# Zero-Coupon Bond Option Pricing

*Change of Numeraire and Measure Theory*

*Complete Solutions with Hull-White Model*

Problem 1

## ZCB Option Pricing Under S-Forward Measure

**Question:**

Let $P(t,T)$ be the price of a zero-coupon bond. Under the risk-neutral measure $\mathbb{Q}$, a call option on this bond with strike $K$ and maturity $S < T$ is priced as:

$$C(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(S)}\max(P(S,T) - K, 0) \mid \mathcal{F}_t\right]$$

Using the change of numeraire theorem, when changing to the $S$-forward measure $\mathbb{Q}^S$, what is $C(t)$?

## Complete Solution

### Step 1: Identify the Original Numeraire

Under the risk-neutral measure $\mathbb{Q}$, the numeraire is the **money market account** $B(t)$, as shown by the discount factor $\frac{B(t)}{B(S)}$ in the pricing formula.

This represents discounting from time $S$ back to time $t$ using the risk-free rate.

### Step 2: Understand the Change of Numeraire

The $S$-forward measure $\mathbb{Q}^S$ is defined by using the **$S$-maturity zero-coupon bond** $P(t,S)$ as the numeraire.

Under this measure, all asset prices are expressed in units of $P(t,S)$ rather than in cash units. This is the natural measure for pricing derivatives with payoffs at time $S$, such as this bond option.

### Step 3: Derive the Radon-Nikodym Derivative

The Radon-Nikodym derivative for changing from $\mathbb{Q}$ to $\mathbb{Q}^S$ at time $S$ is:

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\bigg|_{\mathcal{F}_S} = \frac{P(S,S)/P(0,S)}{B(S)/B(0)}$$

Since $P(S,S)=1$ (a bond is worth 1 at its maturity) and $B(0)=1$, this simplifies to:

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\bigg|_{\mathcal{F}_S} = \frac{1}{P(0,S)\cdot B(S)}$$

### Step 4: Apply the Measure Change Formula

Let $X_S = \frac{B(t)}{B(S)}\max(P(S,T)-K,0)$. The measure change formula states:

$$\mathbb{E}^{\mathbb{Q}}[X_S \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^S}\left[X_S \cdot \frac{d\mathbb{Q}}{d\mathbb{Q}^S}\bigg|_{\mathcal{F}_S} \mid \mathcal{F}_t\right]$$

We multiply by the **inverse** Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S}\bigg|_{\mathcal{F}_S} = P(0,S)\cdot B(S)$$

### Step 5: Substitute and Simplify

Substituting into the measure change formula:

$$C(t) = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{B(t)}{B(S)} \max(P(S,T) - K, 0) \cdot P(0,S) \cdot B(S) \mid \mathcal{F}_t\right]$$

The $B(S)$ terms cancel:

$$C(t) = B(t) \cdot P(0,S) \cdot \mathbb{E}^{\mathbb{Q}^S}\left[\max(P(S,T) - K, 0) \mid \mathcal{F}_t\right]$$

By the relationship between bond prices and the numeraire, $P(t,S) = \mathbb{E}^{\mathbb{Q}}[\frac{B(t)}{B(S)} \mid \mathcal{F}_t]$ and we have $B(t) \cdot P(0,S) = P(t,S) \cdot B(0) = P(t,S)$ (since $B(0) = 1$).

Therefore:

$$C(t) = P(t,S) \cdot \mathbb{E}^{\mathbb{Q}^S}\left[\max(P(S,T) - K, 0) \mid \mathcal{F}_t\right]$$

## Final Answer:

$$C(t) = P(t,S) \cdot \mathbb{E}^{\mathbb{Q}^S}\left[\max(P(S,T) - K, 0) \mid \mathcal{F}_t\right]$$

This is Black's formula for bond option pricing: discount by $P(t,S)$ and take expectation under $\mathbb{Q}^S$ where the forward bond price $\frac{P(t,T)}{P(t,S)}$ is a martingale.

### Key Insight:

This measure is preferred because it eliminates stochastic interest rate discounting and makes the forward bond price a martingale, simplifying pricing to a Black-Scholes-type formula.

Problem 2

# ZCB Option with Hull-White Affine Form - Direct Approach

**Question:**

Starting from the ZCB option pricing formula under the $T$-forward measure:

$$V^{\text{ZCB}}(t_0, T) = P(t_0, T)\mathbb{E}^T\left[\max\left(\tilde{\alpha}\left(e^{\bar{A}_r(\tau)+\bar{B}_r(\tau)r(T)} - K\right), 0\right) \mid \mathcal{F}(t_0)\right]$$

Using the Hull-White affine form, express the option price in terms of the short rate $r(T)$ ready for Black-like pricing.

