Tutorials for Viscoelastic Loading Problems

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1. Introduction

Mass movement is a common phenomenon on the Earth's surface. This may include glaciation and deglaciation processes, groundwater movement, volcano building, and other processes of different temporal and spatial scales. This surface mass movement represents a loading to the Earth's interior, and the Earth's crust and mantle would deform to respond to such a loading. The deformation may manifest itself as surface displacement in vertical and horizontal directions in the vicinity of the load which can be detected via various of geodetic techniques including GPS, INSAR, or gravity measurement. Studies of the loading problems may help understand the loading process and also the deformational properties of the Earth's crust and mantle (e.g., viscosity).

This tutorial discusses how to study basic characteristics of loading problems by using a computer program called citcomVE (Zhong et al., 2003; Zhong and Watts, 2013). CitcomVE solves the conservation equations of the mass and momentum for a viscoelastic Earth's mantle, to obtain the stress and strain fields in the mantle and surface displacement in response to a surface loading, by using a finite element method.

2. Governing equations

The response of an incompressible, viscoelastic medium to loads can be described by the conservation laws of mass and momentum [e.g., Zhong et al., 2003]. The governing equations are

$$u_{i,i} = 0, \tag{1}$$

$$\sigma_{ij,j} - (\rho_0 g u_3)_{,i} = 0, \tag{2}$$

where u_i is the displacement, σ_{ij} is the stress tensor, ρ_0 and g are the background mantle density and gravitational acceleration, and u_3 is the vertical component of the displacement. The second term in (2) represents the restoring force at density interfaces such as the top surface of the deformed plate. We use notation such that $A_{,i}$ represents the derivative of variable A with respect to x_i, and repeated indices indicate summation over those indices.

3. Rheology and Constitutive Laws

The rheological equation for an incompressible Maxwellian media is given by [e.g., Zhong et al., 2003]:

$$\sigma_{ij} + \frac{\eta}{\mu}\frac{d\sigma_{ij}}{dt} = -p\delta_{ij} + \eta\frac{d\varepsilon_{ij}}{dt}, \tag{3}$$

where η and μ are the viscosity and shear modulus respectively, p is the dynamic pressure, δ_{ij} is the Kronecker delta function, and ε_{ij} is the strain tensor that includes both elastic and viscous strains and is related to displacement u_i as $\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ [Zhong et al., 2003].

The mantle viscosity and elastic properties describe how mantle rocks deform or flow under mantle temperature T and pressure P conditions. Two common ways to study the rheology are 1) to examine how rocks deform in the laboratory under conditions as realistic as one can achieve and 2) to model Earth's response to certain forces (e.g., post-glacial rebound or seismology). It is generally agreed that mantle rocks are much weaker and easier to deform under higher T. However, mantle viscosity increases with P. In some part of the mantle, deformation is accommodated via diffusion creep, while dislocation creep is the dominant deformation mechanism in other part of the mantle. For dislocation creep deformation, mantle viscosity also depends on stress and is controlled by what is often called power-law rheology.

In this tutorial, we will use the simplest form of viscosity and shear modulus: either constant viscosity and shear modulus throughout the box, η_0 and μ_0, or some simple layered structure.

4. Boundary Conditions and Initial Conditions.

For the loading calculations presented here, the model is set up for a 2-D Cartesian box (Fig. 1). The boundary conditions for the bottom of the box at z=0 and the two sidewalls of the box are free-slip (i.e., zero shear stress and zero normal displacement), while the top boundary is treated as a deformable free surface that is subjected to time-dependent loads

$$\sigma_{ij}n_j = \sigma_L(x,t)n_i, \tag{4}$$

where n_i is the normal vector of the top surface, and $\sigma_L(x, t)$ represents the pressure produced by the loads (e.g., volcanic material) at the surface at location x and time t.

Although the time-dependent load $\sigma_L(x, t)$ can have any different form. Two simple forms are considered here: 1) a sinusoidal function in time and 2) a step-function in time, both as a sinusoidal function in space or x. More specifically, they are given as

$$\sigma_L(x,t) = A_0 sin(\tfrac{2\pi t}{T_0})cos(\tfrac{2\pi x}{\lambda}), \tag{5}$$

$$\sigma_L(x,t) = A_0 H(t) cos(\tfrac{2\pi x}{\lambda}), \tag{6}$$

for the first and second forms, respectively. In (5) and (6), T_0 and λ are the period and wavelength of the loading, A_0 is the amplitude of the loads, and H(t) is the Heaviside function (i.e., step-function in time, or H(t)=1 for $t \geq 0$, and H(t)=0 for t<0). The load amplitude A_0 should have the unit of pressure (i.e., Pa), and for a layer of ice with thickness d, $A_0 = \rho_{ice} g d$, where ρ_{ice} is the density of ice. Note that in reference to Fig. 1 for a box with dimensionless length L/D (or L in dimensional length), wavelength λ = 2L/D or the box length represents half of the wavelength.

It is also possible to construct a surface load with other spatial dependence, for example, a boxcar load.

