VERROU : panorama of localization method

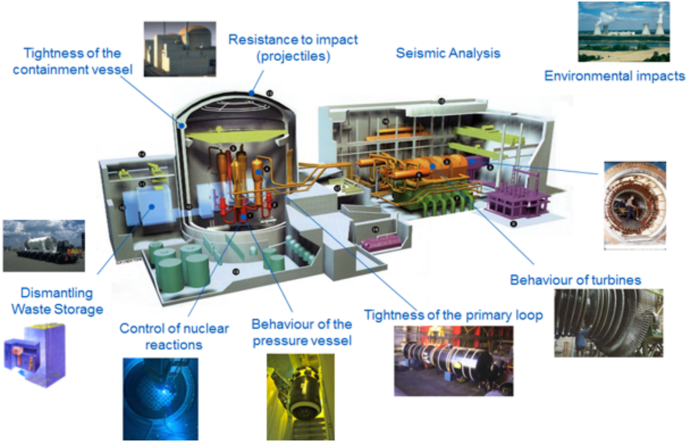
Bruno Lathuilière

Funded by ANR Interflop ANR-20-CE46-0009.

EDF R&D , 30 april 2025

Numerical simulation at EDF

- Guarantee safety
- Improve performances/costs
- Ageing issues



Numerical software at EDF

Large number of in-house codes:

- code_aster (Thermo-mechanic)
- code_saturne (CFD)
- open_telemac (free surface flow)
- salome (Simulation plateform)
- ...

Code properties:

- Huge code base
- Various languages (C/C++, Fortran, Python ...)
- External libraries (MUMPS, numpy ...)
- V&V

Numerical software at EDF

Large number of in-house codes:

- code_aster (Thermo-mechanic)
- code_saturne (CFD)
- open_telemac (free surface flow)
- salome (Simulation plateform)
- ...

Code properties:

- Huge code base
- Various languages (C/C++, Fortran, Python ...)
- External libraries (MUMPS, numpy ...)
- **V&V**

Physical phenomena

↓ Modelisation error

Mathematical model

↓ Discretisation error

Numerical algorithm

↓ Bugs

Implementation

↓ **Floating point error**

Computed solution

VERROU development:

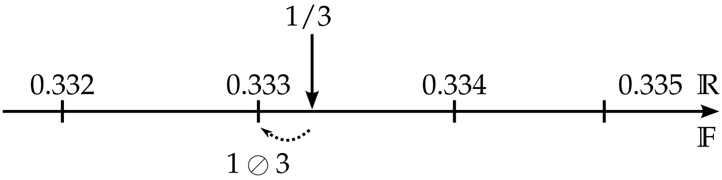
- Binary instrumentation based on valgrind
- Asynchronous stochastic arithmetic
- **Error estimation** and **error localization**

PLAN

- Context
- Verrou and stochastic rounding
- Delta-Debug
- Unstable branches
- Optimization tools
- Conclusions

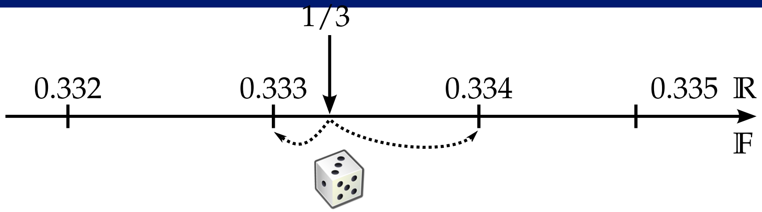
Floating point error

- Floating point representation with limited precision
  - [`float`] binary, 24 significand bits ($\simeq 10^{-7}$)
  - [`double`] binary, 53 significand bits ($\simeq 10^{-16}$)
  - [pedagogic example] decimal, 3 significand digits (% – ‰)

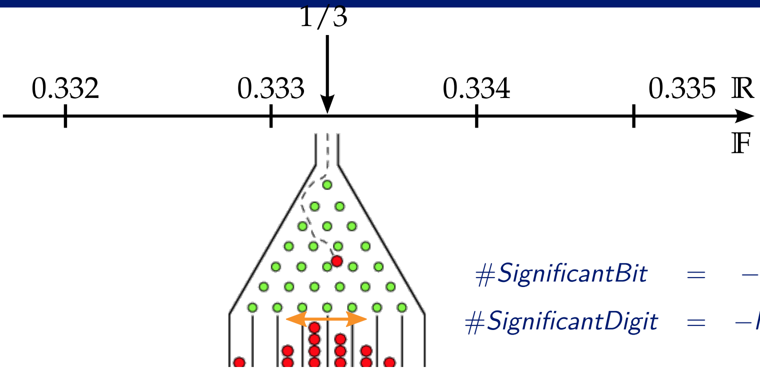


- Floating point computation $\neq$ Real computation
  - rounding error $a \oplus b \neq a + b$
  - associativity loss $(a \oplus b) \oplus c \neq a \oplus (b \oplus c)$

