Exchange Options Pricing

Change of Numeraire and Measure Theory

Complete Solutions with Detailed Derivations

Problem 1

Pricing Under a New Measure: Exchange Options

Question:

An exchange option pays $V(T) = \max(S_1(T) - S_2(T), 0)$. Under the risk-neutral measure $\mathbb{Q}$, the price at time t is:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \cdot \max(S_1(T) - S_2(T), 0) \mid \mathcal{F}_t\right]$$

When changing to the measure $\mathbb{Q}^{S_2}$ induced by using $S_2(t)$ as numeraire, what is the correct result?

Complete Solution

Step 1: Identify the Original Numeraire

Under the risk-neutral measure $\mathbb{Q}$, the numeraire is the **money market account** $B(t) = e^{rt}$, as evidenced by the discount factor $e^{-r(T-t)} = \frac{B(t)}{B(T)}$ in the pricing formula.

Step 2: Understand the Change of Numeraire

When changing to $\mathbb{Q}^{S_2}$, the new numeraire becomes $S_2(t)$. Under this measure, all asset prices are expressed in units of $S_2(t)$ rather than cash.

Step 3: Derive the Radon-Nikodym Derivative

The Radon-Nikodym derivative at time T is:

$$\left.\frac{d\mathbb{Q}^{S_2}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \frac{S_2(T)/S_2(0)}{B(T)/B(0)} = \frac{S_2(T)}{S_2(0)} \cdot e^{-rT}$$

Step 4: Apply the Measure Change Formula

Let $X_T = e^{-r(T-t)} \cdot \max(S_1(T) - S_2(T), 0)$. The measure change formula gives:

$$V(t) = \mathbb{E}^{\mathbb{Q}^{S_2}}\left[X_T \cdot \left.\frac{d\mathbb{Q}}{d\mathbb{Q}^{S_2}}\right|_{\mathcal{F}_T} \mid \mathcal{F}_t\right]$$

where $\frac{d\mathbb{Q}}{d\mathbb{Q}^{S_2}}|_{\mathcal{F}_T} = \frac{S_2(0)}{S_2(T)} \cdot e^{rT}$.

Step 5: Substitute and Simplify

Substituting:

$$V(t) = \mathbb{E}^{\mathbb{Q}^{S_2}}\left[e^{-r(T-t)} \cdot \max(S_1(T) - S_2(T), 0) \cdot \frac{S_2(0)}{S_2(T)} \cdot e^{rT} \mid \mathcal{F}_t\right]$$

Combining exponentials: $e^{-r(T-t)} \cdot e^{rT} = e^{rt}$

$$V(t) = S_2(0)e^{rt} \cdot \mathbb{E}^{\mathbb{Q}^{S_2}}\left[\frac{\max(S_1(T) - S_2(T), 0)}{S_2(T)} \mid \mathcal{F}_t\right]$$

Rewriting: $\frac{\max(S_1(T)-S_2(T),0)}{S_2(T)} = \max\left(\frac{S_1(T)}{S_2(T)} - 1, 0\right)$

Since $S_2(0)e^{rt} = S_2(t)$:

Final Answer:

$$V(t) = S_2(t) \cdot \mathbb{E}^{\mathbb{Q}^{S_2}}\left[\max\left(\frac{S_1(T)}{S_2(T)} - 1, 0\right) \mid \mathcal{F}_t\right]$$

Problem 2

Dynamics Under a New Measure: Correlated Assets

Question:

Consider two tradeable assets $S_1(t)$ and $S_2(t)$ with correlation ρ. Under $\mathbb{Q}$:

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1^{\mathbb{Q}}(t)$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2^{\mathbb{Q}}(t)$$

where $dW_1^{\mathbb{Q}} \cdot dW_2^{\mathbb{Q}} = \rho dt$.

What are the dynamics of $S_1(t)$ under $\mathbb{Q}^{S_2}$ using $S_2(t)$ as numeraire?

