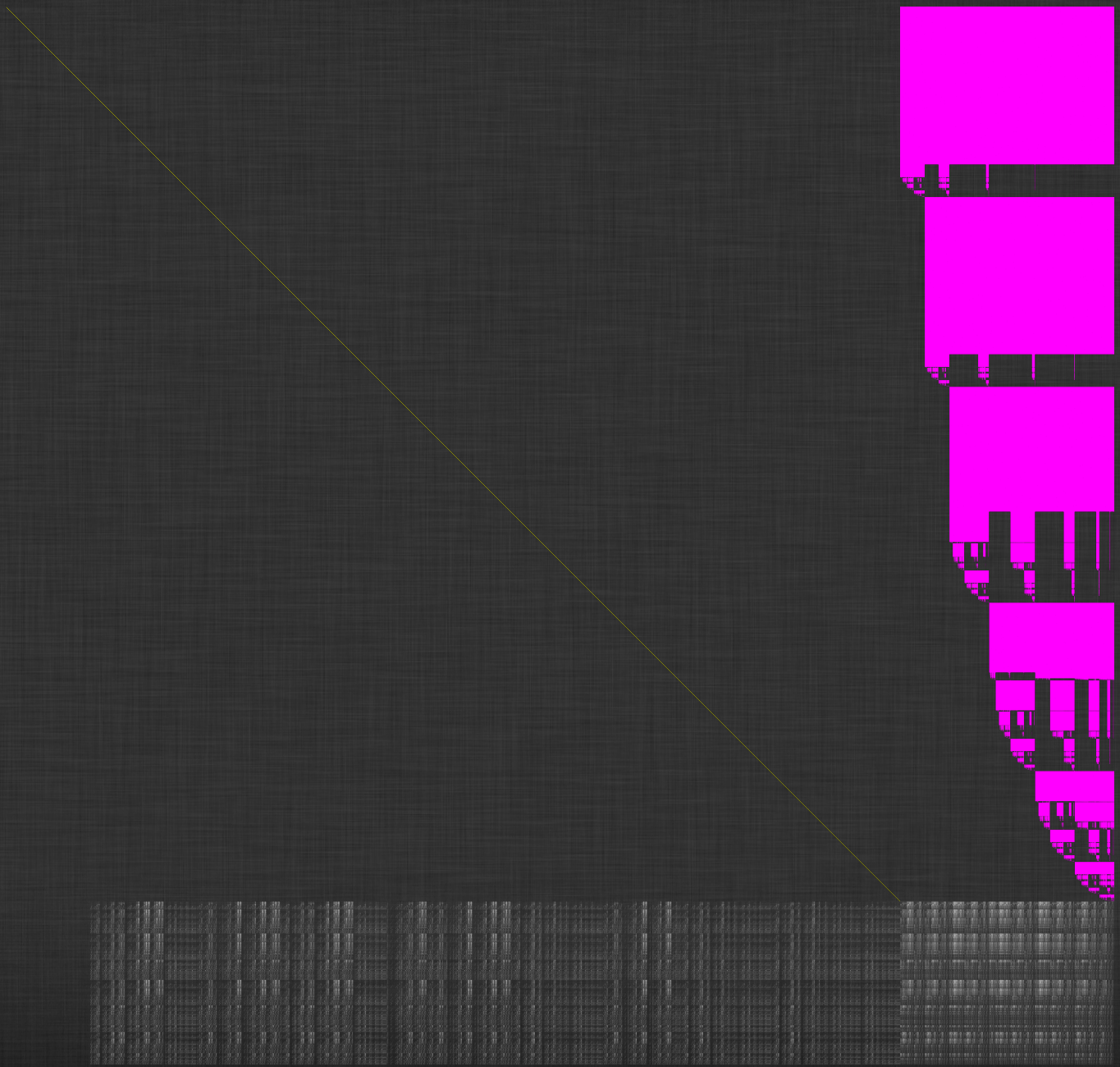
- Dimensions:

full matrix:	$21,182$	$\times$	$22,207$
upper-left:	$17,915$	$\times$	$17,915$
lower-left:	$3,267$	$\times$	$17,915$
upper-right:	$17,915$	$\times$	$4,292$
lower-right:	$3,267$	$\times$	$4,292$

