Department of Mathematics
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2nd Exercise Sheet in „Computer Algebra“

Deadline: Thursday, November 3 2011, 10.00 h

Exercise 1. For an ideal $I \subset K[x_1, \dots, x_n]$ let $I^h := \langle f^h \mid f \in I \rangle \subset K[x_0, \dots, x_n]$. Now let $>$ be a global degree ordering, and let $\{f_1, \dots, f_m\}$ be a Gröbner basis of I. Prove that

$$I^h = \langle f_1^h, \dots, f_m^h \rangle .$$

Exercise 2. Let $A \in \mathrm{GL}(n, \mathbb{Q})$ be a matrix defining, on $\mathrm{Mon}(x_1, \dots, x_n)$, the ordering $>$ and let $I = \langle f_1, \dots, f_m \rangle \subset K[x_1, \dots, x_n]$ be an ideal. Consider the ordering $>_h$ on $\mathrm{Mon}(t, x_1, \dots, x_n)$ defined by the matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

and let $\{G_1, \dots, G_s\}$ be a homogeneous standard basis of $\langle f_1^h, \dots, f_m^h \rangle$, f_i^h, the homogenization of f_i with respect to t. Prove that $\{G_1|_{t=1}, \dots, G_s|_{t=1}\}$ is a standard basis for I.

Exercise 3. Let $>$ be a global monomial ordering on $\mathrm{Mon}(x_1, \dots, x_n)$. Prove the *Product Criterion*: Let $f, g \in K[x_1, \dots, x_n]$ be polynomials such that $\mathrm{lcm}\big(\mathrm{LM}(f), \mathrm{LM}(g)\big) = \mathrm{LM}(f) \cdot \mathrm{LM}(g)$, then

$$\mathrm{NF}\big(\mathrm{spoly}(f, g) \mid \{f, g\}\big) = 0 .$$

Hint: Assume that $\mathrm{LC}(f) = \mathrm{LC}(g) = 1$ and claim that $\mathrm{spoly}(f, g) = -\,\mathrm{tail}(g) f + \mathrm{tail}(f) g$ is a standard representation.

Exercise 4. Assume that $>$ is a global monomial ordering on $K[x_1, \dots, x_n]$. Write a Singular procedure to compute the reduced normal form of a given polynomial f with respect to a given finite list of polynomials G.
Remark: You are not allowed to use the implemented commands `reduce` and `NF`. Don't forget to add at least one example to your procedure.