Power Options Solutions

Change of Numeraire and Measure Theory

Complete Solutions with Detailed Derivations

Problem 1

Pricing Under a New Measure: Power Options

Question:

Under the Black-Scholes model with $dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t)$, $S(0) = S_0$, and savings account $M(t) = e^{rt}$, price the following derivative at time t_0:

$$V(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)} \cdot (S(T)^2 - KS(T))^+ \mid \mathcal{F}_{t_0}\right]$$

When changing to the S-measure (where $S(t)$ is the numéraire), what is $V(t_0)$?

Complete Solution

Step 1: Identify the Original Numeraire

In the given pricing formula, we need to identify the current numeraire under the risk-neutral measure $\mathbb{Q}$.

Under the risk-neutral measure $\mathbb{Q}$, the numeraire is the **money market account** $M(t) = e^{rt}$, as shown by the discount factor $\frac{1}{M(T)} = e^{-rT}$ in the pricing formula.

This represents discounting from time T back to time t_0 using the risk-free rate r
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Step 2: Understand the Change of Numeraire

When changing to the S-measure $\mathbb{Q}^S$, the new numeraire becomes the **stock price** $S(t)$.

The S-measure $\mathbb{Q}^S$ is defined by using the stock price $S(t)$ as the numeraire. Under this measure, all asset prices are expressed in units of the stock rather than in cash units. This is also called the stock measure.

Step 3: Derive the Radon-Nikodym Derivative

The Radon-Nikodym derivative for changing from $\mathbb{Q}$ to $\mathbb{Q}^S$ at time T is given by the ratio of normalized numeraires:

$$\left.\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \frac{S(T)/S(0)}{M(T)/M(0)}$$

Since $M(0) = e^{r\cdot 0} = 1$ and $M(T) = e^{rT}$, this simplifies to:

$$\left.\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \frac{S(T)}{S(0)\cdot e^{rT}}$$

Step 4: Apply the Measure Change Formula

Let $X_T = \frac{1}{M(T)} \cdot (S(T)^2 - KS(T))^+$. The measure change formula states:

$$\mathbb{E}^{\mathbb{Q}}[X_T \mid \mathcal{F}_{t_0}] = \mathbb{E}^{\mathbb{Q}^S}\left[X_T \cdot \left.\frac{d\mathbb{Q}}{d\mathbb{Q}^S}\right|_{\mathcal{F}_T} \mid \mathcal{F}_{t_0}\right]$$

We multiply by the **inverse** Radon-Nikodym derivative when converting from $\mathbb{Q}$ to $\mathbb{Q}^S$:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S}\bigg|_{\mathcal{F}_T} = \frac{S(0)\cdot e^{rT}}{S(T)}$$

Step 5: Substitute and Simplify the Payoff

Substituting into the measure change formula:

$$V(t_0) = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{1}{e^{rT}}\cdot (S(T)^2 - KS(T))^+ \cdot \frac{S(0)\cdot e^{rT}}{S(T)} \mid \mathcal{F}_{t_0}\right]$$

The exponentials cancel:

$$V(t_0) = \mathbb{E}^{\mathbb{Q}^S}\left[S(0)\cdot \frac{(S(T)^2 - KS(T))^+}{S(T)} \mid \mathcal{F}_{t_0}\right]$$

Now factor out $S(T)$ from the payoff. Since $S(T) > 0$, we can write:

$$(S(T)^2 - KS(T))^+ = (S(T)(S(T)-K))^+ = S(T)\cdot (S(T)-K)^+$$

Substituting this factorization:

$$V(t_0) = \mathbb{E}^{\mathbb{Q}^S}\left[S(0)\cdot \frac{S(T)\cdot (S(T)-K)^+}{S(T)} \mid \mathcal{F}_{t_0}\right]$$

Simplifying:

$$V(t_0) = S(0)\cdot \mathbb{E}^{\mathbb{Q}^S}\left[(S(T)-K)^+ \mid \mathcal{F}_{t_0}\right]$$