## Complete Solution

### Step 1: Express ZCB Price Using Affine Form

In the Hull-White model, the zero-coupon bond price has an affine form:

$$P(T, T_S) = e^{\bar{A}_r(\tau)+\bar{B}_r(\tau)r(T)}$$

where $\tau = T_S - T$, and:

$$\bar{B}_r(\tau) = \frac{1}{\lambda}\left(e^{-\lambda\tau} - 1\right)$$

$$\bar{A}_r(\tau) = \lambda\int_0^\tau \theta(T_S - z)\bar{B}_r(z)dz + \frac{\eta^2}{4\lambda^3}\left(e^{-2\lambda\tau}(4e^{\lambda\tau} - 1) - 3\right) + \frac{\eta^2\tau}{2\lambda^2}$$

### Step 2: Set Up Option Payoff

The option payoff at time $T$ is:

$$\max(P(T,T_S)-K,0)=\max\left(e^{\bar{A}_r(\tau)+\bar{B}_r(\tau)r(T)}-K,0\right)$$

Define the modified strike:

$$\hat{K}=Ke^{-\bar{A}_r(\tau)}$$

Then the option condition becomes:

$$e^{\bar{B}_r(\tau)r(T)}>\hat{K}$$

### Step 3: Rewrite with Lognormal Variable

Factoring out $e^{\bar{A}_r(\tau)}$, the scaled option value is:

$$\frac{V^{\mathrm{ZCB}}(t_0,T)}{P(t_0,T)}=e^{\bar{A}_r(\tau)}\cdot\mathbb{E}^T\left[\max\left(e^{\bar{B}_r(\tau)r(T)}-\hat{K},0\right)\mid\mathcal{F}(t_0)\right]$$

**Key Observation:** This is a call option on the random variable $S=e^{\bar{B}_r(\tau)r(T)}$ with strike $\hat{K}$.

Under the $T$-forward measure, $r(T)$ is normally distributed:

$$r(T)\sim\mathcal{N}(\mu_r(T),v_r^2(T))$$

Therefore, $S=e^{\bar{B}_r(\tau)r(T)}$ is **lognormally distributed**, allowing Black-like pricing.

### ZCB Option Formula Ready for Black Pricing:

$$V^{\mathrm{ZCB}}(t_0,T)=P(t_0,T)\cdot e^{\bar{A}_r(\tau)}\cdot\mathbb{E}^T\left[\max\left(e^{\bar{B}_r(\tau)r(T)}-\hat{K},0\right)\right]$$

where $\hat{K}=Ke^{-\bar{A}_r(\tau)}$ and the expectation is over a lognormal variable.

Problem 3

# Analytical Black Formula for Hull-White ZCB Option

**Question:**

Derive the analytical Black-like formula for the ZCB option where $r(T) \sim \mathcal{N}(\mu_r, v_r^2)$.

## Complete Solution

### Step 1: Identify Lognormal Variable

Let $S = e^{\bar{B}_r r(T)} = P(T, T_S)$. Since $r \sim \mathcal{N}(\mu_r, v_r^2)$:

$$F = \exp(\bar{A}_r + \bar{B}_r \mu_r + \tfrac{1}{2}\bar{B}_r^2 v_r^2), \quad \sigma_S = |\bar{B}_r| v_r$$

### Step 2: Apply Black's Formula

For call on lognormal $S = e^{\bar{A}_r + \bar{B}_r r(T)}$ with strike $K$:

$$\mathbb{E}[\max(S - K, 0)] = F\Phi(d_1) - K\Phi(d_2)$$

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma_S^2}{\sigma_S}, \quad d_2 = d_1 - \sigma_S$$

### Step 3: Python Implementation

```
A_r = HW_A(lambd, eta, P0T, T1, T2)
B_r = HW_B(lambd, eta, T1, T2)
mu_r = HW_Mu_FrwdMeasure(P0T, lambd, eta, T1)
v_r = np.sqrt(HWVar_r(lambd, eta, T1))

# Forward price of ZCB
sigma = np.abs(B_r) * v_r
forward = np.exp(A_r + B_r*mu_r + 0.5*B_r**2*v_r**2)

# Black formula
d1 = (np.log(forward/K) + 0.5*sigma**2) / sigma
d2 = d1 - sigma

option_value = forward * norm.cdf(d1) - K * norm.cdf(d2)
zcb_call = P0T(T1) * option_value

# Put option via put-call parity
zcb_put = zcb_call - P0T(T2) + K * P0T(T1)
```

### Final Formula:

$$V^{\text{CALL}}(t_0) = P(t_0, T)[F\Phi(d_1) - K\Phi(d_2)]$$

$$V^{\text{PUT}}(t_0) = V^{\text{CALL}}(t_0) - P(t_0, T_S) + KP(t_0, T)$$