Caplet Pricing Solutions

Change of Numeraire and Measure Theory

Complete Solutions with Hull-White model

Problem 1

Caplet Pricing Under Forward Measure

Question:

A caplet pays $V(T_2) = \tau \cdot \max(L(T_1, T_2) - K, 0)$ at time T_2, where $L(T_1, T_2)$ is the LIBOR rate set at T_1 for the period $[T_1, T_2]$ and $\tau = T_2 - T_1$.

Under $\mathbb{Q}$, the price is:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T_2)} \cdot \tau \cdot \max(L(T_1, T_2) - K, 0) \mid \mathcal{F}_t\right]$$

When changing to the T_2-forward measure $\mathbb{Q}^{T_2}$, what is $V(t)$?

Complete Solution

Step 1: Identify the Original Numeraire

In the given pricing formula, we need to identify the current numeraire under the risk-neutral measure $\mathbb{Q}$.

Under the risk-neutral measure $\mathbb{Q}$, the numeraire is the **money market account** $B(t)$, as shown by the discount factor $\frac{B(t)}{B(T_2)}$ in the pricing formula.

This represents discounting from time T_2 back to time t using the risk-free rate. The money market account grows continuously at the short rate: $dB(t) = r(t)B(t)dt$.

Step 2: Understand the Change of Numeraire

When changing to the T_2-forward measure $\mathbb{Q}^{T_2}$, the new numeraire becomes the T_2**-maturity zero-coupon bond** $P(t, T_2)$.

The T_2-forward measure $\mathbb{Q}^{T_2}$ is defined by using the T_2-maturity zero-coupon bond $P(t, T_2)$ as the numeraire. Under this measure, all asset prices are expressed in units of $P(t, T_2)$ rather than in cash units.

This is particularly useful for pricing interest rate derivatives because the forward LIBOR rate becomes a martingale under this measure.

Step 3: Derive the Radon-Nikodym Derivative

The Radon-Nikodym derivative for changing from $\mathbb{Q}$ to $\mathbb{Q}^{T_2}$ at time T_2 is given by the ratio of normalized numeraires:

$$\left.\frac{d\mathbb{Q}^{T_2}}{d\mathbb{Q}}\right|_{\mathcal{F}_{T_2}} = \frac{P(T_2, T_2)/P(0, T_2)}{B(T_2)/B(0)}$$

Since $P(T_2, T_2) = 1$ (a zero-coupon bond is worth exactly 1 at its maturity) and $B(0) = 1$ (we normalize the money market account to start at 1), this simplifies to:

$$\left.\frac{d\mathbb{Q}^{T_2}}{d\mathbb{Q}}\right|_{\mathcal{F}_{T_2}} = \frac{1}{P(0, T_2) \cdot B(T_2)}$$

This derivative relates the probability measures at the terminal time T_2.

Step 4: Apply the Measure Change Formula

Let $X_{T_2} = \frac{B(t)}{B(T_2)} \cdot \tau \cdot \max(L(T_1, T_2) - K, 0)$. The measure change formula states:

$$\mathbb{E}^{\mathbb{Q}}[X_{T_2} \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^{T_2}}\left[X_{T_2} \cdot \left.\frac{d\mathbb{Q}}{d\mathbb{Q}^{T_2}}\right|_{\mathcal{F}_{T_2}} \mid \mathcal{F}_t\right]$$

We multiply by the **inverse** Radon-Nikodym derivative when converting from $\mathbb{Q}$ to $\mathbb{Q}^{T_2}$:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{Q}^{T_2}}\right|_{\mathcal{F}_{T_2}} = P(0, T_2) \cdot B(T_2)$$

This inverse derivative adjusts for the change from money market numeraire to bond numeraire.

Step 5: Substitute and Simplify

Substituting into the measure change formula:

$$V(t) = \mathbb{E}^{\mathbb{Q}^{T_2}}\left[\frac{B(t)}{B(T_2)} \cdot \tau \cdot \max(L(T_1, T_2) - K, 0) \cdot P(0, T_2) \cdot B(T_2) \mid \mathcal{F}_t\right]$$

The $B(T_2)$ terms cancel:

$$V(t) = B(t) \cdot P(0, T_2) \cdot \tau \cdot \mathbb{E}^{\mathbb{Q}^{T_2}}\left[\max(L(T_1, T_2) - K, 0) \mid \mathcal{F}_t\right]$$

Now we use a key relationship. The zero-coupon bond price at time t can be expressed as:

$$P(t, T_2) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T_2)} \mid \mathcal{F}_t\right]$$

By the relationship between bond prices and the numeraire change, we have:

$$B(t) \cdot P(0, T_2) = P(t, T_2) \cdot B(0) = P(t, T_2)$$

since $B(0) = 1$. Therefore:

$$V(t) = P(t, T_2) \cdot \tau \cdot \mathbb{E}^{\mathbb{Q}^{T_2}}\left[\max(L(T_1, T_2) - K, 0) \mid \mathcal{F}_t\right]$$