Stochastic arithmetic for numerical verification



Stochastic arithmetic for numerical verification



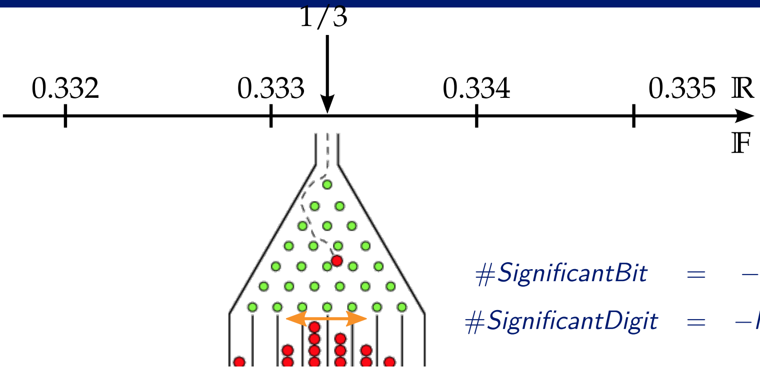
$$\#SignificantBit = -log2\left(\frac{max_i(|X_i - X_{nearest}|)}{|X_{nearest}|}\right)$$

$$\#SignificantDigit = -log10\left(\frac{max_i(|X_i - X_{nearest}|)}{|X_{nearest}|}\right)$$

Instruction	Eval. Nearest	Eval. 1	Eval. 2	Eval. 3
$a = 1/3$	0.333	$0.333_{\downarrow}$	$0.334^{\uparrow}$	$0.334^{\uparrow}$
$b = a \times 3$	0.999	0.999	$1.00_{\downarrow}$	$1.01^{\uparrow}$

$\#SignificantDigit \approx 1.95$

Stochastic arithmetic for numerical verification



$$\#SignificantBit = -log2\left(\frac{max_i(|X_i - X_{nearest}|)}{|X_{nearest}|}\right)$$

$$\#SignificantDigit = -log10\left(\frac{max_i(|X_i - X_{nearest}|)}{|X_{nearest}|}\right)$$

Instruction	Eval. Nearest	Eval. 1	Eval. 2	Eval. 3
$a = 1/3$	0.333	$0.333_{\downarrow}$	$0.334^{\uparrow}$	$0.334^{\uparrow}$
$b = a \times 3$	0.999	0.999	$1.00_{\downarrow}$	$1.01^{\uparrow}$

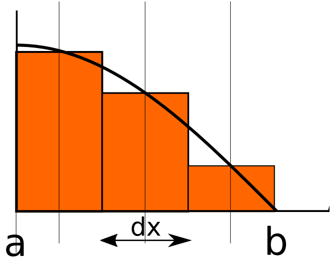
$\#SignificantDigit \approx 1.95$

- Compatible with binary instrumentation (**Verrou** based on valgrind) or LLVM pass (Verificarlo)
- Few false positive detection (due to asynchronous approach and dedicated stochastic rounding mode)

Example : rectangle method

$$\int_a^b f(x)dx \approx \sum_i^n dx.f(x_i)$$

avec $\left| \begin{array}{l} dx = \frac{(b-a)}{n} \\ x_i = a + (i + 0.5)dx \end{array} \right.$



Listing – integrate.hxx

```
template <typename T, class REALTYPE>
REALTYPE integrate (const T& f, REALTYPE a, REALTYPE b, unsigned int n) {
  // Integration step
  const REALTYPE dx = (b - a) / n;
  // Naive integration Loop
  REALTYPE sum = 0.;
  for (REALTYPE x = a + (REALTYPE)0.5 * dx ;
       x < b ;
       x += dx) {
    sum += dx * f(x);
  }
  return sum;
}
```

Example: convergence analysis

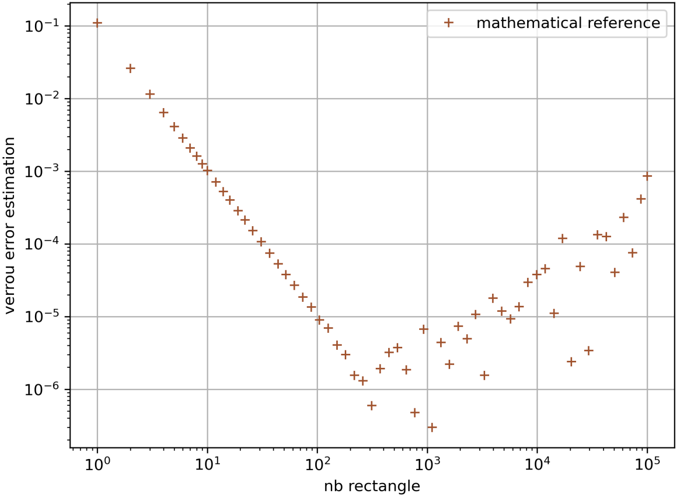
Listing – unitTest.cxx (Verification: $\int_0^{\frac{\Pi}{2}} cos(x)dx = 1$)

```
void testConvergence (const float & step) {
  std::cout << std::scientific << std::setprecision(17);
  const size_t maxN= 100000;
  for (unsigned int n = 1 ; n <= maxN ;
       n = std::max (std::min(maxN, (size_t)(step*n)), n+1)) {

    float res = integrate (std::cos,0., M_PI_2, n);

    float err = std::abs(1 - res);

    // 3 columns output: Nrectangles Result Error
    std::cout << std::setw(10) << n << " " << res << " " << err << std::endl;
  }
}
```