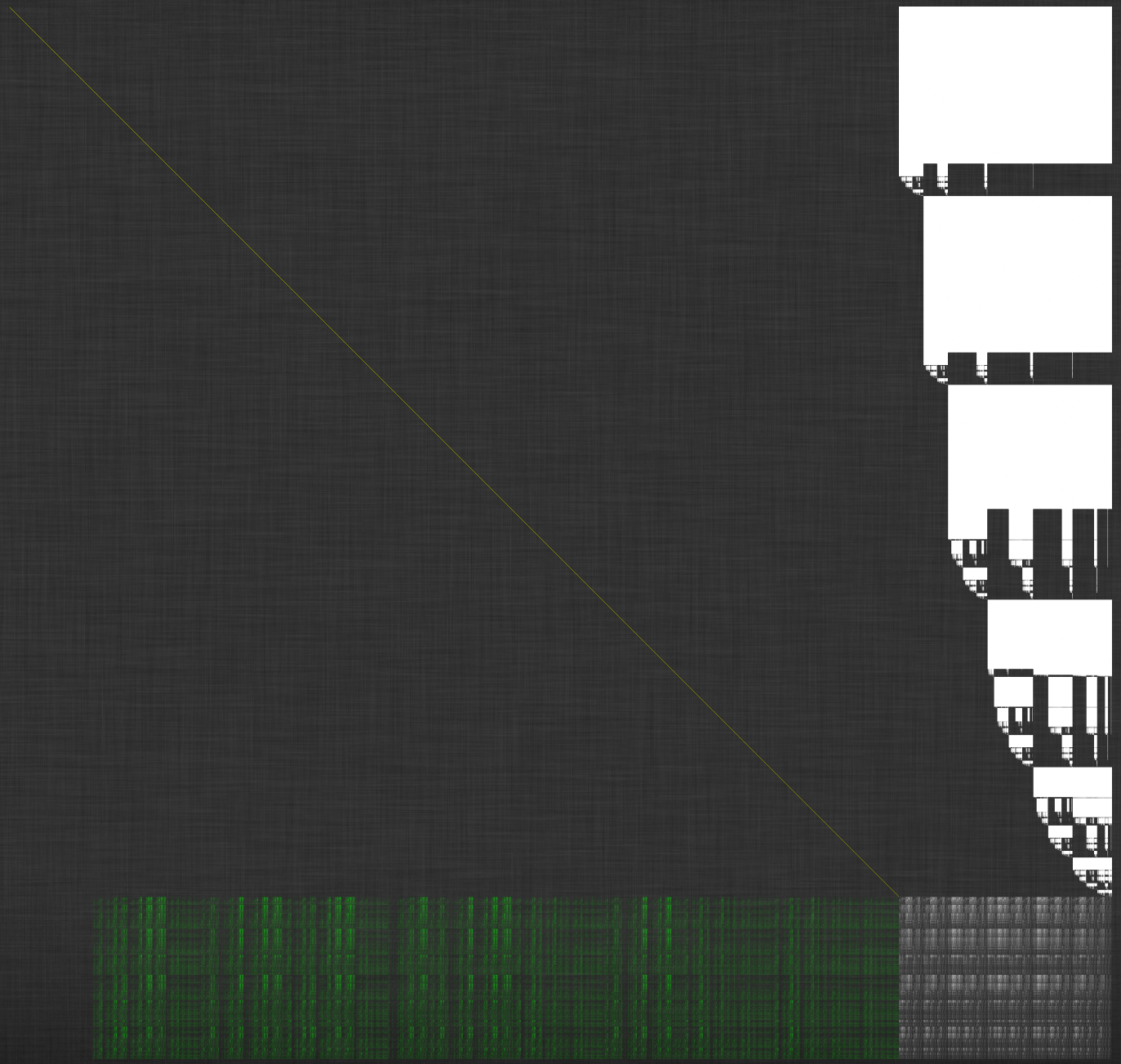
Hybrid Matrix Multiplication $A^{-1}B$



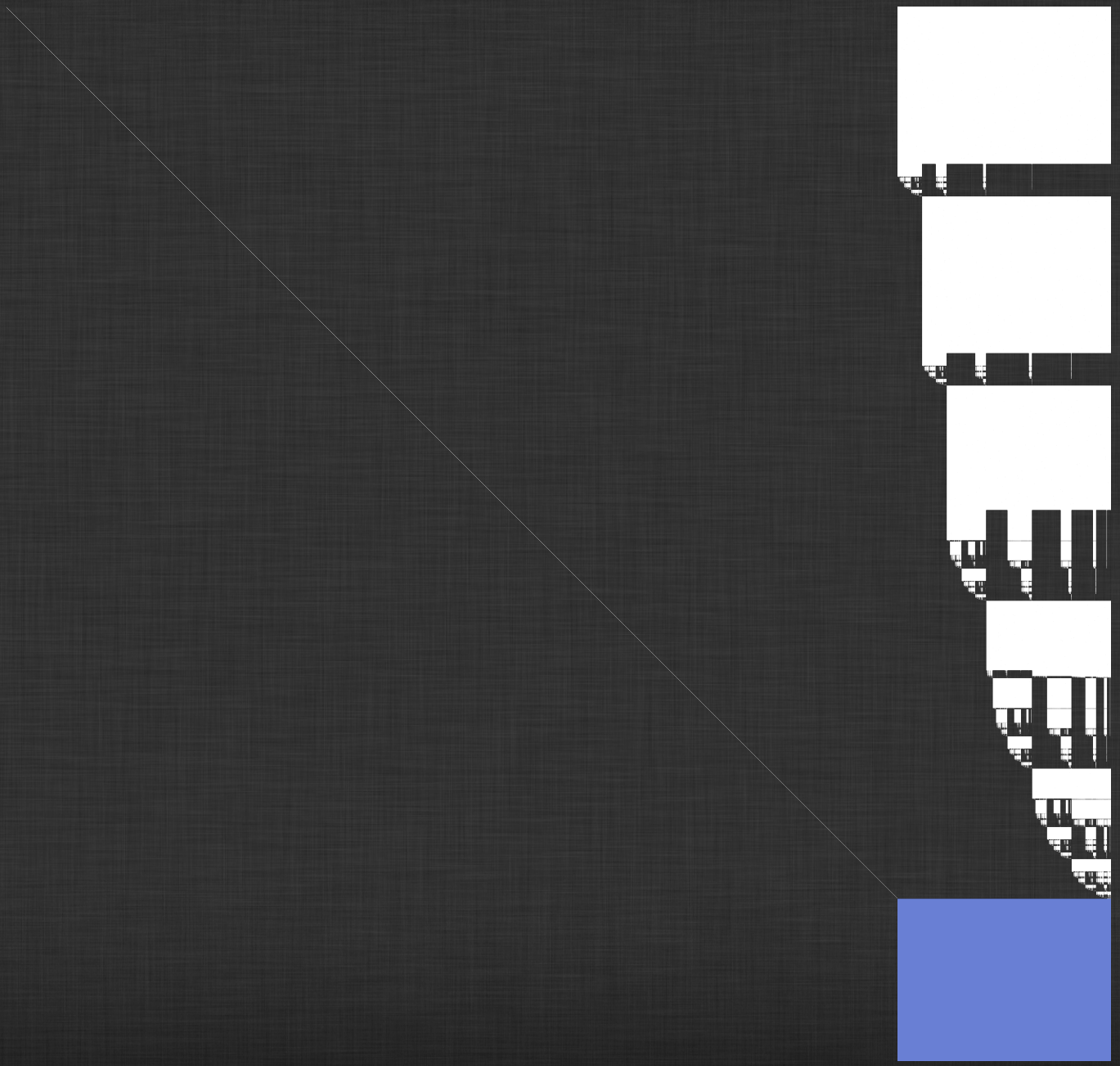
Hybrid Matrix Multiplication $A^{-1}B$