Complete Solution

Step 1: Apply Itô's Lemma to $\frac{B(t)}{S_2(t)}$

Since $\frac{B(t)}{S_2(t)}$ must be a martingale under $\mathbb{Q}^{S_2}$, apply Itô's lemma with $dB(t) = rB(t)dt$:

$$d\left(\frac{B}{S_2}\right) = \frac{B}{S_2}\left[\sigma_2^2 dt - \sigma_2 dW_2^{\mathbb{Q}}\right]$$

Step 2: Find Girsanov Transformation for W_2

For martingale property, under $\mathbb{Q}^{S_2}$: $d\left(\frac{B}{S_2}\right) = -\sigma_2 \frac{B}{S_2} dW_2^{S_2}$

Equating and solving:

$$dW_2^{S_2} = dW_2^{\mathbb{Q}} - \sigma_2 dt$$

Step 3: Apply Itô's Lemma to $\frac{S_1(t)}{S_2(t)}$

For $\frac{S_1}{S_2}$ martingale under $\mathbb{Q}^{S_2}$, using $dS_1 \cdot dS_2 = \rho\sigma_1\sigma_2 S_1 S_2 dt$:

$$d\left(\frac{S_1}{S_2}\right) = \frac{S_1}{S_2}\left[(\sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_1^{\mathbb{Q}} - \sigma_2 dW_2^{\mathbb{Q}}\right]$$

Step 4: Find Girsanov Transformation for W_1

Under $\mathbb{Q}^{S_2}$: $d\left(\frac{S_1}{S_2}\right) = \frac{S_1}{S_2}\left[\sigma_1 dW_1^{S_2} - \sigma_2 dW_2^{S_2}\right]$

Substituting $dW_2^{S_2} = dW_2^{\mathbb{Q}} - \sigma_2 dt$ and equating:

$$\sigma_1 dW_1^{S_2} - \sigma_2 dW_2^{\mathbb{Q}} + \sigma_2^2 dt = (\sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_1^{\mathbb{Q}} - \sigma_2 dW_2^{\mathbb{Q}}$$

Therefore:

$$dW_1^{S_2} = dW_1^{\mathbb{Q}} - \rho\sigma_2 dt$$

Step 5: Derive Dynamics of $S_1(t)$ under $\mathbb{Q}^{S_2}$

Starting with $dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1^{\mathbb{Q}}(t)$

From $dW_1^{S_2} = dW_1^{\mathbb{Q}} - \rho\sigma_2 dt$, we get $dW_1^{\mathbb{Q}} = dW_1^{S_2} + \rho\sigma_2 dt$

Substituting:

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + \sigma_1 S_1(t)(dW_1^{S_2} + \rho\sigma_2 dt) \\ &= rS_1(t)dt + \sigma_1 S_1(t)dW_1^{S_2} + \rho\sigma_1\sigma_2 S_1(t)dt \\ &= (r + \rho\sigma_1\sigma_2)S_1(t)dt + \sigma_1 S_1(t)dW_1^{S_2} \end{aligned}$$

Final Answer:

$$dS_1(t) = (r + \rho\sigma_1\sigma_2)S_1(t)dt + \sigma_1 S_1(t)dW_1^{S_2}(t)$$

The drift adjustment $\rho\sigma_1\sigma_2$ reflects the correlation between the two assets.

Problem 3

Analytical Black-Scholes Formula: Margrabe's Formula

Question:

Given the exchange option pricing formula under the S_2-measure:

$$V(t) = S_2(t) \cdot \mathbb{E}^{\mathbb{Q}^{S_2}}\left[\max\left(\frac{S_1(T)}{S_2(T)} - 1, 0\right) \mid \mathcal{F}_t\right]$$

and the dynamics under $\mathbb{Q}^{S_2}$:

$$dS_1 = (r + \rho\sigma_1\sigma_2)S_1 dt + \sigma_1 S_1 dW_1^{S_2}$$

$$dS_2 = (r + \sigma_2^2)S_2 dt + \sigma_2 S_2 dW_2^{S_2}$$

Derive the analytical Black-Scholes formula (Margrabe's formula) for the exchange option.