1	1.11072075366973877e+00	1.10720753669738770e-01
10	1.00102877616882324e+00	1.02877616882324219e-03
100	1.00001001358032227e+00	1.00135803222656250e-05
1000	9.99992489814758301e-01	7.51018524169921875e-06
10000	1.00000333786010742e+00	3.33786010742187500e-06
100000	9.99139010906219482e-01	8.60989093780517578e-04

Example: convergence analysis

Plot 3rd column (available only for analytical test case)



Example: manual analysis

$ ./unitTest > unittest.output.txt

```
unittest.output.txt
     1  1.11072075366973877e+00  1.10720753669738770e-01
    10  1.00102877616882324e+00  1.02877616882324219e-03
   100  1.00001001358032227e+00  1.00135803222656250e-05
  1000  9.99992489814758301e-01  7.51018524169921875e-06
 10000  1.00000333786010742e+00  3.33786010742187500e-06
100000  9.99139010906219482e-01  8.60989093780517578e-04
```

Example: manual analysis

$ ./unitTest > unittest.output.txt
$ **valgrind --tool=verrou --rounding-mode=random --libm=instrumented** ./unitTest > unittest.output-random_1.txt

unittest.output.txt

1	1.11072075366973877e+00	1.10720753669738770e-01
10	1.00102877616882324e+00	1.02877616882324219e-03
100	1.00001001358032227e+00	1.00135803222656250e-05
1000	9.99992489814758301e-01	7.51018524169921875e-06
10000	1.00000333786010742e+00	3.33786010742187500e-06
100000	9.99139010906219482e-01	8.60989093780517578e-04

unittest.output_random_1.txt

1	1.11072075366973877e+00	1.10720753669738770e-01
10	1.00102853775024414e+00	1.02853775024414062e-03
100	1.00001108646392822e+00	1.10864639282226562e-05
1000	1.00000536441802979e+00	5.36441802978515625e-06
10000	1.00001549720764160e+00	1.54972076416015625e-05
100000	1.00013923645019531e+00	1.39236450195312500e-04

Example: manual analysis

```
$ ./unitTest > unittest.output.txt
$ valgrind --tool=verrou --rounding-mode=random --libm=instrumented ./unitTest > unittest.output-random_1.txt
$ meld unittest.output.txt unittest.output-random_1.txt
```

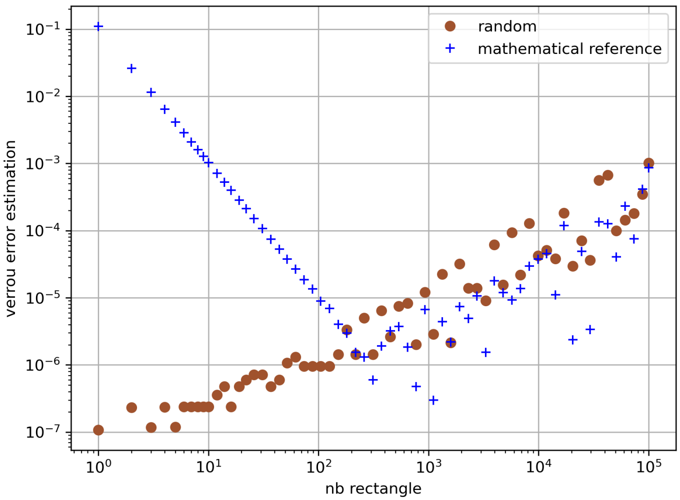
unittest.output.txt

1	1.11072075366973877e+00	1.10720753669738770e-01
10	1.00102877616882324e+00	1.02877616882324219e-03
100	1.00001001358032227e+00	1.00135803222656250e-05
1000	9.99992489814758301e-01	7.51018524169921875e-06
10000	1.00000333786010742e+00	3.33786010742187500e-06
100000	9.99139010906219482e-01	8.60989093780517578e-04

unittest.output_random_1.txt

1	1.11072075366973877e+00	1.10720753669738770e-01
10	1.00102853775024414e+00	1.02853775024414062e-03
100	1.00001108646392822e+00	1.10864639282226562e-05
1000	1.00000536441802979e+00	5.36441802978515625e-06
10000	1.00001549720764160e+00	1.54972076416015625e-05
100000	1.00013923645019531e+00	1.39236450195312500e-04