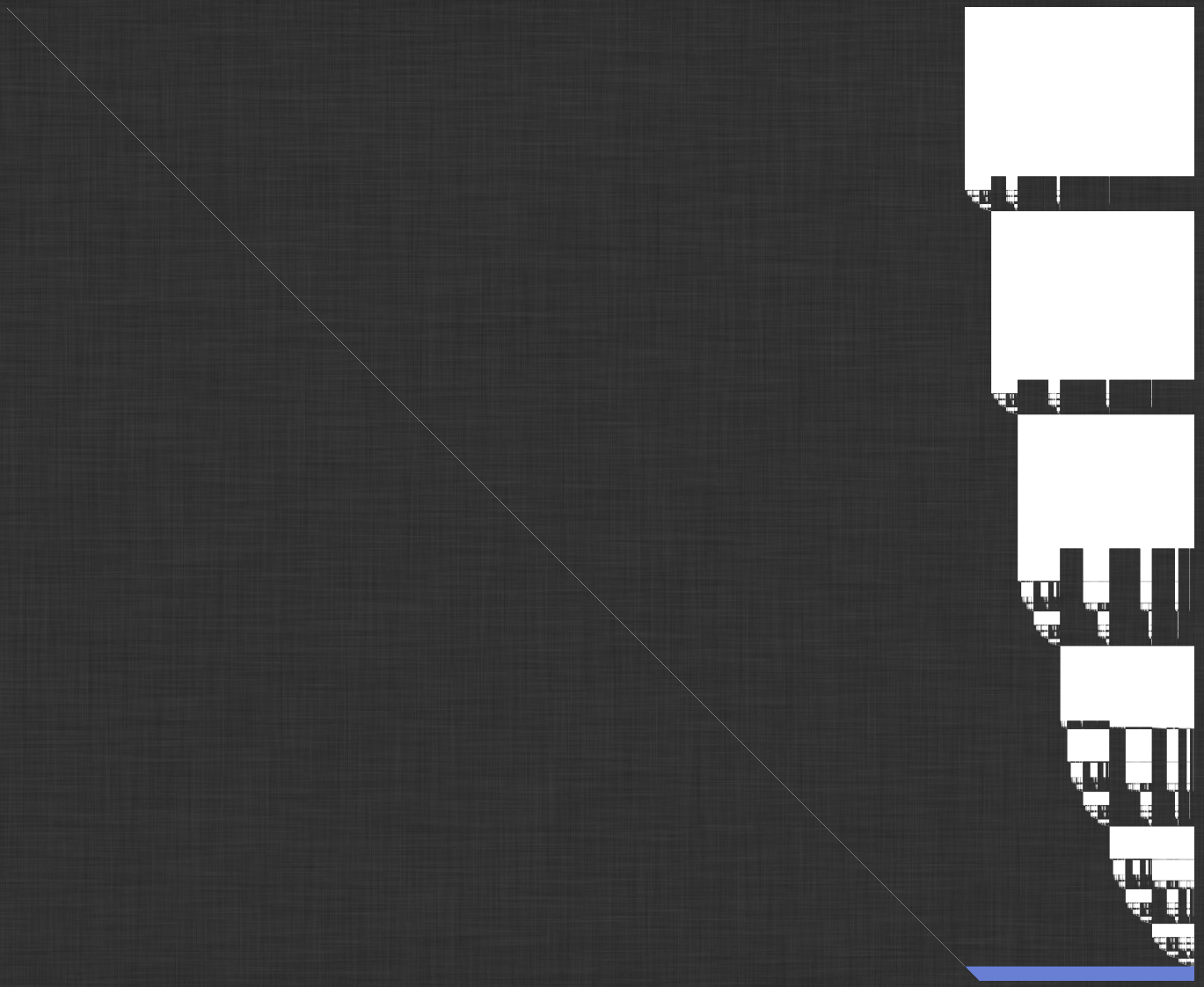
Reduce C to zero



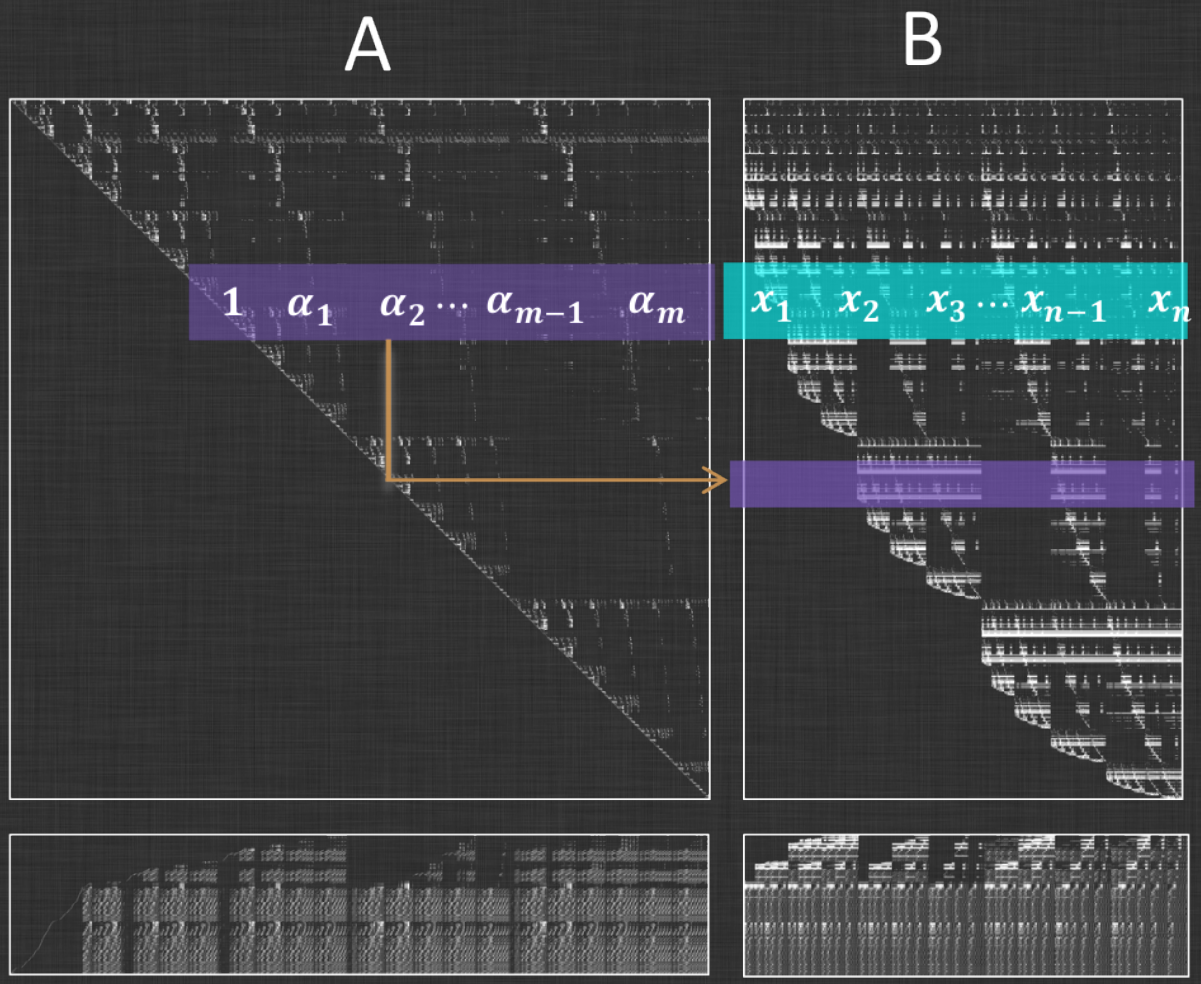
Gaussian Elimination on D



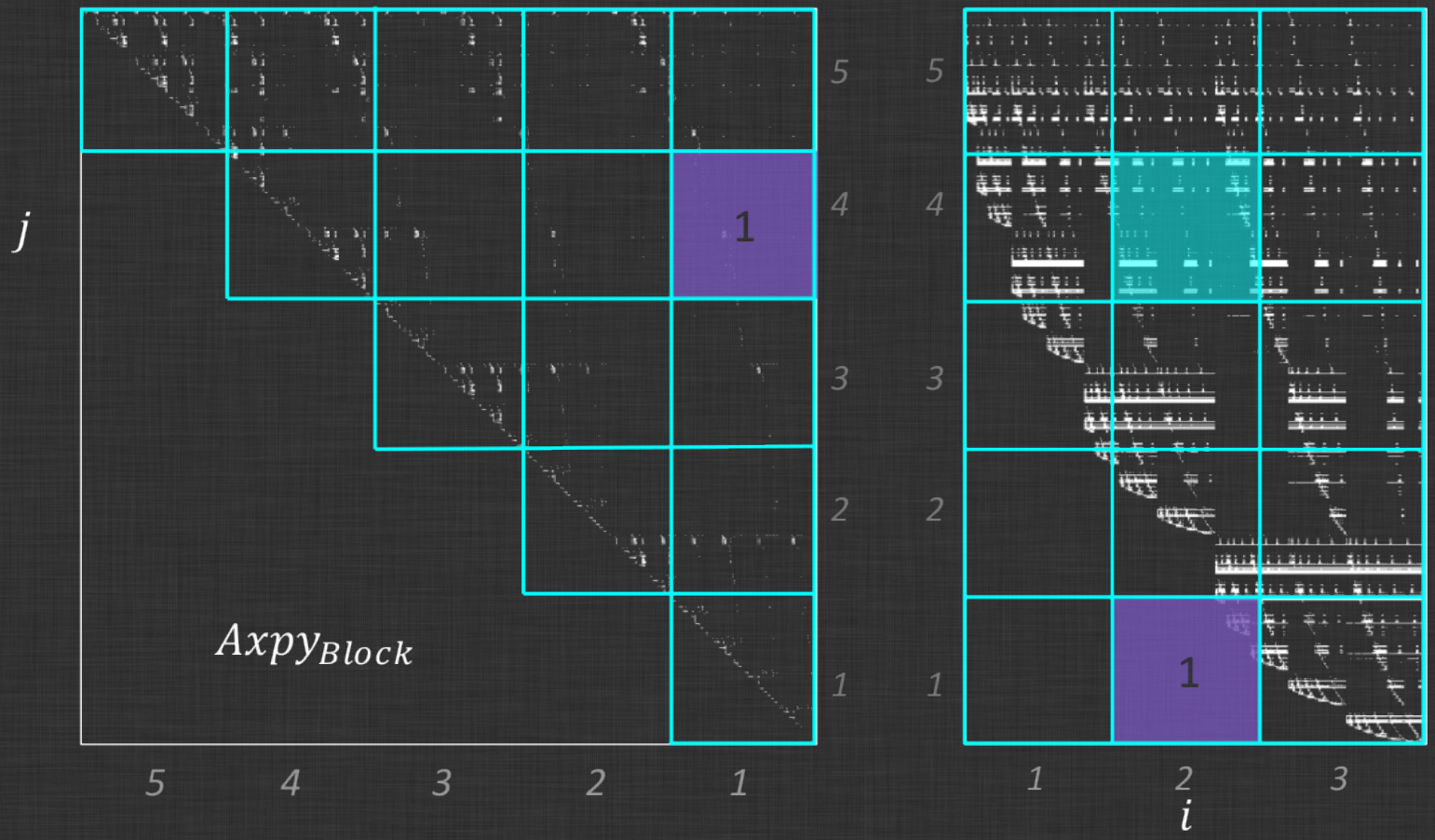
New information



