

Quanto Options pricing

Change of Numeraire and Measure Theory

Complete Solutions with Detailed Derivations

Problem 1

Quanto Call Option Pricing Under Domestic Asset Measure

Question:

A quanto call option pays $V(T) = S(T) \cdot \max(\tilde{S}(T) - K, 0)$, where $S(t)$ is the price of a domestic asset and $\tilde{S}(t)$ is the price of a foreign asset.

Under the domestic risk-neutral measure \mathbb{Q} , the foreign asset follows:

$$d\tilde{S}(t) = r_f \tilde{S}(t) dt + \sigma_{\tilde{S}} \tilde{S}(t) dW^{\mathbb{Q}}(t)$$

The price under \mathbb{Q} is:

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r_d(T-t)} \cdot S(T) \cdot \max(\tilde{S}(T) - K, 0) \mid \mathcal{F}_t \right]$$

When changing to the measure \mathbb{N} induced by using $S(t)$ as numeraire, what is $V(t)$?

Complete Solution

Step 1: Identify the Original Numeraire

Under the domestic risk-neutral measure \mathbb{Q} , the numeraire is the **domestic money market account** $B_d(t) = e^{r_d t}$, as shown by the discount factor $e^{-r_d(T-t)} = \frac{B_d(t)}{B_d(T)}$.

Step 2: Choose the Appropriate Numeraire

For a quanto option with payoff $S(T) \cdot \max(\tilde{S}(T) - K, 0)$, the most suitable numeraire is the **domestic asset** $S(t)$.

Using $S(t)$ as numeraire creates measure \mathbb{N} where $S(T)$ factors out naturally.

Step 3: Derive the Radon-Nikodym Derivative

The Radon-Nikodym derivative is:

$$\frac{d\mathbb{N}}{d\mathbb{Q}} \Big|_{\mathcal{F}_T} = \frac{S(T)/S(0)}{B_d(T)/B_d(0)} = \frac{S(T)}{S(0)} \cdot e^{-r_d T}$$

Step 4: Apply the Measure Change Formula

Using the inverse Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{N}} \Big|_{\mathcal{F}_T} = \frac{S(0)}{S(T)} \cdot e^{r_d T}$$

Step 5: Substitute and Simplify

Substituting:

$$V(t) = \mathbb{E}^{\mathbb{N}} \left[e^{-r_d(T-t)} \cdot S(T) \cdot \max(\tilde{S}(T) - K, 0) \cdot \frac{S(0)}{S(T)} \cdot e^{r_d T} \mid \mathcal{F}_t \right]$$

The $S(T)$ terms cancel and exponentials simplify to $e^{r_d t}$:

$$V(t) = S(0) \cdot e^{r_d t} \cdot \mathbb{E}^{\mathbb{N}} \left[\max(\tilde{S}(T) - K, 0) \mid \mathcal{F}_t \right] = S(t) \cdot \mathbb{E}^{\mathbb{N}} \left[\max(\tilde{S}(T) - K, 0) \mid \mathcal{F}_t \right]$$

Final Answer:

$$V(t) = S(t) \cdot \mathbb{E}^{\mathbb{N}} \left[\max(\tilde{S}(T) - K, 0) \mid \mathcal{F}_t \right]$$

Problem 2

Quanto Adjustment: Foreign Asset Drift Under Domestic Measure

Question:

A quanto option involves a foreign asset $S_f(t)$ and exchange rate X_t . Under \mathbb{Q} :

$$dB_t = r_d B_t dt, \quad \frac{dX_t}{X_t} = (r_d - r_f)dt + \sigma_X dW_X^{\mathbb{Q}}, \quad \frac{dS_f}{S_f} = \mu_f dt + \sigma_f dW_f^{\mathbb{Q}}$$

where $dW_X^{\mathbb{Q}} \cdot dW_f^{\mathbb{Q}} = \rho dt$. The foreign asset in domestic currency is $Y_t = X_t S_f(t)$. What is μ_f ?