Error plot with 20 samples



Delta-debug: *trial and error* search algorithm

Delta-debug search:

```
verrou_dd_line --nruns=5 ddRun.sh ddCmp.py
```

runScript: ddRun.sh

```
#!/bin/bash
OUTDIR=$1
valgrind --tool=verrou --rounding-mode=random --libm=instrumented\
    ./unitTest >${OUTDIR}/res.dat
```

cmpScript: ddCmp.py

```
def extract(rep):
    lines=(open(os.path.join(rep, "res.dat"))).readlines()
    return re.split(" ",lines[-1].strip())[1]

if __name__=="__main__":
    if len(sys.argv)==2: #extract for verrou_plot_stat
        print(extract(sys.argv[1]))
    if len(sys.argv)==3: #cmp for verrou_dd_*
        refValue=float(extract(sys.argv[1]))
        value=float(extract(sys.argv[2]))
        relDist= abs((value -refValue)/refValue)
        if relDist < 1e-5: sys.exit(0)
        else:sys.exit(1)
```

ddmin	filename:line	demangled symbol name
ddmin0	`integrate.hxx:17`	testConvergence(float const &)
ddmin1	`integrate.hxx:16`	testConvergence(float const &)

Valgrind developer point of view:

- need to generate a search space: list of symbols (or line) containing FP operations;
- need to run a specific configuration (set instrumented /not instrumented).

Code coverage for unstable branches detection

nearest coverage		random coverage	
-	:8:template <typename T, class REALTYPE>	-	:8:template <typename T, ✂
6	:9:REALTYPE integrate (const T& f, REALTYPE a, REALTYPE b, unsigned int n) {	6	:9:REALTYPE integrate (co✂
-	:10: // Integration step	-	:10: // Integration step✂
6	:11: const REALTYPE dx = (b - a) / n;	6	:11: const REALTYPE dx =✂
-	:12: // Naive integration Loop	-	:12: // Naive integratio✂
6	:13: REALTYPE sum = 0.;	6	:13: REALTYPE sum = 0.;
6	:14: for (REALTYPE x = a + (REALTYPE)0.5 * dx ;	6	:14: for (REALTYPE x = a✂
111**000**	:15: x < b ;	111**179**	:15: x < b ;
11**0994**	:16: x += dx) {	11**1173**	:16: x += dx) {
11**0994**	:17: sum += dx * f(x);	11**1173**	:17: sum += dx * f(x);✂
-	:18: }	-	:18: }
-	:19: return sum;	-	:19: return sum;
-	:20:}	-	:20:}

- Need to recompile with coverage option (`-fcoverage`).
- Need to rerun delta-debug.
- Need time to interpret results due to false-positive.

BasicBloc coverage

ddmin1 (integrate.hxx:16) | random		nearest	
1	:(?)iostream(74)unitTest.cxx(65)	1	:(?)iostream(74)unitTest.cxx(65)
1	:(?)unitTest.cxx(65)iostream(74)	1	:(?)unitTest.cxx(65)iostream(74)
11**1168**	:(?)cmath(185)	11**0988**	:(?)cmath(185)
6	:(?)integrate.hxx(15)cmath(185)	6	:(?)integrate.hxx(15)cmath(185)
11**1174**	:(?)integrate.hxx(15-17) F?	11**0994**	:(?)integrate.hxx(15-17) F?
4	:(?)iomanip(240)integrate.hxx(11,13-15) F?	4	:(?)iomanip(240)integrate.hxx(11,13-15) F?
1	:(?)iomanip(240)stl_algobase.h(237) ↩ integrate.hxx(11,13-15) F?	1	:(?)iomanip(240)stl_algobase.h(237) ↩ integrate.hxx(11,13-15) F?
	ostream(196) F		ostream(196) F
	ios_base.h(84,88,731)integrate.hxx(11,13-15) F?		ios_base.h(84,88,731)integrate.hxx(11,13-15) F?

✂23 lines skipped ✂

ddmin0 (integrate.hxx:17) | random		nearest	
1	:(?)iostream(74)unitTest.cxx(65)	1	:(?)iostream(74)unitTest.cxx(65)
1	:(?)unitTest.cxx(65)iostream(74)	1	:(?)unitTest.cxx(65)iostream(74)
110988	:(?)cmath(185)	110988	:(?)cmath(185)
6	:(?)integrate.hxx(15)cmath(185)	6	:(?)integrate.hxx(15)cmath(185)
110994	:(?)integrate.hxx(15-17) F?	110994	:(?)integrate.hxx(15-17) F?
4	:(?)iomanip(240)integrate.hxx(11,13-15) F?	4	:(?)iomanip(240)integrate.hxx(11,13-15) F?
1	:(?)iomanip(240)stl_algobase.h(237) ↩ integrate.hxx(11,13-15) F?	1	:(?)iomanip(240)stl_algobase.h(237) ↩ integrate.hxx(11,13-15) F?
	ostream(196) F		ostream(196) F
	ios_base.h(84,88,731)integrate.hxx(11,13-15) F?		ios_base.h(84,88,731)integrate.hxx(11,13-15) F?

