Department of Mathematics
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1st Exercise Sheet in „Computer Algebra“

Deadline: Thursday, 27 October 2011, 10.00 h

Exercise 1. Consider $>_1$, a monomial ordering on $\mathrm{Mon}(x_1, \ldots, x_{n_1})$, and $>_2$, a monomial ordering on $\mathrm{Mon}(y_1, \ldots, y_{n_2})$. Then the *product ordering* or *block ordering* $>$, also denoted by $(>_1, >_2)$ on $\mathrm{Mon}(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$, is defined as

$$x^\alpha y^\beta > x^{\alpha'} y^{\beta'} \quad :\Longleftrightarrow \quad x^\alpha >_1 x^{\alpha'} \text{ or } \left(x^\alpha = x^{\alpha'} \text{ and } y^\beta >_2 y^{\beta'}\right).$$

Given a vector $w = (w_1, \ldots, w_n)$ of integers, we define the *weighted degree* of x^α by

$$w - \deg(x^\alpha) := \langle w, \alpha \rangle := w_1 \alpha_1 + \ldots + w_n \alpha_n,$$

that is, the variable x_i has degree w_i. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha$, we define the weighted degree,

$$w - \deg(f) := \max\{w - \deg(x^\alpha) \mid a_\alpha \neq 0\}.$$

Using the weighted degree in the definition of $>_{dp}$, respectively $>_{ds}$, with all $w_i > 0$, instead of the usual degree, we obtain the *weighted reverse lexicographical ordering* $>_{wp(w_1,\ldots,w_n)}$, respectively the *negative weighted reverse lexicographical ordering* $>_{ws(w_1,\ldots,w_n)}$.

Now determine matrices $A \in \mathrm{GL}(n, \mathbb{R})$ defining the orderings

(a) $>_{ws(5,3,4)}$ on $\mathrm{Mon}(x_1, x_2, x_3)$,

(b) $(>_{dp}, >_{ls})$ on $\mathrm{Mon}(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$ with $n = n_1 + n_2$,

(c) $(>_{ds}, >_{wp(7,1,9)})$ on $\mathrm{Mon}(x_1, \ldots, x_{n_1}, y_1, y_2, y_3)$ with $n = n_1 + 3$.

Exercise 2. Let $>$ be a *non*-well-ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$. Show that there *cannot* exist a normal form on $K[x_1, \ldots, x_n]_>$.

Exercise 3. Let $>$ be any monomial ordering, $R = K[x_1, \ldots, x_n]_>$ and $I \subset R$ be an ideal. Show that if I has a reduced standard basis, then it is unique.

Exercise 4. Write a Singular procedure, having a list $P = ((g_1, h_1), \ldots, (g_r, h_r))$ of pairs of polynomials, an ideal $I = \langle f_1, \ldots, f_s \rangle$ and a polynomial f as input and returning the extendend pair-set $P = P \cup ((f, f_1), \ldots, (f, f_s))$ as output.