Since $S(0) = S(t_0)$ (the stock price at time t_0), we obtain:

$$V(t_0) = S(t_0)\cdot \mathbb{E}^{\mathbb{Q}^S}\left[(S(T)-K)^+ \mid \mathcal{F}_{t_0}\right]$$

Final Answer:

$$V(t_0) = S(t_0)\cdot \mathbb{E}^{\mathbb{Q}^S}\left[(S(T)-K)^+ \mid \mathcal{F}_{t_0}\right]$$

This expresses the power option price under the stock measure, where the quadratic payoff $(S(T)^2 - KS(T))^+$ reduces to $S(t_0)$ times a standard call option payoff.

Problem 2

Dynamics Under a New Measure: Stock Under Stock Measure

Question:

Under the risk-neutral measure $\mathbb{Q}$, a stock has dynamics:

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t)$$

and the money market account satisfies:

$$dB(t) = rB(t)dt$$

What is the dynamics of $S(t)$ under the stock measure $\mathbb{Q}^S$ where $S(t)$ is the numeraire?

Complete Solution

Step 1: Set Up Itô's Lemma for $\frac{B(t)}{S(t)}$

To find the dynamics under the stock measure $\mathbb{Q}^S$, we need $\frac{B(t)}{S(t)}$ to be a martingale. Let $f(B, S) = \frac{B}{S}$. We compute the partial derivatives:

$$\frac{\partial f}{\partial B} = \frac{1}{S}$$

$$\frac{\partial f}{\partial S} = -\frac{B}{S^2}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{2B}{S^3}$$

Step 2: Apply Itô's Lemma

Given $dB(t) = rB(t)dt$ and $dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t)$, and using $(dS)^2 = \sigma^2 S^2 dt$, Itô's lemma gives:

$$df = \frac{\partial f}{\partial B}dB + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2$$

Substituting the partial derivatives:

$$d\left(\frac{B}{S}\right) = \frac{1}{S}(rBdt) - \frac{B}{S^2}(rSdt + \sigma S dW^{\mathbb{Q}}) + \frac{1}{2}\frac{2B}{S^3}(\sigma^2 S^2 dt)$$

Simplifying term by term:

$$= \frac{rB}{S}dt - \frac{rB}{S}dt - \frac{B\sigma}{S}dW^{\mathbb{Q}} + \frac{B\sigma^2}{S}dt$$

$$= \frac{B}{S}\left[\sigma^2 dt - \sigma dW^{\mathbb{Q}}(t)\right]$$

Step 3: Apply Girsanov's Theorem to Remove Drift

Since $\frac{B(t)}{S(t)}$ must be a martingale under $\mathbb{Q}^S$, the drift term must vanish. We can write the dynamics under $\mathbb{Q}^S$ as:

$$d\left(\frac{B}{S}\right) = \frac{B}{S}\left[-\sigma dW^S(t)\right]$$

Comparing with our result from Step 2:

$$\frac{B}{S}\left[-\sigma dW^S\right] = \frac{B}{S}\left[\sigma^2 dt - \sigma dW^{\mathbb{Q}}\right]$$

Dividing both sides by $-\sigma\frac{B}{S}$:

$$dW^S = -\sigma dt + dW^{\mathbb{Q}}$$

Therefore, by Girsanov's theorem:

$$dW^S(t) = dW^{\mathbb{Q}}(t) - \sigma dt$$

Equivalently:

$$dW^{\mathbb{Q}}(t) = dW^S(t) + \sigma dt$$

Step 4: Derive Stock Dynamics Under $\mathbb{Q}^S$

Starting with the stock dynamics under $\mathbb{Q}$:

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t)$$

Substitute $dW^{\mathbb{Q}}(t) = dW^S(t) + \sigma dt$:

$$dS(t) = rS(t)dt + \sigma S(t)\left[dW^S(t) + \sigma dt\right]$$

Expanding:

$$dS(t) = rS(t)dt + \sigma S(t)dW^S(t) + \sigma^2 S(t)dt$$

Combining the drift terms:

$$dS(t) = (r + \sigma^2)S(t)dt + \sigma S(t)dW^S(t)$$

Final Answer:

$$dS(t) = (r + \sigma^2)S(t)dt + \sigma S(t)dW^S(t)$$

Under the stock measure $\mathbb{Q}^S$, the stock's drift increases by σ^2 compared to the risk-neutral measure. The volatility term remains unchanged at $\sigma S(t)dW^S(t)$.