✂23 lines skipped ✂

▶ Can be generated automatically with `post_verrou_dd`

Example : come back to code

Listing – integrate.hxx

```
REALTYPE sum = 0.;
for (REALTYPE x = a + (REALTYPE)0.5 * dx ;
      x < b ;
      x += dx) {
  sum += dx * f(x);
}
return sum;
```

- Convert the floating point loop ($x+=dx$) into an integer loop ($i++$).
- Accumulation in double (*mixed precision*) or compensated algorithm.

Listing – integrate.hxx

```
double sum = 0.;
for (unsigned int i=0 ; i< n ; i++){
  REALTYPE x = a + ((REALTYPE)0.5 +i) * dx ;
  sum += dx * f(x);
}
return (REALTYPE)sum;
```

Partial BasicBloc coverage

ddmin1 (integrate.hxx:16) | random		nearest	
1	:?iostream(74)unitTest.cxx(65)	1	:?iostream(74)unitTest.cxx(65)
1	:?unitTest.cxx(65)iostream(74)	1	:?unitTest.cxx(65)iostream(74)
111168	:?cmath(185)	**110988**	:?cmath(185)
6	:?integrate.hxx(15)cmath(185)	6	:?integrate.hxx(15)cmath(185)
111174	:?integrate.hxx(15-17) F?	**110994**	:?integrate.hxx(15-17) F?
4	:?iomanip(240)integrate.hxx(11,13-15) F?	4	:?iomanip(240)integrate.hxx(11,13-15) F?

✂24 lines skipped ✂

partial (last iter) ddmin1 (integrate.hxx.16) | random		partial (last iter) | nearest	
100062	:?cmath(185)	**99883**	:?cmath(185)
1	:?integrate.hxx(15)cmath(185)	1	:?integrate.hxx(15)cmath(185)
100063	:?integrate.hxx(15-17) F?	**99884**	:?integrate.hxx(15-17) F?
1	:?iomanip(240)stl_algobase.h(237) ↩ integrate.hxx(11,13-15) F?	1	:?iomanip(240)stl_algobase.h(237) ↩ integrate.hxx(11,13-15) F?
1	:?locale_facets.h(874)	1	:?locale_facets.h(874)

✂10 lines skipped ✂

How to specify coverage point

- client request : `VERROU_DUMP_COVER`
- IOMatch script

```
cmatch:     100000 * *
apply: dump_cover
cmatch:      10000 * *
apply: dump_cover
```

Temporal Delta-debug

- verrou_dd_task (with manual definition in source or automatic for python line)
- verrou_dd_stdout (task automatically detected thanks to IOMatch)

```
verrou_dd_stdout --nruns=5 ddRun.sh ddCmp.py
```

```
__verrou__stdout__init__
         1 * *
        10 * *
       100 * *
      1000 * *
     10000 * *
    100000 * *
```

(a) Search space.

ddmin	match line
ddmin0	10000 * *

(b) Result.

- Warning : pay attention to bufferisation.
- It is possible to modify stdout (and so task) thanks a user script.

Help to optimize

Compatible with all localization tools:

- Subnormal numbers : daz/ftz rounding-mode
- Fma : –unfma (compatible with all rounding-modes)
- Mixed precision : –float (compatible with all rounding-modes)

Mixed precision

First of all, check the accuracy with double precision

Two strategies:

Delta-debug (and other localization tools) with `-float` option with the original code in double.

- The result can be checked with `-float -rounding-mode=random` options
- The Delta debug pinpoints operations and not data storage

Delta-debug (and other localization tools) with `-rounding-mode=random` with the code in float.

- Trick: select performance hotspots to rewrite the code.
- Possible to deals with floating point specialized function

Warning:

- Dynamic approach: understanding required to generalize to all datasets.
- Does the search spaces fit to your algorithm?
- More float does not mean necessarily faster:
  - Conversion cost, more subnormal numbers
  - Roof line malediction
- Need to adapt/control convergence criteria
- Read the bibliography: verrou is not able to find a preconditioner or an iterative refinement

Mixed precision search for inner/outer iteration

```
CMD="etest2 100 100 1 res.dat rh.dat
        -e ii -i bicgstab -p ilu -ilu_fill 2
        -print out -eprint out"
valgrind --tool=verrou
         --float=yes
         --exclude=exclude.ex
         $CMD > $1/res.out
```

```
> cat exclude.ex
lis_vector_dot *
```

Stdout extract:

```
...
iteration:    47  relative residual = 2.102138E-10
iteration:    48  relative residual = 2.340582E-11
iteration:    49  relative residual = 7.864099E-12
iteration:    50  relative residual = 4.231160E-12
iteration:    51  relative residual = 2.719284E-12
iteration:    52  relative residual = 8.079880E-13
linear solver status  : normal end

iteration:     2  relative residual = 7.941629E-02
initial vector x      : all components set to 0
precision             : double
linear solver         : BiCGSTAB
preconditioner        : ILU(2)
convergence condition : ||b-Ax||_2 <= 1.0e-12 * ||b-Ax_0||_2
matrix storage format : CSR
iteration:     1  relative residual = 9.351250E-02
iteration:     2  relative residual = 6.529751E-02
iteration:     3  relative residual = 2.403754E-02
iteration:     4  relative residual = 1.016325E-02
...
```