Complete Solution

Step 1: Apply Itô's Lemma to $Y_t = X_t S_f(t)$

Using Itô's product rule with $dX \cdot dS_f = \rho\sigma_X\sigma_f X S_f dt$:

$$dY = Y[(r_d - r_f + \mu_f + \rho\sigma_X\sigma_f)dt + \sigma_X dW_X^{\mathbb{Q}} + \sigma_f dW_f^{\mathbb{Q}}]$$

Step 2: Apply Itô's Lemma to $\frac{Y_t}{B_t}$

Since Y_t is tradeable in domestic currency, $\frac{Y_t}{B_t}$ must be a martingale under \mathbb{Q} :

$$d\left(\frac{Y}{B}\right) = \frac{Y}{B} \left[(-r_f + \mu_f + \rho\sigma_X\sigma_f)dt + \sigma_X dW_X^{\mathbb{Q}} + \sigma_f dW_f^{\mathbb{Q}} \right]$$

Step 3: Apply Martingale Condition

For martingale property, drift must vanish:

$$-r_f + \mu_f + \rho\sigma_X\sigma_f = 0 \implies \mu_f = r_f - \rho\sigma_X\sigma_f$$

Final Answer:

$$\mu_f = r_f - \rho\sigma_X\sigma_f$$

Problem 3

Foreign Asset Dynamics Under Domestic Asset Measure

Question:

Under \mathbb{Q} :

$$dB_t = r_d B_t dt, \quad dS_d = r_d S_d dt + \sigma_d S_d dW_d^{\mathbb{Q}}$$

$$\frac{dX_t}{X_t} = (r_d - r_f)dt + \sigma_X dW_X^{\mathbb{Q}}, \quad dS_f = (r_f - \rho_X \sigma_X \sigma_f) S_f dt + \sigma_f S_f dW_f^{\mathbb{Q}}$$

where $dW_d^{\mathbb{Q}} \cdot dW_f^{\mathbb{Q}} = \rho dt$ and $dW_X^{\mathbb{Q}} \cdot dW_f^{\mathbb{Q}} = \rho_X dt$. What are the dynamics of $S_f(t)$ under \mathbb{Q}^{S_d} ?

Complete Solution

Step 1: Find Girsanov Transformation for W_d

Applying Itô's lemma to $\frac{B_t}{S_d(t)}$:

$$d\left(\frac{B}{S_d}\right) = \frac{B}{S_d} \left[\sigma_d^2 dt - \sigma_d dW_d^{\mathbb{Q}} \right]$$

For martingale under \mathbb{Q}^{S_d} :

$$dW_d^{S_d} = dW_d^{\mathbb{Q}} - \sigma_d dt$$

Step 2: Apply Itô's Lemma to $\frac{Y_t}{S_d}$

Let $Y_t = X_t S_f(t)$. With cross-variation $dY \cdot dS_d = Y S_d \sigma_d \sigma_f \rho dt$:

$$d\left(\frac{Y}{S_d}\right) = \frac{Y}{S_d} [(\sigma_d^2 - \sigma_d \sigma_f \rho) dt + \sigma_X dW_X^{\mathbb{Q}} + \sigma_f dW_f^{\mathbb{Q}} - \sigma_d dW_d^{\mathbb{Q}}]$$

Step 3: Find Girsanov Transformation for W_f

For $\frac{Y_t}{S_d}$ to be martingale, matching coefficients gives:

$$dW_X^{S_d} = dW_X^{\mathbb{Q}}, \quad dW_f^{S_d} = dW_f^{\mathbb{Q}} - \rho\sigma_d dt$$

Step 4: Transform Foreign Asset Dynamics

Starting with $dS_f = (r_f - \rho_X\sigma_X\sigma_f)S_f dt + \sigma_f S_f dW_f^{\mathbb{Q}}$:

Substitute $dW_f^{\mathbb{Q}} = dW_f^{S_d} + \rho\sigma_d dt$:

$$dS_f = (r_f - \rho_X\sigma_X\sigma_f + \rho\sigma_d\sigma_f)S_f dt + \sigma_f S_f dW_f^{S_d}$$

Final Answer:

$$dS_f = (r_f - \rho_X\sigma_X\sigma_f + \rho\sigma_d\sigma_f)S_f dt + \sigma_f S_f dW_f^{S_d}$$

Problem 4

Analytical Black-Scholes Formula for Quanto Options

Question:

Given the quanto option pricing formula under the S_d -measure:

$$V(0) = S_d(0) \cdot \mathbb{E}^{S_d} [\max(S_f(T) - K, 0)]$$

and the foreign asset dynamics under \mathbb{Q}^{S_d} :

$$dS_f = (r_f - \rho_X\sigma_X\sigma_f + \rho\sigma_d\sigma_f)S_f dt + \sigma_f S_f dW_f^{S_d}$$

Derive the analytical Black-Scholes formula for the quanto option price.