$$B \leftarrow A^{-1}B$$



$B \leftarrow A^{-1}B$ – Block Version



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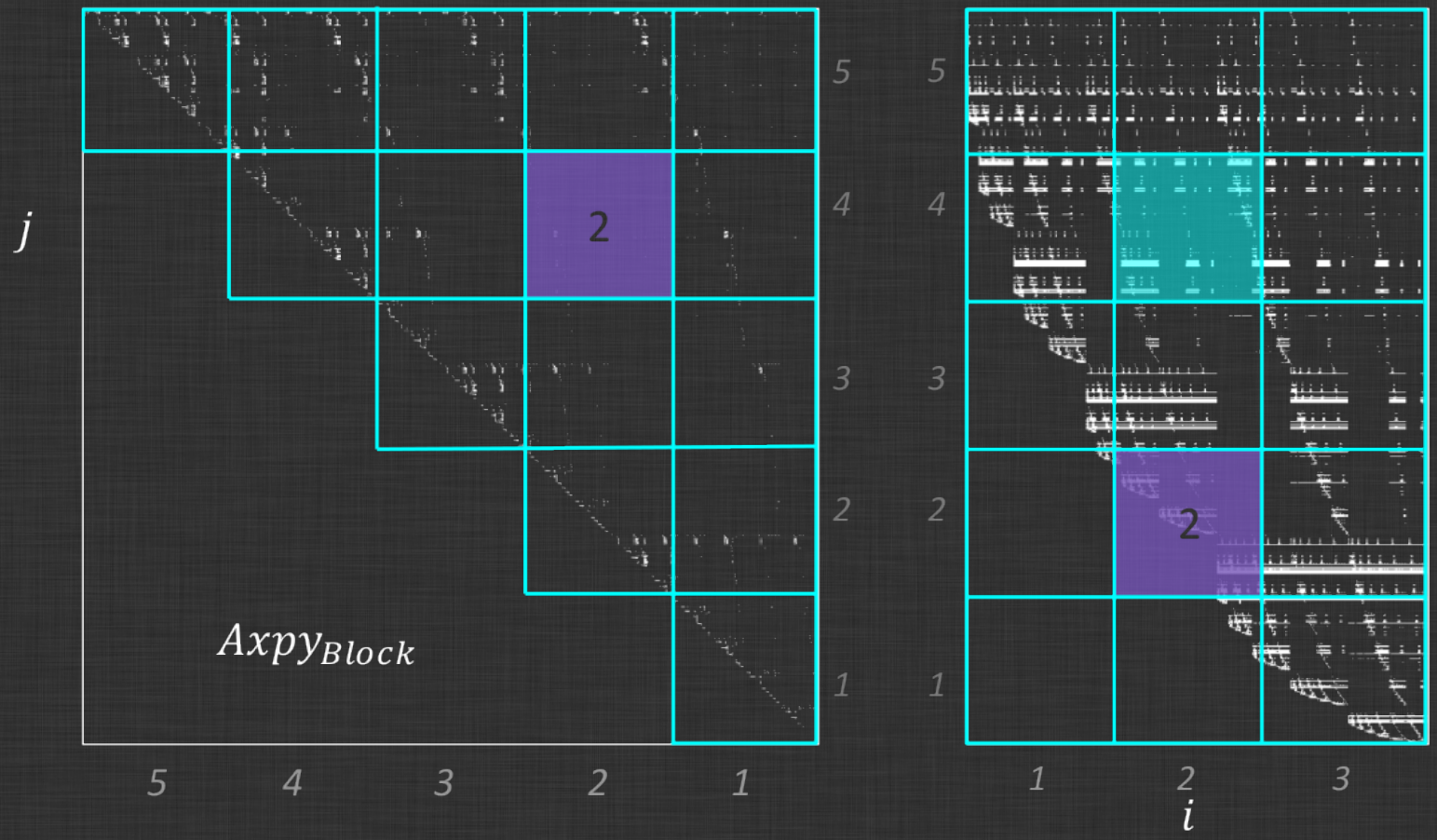