End of stdout

```
Inverse: mode number          = 0
Inverse: eigenvalue           = 1.934862e-03
Inverse: number of iterations = 1000
Inverse: elapsed time         = 3.840000e+02 sec.
Inverse:   preconditioner     = 1.280000e+02 sec.
Inverse:     matrix creation  = 0.000000e+00 sec.
Inverse:   linear solver      = 2.560000e+02 sec.
Inverse: relative residual    = 3.699645e-06
```

search space	ddmin subset	
	all	all \{dot}
outer-0 matrix storage format : CSR	ddmin15	OK
preconditioner : ILU	OK	OK
initial vector x : all components set to 0	OK	OK
outer-0 iteration: 10 relative residual = *	ddmin15	OK
outer-0 iteration: 20 relative residual = *	OK	OK
outer-0 iteration: 30 relative residual = *	OK	OK
outer-0 iteration: 40 relative residual = *	OK	OK
outer-0 linear solver status : normal end	OK	OK
outer iteration: 0 relative residual = *	OK	OK
outer-5 matrix storage format : CSR	ddmin0	ddmin0
outer-5 iteration: 10 relative residual = *	ddmin1	ddmin1
outer-5 iteration: 20 relative residual = *	ddmin16	ddmin2
outer-5 iteration: 30 relative residual = *	ddmin14	OK
outer-5 iteration: 40 relative residual = *	ddmin16	OK
outer-5 linear solver status : normal end	OK	OK
outer iteration: 5 relative residual = *	OK	OK
outer-10 matrix storage format : CSR	ddmin2	ddmin3
outer-10 iteration: 10 relative residual = *	ddmin3	ddmin4
outer-10 iteration: 20 relative residual = *	ddmin4	ddmin5
outer-10 iteration: 30 relative residual = *	ddmin5	ddmin6
outer-10 iteration: 40 relative residual = *	OK	OK
outer-10 linear solver status : normal end	OK	OK
outer iteration: 10 relative residual = *	ddmin6	ddmin7
outer-15 matrix storage format : CSR	ddmin7	ddmin8
outer-15 iteration: 10 relative residual = *	ddmin8	ddmin9
outer-15 iteration: 20 relative residual = *	ddmin9	ddmin10
outer-15 iteration: 30 relative residual = *	ddmin10	ddmin11
outer-15 linear solver status : normal end	ddmin11	ddmin12

Interflop verification tools

Verrou is not the silver bullet tool. As a user I'm interested in other Interflop verification tools :

- **Verificarlo** `https://github.com/verificarlo/verificarlo`
  - Computation time (especially with shared memory)
  - Better Debugger compatibility
- **Pene** `https://github.com/aneoconsulting/PENE`
  - Windows Portability
  - avx512
- **CADNA** `https://cadna.lip6.fr/`
  - Algorithm which take into account accuracy estimation
  - Useful to perform synchronous/asynchronous comparison.
- **FLDLib** `https://github.com/fvedrine/fldlib`
  - High level of confidence for tricky kernel.

Conclusions and perspectives

- VERROU provides an easy floating point error estimation ;
- VERROU provides error localization ;
- VERROU is used with industrial code ;
- VERROU is open-source and available: `https://github.com/edf-hpc/verrou`

Perspectives

- Performance improvements (instrumentation, vectorization, delta-debug parallelism . . .)
- Localization improvements (delta-debug search space, amplification detection . . .)

Merci
Questions?

Performance

type	double		float	
compilation option	O0	O3	O0	O3
tool_none	x6.3	x6.4	x7.0	x8.4
nearest	x10.0	x22.3	x10.6	x27.9
nearest-nc	x10.1	x21.8	x10.4	x27.2
random	x15.4	x41.2	x17.7	x54.1
average	x17.4	x47.5	x21.0	x66.0
random_det	x16.6	x46.0	x19.8	x64.7
random_comdet	x16.8	x47.2	x20.2	x66.8
random_scomdet	x19.4	x56.2	x23.4	x79.8
average_det	x19.2	x53.1	x23.5	x77.5
average_comdet	x20.2	x56.5	x24.9	x85.3
average_scomdet	x20.0	x56.7	x25.3	x85.4
sr_monotonic	x20.4	x57.4	x25.0	x81.5
sr_smonotonic	x20.5	x57.0	x25.3	x83.1

