ARMA Video Notes

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Lesson 6 Part 1 - Intro to AR Models

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$$

x_t is the time series and ϕ are the set of AR parameters and w_t are the unexplained errors (iid). We are looking at an order p AR Process [AR(p)]

Backshift Operator -

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \cdots - \phi_p x_{t-p} = w_t$$

Defining $Bx_t = x_{t-1}$

$$B^p x_t = x_{t-p}$$

We can define the linear process as

$$x_t - \phi_1 B x_t - \phi_2 B^2 x_t - \cdots - \phi_p B^p x_t = w_t$$

We can also factor out x_t

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots$$

AR(1) Process $\rightarrow (1 - \phi_1 B)x_t = w_t$ or $x_t = \phi_1 x_{t-1} + w_t$

Is one point more likely to be closer to its neighbor? We should compute the ACF in order to do so.

Lesson 6 Part 2 - Inverting AR(1) Process

What is the goal?

We have $\Phi(B)x_t = w_t$. We would like to write $x_t = \Phi^{-1}(B)w_t$. This will have some value in the future.

In the simple case AR(1)

Note that $\phi x_{t-1} = \phi(\phi x_{t-2} + w_{t-1})$. We can continue to dig deeper and deeper into this process.

We can then see it as

$$\phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}$$

The first term should approach zero if $|\phi| < 1$ and simply have the second term.

Lesson 6 Part 3 - ACF of AR1 Process

Assuming w_t is iid, then $E[x_t] = 0$

we are going to use the inversion of the AR(1) function in order to calculate the covariance

$$\gamma(h) = E\left[\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j}\right)\left(\sum_{k=0}^{\infty} \phi^k w_{t-k}\right)\right]$$

We just need to take advantage of the fact that the $E[w_i w_j] = 0$ for most instances unless i = j.

As we expand the series and simplify we come to

$$\sigma_w^2 \sum_{j=0}^{\infty} \phi^{h+j} \phi^j = \sigma_w^2 \phi^h \sum_{j=1}^{\infty} \phi^{2j} = \sigma_w^2 \phi^h \frac{1}{1-\phi^2}$$

From this we can get the ACF which is simply $\rho(h) = \phi^h$

Lesson 6 Part 4 - AR(p) Parameters

For what values of ϕ is x_t stationary?

It requires the characteristic polynomial from the backshift operator

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$

For example we have the AR(1) process:

$$\phi(z) = 1 - \phi_1 z \to 1 - \phi_1 z = 0 \to z = \frac{1}{\phi_1}$$

For AR(2) we have a similar process:

$$\phi(z) = 1 - .5z - .5z^2 = 0 \to z = 1 \text{ and } -2$$

The issue is that the roots need to be outside the unit circle and unfortunately the 1 is on the unit circle. Thus it is not stationary

Lesson 6 Part 5 - Intro to MA processes

A moving average function is based on the process of previous noise and variance having an effect on the next value as it moves around a certain average. Previous values do not have an exact effect but previous error does.

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + theta_q w_{t-q}$$

There is a cut off to the correlation between values based on the number of values in the process and does not fade away like the AR process

$$\Theta(B) = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q)$$

with $x_t = \Theta(B)w_t$ the expected value is 0

The autocovariance process is similar to that of the AR process:

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2 & h = 0 \\ \theta\sigma_w^2 & h = 1 \\ 0 & h > 1 \end{cases}$$

We also have

$$\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & h = 1 \\ 0 & h > 1 \end{cases}$$

Lesson 6 Part 6 - Inverting MA Processes

We have

$$\Theta(B) = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q)$$

In general for an MA(q) process $\rho(h) = 0, \quad h > q$

Unlike AR process, MA process are stationary for an value of theta.

But there is an issue of invertibility which makes the process easier to work with.

A model of theta and its reciprocal produces the same ACF because of this we need to be aware of makign the better choice of coefficients and adjust our noise accordingly

We will usually choose the invertible one and also some to the following depiction:

$$w_t = -\theta w_{t-1} + x_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j}$$

This idea of theta being less than 1 absolutely can be generalized to MA(q) processes

Lesson 7 Part 1 - Yule Walker Equations

We need to look at the AR(p) model to start this

We will then get the autocovariance of the AR process which will aid us

We can begin to identify that $\gamma(h) = \phi_1\gamma(h-1) + \phi_2\gamma(h-2)\overset{:}{}$

We can also see that

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \cdots + \phi_p\gamma(p) + E[x_t w_t], \quad E[x_t w_t] = \psi_0 w_t^2 = 1 * w_t^2 = \sigma_w^2$$

Finally we come to

$$\sigma_w^2 = \gamma(0) - \phi_1\gamma(1) - \cdots - \phi_p\gamma(p)$$

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We can put togeter the Yule Walker equations using autocovariance or autocorrelation. Autocorrelation is a bit more clean and thus shall be used :)

$$\hat{R}_p \hat{\phi} = \hat{\rho}_p, \quad \sigma_w^2 = \gamma(\hat{0})(1 - \hat{\rho}_p^T \hat{\phi})$$

Where we have

$$\hat{\phi} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_p \end{bmatrix}, \quad \hat{\rho} = \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(p) \end{bmatrix}$$

$$\hat{R}_p = \begin{bmatrix} 1 & \hat{\rho}(1) & \dots & \hat{\rho}(p-1) \\ \hat{\rho}(1) & 1 & \dots & \hat{\rho}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}(1-p) & \hat{\rho}(2-p) & \dots & 1 \end{bmatrix}$$

Lesson 7 Part 2 - PACF

The partial autocorrelation coefficient of order k, denoted by ρ_k^p and referred to as the PACF, is defined as the correlation coefficient between observations separated by k periods, when we eliminate the linear dependence due to intermediate values

Lesson 7 Part 3 - Nonlinear Least Squares

Non-linear system

$$\begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

This is understood as Ax = b

Lesson 7 Part 4