Problem 3

Analytical Black-Scholes Formula for Power Options

Question:

Given the power option pricing formula under the S-measure:

$$V(0) = S(0) \cdot \mathbb{E}^{\mathbb{Q}^S}[(S(T) - K)^+ \mid \mathcal{F}_0]$$

and the stock dynamics under $\mathbb{Q}^S$:

$$dS(t) = (r + \sigma^2)S(t)dt + \sigma S(t)dW^S(t)$$

Derive the analytical Black-Scholes formula for the power option with payoff $(S(T)^2 - KS(T))^+$.

Complete Solution

Step 1: Identify the Adjusted Drift Rate

Under the S-measure $\mathbb{Q}^S$, the stock $S(t)$ has drift:

$$r_{\text{adjusted}} = r + \sigma^2$$

This adjusted drift accounts for the change of numeraire from the money market account to the stock itself. The additional σ^2 term comes from the Girsanov transformation:

$$dW^S = dW^{\mathbb{Q}} - \sigma dt$$

The volatility remains σ (unchanged under measure change).

Step 2: Express Stock Price at Maturity

Under $\mathbb{Q}^S$, $S(T)$ follows:

$$S(T) = S(0) \exp\left(\left(r + \sigma^2 - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)$$
$$= S(0) \exp\left(\left(r + \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)$$

where $Z \sim \mathcal{N}(0,1)$ under $\mathbb{Q}^S$.

The expectation $\mathbb{E}^S[(S(T) - K)^+]$ is a standard European call option under the modified measure with adjusted drift.

Step 3: Apply Black-Scholes Formula Under $\mathbb{Q}^S$

The expected payoff of a call option under the S-measure is:

$$\mathbb{E}^S[(S(T) - K)^+] = S(0)e^{(r+\sigma^2)T}\Phi(d_1) - K\Phi(d_2)$$

where:

$$d_1 = \frac{\ln(S(0)/K) + (r + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(S(0)/K) + (r + \sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Step 4: Complete Power Option Formula

The power option value at time 0 is:

$$V(0) = S(0) \cdot \mathbb{E}^S[(S(T) - K)^+]$$

Substituting the Black-Scholes formula:

$$V(0) = S(0)\left[S(0)e^{(r+\sigma^2)T}\Phi(d_1) - K\Phi(d_2)\right]$$
$$= S(0)^2 e^{(r+\sigma^2)T}\Phi(d_1) - S(0)K\Phi(d_2)$$

Python Implementation:

```
# Adjusted drift under S-measure: r + 0.5*sigma^2
r_adjusted = r + 0.5 * sigma**2

# Calculate d1 and d2
d1 = (np.log(S_0/K) + (r_adjusted + 0.5*sigma**2)*T) / (sigma*np.sqrt(T))
d2 = d1 - sigma*np.sqrt(T)

# Black-Scholes call under S-measure
# E^S[(S(T) - K)^+] = S_0*exp((r + sigma^2)*T)*N(d1) - K*N(d2)
bs_call = S_0 * np.exp((r + sigma**2)*T) * norm.cdf(d1) - K * norm.cdf(d2)

# Power option value: V(0) = S(0) * E^S[(S(T) - K)^+]
V_0 = S_0 * bs_call
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Final Analytical Formula:

$$V(0) = S(0)^2 e^{(r+\sigma^2)T}\Phi(d_1) - S(0)K\Phi(d_2)$$

where:

$$d_1 = \frac{\ln(S(0)/K) + (r+\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$