Number of samples

Confidence level $1-\alpha$	Probability p								
	0.66	0.75	0.8	0.85	0.9	0.95	0.99	0.995	0.999
0.66	3	4	5	7	11	22	108	216	1079
0.75	4	5	7	9	14	28	138	277	1386
0.8	4	6	8	10	16	32	161	322	1609
0.85	5	7	9	12	19	37	189	379	1897
0.9	6	9	11	15	22	45	230	460	2302
0.95	8	11	14	19	29	59	299	598	2995
0.99	12	17	21	29	44	90	459	919	4603
0.995	13	19	24	33	51	104	528	1058	5296
0.999	17	25	31	43	66	135	688	1379	6905

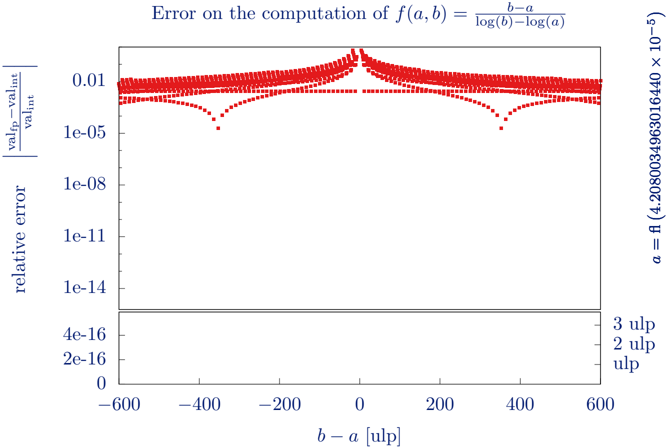
Confidence Intervals for Stochastic Arithmetic, Devan Sohier, Pablo De Oliveira Castro, François Févotte, Bruno Lathuilière, Eric Petit, Olivier Jamond

Bug fix example (1/3)

$$f(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\log(b)-\log(a)} & \text{if not} \end{cases}$$

Empirical study

- outside the code
- around the problematic
- reference = interval arithmetic

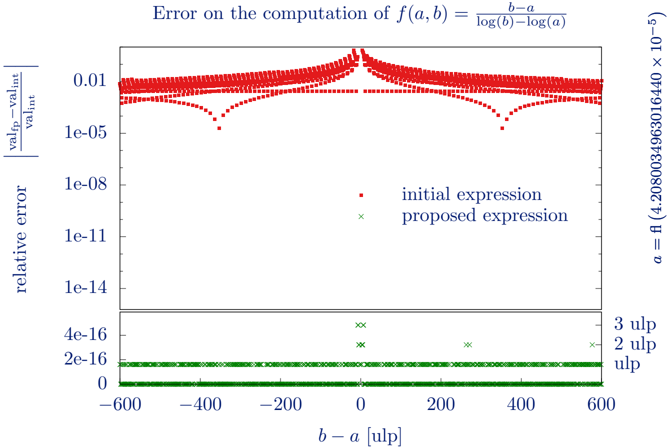


Bug fix example (1/3)

$$f(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\log(b)-\log(a)} & \text{if not} \end{cases} \xrightarrow[\text{manual}]{\text{rewritting}} f(a,b) = \begin{cases} a & \text{if } a = b \\ a\,\frac{\frac{b}{a}-1}{\log(\frac{b}{a})} & \text{if not} \end{cases}$$

Empirical study

- outside the code
- around the problematic
- reference = interval arithmetic



Bug fix example (1/3)

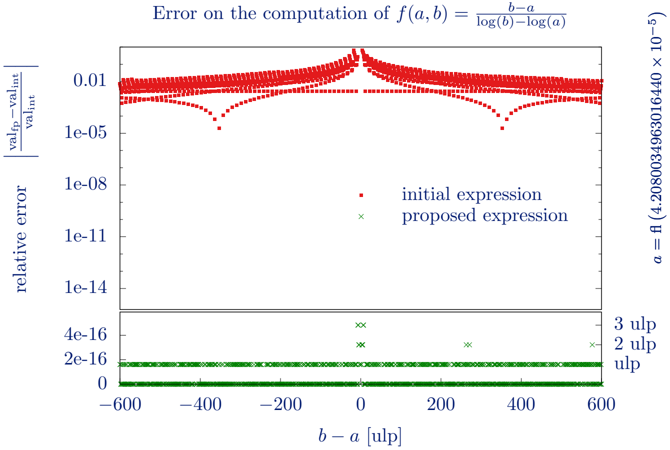
$$f(a,b) = \begin{vmatrix} a & \text{if } a = b \\ \frac{b-a}{\log(b)-\log(a)} & \text{if not} \end{vmatrix} \xrightarrow[\text{manual}]{\text{rewritting}} f(a,b) = \begin{vmatrix} a & \text{if } a = b \\ a\,\frac{\frac{b}{a}-1}{\log(\frac{b}{a})} & \text{if not} \end{vmatrix}$$