Complete Solution

Step 1: Identify the Modified Drift Rate

Under the S_d -measure \mathbb{Q}^{S_d} , the foreign asset $S_f(t)$ has drift:

$$r_{\text{adjusted}} = r_f - \rho_X \sigma_X \sigma_f + \rho \sigma_d \sigma_f$$

This adjusted drift accounts for:

- r_f : Foreign risk-free rate
- $-\rho_X \sigma_X \sigma_f$: Quanto adjustment (correlation with exchange rate)
- $+\rho \sigma_d \sigma_f$: Numeraire adjustment (correlation with domestic asset)

The volatility remains σ_f (unchanged under measure change).

Step 2: Apply the Lognormal Distribution

Under \mathbb{Q}^{S_d} , $S_f(T)$ is lognormally distributed with:

$$S_f(T) = S_f(0) \exp \left(\left(r_{\text{adjusted}} - \frac{1}{2} \sigma_f^2 \right) T + \sigma_f \sqrt{T} Z \right)$$

where $Z \sim \mathcal{N}(0, 1)$ under \mathbb{Q}^{S_d} .

The expectation $\mathbb{E}^{S_d}[\max(S_f(T) - K, 0)]$ is a standard European call option under a measure where the forward price grows at rate r_{adjusted} .

Step 3: Derive the Black-Scholes Formula

The expected payoff of a call option under the modified measure is given by the Black-Scholes formula:

$$\mathbb{E}^{S_d}[\max(S_f(T) - K, 0)] = S_f(0) e^{r_{\text{adjusted}} T} \Phi(d_1) - K \Phi(d_2)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and:

$$d_1 = \frac{\ln\left(\frac{S_f(0)}{K}\right) + \left(r_{\text{adjusted}} + \frac{1}{2}\sigma_f^2\right)T}{\sigma_f\sqrt{T}}$$

$$d_2 = d_1 - \sigma_f\sqrt{T}$$

Step 4: Multiply by Domestic Asset Price

The quanto option value at time 0 is:

$$V(0) = S_d(0) \cdot \mathbb{E}^{S_d}[\max(S_f(T) - K, 0)]$$

Substituting the Black-Scholes formula:

$$V(0) = S_d(0) [S_f(0)e^{r_{\text{adjusted}}T}\Phi(d_1) - K\Phi(d_2)]$$

Step 5: Complete Formula with All Parameters

The complete analytical pricing formula for a quanto call option is:

$$V(0) = S_d(0) \cdot S_f(0)e^{r_{\text{adjusted}}T}\Phi(d_1) - S_d(0) \cdot K\Phi(d_2)$$

where:

$$r_{\text{adjusted}} = r_f - \rho_X\sigma_X\sigma_f + \rho\sigma_d\sigma_f$$

$$d_1 = \frac{\ln(S_f(0)/K) + (r_{\text{adjusted}} + \frac{1}{2}\sigma_f^2)T}{\sigma_f\sqrt{T}}$$

$$d_2 = d_1 - \sigma_f\sqrt{T}$$

Implementation Note: In the provided Python code:

```
# Adjusted drift under S_d measure
r_adjusted = r_f - rho_X * sigma_X * sigma_f + rho * sigma_d * sigma_f

# Black-Scholes d1 and d2
d1 = (np.log(S_f_0 / K) + (r_adjusted + 0.5 * sigma_f**2) * T) / (sigma_f
```

```
d2 = d1 - sigma_f * np.sqrt(T)

# Black-Scholes call under adjusted measure
bs_call = S_f_0 * np.exp(r_adjusted * T) * norm.cdf(d1) - K * norm.cdf(d2)

# Multiply by domestic asset price
V_0 = S_d_0 * bs_call
```

Final Analytical Formula:

$$V(0) = S_d(0) \left[S_f(0) e^{(r_f - \rho_{Xf} \sigma_X \sigma_f + \rho \sigma_d \sigma_f)T} \Phi(d_1) - K \Phi(d_2) \right]$$