Complete Solution

Step 1: Define the Ratio Process

Define the ratio $Y(t) = \frac{S_1(t)}{S_2(t)}$. The exchange option becomes a call option on $Y(T)$ with strike $K = 1$:

$$V(t) = S_2(t) \cdot \mathbb{E}^{\mathbb{Q}^{S_2}}[\max(Y(T) - 1, 0) \mid \mathcal{F}_t]$$

Initial value: $Y(0) = \frac{S_1(0)}{S_2(0)}$

Step 2: Derive Dynamics of the Ratio

Apply Itô's lemma to $Y = \frac{S_1}{S_2}$:

$$dY = \frac{1}{S_2}dS_1 - \frac{S_1}{S_2^2}dS_2 + \frac{S_1}{S_2^3}(dS_2)^2 - \frac{1}{S_2^2}dS_1 \cdot dS_2$$

With $(dS_2)^2 = \sigma_2^2 S_2^2 dt$ and $dS_1 \cdot dS_2 = \rho\sigma_1\sigma_2 S_1 S_2 dt$:

$$dY = Y\left[(r + \rho\sigma_1\sigma_2)dt + \sigma_1 dW_1^{S_2} - (r + \sigma_2^2)dt - \sigma_2 dW_2^{S_2} + \sigma_2^2 dt - \rho\sigma_1\sigma_2 dt\right]$$

Simplifying the drift terms:

$$\text{Drift} = r + \rho\sigma_1\sigma_2 - r - \sigma_2^2 + \sigma_2^2 - \rho\sigma_1\sigma_2 = 0$$

Therefore, $Y(t)$ **is a martingale** under $\mathbb{Q}^{S_2}$:

$$dY = Y\left[\sigma_1 dW_1^{S_2} - \sigma_2 dW_2^{S_2}\right]$$

Step 3: Calculate the Combined Volatility

The volatility of Y is:

$$\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

This comes from:

$$(\sigma_1 dW_1^{S_2} - \sigma_2 dW_2^{S_2})^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)dt$$

Thus, $Y(t)$ follows:

$$dY = \sigma_Y Y dW^{S_2}$$

where W^{S_2} is a Brownian motion under $\mathbb{Q}^{S_2}$.

Step 4: Apply Black-Scholes Formula with Zero Drift

Since Y is a martingale (zero drift), at time T:

$$Y(T) = Y(0)\exp\left(-\frac{1}{2}\sigma_Y^2 T + \sigma_Y\sqrt{T}Z\right)$$

where $Z \sim \mathcal{N}(0,1)$. For a call option on Y with strike $K = 1$:

$$\mathbb{E}^{S_2}[\max(Y(T) - 1, 0)] = Y(0)\Phi(d_1) - 1 \cdot \Phi(d_2)$$

where:

$$d_1 = \frac{\ln(Y(0)/1) + \frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T}}$$

$$d_2 = d_1 - \sigma_Y\sqrt{T} = \frac{\ln(Y(0)/1) - \frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T}}$$

Step 5: Complete Margrabe's Formula

Substituting $Y(0) = \frac{S_1(0)}{S_2(0)}$ and $K = 1$:

$$V(0) = S_2(0)\left[\frac{S_1(0)}{S_2(0)}\Phi(d_1) - \Phi(d_2)\right]$$

$$= S_1(0)\Phi(d_1) - S_2(0)\Phi(d_2)$$

This is **Margrabe's formula** for exchange options.

Python Implementation:

```
# Ratio and strike
Y_0 = S1_0 / S2_0
K = 1.0

# Combined volatility
sigma_Y = np.sqrt(sigma_1**2 + sigma_2**2 - 2*rho*sigma_1*sigma_2)

# Black-Scholes d1 and d2
d1 = (np.log(Y_0/K) + 0.5*sigma_Y**2*T) / (sigma_Y*np.sqrt(T))
d2 = d1 - sigma_Y*np.sqrt(T)

# Call value per unit of S_2
call_value_per_unit = Y_0 * norm.cdf(d1) - K * norm.cdf(d2)

# Full exchange option value
exchange_value = S2_0 * call_value_per_unit
# = S1_0 * norm.cdf(d1) - S2_0 * norm.cdf(d2)
```

Margrabe's Formula:

$$V(0) = S_1(0)\Phi(d_1) - S_2(0)\Phi(d_2)$$

where:

$$\sigma_Y = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

$$d_1 = \frac{\ln(S_1(0)/S_2(0)) + \frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T}}, \quad d_2 = d_1 - \sigma_Y\sqrt{T}$$