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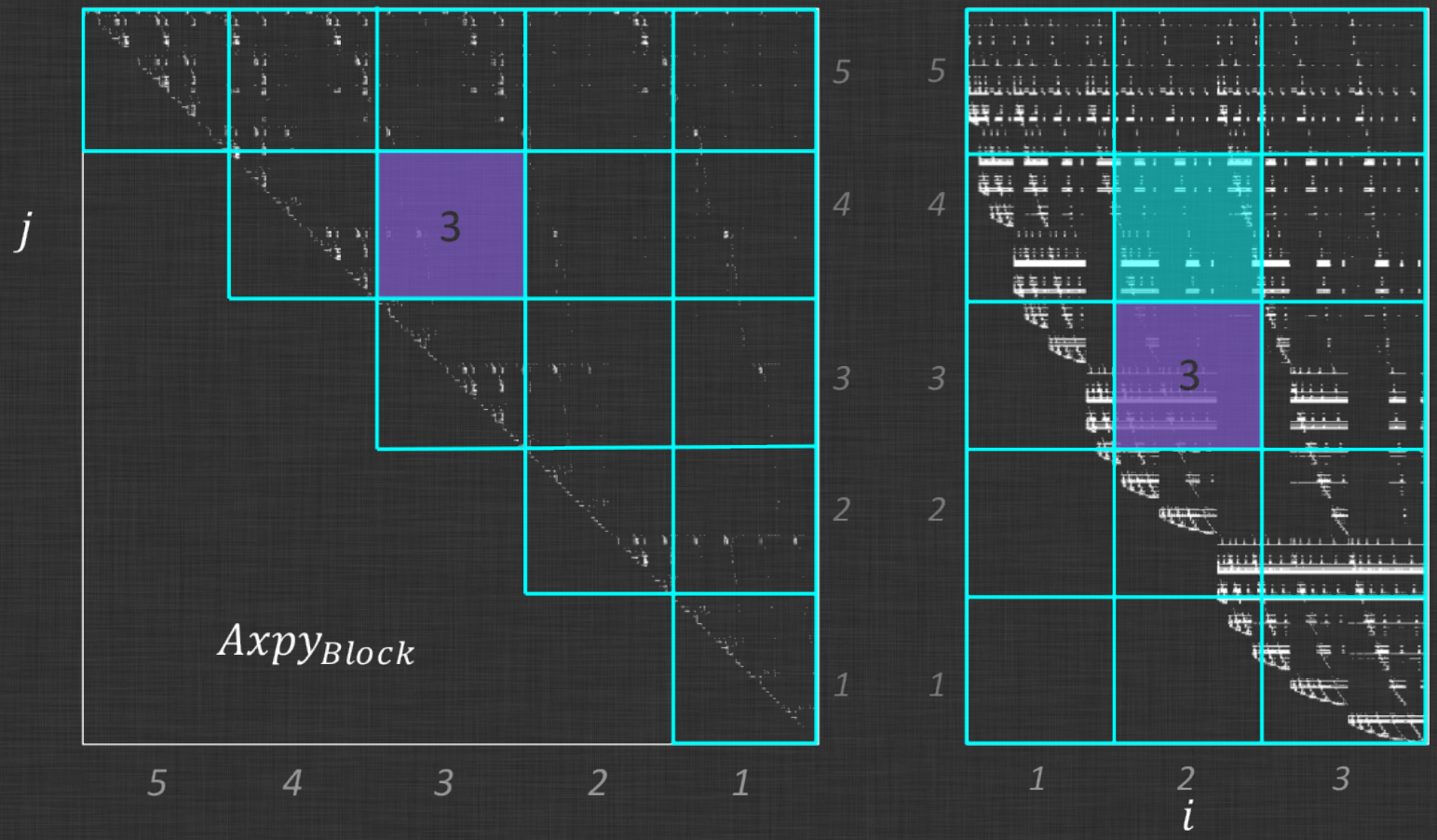

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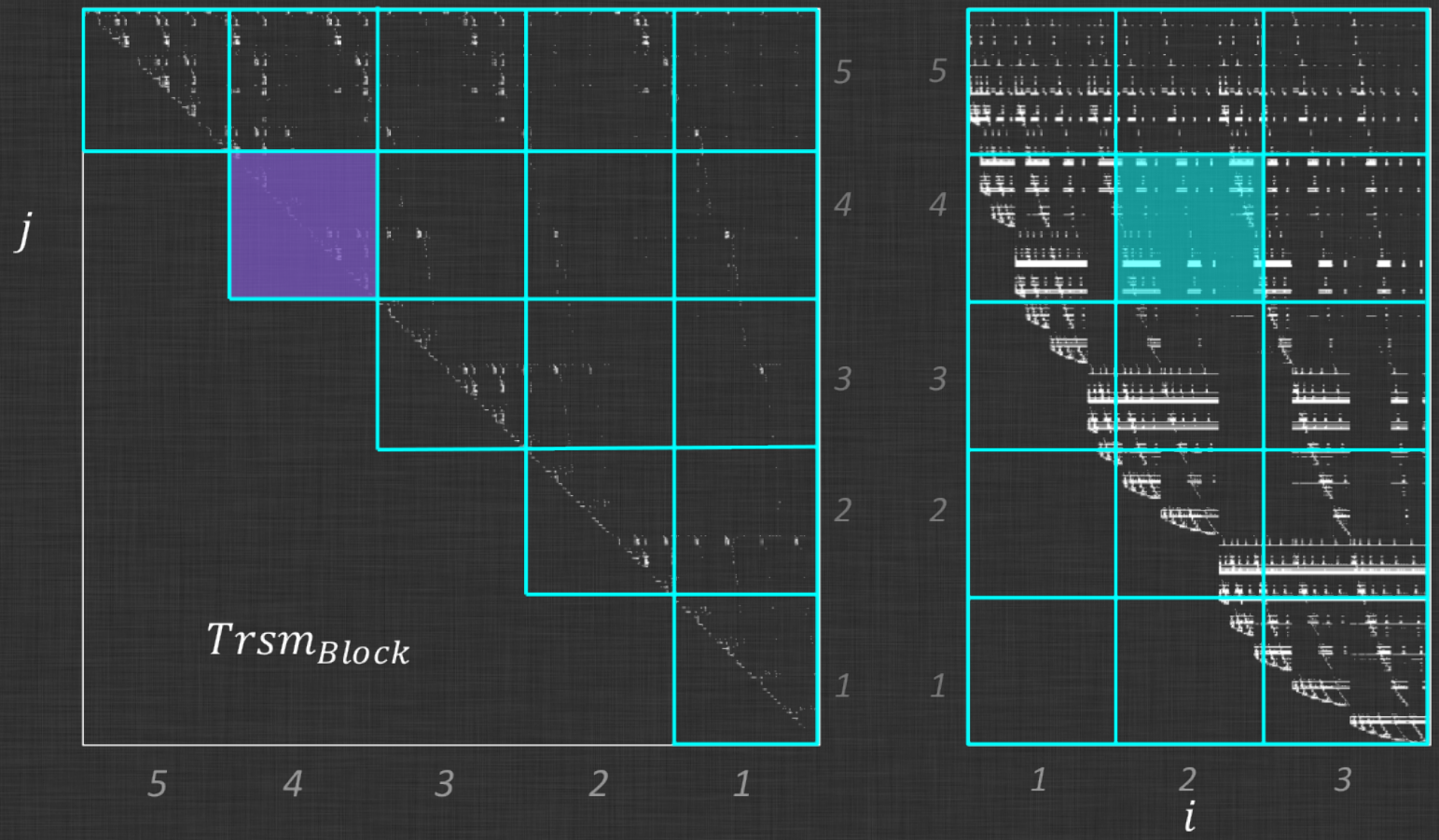

First attempts

2011 – University of Kaiserslautern
Bradford Hovinen – LELA
`https://github.com/Singular/LELA`

2012 – UPMC Paris 6, INRIA PolSys Team
Fayssal Martani – new implementation in LELA
`https://github.com/martani/LELA`

2012-2013 – University of Kaiserslautern
Bjarke Hammersholt Roune – MathicGB
`https://github.com/broune/mathicgb`

2012-2014 – University of Passau
Severin Neumann – parallelGBC
`https://github.com/svrnm/parallelGBC`

- Predicting zero reductions
- Pair set sorting and homogenization
- Finding better reducers
- Fast linear algebra for computing Gröbner bases
- **Modular computations**
- Change of ordering

Computing over the rationals

Coefficient growth over the rationals is problematic.

