

Predicting zero reductions in Gröbner Basis computations

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How to detect zero reductions in advance?

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We can reduce further using z^2g_2 :

$$-y^2z^2 + z^4 + y^2z^2 - z^4 = 0.$$

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Product criterion [1, 2]

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Standard representations of $\text{spol}(g_2, g_1)$ and $\text{spol}(g_3, g_1)$

\implies Standard representation of $\text{spol}(g_3, g_2)$.

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Let $f, g, h \in \mathcal{R}$, $G \subset \mathcal{R}$ finite. If

1. $\text{lt}(h) \mid \text{lcm}(\text{lt}(f), \text{lt}(g))$, and
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Combined implementation of Product and Chain criterion:
Gebauer-Möller Installation [10]

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- ▶ **A signature** of f is given by $s(f) = \text{lt}_{\prec}(\alpha)$ where $f = \bar{\alpha}$.
- ▶ An element $\alpha \in \mathcal{R}^m$ with $\bar{\alpha} = 0$ is called **a syzygy**.

Our example again – with signatures and \prec_{pot}

$$g_1 = xy - z^2, \mathfrak{s}(g_1) = e_1,$$

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Note that $\mathfrak{s}(\text{spol}(g_3, g_1)) = xye_2$ and $\text{lm}(g_1) = xy$.

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Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from P by increasing signature.

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Sketch of proof

1. $s(\alpha - \beta) \prec s(\alpha), s(\beta)$.
2. All S-pairs are handled by increasing signature.
 \implies All relations $\prec s(\alpha)$ are known:

$$\alpha = \beta + \text{elements of smaller signature}$$



Signature-based criteria

S-pairs in signature T

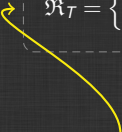
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Define an order \leq on \mathfrak{R}_T and choose the maximal element.

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Revisiting our example with \prec_{pot}

$$\left. \begin{array}{l} g_1 = xy - z^2 \\ g_2 = y^2 - z^2 \end{array} \right\} \Rightarrow \text{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \dots$$

$\mathfrak{s}(\text{spol}(g_3, g_1)) = xye_2$

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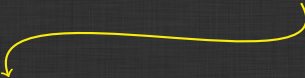
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S-pair fulfilling Product criterion
not detected by Rewrite criterion



Add **one** corresponding syzygy.
(e.g. **SBA** in **Singular**)

Experimental results

Implementation done in **Singular** [4]

Benchmark	STD	SBA \prec_{pot}	SBA \prec_{lt}	
	ZR	ZR	ZR	ZR / PC
cyclic-8	4284	243	771	771 / 0
cyclic-8-h	5843	243	771	771 / 0
eco-11	3476	0	614	614 / 0
eco-11-h	5429	502	629	608 / 0
katsura-11	3933	0	348	304 / 0
katsura-11-h	3933	0	348	304 / 0
noon-9	25508	0	682	646 / 0
noon-9-h	25508	0	682	646 / 0
binomial-6-2	21	6	15	8 / 7
binomial-6-3	20	13	15	9 / 6
binomial-7-3	27	24	21	21 / 0
binomial-7-4	41	16	19	16 / 3
binomial-8-3	53	23	27	27 / 0
binomial-8-4	40	31	26	26 / 0

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Ongoing work:

1. Describe in detail the connection between our conjecture and Moreno-Socías conjecture [12].
2. Try to exploit even more algebraic structures for predicting zero reductions.

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