A (short) survey on signature-based Gröbner Basis Algorithms

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We can reduce further using $z^2 g_2$:

$$\boxed{-y^2z^2 + z^4 + y^2z^2 - z^4 = 0.}$$

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- **A signature** of f is given by $\mathfrak{s}(f) = \mathrm{lt}_{\prec}(\alpha)$ where $f = \overline{\alpha}$.
- An element $\alpha \in \mathscr{R}^m$ with $\overline{\alpha} = 0$ is called **a syzygy**.

Our example again – with signatures and $\prec_{\textbf{pot}}$

$$g_1 = xy - z^2, \; \mathfrak{s}(g_1) = e_1,$$
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Note that $\mathfrak{s}(\operatorname{spol}(g_3, g_1)) = xy\,e_2$ and $\operatorname{lm}(g_1) = xy$.

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$$\operatorname{spol}\left(\overline{\alpha}, \overline{\beta}\right) = a\overline{\alpha} - b\overline{\beta} \implies \operatorname{spair}(\alpha, \beta) = a\alpha - b\beta$$

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Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from P by increasing signature.

Signature-based criteria

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Sketch of proof

1. $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta)$.
2. All S-pairs are handled by increasing signature.
   $\Rightarrow$ All relatons $\prec \mathfrak{s}(\alpha)$ are known:

$$\alpha = \beta + \text{ elements of smaller signature}$$

$\square$

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S-pairs in signature T

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What are all possible configurations to reach signature T?

Define an order $\trianglelefteq$ on $\mathfrak{R}_T$ and choose the maximal element.

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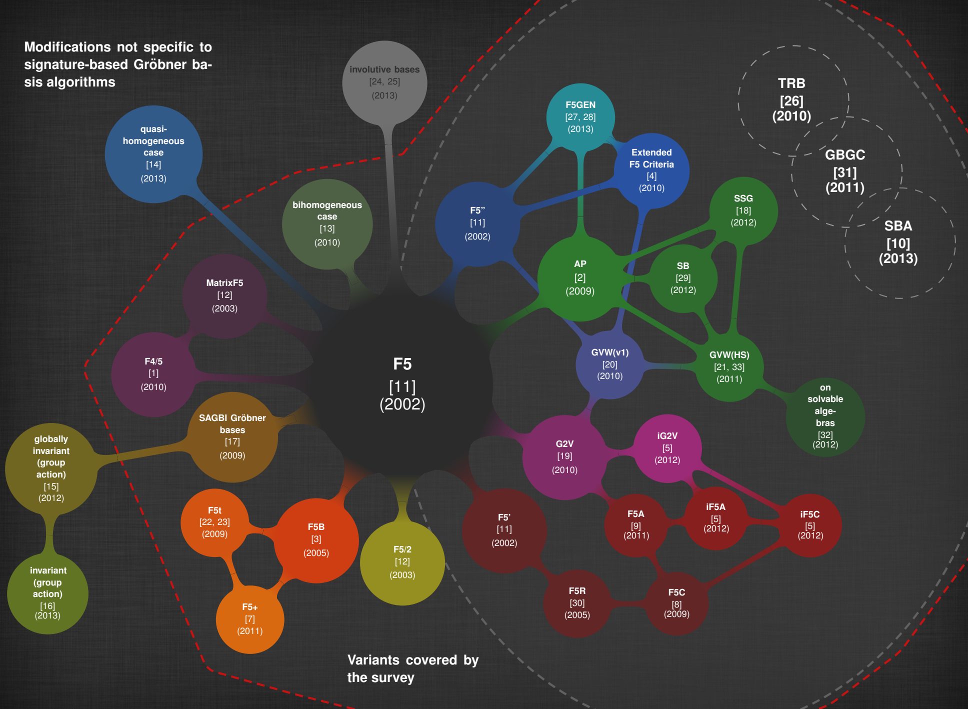
Revisiting our example with $\prec_{\text{pot}}$

$$\mathfrak{s}\left(\text{spol}(g_3, g_1)\right) = xye_2$$

$$\left.\begin{aligned} g_1 &= xy - z^2 \\ g_2 &= y^2 - z^2 \end{aligned}\right\} \Rightarrow \text{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \ldots$$


Modifications not specific to signature-based Gröbner basis algorithms
involutive bases [24, 25] (2013)
quasi-homogeneous case [14] (2013)
bihomogeneous case [13] (2010)
MatrixF5 [12] (2003)
F4/5 [1] (2010)
SAGBI Gröbner bases [17] (2009)
globally invariant (group action) [15] (2012)
invariant (group action) [16] (2013)
F5t [22, 23] (2009)
F5B [3] (2005)
F5+ [7] (2011)
F5/2 [12] (2003)
F5 [11] (2002)
F5GEN [27, 28] (2013)
Extended F5 Criteria [4] (2010)
F5” [11] (2002)
AP [2] (2009)
SSG [18] (2012)
SB [29] (2012)
GVW(v1) [20] (2010)
GVW(HS) [21, 33] (2011)
on solvable algebras [32] (2012)
G2V [19] (2010)
iG2V [5] (2012)
F5’ [11] (2002)
F5A [9] (2011)
iF5A [5] (2012)
iF5C [5] (2012)
F5R [30] (2005)
F5C [8] (2009)
TRB [26] (2010)
GBGC [31] (2011)
SBA [10] (2013)
Variants covered by the survey

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Common choice: By increasing signature

3. Choose a rewrite order $\trianglelefteq$ on $\mathfrak{R}_T$ such that $\alpha \trianglelefteq \beta$:
Common choices:

- $\alpha \in \mathscr{G} \trianglelefteq \beta$ syzygy
- β added to $\mathscr{G}$ after α or $\mathfrak{s}(\alpha)\,\mathrm{lt}\left(\overline{\beta}\right) \prec \mathfrak{s}(\beta)\,\mathrm{lt}(\overline{\alpha})$.

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Generate all possible principal syzygies with α.
(e.g. **GVW**)

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S-pair fulfilling Product criterion not detected by Rewrite criterion

▼

Add one corresponding syzygy.
(e.g. **SB** in **Singular**)

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