In each reduction step $f - \lambda g$

1. $\mathrm{lc}(g)$ needs to be adjusted to meet $\mathrm{lc}(f)$
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1 GB computation over $\mathbb{Q}$ $\longleftrightarrow$ several GB computations over $\mathbb{F}_{p_i}$
[2, 3, 38]

(Simplified) Modular GB computations

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle \subset \mathbb{Q}[x_1, \ldots, x_n]$
Output: Gröbner basis G for I in $\mathbb{Q}[x_1, \ldots, x_n]$

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4. Lift results back to $\mathbb{Q}[x_1, \ldots, x_n]$:
   - Chinese Remainder Theorem:

$$\prod_{i=1}^{s} \mathbb{Z}_{p_i}[x_1, \ldots, x_n] \longrightarrow \mathbb{Z}_N[x_1, \ldots, x_n]$$

$$G_{p_1} \times \cdots \times G_{p_s} \longmapsto G_N$$

   - Use Farey reconstruction to map G_N to G over $\mathbb{Q}$.

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   - ▷ Use Farey reconstruction to map G_N to G over $\mathbb{Q}$.
5. Test G:
   - ▷ Make a fast test if G mod q is a GB for I_q, $q \notin Q$.
   - ▷ Check if $I \subset \langle G \rangle$ (over $\mathbb{Q}$).
   - ▷ Check if G is a GB for $\langle G \rangle$ (over $\mathbb{Q}$).

   If one test fails, go back to (1) and generate more primes $\notin Q$.

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There are several different attempts to handle this problem:

- Using Hilbert functions, statically and dynamically
- Converting a GB G_1 w.r.t. $<_1$ to a GB G_2 w.r.t. $<_2$

1. Hilbert functions, statically

Let us assume homogeneous input $I = \langle f_1, \ldots, f_m \rangle$.

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Sloppy
$H_{\mathscr{R}/I}(d)$ measures how many elements of degree d a GB G for I has.
This number does not depend on the monomial order chosen.

We use this fact in the following way:

1. Get the Hilbert function $H(t) := H_{\mathscr{R}/I}(t)$
   - ▷ Either we have it
   - ▷ or we compute a GB G_1 for I w.r.t. an easy order $<_1$ (e.g. DRL).

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2. Start usual GB computation for I w.r.t. an hard order $<_2$ (e.g. LEX).
   - ▷ Use normal selection strategy (increasing degree)
   - ▷ Whenever a new element f of $\deg(f) = d$ is added to G_2 we check if $H(t) = H_{\mathscr{R}/L(G_2)}(t)$ for all $t \geq d$.

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Use information from G_1 during the computation of G_2.
Goes back to Traverso [47].

Implementation: **stdhilb** in SINGULAR

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Receive a GB for I w.r.t. some order $<$ ($<$ not known beforehand):

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     - ▷ Find a better order $<_{\text{new}}$ such that $H_{\mathscr{R}/L_{<\text{new}}(G_2)}(t)$ is minimal.
     - ▷ Re-reduce G w.r.t. $<_{\text{new}}$ and go on.

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There are several questions to be answered.

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1. It is hard to get the order changing right.

2. Caboara and Perry are improving the idea currently [12].

3. Converting a GB

Let $v \in \mathscr{R}^n$ be a weight vector, $I = \langle f_1, \ldots, f_m \rangle \subset \mathscr{R}$ an ideal, and let G be a GB for I w.r.t. some order $<$.

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- $f \in I$, initial element of f w.r.t. v: $\mathrm{in}_v(f) = t$, $\deg_v(t)$ maximal for all $t \in \mathrm{support}(f)$.
- Initial ideal of I w.r.t. v: $\mathrm{in}_v(I) = \langle \mathrm{in}_v(f_1), \ldots, \mathrm{in}_v(f_m) \rangle$.
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Note

There are only finitely many not equivalent monomial orders on $\mathscr{R}$
$\Longrightarrow$ Gröbner cone is well-defined.

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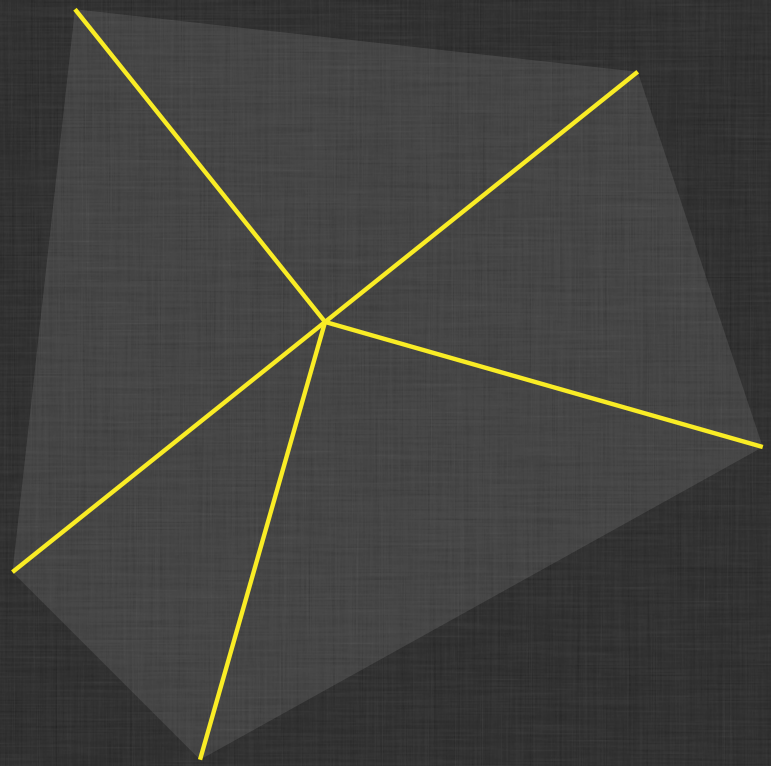

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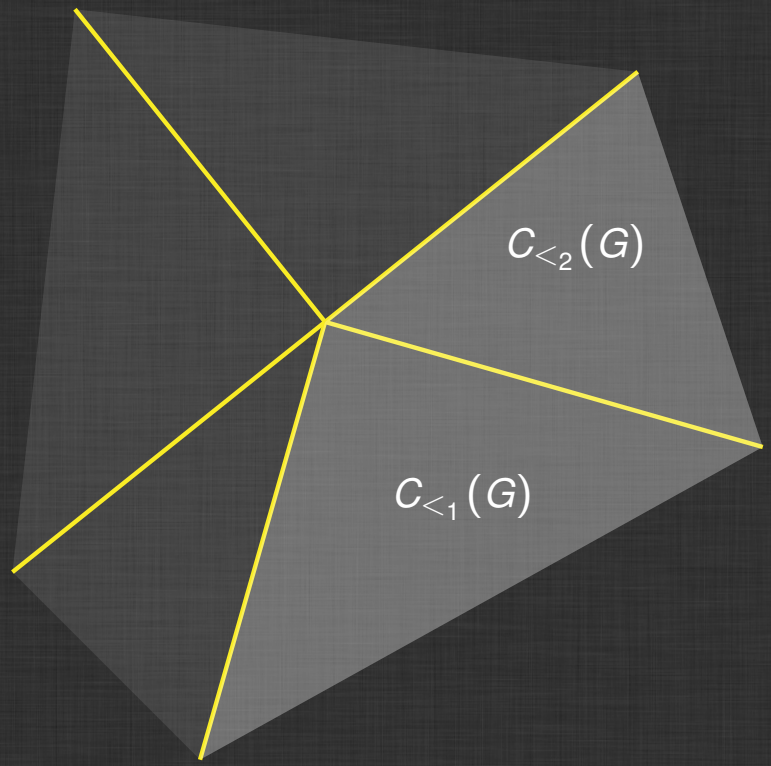