Empirical study

- outside the code
- around the problematic
- reference = interval arithmetic

Proof

- error bounded by 10 ulps

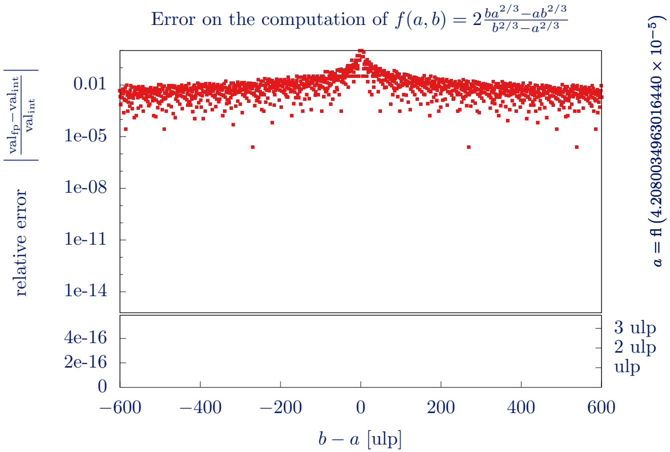


Bug fix example (2/3)

$$f(a,b) = \begin{cases} a & \text{if } a = b \\ 2\frac{ba^{2/3}-ab^{2/3}}{b^{2/3}-a^{2/3}} & \text{if not} \end{cases}$$

Empirical study

- outside the code
- around the problematic
- reference = interval arithmetic



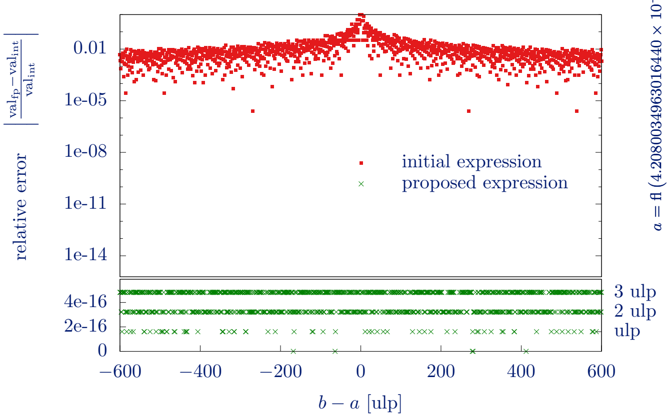
Bug fix example (2/3)

$$f(a,b) = \left| \begin{array}{ll} a & \text{if } a = b \\ 2\frac{ba^{2/3}-ab^{2/3}}{b^{2/3}-a^{2/3}} & \text{if not} \end{array} \right. \quad \xrightarrow[\text{alpha}]{\text{wolfram}} \quad f(a,b) = 2\,\frac{a^{2/3}b^{2/3}}{a^{1/3}+b^{1/3}}$$

Empirical study

- outside the code
- around the problematic
- reference = interval arithmetic

Error on the computation of $f(a,b) = 2\frac{ba^{2/3}-ab^{2/3}}{b^{2/3}-a^{2/3}}$

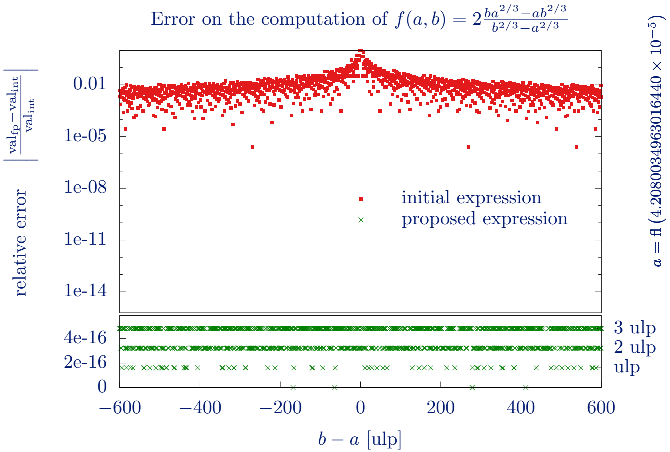


Bug fix example (2/3)

$$f(a,b) = \begin{cases} a & \text{if } a = b \\ 2\frac{ba^{2/3} - ab^{2/3}}{b^{2/3} - a^{2/3}} & \text{if not} \end{cases} \quad \xrightarrow[\text{alpha}]{\text{wolfram}} \quad f(a,b) = 2\,\frac{a^{2/3}b^{2/3}}{a^{1/3} + b^{1/3}}$$

Empirical study

- outside the code
- around the problematic
- reference = interval arithmetic



Bug fix example (3/3)

$$f_n(a,b) = \begin{cases} a & \text{if } a = b \\ (n-1)\dfrac{b^{\frac{1}{n}} - a^{\frac{1}{n}}}{a^{\frac{1}{n}-1} - b^{\frac{1}{n}-1}} & \text{if not} \end{cases} \xrightarrow[\text{rewriting}]{\text{manual}} f_n(a,b) = \frac{n-1}{\sum_{i=1}^{n-1} a^{\frac{i-n}{n}} b^{\frac{-i}{n}}}$$