5. Numerical Methods and Grid

In practice, Equations (1)-(4) are non-dimensionalized using the following characteristic scales: length by D; viscosity by η_0, shear modulus by μ_0, time by $\tau_0=\eta_0/\mu_0$, pressure and stress by μ_0, where D, η_0, and μ_0 are the thickness of the box, reference viscosity, and reference shear modulus. $\tau_0=\eta_0/\mu_0$ is the so-called Maxwell time and is a measure of time-scale over which the stress in viscoelastic media with viscosity η_0 and shear modulus μ_0 relaxes. For example, for $\mu_0=5\times10^{10}$ Pa and $\eta_0=10^{21}$ Pa s, τ_0 is equal to η_0/μ_0 or 2×10^{10} seconds or ~635 years.

CitcomVE code solves the non-dimensional equations for a viscoelastic medium using a finite element method [Zhong et al., 2003; Zhong and Watts, 2013]. The code was modified from convection code Citcom that works for a purely viscous medium [Moresi and Solomatov, 1995]. The key modifications are to adopt a Lagrangian formulation with deformable grid to replace the Eulerian formulation with a fixed grid and to implement a Maxwellian rheology (equation 3) to replace the viscous rheology. Details can be found in Zhong et al. [2003].

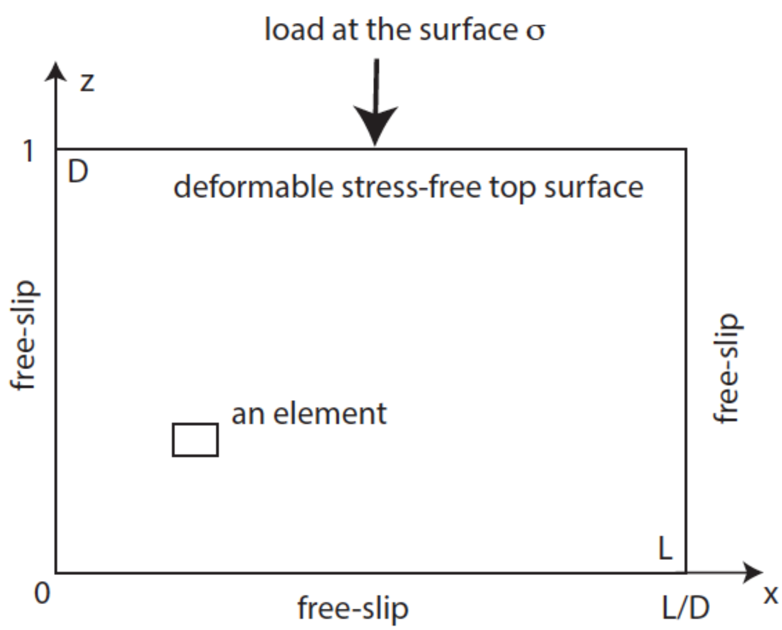

Fig. 1. 2-D Cartesian box model for surface loading problem. The box has length L and height D or non-dimensional length L/D and height 1 (normalized by D).

The box domain is divided into a number of small elements or grids with the numbers of elements elx in x direction and elz in z direction. The numbers of nodes in x and z directions are nox=elx+1 and noz=elz+1, respectively, and the total of number of nodes is nno = nox*noz. The solutions of displacements and stress fields from citcomVE code are given on each grid points or node.

6. Tutorial topics

As an initial try, one may consider a simple step-function loading as given by equation (6). It would be interesting to see how the surface topography or deflection evolves with time in response to this step-function loading, and also how the interior stress evolves with time. Questions to ask include.
1) Does the surface topography reach a steady state?
2) How long does it take for the solution to reach a steady state, if it does?
3) How does the interior stress evolve with time at t=0 and later time?
4) What is the time scale for the stress to evolve?
5) What if you vary the box length?
Make attempt to understand the time-dependent solutions.

Other topics you may explore include using different loading functions (simple functions in space and time) and possibly different viscosity structure (e.g., a layered viscosity with very high viscosity for the top layer). Of course, you can always pose some questions you are interested and explore them.

To explore these questions, you will need to run citcomVE.x code with an input file that captures the specifics of your designed models, and to process the output results from the model runs including make plots.

7. Use the code, input and output files

Use the code

For all the tutorial topics discussed here, we will use a single executable citcomVE.x. In your work directory where the source code is contained, type unix/linux command

```
$make citcomVE.x
```

and citcomVE.x will be generated and be ready to use.

To run CitcomVE calculation, go to subdirectory DATA and issue the following commands

```
$cd DATA
$../citcomVE.x input1
```

where input1 is the input file that is specific to the run. The run may take a few minutes or hours, depending on the number of time steps and number of finite element nodes. The run will produce outputs in a directory specified in the input file (e.g., CASE1 under the work directory).

Input files

We now discuss input file, or the sample file input1. We will only discuss relevant parameters (i.e., those that you may want to vary) from the beginning of the file. The lines highlighted in red are from input1.