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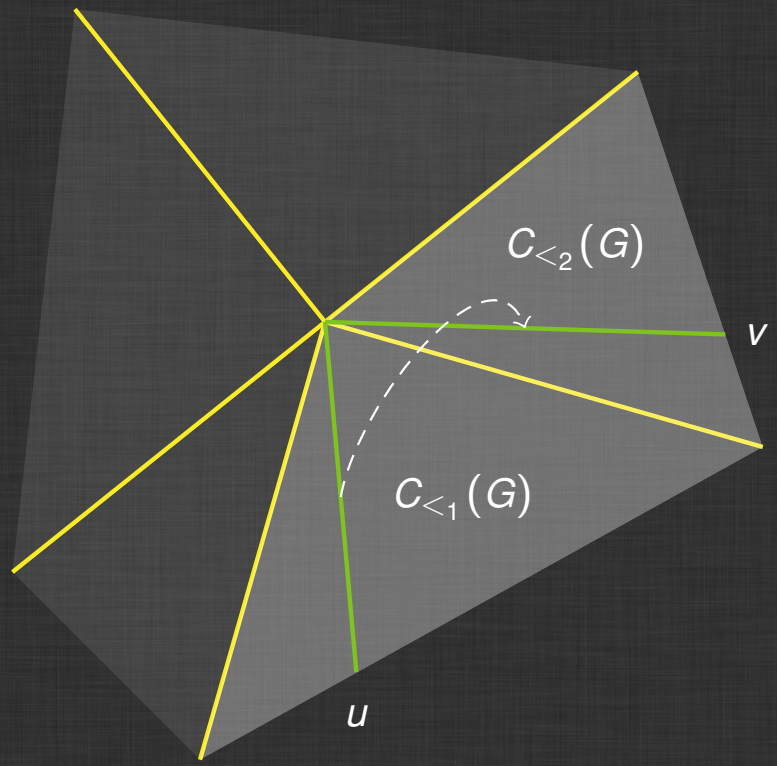

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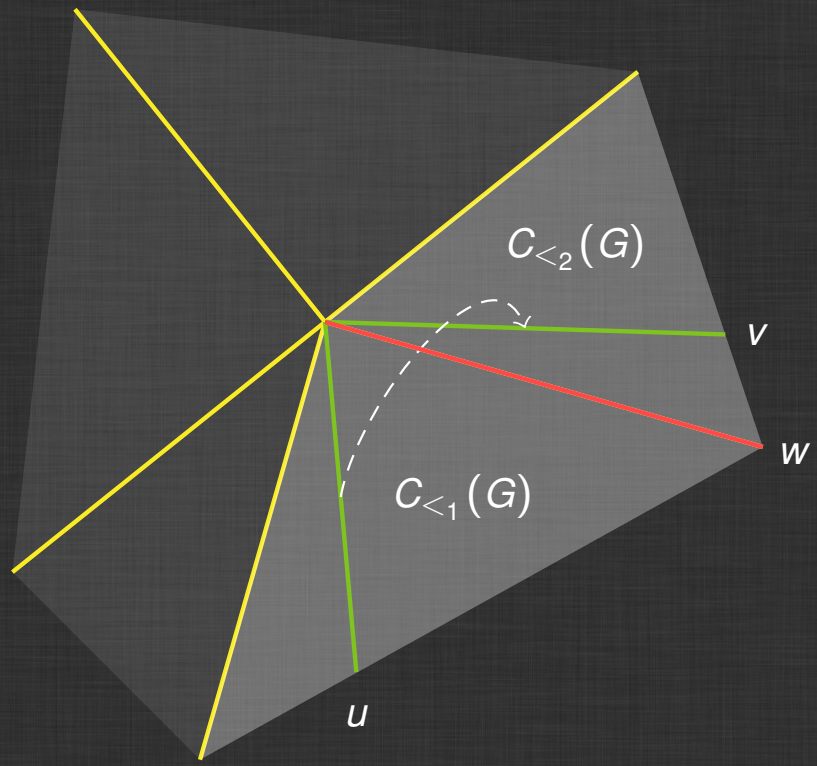

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Lemma

Let $<_1, <_2$ be two different orders such that $C_{<_1}(I) \cap C_{<_2}(I) \neq \emptyset$. Then there exists $w \in C_{<_1}(I) \cap C_{<_2}(I)$ such that $\mathrm{in}_w(I)$ is not monomial.

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Given a reduced $G_{<_1} = \{g_1, \ldots, g_r\}$ (and w), how to get now $G_{<_2}$?

1. Refine w by $<_2$: $(w, <_2)$
2. Compute reduced GB M of $\text{in}_w(G_{<_1})$ w.r.t. $(w, <_2)$ (Nearly monomial!) $m_j \in M$ represented by $m_j = \sum_{i=1}^{r} h_{ij} \text{in}_w(g_i)$.
3. $p_j \leftarrow$ Take m_j and replace $\text{in}_w(g_i)$ by g_i.
4. $G_{(w,<_2)} \leftarrow$ Reduce $\{p_1, \ldots, p_k\}$ w.r.t. $(w, <_2)$.
5. Convert $G_{(w,<_2)}$ to a reduced GB w.r.t. $<_2$.

Conclusions

- Often better complexity when computing GB in DRL and walk to LEX.
- Problem if conversion between not adjacent orders
  $\Longrightarrow$ Several steps from cone to cone
- Note that for more than 2 variables searching good paths is quite hard.
- Kalkbrener [40]: When converting from $G_{<_1}$ to an adjacent $G_{<_2}$ maximal degree of elements in $G_{<_2}$ bounded by

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Implementations: **GBWalk** in SINGULAR, **GFan** library [39]

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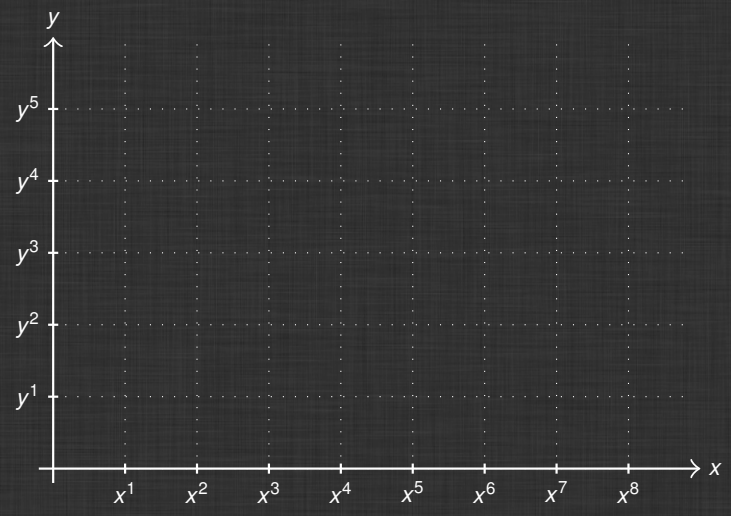