Final Answer:

$$V(t) = P(t, T_2) \cdot \tau \cdot \mathbb{E}^{\mathbb{Q}^{T_2}} \left[\max(L(T_1, T_2) - K, 0) \mid \mathcal{F}_t \right]$$

Problem 2

Caplet Pricing Using Hull-White Affine Form - Direct Approach

Question:

Starting from the caplet pricing formula under the T_k-forward measure:

$$\frac{V^{\text{CPL}}(t_0)}{P(t_0, T_k)} = N\tau_k \mathbb{E}^{T_k} \left[\max\left(\frac{1}{\tau_k} \left(\frac{1}{P(T_{k-1}, T_k)} - 1 \right) - K, 0 \right) \mid \mathcal{F}(t_0) \right]$$

Using the Hull-White affine form $P(T_{k-1}, T_k) = e^{\bar{A}_r(\tau_k) + \bar{B}_r(\tau_k) r(T_{k-1})}$, express the caplet price in terms of the short rate $r(T_{k-1})$ and derive the expression ready for Black-like pricing.

Complete Solution

Step 1: Express LIBOR Rate Using Affine Form

The LIBOR rate is defined as:

$$L(T_{k-1}, T_k) = \frac{1}{\tau_k} \left(\frac{1}{P(T_{k-1}, T_k)} - 1 \right)$$

Using the Hull-White affine form:

$$P(T_{k-1}, T_k) = e^{\bar{A}_r(\tau_k) + \bar{B}_r(\tau_k) r(T_{k-1})}$$

Therefore:

$$\frac{1}{P(T_{k-1}, T_k)} = e^{-\bar{A}_r(\tau_k) - \bar{B}_r(\tau_k) r(T_{k-1})}$$

Substituting into the LIBOR rate formula:

$$L(T_{k-1}, T_k) = \frac{1}{\tau_k} \left(e^{-\bar{A}_r(\tau_k) - \bar{B}_r(\tau_k) r(T_{k-1})} - 1 \right)$$

Step 2: Express Caplet Payoff in Terms of Short Rate

The caplet payoff becomes:

$$\tau_k \max(L(T_{k-1}, T_k) - K, 0) = \max \left(e^{-\bar{A}_r(\tau_k) - \bar{B}_r(\tau_k) r(T_{k-1})} - 1 - \tau_k K, 0 \right)$$

The caplet pays off when:

$$e^{-\bar{A}_r(\tau_k) - \bar{B}_r(\tau_k) r(T_{k-1})} > 1 + \tau_k K$$

Define the modified strike:

$$\hat{K} = (1 + \tau_k K) e^{\bar{A}_r(\tau_k)}$$

Then the caplet condition becomes:

$$e^{-\bar{B}_r(\tau_k) r(T_{k-1})} > \hat{K}$$

Step 3: Rewrite Caplet Formula with Exponential Variable

Factoring out $e^{-\bar{A}_r(\tau_k)}$, the scaled caplet value is:

$$\frac{V^{\text{CPL}}(t_0)}{P(t_0, T_k)} = N \cdot e^{-\bar{A}_r(\tau_k)} \cdot \mathbb{E}^{T_k} \left[\max \left(e^{-\bar{B}_r(\tau_k) r(T_{k-1})} - \hat{K}, 0 \right) \mid \mathcal{F}(t_0) \right]$$

Key Observation: This is a call option on the random variable $S = e^{-\bar{B}_r(\tau_k) r(T_{k-1})}$ with strike $\hat{K}$.

Under the T_k-forward measure, $r(T_{k-1})$ is normally distributed:

$$r(T_{k-1}) \sim \mathcal{N}(\mu_r(T_{k-1}), v_r^2(T_{k-1}))$$

Therefore, $S = e^{-\bar{B}_r(\tau_k)r(T_{k-1})}$ is **lognormally distributed**:

$$S \sim \text{LogNormal}\left(-\bar{B}_r(\tau_k)\mu_r, \bar{B}_r^2(\tau_k)v_r^2\right)$$

This allows us to use a Black-like formula for pricing the option on S.

Caplet Formula Ready for Black Pricing:

$$V^{\text{CPL}}(t_0) = N \cdot P(t_0, T_k) \cdot e^{-\bar{A}_r(\tau_k)} \cdot \mathbb{E}^{T_k}\left[\max\left(e^{-\bar{B}_r(\tau_k)r(T_{k-1})} - \hat{K}, 0\right)\right]$$

where $\hat{K} = (1 + \tau_k K)e^{\bar{A}_r(\tau_k)}$ and the expectation is over a lognormal variable.