Datafile="CASE1"

This specifies where output files are to be stored (e.g., directory CASE1). Notice that you must create directory CASE1 first (e.g., using command: $mkdir CASE1) before running the case (Note that CASE1 already exists when you download the code, but if you want to run another model say CASE2, you will need to create it first).

maxstep=200
storage_spacing=2

They specify the total number of time steps (i.e., 200) and how frequently the outputs (e.g., surface topography) are generated (e.g., every 2 time steps).

dimenx=2.5
dimenz=1.0

They represent dimensionless length and thickness of the box. If you use sinusoidal load function (in space or x), dimenx=2.5 would correspond to wavelength of λ=2.5.

mgunitx=6
mgunitz=3
levels=5

They specify the number of elements in each of x and z directions and multi-grid levels, respectively. The number of element in x or z direction is given by the equation: elx = mgunitx*$2^{(levels-1)}$ or elz = mgunitz*$2^{(levels-1)}$. In this case, elx=96, and elz=48, and the number of nodes in each direction nox=97, noz=49, and the total number of the nodes nno=nox*noz=4753. However, if you would like to small number of nodes, you may reduce levels from 5 to 4 or 3, or reduce mgunitx or mgunitz.

delta_t=2.0e-01

This is the time increment from one time step to the next. The time increment Δt is given as dimensionless time and in terms of Maxwell time τ_0. For μ_0=5x10^{10} Pa and η_0=10^{21} Pa s, τ_0 = 635 years and Δt = 2.0e-1 would correspond to 317.5 years.

For sinusoidal time function loading in equation (5), the following line gives the dimensionless period T_0 in terms of Maxwell time τ_0.

period=40

This period corresponds to $40\tau_0$ or 25,400 years, or about 1/10 of the period of glaciation and deglaciation.

For step-function in time loading as in equation (6), the load is applied at t=0 and is maintained for all the time. In this case, we will set parameter period=0 in the input file. Note that if period is non-zero in input1, then the code will assume that sinusoidal time function loading in equation (5) is used.

In either (5) or (6), the spatial dependence is given by a sinusoidal function of x with wavelength λ. The non-dimensional amplitude of the load is given in terms of equivalent mantle rock with dimensionless thickness, perturbmag, and the non-dimensional half-wavelength is given as ll, as below in input1,

perturbmag=1.0e-4
ll=2.5

Given the default box thickness of 1000 km, this load amplitude is equivalent to 100-meter thick of mantle rock applied to the surface. The sinusoidal function in x direction as in equation (6) still applies. In almost all the cases, the half-wavelength $\lambda/2$ must be equal to box length dimenx. That is, if you change one, you will need to change the other.

The other relevant parameters are self-explanatory and all are in SI units as below

box_height=1.0e6
density=3330.0
shearmodulus=5e10
refvisc=1.0e21
gravacc=10

In most cases, you do not need to change other parameters in the input file.

Output files

We now discuss output files from citcomVE runs. First, the output files are stored in directory with prefix that are specified in input file, as discussed before. For example, “CASE1” in the input file means that the files are stored in directory CASE1. If CASE1 directory does not exist under DATA subdirectory where your input1 file is. You must create directory CASE1 before starting the calculation.

In the directory, you will see typically output files like: velo.XXX, stress.XXX, and topo_s_00XXX.dat, where XXX is time step.

File velo.XXX contains x and z coordinates (in the 2nd and 3rd columns), x and z displacements (in the 4th and 5th columns), for each grid node at time step XXX. The first column is for nodal point. Other columns are not important for now. Also, you may ignore the first line for now (the first number of the first line gives you the total number of nodes for the grid). The coordinates for each node will be useful if you want to plot stress field.

File stress.XXX is for the second invariant of stress tensor, σ_{2nd}, at each node at time step XXX, and $\sigma_{2nd} = [0.5(\tau_{xx}^2 + \tau_{zz}^2 + 2\tau_{xz}^2)]^{1/2}$ is a measure of stress magnitude. The coordinates of each node are given in velo.XXX. Since the grid coordinates do not change significantly with time, and you may use them at any time step, 0 or 200. The first line of this file gives the total number of nodes, time step, and the current time (non-dimensional), and ignore the fourth number. Starting from 2nd line, σ_{2nd} is in the first column, and viscosity is in the third column (ignore the 2nd column for now) for each grid point.

File topo_s_00XXX.dat is for surface incremental displacement and total vertical displacement (i.e., topography) and other information at time step XXX. Note that there is only nox numbers of lines, as it is only for the surface as a function of x. The 1st and 2nd columns are for x coordinate and total vertical displacement (i.e., topography), and 3rd column is for the rate of vertical displacement at this location. You may ignore the 4th column for now.

Of course, it is not easy to see and understand the results by looking at the data files, and graphics are big helps here. Try to port the output data to any plotting package you may have to make plots.

8. References

Moresi, L. N., S. Zhong, and M. Gurnis, The accuracy of finite element solutions of Stokes' flow with strongly varying viscosity, *Phys. Earth Planet. Inter*., 97, 83-94, 1996.

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