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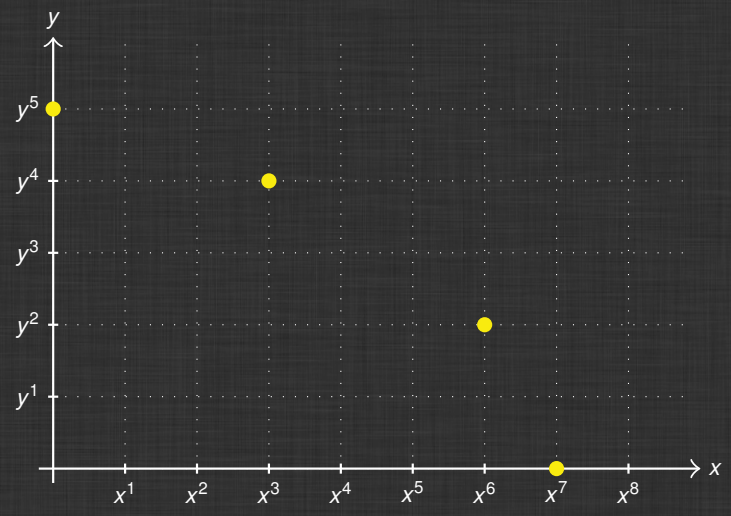

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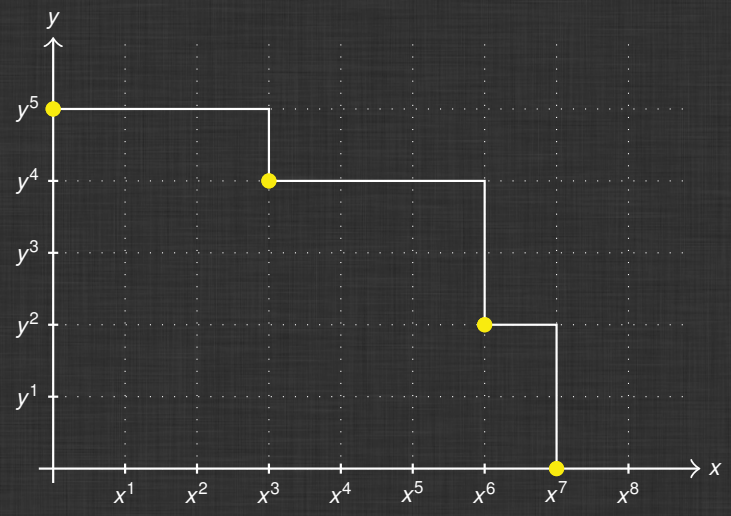

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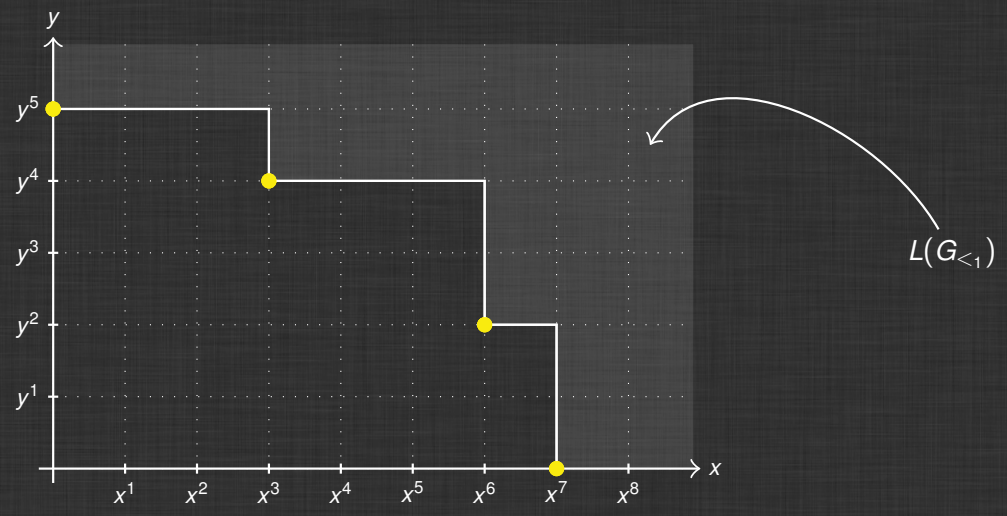

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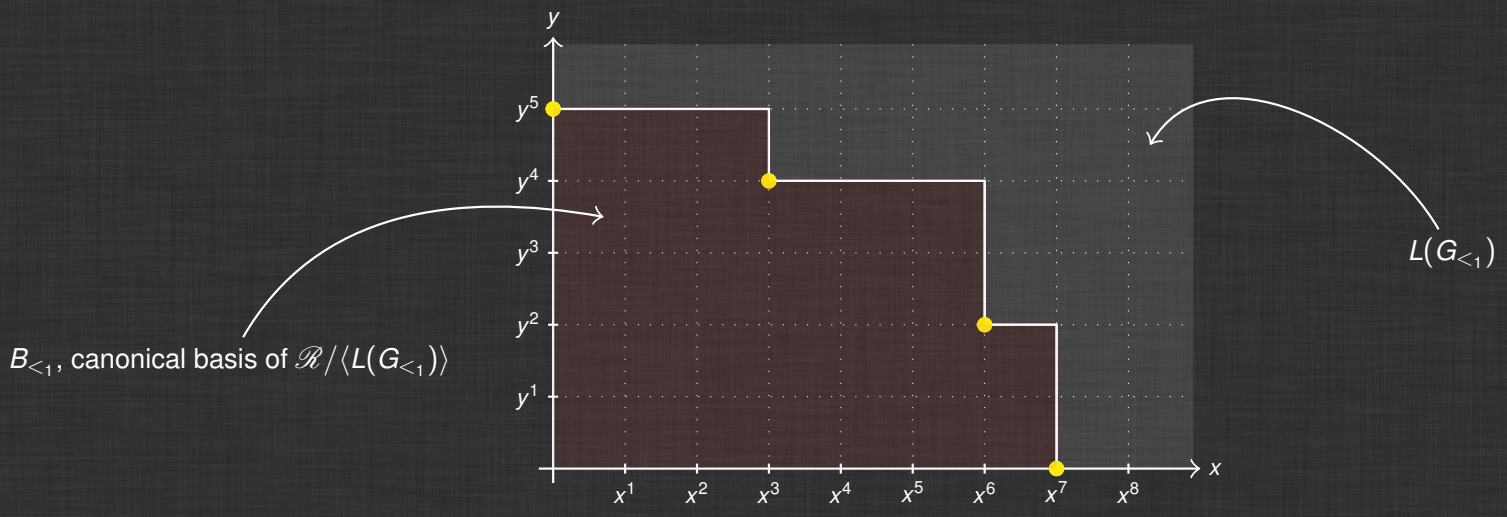

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5. Check if $\mathrm{lt}(g) = x_k^j$ for x_k largest variable w.r.t. $<_2$:
   - If $\mathrm{lt}(g) = x_k^j \Longrightarrow$ Terminate algorithm (Here we need "zero dimensional").
   - Else, take smallest monomial m' such that $m' >_2 m$ and $m' \notin L(G_{<_2})$. Go back to (1).

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Implementations: **FGLM** in nearly all CAS

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