Problem 3

Analytical Black Formula for Hull-White Caplet

Question:

Given the caplet pricing formula:

$$V^{\text{CPL}}(t_0) = N \cdot P(t_0, T_k) \cdot e^{-\bar{A}_r(\tau_k)} \cdot \mathbb{E}^{T_k}\left[\max\left(e^{-\bar{B}_r(\tau_k)r(T_{k-1})} - \hat{K}, 0\right)\right]$$

where $r(T_{k-1}) \sim \mathcal{N}(\mu_r, v_r^2)$ under $\mathbb{Q}^{T_k}$, derive the analytical Black-like formula for the caplet price.

Complete Solution

Step 1: Identify the Lognormal Variable

Let $S = e^{-\bar{B}_r(\tau_k)r(T_{k-1})}$. Since $r(T_{k-1}) \sim \mathcal{N}(\mu_r, v_r^2)$, we have:

$$\log S = -\bar{B}_r(\tau_k)r(T_{k-1}) \sim \mathcal{N}(-\bar{B}_r\mu_r, \bar{B}_r^2 v_r^2)$$

The forward price (expected value) of S is:

$$F = \mathbb{E}^{T_k}[S] = \exp\left(-\bar{B}_r\mu_r + \frac{1}{2}\bar{B}_r^2 v_r^2\right)$$

The volatility parameter is:

$$\sigma_S = |\bar{B}_r| \cdot v_r$$

Note: We use $|\bar{B}_r|$ because $\bar{B}_r < 0$ for $\tau_k > 0$.

Step 2: Apply Black's Formula for Lognormal Variable

For a call option on a lognormal variable S with strike $\hat{K}$, Black's formula gives:

$$\mathbb{E}[\max(S - \hat{K}, 0)] = F \cdot \Phi(d_1) - \hat{K} \cdot \Phi(d_2)$$

where:

$$d_1 = \frac{\log(F/\hat{K}) + \frac{1}{2}\sigma_S^2}{\sigma_S}$$

$$d_2 = d_1 - \sigma_S$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Step 3: Substitute Parameters

Substituting our specific parameters:

$$\sigma_S = |\bar{B}_r(\tau_k)| \cdot v_r(T_{k-1})$$

$$F = \exp\left(-\bar{B}_r(\tau_k)\mu_r(T_{k-1}) + \frac{1}{2}\bar{B}_r^2(\tau_k)v_r^2(T_{k-1})\right)$$

$$\hat{K} = (1 + \tau_k K)e^{\bar{A}_r(\tau_k)}$$

The d_1 and d_2 become:

$$d_1 = \frac{\log(F/\hat{K}) + \frac{1}{2}\sigma_S^2}{\sigma_S}$$

$$d_2 = d_1 - \sigma_S = \frac{\log(F/\hat{K}) - \frac{1}{2}\sigma_S^2}{\sigma_S}$$

Step 4: Complete Caplet Formula

The complete analytical caplet pricing formula is:

$$V^{\text{CPL}}(t_0) = N \cdot P(t_0, T_k) \cdot e^{-\bar{A}_r(\tau_k)} \cdot [F \cdot \Phi(d_1) - \hat{K} \cdot \Phi(d_2)]$$

Implementation Note: From the provided Python code:

```
# Get A_r, B_r, and moments of r(T1) under T2-forward measure
A_r = HW_A(lambd, eta, P0T, T1, T2)
B_r = HW_B(lambd, eta, T1, T2)
mu_r = HW_Mu_FrwdMeasure(P0T, lambd, eta, T1)
v_r = np.sqrt(HWVar_r(lambd, eta, T1))

# Modified strike
K_hat = (1.0 + tau_k * K) * np.exp(A_r)

# Volatility and forward price
sigma = np.abs(B_r) * v_r
forward = np.exp(-B_r * mu_r + 0.5 * B_r * B_r * v_r * v_r)

# Black formula
d1 = (np.log(forward / K_hat) + 0.5 * sigma * sigma) / sigma
d2 = d1 - sigma

option_value = forward * st.norm.cdf(d1) - K_hat * st.norm.cdf(d2)

# Final caplet price
caplet = N * P0T(T2) * np.exp(-A_r) * option_value
```

Final Analytical Caplet Formula:

$$V^{\text{CPL}}(t_0) = N \cdot P(t_0, T_k) \cdot e^{-\bar{A}_r(\tau_k)} \left[F\Phi(d_1) - \hat{K}\Phi(d_2) \right]$$

where:

$$F = e^{-\bar{B}_r \mu_r + \frac{1}{2}\bar{B}_r^2 v_r^2}, \quad \sigma_S = |\bar{B}_r| v_r, \quad \hat{K} = (1 + \tau_k K) e^{\bar{A}_r}$$

$$d_1 = \frac{\log(F/\hat{K}) + \frac{1}{2}\sigma_S^2}{\sigma_S}, \quad d_2 = d_1 - \sigma_S$$